A generalization of the Virasoro algebra to arbitrary dimensions

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Colored tensor models generalize matrix models in higher dimensions. They admit a $1/N$ expansion dominated by spherical topologies and exhibit a critical behavior strongly reminiscent of matrix models. In this paper we generalize the colored tensor models to colored models with generic interaction, derive the Schwinger Dyson equations in the large $N$ limit and analyze the associated algebra of constraints satisfied at leading order by the partition function. We show that the constraints form a Lie algebra (indexed by trees) yielding a generalization of the Virasoro algebra in arbitrary dimensions.

Keywords: Random tensor models, 1/N expansion, critical behavior

I. INTRODUCTION

Random matrix models [1–3] generalize in higher dimensions to random tensor models [4–8] and group field theories [9–12] (see [13–18] for some further developments). The Feynman graphs of GFT in $D$ dimensions are built from vertices dual to $D$ simplices and propagators encoding the gluing of simplices along their boundary. Parallel to ribbon graphs of matrix models (dual to discretized surfaces), GFT graphs are dual to discretized $D$ dimensional topological spaces. Tensor models are notoriously hard to control analytically and one usually resorts to numerical simulations [19–21]. Progress has recently been made in the analytic control of tensor models with the advent of the $1/N$ expansion [22–24] of colored [25–27] tensor models. This expansion synthesizes several evaluations of graph amplitudes [28–36] and provides a straightforward generalization of the familiar genus expansion of matrix models [37, 38] in arbitrary dimension. The coloring of the fields allows one to address previously inaccessible questions like the implementation of the diffeomorphism symmetry [36, 39, 40] or the identification of embedded matrix models [41] in tensor models. The symmetries of generic tensor models have recently been studied using $n$-ary algebras [42, 43].

The critical behavior of matrix models is most conveniently addressed using the loop equations [44–46] in conjunction with the $1/N$ expansion. The loop equations translate in a set of constraints (obeying the Virasoro algebra) satisfied by the partition function and provide the link between matrix models and continuum conformal field theories. Generic matrix models exhibit multi critical points [47] which are at the core of their applications to string theory [48, 49], two dimensional gravity [3], critical phenomena [1, 50, 51], black hole physics [52], etc.

Recently the investigation of the critical behavior of the simplest colored tensor models has been performed [53] mapping the dominant family of graphs (generalizing the planar [37] graphs of matrix models) on certain species of colored trees.

However, up to now, the colored tensor models considered possess only one interaction term (one coupling constant). It is well known that multi critical points for matrix models appear only when one adds multiple interaction terms. A first question we will solve in this paper is to write a colored tensor model with generic interactions. In order to access the critical behavior of such a model one must derive a generalization of the loop equation in higher dimensions. We will derive the
closed set of Schwinger-Dyson Equations (SDE) obeyed by a generic colored tensor models in the large $N$ limit which we translate in constraints on the partition function. We will prove that the constraints form a Lie algebra yielding a higher dimensional generalization of the Virasoro algebra.

This paper is organized as follows. In section II we recall the derivation of the loop equations and the link with the Virasoro algebra in matrix models. In section III we introduce an algebra indexed by colored trees. In section IV we derive the SDEs at leading order in the $1/N$ expansion of colored tensor models and translate them into constraints on the partition function which we identify with the generators of our algebra. Section V draws the conclusions of this work.

II. MATRIX MODELS AND THE VIRASORO ALGEBRA

This section is a quick digest of [44, 46] and presents the classical derivation of the loop equations and their link with the Virasoro algebra in matrix models. In the spirit of our subsequent treatment of colored tensor models, we will start from a colored matrix model [54–56] of three independent non hermitian matrices $M_1, M_2$ and $M_3$, defined by the partition function

$$Z = \int [dM_1][dM_2][dM_3] \ e^{-N \text{Tr}[V(M_1,M_2,M_3)]},$$

$$V(M_1, M_2, M_3) = M_1 M_1^\dagger + M_2 M_2^\dagger + M_3 M_3^\dagger - \lambda M_1 M_2 M_3 - \bar{\lambda} M_1^\dagger M_2^\dagger M_3^\dagger,$$  \hspace{1cm} (1)

with $[dM] = \prod_{a,b} dM_{ab} d\bar{M}_{ab}$. As the integral is Gaussian, one can explicitly integrate over two colors to obtain the partition function as an integral over one matrix

$$Z = \int [dM_3] \ e^{-N \text{Tr}[V(M_3 M_3^\dagger)]}$$

$$V(M_3 M_3^\dagger) = M_3 M_3^\dagger + \ln(\mathbb{I} - \lambda \bar{\lambda} M_3 M_3^\dagger) = M_3 M_3^\dagger + \sum_j \frac{(\lambda \bar{\lambda})^j}{j} (M_3 M_3^\dagger)^j.$$ \hspace{1cm} (2)

To pass from a model with one coupling constant to a generic matrix model, one attributes to every operator in the effective action for the last color an independent coupling constant replacing $(\lambda \bar{\lambda})^j$ by $t_j$

$$Z = \int [dM] e^{-N \text{Tr}[V(M M^\dagger)]}, \quad V(M M^\dagger) = \sum_{j=1}^\infty t_j (M M^\dagger)^j.$$  \hspace{1cm} (3)

The Schwinger-Dyson equations (SDE) of a generic matrix model write

$$0 = \int [dM] \frac{\delta}{\delta M_{ab}} \left( [(M M^\dagger)^n M]_{ab} e^{-N \text{Tr}[V(M M^\dagger)]} \right)$$

$$= \left\langle \sum_{k=0}^n [(M M^\dagger)^k]_{aa} [(M^\dagger M)^{n-k}]_{bb} \right\rangle - N \left\langle \sum_{j=1}^n j t_j [(M M^\dagger)^n M]_{ab} [M^\dagger (M M^\dagger)^{j-1}]_{ab} \right\rangle,$$ \hspace{1cm} (4)

which, summing over $a$ and $b$, becomes

$$\left\langle \sum_{k=0}^n \text{Tr}[(M M^\dagger)^k] \text{Tr}[(M M^\dagger)^{k-n} k] - N \sum_{j=1}^n j t_j \text{Tr}[(M M^\dagger)^{n+j}] \right\rangle = 0.$$ \hspace{1cm} (5)

Every insertion of an operator $\text{Tr}[(M M^\dagger)^j]$ in the correlation function can be re expressed as a derivative of $V(M M^\dagger)$ with respect to $t_j$. Consequently the SDEs become

$$L_n Z = 0, \text{ for } n \geq 0,$$
\[ L_n = N^2 \delta_{0,n} - \frac{2}{N} \frac{\partial}{\partial t_n} + \frac{1}{N^2} \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_k \partial t_{n-k}} + \sum_{j=1}^{\infty} j \ t_j \frac{\partial}{\partial t_{n+j}}, \]  

(6)

where the derivatives w.r.t. \( t_j \), with \( j \leq 0 \) are understood to be omitted. A direct computation (involving some relabeling of discrete sums) shows \([45]\) that the \( L_n \)'s respect the commutation relations of (the positive operators of) the Virasoro algebra

\[ [L_m, L_n] = (m - n) \ L_{m+n} \text{ for } m, n \geq 0. \]  

(7)

Note that as we only deal with \( L_m, m \geq 0 \) we of course do not obtain the central charge term. The key to recovering the Virasoro algebra is the presence of the last term in eq. (6). The truncated operators \( L'_m = L_m - \frac{1}{N^2} \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_k \partial t_{n-k}} \) also respect \([L'_m, L'_n] = (m - n) \ L'_{m+n}\). If one specializes the operators we define below for \( D = 2 \) (and takes into account the cyclicity of the trace), one obtains the operators \( L'_m \).

This classical result is our guide towards deriving SDEs and loop equations for colored tensor models in arbitrary dimensions.

III. A LIE ALGEBRA INDEXED BY COLORED, ROOTED, D-ARY TREES

As the higher dimensional generalization of the Virasoro algebra we obtain is rather non trivial we will first present it in full detail and only later identify it with the algebra of constraints satisfied by the partition function. The operators in our algebra are indexed by colored rooted \( D \)-ary trees. Trees and \( D \)-ary trees are well studied in the mathematical literature \([57]\). The colored rooted \( D \)-ary trees index the leading order in the \( 1/N \) expansion of colored tensor models \([53]\). The colored rooted \( D \)-ary trees index the leading order in the \( 1/N \) expansion of colored tensor models \([53]\).

