EINSTEIN-KÄHLER METRICS ON SYMMETRIC TORIC FANO MANIFOLDS

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Abstract

Let $X$ be a complex toric Fano $n$-fold and $\mathcal{N}(T)$ the normalizer of a maximal torus $T$ in the group of biholomorphic authomorphisms $Aut(X)$. We call $X$ symmetric if the trivial character is a single $\mathcal{N}(T)$-invariant algebraic character of $T$. Using an invariant $\alpha_G(X)$ introduced by Tian, we show that all symmetric toric Fano $n$-folds admit an Einstein-Kähler metric. We remark that so far one doesn’t know any example of a toric Fano $n$-fold $X$ such that $Aut(X)$ is reductive, the Futaki character of $X$ vanishes, but $X$ is not symmetric.

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1 Introduction

Let $X$ be a $n$-dimensional compact complex manifold with positive first Chern class $c_1(X)$, $g = \{g_{i\overline{j}}\}$ a Kähler metric on $X$ such that the corresponding 2-form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{i\overline{j}} dz_i \wedge d\overline{z}_j$$

represents $c_1(X)$. It is well-known that the Ricci curvature of $g$,

$$Ric(g) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} R_{i\overline{j}} dz_i \wedge d\overline{z}_j,$$

also represents $c_1(X)$. The metric $g$ is called Einstein-Kähler if $Ric(g) = \omega_g$.

Let $Aut(X)$ be the group of biholomorphic automorphisms of $X$ and $Lie(Aut(X))$ the Lie algebra of $Aut(X)$. In 1957, Matsushima proved that if $X$ admits an Einstein-Kähler metric then $Aut(X)$ is a reductive algebraic group [9]. In 1983, Futaki introduced a linear function $F_X : Lie(Aut(X)) \to \mathbb{C}$, so called Futaki character, which vanishes provided $X$ admits an Einstein-Kähler metric [7]. Futaki has conjectured that the condition $F_X = 0$ is sufficient for the existence of an Einstein-Kähler metric on $X$. Recently Tian disproved this conjecture [20]. This shows that the problem of finding a sufficient condition for the existence of an Einstein-Kähler metric is rather subtle.

In this paper we restrict ourselves to the case of compact complex manifolds $X$ with positive first Chern class which are toric (see [4, 5, 6, 14]). If $X$ is a toric Fano $n$-fold, then a maximal torus $T \cong (\mathbb{C}^*)^n \subset Aut(X)$ has an open dense orbit $U \cong T \subset X$. Denote by $M \cong \mathbb{Z}^n$ the group of algebraic characters of $T$. Then the Lie algebra $Lie(T)$ of $T$ can be identified with $N \otimes_{\mathbb{Z}} \mathbb{C}$, where $N := Hom(M, \mathbb{Z})$ the dual group. Using the anticanonical embedding $X \hookrightarrow \mathbb{P}^m$ and a Kähler metric $g$ on $X$ induced by the Fubini-Study metric on $\mathbb{P}^m$, we obtain a natural moment map

$$\mu_g : X \to M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$$

whose image is a convex polyhedron $\Delta$. The polyhedron $\Delta$ is reflexive, and $X$ can be recovered from $\Delta$ as projectivization $X = \mathbb{P}_\Delta = Proj S_\Delta$, where $S_\Delta$ is the graded semigroup $\mathbb{C}$-algebra of lattice points in the cone over $\Delta$ (see [2]). We denote by $R(\Delta)$ the set of all $M$-lattice points contained in relative interiors of codimension-1 faces of $\Delta$. It is well-known that $Aut(X)$ is reductive if and only if the set $R(\Delta)$ is centrally symmetric: $R(\Delta) = -R(\Delta)$. It has been shown by Mabuchi that if $Aut(X)$ is reductive, then the Futaki character $F_X$ vanishes if and only if the barycenter $b(\Delta) \in M_{\mathbb{R}}$ of the polyhedron $\Delta$ is zero. Using this result and the complete classification of toric Fano 3-folds due to the first author [1] and
Let $X$ be a smooth projective toric $n$-fold. Denote by $\mathcal{N}(T) \subset Aut(X)$ the normalizer of a maximal torus $T$. The group $\mathcal{N}(T)$ naturally acts on $T$ by conjugations. This induces a linear action of $\mathcal{N}(T)$ on the group of algebraic characters $M = \text{Hom}_{\text{alg}}(T, \mathbb{C}^*)$. Since $T$ acts trivially on $M$, the latter determines a linear representation of the finite group $W(X) := \mathcal{N}(T)/T$ by integral-valued $n \times n$-matrices from $GL(M) \cong GL(n, \mathbb{Z})$. We call $X$ symmetric, if the trivial character is a single $W(X)$-invariant (or, equivalently, $\mathcal{N}(T)$-invariant) algebraic character of $T$:

$$M^{W(X)} := \{ \chi \in M : \chi^g = \chi \text{ for all } g \in W(X) \} = 0.$$ 

Our main result is the following:

