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Self-interacting quantum electron

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Abstract A model of self-interacting quantum non-local Dirac’s electron has been proposed. Its dynamics was revealed by the projective representation of operators corresponding to spin/charge degrees of freedom. Energy-momentum field is described by the system of quasi-linear “field-shell” PDE’s following from the conservation law expressed by the affine parallel transport of the energy-momentum vector field in $\mathbb{C}P(3)$. I discuss here travel-wave solutions of these equations and the “off-shell” dispersion law asymptotically coinciding with the “on-shell” de Broglie dispersion law.

Keywords quantum state · projective representation · self-interaction · mass of electron

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1 Introduction

Statistical analysis of the energy distribution is the base of the black body radiation [1] and the Einstein’s theory of the light emission and absorption [2]. Success of Einstein hypothesis of photons, de Broglie wave concept of particles [3] and Schrödinger’s equation for hydrogen atom [4] established so-called the corpuscular-wave duality of matter. This conceptual line was logically finished by Dirac in his method of the second quantization [5]. This approach is perfectly fits to many-body weakly interacting quantum systems.
and it was assumed that the “corpuscle-wave duality” is universal. This duality may be broken being used in strong interacting quantum particles and even to single particle. Physically it is clear why: quantum particle is self-interacting system and this interaction is at least of the order of its rest mass. Since the nature of the mass is the open problem we do not know the energy distribution in quantum particles up to now. Here I try to show a possible approach to this problem in the framework of simple model of self-interacting quantum electron with possible “unparticle” excitations. “Unparticle” sector of quantum excitations is intensively discussed now in the framework of effective QFT [6, 7, 8].

I should note that Blochintzev about 60 years ago discussed the unparticle sector in the framework of universality of wave - particle “duality” for interacting quantum fields [9, 10]. For such fields the universality is generally broken. Namely, attempt to represent two interacting boson fields as the set of free quantum oscillators leads to two types of oscillators: quantized and non-quantized. The second one arises under simple relation $g > \frac{m_1m_2c^2}{\hbar}$ between coupling constant $g$ and masses $m_1$ and $m_2$ of two scalar fields. For such intensity of coupling we obtain a field with excitation states in two sectors: particle and “unparticle”. Furthermore, the excitations in “unparticle” sector has an imaginary mass and they propagate with group velocity larger than $c$. For self-interacting scalar field of mass $m$ the intensity of self-interaction $g$ leads to breakdown of the universality of the wave - particle “duality” if it is larger than the inverse square of the Compton wavelength: $g > \frac{mc^2}{\hbar^2} = \frac{1}{\lambda_C}$.

Blochintzev’s examples were oversimplified for clarity. I would like to discuss here self-interacting electron in the spirit of reaction $e^- \rightarrow U \rightarrow e^-$. In other words I propose to study the particle/unparticle sectors of matter in wide range of momenta in order to solve the localization problem of the foundations of quantum physics. In order to formulate a robust theory of self-interacting quantum “particles”, say electron, one should analyse the quantum invariants and their relations to space-time symmetries.

Fundamental observation of quantum interference shows that variation in quantum setup leads generally to deformation of interference patterns. Quantum formalism generally shows that two setups $S_1$ and $S_2$ where participated say, electrons, generate two different amplitudes $|S_1>$ and $|S_2>$ of outcome event. There are infinite number of different setups $S_1, S_2, ..., S_p, ...$ and not only in the sense of different space-time position but in the sense of different parameters of fields, using devices, etc. Symmetries relative space-time transformations of whole setup have been studied in ordinary quantum theory. Such symmetries reflects, say, the first order of relativity: the physics is same if any complete setup subject (kinematical, not dynamical!) shifts, rotations, boosts as whole in single Minkowski space-time.

Further thinking leads to conclusion that there is a different type of symmetry (second order of relativity or “super-relativity” [11, 12, 13]). It may be formulated initially on the intuitive level as the invariance of physical properties of “quantum particles”, i.e. their quantum numbers like mass, spin, charge, etc., lurking behind two amplitudes $|S_1>$, $|S_2>$. Say, physical properties of electrons are the same in both setups $S_1$ and $S_2$ but they are hidden in amplitudes of different outcomes. Presumably the invariant content of these
properties may be kept if one makes the infinitesimal variation of some “flexible quantum setup” that may be reached by small variation of some fields or adjustment of tuning devices.

Of course, all non-essential details of real setup should be avoided in the problem where one seeks the invariant properties of quantum objects lurked behind generating amplitudes. Otherwise we will trapped in the Bohr’s tenet of “classical language” leading to mixture of quantum and outlandish classical that is the obstacle for building pure quantum model. This is why the Fock’s principle of “relativity to measuring device” \cite{14} and “functional relativity” \cite{15,16} could not be realized in full measure since there is no and could not be a good mathematical quantum model for classical setup. Therefore a model of “flexible pure quantum setup” with a possibility of infinitesimal variation of some parameter (in my model parameters of SU(N)) should be built. In order to do it one needs to find invariant laws of quantum motions and to provide their classification. If we limit ourself by unitary finite dimension dynamics then group SU(N) acting in CN may be used. This approach was developed is the framework of “local functional relativity” or “super-relativity” \cite{11,12,13}. This is actual physical reason why I use vector fields on CP(N-1) playing the role of local dynamical variables (LDV’s) in order to build flexible quantum reference frame \cite{17}. All arguments given above say that one should use primary functional coordinates in group sub-manifold instead of space-time coordinates. Why?

Coordinates of classical events established by means of classical electromagnetic field is based on the distinguishability, i.e. individualization of material points. However we loss this possibility by means of quantum fields since we no have anymore solid scales and ideal clocks acceptable in the framework of special relativity. The problem of identification is the root problem even in classical physics and its recognition gave to Einstein the key to formalization of the relativistic kinematics and dynamics. Indeed, only assuming the possibility to detect locally the coincidence of two pointwise events of a different nature it is possible to build all kinematic scheme and the physical geometry of space-time \cite{15,19}. As such the “state” of the local clock gives us local coordinates - the “state” of the incoming train. In the classical case the notions of the “clock” and the “train” are intuitively clear and it is assumed that they may be replaced my material points. Furthermore, Einstein especially notes that he does not discuss the inaccuracy of the simultaneity of two approximately coincided events that should be overcome by some abstraction \cite{18}. This abstraction is of course the neglect of finite sizes (and all internal degrees of freedom) of the both real clock and train. It gives the representation of these “states” by mathematical points in space-time. Thereby the local identification of positions of two events is the formal source of the classical relativistic theory. But generally in the quantum case such identification is impossible since the space-time coordinates of quantum particles is state-dependent \cite{20,21}. Hence the quantum identification of particles cannot be done in imitative manner (like in special relativity) and it requires a physically motivated operational procedure with corresponding mathematical description. In order to do it some conservation law in the state space expressing the “self-identification” should be formulated.
There were many attempts to build extended models of quantum “elementary” particles leaving Minkowski space-time structure intact. I will mention here only Schrödinger’s attempt to build stable wave packet as the model of harmonic oscillator (the first example of coherent state) [22], Skyrme’s soliton solution of sin-Gordon equation [23], Dirac’s model of extended electron-muon system [24], and t’ Hooft-Polyakov non-singular monopole solution [25]. Together with these works intending to build the model of non-local quantum particles, it is important to take into account the relativistic quantum non-locality discovered by Newton, Wigner [20] and Foldy-Wouthuysen [26] under attempts to reach in fact the opposite target - to find pointwise localization of relativistic wave functions.

Here I would like develop essentially different theory of non-local quantum electron where space-time structure arises under objective quantum “measurement”. It is state-dependent gauge field theory based on the intrinsically geometric “functional” unification of quantum theory and relativity, so-called “super-relativity” [11, 12, 13, 17, 27, 28]. Quantum state and geometric classification of their motion in projective Hilbert space are primordial elements of new quantum theory. The main assumption is that consistent quantum theory should be based on internal geometry of quantum state space (in my case it is “phase space” $CP(N-1)$ diffeomorphic to coset manifold $G/H = SU(N)/S[U(1) \times U(N-1)]$) and that 4D space-time structure arises only under attempt to “measure” some quantum dynamical variable, i.e. to establish single value for local dynamical variables (LDV) of the model [17]. It means nothing but the physical “formulation” of quantum question unescapably related to local Lorentz structure of 4D dynamical space-time. Thereby the problem of quantum measurement requires the reconstruction of all fundamental notions comprising dynamical space-time structure and geometry of the state space. It turns out that objective quantum measurement is non-distinguishable from space-time structure.

### 2 Coset transformations vs F.-W.

Dirac clearly understood that electron is non-local particle since it has internal structure [24, 29]. Nevertheless, he successfully formulated linear relativistic wave equation for pointwise particle. Next progress should be achieved on the way of the second quantization of the bi-spinor amplitudes in order to take into account self-interaction, creation of pairs, etc. However higher orders of perturbation being formally applied to equations of motion arose under this “stiff” method of quantization lead to divergences [30] even in the case of QED with small coupling fine structure constant $\alpha = e^2/\hbar c \approx 0.007$.

Trying to study non-local structure of electron, I avoid to use the second quantization [5] using instead the smooth vector fields playing the role of LDV’s of Dirac’s electron [11, 17]. The main aim is to get non-linear wave equation expressing conservation law of relativistic energy-momentum operator and to study its lump (soliton-like) solution for the “field shell” associated with the surrounding field of single electron. This equation should have solutions similar to well known t’ Hooft-Polyakov regular monopole solution with finite energy [25] but without additional Higgs fields. It should be proved (it
is not done yet) that “field shell” integrally contains all processes treated in the standard QFT as vacuum polarization, etc.

