Lp-Kato class measures for symmetric Markov processes under heat kernel estimates

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Abstract
In this paper, we establish the coincidence of two classes of Lp-Kato class measures in the framework of symmetric Markov processes admitting upper and lower estimates of heat kernel under mild conditions. One class of Lp-Kato class measures is defined by the pth power of positive order resolvent kernel, another is defined in terms of the pth power of Green kernel depending on some exponents related to the heat kernel estimates. We also prove that functions u such that sup\(x \in E, \int_{B_1(x)} |u|^q dm < \infty\) are of Lp-Kato class if q is greater than a constant related to p and the constants appeared in the upper and lower estimates of the heat kernel. These are complete extensions of some results by Aizenman–Simon and the recent results by the second named author in the framework of Brownian motions on Euclidean space. We further give necessary and sufficient conditions for a Radon measure with Ahlfors regularity to belong to Lp-Kato class. Our results can be applicable to many examples, for instance, symmetric (relativistic) stable processes, jump processes on d-sets, Brownian motions on Riemannian manifolds, diffusions on fractals and so on.

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1 Introduction

Let \( p \in [1, +\infty[ \). A Borel measure \( \mu \) on \( \mathbb{R}^d \) is said to be of \( L^p \)-Kato class (of \( p \)-Kato class in short) \( K^p_d \) if

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} \frac{\mu(dy)}{|x-y|^{(d-2)p}} = 0 \quad \text{for} \quad d \geq 3,
\]

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} (\log |x-y|^{-1})^p \mu(dy) = 0 \quad \text{for} \quad d = 2,
\]

\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq 1} \mu(dy) < +\infty \quad \text{for} \quad d = 1.
\]

We write \( K_d \) instead of \( K^1_d \) for \( p = 1 \). The notion of \( (1-) \)Kato class \( K_d \) was introduced by Kato [23,24] in order to solve the essential self-adjointness of the Schrödinger operator \( -\Delta + V \) on \( C_0^\infty(\mathbb{R}^d) \) (see the survey paper [33] by Simon). Let \( X^w = (\Omega, B_t, \mathbb{P}_t)_{\tau \in \mathbb{R}^d} \) be a \( d \)-dimensional Brownian motion on \( \mathbb{R}^d \). The following theorem was proved by Aizenman–Simon [1] under \( p = 1 \) and noted by the second named author in [30] for general \( p \in [1, +\infty[ \) with \( d - p(d-2) > 0 \):

**Theorem 1.1** ([1, Theorem 1.3(ii)], [30, Example 2.4]) Let \( p \in [1, +\infty[ \) with \( d - p(d-2) > 0 \). Then \( \mu \in K^p_d \) if and only if

\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \to 0 \quad \text{as} \quad t \to 0,
\]

where \( p_t(x, y) := \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{2t}\right) \) is the heat kernel of \( X^w \).

In [28], we extend Theorem 1.1 under \( p = 1 \) in a probabilistic way. Our first main theorem (Theorem 3.1 below) is an extension of Theorem 1.1 for general \( p \in [1, +\infty[ \) under the framework of [28].

The following theorem is proved by Vogt [34, Proposition 2.2]:

**Theorem 1.2** (Vogt [34, Proposition 2.2]) Let \( \gamma \in [0, 2] \). Suppose that a Borel measure \( \mu \) on \( \mathbb{R}^d \) satisfies that there exists \( C > 0 \) such that \( \mu(B_r(x)) \leq Cr^{d-\gamma} \) for all \( x \in \mathbb{R}^d \) and \( r \in [0, +\infty[ \). Then \( \mu \in K^p_d \) with \( p = 1 \) for \( \gamma < 2 \) (\( \gamma \leq 1 \) if \( d = 1 \)).

Our second main theorem (Theorem 3.2 below) is a complete extension of Theorem 1.2 for general \( p \in [1, +\infty[ \).

The following theorems are also shown by Aizenman–Simon [1] under \( p = 1 \) and noted in [30, Example 2.4] for general \( p \in [1, +\infty[ \) with \( d - p(d-2) > 0 \):

**Theorem 1.3** ([1, Theorem 1.4(iii)], [30, Example 2.4]) Let \( p \in [1, +\infty[ \) with \( d - p(d-2) > 0 \). Then \( f \in L^q_{\text{unif}}(\mathbb{R}^d) \) implies \( |f|dm \in K^p_d \) if \( q > d/(d - p(d-2)) \) with \( d \geq 2 \), or \( q \geq 1 \) with \( d = 1 \). Here \( m \) denotes the Lebesgue measure on \( \mathbb{R}^d \) and \( f \in L^q_{\text{unif}}(\mathbb{R}^d) \) means

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\[ \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq 1} |f(y)|^q \, dy < +\infty. \]

Let \( M_{\alpha, q} \) (\( \alpha > 0, q > 1 \)) be the family of measurable functions \( f \) on \( \mathbb{R}^d \) satisfying
\[ \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq 1} \frac{|f(y)|^q}{|x-y|^{d-\alpha}} \, dy < +\infty. \]

\( M_{\alpha, q} \) is called the Schechter class.

**Theorem 1.4** ([1, Propositions 4.1 and 4.2], [30, Example 2.4]) Let \( p \in [1, +\infty[ \) with \( d - p(d-2) > 0 \). Assume \( d \geq 3 \). Then \( f \in M_{\alpha, q} \) implies \( |f| \, dm \in K_d^p \) if \( q > \alpha/(d - p(d-2)) \).

**Remark 1.5** As noted in [30, Example 2.4], there are typos in [1, Propositions 4.1 and 4.2].

In [27], the first named author and Takahashi partially extend Theorem 1.3 for \( p = 1 \) by replacing \( L^q_{\text{unif}}(\mathbb{R}^d) \) (resp. \( K_d \)) with \( L^q(\mathbb{R}^d) \) (resp. \( S_K \)) under Nash-type estimate of semigroup kernel of Markov processes. In [28], they finally extend Theorem 1.3 for \( p = 1 \) in the framework of symmetric Markov processes satisfying conditions (A1), (A2) and (A3) below. However, they did not write down the extension of Theorem 1.4 under \( p = 1 \) in [28] for the limit of the length of [28].

The purpose of this paper is to show that the assertions in Theorems 1.1, 1.2, 1.3 and 1.4 can be extended for general \( p \in [1, +\infty[ \) in the framework of general symmetric Markov processes admitting semigroup kernel with upper and lower estimates under some conditions. These are presented as Theorems 3.1, 3.2, 4.3 and Corollary 4.4 in this paper. Not only these extensions, we provide some useful criteria for measures of \( L^p \)-Kato class in Theorems 4.1, 4.6 and Corollaries 3.3, 4.4 and 4.5. Our results are applicable to many Markov processes, for example, symmetric \( \alpha \)-stable processes, relativistic \( \alpha \)-stable processes, jump type processes on \( d \)-sets, Brownian motions on Riemannian manifolds with Ricci curvature lower bound and positivity of injectivity radius, diffusions on fractals and so on.

The constitution of this paper is as follows. In Sect. 2, we prepare our framework and expose our assumptions. In Sect. 3, we state our main theorems (Theorems 3.1, 3.2 and Corollary 3.3). In Sect. 4, we state Theorems 4.1, 4.3, 4.6 and Corollaries 4.4 and 4.5, which are useful criteria for \( L^p \)-Kato or \( L^p \)-Dynkin classes. Theorem 3.1 (resp. Theorem 3.2, Corollary 4.4, Theorem 4.3) extends Theorem 1.1 (resp. Theorems 1.2, 1.3, 1.4). In Sect. 5, we give the proofs of Theorems 3.1, 3.2, 4.1, 4.3, 4.6 and Corollaries 3.3, 4.4, 4.5. In the last section, we expose concrete examples.

Finally, we announce the paper [26] on the \( L^p \)-Green-tight measures of Kato class. In [26], for transient symmetric Markov processes, we establish the coincidence of two families of \( L^p \)-Green-tight measures of \( L^p \)-Kato class. One is defined to be a subclass of \( S^p_{K} \) and another is defined to be a subclass of \( K_{\nu, \beta}^p \). This is a natural extension of our Theorem 3.1.
2 Preliminary

For real numbers $a, b \in \mathbb{R}$, we set $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. Let $(E, d)$ be a locally compact separable metric space and $m$ a positive Radon measure with full support. Let $E_{\partial} := E \cup \{\partial\}$ be the one-point compactification of $E$. For each $x \in E$ and $r > 0$, denote by $B_r(x) := \{y \in E \mid d(x, y) < r\}$ the open ball with center $x$ and radius $r$. We consider and fix a symmetric regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$. Then there exists a Hunt process $X = (\Omega, \mathcal{F}, \xi, \mathbf{P}_x)$ such that for each Borel $u \in L^2(E; m)$, $T_t u(x) = \mathbf{E}_x[u(X_t)]$ $m$-a.e. $x \in E$ for all $t > 0$, where $(T_t)_{t \geq 0}$ is the semigroup associated with $(\mathcal{E}, \mathcal{F})$. Here $\zeta := \inf\{t \geq 0 \mid X_t = \partial\}$ denotes the life time of $X$. For a Borel set $B$, we denote $\sigma_B := \inf\{t > 0 \mid X_t \in B\}$ (resp. $\tau_B := \inf\{t > 0 \mid X_t \notin B\}$) the first hitting time to $B$ (resp. first exit time from $B$). Further, we assume that there exists a jointly measurable function $p_t(x, y)$ defined for all $(t, x, y) \in (0, +\infty) \times E$ such that $\mathbf{E}_x[u(X_t)] = \int_E p_t(x, y)u(y)m(dy)$ for any $x \in E$, bounded Borel function $u$ and $t > 0$. $p_t(x, y)$ is said to be a semigroup kernel, or sometimes called a heat kernel of $X$ on the analogy of heat kernel of diffusions. Then $P_t$ can be extended to contractive semigroups on $L^p(E; m)$ for $p \geq 1$. The following are well-known:

1. $p_{t+s}(x, y) = \int_E p_s(x, z)p_t(z, y)m(dz)$ for all $x, y \in E$ and $t, s > 0$.
2. $P_t(x, dy) = p_t(x, y)m(dy)$ for all $x \in E$ and $t > 0$.
3. $\int_E p_t(x, y)m(dy) \leq 1$ for all $x \in E$ and $t > 0$.

Throughout this paper, we fix $\nu, \beta \in (0, +\infty]$ and $t_0 \in (0, +\infty]$ and prepare the following assumptions.

(A1) (Life time condition) $X$ has the following property that

$$\lim_{t \to 0} \sup_{x \in E} \mathbf{P}_x(\xi \leq t) =: \gamma \in [0, 1[.$$

In particular, if $X$ is stochastically complete, that is, $X$ is conservative, then this condition is satisfied with $\gamma = 0$.

We fix an increasing positive function $V$ on $[0, +\infty[$.

(A2) (Bishop type inequality) Suppose $r \mapsto V(r)/r^\nu$ is increasing or bounded, and $\sup_{x \in E} m(B_r(x)) \leq V(r)$ for all $r > 0$.

(A3) (Upper and lower estimates of heat kernel) Let $\Phi_i$ $(i = 1, 2)$ be positive decreasing functions defined on $[0, +\infty[$ which may depend on $t_0$ if $t_0 < +\infty$ and assume that $\Phi_2$ satisfies the following condition $H(\Phi_2)$:

$$\int_1^{+\infty} \frac{(V(t) \vee t^\nu)\Phi_2(t)}{t}dt < +\infty$$

and $(\Phi_{E_{\nu, \beta}})$: for any $x, y \in E$, $t \in [0, t_0[$

$$\frac{1}{t^{\nu/\beta}}\Phi_1\left(\frac{d(x, y)}{t^{1/\beta}}\right) \leq p_t(x, y) \leq \frac{1}{t^{\nu/\beta}}\Phi_2\left(\frac{d(x, y)}{t^{1/\beta}}\right).$$
Note that the assumption (A3) is essentially introduced as the hypothesis of [17].

We next introduce the classes of measures dealt with in this paper. Throughout this paper, we consider a constant $p \in [1, +\infty[. $

**Definition 2.1** ($L^p$-Kato class $S^p_K$, $L^p$-Dynkin class $S^p_D$) For a positive Borel measure $\mu$ on $E$, $\mu$ is said to be of $L^p$-Kato ($p$-Kato in short) class relative to $p_t(x, y)$ (write $\mu \in S^p_K$) if

$$\lim_{t \to 0} \sup_{x \in E} \int_E \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) = 0 \quad (2.1)$$

and $\mu$ is said to be of $L^p$-Kato ($p$-Kato in short) class relative to $p_t(x, y)$ of order $\delta \in ]0, 1]$ (write $\mu \in S^p_{K, \delta}$) if

$$\sup_{x \in E} \left( \int_E \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \right)^{1/p} = O(t^\delta) \quad (t \to 0). \quad (2.2)$$

For a positive Borel measure $\mu$ on $E$, $\mu$ is said to be of $L^p$-Dynkin ($p$-Dynkin in short) class relative to $p_t(x, y)$ (write $\mu \in S^p_D$) if

$$\sup_{x \in E} \int_E \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) < +\infty \quad \text{for some } t > 0. \quad (2.3)$$

Clearly, $S^p_{K, \delta} \subset S^p_K \subset S^p_D$. When $p = 1$, we write $S_D$ (resp. $S^1_K$, $S^1_{K, \delta}$) instead of $S^1_D$ (resp. $S^1_K$, $S^1_{K, \delta}$) for simplicity.

