Optimal Methods for Higher-Order Smooth Monotone Variational Inequalities

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Abstract

In this work, we present new simple and optimal algorithms for solving the variational inequality (VI) problem for $p$th-order smooth, monotone operators — a problem that generalizes convex optimization and saddle-point problems. Recent works (Bullins and Lai (2020), Lin and Jordan (2021), Jiang and Mokhtari (2022)) present methods that achieve a rate of $\tilde{O}(\epsilon^{-2/(p+1)})$ for $p \geq 1$, extending results by (Nemirovski (2004)) and (Monteiro and Svaiter (2012)) for $p = 1, 2$. A drawback to these approaches, however, is their reliance on a line search scheme. We provide the first $p$th-order method that achieves a rate of $O(\epsilon^{-2/(p+1)})$. Our method does not rely on a line search routine, thereby improving upon previous rates by a logarithmic factor. Building on the Mirror Prox method of Nemirovski (2004), our algorithm works even in the constrained, non-Euclidean setting. Furthermore, we prove the optimality of our algorithm by constructing matching lower bounds. These are the first lower bounds for smooth MVIs beyond convex optimization for $p > 1$. This establishes a separation between solving smooth MVIs and smooth convex optimization, and settles the oracle complexity of solving $p$th-order smooth MVIs.

1 Introduction

In the variational inequality (VI) problem, given an operator $F : Z \to \mathbb{R}^n$ over a closed convex set $Z \subseteq \mathbb{R}^n$, the goal is to find $z^* \in Z$ that satisfies:

$$\langle F(z) , z^* - z \rangle \leq 0, \quad \forall z \in Z.$$  

This problem captures constrained convex optimization by setting $F$ to be the gradient of the function, as well as min-max problems of the form

$$\min_{x \in X} \max_{y \in Y} \phi(x, y)$$

by setting $F = [\nabla_x \phi, -\nabla_y \phi]^T$ for $z = (x, y)$. The VI problem has proven itself useful across a wide range of applications which include training neural networks [Madry et al., 2018] and generative adversarial networks (GANs) [Goodfellow et al., 2014], signal processing [Liu et al., 2013, Giannakis et al., 2016], as well as game theoretic applications such as finding Nash equilibria [Daskalakis et al., 2011]. In this work, we focus on simple and optimal algorithms for the case of monotone operators and the associated monotone variational inequality (MVI) problem which generalizes convex optimization to the
VI setting. MVIs capture convex-concave saddle point problems, and include applications from robust optimization [Ben-Tal et al., 2009] and zero-sum games [Kroer et al., 2018].

In the special case of convex optimization, restricting to smooth convex functions (with bounded Lipschitz constant of the function gradient) allows us to obtain fast convergent algorithms with an iteration complexity of $O(\varepsilon^{-1/2})$, e.g., Nesterov’s accelerated gradient descent [Nesterov, 1983, 2004], which is optimal in this setting. Analogously, for smooth ($p = 1$) MVIs, the Mirror Prox method of Nemirovski [2004] and the dual extrapolation method of Nesterov [2007] achieve an $O(\varepsilon^{-1})$ iteration complexity, building on the initial extragradient method of Korpelevich [1976]. This rate has been shown to be tight for MVIs, assuming access to only a first order oracle, for smooth convex-concave saddle point problems [Ouyang and Xu, 2021], which, as we have seen, are a special case of the MVI problem.

In the search for better algorithms for convex optimization, recent celebrated works have obtained methods with improved convergence rates of $\tilde{O}(\varepsilon^{-2/(3p+1)})$, where $\tilde{O}(\cdot)$ hides logarithmic factors. Monteiro and Svaiter [2012] show how to achieve an improved convergence rate of $O(\varepsilon^{-2/3})$. For $p$th-order smooth MVIs, recent works [Bullins and Lai, 2020, Lin and Jordan, 2021, Jiang and Mokhtari, 2022] have established convergence rates of $O(\varepsilon^{-2/(p+1)})$, again assuming access to a $p$th-order Taylor series oracle. Note that this rate is strictly worse than that for convex optimization.

A drawback of all these algorithms for higher-order smooth MVIs, including [Monteiro and Svaiter, 2012], is that they require a line search procedure. The first question we address is whether such a line-search is necessary, or if one can design a simpler line-search-free algorithm for $p$th-order smooth MVIs without compromising on the iteration count.

More importantly, there are no matching lower bounds for solving $p$th-order smooth MVIs. Thus, it is unknown whether a convergence rate of $\tilde{O}(\varepsilon^{-2/(p+1)})$ is optimal for $p$th-order smooth MVIs, or if one could hope to achieve better rates, possibly matching those for convex optimization.

**Our Results.** In this work, we provide a simple algorithm for solving $p$th-order smooth MVIs which achieves a rate of $O(\varepsilon^{-2/(p+1)})$ without requiring any line-search procedure, thereby improving upon previous works by a logarithmic factor. Our algorithm builds on the Mirror Prox approach of Nesterov [2004], resulting in a much more simplified analysis compared to the previous line-search-dependent methods. In addition, our algorithm is applicable to both non-Euclidean and constrained settings. Our algorithm requires access to an oracle for solving an MVI subproblem (see Definition 3.1) obtained by regularizing the $p$th-order Taylor series expansion for the operator. This is analogous to the Taylor series oracle from the works on highly-smooth convex optimization [Bubeck et al. 2019, Gasnikov et al. 2019], and identical to the oracle from the Jiang and Mokhtari [2022] work on highly-smooth VIs.

Additionally, we construct a family of hard saddle-point problems, and we show that every algorithm that has access to only a $p$th-order Taylor series oracle will require $\Omega(\varepsilon^{-2/(p+1)})$ iterations to converge. To the best of our knowledge, this is the first lower bound for $p$th-order smooth MVI problems for $p \geq 2$, and it shows that our algorithm is optimal up to constant factors. This effectively settles the oracle complexity of highly-smooth MVIs, and it furthermore establishes a separation from the minimization of highly-smooth convex functions.

**Approximately Solving Subproblems.** The $p$th-order MVI subproblems (Definition 3.1) that need to be solved in our algorithm are identical to those arising in the algorithm from Jiang and Mokhtari [2022].
and when restricted to the case of unconstrained convex optimization with smoothness measured in Euclidean norms, they become identical to those from Bubeck et al. [2019]. In the appendix, we show that it is sufficient to solve the subproblems approximately. Further we show how to solve the subproblem efficiently in the $p = 2$ case.

All previous works on higher-order algorithms [Gasnikov et al., 2019, Bubeck et al., 2019, Jiang and Mokhtari, 2022] assume access to an oracle for solving such subproblems. Even for the special case of unconstrained convex optimization and Euclidean norms, it remains an open problem for how to solve these subproblems for $p \geq 3$.

Independent Work [Lin and Jordan, 2022] A concurrent work by Lin and Jordan [2022] also presents an algorithm for $p$th-order smooth MVIs that does not require a binary search procedure and achieves a rate of $O(\varepsilon^{-2/(p+1)})$. Their work builds on the dual extrapolation method and solves the same subproblems as our algorithm. Their algorithm is also shown to work only for Euclidean norms, although it can possibly be extended to non-Euclidean settings as well. Our results were derived independently and our algorithm works for non-Euclidean settings. Additionally, we include lower bounds which establish that these rates are optimal. Lin and Jordan [2022] also make note of the keen observation by Nesterov [2018, Section 4.3.3] that eliminating the binary search could provide significant practical benefit (relative to the improvements in terms of $\varepsilon$), and thus being able to do so has remained a key open problem.

2 Preliminaries

We let $X, Y, Z \subseteq \mathbb{R}^n$ denote closed convex sets. We use $F : X \rightarrow \mathbb{R}^n$ to denote an operator, $\mathbb{R}^n_k$ to denote the space of $x \in \mathbb{R}^n$ with $x_i = 0, \forall i > k$, and $e_{i,n}$ to denote the all $0$’s vector with $1$ at the $i$th coordinate.

We let $\| \cdot \|$ denote any norm and $d : X \rightarrow \mathbb{R}$ denotes a prox function that is strongly convex with respect to $\| \cdot \|$, i.e.,

$$d(x) - d(y) - \langle \nabla d(y), x - y \rangle \geq \| x - y \|^2.$$  

Let $\omega(x, y)$ denote the Bregman divergence of $d$, i.e.,

$$\omega(x, y) = d(x) - d(y) - \langle \nabla d(y), x - y \rangle \geq \| x - y \|^2. \quad (1)$$

2.1 Standard Results

We first recall several standard results which will be useful throughout the paper, starting with the three point property of the Bregman divergence, which generalizes the law of cosines.

Lemma 2.1 (Three Point Property). Let $\omega(x, y)$ denote the Bregman divergence of a function $d$. The three point property states, for any $x, y, z$,

$$\langle \nabla d(y) - \nabla d(z), x - z \rangle = \omega(x, z) + \omega(z, y) - \omega(x, y).$$

Lemma 2.2 [Tseng 2008]. Let $\phi$ be a convex function, let $x \in X$, and let

$$x_+ = \arg \min_{y \in X} \{ \phi(y) + \omega(y, x) \}.$$  

Then, for all $y \in X$, we have, $\phi(y) + \omega(y, x) \geq \phi(x_+) + \omega(x_+, x) + \omega(y, x_+)$.  

The next lemma follows from the power mean inequality (see Bullins and Lai 2020, Lemma 4.4).

Lemma 2.3. Given $R, \xi_1, \ldots, \xi_T \geq 0$ such that $\sum_{t=1}^T \xi_t^2 \leq R$, we have $\sum_{t=1}^T \xi_t^{-q} \geq \frac{T^{q+1}}{R^q}$.  

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2.2 Monotone Variational Inequalities

In this section, we formally define our problem and some definitions for higher-order derivatives.