The extremal of least action principle, say, solution of ordinary Dirac equation is the plane wave “modulated” by bi-spinor

$$|\Psi(x)| = \left(\begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{array}\right) \exp\left(-\frac{i}{\hbar} P_\mu x^\mu\right).$$  \hspace{1cm} (1)

The plane wave is improper state of the quantum action with an arbitrary mass and momentum connected only by the “on-shell” dispersion law. If we try to find some invariant physics of electron lurked behind states generated by flexible “setup” described by the LDV’s, one should variate LDV in the vicinity of the extremal in functional state space and to put some conservation law of quantum numbers.

Relativistic Klein-Gordon and Dirac wave equations are based on the classical mass-shell relation $p^\mu p_\mu - m^2 c^2 = 0$. The last one is so restrictive that most consequences of the Dirac equations are almost literally coincide with classical equations of motions \[14\]. However further quantum corrections generated by the second quantization destroy in fact this idealized picture: there is a diffusion of the mass-shell due to space-like self-interaction effects. On the other hand the Foldy-Wouthuysen (F.-W.) unitary transformations

$$U = e^{iS} = \cos|P|\theta + \frac{\beta\alpha}{|P|} \sin|P|\theta, \hspace{1cm} U \in SU(4)$$  \hspace{1cm} (2)

\[26\] reveal already the non-local nature of electron wave function without any references to second quantization. It is interesting therefore to get more general relativistic non-linear wave equation which have non-singular localizable solution associated with quantum particles.

F.-W. unitary transformations were invoked to diagonalize Dirac’s hamiltonian in order to separate bi-spinor components with positive and negative energies. Generally, the diagonalization is exactly possible in the case of free electron and approximately - for electron in external fields. This transformation is non-local since it effectively delocalises pointwise electron in a spatial vicinity with the radius $\delta r \approx \frac{\hbar}{mc}$. The mass of electron $m$ is a free parameter of the model.

I should note two peculiarities of the F.-W. transformations. First, these transformations intended to separate “large” and “small” components with positive and negative energies. This “large/small” classification is neither unitary nor scale invariant. Approximate diagonalization chosen by F.-W. is based on the iteration scheme of the Hamiltonian corrections in moving frame in “slowness” parameter $\frac{\hbar}{mc}$ that is scale non-invariant. However, the relation between two radial functions $R(r)/S(r) = \sqrt{\frac{\alpha}{1-\sqrt{1-\alpha}}} \approx \frac{r^2}{2m}$ of the ground state $j = 1/2, l = 0$ of the hydrogen atom \[31\] gives us the example of the invariant dimensionless combinations. Such invariant dimensionless combination is generic with the relation $\pi^1 = \frac{mc}{\sqrt{\alpha^2 + \alpha^2 - \alpha}}$ functionally coinciding with the local projective coordinates $(\pi^1, \pi^2, \pi^3)$ of free electron in $CP(3)$,
i.e. with relative components of bi-spinos of stationary state. It is naturally to use these invariant functional variables in order to establish relation between spatial size of extended quantum electron and their energy-momentum distribution.

Second, partially separated quantum degrees of freedom (spin and charge) and the space-time coordinates lead (under F.-W. transformations) to delocalization of electron as we will discuss below. The analysis of the “delocalization” leads however to some progress in understanding quantum dynamics of electron.

In order to understand the reason of delocalization arose as a result of diagonalization one need to take into account the geometry of \( SU(4) \) group and the Cartan decomposition of \( \text{Alg}SU(4) \) \cite{32,11,12,13}. First of all we shall note that Dirac’s matrices

\[
\begin{align*}
\hat{\gamma}_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
\hat{\gamma}_1 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
\hat{\gamma}_2 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\
\hat{\gamma}_3 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},
\end{align*}
\] (3)

may be represented as linear combinations of “standard” \( SU(4) \) \( \lambda \)-generators \cite{33}

\[
\begin{align*}
\hat{\gamma}_0 &= \hat{\lambda}_3 + \frac{1}{3}[\sqrt{3}\hat{\lambda}_8 - \sqrt{6}\hat{\lambda}_{15}], \\
\hat{\gamma}_1 &= \hat{\lambda}_2 + \hat{\lambda}_{14}, \\
\hat{\gamma}_2 &= \hat{\lambda}_1 - \hat{\lambda}_{13}, \\
\hat{\gamma}_3 &= -\hat{\lambda}_5 + \hat{\lambda}_{12}.
\end{align*}
\] (4)

Since any state \( |S> \) has the isotropy group \( H = U(1) \times U(N) \), only the coset transformations \( G/H = SU(N)/SU(1) \times U(N-1) = C^P(N-1) \) effectively act in \( C^N \). One should remember, however, that the concrete representation of hermitian matrices belonging to subsets \( h \) or \( b \) depends on a priori chosen vector (all “standart” classification of the traceless matrices of Pauli, Gell-Mann, etc., is base on the vector \( (1,0,0,...,0)^T \)). The Cartan’s decomposition of the algebra \( \text{Alg}SU(N) \) is unitary invariant and I will use it instead of Foldy-Wouthuysen decomposition in “even” and “odd” components.

According to Cartan’s classification of \( SU(4) \) generators, there are two types of generators \( \hat{\gamma} \) relative the state vector

\[
|S(x) > = \begin{pmatrix} \psi_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \exp \frac{-i}{\hbar} P_{\mu} x^\mu : (5)
\]

\( \hat{\gamma}_0 \) is generator of the isotropy group \( H = U(1) \times U(3) \) of the \( |S(x) > \) leaving it intact, whereas \( \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3 \) belong to coset transformations \( G/H = \)
$SU(4)/S[U(1) \times U(3)] = CP(3)$ that deform the chosen state. The algebra of generators $AlgSU(N)$ is $\mathbb{Z}_2$-graded in respect with following properties of the commutation relations. $[h_{\downarrow S}, h_{\downarrow S}] \subseteq h_{\downarrow S}$, $[b_{\downarrow S}, h_{\downarrow S}] \subseteq h_{\downarrow S}$, $[b_{\downarrow S}, h_{\downarrow S}] \subseteq b_{\downarrow S}$. One may easy check that for example $\hat{\gamma}_1 \hat{\gamma}_2 \subseteq b_{\downarrow S}$ and

$$\hat{\gamma}_1 \hat{\gamma}_2 - \hat{\gamma}_2 \hat{\gamma}_1 = 2i\left[\frac{1}{3}(\sqrt{3}\hat{\lambda}_8 - \sqrt{6}\hat{\lambda}_{15}) - \hat{\lambda}_3\right] \subseteq h_{\downarrow S}.$$  \hspace{1cm} (6)

Physically it is important to use the Cartan decomposition of unitary group in respect with initially chosen state vector $|S\rangle$. Therefore the parametrization of these decomposition is state-dependent $[h_{\downarrow S}, h_{\downarrow S}] \subseteq h_{\downarrow S}$, $[b_{\downarrow S}, h_{\downarrow S}] \subseteq h_{\downarrow S}$, $[b_{\downarrow S}, h_{\downarrow S}] \subseteq b_{\downarrow S}$ \cite{11,12,13}. It means that physically it is interesting not abstract unitary group relations but realization of the unitary transformations resulting in motion of the pure quantum states represented by rays in projective Hilbert space. Therefore the ray representation of $SU(N)$ in $\mathbb{C}^N$, in particular, the embedding of $H$ and $G/H$ in $G$, is a state-dependent parametrization. The diagonalization of the Dirac’s Hamiltonian is the annihilation of the coset part of the initial Hamiltonian acting on the bi-spinor in $C^4$. Notice, that the “modulated” plane wave (1) belongs to functional space $\mathcal{H} = \mathbb{C}^4 \otimes \mathbb{C}^\infty$ which is the tensor product. But unitary operator (4) capable diagonalize only matrix part acting on the bi-spinor components in $C^4$ and it does not commute with operator of the coordinate \cite{34}

$$<r|\hat{U}|r'\rangle = (2\pi)^{-3} \int \left[\frac{m + E_p}{2E_p} + \frac{\delta p}{\sqrt{2E_p(m + E_p)}}\right] e^{i\mathbf{p}(r-r')} dp.$$ \hspace{1cm} (7)

In other words F.-W. transformations could not leave intact the plane waves thus they create the superposition of improper states in the Hilbert space denoted here as $\mathbb{C}^\infty$. Therefore according to general classification of quantum motion \cite{11} the coset transformation is a quantum analog of force giving deformation of quantum state. This leads to delocalization of electron and I try to develop this result attempting to derive non-linear relativistic quantum field equations in the spirit of approach proposed a few years before \cite{11,13,17}. The main aspiration is to find new non-linear wave equation for energy-momentum of electron moving in dynamical space-time.

3 Energy-momentum operator as a tangent vector to $CP(3)$

Since it is impossible to find the representation capable exactly to diagonalize Hamiltonian with help of global non-Abelian actions of $SU(4)$ in dynamical situation and because even in the case of free electron the diagonalization is achievable only in one sub-space $C^4$ of full state space, let me reformulate the problem as follows.

I will work with Dirac’s operator of energy-momentum

$$\hat{\gamma}^\mu p_\mu = i\hbar \frac{\partial}{\partial x^\mu}.$$ \hspace{1cm} (8)
instead of the Hamiltonian. This combined operator acts in the direct product
\( S = C^4 \times H_D \), where \( H_D \) means a Hilbert space of differentiable func-
tions. Such splitting seems to be artificial and I try to find more flexible
construction of energy-momentum operator. Let's apply to this operator the
similarity transformation (transition to “moving frame” freezing the action
of the differentiation in space-time coordinates) with help the canonical uni-
tary operator. In the case of pseudo-euclidian coordinates \( x^\mu \) it is possible
to use simply the “plane wave” \( U_{\text{gauge}} = \exp(-\frac{i}{\hbar} P_\mu x^\mu) \). But if one uses,
say, spherical coordinates, one needs to use non-Abelian gauge transforma-
tions of \( SU(4) \) in order to convert the operator into a matrix with functional
elements. I will discuss it in a separate paper.

