INTERSECTIONS OF DIAGONAL ORBITS

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Abstract. Let $A \subseteq SL_n(\mathbb{R})$ group of diagonal matrices with positive diagonal, let $ST_n \subseteq X_n := SL_n(\mathbb{R})/ SL_n(\mathbb{Z})$ be the set of stable lattices, and let $WR_n \subseteq X_n$ be the set of well-rounded lattices. We prove that any $A$-orbit in $X_n$ intersects both $ST_n$ and $WR_n$.

1. Introduction

Let $A \subseteq SL_n(\mathbb{R})$ be the diagonal subgroup and let $X_n := SL_n(\mathbb{R})/ SL_n(\mathbb{Z})$ be the space of lattices. It is believed that Minkowski suggested the following conjecture:

**Conjecture 1.1.** For every $\Lambda \in X_n$ and $p \in \mathbb{R}^n$ there exist $a \in A$ and $v \in \Lambda$ such that $\|a(p - v)\| \leq \sqrt{n}/2$.

The conjecture was proved for $n \leq 9$. The first proofs for $n \leq 5$ used the following strategy, known as the Remak-Davenport approach. Define the set of well-rounded lattices $WR_n \subseteq X_n$ as the set of all lattices such that all the Minkowski successive minima are equal. The Remak-Davenport approach states that to prove Minkowski’s conjecture it is enough to prove the following two statements.

$(W_n)$ For every lattice $\Lambda \in X_n$ we have $A\Lambda \cap WR_n \neq \emptyset$.
$(C_n)$ For every $\Lambda \in WR_n$,

$$\sup_{p \in \mathbb{R}^n} \inf_{v \in \Lambda} \|p - v\| \leq \sqrt{n}/2.$$  

The cases $n = 2, 3, 4, 5$ were proven by Minkowski [12], Remak [14], Dyson [3], and Skubenko [16], respectively.

McMullen [10] proved a weaker version of $(W_n)$, that, combined with a result of Birch and Swinnerton-Dyer [1], demonstrated that if $(C_1), (C_2), \ldots, (C_n)$ holds then Minkowski’s Conjecture holds for $n$. Woods [17] proved $(C_n)$ for $n = 6$, and in [5, 6, and 8] Hans-Gill, Kathuria, Raka, and Sehmi proved $(C_n)$ for $n = 7, 8, 9$. In particular, the Minkowski Conjecture indeed holds for $n \leq 9$.  

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Regev, Shapira, and Weiss [13] proved that \((C_n)\) is false for \(n \geq 30\), and therefore the Remak-Davenport approach is bound to fail. Shapira and Weiss [15] suggested a similar approach replacing the set of well-rounded lattices by the set of stable lattices (see Definition 2.1 below).

As for \((W_n)\), McMullen [10] proved that any bounded orbit closure \(A\Lambda \subseteq X\) intersects \(WR_n\). Levin, Shapira, and Weiss [9] proved that every closed orbit \(A\Lambda \subseteq X_n\) intersects \(WR_n\). Shapira and Weiss [15] proved that every orbit closure \(A\Lambda \subseteq X\) intersects the set of stable lattices \(ST_n\), and concluded that the analog of \((C_n)\), when replacing \(WR_n\) by \(ST_n\), implies Minkowski’s Conjecture.

In this paper we prove the following result, which strengthens results in [10], [9], and [15].

\textbf{Theorem 1.2.} For every \(\Lambda \in X_n\) the orbit \(A\Lambda\) intersects \(ST_n\) and \(WR_n\) w.r.t. any norm.

The proof is inspired by [10] and is a combination of a topological claim and some lattice geometry. To state the topological theorem, we need the concept of invariance dimension. Recall that \(\mathbb{R}^n\) acts on its subsets by translations.

\textbf{Definition 1.3.} The \textit{invariance dimension} of a convex open set \(U \subseteq \mathbb{R}^n\) is the dimension of its stabilizer over \(\mathbb{R}^n\), that is,

\[
\text{invdim } U := \dim \text{stab}_{\mathbb{R}^n}(U).
\]

By convention \(\text{invdim } \emptyset := -\infty\).

The topological result that we need and that extends theorem 5.1 in [10] is the following.

\textbf{Theorem 1.4.} Let \(\mathcal{U}\) be an open cover of \(\mathbb{R}^n\). Assume that

1. the cover
\[
\{\text{conv}U : U \in \mathcal{U}\}
\]
is locally finite;\(^1\)
2. for every \(k \leq n\) and \(k\) different sets \(U_1, \ldots, U_k \in \mathcal{U}\) one has
\[
\text{invdim conv}(U_1 \cap U_2 \cap \ldots \cap U_k) \leq n - k.
\]

Then there are \(n + 1\) sets in \(\mathcal{U}\) with nontrivial intersection.

\(^1\)An open cover is \textit{locally finite} if every compact set intersects only finitely many cover elements.
2. Proof of Theorem 1.2

We will provide some notations, most are taken from [10]. Define the Minkowski successive minima of a lattice $\Lambda$ by

$$\lambda_i(\Lambda) := \inf\{r > 0 : \dim \text{span}\{v \in \Lambda : |v| < r\} \geq i\}.$$ 

Let $\text{WR}_n \subseteq X_n$ be the set of all lattices for which all Minkowski successive minima are equal. Although the standard definition of $\text{WR}_n$ uses the euclidean norm $|\cdot|$, here we consider the analogous definition with an arbitrary fixed norm.

