Computer-aided study of double extensions of restricted Lie superalgebras preserving the non-degenerate closed 2-forms in characteristic 2

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Abstract. A Lie (super)algebra with a non-degenerate invariant symmetric bilinear form \( B \) is called a nis-(super)algebra. The double extension \( g \) of a nis-(super)algebra \( a \) is the result of simultaneous adding to \( a \) a central element and a derivation so that \( g \) is a nis-algebra. Loop algebras with values in simple complex Lie algebras are most known among the Lie (super)algebras suitable to be doubly extended. In characteristic 2 the notion of double extension acquires specific features. Restricted Lie (super)algebras are among the most interesting modular Lie superalgebras. In characteristic 2, using Grozman’s Mathematica-based package SuperLie, we list double extensions of restricted Lie superalgebras preserving the non-degenerate closed 2-forms with constant coefficients. The results are proved for the number of indeterminates ranging from 4 to 7 — sufficient to conjecture the pattern for larger numbers. Considering multigradings allowed us to accelerate computations up to 100 times.

Keywords: Restricted Lie superalgebra, characteristic 2, double extension

1 Introduction

1.1 Setting of the problem For a given Lie (super)algebra \( a \) over the ground field \( K \), the notion of double extension \( g := \mathcal{K} \oplus a \oplus \mathcal{K}^* \), where \( \mathcal{K} := Kc \) and \( \mathcal{K}^* := KD \), was recently distinguished, see [MR]. It simultaneously involves
1) a central extension of \( a \) (with center \( \mathcal{K} := Kc \)),
2) a derivation \( D \) of \( a \),
3) a non-degenerate invariant symmetric bilinear form (briefly: NIS) on \( a \) extendable to \( g \). Hereafter any Lie (super)algebra with a NIS is called a nis-(super)algebra.

Most known examples of double extensions are affine Kac–Moody algebras \( g \) over \( \mathbb{C} \) or \( \mathbb{R} \) important in physics and mathematics (Google returns 482K entries); here \( a \) is a loop algebra with values in a simple finite-dimensional Lie
algebra $\mathfrak{s}$. Less known examples: the Lie superalgebra $\mathfrak{gl}(a|a)$ over $\mathbb{C}$, a double extension of $\mathfrak{psl}(a|a)$, and $\mathfrak{gl}(np)$ in characteristic $p > 0$, a double extension of $\mathfrak{psl}(np)$.

In what follows we show that for the double extension to be “interesting”, i.e., not just the direct sum of two ideals ($\mathfrak{a}$ and $\mathfrak{K} \oplus \mathfrak{K}^*$), the following conditions should be satisfied:

a) the derivation $D$ of $\mathfrak{a}$ must be outer for any $p$; if $p = 2$, and $D \in (\text{out } \mathfrak{a})\bar{1}$, then condition $D^2 = 0$ is a must, see [BKLS];

b) the central extension has to be non-trivial, see Subsection 2.4.

The simple Lie (super)algebras with NIS are classified for various types of Lie (super)algebras over the field $\mathbb{K}$ of characteristic $p \neq 2$ in review [BKLS].

The new and most interesting results of [BeB1] are general constructions and examples of double extensions of Lie superalgebras for $p = 2$. Here we recall the main definitions and general results of [BeB1] carefully pointing at the difference between the cases where $p \neq 2$ and the cases where $p = 2$.

In particular, in [BeB1], the double extensions of simple NIS-Lie superalgebras $\mathfrak{h}_{II}^{(1)}(0|4)$ and $\mathfrak{h}_{II}^{(1)}(0|5)$, where $\mathfrak{g}^{(1)}$ is the derived of $\mathfrak{g}$, see eq. (18), were classified. Clearly, $\mathfrak{h}_{II}^{(1)}(0|n)$ has the Poisson Lie superalgebra $\mathfrak{psl}_{II}(0|n)$ as its double extension, but there are more: several new double extensions were found; one of them gave an interpretation of a result in [BGLL1].

P. Deligne advised one of co-authors of this note to consider restricted algebras first of all, as pertaining to geometry, see [LL]. In [BBH], the notion of double extension was extended to restricted Lie algebras. If $p = 2$, there are several notions of restrictedness, see [BLLSq]; here we consider the “classical” one. Vectorial Lie (super)algebras can only be restricted if the shearing vector is equal to $1 := (1, \ldots, 1)$.

Hereafter $\mathbb{K}$ is an algebraically closed field of characteristic 2 and the shearing vector is equal to $1$, so we do not indicate it. For details of description of simple Lie superalgebras we study in what follows, see [BGLLS].

Our results. There are two types of super analogs of the Hamiltonian Lie algebra: series $\mathfrak{h}$, and series $\mathfrak{le}$ introduced in [Le]. If char $\mathbb{K} > 0$, there are several analogs of these series; here we consider the “standard” ones, i.e., with constant coefficients, see [BKLS] Subsection 4.7.12:

a) the restricted Lie (super)algebras $\mathfrak{h}^{(1)}_{B}(a|b)$ on $a$ even and $b$ odd indeterminates $X = (X_1, \ldots, X_{a+b})$ with forms $\mathfrak{B} = II, II, III, I$ and $II$ on the superspace spanned by the $X_i$ (in particular, $\mathfrak{B} = II$ and $I$ on the space);

b) the restricted Lie superalgebras $\mathfrak{le}^{(1)}(a|a)$ on $a$ even and $a$ odd indeterminates.

We sharpen the conjecture from [BeB1] on the shape of double extensions $\mathfrak{h}^{(1)}_{II}(0|n)$ for $n > 5$ and prove it. We computed double extensions of $\mathfrak{h}^{(1)}_{II}(a|b)$ for $4 \leq a + b \leq 6$ and $\mathfrak{h}^{(1)}_{II}(0|7)$, and $\mathfrak{le}^{(1)}(a|a)$ for $a = 2, 3$: this suffices to see the pattern for any $a$ and $b$. The deforms of series $\mathfrak{h}$ and $\mathfrak{le}$ preserving the 2-forms with non-constant coefficients are being considered elsewhere.
1.2 Preliminaries: gradings
A. Lebedev argued that in the modular case, the space of roots, although a particular case of the space of weights, should be considered over \( \mathbb{R} \), not over the ground field \( K \) as weights are considered in any representation except for the adjoint one, see \([BGL, Subsect. 4.3]\). The same applies to the notion of \( \mathbb{Z} \)-graded Lie (super)algebras: all roots of \( \text{der} \, g \), in particular, \( \mathbb{Z} \)-gradings, should be considered over \( \mathbb{R} \). (It is strange that this observation was not made ca 50 years earlier or any time later.) This interpretation of the gradings and roots is implemented in SuperLie; it allowed us to accelerate computations up to 100 times: e.g., the computation time for algebras with 6 indeterminates reduces from 8 hours to 5 minutes.

1.3 Preliminaries: Bilinear form and brackets
A. Lebedev proved \([\text{LeD}]\) that if \( p = 2 \) and \( \dim V \) is odd, there is one equivalence class of the even non-degenerate symmetric bilinear forms on the linear space \( V \), whereas if \( \dim V \) is even, there are two classes:

- **type I** if all elements on the main diagonal of the Gram matrix are zero.
- **type II** the Gram matrix has at least one non-zero element on the main diagonal.