Most delicate point of the construction is as follows. Energy-momentum
variation evoked by internal dynamical structure of electron is independent
on global space-time transformation (being applied to electron’s “center of
mass”) but nevertheless it should be reflected in space-time motion of the
“field-shell” [11]. This may be treated as result of the back-reaction from
quantum dynamics of spin and charge degrees of freedom during “metabolic
time” controlling the motion in the state space \( CP(3) \). “Observable” wave
dynamics arises under the lift in the tangent fibre bundle from the base
manifold \( CP(3) \) into the state-dependent “dynamical space-time” (DST) [12]
that will be discussed below.

I assume that \( P_\mu = P_\mu(\tau) \) is the function of the “metabolic time” \( \tau \) and
consequently the function of state-dependent dynamical space-timecoordinates
that will be introduced only on the stage of “quantum measurement” [13]. Then one has the matrix

\[
U_{\text{gauge}}^{-1} \hat{\gamma}^\mu P_\mu U_{\text{gauge}} = \hat{\gamma}^\mu P_\mu(\tau)
\]

with functional elements, not operator.

Stationary states of the operator (8) are solutions of the Dirac equation.
Their relative bi-spinor components may be marked by the local coordinates

\[
\pi^i_{(j)} = \begin{cases} \frac{\psi^i}{\psi^j} & \text{if } 1 \leq i < j \\ \frac{\psi^{i+1}}{\psi^j} & \text{if } j \leq i < 3 \end{cases}
\]

in the map \( U_j : \{|S \rangle, |\psi\rangle \neq 0 \}, 0 \leq j \leq 3 \). In these coordinates (representation) the improper states like plane waves are simply deleted.

Now, infinitesimal energy-momentum variations evoked by interaction
charge-spin degrees of freedom (burked in \( \hat{\gamma}^\mu \) ) may be expressed in terms
of the local coordinates \( \pi^i \) since there is a diffeomorphism between the space
of the rays \( CP(3) \) and the \( SU(4) \) group sub-manifold of the coset transforma-
tions \( G/H = SU(4)/S[U(1) \times U(3)] = CP(3) \) and the isotropy group
\( H = U(1) \times U(3) \) of some state vector. These diffeomorphism will be ex-
pressed by the coefficient functions of combinations of the \( SU(4) \) generators
of locally unitary transformations that will be defined by equation arose un-
der infinitesimal variation of the energy-momentum

\[
\Phi^i_{\mu}(\gamma_\mu) = \lim_{\epsilon \to 0} \epsilon^{-1} \left\{ \frac{\exp(i \epsilon \hat{\gamma}_\mu)}{\exp(i \epsilon \hat{\gamma}_\mu)} \left[ \psi^i_m \psi^m - \psi^i \right] \right\} = \lim_{\epsilon \to 0} \epsilon^{-1} \left\{ \pi^i(\epsilon \hat{\gamma}_\mu) - \pi^i \right\}
\]
in non-linear local realization of $SU(4)$ \[11\]. Here $\psi^m$, $1 \leq m \leq 4$ are ordinary bi-spinor amplitudes. I calculated twelve coefficient functions $\Phi^i_\mu(\gamma_\mu)$, they are as follows:

\[
\begin{align*}
\Phi^0_0(\gamma_0) &= 0, & \Phi^0_0(\gamma_0) &= -2i\pi^2, & \Phi^0_0(\gamma_0) &= -2i\pi^3; \\
\Phi^1_1(\gamma_1) &= i(\pi^2 - \pi^4\pi^3), & \Phi^2_2(\gamma_1) &= i(\pi^1 - \pi^2\pi^3), & \Phi^3_3(\gamma_1) &= i(1 - (\pi^3)^2); \\
\Phi^2_2(\gamma_2) &= -i(\pi^2 + \pi^4\pi^3), & \Phi^2_2(\gamma_2) &= \pi^1 - \pi^2\pi^3, & \Phi^3_3(\gamma_2) &= -1 - (\pi^3)^2; \\
\Phi^3_3(\gamma_3) &= i(\pi^2 - \pi^4\pi^3), & \Phi^3_3(\gamma_3) &= i(1 - (\pi^2)^2), & \Phi^4_4(\gamma_3) &= -i(\pi^1 + \pi^2\pi^3).
\end{align*}
\] (12)

It is well known that the Abelian gauge invariance leads to conservation of electric charge. The wish to represent this invariance geometrically is so strong that a new “covariant derivative” in Minkowski space-time has been invented. In this picture the 4-potential $A_\mu$ elongating operator of the 4-momentum $p^\mu = i\hbar\frac{\partial}{\partial x^\mu}$ to “covariant derivative” $P_\mu = p_\mu - \not{c}A_\mu$ is treated as a connection in the $U(1)$-fibre bundle. This construction is justified by the invariance of the Klein-Gordon or Dirac equations for the particles “minimally” coupled with external electromagnetic field. The nature of the charge and spin is hidden under such approach. The internal dynamics of these degrees of freedom should be used order to study this nature. It is assumed that state-dependent gauge field $H = S[U(1) \times U(3)]$ is responsible for such dynamics \[11\,[12,13,17].

I would like to discuss here a different realization of Abelian and generalized (non-Abelian) transformations of the state vector leading to different kind of field equations. Ordinary Abelian gauge transformations act as follows

\[
|\Psi> = \psi^a|a> \rightarrow |\Psi'> = e^{i\phi(x)}|\Psi> = e^{i\phi(x)}\psi^a|a> .
\] (13)

Transition to local non-homogeneous projective coordinates

\[
\pi'^{(i)}_{(j)} = \frac{e^{i\phi(x)}\psi^{(i)}}{e^{i\phi(x)}\psi^{(j)}} = \frac{\psi^{(i)}}{\psi^{(j)}} = \pi^{(i)}_{(j)}
\] (14)

leaves the local projective coordinates intact but the wave equation of Dirac being rewritten in terms of local coordinates $\pi'^{(i)}_{(j)}$ is not gauge invariant in the projective representation since the space-time derivatives bring additional terms that cannot be compensated by reasonable transformation of the external potential $A_\mu$. One need to find some wave equation in different space (I will use the projective representation in $CP(3)$) which will be invariant not only relative Abelian but even relative state-dependent non-Abelian $H = S[U(1) \times U(3)]$ transformations of Dirac’s bi-spinor components. The well known F.-W. unitary global transformations from $SU(4)$ served for me as an important allusion \[11\].

Now I will define $\Gamma$-vector field

\[
\Gamma_\mu = \Phi^i_\mu(\gamma_\mu)\frac{\partial}{\partial \pi^i}
\] (15)

and then the energy-momentum operator will be defined as functional vector field

\[
P = P^\mu\Gamma_\mu = P^\mu\Phi^i_\mu(\gamma_\mu)\frac{\partial}{\partial \pi^i} + c.c.
\] (16)
where the ordinary 4-momentum $P^\mu = (E^c - e^c \phi, P^- e^c A) = (\bar{\hbar} \omega^c - e^c \phi, \bar{\hbar} k - e^c A)$ (not operator!) should be identified with the solution of quasi-linear "field-shell" PDE’s for the contravariant components of the energy-momentum tangent vector field in $CP(3)$

$$P^i = P^\mu \Phi_i (\gamma_\mu).$$ (17)

One sees that infinitesimal variation of energy-momentum is represented by the operator of partial differentiation in complex local coordinates $\pi^i$. In some sense this construction is similar to the second quantization scheme, but operators of differentiation act on functions $F = F(\pi^1, \pi^2, \pi^3)$ of local coordinates $\pi^i$ of electronic states. These functions sometimes take the place of "local vacuum" relate to particular physical system and in simple cases they may be found. One of the reasonable equation for the function of the local vacuum $V = V(\pi^1, \pi^2, \pi^3)$ is the equation

$$PV = P^\mu \Gamma_\mu V = P^\mu \Phi_i (\gamma_\mu) \frac{\partial V}{\partial \pi^i} = 0$$ (18)

The assumed separability of external “slow” degrees of freedom and internal “fast” spin and charge quantum degrees of freedom is of course approximate. Uniliteral action is idealization and “slow” degrees of freedom subject to back-reaction from quantum dynamics of spin and charge degrees of freedom. This back-reaction was widely studied [35]. For us will be interesting (as a hint) the generation of pseudo-electric and pseudo-magnetic Berry potentials [36]. I made the three changes in the back-reaction studying:

1. Instead of Berry parameters $X$ of a Hamiltonian $H(X)$ I will use local projective coordinates $(\pi^1, \pi^2, \pi^3)$ of generalized coherent state (GCS) of electron and therefore the quantum metric tensor is the metric tensor of Fubini-Study [32, 11];

2. Iteration procedure of the Foldy-Wouthuysen coset transformations will be replaced by the infinitesimal action of local dynamical variables (LDV) represented by tangent vector fields on $CP(3)$ diffeomorphic to the coset sub-manifold $SU(4)/SU(1) \times SU(3)$;

3. Affine parallel transport of the energy-momentum vector field on $CP(3)$ agrees with Fubini-Study metric will be used instead of “adiabatic renormalization” [39] of the Dirac operator.

Further, I pose the following question: could the curvature of the $CP(3)$ play the role of the source of actual electromagnetic potentials surrounding the self-interacting non-local electron?

In order to study this question one need to find LDV in the local ray representation (in our case it will be local energy-momentum vector field $P^i(\pi^1, \pi^2, \pi^3)$ associated with corresponding generators of $SU(4)$) and further its variation in the vicinity of the stationary state. These LDV’s are example of Dirac’s “q-number” with known commutation rules [11]. But instead of the operator creation of electron $\psi_n = K^*_n$ obeys to Heisenberg equations of motion

$$i\hbar \frac{dK}{dt} = i\hbar \frac{\partial K}{\partial t} + KH - HK = 0,$$ (19)
I will use the equation of the affine parallel transport

\[ \frac{\delta P^i}{\delta \tau} = 0 \]  \hspace{1cm} (20)

of the energy-momentum tangent vector field in \( CP(3) \) as it will be defined below.

