EQUIVARIANT COHOMOLOGY OF WEIGHTED GRASSMANNIANS
AND WEIGHTED SCHUBERT CLASSES

HIRAKU ABE AND TOMOO MATSUMURA

Abstract. In this paper, we study the $T_w$-equivariant cohomology of the weighted Grassmannians $wGr(d, n)$ introduced by Corti-Reid [5], where $T_w$ is the $n$-dimensional torus that naturally acts on $wGr(d, n)$. We introduce the equivariant weighted Schubert classes and, after we show that they form a basis of the equivariant cohomology, we give an explicit formula for the structure constants with respect to this Schubert basis. We also find a linearly independent subset $\{wu_1, \cdots, wu_n\}$ of $\text{Lie}(T_w)^*$ such that those structure constants are polynomials in $wu_i$'s with non-negative coefficients, up to a permutation on the weights.

1. Introduction

The weighted Grassmannian $wGr(d, n)$ introduced and studied by Corti-Reid [5], following the work of Grojnowski, is a projective variety with at worst orbifold singularity with a torus action. It is a generalization of the ordinary Grassmannian and is defined in a weighted projective space by the well-known Plücker relations as weighed homogeneous polynomials with appropriate weights. In this paper, we define the weighted Schubert classes and show that they will form a basis of the equivariant/non-equivariant cohomology of $wGr(d, n)$ over $\mathbb{Q}$-coefficients. Our main goal is to study the structure constants of the cohomology rings with respect to these weighted Schubert classes. The explicit formula of these structure constants for the weighted Grassmannian is derived from any formula for the ordinary Grassmannian (for example, the Knutson-Tao’s puzzle formula [19]), by detouring to the equivariant cohomology of the quasi-projective variety $aPl(d, n)^\times$ defined in the affine space by the Plücker relations. We have found appropriate equivariant parameters in which the equivariant structure constants are polynomials with non-negative rational coefficients when the weights are non-decreasing (hence this implies that the structure constants of the ordinary cohomology are also non-negative). This is an analogue of the equivariant positivity proved by Graham [12], although we do not have the geometric or representation-theoretic interpretation of those parameters, while as, in [12], they are the simple roots in the character group of the maximal torus when we regard the flag varieties as homogeneous varieties.

Below we summarize our results in detail. Recall that the ordinary Grassmannian $Gr(d, n)$ is the space of $d$-dimensional subspaces in the $n$-dimensional complex plane $\mathbb{C}^n$. It can be described as a non-singular projective variety of dimension $d(n - d)$ defined by the well-known homogeneous polynomials, called the Plücker relations. It is embedded in the projective space $P(\mathbb{C}^{\binom{n}{d}})$ where $\binom{n}{d} := \{\lambda_1, \cdots, \lambda_d \mid 1 \leq \lambda_1 < \cdots < \lambda_d \leq n\}$ and $\mathbb{C}^{\binom{n}{d}}$ is the affine space of the plücker coordinates. Let $aPl(d, n)$ be the affine variety in $\mathbb{C}^{\binom{n}{d}}$ defined by the plücker relation and let $aPl(d, n)^\times := aPl(d, n) - \{0\}$. The $(n+1)$-dimensional complex torus $K := (\mathbb{C}^\times)^n \times \mathbb{C}^\times = \mathbb{T} \times \mathbb{C}^\times$ naturally acts on $aPl(d, n)$ and $aPl(d, n)^\times$, through the homomorphism

$$\rho: K \to (\mathbb{C}^\times)^\binom{n}{d}, \quad (t_1, \cdots, t_n, s) \mapsto (st_{\lambda_1})_{\lambda \in \binom{n}{d}},$$

where $t_{\lambda} := t_{\lambda_1} \cdots t_{\lambda_d}$.

The Grassmannian $Gr(d, n)$ is the quotient of $aPl(d, n)^\times$ by the $\mathbb{C}^\times$-action of the last component of $K$. Following [5], we define the weighted Grassmannian $wGr(d, n)$ as the quotient of $aPl(d, n)^\times$ by the locally

Date: May 2, 2014.

2010 Mathematics Subject Classification. Primary: 14N15; Secondary: 55N91, 57R18.

Key words and phrases. weighted Grassmannians, orbifolds, torus actions, equivariant cohomology, structure constants, Schubert calculus.
free action of a “twisted diagonal” \( wD \) in \( K \): for \( w := (w_1, \cdots, w_n) \in (\mathbb{Z}_{\geq 0})^n \) and \( a \in \mathbb{Z}_{\geq 1} \), let

\[
  wD := \{(t^{w_1}, \ldots, t^{w_n}, t^a) \in K \mid t \in \mathbb{C}^\times \}.
\]

The weighted Grassmannian is defined as

\[
  \text{wGr}(d, n) := \text{aPl}(d, n)^x / wD,
\]

together with the residual action of the quotient torus \( T_w := K/wD \). It is a projective variety with at worst orbifold singularities, naturally embedded in the weighted projective space \( \mathbb{P}_w(\mathbb{C}(d)) := (\mathbb{C}(d) - \{0\})/wD \) with the weights

\[
  (w_\lambda := w_{\lambda_1} + \cdots + w_{\lambda_d} + a)_{\lambda \in \{d\}^n}.
\]

In Section 2 we study the analogue of the usual Schubert cell (Bruhat) decomposition for \( \text{wGr}(d, n) \) (Proposition 2.4) and then by the standard argument we show our first result (more precise versions of the claims will be in the main body of the paper). In this paper, all cohomologies are assumed to be the singular cohomologies over \( \mathbb{Q} \)-coefficients unless otherwise specified.

**Proposition A** (Proposition 2.3). The cohomology \( H^* (\text{wGr}(d, n)) \) is concentrated in even degree. As a consequence, the equivariant cohomology \( H^*_T (\text{wGr}(d, n)) \) is a free module over \( H^*(BT_w) \).

In Section 3 we explain the following key isomorphisms among the equivariant cohomology rings of \( \text{Gr}(d, n), \text{aPl}(d, n)^x \) and \( \text{wGr}(d, n) \). The claim follows essentially from the Vietoris-Begle mapping theorem.

**Proposition B** (Proposition 3.1). The pullback maps on the equivariant cohomologies

\[
  H^*_T (\text{Gr}(d, n)) \overset{\pi^*}{\longrightarrow} H^*_K (\text{aPl}(d, n)^x) \overset{\pi_w^*}{\longrightarrow} H^*_T (\text{wGr}(d, n))
\]

are isomorphisms of rings over \( H^*(BT) \) and \( H^*(BT_w) \) respectively.

Having these isomorphisms, we introduce the equivariant Schubert classes \( \tilde{a}S_\lambda \) of \( \text{aPl}(d, n)^x \) to be the image of the usual equivariant Schubert class \( S_\lambda \) in \( H^*_T (\text{Gr}(d, n)) \) under the pullback \( \pi^* \) and define the equivariant weighted Schubert classes \( wS_\lambda \) of \( \text{wGr}(d, n) \) by

\[
  wS_\lambda := (\pi_w^*)^{-1}(\tilde{a}S_\lambda).
\]

The corresponding ordinary cohomology classes for \( \text{Gr}(d, n) \) and \( \text{wGr}(d, n) \) are denoted by \( S_\lambda \) and \( wS_\lambda \) respectively.

In Section 4 we obtain the GKM (Goresky-Kottwitz-Macpherson) descriptions of \( H^*_K (\text{aPl}(d, n)^x) \) and \( H^*_T (\text{wGr}(d, n)) \), following [4], [11] and [14]. We observe that there is the following commutative diagram of injective localization maps

\[
  \begin{array}{ccc}
  H^*_T (\text{Gr}(d, n)) & \longrightarrow & \bigoplus \lambda H^*(BT) \\
  \pi^* & \cong & \cong \\
  H^*_T (\text{aPl}(d, n)^x) & \longrightarrow & \bigoplus \lambda H^*(BK_\lambda) \\
  \pi_w^* & \cong & \cong \\
  H^*_T (\text{wGr}(d, n)) & \longrightarrow & \bigoplus \lambda H^*(BT_w)
  \end{array}
\]

where \( K_\lambda \) is the kernel of \( K \to \mathbb{C}^\times; t \mapsto t_{\lambda} \). The GKM descriptions of \( H^*_K (\text{aPl}(d, n)^x) \) and \( H^*_T (\text{wGr}(d, n)) \) are obtained in Proposition 4.1 and 4.2 from the well-known one for \( H^*_T (\text{Gr}(d, n)) \) by the commutative diagram. Furthermore, the upper triangularity of the image of \( a\tilde{S}_\lambda \) and \( wS_\lambda \) is given in Proposition 4.3 and 4.5 and as a consequence, we have

**Proposition C** (Proposition 4.6). \( \{wS_\lambda\}_\lambda \) is a basis of \( H^*_T (\text{wGr}(d, n)) \) as a module over \( H^*(BT_w) \).
This allows us to define the structure constants \( w_{\lambda\mu}^\nu \) of \( H^*_T(wGr(d,n)) \) by

\[
w\tilde{S}_\lambda \cdot w\tilde{S}_\mu = \sum_{\nu} w_{\lambda\mu}^\nu w\tilde{S}_\nu \quad \text{where } w_{\lambda\mu}^\nu \in H^*(BT_w).
\]

In Section 5 we derive the formula for \( w_{\lambda\mu}^\nu \) and prove the equivariant positivity. Let \( \{y_1, \ldots, y_n, z\} \) be the standard basis of \( \text{Lie}(K)_Z \) and identify \( H^*(BK) \) with \( \mathbb{Q}[y_1, \ldots, y_n, z] \). Since \( T_w \) is a quotient of \( K \), we can regard \( \text{Lie}(T_w)_Z \) as a subspace of \( \text{Lie}(K)_Z \). For each pair \( \alpha = (i, j) \) of integers in \( [n] \) with \( i > j \), let

\[
u_a := y_i - y_j \in \mathbb{Q}[T^*] \quad \text{and} \quad wu_{\alpha} := \frac{(y_i - y_j)^{w_i - w_j}}{wy_{id}} \in \mathbb{Q}[T^*]
\]

where \( id \in \{0\} \) is the unique minimum element in the Bruhat order and \( y_\lambda := y_{\lambda_1} + \cdots + y_{\lambda_d} \). It is easy to see that \( \{wu_{(i+1,j)}, i = 1, \ldots, n-1\} \) is a linearly independent subset of \( \text{Lie}(T_w)_Z \). For simplicity, we let \( u_i := u_{(i+1,i)} \) and \( wu_{i} := wu_{(i+1,i)} \). For each finite collection \( I = \{\alpha_1, \ldots, \alpha_p\} \) of pairs of integers in \( [n] \) as above, let

\[
u_I := \nu_{\alpha_1} \cdots \nu_{\alpha_p} \quad \text{and} \quad wu_{I}^{(r)} = \sum_{1 \leq s_1 < s_2 < \cdots < s_r} \frac{w(\alpha_1)}{w_{id}} \cdots \frac{w(\alpha_r)}{w_{id}} \nu_{\alpha_1} \cdots \nu_{\alpha_r}.
\]

where \( w(\alpha) := w_i - w_j \in \mathbb{Z} \) if \( \alpha = (i, j) \). Here note that \( wu_{I}^{(0)} = \nu_{\alpha_1} \cdots \nu_{\alpha_p} \).

