The Local Approach to Causal Inference under Network Interference

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July 28, 2023

Abstract

We propose a new nonparametric modeling framework for causal inference when outcomes depend on how agents are linked in a social or economic network. Such network interference describes a large literature on treatment spillovers, social interactions, social learning, information diffusion, disease and financial contagion, social capital formation, and more. Our approach works by first characterizing how an agent is linked in the network using the configuration of other agents and connections nearby as measured by path distance. The impact of a policy or treatment assignment is then learned by pooling outcome data across similarly configured agents. We demonstrate the approach by proposing an asymptotically valid test for the hypothesis of policy irrelevance/no treatment effects and deriving finite-sample bounds on the mean-squared error of a k-nearest-neighbor estimator for the average or distributional policy effect/treatment response.

1 Introduction

Economists are often tasked with predicting outcomes under a counterfactual policy or treatment assignment. In many cases, the counterfactual depends on how the agents are linked

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We thank Tim Armstrong, Stephane Bonhomme, Ivan Canay, Ben Golub, Joel Horowitz, Chuck Manski, Roger Moon, Seth Richards-Shubik, Azeem Shaikh, Chris Taber, Alex Torgovitsky and seminar participants at Cornell, Oxford, UCL, UC Boulder, U Rochester, U Glasgow, Laval, UW Madison, USC, Northwestern, U Chicago, TSE for helpful discussions. Research supported by NSF grants SES-2149408 and SES-2149422. This paper supersedes the earlier working paper A Nonparametric Network Regression.
in a social or economic network. A diverse literature on treatment spillovers, social interactions, social learning, information diffusion, disease and financial contagion, social capital formation, and more approaches this problem from a variety of specialized, often highly-parametric frameworks (see for instance Graham 2011; Blume et al. 2010; Athey and Imbens 2017; Jackson et al. 2017; Bramoullé et al. 2020). In this paper, we propose a new framework for causal inference that accommodates many such examples of network interference.

Our main innovation is a nonparametric modeling approach for sparse network data based on local configurations. Informally, a local configuration refers to the features of the network (the agents, their characteristics, treatment statuses, and how they are connected) nearby a focal agent as measured by path distance. The idea is that these local configurations index the different ways in which a policy or treatment assignment can impact the focal agent’s outcome under network interference.

This approach generalizes a developed literature on spillovers and social interactions in which the researcher specifies reference groups or an exposure map that details exactly how agents influence each other (see for instance Manski 1993, 2013; Hudgens and Halloran 2008; Aronow and Samii 2017; Vazquez-Bare 2017; Leung 2019; Arduini et al. 2020; Sävje 2021; Qu et al. 2021). One limitation of this literature is that the effect of a policy or treatment assignment is generally sensitive to how the researcher models the dependence. For example, in the spillovers literature it is often assumed that agents respond to the average treatment of their peers, while in the diffusion literature agents may be informed or infected by any peer. When the researcher is uncertain as to exactly how agents influence each other, misspecification can lead to inaccurate estimates and invalid inferences about the impact of the policy or treatment assignment of interest.

Another limitation of this literature is that it does not generally consider policies that change the structure of the network. Network-altering policies are common in economics. Examples include those that add or remove agents, or connections between agents, from the community (see for instance Ballester et al. 2006; Azoulay et al. 2010; Donaldson and

\[1\] The frameworks of Leung (2019), Sävje (2021) are misspecification-robust in the sense that their inference procedures may still be valid even if the model of interference used to define the estimand is misspecified. Their procedures do not address the issue we highlight here that an estimand based on a misspecified model may fail to accurately describe the impact of the policy of interest.
Such policies may be difficult to evaluate using standard frameworks, which focus on the reassignment of treatment to agents keeping the network structure fixed.

Our methodology addresses these limitations by using local configurations to model network interference. Intuitively, we use the space of local configurations as a “network sieve” that indexes the distribution of agent-specific outcomes associated with a given policy or treatment assignment. A contribution of our work is to formalize this local approach and apply it to causal inference under network interference.

The use of local configurations in economics was proposed by de Paula et al. (2018); Anderson and Richards-Shubik (2019) and related to that of ego-centered networks in sociology (see generally Wasserman and Faust 1994). In their work, local configurations index moment conditions that partially identify the parameters of a strategic network formation model. The researcher has the flexibility to choose the configurations used for this task and can restrict attention to a small number that occur frequently in the data. In our setting, local configurations correspond to fixed counterfactual policies. It is usually the case that no exact instances of a given policy appear in the data and so we substitute outcomes associated with similar but not exactly the same configurations. Formalizing this procedure requires new machinery, which we introduce building on work by Benjamini and Schramm (2001).

We demonstrate our local approach with applications to two causal inference problems. In both problems the researcher starts with a status-quo policy as described by one local configuration and is tasked with evaluating the impact of an alternative policy as described by another local configuration. The researcher has access to data from multiple-networks corresponding to a partial or stratified interference setting. Such designs are common in education, industrial organization, labor, and development economics where the researcher may collect network data on multiple independent schools, markets, firms, or villages. Our framework can also be applied to other settings (for example, one large network), but we leave the details to future work.

The first problem is to test policy irrelevance/no treatment effects. For instance, the status-quo policy may be given by a particular network structure where no agents are treated and the new policy may keep the same connections between agents but have every agent
treated. The hypothesis to be tested is that both policies are associated with the same
distribution of outcomes for one or more agents. We propose an asymptotically valid ran-
domization test for this hypothesis building on work by Canay and Kamat (2018).

The second problem is to estimate average or distributional policy effects/treatment
response. For instance, the status-quo policy may be given by a particular network structure
and the new policy may be one in which a key agent is removed. The policy effect to
be estimated is the expected change in outcome for one or more agents. We propose a
$k$-nearest-neighbors estimator for the policy effect and provide non-asymptotic bounds on
mean-squared error building on work by Döring et al. (2017), which can be used to construct
confidence intervals in the usual way.

Section 2 specifies a general model of network interference. Section 3 introduces the lo-
cal approach. Section 4 describes the applications testing policy irrelevance and estimating
policy effects. Section 5 contains an empirical illustration evaluating the impact of social net-
work structure on favor exchange in the setting of Jackson et al. (2012). Section 6 concludes.
Proof of claims, simulation evidence, and other details can be found in an appendix.

2 Preliminaries

We specify a general model of treatment response under network interference. The model is
used to motivate our local approach in Section 3.

2.1 Terminology and notation

A countable population of agents indexed by $I \subseteq \mathbb{N}$ is linked in a weighted and directed
network. The weight of a link from agent $i$ to $j$ is given by $D_{ij} \in \mathbb{Z}_+ \cup \{\infty\}$. The matrix
$D$ indexed by $I \times I$ with $D_{ij}$ in the $ij$th entry is called the adjacency matrix. We take the
convention that larger values of $D_{ij}$ correspond to weaker relationships between $i$ and $j$. For
instance, $D_{ij}$ might measure the physical distance between agents $i$ and $j$. We suppose that
$D_{ij} = 0$ if and only if $i = j$. When the network is unweighted (agent pairs are either linked
or not) $D_{ij} = 1$ denotes a link and $D_{ij} = \infty$ denotes no link from $i$ to $j$.

A path from $i$ to $j$ is a finite ordered set $\{t_1, ..., t_L\}$ with values in $\mathbb{N}$, $t_1 = i$, $t_L = j$, and
$L \in \mathbb{N}$. The length of the path \(\{t_1, ..., t_L\}\) is given by $\sum_{s=1}^{L-1} D_{t_st_{s+1}}$. The path distance from \(i\) to \(j\) denoted $\rho(i, j)$ is the length of the shortest path from \(i\) to \(j\). That is,

$$
\rho(i, j) := \inf_{\{t_1, ..., t_L\} \text{ s.t. } t_1 = i, t_L = j} \sum_{s=1}^{L-1} D_{t_st_{s+1}}.
$$

If the path distance from \(i\) to \(j\) is finite we say that \(i\) is path connected to \(j\). For any \(i \in \mathcal{I}\) and \(r \in \mathbb{Z}_+\), agent \(i\)’s \(r\)-neighborhood $\mathcal{N}_i(r) := \{j \in \mathcal{I} : \rho(i, j) \leq r\}$ is the collection of agents within path distance \(r\) of \(i\). $N_i(r) := |\mathcal{N}_i(r)|$ is the size of agent \(i\)’s \(r\)-neighborhood (i.e. the number of agents in $\mathcal{N}_i(r)$). For any agent-specific variable (such as an outcome or treatment assignment) $\mathbf{W} := \{W_i\}_{i \in \mathcal{I}}$, $W_i(r) := \sum_{j \in \mathcal{I}} W_j 1\{\rho(i, j) \leq r\}$ is the \(r\)-neighborhood count of $\mathbf{W}$ for agent \(i\). It describes the partial sum of $\mathbf{W}$ for the agents in $\mathcal{N}_i(r)$. $N_i(0) = \{i\}$, $N_i(0) = 1$, and $W_i(0) = W_i$ since $D_{ij} = 0$ if and only if $i = j$.

We assume that the network (adjacency matrix) is \textit{locally finite}. That is, for every \(i \in \mathcal{I}\) and \(r \in \mathbb{Z}_+\), $N_i(r) < \infty$. In words, the assumption is that every \(r\)-neighborhood of every agent contains only a finite number of agents. The assumption is implicit in much of the literature on network interference (including the examples below) where the researcher observes all of the relevant dependencies between agents in finite data. We do not assume a uniform bound on the size of the \(r\)-neighborhoods (c.f. de Paula et al. 2018). The set of all locally finite adjacency matrices is denoted $\mathcal{D}$.

### 2.2 Outcome model

Each agent $i \in \mathcal{I}$ has an outcome $Y_i \in \mathbb{R}$ and treatment assignment $T_i \in \mathbb{R}$. We call these quantities agent-specific variables and denote the corresponding population vectors $\mathbf{Y} = \{Y_i\}_{i \in \mathcal{I}}$ and $\mathbf{T} = \{T_i\}_{i \in \mathcal{I}}$. Adding additional covariates to the model is straightforward, see Appendix A.1. A general outcome model under network interference is

$$
Y_i = f_i(\mathbf{T}, D, U_i) 
$$

where the real-valued function $f_i$ represents unobserved agent-specific policy-relevant heterogeneity and $U_i \in \mathcal{U}$ represents unobserved agent-specific policy-invariant heterogeneity.
for some separable metric space $\mathcal{U}$. These two sources of heterogeneity are distinguished by Assumption 4.1 below. $Y_i$ depends on $T$ and $D$ so that individual $i$'s outcome may vary with any of the treatments or network connections in the population. Let $\mathcal{T}$ denote the set of all population treatment vectors $T$. The counterfactual outcome for agent $i$ at a fixed policy choice $(t, d) \in \mathcal{T} \times \mathcal{D}$ is $f_i(t, d, U_i)$.

Without further assumptions (1) is not informative about policy effects that rely on counterfactual outcomes because the model does not specify how exactly data from one policy can be used to learn about the outcomes associated with a different policy (see Manski 2013, Section 1.2). A common solution to this problem is to impose what Manski (2013) calls a constant treatment response (CTR) assumption. Let $\lambda_i : \mathcal{T} \times \mathcal{D} \rightarrow \mathcal{L}$ be an exposure map that maps treatment and network information into an effective treatment for agent $i$, where $\mathcal{L}$ denotes the set of all effective treatments. Then the CTR assumption is that for $(t, d)$ and $(t', d')$ such that $\lambda_i(t, d) = \lambda_i(t', d')$

$$f_i(t, d, u) = f_i(t', d', u)$$

for all $u \in \mathcal{U}$. In words, the CTR assumption states that for a fixed $u \in \mathcal{U}$, all policies $(t, d)$ associated with the same effective treatment $\lambda_i(t, d)$ generate the same outcome $f_i(t, d, u)$. Note that in the practice the CTR assumption could hold for multiple effective treatment assignments or exposure maps.

