Ensemble learning of linear perceptrons;
Online learning theory

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Abstract
Within the framework of on-line learning, we study the generalization error of an ensemble
learning machine learning from a linear teacher perceptron. The generalization error achieved by
an ensemble of linear perceptrons having homogeneous or inhomogeneous initial weight vectors is
precisely calculated at the thermodynamic limit of a large number of input elements and shows rich
behavior. Our main findings are as follows. For learning with homogeneous initial weight vectors,
the generalization error using an infinite number of linear student perceptrons is equal to only half
that of a single linear perceptron, and converges with that of the infinite case with $O(1/K)$ for a
finite number of $K$ linear perceptrons. For learning with inhomogeneous initial weight vectors, it is
advantageous to use an approach of weighted averaging over the output of the linear perceptrons,
and we show the conditions under which the optimal weights are constant during the learning
process. The optimal weights depend on only correlation of the initial weight vectors.

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I. INTRODUCTION

Many ensemble learning algorithms, such as bagging [1] and the Ada-boost [2] algorithm, try to improve upon the performance of a single learning machine by using many learning machines; such an approach has recently received considerable attention in the field of machine learning.

Theoretical analysis of the generalization error of ensemble learning has been done using statistical mechanics [3, 4]. Sollich analyzed batch-mode ensemble learning with linear perceptrons under the noisy learning condition [4], and demonstrated that the generalization error can be reduced by applying different subsets of entire learning examples. Urbanczik analyzed the generalization error of ensemble learning by using simple perceptrons based on on-line learning [3]. He discussed two types of ensemble learning with $K$ simple perceptrons. In the first scenario, a new simple perceptron obtained by averaging the weight vectors of the $K$ simple perceptrons was used as that of ensemble learning (model 1). In the second, the average of the outputs of the $K$ simple perceptrons was used as an output of the ensemble learning (model 2). Since the output property of the single perceptron is non-linear, it requires $O(e^K)$ calculations to theoretically obtain the generalization error for model 2 where $K$ is the number of simple perceptrons. Urbanczik mainly discussed model 1 to avoid this difficulty and demonstrated that in the limit of $K \rightarrow \infty$, the generalization error of model 2 converges to that of model 1.

When the linear perceptron is employed, models 1 and 2 are identical, and analysis for a finite number of $K$ becomes possible. This is the main reason we employed linear perceptrons in this paper. We assume two initial conditions to calculate the generalization error of the ensemble learning machine: one is that the correlation between the weight vectors of learning machines is homogeneous, and the other is that the correlation is inhomogeneous. In the homogeneous case, the correlation between the weight vectors of learning machines will be uniform and the weight vectors will remain uniform throughout the learning process because of the symmetry of the evolution equation used as the update rule. Thus, a simple average of the $K$ outputs of the learning machine can be used as the ensemble output of the learning machine. We derived the generalization error and found that it consist of two terms; the first depends on the number of learning machines $K$, while the second does not depend on $K$. It will be confirmed that the generalization error is equal to half that of a single learning
machine when $K \to \infty$, and that the generalization error converges into that of the infinite case with $O(1/K)$ when $K$ is finite.

In the inhomogeneous case, the generalization error can be improved by introducing weights to average the $K$ outputs of the learning machines (i.e., to obtain a weighted average rather than a simple average), and adapting the weights to minimize the generalization error (i.e., parallel boosting) \[\text{[7]}\] is required. We also analyze the time dependence of the weights used for averaging.

II. MODEL

A. Network structure and ensemble output

In this paper, we discuss the generalization error of ensemble learning with $K$ linear perceptrons. We assume the teacher and student networks receive $N$ dimensional input $\mathbf{x} = (x_1, \ldots, x_N)$ as shown in Fig. [I] and consider the thermodynamic limit of $N \to \infty$. We also assume that the elements $x_i$ of the independently drawn input $\mathbf{x}$ are uncorrelated random variables with zero mean and $1/N$ variance; that is, the elements are drawn from a probability distribution $P(\mathbf{x})$. The size of $\mathbf{x}$ is then $|\mathbf{x}| = 1$.

$$\langle x_i \rangle = 0, \quad \langle (x_i)^2 \rangle = \frac{1}{N}, \quad |\mathbf{x}| = 1, \quad (1)$$

where $\langle \cdots \rangle$ means averaging over the distribution $P(\mathbf{x})$. 