The Harder-Narasimhan filtration was defined in [7] and described nicely by Grayson [4]. Its construction for standard lattices in $\mathbb{R}^n$ goes as follows. For every discrete subgroup $\Gamma < \mathbb{R}^n$, denote by $\text{covol} \Gamma$ the Euclidean volume of the group span $\Gamma/\Gamma$. By convention $\text{covol}\{0\} := 1$. We associate to $\Gamma$ the point $p_{\Gamma} := (\text{rank}(\Gamma), \log \text{covol} \Gamma) \in \mathbb{R}^2$.

For every lattice $\Lambda \in X_n$ define $S_\Lambda := \{p_\Gamma : \Gamma \leq \Lambda\}$. Denote the extreme points of $\text{conv}(S_\Lambda)$ by $p_0, \ldots, p_k$, and, for each $0 \leq i \leq k$, let $\Gamma_i \leq \Gamma$ satisfy $p_i = p_{\Gamma_i}$. A result of the Harder-Narasimhan filtration states that up to reordering, $\{0\} = \Gamma_0 < \ldots < \Lambda_k = \Lambda$, are of strictly increasing ranks. Furthermore, if $p(\Gamma)$ is an extreme point, then $\Gamma$ is the unique subgroup that is associated to this point. In addition, for every $0 \leq i \leq k$ one has $\text{covol} \Gamma_i \leq 1$. The filtration $\{0\} = \Gamma_0 < \ldots < \Lambda_k = \Lambda$ is called the Harder-Narasimhan Filtration.

**Definition 2.1.** The set of stable lattices $\text{ST}_n$ is the set of all lattices $\Lambda \in X_n$ such that the Harder-Narasimhan filtration of $\Lambda$ contains only $\{0\}$ and $\Lambda$, that is, for every $\Gamma \leq \Lambda$ one has $\text{covol} \Gamma \geq 1$.

**Wedge product geometry.** Denote $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$. A basis for $\wedge^k \mathbb{R}^n$ is given by $e_J := e_{j_1} \wedge \ldots \wedge e_{j_k}$ for $J = \{0 < j_1 < \ldots < j_k \leq n\}$. Its dual basis is denoted $\{\varphi_J : \#J = k\} \subseteq \left(\wedge^k \mathbb{R}^n\right)^*.$

A vector in the $k$’th wedge product is called a $k$-vector. For simplicity, for every $k$-vector $v \in \wedge^k \mathbb{R}^n$ we use the norm

$$\|v\|_{k-\text{vec}} := \max_J |\varphi_J(v)|$$

and

$$\text{supp} \omega := \{J : \varphi_J(\omega) \neq 0\}.$$ 

Do not confuse the arbitrary norm $|\cdot|$ of $\mathbb{R}^n$ with $\|\cdot\|_{1-\text{vec}}$ on $\wedge^1 \mathbb{R}^n \cong \mathbb{R}^n$.

**Measured subspaces.** A $k$-dimensional measured subspace is a real vector subspace $M \subseteq \mathbb{R}^n$ equipped with a nonzero $k$-vector $\det(M) \in \wedge^k M$, chosen up to sign. We denote the set of $k$ dimensional measured subspaces by $\mathfrak{G}_{n,k}$. For every $k$ dimensional measured subspace $M$ we define $\|M\|_{\text{MS}} := \|\det M\|_{k-\text{vec}}$. 
Any discrete subgroup $\Gamma < \mathbb{R}^n$ gives rise to a measured space $M(\Gamma) \in \mathcal{G}_{n, \text{rank } \Gamma}$; the space is $\text{span } \Gamma$, and $\det M(\Gamma) = v_1 \wedge ... \wedge v_k$ for a basis $v_1, ..., v_k$ of $\Gamma$.

For a vector space $V$ we define its support to be $\text{supp}(v_1 \wedge ... \wedge v_k)$ for some (any) basis $v_i$ of $V$, this is well-defined because changing the basis only multiplies $v_1 \wedge ... \wedge v_k$ by a nonzero scalar.

An alternative definition of $\text{supp } v$ is

$$\text{supp } v := \{ J \subseteq \{1, ..., n\} \text{ of size } k : \pi_J|_v \text{ is injective} \},$$

where $\pi_J : \mathbb{R}^n \to \mathbb{R}^n$ is the projection setting all coordinates not in $J$ to 0.

**Flags.**

The main object that we use in the proof is the concept of a measured flag. A measured flag is a sequence of measured spaces $\{0 = v_0 < v_1 < ... < v_l = \mathbb{R}^d\}$.

We impose no restrictions on the volume elements. Denote the set of measured flag by $\mathfrak{F}_n$ and for every measured flag $F = \{0 = v_0 < v_1 < ... < v_l = \mathbb{R}^d\}$ define $\|F\|_F := \max_{l>0} \|v_l\|_{MS}$. We will investigate functions $F : A \to \mathfrak{F}_n$ with the following properties.

**Definition 2.2.** A function $F : A \to \mathfrak{F}_n$ is bounded if

$$\sup_{a \in A} \|F(a)\|_F < \infty.$$

It is lower locally invariant if for every $a \in A$ there is a neighborhood $U \subseteq A$ of the identity matrix such that $a'F(a) \subseteq F(a')$ for every $a' \in U$.

$F$ is discrete if the set

$$\{a^{-1} \det v : a \in A, v \in F(a)\}$$

is discrete in $\bigcup_{k=0}^n \Lambda^k \mathbb{R}^n$.

