From the conformal self-duality equations
to the Manakov-Santini system

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Abstract

Under two separate symmetry assumptions, we demonstrate explicitly how the
equations governing a general anti-self-dual conformal structure in four dimen-
sions can be reduced to the Manakov-Santini system, which determines the three-
dimensional Einstein-Weyl structure on the space of orbits of symmetry. The two
symmetries investigated are a non-null translation and a homothety, which are pre-
viously known to reduce the second heavenly equation to the Laplace’s equation
and the hyper-CR system, respectively. Reductions on the anti-self-dual null-Kähler
condition are also explored in both cases.

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1 Introduction

The integrability of the equations governing the anti-self-dual (ASD) conformal structures in four dimensions and the Einstein-Weyl (EW) structures in three dimensions is well known. The twistor correspondences which reveal their integrability were given by Penrose [23] and Hitchin [17], respectively. The relation between the two geometries was then established by Jones and Tod [18], who proved that any ASD conformal structure with a non-null conformal Killing symmetry gives rise to an EW space, and conversely given an EW space one can always find an ASD spacetime with a conformal Killing vector field whose space of orbits is the EW space one started with. In fact the relation is not one-to-one; different ASD spacetimes can yield the same EW space, and starting from a given EW space one needs to solve the so-called generalised monopole equation, different solutions of which determine different ASD spacetimes.

In [8], Dunajski et al. presented explicit forms of the equations for the ASD and EW structures in specially adapted coordinate systems. Using Cartan’s approach they proved that any Lorentzian EW structure is locally determined by a solution of the Manakov-Santini system, which was first derived in [19] as a generalisation of the dispersionless Kadomtsev-Petviashvili (dKP) equation. They also proved that any ASD conformal structure in neutral signature can locally be written in terms of two functions satisfying a system of two third order partial differential equations (PDEs). We shall call this system, which was derived using the Plebański-Robinson coordinates, the “ASD conformal equations”.

According to the Jones-Tod correspondence, one should be able to obtain the Manakov-Santini (MS) system from the ASD conformal equations under a conformal symmetry assumption. The main aim of this paper is to demonstrate this explicitly. We manage to do this in two cases of symmetry assumptions. The first one is the reduction by a non-null translation and the second is by a homothetic Killing symmetry. These two symmetries are previously known to reduce the second heavenly equation, determining the ASD Ricci-flat metrics, to the Laplace’s equation [14] and the hyper-CR system [4], respectively.

In both cases we also explore symmetry reductions when the ASD conformal class admits a null-Kähler metric. A metric of neutral signature is said to be null-Kähler if it admits a covariantly constant real spinor. It was shown in [5] that any ASD null-Kähler metric with a Killing symmetry preserving the parallel spinor gives rise to an EW space with a parallel weighted vector field. The latter is determined by the dKP equation [4], which is a reduction of the MS system. In the case of the non-null translation, the translation is assumed to be a Killing symmetry of the ASD null-Kähler metric, representing the conformal class. Moreover, it preserves the covariantly constant spinor.
Therefore one expects the ASD null-Kähler condition to give rise to the dKP equation. We demonstrate this explicitly. On the other hand, the homothety is a conformal Killing symmetry, which rescales the ASD null-Kähler metric to another metric, not necessarily null-Kähler. Nevertheless we expected to arrive at some reduction of the MS system, perhaps characterising the EW structure coming from an ASD conformal structure admitting a null-Kähler metric. Instead, we find that the corresponding EW metric is still determined by a general solution of the the full MS system.

This paper is organised as follows. In Section 2 the relevant theorems which form the basis of our work are reviewed. Then symmetry reductions by the non-null translation are investigated in Section 3. There the ASD conformal equations is shown to reduce to the MS system under the symmetry assumption and a Legendre transformation, and the dKP equation is derived directly from the ASD null-Kähler condition. Section 4 explores reductions by the homothetic Killing symmetry that reduces the second heavenly equation to the hyper-CR system. Assuming this conformal symmetry, we derive the MS system from the ASD conformal equations, and show that even with the assumption of a null-Kähler metric in the ASD conformal class, the corresponding EW structure is still governed by the MS system. Lastly, in Section 5 we discuss our attempts to explicitly realise the Bogdanov’s system as another system of equations, other than the MS system, which describes a generic EW structure.

2 Conformal self-duality and Einstein-Weyl equations

Recall that a four-dimensional anti-self-dual (ASD) conformal structure \((M, [g])\) consists of a four-dimensional manifold \(M\) and a conformal class of metrics \([g]\) whose Weyl tensor is anti-self-dual with respect to the Hodge star operator. On the other hand, the Einstein-Weyl (EW) structure in three dimensions \((\mathcal{W}, [h], D)\) consists of a three-dimensional manifold \(\mathcal{W}\), a conformal class of metrics \([h]\) and a torsion-free connection \(D\) compatible with \([h]\), such that \(Dh = \nu \otimes h\) for some one-form \(\nu\), and the symmetrised Ricci tensor of \(D\) is functionally proportional to \(h\). One can regard the EW condition as equations for unknowns \(h\) and \(\nu\), which determine the EW structure.

There are several explicit formulations for the ASD and EW conditions. Here we shall focus on the explicit forms given in the main theorems of [8], which are reviewed as Theorem 2.1 and 2.2 below.
Theorem 2.1 Any ASD conformal structure in signature $(2, 2)$ can be locally represented by a metric $g \in [g]$ of the form

$$g = dW dX + dZdY + F_Y dW^2 - (F_X + G_Y) dW dZ + G_X dZ^2,$$  

(2.1)

where $(W, X, Y, Z)$ are local coordinates on the manifold $M$ and the functions $F$ and $G$ satisfy a coupled system of third order PDEs,

$$\partial_X(Q(F)) - \partial_Y(Q(G)) = 0,$$  

(2.2)

$$\partial_W - F_Y \partial_X + G_Y \partial_Y)Q(G) + (\partial_Z + F_X \partial_X - G_X \partial_Y)Q(F) = 0,$$  

(2.3)

with $Q$ being a second order differential operator given by

$$Q = \partial_W \partial_X + \partial_Z \partial_Y - F_Y \partial_X^2 - G_X \partial_Y^2 + (F_X + G_Y) \partial_X \partial_Y.$$  

(2.4)

Theorem 2.2 Any Lorentzian EW structure in three dimensions can be locally represented by a metric $h \in [h]$ and a one-form $\nu$ of the form

$$h = -(dy - v_x dt)^2 + 4(dx - (u - v_y) dt) dt, \quad \nu = -v_x dx + (4u_x - 2v_{xy} + v_x v_{xx}) dt,$$  

(2.5)

where $(x, y, t)$ are local coordinates on the manifold $W$ and the functions $u$ and $v$ are solutions of the Manakov-Santini system

$$P(u) + u_x^2 = 0, \quad P(v) = 0,$$  

(2.6)

where

$$P = \partial_x \partial_t - \partial_y^2 + (u - v_y) \partial_x^2 + v_x \partial_x \partial_y.$$  

We note here that it was first shown in [7] that any solution of the Manakov-Santini (MS) system (2.6) gives rise to an EW structure, but it was proved in [8] that all EW structures arise in this way.