Under the CTR assumption, we define the function $h : \mathcal{L} \times \mathcal{U} \rightarrow \mathbb{R}$ such that $h(\lambda_i(t, d), u) = f_i(t, d, u)$ for every $u \in \mathcal{U}$, and rewrite the outcome model as

$$Y_i = h(L_i, U_i)$$

where $L_i = \lambda_i(T, D)$. Agent $i$'s outcome now only depends on the policy $(T, D)$ through their effective treatment $G_i$.

With the CTR assumption, (1) may be informative about the policy effect of interest. A policy implies a collection of effective treatments, one for each individual. If the effective treatments of the counterfactual policy are observed in the data, then the researcher might infer the impact of the policy by looking at the associated outcomes. For example,
if $\lambda_i(T, D) = T_i$, then agent $i$’s outcome only depends on their own treatment status. The impact of a policy that treats an untreated agent might then be learned by comparing the outcomes of the treated agents to those of the untreated agents in the data.

Some common choices of $\lambda_i$ and $\mathcal{L}$ from the economics literature are illustrated in the examples below. These choices are however based on restrictive modeling assumptions that when wrong may lead the researcher to mischaracterize the policy effect of interest. Our local approach is instead based on a specific but flexible choice of $\mathcal{L}$ called the space of *rooted networks*. Rooted networks generalize the notion of a local configuration and approximate a large class of effective treatments. We explain this approach in Section 3 below.

### 2.3 Examples

**Example 2.1.** (Neighborhood spillovers): Agents are assigned to either treatment or control status with $T_i = 1$ if $i$ is treated and $T_i = 0$ if $i$ is not. Agent $i$’s outcome depends on their treatment status and the number of treated agents nearby

$$Y_i = Y(T_i, T_i(r), U_i)$$

where $T_i(r) = \sum_{j \in N} T_j 1\{\rho(i, j) \leq r\}$ is the neighborhood count of treatment assigned within some fixed radius $r$ of $i$. See for instance Cai et al. (2015); Leung (2016); He and Song (2018); Viviano (2019). An example policy of interest is the effect of treating every agent in the community versus treating no one, holding the network connections fixed. An effective treatment for $i$ is $\lambda_i(T, D) = (T_i, T_i(r))$. It only depends on the network connections and treatment statuses within path distance $r$ of $i$.

**Example 2.2.** (Social capital formation): Agents leverage their network connections to garner favors, loans, advice, etc. Jackson et al. (2012) specify a model in which one agent performs a favor for another when there is a third agent connected to both to monitor the exchange. We consider a stochastic version of this model where agent $i$’s stock of social
capital depends on their number of monitored connections

\[ Y_i = \left( \sum_{j \in I} 1 \{ N_i(1) \cap N_j(1) \neq \emptyset \} \right) \cdot U_i . \]

Karlan et al. (2009); Cruz et al. (2017) consider related models of social capital. An example policy of interest is the effect of adding additional connections to the network. An effective treatment for \(i\) is

\[ \lambda_i(T, D) = \sum_{j \in I} 1 \{ N_i(1) \cap N_j(1) \neq \emptyset \} . \]

It only depends on the network connections within path distance 2 of agent \(i\).

**Example 2.3.** (Social interactions): Agent \(i\)'s equilibrium outcome depends on the number and length of paths between them and the other agents, the treatment statuses of those other agents, and the average treatment statuses of the agents nearby those other agents

\[ Y_i = \lim_{S \to \infty} \sum_{s=0}^{S} \left[ \delta^s A^*(D)^s (T \beta + T^*(1) \gamma) \right]_i + U_i , \]

where \(T^*_i(1) = T_i(1)/N_i(1)\), \([\cdot]_i\) is the \(i\)th entry of a vector, and the \(ij\)th entry of \(A^*(D)\) is \(A^*_{ij}(D) = 1\{0 < D_{ij} \leq 1\}/N_i(1)\). See for instance Bramoullé et al. (2009); Blume et al. (2010); De Giorgi et al. (2010); Lee et al. (2010); Goldsmith-Pinkham and Imbens (2013). An example policy effect of interest is the average effect of removing a particular agent from the community. See for instance Ballester et al. (2006); Calvó-Armengol et al. (2009); Lee et al. (2021). An effective treatment for \(i\) is

\[ \lambda_i(T, D) = \left\{ [\delta^s A^*(D)^s (T \beta + T^*(1) \gamma)]_i \right\}_{s=0}^{\infty} . \]

There are two key differences between Example 2.3 and Examples 2.1 and 2.2. First, in Example 2.3, the effective treatment for \(i\) depends on all of the treatment statuses and network connections of the agents path connected to \(i\). Leung (2019) argues that this is a common feature of many economic models of network interference. Second, in Example 2.3, the description of the effective treatment depends on model parameters which are unknown to the policy maker. Our local approach is to our knowledge the first to accommodate these two features when modeling causal effects under network interference.
3 The local approach

We start with an informal description of the local configurations that form the basis of our approach. We then formally introduce the construction and model of network interference.

3.1 Informal description

Intuitively, agent $i$’s local configuration refers to the agents within path distance $r$ of $i$, their characteristics, and how they are connected. Larger values of $r$ are associated with more complicated configurations, which give a more precise picture about how $i$ is connected in the network. This idea is illustrated in Figure 1.

Panel (a) depicts twelve agents connected in an unweighted and undirected network with a binary treatment. Agents are either assigned to treatment (red square nodes) or control (blue circle nodes). Panel (b) depicts the local configurations of radius 1 for agents 1 and 2. They are both equivalent to a wheel with the focal untreated agent in the center and three other agents on the periphery, one of which is treated. Panel (c) depicts the local configurations of radius 2 for agents 1 and 2. They are both equivalent to a ring between five untreated agents (one of which is the focal agent) where the focal agent is also connected to a treated agent who is connected to an untreated agent and another agent in the ring adjacent to the focal agent is connected to a treated agent. Panel (d) depicts the local configurations of radius 3 for agents 1 and 2. They are not equivalent because the local configuration for agent 1 contains four treated agents while the local configuration for agent 2 contains only three treated agents.

In this way, one can describe the local configurations for any choice of agent and radius. Since the diameter of the network (the maximum path distance between any two agents) is 7, any local configuration of radius greater than 7 will be equal to the local configuration of radius 7. However, for networks defined on large connected populations, increasing the radius of the local configuration typically reveals a more complicated network structure.

Following Benjamini and Schramm (2001), we call the infinite-radius local configuration a rooted network. Our local approach is based on the observation that, for many models with network interference, an agent’s effective treatment is determined by their rooted network.
Figure 1: Illustration of local configurations.

(a) A network connecting a dozen agents.

(b) The local configurations for agents 1 and 2 associated with radius 1 are equivalent.

(c) The local configurations for agents 1 and 2 associated with radius 2 are equivalent.

(d) The local configurations for agents 1 and 2 associated with radius 3 are not equivalent.
For instance, in the neighborhood spillovers model of Example 2.1, an agent is influenced by their treatment status and the treatment statuses of their \( r \)-neighbors. As a result, two agents with the same local configurations of radius \( r \) have the same effective treatment. In the social interactions model of Example 2.3, an agent’s effective treatment is not necessarily determined by any local configuration of finite radius. This is because an agent is influenced by all of the other agents to which they are path connected. However, the agent’s effective treatment can be arbitrarily well-approximated by a local configuration of finite radius in the sense that for every \( \varepsilon > 0 \) there exists an \( r(\varepsilon) \in \mathbb{Z}_+ \) such that two agents with the same local configuration of radius \( r(\varepsilon) \) will have effective treatments within \( \varepsilon \) of each other. We show this in Example 3.3 below.

This observation motivates our local approach, which is to model network interference using rooted networks. We now formalize the idea.

### 3.2 Formal specification

We define the space of rooted networks building on Benjamini and Schramm (2001); Aldous and Steele (2004) and generalizing the local configurations of Section 3.1. Our main proposal is to use this space to index effective treatments in the outcome model of Section 2.2.

#### 3.2.1 Rooted networks

We maintain the assumptions of Section 2, but introduce some new notation. For an adjacency matrix \( D \in \mathcal{D} \) and treatment assignment vector \( T \in \mathcal{T} \), a network is the triple \((V, E, T)\) where \( V \) is the vertex set (the population of agents represented by \( I \)) and \( E \) is the weighted edge set (represented by the adjacency matrix \( D \)). A rooted network \( G_i = ((V, E, T), i) \) is the triple \((V, E, T)\) with a focal agent \( i \in V \) called the root. Informally, \( G_i \) is the network \((V, E, T)\) “from the point of view” of \( i \).

For any \( r \in \mathbb{Z}_+ \), \( G_i^r \) is the subnetwork of \((V, E, T)\) rooted at \( i \) and induced by the agents within path distance \( r \) of \( i \) as measured by path distance \( \rho \). Formally, \( G_i^r := ((V_i^r, E_i^r, T_i^r), i) \), where \( V_i^r := \mathcal{N}_i(r) := \{ j \in V : \rho(i, j) \leq r \}, E_i^r := \{ e_{jk} \in E : j, k \in V_i^r \}, \) and \( T_i^r := \{ T_j \in T : j \in V_i^r \} \). We say \( G_i^r \) is the rooted network \( G_i \) truncated at radius \( r \). In Section 3.1 we
called this object agent $i$’s local configuration of radius $r$.

For any $\varepsilon \geq 0$, two rooted networks $G_{i_1}$ and $G_{i_2}$ are $\varepsilon$-isomorphic (denoted $G_{i_1} \simeq_\varepsilon G_{i_2}$) if all of their $r$-neighborhoods are equivalent up to a relabeling of the non-rooted agents, but where treatment assignments are allowed to disagree up to a tolerance of $\varepsilon$. Formally, $G_{i_1} \simeq_\varepsilon G_{i_2}$ if for any $r \in \mathbb{Z}_+$ there exists a bijection $f : V_{i_1}^r \leftrightarrow V_{i_2}^r$ such that $f(i_1) = i_2$, $e_{jk} = e_{f(j)f(k)}$, for any $j, k \in V_{i_1}^r$, and $|T_j - T_{f(j)}| \leq \varepsilon$ for any $j \in V_{i_1}^r$.

Two rooted networks that are not $0$-isomorphic are assigned a strictly positive distance inversely related to the largest $r$ and smallest $\varepsilon$ such that they have $\varepsilon$-isomorphic $r$-neighborhoods. Specifically, we define the following distance on the set of rooted networks:

$$d(G_{i_1}, G_{i_2}) := \min \left\{ \inf_{(r, \varepsilon) \in \mathbb{Z}_+ \times \mathbb{R}_+} \{\zeta(r) + \varepsilon : G_{i_1}^r \simeq_\varepsilon G_{i_2}^r\}, 2 \right\}.$$  

where $\zeta(x) = (1 + x)^{-1}$. We demonstrate that $d(\cdot, \cdot)$ is a pseudo-metric in Appendix A.1. The outer minimization is not essential, but taking $d$ to be bounded simplifies later arguments.

Let $\mathcal{G}$ denote the set of equivalence classes of all possible locally finite rooted networks under $d$. We demonstrate that $(\mathcal{G}, d)$ is a separable and complete metric space in Appendix A.1. Following Aldous and Steele (2004), we call the topology on $\mathcal{G}$ induced by $d$ the local topology, and more broadly call modeling on $\mathcal{G}$ the local approach.