A. Colored rooted \( D \)-ary trees: Definitions

A colored rooted \( D \)-ary tree \( T \) with \(|T|\) vertices is a tree with the following properties

- It has a root vertex, denoted \((\ )\), of coordination \( D \).
- It has \(|T| - 1\) vertices of coordination \( D + 1 \) (i.e. each of them has \( D \) descendants).
- It has \((D - 1)|T| + 1\) leaves of coordination 1 (i.e. with no descendants).
- All lines have a color index, 0, 1, \ldots, \( D - 1 \), such that the \( D \) direct descendants (leaves or vertices) of a vertex (or of the root) are connected by lines with different colors.

We will ignore in the following the leaves of the tree, as they can automatically be added once the vertices and lines of the colored tree are known. A crucial fact in the sequel is that a colored rooted \( D \)-ary tree admits a canonical labeling of its vertices. Namely, every vertex can be labeled by the list of colors \( V = (i_1, \ldots, i_n) \) of the lines in the unique path connecting \( V \) to the root \((\ )\). The first color, \( i_1 \), is the color of the line in the path ending on the root (and \( i_n \) is the color of the line ending on \( V \)). For instance the vertex \((0)\) is the descendant connected to the root \((\ )\) by the line of color 0, and the vertex \((01)\) is the descendant of the vertex \((0)\) connected to it by a line of color 1 (see figure 11 for an example of a canonically labeled \( 3 \)-ary tree with \(|T| = 7 \) vertices).

In the sequel “tree” will always mean a colored rooted \( D \)-ary tree. A tree is completely identified by its canonically labeled vertices, hence it is a set \( T = \{(\ ), \ldots\} \). We denote \( i_1, \ldots, i_n \) a list of \( n \)
FIG. 1. A colored rooted $D$-ary tree.

identical labels $i$. We will use the shorthand notation $V$ for the list of labels identifying a vertex. Let a tree $T$, and one of its vertices

$$V = l, \ldots, k, i, \ldots, i,$$

with $k \neq i$. The colors $l, \ldots, k$ might be absent. The **successor of color** $j$ of $V$, denoted $s^j_T[V]$, is the vertex

$$s^j_T[(l, \ldots, k, i, \ldots, i)] = \begin{cases} (l, \ldots, k, i, \ldots, i, j) & \text{if it exists} \\ (l, \ldots, k, i, \ldots, i) & \text{if not} \\ (l, \ldots, k) & \text{if } j = i \end{cases}.$$  

The colored successor functions are cyclical, namely if a vertex does not have a descendant in the tree of color $j$, then its “successor of color $j$” is the first vertex one encounters, when going from $V$ to the root, whose label does not end by $j$. For the example of figure 1 we have.

$$s^0_0[( )] = (0) \quad s^1_0[( )] = ( ) \quad s^2_0[( )] = (2)$$
$$s^0_0[(0)] = (0) \quad s^1_0[(0)] = (01) \quad s^2_0[(0)] = (0)$$
$$s^0_0[(01)] = (01) \quad s^1_0[(01)] = (0) \quad s^2_0[(01)] = (01)$$
$$s^0_0[(2)] = (20) \quad s^1_0[(2)] = (2) \quad s^2_0[(2)] = (22)$$
$$s^0_0[(20)] = (2) \quad s^1_0[(20)] = (20) \quad s^2_0[(20)] = (20)$$
$$s^0_0[(22)] = (22) \quad s^1_0[(22)] = (22) \quad s^2_0[(22)] = (222)$$
$$s^0_0[(222)] = (222) \quad s^1_0[(222)] = (222) \quad s^2_0[(222)] = ( ).$$

We call $V$ the **maximal vertex of color** $i$ in a tree $T$ if

$$V = i, \ldots, i \quad \text{such that} \quad s^i_T[(i, \ldots, i)] = ( ).$$

In figure 1 the maximal vertex of color 2 is $(222)$, the maximal vertex of color 0 is $(0)$ and the maximal vertex of color 1 is the root $( )$ itself.

We define the **branch of color** $i$ of $T$, denoted $T^i$, the tree

$$T^i = \{(X) \mid (i, X) \in T\}.$$  

The branch $T^i$ can be empty. The root of the branch $T^i$, $( ) \in T^i$ corresponds to the vertex $(i) \in T$. The rest of the vertices of $T$ (that is the root $( )$ and all vertices of the form $(k, U) \in T$, $k \neq i$) also form canonically labeled tree $\tilde{T}^i$, the **complement in** $T$ of the branch $T^i$. In figure 1 the
branch of color 2 is the tree $T^2 = \{(\ ), (2), (22), (0)\}$, as all the vertices (2), (22), (222) and (20) belong to $T$. Its complement is $\tilde{T}^2 = \{(\ ), (0), (01)\}$.

Two colored rooted $D$-ary trees $T$ and $T_1$ can be joined (or glued) at a vertex $V \in T$. For all colors $i$, denote the maximal vertices of color $i$ of $T_1$

\[
(i,\ldots,i), \quad s^T_1[(i,\ldots,i)] = (\ ).
\]

The glued tree $T \star_V T_1$, is the tree canonically labeled

\[
T \star_V T_1 = \begin{cases}
(X) & \text{for all } (X) \in T, \ (X) \neq (V,\ldots) \\
(V,Y) & \text{for all } (Y) \in T_1 \\
(V,\underbrace{i,\ldots,i}_{n_i+1},Z) & \text{for all } (V,i,Z) \in T
\end{cases}.
\]

This operation can be seen as cutting all the branches starting at $V$ in $T$, gluing the tree $T_1$ at $V$, and then gluing back the branches at the maximal vertices of the appropriate color in $T_1$. The vertices of $T \setminus (V)$ and $T_1 \setminus (\ )$ map one to one onto the vertices of $(T \star_V T_1) \setminus (V)$, and both $(V) \in T, (\ ) \in T_1$ map to $(V) \in T \star_V T_1$, thus $|T \star_V T_1| = |T| + |T_1| - 1$. An example is given in figure 2, where the leaves are not drawn.

![Figure 2](image.png)

**FIG. 2.** Gluing of two trees at a vertex $T \star_{(2)} T_1$.

For any tree $T$, with maximal vertex of color $i$, $(i,\ldots,i)$, the maximal vertex of color $i$ in the branch $T^i$ will have one less label $(i,\ldots,i)$. One can glue the tree $\{(\ ), (i)\}$ on $(i,\ldots,i) \in T^i$. All the vertices of $T^i$ are unchanged by this gluing, its only effect being to introduce a new vertex, $(i,\ldots,i)$ which becomes the new maximal vertex of color $i$. Subsequently, one can glue the complement of the branch $i$, $\tilde{T}^i$ on this new maximal vertex

\[
T' = \left( T^i \star_{(i,\ldots,i)} \{(\ ), (i)\} \right) \star_{(i,\ldots,i)} \tilde{T}^i.
\]

The two trees $T$ and $T'$ have the same number of vertices $|T'| = |\tilde{T}^i| + |T^i| = |T|$, and the vertices of the initial tree $T$ map one to one on the vertices of the final $T'$. As none of the vertices of $\tilde{T}^i$ starts by a label $i$, it follows that the maximal vertex of color $i$ in $T'$ is $(i,\ldots,i)$. Thus

\[
(i, V) \in T \leftrightarrow (V) \in T', \quad (V) \neq (i,\ldots,i, U),
\]
\[(W) \in \mathcal{T}, \ (W) \neq (i, V) \iff (i, \ldots, i, W) \in \mathcal{T}'. \ \ (16)\]

Most importantly, it is straightforward to check that the mapping is consistent with the successor functions

\[V, W \in \mathcal{T} \iff V', W' \in \mathcal{T}' \text{ such that } s^i_T[V] = W \iff s^i_{T'}[V'] = W'. \ \ (17)\]

We will say that the two trees \(\mathcal{T}\) and \(\mathcal{T}'\) are equivalent, \(\mathcal{T} \sim \mathcal{T}'\), and extended by transitivity \(\sim\) to an equivalence relation between rooted trees. An example is presented in figure 3.

