Interactions of three-dimensional solitons in the cubic-quintic model

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We report results of a systematic numerical analysis of interactions between three-dimensional (3D) fundamental solitons, performed in the framework of the nonlinear Schrödinger equation (NLSE) with the cubic-quintic (CQ) nonlinearity, combining the self-focusing and defocusing terms. The 3D NLSE with the CQ terms may be realized in terms of spatiotemporal propagation of light in nonlinear optical media, and in Bose-Einstein condensates, provided that losses may be neglected. The first part of the work addresses interactions between identical fundamental solitons, with phase shift $\varphi$ between them, separated by a finite distance in the free space. The outcome strongly changes with the variation of $\varphi$: in-phase solitons with $\varphi = 0$, or with sufficiently small $\varphi$, merge into a single fundamental soliton, with weak residual oscillations in it (in contrast to the merger into a strongly oscillating breather, which is exhibited by the 1D version of the same setting), while the choice of $\varphi = \pi$ leads to fast separation between mutually repelling solitons. At intermediate values of $\varphi$, such as $\varphi = \pi/2$, the interaction is repulsive too, breaking the symmetry between the initially identical fundamental solitons, there appearing two solitons with different total energies (norms). The symmetry-breaking effect is qualitatively explained, similar to how it was done previously for 1D solitons. In the second part of the work, a pair of fundamental solitons trapped in a 2D potential is considered. It is demonstrated that they may form a slowly rotating robust “molecule”, if annihilation kicks are applied to them in opposite directions, perpendicular to the line connecting their centers.

The self-focusing cubic nonlinearity is ubiquitous in a large variety of physical media (optics, plasmas, Bose-Einstein condensates, etc.), helping to build solitons in these media. However, while the one-dimensional (1D) solitons are completely stable, they are subject to destructive instabilities in 2D and 3D, induced by the critical and supercritical collapse, which is driven by the same cubic nonlinearity in the multidimensional settings. A well-known solution of the stabilization problem for 2D and 3D solitons is the addition of a self-defocusing quintic term, which arrests the collapse. This term naturally appears in models of optical media (in particular, colloidal suspensions of metallic nanoparticles). Once the stabilization of the solitons is secured, the next natural step is to consider interactions between them, which is the subject of the present work, for the most interesting case of 3D solitons. We perform the analysis by means of systematic numerical simulations and some analytical approximations. First, we consider interactions between identical solitons, with phase shift $\varphi$ between their complex wave functions, assuming that the solitons are separated by a relatively small distance. The outcome of the interaction strongly depends on $\varphi$. First, attraction between in-phase solitons, with $\varphi = 0$, leads to their quick merger into a single soliton, in an almost fundamental form (with weak residual intrinsic oscillations in it, which is different from the outcome of the interaction in 1D, where in-phase solitons merge into a strongly excited breather). On the other hand, the solitons with $\varphi = \pi$ interact repulsively, separating from each other. A noteworthy outcome is produced by the interaction in the intermediate case, with $\varphi = \pi/2$: the pair of initially identical solitons features symmetry breaking between them, producing two solitons with different integral norms. This nontrivial effect is explained by considering the amplitude and phase structure of the two-soliton pair (the symmetry breaking is a consequence of a mismatch between the pair’s “amplitude center” and “phase center”). A challenging situation is addressed in the second part of the work, which deals with a pair of 3D solitons trapped in a common 2D potential (the third spatial direction remains unconfined). It is found that the pair may form a sufficiently robust slowly rotating “molecule”, if appropriate initial conditions are applied. The theoretical results reported in the paper suggest new possibilities for experiments with multidimensional solitons.

