Coartinianess of local homology modules for ideals of small dimension

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Abstract
Let $a$ be an ideal of a commutative noetherian ring $R$ and $M$ an $R$-module with Cosupport in $V(a)$. We show that $M$ is $a$-coartinian if and only if $\text{Ext}_R^i(R/a, M)$ is artinian for all $0 \leq i \leq \text{cd}(a, M)$, which provides a computable finitely many steps to examine $a$-coartinianness. We also consider the duality of Hartshorne’s questions: for which rings $R$ and ideals $a$ are the modules $H_a^i(M)$ $a$-coartinian for all $i \geq 0$; whether the category $\mathcal{C}(R, a)_{a\text{-coa}}$ of $a$-coartinian modules is an Abelian subcategory of the category of all $R$-modules, and establish affirmative answers to these questions in the case $\text{cd}(a, R) \leq 1$ and $\text{dim}R/a \leq 1$.

Key Words: coartinian module; local homology; semi-discrete linearly compact module

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1. Introduction and Preliminaries
The theory of local cohomology has been developed rapidly for the last 60 years after it was introduced by Grothendieck. Let $R$ denote a commutative noetherian ring with identity and $a$ an ideal of $R$. For an $R$-module $M$, the $i$th local cohomology module of $M$ with respect to $a$ is defined as

$$H_a^i(M) = \lim\inf \text{Ext}_R^i(R/a^t, M).$$

Local homology as its duality was initiated by Matlis [11] in 1974. Denote $\Lambda^a(M) = \varprojlim M/a^t M$ is the $a$-adic completion of $M$. Even if $R$ is noetherian, the $a$-adic functor $\Lambda^a(-)$ is neither left nor right exact on the category of all $R$-modules. Hence the computation of the left derived functors $L_a^i(-)$ of the $a$-adic functor $\Lambda^a(-)$ is very difficult and so local homology is not known so much. Moreover, we recall the $i$th local homology module of $M$ is

$$H_a^i(M) = \lim\inf \text{Tor}_i^R(R/a^t, M).$$

Cuong and Nam [4] showed that $\Lambda^a(-)$ has right exactness on the category of semi-discrete linearly compact $R$-modules, which says that $\Lambda^a(M) = H_a^0(M) \cong L_a^0(M)$ and $H_a^i(M) \cong L_a^i(M)$ for $i > 0$ when $M$ is a semi-discrete linearly compact $R$-module.

Let $(R, m, k)$ be a complete local ring, $E$ an injective hull of $k$ over $R$. We recall the following various results due to Matlis and Grothendieck.
Fact 1.1. For an $R$-module $N$, the following conditions are equivalent.
(1) $N$ is artinian.
(2) $N$ is isomorphic to a submodule of a finite direct sum of copies of $E$.
(3) There is a finitely generated $R$-module $M$ such that $N \cong \text{Hom}_R(M, E)$.
(4) $\text{Supp}_R N = \{m\}$ and $\text{Hom}_R(k, N)$ is finitely generated.

Hartshorne [7] called $N$ $m$-cofinite if it satisfies the equivalent conditions of the above fact, and the $m$-cofinite modules form an abelian subcategory of the category of all $R$-modules. For an ideal $a$ of $R$, Hartshorne defined an $R$-module $L$ to be $a$-cofinite, if $\text{Supp}_R(L) \subseteq V(a)$ and $\text{Ext}^i_R(R/a, L)$ is finitely generated for all $i$, which is to generalize condition (4) of the fact. Cofinite module is an important tool in the studying of Huneke’s problems on local cohomology. Hartshorne also posed two questions on cofinite modules:

**Question 1.** Are the local cohomology modules $H^i_a(M)$ $a$-cofinite for every finitely generated $R$-module $M$ and every $i \geq 0$?

**Question 2.** Whether the category $\mathcal{C}(R, a)_{\text{cof}}$ of $a$-cofinite modules is an Abelian subcategory of the category of all $R$-modules?

Question 1 and 2 have been high-profile among researchers. For example, see [1 2 3 5 6 9 10 14 16 23].

Combining Fact 1.1 with Matlis duality, it is clear when $(R, m)$ is a complete local ring, an $R$-module $N$ is finitely generated is equivalent to $\text{Cosupp}_R N \subseteq \{m\}$ and $N/mN$ is artinian. Hence Nam [17] defined $a$-coartinian module which is in some sense dual to the concept of cofinite module. Recall that an $R$-module $M$ is $a$-coartinian if $\text{Cosupp}_R M \subseteq V(a)$, and $\text{Tor}^R_i(R/a, M)$ is artinian for all $i \geq 0$.

The first aim of this paper is to give an equivalent characterization of coartinian modules. More precisely, we show that if $M$ is an $R$-module with $\text{Cosupp}_R M \subseteq V(a)$, then $M$ is $a$-coartinian if and only if $\text{Ext}^i_R(R/a, M)$ is artinian for all $0 \leq i \leq \text{cd}(a, M)$ in section 2.

Nam [18] posed a question on local homology: whether the number of coassociated primes of local homology modules $H^i_a(M)$ always finite for every $i \geq 0$. One can easily observe that if the local homology $R$-module $H^i_a(M)$ is $a$-coartinian, then the set $\text{Coass}_R H^i_a(M)$ and the Bass number $\mu^i_R(\mathfrak{P}, D(H^i_a(M)))$ are finite for any $\mathfrak{P} \in \text{Spec} R$, when $(R, m)$ is local with $D(-) = \text{Hom}_R(-, E(R/m))$ and $\hat{R}$ is the $m$-adic completion of $R$.