4 Quantum boosts and angular velocities

How internal quantum degrees of freedom of the electron may be mapped in dynamical space-time? How to lift it from the base \( CP(3) \) into the tangent fibre bundle? If we assume that internal dynamics is represented by LDV like energy-momentum vector field \( P \) then it is natural to think that process of measurement of value of this LDV should be somehow connected with this mapping. Being geometrically formulated, i.e. in invariant manner, this process will be objective. In such case the “measurement” means only the process of some “projection” in attempt to find single value of physical dynamical variable. It means that objective quantum measurement is in fact embedding of quantum dynamics in space-time. But how we should to do it in invariant, geometric manner? And what is the space-time itself without such measurement?

Absolute values (reference frame independent) of space and time intervals lost their sense in the framework of relativity and only space-time interval has invariant sense. Analogously, one should find invariant values of local dynamical variables in quantum dynamics. Self-conservation of electron may be expressed through affine parallel transported energy-momentum filed \( P^i \) and the “measurement procedure” is necessary for recovering the distribution of energy-momentum in dynamical space-time. Only after definition of the functional reference frame in the state space and its variation during “measurement” - some kind of the D’Alembert’s principle of the “virtual displacements” there is a possibility to establish local space-time relations.

I should note that my construction of the lift with the help of the “qubit spinor” proposed a few years ago \([13]\) is unsatisfactory in some reasons.

First, the necessity of embedding of projective Hilbert space \( CP(N-1) \) in ordinary Hilbert space \( C^N \) or \( C^\infty \) is in a contradiction with the general assumption of the fundamental character of the projective Hilbert space.

Second, under such embedding the choice of the normal \( |N> \) is generally an ambiguous procedure.

Third, the tangent vector \( |\Psi> \) was artificially connected with some “Hamiltonian”, the notion that should be avoided in strictly relativistic description.

I think that after some attempts I found a satisfactory the qubit spinor construction where only Fubini-Study intrinsically invariant relations are used without unnecessary “extraneous” constructions. It is worth while to note that this construction solves “the quantum measurement problem” in natural and objective manner since outcome of “measurement” of LDV \( D^i \)
is provided by spontaneous “falling down” of the “centrum of mass” in the affine gauge potential (see graph in [12,37])

$$I_{mn}^{i} = \frac{1}{2} G^{ip^{*}} \left( \frac{\partial G_{mp^{*}}}{\partial \pi^{n}} + \frac{\partial G_{p^{*}n}}{\partial \pi^{m}} \right) = - \frac{\delta_{m}^{i} \pi^{n^{*}} + \delta_{n}^{i} \pi^{m^{*}}}{1 + \sum |\pi|^{2}}. \quad (21)$$

agrees with Fubini-Study metric

$$G_{ik^{*}} = \left[ \left( 1 + \sum |\pi|^{2} \right) \delta_{ik} - \pi_{i}^{*} \pi_{k} \right] \left( 1 + \sum |\pi|^{2} \right)^{-2} \quad (22)$$

without additional introduction of the Higgs fields.

Let me assume that we would like to measure some LDV $D^{i}$, say corresponding to $\rho$-matrices of charge represented by the formulas (11). I define now two vector fields. One of them is tangent energy-momentum vector field

$$T^{i} = \frac{d\pi^{i}}{d\tau} = \frac{c}{\hbar} \phi^{i}_{\mu} P^{\mu} \quad (23)$$

and the second one is co-vector field

$$V_{i} = \frac{\partial V}{\partial \pi^{i}} \quad (24)$$

associated with the gradient of the “vacuum” functional i.e the first integral of the energy-momentum vector field $V = V(\pi^{1}, \pi^{2}, \pi^{3})$, so that

$$T^{i}V_{i} = P^{\mu} \phi_{\mu}^{i} \left( \gamma_{\mu} \frac{\partial V}{\partial \pi^{i}} \right) = 0. \quad (25)$$

It means that relative this vacuum these two vector fields are orthogonal in the sense of the Fubini-Study metric. Geometrically one may imagine that the “vertical” coordinate along $V_{i}$ marks of the “level of height” on the vacuum surface $V = V(\pi^{1}, \pi^{2}, \pi^{3})$ in the space $C \times CP(3)$ as it will be described somewhere.

Now we are ready to introduce the invariant components $(\eta^{0}, \eta^{1})$ of the qubit spinor associated with “measuring” LDV $D^{i}$. These give the measure of the contribution of the $D^{i}$ in energy-momentum of electron as follows:

$$\eta^{0} = G_{iks} D^{j} V^{k*}, \quad V^{k*} = G^{iks} V_{i} \quad (26)$$

and

$$\eta^{1} = G_{iks} D^{j} T^{k*}. \quad (27)$$

Then from the infinitesimally close GCS $(\pi^{1} + \delta^{1}, ..., \pi^{N-1} + \delta^{N-1})$, whose shift is induced by the interaction used for a measurement, one get a close spinor $(\eta^{0} + \delta \eta^{0}, \eta^{1} + \delta \eta^{1})$ with the components

$$(\eta^{0} + \delta \eta^{0}) = G_{iks} D^{j} \hat{V}^{k*} = G_{iks} D^{j} G^{iks} (V_{j} + \Gamma^{j}_{jm} V_{i} d\pi^{m})$$

$$(\eta^{1} + \delta \eta^{1}) = G_{iks} D^{j} \hat{T}^{k*} = G_{iks} D^{j} (T^{k*} - \Gamma^{k*}_{lm} T^{l*} d\pi^{m*}), \quad (28)$$

where the basis $(\hat{T}^{k*}, \hat{V}^{k*})$ is the parallel transported basis $(T^{k*}, V^{k*})$ from the infinitesimally close point $(\pi^{1} + \delta^{1}, ..., \pi^{N-1} + \delta^{N-1})$ back to the $(\pi^{1}, ..., \pi^{N-1})$.
Notice, that I changed the signs before $\Gamma$ terms since we do infinitesimal parallel transport back to initial point. It is very important that these terms in $\eta^0$ and $\eta^1$ have opposite signs. Taking into account the shape of the affine gauge potential (see graph in [12,37]) one may conclude that there is the natural mechanism (without Higgs fields) of instability relative falling down into the point ($\pi^1 = \pi^2 = \pi^3 = 0$) or into the valley of the affine gauge potential. It means that under “virtual displacement” used for the measurement of LDV $D^i$ spontaneously alive (after elapse of the metabolic time) one of the qubit spinor component $\eta^0$ or $\eta^1$ so that the velocity of the one of them is zero (stationary value) and the velocity of the second one is finite.

Any two infinitesimally close spinors $\eta$ and $\eta + \delta\eta$ may be formally connected with infinitesimal $SL(2,C)$ transformations represented by “Lorentz spin transformations matrix” \[ \hat{L} = \begin{pmatrix} 1 - \frac{i}{2} \delta\tau (\omega_3 + ia_3) & -\frac{i}{2} \delta\tau (\omega_1 + ia_1 - i(\omega_2 + ia_2)) \\ -\frac{i}{2} \delta\tau (\omega_1 + ia_1 + i(\omega_2 + ia_2)) & 1 - \frac{i}{2} \delta\tau (-\omega_3 - ia_3) \end{pmatrix} \].

I have assumed that there is not only formal but dynamical reason, namely: self-interaction of charge with the energy-momentum encoded by the dynamics of two-level system whose components comprise the qubit spinor. Therefore this process may be represented in DST associated with manifold of coordinates in attached Lorentz reference frame. Then “quantum accelerations” $a_1, a_2, a_3$ and “quantum angular velocities” $\omega_1, \omega_2, \omega_3$ may be found in the linear approximation from the equation $\delta\eta = \hat{L}\eta - \eta$, or, strictly speaking, from its consequence - the equations for the velocities $\xi$ of $\eta$ spinor variations

\[ \hat{R} \left( \begin{array}{c} \eta^0 \\ \eta^1 \end{array} \right) = \frac{1}{\delta\tau} (\hat{L} - \hat{1}) \left( \begin{array}{c} \eta^0 \\ \eta^1 \end{array} \right) = \left( \begin{array}{c} \xi^0 \\ \xi^1 \end{array} \right). \] (30)

Now both components $\eta^0$ and $\eta^1$ subject the affine parallel transport back to the initial GCS with velocities: $\xi^0 = \frac{\delta\pi^0}{\delta\tau} = \Gamma \eta^0 \frac{\delta\tau}{\delta\tau}$ and $\xi^1 = \frac{d\eta^1}{d\tau} = -\Gamma \eta^1 \frac{\delta \theta}{\delta\tau}$. If one put $\pi = e^{-i\phi} \tan(\theta/2)$ then $\frac{\delta\theta}{\delta\tau} = \frac{\delta\pi}{\delta\tau} \frac{\delta\theta}{\delta\phi} + \frac{\delta\phi}{\delta\tau}$, where

\[ \frac{\delta\phi}{\delta\tau} = a_3 + (1/2)((a_1 - \omega_2)\cos(\phi) - (a_2 + \omega_1)\sin(\phi)) \tan(\theta/2) \]

\[ -((a_1 + \omega_2)\cos(\phi) - (a_2 - \omega_1)\sin(\phi)) \cot(\theta/2), \] (31)

then one has the linear system of 6 real non-homogeneous equation

\[ \Re(\hat{R}_{00}\eta^0 + \hat{R}_{01}\eta^1) = -\Re(\Gamma \eta^0 \frac{\delta\pi}{\delta\tau}), \]

\[ \Im(\hat{R}_{00}\eta^0 + \hat{R}_{01}\eta^1) = \Im(\Gamma \eta^0 \frac{\delta\pi}{\delta\tau}), \]

\[ \Re(\hat{R}_{10}\eta^0 + \hat{R}_{11}\eta^1) = -\Re(\Gamma \eta^1 \frac{\delta\pi}{\delta\tau}), \]

\[ \Im(\hat{R}_{10}\eta^0 + \hat{R}_{11}\eta^1) = -\Im(\Gamma \eta^1 \frac{\delta\pi}{\delta\tau}). \]
\[
\begin{align*}
\frac{\delta \theta}{\delta \tau} &= F_1, \\
\frac{\delta \phi}{\delta \tau} &= F_2,
\end{align*}
\]  
(32)
giving \(a_Q(\eta^0, \eta^1, \theta, \phi, F_1, F_2, \Omega_Q(\eta^0, \eta^1), \theta, \phi, F_1, F_2)\) as the functions of “measured” components of LDV \((\eta^0, \eta^1)\), the local coordinates of GCS \((\theta, \phi)\) or complex \(\pi\), and 2 real perturbation frequencies \((F_1, F_2)\) of coset deformation acting along some geodesic in \(CP(3)\). Since \(CP(3)\) is totally geodesic manifold \([32]\), each geodesic belongs to some \(CP(1)\) parameterized by single \(\pi = e^{-i\phi} \tan(\theta/2)\) used above.