For the normal shapes of these Gram matrices we take

\[
\Pi_{2n} := \begin{pmatrix} 1_2 & 0 \\ 0 & \Pi_{2n-2} \end{pmatrix} \quad \text{if } B \text{ is of type I and } \dim V = 2n, \quad (1)
\]

\[
\Pi_{2a} := \begin{pmatrix} 0 & 1_a \\ 1_a & 0 \end{pmatrix} \quad \text{if } B \text{ is of type II and } \dim V = 2a, \quad (2)
\]

\[
\Pi_{2a+1} := \begin{pmatrix} 0 & 0 & 1_a \\ 0 & 1 & 0 \\ 1_a & 0 & 0 \end{pmatrix} \quad \text{if } \dim V = 2a + 1 \text{ and } B \text{ of any type.} \quad (3)
\]

We denote the Lie algebra preserving the form \( B \) by \( \mathfrak{o}_B(V) \) or \( \mathfrak{o}_B(a) \) if \( \dim V = a \).

If \( V \) is a superspace, the even form \( B \) on it is the direct sum of the forms on the even and odd parts of \( V \), and hence the non-degenerate symmetric forms can be of the types \( B = II, III, II \), and \( II \) — short for \( I \oplus I \), etc. We denote the ortho-orthogonal Lie superalgebra preserving \( B \) by \( \mathfrak{oo}_B(V) \) or \( \mathfrak{oo}_B(a|b) \) if \( s\dim V = a|b \).

A. Lebedev proved \([\text{LeD}]\) that for any \( p \), there is one equivalence class of non-degenerate odd symmetric bilinear forms \( B \) on \( V \). In this case, \( s\dim V = a|a \) and for the normal shape of \( B \) one can take \( \Pi_{a|a} = \Pi_{2a} \). We denote the Lie superalgebra preserving \( B \) by \( \mathfrak{pe}_B(V) \) or \( \mathfrak{pe}_B(a) \).

- **Define the Poisson bracket** on the space of Grassmann algebra generated by either \( 2n \) odd generators \( \xi_1, \ldots, \xi_n \) and \( \eta_1, \ldots, \eta_n \) (case II), or \( n \) odd generators...
\[ \{ f, g \}_n := \sum \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i} \right); \quad (4) \]

\[ \{ f, g \}_I := \sum \left( \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial \theta_i} \right) \quad \text{for } n \text{ odd} \]

\[ \{ f, g \}_I := \sum \left( \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial \theta_i} \right) \quad \text{for } n \text{ even}. \quad (6) \]

- If, instead of the Grassmann algebra, we consider the algebra of truncated polynomials generated by either \(2n\) even generators \(q_1, \ldots, q_n\) and \(p_1, \ldots, p_n\) (case II), or \(n\) even generators \(z_1, \ldots, z_n\) (case I), the Poisson bracket becomes

\[ \{ f, g \}_n := \sum \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} + \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right); \quad (7) \]

\[ \{ f, g \}_I := \sum \left( \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_i} \right) \quad \text{for } n \text{ odd} \]

\[ \{ f, g \}_I := \sum \left( \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_i} \right) \quad \text{for } n \text{ even}. \quad (9) \]

We call the above (super)spaces with the Poisson brackets the Poisson algebras. Observe that the space of truncated polynomials with the bracket \(\{ \cdot, \cdot \}_I\) is not a Lie algebra, but a Leibniz algebra: \(\{ z_i, z_j \}_I = 1, \text{not } 0\). The quotient modulo center is, however, a Lie algebra. For all the above Poisson brackets and the combinations thereof on \(a\) even and \(b\) odd indeterminates, the quotients modulo center (generated by constants) is the Lie algebra of Hamiltonian vector fields \(\mathfrak{b}_B(a|b)\), where \(B = II, III, III,\) or \(II\).

- We also consider the Buttin superalgebra \(\mathbb{b}(a|a)\); its space is the tensor product of Grassmann algebra by the algebra of truncated polynomials generated by \(n\) even generators \(q_1, \ldots, q_n\) and \(n\) odd generators \(\pi_1, \ldots, \pi_n\) with the Schouten bracket a.k.a. Buttin bracket a.k.a. anti-bracket:

\[ \{ f, g \}_{B,a} := \sum \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial \pi_i} + \frac{\partial f}{\partial \pi_i} \frac{\partial g}{\partial q_i} \right). \]

Set \(\mathfrak{c}(a|a) = \mathbb{b}(a|a)/\mathfrak{c}\) and \(\mathfrak{b}_B(a|b) = \mathfrak{po}_B(a|b)/\mathfrak{c}\), where \(\mathfrak{c}\) is the center (spanned by constants).

On each of the above Poisson (Leibniz) algebras \(\mathfrak{po}_B(a|b)\) and the Buttin algebras \(\mathbb{b}(a|a)\), define NIS by means of the Berezin integral = the coefficient of the highest monomial: \(B(f, g) = \int fg\text{ vol}\), where vol is the volume element. This NIS is of the same parity as the number of odd indeterminates and induces a

\[ \text{The (left) Leibniz algebra } L \text{ satisfies } [x, [y, z]] = [[x, y], z] + [y, [x, z]] \text{ for any } x, y, z \in L; \text{ if in addition } L \text{ is anti-commutative, it is a Lie algebra. Superization is immediate, via the Sign Rule.} \]
NIS of the simple subquotients, $h_k^{(1)}(a|b)$ and $k^{(1)}(a|a)$. Clearly, $h_k^{(1)}(a|b)$ (resp., $k^{(1)}(a|a)$) have the Poisson or Leibniz algebras (resp., the Buttin algebra) as their double extensions; but they also have other double extensions.

For Lie superalgebras with Cartan matrices, see \cite{BGL}; for descriptions in terms of Cartan-Tanaka-Shchepochkina prolongations, see \cite{BGLLS,BGLLS}; for the classification of simple Lie superalgebras, see \cite{BGL,BLLSq}.

2 Background: Double extensions for $p \neq 2$ (after \cite{BKLS,ABB,Be})

2.1 Lemma (On a central extension) (Lemma 3.6, page 73 in \cite{BB}) Let $a$ be a Lie (super)algebra over a field $K$, let $B_a$ be an $a$-invariant NIS on $a$, let $D \in \text{Der} a$ be a derivation such that $B_a$ is $D$-invariant, i.e.,

$$B_a(Da, b) + (-1)^{p(a)p(D)}B_a(a, Db) = 0 \quad \text{for any } a, b \in a.$$  \hspace{1cm} (10)

Then the bilinear form

$$\omega(a, b) := B_a(Da, b)$$  \hspace{1cm} (11)

is a 2-cocycle of the Lie (super)algebra $a$; clearly, $p(\omega) = p(B) + p(D)$.

Thus, under assumptions of Lemma 2.1 we can construct a central extension $a_\omega$ of $a$ with the center spanned by an element $c$ given by cocycle $\omega$ so that $a_\omega / Kc \simeq a$; moreover, we can construct a semidirect sum $g = a_\omega \ltimes K D$.

On the Lie (super)algebra $g$, define a bilinear form $B$ by setting for any $x \in a$

$$B|_a = B_a, \quad B(D, c) = 1, \quad B(c, x) = B(D, x) = B(c, c) = B(D, D) = 0.$$  \hspace{1cm} (12)

2.2 Lemma [On NIS] (For Lie algebras: \cite{K} Exercise 2.10; for Lie superalgebras: \cite{BB} Theorem 1, page 68) The form $B$ defined by (12) is a NIS.

Thus constructed Lie (super)algebra $g$ with NIS $B$ on it is called the double extension or, for emphasis, $D$-extension, of $a$. If the derivation $D$ is inner, then $g$ is a direct sum of its ideal $a$ and a 2-dimensional commutative ideal; this — decomposable — case is not interesting.

2.3 Lie (super)algebra $g$ that can be a double extension of a Lie (super)algebra $a$ Let $g$ be a Lie algebra over any field $K$ or a Lie superalgebra over a field $K$ of characteristic $p \neq 2$; let $B$ be a NIS on $g$, and $c \neq 0$ a central element of $g$.

