Fuzzy Homotopy Analysis Method For Solving Fuzzy Riccati Differential Equation

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Abstract. In this work, we have used fuzzy homotopy analysis method to find the fuzzy series solution (fuzzy semi-analytical solution) of the first order fuzzy Riccati differential equation. The fuzzy approximate-analytical solutions that we obtained during this paper are accurate solutions and very close to the fuzzy exact-analytical solutions. Some numerical results are given to illustrate the method. The obtained numerical results are compared with the exact solutions.

Keywords: Fuzzy homotopy analysis method, Fuzzy Riccati differential equation, Fuzzy series solution.

1. Introduction

Many methods have been developed so far for solving fuzzy initial value problems (FIVPs) of the fuzzy ordinary differential equations. And since the FIVPs have many important applications in various types of sciences, medicine and engineering, these proposed methods included all kinds of the fuzzy numerical solutions, fuzzy exact-analytical solutions and fuzzy semi-analytical solutions. Finding different types of solutions gives more freedom in dealing with the FIVB, because the exact solution may be difficult or non-existent, then resort to the numerical solution or the semi-analytical solution.

The topic of fuzzy semi-analytical methods (fuzzy series method) for solving FIVP has been rapidly growing in recent years, whereas the fuzzy series solutions of FIVP have been studied by several authors during the past few years. Several fuzzy semi-analytical methods have been proposed to obtain the fuzzy series solution of the linear and non-linear FIVB which are mostly first order problems. Some of these methods have been proposed to obtain the fuzzy series solutions of the high order FIVB.

Fuzzy homotopy analysis method was used for the first time to solve the fuzzy differential equations in 2012. Researchers and scientists are continuing to develop this method for solving various types of the fuzzy initial value problems because it represents an efficient and effective technique.

In the following we will review some of the findings of the researchers regarding this method. In 2012, Hashemi, Malekinagad and Marasi [4] suggested and applied the Fuzzy homotopy analysis method for solving a system of fuzzy differential equations with fuzzy initial conditions. In 2013, Abu-Arqub, El-Ajou and Momani [6] studied and developed the Fuzzy homotopy analysis method to obtain the analytical solutions of the fuzzy initial value problems. In 2014, Jameel, Ghoreishi and Ismail [8] introduced and applied the Fuzzy homotopy analysis method to obtain the approximate-analytical solutions of the high order fuzzy initial value problems. In 2015, Al-Jassar [9] introduced
and presented fuzzy semi-analytical methods (including the fuzzy homotopy analysis method) to obtain the numerical and approximate-analytical solutions of the linear and non-linear fuzzy initial value problems. In 2016, Otadi and Mosleh [12] studied and developed fuzzy homotopy analysis method to obtain numerical and approximate-analytical solutions of the hybrid fuzzy differential equations with fuzzy initial conditions. As well, In 2016, Lee, Kumaresan and Ratnavelu [11] suggested a solution of the fuzzy fractional differential equations with fuzzy initial conditions by using fuzzy homotopy analysis method. In 2017, Padma and Kaliyappan [14] introduced and presented fuzzy semi-analytical methods (including the fuzzy homotopy analysis transform method) to obtain the numerical and approximate-analytical solutions of the fuzzy fractional initial value problems. In 2018, Sevindir, Cetinkaya and Tabak [16] introduced and presented fuzzy semi-analytical methods (including the fuzzy homotopy analysis method) to obtain the numerical and approximate-analytical solutions of the first order fuzzy initial value problems. Also, In 2018, Jameel, Saaban and Altaie [15] suggested and applied new concepts for solving first order non-linear fuzzy initial value problems by using fuzzy optimal homotopy asymptotic method. In 2020, Nematallah and Najafi [18] introduced and applied the fuzzy homotopy analysis method to obtain the fuzzy semi-analytical solution of the fuzzy fractional initial value problems based on the concepts of generalized Hukuhara differentiability. As well, In 2020, Ali and Ibraheem [17] studied and developed some fuzzy analytical and numerical solutions of the linear first order fuzzy initial value problems by using fuzzy homotopy analysis method based on the Padé Approximate method.

In this work, we have used fuzzy homotopy analysis method to obtain the fuzzy series solution (fuzzy approximate-analytical solution) of the first order fuzzy Riccati differential equation with real variable coefficients(real-valued function coefficients). The fuzzy semi-analytical solutions that we obtained during this work are accurate solutions and very close to the fuzzy exact-analytical solutions, based on the comparison that we introduced between the results that we obtained and the fuzzy exact-analytical solutions.

2. Basic Definitions In Fuzzy Set Theory
In this section, we will present some of the fundamental basic and primitive concepts related to the fuzzy set theory, which are necessary for understanding this subject.

Definition (1), [1] (Fuzzy Set)
Let X be a classical set of objects, called the universal set, whose generic elements are denoted by x . The membership in a classical subject A of X is often viewed as a characteristic function μA from X onto {0,1} , such that :

\[ μ_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \]  

(1)

{0,1} is called a valuation set. If a valuation set is allowed to be real interval [0,1], then A is called a fuzzy sets (which is denoted in this case by Ã) and μA(x) is the grade of membership of x in Ã. Also, it is remarkable that the closer the value of μA(x) to 1 , the more x belong to Ã. Clearly, Ã is a subset of X that has no sharp boundary. The fuzzy set Ã is completely characterized by the set of pairs:

\[ Ã = \{(x, μ_A(x)) : x \in X; 0 \leq μ_A(x) \leq 1 \} \]  

(2)

Definition (2), [7] (α - Level Set )
The α - level ( or α - cut ) set of a fuzzy set Ã labeled by Aα , is the crisp set of all x in X such that : μA(x) ≥ α ; i.e.
\[ A_{\alpha} = \{ x \in X : \mu_{\alpha}(x) \geq \alpha \ ; \ \alpha \in [0,1] \} \]

**Definition (3), [9] (Fuzzy Number)**
A fuzzy number \( \tilde{u} \) is completely determined by an ordered pair of functions \( (u(\alpha), \overline{u}(\alpha)) \), \( 0 \leq \alpha \leq 1 \), which satisfy the following requirements:

1) \( u(\alpha) \) is a bounded left continuous and non-decreasing function on \([0,1]\).
2) \( \overline{u}(\alpha) \) is a bounded left continuous and non-increasing function on \([0,1]\).
3) \( u(\alpha) \leq \overline{u}(\alpha) \), \( 0 \leq \alpha \leq 1 \).

**Remark (1), [9]:**

1) The crisp number \( u \) is simply represented by:
\[ u(\alpha) = \overline{u}(\alpha) = u \ , \ 0 \leq \alpha \leq 1 \ . \]

2) The set of all the fuzzy numbers is denoted by \( E^1 \).

