Anti-Parity-Time Symmetry in a Single Damping Mechanical Resonator

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(Dated: August 18, 2022)

Abstract

A damping mechanical resonator undergoes a phase transition from an oscillatory motion with damping amplitude (under-damping) to a monotonically damping motion without oscillation (over-damping) across a critical-damping state. However, what kind of symmetry is broken for this phase transition in the damping mechanical resonator is still unclear. Here we discover a hidden symmetry, i.e., anti-parity-time (anti-PT) symmetry, in the effective Hamiltonian of a damping mechanical resonator. We show that the broken of anti-PT symmetry with an exceptional point (EP) yields the phase transition between different damping behaviors, i.e., the over-damping and under-damping across a critical-damping. We propose that the mechanical anti-PT phase transition can be induced by the optical spring effect in a quadratic optomechanical system with a strong driving field, and highly sensitive optomechanical sensing can be realized around the EPs for anti-PT phase transition.

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I. INTRODUCTION

Damping oscillation is one of the most fundamental and important physical processes [1], for such behavior appears in almost all kinds of systems, such as electronic, atomic, mechanical (acoustic) and optical (photonic) resonators. It is well known that a damping resonator undergoes a phase transition from the oscillatory behavior with a damping amplitude (under-damping) to a monotonically damping without oscillation (over-damping) across a critical-damping state with the damping rate as much as twice the frequency of the resonator. But, there is no discussion on whether there is a kind of symmetry that is broken for this phase transition in the damping resonator. According to the quantum theory in open systems [2], the damping effect can be described simply with a damping rate by eliminating the reservoir coupling to the resonator adiabatically; thus the symmetry in the damping oscillator should be discussed based on a non-Hermitian Hamiltonian with damping rate included.

Non-Hermitian physics has attracted intense interest in the past decades due to the entirely real spectrum in systems with Parity-time (PT) symmetry [3–7] and the phase transition with an exceptional point (EP) in the parameter space [8]. With a significant progress in experiment, PT symmetry has been demonstrated in photonic [8–12], acoustic [13–16], electronic [17–19], magnonic [20], atomic [21, 22], and single-spin [23] systems. Non-Hermitian systems showcase a variety of features that may not be available in Hermitian counterparts, such as directional invisibility [24–27], giant enhancement of mechanical gain [28, 29], parameter sensing [30–32], single-mode lasing [33], loss-induced suppression and revival of lasing [34], coherent perfect absorption [35], and robust wireless power transfer [36, 37]. However, the requirement of balanced gain and loss has hindered the possibilities of exhibiting PT symmetry in a single damping resonator.

Anti-PT symmetry is another non-Hermitian symmetry [38] has been a subject of considerable recent theoretical and experimental interest for it can be used to achieve exotic functionalities in pure dissipative systems, such as unidirectional reflectionless [39], nanoparticle sensing [40], topological energy transfer [41]. A tremendous effort has also been witnessed in achieving anti-PT symmetry, which has been demonstrated by using dissipative coupled systems, including atomic systems [42–44], electrical circuits [45], thermal materials [46], optical devices [47–51], mechanical [52] and magnonic systems [53]. Theoretically, anti-PT
symmetry can also be realized in a single mode system containing parametric (nonlinear) driving [54, 55]. Nevertheless, the possibility of achieving anti-$PT$ symmetry in a single damping linear resonator has not been reported yet.

Here, taking mechanical system as a simple example, we find that the effective Hamiltonian of a damping mechanical resonator without any nonlinearity is anti-$PT$ symmetric, and the broken of anti-$PT$ symmetry induces the phase transition from the over-damping to under-damping states, and the EP corresponds to the critical-damping state. This result is general, which can be applied to any resonant systems. We propose that anti-$PT$ phase transition can be observed in a single damping mechanical resonator by optical string effect in a optomechanical system, which provides us a ideal platform to modify the the spring constant as well as the resonant frequency of a damping mechanical resonator through radiation pressure or gradient force [56]. Moreover, we show that anti-$PT$ symmetry in a damping mechanical resonator can be used for ultra-sensitive sensing for the frequency splitting around the EP is highly sensitive to the external perturbation, such as the change of mass induced by small nano-particles. Our work provides an alternative method to achieve ultra-sensitive sensing based on the EPs for the anti-$PT$ phase transition in a single damping mechanical resonator.