**Definition 2.4 (Directional Derivative).** Let $\mathcal{X} \subseteq \mathbb{R}^n$. Consider a $k$-times differentiable operator $F : \mathcal{X} \to \mathbb{R}^n$. For $r \leq k + 1$, we let

$$\nabla^k F(x)[h]^r = \left. \frac{\partial^k}{\partial h^k} \right|_{t_1=0,\ldots,t_r=0} F(x + t_1 h + \cdots + t_r h)$$

denote, for $x, h \in \mathcal{X}$, the $k$th directional derivative of $F$ at $x$ along $h$.

**Definition 2.5 (Monotone Operator).** For $\mathcal{X} \subseteq \mathbb{R}^n$, consider an operator $F : \mathcal{X} \to \mathbb{R}^n$. We say that $F$ is monotone if

$$\forall x, y \in \mathcal{X}, \quad \langle F(x) - F(y), x - y \rangle \geq 0.$$ 

Equivalently, an operator $F$ is monotone if its Jacobian $\nabla F$ is positive semidefinite.

**Definition 2.6 (Higher-Order Smooth Operator).** For $p \geq 1$, an operator $F$ is $p$th-order $L_p$-smooth with respect to a norm $\|\cdot\|$ if the higher-order derivative of $F$ satisfies

$$\|\nabla^{p-1} F(y) - \nabla^{p-1} F(x)\|_* \leq L_p \|y - x\|, \forall x, y \in \mathcal{X},$$

or

$$\|F(y) - T_{p-1}(y; x)\|_* \leq \frac{L_p}{p!} \|y - x\|^p,$$

where we let

$$T_p(y; x) = \sum_{i=0}^p \frac{1}{i!} \nabla^i F(x)[y - x]^i,$$

denote the $p$th-order Taylor expansion of $F$, and we let

$$\|\nabla^{p-1} F(y) - \nabla^{p-1} F(x)\|_* \overset{\text{def}}{=} \max_{h, \|h\| \leq 1} |\nabla^{p-1} F(y)[h]_p - \nabla^{p-1} F(x)[h]_p|$$

denote the operator norm.

For any operator $F$, the variational inequality problem associated with $F$ may ask for two kinds of solutions which we define next.

**Definition 2.7 (Weak and Strong Solutions).** For $\mathcal{X} \subseteq \mathbb{R}^n$ and operator $F : \mathcal{X} \to \mathbb{R}^n$, a strong solution to the variational inequality problem associated with $F$ is a point $x^* \in \mathcal{X}$ satisfying:

$$\langle F(x^*), x^* - x \rangle \leq 0, \quad \forall x \in \mathcal{X}.$$ 

A weak solution to the variational inequality problem associated with $F$ is a point $x^* \in \mathcal{X}$ satisfying:

$$\langle F(x), x^* - x \rangle \leq 0, \quad \forall x \in \mathcal{X}.$$ 

If $F$ is continuous and monotone, then a weak solution is the same as a strong solution.

**Definition 2.8 ($\varepsilon$-Approximate MVI Solution).** Let $\varepsilon > 0$, $\mathcal{X} \subseteq \mathbb{R}^n$, and operator $F : \mathcal{X} \to \mathbb{R}^n$ be monotone, continuous and $p$th-order $L_p$-smooth with respect to a norm $\|\cdot\|$. Our goal is to find an $\varepsilon$-approximate solution to the MVI, i.e., an $x^* \in \mathcal{X}$ satisfying:

$$\langle F(x), x^* - x \rangle \leq \varepsilon, \quad \forall x \in \mathcal{X}.$$
Organization. In Section 3 we present our algorithm and analysis for the MVI problem (Definition 2.8). In Section 4 we present a lower bound for the MVI problem which shows that our rates of convergence are tight up to constant factors. We further show how to solve our MVI subproblem for $p = 2$, the details of which we defer to the appendix.

3 Algorithm

We now present our algorithm for the MVI problem defined in Definition 2.8. Our algorithm is based on a Mirror Prox method and does not require any binary search procedure or solution to an implicit subproblem.

Our algorithm MVI-OPT (Algorithm 1) solves the following subproblem at every iteration.

**Definiton 3.1 (MVI Subproblem).** We assume access to an oracle which, for any $\hat{x} \in \mathcal{X}$, solves the following variational inequality problem:

\[
\text{Find } T(\hat{x}) : \langle U_{p,\hat{x}}(T(\hat{x})), T(\hat{x}) - x \rangle \leq 0, \quad \forall x \in \mathcal{X},
\]

where

\[
U_{p,x}(y) = T_{p-1}(y ; x) + \frac{2L_p}{p^i} \omega(y, x) \frac{\nabla d(y) - \nabla d(x)}{\|\nabla d(y) - \nabla d(x)\|}.
\]

We note that for the case of $\mathcal{X} = \mathbb{R}^n$, $d(x) = \|x\|_2^2$ and $F = \nabla f$, where $f$ is a $p^{th}$-order smooth convex function, the above subproblem is equivalent to the subproblem solved by the algorithm of Bubeck et al. [2019] (up to constant factors), which is known to have optimal iteration complexity for highly-smooth convex optimization. Previous works on higher-order smooth MVIs also solve essentially the same subproblem in their algorithms [Jiang and Mokhtari, 2022]. It has been shown by [Jiang and Mokhtari, 2022, Lemma 7.1] that these subproblems are monotone and are guaranteed to have a unique solution, though efficiently finding such a solution in general remains an open problem, even in the case of convex optimization. We further show in the appendix that it is sufficient to solve these subproblems approximately, and for the case of $p = 2$ we provide an algorithm for solving the associated subproblem.

**Algorithm 1** Algorithm for Higher-Order Smooth MVI Optimization

1. procedure MVI-OPT($x_0 \in \mathcal{X}, K, p$)
2. for $i = 0$ to $i = K$
3. $x_{i+\frac{1}{2}} \leftarrow T(x_i)$
4. $\lambda_i \leftarrow \frac{1}{2} \omega(x_{i+\frac{1}{2}}, x_i)^{-\frac{p-1}{2}}$
5. $x_{i+1} \leftarrow \arg \min_{x \in \mathcal{X}} \left\{ F(x_{i+\frac{1}{2}}, x) - x_i \right\}$
6. return $\hat{x}_K = \frac{\sum_{i=0}^{K} \lambda_i x_{i+\frac{1}{2}}}{\sum_{i=0}^{K} \lambda_i}$

We now move to the analysis of our algorithm. The following lemma, which helps us prove our final rate of convergence, characterizes the iterates and step sizes involved in our algorithm.

**Lemma 3.2.** For any $K \geq 1$ and $x \in \mathcal{X}$, the iterates $x_{i+\frac{1}{2}}$ and parameters $\lambda_i$ satisfy

\[
\sum_{i=0}^{K} \lambda_i \frac{p^i}{L_p} \langle F(x_{i+\frac{1}{2}}), x_{i+\frac{1}{2}} - x \rangle \leq \omega(x, x_0) - \frac{15}{16} \sum_{i=0}^{K} (2\lambda_i)^{-\frac{p}{p-1}}.
\]

**Proof.** For any $i$ and any $x \in \mathcal{X}$, we first apply Lemma 2.2 with $\phi(x) = \lambda_i \frac{p^i}{L_p} \langle F(x_{i+\frac{1}{2}}), x - x_i \rangle$, which gives us

\[
\lambda_i \frac{p^i}{L_p} \langle F(x_{i+\frac{1}{2}}), x_{i+1} - x \rangle \leq \omega(x, x_i) - \omega(x, x_{i+1}) - \omega(x_{i+1}, x_i).
\]

(2)
Additionally, the guarantee of Definition 3.1 with \( x = x_{i+1} \) yields
\[
\langle T_{p-1}(x_{i+\frac{1}{p}}; x_i), x_{i+\frac{1}{p}} - x_{i+1} \rangle \leq \frac{2L_p}{p!} \omega(x_{i+\frac{1}{p}}, x_i) \frac{1}{p} \langle \nabla d(x_i) - \nabla d(x_{i+\frac{1}{p}}), x_{i+\frac{1}{p}} - x_{i+1} \rangle. \tag{3}
\]

Applying the Bregman three point property (Lemma 2.1) and the definition of \( \lambda_i \) to Equation 3 we have
\[
\lambda_i \frac{p!}{L_p} \langle T_{p-1}(x_{i+\frac{1}{p}}; x_i), x_{i+\frac{1}{p}} - x_{i+1} \rangle \leq \omega(x_{i+1}, x_i) - \omega(x_i, x_{i+\frac{1}{p}}) - \omega(x_{i+\frac{1}{p}}, x_i). \tag{4}
\]

Summing Equations 2 and 4, we obtain
\[
\lambda_i \frac{p!}{L_p} \langle F(x_{i+\frac{1}{p}}), x_{i+\frac{1}{p}} - x \rangle + \langle T_{p-1}(x_{i+\frac{1}{p}}; x_i) - F(x_{i+\frac{1}{p}}), x_{i+\frac{1}{p}} - x_{i+1} \rangle \leq \omega(x, x_i) - \omega(x, x_{i+1}) - \omega(x_{i+1}, x_{i+\frac{1}{p}}) - \omega(x_{i+\frac{1}{p}}, x_i). \tag{5}
\]

Now, we obtain
\[
\lambda_i \frac{p!}{L_p} \langle T_{p-1}(x_{i+\frac{1}{p}}; x_i) - F(x_{i+\frac{1}{p}}), x_{i+\frac{1}{p}} - x_i \rangle \\
\begin{align*}
&\geq (i) \lambda_i \frac{p!}{L_p} \left\| T_{p-1}(x_{i+\frac{1}{p}}; x_i) - F(x_{i+\frac{1}{p}}) \right\|_{x_{i+\frac{1}{p}} - x_i} \\
&\geq (ii) \lambda_i \left\| x_{i+\frac{1}{p}} - x_i \right\|_{x_{i+\frac{1}{p}} - x_i} \\
&\geq (iii) - \frac{1}{2} \omega(x_{i+\frac{1}{p}}, x_i) - \frac{2}{p} \omega(x_{i+\frac{1}{p}}, x_i) \frac{1}{p} \omega(x_{i+1}, x_{i+\frac{1}{p}}) \frac{1}{p} \\
&= (iv) - \frac{1}{2} \omega(x_{i+\frac{1}{p}}, x_i) - \omega(x_i, x_{i+\frac{1}{p}}).
\end{align*}
\]

Here, (i) used Hölder’s inequality, (ii) used Definition 2.4, (iii) used the 1-strong convexity of \( \omega \), and (iv) used the inequality \( \sqrt{xy} \leq 2x + \frac{1}{8} y \) for \( x, y \geq 0 \). Combining with 5 and rearranging yields
\[
\lambda_i \frac{p!}{L_p} \langle F(x_{i+\frac{1}{p}}), x_{i+\frac{1}{p}} - x \rangle \leq \omega(x, x_i) - \omega(x, x_{i+1}) - \frac{15}{16} \omega(x_{i+\frac{1}{p}}, x_i).
\]

We observe that \( \omega(x_{i+\frac{1}{p}}, x_i) = (2\lambda_i)^{-\frac{2}{p-1}} \). Applying this fact and summing over all iterations \( i \) yields
\[
\sum_{i=0}^{K} \lambda_i \frac{p!}{L_p} \langle F(x_{i+\frac{1}{p}}), x_{i+\frac{1}{p}} - x \rangle \leq \omega(x, x_0) - \frac{15}{16} \sum_{i=0}^{K} \left(2\lambda_i\right)^{-\frac{2}{p-1}}
\]
as desired.