![FIG. 3. Two equivalent trees \(\mathcal{T} \sim \mathcal{T}'\), with \(\mathcal{T}' = (\mathcal{T} \ast (22) \{((), (2))\}) \ast (222) \tilde{T}^2\).](image)

The equivalence class of a tree \([\mathcal{T}]\) has \(|\mathcal{T}|\) members all obtained by choosing a vertex \(V = i, j, k, \ldots, l\) in \(\mathcal{T}\) performing the elementary operation \(\sim\) on the colors \(i\) followed by \(j\) followed by \(k\) and so on up to \(l\).

**B. Colored rooted D-ary trees: Properties**

In this section we prove a number of lemmas concerning the gluing of trees, \(\ast\). All this properties can be readily understood in terms of the graphical representation of the trees. In the sequel we will deal with three trees \(\mathcal{T}, \mathcal{T}_1\) and \(\mathcal{T}_2\). We denote \((i, \ldots, i)\) the maximal vertex of color \(i\) in \(\mathcal{T}_1\) and \((i, \ldots, i)\) the maximal vertex of color \(i\) in \(\mathcal{T}_2\).

**Lemma 1.** If \((V) = (k, U) \in \mathcal{T}\) then \((\mathcal{T} \ast_V \mathcal{T}_1)^k = \mathcal{T}^k \ast_U \mathcal{T}_1\), and, for \(i \neq k\), \((\mathcal{T} \ast_V \mathcal{T}_1)^i = \mathcal{T}^i\).

**Proof:** The composite tree \(\mathcal{T} \ast_V \mathcal{T}_1\) is

\[\mathcal{T} \ast_V \mathcal{T}_1 = \begin{cases} (i, X) & \text{for all } (i, X) \in \mathcal{T}, \ (i, X) \neq (k, U, \ldots) \\ (k, U, Y) & \text{for all } (Y) \in \mathcal{T}_1 \\ (k, U, i, \ldots, i, Z) & \text{for all } (k, U, i, Z) \in \mathcal{T} \end{cases}. \ \ (18)\]

It follows that

\[i \neq k: \quad (\mathcal{T} \ast_V \mathcal{T}_1)^i = \left\{(X) \text{ for all } (i, X) \in \mathcal{T}\right\} = \mathcal{T}^i, \quad (19)\]
and the branch of color $k$ of $T \ast_V T_1$ is

$$(T \ast_V T_1)^k = \begin{cases}
(X) & \text{for all } (k, X) \in T, (k, X) \neq (k, U, \ldots) \\
(U, Y) & \text{for all } (Y) \in T_1 \\
(U, i, \ldots, i, Z) & \text{for all } (k, U, i, Z) \in T
\end{cases}, \quad (20)$$

As $T^k = \{(X) \mid (k, X) \in T\}$ it follows that $(U) \in T^k$ and

$$(T^k \ast_U T^1) = \begin{cases}
(X) & \text{for all } (X) \in T^k, (X) \neq (U, \ldots) \Leftrightarrow (k, X) \in T, (k, X) \neq (k, U, \ldots) \\
(U, Y) & \text{for all } (Y) \in T_1 \\
(U, i, \ldots, i, Z) & \text{for all } (U, i, Z) \in T^k \Leftrightarrow (k, U, i, Z) \in T
\end{cases}. \quad (21)$$

Lemma 2. For any three trees, $(T \ast_V T_1) \ast_V T_2 = T \ast_V (T_1 \ast_V T_2)$.

Proof: Note that the label of any vertex in $T_1$ starts by the empty label $(\ )$, hence the joining at the root writes

$$T_1 \ast( ) T_2 = \begin{cases}
(Y_2) & \text{for all } (Y_2) \in T_2 \\
(i, \ldots, i, Y_1) & \text{for all } (i, Y_1) \in T_1
\end{cases}. \quad (22)$$

The maximal vertices of color $i$ in $T_1 \ast( ) T_2$ are $(i, \ldots, i)$. Both operations lead to a tree with vertices

$$\begin{cases}
(X) & \text{for all } (X) \in T, (X) \neq (V, \ldots) \\
(V, Y_2) & \text{for all } (Y_2) \in T_2 \\
(V, i, \ldots, i, Y_1) & \text{for all } (i, Y_1) \in T_1 \\
(V, i, \ldots, i, Z) & \text{for all } (V, i, Z) \in T
\end{cases}. \quad (23)$$

Lemma 3. For any two distinct vertices $V \neq W \in T$, we denote $W'$ the image of $W$ in the tree $T \ast_V T_1$ and $V'$ the image of $V$ in $T \ast_W T_2$. Then $(T \ast_V T_1) \ast_W T_2 = (T \ast_V T_2) \ast_V T_1$.

Proof: We distinguish two cases. Either $V$ and $W$ are not ancestor to each other in $T$ (both $(W) \neq (V, \ldots)$ and $(V) \neq (W, \ldots)$). Hence $W' = W$ and $V' = V$. In this case, both operations lead to

$$\begin{cases}
(X) & \text{for all } (X) \in T, (X) \neq (V, \ldots) \text{ and } (X) \neq (W, \ldots) \\
(V, Y_1) & \text{for all } Y_1 \in T_1 \\
(V, i, \ldots, i, Z) & \text{for all } (V, i, Z) \in T \\
(W, Y_2) & \text{for all } Y_2 \in T_1 \\
(W, i, \ldots, i, Z) & \text{for all } (W, i, Z) \in T
\end{cases}. \quad (24)$$
Or one of them (say \( V \)) is ancestor to the other (hence \( (W) = (V,j,A) \)). In this case \( V' = V \), but \( W' = V,j_1,j_2,...,j_n,A \). Both operations lead to the tree

\[
\begin{align*}
(\text{X}) & \quad \text{for all } X \in \mathcal{T}, \ X \neq (V,...) \\
(V,Y_1) & \quad \text{for all } Y_1 \in \mathcal{T}_1 \\
(V,i,...,i,Z) & \quad \text{for all } (V,i,Z) \in \mathcal{T} \text{ with } (V,i,Z) \neq (V,j,A,Z') = (W,Z') \\
(V,j,...,j,A,Y_2) & \quad \text{for all } Y_2 \in \mathcal{T}_2 \\
(V,j,...,j,A,i_1,...,i_{q_1}Z) & \quad \text{for all } (V,j,A,i,Z) = (W,i,Z) \in \mathcal{T} \\
\end{align*}
\]

(25)

\[\square\]

**Lemma 4.** Let \( V \in \mathcal{T} \) and \( W \in \mathcal{T}_1 \) then \((\mathcal{T} \ast_V \mathcal{T}_1) \ast_W \mathcal{T}_2 = \mathcal{T} \ast_V (\mathcal{T}_1 \ast_W \mathcal{T}_2)\), where again \( W' \) denotes the image of \( W \) in the tree \( \mathcal{T} \ast_V \mathcal{T}_1 \).

**Proof:** The image of \( W, W' = V,W \). Both joinings lead to the tree

\[
\begin{align*}
(\text{X}) & \quad \text{for all } X \in \mathcal{T}, X \neq (V,...) \\
(V,Y_1) & \quad \text{for all } Y_1 \in \mathcal{T}_1 \\
(V,W,Y_2) & \quad \text{for all } Y_2 \in \mathcal{T}_2 \\
(V,W,i,...,i,Z) & \quad \text{for all } (W,i,Z) \in \mathcal{T}_1 \\
(V,i,...,i,Z) & \quad \text{for all } (V,i,Z) \in \mathcal{T} \text{ if } (W) \neq (i,...,i) \\
(V,i,...,i,Z) & \quad \text{for all } (V,i,Z) \in \mathcal{T} \text{ if } (W) = (i,...,i) \\
\end{align*}
\]

(26)

\[\square\]

**Lemma 5.** We have \((\mathcal{T}_2 \ast_{(\cdot)} \mathcal{T}_1)^i \sim (\mathcal{T}_1 \ast_{(\cdot)} \mathcal{T}_2)^i\).