In section 3, we consider the following two questions on coartinian modules which are dual to Hartshorne’s questions:

**Question 1’.** For which rings $R$ and ideals $a$ are the modules $H^i_a(M)$ $a$-coartinian for all $i \geq 0$?

**Question 2’.** Whether the category of $a$-coartinian modules is an Abelian subcategory of the category of all $R$-modules?

For the first question, we show that when $(R, m)$ is a local ring with the $m$-adic topology and $M$ a semi-discrete linearly compact $R$-module with $\text{mag}_R(M) \leq 1$, then $H^i_a(M)$ is $a$-coartinian for all $i$. Subsequently, we answer Question 2’ completely in the case $\text{cd}(a, R) \leq 1$ and $\dim R/a \leq 1$. 

Next we recall some notions which will need later.

We write Spec$R$ for the set of prime ideals of $R$, Max$R$ for the set of maximal ideals of $R$. For an ideal $a$ of $R$, set

$$V(a) = \{ p \in \text{Spec}R | a \subseteq p \}.$$

Fix $p \in \text{Spec}R$, let $M_p$ denote the localization of $M$ at $p$.

**Associated prime and Coassociated prime.** Let $M$ be an $R$-module. The set of associated prime ideals of $M$ is denoted by Ass$_R M$ and it is the set of primes $p$ such that there exists a cyclic submodule $N$ of $M$ with $p = \text{Ann}_R N$.

Yassemi [21] introduced the cocyclic modules. An $R$-module $L$ is cocyclic if $L$ is a submodule of $E(R/m)$ for some $m \in \text{Max}R$. A prime ideal $p$ of $R$ is called a coassociated prime of $R$-module $M$ if there exists a cocyclic homomorphic image $L$ of $M$ such that $p = \text{Ann}_R L$, the annihilator of $L$. The set of coassociated prime ideals of $M$ is denoted by Coass$_R M$.

**Support and Cosupport.** The “large” support of an $R$-module $M$ which is denoted by Supp$_R M$ is defined as the set of prime ideals of $p$ such that there is a cyclic submodule $N$ of $M$ with $\text{Ann}_R N \subseteq p$.

The cosupport of $M$ is defined as the set of prime ideals $p$ such that there is a cocyclic homomorphic image $L$ of $K$ with $p \supseteq \text{Ann}_R L$, and denoted this set by Cosupp$_R M$.

**Dimension and Magnitude.** The (Krull) dimension of an $R$-module $0 \neq M$ is

$$\dim_R M = \sup\{ \dim R/p | p \in \text{Supp}_R M \}.$$ 

If $M = 0$, we write $\dim_R M = -\infty$.

Yassemi [22] introduced a dual concept of dimension for modules and called it magnitude of modules and denoted $\text{mag}_R M$.

$$\text{mag}_R M = \sup\{ \dim R/p | p \in \text{Cosupp}_R M \}.$$ 

If $M = 0$, we write $\text{mag}_R M = -\infty$.

**Cohomological dimension and homological dimension.** Let $a$ be an ideal of $R$, $M$ an $R$-module. Recall that the cohomological dimension of $M$ with respect to $a$ is

$$\text{cd}(a, M) = \sup\{ i \geqslant 0 | H^i_a(M) \neq 0 \}.$$ 

The homological dimension of $a$ with respect to $M$ is

$$\text{hd}(a, M) = \sup\{ i \geqslant 0 | H_i^a(M) \neq 0 \}.$$ 

Throughout this paper, $R$ is a commutative noetherian ring with non-zero identity and $a$ an ideal of $R$.

2. Coartinianness of modules

This section will provide a computable finitely many steps to examine $a$-coartinianness for an $R$-module $M$ with Cosupp$_R M \subseteq V(a)$.

**Lemma 2.1.** Let $a$ be an ideal of $R$, $M$ an $R$-module and $s > 0$. If $\text{Ext}_R^i(R/a, H^j_a(M))$ is artinian for $i \geqslant 0$ and $0 \leqslant j \leqslant s - 1$ and $\text{Ext}_R^s(R/a, M)$ is artinian, then $H^s_a(M)$ is artinian.
Proof. There is a spectral sequence
\[ E_2^{pq} = \text{Ext}^p_R(R/\mathfrak{a}, H^q_\mathfrak{a}(M)) \Rightarrow \text{Ext}^{p+q}_R(R/\mathfrak{a}, M). \]
For \( s \geq 0 \), there is a finite filtration
\[ 0 = U^{-1} \subseteq U^0 \subseteq \cdots \subseteq U^{s-1} \subseteq U^s = \text{Ext}^s_R(R/\mathfrak{a}, M), \]
such that \( U^p/U^{p-1} \cong E^{p,s-p}_\infty \) for every \( 0 \leq p \leq s \). As \( \text{Ext}^s_R(R/\mathfrak{a}, M) \) is artinian, we have \( E^{0,s}_\infty \cong U^0/U^1 \) is artinian. Let \( r \geq 2 \), consider the differentials
\[ 0 = E^{-r,s+r-1}_{r} \xrightarrow{d_{r}} E^0_{r,s} \xrightarrow{d_{r,s}} E^{r,s-r+1}_r, \]
where the vanishing comes from the facts that \( E^{-r,s+r-1}_r = 0 \) and \( E^{r,s+r-1}_r \) is a subquotient of \( E^{-r,s+p-1}_p \) for all \( r \geq 2 \). On the other hand, we obtain \( E^{r+1}_0 = \text{Ker}_{d_{r}}^{0,s}/\text{Im}_{d_{r}}^{r,s+r-1} = \text{Ker}_{d_{r}}^{0,s} \) and a short exact sequence
\[ 0 \rightarrow E^0_{r,s} \rightarrow E^0_{r,s} \rightarrow \text{Im}_{d_{r}}^{0,s} \rightarrow 0. \]
As \( s - r + 1 \leq s - 1 \), the hypothesis implies that \( E^{r,s-r+1}_r \) is artinian, consequently \( \text{Im}_{d_{r}}^{0,s} \) is artinian. Note that \( E^0_{r,s} = E^0_{r+1,s} \) for every \( r \geq r_0 = s + 2 \). It follows that \( E^0_{r+1,s} \) is artinian. Hence the above short exact sequence inductively, we conclude that \( E^{0,s}_r = \text{Hom}_R(R/\mathfrak{a}, H^s_\mathfrak{a}(M)) \) is artinian. Hence, \[ \text{[12] Theorem 1.3} \] implies that \( H^s_\mathfrak{a}(M) \) is artinian. \( \square \)

The following Lemma is crucial to the proof of the main result in the section.