The relation between ASD conformal structures and the EW structures is given by the Jones-Tod correspondence [18], which we shall state below for the ASD conformal structures in signature $(2, 2)$ and the Lorentzian EW structures using the notation of [5].
Theorem 2.3 [18, 5] Let \((M, \hat{g})\) be a four-dimensional ASD conformal structure in neutral signature with a non-null conformal Killing vector field \(K\), and let \(\mathcal{W}\) be the space of trajectories of \(K\). An EW structure on \(\mathcal{W}\) is defined by the metric \(h\) and the one-form \(\nu\) given by

\[
h = |K|^{-2} \hat{g} - |K|^{-4} K \odot K, \quad \nu = 2 |K|^{-2} \ast_{\hat{g}} (K \wedge dK),
\]

where \(|K| = \hat{g}^{ab} K_a K_b\), \(K\) is the dual one-form of \(K\) and \(\ast_{\hat{g}}\) is the Hodge star operator with respect to \(\hat{g}\).

Conversely, suppose \(\mathcal{W}\) is a three-dimensional manifold with a Lorentzian EW structure defined by a pair of metric and one-form \((h, \nu)\). Let \(V\) and \(\alpha\) be a function of weight \(-1\) and a one-form on \(\mathcal{W}\), respectively. If \(V\) and \(\alpha\) satisfy the generalised monopole equation

\[
\ast_h \left( dV + \frac{1}{2} \nu V \right) = d\alpha,
\]

where \(\ast_h\) is the Hodge star operator with respect to \(h\), then

\[
g = V h - \frac{1}{V} (d\tau + \alpha)^2
\]

is an ASD metric with a Killing symmetry generated by \(\partial_\tau\).

From these three theorems, one expects that under a non-null symmetry assumption the ASD conformal equations \((2.2, 2.3)\) should reduce to the MS system \((2.6)\) or a special case of it. In the next two sections we shall show that one can arrive at the MS system explicitly, using a symmetry assumption and a Legendre transformation. The use of a Legendre transformation is motivated by the work of Finley and Plebański [14].

3 Non-null translational symmetry reduction - Lineage of the Laplace’s equation

A simplest symmetry to consider is a translation generated by a coordinate vector field. It turns out that such a symmetry is sufficient to reduce the ASD conformal equations \((2.2, 2.3)\) to the full MS system \((2.6)\), rather than a special case of it. In particular, here we shall assume that the ASD metric \((2.1)\) admits the non-null translational Killing symmetry generated by \(\partial_\mathcal{W}\) (or equivalently \(\partial_Z\)). It was previously shown in the aforementioned work of Finley and Plebański [14] that the second heavenly equation [24], which governs
the ASD Ricci-flat metrics, can be linearised to the three-dimensional Laplace’s equation under this symmetry assumption.

### 3.1 From ASD condition to the MS system

Assume that the metric (2.1) admits a non-null translational Killing symmetry generated by \(\partial_W\). Without loss of generality, this implies that the functions \(F\) and \(G\) in the metric (2.1) is independent of the \(W\) coordinate. For convenience, let \(\hat{F} := Q(F)\) and \(\hat{G} := Q(G)\), then equations (2.2) and (2.3) become

\[
\hat{F}_X - \hat{G}_Y = 0, \tag{3.9}
\]

\[
-F_Y \hat{G}_X + G_Y \hat{G}_Y + \hat{F}_Z + F_X \hat{F}_X - G_X \hat{F}_Y = 0, \tag{3.10}
\]

respectively, where

\[
\hat{F} = F_{ZY} - F_Y F_{XX} - G_X F_{YY} + (F_X + G_Y) F_{XY}, \tag{3.11}
\]

\[
\hat{G} = G_{ZY} - F_Y G_{XX} - G_X G_{YY} + (F_X + G_Y) G_{XY}. \tag{3.12}
\]

Equation (3.9) implies the existence of a function \(u(X,Y,Z)\) such that \(u_Y = \hat{F}\) and \(u_X = \hat{G}\). Working with the variable \(u\), equation (3.9) is satisfied identically and (3.10) becomes

\[
-F_Y u_{XX} + F_X u_{XY} + G_Y u_{XY} - G_X u_{YY} + u_{YZ} = 0.
\]

The above equation can be written in terms of three-forms as

\[
dF \wedge du_X \wedge dZ + du_Y \wedge (dG \wedge dZ + dX \wedge dY) = 0. \tag{3.13}
\]

To facilitate the calculation, let us write the remaining two equations in terms of three-forms. Equations (3.11) and (3.12) are equivalent to

\[
-du \wedge dX \wedge dZ - dF_Y \wedge dX \wedge dY + F_Y dF_X \wedge dY \wedge dZ
- G_X dF_Y \wedge dZ - (F_X + G_Y) dF_Y \wedge dY \wedge dZ = 0, \tag{3.14}
\]

and

\[
du \wedge dY \wedge dZ - dG_Y \wedge dX \wedge dY + F_Y dG_X \wedge dY \wedge dZ
- G_X dG_Y \wedge dZ - (F_X + G_Y) dG_Y \wedge dY \wedge dZ = 0, \tag{3.15}
\]

respectively.
Now we shall use a Legendre transformation to show that the system \((3.13, 3.14, 3.15)\) is equivalent to the MS system \((2.6)\). Note that one can assume that \(F_Y \neq 0\) throughout a neighbourhood of interest. Otherwise, the Killing vector field \(\partial_W\) would become null. Introducing new independent variables \((x, y, t)\), where \(x = -F, y = X\) and \(t = -Z\), we shall now write the system \((3.13, 3.14, 3.15)\) in \((x, y, t)\) coordinates as follows.

