Cardy-Verlinde Formula and entropy bounds in Kerr-Newman-AdS$_4$/dS$_4$ black holes backgrounds

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Abstract

The Cardy-Verlinde formula is further verified by using the Kerr-Newman-AdS$_4$ and Kerr-Newman-dS$_4$ black holes. In the Kerr-Newman-AdS$_4$ spacetime, we find that, for strongly coupled CFTs with AdS duals, to cast the entropy of the CFT into the Cardy-Verlinde formula the Casimir energy must contains the terms $-n \left( J \Omega_H + Q \Phi + Q \Phi_0 \right)$, which associate with rotational and electric potential energies, and the extensive energy includes the term $-Q \Phi_0$. For the Kerr-Newman-dS$_4$ black hole, we note that the Casimir energy is negative but the extensive energy is positive on the cosmological horizon; while the Casimir energy is positive but the extensive energy is negative on the event horizon (the definitions for the two energies possess the same forms as the corresponding quantities of the Kerr-Newman-AdS$_4$ black hole). Thus we have to take the absolute value of the Casimir (extensive) energy in the Cardy-Verlinde formula for the cosmological (event) horizon. The result for the Kerr-Newman-dS$_4$ spacetime provides support of the dS/CFT correspondence. Furthermore, we also obtain the Bekenstein-Verlinde-like entropy bound for the Kerr-Newman-AdS$_4$ black hole and the D-bound on the entropy of matter system in Kerr-Newman-dS$_4$ spacetime. We find that both the bounds are tightened by the electric charge.

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I. INTRODUCTION

Erik Verlinde [1] proposed in a recent paper that the entropy of the conformal field theory (CFT) in a spacetime

\[ ds^2 = -dt^2 + R^2 d\Omega_n^2, \]

(1.1)
can be related to its total energy \( E \), Casimir energy \( E_c \), and radius \( R \) of the unit sphere \( S^n \), which can be explicitly expressed as

\[ S = \frac{2\pi R \sqrt{a_1 b_1} \sqrt{E_c(2E - E_c)}}{\sqrt{E_c}}, \]

(1.2)

where \( a_1 \) and \( b_1 \) are two positive coefficients which are independent of \( R \) and \( S \). For strong coupled CFT’s with AdS dual, the value of \( \sqrt{a_1 b_1} \) is fixed to \( n \) exactly. The expression (1.2) is referred to as Cardy-Verlinde formula since it was generalized from the \((1+1)\)-dimensional Cardy formula [2] to the case in arbitrary dimensions.

From the Cardy-Verlinde formula (1.2) Verlinde postulated the Bekenstein-Verlinde entropy bound [1]

\[ \frac{S}{2\pi RE} \leq \frac{1}{\sqrt{a_1 b_1}}. \]

(1.3)

It is shown that CFTs possessing AdS duals satisfy a version of the Bekenstein entropy bound [3]

\[ \frac{S}{2\pi RE} \leq 1. \]

(1.4)

Witten [4] pointed out that the energy, temperature, and entropy of the CFT can be identified with corresponding quantities of the black hole whose boundary provides a space in which the CFT resides. Furthermore, conformal invariance can be invoked to prove that the Bekenstein-Hawking entropy of the black hole scales with volume of the event horizon. However, we need additional information to fix the proportional constant between the entropy and the event horizon volume. It is the Cardy-Verlinde formula that provides this additional information.

