Quasi-regular Dirichlet forms and the obstacle problem for elliptic equations with measure data

by

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Abstract. We consider the obstacle problem with irregular barriers for semilinear elliptic equations involving measure data and an operator corresponding to a general quasi-regular Dirichlet form. We prove existence and uniqueness of a solution as well as its representation as an envelope of supersolutions to some related partial differential equation. We also prove regularity results for the solution and the Lewy–Stampacchia inequality.

1. Introduction. Let \( E \) be a Luzin space (i.e. the image of a Polish space under an injective continuous mapping), \( m \) be a \( \sigma \)-finite positive measure on \( \mathcal{B}(E) \) and let \((L, D(L))\) be a Dirichlet operator associated with some quasi-regular (possibly non-symmetric) Dirichlet form \((\mathcal{E}, D[\mathcal{E}])\) on \( L^2(E; m) \). In the present paper, we investigate the obstacle problem of the form

\[
\begin{aligned}
-Lu &\leq f(\cdot, u) + \mu &\text{on } \{u > h_1\}, \\
-Lu &\geq f(\cdot, u) + \mu &\text{on } \{u < h_2\}, \\
h_1 &\leq u \leq h_2 &\text{m-a.e.,}
\end{aligned}
\]

(1.1)

where \( \mu \) is a smooth measure (if \( \mu \) is bounded this means that \( \mu \) charges no \( \mathcal{E} \)-exceptional sets; for the general definition see Section 2), and \( f : E \times \mathbb{R} \rightarrow \mathbb{R} \) and \( h_1, h_2 \) are measurable functions on \( E \) such that \( h_1 \leq h_2 \) \( m \)-a.e. We also consider the one-sided problem, i.e. we allow \( h_1 \equiv -\infty \) or \( h_2 \equiv +\infty \).

The class of operators associated with quasi-regular Dirichlet forms is quite wide. It includes local operators in divergence form, \( \alpha \)-Laplacian type operators, Ornstein–Uhlenbeck type operators in Hilbert spaces and others (see, e.g., [26, 30, 37, 40, 46] for concrete examples). We think that the fact

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that nonlocal operators fit into our general framework is of special interest, because problem (1.1) with nonlocal operators and measure data is considered here for the first time.

For an overview of numerous applications of obstacle problem for elliptic and parabolic PDEs we refer the reader to [57] and the references therein. In recent years nonlocal models attracted quite a lot of interest because it was observed that equations with nonlocal Lévy type operators describe some phenomena better than local equations (see, e.g., [19, 29]). The applications we have in mind include population biology models, models of mathematical finance involving jump processes and some interacting particles models with repulsive/attractive interaction potentials. In all those models the obstacle problem with rough data naturally appears. In population models it is well known (see [23, 24]) that solutions of steady-state predator-pray models with sufficiently large/small appropriate parameters behave like solutions of certain free boundary problems which may be equivalently formulated as an obstacle problem of the form (1.1) with a merely measurable barrier. In these models, \( L \) describes the dispersal of animals, \( f \) describes the growth of the population and \( \mu \) is the harvesting distribution. In the theory of option pricing, the fair price of some derivative contracts is of the form

\[
(1.2) \quad u(x) = \sup_{\tau \geq 0} \inf_{\sigma \geq 0} E_x \left( \int_0^{\tau \wedge \sigma} f(\cdot, u)(X_r) \, dr \right. \\
+ \left. \int_0^{\tau \wedge \sigma} dA^\mu_r + h_1(X_\tau)1_{\{\tau < \sigma\}} + h_2(X_\sigma)1_{\{\sigma \leq \tau\}} \right),
\]

where \( X \) is a process with generator \( L \) starting from \( x \) at time 0. The process \( X \) describes the evolution of stock prices, \( f \) generates the nonlinear expectation (see [25]), the additive functional \( A^\mu \) (generated by a smooth measure \( \mu \)) is the running cost or profit, and \( h_1(X), h_2(X) \) are pay-off processes (such a situation appears for instance when considering American options or Israeli options). Since the 70s, connections of value functions of the form (1.2) with obstacle problems with one and two barriers have been intensively studied in the literature (see, e.g., [6, 28, 64]). It is worth noting here that in some applications (for instance to digital options, see [28]) the functions \( h_1, h_2 \) are assumed to be merely measurable. In the interacting particles models, for a given Green function \( G \) and a positive function \( W_a \), we are looking for a local minimizer for the interacting energy

\[
(1.3) \quad E[\gamma] = \frac{1}{2} \int_E \int_E (G(x, y) + W_a(x - y)) \gamma(dy) \gamma(dx)
\]

in the class of probability measures \( \gamma \) on \( E \). It is known (see, e.g., [16] for the case of Riesz’s potentials) that such a minimizer is the second component
In the paper, we assume that this condition is equivalent to the existence of a strictly positive $g \in L^1(E;m)$ (see (1.6)) of the local solution to (1.1) with the operator $L$ associated with the Green function $G$ and $\mu = -LW_a \gamma$.

In this paper, we impose very weak conditions on $E$ and the data $\mu, f, h_1, h_2$. To formulate them, let us first recall that the operator $(L, D(L))$ and its adjoint operator $(\hat{L}, D(\hat{L}))$ are generators of $C_0$-semigroups of contractions $\{T_t, t \geq 0\}, \{\hat{T}_t, t \geq 0\}$ on $L^p(E;m)$ for every $p \geq 1$. Let $\{G_\alpha, \alpha > 0\}$ (resp. $\{\hat{G}_\alpha, \alpha > 0\}$) be the resolvent of $\{T_t, t \geq 0\}$ (resp. $\{\hat{T}_t, t \geq 0\}$). For positive $f \in L^p(E;m)$ we set

$$Gf = \sup_{n \geq 1} G_{1/n} f, \quad \hat{G}f = \sup_{n \geq 1} \hat{G}_{1/n} f.$$ 

In the paper, we assume that $E$ satisfies the strong sector condition (see Section 2) and it is transient, i.e. $Gf$ is finite m-a.e. for some strictly positive $f \in L^1(E;m)$ (and hence for every $f \in L^1(E;m)$). It is known (see [26]) that this condition is equivalent to the existence of a strictly positive $g \in L^1(E;m)$ such that

$$\int_E |u| g \, dm \leq E(u,u)^{1/2}, \quad u \in D[E].$$

As for $\mu$, we assume that it belongs to the class

$$\mathcal{M}_0 = \{\mu : |\mu| \text{ is smooth and } \hat{G} \phi \cdot \mu \in \mathcal{M}_{0,b} \text{ for some } \phi \in L^1(E;m) \text{ such that } \phi > 0 \text{ m-a.e.} \},$$

considered in [40]. Here $|\mu|$ denotes the variation of $\mu$, and $\mathcal{M}_{0,b}$ is the space of all finite smooth signed measures on $\mathcal{B}(E)$. Of course, the class $\mathcal{M}_0$ depends on the structure of $E$, but by [50 Corollary 1.3.6], we always have $\mathcal{M}_{0,b} \subset \mathcal{M}_0$. In general, the inclusion is strict. For instance, if $d \geq 3$ and $L = \Delta^{1/2}$ with $\alpha \in (0,2]$ on a bounded smooth domain $D \subset \mathbb{R}^d$, then by [43] there exist $c_1, c_2 > 0$ such that

$$c_2 \delta^{\alpha/2}(x) \leq G_1(x) \leq c_2 \delta^{\alpha/2}(x), \quad x \in D,$$

where $\delta$ is the distance to the boundary of $D$. Consequently, in that case $\mathcal{M}_0$ includes Radon measures of infinite total variation. In particular, we have $L^1(D;\delta^{\alpha/2}(x) \, dx) \subset \mathcal{M}_0$. In recent years elliptic equations involving the Laplace operator and $L^1(D;\delta(x) \, dx)$ data were considered by many authors (see, e.g., [54, 55] and references therein). Note that it may also happen that $\mathcal{M}_0$ includes nowhere Radon measures (see Example 3.3). If the resolvent of the operator $(L, D(L))$ is strongly Feller (i.e. $G_\alpha(\mathcal{B}_b(E)) \subset C_b(E)$), then $(L, D(L))$ has the Green function, i.e. there exists $r \in \mathcal{B}^+(E) \times \mathcal{B}^+(E)$ such that

$$Gf = \int_E r(\cdot, y) f(y) \, m(dy), \quad f \in L^1(E;m),$$
and moreover
\[ M_0 \supset \{ \mu \text{ a Borel measure on } E : \int_E r(x, y) |\mu|(dy) < \infty, \, x \in E \}. \]

The inclusion above can be replaced by equality if we additionally assume that \( \mu \) is smooth and we replace “for every” by “quasi every” (with respect to the capacity associated with \( E \)). The characterization of \( M_0 \) in this spirit is also possible for a general operator \((L, D(L))\) but to state it requires the introduction of the notion of positive additive functional (see Section 3).

The function \( f : E \times \mathbb{R} \to \mathbb{R} \) is assumed to be continuous and nonincreasing with respect to the second variable. We also assume that \( f(\cdot, 0) \cdot m \in M_0 \) and for every \( y \in \mathbb{R}, \, f(\cdot, y) \) is quasi-integrable (a weaker condition than integrability, see Section 2). These assumptions on \( f \) were used previously in many papers devoted to linear and nonlinear equations involving measure data and local operators but with \( f(\cdot, y) \in L^1(E; m) \) (see, e.g., [3, 10]). Semilinear elliptic equations with quasi-integrable data and local operators were considered in [49]. Equations with quasi-integrable data and nonlocal operators were considered for the first time in [37] (see also [40]).

In the paper we do not impose any regularity assumption on the barriers \( h_1, h_2 \). Therefore to guarantee the existence of a solution we have to assume that they satisfy some kind of separation condition. Roughly speaking, our condition says (see Section 3) that between the barriers one can find some function \( v \) such that \( v \) is a difference of two natural potentials and \( f(\cdot, v) \cdot m \in M_0 \). For instance, this condition is satisfied if \( h_1 \leq \varphi(w) \leq h_2 \) and \( f(\cdot, \varphi(w)) \in L^1(E; m) \) for some \( w \in D(L) \) and \( \varphi \) being the difference of two convex functions and such that \( \varphi(0) = 0 \).

Since our data are irregular, the classical approach to (1.1) via variational inequalities (see [2, 12, 61]) does not apply (see, however, [13] for the case \( L = \Delta \)). In the present paper by a solution to (1.1) we understand a solution of the complementary system (see [2, 34]) associated with (1.1). Roughly speaking it is a pair \((u, \nu)\) consisting of a quasi-continuous function \( u \) on \( E \) and a measure \( \nu \in M_0 \) such that

\[
\begin{cases}
-Lu = f(\cdot, u) + \mu + \nu, \\
h_1 \leq u \leq h_2 \quad m\text{-a.e.}, \\
\int_E (u - h_1) \, d\nu^+ = \int_E (h_2 - u) \, d\nu^- = 0,
\end{cases}
\tag{1.7}
\]

where \( \nu^+, \nu^- \) denote the positive and negative parts in the Jordan decomposition of \( \nu \).

The obstacle problem with irregular data is the subject of intensive study. Most of the available results are formulated in the language of differential inclusions (when \( L \) is a general accretive or completely accretive operator) or in the language of entropy or renormalized solutions (when \( L \) is a nonlinear
Leray–Lions type operator; when \( L \) is a linear Leray–Lions type operator, one can use an equivalent notion of Stampacchia’s solution by duality).

The paper by Brezis and Strauss [14] is the first paper devoted to problems of type (1.1) with \( L^1 \) data. More precisely, in [14] the differential inclusions of the form

\[
-\lambda u - Au + \beta(x,u) \ni \mu
\]

are considered. Here \( \lambda \geq 0, \mu \in L^1(E;m), A \) is an operator with sub-Markovian resolvent such that \( D(A) \subset L^1(E;m) \), and for fixed \( x \in \mathbb{R} \), \( \beta(x,\cdot) \) is a maximal monotone graph on \( \mathbb{R} \times \mathbb{R} \). Note that if we define \( \beta \) by

\[
D(\beta(x,\cdot)) = [h_1(x),h_2(x)], \quad \beta(x,y) = \begin{cases} [0,\infty), & y = h_1(x), \\ \{0\}, & h_1(x) < y < h_2(x), \\ (-\infty,0], & y = h_2(x), \end{cases}
\]

then (1.8) reduces to the obstacle problem with \( L = \lambda + A \) and with barriers \( h_1 \) and \( h_2 \). In fact, in [14] equation (1.8) with \( \lambda = 0 \) and \( \beta \) not depending on \( x \) is considered, so the results of [14] apply to obstacle problems with constant barriers. As for \( A \), in [14] it is assumed that

\[
\|u\|_{L^1} \leq c\|Au\|_{L^1}, \quad u \in D(A).
\]

The above conditions guarantee that the solution \( u \) to (1.8) belongs to \( D(A) \subset L^1(E;m) \). Consequently, if we set

\[
w := \mu + \lambda u + Au,
\]

then \( w \in L^1(E;m) \) (of course \( w \in \beta(u) \) a.e.). By the monotonicity of \( \beta \), for every function \( v \) on \( E \) such that \( h_1 \leq v \leq h_2 \), we have

\[
\int_E (u-v)w \, dm = \int_E (u-v)(w-0) \, dm \leq 0
\]

since \( 0 \in \beta(v) \) a.e. In other words, the pair \( (u,w \cdot m) \) is a solution to (1.7) with \( L = \lambda + A \).

If \( \beta \) depends on \( x \), then depending on the regularity of \( \beta \) with respect to \( x \), one can consider the so-called strong or generalized solutions to (1.8). Hence, in the case where \( \beta \) is given by (1.9), the concept of solution depends on the regularity of barriers (see [62, 63]). Roughly speaking, strong solution corresponds to the case when the reaction measure \( \nu \) (or equivalently \( w \)) is absolutely continuous with respect to \( m \). Generalized solutions to (1.8) with \( \mu \in L^1(E;m) \) were considered in [5, 62]. In [62] problem (1.8) with a linear Leray–Lions type operator \( A \) is considered. It is shown there that in general \( w \) is a measure and for every function \( v \) on \( E \) such that \( h_1 \leq v \leq h_2 \),

\[
\int_E (u-v) \, dw \leq 0.
\]
Therefore also in case \( \beta \) depends on \( x \) problem (1.8) can be rewritten in the form (1.7) (see also [2, Theorem 3.2]).

The obstacle problem of the form (1.1) with a nonlinear Leray–Lions type operator \( L \) and \( \mu \in L^1(E;m) \) was considered in [9, 11]. In both papers the problem is studied in the setting of entropy solutions introduced in [3] (for a closely related notion of renormalized solution see [22]).

To our knowledge, first results concerning (1.1) with “true” measure data were obtained in [21] by using Stampacchia’s approach by duality (see also [20]). In [21] the obstacle problem with one lower barrier \( h_1 \) (i.e. \( h_2 \equiv +\infty \)) is considered and it is assumed that \( L \) is a uniformly elliptic divergence form operator. The results of [21] were extended in [45] to the case of a nonlinear Leray–Lions type operator \( L \). In [45] the setting of renormalized solutions is used.

