Asymptotics of the Heat Kernel on Rank 1 Locally Symmetric Spaces

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Abstract

We consider the heat kernel (and the zeta function) associated with
Laplace type operators acting on a general irreducible rank 1 locally
symmetric space $X$. The set of Minakshisundaram-Pleijel coefficients
$\{A_k(X)\}_{k=0}^{\infty}$ in the short-time asymptotic expansion of the kernel is
calculated explicitly.

1 Introduction

In the theory of quantum fields on curved background spaces, the short-time
expansion of the heat kernel plays an extremely important role. In particular
situations, for example, the coefficients in the expansion control the one-loop
divergences of the effective action, and related quantities such as the stress energy momentum tensor. Some of these coefficients have been determined and appear in the physics and mathematical literature. Note the references [1, 2, 3, 4, 5, 6] for closed Riemannian manifolds and [7, 8] for manifolds with a smooth boundary. The literature on these matters is very vast.

In Refs. [1, 2, 3], R. Miatello studies the case of a closed locally symmetric rank 1 manifold \(X\), using the representation theory of the group of isometries of \(X\). We consider the same case in the present paper, but we use the spectral zeta function of \(X\). By our approach we determine the expansion coefficients immediately and explicitly (essentially in one step), given the results of [9]. Recently the topological Casimir energy [10], the one-loop effective action, and the multiplicative and conformal anomaly [11, 12] associated with Laplace type operators on \(X\), and their product, have been analysed also by use of the spectral zeta function.

The paper is organized as follows. In Sect. 2 we define the spectral zeta function \(\zeta_{\Gamma}(s; \chi)\) of \(X\) corresponding to a finite-dimensional representation \(\chi\) of the fundamental group \(\Gamma\) of \(X\). The residues of \(\zeta_{\Gamma}(s; \chi)\) and special values of this zeta function, which relate to the expansion coefficients, are provided by Theorems 2.1 and 2.2. In Sect. 3 we consider the asymptotic expansion of the heat kernel (as \(t \to 0^+\)), and compute all the expansion coefficients in closed form in the main theorem, Theorem 3.1. Sect. 4 contains some remarks in summary. We include an Appendix with information supplementary to Theorems 2.1, 2.2 and 3.1.

2 The Spectral Zeta Function

We shall be working with an irreducible rank 1 symmetric space \(M = G/K\) of non-compact type. Thus \(G\) will be a connected non-compact simple split rank 1 Lie group with finite center and \(K \subset G\) will be a maximal compact subgroup [13]. Let \(\Gamma \subset G\) be a discrete, co-compact torsion free subgroup. Then \(X = X_\Gamma = \Gamma\backslash M\) is a compact Riemannian manifold with fundamental group \(\Gamma\); namely \(X\) is a compact locally symmetric space. Given a finite-dimensional unitary representation \(\chi\) of \(\Gamma\) there is the corresponding vector bundle \(V_\chi \leftrightarrow X\) over \(X\) given by \(V_\chi = \Gamma\backslash (M \otimes F_\chi)\), where \(F_\chi\) (the fibre of \(V_\chi\)) is the representation space of \(\chi\) and where \(\Gamma\) acts on \(M \otimes F_\chi\) by the rule \(\gamma \cdot (m, f) = (\gamma \cdot m, \chi(\gamma)f)\) for \((\gamma, m, f) \in (\Gamma \otimes M \otimes F_\chi)\). Let \(\Delta_\Gamma\)
be the Laplace-Beltrami operator of $X$ acting on smooth sections of $V_{\chi}$; we obtain $\Delta_\Gamma$ by projecting the Laplace-Beltrami operator of $M$ (which is $G-$invariant and thus $\Gamma-$invariant) to $X$. As $X$ is compact we can consider the spectrum \( \{\lambda_j = \lambda_j(\chi), n_j = n_j(\chi)\}_{j=0}^\infty \) of $-\Delta_\Gamma$, where $n_j$ is the (finite) multiplicity of the eigenvalue $\lambda_j$. We use the minus preceding $\Delta_\Gamma$ to have the $\lambda_j \geq 0$: $0 = \lambda_0 < \lambda_1 < \lambda_2 ...; \lim_{j \to \infty} \lambda_j = \infty$.