We first introduce \( K_{\lambda, \eta}^\nu \in \mathbb{Q}[T^*_w] \) as the coefficient for the following product.

\[
(a\tilde{S}_{\div})^\nu \tilde{S}_\eta = \sum_{\nu} K_{\lambda, \eta}^\nu a\tilde{S}_\nu.
\]

The explicit formula for \( K_{\lambda, \eta}^\nu \) is given in Lemma 5.3. To obtain the formula for \( w_{\lambda\mu}^\nu \), we use the well-known fact that the equivariant Schubert structure constant \( c_{\lambda\mu}^\nu \) for \( H^*_T(Gr(d,n)) \) is an element of \( \mathbb{Z}[u_1, \ldots, u_{n+1}] \).

\[
c_{\lambda\mu}^\nu = \sum_{|I|=l(\lambda)+l(\mu)-l(\nu)} c(\lambda, \mu, \nu; I)\nu_I, \quad c(\lambda, \mu, \nu; I) \in \mathbb{Z}_{\geq 0}
\]

where \( I \) runs over collections of pairs \( (i, j) \) of integers in \( [n] \) with \( i > j \) as above. For example, Knutson-Tao [19] computed the number \( c(\lambda, \mu, \nu; I) \) in terms of the equivariant puzzles.

The following is our main theorem.

**Theorem E** (Theorem 5.3, 5.7). Let \( \lambda, \mu, \nu \in \{0\}_n \), then

\[
w_{\lambda\mu}^\nu = \sum_{\nu_\eta \geq \lambda, \mu} \sum_{I : \nu(\lambda, \mu, \nu; I)} c(\lambda, \mu, \nu; I)K_{\lambda, \eta}^\nu wu_I^{(r)}.
\]

Moreover, if \( w_1 \leq w_2 \leq \cdots \leq w_n \), then \( w_{\lambda\mu}^\nu \) is a polynomial in \( wu_1, \ldots, wu_{n+1} \) with non-negative coefficients.

It is worth noting that a permutation of the index \( (1, \cdots, n) \) can change the order of the weights \( \{w_1, \cdots, w_n\} \) into a non-decreasing order without changing the space \( wGr(d,n) \) up to isomorphisms. Therefore we can always find a Schubert basis that satisfies the positivity.

The ordinary structure constants \( w_{\lambda\mu} \) are given by the non-equivariant limit \( wu_1 = \cdots = wu_{n+1} = 0 \). Thus we obtain the formula for \( w_{\lambda\mu} \) too, particularly in terms of a specialization of the equivariant structure constants \( c_{\lambda\mu}^\nu \) for \( Gr(d,n) \) computed in [19].

**Corollary G** (Corollary 5.3 below) Let \( \lambda, \mu, \nu \in \{0\}_d \). The structure constant \( w_{\lambda\mu}^\nu \) is given by

\[
w_{\lambda\mu}^\nu = \sum_{\nu \geq \eta \geq \lambda, \mu} \sum_{I : \nu(\lambda, \mu, \nu; I)} c_{\lambda\mu}^\nu(u_i = w_{i+1} - w_i, i = 1, \ldots, n - 1)
\]

if \( l(\lambda) + l(\mu) = l(\nu) \) and \( w_{\lambda\mu}^\nu = 0 \) if otherwise. Furthermore, if \( w_1 \leq w_2 \leq \cdots \leq w_n \), \( w_{\lambda\mu}^\nu \) is non-negative.
We conclude Section 5 by including the examples of wGr(1, n) and wGr(2, 4). The space wGr(1, n) is the well-known weighted Projective space and its integral cohomology are first studied by Kawasaki [16] and its equivariant cohomology by Bahri-Franz-Ray [2] and Tymoczko [26]. We discuss the relation of our Schubert basis to Kawasaki’s basis over Z-coefficients at Example 5.11.

2. Weighted Grassmannians and Schubert cell decomposition

In this section, we recall the definition of the weighted Grassmannian wGr(d, n), following [5]. We study the coordinate charts and obtain a quasi-cell decomposition which generalizes the usual Schubert cell decomposition of the ordinary Grassmannian Gr(d, n). This allows us to show that the odd degree classes of the rational cohomology of wGr(d, n) vanish and also the equivariant cohomology is a free module over a polynomial ring.

For positive integers d and n such that d < n, let \([n] := \{1, \cdots, n\}\), and

\[
\{\lambda\} \ := \{\lambda \in [n] \mid |\lambda| = d\}.
\]

We denote the elements of \(\lambda\) by \(\lambda_1, \cdots, \lambda_d\) where \(\lambda_1 < \cdots < \lambda_d\). For \(\lambda, \mu \in \{\lambda\}\), we define the Bruhat order by

\[
\lambda \geq \mu \quad \text{if} \quad \lambda_i \leq \mu_i \text{ for all } i = 1, \cdots, d.
\]

An inversion \((k, l)\) of \(\lambda\) is a pair of \(k \in \lambda\) and \(l \notin \lambda\) such that \(k < l\). Let \(\text{inv}(\lambda)\) be the set of all inversions of \(\lambda\). The length \(l(\lambda)\) of \(\lambda\) is defined to be the cardinality of \(\text{inv}(\lambda)\). For each \((k, l) \in \text{inv}(\lambda)\), let \((k, l)\lambda\) be the element of \(\{\lambda\}\) obtained by replacing \(k\) in \(\lambda\) by \(l\). We say that \(\lambda\) covers \(\mu\) if \(|\lambda| \geq |\mu|\) and \(l(\lambda) = l(\mu) + 1\), and denote \(\lambda \to \mu\).

Remark 2.1. For each \(\lambda \in \{\lambda\}\), we can consider the sequence of ones and zeros such that one for each \(\lambda_i\)-th position \((i = 1, \cdots, d)\) and zeros for the other positions. Obviously this is a bijection, and we can identify \(\{\lambda\}\) and the set of sequences of \(d\) zeros and \(n - d\) ones by this rule. This will help us to compare our notation and the notations in [19].

2.1. The weighted Grassmannian. Let \(\mathbb{C}^n\) be the complex n-plane and \(\wedge^d \mathbb{C}^n\) its d-th exterior product. The standard representation of \(\text{GL}_n(\mathbb{C})\) on \(\mathbb{C}^n\) canonically induces the representation of \(\text{GL}_n(\mathbb{C})\) on \(\wedge^d \mathbb{C}^n\) and hence a linear \(\text{GL}_n(\mathbb{C})\)-action on \(\mathbb{P}(\wedge^d \mathbb{C}^n)\). Through the Plücker embedding, the Grassmannian Gr(d, n) of d-dimensional subspaces in \(\mathbb{C}^n\) can be identified with the GL\(_n(\mathbb{C})\)-orbit \(\text{GL}_n(\mathbb{C}) \cdot [e_1 \wedge \cdots \wedge e_d]\) in \(\mathbb{P}(\wedge^d \mathbb{C}^n)\). Consider the affine cone of Gr(d, n) in \(\wedge^d \mathbb{C}^n\)

\[
a\text{Pl}(d, n) := \text{GL}_n(\mathbb{C}) \cdot [e_1 \wedge \cdots \wedge e_d].
\]

Let \(T := (\mathbb{C}^\times)^n\) and identify it with the diagonal matrices of \(\text{GL}_n(\mathbb{C})\). The quasi-affine variety \(a\text{Pl}(d, n)^\times := a\text{Pl}(d, n) - \{0\}\) is preserved under the action of \(K := T \times \mathbb{C}^\times\) on \(\wedge^d \mathbb{C}^n\) where the first factor acts through the \(\text{GL}_n(\mathbb{C})\)-action and the second factor by the scalar multiplication.

Definition 2.2 (Corti-Reid [3]). Let \(w := (w_1, \cdots, w_n) \in (\mathbb{Z}_{\geq 0})^n\) and \(a \in \mathbb{Z}_{\geq 1}\). Consider the subgroup of \(K\) and the corresponding quotient group:

\[
wD := \{ (t^{w_1}, \cdots, t^{w_n}, t^a) \in K \mid t \in \mathbb{C}^\times \} \quad \text{and} \quad Tw := K/wD.
\]

The weighted Grassmannian wGr(d, n) is the projective variety with at worst orbifold singularities, defined by

\[
w\text{Gr}(d, n) := a\text{Pl}(d, n)^\times / wD.
\]

The quotient map \(\pi_w : a\text{Pl}(d, n)^\times \to w\text{Gr}(d, n)\) is equivariant with respect to the quotient homomorphism \(\kappa_w : K \to Tw\).

When \(w = (0, \cdots, 0)\) and \(a = 1\), wGr(d, n) becomes the usual Grassmannian Gr(d, n) and \(Tw\) is canonically identified with \(T\). In this case, we denote the above quotient maps by \(\kappa : K \to T\) and \(\pi : a\text{Pl}(d, n)^\times \to \text{Gr}(d, n)\) respectively.
2.2. The Charts for $\text{aPl}(d,n)^\times$ and $\text{wGr}(d,n)$. The standard basis $\{e_i, i \in [n]\}$ of $\mathbb{C}^n$ induces the standard basis $\{e_{\lambda} := e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d}, \lambda \in \{\frac{n}{d}\}\}$ of $\wedge^d \mathbb{C}^n$. We identify $\wedge^d \mathbb{C}^n$ with a coordinate space $\mathbb{C}^{\binom{n}{d}}$ where a vector $x = \sum x_\lambda e_\lambda$ corresponds to the coordinate vector $x = (x_\lambda)_{\lambda \in \{\frac{n}{d}\}}$. For each $(t,s) \in K = T \times \mathbb{C}^\times$, its action on $e_\lambda$ is given by
\[
(t,s) \cdot e_\lambda = (st_\lambda)e_\lambda \quad \text{where} \quad t_\lambda := \prod_{i \in \lambda} t_i.
\]
Let $U^\lambda := \{[x] \in \text{Gr}(d,n) \mid x_\lambda \neq 0\}$. It is well-known that $U^\lambda$ is $T$-equivariantly isomorphic to the complex affine space $\mathbb{C}^{d(n-d)}$ with a linear $T$-action. It is a $T$-invariant affine chart of $[e_\lambda]$, and these form an affine open cover of $\text{Gr}(d,n)$.