**Theorem 2.3.** For any discrete bounded lower locally invariant $F$ there is a point $a \in A$ such that $F(a)$ is the trivial flag $\{0 < \mathbb{R}^n\}$.

**Proof of Theorem 1.2 using Theorem 2.3.** Fix a lattice $\Lambda_0$ and let $B(r) \subseteq \mathbb{R}^n$ be the ball of radius $r$ with respect to the norm $| \cdot |$. For every lattice $\Lambda \in X_n$ define the Minkowski measured flag by

$$F_{\text{Mink}}(\Lambda) := \{ \text{span } B(r) \cap \Lambda : r > 0 \}$$

and for every $v \in F(\Lambda)$ the volume element is given by $M(v \cap \Lambda)$. By Minkowski’s second theorem one can see that there is a constant $C_n$ depending only on $n$ such that $\|F(\Lambda)\|_F \leq C_n$.

We will prove that for some $a \in A$ we have $a\Lambda_0 \in \mathbb{W} \mathbb{R}_n$. We apply Theorem 2.3 to the flag

$$F(a) := F_{\text{Mink}}(a\Lambda_0).$$
By the previous discussion this flag is bounded. It is discrete since \( \bigcup \Lambda_0^k \) is discrete. It is lower locally invariant by the definition of \( F \). The result follows since \( WR_n = \{ \Lambda \in X_n : F(\Lambda) = \{0 < \mathbb{R}^n\} \} \). A similar proof, using the Harder-Narasimhan filtration instead of the Minkowski measured flag, shows that \( ST_n \) intersect every \( A \) orbit.

The rest of this section is dedicated to the proof of Theorem 2.3 Using Theorem 1.4.

Denote \([a] = \{1, \ldots, a\}\). We will prove the following simple observation.

**Lemma 2.4.** For every flag \( F = \{0 = v_0 < v_1 < \ldots < v_l = \mathbb{R}^n\} \) there exist a permutation \( \sigma \) of \([n]\) such that \( \sigma([\dim v_i]) \in \text{supp} v_i \) for every \( 0 \leq i \leq l \).

**Proof.** Without loss of generality add some subspaces to the flag and assume that \( l = n \), that is, all dimensions appear in \( F \) and \( \dim v_i = i \) for every \( 0 \leq i \leq n \). Recall that for every \( J \subseteq [n] \) we denoted by \( \pi_J : \mathbb{R}^n \to \mathbb{R}^n \) setting all coordinates not in \( J \) to 0, which has rank \#J.

We construct the permutation \( \sigma \) inductively. At the \( k' \)th stage we will construct \( \sigma(k) \) such that \( \pi_{\sigma([k])} \mid v_k \) is a bijection (for \( k = 0 \) this assumption is vacuous). Suppose by induction for some \( J = \sigma([k]) \) we have that \( \pi_J \mid v_k \) is a bijection. We will show that there exist \( j' \notin J \) such that \( \pi_{J \cup \{j'\}} \mid v_{k+1} \) is a bijection and define \( \sigma(k + 1) = j' \). Since \( \dim v_{k+1} > k \) there is a nontrivial vector \( v \in \ker \pi_J \mid v_{k+1} \). Since \( v \in \ker \pi_J \) all its \( J \) coordinates vanish. Since it is nontrivial, there is \( j' \) such that the \( j' \) coordinate of \( v \) is nontrivial. Denote \( J' := J \cup \{j'\} \). Since the \( j' \) coordinate of \( v \) is nontrivial, \( \pi_{J'}(v) \neq 0 \). But \( \pi_J(v) = \pi_J \circ \pi_{J'}(v) = 0 \) and hence \( k = \dim \pi_J(v_{k+1}) < \dim \pi_{J'}(v_{k+1}) \). Therefore \( \pi_{J'} \mid v_{k+1} \) is a bijection, as desired.

**Convex sets**

**Lemma 2.5.** If \( \emptyset \neq U_1 \subseteq U_2 \subseteq \mathbb{R}^n \) are open convex sets then \( \text{invdim} U_1 \leq \text{invdim} U_2 \).

**Proof.** Assume without loss of generality that \( 0 \in U_1 \). Since for every open convex set \( U \) that contains 0 we have

\[
\text{stab}_{\mathbb{R}^n} U = \{ v \in \mathbb{R}^n : \mathbb{R} v \subseteq U \},
\]

the result follows.

Define

\[
\exp : \mathbb{R}^{n-1}_0 := \left\{ (x_1, \ldots, x_n) : \sum_{i=1}^n x_i = 0 \right\} \to A
\]

\[
(x_1, \ldots, x_n) \mapsto \text{diag}(\exp x_1, \ldots, \exp x_n),
\]
and \( \log : A \to \mathbb{R}^{n-1}_0 \) be the inverse function. We will identify \( A \) and \( \mathbb{R}^{n-1}_0 \) using this transformation and push all the notions of convexity that are defined on \( \mathbb{R}^{n-1}_0 \) to \( A \).

Since the exponential function \( x \mapsto e^x \) is convex, and since maximum preserves convexity, the function \( a \mapsto \|aM\|_{MS} \) is a convex function for all \( M \in \mathcal{G}_{n,k} \) and so is \( a \mapsto \|aF\|_F \) for all \( F \in \mathcal{F}_n \).