The Cardy-Verlinde formula is very useful, but it has not been proved for all CFTs exactly yet. Therefore, the study of validity of the Cardy-Verlinde formula for every typical spacetimes has attracted much attention recently [1], [3] - [22]: Verlinde [1] checked the formula (1.2) using the AdS Schwarzschild black holes in various dimensions and found it holds exactly; Cai [3] studied the AdS Reissner-Nordström black holes in arbitrary dimensions and the AdS black holes in higher derivative gravity, he showed that the Cardy-Verlinde formula is valid for the AdS Reissner-Nordström black hole if we subtract the electric potential energy from the Casimir and total energies, but it is invalid for the AdS black holes in higher derivative gravity; Birmingham and Mokhtar [8] found that the Cardy-Verlinde formula holds for the Taub-Bolt-AdS spacetimes at high temperature even though these spaces possess some special thermodynamical properties; Danielsson [9], Cai [8], Medved [10], Ogushi [11] et al attempt to generalize the formula to the case of the de Sitter (dS)
black holes. Such studies were motivated by the observational evidence that our universe has positive cosmological constant [23–25] and an interesting proposal for dS/CFT correspondence [26]; Klemm, Petkou, and Siopsis [13] found that the Cardy-Verlinde formula holds for the one rotation parameter Kerr-AdS$_n$ black holes whose boundary is a rotating Einstein universe. However, to my best knowledge, at the moment the question whether or not the Cardy-Verlinde formula can be used for the Kerr-Newman-AdS$_4$ and the Kerr-Newman-dS$_4$ black holes still remains open. The main aim of this paper is to settle the question. Another purpose is to study Bekenstein-Verlinde entropy bound for the Kerr-Newman-AdS$_4$ black hole and D-bound [27,28] for the Kerr-Newman-dS$_4$ black hole.

The paper is organized as follows. In Sec. II, the Cardy-Verlinde formula and Bekenstein-Verlinde-like entropy bound for the Kerr-Newman-AdS$_4$ black hole are studied. In Sec. III, the Cardy-Verlinde formulae on the cosmological and event horizons of the Kerr-Newman-dS$_4$ black hole are considered, and the D-bound on the entropy of matter system in the Kerr-Newman-dS$_4$ spacetime is investigated. Some discussions and conclusions are presented in the last section.

II. THE KERR-NEWMAN-ADS$_4$ BLACK HOLE

The Einstein action with a negative cosmological constant along with the Gibbons-Hawking boundary term on a region of spacetime $\Sigma$, which with boundary $\partial \Sigma$, takes the follows form [24]

$$I = \frac{1}{8\pi G(d)} \left( \int_{\Sigma} d^{d}x \sqrt{-g} \left[ R - F^2 - 2\Lambda \right] - \int_{\partial\Sigma} d^{d-1}x \sqrt{-\gamma}K \right), \tag{2.1}$$

where $R$ is the Ricci scalar, $F_{\mu\nu}$ is the electromagnetic field, $\Lambda$ is the cosmological constant, which defines a natural length scale as $\Lambda = \frac{(d-1)(d-2)}{2\ell^2}$, $\gamma_{ab}$ is the induced metric on the timelike boundary, and $K$ is the trace of the extrinsic curvature $K^{ab}$ of the boundary.

The usual charged rotating Kerr-Newman-AdS$_4$ black hole obtained from the action (2.1) reads in the Boyer-Lindquist coordinates [30,31]

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left( dt - \frac{a}{\Xi} \sin^2 \theta d\phi \right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left[ a dt - \frac{(r^2 + a^2)}{\Xi} d\phi \right]^2, \tag{2.2}$$

with

$$\Delta_r = (r^2 + a^2) \left( 1 + \frac{r^2}{\ell^2} \right) - 2Mr + q^2,$$

$$\Delta_\theta = 1 - \frac{a^2 \cos^2 \theta}{\ell^2},$$

$$\Xi = 1 - \frac{q^2}{\ell^2},$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta. \tag{2.3}$$
where the parameters $M$, $a$, and $q$ are related to the mass, angular momentum, and electric charge parameters of the black hole, respectively. The event horizon is located at $r_+$, which is defined as the largest root of the $\Delta_r = 0$. The mass parameter $M$ in the spacetime (2.2) and (2.3) can be written as

$$ M = \frac{(r_+^2 + a^2)(r_+^2 + \ell^2) + q^2 \ell^2}{2r_+ \ell^2}. \quad (2.4) $$

The asymptotic AdS nature of the spacetime (2.2) can be exhibited by introducing new coordinates

$$ t = t, \quad \Xi y^2 \sin^2 \Theta = (r^2 + a^2) \sin^2 \theta, \quad y \cos \Theta = r \cos \theta, \quad \Phi = \phi + \frac{a}{\ell^2} t. \quad (2.5) $$