Lemma 2.2. Let \( M \) be an \( R \)-module. Then the following conditions are equivalent.
1. \( H^i_\mathfrak{a}(M) \) is artinian for all \( i \geq 0 \).
2. \( \text{Ext}^i_R(R/\mathfrak{a}, M) \) is artinian for all \( i \geq 0 \).
3. \( \text{Ext}^i_R(N, M) \) is artinian for every finitely generated \( R \)-module \( N \) with \( \text{Supp}_R N \subseteq V(\mathfrak{a}) \) and every \( i \geq 0 \).
4. \( \text{Ext}^i_R(R/\mathfrak{a}, M) \) is artinian for all \( 0 \leq i \leq \text{cd}(\mathfrak{a}, M) \).

Proof. (1) \( \Rightarrow \) (2) Holds by \[ \text{[1] Proposition 3.3}. \]
(2) \( \Rightarrow \) (3) Let \( N \) be a finitely generated \( R \)-module with \( \text{Supp}_R N \subseteq V(\mathfrak{a}) \). We argue by induction on \( i \). Let \( i = 0 \), we show \( \text{Hom}_R(N, M) \) is artinian. Using the improved consequence of Gruson’s Theorem \[ \text{[1] Lemma 2.2}, \] there is a finite filtration
\[ 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{t-1} \subseteq N_t, \]
such that \( N_j/N_{j-1} \) is a quotient of \( R/\mathfrak{a} \) for \( 1 \leq j \leq t \). For \( 1 \leq j \leq t \), the exact sequence
\[ 0 \rightarrow K_j \rightarrow R/\mathfrak{a} \rightarrow N_j/N_{j-1} \rightarrow 0, \]
induces an exact sequence
\[ 0 \rightarrow \text{Hom}_R(N_j/N_{j-1}, M) \rightarrow \text{Hom}_R(R/\mathfrak{a}, M). \]
It implies that $\text{Hom}_R(N_j/N_{j-1}, M)$ is artinian. Now the exact sequence

$$0 \to N_{j-1} \to N_j \to N_j/N_{j-1} \to 0,$$

yields the exact sequence

$$0 \to \text{Hom}_R(N_j/N_{j-1}, M) \to \text{Hom}_R(N_j, M) \to \text{Hom}_R(N_{j-1}, M).$$

Using the above exact sequences successively, we have $\text{Hom}_R(N, M)$ is artinian. Now assume that the result holds for less than $i$, it suffices to show $\text{Ext}^i_R(N, M)$ is artinian. For $1 \leq j \leq t$, there is an exact sequence

$$\text{Ext}^j_R(K_j, M) \to \text{Ext}^j_R(N_j/N_{j-1}, M) \to \text{Ext}^j_R(R/\mathfrak{a}, M).$$

The induction hypothesis shows that $\text{Ext}^j_R(K_j, M)$ is artinian since $K_j$ is finitely generated and $\text{Supp}_RK_j \subseteq V(\mathfrak{a})$. From the above exact sequence we get $\text{Ext}^j_R(N_j/N_{j-1}, M)$ is artinian.

Note that there is also an exact sequence

$$\text{Ext}^j_R(N_j/N_{j-1}, M) \to \text{Ext}^j_R(N_j, M) \to \text{Ext}^j_R(N_{j-1}, M).$$

A successive use of the above exact sequences implies that $\text{Ext}^j_R(N, M)$ is artinian.

(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (1) Since $H^i_a(M) = 0$ is artinian for $i > \text{cd}(a, M)$, we use induction on $s = \text{cd}(a, M)$ to show that if $\text{Ext}^i_R(R/\mathfrak{a}, M)$ is artinian for $0 \leq i \leq s$, then $H^i_a(M)$ is artinian for $0 \leq i \leq s$. If $s = 0$, then $\text{Hom}_R(R/\mathfrak{a}, M)$ is artinian, it implies that $\text{Hom}_R(R/\mathfrak{a}, H^0_a(M))$ is artinian because $H^0_a(M) \cong \Gamma_a(M)$ is a submodule of $M$. Therefore, it follows from [12, Theorem 1.3] that $H^0_a(M)$ is artinian since $H^0_a(M)$ is $a$-torsion. Now assume that $s > 0$ and the result holds for $s - 1$. Then $H^s_a(M)$ is artinian for $0 \leq j \leq s - 1$. So $\text{Ext}^i_R(R/\mathfrak{a}, H^s_a(M))$ is artinian for $i \geq 0$ and $0 \leq j \leq s - 1$, and $\text{Ext}^s_R(R/\mathfrak{a}, M)$ is artinian. Hence Lemma 2.1 implies that $H^s_a(M)$ is artinian.

\[\square\]

**Theorem 2.3.** Let $\mathfrak{a}$ be an ideal of $R$ and $M$ an $R$-module with $\text{Cosupp}_RM \subseteq V(\mathfrak{a})$. Then the following conditions are equivalent.

1. $M$ is $\mathfrak{a}$-coartinian.
2. $\text{Ext}^i_R(R/\mathfrak{a}, M)$ is artinian for all $0 \leq i \leq \text{cd}(a, M)$.

