SPECTRA OF LOCALLY MATRIX ALGEBRAS

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Abstract. We describe spectra of associative (not necessarily unital and not necessarily countable-dimensional) locally matrix algebras. We determine all possible spectra of locally matrix algebras and give a new proof of Dixmier-Baranov Theorem. As an application of our description of spectra we determine embeddings of locally matrix algebras.

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1. Introduction

Let $\mathbb{F}$ be a ground field. Recall that an associative $\mathbb{F}$–algebra $A$ is called a locally matrix algebra (see, [10]) if for an arbitrary finite subset of $A$ there exists a subalgebra $B \subset A$ containing this subset and such that $B$ is isomorphic to a matrix algebra $M_n(\mathbb{F})$ for some $n \geq 1$. In what follows we will sometimes identify $B$ and $M_n(\mathbb{F})$, that is, assume that $M_n(\mathbb{F}) \subset A$. We call a locally matrix algebra unital if it contains $1$.

Let $A$ be a countable–dimensional unital locally matrix algebra. In [7], J.G. Glimm defined the Steinitz number $\text{st}(A)$ of the algebra $A$ and proved that $A$ is uniquely determined by $\text{st}(A)$. J. Dixmier [5] showed that non-unital countable–dimensional locally matrix algebras over the field of complex numbers can be parametrized by pairs $(s, \alpha)$, where $s$ is a Steinitz number and $\alpha$ is a nonnegative real number. A.A. Baranov [1] extended this parametrization to locally matrix algebras over arbitrary fields.

In [2], we defined the Steinitz number $\text{st}(A)$ for a unital locally matrix algebra $A$ of an arbitrary dimension. We showed that for a unital locally matrix algebra $A$ of dimension $> \aleph_0$ the Steinitz number $\text{st}(A)$ no longer determines $A$; see, [3, 4]. However, it determines the universal elementary theory of $A$ [3].
In this paper for an arbitrary (not necessarily unital and not necessarily countable-dimensional) locally matrix algebra $A$, we define a subset of $\mathbb{SN}$ that we call the spectrum of $A$ and denote as $\text{Spec}(A)$. We determine all possible spectra of locally matrix algebras and give a new proof of Dixmier-Baranov Theorem. As an application of our description of spectra we determine embeddings of locally matrix algebras.

2. Spectra of locally matrix algebras

Let $\mathbb{P}$ be the set of all primes and $\mathbb{N}$ be the set of all positive integers. A Steinitz number (see, [11]) is an infinite formal product of the form

$$\prod_{p \in \mathbb{P}} p^{r_p},$$

where $r_p \in \mathbb{N} \cup \{0, \infty\}$ for all $p \in \mathbb{P}$.

Denote by $\mathbb{SN}$ the set of all Steinitz numbers. Notice, that the set of all positive integers $\mathbb{N}$ is subset of $\mathbb{SN}$. The numbers $\mathbb{SN} \setminus \mathbb{N}$ are called infinite Steinitz numbers.

Let $A$ be a locally matrix algebra with a unit 1 over a field $F$ and let $D(A)$ be the set of all positive integers $n$ such that there is a subalgebra $A'$, $1 \in A' \subseteq A$, $A' \cong M_n(F)$. Then the least common multiple of the set $D(A)$ is called the Steinitz number of the algebra $A$ and denoted as $\text{st}(A)$; see, [2].

Now let $A$ be a (not necessarily unital) locally matrix algebra. For an arbitrary idempotent $0 \neq e \in A$ the subalgebra $eAe$ is a unital locally matrix algebra. That is why we can talk about its Steinitz number $\text{st}(eAe)$. The subset

$$\text{Spec}(A) = \{ \text{st}(eAe) \mid e \in A, e \neq 0, e^2 = e \},$$

where $e$ runs through all nonzero idempotents of the algebra $A$, is called the spectrum of the algebra $A$.

For a Steinitz number $s$ let $\Omega(s)$ denote the set of all natural numbers $n \in \mathbb{N}$ that divide $s$.

For a Steinitz numbers $s_1$, $s_2$ we say that $s_1$ finitely divides $s_2$ if there exists $b \in \Omega(s_2)$ such that $s_1 = s_2/b$ (we denote: $s_1 \mid_{\text{fin}} s_2$).

Steinitz numbers $s_1$, $s_2$ are rationally connected if $s_2 = q \cdot s_1$, where $q$ is some rational number.

We call a subset $S \subset \mathbb{SN}$ saturated if

1) any two Steinitz numbers from $S$ are rationally connected;
2) if $s_2 \in S$ and $s_1 \mid_{\text{fin}} s_2$ then $s_1 \in S$;
3) if $s, ns \in S$, where $n \in \mathbb{N}$, then $is \in S$ for any $i$, $1 \leq i \leq n$. 
**Theorem 1.** For an arbitrary locally matrix algebra $A$ its spectrum is a saturated subset of $\mathbb{S}\mathbb{N}$.

Let us consider examples of saturated subsets of $\mathbb{S}\mathbb{N}$.

**Example 1.** For an arbitrary natural number $n$ the set $\{1, 2, \ldots, n\}$ is saturated.

**Example 2.** Let $s$ be a Steinitz number. The set
$$\mathcal{S}(\infty, s) := \left\{ \frac{a}{b} \cdot s \mid a, b \in \mathbb{N}, b \in \Omega(s) \right\}$$
is saturated. For an arbitrary Steinitz number $s' \in \mathcal{S}(\infty, s)$ we have $\mathcal{S}(\infty, s) = \mathcal{S}(\infty, s')$. If $s \in \mathbb{N}$ then $\mathcal{S}(\infty, s) = \mathbb{N}$.

**Example 3.** Let $r$ be a real number, $1 \leq r < \infty$. Let $s$ be an infinite Steinitz number. The set
$$\mathcal{S}(r, s) = \left\{ \frac{a}{b} s \mid a, b \in \mathbb{N}; b \in \Omega(s), a \leq rb \right\}$$
is saturated.

**Example 4.** Let $s$ be an infinite Steinitz number and let $r = u/v$ be a rational number; $u, v \in \mathbb{N}, v \in \Omega(s)$. Then the set
$$\mathcal{S}^+(r, s) = \left\{ \frac{a}{b} s \mid a, b \in \mathbb{N}; b \in \Omega(s), a < rb \right\}$$
is saturated.

**Theorem 2.** Every saturated subset of $\mathbb{S}\mathbb{N}$ is one of the following sets:

1) $\{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, or $\mathbb{N}$;
2) $\mathcal{S}(\infty, s)$, $s \in \mathbb{S}\mathbb{N} \setminus \mathbb{N}$;
3) $\mathcal{S}(r, s)$, where $s \in \mathbb{S}\mathbb{N} \setminus \mathbb{N}$, $r \in [1, \infty)$;
4) $\mathcal{S}^+(r, s)$, where $s \in \mathbb{S}\mathbb{N} \setminus \mathbb{N}$, $r = u/v$, $u \in \mathbb{N}, v \in \Omega(s)$.

**Remark 1.** The real number $r$ above is the inverse of the density invariant of Dixmier–Baranov.

**Theorem 3.** (1) For any saturated subset $S \subseteq \mathbb{S}\mathbb{N}$ there exists a countable-dimensional locally matrix algebra $A$ such that $\text{Spec}(A) = S$.