**Remark (2), [13]:**
The distance between two arbitrary fuzzy numbers \( \tilde{u} = (u, \overline{u}) \) and \( \tilde{v} = (v, \overline{v}) \) is given as:
\[
D(\tilde{u}, \tilde{v}) = \left[ \int_0^1 (u(\alpha) - v(\alpha))^2 d\alpha + \int_0^1 (\overline{u}(\alpha) - \overline{v}(\alpha))^2 d\alpha \right]^{1/2} \]
\[ (E^1, D) \text{ is a complete metric space .} \]

**Definition (4), [9] (Fuzzy Function )**
A mapping \( F : T \to E^1 \) for some interval \( T \subseteq E^1 \) is called a fuzzy function or fuzzy process with non-fuzzy variable (crisp variable), and we denote \( \alpha \) - level sets by:
\[ [F(t)]_{\alpha} = [\underline{F}(t; \alpha) \ , \ \overline{F}(t; \alpha)] \]

Where \( t \in T \), \( \alpha \in [0,1] \). That is to say, the fuzzifying function is a mapping from a domain to a fuzzy set of range. Fuzzifying function and the fuzzy relation coincide with each other in the mathematical manner. We refer to \( \underline{F} \) and \( \overline{F} \) as the lower and upper branches on \( F \).

**Definition (5), [9] ( H-Difference )**
Let \( u \), \( v \in E^1 \). If there exist \( w \in E^1 \) such that \( u = v + w \) then \( w \) is called the H-difference (Hukuhara-difference) of \( u \), \( v \) and it is denoted by \( w = u \ominus v \).

In this work the \( \ominus \) sign stands always for H-difference, and let us remark that \( u \ominus v \neq u + (-1) v \).

**Definition (6), [13] ( Fuzzy Derivative )**
Let \( F : T \to E^1 \) for some interval \( T \subseteq E^1 \) and \( t_0 \in T \). We say that \( F \) is H-differential (Hukuhara-differential) at \( t_0 \) if there exists an element \( F'(t_0) \in E^1 \) such that for all \( h > 0 \) (sufficiently small), \( \exists F(t_0 + h) \ominus F(t_0) \subset F'(t_0) \) and the limits (in the metric D)
\[ \lim_{h \to 0} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0) \]
Then \( F'(t_0) \) is called the fuzzy derivative (H-derivative) of \( F \) at \( t_0 \). where \( D \) is the distance between two fuzzy numbers.

**Theorem (1), [13]:**
Let \( F : T \rightarrow E^1 \) for some interval \( T \subseteq E^1 \) be a Hukuhara differentiable function at \( t \in T \) and denote

\[
[F(t)]_\alpha = [F(t; \alpha), \bar{F}(t; \alpha)] \forall \alpha \in [0,1]
\]

Then the boundary functions \( F(t; \alpha) \), \( \bar{F}(t; \alpha) \) are both Hukuhara differentiable functions and

\[
[F'(t)]_\alpha = [F'(t; \alpha), \bar{F}'(t; \alpha)] \forall \alpha \in [0,1].
\]

### 3. Fuzzy Riccati Differential Equation
A fuzzy Riccati differential equation of the first order is in the form:

\[
x'(t) = f(t, x(t)) = q_0(t) + q_1(t)x(t) + q_2(t)x^2(t), t \in [t_0, h]
\]

with the fuzzy initial condition:

\[
x(t_0) = x_0
\]

where:

- \( x \) is a fuzzy function of the crisp variable \( t \) and \( f(t, x(t)) \) is a fuzzy function of the crisp variable \( t \)
- and the fuzzy variable \( x \) while \( x' \) is the fuzzy derivative of \( x \) and \( x_0 \) is a fuzzy number.
- As well, \( q_0(t) \), \( q_1(t) \) and \( q_2(t) \) are real valued functions of the real variable \( t \).

The main idea in solving the fuzzy Riccati differential equation is to convert it into a system of non-fuzzy (crisp) differential equations, and then solve this system by the known and commonly used methods of solving the non-fuzzy differential equations.

Now it is possible to replace (10) by the following equivalent system of the first order crisp ordinary differential equations\[13\] :

\[
x'(t) = f(t, x) = F(t, \underline{x}, \bar{x}), \underline{x}(t_0) = \underline{x}_0
\]

\[
\bar{x}'(t) = \bar{f}(t, x) = G(t, \underline{x}, \bar{x}), \bar{x}(t_0) = \bar{x}_0
\]

Where

\[
F(t, \underline{x}, \bar{x}) = \min \left\{ f(t, u) : u \in [\underline{x}, \bar{x}] \right\}
\]

\[
G(t, \underline{x}, \bar{x}) = \max \left\{ f(t, u) : u \in [\underline{x}, \bar{x}] \right\}
\]

The parametric form of system (13 - 14) is given by:

\[
x'(t, \alpha) = F[t, \underline{x}(t, \alpha), \bar{x}(t, \alpha)], \underline{x}(t_0, \alpha) = \underline{x}_0(\alpha)
\]

\[
\bar{x}'(t, \alpha) = G[t, \underline{x}(t, \alpha), \bar{x}(t, \alpha)], \bar{x}(t_0, \alpha) = \bar{x}_0(\alpha)
\]

where \( t \in [t_0, h] \) and \( \alpha \in [0,1] \).

The following theorem ensures the existence and uniqueness of the fuzzy solution of the first order fuzzy Riccati differential equation.
Theorem(2), [2]:
If we return to problem (10),
\[ x'(t) = f(t, x(t)), \quad t \in [t_0, h]. \]
Let \( f : T \rightarrow E^1 \) be a continuous fuzzy function, \( T = [t_0, h] \) and assume that there exist a real number \( k > 0 \) such that
\[ D\left( f(t, z), f(t, w) \right) \leq k D(z, w) \] (17)
For all \( t \in T \) and all \( z, w \in E^1 \).
Then the above fuzzy differential equation has a unique fuzzy solution on \( T \).

4. Fuzzy Homotopy Analysis Method
Fuzzy homotopy analysis method is one of the fuzzy semi-analytical methods used to obtain the fuzzy series solution (fuzzy semi-analytical solution) of the FIVBs. This technique utilizes homotopy in order to generate a convergent fuzzy series of fuzzy linear equations from fuzzy non-linear ones. This means that this technique is based on generating a convergent fuzzy series of fuzzy solutions to approximate the fuzzy analytical solution of the FIVB.
The basic mathematical concepts of the fuzzy homotopy analysis method are the same as the basic mathematical concepts of the homotopy analysis method, but with the use of the concepts of fuzzy theory. This means that solving any FIVB by using fuzzy homotopy analysis method is based on converting the FIVB into a system of non-fuzzy initial value problems by using the steps that we explained in section(3), and then using the homotopy analysis method to solve this system.
The fuzzy homotopy analysis method provides us with both the freedom to choose proper base fuzzy functions for approximating a non-linear fuzzy problem and a simple way to ensure the convergence of the fuzzy series solution.