To be sure, the $\varepsilon$-isomorphism used to define the network distance $d$ could be modified to fit the researcher’s application. For instance, economic theory may suggest that two agents are qualitatively similar if their rooted networks are similar under edit distance (i.e. they can be made $\varepsilon$-isomorphic by adding or deleting a small number of agents or links). In this case the researcher may wish to relax the $\varepsilon$-isomorphism to allow for such discrepancies. The results developed in Section 4 continue to hold mutatis mutandis with alternative distance metrics. We view our proposed metric as one relatively simple albeit conservative option.

### 3.2.2 Outcome model revisited

Recall that in Section 2 we specified the outcome model

$$Y_i = h(L_i, U_i)$$

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where \((L_i, U_i) \in \mathcal{L} \times \mathcal{U}\) and \(\mathcal{L}\) is a collection of effective treatments. We propose taking \(\mathcal{L}\) to be the space of rooted networks \(\mathcal{G}\) so that \(L_i = G_i\) is the rooted network of agent \(i\). In Section 3.3 we show that for each example of Section 2.3 \(G_i\) is in fact an effective treatment for \(i\).

We endow \(\mathcal{G} \times \mathcal{U}\) with the usual product topology, define probability measures on the corresponding Borel sigma-algebra, and associate a stochastic rooted network and error pair \((G_i, U_i)\) with a probability measure \(\mu\). For now we take \(\mu\) as arbitrary and fixed by the researcher. We motivate a specific choice of \(\mu\) in the context of a multiple-networks setting in Section 4.2 below.

Our main objects of interest are the average structural function (ASF) and distributional structural function (DSF) that describe the outcome for \(i\) associated with a policy that sets the rooted network to some \(g \in \mathcal{G}\). That is, respectively,

\[
h(g) = E[h(g, U_i)] \quad \text{and} \quad h_y(g) = E[\mathbb{1}\{h(g, U_i) \leq y\}]
\]

where the expectation refers to the marginal distribution of \(U_i\) under \(\mu\). See for instance Blundell and Powell (2003). These functions can be used to estimate and conduct inferences about many causal effects of interest. For example, the average treatment effect (ATE) associated with a policy that alters agent \(i\)'s rooted network from \(g\) to \(g'\) is described by \(h(g') - h(g)\). Under appropriate continuity assumptions, \(h(g)\) and \(h_y(g)\) can be approximated by averaging the outcomes corresponding to rooted networks that are close to \(g\) under \(d\).

We demonstrate this idea with applications to testing policy irrelevance/no treatment effects and estimating policy effects/treatment response in Sections 4.3 and 4.4 below.

This ATE is closely related to the average exposure effect of Leung (2019); Sävje et al. (2021). These authors consider inference in the setting where the exposure map chosen by the researcher is potentially invalid in the sense that the CTR assumption of Section 2.2 is false. A limitation of their approach is that an exposure effect based on a misspecified exposure map may not characterize any policy of interest. We limit the scope for misspecification by using a general class of exposure maps indexed by the space of rooted networks.

We could alternatively consider conditional treatment effect parameters. For instance,
the researcher may be interested in a policy that alters treatment assignment conditional on
the structure of the network. Or the policy may alter the network connections conditional
on some observed agent attributes. In such cases the researcher may wish to consider a
conditional average treatment effect where the researcher conditions on the part of the rooted
network not altered by the policy. Such parameters may in some cases be identified under
weaker assumptions than those required to identify the above unconditional ATE. This is
discussed following Assumption \[1.1\] below.

Another potential modification concerns settings in which the policy of interest is asso-
ciated with a stochastic network. For example, the researcher might want to evaluate the
impact of a social program that affects outcomes by altering agent incentives to form links
in the network in a non-deterministic way. One could model such policies by first charac-
terizing the average effect for each rooted network, weight by the policy-specific distribution
of rooted networks (as determined by some network formation model), and then compare
across policies. We leave a formal study of such extensions to future work.

### 3.3 Examples revisited

We show that rooted networks serve the role of effective treatments in the Section 2 examples.

**Example 3.1.** Recall that in the treatment spillovers model of Example 2.1 an effective
treatment is \(\lambda_i(T, D) = (T_i, T_i(r))\). Let \(G_i\) be agent \(i\)'s rooted network. \(T_i\)
associated with agent \(i\) and so is determined by \(G_i^0\). \(T_i(r)\) counts the number of treated agents within
distance \(r\) of \(i\) and so is determined by \(G_i^r\). As a result, \(\lambda_i(T, D)\) is determined by \(G_i\). It
follows that \(Y_i = h(G_i, U_i)\) for some \(h\).

**Example 3.2.** Recall that in the social capital formation model of Example 2.2 an effective
treatment is \(\lambda_i(T, D) = \sum_{j \in I} 1\{N_i(1) \cap N_j^1(1) \neq \emptyset\}\). Let \(G_i\) be \(i\)'s rooted network. Agent
\(i\)'s effective treatment is the number of agents of path distance 2 from \(i\), which is determined
by \(G_i^2\). It follows that \(Y_i = h(G_i, U_i)\) for some \(h\).

**Example 3.3.** Recall that in the social interactions model of Example 2.3 an effective treat-
ment is \(\lambda_i(T, D) = \{[\delta^s A^s(D)^s (T \beta + T^s(1)\gamma)]_i\}_{s=0}^\infty\). Let \(G_i\) be agent \(i\)'s rooted network.
The $i$th entry of $\delta^s A^*(D)^s (T\beta + T^*(1)\gamma)$ only depends on the treatment statuses and connections of agents within path-distance $s+1$ of $i$ and so is determined by $G_i^{s+1}$. It follows that $Y_i = \lim_{S \to \infty} \sum_{s=1}^{S} h_s(G_i^s, U_i) = h(G_i, U_i)$ for some functions $h_s$ and $h$. Unlike the effective treatment described in Example 2.3, our rooted networks do not depend on the unknown model parameters.

4 Two applications to causal inference

We apply the local approach framework of Section 3 to two causal inference problems. In both problems, the researcher begins with a rooted network associated with a status-quo policy and is tasked with evaluating the impact of an alternative. The researcher has access to data on outcomes and policies from multiple independent communities or clusters such as schools, villages, firms, or markets. This data structure, described in Section 4.1, is not crucial to our methodology, but simplifies the analysis. Applications to alternative settings, for example with data from one large network or endogenous policies is straightforward, but we leave the details to future work.

Our first application, described in Section 4.2, is to testing policy irrelevance. We test the hypothesis that the rooted networks associated with two policies generate the same distribution of outcomes. Our second application, described in Section 4.3, is to estimating policy effects. We construct a $k$-nearest-neighbors estimator for the structural function of Section 3.2.2 and provide non-asymptotic bounds on estimation error that can be used to construct confidence intervals in the usual way. These applications contrast a literature studying the magnitude of any potential spillover effects or testing the hypothesis of no spillovers (see for instance [Aronow 2012] [Athey et al. 2018] [Hu et al. 2021] [Savje et al. 2021]).

4.1 Multiple-networks setting

A random sample of communities (clusters) are indexed by $c \in [C] := 1, \ldots, C$. Each $c \in [C]$ is associated with a finite collection of $m_c$ observations $\{W_{ic}\}_{i \in [m_c]}$ where $m_c$ is a random positive integer, $W_{ic} := (Y_{ic}, G_{ic})$, and $Y_{ic} := h(G_{ic}, U_{ic})$ for some unobserved error $U_{ic}$. Intuitively, each community $c$ is represented by an initial network connecting $m_c$ agents and
the $m_c$ rooted networks refer to this initial network rooted at each agent in the community. Let $W_c := \{W_{ic}\}_{i \in [m_c]}$. We impose the following assumptions on $\{(G_{ic}, U_{ic})\}_{i \in [m_c], c \in [C]}$.

**Assumption 4.1.**

(i) $\{W_c\}_{c \in [C]}$ are independent and identically distributed (across communities).

(ii) $\{U_{ic}\}_{i \in [m_c], c \in [C]}$ are identically distributed (within and across communities).

(iii) For any measurable $f$, $i \in [m_c]$, and $c \in [C]$,

$$E[f(G_{ic}, U_{ic})|G_{1c}, ..., G_{m_c,c}] = E[f(G_{ic}, U)|G_{ic}],$$

where $U$ is an independent copy of $U_{ic}$ (i.e. $U$ has the same marginal distribution as $U_{ic}$ but is independent of $\{W_c\}_{c \in [C]}$).

Assumption 4.1 (i) is what makes our analysis “multiple-networks.” It states that the networks and errors are independent and identically distributed across communities. We use this independence across communities to characterize the statistical properties of our test procedure and estimator below. We do not make any restrictions on the dependence structure between observations within a community. This allows for arbitrary dependence within a community; see for instance Example 2.3. Weakening the independence assumption (for instance, considering dependent data from one large community) would require additional assumptions about the intra-community dependence structure which we leave to future work.

Assumption 4.1 (ii) fixes the marginal distribution of the errors. It is used to define the policy effect of interest. The relevant structural function is defined as the expected outcome over the homogeneous marginal distribution of $U_{ic}$ for a fixed rooted network. This assumption can be dropped by defining the expectation to be with respect to the (mixture) distribution of $U_{ic,c}$ generated by drawing $\iota_c$ uniformly at random from $[m_c]$.

Assumption 4.1 (iii) states that the rooted networks are exogenous (i.e. the errors are policy-irrelevant). We require that the conditional distribution of $(G_{ic}, U_{ic})$ given $G_{1c}, ..., G_{m_c,c}$ is equal to the conditional distribution of $(G_{ic}, U)$ given $G_{ic}$, where $U$ is an independent copy of $U_{ic}$. Exogeneity is a strong assumption, but allows us to approximate the unknown policy
functions using sample averages. It is often assumed in the literature cited in Section \ref{sec:overview}.

The study of endogenous rooted networks where the policy \((T, D)\) is potentially related to the errors \(U\) is left to future work. If the policy maker is interested in identifying a conditional average treatment effect as described in the discussion following the definition of \(h(g)\) in Section \ref{sec:causal}, then Assumption 4.1 (iii) can be weakened to a conditional exogeneity assumption where the structural errors are independent of the effective treatments conditional on the relevant part of the rooted network.

### 4.2 Testing policy irrelevance

The policy maker begins with a status-quo community policy described by a treatment and network pair \((t, d)\), and proposes an alternative \((t', d')\). The researcher is tasked with testing whether the two policies are associated with the same distribution of outcomes for an agent whose effective treatment under the status-quo is described by the rooted network \(g \in \mathcal{G}\) and whose effective treatment under the alternative is described by the rooted network \(g' \in \mathcal{G}\). One can extend our procedure to test policy irrelevance for multiple agents via a multiple testing procedure.

Following Section 3.2, the potential outcomes under \(g\) and \(g'\) are given by \(h(g, U)\) and \(h(g', U)\) respectively for some error \(U\). The hypothesis of policy irrelevance is

\[
H_0 : h_y(g) = h_y(g') \text{ for every } y \in \mathbb{R}
\]  

(2)

where \(h_y(g) := E \{ h(g, U) \} \leq y \). Under Assumption 4.1 (iii), \(h_y(g)\) describes the conditional distribution of \(Y_i\) given \(G_i \sim g\).

Our test procedure is described in Section \ref{sec:testing}. Intuitively, it compares the empirical distribution of outcomes for agents in the data whose rooted networks are most similar to \(g\) to the analogous distribution for \(g'\). Asymptotic validity follows two additional assumptions.