I. INTRODUCTION

Cubic nonlinearity, which represents the Kerr effects, is a ubiquitous feature of a great variety of optical media. It is commonly known that the interplay of the cubic self-focusing with the group-velocity dispersion or spatial diffraction gives rise to effectively one-dimensional (1D) solitons in the temporal and spatial domains, respectively \[1\]. The same type of the nonlinearity, which represents attractive interaction between atoms in Bose-Einstein condensates (BECs), helps to create stable effec-
tively 1D matter-wave solitons \[2\]. A natural extension of these well-known results is building multidimensional solitons \[3\] (in particular, spatiotemporal solitons in optics \[4, 3\], alias “light bullets” \[3\]). However, a well-known problem is that, while formal soliton solutions to two- and three-dimensional (2D and 3D) nonlinear Schrödinger equations (NLSEs) can be easily found in the numerical form, they are completely unstable due to the presence of the critical (2D) and supercritical (3D) collapse in the same equation.

The simplest possibility to arrest the collapse and stabilize the multidimensional solitons is using a medium with nonlinear response of the cubic-quintic (CQ) type, which includes a self-defocusing fifth-order term. The CQ nonlinearity may be realized in BEC too, with the quintic term provided by repulsive three-body collisions in a relatively dense condensate \[14, 20, 28\], although this interpretation is hampered by the fact that the three-body interactions contribute to losses in the condensate \[21, 22\].

As concerns the theoretical analysis, the stability of isotropic fundamental solitons in 2D and 3D media with the CQ nonlinearity is obvious \[22, 28\]. A challenging issue for this model is a possibility of the existence of stable 2D and 3D solitons with embedded vorticity, as the saturation of the nonlinearity (in particular, provided by the quintic self-defocusing) does not, generally, secure the stability of vortex (ring-shaped) solitons against ring-splitting perturbations \[26\]. The partial stability of 2D solitary vortices with topological charges \(S = 1, 2, 3\) was discovered in direct simulations and rigorously investigated in Refs. \[29\] and \[31\], respectively (stability regions for \(S > 3\) exist too, but they are extremely narrow); see also Refs. \[32\] and \[33\]. In 3D, stability regions for toroidal solitons with \(S = 1\) in single- and two-component NLSE systems with the CQ nonlinearity were found, respectively, in Refs. \[34\] and \[35\] (the stability of vortex solitons in the two-component 2D model was considered in Ref. \[35\]). Related results were produced by simulations of the evolution of multi-soliton 2D \[36\] and 3D \[37\] clusters carrying an overall angular momentum, which demonstrate slow merger or expansion of the cluster \[37\].

The objective of the present work is to consider interactions of 3D fundamental solitons in the framework of the NLSE with the CQ nonlinearity, which is a relevant problem for the above-mentioned physical realizations of the model. Thus far, soliton-soliton collisions in the CQ model were chiefly studied in the 1D setting \[38\] (some results were also reported for interactions of 2D solitons \[39\]). By means of systematic simulations of the 3D equation, we demonstrate that soliton-soliton interactions are essentially inelastic: depending on the initial phase difference, \(\varphi\), two identical solitons merge into a single one at \(\varphi = 0\) (on the contrary to the previously known results for the inelastic interaction of 1D solitons, the merger creates a nearly stationary fundamental soliton, rather than a strongly excited breather); the solitons bounce back at \(\varphi = \pi\); and symmetry breaking takes place at intermediate values of \(\varphi\), producing a pair of separating fundamental solitons with unequal energies (norms). In addition to that, we also construct a slowly rotating “two-soliton molecule” (this concept is known in 1D models \[40, 41\]), placing a 3D soliton pair, with angular momentum imparted to it, in a 2D trapping potential. The latter result suggests a possibility of the creation of a such a specific rotating bound state in the experiment.

The rest of the paper is organized as follows. The model is formulated in Section II. Numerical results for the soliton-soliton interactions in free space are summarized in Section III, the formation of the rotating “soliton molecule” is reported in Section IV, and the paper is concluded by Section V.