5. Description of The Method
To describe the basic mathematical ideas of the fuzzy homotopy analysis method, we consider the following FIVB:
\[ [N(x(t))]_\alpha = 0 \] (18)
Where \( N \) is the fuzzy non-linear operator, \( t \) denotes the independent crisp variable, \( x(t) \) is an unknown fuzzy function.
By the concepts of section(3), We can conclude that:
\[ [N(x(t))]_\alpha = [ [N(x(t))]_{\alpha L}, [N(x(t))]_{\alpha U} ] \] (19)
Since \( 0 = [0, 0] \), we can get:
\[ [N(x(t))]_{\alpha L} = 0 \] (20i)
\[ [N(x(t))]_{\alpha U} = 0 \] (20ii)
Now, we construct the zero-order fuzzy deformation equation:
\[ [(1 - w)L (\theta(t; w) - x_0(t))]_\alpha = [w h N(\theta(t; w))]_\alpha \] (21)
Where \( w \in [0, 1] \) is the homotopy embedding parameter, \( h \in [-1, 0] \) is the convergence control parameter, \( L \) is the fuzzy linear operator, \( x_0(t) \) is the fuzzy initial guess of \( x(t) \) and \( \theta(t; w) \) is a fuzzy function.
By the concepts of section(3), We can get:

\[(1 - w)L\left( [\theta(t; w)]^L - [x_0(t)]^L \right) = w\ h( [N(\theta(t; w))]^L ) \tag{22i}\]

\[(1 - w)L\left( [\theta(t; w)]^U - [x_0(t)]^U \right) = w\ h( [N(\theta(t; w))]^U ) \tag{22ii}\]

Obviously, when \( w = 0 \) and \( w = 1 \), both

\[[\theta(t; 0)]^L_\alpha = [x_0(t)]^L_\alpha \tag{23}\]

\[[\theta(t; 1)]^L_\alpha = [x(t)]^L_\alpha \tag{24}\]

Hold, therefore when \( w \) is increasing from 0 to 1, the fuzzy solutions \([\theta(t; w)]^L_\alpha\) and \([\theta(t; w)]^U_\alpha\) varies from the fuzzy initial guess \([x_0(t)]^L_\alpha\) to the fuzzy solution \([x(t)]^L_\alpha\).

Thus, we have:

\[[\theta(t; 0)]^L_\alpha = [x_0(t)]^L_\alpha \tag{25i}\]

\[[\theta(t; 0)]^U_\alpha = [x_0(t)]^U_\alpha \tag{25ii}\]

and

\[[\theta(t; 1)]^L_\alpha = [x(t)]^L_\alpha \tag{26i}\]

\[[\theta(t; 1)]^U_\alpha = [x(t)]^U_\alpha \tag{26ii}\]

By expanding \([\theta(t; w)]^L_\alpha\) in Taylor series with respect to \( w \), one has:

\[[\theta(t; w)]^L_\alpha = [x_0(t)]^L_\alpha + \sum_{m=1}^{\infty} [x_m(t)w^m]_\alpha \tag{27}\]

Where

\[[x_m(t)]^L_\alpha = \frac{1}{m!} \left. \frac{\partial^m [\theta(t; w)]^L_\alpha}{\partial w^m} \right|_{w=0} \tag{28}\]

By the concepts of parametric form in section(3), We can conclude that:

\[[\theta(t; w)]^L_\alpha = [x_0(t)]^L_\alpha + \sum_{m=1}^{\infty} [x_m(t)]^L_\alpha w^m \tag{29i}\]

\[[\theta(t; w)]^U_\alpha = [x_0(t)]^U_\alpha + \sum_{m=1}^{\infty} [x_m(t)]^U_\alpha w^m \tag{29ii}\]

Where

\[[x_0(t)]^L_\alpha = \frac{1}{m!} \left. \frac{\partial^m [\theta(t; w)]^L_\alpha}{\partial w^m} \right|_{w=0} \tag{30i}\]

\[[x_0(t)]^U_\alpha = \frac{1}{m!} \left. \frac{\partial^m [\theta(t; w)]^U_\alpha}{\partial w^m} \right|_{w=0} \tag{30ii}\]

If the fuzzy linear operator, the fuzzy initial guess, the auxiliary parameter \( h \), and the auxiliary fuzzy function are so properly chosen, then the fuzzy series \((27)\) converges at \( w = 1 \), and one has:
\[ \theta(t; 1)_{\alpha} = [x(t)]_{\alpha} = [x_0(t)]_{\alpha} + \sum_{m=1}^{\infty} [x_m(t)]_{\alpha} \]  

(31)

Where

\[ \theta(t; 1)_{\alpha}^{L} = [x(t)]_{\alpha}^{L} = [x_0(t)]_{\alpha}^{L} + \sum_{m=1}^{\infty} [x_m(t)]_{\alpha}^{L} \]  

(32i)

\[ \theta(t; 1)_{\alpha}^{U} = [x(t)]_{\alpha}^{U} = [x_0(t)]_{\alpha}^{U} + \sum_{m=1}^{\infty} [x_m(t)]_{\alpha}^{U} \]  

(32ii)

Which must be one of the fuzzy solutions of the problem (18).

If \( h = -1 \), (21) becomes

\[ [(1 - w)L(\theta(t; w) - x_0(t))]_{\alpha} + [w N(\theta(t; w))]_{\alpha} = 0 \]  

(33)

Where

\[ (1 - w)L(\theta(t; w)]_{\alpha}^{L} - [x_0(t)]_{\alpha}^{L} + w [N(\theta(t; w))]_{\alpha}^{L} = 0 \]  

(34i)

\[ (1 - w)L(\theta(t; w)]_{\alpha}^{U} - [x_0(t)]_{\alpha}^{U} + w [N(\theta(t; w))]_{\alpha}^{U} = 0 \]  

(34ii)

Which is used mostly in the fuzzy homotopy analysis method.

We define the fuzzy vectors

\[ [\bar{x}_i]_{\alpha} = \{ [x_0(t)]_{\alpha}, [x_1(t)]_{\alpha}, [x_2(t)]_{\alpha}, ..., [x_i(t)]_{\alpha} \} \]  

(35)

Where

\[ [\bar{x}_i]_{\alpha}^{L} = \{ [x_0(t)]_{\alpha}^{L}, [x_1(t)]_{\alpha}^{L}, [x_2(t)]_{\alpha}^{L}, ..., [x_i(t)]_{\alpha}^{L} \} \]  

(36i)

\[ [\bar{x}_i]_{\alpha}^{U} = \{ [x_0(t)]_{\alpha}^{U}, [x_1(t)]_{\alpha}^{U}, [x_2(t)]_{\alpha}^{U}, ..., [x_i(t)]_{\alpha}^{U} \} \]  

(36ii)

Now, by differentiating (21) \( m \) times with respect to the parameter \( w \) and then setting \( w = 0 \) and finally dividing them by \( m! \), we have the \( m \)-th order fuzzy deformation equation:

\[ L([x_m(t)]_{\alpha} - \chi_m[x_{m-1}(t)]_{\alpha}) = h([R_m(\bar{x}_{m-1})]_{\alpha}) \]  

(37)

Where

\[ [R_m(\bar{x}_{m-1})]_{\alpha} = \frac{1}{(m-1)!} \frac{\partial^{m-1}[N(\theta(t; w))]_{\alpha}}{\partial w^{m-1}} \bigg|_{w=0} \]  

(38)

By the concepts of parametric form in section (3), We get:

\[ L([x_m(t)]_{\alpha}^{L} - \chi_m[x_{m-1}(t)]_{\alpha}^{L}) = h([R_m(\bar{x}_{m-1})]_{\alpha}^{L}) \]  

(39i)

\[ L([x_m(t)]_{\alpha}^{U} - \chi_m[x_{m-1}(t)]_{\alpha}^{U}) = h([R_m(\bar{x}_{m-1})]_{\alpha}^{U}) \]  