**Theorem 1.1** Let $X$ be a symmetric toric Fano $n$-fold. Then $X$ admits an Einstein-Kähler metric.

It follows immediately from the definition of symmetric Fano manifolds that if $X = \mathbb{P}_\Delta$ is symmetric, then the barycenter of $\Delta$ is zero. By theorem of Matsushima [4], one also gets:

**Corollary 1.2** If $X = \mathbb{P}_\Delta$ is a symmetric toric Fano $n$-fold, then

$$R(\Delta) = -R(\Delta).$$

It would be interesting to know whether there exists a direct proof of 1.2 without using [4]. We remark our theorem covers all already known examples of toric Fano $n$-folds ($n \leq 4$) whose Futaki character vanish and whose authomorphism group is reductive. It would be interesting to know whether there exists an example of a toric Fano $n$-fold $X$ such that $F_X = 0$, $Aut(X)$ is reductive, but $X$ is not symmetric. Moreover, it is still unknown whether the condition $F_X = 0$ and \{Aut($X$) is reductive\} is sufficient for the existence of an Einstein-Kähler metric on toric Fano manifolds of arbitrary dimension $n$.

The paper is organized as follows. In Section 2 we remind the definition of the invariant $\alpha_G(X)$ introduced by Tian and its connection to solutions of complex Monge-Ampère equations obtained by the continuity method. In Section 3 we give a
proof of Theorem 1.1. In Section 4 we discuss several series of examples of symmetric toric Fano manifolds which include all examples of Einstein-Kähler toric Fano manifolds of dimension \( n \leq 4 \).

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\section{Tian invariant \( \alpha_G(X) \)}

Let \( X \) be a \( n \)-dimensional compact complex manifold with positive first Chern class \( c_1(X) \) and \( G \) a compact subgroup of \( Aut(X) \). Choose a \( G \)-invariant Kähler metric \( g = \{ g_{i\bar{j}} \} \) on \( X \) such that

\[
\omega_g = \frac{-1}{2\pi} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j
\]

represents \( c_1(X) \). One has a natural \( G \)-invariant volume form \( dV_g \) on \( X \)

\[
dV_g := \frac{\omega_g^n}{n!}, \quad Vol_g(X) := \int_X dV_g = \frac{c_1^n(X)}{n!}.
\]

It is well-known that the problem of finding an Einstein-Kähler metric on \( X \) is equivalent to solving the following complex Monge-Ampère equation for smooth real-valued functions \( \varphi \) on \( X \):

\[
\det \left( g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right) = \det(g_{i\bar{j}}) e^{F-t\varphi}, \quad \forall t \in [0, 1] \tag{1}
\]

where the smooth real-valued function \( F \) is defined by the conditions:

\[
\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} = R_{i\bar{j}} - g_{i\bar{j}}, \quad \int_X e^F dV_g = Vol_g(X).
\]

If \( \varphi \) is a solution of (1) for \( t = 1 \), then

\[
g'_{i\bar{j}} := g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}
\]

is an Einstein-Kähler metric on \( X \). By famous theorem of Yau, there exists always a solution of (1) for all \( t \in [0, \varepsilon) \) if \( \varepsilon \) is sufficiently small. Using the continuity method, one can show that the existence of a solution \( \varphi \) for \( t = 1 \) is equivalent to zero-order \textit{a priori} estimates of \( \varphi \).

Let us recall the definition of an invariant \( \alpha_G(X) \) introduced by Tian [18]:

\[
\alpha_G(X) := \frac{\int_X \omega^n_g}{\int_X dV_g}.
\]
**Definition 2.1** Let $P_G(X,g)$ be the set of all $C^2$-smooth $G$-invariant real-valued functions $\phi$ such that $\sup_X \phi = 0$ and

$$\omega_g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi$$

is a nonnegative $(1,1)$-form. Then **Tian invariant** $\alpha_G(X)$ is defined as supremum of all $\lambda > 0$ such that

$$\int_X e^{-\lambda \phi} dV_g \leq C(\lambda) \quad \forall \phi \in P_G(X,g),$$

where $C(\lambda)$ is a positive constant depending only on $\lambda$, $g$ and $X$.

**Remark 2.2** It is easy to show that $\alpha(X)$ doesn’t depend on the choice of a $G$-invariant metric $g$. Moreover, $\alpha_G(X)$ doesn’t change if in the above definition we replace $P_G(X,g)$ by a smaller subset consisting of all $C^\infty$-smooth $G$-invariant real-valued functions $\phi$ such that $\sup_X \phi = 0$ and

$$\omega_g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi$$

is a positive definite $(1,1)$-form (see [19]).

Deriving a zero-order *a priori* estimate for the solutions of (1), Tian has proved the following important result ([18], Theorems 2.1 and 4.1):

**Theorem 2.3** Let $X$ be a Fano $n$-fold and $G \subset Aut(X)$ is a compact subgroup such that

$$\alpha_G(X) > \frac{n}{n+1}.$$ 

Then $X$ admits an Einstein-Kähler metric.

