Multilinearity of the covariances in one-dimensional models out of equilibrium

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Abstract
We consider non-equilibrium steady states of one-dimensional models of heat conduction (or wealth exchange) which are coupled to boundary reservoirs. We give necessary and sufficient conditions under which the covariances in the stationary measure are multilinear. The conditions depend only on the first two moments of the reservoir measures and the redistribution parameter. In particular we obtain the first example of a multilinear covariances in the absence of product stationary measures.

Keywords: non-equilibrium steady states, multilinearity, two-point functions

1. Introduction

One of the main issues of models in equilibrium statistical mechanics is to explain phase transitions and critical phenomena. In some aspects, especially regarding critical properties, the macroscopic behaviour does not depend much on the microscopic one which leads to universal behaviour for different systems at large scales.

In non-equilibrium systems a greater variety of phenomena occurs and in some sense they are more sensitive to microscopic information. A unified thermodynamical theory is much more difficult to obtain and different attempts were made.

The simplest situation where one can create a so-called non-equilibrium steady states (NESS) is to consider a system coupled at the boundaries in with different reservoirs creating currents. To construct such models, one usually starts from a simple bulk model to which boundary terms are added. By simple bulk model here we mean a reversible Markovian conservative dynamics (such as the simple symmetric exclusion process (SEP)) having simple stationary measures e.g. of product nature. Upon coupling such a system to different reservoirs, the nature of the stationary measures changes dramatically, i.e., NESS thus created are

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far from product, and generically show long-range correlations. In some special systems, these correlations are accessible analytically [7]. Such models could be considered as the non-equilibrium analogue of exactly solvable models in equilibrium (such as the Ising model). On the macroscale, for a large class of one-dimensional conservative Markov dynamics, it is believed [1] that the covariances in NESS are multilinear. On the microscale this multilinearity is rare and a possible indication of exact solvability e.g. in the sense of the existence of a matrix ansatz solution [4].

In [3] a class of wealth distribution models with two agents in equilibrium is considered. The authors showed under which conditions on the redistribution measure and transition operator there exist stationary product measures and duality functions. At each time step two agents are redistributing a random amount $\epsilon$ of their wealth and a constant part $\lambda \in [0, 1]$ is retained. It turns out that there exist only stationary product measures for the trivial case $\lambda = 0$ and the law of $\epsilon$ is independent of the amount of total wealth and in particular is Beta$(k, k)$ distributed.

In [1] the authors study particle systems out of equilibrium. Multilinearity of the covariances is obtained for the Kipnis–Marchioro–Presutti (KMP) model of heat conduction [1] and for the boundary driven SEP in [7]. The covariances turn out to be negatively correlated for the SEP, resp. positively correlated for KMP, while the asymptotic behaviour of the absolute value of the off-diagonal terms is essentially the same in the large $N$ limit and given by the Green’s function of the Dirichlet Laplacian. The KMP model as well as its generalization studied in [2] are special cases of the general model which we consider here. Inspired by both models of KMP type and models of wealth redistribution, our model has two essential parameters: $\lambda$ (propensity) and $\epsilon$ (redistribution parameter). For $\lambda = 0$ and $\epsilon$ uniformly distributed, we recover the KMP model.

An example where one does not obtain multilinearity for a model of heat conduction can be found in [6]. The authors study the Brownian momentum process which is weakly coupled to heat baths and in particular the NESS and its proximity to the local equilibrium measure in terms of the coupling strength. For three- and four-site systems, they obtain covariances and show it is generically not multilinear.

Here we are interested in the question under which conditions we have multilinear covariances in this class of models. The conditions will only include a relation between the moments of the reservoir laws, moments of the redistribution parameter and the constant $\lambda$. The result presents a first example where one can obtain multilinear covariances even in the absence of product stationary measures.

The rest of our paper is organized as follows. In section 2 we will provide all necessary definitions. Section 3 deals with the main result and finally we discuss in section 4 some generalizations. Detailed computations can furthermore be found in the appendix.