The infinitesimal transition from one GCS of the electron to another is now accompanied by dynamical transition from one Lorentz frame to another. Thereby, infinitesimal Lorentz transformations define infinitesimal “dynamical space-time” coordinates variations. It is convenient to take Lorentz transformations in the following form

\[
\begin{align*}
ct' &= ct + (xa_Q)\delta \tau \\
x' &= x + cta_Q\delta \tau + (\Omega_Q \times x)\delta \tau
\end{align*}
\]  
(33)
where I put \(a_Q = (a_1/c, a_2/c, a_3/c)\), \(\Omega_Q = (\omega_1, \omega_2, \omega_3)\) \([40]\) in order to have for \(\tau\) the physical dimension of time. The expression for the “4-velocity” \(V^\mu\) is as follows

\[
V^\mu_Q = \frac{\delta x^\mu}{\delta \tau} = (xa_Q, cta_Q + \Omega_Q \times x).
\]  
(34)
The coordinates \(x^\mu\) of imaging point in dynamical space-time serve here merely for the parametrization of the energy-momentum distribution in the “field shell” arising under “morphogenesis” described by quasi-linear field equations \([11, 13]\) in DST.

5 Perturbation theory in vicinity of degeneration and affine gauge fields in \(CP(3)\)

Since the system of eigen-vectors belonging to degenerated eigenvalue is defined up to unitary transformations, the approximate calculation of eigenvalues and corresponding eigen-state vectors in the conditions of degeneration is the natural place for application of geometry of unitary group. For example, the solution of the problem of small denominators arising in the framework of perturbation theory is based in fact on the geometry of \(CP(1)\), see for example \([14]\).

Pseudo-electric and pseudo-magnetic fields arose as gauge fields with singular potentials at the degeneration points of Hamiltonian spectrum \([30]\). The structure of degeneration is unstable relative small perturbation of the Hamiltonian and hence could not serve as a source of real electromagnetic potentials. I would like to study the nature of affine unitary gauge fields arose under breakdown (reconstruction) of global \(G = SU(4)\) symmetry of degenerated bi-spinors states \(|2S_{1/2}\rangle\) and \(|2P_{1/2}\rangle\) to the local gauge group \(H = S[U(1) \times U(3)]\) acting by state-dependent generators on “phase space”
Deformation of quantum state under action of the geodesic flow is treated here as morphogenesis of a new quantum orbital.

The state vector (1) may be expressed in local coordinates as follows: for $a = 1$ one has

$$
\psi^1(\pi_1^{(p)}, \pi_2^{(p)}, \pi_3^{(p)}) = e^{i\alpha_1} (1 + \sum_{s=1}^3 |\pi_s^{(p)}|^2)^{-1/2}
$$

and for $2 \leq a = i \leq 4$ one has

$$
\psi^i(\pi_1^{(p)}, \pi_2^{(p)}, \pi_3^{(p)}) = e^{i\alpha_1} \pi_i^{(p)} (1 + \sum_{s=1}^3 |\pi_s^{(p)}|^2)^{-1/2}.
$$

Notice, the local coordinates in this case (without external fields) do not contain space-time degrees of freedom. It is interesting to restore these degrees of freedom in state-dependent dynamical space-time for self-interacting electron that presumably should generate surrounding electromagnetic field.

I will use the Berry’s formula for 2-form as a hint [36]. Being applied to state vector $|\Psi(x)\rangle$ in the local coordinates $\pi^i$, one has antisymmetric second-rank tensor

$$
V^i_{\ell k}(\pi^i) = \Im \sum_{a=1}^4 \left\{ \frac{\partial \psi^{a*}}{\partial \pi^\ell} \frac{\partial \psi^a}{\partial \pi^k} - \frac{\partial \psi^{a*}}{\partial \pi^k} \frac{\partial \psi^a}{\partial \pi^\ell} \right\}
$$

$$
= -\Im [(1 + \sum |\pi^s|^2) \delta_{ik} - \pi^i \pi^k] (1 + \sum |\pi^s|^2)^{-2} = -\Im G^i_{\ell k}. \tag{37}
$$

It is simply the imaginary part of the Fubini-Study quantum metric tensor. There are following important differences between original Berry's formula referring to arbitrary parameters and this 2-form in local coordinates inherently related to eigen-problem.

1. The $V^i_{\ell k}(\pi^i) = \Im i G^i_{\ell k}$ is the singular-free expression.
2. It does not contain two eigen-values, say, $E_n, E_m$ explicitly, but implicitly $V^i_{\ell k} = \Im i G^i_{\ell k}$ depends locally on the choice of single $\lambda_\ell$ through the dependence in local coordinates $\pi^{(p)}$. Even in the case of degenerated eigenvalue, the reason of the anholonomy lurks in the curvature of $CP(3)$ and therefore it has intrinsically invariant and stable character.
3. It is impossible of course directly identify $V^i_{\ell k} = \Im i G^i_{\ell k}$ with the electromagnetic tensor $F_{ij} = A_{j,i} - A_{i,j}$. But I try to understand how the geometry of $CP(3)$ generates electromagnetic potentials in terms of “filed-shell” equations for energy-momentum [11].

6 Derivation of the “field-shell” equations for non-local quantum electron

The “field-shell” equations were derived as the consequence of the conservation law of energy momentum [11, 12, 13]. Since I use operator of energy-momentum as a tangent vector field to $CP(3)$ instead of Lagrangian, I will use a complex differentiation instead of variation. In fact $CP(3)$ takes the
place of finite dimension functional space. Differential equations arose in the section of tangent fibre bundle over $CP(3)$. It leads naturally to some “lump” solutions which should be carefully studied. I show here some preliminary results promising a progress in understanding structure of quantum electron. In particular it is clear that quantum nature of derived field quasi-linear PDE’s (without references to classical analogy) could shed the light on the their generic connection with Hamilton-Jacobi classical equations and de Broglie-Schrödinger optics-mechanics analogy.

Quantum lump of non-local electron should presumably serve as extended source of electromagnetic field. This lump may be mapped onto dynamical space-time if one assumes that transition from one GCS of the electron to another is accompanied by dynamical transition from one Lorentz frame to another. Thereby, infinitesimal Lorentz transformations define small “dynamical space-time” coordinates variations (31) parameterizing energy-momentum distribution.

The conservation law of the energy-momentum vector field in $CP(3)$ during evolution will be expressed by the equation of the affine parallel transport

\[
\frac{\delta P^i}{\delta \tau} = \frac{\delta (P^\mu \Phi^i_{\mu}(\gamma))}{\delta \tau} = 0,
\]

which is equivalent to the following system of four coupled quasi-linear PDE for dynamical space-time distribution of energy-momentum “field shell” of quantum state and ordinary differential equations for relative amplitudes

\[
V_Q\left(\frac{\partial P^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda} P^\lambda\right) = \frac{c}{\hbar} \left(\Gamma^m_{\mu n} \Phi^n_{\mu}(\gamma) + \frac{\partial \Phi^n_{\mu}(\gamma)}{\partial \pi^n}\right) P^\nu P^\mu, \quad \frac{d\pi^k}{d\tau} = \frac{c}{\hbar} \Phi^k_{\mu} P^\mu,
\]

which is in fact the equations of characteristic for linear “super-Dirac” equation

\[
i P^\mu \Phi^i_{\mu}(\gamma) \frac{\partial \Psi}{\partial \pi^i} = mc \Psi
\]

that supposes ODE for single “total state function”

\[
i \hbar \frac{d\Psi}{d\tau} = mc^2 \Psi
\]

with the solution for variable mass $m(\tau)$

\[
\Psi(T) = \Psi(0) e^{-\frac{ic}{\hbar} \int_0^T m(\tau)d\tau}.
\]

In this article I will discuss only the “field-shell” equations (39) and their consequences.