II. THE MODEL

The NLSE for local amplitude \(\Psi\) of the electromagnetic wave in an optical waveguide, or the mean-field wave function in BEC, with the CQ nonlinearity in the 3D space \((x, y, z)\), and 2D trapping potential \(V(x, y)\), if any, is written, in the scaled form, as

\[
\begin{aligned}
\imath \frac{\partial \Psi}{\partial t} + \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Psi + \left( |\Psi|^2 - |\Psi|^4 \right) + V(x, y) \Psi &= 0. \\
\end{aligned}
\tag{1}
\]

In terms of the spatiotemporal propagation in optics, evolution variable \(t\) is actually the propagation distance, \(x\) and \(y\) are transverse coordinates, and \(z\) the temporal coordinate (reduced time) \[1\]. In addition to the Hamiltonian, Eq. (1) conserves the integral norm, which is scaled energy \((E)\) in the optical model, or scaled number of atoms in BEC:

\[
E = \iiint |\Psi(x, y, z)|^2 dxdydz. \tag{2}
\]

First we address the free-space case, with \(V(x, y) = 0\), while the dynamics of a rotating bi-soliton “molecule”, trapped in the 2D harmonic-oscillator potential,

\[
V(x, y) = V_0 \left[ (x - x_c)^2 + (y - y_c)^2 \right], \tag{3}
\]

where \((x_c, y_c)\) are coordinates of the center of the integration domain (see, e.g., Fig. 2 below), is considered separately in Section IV. In the absence of any potential (free space), Eq. (1) conserved the linear and angular
momenta, together with the Hamiltonian and $E$. In the presence of the isotropic trapping potential \([3], \text{Eq. (1)}\), still conserved the $z$-component of the angular momentum.

In terms of optics, stationary solutions for isotropic fundamental solitons in the free space, with propagation constant $K > 0$ (in terms of BEC, $-K$ is the chemical potential), are looked for in the usual form,

$$
\Psi (x, y, z, t) = e^{iKt}U \left( r = \sqrt{(x-x_c)^2 + (y-y_c)^2 + z^2} \right),
$$

with real radial amplitude function, $U(r)$, satisfying equation

$$
KU = \frac{d^2U}{dr^2} - \frac{2}{r} \frac{dU}{dr} + U^3 - U^5,
$$

subject to boundary conditions $dU/dr(r = 0) = 0$, $U(r) \sim \exp \left(-\sqrt{Kr}\right)$ at $r \to \infty$.

A set of numerically generated profiles $U(r)$ (actually, they were obtained, although not displayed, in Ref. \[33\]) is presented in Fig. 1. They were produced by a numerical method based on Newton-Raphson iterations \[42\], implemented in the Cartesian coordinates in the 3D domain of size $L \times L \times L$, where $L = 110$ was sufficient to accommodate all the solitons and two-soliton complexes considered in this work. The stationary solutions were obtained with relative accuracy $10^{-8}$. Simulations of the evolution of 3D states were then performed by means of the well-known split-step algorithm \[13\].

Equation (5) generated 3D fundamental solitons at $K < K_{\text{max}} = 3/16$, with $K_{\text{max}}$ determined by the exact soliton solution for the 1D NLSE with the CQ nonlinear terms \[13, 42\]; this happens because very broad 3D solitons become quasi-one-dimensional in the limit of $K \to 3/16$. In this limit, the solitons develop a flat-top shape with a diverging radial size, $R \simeq (1/\sqrt{3}) \ln \left((3 - 16K)^{-1}\right)$, and the asymptotic value of the amplitude in the flat-top area, $U(K = 3/16) = \sqrt{3}/2$. As shown in Ref. \[33\], the fundamental 3D solitons are stable at $K > K_{\text{min}} \approx 0.02$, hence all the solitons whose profiles are displayed in Fig. 1 are stable, except for the one corresponding to $K = 0.012$.

Lastly, it is relevant to mention that the shape of the fundamental solitons which do not yet feature an extended flat-top profile, may be accurately predicted by the variational approximation based on the Gaussian ansatz, $U(r) = A \exp \left(-r^2/W^2\right)$, with constants $A$ and $W$ representing the soliton’s amplitude and width. These results are not displayed here in detail, as the approximation is similar to that developed previously for the NLSE with the CQ nonlinearity in 1D \[20, 40\], 2D \[23\], and 3D \[25\] settings.

