A Note on Order and Index Reduction for Descriptor Systems

Martin J. Corless and Robert N. Shorten

Abstract—We present order reduction results for linear time invariant descriptor systems. Results are given for both forced and unforced systems as well methods for constructing the reduced order systems. Our results establish a precise connection between classical and new results on this topic, and lead to an elementary construction of quasi-Weierstrass forms for a descriptor system. Examples are given to illustrate the usefulness of our results.

Index Terms—Descriptor systems, system order reduction, quasi-Weierstrass form

I. INTRODUCTION

Descriptor systems have been widely studied in the mathematics and engineering literature for several decades [2], [3], [4]. Recently, they have also become very popular in the mainstream control engineering literature, especially in the context of switching and hybrid dynamical systems [3], [6], [7], [8], [9], [10], [11], motivated in part, by the fact that descriptor systems provide a natural framework to model and analyse many dynamic systems with algebraic constraints (for example, a mechanical system with coordinate constraints) [13]. Formally, a descriptor characterization of a dynamical system consists of a combination of differential equations and algebraic equations, that coupled together describe the dynamics of the system under study. Even though this formalisation is convenient for many physical and man-made dynamic systems, the analysis of such systems requires bespoke techniques when compared with conventional systems. Our interest in this paper concerns linear time invariant descriptor systems, and methods for characterising the qualitative properties of these systems in terms of lower order systems. As a special case we also consider reduction methods that yield a standard system; that is, a system described only by standard differential equations and no algebraic equations. Our motivation is deriving these tools is that reduced order characterisations are often useful than the corresponding original descriptor characterisations due to their compatibility with the broad portfolio of existing results in Systems Theory which characterise the properties of ordinary differential equations. This work builds on our previous works on the topic. Order reduction ideas based on full rank decompositions were first introduced in [15] and [16]. These results were developed further in [12] and [13]. The present paper extends our prior work fundamentally in a number of ways. In the original work, one could (in one reduction step) only reduce a system to one whose index was one less than the index of the original system; here one can reduce all the way to an index zero system (standard system) in one step. Second, systems with inputs are considered. Third, missing links to established and classical descriptor results are established, revealing the utility of the approach advocated here. Finally, new reduced order forms are also introduced that are not considered in these previous papers.

Specifically our contributions may be summarized as follows.

(a) We consider first systems with no input. It is known that, subject to some constraints, such a system can be equivalently represented by a lower order standard system. Since the order of a standard system cannot be reduced, this is the lowest order that can be achieved for the original descriptor system. There are situations where it is advantageous to obtain an equivalent system description of lower order but not necessarily of minimal lower order. This occurs, for example in analyzing switching linear descriptor systems [12], [13]. Our first set of results is to demonstrate how one can readily obtain various equivalent system descriptions of lower order for a linear descriptor system.

(b) We also give a simple procedure to reduce a descriptor system to an equivalent standard system.

Note that, although there are many results in the literature for reducing a descriptor system to a standard system (see [1], for one of the earliest results) there are very few results on reducing to a lower order descriptor system, with the notable exception of [12], and the results therein reduce the index of the system by one. The results in this present paper allow one to reduce a descriptor system to a lower order system of any lower index.

(c) In the second part of our paper we consider systems with inputs and obtain two coupled reduced order systems associated with the original system in descriptor form. These two systems lead directly to the celebrated quasi-Weierstrass form [14] of the original system, but in an elementary manner when compared with existing literature. Recall the quasi-Weierstrass form gives rise a form that consists of two subsystems which together are equivalent to the original system. One of these subsystems is a standard system whereas the other is very special type of descriptor system called a pure descriptor system. As stated our derivation provides a simple way of constructing a quasi-Weierstrass form for a linear descriptor system, and relates our approach to existing mathematical results on Descriptor systems.

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Our paper is structured as follows. We present preliminary material in Section 2. Our main results are derived in Sections 3 and 4. Examples illustrating the utility of our results are also given on Section 4.

II. PREAMBLE - DESCRIPTOR SYSTEMS

Consider a linear time-invariant (LTI) system described by the differential algebraic equation (DAE)

$$ E\dot{x} = Ax $$

where $x(t) \in \mathbb{C}^n$ is the system state at time $t \in \mathbb{R}$ and $E, A \in \mathbb{C}^{n \times n}$. When $E$ is nonsingular, this system is also described by the standard system $\dot{x} = E^{-1}Ax$. If $E$ is singular, then both algebraic equations and differential equations describe the behavior of the system, and the system is known as a descriptor system.

We say that system (1) or $(E, A)$ is regular if the polynomial det$(sE - A)$ is nonzero, that is, there exists $\lambda \in \mathbb{C}$ such that $\lambda E - A$ is nonsingular. For such a scalar $\lambda$, we can rewrite system (1) as

$$ E\dot{x} = (A - \lambda E)x + \lambda Ex $$

and pre-multiply by $(A - \lambda E)^{-1}$ to obtain

$$ \dot{x} = (I + \lambda F)x $$

where

$$ F := (A - \lambda E)^{-1}E $$

We will find this system description useful for several purposes, in particular for reducing system (1) to a system of lower order, that is, lower state dimension.