A. Cardy-Verlinde formula

Euclideanizing the metric (2.2) and identifying $\tau \sim \tau + \beta_{HK}$ and $\phi \sim \phi + i \beta_{HK} \Omega_H$, we can get the inverse Hawking temperature

$$ \beta_{HK} = \frac{4 \pi r_+ \ell^2 (r_+^2 + a^2)}{3r_+^4 + r_+^2 (a^2 + \ell^2) - (a^2 + q^2) \ell^2}, \quad (2.6) $$

and the angular velocity of the event horizon

$$ \Omega_H = \frac{a \Xi}{r_+^2 + a^2}. \quad (2.7) $$

The Bekenstein-Hawking entropy can be easily obtained form metric (2.2)

$$ S = \frac{\pi (r_+^2 + a^2)}{\Xi}. \quad (2.8) $$

In order to study Cardy-Verlinde formula, we will use the energy $E$, angular momentum $J$, and electric charge $Q$, et al. The quantities for the Kerr-Newman-AdS$_4$ black hole can be work out from the action and metric [32–39]. However, in general the action (2.1) is divergent for the spacetime. Several kinds of methods for treating the problem were proposed [33–40]. Two attractive solutions to eliminate the divergence in anti-de Sitter spacetimes are the conformal and counterterm methods [33]. For the Kerr-Newman-AdS$_4$ black hole, the conserved charges, such as the mass and angular momentum, calculated with either approach will yield the same values.

Starting from the metric (2.2), actions (2.1), and counterterm action $I_{ct}$, Caldarelli, Cognola, and Klemm [41] have computed the mass $\mathcal{M}$, the angular momentum $J_\phi$, the electric charge $Q$, and the electric potential $\Phi$ of the Kerr-Newman-AdS$_4$ black hole, which are given by [41]
\[ \mathcal{M} = \frac{M}{\Xi}, \]
\[ \mathcal{J}_\phi = \frac{Ma}{\Xi^2}, \]
\[ Q = \frac{q}{\Xi}, \]
\[ \Phi_q = \frac{qr_+}{r_+^2 + a^2}. \] 

We define an electric potential when the rotation goes to be zero as
\[ \Phi_{q_0} = \lim_{a \to 0} \Phi_q = \frac{q}{r_+}. \] 

We now use above thermodynamical quantities to discuss the question of whether the entropy of the event horizon for the Kerr-Newman-AdS\textsubscript{4} black hole can be described by the Cardy-Verlinde formula. According to the AdS/CFT duality conjecture \[ [4, 12, 13] \], the above thermodynamical quantities are associated to a strongly coupled CFT residing on the conformal boundary of the spacetime \( (2.2) \). The boundary spacetime in which the boundary CFT resides can be obtained from the bulk metric, up to a conformal factor. We can rescale the boundary metric so that the finite volume has a radius \( R \). By standard technique we know that the boundary line element obtained from the metric \( (2.2) \) can be expressed as
\[ ds_b^2 = \lim_{r \to \infty} \frac{R^2}{r^2} ds^2 = \frac{-R^2}{\ell^2} dt^2 + \frac{2aR^2 \sin^2 \theta}{\ell^2 \Xi} dt d\phi + \frac{R^2}{\Delta_\theta} d\theta^2 + \frac{R^2 \sin^2 \theta}{\Xi} d\phi^2, \] which is a rotating Einstein universe.

By the standard technique we know that the temperature \( T \), the energy \( E \), angular momentum \( \mathcal{J} \), the electric potential \( \Phi \), and the non-rotational electric potential \( \Phi_0 \) of the CFT must be rescaled by a factor of \( \frac{\ell}{R} \), i.e.,
\[ T = \frac{\ell}{R} T_{HK}, \quad E = \frac{\ell}{R} \mathcal{M}, \quad \mathcal{J} = \frac{\ell}{R} \mathcal{J}_\phi, \quad \Phi = \frac{\ell}{R} \Phi_q, \quad \Phi_0 = \frac{\ell}{R} \Phi_{q_0}. \] However, the horizon angular velocity and the entropy are still given by Eqs. \( (2.7) \) and \( (2.8) \), respectively.