II. MECHANICAL ANTI-$PT$ SYMMETRY

We consider a mechanical resonator with mass $m$ and spring constant $k$, which can be described by a Hamiltonian $H_m = p^2/(2m) + kx^2/2$, with the displacement $x$ and momentum $p$. The resonant frequency of the mechanical resonator is $\omega_m = \sqrt{k/m}$, and $H_m$ can be rewritten as $H_m = \hbar \omega_m (Q^2 + P^2)/2$ with the dimensionless displacement $Q = x/\sqrt{\hbar/(m\omega_m)}$ and momentum $P = p/\sqrt{\hbar m \omega_m}$. The equations of motion for the mean values of dimensionless displacement $\langle Q \rangle$ and momentum $\langle P \rangle$ are given by $d\langle Q \rangle/dt = \omega_m \langle P \rangle$ and $d\langle P \rangle/dt = -\omega_m \langle Q \rangle - \gamma_m \langle P \rangle$, where $\gamma_m$ is damping rate induced by the coupling to the thermal reservoir [57]. It is well known that the damping mechanical resonator exhibits under-damping, critical-damping, and over-damping states, corresponding to $\omega_m > \gamma_m/2$, $\omega_m = \gamma_m/2$, and $\omega_m < \gamma_m/2$. The dynamic behaviors for these three different states are shown in Figs. 1(a)-1(c), respectively.

Here, we find that the effective Hamiltonian of a damping mechanical resonator is anti-
FIG. 1: (Color online) Three states of a damping mechanical resonator: (a) under-damping \((\gamma_m/\omega_m = 1/2)\), (b) critical-damping \((\gamma_m/\omega_m = 2)\), and (c) over-damping \((\gamma_m/\omega_m = 4)\). Eigenvalues of the effective Hamiltonian \(H_{\text{eff}}\) with exceptional point (EP) at \(\gamma_m/\omega_m = 2\): (d) real parts and (e) imaginary parts. (f) The correspondence between the dynamic states and the sorts of symmetry for a damping mechanical resonator.

\(\mathcal{PT}\) symmetric, and the over-, under- and critical-damping states are associated with the phases of anti-\(\mathcal{PT}\) symmetry, anti-\(\mathcal{PT}\) broken, and EP, respectively. To reveal the anti-\(\mathcal{PT}\) symmetry in a damping mechanical resonator, we rewrite the mechanical displacement \(Q\) and momentum \(P\) as \(Q = (b^\dagger + b)/\sqrt{2}\) and \(P = i (b^\dagger - b)/\sqrt{2}\), then the equations of motion are rewritten as \(id(\langle b\rangle, \langle b^\dagger\rangle)^T/dt = H_{\text{eff}}(\langle b\rangle, \langle b^\dagger\rangle)^T\), with the effective Hamiltonian

\[
\frac{H_{\text{eff}}}{\hbar} = \begin{pmatrix}
\omega_m - i \frac{\gamma_m}{2} & i \frac{\gamma_m}{2} \\
-i \frac{\gamma_m}{2} & -\omega_m - i \frac{\gamma_m}{2}
\end{pmatrix}.
\]  

(1)

It is easy to verify that this Hamiltonian Eq. (1) is anti-\(\mathcal{PT}\)-symmetric, i.e., \((\mathcal{PT})H_{\text{eff}}(\mathcal{PT})^{-1} = -H_{\text{eff}}\), with the parity operation \(P \equiv \sigma_x\) for switching \(b \leftrightarrow b^\dagger\), and the time-reversal operation \(\mathcal{T}\) for complex conjugation. The eigenvalues of the effective
Hamiltonian $H_{\text{eff}}/\hbar$ are given by

$$\lambda_\pm = -i\frac{\gamma_m}{2} \pm i\sqrt{\left(\frac{\gamma_m}{2}\right)^2 - \omega_m^2},$$

(2)
corresponding to eigenstate $\Psi_\pm = (\beta_\pm, \beta'_\pm)^T$, with coefficients

$$\frac{\beta_\pm}{\beta'_\pm} = -i\frac{2\omega_m}{\gamma_m} \pm \sqrt{1 - \left(\frac{2\omega_m}{\gamma_m}\right)^2}.$$  