**Proof.** According to [11, Proposition 3.3], the $R$-module $\text{Ext}^i_R(R/\mathfrak{a}, M)$ is artinian if and only if $\text{Tor}^i_R(R/\mathfrak{a}, M)$ is artinian for each $i \geq 0$. Therefore, the proof is clear by Lemma 2.2. \[\square\]

**Lemma 2.4.** Let $f: M \to N$ be an $R$-homomorphism. If both $\text{Tor}^i_R(R/\mathfrak{a}, \ker(f))$ and $\text{Tor}^i_R(R/\mathfrak{a}, \text{coker}(f))$ are artinian, then $\ker(\text{Tor}^i_R(R/\mathfrak{a}, f))$ and $\text{coker}(\text{Tor}^i_R(R/\mathfrak{a}, f))$ are also artinian, for all $i \geq 0$.

**Proof.** It is clear by [14, Lemma 3.1]. \[\square\]

The following proposition is very useful to decide whether a module is coartinian.

**Proposition 2.5.** Suppose that $x \in \mathfrak{a}$ and $\text{Cosupp}_RM \subseteq V(\mathfrak{a})$. If $(0 :_M x)$ and $M/xM$ are both $\mathfrak{a}$-coartinian, then $M$ is also $\mathfrak{a}$-coartinian.
Proof. Consider $R$-homomorphism $x : M \to M$. Then $\ker x = (0 : M, x)$, $\coker x = M/xM$. Since $\ker x$ and $\coker x$ are $\mathfrak{a}$-coartinian, we have $\Tor_i^R(R/\mathfrak{a}, \ker x)$ and $\Tor_i^R(R/I, \coker x)$ are artinian for all $i$. Apply Lemma 2.4 to $x$, $\ker \Tor_i^R(R/\mathfrak{a}, x)$ and $\coker \Tor_i^R(R/\mathfrak{a}, x)$ are artinian for all $i$. While $x \in \mathfrak{a}$, $\Tor_i^R(R/\mathfrak{a}, x) = 0$ for all $i$, hence $\ker \Tor_i^R(R/\mathfrak{a}, x) = \Tor_i^R(R/\mathfrak{a}, M)$ is artinian. Thus $M$ is also $\mathfrak{a}$-coartinian.

□

3. Two questions on $\mathfrak{a}$-coartinian modules

In this section, we give an affirmative answer for these two questions in introduction when $\text{cd}(\mathfrak{a}, R) \le 1$ or $\dim R/\mathfrak{a} \le 1$ in the category of semi-discrete linearly compact $R$-modules.

We begin by recalling definition of linearly compact modules that we will use. Let $M$ be a topological $R$-module. $M$ is said to be linearly topologized if $M$ has a base of neighborhoods of the zero element $\mathcal{M}$ consisting of submodules. $M$ is called Hausdorff if the intersection of all the neighborhoods of the zero element is 0. A Hausdorff linearly topologized $R$-module $M$ is said to be linearly compact if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in $M$ which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection (see [10]). A Hausdorff linearly topologized $R$-module $M$ is called semi-discrete if every submodule of $M$ is closed. The class of semi-discrete linearly compact $R$-modules contains all artinian $R$-modules.

Lemma 3.1. Let $M$ be a linearly compact $R$-module, then $H_i^0(M) = 0$ for all $i > \text{mag}_R(M)$.

Proof. It follows from [8] Theorem 2.8. □

Lemma 3.2. Let $M$ be a linearly compact $R$-module such that $\Tor_i^R(R/\mathfrak{a}, M)$ is artinian for every $i$. If $s$ is an integer such that $H_i^s(M)$ is $\mathfrak{a}$-coartinian for all $i \neq s$, then $H_i^s(M)$ is $\mathfrak{a}$-coartinian.

Proof. Since $M$ is linearly compact, then $\text{Cosupp}_R H_i^s(M) \subseteq V(\mathfrak{a})$. Hence we only need to show that $\Tor_j^R(R/\mathfrak{a}, H_i^s(M))$ is artinian for all $j \ge 0$. Note that there is a spectral sequence

$$E^2_{p,q} = \Tor_p^R(R/\mathfrak{a}, H_q^s(M)) \Rightarrow \Tor_{p+q}^R(R/\mathfrak{a}, M).$$

We argue by induction on $j$. Let $j = 0$, we show that $R/\mathfrak{a} \otimes_{R} H_i^s(M)$ is artinian. There exists a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \cdots \subseteq U^s = \Tor_i^R(R/\mathfrak{a}, M),$$

such that $U^p/U^{p-1} \cong E_{p-s}^{\infty}$ for every $0 \le p \le s$. As $\Tor_i^R(R/\mathfrak{a}, M)$ is artinian, we conclude that $E_0^{\infty} \cong U^0/U^{-1}$ is artinian. Assume that $r \ge 2$, consider the differentials