### 2. Notation and definitions

We consider one-dimensional interacting models on the set $\{1, \ldots, N\}$ coupled to boundary reservoirs. The sites $1, \ldots, N$ can be interpreted as particles or agents. The boundary sites are interacting with reservoirs which are represented by the ghost sites 0 and $N + 1$. Let $\Omega = S^N = [0, \infty)^N$ denote the state space of the model. Each element $\mathbf{x} = (x_1, \ldots, x_N) \in \Omega$ can be seen as e.g. the wealths or energies at vertices 1, ..., $N$, see [3, 5].
For some random $\epsilon \in [0, 1]$ and constant $\lambda \in (0, 1)$ we can define a map $T_{\lambda, \epsilon}^{k,l}: \Omega \to \Omega$ for some $k, l \in \{1, \ldots, N\}$ with $k < l$ by

$$
T_{\lambda, \epsilon}^{k,l}(\mathbf{x}) = \begin{cases} x_i, & \text{if } i \in \{1, \ldots, k-1\} \\
\lambda x_k + \epsilon(1-\lambda)(x_k + x_l), & \text{if } i = k, \\
\lambda x_l + (1-\epsilon)(1-\lambda)(x_k + x_l), & \text{if } i = N.
\end{cases}
$$

(1)

Let us remark that if $\lambda = 0$ and $\nu$ is the uniform distribution, our model corresponds to the KMP model whereas for $\lambda = 0$ and $\nu$ is Beta-distributed, our model corresponds to the generalized KMP model in [2].

At each time step some random amount of energy or wealth is exchanged. During the exchange some fixed amount $\lambda$ is kept while some random amount (modeled by $\epsilon$) is exchanged.

We remark that the map conserves the total wealth (energy) $s = x_1 + \ldots + x_N$.

Let $\nu$ be the law of $\epsilon \in [0, 1]$ which will be called the redistribution parameter. We further assume (A1) that $\nu$ is symmetric and its first moment is equal to $\frac{1}{2}$ and that it has a second moment (A2) which will be abbreviated by $\int e(1-e)\nu(de) = \alpha$. Note that it follows trivially that by (A1), $\alpha \leq \frac{1}{4}$.

The stochastic dynamics is then described as follows. In the bulk every site $2 \leq i \leq N - 1$ has an exponential clock with parameter 1. When it rings, the site exchanges energy (or wealth) with its right or left neighbour with equal probability according to rule (1). Moreover the sites $i$ and $N$ exchange only with 2 or $N - 1$ in the bulk dynamics. On the boundary the site 1 exchanges with the boundary reservoir at site 0 according to rule (1). $x_0$ is distributed according to the left reservoir measure $\mu_L$. Respectively the boundary site $N$ exchanges its wealth or energy with the boundary reservoir at site $N + 1$ where $x_{N+1}$ is distributed according to the right reservoir measure $\mu_R$.

The generator of the model can be expressed as the sum of the bulk generator describing the dynamics in the bulk and boundary generators which describe the interaction with the reservoirs

$$
\mathcal{L} = \mathcal{L}_L + \mathcal{L}_b + \mathcal{L}_R.
$$

(2)

Let $f$ be some bounded continuous function, $f: \Omega \to \mathbb{R}$, then

$$
\mathcal{L}_b(f(\mathbf{x})) = \sum_{k,l \in \{1,\ldots,N\}} p(k,l) \left[ \int_0^1 \int f \left( T_{\lambda, \epsilon}^{k,l}(x_1, \ldots, x_N) \right) \nu(de) - f(x_1, \ldots, x_N) \right],
$$

