The Dirichlet Problem for a Class of Prescribed Curvature Equations

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Abstract
In this paper, we consider the Dirichlet problem for a class of prescribed curvature equations. Both degenerate and non-degenerate cases are considered. The existence of the $C^{1,1}$ regular graphic hypersurfaces with prescribing a class of curvatures and constant boundary is proved for the degenerate case.

Keywords Prescribed curvature equations · Degenerate · $C^{1,1}$ regularity

1 Introduction

One of classic problems in differential geometry is to find hypersurfaces with prescribed curvatures and boundary data such as Plateau problem and its fully non-linear generalizations [15, 16]. If the boundary is the graph of a given function $\varphi : \partial \Omega \to \mathbb{R}$, where $\Omega$ is a domain in $\mathbb{R}^n$, such problems can be simplified to find a graphic hypersurfaces in $\mathbb{R}^{n+1}$ which are equivalent to solving Dirichlet problems of the form

$$\begin{cases}
  f(\kappa[M_u]) = \psi & \text{in } \Omega, \\
  u = \varphi & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

where $M_u = \{(x, u(x)); x \in \Omega\}$ is the graphic hypersurface defined by the function $u$, $\kappa[M_u] = (\kappa_1, \ldots, \kappa_n)$ are the principal curvatures of $M_u$ and $f$ is the curvature function. In particular, when $f = \kappa_1 + \cdots + \kappa_n$, $f = \sum_{i<j} \kappa_i \kappa_j$ and $f = \kappa_1 \cdots \kappa_n$,
are prescribed mean curvature, scalar curvature and Gauss curvature equations, respectively.

We call a $C^2$ regular hypersurface $M \subset \mathbb{R}^{n+1}$ $(\eta, n)$-convex if its principal curvature vector $\kappa(X) \in \Gamma$, where $\Gamma$ is a symmetric cone defined by

$$\Gamma := \{ \kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n : \lambda_i = \sum_{j \neq i} \kappa_i > 0, i = 1, \ldots, n \}$$

for all $X \in M$. A $C^2$ function $u : \Omega \rightarrow \mathbb{R}$ is called admissible if its graph is $(\eta, n)$-convex.

Such hypersurface is useful to describe the boundaries of Riemannian manifolds which have the homotopy type of a CW-complex [37, 45] and were studied by Sha [37, 38], Wu [45] and Harvey-Lawson [25] intensively. It is worth mentioning that an $(\eta, n)$-convex hypersurface was called $(n-1)$-convex in these references. Let $H(X)$ be the mean curvature of $M$ at $X \in M$. Define the $(0, 2)$-tensor field $\eta$ on $M$ by

$$\eta = Hg - h,$$

where $g$ and $h$ are the first and second fundamental forms of $M$, respectively. Obviously a hypersurface $M$ is $(\eta, n)$-convex if and only if $\eta$ is positive definite at each point of $M$. To measure the $(\eta, n)$-convexity, it is natural to introduce the $(\eta, n)$-curvature at $X \in M$: $K_\eta(X) := \lambda_1(X) \cdots \lambda_n(X)$. It is clear that

$$K_\eta(X) = \det(g^{-1}(\eta(X))).$$

In this paper, we are concerned with the existence of graphic $(\eta, n)$-convex hypersurfaces with prescribed $(\eta, n)$-curvature and boundary. In particular, we consider the Dirichlet problem

$$\begin{cases}
K_\eta[M_u] = \psi(X, \nu(X)) & X = (x, u(x)), x \in \Omega, \\
u(x) = 0 & x \in \partial \Omega,
\end{cases} \quad (1.2)$$

where $\nu(X)$ denotes the upward unit normal vector to $M_u$ at $X \in M_u$. In the current work, the function $\psi$ is allowed to vanish somewhere so that the Eq. (1.2) is degenerate. Our main results are stated in the following theorem.

**Theorem 1.1** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth strictly convex boundary $\partial \Omega$. Suppose

$$\psi^{1/(n-1)} = \psi^{1/(n-1)}(x, z, \nu) \in C^{1,1}(\mathbb{R}^n \times \mathbb{R} \times S^n) \geq 0 \quad (1.3)$$

and $\psi_z \geq 0$. Assume that that there exists an admissible subsolution $u \in C^{1,1}(\overline{\Omega})$ satisfying $\kappa[M_u] \in \Gamma$ and

$$\begin{cases}
K_\eta[M_u] \geq \psi(X, \nu(X)) & X = (x, u(x)), x \in \Omega, \\
u(x) = 0 & x \in \partial \Omega,
\end{cases} \quad (1.4)$$
where \( v(X) \) denotes the upward unit normal vector to \( M_u \) at \( X \in M_u \). In addition, there exists a function \( \psi = \psi(x, z) \in C^0(\overline{\Omega} \times \mathbb{R}) \geq 0 \) satisfying \( \psi \neq 0 \) on \( \overline{\Omega} \times [-\epsilon, 0] \) for any \( \epsilon > 0 \) such that

\[
\psi(x, z, v) \geq \psi(x, z) \quad \text{for} \quad (x, z, v) \in \overline{\Omega} \times \mathbb{R} \times S^n. \tag{1.5}
\]

Then there exists a unique admissible solution \( u \in C^{1,1}(\overline{\Omega}) \) of (1.2).

To prove Theorem 1.1, we need the solvability of the non-degenerate Eq. (1.2) which is stated in the next theorem.

**Theorem 1.2** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth strictly convex boundary \( \partial \Omega \). Suppose \( \psi = \psi(x, z, v) \in C^\infty(\overline{\Omega} \times \mathbb{R} \times S^n) > 0 \) and \( \psi_z \geq 0 \). Assume that there exists a subsolution \( u \) as in Theorem 1.1. Then there exists a unique admissible solution \( u \in C^\infty(\overline{\Omega}) \) of (1.2).

The non-degenerate prescribed curvature equations have received extensively studies. In particular, Caffarelli-Nirenberg-Spruck [6] studied the non-degenerate prescribed curvature equations of general form with \( \psi \) independent of \( u \) and \( \nu \) and \( \varphi \equiv \text{constant} \) in strictly convex domains. Ivochkina [27, 28] studied the Dirichlet problem for the prescribed curvature equation

\[
\sigma_k(\kappa) = \psi(x, u), \tag{1.6}
\]

where \( \sigma_k \) is the \( k \)-th elementary symmetric function

\[
\sigma_k(\kappa) = \sum_{i_1 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k}, \quad \text{for} \quad k = 1, 2, \cdots, n,
\]

and her results were generalized by Lin-Trudinger [33] and Ivochkina-Lin-Trudinger [29] to the equation

\[
(\sigma_k/\sigma_l)(\kappa) = \psi(x, u). \tag{1.7}
\]

The reader is referred to [5, 7, 14, 18, 21, 22, 35, 36, 39] for more researches about non-degenerate curvature equations.

The degenerate curvature equations arise naturally in many geometric problems such as the degenerate Weyl problem studied by Guan-Li [19] and Hong-Zuily [26], where the equations can be viewed as some two-dimensional degenerate Monge–Ampère type curvature equation. Guan-Li [20] considered the prescribed Gauss curvature measure problem, which also can be rewritten as some degenerate Monge–Ampère type curvature equation. As we know, the best regularity one can expect for degenerate curvature equations is \( C^{1,1} \). Recently, Jiao-Wang [31] proved the existence of solutions in \( C^{1,1}(\overline{\Omega}) \) for degenerate prescribed \( k \)-curvature Eq. (1.7). Guan-Zhang [24] considered another class of curvature type equations which is the combination of \( \sigma_k \).
When \( n = 2 \), Eq. (1.2) is the classic prescribed Gauss curvature equation. For general \( n \),

\[
K_\eta[M] = K_\eta(\kappa) = \sum_{i=2}^{n} \sigma_1(\kappa)^{n-i} \sigma_i(\kappa).
\] (1.8)

In [7] Chu-Jiao considered the curvature estimates for star-shaped hypersurfaces \( M \), i.e., \( M \) can be represented by a radial graph of positive function on \( \mathbb{S}^n \), satisfying the equation

\[
\sigma_k(\lambda(\eta(X))) = f(X, \nu(X)), \quad \text{for} \; X \in M,
\] (1.9)

where \( \lambda(\eta) \) denote the eigenvalues of \( \eta \) with respect to the metric \( g \). When \( k = n \), \( \sigma_k(\lambda(\eta)) \) is the \((\eta, n)\)-curvature of \( M \). It is an interesting problem to consider the Dirichlet problem (1.2) with \( K_\eta[M_u] \) replaced by \( \sigma_k(\lambda(\eta)) \). Recently, Yuan [46] proved an inequality for concave functions which may be applied to derive the second-order interior estimates for more general equations whose elliptic cones are not the positive cone \( \Gamma_n \).