We define the Casimir energy of the CFT as
\[ E_c = n \left( E + pV - TS - \mathcal{J} \Omega_H - \frac{Q\Phi}{2} - \frac{Q\Phi_0}{2} \right), \] where the pressure is defined as \( p = -\left( \frac{\partial E}{\partial V} \right)_{S, \mathcal{J}, Q} \) and \( n = 2 \). Substituting the corresponding quantities into Eq. \( (2.13) \), we get
\[ E_c = \frac{(r_+^2 + a^2)\ell}{R\Xi r_+}. \]

We also define the extensive energy as
\[ E_{ext} = 2 \left( E - \frac{Q\Phi_0}{2} \right) - E_c. \]
which is in this case

\[ E_{\text{ext}} = \frac{(r_+^2 + a^2)r_+}{\ell R \Xi}. \]  

(2.16)

In order to further understand the strongly coupled CFTs with AdS duals, we can write

\[ 2 \left( E - \frac{Q\Phi_0}{2} \right) R = \frac{n}{2\pi} \frac{SR}{r_+} \left( 1 + \frac{1}{\Delta^2} \right), \]  

(2.17)

and define the “Casimir entropy” by

\[ S_c = \frac{2\pi n}{E_c} R = \frac{SR}{r_+}. \]  

(2.18)

It is obvious that these results possess exactly the behavior of a two-dimensional CFT [13] with characteristic scale \( R \), temperature \( \tilde{T} = 1/(2\pi \Delta) = r_+/(2\pi r^2) \), and central charge proportional to \( SR/r_+ \) (Eq. (2.17) takes the same form as Eq. (17) in Ref. [13], and the Casimir energy \( S_c \) is essentially proportional to the central charge).

With the Casimir energy (2.14), and the extensive energy (2.16), we find that the entropy of the CFT on the horizon can be expressed as

\[ S = \frac{2\pi R}{n} \sqrt{E_c \left[ 2(E - E_Q) - E_c \right]} = \frac{\pi(r_+^2 + a^2)}{\Xi}, \]  

(2.19)

where \( E_Q \) is electric potential energy, which is defined as

\[ E_Q = \frac{Q\Phi_0}{2}. \]  

(2.20)

The difference between expression (2.19) and the standard Cardy-Verlinde formula \( S = \frac{2\pi R}{n} \sqrt{E_c(2E - E_c)} \) is that the electric potential energy \( E_Q \) emerges in (2.19), which must be included in the extensive energy (2.13) to cast the entropy of the CFT into the form of the Cardy-Verlinde formula. We should note that the rotational and electric potential energies were also contained in the Casimir energy (2.13) for case of the Kerr-Newman-AdS\(_4\) black hole. We learn that from Eqs. (2.8) and (2.19) the entropy of the CFT agrees with the Bekenstein-Hawking entropy of the Kerr-Newman-AdS\(_4\) black hole. The result provides a conformal field theory interpretation of the Bekenstein-Hawking entropy. When the rotation parameter \( a \) tends to zero the expressions (2.13) and (2.19) reduce to results of the AdS Reissner-Nordström black hole obtained by Cai in Ref. [5].

**B. Bekenstein-Verlinde-like entropy bound**

Now we study entropy bound of the Kerr-Newman-AdS\(_4\) black hole. As in Ref. [4] we can define the “Bekenstein-Hawking energy” corresponding to the Bekenstein-Hawking entropy (2.8) by
\[ E_{BH} = \frac{nS}{2\pi R} = \frac{(r_+^2 + a^2)}{\Xi R}. \]  

(2.21)

From Eqs. (2.4), (2.8), (2.9), and (2.21) we find that the energy of the black hole can be expressed as

\[ E = \frac{1 + \Delta^2}{2\Delta} E_{BH} + \frac{Q\Phi_0}{2} \]

\[ = \frac{1 + \Delta^2}{2\Delta} E_{BH} + E_Q, \]

(2.22)

where \( \Delta = \frac{R}{r_+} \) and \( R = \ell \). We now rewrite the expression as

\[ E - E_Q = \frac{1 + \Delta^2}{2\Delta} E_{BH}, \]

(2.23)

and define a entropy \( S_B \) which relates to the energy \( E - E_Q \) as

\[ S_B = \frac{2\pi}{n} R (E - E_Q). \]

(2.24)

Substituting Eqs. (2.21) and (2.23) into (2.24) we get

\[ S_B = \frac{2\pi}{n} R E_{BH} \frac{1 + \Delta^2}{2\Delta} \]

\[ = \frac{1 + \Delta^2}{2\Delta} S. \]