FIG. 1: Network structure of teacher and student networks, all having the same network structure.
The teacher is a linear simple perceptron (as shown) that outputs \( v(m) \) for \( N \) dimensional input \( \mathbf{x}(m) = (x_1(m), \ldots, x_N(m)) \) at \( m \)th learning iterations.

\[
v(m) = \sum_{i=1}^{N} B_i x_i(m) = \mathbf{B} \cdot \mathbf{x}(m), \quad (2)
\]

\[
\mathbf{B} = (B_1, \ldots, B_N), \quad (3)
\]

Each element \( B_i \) of the teacher weight vector \( \mathbf{B} \) is drawn from the probability distribution of zero mean and unit variance, and is fixed throughout the learning process. In this case, the size of the teacher weight vector is \( \sqrt{N} \),

\[
\langle B_i \rangle = 0, \quad \langle (B_i)^2 \rangle = 1, \quad |\mathbf{B}| = \sqrt{N}. \quad (4)
\]

\( K \) linear simple perceptrons are used as the student networks that compose the ensemble learning machine. Each student network has the same architecture as the teacher network and outputs \( u_k(m) \) for the \( N \) dimensional input \( \mathbf{x}(m) \).

\[
u_k(m) = \sum_{i=1}^{N} J^k_i(m)x_i(m) = \mathbf{J}^k(m) \cdot \mathbf{x}(m) \quad (5)
\]

\[
\mathbf{J}^k(m) = (J^k_1(m), \ldots, J^k_N(m)) \quad (6)
\]

where \( u_k(m) \) denotes \( k \)th student output and \( \mathbf{J}^k(m) \) denotes weight vector of \( k \)th student. The student weight vector is changed through the learning process, so the size of \( \mathbf{J}^k(m) \) is assumed to be \( |\mathbf{J}^k(m)| = l_k(m) \sqrt{N} \) and the size of \( l_k(m) \) is \( O(1) \). We call \( l_k(m) \) the length of the student weight vector \( \mathbf{J}^k(m) \) at \( m \)th learning iterations.

The ensemble output of the student networks \( \mathbf{u}(m) \) is given by the weighted average of each student network output with the weight for averaging \( C_k(m) \),

\[
\mathbf{u}(m) = \sum_{k=1}^{K} C_k(m)u_k(m) = \sum_{k=1}^{K} C_k(m)\mathbf{J}^k(m) \cdot \mathbf{x}(m), \quad (7)
\]

\[
\sum_{k=1}^{K} C_k(m) = 1. \quad (8)
\]

Bagging is a form of ensemble learning using a fixed uniform weight for averaging so that \( C_k(m) = 1/K \) throughout the learning process while parallel boosting uses the weighted average.
B. Learning algorithm

Learning is defined as a student network modifying the weight vector to make the output $u_k(m)$ approach the teacher output $v(m)$ for an given input $x(m)$. We use the gradient descent algorithm to modify the student’s weight vector $J^k(m)$. An identical input $x(m)$ is applied to all student networks in the same order. Therefore, all the student networks can independently learn the relation between the input $x(m)$ and the target $v(m)$. The learning equation is

$$J^k(m + 1) = J^k(m) + (v(m) - u_k(m))x(m),$$

where $m$ denotes the iteration number. As shown in Eq. (9), the weight $J^k(m)$ is updated by using single input $x(m)$, and then is not used again after the learning. This is called on-line learning. According to the above formulation, the weight vector $J^k(m)$ is statistically independent of a new learning input $x(m)$ and this makes analysis easier.

III. THEORY

In this paper, we consider the thermodynamic limit of $N \to \infty$ to analyze the dynamics of the present learning system using statistical mechanics. In the following sections, the iteration number $m$ is neglected to simplify notation of equations.