The consistency space $\mathcal{C} = \mathcal{C}(E, A)$ for system (1) or $(E, A)$ is the set of all initial states $x_0 \in \mathbb{C}^n$ for which equation (1) has a classical (that is, differentiable) solution $x(\cdot) : [0, \infty) \to \mathbb{C}^n$ with the initial condition $x(0) = x_0$. We can characterize this with the following concept. The index of a matrix $F \in \mathbb{C}^{n \times n}$ is the smallest nonnegative integer $k^*$ for which $\text{rank}(F^{k+1}) = \text{rank}(F^k)$ where rank denotes the rank of a matrix; this index is zero for a nonsingular matrix. Note that the index of $F$ is also the smallest nonnegative integer $k^*$ for which $\mathcal{R}(F^{k+1}) = \mathcal{R}(F^k)$ where $\mathcal{R}$ denotes the image or range of a matrix. Also $\mathcal{R}(F^k) = \mathcal{R}(F^{k^*})$ for all $k \geq k^*$ and $\mathcal{R}(F^k) \supset \mathcal{R}(F^{k^*})$ for $k \leq k^*$. If $F^{k^*} = 0$ we say that $F$ is nilpotent.

**Remark 1** It can readily be shown that, for any $k = 0, 1, 2, \ldots$, the subspace $\mathcal{R}(F^k)$ is the same for all $\lambda$ for which $\lambda E - A$ is nonsingular [2]; hence the index of $F$ is the same for all $\lambda$ for which $\lambda E - A$ is nonsingular; we call this the index of system (1) or $(E, A)$. It is also shown in [2] that $\mathcal{C}(E, A) = \mathcal{R}(F^k)$ for $k \geq k^*$ where $k^*$ is the index of $F$ and for all $\lambda$ for which $\lambda E - A$ is nonsingular.

**Remark 2** Since $\mathcal{R}(F^{k+1}) = \mathcal{R}(F^k) = \mathcal{C}$ we see that $F'\mathcal{C} = \mathcal{C}$. This implies that $F$ is a one-to-one mapping of $\mathcal{C}$ onto itself; hence the kernel of $F$ and $\mathcal{C}$ intersect only at zero. Note that $\mathcal{C} = \{0\}$ if and only if $F$ is nilpotent; in this case we say that the system is a pure descriptor system and the only differentiable solution is the zero solution $x(t) \equiv 0$. If $\mathcal{C} \neq \{0\}$, we let $G$ be the inverse of the map $F$ restricted to $\mathcal{C}$, that is, $GFx = x$ and $FGx = x$ when $x \in \mathcal{C}$. When the solution $x(t)$ is in $\mathcal{C}$ for all $t$ then so is $\dot{x}(t)$; hence multiplying (2) by $G$ results in

$$ \dot{x} = \hat{A}x $$

where $\hat{A} = G + \lambda I$. Also multiplying (4) by $F$ results in (2). Thus (4) is equivalent to (2); hence (4) and (1) are equivalent. Thus the restriction of the descriptor system to its consistancy space is equivalent to the standard system (1) where $x(t)$ is in $\mathcal{C}$.

III. REDUCING A DESCRIPTOR SYSTEM

Our first main result, Lemma 3, shows how to simply reduce system (1) to an equivalent system of lower order and lower index. It requires the following concepts and lemmas. For a full column rank matrix $X$, the matrix $X^\dagger$ denotes any left-inverse of $X$, that is, it satisfies $X^\dagger X = I$

where $I$ is an identity matrix. For example, $X^\dagger = (X'X)^{-1}X'$.

We need the following result for an arbitrary $n \times m$ matrix $F$.

**Lemma 1**: Suppose $F \in \mathbb{C}^{n \times n}$, $F^k \neq 0$ for some integer $k \geq 1$ and $X$ is a matrix of full column rank whose range equals that of $F^k$. Then, for any integer $l \geq 0$,

$$ F^lX = XF^l \quad \text{where} \quad F = X^\dagger FX $$

**Proof**: Clearly it holds for $l = 0$. We now prove, by induction that it holds for any $l \geq 1$. We first show that (5) holds for $l = 1$, that is, $FX = X\hat{F}$. By assumption, $\mathcal{R}(X) = \mathcal{R}(F^k)$; thus

$$ \mathcal{R}(FX) = \mathcal{R}(F^{k+1}) \subset \mathcal{R}(F^k) = \mathcal{R}(X) $$

that is $\mathcal{R}(FX) \subset \mathcal{R}(X)$. So $FX = X\hat{F}$ for some matrix $\hat{F}$. Multiplying both sides of this equation by any left-inverse $X^\dagger$ of $X$ yields $\hat{F} = X^\dagger FX$. Now suppose that (5) holds for some integer $l \geq 1$. Then

$$ F^{l+1}X = FF^lX = FX\hat{F}^l = X\hat{F}\hat{F}^l = X\hat{F}^{l+1} $$

Thus, (5) holds with $l$ replaced with $l + 1$. By induction, it holds for all $l \geq 1$. QED