The spectral zeta function $\zeta_\Gamma(s; \chi)$ of $X_\Gamma$ of Minakshisundaram-Pleijel type [14], which we shall consider is defined by

$$
\zeta_\Gamma(s; \chi) = \sum_{j=1}^\infty \frac{n_j(\chi)}{\lambda_j(\chi)^s},
$$

(2.1)

for $\Re s >> 0$. $\zeta_\Gamma(s; \chi)$ is a holomorphic function on the domain $\Re s > d/2$, where $d$ is the dimension of $M$, and by general principles $\zeta_\Gamma(s; \chi)$ admits a meromorphic continuation to the full complex plane $\mathbb{C}$. However since the manifold $X_\Gamma$ is quite special it is desirable to have the meromorphic continuation of $\zeta_\Gamma(s; \chi)$ in an explicit form, for example in terms of the structure of $G$ and $\Gamma$. Using the Selberg trace formula and the $K$-spherical harmonic analysis of $G$, we have obtained such a form in [9]; also see Refs. [15, 10]. In particular we can obtain the residues of $\zeta_\Gamma(s; \chi)$, and compute the special values $\zeta_\Gamma(-n; \chi)$, $n = 0, 1, 2, ...$ – results which play a decisive role in the present work. To state these results we introduce further notation.

Up to local isomorphism we can represent $M = G/K$ by the following quotients:

$$
M = \begin{bmatrix}
SO_1(n,1)/SO(n) & (I) \\
SU(n,1)/U(n) & (II) \\
SP(n,1)/\left(\text{SP}(n) \otimes \text{SP}(1)\right) & (III) \\
F_{4(-20)}/\text{Spin}(9) & (IV)
\end{bmatrix},
$$

(2.2)

where $d = n, 2n, 4n, 16$ respectively. We shall need the real number $\rho_0$ which corresponds to $1/2$ the sum of the positive real restricted roots of $G$ with respect to a nilpotent factor in an Iwasawa decomposition of $G$. $\rho_0$ is given by $\rho_0 = (n - 1)/2, n, 2n + 1, 11$ respectively in the cases (I) to (IV). For details on these matters the reader may consult [13], and also the Appendix in [11].

The spherical harmonic analysis on $M$ is controlled by Harish-Chandra’s Plancherel density $|C(r)|^{-2}$, a function on the real numbers $\mathbb{R}$, computed by
Miatello [1, 2, 3], and others, in the rank 1 case we are considering. We choose a normalization of Haar measure on $G$ however which differs from that of [1, 2, 3]; see Ref. [9]. For a suitable constant $C_G$ depending only on $G$, and for a suitable even polynomial $P(r)$ of degree $d - 2$ for $G \neq SO_1(n, 1)$ with $n$ odd, and of degree $d - 1 = 2m$ for $G = SO_1(2m + 1, 1)$, $|C(r)|^{-2}$ is given by

$$ |C(r)|^{-2} = \begin{cases} 
C_G \pi r P(r) \tanh(\pi r) & \text{for } G = SO_1(2m, 1), \\
C_G \pi r P(r) \tanh(\pi r/2) & \text{for } G = SU(n, 1), \text{ } n \text{ odd or } G = SP(n, 1), \text{ } F_4(-20), \\
C_G \pi r P(r) \coth(\pi r/2) & \text{for } G = SU(n, 1), \text{ } n \text{ even}, \\
C_G \pi r P(r) & \text{for } G = SO_1(2m + 1, 1)
\end{cases}. $$

(2.3)

The value of $C_G$ and the explicit form of $P(r)$ is given in the Appendix. For real hyperbolic space $M = SO_1(2m, 1)/SO(2m)$ of even dimension $2m$, for example, $P(r)$ is given by

$$ P(r) = \prod_{j=0}^{m-2} \left[ r^2 + \frac{(2j + 1)^2}{4} \right]. $$

(2.4)

The coefficients of $P(r)$ will be denoted by $a_{2j}$:

$$ P(r) = \sum_{j=0}^{d/2-1} a_{2j} r^{2j} \quad \text{for } G \neq SO_1(2m + 1, 1), $$

$$ = \sum_{j=0}^{m} a_{2j} r^{2j} \quad \text{for } G = SO_1(2m + 1, 1). $$

(2.5)

We denote by $Vol(\Gamma \backslash G)$ the $G-$ invariant volume of $\Gamma \backslash G$ induced by Haar measure on $G$.