**Proof:** We have

\[
\mathcal{T}_2 \ast_{(\cdot)} \mathcal{T}_1 = \begin{cases} 
(k,Y_1) & \text{for all } (k,Y_1) \in \mathcal{T}_1 \\
(i,...,i,Y_2) & \text{for all } (i,Y_2) \in \mathcal{T}_2 \\
n_{i+1} & 
\end{cases}
\]

(27)

hence

\[
(\mathcal{T}_2 \ast_{(\cdot)} \mathcal{T}_1)^i = \begin{cases} 
(Y_1) & \text{for all } (i,Y_1) \in \mathcal{T}_1 \\
(i,...,i,Y_2) & \text{for all } (i,Y_2) \in \mathcal{T}_2 \\
n_i & 
\end{cases}
\]

(28)

Note that the vertices \((Y_1) \in \mathcal{T}_1^i\) start necessarily by at most \(n_i - 1\) labels \( i \). The maximal vertex of color \( i \) in \((\mathcal{T}_2 \ast_{(\cdot)} \mathcal{T}_1)^i\) is \((i,...,i)\). The tree \((\mathcal{T}_2 \ast_{(\cdot)} \mathcal{T}_1)^i\) is equivalent with \( \mathcal{T}' \) obtained by mapping the vertices

\[
(i,V) \in (\mathcal{T}_2 \ast_{(\cdot)} \mathcal{T}_1)^i \rightarrow (V) \in \mathcal{T}' \quad (W) \in \mathcal{T}, (W) \neq (i,V) \rightarrow (i,...,i,W).
\]

(29)
Iterating \( n_i \) times we see that all the vertices belonging initially to \( T_i^1 \) will, at one step, acquire \( n_i + q_i - 1 \) labels \( i \) and lose a label \( i \) at all the other \( n_i - 1 \) steps. The vertices belonging to \( T_2 \) will only lose a label \( i \) at all \( n_i \) steps. Thus \( (T_2 \circ \cdots \circ T_1)^i \) is equivalent to the tree

\[
\begin{cases}
(i, \ldots, i, Y_1) & \text{for all } Y_1 \in T_1^i \\
(Y_2) & \text{for all } Y_2 \in T_2^i
\end{cases}
\]

(30)

which we recognize by eq. (28) to be \((T_1 \circ \cdots \circ T_2)^i\).

\[
\square
\]

C. The Lie algebra indexed by colored rooted D-ary trees

We now define a Lie algebra of operators indexed by the trees. We associate to every tree a variable \( t_T \), and we denote \( |R_i| \) the coordination of the root of \( T_i \). Consider the operators \( L_{T_i} \) defined as

\[
L_{T_1} = (-)^{|R_1|} N^{D-D|R_i|} \frac{\partial |R_1|}{\prod_{i: \not T_1 \neq T_1} \partial t_{T_1}} + \sum_T t_T \sum_{V \in T} \frac{\partial f}{\partial t_{T \ast V \ast T_1}},
\]

\[
L_{\{()\}} = N^D \sum_T t_T \sum_{V \in T} \frac{\partial f}{\partial t_T},
\]

(31)

where \( N \) is some parameter (destined to become the large \( N \) parameter of the tensor model). We will not consider the most general domain of this operators. Namely, in stead of defining them for arbitrary functions \( f(t_T) \) we restrict their domain to class functions \( f(t_{[T]}) \), with \( t_{[T]} = \sum_{T' \sim T} t_{T'} \).

**Theorem 1.** When restricted to class functions, the operators \( L_T \) form a Lie algebra with commutator

\[
[L_{T_2}, L_{T_1}] f(t_{[T]}) = \sum_{V \in T_2} L_{T_2 \ast V \ast T_1} f(t_{[T]}) - \sum_{V \in T_1} L_{T_1 \ast V \ast T_2} f(t_{[T]}).
\]

**Proof:** We start by evaluating \( L_{T_2} L_{T_1} f \). We have

\[
L_{T_2} L_{T_1} f = \left[ (-)^{|R_2|} N^{D-|R_2|D} \frac{\partial |R_2|}{\prod_T \partial t_{T_2}^j} \sum_T t_T \sum_{V \in V_T} \frac{\partial f}{\partial t_{T \ast V \ast T_2}} \right] \left( \begin{array}{l}
\frac{\partial |R_1|}{\prod_T \partial t_{T_1}^j} \sum_T t_T \sum_{V \in V_T} \frac{\partial f}{\partial t_{T \ast V \ast T_1}} \\
+ \frac{\partial |R_1| + |R_2|}{\prod_T \partial t_{T_1}^j \prod_T \partial t_{T_2}^j} f \sum_k \sum_{V \in V_k} \frac{\partial f}{\partial t_{T \ast V \ast T_1}} \frac{\partial f}{\partial t_{T \ast V \ast T_2}} \\
+ (-)^{|R_1| + |R_2|} N^{2D-D(|R_1| + |R_2|)} \frac{\partial |R_1| + |R_2|}{\prod_T \partial t_{T_1}^j \prod_T \partial t_{T_2}^j} f \sum_k \sum_{V \in V_k} \frac{\partial f}{\partial t_{T \ast V \ast T_1}} \frac{\partial f}{\partial t_{T \ast V \ast T_2}} \\
\end{array} \right)
\]

(33)
hence the commutator is

\[
\left[ \mathcal{L}_{T_2}, \mathcal{L}_{T_1} \right] f = (-)^{|R_2|} N^{D-|R_2|} \sum_{k} \sum_{V \in T_2^k} \frac{\partial^{|R_2|}}{\partial t_{T_2} \partial t_{T_2^k \star V T_1}} f
\]

\[
- (-)^{|R_1|} N^{D-|R_1|} \sum_{k} \sum_{V \in T_1^k} \frac{\partial^{|R_1|}}{\partial t_{T_1} \partial t_{T_1^k \star V T_2}} f
\]

\[
+ \sum_{T} t_T \sum_{V \in T} \sum_{V' \in T \star V T_2} \frac{\partial f}{\partial t(T_{V' T_2} \star V T_1)} - \sum_{T} t_T \sum_{V \in T} \sum_{V' \in T \star V T_2} \frac{\partial f}{\partial t(T_{V' T_2} \star V T_1)},
\]

where, by a slight abuse of notations, we denote \( V \) and \( V' \) also the images of \( V \) and \( V' \) under \( \star \) operations. By lemma 3, the terms in the third line cancel (after exchanging \( V \) and \( V' \) in the second term). Using lemma 4, the terms in the last line recombine with the ones in the first two lines. Finally, the terms in the fourth line rewrite using lemma 2. We thus obtain

\[
\left[ \mathcal{L}_{T_2}, \mathcal{L}_{T_1} \right] f = \sum_{V \in T_2} \frac{\mathcal{L}_{T_2 \star V T_1} f - \mathcal{L}_{T_1 \star V T_2} f}{\mathcal{L}_{T_2 \star V T_1} f} \]

\[
+ \sum_{T} t_T \sum_{V \in T} \left( \frac{\partial f}{\partial t(T_{V T_1} \star V T_1)} - \frac{\partial f}{\partial t(T_{V T_1} \star V T_2)} \right),
\]

Note that in both trees \( T_1 \star (\_ T_2 \) and \( T_2 \star (\_ T_1 \), the root has a nonempty branch of color \( i \) if at least one of \( T_1^i \) or \( T_2^i \) is non-empty. The two roots have then equal coordination denoted \( |R_{12}| \). Adding and subtracting, the commutator becomes

\[
\left[ \mathcal{L}_{T_2}, \mathcal{L}_{T_1} \right] f = \sum_{V \in T_2} \mathcal{L}_{T_2 \star V T_1} f - \sum_{V \in T_1} \mathcal{L}_{T_1 \star V T_2} f - N^{D-D|R_{12}|} \left( \frac{\partial^{|R_{12}|}}{\prod_{T(T_{V T_1} \star T_2)} t_{T_{V T_1}}^{i}} - \frac{\partial^{|R_{12}|}}{\prod_{T(T_{V T_1} \star T_2)^i}} \right),
\]

and the last term cancels due to lemma 5 and taking into account that \( f \) is a class function \( f([T]) \), thus \( \partial_{T} f = \partial_{T'}, f \) if \( T \sim T' \).