The corresponding Hessian type equation of (1.2),

\[
det(\Delta u I - D^2 u) = \psi(x, u, Du)
\] (1.10)

is called \((n-1)\) Monge–Ampère equation. Recently, Dinew [8], Chu-Jiao [7] studied the interior estimates for some Hessian type equations including (1.10) and its Dirichlet problem was studied later [30]. The complex analog of (1.10) has been studied extensively since it is related to the Gauduchon conjecture (see [11, §IV.5]) which was solved by Székelyhidi-Tosatti-Weinkove [41] in complex geometry. For more references, the reader is referred to [9, 10, 34, 40, 42, 43] and references therein.

Now we make some remarks on the conditions of Theorem 1.1. In general, the Dirichlet problem is not always solvable without the existence of a subsolution. In this paper, we only use the subsolution to derive the lower bound of \( u \) and the gradient estimates on the boundary. Compared with Hessian type equations, the subsolution is not as powerful to construct barriers because the principal curvatures are eigenvalues of a much more complicated matrix. We make use of the strict convexity of \( \Omega \) to construct suitable barriers for second-order estimates on the boundary. However, more natural condition on the domain \( \Omega \) should be that there exists a positive constant \( K \) such that

\[
(\kappa_1^b(x), \ldots, \kappa_{n-1}^b(x), K) \in \Gamma
\]

for any \( x \in \partial \Omega \), where \( \kappa_1^b(x), \ldots, \kappa_{n-1}^b(x) \) are the principal curvatures of \( \partial \Omega \) at \( x \). Such domain is called admissible. It is of interest to ask if we can establish the second boundary estimates in admissible domains. For non-negative functions, the condition

\[
\psi^{1/(n-1)} \in C^{1,1}(\overline{\Omega} \times \mathbb{R} \times \mathbb{S}^n)
\] (1.11)
is a little weaker than that $\psi^{1/n} \in C^{1,1}(\overline{\Omega} \times \mathbb{R} \times S^n)$. Guan-Trudinger-Wang [23] proved the $C^{1,1}$ regularity for the solution of degenerate Monge–Ampère equation

$$\det(D^2u) = \psi(x) \geq 0 \text{ in } \Omega$$

with $u = \varphi$ on $\partial\Omega$ under the condition $\psi^{1/(n-1)} \in C^{1,1}(\Omega)$. Examples of Wang [44] show that the condition is optimal. Note that the assumption (1.3) in Theorem 1.1 is stronger than (1.11). Let $\tilde{\psi} := \psi^{1/(n-1)}$. Indeed, (1.3) implies (1.11) and that

$$|D\tilde{\psi}(x, z, \nu)| \leq C\mu_0 \sqrt{\tilde{\psi}(x, z, \nu)} \quad (1.12)$$

for any $(x, z, \nu) \in \overline{\Omega} \times [-\mu_0, \mu_0] \times S^n$ and some positive constant $C\mu_0$ depending only on $\|\tilde{\psi}\|_{C^{1,1}[\overline{\Omega} \times [-\mu_0, \mu_0] \times S^n]}$ (seeing Lemma 3.1 in [2]) which is only used to derive the gradient estimates. It is easy to see (1.12) yields that

$$\psi^{1/n} \in C^{1}(\overline{\Omega} \times \mathbb{R} \times S^n). \quad (1.13)$$

The existence of the function $\tilde{\psi}$ satisfying the assumptions in Theorem 1.1 is only used to establish the estimates for normal-normal second-order derivatives. If $\psi$ does not depend on $\nu$, we do not need the existence of $\tilde{\psi}$ in Theorem 1.1.

The key step to prove Theorem 1.1 is the establishment of the a priori $C^2$ estimates for non-degenerate Eq. (1.2) independent of the lower bound of $\psi$. Thus, the existence of solutions in $C^{1,1}(\overline{\Omega})$ can be proved by a non-degenerate approximation. The main parts of this work are the global and boundary estimates for second-order derivatives. Since the function $\psi$ may depend on the unit normal $\nu$, there are more troublesome terms when we differentiating the Eq. (1.2). We shall apply an idea of [7] to overcome this by using some properties of $\sigma_{n-1}$ in the global estimates. Another difficulty in global estimates is that we cannot use the concavity of $\sigma^{1/n}$ directly since we only assume (1.11). For this we use an idea of [31] to apply a lemma from the concavity of $(\sigma_k/\sigma_1)^{1/(k-1)}$ which was proved by Guan-Li-Li [18]. The main challenge for the boundary estimates is from the degeneracy of the equation. The key is a calculation of the linearized operator acting on the tangential derivatives of the solution, namely, Lemma 5.3. We remark that the special structure of the $(\eta, n)$-curvature plays an important role in the proof of Lemma 5.3.

It is worth mentioning that, in the non-degenerate case, if $\psi \equiv \psi(x, u) > 0$ depends only on $X = (x, u(x)) \in M_\eta$, the second-order boundary estimates can be obtained by an easy generalization of [6]. However, when $\psi$ also depends on $\nu$, some uncontrollable terms arise when the linearized operator acts on their barrier (seeing (4.6) in [6]). We do not have any idea to generalize the method of [6] in such case yet.

This paper is organized as follows. In Sect. 2 some preliminaries are provided. The $C^1$ estimates are established in Sect. 3. Sections 4 and 5 are devoted to the global and boundary estimates for second-order derivatives, respectively.
2 Preliminaries

Throughout this paper, $\phi_i = \frac{\partial \phi}{\partial x_i}$, $\phi_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$, $D\phi = (\phi_1, \cdots, \phi_n)$ and $D^2\phi = (\phi_{ij})$ denote the ordinary first and second-order derivatives, gradient and Hessian of a function $\phi \in C^2(\Omega)$, respectively.

A graphic hypersurface $M_u$ in $\mathbb{R}^{n+1}$ is a codimension one submanifold which can be written as a graph

$$M_u = \{X = (x, u(x))| x \in \mathbb{R}^n\}.$$ 

Let $\epsilon_{n+1} = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$, then the height function of $M_u$ is $u(x) = (X, \epsilon_{n+1})$. It’s easy to see that the induced metric and second fundamental form of $M$ are given by

$$g_{ij} = \delta_{ij} + u_i u_j, \quad 1 \leq i, j \leq n,$$

and

$$h_{ij} = \frac{u_{ij}}{\sqrt{1 + |Du|^2}},$$

while the upward unit normal vector field to $M$ is

$$v = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}.$$

By straightforward calculations, we have the principle curvatures of $M_u$ are eigenvalues of the matrix

$$\frac{1}{w}(I - \frac{Du \otimes Du}{w^2})D^2u$$

or the symmetric matrix $A[u] = (a_{ij})$:

$$a_{ij} = \frac{1}{w}\gamma_{ik}u_{kl}\gamma^{lj}, \quad (2.1)$$

where $\gamma_{ik} = \delta_{ik} - \frac{a_i a_k}{w(1+w)}$ and $w = \sqrt{1 + |Du|^2}$. Note that $(\gamma^{ij})$ is invertible with inverse $\gamma_{ij} = \delta_{ij} + \frac{u_i u_j}{1+w}$, which is the square root of $(g_{ij})$.

For $r \in S^{n \times n}$ and $p \in \mathbb{R}^n$, define

$$\lambda(r, p) = \lambda \left((I - \frac{p \otimes p}{1 + |p|^2})r\right)$$

and

$$S_k(r, p) = \sigma_k(\lambda(r, p)).$$
As in [29], we introduce the notations as follows. For $p \in \mathbb{R}^n$, $i = 1, \ldots, n$, let $p(i)$ be the vector obtained by setting $p_i = 0$, $r(i)$ the matrix obtained by setting the $i^{th}$ row and column to zero and $r(i, i)$ the matrix obtained by setting $r_{ii} = 0$. Denote

$$S_{k,i}(r, p) = S_k(r(i), p(i)).$$

Ivochkina-Lin-Trudinger [29] proved the formula

$$S_k(r, p) = \frac{1 + |p(i)|^2}{1 + |p|^2} r_{ii} S_{k-1;i}(r, p) + O(|r(i, i)|^k) \tag{2.2}$$

for all $1 \leq i \leq n$ and $1 \leq k \leq n$ and $S_0$ is defined by $S_0 \equiv 1$. Combining (1.8) and (2.2), we have

$$K_\eta(M_u) = \frac{1}{w^n} \sum_{i=2}^{n} S_1^{n-i}(D^2 u, Du) S_i(D^2 u, Du)$$

$$= \frac{1}{w^n} \left( \frac{1 + |Du(n)|^2}{1 + |Du|^2} \right)^{n-1} S_{1:n}(D^2 u, Du) u_{nn}^{n-1} + \sum_{i=1}^{n-2} P_i u_{nn}^i + P_0, \tag{2.3}$$

where $P_i$ depend only on $u_{\alpha\beta}$ ($\alpha + \beta < 2n$) and $Du$. For $\kappa \in \mathbb{R}^n$ let

$$\lambda_i = \sum_{j \neq i} \kappa_j, i = 1, \ldots, n. \tag{2.4}$$

Define the function $f(\kappa)$ on the cone $\Gamma$ by

$$f(\kappa) := \lambda_1 \cdots \lambda_n. \tag{2.5}$$

The function $f$ satisfies the following properties which may be used in the following sections. Their proofs can be found in [30].