Quite recently first papers devoted to semilinear elliptic equations involving measure data and nonlocal operators (mostly fractional Laplacian) appeared (see, e.g., [1, 17, 33, 44]). General results on existence, uniqueness and regularity of solutions of such equations with the operator \( L \) corresponding to a Dirichlet form were proved in [37, 40] (see also [39]) in case \( \mu \) is a smooth measure, and in [36] for a general Borel measure \( \mu \). However, to our knowledge, there are no results on the obstacle problem (1.1) with true measure data and a nonlocal operator \( L \). Therefore all the results of the present paper are new in case \( L \) is nonlocal and \( \mu \) is a “true” measure. It is worth mentioning, however, that they are new even if \( \mu \in L^1(E;m) \), because as compared with papers devoted to problem (1.8) we also consider the case \( \lambda = 0 \) and we do not assume (1.10). Also note that in general, our solutions are not even locally integrable, so need not satisfy the condition

\[
\int_E (u - k)^+ \, dm < \infty \quad \text{for some } k > 0,
\]

which is the minimal requirement on \( u \) when one investigates (1.8) in the setting of completely accretive operators (see [4]).

In general, under weak assumptions on \( f, \mu \) described above the solution \( u \) to (1.1) may be very irregular. Therefore the problem of making sense of the first equation in (1.7) arises. Following [37, 40] we address it by using stochastic analysis methods. Namely, by a solution of the first equation in (1.7) we mean a function \( u : E \to \mathbb{R} \) satisfying for quasi-every (q.e. for short) \( x \in E \) the following generalized Feynman–Kac formula:

\[
(1.12) \quad u(x) = E_x \left[ \int_0^\zeta f(X_t, u(X_t)) \, dt + E_x \int_0^\zeta dA^\mu_t + E_x \int_0^\zeta dA^n_t \right].
\]

Here \( M = (X, P_x) \) is a special standard process with life-time \( \zeta \) associated with the form \( (\mathcal{E}, D[\mathcal{E}]) \), \( E_x \) is the expectation with respect to \( P_x \), and \( A^\mu, A^n \)
are continuous additive functionals of $\mathcal{M}$ in the Revuz correspondence with $\mu$ and $\nu$, respectively.

It is worth remarking that in the important case where $\mu, \nu \in \mathcal{M}_{0,b}$, the probabilistic definition (1.12) can be rephrased in purely analytical terms. Namely, under these assumptions on $\mu, \nu$ (1.12) is equivalent to saying that for any $\phi \in L^1(E; m)$ with $\| \hat{G}\phi \|_\infty < \infty$,

$$
(u, \phi) = (f(\cdot, u), \hat{G}\phi) + \int_E \hat{G}\phi \; d\mu + \int_E \hat{G}\phi \; d\nu
$$

(see [40]). Note that (1.13) is a generalization of Stampacchia’s definition by duality introduced in [60] for solutions of uniformly elliptic PDEs with measure data. Another equivalent definition is given in [39], where it is shown that (1.12) is satisfied if and only if $u$ is a renormalized solution to the first equation of (1.7), i.e. $u$ is quasi-continuous, $f(\cdot, u) \in L^1(E; m)$, $T_k(u) := (u \wedge k) \vee (-k)$ belongs to the extended Dirichlet space $D_e[E]$ (see Section 2 for the definition) and

$$
\mathcal{E}(T_k u, v) = \int_E f(\cdot, u) v \; dm + \int_E v \; d\mu + \int_E v \; d\nu + \int_E v \; d\nu_k
$$

for some sequence $\{\nu_k\}$ of bounded smooth measures on $E$ such that $\|\nu_k\| \to 0$ as $k \to \infty$, where $\| \cdot \|$ stands for the total variation norm on the space of signed Borel measures on $E$. The concept of renormalized solutions to elliptic equations with measure data and local operators of Leray–Lions type was introduced in [22].

Our main result on existence and uniqueness of solutions of the complementary system (1.7) is first proved for one reflecting barrier in Section 3 and then for two barriers in Section 4. It is worth mentioning that in both cases we give necessary and sufficient conditions on barriers $h_1, h_2$ under which there exists a solution $u$ of (1.7) with $f, \mu$ satisfying our assumptions. We also prove that $u$ is an envelope of supersolutions of some partial differential equation related to (1.7). More precisely, we show that

$$
u^{-} = \text{quasi-essinf}\{v \geq h_1 \text{ a.e., } v \text{ is a supersolution of PDE}(f + d\mu - d\nu^{-})\},$$

where as before $\nu^{-}$ denotes the negative part of the reaction measure. A result similar to (1.15) was proved in [38] for evolution obstacle problems involving divergence form operators.

In case $\mu \in \mathcal{M}_{0,b}$, $f(\cdot, 0) \in L^1(E; m)$ and the barriers satisfy some additional regularity condition, we show that $\nu \in \mathcal{M}_{0,b}$. When combined with the regularity results proved in [37, 40], this implies that for every $k \geq 0$ the truncation $T_k(u)$ of $u$ at level $k$ belongs to the extended Dirichlet space $D_e[E]$ and

$$
\mathcal{E}(T_k(u), T_k(u)) \leq 2k(\|\mu\| + \|\nu\| + \|f(\cdot, 0)\|_{L^1(E; m)}).
$$
Moreover, we show that if $u$ is a solution to \eqref{1.1} and $\mu \in D_e'[\mathcal{E}]$, where $D_e'[\mathcal{E}]$ is the dual of $D_e[\mathcal{E}]$, and moreover $f(\cdot, u) \in D_e'[\mathcal{E}]$ and there exists $v = R\lambda$ for some $\lambda \in D_e'[\mathcal{E}]$ (in the case of $h_2 \equiv \infty$ it is enough to assume that $v \in D_e[\mathcal{E}]$) such that $h_1 \leq v \leq h_2$, then $u \in D_e[\mathcal{E}]$, $\nu \in D_e'[\mathcal{E}]$ and $(u, \nu)$ is the unique pair in $D_e[\mathcal{E}] \times D_e'[\mathcal{E}]$ such that

\begin{align}
E(u, \eta) &= \int_E f(\cdot, u) \eta \, dm + \int_E \eta \, d\mu + \int_E \eta \, d\nu, \quad \eta \in D_e[\mathcal{E}], \tag{1.16} \\
\int_E (u - h_1) \, d\nu^+ &= \int_E (h_2 - u) \, d\nu^- = 0, \quad h_1 \leq u \leq h_2 \quad \text{q.e.} \tag{1.17}
\end{align}

This definition of a solution is equivalent to the variational formulation, i.e. finding $u \in D_e[\mathcal{E}]$ such that $\psi_1 \leq u \leq \psi_2$ $m$-a.e. and

\begin{align}
E(u, u - \eta) \leq \int_E f(\cdot, u)(u - \eta) \, dm + \int_E (u - \eta) \, d\mu, \quad \eta \in D_e[\mathcal{E}], \psi_1 \leq \eta \leq \psi_2. \tag{1.18}
\end{align}

It is enough to put $u - \eta$ as a test function in \eqref{1.16} and apply \eqref{1.17}. Note here that in general it is not true that $L^2(E; m)$ is a subset of $D_e'[\mathcal{E}]$.

In Section \ref{Section 5}, we prove a Lewy–Stampacchia type inequality, which is known to be useful in the study of regularity of solutions of \eqref{1.7}. If one of the barriers, say $h_1$, is a difference of two natural potentials, then

$$
\nu^+ \leq \mathbf{1}_{\{u = h_1\}}(f(\cdot, h_1) + \mu + Lh_1)^- \cdot m.
$$

Note that even in the case of local operators there are only a few results of this type for two-sided obstacle problems (see \cite{Bouchut2009, Bouchut2011, Cassier2013}). We also prove some stability result which in particular implies that probabilistic solutions to \eqref{1.1} are pointwise limits of analytic solutions.

\section{Preliminaries.} For the convenience of the reader and to fix notation, in this section we provide some basic information on Dirichlet spaces and associated Markov processes. For more details we refer the reader to monographs \cite{Bouchut2009, Cassier2013} (quasi-regular Dirichlet forms) and \cite{Cassier2011, Cassier2013} (regular Dirichlet forms).

In the whole paper $E$ is a Luzin space and $m$ is a positive $\sigma$-finite measure on the $\sigma$-field $\mathcal{B}(E)$ of Borel subsets of $E$.

Let $D[\mathcal{E}]$ be a dense linear subspace of $L^2(E, m)$ and $\mathcal{E} : D[\mathcal{E}] \times D[\mathcal{E}] \to \mathbb{R}$ a bilinear form.

We say that $(\mathcal{E}, D[\mathcal{E}])$ is positive if $\mathcal{E}(u, u) \geq 0$ for $u \in D[\mathcal{E}]$. A positive definite form $(\mathcal{E}, D[\mathcal{E}])$ is called a coercive closed form if

(a) $(\tilde{\mathcal{E}}, D[\mathcal{E}])$ is a symmetric closed form on $L^2(E; m)$, where $\tilde{\mathcal{E}}$ denotes the symmetric part of $\mathcal{E}$, i.e. $\tilde{\mathcal{E}}(u, v) = \frac{1}{2}(\mathcal{E}(u, v) + \mathcal{E}(v, u))$ for $u, v \in D[\mathcal{E}]$, 

\begin{align}
\int_E (u - \eta) \, d\nu^+ &= \int_E (h_2 - u) \, d\nu^- = 0, \quad h_1 \leq u \leq h_2 \quad \text{q.e.} \
\int_E f(\cdot, u)(u - \eta) \, dm + \int_E (u - \eta) \, d\mu, \quad \eta \in D_e[\mathcal{E}], \psi_1 \leq \eta \leq \psi_2.
\end{align}
(b) \((\mathcal{E}, D[\mathcal{E}])\) satisfies the weak sector condition, i.e. there exists \(K > 0\) such that
\[
|\mathcal{E}_1(u, v)| \leq K \mathcal{E}_1(u, u)^{1/2} \mathcal{E}_1(v, v)^{1/2}, \quad u, v \in D[\mathcal{E}].
\]
Here and henceforth,
\[
\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v), \quad u, v \in D[\mathcal{E}]
\]
for \(\alpha > 0\). A form \((\mathcal{E}, D[\mathcal{E}])\) is said to satisfy the strong sector condition if there is \(K > 0\) such that
\[
|\mathcal{E}(u, v)| \leq K \mathcal{E}(u, u)^{1/2} \mathcal{E}(v, v)^{1/2}, \quad u, v \in D[\mathcal{E}].
\]
Note that symmetric forms satisfy the strong sector condition with \(K = 1\) by Schwarz’s inequality.

We say that \((\mathcal{E}, D[\mathcal{E}])\) is a Dirichlet form if it is a closed coercive form and for all \(u \in D[\mathcal{E}]\) we have \(u^+ \land 1 \in D[\mathcal{E}]\) and
\[
\mathcal{E}(u + u^+ \land 1, u - u^+ \land 1) \geq 0, \quad \mathcal{E}(u - u^+ \land 1, u + u^+ \land 1) \geq 0.
\]

For a Dirichlet form \((\mathcal{E}, D[\mathcal{E}])\) there exists a unique operator \((L, D(L))\) on \(L^2(E; m)\) (sometimes called the Dirichlet operator) such that
\[
D(L) \subset D[\mathcal{E}], \quad \mathcal{E}(u, v) = (-Lu, v), \quad u \in D(A), v \in D[\mathcal{E}].
\]

By \(\{G_\alpha\}_{\alpha > 0}\) (resp. \(\{T_t\}_{t > 0}\)) we will denote the strongly continuous contraction resolvent (resp. semigroup) generated by \((L, D(L))\) (see [46, Chapter I]).

Given \(F \in \mathcal{B}(E)\) we set \(D[\mathcal{E}]|_F = \{u \in D[\mathcal{E}] : u = 0\} \text{ on } F^c \text{ m-a.e.}\). An increasing sequence \(\{F_k\}\) of closed subsets of \(E\) is called an \(\mathcal{E}\)-nest if \(\bigcup_{k \geq 1} D[\mathcal{E}]|_{F_k}\) is dense in \(D[\mathcal{E}]\) with respect to the norm \(\mathcal{E}_1^{1/2}\). A set \(N\) is an \(\mathcal{E}\)-exceptional set if \(N^c \subset \bigcap_{k \geq 1} F_k^c\) for some \(\mathcal{E}\)-nest \(\{F_k\}\). We say that a property in \(E\) holds q.e. if it holds outside some exceptional set. By [46, Theorem III.2.11] (see also [46, Exercise III.2.3]), every Borel \(\mathcal{E}\)-exceptional set is of \(m\)-measure zero. Consequently, if some property holds q.e., it holds \(m\)-a.e. For equivalent definitions of \(\mathcal{E}\)-nest and \(\mathcal{E}\)-exceptional set, expressed in terms of some capacity associated with \((\mathcal{E}, D[\mathcal{E}])\), we refer the reader to [46, Section III.2].

For a given nest \(\{F_k\}\) we set
\[
C(\{F_k\}) = \{f : E \to \mathbb{R} : f|_{F_k} \text{ is continuous for every } k \geq 1\}.
\]

Similarly we define sets \(L(\{F_k\}), U(\{F_k\})\) replacing in the above definition the word “continuous” by “lower semicontinuous” (l.s.c.) and “upper semicontinuous” (u.s.c.), respectively. We say that a function \(u\) on \(E\) is \(\mathcal{E}\)-quasi-continuous (resp. \(\mathcal{E}\)-l.s.c., \(\mathcal{E}\)-u.s.c.) if there exists an \(\mathcal{E}\)-nest \(\{F_k\}\) such that \(u \in C(\{F_k\})\) (resp. \(u \in L(\{F_k\}), u \in U(\{F_k\})\) ).
A Dirichlet form \((\mathcal{E}, D[\mathcal{E}])\) on \(L^2(E; m)\) is called quasi-regular if
(a) there exists an \(\mathcal{E}\)-nest \(\{F_k\}\) consisting of compact sets,
(b) there exists an \(\mathcal{E}_{1/2}\)-dense subset of \(D[\mathcal{E}]\) whose elements have \(\mathcal{E}\)-quasi-continuous \(m\)-versions,
(c) there exist a sequence \(\{u_n\}\subset D[\mathcal{E}]\) of \(\mathcal{E}\)-quasi-continuous functions and an \(\mathcal{E}\)-exceptional set \(N\subset E\) such that \(\{u_n\}\) separates the points of \(E\setminus N\).

Let \((\mathcal{E}, D[\mathcal{E}])\) be a quasi-regular Dirichlet form on \(L^2(E; m)\). Adjoin \(\Delta\) as an extra point to \(E\) and set \(E_\Delta = E \cup \Delta\). It is known (see [46, Chapter IV]) that there exists an \(m\)-tight special standard process \(\mathbf{M} = (\Omega, \mathcal{F}, \{X_t\}_{t \geq 0}, \{P_x\}_{x \in E_\Delta})\) with life time \(\zeta\) properly associated with the form \((\mathcal{E}, D[\mathcal{E}])\), i.e. for every \(t > 0\) and \(f \in \mathcal{B}_b(E) \cap L^2(E; m)\),
\[
T_t f(x) = E_x f(X_t)
\]
for \(m\)-a.e. \(x \in E\) and \(x \mapsto E_x f(X_t)\) is \(\mathcal{E}\)-quasi-continuous. Note that \(X_t = \Delta\) for \(t \geq \zeta\) and that throughout we adopt the convention that each function \(f\) on \(E\) is extended to \(E_\Delta\) by putting \(f(\Delta) = 0\). We denote by \(\mathcal{T}\) the set of all stopping times with respect to \(\mathcal{F}\). In particular \(\zeta \in \mathcal{T}\).