The quotient map $\pi : \text{aPl}(d,n)^\times \to \text{Gr}(d,n)$ is a $\kappa$-equivariant $\mathbb{C}^\times$-principal bundle. The preimage of $U^\lambda$ is
\[
aU^\lambda := \{x \in \text{aPl}(d,n)^\times \mid x_\lambda \neq 0\}.
\]
and then a $\kappa$-equivariant trivialization is given by
\[
\psi_\lambda : aU^\lambda \to U^\lambda \times \mathbb{C}^\times; \quad x \mapsto (\pi(x), x_\lambda)
\]
where the $K$-action on $U^\lambda \times \mathbb{C}^\times$ is defined by $(t,s) \cdot ([x], y) := ([tx], st_\lambda y)$. Indeed, the inverse map is given by
\[
U^\lambda \times \mathbb{C}^\times \to aU^\lambda; \quad ([x], t) \mapsto (tx^{-1}_\lambda x\eta)_{\eta \in \{\frac{n}{d}\}}.
\]
This $aU^\lambda$ plays a role of a $K$-invariant chart of $e_\lambda$ in $\text{aPl}(d,n)^\times$.

The quotient $wU^\lambda := aU^\lambda/wD$ gives us a $T_w$-equivariant open neighborhood of $[e_\lambda] \in \text{wGr}(d,n)$ and $\psi_\lambda$ induces an equivariant isomorphism
\[
\overline{\psi}_\lambda : wU^\lambda \xrightarrow{\sim} (U^\lambda \times \mathbb{C}^\times)/wD.
\]
Let
\[
w_\lambda := a + \sum_{i=1}^d w_{\lambda_i} \quad \text{for each } \lambda \in \{\frac{n}{d}\}.
\]
Then the finite cyclic subgroup of $wD$
\[
G_\lambda = \{(t^{w_{\lambda_1}}, \cdots, t^{w_{\lambda_d}}, t^{w_\lambda}) \in wD \mid t \in \mathbb{C}^\times \text{ and } t^{w_\lambda} = 1\}
\]
acts on the second factor of $U^\lambda \times \mathbb{C}^\times$ trivially. Hence the image of the isomorphism $\overline{\psi}_\lambda$ is equivariantly homeomorphic to $U^\lambda/G_\lambda$.

2.3. The Schubert cell decompositions. For each $\lambda \in \{\frac{n}{d}\}$, we have the Schubert cell $\Omega^\circ_\lambda$ and the Schubert variety $\Omega_\lambda$ in $\text{Gr}(d,n)$ (See [10] or [19]). Define
\[
a\Omega^\circ_\lambda := \pi^{-1}(\Omega^\circ_\lambda) \quad \text{and} \quad a\Omega_\lambda := \pi^{-1}(\Omega_\lambda).
\]
Under the chart $\psi_\lambda$, we have $a\Omega^\circ_\lambda \cong \mathbb{C}^\times \times \Omega^\circ_\lambda$ and so its complex codimension in $\text{aPl}(d,n)^\times$ is the length $l(\lambda)$. The irreducibility of $a\Omega_\lambda$ follows from the irreducibility of $\Omega_\lambda$ and the fiber $\mathbb{C}^\times$ of the bundle $\pi$. Therefore the closure of the open subset $a\Omega^\circ_\lambda \subset a\Omega_\lambda$ coincides with $a\Omega_\lambda$. The usual Schubert cells decomposition $\text{Gr}(d,n) = \bigsqcup_{\lambda \in \{\frac{n}{d}\}} \Omega^\circ_\lambda$ induces the $K$-invariant decomposition
\[
\text{aPl}(d,n)^\times = \bigsqcup_{\lambda \in \{\frac{n}{d}\}} a\Omega^\circ_\lambda.
\]
By the $K$-invariany, it descends to the quotient $\text{wGr}(d,n)$ and gives the $T_w$-invariant decomposition:

**Proposition 2.3.**
\[
\text{wGr}(d,n) = \bigsqcup_{\lambda \in \{\frac{n}{d}\}} w\Omega^\circ_\lambda \quad \text{where} \quad w\Omega^\circ_\lambda := a\Omega^\circ_\lambda/wD.
\]
Under the chart $\overline{\psi}_\lambda$, $w\Omega^\circ_\lambda \cong \Omega^\circ_\lambda/G_\lambda$. 
We call this decomposition a quasi-cell decomposition because each “cell” is homeomorphic to an Euclidean space modulo a finite group.

2.4. Vanishing of the odd degree. The argument of Appendix B in [10] can be applied to the quasi-cell decomposition, and we obtain

\[ \overline{\Pi}_i(w\Gr(d,n)) \cong \bigoplus_{2\dim w\Omega^\lambda_i = i} \overline{\Pi}_i(w\Omega^\lambda_i) \]

where \( \overline{\Pi}_* \) is the rational Borel-Moore homology. We have

\[ \overline{H}_i(w\Omega^\lambda_i) = \overline{H}_i(\Omega^\lambda_i/G_\lambda) \cong H^{2\dim \Omega^\lambda_i - i}(\Omega^\lambda_i/G_\lambda) \cong H^{2\dim \Omega^\lambda_i - i}(\Omega^\lambda_i)^G \]

where the second equality follows from the rational Poincaré duality (c.f. [3] Prop 13.A.4, Appendix, Chap. 13) and the third equality is well-known (see [2]). Since the \( G_\lambda \) acts on \( \Omega^\lambda_i \) through the action of a connected group, \( G_\lambda \) acts on \( H^\ast(\Omega^\lambda_i) \) trivially. Therefore after we apply the Poincaré duality again, we obtain

**Proposition 2.4.**

\[ H^i(w\Gr(d,n)) \cong \begin{cases} \bigoplus_{\lambda(i) = i} \mathbb{Q} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases} \]

Recall that the equivariant cohomology for the \( T_w \)-action on \( w\Gr(d,n) \) is defined as the cohomology of the Borel construction, i.e. the total space of the fibration

\[ w\Gr(d,n) \xrightarrow{\zeta} ET_w \times_{T_w} w\Gr(d,n) \rightarrow BT_w, \]

where \( ET_w \rightarrow BT_w \) is a universal principal \( T_w \)-bundle with a contractible total space and \( ET_w \times_{T_w} w\Gr(d,n) := (ET_w \times w\Gr(d,n))/T_w \). The pullback along the projection makes \( H^\ast_{T_w}(w\Gr(d,n)) \) an \( H^\ast(BT_w) \)-module. Since the fiber is path-connected, the vanishing of odd degree classes implies that the associated Serre spectral sequence collapses at \( E_2 \)-stage. Thus we have

**Proposition 2.5.** As \( H^\ast(BT_w) \)-modules,

\[ H^\ast_{T_w}(w\Gr(d,n)) \cong H^\ast(BT_w) \otimes_{\mathbb{Q}} H^\ast(w\Gr(d,n)). \]

In particular, \( H^\ast_{T_w}(w\Gr(d,n)) \) is a free module over \( H^\ast(BT_w) \).

3. Equivariant Weighted Schubert Classes

In this section, we observe that the rational equivariant cohomologies \( H^\ast_K(a\Pl(d,n)^\times) \), \( H^\ast_T(w\Gr(d,n)) \), and \( H^\ast_T(Gr(d,n)) \) are all isomorphic as rings, while they are modules over different polynomial rings. We define the equivariant weighted Schubert classes \( \widetilde{S}^w_\lambda \) in \( H^\ast_{T_w}(w\Gr(d,n)) \) using these ring isomorphisms.

The quotient maps from \( a\Pl(d,n)^\times \) to \( w\Gr(d,n) \) and \( Gr(d,n) \), and from \( K \) to \( T_w \) and \( T \), induce the following commutative diagram of the Borel constructions:

\[ \begin{array}{cccccc}
ET \times_{T_w} Gr(d,n) & \xrightarrow{\pi} & EK \times_K a\Pl(d,n)^\times & \xrightarrow{\pi_w} & ET_w \times_{T_w} w\Gr(d,n) \\
\downarrow & & \downarrow & & \downarrow \\
BT \xrightarrow{\kappa} & BK \xrightarrow{\kappa_w} & BT_w
\end{array} \]

By the functoriality, the pullback maps on cohomologies

\[ \pi^* : H^\ast_T(Gr(d,n)) \rightarrow H^\ast_K(a\Pl(d,n)^\times) \]

and

\[ \pi_w^* : H^\ast_{T_w}(w\Gr(d,n)) \rightarrow H^\ast_K(a\Pl(d,n)^\times) \]

are homomorphism of rings over the polynomial rings \( H^\ast(BT) \) and \( H^\ast(BT_w) \) respectively. The proof of the following proposition is postponed until after we define the weighted Schubert classes.
Proposition 3.1. The maps $\pi^*$ and $\pi_w^*$ are isomorphisms as rings over the polynomial rings $H^*(BT)$ and $H^*(BT_w)$ respectively.

Since each $a\mathfrak{f}_\lambda$ is a closed $K$-invariant irreducible subvariety in a non-singular quasi-projective $K$-variety $\text{aPl}(d, n)^\times$, the equivariant Gysin map $H^*_K(a\Omega) \rightarrow H^*_{K}^+(\text{aPl}(d, n)^\times)$ (c.f. [10], Appendix B) defines the cohomology class $[a\Omega]_K$ in $H^*_K(\text{aPl}(d, n)^\times)$ associated to $a\Omega$ as the image of $1 \in H^*_K(a\Omega)$:

$$a\mathcal{S}_\lambda := [a\Omega]_K \in H^*_K(\text{aPl}(d, n)^\times).$$

Since $a\Omega = \pi^{-1}(\Omega)$ and $\pi : \text{aPl}(d, n)^\times \rightarrow \text{Gr}(d, n)$ is an equivariant fiber bundle with respect to the quotient $\kappa : K \rightarrow T$, we actually have

$$a\mathcal{S}_\lambda = \pi^*(\mathcal{S}_\lambda)$$

where $\mathcal{S}_\lambda = [\Omega]_T$ is the $T$-equivariant Schubert class in $H^*_T(\text{Gr}(d, n))$ (c.f. Appendix B-(8) in [10]).

Definition 3.2. Define the $T_w$-equivariant weighted Schubert class corresponding to $\lambda$ by $w\mathcal{S}_\lambda := (\pi_w^*)^{-1}(a\mathcal{S}_\lambda) \in H^*_{T_w}(w\text{Gr}(d, n))$.

