LAZARSFELD-MUKAI BUNDLES ON K3 SURFACES ASSOCIATED WITH A PENCIL COMPUTING THE CLIFFORD INDEX

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Abstract. Let \( X \) be a smooth projective K3 surface over the complex numbers and let \( C \) be an ample curve on \( X \). In this paper we will study the semistability of the Lazarsfeld-Mukai bundle \( E_{C,A} \) associated to a line bundle \( A \) on \( C \) such that \(|A|\) is a pencil on \( C \) and computes the Clifford index of \( C \). We give a necessary and sufficient condition for \( E_{C,A} \) to be semistable.

1. Introduction

Let \( X \) be a smooth projective K3 surface over the complex numbers and \( C \) be a smooth projective ample curve in \( X \). Given a globally generated line bundle \( A \) on \( C \), the Lazarsfeld-Mukai bundle \( E_{C,A} \) is defined as the dual of the kernel of the evaluation map

\[
H^0(A) \otimes \mathcal{O}_X \to \iota_*(A),
\]

where \( \iota : C \to X \) is the inclusion. To study the behavior of certain invariants of a curve \( C \) along a linear system (for example Clifford index, gonality, gonality sequence etc) it has been essential to investigate the properties of the Lazarsfeld-Mukai bundles. Several authors have studied the Lazarsfeld-Mukai bundle in several contexts for example see [2], [4], [8], [7] and the references therein.

The study of the stability (Gieseker or slope) of vector bundles with respect to a given ample line bundle on algebraic varieties is a very active topic in algebraic geometry. The purpose of this paper is to study the stability of the Lazarsfeld-Mukai bundle \( E_{C,A} \) associated to a curve \( C \) on a K3 surfaces \( X \) and a globally generated line bundle \( A \) on \( C \). In [7] Margherita Lelli-Chiesa proved that if \( C \) is a \( \lfloor \frac{g+3}{2} \rfloor \)-gonal curve of genus \( g \) and Clifford dimension one and degree of \( A = d \) satisfies \( \rho(g, 1, d) = 2d - g - 2 > 0 \), then \( E_{C,A} \) is stable with respect to \( \mathcal{O}_X(C) \). In [5], Watanable has shown that if \( E_{C,A} \) is not slope semistable with respect to \( \mathcal{O}_X(C) \), the maximal destabilizing subsheaf of it contains an initialized and ACM line bundle with respect to \( \mathcal{O}_X(C) \), which gives a sufficient condition for \( E_{C,A} \) to be \( \mathcal{O}_X(C) \)-slope semistable and gave some examples.

Key words and phrases. K3 surface, Lazarsfeld-Mukai bundles, Clifford index.
In this article, we will show that if the Clifford index of \( C \) is strictly smaller than \( \left\lfloor \frac{g-1}{2} \right\rfloor - 1 \) and \( A \) is a pencil on \( C \) computing the Clifford index of \( C \), then \( E_{C,A} \) is never \( O_X(C) \)-slope stable. In fact we shall show that if \( E_{C,A} \) is semistable then it is free, that is, \( E_{C,A} \) is a direct sum of line bundles of the same slope. Hence \( C \) must have a decomposition of the form \( C \sim 2D \), where \( D \) is an effective divisor. More precisely we will prove the following theorem:

**Theorem 1.1.** Let \( C \) be a smooth projective curve on a smooth projective \( K3 \) surface \( X \) of Clifford index \( < \left\lfloor \frac{g-1}{2} \right\rfloor - 1 \). Let \( A \) be a globally generated pencil on \( C \) computing the Clifford index of \( C \). If \( E_{C,A} \) is semistable then it splits as a direct sum of line bundles of the same slope with respect to \( O_X(C) \) and \( C \) decomposes as \( C \sim 2D \) for some effective divisor \( D \) and the Clifford index of \( C \), is \( \left\lfloor \frac{g-1}{2} \right\rfloor - 2 \). In particular \( E_{C,A} \) can never be stable. Furthermore, if the Clifford index of \( C \), is strictly smaller that \( \left\lfloor \frac{g-1}{2} \right\rfloor - 2 \), then \( E_{C,A} \) is never semi-stable. Conversely, if the Clifford index of \( C \), is equals to \( \left\lfloor \frac{g-1}{2} \right\rfloor - 2 \) and \( C \) can be decomposed as \( C \sim 2D' \) but \( C \sim 2D' + kE + k\Delta \), where \( k \) is a positive integer, \( E \) is a smooth elliptic curve and \( \Delta \) is a \((-2)\) curve, then for a pencil \( A \) computing Clifford index of \( CE_{C,A} \) is semistable.