We say that a positive measure \(\mu\) on \(\mathcal{B}(E)\) is \(\mathcal{E}\)-smooth if \(\mu(N) = 0\) for every \(\mathcal{E}\)-exceptional set \(N \in \mathcal{B}(E)\) and there exists an \(\mathcal{E}\)-nest \(\{F_k\}\) of compact subsets of \(E\) such that \(\mu(F_k) < \infty\) for \(k \geq 1\). The set of all \(\mathcal{E}\)-smooth measures on \(\mathcal{B}(E)\) will be denoted by \(S\). We denote by \(\mathcal{M}_{0, b}\) the set of bounded Borel measures \(\mu\) on \(E\) such that \(|\mu| \in S\).

In this paper, we frequently use the notion of additive functional (AF for short) of \(\mathbf{M}\) (for the definition see [26, Section 5.1]). We say that an AF \(A\) of \(\mathbf{M}\) is positive (resp. continuous) if \(A_t \geq 0\) for \(t \geq 0\) \(P_x\)-a.s. (resp. \(t \mapsto A_t\) is continuous on \([0, \infty)\) \(P_x\)-a.s.) for q.e. \(x \in E\). We say that a process \(A\) is a martingale AF of \(\mathbf{M}\) if \(A\) is an AF of \(\mathbf{M}\) and it is a martingale with respect to \(\mathcal{F}\) under the measure \(P_x\) for q.e. \(x \in E\).

It is known (see [46, Theorem VI.2.4]) that there is a one-to-one correspondence between \(\mathcal{E}\)-smooth measures and positive continuous additive functionals (PCAFs) of \(\mathbf{M}\). This correspondence, called the Revuz correspondence, can be expressed as
\[
\lim_{t \searrow 0} E_m\left(\frac{1}{t} \int_0^t f(X_s) \, dA_s\right) = \int_E f \, d\mu, \quad f \in \mathcal{B}^+(E),
\]
where \(E_m\) denotes the expectation with respect to the measure \(P_m(\cdot) = \int_E P_x(\cdot) \, m(dx)\). For an \(\mathcal{E}\)-smooth measure \(\mu\) we denote by \(A^\mu\) the unique PCAF of \(\mathbf{M}\) associated with \(\mu\). We also set, for \(\mu \in S\),
\[
R_\mu(x) = E_x \int_0^\zeta dA^\mu_x, \quad x \in E.
\]
We say that a form \((\mathcal{E}, D[\mathcal{E}])\) is transient if the associated semigroup \(\{T_t\}_{t > 0}\) is transient, i.e. \(G\phi\) is finite \(m\)-a.e. for every nonnegative \(\phi \in L^1(E; m)\). Equivalently (see [31 Corollary 3.5.34]), the form is transient if there exists a strictly positive \(g \in L^1(E; m)\) such that (1.4) is satisfied.

For a coercive closed form \((\mathcal{E}, D[\mathcal{E}])\) we define \(D_e[\mathcal{E}]\) as follows: \(D_e[\mathcal{E}]\) is the family of all functions \(u\) on \(E\) for which there exists an \(\mathcal{E}\)-Cauchy sequence (i.e. a Cauchy sequence \(\{u_n\} \subset D[\mathcal{E}]\) with respect to the norm generated by the inner product \(\hat{\mathcal{E}}\)) such that \(u_n \to u\) \(m\)-a.e. \(\{(u_n)\}\) is called an approximating sequence for \(u\). It is known that if \((\mathcal{E}, D[\mathcal{E}])\) is transient then for each fixed \(u \in D_e[\mathcal{E}]\) the limit of \(\{\mathcal{E}(u_n, u_n)\}\) is independent of the approximating sequence for \(u\). We set \(\mathcal{E}(u, u) = \lim_{n \to \infty} \mathcal{E}(u_n, u_n)\). By [26 Lemma 1.5.5], the pair \((\hat{\mathcal{E}}, D_e[\mathcal{E}])\) is a Hilbert space. By [41 Remark 2.2], each \(u \in D_e[\mathcal{E}]\) has an \(m\)-version which is quasi-continuous. From now on for given \(u \in D_e[\mathcal{E}]\) we always consider its quasi-continuous \(m\)-version.

We denote by \(\| \cdot \|_\mathcal{E}\) the norm generated by \(\hat{\mathcal{E}}\) and by \(\| \cdot \|_{\mathcal{E}'}\) the norm on its dual space. If \((\mathcal{E}, D[\mathcal{E}])\) is transient, then by [41 Lemma 2.1], for every \(\mu \in S\) there exists an \(\mathcal{E}\)-nest \(\{F_k\}\) such that \(1_{F_k} \cdot \mu \in D'_e[\mathcal{E}]\). If, in addition, \((\mathcal{E}, D[\mathcal{E}])\) satisfies the strong sector condition, then by [40 Lemma 2.4], if \(\mu \in D'_e[\mathcal{E}]\), then \(u := R\mu \in D_e[\mathcal{E}]\) and

\[
(2.2) \quad \mathcal{E}(u, \eta) = \int_E \eta \, d\mu, \quad \eta \in D_e[\mathcal{E}].
\]

A nonnegative measurable function \(u : E \to \mathbb{R}\) is called \(\mathcal{E}\)-excessive if \(T_t u \leq u\) for \(t \geq 0\) \(m\)-a.e. We say that \(u\) is an \(\mathcal{E}\)-natural potential if there exists a positive \(\mu \in M_0\) such that \(u = R\mu\) \(q\)-e. A function \(f : E \to \mathbb{R}\) is called \(\mathcal{E}\)-quasi-integrable (\(f \in qL^1(E; m)\) in notation) if \(A^{\|f\|_m}\) is a finite AF of \(M\). We say that \(f : E \to \mathbb{R}\) is locally \(\mathcal{E}\)-quasi-integrable if \(A^{\|f\|_m}\) is an AF of \(M\).

In [49] the notion of quasi-integrability was considered in the case of the Laplace operator. Our notion of quasi-integrability is more general (since it applies to a wider class of operators), but at the same time it is stronger than the notion introduced in [49] in the particular case of the Laplace operator. As a matter of fact, the quasi-integrability introduced in [49] coincides with the local quasi-integrability considered in the paper [35] devoted to elliptic systems involving the Laplace operator (see comments following [35 Remark 2.3]). Note also that for the Laplace operator the life-time \(\zeta\) of the associated process is predictable. Therefore the results of [35] suggest that in the case of operators associated with a quasi-regular Dirichlet form for which the life-time of the associated process is predictable (e.g. a regular Dirichlet form without killing part) the main results of our paper hold true if in their assumptions we replace quasi-integrability by local quasi-integrability.

Throughout, if there is no ambiguity, we drop the letter \(\mathcal{E}\) from the notation. For instance, instead of writing \(\mathcal{E}\)-quasi-continuous, \(\mathcal{E}\)-smooth, etc. we
simply write quasi-continuous, smooth, etc. By \( \rightarrow_p \) we denote convergence in probability \( P \). Finally, \( x^+ = \max(x, 0), x^- = \max(-x, 0) \).

3. One-sided obstacle problem. From now on, \((\mathcal{E}, D[\mathcal{E}])\) is a transient quasi-regular Dirichlet form that satisfies the strong sector condition, \( f : E \times \mathbb{R} \to \mathbb{R} \) and \( h, h_1, h_2 : E \to \mathbb{R} \) are measurable functions and \( \mu \) is a measure on \( \mathcal{B}(E) \) such that \( |\mu| \in S \).

Given \( \mu \in S \) we define the 0-order potential operator by putting

\[
R\mu(x) = E_x^t |A|^\mu \text{ d}A_t^\mu \quad \text{for q.e. } x \in E.
\]

for q.e. \( x \in E \). In the important case where \( \mu = f \cdot m \) for some \( f \in L^1(E; m) \) the AF associated with \( \mu \) has the form \( A^\mu_t = \int_0^t f(X_r) \text{ d}r \), \( t \geq 0 \) (see [13 Theorem A.3.5] and remarks following it). Consequently, with our convention that \( f(\Delta) = 0 \), in that case we have

\[
R\mu(x) = E_x^\infty |f(X_t)| \text{ d}t
\]

for q.e. \( x \in E \). From this and (2.1) it follows that

\[
R\mu = Gf \quad \text{m.a.e.}
\]

The above relation may be easily extended to \( f \in \mathcal{B}^+(E) \) by approximation.

We will need the following hypotheses:

(H1) \( y \mapsto f(x, y) \) is nonincreasing for every \( x \in E \),

(H2) \( y \mapsto f(x, y) \) is continuous for every \( x \in E \),

(H3) \( x \mapsto |f(x, y)| \in qL^1(E; m) \) for every \( y \in \mathbb{R} \),

(H4) \( R|f(\cdot, 0)| + R|\mu| < \infty \) m-a.e.,

(H5) there exists \( v : E \to \mathbb{R} \) such that \( v \) is a difference of natural potentials and m-a.e.,

\[
v \geq h, \quad Rf^-(\cdot, v) < \infty,
\]

(H6) there exists \( v : E \to \mathbb{R} \) such that \( v \) is a difference of natural potentials and m-a.e.,

\[
h_1 \leq v \leq h_2, \quad R|f(\cdot, v)| < \infty.
\]

Remark 3.1. (i) Let \( h \in \mathcal{B}(E) \). If \( C = \{ u \in D(\mathcal{E}) : u \geq h \} \neq \emptyset \), then there exists the smallest natural potential \( v \geq h \). This is a consequence of the Lax–Milgram theorem (see [46, Proposition III.1.5]). Therefore, if \( C \neq \emptyset \) and \( f^-(\cdot, v) \in L^1(E; m) \), then (H5) is satisfied.

(ii) In practice, an effective criterion ensuring (H6) is the following:

(a) \( f^+(\cdot, h_1), f^-(\cdot, h_2) \in L^1(E; m) \),

(b) there exist \( w \in D(L) \) and \( \varphi \) that is a difference of convex functions with \( \varphi(0) = 0 \) such that \( h_1 \leq \varphi(w) \leq h_2 \).
By the Tanaka–Meyer formula (see \cite[Theorem IV.70]{53}), if (b) is satisfied, then $\phi(w)$ is a difference of natural potentials.

Let us define the class $M_0$ by (1.5). In \cite{40} it is shown that $M_0$ can be equivalently defined as

\begin{equation}
M_0 = \{ \mu : |\mu| \in S, R|\mu| < \infty \text{ m-a.e.} \}.
\end{equation}

Note that from \cite[Corollary 1.3.6]{50} it follows immediately that $M_{0,b} \subset M_0$. Thus, in particular (H4)–(H6) are satisfied if $f(\cdot, 0), f^-(\cdot, v) \in L^1(E; m)$, $f(\cdot, v) \in L^1(E; m)$ and $\mu \in M_{0,b}$. In general, the inclusion is strict as the following examples show.

**Example 3.2.** Let $\alpha \in (0, 2)$, $d \geq 3$, and let $D \subset \mathbb{R}^d$ be an open bounded set with smooth boundary. Consider the form $(\mathcal{E}_D, D[\mathcal{E}_D])$ associated with the $\alpha$-Laplace operator $\Delta^\alpha_D$ on $D$ with zero Dirichlet boundary conditions (see, e.g., \cite[Section 6.3]{40}). The form $\mathcal{E}_D$ can be constructed as follows. We first consider the form $(\mathcal{E}, D[\mathcal{E}])$ associated with $\Delta^\alpha_D$ on $\mathbb{R}^d$, i.e.

$$
\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \hat{u}(x) \hat{v}(x) \psi(x) \, dx, \quad u, v \in D[\mathcal{E}],
$$

where $\psi(x) = |x|^\alpha/2$ for $x \in \mathbb{R}^d$ and $\hat{u}, \hat{v}$ denote the Fourier transforms of $u$ and $v$, and

$$
D[\mathcal{E}] = \left\{ w \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\hat{w}(x)|^2 \psi(x) \, dx < \infty \right\}
$$

(see \cite[Example 1.4.1]{26}). Next we set

$$
\mathcal{E}_D(u, v) = \mathcal{E}(u, v), \quad u, v \in D[\mathcal{E}_D] := \{ w \in D[\mathcal{E}] : w = 0 \text{ q.e. on } \mathbb{R}^d \setminus D \},
$$

that is, $(\mathcal{E}_D, D[\mathcal{E}_D])$ is the part of $(\mathcal{E}, D[\mathcal{E}])$ on $D$. By \cite[Theorems 4.4.3, 4.4.4]{26}, $(\mathcal{E}_D, D[\mathcal{E}_D])$ is again a regular symmetric transient Dirichlet form, so it generates a Dirichlet operator which we denote by $\Delta^\alpha_D$. Note that from the definition of $D[\mathcal{E}_D]$ it follows that in the case of the nonlocal operator $\Delta^\alpha_D$, zero Dirichlet boundary condition in fact means zero exterior condition. By \cite[Proposition 4.9]{43}, there exist constants $0 < c_1 < c_2$ depending only on $d, \alpha, D$ such that (1.6) is satisfied with $\delta(x) = \text{dist}(x, \partial D)$ and $G$ associated with $(\mathcal{E}_D, D[\mathcal{E}_D])$. Therefore from (1.5) it immediately follows that $L^1(D; \delta^\alpha/2(x) \, dx) \subset M_0$.

**Example 3.3.** Let $(\mathcal{E}, D[\mathcal{E}])$ be a regular symmetric Dirichlet form on $L^2(E; m)$ and let $\mu \in S$. Consider the form $(\mathcal{E}^\mu, D[\mathcal{E}^\mu])$, the perturbation of $(\mathcal{E}, D[\mathcal{E}])$ by $\mu$, which is defined by

$$
\mathcal{E}^\mu(u, v) = \mathcal{E}(u, v) + \int_E uv \, d\mu, \quad u, v \in D[\mathcal{E}^\mu] := \left\{ u \in D[\mathcal{E}] : \int_E |u|^2 \, d\mu < \infty \right\}.
$$
It is known (see [46, Section IV.4(c)] and [26, Section 6.1]) that \((\mathcal{E}^\mu, D[\mathcal{E}^\mu])\) is a quasi-regular Dirichlet form on \(L^2(E; m)\) and the 0-order potential operator \(R^\mu\) associated with \((\mathcal{E}^\mu, D[\mathcal{E}^\mu])\) has the form

\[
R^\mu \nu(x) = E_x \int_0^\infty e^{-A^\mu_t} \, dA^\nu_t
\]

for \(\nu \in S\) (here \(A^\mu, A^\nu\) are PCAFs of the process \(\mathbf{M}\) associated with \((\mathcal{E}, D(\mathcal{E}))\) in the Revuz correspondence with \(\mu\) and \(\nu\), respectively). In particular,

\[
R^\mu \mu(x) = E_x \int_0^\infty e^{-A^\mu_t} \, dA^\mu_t.
\]

The last integral is less than or equal to 1. Since by [46, Lemma IV.4.5] the measure \(\mu\) is smooth with respect to the perturbed form \((\mathcal{E}^\mu, D[\mathcal{E}^\mu])\), it follows from (3.1) that \(\mu\) belongs to the class \(\mathbb{M}_0(\mathcal{E}^\mu)\) defined for the form \((\mathcal{E}^\mu, D[\mathcal{E}^\mu])\). This shows that even nowhere Radon measures may belong to the class \(\mathbb{M}_0\) (for construction of smooth nowhere Radon measure see [46, Section IV.4(c)]).