(3)

where $p(k,l)$ is the transition probability of a symmetric nearest neighbour random walk with closed boundary conditions given by
The generators of the reservoirs are defined as

\[
\mathcal{L}_L(f(\mathbf{x})) = \int_0^\infty \int_0^1 f'(\lambda x_1 + e(1-\lambda)(x_0 + x_1), x_2, \ldots, x_N) \nu(\dd \mu_L(\dd x_0)) - f(\mathbf{x})
\]

\[
\mathcal{L}_R(f(\mathbf{x})) = \int_0^\infty \int_0^1 f(x_1, \ldots, x_{N-1}, \lambda x_1 + e(1-\lambda)(x_N + x_{N+1})) \nu(\dd \mu_R(\dd x_{N+1})) - f(\mathbf{x})
\]

\[
\mu_L, \mu_R \text{ denote the distributions of the reservoirs at the ghost sites 0 resp. } N + 1. \text{ In more physical language, the evolution of the time-dependent probability density of the Markov process } f_t(\mathbf{x}) \text{ is then }
\]

\[
\frac{\partial}{\partial t} f_t(\mathbf{x}) = \mathcal{L}^* (f_t(\mathbf{x}))
\]

where \( \mathcal{L}^* \) denotes the adjoint operator of \( \mathcal{L} \) defined in (2)–(5).

We abbreviate their first moments by

\[
T_L := \int_0^\infty x_0 \mu_L(\dd x_0), \quad T_R := \int_0^\infty x_{N+1} \mu_R(\dd x_{N+1})
\]

resp. the second moments by

\[
L^2 := \int_0^\infty x_0^2 \mu_L(\dd x_0), \quad R^2 := \int_0^\infty x_{N+1}^2 \mu_R(\dd x_{N+1}).
\]

We remark that at this point we only assume that the first and second moments of the reservoir measures exist.

We call a probability measure \( \mu \in \mathcal{P}(\Omega) \) on \( \Omega \) stationary if and only if for \( f \) bounded and continuous on \( \Omega \):

\[
\int_\Omega \mathcal{L}(f(\mathbf{x})) \mu(\mathbf{x}) = 0.
\]

For \( i < j \) and \( i, j \in \{0, \ldots, N + 1\} \) let us define \( \rho_N(i, j) \) by \( \rho_N(i, j) := \int_\Omega x_i x_j \mu(\mathbf{x}) \), \( E_N(i) := \int_\Omega x_i \mu(\mathbf{x}) \) and the covariance of \( \mu \) by

\[
C_N(i, j) := \rho_N(i, j) - E_N(i)E_N(j).
\]

We set

\[
C_N(0, 0) := \int_0^\infty (x_0 - T_L)^2 \mu_0(\dd x_0)
\]
resp. for
\[ C_N(N+1,N+1) := \int_0^\infty (x_{N+1} - T_R)^2 \mu_{N+1}(dx_{N+1}). \] (11)

Note that this is equivalent to saying \( L^2 = \rho_N(0,0) \) resp. \( R^2 = \rho_N(N+1,N+1) \). For \( 0 \leq i \leq N \) and \( 1 \leq j \leq N + 1 \) we set \( C_N(0,j) := C_N(i,N+1) := 0 \).

We say that functions \((\xi(i,j))_{i,j}\) satisfy the multilinearity ansatz if and only if we can find some coefficients \( a, b, c, d, e, f, g \) such that for \( i, j \in \{0, \ldots, N + 1\} \) and \( i < j \)
\[
\begin{cases}
\xi(i, j) = a + bi + cj + dij, & \text{if } i < j, \ i = 1, \ldots, N - 1, \ j = 2, \ldots, N, \\
\xi(i, i) = e + fi + gi^2, & \text{if } i = j, \ i = 1, \ldots, N.
\end{cases}
\] (12)

3. Result

We present our main theorem.

**Theorem 3.1.** The functions \((\rho_N(i,j))_{i,j}\) are multilinear with coefficients given by
\[
a := T_L^2,
b := \frac{(T_R - T_L)(N + 1)(\lambda + 2\alpha(1 - \lambda))T_L + (1 - \lambda)(1 - 4\alpha)T_R}{(N + 1)(1 + N\lambda + 2(N - 1)(1 - \lambda)\alpha)},
c := \frac{(T_R - T_L)T_L}{N + 1},
d := \frac{(\lambda + 2\alpha(1 - \lambda))(T_L - T_R)^2}{(N + 1)(1 + N\lambda + 2(N - 1)(1 - \lambda)\alpha)},
f := \frac{((1 - 2\alpha(1 - \lambda))(T_R - T_L)\left[\left(1 + (2N + 1)\lambda + 4N\alpha(1 - \lambda)\right)T_L + (1 - 4\alpha)(1 - \lambda)T_R\right]}{(N + 1)(1 + N\lambda + 2(N - 1)(1 - \lambda)\alpha)} - \frac{(\lambda + 2\alpha(1 - \lambda))^2}{(N + 1)(1 + N\lambda + 2(N - 1)(1 - \lambda)\alpha)},
g := \frac{((1 - 2\alpha(1 - \lambda))(T_L - T_R)^2}{(N + 1)(1 + N\lambda + 2(N - 1)(1 - \lambda)\alpha)}.
\] (13)