For $\alpha > 0$, we denote the $\alpha$-order resolvent kernel by

$$r_\alpha(x, y) := \int_0^\infty e^{-\alpha t} p_t(x, y) dt.$$ 

The following are proved by the second named author in [30, Proposition 2.7 and Corollary 2.8] extending [27, Lemma 3.1].

**Lemma 2.2** ([30, Propositions 2.7, 2.11 and Corollary 2.8]) Let $\delta \in ]0, 1]$. $\mu \in S^p_K$ (resp. $\mu \in S^p_{K, \delta}$) is equivalent to

$$\lim_{\alpha \to \infty} \sup_{x \in E} \int_E r_\alpha(x, y) p \mu(dy) = 0 \quad (2.4)$$

$$(\text{resp. } \sup_{x \in E} \left( \int_E r_\alpha(x, y) p \mu(dy) \right)^{1/p} = O(\alpha^{-\delta}) \quad (\alpha \to \infty)) \quad (2.5)$$

and $\mu \in S^p_D$ is equivalent to

$$\sup_{x \in E} \int_E r_\alpha(x, y) p \mu(dy) < +\infty \quad \text{for some } \alpha > 0. \quad (2.6)$$
Lemma 2.3 ([30, Proposition 2.7], see also [27, Lemma 3.2],[2]) The following are equivalent to each other:

1. \( \mu \in S_p^p \),

2. \( \sup_{x \in E} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) < +\infty \) for any \( t > 0 \).

3. \( \sup_{x \in E} \int_{E} r_\alpha(x, y)^p \mu(dy) < +\infty \) for any \( \alpha > 0 \).

Definition 2.4 (Dynkin class \( D_{\nu,\beta}^p \)) Fix \( \nu > 0 \) and \( \beta > 0 \). For a positive Borel measure \( \mu \) on \( E \), \( \mu \) is said to be of \( L^p \)-Dynkin (\( p \)-Dynkin in short) class relative to Green kernel (write \( \mu \in D_{\nu,\beta}^p \)) if

\[
\sup_{x \in E} \int_{d(x,y) < r} G(x, y)^p \mu(dy) < +\infty \quad \text{for some} \quad r > 0 \quad \text{for} \quad \nu \geq \beta,
\]

\[
\sup_{x \in E} \int_{d(x,y) \leq 1} \mu(dy) < +\infty \quad \text{for} \quad \nu < \beta,
\]

where \( G(x, y) := G(d(x, y)) \) with

\[
G(r) := \begin{cases} 
    r^{\beta - \nu} & \text{if} \quad \nu > \beta, \quad r \in ]0, +\infty[, \\
    \log(r^{-1}) & \text{if} \quad \nu = \beta, \quad r \in ]0, 1[.
\end{cases}
\]

When \( p = 1 \), we write \( D_{\nu,\beta} \) instead of \( D_{\nu,\beta}^1 \).

Definition 2.5 (Kato class \( K_{\nu,\beta}^p \)) Fix \( \nu > 0 \) and \( \beta > 0 \). For a positive Borel measure \( \mu \) on \( E \), \( \mu \) is said to be of \( L^p \)-Kato (\( p \)-Kato in short) class relative to Green kernel (write \( \mu \in K_{\nu,\beta}^p \)) if

\[
\limsup_{r \to 0} \sup_{x \in E} \int_{d(x,y) < r} G(x, y)^p \mu(dy) = 0 \quad \text{for} \quad \nu \geq \beta,
\]

\[
\sup_{x \in E} \int_{d(x,y) \leq 1} \mu(dy) < +\infty \quad \text{for} \quad \nu < \beta,
\]

where \( G(x, y) \) is the function appeared above. When \( p = 1 \), we write \( K_{\nu,\beta} \) instead of \( K_{\nu,\beta}^1 \).

Lemma 2.6 If \( \mu \in D_{\nu,\beta}^p \), then \( \sup_{x \in E} \mu(B_r(x)) < +\infty \) for small \( r \in ]0, e^{-1}[ \). In particular, every \( \mu \in D_{\nu,\beta}^p \) is a Radon measure.

Proof The assertion is clear from \( \mu(B_r(x)) \leq \frac{1}{G(r)^p} \int_{B_r(x)} G(x, y)^p \mu(dy) \) for \( r \in ]0, e^{-1}[ \) with \( \nu \geq \beta \). The case for \( \nu < \beta \) is trivial. \( \square \)

Lemma 2.7 For \( 1 \leq p_1 \leq p_2 \), we have \( D_{\nu,\beta}^{p_2} \subset D_{\nu,\beta}^{p_1} \) and \( K_{\nu,\beta}^{p_2} \subset K_{\nu,\beta}^{p_1} \). In particular, \( D_{\nu,\beta}^p \subset D_{\nu,\beta} \) and \( K_{\nu,\beta}^p \subset K_{\nu,\beta} \) hold.

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Proof. When $\nu < \beta$, $K_{v,\beta}^p = D_{v,\beta}$ is independent of $p$. So we may assume $\nu \geq \beta$. Let $1 \leq p_1 \leq p_2$ and take $\mu \in D_{v,\beta}^{p_2}$. Then

$$
\int_{B_r(x)} G(x, y)^{p_1} \mu(dy) \\
\leq \left( \int_{B_r(x)} G(x, y)^{p_2} \mu(dy) \right)^{\frac{p_1}{p_2}} \left( \mu(B_r(x)) \right)^{1-\frac{p_1}{p_2}} \\
\leq \left( \int_{B_r(x)} G(x, y)^{p_2} \mu(dy) \right)^{\frac{p_1}{p_2}} \left( \frac{1}{G(r)^{p_2}} \int_{B_r(x)} G(x, y)^{p_2} \mu(dy) \right)^{1-\frac{p_1}{p_2}} \\
= \frac{1}{G(r)^{p_2-p_1}} \int_{B_r(x)} G(x, y)^{p_2} \mu(dy), \quad r \in ]0, e^{-1}[,
$$

which implies $\nu \in D_{v,\beta}^{p_2}$. Since $\lim_{r \to 0} 1/G(r) = 0$, we obtain the inclusion $K_{v,\beta}^{p_2} \subset K_{v,\beta}^{p_1}$. \qed

Definition 2.8 (Measures of finite energy integrals, $S_0, S_{00}$; cf. [15]) A Radon measure $\mu$ on $E$ is said to be of finite energy integral with respect to $(E, F)$ (write $\mu \in S_0$) if there exists $C > 0$ such that

$$
\int_E |v|d\mu \leq C\sqrt{\mathcal{E}_1(v, v)}, \quad \text{for any} \quad v \in F \cap C_0(E).
$$

In that case, for every $\alpha > 0$, there exists $U_\alpha \mu \in F$ such that

$$
\mathcal{E}_\alpha(U_\alpha \mu, v) = \int_E v(x)\mu(dx), \quad \text{for any} \quad v \in F \cap C_0(E).
$$

Moreover we write $\mu \in S_{00}$ if $\mu(E) < +\infty$ and $U_\alpha \mu \in F \cap L^\infty(E; m)$ for some/all $\alpha > 0$.

Definition 2.9 (Smooth measures, $S$; cf. [15]) A Borel measure $\mu$ on $E$ is said to be a smooth measure with respect to $(E, F)$ (write $\mu \in S$) if $\mu$ charges no exceptional set and there exists a generalized nest $\{F_n\}$ of closed sets such that $\mu(F_n) < +\infty$ for each $n \in \mathbb{N}$.

Definition 2.10 (Smooth measures in the strict sense, $S_1$; cf. [15]) A Borel measure $\mu$ on $E$ is said to be a smooth measure in the strict sense with respect to $(E, F)$ (write $\mu \in S_1$) if there exists an increasing sequence $\{E_n\}$ of Borel sets such that $E = \bigcup_{n=1}^\infty E_n$, and for any $n \in \mathbb{N}$, $1_{E_n} \mu \in S_00$ and $P_x(\lim_{n \to \infty} \sigma_{E \setminus E_n} \geq \xi) = 1$ for any $x \in E$.

Remark 2.11 It is shown in [30, Proposition 2.6] that $S_K^p \subset S_D^p \subset S_1$. 

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3 Main theorems

Now we are ready to state the main theorems. Our first main theorem is a complete extension of Theorem 1.1 and [28, Theorem 3.2]. This is the most important theorem in this paper.

**Theorem 3.1** Let \( p \in [1, +\infty] \), Suppose that (A1), (A2) and (A3) hold. Then we have \( S_K^p = K_{v,\beta}^p \) and \( S_D^p = D_{v,\beta}^p \). Moreover, \( \mu \in K_{v,\beta}^p \) implies that

\[
\sup_{x \in E} \mu(B_R(x)) < +\infty \quad \text{for all} \quad R > 0.
\]  \quad (3.1)

For \( v < \beta \), we have \( S_D^p = K_{v,\beta}^p = S_D = K_{v,\beta} \) and \( \mu \in K_{v,\beta}^p \) is equivalent to (3.1).

Our second main theorem gives a criterion for \( L^p \)-Kato and \( L^p \)-Dynkin class measures based on the decay rate of the measures of balls.

**Theorem 3.2** Let \( \mu \) be a Radon measure, \( p \in [1, +\infty] \) and \( \eta \in ]0, v] \). Suppose that (A2) and (A3) hold.

1. If there exist constants \( r_0, C_2 > 0 \) such that \( \mu(B_r(x)) \leq C_2 r^n \) for any \( x \in E \) and \( r \in ]0, r_0] \) and \( \eta - p(v - \beta) > 0 \) holds, then \( \mu \in K_{v,\beta}^p \).
2. If there exist \( x_0 \in E \) and constants \( r_0, C_1 > 0 \) such that \( C_1 r^n \leq \mu(B_r(x_0)) \) for any \( r \in ]0, r_0] \) and \( \mu \in D_{v,\beta}^p \) holds, then \( \eta - p(v - \beta) \geq 0 \).
3. If there exist \( x_0 \in E \) and constants \( r_0, C_1, C_2 > 0 \) such that \( C_1 r^n \leq \mu(B_r(x_0)) \leq C_2 r^n \) for any \( r \in ]0, r_0] \) and \( \mu \in D_{v,\beta}^p \) holds, then \( \eta - p(v - \beta) > 0 \).

In particular, if \( \mu \) satisfies the Ahlfors regularity, i.e., \( C_1 r^n \leq \mu(B_r(x)) \leq C_2 r^n \) for all \( x \in E \) and \( r \in ]0, r_0] \) with some \( r_0, C_1, C_2 > 0 \), then the following are equivalent:

1. \( \mu \in K_{v,\beta}^p \).
2. \( \mu \in D_{v,\beta}^p \).
3. \( \eta - p(v - \beta) > 0 \).

**Corollary 3.3** Let \( p \in [1, +\infty] \). Suppose that (A1), (A2) and (A3) hold. Then the following are equivalent:

1. \( m \in K_{v,\beta}^p = S_K^p \).
2. \( m \in D_{v,\beta}^p = S_D^p \).
3. \( \eta - p(v - \beta) > 0 \).

4 Criteria for \( L^p \)-Kato and \( L^p \)-Dynkin classes

In this section, we give other criteria for \( L^p \)-Kato and \( L^p \)-Dynkin classes.

**Theorem 4.1** Let \( p \in [1, +\infty] \). Suppose that (A3) and \( v \geq \beta \) hold. Then the following are equivalent:

1. \( \mu \in K_{v,\beta}^p \).
2. For any \( \alpha > 0 \), \( \lim_{r \to 0} \sup_{x \in E} \int_{B_r(x)} r_\alpha(x, y)^p \mu(dy) = 0 \).
3. For some \( \alpha > 0 \), \( \lim_{r \to 0} \sup_{x \in E} \int_{B_r(x)} r_\alpha(x, y)^p \mu(dy) = 0 \).
4. For any \( t > 0 \), \( \lim_{r \to 0} \sup_{x \in E} \left( \int_0^t p_s(x, y)ds \right)^p \mu(dy) = 0 \).
(5) For some $t > 0$, \( \lim_{r \to 0} \sup_{x \in E} \int_{B_r(x)} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) = 0. \)

Moreover, the following are equivalent:

(1') \( \mu \in D_{v, \beta}^p \)

(2') For any \( \alpha > 0 \), \( \sup_{x \in E} \int_{B_r(x)} r_\alpha(x, y)^p \mu(dy) < +\infty \) for some \( r > 0 \).

(3') For some \( \alpha > 0 \), \( \sup_{x \in E} \int_{B_r(x)} r_\alpha(x, y)^p \mu(dy) < +\infty \) for some \( r > 0 \).

(4') For any \( t > 0 \), \( \sup_{x \in E} \int_{B_r(x)} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) < +\infty \) for some \( r > 0 \).

(5') For some \( t > 0 \), \( \sup_{x \in E} \int_{B_r(x)} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) < +\infty \) for some \( r > 0 \).