We now state and prove our main theorem.

**Theorem 3.3.** Let \( \varepsilon > 0 \), \( p \geq 1 \) and \( \mathcal{X} \subseteq \mathbb{R}^n \) be any closed convex set. Let \( F : \mathcal{X} \to \mathbb{R}^n \) be an operator that is \( p \)-th order \( L_p \)-smooth with respect to an arbitrary norm \( \|\cdot\| \). Let \( \omega(\cdot, \cdot) \) denote the Bregman divergence of a function that is strongly convex with respect to the same norm \( \|\cdot\| \). Algorithm 7 returns \( \hat{x} \) such that \( \forall x \in \mathcal{X} \),
\[
\langle F(x), \hat{x} - x \rangle \leq \varepsilon,
\]
in at most
\[
16 \left( \frac{2L_p}{p!} \right)^{2/p+1} \frac{\omega(x, x_0)}{\varepsilon^{2/p+1}}
\]
calls to an oracle that solves the subproblem defined in Definition 3.1.
Proof. Let $S_K = \sum_{i=0}^{K} \lambda_i$. We first note that, $\forall x \in \mathcal{X}$,
\[
\langle F(x), \hat{x} - x \rangle = \sum_{i=0}^{K} \frac{\lambda_i}{S_K} \langle F(x), x_{i+\frac{1}{q}} - x \rangle \\
\leq \sum_{i=0}^{K} \frac{\lambda_i}{S_K} \langle F(x_{i+\frac{1}{q}}), x_{i+\frac{1}{q}} - x \rangle. \\
\text{ (From monotonicity of } F \text{)}
\]
\[
\leq \frac{L_p}{S_K p!} \omega(x, x_0). \\
\text{ (From Lemma 3.2 and } \omega(x, y) \geq 0, \forall x, y)\]

It is now sufficient to find a lower bound on $S_K$. We will use Lemma 2.3 for $q = p - 1$, $\xi_i = (2\lambda_i)^{-\frac{1}{p-1}}$. Observe from Lemma 3.2 that
\[
\sum_{i=1}^{K} \xi_i^2 = \sum_{i=0}^{K} (2\lambda_i)^{-\frac{2}{p-1}} \leq \frac{16}{15} \omega(x, x_0) = \frac{16}{15} R^2.
\]

Now, Lemma 2.3 gives
\[
2S_K = 2 \sum_{i=0}^{K} \lambda_i = \sum_{i=0}^{K} \xi_i^{-(p-1)} \geq \left(\frac{K+1}{16} \right)^{\frac{p}{p-1}} R^2.
\]

We thus have for all $x \in \mathcal{X}$,
\[
\langle F(x), \hat{x} - x \rangle \leq \frac{2L_p}{p!} \left(\frac{16}{p^2} \right)^{\frac{p}{p-1}} \omega(x, x_0)^{\frac{2}{p-1}},
\]
which gives an $\varepsilon$ approximate solution after $\frac{16}{15} \cdot \left(\frac{2L_p}{p!} \right)^{\frac{1}{p-1}} \omega(x, x_0)$ iterations. \qed

4 Lower Bound for Higher-Order Smooth Variational Inequalities

In this section, we prove a lower bound for the monotone variational inequality problem, for $p^\text{th}$-order smooth monotone operators $F$, when finding an $\varepsilon$-approximate MVI solution, i.e., finding $z^* \in \mathcal{Z}$ such that, for $\varepsilon > 0$ and closed convex set $\mathcal{Z} \subseteq \mathbb{R}^n$,
\[
\langle F(z), z^* - z \rangle \leq \varepsilon, \quad \forall z \in \mathcal{Z}. \tag{6}
\]

Our analysis and hard instances are inspired by the constructions of Nesterov [2021] and Ouyang and Xu [2021].

Oracle for Computing Iterates. We define the following model for computing iterates. For a $p^\text{th}$-order smooth operator $F$, consider methods which at every iteration compute stationary points of the following family of higher-order tensor polynomial for some $a \in \mathbb{R}^p, \gamma \in \mathbb{R}, m > 1$:
\[
\Phi_{a, \gamma, m}(h) = \sum_{i=0}^{p-1} a_i \nabla^i F(z)[h]^{i+1} + \gamma \|h\|_2^m. \tag{7}
\]

Let $\Gamma_{z, F}(a, \gamma, m)$ denote the set of all stationary points of the above polynomial. Define the linear subspace
\[
S_F(z) = \text{span}\{\Gamma_{z, F}(a, \gamma, m) : a \in \mathbb{R}^p, \gamma > 0, m > 1\}.
\]

Assumption 4.1. For a $p^\text{th}$-order smooth operator $F$, we consider methods that generate a sequence of points $\{z_k\}_{k \geq 0} \in \mathcal{Z}$ satisfying
\[
z^{(k+1)} \in z^{(0)} + \sum_{i=1}^{k} S_F(z^{(i)}).
\]
Hard Instance. We will work with the following family of saddle point problems parameterized by \( t \in \{1, 2, \ldots, n - 1\} \),

\[
\begin{align*}
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \zeta_t(x, y) &= f_t(x) + \langle A_t x - b_t, y \rangle, \\
\end{align*}
\]

for closed convex sets \( \mathcal{X} \subseteq \mathbb{R}^n \) and \( \mathcal{Y} \subseteq \mathbb{R}^m \), \( m \leq n \) and, \( p^{th} \)-order smooth, convex function \( f_t \), matrix \( A_t \in \mathbb{R}^{m \times n} \), vector \( b_t \in \mathbb{R}^m \). We prove that these problems require at least \( t^{-(p+1)/2} \) iterations to converge.

We are now ready to state our lower bound.

We note that Problem (8) is a special case of Problem (6) for \( \mathcal{X} = \mathcal{X} \times \mathcal{Y} \), and

\[
F = \begin{bmatrix} \nabla f_t + A_t^\top y \\ A_t x - b_t \end{bmatrix}.
\]

We now define the function \( f_t \), matrix \( A_t \) and vector \( b_t \) similar to Nesterov [2021] and Ouyang and Xu [2021]. For \( t \in \{1, \ldots, n - 1\} \),

\[
f_t(x) = \frac{L_f}{(p+1)!} \left( \sum_{i=1}^{t} B_i x_i^{p+1} + \sum_{i=t+1}^{n} |x_i|^{p+1} \right) - \frac{1}{p!} \left( L_f + \frac{L_A}{2} \right) x \cdot e_{1,n}.
\]

\[
A_t = \frac{L_A}{p!} \begin{bmatrix} B_t & \bar{0} \\ \bar{0} & G \end{bmatrix}, \quad b_t = \frac{L_A}{p!} \begin{bmatrix} 1_t \\ \bar{0} \end{bmatrix}.
\]

Here \( L_A \geq 0, L_f > 0 \) and \( L_f \geq L_A \). For \( m < n, A \in \mathbb{R}^{m \times n}, G \in \mathbb{R}^{(m-t) \times (m-t)} \) is a full rank matrix s.t. \( \|G\| = 2 \), and \( B_t \in \mathbb{R}^{t \times t} \) is defined as

\[
B_t = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}.
\]

We note that \( f_t \) is \( L_f \cdot \|B_t\|^{p+1} \leq 2^{p+1}L_f, p^{th} \)-order smooth and \( \|A\| = \frac{2}{p}L_A \).

Before we state our main result, we define sets \( \mathcal{X}, \mathcal{Y} \) and the primal and dual problems associated with Problem (8).

\[
\mathcal{X} = \{ x \in \mathbb{R}^n : \|x\|_2^2 \leq R_X^2 = 3(t+1)^3 \}, \quad \mathcal{Y} = \{ y \in \mathbb{R}^m : \|y\|_2^2 \leq R_Y^2 = t + 1 \}.
\]

The associated primal and dual problems are defined as,

\[
\begin{align*}
\min_{x \in \mathcal{X}} \phi_t(x) &= f_t(x) + \max_{y \in \mathcal{Y}} \langle A_t x - b_t, y \rangle \\
\max_{y \in \mathcal{Y}} \psi_t(y) &= \langle A_t x - b_t, y \rangle + \min_{x \in \mathcal{X}} f_t(x).
\end{align*}
\]

We are now ready to state our lower bound.