$$f_i(\kappa) = \frac{\partial f(\kappa)}{\partial \kappa_i} > 0, \text{ in } \Gamma, i = 1, \ldots, n; \tag{2.6}$$

$$f^{1/n}(\kappa) \text{ is concave in } \Gamma; \tag{2.7}$$

$$f > 0 \text{ in } \Gamma \text{ and } f = 0 \text{ on } \partial \Gamma; \tag{2.8}$$

$$f_j(\kappa) \geq \delta_0 \sum_i f_i(\kappa), \text{ if } \kappa_j < 0, \forall \kappa \in \Gamma \tag{2.9}$$
for some positive constant $\delta_0$ depending only on $n$ and for any constant $A > 0$ and any compact set $K$ in $\Gamma$ there is a number $R = R(A, K)$ such that
\begin{equation}
 f(\kappa_1, \ldots, \kappa_{n-1}, \kappa_n + R) \geq A, \text{ for all } \kappa \in K.
\end{equation}
(2.10)

We also need some algebraic equalities and inequalities of $\sigma_k$ (seeing [32]):
\begin{align*}
\sum_i \sigma_{k-1;i}(\kappa) &= (n - k + 1)\sigma_{k-1}(\kappa), \\
\sum_i \sigma_{k-1;i}(\kappa)\kappa_i &= k\sigma_k(\kappa),
\end{align*}
and
\begin{align*}
\sigma_{k-1}(\kappa) &\geq c_0\sigma_1^{1/(k-1)}(\kappa)\sigma_k^{1/(k-1)}(\kappa) \\
\sigma_1(\kappa) &\geq c_0\sigma_k^{1/k}(\kappa)
\end{align*}
(2.11) (2.12)
for any $\kappa \in \Gamma_k$ and some positive constant $c_0$ depending only on $n$ and $k$. Note that the last two inequalities are consequences of the Newton-Maclaurin inequalities.

Now, let $\{e_1, e_2, \ldots, e_n\}$ be a local orthonormal frame on $T M\nu$. We will use $\nabla$ to denote the induced Levi-Civita connection on $M$. For a function $v$ on $M\nu$, we denote $\nabla_i v = \nabla_{e_i} v$, $\nabla_{ij} v = \nabla^2 v(e_i, e_j)$, etc in this paper. Thus, we have
\begin{equation}
|\nabla u| = \sqrt{g^{ij} u_i u_j} = \frac{|Du|}{\sqrt{1 + |Du|^2}}.
\end{equation}

Using normal coordinates, we also need the following well known fundamental equations for a hypersurface $M$ in $\mathbb{R}^{n+1}$:
\begin{align*}
\nabla_{ij} X &= h_{ij} v \quad \text{(Gauss formula)} \\
\nabla_i v &= - h_{ij} e_j \quad \text{(Weigarten formula)} \\
\nabla_k h_{ij} &= \nabla_{j} h_{ik} \quad \text{(Codazzi equation)} \\
R_{ijkl} &= h_{is} h_{jt} - h_{it} h_{js} \quad \text{(Gauss equation),}
\end{align*}
(2.13)
where $h_{ij} = \langle \nabla_{e_i} e_j, v \rangle$, $R_{ijkl}$ is the $(4, 0)$-Riemannian curvature tensor of $M$, and the derivative here is covariant derivative with respect to the metric on $M$. Therefore, the Ricci identity becomes,
\begin{equation}
\nabla_{ij} h_{st} = \nabla_{st} h_{ij} + (h_{mi} h_{sj} - h_{mj} h_{si})h_{mi} + (h_{mt} h_{ij} - h_{mj} h_{it})h_{ms}.
\end{equation}
(2.14)

### 3 $C^1$ Estimates

In this and the following sections, we assume $\psi > 0$ and establish the $C^2$ estimates independent of $\inf \psi$. 

\[ \text{Springer} \]
In this section, we consider the $C^1$ estimates for the admissible solution of (1.2). Since $M_u$ is $(\eta, n)$-convex, we find the mean curvature of $M_u$, $H[M_u] > 0$. It follows from the maximum principle that $u \leq 0$ in $\Omega$ since $u = 0$ on $\partial \Omega$. By the maximum principle again we get $u \geq 0$ in $\Omega$. It follows that
\[
\sup_{\Omega} |u| + \sup_{\partial \Omega} |Du| \leq C,
\]
for some positive constant $C$ depending only on $\|u\|_{C^1(\Omega)}$. Next we establish the global gradient estimates to prove

**Theorem 3.1** Let $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ be an admissible solution of (1.2). Suppose $\psi \geq 0$. Then there exists a positive constant $C$ depending only on $n$, $\|u\|_{C^0(\overline{\Omega})}$ and $\|\psi^{1/\mu_0}\|_{C^1(\overline{\Omega} \times [-\mu_0, \mu_0] \times \mathbb{S}^n)}$ but independent of $\inf \psi$ such that
\[
\sup_{\Omega} |Du| \leq C(1 + \sup_{\partial \Omega} |Du|),
\]
where $\mu_0 := \|u\|_{C^0(\overline{\Omega})}$.

**Proof** Suppose $\epsilon_1, \ldots, \epsilon_{n+1}$ is a standard basis of $\mathbb{R}^{n+1}$. Let
\[
Q = \log w + B\langle X, \epsilon_{n+1} \rangle = \log w + Bu,
\]
where $w = \frac{1}{\langle \nu, \epsilon_{n+1} \rangle} = \sqrt{1 + \|Du\|^2}$ and $B$ is a positive constant sufficiently large to be determined later. Suppose the maximum value of $Q$ is achieved at an interior point $x_0 \in \Omega$. We rotate $\epsilon_1, \ldots, \epsilon_n$ to satisfy, at $x_0$, $u_1 = |Du|$, $u_j = 0$ for $j \geq 2$. Define $e_i = \gamma^{i \bar{s}} \tilde{\delta}_s$, where $\tilde{\delta}_s := \epsilon_s + u_s \epsilon_{n+1}$ for $1 \leq s \leq n$, $i = 1, \ldots, n$. It is clear that \{e_1, e_2, \cdots, e_n\} is an orthonormal frame on $M_u$ and at $X_0 = (x_0, u(x_0))$, satisfy
\[
\nabla_1 u = \frac{|Du|}{w} = |\nabla u|, \nabla_i u = u_i = 0, \text{ for } i \geq 2.
\]
We may further rotate $\epsilon_2, \ldots, \epsilon_n$ such that $\{u_{ij}\}_{i,j \geq 2}$ is diagonal at $x_0$. We have
\[
\nabla_i w = \frac{1}{\langle \nu, \epsilon_{n+1} \rangle^2} h_{im} \nabla_m u.
\]
Thus, at the maximum point $x_0$ of $Q$, we get
\[
\nabla_i Q = wh_{i1} \nabla_1 u + B \nabla_i u = 0.
\]
Taking $i = 1$ and $i \geq 2$, respectively, we get, at $X_0$,
\[
wh_{11} = -B, h_{i1} = 0.
\]
Since at \( X_0 \),
\[
    h_{11} = \frac{u_{11}}{w}, \quad h_{1i} = \frac{u_{1i}}{w} \quad \text{and} \quad h_{ij} = \frac{u_{ij}}{w} \quad \text{for} \ i, j \geq 2,
\]
the matrix \( \{h_{ij}\} \) is diagonal at \( X_0 \). At \( X_0 \), we have
\[
    \nabla_{ii} Q = w^2 (h_{im\nabla_m u})^2 + w \nabla_i h_{11} \nabla_1 u + w h_{im\nabla_m u} + B \nabla_{ii} u \leq 0. \tag{3.4}
\]
Let \( \{y_k\}_{k=1}^n \) denote the standard local coordinate system on \( \mathbb{S}^n \) near \( (0, \ldots, 0, 1) \). Let
\[
    F^{ij} = \frac{\partial f(\lambda(h))}{h_{ij}}.
\]
By Codazzi equations, Weingarten formula and differentiating the Eq. (1.2), we have, at \( X_0 \),
\[
    F^{ii} \nabla_i h_{11} = F^{ii} \nabla_1 h_{1i} = c_1(\psi) = \psi_{x_j} \nabla_1 x_j + \psi_u \nabla_1 u + \partial y_k \psi \nabla_1 v_k
    = \frac{\psi_{x_1}}{w} + \psi_u \nabla_1 u - \frac{\partial y_k \psi h_{11}}{w} \tag{3.5}
    = \psi_{x_1} + \psi_u \nabla_1 u + \frac{B \partial y_k \psi}{w^2}.
\]
Using (1.13), (3.4), (3.5), \( \psi_u \geq 0 \) and that \( \nabla_{ij} u = \frac{h_{ij}}{w} \), we obtain, at \( X_0 \),
\[
    0 \geq w^2 F^{11} h_{11}^2 (\nabla_1 u)^2 + w \nabla_1 u(e_1 \psi) + F^{ii} h_{1i}^2 + \frac{n B \psi}{w}
    \geq F^{11} (\nabla_1 u)^2 B^2 + \psi_{x_1} \nabla_1 u + \frac{B \nabla_1 u \partial y_1 \psi}{w} \tag{3.6}
    \geq B^2 F^{11} (\nabla_1 u)^2 - C \left(1 + \frac{B}{w}\right) \psi^{1-1/n}.
\]
By (2.9) and that \( \kappa_1 \) is negative, we get
\[
    F^{11} \geq \delta_0 \sum F^{ii} = \delta_0 (n - 1) \sigma_{n-1}(\eta) \geq \delta_1 \psi^{1-1/n}
\]
for some positive constant \( \delta_1 \) depending only on \( n \). Therefore, from (3.6), we have
\[
    B^2 \delta_1 |\nabla u|^2 - C \left(1 + \frac{B}{w}\right) \leq 0.
\]
Since we can assume \( w \) is sufficiently large, using \( \frac{1}{w} + |\nabla u|^2 = 1 \), we can assume \( |\nabla u|^2 \geq 1/2 \). We obtain the desired estimate by fixing \( B \) sufficiently large. \( \square \)

