A PATH INTEGRAL METHOD FOR COARSE-GRAINING NOISE IN
STOCHASTIC DIFFERENTIAL EQUATIONS WITH MULTIPLE
TIME SCALES

TOBIAS SCHÄFER∗ AND RICHARD O. MOORE†

Abstract. We present a new path integral method to analyze stochastically perturbed ordinary
differential equations with multiple time scales. The objective of this method is to derive from
the original system a new stochastic differential equation describing the system’s evolution on slow
time scales. For this purpose, we start from the corresponding path integral representation of the
stochastic system and apply a multi-scale expansion to the associated path integral kernel of the
corresponding Lagrangian. As a concrete example, we apply this expansion to a system that arises
in the study of random dispersion fluctuations in dispersion-managed fiber optic communications.
Moreover, we show that, for this particular example, the new path integration method yields the
same result at leading order as an asymptotic expansion of the associated Fokker-Planck equation.

1. Introduction. Physical phenomena exhibiting scale separation are ubiquitous in a wide range of fields, most obviously in those permitting a description via
continuum mechanics such as fluid dynamics, electromagnetism, and material science. The ability to separate the (macroscopic) evolution on slow time scales from the (mi-
croscopic) evolution on fast time scales is key to understanding a system’s behavior
or to performing numerical simulations in an efficient way. Many mathematical tech-
niques have been developed for explicitly performing this separation of scales in the
absence of stochastic fluctuations, e.g. multi-scale expansions [1], Lie transform [2],
and renormalization group methods [3]. In a typical multi-scale analysis, we start
from a given system, which we write here as an ordinary differential equation for a
vector \( x = x(t) \),

\[
\frac{dx}{dt} = f(x, t, \epsilon),
\]

where \( \epsilon \) is a small parameter. Upon introducing, for instance, two time scales \( t_0 = t \)
and \( t_1 = \epsilon^\gamma t \) (the actual power of \( \gamma > 0 \) is selected by the problem), we derive
systematically a new ordinary differential equation

\[
\frac{dX}{dt_1} = F(X, t_1),
\]

describing the evolution of the system on the slow time scale represented by \( t_1 \) through
\( X = X(t_1) \). In this article, we are interested in the effect of stochastic perturbations
on this description over the coarse-grained (slow) scales. Intuitively we expect to
derive from the microscopic SDE given by

\[
\frac{dx}{dt} = f(x, t, \epsilon) + g(x, t, \epsilon)\xi(t),
\]

where \( \xi \) represents white noise, a new stochastic differential equation of the form

\[
\frac{dX}{dt_1} = F(X, t_1) + G(X, t_1)\Xi(t_1),
\]

∗ Department of Mathematics, The College of Staten Island, City University of New York, New
York (tobias@math.csi.cuny.edu).
† Department of Mathematics, New Jersey Institute of Technology, Newark, New Jersey
(rmoore@njit.edu).
describing the slow scale evolution, where $\Xi$ represents white noise on the slow time scale $t_1$. Our intent is not to explore under what circumstances or over what range of parameters such an approximation is valid; rather, we present a practical method based on path integrals that can be used for explicitly calculating the coarse-grained stochastic differential equation from the original system, implicitly assuming that such an approach is valid.

For this purpose, we now state the problem in a slightly more formal way: Assume a system of stochastic ordinary differential (Langevin) equations expressed by

$$\dot{x} = f(x, t; \epsilon) + \sqrt{\epsilon}g(x, t)\xi(t), \quad x(0) = a$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, $f : [\mathbb{R}^n \times \mathbb{R}^+] \to \mathbb{R}^n$, $g : [\mathbb{R}^n \times \mathbb{R}^+] \to \mathbb{R}^n \times m$, and $f$ is assumed to be sufficiently differentiable in $\epsilon$. The random term, $\xi(t)$, is assumed to be $m$-dimensional delta-correlated white noise, i.e., with

$$\langle \xi_p(t)\xi_q(t') \rangle = \delta(t - t')\delta_{pq}$$

where $\langle \cdot \rangle$ denotes the ensemble average, and Gaussian-distributed with a probability density given by

$$p(\xi) = \frac{1}{(2\pi)^{m/2}} \exp\left(-\frac{1}{2} \xi \cdot \xi\right).$$

We choose to take the Stratonovich interpretation of Eqn. 1.1 given our interest in modeling physical processes; however, this is simply a convention given the general form of Eqn. 1.1. Further assumptions will be indicated as they become necessary.

The goal is to obtain another stochastic ODE that captures the evolution of $x$ on a slow scale when $0 < \epsilon \ll 1$, i.e., to “scale up” the noise from its microscopic representation to a macroscopic noise that reflects the fast microscopic response of $x$ in addition to the noise itself. The ability to integrate a macroscopic stochastic equation on a much slower time scale, where the microscopic trajectories can be reconstructed approximately from the macroscopic trajectories, is of obvious benefit from the perspectives of asymptotic analysis and numerical generation of the statistical properties of $x$.

For the reader who is less acquainted with the scaling of small parameters in stochastic equations, we comment briefly on the factor $\sqrt{\epsilon}$ in the problem (1.1) in the context of an expansion using multiple time scales. A naive expansion of $x$ in Eqn. 1.1 in powers of $\sqrt{\epsilon}$ is quickly seen to become disordered when $t \sim 1/\epsilon$ ($t \sim 1/\sqrt{\epsilon}$).

A standard method that is widely used in the deterministic case [1] to address this onset of secularity is the use of multiple scales in time, i.e., let

$$x = x^{(0)}(t_0, t_1, \ldots) + \epsilon x^{(1)}(t_0, t_1, \ldots) + \ldots,$$

where $t_j = \epsilon^j t$. Whereas the dependence of $x^{(0)}$, $x^{(1)}$, etc., on the “fast” time variable $t_0$ alone is insufficient to prevent this expansion of $x$ from becoming disordered, their dependence on the “slow” time variables $t_1$, etc., allows sufficient freedom to suppress secular growth up to times of order $1/\epsilon$. Before proceeding, we have to justify the particular expansion suggested in Eqn. 1.4. For simplicity, we do this in a one-dimensional context; the argument generalizes in an obvious way to higher dimensions.