The following decomposition is useful in some of the results of this paper. Consider any non-zero matrix $M \in \mathbb{C}^{n \times n}$. A pair of matrices $(X, Y)$ is a full rank decomposition of $M$ if $X$ and $Y$ have maximum column rank and

$$ M = XY' $$

If $r$ is the rank of $M$ then $r \leq n$ and $X, Y \in \mathbb{C}^{n \times r}$. Clearly, $X$ and $M$ have the same range while $Y$ and $M'$ have the same range. Also,

$$ X = MY' \quad \text{and} \quad Y = M'X' $$

**Lemma 2**: Suppose $F \in \mathbb{C}^{n \times n}$, $F^k \neq 0$ for some integer $k \geq 1$ and $X, Y$ is a full rank column rank decomposition of $F^k$. Then,

$$ X^\dagger FX = Y'F'Y' =: \hat{F} $$

where $\hat{F} = (X'X)^{-1}X'F$. QED
and for any integer \( l \geq 0 \),
\[
F^{l+k} = XF^lY'
\]
(9)

Proof. Since \((X,Y)\) is a full rank column rank decomposition of \( F^k \),
\[
F^k = XY'
\]
(10)
where \( X,Y \) are full column rank matrices and the range of \( X \) equals that of \( F^k \). Thus \( X = F^kY' \) and
\[
X^\dagger FX = X^\dagger F^kY'Y'' = X^\dagger XY'Y'' = Y'FY' \]

Consider any integer \( l \geq 0 \). According to Lemma [1], \( XF^l = F^lX \); hence
\[
\tilde{F}^l = X^\dagger F^lX
\]
(11)
Post-multiplying both sides of (11) by \( Y' \) and using (10):
\[
\tilde{F}^lY' = X^\dagger F^lXY' = X^\dagger F^kF^l = X^\dagger F^{l+k}
\]
(12)
Since \( \mathcal{R}(F^{l+k}) \subset \mathcal{R}(F^k) \subset \mathcal{R}(X) \), there exists a matrix \( Y_l \) such that
\[
F^{l+k} = XY_l
\]
(13)
hence \( X^\dagger F^{l+k} = X^\dagger XY_l = Y_l' \). It now follows from (12) that \( Y_l' = \tilde{F}^lY' \). Combining this with (13) yields the desired result, \( F^{l+k} = XF^lY' \). QED

We now obtain our first reduction result.

Lemma 3: Consider a regular descriptor system described by (1) and any \( \lambda \in \mathbb{C} \) for which \( \lambda E - A \) nonsingular. For any integer \( k \geq 1 \) with \( F^k \neq 0 \), where \( F \) is given by (3), let \( X \) be any matrix of full column rank whose range equals that of \( F^k \). Then, \( x(\cdot) \) is a differential to \( \tilde{F}^l \) if and only if
\[
x = Xz
\]
(14) and \( z(\cdot) \) is a differential to
\[
\tilde{F}z = (I + \lambda F)z
\]
(15)
where
\[
\tilde{F} := X^\dagger FX
\]
(16)
Moreover \( z = X^\dagger x \) and the index of (15) is \( \max\{k^* - k, 0\} \) where \( k^* \) is the index of (1).

Proof. When \( x(\cdot) \) is a differential to (11) we have \( x(t) \in \mathcal{C} \) where \( \mathcal{C} \) is the consistency space of \((E,A)\). Since \( \mathcal{C} \subset \mathcal{R}(F^k) \) it follows that \( \mathcal{C} \subset \mathcal{R}(X) \). Hence, \( x = Xz \) and \( z \) is uniquely given by \( z = X^\dagger x \). As shown earlier, \( x(\cdot) \) is a differential to (11) if and only if it a solution of (4) which is equivalent to
\[
FXz = (I + \lambda F)Xz
\]
(17)
It follows from Lemma [1] that \( FX = XF \) where \( F \) is given by (16). Thus (15) is equivalent to
\[
X\tilde{F}z = X(I + \lambda \tilde{F})z
\]
(18)
Since \( X \) has maximum column rank, (13) is equivalent to (15).
To obtain the index of (15), choose any matrix \( Y \) such that \((X,Y)\) is a full rank decomposition of \( F^k \). Recall from Lemma [2] that for any \( l \geq 0 \), \( F^{l+k} = X\tilde{F}^lY' \). Since \( X \) has maximum column rank the matrices \( F^{l+k} \) and \( X\tilde{F}^lY' \) have the same rank. Since \( Y' \) has maximum row rank the matrices \( \tilde{F}^lY' \) and \( FX \) have the same range; hence \( F^{l+k} \) and \( F^l \) have the same rank. It now follows that if \( k \leq k^* \) then the index \( l^* \) of (15) is \( k^* - k \) and if \( k > k^* \) we have \( l^* = 0 \). QED

Remark 3 For a descriptor system with singular \( E \), the rank of the matrix \( F \) is less than \( n \); thus the rank of \( F^k \) and, hence, \( X \) is less than \( n \). Since \( X \) has maximum column rank this tells us that the state \( z \) of the new system in (17) is in \( \mathbb{C}^m \) with \( m < n \). Hence (15) is an equivalent reduced order version of the original system (11).