(3)

As shown in Figs. 1(d) and 1(e), there are two different phases for the damping mechanical resonator with phase transition occurring at $\gamma_m/\omega_m = 2$. In the anti-$\mathcal{PT}$-symmetric phase with $\gamma_m/\omega_m > 2$, $\Psi_\pm$ is also the eigenstates of the parity-time operator $\mathcal{PT}$, corresponding to the well known over-damping state for $\omega_m < \gamma_m/2$. In the anti-$\mathcal{PT}$ broken phase with $\gamma_m/\omega_m < 2$, $\Psi_\pm$ is no longer the eigenstates of the parity-time operator $\mathcal{PT}$, corresponding to the well known under-damping state for $\omega_m > \gamma_m/2$. Moreover, the critical point $\gamma_m/\omega_m = 2$ for anti-$\mathcal{PT}$ phase transition, i.e., EP, corresponds to the well known critical-damping state for $\omega_m = \gamma_m/2$. The correspondence between the dynamic states and the sorts of symmetry for a damping mechanical resonator is demonstrated in Fig. 1(f). As the symmetry of the damping resonator depends on the rate of $\gamma_m/\omega_m$, anti-$\mathcal{PT}$ phase transition can be realized by adjusting the decay rate $\gamma_m$ or the resonant frequency $\omega_m$ of the damping mechanical resonator.

III. OPTOMECHANICAL INDUCED ANTI-$\mathcal{PT}$ PHASE TRANSITION

Optomechanical system, that a mechanical resonator coupling to a optical mode through radiation pressure or gradient force [56], provides us a ideal platform to observe anti-$\mathcal{PT}$ phase transition via regulating the resonant frequency $\omega_m$ of a mechanical resonator based on optical spring effect. Specifically, we consider a mechanical resonator quadratically coupled to an optical mode ($A$ and $A^\dagger$, with frequency $\omega_c$) through optomechanical interaction, and the optical mode is driven resonantly by an external field with the strength $\Omega$ and frequency $\omega_L = \omega_c$. In the rotating reference frame with frequency $\omega_L$ of the external optical driving field, the system can be described by a Hamiltonian

$$H_{\text{OM}} = \frac{p^2}{2m} + \frac{1}{2}(k + 4m\omega_m gA^\dagger A)x^2 + \hbar\Omega (A^\dagger + A),$$

(4)
where \(4m\omega_mA^\dagger A\) is the spring effect induced by the optical mode with the quadratic optomechanical coupling strength \(g\).

The quadratic optomechanical interaction has been demonstrated in various cavity-optomechanical systems, including mechanical resonator (membrane [58–60], nanosphere [61–63], cold atoms [64]) trapped in Fabry-Perot cavities, or coupled to whispering-gallery-mode [65–67] or photonic crystal cavities [68, 69]. Without loss of generality, in calculations, we will take the experimental parameters in a planar silicon photonic crystal optomechanical cavity [69] with quadratic optomechanical coupling strength \(g/2\pi = 245\) Hz, mechanical resonance frequency \(\omega_m/2\pi = 8.7\) MHz (quality factor \(Q_m = 10^4\)), and optical damping rates \(\gamma_c/2\pi = 5\) GHz, i.e., the system works in the sideband unresolved regime with \(\gamma_c \gg \omega_m\).

Radiation pressure-induced buckling transitions have been reported in the optomechanical system with a dielectric membrane in the middle of a symmetrical optical cavity [70]. These transitions can be understood from the effective potential energy of the mechanical mode which changes smoothly from a single-well to a multi-well potential with the increasing of the optical driving power. The mean values in the steady states (i.e., evolution time \(t \gg 1/\gamma_c\)) \(\langle A \rangle = \alpha_s\), \(\langle P \rangle = P_s\), and \(\langle Q \rangle = Q_s\) can be obtained analytically from the dynamic equations (See Ref. [71] for more details). Two different results appear when the coupling \(g\) is positive \(g > 0\) or negative \(g < 0\). If \(g > 0\), we always have \(\alpha_s = -i2\Omega/\gamma_c\) and \(Q_s = 0\). If \(g\) is negative \(g < 0\), then we have \(\alpha_s = -i2\Omega/\left(\gamma_c + i4gQ_s^2\right)\) and