Remark 4.2 Theorem 4.1 is a complete extension of [28, Theorem 3.1]. The equivalence (1)–(5) in Theorem 4.1 does not hold for \( \nu < \beta \) in general. In fact, for 1-dimensional Brownian motion \( X^w \), we see that \( \mu = \delta_0 \in K_1 = K_1^p \) does not satisfy the conditions (2), (3) in Theorem 4.1 because of \( r_\alpha(x, y) = e^{-r_2|x-y|}/\sqrt{2\alpha} \).

Next theorem is a generalization of Theorem 1.4 (i.e., [1, Propositions 4.1 and 4.2]), which does not treat the case \( q = 1 \).

Theorem 4.3 Let \( p \in [1, +\infty[ \) with \( \nu - p(\nu - \beta) > 0 \) and fix \( \alpha > 0 \). Suppose that (A2) and assume \( \nu \geq \alpha \) and \( q \geq 1 \), or \( \nu < \alpha \) and \( q > \alpha/\nu \). Then for any \( f \in M_{\alpha, q} \), we have \( |f| dm \in K_{\nu, \beta}^p \) if \( q > \alpha/(\nu - p(\nu - \beta)) \), where \( f \in M_{\alpha, q} \) means

\[
\sup_{x \in E} \int_{d(x, y) \leq 1} \frac{|f(y)|^q}{d(x, y)^{\nu-\alpha}} m(dy) < +\infty.
\]

By setting \( \nu = \alpha \) in Theorem 4.3, we have the following corollary, which is a complete extension of Theorem 1.3 and [28, Theorem 3.3].

Corollary 4.4 Let \( p \in [1, +\infty[ \) with \( \nu - p(\nu - \beta) > 0 \). Suppose that (A2) holds. Then for any \( f \in L^q_{\text{unif}}(E; m) \), we have \( |f| dm \in K_{\nu, \beta}^p \) if \( q > \nu/(\nu - p(\nu - \beta)) \) with \( \nu \geq \beta \), or if \( q \geq 1 \) with \( \nu < \beta \). Here \( f \in L^q_{\text{unif}}(E; m) \) means

\[
\sup_{x \in E} \int_{d(x, y) \leq 1} |f(y)|^q m(dy) < +\infty.
\]

The following is an easy consequence of Corollary 4.4.

Corollary 4.5 Let \( p \in [1, +\infty[ \) with \( \nu - p(\nu - \beta) > 0 \). Suppose that (A2) holds. For any fixed point \( o \in E \), we have \( d(\cdot, o)^{-\gamma} m \in K_{\nu, \beta}^p \) if \( \gamma \in [0, \nu - p(\nu - \beta)] \) with \( \nu \geq \beta \), or if \( \gamma \in [0, 1] \) with \( \nu < \beta \). In particular, \( m \in K_{\nu, \beta}^p \) always holds.

Finally, we give the following theorem without assuming the lower estimate of the heat kernel.
Theorem 4.6  Let $p \in [1, +\infty]$. Assume that there exist constants $C, r_0 > 0$ and $\eta \in ]0, v]$ satisfying $\eta - p(v - \beta) > 0$ such that $\mu(B_r(x)) \leq Cr^n$ for any $x \in E$ and $r \in ]0, r_0]$. Suppose that (A3) holds without assuming the lower estimate of $(\Phi_{E,v,\beta})$. More precisely, there exists a positive decreasing function $\Phi_2$ on $]0, +\infty[$ which may depend on $t_0$ if $t_0 < +\infty$ and $\Phi_2$ satisfies

$$
\int_1^{+\infty} t^{v-1} \Phi_2(t) dt < +\infty
$$

and for any $x, y \in E$ and $t \in ]0, t_0[$

$$
p_t(x, y) \leq \frac{1}{t^{v/\beta}} \Phi_2\left(\frac{d(x, y)}{t^{1/\beta}}\right).
$$

Then we have the following:

1. Suppose $\mu(E) < +\infty$. Then $\mu \in S_{K}^{p,\delta}$ for any $\delta \in ]0, \frac{\eta - p(v - \beta)}{p\beta}[$.

2. Suppose $p > 1$ and $\mu \in S_{D}$. Then $\mu \in S_{K}^{p,\delta}$ holds for any $\delta \in ]0, \frac{\eta - p(v - \beta)}{p\beta}[$ (resp. $\delta \in ]0, (p - 1) \cdot \frac{\eta - p(v - \beta)}{p\beta}[$) under $p \in [\frac{\eta + \beta}{v}, \frac{\eta}{v - \beta}[$ (resp. $p \in [1, \frac{\eta}{v}[$).

Here $\frac{\eta}{v - \beta}_+ := \frac{\eta}{v - \beta}$ for $v > \beta$, and $\frac{\eta}{v - \beta}_+ := +\infty$ for $v \leq \beta$.

3. Suppose $p > 1$ and $\mu \in S_{D}$. Assume further that there exists $C > 0$ such that $\sup_{x \in E} \mu(B_r(x)) \leq Cr^n$ holds for all $r \in ]0, +\infty[$, or

$$
\int_1^{+\infty} u^{v+\gamma-1} \Phi_2(u) du < +\infty
$$

holds for any $\gamma > 0$. Then $\mu \in S_{K}^{p,\delta}$ for any $\delta \in ]0, \frac{\eta - p(v - \beta)}{p\beta}[$.

5 Proofs of theorems and corollaries

5.1 Proof of Theorem 4.1

To prove Theorem 4.1, we begin with auxiliary lemmas.

Lemma 5.1 ([28, Lemma 4.1]) Under (A3), there exists $C'_{v,\beta,t_0} > 0$ such that for any $t \in ]0, t_0[$ ($t \in ]0, +\infty[$ if $t_0 = +\infty$)

1. for $v < \beta$ and $x, y \in E$ with $d(x, y)^\beta < t$, we have

$$
\int_0^t p_s(x, y) ds \geq C'_{v,\beta,t_0} t^{1-\frac{v}{\beta}},
$$

2. for $v = \beta$ and $x, y \in E$ with $d(x, y)^{\beta/2} < t < 1/2$, we have

$$
\int_0^t p_s(x, y) ds \geq C'_{v,\beta,t_0} \log(d(x, y)^{-1}),
$$
(3) for $\nu > \beta$ and $x, y \in E$ with $d(x, y)^{\beta} < t$,
\[
\int_0^t p_s(x, y)ds \geq C'_{v,\beta,t_0}d(x, y)^{\beta - \nu}.
\]

**Lemma 5.2** ([28, Lemma 4.2]) Under (A3), for any $t \in [0, +\infty[$, there exists $C'_{v,\beta,t_0,t} > 0$ such that for $x, y \in E$ with $d(x, y)^{\beta} < t$, we have
\[
\int_0^t p_s(x, y)ds \geq C'_{v,\beta,t_0,t}.
\]

**Lemma 5.3** ([28, Lemma 4.3]) Under (A3), there exists $C_{v,\beta,t_0} > 0$ such that for any $t \in [0, t_0]$ ($t \in [0, +\infty[$ if $t_0 = +\infty$)

1. for $\nu < \beta$ and $x, y \in E$, we have
\[
\int_0^t p_s(x, y)ds \leq C_{v,\beta,t_0}t^{1-v/\beta},
\]

2. for $\nu = \beta$ and $x, y \in E$ with $d(x, y)^{\beta} \vee t < 1/2$, we have
\[
\int_0^t p_s(x, y)ds \leq C_{v,\beta,t_0}\log(d(x, y)^{-1}),
\]

3. for $\nu > \beta$ and $x, y \in E$, we have
\[
\int_0^t p_s(x, y)ds \leq C_{v,\beta,t_0}d(x, y)^{\beta - \nu}.
\]

**Proof of Theorem 4.1** The implications $(2) \implies (3) \implies (4) \implies (5)$ and $(2') \implies (3') \implies (4') \implies (5')$ are trivial in view of the estimate
\[
\int_0^t p_s(x, y)ds \leq e^{at}r_\alpha(x, y).
\]

First we show $(5) \implies (1)$. Suppose $(5)$. Then
\[
\lim_{r \to 0} \sup_{x \in E} \int_{B_r(x)} \left(\int_0^{t_1} p_s(x, y)ds\right)^p \mu(dy) = 0
\]
holds for some $t_1 > 0$. We may assume $t_0 < 1/2 \wedge t_1$. By Lemma 5.1, we see that for $d(x, y) < r < t_0^{2/\beta}/2$
\[
C'_{v,\beta,t_0} G(x, y) \leq \int_0^{t_0} p_s(x, y)ds \leq \int_0^{t_1} p_s(x, y)ds.
\]
Then
\[
\left( C'_{\nu, \beta, t_0} \right)^p \limsup_{r \to 0} \int_{B_r(x)} G(x, y)^p \mu(\mathrm{d}y)
\leq \limsup_{r \to 0} \int_{B_r(x)} \left( \int_0^{t_1} p_s(x, y) \mathrm{d}s \right)^p \mu(\mathrm{d}y) = 0.
\]

Thus we have (1). The proof of (5')\(\Rightarrow\) (1') is similar. Next we show (1)\(\Rightarrow\) (2). Owing to the estimate in the proof of Lemma 5.3(3), for \(\nu > \beta\) we have

\[
r_{\alpha}(x, y) = \sum_{k=0}^{\infty} \int_{k_0}^{(k+1)l_0} e^{-\alpha s} p_s(x, y) \mathrm{d}s
\leq \sum_{k=0}^{\infty} e^{-\alpha k_0} \int_0^{t_0} p_{s+k_0}(x, y) \mathrm{d}s
= \sum_{k=0}^{\infty} e^{-\alpha k_0} \int_0^{t_0} \int_E p_{k_0}(x, z) p_s(z, y) m(\mathrm{d}z) \mathrm{d}s
\leq \sum_{k=0}^{\infty} e^{-\alpha k_0} \int_E p_{k_0}(x, z) \frac{M_{\nu, \beta}}{d(z, y)^{\nu-\beta}} m(\mathrm{d}z),
\]

where \(M_{\nu, \beta} := \beta \int_0^\infty u^{\nu-\beta-1} \Phi_2(u) \mathrm{d}u < +\infty\). Hence

\[
\left( \int_{B_r(x)} r_{\alpha}(x, y)^p \mu(\mathrm{d}y) \right)^{\frac{1}{p}}
\leq \sum_{k=0}^{\infty} e^{-\alpha k_0} M_{\nu, \beta} \left( \int_{B_r(x)} \left( \int_E p_{k_0}(x, z) \cdot \frac{m(\mathrm{d}z)}{d(z, y)^{\nu-\beta}} \right)^p \mu(\mathrm{d}y) \right)^{\frac{1}{p}}
\leq \sum_{k=0}^{\infty} e^{-\alpha k_0} M_{\nu, \beta} \left\{ \left( \int_{B_r(x)} \left( \int_{B_{2r}(x)} p_{k_0}(x, z) \cdot \frac{m(\mathrm{d}z)}{d(z, y)^{\nu-\beta}} \right)^p \mu(\mathrm{d}y) \right)^{\frac{1}{p}}
+ \left( \int_{B_r(x)} \left( \int_{B_{2r}(x)^c} p_{k_0}(x, z) \cdot \frac{m(\mathrm{d}z)}{d(z, y)^{\nu-\beta}} \right)^p \mu(\mathrm{d}y) \right)^{\frac{1}{p}} \right\}.
\]

The first term in the curly brackets of the right-hand side is bounded from above by

\[
\left( \int_{B_{2r}(x)} \int_{B_{2r}(z)} p_{k_0}(x, z) \frac{m(\mathrm{d}z)}{d(z, y)^{\nu-\beta}} \mu(\mathrm{d}y) \right)^{\frac{1}{p}}
\leq \left( \int_{B_{2r}(x)} \int_{B_{3r}(z)} p_{k_0}(x, z) \frac{\mu(\mathrm{d}y)}{d(z, y)^{\nu-\beta}} m(\mathrm{d}z) \right)^{\frac{1}{p}}.
\[ \left( \sup_{z \in E \int_{B_r(z)} d(z, y)^p} \frac{\mu(dy)}{d(z, y)^p(v-\beta)} \right)^{\frac{1}{p}} \]

and the second term is bounded from above by

\[ \left( \int_{B_r(x)} \frac{1}{r^{p(v-\beta)}} \left( \int_{B_r(y)^c} p_{k0}(x, z) m(dz) \right)^p \mu(dy) \right)^{\frac{1}{p}} \leq \left( \int_{B_r(x)} \frac{\mu(dy)}{d(x, y)^p(v-\beta)} \right)^{\frac{1}{p}}. \]

Thus, for \( v > \beta \)

\[ \int_{B_r(x)} r_\alpha(x, y)^p \mu(dy) \leq \left( \frac{2M \nu, \beta}{1 - e^{-\alpha t_0}} \right)^p \sup_{x \in E \int_{B_r(x)}} G(x, y)^p \mu(dy). \]  \hspace{1cm} (5.1)

Suppose \( v = \beta \). We can see that for \( d(y, z) \geq t_0^{1/\beta} \),

\[ \beta \int_{d(z, y)/t_0^{1/\beta}}^{\infty} u^{-1} \Phi_2(u) du \leq \beta \int_1^{\infty} u^{-1} \Phi_2(u) du \leq \beta \int_1^{\infty} u^{v-1} \Phi_2(u) du \]

and for \( d(y, z) < t_0^{1/\beta} (< 1) \),

\[ \beta \int_{d(z, y)/t_0^{1/\beta}}^{\infty} u^{-1} \Phi_2(u) du = \beta \int_1^{1/d(z, y)/t_0^{1/\beta}} u^{-1} \Phi_2(u) du + \beta \int_1^{\infty} u^{-1} \Phi_2(u) du \leq \beta \Phi_2(0) \log(d(y, z)^{-1}) + \beta \int_1^{\infty} u^{v-1} \Phi_2(u) du. \]

Then we have that

\[ r_\alpha(x, y) = \sum_{k=0}^{\infty} e^{-\alpha k t_0} \int_E p_{k0}(x, z) \left( \int_0^{t_0} p_s(z, y) ds \right) m(dz) \]

\[ \leq \sum_{k=0}^{\infty} e^{-\alpha k t_0} \int_E p_{k0}(x, z) \left( \beta \int_{d(z, y)/t_0^{1/\beta}}^{\infty} u^{-1} \Phi_2(u) du \right) m(dz) \]

\[ \leq \sum_{k=0}^{\infty} e^{-\alpha k t_0} \left( 2 \beta \int_1^{\infty} u^{v-1} \Phi_2(u) du \right) \]

\[ + \beta \Phi_2(0) \int_{d(y, z) < t_0^{1/\beta}} p_{k0}(x, z) \log(d(y, z)^{-1}) m(dz) \].

