PERELMAN’S W-FUNCTIONAL AND
STABILITY OF KÄHLER-RICCI FLOW

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0. Introduction.

In this note we study the second variation of Perelman’s W-functional on the space of Kähler metrics at a Kähler-Ricci soliton and its applications. A Kähler metric $g_{KS}$ on a compact manifold $M$ is called a (shrinking) Kähler-Ricci soliton if its Kähler form $\omega_{KS}$ satisfies the equation

$$\text{Ric}(\omega_{KS}) - \omega_{KS} = L_X \omega_{KS},$$

where $\text{Ric}(\omega_{KS})$ is the Ricci form of $g_{KS}$ and $L_X \omega_{KS}$ denotes the Lie derivative of $\omega_{KS}$ along a holomorphic vector field $X$ on $M$. If $X = 0$, then $g_{KS}$ is a Kähler-Einstein metric with positive scalar curvature. We will show that the second variation of Perelman’s W-functional is non-positive in the space of Kähler metrics with $2\pi c_1(M)$ as Kähler class. Furthermore, if $(M, g_{KS})$ is a Kähler-Einstein manifold, then the second variation is non-positive in the space of Kähler metrics with Kähler classes cohomologous to $2\pi c_1(M)$ (complex structures on $M$ may vary). This implies that Perelman’s W-functional is stable in the sense of variations. We will also study the kernel of elliptic operators which arise from the second variation. As an application, we will prove a stability theorem about Kähler-Ricci flow near a Kähler-Einstein metric.

The organization of this paper is as follows: In Section 1, we review Perelman’s W-functional and give a formula for its second variation. In Section 2, we compute the second variation of W-functional on the space of Kähler metrics with Kähler class $2\pi c_1(M)$ on a fixed complex manifold $M$. In Section 3, we extend our calculations to possible varying complex structures on a Kähler-Einstein manifold. In Section 4, a stability theorem about Kähler-Ricci flow will be proved.

The result of this problem was discussed in the lectures by the first named author in the Clay summer school on Ricci Flow and Geometrization in the summer of 2005.

1. The second variation formula of $\lambda(g)$.

Recently, G. Perelman introduced a functional on a compact differential manifold $M$ of dimension $n$ [Pe],

$$W(g, f, \tau) = (4\pi \tau)^{-n} \int_M [\tau (R(g) + |Df|^2) + f - n] e^{-f} dV_g,$$  \hspace{1cm} (1.1)

1991 Mathematics Subject Classification. Primary: 53C25; Secondary: 32J15, 53C55, 58E11.

* Partially supported by NSF10425102 in China

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where $R(g)$ denotes the scalar curvature of a Riemannian metric $g$, $f$ is a smooth function and $\tau$ is a constant. Furthermore, we may normalize the triple $(g, f, \tau)$ so that

$$\left(4\pi\tau\right)^{-n} \int_M e^{-f} dV_g \equiv 1.$$  

In our case, we will further normalize the volume of $g$, i.e.,

$$\int_M dV_g \equiv 1. \tag{1.2}$$

Then the $W$-functional can be reduced to the following functional on a pair $(g, f)$,

$$W(g, f) = \int_M \left[(R(g) + |Df|^2) + f\right] e^{-f} dV_g, \tag{1.1'}$$

where $(g, f)$ satisfies

$$\int_M e^{-f} dV_g = \int_M dV_g \equiv 1. \tag{1.3}$$

For any Riemannian metric $g$ with normalized volume (1.2), we define

$$\lambda(g) = \inf_f \{W(g, f) \mid f \text{ satisfies (1.3)}\}.$$ 

The number $\lambda(g)$ can be attained by some $f$. In fact, such a $f$ is a solution of the equation,

$$2\Delta f + f - |Df|^2 + R = \lambda(g). \tag{1.4}$$

As in [Pe], we have the first variation of $\lambda(g)$,

$$\delta\lambda(g) = - \int_M <\delta g, \text{Ric}(g) - g + D^2 f > e^{-f} dV_g, \tag{1.5}$$

where $\text{Ric}(g)$ denotes the Ricci tensor of $g$ and $D^2 f$ is the Hessian of $f$. It follows from (1.5) that $g$ is a critical metric of $\lambda(g)$ if and only if $g$ is a gradient shrinking Ricci soliton, i.e., the metric $g$ satisfies,

$$\text{Ric}(g) - g = -D^2 f,$$

for some smooth function $f$.

By the standard computation, one can easily get the second variation of $\lambda(g)$ at a critical point, i.e., a gradient shrinking Ricci soliton $g_{RS}$. This is given in the following proposition (also see [CHI]).

**Proposition 1.1.** Let $(g_{RS}, f)$ be a gradient shrinking Ricci soliton on $M$. Let $L_0$ and $L'$ be defined on the space of symmetric tensors of rank 2, respectively, by

$$L_0 h = -\frac{1}{2} D^* D h + \text{Rm}(h, \cdot) + \frac{1}{2} (D^2 f \cdot h + h \cdot D^2 f)$$

where $D^* D h + \text{Rm}(h, \cdot)$ is the trace-free Ricci curvature and $D^2 f$ is the Hessian of $f$.
and
\[
L'h = \Delta(tr(h)) + tr(h) + < h, D^2 f >_{g_{RS}} - \text{div}(\text{div} h)
\]
\[
+ < \text{div} h, Df >_{g_{RS}} - \frac{1}{2} < D(\Delta(tr(h))), Df >_{g_{RS}} - < Df, Df >_h,
\]
where \( < Df, Df >_h = \sum_i h^{ij} f_i f_j \) and \( h^{ij} = \sum_{kl} g^{ik} g^{jl} h_{kl} \). Then we have
\[
\delta^2 \lambda(g)(h, h) = (h, Lh)_g_{RS} = \int_M < h, Lh >_{g_{RS}} dV_{g_{RS}},
\]
(1.6)
where \( L \) is defined by
\[
Lh = L_0 h + \text{div}^* \cdot \text{div} h + \frac{1}{2} D^2(tr(h)) - D^2(P^{-1} \cdot L'(h)),
\]
and \( P \) is given by
\[
P\psi = 2\Delta \psi + 2 < Df, D\psi >_{g_{RS}} + \psi, \forall \psi \in C^\infty(M).
\]