\[
\begin{align*}
Q_s^2 &= \begin{cases} 
0 & \Omega \leq \Omega_c, \\
\frac{1}{g\omega_m} \left(\Omega^2 - \Omega_c^2\right) & \Omega > \Omega_c,
\end{cases} 
\end{align*}
\]

with the appearance of spontaneous symmetry broken (SPB) phase transition at the critical driving strength \(\Omega_c = \sqrt{-\gamma_c^2\omega_m/(16g)}\), as shown Fig. 2(a). The effective potential energy of the mechanical mode can be obtained analytically by eliminated the optical freedom adiabatically under the conditions that \(\omega_m \ll \gamma_c\), as \(U_{\text{eff}} = \frac{1}{2}\omega_m \langle Q \rangle^2 + \frac{2\Omega^2}{\gamma_c} \arctan \left(\frac{4g\langle Q \rangle^2}{\gamma_c^2}\right)\). Here the second term describes the optical spring effect induced by the quadratic optomechanical interaction, which is negative for negative \(g\). Then under the critical driving for \(0 < \Omega \leq \Omega_c\), the effective spring constant \(k_{\text{eff}} = k(1 - \Omega^2/\Omega_c^2)\) monotonously decreases (i.e., the spring softens) with the driving strength \(\Omega\). When the driving strength \(\Omega\) exceeds the critical point (CP) \(\Omega_c\), the effective potential energy of the mechanical mode becomes a double well,
i.e., the parity symmetry of the ground states of the mechanical resonator is broken when $\Omega > \Omega_c$. Apart from SPB phase transition, here we propose the observation of mechanical anti-$\mathcal{PT}$ phase transitions in both the regimes of under and over the critical driving $\Omega_c$.

Under the critical driving for $0 < \Omega \leq \Omega_c$, as the effective frequency $\omega_{\text{eff}} = \sqrt{k_{\text{eff}}/m}$ with the effective spring constant $k_{\text{eff}} = k(1 - \Omega^2/\Omega_c^2)$, the first EP (EP1) appears around the point $\omega_{\text{eff}} = \gamma_m/2$ with the optical driving strength

$$
\Omega_{\text{EP1}}^2 = \Omega_c^2 \left[ 1 - \left( \frac{\gamma_m}{2\omega_m} \right)^2 \right].
$$

EP1 corresponds to the transition from anti-$\mathcal{PT}$-broken to anti-$\mathcal{PT}$-symmetric phase. EP1 can also be obtained from the eigenvalues of the effective Hamiltonian Eq. (2) with $\omega_m$ replaced by $\omega_{\text{eff}}$, as

$$
\lambda_{\pm} = -i\frac{\gamma_m}{2} \pm i \sqrt{\left( \frac{\gamma_m}{2} \right)^2 - \omega_m^2 \left( 1 - \frac{\Omega^2}{\Omega_c^2} \right)}.
$$

The eigenvalues are shown in Figs. 2(b) and 2(c), which is consistent well with the analytical results.

FIG. 2: (Color online) (a) Mean value of the mechanical displacement operator in the steady states $Q_s$ versus the driving strength $\Omega$. Eigenvalues of the effective mechanical Hamiltonian $\lambda_{m,\pm}$ versus the driving strength $\Omega$: (b) real parts and (c) imaginary parts.
Over the critical driving for $\Omega > \Omega_c$, the position of $Q_s = 0$ become unstable [see the dashed curves in Fig. 2(c)], and two new stable positions $Q_s = \pm [(\Omega_c^2 - \Omega^2)/(g\omega_m)]^{1/4}$ appear. Around the new stable positions, we have the effective spring constant $k_{\text{eff}} = 4k(1 - \Omega^2_c/\Omega^2)$, and the eigenvalues of the effective Hamiltonian

$$\lambda_\pm = -i\gamma_m^2 \pm i\sqrt{\left(\frac{\gamma_m^2}{2}\right)^2 - 4\omega_m^2 \left(1 - \frac{\Omega_c^2}{\Omega^2}\right)}. \tag{8}$$

So the driving strength for the second EP (EP2) in the SPB regime can be given by

$$\Omega_{\text{EP2}}^2 = \Omega_c^2 \left[1 - \left(\frac{\gamma_m^2}{2\omega_m}\right)^2\right]^{-1}. \tag{9}$$

Different from the EP1, EP2 is corresponding to the transition from anti-$\mathcal{PT}$-symmetric to anti-$\mathcal{PT}$-broken phase as the driving power increases.