**FIG. 1.** (Color online) A set of radial profiles of the 3D isotropic solitons, found in the numerical form at values of the propagation constant, $K$, indicated in the figure.

### III. INTERACTIONS OF THREE-DIMENSIONAL SOLITONS

The interaction between two solitons in the framework of Eq. (1) is initiated by input

$$
\Psi_0 (x, y, z) = \exp(i\varphi/2)U \left( x - x_c - d/2\sqrt{2}, y - y_c - d/2\sqrt{2}, z \right) + \exp(-i\varphi/2)U \left( x - c_c + d/2\sqrt{2}, y - y_c + d/2\sqrt{2}, z \right),
$$

where $U \left( x - x_c, y - y_c \right)$ are the stationary solitons [see Eq. (4)], whose radial profiles are shown in Fig. 1, with separation $d$ between their centers [directed along the diagonal in the $(x, y)$ plane], and phase shift $\varphi$ between them. Generic outcomes of the interaction may be adequately displayed for soliton pairs with $K = 0.062$, separated by distance $d = 10$, by varying phase $\varphi$.

**A. The merger of in-phase solitons ($\varphi = 0$) and rebound of out-of-phase ones ($\varphi = \pi$)**

Figure 2 shows a typical example of the interaction of two identical in-phase solitons, with zero phase difference. It is known that the sign of the interaction between fundamental 3D solitons is attractive for $\varphi = 0$ in input \[6, 17\]. In accordance with this, the in-phase solitons attract each other, quickly merging into a single one. Figure 3 demonstrates that the soliton produced by merger is a nearly isotropic state (although the input is obviously anisotropic), whose shape is very close to that of the stationary fundamental soliton with the same value of total energy \[6\]. The simulations demonstrate residual oscillations of the local power at the center of the emerging sin-
FIG. 2. (Color online) A typical example of the merger of two in-phase solitons (with phase shift $\varphi = 0$), initially separated by distance $d = 10$ in the diagonal direction, as per Eq. (6), the propagation constant of each soliton being $K = 0.062$. In this figure, and in Figs. 3, 5, and 9 below, the density profile is displayed in the midplane, $z = 0$.

In-phase solitons governed by the one-dimensional NLSE with the CQ nonlinearity [Eq. (1) in which terms $(\partial^2/\partial y^2 + \partial^2/\partial z^2) \Psi$ are dropped] also interact attractively, which leads to their merger (which is possible in the framework of the nonintegrable equation), but an essential difference from the 3D model is that, as shown in Fig. 1, the merger of 1D solitons produces a nonstationary breather, rather than a nearly-fundamental soliton. The breather is a robust state which persists in the course of indefinitely long simulations. An explanation to this drastic difference is that the 1D equation is relatively close to the integrable cubic NLSE, which has well-known exact solutions in the form of breathers (higher-order solitons) [48]. On the other hand, the 3D NLSE with the CQ nonlinearity is very far from any integrable limit, hence the 3D equation gives rise to the quick fusion of the colliding fundamental solitons into a new one, rather than forming a complex mode with strong inner vibrations.

On the other hand, the interaction between out-of-phase fundamental 3D solitons is repulsive [47]. In accordance with this expectation, the simulations demonstrate, in Fig. 4, that the solitons with phase shift $\varphi = \pi$ bounce back from each other, without any conspicuous excitation of internal oscillations in either soliton. Simulations of the 1D version of the NLSE with the CQ nonlinear terms also reveal fast separation of the solitons (not shown here, as the result is quite obvious).
The repulsive interaction of two out-of-phase 3D solitons, with phase difference $\varphi = \pi$, initial distance $d = 10$, and propagation constant $K = 0.062$ of each soliton in the initial state.