***Proof.*** According to [Be], we have
\[
\delta \text{Ric}(g)(h) = \frac{1}{2} D^* Dh - \text{Rm}(h, \cdot) + \frac{1}{2} (\text{Ric} \cdot h + h \cdot \text{Ric})
\]
\[
- \text{div}^* \cdot \text{div} h - \frac{1}{2} D^2(tr(h)).
\]
On the other hand, by (1.4), one gets an equation for \( \delta f \),
\[
P(\delta f) = L'h.
\]
Combining these two, we obtain (1.6). \qed

2. The case for a fixed complex structure.

In this section, we compute the second variation of \( \lambda(g) \) restricted to the space of Kähler metrics with Kähler forms in \( 2\pi c_1(M) > 0 \). This variation is computed at a (shrinking) Kähler-Ricci soliton and contains more information than that in the real case.

Let \((g_{KS}, X)\) be a Kähler Ricci soliton with its Kähler form \( \omega_{KS} \) in \( 2\pi c_1(M) \) on a compact complex manifold \((M, J)\), where \( J \) denotes a complex structure of \( M \). We consider all Kähler metrics \( g \) with Kähler forms in \( 2\pi c_1(M) \). Without loss of generality, we may assume that \( g = g_t = g_{KS} + th_0 \) is a family of such Kähler metrics, where
\[
h_0 = \sum_{ij} \partial_i \partial_j \psi dz^i \otimes d\overline{z}^j
\]
for some real-valued smooth function \( \psi \). We shall compute \( \frac{d^2 \lambda(g_t)}{dt^2} \)|\(t=0\). Put
\[
P_0 \psi = 2\Delta \psi + \psi - (X + \overline{X})(\psi),
\]
\[
L_1 \psi = \Delta \psi + \psi - X(\psi),
\]
and
\[
L'_1 \psi = \Delta \psi - X(\psi).
\]
Lemma 2.1. Let $g_t = g_{KS} + t h_0$ be a family of Kähler metrics as above and $f = f_t$ be a family of smooth functions which are solutions of (1.4) associated to $g_t$. Let $u = \frac{df}{dt} |_{t=0}$. Then

$$P_0(u - X(\psi)) = (L'_1 \cdot L_1)(\psi).$$

Proof. Differentiating relations (1.4) at $t = 0$, we have

$$P_0(u) = 2 \Delta u + u - 2\text{re}(X(u))$$
$$= \Delta^2 \psi + <\text{Ric}(g_{KS}), \sqrt{-1} \partial \bar{\partial} \psi > + \psi \frac{1}{2} (2 f_j j - f_j f_j).$$

It follows

$$P_0(u) = \Delta^2 \psi + \Delta \psi + \psi \frac{1}{2} (f_j j - f_j f_j)$$
$$= \Delta^2 \psi + \Delta \psi + \Delta (X(\psi)) - X(\Delta \psi) - X(X(\psi))$$
$$= (\Delta - X)(\Delta \psi + \psi - X(\psi)) + 2 \Delta (X(\psi)) + X(\psi) - X(X(\psi)) - \bar{X}(X(\psi))$$
$$= [(\Delta - X) \cdot L_1](\psi) + P_0(X(\psi)).$$

The lemma follows. □

Proposition 2.1. Let $g_t = g_{KS} + t h_0$ be a family of Kähler metrics as in Lemma 2.1. Then

$$\frac{d^2 \lambda(g_t)}{dt^2} |_{t=0} = \int_M \psi \times [P_0^{-1} \cdot (L'_1 L'_1) \cdot (L_1 L_1)](\psi) e^{-f} \omega_{KS}^n \leq 0. \quad (2.1)$$

Moreover the equality in (2.1) holds if and only if $\psi = \theta_v + \bar{\theta}_v$ for some holomorphic vector field $v$ on $M$, where $\theta_v$ is a potential associated to $v$ defined by

$$i_v(\omega_{KS}) = \sqrt{-1} \partial \bar{\partial} \theta_v.$$

Proof. First we see

$$\frac{d \lambda(g_t)}{dt} = - \int_M <\sqrt{-1} \partial \bar{\partial} \psi + \theta, \text{Ric}(\omega) - \omega + \sqrt{-1} \partial \bar{\partial} f > e^{-f} \omega^n.$$

Since

$$\frac{d \text{Ric}(g_t)}{dt} |_{t=0} = - \sqrt{-1} \partial \bar{\partial} (\Delta \psi),$$