The invariance of the form $B$ implies that $B(c, [x, z]) = 0$ for any $x, z \in g$, i.e., the space $c^\perp$ contains the commutant $g^{(1)} = [g, g]$ of $g$, and hence is an ideal. Since the form $B$ is non-degenerate, the codimension of this ideal is equal to 1.

If $B(c, c) \neq 0$, then the Lie (super)algebra $g$ is just a direct sum $g = Kc \oplus c^\perp$.

Theorem Let $g$ be a Lie algebra over any field $K$ or a Lie superalgebra over a field $K$ of characteristic $p \neq 2$; let $B$ be a NIS on $g$, and $z(g)$ the center of $g$. If $g^{(1)} \cap z(g) \neq 0$, then $g$ is a double extension of a Lie (super)algebra $a$. 
2.4 Central extension must be nontrivial for DE to be indecomposable

Let $g$ be a finite-dimensional Lie (super)algebra, and $B$ a NIS on it. Let $D \in \text{der } g$ preserve $B$, and the 2-cocycle $\omega(a, b) := B(Da, b)$ be trivial, i.e., there exists $\alpha \in g^*$ such that $\omega(a, b) = \alpha([a, b])$ for all $a, b \in g$. Let us prove that in this case $D$ is an inner derivation, so the double extension is decomposable.

Since $B$ is non-degenerate, there exists an $x \in g$ such that $\alpha(y) = B(x, y)$ for all $y \in g$. Then

$$B(Da, b) = \alpha([a, b]) = B(x, [a, b]) = B([x, a], b) \quad \text{for all } a, b \in g.$$  

For this $x$, we have $Da = [x, a]$ for all $a \in g$, q.e.d.

2.5 Remark on affine Kac–Moody (super)algebras

Let $a$ be a simple finite-dimensional Lie (super)algebra with a NIS $B$. Let $a^{(1)} := a \otimes \mathbb{C}[x^{-1}, x]$, where $x = \exp(i\varphi)$ for the angle parameter $\varphi$ on the circle, be the $a$-valued Lie (super)algebra of loops expandable into Laurent polynomials. It is easy to see that for any $n \in \mathbb{Z}$, the bilinear form

$$B_n(f, g) := \text{Res } f(x)g(\mathbb{C})$$

is a NIS on $a^{(1)}$. The non-trivial central extension given by the cocycle with values in $\mathbb{C}$

$$\omega(f, g) := \text{Res } f(x)g(\mathbb{C}) = B_0(f, \frac{d}{dx} g)$$

and $D_1 = x\frac{d}{dx}$, where $D_n = x^n \frac{d}{dx}$, make $g := (\mathbb{C} \otimes a^{(1)}) \ltimes \mathbb{C}D_1$ a double extension of $a^{(1)}$ called affine Kac–Moody Lie (super)algebra. If $a$ has Cartan matrix, then $g$ also has Cartan matrix; the Dynkin graph of $g$ is the extended Dynkin graph of $a$.

The above description made us wonder: (1) In the case where $\text{der } a = a$ for simplicity, the space of outer derivations of $a^{(1)}$ is $\text{vect}(1) = \text{der } \mathbb{C}[x^{-1}, x]$. What is so special in $D_1$ to be selected for the role of outer derivation of $a^{(1)}$? (2) Why instead of the cocycle (11) with $D = D_1$, everybody uses eq. (14) with $D = D_0$?

Let $f = x^s$ and $g = x^t$ be $a$-valued functions with values equal to $u$ and $v$, respectively. Then condition (10) for $B_0$, see (13), turns into

$$(s + t) \text{Res } b(u, v)x^{n+t-1}dx = 0$$

true if either $n + s + t \neq 0$, or $s + t = 0$. Therefore, $D_n$ preserves NIS $B_{-n}$, see (13), on $a^{(1)}$, and the cocycle (14) can be obtained as

$$\omega(f, g) := B_n(f, D_{-n}g) \quad \text{for any } n \in \mathbb{Z}.$$
3 Background: Double extensions for $p = 2$ (after [BeB1])

3.1 Quadratic and bilinear forms when $p = 2$ A given map $q : V \to \mathbb{K}$, where $V$ is a $\mathbb{K}$-vector space, is called a quadratic form if

$$q(\lambda v) = \lambda^2 q(v)$$

for any $\lambda \in \mathbb{K}$ and for any $v \in V$, and the map

$$(u, v) \mapsto B_q(u, v) := q(u + v) - q(u) - q(v)$$

is bilinear.

The form $B_q$ is called the polar form of $q$. Recently, Lebedev classified non-degenerate bilinear forms over a perfect field $\mathbb{K}$ (i.e., such that $\mathbb{K}^2 = \mathbb{K}$), see [LeD]. This is a non-trivial result not related with a well-known classification of quadratic forms in any characteristic, because in characteristic 2, each of the arrows

$$q \leftrightarrow B_q$$

is not necessarily onto, has a kernel, and two quadratic forms with different Arf invariants may have identical polar forms.

3.2 Lie superalgebras for $p = 2$ For $p \neq 2$, superization of many notions of Linear Algebra is performed, as is now well-known, with the help of the Sign Rule. If $p = 2$, additional conditions appear. We recall basic definitions retaining the minus sign from definitions for $p \neq 2$: for clarity.

- A Lie superalgebra in characteristic 2 is a superspace $g = g_0 \oplus g_1$ over a field $\mathbb{K}$ such that the even part $g_0$ is a Lie algebra, the odd part $g_1$ is a $g_0$-module made two-sided by anti-symmetry, and on the odd part $g_1$ a squaring is defined as a map

$$s_g : g_1 \to g_0$$

given by $f \mapsto f^2$ such that

$$(\lambda f)^2 = \lambda^2 f^2$$

for any $f \in g_1$ and $\lambda \in \mathbb{K}$, and the map

$$[f, g] := (f + g)^2 - f^2 - g^2$$

for any $f, g \in g_1$ is a bilinear form on $g_1$ with values in $g_0$.

The bracket on $g_0$, as well as the action of $g_0$ on $g_1$, is denoted also by the same symbol $[\cdot, \cdot]$, or $[\cdot, \cdot]_g$ for clarity. The Jacobi identities for three even elements, and involving one or two odd elements, are the same as for $p \neq 2$; the following new identity replaces the one with three odd elements: it involves the squaring:

$$[f^2, g] = [f, [f, g]]$$

for any $f \in g_1$ and $g \in g$.

- Desuperizing (retain only the bracket) we get a $\mathbb{Z}/2$-graded Lie algebra $g$.
- For any Lie superalgebra $g$ in characteristic 2, its derived algebras are

$$g^{(0)} := g, \quad g^{(i+1)} = [g^{(i)}, g^{(i)}] + \mathbb{K}\{f^2 \mid f \in (g^{(i)})_1\}.$$  

A linear map $D : g \to g$ is called a derivation of the Lie superalgebra $g$ if

$$D([f, g]) = [D(f), g] + [f, D(g)]$$

for any $f, g \in g$

and, additionally (generalization of (17), where $D = \text{ad}_g$),

$$D(f^2) = [D(f), f]$$

for any $f \in g_1$. 
We denote the space of all derivations of \( g \) by \( \text{der}(g) \).