As pointed out earlier, the explicit meromorphic structure of the zeta function $\zeta_{\Gamma}(s; \chi)$ of (2.1) is worked out in [3] in terms of the spherical harmonic analysis of $G$ and $\Gamma-$ structure; see Theorems 4.2, 5.1 there; also compare Theorems 5.2, equation (6.1), and Theorem 6.9 of [10]. In particular, apart from the case $G = SO_1(n, 1)$ with $n$ odd (a case which we treat separately), $\zeta_{\Gamma}(s; \chi)$ is holomorphic except for possibly simple poles at $s = 1, 2, ..., d/2$. By Theorem 5.1 of [3], or by the results stated in [10] we
can compute the residues at these points \( s = 1, 2, \ldots, d/2 \). The results are the following, where we omit the cotangent case.

**Theorem 2.1** Apart from the cases \( SO_1(\ell, 1) \), \( SU(q, 1) \) with \( \ell \) odd and \( q \) even, the residue of \( \zeta_t(s; \chi) \) at \( s = m \) (for \( m \) an integer, \( 1 \leq m \leq d/2 \)) equals

\[
\frac{1}{4} \chi(1) \text{Vol}(\Gamma \backslash G) C_G \frac{d/2-m}{j=0} (-1)^j \binom{m+j-1}{j} \rho_0^{2j} a_{2(m+j-1)},
\]

given the preceding notation. Also for \( n = 1, 2, \ldots, \)

\[
\zeta_t(-n; \chi) = \frac{1}{4} \chi(1) \text{Vol}(\Gamma \backslash G) C_G \frac{d/2-1}{j=0} (-1)^j j! \rho_0^{2(j+n+1)} a_{2j} + 2 \sum \frac{(-1)^k n!}{(n-k)!} b_{k+1}(j) a_{2j},
\]

where

\[
b_p(j) \overset{\text{def}}{=} \left[ 2^{1-2(p+j)} - 1 \right] \left[ \frac{\pi}{a(G)} \right]^{2(p+j)} \frac{(-1)^j B_{2(p+j)}}{2(p+j)(p-1)!},
\]

for \( p = 1, 2, \ldots, B_r \) the \( r \)-th Bernoulli number, and for

\[
a(G) \overset{\text{def}}{=} \begin{cases} 
\pi & \text{if } G = SO_1(\ell, 1) \text{ with } \ell \text{ even,} \\
\frac{\pi}{2} & \text{if } G = SU(q, 1) \text{ with } q \text{ odd} \\
& \text{or } G = SP(\ell, 1), \text{ any } \ell, \text{ } F_4(-20) \\
\end{cases}
\]

\( \zeta_t(0; \chi) = -n_0(\chi)+ \) (the right hand side of Eq. (2.7) evaluated at \( n = 0 \)).

Now we consider the case \( G = SO_1(\ell, 1) \) with \( \ell \) odd. By the results of [4], for \( G = SO_1(2n+1, 1) \) \( \zeta_t(s; \chi) \) has at most a simple pole at the points \( s = d/2 - k, k = 0, 1, 2, \ldots \). Moreover

**Theorem 2.2** For \( G = SO_1(2n+1, 1) \) the residue of \( \zeta_t(s; \chi) \) at \( s = d/2 - k \) (where \( d/2 = n + 1/2, k = 0, 1, 2, \ldots \)) equals
\[
\frac{1}{4} \chi(1) \Vol(\Gamma \setminus G) C_G \sum_{j=0}^{n} \frac{(-1)^{j+n+k} \rho_0^{2(j+k-n)} \Gamma(j + \frac{1}{2}) a_{2j}}{(j-n+k)! \Gamma(n + \frac{1}{2} - k)}, \tag{2.10}
\]
for \( k \geq n \), and equals
\[
\frac{1}{4} \chi(1) \Vol(\Gamma \setminus G) C_G \sum_{j=0}^{k} \frac{(-1)^j \rho_0^{2j} \Gamma(n - k + j + \frac{1}{2}) a_{2(n-k+j)}}{j! \Gamma(n + \frac{1}{2} - k)}, \tag{2.11}
\]
for \( 0 \leq k < n \). Here \( \rho_0 = n \). Also \( \zeta_{\Gamma}(0; \chi) = -n_0(\chi) \), whereas \( \zeta_{\Gamma}(-k; \chi) = 0 \) for \( k = 1, 2, \ldots \).

In Theorems 2.1 and 2.2 the constant \( C_G \) is given in the Appendix.