First, for equation \((3.13)\), one needs expressions of \(u_X\) and \(u_Y\) in \((x, y, t)\) coordinates. Suppose \(\hat{u}(x, y, t)\) is a function defined by \(u(X, Y, Z) = \hat{u}(x(X, Y, Z), y(X), t(Z))\). Then by the Chain rule, \(u_X = \hat{u}_y y + \hat{u}_x x\) and \(u_Y = \hat{u}_x x\). Now, let \(Y = \Phi(x, y, t)\), the derivatives \(x_X\) and \(x_Y\) can be found by differentiating \(Y = \Phi(x(X, Y, Z), y(X), t(Z))\) with respect to \(Y\) and \(X\), respectively. This gives \(x_X = -\frac{\Phi_y}{\Phi_x}\) and \(x_Y = \frac{1}{\Phi_x}\). For convenience, from now we shall abuse the notation and drop the hat from \(\hat{u}\), thus we have

\[
 u_X = u_y - \frac{u_x \Phi_y}{\Phi_x} \quad \text{and} \quad u_Y = \frac{u_x}{\Phi_x},
\]

where \(u\) on the right-hand side of each equation refers to the function \(u\) expressed in the \((x, y, t)\) coordinates.

Hence \((3.13)\) is given in \((x, y, t)\) coordinates by

\[
 dx \wedge d\left(u_y - \frac{u_x \Phi_y}{\Phi_x}\right) \wedge dt + d\left(\frac{u_x}{\Phi_x}\right) \wedge (dt \wedge dG + dy \wedge d\Phi) = 0,
\]

which is equivalent to

\[
 -u_{yy} + u_{xt} + \left(\frac{G_y - \Phi_x}{\Phi_x}\right) u_{xx} + \left(\frac{\Phi_y - G_x}{\Phi_x}\right) u_{xy} + u_x \left(\partial_x \left(\frac{G_y - \Phi_t}{\Phi_x}\right) + \partial_y \left(\frac{\Phi_y - G_x}{\Phi_x}\right)\right) = 0.
\]

Note that the condition \(F_Y \neq 0\) guarantees that \(dx \wedge dy \wedge dt \neq 0\).

We shall now show that \((3.16)\) (which is \((3.13)\) in \((x, y, t)\) coordinates) is equivalent to the first equation of the MS system \((2.6)\) provided that \((3.14)\) holds. Writing \((3.14)\) in \((x, y, t)\) coordinates yields

\[
 u_x dx \wedge dy \wedge dt + \frac{1}{\Phi_x} d\left(\frac{\Phi_y}{\Phi_x}\right) \wedge d\Phi \wedge dt \\
 + d\left(\frac{1}{\Phi_x}\right) \wedge \left(dy \wedge d\Phi - \frac{\Phi_y}{\Phi_x} d\Phi \wedge dt - \left(G_y - G_x \frac{\Phi_y}{\Phi_x}\right) dy \wedge dt - \frac{G_x}{\Phi_x} d\Phi \wedge dt\right) = 0,
\]

where we have used \(x_X = -\frac{\Phi_y}{\Phi_x}, x_Y = \frac{1}{\Phi_x}, G_X = G_y - G_x \frac{\Phi_y}{\Phi_x}\) and \(G_Y = \frac{G_x}{\Phi_x}\).
Since $dx \wedge dy \wedge dt \neq 0$, this implies

$$u_x = \partial_x \left( \frac{G_y - \Phi_t}{\Phi_x} \right) + \partial_y \left( \frac{\Phi_y - G_x}{\Phi_x} \right). \quad (3.17)$$

Integrating (3.17) with respect to $x$ gives

$$u = \frac{G_y - \Phi_t}{\Phi_x} + \int \partial_y \left( \frac{\Phi_y - G_x}{\Phi_x} \right) dx.$$  

As all functions involved are assumed to be real analytic and can thus be differentiated under the integral sign, we have

$$u = \frac{G_y - \Phi_t}{\Phi_x} + v_y, \quad \text{where} \quad v = \int \left( \frac{\Phi_y - G_x}{\Phi_x} \right) dx, \quad \text{and thus} \quad v_x = \frac{\Phi_y - G_x}{\Phi_x}. \quad (3.18)$$

With (3.17) and (3.18), equation (3.16) becomes

$$u_{xt} - u_{yy} + (u - v_y) u_{xx} + v_x u_{xy} + u_x^2 = 0,$$

which is the first equation of (2.6).

The second equation of the MS system (2.6) arises from (3.15). In the coordinates $(x, y, t)$, (3.15) becomes

$$- u_y + \frac{\Phi_y}{\Phi_x} u_x + \frac{G_{xx} \Phi_t}{\Phi_x^2} - \frac{G_x \Phi_{xx} \Phi_t}{\Phi_x^3} - \frac{G_{xx} G_y}{\Phi_x^2} + \frac{G_x \Phi_{xx} G_y}{\Phi_x^3} - \frac{G_x}{\Phi_x} \left( \frac{\Phi_y - G_x}{\Phi_x} \right)_y - \frac{G_{xy} \Phi_y}{\Phi_x^2} + \frac{G_{yy} G_y}{\Phi_x} - \frac{G_{xt}}{\Phi_x} + \frac{G_x \Phi_{xt}}{\Phi_x^2} = 0.$$  

With $u$ and $v_x$ given in (3.18), the above equation can be rearranged to give

$$v_{xt} - v_{yy} + (u - v_y) v_{xx} + v_x v_{xy} = 0,$$

which is the second equation of (2.6).

The ASD metric (2.1) can be written in the form (2.8), where the three-dimensional EW metric $h$ is given by (2.5) as follows. First, let $w = W$, in the new coordinates $(w, x, y, t)$, the metric (2.1) is given by

$$g = dwdy - d\Phi dt - \frac{1}{\Phi_x} dw^2 + \left( \frac{\Phi_y + G_x}{\Phi_x} \right) dwdt + \left( G_y - \frac{\Phi_y G_x}{\Phi_x} \right) dt^2,$$
where we recall that $dY = d\Phi(x, y, t)$ and the derivatives $F_X, F_Y, G_X, G_Y$ as functions of $(x, y, t)$ can be found using the chain rule.