Were $\xi(t)$ a deterministic forcing term, we would ordinarily expand $x$ and $t$ in powers of $\sqrt{\epsilon}$. In this case, however, $\xi(t)$ is formally the time-derivative of a Wiener process $W(t)$, with Eqn. 1.1 more appropriately expressed in the form

$$dx = f(x, t; \epsilon) \, dt + \sqrt{\epsilon}g(x, t) \, dW,$$
where \( \langle W(t)W(t') \rangle = \min(t, t') \), i.e., with \( \langle dW^2 \rangle \equiv \langle (W(dt) - W(0))^2 \rangle = dt \). As \( x(t) \)
 is a stochastic process, we wish to track all of its moments, as an alternative to the
direct evolution of the probability density function via the Fokker-Planck equation
(see Sec. 4). In a time step of \( dt \), we have

\[
\langle x(t + dt) - x(t) \rangle = \langle dx \rangle = \langle f(x, t; \epsilon) \rangle dt,
\]

\[
\langle x^2(t + dt) - x^2(t) \rangle = 2 \langle dx \rangle + \langle dx^2 \rangle \\
= 2 \langle f(x, t; \epsilon) \rangle dt + \epsilon \langle g^2(x, t) \rangle dt,
\]

so that all terms of \( \mathcal{O}(\epsilon^{j/2}) \) vanish for \( j \) odd. An expansion in powers of \( \epsilon \) is therefore
self-consistent. We also see from this consideration that the influence of the stochastic
driving term in Eqn. 1.1 is first felt at \( \mathcal{O}(\epsilon) \), suggesting that the term of \( \mathcal{O}(\sqrt{\tau} dW) \)
mimics a term of \( \mathcal{O}(\epsilon dt) \).

The paper is organized as follows: In Sec. 2 we present the path integral method
for deriving the slow stochastic dynamics from the original system. In Sec. 3 we
apply this method to a particular problem in fiber optics. In what follows, we show
that the new method, for the chosen example, yields the same result as an asymp-
totic expansion of the associated Fokker-Planck equation (Sec. 4). Conclusions are
presented in Sec. 5.

2. Averaging of path integral kernels. Instead of considering the stochastic
differential Eqn. 1.1 directly we can consider the probability density \( p(x, t) \)
corresponding to the stochastic process \( x \) and write down a path integral representation
for \( p(x, t) \) [4–6]:

\[
p(x, t) = \int_{C(x, t|a, 0)} \mathcal{D}x(\tau) \ e^{-\int_0^t L(x(\tau), \dot{x}(\tau), \tau) \ d\tau} = \int_{C(x, t|a, 0)} \mathcal{D}x(\tau) \mathcal{P}(x(\tau)) \tag{2.1}
\]

with the Lagrangian \( L \) given by

\[
L = \frac{1}{2\epsilon} \sum_{i=1}^{n} \hbar^2 \left( \dot{x}_k - \left( f_k + \epsilon s \frac{\partial g_{kj}}{\partial x_l} g_{lj} \right) \right)^2, \tag{2.2}
\]

written according to Einstein’s summation notation (see Appendix). Here we set \( s = 1/2 \) according to the Stratonovich interpretation of Eqn. 1.1. In cases where \( h = g^{-1} \)
does not exist, it is necessary to consider a regularized path integral representation.
A short derivation for both cases can be found in the appendix. In Eqn. 2.1 we take
all paths \( \Gamma \in C(x, t|a, 0) \) into account, that lead from the initial point \( (a, 0) \) to the
final point \( (x, t) \). We want to develop a multi-scale technique on the level of this
path integral representation and assume a periodicity on the fast time scale \( t_0 \)
with a period \( t^* \). Assume that the final time \( t = (K + 1)t^* \), meaning that we propagate
for \( K + 1 \) periods and with \( K \) intermediate times \( t_{ok} = kt^* \). The fast time scale
\( t_0 = t \) characterizes the system’s evolution within one period, e.g. for a time interval
\([t_{ok}, t_{ok} + t^*] \). The evolution on the time scale \( t_1 = ct \) characterizes the slow evolution
of the system. For a given path \( \Gamma \in C(x, t|a, 0) \), we take \( X_k = x_k = x(kt^*) \), \( 0 \leq k \leq K \)
as sample points of the process \( x \) at times \( t_{1k} = ckt^* \) and we are looking for a slowly
varying process \( X = X(t_1) \) as a continuum limit of this sampled process \( (X_k) \). We
can write down the transition probability \( p(x, t) \equiv p(x, t|a, 0) \) as a \( K \)-dimensional
integral

\[ p(x, t|a, 0) = \int \left( \prod_{k=0}^{K} \left( \begin{array}{c} K \\ k \end{array} \right) p(x_{k+1}, (k+1)t^*|x_k, kt^*) \right) \, dx_1...dx_K \]

\[ = \int \left( \prod_{k=0}^{K} \tilde{p}(X_{k+1}, t_{1(k+1)}|X_k, t_{1k}) \right) \, dX_1...dX_K \]

with

\[ \tilde{p}(X_{k+1}, t_{1(k+1)}|X_k, t_{1k}) = p(x_{k+1}, (k+1)t^*|x_k, kt^*). \quad (2.3) \]

In order to use the path integral representation given in Eqn. 2.1 we can write any path \( \Gamma \in \mathcal{C}(x, t|a, 0) \) as

\[ \Gamma = \sum_{k=0}^{K} \Gamma_k, \quad \Gamma_k \in \mathcal{C}(x_{k+1}, (k+1)t^*|x_k, kt^*). \]

The following Figure 2.1 illustrates this decomposition of a path into polygons connecting the start- and endpoints of the spans and the paths within one period. The

**Fig. 2.1. A path from \((a, 0)\) to \((x, t)\) can be decomposed into a polygon connecting the beginning and the endpoints of the periods and paths within the periods. The short-time propagator averages over all paths within one period. As an example, the paths from time \(kt^*\) to time \((k+1)t^*\) are drawn as solid lines.**

transition probability \(p(x_{k+1}, (k+1)t^*|x_k, kt^*)\) from the beginning of a period to its end is then given by the path integral

\[ p(x_{k+1}, (k+1)t^*|x_k, kt^*) = \int_{\mathcal{C}(x_{k+1}, (k+1)t^*|x_k, kt^*)} \mathcal{P}(x(\tau)) \, Dx(\tau) \quad (2.4) \]
and in the continuum limit for the slow scale $t_1$, the polygons connecting $(a,0)$ with $(x,t)$ through $K$ intermediate points become paths $X(t_1)$ and the probability distribution for the polygons becomes the probability distribution $\tilde{P}(X(t_1))$ of these paths

$$
\lim_{K \to \infty, \delta \to 0} \left( \prod_{k=1}^{K} \tilde{p}(X_{k+1}, t_{1(k+1)} | X_k, t_{1k}) \right) = \tilde{P}(X(t_1)).
$$