(39ii)

Where

\[ [R_m(\bar{x}_{m-1})]_{\alpha}^{L} = \frac{1}{(m-1)!} \frac{\partial^{m-1}[N(\theta(t; w))]_{\alpha}^{L}}{\partial w^{m-1}} \bigg|_{w=0} \]  

(40i)

\[ [R_m(\bar{x}_{m-1})]_{\alpha}^{U} = \frac{1}{(m-1)!} \frac{\partial^{m-1}[N(\theta(t; w))]_{\alpha}^{U}}{\partial w^{m-1}} \bigg|_{w=0} \]  

(40ii)
\[ x_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{41} \]

6. Applied Example

In this section, one fuzzy problem has been solved in order to clarify the efficiency and accuracy of the method. According to Liao’s book [3], the optimal value of \( h \) was found to be approximately \(-1 \leq h < 0\). In addition, the practical examples in [3,5,10] showed that the optimal value of \( h \) can be determined while solving the problem by experimenting with a number of different values of \( h \). The optimal value of \( h \) depends greatly on the nature of the problem, but still \( h = -1 \) is an optimal value and achieves a rapid convergence.

**Example (1):** Consider first order fuzzy Riccati differential equation:

\[ x'(t) = x^2(t) + 1, \]

subject to the fuzzy initial condition:

\[ [x(0)]_\alpha = [0.1 \alpha - 0.1, 0.1 - 0.1 \alpha], \alpha \in [0,1]. \]

**Solution:**

The fuzzy linear operator is:

\[ [L(\theta(t;w))]_\alpha = \left[ [L(\theta(t;w))]_\alpha^L, [L(\theta(t;w))]_\alpha^U \right] \]

Where

\[ [L(\theta(t;w))]_\alpha^L = \left[ \frac{\partial \theta(t;w)}{\partial t} \right]_\alpha^L \]

\[ [L(\theta(t;w))]_\alpha^U = \left[ \frac{\partial \theta(t;w)}{\partial t} \right]_\alpha^U \]

We define the fuzzy non-linear operator as:

\[ [N(\theta(x;w))]_\alpha = \left[ [N(\theta(x;w))]_\alpha^L, [N(\theta(x;w))]_\alpha^U \right] \]

Where

\[ [N(\theta(x;w))]_\alpha^L = \left[ \frac{\partial [\theta(x;w)]_\alpha^L}{\partial t} \right] - \left[ (\theta(x;w))^2 \right]_\alpha^L - 1 \]

\[ [N(\theta(x;w))]_\alpha^U = \left[ \frac{\partial [\theta(x;w)]_\alpha^U}{\partial t} \right] - \left[ (\theta(x;w))^2 \right]_\alpha^U - 1 \]

The fuzzy series solution is:

\[ [x(t)]_\alpha = \left[ [x(t)]_\alpha^L, [x(t)]_\alpha^U \right] \]

Where

\[ [x(t)]_\alpha^L = [x_0(t)]_\alpha^L + [x_1(t)]_\alpha^L + [x_2(t)]_\alpha^L + [x_3(t)]_\alpha^L + \ldots \]

\[ [x(t)]_\alpha^U = [x_0(t)]_\alpha^U + [x_1(t)]_\alpha^U + [x_2(t)]_\alpha^U + [x_3(t)]_\alpha^U + \ldots \]
By Taylor series expansion, the fuzzy initial approximation is

\[ [x_0(t)]_\alpha = [ [x_0(t)]^L_\alpha, [x_0(t)]^U_\alpha ] \]

Where

\[ [x_0(t)]^L_\alpha = 0.1 \alpha - 0.1 \]
\[ [x_0(t)]^U_\alpha = 0.1 - 0.1 \alpha \]

To find \[ [x_1(t)]_\alpha = [ [x_1(t)]^L_\alpha, [x_1(t)]^U_\alpha ] \:

From (29), we can find

\[ [\theta(t; w)]^L_\alpha = [x_0(t)]^L_\alpha + w[x_1(t)]^L_\alpha \]
\[ [\theta(t; w)]^U_\alpha = [x_0(t)]^U_\alpha + w[x_1(t)]^U_\alpha \]

From (37), we can find

\[ L([x_1(t)]^L_\alpha - 0) = h[R_1]^L_\alpha \]
\[ L([x_1(t)]^U_\alpha - 0) = h[R_1]^U_\alpha \]

Then from (38), we can get:

\[ [R_1]^L_\alpha = [N(\theta(t; w))]^L_{\alpha|w=0} \]
\[ [R_1]^U_\alpha = [N(\theta(t; w))]^U_{\alpha|w=0} \]

Then, we apply the following steps:

\[ \frac{\partial[\theta(t;w)]^L_\alpha}{\partial t} = \frac{\partial[x_0(t)]^L_\alpha}{\partial t} + w \frac{\partial[x_1(t)]^L_\alpha}{\partial t} \]
\[ \frac{\partial[\theta(t;w)]^U_\alpha}{\partial t} = \frac{\partial[x_0(t)]^U_\alpha}{\partial t} + w \frac{\partial[x_1(t)]^U_\alpha}{\partial t} \]
\[ [N(\theta(x; w))]^L_{\alpha|t} = \frac{\partial[x_0(t)]^L_\alpha}{\partial t} + w \frac{\partial[x_1(t)]^L_\alpha}{\partial t} - \left( [x_0(t)]^L_\alpha + w[x_1(t)]^L_\alpha \right)^2 - 1 \]
\[ [N(\theta(x; w))]^U_{\alpha|t} = \frac{\partial[x_0(t)]^U_\alpha}{\partial t} + w \frac{\partial[x_1(t)]^U_\alpha}{\partial t} - \left( [x_0(t)]^U_\alpha + w[x_1(t)]^U_\alpha \right)^2 - 1 \]
\[ [R_1]^L_\alpha = \frac{\partial[x_0(t)]^L_\alpha}{\partial t} - \left( [x_0(t)]^L_\alpha \right)^2 - 1 \]
\[ [R_1]^U_\alpha = \frac{\partial[x_0(t)]^U_\alpha}{\partial t} - \left( [x_0(t)]^U_\alpha \right)^2 - 1 \]
\[ [R_1]^L_\alpha = -0.01 \alpha^2 + 0.02 \alpha - 1.01 \]
\[ [R_1]_a^U = -0.01 \alpha^2 + 0.02 \alpha - 1.01 \]
\[ L[x_1(t)]_a^L = -0.01 \alpha^2 h + 0.02 a h - 1.01 h \]
\[ L[x_1(t)]_a^U = -0.01 \alpha^2 h + 0.02 a h - 1.01 h \]
\[ [x_1(t)]_a^L = \int (-0.01 \alpha^2 h + 0.02 a h - 1.01 h) \, dt \]
\[ [x_1(t)]_a^U = \int (-0.01 \alpha^2 h + 0.02 a h - 1.01 h) \, dt \]
\[ [x_1(t)]_a^L = -0.01 \alpha^2 h + 0.02 a h - 1.01 h \]
\[ [x_1(t)]_a^U = -0.01 \alpha^2 h + 0.02 a h - 1.01 h \]