**Proof of Theorem 2.3 using Theorem 1.4.** Assume to the contrary that \( F : A \to \mathbb{R}^n \) is discrete, lower locally invariant, nowhere trivial, and bounded by \( c_F > 0 \). Construct the following cover of \( \mathbb{R}^{n-1}_0 \). For every \( 0 < k < n \) and \( k \)-dimensional measured space \( v \) define \( U_v := \{a \in A : av \in F(a)\} \). Let \( \mathcal{U} \) be the collection of sets \( \{U_v\} \), where \( v \) ranges over all \( k \)-dimensional measured spaces with \( 0 < k < n \). Since \( F \) is nowhere trivial, we deduce that \( \mathcal{U} \) is a cover of \( A \).

To use Theorem 1.4 we need to prove that its Conditions (1) and (2) holds. To prove that Condition (1) holds, let \( \mathcal{U}' \) be the collection of sets \( U'_v := \{a \in A : \|av\|_{MS} \leq c_F\} \). Since \( F \) is bounded by \( c_F \), we have \( U'_v \supseteq U_v \) for every measured space \( v \). Consequently, \( \mathcal{U}' \) is a cover, and since \( F \) is discrete, it is locally finite. Hence \( \mathcal{U} \) is locally finite as well.

To prove Condition (2) we will classify intersection of elements in \( \mathcal{U} \). Let \( U_{v_1}, U_{v_2}, ..., U_{v_l} \) be elements of \( \mathcal{U} \) that have a nontrivial intersection \( V \neq \emptyset \). For all \( a \in V \) we have \( av_1, ..., av_l \in F(a) \), and hence \( v_1, ..., v_l \) form a flag. Assume without loss of generality that \( 0 < v_1 < v_2 < ... < v_l < \mathbb{R}^n \). By Lemma 2.3 there exists a permutation \( \sigma : [n] \to [n] \) such that \( \sigma([\dim v_k]) \in \text{supp } v_k \). Assume without loss of generality that \( \sigma \) is the identity permutation. Note that for all \( 1 \leq k \leq l \), \( \bar{x} \in \mathbb{R}^{n-1}_0 \) one has

\[
\varphi_{\dim v_k}(\exp(\bar{x})v_k) = \exp(\psi_{\dim v_k}(\bar{x}))\varphi_{\dim v_k}(v_k),
\]

where

\[
\psi_m : \mathbb{R}^{n-1}_0 \to \mathbb{R},
\]

\[
\bar{x} = (x_1, ..., x_n) \mapsto x_1 + ... + x_m.
\]

Denote \( c_k := |\varphi_{\dim v_k}(v_k)| \). For every \( \bar{x} \in \log V \) one has

\[
c_F > \|F(\exp(\bar{x}))\|_F \geq \max_{k=1}^l \|\exp(\bar{x})v_k\|_{MS} \geq \max_{k=1}^l \exp(\psi_{\dim v_k}(\bar{x}))c_k,
\]

and hence the set \( \log V \) is contained in \( P := \bigcap_{k=1}^l \psi_k^{-1}(\infty, \log c_F - \log c_k) \). Since the functionals \( \psi_k \) are linearly independent, the set \( P \) satisfies \( \text{invdim } P = n - 1 - l \), and hence \( \text{invdim } \text{conv}(V) \leq n - 1 - l \).

We proved that the conditions of Theorem 1.4 holds, and therefore the conclusion is as well: there is a nontrivial intersection of \( n \) sets of \( \mathcal{U} \). As shown
above, this intersection corresponds to a nontrivial flag with \( n \) nontrivial elements, which is a contradiction. \( \square \)

3. Proof of Theorem 1.4

3.1. Sketch of proof. The proof of Theorem 1.4 is a modified version of the proof of Theorem 5.1 in [10]. The main steps of the two proofs are the following:

1. We construct a complex of presheaves

\[
\mathcal{F} : 0 \xrightarrow{d} \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \cdots
\]

on \( \mathbb{R}^n \) such that the \( n \)'th cohomology of \( \mathbb{R}^n \) w.r.t. \( \mathcal{F} \), denoted \( H^n_\mathcal{F}(\mathbb{R}^n) \), is nontrivial. We select a family \( \mathcal{E} \) of open subsets of \( \mathbb{R}^n \) and calculate their \( \mathcal{F} \)-cohomologies.

2. Using conditions (1) and (2) we construct for every set of the form \( V := U_1 \cap U_2 \cap \cdots \cap U_k \) a nice set \( V \subseteq E(V) \in \mathcal{E} \) for which the \( (n - k) \) \( \mathcal{F} \)-cohomology is trivial, and such that whenever \( V_1 \subseteq V_2 \) we have \( E(V_1) \subseteq E(V_2) \).

3. We complete the proof using some cohomological algebra. We construct a Čech-deRham double complex \( \mathcal{A} \) using \( \mathcal{F} \) and \( \mathcal{U} \). We prove exactness in the Čech direction, and conclude that the \( \mathcal{F} \)-cohomology of \( \mathbb{R}^n \) is equal to the total cohomology of \( \mathcal{A} \). We cover \( \mathcal{A} \) by a double complex \( \mathcal{B} \), built with \( E \) instead of the intersections themselves. We show that the restriction map \( \mathcal{B} \rightarrow \mathcal{A} \) is onto on the cohomologies. Then we show that \( \mathcal{B} \) is exact in the \( \mathcal{F} \) direction on the \( n \)'th level, and hence any element in the \( n \) cohomology class of \( \mathcal{B} \) can be represented in the class that represents \( E \) of intersection of \( n + 1 \) elements. Since there is a nontrivial \( n \) dimensional \( \mathcal{F} \) cohomology class in \( \mathbb{R}^n \) there is an nonempty intersection of \( n + 1 \) elements of \( U \).