(2) If $A, B$ are countable-dimensional locally matrix algebras and $\text{Spec}(A) = \text{Spec}(B)$ then $A \cong B$.

**Remark 2.** The part (2) of Theorem 3 is a new proof of Dixmier–Baranov Theorem.

Which spectra above correspond to unital algebras?
Theorem 4. A locally matrix algebra $A$ is unital if and only if $\text{Spec}(A) = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$, or $\text{Spec}(A) = S(r, s)$, where $s \in \mathbb{N} \setminus \mathbb{N}$, $r = u/v$, $u, v \in \mathbb{N}$, $v \in \Omega(s)$.

Proof of Theorem 4. In what follows, we assume that $A$ is a locally matrix $\mathbb{F}$-algebra. Recall the partial order on the set of all idempotents of $A$ : for idempotents $e$, $f \in A$ we define $e \geq f$ if $f \in eAe$.

We claim that for arbitrary idempotents $e_1$, $e_2 \in A$ there exists an idempotent $e_3 \in A$ such that $e_1 \leq e_3$, $e_2 \leq e_3$. Indeed, there exists a subalgebra $A' \subset A$ such that $e_1$, $e_2 \in A'$ and $A' \cong M_n(\mathbb{F})$, $n \geq 1$. Let $e_3$ be the identity element of the subalgebra $A'$. Then $e_1 \leq e_3$, $e_2 \leq e_3$.

Now suppose that the locally matrix algebra $A$ is unital. Let $a \in A$. Choose a subalgebra $A' \subset A$ such that $1 \in A'$, $a \in A'$ and $A' \cong M_n(\mathbb{F})$, $n \geq 1$. Let $r$ be the range of the matrix $a$ in $A'$. Let

$$r(a) = \frac{r}{n}, \quad 0 \leq r(a) \leq 1.$$ 

V.M. Kurochkin noticed that the number $r(a)$ does not depend on a choice of a subalgebra $A'$. We call $r(a)$ the relative range of the element $a$. In [4], we showed that if $A$ is a unital locally matrix algebra and $e \in A$ is an idempotent, then $\text{st}(eAe) = r(e) \cdot \text{st}(A)$.

Now let $A$ be a not necessarily unital locally matrix algebra. Let $e_1$, $e_2 \in A$ be idempotents. Choose an idempotent $e_3 \in A$ such that $e_1 \leq e_3$, $e_2 \leq e_3$, i.e. $e_1$, $e_2 \in e_3Ae_3$. Let $q_1$, $q_2$ be relative ranges of the idempotents $e_1$, $e_2$ in unital locally matrix algebra $e_3Ae_3$. Then

$$\text{st}(e_1Ae_1) = q_1 \text{st}(e_3Ae_3), \quad \text{st}(e_2Ae_2) = q_2 \text{st}(e_3Ae_3).$$

This implies that the Steinitz numbers $\text{st}(e_1Ae_1)$, $\text{st}(e_2Ae_2)$ are rationally connected. We have checked the condition 1) from the definition of saturated sets.

Let $0 \neq e \in A$ be an idempotent. Let $s_2 = \text{st}(eAe)$, $k \in \Omega(s_2)$ and let $s_1 = s_2/k$. The unital locally matrix algebra $eAe$ contains a subalgebra $e \in M_k(\mathbb{F}) \subset eAe$. Consider the matrix unit $e_{11}$ of the algebra $M_k(\mathbb{F})$. The relative range of the idempotent $e_{11}$ in the unital algebra $eAe$ is equal to $1/k$. Hence

$$\text{st}(e_{11}Ae_{11}) = \frac{1}{k} \text{st}(eAe) = s_1, \quad s_1 \in \text{Spec}(A).$$

We have checked the condition 2).

Now let $n \geq 1$. Suppose that Steinitz numbers $s$ and $ns$ lie in $\text{Spec}(A)$. It means that there exist idempotents $e_1$, $e_2 \in A$ such that $s = \text{st}(e_1Ae_1)$, $ns = \text{st}(e_2Ae_2)$. There exists a matrix subalgebra $M_k(\mathbb{F}) \subset A$ that contains $e_1$ and $e_2$. As above, let $e_3$ be the identity element of the algebra $M_k(\mathbb{F})$. Let $\text{rk}(e_i)$ be the range of the idempotent
in the matrix algebra $M_k(\mathbb{F})$. We have

$$s = \frac{\text{rk}(e_1)}{k} \cdot \text{st}(e_3 Ae_3), \quad n \ s = \frac{\text{rk}(e_2)}{k} \cdot \text{st}(e_3 Ae_3),$$

which implies $\text{rk}(e_2) = n \cdot \text{rk}(e_1)$. In particular, $n \cdot \text{rk}(e_1) \leq k$. Let $1 \leq i \leq n$. Consider the idempotent

$$e = \text{diag} \left( \underbrace{1, 1, \ldots, 1}_i, 0, 0, \ldots, 0 \right)$$

in the matrix algebra $M_k(\mathbb{F})$. We have

$$\text{st}(e Ae) = \frac{i \cdot \text{rk}(e_1)}{k} \cdot \text{st}(e_3 Ae_3) = i \cdot \text{st}(e_1 Ae_1) = is.$$

We showed that $is \in \text{Spec}(A)$. Hence $\text{Spec}(A)$ is a saturated subset of $SN$. It completes the proof of Theorem 1. □

3. Classification of saturated subsets of $SN$

Our aim in this section is to classify all saturated subsets of $SN$. We remark that if at least one Steinitz number from a saturated set $S$ is infinite then by the condition 1) all Steinitz numbers from $S$ are infinite.

Let $S$ be a saturated subset of $SN$. For a Steinitz number $s \in S$ and for a natural number $b \in \Omega(s)$ let

$$r_s(b) = \max \left\{ i \geq 1 \mid i \cdot \frac{s}{b} \in S \right\}.$$

**Lemma 1.** If there exists a Steinitz number $s_0 \in S$ and a natural number $b_0 \in \Omega(s_0)$ such that $r_{s_0}(b_0) = \infty$ then for any $s \in S$ and any $b \in \Omega(s)$ we have $r_s(b) = \infty$.

**Proof.** Let us show at first that $r_{s_0}(b) = \infty$ for any $b \in \Omega(s_0)$. Indeed, there exists a natural number $c \in \Omega(s_0)$ such that both $b_0$ and $b$ divide $c$. Then for an arbitrary $i \geq 1$ we have

$$i \cdot \frac{s_0}{b_0} = \left( i \cdot \frac{c}{b_0} \right) \cdot \frac{s_0}{c} \in S.$$

This implies that $r_{s_0}(c) = \infty$. Hence,

$$i \cdot \frac{s_0}{b} = \left( i \cdot \frac{c}{b} \right) \cdot \frac{s_0}{c} \in S,$$

which proves the claim.

Now choose an arbitrary Steinitz number $s \in S$. By the condition 1), the Steinitz numbers $s$ and $s_0$ are rationally connected, i.e. there exist $a \in \mathbb{N}$, $b \in \Omega(s_0)$ such that $s = (a/b) \cdot s_0$. By condition 2), $s_0/b \in S$. Choose a natural number $c \in \Omega(s_0/b)$. Then $c \in \Omega(s)$ and
$bc \in \Omega(s_0)$. For an arbitrary $i \geq 1$ we have $i \cdot s/c = i \cdot a \cdot s_0/(bc) \in S$ since $r_{s_0}(bc) = \infty$. This implies $r_s(c) = \infty$ and completes the proof of the lemma.