If and only if \( L^2 = L^2(\alpha, \lambda) \) and \( R^2 = R^2(\alpha, \lambda) \) are chosen in the following way
\[
L^2 := \frac{(1 - 2\alpha(1 - \lambda))}{(\lambda + 2\alpha(1 - \lambda))} T_L^2 + \frac{\alpha(1 - \lambda)(1 - 2\alpha(1 - \lambda))(T_L - T_R)^2}{(N + 1)(1 + N\lambda + 2(N - 1)(1 - \lambda)\alpha)(\lambda + 2\alpha(1 - \lambda))},
\]
\[
R^2 := \frac{(1 - 2\alpha(1 - \lambda))}{(\lambda + 2\alpha(1 - \lambda))} T_R^2 + \frac{\alpha(1 - \lambda)(1 - 2\alpha(1 - \lambda))(T_L - T_R)^2}{(N + 1)(1 + N\lambda + 2(N - 1)(1 - \lambda)\alpha)(\lambda + 2\alpha(1 - \lambda))}.
\] (14)
In particular the covariances also multilinear and are equal to

\[
C_N(i, j) = \begin{cases}
\frac{(1 - 4\alpha)(1 - \lambda)(T_L - T_R)^2}{(1 + \lambda N + 2(N - 1)(1 - \lambda)\alpha)} i N + 1 \left(1 - \frac{j}{N + 1}\right) \\
\lambda + 2\alpha(1 - \lambda) + \frac{\alpha(1 - 2\alpha(1 - \lambda))(1 - \lambda)(T_L - T_R)^2}{(\lambda + 2\alpha(1 - \lambda))[1 + \lambda N + 2(N - 1)(1 - \lambda)\alpha](N + 1)}
\end{cases}
\]

\[
\times \left(\frac{1}{N + 1} + \frac{(1 - 4\alpha)(1 - \lambda)(T_L - T_R)^2 N}{(1 + \lambda N + 2(N - 1)(1 - \lambda)\alpha)^2}\right) \left(N + 1\right)^2; \quad 0 \leq i = j \leq N + 1.
\]

**Proof.** We are looking for expressions for \(\rho_{ij}(i, j)\) for \(i, j \in \{0, \ldots, N + 1\}\). In the bulk for \(i \leq j\), \(\rho_{ij}(i, j)\) have to satisfy the following set of equations (see appendix), we will use for convenience the abbreviation \(\rho_{ij}\) for \(\rho_{ij}(i, j)\):

\[
\begin{cases}
-2(1 - A)\rho_{i,i} + A\left(\rho_{i-1,i-1} + \rho_{i+1,i+1}\right) + B\left(\rho_{i+1,i} + \rho_{i-1,i}\right) = 0, & \text{if } i = 2, \ldots, N - 1, \\
\rho_{i+1,j} + \rho_{i-1,j} + \rho_{i,j+1} + \rho_{i,j-1} - 4\rho_{ij} = 0, & \text{if } i = 2, \ldots, N - 3, j = \ldots, N - 1, \\
C\left(\rho_{i,i} + \rho_{i+1,i+1}\right) + \frac{1}{2}\left(\rho_{i-1,i+1} + \rho_{i,i+2}\right) - (1 + B)\rho_{i,i+1} = 0, & \text{if } i = 2, \ldots, N - 2,
\end{cases}
\]

with \(A, B, C\) defined in the previous section as

\[
A = \left(\frac{1}{2} - \alpha\right)(1 - \lambda) \\
B = 1 - 2\alpha(1 - \lambda) \\
C = \frac{\lambda}{2} + \alpha(1 - \lambda).
\]