Completing the square and using (3.18) gives

$$g = \frac{\Phi_x}{4}
\left((dy - v_x dt)^2 - 4(dx - (u - v_y) dt) dt + \frac{\Phi_x}{4} \left(\frac{\Phi_y + G_x}{\Phi_x} dt + dy\right)\right)^2,$$

which is of the form (2.8), where $V = \frac{\Phi_x}{4}$, $\alpha = -\frac{\Phi_x}{4} \left(\frac{\Phi_y + G_x}{\Phi_x} dt + dy\right)$ and the EW metric $h$ is given by

$$h = (dy - v_x dt)^2 - 4(dx - (u - v_y) dt) dt,$$

which is minus the metric given in (2.5).

To end this section, let us note on what happens when $F_Y = 0$. It follows that the Killing vector field $K = \partial_W$ becomes null. Then the space of trajectories inherits a degenerate conformal structure and we no longer have an EW structure. This situation was investigated by Dunajski and West [10]. Instead of the space of trajectories, they considered the two-dimensional space of ASD totally null surfaces in the manifold, called $\beta$-surfaces, containing the null Killing vector field, and showed that it admits a natural projective structure. Local expressions for a general conformal structure admitting a null conformal Killing vector field were also derived in [11]. The form of a representative metric $g \in [g]$ was given in two cases; depending on whether the twist vanishes. Recall that the twist is given by $\mathbb{K} \wedge d\mathbb{K}$, where $\mathbb{K} := g(K, \cdot)$. In the non-twisting case, the ASD condition is completely solvable and the metric $g$ is given in terms of arbitrary functions of two variables. If the twist is nonzero, then the ASD condition reduces to a linear PDE.

Let us now look back the ASD conformal equations (3.9) - (3.12). If $F_Y = 0$, it follows from (3.11) that $\widehat{F} = 0$, and (3.9) and (3.10) reduce to one equation, $\widehat{G}_Y = 0$. Together with (3.12), this gives

$$G_{ZYY} - G_X G_{YY} + (F_X + G_Y) G_{XY} = 0,$$  (3.19)

where $F$ is an arbitrary function of $X$ and $Z$. Now, it can be shown that the twist $\mathbb{K} \wedge d\mathbb{K}$ vanishes if and only if $G_{YY} = 0$.

Hence in the non-twisting case, with $G_{YY} = 0$, equation (3.19) is satisfied trivially, and the metric (2.11) is given by

$$g = dW dX + dZ dY + (A + B) dW dZ + (Y B_X + C) dZ^2,$$  (3.20)
for arbitrary functions $A(X, Z)$, $B(X, Z)$ and $C(X, Z)$.

In the twisting case, where $G_{YY} \neq 0$, a theorem in [10] guarantees that equation (3.19) can be linearised. It is interesting to find an exact transformation which puts the metric (2.11) in the local form of [10] and read off the projective structure. We leave this for future work.

### 3.2 From ASD Null-Kähler condition to the dKP equation

In this section we shall assume that the ASD conformal class admits a null-Kähler metric. Recall that a metric of signature $(+ + --)$ is called a null-Kähler metric if it admits a covariantly constant real spinor. ASD null-Kähler metrics were studied in [5]. In particular it was shown that all ASD null-Kähler metrics can locally be written in the form

$$g = dW dX + dZ dY - \theta_{YY} dW^2 + 2 \theta_{XY} dW dZ - \theta_{XX} dZ^2,$$

(3.21)

where function $\theta$ satisfies a fourth order PDE.

Now the form (3.21) can be obtained from the representative metric $g$ (2.1) in Theorem 2.1 via the ansatz [8]

$$F = -\theta_Y, \quad G = -\theta_X.$$  

(3.22)

It also turns out that, via this ansatz, the ASD conformal equations (2.2, 2.3) reduces to the fourth order PDE for the null-Kähler condition. To see this, note that with the ansatz (3.22), $Q(F)$ and $Q(G)$ can be written as

$$Q(F) = -\partial_Y f, \quad Q(G) = -\partial_X f, \quad \text{where} \quad f = \theta_{XX} + \theta_{YZ} + \theta_{XY} \theta_{YY} - \theta_{XX}^2.$$  

(3.23)

Then one of the ASD conformal equations, (2.2), is then satisfied trivially by the commutativity of partial derivatives, and (2.3) becomes

$$\tilde{Q}(f) = 0, \quad \text{where} \quad \tilde{Q} = \partial_W \partial_X + \partial_Z \partial_Y + \theta_{YY} \partial_X^2 + \theta_{XX} \partial_Y^2 - 2 \theta_{XY} \partial_X \partial_Y,$$  

(3.24)

which is the fourth order equation in $\theta$ determining the ASD null-Kähler metric found in [5].

As mentioned in the Introduction, any ASD null-Kähler metric with symmetry preserving the parallel spinor determines an EW structure with a parallel weighted vector field, via the Jones-Tod correspondence. The latter in turn is governed by a solution of the dKP equation [4]. To be precise, we have

**Theorem 3.1** [4] *Any three-dimensional Lorentzian EW structure which admits a covariantly constant weighted vector field can be locally represented by a metric $h \in [h]$ and*
a one-form $\nu$ of the form

$$h = dy^2 - 4dx dt - 4f dt^2, \quad \nu = -4f_x dt,$$  \hspace{1em} (3.25)

where the function $f(x, y, t)$ satisfies the dKP equation

$$(f_t - f f_x)_x = f_{yy}.$$  \hspace{1em} (3.26)

An explicit local expression of an ASD null-Kähler metric with symmetry in terms of solutions of the dKP equation and its linearisation (in ‘potential’ form) is given in \[5\]. The proof is based on the Jones-Tod correspondence and the twistor correspondences for the ASD null-Kähler and the EW metrics. However, here we shall show that one can obtain the dKP equation directly from the fourth order PDE (3.24) using a non-null translational Killing symmetry and a Legendre transformation, in a similar way as in Section 3.1.