If a saturated set satisfies the assumptions of Lemma 1 then it is referred to as a set of infinite type. Otherwise, we talk about a saturated set of finite type.

**Lemma 2.** 1) For an arbitrary Steinitz number $s_0 \in \mathbb{SN}$ the set

$$S(\infty, s_0) := \left\{ \frac{a}{b} \cdot s_0 \mid a \in \mathbb{N}, b \in \Omega(s_0) \right\}$$

is a saturated set of infinite type.

2) If $S$ is a saturated set of infinite type, then for an arbitrary Steinitz number $s \in S$ we have $S = S(\infty, s)$.

**Proof.** We have to show that the set $S(\infty, s_0)$ satisfies the conditions 1), 2), 3). The condition 1) is obvious. Let $s = (a/b) \cdot s_0, b \in \Omega(s_0)$. Without loss of generality, we assume that $a$ and $b$ are coprime. Let $c \in \Omega(s)$ and let $d = \gcd(c, a)$ be the greatest common divisor of $a$ and $c$, $a = a'd$, $c = c'd$, the numbers $a'$, $c'$ are coprime. Then $a \cdot s_0/(bc) = a' \cdot s_0/(bc')$, which implies that $dc' \in \Omega(s_0)$. Hence

$$\frac{s}{c} = \frac{a}{bc} \cdot s_0 = \frac{a'}{bc'} \cdot s_0 \in S(\infty, s_0).$$

We have checked the condition 2).

Let us check the condition 3). Choose $s = (a/b) \cdot s_0 \in S(\infty, s_0)$, $b \in \Omega(s_0)$. Let $c \in \Omega(s)$. We need to check that for any $i \geq 1$

$$i \cdot \frac{s}{c} = \frac{ia}{bc} \cdot s_0 \in S(\infty, s_0).$$

Let $a/(bc) = a'/b'$, where the natural numbers $a'$, $b'$ are coprime. Since

$$\frac{a}{bc} \cdot s_0 = \frac{s}{c} \in \mathbb{SN}$$

it follows that $b' \in \Omega(s_0)$. Hence, $i \cdot (a'/b') \cdot s_0 \in S(\infty, s_0)$, which implies that $S(\infty, s_0)$ satisfies the condition 3) and therefore is saturated.

Let $S$ be a saturated subset of $\mathbb{SN}$ of infinite type. Choose $s_0 \in S$. Our aim is to show that $S = S(\infty, s_0)$. Since the subset $S$ is of infinite type it follows that $r_s(b) = \infty$ for any $s \in S, b \in \Omega(s)$. In particular,

$$S(\infty, s_0) = \left\{ \frac{a}{b} \cdot s_0 \mid s \in \Omega(s_0) \right\} \subseteq S.$$
Now let $S \subset SN$ be a saturated subset of finite type, that is, for any $s \in S$, $d \in \Omega (s)$ we have

$$r_s(b) = \max \left\{ i \in \mathbb{N} \mid i \cdot \frac{s}{b} \in S \right\} < \infty.$$ 

By the condition 3),

$$\left\{ i \in \mathbb{N} \mid i \cdot \frac{s}{b} \in S \right\} = [1, r_s(b)].$$

Since $b \cdot (s/b) \in S$ it follows that $b \leq r_s(b)$. Choose a Steinitz number $s \in S$ and two natural numbers $b, c \in \Omega(s)$ such that $b$ divides $c$. If $i \cdot (s/b) \in S$ then $(ic/b) \cdot (s/c) \in S$. Hence $r_s(b) \cdot (c/b) \leq r_s(c)$. In other words,

$$\frac{r_s(b)}{b} \leq \frac{r_s(c)}{c}.$$ 

Let $i \in \mathbb{N}$, $s/c \in S$ and let $k$ be a maximal nonnegative integer such that $k \cdot (c/b) \leq i$. By the condition 3), $k \cdot (c/b) \cdot (s/c) \in S$, hence $k \cdot (s/b) \in S$. So, $k \leq r_s(b)$. We proved that

$$\left\lfloor \frac{r_s(c)}{c/b} \right\rfloor \leq r_s(b).$$

The inequalities [1], [2] imply

$$\left\lfloor \frac{r_s(c)}{c/b} \right\rfloor \leq r_s(b) \leq \frac{r_s(c)}{c/b}.$$ 

Hence

$$r_s(b) = \left\lfloor \frac{r_s(c)}{c/b} \right\rfloor.$$ 

In particular,

$$\frac{r_s(c)}{c/b} - 1 < r_s(b), \quad \frac{r_s(c)}{c/b} < r_s(b) + 1.$$ 

Dividing by $b$, we get

$$\frac{r_s(b)}{b} \leq \frac{r_s(c)}{c} < \frac{r_s(b)}{b} + \frac{1}{b}.$$

**Lemma 3.** Let $S \subset SN$ be a saturated subset of finite type and let $s \in S$ be an infinite Steinitz number. Then there exists a limit

$$r_S(s) = \lim_{\substack{b \in \Omega(s) \\ b \to \infty}} \frac{r_s(b)}{b}, \quad 1 \leq r_S(s) < \infty.$$ 

If the set $S$ is fixed then we denote $r_S(s) = r(s)$. 
Remark 3. The limit \( r(s) \) is equal to the inverse of the density invariant of Dixmier–Baranov ([1, 5]).

The proof of Lemma 3. The set \( \{ r_s(b)/b \mid b \in \Omega(s) \} \) is bounded from above. Indeed, choose \( b_0 \in \Omega(s) \). For an arbitrary \( b \in \Omega(s) \) there exists \( c \in \Omega(s) \) that is a common multiple for \( b_0 \) and \( b \). Then by (1) and (4),

\[
\frac{r(b)}{b} \leq \frac{r(c)}{c} < \frac{r(b_0)}{b_0} + \frac{1}{b_0}.
\]

Let

\[
r = r(s) = \sup \left\{ \frac{r_s(b)}{b} \mid b \in \Omega(s) \right\}.
\]

Clearly, \( 1 \leq r < \infty \). Choose \( \varepsilon > 0 \). Let \( N(\varepsilon) = [2r/\varepsilon] + 1 \). We will show that for any \( b \in \Omega(s) \), \( b \geq N(\varepsilon) \), we have \( r - \varepsilon < r_s(b)/b \).

Indeed, let \( b \in \Omega(s) \), \( b \geq N(\varepsilon) > 2r/\varepsilon \). Then \( 1/b < \varepsilon/(2r) \leq \varepsilon/2 \).

There exists a natural number \( b_0 \in \Omega(s) \) such that \( r - \varepsilon/2 < r_s(b_0)/b_0 \).

Let \( c \in \Omega(s) \) be a common multiple of \( b_0 \) and \( b \). Then (4) implies

\[
\frac{r(b)}{b} > \frac{r(c)}{c} - \frac{1}{b} \geq \frac{r(b_0)}{b_0} - \frac{1}{b} > r - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = r - \varepsilon.
\]

So,

\[
r = \lim_{b \in \Omega(s)} \frac{r_s(b)}{b}
\]

and this completes the proof of the lemma. \( \square \)

Lemma 4. Let \( s, s' \in S \) be infinite Steinitz numbers, \( s' = (a/b) \cdot s; a, b \in \mathbb{N}; b \in \Omega(s) \). Then \( r(s') = (a/b) \cdot r(s) \).