We denote by \(\mathcal{G}_E^p\) the set of all quasi-continuous functions on \(E\) such that for q.e. \(x \in E\),

\[
E_x \sup_{t \geq 0} |u(X_t)|^p < \infty.
\]

DEFINITION 3.4. We say that \(u : E \to \mathbb{R}\) is a solution of PDE\((f + d\mu)\) if

(a) \(u\) is quasi-continuous and \(f(\cdot, u) \cdot m \in \mathbb{M}_0\),

(b) for q.e. \(x \in E\),

\[
u(x) = E_x \left( \int_0^\zeta f(X_t, u(X_t)) \, dt + \int_0^\zeta dA^\mu_t \right).
\]

DEFINITION 3.5. A pair \((u, \nu)\) is a solution of OP\((f + d\mu, h)\) if

(a) \(u\) is quasi-continuous and \(\nu, f(\cdot, u) \cdot m \in \mathbb{M}_0\),

(b) for q.e. \(x \in E\),

\[
(3.2) \quad u(x) = E_x \left( \int_0^\zeta f(X_t, u(X_t)) \, dt + \int_0^\zeta dA^\mu_t + \int_0^\zeta dA^\nu_t \right),
\]

(c) \(u(x) \geq h(x)\) for \(m\)-a.e. \(x \in E\),

(d) for q.e. \(x \in E\),

\[
E_x \int_0^\zeta (u(X_t) - h^*(X_t)) \, dA^\nu_t = 0
\]

for every quasi-continuous function \(h^*\) on \(E\) such that \(h \leq h^* \leq u\) \(m\)-a.e.
Remark 3.6. (i) By the Revuz duality, condition (d) is equivalent to the following condition:

$$\int_{E} (u - h^*) \, d\nu = 0$$

for every quasi-continuous function $h^*$ such that $h \leq h^* \leq u$ m.a.e. A standard argument shows that in fact one can replace $h^*$ by any quasi-u.s.c. $h^{**}$ such that $h \leq h^{**} \leq u$ m.a.e.

(ii) Let $\hat{h}$ be a quasi-u.s.c. regularization of $h$, i.e.

$$\hat{h} = \text{quasi-essinf} \{ \eta \geq h \text{ m.a.e.} : \eta \text{ is quasi-u.s.c.} \}.$$ 

Then $(u, \nu)$ is a solution of $\text{OP}(f + d\mu, h)$ if and only if it is a solution of $\text{OP}(f + d\mu, \hat{h})$. Indeed, if $(\hat{u}, \hat{\nu})$ is a solution of $\text{OP}(f + d\mu, \hat{h})$ then of course $\hat{u} \geq h$ m.a.e. Furthermore, for every quasi-u.s.c. $h^*$ such that $h \leq h^* \leq \hat{u}$ m.a.e. we have

$$\int_{E} (\hat{u} - h^*) \, d\hat{\nu} \leq \int_{E} (\hat{u} - \hat{h}) \, d\hat{\nu} = 0$$

since $\hat{h} \leq h^*$ q.e. Therefore $(\hat{u}, \hat{\nu})$ is a solution of $\text{OP}(f + d\mu, h)$. Now assume that $(u, \nu)$ is a solution of $\text{OP}(f + d\mu, h)$. Then $\hat{h} \leq u$ q.e. since $u$ is quasi-continuous, and

$$\int_{E} (u - \hat{h}) \, d\nu = 0$$

as $\hat{h}$ is quasi-u.s.c. and $h \leq \hat{h}$ m.a.e. So $(u, \nu)$ is a solution of $\text{OP}(f + d\mu, \hat{h})$. It follows that without loss of generality we can confine ourselves to considering quasi-u.s.c. barriers. Moreover, if $h$ is quasi-u.s.c. then the minimality condition (d) reduces to

$$\int_{E} (u - h) \, d\nu = 0.$$ 

In the proof of Theorem 3.8 we will use the form $(\mathcal{E}^#, D[\mathcal{E}^#])$, which is described in detail in [46, Theorem VI.1.2]. Here let us only mention that $E^#$ is a local compactification of $E$ and $(\mathcal{E}^#, D[\mathcal{E}^#])$ is a regular Dirichlet form on $L^2(E^#; m^#)$, which is an extension of the form $(\mathcal{E}, D[\mathcal{E}])$. Since $(\mathcal{E}^#, D[\mathcal{E}^#])$ is regular, one can associate with it a Hunt process $\mathbf{M}^# = (\Omega^#, \mathcal{F}^#, \{X^#_t\}_{t \geq 0}, \{P^#_x\}_{x \in E^#_\Delta})$ with life time $\zeta^#$. The process $\mathbf{M}^#$, being a Hunt process, is a special standard process, and moreover its trajectories have left limits on $(0, +\infty)$. $\mathbf{M}^#$ is a standard extension of $\mathbf{M}$, i.e. $P^#_x = P^#_x$, $X^#_t = X^#_t$, $t \geq 0$, $P^#_x$-a.s. for every $x \in E$ and $P^#_x = \delta_x$, $X^#_t = x$, $t \geq 0$, for every $x \in E^# \setminus E$. Given $u : E \to \mathbb{R}$ we will denote by $u^#$ its extension to $E^#$ defined as $u^#(x) = u(x)$ for $x \in E$ and $u^#(x) = 0$ for $x \in E^# \setminus E$.

The above procedure of regularization of quasi-regular Dirichlet form and associated Markov process is called the transfer method in [46]. In what
follows, we use this procedure without explicit mention when we cite some results from [26] or other papers dealing with regular Dirichlet forms (and non-quasi-regular forms).

**Lemma 3.7.** Suppose that $\mu \in M_0$ and $u$ is a quasi-continuous function on $E$ such that

\[(3.3)\quad u(x) = E_x \int_0^\zeta dA_t^\mu\]

for q.e. $x \in E$. Then there exists a martingale AF $M$ of $\mathcal{M}$ such that for q.e. $x \in E$,

\[(3.4)\quad u(X_t) = \int_0^\zeta dA_t^\mu - \int_0^\zeta dM_t, \quad t \in [0, \zeta], \ P_x\text{-a.s.}\]

**Proof.** By the transfer method, we may assume that $\mathcal{M}$ is a Hunt process. By [26, Theorem 4.1.1], there exists a properly exceptional set $N \subset E$ such that (3.3) holds for $x \in E \setminus N$. Using the Markov property and additivity of $A^\mu$ we conclude from (3.3) that

\[(3.5)\quad u(X_t) = E_{X_0} A_t^\mu = E_x (A_t^\mu \circ \theta_t | \mathcal{F}_t) = E_x \left( \int_0^\zeta dA_r \bigg| \mathcal{F}_t \right) - A_t^\mu, \quad t \geq 0,

for every $x \in E \setminus N$. In the above equation, $\theta$ is the shift operator on $\Omega$, that is, $X_t(\theta s \omega) = X_{s+t}(\omega)$ for $\omega \in \Omega$ and $s, t \geq 0$. Set

\[(3.6)\quad M_t = u(X_t) - u(X_0) + A_t^\mu, \quad t \geq 0.

Clearly $M$ is an additive functional. By quasi-continuity of $u$ and [26 Theorem 4.6.1], it is a càdlàg process. By (3.5), $E_x M_t = 0$ for $t \geq 0$ and $x \in E \setminus N$. Thus $M$ is a martingale AF of $\mathcal{M}$. From (3.6) we get (3.4). \hfill \blacksquare

Now we will prove the main result of this section. Besides the existence of a solution $(u, \nu)$ to OP$(f + d\mu, h)$ we will show that $(u, \nu)$ can be approximated by a solution $u_n$ to penalized PDE (3.7) with $\nu_n$ defined via $u_n$ and $h$. This approximation is important in analysis of various properties of $u, \nu$ and in numerical methods. We show the convergence of $u_n$ in the metric of the space $\mathcal{S}^c_q$, which implies the convergence of the measures $\nu_n$ to $\nu$ in the sense that $A^{\nu_n} \to A^\nu$ in $\mathcal{S}^c_q$ (clearly this convergence is stronger than weak convergence since it preserves smoothness of measures). Note here that in many applications the information about the measure $\nu$ is crucial. As already mentioned in the Introduction, $\nu$ can be interpreted as a local minimizer of the interacting energy (1.3). Moreover, in applications to mathematical finance, the AF $A^\nu$ generated by $\nu$ can be interpreted in some models as the so-called early exercise premium (see [42]). As a by-product, we also get a
probabilistic interpretation of solutions to OP($f + d\mu, h$). This result is a basis for probabilistic numerical methods (Monte Carlo methods) and optimal stopping theory, which links value functions of type (1.2) with solutions to OP($f + d\mu, h$).

Let $u$ be a real function on $E$. From now on,

$$f_u(x) := f(x, u(x)), \quad x \in E.$$  

**Theorem 3.8.** Assume (H1)–(H4). Then there exists a solution $(u, \nu)$ of OP($f + d\mu, h$) if and only if (H5) is satisfied. Moreover, if (H5) is satisfied, then $u \in \mathcal{G}_c^q$ for $q \in (0, 1)$, $u_n \to u$ in $\mathcal{G}_c^q$ for $q \in (0, 1)$ and $u_n \nearrow u$ q.e., where $u_n$ is the unique solution of the problem

(3.7) $$-Lu_n = f(\cdot, u_n) + \mu + \nu_n$$

with $\nu_n = n(u_n - h)^- \cdot m$.

**Proof.** The necessity of (H5) follows from the fact that $u$ defined by (3.2) is a difference of natural potentials. To prove that (H5) is sufficient let us first note that from [40, Theorem 3.8] (see also [37, Theorem 4.7]) it follows that for each $n \in \mathbb{N}$ there exists a unique solution $u_n$ of (3.7). Moreover, by [37, Proposition 4.9], $u_n \leq u_{n+1}$ q.e. By (H5) there exists $\lambda \in \mathcal{M}_0$ such that $-Lv = \lambda$ and $f^-(\cdot, v) \in \mathcal{M}_0$. Hence

$$-Lv = \lambda + f_v + f_v^- - f_v^+.$$  

Let $\overline{v}$ be a solution of

$$-L\overline{v} = \lambda^+ + f_{\overline{v}} + f_{\overline{v}}^- + \mu^+.$$  

By [37, Proposition 4.9], $v \leq \overline{v}$ q.e. Consequently, $h \leq \overline{v}$ q.e. From this we conclude that

$$-L\overline{v} = \lambda^+ + f_{\overline{v}} + f_{\overline{v}}^- + \mu^+ + n(\overline{v} - h)^-.$$  

By [37, Proposition 4.9] again, for every $n \in \mathbb{N}$,

(3.8) $$u_n \leq \overline{v} \quad \text{q.e.}$$

Set $u = \sup_{n \geq 1} u_n$ and

$$v_n(x) = -E_x \int_0^\zeta f(X_t, u_n(X_t)) dt - E_x \int_0^\zeta dA_t^\mu.$$  

Since $u_n \leq u_{n+1}$ q.e., it follows from (H1) that $v_n \leq v_{n+1}$ q.e. For $n \in \mathbb{N}$ set

$$w_n(x) = u_n(x) + v_n(x).$$  

Then

$$w_n(x) = E_x \int_0^\zeta dA_t^\nu.$$
From this we see that $w_n$ is a natural potential. In particular, $w_n$ is an excessive function. Therefore $w$ defined as

$$w(x) = \sup_{n \geq 1} w_n(x) \quad \text{for q.e. } x \in E$$

is excessive too (see [7 Proposition 1.2.1]), and hence quasi-continuous (see [26 Theorems A.2.7 and 4.6.1]). By (3.8), (H1), (H2) and the Lebesgue dominated convergence theorem, we have

$$v_n(x) \to -E_x \int_0^\zeta f(X_t, u(X_t)) \, dt - E_x \int_0^\zeta dA^\mu_t.$$ (3.9)

Hence

$$w(x) = u(x) - E_x \int_0^\zeta f(X_t, u(X_t)) \, dt - E_x \int_0^\zeta dA^\mu_t + E_x \int_0^\zeta dA^\nu_t$$

for q.e. $x \in E$. From the above equation, (3.8), quasi-continuity of $w$ and [8, Theorem VI.4.22] we conclude that $w$ is a natural potential. Therefore there exists a smooth measure $\nu$ such that for q.e. $x \in E$,

$$w(x) = E_x \int_0^\zeta dA^\nu_t.$$ (3.10)

Hence

$$u(x) = E_x \int_0^\zeta f(X_t, u(X_t)) \, dt + E_x \int_0^\zeta dA^\mu_t + E_x \int_0^\zeta dA^\nu_t$$

for q.e. $x \in E$. By Lemma 3.7 there exists a martingale AF $M$ of $\mathbb{M}$ such that

$$u(X_t) = \int_{T_0}^{\zeta} f_u(X_r) \, dr + \int_{T_0}^{\zeta} dA^\mu_r + \int_{T_0}^{\zeta} dA^\nu_r + \int_{T_0}^{\zeta} dM_r, \quad 0 \leq t \leq \zeta, \ P_x\text{-a.s.}$$

for q.e. $x \in E$. Since $u_n, u$ are quasi-continuous and we know that $u_n \to u$ and $u_n \leq u_{n+1}$ q.e., we see that $u^\#, u_n^\#$ are $\mathcal{E}^\#$-quasi-continuous, $u_n^\# \to u^\#$ and $u_n^\# \leq u_{n+1}^\#$, $\mathcal{E}^\#$-q.e. Therefore by [46 Theorem IV.5.29], $u_n^\#(X_t^\#) \to u^\#(X_t^\#), t \geq 0$, and $u_n^\#(X_t^\#) \to u^\#(X_t^\#), t \geq 0, P_x^\#\text{-a.s.}$ for $\mathcal{E}^\#$-q.e. $x \in E^\#$. By [46 Proposition V.2.28] (see also [46 Proposition V.2.12]), $u_n^\#(X_{t-}^\#) = (u_n^\#(X_{t+}^\#))_-$ and $u^\#(X_{t-}^\#) = (u^\#(X_{t+}^\#))_-$ for $t \geq 0$. Therefore by Dini’s theorem, for every $T > 0$,

$$\sup_{t \leq T} |u_n^\#(X_t^\#) - u^\#(X_t^\#)| \to_{P_x^\#} 0$$

for $\mathcal{E}^\#$-q.e. $x \in E$, which implies that

$$\sup_{t \leq T} |u_n(X_t) - u(X_t)| \to_{P_x} 0$$ (3.10)
Dirichlet forms and the obstacle problem

for $\mathcal{E}$-q.e. $x \in E$. Since the finite variation parts of the semimartingales $u_0(X)$ and $u(X)$ are continuous, $u_0(X), u(X)$ are special semimartingales (see [53 Theorem III.34]). Therefore there exists an increasing sequence $\{\tau_k\} \subset T$ such that $\tau_k \nearrow \infty$, and

$$E_x \sup_{t \leq \tau_k} |u(X_t)| + E_x \sup_{t \leq \tau_k} |u_0(X_t)| < \infty, \quad k \geq 1.$$  