#### 4.2.1 Assumptions

We assume that \(g\) and \(g'\) are supported in the data and \(h\) satisfies certain continuity conditions. We do not believe either assumption to be restrictive in practice.
Assumption 4.2. For any $\ell > 0$ and $g_0 \in \{g, g'\}$, $\psi_{g_0}(\ell) := P\left(\min_{i \in [m_c]} d(G_{ic}, g_0) \leq \ell\right) > 0$.

The function $\psi_g(\ell)$ measures the probability that there exists an agent from a randomly drawn community whose rooted network is within distance $\ell$ of $g$ under $d$. Assumption 4.2 states that for a randomly drawn community there exists a rooted network within distance $\ell$ of $g$ or $g'$ with positive probability. It implies that as the number of communities $C$ grows, the researcher will eventually observe rooted networks that are arbitrarily close to $g$ or $g'$.

Assumption 4.3. For every $\tilde{g} \in \mathcal{G}$, the distribution of $h(\tilde{g}, U)$ is either continuous or discrete with finite support. For every $y \in \mathbb{R}$, $h_y(\tilde{g})$ is continuous in $\tilde{g}$.

Assumption 4.3 states that the distribution of outcomes associated with agents whose rooted networks are close to $\tilde{g}$ approximate the distribution of outcomes at $\tilde{g}$. These continuity assumptions are satisfied by the three examples of Section 2.3.

4.2.2 Test procedure

We propose an approximate permutation test of $H_0$ building on Canay et al. (2017); Canay and Kamat (2018). The test procedure is described in Algorithm 4.1. We assume that $C$ is even to simplify notation. When determining the closest agent in Step 1 or reordering the vectors in Step 2, ties are broken uniformly at random.

Algorithm 4.1. Input: data $\{W_c\}_{c \in [C]}$ partitioned into two disjoint sets of of size $C/2$ labelled $W_1$ and $W_2$, and parameters $q \leq C/2$, $\alpha \in [0, 1]$. Output: a rejection decision.

1. For every $c \in W_1$, let $i_c(g) := \arg\min_{i \in [m_c]} \{d(G_{ic}, g)\}$ be the agent in $c$ whose rooted network $G_{ic}$ is closest to $g$ and $W_c(g) := (Y_c(g), G_c(g)) := (Y_{i_c(g)c}, G_{i_c(g)c})$ be $i_c(g)$’s outcome and rooted network. Similarly define $W_c(g') := (Y_c(g'), G_c(g'))$ for every $c \in W_2$.

2. Reorder $\{W_c(g)\}_{c \in W_1}$ and $\{W_c(g')\}_{c \in W_2}$ so that the entries are increasing in $d(G_c(g), g)$ and $d(G_c(g'), g')$ respectively. Denote the first $q$ elements of the reordered $\{W_c(g)\}_{c \in W_1}$

$$W^*(g) := (W^*_1(g), W^*_2(g), \ldots, W^*_q(g)),$$
where \( W^*_c(g) = (Y^*_c(g), G^*_c(g)) \). Similarly define \( W^*(g') \). Collect the 2q outcomes of \( W^*(g) \) and \( W^*(g') \) into the vector

\[
S_C := (S_{C,1}, ..., S_{C,2q}) := (Y^*_1(g), ..., Y^*_q(g), Y^*_1(g'), ..., Y^*_q(g')).
\]

3. Define the Cramer von Mises test statistic

\[
R(S_C) = \frac{1}{2q} \sum_{j=1}^{2q} \left( \hat{F}_1(S_{C,j}; S_C) - \hat{F}_2(S_{C,j}; S_C) \right)^2,
\]

where

\[
\hat{F}_1(y; S_C) = \frac{1}{q} \sum_{j=1}^{q} 1\{S_{C,j} \leq y\} \quad \text{and} \quad \hat{F}_2(y; S_C) = \frac{1}{q} \sum_{j=q+1}^{2q} 1\{S_{C,j} \leq y\}.
\]

4. Let \( H \) be the set of all permutations \( \pi = (\pi(1), ..., \pi(2q)) \) of \( \{1, ..., 2q\} \) and

\[
S^\pi_C := (S_{C,\pi(1)}, ..., S_{C,\pi(2q)}).
\]

Reject \( H_0 \) if the p-value \( p \leq \alpha \) where

\[
p := \frac{1}{|H|} \sum_{\pi \in H} 1\{R(S^\pi_C) \geq R(S_C)\}.
\]

This p-value in Step 4 may be difficult to compute when \( H \) is large. Theorem 4.1 below continues to hold if \( H \) is replaced by \( \hat{H} \) where \( \hat{H} = \{\pi_1, ..., \pi_B\} \), \( \pi_1 \) is the identity permutation and \( \pi_2, ..., \pi_B \) are drawn independently and uniformly at random from \( H \). Such sampling is standard in the literature, see also Canay and Kamat (2018), Remark 3.2.

The test presented in Algorithm 4.1 is non-randomized in the sense that the decision to reject the null hypothesis is a deterministic function of the data. This leads to a test which is potentially conservative. One could alternatively consider a non-conservative version of this test which is randomized. See Lehmann and Romano (2006), Section 15.2.

In practice we partition \( \{W_c\}_{c \in [C]} \) into two disjoint sets iteratively in the following way. We first select the community which contains the observation closest to \( g \), add it to \( W_1 \), and remove it from the pool of candidate communities. We then select the community which contains the observation closest to \( g' \), add it to \( W_2 \), and remove it from the pool of
candidate communities. We continue this process, alternating between \( g \) and \( g' \), until the pool of candidate communities is exhausted.

### 4.2.3 Asymptotic validity

If the entries of \( Y^*(g) = (Y^*_1(g), \ldots, Y^*_q(g)) \) and \( Y^*(g') = (Y^*_1(g'), \ldots, Y^*_q(g')) \) were identically distributed to \( h(g,U) \) and \( h(g',U) \) respectively, then the test described in Algorithm 4.1 would control size in finite samples following standard arguments (e.g. Lehmann and Romano 2006, Theorem 15.2.1). However, since the rooted networks corresponding to \( Y^*_j(g) \) and \( Y^*_j(g') \) are not exactly \( g \) and \( g' \), such an argument cannot be directly applied.

Our assumptions instead imply that in an asymptotic regime where \( q \) is fixed and \( C \to \infty \), the entries of \( Y^*(g) \) and \( Y^*(g') \) are approximately equal in distribution to \( h(g,U) \) and \( h(g',U) \). A fixed \( q \) rule is appropriate in our setting because the quality of the nearest neighbors to \( g \) or \( g' \) may degrade rapidly with \( q \). Our simulation evidence in Appendix \( C \) suggests that \( q = 5 \) works well in practice.

We demonstrate that the test procedure in Algorithm 4.1 is asymptotically valid building on work by Canay et al. (2017); Canay and Kamat (2018). Finite-sample behavior is examined via simulation in Appendix \( C \).

**Theorem 4.1.** Under Assumptions 4.1, 4.2 and 4.3, the test described in Algorithm 4.1 is asymptotically \( (C \to \infty, q \text{ fixed}) \) level \( \alpha \).

We believe that it is possible to modify Assumption 4.2 and the proof of Theorem 4.1 so that the test is also asymptotically valid in the case of a fixed number of communities and a growing number of agents within each community. However we do not pursue this here.

### 4.3 Estimating Policy Effects

The policy maker begins with a status-quo community policy described by a treatment and network pair \((t,d)\), and proposes an alternative \((t',d')\). The researcher is tasked with estimating the expected effect of the policy change for an agent whose effective treatment under the status-quo is described by the rooted network \( g \in \mathcal{G} \) and whose effective treatment under the alternative is described by the rooted network \( g' \in \mathcal{G} \). Following Section 3.2, the
potential outcomes under policies \(g\) and \(g'\) are described by \(h(g, U)\) and \(h(g', U)\) respectively for some error \(U\). The object of interest is

\[
h(g') - h(g)
\]

where \(h(g) = E[h(g, U)]\). Alternative effects (e.g. distributional effects) based on other features of the conditional distribution of outcomes can be estimated analogously.

The proposed estimator for the policy effect is described in Section 4.3.2. Intuitively, it compares the average outcome of the \(k\) agents whose rooted networks are most similar to \(g\) to the analogous average for \(g'\). Non-asymptotic bounds on mean-squared error follow three additional assumptions\(^2\)

### 4.3.1 Assumptions

We impose smoothness conditions on the model parameters and bound the variance of the outcome. We do not believe these assumptions to be restrictive in practice.

**Assumption 4.4.** For \(g_0 \in \{g, g'\}\), \(\psi_{g_0}(\ell) := P\left(\min_{i \in [m_c]} d(G_{ic}, g_0) \leq \ell\right)\) is continuous at every \(\ell \geq 0\).

Recall that \(\psi_g(\ell)\) measures the probability that there exists an agent from a randomly drawn community whose rooted network is within \(\ell\) of \(g\). Assumption 4.4 states that \(\psi_g(\ell)\) and \(\psi_{g'}(\ell)\) are continuous in \(\ell\). It justifies a probability integral transform used to characterize the bias of the estimator. The assumption can be guaranteed by adding a randomizing component to the metric (see the discussion following equation (19) in [Györfi and Weiss 2020](#)), which allows for \(G_{ic}\) to be discrete.

**Assumption 4.5.** For \(g_0 \in \{g, g'\}\) there exists an increasing function \(\phi_{g_0} : \mathbb{R}_+ \to \mathbb{R}_+\) such that \(\phi_{g_0}(x) \to \phi_{g_0}(0) = 0\) as \(x \to 0\) and for every \(\tilde{g} \in \mathcal{G}\)

\[
|h(g_0) - h(\tilde{g})| \leq \phi_{g_0}(d(g_0, \tilde{g}))
\]

\(^2\)These assumptions are also sufficient to construct an asymptotically normal estimator for \(h(g)\), which could be used for inference. An example is the estimator we propose below. In general we expect the asymptotic bias of this estimator to be non-negligible. Accordingly, one could use our characterization of the bias in Theorem 4.2 combined with the approach advocated in [Armstrong and Kolesár 2020 2021](#) to construct valid confidence bounds.
Assumption 4.5 states that $h$ has a modulus of continuity $\phi_g$ at $g$. Such a smoothness condition is standard in the nonparametric estimation literature. A model of network interference implies a specific choice of $\phi_g$. The three examples of Section 2.3 all satisfy the assumption with $\phi_g(x) = C^{(x-1)/x}$ for some $C > 1$ potentially depending on parameters of the model but not $g$. See Appendix Section B for details.

**Assumption 4.6.** For every $\tilde{g} \in \mathcal{G}$, $\sigma^2(\tilde{g}) := E[(h(\tilde{g}, U_{ic}^g) - h(\tilde{g}))^2] \leq \sigma^2$.

Assumption 4.6 bounds the variance of the outcome variable and is also standard in the nonparametric estimation literature.

### 4.3.2 Estimator

Let $W_c(g)$ be the observation from $\{W_{ic}\}_{1 \leq i \leq m_c}$ whose value of $G_{ic}$ is closest to $g$ and order $\{W_c(g)\}_{1 \leq c \leq C}$ to be increasing in $d(G_c(g), g)$ (ties are broken uniformly at random—see Algorithm 4.1, Step 2). Denote the first $k$ elements $W_1^*(g), W_2^*(g), \ldots, W_k^*(g)$. The proposed estimator for $h(g)$ is then the average of the outcomes $\{Y_j^*(g)\}_{j=1}^k$ associated with $\{W_j^*(g)\}_{j=1}^k$

$$\hat{h}(g) := \frac{1}{k} \sum_{j=1}^k Y_j^*(g)$$

and the analogous estimator for $h(g') - h(g)$ is

$$\hat{h}(g') - \hat{h}(g) := \frac{1}{k} \sum_{j=1}^k (Y_j^*(g') - Y_j^*(g)).$$

The results to follow are valid for any choice of $k$. In practice the researcher may wish to choose $k$ via cross-validation.