Proof. It is sufficient to show that if \( s, ms \in S; m \in \mathbb{N} \), then \( m \cdot r(ms) = r(s) \).

Suppose that \( b \in \Omega(s) \) and \( i \cdot (ms/b) \in S \). Then \( i \cdot m \cdot (s/b) \in S \). Hence \( r_{ms}(b) \cdot m \leq r_s(b) \) and, therefore, \( r(ms) \cdot m \leq r(s) \).

On the other hand, if \( i \cdot (s/b) \in S \) then \( [i/m] \cdot m \leq i \) and, therefore, \( [i/m] \cdot m \cdot (s/b) \in S \). We showed that

\[
\left[ \frac{r_s(b)}{m} \right] \leq r_{ms}(b), \quad \frac{r_s(b)}{m} - 1 < r_{ms}(b),
\]

\[
\frac{1}{m} \cdot \frac{r_s(b)}{b} - \frac{1}{b} < \frac{r_{ms}(b)}{b}.
\]

Assuming \( b \to \infty \) we get \( (1/m) \cdot r(s) \leq r(ms) \), which completes the proof of the lemma. \( \square \)
In the inequality \[4\], let \( c \to \infty \). Then
\[
\frac{r_s(b)}{b} \leq r(s) \leq \frac{r_s(b)}{b} + \frac{1}{b}, \quad r_s(b) \leq r(s) b \leq r_s(b) + 1.
\]
If the number \( r(s) \) is irrational then \( r_s(b) = \lfloor r(s)b \rfloor \) for all \( b \in \Omega(s) \).

Now suppose that the number \( r = r_s(b) \) is rational; \( r = u/v \); \( u, v \) are coprime. If a number \( b \in \Omega(s) \) is not a multiple of \( v \) then, as above, \( r_s(b) = \lfloor (u/v) \cdot b \rfloor \). If \( b \) is a multiple of \( v \) then
\[
r_s(b) = \begin{cases} \frac{r b}{v} & \text{or} \\ \frac{r b}{v} - 1 \end{cases}.
\]

**Lemma 5.** If at least for one number \( b_0 \in \Omega(s) \cap v\mathbb{N} \) we have \( r_s(b_0) = rb_0 \) then for all \( b \in \Omega(s) \cap v\mathbb{N} \) we have \( r_s(b) = rb \).

**Proof.** Let \( b, c \in \Omega(s) \cap v\mathbb{N} \) and \( b \) divides \( c \). If \( r_s(b) = rb \) then by the inequality \[1\], we have
\[
r = \frac{r_s(b)}{b} \leq \frac{r_s(c)}{c},
\]
which implies \( r_s(c) = rc \). On the other hand, if \( r_s(c) = rc \) then by the inequality \[4\],
\[
r = \frac{r_s(c)}{c} < \frac{r_s(b)}{b} + \frac{1}{b},
\]
which implies \( r_s(b) > rb - 1 \). Hence \( r_s(b) = rb \). We showed that \( r_s(b) = rb \) if and only if \( r_s(c) = rc \).

Now choose \( b_1, b_2 \in \Omega(s) \cap v\mathbb{N} \) and suppose that \( r_s(b_1) = rb_1 \). There exists \( c \in \Omega(s) \cap v\mathbb{N} \) such that both \( b_1 \) and \( b_2 \) divide \( c \). In view of the above, \( r_s(b_1) = rb_1 \) implies \( r_s(c) = rc \) which implies \( r_s(b_2) = rb_2 \). This completes the proof of the lemma. \( \square \)

Recall that for an infinite Steinitz number \( s \) and a real number \( r \), \( 1 \leq r < \infty \),
\[
S(r, s) = \left\{ \frac{a}{b} \ s \bigg| a, b \in \mathbb{N}; b \in \Omega(s), a \leq rb \right\},
\]
\[
S^+(r, s) = \left\{ \frac{a}{b} \ s \bigg| a, b \in \mathbb{N}; b \in \Omega(s), a < rb \right\}.
\]
If \( r \) is an irrational number or \( r = u/v \), the integers \( u, v \) are coprime and \( v \notin \Omega(s) \) then \( S(r, s) = S^+(r, s) \). If \( r = u/v \), \( v \in \Omega(s) \) then \( S^+(r, s) \subsetneq S(r, s) \).

**Lemma 6.** The subsets \( S(r, s) \) and \( S^+(r, s) \) are saturated.
We showed that the set $c \in \Omega(s)$. We need to show that $(a \cdot s)/(b \cdot c) \in S(r, s)$ (respectively, $(a \cdot s)/(b \cdot c) \in S^+(r, s)$). Let $d = \gcd(a, c)$, $a = da'$, $c = dc'$. Then

$$\frac{a \cdot s}{b \cdot c} = \frac{a'}{b \cdot c'} \quad s \in \mathbb{N}.$$ 

Since the number $bc'$ is coprime with $a'$ it follows that $bc' \in \Omega(s)$. The inequality $a' \leq rbc'$ (respectively $a' < rbc'$) is equivalent to the inequality $a \leq rbc$ (respectively $a < rbc$). The latter inequality follows from $a \leq rb$ (respectively $a < rb$). The condition 2) is verified.

Let us check the condition 3). As above, we assume that $a$, $b$ are coprime natural numbers, $b \in \Omega(s)$ and $a/b \in S(r, s)$ (respectively $a/b \in S^+(r, s)$). Let $c \in \Omega((a/b) \cdot s)$, $\gcd(a, c) = d$, $a = da'$, $c = dc'$. We have shown above that $bc' \in \Omega(s)$. Let $n \in \mathbb{N}$ and $n \cdot (as/(bc)) \in S(r, s)$ (respectively $n \cdot (as/(bc)) \in S^+(r, s)$). Then $na' \leq rbc'$ (respectively $na' < rbc'$). This immediately implies that for any $i, 1 \leq i \leq n$, we have $ia' \leq rbc'$ (respectively $ia' < rbc'$). Hence, $i \cdot (as/b) \in S(r, s)$ (respectively $i \cdot (as/b) \in S^+(r, s)$).

**Lemma 7.** Let $r = u/v$, where $u$, $v$ are coprime natural numbers. Let $s$ be an infinite Steinitz number and $v \in \Omega(s)$. Then the set $S^+(r, s)$ is not equal to any of the sets $S(r', s')$, $r' \in [1, \infty)$, $s' \in \mathbb{N}$.

**Proof.** Let $s_2 \in S(r, s_1)$ (respectively $s_2 \in S^+(r, s_1)$). Then $s_2 = (a/b) \cdot s_1$, where $a, b \in \mathbb{N}$, $b \in \Omega(s_1)$. By Lemma 4

$$S(r, s_1) = S\left(\frac{b}{a}, s_2\right) \quad \text{(respectively } S^+(r, s_1) = S^+\left(\frac{b}{a}, s_2\right)\text{)}.$$ 

We showed that the set $S(r, s)$ (respectively $S^+(r, s)$) is determined by any Steinitz number $s' \in S(r, s)$ (respectively $s' \in S^+(r, s)$) with an appropriate recalibration of $r$.