7 Solutions of “field-shell” equations and dispersion law for self-interacting electron

I will discuss now the solution of the “field-shell” equations (39). The theory of these equations is well known. Particularly, our system is the system with
identical principle part $V^\mu_Q$ which is properly discussed in the Application 1 to the Chapter II [38]. Initially, I will neglect for simplicity by the term $\Gamma^\nu_{\mu\lambda}P^\lambda$ associated with the space-time curvature which cannot spoil the general discussion and that will play an important role. Hereby, one has the quasi-linear PDE system

$$V^\mu_Q \frac{\partial P^\nu}{\partial x^\mu} = -\frac{c}{\hbar} (\Gamma^m_{mn} \Phi^m_n(\gamma) + \frac{\partial \Phi^n_m(\gamma)}{\partial \pi^n} ) P^\nu P^\mu$$  \hspace{1cm} (43)$$

for which we will build characteristics for the system of implicit solutions for 4+4 extended variables

$$\phi_1(x^0, x^1, x^2, x^3, P^0, P^1, P^2, P^3) = c_1;$$
$$\phi_2(x^0, x^1, x^2, x^3, P^0, P^1, P^2, P^3) = c_2;$$
$$\phi_3(x^0, x^1, x^2, x^3, P^0, P^1, P^2, P^3) = c_3;$$
$$\phi_4(x^0, x^1, x^2, x^3, P^0, P^1, P^2, P^3) = c_4.$$  \hspace{1cm} (44)$$

Differentiation of $\phi_\mu$ in $x^\nu$ gives

$$\frac{\partial \phi_\mu}{\partial x^\nu} + \frac{\partial \phi_\mu}{\partial P_\lambda} \frac{\partial P_\lambda}{\partial x^\nu} = 0.$$  \hspace{1cm} (45)$$

This equation being multiplied by $\frac{\delta x^\nu}{\delta \tau} = V^\nu_Q$ gives the equation

$$\frac{\partial \phi_\mu}{\partial x^\nu} \frac{\delta x^\nu}{\delta \tau} + \frac{\partial \phi_\mu}{\partial P_\lambda} \frac{\delta P_\lambda}{\partial x^\nu} \frac{\delta x^\nu}{\delta \tau} = 0$$  \hspace{1cm} (46)$$

or

$$\frac{\partial \phi_\mu}{\partial x^\nu} V^\nu_Q + \frac{\partial \phi_\mu}{\partial P_\lambda} (V^\nu_Q \frac{\partial P_\lambda}{\partial x^\nu}) = 0.$$  \hspace{1cm} (47)$$

Redefinition of the coefficients $C^{\nu+\lambda} := -\frac{c}{\hbar} (\Gamma^m_{mn} \Phi^m_n(\gamma) + \frac{\partial \Phi^n_m(\gamma)}{\partial \pi^n} ) P^\lambda P^\mu$ and variables $x^{\nu+\lambda} := P^\lambda$ gives a possibility to rewrite this equation for any $\phi = \phi_\mu$ as follows

$$\sum_{\kappa=1}^8 C^{\kappa} \frac{\partial \phi}{\partial x^\kappa} = 0.$$  \hspace{1cm} (48)$$

Then one has the system of eight ODE’s of characteristics

$$\frac{\delta x^\nu}{\delta \tau} = V^\nu_Q,$$
$$\frac{\delta P^\nu}{\delta \tau} = -\frac{c}{\hbar} (\Gamma^m_{mn} \Phi^m_n(\gamma) + \frac{\partial \Phi^n_m(\gamma)}{\partial \pi^n} ) P^\nu P^\mu.$$  \hspace{1cm} (49)$$

Let me represent and discuss solutions of these equations. I will start with first system written as follows:

$$\frac{\delta x^0}{V_Q} = \frac{\delta x^1}{V_Q} = \frac{\delta x^2}{V_Q} = \frac{\delta x^3}{V_Q} = d\tau.$$  \hspace{1cm} (50)$$
The integration of this system gives 6 invariant quadratic forms

\[
\begin{align*}
& a_2 x^0 x_1 + \frac{\omega_3}{2} (x^1)^2 - \omega_1 x^1 x^3 - a_1 x^0 x^2 - \omega_2 x^2 x^3 + \frac{\omega_3}{2} (x^2)^2 = C_{12}, \\
& a_3 x^0 x_1 - \frac{\omega_2}{2} (x^1)^2 + \omega_1 x^1 x^2 - a_1 x^0 x^3 + \omega_3 x^2 x^3 - \frac{\omega_2}{2} (x^3)^2 = C_{13}, \\
& a_3 x^0 x_2 + \frac{\omega_1}{2} (x^2)^2 - \omega_2 x^1 x^2 - a_2 x^0 x^3 - \omega_3 x^1 x^3 + \frac{\omega_1}{2} (x^3)^2 = C_{23}, \\
& \omega_2 x^0 x_3 + \frac{\omega_1}{2} (x^0)^2 - \omega_3 x^2 x^0 - \frac{\omega_2}{2} x^1 x^2 - a_2 x^0 x^1 - a_3 x^2 x^1 = C_{01}, \\
& \omega_3 x^0 x_1 + \frac{\omega_2}{2} (x^0)^2 - \omega_1 x^3 x^0 - \frac{\omega_2}{2} (x^2)^2 - a_1 x^2 x^1 - a_3 x^3 x^2 = C_{02}, \\
& \omega_1 x^0 x_2 + \frac{\omega_3}{2} (x^0)^2 - \omega_2 x^1 x^0 - \frac{\omega_3}{2} (x^3)^2 - a_1 x^3 x^1 - a_2 x^2 x^2 = C_{03}. 
\end{align*}
\]

They are cones, hyperboloid, etc, whose parameters are defined by quantum boosts and rotations that given by the system (32). Potentially they are important for analysis of the conditions of excitations of electron and its decay. The question whether it is possible to connect them with the short-range geometry and topology (like the space-time foam) of the dynamical space-time is open and requires additional studying.

Let me to integrate now a “cross” combination of the characteristic equations from the first and the second system. One of the combination is as follows

\[
\frac{\delta x^0}{V_0} = \frac{\delta P^0}{P^0 (L_0 P^0 + L_1 P^1 + L_2 P^2 + L_3 P^3)},
\]

(52)

where \( L_\mu = -\frac{\kappa}{2} (F_{\mu \nu} F^\nu_\alpha + \frac{\partial \phi^\mu_\alpha (\gamma)}{\partial x^\alpha}) \). If \( L_0 P^0 < 0 \) then one has implicit solution

\[
\frac{x^0}{a_\alpha x^\alpha} + T^0 = - \frac{2}{L_0 P^0} \tanh^{-1} (1 + \frac{2 L_0 P^0}{L_\alpha P^\alpha}),
\]

(53)

where \( T^0 \) is an integration constant. Explicit solution for energy is the kink

\[
P^0 = \frac{L_\alpha P^\alpha}{2 L_0} [\tanh ((-\frac{x^0}{a_\alpha x^\alpha} + T^0) \frac{L_\alpha P^\alpha}{2}) - 1].
\]

(54)

If I put \( L_\alpha P^\alpha = 1, L_0 = 1, V = a_\alpha x^\alpha = 0.6 \) the kink solution may be represented by the graphic in Fig. 1.

This solution represent the lump of electron self-interacting through electromagnetic-like field in the co-moving Lorentz reference frame. The nature of this field will be discussed in separate article. The lump “modulates” the ordinary plane wave essentially only in the vicinity of core of the lump. In the standard QED self-interacting effects are treated as a polarization of the vacuum. It the present picture the lump is dynamically self-supporting by outwards and inwards waves whose characteristics are represented by equations (49).
Fig. 1 The kink solution.

Let's discuss now the first integrals concerning energy-momentum relationship. I will use following denotations: \( \tilde{E} := L_0 P^0 \), \( \tilde{P} := L_\alpha P^\alpha = P^1 + P^2 + P^3 \). Then it is easy to find 3 first integrals

\[
(\tilde{E} + \tilde{P} - \tilde{P}^1)[C_{p0p1} + \frac{1}{\tilde{P}} \tanh^{-1}(\frac{2\tilde{E}}{\tilde{P}} + 1)] = \tanh^{-1}(\frac{\tilde{E} + \tilde{P} + \tilde{P}^1}{\tilde{E} + \tilde{P} - \tilde{P}^1}),
\]

\[
(\tilde{E} + \tilde{P} - \tilde{P}^2)[C_{p0p2} + \frac{1}{\tilde{P}} \tanh^{-1}(\frac{2\tilde{E}}{\tilde{P}} + 1)] = \tanh^{-1}(\frac{\tilde{E} + \tilde{P} + \tilde{P}^2}{\tilde{E} + \tilde{P} - \tilde{P}^2}),
\]

\[
(\tilde{E} + \tilde{P} - \tilde{P}^3)[C_{p0p3} + \frac{1}{\tilde{P}} \tanh^{-1}(\frac{2\tilde{E}}{\tilde{P}} + 1)] = \tanh^{-1}(\frac{\tilde{E} + \tilde{P} + \tilde{P}^3}{\tilde{E} + \tilde{P} - \tilde{P}^3}).
\]

(55)

Assuming \( C_{p0p1} = C_{p0p2} = C_{p0p3} = \frac{1}{K} \) one finds after summation

\[
(3\tilde{E} + 2\tilde{P})[\frac{1}{K} + \frac{1}{\tilde{P}} \tanh^{-1}(\frac{2\tilde{E}}{\tilde{P}} + 1)] = \tanh^{-1}(\frac{3\tilde{E} + 2\tilde{P}}{\tilde{E} + \tilde{P} - \tilde{P}^1} + \frac{1}{\tilde{P}} \tanh^{-1}(\frac{2\tilde{E}}{\tilde{P}} + 1))
\]

\[
= \tanh^{-1}(\frac{3\tilde{E} + 2\tilde{P}}{\tilde{E} + \tilde{P} - \tilde{P}^1}) + \tanh^{-1}(\frac{3\tilde{E} + 2\tilde{P}}{\tilde{E} + \tilde{P} - \tilde{P}^2}) + \tanh^{-1}(\frac{3\tilde{E} + 2\tilde{P}}{\tilde{E} + \tilde{P} - \tilde{P}^3}).
\]

(56)

After some simplification one may write the “dispersion law” in implicit form as follows:

\[
\tanh\{(3\tilde{E} + 2\tilde{P})[\frac{1}{K} + \frac{1}{\tilde{P}} \tanh^{-1}(\frac{2\tilde{E}}{\tilde{P}} + 1)]\} = \frac{(\tilde{E} + \tilde{P})^3 + \tilde{P}^1 \tilde{P}^2 \tilde{P}^3}{(\tilde{E} + \tilde{P})^3 - \tilde{P}^1 \tilde{P}^2 \tilde{P}^3}.
\]

(57)

The physical sense of this “dispersion law” is not clear yet.

Integration of the general combinations (including (52))

\[
\frac{\delta x^\nu}{V_0^\nu} = \frac{\delta P^\nu}{P^\nu(L_0 P^0 + L_1 P^1 + L_2 P^2 + L_3 P^3)},
\]

(58)

gives four equations

\[
\frac{x^0 + a_\alpha x^\alpha T_0}{a_\alpha x^\alpha} = -\frac{2}{L_\alpha P^\alpha} \tanh^{-1}(1 + \frac{2L_0 P^0}{L_\alpha P^\alpha})).
\]
that constitute more general kink-solutions than were discussed above.