(2.25)

Furthermore, since we are above the Hawking-Page transition point [44], we have

\[ r_+ \geq R, \quad \frac{1 + \Delta^2}{2\Delta} \geq 1. \]

(2.26)

Thus we obtain

\[ E_{BH} \leq (E - E_Q), \]

\[ S \leq S_B = \frac{2\pi}{n} R \left( E - \frac{q^2}{2\Xi r_+} \right), \]

(2.27)

where the equality holds when the Hawking-Page phase transition is reached. When \( q = 0 \) and \( a = 0 \) the entropy bound (2.27) reproduces precisely the Bekenstein-Verlinde one (1.3) for the nonrotating neutral object. It is obvious that the bound is tightened by the electric charge \( q \).

III. THE KERR-NEWMAN-DS\(_{4}\) BLACK HOLE

Strominger [26] proposed the dS/AdS correspondence which says that there is a dual between quantum gravity on a dS space and a Euclidean CFT on a boundary of the dS
space. Although much effort was spent recently, we know that the dS/CFT correspondence so far acquired quite incomplete. Thus, the study in this section only provide support of the dS/CFT correspondence.

The metric of the Kerr-Newman-dS\textsubscript{4} black hole can be obtained from the (2.2) by replacing $\ell^2$ with $-\ell^2$, which can be written as

$$d s^2 = -\frac{\Delta_r}{\rho^2} \left( d t - \frac{a}{\Xi} \sin^2 \theta d \phi \right)^2 + \frac{\rho^2}{\Delta_r} d r^2 + \frac{\rho^2}{\Delta_\theta} d \theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left[ a d t - \frac{(r^2 + a^2)}{\Xi} d \phi \right]^2,$$ \hspace{1cm} (3.1)

with

$$\Delta_r = (r^2 + a^2) \left(1 - \frac{r_c^2}{\ell^2}\right) - 2Mr + q^2,$$

$$\Delta_\theta = 1 + \frac{a^2 \cos^2 \theta}{\ell^2},$$

$$\Xi = 1 + \frac{a^2}{\ell^2},$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta.$$ \hspace{1cm} (3.2)

where the parameters $M$, $a$, $q$ have the same means as the case of the Kerr-Newman-AdS\textsubscript{4} black hole. The mass parameter $M$ in the spacetime (3.1) can be expressed as

$$M = -\frac{(r_c^2 + a^2)(r_c^2 - \ell^2) - q^2 \ell^2}{2r_c \ell^2}. \hspace{1cm} (3.3)$$

A. Cardy-Verlinde formula on cosmological horizon

The thermodynamical quantities associated the cosmological horizon are

$$\beta_c = 1 - \frac{4\pi r_c \ell^2 (r_c^2 + a^2)}{3r_c^4 + r_c^2 (a^2 - \ell^2) + (a^2 + q^2)\ell^2},$$

$$\mathcal{M} = -\frac{M}{\Xi},$$

$$\Omega_c = -\frac{a\Xi}{r_c^2 + a^2},$$

$$\mathcal{J}_\phi = \frac{Ma}{\Xi^2},$$

$$Q = \frac{q}{\Xi},$$

$$\Phi_q = -\frac{q r_c}{r_c^2 + a^2},$$

$$\Phi_{q0} = -\lim_{a \to 0} \Phi_q = -\frac{q}{r_c}.$$ \hspace{1cm} (3.4)
while the entropy is
\[ S = \frac{\pi (r_c^2 + a^2)}{\Xi}. \]  

(3.5)

The temperature \( T \), the energy \( E \), angular momentum \( J \), the electric potential \( \Phi \), and the non-rotational electric potential \( \Phi_0 \) of the CFT must be rescaled by a factor of \( \frac{\ell}{R} \) as.