### 3 Main theorem

Throughout this section we use standard notations from the theory of toric varieties (see e.g. [4]). Let $M$ be a free abelian group of rank $n$, $N = Hom(M, \mathbb{Z})$ the dual group, $M_\mathbb{R} := M \otimes \mathbb{Z} \mathbb{R}$, $N_\mathbb{R} := N \otimes \mathbb{Z} \mathbb{R}$. Denote by $(*,*) : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ the canonical nondegenerate pairing. Let $X = X_\Sigma$ be a smooth projective toric $n$-fold defined by a complete fan $\Sigma$ of regular cones $\sigma \subset N_\mathbb{R}$. Then a maximal torus $T \subset Aut(X)$ acting on $X$ has an open dense orbit $U \subset X$. The normalizer $N(T) \subset Aut(X)$ of $T$ has a natural action on $U$. Let us set $W(X) := N(T)/T$. By functorial properties of toric varieties (see [4], §5), one immediately obtains:

**Proposition 3.1** Let $X = X_\Sigma$ be a smooth projective toric $n$-fold defined by a complete regular polyhedral fan $\Sigma$. Then the group $W(X)$ is isomorphic to the finite group of all symmetries of $\Sigma$, i.e., $W(X)$ is isomorphic to a subgroup of $GL(M) \cong GL(n, \mathbb{Z})$ consisting of all elements $\gamma \in GL(M)$ such that $\gamma(\Sigma) = \Sigma$. 

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Since the open subvariety \( U \subset X \) is a principal homogeneous space of \( T \), we can identify \( U \) with \( T \) by choosing an arbitrary point \( x_0 \in U \). This identification defines a splitting of the short exact sequence

\[
1 \to T \to \mathcal{N}(T) \to \mathcal{W}(X) \to 1,
\]
i.e., an embedding \( \mathcal{W}(X) \hookrightarrow \mathcal{N}(T) \subset \text{Aut}(X) \). We denote by \( \mathcal{W}(X, x_0) \) the image of \( \mathcal{W}(X) \) in \( \text{Aut}(X) \) under this embedding. Denote by \( \mathcal{K}(T) \cong (S^1)^n \) the maximal compact subgroup in \( T \). In the sequel we shall use the canonical isomorphism \( T/\mathcal{K}(T) \cong N_\mathbb{R} \) and the isomorphism \( U/\mathcal{K}(T) \cong N_\mathbb{R} \) which identifies the orbit \( \mathcal{K}(T)x_0 \) with the zero element \( 0 \in N_\mathbb{R} \). The last isomorphism shows that the \( \mathcal{N}(T) \)-action on \( U \) descends to a linear action of \( \mathcal{W}(X) \) on \( N_\mathbb{R} \). If one chooses an integral basis \( e_1, \ldots, e_n \) of \( N \) and the dual basis \( e_1^*, \ldots, e_n^* \) of \( M \), then the induced isomorphisms \( N_\mathbb{R} \cong \mathbb{R}^n \), \( M_\mathbb{R} \cong \mathbb{R}^n \) and \( T \cong (\mathbb{C}^*)^n \) allow to introduce affine logarithmic coordinates \( y_i = \log |z_i| \) (\( i = 1, \ldots, n \)) on \( N_\mathbb{R} \), where \( z_1, \ldots, z_n \) the standard holomorphic coordinate system on \( (\mathbb{C}^*)^n \). We choose \( G \) to be the maximal compact subgroup in \( \mathcal{N}(T) \) generated by \( \mathcal{W}(X, x_0) \) and \( \mathcal{K}(T) \), so that we have the short exact sequence

\[
1 \to \mathcal{K}(T) \to G \to \mathcal{W}(X) \to 1.
\]

Now we assume that a projective toric \( n \)-fold \( X \) has positive first Chern class. In this case, one obtains a convex \( \mathcal{W}(X) \)-invariant polyhedron \( \Delta \subset M_\mathbb{R} \) defined by the affine linear inequalities \( \langle y, e \rangle \leq 1 \) where \( e \) runs over all primitive integral generators \( e \) of 1-dimensional cones \( \sigma = \mathbb{R}_{\geq 0} e \in \Sigma \). Let \( L(\Delta) = \{v_0, v_1, \ldots, v_m\} := M \cap \Delta \). Then \( v_0, v_1, \ldots, v_l \) determine algebraic characters \( \chi_i : T \to \mathbb{C}^* \) of \( T \) (\( i = 0, \ldots, m \)). Moreover, we have

\[
|\chi_i(x)| = e^{\langle v_i, y \rangle}, \quad i = 0, \ldots, m,
\]
where \( y \) is the image of \( x \) under the canonical projection \( \pi : T \to N_\mathbb{R} \). Let us define the function \( u : U \to \mathbb{R} \) as follows:

\[
u := \log(\sum_{i=0}^{m} |\chi_i(x)|), \quad x \in U \approx T.
\] (2)