we have

$$\frac{d^2 \lambda(g_t)}{dt^2} |_{t=0}$$
$$= \int_M <\partial \bar{\partial} \psi, \partial \bar{\partial} (\Delta \psi + \psi - \frac{df_t}{dt} |_{t=0}) > e^{-f} \omega_{KS}^n$$
$$= \int_M \sum \psi_{ij}(\Delta \psi + \psi - u) e^{-f} \omega_{KS}^n.$$
Integrating by parts, we get
\[
\frac{d^2 \lambda(g_t)}{dt^2}\bigg|_{t=0} = \int_M (\triangle \psi + \psi - u)[\triangle^2 \psi + X(X(\psi)) - 2\text{Re}(X(\triangle \psi)) - \psi_{ij} f_{ji}] e^{-f} \omega^n_{KS}
\]
\[
= \int_M (\triangle \psi + \psi - u)[\triangle(\triangle \psi - X(\psi)) - X(\triangle \psi - X(\psi))] e^{-f} \omega^n_{KS}
\]
\[
= \int_M (\triangle \psi + \psi - u)[(\triangle - X)(\triangle \psi - X(\psi))] e^{-f} \omega^n_{KS}
\]
\[
= \int_M [(\triangle - X)(\triangle \psi + \psi - X(\psi)) + (\triangle - X)(X(\psi) - u)]
\]
\[
\times (\triangle \psi - X(\psi)) e^{-f} \omega^n_{KS}.
\]
By Lemma 2.1, we derive from (2.2),
\[
\frac{d^2 \lambda(g_t)}{dt^2}\bigg|_{t=0} = \int_M [P(u - X(\psi)) + (\triangle - X)(X(\psi) - u)] \times (\triangle \psi - X(\psi)) e^{-f} \omega^n_{KS}
\]
\[
= \int_M [(\triangle - X)(u - X(\psi)) + (u - X(\psi))] \times (\triangle \psi - X(\psi)) e^{-f} \omega^n_{KS}
\]
\[
= \int_M \mathcal{L}_1 (u - X(\psi)) \times (\triangle \psi - X(\psi)) e^{-f} \omega^n_{KS}
\]
\[
(2.3)
\]
\[
= \int_M P^{-1}_0 \cdot [(L_1' L_1) \psi] \times (L_1' L_1)(\psi) e^{-f} \omega^n_{KS}
\]
\[
= \int_M \mathcal{L}_1 \cdot P^{-1}_0 \cdot [(L_1' L_1)(\psi) \times \psi e^{-f} \omega^n_{KS}.
\]
Since any two operators of $P_0, L_1, L_1'$ commute, $P^{-1}_0$ commutes with $L_1, L_1'$. Thus we have
\[
\frac{d^2 \lambda(g_t)}{dt^2}\bigg|_{t=0} = \int_M \psi \times [P^{-1}_0 \cdot (\mathcal{L}_1' L_1') \cdot (\mathcal{L}_1 L_1)](\psi) e^{-f} \omega^n_{KS}.
\]
Note that $P_0, L', \mathcal{L}'$ are all elliptic, so does $P^{-1}_0 \cdot (\mathcal{L}_1' L_1')$. This shows that
\[
\frac{d^2 \lambda(g_t)}{dt^2}\bigg|_{t=0} \leq 0
\]
and the equality holds if and only if
\[
\mathcal{L}_1 L_1(\psi) = 0.
\]
Then Proposition 2.1 will be completed from the next lemma. □
Lemma 2.2. The operator $\overline{L}_1L_1$ is real and nonnegatively definite. Moreover, there is an isomorphism between $\ker(\overline{L}_1L_1)$ and the linear space $\eta(M)$ of holomorphic vector fields on $(M, J)$ given by a relation $\psi = \theta_v + \overline{\theta}_{v'}$ for some $v \in \eta(M)$.

Proof. It suffices to prove the second part of the Lemma. This follows from an argument in Appendix of [TZ1]. In fact

$$\overline{L}_1L_1\psi = 0$$

implies

$$L_1\psi = \theta_u$$

for some $u \in \eta(M)$. From the proof of Lemma A.2 in [TZ1], we see that

$$\psi = \theta_v + \overline{\theta}_{v'}$$

for some $v, v' \in \eta(M)$. Since $\psi$ is real-valued function, $v'$ must be equal to $v$. Thus the lemma is true. □

The formula (2.1) in Proposition 2.1 can be generalized to the variation of any Kähler metrics $g$ for the fixed complex structure if the underlying manifold $M$ is Kähler-Einstein with positive scalar curvature as follows. Let $g_{KE}$ be a Kähler-Einstein metric on $(M, J)$ and

$$g_t = g_{KE} + t(\theta + \sum_{ij} \partial_i \partial_j \psi dz^i \otimes d\overline{z}^j) \quad (2.4)$$

be a family of Kähler metrics with

$$\int_M \omega^n_t = \int_M \omega^n_{KE}, \quad (2.5)$$

where $\theta$ be a hermitian and symmetric tensor with respect to the complex structure $J$. It is easy to see that condition (2.5) means

$$\int_M \text{tr}_{\omega_{KE}}(\theta) \omega^n_{KE} = 0.$$

Without loss of generality, we may further assume that the corresponding (1,1)-form of $\theta$ is harmonic associated to the metric $\omega_{KE}$. This implies

$$d[\text{tr}_{\omega_{KE}}(\theta)] = 0.$$

Thus we get

$$\text{tr}_{\omega_{KE}}(\theta) = 0. \quad (2.6)$$

Proposition 2.2. Let $(M, J)$ be a Kähler-Einstein manifold with positive first Chern class and $g_{KE}$ be a Kähler-Einstein metric on $M$. Let $g_t$ be a family of Kähler metrics of the form (2.4) and satisfying (2.5). Then

$$\frac{d^2 \lambda(g_t)}{dt^2} \bigg|_{t=0} = \int_M \|\theta\|^2 \omega^n_{KE} + \int_M (< D^* D\psi, (P_0')^{-1}(D^* D\psi) > \omega^n_{KE}, \quad (2.7)$$

where $\lambda(g)$ is the Kähler potential of $g$. The right-hand side of (2.7) is a polynomial in $\|\theta\|^2$. The leading term of this polynomial is $\|\theta\|^2 \omega^n_{KE}$.
where $P'_0$ and $D$ are defined, respectively, by
\[ P'_0 \psi = 2 \Delta_{KE} \psi + \psi \]
and
\[ D \psi = \sum \psi_{i \bar{j}} dz^i d\bar{z}^j, \]
and $D^*$ is the adjoint operator of $D$.