In the same way to obtain (5.1), we have that

\[ \int_{B_r(x)} \left( \beta \Phi_2(0) \int_{d(y, z) < t_0^{1/\beta}} p_{k0}(x, z) \log(d(y, z)^{-1}) m(dz) \right)^p \mu(dy) \]
\[
\leq (2 \beta \Phi_2(0))^p \sup_{x \in E} \int_{B_\epsilon(x)} G(x, y)^p \mu(dy).
\]

Hence, for \( \nu = \beta \) with \( r < e^{-1} \)
\[
\int_{B_r(x)} r^\alpha(x, y)^p \mu(dy) \leq \left( \frac{2C_1 + 2C_2}{1 - e^{-\alpha t_0}} \right)^p \sup_{x \in E} \int_{B_\epsilon(x)} G(x, y)^p \mu(dy),
\]
where \( C_1 := \beta \int_1^\infty u^{\nu-1} \Phi_2(u) du \) and \( C_2 := \beta \Phi_2(0) \). Here we use that for \( e < r^{-1} \)
\[
\mu(B_r(x)) \leq (\log r^{-1})^p \mu(B_r(x)) \leq \int_{B_r(x)} (\log d(x, y)^{-1})^p \mu(dy).
\]

Therefore we obtain that for a constant \( D_{\nu, \beta, \alpha, t_0} > 0 \)
\[
\limsup_{r \to 0} \sup_{x \in E} \int_{B_r(x)} r^\alpha(x, y)^p \mu(dy) \leq D_{\nu, \beta, \alpha, t_0} \limsup_{r \to 0} \sup_{x \in E} \int_{B_r(x)} G(x, y)^p \mu(dy) = 0,
\]
which implies the desired assertion. The proof of (1')\(\Rightarrow\)(2') is also similar. \( \square \)

### 5.2 Proof of Theorem 3.1

To prove Theorem 3.1, we begin with auxiliary lemmas.

**Lemma 5.4** Under (A3), we have \( S^p_D \subset D^p_{\nu, \beta} \) and \( S^p_K \subset K^p_{\nu, \beta} \). Moreover, \( \mu \in S^p_D \) implies \( \sup_{x \in E} \mu(B_R(x)) < +\infty \) for all \( R > 0 \). In particular, \( S^p_D \subset K^p_{\nu, \beta} \) if \( \nu < \beta \).

**Proof** Take \( \mu \in S^p_D \). By Lemma 5.1, for \( \nu > \beta \) (resp. \( \nu = \beta \)) with \( r := t^{1/\beta} \) (resp. \( r := t^{2/\beta} \)), we have
\[
\int_E \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \geq \left( C'_{\nu, \beta, t_0} \right)^p \int_{d(x, y) < r} G(x, y)^p \mu(dy).
\]
Then we see \( \mu \in D^p_{\nu, \beta} \). Hence, for \( \mu \in S^p_K \), we have
\[
\limsup_{r \downarrow 0} \sup_{x \in E} \int_{d(x, y) < r} G(x, y)^p \mu(dy) \leq \frac{1}{\left( C'_{\nu, \beta, t_0} \right)^p} \limsup_{t \downarrow 0} \int_E \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) = 0.
\]
Take \( \mu \in S^p_D \). By using Lemma 5.2,
\[
\int_E \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \geq \int_{d(x, y) < t^{1/\beta}} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \geq \left( C'_{v, \beta, t_0, t} \right)^p \int_{d(x, y) < t^{1/\beta}} \mu(dy).
\]
So it suffices to apply Lemma 2.3 with \( t = R^\beta \).

**Lemma 5.5** Under (A3), we have \( D^p_{v, \beta} \subset S_1 \).

**Proof** Recall that every \( \mu \in D^p_{v, \beta} \) is a positive Radon measure on \( E \), that is, \( \mu(K) < +\infty \) for each compact set \( K \). It suffices to show that for each compact set \( K, \mu \in D^p_{v, \beta} \) implies \( 1_K \mu \in S^p_D \). In fact, \( 1_K \mu \in S^p_D \) implies \( 1_K \mu \in S_0 \), hence \( \mu \in S_1 \) by [15, Theorem 5.1.7(iii)].

By Lemma 5.3(1), for \( v < \beta \), we have
\[
\int_K \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \leq \left( C_{v, \beta, t_0} \right)^p \mu(K) t^{(1-v/\beta)}.
\]
Then we obtain \( 1_K \mu \in S^p_D \) in this case. For \( v \geq \beta \) with \( t = r^\beta < 1/2 \) and \( r < 1/e \), we have from Lemma 5.3(2),(3),
\[
\int_{K \cap B_r(x)} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \leq \left( C_{v, \beta, t_0} \right)^p \int_{d(x, y) < r} G(x, y)^p \mu(dy)
\]
and
\[
\int_{K \cap B_r(x)^c} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \leq \left( \beta r^\beta \int_{r/1^{1/\beta}}^{\infty} u^{v-1} \Phi_2(u) du \right)^p \mu(K)
\]
\[
= \left( \beta r^\beta \int_{1}^{\infty} u^{v-1} \Phi_2(u) du \right)^p \mu(K) < +\infty.
\]
Thus there exists \( t \in (0, t_0 \wedge 1/2) \) such that
\[
\sup_{x \in E} \int_E \left( \int_0^t p_s(x, y) ds \right)^p 1_K(y) \mu(dy) < +\infty,
\]
which implies \( 1_K \mu \in S^p_D \).

**Proof of Theorem 3.1** The assertion \( K^{p,v}_{v, \beta} = S^p_K \) of Theorem 3.1 is proved under \( p = 1 \) by [28, Theorem 3.2]. That is, we already know \( K^{p,v}_{v, \beta} = S_K \subset S_D \) under (A1), (A2) and (A3). We do not prove the coincidence \( D^p_{v, \beta} = S^p_D \) under (A1), (A2) and (A3) in [28, Theorem 3.2]. But the method of the proof of [28, Theorem 3.2] still works to
prove $\nu, \beta = S_D$. We omit the details but note that its proof can be achieved by use of the estimates

$$\sup_{x \in E} \mathbb{E}_x[A_{x(B_t(s))}^\mu] \leq \begin{cases} C_{\nu, \beta, t_0} \sup_{x \in E} \int_{B_r(x)} G(x, y) \mu(dy), & \nu \geq \beta, \\ C_{\nu, \beta, t_0} \sup_{x \in E} \mu(B_1(x)), & \nu < \beta \end{cases}$$

for $s \in [0, t_0 \wedge 1/2]$ and

$$\sup_{x \in E} \mathbb{P}_x(T_k < s_0) < (\gamma + \epsilon)^k.$$

Here $A^\mu$ is the positive continuous additive functional associated to $\mu$, $\gamma$ is the constant appeared in (A1) satisfying $\gamma + \epsilon < 1$ for some $\epsilon > 0$, and $(T_k)_{k \geq 0}$ is the sequence of stopping times defined by $T_0 = 0, T_{k+1} = T_k + (\tau_{B_{\epsilon}(x_0) \wedge s}) \circ \theta_{T_k}$ (see [28, pp. 102]).

By Lemma 2.7, we have $D_{\nu, \beta} \subset D_{\nu, \beta} = S_D$ and $K_{\nu, \beta} \subset K_{\nu, \beta} = S_K$. From Lemma 5.4, we also have $S_{\nu, \beta}^D \subset D_{\nu, \beta}$ and $S_{\nu, \beta}^K \subset K_{\nu, \beta}$. So it suffices to show $D_{\nu, \beta} \subset S_D$ and $K_{\nu, \beta} \subset S_K$. Take $\mu \in K_{\nu, \beta}$ and fix $r > 0$. Note that for $d(x, y) \geq r^{1/\beta}$,

$$\int_0^t p_s(x, y) ds \leq \int_{(r/t)^{1/\beta}}^{\infty} u^{-1} \Phi_2(u) du \longrightarrow 0 \quad \text{as} \quad t \rightarrow 0,$$

because $u^{-1} \Phi_2(u)$ is integrable on $[1, +\infty)$ under the condition (A2). By combining this with the fact $\mu \in K_{\nu, \beta} \subset S_D$ as noted above, we have

$$\sup_{x \in E} \int_{d(x, y) \geq r^{1/\beta}} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy)$$

$$\leq \left( \int_{(r/t)^{1/\beta}}^{\infty} u^{-1} \Phi_2(u) du \right)^{p-1} \sup_{x \in E} \int E \left( \int_0^t p_s(x, y) ds \right) \mu(dy) \longrightarrow 0 \quad \text{as} \quad t \rightarrow 0.$$

Hence,

$$\lim_{t \rightarrow 0} \sup_{x \in E} \int E \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy)$$

$$\leq \lim_{t \rightarrow 0} \sup_{x \in E} \int_{d(x, y) < r^{1/\beta}} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy)$$

$$+ \lim_{t \rightarrow 0} \sup_{x \in E} \int_{d(x, y) \geq r^{1/\beta}} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy)$$

$$\leq \sup_{x \in E} \int_{d(x, y) < r^{1/\beta}} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy),$$

and the right-hand side of the above inequality goes to zero as $r \rightarrow 0$ by Theorem 4.1, which concludes $\mu \in S_K^P$. The proof of $\mu \in D_{\nu, \beta} \subset S_D \implies \mu \in S_D^P$ is similar.
Therefore we obtain $S^p_D = D^p_{v, \beta}$ and $S^p_K = K^p_{v, \beta}$ under (A1), (A2) and (A3). Finally, we prove the rest assertions. As proved in Lemma 5.4 (see [28, Lemma 4.4]), every $\mu \in S^p_D$ satisfies (3.1) under (A3). The same holds for $\mu \in K^p_{v, \beta}$ under (A1), (A2) and (A3). When $\nu < \beta$, we have $K^p_{v, \beta} = K_{v, \beta} = S_D = S_K = S_K^p = S_D^p$ in view of Lemma 5.4 under the above assumptions. In this case, any Radon measure $\mu$ satisfying (3.1) always belongs to $K^p_{v, \beta}$.

5.3 Proofs of Theorem 4.3, Corollaries 4.4 and 4.5

**Proof of Theorem 4.3** Suppose $v - p(v - \beta) > 0$. When $\nu \geq \beta$, take $\gamma > 0$ such that $\alpha / q < \gamma < v - p(v - \beta)$. Then the conclusion immediately follows from Lemmas 5.6 and 5.7 stated below.

