Nonextensive thermodynamics with finite chemical potential, hadronic matter and protoneutron stars

To cite this article: Airton Deppman et al 2015 J. Phys.: Conf. Ser. 607 012007

View the article online for updates and enhancements.

Related content
- Power-law distributions in protoneutron stars
  G Gervino, A Lavagno and D Pigato
- Nonextensive quantum H-theorem
  R. Silva, D. H. A. L. Anselmo and J. S. Alcaniz
- The equation of state at finite temperature: Structure and composition of protoneutron stars
  G F Burgio, M Baldo, H Chen et al.

Recent citations
- Non-extensive thermodynamics and neutron star properties
  Débora P. Menezes et al
Nonextensive thermodynamics with finite chemical potential, hadronic matter and protoneutron stars

Airton Deppman,1 Eugenio Megías,2, 3 and Débora P. Menezes4, 5

1Instituto de Física, Universidade de São Paulo - Rua do Matão Travessa R N°.187 CEP 05508-090 Cidade Universitaria, São Paulo - Brasil email: deppman@if.usp.br
2Grup de Física Teorica and IFAE, Departament de Física, Universitat Autonoma de Barcelona, Bellaterra E-08193 Barcelona, Spain
3Max-Planck-Institut fur Physik (Werner-Heisenberg-Institut), Fohringer Ring 6, D-80805, Munich, Germany email: emegias@mpp.mpg.de
4Departamento de Física, CFM, Universidade Federal de Santa Catarina, CP 476, CEP 88.040-900 Florianopolis - SC - Brazil
5Departamento de Física Aplicada, Universidad de Alicante, Ap. Correus 99, E-03080, Alicante, Spain email: debora.p.m@ufsc.br

The nonextensive thermodynamics of an ideal gas composed by bosons and/or fermions is derived from its partition function for systems with finite chemical potentials. It is shown that the thermodynamical quantities derived in the present work are in agreement with those obtained in previous works when \( \mu \leq m \). However some inconsistencies of previous references are corrected when \( \mu > m \). A discontinuity in the first derivatives of the partition function and its effects are discussed in details. We show that at similar conditions, the nonextensive statistics provide a harder EOS than that provided by the Boltzmann-Gibbs statistics. This fact induced us to consider an application to a proto-neutron star, which is discussed for different assumptions.

PACS numbers: 05.70.Ce, 95.30.Tg, 26.60.-c
I. INTRODUCTION

One of the main objectives in the study of ultrarelativistic collisions is the investigation of the properties of the quark-gluon plasma. Thermodynamical aspects of the plasma are of specially interest due to the possibility of studying the deconfinement process. The nonextensive statistics proposed by C. Tsallis in 1988 [1] was introduced in High Energy Physics (HEP) by Bediaga, Curado and Miranda (BCM) [2] where the transverse momentum ($p_T$) distribution from Hagedorn’s theory [3] was formally modified by the introduction of the q-exponential function commonly used in the Tsallis statistics. The resulting formula describes the whole $p_T$-distribution obtained in HEP experiments.

The nonextensive formula for $p_T$-distribution has been used in a large number of works, all of them reporting good agreement with experiment [4–8]. In Ref. [10] it has been shown that Hagedorn’s theory can be generalized to include the nonextensive statistics [9, 11–15] self-consistently. The resulting theory shows that there must exist a limiting temperature, $T_o$, and also an entropic index, $q_o$, which characterizes the hadronic systems. The theory gives a new formula for the hadronic mass spectrum in terms of $T_o$ and $q_o$, which can describe quite well the known spectrum of hadrons with masses from the pion mass up to $\sim 2.5$ GeV [16]. In this way the nonextensive self-consistent theory [10] presents much more restrictive conditions to the applicability of the Tsallis statistics in HEP. Several analyses of experimental data [16–19] have shown that the theoretical predictions are in agreement with the experimental findings.

A study of the nonextensive thermodynamics of hadronic matter for null chemical potential ($\mu$) and its comparison to Lattice-QCD data was already presented in Ref. [20] showing good agreement between LQCD and the thermodynamical theory. In this work we discuss the extension of the thermodynamics theory to finite chemical potential, which is of importance in the study of nucleus-nucleus collisions and of astrophysical massive objects.

The paper is organized as follows: in section 2 we introduce the partition function and show that it is in agreement with the $p_T$-distribution used in previous works [2, 10, 19, 21]; in section 3 we derive the thermodynamical functions of interest and establish the phase transition line between confined and deconfined regimes in the $T \times \mu$ diagram; in section 4 we discuss our results and in section 5 we present our conclusions.

The paper is organized as follows: in section II we introduce the partition function and show that it is in agreement with the $p_T$-distribution used in previous works [2, 10, 19, 21]; in section III we derive the thermodynamical functions of interest and establish the phase transition line between confined and deconfined regimes in the $T \times \mu$ diagram; in section IV we apply our results to the neutron star problem and finally we present our conclusions in section V. A full description of the thermodynamical theory developed here can be found in Ref. [22], and an application of the theory for (proto)neutron stars which is based on constant entropy can be found in Ref. [23] (see also Ref. [24]).