**Proof.** By
\[ \frac{d\lambda(g_t)}{dt} = -\int_M \langle -\sqrt{-1} \partial \bar{\partial} \psi + \theta, \text{Ric}(\omega) - \omega + \sqrt{-1} \partial \bar{\partial} f \rangle e^{-f} \omega^n \]
and
\[ \frac{d\text{Ric}(g_t)}{dt} \bigg|_{t=0} = -\sqrt{-1} \partial \bar{\partial} (\text{tr} \theta + \Delta \psi) = -\sqrt{-1} \partial \bar{\partial} (\Delta \psi), \]
we have
\[ \frac{d^2 \lambda(g_t)}{dt^2} \bigg|_{t=0} = \int_M < \partial \bar{\partial} \psi + \theta, \partial \bar{\partial} (\Delta \psi + \psi - \frac{df_t}{dt} \bigg|_{t=0}) + \theta > \omega^n_{KE}. \]

Taking the integral by parts, we get
\[ \frac{d^2 \lambda(g_t)}{dt^2} \bigg|_{t=0} = \int_M \| \theta \|^2 \omega^n_{KE} \]
\[ + \int_M < \partial \bar{\partial} \psi, \partial \bar{\partial} (\Delta \psi + \psi - \frac{df_t}{dt} \bigg|_{t=0}) > \omega^n_{KE}. \]

(2.8)

Since $f = f_t$ satisfies
\[ 2 \Delta f + f - |Df|^2 + R = \lambda(g), \]
differentiating at $t$ on the both sides, we get
\[ P'_0 \frac{df_t}{dt} \bigg|_{t=0} = 2 \Delta \frac{df_t}{dt} \bigg|_{t=0} + \frac{df_t}{dt} \bigg|_{t=0} \]
\[ = \Delta^2 \psi + \Delta \psi + \Delta (\text{tr} \theta) + \text{tr} \theta \]
\[ = D^* D \psi. \]

Note that $P'_0$ is invertible since the first non-zero eigenvalue is 1 on the Kähler-Einstein manifold. Using this relation, we obtain
\[ \int_M < \partial \bar{\partial} \psi, \partial \bar{\partial} (\Delta \psi + \psi - \frac{df_t}{dt} \bigg|_{t=0}) > \omega^n_{KE} \]
\[ = \int_M < D^* D \psi, (P'_0)^{-1} (D^* D \psi) > \omega^n_{KE}. \]

(2.9)

Thus combining (2.8) and (2.9), we prove the proposition. \(\square\)

The following corollary shows that Proposition 2.1 is not true in general if Kähler metrics are not fixed in the Kähler class $2\pi c_1(M)$. 7
Corollary 2.1. Let \((M, J)\) be a Kähler-Einstein manifold with positive first Chern class. Suppose that \(\dim H^{1,1}(M, J) \geq 2\). Then \(\delta^2 \lambda(g)(h, h)\) is not non-positive at a Kähler-Einstein metric \(g_{KE}\) for the variation of general Kähler metrics and so \(g_{KE}\) is not a local maximum of \(\lambda(g)\) in the total space of Kähler metrics.

**Proof.** Let \(\omega'\) be another harmonic \((1,1)\)-form of \((M, J)\) which is not a multiple of \(c_1(M)\). Then there are two number \(a\) and \(b\) such that

\[
an + b \operatorname{tr}_{\omega_{KE}} \omega' = 0.
\]

Let \(\theta = a \omega' + b \omega_{KE}\) and \(h_0\) be its corresponding hermitian and symmetric tensor. Then

\[
\int_M \operatorname{tr}_{\omega_{KE}} (\theta) \omega_{KE}^n = 0.
\]

Thus by Proposition 2.2, we have

\[
\delta^2 \lambda(g)(h_0, h_0) = \int_M \|\theta\|^2 \omega_{KE}^n > 0.
\]

This implies that \(\delta^2 \lambda(g)\) is not non-positive in the direction of \(h_0\), so the corollary is true. \(\square\)

3. The case for varying complex structures.

In this section, let \((M, g_{KE}, J_0)\) be a Kähler-Einstein manifold, we will study the second variation of \(\lambda(g)\) at \(g_{KE}\) when restricted Kähler metrics with Kähler forms cohomologous to \(2\pi c_1(M)\). Set

\[
\mathcal{W} = \{ h | \text{there is a family of Kähler metrics } (g_t, J_t)(0 \leq t \leq \epsilon) \text{ such that } h = \left. \frac{dg_t}{dt} \right|_{t=0}, \ (g_0, J_0) = (g_{KE}, J_0), \text{ and } [\omega_t] = 2\pi c_1(M) \}.
\]

Here \(J_t\) denotes a family of complex structures on \(M\) and \(\omega_t\) denotes the Kähler form of \(g_t\). We shall prove

**Theorem 3.1.** The operator \(L\) defined in Proposition 1.1 is non-positive on \(\mathcal{W}\). Namely, for any \(h \in \mathcal{W}\), we have

\[
\delta^2 \lambda(g_{KE})(h) = (h, Lh)_{g_{KE}} = \int_M <h, Lh>_{\omega_{KE}} \omega_{KE}^n \leq 0.
\]

Moreover, there exists an isomorphism

\[
i : \ker(L) \rightarrow \eta(M, J_0) + H^1(M, J_0, \Theta),
\]

where \(\ker(L)\) denotes the kernel of \(L\), and \(\eta(M, J_0)\) is the space of holomorphic vector fields associated to the complex structure \(J_0\) on \(M\) and \(H^1(M, J_0, \Theta)\) is the Čech cohomology class associated to the infinitesimal deformation of complex structures on \(M\) \([\text{Kod}]\).
Remark 3.1. According to Corollary 2.1 in Section 2, we see that Theorem 3.1 is not true in general for Kähler metrics without the assumption that the Kähler class $2\pi c_1(M)$ is fixed. The same observation was made in [CHI] in the Riemannian case.