It remains to show the assertion in the case $v < \beta$. First we assume $v \geq \alpha$. When $q \in ]1, +\infty[$, Hölder’s inequality gives that

$$
\sup_{x \in E} \int_{d(x, y) \leq 1} |f(y)|m(dy)
\leq \left( \sup_{x \in E} \int_{d(x, y) \leq 1} d(x, y)^{\frac{\nu - q}{q - 1}} m(dy) \right)^{\frac{q - 1}{q}} \left( \sup_{x \in E} \int_{d(x, y) \leq 1} \frac{|f(y)|^q}{d(x, y)^{\nu - \alpha}} m(dy) \right)^{\frac{1}{q}}
$$

When $q = 1$, we have

$$
\sup_{x \in E} \int_{d(x, y) \leq 1} |f(y)|m(dy) = \sup_{x \in E} \int_{d(x, y) \leq 1} \frac{|f(y)|}{d(x, y)^{\nu - \alpha}} d(x, y)^{\nu - \alpha} m(dy)
\leq \sup_{x \in E} \int_{d(x, y) \leq 1} \frac{|f(y)|}{d(x, y)^{\nu - \alpha}} m(dy) < +\infty.
$$

Next we assume $v < \alpha$. For $q \in ]\alpha / v, +\infty[$, Hölder’s inequality gives that

$$
\sup_{x \in E} \int_{d(x, y) \leq 1} |f(y)|m(dy)
\leq \left( \sup_{x \in E} \int_{d(x, y) \leq 1} d(x, y)^{\frac{v - q}{q - 1}} m(dy) \right)^{\frac{q - 1}{q}} \left( \sup_{x \in E} \int_{d(x, y) \leq 1} \frac{|f(y)|^q}{d(x, y)^{\nu - \alpha}} m(dy) \right)^{\frac{1}{q}}.
$$

The quantity in the first parentheses of the right-hand side is bounded from above by

$$
\sup_{x \in E} \sum_{k=0}^{\infty} \int_{\frac{1}{2} \leq d(x, y) < \frac{1}{2^k}} d(x, y)^{\frac{v - q}{q - 1}} m(dy)
\leq C \sum_{k=0}^{\infty} (2^{k+1})^{\frac{q - 1}{q - 1}} 2^{-kv} = C \cdot 2^{\frac{q - 1}{q - 1}} \sum_{k=0}^{\infty} 2^{-k \frac{q - 1}{q - 1}} < +\infty.
$$
Hence $|f| \, dm \in K^{p}_{v, \beta}$. \hfill \Box

The following lemmas are extensions of [1, Proposition 4.1] and the inclusion below [1, Proposition 4.1], in which the results are obtained for the case of $p = 1$ and for a Brownian motion on $\mathbb{R}^d$ with $d \geq 3$:

**Lemma 5.6** Assume (A2) and let $\alpha, \gamma > 0$ and $q \geq 1$ with $\alpha < \gamma q$. Then it holds that $M_{\alpha, q} \subseteq M_{\gamma, 1}$.

**Proof** The conclusion is trivial when $q = 1$. Suppose $q > 1$ and set $a := v + \frac{q}{q-1}(\gamma - v + \frac{\nu - \alpha}{q}) > 0$. Consider the integral $\int_{d(x,y) \leq 1} d(x,y)^{a-v}m(dy)$. Note that, by (A2) there exists $C > 0$ such that $\sup_{x \in E} m(B_r(x)) \leq V(r) \leq Cr^{\nu}$ for any $r \in [0, 2]$. When $a \geq v$, the integral is bounded from above by $m(B_1(x)) \leq V(2) < +\infty$. When $a < v$, the integral is bounded from above by

$$
\sum_{k=0}^{\infty} \int_{2^{-(k+1)} < d(x,y) \leq 2^{-k}} d(x,y)^{a-v}m(dy) \leq \sum_{k=0}^{\infty} 2^{-(k+1)(a-v)}m(B_{2^{-k}}(x)) \leq C2^{-(a-v)} \sum_{k=0}^{\infty} 2^{-ak} < +\infty.
$$

Now, let $f \in M_{\alpha, q}$. By Hölder’s inequality we have

$$
\int_{d(x,y) \leq 1} \left| \frac{f(y)}{d(x,y)^{\nu-\gamma}} m(dy) \right| d(x,y)^{\nu-\gamma}m(dy) = \int_{d(x,y) \leq 1} \frac{|f(y)|}{d(x,y)^{\nu-\gamma}} d(x,y)^{\nu-\gamma+\frac{\nu-\alpha}{q}}m(dy)
\leq \left( \int_{d(x,y) \leq 1} \frac{|f(y)|^{q}}{d(x,y)^{\nu-\alpha}} m(dy) \right)^{\frac{1}{q}} \left( \int_{d(x,y) \leq 1} d(x,y)^{a-v}m(dy) \right)^{\frac{q-1}{q}},
$$

which concludes $f \in M_{\gamma, 1}$. \hfill \Box

**Lemma 5.7** Let $p \in [1, +\infty]$ and $\nu \geq \beta$ with $v - p(v - \beta) > 0$. For any $f \in M_{\gamma, 1}$ ($\gamma > 0$), we have $|f| \, dm \in K^{p}_{v, \beta}$ if $\gamma < v - p(v - \beta)$.

**Proof** Take $f \in M_{\gamma, 1}$. When $v > \beta$, we have for $r < 1$

$$
\int_{B_r(x)} \frac{|f(y)|}{d(x,y)^{p(v-\beta)}} m(dy) = \int_{B_r(x)} \frac{|f(y)|}{d(x,y)^{v-\gamma}} d(x,y)^{v-\gamma-p(v-\beta)} m(dy) \leq r^{v-\gamma-p(v-\beta)} \int_{B_r(x)} \frac{|f(y)|}{d(x,y)^{v-\gamma}} m(dy)
$$

and hence $|f| \, dm \in K^{p}_{v, \beta}$. \hfill \Box

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When \( \nu = \beta \), note that the function \( r^{\nu - \gamma}(\log r^{-1})^p \) is monotonically increasing for sufficiently small \( r > 0 \) and converges to 0 as \( r \downarrow 0 \). Then we have for such \( r \),

\[
\int_{B_r(x)} (\log d(x, y)^{-1})^p |f(y)| m(dy) = \int_{B_r(x)} \frac{|f(y)|}{d(x, y)^{\nu - \gamma}} d(x, y)^{\nu - \gamma}(\log d(x, y)^{-1})^p m(dy) \leq r^{\nu - \gamma}(\log r^{-1})^p \int_{B_r(x)} \frac{|f(y)|}{d(x, y)^{\nu - \gamma}} m(dy)
\]

and hence \( |f|dm \in K_{\nu, \beta}^p \).

\[\square\]

Proof of Corollary 4.4 The assertion of Corollary 4.4 is a special case of the assertion of Theorem 4.3 by setting \( \nu = \alpha \). So there is no need to show the proof.

\[\square\]

Proof of Corollary 4.5 Suppose \( \nu - p(\nu - \beta) > 0 \). It suffices to prove that \( d(\cdot, o)^{-\gamma} \in L^q_{\text{unif}}(E; m) \) for any \( q \in ](\nu - p(\nu - \beta)), \nu/\gamma[ \) with \( \nu \geq \beta \), and for \( q \in [1, \nu/\gamma[ \) with \( \nu < \beta \). By (A2), there exists \( C > 0 \) such that \( \sup_{x \in E} m(B_r(x)) \leq V(r) \leq Cr^\nu \) for any \( r \in ]0, 3[ \).

\[
\sup_{d(x, o) \geq 2} \int_{d(x, y) \leq 1} d(y, o)^{-q \gamma} m(dy) \leq \sup_{d(x, o) \geq 2} \int_{d(x, y) \leq 1, d(y, o) \geq 1} d(y, o)^{-q \gamma} m(dy) \leq \sup_{x \in E} m(B_1(x)) < +\infty.
\]

On the other hand,

\[
\sup_{d(x, o) < 2} \int_{d(x, y) \leq 1} d(y, o)^{-q \gamma} m(dy) \leq \int_{d(y, o) < 3} d(y, o)^{-q \gamma} m(dy) \leq \sum_{k=0}^{\infty} \int_{d(y, o) < \frac{3}{2^k}} d(y, o)^{-q \gamma} m(dy) \leq 2C \cdot 3^{\nu-q \gamma} \sum_{k=0}^{\infty} \left( \frac{1}{2^{\nu-q \gamma}} \right)^k < +\infty.
\]

Therefore

\[
\sup_{x \in E} \int_{d(x, y) \leq 1} d(y, o)^{-q \gamma} m(dy) < +\infty.
\]

\[\square\]

5.4 Proofs of Theorem 3.2, Corollary 3.3, and Theorem 4.6

Proof of Theorem 3.2 First we prove (1).
(Case I) $v < \beta$: In this case, we can directly check $\mu \in K_{v, \beta}^p$ by

$$\sup_{x \in E} \mu(B_r(x)) \leq C r^\eta \quad \text{for} \quad r \in [0, r_0].$$

(Case II) $v = \beta$: In this case,

$$\int_0^\infty u^{v-(1-\delta)\beta-1} \Phi_2(u) du < +\infty \quad (5.2)$$

for any $\delta \in [0, 1]$. By the upper estimate, we have

$$\int_0^t p_s(x, y) ds \leq \frac{\beta}{d(x, y)^{v-\beta}} \int_0^\infty u^{v-\beta-1} \Phi_2(u) du$$

$$\leq \frac{t^\delta}{d(x, y)^{(1-\delta)\beta}} \left( \beta \int_0^\infty u^{v-(1-\delta)\beta-1} \Phi_2(u) du \right)$$

and hence, for $r \in [0, r_0]$ and $t \in [0, t_0]$,

$$\sup_{x \in E} \int_{d(x, y) \leq r} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy)$$

$$\leq t^{p\delta} \left( \beta \int_0^\infty u^{v-(1-\delta)\beta-1} \Phi_2(u) du \right)^p \sup_{x \in E} \int_{d(x, y) \leq r} \frac{1}{d(x, y)^{(1-\delta)\beta}} \mu(dy)$$

$$= t^{p\delta} \left( \beta \int_0^\infty u^{v-(1-\delta)\beta-1} \Phi_2(u) du \right)^p$$

$$\times \sup_{x \in E} \sum_{k=0}^\infty \int_{\frac{x}{2^{k+1}}}^{\frac{x}{2^k}} \frac{1}{d(x, y)^{(1-\delta)\beta}} \mu(dy)$$

$$\leq t^{p\delta} \left( \beta \int_0^\infty u^{v-(1-\delta)\beta-1} \Phi_2(u) du \right)^p$$

$$\times C_2 2^{p(v-\beta)+\delta p\beta} \left( \sum_{k=0}^\infty 2^{-k(\eta-p(v-\beta)-\delta p\beta)} \right) r^{\eta-p(v-\beta)-\delta p\beta}. \quad (5.3)$$

This goes to 0 as $r \to 0$ for a constant $\delta \in [0, 1]$ satisfying $\eta - p(v - \beta) > \beta p\delta$. Hence $\mu \in K_{v, \beta}^p$ by Theorem 4.1.

(Case III) $v > \beta$: In this case, (5.2) holds for any $\delta \in [0, 1]$. So (5.3) goes to 0 as $r \to 0$ for a constant $\delta \in [0, 1]$ satisfying $\eta - p(v - \beta) > \beta p\delta$. Hence $\mu \in K_{v, \beta}^p$ by Theorem 4.1.

Next we prove (2). Assume $\eta - p(v - \beta) < 0$. Then it implies $v > \beta$. Suppose that there exist $x_0 \in E$ and $r_0$, $C_1 > 0$ such that $\mu(B_r(x_0)) \geq C_1 r^\eta$ for all $r \in [0, r_0]$. Then, for any $s \in [0, +\infty[ \text{ and } t \in [0, t_0], \)
This implies $\mu \notin D_{v, \beta}^\rho$ under $\eta - p(v - \beta) < 0$ by Theorem 4.1 (1') $\iff$ (5'). Note here that $\int_0^\infty u^{-1} \Phi_1(u) \, du \leq \int_0^\infty u^{-1} \Phi_2(u) \, du < +\infty$.

Finally we prove (3). Assume $\eta - p(v - \beta) \leq 0$. Then it implies $v > \beta$. Suppose that there exist $x_0 \in E$ and $r_0, C_1, C_2 > 0$ such that $C_1 r_0^{\eta} \leq \mu(B_r(x_0)) \leq C_2 r^{\eta}$ for all $r \in [0, r_0]$. Take a large $\ell > 0$ so that $C_1 > C_2/\ell^{\eta}$. Then, for any $s \in [0, +\infty[, \ r \in [0, s \wedge r_0]$ and $t \in [0, t_0]$,

$$\sup_{x \in E} \int_{d(x, y) \leq s} \left( \int_0^t p_s(x, y) \, ds \right)^p \mu(dy) \geq \left( \beta \int_{r_0^{1/\beta}}^\infty u^{-\beta - 1} \Phi_1(u) \, du \right)^p \int_{d(x_0, y) \leq r} \frac{1}{d(x_0, y)^{p(v - \beta)}} \mu(dy)$$

$$= \left( \beta \int_{r_0^{1/\beta}}^\infty u^{-\beta - 1} \Phi_1(u) \, du \right)^p \sum_{k=0}^{\infty} \int_{r/\ell^{k+1} < d(x_0, y) \leq r/\ell^k} \frac{1}{d(x_0, y)^{p(v - \beta)}} \mu(dy)$$

$$\geq \left( \beta \int_{r_0^{1/\beta}}^\infty u^{-\beta - 1} \Phi_1(u) \, du \right)^p \left( C_1 - \frac{C_2}{\ell^{\eta}} \right) \left( \sum_{k=0}^{\infty} \ell^{-k(\eta - p(v - \beta))} \right)^p \eta - p(v - \beta)$$

$$= +\infty.$$ (5.5)

This implies $v \notin D_{v, \beta}^\rho$ under $\eta - p(v - \beta) \leq 0$ by Theorem 4.1 (1') $\iff$ (5').

**Proof of Corollary 3.3** It suffices to consider $\eta = v$. In view of [28, Corollary 4.1], if further (A1) is satisfied, then the Ahlfors regularity holds. This implies the assertion.