**Theorem 4.2.** Let \( p \geq 2, 1 \leq t \leq \frac{n-1}{2} \), \( L_f > 0, L_A \geq 0 \) and \( L_f \geq L_A \). Let \((\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{Y}\) be the output after \( t \) iterations of a method \( M \) that satisfies Assumption 4.1 when applied to Problem (8) for \( \zeta_{2t+1} \). Then,

\[
\phi_{2t+1}(\bar{x}) - \psi_{2t+1}(\bar{y}) \geq \frac{1}{10 \cdot 3^{(p+2)/2} (p+1)!} \frac{R_X^{p+1}}{(t+1)^{p+2}} + \frac{L_A}{p!} \frac{R_X R_Y^{(p+1)}}{\sqrt{3(t+1)^{p+1}}},
\]
4.1 A Lower Bound for Highly-Smooth Saddle-Point Problems

We now work towards proving Theorem 4.2. We rely on the following lemmas, whose proofs can be found in Appendix A. We begin by characterizing the iterates produced by a method $\mathcal{M}$ satisfying Assumption 4.1 when applied to the primal problem (11).

**Lemma 4.3.** Any method $\mathcal{M}$ satisfying Assumption 4.1 applied to the Primal Problem (11) for $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y}$ as defined in (10), starting from $x^{(0)} = 0$ generates points $\{x^{(k)}\}_{k \geq 0}$ satisfying

$$x^{(k+1)} \in \sum_{i=0}^{k} S_{\nabla \phi}(x^{(i)}) \subseteq \mathbb{R}^n_{k+1}, \quad 0 \leq k \leq t - 1.$$  

Next, we compute the values of the optimizer and the optimum of Problem (8).

Our final lemma, before we prove our main result, bounds the minimum values of the function $f_{2t+1}$ and the norm $\|A_{2t+1}x - b_{2t+1}\|_2$, which we will need to prove the final bound.

**Lemma 4.5.** For $f_{2t+1}, A_{2t+1}, b_{2t+1}$ as defined above, the following holds,

$$\min_{x \in \mathbb{R}^n_{t}} f_{2t+1}(x) \geq \frac{pL_f}{(p+1)!} \left( \frac{3}{2} \right)^{1 + \frac{1}{p}} t, \text{ and,}$$

$$\min_{x \in \mathbb{R}^n_{t}} \|A_{2t+1}x - b_{2t+1}\|_2 \geq \frac{L_A}{p!} (t + 1).$$

We are now ready to prove Theorem 4.2.

**Proof of Theorem 4.2**

Proof. We first claim that it is sufficient to lower bound $\min_{x \in \mathbb{R}^n_{t}} \phi_{2t+1}(x) - \phi^*_{2t+1}$. To see this, first note that since $\bar{y} \in \mathcal{Y}$, and $\psi_{2t+1}(\bar{y})$ is the dual objective, from weak duality,

$$\psi_{2t+1}(\bar{y}) \leq \psi^*_{2t+1} \leq \phi^*_{2t+1}.$$  

From Lemma 4.3 after $t$ iterations all iterates produced by $\mathcal{M}$ when applied to the problem $\min_{x \in \mathbb{R}^n_{t}} \phi_{2t+1}(x)$ must belong to the space $\mathbb{R}^n_{t}$. We now have the following,

$$\phi_{2t+1}(\bar{x}) - \psi_{2t+1}(\bar{y}) \geq \phi_{2t+1}(\bar{x}) - \phi^*_{2t+1} \geq \min_{x \in \mathbb{R}^n_{t}} \phi_{2t+1}(x) - \phi^*_{2t+1},$$

which proves our claim. In the remaining proof, we will focus on lower bounding $\min_{x \in \mathbb{R}^n_{t}} \phi_{2t+1}(x) - \phi^*_{2t+1}$.

Since $\mathcal{Y}$ is a Euclidean ball,

$$\max_{y \in \mathcal{Y}} \langle A_{2t+1}x - b_{2t+1}, y \rangle = \mathcal{R}_{\mathcal{Y}} \|A_{2t+1}x - b_{2t+1}\|_2,$$

which gives us $\phi_{2t+1}(x) = f_{2t+1}(x) + \mathcal{R}_{\mathcal{Y}} \|A_{2t+1}x - b_{2t+1}\|_2$. 

\[
\min_{x \in \mathbb{R}^n_t} \phi(x) - \phi^* \geq \min_{x \in \mathbb{R}^n_t} f_{2t+1}(x) + \min_{x \in \mathbb{R}^n_t} \mathcal{R}_Y \|A_{2t+1}x - b_{2t+1}\|_2 - \phi^*
\]
\[
\geq -\frac{p L_f}{(p+1)!} \left( \frac{3}{2} \right)^{1+\frac{p}{2}} t + \mathcal{R}_Y \frac{L_A}{p!} (t + 1) + \frac{R_{X} L_f + R_{A}}{p!} (2t + 1)
\]

(Using the lower bound on the first two terms from lemma 4.5 and value of \(\phi^*\) from Lemma 4.4)

\[
= \left( \frac{p L_f}{10(p+1)!} + \frac{R_{A}}{p!} \right) (t + 1) + \mathcal{R}_Y \frac{L_A}{p!} (t + 1)
\]

(Since for \(p \geq 2\), \(2 - (1.5)^{1+\frac{p}{2}} \geq 2 - 1.5^{1.5} \geq 0.1\))

\[
\geq \frac{p L_f}{10(p+1)!} (t + 1) + \mathcal{R}_Y \frac{L_A}{p!} (t + 1)
\]

\[
\geq \frac{p L_f}{10 \cdot 3^{1+\frac{p}{2}} (p+1)!} \left( \frac{\mathcal{R}_{X}^{p+1}}{(t + 1)^{\frac{p+1}{2}}} + \frac{L_A}{p!} \sqrt{\mathcal{R}_X \mathcal{R}_Y^p} \right)
\]

The last inequality follows from the fact \(\mathcal{R}_{X} = \sqrt{3}(t + 1)^{3/2}\) and \(\mathcal{R}_Y = \sqrt{t + 1}\). This concludes the proof of the theorem.

## 5 Conclusions

In this paper, we have presented an algorithm for solving \(p\)-order smooth MVI problems that converges at a rate of \(O(\varepsilon^{-2/(p+1)})\), without any line search as required by previous methods. Our algorithm is simple and can be applied to constrained and non-Euclidean settings. Our algorithm requires solving an MVI subproblem in every iteration obtained by regularizing the \(p\)-order Taylor expansion of the operator.

The MVI subproblems solved by our algorithm in each iteration are the same as those arising in previous works, and when restricted to the case of unconstrained convex optimization and Euclidean norms, they become identical to those from optimal higher-order smooth convex optimization algorithms. We further demonstrate in the appendix that it is sufficient to solve these subproblems approximately, and give an efficient algorithm for solving them for \(p = 2\). Solving these subproblems efficiently for \(p \geq 3\) is an open problem even for the special case of unconstrained convex optimization with Euclidean norms.

Finally, we provide a lower bound that matches the above rate up to constant factors, thus showing that our algorithm is optimal. This settles the oracle complexity of solving highly-smooth MVI's, and establishes a gap between the rates achievable for highly-smooth convex optimization and those for highly-smooth MVI's.
References

N. Agarwal and E. Hazan. Lower bounds for higher-order convex optimization. In Conference On Learning Theory, pages 774–792. PMLR, 2018.

Y. Arjevani, O. Shamir, and R. Shiff. Oracle complexity of second-order methods for smooth convex optimization. Mathematical Programming, 178(1):327–360, 2019.

A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. Robust optimization. In Robust optimization. Princeton university press, 2009.

S. Bubeck, Q. Jiang, Y. T. Lee, Y. Li, and A. Sidford. Near-optimal method for highly smooth convex optimization. In Conference on Learning Theory, pages 492–507. PMLR, 2019.

B. Bullins and K. A. Lai. Higher-order methods for convex-concave min-max optimization and monotone variational inequalities. arXiv preprint arXiv:2007.04528, 2020.

Y. Carmon, A. Jambulapati, Q. Jiang, Y. Jin, Y. T. Lee, A. Sidford, and K. Tian. Acceleration with a ball optimization oracle. Advances in Neural Information Processing Systems, 33:19052–19063, 2020.

C. Daskalakis, A. Deckelbaum, and A. Kim. Near-optimal no-regret algorithms for zero-sum games. In Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms, pages 235–254. SIAM, 2011.

A. Gasnikov, P. Dvurechensky, E. Gorbunov, E. Vorontsova, C. A. Uribe, B. Jiang, H. Wang, S. Zhang, S. Bubeck, et al. Near optimal methods for minimizing convex functions with lipschitz p-th derivatives. In Conference on Learning Theory, pages 1392–1393. PMLR, 2019.

G. B. Giannakis, Q. Ling, G. Mateos, I. D. Schizas, and H. Zhu. Decentralized learning for wireless communications and networking. In Splitting Methods in Communication, Imaging, Science, and Engineering, pages 461–497. Springer, 2016.

I. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, D. Warde-Farley, S. Ozair, A. Courville, and Y. Bengio. Generative adversarial nets. Advances in neural information processing systems, 27, 2014.

R. Jiang and A. Mokhtari. Generalized optimistic methods for convex-concave saddle point problems. arXiv preprint arXiv:2202.09674, 2022.

G. M. Korpelevich. The extragradient method for finding saddle points and other problems. Matecon, 12:747–756, 1976.

C. Kroer, G. Farina, and T. Sandholm. Solving large sequential games with the excessive gap technique. Advances in neural information processing systems, 31, 2018.

T. Lin and M. Jordan. Monotone inclusions, acceleration and closed-loop control. arXiv preprint arXiv:2111.08093, 2021.

T. Lin and M. I. Jordan. Perseus: A simple high-order regularization method for variational inequalities. arXiv preprint arXiv:2205.03202, 2022.

Y.-F. Liu, Y.-H. Dai, and Z.-Q. Luo. Max-min fairness linear transceiver design for a multi-user mimo interference channel. IEEE Transactions on Signal Processing, 61(9):2413–2423, 2013.

A. Madry, A. Makelov, L. Schmidt, D. Tsipras, and A. Vladu. Towards deep learning models resistant to adversarial attacks. In International Conference on Learning Representations, 2018.
A. Nemirovski. Prox-method with rate of convergence o (1/t) for variational inequalities with lipschitz continuous monotone operators and smooth convex-concave saddle point problems. SIAM Journal on Optimization, 15(1):229–251, 2004.