Let $S = S(r_1, s_1) = S^+(r_2, s_2)$. Choosing an arbitrary Steinitz number $s \in S$ we get $S(r_1, s) = S^+(r_2, s)$ for some $r_1', r_2' \in [1, \infty)$. The number $r_2' = u/v$ is rational, $\gcd(u, v) = 1$ and $v \in \Omega(s)$.

The number $r$ is uniquely determined by a saturated subset $S$ and a choice of $s \in S$. Hence $r_1' = r_2'$. Now it remains to notice that for a rational number $r = u/v$, $\gcd(u, v) = 1$ and an infinite Steinitz number $s$, such that $v \in \Omega(s)$, we have $S(r, s) \neq S^+(r, s)$. This completes the proof of the lemma. 

**Lemma 8.** Let $S \subset \mathbb{SN} \setminus \mathbb{N}$ be a saturated subset of finite type, $s \in S$, $r = r_S(s) \in [1, \infty)$. Then $S = S(r, s)$ or $S = S^+(r, s)$. 

Proof. Recall that for a natural number \( b \in \Omega(s) \) we defined
\[
\rho_s(b) = \max \left\{ i \in \mathbb{N} \mid i \frac{s}{b} \in S \right\}.
\]
We showed that if \( r \) is an irrational number or \( r = u/v; u, v \) are coprime and \( v \notin \Omega(s) \) then \( \rho_s(b) = \lfloor rb \rfloor \) for an arbitrary \( b \in \Omega(s) \).

An arbitrary Steinitz number \( s' \in S \) is representable as \( s' = \frac{a}{b} \cdot s \), where \( a, b \) are coprime natural numbers. Clearly, \( b \in \Omega(s) \) and \( a \leq \rho_s(b) = \lfloor rb \rfloor \). That is why in the case when \( r \) is irrational or \( r = u/v, \gcd(u, v) = 1, v \notin \Omega(s) \) we have \( S = S(r, s) = S^+(r, s) \).

Suppose now that \( r = u/v, \gcd(u, v) = 1, v \in \Omega(s) \). If \( b \in \Omega(s) \setminus v\mathbb{N} \) then as above \( \rho_s(b) = \lfloor rb \rfloor \). By Lemma 5 either for all \( b \in \Omega(s) \cap v\mathbb{N} \) we have \( \rho_s(b) = rb \) or for all \( b \in \Omega(s) \cap v\mathbb{N} \) we have \( \rho_s(b) = rb - 1 \). In the first case \( S = S(r, s) \), in the second case \( S = S^+(r, s) \).

**Lemma 9.** Let \( S \subseteq \mathbb{N} \) be a saturated subset. Then either \( S = \{1, 2, \ldots, n\} \) for some \( n \in \mathbb{N} \) or \( S = \mathbb{N} \).

*Proof.* First, notice that the subsets \( \{1, 2, \ldots, n\} \) and \( \mathbb{N} \) are saturated.

Now let \( S \subseteq \mathbb{N} \) be a saturated subset. If \( n \in S \) then \( n \in \Omega(n) \) and \( n \cdot (n/n) \in S \). By the condition 3), all natural numbers \( i = i \cdot (n/n), 1 \leq i \leq n, \) lie in \( S \). This implies the assertion of the lemma. \( \square \)

Now, Theorem 2 follows from Lemmas 8, 9.

4. **Countable-dimensional locally matrix algebras**

For an algebra \( A \) and an idempotent \( 0 \neq e \in A \) we call the subalgebra \( eAe \) a corner of the algebra \( A \).

Let \( A_1 \subseteq A_2 \subseteq \cdots \) be an ascending chain of unital locally matrix algebras, \( A_i \) is a corner of the algebra \( A_{i+1}, i \geq 1, \)
\[
A = \bigcup_{i=1}^{\infty} A_i.
\]
Clearly, \( \text{Spec}(A_1) \subseteq \text{Spec}(A_2) \subseteq \cdots \).

**Lemma 10.**
\[
\text{Spec}(A) = \bigcup_{i=1}^{\infty} \text{Spec}(A_i).
\]

*Proof.* For an arbitrary idempotent \( e \in A_i \) we have \( eA_i e = eAe \), hence \( \text{Spec}(A_i) \subseteq \text{Spec}(A) \). On the other hand, an arbitrary idempotent \( e \in A \) lies in one of the subalgebras \( A_i \). Hence \( \text{st}(eAe) = \text{st}(eA_i e) \in \text{Spec}(A_i) \). \( \square \)
Proof of Theorem 3 (1). To start with we notice that \( \{1, 2, \ldots, n\} = \text{Spec}(M_n(\mathbb{F})) \). Let \( s \) be a Steinitz number. In [2], we showed that there exists a unital locally matrix algebra \( A \), \( \dim_{\mathbb{F}} A \leq \aleph_0 \), such that \( \text{st}(A) = s \). Consider the algebra \( M_\infty(A) \) of infinite \( N \times N \)-matrices, having finitely many nonzero entries. The algebra \( M_n(A) \) of \( n \times n \)-matrices over \( A \) is embedded in \( M_\infty(A) \) as a northwest corner,

\[
M_1(A) \subset M_2(A) \subset \cdots, \quad M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A).
\]

In particular, it implies that \( M_\infty(A) \) is a locally matrix algebra. We will show that

\[
(5) \quad \text{Spec}(M_\infty(A)) = S(\infty, s).
\]

Indeed, by Lemma 10

\[
\text{Spec}(M_\infty(A)) = \bigcup_{n=1}^{\infty} \text{Spec}(M_n(A)).
\]

We have \( \text{st}(M_n(A)) = ns \). In [4], we showed that

\[
\text{Spec}(M_n(A)) = \left\{ \frac{a}{b} \mid b \in \Omega(ns), a, b \in \mathbb{N}; 1 \leq a \leq b \right\}.
\]

This implies \( \text{Spec}(M_n(A)) \subseteq S(\infty, s) \). A Steinitz number \( (a/b) \cdot s \), \( b \in \Omega(s) \) lies in \( \text{Spec}(M_n(A)) \) provided that \( a/b \leq n \). This completes the proof of [5]. In particular, \( \text{Spec}(M_\infty(\mathbb{F})) = \mathbb{N} \).