On the boundaries we need the following conditions to be satisfied. First for the left reservoir \((i = 1)\):

\[
\[
\begin{align*}
-2(1 - A)\rho_{1j} + A\left(L^2 + \rho_{2j}\right) + B\left(\rho_{1,2} + T_L E_N(1)\right) &= 0, \quad \text{if } j = 1, \\
\rho_{2,j} + T_L E_N(j) + \rho_{1,j-1} + \rho_{1,j+1} - 4\rho_{j,j} &= 0, \quad \text{if } j = 3, \ldots, N - 1, \\
C\left(\rho_{11} + \rho_{22}\right) + \frac{1}{2}\left(\rho_{15} + T_L E_N(2)\right) - (1 + B)\rho_{12} &= 0, \quad \text{if } j = 2,
\end{align*}
\]

and second on the right for \(j = N\),

\[
\begin{align*}
-2(1 - A)\rho_{NN} + A\left(\rho_{N-1,N-1} + R^2\right) \\
+ B\left(\rho_{N-1,N} + T_N E_N(N)\right) &= 0, \quad \text{if } i = N, \\
\rho_{i+1,N} + \rho_{i-1,N} + T_R E_N(i) + \rho_{i,N-1} - 4\rho_{iN} &= 0, \quad \text{if } i = 2, \ldots, N - 2, \\
C\left(\rho_{N-1,N-1} + \rho_{NN}\right) + \frac{1}{2}\left(\rho_{N-2,N} + T_R E_N(N - 1)\right) \\
- (1 + B)\rho_{N-1,N} &= 0, \quad \text{if } i = N - 1,
\end{align*}
\]

and

\[E_N(i) = T_L \left(1 - \frac{i}{N + 1}\right) + T_R \frac{i}{N + 1} \quad i = 0, \ldots, N + 1.\]  

We obtained the result by some tedious computations and its correctness can be checked directly by substituting it into the above equations (3) and (19). \(\square\)

4. Conclusion and discussion

In this note we obtained conditions under which two-point functions in the non-equilibrium case will be multilinear for some class of one-dimensional interacting models of wealth (heat) transport. In particular we present a first example of models for which the two-point function is multilinear when the stationary measure is not product.

Let us make some final remarks.

Remark 1. For \(\lambda = 0\) and \(\alpha = \frac{1}{6}\) we find the covariances obtained for the KMP model in [1].

Remark 2. For a non-degenerate distribution of the redistribution parameter \(\epsilon\), \(\alpha < \frac{1}{4}\), hence the occupation variables are always positively correlated. If \(\nu = \delta_{1/2}\) is a degenerate measure, \(\alpha = \frac{1}{4}\), and the covariances are 0.

Remark 3. We tried to apply the multilinearity ansatz in the case that the reservoirs are additionally depending on factors \(\chi_L\) and \(\chi_R\), hence the temperature profile is not entirely linear. The boundary generators are then given by
It turns out that the two-point functions will never be multilinear as long as \( \gamma_L \neq 1 \) or \( T_L \neq T_R \).

**Remark 4.** Further we tried to find out under what conditions three-point function might be multilinear. Unfortunately this problem is very complex and we could not find a general solution to this problem even for \( N = 6 \).

**Appendix**

**A.1. Generator for the two-point function**

In the following we determine the generator \( \mathcal{L} \) (see (2)) of the two-point function \( f_\mu(x) := x, x_j \). It will also depend on the one-point linear functions on the borders \( f_\mu(x) := x_j \).

Recall that the transition operator \( T_{\lambda, \epsilon}^{ij} \) was defined in (1), \( \epsilon \) satisfies (A1) and (A2) and the reservoirs have finite first and second moments.. We will distinguish three cases. Case I represents \( i = j \), case II \( |i - j| > 1 \) and finally case II \( |i - j| = 1 \).