Since $u_0 \leq u_n \leq u$ for $n \geq 1$, (H1) implies that for q.e. $x \in E$,

$$E_x \int_0^{\tau_k} dA^\nu_t \leq E_x \sup_{t \leq \tau_k} |u(X_t)| + E_x \sup_{t \leq \tau_k} |u_0(X_t)| + E_x \int_0^{\tau_k} |f_u(X_t)| dt$$

$$+ E_x \int_0^{\tau_k} |f_{u_0}(X_t)| dt + E_x \int_0^{\tau_k} dA^\mu_t.$$  

When combined with (3.10), this implies that for every $T > 0$,

$$[u_n(X) - u(X)]_T = [M^n - M]_T \to P_x 0$$

(see [32 Theorem 1.8]), which is equivalent (since $\sup_{t \leq \tau_k} |\Delta M^n_t|$ is uniformly integrable with respect to $n$) to

$$\sup_{t \leq T} |M^n_t - M_t| \to P_x 0.$$  

By (3.8), (H1), (H2) and the Lebesgue dominated convergence theorem, we get

$$E_x \int_0^\zeta |f_{u_n}(X_t) - f_u(X_t)| dt \to 0.$$  

From (3.10), (3.12) and (3.13) it follows that for every $T > 0$,

$$\sup_{t \leq T} \left| \int_0^t dA^\nu_t - \int_0^t dA^\mu_t \right| \to P_x 0$$

for q.e. $x \in E$. Observe that by (3.11),

$$E_x \int_0^{\tau_k} (u_n(X_t) - h(X_t))^+ dt \to 0$$

for q.e. $x \in E$, which together with (3.10) implies that $u \geq h$ m-a.e. Finally, let $h^*$ be a quasi-continuous function such that $h \leq h^* \leq u$ m-a.e. Then by (3.10) and (3.14), for every $T > 0$ we have

$$\int_0^T (u_n(X_t) - h^*(X_t))^+ dA^\nu_t \to P_x 0$$

for q.e. $x \in E$. Observe that by (3.11),

$$E_x \int_0^{\tau_k} (u_n(X_t) - h(X_t))^+ dt \to 0$$

for q.e. $x \in E$, which together with (3.10) implies that $u \geq h$ m-a.e. Finally, let $h^*$ be a quasi-continuous function such that $h \leq h^* \leq u$ m-a.e. Then by (3.10) and (3.14), for every $T > 0$ we have

$$\int_0^T (u_n(X_t) - h^*(X_t))^+ dA^\nu_t \to P_x 0$$

for q.e. $x \in E$. Observe that by (3.11),
On the other hand,
\[
\begin{align*}
T_0 \left( u_n(X_t) - h^*(X_t) \right)^+ dA_t^{\nu^n} \\
= n \int_0^T \left( u_n(X_t) - h^*(X_t) \right)^+ (u_n(X_t) - h(X_t) - dt \leq 0,
\end{align*}
\]
which implies that
\[
\begin{align*}
T_0 \left( u_n(X_t) - h^*(X_t) \right) dA_t^{\nu} = 0 \quad P_x\text{-a.s.}
\end{align*}
\]
since \( h^* \leq u \) q.e. Therefore, \((u, \nu)\) is a solution to \(\text{OP}(f + d\mu, h)\). By \cite{40} Theorem 3.8], \(u_n, \tilde{v} \in \mathcal{G}_c^q\) for every \(q \in (0, 1)\). From this, \(3.8\) and \(3.10\), we conclude that \(u \in \mathcal{G}_c^q\) for \(q \in (0, 1)\), and \(u_n \to u\) in \(\mathcal{G}_c^q\) for \(q \in (0, 1)\). This completes the proof.

**Corollary 3.9.** Assume (H1)–(H5) and retain the notation from Theorem 3.8. Then for every \(q \in (0, 1)\),
\[
\lim_{n \to \infty} E_x \sup_{t \geq 0} |A_t^{\nu^n} - A_t^{\nu^q}|^q \to 0 \quad \text{for q.e. } x \in E.
\]

**Proof.** Follows from \(3.14\) and \(3.11\). \(\blacksquare\)

In what follows we denote by \(\| \cdot \|\) the total variation norm on the space of signed Borel measures on \(E\).

**Proposition 3.10.** Assume (H1)–(H5). Let \((u, \nu)\) be a solution of \(\text{OP}(f + d\mu, h)\). Then
\[
\|\nu\| \leq 2(\|\mu\| + \|f_0\| + \|\lambda^+\| + \|f_v^-\|)
\]
with \(\lambda = -Lv\), where \(v\) is the function from condition (H5).

**Proof.** Let \(\tilde{v}\) be as in the proof of Theorem 3.8. By \(3.8\),
\[
E_x \int_0^\zeta dA_t^{\nu} \leq E_x \int_0^\zeta dA_t^{\mu^-} + E_x \int_0^\zeta f_u^- (X_t) dt + E_x \int_0^\zeta dA_t^{\lambda^+} + E_x \int_0^\zeta f_v^+ (X_t) dt + E_x \int_0^\zeta f_v^- (X_t) dt.
\]
By \cite{40} Lemma 2.6 (see also \cite{37} Lemma 5.4)],
\[
\|\nu\| \leq \|\mu^-\| + \|f_u^-\| + \|\lambda^+\| + \|f_v^+\| + \|f_v^-\|.
\]
By (H1) and \(3.8\), \(f_u^- \leq f_v^-\). Therefore
\[
\|\nu\| \leq \|\mu^-\| + \|\lambda^+\| + \|f_v^-\|.
\]
Since by [40, Proposition 3.10], \( \|f_{\bar{v}}\| \leq \|\lambda^+\| + \|f_v^-\| + \|\mu^+\| + \|f_0\| \), the desired estimate follows. □

For \( k \geq 0 \) we define the truncation operator \( T_k : \mathbb{R} \to \mathbb{R} \) as
\[
T_k(y) = \min\{\max\{-k, y\}, k\}, \quad y \in \mathbb{R}.
\]

**Proposition 3.11.** Assume (H1)–(H5). Let \((u, \nu)\) be a solution of \( \text{OP}(f + d\mu, h) \). If \( f - v, \mu, \lambda +, f_0 \in M_{0,b} \), then \( \nu \in M_{0,b} \), \( T_k(u) \in D_{e}[E] \) for every \( k \geq 0 \), and
\[
E(T_k(u), T_k(u)) \leq 2k(\|\mu\| + \|\nu\| + \|f_0\|), \quad k \geq 0.
\]

**Proof.** Follows from Proposition 3.10 and [40, Proposition 3.10, Theorem 4.2]. □

The uniqueness of solutions of the obstacle problem follows from the following comparison result, in which we assume that \( f_1, f_2 : E \times \mathbb{R} \to \mathbb{R} \) and \( h_1, h_2 : E \to \mathbb{R} \) are measurable and \( \mu_1, \mu_2 \in \mathcal{M}_0 \).

**Proposition 3.12.** Assume that \((u_i, \nu_i), i = 1, 2,\) is a solution of \( \text{OP}(f_i + d\mu_i, h_i) \). If
\[
d\mu_1 \leq d\mu_2, \quad h_1 \leq h_2 \text{ m-a.e.}
\]
and either
\[
f_1 \text{ satisfies (H1) and } f_1(\cdot, u_2) \leq f_2(\cdot, u_2) \text{ m-a.e.}
\]
or
\[
f_2 \text{ satisfies (H1) and } f_1(\cdot, u_1) \leq f_2(\cdot, u_1) \text{ m-a.e.},
\]
then \( u_1 \leq u_2 \) q.e. Moreover, if \( h_1 = h_2 \) and \( f_1, f_2 \) satisfy (H1), then \( d\nu_1 \geq d\nu_2 \).

**Proof.** Suppose that \( f_1 \) satisfies (H1) and \( f_1(\cdot, u_2) \leq f_2(\cdot, u_2) \) m-a.e. Since the Revuz correspondence is one-to-one, we have
\[
\int_0^t f_1(X_r, u_2(X_r)) \, dr \leq \int_0^t f_2(X_r, u_2(X_r)) \, dr, \quad \int_0^t dA^\mu_{r1} \leq \int_0^t dA^\mu_{r2}, \quad t \geq 0.
\]
By the definition of a solution to the obstacle problem and Lemma 3.7 there exist martingale AFs \( M^1, M^2 \) of \( \mathcal{M} \) such that
\[
u_i(X_t) = \int_0^\zeta f_i(X_r, u_i(X_r)) \, dr + \int_0^\zeta dA^\mu_{r1} + \int_0^\zeta dA^\nu_{r1} + \int_0^\zeta dM^i_r, \quad 0 \leq t \leq \zeta,
\]
P_x-a.s., \( i = 1, 2 \), for q.e. \( x \in E \). By the Tanaka–Meyer formula (see, e.g., [53]...
Theorem IV.70], for every \( \tau \in \mathcal{T} \) we have
\[
(u_1 - u_2)^+(X_t) \leq (u_1 - u_2)^+(X_{\tau})
+
\int_t^\tau (f_1(X_r, u_1(X_r)) - f_2(X_r, u_2(X_r))) 1_{\{u_1 > u_2\}}(X_r) \, dr
+
\int_t^\tau 1_{\{u_1 > u_2\}}(X_r) d(A_{\nu_1}^\tau - A_{\nu_2}^\tau)
-
\int_t^\tau 1_{\{u_1 > u_2\}}(X_r) dA_r^\tau
\]
\[=: \sum_{i=1}^6 I_i(t, \tau).\]

Observe that \( I_2(t, \tau) \leq 0 \) by the assumptions on \( f_1, f_2 \). Since \( h_1 \leq u_1 \wedge u_2 \leq u_1 \),
\[
I_4(t, \tau) = \int_t^\tau (u_1 - u_2)^{-1} \cdot (u_1 - u_1 \wedge u_2)(X_r) 1_{\{u_1 > u_2\}}(X_r) dA_r^{\nu_1} = 0.
\]

It is also clear that \( I_3(t, \tau) \leq 0 \) and \( I_5(t, \tau) \leq 0 \). Let \( \{\tau_k\} \subset \mathcal{T} \) be a fundamental sequence for the martingale \( M^1 - M^2 \). Then by the above estimates,
\[
E_x(u_1 - u_2)^+(X_{t\wedge}) \leq E_x(u_1 - u_2)^+(X_{\tau})
\]
for q.e. \( x \in E \). From this and the fact that \( u_1, u_2 \) are differences of natural potentials we conclude that \( u_1 \leq u_2 \) q.e. Now assume that \( h_1 = h_2 \). By Corollary 3.9 for every \( T > 0 \),
\[
\sup_{t \leq T} |A_{\nu_1}^t - A_{\nu_2}^t| + \sup_{t \leq T} |A_{\nu_1}^t - A_{\nu_2}^t| \to_{P_x} 0
\]
for q.e. \( x \in E \), where \( u_n^i \) is a solution of
\[
-Lu_n^i = f_i(x, u_n^i) + \mu_i + n(u_n^i - h_1)^-
\]
and \( \nu_n^i = n(u_n^i - h_1)^- \cdot m \). By [37, Proposition 4.9], \( u_1^n \leq u_2^n \) q.e., which implies the second assertion of the proposition.

**Corollary 3.13.** Under (H1) there exists at most one solution of \( \text{OP}(f + d\mu, h) \).

In the case where \( L \) is a uniformly elliptic divergence form operator with zero Dirichlet boundary conditions the existence and uniqueness of a solution \((u, \nu)\) to the problem (1.1) (in the sense of the definition of the present paper) was proved in [58]. In [58] it is assumed that \( h \) is quasi-continuous, \( \mu \in M_{0,b} \) and \( f \) satisfies (H1), (H2) and some integrability conditions slightly stronger
than (H3)–(H5). Note also that in the special case considered in [58], \( u \) is an entropy solution of (1.1).

**Definition 3.14.** We say that \( v \) is a supersolution of PDE\((f + d\mu)\) if there exists a positive \( \lambda \in \mathbb{M}_0 \) such that \( v \) is a solution of PDE\((f + d\mu + d\lambda)\).

**Proposition 3.15.** Assume (H1)–(H4) and suppose \( u \) is a solution of \( \text{OP}(f + d\mu, h) \). Then

\[
 u = \text{quasi-essinf} \{ v \geq h \text{ m-a.e.} : v \text{ is a supersolution of PDE}(f + d\mu) \}.
\]

**Proof.** Let \( v \) be a supersolution of PDE\((f + d\mu)\) and \( v \geq h \text{ m-a.e.} \). Then

\[
 -Lv = f(\cdot, v) + \mu + \lambda + n(v - h)^-.
\]

Denote by \( u_n \) the solution of

\[
 -Lu_n = f(\cdot, u_n) + \mu + n(u_n - h)^-.
\]

By [37, Proposition 4.9], \( u_n \leq v \). Since we know that \( u_n \not\rightarrow u \) q.e., the desired assertion follows.

**Proposition 3.16.** Let \((u, \nu)\) be a solution to \( \text{OP}(d\mu, h) \). Assume that \( \mu \in D'_{\mathcal{E}} \), and there exists \( v \in D_\mathcal{E} \) such that \( v \geq h \). Then \( u \in D_\mathcal{E} \), \( \nu \in D'_{\mathcal{E}} \) and \((u, \nu)\) is the unique pair in \( D_\mathcal{E} \times D'_{\mathcal{E}} \) such that

\[
 \mathcal{E}(u, \eta) = \int_\mathcal{E} \eta d\mu + \int_\mathcal{E} \eta d\nu, \quad \eta \in D_\mathcal{E}, \, u \geq h \text{ a.e.}
\]

and

\[
 \int_\mathcal{E} (u - \eta) d\nu \leq 0, \quad \eta \in D_\mathcal{E}, \, \eta \geq h \text{ a.e.}
\]

Moreover,

\[
 \|\nu\|_{\mathcal{E}'} \leq 3(\|v\|_\mathcal{E} + \|\mu\|_{\mathcal{E}'}).
\]

**Proof.** By Theorem 3.8, \( u_n \not\rightarrow u \), where

\[
 -Lu_n = \mu + \nu_n, \quad \nu_n = n(u_n - h)^- \cdot m.
\]

By the definition of a solution to (3.18),

\[
 u_n = R\mu + R\nu_n.
\]

Let \( \{F_k\} \) be an \( \mathcal{E} \)-nest such that \( \nu_n = 1_{F_k} \cdot \nu_n \in D'_\mathcal{E} \), and let

\[
 u^k_n = R\mu + R\nu^k_n.
\]

By [2.2], \( u^k_n \in D_\mathcal{E} \) and

\[
 \mathcal{E}(u^k_n, \eta) = \int_\mathcal{E} \eta d\mu + \int_\mathcal{E} \eta d\nu^k_n, \quad \eta \in D_\mathcal{E}.
\]

Setting \( \eta = u^k_n - v \) and using the fact that \( \int_\mathcal{E} (u^k_n - v) d\nu^k_n \leq 0 \) we easily get

\[
 \|u^k_n\|_{\mathcal{E}} \leq 2(\|v\|_{\mathcal{E}} + \|\mu\|_{\mathcal{E}'})
\].
Let $\eta \in D_e[\mathcal{E}]$ be a positive function. Then

\begin{equation}
\int_E \eta \, d\nu_n^k = \mathcal{E}(u_n^k, \eta) - \int_E \eta \, d\mu \leq \|u_n^k\|_\mathcal{E} \|\eta\|_\mathcal{E} + \|\eta\|_\mathcal{E} \|\mu\|_{\mathcal{E}'}.
\end{equation}