II. PARTITION FUNCTION FOR HADRONIC MATTER

The q-exponential function is defined as

\[
\begin{align*}
\epsilon_q^{(+)}(x) &= \left[1 + (q-1)x\right]^{1/(q-1)} , \quad x \geq 0 \\
\epsilon_q^{(-)}(x) &= \frac{1}{\epsilon_q^{(+)}(x)} = \left[1 + (1-q)x\right]^{1/(1-q)} , \quad x < 0.
\end{align*}
\]

We define the partition function for a nonextensive ideal quantum gas as

\[
\ln_q \Xi(V,T,\mu) = \left\{ \begin{array}{ll}
\frac{V}{(2\pi)^7} \int_{m}^{\infty} \log_+ \frac{\epsilon_q^{(+)}(x)+\xi}{\epsilon_q^{(+)}(x)} \\
\frac{V}{(2\pi)^7} \int_{m}^{\infty} \log_- \frac{\epsilon_q^{(-)}(x)+\xi}{\epsilon_q^{(-)}(x)}
\end{array} \right.,
\]

where $x = \beta(\sqrt{p^2 + m^2} - \mu)$, $m$ is the hadron mass, $\beta = 1/T$ is the inverse of temperature, and $\mu$ the chemical potential with $\xi = 1$ for fermions and $\xi = -1$ for bosons. In the limit $q \to 1$ the partition functions defined above reduces to the well-known Fermi-Dirac and Bose-Einstein partition functions for fermions and bosons, respectively. Below we show that Equations 2 define in fact the partition function for the ideal quantum gas in Tsallis statistics.

III. THERMODYNAMICAL FUNCTIONS

To show that the partition function defined in Eq. (2) corresponds to the one for a quantum gas in non extensive statistics we derive the occupation number, average number of particles, energy density and the entropy, and show that the results are identical to the ones obtained in Refs [19, 21] in the sector $x \geq 0$. However, some differences are found and discussed for $x < 0$. 

2
A. Derivation of thermodynamic quantities from the partition function

In this section we will derive from the partition functions defined in Eq. (2) the occupation number and the entropy and show that the results are identical to the results obtained in Refs [19, 21] for quantum gases in nonextensive statistics.

The occupation number can be obtained through the relations

\[ n(x) = \beta^{-1} \frac{\partial}{\partial \mu} \ln \Xi, \]

resulting in

\[ n_q^{(+,-)}(p) = \left( \frac{1}{e_q^{(+,-)}(x) - \xi} \right)^{\frac{1}{q}}. \]

This result is identical to those obtained by CMP [21, 25] and by Cleymans and Worku [19]. In addition, observe that for \( \mu = 0 \) and \( \varepsilon \) sufficiently high it reduces to the equation used by BCM in Ref. [2] to describe the HEP \( p_T \)-distributions. Thus we can conclude that the occupation number derived from the partition functions defined in Eq. (2) can correctly describe the results obtained in ultrarelativistic collisions.

The entropy can be obtained through the relations

\[ S = \beta^2 \frac{\partial}{\partial \beta} \left( \frac{\ln \Xi_q}{\beta} \right), \]

resulting

\[ S = V \int \frac{d^3 p}{(2\pi)^3} \sum_{r=\pm} \Theta(r x) \left[ -[\tilde{n}_q^{(r)}(x)]^{\frac{1}{q}} \log_q^{(-r)} (\tilde{n}_q^{(r)}(x)) + \xi[1 + \xi \tilde{n}_q^{(r)}(x)]^{\frac{1}{q}} \log_q^{(-r)} \left( 1 + \xi \tilde{n}_q^{(r)}(x) \right) \right], \]

where we have defined \( \tilde{n}_q^{(r)}(x) \equiv [n_q^{(r)}(x)]^{1/q} \), which is identical to the CMP entropy defined in Ref [21].

With this results we conclude that the partition function in Eq. (2) does represent the relevant function for an ideal quantum gas in Tsallis statistics. Observe that for \( \mu = 0 \) and \( \varepsilon \) sufficiently high the partition functions defined here are similar to those used in Ref. [20] to extend Hagedorn’s theory to nonextensive statistics. Therefore definitions given here are in accordance with the nonextensive self-consistent thermodynamics. In the following we explore some of the features of the thermodynamical systems described by Eq. (2).

B. Thermodynamical properties of hadronic systems

Before studying the properties of a thermodynamically equilibrated hadronic system in the non extensive thermodynamics we have to find the region where this system can exist. In fact, due to the transition from confined to deconfined regimes, the hadronic matter can be found only below the phase transition line. To find this line we use a result obtained by Cleymans and Redlich [26] where they pointed out that the transition line can be determined by the condition that \( \langle E \rangle / \langle N \rangle = 1 \) GeV. This result was obtained through a systematic analysis of particle yields from HEP experiments, where the authors used the Boltzmann’s statistics in the formulation of their thermodynamics. But since yields are calculated by the integration over all energy states, and considering that the differences between the distributions obtained through Boltzmann statistics and Tsallis statistics are relevant only in the high energy tail, we can assume with some confidence that the same relation holds in the non extensive case. Of course this assumption must be checked by a similar analysis of experimental data, now using the non extensive formulas derived here, a task that is beyond the scope of the present work. However we present below some evidences that this hypothesis is correct.