Since \( \mathcal{F} \) is not a sheaf, some work is needed to achieve exactness.

The differences between the proof of Theorem 1.4 and of McMullen are the following:

- McMullen uses the complex of bounded forms while we use the complex of boundedly supported forms.
- For the family \( \mathcal{E} \) McMullen uses cylinders, while we use convex sets.
- The cohomology calculation is different: McMullen calculates it directly while we use the Mayer-Vietoris sequence.
- The Čech-deRham double complex is different: McMullen used direct sum of normed spaces while we use standard direct sum.
3.2. Boundedly supported forms. Denote by $B(r) \subseteq \mathbb{R}^n$ the open ball of radius $r$ around 0. For every open set $U \subseteq \mathbb{R}^n$ denote by $\Omega^k(U)$ the set of $k$-forms on $U$ and by $\Omega^k_{bs}(U)$ the set of $k$-forms on $U$ that vanish outside $B(r)$ for some $r > 0$. Recall the differential transformation $d = d_k : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$. Denote by $H^*(U)$ the $\Omega^*(U)$-cohomology group and by $H^*_{bs}(U)$ the $\Omega^*_{bs}(U)$-cohomology group.

**Definition 3.1.** For every convex open set $U \subseteq \mathbb{R}^n$ we define $\deg U$ as follows. If the projection of $U$ to $\mathbb{R}^n/\text{stab}_{\mathbb{R}^n}(U)$ is bounded then $\deg U := \text{invdim}(U)$; otherwise, $\deg U := -\infty$.

For example, the convex region $U_0 \subseteq \mathbb{R}^2$ bounded by a parabola satisfies $\text{invdim} U_0 = 0$ and $\deg U_0 = -\infty$, and the open cylindrical neighborhood of a line $U_1 \subseteq \mathbb{R}^3$ satisfies $\text{invdim} U_1 = \deg U_1 = 1$.

**Lemma 3.2.** If $U \subseteq \mathbb{R}^n$ is an unbounded open convex set and $\text{invdim} U = 0$, then there is a functional $\varphi$ such that $\{x \in U : \varphi(x) < r\}$ is bounded for every $r > 0$. (3.1)

**Proof.** Assume without loss of generality that $0 \in U$. Denote by $A = A(U) := \{x \in \mathbb{R}^n : \forall \lambda > 0, \lambda x \in U\}$ the union of all rays from 0 that are contained in $U$. Note that $A = \bigcap_{\lambda > 0} \lambda U$ is the intersection of convex sets and hence convex. Since $\frac{1}{2} U \subseteq U$ we have $A = \bigcap_{\lambda > 0} \lambda \bar{U}$, and hence $A$ is closed. Let $S^{n-1}$ be the $n-1$ unit sphere and denote $C = C(U) := S^{n-1} \cap A$. Since

$$C = \bigcap_{\lambda > 0} (S^{n-1} \cap \lambda \bar{U})$$

is the intersection of nonempty compact sets that decrease as $\lambda$ goes to 0, it is nonempty. We argue that $0 \notin \text{conv}(C)$. Indeed if $0 \in \text{conv}(C)$ then there exist $l > 0$, $v_1, ..., v_l \in C$ and positive $\alpha_1, ..., \alpha_l$ such that $\sum_{i=1}^{l} \alpha_i v_i = 0$. Since $U$ is convex it follows that $V := \text{span}\{v_1, ..., v_l\} \subseteq U$, and hence $U$ is invariant to translations by vectors in $V$, which contradicts the assumption that $\text{invdim} U = 0$. Hence, $0 \notin \text{conv} C$, and there exists a functional $\varphi \in (\mathbb{R}^n)^*$ such that $\varphi|_C > 1$. We will show that $\varphi$ satisfies Equation (3.1). Otherwise, there exists $r > 0$ such that the set $U' := \{x \in U : \varphi(x) < r\}$ is unbounded. In particular

$$\emptyset \neq C(U') \subseteq C(U) = C.$$ (3.2)
On the other hand
\[ C(U') \subseteq A(U') = \bigcap_{\lambda > 0} \lambda U' \subseteq \bigcap_{\lambda > 0} \{ x \in U : \varphi(x) < r \} = \{ x \in U : \varphi(x) \leq 0 \}, \]
which, together with Equation (3.2), contradicts \( \varphi|_C > 1 \). Therefore \( U' \) is bounded, as desired.

\[ \square \]