(2.5)

Thus we obtain the final representation of the transition probability on the slow scales:

$$
p(x,t) = \tilde{p}(X, t_1 | a, 0) = \int_{C(X, t_1 | a, 0)} \tilde{P}(X(\tau_1)) dX(\tau_1)
$$

(2.6)

The last three equations are the mathematical formulation of the intuitive approach: First, one considers the dynamics of the system on the fast scales. The resulting path integral yields the propagator of the system from the beginning of a period to the end of the period. Combining these propagators leads to the probability of the path on the slow scales. Therefore, as an abbreviation for combining Eqn. 2.4 and Eqn. 2.6 we can write the transition probability $p$ as a hierarchy of path integrals:

$$
p(x,t | a, 0) = \int \ldots \left( \lim \prod \int \mathcal{P}(X(\tau_0)) dX(\tau_0) \right) dX(\tau_1)
$$

(2.7)

This method can be generalized in an obvious way: Introducing $J$ time scales $t_j = \epsilon^j t$, the product of the propagators on the time scale $t_j$ leads to the probability of a path on the time scale $t_{j+1}$. Using the same abbreviated notation as in Eqn. 2.7 we can represent $p = p(x,t | a, 0)$ as

$$
p = \int \ldots \left( \lim \prod \int \left( \lim \prod \int \mathcal{P}(X(\tau_0)) dX(\tau_0) \right) dX(\tau_1) \right) \ldots dX(\tau_J)
$$

(2.8)

Once we have obtained the path integral representation on the slow scale with an averaged kernel

$$
\tilde{P}(X(\tau_1)) = e^{- \int_{\tau_{10}}^{\tau_1} \tilde{L}(X(\tau_1), \dot{X}(\tau_1), \tau_1) d\tau_1}
$$

(2.9)

we can also convert back in order to find the corresponding stochastic equation on the slow time scale.

3. Dispersion-managed fiber-optic communications. To demonstrate the application of the method described above, we consider the propagation of an electromagnetic field through an optical fiber with a periodic, piecewise-constant coefficient of dispersion. The model equation for this process is the dispersion-managed nonlinear Schrödinger (DMNLS) equation given by

$$
i E_z + d(z) E_{tt} + \epsilon |E|^2 E = 0,
$$

(3.1)

where $z$, $t$, $E(t,z)$, $d(z)$ are dimensionless quantities representing distance down the fiber (the “time-like” evolution variable in signaling coordinates), time (the transverse variable in signaling coordinates), the electric field envelope and the dispersion coefficient. The scaling parameter $0 < \epsilon \ll 1$ reflects the typical case of a weak nonlinearity relative to the dispersion strength at any point in the fiber. Under typical deterministic dispersion management, the fiber dispersion has a small positive
(anomalous) mean value and $O(1)$ local mean-zero value, i.e., $d(z) = \epsilon d_{av} + d_0(z)$, where, for instance,

$$d_0(z) = \begin{cases} +\hat{d} & 0 \leq z < 1/4 \\ -\hat{d} & 1/4 \leq z < 3/4 \\ +\hat{d} & 3/4 \leq z < 1 \end{cases}$$  \hspace{1cm} (3.2)$$

and where $d_0(z)$ has unit periodicity. Here, we consider the case where $d(z)$ has an additional random component, giving $d(z) = \epsilon d_{av} + d_0(z) + \sqrt{\epsilon D} \xi(z)$, where $\xi(z)$ is taken to be $\delta$-correlated white noise with unit strength, i.e.,

$$<\xi(z)\xi(z')> = \delta(z-z').$$  \hspace{1cm} (3.3)$$

For more details on the derivation of this model from Maxwell’s equations, refer to [7, 8]. It has already been demonstrated [9, 10] that the effect of this random dispersion on single periodic solutions (referred to as “dispersion-managed solitons”) of Eqn. 3.1 can effectively be captured by the two-dimensional system of stochastic ordinary differential equations given by

$$\frac{dT}{dz} = 4d(z)M \quad \text{and}$$

$$\frac{dM}{dz} = \frac{C_1d(z)}{4T^3} - \epsilon C_2 \frac{\partial h}{\partial \Omega}$$

where $T(z)$ and $M(z)$ represent the width and chirp (i.e., the quadratic component of the phase) of the soliton. For derivations of this system using a variational principle, by applying the lens transform followed by expansion in Gauss-Hermite modes, or using moments with a closure condition, see [11, 12]. Constants $C_1$ and $C_2$ depend on the particular shape assumed for the soliton. When $\epsilon = 0$, these equations simply reflect the linear Schrödinger equation with mean-zero dispersion, and all initial conditions, in particular those of the form \{T(0), M(0)\} = \{T_0, 0\}, yield periodic solutions. When $\epsilon$ is finite and $D = 0$ (i.e., the dispersion is deterministic and periodic), the nonlinearity and finite-mean dispersion “select” a member of this one-dimensional family of periodic solutions that persists for $\epsilon > 0$ [13]. When $\epsilon > 0$ and $D = O(1)$, the width and chirp of this soliton undergo a random walk in addition to the “breathing” induced by the deterministic component of the dispersion map. We now seek to apply the methods discussed in the previous sections to determine the properties of this random walk.

Before we proceed, it is useful to note that the action-angle coordinates [14] in the $\epsilon = 0$ case also simplify analysis for the finite-$\epsilon$ case. Letting $\Omega = C_1/T^2 + 16M^2$ and $\beta = 4TM$ gives

$$\frac{d\Omega}{dz} = 3\epsilon \frac{\partial h}{\partial \beta} \quad \text{and}$$

$$\frac{d\beta}{dz} = d_0(z)\Omega - \epsilon \frac{\partial h}{\partial \Omega}$$  \hspace{1cm} (3.4)$$

where

$$h(\Omega, \beta) = \frac{2C_2\Omega^{3/2}}{3(C_1 + \beta^2)^{1/2}} - \frac{1}{2} d_{av}\Omega^2.$$  \hspace{1cm} (3.5)$$
An initial condition in Eqns. 3.4 equal to \{T_0, 0\} corresponds to an initial condition in Eqns. 3.4 of \{\Omega_0, 0\}, where \(\Omega_0 = C_1/T_0^2\).