\( \square \)
IV. SCHWINGER DYSON EQUATIONS IN THE LARGE N LIMIT OF COLORED TENSOR MODELS

In this section we first recall the independent identically distributed (i.i.d.) colored tensor model and its $1/N$ expansion. We then generalize it to a colored model with a generic potential, derive, in the large $N$ limit the SDEs of the model and translate them into a set of equations (involving the operators $\mathcal{L}_{T_i}$) for the partition function $Z$. We will closely parallel the derivation of the loop equations in section II.

A. The i.i.d. colored tensor models with one coupling

We denoted $\vec{n}_i$, for $i = 0, \ldots, D$, the $D$-tuple of integers $\vec{n}_i = (n_{i,1}, \ldots, n_{i,0}, \ldots, n_{i,D-1})$, with $n_{i,k} = 1, \ldots, N$. This $N$ is the size of the tensors and the large $N$ limit defined in \cite{22,24} represents the limit of infinite size tensors. We set $n_{ij} = n_{ji}$. Let $\psi_{\vec{n}_i}^i, \bar{\psi}_{\vec{n}_i}^i$, with $i = 0, \ldots, D$, be $D+1$ couples of complex conjugated tensors with $D$ indices. The independent identically distributed (i.i.d.) colored tensor model in dimension $D$ \cite{24,27} is defined by the partition function

$$e^{-NDF_N(\lambda, \bar{\lambda})} = Z_N(\lambda, \bar{\lambda}) = \int d\psi d\bar{\psi} e^{-S(\psi, \bar{\psi})},$$

$$S(\psi, \bar{\psi}) = \sum_{i=0}^{D} \sum_{\vec{n}} \psi_{\vec{n}_i}^i \bar{\psi}_{\vec{n}_i}^i + \frac{\lambda}{N^{D(D-1)/2}} \sum_{\vec{n}} \prod_{i=0}^{D} \psi_{\vec{n}_i}^i + \frac{\bar{\lambda}}{N^{D(D-1)/2}} \sum_{\vec{n}} \prod_{i=0}^{D} \bar{\psi}_{\vec{n}_i}^i. \quad (38)$$

$\sum_{\vec{n}}$ denotes the sum over all indices $n_{ij}$ from 1 to $N$. The tensor indices $n_{ij}$ need not be simple integers (they can for instance index the Fourier modes of an arbitrary compact Lie group, or even of a finite group of large order \cite{58}). Rescaling $\psi_{\vec{n}_i}^i = N^{-D/4} P_{\vec{n}_i}^i$ leads to

$$S(\bar{P}, P) = N^{D/2} \left( \sum_{i=0}^{D} \sum_{\vec{n}} P_{\vec{n}_i}^i \bar{P}_{\vec{n}_i}^i + \lambda \sum_{\vec{n}} \prod_{i=0}^{D} P_{\vec{n}_i}^i + \bar{\lambda} \sum_{\vec{n}} \prod_{i=0}^{D} \bar{P}_{\vec{n}_i}^i \right). \quad (39)$$

The partition function of equation \cite{38} is evaluated by colored stranded Feynman graphs \cite{25,27}. The tensors have no symmetry properties under permutations of their indices (i.e. all $\psi_{\vec{n}_i}^i, \bar{\psi}_{\vec{n}_i}^i$ are independent). The colors $i$ of the fields $\psi^i, \bar{\psi}^i$ induce important restrictions on the combinatorics of stranded graphs. We have two types of vertices, say one of positive (involving $\psi$) and one of negative (involving $\bar{\psi}$). The lines always join a $\psi^i$ to a $\bar{\psi}^i$ and possess a color index. Any Feynman graphs $\mathcal{G}$ of this model is a simplicial pseudo-manifold \cite{26} and the colored tensor models provide a statistical theory of random triangulations in dimensions $D$, generalizing random matrix models. The tensor indices $n_{ijk}$ are preserved along the strands. The amplitude of a graph with $2p$ vertices and $F$ faces (closed strands) is \cite{24}

$$A(\mathcal{G}) = (\lambda \bar{\lambda})^p N^{-p\frac{D(D-1)}{2} + F}. \quad (40)$$

The $n$-bubbles of the graph are the maximally connected subgraphs made of lines with $n$ fixed colors. For instance, the $D$-bubbles are the maximally connected subgraphs containing all but one of the colors. They are associated to the $0$ simplices (vertices) of the pseudo-manifold. We label $\mathcal{B}^n_{\rho}$ the $D$-bubbles with colors $\{0, \ldots, D\} \setminus \{i\}$ (and $\rho$ labels the various bubbles with identical colors). We denote $\mathcal{B}^{[D]}$ the total number of $D$ bubbles, which respects \cite{24}

$$p + D - \mathcal{B}^{[D]} \geq 0. \quad (41)$$
where \( p \) is half the number of vertices of the graph.

A second class of graphs crucial for the \( 1/N \) expansion of the colored tensor model are the jackets \([22, 24, 31]\).

**Definition 1.** Let \( \tau \) be a cycle on \( \{0, \ldots, D\} \). A colored jacket \( J \) of \( \mathcal{G} \) is the ribbon graph made by faces with colors \((\tau^q(0), \tau^{q+1}(0))\), for \( q = 0, \ldots, D \), modulo the orientation of the cycle.

A jacket \( J \) of \( \mathcal{G} \) contains all the vertices and all the lines of \( \mathcal{G} \) (hence \( J \) and \( \mathcal{G} \) have the same connectivity), but only a subset of faces. The jackets (further studied in \([35, 41]\)) are ribbon graphs, completely classified by their genus \( g_J \). For a colored graph \( \mathcal{G} \) we define its degree \([23, 24]\).

**Definition 2.** The degree \( \omega(\mathcal{G}) \) of a graph is the sum of genera of its jackets, \( \omega(\mathcal{G}) = \sum_J g_J \).

The number of faces of a graph evaluates as a function of its degree \([23, 24]\):

\[
\omega(\mathcal{G}) = \frac{(D-1)!}{2} \left( p + D - B^{[D]} \right) + \sum_{i, \rho} \omega(B^i_{(\rho)}) .
\]

\[
\frac{2}{(D-1)!} \omega(\mathcal{G}) = \frac{D(D-1)}{2} p + D - \mathcal{F} . \tag{42}
\]

The \( 1/N \) expansion of the colored tensor model is encoded in the remark that \( \omega(\mathcal{G}) \), which is a positive number, has exactly the combination of \( p \) and \( \mathcal{F} \) appearing in the amplitude of a graph \([40]\), thus

\[
A(\mathcal{G}) = (\lambda \bar{\lambda})^p N^{D-1} \frac{1}{p} \omega(\mathcal{G}) . \tag{43}
\]

The free energy \( F_N(\lambda, \bar{\lambda}) \) of the model admits then an expansion in the degree

\[
F_N(\lambda, \bar{\lambda}) = F_\infty(\lambda, \bar{\lambda}) + O(N^{-1}) , \tag{44}
\]

where \( F_\infty(\lambda, \bar{\lambda}) \) is the sum over all graphs of degree 0. The degree plays in dimensions \( D \geq 3 \) the role played by the genus in matrix models, and in particular degree 0 graphs are spheres \([23]\).

**Lemma 6.** If the degree vanishes (i.e. all jackets of \( \mathcal{G} \) are planar) then \( \mathcal{G} \) is dual to a \( D \)-sphere.

We conclude this section with the following lemma.