**Now, to find** \([x_2(t)]_a = [ [x_2(t)]_a^L, [x_2(t)]_a^U ]:

From (29), we can find
\[
[\theta(t; w)]_a^L = [x_0(t)]_a^L + w[x_1(t)]_a^L + w^2[x_2(t)]_a^L \\
[\theta(t; w)]_a^U = [x_0(t)]_a^U + w[x_1(t)]_a^U + w^2[x_2(t)]_a^U
\]

From (37), we can find
\[
L([x_2(t)]_a^L - [x_1(t)]_a^L) = h[R_2]_a^L \\
L([x_2(t)]_a^U - [x_1(t)]_a^U) = h[R_2]_a^U
\]

Then from (38), we can get:
\[
[R_2]_a^L = \left. \frac{\partial [N(\theta(t; w))]_a^L}{\partial w} \right|_{w=0} \\
[R_2]_a^U = \left. \frac{\partial [N(\theta(t; w))]_a^U}{\partial w} \right|_{w=0}
\]

Then, we apply the following steps:
\[
\left. \frac{\partial [\theta(t; w)]_a^L}{\partial t} \right|_a = \frac{\partial [x_0(t)]_a^L}{\partial t} + w \frac{\partial [x_1(t)]_a^L}{\partial t} + w^2 \frac{\partial [x_2(t)]_a^L}{\partial t} \\
\left. \frac{\partial [\theta(t; w)]_a^U}{\partial t} \right|_a = \frac{\partial [x_0(t)]_a^U}{\partial t} + w \frac{\partial [x_1(t)]_a^U}{\partial t} + w^2 \frac{\partial [x_2(t)]_a^U}{\partial t} \\
[N(\theta(x; w))]_a^L = \frac{\partial [x_0(t)]_a^L}{\partial t} + w \frac{\partial [x_1(t)]_a^L}{\partial t} + w^2 \frac{\partial [x_2(t)]_a^L}{\partial t} - \left( [x_0(t)]_a^L + w[x_1(t)]_a^L + w^2[x_2(t)]_a^L \right)^2 - 1 \\
[N(\theta(x; w))]_a^U = \frac{\partial [x_0(t)]_a^U}{\partial t} + w \frac{\partial [x_1(t)]_a^U}{\partial t} + w^2 \frac{\partial [x_2(t)]_a^U}{\partial t} - \left( [x_0(t)]_a^U + w[x_1(t)]_a^U + w^2[x_2(t)]_a^U \right)^2 - 1
\]
\[
\frac{\partial [\theta(t; w)]^L_\alpha}{\partial w} = \frac{\partial [x_0(t)]^L_\alpha}{\partial t} + 2w \frac{\partial [x_2(t)]^L_\alpha}{\partial t} - 2([x_0(t)]^L_\alpha + w[x_1(t)]^L_\alpha + w^2[x_2(t)]^L_\alpha)([x_1(t)]^L_\alpha + 2w[x_2(t)]^L_\alpha) \\
\frac{\partial [\theta(t; w)]^U_\alpha}{\partial w} = \frac{\partial [x_0(t)]^U_\alpha}{\partial t} + 2w \frac{\partial [x_2(t)]^U_\alpha}{\partial t} - 2([x_0(t)]^U_\alpha + w[x_1(t)]^U_\alpha + w^2[x_2(t)]^U_\alpha)([x_1(t)]^U_\alpha + 2w[x_2(t)]^U_\alpha) \\
[R_2]^L_\alpha = \frac{\partial [x_1(t)]^L_\alpha}{\partial t} - 2[x_0(t) x_1(t)]^L_\alpha \\
[R_2]^U_\alpha = \frac{\partial [x_1(t)]^U_\alpha}{\partial t} - 2[x_0(t) x_1(t)]^U_\alpha \\
[R_2]^L_\alpha = -0.01 \alpha^2 h + 0.02 ah - 1.01 h + 0.002 \alpha^3 ht - 0.006 \alpha^2 ht + 0.206 aht - 0.202 ht \\
[R_2]^U_\alpha = -0.01 \alpha^2 h + 0.02 ah - 1.01 h + 0.002 \alpha^3 ht - 0.006 \alpha^2 ht - 0.206 aht + 0.202ht \\
L([x_2(t)]^L_\alpha^t - [x_1(t)]^L_\alpha) = -0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 + 0.002 \alpha^3 h^2 t - 0.006 \alpha^2 h^2 t + 0.206 \alpha h^2 t - 0.202 h^2 t \\
L([x_2(t)]^U_\alpha^t - [x_1(t)]^U_\alpha) = -0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 - 0.002 \alpha^3 h^2 t + 0.006 \alpha^2 h^2 t - 0.206 \alpha h^2 t + 0.202 h^2 t \\
[x_2(t)]^L_\alpha^t - [x_1(t)]^L_\alpha = \int (-0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 + 0.002 \alpha^3 h^2 t - 0.006 \alpha^2 h^2 t + 0.206 \alpha h^2 t - 0.202 h^2 t) \, dt \\
[x_2(t)]^U_\alpha^t - [x_1(t)]^U_\alpha = \int (-0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 - 0.002 \alpha^3 h^2 t + 0.006 \alpha^2 h^2 t - 0.206 \alpha h^2 t + 0.202 h^2 t) \, dt \\
[x_2(t)]^L_\alpha^t - [x_1(t)]^L_\alpha = -0.01 \alpha^2 h^2 t + 0.02 ah^2 t - 1.01 h^2 t + 0.001 \alpha^3 h^2 t^2 - 0.003 \alpha^2 h^2 t^2 + 0.103 \alpha h^2 t^2 - 0.101 h^2 t^2 \\
[x_2(t)]^U_\alpha^t - [x_1(t)]^U_\alpha = -0.01 \alpha^2 h^2 t + 0.02 ah^2 t - 1.01 h^2 t - 0.001 \alpha^3 h^2 t^2 + 0.003 \alpha^2 h^2 t^2 - 0.103 \alpha h^2 t^2 + 0.101 h^2 t^2 \\
[x_2(t)]^L_\alpha = -0.01 \alpha^2 h^2 t + 0.02 ah^2 t - 1.01 h^2 t + 0.001 \alpha^3 h^2 t^2 - 0.003 \alpha^2 h^2 t^2 + 0.103 \alpha h^2 t^2 - 0.101 h^2 t^2 - 0.01 \alpha h^2 t + 0.02 aht - 1.01 ht \\
[x_2(t)]^U_\alpha = -0.01 \alpha^2 h^2 t + 0.02 ah^2 t - 1.01 h^2 t - 0.001 \alpha^3 h^2 t^2 + 0.003 \alpha^2 h^2 t^2 - 0.103 \alpha h^2 t^2 + 0.101 h^2 t^2 - 0.01 \alpha h^2 t + 0.02 aht - 1.01 ht \\
Now, to find \[x_3(t)]_\alpha = \left[[x_3(t)]^L_\alpha, [x_3(t)]^U_\alpha\right] \\
From (29), we can find \[\theta(t; w)]_\alpha = [x_0(t)]^L_\alpha + w[x_1(t)]^L_\alpha + w^2[x_2(t)]^L_\alpha + w^3[x_3(t)]^L_\alpha \\
[\theta(t; w)]_\alpha = [x_0(t)]^U_\alpha + w[x_1(t)]^U_\alpha + w^2[x_2(t)]^U_\alpha + w^3[x_3(t)]^U_\alpha \\
From (37), we can find
\[ L \left( [x_3(t)]_\alpha^L - [x_2(t)]_\alpha^L \right) = h[R_3]_\alpha^L \]

\[ L \left( [x_3(t)]_\alpha^U - [x_2(t)]_\alpha^U \right) = h[R_3]_\alpha^U \]