8 Stability of energy-momentum characteristics

Let me discuss the stability of energy-momentum characteristics given by the system of four ODE’s

\[
\frac{\delta P^\lambda}{\delta \tau} = -V^\mu_{Q} \Gamma^\nu_{\mu \lambda} P^\lambda - \frac{c}{\hbar} (\Gamma^m_{mn} \phi^p_n(\gamma) + \frac{\partial \phi^p_n(\gamma)}{\partial \pi^m}) P^{\nu} P^{\mu} .
\] (60)

If seriously to treat these characteristics as trajectories of electrons then their stability in DST is essential problem. Self-interaction electron is represented here as dynamical field system whose equilibrium is provided by the counterbalance of outward and inward quasi-spherically symmetrical oscillations.

The standard approach to stability analysis instructs us to find the stationary points. The stationary condition

\[
\frac{\delta P^\lambda}{\delta \tau} = 0
\] (61)

leads to the system of algebraic equations

\[
V^\mu_{Q} \Gamma^\nu_{\mu \lambda} P^\lambda + \frac{c}{\hbar} (\Gamma^m_{mn} \phi^p_n(\gamma) + \frac{\partial \phi^p_n(\gamma)}{\partial \pi^m}) P^{\nu} P^{\mu} = 0.
\] (62)

Let me to investigate initially the simplified case neglecting the space-time connection term \( V^\mu_{Q} \Gamma^\nu_{\mu \lambda} P^\lambda \). This gives us more simple equations for stationary points

\[
(\Gamma^m_{mn} \phi^p_n(\gamma) + \frac{\partial \phi^p_n(\gamma)}{\partial \pi^m}) P^{\nu} P^{\mu} = 0.
\] (63)

telling us that in the non-trivial case (i.e. exclusion of condition \( P^{\nu} = 0 \)), one has the equation of hyper-plane

\[
(\Gamma^m_{mn} \phi^p_n(\gamma) + \frac{\partial \phi^p_n(\gamma)}{\partial \pi^m}) P^{\mu}_0 = 0
\] (64)
“rotating” with variation of local coordinates $\pi^i$. The probing solution in the vicinity of the stationary points $P_0^\mu$ is as follows

$$P^\mu(\tau) = P_0^\mu + p^\mu e^{-\tau}.$$ (65)

This solution being substituted in the equation

$$\frac{\delta P^\nu}{\delta \tau} = -\frac{\hbar}{c}(\Gamma_{mn}^m P_0^p(\gamma) + \frac{\partial \Phi_\mu^n(\gamma)}{\partial\pi^n}) P^\nu P^\mu.$$ (66)

leads to the linear system

$$\frac{\hbar \omega}{c} p^\nu + (\Gamma_{mn}^m P_0^p(\gamma) + \frac{\partial \Phi_\mu^n(\gamma)}{\partial\pi^n}) p^\mu P^\nu = 0.$$ (67)

The determinant of this homogeneous linear system is as follows

$$D = \left( \frac{\hbar \omega}{c} \right)^4 + \left( \frac{\hbar \omega}{c} \right)^3 \left( \Gamma_{mn}^m P_0^p(\gamma) + \frac{\partial \Phi_\mu^n(\gamma)}{\partial\pi^n} \right) P^\mu_0.$$ (68)

The determinant being equalizing to zero together with the hyper-plane equation gives $\omega = 0$. It means that the mass of self-interacting quantum electron is zero for trivial space-time flat connection.

I would like to compare the “off-shell” dispersion law (54) with the de Broglie “on-shell” dispersion law. The result has been shown in Fig. 2. It

is clearly seen that it traverses below the asymptote $E = cP$, whereas de Broglie “on-shell” dispersion law $P^\mu P_\mu - m^2 c^2 = 0$ traverses above it. The comparison of our dispersion law with Blochintzev spectrum $E^2 = P^2 - m^2 c^2$ is shown Fig. 3. The approximate expression for dispersion in the vicinity of zero is as follows

$$E_{app} = \frac{c}{2} P + \frac{c^2}{4\hbar}(T_0 + \frac{x_0}{V})P^2$$

$$= \frac{c^2}{4\hbar} (T_0 + \frac{x_0}{V}) \left[ (P + \frac{\hbar}{c} \frac{V}{x_0 + VT_0})^2 - \left( \frac{\hbar}{c} \frac{V}{x_0 + VT_0} \right)^2 \right]$$ (69)
Fig. 3 “Off-shell” dispersion law (red) in comparison with Blochintzev dispersion law of two strongly coupled linear oscillators (green).

Fig. 4 Approximate square dispersion law (red) in comparison with “off-shell” dispersion law (green) in the vicinity of zero.

and it is depicted in Fig. 4. The minimum of the energy is as follows

$$E_{\text{min}} = -\frac{\hbar}{4} \frac{V}{x_0 + V T_0}$$  \hspace{1cm} (70)$$

at the momentum

$$P_{\text{min}} = -\frac{\hbar}{c} \frac{V}{x_0 + V T_0}.$$  \hspace{1cm} (71)$$

The group velocity of propagation shows that there is the “zone” of the wave vectors where one has a space-like leakage of self-interacting field, see Fig. 5. Probably it may be related to the quantum entanglement but this phenomenon may be naturally realized in the context of multi-kink solutions and it will be studied later. Only asymptotically it tends to the velocity of light together with phase velocity, see Fig. 6. It really looks like in “unparticle” sector of Blochintzev: phase velocity is always smaller than $c$, but the behavior of group velocity is more complicated.
I would like to note that \( V = x^{\alpha} a_{\alpha} = x a_Q \) is the time component of the “4-velocity” (34) proportional to the “quantum acceleration” \( a_Q \). This boost parameter together with angular velocity \( \omega_Q \) is the solution of the non-homogeneous system of linear equations (32) expressing the condition of measurability of LDV. In fact it is the condition of the existence the number value of LDV expressed by complex vector field in \( CP(3) \). The boost and angular rotation parameters are proportional to affine connection associated with gauge potential in \( CP(1) \subseteq CP(3) \). It gives a hope to treat the expression \( (P + \frac{2}{c} x_0 + V x_0) \) in \( E_{app} \) as the momentum with additional “electromagnetic-like” potential generated by the geometry of \( SU(4) \) sub-manifold \( CP(3) \). The complex scalar potential of electromagnetic field generated by the logarithm of dimensionless projective coordinate has been already discussed [39]. The authors treated electric field as a generator of a boost and magnetic field as the generator of rotations. The quantum conditions of measurability of LDV.
The trivial solution whereas (60). Then we come to the homogeneous linear system responding to finite mass of the electron one should analyse the full equation space-time. It is a consequence of simplified condition of characteristic state "electromagnetic-like" field. This question is not properly understood yet and it requires an additional investigation concerning the sectional curvature of the $CP(3)$.

The gapless dispersion law discussed above arose in the flat Minkowski space-time. It is a consequence of simplified condition of characteristic stability when space-time connection is vanished. Therefore in order to find the "optical" dispersion law with a mass-gap and state-dependent corresponding to finite mass of the electron one should analyse the full equation (60). Then we come to the homogeneous linear system

$$\frac{\hbar \omega}{c} \rho' + \frac{\hbar}{c} V^\mu \Gamma_{\mu \lambda} P^\lambda + (\Gamma_{mn} \Phi^m_{\mu}(\gamma) + \frac{\partial \Phi^m_{\mu}(\gamma)}{\partial \pi^n}) P^\nu_{0} = 0. \quad (72)$$

The determinant of this system is as follows

$$D_1 = (\frac{\hbar \omega}{c})^4 + \alpha (\frac{\hbar \omega}{c})^3 + \beta (\frac{\hbar \omega}{c})^2 + \gamma (\frac{\hbar \omega}{c}) + \delta, \quad (73)$$

with complicated coefficients $\alpha, \beta, \gamma, \delta$. I put $K_\alpha' = \frac{\hbar}{c} V^\mu \Gamma_{\mu \lambda}$ and $M_{\mu}' = (\Gamma_{mn} \Phi^m_{\mu}(\gamma) + \frac{\partial \Phi^m_{\mu}(\gamma)}{\partial \pi^n}) P^\nu_{0}$ then one may find that

$$\alpha = Tr(K_\alpha') + Tr(M_{\mu}') \quad (74)$$

and

$$\beta = K_0'(L_1 P_1 + L_2 P_2 + L_3 P_3) + K_1'(L_0 P_0 + L_2 P_2 + L_3 P_3) + K_2'(L_1 P_1 + L_0 P_0 + L_3 P_3) + K_3'(L_1 P_1 + L_2 P_2 + L_0 P_0)$$

$$- K_1' L_0 P_0 - K_2' L_1 P_1 - K_3' L_2 P_2 - K_1' L_0 P_0 - K_2' L_1 P_1 - K_3' L_2 P_2 - K_0' L_3 P_3, \quad (75)$$

whereas $\gamma, \delta$ have higher order in $G_N$ and they may be temporarily discarded in our approximate dispersion law. This dispersion law may be written as follows

$$\left(\frac{\hbar \omega}{c}\right)^2 (\frac{\hbar \omega}{c})^2 + \alpha \left(\frac{\hbar \omega}{c}\right) + \beta = 0. \quad (76)$$

The trivial solution $\omega_{1,2} = 0$ has already been discussed. Two non-trivial solutions in weak gravitation field when $\alpha^2 \gg \beta$ are given by the equations

$$\hbar \omega_{3,4} = \alpha - \pm \sqrt{\frac{1 - \frac{4 \beta}{\alpha}}{2}} \approx \alpha - \pm \left(1 - \frac{4 \beta}{\alpha}\right), \text{ (77)}$$

$$\hbar \omega_3 = -\frac{c \beta}{\alpha}, \quad \hbar \omega_4 = -\alpha + \frac{c \beta}{\alpha}.$$
9 Conclusion

Primarily, there were two mathematical approaches to the formulation of quantum theory. The first one (developed by Hiesenberg) makes accent on the non-commutative character of new “quantized” dynamical variables whereas the second one (developed by Schrödinger) replaces ordinary differential Hamilton’s equations of classical dynamics by linear differential equations in partial derivatives associated with Hamilton-Jacobi equation \[4\]. Both approaches are equivalent in the framework of so-called optics-mechanics analogy and comprise of the fundament of modern quantum mechanics. This analogy, however, is limited by itself in very clear reasons: mechanics is merely a coarse approximation (even being generalized to many-dimension dynamics of Hertz) and the “optics” of the action waves is too tiny for description of complicated structure of “elementary” quantum particles. It is was realized already under the first attempts to synthesize relativistic and quantum principles.