This definition coincides with the usual equivariant Schubert class for the ordinary Grassmannian when the weights are trivial. We denote by $\mathcal{S}_\lambda$ and $w\mathcal{S}_\lambda$ the corresponding classes in ordinary cohomologies $H^*(\text{Gr}(d, n))$ and $H^*(w\text{Gr}(d, n))$ respectively.

Proof of Proposition 3.1. By an elementary application of the Vietoris-Begle mapping theorem (c.f. [24] Thm.15, Sec.9, Chap.d), we have the following lemma.

Lemma 3.3. Let $M$ be a compact manifold with a smooth action of a compact torus $K$. Let $G \subset K$ be a subtorus that acts on $M$ with finite stabilizers. Let $\mathcal{T} := K/G$. Then the natural map $\theta : EK \times_K M \rightarrow ET \times_\mathcal{T} (M/G)$ induces an isomorphism of rings over $H^*(BT)$ on the rational equivariant cohomology $\theta^* : H^*_T(M/G) \rightarrow H^*_K(M)$.

Thus, we need to prepare only a description of $w\text{Gr}(d, n)$ as the quotient of a compact space by a real torus. Let $K^R, wD^R, T^R$ and $T^R_w$ be the real tori in $K, wD, T$ and $T_w$ respectively. Recall that we have a natural isomorphism $H^*_K(Y) \cong H^*_K(Y)$ for any $K$-space $Y$. Since the $wD^R$-action on $C^{\{d\}}_{\{n\}}$ factors through the canonical $(S^1)^{\{d\}}$-action, it is Hamiltonian with the standard moment map. Since $\text{aPl}(d, n)^\times$ is a $wD^R$-invariant symplectic manifold of $C^{\{d\}}_{\{n\}}$, there is the induced moment map $\Psi : \text{aPl}(d, n)^\times \rightarrow \mathbb{R}$ ; $x \mapsto -\frac{1}{2} \sum_{\lambda \in \{\lambda\}} d \cdot w\lambda |x\lambda|^2$

where the integer $w\lambda$ is defined at [25]. For a regular value $\xi$, the preimage $M := \Psi^{-1}(\xi)$ is a compact $K^R$-invariant submanifold of $\text{aPl}(d, n)^\times$. Moreover there is a $K^R$-equivariant deformation retraction from $\text{aPl}(d, n)^\times$ to $M$ given by the homotopy $F : \text{aPl}(d, n)^\times \times I \rightarrow \text{aPl}(d, n)^\times ; (x, s) \mapsto \left((s\sqrt{\xi/\Psi(x)} + (1 - s))x\lambda\right)_{\lambda \in \{\lambda\}}$.

Thus, the inclusion $\iota : M \hookrightarrow \text{aPl}(d, n)^\times$ induces the isomorphism:

$$\iota^* : H^*_K(M) \rightarrow H^*_K(\text{aPl}(d, n)^\times).$$

(3.1)

Passing to the quotients, we obtain an equivariant map $\pi : M/wD^R \rightarrow w\text{Gr}(d, n)$ with respect to the inclusion $T^R_w \hookrightarrow T_w$. This map can be shown to be a homeomorphism by a direct computation (See also [14], Theorem 7.4]). Hence, we obtain the isomorphism:

(3.2) $\pi^* : H^*_T(M/wD^R) \rightarrow H^*_T(w\text{Gr}(d, n))$.

1Here we identify $\text{Lie}(wD) \cong \text{Lie}(C^\infty) = \mathbb{R}$ by the map $S^1 \rightarrow wD(t \mapsto (t^{d\lambda_1 + a}, \ldots, t^{d\lambda_n + a})).$
Let $\theta : ET \times_T M \to ET_w \times_{T_w} M/\text{wD}^R$ be a map induced by the quotient maps $M \to M/\text{wD}^R$ and $K^R \to T_w^R$. Then we have the following commutative diagram.

\begin{equation}
(3.3) \quad H^*_w(\text{wGr}(d,n)) \xrightarrow{\pi_w^*} H^*_K(\text{aPl}(d,n)^\times) \cong \gamma^* \quad \cong \quad H^*_T w(M) \leftarrow \theta^* \quad H^*_K(M)
\end{equation}

Thus $\pi_w^*$ is an isomorphism if $\theta^*$ is an isomorphism, which follows from Lemma 3.3. 

\section{GKM Descriptions and Schubert Classes}

In this section, we study the combinatorial presentations of $H^*_w(\text{wGr}(d,n))$ and $H^*_K(\text{aPl}(d,n)^\times)$, known as the GKM theory developed in [4] and [11]. This allows us, in particular, to show that the equivariant weighted Schubert classes $w$ is an isomorphism if $	heta^*$ is an isomorphism, which follows from Lemma 3.3.

Recall that $H^*(BK)$ can be canonically identified with the symmetric algebra $\text{Sym}(\text{Lie}(K)^\times \otimes \mathbb{Q})$ where $\text{Lie}(K)^\times$ is the space of $\mathbb{Z}$-linear functions on the integral lattice $\text{Lie}(K)^\times \subset \text{Lie}(K)$. Since $K = (\mathbb{C}^*)^n \times \mathbb{C}^\times$ is a standard torus, we can take the standard $\mathbb{Z}$-basis $\{y_1, \ldots, y_n, z\}$ of $\text{Lie}(K)^\times$. Hence we let

$$Q[K^\times] := H^*(BK) = \text{Sym}(\text{Lie}(K)^\times \otimes \mathbb{Q}) = Q[y_1, \ldots, y_n, z].$$

Since $T_w$ is a quotient of $K$, we identify $\text{Lie}(T_w)^\times_\mathbb{Z}$ with its image in $\text{Lie}(K)^\times$ and it is easy to see that

$$y_i^w := y_i - \frac{w_i}{a^i} z, \quad i = 1, \ldots, n,$$

form a basis of $\text{Lie}(T_w)^\times_\mathbb{Z} \otimes \mathbb{Q}$. We let

$$Q[T^\times_\mathbb{Z}] := H^*(BT_w) = \text{Sym}(\text{Lie}(T_w)^\times_\mathbb{Z} \otimes \mathbb{Q}) = Q[y_1^w, \ldots, y_n^w] \subset Q[K^\times].$$

We use the following notation in the rest of the paper: for each $\lambda \in \{\underline{d}\}$, let

\begin{equation}
(4.1) \quad y_{\lambda} := \sum_{i \in \lambda} y_i \in Q[T^\times] \quad \text{and} \quad y_{\lambda}^w := \sum_{i \in \lambda} y_i^w \in Q[T^\times_\mathbb{Z}].
\end{equation}

The $T$-fixed points in $\text{Gr}(d,n)$ are the points $[e_{\mu}]$, $\mu \in \{\underline{d}\}$ and the cohomology $H_T^*([e_{\mu}])$ is identified with $Q[T^\times]$. The restriction map to the fixed points

\begin{equation}
(4.2) \quad H_T^*(\text{Gr}(d,n)) \to \bigoplus_{\mu \in \{\underline{d}\}} Q[T^\times]; \quad \gamma \mapsto (\gamma|_{\mu})_{\mu \in \{\underline{d}\}}
\end{equation}

is injective and the image is given by (see [10])

\begin{equation}
(4.3) \quad \left\{ \alpha = (\alpha(\mu))_{\mu \in \{\underline{d}\}} \in \bigoplus_{\mu \in \{\underline{d}\}} Q[T^\times] \bigg| \alpha(\lambda) - \alpha(\mu) \text{ is divisible by } y_{\lambda} - y_{\mu} \text{ for any } \lambda \text{ and } \mu \text{ such that } |\lambda \cap \mu| = d - 1 \right\},
\end{equation}

The fixed points of the $T_w$-action on $\text{wGr}(d,n)$ are again the images of $e_{\mu}$ in $\text{wGr}(d,n)$ and we also denote it by $[e_{\mu}]$. By identifying $H^*_w([e_{\mu}]) \cong Q[T^\times_\mathbb{Z}]$, we have the restriction map

\begin{equation}
(4.4) \quad H^*_w(\text{wGr}(d,n)) \to \bigoplus_{\mu \in \{\underline{d}\}} Q[T^\times_\mathbb{Z}]; \quad \gamma \mapsto (\gamma|_{\mu})_{\mu \in \{\underline{d}\}}.
\end{equation}

For $\text{aPl}(d,n)^\times$, we restrict $H^*_K(\text{aPl}(d,n)^\times)$ to the complex 1-dimensional orbits of $K$, which are given by $\mathbb{C}^\times e_{\mu}$. The isotropy subgroup $K_\mu$ at $e_{\mu}$ of the $K$-action is the kernel of the map $K \to \mathbb{C}^\times$ sending $(t_1, \ldots, t_n, s)$ to $s \cdot t_\mu$. It is connected and the inclusion $K_\mu \hookrightarrow K$ induces the isomorphism $\text{Lie}(K_\mu)^\times_\mathbb{Z} \cong \text{Lie}(K)^\times_\mathbb{Z}/(y_{\mu} + z)$. Thus

$$H^*_K(\mathbb{C}^\times e_{\mu}) \cong H^*_K(e_{\mu}) \cong Q[K^\times_\mathbb{Z}] \cong Q[K^\times]/(y_{\mu} + z).$$
This follows from a straightforward computation. Indeed, we have

\[ H^*_K(\text{aPl}(d, n)^\times) \rightarrow \bigoplus_{\mu \in \{\bar{d}\}} \mathbb{Q}[K^*_\mu], \quad P \mapsto (P|_\mu)_{\mu \in \{\bar{d}\}}. \]

Putting (4.2, 4.4, 4.5) together with \( \pi^* \) and \( \pi^*_w \), we have the following commutative diagram

\[
\begin{array}{ccc}
H^*_T(\text{Gr}(d, n)) & \xrightarrow{\pi^*} & \bigoplus_{\mu} \mathbb{Q}[T^*] \\
\cong & & \cong \\
H^*_K(\text{aPl}(d, n)^\times) & \xrightarrow{\kappa} & \bigoplus_{\mu} \mathbb{Q}[K^*_\mu] \\
\cong & & \cong \\
H^*_T(w\text{Gr}(d, n)) & \xrightarrow{\kappa_w} & \bigoplus_{\mu} \mathbb{Q}[T^*_w]
\end{array}
\]

where the right vertical maps are induced from \( \kappa_\mu : K_\mu \rightarrow K \rightarrow T \) and \( \kappa_{w\mu} : K_\mu \rightarrow K \rightarrow T_w \) and they are isomorphisms because \( \kappa_\mu \) and \( \kappa_{w\mu} \) have finite kernels. The following are obtained by translating (4.3) to \( H^*_K(\text{aPl}(d, n)^\times) \) and \( H^*_T(w\text{Gr}(d, n)) \) via this diagram.