$$E_{r,s-r+1}^r \xrightarrow{d_{r,s-r+1}} E_{0,s}^r \xrightarrow{d_{0,s}} E_{-r,s+r-1}^r = 0.$$
Since $H_i^a(M)$ is $a$-coartinian for $i \neq s$, $E_{r,s-r+1}^2$ is artinian. On the other hand, $E_{r,s-r+1}^r$ is a subquotient of $E_{r,s-r+1}^2$, it follows that $E_{r,s-r+1}^r$ is artinian and consequently $\text{Im}d_{r,s-r+1}^r$ is also artinian. We obtain a short exact sequence
\[ 0 \to \text{Im}d_{r,s-r+1}^r \to E_{0,s}^r \to E_{0,s}^{r+1} \to 0.\]
There is an integer $r_0 \geq 2$, such that $E_{r,s-r+1}^r = 0$ for all $r \geq r_0$, hence $E_{0,s}^{r+1} \cong E_{0,s}^\infty$ for every $r \geq r_0$. It follows that $E_{0,s}^{r_0+1}$ is artinian. Now the short exact sequence implies that $E_{0,s}^r$ is artinian. Using the short exact sequence inductively, we conclude that $E_{0,s}^r = R/a \otimes H_s^a(M)$ is artinian. Now suppose that the result holds for less than $j$, we show that $E_{j,s}^2 = \text{Tor}_j^R(R/a, H_s^a(M))$ is artinian. Let $r \geq 2$, consider the following differentials
\[ E_{j+r,s-r+1}^r \xrightarrow{d_{j+r,s-r+1}^r} E_{j,s}^r \xrightarrow{d_{j,s}^r} E_{j-r,s+r-1}^r.\]
Since $H_s^a(M)$ is $a$-coartinian for $i \neq s$, $E_{j+r,s-r+1}^2$ and $E_{j-r,s+r-1}^2$ are artinian, while $E_{j+r,s-r+1}^r$ is a subquotient of $E_{j+r,s-r+1}^2$ and $E_{j-r,s+r-1}^r$ is a subquotient of $E_{j-r,s+r-1}^2$. Hence $E_{j+r,s-r+1}^r$ and $E_{j-r,s+r-1}^r$ are artinian, and thus $\text{Im}d_{j+r,s-r+1}$ and $\text{Im}d_{j,s}^r$ are artinian. There is $r_1 \geq 2$, such that $E_{j+r,s-r+1}^r = 0$, hence we have an exact sequence
\[ 0 \to E_{j,s}^{r_1} \to E_{j-r_1,s+r_1-1}^{r_1},\]
and since $E_{j-r_1,s+r_1-1}^{r_1}$ is artinian, it follows that $E_{j,s}^{r_1}$ is artinian. On the other hand, there are two short exact sequences
\[ 0 \to \text{Ker}d_{j,s}^r \to E_{j,s}^r \to \text{Im}d_{j,s}^r \to 0,\]
\[ 0 \to \text{Im}d_{j+r,s-r+1} \to \text{Ker}d_{j,s}^r \to E_{j,s}^{r+1} \to 0.\]
These two short exact sequences imply that $E_{j,s}^{r+1}$ is artinian. Using the short exact sequences inductively, we have $E_{j,s}^2 = \text{Tor}_j^R(R/a, H_s^a(M))$ is artinian. \hfill \Box

A sequence of elements $x_1, \ldots, x_n$ in $R$ is said to be an $M$-coregular sequence if $(0 :_M (x_1, \ldots, x_{i-1})) \to (0 :_M (x_1, \ldots, x_{i-1}))$ is surjective and $(0 :_M (x_1, \ldots, x_i)) \neq 0$ for $i = 1, \ldots, n$. We denote by $\text{width}_a(M)$ the supremum of the lengths of all maximal $M$-coregular sequences in the ideal $a$. Following [4], if $M$ is a semi-discrete linearly compact $R$-module, then
\[ \text{width}_a M = \inf \{ i \geq 0 | H_i^a(M) \neq 0 \}. \]

**Corollary 3.3.** If $M$ is a semi-discrete linearly compact $R$-module such that $H_i^a(M) \neq 0$ for all $i \neq s$. i.e. $\text{width}_a M = \text{hd}(a, M) = s$, then $H_i^a(M)$ is $a$-coartinian.

**Proof.** It is clear from Lemma [3.2] \hfill \Box

**Corollary 3.4.** If $\text{hd}(a, M) \leq 1$, then $H_i^a(M)$ is $a$-coartinian for all $i$ and every artinian $R$-module $M$.  

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Proof. Since $\text{hd}(a, M) \leq 1$, $H_i^a(M) = 0$ for $i > 1$. On the other hand, $H_0^a(M) = \Lambda^a(M)$ is artinian because $M$ is artinian. Hence $H_0^a(M)$ is $a$-coartinian and therefore $H_1^a(M)$ is also $a$-coartinian by Lemma 3.2.

**Lemma 3.5.** Let $M$ be a semi-discrete linearly compact $R$-module with $\text{mag}_R M = 0$. Then $H_i^a(M)$ is $a$-coartinian for $i \geq 0$.

Proof. Since $M$ is semi-discrete linearly compact, we have $\text{Cosupp}_R H_i^a(M) \subseteq V(a)$ and $H_i^a(M) = 0$ for $i > 0$ by Lemma 3.1. We only need to show that $H_0^a(M)$ is $a$-coartinian. The proof will be divided into two steps.

**Step 1.** Assume that $M$ is an artinian module. It consequently follows from Lemma 3.2 that $H_0^a(M)$ is $a$-coartinian.

**Step 2.** Suppose that $M$ is a semi-discrete linearly compact $R$-module, $\mathcal{M}$ be a base consisting of neighborhood of the zero element of $M$. By [10, 5.5], $M \cong \varprojlim_{U \in \mathcal{M}} M/U$ and $M/U$ is an artinian $R$-module with $\text{mag}_R M/U = 0$ for all $U \in \mathcal{M}$. As $\text{mag}_R M = 0$ and $M$ is semi-discrete linearly compact, $\text{Ndim} M = \text{mag}_R M = 0$, that is, $M$ is finitely generated. Hence $\varprojlim_{U \in \mathcal{M}} M/U$ is finitely generated. As $H_0^a(M/U)$ is $a$-coartinian for all $U \in \mathcal{M}$, it follows from [17, Proposition 3.4] that $H_0^a(M) \cong H_0^a(\varprojlim_{U \in \mathcal{M}} M/U) \cong \varprojlim_{U \in \mathcal{M}} H_0^a(M/U)$ is $a$-coartinian. The proof is complete.