Example 1: To illustrate Lemma 3 consider a descriptor system described by (1) with
\[
E = \begin{pmatrix} 2 & -2 & -2 \\ 2 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}
\]
Since \( A \) is non-singular, we can consider \( \lambda = 0 \); hence
\[
F = A^{-1}E = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}
\]
The rank of \( F \) is two whereas that of
\[
F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & 0 \end{pmatrix}
\]
(19)
and \( F^3 \) is one. Thus this is an index two system whose consistency space is the range of \( F^2 \). Considering \( k = 1 \), the full column rank matrix
\[
X = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 \end{pmatrix}
\]
has the same range as that of \( F \). Hence this system can be described by \( x = Xz \) and \( \tilde{F}z = z \) where \( z = X^\dagger x \) and
\[
\tilde{F} = X^\dagger FX = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix}
\]
which is an index one matrix. Considering \( k = 2 \), the range of full column rank matrix
\[
X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
(20)
is the same as that of \( F^2 \) and is the consistency space. Here \( \tilde{F} = X^\dagger FX = -2 \). Hence the original descriptor system can be described by the standard system
\[
-2z = z
\]
and \( x = Xz = [0 \ z \ z]^T \). Also \( z = X^\dagger x = (x_1 + x_2)/2 \).

We now obtain the following result for an arbitrary \( n \times n \) matrix \( F \). This shall be used to obtain another reduction result; namely, Lemma 5
Lemma 4: Suppose that $F \in \mathbb{C}^{n \times n}$ has index $k^*$ and $Y$ is a matrix whose range is the same as that of $F^k$ for some integer $k \geq 1$. Then, $Y'F^l x \neq 0$ for all nonzero $x \in \mathcal{R}(F^k)$ and all nonnegative integers $l$.

Proof. Consider any nonnegative integer $l$. Suppose that $Y'F^l x = 0$ for some $x \in \mathcal{R}(F^k)$. Since the range of $Y$ is the same as that of $F^k$, $F^k = Y \hat{X}'$ for some matrix $\hat{X}$ and $F^k = XY'$. Hence,

$$0 = \hat{X}Y'F^l x = F^k F^l x = F^{k+l} x$$  (21)

Since $F$ has index $k^*$,

$$\mathcal{R}(F^k) = \mathcal{R}(F^{k+l+k^*}) = F^{k+l} \mathcal{R}(F^k)$$

thus, $F^{k+l} \mathcal{R}(F^k) = \mathcal{R}(F^k)$. This implies that $F^{k+l}$ is a one-to-one mapping of $\mathcal{R}(F^k)$ onto itself; hence the kernel of $F^{k+l}$ and $\mathcal{R}(F^k)$ intersect only at zero. Now (21) implies that $x = 0$.

We now obtain a second reduction result.

Lemma 5: Consider a regular descriptor system described by (1) and any $\lambda \in \mathbb{C}$ for which $\lambda E - A$ nonsingular. For any integer $k \geq 1$ with $F^k \neq 0$, where $F$ is given by (3), let $Y$ be any matrix of maximum column rank whose range is the same as that of $F^k$. Then, there is a matrix $H$ such that $x(\cdot)$ is a differentiable solution to (1) if and only if

$$x = H z$$  (22)

and $z(\cdot)$ is a differentiable solution to

$$\hat{F} \dot{z} = (I + \lambda \hat{F}) z$$  (23)

where

$$\hat{F} = Y' F^\dagger Y'$$  (24)

Moreover

$$z = Y' x$$  (25)

and the index of (23) is max$\{k^*-k,0\}$ where $k^*$ is the index of (1).

Proof. As shown earlier, $x(\cdot)$ is a differentiable solution to (1) if and only if it is a solution of (2). Introducing $\hat{x} = F^k x$ we obtain that

$$\dot{\hat{x}} = (I + \lambda F) \hat{x}$$  (26)

Using Lemma 3, $\hat{x}(\cdot)$ is a differentiable solution to (26) if and only if

$$\hat{x} = X z$$  (27)

and $z(\cdot)$ is a differentiable solution to

$$\hat{F} \dot{z} = (I + \lambda \hat{F}) z$$  (28)

where

$$\hat{F} = X^\dagger F X = Y' F Y'^\dagger$$

The second equality comes from Lemma 2. The index of (28) is max$\{k^*-k,0\}$ where $k^*$ is the index of (1) and

$$z = X^\dagger \hat{x} = X^\dagger F^k x = X^\dagger X Y' x = Y' x$$

Lemma 3 tells us that the kernel of $Y'$ and $\mathcal{C}$ intersect only at zero, there is a unique matrix $H$ such that (22) holds. QED

Example 2: To illustrate Lemma 5 recall the system in Example 1. We see that

$$Y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}$$

is a full column rank matrix whose range is the same as that of $F'$. Hence this system can be described by $\hat{F} \dot{z} = z$ where $z = Y' x$ and

$$\hat{F} = Y' F Y'^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

which is an index one matrix. Since $z_2 = x_2$ and $x$ must be in the range of the matrix $X$ in (20) (the consistency space), we must have $x = [0 \ z_2 \ z_3]^T$. Considering $k = 2$ the full column rank matrix

$$Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

has the same range as that of $F'^2$. Here $\hat{F} = Y' F Y'^\dagger = -2$. Hence the original descriptor system can be described by the standard system $-2 \dot{z} = z$ and $\dot{z} = Y' x = x_2$. Since $x_2 = z$ and $x$ must be in the range of the matrix $X$ in (20), we must have $x = [0 \ z \ z]^T$.