Since \( u \) is \( \mathcal{K}(T) \)-invariant, \( u \) descends to a function \( \tilde{u} : N_\mathbb{R} \to \mathbb{R} \) defined as

\[
\tilde{u} := \log(\sum_{i=0}^{m} e^{\langle v_i, y \rangle}), \quad y \in N_\mathbb{R}.
\] (3)

Since \( L(\Delta) \) is \( \mathcal{W}(X) \)-invariant, one obtains the following \( G \)-equivariant moment map

\[
\mu_{\tilde{u}} : N_\mathbb{R} \to M_\mathbb{R},
\]

\[
y = (y_1, \ldots, y_n) \mapsto \text{Grad} \tilde{u} := \left( \frac{\partial \tilde{u}}{\partial y_1}(y), \ldots, \frac{\partial \tilde{u}}{\partial y_n}(y) \right)
\]
which is a diffeomorphism of \( N_\mathbb{R} \) with the interior of the polyhedron \( \Delta \).
Consider the $G$-invariant hermitian metric $g = \{g_{ij}\}$ on $X$ such that the restriction of the corresponding to $g$ differential 2-form on $U$ is defined by

$$\omega_g = \frac{-1}{2\pi} \partial \bar{\partial} u.$$ 

We remark that the metric $g$ is exactly the pull-back of the Fubuni-Study metric form $\mathbb{P}^m$ with respect to the anticanonical embedding $X \hookrightarrow \mathbb{P}^m$ defined by the algebraic characters $\chi_0, \chi_1, \ldots, \chi_m$. Then the restriction of the moment $\mu_g : X \rightarrow M_R$ to $U$ is exactly the composition of the canonical projection $\pi : T \rightarrow N_R$ and $\mu_{\tilde{u}} : N_R \rightarrow M_R$. In particular, $\Delta = \mu_g(X)$.

Using the above considerations, one can derive from the complex Monge-Ampère equation (1) for a $G$-invariant function $\varphi : X \rightarrow \mathbb{R}$ the real Monge-Ampère equation

$$\det \left( \frac{\partial^2 (\tilde{u} + \tilde{\varphi})}{\partial y_i \partial y_j} \right) = \exp(-\tilde{u} - t \tilde{\varphi}), \quad \forall t \in [0, 1], \quad (4)$$

where $\tilde{\varphi}$ is a smooth $\mathcal{W}(X)$-invariant real-valued function on $N_R$ obtained as descent of $\varphi|_U$ to $N_R$.

**Proposition 3.2** Let $X$ be a toric Fano $n$-fold with $G$-action as above. Denote by $dy$ the volume $n$-form on $N_R(\cong \mathbb{R}^n)$ corresponding to the Haar measure on $N_R$ normalized by the lattice $N \subset N_R$. Let $\tilde{\alpha}_G(X)$ be the supremum of all $\lambda > 0$ such that

$$\int_{N_R} e^{-\lambda \tilde{u} - \bar{\tilde{u}}} dy \leq \tilde{C}(\lambda), \quad \forall \tilde{\varphi} \in P_G(N_R, \tilde{u}),$$

where $P_G(N_R, \tilde{u})$ is the set of all $C^2$-smooth $\mathcal{W}(X)$-invariant functions $\tilde{\varphi} : N_R \rightarrow \mathbb{R}$ such that $\tilde{u} + \tilde{\varphi}$ is upper convex, $\sup_X \tilde{\varphi} = 0$, and $|\tilde{\varphi}|$ is bounded on the whole $N_R$. Then

$$\tilde{\alpha}_G(X) \leq \alpha_G(X).$$

**Proof.** Let $\phi$ be an element of $P_G(X, g)$. Since $\phi$ is $\mathcal{K}(T)$-invariant, the restriction of $\phi$ to $U$ descends to a smooth $C^2$-function real-valued $\tilde{\phi}$ on $N_R(\cong U/\mathcal{K}(T))$. Moreover, it follows from $G$-variance of $\phi$ that $\tilde{\phi}$ is invariant under the finite group $\mathcal{W}(X)$ acting linearly on $N_R$. The nonnegativity of the $(1, 1)$-form

$$\omega_g + \frac{-1}{2\pi} \partial \bar{\partial} \phi = \frac{-1}{2\pi} \partial \bar{\partial} (\tilde{u} + \phi)$$

immediately implies that the matrix

$$\left( \frac{\partial^2 (\tilde{u} + \tilde{\phi})}{\partial y_i \partial y_j} \right)$$

is nonnegative definite, i.e., $\tilde{u} + \tilde{\phi}$ is an upper convex function on $N_R$. Let $d\theta$ be a volume $n$-form defining the canonically normalized Haar measure on the compact group $\mathcal{K}(T)$. We remark that the restriction of the volume $2n$-form $dV_g$ to
\[ U \cong T \text{ equals } he^{-u}dyd\theta, \text{ where } h \text{ is a smooth real-valued bounded function on } X. \] 

Therefore, the inequality 
\[ \int_X e^{-\lambda \phi}dV \leq C(\lambda) \quad \forall \phi \in P_G(X, g) \]

immediately follows from
\[ \int_{N_{\mathbb{R}}} e^{-\lambda \tilde{u}}dy \leq \tilde{C}(\lambda) \quad \forall \tilde{\phi} \in P_G(N_{\mathbb{R}}, \tilde{u}). \]

Thus, we have \( \tilde{\alpha}_G(X) \leq \alpha_G(X). \)

Proposition 3.3 Let \( X = \mathbb{P}_{\Delta} \) be a toric Fano \( n \)-fold and \( \tilde{u} \) the function defined by (3). Choose \( \tau \) to be an arbitrary positive real number. Then

\[ \int_{N_{\mathbb{R}}} e^{-\tau \tilde{u}}dy \leq \frac{\nu(\Delta)}{\tau^n}, \]

where \( \nu(\Delta) \) is the number of vertices of \( \Delta \).