The following result answers Question 1’.

**Theorem 3.6.** Let $(R, m)$ be a local ring with the $m$-adic topology and $M$ a semi-discrete linearly compact $R$-module with $\text{mag}_R(M) \leq 1$. Then $H_i^a(M)$ is $a$-coartinian for all $i$.

Proof. Since $M$ is semi-discrete linearly compact, $\text{Cosupp}_R H_i^a(M) \subseteq V(a)$ and $H_i^a(M) = 0$ for $i > 1$ by Lemma 3.1. If $\text{mag}_R M = 0$, the result holds by Lemma 3.5. If $\text{mag}_R M = 1$, $H_0^a(M)$ is $a$-coartinian by [17, Theorem 4.14]. We only need to show that $H_0^a(M)$ is $a$-coartinian. Note that $M$ is semi-discrete linearly compact, from [24] there is a short exact sequence

$$0 \to N \to M \to B \to 0,$$

where $N$ is finitely generated and $B$ is artinian. Hence we get a long exact sequence of local homology modules

$$\cdots \to H_1^a(B) \to H_0^a(N) \to H_0^a(M) \to H_0^a(B) \to 0.$$

Since $B$ is artinian, we get $H_0^a(B)$ is $a$-coartinian. On the other hand, as $N$ is finitely generated and linearly compact, we have $H_0^a(N)$ is $a$-coartinian by Lemma 3.3. Hence $H_0^a(M)$ is $a$-coartinian.

**Lemma 3.7.** Let $(R, m)$ be a local ring and $M$ a finitely generated $R$-module. Let $a$ be an ideal of $R$ such that the $R$-module $M/aM$ is artinian. Then $V(a) \cap \text{Ass}_R M \subseteq V(m)$.

Proof. Let $p \in V(a) \cap \text{Ass}_R M$, and let

$$0 = T \cap S_1 \cap \cdots \cap S_n$$
be a minimal primary decomposition of 0 in $M$, where $T$ is a $p$-primary submodule of $M$ and $S_i$ is a $p_i$-primary submodule of $M$ for $i = 1, \cdots, n$. Since $T$ is $p$-primary, it follows that there exists a positive integer $t$ such that $p'(M/T) = 0$, and so $a'(M/T) = 0$ as $a \subseteq p$. Since $M/aM$ is artinian, $M/a'M$ is artinian. On the other hand, there is a short exact sequence

$$0 \to T/a'M \to M/a'M \to M/T \to 0.$$ 

Hence $M/T$ has finite length, and so $p = m$. Therefore, $V(a) \cap \text{Ass}_RM \subseteq V(m)$ as required. \qed

**Lemma 3.8.** Let $M$ be a semi-discrete linearly compact $R$-module with $\text{mag}_RM = 0$ and $\text{Cosupp}_RM \subseteq V(a)$. Then $M$ is $a$-coartinian if and only if $M/aM$ is artinian.

**Proof.** $\Rightarrow$ It is clear.

$\Leftarrow$ Note that $\text{Cosupp}_RM/aM \subseteq \text{Cosupp}_RM \cap V(a) = \text{Cosupp}_RM$. Since $\text{mag}_RM = 0$, one has $\text{mag}_RM/aM = 0$. As $M/aM$ is semi-discrete linearly compact, $\text{Ndim}_M/aM = \text{mag}_RM/aM = 0$, that is, $M/aM$ is finitely generated. Thus $M/aM$ has finite length. It follows that $M$ is $a$-coartinian by [13, Lemma 3.5]. \qed

Recall the arithmetic rank of an ideal $b$ in $R$, denoted by $\text{ara}(b)$, is the least number of elements of $R$ required to generate an ideal which has the same radical as $b$, i.e.,

$$\text{ara}(b) = \min\{n \in \mathbb{N}_0 : \exists b_1, \cdots b_n \in R \text{ with } \text{Rad}(b_1, \cdots b_n) = \text{Rad}(b)\}.$$ 

Let $M$ be an $R$-module. The arithmetic rank of an ideal $b$ in $R$ with respect to $M$, denoted by $\text{ara}_M(b)$, is defined by the arithmetic rank of an ideal $b + \text{Ann}_RM/\text{Ann}_RM$ in the ring $R/\text{Ann}_RM$.

The next is the second main result, which is useful in answering Question 2'.

**Theorem 3.9.** Let $M$ be a semi-discrete linearly compact $R$-module with $\text{mag}_RM \leq 1$ and $\text{Cosupp}_RM \subseteq V(a)$. Then $M$ is $a$-coartinian if and only if $M/aM$ and $\text{Tor}_1^R(R/a, M)$ are artinian.

**Proof.** $\Rightarrow$ It is clear.

$\Leftarrow$ When $\text{mag}_RM = 0$, it is by Lemma 3.8. We may assume $\text{mag}_RM = 1$ and use induction on

$$t = \text{ara}_M(a) = \text{ara}(a + \text{Ann}_RM/\text{Ann}_RM)$$

to prove that $M$ is $a$-coartinian. If $t = 0$, then it follows from definition that $a^n \subseteq \text{Ann}_RM$ for some positive integer $n$, and $M/a^nM \cong M$. Since $M/aM$ is artinian, we get $M/a^nM \cong M$ is artinian. Hence $M$ is $a$-coartinian. So assume that $t > 0$, and the result has been proved for $i \leq t - 1$. Let

$$\mathcal{T} := \{p \in \text{Cosupp}_RM \mid \text{dim}R/p = 1\}$$

As $\text{Coass}_RM/aM = V(a) \cap \text{Coass}_RM = \text{Coass}_RM$ and $M/aM$ is artinian, it follows that the set $\text{Coass}_RM$ is finite. Hence $\mathcal{T}$ is finite. Moreover, as for each $p \in \mathcal{T}$, $\text{Cosupp}_{R_p}\text{Hom}_R(R_p, M)$
⊆ V(pR_p), one has mag_{R_p} Hom_{R_p}(R_p, M) = 0. Since Hom_{R_p}(R_p, M) is a semi-discrete linearly compact R-module, there is a short exact

0 → K → Hom_{R_p}(R_p, M) → A → 0,

where K is a finitely generated R-module and A is an artinian R-module. It induces an exact sequence

0 → Hom_{R_p}(R_p, K) → Hom_{R_p}(R_p, Hom_{R_p}(R_p, M)) → Hom_{R_p}(R_p, A) → 0.