Substituting Eqs. (3.4) and (3.5) into the Casimir energy \( E_c = n(E + pV - TS - J\Omega_c - Q\Phi/2 - Q\Phi_0/2) \), we get
\[ E_c = \frac{(r_c^2 + a^2)}{R^2r_c}, \]

(3.6)

while the extensive energy \( 2(E - Q\Phi_0/2) - E_c \) becomes
\[ E_{ext} = \frac{(r_c^2 + a^2)r_c}{\ell R^2}. \]

(3.7)

Thus it is easy to see that the entropy of the CFT on the cosmological horizon is given by
\[ S = \frac{2\pi R}{n} \sqrt{|E_c|[2(E - E_Q) - E_c]}, \]

(3.8)

The only one difference between Eqs. (2.19) and (3.8) is that we take the absolute value of the Casimir energy in Eq. (3.8).

**B. Cardy-Verlinde formula on black hole horizon**

The thermodynamical quantities associated the black hole horizon are given by
\[ \beta_{HK} = -\frac{4\pi r_+\ell^2(r_+^2 + a^2)}{3r_+^4 + r_+^2(a^2 - \ell^2) + (a^2 + q^2)\ell^2}, \]
\[ \mathcal{M} = \frac{M}{\Xi}, \]
\[ \Omega_H = \frac{a\Xi}{r_+^2 + a^2}, \]
\[ J_\phi = \frac{Ma}{\Xi^2}, \]
\[ Q = \frac{q\Xi}{\Xi}, \]
\[ \Phi_q = \frac{qr_+}{r_+^2 + a^2}, \]
\[ \Phi_{q0} = \lim_{a\to0} \Phi_q = \frac{q}{r_+}. \]

(3.9)

The entropy is
\[ S = \frac{\pi (r_+^2 + a^2)}{\Xi}. \]

(3.10)
The temperature $T$, the energy $E$, angular momentum $J$, the electric potential $\Phi$, and the non-rotational electric potential $\Phi_0$ of the CFT must be rescaled by a factor of $\frac{\ell}{R}$ as we did in the case of the Kerr-Newman-AdS$_4$ black hole.

Substituting the thermodynamical quantities listed in Eqs. (3.9) and (3.10) into definition of the Casimir energy

$$E_c = n \left( E + pV - TS - J\Omega_H - \frac{Q\Phi}{2} - \frac{Q\Phi_0}{2} \right),$$

we get

$$E_c = \frac{(r_+^2 + a^2)\ell}{R\Xi r_+},$$

(3.12)

while the extensive energy

$$E_{\text{ext}} = 2 \left( E - \frac{Q\Phi_0}{2} \right) - E_c,$$

(3.13)

takes the value

$$E_{\text{ext}} = -\frac{(r_+^2 + a^2)r_+}{\ell R\Xi}.$$  

(3.14)

We note that the extensive energy is negative now. With the Casimir energy (3.12) and the extensive energy (3.14), the entropy of the CFT on the event horizon can be cast to

$$S = \frac{2\pi R}{n} \sqrt{E_c|2(E - E_Q) - E_c|}$$

$$= \frac{\pi(r_+^2 + a^2)}{\Xi},$$

(3.15)

where $E_Q = \frac{Q\Phi_0}{2}$. The only one difference between the Cardy-Verlinde formulae (2.19) and (3.15) is that we take the absolute value of the extensive energy in Eq. (3.13).

C. D-bound of entropy

By applying the classical Geroch process to the cosmological horizon, Bousso [27,28] proposed the D-bound on the entropy of matter system in the Schwarzschild-de Sitter space. The D-bound shows that the entropy of objects in dS space is bounded by the difference of the entropies in the exact dS space and in the asymptotically dS space

$$S_m \leq S_0 - S_c,$$

(3.16)

where $S_0$ is the entropy of the exact dS space and $S_c$ is the cosmological horizon entropy when the matter is present.

We now apply the D-bound to the Kerr-Neuam-dS$_4$ black hole. The cosmological horizon radius is determined by the maximal root of the function
\[(1 + \frac{a^2}{r_c^2})\left(1 - \frac{r_c^2}{r_0^2}\right) - \frac{2M}{r_c} + \frac{q^2}{r_c^2} = 0, \quad (3.17)\]

where \(r_0 \equiv \sqrt{\ell}^2\). The equation can be rewritten as

\[\frac{r_0^2}{r_c^2 + a^2} = \left(1 - \frac{2M}{r_c} + \frac{q^2 + a^2}{r_c^2}\right)^{-1} \approx 1 + \frac{2M}{r_c} - \frac{q^2 + a^2}{r_c^2}, \quad (3.18)\]

where we use the large cosmological horizon limit:

\[\frac{2M}{r_c} \ll 1, \quad \frac{q^2}{r_c^2} \ll 1, \quad \frac{a^2}{r_c^2} \ll 1. \quad (3.19)\]