### 4.3.3 Bound on estimation error

We derive a finite-sample bound on the mean-squared error of $\hat{h}(g)$ building on work by Biau and Devroye (2015); Döring et al. (2017); Győrfi and Weiss (2020).
Theorem 4.2. Under Assumptions 4.1, 4.4, 4.5, and 4.6

\[ E \left[ (\hat{h}(g) - h(g))^2 \right] \leq \frac{\sigma^2}{k} + E \left[ \varphi_g(U_{(k,C)})^2 \right] , \]

where \( \varphi_g(x) = \phi_g \circ \psi^\dagger_g(x) \), \( \psi^\dagger_g : [0, 1] \rightarrow \mathbb{R}_+ \) refers to the upper generalized inverse

\[ \psi^\dagger_g(x) = \sup \{ \ell \in \mathbb{R}_+ : \psi_g(\ell) \leq x \} , \]

and \( U_{(k,C)} \) is distributed \( \text{Beta}(k, C - k + 1) \).

The bound in Theorem 4.2 features a familiar bias-variance decomposition. It establishes consistency of \( \hat{h}(g) \) in well-behaved settings. For example if (for \( g \) fixed) \( \varphi_g(\cdot) \) is bounded, continuous at zero, \( \varphi_g(0) = 0 \), and \( k \rightarrow \infty, k/C \rightarrow 0 \) as \( C \rightarrow \infty \) then

\[ \sigma^2/k \rightarrow 0 \text{ and } E[\varphi_g(U_{(k,C)})^2] = E \left[ (\phi_g \circ \psi^\dagger_g(U_{(k,C)}))^2 \right] \rightarrow 0 . \]

The variance component \( \sigma^2/k \) is standard and decreases as \( k \) grows large. In contrast, the bias component \( E[\varphi_g(U_{(k,C)})^2] \) and its relationship with \( k \) and \( C \) are difficult to characterize without further information about \( \phi_g \) and \( \psi_g \). Intuitively, the first controls the smoothness of the regression function \( h(g) \) and the second controls the quality of the nearest-neighbors that make up \( \hat{h}(g) \) in terms of proximity to \( g \). We provide further discussion of how the bound depends on features of these parameters in Appendix B.

Theorem 4.2 has the immediate corollary

Corollary 4.1. Suppose the hypothesis of Theorem 4.2. Then

\[ E \left[ (\hat{h}(g') - \hat{h}(g) - (h(g') - h(g)))^2 \right] \leq \frac{4\sigma^2}{k} + 4E \left[ \varphi_{g\vee g'}(U_{(k,C)})^2 \right] , \]

where \( \varphi_{g\vee g'} := \varphi_g \vee \varphi_{g'} \).

Corollary 4.1 can be directly applied to bound large deviations in policy effects following, for example, the framework of In this setting, the researcher explicitly accounts for the bias using bounds on the deviations of the \( h \) function along the lines of our Assumption 4.5.
5 Empirical illustration

We illustrate our local approach with a study of the influence of network structure on favor exchange in the empirical setting of Jackson et al. (2012). In their work, Jackson et al. (2012) specify an equilibrium model of favor exchange where one agent is willing to perform a favor for another if there is a third agent who has a social connection to both agents and can monitor the exchange. Intuitively, the social norm is for agents to be altruistic and provide favors for each other and the third agent exerts social pressure on the other two agents to conform to this norm. Without this social pressure, agents have an incentive to ignore the social norm and not provide favors for each other. When such a norm enforcing third agent exists they say that the favor exchange relationship between the first two agents is supported. The notion of support is extended to agents by counting the number of supported links adjacent to that agent and we refer to this statistic as agent support.

Support contrasts alternative measures of social cohesion such as clustering. An agent’s clustering coefficient measures the fraction of agent pairs connected to the agent that are also connected to each other. One might think that high levels of agent clustering also drives favor exchange between agents. However, in the Jackson et al. (2012) model, the two are unrelated conditional on support. Intuitively, the decision for agents to exchange favors is made on the extensive margin in that it only depends on whether or not their relationship is monitored. Adding additional monitors may increase the level of clustering, but under the theory does not impact favor exchange all else equal. The authors take the existence of low-clustering high-support favor exchange networks in real-world network data as corroborating evidence for their model.

We apply our local approach to evaluate the influence of support and clustering on the number of favors exchanged directly. Following Jackson et al. (2012), we consider data from 75 rural villages in Karnakata, India and construct our favor exchange variable (the outcome) and monitoring networks as described in their footnote 39. Specifically, we use their hedonic network to construct the monitoring network and their physical favors network to define the

---

3Strictly speaking, Jackson et al. (2012) only define support for pairs of agents that exchange favors. That is, they say that the relationship between two agents is supported if they exchange favors and their relationship is monitored. We instead say that a relationship is supported if it is monitored, regardless of whether or not the relevant agents exchange favors.
Figure 2: These three networks constitute the “cutlery ensemble.” We use them to describe three different ways that agents can monitor each other in a social network. The first network on the left is the “knife” network rooted at agent $\alpha$. The second network in the middle is the “fork” network rooted at agent $\beta$. The third network on the right is the “spoon” network rooted at the agent $\gamma$.

number of favors exchanged. The data was originally collected by [Banerjee et al. (2013)] to study the diffusion of information about a microfinance program.

To be sure, the social network links are not randomly assigned in this setting and our causal interpretation hinges on the validity of our Assumption 4.1. In particular, we assume that any other driver of favor exchange in the village are unrelated to the structure of the monitoring network. To the extent that this exogeneity assumption is violated, the associations inferred from our methodology may not be causal.

Below we test the hypothesis of policy irrelevance and estimate a policy effect for two pairs of rooted networks describing three different ways that agents can monitor each other in the hedonic network. The networks we chose are given by the “cutlery ensemble” of rooted networks depicted in Figure 2. In the knife network, the root agent $\alpha$ has one supported relationship and a clustering coefficient of 0. In the fork network, the root agent $\beta$ has two supported relationships and a clustering coefficient of 0. In the spoon network, the root agent $\gamma$ has three supported relationships and a clustering coefficient of 1. We chose these networks for our illustration because they are simple to describe and have varying degrees of support and clustering.

Our first policy comparison is the level of favor exchange in the fork network to that of the knife network. Specifically, we compare the distributions of $Y_{\alpha}$ and $Y_{\beta}$ where the two random variables refer to the number of favors performed by $\alpha$ and $\beta$ respectively. Intuitively, this comparison measures the effect of taking $\alpha$’s community as given by the knife network, adding an additional fourth agent, and connecting them to the center agent of the knife. This policy change increases the support of the root agent from 1 to 2 but does not impact
the clustering coefficient of the root.

We test the hypothesis that $Y_\alpha$ and $Y_\beta$ are equal in distribution using the approximate randomization test described in Section 4.2 with $q = 5$. We obtain a p-value of 1.0, which we interpret as providing no evidence against the hypothesis. We estimate the expected difference in favors performed using the $k$-nearest neighbors procedure described in Section 4.3 with $k = 10, 20, 50$. We compute a difference of 0 favors with $k = 10$, $-0.05$ favors with $k = 20$, and $0.06$ favors with $k = 50$, which we interpret as being qualitatively similar to an expected policy effect of 0.

Our second policy comparison is the level of favor exchange in the spoon network to that of the fork network (i.e. we compare the distributions of $Y_\beta$ and $Y_\gamma$). Intuitively, this comparison measures the effect of taking $\beta$’s community as given by the fork network and adding a social connection between agent $\beta$ and the other “prong” of the fork directly below $\beta$. This policy change increases the support of the root agent from 2 to 3 and the clustering coefficient of the root agent from 0 to 1.

We test the hypothesis that $Y_\beta$ and $Y_\gamma$ are equal in distribution with $q = 5$ and find a p-value of 0.078, which we interpret as providing strong evidence against the hypothesis. We estimate the expected difference in favors performed with $k = 10, 20, 50$ and compute a difference of 0.5 favors with $k = 10$, 0.3 favors with $k = 20$, and 0.55 favors with $k = 50$, which we interpret as being qualitatively different from an expected policy effect of 0.

Ultimately, we view these empirical results as challenging the view that clustering does not play a role in favor exchange. This is because the first policy comparison (between knife and fork) constitutes an increase in support holding clustering fixed, but we found no evidence of a change in favor exchange. The second policy comparison (between spoon and fork) constitutes both an increase in support and clustering and we did find evidence of an increase in favor exchange. Our conclusion is that there may still be some role for clustering to explain altruism in real-world networks.\footnote{We thank Ben Golub for helping us interpret these results in the context of Jackson et al. (2012)’s model.}
6 Conclusion

This paper proposes a new nonparametric modeling framework for causal inference under network interference. Rooted networks serve the role of effective treatments in that they index the ways in which the treatment assignments and network structure can influence the agent outcomes. We demonstrate the approach with a test for the hypothesis of policy irrelevance, an estimation strategy for average or distributional policy effects, and an empirical illustration studying the effect of network structure on social capital formation.

Much work remains to be done, as indicated in the discussions of the assumptions and results in the sections above. Other potential directions for future work include considering the problem of policy learning under network interference (see for example Ananth 2020; Viviano 2019; Kitagawa and Wang 2020), applying the framework to identify and estimate the parameters of a strategic network formation model (as in for instance de Paula et al. 2018), or semiparametrically estimating average treatment effects for binary treatment (as in, for instance Abadie and Imbens 2006). Ultimately we see our work as one step in a direction allowing for the more flexible econometric modeling of sparse network structures.

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### A Proof of claims

#### A.1 Properties of \((G, d)\)

The results of this section consider a more general notion of rooted network space than given in Section 3.2.1. Specifically, each agent \(i\) is associated with \(P\) covariates \(C_i \in \mathbb{R}^P\) where the \(p\)th entry of \(C_i\) is given by \(c_{ip}\) and \(C = \{C_i\}_{i \in I}\). This is as opposed to Section 3.2.1 where each agent \(i\) is associated with just one covariate \(T_i \in \mathbb{R}\) called treatment assignment. The definition of the \(\epsilon\)-isomorphism in Section 3.2.1 used to define the rooted network distance in this more general case is modified so that \(|c_{jp} - c_{f(j)p}| < \epsilon\) for all \(p \in [P]\). Otherwise the definition of the rooted network space \((G, d)\) is identical.

**Proposition A.1.** The rooted network distance \(d\) is a pseudometric on the set of locally-finite rooted networks.

**Proof.** To be a pseudo-metric on the set of rooted networks, \(d\) must satisfy (i) the identity condition \(d(g, g) = 0\), (ii) the symmetry condition \(d(g, g') = d(g', g)\), and (iii) the triangle inequality \(d(g, g'') \leq d(g, g') + d(g', g'')\) for any three rooted networks \(g, g', g''\). Conditions (i) and (ii) follow immediately from the definition of \(d\). We demonstrate (iii) below.
Fix an arbitrary $g,g',g''$ and denote the vertex sets, edge sets, and agent-specific covariates associated with $g$ by $(V(g), E(g), C(g))$. Suppose $d(g,g') + d(g',g'') = \eta$. Recall $\zeta(x) = (1 + x)^{-1}$. Then by the definition of $d$, for every $\nu > 0$ there exist $\varepsilon, \varepsilon' \in \mathbb{R}_+$ and $r, r' \in \mathbb{Z}_+$ with $\zeta(r) + \varepsilon + \zeta(r') + \varepsilon' < \eta + \nu$ such that for every $x, x' \in \mathbb{Z}_+$ there exists root-preserving bijections $f_x : V(g'')^x \leftrightarrow V(g''')^x$ and $f'_x : V(g''')^x \leftrightarrow V(g''''')^x$ such that $e_{jk}(g'') = e_{f_x(j)f_x(k)}(g''')$, $|c_{jp}(g'') - c_{f_x(j)p}(g''')| \leq \varepsilon$, for every $j, k \in V(g'')^x$, $p \in [p]$, and $e_{jk}(g'''') = e_{f'_x(j)f'_x(k)}(g''''')$, $|c_{jp}(g'''') - c_{f'_x(j)p}(g''''')| \leq \varepsilon$, for every $j, k \in V(g''''')^x$, $p \in [p]$.