**Proposition 4.1** (GKM for wGr\((d, n)\)). The restriction map (4.4) is injective and the image is given by

\[
\left\{ \alpha \in \bigoplus_{\mu \in \{\bar{d}\}} \mathbb{Q}[T^*_w] \mid \alpha(\lambda) - \alpha(\mu) \text{ divisible by } w_\mu y^w_\lambda - w_\lambda y^w_\mu \right\}
\]

for any \( \lambda \) and \( \mu \) such that \(|\lambda \cap \mu| = d - 1\)

where \( y^w_\mu \) is defined in (4.1).

**Proposition 4.2** (GKM for aPl\((d, n)^\times\)). The restriction map (4.5) is injective and the image is given by

\[
\left\{ P \in \bigoplus_{\mu \in \{\bar{d}\}} \mathbb{Q}[K^*_\mu] \mid P(\lambda) = P(\mu) \text{ in } \mathbb{Q}[K^*]/\langle y_\lambda + z, y_\mu + z \rangle \right\}
\]

for any \( \lambda \) and \( \mu \) such that \(|\lambda \cap \mu| = d - 1\).

**Proof of Proposition 4.2 and Proposition 4.1**

The injectivity of the maps (4.4) and (4.5) follows from the injectivity of the map (4.2) by the commutativity of the diagram (4.6). It remains to check that the GKM conditions are equivalent under the isomorphisms \( \kappa^* \) and \( \kappa^*_w \). We prove it for \( \kappa_w \) because \( \kappa \) is a special case of \( \kappa_w \). First note that, in Proposition 4.1, \( \alpha(\lambda) - \alpha(\mu) \) is divisible by \( w_\mu y^w_\lambda - w_\lambda y^w_\mu \) if and only if \( \alpha(\lambda) - \alpha(\mu) = 0 \) in \( \mathbb{Q}[T^*_w]/(y_\lambda + z, y_\mu + z) \). Therefore the GKM conditions are equivalent under \( \kappa^*_w \) if \( \kappa^*_w \) and \( \kappa^*_w \) induce the isomorphism

\[
\frac{\mathbb{Q}[T^*_w]}{(w_\mu y^w_\lambda - w_\lambda y^w_\mu)} \rightarrow \frac{\mathbb{Q}[K^*]}{(y_\lambda + z, y_\mu + z)}, \quad f \mapsto \kappa^*_w(f) = \kappa^*_w(f).
\]

This follows from a straightforward computation. Indeed, we have

\[
w_\lambda y^w_\mu - w_\mu y^w_\lambda = w_\lambda(y_\mu + z) \quad \text{in } \text{Lie}(K)_{\mathbb{Q}}^*/(y_\lambda + z)
\]

and therefore the linear isomorphism \( \kappa^*_w : \text{Lie}(T_w)_{\mathbb{Q}}^*/(y_\lambda + z) \rightarrow \text{Lie}(K)_{\mathbb{Q}}^*/(y_\lambda + z) \) induces the linear isomorphism

\[
\text{Lie}(T_w)_{\mathbb{Q}}^*/(w_\lambda y^w_\mu - w_\mu y^w_\lambda) \cong \text{Lie}(K)_{\mathbb{Q}}^*/(y_\lambda + z, y_\mu + z).
\]

\( \Box \)

**Remark 4.3.** Proposition 4.2 can be shown directly from Theorem 5.5 in [14] by using the description of wGr\((d, n)\) as the symplectic quotient of aPl\((d, n)^\times\) by the real torus \( d^8 \) explained in Section 3.

It is known that \( \bar{S}_\lambda|_\mu = \prod_{(k, l) \in \text{inv}(\lambda)} (y_{(k, l)} - y_\lambda) \) and \( \bar{S}_\lambda|_\mu = 0 \) for all \( \mu \not\subseteq \lambda \) (c.f. [19]). From this fact, together with the diagram (4.6) and \( \pi^*(\bar{S}_\lambda) = a\bar{S}_\lambda \), we have
Proposition 4.4.

\[ a\overline{S}_\lambda|_\mu = \begin{cases} 0 & \text{if } \mu \nleq \lambda, \\ \prod_{(k,l) \in \text{inv}(\lambda)} (y_{(k,l)\lambda} + z) & \text{if } \mu = \lambda, \end{cases} \quad \text{in } \mathbb{Q}[K^*/(y_\mu + z)]. \]

The next proposition is now immediate from Proposition 4.4, the definition \( w\overline{S}_\lambda = (\pi_w^*)^{-1}(a\overline{S}_\lambda) \) and (4.8).

Proposition 4.5.

\[ w\overline{S}_\lambda|_\mu = \begin{cases} 0 & \text{if } \mu \nleq \lambda, \\ \prod_{(k,l) \in \text{inv}(\lambda)} \left( y_{(k,l)\lambda}^w - \frac{w_{(k,l)\lambda}}{w_\lambda} y_{(k,l)\lambda}^w \right) & \text{if } \mu = \lambda. \end{cases} \]

Having the upper-triangularity of the weighted Schubert classes as above, the proof of [19, Proposition 1] can be applied words by words to obtain

Proposition 4.6. \( \{w\overline{S}_\lambda\}_\lambda \) is an \( H^*(BT_w) \)-module basis of \( H^*_{T_w}(w\text{Gr}(d,n)) \).

Example 4.7. The followings is \( w\overline{S}_{14} \) in \( H^*_{T_w}(w\text{Gr}(2,4)) \):

\[
(y_{23}^w - \frac{w_{23}}{w_1} y_{12}^w)(y_{24}^w - \frac{w_{24}}{w_1} y_{12}^w)\]

\[
(y_{23}^w - \frac{w_{23}}{w_1} y_{13}^w)(y_{34}^w - \frac{w_{34}}{w_1} y_{13}^w)\]

\[
(y_{24}^w - \frac{w_{24}}{w_1} y_{13}^w)(y_{34}^w - \frac{w_{34}}{w_1} y_{14}^w)\]

where the vertices are the elements of \( \{\frac{3}{2}\} \) and there is an edge for each pair of \( \lambda \) and \( \mu \) satisfying \( |\lambda \cap \mu| = 1 \).

5. Structure Constants and Positivity

Since \( \{w\overline{S}_\lambda\}_\lambda \) is an \( H^*(BT_w) \)-module basis of \( H^*_{T_w}(w\text{Gr}(d,n)) \), we can expand their pairwise cup product uniquely over \( H^*(BT_w) \):

\[
w\overline{S}_\lambda w\overline{S}_\mu = \sum_\nu w\overline{c}_\lambda^\nu w\overline{S}_\nu \quad \text{where } w\overline{c}_\lambda^\nu \in H^*(BT_w).
\]

In [12] and [19], it is shown that we can express \( \overline{c}_\lambda^\nu \) as a polynomial in \( u_i \)'s with non-negative coefficients where \( u_i := y_{i+1} - y_i \in \text{Lie}(T)_Z^* \) for each \( i = 1, \cdots, n - 1 \). In this section, we derive a formula for \( w\overline{c}_\lambda^\nu \) from any given formula for \( \overline{c}_\lambda^\nu \). In particular, the formula of \( w\overline{c}_\lambda^\nu \) is expressed in terms an independent subset \( \{w_{u_i}\}_{i=1,\cdots,n-1} \) of \( \text{Lie}(T_w)^*_Z \otimes \mathbb{Q} \) in such a way that the positivity of \( w\overline{c}_\lambda^\nu \) implies the positivity of \( w\overline{c}_\lambda^\nu \) with respect to \( \{w_{u_i}\}_{i=1,\cdots,n-1} \). Moreover, a manifestly positive formula for the structure constants \( \{w\overline{c}_\lambda^\nu\} \) of the ordinary cohomology \( H^*(w\text{Gr}(d,n)) \) is also obtained by specializing the one for \( w\overline{c}_\lambda^\nu \) at \( wu_1 = \cdots = wu_{n-1} = 0 \).

5.1. Equivariant Structure Constants. We start with the following lemma which describes the divsor Schubert class.

Lemma 5.1. Let \( \text{id} \) be the unique minimum in \( \{\frac{n}{2}\} \) with respect to the Bruhat order and \( \text{div} \) the unique element with \( l(\text{id}) = 1 \). We have \( a\overline{S}_{\text{div}} = (y_{\text{id}} + z) \cdot 1 \).
By applying Proposition 5.2 repeatedly, we can compute $K_{\nu}^\mu = (y_{id} - y_{\mu}) (y_{id} - y_{\mu}) \in \mathbb{Q}[K^*]/(y_{\mu} + z)$. Therefore the claim holds: $a\tilde{S}_{div} = (y_{id} + z) a\tilde{S}_{id} = (y_{id} + z) \cdot 1$.

Now we obtain the following expansion formula for the product $a\tilde{S}_{div} a\tilde{S}_\lambda$ over $\mathbb{Q}[T^*]_w$.

**Proposition 5.2. (The weighted Pieri-rule)**

$$a\tilde{S}_{div} a\tilde{S}_\lambda = \left( y_{id}^w - \frac{w_{id}}{w_{\lambda}} y_{\lambda}^w \right) a\tilde{S}_\lambda + \sum_{\lambda' \rightarrow \lambda} \frac{w_{id}}{w_{\lambda}} a\tilde{S}_{\lambda'}$$

**Proof.** By the isomorphism $\pi^*$, the equivariant Pieri-rule given in [19, Proposition 2] implies

$$a\tilde{S}_{div} a\tilde{S}_\lambda = (y_{id} - y_{\lambda}) a\tilde{S}_\lambda + \sum_{\lambda' \rightarrow \lambda} a\tilde{S}_{\lambda'}.$$  

Together with Lemma 5.1, we obtain

$$(5.3) 
0 = -(y_{\lambda} + z) a\tilde{S}_\lambda + \sum_{\lambda' \rightarrow \lambda} a\tilde{S}_{\lambda'}.$$  

Multiply both sides of this equation by $\frac{w_{id}}{w_{\lambda}}$, and then again by Lemma 5.1, we get

$$a\tilde{S}_{div} a\tilde{S}_\lambda = (y_{id} + z) a\tilde{S}_\lambda - \frac{w_{id}}{w_{\lambda}} (y_{\lambda} + z) a\tilde{S}_\lambda + \sum_{\lambda' \rightarrow \lambda} \frac{w_{id}}{w_{\lambda}} a\tilde{S}_{\lambda'}.$$  

Since $y_{\nu} + z = y_{id}^w + \frac{w_{id}}{w_{\lambda}} z$ in $\text{Lie}(K^*)_Q$ for all $\nu$, the terms with $z$ cancel and the claim follows.