**Lemma 7.** Let \( \mathcal{G} \) be a graph (with colors \( 0, \ldots, D \) and \( B^{\hat{D}}_{(\rho)} \) its \( D \)-bubbles with colors \( 0, \ldots, D - 1 \). Then

\[
\omega(\mathcal{G}) \geq D \sum_{\rho} \omega(B^{\hat{D}}_{(\rho)}) . \tag{45}
\]

**Proof:** Consider a jacket \( J \) of \( \mathcal{G} \). By eliminating the color \( D \) in its associated cycle we obtain a cycle over \( 0, \ldots, D - 1 \) associated to a jacket \( J^{\hat{D}}_{(\rho)} \) for each of its bubbles. As graphs, \( J^{\hat{D}}_{(\rho)} \) are one to one with disjoint subgraphs of \( J \) (obtained by deleting the lines of color \( D \) and joining the strands \((\pi^{-1}(D), D)\) and \((D, \pi(D))\) in mixed faces corresponding to \((\pi^{-1}(D), \pi(D))\) in \( J^{\hat{D}}_{(\rho)} \)[23]), consequently

\[
g_J \geq \sum_{(\rho)} g_{J^{\hat{D}}_{(\rho)}} . \tag{46}
\]

Every jacket \( J^{\hat{D}}_{(\rho)} \) is obtained as subgraph of exactly \( D \) distinct jackets \( J \) (corresponding to inserting the color \( D \) anywhere in the cycle associated to \( J^{\hat{D}}_{(\rho)} \)). Summing over all jackets of \( \mathcal{G} \) we obtain

\[
\sum_J g_J \geq D \sum_{\rho} \sum_{J^{\hat{D}}_{(\rho)}} g_{J^{\hat{D}}_{(\rho)}} . \tag{47}
\]

\(\square\)
B. Leading order graphs

The leading order graphs of the colored tensor model have been analyzed in detail in [53]. We present below reader’s digest of these results. We are interested in understanding in more depth the structure of leading order vacuum graphs in $D$ dimensions. Leading order vacuum graph can be obtained from leading order two point graphs by reconnecting the two external lines (and conversely, cutting any line in a leading order vacuum graph leads to a leading order two point graph). We detail below the two point graphs.

A $D$-bubble with two vertices $B^i_{\rho}$ has $\frac{D(D-1)}{2}$ internal faces, hence, by equation (42), the degree (and the topology) of a graph $G$ and of the graph $G / B^i_{\rho}$ obtained by replacing $B^i_{\rho}$ with a line of color $i$ (see figure 4) are identical.\footnote{This elimination is a 1-Dipole contraction for one of the two lines of color $i$ touching $B^i_{\rho}$ [24].}

It can be shown [53] that, for $D \geq 3$, a leading order two point graph must possess a $D$-bubble with exactly two vertices. Eliminating this bubble, we obtain a leading order graph having two less vertices. The new graph must in turn possess a bubble with two vertices, which we eliminate, and so on. It follows that the leading order 2-point graphs must reduce after a sequence of eliminations of $D$-bubbles to the graph with a single $D$-bubble and only two vertices of figure 4. It is more useful to take the reversed point of view and start with the graph of figure 4 and insert $D$-bubbles with two vertices on its lines. This insertion procedure preserves colorability, degree and topology. The leading order 2-point connected graphs (with external legs of color say $D$) are in one to one correspondence with colored rooted $(D+1)$-ary trees.

Order $(\lambda\bar{\lambda})$: The lowest order graph consists in exactly one $D$-bubble with two vertices (and external lines say of color $D$). We represent this graph by the tree with only the root vertex ( ) decorated with $(D+1)$ leaves. The leaves have colors $0, \ldots, D$. On $G$, we consider “active” all lines of colors $j \neq D$ and the line of color $D$ touching the vertex $\lambda$. They correspond to the leaves of the vertex. See figure 5 where the vertex $\lambda$ is dotted and the inactive line is represented as dashed.

Order $(\lambda\bar{\lambda})^2$: At second order we have $D+1$ graphs contributing. They come from inserting a $D$-bubble with two vertices on any of the $D+1$ active lines of the first order graph. All the interior lines of the new $D$-bubble are active, and so is the exterior line touching its vertex $\lambda$. Say we insert the new bubble on the active line of color $j$. This graph corresponds to the tree $\{(\ ),(j)\}$, see figure 6 for the case $j = 0$.

Order $(\lambda\bar{\lambda})^{p+1}$: We obtain the graphs at order $p+1$ by inserting a $D$-bubble with two vertices on any of the active lines of a graph at order $p$. The interior lines (and the exterior line touching the
vertex $\lambda$) of the new bubble are active. We represent this by connecting a vertex of coordination $D + 2$, with $D + 1$ active leaves, on one of the active leaves of a tree at order $p$. The new tree line inherits the color of the active line on which we inserted the $D$-bubble. At order $(\lambda \bar{\lambda})^p$ we obtain contributions from all rooted colored $(D + 1)$-ary trees with $p$ vertices.

Our tree is a colored version of Gallavotti-Nicolo [59] tree. The vertices of the tree represent certain subgraphs of $\mathcal{G}$. We call them melons $\mathcal{M}$ and we identify them as the 1-particle irreducible (1PI) amputated 2-point sub-graphs of $\mathcal{G}$. The intuitive picture is that a melon is itself made of melons within melons. The $D$-bubbles with only two vertices are obviously the smallest melons. The largest melon is the graph itself.

A rooted tree is canonically associated to a partial order. The partial ordering corresponding to the tree we have introduced is

$$\mathcal{M}_1 \geq \mathcal{M}_2 \quad \text{if} \quad \left\{ \begin{array}{l}
\text{either} \quad \mathcal{M}_1 \supset \mathcal{M}_2 \\
\exists \mathcal{N}_\rho \text{, } \mathcal{M}_1 \cup \left( \cup \mathcal{N}_\rho \right) \cup \mathcal{M}_2 \text{ is a 2-point amputated connected sub-graph of } \mathcal{G} \\
\text{with external points } \bar{\lambda} \in \mathcal{M}_1 \text{ and } \lambda \in \mathcal{M}_2
\end{array} \right.,$$  \hspace{1cm} (48)

and $\geq$ is transitive.

The line connecting $\mathcal{M}$ towards the root on the tree (i.e. going to a greater melon) inherits the color of the exterior half-lines of $\mathcal{M}$. An example in $D = 3$ is given in figure 7 where the dotted vertices of $\mathcal{G}$ are $\lambda$, the inactive line of $\mathcal{G}$ is dashed and the active leaves are implicit. We identify the melons by their external point $\bar{\lambda}$. Since the active external line of a melon is always chosen to be the one touching the vertex $\lambda$, the root melon in an arbitrary graph is the one containing the external point $\bar{\lambda}$, e.g. $\mathcal{M}_1$ in figure 7. Note that $\mathcal{M}_3 \supset \mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_6, \mathcal{M}_7$, hence it is their ancestor. Also $\mathcal{M}_3 \cup \mathcal{M}_8 \cup \mathcal{M}_{10}$ forms a two point function with external point $\bar{\lambda} \in \mathcal{M}_3$ and as $\mathcal{M}_9 \subset \mathcal{M}_{10}$, the melon $\mathcal{M}_3$ is the ancestor of $\mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_6, \mathcal{M}_7, \mathcal{M}_8, \mathcal{M}_9, \mathcal{M}_{10}$.

The vacuum leading order graphs (also called melonic) are obtained by reconnecting the two exterior half lines of a melonic two point graphs with a line. Their amplitude is $N^D$. If a graph is a melonic graph with $D + 1$ colors, all its $D$ bubbles are melonic graphs with $D$ colors. This is easy to see, as the reduction of a $D$ bubble with two vertices represents the reduction of a $D − 1$ bubble with two vertices for all $D$ bubbles which contain the two vertices. When reducing the graph to its root melon, one by one all its $D$-bubbles reduce to $D$-bubbles with two vertices.

Moreover, the $D$-ary trees of the $D$-bubbles are trivially obtained from the tree of the graph by deleting all lines (and leaves) of color $D$. We will be needing below the following obvious fact: given a melonic graph and one of its $D$ bubbles (say $\mathcal{B}_{(1)}^D$), all the lines of color $D$ connecting on it either separate it from a different $D$-bubble (and are tree lines in the associated colored rooted tree) or they connect the two external points of a 1PI amputated two point subgraph with $D − 1$ colors of $\mathcal{B}_{(1)}^D$ (i.e. they connect the two external points of a melon in $\mathcal{B}_{(1)}^D$, in which case they are leaves of the associated tree.
C. From one to an infinity of coupling constants

Inspired by section II, we generalize the colored tensor model with one coupling to a model with an infinity of couplings and derive the SDEs of the general model. First we integrate all colors save one, and second we “free” the couplings of the operators in the effective action for the last color.