As before, we let $J_t(0 \leq t \leq \epsilon)$ be a family of complex structures on a Kähler-Einstein manifold $M$ with $J_0 = J$. Then according to [Koi], one can decompose $\frac{dJ_t}{dt}|_{t=0}$ as a direct product into

$$\frac{dJ_t}{dt}|_{t=0} = L_Z J + I_E$$

with

$$\int_M <L_Z J, I_E> \omega_{KE}^n = 0,$$

where $L_Z$ denotes the Lie derivative along a vector field $Z$ on $M$ and $I_E$ is the part of an essential infinitesimal deformation of complex structures on $M$. If we let $h'$ be a covariant tensor of rank 2 defined by

$$h'(X,Y) = \omega_{KE}(X,I_E Y), \quad (3.2)$$

then $h'$ is anti-hermitian, and so it is a real part of some $(0, 2)$-type tensor $I = I_{ij} d\bar{z}^i \otimes dz^j$, i.e.,

$$h' = \text{Re}(I).$$

Moreover, $I$ satisfies ([Koi]),

$$\nabla_k I_{ij} = \nabla_i I_{jk}, \quad \forall \ i, j, k, \quad (3.3)$$

and

$$\sum_j \nabla_j I_{ij} = 0, \quad \forall \ i. \quad (3.4)$$

The relations (3.3) and (3.4) imply that the complexification of $I_E$ is a $\bar{\partial}$-closed, $(0, 1)$-form with values in $\Theta_{J_0}$. This defines a Čech cohomology class in $H^1(M, \Theta_{J_0})$, where $\Theta_{J_0}$ denotes the $(1, 0)$-typed tangent sheaf associated to $J_0$ on $(M, J_0)$ [Kod]. (3.4) also implies

$$\text{div} h' = \sum_{\alpha} D_{e_{\alpha}} h'(\cdot, e_{\alpha}) = 0. \quad (3.5)$$

Let $\rho_t$ be an one-parameter diffeomorphisms group generated by the vector $-Z$ and $\rho_t^* g_t$ be a family of induced Riemannian metrics, where

$$g_t(X,Y) = \omega_t(X, J_t Y).$$

Then

$$(\rho_t)^* g_t(X,Y) = g_t((\rho_t)_* X, (\rho_t)_* Y) = \omega_t((\rho_t)_* X, J_t((\rho_t)_* Y))$$

$$= (\rho_t)^* \omega_t(X, ((\rho_t)^* J_t) Y).$$
It follows
\[
\tilde{h}(X, Y) = \frac{d((\rho_t)^*g_t)}{dt}|_{t=0}(X, Y)
\]
\[
= \frac{d((\rho_t)^*\omega_t)}{dt}|_{t=0}(X, JY) + \omega_{KE}(X, \frac{d((\rho_t)^*J_t)}{dt}|_{t=0} Y)
\]
\[
= \frac{d((\rho_t)^*\omega_t)}{dt}|_{t=0}(X, JY) + \omega_{KE}(X, J_E Y).
\]

(3.6)

Set
\[\mathcal{W}_0 = \{h \text{ is a covariant symmetric tensor of rank 2 such that } \ Lh = 0 \text{ and } \text{div}(h) = 0\}\.\]

**Lemma 3.1.** Let \(h'\) be the covariant 2-tensor in (3.2). Then \(h' \in \mathcal{W}_0\).

*Proof.* By a direct computation, it was showed in [Koi] that (3.3) and (3.4) implies,
\[
L_0 h' = \frac{1}{2} D^* D h' - \text{Rm}(h', \cdot) = 0.
\]

(3.7)

On the other hand, one can decompose \(h'\) into a symmetric part \(b\) and an anti-symmetric part \(a\) which is orthogonal in the sense of inner product
\[
(a, b)_{\omega_{KE}} = \int_M <a, b>_{\omega_{KE}} \omega^n_{KS}.
\]

Since the operator \(L_0\) keeps the symmetry and anti-symmetry, the anti-symmetric part \(a\) of \(h'\) also satisfies equation (3.7), and consequently, \(a\) is parallel, i.e.,
\[
D a = 0.
\]

By using the Ricci identity
\[
D^i D_i a - D^i D_i a = 2a,
\]
we see that \(a = 0\). This implies that \(h'\) is a symmetric 2-tensor. By using (3.5) and the fact \(\text{tr}_{\omega_{KE}} h' = 0\), we also get
\[
L h' = L_0 h' = 0.
\]

Hence, we have \(h' \in \mathcal{W}_0\). \(\square\)

**Lemma 3.2.** Assume that \(\omega_t \in 2\pi c_1(M)\). Then there is a smooth real-valued function \(\psi\) on \(M\) such that
\[
\frac{d((\rho_t)^*\omega_t)}{dt}|_{t=0} = \sqrt{-1} \partial \bar{\partial} \psi.
\]

*Proof.* Let \(\theta_1\) and \(\theta_2\) be in \(A^{1,1}(M, J)\) and \(A^{2,0}(M, J)\) respectively, such that
\[
\frac{d((\rho_t)^*\omega_t)}{dt}|_{t=0} = \theta_1 + \text{Re}(\theta_2).
\]
Clearly, tensor $h_1$ defined by $h_1(X,Y) = \theta_1(X,JY)$ is symmetric and hermitian and tensor $h_2$ defined by $h_2(X,Y) = \text{Re}(\theta_2)(X,JY)$ is anti-symmetric. Since $h$ and $h'$ are both symmetric according to $h \in \mathcal{W}$ and Lemma 3.1, we see that $\text{Re}(\theta_2)$ must vanish. Thus we get

$$\frac{d[(\rho_t)^*\omega_t]}{dt}|_{t=0} = \theta_1.$$ 

Note that $\theta_1$ is also an exact form because of $\omega_t \in 2\pi c_1(M)$. Therefore the lemma is true. □