**Notations and Conventions**

We work over the complex number field \( \mathbb{C} \). Surfaces and curves are smooth and projective. For a curve \( C \), we denote by \( K_C \) the canonical line bundle of \( C \). For a line bundle \( L \) on a smooth projective variety \( X \), we denote by \(|L|\) the linear system defined by \( L \), i.e., \(|L| = \mathbb{P}(H^0(L)^*)\).

For a line bundle \( A \) on a curve \( C \), the Clifford index of \( A \) is defined as follows:

\[
\text{Cliff}(A) := \text{degree}(A) - 2\dim(|A|).
\]

The Clifford index of a curve \( C \) is defined as follows;

\[
\text{Cliff}(C) := \min\{\text{Cliff}(A) | h^0(A) \geq 2, h^1(A) \geq 2\}.
\]

Clifford’s theorem states that \( \text{Cliff}(C) \geq 0 \) with equality if and only if \( C \) is hyperelliptic, and similarly \( \text{Cliff}(C) = 1 \) if and only if \( C \) is trigonal or a smooth plane quintic. At the other extreme, if \( C \) is a general curve of genus \( g \) then \( \text{Cliff}(C) = [(g-1)/2] \), and in any event \( \text{Cliff}(C) \leq [(g-1)/2] \). We say that a line bundle \( A \) on \( C \) contributes to the Clifford index of \( C \) if \( A \) satisfies the inequalities in the definition of \( \text{Cliff}(C) \); it computes the Clifford index of \( C \) if in addition \( \text{Cliff}(C) = \text{Cliff}(A) \).

### 2. Linear System on K3 Surfaces

In this section we recall some classical results about line bundles and divisors on K3 surfaces.

**Proposition 2.1.** Let \( D \) be a non-zero effective divisor with \( D^2 \geq 0 \) on a K3 surface \( X \). Then one can write \( D \sim D' + \Delta \), where \( \Delta \) is the fixed component of \( D \) and \( D' \) is base point free. Let \( \Delta_1, \Delta_2, ..., \Delta_n \) be the connected reduced
components of $\Delta$. Then one of the following holds:

(i) there exists an elliptic curve $E$, such that $D' \sim kE$, for some integer $k \geq 2$ and there exist one and only one connected reduced component $\Delta_1$ of $\Delta$ such that $E \cdot \Delta_1 = 1$ and $\Delta_i \cdot E = 0$, for $i \neq 1$.

(ii) $D'$ is an irreducible and $D' \cdot \Delta_i = 0$ or $1$, $i = 1, 2, \ldots, n$.

Proof. See [9, 2.7].

Remark 2.2. (a) Note that any connected reduced component of $\Delta$ has self intersection number $\leq -2$. Thus if $\Delta_1$ is a connected reduced component of $\Delta$, and the second case occurs then $D \cdot \Delta_1 < 0$. Thus if $D$ is nef, then we get a contradiction. In other words, if $D$ is nef then case (ii) does not occur. Similarly, in case (i), one can see that if $D$ is nef, then $D \sim kE + \Delta$, where $k \geq 2$ and $\Delta$ is an irreducible $(-2)$ curve.

(b) Further more if $D^2 = 0$, then $\Delta = 0$, hence $D \sim kE$ for some non-negative integer $k$.