As pointed out, we will discuss learning based on on-line learning. In on-line learning, the input $x$ is not used after the learning and the weight vector $J^k$ is statistically independent of a new learning input. Note that the distribution of the normalized output of the student network $\tilde{u}_k = u_k/l_k$ obeys the Gaussian distribution of zero mean and unit variance at the thermodynamic limit of $N \to \infty$. For the same reason, the distribution of teacher output $v$ obeys the Gaussian distribution of zero mean and unit variance at the thermodynamic limit. Thus, the distribution $P(v, \{\tilde{u}_k\})$ of $v$ and $\{\tilde{u}_k\}$ is
\[
P(v, \{\tilde{u}_k\}) = \frac{1}{\sqrt{(2\pi)^{K+1}}} \exp \left( -\frac{1}{2} (v, \{\tilde{u}_k\})^T \Sigma^{-1} (v, \{\tilde{u}_k\}) \right)
\]  \hspace{1cm} (10)

\[
\Sigma = \begin{pmatrix}
1 & R_1 & \ldots & R_{K-1} & R_K \\
R_1 & 1 & q_{1,2} & \ldots & q_{1,K-1} & q_{1,K} \\
\vdots & q_{2,1} & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
R_{K-1} & q_{K-1,1} & \ldots & 1 & q_{K-1,K} \\
R_K & q_{K,1} & \ldots & q_{K,K-1} & 1
\end{pmatrix}
\]  \hspace{1cm} (11)

Here, \( R_k \) is the overlap between the teacher weight vector \( B \) and the student weight vector \( J^k \) (i.e., it is the direction cosine of \( B \) and \( J^k \)), and \( q_{kk'} \) is the overlap between two student weight vectors \( J^k \) and \( J^{k'} \). These two overlaps, are defined as

\[
R_k = \frac{B \cdot J^k}{|B| \cdot |J|} = \frac{1}{N l_k} \sum_{j=1}^{N} B_j J_j,
\]  \hspace{1cm} (12)

\[
q_{kk'} = \frac{J^k \cdot J^{k'}}{|J^k| \cdot |J^{k'}|} = \frac{1}{N l_k l_{k'}} \sum_{j=1}^{N} J^k_j J^{k'}_j.
\]  \hspace{1cm} (13)

\( R_k \) and \( q_{kk'} \) are the order parameters of the present learning system.

**A. Generalization error**

The squared error of the teacher output \( v \) and the ensemble output of the student networks \( \overline{\pi} \) is used to evaluate the student network’s performance.

\[
\epsilon = \frac{1}{2} (v - \overline{\tau})^2 = \frac{1}{2} \left( B \cdot \mathbf{x} - \sum_{k=1}^{K} C_k J^k \cdot \mathbf{x} \right)^2
\]  \hspace{1cm} (14)

Here, \( C_k \) is a weight for averaging and the ensemble output of the student networks is a weighted average of each student network’s output. The generalization error \( \epsilon_g \) is given by squared error \( \epsilon \) in Eq. (14) averaged over the possible input \( \mathbf{x} \) drawn from the Gaussian distribution \( P(\mathbf{x}) \) of zero mean and \( 1/N \) variance.
\[ \epsilon_g = \int dx P(x) \epsilon \]
\[ = \int dx P(x) \frac{1}{2} \left( \mathbf{B} \cdot \mathbf{x} - \sum_{k=1}^{K} C_k \mathbf{J}^k \cdot \mathbf{x} \right)^2 \] (15)

For on-line learning at the thermodynamic limit, as mentioned, the distributions of the stochastic variable \( \tilde{u}_k = (\mathbf{J}^k \cdot \mathbf{x})/l_k \) and \( v = \mathbf{B} \cdot \mathbf{x} \) obey the Gaussian distribution of zero mean and unit variance. Hence, the generalization error can be rewritten using Eqs. (10), (11),

\[ \epsilon_g = \int dv \prod_{k=1}^{K} d\tilde{u}_k P(v, \{\tilde{u}_k\}) \frac{1}{2} \left( v - \sum_{k=1}^{K} C_k u_k \right)^2 . \] (16)

This equation is the \((K+1)\)th Gaussian integral with \(\{\tilde{u}_k\}\) and \(v\), and it enables us to make the calculation. The result is shown by the following equation.

\[ \epsilon_g = \frac{1}{2} \left\{ 1 - 2 \sum_{k=1}^{K} C_k R_k l_k + \sum_{k=1}^{K} \sum_{k'=1}^{K} C_k C_{k'} q_{kk'} l_k l_{k'} \right\} \] (17)

Consequently, the dynamics of the generalization error is calculated by substituting the time step value of \( l_k, R_k, \) and \( q_{kk'} \) into Eq. (17). Therefore, we solve the dynamics of \( l_k, R_k, \) and \( q_{kk'} \) in the next subsection. Here, \( l_k, R_k, \) and \( q_{kk'} \) are macroscopic parameters that represent the system dynamics.