## 3 The Heat Kernel Coefficients

The object of interest is the heat kernel \( \omega_{\Gamma}(t; \chi) \) defined for \( t > 0 \) by
\[
\omega_{\Gamma}(t; \chi) = \sum_{j=0}^{\infty} n_j(\chi) e^{-\lambda_j(\chi)t}. \tag{3.1}
\]
If \( h_t \) is the fundamental solution of the heat equation on \( M \), then \( h_t \) and \( \omega_{\Gamma}(t; \chi) \) are related by the Selberg trace formula (cf. [9])
\[
\omega_{\Gamma}(t; \chi) = \chi(1) \Vol(\Gamma \setminus G) h_t(1) + \theta_{\Gamma}(t; \chi), \tag{3.2}
\]
where the theta function \( \theta_{\Gamma}(t; \chi) \) is given by Eq. (4.18) of [9] (for \( b = 0 \) there) and where
\[
h_t(1) = \frac{1}{4\pi} e^{-\rho_0^2} \int_{\mathbb{R}} e^{-r^2 |C(r)|^{-2}} dr. \tag{3.3}
\]

We shall not need the result (3.2). Our goal is to compute explicitly all of the coefficients \( A_k = A_k(\Gamma, \chi) \) in the asymptotic expansion
\[
\omega_{\Gamma}(t; \chi) \simeq (4\pi t)^{-d/2} \sum_{k=0}^{\infty} A_k t^k, \quad \text{as } t \to 0^+. \tag{3.4}
\]

Now \( \zeta_{\Gamma}(s; \chi) \) and \( \omega_{\Gamma}(t; \chi) \) are related by the Mellin transform:
\[ \zeta_{\Gamma}(s; \chi) = \frac{\mathfrak{M}[\omega_{\Gamma}](s)}{\Gamma(s)} = \frac{1}{\Gamma(s)} \int_0^{\infty} \omega_{\Gamma}(t; \chi)t^{s-1}dt, \quad \text{for} \quad \Re s > \frac{d}{2}. \quad (3.5) \]

Moreover one knows by abstract generalities (cf. [14, 16] for example) that the coefficients \( A_k \) are related to residues and special values of \( \zeta_{\Gamma}(s; \chi) \). Namely for \( m \) an integer with \( 1 \leq m \leq d/2 \), for \( d \) even

\[ A_{\frac{d}{2} - m} = (4\pi)^{d/2} \Gamma(m) \times \text{[residue of \( \zeta_{\Gamma}(s; \chi) \) at \( s = m \)]}. \quad (3.6) \]

Also for a positive integer \( n \)

\[ A_{\frac{d}{2} + n} = \frac{(-1)^n(4\pi)^{d/2}}{n!} \zeta_{\Gamma}(-n; \chi), \quad (3.7) \]

whereas

\[ A_{\frac{d}{2}} = (4\pi)^{d/2} [n_\omega(\chi) + \zeta_{\Gamma}(0; \chi)]. \quad (3.8) \]

For \( G = SO_1(2n + 1, 1) \) (the only case in which \( d \) is odd) we have for \( k = 0, 1, 2, ... \)

\[ A_k = (4\pi)^{d/2} \Gamma\left( \frac{d}{2} - k \right) \times \text{[residue of \( \zeta_{\Gamma}(s; \chi) \) at \( s = \frac{d}{2} - k \)]}; \quad (3.9) \]

\( d/2 = n + 1/2 \). Hence by Eqs. (3.6) - (3.9), and Theorems 2.1, 2.2 we obtain the following main result.

**Theorem 3.1** The heat kernel \( \omega_{\Gamma}(t; \chi) \) in (3.1) admits an asymptotic expansion (3.4). More precisely, given any non-negative integer \( N \) one has

\[ \lim_{t \to 0^+} \left[ (4\pi t)^{d/2} \omega_{\Gamma}(t; \chi) - \sum_{k=0}^{N} A_k(\Gamma, \chi)t^k \right] t^{-N} = 0 \quad (3.10) \]

where, apart from the cotangent case in (2.3) (i.e. the case \( G = SU(q, 1) \) with \( q \) even), the coefficients \( A_k(\Gamma, \chi) = A_k(X_{\Gamma}) \) are given as follows.