Let $K_{1,\nu}^\mu$ be the coefficient in $\mathbb{Q}[T^*]_w$ for the following product

$$(a\tilde{S}_{div})^r a\tilde{S}_\mu = \sum_{\nu} K_{1,\nu}^\mu a\tilde{S}_\nu.$$  

By applying Proposition 5.2 repeatedly, we can compute $K_{1,\nu}^\mu$ explicitly. For example, for $r = 2$, we have

$$(a\tilde{S}_{div})^2 a\tilde{S}_\eta = a\tilde{S}_{div} \left( \left( y_{id}^w - \frac{w_{id}}{w_{\eta}} y_{\eta}^w \right) a\tilde{S}_\eta + \sum_{\eta' \rightarrow \eta} \frac{w_{id}}{w_{\eta}} a\tilde{S}_{\eta'} \right)$$

$$= \left( y_{id}^w - \frac{w_{id}}{w_{\eta}} y_{\eta}^w \right)^2 a\tilde{S}_\eta + \sum_{\eta' \rightarrow \eta} \left( \left( y_{id}^w - \frac{w_{id}}{w_{\eta'}} y_{\eta'}^w \right) y_{id}^w + \frac{w_{id}}{w_{\eta}} \left( y_{id}^w - \frac{w_{id}}{w_{\eta}} y_{\eta}^w \right) \right) a\tilde{S}_{\eta'} + \sum_{\eta' \rightarrow \eta' \rightarrow \eta} \frac{w_{id}}{w_{\eta}} \frac{w_{id}}{w_{\eta'}} a\tilde{S}_{\eta''}.$$  

The general formula for $K_{1,\nu}^\mu$ is recorded without proof as follows.

**Lemma 5.3.** If $\nu \not\geq \mu$, $K_{1,\nu}^\mu = 0$. If $\nu \geq \mu$,

$$(5.4) 
K_{1,\nu}^\mu = \sum_{\nu' \leq \nu' \rightarrow \nu' \rightarrow \cdots \rightarrow \nu' \rightarrow \eta} \frac{w_{\nu}}{w_{\eta}} \prod_{q=0}^{l} \frac{w_{id}}{w_{\nu_q}} \left( y_{id}^w - \frac{w_{id}}{w_{\nu_q}} y_{\nu_q}^w \right)^{j_q}.$$  

where $l := l(\nu) - l(\mu)$ and $J$ runs over all sequences $(j_0, \cdots, j_l)$ of non-negative integers satisfying $j_0 + \cdots + j_l = r - l$. In particular,

$$K_{1,\eta}^\eta = \left( y_{id} - \frac{w_{id}}{w_{\eta}} y_{\eta}^w \right)^r.$$  

For each pair \( \alpha = (i, j) \) of integers in \([n]\) such that \( i > j \), let

\[
\begin{align*}
\nu_\alpha & := y_i - y_j \in \mathbb{Q}[T^*], \\
\nu_{u_\alpha} & := (y_i^w - y_j^w) - \frac{w_i - w_j}{w_{id}} y_{id} \in \mathbb{Q}[T_w^*], \\
w(\alpha) & := w_i - w_j \in \mathbb{Q}.
\end{align*}
\]

(5.5)

For simplicity, let \( u_i := u_{(i+1, i)} \) and \( w_i := u_{(i+1, i)} \) for \( i = 1, \ldots, n - 1 \). We can easily check that \( \{ w_{u_1}, \ldots, w_{u_{n-1}} \} \) is linearly independent in \( \text{Lie}(T_w)^* \otimes \mathbb{Q} \) and each \( w_{u_\alpha} \) is a linear combination of \( w_i \)'s with non-negative coefficients.

The next proposition gives the essential equation to relate the \( \mathbb{Q}[T^*] \)-action to the \( \mathbb{Q}[T_w^*] \)-action in \( H^*_T(\text{aPl}(d, n)^\times) \) and it follows from Lemma \([51] \) immediately.

**Proposition 5.4.** In \( H^*_T(\text{aPl}(d, n)^\times) \), we have

\[
y_i \cdot 1 = \left( y_i^w - \frac{w_i}{w_{id}} y_{id} \right) \cdot 1 + \frac{w_i}{w_{id}} a \mathcal{S}_{\text{div}} \quad \text{and} \quad u_\alpha \cdot 1 = w_{u_\alpha} \cdot 1 + \frac{w(\alpha)}{w_{id}} a \mathcal{S}_{\text{div}}.
\]

Let \( Q \) be a formal variable. For each finite collection \( I = \{ \alpha_1, \ldots, \alpha_p \} \) of pairs of integers in \([n]\) as above, define \( w_{u_I}^{(0)}, w_{u_I}^{(1)}, \ldots, w_{u_I}^{(p)} \in \mathbb{Q}[T_w^*] \) by

\[
\left( \frac{w(\alpha_1)}{w_{id}} Q \right) \cdots \left( \frac{w(\alpha_p)}{w_{id}} Q \right) = \sum_{r=0}^{p} w_{u_I}^{(r)} Q^r
\]

Explicitly, we have

\[
w_{u_I}^{(r)} = \sum_{1 \leq s_1 < \cdots < s_r \leq p} \frac{w(\alpha_{s_1})}{w_{id}} \cdots \frac{w(\alpha_{s_r})}{w_{id}} \frac{w_{u_{\alpha_1}} \cdots w_{u_{\alpha_r}}}{w_{u_{\alpha_1}} \cdots w_{u_{\alpha_r}}}.
\]

For example,

\[
w_{u_I}^{(0)} = w_{u_{\alpha_1}} \cdots w_{u_{\alpha_p}} \quad \text{and} \quad w_{u_I}^{(p)} = \frac{w(\alpha_1)}{w_{id}} \cdots \frac{w(\alpha_p)}{w_{id}}.
\]

Also note that, if \( w_1 = \cdots = w_n = 0 \), then \( w_{u_I}^{(r)} = 0 \) for \( r \geq 1 \). In this case, we denote \( u_I := w_{u_I}^{(0)} = \prod_{\alpha \in I} u_{\alpha} \).

It is known that the equivariant Schubert structure constant \( \tilde{c}_{\lambda \mu}^\nu \) for \( H^*_T(\text{Gr}(d, n)) \) is an element of \( \mathbb{Z}[u_1, \ldots, u_{n-1}] \)

\[
\tilde{c}_{\lambda \mu}^\nu = \sum_{|I| = |(\lambda) + (\mu) - (\nu)|} c(\lambda, \mu, \nu; I) u_I. \quad c(\lambda, \mu, \nu; I) \in \mathbb{Z}
\]

where \( I \) runs over collections of pairs \( (i, j) \) of integers in \([n]\) with \( i > j \) as above. For example, Knutson-Tao (\([119]\)) computed the number \( c(\lambda, \mu, \nu; I) \) in terms of the equivariant puzzles: with their notations, we have

\[
c(\lambda, \mu, \nu; I) = |\{ \text{equivariant puzzles } P \mid \partial P = \Delta_{\lambda \mu}^\nu \text{ and } \text{wt}(P) = u_I \}|.
\]

Now we state the main theorem of this section.

**Theorem 5.5.** Let \( \lambda, \mu, \nu \in \{n\}_d \), then

\[
w \tilde{c}_{\lambda \mu}^\nu = \sum_{\nu \geq \eta \geq \lambda, \mu} \sum_{I} \sum_{r=0}^{|I|} c(\lambda, \mu, \eta; I) K_{1, \nu}^{\nu} w_{u_I}^{(r)}
\]

where \( I = \{ \alpha_1, \cdots, \alpha_{|I|} \} \) runs over collections of pairs \( (i, j) \) of integers in \([n]\) with \( i > j \).
Proof. Since the map \( \pi^* \) is an isomorphism of rings over \( \mathbb{Q}[T^*] \), we have \( a S_\lambda a S_\mu = \sum \nu \tilde{c}_{\lambda \mu}^\nu a S_\nu \) in \( H_K(a Pl(d, n))^\times \). Lemma 5.3 allows us to write

\[
c_{\lambda \mu}^\eta = \sum_I c(\lambda, \mu, \eta; I) \left( w_{\alpha_1} + \frac{w(\alpha_1)}{w_{id}} a_{\text{div}} \right) \cdots \left( w_{\alpha_p} + \frac{w(\alpha_p)}{w_{id}} a_{\text{div}} \right) = \sum_I \sum_{r=0}^p c(\lambda, \mu, \eta; I) w_{\mu}^{(r)} (a S_{\text{div}})^r,
\]

where we denoted \( p := |I| \). Therefore,

\[
a S_\lambda a S_\mu = \sum_{\eta \geq \lambda, \mu} \sum_I |I| c(\lambda, \mu, \eta; I) w_{\mu}^{(r)} (a S_{\text{div}})^r a S_\eta
\]

\[
= \sum_{\eta \geq \lambda, \mu} \sum_I |I| c(\lambda, \mu, \eta; I) w_{\mu}^{(r)} \sum_{\nu \geq \eta} K^{(r)}_{\nu \eta} a S_{\nu}
\]

\[
= \sum_{\nu} \left( \sum_{\nu \geq \lambda, \mu} \sum_I |I| c(\lambda, \mu, \eta; I) w_{\mu}^{(r)} K^{(r)}_{\nu \eta} \right) a S_{\nu}.
\]

Since the coefficients are in \( \mathbb{Q}[T_w^*] \), this proves the desired formula. \( \square \)

Remark 5.6. From the equivariant weighted Pieri rule, we can derive a recursive formula for the structure constants \( \tilde{c}_{\lambda \mu}^\nu \), in the exactly same way shown in \[19\] Theorem 3:

\[
( w_{\text{div}} \nu - w_{\text{div}} \lambda ) w^{\nu}_{\lambda \mu} = \left( \sum_{\lambda' \rightarrow \lambda} \frac{w_{id}}{w_{\lambda}} w^{\nu}_{\lambda' \mu} - \sum_{\nu \rightarrow \nu'} \frac{w_{id}}{w_{\nu'}} \tilde{c}_{\lambda \mu}^{\nu'} \right).
\]

However this equation plays no role in the derivation of our main formula \(5.6\), while the recursive formula in \[19\] plays a crucial role in their process of obtaining the original puzzle formula for \( \tilde{c}_{\lambda \mu}^\nu \).