Remark 4 Suppose that $(X,Y)$ is a full rank decomposition of the matrix $F$ in (3). Then $F = X Y'$. Considering the result in Lemma 3 for $k = 1$, we see that the matrix $\hat{F}$ in (16) is given by

$$\hat{F} = X^\dagger F X = X^\dagger X Y' X = Y' X$$

This along with Lemma 3 and and $\lambda = 0$ captures a corresponding result in (12) when $A$ is nonsingular.

Application to switching linear systems

The above results can be useful in reducing a switching descriptor system to a lower order system. To illustrate, consider a switching descriptor system described by

$$E_{\sigma(t)} \dot{x} = A_{\sigma(t)} x$$  (29)

where $\sigma(t) \in \{1,2,\ldots,N\}$ and $E_i A_i \in \mathbb{C}^{n \times n}$ for $i = 1,2,\ldots,N$. Suppose that for some $\lambda \in \mathbb{C}$ and for each $i$ there exists $k_i$ such that the range of $F_i^{k_i}$ is the same for all $i$ where $F_i = (A_i - \lambda E_i)^{-1} E_i$. Recalling Lemma 3 let $X$ be any matrix of maximum column rank whose range is the same as that of $F_i^{k_i}$ for all $i$. Then, $x(\cdot)$ is a differentiable solution to (29) if and only if $x = X z$ and $z(\cdot)$ is a differentiable solution to the lower order switching system

$$\hat{F}_{\sigma(t)} \dot{z} = (I + \lambda \hat{F}_{\sigma(t)}) z$$  (30)

where $\hat{F}_i := X^\dagger F_i X$. Moreover $z = X^\dagger x$. 


IV. Equivalent Standard Systems

We have already seen that (1) is equivalent to a standard system on the consistency space. Here we provide simple characterizations of reduced order standard systems which are equivalent to (1). Lemma 3 leads to the following result which yields an equivalent lower order standard system for the original descriptor system (1).

**Corollary 1:** Consider a regular non-pure descriptor system described by (1) and any \( A \in \mathbb{C} \) for which \( A - \lambda E \) nonsingular. With \( F \) given by (3) let \( X \) be any full column rank matrix whose range is the same as that of \( F^k \) for some integer \( k \geq k^* \) where \( k^* \) is the index of \( (E,A) \). Then \( X^tFX \) is nonsingular and \( x(\cdot) \) is a differentiable solution to (1) if and only if
\[ x = Xz \] (31)
and \( z(\cdot) \) is a differentiable solution to
\[ \dot{z} = \tilde{A}z \] (32)
where
\[ \tilde{A} = (X^tFX)^{-1} + \lambda I \] (33)
Moreover
\[ z = X^tx \] (34)

When \( A \) is invertible, one can choose \( \lambda = 0 \). In this case, we obtain the following simpler expressions:
\[ F = A^{-1}E, \quad \tilde{A} = (X^tA^{-1}EX)^{-1} \] (35)
Lemma 5 leads to the following result which yields another equivalent lower order standard system for the original descriptor system (1).

**Corollary 2:** Consider a regular non-pure descriptor system described by (1) and any \( A \in \mathbb{C} \) for which \( \lambda E - A \) nonsingular. With \( F \) given by (3), let \( Y \) be any matrix of maximum column rank whose range is the same as that of \( F^k \) for some integer \( k \geq k^* \) where \( k^* \) is the index of \( (E,A) \). Then \( Y^tFY^t \) is nonsingular and \( x(\cdot) \) is a differentiable solution to (1) if and only if
\[ x = Y^tY^t^*z \] (36)
and \( z(\cdot) \) is a differentiable solution to
\[ \dot{z} = \tilde{A}z \] (37)
where
\[ \tilde{A} = (Y^tFY^t)^{-1} + \lambda I \] (38)
Moreover
\[ z = Y^tx \] (39)

**Proof.** We just need to show that \( H = Y^t^* \). Since \( Y^tx \neq 0 \) holds for all \( x \) in the consistency space \( \mathcal{C} \) of (1), it follows that \( \{ z : z = Y^tx \text{ and } x \in \mathcal{C} \} = \mathcal{C}^m \) where \( m \) equals the dimension of \( \mathcal{C} \) and the number of columns of \( Y \). Using (25) and (22) we now obtain that \( z = Y^tHz \) for all \( z \in \mathcal{C}^m \). Hence \( Y^tH = I \) from which it follows that \( H = Y^t^* \). QED

When \( A \) is invertible, one can choose \( \lambda = 0 \). In this case, we have the simpler expressions:
\[ F = A^{-1}E, \quad \tilde{A} = (Y^tA^{-1}EY^t)^{-1} \] (40)
The following result leads to further expressions for \( \tilde{A} \).

**Lemma 6:** Suppose that \( F \in \mathbb{C}^{n \times n} \) is a matrix which is not nilpotent, has index \( k^* \) and \( X \) and \( Y \) are full column rank matrices whose ranges are the same as that of \( F^k \) and \( F^{k^*} \), respectively, for some integer \( k \geq k^* \). Then, \( Y^tFY^t \) is nonsingular for every nonnegative integer \( l \).