According to Lemma 3.1 and 3.2, we see that $h_1$ and $h'$ are hermitian and anti-hermitian symmetric tensors respectively. Then $<h_1, h'>_{\omega_{KE}} = 0$. Thus we have

**Lemma 3.3.**

$$(h_1, h')_{\omega_{KE}} = 0.$$ 

Combining Lemma 3.2 and 3.3, we get

**Proposition 3.1.**

$$\mathcal{W}/\text{diff}(M) \cong A^{1,1}(M, J_0) \bigoplus H^1(M, \Theta_{J_0}).$$

**Proof of Theorem 3.1.** By Lemma 3.2, we have

$$\tilde{h}(X,Y) = h_1(X,Y) + h'(X,Y).$$

Since $(h_1, h')_{\omega_{KE}} = 0$ by Lemma 3.3, we see that

$$(\tilde{h}, L\tilde{h}) = (h_1, Lh_1) + (h', Lh') = (h_1, Lh_1). \quad (3.8)$$

The last equality follows from Lemma 3.1. On the other hand, by Lemma 3.2, we see that there is a smooth real-valued function $\psi$ such that

$$h_1(X,Y) = \sqrt{-1} \sum_{i,j} \partial_i \partial_j^\top \psi dz^i \wedge d\bar{z}^j(X,JY).$$

Then according to Proposition 2.2, we have

$$(h_1, Lh_1) = \frac{d^2\lambda(g_t)}{dt^2}|_{t=0} \leq 0,$$

where $g_t$ is a family Kähler metrics defined as in Lemma 2.1 with $h_0$ replaced by $h_1$. Thus

$$(\tilde{h}, L\tilde{h}) = (h_1, Lh_1) \leq 0.$$ 

Since $W$-functional is invariant under diffeomorphisms, we obtain

$$(h, Lh) = (\tilde{h}, L\tilde{h}) \leq 0.$$  

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By relation (3.8) and Proposition 2.1, \( \tilde{h} \) is a kernel of \( L \) iff \( \tilde{h} = h_1 + h' \), where \( h' \) is defined by (3.2) and
\[
h_1 = \text{Re} \left( \sum_{i,j} \partial_i \partial_j \psi dz^i \otimes d\bar{z}^j \right)
\]
for some real-valued function \( \psi \) which satisfies
\[
\psi = \theta_v + \bar{\theta}_v,
\]
where \( v \in \eta(M) \). Thus the operator \( L \) induces an injective homomorphism
\[
i : \text{Ker}(L) \to \eta(M,J_0) + H^1(M, \Theta_{J_0}).
\]
It is clear that \( i \) is surjective by (3.5) and (3.7) and the fact that \( \lambda(g) \) is invariant under the holomorphic transformations. Therefore the theorem is true. \( \square \)

**Remark 3.2.** The relation (3.1) can be also obtained by showing that \( g_{KE} \) is a global minimizer of \( \lambda(g) \) in the total space of Kähler metrics \( (g, J) \) with Kähler classes cohomologous to \( 2\pi c_1(M) \). One can show that
\[
\lambda(g_{KE}) = W(g_{KE}, 1) = \int_M \omega_{KE}^n
\]
\[
= W(g, 1) \geq \inf_f \{ W(g, f) \mid f \text{ satisfies } (1.3) \} = \lambda(g).
\]

But Theorem 3.1 determines more explicitly the kernel of the elliptic operator which arises from the second variation. In fact, we conjecture that there exists an analogous version of Theorem 3.1 for general Kähler-Ricci solitons.

4. Stability of the Kähler-Ricci flow.

In this section, we apply the above computation on \( W \)-functional to studying the stability of Kähler-Ricci flow near a Kähler-Einstein metric on a compact complex manifold \( M \) with \( c_1(M) > 0 \). In case of \( c_1(M) = 0 \) or \( c_1(M) < 0 \), we refer readers to [S2] and [DWW].

The Ricci flow was first introduced by R. Hamilton in [Ha]. If the underlying manifold \( M \) is Kähler with positive first Chern class, it is more natural to study the following Kähler-Ricci flow (normalized),
\[
\begin{aligned}
\frac{\partial g(t, \cdot)}{\partial t} &= -\text{Ric}(g(t, \cdot)) + g(t, \cdot), \\
g(0, \cdot) &= g.
\end{aligned}
\]

It can be shown that (4.1) preserves the Kähler class. Moreover, (4.1) has a global solution \( g(t) \) for any \( t > 0 \) ([Ca]). The difficulty is to study the limiting behavior of \( g(t) \) as \( t \) tends to \( \infty \) (cf. [CT1], [CT2], [TZ2], etc.).

The following lemma is a direct corollary of Remark 3.2 by using the Kähler-Ricci flow.
Lemma 4.1. Let \((g_{KE}, J_0)\) and \((g, J)\) be a Kähler-Einstein metric and a Kähler metric on \(M\) with both Kähler forms cohomologous to \(2\pi c_1(M)\). Suppose that \(\lambda(g) \geq \lambda(g_{KE})\). Then \(g\) is a Kähler-Einstein metric with respect to the complex structure \(J\).