5.2. Positivity. The equivariant positivity of \[12\] guarantees that the structure constants \( \tilde{c}_{\lambda \mu}^\nu \) for \( \text{Gr}(d,n) \) are polynomials in \( u_1, \cdots, u_{n-1} \) with non-negative coefficients. In analogy to this fact, we prove the following equivariant positivity theorem for \( \text{wGr}(d,n) \).

Theorem 5.7. If \( w_1 \leq w_2 \leq \cdots \leq w_n \), then \( w^{\nu}_{\lambda \mu} \) is a polynomial in \( w_{u_1}, \cdots, w_{u_{n-1}} \) with non-negative coefficients.

Proof. We look at Theorem 5.5. By the assumption, it is clear that \( w_{\mu}^{(r)} \) is a polynomial in \( w_{u_1}, \cdots, w_{u_{n-1}} \) with non-negative coefficients. For the positivity of \( K^{(r)}_{\nu \eta} \), it suffices to show that \( y_{\nu} - \frac{w_{id}}{w_{\nu}} y_{\nu} \) is a polynomial in \( w_{u_1}, \cdots, w_{u_{n-1}} \) with non-negative coefficients for all \( \nu \neq \text{id} \). There exists \( a \in [n] \) such that \( (a, a+1) = \nu' \) and \( l(\nu') = l(\nu) - 1 \). The straightforward computation shows

\[
y_{\nu} - \frac{w_{id}}{w_{\nu}} y_{\nu} = \frac{w_{\nu'}}{w_{\nu}} \left( y_{\nu} - \frac{w_{id}}{w_{\nu'}} y_{\nu} \right) + \frac{w_{id}}{w_{\nu}} y_{u_a}
\]

Therefore the claim follows by the induction on the length of \( \nu \). \( \square \)

Remark 5.8. Our positivity theorem holds for all weighted Grassmannians in the following sense: for a given \( \text{wGr}(d, n) \) with the weight \( w = (w_1, \cdots, w_n) \), we can always perform a permutation on the basis \( \{ e_1, \cdots, e_n \} \) of \( \mathbb{C}^n \) so that the new order on the weight is non-decreasing. Then we can re-define the Schubert classes \( \{ w S_\lambda \} \) to make sure that the structure constants are positive.
5.3. Structure Constants for Ordinary Cohomology. For each \( \lambda \in \{0\} \), define \( wS_\lambda := \zeta^*(w\tilde{S}_\lambda) \in H^*(wGr(d,n)) \) where \( \zeta^* \) is the pullback along an inclusion as a fiber \( \zeta : wGr(d,n) \to T_w \times T_w \) \( wGr(d,n) \). Equivalently, this can be defined by \( wS_\lambda := (\pi_w^*)^{-1}[a\Omega_\lambda]_{wD} \) under the map \( \pi_w^* : H^*_w(aPl(d,n)) \to H^*(wGr(d,n)) \).

Corollary 5.9. Let \( \lambda, \mu, \nu \in \{0\} \). The structure constant \( wc^\nu_{\lambda,\mu} \) is given by

\[
wc^\nu_{\lambda,\mu} = \sum_{\nu \geq \eta \geq \lambda, \mu} \sum_{\nu_1, \nu_2, \ldots, \nu_{n-1}} c^\eta_{\lambda, \mu} (w_i = w_{i+1} - w_i, i = 1, \ldots, n-1)
\]

if \( l(\lambda) + l(\mu) = l(\nu) \) and \( wc^\nu_{\lambda,\mu} = 0 \) if otherwise. Furthermore, if \( w_1 \leq w_2 \leq \cdots \leq w_n \), \( wc^\nu_{\lambda,\mu} \) is non-negative.

Proof. After the evaluation, each term in (5.6) can survive only if \( l := l(\nu) - l(\eta) = r = |I| \):

\[
wc^\nu_{\lambda,\mu} |_{wu_1 = \cdots = wu_{n-1} = 0} = \sum_{\nu \geq \eta \geq \lambda, \mu} \sum_{|I| = l} c(\lambda, \mu, \eta; I)K^\nu_I wu_I
\]

\[
= \sum_{\nu \geq \eta \geq \lambda, \mu} \sum_{|I| = l} \sum_{\nu_1, \nu_2, \ldots, \nu_{n-1}} c(\lambda, \mu, \eta; I) \frac{w(\alpha_1) \cdots w(\alpha_l)}{w_{\nu_1} \cdots w_{\nu_l}}
\]

\[
= \sum_{\nu \geq \eta \geq \lambda, \mu} \sum_{\nu_1, \nu_2, \ldots, \nu_{n-1}} c^\eta_{\lambda, \mu} (w_i = w_{i+1} - w_i, i = 1, \ldots, n-1)
\]

The positivity is a direct consequence of the equivariant positivity (Theorem 5.7).

5.4. Examples.

Example 5.10 (Weighted Projective Space \( wGr(1,n) \)). The weighted projective space \( \mathbb{C}P_b = \mathbb{C}P_{b_1, \ldots, b_n} \) is the quotient of \( \mathbb{C}^n \setminus \{0\} \) by one dimensional torus \( D_b := \{(s^{b_1}, \ldots, s^{b_n}) \mid s \in \mathbb{C}^x \} \subset R := (\mathbb{C}^x)^n \) where \( R \) acts on \( \mathbb{C}^n \setminus \{0\} \) in the standard way. Let \( R_b := R/D_b \) and \( \{z_1, \ldots, z_n\} \) the standard basis of \((\text{Lie } R)_L^*\). Then it is well-known that the \( R_b \)-equivariant cohomology of \( \mathbb{C}P_{b_1, \ldots, b_n} \) is the corresponding Stanley-Reisner ring by regarding \( \mathbb{C}P_{b_1, \ldots, b_n} \) as a toric variety (c.f. [2]):

\[
H^*_R_b (\mathbb{C}P_b) = \frac{\mathbb{Q}[z_1, \ldots, z_n]}{(z_1 \cdots z_n)}.
\]

In our notation, \( wGr(1,n) = \mathbb{C}P_b \) where \( b_i = w_i + a \) and \( K \) is related to \( R \) via the map

\[
K \to R: (t_1, \ldots, t_n, s) \mapsto (st_1, \ldots, st_n).
\]

Therefore we can identify \( H^*_R (\mathbb{C}P_b) \) with a subring of \( H^*_T (wGr(1,n)) \) by \( z_i = y_i + z \). With this identification, the Schubert classes \( wS_\lambda \) are given by

(5.8) \[
\begin{align*}
\tilde{wS}_{\{n\}} &= 1, \quad \tilde{wS}_{\{n-1\}} = z_n, \quad \cdots, \quad \tilde{wS}_{\{k\}} = z_{k+1} \cdots z_n, \quad \cdots, \quad \tilde{wS}_{\{1\}} = z_2 \cdots z_n.
\end{align*}
\]

where the Bruhat order is \( \{n\} \leq \cdots \leq \{1\} \). The equivariant weighted Pieri rule gives

\[
w\tilde{S}_{\{n-1\}} \cdot w\tilde{S}_{\{k\}} = \left( z_n - \frac{b_n}{b_k} \right) w\tilde{S}_{\{k\}} + \frac{b_n}{b_k} w\tilde{S}_{\{k-1\}}.
\]

This is actually obvious in the presentation \( \mathbb{Q}[z_1, \ldots, z_n]/(z_1 \cdots z_n) \).

Example 5.11 (Relation to the work of Kawasaki [16]). In this example, all cohomologies are over \( \mathbb{Z} \)-coefficients. The integral cohomology of the weighted projective space \( H^*(\mathbb{C}P_b; \mathbb{Z}) \) is known to be a free \( \mathbb{Z} \)-module generated by the Kawasaki basis [16]. Namely, the map

\[
\zeta : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\}, \quad (x_1, \ldots, x_n) \mapsto (x_1^{b_1}, \ldots, x_n^{b_n})
\]
induces a map \( \tilde{\zeta} : \mathbb{C}P^{n-1} \to \mathbb{C}P_b \) and the inclusion \( \tilde{\zeta}^* : H^*(\mathbb{C}P_b; \mathbb{Z}) \to H^*(\mathbb{C}P^{n-1}; \mathbb{Z}) \). Following \([7]\), we represent \( H^*(\mathbb{C}P^{n-1}; \mathbb{Z}) \) as

\[
\frac{\mathbb{Z}[z_1, \ldots, z_n]}{(z_1 \cdots z_n, u_1, \ldots, u_{n-1})}
\]

where \( u_i := z_{i+1} - z_i \). Then after we identity \( H^*(\mathbb{C}P_b; \mathbb{Z}) \) with the image of \( \tilde{\zeta}^* \), it is a free \( \mathbb{Z} \)-module generated by

\[
\gamma_1 := l^b_1 = 1, \quad \gamma_2 := l^b_2 z_2, \quad \gamma_3 := l^b_3 z_2 z_3, \quad \ldots , \quad \gamma_n := l^b_n z_2 \cdots z_n
\]

where

\[
l^b_k := \text{l.c.m. of } \left\{ \frac{b_{i_1} \cdots b_{i_k}}{\gcd(b_{i_1}, \ldots, b_{i_k})} \right\} 1 \leq i_1 < \cdots < i_k \leq n.\]

On the other hand, the cohomology \( H^*_D_b(\mathbb{C}^n \setminus \{0\}; \mathbb{Z}) \) is known to be

\[
\frac{\mathbb{Z}[z_1, \ldots, z_n]}{(z_1 \cdots z_n, u_1^b, \ldots, u_{n-1}^b)}
\]

where \( \{u_i^b\} \) is a \( \mathbb{Z} \)-basis of \( \text{Lie}(\check{R}_b)^*_Z \subset (\text{Lie} R)^*_Z \) (c.f. Example 6.1 \([22]\)). We can regard our non-equivariant Schubert classes \( wS_{k} \) as the monomial \( z_{k+1} \cdots z_n \) in \( H^*_D_b(\mathbb{C}^n \setminus \{0\}; \mathbb{Z}) \). Together with the homomorphism

\[
R \to R ; (s_1, \ldots , s_n) \mapsto (s_1^{b_1}, \ldots , s_n^{b_n}),
\]

\( \zeta \) induces a map

\[
\omega : \mathbb{C}P^{n-1} \to ED_b \times D_b (\mathbb{C}^n \setminus \{0\})
\]

and the pullback \( \omega^* \) is given by

\[
\omega^* : \frac{\mathbb{Z}[z_1, \ldots, z_n]}{(z_1 \cdots z_n, u_1, \ldots, u_{n-1})} \to \frac{\mathbb{Z}[z_1, \ldots, z_n]}{(z_1 \cdots z_n, u_1^b, \ldots, u_{n-1}^b)}; \quad z_i \mapsto b_i z_i.
\]