Y. Nesterov. A method for unconstrained convex minimization problem with the rate of convergence o(1/k^2). In Doklady an ussr, volume 269, pages 543–547, 1983.

Y. Nesterov. Introductory lectures on convex optimization: A basic course, ser. Mathematics and its applications. Kluwer Academic Publishers, 2004.

Y. Nesterov. Cubic regularization of newton’s method for convex problems with constraints. Technical report, CORE, 2006.

Y. Nesterov. Dual extrapolation and its applications to solving variational inequalities and related problems. Mathematical Programming, 109(2):319–344, 2007.

Y. Nesterov. Lectures on convex optimization, volume 137. Springer, 2018.

Y. Nesterov. Implementable tensor methods in unconstrained convex optimization. Mathematical Programming, 186(1):157–183, 2021.

Y. Nesterov and B. T. Polyak. Cubic regularization of newton method and its global performance. Mathematical Programming, 108(1):177–205, 2006.

Y. Ouyang and Y. Xu. Lower complexity bounds of first-order methods for convex-concave bilinear saddle-point problems. Mathematical Programming, 185(1):1–35, 2021.

C. Song, Y. Jiang, and Y. Ma. Unified acceleration of high-order algorithms under general Hölder continuity. SIAM Journal on Optimization, 31(3):1797–1826, 2021.

P. Tseng. Accelerated proximal gradient methods for convex optimization. Technical report, University of Washington, Seattle, 2008.

A. Proofs from Section 4

Lemma 4.3. Any method \( \mathcal{M} \) satisfying Assumption \ref{assumption} applied to the Primal Problem \ref{primal_problem} for \( \mathcal{X} = \mathbb{R}^n \) and \( \mathcal{Y} \) as defined in \ref{dual_problem}, starting from \( \mathbf{x}^{(0)} = 0 \) generates points \( \{\mathbf{x}^{(k)}\}_{k \geq 0} \) satisfying

\[
\mathbf{x}^{(k+1)} \in \sum_{i=0}^{k} S_{\nabla \phi_t}(\mathbf{x}^{(i)}) \subseteq \mathbb{R}_k^n, \quad 0 \leq k \leq t - 1.
\]

Proof. We first prove that \( \mathbf{x} \in \mathbb{R}_k^n \) implies \( S_{\nabla \phi_t}(\mathbf{x}) \subseteq \mathbb{R}_k^n \). Since the space \( S_{\nabla \phi_t}(\mathbf{x}) \) is defined by the span of the stationary points of a polynomial defined by the directional derivatives of \( \phi \), we first compute all directional derivatives. For simplicity of notation we let

\[
C_t = \begin{bmatrix} B_t & 0 \\ 0 & I_{n-t} \end{bmatrix},
\]

so that \( f_t(\mathbf{x}) = \...\)
\[
\frac{L_f}{(p+1)!} ||C_i x||^{p+1}_{p+1} - \frac{1}{p!} \left( L_f + \frac{L_A}{2} \right) x \cdot e_{1,n}. \]
We can explicitly compute \( \max_{y \in Y} \langle A x - b, y \rangle = R_\gamma \| b - A x \|_2. \)
We thus have,
\[
\nabla \phi_t(x)[h] = \nabla f_t(x)^T h + R_\gamma \frac{A_i^\top b_t - A_i^\top A_i x}{\| b_t - A_i x \|_2} \cdot h \\
= \frac{L_f}{p!} (C_i x)^\top \text{DIAG}(||C_i x||^{p-1}) C_i x + \frac{1}{p!} \left( L_f + \frac{L_A}{2} \right) h_1 - R_\gamma \frac{A_i^\top b_t - A_i^\top A_i x}{\| b_t - A_i x \|_2} \cdot h.
\]
For \( 2 \leq i \leq p - 1, \)
\[
\nabla^i \phi_t(x)[h]^i = \nabla^i f_t(x)[h]^i + R_\gamma \cdot \nabla^i \left( \| b_t - A_i x \|_2 \right)[h]^i.
\]
From the proof of Lemma 2 of [Nesterov, 2021],
\[
\nabla^j f_t(x)[h]^j = \sum_{i=1}^k d_{i,j}(e_{i,n}, C_i h)^j, \quad 2 \leq j \leq p.
\]
Here \( d_{i,j} \) are defined for \( i = 1, \ldots, n, j = 1 \ldots p \) and is some scalar function of \( C_i x \). We next compute \( \nabla^i \left( \| b_t - A_i x \|_2 \right)[h]^i \).
Let \( h(v) = \| v \|_2 \) and \( v(x) = b_t - A_i x \) so that \( \| b_t - A_i x \|_2 = h \circ v(x) \). In order to compute these higher order directional derivatives, we will use Faà di Bruno’s formula. Since \( \nabla_x v = 0 \) for \( i \geq 2 \) and \( A_i \) for \( i = 1 \), the higher order derivatives of our function are as,
\[
\nabla^i \left( \| b_t - A_i x \|_2 \right)[h]^i = \left( \nabla^i h \circ v \right) (\nabla_x v)^{\otimes i} |h|^i.
\]
We can recursively define the derivatives as follows. For any \( i \leq p - 1 \)
\[
[\nabla^i h(v)]_{j_1 \neq j_2 \neq \ldots \neq j_i} = (-1)^{i+1} \cdot \sum_{\| v \|_2^{2i-1}}^{|v|_2} \cdot \frac{v_j_1 v_j_2 \cdots v_j_i}{\| v \|_2^{2i-1}}.
\]
All other permutations of \( j_1, \ldots, j_i \) would give \( [\nabla^i h(v)]_{j_1, \ldots, j_i} \) that has a similar structure as above i.e., a multinomial expression of the coordinates of \( v \). We can thus compute for \( c_{i,j} \), \( 1 \leq i \leq n, 1 \leq j \leq p \) which are functions of \( A_i^\top (b - A x) \),
\[
\nabla^j \left( \| b_t - A_i x \|_2 \right)[h]^j = \sum_{i=1}^k c_{i,j}(e_{i,n}, h)^j.
\]
Here, the sum is only from \( i = 1 \) to \( k \) since if \( x \in \mathbb{R}^n \) then \( A_i^\top (b - A x) \in \mathbb{R}^n \).

The gradients of these derivatives with \( h \) are,
\[
\nabla_h \nabla \phi_t(x)[h] = \frac{L_f}{p!} C_i^\top \text{DIAG}(||C_i x||^{p-1}) C_i x - \frac{1}{p!} \left( L_f + \frac{L_A}{2} \right) e_{1,n} + R_\gamma \frac{A_i^\top b_t - A_i^\top A_i x}{\| b_t - A_i x \|_2} \in \mathbb{R}^p_{k+1}.
\]
\[
\nabla_h \nabla^j \phi_t(x)[h]^j = \sum_{i=1}^k j c_{i,j}(e_{i,n}, h)^j e_{i,n}, \quad 2 \leq j \leq p.
\]
Since \( C_i x, A_i x \in \mathbb{R}^n, \nabla_h \nabla^j \phi_t(x)[h]^j \in \mathbb{R}^p_{k+1}. \) Since the regularizer in \( \| \| \) is in the euclidean norm, all the stationary points of this function belong to \( \mathbb{R}^p_{k+1} \) and as a result \( S_{\nabla \phi_t(x)} \subseteq \mathbb{R}^p_{k+1}. \)
It remains to prove \( x^{(k)} \in \mathbb{R}^n \), which we show by induction. For \( k = 0 \), \( x^{(0)} = 0 \),

\[
\nabla_h \nabla_x \phi_t(x^{(0)}) = -\frac{1}{p!} \left( L_f + \frac{L_A}{\sqrt{2}} \right) e_{1,n} + R_\gamma \, A^\top b_1,
\]

and since for \( x^{(0)} = 0 \), \( c_{i,j} \)'s are a function of \( A^\top b \in \mathbb{R}^n \),

\[
\nabla_h \nabla_x \phi_t(x^{(0)}[h]) = \text{constant} \cdot h_1 e_{1,n}, 2 \leq i \leq p - 1.
\]

All the above derivatives are in \( \mathbb{R}^n \) which gives us \( x^{(1)} \in \mathbb{R}^n \) by Assumption 1.1. Now, assume \( x^{(i)} \in \mathbb{R}^n \) for all \( 1 \leq i \leq k \). Since we have shown that \( S_{\nabla \phi_t}(x^{(k)}) \subseteq \mathbb{R}^n_{k+1} \), again from Assumption 1.1, \( x^{(k+1)} \in \mathbb{R}^n_{k+1} \).

**Lemma 4.4.** For Problem 8 with \( \mathcal{X}, \mathcal{Y} \) as defined in 10, the optimal solution is given by

\[
x_{2t+1}^* = \begin{cases} 
(2t+1) - i + 1 & \text{if } 1 \leq i \leq 2t + 1, \\
0 & \text{otherwise,} \end{cases} \quad y_{2t+1}^* = \frac{1}{2} \left[ \begin{array}{c} 1_{2t+1} \\
0 \end{array} \right],
\]

and the optimal objective value is

\[
\zeta_{2t+1}^* = -\frac{p! L_f + \frac{L_A}{p}}{p!} (2t + 1).
\]

**Proof.** The optimality condition is that there exist \( x^* \in \mathcal{X} \) and \( y^* \in \mathcal{Y} \) such that, for all \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \),

\[
(\nabla f_{2t+1}(x^*) + A_{2t+1}^\top y^*, x^* - x) \leq 0, \quad (A_{2t+1}^\top x^* - b_{2t+1}, y^* - y) \leq 0.
\]

Since \( A_{2t+1}^\top x_{2t+1}^* = b_{2t+1} \), the second condition is satisfied. We note that

\[
\nabla f_{2t+1}(x_{2t+1}^*) = \left[ \frac{L_f}{p!} B^\top \text{Diag}(|Bx_{2t+1}^*|^{p-1}) Bx_{2t+1}^* - \frac{1}{p!} \left( L_f + \frac{L_A}{2} \right) e_{1,2t+1} \right] = -A_{2t+1}^\top y_{2t+1}^*.
\]

Therefore, the first condition also holds and \( x_{2t+1}^* \in \mathcal{X}, y_{2t+1}^* \in \mathcal{Y} \) is an optimizer. Evaluating the function value at this point gives us the value of \( \zeta^* \).