**A.1.1. Temperature profile.** We will determine the closed form of the linear functions corresponding to the temperature profile of the system. Let \( \mu \) be a stationary measure. We calculate the density profile \( E_N(i) := \int_{\Omega} x_i \mu(dx) \) by solving

\[
\int_{\Omega} \mathcal{L}(x)_i \mu(dx) = 0
\]

for \( i = 1, \ldots, N \). Let \( i = 2, \ldots, N - 1 \) then we need to solve

\[
\int_{\Omega} \mathcal{L}(x)_i \mu(dx) = \frac{1 - \lambda}{2} (E_N(i + 1) + E_N(i - 1) - 2E_N(i)) = 0
\]

for \( i = 1 \) and \( i = N \) we have

\[
\int_{\Omega} \mathcal{L}(x)_1 \mu(dx) = \frac{1 - \lambda}{2} (E_N(2) + T_L - 2E_N(1)) = 0
\]

\[
\int_{\Omega} \mathcal{L}(x)_N \mu(dx) = \frac{1 - \lambda}{2} (T_R + E_N(N - 1) - 2E_N(N)) = 0.
\]

We can compute explicitly the closed-form expression, namely

\[
E_N(i) = T_L \left( 1 - \frac{i}{N + 1} \right) + T_R \frac{i}{N + 1} \quad i = 0, \ldots, N + 1.
\]

Note that the linear profile is the same as in [1] for the SEP and KMP models.

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A.1.2. Case i = j. We will determine the generator
\[ \mathcal{L}(f_i(x)) = \mathcal{L}_L(f_{i1}(x)) + \mathcal{L}_b(f_{ii}(x)) + \mathcal{L}_R(f_{NN}(x)) \]
in the case that \( i = j \). Let us first fix \( i = 2, \ldots, N - 1 \). First we calculate the generator in the bulk \( \mathcal{L}_b(f_{ii}(x)) \).

\[
\mathcal{L}_b(f_{ii}(x)) = \sum_{k,l \in \{1, \ldots, N\}} p(k, l) \left[ \int_0^1 f \left( T_{k,l} f_{ii}(x) \right) \nu(x) - f_{ii}(x) \right] = \left[ f_{ii} \left( T_{i,i+1} f_{ii}(x) \right) - f_{ii}(x) \right] + \frac{1}{2} \left[ f_{ii} \left( T_{i,i-1} f_{ii}(x) \right) - f_{ii}(x) \right]
\]

\[
= f_{ii} \left( T_{i,i+1} f_{ii}(x) \right) + f_{ii} \left( T_{i,i-1} f_{ii}(x) \right) - 2f_{ii}(x). \tag{24}
\]

The boundary generators are obtained in the following way, let \( i = 1 \):

\[
\mathcal{L}_L(f_{i1}(x)) = f_{i1} \left( T_{0,1} f_{i1}(x) \right) - f_{i1}(x) + f_{i2} \left( T_{1,2} f_{i1}(x) \right) - f_{i1}(x),
\]

where

\[
f_{i1} \left( T_{0,1} f_{i1}(x) \right) = \int_0^\infty \int_0^1 \left( \lambda x_1 + \epsilon(1 - \lambda)(x_1 + x_0) \right) \nu(x) \mu_L(\text{d}x_0)
\]

\[
= \lambda x_1 + \frac{1}{2}(1 - \lambda)(x_1 + T_L)
\]

\[
= \frac{1 + \lambda}{2} x_1 + \frac{1}{2}(1 - \lambda) T_L. \tag{25}
\]

and analogously for the right boundary generator \( \mathcal{L}_R \) with \( T_L \) replaced by \( T_R \) and \( x_1 \) by \( x_N \). Hence for \( i = 1, \ldots, N \) the generator \( \mathcal{L} \) acting on \( f_{ii}(x) \) is equal to

\[
\mathcal{L}(f_{ii}(x)) = \mathcal{L}_L(f_{i1}(x)) + \mathcal{L}_b(f_{ii}(x)) + \mathcal{L}_R(f_{NN}(x))
\]