B. Symmetry breaking in the interaction of identical 3D solitons with intermediate values of the phase difference.

The lowest-order approximation for the effective potential of the interaction of fundamental 3D solitons gives zero for phase difference $\varphi = \pi/2$ between them [47]. On the other hand, direct simulations, displayed in Fig. 6, reveal repulsion between the solitons, which is coupled to the symmetry breaking, that takes place at the initial stage of the interaction: the separating solitons carry obviously different energies (2), although input (6) was composed of identical solitons. Further, Fig. 7 corroborates that the asymmetric solitons appear as almost unperturbed fundamental ones.

The explanation of the symmetry breaking in soliton-soliton collisions was proposed (in terms of 1D solitons) in Ref. [38]. Namely, the interaction gives rise to opposite velocity vectors of the two solitons, $\pm v$. It is well known that moving solitons in the NLSE acquire the phase structure, represented by factors $\exp(\pm iv \cdot r)$ in the solutions, where $r \equiv \{x - x_c, y - y_c\}$. Then, before the solitons, which were originally introduced as per Eq. (6), separate, their coherent superposition gives rise to factor $\cos(v \cdot r + \varphi)$. As follows from here, there is a mismatch between the “amplitude center”, located at $r = 0, z = 0$, and the “phase center”, which is shifted to $r_0 = -\varphi v/v_0^2$. The mismatch qualitatively explains the breaking of the spatial symmetry by the interaction between the solitons.

Results of the systematic study of this effect are summarized in Fig. 8, which displays evolution of the symmetry-breaking measure,

$$\Delta(t) \equiv \left| E_1(t) - E_2(t) \right| / \left| E_1(t) + E_2(t) \right|,$$  \hspace{0.5cm} (7)

in the course of the collision, where $E_{1,2}(t)$ are energies computed separately for the two solitons. This definition is formal for the solitons merging into a single one (see Fig. 2), but it makes sense in the case of the symmetry-breaking interaction because, as seen, e.g., in Fig. 6 in such a case the solitons always remain separated. In the limit of $t \to \infty$, $\Delta(t)$ takes a final value corresponding to the pair of far separated solitons. As explained in the caption to Fig. 8, $\Delta = 1$ actually corresponds to the merger of the pair into a single soliton, which does not break the spatial symmetry [as mentioned above, definition (7) does not adequately measure the symmetry breaking for merging solitons]. Therefore, the actual symmetry-breaking maximum in data displayed in Fig. 8 may be identified as $\Delta \approx 0.76$, achieved at $\varphi = \pi/2$. It is assumed that there is a critical value, $\varphi_{cr}$, between $\varphi = \pi/4$ and $\pi/2$, at which the merger becomes
FIG. 8. (Color online) The symmetry-breaking measure, defined as per Eq. (4), is plotted versus $t$ for soliton-soliton interactions at several values of the initial phase shift, $\varphi$ (which appears as $\phi$ in this figure), between the initially identical interacting solitons. The line with symbols, corresponding to $\varphi = 0$, represents the quick merger of in-phase solitons into a single one, as shown in Fig. 1. The line corresponding to $\varphi = \pi/4$ also implies the merger, which occurs at $t \approx 7$. The merger formally corresponds to $\Delta = 1$ (because the single soliton is produced by the interaction), although the spatial symmetry actually remains unbroken in this case. The interacting solitons remain separated at $\varphi > \pi/4$. The out-of-phase soliton pair ($\varphi = \pi$) separate without symmetry breaking, keeping $\Delta \equiv 0$.

Incomplete and a second soliton will appear in the outcome of the interaction. Accurate identification of $\varphi_{cr}$ is a challenging objective for the 3D simulations.

IV. Rotating Two-Soliton Molecules Trapped in the External Potential

In the free space, the NLSE with the CQ nonlinearity does not produce persistent bound states of 3D solitons (such as “molecules” or ring-shaped “necklaces” (37)). Here, we aim to demonstrate that a rotating two-soliton “molecule” may be formed, in the 3D space, with the help of the 2D trapping potential (8). To this end, centers of two identical solitons, with phase shift $\varphi$ between them, were initially placed at points $(x - x_c = \pm d/2, y - y_c = z = 0)$, imparting to them initial velocities $\pm v_0$ in the direction of $y$ [i.e., multiplying the wave functions of individual solitons by $\exp(\pm iv_0 y)$].