**Lemma 4.5.** For \( f_{2t+1}, A_{2t+1}, b_{2t+1} \) as defined above, the following holds,

\[
\min_{x \in \mathbb{R}^n} f_{2t+1}(x) \geq \frac{p! L_f}{(p+1)!} \left( \frac{3}{2} \right)^{1+\frac{t}{p}} t, \quad \text{and,}
\]

\[
\min_{x \in \mathbb{R}^n} \|A_{2t+1}^\top x - b_{2t+1}\|_2 \geq \frac{L_A}{p!} (t + 1).
\]

**Proof.** Since \( x \in \mathbb{R}^n \), from the definition of \( f_{2t+1} \), we have \( f_i = f_{2t+1} \). Therefore, it is sufficient to look at the optimizer of \( \min_{x \in \mathbb{R}^n} f_i(x) \). Let \( x = (u^\top, v^\top)^\top, u \in \mathbb{R}^t, v \in \mathbb{R}^{n-t} \). The KKT condition is, \( \nabla f_i(x) = 0 \), i.e.,

\[
\frac{L_f}{p!} B^\top \text{Diag}(|Bu|^{|p-1|}) Bu - \frac{1}{p!} \left( L_f + \frac{L_A}{2} \right) e_{1,t} = 0,
\]

and,

\[
\frac{L_f}{p!} \text{Diag}(|v^*|^{p-1}) v^* = 0.
\]

We thus have \( v^* = 0 \), and,

\[
L_f |Bu|^p \text{sign}(Bu^*) = \left( L_f + \frac{L_A}{2} \right) 1_t,
\]

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or,

$$Bu^* = \left(1 + \frac{L_A}{2L_f}\right)^{1/p} 1_t, \quad u_1 = \left(1 + \frac{L_A}{2L_f}\right)^{1/p} t.$$ 

Plugging these values back in the main objective gives,

$$f^*_i = \frac{L_f}{(p+1)!} \left(1 + \frac{L_A}{2L_f}\right)^{1+p} t - \frac{1}{p!} \left(L_f + \frac{L_A}{2}\right) \left(1 + \frac{L_A}{2L_f}\right)^{1+p} t$$

$$= -\frac{pL_f}{(p+1)!} \left(1 + \frac{L_A}{2L_f}\right)^{1+p} t$$

Since $L_f \geq L_A$, the above reduces to,

$$f^*_i \geq -\frac{pL_f}{(p+1)!} \left(1 + \frac{1}{2}\right)^{1+p} t$$

We next bound min$_{x \in \mathbb{R}^n} \|A_{2t+1} x - b_{2t+1}\|_2$.

Since for any $x \in \mathbb{R}^n$, only the first $t$ entries can be non-zero, $(A_{2t+1} x - b_{2t+1})_i = (b_{2t+1})_i = 1$, for $i \in [t+1, 2t+1]$. We thus have,

$$\min_{x \in \mathbb{R}^n} \|A_{2t+1} x - b_{2t+1}\|_2 \geq \frac{L_A}{p!} \sqrt{t+1}$$

$$\geq \frac{L_A}{p!} \frac{t+1}{\sqrt{3(t+1)^{2+\frac{1}{2}}}}$$

$$= \frac{L_A}{p!} \frac{t+1}{\sqrt{3(t+1)^{2+\frac{1}{2}}}}.$$ 

\[ \square \]

### B Approximate MVI Solution

We now show we may handle approximation errors within the standard VI analysis.

**Algorithm 2** Algorithm for Higher-Order Smooth MVI Optimization (Approximate Subproblem Solve)

```
1: procedure MVI-OPT-APPROX(x₀ ∈ X, K, p, δ)
2:   for i = 0 to i = K do
3:     xᵢ₊½ ← APPROX-VI-SOLVE$_{p, δ}$(xᵢ)
4:     λᵢ ← \( \frac{1}{L_p} \omega(xᵢ₊½, xᵢ) \)⁻¹
5:     xᵢ₊₁ ← arg min$_{x \in X} \{ (F(xᵢ₊½), x - xᵢ₊½) + \frac{L_p}{2L_p} \omega(x, xᵢ) \}$
6:   return \( \hat{x}_K = \frac{\sum_{i=0}^K \lambda_i xᵢ₊½}{\sum_{i=0}^K \lambda_i} \)
```

To begin, we need to establish a variant of Lemma 3.2 that is specific to the case where we only have an approximate solution. Note that the proof remains nearly the same as before.

**Lemma B.1.** Suppose, for any \( \hat{x} \in X \), APPROX-VI-SOLVE$_{p, δ}$(\( \hat{x} \)) outputs a \( δ \)-approximate solution to the regularized $p^{th}$-order MVI given in Definition 3.1. Then, for any $K \geq 1$ and $x \in X$, the iterates $xᵢ₊½$ and parameters $λᵢ$ in Algorithm 2 satisfy

$$\frac{p!}{L_p} \sum_{i=0}^K \lambda_i(F(xᵢ₊½), xᵢ₊½ - x) - δ \leq \omega(x, x₀) - \frac{15}{16} \sum_{i=0}^K (2λᵢ)^{-\frac{2}{3}}.$$ 

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Proof. For any \( i \) and any \( x \in \mathcal{X} \), we first apply Lemma \ref{lem:approx-vi} with \( \phi(x) = \lambda_i \frac{p}{L_p} (F(x_{i+\frac{1}{2}}), x - x_i) \), which gives us

\[
\lambda_i \frac{p}{L_p} (F(x_{i+\frac{1}{2}}), x_{i+1} - x) \leq \omega(x, x_i) - \omega(x, x_{i+1}) - \omega(x_{i+1}, x_i). \tag{13}
\]

Additionally, by assumption, the guarantee of the output of APPROX-VI-SOLVE is such that

\[
\left\langle T_{p-1}(x_{i+\frac{1}{2}}; x_i), x_{i+\frac{1}{2}} - x_{i+1} \right\rangle \leq \frac{2}{p} \frac{1}{L_p} \omega(x_{i+\frac{1}{2}}, x_i) \frac{p-1}{p} \left\langle \nabla d(x_i), \nabla d(x_{i+\frac{1}{2}}), x_{i+\frac{1}{2}} - x_{i+1} \right\rangle + \delta. \tag{14}
\]

Applying the Bregman three point property (Lemma \ref{lem:three-point}) and the definition of \( \lambda_\epsilon \) to Equation \ref{14} we have

\[
\lambda_i \frac{p}{L_p} \left\langle T_{p-1}(x_{i+\frac{1}{2}}; x_i), x_{i+\frac{1}{2}} - x_{i+1} \right\rangle \leq \omega(x_{i+1}, x_i) - \omega(x_{i+1}, x_{i+\frac{1}{2}}) - \omega(x_{i+\frac{1}{2}}, x_i) + \lambda_i \frac{p}{L_p} \delta. \tag{15}
\]

Summing Equations \ref{13} and \ref{15} we obtain

\[
\lambda_i \frac{p}{L_p} \left( (F(x_{i+\frac{1}{2}}), x_{i+\frac{1}{2}} - x) + \left\langle T_{p-1}(x_{i+\frac{1}{2}}; x_i) - F(x_{i+\frac{1}{2}}), x_{i+\frac{1}{2}} - x_{i+1} \right\rangle \right)
\leq \omega(x, x_i) - \omega(x, x_{i+1}) - \omega(x_{i+1}, x_{i+\frac{1}{2}}) - \omega(x_{i+\frac{1}{2}}, x_i) + \lambda_i \frac{p}{L_p} \delta. \tag{16}
\]

Now, we obtain

\[
\lambda_i \frac{p}{L_p} \left\langle T_{p-1}(x_{i+\frac{1}{2}}; x_i) - F(x_{i+\frac{1}{2}}), x_{i+\frac{1}{2}} - x_{i+1} \right\rangle
\geq \lambda_i \frac{p}{L_p} \left\| T_{p-1}(x_{i+\frac{1}{2}}; x_i) - F(x_{i+\frac{1}{2}}) \right\| \left\| x_{i+\frac{1}{2}} - x_{i+1} \right\|
\geq \lambda_i \frac{p}{L_p} \frac{p}{2} \omega(x_{i+\frac{1}{2}}, x_i) \frac{p-1}{p} \omega(x_{i+1}, x_{i+\frac{1}{2}}) \frac{p}{p-1}
= \lambda_i \frac{p}{L_p} \frac{p}{2} \omega(x_{i+\frac{1}{2}}, x_i) \frac{p}{p-1} \omega(x_{i+1}, x_{i+\frac{1}{2}}) \frac{p}{p-1}
\geq -\frac{1}{16} \lambda_i \frac{p}{L_p} \omega(x_{i+\frac{1}{2}}, x_i) - \omega(x_{i+1}, x_{i+\frac{1}{2}}).
\]

Here, \( (i) \) used Hölder’s inequality, \( (ii) \) used Definition \ref{def:strong-convex}, \( (iii) \) used the 1-strong convexity of \( \omega \), and \( (iv) \) used the inequality \( \sqrt{ab} \leq 2x + \frac{1}{2} \delta \) for \( x, y \geq 0 \). Combining with \ref{16} and rearranging yields

\[
\lambda_i \frac{p}{L_p} (F(x_{i+\frac{1}{2}}), x_{i+\frac{1}{2}} - x) \leq \omega(x, x_i) - \omega(x, x_{i+1}) - \frac{15}{16} \omega(x_{i+\frac{1}{2}}, x_i) + \lambda_i \frac{p}{L_p} \delta. \tag{17}
\]

We observe that \( \omega(x_{i+\frac{1}{2}}, x_i) = (2\lambda_i)^{-\frac{2}{p-1}} \). Applying this fact and summing over all iterations \( i \) yields

\[
\sum_{i=0}^{K} \lambda_i \frac{p}{L_p} (F(x_{i+\frac{1}{2}}), x_{i+\frac{1}{2}} - x) = -\frac{p}{L_p} \delta \sum_{i=1}^{K} \lambda_i \leq \omega(x, x_0) - \frac{15}{16} \sum_{i=0}^{K} (2\lambda_i)^{-\frac{2}{p-1}},
\]

as desired.