In the following we assume that the entropic index, \( q_o \), is a fixed property of the hadronic matter with its value determined in the analysis of \( p_T \)-distributions and in the study of the hadronic mass spectrum in Ref. [16], so we set \( q_o = 1.14 \), although in some cases we analyze the behavior of some quantities for different values of \( q \). In the following we refer to Boltzmann-Gibbs statistics also by \( q = 1 \).

The system of interest here is a gas composed by different hadronic species in thermodynamical and chemical equilibrium. The partition function is then given by

\[ \ln \Xi_q(V, T, \{ \mu \}) = \sum_i \ln \Xi_q(V, T, \mu_i), \]

(7)
where $\mu_i$ refers to the chemical potential for the $i$-th hadron. The lowest-lying hadrons considered in our calculations are taken from the Particle Data Group [27], and some of them are presented in Tables I and II. The total numbers of hadronic states considered are 808 for mesons and 1168 for baryons (+ anti-baryons), including degeneracies, which correspond to a maximum value of the mass about 11 GeV and 5.8 GeV respectively. The computation will be performed by restricting the ensemble summation in Eq. (7) to zero strangeness (see [26] for details). We will focus first on a study of the phase transition by assuming $\mu = 0$ for all mesons and considering that all baryons have the same chemical potential value $\mu_B \neq 0$. After that, the effect of a nonzero value of the chemical potential for pions will be explored as well.

The phase transition line obtained as described above is reported in Fig. 1 (left), and it is compared to the Boltzmann's result, corresponding to $q \to 1$. For the sake of clarity we refer to the temperature obtained with BG statistics as $T$ and to the temperature obtained with Tsallis statistics as $\tau$. The relation between $\tau$ and $T$ was already investigated in Ref. [20], and it was found that for a fixed chemical potential there is a linear correspondence between both quantities. The relation between both quantities when varying $\mu_B$ is shown in Fig. 1 (right).

![Diagram of T vs. $\mu_B$](image1.png)

![Diagram of $\tau$ vs. $\mu_B$](image2.png)

Figure 1. Left: Chemical freeze-out line $T = T(\mu_B)$ obtained by assuming $\langle E \rangle / \langle N \rangle = 1$ GeV. We plot as continuous lines from top to bottom the result using: Boltzmann-Gibbs statistics, Tsallis statistics with $q = 1.02$, and with $q = 1.14$. Dashed red line corresponds to the curve with $q = 1.14$ multiplied by the factor 3.65. Right: Quotient between physical temperature $T$ obtained with BG statistics, and the effective temperature $\tau$ obtained with Tsallis statistics and $q = 1.14$, as a function of the baryonic chemical potential. We consider $\mu = 0$ for mesons in both figures.

| Mesons | Mass | $S$ | $g$ | Mesons | Mass | $S$ | $g$ |
|--------|------|-----|----|--------|------|-----|----|
| $\pi^0$ | 134.98 | 0 | 1 | $K^+$ | 493.68 | 1 | 1 |
| $\pi^+, \pi^-$ | 139.57 | 0 | 2 | $K^0$ | 497.67 | 1 | 1 |
| $\eta$ | 547.3 | 0 | 1 | $K^*(892)^+$ | 891.66 | 1 | 3 |
| $\rho(770)$ | 771.1 | 0 | 9 | $K^*(892)^0$ | 896.1 | 1 | 3 |
| $\omega(782)$ | 782.57 | 0 | 3 | $K_1(1270)$ | 1273 | 1 | 6 |
| $\eta'(958)$ | 957.78 | 0 | 1 | $K_1(1400)$ | 1402 | 1 | 6 |
| $f_0(980)$ | 980 | 0 | 1 | $K_0^*(1430)$ | 1412 | 1 | 2 |
| $a_0(980)$ | 984.7 | 0 | 3 | $K^*(1410)$ | 1414 | 1 | 6 |
| $\phi(1020)$ | 1019.46 | 0 | 3 | $K^*_2(1430)^+$ | 1425.6 | 1 | 5 |
| $h_1(1170)$ | 1170 | 0 | 3 | $K^*_2(1430)^0$ | 1432.4 | 1 | 5 |
| $b_1(1235)$ | 1229.5 | 0 | 9 | $K(1460)$ | 1460 | 1 | 10 |