**Theorem 3.3.** For every convex open set \( U \subseteq \mathbb{R}^n \) and every \( k \geq 0 \) we have
\[
H^k_{bs}(U) \cong \begin{cases} \mathbb{R} & k = \deg U, \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** We will prove the claim by induction on \( \text{inhdim} U \). Assume first that \( \text{inhdim} U = 0 \). If \( U \) is bounded, we have \( \Omega^*_s(U) = \Omega^s(U) \), and the claim holds since \( U \) is convex. Assume now that \( U \) is unbounded and define \( \Omega^k_o(U) := \Omega^k(U)/\Omega^k_{bs}(U) \). Let \( \varphi \) be functional satisfying Equation (3.1). Choose \( \omega \in \Omega^k(U) \) that represents a cocycle in \( \Omega^k_o(U) \), the \( \Omega^k_o \)-cohomology of \( U \). Then there is \( r > 0 \) such that \( d\omega \) vanishes on \( V := U \cap \varphi^{-1}(r, \infty) \). Since \( V \) is a nonempty convex set, \( H^k(V) = \begin{cases} \mathbb{R} & k = 0, \\ 0 & \text{otherwise}. \end{cases} \), and hence there exists \( \varpi \in \Omega^{k-1}(V) \) such that
\[
\omega = \begin{cases} \text{const} & \text{if } k = 0, \\ d\varpi & \text{otherwise}. \end{cases}
\]
One can find a \((k-1)\)-form \( \varpi' \in \Omega^{k-1}(U) \) that agrees with \( \varpi \) on \( U \cap \varphi^{-1}(r+1, \infty) \), and thus \([\omega] \in H^k_o(U)\) is either trivial, or equivalent to the constant function if \( k = 0 \). One can see that \( H^0_o(U) \cong \mathbb{R} \), since the constant functions in \( \Omega^0_o(U) \) generate a nontrivial class. By definition the following is a short exact sequence of complexes:
\[
0 \longrightarrow \Omega^*_bs(U) \longrightarrow \Omega^s(U) \longrightarrow \Omega^*_o(U) \longrightarrow 0.
\]
By the snake lemma the following is a long exact sequence of cohomologies:
\[
0 \rightarrow H^0_{bs}(U) \rightarrow H^0(U) \rightarrow H^0_o(U) \rightarrow H^1_{bs}(U) \rightarrow H^1(U) \rightarrow H^1_o(U) \rightarrow \ldots
\]
Note that the arrow \( H^k(U) \rightarrow H^k_o(U) \) is an isomorphism for every \( k \). For \( k = 0 \) the two groups are isomorphic to \( \mathbb{R} \) and the arrow is a monomorphism. For \( k > 0 \) both are trivial. Therefore, all the cohomologies in the sequence \( H^*_o(U) \) are 0, and the proof for the case \( \text{inhdim} U = 0 \) is complete.

For the induction step, suppose \( \text{inhdim} U = k > 0 \). Assume without loss of generality that \( U = \mathbb{R}^k \times U' \) for \( U' \subseteq \mathbb{R}^{n-k} \) with \( \text{inhdim} U' = 0 \). Write \( U = U_1 \cup U_2 \) where \( U_1 := U \cap \{ x_1 \geq -1 \} \) and \( U_2 := U \cap \{ x_1 \leq 1 \} \). Denote
\[ V := U_1 \cap U_2. \] Note that \( \text{invdim} U_1 = \text{invdim} U_2 = \text{invdim} V = k - 1, \) \( \deg U_1 = \deg U_2 = -\infty, \) and \( \deg V = \deg U - 1. \) Note that

\[
\begin{array}{c}
0 \longrightarrow \Omega_{bs}^0(U) \xrightarrow{\alpha \mapsto \alpha^0} \Omega^*_0(U_1) \times \Omega^*_0(U_2) \xrightarrow{(\alpha, \beta) \mapsto \alpha - \beta} \Omega^*_0(V) \longrightarrow 0
\end{array}
\]

is a short exact sequence of complexes and by the snake lemma there is a long exact sequence of cohomologies

\[
0 \to H^0_{bs}(U) \to H^0_{bs}(U_1) \oplus H^0_{bs}(U_2) \to H^0_{bs}(V) \to H^1_{bs}(U) \to H^1_{bs}(U_1) \oplus H^1_{bs}(U_2) \to H^1_{bs}(V) \to \ldots
\]

Since \( H^*_{bs}(U_1) \) and \( H^*_{bs}(U_2) \) vanish we conclude that \( H^l_{bs}(U) \cong H^l_{bs}(V) \) for every \( l \geq 1, \) as desired. \( \square \)

3.3. Complexes. A double complex is a collection of Abelian groups \( \{ C^{p,q} \}_{p,q \geq 0} \) with two maps

\[
d : \bigoplus_{p,q \geq 0} C^{p,q} \to \bigoplus_{p,q \geq 0} C^{p,q+1}, \quad \delta : \bigoplus_{p,q \geq 0} C^{p,q} \to \bigoplus_{p,q \geq 0} C^{p+1,q},
\]

defined by the restrictions

\[
d|_{C^{p,q}} = d_{p,q} : C^{p,q} \to C^{p,q+1}, \quad \delta|_{C^{p,q}} = \delta_{p,q} : C^{p,q} \to C^{p+1,q},
\]

which are differentials and commute:

\[ \delta^2 = d^2 = \delta d - d \delta = 0. \]

We say that the degree of \( C^{p,q} \) is \( p + q \) and define the total complex of \( C \) by \( C^r := \bigoplus_{p+q=r} C^{p,q} \) and

\[
D : \bigoplus_{r \geq 0} C^r \to \bigoplus_{r \geq 0} C^{r+1},
\]

defined by the restrictions \( D|_{C^r} = D_r : C^r \to C^{r+1}, \) which in turn is defined by \( D_r|_{C^{p,q}} = (-1)^q \delta_{p,q} + d_{p,q}. \) One can verify that \( D^2 = 0. \) The total cohomologies of the double complex are \( H^*_C := \ker D_r/\text{Im} D_{r-1}. \)

Lemma 3.4. If \( \delta \) is exact at all groups of degree \( r, \) then any \( \alpha \in H^*_C \) has a representative \( a \in C^0. \)