We now state and prove the main theorem of this section.

**Theorem B.2.** Let \( \varepsilon > 0, p \geq 1, \delta \leq \frac{\varepsilon}{2} \) and let \( \mathcal{X} \subseteq \mathbb{R}^n \) be any closed convex set. Let \( F : \mathcal{X} \to \mathbb{R}^n \) be an operator that is \( p \)-th order \( L_p \)-smooth with respect to an arbitrary norm \( \| \cdot \| \). Let \( \omega(\cdot, \cdot) \) denote the Bregman
divergence of a function that is strongly convex with respect to the same norm \( \| \cdot \| \). Algorithm 3 returns \( \hat{x} \) such that \( \forall x \in \mathcal{X} \),
\[
\langle F(x), \hat{x} - x \rangle \leq \varepsilon,
\]
in at most
\[
\frac{16}{15} \left( \frac{4L_p}{p!} \right)^{2/\mu(p+1)} \frac{\omega(x, x_0)}{\varepsilon^{2/(p+1)}}
\]
calls to an APPROX-VI-SOLVE subroutine.

Proof. Let \( S_K = \sum_{i=0}^{K} \lambda_i \). We first note that, \( \forall x \in \mathcal{X} \)
\[
\langle F(x), \hat{x} - x \rangle - \delta = \sum_{i=0}^{K} \frac{\lambda_i}{S_K} \langle F(x), x_{i+\frac{1}{2}} - x \rangle - \delta
\]
\[
\leq \sum_{i=0}^{K} \frac{\lambda_i}{S_K} \langle F(x_{i+\frac{1}{2}}), x_{i+\frac{1}{2}} - x \rangle - \delta, \quad \text{(From monotonicity of} F \)
\]
\[
\leq \frac{L_p}{SKP!} \omega(x, x_0), \quad \text{(From Lemma B.1 and} \lambda_i \geq 0, \forall i \)
\]

It is now sufficient to find a lower bound on \( S_K \). We will use Lemma B.3 for \( q = p - 1, \xi_i = (2\lambda_i)^{-\frac{1}{p+1}} \).
Observe from Lemma B.1 that
\[
\sum_{i=1}^{K} \xi_i^2 = \sum_{i=0}^{K} (2\lambda_i)^{-\frac{1}{p+1}} \leq \frac{16}{15} \omega(x, x_0) = \frac{16}{15} R^2.
\]

Now, Lemma B.3 would give
\[
2S_K = 2 \sum_{i=0}^{K} \lambda_i = 2 \sum_{i=0}^{K} \xi^{-(p+1)} \geq \frac{(K + 1)^{\frac{1}{p+1}}}{\left(\frac{16}{15} R^2\right)^{\frac{1}{p+1}}}
\]
We thus have for all \( x \in \mathcal{X} \),
\[
\langle F(x), \hat{x} - x \rangle - \delta \leq \frac{2L_p}{p!} \frac{\left(\frac{16}{15} R^2\right)^{\frac{1}{p+1}} \omega(x, x_0)^{\frac{1}{p+1}}}{(K + 1)^{\frac{1}{p+1}}},
\]
which gives an \( \varepsilon \) approximate solution after \( \frac{16}{15} \cdot \left( \frac{4L_p}{p!\varepsilon} \right)^{\frac{1}{p+1}} \omega(x, x_0) \) iterations. \[\square\]

C Solving the Subproblem for \( p = 2 \)

Following along the lines of previous work on solutions to trust region/cubic regularization subproblems [Nesterov and Polyak, 2006, Carmon et al., 2020], we now show how our VI subproblem may be approximately solved in the unconstrained Euclidean setting for \( p = 2 \). Thus, in the case where \( \mathcal{X} = \mathbb{R}^n \) and \( d(x) = \|x\|^2 \), we have
\[
U_{2,x}(y) = T_{1}(y; x) + 2L_2\|y - x\|(y - x)
\]
\[
= F(x) + \nabla F(x)(y - x) + 2L_2\|y - x\|(y - x),
\]
and so for any \( \hat{x} \in \mathbb{R}^n \), our subproblem is to find \( T(\hat{x}) \in \mathbb{R}^n \) such that
\[
\langle F(\hat{x}) + \nabla F(\hat{x})(T(\hat{x}) - \hat{x}) + 2L_2\|T(\hat{x}) - \hat{x}\|(T(\hat{x}) - \hat{x}), T(\hat{x}) - x \rangle \leq 0, \quad \forall x \in \mathbb{R}^n. \quad (18)
\]

To begin, we characterize the solution to this VI via the following lemma.
Lemma C.1. There exists a unique $\lambda^* \geq 0$ such that $T(\hat{x}) = \hat{x} - (\nabla F(\hat{x}) + \lambda^* I)^{-1} F(\hat{x})$ is a solution to (IS). Furthermore, $\frac{\lambda^*}{2 L_2^2} = \| (\nabla F(\hat{x}) + \lambda^* I)^{-1} F(\hat{x}) \|$.

Proof. The lemma follows from KKT optimality conditions. Let $\hat{x} \in \mathbb{R}^n$, and consider the auxiliary functions

$$\Phi(y, \lambda) \overset{\text{def}}{=} \left[ F(\hat{x}) + \nabla F(\hat{x})(y - \hat{x}) + \frac{2}{3} \lambda(y - \hat{x}), \frac{1}{3} \| y - \hat{x} \|^2 \right]^T$$

and $h(y, \lambda) \overset{\text{def}}{=} \frac{y}{2} L_2^2 \| y - \hat{x} \|^2 - \frac{1}{2} \lambda^2$. Note that a solution to $\Phi(y^*, \lambda^*) + \nabla h(y^*, \lambda^*) \nu^* = 0$

$h(y^*, \lambda^*) = 0$, for some Lagrange multiplier $\nu^*$. Equivalently, we have

$$F(\hat{x}) + \nabla F(\hat{x})(y^* - \hat{x}) + \frac{2}{3} \lambda^*(y - \hat{x}) + 9 L_2^2 \nu^*(y^* - \hat{x}) = 0$$

$$\frac{1}{3} \| y^* - \hat{x} \|^2 - \nu^* \lambda^* = 0$$

$$\frac{9}{2} L_2^2 \| y^* - \hat{x} \|^2 - \frac{1}{2} \lambda^* = 0,$$

Combining the last two equations gives us that $\nu^* = \frac{\lambda^*}{2 L_2^2}$, and so we may equivalently rewrite the system as:

$$F(\hat{x}) + (\nabla F(\hat{x}) + \lambda^* I)(y^* - \hat{x}) = 0$$

$$\frac{9}{2} L_2^2 \| y^* - \hat{x} \|^2 - \frac{1}{2} \lambda^* = 0.$$

Finally, solving for the first equation gives $(y^* - \hat{x}) = -(\nabla F(\hat{x}) + \lambda^* I)^{-1} F(\hat{x})$, and so $y^* = \hat{x} - (\nabla F(\hat{x}) + \lambda^* I)^{-1} F(\hat{x})$.

We now want to establish how closely we need to approximate $\lambda^*$ for a sufficiently accurate solution.

Lemma C.2. Let $\lambda^* \geq 0$ be such that $T(\hat{x}) = \hat{x} - (\nabla F(\hat{x}) + \lambda^* I)^{-1} F(\hat{x})$ is a solution to (IS), and suppose that, for $\mu > 0$, for all $x \in \mathbb{R}^n$, $x^T \nabla F(\hat{x}) x \geq \mu$. Then, for any $\lambda$ such that $|\lambda - \lambda^*| \leq \frac{\delta}{\| F(\hat{x}) \|}$, we have that

$$\left\| (\nabla F(\hat{x}) + \lambda I)^{-1} F(\hat{x}) - (\nabla F(\hat{x}) + \lambda^* I)^{-1} F(\hat{x}) \right\| \leq \delta.$$
Algorithm 3

Approximate Solver for Second-Order MVI Subproblem

Proof. Let \( \lambda > 0 \). We first note that

\[
\left\| (\nabla F(\tilde{x}) + \lambda I)^{-1} F(\tilde{x}) - (\nabla F(\tilde{x}) + \lambda^* I)^{-1} F(\tilde{x}) \right\|
\]

\[
= \left\| (\nabla F(\tilde{x}) + \lambda I)^{-1} - (\nabla F(\tilde{x}) + \lambda^* I)^{-1} \right\| F(\tilde{x})
\]

\[
= \left\| (\nabla F(\tilde{x}) + \lambda I)^{-1} - (\nabla F(\tilde{x}) + \lambda - \lambda^* I + \lambda^* I)^{-1} \right\| F(\tilde{x})
\]

\[
= \left\| (\nabla F(\tilde{x}) + \lambda I)^{-1} - (\nabla F(\tilde{x}) + \lambda I)^{-1} \left( \frac{1}{\lambda - \lambda^*} I + (\nabla F(\tilde{x}) + \lambda I)^{-1} \right)^{-1} (\nabla F(\tilde{x}) + \lambda I)^{-1} \right\| F(\tilde{x})
\]

\[
- (\nabla F(\tilde{x}) + \lambda I)^{-1} F(\tilde{x})
\]

\[
= \left\| \nabla F(\tilde{x}) + \lambda I \right\|^{-1} \left( \frac{1}{\lambda - \lambda^*} I + (\nabla F(\tilde{x}) + \lambda I)^{-1} \right)^{-1} (\nabla F(\tilde{x}) + \lambda I)^{-1} F(\tilde{x})
\]

\[
\leq |\lambda - \lambda^*| \left\| (\nabla F(\tilde{x}) + \lambda I)^{-1} \right\|^2 \| F(\tilde{x}) \|
\]

\[
\leq \delta,
\]

where the final inequality follows from the bound on \( |\lambda - \lambda^*| \).