Analysis of the foundations of quantum theory and relativity shows that there are two types of symmetries. One of them is symmetries relative space-time transformations of whole setup reflects, say, the first order of relativity. Different type of (state-dependent) symmetries is realized in the quantum state space relative local infinitesimal variation of flexible setup (second order of relativity or “super-relativity” \[11,12,13\]). Gauge invariance is a particular case of this type of symmetry. Analysis shows that it is impossible to use ordinary primordial elements like particles, material points, etc., trying to build consistent theory. Even space-time cannot conserve its independent and a priori structure. Therefore the unification of relativity and quantum principles may be formalized if one uses new primordial elements and the classification of their motions: rays of quantum states instead of material points (particles) and complex projective Hilbert state space \(CP(N - 1)\) where move these states under action of unitary group \(SU(N)\) instead of space-time \[12,13\].

Then:

1. Dynamical variables are in fact the generators of the group of symmetry and their non-commutative character is only a consequence of the curvature of the group manifold \[40\]. State-dependent realization of \(SU(N)\) generators as vector fields on \(CP(N - 1)\) evidently reveals the non-trivial global geometry of \(SU(N)\) and it coset sub-manifold \[12,17\].

2. Attempts “to return” in the Minkowski space-time (after second quantization) from the Schrödinger’s configuration space is successful for statistical aims but they are contradictable on the fundamental level of single quantum particle (which without any doubt does exist!) and therefore should be revised. In fact initially one should delete global space-time by transition to “co-moving frame” and after virtual infinitesimal displacement of generalized coherent state (GCS) of electron to restore state-dependent local dynamical space-time.

3. The physically correct transition from quantum to classical mechanics arose as a serious problem immediately after the formulation of “wave mechanics” of Schrödinger \[22\]. The failure to build stable wave packet for
single electron from solutions of linear PDE’s lead to statistical interpretation of the wave function. The further progress in the theory of non-linear PDE’s like sin-Gordon or KdV renewed generally the old belief in possibility to return to non-singular quantum particles [41].

The revision mentioned above (see point (2)) proposed here intended to derive new non-linear quantum equations for self-interacting non-local electron. Notice, that new field equations could not contain arbitrary potential as it was in the case of Schrödinger or Dirac equations. This potential should be generated by the spin/charge self-interaction. One of the consistent way is to use quasi-linear field PDE’s following from conservation law that has been already discussed [11, 12, 13, 17, 27, 28]. It is provided by state-dependent local non-Abelian “chiral” gauge field acting on $CP(3)$ as a tangent vector fields.

Perturbation of generalized coherent state of $G = SU(4)$ of the electron is studied in the vicinity of the stationary degenerated state given by ordinary (not secondly quantized) Dirac’s equation. This perturbation is generated by coset transformations $G/H = SU(4)/SU(1) \times U(3) = CP(3)$ as an analog of the infinitesimal F.-W. transformations [11]. Self-interaction arose due to the curvature of the projective Hilbert space $CP(3)$ and the state-dependent dynamical space-time (DST) is built during “objective quantum measurement” [13].

10 Summary

A new model of non-local self-interacting quantum electron has been proposed. Such self-interaction is provided by the spin-charge quantum dynamics. The non-linear realization of $\gamma$-matrices of Dirac by the tangent vector fields to $CP(3)$ is used instead of the second quantization. The back-reaction of the internal dynamics reflects in “slow” accelerated motion of attached “Lorentz reference frame” introducing state-dependent dynamical space-time coordinates. “Field-shell” of energy-momentum distribution is described by the system of quasi-linear PDE’s. These are the consequence of the conservation law of energy-momentum vector field expressed by affine parallel transport in $CP(3)$ agrees with Fubini-Study metric.

“Off-shell” dispersion law, group and phase velocities asymptotically coincide with de Broglie “on-shell” dispersion law. These excitations of self-interacting electron pose a lot of interesting questions. For example:

1. The general dispersion law (73) probably may be related to the problem of the lepton generations (electron, muon, tauon). The detailed numerical analysis should give reply on this question.

2. The generation of electromagnetic-like field by the coset transformations of manifold $CP(3)$ is also questionable.

3. Self-interacting electron sharply concentrated in the the area with linear size of the order of Compton wave-length. New calculation of the Lamb shift in the framework of non-local electron will be done. Probably it may avoid divergences without renormalization procedure.
These acute old problems could not be solved separately and they require some general framework. Here I would like to give outlook (without technical details given above in “Conclusion”) of my approach to these topics.

1. The main my target is to build simplest unification of quantum and relativity principles. It means that minimal number of primordial elements and postulates should be used and only standard mathematical operations are eligible. In my case only rays of quantum states are primordial elements and the assumption about unitary character of their dynamics have been used. But these simplest assumptions lead to deep reconstruction in the spirit of “Deterministic underlying theory: We suspect that our world can be understood by starting from a pre-quantized classical, or ontological, system” [42]. In my model the deterministic underlying theory is rooted in complex projective Hilbert space $CP(N - 1)$.

2. The most fundamental difficulty, in fact the stumbling-block on the way of unification of quantum and relativity principles, is the localization problem [43,44,45]. This problem is so acute that it evokes the new concept of space-time as some sub-manifold of a Hilbert space [15] and a new principle of the “functional relativity” in this space [16].

Localization being treated as ability of coordinate description of an object in classical relativity closely connected with operational identification of “events” [18]. It is tacitly assumed that all classical objects (frequently represented by material points) are self-identical and they can not disappear because of the energy-momentum conservation law. However the quantum identification and therefore localization of particles cannot be done in imitative manner (like in special relativity) and it requires a physically motivated operational procedure with corresponding mathematical description. In order to do it some conservation law in the state space expressing the “self-identification” or “self-conservation” should be formulated [11,12,13].

It turns out that QFT dictates the necessity to reformulate QM according to new principle of invariance as well as electromagnetic theory insisted to reformulate the Newton’s kinematic and dynamics in relativistic manner. This is the true reason why I called this theory “super-relativity”. Formally it means that the complex projective geometry lies in the base of quantum theory and specific geometric factors associated with covariant derivatives in $CP(N - 1)$ contain well known relativistic factors in “rigid-body” approximations.

3. Two simple observations I’ve put in the base of my theory:

A. Quantum interference phenomenon shows the symmetries relative space-time transformations of whole setup. These have been studied in ordinary quantum theory. Such symmetries reflects, say, the first order of relativity: the physics is same if any complete setup subject (kinematical, not dynamical!) shifts, rotations, boosts as whole in single Minkowski space-time.

B. There is however a different type of tacitly assumed symmetry that may be formulated on the intuitive level as the invariance of physical properties of “quantum particles”, i.e. the invariance of their quantum numbers like mass, spin, charge, etc., relative variation of quantum amplitude. Say, physical properties of electrons are the same in two setups $S_1$ and $S_2$. I postulated that the invariant content of these properties may be kept if one makes
the infinitesimal variation of some “flexible quantum setup” reached by small variation of some fields or adjustment of tuning devices; it comprises of the second order of relativity or “super-relativity” \[11,12,13\].

This “flexible quantum setup” is invariant construction (in the sense of Fubini-Study metric) since it is built from the tangent vector fields on \(CP(N - 1)\) manifold represented by the generators of \(SU(N)\) expressed in the local coordinates \(\pi'\). These generators are state-dependent dynamical variables \[17\] similar those arose in Weinberg’s attempt generalization of the ordinary QM \[46\]. The non-bilinear character of corresponding observables is naturally provided here by the state-dependent character of the local dynamical variables (LDV’s).

It is possible to say that “super-relativity” is the local version of the “functional relativity” under some reservations in the choice of Hilbert space and the space-time construction.

4. There is a different approach to extended quantum objects studying in the framework of the Thermo Field Dynamics (TFD). These arise due to boson condensation providing the dynamical reconstruction of symmetry \[47\]. Dynamical reconstruction of \(SU(N)\) symmetry has been already discussed in the context of projective representation of \(SU(N)\) generators \[27\]. TFD assumes the applicability of the second quantization scheme. It has been shown by Blochintzev \[9,10\] that there is an essential problem in the application of this method for strongly interacting and self-interacting fields.

Since I investigate self-interacting electron it would be reasonable to start from its relativistic quantum (not secondly quantized) model (according to “A”) and to study their quantum invariants under infinitesimal deformations of amplitudes (according to “B”). I used for this aim the Dirac’s electron and the geometric features of the Foldy-Wouthuysen unitary transformations from \(SU(4)\).

5. The main my aim was to get new non-linear wave equations and to study its lump (soliton-like) solution for the “field shell” associated with the surrounding field of single electron. Dirac’s equations are the first order linear system of PDE’s. The most close to such equations are quasi-linear first order PDE’s the theory of which is properly developed \[55\]. These equations very naturally follow from the conservation law of energy-momentum field expressed by the affine parallel transport of the energy momentum vector field in \(CP(3)\) agrees with Fubini-Study metric \[11\]. Since all quantum dynamics concentrated now in \(CP(3)\) base manifold and dynamical space-time arises merely for the parametrization of section in tangent fibre bundle, the differentiation in complex local coordinates \(\pi'\) has been used instead of variation procedure.

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