Table I. List of the lowest-lying mesons used in Eq. (7). We include some of their properties: mass, strangeness ($S$) and degeneracy ($g$). The corresponding anti-mesons with $S = -1$ are not shown in the table but they are considered as well in the computation. The dots indicate that heavier mesons are included in the computation, although they are not explicitly shown in this table due to lack of space.
Table II. List of the lowest-lying baryons used in Eq. (7). We show only the baryons with baryonic number $B = 1$. The corresponding anti-baryons $B = -1$ and $S = 0, 1, 2$ and 3 are considered as well in the computation.

| Baryons | Mass | $g$ | $S$ | Baryons | Mass | $g$ |
|---------|------|-----|-----|---------|------|-----|
| $p$     | 938.27 | 0   | 2   | $\Lambda^0$ | 1115.68 | -1  |
| $n$     | 939.56 | 0   | 2   | $\Sigma^+$ | 1189.37 | -1  |
| $\Delta (1232)$ | 1232 | 0   | 16  | $\Sigma^0$ | 1192.64 | -1  |
| $N (1440)$ | 1440 | 0   | 4   | $\Sigma^-$ | 1197.45 | -1  |
| $N (1520)$ | 1520 | 0   | 8   | $\Sigma'^+$ | 1382.8 | -1  |
| $N (1535)$ | 1535 | 0   | 4   | $\Sigma''^0$ | 1383.7 | -1  |
| $\Delta (1600)$ | 1600 | 0   | 16  | $\Sigma^-$ | 1387.2 | -1  |
| $\Delta (1620)$ | 1620 | 0   | 8   | $\Lambda (1405)$ | 1406 | -1  |
| $\Xi^0$ | 1314.83 | -2  | 2   | $\Omega^-$ | 1672.45 | -3  |
| $\Xi^-$ | 1321.31 | -2  | 2   |
| $\Xi^{*0}$ | 1531.8 | -2  | 4   |
| $\Xi^{*-}$ | 1535 | -2  | 4   |

We observe in Fig. 1 (left) that for both $q = 1$ and $q = 1.14$ we obtain lines with similar shapes, but $r$ is always lower than $T$. The ratio between both temperatures is practically constant up to $\mu_B \simeq 0.8$ GeV, as can be seen in Fig. 1 (right), and a consequence is that the traced line which results from the multiplication of the results with $q = 1.14$ by a constant factor equal to 3.65 almost reproduces the result with BG statistics in Fig. 1 (left). We also include in Fig. 1 (left) the results for an intermediate value of $q = 1.02$, which shows that the phase transition line smoothly approaches that obtained for Boltzmann statistics as $q$ decreases towards the unity. All curves in Fig. 1 (left) show an inflexion for $\mu_B = 0$ GeV which is related to the sharp increase in the baryon density as the baryonic chemical potential approaches the proton/neutron mass $m_{p,n} \simeq 0.94$ GeV. This increase is displayed in Fig. 2, where we plot the proton and neutron densities as a function of $\mu_B$ in Tsallis statistics. A similar behaviour is observed in Boltzmann-Gibbs statistics.

![Figure 2. Density of protons and neutrons as a function of the baryonic chemical potential, in Tsallis statistics with $q = 1.14$. We consider $T = 20$ MeV. Dashed green line corresponds to the density of protons, while continuous red line is the density of neutrons.](image)

For $\mu_B = 0$ the effective temperature is $T_0 = 45.6$ MeV for $q = 1.14$, which is not in agreement with the value $T_0 = (60.7 \pm 0.5)$ MeV found in the analysis of the $pT$-distributions [16]. This disagreement can be related to the value adopted for $\langle E \rangle / \langle N \rangle$, which still must be checked by analysis of experimental data with the non extensive statistic. In order to provide an estimate of the sensitivity of the value of $T_0$ when changing $q$ it is worth mentioning that the effective temperature for a slightly smaller value of the entropic index, $q = 1.12$, is $T_0 = 61.0$ MeV, which is in
agreement with that reference.

The transition line determines the region where the confined states exist (below the line), and the region where one expects to find the quark-gluon plasma (above the line).

Figure 3. Left: Pressure as a function of temperature. The baryonic chemical potential is kept fixed to the value $\mu_B = 0.8$ GeV. Right: Pressure as a function of energy density when changing the baryonic chemical potential in the range $0$ GeV < $\mu_B < 0.9$ GeV. The temperature is kept fixed to the value $T = 20$ MeV. In both figures we plot as a dashed blue line the result in Boltzmann-Gibbs statistics, and as a continuous red line the result in Tsallis statistics with $q = 1.14$.

Figure 4. Chemical freeze-out line $T = T(\mu_B)$ obtained by assuming $\langle E \rangle / \langle N \rangle = 1$ GeV. Left: Boltzmann-Gibbs statistics. Right: Tsallis statistics with $q = 1.14$. We show the result in different cases, including: i) “(anti)protons + (anti)neutrons”, ii) “(anti)protons + (anti)neutrons + pions” with $\mu_\pi = 0$, $\mu_{\pi^+} = m_{\pi^+}$ or $\mu_{\pi^+} = -m_{\pi^+}$, iii) all hadrons with $\mu_{\text{mesons}} = 0$ (as in Fig. 1). In ii) we consider $\mu_{\pi} = \mu_p - \mu_{\pi^+}$, see Eq. (13), and in this case it is represented in the horizontal axis $\mu_p$ when $\mu_{\pi^+} = m_{\pi^+}$, and $\mu_{\pi}$ when $\mu_{\pi^+} = -m_{\pi^+}$.