Proof. Let \( \alpha \in C^r \) for which \( D\alpha = 0. \) We will find \( \beta \in C^{r-1} \) such that \( \alpha + D\beta \in C^0. \) Assume that

\[ \alpha = \sum_{p+q=r, p \leq l} a^{p,q} \in \bigoplus_{p+q=r, p \leq l} C^{p,q}, \quad (3.3) \]

where \( a^{p,q} \in C^{p,q}, \) \( a^{l+1-r} \neq 0, \) and \( l > 0. \) We will show that there is \( \beta \in C^{r-1} \) that satisfies \( \alpha + D\beta \in \bigoplus_{p+q=r, p \leq l-1} C^{p,q}. \) Iterating this process yields the desired result.
Since $D\alpha = 0$ and $l$ is the maximal index in the right-most term in Equation (3.3), we deduce that $\delta \alpha^{l,r-l} = 0$. Since $\delta$ is exact, there is $\beta \in C^{l-1,r-l}$ that satisfies $(-1)^{r-l}\delta \beta + \alpha^{l,r-l} = 0$. Therefore

$$\alpha + D\beta \in \bigoplus_{p+q=r,p\leq l-1} C^{p,q}.$$  

\[\square\]

**Remark 3.5.** The Proof of Lemma 3.4 is valid as soon as $D\alpha \in C^{0,r+1}$.

Define $C^{-1,q} = \ker \delta_{0,q}$. This construction has the following meaning: one can extend $C$ to a double complex with the new cells $C^{-1,q}$. Note that the image of the restriction $D|C^{-1,q} = d|C^{-1,q}$ lies in $C^{-1,q+1}$, and hence $C^{-1,q}$ is a complex. We denote its cohomologies by $H_{C,d}$. Note that there is an inclusion map $C^{-1,q} \hookrightarrow C^{r}$ which induces a map $H_{C,d}^{r} \rightarrow H_{C}^{r}$. 

**Corollary 3.6.** If $\delta$ is exact then the map $H_{C,d}^{r} \rightarrow H_{C}^{r}$ is an isomorphism.

**Proof.** By Lemma 3.4 the map $H_{C,d}^{r} \rightarrow H_{C}^{r}$ is onto. We will show that this map is one to one. Assume $[\alpha^{0,r}] \in H_{C,d}^{r}$ vanishes in $H_{C}^{r}$; that is, there exists $\beta \in C^{r-1}$ such that $D\beta = \alpha^{0,r}$. By Lemma 3.4 and Remark 3.5 there exists $\gamma \in C^{r-2}$ such that $\beta^{0,r-1} = D\gamma + \beta \in C^{0,r-1}$. Thus, $\alpha^{0,r} = D\beta = D\beta^{0,r-1} = \delta \beta^{0,r-1}$. Since $D\beta^{0,r-1} = \alpha^{0,r-1}$, one has $\delta \beta^{0,r-1} = 0$, and thus $\alpha^{0,r}$ is trivial in $H_{C,d}^{r}$. 

\[\square\]

### 3.4. The Čech-De Rham double complex.

We will start this section by defining the Čech-De Rham double complex. Let $\mathcal{U}$ be an open cover of $\mathbb{R}^{n}$ that satisfies the conditions of Theorem 1.4. Choose an arbitrary order on the set $\mathcal{U}$.

Consider the following double complex:

$$\mathcal{A}^{p,q} = C^{p}(\mathcal{U}, \Omega^{q}_{\text{bs}}) := \bigoplus_{J \subseteq \mathcal{U}, \# J = p+1} \Omega^{q}_{\text{bs}}(U_{J}),$$

where $U_{J} := \bigcap_{U \in J} U$. We think of this direct sum as a subset of the direct product, and write its elements in coordinate form.

The differential $d = d_{p,q} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p+1,q}$ is the one defined on forms, and the differential

$$\delta = \delta_{p,q} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p+1,q}$$

$$(\omega_{J})_{\# J = p+1} \mapsto (\omega'_{J'})_{\# J' = p+2}$$
is the one defined by

\[ \omega_J \in \Omega_{bs}^q(U_J), \quad \omega_{J'} := \sum_{U \in J'} (-1)^{[U:J']} \omega_{J'-U} \in \Omega_{bs}^q(U_{J'}), \]

where \([U : J]\) is the index of \(U\) in \(J\) by the order induced from \(\mathcal{U}\); it is 0 if \(U\) is the smallest element in \(J\) and \(p\) if it is the largest. Because only finitely many \(\omega_J\) are nonzero and they all have bounded support, every \(\omega_{J'}\) vanishes outside a bounded set. Since \(\mathcal{U}\) is locally finite, only finitely many \(U_{J'}\)-s intersect any bounded set, and hence only finitely many \(\omega_{J'}\)-s are nonzero. \(A^{p,q}\) is the Čech-De Rham double complex. One can verify that \(\delta^2 = 0\) and that \(\delta\) and \(d\) commute.

One property of the Čech-De Rham double complex is that \(\delta\) is exact.