**B. Dynamics of order parameters**

We first derive the dynamics of the length of student weight vector \( l_k \). To obtain the differential equation of \( l_k \), we square both sides of equation (9). We then average the term of equation (9) by the distribution of \( P(v, \{\tilde{u}_k\}) \). Note that \( \mathbf{x} \) and \( \mathbf{J}^k \) are random variables, so the equation becomes a random recurrence formula. We formulate the size of the weight vectors to be \( O(\sqrt{N}) \), and the size of input \( \mathbf{x} \) is \( O(1) \), so the length of student weight vector \( l_k \) has a self-averaging property. Here, we rewrite \( m \) as \( m = Nt \), and represent the learning process using continuous time \( t \). We obtain the deterministic differential equation of \( l_k \) at
the thermodynamic limit,
\[
dl_k \frac{dt}{dt} = \frac{1 - l_k^2}{2l_k}
\] (18)

Note that \(l_k = 1\) is a stable fixed point of this equation. Next, we derive the differential equation of the overlap \(R_k\) between the teacher weight vector \(B\) and the student weight vector \(J^k\). The differential equation of overlap \(R_k\) is derived by calculating the product of \(B\) and Eq. (9), and then averaging the term of the equation by the distribution of \(P(v, \{\tilde{u}_k\})\). The overlap \(R_k\) also has a self-averaging property, and at the thermodynamic limit, the differential equation of \(R_k\) is then obtained through a calculation similar to that used for \(l_k\).

\[
dl_k \frac{dt}{dt} = \frac{1}{l_k} - \frac{R_k}{2} \left(1 + \frac{1}{l_k^2}\right)
\] (19)

To calculate the generalization error in Eq. (17) we have to obtain the differential equation of overlap \(q_{kk'}\) defined by Eq. (13). The overlap \(q_{kk'}\) also has a self-averaging property, so we can derive the differential equation at the thermodynamic limit. We define the overlap \(q_{kk'} = (J^k \cdot J^{k'})/Nl_k l_{k'}\), the differential equation is derived by calculating the product of Eq. (9) for \(J^k\) and the equation for \(J^{k'}\), and we obtain the deterministic differential equation below.

\[
dl_k \frac{dt}{dt} = \frac{1}{l_k l_{k'}} - \frac{q_{kk'}}{2} \left(\frac{1}{l_k^2} + \frac{1}{l_{k'}^2}\right)
\] (20)

Equations (18), (19), and (20) form the simultaneous differential equations.

IV. RESULTS

A. Homogeneous correlation of initial weight vectors

The component of independently drawn teacher weight vector \(B\) is as shown in Sec. II A and the component of the student’s weight vector \(J_i^k\), \(i = 1 \sim N\) is initialized by being drawn from independent random variables with zero mean and unit variance in this section. In this case, the initial values of the order parameters are

\[
l_k(0) = 1, \quad R_k(0) = 0, \quad q_{kk'}(0) = 0.
\] (21)
From the symmetry of the evolution equation for updating the weight vector,

\[ l_k(t) \rightarrow l(t), \quad R_k(t) \rightarrow R(t), \quad q_{kk'}(t) \rightarrow q(t) \tag{22} \]

are obtained. The dynamics of order parameters \( l(t), R(t), \) and \( q(t) \) are derived as Eqs. (23), (24), and (25) by substituting the above conditions into Eqs. (18), (19), and (20),

\[
\frac{dl(t)}{dt} = \frac{1 - l(t)}{2l(t)}, \tag{23}
\]
\[
\frac{dR(t)}{dt} = 1 - R(t), \tag{24}
\]
\[
\frac{dq(t)}{dt} = 1 - q(t). \tag{25}
\]

Note that \( l(t) = 1 \) is the stable fixed point of the dynamics of \( l(t) \) and is given by solving \( dl/dt = 0 \).