In Fig. 3 (left) we plot the pressure as a function of the temperature for the chemical potentials fixed at the value $\mu_B = 0.8$ GeV for the baryonic chemical potential and $\mu = 0$ for all mesons with $q = 1$ and $q = 1.14$. We observe that the pressure increases faster in the non extensive case, $q = 1.14$, than in the extensive one, $q = 1$. In Fig. 3 (right) we show the results for the pressure as a function of the energy density for $T = 20$ MeV. Also here we observe that the pressure increases faster in the case of $q = 1.14$ in comparison with the case $q = 1$. From Fig. 3 one can conclude that the EOS for hadronic matter obtained with the Tsallis statistics is harder than the one obtained from the BG case.

Up to now we have studied the equation of state including the spectrum of hadrons in Table I and II. It would be interesting to analyze also the case in which only protons, neutrons and possibly pions contribute to the equation of state, all of them with nonzero chemical potential. We have studied the finite pion chemical potential in two different cases: a) $\mu_{\pi^+} = m_{\pi^+} = -\mu_{\pi^-}$, b) $\mu_{\pi^+} = -m_{\pi^+} = -\mu_{\pi^-}$; and $\mu_{\pi^0} = 0$, $\mu_{\pi} = \mu_p - \mu_{\pi^+}$ in both cases. These values are motivated in section IV. The phase transition lines in Boltzmann-Gibbs and in Tsallis statistics in the regime of high baryonic chemical potential are plotted in Fig. 4. Note that the effect of finite pion chemical potential is to increase the transition line to higher values of temperature, and the curves with $\mu_{\pi^+} = m_{\pi^+}$ and $\mu_{\pi^+} = -m_{\pi^+}$ coincide, as we have not introduced electrons in the computation (see section IV for a discussion including electrons). We show in Fig. 5 the result for the equation of state considering zero and finite pion chemical potential. The effect of pions is to increase the values for the pressure with respect to the case with only protons and neutrons. When considering a
nonzero value for the pion chemical potential this leads to a noticeable effect on the EOS, as it becomes harder either in Boltzmann-Gibbs or in Tsallis statistics, see Fig. 5 (right).

The results presented in this section make us to consider an application to stellar matter, where high pressures are necessary to compensate for the gravitational force so that protoneutron star stability and correct macroscopic properties are attained.

IV. APPLICATION TO (PROTO)NEUTRON STARS

During the evolution process of a protoneutron star, neutrinos are believed to be trapped for a few seconds and then leave the star carrying huge amounts of energy. This process is known as deleptonization. The entropy then decreases and the gas pressure due to their small masses. One also needs to make sure that the neutron stars are electrically neutral.

For the sake of simplicity let us initially think about a neutron star composed only by protons, neutrons, electrons and muons. The star is not static, it is indeed a dynamically equilibrated system because protons and neutrons are necessary to compensate for the gravitational force so that protoneutron star stability and correct macroscopic properties are attained.

\[
\begin{align*}
  n &\rightarrow p + e^- + \bar{\nu}_e \\
  p &\rightarrow n + e^+ + \nu_e
\end{align*}
\]

and the dynamical equilibrium between the relative number of neutrons and protons is regulated by their respective chemical potentials, which must satisfy the relation

\[
\mu_n = \mu_p + \mu_e \quad \mu_c = \mu_\mu 
\]
where it was assumed that the positron chemical potential is opposite to the electron chemical potential. The equations above cannot fully determine \( \mu_n \), so this is a free parameter which is in fact the one which regulates the baryonic density in the star, given by

\[
\rho = \rho_p + \rho_n,
\]

where \( \rho_p \) and \( \rho_n \) are respectively the proton and neutron densities. As the star has to be electrically neutral, the condition that

\[
\rho_p = \rho_e + \rho_\mu
\]

is also enforced. Therefore our first calculation is for a neutron-proton-lepton star with leptons being included in our EOS by using the standard BG statistics for a fermionic free gas. Notice that we have two independent chemical potentials, \( \mu_n \) and \( \mu_e \), which are constrained by Eqs. (9) and (11). It is also well known that the macroscopic star properties cannot be explained in terms of free Fermi gases only. In order to attain a reasonable stellar mass, all models have to take into account the nuclear interaction, generally described by mediating mesons of Yukawa type interactions [32]. In [33], protoneutron stars with a fixed entropy per baryon \( s = 2 \) were studied within quantum hadrodynamics extended to consider also nonextensive statistical effects. The difference between the usual non-linear Walecka type models [30, 32, 34] and the work developed in [33] is only due to the different fermion distribution functions for hadrons (the paper does not clarify whether the Tsallis statistics is used for leptons as well). Bearing in mind that, apart from the fact that the equations of motion in a mean field approximation are solved in a self-consistent way, the influence of the nuclear interaction, mediated by the meson fields, is approximately the same in both approaches. Hence, the main influence of the non-extensive statistics resides in the gas contribution, which we analyse next.