From (3.19) it is clear that $u_n^k \to u_n$ q.e. as $k \to \infty$. Since $(\mathcal{E}, D_e[\mathcal{E}])$ is a Hilbert space, it follows from this and (3.21) that $u_n^k \to u_n$ weakly in $(\mathcal{E}, D_e[\mathcal{E}])$ as $k \to \infty$. On the other hand, \( \int_E \eta \, d\nu_n^k \to \bigcup_{k=1}^{\infty} F_k \eta \, d\nu_n = \int_E \eta \, d\nu_n \), the equality being a consequence of the fact that $E \setminus \bigcup_{k=1}^{\infty} F_k$ is $\mathcal{E}$-exceptional. Therefore letting $k \to \infty$ in (3.20) shows that

\begin{equation}
\mathcal{E}(u_n, \eta) = \int_E \eta \, d\mu + \int_E \eta \, d\nu_n, \quad \eta \in D_e[\mathcal{E}].
\end{equation}

Furthermore, by (3.21) and (3.22),

\begin{equation}
\|u_n\|_\mathcal{E} \leq 2(\|v\|_\mathcal{E} + \|\mu\|_{\mathcal{E}'}), \quad \int_E \eta \, d\nu_n \leq \|u_n\|_\mathcal{E} \|\eta\|_\mathcal{E} + \|\eta\|_\mathcal{E} \|\mu\|_{\mathcal{E}'}.
\end{equation}

Similarly, since $u_n \not\to u$, it follows from the first inequality in (3.24) that \{u_n\} is weakly convergent in $(\mathcal{E}, D_e[\mathcal{E}])$ to $u \in D_e[\mathcal{E}]$. From (3.24) it also follows that, up to a subsequence, \{\nu_n\} is weakly convergent in $(\mathcal{E}, D_e'[\mathcal{E}])$ to $\tilde{\nu} \in D_e'[\mathcal{E}]$. Letting $n \to \infty$ in (3.23) we obtain the variational equality in (3.16) with $\nu$ replaced by $\tilde{\nu}$. By virtue of (2.2) this implies that

\[ u = R\mu + R\tilde{\nu} \quad \text{q.e.}, \]

so $R\nu = R\tilde{\nu}$, q.e., which forces $\tilde{\nu} = \nu$. From this and (3.24), we see that $\nu$ satisfies (3.17). The other properties of $(u, \nu)$ formulated in (3.16) follow from the definition of a solution of OP($d\mu, h$).

4. Two-sided obstacle problem

DEFINITION 4.1. We say that a pair $(u, \nu)$ is a solution of OP($f + d\mu, h_1, h_2$) if

(a) $u$ is quasi-continuous and $\nu \in M_0$, $f(\cdot, u) \cdot m \in M_0$,
(b) for q.e. $x \in E$,

\[ u(x) = E_x \left( \int_0^\zeta f(X_t, u(X_t)) \, dt + \int_0^\zeta dA^\mu_t + \int_0^\zeta dA^\nu_t \right). \]

(c) $h_1(x) \leq u(x) \leq h_2(x)$ for m-a.e. $x \in E$,
(d) for q.e. $x \in E$,

\[ E_x \left( u(X_t) - h_1^*(X_t) \right) dA_t^{\nu^+} = E_x \left( h_2^*(X_t) - u(X_t) \right) dA_t^{\nu^-} = 0 \]

for any quasi-continuous functions $h_1^*, h_2^*$ on $E$ such that $h_1 \leq h_1^* \leq u \leq h_2^* \leq h_2$ m-a.e.
Proposition 4.2. Let \((u_i, \nu_i), \ i = 1, 2,\) be a solution of \(OP(f_i + d\mu_i, h_i^1, h_i^2)\). Assume that
\[
d\mu_1 \leq d\mu_2, \quad h_1^1 \leq h_1^2, \quad h_2^1 \leq h_2^2 \quad \text{m-a.e.}
\]
and either
\[
f_1 \text{ satisfies (H1) and } f_1(\cdot, u_2) \leq f_2(\cdot, u_2) \quad \text{m-a.e.}
\]
or
\[
f_2 \text{ satisfies (H1) and } f_1(\cdot, u_1) \leq f_2(\cdot, u_1) \quad \text{m-a.e.}
\]
Then \(u_1(x) \leq u_2(x)\) for q.e. \(x \in E\).

Proof. Since the Revuz correspondence is one-to-one,
\[
\int_0^t f_1(X_r, u_2(X_r)) \, dr \leq \int_0^t f_2(X_r, u_2(X_r)) \, dr, \quad \int_0^t dA_{r}^{\mu_1} \leq \int_0^t dA_{r}^{\mu_2}, \quad t \geq 0.
\]
By the definition of solution to the obstacle problem and Lemma 3.7 there exist martingale AFs \(M^1, M^2\) of \(\mathbf{M}\) such that
\[
u_i(X_t) = \int_0^t f_i(X_r, u_i(X_r)) \, dr + \int_0^t dA_{r}^{\mu_i} + \int_0^t dM_{r}, \quad 0 \leq t \leq \zeta,
\]
P\text{.e.}, \(i = 1, 2,\) for q.e. \(x \in E\). By the Tanaka–Meyer formula (see [53 Theorem IV.70]), for every \(\tau \in \mathcal{T}\),
\[
(u_1 - u_2)^+(X_{\tau}) \leq (u_1 - u_2)^+(X_{\tau})
\]
\[
\begin{align*}
+ & \int_0^\tau 1_{\{u_1 > u_2\}}(X_r)(f_1(X_r, u_1(X_r)) - f_2(X_r, u_2(X_r))) \, dr \\
+ & \int_0^\tau 1_{\{u_1 > u_2\}}(X_r) (A_{r}^{\mu_1} - A_{r}^{\mu_2}) \, dr + \int_0^\tau 1_{\{u_1 > u_2\}}(X_r) dA_{r}^{\nu_1} \\
- & \int_0^\tau 1_{\{u_1 > u_2\}}(X_r) dA_{r}^{\nu_2} - \int_0^\tau 1_{\{u_1 > u_2\}}(X_{\tau-}) d(M_{r}^1 - M_{r}^2)
\end{align*}
\]
\[
=: \sum_{i=1}^6 I_i(t, \tau).
\]
It is easy to see that \(I_2(t, \tau) \leq 0\) and \(I_3(t, \tau) \leq 0\). By the minimality of \(\nu_1, \nu_2\) (condition (d) in the definition of a solution of the obstacle problem), we have
\[
I_4(t, \tau) \leq \int_0^\tau 1_{\{u_1 > u_2\}}(u_1 - u_2)^{-1}(u_1 - u_1 \wedge u_2) dA_{r}^{\nu_1^+} = 0,
\]
\[
I_5(t, \tau) \leq \int_0^\tau 1_{\{u_1 > u_2\}}(u_1 - u_2)^{-1}(u_1 \vee u_2 - u_2) dA_{r}^{\nu_2^-} = 0.
\]

The rest of the proof runs as in the proof of Proposition 3.12.
COROLLARY 4.3. Under (H1) there is at most one solution of OP($f + d\mu$, $h_1, h_2$).

Below we give the main theorem of this section. We give an existence result for (4.1) and show the convergence of two penalization schemes. In the first one, we approximate the solution $(u, \nu)$ to OP($f + d\mu, h_1, h_2$) by solutions $u_n$ to PDE (4.1) (with $n = k$). In (4.1), a measure $\nu_n$ with density (with respect to $m$) defined via $u_n$ and $h_1, h_2$ appears. The convergence of $\nu_n$ to $\nu$ is in the same metric as in the case of one barrier, i.e. $A^{\nu_n} \to A^{\nu}$ in $\mathcal{G}^q_c$. In the second penalization scheme, we approximate $u$ by the first component of the solution $(u_k, \alpha_k)$ to the obstacle problem (4.2) with one lower barrier $h_1$, and we approximate $\nu$ by measures $\nu_k$ defined as the sum of $\alpha_k$ and a measure with density (with respect to $m$) defined via $u_k, h_2$. The advantage of the second penalization is that \{u_k\} is monotone, and we have stronger convergence of the approximation measures $\nu_k$ (see Corollary 4.5). As in the case of one barrier, as a by-product we also get a probabilistic representation of solutions.

THEOREM 4.4. Assume (H1)–(H4). Then there exists a solution $(u, \nu)$ of OP($f + d\mu, h_1, h_2$) if and only if (H6) is satisfied. Moreover, if (H6) is satisfied, then $u \in \mathcal{G}^q_c$ for $q \in (0, 1)$ and

(i) if $u_{n,k}$ is a solution of the equation

\begin{equation}
-Lu_{n,k} = f(\cdot, u_{n,k}) + \mu + n(u_{n,k} - h_1)^- - k(u_{n,k} - h_2)^+,
\end{equation}

then $u_{n,k} \to u$ q.e. and in $\mathcal{G}^q_c$ for $q \in (0, 1)$ as $n, k \to \infty$,

(ii) if $(u_k, \alpha_k)$ is a solution of the obstacle problem

\begin{equation}
-Lu_k = f(\cdot, u_k) + \mu + \alpha_k - k(u_k - h_2)^+,
\end{equation}

then $u_k \searrow u$ q.e. and in $\mathcal{G}^q_c$ for $q \in (0, 1)$ as $k \to \infty$.

Proof. The necessity is clear. To prove that (H6) is sufficient let us first observe that by Proposition 3.12 $u_k \geq u_{k+1}$ and $d\alpha_k \leq d\alpha_{k+1}$. By (H6) there exist a function $v$ and a measure $\lambda \in \mathcal{M}_0$ such that

\begin{equation}
-Lv = \lambda, \quad f(\cdot, v) \in \mathcal{M}_0, \quad h_1 \leq v \leq h_2 \quad \text{m-a.e.}
\end{equation}

Hence

\begin{equation}
-Lv = f(\cdot, v) + (\lambda^+ + f^-(\cdot, v)) - (\lambda^- + f^+(\cdot, v)) + n(v - h_1)^- - k(v - h_2)^+.
\end{equation}

Let $\overline{v}_n$ be a solution of the equation

\begin{equation}
-L\overline{v}_n = f(\cdot, \overline{v}_n) - \lambda^- - f^+(\cdot, v) - \mu^- + n(\overline{v}_n - h_1)^-.
\end{equation}

By Proposition 3.12 $\overline{v}_n \leq v$ q.e., and consequently $\overline{v}_n \leq h_2$ m-a.e. Therefore

\begin{equation}
-L\overline{v}_n = f(\cdot, \overline{v}_n) - \lambda^- - f^+(\cdot, v) - \mu^- + n(\overline{v}_n - h_1)^- - k(\overline{v}_n - h_2)^+.
\end{equation}
By Proposition 3.12 again, \( u_{n,k} \geq \overline{v}_n \) q.e., which implies that
\[
(4.3) \quad n(u_{n,k} - h_1)^- \leq n(\overline{v}_n - h_1)^-.
\]

By Theorem 3.8, \( \overline{v}_n \nearrow \overline{v} \) q.e. where \((\overline{v}, \nu)\) is a solution of the obstacle problem
\[-L\overline{v} = f(\cdot, \overline{v}) - \lambda^- - f^+(\cdot, v) - \mu^- + \nu, \quad \overline{v} \geq h_1.\]

Hence
\[
(4.4) \quad E_x \int_0^\zeta d\overline{A}^\nu_t \to E_x \int_0^\zeta d\overline{A}_t
\]
for q.e. \( x \in E \), where \( \overline{v}_n = n(\overline{v}_n - h_1)^- \cdot m \). Write \( \alpha_{n,k} = n(u_{n,k} - h_1)^- \cdot m \). By (4.3), \( E_x \int_0^\zeta d\alpha_{n,k} \leq E_x \int_0^\zeta d\overline{A}^\nu_t \), whereas by Theorem 3.8, \( E_x \int_0^\zeta d\alpha_{n,k} \to E_x \int_0^\zeta d\alpha_t \) for q.e. \( x \in E \), where \( \alpha_k \) is defined in (ii). Therefore
\[
(4.5) \quad E_x \int_0^\zeta d\alpha_t \leq E_x \int_0^\zeta d\overline{A}_t
\]
for q.e. \( x \in E \). Since \( d\alpha_k \leq d\alpha_{k+1} \).
\[
(4.6) \quad d\alpha_t \leq d\alpha_{t+1} \quad P_x\text{-a.s.}
\]

Set \( A_t = \sup_{k \geq 1} A_{\alpha_t}^k \). By [51, Lemma 3.2], \( A \) is a càdlàg process. Consequently, it is a positive additive functional as an increasing limit of additive functionals. Thus, \( w := EA_\zeta \) is an excessive function (see [8, Proposition IV.2.4]). Consequently, by [26, Theorem A.2.7], \( w \) is finely-continuous. Therefore, by [26, Theorem 4.6.1], \( w \) is quasi-continuous. This implies that \( A \) is a continuous AF. Therefore there exists a smooth measure \( \alpha \) such that \( A = A^\alpha \). Moreover, by (4.4) and (4.5), \( \alpha \in \mathcal{M}_0 \). By (4.6) and Dini’s theorem, for every \( T > 0 \),
\[
(4.7) \quad \sup_{t \leq T} |A_{\alpha_t}^k - A_{\alpha_t}^\alpha| \to P_x 0
\]
for q.e. \( x \in E \). Let \( u(x) = \inf_{k \geq 1} u_k(x) \), where \( u_k \) is defined in (ii). Thanks to (4.7) we may now repeat arguments from the proof of Theorem 3.8 to show that \( u \) is quasi-continuous, and moreover the following hold:
\[
E_x \int_0^\zeta |f_{u_k}(X_t) - f_u(X_t)| dt \to 0
\]
for q.e. \( x \in E \), there exists a nonnegative measure \( \delta \in \mathcal{M}_0 \) such that for every \( T > 0 \),
\[
(4.8) \quad \sup_{t \leq T} |A_{\alpha_t}^\delta - A_t^\delta| \to P_x 0
\]
for q.e. \( x \in E \), where \( \delta_k = k(u_k - h_2)^+ \cdot m \),

\[(4.9) \quad \sup_{t \leq T} |u_k(X_t) - u(X_t)| \to_P \, 0 \]

for q.e. \( x \in E \), and finally,

\[(4.10) \quad u(x) = E_x \int_0^\zeta f_u(X_t) \, dt + E_x \int_0^\zeta dA_t^\mu + E_x \int_0^\zeta dA_t^\alpha - E_x \int_0^\zeta dA_t^\delta \]

for q.e. \( x \in E \). By (4.5), \( u \geq h_1 \) m.a.e. By the definition of a solution of the obstacle problem,

\[u_k(x) = E_x \int_0^\zeta f_{u_k}(X_t) \, dt + E_x \int_0^\zeta dA_t^\mu + E_x \int_0^\zeta dA_t^{\alpha k} - E_x \int_0^\zeta dA_t^{\delta k} \]

for q.e. \( x \in E \). From the above equation, (4.10) and the convergence results for \( u_k, f_{u_k}, A^{\alpha k} \) we have already proved, we conclude that

\[(4.11) \quad E_x \int_0^\zeta dA_t^{\delta k} \to E_x \int_0^\zeta dA_t^\delta \]

for q.e. \( x \in E \), which implies that \( u \leq h_2 \) m.a.e. Using (4.7)–(4.9) we can show in the same way as in the proof of minimality of the measure \( \nu \) in Theorem 3.8 that for every quasi-continuous \( h_1^*, h_2^* \) such that \( h_1 \leq h_1^* \leq u \leq h_2^* \leq h_2 \) m.a.e. we have

\[E_x \int_0^\zeta (h_2^*(X_t) - u(X_t)) \, dA_t^\delta = E_x \int_0^\zeta (u(X_t) - h_1^*(X_t)) \, dA_t^\alpha = 0 \]

for q.e. \( x \in E \). Of course, putting \( \nu = \delta - \alpha \) yields the above equation with \( \nu^- \) in place of \( \delta \) and \( \nu^+ \) in place of \( \alpha \). Thus the pair \( (u, \nu) \) is a solution of \( \text{OP}(f + d\mu, h_1, h_2) \). Observe that

\[(4.12) \quad w_n \leq u_{n,k} \leq u_k \quad \text{q.e.,} \]

where \( (w_n, \beta_n) \) is a solution of the obstacle problem

\[-Lw_n = f(\cdot, w_n) + n(w_n - h_1)^- + \mu - \beta_n, \quad w_n \leq h_2. \]