Combining (3.1) and (3.2), we obtain the \textit{a priori} \( C^1 \) estimates.
4 Global Estimates for Second-Order Derivatives

In this section, we deal with the global estimates for second-order derivatives. Since when \( n = 2 \), (1.2) becomes classic two-dimensional Monge–Ampère equation or a 2-Hessian equation and the second-order estimates follows by Proposition 2.3 in [22], it suffices to consider the case when \( n \geq 3 \). Therefore, in the following of this section, we assume \( n \geq 3 \).

**Theorem 4.1** Suppose (1.11) holds. Let \( u \in C^4(\Omega) \cap C^2(\overline{\Omega}) \) be an admissible solution of (1.2). Then there exists a positive constant \( C \) depending on \( n, \|u\|_{C^1(\overline{\Omega})} \) and \( \|\psi^{1/(n-1)}\|_{C^2(\overline{\Omega} \times [-\mu_0, \mu_0] \times \mathbb{S}^n)} \) satisfying

\[
\sup_{\Omega} |D^2u| \leq C \left( 1 + \sup_{\partial\Omega} |D^2u| \right),
\]

where \( \mu_0 := \|u\|_{C^0(\overline{\Omega})} \).

By Lemma 3.2 of [18], we have the following lemma.

**Lemma 4.2** Let \( \alpha = \frac{1}{n-1} \). Then we have

\[
\sum_s \sigma_n^{ij,pq} \nabla_s \eta_{ij} \nabla_s \eta_{pq} \leq (1 - \alpha) \frac{\|\nabla \psi\|^2}{\psi} + 2\alpha \frac{\langle \nabla \psi \cdot \nabla H \rangle}{H} - (1 + \alpha) \frac{\psi |\nabla H|^2}{H^2},
\]

where

\[
\sigma_n^{ij,pq} = \frac{\partial^2 \sigma_n(\lambda(\eta))}{\partial \eta_{ij} \partial \eta_{pq}},
\]

\( \nabla \psi \) denotes the covariant gradient of \( \psi \) when \( \psi = \psi(X, \nu(X)) \) is only regarded as the function of \( X \in M_u \) and \( H = H(X) \) is the mean curvature of \( M_u \) at \( X \in M_u \).

**Proof of Theorem 4.1** Let \( v := 1/w = \langle v, \epsilon_{n+1} \rangle \). There exists a positive constant \( a \) depending only on \( \|Du\|_{C^0(\overline{\Omega})} \) such that \( v \geq 2a \). As in [31], we consider the test function

\[
V(X, \xi) := (v - a)^{-1} \exp \left\{ \frac{\delta}{2} |X|^2 \right\} H,
\]

where \( X \in M_u, \xi \in T_X M_u \) is a unit vector and \( \delta \) is a positive constant to be determined later. Suppose that the maximum value of \( V \) is achieved at a point \( X_0 = (x_0, u(x_0)) \in M, x_0 \in \Omega \) and \( \xi_0 \in T_{X_0} M_u \). We choose a local orthonormal frame \( \{e_1, e_2, \cdots, e_n\} \) near \( X_0 \) such that \( \xi = e_1(X_0) \),

\[
h_{ij} = \delta_{ij}h_{ii} \quad \text{and} \quad h_{11} \geq h_{22} \geq \cdots \geq h_{nn} \quad \text{at} \quad X_0.
\]
Therefore, at $X_0$, taking the covariant derivatives twice with respect to

$$
\log H + \log(v - a)^{-1} + \frac{\delta}{2} |X|^2,
$$

we have

$$
- \frac{\nabla_i v}{v - a} + \delta \langle X, e_i \rangle + \frac{\nabla_i H}{H} = 0, \; i = 1, \ldots, n \tag{4.3}
$$

and

$$
0 \geq F_{ii} \left\{ - \frac{\nabla_{ii} v}{v - a} + \frac{(\nabla_i v)^2}{(v - a)^2} + \delta(1 + \kappa_i \langle X, v \rangle) + \frac{\nabla_{ii} H}{H} - \frac{(\nabla_i H)^2}{H^2} \right\}, \tag{4.4}
$$

where $\kappa_1, \ldots, \kappa_n$ are principal curvatures of $M_u$ at $X_0$ and

$$
F_{ij} = \frac{\partial f(\lambda(h))}{\partial h_{ij}}.
$$

Let

$$
\hat{F}_{ij} := \frac{\partial \sigma_n(\lambda(\eta))}{\partial \eta_{ij}} \text{ and } f_i = \frac{\partial f(\kappa)}{\partial \kappa_i}.
$$

It is easy to see

$$
F_{ij} = \sum_{s=1}^n \hat{F}_{ss} \delta_{ij} - \hat{F}_{ij} \text{ at } X_0.
$$

By the Weingarten equation, we have

$$
\nabla_i v = -h_{im} \langle e_m, e_{n+1} \rangle = -h_{im} \nabla_m u \text{ and } F_{ii} (\nabla_i v)^2 \leq C f_i \kappa_i^2. \tag{4.5}
$$

Next, by Gauss formula, the Eqs. (1.2) and (1.11), we have

$$
F_{ii} \nabla_{ii} v = -F_{ii} \nabla_m h_{ii} \nabla_m u - f_i \kappa_i^2 v = -vf_i \kappa_i^2 - (\nabla \psi, \nabla u) \leq -vf_i \kappa_i^2 + CH \psi^{1-1/(n-1)}. \tag{4.6}
$$

where the positive constant $C$ depends on $\|\psi^{1/(n-1)}\|_{C^1}$ and $\|u\|_{C^1(\Omega)}$. By (2.14) and differentiating the Eq. (1.2) twice, we find

$$
F_{ii} \nabla_{ii} H = \sum_{s=1}^n F_{ii} \nabla_{ss} h_{ii} - Hf_i \kappa_i^2 + n\psi \sum_{s=1}^n h_{ss}^2
\geq \sum_{s=1}^n \nabla_{ss} \psi - \sum_{s=1}^n \sigma_{n}^{ij, pq} \nabla_s \eta_{ij} \nabla_s \eta_{pq} - Hf_i \kappa_i^2 + \psi H^2. \tag{4.7}
$$
Let \( \alpha := \frac{1}{n-1} \). Using (1.11) and Codazzi equation, we have

\[
\sum_{s=1}^{n} \nabla_{ss} \psi \geq (1 - \alpha) \frac{\lvert \nabla \psi \rvert^2}{\psi} - C(\lvert \nabla X \rvert^2 + \lvert \nabla v \rvert^2) \psi^{1-\alpha} + H \psi X_k v_k - \sum_{s=1}^{n} \nabla_s h_{sj}(d_v \psi)(e_j)
\]

\[
\geq (1 - \alpha) \frac{\lvert \nabla \psi \rvert^2}{\psi} - CH^2 \psi^{1-\alpha} - \nabla_j H(d_v \psi)(e_j). \tag{4.8}
\]

By (1.11), (4.3) and (4.5), we get

\[
\frac{\nabla_j H(d_v \psi)(e_j)}{H} = \left( \frac{\nabla_i v}{v - a} - \delta \langle X, e_i \rangle \right) (d_v \psi)(e_j) \leq C(\psi + \delta) \psi^{1-\alpha}. \tag{4.9}
\]

Next, by (4.3) and Cauchy–Schwarz inequality, we have, for any \( \epsilon > 0 \),

\[
\frac{1}{H^2} F^{ii}(\nabla_i H)^2 \leq (1 + \epsilon) \frac{F^{ii}(\nabla_i v)^2}{(v - a)^2} + C(1 + 1/\epsilon)\delta^2 \sum F^{ii}. \tag{4.10}
\]