The frequency bifurcates around the EPs in the mechanical anti-$\mathcal{PT}$ system provides us a sensitive way to detect the small variations of the parameters, such as the frequency shift of the mechanical resonator $\omega_m$ caused by external perturbation, which can be used for ultra-sensitive sensing, such as the mass sensing of small nano-particles.

**IV. OPTOMECHANICAL ANTI-$\mathcal{PT}$ SENSOR**

High sensitivity is a long-term pursue goal due to the vital importance in both fundamental and applied physics. We noted that ultrasensitive sensing based on the EP have been studied both theoretically [40, 72–74] and experimentally [31, 49]. However, the proposal of ultrasensitive nanoparticle sensing based on the EPs in a single damping mechanical resonator has not reported yet. Without loss of generality, we consider ultrasensitive sensing on the frequency perturbation of the mechanical resonator $\omega_m' = \omega_m + \delta$, which is one of the most important parameters for optical sensing.

The sensitivity of the frequency splitting $\omega_m,\pm$ on the mechanical resonator $\omega_m$ can be described by the derivative of $\omega_m,\pm$ with respect to $\omega_m$ as

$$\frac{d\omega_m,\pm}{d\omega_m} = \begin{cases} \pm \frac{\omega_m}{\omega_m,\pm} \left(1 - \frac{\Omega_{\text{EP1}}^2}{2\omega_m^2}\right) & \Omega < \Omega_{\text{EP1}}, \\
0 & \Omega_{\text{EP1}} < \Omega < \Omega_{\text{EP2}}, \\
\pm \frac{4\omega_m}{\omega_m,\pm} \left(1 - \frac{3\Omega_{\text{EP2}}^2}{2\omega_m^2}\right) & \Omega > \Omega_{\text{EP2}}. \end{cases} \tag{10}$$
FIG. 3: (Color online) (a) The sensitivity $|d\omega_{m,\pm}/d\omega_m|$ versus the driving strength $\Omega$. (b) Dependence of frequency splitting $\omega_{m,\pm}$ corresponding to EP1 (blue dashed curves for $\Omega = \Omega_{EP1}$) and EP2 (red solid curves for $\Omega = \Omega_{EP2}$) on the frequency perturbation $\delta$.

The sensitivity $|d\omega_{m,\pm}/d\omega_m|$ are shown in Fig. 3(a). It is clear that the sensitivity is enhanced sharply as the driving strength coming close to the EPs, and it becomes divergent at the two EPs as $|\omega_{m,\pm}| \to 0$.

As the frequency splitting $\omega_{m,\pm}$ dependencies on the value of the frequency perturbation $\delta$, we show the dependence of frequency splitting $\omega_{m,\pm}$ on the frequency perturbation $\delta$ in Fig. 3(b), corresponding to EP1 (dashed curves) and EP2 (solid curves). Around the two EPs, the frequency splitting $\omega_{m,\pm}$ is given by

$$\frac{\omega_{m,\pm}}{\gamma_m} \approx \begin{cases} \pm \sqrt{\frac{\omega_m}{\gamma_m}} \sqrt{\frac{\delta}{\gamma_m}} \Omega = \Omega_{EP1}, \\ \pm 2 \sqrt{\frac{\omega_m}{\gamma_m}} \sqrt{-\frac{\delta}{\gamma_m}} \Omega = \Omega_{EP2}. \end{cases}$$

As $\omega_{m,\pm}/\gamma_m$ depends on the square root of $\delta/\gamma_m$, the sensitivity can be enhanced sharply when $|\delta/\gamma_m| \ll 1$. Moreover, there is an enhancement factor $\sqrt{\omega_m/\gamma_m}$ in front of the frequency perturbation $\delta$, so that the sensitivity becomes even higher for a mechanical resonator with higher quality.