An even linear map \( \rho : g \rightarrow \mathfrak{gl}(V) \) is a representation of the Lie superalgebra \( g \) in the superspace \( V \), called \( g \)-module, if

\[
\rho([f, g]) = \rho(f)\rho(g) \quad \text{for any } f, g \in g; \text{ and } \rho(f^2) = (\rho(f))^2 \quad \text{for any } f \in \mathfrak{g}_1. \tag{21}
\]

We say that a bilinear form \( B \) on \( g \) is symmetric if

\[
B(f, g) = B(g, f) \quad \text{for any } f, g \in g; \text{ and, additionally, } B(f, f) = 0 \quad \text{for any } f \in \mathfrak{g}_1. \tag{22}
\]

We say that \( D \) preserves \( B \) if

\[
B(D(f), g) + B(f, D(g)) = 0 \quad \text{for any } f, g \in g; \text{ and, additionally, } B(D(f), f) = 0 \quad \text{for any } f \in \mathfrak{g}_1 \text{ and any } g \in g. \tag{22}
\]

We denote the nis-superalgebra with a NIS \( B \) by \((g, B)\). A nis-superalgebra \((g, B)\) is said to be decomposable if it can be decomposed into direct sums of ideals, namely \( g = \bigoplus I_i \), such that all \( I_i \) are orthogonal to each other.

### 3.3 The case where \( B_a \) is even

Here \( g := \mathcal{H} \oplus a \oplus \mathcal{H}^* \) as spaces, where \( \mathcal{H} := \mathbb{K}c \) and \( \mathcal{H}^* := \mathbb{K}D \).

#### D\( \mathfrak{h} \)-extensions. The quadratic form \( q \)

**Theorem.** Let \((a, B_a)\) be a nis-superalgebra, \( p(B_a) = 0 \), let \( D \in \text{der}_a(a) \) preserve \( B_a \). Let \( q : a_1 \rightarrow \mathbb{K} \) be a quadratic form; let its polar form \( B_q \) satisfy

\[
B_a(a, D(b)) = B_q(a, b) \quad \text{for any } a, b \in a_1. \tag{23}
\]

Then there exists a nis-superalgebra structure on \( g \), defined as follows, cf. Subsection 2.1 and (12). The squaring is given by

\[
s_g(a) := s_a(a) + q(a)c \quad \text{for any } a \in \mathfrak{g}_1 \text{ (} = a_1\). \tag{24}
\]

The bracket on \( g \) is defined for any \( a, b \in a \) as follows:

\[
[a, b]_g := [a, b]_a + B_a(D(a), b)c, \quad [D, a]_g := D(a) \text{ and } [c, g]_g = 0. \tag{24}
\]

The NIS \( B \) on \( g \) is given by formulas (12) with one modification: \( B(D, D) \) can be arbitrary.

What is the difference of this Theorem from its analog for \( p \neq 2 \)?

**First,** the extra condition in eq. (22): \( B_a(D(a), a) = 0 \). For \( p \neq 2 \), it follows from eq. (22), whereas for \( p = 2 \) it should be required as a part of invariance condition.

**Second,** the reasons for the quadratic form \( q \) to appear. Indeed, for \( p \neq 2 \), to determine a central extension, we only need a cocycle, whereas for \( p = 2 \), we have to adjust the squaring to match this cocycle and this is precisely what the form \( q \) is needed for.
Everything else is the same as for $p \neq 2$, same as with the desuperisations of $a$, when only even derivations and forms remain, i.e., no extra conditions.

We call the Lie superalgebra $(\mathfrak{g}, B)$ constructed in Theorem 3.3.1 a $D_0$-extension of $(a, B_a)$ by means of $D$ and $q$.

Now, let the “special center” of $\mathfrak{g}$ relative to $B_\mathfrak{g}$ be

$$\mathfrak{z}_s(\mathfrak{g}) := \mathfrak{z}(\mathfrak{g}) \cap s_\mathfrak{g}(\mathfrak{g}_1)^\perp,$$

where $s_\mathfrak{g}(\mathfrak{g}_1) := \text{Span}\{x^2 \mid x \in \mathfrak{g}_1\}$. \hfill (25)

Observe that $\mathfrak{z}_s(\mathfrak{g})_1 = \mathfrak{z}(\mathfrak{g})_1$ and $\mathfrak{z}_s(\mathfrak{g})_0 = \mathfrak{z}(\mathfrak{g})_0 \cap s_\mathfrak{g}(\mathfrak{g}_1)^\perp$. Moreover, $\mathfrak{z}_s(\mathfrak{g})$ is not necessarily an ideal.

3.3.2. Proposition. Let $(\mathfrak{g}, B_\mathfrak{g})$ be an indecomposable nis-superalgebra. Let $\mathfrak{z}_s(\mathfrak{g})_0 \neq \{0\}$. Then $(\mathfrak{g}, B_\mathfrak{g})$ is obtained as an $D_0$-extension from a nis-superalgebra $(a, B_a)$.

What is the difference of the case where $p = 2$ from that where $p \neq 2$? The “indecomposability” condition can be weakened to the existence of a nonzero even element $e$ belonging to the intersection of the center of $\mathfrak{g} \cap [\mathfrak{g}, \mathfrak{g}] \cap s_\mathfrak{g}(\mathfrak{g}_1)^\perp$. Now we see that the only extra condition is related to squaring and is precisely due to the fact that $c^\perp$ must be an ideal.

$D_1$-extensions. The element $A \in a_0$

3.3.3. Theorem. Let $(\mathfrak{a}, B_a)$ be a nis-superalgebra, $p(B_a) = 0$. Let $D \in \text{der}_1(\mathfrak{a})$ preserve $B_a$; let $A \in a_0$ satisfy the following conditions (see [ABB1Be]):

$$D^2 = \text{ad}_A; \quad D(A) = 0. \hfill (26)$$

Then there exists a nis-superalgebra structure on $\mathfrak{g}$, defined as follows. The squaring is given by

$$s_\mathfrak{g}(ra + a + tD) := s_\mathfrak{a}(a) + tD(a) + t^2 A \quad \text{for any } a \in a_1 \text{ and } r, t \in \mathbb{K}.$$

The bracket is given by eq. (24). The NIS $B$ on $\mathfrak{g}$ is given by formulas (25).

We call the nis-superalgebra $(\mathfrak{g}, B)$ constructed in Theorem 3.3.3 a $D_1$-extension of $(a, B_a)$ by means of $D$ and $A$.

3.3.3a. Proposition. Let $(\mathfrak{g}, B)$ be a nis-superalgebra. Define the cone

$$\mathcal{C}(\mathfrak{g}, B) := \{x \in \mathfrak{g}_1 \mid B(s_\mathfrak{g}(x), s_\mathfrak{g}(y)) = 0 \quad \text{for any } y \in \mathfrak{g}_1\}.$$ 

Let $\mathfrak{z}(\mathfrak{g})_1 \cap \mathcal{C}(\mathfrak{g}, B) \neq \{0\}$. Then $(\mathfrak{g}, B)$ is obtained from a nis-superalgebra $(a, B_a)$ as either a $D_0$-extension, or a $D_1$-extension.

3.4 The case where $B_a$ is odd Here $\mathfrak{g} := \mathcal{X} \oplus a \oplus \mathcal{X}^*$ as spaces, where $\mathcal{X} := \mathbb{K}c$ and $\mathcal{X}^* := \mathbb{K}D$. 
3.4.1. Theorem. Let \((a, B_a)\) be a nis-superalgebra, \(p(B_a) = \bar{1}\). Let \(D \in \text{Der}_0(a)\) preserve \(B_a\). Then there exists a nis-superalgebra structure on \(g\), defined as follows. The squaring is given by

\[ s_g(a + \mu c) := s_a(a) \quad \text{for any } a \in a_1 \text{ and } \mu \in \mathbb{K}. \]

The bracket is given by eq. (24). The NIS \(B\) on \(g\) is given by formulas (12).

We call the nis-superalgebra \((g, B)\) constructed in Theorem 3.4.1 the \(D_0\)-extension of \((a, B_a)\) by means of \(D\).

3.4.1a. Proposition. Let \((g, B)\) be an indecomposable nis-superalgebra, \(p(B) = \bar{1}\). If \(s_g(s(g)_{\bar{1}}) \cap s_g(s(g)_{\bar{1}})^\perp \neq \{0\}\), then \((g, B)\) is obtained from a nis-superalgebra \((a, B_a)\) as either a \(D_0\)-extension, or a \(D_1\)-extension.