In the absence of the trapping potential, the so built soliton pair in the free space could perform the rotation by $180^\circ$ and would then break up into separating solitons. On the other hand, Figs. 9 and 10 demonstrate that, even a shallow trapping potential (8), with strength $V_0 = 10^{-4}$, helps to build a rotating “molecule” with a quasi-stationary form. This example is displayed for the soliton pair with initial phase difference $\varphi = \pi/2$ and separation $d = 10$ along the $x$ axis, the angular momentum being imparted by the transverse kick with velocities $\pm 0.3$ in the $y$ direction. It is seen that, in accordance to the sign of the kick, the soliton pair starts the rotation in the counter-clockwise direction, with period $T \approx 150$. It is relevant to mention that, as shown above, in physical units corresponding to typical “light bullets”, $T = 150$ corresponds to the propagation distance $\sim 1$ cm in the bulk optical waveguide.

FIG. 9. (Color online) This figure and the following one display the evolution of the soliton pair, set in the counter-clockwise quasi-stationary rotational motion by initial velocities $\pm 0.3$ applied in the $y$ direction, perpendicular to the line connecting centers of the solitons, separated by initial distance $d = 10$, with phase shift $\varphi = \pi/2$ between them, in trapping potential (8) with strength $V_0 = 10^{-4}$. The rotation period is $T \approx 150$.

FIG. 10. The continuation of Fig. 9.
in the previous section (see Fig. 4), but the symmetry is partly restored, from time to time. Similar results were observed at other values of the parameters, although the collection of systematic data is hampered by fact that these fully 3D solitons are quite heavy. An alternative approach to the search of rotating two-soliton bound states may be based on looking for stationary solutions to Eq. (1) with potential (3) in a rotating reference frame. Solution of this numerical problem is beyond the framework of the present paper.

V. CONCLUSION

In this work, we aimed to perform the systematic numerical analysis of interactions between identical 3D fundamental solitons in the NLSE (nonlinear Schrödinger equation) with the CQ nonlinearity, which is a combination of self-focusing cubic and defocusing quintic terms. The model applies to the spatiotemporal propagation in nonlinear optics and, under certain conditions, to BEC. First, the interactions between two solitons with phase shift $\varphi$ in the free space, separated by distance $d$, were systematically simulated. The outcome of the interaction strongly depends on $\varphi$, leading to the merger of two solitons into one at $\varphi = 0$ (and at relatively small nonzero values of $\varphi$, such as $\varphi = \pi/4$), and the mutual rebound of the out-of-phase solitons at $\varphi = \pi$. A noteworthy finding is that the the in-phase fundamental solitons merge into one in a nearly stationary form, without conspicuous inner vibrations, unlike the results for the 1D version of the model, where the solitons feature fusion into a strongly excited breather. At intermediate values of the phase difference, such as $\varphi = \pi/2$, the solitons separate, interacting repulsively, but, unlike the case of $\varphi = \pi$, the interaction breaks the symmetry between the originally identical solitons, producing two far separated ones with essentially different energies (norms). The symmetry-breaking effect may be qualitatively explained by means of an argument similar to one which was previously used for the explanation of the symmetry breaking in the interaction of 1D solitons [8]. In the other part of the present work, it is demonstrated that a pair of 3D solitons trapped in the 2D potential may form a slowly rotating “molecule”, by initially kicking the solitons in the opposite directions, perpendicular to the line connecting their centers.

As an extension of the work, it may be interesting to systematically study interactions of stable vortex solitons with topological charges $S_{1,2} = \pm 1$, which exist in the same 3D model [5]. In that case, the result should depend, in particular, on the relative sign of $S_1$ and $S_2$. As mentioned above, it may also be relevant to develop a more comprehensive analysis of the rotating bi-soliton “molecules” trapped in the external potential (in particular, looking for them as stationary two-soliton complexes in the rotating reference frame).

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