Intuitively, we expect the solutions of the stochastic dynamical system presented by Eqns. (3.4) to experience scale-separation as in the deterministic case. This separation of scales should result in fast variations of the probability densities for \(\beta\) and \(\Omega\) within a span and a slow evolution of both quantities over many periods visible in the corresponding Poincaré sections. Numerical simulations confirm this intuition. Fig. 3.1 and fig. 3.2 present the evolution of both probability densities.

![Fig. 3.1. Evolution of the probability density of \(\Omega\)](image)

We start our multi-scale analysis of the system by immediately writing Eqns. 3.4 in the form of Eqn. 1.1, with \(x = (\Omega, \beta)^T\) and \(t = z\). Thus,

\[
\begin{align*}
f(x, z; \epsilon) &= \begin{pmatrix} \frac{\partial h}{\partial \beta} - \frac{\partial h}{\partial \Omega} \frac{\partial \beta}{\partial \Omega} & \frac{\partial h}{\partial \Omega} \end{pmatrix} + \epsilon \begin{pmatrix} \frac{\partial h}{\partial \beta} - \frac{\partial h}{\partial \Omega} \frac{\partial \beta}{\partial \Omega} & \frac{\partial h}{\partial \Omega} \end{pmatrix} + \epsilon \begin{pmatrix} \frac{\partial h}{\partial \beta} - \frac{\partial h}{\partial \Omega} \frac{\partial \beta}{\partial \Omega} & \frac{\partial h}{\partial \Omega} \end{pmatrix}, \\
g(x, z) &= \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix}.
\end{align*}
\]

where we write \(\xi(z) = (\xi_1(z) \xi_2(z))^T\) formally even though only \(\xi_2\) is relevant. The initial condition is simply \(a = (\Omega_0, 0)^T\). Note that the behavior at leading order is easily solved to yield

\[
x^{(0)} = \begin{pmatrix} 1 \\
R(z_0) \end{pmatrix} y(z_1) \quad \text{and} \quad R(z_0) = \int_0^{z_0} d_0(\zeta) d\zeta.
\]

In order to obtain coarse-grained equations from the original system given by Eqn. 3.4, we first express the transition probability in the form of a path integral, i.e.,

\[
p(x, z) = \int_{C[\beta, z|\beta_0, 0]} D\beta(\zeta) e^{-\frac{1}{\epsilon} \int_0^z \frac{1}{8\epsilon} (\beta - d_0(\zeta)\Omega + \epsilon \frac{\partial h}{\partial \Omega} \Omega)^2} \delta \left( \Omega(z) - \Omega_0 - \int_0^z 3\epsilon \frac{\partial h}{\partial \beta} d\zeta \right),
\]

7
Fig. 3.2. Evolution of the probability density of $\beta$

with the function $h$ given by Eqn. 3.5. A short derivation of this representation is found in the appendix. For the short-time propagator we note in particular that, since $z_{0k} = k$,

$$p(x_k, k + 1| x_k, k) = \int_{C_{[\beta_{k+1}, k+1|\beta_k, k]}} D\beta(\zeta) \, e^{-\frac{1}{2} \int_{k+1}^{k+1} \frac{1}{\epsilon} (\beta - d_{0k}(\zeta)\Omega(\zeta) + \epsilon \overline{\Phi})^2 d\zeta} \times \delta (\Omega(k + 1) - \Omega(k) - \int_{k}^{k+1} 3\epsilon \frac{\partial h}{\partial \beta} d\zeta).$$

(3.10)

We wish to apply a semi-classical approximation to this short-term propagator by finding the minimal trajectory using the associated Euler-Lagrange equations [5]. For this purpose, we consider a multi-scale expansion

$$\Omega = \Omega^{(0)}(z_0, z_1) + \epsilon \Omega^{(1)}(z_0, z_1) + \ldots$$

(3.11)

$$\beta = \beta^{(0)}(z_0, z_1) + \epsilon \beta^{(1)}(z_0, z_1) + \ldots$$

(3.12)

and find at the leading order

$$\Omega^{(0)}(z_0, z_1) = \bar{\Omega}(z_1) \quad \text{and} \quad \beta^{(0)}(z_0, z_1) = R(z_0)\bar{\Omega}(z_1) + \bar{\beta}(z_1).$$

(3.13)

Therefore, within one period, $\Omega^{(0)}$ can be considered to be constant and the constraint $\dot{\Omega} = 3\epsilon \partial h / \partial \beta$ yields an evolution equation for the $z_1$ dependence

$$\bar{\Omega}' \equiv \frac{d\bar{\Omega}}{dz_1} = \int_{k}^{k+1} 3 \frac{\partial h}{\partial \beta} dz_0$$

(3.14)

as well as the solution of the first-order correction $\Omega^{(1)}$

$$\Omega^{(1)}(z_0, z_1) = \int_{k}^{z_0} 3 \frac{\partial h}{\partial \beta} dz - \bar{\Omega}'(z_1) z_0.$$

(3.15)
Here, we have applied standard deterministic multi-scale analysis - consistent with the fact that the evolution equation of \( \Omega \) does not contain any further source of noise. We can now rewrite the Lagrangian in the kernel of the above path integral given in Eqn. \[3.10\] by noting that

\[
\dot{\beta} - d_0(\zeta)\Omega(\zeta) - \epsilon \frac{\partial h}{\partial \Omega} = \epsilon \left( R(\zeta)\Omega' + \dot{\beta}' + \frac{\partial \beta^{(1)}(\zeta)}{\partial \zeta} - d_0(\zeta)\Omega^{(1)} - \frac{\partial h}{\partial \Omega} \right)
\]

\[= \epsilon \left( R_0(\zeta)\Omega' + \dot{\beta}' + \frac{\partial \beta^{(1)}(\zeta)}{\partial \zeta} - d_0(\zeta) \int_k^\zeta 3 \frac{\partial h}{\partial \beta} d\tilde{z} + d_0(\zeta)\Omega' \zeta + \frac{\partial h}{\partial \Omega} \right)\]

In this way, the Lagrangian depends on \( \beta^{(1)}(\zeta) \) and \( \dot{\beta}^{(1)}(\zeta) = \partial \beta^{(1)}/\partial \zeta \), or more formally written as

\[L = L(\beta^{(1)}, \dot{\beta}^{(1)}, \zeta)\]

and the corresponding Euler-Lagrange equation is given by

\[
\frac{d}{d\zeta} \frac{\partial L}{\partial \dot{\beta}^{(1)}} - \frac{\partial L}{\partial \beta^{(1)}} = 0 \quad (3.16)
\]