In Fig. 6 we show how the electron chemical potential varies with the neutron chemical potential in different situations: 1) neutron, proton and leptons described by free Fermi gases, 2) neutrons and protons described by nonextensive thermodynamics with \( q = 1.14 \), following the analysis performed in Ref. [16], and leptons described by a free Fermi gas, and 3) neutrons and protons described by non extensive thermodynamics with \( q = 1.5 \) and leptons described by a free Fermi gas. One can see that electron chemical potential increases with the increase of the \( q \) value, resulting in different amounts of particles inside the star, which are depicted in Fig. 7. One can observe that there is a discontinuity, which appears at larger neutron chemical potentials as \( q \) increases and that reflects itself in the particle fractions. After the discontinuity, the resulting electron chemical potential tends to the results obtained with the BG statistics. The particle density integrals run from zero to infinity and it can be divided into two integrals, one running from zero to \( \sqrt{\mu_b^2 - m_b^2} \), where \( b = p, n \) and the other running from this point to infinity. The discontinuities take place exactly at the point where the momentum reaches the values \( \sqrt{\mu_b^2 - m_b^2} \) and it coincides for protons and neutrons. The effects of these discontinuities have already been discussed in the last section. We define the particle fraction as \( Y_i = n_i / \rho \), where \( \rho \) is the baryonic density. From Fig. 7, we can see that the particle fractions change considerably from one case to the other, when we move from the BG statistics to the non-extensive statistics.

In the simplest mean-field approximation to neutron star matter using the non-linear Walecka model [32], the pion contribution vanishes and, therefore, they are not considered as constituent particles of the star. In more sophisticated methods, as the Hartree-Fock approximation, they can be introduced. In our approach, it is easy to include pions,
since they are automatically considered in the thermodynamics at high energies, so we have decided to study also the role of pions in the neutron star stability in our second calculation where the star is composed by neutrons, protons, pions and leptons. Pions usually decay into muons through the weak interaction, but in a dense hadronic medium as a neutron star it is more probable that they interact with protons and neutrons through the strong interaction. Therefore, we next check whether the pions can be partially responsible for the nuclear forces acting in the star, in such a way that the number of pions is regulated by their chemical potential and in turn, the pions regulate the intensity of the nuclear force. It is important to emphasize that we do not expect the pions to completely mimic the role of the nuclear interaction in this simple picture, since we know, from the previous sections, that the non-extensive thermodynamics acts as a small perturbation or correction in the system.

The pionic chemical potential can vary inside the limit $\mu_\pi < m_\pi$ and for positive and negative pions the relation $\mu_{\pi^+} = -\mu_{\pi^-}$ must be satisfied since one is the antiparticle of the other. For the same reason the neutral pions have null chemical potential, $\mu_{\pi^0} = 0$. In the nuclear interaction pions can convert protons into neutrons and vice-versa, through the processes

$$\begin{cases} n + \pi^+ \rightarrow p, \\ p + \pi^- \rightarrow n, \end{cases} (12)$$

which lead to the relation

$$\mu_\pi = \mu_p - \mu_{\pi^+} (13)$$

and the total electric charge has to be kept neutral so that

$$\rho_p + \rho_{\pi^+} = \rho_{\pi^-} + \rho_e + \rho_\mu (14)$$

with

$$-m_{\pi^+} \leq \mu_{\pi^+} \leq m_{\pi^+}. (15)$$

Once again, the leptons are included in our EOS by using the standard BG statistics for a fermionic free gas, and then we can calculate all the thermodynamical quantities of interest.

This is a quite simple and naive model, and can be improved in several ways, but it will be enough for a study of the role played by non extensive thermodynamics in neutron stars. In particular, despite the fact that the pions appear as intermediate particles for the nuclear force, we consider below just their on-shell mass and we have taken the pion mass equal to 135 MeV for the three species. Also other hadrons, as baryonic resonances heavier than proton and neutron could be present in the star but they are not included in our model.

In Fig. 5, we can see that the EOS obtained with a pion chemical potential of the order of the pion mass is the hardest of all. For this reason, we have considered three different possibilities for the pion chemical potential: i) the first one is to assume that $\mu_{\pi^-} = \mu_e$ or analogously that $\mu_{\pi^+} = -\mu_e$, what can be inferred directly from the comparison of Eqs. (9) and (13). The second possibility is to fix the pion chemical potential as the pion mass and we have taken the two limits of Eq. (15): ii) $\mu_{\pi^-} = m_\pi$ ($\mu_{\pi^+} = -m_\pi$) and iii) $\mu_{\pi^-} = -m_\pi$ ($\mu_{\pi^+} = m_\pi$). For the
sake of comparison, the BG results are also plotted for case i). We can see that in this case, the electron chemical potential increases until it reaches the pion mass value and then it is kept constant, due to the fact that the negative pions start to replace the electrons. In this case, zero charged and positive pions are almost inexistent. The difference between the BG and non-extensive statistics resides again at very low densities, where the appearance of pions is slightly quenched if Tsallis statistic is used, as can be seen from the top graphs in Fig 8. For this study $q = 1.14$. From Fig 8, one sees that, in these cases, the amount of pions decreases as the density of the system increases, contrary to what happens in case i). Hence, the pion contribution to the system can vary considerably, depending on the way its chemical potential is fixed. These differences in the constitution of the star at low densities are very important because they may influence the existence of the pasta phase [35] and control neutrino mean free paths through the diffusion coefficients [36], with possible consequences in the protoneutron star cooling mechanism.