Then from (38), we can get:

\[ [R_3]_\alpha^L = \frac{1}{2} \left. \frac{\partial^2 \left( N[\theta(x; w)]_\alpha \right) \partial w^2 \right|_{w=0} \]

\[ [R_3]_\alpha^U = \frac{1}{2} \left. \frac{\partial^2 \left( N[\theta(x; w)]_\alpha \right) \partial w^2 \right|_{w=0} \]

Then, we apply the following steps:

\[ \frac{\partial[\theta(t; w)]_\alpha^L}{\partial t} = \frac{\partial[x_0(t)]_\alpha^L}{\partial t} + w \frac{\partial[x_1(t)]_\alpha^L}{\partial t} + w^2 \frac{\partial[x_2(t)]_\alpha^L}{\partial t} + w^3 \frac{\partial[x_3(t)]_\alpha^L}{\partial t} \]

\[ \frac{\partial[\theta(t; w)]_\alpha^U}{\partial t} = \frac{\partial[x_0(t)]_\alpha^U}{\partial t} + w \frac{\partial[x_1(t)]_\alpha^U}{\partial t} + w^2 \frac{\partial[x_2(t)]_\alpha^U}{\partial t} + w^3 \frac{\partial[x_3(t)]_\alpha^U}{\partial t} \]

\[ [N(\theta(x; w))]_\alpha^L = \frac{\partial[x_0(t)]_\alpha^L}{\partial t} + w \left[ \frac{\partial[x_1(t)]_\alpha^L}{\partial t} \right]^L + w^2 \left[ \frac{\partial[x_2(t)]_\alpha^L}{\partial t} \right]^L + w^3 \left[ \frac{\partial[x_3(t)]_\alpha^L}{\partial t} \right]^L - ([x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L + w^2[x_2(t)]_\alpha^L + w^3[x_3(t)]_\alpha^L)^2 - 1 \]

\[ [N(\theta(x; w))]_\alpha^U = \frac{\partial[x_0(t)]_\alpha^U}{\partial t} + w \left[ \frac{\partial[x_1(t)]_\alpha^U}{\partial t} \right]^U + w^2 \left[ \frac{\partial[x_2(t)]_\alpha^U}{\partial t} \right]^U + w^3 \left[ \frac{\partial[x_3(t)]_\alpha^U}{\partial t} \right]^U - ([x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U + w^2[x_2(t)]_\alpha^U + w^3[x_3(t)]_\alpha^U)^2 - 1 \]

\[ \frac{\partial[N(\theta(t; w))]_\alpha^L}{\partial w} = \frac{\partial[x_0(t)]_\alpha^L}{\partial t} + 2w \frac{\partial[x_0(t)]_\alpha^L}{\partial t} + 3w^2 \frac{\partial[x_3(t)]_\alpha^L}{\partial t} - 2([x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L + w^2[x_2(t)]_\alpha^L + w^3[x_3(t)]_\alpha^L)([x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L + w^2[x_2(t)]_\alpha^L + w^3[x_3(t)]_\alpha^L) \]

\[ \frac{\partial[N(\theta(t; w))]_\alpha^U}{\partial w} = \frac{\partial[x_0(t)]_\alpha^U}{\partial t} + 2w \frac{\partial[x_0(t)]_\alpha^U}{\partial t} + 3w^2 \frac{\partial[x_3(t)]_\alpha^U}{\partial t} - 2([x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U + w^2[x_2(t)]_\alpha^U + w^3[x_3(t)]_\alpha^U)([x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U + w^2[x_2(t)]_\alpha^U + w^3[x_3(t)]_\alpha^U) \]

\[ \frac{\partial^2[N(\theta(t; w))]_\alpha^L}{\partial w^2} = 2 \frac{\partial[x_0(t)]_\alpha^L}{\partial t} + 6w \frac{\partial[x_3(t)]_\alpha^L}{\partial t} - 2([x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L + w^2[x_2(t)]_\alpha^L + w^3[x_3(t)]_\alpha^L)([x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L + w^2[x_2(t)]_\alpha^L + w^3[x_3(t)]_\alpha^L) \]

\[ \frac{\partial^2[N(\theta(t; w))]_\alpha^U}{\partial w^2} = 2 \frac{\partial[x_0(t)]_\alpha^U}{\partial t} + 6w \frac{\partial[x_3(t)]_\alpha^U}{\partial t} - 2([x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U + w^2[x_2(t)]_\alpha^U + w^3[x_3(t)]_\alpha^U)([x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U + w^2[x_2(t)]_\alpha^U + w^3[x_3(t)]_\alpha^U) \]

\[ [R_3]_\alpha^L = \frac{\partial[x_0(t)]_\alpha^L}{\partial t} - 2[x_0(t)x_2(t)]_\alpha^L - [(x_1(t))]_\alpha^L \]

\[ [R_3]_\alpha^U = \frac{\partial[x_0(t)]_\alpha^U}{\partial t} - 2[x_0(t)x_2(t)]_\alpha^U - [(x_1(t))]_\alpha^U \]
\[
[R_3]_U^L = -0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 + 0.004 \alpha^3 h^2 t - 0.012 \alpha^2 h^2 t + 0.412 ah^2 t - 0.404 h^2 t - 0.01 \alpha^2 h + 0.02 ah - 1.01 h - 0.0003 \alpha^4 h^2 t^2 + 0.0012 \alpha^3 h^2 t^2 - 0.0418 \alpha^2 h^2 t^2 + 0.0812 ah^2 t^2 + 0.002 \alpha^3 h - 0.006 \alpha^2 h t + 0.206 ah t - 1.0403 h^2 t^2 - 0.202 h t
\]

\[
[R_3]_U^T = -0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 - 0.004 \alpha^3 h^2 t + 0.012 \alpha^2 h^2 t - 0.412 ah^2 t + 0.404 h^2 t - 0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 - 0.0003 \alpha^4 h^2 t^2 + 0.0012 \alpha^3 h^2 t^2 - 0.0418 \alpha^2 h^2 t^2 + 0.0812 ah^2 t^2 + 0.002 \alpha^3 h^2 t - 0.006 \alpha^2 h^2 t + 0.206 ah^2 t - 1.0403 h^2 t^2 - 0.202 h^2 t
\]

\[
L([x_3(t)]_U^L - [x_2(t)]_U^L) = -0.01 \alpha^2 h^3 + 0.02 ah^3 - 1.01 h^3 + 0.004 \alpha^3 h^3 t - 0.012 \alpha^2 h^3 t + 0.412 ah^3 t - 0.404 h^3 t - 0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 + 0.0012 \alpha^3 h^2 t^2 - 0.0418 \alpha^2 h^2 t^2 + 0.0812 ah^2 t^2 - 1.0403 h^2 t^2 + 0.006 \alpha^2 h^2 t - 0.206 ah^2 t + 0.202 h^2 t - 0.0003 \alpha^4 h^2 t^2 - 0.0002 \alpha^3 h^2 t^2
\]