Since Hom_{R_p}(R_p, K) is a semi-discrete linearly compact R-module, there is a short exact sequence

0 → K → Hom_{R_p}(R_p, M) → A → 0,

where K is finitely generated, we have Hom_{R_p}(R_p, K) = 0. Since Hom_{R_p}(R_p, M) is a semi-discrete linearly compact R-module, there is a short exact sequence

0 → Hom_{R_p}(R_p, K) → Hom_{R_p}(R_p, Hom_{R_p}(R_p, M)) → Hom_{R_p}(R_p, A) → 0.

Since K is finitely generated, we have Hom_{R_p}(R_p, K) = 0. Thus Hom_{R_p}(R_p, M) ∼= Hom_{R_p}(R_p, A).

It follows from [13, Theorem 3.2] that Hom_{R_p}(R_p, A) is annihilated by pR_p, then Hom_{R_p}(R_p, A) is artinian by [20, Theorem 7.30]. Hence Hom_{R_p}(R_p, M)/aR_pHom_{R_p}(R_p, M) is artinian. Let

T = \{p_1, \ldots, p_s\}.

By Lemma 3.7, we have

V(aR_{p_j}) \cap Ass_{R_{p_j}} Hom_{R_{p_j}}(R_{p_j}, M) ⊆ V(p_jR_{p_j}),

for all j = 1, 2, \ldots, s. Next, let

U := \bigcup_{j=1}^{n} \{q ∈ \text{Spec} R \mid qR_{p_j} ∈ Ass_{R_{p_j}} Hom_{R_{p_j}}(R_{p_j}, M)\}.

Then it is easy to see that U ∩ V(a) ⊆ T. On the other hand, since t = ara_M(a) ≥ 1, there exists elements y_1, \ldots, y_t ∈ a such that

Rad(a + Ann_{R}M/Ann_{R}M) = Rad((y_1, \ldots, y_t) + Ann_{R}M/Ann_{R}M).

Now, as a ∉ \bigcup_{q\in U \setminus V(a)} q, it follows that (y_1, \ldots, y_t) + Ann_{R}M ∉ \bigcup_{q\in U \setminus V(a)} q. On the other hand, for each q ∈ U, we have

qR_{p_j} ∈ Ass_{R_{p_j}} Hom_{R_{p_j}}(R_{p_j}, M),

for some integer 1 ≤ j ≤ n. Hence

Ann_{R}(M)R_{p_j} ⊆ Ann_{R_{p_j}} Hom_{R_{p_j}}(R_{p_j}, M) ⊆ qR_{p_j}

and so Ann_{R}M ⊆ q. Consequently, it follows from

Ann_{R}M ⊆ \bigcap_{q\in U \setminus V(a)} q

that (y_1, \ldots, y_t) ∉ \bigcup_{q\in U \setminus V(a)} q. By [13, Theorem 16.8] there is a ∈ (y_2, \ldots, y_t) such that y_1 + a ∉ \bigcup_{q\in U \setminus V(a)} q. Let x := y_1 + a. Then x ∈ a and

Rad(a + Ann_{R}M/Ann_{R}M) = Rad((x, y_2, \ldots, y_t) + Ann_{R}M/Ann_{R}M).

Next, let N := M/xM. Then, it is easy to see that

ara_N(a) = ara(a + Ann_{R}N/Ann_{R}N) ≤ t − 1.
(note that $x \in \text{Ann}_R N$), and
\[ \text{Rad}(a + \text{Ann}_R N/\text{Ann}_R N) = \text{Rad}((y_2, \cdots, y_t) + \text{Ann}_R N/\text{Ann}_R N). \]

Now, the exact sequence
\[ 0 \rightarrow xM \rightarrow M \rightarrow N \rightarrow 0 \]
induces an exact sequence
\[ \text{Tor}^1_R(R/a, M) \rightarrow \text{Tor}^1_R(R/a, N) \rightarrow xM/axM \rightarrow M/aM \rightarrow N/aN \rightarrow 0, \]
which implies that the $R$-modules $N/aN$ and $\text{Tor}^R(R/a, N)$ are artinian. Consequently, by the inductive hypothesis, the $R$-module $N$ is $a$-coartinian. Moreover, the exact sequence
\[ 0 \rightarrow xM \rightarrow M \rightarrow N \rightarrow 0 \]
induces an exact sequence
\[ \text{Tor}^2_R(R/a, N) \rightarrow \text{Tor}^1_R(R/a, xM) \rightarrow \text{Tor}^1_R(R/a, M), \]
which implies that the $R$-module $\text{Tor}^1_R(R/a, xM)$ is artinian. Also, from the exact sequence
\[ 0 \rightarrow (0 :_M x) \rightarrow M \rightarrow xM \rightarrow 0, \]
let $L := (0 :_M x)$, we get the exact sequence
\[ \text{Tor}^1_R(R/a, xM) \rightarrow L/aL \rightarrow M/aM, \]
which implies that the $R$-module $L/aL$ is artinian. On the other hand, since $M$ is semi-discrete linearly compact, $L$ is also semi-discrete linearly compact. It follows from [18, Lemma 3.5] that $L$ is $a$-coartinian. Since the $R$-modules $L = (0 :_M x)$ and $M/xM$ are $a$-coartinian, it follows from Proposition [2.5] that $M$ is $a$-coartinian. This completes the inductive step. □

**Corollary 3.10.** Let $a$ be an ideal of $R$, $\mathcal{C}(R, a)^1_{\mathrm{coa}}$ denote the category of semi-discrete linearly compact $a$-coartinian $R$-modules $M$ with $\text{mag}_R M \leq 1$. Then $\mathcal{C}(R, a)^1_{\mathrm{coa}}$ is an Abelian category.