with

\[
\mathcal{L}_L(f_{i1}(x)) = (1 - \lambda) \left[ -2(1 - A) f_{i1}(x) + A \left( L^2 + f_{i2}(x) \right) + B \left( f_{i2}(x) + T_L f_{i1}(x) \right) \right]
\]

\[
\mathcal{L}_b(f_{ii}(x)) = (1 - \lambda) \left[ -2(1 - A) f_{ii}(x) + A \left( f_{i-1,i}(x) + f_{i+1,i}(x) \right) + B \left( f_{i+1,i}(x) + f_{i-1,i}(x) \right) \right]
\]

\[
\mathcal{L}_R(f_{NN}(x)) = (1 - \lambda) \left[ -2(1 - A) f_{NN}(x) + A \left( f_{N-1,N}(x) + R^2 \right) + B \left( f_{N-1,N}(x) + T_R f_{NN}(x) \right) \right] \tag{26}
\]

with coefficients given by

\[
A = A(\alpha, \lambda) = \left( \frac{1}{2} - \alpha \right)(1 - \lambda)
\]

\[
B = B(\alpha, \lambda) = \lambda + (1 - 2\alpha)(1 - \lambda). \tag{27}
\]
A.1.3. Case II: \(|i-j| > 1\). We calculate \(\mathcal{L}(f_j(x))\) for \(f_j(x) = x_i x_j\). Let us first assume \(i < j\), \(i = 2, \ldots, N-2\) and \(j = 4, \ldots, N-1\). We can easily see, doing similar calculations as in the first case, that
\[
\mathcal{L}_b(f_j(x)) = f_j \left( T^{i,j+1}(x) \right) + f_j \left( T^{i-1,j}(x) \right) + f_j \left( T^{j+1,i}(x) \right) + f_j \left( T^{j-1,i}(x) \right) - 4f_j(x). \tag{28}
\]

At the left reservoir \(i = 1, 2 < j\) we have
\[
\mathcal{L}_L(f_{i,j}(x)) = f_j \left( T^{1,2}(x) \right) + f_j \left( T^{0,1}(x) \right) + f_j \left( T^{i,j+1}(x) \right) + f_j \left( T^{j-1,i}(x) \right) - 4f_j(x)
\]
and for the right boundary \(j = N, i < N - 1\)
\[
\mathcal{L}_R(f_{i,N}(x)) = f_{i,N} \left( T^{i,i+1}(x) \right) + f_{i,N} \left( T^{i-1,i}(x) \right) + f_{i,N} \left( T^{N,N+1}(x) \right) + f_{i,N} \left( T^{N-1,N}(x) \right) - 4f_{i,N}(x).
\]

It follows for \(i < j, i = 2, \ldots, N - 2\) and \(j = 4, \ldots, N - 1\) that the generator \(\mathcal{L}\) acting on \(f_j(x)\) with \(|i-j| > 1\) can be written as
\[
\mathcal{L}(f_j(x)) = \mathcal{L}_L(f_j(x)) + \mathcal{L}_b(f_j(x)) + \mathcal{L}_R(f_{i,j}(x))
\]
\[
\mathcal{L}_L(f_j(x)) = \frac{(1 - \lambda)}{2} \left[ f_{i,j}(x) + T_{i,j}f_j(x) + f_{i,j+1}(x) + f_{i,j-1}(x) - 4f_{i,j}(x) \right]
\]
\[
\mathcal{L}_b(f_j(x)) = \frac{(1 - \lambda)}{2} \left[ f_{i+1,j}(x) + f_{i-1,j}(x) + f_{i,j+1}(x) + f_{i,j-1}(x) - 4f_{i,j}(x) \right] + f_{i,j,N}(x) + f_{i,j-1}(x)T_R + f_{i,j,N-1}(x) - 4f_{i,j,N}(x).
\]