Let $r'' = \min(r, r')$. We construct a root-preserving bijection $f''_x : V(g''')^x \leftrightarrow V(g''''')^x$ for every $x'' \in \mathbb{Z}_+$ such that $e_{jk}(g''''') = e_{f''_x(j)f''_x(k)}(g''''')$ and $|c_{jp}(g''''') - c_{f''_x(j)p}(g''''')}| \leq \varepsilon + \varepsilon'$ for every $j, k \in V(g''''')^x$, $p \in [p]$. It follows from the definition of $d$ and $\zeta$ that $d(g,g'') \leq \zeta(r'') + \varepsilon + \varepsilon' \leq \zeta(r) + \zeta(r') + \varepsilon + \varepsilon' < \eta + \nu$.

Since $\nu > 0$ was arbitrary, (iii) follows.

To construct such a bijection $f''_x$, we rely on the fact that for any rooted network $g$ and $a, b \in \mathbb{R}_+$, $V(g^a)^b = V(g)^a = V(g^{\min(a,b)})$. This implies that for any $x'' \leq r''$, $V(g)^{x''} = V(g^{x''})^{r''}$, $V(g^{x''})^{r''} = V(g)^{r''} = V(g^{r''})^{x''}$, and $V(g^{r''})^{x''} = V(g^{r''})^{x''}$, and $V(g^{r''})^{x''} = V(g^{r''})^{x''}$. For any $x'' > r''$, $V(g^{x''})^{r''}$ and $V(g^{r''})^{x''}$ are both equal to $V(g^{r''})^{x''}$. Furthermore by construction $e_{jk}(g''') = e_{f''_x(j)f''_x(k)}(g''''')$ for every $j, k \in V(g''')^{x''}$, and by the triangle inequality for $|\cdot|$ we obtain $|c_{jp}(g''') - c_{f''_x(j)p}(g''''')| \leq \varepsilon + \varepsilon'$ for every $j \in V(g''')^{x''}$ and $p \in [p]$.

**Proposition A.2.** $(\mathcal{G}, d)$ is a separable metric space.

**Proof.** To be separable, $\mathcal{G}$ must have a countable dense subset. Let $\tilde{\mathcal{G}}$ be the subset of rooted networks with finite vertex sets and rational-valued covariates. $\tilde{\mathcal{G}}$ is countable because it is the countable union of countable sets. The claim follows by showing that $\tilde{\mathcal{G}}$ is dense in $\mathcal{G}$.

To demonstrate that $\tilde{\mathcal{G}}$ is dense in $\mathcal{G}$, we fix an arbitrary rooted network $g \in \mathcal{G}$ and $\eta > 0$, and show that there exists a rooted network $g' \in \tilde{\mathcal{G}}$ such that $d(g,g') \leq \eta$. Choose any $r \in \mathbb{Z}_+$, $\varepsilon \in \mathbb{R}_+$ such that $\zeta(r) + \varepsilon \leq \eta$. Define $g'$ to be the rooted network on the
same set of agents and edge weights as \( g' \), but with rational-valued covariates chosen to be uniformly within \( \varepsilon \) of their analogues in \( g \). Since \( g \) is locally finite, \( g' \) has a finite vertex set, and so \( g' \in \tilde{G} \). Furthermore \( g' \) and \( g' \) are \( \varepsilon \)-isomorphic by construction. It follows that 
\[
d(g, g') \leq \zeta(r) + \varepsilon \leq \eta.
\]

\( \square \)

**Proposition A.3.** \((\mathcal{G}, d)\) is a complete metric space.

**Proof.** To be complete, every Cauchy sequence in \( \mathcal{G} \) must have a limit that is also in \( \mathcal{G} \). Let \( g_n \) be a Cauchy sequence in \( \mathcal{G} \). That is, for every \( \eta > 0 \) there exists an \( m \in \mathbb{N} \) such that 
\[
d(g_{m'}, g_{m''}) \leq \eta \quad \text{for every } m', m'' \in \mathbb{N} \text{ with } m', m'' \geq m.
\]

Fix an arbitrary \( r \in \mathbb{Z}_+ \) and let \( g^r_n \) be the sequence of rooted networks formed by taking \( g_n \) and replacing each rooted network in the sequence with its truncation at radius \( r \). Since \( \mathcal{G} \) is locally finite, the vertex set of each element of \( g^r_n \) is finite. Let \( N^r_n := |V(g^r_n)| \) denote the sequence of vertex set sizes. Since \( g_n \) is a Cauchy sequence, \( g^r_n \) is a Cauchy sequence, and so the entries of \( N^r_n \) must also be uniformly bounded. Let \( N^r := \limsup_{n \to \infty} N^r_n < \infty \).

Let \( g^r_{s(n)} \) be an infinite subsequence of \( g^r_n \) such that \( N^r_{s(n)} = V \) for some \( V \in \mathbb{N} \) and every \( n \in \mathbb{N} \). Such a subsequence and choice of \( V \) must exist because \( N^r_n \) takes only finitely many values. Since \( g^r_n \) is a Cauchy sequence, \( g^r_{s(n)} \) is a Cauchy sequence, and so for every \( \eta > 0 \) there exists an \( m \in \mathbb{N} \) such that for every \( m', m'' \in \mathbb{N} \) with \( m', m'' \geq m \) there exists a bijection 
\[
f : V(g^r_{s(m')}) \leftrightarrow V(g^r_{s(m''}) \quad \text{with } e_{f(j)f(k)}(g^r_{s(m')}) = e_{f(j)f(k)}(g^r_{s(m'')}) \quad \text{for every } j, k \in V(g^r_{s(m')}) \text{ and } |e_{f(j)p}(g^r_{s(m')}) - e_{f(j)p}(g^r_{s(m'')})| \leq \eta \quad \text{for every } j \in V(g^r_{s(m')}) \text{ and } p \in [P].
\]

Let the rooted networks in the sequence \( \tilde{g}^r_{s(n)} \) be 0-isomorphic to those in \( g^r_{s(n)} \) but with agents ordered in a canonical way such that for any \( j \in [V] \) and \( p \in [P] \) the corresponding sequences of covariates \( e_{jp}(\tilde{g}^r_{s(n)}) \) are Cauchy sequences in \( \mathbb{R} \) and the corresponding sequence of edges \( e_{jk}(\tilde{g}^r_{s(n)}) \) are constant. The resulting sequence of covariate matrices are Cauchy sequences with respect to the max-norm in \( \mathbb{R}^{V \times P} \), and so converge to a unique limit \( C \) because finite-dimensional Euclidean space is complete. Let \( E = e_{jk}(\tilde{g}^r_{s(n)}) \) be the constant (in \( n \)) edge set.

Let \( g^r_\infty \) be the rooted network with vertex set \([V]\), edge set \( E \) and covariate set \( C \). Then \( g^r_\infty \) is a limit for the sequence \( g^r_{s(n)} \) by construction. It is also a limit for the sequence \( g^r_n \) because \( g^r_{s(n)} \) is a subsequence of \( g^r_n \) and \( g^r_n \) is a Cauchy sequence. Since \( r \in \mathbb{Z}_+ \) was arbitrary,
$g_r^*$ converges to $g_{\infty}^r \in \mathcal{G}$ for any $r \in \mathbb{Z}_+$. Define $g$ to be the rooted network such that $g^r = g_{\infty}^r$ for every $r \in \mathbb{Z}_+$. Then $g$ is the limit of $g_n$ and the claim follows.

A.2 Theorem 4.1

We verify the assumptions of Theorem 4.2 in Canay and Kamat (2018). Specifically, we verify their Assumption 4.5 in the case where $h(g,U)$ is continuous and their Assumption 4.6 in the case where $h(g,U)$ is discrete. Their assumptions 4.5 and 4.6 parts (i), (ii) and (iii) follow from our Lemma A.2 below along with our Assumption 4.3 that $h(g,U)$ is either continuous or discrete. Assumption 4.5 part (iv) follows from our choice of test statistic in Step 4 of Algorithm 4.1.

Our proof of Lemma A.2 relies on the following Lemma A.1. Lemma A.1 is also used to demonstrate Theorem 4.2 below.

Lemma A.1. Let Assumption 4.1 hold and for any measurable $f : \mathbb{R} \times \mathcal{G} \rightarrow \mathbb{R}$ define

$$r(g) = E[f(h(g,U_{ic}),g)].$$

Then for any $g \in \mathcal{G}$, the entries of

$$(W_1^*(g),\ldots,W_C^*(g))$$

are independent conditional on $G_1(g),\ldots,G_C(g)$ and for every $1 \leq j \leq C$

$$E[f(W_j^*(g))|G_1(g),\ldots,G_C(g)] = r(G_j^*(g)).$$

Proof. Proposition 8.1 of Biau and Devroye (2015) directly implies the first claim that the elements of $(W_1^*(g),\ldots,W_C^*(g))$ are independent conditional on $G_1(g),\ldots,G_C(g)$. For the second claim, note that by a similar argument to the second statement in Proposition 8.1 of Biau and Devroye (2015),

$$E[f(W_j^*(g))|G_1(g),\ldots,G_C(g)] = \tilde{r}_g(G_j^*(g)).$$
where $\tilde{r}_g(\bar{g}) = E[f(W_c(g))|G_c(g) \simeq_0 \bar{g}]$. Next we show that $\tilde{r}_g(\bar{g}) = r(\bar{g})$ (up to $\nu$-null sets, where $\nu$ is the pushforward measure induced by $G_c(g)$). Once we have shown that, then it will follow that $\tilde{r}_g(G_j^*(g)) = r(G_j^*(g))$ (since the pushforward measure induced by $G_j^*(g)$ is dominated by $\nu$ by construction) which demonstrates the claim.

Let $\tau$ be the random index such that $W_{\tau,c} = W_c(g)$, then

$$E[f(W_c(g))|G_{1c}, \ldots, G_{m_{c,c}}] = \sum_{i=1}^{m_c} 1\{\tau = i\} E[f(W_{i,c})|G_{1c}, \ldots, G_{m_{c,c}}]$$

$$= \sum_{i=1}^{m_c} 1\{\tau = i\} E[f(h(G_{ic}, U_{ic}), G_{ic})|G_{1c}, \ldots, G_{m_{c,c}}]$$

$$= \sum_{i=1}^{m_c} 1\{\tau = i\} E[f(h(G_{ic}, U), G_{ic})|G_{ic}]$$

$$= \sum_{i=1}^{m_c} 1\{\tau = i\} r(G_{ic})$$

$$= r(G_c(g)),$$

where the first equality follows from the fact that $\tau$ is a function of $G_{1c}, \ldots, G_{m_{c,c}}$, the second follows from the definition of $W_{ic}$, and the third and fourth equalities follow from Assumption 4.1(iii). By the law of iterated expectations and the fact that $G_c(g)$ is a measurable function of $G_{1c}, \ldots, G_{m_{c,c}}$ it follows that

$$\tilde{r}_g(G_c(g)) = E[E[f(W_c(g))|G_{1c}, \ldots, G_{m_{c,c}}]|G_c(g)] = r(G_c(g)),$$

and so $\tilde{r}_g(\bar{g}) = r(\bar{g})$ (up to $\nu$-null sets).

\[\square\]

**Lemma A.2.** Under Assumptions 4.1, 4.2 and 4.3 and the null hypothesis of policy irrelevance \[\square\]

\[
S_C \overset{d}{\rightarrow} S
\]

where $S = (S_1, \ldots, S_{2q})$ is a random vector with independent and identically distributed entries equal in distribution to $h(g, U)$. For any permutation $\pi \in \Pi$

\[
S^{\pi} \overset{d}{=} S
\]

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so that we satisfy (i) and (ii) of Assumptions 4.5 and 4.6 of Canay and Kamat (2018).