3.4.1b. Theorem. Let \((a, B_a)\) be a nis-superalgebra, \(p(B_a) = \bar{1}\). Let \(A \in a_{\bar{0}}\) and \(D \in \text{Der}_1(a)\) preserving \(B_a\) satisfy the conditions (26):

Let \(q\) be a quadratic form on \(a_{\bar{1}}\) such that \(B_{q}(a, b) = B_a(D(a), b)\).

Then there exists a nis-superalgebra structure on \(g\), defined as follows. The squaring is given by

\[ s_g(a + \mu D) := s_a(a) + (\mu^2 m + q(a))c + \mu^2 A + \mu D(a) \quad \text{for any } a \in a_1 \text{ and } \mu \in \mathbb{K}, \text{ and some } m \in \mathbb{K}. \]

The bracket is given by eq. (24). The NIS \(B\) on \(g\) is given by formulas (12).

We call the nis-superalgebra \((g, B)\) constructed in Theorem 3.4.1b a \(D_1\)-extension of \((a, B_a)\) by means of \(D, q, A, m\).

3.4.1c. Proposition. Let \((g, B)\) be an indecomposable nis-superalgebra, \(p(B) = \bar{1}\). If \(s_g(s(g)_{\bar{0}}) \neq \{0\}\), then \((g, B)\) is obtained as a \(D_1\)-extension from a nis-superalgebra.

3.5 Summary: to construct a double extension of a nis-(super)algebra \(a\), we need \(q, A, m\)

\[
\begin{array}{c|cc}
D & \text{even} & \text{odd} \\
\hline
\text{even} & q & - \\
\text{odd} & A & q, A, m \\
\end{array}
\]

(27)

4 Isomorphisms, and equivalence classes of derivations (after [BeB1])

For a nis-superalgebra \(a\) with NIS \(B_a\), denote by \((g, B_g)\) (resp. \((\tilde{g}, \tilde{B}_g)\)) the double extension of \(a\) by means of a derivation \(D\) (resp. \(\tilde{D}\)), i.e., \(g := \mathcal{K}c \oplus a \oplus \mathcal{K}^*\), where \(\mathcal{K} = \mathbb{K}c\) and \(\mathcal{K}^* = \mathbb{K}D\) (resp. \(\tilde{g} := \mathcal{K}c \oplus a \oplus \mathcal{K}^*\), \(\tilde{g} := \mathbb{K}c\) and \(\mathcal{K}^* = \mathbb{K}D\)).
If $p = 2$, we additionally require a quadratic form $q$ (resp. $\tilde{q}$) if the center $c$ is even, while in the case of $D_1$-extensions, we need, moreover, an element $A \in a_0$, and sometimes $m \in K$ (resp. $\tilde{A}$ and $\tilde{m}$) described in the previous section.

An isomorphism of of nis-superalgebras $\pi : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ is a nis-preserving isomorphism of Lie superalgebra structures:

$$\pi([f, g]) = [\pi(f), \pi(g)], \quad B_{\tilde{q}}(\pi(f), \pi(g)) = B_q(f, g) \quad \text{for any } f, g \in \mathfrak{g}.$$ 

### 4.1 The case where $B_a$ is even

#### 4.1.1. Theorem

Let $p(D) = p(\tilde{D}) = 0$. Let $\pi_0$ be an automorphism of $(a, B_a)$. Let $\lambda \in K^\times$ and let $y \in a_0$ satisfy the following conditions:

$$\begin{align*}
\tilde{q}(a) &= \lambda q \circ \pi_0^{-1}(a) + B_a(y, s_a \circ \pi_0^{-1}(a)) \quad \text{for any } a \in a_1; \\
\pi_0^{-1}\tilde{D}\pi_0 &= \lambda D + \text{ad}_y \quad \text{for any } a \in a_0; \\
B_q(D, D) &= \lambda^{-2}(B_a(y, y) + B_q(\tilde{D}, \tilde{D})).
\end{align*}$$

Then there exists an isomorphism $\pi : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ given by the formulas

$$\begin{align*}
\pi(a) &= \pi_0(a) + B_a(y, a)c \quad \text{for any } a \in a; \\
\pi(c) &= \lambda c; \\
\pi(D) &= \lambda^{-1}(\tilde{D} + \pi_0(y)) + \nu c, \quad \text{where } \nu \in K \text{ is arbitrary}.
\end{align*}$$

If $[D] = [\tilde{D}]$ in $H^1_{\text{nis}}(a; a)$, then they define isomorphic double extensions of $a$.

#### 4.1.2. Theorem

Let $p(D) = p(\tilde{D}) = 1$. Let $\pi_0$ be an automorphism of $(a, B_a)$. Let $\lambda \in K^\times$ and $y \in a_1$ satisfy the following condition:

$$\pi_0^{-1}\tilde{D}\pi_0 = \lambda D + \text{ad}_y \quad \text{on } a.$$ 

Then there exists an isomorphism $\pi : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ given by the formulas

$$\begin{align*}
\pi(a) &= \pi_0(a) + B_a(y, a)c \quad \text{for any } a \in a_1; \\
\pi &= \pi_0 \quad \text{on } a_0; \\
\pi(c) &= \lambda c; \\
\pi(D) &= \lambda^{-1}(\tilde{D} + \pi_0(y)) + \nu c, \quad \text{where } \nu \text{ is arbitrary}; \\
A &= \lambda^2\pi_0(A) + s_a(\pi_0(y)) + \lambda\pi_0(D(y)).
\end{align*}$$

If $[D] = [\tilde{D}]$ in $H^1_{\text{nis}}(a; a)$, then they define isomorphic double extensions of $a$.

### 4.2 The case where $B_a$ is odd

#### 4.2.1. Theorem

Let $p(D) = p(\tilde{D}) = 1$. Let $\pi_0$ be an automorphism of $(a, B_a)$. Let $\lambda \in K^\times$ and $y \in a_1$ satisfy the following conditions:

$$\begin{align*}
\tilde{q}(a) &= \lambda q \circ \pi_0^{-1}(a) + B_a(y, s_a \circ \pi_0^{-1}(a)) \quad \text{for any } a \in a_1; \\
\pi_0^{-1}\tilde{D}\pi_0 &= \lambda D(a) + \text{ad}_y(a) \quad \text{for any } a \in a_0; \\
\tilde{A} &= \lambda^2\pi_0(A) + \lambda\pi_0(D(y)) + s_a(\pi_0(y)); \\
\tilde{m} &= \lambda^2\tilde{q}(y) + \lambda B_a(y, s_a(y) + \lambda A) + \lambda^3 m.
\end{align*}$$
Then there exists an isomorphism $\pi : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ given by the formulas
\[
\pi(a) = \pi_0(a) + B_a(y, a)\tilde{c} \quad \text{for any } a \in \mathfrak{a};
\]
\[
\pi(c) = \lambda\tilde{c};
\]
\[
\pi(D) = \lambda^{-1}(\tilde{D} + \pi_0(y)).
\]  

If $[D] = [\tilde{D}]$ in $H^1_1(\mathfrak{a}; \mathfrak{a})$, then they define isomorphic double extensions of $\mathfrak{a}$.

4.2.2. Theorem. Let $p(D) = p(\tilde{D}) = 0$. Let $\pi_0$ be an automorphism of $(\mathfrak{a}, B_{\mathfrak{a}})$. Let $\lambda \in \mathbb{K}^\times$ and $y \in a_0$ satisfy the following conditions:
\[
\pi_0^{-1}\tilde{D}\pi_0(a) = \lambda D(a) + \text{ad}_y(a) \quad \text{for any } a \in \mathfrak{a}.
\]

Then there exists an isomorphism $\pi : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ given by the formulas [25]. If $[D] = [\tilde{D}]$ in $H^1_0(\mathfrak{a}; \mathfrak{a})$, then they define isomorphic double extensions of $\mathfrak{a}$.