\[
L([x_3(t)]_U^T - [x_2(t)]_U^T) = -0.01 \alpha^2 h^3 + 0.02 ah^3 - 1.01 h^3 + 0.004 \alpha^3 h^3 t - 0.012 \alpha^2 h^3 t + 0.412 ah^3 t - 0.404 h^3 t - 0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 + 0.0012 \alpha^3 h^2 t^2 - 0.0418 \alpha^2 h^2 t^2 + 0.0812 ah^2 t^2 + 0.002 \alpha^3 h^2 t - 0.006 \alpha^2 h^2 t + 0.206 ah^2 t - 1.0403 h^2 t^2 - 0.202 h^2 t^2
\]

\[
[x_3(t)]_U^L - [x_2(t)]_U^L = \int (-0.01 \alpha^2 h^3 + 0.02 ah^3 - 1.01 h^3 + 0.004 \alpha^3 h^3 t - 0.012 \alpha^2 h^3 t + 0.412 ah^3 t - 0.404 h^3 t - 0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 + 0.0012 \alpha^3 h^2 t^2 - 0.0418 \alpha^2 h^2 t^2 + 0.0812 ah^2 t^2 + 0.002 \alpha^3 h^2 t - 0.006 \alpha^2 h^2 t + 0.206 ah^2 t - 1.0403 h^2 t^2 - 0.202 h^2 t^2) dt
\]

\[
[x_3(t)]_U^T - [x_2(t)]_U^T = \int (-0.01 \alpha^2 h^3 + 0.02 ah^3 - 1.01 h^3 - 0.004 \alpha^3 h^3 t + 0.012 \alpha^2 h^3 t - 0.412 ah^3 t + 0.404 h^3 t - 0.01 \alpha^2 h^2 + 0.02 ah^2 - 1.01 h^2 + 0.0012 \alpha^3 h^2 t^2 - 0.0418 \alpha^2 h^2 t^2 + 0.0812 ah^2 t^2 + 0.006 \alpha^2 h^2 t - 0.206 ah^2 t + 0.202 h^2 t - 0.0003 \alpha^4 h^2 t^2 - 0.0002 \alpha^3 h^2 t^2) dt
\]

\[
[x_3(t)]_U^L - [x_2(t)]_U^L = -0.01 \alpha^2 h^2 t + 0.02 ah^2 t - 1.01 h^3 t + 0.002 \alpha^3 h^3 t^2 - 0.006 \alpha^2 h^3 t^2 + 0.206 ah^2 t^2 - 0.202 h^2 t^2 - 0.01 \alpha^2 h^2 t + 0.02 ah^2 t - 1.01 h^2 t - 0.0001 \alpha^4 h^2 t^3 + 0.0004 \alpha^3 h^2 t^3 + 0.01207 ah^2 t^3 + 0.0013 \alpha^2 h^2 t^3 + 0.0003 \alpha^4 h^2 t^3 - 0.0001 \alpha^3 h^2 t^3 - 0.0001 \alpha^2 h^2 t^3
\]

\[
[x_3(t)]_U^T = -0.01 \alpha^2 h^3 t + 0.02 ah^3 t - 1.01 h^3 t + 0.002 \alpha^3 h^3 t^2 - 0.006 \alpha^2 h^3 t^2 + 0.206 ah^3 t^2 - 0.202 h^3 t^2 - 0.01 \alpha^2 h^2 t + 0.04 ah^2 t - 2.02 h^2 t - 0.0001 \alpha^4 h^3 t^3 + 0.0004 \alpha^3 h^3 t^3 - 0.01207 ah^3 t^3 + 0.0013 \alpha^2 h^3 t^3 + 0.0003 \alpha^4 h^3 t^3 - 0.01207 ah^3 t^3 + 0.0013 \alpha^2 h^3 t^3 + 0.0003 \alpha^3 h^3 t^3 - 0.0001 \alpha^3 h^2 t^3 + 0.02 ah^2 t + 1.01 h^2 t
\]

\[
[x_3(t)]_U^L = -0.01 \alpha^2 h^2 t + 0.02 ah^2 t - 1.01 h^2 t - 0.002 \alpha^3 h^3 t^2 + 0.006 \alpha^2 h^3 t^2 - 0.206 ah^2 t^2 + 0.202 h^2 t^2 - 0.01 \alpha^2 h^2 t + 0.04 ah^2 t - 2.02 h^2 t + 0.0004 \alpha^3 h^3 t^3 - 0.01207 ah^3 t^3 + 0.0013 \alpha^2 h^3 t^3 + 0.0003 \alpha^4 h^3 t^3 - 0.01207 ah^3 t^3 + 0.0013 \alpha^2 h^3 t^3 + 0.0003 \alpha^3 h^3 t^3 - 0.0001 \alpha^4 h^3 t^3 - 0.0001 \alpha^3 h^2 t^3 + 0.02 ah^2 t + 1.01 h^2 t
\]

Then the fuzzy series solution is:
\[
[x(t)]_t = \begin{bmatrix} [x(t)]^L_t, [x(t)]^U_t \end{bmatrix}
\]

Where

\[
[x(t)]^L_t = 0.1 \alpha - 0.1 - 0.03 \alpha^2 ht + 0.06 aht - 3.03 ht - 0.03 \alpha^2 h^2 t + 0.06 ah^2 t - 3.03 h^2 t - 0.001 \alpha^2 h^2 t^2 - 0.003 \alpha^2 h^2 t^2 + 0.103 ah^2 t^2 + 0.101 h^2 t^2 - 0.01 \alpha^2 h^3 t - 0.02 ah^3 t - 1.01 h^3 t - 0.002 \alpha^2 h^3 t^2 + 0.006 \alpha^2 h^3 t^2 - 0.206 ah^3 t^2 + 0.202 h^3 t^2 + 0.0004 \alpha^3 h^3 t^3 - 0.01393 \alpha^3 h^3 t^3 + 0.02707 ah^3 t^3 - 0.34677 h^3 t^3 - 0.0001 \alpha^4 h^3 t^3 + \ldots
\]

\[
[x(t)]^U_t = 0.1 - 0.1 \alpha - 0.03 \alpha^2 ht + 0.06 aht - 3.03 ht - 0.03 \alpha^2 h^2 t + 0.06 ah^2 t - 3.03 h^2 t + 0.001 \alpha^2 h^2 t^2 - 0.003 \alpha^2 h^2 t^2 + 0.103 ah^2 t^2 - 0.101 h^2 t^2 - 0.01 \alpha^2 h^3 t + 0.02 ah^3 t - 1.01 h^3 t + 0.002 \alpha^2 h^3 t^2 - 0.006 \alpha^2 h^3 t^2 + 0.206 ah^3 t^2 - 0.202 h^3 t^2 - 0.0001 \alpha^4 h^3 t^3 + 0.0004 \alpha^3 h^3 t^3 - 0.01393 \alpha^3 h^3 t^3 + 0.02707 ah^3 t^3 - 0.34677 h^3 t^3 + \ldots
\]