Combining (4.2)-(4.10), we obtain

\[
0 \geq \frac{a}{v - a} f_i \kappa_i^2 - \epsilon \frac{F^{ii}(\nabla_i v)^2}{(v - a)^2} - CH \psi^{1-\alpha} - 2\alpha \frac{\langle \nabla \psi, \nabla \psi \rangle}{H^2}
\]

\[
+ \left( \delta - C(1 + 1/\epsilon)\delta^2 \right) \sum F^{ii} \geq \left( \frac{a}{v - a} - \frac{C\epsilon}{(v - a)^2} \right) f_i \kappa_i^2 + \left( \delta - C(1 + 1/\epsilon)\delta^2 \right) \sum F^{ii} - CH \psi^{1-\alpha} \tag{4.11}
\]

provided \( H \) is sufficiently large. Taking sufficiently small \( \epsilon \) to satisfy \( C\epsilon < a(v - a)/2 \) and \( \delta \) sufficiently small such that \( C(1 + 1/\epsilon)\delta^2 < \delta/2 \) in (4.11), we get

\[
0 \geq \frac{\delta}{2} \sum F^{ii} + \frac{a}{2(v - a)} f_i \kappa_i^2 - CH \psi^{1-\alpha} \geq \frac{\delta}{2} \sum F^{ii} + \delta_0 f_i \kappa_i^2 - CH \psi^{1-\alpha}, \tag{4.12}
\]

where \( \delta_0 := \frac{a}{2(\max_{\Omega_1} v - a)} > 0. \)

We consider two cases. The positive constant \( \epsilon_0 \) will be determined later.

**Case 1** \( \lvert h_{ii} \rvert \leq \epsilon_0 h_{11} \) for all \( i \geq 2 \).

In this case, we have

\[
[1 - (n - 2)\epsilon_0] h_{11} \leq \eta_{22} \leq \cdots \leq \eta_{nn} \leq [1 + (n - 2)\epsilon_0] h_{11}.
\]
It then follows that
\[
\sigma_{n-1}(\eta) \geq \eta_{22} \cdots \eta_{nn} \geq (1 - (n - 1)\epsilon_0)^{n-1} h_{11}^{n-1}.
\]

Choosing \(\epsilon_0\) sufficiently small and using \(n \geq 3\),
\[
\sigma_{n-1}(\eta) \geq \frac{h_{11}^{n-1}}{2} \geq \frac{h_{11}^2}{2} \geq \frac{H^2}{2n^2}.
\] (4.13)

We obtain, at \(X_0\),
\[
\sum_i F_{ii} = (n - 1) \sum_i \hat{F}_{ii} = (n - 1)\sigma_{n-1}(\eta) \geq \delta_1 H^2
\] (4.14)
for some positive constant \(\delta_1 = \frac{n-1}{2n^2}\). Thus, by (4.12) and (4.14), we have
\[
H \leq \frac{2C}{\delta \delta_1} (\sup \psi)^{1-\alpha}
\]
and (4.1) is proved.

**Case 2** \(h_{22} > \epsilon_0 h_{11}\) or \(h_{nn} < -\epsilon_0 h_{11}\).

By the definitions of \(F_{ii}\) and \(\hat{F}_{ii}\) and (2.11),
\[
F_{22} = \sum_{i \neq 2} \hat{F}_{ii} \geq \hat{F}_{11} \geq \frac{1}{n} \sum_i \hat{F}_{ii} = \frac{1}{n} \sigma_{n-1}(\eta) \geq \frac{c_0}{n} \sigma_1(\eta) \sigma_{n-1}^{1-\alpha}(\eta) = \delta_2 H^\alpha \psi^{1-\alpha},
\] (4.15)
where \(\delta_2 = \frac{c_0(n-1)^\alpha}{n}\). Similarly, we have
\[
F_{nn} \geq \delta_2 H^\alpha \psi^{1-\alpha}.
\]

In this case, we have
\[
\frac{f_i}{k_i^2} = F_{ii} h_{ii}^2 \geq F_{22} h_{22}^2 + F_{nn} h_{nn}^2 \geq \epsilon_0^2 \delta_2 h_{11}^2 \geq \frac{\epsilon_0^2 \delta_2}{n^2} H^{2+\alpha} \psi^{1-\alpha}.
\]

Thus, by (4.12), we obtain
\[
H \leq \left( \frac{Cn^2}{\delta_0 \delta_2 \epsilon_0^2} \right)^{1/(1+\alpha)}
\]
and Theorem 4.1 follows immediately. \(\square\)
5 Boundary Estimates for Second-Order Derivatives

In this section, we establish the boundary estimates of second-order derivatives.

**Theorem 5.1** Suppose $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth strictly convex boundary $\partial \Omega$. Assume that (1.5) and (1.11) hold. Let $u \in C^3(\Omega)$ be an admissible solution of (1.2). Then there exists a positive constant $C$ depending only on $\|u\|_{C^1(\Omega)}$, $\|\psi^{1/(n-1)}\|_{C^1(\Omega \times [-\mu_0, \mu_0] \times S^n)}$ and $\partial \Omega$ satisfying

$$\max_{\partial \Omega} |D^2 u| \leq C,$$

where $\mu_0 := \|u\|_{C^0(\Omega)}$.

To prove (5.1), we consider an arbitrary point $x_0 \in \partial \Omega$. Without loss of generality, we may assume that $x_0$ is the origin and that the positive $x_n$-axis is in the interior normal direction to $\partial \Omega$ at the origin. Suppose near the origin, the boundary $\partial \Omega$ is given by

$$x_n = \rho(x') = \frac{1}{2} \sum_{\alpha < n} \kappa^b_{\alpha} x^2_\alpha + O(|x'|^3),$$

where $\kappa^b_1, \ldots, \kappa^b_{n-1}$ are the principal curvatures of $\partial \Omega$ at the origin and $x' = (x_1, \ldots, x_{n-1})$. Differentiating the boundary condition $u = 0$ on $\partial \Omega$ twice, we can find a constant $C$ depending on $\|u\|_{C^1(\Omega)}$ satisfying

$$|u_{\alpha\beta}(0)| \leq C \text{ for } 1 \leq \alpha, \beta \leq n-1.$$ 

Since $\nu = \left(\frac{-Du, 1}{w}\right)$, $\psi$ can be regarded as a function of $x, u$ and $Du$. In the following, we denote

$$\psi(x, u, Du) = \psi(X, v(X)) = \psi \left(x, u, \left(\frac{-Du, 1}{w}\right)\right)$$

and the Eq. (1.2) can be written as

$$G(D^2 u, Du) := f(\lambda(A[u])) = \psi(x, u, Du),$$

where $G = G(r, p)$ is viewed as a function of $(r, p)$ for $r \in S^{n \times n}$ and $p \in \mathbb{R}^n$. Define

$$G^{ij} = \frac{\partial G}{\partial r_{ij}}(D^2 u, Du), \quad G^i = \frac{\partial G}{\partial p_i}(D^2 u, Du), \quad \psi_{ui} = \frac{\partial \psi}{\partial u_i}(x, u, Du)$$

and the linearized operator by

$$L = G^{ij} \partial_{ij} - \psi_{ui} \partial_i.$$
The following lemma was proved in [17].

**Lemma 5.2** We have

\[ G^s = -\frac{u_s}{w} \sum_i f_i \kappa_i - \frac{2}{w(1+w)} \sum_{i,j} F^{ij} a_{i1} (wu_i \gamma^{sj} + u_j \gamma^{is}), \]  

(5.6)

where

\[ a_{ij} = \frac{1}{w} \gamma^{ik} u_{kl} \gamma^{lj}, \]

\[ \kappa = \lambda(\{a_{ij}\}), \]

\[ f_i = \frac{\partial f(\kappa)}{\kappa_i} \]

and

\[ F^{ij} = \frac{\partial f(\lambda(A[u]))}{\partial a_{ij}}. \]

Next, we establish the estimate

\[ |u_{\alpha n}(0)| \leq C \text{ for } 1 \leq \alpha \leq n - 1. \]  

(5.7)

Since \( \Omega \) is uniformly convex with smooth boundary, there exists a smooth strictly convex function \( v_0 \) satisfying \( v_0 = 0 \) on \( \partial \Omega \) and \( v_0 \leq 0 \) in \( \Omega \). Clearly there exists a positive constant \( \theta \) such that

\[ D^2 v_0 \geq 2\theta I \text{ in } \Omega. \]

Define

\[ v = v_0 - \frac{\theta}{2} |x|^2. \]

(5.8)

Let \( \Omega_\delta := \{ x \in \Omega : |x| < \delta \} \). We have \( D^2 v \geq \theta I \) in \( \Omega \) and

\[ v \leq -\frac{\theta \delta^2}{2}, \quad \text{on } \partial \Omega_\delta \cap \Omega \]

(5.9)

\[ v \leq -\frac{\theta |x|^2}{2}, \quad \text{on } \partial \Omega_\delta \cap \partial \Omega. \]

Furthermore,

\[ \frac{1}{w} \{ \gamma^{is} (v_{st} - \theta \delta_{st}) \gamma^{jt} \} \text{ is positive definite.} \]

(5.10)

Since \( \Delta v_0 \geq 2n\theta \), by Hopf’s lemma we have

\[ \frac{\partial v_0}{\partial x_n}(0) = -2\theta_0 < 0 \text{ for some positive constant } \theta_0 \]

and hence

\[ v_n \leq -\theta_0 \text{ on } \overline{\Omega_\delta} \]

(5.11)
provided \( \delta \) is small enough. As [27], we consider

\[
W := \nabla' \alpha u - \frac{1}{2} \sum_{1 \leq \beta \leq n-1} u_\beta^2
\]

defined on \( \overline{\Omega}_\delta \) for some small \( \delta \), where

\[
\nabla' \alpha u := u \alpha + \rho \alpha u_n, \text{ for } 1 \leq \alpha \leq n - 1.
\]

Similar to Lemma 3.3 of [31], we need the following lemma.