Proposition 2.3. Let $L$ be a line bundle on a K3 surface $X$ such that $|L| \neq \emptyset$ and such that $|L|$ has no fixed components. Then either

(i) $L^2 > 0$, and the generic member of $|L|$ is an irreducible curve of arithmetic genus $\frac{1}{2}(L.L) + 1$. In this case $h^1(L) = 0$, or

(ii) $L^2 = 0$, then $L \cong (O_X(E))^\oplus k$, where $k$ is an integer $\geq 1$ and $E$ an irreducible curve of arithmetic genus 1. In this case $h^1(L) = k$ and every member of $|L|$ can be written as a sum $E_1 + E_2 + \ldots + E_k$, where $E_i \in |E|$ for $i = 1, 2, \ldots, k$.

Proof. See [9, Proposition 2.6].

Proposition 2.4. Let $L$ be a line bundle on a K3 surface $X$ such that $|L| \neq \emptyset$. Then $|L|$ has no base points outside its fixed components.

Proof. See [9, Corollary 3.2].

Theorem 2.5. Let $|L|$ be a complete linear system on a K3 surface $X$, without fixed components, and such that $L^2 \geq 4$. Then $L$ is hyperelliptic only in the following cases:

(i) There exists an irreducible curve $E$ such that $p_a(E) = 1$ and $E.L = 1$ or 2.

(ii) There exists an irreducible curve $B$ such that $p_a(B) = 2$ and $L \cong O_X(2B)$.

Proof. See [9, Theorem 5.2].

3. Structure of Lazarsfeld-Mukai bundles

In this section we recall the basic properties of the bundle $E_{C,A}$ of Lazarsfeld [6], associated to an irreducible smooth curve $C$ in $X$ and a globally generated line bundle $A$. 

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Let $X$ be a $K3$ surface. Let $C$ be an irreducible smooth curve in $X$ and let $A$ be a globally generated line bundle on $C$. Viewing $A$ as a sheaf on $X$, consider the evaluation map:

$$H^0(C, A) \otimes O_X \rightarrow A.$$ 

Let $F_{C,A}$ be its kernel and $E_{C,A} := F^*_{C,A}$. Then $F_{C,A}$ fits in the following exact sequence on $X$.

$$0 \rightarrow F_{C,A} \rightarrow H^0(C, A) \otimes O_X \rightarrow A \rightarrow 0.$$ 

It is easy to check that $F_{C,A}$ is locally free. Dualizing the above exact sequence one gets

$$0 \rightarrow H^0(C, A)^* \otimes O_X \rightarrow E_{C,A} \rightarrow O_C(C) \otimes A^* \rightarrow 0.$$ 

Then it is easy to check that the following properties hold:

**Lemma 3.1.**
1. Rank of $E_{C,A} = h^0(C, A)$.
2. $\det(E_{C,A}) = O_X(C)$.
3. $c_2(E_{C,A}) = \deg(A)$.
4. $h^0(X, E_{C,A}^*) = h^1(X, E_{C,A}^*) = 0$.
5. $E_{C,A}$ is generated by its global sections off a finite set.
6. If $\rho(g,d,r) = g - (r + 1)(g - d + r) < 0$, then $E_{C,A}$ is non-simple.

Furthermore if $E_{C,A}$ is of rank 2, that is, $|A|$ is a pencil then $E_{C,A}$ has the following characterization.

**Lemma 3.2.** Let $F$ be a non-simple vector bundle of rank 2 on $X$. There exists line bundles $M, N$ on $X$ and a zero-dimensional subscheme $Z \subset X$ such that $F$ fits in an exact sequence

$$0 \rightarrow M \rightarrow F \rightarrow N \otimes I_Z \rightarrow 0$$

and either

(a) $M \geq N$ or
(b) $Z$ is empty and the sequence splits, $F \cong M \oplus N$.

**Proof.** See [3, Lemma 4.4].

4. **The main theorem**

In this section we will prove the main theorem.