The fuzzy series solution at \( h = -1 \), will be

\[
[x(t)]_t = \begin{bmatrix} [x(t)]^L_t, [x(t)]^U_t \end{bmatrix}
\]

Where

\[
[x(t)]^L_t \approx 0.1 \alpha - 0.1 + 0.01 \alpha^2 t - 0.02 \alpha t + 1.01 t + 0.001 \alpha^3 t^2 - 0.003 \alpha^2 t^2 + 0.103 \alpha t^2 - 0.101 t^2 - 0.0004 \alpha^3 t^3 + 0.01393 \alpha^2 t^3 - 0.02707 at^3 + 0.34677 t^3 + 0.0001 \alpha^4 t^3 + \ldots
\]

\[
[x(t)]^U_t \approx 0.1 - 0.1 \alpha + 0.01 \alpha^2 t - 0.02 \alpha t + 1.01 t - 0.001 \alpha^3 t^2 + 0.003 \alpha^2 t^2 - 0.103 \alpha t^2 + 0.101 t^2 + 0.0001 \alpha^4 t^3 - 0.0004 \alpha^3 t^3 + 0.01393 \alpha^2 t^3 - 0.02707 at^3 + 0.34677 t^3 + \ldots
\]

7. Discussion

When solving a fuzzy Riccati differential equation by using the fuzzy homotopy analysis method, the accuracy of the results depends greatly on the value of \( h \), other factors also affect, including : the number of terms of the solution series, the value of the constant \( \alpha \) and the period to which the variable \( t \) belongs. The fuzzy semi-analytical solutions that we obtained during this work are accurate solutions and very close to the fuzzy exact-analytical solutions, based on the comparison that we will make between the results that we obtained and the fuzzy exact-analytical solutions to the chosen problem.

If we go back to example (1):

\[ x'(t) = x^2(t) + 1, \quad t \in [0, 0.2] \]

The fuzzy exact-analytical solution for this problem is:

\[
[x(t)]_t = \begin{bmatrix} [x(t)]^L_t, [x(t)]^U_t \end{bmatrix}
\]

Where

\[
[x(t)]^L_t = tan(t + tan^{-1}(0.1\alpha - 0.1))
\]

\[
[x(t)]^U_t = tan(t + tan^{-1}(0.1 - 0.1\alpha))
\]

While the fuzzy semi-analytical solution that we got (at \( h = -1, \alpha = 0.2 \)) is
\[ [x(t)]_{\alpha} = [ [x(t)]_{L\alpha} , [x(t)]_{U\alpha} ] \]

Where

\[ [x(t)]_{L\alpha} = -0.08 + 1.0064 t - 0.080512 t^2 + 0.341910 t^3 + \ldots \]

\[ [x(t)]_{U\alpha} = 0.08 + 1.0064 t + 0.080512 t^2 + 0.341910 t^3 + \ldots \]

We test the accuracy of the obtained solutions by computing the absolute errors

\[ [\text{error}]_{L\alpha}^L = | [x_{\text{exact}}(t)]_{L\alpha} - [x_{\text{series}}(t)]_{L\alpha} | \]

\[ [\text{error}]_{U\alpha}^L = | [x_{\text{exact}}(t)]_{U\alpha} - [x_{\text{series}}(t)]_{U\alpha} | \]

The following tables provide a comparison between the fuzzy exact-analytical solution and the fuzzy semi-analytical solution for this problem

**Table 1.** Comparison of the results of example(1) , \( \alpha = 0.2 \).

| \( t \) | \([x_{\text{exact}}(t)]_{L\alpha}\) | \([x_{\text{series}}(t)]_{L\alpha}\) | \([\text{error}]_{L\alpha}^L\) | \([x_{\text{exact}}(t)]_{U\alpha}\) | \([x_{\text{series}}(t)]_{U\alpha}\) | \([\text{error}]_{U\alpha}^L\) |
|------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0    | -0.079999999 | -0.08       | 7.17e-10    | 0.079999999 | 0.08        | 7.17e-10    |
| 0.02 | -0.059901477 | -0.059901469| 7.00e-9     | 0.100162948 | 0.100162940 | 8.00e-9     |
| 0.04 | -0.039851060 | -0.039850936| 1.23e-7     | 0.120406853 | 0.120406701 | 1.52e-7     |
| 0.06 | -0.019832584 | -0.019831990| 5.94e-7     | 0.140748507 | 0.140747695 | 8.12e-7     |
| 0.08 | 0.000170015  | 0.000171781 | 1.77e-6     | 0.161205022 | 0.161202334 | 2.69e-6     |
| 0.10 | 0.020172750  | 0.020176790 | 4.04e-6     | 0.181793889 | 0.181787030 | 6.86e-6     |
| 0.12 | 0.040191635  | 0.040199447 | 7.81e-6     | 0.202533040 | 0.202518193 | 1.48e-5     |
| 0.14 | 0.060242734  | 0.060256165 | 1.34e-5     | 0.223449011 | 0.223412236 | 2.87e-5     |
| 0.16 | 0.080342214  | 0.080363356 | 2.11e-5     | 0.244536513 | 0.244485570 | 5.09e-5     |
| 0.18 | 0.100506401  | 0.100537430 | 3.10e-5     | 0.265839503 | 0.265754607 | 8.49e-5     |
| 0.2  | 0.120751827  | 0.120794800 | 4.30e-5     | 0.287370261 | 0.287235760 | 1.35e-4     |

Also, the fuzzy semi-analytical solution that we got (at \( h = -1, \alpha = 0.5 \)) is

\[ [x(t)]_{\alpha} = [ [x(t)]_{L\alpha} , [x(t)]_{U\alpha} ] \]

Where

\[ [x(t)]_{L\alpha} = -0.05 + 1.0025 t - 0.050125 t^2 + 0.336674 t^3 + \ldots \]

\[ [x(t)]_{U\alpha} = 0.05 + 1.0025 t + 0.050125 t^2 + 0.336674 t^3 + \ldots \]

**Table 2.** Comparison of the results of example(1) , \( \alpha = 0.5 \).

| \( t \) | \([x_{\text{exact}}(t)]_{L\alpha}\) | \([x_{\text{series}}(t)]_{L\alpha}\) | \([\text{error}]_{L\alpha}^L\) | \([x_{\text{exact}}(t)]_{U\alpha}\) | \([x_{\text{series}}(t)]_{U\alpha}\) | \([\text{error}]_{U\alpha}^L\) |
|------|-------------|-------------|-------------|-------------|-------------|-------------|

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8. Conclusion

In this work, we study the fuzzy approximate-analytical solutions of the first order fuzzy Riccati differential equation. Obviously the accuracy of the results that can be obtained when solving using fuzzy homotopy analysis method, these results may improve further when increasing the number of terms of the solution series or using another value for the parameter $h$. The value of the variable $t$ greatly affects the accuracy of the results, if the value of the variable $t$ is close to the initial value, the results will be more accurate. Also, the value of the constant $\alpha$ greatly affects the accuracy of the results. Certainly, the best value of $\alpha$ cannot be determined, as it changes from one problem to another.

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