Proof. Let \( \nu(\Delta) = l \). Denote by \( w_1, \ldots, w_l \) all vertices of \( \Delta \). It follows from the formula (3) that for all \( y \in N_{\mathbb{R}} \) we have
\[ \tilde{u}(y) > \langle w_j, y \rangle, \quad j = 1, \ldots, l, \]

and hence \( \tilde{u}(y) > \overline{u}(y) \), where \( \overline{u} := \max_{j=1,\ldots,l} \langle w_j, y \rangle \). Therefore, we obtain
\[ \int_{N_{\mathbb{R}}} e^{-\tau \tilde{u}}dy \leq \int_{N_{\mathbb{R}}} e^{-\tau \overline{u}}dy. \]

It follows from definition of \( \Delta \) that \( l \) is exactly the number of \( n \)-dimensional cones \( \sigma_1, \ldots, \sigma_l \) in the fan \( \Sigma \) defining \( X \). Moreover, \( \overline{u} \) is a continuous piecewise linear function whose restriction to \( \sigma_j \) equals \( \langle w_j, y \rangle \). On the other hand,
\[ \int_{\sigma_j} e^{-\tau \overline{u}}dy = \int_{\mathbb{R}_{\geq 0}^n} e^{-\tau (y_1 + \cdots + y_n)}dy_1 \cdots dy_n = \prod_{i=1}^n \left( \int_{\mathbb{R}_{\geq 0}} e^{-\tau y_i}dy_i \right) = \frac{1}{\tau^n}, \]

since every \( n \)-dimensional cone \( \sigma_j \in \Sigma \) (\( j = 1, \ldots, l \)) is generated by a basis of the lattice \( N \). Using \( N_{\mathbb{R}} = \sigma_1 \cup \cdots \cup \sigma_l \) together with (3) and (5), we come to the required inequality. \( \Box \)

The next statement plays the crucial role in the proof of Theorem 1.1:

Theorem 3.4 Let \( X = \mathbb{P}_{\Delta} \) be a symmetric toric Fano \( n \)-fold and \( \tilde{\phi} \) is an arbitrary function from \( P_G(N_{\mathbb{R}}, \tilde{u}) \). Then
\[ \tilde{u}(y) + \tilde{\phi}(y) \geq 0 \quad \forall y \in N_{\mathbb{R}}. \]
Proof. Let \( \tilde{\phi} \) be an arbitrary function from \( P_G(N_R, \check{u}) \). Consider the following moment map:

\[
\mu_{\check{u} + \tilde{\phi}} : N_R \to M_R, \\
y = (y_1, \ldots, y_n) \mapsto \text{Grad} (\check{u} + \tilde{\phi})(y) := \left( \frac{\partial (\check{u} + \tilde{\phi})}{\partial y_1}(y), \ldots, \frac{\partial (\check{u} + \tilde{\phi})}{\partial y_n}(y) \right).
\]

First of all we show that \( \mu_{\check{u} + \tilde{\phi}}(N_R) \subset \Delta \). Let \( z = \mu_{\check{u} + \tilde{\phi}}(y') \) for some \( y' \in N_R \). It follows from the convexity of \( \check{u} + \tilde{\phi} \) that for all \( y \in N_R \) one has

\[
\check{u}(y) + \tilde{\phi}(y) \geq \langle z, y - y' \rangle + \check{u}(y') + \tilde{\phi}(y').
\]

In other words, the function \( \check{u}(y) + \tilde{\phi}(y) - \langle z, y \rangle \) attains the global minimum at \( y' \in N_R \). Let \( \{w_1, \ldots, w_l\} \) be the set of all vertices of \( \Delta \) and \( \overline{\mu} := \max_{j=1,\ldots,l} \langle w_j, y \rangle \) the piecewise linear function as in the proof of [3.3]. Using obvious inequalities

\[
\log l + \overline{\mu} \geq \check{u} \geq \overline{\mu}
\]

and the fact that \( \tilde{\phi} \) is globally bounded on \( N_R \), we conclude that the piecewise linear function \( \overline{\mu}(y) - \langle z, y \rangle \) is bounded from below on the whole \( N_R \). The latter is possible only if \( \overline{\mu}(y) - \langle z, y \rangle \geq 0 \) for all \( y \in N_R \). Since \( \overline{\mu}(e) = 1 \) for all primitive integral generators of 1-dimensional cones \( \sigma \in \Sigma \), we obtain that for all these generators holds \( \langle z, e \rangle \leq 1 \), i.e., \( z \in \Delta \).