Making the effect of EP enhancement more clearly, we can compare the frequency splitting $\omega_{m,\pm}$ for the driving strength around the EPs with the results for the driving strength far
away from the EPs. The frequency splitting $\omega_{m,\pm}$ for the driving strength far away from the EPs are given approximately as

$$\omega_{m,\pm} \approx \begin{cases} \pm (\omega_m + \delta) & \Omega \ll \Omega_{EP1}, \\ \pm 2(\omega_m + \delta) & \Omega \gg \Omega_{EP2}. \end{cases}$$  \hfill (12)

There are two differences in expressions of the frequency splitting $\omega_{m,\pm}$ for the driving strength close to and far away from the EPs. First, $\omega_{m,\pm}$ depends on the square root of the frequency perturbation $\delta$ for the driving strength close to the EPs, while $\omega_{m,\pm}$ is linearly dependent on the frequency perturbation $\delta$ for the driving strength far away from the EPs. Second, there is an enhancement factor $\sqrt{\omega_m/\gamma_m}$ in the expressions of $\omega_{m,\pm}$ for the driving strength close to the EPs. Based on these two points, ultra-sensitive sensing can be realized based on the frequency splitting around the EP for the driving strength close to the EPs.

As a specific application, let us consider the detection of the mass of a nanoparticle as a simple example for ultra-sensitive sensing based on the EPs. Consider a nanoparticle with mass $m_p$ adheres to the mechanical resonator, then the frequency of the mechanical resonator is shifted from $\omega_m$ to $\omega'_m = \omega_m + \delta$ with $\delta \approx -\omega_m^2 m_p/m$. With a driven strength of the driving field close to the EP2, i.e., $\Omega \approx \Omega_{EP2}$, we have

$$\frac{\omega_{m,\pm}}{\gamma_m} \approx \pm \sqrt{2}\frac{\omega_m}{\gamma_m} \sqrt{\frac{m_p}{m}}.$$ \hfill (13)

The frequency splitting can be resolved, i.e., $|\omega_{m,+} - \omega_{m,-}| > \gamma_m$, when the mass of the nanoparticle $m_p/m \geq 1/(8Q_m^2)$ for $Q_m \equiv \omega_m/\gamma_m$. Taking the experimental parameters [69]: mechanical mass $m = 3.6 \text{ pg}$ and Q-factor $Q_m = 7 \times 10^5$, we can detect a nanoparticle with mass as small as $m_p \approx 1 \times 10^{-24} \text{ g}$, which is much smaller than the mass sensitivity ever reported in the optomechanical systems [75–81].

V. DISCUSSIONS AND CONCLUSIONS

In summary, we discovered the inherent anti-PT symmetry in a damping mechanical resonator, which provided new perspectives on the dynamic behaviors of a damping mechanical resonator, and similar results can be observed in any resonant systems. Our work indicates that anti-PT symmetry is commonly presented and can be realized in a simple damping resonator, without deliberate design as done in previous studies. Damping resonator with anti-PT symmetry could be a useful element for more complex applications,
such as high-order EPs [82–85] and non-Hermitian topological phases [86]. In addition, we proposed to observe mechanical anti-\(\mathcal{PT}\) phase transition by optical spring effect in a quadratic optomechanical system, and the frequency bifurcation around the EPs in the mechanical anti-\(\mathcal{PT}\) symmetric system provides us an effective way for ultra-sensitive sensing, such as mass sensing.

**Acknowledgement.**—X.-W.X. was supported by the National Natural Science Foundation of China (NSFC) (Grant No. 12064010), and Natural Science Foundation of Hunan Province of China (Grant No. 2021JJ20036). J.-Q.L. was supported in part by the NSFC (Grants No. 12175061 and No. 11935006) and the Science and Technology Innovation Program of Hunan Province (Grants No. 2021RC4029, No. 2017XXK2018, and No. 2020RC4047). H.J. was supported by the NSFC (Grants No. 11935006 and No. 11774086) and the Science and Technology Innovation Program of Hunan Province (Grant No. 2020RC4047). L.-M.K. was supported by the NSFC (Grants No. 1217050862, 11935006 and No. 11775075) and the Science and Technology Innovation Program of Hunan Province (Grant No. 2020RC4047).

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