To see this it is enough to observe that

\[-Lu_k = f(\cdot, u_k) + n(u_k - h_1)^- - k(u_k - h_2)^+ + \mu + \alpha_k, \]

\[-Lw_n = f(\cdot, w_n) + n(w_n - h_1)^- - k(w_n - h_2)^+ + \mu - \beta_n, \]

and apply Proposition 3.12. By the same method as in the case of \( \{u_k\} \), one can show that the limit of \( \{w_n\} \) is the first component of the solution of \( \text{OP}(f + d\mu, h_1, h_2) \). Hence, by Corollary 4.3, \( w_n \to u \) q.e. Finally, observe
By Proposition 3.12, for every $q \in (0, 1)$,
\[ E_x \sup_{t \geq 0} |A_{t+}^\beta - A_t^\alpha|^q + E_x \sup_{t \geq 0} |A_{t+}^{\alpha_k} - A_t^\alpha|^q \to 0 \]
for q.e. $x \in E$. Moreover, by the Tanaka–Meyer formula (see [53] Theorem IV.70),
\[ |u_k(X_t)| \leq E_x \left( \int_0^\zeta |f(X_t, 0)| \, dt + \int_0^{\zeta} |dA_{t+}^\mu| + \int_0^{\zeta} |dA_{t+}^\nu| \right) \cdot \mathcal{F}_t. \]
Therefore by [15] Lemma 6.1, for every $q \in (0, 1)$,
\[ E_x \sup_{t \geq 0} |u_k(X_t)|^q \leq (1 - q)^{-1} \left[ E_x \left( \int_0^\zeta |f(X_t, 0)| \, dt + \int_0^{\zeta} |dA_{t+}^\mu| + \int_0^{\zeta} |dA_{t+}^\nu| \right) \right]^q. \]
From this we conclude that $u_n \to u$ in $G_c^q$ for $q \in (0, 1)$. In the same manner we can see that $w_n \to u$ in $G_c^q$ for $q \in (0, 1)$, which when combined with (4.12) implies that $u_{n,k} \to u$ in $G_c^q$ for $q \in (0, 1)$.

**Corollary 4.5.** Assume (H1)–(H4), (H6) and retain the notation from Theorem 4.4 and its proof. Then for every $q \in (0, 1)$, and for q.e. $x \in E$,
(i) $E_x \sup_{t \geq 0} |A_{t+}^{\alpha_k} - A_t^{\nu+}|^q + E_x \sup_{t \geq 0} |A_{t+}^\delta - A_t^\nu|^q \to 0$ as $k \to \infty$,
(ii) $E_x \sup_{t \geq 0} |A_{t+}^{\nu^n} - A_t^{\nu}|^q \to 0$ as $n \to \infty$, where $\nu_n = n(u_{n,n} - h_1)^- - n(u_{n,n} - h_2)^+.$

**Proof.** (i) One can regard $(u, \nu^-)$ as a solution of OP$(f + d\mu + d\nu^+, h_2)$ (with upper barrier). Therefore by Theorem 3.8 $y_k \searrow u$ q.e., where
\[ -Ly_k = f(\cdot, y_k) - k(y_k - h_2)^+ + \nu^+ + \mu, \]
and for every $q \in (0, 1)$,
\[ E_x \sup_{t \geq 0} |A_{t+}^\delta - A_t^{\nu^-}|^q \to 0 \]
for q.e. $x \in E$, where $\beta_k = k(y_k - h_2)^+$. Since $y_k \searrow u$, $y_k \geq h_1$ q.e. Therefore
\[ -Ly_k = f(x, y_k) + n(y_k - h_1)^- - k(y_k - h_2)^+ + \nu^+ + \mu. \]
By Proposition 3.12 $y_k \geq u_{n,k}$ q.e., so $k(u_{n,k} - h_2)^+ \leq k(y_k - h_2)^+$. By (4.13) and the convergence of $\{A_{t+}^{\alpha_n,k}\}$ shown in the proof of Theorem 3.8 $dA^\alpha \leq dA^{\nu+}$, which implies that $dA^\alpha \leq dA^\nu$. The same reasoning applied to the measure $\delta$ shows that $d\delta \leq d\nu^-$. From this and minimality of the Jordan decomposition of measure $\nu$ we conclude that $\alpha = \nu^+$, $\delta = \nu^-$. 

(ii) By Theorem 4.4 $u_{n,n} \to u$ in $G_c^q$ for every $q \in (0, 1)$. By (4.12), $w_1 \leq u_{n,n} \leq w_1$, $n \geq 1$. The rest of the proof of (ii) is analogous to that of Corollary 3.9.
Proposition 4.6. Assume that (H1)–(H4) and (H6) are satisfied and let \((u, \nu)\) be a solution of \(\text{OP}(f + d\mu, h_1, h_2)\). Then
\[
\|\nu^+\| \leq 4(\|\mu\| + \|f_0\| + \|\lambda^+\| + \|f_v^-\|), \\
\|\nu^-\| \leq 4(\|\mu\| + \|f_0\| + \|\lambda^-\| + \|f_v^+\|)
\]
with \(\lambda = -Lv\), where \(v\) is the function from condition (H6).

Proof. From (4.5), (4.6) and [40] Lemma 2.6 we deduce that \(\|\alpha\| \leq \|\nu\|\). Hence \(\|\nu^+\| \leq \|\nu\|\) since \(\alpha = \nu^+\) by Corollary 4.5 On the other hand, by Proposition 3.10
\[
\|\nu\| \leq 2(\|\lambda^+\| + \|f_v^-\| + \|\mu^-\| + \|f_0\| + \|\lambda^+\| + \|f_v^-\|),
\]
which proves the desired inequality for \(\nu^+\). The inequality for \(\nu^-\) can be proved in much the same way. ■

Proposition 4.7. Assume that (H1)–(H4) and (H6) are satisfied and let \((u, \nu)\) be a solution of \(\text{OP}(f + d\mu, h_1, h_2)\). If \(\lambda, f_v, f_0, \mu \in \mathcal{M}_{0,b}\), then \(\nu \in \mathcal{M}_{0,b}\), \(T_k(u) \in D_{e}[\mathcal{E}]\) for every \(k \geq 0\) and (3.15) is satisfied.

Proof. This follows from Proposition 4.6 and [40] Proposition 3.10, Theorem 4.2. ■

Proposition 4.8. Let \((u, \nu)\) be a solution to \(\text{OP}(d\mu, h_1, h_2)\). Assume that there exists \(v\) such that \(h_1 \leq v \leq h_2\) and \(v = R\lambda\) for some \(\lambda\) such that \(|\lambda| \in D'_c[\mathcal{E}]\). Then \(u \in D_{e}[\mathcal{E}], \nu \in D'_c[\mathcal{E}]\) and \((u, \nu)\) is the unique pair in \(D_{e}[\mathcal{E}] \times D'_c[\mathcal{E}]\) such that
\[
\mathcal{E}(u, \eta) = \int_E \eta d\mu + \int_E \eta d\nu, \quad \eta \in D_{e}[\mathcal{E}], \quad h_1 \leq u \leq h_2 \text{ a.e.}
\]
and
\[
\int_E (u - \eta) d\nu \leq 0, \quad \eta \in D_{e}[\mathcal{E}], \quad h_1 \leq \eta \leq h_2 \text{ a.e.}
\]

Proof. Since \(|\lambda| \in D'_c[\mathcal{E}]\), we have \(v \in D_{e}[\mathcal{E}]\). With the notation of Theorem 4.4 (with \(f \equiv 0\)), by (4.3) we have
\[
\|\alpha_{n,k}\|_{\mathcal{E}'} \leq \|\tilde{\nu}_n\|_{\mathcal{E}'}, \quad n, k \geq 1.
\]
Of course \((\tilde{u}_n, \tilde{\nu}_n)\) is a solution to \(\text{OP}(-d\mu^- - d\lambda^-, h - (u_n - h)^-)\), so by Proposition 3.16
\[
\|\tilde{\nu}_n\|_{\mathcal{E}'} \leq 3(\|\mu^-\|_{\mathcal{E}'} + \|\lambda^-\|_{\mathcal{E}'} + \|v\|_{\mathcal{E}}).
\]
Since \(\|R\beta\|_{\mathcal{E}} \leq \|\beta\|_{\mathcal{E}'}\) for every \(\beta \in D'_c[\mathcal{E}]\), from the above inequalities it follows that
\[
\|R\alpha_{n,k}\|_{\mathcal{E}} \leq 3(\|\mu^-\|_{\mathcal{E}'} + \|\lambda^-\|_{\mathcal{E}'} + \|v\|_{\mathcal{E}}).
\]
By Theorem 4.4 and Corollary 4.5 we have \( R\alpha_{n,k} \to R\alpha_n \) as \( k \to \infty \) and \( R\alpha_n \not\to R\nu^+ \) as \( n \to \infty \). Hence we get
\[
\|R\nu^+\| \leq 3(\|\mu^-\|\mathcal{E}^c + \|\lambda^-\|\mathcal{E}^c + \|v\|\mathcal{E}).
\]
This implies that \( \nu^+ \in D'_e[\mathcal{E}] \). Of course the pair \((-u, \nu^-)\) is a solution to \( \text{OP}(-d\mu-d\nu^+, -h_2) \), so the desired result follows from Proposition 3.16. 

**Proposition 4.9.** Assume (H1)–(H4). If \((u, \nu)\) is a solution of \( \text{OP}(f + d\mu, h_1, h_2) \), then \( u \) admits representation (1.15).

**Proof.** Let \( v \) be a supersolution of \( \text{PDE}(f + d\mu - d\nu^-) \) such that \( v \geq h_1 \text{ m-a.e.} \). Then there exists a nonnegative measure \( \lambda \in \mathbb{M}_0 \) such that
\[
-Lv = f(x, v) + \mu - \nu^- + \lambda.
\]
Since \( v \geq h_1 \text{ m-a.e.} \),
\[
-Lv = f(x, v) + \mu + n(v - h_1)^- - \nu^- + \lambda.
\]
Observe that the pair \((u, \nu^+)\) is a solution of \( \text{OP}(f + d\mu - d\nu^-, h_1) \). Therefore, by Theorem 3.8 \( u_n \not\geq u \text{ q.e.} \), where
\[
-Lu_n = f(x, u_n) + \mu + n(u_n - h_1)^- - \nu^-.
\]
By Proposition 3.12 \( u_n \leq v \text{ q.e.} \), which implies that \( u \leq v \text{ q.e.} \). 

**5. Lewy–Stampacchia type inequality and stability results.** In this section, we prove a Lewy–Stampacchia type inequality in our general framework and give some stability results for solutions. In the case of one barrier and regular data, inequalities of such type for nonlocal operators (on \( \mathbb{R}^n \)) were proved in [59] (see also the recent papers [27, 52] for an abstract Lewy–Stampacchia inequality and for the same type of inequality in the Heisenberg group).

Let us stress that the measures \( f_{h_1} \cdot m, \mu, Lh_1 \) and \( \nu \) in the theorem below need not be finite.

**Theorem 5.1.** Let \( \mu \in \mathbb{M}_0 \) and suppose that \((u, \nu)\) is a solution of \( \text{OP}(f + d\mu, h_1, h_2) \). If \( h_1 \) is a difference of natural potentials, then
\[
(5.1) \quad \nu^+ \leq 1_{\{u = h_1\}} \cdot (f_{h_1} \cdot m + \mu + Lh_1)^-.
\]

**Proof.** By the assumption on the barrier \( h_1 \), there is a measure \( \alpha \in \mathbb{M}_0 \) such that for q.e. \( x \in E \),
\[
h_1(x) = E_x \zeta \int_0^\zeta dA^\alpha_t.
\]
Therefore, by Lemma 3.7, there exists a martingale AF \( M^1 \) of \( \mathbb{M} \) such that
\[
h_1(X_t) = \int_0^\zeta dA^\alpha_t - \int_0^t dM^1_t, \quad t \in [0, \zeta].
\]
By the Tanaka–Meyer formula (see [53, Theorem IV.70]),
\[(u - h_1)^+(X_t) = (u - h_1)^+(X_0) - \int_0^t 1_{\{u > h_1\}}(X_r)f_u(X_r)\,dr\]

\[-\int_0^t 1_{\{u > h_1\}}(X_r)\,d(A^{\nu+}_r + A^\mu_r - A^\alpha_r) + \int_0^t 1_{\{u > h_1\}}(X_r)\,dA^-_r\]

\[-\frac{1}{2}L^0_t(Y) + J^+_t + \int_0^t 1_{\{u > h_1\}}(X_r)\,d(M_r - M^1_r),\]

where

\[J^+_t = \sum_{0<s\leq t} (\varphi(Y_s) - \varphi(Y_{s-}) - \varphi'(Y_{s-})\Delta Y_s), \quad Y_t = (u - h_1)(X_t), \quad \varphi(x) = x^+,\]

\[\varphi'\] denotes the left derivative of \(\varphi\), and \(L^0(Y)\) is the local time of \(Y\) at 0. Since \(Y_t \geq 0\) for \(t \geq 0\), we conclude from the above equations that

\[0 = \int_0^t 1_{\{u = h_1\}}(X_r)f_{h_1}(X_r)\,dr + \int_0^t 1_{\{u = h_1\}}(X_r)\,d(A^{\nu+}_r + A^\mu_r - A^\alpha_r)\]

\[-\int_0^t 1_{\{u = h_1\}}(X_r)\,dA^-_r + \frac{1}{2}L^0_t(Y) + J^+_t - \int_0^t 1_{\{u = h_1\}}(X_r)\,d(M_r - M^1_r).\]

Since \(\int_0^t dA^{\nu+}_r = \int_0^t 1_{\{u = h_1\}}(X_r)\,dA^{\nu+}_r\), we have

\[\frac{1}{2}L^0_t(Y) + J^{+,p}_t + \int_0^t dA^{\nu+}_r = -\int_0^t 1_{\{u = h_1\}}(X_r)f_{h_1}(X_r)\,dr\]

\[+ \int_0^t 1_{\{u = h_1\}}(X_r)\,d(A^{\nu-}_r - A^\mu_r + A^\alpha_r),\]

where \(J^{+,p}_t\) is the dual predictable projection of the process \(J^+_t\). Since \(dA^{\nu+}\), \(dA^{\nu-}\) are orthogonal, \(\int_0^t 1_{\{u = h_1\}}(X_r)\,dA^{\nu-}_r = 0\). Therefore

\[dA^{\nu+}_t \leq 1_{\{u = h_1\}}(X_t)(-f_{h_1}(X_t)\,dt - dA^\mu_t + dA^\alpha_t)^+\]

\[= 1_{\{u = h_1\}}(X_t)(f_{h_1}(X_t)\,dt + dA^\mu_t - dA^\alpha_t)^-,

which combined with Revuz duality implies (5.1). \(\blacksquare\)

**Proposition 5.2.** Assume that \(\mu_n, \mu \in \mathbb{M}_0\) and \(f, \nu\) satisfy (H1). Let \((u_n, \nu_n), \ (u, \nu)\) be solutions of \(\text{OP}(f_n + d\mu_n, h_1, h_2)\) and \(\text{OP}(f + d\mu, h_1, h_2)\), respectively. If

\[(5.2) \quad R|\mu_n - \mu| \to 0, \quad R|f_n(\cdot, u) - f(\cdot, u)| \to 0 \quad \text{m-a.e.},\]

then \(u_n \to u \text{ m-a.e.}\).