Consider now a saturated subset \( S = S(r, s) \) or \( S = S^+(r, s), 1 \leq r < \infty \), where \( s \) is an infinite Steinitz number. Choose a sequence \( b_1, b_2, \ldots \in \Omega(s) \) such that \( b_i \) divides \( b_{i+1}, i \geq 1 \), and \( s \) is the least common multiple of \( b_i, i \geq 1 \). There exists a unique (up to isomorphism) unital countable-dimensional locally matrix algebra \( A_{s/b_i} \) such that \( \text{st}(A_{s/b_i}) = s/b_i \). Let \( A_i = M_{r_i(b_i)}(A_{s/b_i}) \). We have

\[
\text{st}(A_{s/b_i}) = \text{st}\left( (M_{b_{i+1}/b_i}(A_{s/b_{i+1}})) \right) = s/b_{i+1}.
\]

Hence, by Glimm’s Theorem, \( A_{s/b_i} \simeq M_{b_{i+1}/b_i}(A_{s/b_{i+1}}) \) and, therefore,

\[
A_i = M_{r_i(b_i)} \left( A_{s/b_i} \right) \simeq M_{r_i(b_i)} \left( A_{s/b_{i+1}} \right).
\]

By the inequality [1], \( r_i(b_i) \cdot (b_{i+1}/b_i) \leq r_s(b_{i+1}) \). Hence, the algebra \( A_i \) is embeddable in the algebra \( A_{i+1} \) as a northwest corner. Let

\[
A = \bigcup_{i=1}^{\infty} A_i.
\]
We will show that Spec\((A) = S\). Let \(0 \neq e \in A\) be an idempotent. Then \(e \in A_i\) for some \(i \geq 1\). In \([4]\), we showed that
\[
\text{st}(eA_i e) = \frac{a}{b} \text{st}(A_i),
\]
where \(a, b \in \mathbb{N}; a, b\) are coprime natural numbers; \(b \in \Omega(\text{st}(A_i)), a \leq b\).

Furthermore,
\[
\text{st}(A_i) = r_s(b_i) \frac{s}{b_i}, \quad \text{st}(eA_i e) = \frac{a}{b} r_s(b_i) \frac{s}{b_i}.
\]

Let \(d = \gcd(b, r_s(b_i)), b = db', r_s(b_i) = d \cdot r_s(b_i)'. \) So,
\[
\text{st}(eA_i e) = \frac{a \cdot r_s(b_i)'}{b'} \cdot \frac{s}{b_i} \in \mathbb{SN}.
\]

This implies that \(b' \in \Omega(s/b_i)\). Therefore, \(b'b_i \in \Omega(s)\). To show that \(\text{st}(eA_i e)\) lies in \(S(r, s)\) (respectively \(S^+(r, s)\)) we need to verify that \(a \cdot r_s(b_i) \leq rb'b_i\) (respectively \(a \cdot r_s(b_i)' < rb'b_i\)). Multiplying both sides of the inequality by \(d\) we get \(a \cdot r_s(b_i) \leq rbb_i\) (respectively \(a \cdot r_s(b_i)' = rbb_i\)). This inequality holds since \(a \leq b\) and \(r_s(b_i) \leq r \cdot b_i\) (respectively \(a \leq b\) and \(r_s(b_i)' < r \cdot b_i\)). We proved that \(\text{Spec}(A) \subseteq S\).

Let us show that \(S \subseteq \text{Spec}(A)\). Consider a Steinitz number \((a/b) \cdot s \in S\), where \(a, b \in \mathbb{N}; b \in \Omega(s), a \leq rb\) in the case \(S = S(r, s)\) or \(a < rb\) in the case \(S = S^+(r, s)\).

There exists a member of our sequence \(b_i\) such that \(b\) divides \(b_i\), \(b_i = k \cdot b, k \in \mathbb{N}\). Then \((a/b) \cdot s = (ak/b_i) \cdot s\).

We will show that \(ak \leq r_s(b_i)\). Indeed, multiplying both sides of the inequality by \(b\) we get \(ab_i \leq r_s(b_i)b\). Let \(S = S(r, s)\). Then \(a \leq rb\). Since \(a \in \mathbb{N}\) it implies \(a \leq [rb]\). Furthermore, \(r_s(b_i) = [rb_i] = [rbk]\). So, it is sufficient to show that \([rb]k \leq [rbk]\). This inequality holds since \([rb]k\) is an integer and \([rb]k \leq [rbk]\).

Now suppose that \(S = S^+(r, s)\). Then \(a < rb\),
\[
r_s(b_i) = \begin{cases} 
[r b_i], & \text{if } r b_i \notin \mathbb{N}, \\
rb_i - 1, & \text{if } r b_i \in \mathbb{N}.
\end{cases}
\]

There are three possibilities:

1) \(rb \in \mathbb{N}\) and, therefore, \(rb_i \in \mathbb{N}\). In this case \(a \leq rb - 1, r_s(b_i) = rb_i - 1\). We have \(ab_i \leq (rb - 1)b_i \leq (rb - 1)b = r_s(b_i)b\);

2) \(rb \notin \mathbb{N}\), but \(rb_i \in \mathbb{N}\). In this case \(a \leq [rb], r_s(b_i) = rb_i - 1\), we have \(ab_i \leq [rb]b_i, r_s(b_i)b = (rb_i - 1)b\). Hence, it is sufficient to show that \([rb]k \leq rb_i - 1 = rbk - 1\). The number \([rb]k\) is an integer and \([rb]k < rbk\) since \([rb] < rb\). This implies the claimed inequality;
3) \(rb_i \notin \mathbb{N}\) and, therefore, \(rb \notin \mathbb{N}\). In this case \(ab_i \leq [rb]b_k\), \(r_s(b_i)b = [rbk]b\) and it remains to notice that \([rb]k \leq [rbk]\).

We showed that both for \(S = S(r, s)\) and for \(S = S^+(r, s)\) there holds the inequality \(ak \leq r_s(b_i)\).

Recall that \(A_i = M_{r_s(b_i)}(A_{s/b_i})\). Consider the north–east corner \(M_{ak}(A_{s/b_i})\) of the algebra \(A_i\). We have

\[
\text{st} \left( M_{ak} \left( A_{s/b_i} \right) \right) = a k \cdot \frac{s}{b_i} = \frac{a}{b} s,
\]

and, therefore, \(S \subseteq \text{Spec}(A)\). This completes the proof of Theorem 3(1). \(\Box\)

For the proof of Theorem 3(2) we will need several lemmas on extension of isomorphisms.

\textbf{Lemma 11.} Let \(A\) be a locally matrix algebra and let \(A_1\) be a subalgebra of \(A\) such that \(A_1 \cong M_n(\mathbb{F})\). Then every automorphism of the algebra \(A_1\) extends to an automorphism of the algebra \(A\).

\textit{Proof.} Let \(e\) be the identity element of the subalgebra \(A_1\). Then the corner \(eAe\) is a unital locally matrix algebra. Let \(C\) be the centraliser of the subalgebra \(A_1\) in \(eAe\). By Wedderburn’s Theorem [see, \cite{6, 8}] we have \(eAe = A_1 \otimes \mathbb{F} C\). An arbitrary automorphism \(\varphi\) of the subalgebra \(A_1\) is inner, that is, there exists an invertible element \(x\) of the subalgebra \(A_1\) such that \(\varphi(a) = x^{-1}ax\) for all elements \(a \in A_1\). Conjugation by the element \(x \otimes e\) extends \(\varphi\) to an automorphism of the algebra \(eAe\). Consider the Peirce decomposition

\[
A = eAe + eA(1 - e) + (1 - e)Ae + (1 - e)A(1 - e),
\]

and the mapping

\[
\tilde{\varphi}: A \ni a \mapsto x^{-1}ax + x^{-1}a(1 - e) + (1 - e)ax + (1 - e)a(1 - e).
\]

The mapping \(\tilde{\varphi}\) extends \(\varphi\) and \(\tilde{\varphi} \in \text{Aut}(A)\). This completes the proof of the lemma. \(\Box\)

\textbf{Lemma 12.} Let \(A\) be a unital locally matrix algebra with an idempotent \(e \neq 0\). Then an arbitrary automorphism of the corner \(eAe\) extends to an automorphism of the algebra \(A\).