**Lemma 4.1.** Let $C$ be an ample curve of genus $g$ on $X$ such that $C$ can be decomposed as $C \sim C_1 + C_2$ such that $C_1.C = C_2.C$ with $C_1.C_2 \leq \left\lfloor \frac{g-1}{2} \right\rfloor$. If $C$ has another decomposition $C \sim D_1 + D_2$ with $D_1.D_2 \leq \left\lfloor \frac{g-1}{2} \right\rfloor$ such that $D_1.C > D_2.C$, then $D_1.D_2 \leq C_1.C_2$, and the equality holds if and only if $D_1 \sim D_2 + kE + k\Delta$ where $k$ is a positive integer, $E$ is an elliptic curve and $\Delta$ is either zero or a connected reduced $(-2)$ curve.
Proof. Note that since $C_1C_2 \leq \lfloor \frac{g-1}{2} \rfloor$, $(C_1 - C_2)^2 = C_1^2 + C_2^2 - 2C_1C_2 = C_2^2 - 4C_1C_2 \geq 0$. Thus $O_X(C_1 - C_2)$ has a section up to exchanging $C_1$ and $C_2$. Then we have two possibilities. (i) $C_1 - C_2 \sim 0$, then $C_1 - C_2 \sim D$ for some effective divisor $D$. Or (ii), $C_1 - C_2 \sim 0$.

If $D$ is non-zero effective then since $C$ is ample, $C.D > 0$, a contradiction. Thus $C_1 \sim C_2$. Since $D_1.D_2 \leq \lfloor \frac{g-1}{2} \rfloor$ and $D_1 \sim D_2$, as before $D_1 - D_2 \sim D$ for some non-zero effective divisor $D$. Thus we have

$$2D_1 \sim C + D \text{ and } 2D_2 \sim C - D,$$

which gives that $D_1^2 + D_2^2 = \frac{C_1^2}{2} + \frac{D^2}{2}$. Thus we have

$$C_1^2 = D_1^2 + D_2^2 + 2D_1.D_2$$

$$= \frac{C_1^2}{2} + \frac{D^2}{2} + 2D_1.D_2 \text{ which implies}$$

$$\frac{C_1^2}{2} - \frac{D^2}{2} = 2D_1.D_2. \quad (4)$$

On the other hand, $\frac{C_1^2}{2} = 2C_1C_2$. Thus we have

$$2C_1C_2 = 2D_1.D_2 + \frac{D^2}{2}. \quad (5)$$

Since $D^2 \geq 0$, $D_1.D_2 \leq C_1C_2$ and equality holds if and only if $D^2 = 0$. By Remark 2.2, $D \sim kE + k\Delta$ ($\Delta$ could be 0) where $k$ is a positive integer and $E$ is a smooth elliptic curve and $\Delta$ is a $(-2)$ curve, which concludes the Lemma.

Lemma 4.2. Let $C$ be an ample curve on a K3 surface $X$ with Clifford index $\leq \lfloor \frac{g-1}{2} \rfloor$. Let $A$ be a globally generated pencil on $C$ computing the Clifford index. Then $E_{C,A}$ fits in an exact sequence of the form

$$0 \to M \to E_{C,A} \to N \to 0$$

where $M, N$ are line bundles with $h^0(M), h^0(N) \geq 2$ and $N$ is base point free.

Proof. Let $A$ be a line bundle on $C$ of degree $d$ such that $h^0(A) = 2$ and $A$ computes the Clifford index of $C$. Note that $c = d - 2$. By Riemann-Roch, we have $h^0(K_C \otimes A^*) = g - c - 1$. Thus from the exact sequence (2), we have

$$h^0(E_{C,A}) = 2 + h^0(K_C \otimes A^*) = g + 1 - c.$$

On the other hand, by Lemma 3.2, there are line bundles $M, N$ satisfying the hypothesis of the Lemma and a zero-dimensional subscheme $Z \subset X$ such that $E_{C,A}$ fits in the following exact sequence:

$$0 \to M \to E_{C,A} \to N \otimes \mathcal{I}_Z \to 0.$$

Note that since $E_{C,A}$ is globally generated, $N$ is also globally generated, hence $h^0(N) \geq 2$. If $Z$ is empty, then $M$ is also globally generated and if
Z is non-empty, then $M \geq N$. Thus in any case $h^0(M), h^0(N) \geq 2$. Let us assume that $Z$ is non-empty. It is easy to see that $M|_C$ computes the Clifford index of $C$. Thus we have

$$c = M.C + 2 - 2h^0(M|_C)$$

which gives $h^0(M|_C) = \frac{M.C}{2} + 1 - \frac{c}{2}$.