Next, we can easily solve the above equations analytically,

\[
l(t) = 1, \tag{26}
\]
\[
R(t) = 1 - \exp(-t), \tag{27}
\]
\[
q(t) = 1 - \exp(-t). \tag{28}
\]

From Eqs. (27) and (28), \( q(t) = R(t) \). Since the order parameters do not depend on \( K \), the optimal weights for averaging should be \( C_k = 1/K \). By substituting Eqs. (26), (27), and (28) for Eq. (17), the generalization error for \( K \) student networks is rewritten as

\[
\epsilon^K_g(t) = \frac{1}{2} \left( \frac{(l(t))^2(1 - q(t))}{K} + (q(t) - R(t)^2)(l(t))^2 + (R(t)l(t) - 1)^2 \right) \tag{29}
\]
\[
= \frac{1}{2} \left\{ \frac{1 - R(t)}{K} + (1 - R(t)) \right\} \tag{30}
\]
\[
= \frac{1}{2} \left\{ \frac{\exp(-t)}{K} + \exp(-t) \right\} \tag{31}
\]

\( \epsilon^K_g(t) \) denotes the generalization error with \( K \) student networks. The first term on the left side of this equation depends on the number of student networks \( K \) and becomes negligible when \( K \) goes to infinity. The second term does not depend on \( K \), so it remains and cannot
(a) Theoretical results. 

FIG. 2: Dependence of the ensemble learning generalization error on the number of student networks $K$.

be ignored. Substituting $K = 1$ for Eq. (30), we show that the generalization error for a single student network is $\epsilon_1^g(t) = 1 - R(t)$, and this error is identical to the generalization error of a simple perceptron [5]. From Eq. (30), when $K$ becomes infinite, the generalization error of an ensemble of $K$ student networks asymptotically converges to $(1 - R(t))/2$. Hence, the generalization error of an ensemble of $K$ student networks converges to half that of a single student network at the limit of $K$ going to infinity. Note that Eqs. (30) and (31) depict dynamics of the generalization error of the ensemble because $l(t) = 1$ is the fixed point of the learning process as shown in the formulation of the initialization given in this subsection.

Figure 2 shows the $K$ dependence of the generalization error of the ensemble: (a) shows theoretical results obtained using Eq. (31), and (b) shows the results obtained through a numerical simulation. In these figures, the horizontal axis is time $t = m/N$, and the unit time corresponds to the time needed to feed in $N$ inputs $x$. The vertical axis is the generalization error $\epsilon_g$.

First, the theoretical results for $K = 1, 3, 10,$ and 10000 are shown in Fig. 2(a) as the diagonal lines. The generalization error for larger $K$ (the lower lines) shifted and converged to half of the generalization error for $K = 1$ with respect to the order of $1/K$. Next, the simulation results for $K = 1, 3,$ and 10 are shown in Fig. 2(b). These results agree with the theoretical results, confirming the validity of the theoretical results. Hence, in the following
analysis, we show only the theoretical results.

**B. Inhomogeneous correlation of initial weight vectors**

When the correlation between student weight vectors is inhomogeneous, for instance, if the number of student networks is $K=3$, and the initial student weight vector $J^1(0) = J^2(0)$ and $J^3(0)$ is independent of $J^1(0) = J^2(0)$, it seems natural to select weights for averaging where $C_1 = C_2 = 0.25$ and $C_3 = 0.5$, instead of using the uniform value of $C_1 = C_2 = C_3 = 1/3$. This consideration suggests that the optimal generalization error can be obtained by using a weighted average of the student outputs when the correlation between students $q_{kk'}$ is inhomogeneous. This method is called “parallel boosting”[7].

Because the student weight vectors are inhomogeneous, overlap $R_k$ and overlap $q_{kk'}$ differ from subscript $k$ or $k'$, and a weight for averaging $C_k$ will depend on $R_k$ and $q_{kk'}$ to reduce the generalization error. We assume that the length of the student weight vector $l_k(0) = 1$, thus it is identical for subscript $k$.

The optimal weights $C_k$ satisfy the following condition,

$$\frac{\partial \epsilon_g}{\partial C_k} = 0. \quad (32)$$

Since $\epsilon_g$ is quadratic (second-order) function with respect to $C_k$, Eq. (32) becomes a linear equation, and we can easily obtain the optimal $C_k$ when the order parameter $l_k(t)$, $R_k(t)$ and $q_{kk'}(t)$ are given. Order parameters $R_k(t)$ and $q_{kk'}(t)$ are time-dependent parameters, so the optimal weights for average $C_k$ given by Eq. (32) generally become time-dependent $C_k(t)$. In this case, we assumed $l_k(0) = 1$, which then means $l_k(t) = 1$ (see Eq. (23)). Substituting $l_k(t) = 1$ and $\sum_{k=1}^K C_k(t) = 1$ to Eq. (17), we obtain the generalization error as

$$\epsilon_g(t) = 1 - \sum_{k=1}^{K-1} C_k(t)(R_k(t) - R_K(t)) - R_K(t)$$

$$+ \sum_{k=1}^{K-1} C_k^2(t)(1 - q_{kk}(t)) - \sum_{k=1}^{K-1} C_k(t)(1 - q_{kk}(t))$$

$$+ \sum_{k=1}^{K-1} \sum_{k'=2}^{K-1} C_k(t)C_{k'}(t)(1 + q_{kk'}(t) - q_{kk}(t) - q_{k'K}(t)). \quad (33)$$