Algorithm 3 Approximate Solver for Second-Order MVI Subproblem

1: procedure APPROX-SO-VI-SOLVE(\( \tilde{x} \in \mathbb{R}^n, \delta \in (0, 1) \))
2: \( l = 0, u = \frac{\|F(\tilde{x})\|^2}{\delta}, \nu = \frac{\delta u^2}{\|F(\tilde{x})\|}, \lambda = \frac{l + u}{2}, \lambda^- = \lambda - \nu \)
3: while not \( \left\| (\nabla F(\tilde{x}) + \lambda I)^{-1} F(\tilde{x}) \right\| \leq \lambda \) and \( \left\| (\nabla F(\tilde{x}) + \lambda^- I)^{-1} F(\tilde{x}) \right\| > \lambda^- \) do
4: if \( \lambda \leq \frac{\delta u^2}{\|F(\tilde{x})\|} \) then
5: Break
6: if \( \left\| (\nabla F(\tilde{x}) + \lambda I)^{-1} F(\tilde{x}) \right\| \leq \lambda \) then
7: \( u = \lambda, \lambda = \frac{l + u}{2}, \lambda^- = \lambda - \nu \)
8: else
9: \( l = \lambda, \lambda = \frac{l + u}{2}, \lambda^- = \lambda - \nu \)
10: return \( \tilde{x} - (\nabla F(\tilde{x}) + \lambda I)^{-1} F(\tilde{x}) \)

Next we want to ensure that our subproblem solver routine APPROX-SO-VI-SOLVE (Algorithm 3) can find a solution that approximates the exact solution to sufficient accuracy.

Theorem C.3. Let \( \delta \in (0, 1) \). The output of APPROX-SO-VI-SOLVE (Algorithm 3) given as \( \tilde{T}(\tilde{x}) = \tilde{x} - (\nabla F(\tilde{x}) + \lambda I)^{-1} F(\tilde{x}) \) is such that

\[
\langle U_{2,\tilde{x}} \tilde{T}(\tilde{x}), \tilde{T}(\tilde{x}) - \tilde{x} \rangle \leq \delta \left( \frac{L_2}{2} + \| \nabla U_{2,\tilde{x}} \tilde{T}(\tilde{x}) \| \right) \| \tilde{T}(\tilde{x}) - \tilde{x} \|, \quad \forall \tilde{x} \in \mathbb{R}^n.
\]

In addition, the total computational cost is at most the cost of a single Schur decomposition, which takes \( n^\omega \) time, where \( \omega \approx 2.3728 \) is the matrix multiplication constant, plus \( O \left( \log \left( \frac{\|F(\tilde{x})\|}{\mu \delta} \right) \right) \) calls to a linear system solver in a quasi-upper-triangular system, each of which requires \( O(n) \) time.

Proof. Note that, by monotonicity of \( \| (\nabla F(\tilde{x}) + \lambda I)^{-1} F(\tilde{x}) \| \) in \( \lambda \), along with uniqueness of \( \lambda^* \), if it is the case that the conditions of the while loop in Algorithm 3 are not met (and so we break), then we know that
λ^− ≤ λ^∗ ≤ λ. Thus, since |λ - λ^−| ≤ \frac{\delta \mu^2}{\|F(x)\|}$, it follows that |λ - λ^∗| ≤ \frac{\delta \mu^2}{\|F(x)\|}$. If, on the other hand, we break out of the while loop due to λ ≤ \frac{\delta \mu^2}{\|F(x)\|}$ (which will happen after at most $O\left(\log\left(\frac{\|F(x)\|}{\mu^2}\right)\right)$ iterations of the loop), we know that |λ - λ^∗| ≤ \frac{\delta \mu^2}{\|F(x)\|}$. Furthermore, we may precompute a Schur decomposition of $\nabla F(\hat{x}) = QUQ^{-1}$, whereby $U$ is quasi-upper-triangular (since $\nabla F(\hat{x})$ has all real entries), which means that $U$ is a block diagonal matrix with block size at most $2 \times 2$. It follows that, for any λ, solving a system in $\nabla F(\hat{x}) + \lambda I = Q(U + \lambda I)Q^{-1}$ can be done in $O(\log\|F(x)\|)$ time, and so the total computational cost will be at most $n^ω + O\left(n \log\left(\frac{\|F(x)\|}{\mu^2}\right)\right)$. Now, by Lemma [C.2] we know that APPROX-SO-VI-SOLVE outputs $\hat{T}(\hat{x}) = \hat{x} - (\nabla F(\hat{x}) + \lambda^∗ I)^{-1}F(\hat{x})$ such that

$$\|\hat{T}(\hat{x}) - \hat{T}(\hat{x})\| \leq δ,$$

where we let $T(\hat{x}) = \hat{x} - (\nabla F(\hat{x}) + \lambda^∗ I)^{-1}F(\hat{x})$. By optimality conditions for this unconstrained problem, we know that $U_{2,\hat{x}}(T(\hat{x})) = 0$. We now note that, for all $x \in \mathbb{R}^n$,

$$\langle U_{2,\hat{x}}(\hat{T}(\hat{x})), \hat{T}(\hat{x}) - x \rangle = \langle U_{2,\hat{x}}(\hat{T}(\hat{x})), \hat{T}(\hat{x}) - x \rangle \leq \|U_{2,\hat{x}}(\hat{T}(\hat{x}))\| \|\hat{T}(\hat{x}) - x\| = \|\hat{T}(\hat{x}) - x\| + \|\nabla U_{2,\hat{x}}(\hat{T}(\hat{x}))(\hat{T}(\hat{x}) - x)\| \|\hat{T}(\hat{x}) - x\|$$

$$\leq \left(\|U_{2,\hat{x}}(\hat{T}(\hat{x}))\| + \|\nabla U_{2,\hat{x}}(\hat{T}(\hat{x}))(\hat{T}(\hat{x}) - x)\|\right) \|\hat{T}(\hat{x}) - x\| \leq \left(\frac{L_2}{2} \left\|\hat{T}(\hat{x}) - T(\hat{x})\right\|^2 + \|\nabla U_{2,\hat{x}}(\hat{T}(\hat{x}))(\hat{T}(\hat{x}) - T(\hat{x}))\| \|\hat{T}(\hat{x}) - x\| \right) \|\hat{T}(\hat{x}) - x\|$$

$$\leq \left(\frac{L_2}{2} \delta^2 + \|\nabla U_{2,\hat{x}}(\hat{T}(\hat{x}))\| \right) \|\hat{T}(\hat{x}) - x\| \leq \delta \left(\frac{L_2}{2} + \|\nabla U_{2,\hat{x}}(\hat{T}(\hat{x}))\|\right) \|\hat{T}(\hat{x}) - x\|,$$

which completes the proof. \qed

Now that we have established all of the prerequisite results, we may state and prove our main theorem concerning how to instantiate our method for the unconstrained Euclidean case, for $p = 2$.

**Theorem C.4.** Let $\varepsilon > 0$, and let $\mathcal{X} = \mathbb{R}^n$. Let $F : \mathcal{X} \to \mathbb{R}^n$ be an operator that is second-order $L_2$-smooth with respect to the $L_2$ norm $\|\cdot\|$. Let $\omega(\cdot, \cdot)$ denote the Bregman divergence of a function that is strongly convex with respect to the same norm $\|\cdot\|$. Furthermore, suppose we are given $\Gamma, \Lambda, \Pi, \mu$ such that, for all iterates $x_i, x_{i+\frac{1}{2}}$ throughout the execution of Algorithm 3, $\|\nabla U_{2,\hat{x}, r}(x_{i+\frac{1}{2}})\| \leq \Gamma$, $\|x_{i+\frac{1}{2}} - x_{i+1}\| \leq \Lambda$, $\|F(x_i)\| \leq \Pi$, and $x^\top \nabla F(x_i) x \geq \mu$ for all $x \in \mathcal{X}$. In addition, let $\delta = \frac{1}{2(2L_2 + 1)}$. Then, Algorithm 3 whereby APPROX-VI-SOLVE is instantiated by APPROX-SO-VI-SOLVE (Algorithm 3), returns $\hat{x}$ such that for all $x \in \mathcal{X}$,

$$\langle F(x), \hat{x} - x \rangle \leq \varepsilon,$$

in at most

$$\frac{16}{15} \left(\frac{2L_2}{2L_2^2 + 3}\right)^{2/3} \frac{\omega(x, x_0)}{\varepsilon^{2/3}}$$
calls to \textsc{Approx-SO-VI-Solve} (Algorithm \ref{alg:sovi}), each of which requires a single Schur decomposition and \(O\left(\log\left(\frac{(L_2+\Gamma)\Pi}{\mu \varepsilon}\right)\right)\) calls to a linear system solver in a quasi-upper-triangular system, for a total computational cost of \(n^\omega + \tilde{O}(n)\), where \(\omega \approx 2.3728\) is the matrix multiplication constant.

Proof. Invoking Theorem \ref{thm:main} with our choice of \(\delta = \frac{\varepsilon}{2\lambda(L_2+\Gamma)}\) implies that, for any iteration \(i\), the output of Algorithm \ref{alg:sovi} is such that
\[
\langle U_2, x_i \rangle, \tilde{T}(x_i) - x_i \rangle \leq \frac{\varepsilon}{2}, \quad \forall x \in \mathbb{R}^n.
\]
The rest follows from Theorem \ref{thm:accuracy}.

Furthermore, the total number of calls to a linear system solver in a quasi-upper-triangular system is bounded \(O\left(\log\left(\frac{(L_2+\Gamma)\Pi}{\mu \varepsilon}\right)\right)\), which follows from Theorem \ref{thm:main} combined with our choice of \(\delta\). \qed