On the other hand, by Riemann-Roch, we have

$$h^0(N|_C) = N.C + 1 - g + h^1(N|_C) = N.C + 1 - g + h^0(M|_C)$$

$$= N.C + 1 - g + \frac{M.C}{2} + 1 - \frac{c}{2}.$$

Since $E_{C,A}$ is globally generated off a finite set and by Proposition 2.4, $N$ cannot have base points outside a fixed component, $N$ is base point free. Thus if $Z$ is nonempty, then $h^0(N \otimes \mathcal{I}_Z) < h^0(N)$. Thus we have,

$$g + 1 - c = h^0(E_{C,A})$$

$$\leq h^0(M) + h^0(N \otimes \mathcal{I}_Z)$$

$$< h^0(M) + h^0(N)$$

$$\leq h^0(M|_C) + h^0(N|_C)$$

$$= \frac{M.C}{2} + 1 - \frac{c}{2} + N.C + 1 - g + \frac{M.C}{2} + 1 - \frac{c}{2}$$

$$= M.C + N.C + 3 - g - c$$

$$= (M + N).C + 3 - g - c$$

$$= C^2 + 3 - g - c$$

$$= 2g - 2 + 3 - g - c$$

$$= g + 1 - c,$$

a contradiction. Thus $E_{C,A}$ fits in the following exact sequence

(7) \hspace{1cm} 0 \to M \to E_{C,A} \to N \to 0.

\[\square\]

**Remark 4.3.** Note that if the sequence, in the above Lemma does not split, then $M \geq N$. Thus $(M - N).C \geq 0$ for any irreducible curve $C$. In other words, $M \otimes N^*$ is nef.

**Proof of the Theorem 1.1:**

By Lemma 4.2, $E_{C,A}$ fits in an exact sequence of the form 7. If $M \sim N$ then, $h^1(M \otimes N^*) = h^1(O_X) = 0$. Hence the sequence splits and we are done. Let us assume $M \not\sim N$. Note that $M.N = c_2(E_{C,A}) = d$. 

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Now
\[ (8) \]
\[
c_1(M \otimes N^*)^2 = M^2 + N^2 - 2M.N = M^2 + N^2 + 2M.N - 4M.N
\]
\[
= c_1(M \otimes N)^2 - 4d
\]
\[
= C^2 - 4d
\]
\[
= 2g - 2 - 4d \geq 0, \text{ since } c = d - 2 < \left\lfloor \frac{g-1}{2} \right\rfloor - 1.
\]

Since \( M \geq N \), the Euler characteristic computation says that \( M \otimes N^* \) has a section. In other words, \( |M \otimes N^*| \) contains an effective divisor.

**Claim:** \( h^1(M \otimes N^*) = 0 \)

**Proof of the claim:**

By Remark 4.3, \( M \otimes N^* \) is nef. Thus if \( M \otimes N^* \) is not base point free, then by Remark 2.2, there exists a smooth elliptic curve \( E \) and a rational curve \( \Gamma \) such that \( M \otimes N^* \cong \mathcal{O}_X(kE + \Gamma) \), where \( k \) is an integer \( \geq 2 \) and \( E, \Gamma = 1 \).

But \( h^0(\mathcal{O}_X(kE + \Gamma)) = k + 1 \). Thus the Euler characteristic computation says that \( h^1(\mathcal{O}_X(kE + \Gamma)) = 0 \).

Let us assume \( M \otimes N^* \) is base point free.

If \( c_1(M \otimes N^*)^2 > 0 \), then by Proposition 2.3, \( h^1(M \otimes N^*) = 0 \) and we are done in this case. If \( c_1(M \otimes N^*)^2 = 0 \), then by Proposition 2.3, \( M \otimes N^* \cong \mathcal{O}_X(kE) \), where \( k \) is a positive integer and \( h^1(M \otimes N^*) = k - 1 \).

Note that \( M \otimes 2 = \mathcal{O}_X(C + kE) \). Thus we have \( 2M.C = C^2 + kC.E > 2g - 2 \).