**Proof.** Consider any nonnegative integer \( l \) and suppose that \( Y^tFY^tXz = 0 \). Since the vector \( Xz \) is in \( \mathcal{S}(F^k) \) and \( k \geq k^* \), this vector is in \( \mathcal{S}(F^{k^*}) \). It now follows from Lemma 4 that \( Xz = 0 \).

Since \( X \) has maximum column rank we obtain that \( z \) is zero. With \( Y \) and \( X \) having the same dimensions, \( Y^tFY^tX \) is square. Thus \( Y^tFY^tX \) is nonsingular. QED

**Remark 5** Consider a non-pure system described by (1). Then \( F^k \neq 0 \) for every nonnegative integer \( k \) where \( F \) is given by (3). Suppose that \( X \) and \( Y \) are full column rank matrices whose ranges are the same as that of \( F^k \) and \( F^{k^*} \), respectively, where \( k \geq k^* \) and \( k^* \) is the index of \( F \). Then, the above result tells us that \( Y^tX \) is invertible. Since \( (Y^tX)^{-1}Y^tX = I \), a left-inverse of \( X \) is given by
\[ X^{-1} = (Y^tX)^{-1}Y^t \] (41)
Hence
\[ X^tFX = (Y^tX)^{-1}Y^tFX \] (42)
and the matrix in (32) is given by
\[ \tilde{A} = (Y^tFX)^{-1}Y^tX + \lambda I \] (43)
Since, \( (Y^tX)^{-1}X^tY = I \), a left-inverse of \( Y \) is given by
\[ Y^t = (Y^tX)^{-1}X^t \] (44)
Hence \( Y^tFY^t = Y^tFX(Y^tX)^{-1} \) and the matrix in (32) is given by
\[ \tilde{A} = Y^tX(Y^tFX)^{-1} + \lambda I \] (45)

An equivalent full order standard system on the consistency space: Using the results in Corollary 1 or Corollary 2 we can obtain a standard system which is equivalent to the original descriptor system and has the same state as the original system.

**Lemma 7:** Consider a non-pure system described by (1).

Suppose that \( X \) and \( Y \) are full column rank matrices whose ranges are the same as that of \( F^k \) and \( F^{k^*} \), respectively, where \( k \geq k^* \) and \( k^* \) is the index of \( F \). Then, \( Y^tX \) and \( Y^tFX \) are nonsingular and \( x(\cdot) \) is a differentiable solution to (1) if and only if \( x(t) \) is in the range of \( X \) and
\[ \dot{x} = \tilde{A}x \] (46)
where
\[ \tilde{A} = X(Y^tFX)^{-1}Y^t + \lambda X(Y^tX)^{-1}Y^t \] (47)

**Proof.** Lemma 6 tells us that \( Y^tX \) and \( Y^tFX \) are nonsingular. It follows from (31), (32) and (34) that the behavior of \( x \) is
described by (46) with \( \hat{A} = XAX^† \). Recalling (43) and (42) we see that
\[
\hat{A} = X(YFX)^†(YFX)(YFX)^†Y' + \lambda X(YFX)^†Y' = X(YFX)^†Y' + \lambda X(YFX)^†Y'
\]
One obtains the same result using (36), (37) and (39) along with (44) and (43). QED

When \( E \) is invertible, consider any \( \lambda \) for which \( A - \lambda E \) is invertible. In this case the index \( k' \) of \( F = (A - \lambda E)^{−1}E \) is zero. Hence \( X \) and \( Y \) are invertible one can readily show that \( \hat{A} = E^{-1}A \). When \( A \) is invertible, one can choose \( \lambda = 0 \). In this case, \( F = A^{-1}E \) and
\[
\hat{A} = X(YA^{-1}EX)^{-1}Y'
\]

V. SYSTEMS WITH INPUTS
We now consider systems with inputs described by
\[
E\dot{x} = Ax + Bu
\]
where \( u(t) \in \mathbb{C}^m \) is the system input and \( B \in \mathbb{C}^{n \times m} \). When \( u = 0 \), a classical solution to (49) is constrained to the consistency space associated with (49). When \( u \neq 0 \) this is not necessarily the case and we need further analysis. When \((E,A)\) is regular, there exists \( \lambda \in \mathbb{C} \) such that \( A - \lambda E \) is nonsingular and, following the derivation of (2), we see that (49) is equivalent to
\[
F\dot{x} = (I + \lambda F)x + Gu
\]
where \( F \) is given by (3) and
\[
G := (A - \lambda E)^{−1}B
\]
Using the following corollary to Lemma 1 we can obtain our first result, Lemma 8.

**Corollary 3:** Suppose \( F \in \mathbb{C}^{n \times n} \), \( F^k \neq 0 \) for some integer \( k \geq 1 \) and \( Y \) is a matrix of full column rank whose range equals that of \( F^k \). Then, for any integer \( l \geq 1 \),
\[
YF^l = F^lY'
\]
where \( F = YFY' \) for \( l = 1, 2, \ldots \) where \( \hat{F} = YFY' \).