**Proof of Theorem 4.6** We will estimate the inner integral

$$\int_{d(x, y) \leq r} \left( \int_0^t p_s(x, y) \, ds \right)^p \mu(dy)$$

and the outer integral

$$\int_{d(x, y) > r} \left( \int_0^t p_s(x, y) \, ds \right)^p \mu(dy)$$

respectively.
(1) Suppose $\mu(E) < +\infty$. Let $0 < \delta < \frac{\eta - p(v - \beta)}{p\beta}$ (\(\leq 1\)) and fix $r \in [0, r_0]$. We have for $t < 1 \land t_0 \land r\beta$
\[
\int_0^t s^{-v/\beta} \Phi_2(r s^{-1/\beta}) ds = \beta r^{\beta - v} \int_r^{t r/\beta} u^{v - 1/\beta} \Phi_2(u) du \\
\leq \{ \beta \int_1^\infty u^{v - 1} \Phi_2(u) du \} r^{-v} t
\]
and then
\[
\sup_{x \in E} \int_{d(x, y) > r} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \\
\leq \mu(E) \left\{ \beta \int_1^\infty u^{v - 1} \Phi_2(u) du \right\} r^{-p_v} t^p
\]
As for the inner integral, take $\xi > 0$ such that
\[
\frac{\beta - v}{\beta} < \xi \quad \text{and} \quad \delta < \xi < \frac{\eta - p(v - \beta)}{p\beta}
\]
(the first assertion automatically holds when $v \geq \beta$). We note that (5.3) with replacing $\delta$ with $\xi$ still holds in the case $v \neq \beta$ because of $(\beta - v)/\beta < \xi$. Hence we have for $t < 1 \land t_0$
\[
\sup_{x \in E} \int_{d(x, y) \leq r} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \leq C r^{\eta - p(v - \beta) - p\xi} t^p
\]
\[
\leq C r^{\eta - p(v - \beta) - p\xi \beta} t^p \delta. \quad (5.6)
\]
Therefore we conclude that $\mu \in S_{K}^{p, \delta}$.

(2) Suppose $p > 1$ and $\mu \in S_D$. The inner integral estimate (5.7) still holds in this case, and it remains to calculate the outer integral. If $p \geq \frac{\eta + \beta}{v} (> 1)$ and $\delta < \frac{\eta - p(v - \beta)}{p\beta}$, we have $p\delta \leq p - 1$. If $p > 1$ and $\delta < \frac{p - 1}{p} \cdot \frac{\eta - p(v - \beta)}{p\beta}$, we also have $p\delta \leq p - 1$. Hence, for both cases, by a similar calculation as that to obtain (5.6), we have for $t < 1 \land t_0 \land r\beta$
\[
\sup_{x \in E} \int_{d(x, y) > r} \left( \int_0^t p_s(x, y) ds \right)^p \mu(dy) \\
\leq \left\{ \sup_{x \in E} \int_{E} \left( \int_0^1 p_s(x, y) ds \right) \mu(dy) \right\} \left\{ \beta \int_1^\infty u^{v - 1} \Phi_2(u) du \right\} r^{-(p - 1)} t^{p - 1}
\]
\[
\leq \left\{ \sup_{x \in E} \int_{E} \left( \int_0^1 p_s(x, y) ds \right) \mu(dy) \right\} \left\{ \beta \int_1^\infty u^{v - 1} \Phi_2(u) du \right\} r^{-(p - 1)} t^{p \delta}. \quad (5.8)
\]
Therefore we conclude that $\mu \in S_{K}^{p,\delta}$.

(3) Suppose $p > 1$ and $\mu \in S_D$ and assume that there exists $C > 0$ such that $\sup_{x \in E} \mu(B_r(x)) \leq Cr^n$ for all $r \in [0, +\infty[$. Under this condition, the calculation to obtain (5.3) remains valid for all $r \in [0, +\infty[$. Let $0 < \delta < \frac{\eta - p(v - \beta)}{p\beta} (< 1)$ and take $\xi > 0$ such that

$$\frac{\beta - v}{\beta} < \xi, \quad \frac{\xi}{\beta} < \frac{\eta - p(v - \beta)}{p\beta} \quad\text{and}\quad \frac{p}{p - 1} \delta < \frac{\eta - p(v - \beta) - p\beta \xi}{\eta - p(v - \beta) - p\beta \delta}$$

(the first assertion automatically holds when $v \geq \beta$). Set $r = t^{-\alpha}$, where $\alpha = \frac{p(\xi - \delta)}{\eta - p(v - \beta) - p\beta \xi} > 0$. Then, by the same calculation as that to obtain (5.8), we have for $t < 1$ and

$$\sup_{x \in E} \int_{d(x,y) > r} \left( \int_0^t p_s(x,y) \, ds \right)^p \mu(dy)$$

$$\leq \left\{ \sup_{x \in E} \int_{d(x,y) > r} \left( \int_0^1 p_s(x,y) \, ds \right) \mu(dy) \right\} \left\{ \beta \int_1^\infty u^{v-1} \Phi_2(u) \, du \right\}^{p-1} r^{(p-1)\beta} t^{-p\delta}$$

$$\leq \left\{ \sup_{x \in E} \int_{d(x,y) > r} \left( \int_0^1 p_s(x,y) \, ds \right) \mu(dy) \right\} \left\{ \beta \int_1^\infty u^{v-1} \Phi_2(u) \, du \right\}^{p-1} t^{p\delta},$$

where we used

$$(p - 1)(\alpha \beta + 1) = (p - 1) \frac{\eta - p(v - \beta) - p\beta \delta}{\eta - p(v - \beta) - p\beta \xi} > p\delta.$$ 

The inner integral estimate follows from the middle side of (5.7) since

$$r^{\eta - p(v - \beta) - p\xi \beta} t^{p\xi} = t^{-\alpha (\eta - p(v - \beta) - p\xi \beta)} t^{p\xi} = t^{p\delta}.$$ 

Therefore we conclude that $\mu \in S_{K}^{p,\delta}$.

We next assume that $\int_1^\infty u^{v+\gamma-1} \Phi_2(u) \, du < +\infty$ holds for any $\gamma > 0$. The inner integral estimate (5.7) still holds in this case. As for the outer integral, take $\gamma > 0$ such that $\frac{\beta + \gamma}{\beta} > \frac{p}{p-1} \delta$. Then, by a similar calculation as that to obtain (5.6), we have

$$\sup_{x \in E} \int_{d(x,y) > r} \left( \int_0^t p_s(x,y) \, ds \right)^p \mu(dy)$$

$$\leq \left\{ \sup_{x \in E} \int_{d(x,y) > r} \left( \int_0^1 p_s(x,y) \, ds \right) \mu(dy) \right\}$$

$$\times \left\{ \beta \int_1^\infty u^{v+\gamma-1} \Phi_2(u) \, du \right\}^{p-1} r^{-(p-1)(\beta + \gamma)} t^{(p-1)\frac{\beta + \gamma}{p}}$$

$$\leq \left\{ \sup_{x \in E} \int_{d(x,y) > r} \left( \int_0^1 p_s(x,y) \, ds \right) \mu(dy) \right\}.$$
\[ \times \left\{ \beta \int_1^\infty u^{p-1} \Phi_2(u) du \right\}^{p-1} r^{-(p-1)(\beta+\gamma)} t^{p\delta}. \]

Therefore we conclude that \( \mu \in S_{K}^{p,\delta}. \)

\[ \square \]

6 Examples

Example 6.1 (Brownian motions on \( \mathbb{R}^d \)) Let \( X^w = (\Omega, B_t, P_x)_{x \in \mathbb{R}^d} \) be a \( d \)-dimensional Brownian motion on \( \mathbb{R}^d \). Consider \( p \in [1, +\infty[. \) We say that \( \mu \in K^p \) (or \( \mu \in K_{p,2} \)) if and only if

\[ \lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} \frac{\mu(dy)}{|x-y|^{(d-2)p}} = 0 \quad \text{for} \quad d \geq 3, \]

\[ \lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} (\log |x-y|^{-1})^p \mu(dy) = 0 \quad \text{for} \quad d = 2, \]

\[ \sup_{x \in \mathbb{R}^d} \int_{|x-y|\leq1} \mu(dy) < +\infty \quad \text{for} \quad d = 1. \]

We write \( K^p \) instead of \( K^1_p \) for \( p = 1 \). As in Sect. 1, we have \( K^p = S^p_K \) by [30, Example 2.4] or its extension Theorem 3.1.

The \( d \)-dimensional Lebesgue measure \( m \) belongs to \( K^p \) if and only if \( p \in [1, d/(d-2)_+[, \) by Theorem 3.2 or Corollary 4.5, where \( d/(d-2)_+ := d/(d-2) \) if \( d \geq 3, d/(d-2)_+ := +\infty \) if \( d = 1, 2 \). For any non-negative bounded \( g \in L^1(\mathbb{R}^d) \) the finite measure \( g m \) also belongs to \( S^p_K \) for \( 0 < \delta < (d - p(d-2))/2p \) under \( p \in [1, d/(d-2)_+[, \) by Theorem 4.6(1). Moreover, \( m \in S^p_K \) for \( 0 < \delta < (d - p(d-2))/2p \) under \( p \in ]1, d/(d-2)_+[, \) by Theorem 4.6(3).

The surface measure \( \sigma_R \) on the \( R \)-sphere \( \partial B_R(0) \) satisfies that \( \sigma_R(B_r(x)) \leq C_2 r^{d-1} \) for any \( x \in \mathbb{R}^d \) and \( r > 0 \) with some \( C_2 > 0 \), and \( \sigma_R(B_r(x)) \geq C_1 r^{d-1} \) for any \( x \in \partial B_R(0) \) and \( r \in ]0, r_0[ \) with some \( C_1, r_0 > 0 \). Then we can conclude that \( \sigma_R \in K^p \) holds if and only if \( p \in [1, (d-1)/(d-2)_+[, \) under \( d \geq 2 \) by Theorems 3.1 and 3.2, where \( (d-1)/(d-2)_+ := (d-1)/(d-2) \) if \( d \geq 3, (d-1)/(d-2)_+ := +\infty \) if \( d = 2 \). Moreover, \( \sigma_R \in S^p_K \) holds for \( 0 < \delta < ((d-1) - p(d-2))/2p \) under \( p \in ]1, (d-1)/(d-2)_+[, \) with \( d \geq 2 \) by Theorem 4.6(1). By Corollary 4.4, we also have that \( f \in L^q(\mathbb{R}^d) \) implies \( |f|dm \in S^p_K \) if \( q > d/(d - p(d-2)) \) with \( d \geq 2, \) or \( q \geq 1 \) with \( d = 1 \).

Example 6.2 (Symmetric relativistic \( \alpha \)-stable process, symmetric \( \alpha \)-stable process) Take \( 0 < \alpha < 2 \) and \( m \geq 0 \). Let \( X = (\Omega, X_t, P_x) \) be a Lévy process on \( \mathbb{R}^d \) with

\[ E_0 \left[ e^{\sqrt{-1}(\xi, X_t)} \right] = \exp \left( -t \left\{ (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m \right\} \right). \]
If $m > 0$, it is called the relativistic $\alpha$-stable process with mass $m$ (see [8]). In particular, if $\alpha = 1$ and $m > 0$, it is called the free relativistic Hamiltonian process (see [4,9,20]). When $m = 0$, $X$ is nothing but the usual (rotationally) symmetric $\alpha$-stable process. It is known that $X$ is transient if and only if $d > 2$ under $m > 0$ or $d > \alpha$ under $m = 0$, and $X$ is a doubly Feller conservative process.