When integrating all colors save one the partition function becomes

$$ Z = \int d\psi_D d\bar{\psi}_D \ e^{-S^D(\psi_D, \bar{\psi}_D)} $$

$$ S^D(\psi_D, \bar{\psi}_D) = \sum_{\mathcal{B}} \tilde{\psi}_{\bar{n}_D} \psi_{\bar{n}_D} + \sum_{\mathcal{B}} (\lambda \bar{\lambda})^p \ Tr_{\mathcal{B}_{\bar{n}}} [\tilde{\psi}_{\bar{n}}, \psi_{\bar{n}}] N^{-(D-1)p + \mathcal{F}_{\mathcal{B}}} \text{Sym}(\mathcal{B}) $$

where the sum over $\mathcal{B}$ runs over all connected vacuum graphs with colors $0, \ldots D - 1$ (i.e. over all the possible $D$-bubbles with colors $0, \ldots D - 1$) and $p$ vertices. The operators $Tr_{\mathcal{B}_{\bar{n}}} [\tilde{\psi}_{\bar{n}}, \psi_{\bar{n}}]$ in the effective action for the last color are tensor network operators. Every vertex of $\mathcal{B}$ is decorated by a tensor $\psi_{\bar{n}}$ or $\bar{\psi}_{\bar{n}}$, and the tensor indices are contracted as dictated by the graph $\mathcal{B}$. We denote $v, \bar{v}$ the positive (resp. negative) vertices of $\mathcal{B}$, and $l_{v, \bar{v}}$ the lines (of color $i$) connecting the positive vertex $v$ with the negative vertex $\bar{v}$. The operators write

$$ Tr_{\mathcal{B}_{\bar{n}}} [\tilde{\psi}_{\bar{n}}, \psi_{\bar{n}}] = \sum_{n} \left( \prod_{v, \bar{v} \in \mathcal{B}} \tilde{\psi}_{\bar{n}_v} \psi_{n_v} \right) \left( \prod_{i=0}^{D-1} \prod_{l_{v, \bar{v}} \in \mathcal{B}} \delta_{n_v, \bar{n}_{\bar{v}}} \right), $$

where all indices $n$ are summed. Note that, as all vertices in the bubble belong to an unique line of a given color, all the indices of the tensors are paired. The scaling with $N$ of an operator computes in terms of its degree

$$ N^{-(D-1)p + \frac{(D-1)(D-2)}{2} \omega(\mathcal{B})} = N^{-(D-1)p + \frac{2}{(D-2)!} \omega(\mathcal{B})}, $$

thus the effective action for the last color writes (dropping the index $D$)

$$ S^D(\psi, \bar{\psi}) = \sum \tilde{\psi}_{\bar{n}} \ \psi_{\bar{n}} + N^{D-1} \sum_{\mathcal{B}} (\lambda \bar{\lambda})^p \text{Sym}(\mathcal{B}) N^{-(D-1)p - \frac{2}{(D-2)!} \omega(\mathcal{B})} Tr_{\mathcal{B}} [\tilde{\psi}, \psi], $$
Attributing to each operator its coupling constant and rescaling the field to $T = \psi N^{-\frac{D-1}{2}}$, we obtain the partition function of colored tensor model with generic potential

$$Z = e^{-NDF(\rho_\beta)} = \int d\tilde{T} dT \ e^{-N^{D-1} S(\tilde{T}, T)} ,$$

$$S(\tilde{T}, T) = \sum \tilde{T}_\alpha T_\alpha + \sum_B t_B \ N^{-\frac{D^2}{2(D-2)}\omega(B)} \ \text{Tr}_B[\tilde{T}, T] . \quad (53)$$

It is worth noting that, although in the end we deal with an unique tensor $T$, the colors are crucial to the definition of the tensor network operators in the effective action. The initial vertex of the tensor model described a $D$ crucial to the definition of the tensor network operators in the effective action. The initial vertex of the tensor model described a $D$ graphs with network operators act as effective vertices (for instance each comes with its own coupling constant). When evaluating amplitudes of graphs obtained by integrating the last tensor $T$, we bring a building a $T$ operators. The graphs $G$ contributing to the connected multi bubble correlations are connected vacuum graphs with $D + 1$ colors. The effective vertices are the subgraphs with colors $0, \ldots, D - 1$, and encode the connectivity of the tensor network operators.

The partition function of eq. (53) provides a natural set of observables of the model: the multi bubble correlations defined as

$$\left\langle \text{Tr}_{B(1)}[\tilde{T}, T] \ \text{Tr}_{B(2)}[\tilde{T}, T] \ldots \ \text{Tr}_{B(\rho)}[\tilde{T}, T] \right\rangle = \prod_{i=1}^{\rho} \left(-N^{-\left[D-1-\frac{D^2}{2(D-2)}\omega(B(i))\right]} \ \frac{\partial}{\partial \theta_{B(i)}}\right) Z . \quad (54)$$

When introducing an infinity of coupling constants, we did not change the scaling with $N$ of the operators. The graphs $\mathcal{G}$ contributing to the connected multi bubble correlations are connected vacuum graphs with $D + 1$ colors and with $\rho$ marked subgraph corresponding to the insertions $\text{Tr}_{B(\rho)}[\tilde{T}, T]$. Taking into account the scaling of the insertions, the global scaling of such graphs is

$$\left\langle \text{Tr}_{B(1)}[\tilde{T}, T] \ \text{Tr}_{B(2)}[\tilde{T}, T] \ldots \ \text{Tr}_{B(\rho)}[\tilde{T}, T] \right\rangle_c \leq N^{D-\frac{D^2}{2(D-2)}\omega(\mathcal{G})} N^{-\rho(D-1)+\sum_{i=1}^{\rho} \frac{D^2}{2(D-2)}\omega(B(i))} \leq N^{D-\rho(D-1)-\frac{D^2}{2(D-2)}\omega(\mathcal{G})} , \quad (55)$$

where we use lemma [7].

In the large $N$ limit, the connected correlations receiving contributions from graphs of degree 0 (melonic graphs) dominate the multi bubble correlations. All their bubbles are necessarily melonic, in particular the insertions $\text{Tr}_{B(\rho)}[\tilde{T}, T]$. As we have seen in section [IV B], the melonic $D$-bubbles (i.e. melonic graphs with $D$ colors) are one to one with colored rooted $D$-ary trees $T$. The tensor network operators, eq. (50), of melonic bubbles can be written directly in terms of $T$. When building a $D$-bubble starting from $T$, each time we insert a melon corresponding to a vertex $V \in T$ we bring a $T$ and a $\tilde{T}$ tensor for the two external points of the melon. We denote the indices of $T$ by $\tilde{n}_V$ and the ones of $\tilde{T}$ by $\tilde{\bar{n}}_V$, and we get

$$\text{Tr}_B[\tilde{T}, T] \equiv \text{Tr}_T[T, T] = \prod_{V \in T} \left(T_{\tilde{n}_V} \ T_{\tilde{\bar{n}}_V} \ T_{\tilde{n}_V} \prod_{i=0}^{D-1} \delta_{s_i \tilde{\bar{n}}_V} \right) , \quad (56)$$

where $s_i^T$ is exactly the colored successor function defined in section [III]. As this operator depends exclusively of the successor functions, it is an invariant for an equivalence class of trees $T \sim T' \Rightarrow \text{Tr}_T[T, T] = \text{Tr}_{T'}[T, T]$, hence the action and the partition function depend only on the
class variables $t_{[\mathcal{T}]} = \sum_{\mathcal{T}' \sim \mathcal{T}} t_{\mathcal{T}'}$. Taking into account that the melonic bubbles have degree 0 (and redefining the coupling of the tree $\mathcal{T} = \{ ( ) \}$), the action writes

$$S(\bar{T}, T) = \sum_{\mathcal{T}} t_{\mathcal{T}} \text{Tr}_{\mathcal{T}}[\bar{T}, T] + S^r(\bar{T}, T),$$

where $S^r$ correspond to non melonic bubbles.

## D. Schwinger Dyson equations

Consider a melonic bubble corresponding to the tree $\mathcal{T}_1$ with root $( )_1$. We denote

$$\delta_{\bar{n}, n}^{\mathcal{T}_1} = \prod_{V \in \mathcal{T}_1} \delta_{n_V, \bar{n}_V}^{T_{\mathcal{T}_1}}.$$  

The SDEs are deduced starting from the trivial equality

$$S(\bar{T}, T) = \sum_{\mathcal{T}} t_{\mathcal{T}} \text{Tr}_{\mathcal{T}}[\bar{T}, T] + S^r(\bar{T}, T),$$

where $S^r$ correspond to non melonic bubbles.