Since \( \sup_{N_R} \tilde{\phi} = 0 \), there exists a sequence \( \{q_k\}_{k \geq 1} \) of points \( q_k \in N_R \) such \( -1/k \leq \tilde{\phi}(q_k) \leq 0 \). Denote \( z_k = \mu_{\check{u} + \tilde{\phi}}(q_k) \). Since all \( z_k \) belong to \( \Delta \), we can assume without loss of generality that

\[
\lim_{k \to \infty} z_k = z \in \Delta
\]

(otherwise one chooses an appropriate subsequence of \( \{q_k\}_{k \geq 1} \)). It follows from the convexity of \( \check{u} + \tilde{\phi} \) that for all \( y \in N_R \) and all \( k \geq 1 \) one has

\[
\check{u}(y) + \tilde{\phi}(y) - \langle z_k, y \rangle \geq \check{u}(q_k) + \tilde{\phi}(q_k) - \langle z_k, q_k \rangle.
\]

Now we remark that \( \check{u}(q_k) \geq \overline{\mu}(q_k) \geq \langle z_k, q_k \rangle \) for all \( k \geq 1 \), because \( z_k \) is contained in \( \Delta \). Therefore, we have

\[
\check{u}(y) + \tilde{\phi}(y) - \langle z_k, y \rangle \geq -1/k, \quad \forall y \in N_R.
\]

Taking limit \( k \to \infty \), we obtain

\[
\check{u}(y) + \tilde{\phi}(y) - \langle z, y \rangle \geq 0, \quad \forall y \in N_R.
\]  

(8)

We set \( r := |\mathcal{W}(X)| \) and consider the points \( z, \gamma_1 z, \ldots, \gamma_{r-1} z \), where \( \{\gamma_1, \ldots, \gamma_{r-1}\} \) the set of all elements of \( \mathcal{W}(X) \) which are different from the identity. Since \( \check{u} + \tilde{\phi} \) is \( \mathcal{W}(X) \)-invariant, we obtain from (8) \( r - 1 \) additional inequalities:

\[
\check{u}(y) + \tilde{\phi}(y) - \langle \gamma_j z, y \rangle \geq 0, \quad \forall y \in N_R, \quad j = 1, \ldots, r - 1.
\]  

(9)
Now we remark that
\[ z' := z + \sum_{j=1}^{r-1} \gamma_j z \]
is obviously $\mathcal{W}(X)$-invariant. Using the fact that $X$ is a symmetric Fano $n$-fold, we conclude that $z' = 0$. Summing the inequalities in (8) and (9), we obtain
\[ \tilde{u}(y) + \tilde{\phi}(y) \geq 0 \quad \forall y \in N_R. \]

\[ \blacksquare \]

**Proof of Theorem 1.1.** Choose arbitrary $\lambda \in (0, 1)$ and $\tilde{\phi} \in P_G(N_R, \tilde{u})$. Using 3.4 and 3.3, we obtain
\[
\int_{N_R} e^{-\lambda \tilde{\phi} - \tilde{u}} dy = \int_{N_R} e^{-\lambda (\tilde{\phi} + \tilde{u})} e^{(\lambda - 1)\tilde{u}} dy \leq \sup_{N_R} \left\{ e^{-\lambda (\tilde{\phi} + \tilde{u})} \right\} \int_{N_R} e^{(\lambda - 1)\tilde{u}} dy \leq \frac{v(\Delta)}{(1 - \lambda)^n}.\]

Therefore, $\tilde{\alpha}_G \geq 1$. By 3.2 and 2.3, we conclude that $X$ admits an Einstein-Kähler metric. \[ \blacksquare \]

### 4 Some examples

In this section we consider series of examples of symmetric toric Fano $n$-folds which include many already known examples of toric Einstein-Kähler manifolds.

**Example 4.1** Let $V_k$ smooth projective toric Fano $n$-fold ($n = 2k$) defined by a fan $\Sigma$ of regular polyhedral cones whose generators are $\pm e_1, \ldots, \pm e_n, \pm (e_1 + \cdots + e_n)$, where $e_1, \ldots, e_n$ is an integral basis of the lattice $N$. The toric Fano $n$-fold $V_k$ has been introduced by Voskresensky and Klyachko [22]. Since the corresponding polyhedron $\Delta = \Delta(V_k)$ is centrally symmetric, $V_k$ is a symmetric toric Fano $n$-fold (see 3.1). We remark that $V_1$ is $\mathbb{P}^2$ with 3 points blown-up. The existence of an Einstein-Kähler metric on $V_1$ was proved by Siu [17], Tian-Yau [21], and Nadel [10]. The existence of an Einstein-Kähler metric on the 4-fold $V_2$ was proved by Nakagawa in [11] using results of Nadel [10].