**Lemma 5.3** If \( \delta \) is sufficiently small, we have

\[
LW \leq C \left( \psi^{1-1/(n-1)} + \psi |DW| + \sum_i G^{ii} + G^{ij} W_i W_j \right), \tag{5.12}
\]

where \( C \) is a positive constant depending on \( n \), \( u \in C^1(\overline{\Omega}), \| \psi^{1/(n-1)} \|_{C^1(\overline{\Omega} \times [-\mu_0, \mu_0] \times \mathbb{S}^n)} \) and \( \partial \Omega \), where \( \mu_0 = \| u \|_{C^0(\Omega)}. \)

**Proof** We first note that, by differentiating the Eq. (5.4),

\[
G^{ij} W_{ij} + G^s W_s \leq \nabla' \psi - \sum_{\beta \leq n-1} u_\beta \psi_\beta + 2G^{ij} u_{ni} \rho \alpha j \\
- \sum_{\beta \leq n-1} G^{ij} u_\beta i u_\beta j + u_n G^{ij} \rho \alpha ij + u_n G^s \rho \alpha s. \tag{5.13}
\]

By (1.11) and (5.13), we obtain

\[
LW + G^s W_s \leq C \psi^{1-1/(n-1)} + 2G^{ij} u_{ni} \rho \alpha j \\
- \sum_{\beta \leq n-1} G^{ij} u_\beta i u_\beta j + u_n G^{ij} \rho \alpha ij + u_n G^s \rho \alpha s. \tag{5.14}
\]

We have

\[
G^{ij} = \frac{1}{w} \sum_{s,t} F^{st}_{ij} \gamma^i_s \gamma^j_t \text{ and } u_{ij} = w \sum_{s,t} \gamma_{is} \alpha_{st} \gamma_{tj}.
\]

It follows that

\[
\sum_{\beta \leq n-1} G^{ij} u_{\beta i} u_{\beta j} = w \sum_{\beta \leq n-1} \sum_{s,t} F^{ij}_{\beta s} \gamma^\beta i \delta_{si} \alpha_{tj}.
\]
By [17], we can find an orthogonal matrix $B = (b_{ij})$ that can diagonalize $(a_{ij})$ and $(F^{ij})$ at the same time:

$$F^{ij} = \sum_s b_{is} f_s b_{js} \text{ and } a_{ij} = \sum_s b_{is} \kappa_i b_{js}. $$

Therefore, we get

$$\sum_{\beta \leq n-1} G^{ij} u_{\beta i} u_{\beta j} = w \sum_{\beta \leq n-1} \sum_i \left( \sum_s \gamma_{\beta s} b_{si} \right)^2 f_i \kappa_i^2. $$

Let the matrix $\eta = (\eta_{ij}) = (\sum_s \gamma_{is} b_{sj})$. We find $\eta \cdot \eta^T = g$ and $|\det(\eta)| = \sqrt{1 + |Du|_1^2}$. Therefore, we obtain

$$\sum_{\beta \leq n-1} G^{ij} u_{\beta i} u_{\beta j} = w \sum_{\beta \leq n-1} \sum_i \eta_{\beta i}^2 f_i \kappa_i^2. \quad (5.15)$$

We have

$$G^{ij} u_{ni} \rho_{\alpha j} = \sum_i f_i \kappa_i b_{si} \gamma^{js} b_{ti} \gamma_{nt} \rho_{\alpha j} \leq C \sum_i f_i |\kappa_i|. \quad (5.16)$$

For any indices $j, t$, we have

$$F^{ij} a_{it} = \sum_{i,s,p} b_{is} f_s b_{jp} \kappa_{ip} b_{tp} = \sum_i f_i \kappa_i b_{ji} b_{ti}. $$

Thus, by (5.6), we find

$$|G^s \rho_{as}| \leq C \sum_i f_i |\kappa_i|. \quad (5.17)$$

Combining (5.14)-(5.17), we obtain

$$LW + G^s W_s \leq C \left( \psi^{1-1/(n-1)} + \sum_i G^{ii} + \sum_i f_i |\kappa_i| \right) - w \sum_{\beta \leq n-1} \sum_i \eta_{\beta i}^2 f_i \kappa_i^2. \quad (5.18)$$
Now we consider the term $G^s W_s$. We have, by (5.6) and the definition of the matrix $(b_{ij})$,

$$-G^s W_s = \frac{1}{w} \sum_s \left( n\psi u_s + 2 \sum_{t,i} f_i \kappa_i (b_{ti} u_t) \gamma^{sl} b_{li} \right) W_s \leq C \psi |DW| + \frac{2}{w} \sum_{t,i} f_i \kappa_i (b_{ti} u_t) \gamma^{sl} b_{li} W_s. \quad (5.19)$$

We divide into two cases to further discuss: (a) $\sum_{\beta \leq n-1} \eta_{\beta i}^2 \geq \epsilon$ for all $i$; and (b) $\sum_{\beta \leq n-1} \eta_{\beta r}^2 < \epsilon$ for some index $1 \leq r \leq n$, where $\epsilon$ is some positive constant, which will be determined later.

For the case (a), by (5.15), we have

$$\sum_{\beta \leq n-1} G^{ij} u_{\beta i} u_{\beta j} \geq \epsilon \sum_i f_i \kappa_i^2.$$  

By Cauchy–Schwarz inequality, we get

$$\frac{2}{w} \kappa_i (b_{ti} u_t) \gamma^{sl} b_{li} W_s \leq \frac{\epsilon}{2} \kappa_i^2 + \frac{C}{\epsilon} (\gamma^{sl} b_{li} W_s)^2. \quad (5.20)$$

It follows that

$$-G^s W_s \leq C \psi |DW| + \frac{\epsilon}{2} f_i \kappa_i^2 + \frac{C}{\epsilon} G^{ij} W_i W_j.$$

It is clear that, for any $\epsilon_1 > 0$,

$$\sum_i f_i |\kappa_i| \leq \frac{1}{\epsilon_1} \sum_i f_i + \epsilon_1 \sum_i f_i \kappa_i^2 \leq C \left( \frac{1}{\epsilon_1} \sum_i G^{ii} + \epsilon_1 \sum_i f_i \kappa_i^2 \right).$$

Combining the previous four inequalities with (5.18), (5.12) follows.

For the case (b), as in [27], we have

$$1 \leq \det(\eta) \leq \eta_{rr} \det(\eta') + C_1 \epsilon \leq \sqrt{1 + \mu_1^2} |\det(\eta')| + C_1 \epsilon,$$

where $\eta' := (\eta_{\alpha \beta})_{\alpha \neq n, \beta \neq r}$, $\mu_1 := \|Du\|_{C^0(\Omega)}$ and $C_1$ is a positive constant depending only on $n$ and $\mu_1$. We choose sufficiently small $\epsilon$ satisfying $C_1 \epsilon < \frac{1}{7}$. Thus, we get

$$|\det(\eta')| \geq \frac{1}{2\sqrt{1 + \mu_1^2}}.$$
On the other hand, for any fixed \( \alpha \neq r \), we have
\[
| \det(\eta^\prime) | \leq C \sum_{\beta \neq n} |\eta_{\beta \alpha}|.
\]
for some positive constant \( C \) depending only on \( n \) and \( \mu_1 \). Therefore, combining the above two inequalities, we get, for any \( i \neq r \),
\[
\sum_{\beta \leq n-1} \eta_{\beta i}^2 \geq c_1
\]
for some positive constant \( c_1 \) depending on \( \|u\|_{C^1(\Omega)} \), which implies, in view of (5.15),
\[
\sum_{\beta \leq n-1} G^{ij}_{\beta i} u_{\beta j} \geq c_1 \sum_{i \neq r} f_i \kappa_i^2.
\] (5.21)

If \( \kappa_r \leq 0 \), by Lemma 2.20 of [13], we have
\[
\sum_{i \neq r} f_i \kappa_i^2 \geq \frac{1}{n+1} \sum_{i=1}^n f_i \kappa_i^2.
\]
Thus (5.12) follows using a similar argument as the Case (a).