Proof. Lemma A.1 implies that the entries of
\[(W^*(g), W^*(g')) := (W^*_1(g), \ldots, W^*_q(g), W^*_1(g'), \ldots, W^*_q(g'))\]
are independent conditional on \(\{G_c(g)\}_{c \in D_1}\) and \(\{G_c(g')\}_{c \in D_2}\), and that the conditional distribution functions of \(Y^*_j(g)\) and \(Y^*_j(g')\) are given by \(h_y(G^*_j(g))\) and \(h_y(G^*_j(g'))\) respectively. It follows by the law of iterated expectations that
\[
P\left(Y^*_1(g) \leq y_1, \ldots, Y^*_q(g) \leq y_q, Y^*_1(g') \leq y_{q+1}, \ldots, Y^*_q(g') \leq y_{2q}\right) = E\left[\prod_{j=1}^{q} h_{y_j}(G^*_j(g)) \prod_{j=1}^{q} h_{y_{j+q}}(G^*_j(g'))\right].
\]

We first show that Assumption 4.2 implies that \(G^*_j(g) \overset{p}{\rightarrow} g\) and \(G^*_j(g') \overset{p}{\rightarrow} g'\) for every \(j\) as \(C \to \infty\). To see this, fix \(\epsilon > 0\) and write
\[
\{d\left(G^*_j(g), g\right) > \epsilon\} = \left\{ \frac{1}{C} \sum_{c=1}^{C} 1\{G^*_c(g) \in B_{g,\epsilon}\} < \frac{j}{C} \right\},
\]
where \(B_{g,\epsilon} = \{\hat{g} \in G : d(g, \hat{g}) \leq \epsilon\}\). By the law of large numbers and Assumption 4.2
\[
\frac{1}{C} \sum_{c=1}^{C} 1\{G^*_c(g) \in B_{g,\epsilon}\} \overset{p}{\rightarrow} P\left(G^*_c(g) \in B_{g,\epsilon}\right) > 0,
\]
and since \(j/C \leq q/C \to 0\), it follows that \(P\left(d\left(G^*_j(g), g\right) > \epsilon\right) \to 0\) as \(C \to \infty\).

Assumption 4.3 and the continuous mapping theorem imply that
\[h_y(G^*_j(g)) \overset{p}{\rightarrow} h_y(g)\]
and
\[h_y(G^*_j(g')) \overset{p}{\rightarrow} h_y(g'),\]
for every \(j\) and \(y \in \mathbb{R}\), and so it follows from the dominated convergence theorem that
\[
E\left[\prod_{j=1}^{q} h_{y_j}(G^*_j(g)) \prod_{j=1}^{q} h_{y_{j+q}}(G^*_j(g'))\right] \to \prod_{j=1}^{q} h_{y_j}(g) \prod_{j=1}^{q} h_{y_{j+q}}(g').
\]
The claim follows from the fact that \( h_y(g) = h_y(g') \) under the null hypothesis \([2]\).

### A.3 Theorem 4.2

**Proof.** Our proof follows that of [Döring et al. (2017)](#), Theorem 6. Let \( G_c(g) \) be the rooted network corresponding to \( W_c(g) \). By Lemma [A.1], the entries of

\[
(W_1^*(g), \ldots W_k^*(g))
\]

are independent conditional on \( G_1(g), \ldots, G_C(g) \) with

\[
E[Y_j^*(g)|G_1(g), \ldots G_C(g)] = h(G_j^*(g)) .
\]

Let \( \bar{h}(g) = E[\hat{h}(g)|G_1(g), \ldots G_C(g)] = \frac{1}{k} \sum_{j=1}^{k} h(G_j^*(g)) \). Decomposing the mean-squared error gives

\[
E[(\hat{h}(g) - \bar{h}(g))^2] = E[(\hat{h}(g) - \bar{h}(g))^2] + E[(\bar{h}(g) - h(g))^2] ,
\]

and the claim follows by bounding \( E[(\hat{h}(g) - \bar{h}(g))^2] \leq \sigma^2/k \) and \( E[(\bar{h}(g) - h(g))^2] \leq E[\varphi^2(U_{k,T})^2] \).

To bound the first term, write

\[
E[(\hat{h}(g) - \bar{h}(g))^2|G_1(g), \ldots, G_C(g)] = E \left[ \left( \frac{1}{k} \sum_{j=1}^{k} (Y_j^*(g) - h(G_j^*(g))) \right)^2 \right| G_1(g), \ldots, G_C(g)
\]

\[
= \frac{1}{k^2} \sum_{j=1}^{k} \sigma^2(G_j^*(g)) ,
\]

where \( \sigma^2(\tilde{g}) = E[(h(\tilde{g}, U_{ic}) - h(\tilde{g}))^2] \) and the second equality follows from Lemma [A.1] because \( \{Y_j^*(g) - h(G_j^*(g))\}_{j=1}^{k} \) are independent and mean zero conditional on \( \{G_c(g)\}_{i=1}^{C} \) and

\[
E \left[ (Y_j^*(g) - h(G_j^*(g)))^2 \right| G_1(g), \ldots, G_C(g) = \sigma^2(G_j^*(g)).
\]

By Assumption [4.6], \( \sigma^2(G_j^*(g)) \leq \sigma^2 \) and so

\[
E[(\hat{h}(g) - \bar{h}(g))^2|G_1(g), \ldots G_C(g)] \leq \sigma^2/k .
\]

\( E[(\hat{h}(g) - \bar{h}(g))^2] \leq \sigma^2/k \) follows from the law of iterated expectations.
To bound the second term, Assumption 4.5 and the definition of the upper generalized inverse implies

\[ |h(g) - h(\tilde{g})| \leq \varphi_g(\psi_g(d(g, \tilde{g}))) \]

and so

\[
\begin{align*}
(\bar{h}(g) - h(g))^2 & \leq \left( \frac{1}{k} \sum_{j=1}^{k} |h(g) - h(G_j^*(g))| \right)^2 \\
& \leq \left( \frac{1}{k} \sum_{j=1}^{k} \varphi_g(\psi_g(d(g, G_j^*(g)))) \right)^2 \\
& \leq \varphi_g(\psi_g(d(g, G_k^*(g))))^2.
\end{align*}
\]

Under Assumption 4.4, the probability integral transform implies

\[ \psi_g(d(g, G_k^*(g))) \overset{d}{=} U(k, C), \]

where \( U(k, C) \) is the \( k \)th order statistic from a sequence of \( C \) independent and identically distributed standard uniform random variables, and so is distributed \( Beta(k, C + 1 - k) \). As a result

\[ \varphi_g(\psi_g(d(g, G_k^*(g))))^2 \overset{d}{=} \varphi_g(U(k, C))^2 \]

and so

\[ E[(\bar{h}(g) - h(g))^2] \leq E[\varphi_g(U(k, C))^2]. \]

\[ \square \]

B Interpreting the bias

The bound on estimation error described in Theorem 4.2 of Section 4.3.3 contains a variance term and a bias term. The variance term is standard. The bias term is new and so in this section we characterize how its features depend on the components \( \phi_g \) and \( \psi_g \). Intuitively, the first controls the smoothness of the regression function \( h(g) \) and the second controls the quality of the nearest-neighbors that make up \( \bar{h}(g) \) in terms of proximity to \( g \). Supporting
simulation evidence can be found in Appendix Section C below.

The continuity parameter $\phi_g$ is often relatively easy to characterize using economic theory because many models of network interference give an explicit bound. In particular, for our three examples in Section 2.3, $\phi_g(x)$ quickly converges to 0 with $x$ for any $g \in \mathcal{G}$. In the neighborhood spillovers model of Example 2.1 with binary treatments and uniformly bounded expected outcomes, $\phi_g(x) \leq M \mathbb{1}\{x > \frac{1}{1+r}\}$ where $M = \sup_{g \in \mathcal{G}} 2h(g)$, because if $d(g, \tilde{g}) \leq \frac{1}{1+r}$ then $h(g) = h(\tilde{g})$. Similarly, in the social capital formation model of Example 2.2 with uniformly bounded 2-neighborhoods, $\phi_g(x) \leq M \mathbb{1}\{x > \frac{1}{3}\}$ where $M$ bounds the number of agents within path distance 2 of the root agent. In both of these examples, the function $\phi_g(x)$ is flat in a neighborhood of 0 and so there is no asymptotic bias ($C \to \infty$).

In the linear-in-means peer effects model of Example 2.3 with binary treatments and uniformly bounded 1-neighborhood treatment counts, $\phi_g(x) \leq \frac{M(\delta \rho)(1-x)}{1-\delta \rho} \frac{1}{x}$ where $\rho$ bounds the spectral radius of $A^*(D)$, $|\delta \rho| < 1$ by assumption, and $M = \sup_{i \in V(\tilde{g}), \tilde{g} \in \mathcal{G}} 2|T_i \beta + T_i^*(1) \gamma|$. This is because if $d(g, \tilde{g}) \leq \frac{1}{1+r}$ then $h^s(g) = h^s(\tilde{g})$ for every $s \leq r$ and the remainder term in the policy function $|\sum_{s=r+1}^{\infty} \delta^s A^*(D)^s(\mathbf{T} \beta + \mathbf{T}^*(1) \gamma)|_i \leq C \sum_{s=r+1}^{\infty} \delta^s \rho^s \leq \frac{M(\delta \rho)^r}{1-\delta \rho}$ for any $g \in \mathcal{G}$. In contrast to the first two examples, in this example the function $\phi_g(x)$ is not necessarily flat in a neighborhood of 0. However, it is close to flat in the sense that its slope can be made arbitrarily uniformly close to 0 in an open neighborhood of 0.

In contrast to these explicit bounds on $\phi_g$, we are not aware of any similarly convenient way to analytically characterize the regularity parameter $\psi_g$ even for relatively simple network formation models. If the network is sparse or has a predictable structure (agents interact in small groups or on a regular lattice), then the rooted network variable may essentially act like a discrete random variable, and so $\psi_g(\ell)$ may be uniformly bounded away from 0 for any fixed $\ell > 0$. For example, the model of [Jackson et al., 2012] implies that certain social networks should in equilibrium be described by a union of completely connected subgraphs. If agent degree is also bounded, then this model has $\psi_g(\ell)$ uniformly bounded away from 0 for any fixed $\ell > 0$ and so there is no asymptotic bias ($C \to \infty$).

Irregular network formation models are more common empirically, however. For example, a large literature considers models of network formation in which connections between agents are conditionally independent across agent-pairs. Examples include the Erdős-Renyi
model, latent space model, and stochastic blockmodel (see broadly Graham 2019). For such models, the associated $\psi_g$ function can change dramatically with the model parameters. We demonstrate some example $\psi_g$ functions for the special case of the Erdős-Renyi model in Section C.4 but leave a more detailed study to future work. The Erdős-Renyi model is chosen not because it generates realistic network formation patterns but it is close to uniformly distributed on the set of networks conditional on the number of vertices. Such a distribution provides a relatively unfavorable $\psi_g$ in the sense that all possible rooted networks occur in this model each with relatively low probability.