**Example 4.2** Let $k, m$ be integers satisfying the condition $1 \leq k \leq m$. Denote by $S_{m,k}$ toric Fano $n$-fold ($n = 2m + 1$) which is the projectivization $\mathbb{P}(E)$ of the split bundle $E = \mathcal{O} \oplus \mathcal{O}(k, -k)$ over $\mathbb{P}^m \times \mathbb{P}^m$. This toric manifold is defined by a fan $\Sigma$ whose cones have the following $2m + 4$ generators:
\[ e_1, \ldots, e_{2m}, \pm e_{2m+1}, -(e_1 + e_2 + \cdots + e_m + ke_{2m+1}), \]
\[ -(e_{m+1} + e_{m+2} + \cdots + e_{2m} - ke_{2m+1}), \]
where \( e_1, \ldots, e_{2m+1} \) is an integral basis of \( N \). There exist an authomorphisms \( \alpha \) of \( \Sigma \) of order \( m + 1 \) such that

\[
\alpha(e_{2m+1}) = e_{2m+1}, \quad \alpha(e_i) = e_{i+1}, \quad \alpha(e_{i+m}) = e_{i+m+1}, \quad i = 1, \ldots, m - 1;
\]

\[
\alpha(e_m) = -(e_1 + \ldots + e_m + ke_{2m+1}), \quad \alpha'(e_{2m}) = -(e_{m+1} + \ldots + e_{2m} - k e_{2m+1}).
\]

There exists an authomorphism \( \beta \) of order 2 defined by

\[
\beta(e_{2m+1}) = -e_{2m+1}, \quad \beta(e_i) = e_{i+m}, \quad \beta(e_{i+m}) = e_i, \quad i = 1, \ldots, m.
\]

The common fix point set of \( \alpha \) and \( \beta \) is exactly \( 0 \in \mathbb{N}_\mathbb{R} \). By [3,1], \( S_{m,k} \) is a symmetric toric Fano \( n \)-fold.

The Einstein-Kähler manifold \( S_{m,k} \) was discovered by Sakane [15]. The existence of an Einstein-Kähler metric on \( S_{m,k} \) was obtained by Mabuchi using another method (see (10.3.2) in [3]). We remark that \( S_{m,1} \) is isomorphic to \( \mathbb{P}^{2m+1} \) blown-up at two skew \( m \)-dimensional subspaces. The existence of an Einstein-Kähler metric on \( S_{m,1} \) was proved independently by Nadel ([10], Example 6.4).

**Example 4.3** Choose integers \( k, m \) such that \( 0 \leq k \leq m \). In [12], Nakagawa introduced a toric Fano \( n \)-fold \( X_{m,k} \) \( (n = 2m + 2) \) defined by a fan \( \Sigma \) whose \( 2m + 8 \) generators are

\[
e_1, \ldots, e_{2m}, \pm e_{2m+1}, \pm e_{2m+2}, \pm(e_{2m+1} + e_{2m+2}),
\]

\[
-(e_1 + \ldots + e_m - k e_{2m+1}), \quad -(e_{m+1} + \ldots + e_{2m} + k e_{2m+1}).
\]

There exist an authomorphism \( \alpha \) of \( \Sigma \) of order \( m + 1 \) such that

\[
\alpha(e_{2m+1}) = e_{2m+1}, \quad \alpha(e_{2m+2}) = e_{2m+2},
\]

\[
\alpha(e_i) = e_{i+1}, \quad \alpha(e_{i+m}) = e_{i+m+1}, \quad i = 1, \ldots, m - 1;
\]

\[
\alpha(e_m) = -(e_1 + \ldots + e_m - k e_{2m+1}), \quad \alpha(e_{2m}) = -(e_{m+1} + \ldots + e_{2m} + k e_{2m+1}).
\]

On the other hand, there exists an authomorphism \( \beta \) of \( \Sigma \) of order 2 defined by

\[
\beta(e_i) = e_{i+m}, \quad \beta(e_{i+m}) = e_i, \quad i = 1, \ldots, m;
\]

\[
\beta(e_{2m+1}) = -e_{2m+1}, \quad \beta(e_{2m+2}) = -e_{2m+2}.
\]

The common fix point set of \( \alpha \) and \( \beta \) is exactly \( 0 \in \mathbb{N}_\mathbb{R} \). By [3,1], \( X_{m,k} \) is a symmetric toric Fano \( n \)-fold. The existence of an Einstein-Kähler metric on \( X_{m,k} \) was proved by Nakagawa in [11] using results of Nadel [12].
Example 4.4 Let $W_m$ be $\mathbb{P}^m \times \mathbb{P}^m$ blown-up along $m+1$ codimension-2 subvarieties $Z_i \cong \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$ defined by the equations $z_i = 0$, $z_i' = 0$ ($i = 0, 1, \ldots, m$), where $(z_0 : z_1 : \cdots : z_m)$ and $(z_0' : z_1' : \cdots : z_m')$ are homogeneous coordinates on two $\mathbb{P}^m$'s. The toric manifold $W_m$ is determined by a $2m$-dimensional fan $\Sigma \subset N_\mathbb{R}$ whose cones have the following $3m+3$ generators