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form on $L^2(\mathbb{R}^d)$ associated with $X$. Using Fourier transform $\hat{f}(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i(x,y)} f(y) dy$, it follows from [15, Example 1.4.1] that

\[
\mathcal{F}(f, g) = \int_{\mathbb{R}^d} \hat{f}(\xi) \overline{\hat{g}}(\xi) \left( (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m \right) d\xi \quad \text{for} \quad f, g \in \mathcal{F}.
\]

It is shown in [12] that the corresponding jumping measure $J$ of $(\mathcal{E}, \mathcal{F})$ satisfies

\[
J(dx\,dy) = J_m(x,y) dx\,dy \quad \text{with} \quad J_m(x,y) = A(d, -\alpha) \frac{\Phi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}},
\]

where $A(d, -\alpha) = \frac{\alpha^{2+d+\alpha} \Gamma(d+\alpha)}{2^{d+1} \pi^{d/2} \Gamma(1-\frac{d}{2})}$, and $\Phi(r) := I(r)/I(0)$ with

\[
I(r) := \int_0^\infty s^{\frac{d+\alpha}{2}} e^{-s \frac{r}{2} - s/2} ds
\]

is a decreasing function satisfying $\Phi(r) \asymp e^{-r(1 + r^{(d+\alpha-1)/2})}$ near $r = +\infty$, and $\Phi(r) = 1 + \Phi''(0)r^2/2 + o(r^3)$ near $r = 0$. In particular,

\[
\mathcal{F} = \left\{ f \in L^2(\mathbb{R}^d) \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x) - f(y)|^2 J_m(x,y) dx\,dy < +\infty \right. \right\},
\]

\[
\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) J_m(x,y) dx\,dy \quad \text{for} \quad f, g \in \mathcal{F}.
\]

Let $p_t(x,y)$ be the heat kernel of $X$. The following global heat kernel estimate is proved in [7, Theorem 2.1]: there exist $C_1, C_2 > 0$ such that

\[
C_2^{-1} \Phi_1^{m/C_1}(t,x,y) \leq p_t(x,y) \leq C_2 \Phi_1^{m/C_1}(t,x,y), \quad (6.1)
\]

where
\( \Phi_C^m(t, x, y) := \begin{cases} t^{-d/\alpha} \wedge t J_m(x, y), \\ m^{d/\alpha - d/2} t^{-d/2} \exp \left( -C^{-1}(m^{1/\alpha}|x - y| \wedge m^{2/\alpha - 1} \frac{|x - y|^2}{t}) \right) \end{cases} \quad t \in [0, 1/m], \\
\quad t \in ]1/m, +\infty[. \]

In particular, we have

\[
C_2^{-1}(t^{-d/\alpha} \wedge t J_m(x, y)) \leq p_1(x, y) \leq C_2(t^{-d/\alpha} \wedge t J_m(x, y)) \quad \text{for } t \in [0, 1/m].
\]

(6.2)

It is shown in [11, Theorem 1.2 and Example 2.4] or [5,6, Theorem 1.2] that \( p_1(x, y) \) is jointly continuous in \( (t, x, y) \in ]0, +\infty[ \times \mathbb{R}^d \times \mathbb{R}^d \). The \( \beta \)-order resolvent kernel \( r_\beta(x, y) \in ]0, +\infty[ \) is also continuous in \( (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \). Consider \( p \in ]1, +\infty[ \).

We say that \( \mu \in K_{d,\alpha}^p \) if and only if

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x - y| < r} \frac{\mu(dy)}{|x - y|^{p(d - \alpha)}} = 0 \quad \text{for } d > \alpha,
\]

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x - y| < r} (\log |x - y|^{-1})^p \mu(dy) = 0 \quad \text{for } d = \alpha (= 1),
\]

\[
\sup_{x \in \mathbb{R}^d} \int_{|x - y| \leq 1} \mu(dy) < +\infty \quad \text{for } \alpha > d (= 1).
\]

Then we have \( K_{d,\alpha}^p = S_K^p \) by Theorem 3.1.

Consequently, the \( d \)-dimensional Lebesgue measure \( m \) belongs to \( K_{d,\alpha}^p = S_K^p \) if and only if \( \alpha > \frac{(p-1)d}{p} \) by Corollary 4.5 or Theorem 3.2, and for any non-negative bounded \( g \in L^1(\mathbb{R}^d) \) the finite measure \( g m \) also belongs to \( S_K^{p,\delta} \) for \( 0 < \delta < 1 - \frac{(p-1)d}{p\alpha} \) under \( p \in ]1, d/(d - \alpha) + [ \) by Theorem 4.6(1). Moreover, \( m \in S_K^{p,\delta} \) for \( 0 < \delta < 1 - \frac{(p-1)d}{p\alpha} \) under \( p \in ]1, d/(d - \alpha) + [ \) by Theorem 4.6(3). Here \( d/(d - \alpha)_+ := d/(d - \alpha) \) if \( d > \alpha \) and \( d/(d - \alpha)_+ := +\infty \) if \( d \leq \alpha \). The surface measure \( \sigma_R \) on the \( R \)-sphere \( \partial B_R(0) \) satisfies that \( \sigma_R(B_R(x)) \leq C_2 r^{d-1} \) for any \( x \in \mathbb{R}^d \) and \( r > 0 \) with some \( C_2 > 0 \), and \( \sigma_R(B_r(x)) \geq C_1 r^{d-1} \) for any \( x \in \partial B_R(0) \) and \( r \in ]0, r_0] \) with some \( C_1, r_0 > 0 \). Then we can conclude that \( \sigma_R \in K_{d,\alpha}^p = S_K^p \) holds if and only if \( \alpha > \frac{(p-1)d+1}{p} \) under \( d > \alpha \) by Theorems 3.1 and 3.2, and \( \sigma_R \in S_K^{p,\delta} \) holds for \( 0 < \delta < 1 - \frac{(p-1)d+1}{p\alpha} \) under \( p \in ]1, (d - 1)/(d - \alpha) \) with \( d \geq 2 \) by Theorem 4.6(1). By Corollary 4.4, we also have that \( f \in L^q_{\text{unif}}(\mathbb{R}^d) \) implies \( |f| dm \in S_K^p \) if \( q > d/(d - p(d - \alpha)) \) with \( d \geq \alpha \), or \( q \geq 1 \) with \( d < \alpha \).

We finally expose propositions on \( K_{d,\alpha}^p \):

**Proposition 6.3** Let \( f \) be a \([0, +\infty]\)-valued function on \([0, +\infty]\). Suppose that \(|V(x)|m(dx) \in K_{d,\alpha}^p \) with \( V(x) := f(|x|) \). Then we have

\[
\int_0^R r^{d-p(d-\alpha)-1} f(r) dr < +\infty \quad \text{for some } R > 0 \quad \text{if } d > \alpha,
\]
\[
\int_0^R (\log(r^{-1}))^p f(r) dr < +\infty \quad \text{for some} \quad R \in ]0, 1/e[ \quad \text{if} \quad d = \alpha = 1,
\]
\[
\int_0^R f(r) dr < +\infty \quad \text{for some} \quad R > 0 \quad \text{if} \quad \alpha > d = 1.
\]

If further \( f \) is decreasing on \([0, +\infty[\) and vanishes at infinity, then the converse holds.

**Proof** Suppose \( |V(x)|m(dx) \in K^p_{d,\alpha} \). From Theorem 3.1, we have
\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y|<R} |V(y)| dy < +\infty
\]
for any \( R \in ]0, +\infty[ \). Then we see that for any \( R \in ]0, +\infty[ \) (\( R \in ]0, 1/e[ \) if \( d = \alpha = 1 \),
\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y|<R} \frac{|V(y)|}{|x - y|^{p(d-\alpha)}} dy < +\infty \quad \text{if} \quad d > \alpha,
\]
\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y|<R} (\log |x - y|^{-1})^p |V(y)| dy < +\infty \quad \text{if} \quad d = \alpha (= 1),
\]
\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y|<R} |V(y)| dy < +\infty \quad \text{if} \quad \alpha > d = 1.
\]

Hence we have the assertion. Suppose the converse with the decrease of \( f \). Then the symmetric decreasing rearrangement \( V^* \) of \( V \) equals to \( |V| \) (see Chapter 3 in Lieb and Loss [29]). The simplest rearrangement inequality (see [29, Theorem 3.4]) tells us that
\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} \frac{|V(y)|}{|x - y|^{p(d-\alpha)}} dy = \int_{|y|<r} \frac{|V(y)|}{|y|^{p(d-\alpha)}} dy = (d \cdot \omega_d) \int_0^r s^{d-p(d-\alpha)-1} f(s) ds
\]
if \( d > \alpha \), and that
\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} \log(|x - y|^{-1})^p |V(y)| dy = \int_{|y|<r} (\log |y|^{-1})^p |V(y)| dy = 2 \int_0^r (\log s^{-1})^p f(s) ds
\]
if \( d = \alpha = 1 \), which tends to 0 as \( r \to 0 \), respectively. Here \( \omega_d \) is the volume of the unit ball \( B_1(0) \). We also have
\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y|<R} |V(y)| dy = \int_{|y|<R} |V(y)| dy = 2 \int_0^R f(s) ds < +\infty
\]
if $\alpha > d = 1$. Then $|V(x)|m(dx) \in K^p_{d,\alpha}$. \hfill $\square$

**Proposition 6.4** Let $V$ be a measurable function satisfying

$$
\int_a^\infty m(|V| \geq t)^{d-\rho(d-\alpha)} \, dt < +\infty \quad \text{for some} \quad a > 0 \quad \text{if} \quad d > \alpha,
\int_a^\infty F(m(|V| \geq t)) \, dt < +\infty \quad \text{for some} \quad a > 0 \quad \text{if} \quad d = \alpha = 1,
\int_a^\infty m(|V| \geq t) \, dt < +\infty \quad \text{for some} \quad a > 0 \quad \text{if} \quad \alpha > d = 1.
$$

Then $|V(x)|m(dx) \in K^p_{d,\alpha}$. Here $m$ is the Lebesgue measure on $\mathbb{R}^d$ and $F$ is a function defined by $F(s) := \int_0^{s/2} (\log^+ u^{-1})^p \, du$ with $\log^+ u := (\log u) \vee 0$.

**Proof** The proof is similar with [1, Theorem 4.12]. We only prove the case $d > \alpha$. We may assume $|V| = V^*$, where $V^*$ is the symmetric decreasing rearrangement of $V$, because $m(|V| > t) = m(V^* > t)$ for all $t > 0$. Hence, there exists a decreasing function $f$ on $]0, +\infty[$ such that $V(x) = f(|x|)$ and $V$ is lower semi-continuous. Let $f^{-1}(t) := \sup\{s > 0 \mid f(s) > t\}$ be the right continuous inverse of $f$, which is also a decreasing function. We may assume $0 < f^{-1}(t) < +\infty$ for any $t > 0$, that is, $f$ has an infinite limit at origin and no positive limit at infinity, because $\alpha$ specified in the condition can be taken to be arbitrarily large. When $f$ has a finite limit at origin, $V$ is essentially bounded, which implies $|V(x)|m(dx) \in K^p_{d,\alpha}$. First we assume that $f$ is continuous, but we do not assume the strict decrease of $f$. The continuity of $f$ yields that $b = f(f^{-1}(b))$ for any $b > 0$ and we see $f^{-1}(f(r)) \leq r$ for any $r > 0$. Then for $a < A$

$$
\int_{f^{-1}(a)}^{f^{-1}(A)} r^{d-\rho(d-\alpha)} f(r) \, dr \leq \int_{f^{-1}(a)}^{f^{-1}(A)} (f^{-1}(f(r)))^{d-\rho(d-\alpha)} f(r) \, dr \leq \int_a^A (f^{-1}(t))^{d-\rho(d-\alpha)} \, dt \leq \int_a^\infty (f^{-1}(t))^{d-\rho(d-\alpha)} \, dt < +\infty,
$$

because $m(|V| \geq t) = \omega_d(f^{-1}(t))^d$. By way of the integration by parts formula for Riemann-Stieltjes integrals,

$$
(d - p(d-\alpha)) \int_0^{f^{-1}(a)} f(r) r^{d-\rho(d-\alpha)-1} \, dr \leq a(f^{-1}(a))^{d-\rho(d-\alpha)} + \int_a^\infty (f^{-1}(t))^{d-\rho(d-\alpha)} \, dt < +\infty. \quad (6.3)
$$
Next we show (6.3) for general \( f \). Note that \( V^*(x) = f(|x|) \) is lower semi-continuous (lower semi-continuity of \( f \) is clarified later). We set \( f_n(t) := \inf \{ f(s) + n|s-t| \mid s \in [0, +\infty] \} \). Then \( \{ f_n \} \) is an increasing sequence of nonnegative \( n \)-Lipschitz function on \([0, +\infty[\). We then see that

\[
f_n(|x|) = \inf \left\{ f(|z|) + n|z| - |x| \mid z \in \mathbb{R}^d \right\}.
\]

Hence \( f_n(|x|) \) converges to \( V^*(x) = f(|x|) \) as \( n \to \infty \), because of the lower semi-continuity of \( V^* \), consequently, \( f_n \) converges to \( f \) as \( n \to \infty \), hence \( f \) is lower semi-continuous. Indeed, we may consider the case \( f(|x|) > 0 \) and suppose the existence of \( \varepsilon > 0 \) such that \( f_n(|x|) < f(|x|) - \varepsilon(>0) \) for all \( n \in \mathbb{N} \). Then there exists \( z_n \in \mathbb{R}^d \) with \( f(|z_n|) + n||z_n| - |x|| < f(|x|) - \varepsilon \). From this, we see \( |z_n| \to |x| \) as \( n \to \infty \) and may assume the existence of \( z \in \mathbb{R}^d \) with \( z_n \to z \) as \( n \to \infty \) by taking a subsequence. Hence, we obtain a contradiction \( f(|x|) = f(|z|) \leq \lim\inf_{n \to \infty} f(|z_n|) \leq f(|x|) - \varepsilon \). We set \( g_n(t) := \inf_{s \in [0,t]} f_n(s) \). Then \( g_n \) is a decreasing continuous function vanishing at infinity. We see that \( \uparrow \lim_{n \to \infty} g_n = f \). We also have that \( \{ g_n^{-1} \} \) is an increasing sequence and converges to \( f^{-1} \) as \( n \to \infty \) at each point. Since (6.3) holds for \( g_n, g_n^{-1} \), it holds for \( f, f^{-1} \). Therefore the simplest rearrangement inequality shows that for \( r > 0 \)

\[
\sup_{x \in \mathbb{R}^d} \int_{|y| < r} \frac{|V(y)|}{|x - y|^{\alpha}} \, dy \leq \int_{|y| < r} \frac{|V^*(y)|}{|y|^{\alpha}} \, dy = (d \cdot \omega_d) \int_0^r s^{d - \alpha - 1} f(s) \, ds
\]

and the right-hand side goes to zero as \( r \to 0 \).

\[\square\]

Corollary 6.5 Let \( d > \alpha \) and \( G \) a positive increasing function on \([0, +\infty[\) satisfying

\[
\int_a^\infty (G'(s))^{1 - \frac{d}{\alpha} - 1} \, ds < +\infty \quad \text{for some} \quad a > 0.
\]

Suppose that \( \int_{\mathbb{R}^d} G(|V(x)|) \, m(dx) < +\infty \). Then \( |V(x)| \, m(dx) \in K_{\alpha}^p \).