A.1.4. Case III: \(|i-j| = 1\). Finally we calculate \(\mathcal{L}(f_{i,i+1}(x))\) for the off-diagonal elements \(f_{i,i+1}(x) = x_i x_{i+1}\). In this case
\[
\mathcal{L}(f_{i,i+1}(x)) = \mathcal{L}_L(f_{i,i+1}(x)) + \mathcal{L}_b(f_{i,i+1}(x)) + \mathcal{L}_R(f_{N-i,N}(x)).
\]

Fix \(i = 2, \ldots, N - 2\). It is easy to verify that
\[
\mathcal{L}_b(f_{i,i+1}(x)) = f_{i,i+1} \left( T^{i,i+1}(x) \right) + f_{i,i+1} \left( T^{i-1,i}(x) \right) + f_{i,i+1} \left( T^{i+1,i+2}(x) \right) - 3f_{i,i+1}(x). \tag{30}
\]

At the left boundary for \(i = 1\) the generator acting on \(f_{i,i+1}(x)\) is given by
\[
\mathcal{L}_L(f_{1,i+1}(x)) = f_{1,i+1} \left( T^{1,2}(x) \right) + f_{1,i+1} \left( T^{0,1}(x) \right) + f_{1,i+1} \left( T^{2,3}(x) \right) - 3f_{1,i+1}(x)
\]
and for \(i = N - 1\)
\[
\mathcal{L}_R(f_{N-i,N}(x)) = f_{N-i,N} \left( T^{N-1,N}(x) \right) + f_{N-i,N} \left( T^{N-2,N-1}(x) \right) + f_{N-i,N} \left( T^{N,N+1}(x) \right) - 3f_{N-i,N}(x).
\]
which yields
\begin{align}
\mathcal{L}_L(f_{i+1}(\mathbf{x})) &= (1 - \lambda) \left[ C(f_{i+1}(\mathbf{x}) + f_{i+2}(\mathbf{x})) + \frac{1}{2} \left( f_{i+1}(\mathbf{x}) + T_{i+1} f_{i+1}(\mathbf{x}) \right) - (1 + B)f_{i+2}(\mathbf{x}) \right] \\
\mathcal{L}_R(f_{i-1}(\mathbf{x})) &= (1 - \lambda) \left[ C(f_{i-1}(\mathbf{x}) + f_{i-1,i+1}(\mathbf{x})) + \frac{1}{2} \left( f_{i-1,i+1}(\mathbf{x}) - (1 + B)f_{i-1}(\mathbf{x}) \right) \right] \\
\mathcal{L}_R(f_{N-1,N}(\mathbf{x})) &= (1 - \lambda) \left[ C(f_{N-1,N-1}(\mathbf{x}) + f_{NN}(\mathbf{x})) + \frac{1}{2} \left( f_{N-2,N}(\mathbf{x}) + T_{N-1} f_{N-1}(\mathbf{x}) - (1 + B)f_{N-1,N}(\mathbf{x}) \right) \right]
\end{align}

and
\[ C \equiv C(\alpha, \lambda) = \frac{\lambda}{2} + \alpha(1 - \lambda). \]

References

[1] Bertini L, de Sole A, Gabrielli D, Jona-Lasino G and Landim C 2007 Stochastic interacting particle systems out of equilibrium J. Stat. Mech. P07014

[2] Carinci G, Giardinà C, Giberti C and Redig F 2013 Duality for stochastic models of transport J. Stat. Phys. 152 657–97

[3] Cirillo P, Redig F and Ruszel W M 2014 Duality and stationary distributions of wealth distribution models J. Phys. A: Math. Theor. 47 085203

[4] Derrida B 1996 Systems out of equilibrium: some exact solvable models Proc. Statphys 19 (Xiamen, China) ed H Bailin (Singapore: World Scientific) pp 243–53

[5] Kipnis C, Marchioro C and Presutti E 1982 Heat flow in an exactly solvable model J. Stat. Phys. 27 65–74

[6] Redig F and Vafayi K 2011 Weak coupling limits in a stochastic model of heat conduction J. Math. Phys. 52 9

[7] Spohn H 1983 Long range correlations for stochastic lattice gases in a nonequilibrium steady state J. Phys. A: Math. Gen. 16 4275–91