Proof. Let \(g(t, \cdot)\) be a family of evolved Kähler metrics of equation (4.1) with the initial metric \(g\). Then by the monotonicity of \(\lambda(g)\) along the Ricc flow [Pe], we have
\[
\lambda(g(t, \cdot)) \geq \lambda(g), \quad \forall t > 0.
\]
Thus by (3.9) in Remark 3.2 and condition in the lemma, we see
\[
\lambda(g(t, \cdot)) = \lambda(g_{KE}), \quad \forall t > 0.
\]
Hence by (1.5) in Section 1, it follows that \(g(t, \cdot)\) are all Ricci solitons. Since \(g(t, \cdot)\) are Kähler, one can further prove that \(g(t, \cdot)\) are in fact all Kähler-Ricci solitons (cf. [S1]). By the uniqueness of Kähler-Ricci solitons [TZ1], \(g(t, \cdot)\) are all same, and consequently \(g\) is a Kähler-Ricci soliton with respect to some holomorphic vector field \(X\) induced by a Hamiltonian function \(f\) which satisfies (1.4) in Section 1. Thus
\[
\lambda(g) = W(g, f) = \lambda(g_{KE}) = W(g, 1).
\]
This implies \(f = 1\), so \(g\) is a Kähler-Einstein metric with respect to the complex structure \(J\). □

By using the functional \(\lambda(g)\), we also have

Lemma 4.2. Let \((g_{KE}, J_0)\)and \((g_{KS}, J)\) be a Kähler-Einstein metric and a Kähler-Ricci soliton on \(M\) with both Kähler forms cohomologous to \(2\pi c_1(M)\). Suppose that there exists a smooth path of Kähler-Ricci solitons \((g_t, J_t)\) \((t \in [0, 1])\) on \(M\) with all Kähler forms cohomologous to \(2\pi c_1(M)\), which connects \((g_{KE}, J_0)\)and \((g_{KS}, J)\). Then \((g_{KS}, J)\) must be a Kähler-Einstein metric.

Proof. On contrary, we suppose that \((g_{KS}, J)\) is a nontrivial Kähler-Ricci soliton with respect to a holomorphic vector field \(X\) on \(M\). Then \(X\) is a gradient vector field of a smooth function \(f\) on \(M\), which is a solution of (1.4) with respect to the metric \(g_{KS}\). Thus
\[
\lambda(g_{KS}) = W(g_{KE}, f)) = \inf_{f'}\{W(g, f')| f' \text{ satisfies (1.3)}\}
\]
\[
< W(g_{KS}, 1) = \lambda(g_{KE}).
\]
On other hand, by (1.5), we have
\[
\frac{d}{dt}\lambda(g_t) = 0, \quad \forall t \in [0, 1].
\]
It follows
\[
\lambda(g_{KE}) = \lambda(g_{KS}).
\]
Hence we get a contradiction. □

Now let us to state our stability theorem on the Kähler-Ricci flow. For simplicity, we assume that the Kähler-Einstein metric is isolated on a small deformation of complex structures of \(M\). Let \((g, J)\) and \((g', J')\) be two Kähler metrics on \(M\). Denote \(\|J - J'\|_{C^k(M)}\) to be the \(C^k\)-norm of the difference between complex structures \(J_0\) and \(J\) as elements in \(\text{Hom}(TM, TM)\) and \(\|g - g'\|_{C^k(M)}\) to be the \(C^k\)-norm of the difference between metrics \(g\) and \(g'\). Then we have
Theorem 4.1. Let \((g_{KE}, J_0)\) and \((g, J)\) be a Kähler-Einstein metric and a Kähler metric on \(M\) with both Kähler forms cohomologous to \(2\pi c_1(M)\). Furthermore, we assume that

\[
\|g - g_{KE}\|_{C^3(M)} \leq \delta
\]

and

\[
\|J - J_0\|_{C^3(M)} \leq \delta
\]

for some \(\delta > 0\). Suppose that there is no other Kähler-Einstein metric in a small neighborhood of \((g_{KE}, J_0)\). Then the Kähler-Ricci flow with the initial metric \(g\) converges to a Kähler-Ricci soliton \(g_{KS}\) in the sense of Cheeger-Gromov as long as \(\delta\) is small. Moreover, in any small deformation of complex structures of \(M\) near \(J_0\), there exists a complex structure \(J\) such that for any initial Kähler metric \((g, J)\) satisfying (4.2) and (4.3) the Kähler-Ricci flow converges to a Kähler-Ricci soliton with a different complex structure in the sense of Cheeger-Gromov as long as \(\delta\) is small. The latter means that the complex structure \(J\) will jump to another complex structure along the convergence of Kähler-Ricci flow.

Proof. Let \(g(t, \cdot)\) be a global solution of (4.1) with the initial metric \(g\). We claim that for any \(\epsilon\) there exists \(\delta = \delta(\epsilon) > 0\) such that if \((g, J)\) satisfies (4.2) and (4.3), then

\[
\|g(t, \cdot) - g_{KE}\|_{C^3_{CG}(M)} \leq \epsilon, \forall t > 0,
\]

where \(\|g - g'\|_{C^k_{CG}(M)}\) denotes the \(C^k\)-norm of the difference between metrics \(g\) and \(g'\) in the sense of Cheeger-Gromov (cf. [GW]). On contrary, if the claim is false, we can find a sequence of Kähler metrics \((g_i, J_i)\) on \(M\) with property:

\[
\|g_i - g_{KE}\|_{C^3_{CG}(M)} \rightarrow 0, \text{ as } i \rightarrow \infty, \text{ as } i \rightarrow \infty,
\]

and

\[
\|J_i - J_0\|_{C^3(M)} \rightarrow 0, \text{ as } i \rightarrow \infty,
\]

and there exist an \(\epsilon_0 > 0\) and a sequence of Kähler metrics \((g_i(t_i, \cdot), J_i)\) such that

\[
\|g_i(t_i, \cdot) - g_{KE}\|_{C^3_{CG}(M)} \geq \epsilon_0,
\]

where \(g_i(t, \cdot)\) is the solution of the Kähler-Ricci flow with the initial metric \(g_i\). By the stability of the Ricci flow for the finite time, we may further assume that \(g_i\) satisfies

\[
2\epsilon_0 \geq \|g_i(t, \cdot) - g_{KE}\|_{C^3_{CG}(M)} \geq \epsilon_0, \quad \forall t \in [t_i - 1, t_i + 1].
\]