Assume that the metric $g$ (3.21) admits a Killing symmetry generated by $\partial_Y$, which preserves the null-Kähler two form $dW \wedge dZ$ corresponding to the parallel spinor. Then without loss of generality one can assume that $\theta$ is independent of $W$, and equation (3.24) can be written as a coupled system of equations as

$$f_{YZ} + \theta_{YY} f_{XX} + \theta_{XX} f_{YY} - 2\theta_{XY} f_{XY} = 0$$  \hspace{1em} (3.27)

$$f - (\theta_{YZ} + \theta_{XX} \theta_{YY} - \theta_{XY}^2) = 0.$$  \hspace{1em} (3.28)

The system (3.27, 3.28) is equivalent to

$$df_Y \wedge dX \wedge dY + df_X \wedge d\theta_Y \wedge dZ - df_Y \wedge d\theta_X \wedge dZ = 0$$  \hspace{1em} (3.29)

$$f \, dX \wedge dY \wedge dZ - d\theta_X \wedge d\theta_Y \wedge dZ + d\theta_Y \wedge dY \wedge dX = 0.$$  \hspace{1em} (3.30)

Now, introducing new variables $(x, y, t)$, where $x = \theta_Y, y = X, t = -Z$, and a new function $H(x, y, t) = \theta - xY$, we have

$$dH = \theta_X dy - Y dx - \theta_Z dt,$$

where we have used $dy = dX$ and $dt = -dZ$. This implies that

$$\theta_X = H_y, \quad Y = -H_x, \quad \theta_Z = -H_t.$$  \hspace{1em} (3.31)
Making the substitution according to (3.31), equation (3.30) gives

\[ f = \frac{H_{xt} - H_{yy}}{H_{xx}}, \quad (3.32) \]

and (3.29) becomes

\[ \begin{align*}
  d \left( \frac{f_x}{H_{xx}} \right) \wedge dy \wedge dH_x - d \left( \frac{f_y - f_x H_{xy}}{H_{xx}} \right) \wedge dx \wedge dt - d \left( \frac{f_x}{H_{xx}} \right) \wedge dH_y \wedge dt &= 0, \\
\end{align*} \quad (3.33) \]

where we have used \( f_Y = -\frac{f_x}{H_{xx}} \) and \( f_X = f_y - \frac{f_x H_{xy}}{H_{xx}} \). Then making use of (3.32), equation (3.33) yields the dKP equation (3.26), as required.

Note that imposing the constraint \( v = 0 \) and letting \( u = -f \) reduces the MS system (2.6) to the dKP equation (3.26).

To see that the corresponding EW metric is of the form (3.25), first let \( w = W \) and write the ASD null-Kähler metric (3.21) in the \((w, x, y, t)\) coordinates as

\[ g = dwdy + dt(H_{xy}dy + H_{xx}dx + H_{xt}dt) - \left( H_{yy} - \frac{H_{xy}^2}{H_{xx}} \right) dt^2 + 2 \frac{H_{xy}}{H_{xx}} dt dw + \frac{1}{H_{xx}} dw^2, \]

where we have used \( \theta_{YY} = -\frac{1}{H_{xx}}, \theta_{XX} = H_{yy} - \frac{H_{xy}^2}{H_{xx}} \) and \( \theta_{XY} = -\frac{H_{xy}}{H_{xx}} \).

Using (3.32), the metric can be rearranged to be in the form

\[ g = V \left( dy^2 - 4 dx dt - 4 f dt^2 \right) - \frac{1}{V} \left( d \left( \frac{w}{2} \right) + \alpha \right)^2, \quad (3.34) \]

where \( V = -\frac{H_{xx}}{4} \) and \( \alpha \) is a one-form given by \( \alpha = \frac{H_{xy}}{2} dt - V dy \). Thus, comparing (3.34) with (2.8), one sees that the EW metric on the space of orbits of symmetry is indeed of the form (3.25).

Let us now end this section by noting that a solution of the ASD null-Kähler equation (3.24) is the trivial solution \( f = 0 \), in which case \( \theta \) satisfies the second heavenly equation

\[ \theta_{XW} + \theta_{YZ} + \theta_{XX} \theta_{YY} - \theta_{XY}^2 = 0, \quad (3.35) \]

and the metric is Ricci-flat. The second heavenly equation was first introduced by Plebański [24] as a form of the ASD Ricci-flat condition. Any ASD Ricci-flat metric
in the neutral signature can be written locally in the form of (3.21), with \( \theta \) a solution of the second heavenly equation (3.35). The idea of using the Legendre transformation in our calculation comes from the work of Finley and Plebański [14]. There, it was shown that if an ASD Ricci-flat metric admits a non-null translational Killing symmetry, then the metric can be put in the Gibbons-Hawking ansatz [16]. Moreover, using a Legendre transformation, the second heavenly equation (3.35) can be linearised to the three-dimensional Laplace’s equation

\[
H_{yy} - H_{xt} = 0.
\] (3.36)

4 Homothetic symmetry reduction - Lineage of hyper-CR Einstein-Weyl equations

In this section, we investigate the symmetry reduction by a particular homothetic Killing symmetry. Our interest in this conformal Killing symmetry comes from the fact that it reduces the second heavenly equation - ASD Ricci-flat condition - to the so-called hyper-CR system, which is another reduction of the MS system.

First, recall that the hyper-CR Einstein-Weyl structures are characterised by the absence of the derivative with respect to the spectral parameter in the Einstein-Weyl Lax pair. The name comes from the fact that they arise on the space of orbits of a tri-holomorphic conformal Killing vector field on a hyper-Hermitian manifold [15] (see also references in physics literature such as [21, 2]). Now, as the conformal Killing vector field preserves the hyperboloid of complex structures, these descend to a hyperboloid of Cauchy-Riemann structures on the EW space, thus the name hyper-CR. Via the Jones-Tod correspondence, a special solution to the generalised monopole equation (2.7) can be chosen to construct a hyper-Kähler metric from a hyper-CR EW metric via (2.8). On the other hand, the hyper-CR EW metrics were obtained as reductions of hyper-Kähler metrics in [9].

Minor sign changes can be applied to obtain analogous results for Lorentzian signature. It was shown in [6] that any Lorentzian hyper-CR EW structure is locally determined by solutions to a pair of quasi-linear PDEs of hydrodynamic type [11, 12, 20, 22]

\[
\begin{align*}
a_t + b_y + ab_x - ba_x &= 0, \\
a_y + b_x &= 0,
\end{align*}
\] (4.1)
which we shall call the hyper-CR system. Via the symmetry reduction of a pseudo-
hyper-Kähler metric by a tri-holomorphic homothetic Killing vector field, it was shown
that the second heavenly equation (3.35) reduces to the hyper-CR system (4.1), and the
EW metric \( h \) and the associated one-form \( \nu \) are locally of the form

\[
h = (dy - adt)^2 - 4(dx - ady + bdt)dt, \quad \nu = a_x dy + (a a_x + 2a_y) dt.
\]

Note that (4.1) is a reduction of the MS system under the constraint \( u = 0 \) \[8\] and setting
\( a = -v_x \) and \( b = v_y \).