5 The exceptional cases: $\mathfrak{a} = \mathfrak{h}_{\mathfrak{B}}^{(1)}(a|b)$ for $a + b = 4$ and $\mathfrak{a} = \mathfrak{ic}^{(1)}(2|2)$

Recall that the $\mathbb{Z}$-grading of a given vectorial Lie superalgebra is called standard if the degree of each indeterminate $X_i$ is equal to 1. This grading is given by the Euler operator $\sum X_i \partial X_i$. The roots are given with respect to the maximal torus, spanned by $q_i p_i$, $\eta_i \xi_i$, and $q_i \pi_i$ in the cases $p_\mathfrak{H}(a|0)$, $p_\mathfrak{H}(0|a)$, and $\mathfrak{ic}(a|a)$, respectively. According to Subsection 1.2 the degree and roots are considered over $\mathbb{R}$, assuming that the $i$th coordinate of the weight of $\xi_i$, $\pi_i$, and $p_i$ is equal to 1, whereas the $i$th coordinate of the weight of $\eta_i$, and $q_i$ is equal to $-1$.

The standard $\mathbb{Z}$-grading of $\mathfrak{a} = \mathfrak{h}_{\mathfrak{B}}^{(1)}(a|b)$ (resp. $\mathfrak{ic}^{(1)}(a|a)$) is symmetric, i.e.,
\[
a = \bigoplus_{-1 \leq i \leq 1} a_i \quad \text{with } a_{-1} \simeq a_1 \quad \text{(resp. } a_{-1} \simeq \Pi(a_1)^*)\text{ as } \mathfrak{a}_0\text{-modules},
\]  

5.1 Outer derivations Let the superscript of the derivation $D$ be its degree, let the subscript be its weight or a monomial $(x$ or $b$) or a label $\theta$.

$\mathfrak{h}_H^{(1)}(0|4)$ One outer derivation in each of degrees $\pm 2$ and 5 derivations of degree 0, whose weights are $(\pm 2, 0)$, $(0, \pm 2)$ and $(0, 0)$, see [LeB].

$\mathfrak{h}_H^{(1)}(4|0)$ One outer derivation in each of degrees $\pm 2$ and 5 derivations of degree 0, whose weights are $(\pm 2, 0)$, $(0, \pm 2)$ and $(0, 0)$. This answer is the desuperization of the case $\mathfrak{h}_H^{(1)}(0|4)$; the cocycles are identical in shape to those of $\mathfrak{h}_H^{(1)}(0|4)$.

$\mathfrak{h}_{H\mathfrak{H}}^{(1)}(2|2)$ One outer derivation in each of degrees $\pm 2$ and 5 derivations of degree 0, whose weights are $(\pm 2, 0)$, $(0, \pm 2)$ and $(0, 0)$. This answer is a partial desuperization of the case $\mathfrak{h}_H^{(1)}(0|4)$; the cocycles are identical in shape to those of $\mathfrak{h}_H^{(1)}(0|4)$.
One outer derivation in each of degrees ±2. The apparently missing Euler operator is in this case an inner derivation. The same reason causes absence of certain derivations in cases \( h_I^{(4)}(0|4) \) and \( h_I^{(4)}(4|0) \), and in this case as compared with the case \( h_I^{(4)}(0|4) \).

\( h_I^{(1)}(0|4) \) One outer derivation in each of degrees ±2 and 4 derivations of degree 0, whose weights are ±2 and two derivations of weight 0.

\( h_I^{(1)}(4|0) \) The answer is the desuperization of the case \( h_I^{(1)}(0|4) \); derivations of the same shape.

\( \mathfrak{l}e^{(1)}(2|2) \) The shape of derivations is identical to those for \( h_{II}^{(1)}(0|4) \); but \( D^{(0)}_b \) are odd.

### 5.2 The double extensions

**Case** \( \mathfrak{h} \). For the proof for the cocycle \( D^{(-2)} \) in case \( a = h_I^{(1)}(0|4) \), see [BeB1]; the result for \( D^2 \) is isomorphic due to the symmetry \( (29) \). The idea of the proof is identical for the other derivations. The condition \( B_a^2 D^{(2)} = D^{(2)} B_a \) is easily seen. Besides, since \( D^{(2)} \) acts by zero on \( a_0 \), it follows that

\[
B_a(D^{(2)}(f), f) = 0 \quad \text{for any } f \in a_0.
\]

Tables (31) and (30) show existence of the quadratic form \( q \) (resp. element \( A \)) associated with each derivation \( D \), and names of the respective double extension \( \text{DE} \). In the 6th column of (31) \( B = II, \) or \( III, \) or \( IIII. \)

| Derivation \( D^{(0)}_b \) | \( q \) | \( \text{DE of } h_I^{(1)}(0|4) \) | \( \text{DE of } h_I^{(1)}(4|0) \) |
|---|---|---|---|
| \( D^{(0)}_b \) | yes | \( \mathfrak{po}_I(0|4) \) | \( \mathfrak{po}_I(4|0) \) |
| \( D^{(0)}_a \) | yes | \( \mathfrak{po}_I(0|4) \) | \( \mathfrak{po}_I(4|0) \) |
| \( D^{(0)}_d \) | – | – | – |
| \( D^{(2)} \) | yes | \( \mathfrak{po}_I(0|4) \) | \( \mathfrak{po}_I(4|0) \) |

\[
D^{(0)}_b \quad q \quad h_{II}^{(1)}(0|4) \quad h_{II}^{(1)}(4|0) \quad h_{II}^{(1)}(2|2) \quad h_{II}^{(1)}(2|2) \quad A \quad \mathfrak{l}e^{(1)}(2|2)
\]

| \( D^{(0)}_b \) | yes | \( \mathfrak{po}_{II}(0|4) \) | \( \mathfrak{po}_{II}(4|0) \) | \( \mathfrak{po}_{II}(2|2) \) | – | 0 | \( b^{(2|2)} \) |
| \( D^{(0)}_a \) | yes | \( \mathfrak{gl}(2|2) \) | \( \mathfrak{gl}(4|0) \) | \( \mathfrak{gl}(2|2) \) | – | – | – |
| \( D^{(2)} \) | yes | \( \mathfrak{po}_{II}(0|4) \) | \( \mathfrak{po}_{II}(4|0) \) | \( \mathfrak{po}_{II}(2|2) \) | \( \mathfrak{po}_{II}(2|2) \) | 0 | \( b^{(2|2)} \) |

**Case** \( \mathfrak{l}e. \) Compare with the general case in Subsection 6.13.

### 5.2.1. Lemma

**Let the super-rank of the operator** \( A \) **in the superspace** \( V \) **be the super dimension of the superspace** \( V/ \text{Ker} A \).

1) **Because the super-rank of** \( D^{(0)}_b \) **is** \( (2, 2) \) **and no element in the corresponding** \( \mathfrak{po} \) **and** \( \mathfrak{gl} \) **has such rank, it follows that** \( \mathfrak{po} \) **is not isomorphic to the other two double extensions.**

2) **Because** \( \text{rk} D^{(0)}_b = 8 \) **in** \( \mathfrak{po} \) **and no element in** \( \mathfrak{po}_I \) **and in** \( \mathfrak{po}_II \) **has such rank, it follows that** \( \mathfrak{po}_I \) **is not isomorphic to the other two double extensions.**

3) **Because** \( \text{rk} \text{ad}_{\pi_1} = \text{rk} \text{ad}_{\pi_2} = 7 \) **in** \( b^{(2|2)} \) **and no element in** \( b^{(2|2)} \) **has such rank,** \( b^{(2|2)} \neq b^{(2|2)} \).
6 The general cases. Outer derivations

Let $\hat{X}$ be the product of all indeterminates. Derivations of $\mathfrak{a}$ will be called equivalent if they lie on one orbit of the group of automorphisms of \( \text{der } \mathfrak{a} \).