$$e_1, \ldots, e_{2m}, -(e_1 + \ldots + e_m), -(e_{m+1} + \ldots + e_{2m}), -(e_1 + \ldots + e_{2m}),$$

$$e_i + e_{i+m}, \ i = 1, \ldots, m,$$

where $e_1, \ldots, e_{2m}$ is an integral basis of $N$. There exists an automorphism $\alpha$ of $\Sigma$ of order $m + 1$ such that

$$\alpha(e_i) = e_{i+1}, \ \alpha(e_{i+m}) = e_{i+m+1}, \ i = 1, \ldots, m - 1,$$

$$\alpha(e_m) = -(e_1 + \ldots + e_m), \ \alpha(e_{2m}) = -(e_{m+1} + \ldots + e_{2m}).$$

On the other hand, there exists an automorphism $\beta$ of $\Sigma$ of order 2 defined by

$$\beta(e_i) = e_{i+m}, \ \beta(e_{i+m}) = e_i, \ i = 1, \ldots, m.$$ 

The common fix point set of $\alpha$ and $\beta$ is exactly $0 \in N_\mathbb{R}$. By [1], $W_m$ is a symmetric toric Fano $n$-fold ($n = 2m$).

We remark that $W_1 = V_1$ is again $\mathbb{P}^2$ with 3 points blown-up. The toric Fano 4-fold $W_2$ is exactly the single one missed in the table [3]. In particular, we come to conclusion that there exist exactly 12 different Einstein-Kähler toric Fano 4-folds (cf. [12, 13] and [16], Example 4.7).

We remark that any Einstein-Kähler toric Fano manifold $X$ of dimension $n \leq 4$ which can not be decomposed into a product of lower dimensional varieties is either a projective space, or one of the toric Fano manifolds from 4.1-4.4.

References

[1] V. Batyrev, Toroidal Fano 3-folds, Math. USSR, Izv. 19 (1982), 13-25.

[2] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Alg. Geom. 3, No.3 (1994), 493-535.

[3] V. Batyrev, On the classification of toric Fano 4-folds, [math.AG/9801107].

[4] V. I. Danilov, Geometry of toric varieties, Russ. Math. Surv. 33, No.2 (1978), 97-154.

[5] G. Ewald, Combinatorial convexity and algebraic geometry, Graduate Texts in Mathematics, 168, New York, Springer (1996).

[6] W. Fulton, Introduction to Toric Varieties, Ann. of Math. Studies 131, Princeton Univ. Press, 1993.
[7] A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. 73 (1983), 437-443.

[8] T. Mabuchi, *Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties*, Osaka J. Math. 24 (1987), 705-737.

[9] Y. Matsushima, *Sur la structure du groupe d’homéomorphismes analytiques d’une certaine variété kaehlerienne*, Nagoya Math. J. 11 (1957), 145-150.

[10] A. Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. Math., II. Ser. 132, No.3 (1990), 549-596.

[11] Y. Nakagawa, *Einstein-Kähler toric Fano fourfolds*, Tohoku Math. J., II. Ser. 45, No.2 (1993), 297-310.

[12] Y. Nakagawa, *Classification of Einstein-Kähler toric Fano fourfolds*, Tohoku Math. J., II. Ser. 46, No.1 (1994), 125-133.

[13] Y. Nakagawa, *Combinatorial Formulae for Futaki Characters and Generalized Killing Forms on Toric Fano Orbifolds*, Preprint 1997.

[14] T. Oda, *Convex Bodies and Algebraic Geometry - An introduction to the theory of toric varieties*, Ergebnisse Math. Grenzgeb. (3), Vol. 15, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988.

[15] Y. Sakane, *Examples of compact Einstein Kähler manifolds with positive Ricci tensor*, Osaka J. Math. 23 (1986), 585-616.

[16] H. Sato, *Toward the classification of higher-dimensional toric Fano varieties*, Preprint, November 29, 1998.

[17] Y.-T. Siu, *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group*, Ann. Math., II. Ser. 127, No.3 (1988), 585-627.

[18] G. Tian, *On Kähler-Einstein metrics on certain Kaehler manifolds with $c_1(M) > 0$*, Invent. Math. 89 (1987), 225-246.

[19] G. Tian, *Kähler-Einstein metrics on algebraic manifolds*, Proc. Int. Congr. Math., Kyoto/Japan 1990, Vol. I (1991), 587-598.

[20] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. 130, No.1 (1997), 1-37.

[21] G. Tian, S.-T. Yau, *Kähler-Einstein metrics on complex surfaces with $C_1 > 0$*, Commun. Math. Phys. 112 (1987), 175-203.

[22] V. E. Voskresenskij, A.A. Klyachko, *Toroidal Fano varieties and root systems*, Math. USSR, Izv. 24 (1985), 221-244.
[23] K. Watanabe, M. Watanabe, *The classification of Fano 3-folds with torus embeddings*, Tokyo J. Math. 5 (1982), 37-48.