We shall now lift the hyper-Kähler, i.e. ASD Ricci-flat, condition and apply this
symmetry reduction to a general ASD conformal structure. It turns out that one also
obtains the Manakov-Santini system, rather than a special case of it, from this reduction.
Moreover, assuming this conformal symmetry on an ASD conformal class admitting a
null-Kähler metric, we recover the MS system from the fourth order PDE (3.21).

### 4.1 From ASD condition to the MS system

Without loss of generality, a tri-holomorphic homothetic Killing vector field \( K \) to a
pseudo-hyper-Kähler metric, of the form (3.21), is given by

\[
K = Z \frac{\partial}{\partial Z} + X \frac{\partial}{\partial X}.
\]  

(4.2)

Now, suppose a general ASD metric (2.1)

\[
g = dWdX + dZdY + F_Y dW^2 - (F_X + G_Y) dWdZ + G_X dZ^2
\]

admits a conformal symmetry generated by the homothety (4.2). It follows that

\[
K(F_Y) = F_Y, \quad K(G_X) = -G_X, \quad K(F_X + G_Y) = 0.
\]  

(4.3)

With gauge freedom, it is possible to reduce (4.3) to

\[
K(F) = F, \quad K(G) = 0,
\]

whose general solutions are given by

\[
F = Z P \left( \frac{X}{Z}, Y, W \right), \quad G = R \left( \frac{X}{Z}, Y, W \right)
\]

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for arbitrary functions $P$ and $R$.

Following the change of variables used for the pseudo-hyper-Kähler case in [6], let $U = \frac{X}{Z}$ and $S = \ln Z$, then $K = \frac{\partial}{\partial S}$. One can write the ASD conformal equations (2.2, 2.3) in the new variables $(U, Y, W)$ as follows.

First, let $\hat{F} := Q(F)$ and $\hat{G} := e^S Q(G)$, where $Q$ is the differential operator given by (2.4). Notice that $\hat{F}$ and $\hat{G}$ depend only on $(U, Y, W)$, and equations (2.2) and (2.3) become

\[
\hat{F}_U - \hat{G}_Y = 0, \quad (4.4)
\]

\[
\hat{G}_W - P_Y \hat{G}_U + R_Y \hat{G}_Y + (P_U - U) \hat{F}_U - R_U \hat{F}_Y = 0, \quad (4.5)
\]

respectively.

Equation (4.4) implies the existence of a function $u(U, Y, W)$ such that $\hat{F} = u_Y$, $\hat{G} = u_U$, and (4.4) is satisfied trivially. Then, in terms of differential forms, (4.5) becomes

\[
du \wedge dU \wedge dY - du \wedge dP \wedge dW + du \wedge dR \wedge dW + Udu \wedge dU \wedge dW = 0. \quad (4.6)
\]

Moreover, the definitions of $\hat{F}$ and $\hat{G}$, which becomes

\[
\hat{F} = P_{UW} + P_Y - UP_{UY} - P_Y P_{UU} - R_U P_{YY} + (P_U + R_Y) P_{UY},
\]

\[
\hat{G} = R_{UW} - U R_{UY} - P_Y R_{UU} - R_U R_{YY} + (P_U + R_Y) R_{UY},
\]

yield

\[
du \wedge dU \wedge dW = dP_U \wedge dY \wedge dU + dP \wedge dU \wedge dW - UdP_U \wedge dU \wedge dW
\]

\[
- dP \wedge dP_U \wedge dW + dR \wedge dP_Y \wedge dW, \quad (4.7)
\]

\[
du \wedge dY \wedge dW = dR_U \wedge dU \wedge dY - UdR_U \wedge dW \wedge dU
\]

\[
+ dP \wedge dR_U \wedge dW - dR \wedge dR_Y \wedge dW, \quad (4.8)
\]

respectively.

Now, using the change of variables and the Legendre transform similar to those in Section 3.1, the system (4.6, 4.7, 4.8) can be shown to be equivalent to the MS system (2.6) as follows.

First, assume that $R_U \neq 0$ throughout a neighbourhood of interest. For if $R_U = 0$, the conformal Killing vector field $K$ becomes null and the conformal structure on the space of orbits is degenerate. Then introduce new variables $(x, y, t)$, where $x = -R(U, Y, W)$,
\[ y = Y \text{ and } t = -W. \text{ Let } U = \Phi(x, y, t). \text{ In these new variables, (4.6) becomes} \]
\[
d \left( \frac{u_x}{\Phi_x} \right) \wedge (d\Phi \wedge dy + dP \wedge dt) + d \left( u_y - u_x \frac{\Phi_y}{\Phi_x} \right) \wedge dx \wedge dt - \Phi d \left( \frac{u_x}{\Phi_x} \right) \wedge d\Phi \wedge dt = 0, \]

which is equivalent to
\[
u_{xt} - u_{yy} + \left( \frac{P_y - \Phi_t - \Phi \Phi_y}{\Phi_x} \right) u_{xx} + \left( \Phi + \frac{\Phi_y - P_x}{\Phi_x} \right) u_{xy} \]
\[+ \left( \partial_x \left( \frac{P_y - \Phi_t - \Phi \Phi_y}{\Phi_x} \right) + \partial_y \left( \Phi + \frac{\Phi_y - P_x}{\Phi_x} \right) \right) u_x = 0. \tag{4.9} \]

Note that \( dx \wedge dy \wedge dt \neq 0 \) is guaranteed by the assumption that \( RU \neq 0 \).

Similarly, (4.8) gives
\[
u_x = \partial_x \left( \frac{P_y - \Phi_t - \Phi \Phi_y}{\Phi_x} \right) + \partial_y \left( \Phi + \frac{\Phi_y - P_x}{\Phi_x} \right). \tag{4.10} \]

Integrating the above equation with respect to \( x \), we have
\[
u = \frac{P_y - \Phi_t - \Phi \Phi_y}{\Phi_x} + \partial_y \int \left( \Phi + \frac{\Phi_y - P_x}{\Phi_x} \right) dx, \]
where we have used the assumption that all functions involved are real analytic and thus can be differentiated under the integral sign. This implies that
\[
u = \frac{P_y - \Phi_t - \Phi \Phi_y}{\Phi_x} + v_y \tag{4.11} \]

for a function \( v(x, y, t) \) defined by
\[
v = \int \left( \Phi + \frac{\Phi_y - P_x}{\Phi_x} \right) dx, \text{ and thus } v_x = \Phi + \frac{\Phi_y - P_x}{\Phi_x}. \tag{4.12} \]

Making substitutions from (4.10)-(4.12), equation (4.9) becomes
\[
u_{xt} - u_{yy} + (\nu - v_y) u_{xx} + v_x u_{xy} + u_x^2 = 0, \]
which is the first equation of the MS system (2.6).