6.1 \( \text{out}(\mathfrak{h}^{(1)}_{\mathfrak{II}}(0|2n)) \) Notation for convenience (tags “(\text{c} odd)” are used in Subsection 6.13):

1. Basis: \( B = \{ \eta_1, \ldots, \eta_n; \xi_1, \ldots, \xi_n \} \).
2. The set of products of any \( k \) basis elements (Choose):
   \[ \text{Ch}_B(k) = \{ x_1 \ldots x_k \mid x_i \in B \}; \text{Ch}_\xi(k) = \{ x_1 \ldots x_k \mid x_i \in \{ \xi_1, \ldots, \xi_n \} \}; \text{Ch}_\eta(k) = \{ x_1 \ldots x_k \mid x_i \in \{ \eta_1, \ldots, \eta_n \} \}. \]
   For example,
   \[ \text{Ch}_B(2) = \{ \eta_\eta, \xi_\xi_j \mid \text{for all } i \neq j \leq n \} \cup \{ \eta_\xi_i \mid i, j \leq n \}. \]
3. Index of elements: \( \text{Ind}(x) = \) The set of indices of \( x \) in terms of \( \eta, \xi, \).
   For example, \( \text{Ind}(\eta_\eta, \xi_\xi_3) = \{ 1, 3 \}, \text{Ind}(\eta_\eta \xi_\xi_5 \xi_\xi_7) = \{ 2, 5, 7 \} \).
4. The switch symbol: \( S(\xi_i) = \eta_i \) and \( S(\eta_i) = \xi_i \).
5. Let \( O \) denote the set of all monomials of odd degree.

   (a) \( \text{Deg}=0 \). There are \( 2n \) equivalent derivations: for any \( b \in B \), we have
   \[ D^{(0)}_b := \sum_{0 \leq i \leq 2n-2} \sum_{x \in \text{Ch}(i)} (bx) \otimes (S(b)x). \]  \( (\text{c} : \text{always odd}) \)  \( (32) \)

   (b) \( \text{Deg}=0 \). One particular derivation — the Euler operator
   \[ D^{(0)}_b = \sum_{1 \leq i \leq n-1} \sum_{0 \leq j \leq 2i+1} \sum_{x \in \text{Ch}(2i)} \sum_{y \in \text{Ch}(j)} (xy) \otimes (\bar{xy}). \]  \( (33) \)

   (c) \( \text{Deg}= 2n-2 \).
   \[ D^{(2n-2)} := \sum_{x \in B} \frac{\partial \hat{X}}{\partial x} \otimes (\hat{S}(x)). \]  \( (\text{c} : \text{odd with } n) \)  \( (34) \)

6.2 \( \text{out}(\mathfrak{h}^{(1)}_{\mathfrak{II}}(2n|0)) \) The desuperization of the case \( \text{out}(\mathfrak{h}^{(1)}_{\mathfrak{II}}(0|2n)) \); the same cocycles.

6.3 \( \text{out}(\mathfrak{h}^{(1)}_{\mathfrak{II}}(2a|2b)) \) A partial desuperization of \( \text{out}(\mathfrak{h}^{(1)}_{\mathfrak{II}}(0|2a+2b)) \); the same cocycles.

6.4 \( \text{out}(\mathfrak{h}^{(1)}_{\mathfrak{II}}(0|2n)) \) Cocycles:

   (a) \( \text{Deg}=0 \). There are \( 2n \) equivalent derivations: for any \( b \in B \), we have
   \[ D^{(0)}_b := \sum_{0 \leq i \leq 2n-2} \sum_{x \in \text{Ch}(i)} (bx) \otimes (S(b)x). \]  \( (35) \)

   (b) \( \text{Deg}=0 \). One particular derivation (the apparent asymmetry of \( \theta_1 \) and \( \theta_2 \) is due to SuperLie’s aesthetic criteria)
   \[ D^{(0)}_b = \sum_{x \in C(\theta_1)} \theta_1 x \otimes (\hat{\theta_1} x). \]  \( (36) \)
(c) Deg=0. Another particular derivation — the Euler operator
\[ D_0^{(0)} = \sum_{x \in O} x \otimes (x). \] (37)

(d) Deg = 2n - 2. See eq. (34).

6.5 out(h^{(1)}_I(2n|0)) The desuperization of the case out(h^{(1)}_I(0|2n)); the same cocycles.

6.6 out(h^{(1)}_I(2a|2b)) Cocycles:
(a) Deg = 2a + 2b - 2. See eq. (34).

6.7 out(h^{(1)}_I(0|2n + 1)) Notation for convenience:
1. Basis: \( B = \left\{ \eta_1, \ldots, \eta_n; \theta; \xi_1, \ldots, \xi_n \right\} \) or \( B = \left\{ \xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n \right\} \).
2. Let \( \hat{C}(x) \) denote the set of all monomials each of which is a multiple of \( x \);
   let \( \hat{C}(x) \) denote the set of all monomials each not a multiple of \( x \).
3. The switch symbol: \( S(\xi_i) = \eta_i \) and \( S(\eta_i) = \xi_i \), whereas \( S(\theta) = \theta \).
4. Let \( O \) denote the set of all monomials of odd degree.

Cocycles:
(a) Deg=0. There are \( 2n \) equivalent derivations: of weight \( 2w \) for any \( x \in \bar{B} \) of weight \( w \),
\[ D_w^{(0)} := \sum_{y \in \hat{C}(x) \cap \hat{C}(S(x))} xy \otimes (S(x)y). \] (38)
(b) Deg=0. One particular derivation — the Euler operator, see eq. (37).
(c) Deg = 2n - 1: See eq. (34).

6.8 out(h^{(1)}_I(2a + 1|2b + 1)) Cocycles:
(a) Deg=0. A particular derivation \( D_0^{(0)} \), see eq. (38).
(b) Deg=0. Another particular derivation — the Euler operator, see eq. (37).
(c) Deg = 2a + 2b. See eq. (34).

6.9 out(h^{(1)}_I(2a|2b)) Cocycles:
(a) Deg=0. There are \( 2b \) equivalent derivations, see eq. (38).
(b) Deg=0. A particular derivation \( D_0^{(0)} \), see eq. (38).
(c) Deg = 0. Another particular derivation — the Euler operator, see eq. (37).
(d) Deg = 2a + 2b - 2. See eq. (34).

6.10 out(h^{(1)}_I(2a|2b)) Same cocycles as for out(h^{(1)}_I(2b|2a)), but with \( p, q \) and \( \xi, \eta \) interchanged in all cocycles.

6.11 out(h^{(1)}_I(2a|2b + 1)) A partial desuperization of h^{(1)}_I(0|2n + 1).

6.12 out(h^{(1)}_I(2a + 1|2b)) Same cocycles as for out(h^{(1)}_I(2b|2a + 1)), but with \( p, q \) and \( \xi, \eta \) interchanged in all cocycles.

6.13 out(le^{(1)}(n|n)) Same cocycles as for out(h^{(1)}_I(2n|0)) with \( q, \pi \) replacing \( q, p \); odd cocycles are marked in Subsection 6.1. Observe that \( p(B) \equiv n \pmod{2} \), so there are no \( D_1 \)-extensions for \( n \) odd, and no \( D_0 \)-extensions for \( n \) even, since the center \( c \) is always odd and \( p(c) = p(B) + p(D) \).
7 The general cases. Double extensions

Note: in all cases of \( h \) series, \( p(B_a) \) is congruent to the parity of the number of odd indeterminates; \( p(B_a) \equiv a + 1 \) for \( \mathfrak{a}(a|a) \). Let \( X \) be the product of all indeterminates.