We now turn our attention to the different EOS obtained so far. In Fig. 9 we plot the EOS for both systems studied in the present work, i.e., with and without pions. The results for the BG statistics is shown for comparison. The curves are very similar, but one can check that the inclusion of pions makes the EOS softer in both statistics as far as case i) for the definition of the pion chemical potential is used. In the other situations (case ii) and iii)) the EOS are practically coincident. We conclude that the enforcement of charge neutrality and $\beta$-equilibrium stability conditions wash out completely the hardening effect of the pions on the EOS. The difference in the values of the pressure and energy density as compared with the ones presented in Fig. 5 are also caused by the inclusion of the leptons and the stellar matter conditions.

The Tolman-Oppenheimer-Volkof (TOV) equations [37] are the differential equations for the structure of a static, spherically symmetric, relativistic star in hydrostatic equilibrium. When solving the TOV equations we get the results for some of the macroscopic characteristics of the star, particularly its maximum mass and its corresponding radius. With the EOS obtained in the present work, all the results lie around $0.7\,M_\odot$ and 8 Km for the maximum stellar masses and respective radii. Those are the well known values obtained with EOS described by a free Fermi gas and reflect the fact that the nuclear interaction is not taken into account. These values for the stellar masses are well below

![Figure 8. Particle fractions as a function of the baryonic density at $\tau = 20$ MeV. Top graphs show the results for i) $\mu_{\pi^-} = \mu_e$ for left) BG and right) Tsallis statistics ($q = 1.14$). Bottom graphs are shown for Tsallis statistics ($q = 1.14$) for left ii) $\mu_{\pi^-} = m_\pi$ and right iii) $\mu_{\pi^-} = -m_\pi$.](image-url)
the expected value of 1.4 or even 2 \( M_\odot \) and is in accordance with the analysis made above, where it was evidenced that in this situation both BG and Tsallis statistics yield similar pressures from the hadronic contribution.

V. CONCLUSIONS

In this work we developed the non extensive thermodynamics for an ideal quantum gas for both bosons and fermions from the partition function defined here for the first time. Then we showed that the partition function and the thermodynamics derived from it is equivalent to the thermodynamics derived from the entropy proposed by Conroy, Miller and Plastino [21], and also by Cleymans and Worku [19] when \( \mu \leq m \). For \( \mu > m \) and for fermions some inconsistencies of previous references were addressed, and our result is fully thermodynamically consistent. In the limit of high energies, the partition function is in accordance with that proposed in Ref. [10], and thus the partition function defined here is self-consistent in the sense proposed by Hagedorn.

Some thermodynamical functions are derived from the partition function for different hadronic systems (proton-neutron system and proton-neutron-pion-lepton systems) in different conditions. Particularly we analyze how pressure and energy densities vary when the entropic index or the chemical potentials vary. A discussion about a discontinuity observed in the first derivatives of the partition function is done for the first time.

We apply our formalism to study the neutron star stability under several conditions. To this end electron and muon gases in the Boltzmann-Gibbs statistics are included in the system. Our results show that a simple thermodynamical description of the neutron star matter within the non extensive statistics cannot explain the star stability as observed in nature, as expected, since nuclear interaction was not included. Nevertheless, we have seen that the Tsallis statistics modifies substantially the particle constitution at low densities and this fact deserves more investigation towards the analysis of the potentialities of a non extensive thermodynamical approach. In [33], where the non-linear Walecka model was used, the equation of state becomes harder for \( q \) values larger than one, with the obvious consequence that the maximum stellar mass is higher than in the usual case. This behavior seems to be different from the one we have obtained, since in the present work, the EOS show very little sensitivity to the increase of \( q \). In [38], a simple model for quark stars was used and, for a fixed temperature, the maximum stellar mass was shown to decrease for \( q \) values larger than one, a result which is opposite to the one found in [33]. These somewhat contradictory results deserve better investigation. Moreover, many perspectives for improvements in the simple model proposed here can be done, as the inclusion of other hadrons as the \( \Lambda \) particles, for instance, and the description of leptons also by the non extensive statistics with a \( q \) value different from the one used for hadrons.