Since \( \beta^{(1)} \) is cyclic, we immediately find

\[
R\Omega' + \dot{\beta}' + \frac{\partial \beta^{(1)}(\zeta)}{\partial \zeta} - d(\zeta) \int_k^\zeta 3 \frac{\partial h}{\partial \beta} d\tilde{z} + d(\zeta)\Omega' \zeta + \frac{\partial h}{\partial \Omega} = C \quad (3.17)
\]

The value of the constant \( C \) is found by the boundary conditions on \( \beta^{(1)} \) and for \( \beta^{(1)}(k + 1, z_1) = \beta^{(1)}(k, z_1) \) we find by integration of the above equation over the fast variable \( s \) immediately

\[
C = \beta' + \int_0^1 \left( 3 \frac{\partial h}{\partial \beta} + \frac{\partial h}{\partial \Omega} \right) d\zeta
\]

which finally lets the averaged Lagrangian become

\[
\bar{L} = \frac{1}{2} \frac{1}{\Omega^2} \left[ \beta' + \int_k^{k+1} \left( 3 \frac{\partial h}{\partial \beta} + \frac{\partial h}{\partial \Omega} \right) d\zeta \right]^2
\]

(3.19)

This, however corresponds again exactly to a stochastic equation with a noise process \( \xi = \xi(z_1) \) on the slow scale given by

\[
\dot{\beta}' = \Omega\xi(z_1) - \int_0^1 \left( 3 \frac{\partial h}{\partial \beta} + \frac{\partial h}{\partial \Omega} \right) d\zeta
\]

(3.20)

and together with Eqn. \[3.15\] we have found the coarse-grained system of stochastic differential equations. Note that in the present case, it is possible to solve all integrals analytically such that we can rewrite the system in terms of the slow variable \( y \) with \( y_1 = \Omega \) and \( y_2 = \beta \) as

\[
\frac{\partial}{\partial z_1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \Gamma_1(y_1, y_2) \\ \Gamma_2(y_1, y_2) \end{pmatrix} + \begin{pmatrix} 0 \\ y_1 \Xi(z_1) \end{pmatrix}
\]

(3.21)
The slow-scale white noise process Ξ is simply given by
\[ Ξ(z_t) = \left( \int_0^{z_t} ξ(z_0) \, dz_0 \right) = \left( \int_0^{z_t} ξ(z_1) \right). \] (3.22)

The functions Γ₁ and Γ₂ can be expressed as
\[ Γ_1(y) = \frac{6}{d y_1} \left[ \frac{∂H}{∂β}(y_1, y_2 + \frac{1}{4} \hat{d} y_1) - \frac{∂H}{∂β}(y_1, y_2 - \frac{1}{4} \hat{d} y_1) \right] \] (3.23)
\[ = \frac{4C_2 \sqrt{y_1}}{d} \left( C_1 + (y_2 + \frac{1}{4} \hat{d} y_1)^2 \right)^{1/2} - \left( C_1 + (y_2 - \frac{1}{4} \hat{d} y_1)^2 \right) \] (3.24)
and
\[ Γ_2(y) = \frac{3}{d y_1} \left[ H(y_1, y_2 + \frac{1}{4} \hat{d} y_1) - H(y_1, y_2 + \frac{1}{4} \hat{d} y_1) \right] \] (3.25)
\[ - \frac{3}{2y_1} \left[ \frac{∂H}{∂β}(y_1, y_2 + \frac{1}{4} \hat{d} y_1) + \frac{∂H}{∂β}(y_1, y_2 - \frac{1}{4} \hat{d} y_1) \right] + d_{av} y_1 \] (3.26)

with
\[ H(Ω, β) \equiv \frac{2}{3} C_2 Ω^{3/2} \sinh^{-1}(β/√C_1), \] (3.27)

Although the above analysis was carried out for a particular example, it shows the steps that are necessary to apply and modify the method for other cases. Note that it is not a requirement that the occurring integrals can be solved analytically. Moreover, using the semi-classical method in order to calculate the short-term propagator is appropriate in this case since the evolution of the system is linear on a short time scale. For systems that show nonlinear behavior on short times, other methods of evaluating the corresponding path integral can be applied.

4. Perturbative analysis of Fokker-Planck equation. In this section we show that a perturbative analysis of the corresponding Fokker-Planck equation will yield for the particular example exactly the same result as the path integral based method. The idea to perform asymptotic analysis on the level of the Fokker-Planck equation in order to characterize the behavior of the associated stochastic process has been applied very successfully in a variety of contexts [15]. Since the Fokker-Planck equation is deterministic, we can use the classic theory of deterministic multi-scale expansions, adapted to the Fokker-Planck equation.

Hence, instead of considering the path integral representation of the stochastic process \( x(t) \), we now are considering the evolution equation of the probability density \( p(x, t) \) given by the Fokker-Planck equation
\[ \frac{∂p}{∂t} = - \frac{∂}{∂x_i} (f_i p) + \frac{1}{2} \epsilon \frac{∂}{∂x_i} \left( g_{ik} \frac{∂}{∂x_j} (g_{jk} p) \right), \] (4.1)
with initial condition \( p(x, 0) = \prod_{i=1}^{n} \delta(x_i - a_i) \). Note that, as before, we are using Einstein’s summation convention and the Stratonovich interpretation of Eqn. [16].

Again, the secular growth we obtain when trying a naive perturbation expansion for \( p \) suggests we adopt a multiple-scales approach:

\[
p(x, t) = p^{(0)}(x, t_0, t_1, \ldots) + \epsilon p^{(1)}(x, t_0, t_1, \ldots) + \ldots,
\]

where \( t_j = \epsilon^j t \). In this case, we have no ambiguity in the scale at which the effect of randomness is felt, since the Fokker-Planck equation is completely deterministic.