\textbf{Proof} The proof is quite similar as in [1, Corollary 4.13]. We omit it. \[\square\]

Remark 6.6 The assertions in Propositions 6.3, 6.4 and Corollary 6.5 for \( \alpha = 2 \) remain valid in the framework of \( d \)-dimensional Brownian motion \( X^w \) in Example 6.1.

Example 6.7 (Jump type processes on \( d \)-sets, cf. Chen and Kumagai [10]) Let \( F \) be a closed subset of \( \mathbb{R}^n \) with \( 0 < d \leq n \) and \( n \geq 2 \). Denote the Euclidean open ball by \( B_r(x) \). A positive Borel measure \( m \) with support \( F \) is called a \( d \)-measure on \( F \) (see [22, pp. 28]) if there exists \( C > 0 \) such that

\[
C^{-1} r^d \leq m(B_r(x)) \leq Cr^d \quad \text{for any} \quad x \in F \quad \text{and} \quad r \in [0, 1].
\]

(6.4)
A closed non-empty subset $F$ of $\mathbb{R}^n$ is said to be a $d$-set ($0 < d \leq n$) if there exists a $d$-measure on $F$. For $\alpha \in [0, 2[$, consider the following Dirichlet form $\mathcal{E}^{(\alpha)}$, $\mathcal{F}^{(\alpha)}$:

$$\mathcal{F}^{(\alpha)} := \left\{ u \in L^2(F; m) \mid \int_F \int_F |u(x) - u(y)|^2 \frac{m(dx)m(dy)}{|x - y|^{d+\alpha}} < +\infty \right\},$$

$$\mathcal{E}^{(\alpha)}(u, v) := \frac{1}{2} \int_F \int_F \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} \frac{m(dx)m(dy)}{|x - y|^{d+\alpha}}, \quad u, v \in \mathcal{F}^{(\alpha)}.$$

Under the condition that for some $C > 0$ with

$$m(B_r(x)) \leq Cr^d \quad \text{for all } x \in F \quad \text{and } r > 0,$$  \hfill (6.5)

Chen and Kumagai [10] show that the jump type process associated with $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ admits a semigroup kernel possessing the following upper and lower estimates: there exist $C_i = C_i(\alpha, d) > 0, i = 1, 2$ such that for all $(t, x, y) \in [0, 1] \times F \times F$

$$\frac{C_1}{t^{d/\alpha}} \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha} \leq p_t(x, y) \leq \frac{C_2}{t^{d/\alpha}} \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha}.$$

Then our assumptions (A2) and (A3) are satisfied in this context. By [28, Lemma 2.1], assumption (A1) follows from (A3) and the lower estimate of (6.4). Hu and Kumagai [21] extend the result in [10] replacing the embedding condition into $\mathbb{R}^n$ by a condition on the extension to a metric space with scaling property, however, they assume stronger assumption that for some $C > 0$ with

$$C^{-1}r^d \leq m(B_r(x)) \leq Cr^d \quad \text{for all } x \in F \quad \text{and } r \in [0, \text{diam}(F)].$$  \hfill (6.6)

Hence one can apply our results under (6.6).

**Example 6.8** (Riemannian manifolds with lower Ricci curvature bound) Let $(M, g)$ be a $d$-dimensional smooth complete Riemannian manifold with $\text{Ric}_M \geq (d - 1)\kappa_1$, $\kappa_1 \in \mathbb{R}$. Let $m$ be the volume measure induced from $g$ and set $V(x, r) := m(B_r(x))$. Under $\text{Ric}_M \geq (d - 1)\kappa_1$, we have that the Bishop inequality $V(x, r) \leq V_{\kappa_1}(r)$ and the Bishop-Gromov inequality $V(x, R)/V_{\kappa_1}(R) \leq V(x, r)/V_{\kappa_1}(r), 0 < r < R$ hold. Consequently, we have the volume doubling condition $\sup_{x \in M} V(x, 2r)/V(x, r) < +\infty$ and $\int_1^\infty \frac{\log V(x, s)}{s} ds = +\infty$ which implies the stochastic completeness of the Brownian motion on $(M, g)$ (see [16]). In particular, the condition (A1) holds. Here $V_{\kappa}(r)$ defined by $V_{\kappa}(r) := c_d \int_0^r S_{\kappa}(s)^{d-1} ds$ with $S_{\kappa}(s) := \frac{\sin \sqrt{s} \sqrt{\kappa}}{\sqrt{\kappa}}$ if $\kappa > 0$, $S_0(s) = s$, $S_{\kappa}(s) := \frac{\sinh \sqrt{s} \sqrt{-\kappa}}{\sqrt{-\kappa}}$ if $\kappa < 0$, is the volume of the ball with radius $r$ in the space form of constant sectional curvature $\kappa$ and $c_d$ is the volume of the unit ball in $\mathbb{R}^d$. We also have the scale invariant weak Poincaré inequality (depending on $\kappa_1$ if $\kappa_1 < 0$) (see Saloff-Coste [31] or [32, Theorem 5.6.5]), which implies the weak form of the weak Poincaré inequality (see [32, Theorem 5.5.1(i)]). Then the heat kernel $p_t(x, y)$ of $(M, g)$ satisfies the following Li-Yau’s estimate (see [32, Theorems 5.5.1 and 5.5.3], cf. [18, Springer].
Theorems 6.1 and 6.2): for each $T > 0$ there exist $C_i = C_i(T) > 0$, $i = 1, 2, 3, 4$ such that for $(t, x, y) \in [0, T] \times M \times M$

$$
\frac{C_3}{V(y, \sqrt{t})} \exp \left( -C_1 \frac{d(x, y)^2}{t} \right) \leq p_t(x, y) \leq \frac{C_4}{V(y, \sqrt{t})} \exp \left( -C_2 \frac{d(x, y)^2}{t} \right).
$$

The Bishop inequality tells us that (A2) holds. Further we assume that the injectivity radius of $M$ (write $\text{inj}_M$) is positive. Then we have the following (see the proof of [19, Lemma 5] and [13, Proposition 14]. Though the framework of [13] is restricted to compact Riemannian manifolds, the argument in [19] remains valid): there exists $C_d \in [0, +\infty]$ such that for any $r \in [0, \text{inj}_M/2[$ and $x \in M$,

$$
V(x, r) \geq C_dr^d.
$$

Hence for a small time $t_0 > 0$, we have the Nash-type estimate: for any $t \in [0, t_0[$, $\sup_{x,y\in E} p_t(x, y) \leq C_4 t^{-d/2}$, which gives a Sobolev inequality under $d \geq 3$ (see [19] again). Then [27, Theorem 2.1] holds. Also we have for any $t \in [0, t_0[$, $x, y \in M$

$$
\frac{C_3}{t^{d/2}} \exp \left( -C_1 \frac{d(x, y)^2}{t} \right) \leq p_t(x, y) \leq \frac{C_4}{t^{d/2}} \exp \left( -C_2 \frac{d(x, y)^2}{t} \right).
$$

Since $[0, +\infty[ \ni x \mapsto \sinh x/x$ is increasing, $s \mapsto V_k_1(s)/s^{d-1}$ is increasing for $\kappa_1 \leq 0$, hence $s \mapsto V_k_1(s)/s^d$ is so. For $\kappa_1 > 0$, $s \mapsto V_k_1(s)/s^d$ is bounded. We can confirm that

$$
\int_1^{+\infty} \frac{(V_k(s) \vee s^d)}{s} e^{-C_2s^2} ds < +\infty.
$$

Then (A3) holds. Therefore Theorems 3.1, 3.2, 4.1, 4.3, 4.6 and Corollaries 3.3, 4.4, 4.5 hold. In particular, $m \in S^K_p = K_p^\delta$ if and only if $d - p(d - 2) > 0$ by Corollary 4.5, and $m \in S^K_p,^\delta$ holds for $\delta \in [0, (d - p(d - 2))/2p[$ under $p \in [1, d/(d - 2)]_+$ by the latter half of Theorem 4.6(3).

Example 6.9 (Nested fractals; cf. [14,25]) The heat kernel of diffusion processes on the unbounded nested fractal $\tilde{K}$ constructed by Kumagai [25] has the following upper and lower estimates: there exist $C_i > 0$, $i = 1, 2, 3, 4$ such that for any $(t, x, y) \in [0, +\infty[ \times \tilde{K} \times \tilde{K}$

$$
\frac{C_3}{t^{d_f/d_w}} \exp \left( -C_1 \left( \left( \frac{d(x, y)}{t^{1/d_w}} \right)^{d_w/d_f - 1} \right) \right) \leq p_t(x, y) \leq \frac{C_4}{t^{d_f/d_w}} \exp \left( -C_2 \left( \left( \frac{d(x, y)}{t^{1/d_w}} \right)^{d_w/d_f - 1} \right) \right).
$$

Here $d_f$ is the Hausdorff dimension of $\tilde{K}$, $d_w$ is called the walk dimension, $d_f$ is a different constant from $d_w$. We consider $p \in [1, +\infty[$. It is known that $d_f = d_w$ if $\tilde{K}$ is the (unbounded) Sierpiński Gasket. In general $d_f < d_w$. Hence $\mu \in S^K_p$ if and only if $\sup_{x \in \tilde{K}} \mu((y \in \tilde{K} : d(x, y) \leq 1)) < +\infty$ and $L^q_{\text{uni}}(\tilde{K}; \mu_{\tilde{K}}) \subset S^K_p$ for $q \geq 1$, where $\mu_{\tilde{K}}$ is the Hausdorff measure on $\tilde{K}$. In particular, $\mu_{\tilde{K}} \in S^K_p$ for $p \in [1, +\infty[$.
Example 6.10 (Sierpiński Carpet; cf. [3]) On the unbounded Sierpiński carpet $F$, the heat kernel of diffusion processes exists and admits the following upper and lower estimates proved by Barlow and Bass [3]: there exist constants $C_i > 0, i = 1, 2, 3, 4$ such that for any $(t, x, y) \in ]0, +\infty[ \times F \times F$

$$\frac{C_3}{t^{d_f}} \exp \left( -C_1 \left( \frac{|x-y|}{t^{1/d_w}} \right)^{d_{w}^{-1}} \right) \leq p_t(x, y) \leq \frac{C_4}{t^{d_w}} \exp \left( -C_2 \left( \frac{|x-y|}{t^{1/d_w}} \right)^{d_{w}^{-1}} \right).$$

Here $|x-y|$ denotes the Euclidean norm of $x-y$ in $\mathbb{R}^n$, $d_f$ is the Hausdorff dimension of $F$, $d_w$ is called the walk dimension and $d_s := 2d_f/d_w$ is called the spectral dimension of $F$. They have the relation $1 < d_s \leq d_f < n$, where $n$ is the dimension of the Euclidean space in which $F$ is embedded. Thus we have $2 \leq d_w \leq 2n$.

Take $p \in [1, +\infty[$. We say that $\mu \in K_{d_f,d_w}^P$ if and only if

$$\lim_{r \to 0} \sup_{x \in F} \int_{\{ y \in F : |x-y| < r \}} \frac{\mu(dy)}{|x-y|^{p(d_f-d_w)}} = 0, \quad d_s > 2,$$

$$\lim_{r \to 0} \sup_{x \in F} \int_{\{ y \in F : |x-y| < r \}} (\log |x-y|)^{-1} \mu(dy) = 0, \quad d_s = 2,$$

$$\sup_{x \in F} \int_{\{ y \in F : |x-y| \leq 1 \}} \mu(dy) < +\infty, \quad d_s < 2.$$  

Let $\mu_F$ be the Hausdorff measure on $F$. In this case, Ahlfors regularity holds in the following sense that there exists $C > 0$ such that $C^{-1}r^{d_f} \leq \mu_F(B_r(x)) \leq Cr^{d_f}$ for all $r \in ]1, +\infty[ \ (\text{see [3, Lemma 2.3(f)]}).$ Then $(A1)$ is satisfied by [28, Lemma 2.1]. By [28, Remark 2.1], $(A2)$ holds by taking $V(r) := C r^{d_f}$. We then have $K_{d_f,d_w}^P = S_K^P$ and $L_{\text{unif}}^q(F; \mu_F) \subset S_K^P$ if $q > d_s/(d_s - p(d_s - 2))$ with $d_s \geq 2$, or $q \geq 1$ with $d_s < 2$. In particular, $\mu_F \in S_K^P$ for $p \in [1, +\infty[$ with $d_s - p(d_s - 2) > 0$.

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