The second line in eq. (60) represents graphs in which a line of color $D$ on a melonic bubble connects the $\bar{T}$ on the root melon $( )$ to a $\bar{T}$ on a distinct melon $V_2$. Hence it can not be a melon (see the end of section IV B). The last term represents a melonic bubble connected through a line to a non melonic bubble (coming from $\frac{\delta S^r}{\delta T_{\bar{P}}}$). Thus it can not be a melon either. Taking into account that we have one line explicit in both graphs, (hence a factor $N^{-(D-1)}$), and that the scaling of $\text{Tr}_{\mathcal{T}}[T, \bar{T}]$ is $N^{D-1}$, in both cases the correlations scale at most like

$$\frac{1}{Z} \langle \ldots \rangle \leq N^{D-\frac{2}{(D-1)^2}}.$$

The first term in eq. (60) factors over the branches $\mathcal{T}_1$ of $\mathcal{T}$. We denote $( )_{1,i}$ the root of the branch $\mathcal{T}_1^i$. Recall that, for a non empty branch $\mathcal{T}^i$, the vertex $s^i_{\mathcal{T}^i}[ ( )_1] = (i) \in \mathcal{T}$ maps on the root $( )_{1,i} \in \mathcal{T}^i$. For each branch we evaluate

$$\sum_{n_{\mathcal{T}^i}^{(1)}, \bar{n}_{\mathcal{T}^i}^{(1)}} \delta_{n_{\mathcal{T}^i}^{(1)}, \bar{n}_{\mathcal{T}^i}^{(1)}}^{(1)} \delta_{n_{\mathcal{T}^i}^{(1)}, \bar{n}_{\mathcal{T}^i}^{(1)}}^{(1)} \delta_{n_{\mathcal{T}^i}^{(1)}, \bar{n}_{\mathcal{T}^i}^{(1)}}^{(1)}$$

$$= \begin{cases} N & \text{if } s^i_{\mathcal{T}^i}[ ( )_1] = ( )_1 \\ \delta_{n_{\mathcal{T}^i}^{(1)}, \bar{n}_{\mathcal{T}^i}^{(1)}}^{(1)} & \text{if not} \end{cases},$$

(62)
thus, denoting $|R_1|$ the coordination of the root $( )_1 \in T_1$, we get

$$
\sum_{\bar{n}, \tilde{n}, \bar{R}_1} \delta_{\bar{n}, \bar{R}_1} \delta_{\bar{T}_1} \delta_{n, \bar{n}} = N^{D-|R_1|} \prod_{i=0}^{D-1} \delta_{T_1}^{t_i} = 0.
$$

(63)

The third term in eq. (60) computes

$$
\sum_{\bar{n}_{\{i\}} \bar{n}, \bar{R}_1, \bar{n}_V} \delta_{\bar{n}_{\{i\}} \bar{n}, \bar{R}_1} \delta_{n, \bar{n}} \delta_{n, \bar{n}} D_{\bar{T}_1}^{T} = \sum_{\bar{n}_{\{i\}} \bar{n}_V} \delta_{\bar{n}_{\{i\}} \bar{n}_V} D_{\bar{T}_1}^{T} \prod_{i=0}^{D-1} \delta_{n_{\{i\}}} \delta_{n_{\{i\}}} \delta_{n, \bar{n}} \delta_{n, \bar{n}} = \prod_{i=0}^{D-1} \delta_{n_{\{i\}}} \delta_{n, \bar{n}} \delta_{n, \bar{n}} = 0,
$$

(64)

hence the SDEs write, for every rooted tree $T_1$, with $|R_1|$ non empty branches starting from the root

$$
N^{D-|R_1|} \left( \prod_{i=0}^{D-1} Tr_{T_1}^{T} \right) - N^{D-1} \sum_{V \in T} \left( Tr_{T \times V}^{T} \right) = \langle \ldots \rangle
$$

(-)^{|R_1|} N^{D-|R_1|} \left( \frac{\partial |R_1|}{\prod_{i, T_i \neq T_1} \partial t_i} \right) Z + \sum_{T} t \sum_{V \in T} \frac{\partial}{\partial t_{T \times V}} Z = \langle \ldots \rangle,
$$

(65)

where $\langle \ldots \rangle$ denotes the non melonic terms of eq. (61). Taking into account the definition of $L_{T_1}$ in eq. (61), we obtain

$$
L_{T_1} Z = \langle \ldots \rangle \Rightarrow \lim_{N \to \infty} \left( N^{-D} \frac{1}{Z} L_{T_1} Z \right) = 0, \ \forall T_1.
$$

(66)

Recall that $Z = e^{-N^D F(t)}$ depends only on class variables $t_{T_1}$. At leading order in $1/N$ only melonic graph contribute to the free energy $F(t_B)$, hence $\lim_{N \to \infty} F(t_B) = F_\infty([t_{T_1}])$. The SDEs at leading order imply

$$
\prod_{i=0}^{D-1} \left( \frac{\partial F_\infty(t_{T_1})}{\partial t_i} \right) - \sum_{T} t \sum_{V \in T} \frac{\partial F_\infty(t_{T_1})}{\partial t_{T \times V}} = 0, \ \forall T_1.
$$

(67)

The most useful way to employ the SDEs is the following. Consider a class function $\tilde{Z} = e^{-N^D \tilde{F}}$ satisfying the constraints at all orders in $N$, $L_{T_1} \tilde{Z} = 0$. Its free energy in the large $N$ limit, $\tilde{F}_\infty(t_{T_1}) = \lim_{N \to \infty} \tilde{F}(t_{T_1})$ respects eq. (67), hence $\tilde{F}_\infty(t_{T_1}) = F_\infty(t_{T_1})$, that is the $N \to \infty$ limit of $\tilde{Z}$ and $Z$ coincide.

Note that an SDE at all orders can be derived for the trivial insertion

$$
\sum_{\bar{B}} \int \frac{\delta}{\delta t_{\bar{B}}} \left[ t_{\bar{B}} e^{-N^D-1} \delta \right] = 0 \Rightarrow \left( N^D + \sum_{\bar{B}} |B| \frac{\delta}{\delta t_{\bar{B}}} \right) Z = 0,
$$

(68)

where $|B|$ denotes the number of vertices of the bubble $B$. The above operator, which at leading order is $L_{(1)}$, should be identified with the generator of dilations [60].
V. CONCLUSIONS

We have generalized the colored tensor models to colored tensor models with generic interactions, derived the Schwinger Dyson equations at leading order and established that (at leading order) the partition function satisfies a set of constraints forming a Lie algebra. Much remains to be done in order to fully characterize the critical behavior of the colored tensor models. We present below a non-exhaustive list of topics one needs to address.

First, although the algebra of melonic bubbles observable closes at leading order in $1/N$, it does not close at all orders (in contrast with matrix models, for which the algebra of loop observables closes at all orders). Neither does the algebra of tensor networks corresponding to all bubbles. Indeed, if one attempts to derive the full SDEs, one generates terms associated to the addition of lines of color $D$ on the $D$-bubbles with colors $0, \ldots, D-1$. In order to obtain a full set of observables, one must also include tensor network operators for the corresponding graphs. The full SDEs can be derived, but their algebra is somewhat more involved than the one at leading order.

A second line of inquiry is to study the algebra of constraints $L_T$. As colored rooted $D$-ary trees can be indexed in many alternative ways, it is yet unclear whether this algebra is an entirely new one or some relabeling of an already known algebra. One should study in the future its (unitary) representations, central extension, etc. Although we do not yet know what is the continuum symmetry this algebra encodes, as the generator of dilations is one of its generators, we expect the continuum theory to be scale invariant.

Third, the equation (67) completely defines the free energy at leading order. One can easily write a solution of this equation as a perturbation series in the coupling constants. The perturbative solution is ill-adapted to the study of $F_\infty(t_\cal T)$. The differential equation (67) constitutes a much better starting point for the study of the leading order multi-critical behavior of generic colored tensor models in arbitrary dimensions.

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