Acknowledgments

This work has been supported by Plan Nacional de Altas Energías (FPA2011-25948), Junta de Andalucía grant FQM-225, Generalitat de Catalunya grant 2014-SGR-1450, Spanish MINECO’s Consolider-Ingenio 2010 Programme CPAN (CSD2007-00042), Centro de Excelencia Severo Ochoa Programme grant SEV-2012-0234, by CNPq (Brazil) and FAPESC (Brazil) under project 2716/2012, TR 201200344, and FAPESP (Brazil) under grant 2013/24468-1. A.D. acknowledges the support from CNPq under grant 305639/2010-2. E.M. would like to thank the Instituto de
Física of the Universidade de São Paulo for their hospitality and support during the completion of parts of this work. The research of E.M. has been supported by the Juan de la Cierva Program of the Spanish MINECO, and by the European Union under a Marie Curie Intra-European Fellowship (FP7-PEOPLE-2013-IEF). D.P.M. acknowledges the warm hospitality at the Universidad de Alicante, where parts of this work were carried out.

[1] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[2] I. Bediaga, E.M.F. Curado e J.M. de Miranda, Physica A 286 (2000) 156.
[3] R. Hagedorn, Nuovo Cimento Suppl. 3 (1965) 147.
[4] B. Abelev et al. (ALICE Collaboration), arXiv:1205.5724v1 (2012).
[5] B. Abelev et al. (ALICE Collaboration), Eur. Phys. J. C 72 (2012) 2183.
[6] K. Aamodt et al. (ALICE Collaboration), Eur. Phys. J. C 71 (2011) 1655.
[7] V. Khachatryan et al. (CMS Collaboration), JHEP 05 (2011) 064.
[8] G. Aad et al. (ATLAS Collaboration), Nucl. Phys. B 850 (2011) 387-444.
[9] R. Hagedorn, Lect. Notes Phys. 221, 53 (1985).
[10] A. Deppman, Physica A 391 (2012) 6380.
[11] N. O. Agasian, Phys. Lett. B 519, 71 (2001).
[12] W. Broniowski, W. Florkowski and L. Y. Glozman, Phys. Rev. D 70, 117503 (2004).
[13] E. Megias, E. Ruiz Arriola and L. L. Salcedo, Phys. Rev. Lett. 109, 151601 (2012).
[14] E. Megias, E. Ruiz Arriola and L. L. Salcedo, Nucl. Phys. Proc. Suppl. 234, 313 (2013).
[15] E. Megias, E. Ruiz Arriola and L. L. Salcedo, Phys. Rev. D 89, 076006 (2014).
[16] L. Marques, E. Andrade-II e A. Deppman, Phys. Rev. D 87 (2013) 114022.
[17] I. Sena e A. Deppman, Eur. Phys. J. A 49 (2013) 17.
[18] I. Sena e A. Deppman, AIP Conf. Proc. 1520, (2013) 172.
[19] J. Cleymans e D. Worku, J. Phys. G: Nucl. Part. Phys. 39 (2012) 025006.
[20] A. Deppman, J. Phys. G: Nucl. Part. Phys. 41, 055108 (2014).
[21] J.M. Conroy, H.G. Miller and A.R. Plastino, Physics Letters A 374 (2010) 45814584.
[22] E. Megias, D.P. Menezes and A. Deppman, arXiv:1312.7134, accepted for publication in Physica A (2014).
[23] D.P. Menezes, A. Deppman, E. Megias and L.B. Castro, arXiv:1410.2264, submitted for publication in Phys. Rev. C.
[24] E. Megias, D. P. Menezes and A. Deppman, arXiv:1407.8044.
[25] A.M. Teweldeberhan, A.R. Plastino and H. Miller, Physics Letters A 343 (2005) 71-78.
[26] J. Cleymans, K. Redlich, Phys. Rev. C 60 (1999) 054908.
[27] J. Beringer et al., Phys. Rev. D 86 (2012) 010001.
[28] J. M. Lattimer and M. Prakash, Science 304, 536 (2004).
[29] W. Keil and H. Janka, Astron. Astrophys. 296, 145 (1995).
[30] D.P. Menezes and C. Providência, Phys. Rev. C 68 (2003) 035804; Phys. Rev. C 69 (2004) 045801;
[31] A.M.S. Santos and D.P. Menezes, Phys. Rev. C 69 (2004) 045803.
[32] B. D. Serot and J. D. Walecka, Adv. Nucl. Phys. 16, 1 (1986); J. Boguta and A. R. Bodmer, Nucl. Phys. A 292, 413 (1977).
[33] G. Gervino, A. Lavagno and D. Pigato, J. Physics: Conference Series 442 (2013) 012065.
[34] M.D. Alloy and D.P. Menezes, Phys. Rev. C 83, 035803 (2011).
[35] R.C. Tolman, Phys. Rev. 55, 364 (1939); J.R. Oppenheimer and G.M. Volkoff, Phys. Rev. 55, 374 (1939).
[36] M. Razeira, B.E.J. Bodmann and C.A.Z. Vasconcellos, Int. J. Mod. Phys. D, 16 (2007) 365.