**Theorem 3.7.** The differential \(\delta\) is exact.

**Proof.** For every collection of sets \(J\) and sets \(U \in J, V \notin J\) denote \(J + V := J \cup \{V\}\) and \(J - U := J \setminus \{U\}\). Choose a partition of unity \(\{\rho_U\}_{U \in \mathcal{U}}\). Let

\[ \omega = (\omega_J)_{\#J = p+1} \in C^p(\mathcal{U}, \Omega_{bs}^q) \]

such that only finitely many \(\omega_J\)-s are nonzero. As in \[2\] Prop 8.5] we define

\[ T : C^p(\mathcal{U}, \Omega_{bs}^q) \rightarrow C^{p-1}(\mathcal{U}, \Omega_{bs}^q) \]

\[ \omega \mapsto (\omega_{J'})_{\#J = p} \in C^{p-1}(\mathcal{U}, \Omega_{bs}^q), \]

where

\[ \omega_{J'} := \sum_{V \notin J} (-1)^{[V:J+V]} \rho_V \omega_{J+V}. \]

Because \(\mathcal{U}\) is locally finite, only finitely many \(\omega_{J'}\)-s are nonzero.
Note that
\[
\delta T \omega = \delta \left( \sum_{V \in \Omega \setminus J} (-1)^{[V:J+V]} \rho_{V} \omega_{J+V} \right)_{\#J=p}
\]
\[
= \left( \sum_{U \in J} (-1)^{[U:J]} \left( \sum_{V \in J-U} (-1)^{[V:J-U+V]} \rho_{V} \omega_{J-U+V} \right) \right)_{\#J=p+1}
\]
\[
= \left( \sum_{V=U \in J} (-1)^{[U:J]} (-1)^{[V:J]} \rho_{V} \omega_{J} \right)_{\#J=p+1}
\]
\[
+ \left( \sum_{U \in J} (-1)^{[U:J]} \left( \sum_{V \notin J} (-1)^{[V:J-U+V]} \rho_{V} \omega_{J-U+V} \right) \right)_{\#J=p+1}
\]
\[
= \left( \sum_{V \in J} \rho_{V} \omega_{J} \right)_{\#J=p+1}
\]
\[
+ \left( \sum_{V \notin J} \rho_{V} \omega_{J} \right)_{\#J=p+1}
\]
\[
- \left( \sum_{V \notin J} \rho_{V} (-1)^{[V:J+V]} \left( \sum_{U \in J+V} (-1)^{[U:J+V]} \omega_{J-U+V} \right) \right)_{\#J=p+1}
\]
\[
= \omega - T \delta \omega
\]
Therefore, if \( \omega \in \ker \delta \) then \( \omega = \delta T \omega \), and hence \( \delta \) is exact. \( \square \)

Note also that \( \ker \delta_{0,r} \) represents forms on \( \mathcal{U} \)-elements that agree on pairwise intersections, and hence \( \ker \delta_{0,r} \simeq \Omega_{bs}^{r}(\mathbb{R}^{n}) \). From Corollary 3.6 we deduce that \( H_{A}^{r} \simeq H_{bs}^{r}(\mathbb{R}^{n}) \).

Define the following double complex :
\[
\mathcal{B}^{p,q} := \bigoplus_{J \subseteq \mathcal{U}, \#J=p+1} \Omega_{bs}^{q}(\text{conv} U_{J}),
\]
and define \( d, \delta \), and \( D \) as for the double complex \( \mathcal{A} \). Denote the direct sum of the restriction transformations by \( \text{res} : \mathcal{B}^{p,q} \rightarrow \mathcal{A}^{p,q} \). Since \( \text{res} \) commutes with \( d, \delta \), and \( D \) it define a map \( \text{res}_{*} : H_{B}^{r} \rightarrow H_{A}^{r} \).

**Proposition 3.8.** The map \( \text{res}_{*} \) is onto.

**Proof.** Let \( \alpha \in H_{A}^{r} \). Since \( \delta \) is exact and by Corollary 3.6 we have \( H_{A}^{r} \simeq H_{d,A}^{r} \simeq H_{bs}^{r}(\mathbb{R}^{d}) \), and therefore the class \( \alpha \) corresponds to a class \([\omega] \in H_{bs}^{r}(\mathbb{R}^{d})\). Choosing \( \beta := (\omega |_{\text{conv} U})_{U \in \mathcal{U}} \in \mathcal{B}^{0,r} \) we get \( \alpha = [\text{res} \beta] \) and \( \delta \beta = d \beta = D \beta = 0 \). In particular, \( \alpha \in \text{Im} \text{res}_{*} \). \( \square \)

**Proposition 3.9.** At the groups \( \mathcal{B}^{p,q} \) of degree \( n \) the differential \( d \) is exact.
Proof. It is enough to show that if $p + q = n$ and $J \subseteq \mathfrak{U}$ is of size $p + 1$, then $H^q_{bs}(\text{conv}U_J) = 0$. By Theorem 3.3, the only nontrivial cohomology of $\text{conv}U_J$ may be at rank $\text{invdim} \text{conv}U_J$, and by the assumptions of Theorem 1.4 $\text{invdim} \text{conv}U_J \leq n - (p + 1) = q - 1$. Thus the $q$ boundedly supported cohomology of $\text{conv}U_J$ is trivial, as desired. \hfill \Box

Proof of Theorem 1.4. Since $\text{deg} \mathbb{R}^n = n$ it follows that $H^n_{bs}(\mathbb{R}^n) \cong \mathbb{R} \not\cong 0$. Since $H^n_{bs}(\mathbb{R}^n) \cong H_A^n = 0$. Since $\text{res}_*$ is onto it follows that $H^n_B \not\cong 0$. By Lemma 3.4 and the exactness of $d$ at the groups $B^p,q$ of degree $n$, we get that $B^{n,0} \not\cong 0$. Thus, for some $J \subseteq \mathfrak{U}$ of size $n + 1$ the set $U_J$ is nonempty. \hfill \Box

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