Since \( \tilde{\zeta} \) factors through \( \omega \) and the projection \( \pi : ED_b \times D_b \mathbb{C}^n \setminus \{0\} \to \mathbb{C}P_b \) and by the fact that \( H^*_D_b(\mathbb{C}^n \setminus \{0\}; \mathbb{Z}) \) has no \( \mathbb{Z} \)-torsions in the degrees between 0 and \( 2(n-1) \) (see Theorem 4.2 \([13]\)), we can conclude that the pullbacks of the Kawasaki’s basis along the projection \( \pi \) are the following multiples of our Schubert classes:

\[
\pi^*(\gamma_1) = aS_{(n)} \quad \text{and} \quad \pi^*(\gamma_k) = \frac{l^b_k}{b_{n-k+2} b_{n-k+3} \cdots b_n} aS_{(n-k+1)}, \quad k = 2, \cdots , n.
\]

**Example 5.12** (wGr(2, 4)). Here we demonstrate the computation of the product \( w\tilde{S}_{23}w\tilde{S}_{23} \). By the upper triangularity of the GKM description of \( w\tilde{S}_{23} \), the product must be written by

\[
w\tilde{S}_{23}w\tilde{S}_{23} = w\tilde{c}^{24}_{23, 23} w\tilde{S}_{23} + w\tilde{c}^{13}_{23, 23} w\tilde{S}_{13} + w\tilde{c}^{12}_{23, 23} w\tilde{S}_{12}.
\]

We can compute these coefficients from the formula of the following product for ordinary Grassmannian

\[
\tilde{S}_{23} \tilde{S}_{23} = (y_4 - y_2)(y_4 - y_3)\tilde{S}_{23} + (y_4 - y_3)\tilde{S}_{13} + \tilde{S}_{12}.
\]

That is, we have

\[
c(23, 23, 23; I) = \begin{cases} 1 & \text{if } I = \{(4, 2), (4, 3)\}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
c(23, 23, 13; I) = \begin{cases} 1 & \text{if } I = \{(4, 3)\}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
c(23, 23, 12; I) = \begin{cases} 1 & \text{if } I = \emptyset, \\ 0 & \text{otherwise}, \end{cases}
\]
and other $c(23, 23, \eta, I)$'s are zero. Here are the computation:

$$\begin{align*}
\text{w}_{c_{23,23}} & = \left( y_4^w - y_2^w - \frac{w_4 - w_2}{w_{13}} \right) \left( y_4^w - y_3^w - \frac{w_4 - w_3}{w_{13}} \right) \\
& + \left( y_4^w - \frac{w_{13} y_4^w}{w_{23}} \right) \left( \frac{w_4 - w_2}{w_{13}} \right) \left( \frac{w_4 - w_3}{w_{13}} \right) + \frac{w_4 - w_3}{w_{13}} \left( \frac{w_4 - y_2^w - w_4 - w_2}{w_{13}} \right) \\
& + \left( y_4^w - \frac{w_{13} y_4^w}{w_{23}} \right) \left( \frac{w_4 - w_2}{w_{13}} \right) \left( \frac{w_4 - w_3}{w_{13}} \right) + \frac{w_4 - w_3}{w_{13}} \left( \frac{w_4 - y_2^w - w_4 - w_2}{w_{13}} \right) \\
& + \frac{w_4 - w_3}{w_{13}} \left( \frac{w_4 - w_3}{w_{13}} \right)
\end{align*}$$

$$\begin{align*}
\text{w}_{c_{23,23}} & = y_4^w - y_3^w - \frac{w_4 - w_3}{w_{13}} \left( y_4^w - y_2^w - \frac{w_4 - w_3}{w_{13}} \right) + \frac{w_4 - w_3}{w_{13}} \left( y_4^w - y_2^w - \frac{w_4 - w_3}{w_{13}} \right) + \frac{w_4 - w_3}{w_{13}} \left( y_4^w - y_2^w - \frac{w_4 - w_3}{w_{13}} \right)
\end{align*}$$

$$\begin{align*}
\text{w}_{c_{23,23}} & = 1 + \frac{w_4 - w_3}{w_{13}} + \frac{w_4 - w_2}{w_{13}} \frac{w_4 - w_3}{w_{13}}
\end{align*}$$

Similarly we can also work out

$$\begin{align*}
\text{w}_{c_{23,14}} & = \text{w}_{c_{23,14}} \cdot \text{w}_{c_{23,14}} \cdot \text{w}_{c_{23,14}} \cdot \text{w}_{c_{23,14}}
\end{align*}$$

from $\tilde{S}_{23} \tilde{S}_{14} = (y_4 - y_1) \tilde{S}_{13}$ where

$$\begin{align*}
\text{w}_{c_{23,14}} & = \left( y_4^w - y_2^w - \frac{w_4 - w_1}{w_{13}} \right) + \left( y_4^w - \frac{w_{13} y_4^w}{w_{23}} \right) - \frac{w_4 - w_1}{w_{13}}
\end{align*}$$

$$\begin{align*}
\text{w}_{c_{23,14}} & = \frac{w_4 - w_1}{w_{13}}
\end{align*}$$

Funding

This work was supported by JSPS Research Fellowships for Young Scientists to H.A; and the National Research Foundation of Korea (NRF) grants funded by the Korea government (MEST) [2012-000795, 2011-0001181] to T.M.

Acknowledgment. The first author is particularly indebted to Takashi Otofuji for many valuable discussions and helpful suggestions. The second author is grateful to Allen Knutson for teaching him about the subject and many inspirational discussions. He also would like to express his gratitude to the Algebraic Structure and its Application Research Institute at KAIST for providing him an excellent research environment in 2011-2012. The authors also would like to thank the referee for a thorough reading that improved the paper greatly, as well as Mikiya Masuda, Shintaro Kuroki, Hiroaki Ishida and Yukiko Fukukawa for organizing the international conference Toric Topology in Osaka 2011 which made our collaboration possible.

References

[1] Adem, A., Leida, J., and Ruan, Y. Orbifolds and stringy topology, volume 171 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2007.

[2] Bahri, A., Franz, M., and Ray, N. “The equivariant cohomology ring of weighted projective space.” Math. Proc. Cambridge Philos. Soc., 146(2): 395 - 405, 2009.

[3] Borel, A., Bredon, G., Floyd, E. E., Montgomery, P., and Palais, R. Seminar on transformation groups, Annals of Math. Studies 46. Princeton, New Jersey: Princeton University Press 1960.

[4] Chang, T., and Skjelbred, T. “The topological schur lemma and related results.” Ann. of Math., 100(2): 307 - 321, 1974.

[5] Corti, A., and Reid, M. “Weighted Grassmannians.” In Algebraic geometry, pages 141 - 163. de Gruyter, Berlin, 2002.

[6] Cox, D., Little, J., and Schenck, H. “Toric Varieties”. Graduate Studies in Mathematics, 124. American Mathematical Society (2011).

[7] Davis, M.W., and Januszkiewicz, T. “Convex polytopes, Coxeter orbifolds and torus actions.” Duke Math. J., 62(2):417 - 451, 1991.
17

[8] Dold, A. “Partitions of unity in the theory of fibrations.” Ann. of Math. (2), 78: 223 - 255, 1963.
[9] Edidin, D. “Equivariant geometry and the cohomology of the moduli space of curves” in Handbook of Moduli: Volume I, 331-414, editors Gavril Farkas and Ian Morrison, Adv. Lect. Math. (ALM), 24, Int. Press, Somerville, MA, 2013
[10] Fulton, W. Young tableaux: With applications to representation theory and geometry., volume 35 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997.
[11] Goresky,M., Kottwitz, R., and MacPherson, R. “Equivariant cohomology, Koszul duality, and the localization theorem.” Invent. Math., 131(1): 25 - 83, 1998.
[12] Graham, W. “Positivity in equivariant Schubert calculus.” Duke Math. J., 109(3): 599 - 614, 2001.
[13] Holm, T.S. “Orbifold cohomology of abelian symplectic reductions and the case of weighted projective spaces.” In Poisson geometry in mathematics and physics, volume 450 of Contemp. Math., pages 127 - 146.Amer. Math. Soc., Providence, RI, 2008.
[14] Holm, T.S., and Matsumura, T. “Equivariant cohomology for Hamiltonian torus actions on symplectic orbifolds.” Transform. Groups, 17(3): 717 – 746, 2012.
[15] Kawakubo, K. The theory of transformation groups. The Clarendon Press Oxford University Press, New York, 1991.
[16] Kawasaki, T. “Cohomology of twisted projective spaces and lens complexes.” Math. Ann., 206: 243 - 248,1973.
[17] Kirwan, F.C. Cohomology of quotients in symplectic and algebraic geometry, volume 31 of Mathematical Notes. Princeton University Press, Princeton, NJ, 1984.
[18] Kleiman, S.L., and Laksov, D. “Schubert calculus.” Amer. Math. Monthly, 79: 1061 - 1082, 1972.
[19] Knutson, A., and Tao, T. “Puzzles and (equivariant) cohomology of Grassmannians.” Duke Math. J.,119(2):221-260, 2003.
[20] Knutson, A., Tao, T., and Woodward, C. “The honeycomb model of GLn(C) tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone.” J. Amer. Math. Soc., 17(1): 49 - 194, 2004.
[21] Lerman, E., and Malkin, A. “Hamiltonian group actions on symplectic Deligne-Mumford stacks and toric orbifolds.” Adv. Math., 229(2): 984 - 1000, 2012.
[22] Luo, S., Matsumura, T., and Moore, F.W. “Moment Angle Complexes and Big Cohen-Macaulayness.” arXiv:1205.1566.
[23] Molev, A.I., and Sagan, B.E. “A Littlewood-Richardson rule for factorial Schur functions.” Trans. Amer. Math. Soc., 351(11): 4429 - 4443, 1999.
[24] Okounkov, A. “Quantum immanants and higher Capelli identities.” Transform. Groups, 1(1-2): 99 - 126, 1996.
[25] Spanier, E.H. Algebraic topology. Springer-Verlag, New York, 1981. Corrected reprint.
[26] Tymoczko, J.S. “Equivariant structure constants for ordinary and weighted projective space.” math.AT/0806.3588.

Hiraku Abe, Osaka City University Advanced Mathematical Institute, 3-3-138 Sugimoto, Sumiyoshi-ku Osaka 558-8585 JAPAN
E-mail address: hirakuabe@globe.ocn.ne.jp

Tomoo Matsumura, Department of Mathematical Sciences, Algebraic Structure and its Applications Research Center, KAIST, 291 Daehak-ro Yuseong-gu, Daejeon 305-701, SOUTH KOREA
E-mail address: toomomatsumura@kaist.ac.kr