**Lemma 8:** Consider a regular descriptor system described by (49) and any \( \lambda \in \mathbb{C} \) for which \( \lambda E - A \) nonsingular. For any integer \( k \geq 1 \) with \( F^k \neq 0 \), where \( F \) is given by (3), let \( Y \) be any matrix of maximum column rank whose range is the same as that of \( F^k \). Suppose \( x(\cdot) \) is any differentiable solution to (49) and let
\[
z_1 = Y'x
\]
Then \( z_1(\cdot) \) is a differentiable solution to
\[
\hat{F}z_1 = (I + \lambda \hat{F})z_1 + \tilde{G}_1u
\]
where
\[
\hat{F} = YFY', \quad \tilde{G}_1 = Y'G
\]

**Proof.** As shown above, \( x(\cdot) \) is a differentiable solution to (49) if and only if it a solution to (50). Hence
\[
YF'\dot{x} = Y'(I + \lambda F)x + Y'Gu
\]
Corollary 3 tells us that \( Y'F' = FY' \) where \( F = YFY' \); hence
\[
\hat{F}z_1 = (I + \lambda \hat{F})z_1 + \tilde{G}_1u
\]
where \( z_1 = Y'x \). QED

**Remark 6** If \( k \geq k^* \) in the above lemma, where \( k^* \) is the index of \((E,A)\), then \( YFY' \) is nonsingular; hence (54) is equivalent to the standard system
\[
\dot{z}_1 = \hat{A}z_1 + \hat{B}_1u
\]
where
\[
\hat{A} = (YFY')^{-1} + \lambda I, \quad \hat{B}_1 = (YFY')^{-1}Y'G
\]
With a nonzero input \( u \), the state \( x \) is not confined to the consistency space and we cannot recover \( x \) from \( z_1 \). So, now we proceed to obtain another reduced order system which contains further information on \( x \). To achieve this, need the following result for an arbitrary square matrix \( F \); this result is analogous to Lemma 1.

**Lemma 9:** Suppose that \( F \in \mathbb{C}^{n \times n} \) is singular and \( V \) is any matrix of maximum column rank whose range equals the kernel of \( F^k \) for some integer \( k \geq 1 \). Then, for any integer \( l \geq 1 \),
\[
F^lV = VN^l
\]
where \( N = V^†FV \). Moreover \( N^k = 0 \).

**Proof.** We prove this by induction. We first show that (58) holds for \( l = 1 \), that is, \( FV = VN \). If \( v \) is the range of \( V \), then \( FV = 0 \). Thus \( F^k(FV) = F(F^kV) = 0 \); this implies that \( FV \) is in the kernel of \( F^k \) and, hence, it is in the range of \( V \). Thus \( \mathcal{R}(FV) \subseteq \mathcal{R}(V) \). This means that \( FV = VN \) for some matrix \( N \). Multiplying both sides of this equation by \( V^† \) results in \( N = V^†FV \). Thus, (58) holds for \( l = 1 \). Now suppose that for some integer \( l^* \geq 1 \), (58) holds with \( l = l^* \). Then
\[
F^{l^*+1}V = FF^{l^*}V = FVN^{l^*} = VNN^{l^*} = VN^{l^*+1}
\]
Thus (58) holds with \( l = l^* + 1 \). By induction, it holds for all \( l \geq 1 \). It follows from (58) that \( F^kV = VN^k \); hence \( N^k = V^†F^kV \). Since the range of \( V \) is the kernel of \( F^k \), \( F^kV = 0 \); thus \( N^k = 0 \). QED

The following result is a simple corollary to Lemma 9.

**Corollary 4:** Suppose that \( F \in \mathbb{C}^{n \times n} \) is singular and \( W \) is any matrix of maximum column rank whose range equals the kernel of \( F^k \) for some \( k \geq 1 \). Then, for any integer \( l \geq 1 \),
\[
W^lV = N^lW
\]
where \( N = WFW^† \). Moreover \( N^k = 0 \).

Using Corollary 4 we obtain another reduced order subsystem associated with descriptor system (49).
Lemma 10: Consider a regular descriptor system described by (49) and any $\lambda \in \mathbb{C}$ for which $\lambda E - A$ nonsingular. For any integer $k \geq 1$ let $W$ be any matrix of maximum column rank whose range is the same as that of the kernel of $F^k$ with $F$ given by (3). Suppose $x(\cdot)$ is a differentiable solution to (49) and let

$$z_2 = W'x$$

(60)

Then $z_2(\cdot)$ is a differentiable solution to

$$\tilde{N}z_2 = z_2 + \tilde{B}_2u$$

(61)

where

$$\tilde{N} = (I + \lambda W'FW'')^{-1}W'FW', \quad \tilde{B}_2 = (I + \lambda W'FW'')^{-1}W'G$$

(62)

and $\tilde{N}^k = 0$.