We learn from Eqs. (3.18) and (3.19) that

\[r_0^2 - \frac{r_c^2 + a^2}{\Xi} \leq 2r_c\left[\frac{M}{\Xi} - \frac{q^2}{2r_c\Xi}\right]. \quad (3.20)\]

By using Eqs. (3.16) and (3.20), we obtain an entropy bound of the rotating charged object in the Kerr-Newman-dS\(_4\) space:

\[S \leq S_B = 2\pi R\left(\frac{M}{\Xi} - \frac{q^2}{2R\Xi}\right), \quad (3.21)\]

where we replace \(r_0\) by \(R\). When the rotational parameter \(a = 0\) the bound (3.21) reduces to the D-bound for the Reissner-Norström de-Sitter space \([45]\); and when \(a = 0\) and \(q = 0\) it reproduces precisely the Bekenstein entropy bound (1.4).

**IV. CONCLUSION AND DISCUSSION**

Starting from the four dimensional Kerr-Newman-AdS and Kerr-Newman-dS black holes, we further verify the Cardy-Verlinde formula which relates the entropy of the conformal field theory to its Casimir energy \(E_c\), the energy \(E\) (or the extensive energy), and the radius \(R\) of the unit sphere \(S^n\). For strongly coupled CFTs with AdS duals, we find that in order to apply the Cardy-Verlinde formula to case of the Kerr-Newman-AdS\(_4\) black hole, the Casimir energy (2.13) must contains the terms \(-n\left(J\Omega_H + \frac{Q^2}{2} + \frac{Q\Phi_0}{2}\right)\), which relates to rotational and electric potential energies of the CFT. The difference between expression (2.19) and the standard Cardy-Verlinde formula (1.2) is that the non-rotational electric potential energy \(E_Q = \frac{Q\Phi_0}{2}\) emerges in the formula (2.19). When the rotation parameter vanishes the expressions (2.13) and (2.19) reduce to results of the AdS Reissner-Nordström black hole obtained in Ref. \([5]\).

For the Kerr-Newman-dS\(_4\) black hole, the definitions of the Casimir and the extensive energies possess the same form as that of the Kerr-Newman-AdS\(_4\) black hole. However, we note that the Casimir energy (3.6) is negative but the extensive energy (3.7) is positive.
on the cosmological horizon; while the Casimir energy (3.12) is positive but the extensive energy (3.14) is negative on the event horizon. Thus we must take the absolute value of the Casimir energy in the Cardy-Verlinde formula (3.8) and absolute value of the extensive energy in the Cardy-Verlinde formula (3.15). As the rotation parameter $a$ goes to zero the results reduce to that of the Reissner-Nordström-dS black hole found by Cai in Ref. [22]. The results provide support of the dS/AdS correspondence.

The fact that the entropy of the CFT agrees precisely with the Bekenstein-Hawking entropy also provides a conformal field theory interpretation of the Bekenstein-Hawking entropies of the Kerr-Newman-AdS$_4$ and Kerr-Newman-dS$_4$ black holes. The result supports the conclusion of the reference [46], which says that for the Kerr-Newman-AdS and Kerr-Newman-dS black holes the statistical entropies obtained from the density of states determined by conformal field theory method agree with their Bekenstein-Hawking entropies.

We also study the entropy bounds of these spacetimes. The Bekenstein-Verlinde-like entropy bound (2.27) for the Kerr-Newman-AdS$_4$ black hole is obtained by comparing different kinds of the energies. It is obvious that the bound is tightened by the electric charge. For the non-rotating neutral object the entropy bound (2.27) reduces to the Bekenstein-Verlinde one [1]. On the other hand, the D-bound on the entropy of matter system in Kerr-Newman-dS$_4$ space is also found, which is described by formula (3.21). The bound is also tightened by the charge. For the non-rotating neutral object the bound (3.21) reproduces precisely the Bekenstein entropy bound.

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