Hence, in the following, we may assume \( \kappa_r > 0 \). Without loss of generality, assume \( r = 1 \). Now we consider two cases.

**Case (b-1).** \( |\kappa_i| \leq \epsilon_0 \kappa_1 \) for all \( i \geq 2 \), where the positive constant \( \epsilon_0 \) will be chosen later.

In this case, as in Sect. 4, we find,
\[
(1 - (n - 2)\epsilon_0)\kappa_1 \leq \lambda_i \leq (1 + (n + 2)\epsilon_0)\kappa_1, \text{ for } i \geq 2,
\]
where \( \lambda_1, \ldots, \lambda_n \) are defined in (2.4). By the Eq. (1.2), fixing the constant \( \epsilon_0 \) sufficiently small,
\[
\lambda_1 = \frac{\psi}{\lambda_2 \cdots \lambda_n} \leq C \kappa_1^{1-n} \psi.
\]
It follows that
\[
\sigma_{n-1;i}(\lambda) = \prod_{j \neq i} \lambda_j \leq C \kappa_1^{n-2} \kappa_1^{1-n} \psi = C \kappa_1^{-1} \psi, \text{ for } i \geq 2.
\]
Therefore,
\[
f_1(\kappa) = \sum_{i \neq 1} \sigma_{n-1;i}(\lambda) \leq C \kappa_1^{-1} \psi
\]
and
\[ f_1 \kappa_1 \leq C \psi. \]

By (5.19) and (5.20) and the Cauchy-Schwarz inequality, we have, for any \( \epsilon > 0 \),
\[
-G^s W_s \leq C \psi |DW| + \frac{2}{w} \sum_{i \neq 1} f_i \kappa_i (b_{i1} u_t) \gamma^{sl} b_{11} W_s
\]
\[
\leq C \psi |DW| + C \sum_{i \neq 1} f_i |\kappa_i| \gamma^{sl} b_{11} W_s
\]
\[
\leq C \psi |DW| + \epsilon \sum_{i \neq 1} f_i \kappa_i^2 + \frac{C}{\epsilon} \sum_{i=1}^n f_i \gamma^{sl} b_{11} W_s \gamma^{tk} b_{ki} W_t
\]
\[
\leq C \psi |DW| + \epsilon \sum_{i \neq 1} f_i \kappa_i^2 + \frac{C}{\epsilon} G^{ij} W_i W_j. \tag{5.22}
\]

Using (5.21), (5.12) is proved by fixing \( \epsilon \) sufficiently small.

**Case (b-2) \(|\kappa_{i_0}| > \epsilon_0 \kappa_1 \) for some \( i_0 \geq 2 \),**

First we find
\[
\gamma^{sl} b_{11} W_s = \gamma^{sl} b_{11} \left( u_{\alpha s} + \rho_\alpha u_{ns} - \sum_{\beta \leq n-1} u_\beta u_\beta s + \rho_\alpha u_n \right)
\]
\[
= w \left( \eta_{\alpha 1} + \rho_\alpha \eta_{11} - \sum_{\beta \leq n-1} u_\beta \eta_{\beta 1} \right) \kappa_1 + \gamma^{sl} b_{11} \rho_\alpha u_n.
\]

It follows that
\[
| \gamma^{sl} b_{11} W_s | \leq C w (\epsilon + |\rho_\alpha|) \kappa_1 + C. \tag{5.23}
\]

It is obvious that
\[
f_1 \kappa_1 = n \psi - \sum_{i \neq 1} f_i \kappa_i.
\]

Then we have
\[
\frac{2}{w} f_1 \kappa_1 \left( \sum_{i} (b_{i1} u_t) \gamma^{sl} b_{11} W_s \right) = \frac{2}{w} (n \psi - \sum_{i \neq 1} f_i \kappa_i) \left( \sum_{i} (b_{i1} u_t) \gamma^{sl} b_{11} W_s \right)
\]
\[
\leq C \psi |DW| + C (\epsilon + |\rho_\alpha|) \sum_{i \neq 1} f_i |\kappa_i| \kappa_1 + C \psi + C \sum_{i \neq 1} f_i |\kappa_i|
\]
\[
\leq C \psi |DW| + \left( \epsilon_0^{-1} C (\epsilon + |\rho_\alpha|) + \epsilon_1 \right) \sum_{i \neq 1} f_i \kappa_i^2 + C \psi + \frac{C}{\epsilon_1} \sum_{i \neq 1} f_i.
\]
for any $\epsilon_1 > 0$. We can choose sufficiently small $\delta \epsilon$ and $\epsilon_1$ satisfying

$$\left( \epsilon_0^{-1} C(\epsilon + |\rho_\alpha|) + \epsilon_1 \right) < \frac{c_1}{4}.$$ 

Therefore, (5.12) follows by (5.21) as in Case (b-1).

To proceed we consider the following function on $\Omega_\delta$, for sufficiently small $\delta$,

$$\Psi := v - td + \frac{N}{2} d^2,$$  \hspace{1cm} (5.24)

where $v(x)$ is defined by (5.8), $d(x) := \text{dist}(x, \partial \Omega)$ is the distance from $x$ to the boundary $\partial \Omega$, $t, N$ are two positive constants to be determined later. We first suppose $\delta < 2t/N$ so that

$$-td + \frac{N}{2} d^2 \leq 0 \text{ on } \Omega_\delta.$$  \hspace{1cm} (5.25)

Let

$$\tilde{W} := 1 - \exp(-bW).$$  \hspace{1cm} (5.26)

By (5.12), we can choose the constant $b$ sufficiently large such that

$$L \tilde{W} \leq C(\psi^{1-1/(n-1)} + \psi |D\tilde{W}| + \sum_i G^{ii}).$$  \hspace{1cm} (5.27)

We consider the function

$$\Phi := R\Psi - \tilde{W},$$

where $R$ is a positive constant sufficiently large to be chosen. We shall prove

$$\Phi \leq 0 \text{ on } \overline{\Omega_\delta}$$  \hspace{1cm} (5.28)

by choosing suitable constants $\delta, t, N$ and $R$. We first deal with the case that the maximum of $\Phi$ is attained at an interior point $x_0 \in \Omega_\delta$. Now we consider two cases:

(i) $\psi(x_0, u(x_0), Du(x_0)) \geq \epsilon_0$ and (ii) $\psi(x_0, u(x_0), Du(x_0)) < \epsilon_0$, where $\epsilon_0$ is a positive constant to be determined later.

**Case (i)** Since $f^{1/n}$ is concave in $\Gamma$ and homogeneous of degree one and $|Dd| \equiv 1$ on the boundary $\partial \Omega$, we have, by (2.10),

$$\frac{1}{n} \psi^{1-1} G^{ij}(D^2 v - \frac{\theta}{4} I + N Dd \otimes Dd)_{ij} \geq G^{1/n}(D^2 v - \frac{\theta}{4} I + N Dd \otimes Dd, Du) \geq \mu(N).$$

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at $x_0$, where $\lim_{N \to +\infty} \mu(N) = +\infty$. Consequently, at $x_0$ we have

$$G^{ij} \Psi_{ij} \geq n \epsilon_0^{1/n} \mu(N) + \frac{\theta}{4} \sum_i G^{ii} + (N d - t) G^{ij} d_{ij} \geq 2 \mu_1(N) + (\frac{\theta}{4} - C N \delta - C t) \sum_i G^{ii},$$

where $\mu_1(N) := n \epsilon_0^{1/n} \mu(N)/2$. Note that

$$|D\Psi| = |Dv - tDd + NdDd| \leq C (1 + t) + C \delta N \leq \mu_1(N)^{1/2} \text{ in } \Omega_\delta,$$

if $N$ is sufficiently large and $\delta < \sqrt{\mu_1(N)/2CN}$. Therefore, we further take $\delta$ and $t$ sufficiently small such that $C N \delta + C t < \theta/8$. We thus obtain, at $x_0$,

$$G^{ij} \Psi_{ij} \geq \mu_1(N) + \mu_1(N)^{1/2} |D\Psi| + \frac{\theta}{8} \sum_i G^{ii}. \quad (5.29)$$

Therefore, by (5.27) and (5.29) we have, at $x_0$,

$$0 \geq L \Phi \geq R \mu_1(N) + R \mu_1(N)^{1/2} |D\Psi| - R \psi_u \Psi_i + \frac{R \theta}{8} \sum_i G^{ii} - C \left(1 + |D\tilde{W}| + \sum_i G^{ii}\right) \geq R \mu_1(N) - C + R(\mu_1(N)^{1/2} - C) |D\Psi| + \left(\frac{R \theta}{8} - C\right) \sum_i G^{ii} > 0$$

provided $N$ and $R$ are sufficiently large which is a contradiction.