**Proof.** Let $M, N \in \mathcal{C}(R, a)^1_{\mathrm{coa}}$ and $f: M \rightarrow N$ be an $R$-homomorphism. It is enough to show that the $R$-modules $\ker f$ and $\text{coker} f$ are $a$-coartinian. To this end, the exact sequence
\[ 0 \rightarrow \text{Im} f \rightarrow N \rightarrow \text{coker} f \rightarrow 0, \]
induces a long exact sequence
\[ \text{Tor}^R(R/a, N) \rightarrow \text{Tor}^R(R/a, \text{coker} f) \rightarrow \text{Im} f/a\text{Im} f \rightarrow N/aN \rightarrow \text{coker} f/ac\text{coker} f \rightarrow 0. \]
It implies that $\text{coker} f/ac\text{coker} f$ and $\text{Tor}^R(R/a, \text{coker} f)$ are artinian. Therefore $\text{coker} f$ is $a$-coartinian by Theorem [3.9]. Now, the assertion follows from the following exact sequences
\[ 0 \rightarrow \ker f \rightarrow M \rightarrow \text{Im} f \rightarrow 0, \]
and
\[ 0 \to \text{Im} f \to N \to \text{coker} f \to 0. \]

Now, we begin to prove the last main results of this section which answers the second question on \( a \)-coartinian \( R \)-modules. We denote \( C(R, a)_{\text{coa}} \) the category of semi-discrete linearly compact \( a \)-coartinian \( R \)-modules.

**Corollary 3.11.** Let \( a \) be an ideal of \( R \) with \( \dim R/a \leq 1 \). Then \( C(R, a)_{\text{coa}} \) is an Abelian subcategory of the category of all \( R \)-modules.

**Proof.** Let \( M \) be an arbitrary \( R \)-module in \( C(R, a)_{\text{coa}} \). Since \( \dim R/a \leq 1 \) and \( \text{Cosupp}_R M \subseteq V(a) \), we have \( \text{mag}_R M \leq 1 \). Hence the result holds by Corollary 3.10. \( \square \)

**Theorem 3.12.** Let \( a \) be an ideal of \( R \) such that \( \text{cd}(a, R) \leq 1 \). Then \( C(R, a)_{\text{coa}} \) is an Abelian subcategory of the category of all \( R \)-modules.

**Proof.** Let \( M, N \in C(R, a)_{\text{coa}} \) and \( f: M \to N \) be an \( R \)-homomorphism. It is enough to show that the \( R \)-modules \( \ker f \) and \( \text{coker} f \) are \( a \)-coartinian. To this end, the exact sequence
\[ 0 \to \text{Im} f \to N \to \text{coker} f \to 0, \]
induces an exact sequence
\[ 0 \to \text{Hom}_R(R/a, \text{Im} f) \to \text{Hom}_R(R/a, N). \]
Since \( N \) is \( a \)-coartinian, then \( \text{Hom}_R(R/a, N) \) is artinian by Theorem 2.3 it implies that \( \text{Hom}_R(R/a, \text{Im} f) \) is also artinian. Now, the exact sequence
\[ 0 \to \ker f \to M \to \text{Im} f \to 0, \]
induces an exact sequence
\[ 0 \to \text{Hom}_R(R/a, \ker f) \to \text{Hom}_R(R/a, M) \to \text{Hom}_R(R/a, \text{Im} f) \]
\[ \to \text{Ext}^1_R(R/a, \ker f) \to \text{Ext}^1_R(R/a, M). \]
From the hypothesis, \( \text{Hom}_R(R/a, M) \) and \( \text{Ext}^1_R(R/a, M) \) are artinian, which implies that the \( R \)-modules \( \text{Hom}_R(R/a, \ker f) \) and \( \text{Ext}^1_R(R/a, \ker f) \) are artinian. Therefore, it follows from [19, Lemma 2.1] and Theorem 3.9 that the \( R \)-module \( \ker f \) is \( a \)-coartinian. Now, the following exact sequences
\[ 0 \to \ker f \to M \to \text{Im} f \to 0, \]
\[ 0 \to \text{Im} f \to N \to \text{coker} f \to 0, \]
yields the desired assertion. \( \square \)

**Corollary 3.13.** Let \( a \) be an ideal of \( R \) such that \( \dim R/a \leq 1 \) or \( \text{cd}(a, R) \leq 1 \) and \( M \) an \( a \)-coartinian \( R \)-module. Then the \( R \)-module \( \text{Ext}^i_R(N, M) \) is \( a \)-coartinian, for all finitely generated \( R \)-modules \( N \) and all integers \( i \geq 0 \).
Proof. Since $N$ is finitely generated, we get $N$ has a free resolution $\cdots \to F_i \to F_{i-1} \to \cdots \to F_1 \to F_0 \to 0$ with each $F_i$ is finitely generated free $R$-module. Now the assertion follows using Corollary 3.11 or Theorem 3.12 and computing the $R$-module $\text{Ext}^i_R(N, M)$, using this free resolution. \hfill \square

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