Proof. As shown earlier, $x(\cdot)$ is a differentiable solution to (49) if and only if it a solution to (60). Hence

$$W'F\dot{x} = W'(I + \lambda F)x + W'Gu$$

From Corollary 4 $W'F = NW'$ where $N = W'FW''$ and $N^k = 0$; hence

$$Nz_2 = (I + \lambda N)z_2 + W'Gu$$

where $z_2 = W'x$. Since $N^k = 0$ the eigenvalues of $N$ are zero; hence the eigenvalues of $I + \lambda N$ are one, so $I + \lambda N$ is invertible and we obtain the desired result that

$$(I + \lambda N)^{-1}Nz_2 = z_2 + (I + \lambda N)^{-1}W'Gu$$

Since $(I + \lambda N)^{-1}$ and $N$ commute, $\tilde{N} = (I + \lambda N)^{-1}N$ and $N^k = 0$, it follows that $\tilde{N}^k = (I + \lambda N)^{-k}N^k = 0$. QED

A. Quasi-Weierstrass form

We have obtained two subsystems (54) and (61) associated with the original descriptor system (49). In order for these two subsystems to completely describe the behavior of the original system, we need the matrix $[Y\ W]$ to be nonsingular. This turns out to be the case if we consider $k \geq k^*$, the index of the original system. To prove this we first obtain the following result for an arbitrary square matrix.

Lemma 11: Suppose $F \in \mathbb{C}^{n \times n}$ is singular, is not nilpotent, has index $k^*$ and $V, W$ are any matrices of maximum column rank whose ranges are the kernels of $F^k$ and $F^{k^*}$, respectively, for some $k \geq k^*$. Then $VW$ is nonsingular.

Proof. To show that $VW$ is nonsingular, suppose $VWz = 0$. Then $Wz$ is in the orthogonal complement of the range of $V$ which equals the range of $F^{k^*}$. Hence $Wz = Y\xi$ for some vector $\xi$ where $Y$ is a full column rank matrix whose range equals that of $F^{k^*}$. Let $X$ be a full column rank matrix whose range equals that of $F^k$. Then the range of $X$ equals the orthogonal complement of the range of $W$ and $XY\xi = X'Wz = 0$. Lemma 6 tells us that $XY' = (Y'X)'$ is nonsingular. Thus $\xi$ is zero and since $W$ has maximum column rank, $z = 0$. This implies that $VW$ is nonsingular. QED

We can now prove that $T = [Y\ W]$ is invertible for $k \geq k^*$.

Lemma 12: Suppose $F \in \mathbb{C}^{n \times n}$ is singular, is not nilpotent and has index $k^*$. For any $k \geq k^*$, let $X$ and $Y$ be any matrices of maximum column rank whose ranges are the same as that of $F^k$ and $F^{k^*}$, respectively, and let $V$ and $W$ be any matrices of maximum column rank whose ranges are the kernels of $F^k$ and $F^{k^*}$, respectively. Then $[Y\ W]$ is nonsingular with inverse

$$\begin{bmatrix} (X'Y)^{-1}X' \\ (V'W)^{-1}V' \end{bmatrix}$$

(63)

Proof. Since $k \geq k^*$, where $k^*$ is the index of $F$, we know from Lemma 6 and Lemma 11 that $X'Y$ and $V'W$ are nonsingular. Since the range of $W$ is the kernel of $F^{k^*}$ we have $F^{k^*}W = 0$; hence $W'F = 0$. Since the range of $X$ is $F^k$, we must have $X'W = (W'X)' = 0$. Using the same reasoning we also have $Y'V = 0$. Hence

$$\begin{bmatrix} (X'Y)^{-1}X' \\ (V'W)^{-1}V' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

QED

Using the above lemma along with Remark 6 and Lemma 10 we obtain a decomposition of the original system into a standard system and a pure descriptor system. This decomposition is obtained in (49) and is referred to as a quasi-Weierstrass form of (49). The derivation in (14) is based on the Wong sequences presented in (17). We believe the derivation here is more elementary. Also, one may simply compute the matrices involved here by performing a singular value decomposition of $F^k$ where $k$ is greater than or equal to the index of $(E,A)$; see Remark 7 below.

Theorem 1: Consider a regular non-pure descriptor system of index $k^*$ described by (49) with $E$ singular and any $\lambda \in \mathbb{C}$ for which $\lambda E - A$ nonsingular. With $F$ given by (3) and for any integer $k \geq k^*$, let $X$ and $Y$ be any matrices of maximum column rank whose ranges are the same as that of $F^k$ and $F^{k^*}$, respectively, and let $V$ and $W$ be any matrices of maximum column rank whose ranges equal the kernels of $F^k$ and $F^{k^*}$, respectively. Then $x(\cdot)$ is a differentiable solution to (49) if and only if

$$x = X(Y'X)^{-1}z_1 + V(W'V)^{-1}z_2$$

(64)

and

$$\dot{z}_1 = \tilde{A}z_1 + \tilde{B}_1u$$

(65)

$$\dot{z}_2 = z_2 + \tilde{B}_2u$$

(66)

where $\tilde{A}$ and $\tilde{B}_1$ are given by (57) while $\tilde{N}$ and $\tilde{B}_2$ are given by (62). Moreover

$$\tilde{N}^k = 0$$

and

$$z_1 = Y'x, \quad z_2 = W'x$$
Described by the difference algebraic equation, this paper can be applied to discrete-time descriptor systems. Clearly the results of this paper are only concerned with the pair \((E,A)\) and to obtain discrete-time results just replace \(x\) with \(x(t+1)\).

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VI. Conclusions

In this paper we have obtained order and index reduction results for linear time invariant descriptor systems. Results are given for both forced and unforced systems as well methods for constructing the reduced order systems. Results are also derived that relate our results to existing results in the literature. Future work will consider developing similar results for classes of nonlinear descriptor systems.