Lastly, through similar procedure, equation (4.7) can be realised as the second equation of the MS system,
\[
u_{xt} - v_{yy} + (\nu - v_y) v_{xx} + v_x v_{xy} = 0. \tag{4.13} \]
In the adapted coordinates \((S, U, Y, W)\), the metric \((2.1)\) becomes

\[
g = e^S \left( dW dU + dS dY + P_Y dW^2 + (U - P_U - R_Y) dW dS + R_U dS^2 \right).
\]

Now let \(s = S\), and with the change of variables \(y = Y, t = -W, x = -R(U, Y, W)\) and letting \(U = \Phi(x, y, t)\) as before, the conformal metric \(\hat{g} = e^{-S} g\) is given by

\[
\hat{g} = -dtd\Phi + dsdy + \left( P_y - \frac{P_x \Phi_y}{\Phi_x} \right) dt^2 + \left( \frac{P_x + \Phi_y}{\Phi_x} - \Phi \right) dtds - \frac{1}{\Phi_x} ds^2.
\]

This can be rearranged in the form \((2.8)\)

\[
\hat{g} = V h - \frac{1}{V} (d\tau + \alpha)^2,
\]

where \(\tau = \frac{s}{2}, V = \frac{\Phi_x}{4}\) and \(\alpha = (P_x + \Phi_y - \Phi \Phi_x)dt + \Phi_x dy\). Then, using \((4.11)\) and \((4.12)\), one obtains the EW metric \(h\) precisely of the form \((2.5)\) albeit the minus sign.

Let us now comment on the case \(R_U = 0\). The homothety \(K (4.2)\) is then null with respect to the conformal class, and the space of \(\beta\) surfaces admits a natural projective structure \([10]\), as discussed at the end of Section \(3.1\). One finds that the ASD conformal equations reduce to a single third order PDE

\[
L(P_{UU}) = 0, \quad \text{where} \quad L = \partial_W + (P_U + R_Y - U) \partial_Y - P_Y \partial_U. \quad (4.13)
\]

The work of \([10]\) guarantees that equation \((4.13)\) is completely solvable if the twist of \(K\) vanishes, and linearisable if it does not. In our coordinate system, it can be shown that the twist of \(K\) is zero if and only if \(P_{UU} = 1\). Thus, in the non-twisting case it is clear that \((4.13)\) is satisfied trivially and the conformal class can be represented by a metric of the form \((3.20)\), upon renaming the coordinates. We leave the linearisation of \((4.13)\) in the twisting case for future work.

### 4.2 Conformally Null-Kähler condition

We shall now assume that the ASD conformal class \([g]\) admits a null-Kähler metric. Thus \([g]\) can be represented by the a null-Kähler metric \(g\) of the form \((3.21)\)

\[
g = dW dX + dZ dY - \theta_{YY} dW^2 + 2\theta_{XY} dW dZ - \theta_{XX} dZ^2,
\]
where we recall that $\theta$ satisfies a fourth order PDE, which can be written as a coupled system

\begin{align}
    f_{WX} + f_{YZ} + \theta_{YY} f_{XX} + \theta_{XX} f_{YY} - 2\theta_{XY} f_{XY} &= 0 \quad (4.14) \\
    f - (\theta_{XW} + \theta_{YZ} + \theta_{XX} \theta_{YY} - \theta_{XY}^2) &= 0. \quad (4.15)
\end{align}

The assumption of the homothetic symmetry generated by the vector field (4.2), $K = Z \frac{\partial}{\partial Z} + X \frac{\partial}{\partial X}$, then implies that

\begin{align*}
    K(\theta_{XX}) = -\theta_{XX}, \quad K(\theta_{YY}) = \theta_{YY}, \quad K(\theta_{XY}) = 0.
\end{align*}

With gauge freedom, the system reduces to $K(\theta) = \theta$, whose general solution is given by

\begin{align*}
    \theta = Z P \left( \frac{X}{Z}, Y, W \right) \quad \text{for an arbitrary function } P.
\end{align*}

Then, using the same adapted coordinates $(S, U, Y, W)$ defined in Section 4.1, the system (4.14, 4.15) becomes

\begin{align}
    f_{UU} - U f_{UY} + P_Y f_{UU} + P_{UU} f_{YY} - 2P_{UY} f_{UY} &= 0 \quad (4.16) \\
    f - (P_{UW} + P_Y - UP_{UY} + P_{UU} P_{YY} - P_{UY}^2) &= 0. \quad (4.17)
\end{align}

Following the same procedure as in Section 3.2, we introduce new variables $x = P_U$, $y = Y$, $t = -W$ and a function $H(x, y, t) = P - xU$, which satisfies $H_t = -P_W$, $H_y = P_Y$ and $H_x = U$. It turns out surprisingly that we recover the full MS system, rather than a reduction of it. That is, the system (4.16, 4.17) is equivalent to the coupled system

\begin{align}
    f_{yy} - f_{xt} + H_x f_{xy} - H_y f_{xx} + (f f_x)_x &= 0 \\ 
    H_{yy} - H_{xt} + H_x H_{xy} - H_y H_{xx} + f H_{xx} &= 0, \quad (4.18, 4.19)
\end{align}

which is precisely the MS system (2.6) upon renaming $f = -u$ and $H = -v$.

5 Discussion: Einstein-Weyl equations via Bogdanov’s system

In this last section we discuss another local representation of the Einstein-Weyl equations, other than the MS system and its reductions. It was suggested in [8] that a general solution of the Bogdanov’s system should also locally determines a generic EW structure. It is therefore natural to expect that the Bogdanov’s system should arise from the ASD
conformal equations \((2.2, 2.3)\) under a symmetry assumption. Our attempts to achieve this goal have not been successful so far. Nevertheless, let us present our exploration of a special case, namely the ASD null-Kähler condition.