The probability density \( p \) captures all moments of \( x(t) \). The leading order of this expansion satisfies

\[
\frac{\partial p^{(0)}}{\partial t_0} = - \frac{\partial}{\partial x_i} \left[ f_i(x, t_0; 0)p^{(0)} \right],
\]

so that along characteristics \( x = x_c(t_0; y) \) where

\[
\frac{dx_c}{dt_0} = f(x_c(t_0), t_0; 0) \quad \text{and} \quad x_c(0) = y,
\]

we have that

\[
\frac{dp^{(0)}}{dt_0} = - \left[ \frac{\partial}{\partial x_i} f_i(x_c(t_0), t_0; 0) \right] p^{(0)}.
\]

The \( t_0 \) dependence has been written as a full derivative here to denote the fact that it is taken along characteristics; naturally, \( p \) also depends on the slower time scale \( t_1 \). The solution is then

\[
p^{(0)} = P(y, t_1, \ldots) \exp \int_0^{t_0} - \frac{\partial}{\partial x_i} f_i(x_c(\zeta), \zeta; 0) \, d\zeta
\]

with initial condition

\[
P(y, 0) = \prod_{i=1}^{n} \delta(y_i - a_i).
\]

The next order is

\[
L p^{(1)} = - \frac{\partial p^{(0)}}{\partial t_1} - \frac{\partial}{\partial x_i} \left( \frac{\partial f_i}{\partial \epsilon} \bigg|_{\epsilon=0} p^{(0)} \right) + \frac{1}{2} \frac{\partial}{\partial x_i} \left( g_{ik} \frac{\partial}{\partial x_j} (g_{jk} p^{(0)}) \right)
\]

where

\[
L = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial t_1} [f_i(x, t; 0) \delta],
\]

with adjoint operator

\[
L^\dagger = - \frac{\partial}{\partial t_0} - f_i(x, t; 0) \frac{\partial}{\partial x_i}.
\]

By comparison with the leading order, it is clear that \( \ker L^\dagger \) is spanned by functions that are constant in \( t_0 \) along characteristics \( x_c(t_0; y) \), i.e., any square-integrable function \( \phi(y) \) (in fact, we have thus far omitted a discussion of the appropriate boundary conditions at the initial and final times).
conditions, but for simplicity we will assume \( t^* \)-periodicity in \( t_0 \) and sufficiently fast decay in \(|x|\). The Fredholm alternative therefore requires that

\[
0 = \int_0^{t^*} dt_0 \int d^n x \phi(y) \left[ -\frac{\partial P}{\partial \xi_1} \exp \int_0^{t_0} -\frac{\partial}{\partial x_1} f_i(x_c(\zeta), \zeta; 0) \, d\zeta 
- \frac{\partial}{\partial x_1} \left( \frac{\partial f_i}{\partial \xi_1} \right) \bigg|_{\xi=0} P(y, t_1) \exp \int_0^{t_0} -\frac{\partial}{\partial x_1} f_j(x_c(\zeta), \zeta; 0) \, d\zeta 
+ \frac{1}{2} \frac{\partial}{\partial x_1} \left( g_{ik} \frac{\partial}{\partial x_j} (g_{jk} P(y, t_1)) \right) \exp \int_0^{t_0} -\frac{\partial}{\partial x_1} f_l(x_c(\zeta), \zeta; 0) \, d\zeta \right]
\]

(4.11)

for all members \( \phi(y) \) of the admissible class of functions. This clearly leads to an IBVP for \( P(y, t_1) \) which is now, depending on the nonempty elements of the matrix \( g(x, t) \), parabolic rather than hyperbolic. The particular form of this equation is seen most easily when the characteristic coordinate transformation \( x = x_c(t_0; y) \) is invertible, as we can then write down directly the transformed equation in \( y \) and \( t_1 \). This will be the case for our current example, as we will see below.

In application the concrete problem given by Eqns (3.4) we note immediately from Eqn. (3.6) that

\[
\frac{\partial}{\partial x_i} f_i(x, z; 0) = 0,
\]

(4.12)

which greatly simplifies matters. As before, the leading order characteristics are given by

\[
x_c(z_0) = \begin{pmatrix} 1 \\ R(z_0) \\ 0 \\ 1 \end{pmatrix} y,
\]

(4.13)

with the leading-order solution simply given by

\[
p^{(0)}(x, z_0, z_1, \ldots) = P(y, z_1, \ldots).
\]

(4.14)

After using the above characteristics to change variables from \( x \) to \( y \), the dependence of \( P \) on \( y \) and \( z_1 \), as imposed by the Fredholm alternative condition in Eqn. (4.11) is given by

\[
\frac{\partial P}{\partial z_1} = \int_0^1 d z_0 \left\{ -3 \left( \frac{\partial}{\partial y_1} - R(z_0) \frac{\partial}{\partial y_2} \right) \left[ \frac{\partial h}{\partial \beta} (y_1, y_2 + R(z_0) y_1) P(y, z_1) \right] 
+ \frac{\partial}{\partial y_2} \left[ \frac{\partial h}{\partial \beta} (y_1, y_2 + R(z_0) y_1) P(y, z_1) \right] + \frac{1}{2} y_1^2 \frac{\partial^2 P}{\partial y_2^2} \right\}
\]

\[
= -d_{\alpha} y_1 \frac{\partial P}{\partial y_2} - 3 \frac{\partial}{\partial y_1} \int_0^1 \frac{\partial^2 H}{\partial \beta^2} (\cdot, \cdot) P(y, z_1) \, d z_0
\]

\[
+ \frac{\partial}{\partial y_2} \int_0^1 \left[ \frac{\partial^2 H}{\partial \beta^2} (\cdot, \cdot) + 3 R(z_0) \frac{\partial^2 H}{\partial \beta^2} (\cdot, \cdot) \right] P(y, z_1) \, d z_0 + \frac{1}{2} y_1^2 \frac{\partial^2 P}{\partial y_2^2},
\]

(4.15)

where the derivatives of \( H(\Omega, \beta) \) are evaluated along the characteristics, that can be written explicitly as \( \Omega = y_1, \beta = y_2 + R(z_0) y_1 \). Finally, after some algebra we observe that the above evaluates to

\[
\frac{\partial P}{\partial z_1} = -\frac{\partial}{\partial y_1} (\Gamma_1 P) - \frac{\partial}{\partial y_2} (\Gamma_2 P) + \frac{1}{2} y_1^2 \frac{\partial^2 P}{\partial y_2^2},
\]

(4.16)
which is exactly the Fokker-Planck equation for the slow process given by Eq. 3.21 obtained in Sec. 3.

5. Conclusion. We have presented a new method to derive equations that describe the slow evolution of a system of stochastic ordinary differential equations. The conceptual basis of the method is the expression of state transition probabilities as path integrals, where a separation of scales allows a decomposition of the paths into self-similar segments. By performing a multi-scale expansion of the Lagrangian in each segment, we obtain an effective path integral that represents the coarse-grained noise process relevant to the slow dynamics of the system. From the appropriate limit of this path integral (i.e., refinement of the path partition), we obtain a new system of stochastic differential equations that describe behavior over long time scales. We have applied our method to a stochastic system arising in nonlinear optics, and have shown that our method produces a leading-order result that is identical to that obtained from a standard asymptotic analysis of the singularly perturbed Fokker-Planck equation associated with the stochastic system. The conditions necessary for this agreement to hold is a topic of ongoing research.