On the other hand, since \( E_{C,A} \) is semistable with respect to \( \mathcal{O}_X(C) \), \( M.C \leq g - 1 \) which is a contradiction. Thus we have \( h^1(M \otimes N^*) = 0 \), in other words, the sequence (7) splits and \( E_{C,A} \cong M \oplus N \).

Since \( E_{C,A} \) is semistable, \( M.\mathcal{O}_X(C) = N.\mathcal{O}_X(C) \). Thus \( C \) has a decomposition of the form \( C \sim C_1 + C_2 \), where \( \mathcal{O}_X(C_1) = M \) and \( \mathcal{O}_X(C_2) = N \) and \( C.C_1 = C.C_2 \). From the proof of Lemma 4.1, we have \( C_1 \sim C_2 \). Thus we have \( C \sim 2D \) for some effective divisor \( D \) and

\[
\text{Cliff}(C) = d - 2 = M.N - 2 = D^2 - 2 = \frac{2g - 2}{4} - 2 = \frac{g - 1}{2} - 2.
\]

That proves the first part of the Theorem 1.1.

Conversely, let \( C \sim 2D \) but \( C \sim 2D' + kE \) for any positive integer \( k \) and for any elliptic curve \( E \) and \( \text{Cliff}(C) = \left\lceil \frac{g-1}{2} \right\rceil - 2 \). Let \( A \) be a globally generated pencil computing the Clifford index of \( C \). Let us assume that \( E_{C,A} \) is not semistable. If the subbundle \( M \) in 7 is not destabilizing, that is \( M.\mathcal{O}_X(C) \leq g - 1 \), then from the first part of the proof one can see that the sequence 7 splits. Thus \( E_{C,A} = M \oplus N \) and \( N \) destabilizes \( E_{C,A} \). Therefore in any case either \( M.\mathcal{O}_X(C) \) or \( N.\mathcal{O}_X(C) \) is bigger than or equals to \( g \). Write \( M = \mathcal{O}_X(D_1) \) and \( N = \mathcal{O}_X(D_2) \), where \( D_1, D_2 \) are effective divisors.

Without loss of generality we assume, that \( C.D_1 > C.D_2 \). Thus we have a decomposition of \( C \) of the form \( C \sim D_1 + D_2 \) with \( C.D_1 > C.D_2 \). By Lemma 4.1, we have \( D_1.D_2 \leq D^2 = \frac{g-1}{2} \). If \( D_1.D_2 < \frac{g-1}{2} \), then \( \text{Cliff}(C) < \frac{g-1}{2} - 2 \),
a contradiction. Thus \( D_1 \cdot D_2 = D^2 \). Again by Lemma 4.1, this can happen if and only if \( C \sim 2D' + kE + k\Delta \) for some positive integer \( k \), where \( E \) is an elliptic curve and \( \Delta \) is a \((-2)\) curve. But by hypothesis, that is not possible. Hence we get a contradiction. Therefore \( E_{C,A} \) is semistable.

5. Examples

Let \( X \) be the K3 surface given by a smooth quartic hypersurface in \( \mathbb{P}^3 \). Let \( C \) be a quadric hypersurface section. In other words, \( C \) is a complete intersection of two hypersurfaces of degree 4 and 2 respectively. Clearly \( C \) is an ample curve in \( X \). Then we have the following facts [1, p.199, F-2]:

- \( W^1_3(C) = \emptyset \)
- \( W^1_4(C) \neq \emptyset \)
- \( W^3_8(C) \neq \emptyset \)
- \( W^3_8(C) - W^2_2(C) \subset W^1_4(C) \)
- \( W^2_7(C) = W^2_8(C) - W^1_1(C) \).

Thus the Clifford index of \( C \) is 2 and computed by a line bundle \( A \) of degree 4. Also note that the genus of the curve \( C \) is 9. Thus the Clifford index of \( C \) satisfies the hypothesis of the Theorem 1.1. It is easy to check that \( C \) has a decomposition as \( C \sim 2D \) where \( D \) is hyperplane section of \( X \). Thus \( C \) and \( A \) satisfies the hypothesis of the Theorem 1.1. Hence \( E_{C,A} \) is semistable.

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