In particular, curvatures \(Rm(g_i(t_i, \cdot))\) are uniformly bounded. Thus by the higher order estimates for the Ricci flow [Sh],

\[
\|Rm(g_i(t_i, \cdot))\|_{C^{k-2, \alpha}(M, g_i(t_i, \cdot))} \leq C_k, \quad k \geq 2.
\]

Hence by the compactness theorem of Cheeger-Gromov (cf. [GW]), there is a subsequence \(g_{i_k}(t_k, \cdot)\) of \(g_i(t_i, \cdot)\) which converges to a metric \(g_\infty\) on \(M\) in the \(C^{k, \alpha}\)-topology. Clearly, by (4.7), we have

\[
\|g_\infty - g_{KE}\|_{C^3_{CG}(M)} \geq \epsilon_0 \geq \frac{\epsilon_0}{2}.
\]
Furthermore, by (4.6), there are diffeomorphisms $\Phi_{ik}$ on $M$ such that $(\Phi_{ik}^{-1})_* \cdot (\Phi_{ik})_*$ converge a complex structure $J_\infty$ on $M$ in the $C^{2,\alpha}$-topology. By the regularity theory for the $\overline{\partial}$-equation, $J_\infty$ is analytic modulo a diffeomorphism. Clearly, $g_\infty$ is Kähler with respect to $J_\infty$. For simplicity, we may assume that both $J_0$ and $J_\infty$ are analytic.

By the monotonicity of $\lambda(g)$ along the flow, we have
$$\lambda(g_{ik}) \leq \lambda(g_{ik}(t_{ik}, \cdot)).$$
It follows
$$\lambda(g) \geq \lim_{i_k} \lambda(g_{ik}) = \lambda(g_{KE}).$$
Thus by Lemma 4.1, we see that $g_\infty$ is a Kähler-Einstein metric on $(M, J_\infty)$. On the other hand, by (4.6) and (4.8), it is easy to see that
$$\|J_\infty - J_0\|_{C^{2,\alpha}(M)} \leq \eta(\epsilon_0),$$
where the constant $\eta(\epsilon_0) \to 0$ as $\epsilon_0 \to 0$. It follows
$$\|J_\infty - J_0\|_{C^k(M)} \leq C_k(\epsilon_0). \quad (4.10)$$
Then the assumptions in the theorem implies that $J_\infty$ is induced from $J_0$ and $g_\infty = \Phi^* g_{KE}$ for a diffeomorphism $\Phi$ on $M$. This is contradict to (4.9). Thus the claim is true.

By (4.4) at the above claim and the higher order estimates for the Ricci flow [Sh], we have
$$\|\text{Rm}(g(t, \cdot))\|_{C^{k-2,\alpha}(M, g(t, \cdot))} \leq C_k(\epsilon), \forall \ t > 0.$$  
This implies that for any sequence of $g(t, \cdot)$ there exists a subsequence which converges to a metric on $M$ in the $C^k,\alpha$-topology. By the monotonicity of $\lambda(g)$ it is easy to see that $\lambda(g)$ is uniformly bounded. Thus by (1.5), one sees that there is a sequence of $g(t, \cdot)$ which converges to a gradient shrinking soliton $g_{KR}$ in the sense of Cheeger-Gromov. Since $g_{KR}$ is Kähler, one can further prove that $g_{KR}$ is in fact a Kähler-Ricci soliton with respect to some complex structure $J'$ (cf. [S1]). The first part of the theorem is proved.

On the other hand, by Lemma 4.2, there exists a complex structure $J$ in any small deformation of complex structures of $M$ near $J_0$ such that $(M, J)$ does not admit any Kähler-Ricci soliton. Thus the complex structure of limiting Kähler-Ricci soliton of the Kähler-Ricci flow with an initial Kähler metric $(g, J)$, which satisfies (4.2) and (4.3) must be different to $J$. This completes the theorem. \[\square\]

**Remark 4.1.** According to the proof of Theorem 4.1, one can further prove: if in addition, there is no any Kähler-Ricci soliton in a small deformation of complex structures of $M$ near $J_0$, then the Kähler-Ricci flow with the initial metric $(g, J)$ which satisfies (4.2) and (4.3) converges globally to the Kähler-Einstein metric $g_{KE}$ in the sense of Cheeger-Gromov as long as $\delta$ is small. The latter means that the complex structure $J$ will jump to $J_0$ along the convergence of Kähler-Ricci flow.

An example which satisfies the assumptions in Theorem 4.1 is the Mukai-Umemura’s Fano 3-fold $M$. This is a compactification of the quotient $SL(2, \mathbb{C})/\Gamma$, where $\Gamma$ is the
icosahedral group. It was proved by S. Donaldson that $M$ admits a Kähler-Einstein metric $g_{KE}$ [Do]. By extending the method in [Ti], C. Arezzo proved that there are no other Kähler-Einstein metrics in any complex deformation of $M$ [Ar]. Thus according to Theorem 4.1, the Kähler-Ricci flow deforms any Kähler metric $g$ near $g_{KE}$ with its Kähler class cohomologous to $2\pi c_1(M)$ to $g_{KE}$ or a Kähler-Ricci soliton in the sense of Cheeger-Gromov.

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