Acknowledgments. The work of T. Schäfer was supported by CUNY Research Foundation through the grant PSCREG-38-860 and through the Center of Interdisciplinary Applied Mathematics and Computational Sciences at the College of Staten Island. R. O. Moore gratefully acknowledges support from the NSF through grant NSF-DMS 0511091.
Appendix A. Path integral representation.

In this appendix we give a brief overview of how to derive a path integral representation of a stochastic system following [17]. We start with a stochastic equation of the form (1.1),

\[ \dot{x} = f(x, t; \epsilon) + \sqrt{\eta} g(x, t) \xi(t), \quad x(0) = a, \quad (A.1) \]

where we have denoted the noise strength by \( \eta \) instead of \( \epsilon \) to reflect the fact that noise need not be small in this formalism. We take (A.1) to be the limit of difference equations for \( \Delta x^{(k)}_i \) (where the subscript and superscript now represent the component index and the time increment, respectively) where

\[ \Delta x^{(k)}_i = f^{(k)}_i \Delta t + \sqrt{\eta} \left( g^{(k)}_{ij} + s \frac{\partial g^{(k)}_{ij}}{\partial x^l} \Delta x^{(k)}_l \right) \Delta w^{(k)}_j. \quad (A.2) \]

The time increments \( \Delta t \) identify a regular partition of interval \([0, t]\) with \( K \) intermediate points. All terms in the stochastic difference equation are evaluated at the pre-point, hence

\[ f^{(k)}_i = f_i (x^{(k-1)}, t^{(k-1)}), \quad g^{(k)}_{ij} = g_{ij} (x^{(k-1)}, t^{(k-1)}), \]

and the parameter \( s \) accounts for the different possible definitions of the stochastic integral: a choice of \( s = 0 \) corresponds to the Ito interpretation of (A.1) and a choice of \( s = 1/2 \) corresponds to the Stratonovich interpretation. We can rewrite \( \Delta x^{(k)}_i \) using the fact that the Brownian increments \( \Delta w^{(k)}_j \) are uncorrelated and of order \( \sqrt{\Delta t} \). This yields

\[ \Delta x^{(k)}_i = \left( f^{(k)}_i + \eta s \frac{\partial g^{(k)}_{ij}}{\partial x^l} \Delta x^{(k)}_l \right) \Delta t + \sqrt{\eta} g^{(k)}_{ij} \Delta w^{(k)}_j \quad (A.3) \]

Defining now the inverse of \( g \) to be \( h = g^{-1} \), we can solve for the \( \Delta w^{(k)}_i \) and obtain

\[ \Delta w^{(k)}_i = \frac{h^{(k)}_{ij}}{\sqrt{\eta}} \left( \Delta x^{(k)}_j - \left( f^{(k)}_j + \eta s \frac{\partial g^{(k)}_{ij}}{\partial x^l} \Delta x^{(k)}_l \right) \Delta t \right) \quad (A.4) \]

Note that if \( g \) is not invertible, an appropriate regularization has to be chosen (see Appendix B). In this section, however, we will simply take \( g \) to be nonsingular for all times \( t \) and all points \( x \).

The joint probability distribution of a path \((w) \equiv (w^{(1)}, ..., w^{(K+1)})\) is given by

\[ p((w)) = \frac{1}{\sqrt{2\pi \Delta t}^n} \prod_{k=1}^{K+1} \prod_{i=1}^n \Delta w^{(k)}_i e^{-\frac{1}{\eta \Delta t} (\Delta w^{(k)}_i)^2}. \quad (A.5) \]

The derivation of a path integral representation for the stochastic differential equation (A.1) amounts to performing a coordinate transformation in the above probability density from the path \((w)\) to the path \((x)\) given by \((x) \equiv (x^{(1)}, ..., x^{(K+1)})\). The determinant \(|J|\) of the Jacobian of this transformation is simple, since the discretization is based on the pre-point. We find

\[ |J| = \prod_{k=1}^{K+1} \begin{vmatrix} \frac{\partial x^{(k)}_1}{\partial x^{(k)}_1} & \cdots & \frac{\partial x^{(k)}_1}{\partial x^{(k)}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^{(k)}_n}{\partial x^{(k)}_1} & \cdots & \frac{\partial x^{(k)}_n}{\partial x^{(k)}_n} \end{vmatrix} = \prod_{k=1}^{K+1} \eta^{-n/2} \det([h^{(k)}_{ij}]) \quad (A.6) \]
We now define the action $S$ as the following limit for $K \to \infty$ and $\Delta t \to 0$:

$$
S = \lim_{K \to \infty, t \to 0} \frac{1}{2\Delta t} \sum_{k=1}^{K+1} \sum_{i=1}^{n} (\Delta w_i^{(k)})^2
$$

$$
= \lim_{K \to \infty, t \to 0} \frac{1}{2\Delta t} \sum_{k=1}^{K+1} \sum_{i=1}^{n} \left( \left( f_i^{(k)} + \eta \frac{\partial g_{ij}^{(k)}}{\partial x_l} \partial g_{ij}^{(k)} \right) \Delta t \right)^2
$$

$$
= \frac{1}{2\eta} \int_0^t \sum_{i=1}^{n} h_{ik}^2 \left( \dot{x}_k - \left( f_k + \eta \frac{\partial g_{kj}}{\partial x_l} \right) \right)^2 d\tau,
$$

where we have suppressed the arguments $(x(\tau), \tau)$ in $h_{ik}$, $f_k$, etc., for simplicity of presentation. Defining

$$
\mathcal{D}x(\tau) = \lim_{K \to \infty, t \to 0} \prod_{k=1}^{K+1} \frac{1}{\sqrt{2\pi \Delta t}} \frac{1}{\eta^{n/2}} \det([h_{ij}]) \prod_{i=1}^{n} dx_i^{(k)}
$$

we can write the path integral representation for the transition probability from the initial condition $x(0) = a$ to the point $x(t) = x$ as

$$
p(x, t) = \int_{C(x, t|a, 0)} \mathcal{D}x(\tau) e^{-L(x(\tau), \dot{x}(\tau), \tau)}, \quad (A.7)
$$

where $C(x, t|a, 0)$ denotes the set of all possible paths that connect the point $(a, 0)$ to the point $(x, t)$.