For all \( G \) except \( G = SO_1(\ell, 1), SU(q, 1) \) with \( \ell \) odd and \( q \) even
\[ A_k(\Gamma, \chi) = (4\pi)^{\frac{d-1}{2}} \chi(1) \text{Vol}(\Gamma \backslash G) C_G \pi \sum_{\ell=0}^{k} \frac{(-\rho_0^2)^{k-\ell}}{(k-\ell)!} \frac{d}{2} - (\ell + 1)! a_{2j}^{k}(\ell + 1) \]

for \( 0 \leq k \leq \frac{d}{2} - 1 \),

\[ A_{\frac{d}{2} + n}(\Gamma, \chi) = (-1)^n (4\pi)^{\frac{d-1}{2}} \chi(1) \text{Vol}(\Gamma \backslash G) C_G \pi \left[ \sum_{j=0}^{n-1} (-1)^{j+1} \frac{2(n+j)}{(n+1+j)!} \right] \]

\[ + 2 \sum_{j=0}^{n} \sum_{\ell=0}^{\min(k,n)} (-1)^{\ell} \frac{b_{\ell+1}(j)a_{2j}}{(n-\ell)!} \]

for \( n = 0, 1, 2, ... \),

\[ A_k(\Gamma, \chi) = \pi (4\pi)^{n-\frac{1}{2}} \chi(1) \text{Vol}(\Gamma \backslash G) C_G \sum_{\ell=0}^{\min(k,n)} \frac{(-n^2)^{k-\ell} \Gamma \left( n - \ell + \frac{1}{2} \right) a_{2(n-\ell)}}{(k-\ell)!} \]

\[ \text{or} \]

\[ A_k(\Gamma, \chi) = \pi^{3/2} (4\pi)^{n-\frac{1}{2}} \chi(1) \text{Vol}(\Gamma \backslash G) C_G \sum_{\ell=0}^{\min(k,n)} \frac{(-\rho_0^2)^{k-\ell} [2(n-\ell)]! a_{2(n-\ell)}}{(k-\ell)! (n-\ell)! 2^{2(n-\ell)}} \]

using that \( \Gamma(m + 1/2) = \pi^{1/2}(2m)! \left[ 2^{2m} m! \right]^{-1} \).

### 4 Conclusions

Using results on the meromorphic structure of the zeta function of a rank 1 locally symmetric space \( X \), we have obtained in a quick computation all of the Minakshisundaram-Pleijel coefficients (in closed form) in the
short-time asymptotic expansion of the heat kernel on $X$. Our method differs markedly from that of [1, 2, 3]. Besides their mathematical interest these coefficients play an important role in quantum loop effects (such as the conformal anomaly), and in field theory, quantum gravity, and cosmology [17, 18].

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6 Appendix

The constant $C_G$ in Eq. (2.3) and the Miatello coefficients $a_{2j}$ of the polynomials $P(r)$ in Eq. (2.5) appear in the statements of Theorems 2.1, 2.2 and 3.1. $C_G$ and $P(r)$ for the various rank 1 simple groups $G$ of this paper are given in the following table.

| $G$     | $C_G$                                                                 | $P(r)$                                                                 |
|---------|----------------------------------------------------------------------|----------------------------------------------------------------------|
| $SO_1(n, 1)$, $n \geq 2$ | $2^{2n-4} \Gamma \left( \frac{n}{2} \right)^2 \left[ \prod_{j=0}^{m-2} \left( r^2 + \left( \frac{(2j+1)^2}{4} \right) \right) \right]$, $n = 2m$ | $\prod_{j=0}^{m-1} \left[ r^2 + j^2 \right]$, $n = 2m + 1$ |
| $SU(n, 1)$, $n \geq 2$ | $2^{2n-1} \Gamma(n)^2 \left[ \prod_{j=1}^{n-1} \left( r^2 + \frac{(n-2j)^2}{4} \right) \right]$ | $\prod_{j=3}^{n+1} \left( r^2 + \left( n - j + \frac{3}{2} \right)^2 \right)$ $\prod_{j=0}^{4} \left( r^2 + \left( \frac{2j+1}{2} \right)^2 \right)$ |
| $SP(n, 1)$, $n \geq 2$ | $2^{4n+1} \Gamma(2n)^2 \left[ \prod_{j=3}^{n+1} \left( r^2 + \left( n - j + \frac{3}{2} \right)^2 \right) \right]$ | $\prod_{j=0}^{4} \left( r^2 + \left( \frac{2j+1}{2} \right)^2 \right)$ |
| $F_4(-20)$ | $2^{21} \Gamma(8)^2 \left[ \prod_{j=0}^{4} \left( r^2 + \left( \frac{2j+1}{2} \right)^2 \right) \right]$ | $\prod_{j=0}^{4} \left( r^2 + \left( \frac{2j+1}{2} \right)^2 \right)$ |
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