**Proof.** By the definition of a solution to the obstacle problem and Lemma 3.7 there exist martingale AFs \(M, M^n\) of \(\mathbb{M}\) such that for q.e. \(x \in E\),
\[ u(X_t) = \int_0^\zeta f(X_r, u(X_r)) \, dr + \int_0^\zeta dA_r^\mu + \int_0^\zeta dA_r^\nu + \int_0^\zeta dM_r, \quad 0 \leq t \leq \zeta, \]

\[ u_n(X_t) = \int_0^\zeta f_n(X_r, u_n(X_r)) \, dr + \int_0^\zeta dA_r^{\mu_n} + \int_0^\zeta dA_r^{\nu_n} + \int_0^\zeta dM_r^n, \quad 0 \leq t \leq \zeta, \]
P_{x,\text{a.s.}}. By the Tanaka–Meyer formula, (H1) and the minimality conditions for \( \nu_n \) and \( \nu \) we have

\[ |u_n(x) - u(x)| \leq E_x \int_0^\zeta |f_n(\cdot, u) - f(\cdot, u)|(X_r) \, dr + E_x \int_0^\zeta dA_r^{\mu_n - \mu} \]

\[ = R|f_n(\cdot, u) - f(\cdot, u)|(x) + R|\mu_n - \mu|(x) \]

for q.e. \( x \in X \). From this and (5.2), \( u_n \to u \text{ m-a.e.} \)

**Remark 5.3.** If \( \mu_n \to \mu \) in the total variation norm and \( f_n(\cdot, u) \to f(\cdot, u) \)
in \( L^1(E; m) \), then assumption (5.2) is satisfied for some subsequence of \( \{n\} \).

Indeed, since \( E \) is transient, there exists a strictly positive \( \eta \in B_b(E) \) such that \( \|\hat{G}\eta\|_{\infty} < \infty \) (see [50] Theorem 1.3.4]). Therefore

\[ \int_E \eta R|\mu_n - \mu| \leq \|\hat{G}\eta\|_{\infty}|\mu_n - \mu|(E), \]

and

\[ \int_E \eta R|f_n(\cdot, u) - f(\cdot, u)| \leq \|\hat{G}\eta\|_{\infty}\|f_n - f\|_{L^1}, \]

from which the desired result follows.

**Remark 5.4.** Let \((u, \nu)\) be a solution to \( \text{OP}(f + d\mu, h_1, h_2) \). Assume that there exists \( v \) such that \( h_1 \leq v \leq h_2 \) and \( v = R\lambda \) for some \( \lambda \) such that \( |\lambda| \in D'_e[\mathcal{E}] \) (in the case where \( h_2 \equiv +\infty \) it is enough to assume that there exists \( v \in D_e[\mathcal{E}] \) such that \( v \geq h_1 \)). Let \( g \) be a strictly positive function such that \( g \in D'_e[\mathcal{E}] \) and let \( \{F_n\} \) be a nest such that \( \mu_n := 1_{F_n} \cdot \mu \in D'_e[\mathcal{E}] \). For \( n \in \mathbb{N} \) set

\[ f_n(x, y) = \frac{ng(x)}{1 + ng(x)}(f \wedge n)(x, y), \quad x \in E, \ y \in \mathbb{R}. \]

By Theorem 4.4 and Proposition 4.8 there exists a unique solution \( u_n \) of the variational inequality (1.16) with \( f, \mu \) replaced by \( f_n, \mu_n \), and moreover \( u_n \) coincides with solution to \( \text{OP}(f_n + d\mu_n, h_1, h_2) \). By Proposition 5.2 and Remark 5.3 up to subsequence, \( u_n \to u \text{ m-a.e.} \) This shows that each solution to (1.1) may be approximated by solutions to variational inequalities.

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References

[1] N. Alibaud, B. Andreianov and M. Bendahmane, Renormalized solutions of the fractional Laplace equation, C. R. Math. Acad. Sci. Paris 348 (2010), 759–762.
[2] H. Attouch et C. Picard, Problèmes variationnels et théorie du potentiel non linéaire, Ann. Fac. Sci. Toulouse Math. 1 (1979), 89–136.
[3] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J.-L. Vázquez, $L^1$-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 22 (1995), 241–273.
[4] P. Bénilan and M. G. Crandall, Completely accretive operators, in: P. Clement et al. (eds.), Semigroup Theory and Evolution Equations, Dekker, New York, 1991, 41–76.
[5] P. Bénilan et P. Wittbold, Absorptions non linéaires, J. Funct. Anal. 114 (1993), 59–96.
[6] A. Bensoussan and J.-L. Lions, Applications of Variational Inequalities in Stochastic Control, North-Holland, Amsterdam, 1982.
[7] L. Beznea and N. Boboc, Potential Theory and Right Processes, Kluwer, Dordrecht, 2004.
[8] M. R. Blumenthal and R. K. Getoor, Markov Processes and Potential Theory, Dover Publ., New York, 2007.
[9] L. Boccardo and G. R. Cirmi, Existence and uniqueness of solution of unilateral problems with $L^1$-data, J. Convex Anal. 6 (1999), 195–206.
[10] L. Boccardo et T. Gallouët, Non-linear elliptic and parabolic equations involving measure data, J. Funct. Anal. 8 (1989), 149–169.
[11] L. Boccardo et T. Gallouët, Problèmes unilatéraux avec données dans $L^1$, C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), 617–655.
[12] H. Brezis, Problèmes unilatéraux, J. Math. Pures Appl. 51 (1972), 1–168.
[13] H. Brezis and S. Serfaty, A variational formulation for the two-sided obstacle problem with measure data, J. Funct. Anal. 8 (1992), 357–374.
[14] H. Brezis and W. A. Strauss, Semi-linear second-order elliptic equations in $L^1$, J. Math. Soc. Japan 25 (1973), 565–590.
[15] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica, $L^p$ solutions of backward stochastic differential equations, Stochastic Process. Appl. 108 (2003), 109–129.
[16] J. A. Carrillo, M. G. Delgadino and A. Mellet, Regularity of local minimizers of the interaction energy via obstacle problems, Comm. Math. Phys. 343 (2016), 747–781.
[17] H. Chen and L. Véron, Semilinear fractional elliptic equations involving measures, J. Differential Equations 257 (2014), 1457–1486.
[18] Z.-Q. Chen and M. Fukushima, Symmetric Markov Processes, Time Change, and Boundary Theory, Princeton Univ. Press, Princeton, NJ, 2012.
[19] R. Cont and P. Tankov, Financial Modelling with Jump Processes, Financial Math. Ser., Chapman & Hall/CRC, Boca Raton, FL, 2004.
[20] P. Dall’Aglio and G. Dal Maso, Some properties of the solutions of obstacle problems with measure data, Ricerche Mat. 48 (1999), 99–116.
[21] P. Dall’Aglio and C. Leone, Obstacles problems with measure data and linear operators, Potential Anal. 17 (2002), 45–64.
[22] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28 (1999), 741–808.
[23] E. Dancer and Y. Du, On a free boundary problem arising from population biology, Indiana Univ. Math. J. 52 (2003), 51–67.
E. N. Dancer, D. Hilhorst, M. Mimura and L. A. Peletier, *Spatial segregation limit of a competition-diffusion system*, Eur. J. Appl. Math. 10 (1999), 97–115.

R. Dumitrescu, M.-C. Quenez and A. Sulem, *Generalized Dynkin games and doubly reflected BSDEs with jumps*, Electron. J. Probab. 21 (2016), art. 64, 32 pp.

M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter, Berlin, 1994.

N. Gigli and S. Mosconi, *The abstract Lewy–Stampacchia inequality and applications*, J. Math. Pures Appl. 104 (2015), 258–275.

M. Grigorova, P. Imkeller, Y. Ouknine and M.-C. Quenez, *Doubly reflected BSDEs and \( \mathcal{E}^f \)-Dynkin games: beyond the right-continuous case*, Electron. J. Probab. 23 (2018), art. 122, 38 pp.

N. E. Humphries, N. Queiroz, J. R. M. Dyer, N. G. Pade, M. K. Musyl, K. M. Schaefer, D. W. Fuller, J. M. Brunnschweiler, T. K. Doyle, J. D. R. Houghton, G. C. Hays, C. S. Jones, L. R. Noble, V. J. Wearmouth, E. J. Southall and D. W. Sims, *Environmental context explains Lévy and Brownian movement patterns of marine predators*, Nature 465 (2010), 1066–1069.

N. Jacob, *Pseudo-Differential Operators and Markov Processes. Vol. I: Fourier Analysis and Semigroups*, Imperial College Press, London, 2001.

N. Jacob, *Pseudo-Differential Operators and Markov Processes. Vol. II: Generators and Their Potential Theory*, Imperial College Press, London, 2002.

J. Jacod, *Convergence en loi de semimartingales et variation quadratique*, in: Lecture Notes in Math. 850, Springer, 1981, 547–560.

K. H. Karlsen, F. Petitta and S. Ulusoy, *A duality approach to the fractional Laplacian with measure data*, Publ. Mat. 55 (2011), 151–161.

D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

T. Klimsiak, *Semilinear elliptic systems with measure data*, Ann. Mat. Pura Appl. 194 (2015), 55–76.

T. Klimsiak, *Reduced measures for semilinear elliptic equations involving Dirichlet operators*, Calc. Var. Partial Differential Equations 55 (2016), no. 4, art. 78, 27 pp.

T. Klimsiak and A. Rozkosz, *Dirichlet forms and semilinear elliptic equations with measure data*, J. Funct. Anal. 265 (2013), 890–925.

T. Klimsiak and A. Rozkosz, *Obstacle problem for semilinear parabolic equations with measure data*, J. Evol. Equations 15 (2015), 457–491.

T. Klimsiak and A. Rozkosz, *Renormalized solutions of semilinear equations involving measure data and operator corresponding to Dirichlet form*, Nonlinear Differential Equations Appl. 22 (2015), 1911–1934.

T. Klimsiak and A. Rozkosz, *Semilinear elliptic equations with measure data and quasi-regular Dirichlet forms*, Colloq. Math. 145 (2016), 35–67.

T. Klimsiak and A. Rozkosz, *On the structure of bounded smooth measures associated with quasi-regular Dirichlet form*, Bull. Polish Acad. Sci. Math. 65 (2017), 45–56.

T. Klimsiak and A. Rozkosz, *The valuation of American options in a multidimensional exponential Lévy model*, Math. Finance 28 (2018), 1107–1142.

T. Kulczycki, *Properties of Green function of symmetric stable processes*, Probab. Math. Statist. 17 (1997), 339–364.

T. Kuusi, G. Mingione and Y. Sire, *Nonlocal equations with measure data*, Comm. Math. Phys. 337 (2015), 1317–1368.

C. Leone, *Obstacle problems for monotone operators with measure data*, in: Free Boundary Problems, Int. Ser. Numer. Math. 154, Birkhäuser, Basel, 2007, 291–305.
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[46] Z. Ma and M. Röckner, *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*, Springer, Berlin, 1992.

[47] A. Mokrane and F. Murat, *The Levy–Stampacchia inequality for bilateral problems*, Ricerche Mat. 53 (2004), 139–182.

[48] A. Mokrane, Y. Tahraoui and G. Vallet, *On Levy–Stampacchia inequalities for a pseudomonotone elliptic bilateral problem in variable exponent Sobolev spaces*, Mediterr. J. Math. 16 (2019), art. 64, 22 pp.

[49] L. Orsina and A. C. Ponce, *Semilinear elliptic equations and systems with diffuse measures*, J. Evol. Equations 8 (2008), 781–812.

[50] Y. Oshima, *Semi-Dirichlet Forms and Markov Processes*, de Gruyter, Berlin, 2013.

[51] S. Peng and M. Xu, *The smallest g-supercritical and reflected BSDE with single and double L² obstacles*, Ann. Inst. H. Poincaré Probab. Statist. 41 (2005), 605–630.

[52] A. Pinamonti and E. Valdinoci, *A Levy–Stampacchia estimate for variational inequalities in the Heisenberg group*, Rend. Istit. Mat. Univ. Trieste 45 (2013), 1–22.

[53] P. Protter, *Stochastic Integration and Differential Equations*, 2nd ed., Springer, Berlin, 2004.

[54] P. Quittner and Ph. Souplet, *A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces*, Arch. Ration. Mech. Anal. 174 (2004), 49–81.

[55] J. M. Rakotoson, *A sufficient condition for a blow-up in the space of absolutely continuous functions for the very weak solution*, Appl. Math. Optim. 73 (2016), 153–163.

[56] J. F. Rodrigues and R. Teymurazyan, *On the two obstacles problem in Orlicz–Sobolev spaces and applications*, Complex Var. Elliptic Equations 56 (2011), 769–787.

[57] X. Ros-Oton, *Obstacle problems and free boundaries: an overview*, SeMA J. 75 (2018), 399–419.

[58] A. Rozkosz and L. Slomiński, *Stochastic representation of entropy solutions of semilinear elliptic obstacle problems with measure data*, Electron. J. Probab. 17 (2012), art. 40, 27 pp.

[59] R. Servadei and E. Valdinoci, *Lewy–Stampacchia type estimates for variational inequalities driven by (non)local operators*, Rev. Mat. Iberoamer. 29 (2013), 1091–1126.

[60] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) 15 (1965), 189–258.

[61] G. Stampacchia, *Équations elliptiques du second ordre à coefficients discontinus*, Presses Univ. de Montréal, Montreal, Que., 1966.

[62] P. Wittbold, *Nonlinear diffusion with absorption*, Potential Anal. 7 (1997), 437–465.

[63] J. Voigt, *Absorption semigroups*, J. Operator Theory 20 (1988), 117–131.

[64] J. Zabczyk, *Stopping games for symmetric Markov processes*, Probab. Math. Statist. 4 (1984), 185–196.

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