**Appendix B. Regularized path integral representation of the $(\Omega, \beta)$ equations.** In this section we show how to derive a path integral representation in $(\Omega, \beta)$ for the particular system given by Eqns. (3.4). For simplicity we rewrite these equations as

$$
\frac{d\Omega}{dz} = f_1(\Omega, \beta, z) + \kappa \sqrt{\eta} \xi_1(z) \quad (B.1)
$$

$$
\frac{d\beta}{dz} = f_2(\Omega, \beta, z) + \Omega \sqrt{\eta} \xi_2(z) \quad (B.2)
$$

where we obtain $[3.4]$ for $\kappa = 0$. Similar regularization techniques are well-established, e.g., in the case of Brownian motion [5]. The matrices $g$ and $h$ in $[A.4]$ are given by

$$
g = \begin{pmatrix} \kappa & 0 \\ 0 & \Omega \end{pmatrix}, \quad h = \begin{pmatrix} \kappa^{-1} & 0 \\ 0 & \Omega^{-1} \end{pmatrix} \quad (B.3)
$$

such that

$$
h_{ik} \frac{\partial g_{kj}}{\partial x_l} \partial g_{lj} = 0 \quad (B.4)
$$
for $i = 1, 2$. With $x = (\Omega, \beta)^T$ and $a = (\Omega_0, \beta_0)^T$, we can write the corresponding path integral representation of \[(B.1, B.2)\] as
\[
p(x, z) = \int_{C_{[x, z|a, 0]}} \prod_{\zeta = 0}^{\tilde{z}} \frac{d\Omega(\zeta)}{\sqrt{2\pi d\zeta}} \frac{d\beta(\zeta)}{\sqrt{2\pi d\zeta}} \sqrt{\frac{1}{\kappa}} \frac{1}{\sqrt{\eta}} e^{-\frac{i}{\eta} \int_0^z \frac{1}{\zeta} (\dot{\Omega} - f_1)^2 d\zeta} e^{-\frac{i}{\kappa} \int_0^z \frac{1}{\zeta^2} (\dot{\beta} - f_2)^2 d\zeta}.
\] (B.5)

In order to take the limit $\kappa \rightarrow 0$, we note that
\[
\lim_{\kappa \rightarrow 0} \prod_{\zeta = 0}^{\tilde{z}} \frac{1}{\sqrt{2\pi d\zeta}} \frac{1}{\sqrt{\eta}} e^{-\frac{i}{\eta} \int_0^z \frac{1}{\zeta} (\dot{\Omega} - f_1)^2 d\zeta} = \prod_{\zeta = 0}^{\tilde{z}} \frac{1}{d\zeta} \delta(\dot{\Omega} - f_1),
\] (B.6)
as can be seen by straightforward calculations using the same discretization as in the derivation of the path integral representation. Integration over $d\Omega(\zeta)$ yields then
\[
p(x, z) = \int_{C_{[\beta, z|\beta_0, 0]}} \prod_{\zeta = 0}^{\tilde{z}} \frac{d\beta(\zeta)}{\sqrt{2\pi d\zeta}} \frac{1}{\sqrt{\eta}} e^{-\frac{i}{\eta} \int_0^z \frac{1}{\zeta} (\dot{\beta} - f_2)^2 d\zeta} 
\times \delta \left( \Omega(z) - \Omega_0 - \int_0^z f_1(\Omega(\zeta), \beta(\zeta), \zeta) d\zeta \right)
\]
or, written in a more compact way,
\[
p(x, z) = \int_{C_{[\beta, z|\beta_0, 0]}} D\beta(\zeta) e^{-\frac{i}{\eta} \int_0^z \frac{1}{\zeta} (\dot{\beta} - f_2)^2 d\zeta} \delta \left( \Omega(z) - \Omega_0 - \int_0^z f_1(\Omega(\zeta), \beta(\zeta), \zeta) d\zeta \right).
\] (B.7)

REFERENCES

[1] M. H. Holmes. Introduction to Perturbation Methods. Springer, New York, 1995.
[2] A. H. Nayfeh. Perturbation Methods. Wiley-International, New York, 1973.
[3] L. Y. Chen, N. Goldenfeld, and Y. Oono. Renormalization group theory for global asymptotic analysis. Phys. Rev. Lett., 73:1311–1315, 1994.
[4] R. P. Feynman and A. R. Hibbs. Quantum Mechanics and Path Integrals. McGraw-Hill, New York, 1965.
[5] M. Chaichian and A. Demichev. Path Integrals in Physics: Volume I Stochastic Processes and Quantum Mechanics. IOP, London, 2001.
[6] P. Arnold. Symmetric path integrals for stochastic equations with multiplicative noise. Phys. Rev. E, 61:6099–6102, 2000.
[7] A. C. Newell and J. V. Moloney. Nonlinear Optics. Addison-Wesley, Redwood City, CA, 1992.
[8] G. P. Agrawal. Nonlinear Fiber Optics. Academic Press, San Diego, 1995.
[9] F. Kh. Abdullaev, J. C. Bronski, and G. Papanicolaou. Soliton perturbations and the random Kepler problem. Physica D, 135:369–386, 2000.
[10] T. Schäfer, R. O. Moore, and C. K. R. T. Jones. Pulse propagation in media with deterministic and random dispersion variations. Optics Communications, 214:353–362, 2002.
[11] S. K. Turitsyn. Breathing self-similar dynamics and oscillatory tails of the chirped dispersion-managed soliton. Phys. Rev. E, 58:R1256–R1259, 1998.
[12] T. I. Lakoba and D. J. Kaup. Hermite-Gaussian expansion for pulse propagation in strongly dispersion managed fibers. Phys. Rev. E, 58:6728–6741, 1998.
[13] V. Cautaerts, A. Maruta, and Y. Kodama. On the dispersion managed soliton. Chaos, 10:515–528, 2000.
[14] S. K. Turitsyn, A. B. Aceves, C. K. R. T. Jones, and V. Zharnitsky. Average dynamics of the optical soliton in communication lines with dispersion management: Analytical results. Phys. Rev. E, 58:R48–R52, 1998.
[15] G. C. Papanicolaou. Introduction to the asymptotic analysis of stochastic equations. Lectures in Applied Mathematics, R. C. DiPerna ed., 16:106–147, 1977.
[16] C. W. Gardiner. Handbook of Stochastic Methods. Springer, Heidelberg, 1985.
[17] F. Langouche, D. Roekaerts, and E. Tirapegui. Functional integration and semiclassical expansions. Kluwer, Boston, 1982.