\textit{Proof.} Suppose at first that an automorphism \(\varphi\) of the algebra \(eAe\) is inner, and there exists an element \(x_e \in eAe\) that is invertible in the algebra \(eAe\) such that \(\varphi(a) = x_e^{-1}ax_e\) for an arbitrary element \(a \in eAe\). The element \(x = x_e + (1 - e)\) is invertible in the algebra \(A\). So, conjugation by the element \(x\) extends \(\varphi\).
Now let $\varphi$ be an arbitrary automorphism of the corner $eAe$. Let $A_1 \subseteq A$ be a subalgebra such that $1, e \in A_1$ and $A_1 \cong M_m(\mathbb{F})$ for some $m \geq 1$. Consider $A_2 \subseteq A$ such that $A_1 \subseteq A_2$, $\varphi(eA_1e) \subseteq A_2$ and $A_2 \cong M_n(\mathbb{F})$ for some $n \geq 1$. Consider the embedding

$$\varphi : eA_1e \rightarrow \varphi(eA_1e) \subseteq eA_2e$$

that preserves the identity element $e$. By Skolem–Noether Theorem (see, [6]) there exists an invertible element $x_e \in eA_2e$ such that $\varphi(a) = x_e^{-1}ax_e$ for an arbitrary element $a \in eA_1e$.

As noticed above, there exists an automorphism $\psi$ of the algebra $A$ that extends the automorphism $eAe \rightarrow eAe$, $a \mapsto x_e^{-1}ax_e$. The composition $\psi^{-1} \circ \varphi$ leaves all elements of the algebra $eA_1e$ fixed. Since it is sufficient to prove that the automorphism $\psi^{-1} \circ \varphi \in \text{Aut}(eAe)$ extends to an automorphism of $A$ we will assume without loss of generality that the automorphism $\varphi \in \text{Aut}(eAe)$ fixes all elements of $eA_1e$.

Let $C$ be the centraliser of the subalgebra $A_1$ in $A$. Then $A = A_1 \otimes \mathbb{F} C$ and $eAe = eA_1e \otimes \mathbb{F} C$. Since the subalgebra $eA_1e \otimes \mathbb{F} C$ is the centraliser of $eA_1e \otimes \mathbb{F} C$ in the algebra $eAe$ it follows that $e \otimes \mathbb{F} C$ is invariant with respect to the automorphism $\varphi$. Hence, there exists an automorphism $\theta \in \text{Aut}(C)$ such that $\varphi(a \otimes c) = a \otimes \theta(c)$ for all elements $a \in eAe$, $c \in C$. So, the automorphism $\varphi(a \otimes c) = a \otimes \theta(c)$, $a \in A_1$, $c \in C$, extends $\varphi$. This completes the proof of the lemma. \hfill $\square$

**Lemma 13.** Let $A$ be a unital locally matrix algebra with nonzero idempotents $e_1$, $e_2$. An arbitrary isomorphism $\varphi : e_1Ae_1 \rightarrow e_2Ae_2$ extends to an automorphism of the algebra $A$.

**Proof.** There exists a subalgebra $A_1 \subseteq A$ such that $1, e_1, e_2 \in A_1$ and $A_1 \cong M_n(\mathbb{F})$ for some $n \geq 1$. Let $r_i$ be the matrix range of the idempotent $e_i$ in $A_1$, $i = 1, 2$. In [4], it was shown that

$$\text{st} (e_1 A e_1) = \frac{r_1}{n} \cdot \text{st}(A), \quad \text{st} (e_2 A e_2) = \frac{r_2}{n} \cdot \text{st}(A).$$

Since $e_1Ae_1 \cong e_2Ae_2$ it follows that $r_1 = r_2$. In the matrix algebra $M_n(\mathbb{F})$ any two idempotents of the same range are conjugate via an automorphism. Hence, the idempotents $e_1$, $e_2$ are conjugate via an automorphism of $A_1$. By Lemma [1] an arbitrary automorphism of $A_1$ extends to an automorphism of the algebra $A$. Now the assertion of the lemma follows from Lemma [2]. \hfill $\square$

**Lemma 14.** Let $A, B$ be isomorphic unital locally matrix algebras with nonzero idempotents $e \in A$, $f \in B$. An arbitrary isomorphism $eAe \rightarrow fBf$ extends to an isomorphism $A \rightarrow B$. 


Proof. Let $\varphi : A \to B$, $\psi : eAe \to fBf$ be isomorphisms. Then

$$\varphi^{-1} \circ \psi : eAe \to \varphi^{-1}(f)A \varphi^{-1}(f)$$

is an isomorphism of two corners of the algebra $A$. By Lemma 13, $\varphi^{-1} \circ \psi$ extends to an automorphism $\chi$ of the algebra $A$, the isomorphism $\varphi \circ \chi$ extends $\psi$.

**Lemma 15.** Let $A$ be a unital locally matrix algebra and let $s_1$, $s_2$ be Steinitz numbers from Spec($A$). Suppose that $s_2/s_1 > 1$. Let $e_1 \in A$ be an idempotent such that $\text{st}(e_1Ae_1) = s_1$. Then there exists an idempotent $e_2 > e_1$ such that $\text{st}(e_2Ae_2) = s_2$.

Proof. Since $s_2 \in \text{Spec}(A)$ there exists an idempotent $e' \in A$ such that $\text{st}(e'Ae') = s_2$. Choose a subalgebra $A_1 \subseteq A$ such that $e_1$, $e' \in A_1$ and $A_1 \cong M_n(\mathbb{F})$.

Let $r_1$, $r_2$ be the matrix ranges of $e_1$, $e'$ in $M_n(\mathbb{F})$, respectively. In $[4]$, it was shown that

$$\text{st}(e_1 A e_1) = s_1 = \frac{r_1}{n} \text{st}(A), \quad \text{st}(e' A e') = s_2 = \frac{r_2}{n} \text{st}(A).$$

Hence $r_2 > r_1$. Since every idempotent in the algebra $M_n(\mathbb{F})$ is diagonalizable there exist automorphisms $\varphi$, $\psi$ of the algebra $A_1$ such that $\psi(e') > \varphi(e_1)$. By Lemma 11, the automorphisms $\varphi$, $\psi$ extend to automorphisms $\tilde{\varphi}$, $\tilde{\psi}$ of the algebra $A$, respectively.

Let $e_2 = \tilde{\varphi}^{-1}(\psi(e'))$. Then $e_2 > e_1$ and $\text{st}(e_2Ae_2) = s_2$, which completes the proof of the lemma. $\square$

**Proof of Theorem 3 (2).** Let $A$, $B$ be countable–dimensional locally matrix algebras, Spec($A$) = Spec($B$). Choose bases $a_1$, $a_2$, ... and $b_1$, $b_2$, ... in the algebras $A$, $B$, respectively.

We will construct ascending chains of corners $\{0\} = A_0 \subset A_1 \subset A_2 \subset \cdots$ in the algebra $A$ and $\{0\} = B_0 \subset B_1 \subset B_2 \subset \cdots$ in the algebra $B$, such that

$$\bigcup_{i=0}^{\infty} A_i = A, \quad \bigcup_{i=0}^{\infty} B_i = B$$

and $a_1$, ..., $a_i \in A_i$, $b_1$, ..., $b_i \in B_i$, $\text{st}(A_i) = \text{st}(B_i)$ for all $i \geq 1$.