The Bogdanov’s system \(\Pi\) is given by

\[
\begin{align*}
(e^{-\phi})_{tt} &= m_t \phi_{xz} - m_x \phi_{zt}, \\
mt e^{-\phi} &= m_x m_{zt} - m_t m_{xz}.
\end{align*}
\] (5.1)

It is regarded as a two-component generalisation of the \(SU(\infty)\)-Toda equation: Setting \(m = t\), then the second equation is satisfied trivially and the first equation becomes the \(SU(\infty)\)-Toda equation

\[
(e^{-\phi})_{tt} = \phi_{xx}.
\] (5.2)

The reduction suggests a Killing symmetry to be assumed on the ASD conformal structure. This is again motivated by the work of Finley and Plebański \(\Pi\), which shows that under a certain Killing symmetry assumption the second heavenly equation, governing the ASD hyper-Kähler structure, reduces to the \(SU(\infty)\)-Toda equation. Therefore one expects that assuming the same symmetry on an ASD conformal structure should give rise to the Bogdanov’s system or a special case of it.

In what follows, we shall extend the reduction procedure in \(\Pi\), to apply it to an ASD null-Kähler metric. As a result, a coupled system of equations governing the corresponding EW structure is obtained.

Assume that a general ASD null-Kähler metric \((3.21)\) admits a Killing symmetry generated by the vector field \(K = W \partial_W - X \partial_X\). Note that \(K\) does not preserve the null-Kähler two form \(dW \wedge dZ\), corresponding to the parallel spinor. Therefore we do not expect to recover the dKP equation. The Killing equations are given by

\[
\begin{align*}
K(\theta_{XX}) &= 0, \\
K(\theta_{XY}) &= -\theta_{XY}, \\
K(\theta_{YY}) &= -2\theta_{YY},
\end{align*}
\]

which, using the gauge freedom, can be simplified to

\[
K(\theta) = -2\theta.
\] (5.3)

The general solution of \((5.3)\) is

\[
\theta = W^{-2} P(WX, Y, Z).
\] (5.4)
Let $V = WX$, then the ASD null-Kähler condition (4.14, 4.15) becomes

\[
\hat{f} - (VP_{VV} + P_{VV}P_{VY} - P_{VY}^2 - P_V + P_{YZ}) = 0 \quad (5.5)
\]

\[
V\hat{f}_{VV} - \hat{f}_V + \hat{f}_{YZ} + P_{YY}\hat{f}_{VV} + P_{VY}\hat{f}_{VV} - 2P_{VV}\hat{f}_{YY} = 0, \quad (5.6)
\]

where $\hat{f} = W^2 f$. Note that (5.5) implies that $\hat{f}$ is a function of $(V, Y, Z)$ only.

Equations (5.5) and (5.6) are equivalent to

\[
\hat{f}dY \wedge dV \wedge dZ = dP_Y \wedge dP_Y \wedge dZ + dP_Y \wedge dY \wedge dY - VdP_Y \wedge dY \wedge dZ - P_VdY \wedge dV \wedge dZ, \quad (5.7)
\]

\[
dP_Y \wedge d\hat{f}_V \wedge dZ + d\hat{f}_V \wedge dP_Y \wedge dZ + d\hat{f}_V \wedge dY \wedge dV - Vd\hat{f}_V \wedge dY \wedge dZ - \hat{f}_VdY \wedge dV \wedge dZ = 0, \quad (5.8)
\]

respectively. Now, letting

\[
\gamma = -dP_Y + P_VdZ - VdY, \quad (5.9)
\]

equation (5.7) becomes

\[
\gamma \wedge d\gamma = -\hat{f} dY \wedge dV \wedge dZ. \quad (5.10)
\]

If $f$ and thus $\hat{f}$ are identically zero, the metric (3.21) is Ricci-flat and, as shown in [14], is determined by the $SU(\infty)$-Toda equation. To see this, note that when $f = 0$, (5.10) becomes

\[
\gamma \wedge d\gamma = 0,
\]

where $\gamma$ is given by (5.9). By the Frobenius’ theorem this implies the existence of functions $R, S$ such that

\[
\gamma = RdS. \quad (5.11)
\]

Equating (5.11) and (5.9) and introducing new coordinates $(t, x, z)$ where $t = V$, $x = S(Y, V, Z)$ and $z = -Z$, it can be shown that

\[
R_{tt} + (\ln R)_{xz} = 0,
\]

which gives the $SU(\infty)$-Toda equation (5.2) upon setting $R = e^{-\phi}$, as shown in [14].

For the general case where $f$, and thus $\hat{f}$, are nowhere vanishing in some neighbourhood, equation (5.10) and the Frobenius’ theorem imply the existence of functions $R, S, H$.
of variables \((Y, V, Z)\) such that

\[
\gamma = dH + RdS = -dP_Y + P_VdZ - VdY. \tag{5.12}
\]

Guided by the Ricci-flat case, we introduce new coordinates \((t, x, z)\), where \(t = V, x = S(Y, V, Z)\) and \(z = -Z\), as before. Our aim is to transform the ASD null-Kähler conditions \((5.5, 5.6)\), for two unknowns \(\hat{f}\) and \(P(V, Y, Z)\), to a system of equations for \(R(x, z, t)\) and \(H(x, z, t)\).

After some manipulation, we find that \((5.5, 5.6)\) reduces to a complicated system of equation for \(R\) and \(H\).

That is,

\[
R_{tt} + \left( \frac{R_z - H_t R_t}{R + H_x} \right)_x = 0, \tag{5.13}
\]

\[
A_t R_z - (B + A_z) R_t = H_t B_x + (R + H_x) B_t = 0, \tag{5.14}
\]

where \(A = -\frac{\hat{f}_x}{R_t}\), \(B = \hat{f}_t + \frac{\hat{f}_x}{R_t} \left( \frac{R_z - H_t R_t}{R + H_x} \right), \quad \hat{f} = H_z - \frac{H_t R_z}{R_t} \).

However, we have so far not been able to realise this system as the Bogdanov’s system \((5.1)\) or a special case of it.

The corresponding EW metric in terms of solutions of \((5.13, 5.14)\) is given by

\[
h = \left( dt^2 + 4H_t dt dz + \frac{4H_t R_z}{R_t} dz^2 \right) + 4(R + H_x) dx dz, \tag{5.15}
\]

A possible step forward could be to compare \((5.15)\) with the general expression for an EW metric \([13]\) in terms of a solution to the Bogdanov’s system \((5.1)\), that is

\[
h = (m_x dx + m_t dt)^2 + 4 e^{-\phi} m_t dx dz,
\]

and realise a transformation between the two. This problem still remains open.

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