C Simulation evidence

We provide simulation evidence characterizing some of the finite sample properties of our test procedure and estimator. Section C.1 describes the simulation design, Section C.2 gives the results for the first application testing policy irrelevance, and Section C.3 gives the results for the second application estimating policy effects.

C.1 Simulation design

We simulate data from $C$ communities, where $C$ is specified below. Each community contains 20 agents. Links between agents are drawn from an Erdős-Renyi model with parameter 0.1. That is, links between agent-pairs are independent and identically distributed $Bernoulli(0.1)$ random variables. The Erdős-Renyi model is chosen not because it generates realistic-looking network data (see for instance Jackson and Rogers 2007) but because a large class of rooted networks occur with non-trivial probability. This design choice is unfavorable to our method, which prefers models that reliably generate a small number of rooted network motifs. One can think of the Erdős-Renyi model as the policy maker assigning network connections to agents in each community completely at random. We discuss some features of the Erdős-Renyi model in more detail in Section C.4 below.

Outcomes are generated for each agent $i \in [20]$ in each community $c \in [C]$ according to
the model

\[ Y_{ic} = \alpha_1 f(G_{ic}^1) + \alpha_2 f(G_{ic}^2) + U_{ic} \]

where \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \) is specified below,

\[ f(g) := \text{deg}(g) + 2\text{clust}(g), \]

\[ \text{deg}(g) := \frac{1}{|V(g)|} \sum_{i \in V(g)} \sum_{j \in V(g)} 1\{ij \in E(g)\} \]

measures the average degree of the network \((V(g), E(g))\),

\[ \text{clust}(g) := \frac{1}{|V(g)|} \sum_{i \in V(g)} \sum_{j \in V(g)} \sum_{k \in V(g)} 1\{ij, ik, jk \in E(g)\} \]

measures the average clustering of the network \((V(g), E(g))\), and \( U_{ic} \sim U[-5, 5] \) independent of \( G_{ic} \). Our focus on degree and clustering statistics is meant to mimic the first two examples of Section 2.3 and the empirical example of Section 5. That is, these network statistics are determined by the rooted network truncated at the first or second neighborhood.

C.2 Testing policy irrelevance

We first evaluate how the test procedure outlined in Algorithm 4.1 of Section 4.2.2 controls size when the null hypothesis of policy irrelevance is true. The choice of \( \alpha \) we consider is \((0, 2)\). The choice of rooted networks (policies) we consider is represented by \( g_1 \) and \( g_2 \) in Figure 2. These networks are chosen because under the model in Section 5.1 the conditional distribution of outcomes associated with \( g_1 \) and \( g_2 \) are the same, but the conditional distribution of outcomes associated with \( g_1^3 \) and \( g_1^4 \) are very different. The idea of this design is to illustrate the potential size distortion due to the fact that the permutation test is approximate. This can be seen in Table 1.

Columns 2-4 of Table 1 depict the results of 1000 simulations for \( C \in \{20, 50, 100, 200\} \) communities and test parameters \( q \in \{5, 8, 10\} \) and \( \alpha = 0.05 \). The results show that the test rejects the null hypothesis with probability approximately equal to \( \alpha \) when \( q \) is small.
Figure 3: Four rooted networks truncated at radius 2, labeled \( g_1 \) to \( g_4 \).

\((q = 5)\) or \( C \) is large \((C = 200)\). Size distortion occurs when \( q \) is large and \( C \) is small \((q \geq 8 \text{ and } C \leq 50)\).

To evaluate the power properties of the test procedure, we consider two rooted networks \( g_3 \) and \( g_4 \) that are associated with two different conditional distributions of outcomes under the model in Section 5.1. These two networks are shown in Figure 2.

Columns 5-7 depict the results for the same simulations as Columns 2-4 but for the test based on \( g_3 \) and \( g_4 \) instead of \( g_1 \) and \( g_2 \). The results show that the test correctly rejects the null hypothesis with probability greater than \( \alpha \). The probability of rejection generally increases with \( q \) (except for \( C = 20 \)) at the cost of potential size distortions. Overall, our results suggest that unless a researcher has additional information about the structure of network interference relative to the quality of matches, they should not choose \( q \) to be too large. This is why we use \( q = 5 \) in the empirical example of Section 5.

Table 1: Rejection probabilities: \( \alpha = 0.05 \)

\[ \begin{array}{cccc|cccc}
\text{ } & H_0 : g_1 =_d g_2 & & & H_0 : g_3 =_d g_4 & & \\
q & q & q & q & q & q & q & \text{ } \text{ } \\
C & 5 & 8 & 10 & 5 & 8 & 10 & \\
20 & 6.8 & 9.0 & 8.3 & 18.6 & 20.4 & 19 & \\
50 & 4.2 & 5.8 & 8.6 & 21.7 & 33.5 & 42.4 & \\
100 & 3.6 & 4.7 & 5.7 & 23.7 & 35.9 & 42.6 & \\
200 & 4.9 & 4.2 & 4.6 & 27.4 & 40.5 & 48.6 & \\
\end{array} \]

C.3 Estimating policy effects

We study the mean-squared error of the \( k \)-nearest-neighbor estimator for the policy function \( h \) given in Section 4.3.2 for four rooted networks and \( \alpha \in \{(1, 0), (1, 1/2)\} \). Under \( \alpha = (1, 0) \) the distribution of outcomes depends on the features of the network within radius 1 of the
root. Under $\alpha = (1, 1/2)$ the distribution of outcomes also depends on the features of the network within radius 2 of the root.

The choice of rooted networks we consider is represented by $g_3$ to $g_6$ in Figure 4. Networks $g_3$ and $g_5$ depict two wheels with the rooted agent on the periphery. These networks have moderate average degree and no average clustering: $(3/2, 0)$ and $(5/3, 0)$ respectively. Network $g_4$ depicts a closed triangle connected to a single agent. This network has moderate average degree and high average clustering $(2, 7/12)$. Finally, network $g_6$ depicts a closed triangle connected to a wheel with the rooted agent both on the periphery of the wheel and part of the triangle. This network has moderate average degree and average clustering $(2, 1/3)$. The results of the simulation are shown in Table 2. As suggested by Theorem 4.2, mean-squared error is generally decreasing with $C$ for a fixed choice of $k$. In addition, mean-square error is generally smaller for the $\alpha = (1, 0)$ experiment than it is for the $\alpha = (1, 0.5)$ experiment for a fixed choice of $C$ and $k$. The effect is more pronounced for the networks $g_4$ and $g_6$, for which we typically observe fewer good matches in the data compared to $g_3$ and $g_5$ (we quantify this observation by estimating $\psi_g$ for each of the four networks in Section C.4). This is also consistent with Theorem 4.2.

Fixing $C$ and comparing across $k$, we expect a bias-variance trade-off. For networks $g_3$ and $g_5$, there is no meaningful bias in the estimated policy function because $f(\tilde{g}^1)$ and $f(\tilde{g}^2)$ are similar and $\psi_{\tilde{g}}(1) \approx 1$ for $\tilde{g} = g_3, g_5$ (see Section 5.4 below). As a result, it is optimal to use the nearest-neighbor from every community in $[C]$ (i.e. choose $k = C$). In contrast for $g_4$ and $g_6$ the rooted networks of the nearest neighbors in each community may be very different from the relevant policies, and so setting $k = C$ can lead to an inflated mean-squared error.

We conclude that unless the researcher has additional information about the structure of network interference or the density of the policies of interest, $k$ should not be large relative to the sample size. This is also consistent with our findings for the test of policy irrelevance above.

C.4 Measuring network regularity

In Section 4.3 we identified the function $\psi_g(\ell)$ as a key determinant of the estimation bias in Theorem 4.2. This function measures the probability that the nearest-neighbor of $g$ from
Figure 4: Four rooted networks truncated at radius 2, labeled $g_3$ to $g_6$.

Table 2: Estimated MSEs
(1,000 Monte Carlo iterations)

|     | $C = 20$ |      |      | $C = 50$ |      |      | $C = 100$ |      |      |
|-----|----------|------|------|----------|------|------|-----------|------|------|
| $\alpha$ | $g_3$ | 5   | 10  | 20   | 5   | 10  | 20  | 50  | 5   | 10  | 20  | 50  | 50  | 75  | 100 |
| (1,0) |        |      |      |        |      |      |        |      |      |        |      |      |        |      |      |      |
| $g_3$ | 1.71   | 0.84 | 0.41 | 1.69  | 0.86 | 0.43 | 0.18 | 1.63 | 0.81 | 0.41 | 0.17 | 0.11 | 0.08 |
| $g_4$ | 1.77   | 1.53 | 3.27 | 1.65  | 0.89 | 0.55 | 3.02 | 1.64 | 0.84 | 0.42 | 0.64 | 1.92 | 2.97 |
| $g_5$ | 1.7    | 0.84 | 0.42 | 1.75  | 0.84 | 0.44 | 0.17 | 1.66 | 0.85 | 0.42 | 0.17 | 0.12 | 0.09 |
| $g_6$ | 1.77   | 0.98 | 1.41 | 1.6   | 0.81 | 0.42 | 1.17 | 1.74 | 0.84 | 0.44 | 0.2  | 0.59 | 1.09 |
| (1,0.5)|      |      |      |        |      |      |        |      |      |        |      |      |        |      |      |      |
| $g_3$ | 1.71   | 0.84 | 0.41 | 1.69  | 0.86 | 0.43 | 0.18 | 1.63 | 0.81 | 0.41 | 0.17 | 0.11 | 0.09 |
| $g_4$ | 1.85   | 2.15 | 5.37 | 1.66  | 0.9  | 0.72 | 5.09 | 1.65 | 0.86 | 0.44 | 1.1  | 3.32 | 5.04 |
| $g_5$ | 1.7    | 0.84 | 0.42 | 1.75  | 0.84 | 0.44 | 0.17 | 1.66 | 0.86 | 0.42 | 0.17 | 0.12 | 0.09 |
| $g_6$ | 1.79   | 1.05 | 2.07 | 1.6   | 0.81 | 0.42 | 1.81 | 1.75 | 0.84 | 0.44 | 0.21 | 0.87 | 1.73 |

A randomly drawn network is within distance $\ell$ of $g$ under $d$.

Figure 5 displays estimates of the $\psi_g$ function for the four rooted networks considered in the simulation design of Section C.3. The figures were constructed by generating 3000 Erdős-Rényi(0.1) random graphs with 20 nodes each and recording the distances of the nearest neighbor to $g$ in each graph.

The results indicate that the nearest neighbor of $g_3$ always matches at least at a radius of 1, and often matches at a radius of 2. In contrast $g_4$ is only matched at a radius of 1 in 40% of networks, and is almost never matched at a radius of 2. Intuitively, networks $g_4$ and $g_6$ are rare because triadic closure is uncommon under the random graph model. The network $g_5$ is also relatively rare because the coincidence of five agents linked to a common agent is uncommon for such a sparse random network.

We remark that these results are the extreme case of a model of pure statistical noise. Strategic interaction between agents should in principle discipline the regularity of the network, particularly if only a small number of configurations are consistent with equilibrium linking behavior. Characterizing $\psi_g$ for such strategic network formation models is an important area for future work.
Figure 5: Estimated $\psi_g(\cdot)$ for $g_3$ to $g_6$ (from left to right, top to bottom).