**Case (ii)** Let $H$ denote the mean curvature of $M_u$ at $X_0 = (x_0, u(x_0))$. We consider two cases: (ii-a) $H \leq A$ and (ii-b) $H > A$, where $A$ is a positive constant sufficiently large to be chosen.

**Case (ii-a)** Let

$$\tilde{a}_{ij} = w^{-1} g^{ij} u_{ij}.$$

First we note that $|\tilde{a}_{ij}(x_0)| \leq \hat{C} H \leq \hat{C} A$, $i, j = 1, \ldots, n$, for some positive constant $\hat{C}$ depending only on $n$ since $g^{-1} \eta$ is positive definite. We see, at $x_0$,

$$0 = \Phi_i = (R \Psi - \tilde{W})_i = R(v_i - t d_i + N d d_i) - b \exp\{-bW\} \left(u_{ai} + \rho_a u_{ni} + \rho_{ai} u_n - \sum_{\beta \leq n-1} u_{\beta} u_{\beta i}\right).$$
for $i = 1, \ldots, n$, Therefore, we get

\[
\tilde{a}_{n\alpha} + \rho_{\alpha} \tilde{a}_{nn} + w^{-1} g^{ni} \rho_{i\alpha} u_n - \sum_{\beta \leq n-1} u_{\beta} \tilde{a}_{n\beta} \\
= w^{-1} g^{ni} \left( u_{i\alpha} + \rho_{\alpha} u_{in} + \rho_{i\alpha} u_n - \sum_{\beta \leq n-1} u_{\beta} u_{i\beta} \right) \\
= Rb^{-1} \exp(bW) w^{-1} \left( g^{ni} v_i - t g^{ni} d_i + N d g^{ni} d_i \right).
\]

Using (5.11) and the fact $v_\gamma(0) = d_\gamma(0) = 0$ for $1 \leq \gamma \leq n - 1$, we see at $x_0$, if we let $\delta$ and $t$ be sufficiently small,

\[
\tilde{a}_{n\alpha} \leq -\hat{c}R + C,
\]

where $\hat{c}$ is some positive constant depending on $\theta_0$ and $\|u\|_{C^1(\overline{\Omega})}$. We then get a contradiction if $R \gg A$ is sufficiently large since $|\tilde{a}_{n\alpha}(x_0)| \leq \hat{C}A$.

**Case (ii-b)** Note that, by (2.11),

\[
\sum G^{ii} = \frac{1}{w} g^{ij} F^{ij} \geq \gamma_1 \sigma_{n-1}(\eta) \geq \gamma_2 H^{1/(n-1)} \psi^{1-1/(n-1)} \geq \gamma_2 A^{1/(n-1)} \psi^{1-1/(n-1)}
\]

for some positive constant $\gamma_2$ depending only on $n$ and $\|u\|_{C^1(\overline{\Omega})}$. By (1.11), we find

\[
|\psi_{ui}| \leq C \psi^{1-1/(n-1)}, \text{ for all } i = 1, \ldots, n.
\]

Similar to (5.29), we have

\[
G^{ij} \Psi_{ij} \geq \frac{\theta}{8} \sum G^{ii}
\]

and

\[
L \Psi \geq \frac{\theta}{8} \sum G^{ii} - C \psi^{1-1/(n-1)} \\
\geq \frac{\theta}{16} \sum G^{ii} + \left( \frac{\theta \gamma_2}{16} A^{1/(n-1)} - C \right) \psi^{1-1/(n-1)} \\
\geq \frac{\theta}{16} \sum G^{ii} + \frac{\theta \gamma_2}{32} A^{1/(n-1)} \psi^{1-1/(n-1)}
\]

(5.30)

by fixing $A$ sufficiently large. Next, we see

\[
D\tilde{W}(x_0) = RD\Psi(x_0).
\]
Combining (5.27) and (5.30), we have, at $x_0$,

$$0 \geq L\Phi \geq \left(\frac{\theta R}{16} - C\right) \sum G^{ii} + \left(\frac{\theta \gamma_2 R}{32} A^{1/(n-1)} - C\right) \psi^{1-1/(n-1)} - C\psi > 0$$

if we fix $R$ sufficiently large and $\epsilon_0$ sufficiently small, which is a contradiction.

In view of Case (i) and Case (ii), the function $\Phi$ cannot attain its maximum at an interior point of $\Omega_\delta$ when $R$, $N$ are large enough and $\delta$, $t$ are small enough. By (5.9) and (5.25), we can fix $\delta$ smaller and $R$ larger such that $\Phi < 0$ on $\partial \Omega_\delta$. Thus, (5.28) is proved. Consequently, we have $(R\Psi - \tilde{W})_n(0) \leq 0$ since $(R\Psi - \tilde{W})(0) = 0$. Therefore, we obtain

$$u_{n\alpha}(0) \geq -C,$$

The above arguments also hold with respect to $-\nabla'_\alpha u - 1/2 \sum_{\beta \leq n-1} u^2_\beta$. Hence, we obtain (5.7).

It suffices to establish the upper bound of $u_{nn}(0)$ since $H[M_u] > 0$. By (1.5) and (2.12), we have

$$H[M_u] = \sigma_1(\kappa[M_u]) = \frac{1}{n-1} \sigma_1(\eta) \geq c_0 \sigma_n^{1/n}(\eta) \geq c_0 \psi^{1/n}$$

(5.31)

for some $c_0$ depending only on $n$. Since $\psi \neq 0$ on $\inf u$, there exists a point $x_0 \in \Omega$ such that

$$\psi(x_0, u(x_0)) \geq \delta_0 > 0$$

for some positive constant $\delta_0$. By the continuity of $\psi$ and that we have establish the $C^1$ estimates, there exists a neighborhood $U$ of $x_0$ such that

$$\inf_{U \cap \Omega} \psi(x, u(x)) \geq \frac{\delta_0}{2}.$$

Fix $U_1 \subset \subset U$. As [31], let $\overline{\psi}$ be a smooth function such that $\overline{\psi} = c_0(\delta_0/2)^{1/n}$ in $U_1$, $\overline{\psi} \leq c_0(\delta_0/2)^{1/n}$ in $U - U_1$ and $\overline{\psi} = 0$ outside $U$.

Next we prove

$$u_n(0) \leq -\gamma_1$$

(5.32)

for some uniform positive constant $\gamma_1$.

By Theorem 16.10 of [12], there exists a unique solution $\overline{u}$ of the prescribed mean curvature equation

$$\sigma_1(\kappa[M_{\overline{u}}]) = \epsilon_2 \overline{\psi} \text{ in } \Omega$$
with $\overline{u} = 0$ on $\partial \Omega$, if the positive constant $\epsilon_2 < 1$ is sufficiently small. Since the above equation is uniform elliptic, by Hopf’s lemma, we see $\overline{u} < 0$ in $\Omega$ and $\overline{u}_\mu < 0$ on $\partial \Omega$, where $\mu$ is the unit interior normal with respect to $\partial \Omega$. Since $\partial \Omega$ is compact, there exists a uniform constant $\gamma_1 > 0$ such that $\overline{u}_\mu \leq -\gamma_1$ on $\partial \Omega$. By (5.31), we have

$$\sigma_1(\kappa [M_u]) \geq \sigma_1(\kappa [M_{\overline{u}}]) \text{ in } \Omega.$$ 

Therefore, by the maximum principle and $u = \overline{u} = 0$ on $\partial \Omega$, we find

$$u \leq \overline{u} \text{ in } \Omega$$

and $u_n(0) \leq \overline{u}_n(0) \leq -\gamma_1$ which is (5.32).

It is clear that, at the origin,

$$u_{\alpha\beta} = -(u_n)\kappa^b_{\alpha\beta}, \text{ for } 1 \leq \alpha, \beta \leq n - 1$$

and

$$g^{ij} = \delta_{ij} - \frac{|D u|^2}{w^2} \delta_{in} \delta_{jn}.$$ 

It follows that, at the origin,

$$S_{1,n}(D^2 u, Du) = \sum_{\alpha \leq n-1} u_{\alpha\alpha} = -u_n \sigma_1(\kappa^b) \geq \gamma_2 > 0$$

for some uniform positive constant $\gamma_2$ by (5.32). Thus, by the Eqs. (1.2), (3.4), (5.3) and (5.7), we get upper bound of $u_{nn}(0)$. Theorem 5.1 is proved.

Now we prove Theorem 1.2. We note that we have also established the $C^2$ estimates for (1.2) when it is non-degenerate. The $C^{2,\alpha}$ estimates can be established by Evans-Krylov theory since the Eq. (1.2) is concave uniformly elliptic with respect to admissible solutions in the non-degenerate case. Higher order estimates can be derived using Schauder theory. Then Theorem 1.2 can be proved by standard arguments using the continuity method. The reader is referred to [6] for details.

Finally, Theorem 1.1 can be proved by approximation as in [31].

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Declarations

Conflict of interest The authors declare that there is no conflict of interest.

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