Equations (19) and (20) can then be solved analytically,
\[ R_k(t) = 1 - (1 - R_k(0)) \exp(-t), \]  
(34)  
\[ q_{kk'}(t) = 1 - (1 - q_{kk'}(0)) \exp(-t), \]  
(35)

where \( R_k(0) \) and \( q_{kk'}(0) \) are the initial values of \( R_k(t) \) and \( q_{kk'}(t) \), respectively. The dynamics of the generalization error is given by substituting Eqs. (34) and (35).

\[
\epsilon_g(t) = \exp(-t) \left\{ 1 - R_K(0) - \sum_{k=1}^{K-1} C_k(t) (R_k(0) - R_K(0)) + \sum_{k=1}^{K-1} C_k^2(t) (1 - q_{kK}(0)) - \sum_{k=1}^{K-1} C_k(t) (1 - q_{kK}(0)) + \sum_{k=1}^{K-1} \sum_{k'=2}^{K-1} C_k(t) (1 + q_{kk'}(0) - q_{kK}(0) - q_{k'K}(0)) \right\}
\]  
(36)

As we mentioned, the optimal weight for average \( C_k \) generally depends on time \( t \) because \( \epsilon_g \) is a function of time \( t \). However, in this case, \( \epsilon_g \) only depends on the initial value of order parameters \( R_k(0) \) and \( q_{kk'}(0) \) as shown in Eq. (36). \( \partial \epsilon_g / \partial C_k \) is also independent of time \( t \). Thus the optimal weight for average \( C_k \) does not depend on time \( t \) in this case.

Figure 3 shows the ratio of the generalization error with and without parallel boosting for \( K = 3 \) student networks. Two of the student networks were identical (\( J^1(0) = J^2(0) \)), while \( J^3(0) \) was independent of the others. The generalization error using parallel boosting is denoted as \( \epsilon_g^{\text{PB}} \), and that using bagging is denoted as \( \epsilon_g^{\text{B}} \) in this figure. As the figure shows, parallel boosting is effective and the generalization error ratio with and without parallel boosting was the same throughout the learning process. The ratio of \( \epsilon_g^{\text{PB}} / \epsilon_g^{\text{B}} \) was about 0.96.

V. CONCLUSION

We have analyzed the generalization error of an ensemble of linear perceptrons within the framework of on-line learning. Weights for averaging were introduced. We then derived simultaneous differential equations for the order parameters \( l_k, R_k, \) and \( q_{kk'} \) to calculate the generalization error. Here, \( l_k \) was the length of the student weight vector \( J^k \), \( R_k \) was the
overlap between the teacher weight vector $B$ and the student weight vectors $J^k$, and $q_{kk'}$ was the overlap between two student weight vectors $J^k$ and $J^{k'}$. We have assumed two initial conditions to calculate the generalization error of the ensemble of linear perceptrons: one was that the correlation between the weight vectors of linear perceptrons was homogeneous, and the other was that the correlation was inhomogeneous.

In the homogeneous case, simple averaging over the $K$ outputs of the linear perceptrons was valid to obtain the ensemble output of the linear perceptrons. We found that the generalization error was equal to half that of a single linear perceptron when the number of linear perceptrons $K$ became infinite, and that the generalization error converged into that of the infinite case with $O(1/K)$ when the number of linear perceptrons was finite.

In the inhomogeneous case, the generalization error was improved by introducing the weights for averaging over the $K$ outputs of the linear perceptrons. Order parameters $R_k$ and $q_{kk'}$ are time dependent, so one might think the weight for averaging over the $K$ outputs of the linear perceptrons might be time dependent. However, we found the weights were not time dependent when the initial value of the weight length $l_k(0) = 1$, and they only depended on the initial correlation between the weight vectors of the linear perceptrons. We also carried out numerical simulations whose results agreed with the theoretical results, thus confirming the validity of the theoretical analysis.

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