Suppose that corners $\{0\} = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$, $\{0\} = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n$ have already been selected, $n \geq 0$. There exist corners $A' \subset A$, $B' \subset B$ in the algebras $A$, $B$, respectively, such that $A_n \subset A'$, $a_{n+1} \in A'$ and $B_n \subset B'$, $b_{n+1} \in B'$. The Steinitz numbers $\text{st}(A')$, $\text{st}(B')$ lie in the same saturated subset of $\mathbb{SN}$, therefore, they are rationally connected.
Suppose that $\text{st}(B') \geq \text{st}(A')$. Let $e'$ be an idempotent of the algebra $A$ such that $A' = e' Ae'$. The Steinitz number $\text{st}(B')$ lies in $\text{Spec}(A)$. Hence, by Lemma 15 there exists an idempotent $e \in A$ such that $e \geq e'$ and $\text{st}(e Ae) = \text{st}(B')$. Choose $A_{n+1} = e Ae$, $B_{n+1} = B'$. The chains $\{0\} = A_0 \subset A_1 \subset A_2 \subset \cdots$ and $\{0\} = B_0 \subset B_1 \subset B_2 \subset \cdots$ have been constructed.

By Lemma 14 every isomorphism $A_i \to B_i$ extends to an isomorphism $A_{i+1} \to B_{i+1}$. This gives rise to a sequence of isomorphisms $\varphi_i : A_i \to B_i$, $i \geq 0$, where each $\varphi_{i+1}$ extends $\varphi_i$. Taking the union $\cup_{i \geq 0} \varphi_i$ we get an isomorphism from the algebra $A$ to the algebra $B$. This completes the proof of theorem. \qed

Proof of Theorem 4. It is easy to see that a locally matrix algebra $A$ is a unital if and only if the set of idempotents of $A$ has a largest element: an identity. This is equivalent to $\text{Spec}(A)$ containing a largest Steinitz number. Among saturated sets of Steinitz numbers only $\{1, 2, \ldots, n\}$ and $S(r, s), s \in \mathbb{N}\setminus\mathbb{N}, r = u/v; u$ and $v$ are coprime natural numbers, $v \in \Omega(s)$, satisfy this assumption. \qed

5. Embeddings of locally matrix algebras

Lemma 16. Let $S_1, S_2$ be saturated sets of Steinitz numbers. Then either $S_1 \cap S_2 = \emptyset$ or one of the sets $S_1, S_2$ contains the other one.

Proof. Let $s \in S_1 \cap S_2$. If $s \in \mathbb{N}$ then, by Lemma 9 each set $S_i$ is either a segment $[1, n], n \geq 1$, or the whole $\mathbb{N}$. In this case the assertion of the lemma is obvious.

Suppose that the number $s$ is infinite. Then by Theorem 2 $S_i = S(r_i, s)$ or $S_i = S^+(r_i, s)$, where $r_i = r_{S_i}(s) \in [1, \infty) \cup \{\infty\}, i = 1, 2$. Clearly, if $r_{S_1}(s) < r_{S_2}(s)$ then $S_1 \subseteq S_2$. If $r_{S_1}(s) = r_{S_2}(s)$ then

$$S_1, S_2 = \begin{bmatrix} S(r, s) \\ S^+(r, s) \\ S(\infty, s) \end{bmatrix}$$

and $S^+(r, s) \subseteq S(r, s) \subset S(\infty, s)$ for any $r \in [1, \infty)$. This completes the proof of the lemma. \qed

Let $A$ be a locally matrix algebra. A subalgebra $B \subseteq A$ is called an approximative corner of $A$ if $B$ is the union of an increasing chain of corners. In other words, there exist idempotents $e_0, e_1, e_2, \ldots$ such that

$$e_0 A e_0 \subseteq e_1 A e_1 \subseteq e_2 A e_2 \subseteq \cdots, \quad B = \bigcup_{i = 0}^{\infty} e_i A e_i.$$
It is easy to see that an approximative corner of a locally matrix algebra is a locally matrix algebra.

**Theorem 5.** Let $A$, $B$ be countable-dimensional locally matrix algebras. Then $B$ is embeddable in $A$ as an approximative corner if and only if $\text{Spec}(B) \subseteq \text{Spec}(A)$.

**Proof.** If $B$ is an approximative corner of $A$ then every corner of $B$ is a corner of $A$, hence $\text{Spec}(B) \subseteq \text{Spec}(A)$.

Suppose now that $\text{Spec}(B) \subseteq \text{Spec}(A)$. If the algebra $B$ is unital then it embeds in the algebra $A$ as a corner. Indeed, the embedding $\text{Spec}(B) \subseteq \text{Spec}(A)$ implies that there exists an idempotent $e \in A$ such that $\text{st}(B) = \text{st}(eAe)$. By Glimm’s Theorem [7], we have $B \cong eAe$.

Suppose now that the algebra $B$ is not unital. Then there exists a sequence of idempotents $0 = f_0, f_1, f_2, \ldots$ of algebra $B$ such that

$$\{0\} = f_0 \leq f_1 \leq f_2 \leq \cdots \subseteq \bigcup_{i=0}^{\infty} f_i \leq B.$$

We will construct a sequence of idempotents $e_0, e_1, e_2, \ldots$ in the algebra $A$ such that

$$e_0 A e_0 \subseteq e_1 A e_1 \subseteq e_2 A e_2 \subseteq \cdots, \quad \text{st}(f_i B f_i) = \text{st}(e_i A e_i)$$

for an arbitrary $i \geq 0$. Let $e_0 = 0$. Suppose that we have already selected idempotents $e_0, e_1, \ldots, e_n \in A$ such that $e_0 A e_0 \subseteq e_1 A e_1 \subseteq \cdots \subseteq e_n A e_n$ and $\text{st}(e_i A e_i) = \text{st}(f_i B f_i), \; 0 \leq i \leq n$. We have

$$\text{st}(f_{n+1} B f_{n+1}) > \text{st}(f_n B f_n) = \text{st}(e_n A e_n)$$

and $\text{st}(f_{n+1} B f_{n+1}) \in \text{Spec}(A)$. By Lemma [15] there exists an idempotent $e_{n+1} \in A$ such that $e_n A e_n \subseteq e_{n+1} A e_{n+1}$ and $\text{st}(e_{n+1} A e_{n+1}) = \text{st}(f_{n+1} B f_{n+1})$, which proves existence of a sequence $e_0, e_1, e_2, \ldots$.

The union

$$A' = \bigcup_{i=0}^{\infty} e_i A e_i$$

is an approximative corner of the algebra $A$. By Glimm’s Theorem [7], $e_i A e_i \cong f_i B f_i, \; i \geq 1$. By Lemma [10]

$$\text{Spec}(A') = \bigcup_{i=1}^{\infty} \text{Spec}(e_i A e_i) \quad \text{and} \quad \text{Spec}(B) = \bigcup_{i=1}^{\infty} \text{Spec}(f_i B f_i).$$

Hence $\text{Spec}(B) = \text{Spec}(A')$. By Theorem [8](2), we have $A' \cong B$, which completes the proof of the theorem. □
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