7.1 The double extensions of \( a = h^{(1)}(0|2n + 1) \), \( h^{(1)}_{II}(2a|2b + 1) \), \( h^{(1)}_{II}(2a + 1|2b) \) for \( a + b = n > 2 \)

The \( D_0 \)-extension The derivation \( D_0^{(0)} \) does not preserve the NIS \( B \). Indeed,

\[
B_a(D_0^{(0)}(\theta), \frac{\partial X}{\partial \theta}) = 1 \text{ while } B_a(\theta, D_0^{(0)}(X)) = 0.
\]

Now, the derivations \( D_x^{(0)} \) preserve the bilinear form \( B_a \), and the proof is similar to that for \( h^{(1)}(0|4) \) in Subsection 5.2: all respective double extensions are isomorphic. The double extension by means of the derivation \( D_x^{(0)} \) for any \( x \in \bar{B} \), see Subsection 6.7, is a Lie superalgebra denoted by \( \bar{p} \circ(0|5) \) in [BeB1] for \( n = 2 \):

| Derivation | preserves \( B_a \) | \( s_a(c) \) | Extension | Extension |
|-------------|-----------------|------------|-----------|-----------|
| \( D_x^{(0)} \) | Yes             | 0          | \( \bar{p} \circ(0|2n + 1) \) | \( \bar{p} \circ_{II}(2a|2b + 1) \) |
| \( D_0^{(0)} \) | No              | –          | –         | –         |

(39)

The \( D_1 \)-extension In a lexicographically ordered basis on \( h^{(1)}(0|5) \) the Gram matrix of NIS is \( B = \text{antidiag}(1, \ldots, 1) \). The condition \( B_a^T D^{(2n - 1)} = D^{(2n - 1)} B_a \) is easy to see. Besides, since \( D^{(3)}|_{h_0} = 0 \), it follows that

\[
B_a(D^{(2n - 1)}(f), f) = 0 \text{ for any } f \in a_0.
\]

Now, since \( 0 = (D^{(2n - 1)})^2 = \text{ad}_A \), it follows that \( A = 0 \) since \( a \) has no center. We have, therefore, a parametric family of double extensions by means of \( D^{(2n - 1)} \), \( q \).

\[
A = 0 \text{ and } m \in \mathbb{K} \text{ (see Theorem 3.4.11). It is proved in [BeB1], that } \bar{p} \circ(0|5; m) \text{ for } m \neq 0 \text{ is not isomorphic to } \bar{p} \circ(0|5; 0) := \bar{p} \circ(0|5), \text{ whereas } \bar{p} \circ(0|5; m) \simeq \bar{p} \circ(0|5; \tilde{m}) \text{ for any pair } m \tilde{m} \neq 0. \text{ The same arguments are true for any } n > 1. \text{ Table (40)} \]

summarizes these results for any \( m \in \mathbb{K}^\times \):

| \( D \) | \( q(a) \) | \( s_a(D) \) | Extension | Extension |
|--------|-----------|-------------|-----------|-----------|
| \( D^{(2n-1)} \) | yes | 0 | \( \bar{p} \circ(0|2n + 1) \) | \( \bar{p} \circ_{II}(2a|2b + 1) \) |
| \( D^{(2n-2)} \) | mc | \( \bar{p} \circ(0|2n + 1; m) \) | \( \bar{p} \circ_{II}(2a|2b + 1; m) \) | \( \bar{p} \circ_{II}(2a + 1|2b; m) \) |

(40)

7.2 The double extensions of \( a = h^{(1)}_{II}(0|2n) \) for \( n > 2 \) and its desuperrizations

| \( D \) | \( q \) | \( h^{(1)}_{II}(0|2n) \) | \( h^{(1)}_{II}(2n|0) \) | \( h^{(1)}_{II}(2a|2b) \) | \( h^{(1)}_{II}(2a + 1|2b + 1) \) |
|--------|-------|----------------|----------------|----------------|----------------|
| \( D_0^{(0)} \) | yes | \( \bar{p} \circ_{II}(0|2n) \) | \( \bar{p} \circ_{II}(2n|0) \) | \( \bar{p} \circ_{II}(2a|2b) \) | \( \bar{p} \circ_{II}(2a + 1|2b + 1) \) |
| \( D_0^{(0)} \) | no | – | – | – | – |
| \( D^{(2n-2)} \) | yes | \( \bar{p} \circ_{II}(0|2n) \) | \( \bar{p} \circ_{II}(2n|0) \) | \( \bar{p} \circ_{II}(2a|2b) \) | \( \bar{p} \circ_{II}(2a + 1|2b + 1) \) |

(41)
7.3 The double extensions of \( a = h_I^{(1)}(0|2n) \) for \( n > 2 \) and its desuperizations

Clearly, the results in cases III and II are obtained from one another.

\[
\begin{array}{c|c|c|c|c|c}
D & q & h_I^{(1)}(0|2n) & h_I^{(1)}(2n|0) & h_{II}^{(1)}(2n|2b) & h_{II}^{(1)}(2a|2b) \\
\hline
D_0^{(0)} & yes & \hat{p}_{II}(0|2n) & \hat{p}_{II}(2n|0) & \hat{p}_{II}(2n|2b) & \hat{p}_{II}(2a|2b) \\
D_0^{(0)} & yes & \hat{q}_{II}(0|2n) & \hat{q}_{II}(2n|0) & \hat{q}_{II}(2n|2b) & \hat{q}_{II}(2a|2b) \\
D_0^{(0)} & yes & \hat{p}_{II}(0|2n) & \hat{p}_{II}(2n|0) & \hat{p}_{II}(2n|2b) & \hat{p}_{II}(2a|2b) \\
D_0^{(0)} & yes & \hat{p}_{II}(0|2n) & \hat{p}_{II}(2n|0) & \hat{p}_{II}(2n|2b) & \hat{p}_{II}(2a|2b) \\
D_0^{(0)} & yes & \hat{p}_{II}(0|2n) & \hat{p}_{II}(2n|0) & \hat{p}_{II}(2n|2b) & \hat{p}_{II}(2a|2b) \\
\end{array}
\]

7.4 The double extensions of \( a = \mathfrak{gl}^{(1)}(n|n) \)

\[
\begin{array}{c|c|c|c|c|c|c}
D & A(\mathfrak{gl}^{(1)}(2n|2n)) & A(\mathfrak{gl}^{(1)}(2n+1|2n+1)) \\
\hline
D_0^{(0)} & 0 & \hat{b}(2n|2n) & 0 \\
D_0^{(0)} & 0 & \hat{b}(2n|2n) & 0 \\
D_0^{(0)} & 0 & \hat{b}(2n|2n) & 0 \\
\end{array}
\]

7.4.1. Lemma. 1) We have \( \text{rk } D^{(2n-2)} = 2n \) in \( \hat{p}_{o} \), and there is no such element in both \( \hat{p}_{o} \) and \( \hat{p}_{o} \). The rank of \( D_0^{(0)} \) in \( \hat{p}_{o} \) is \( 2^{2n-1} \neq 2n \) for \( n > 1 \), and there is no such element in \( \hat{p}_{o} \), so \( \hat{p}_{o} \) and \( \hat{p}_{o} \) are non-isomorphic double extensions and each of them is not isomorphic to \( \hat{p}_{o} \).

2) Because \( \text{rk ad}_p = \text{rk ad}_q = 2^{2n-1} - 1 \) in \( b(2n|2n) \) and no element in \( b(2n|2n) \) has odd rank, \( b(2n|2n) \neq b(2n|2n) \).

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