PROJECTIVE VARIETIES WITH
MANY DEGENERATE SUBVARIETIES

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INTRODUCTION

Let \( X \subset \mathbb{P}^N \) be a projective non-degenerate variety of dimension \( n \). Then the hyperplane sections of \( X \) form a family of dimension \( N \) of varieties of dimension \( n - 1 \), each one generating a \( \mathbb{P}^{N-1} \). The sections of \( X \) with the codimension two linear subspaces \( L \) form a family, parametrized by the Grassmannian \( G(N-2, N) \); if \( L \) varies in a suitable open subset of \( G(N-2, N) \) then each section has dimension \( n - 2 \) and generates \( L \). Some particular codimension two linear space may possibly have improper intersection with \( X \), so that \( X \) could contain some subvarieties of dimension \( n - 1 \) generating a \( \mathbb{P}^{N-2} \). More generally, \( X \) could have some “degenerate” subvarieties of codimension \( h, h \geq 1 \), i.e. subvarieties of dimension \( n - h \), contained in a linear space of dimension at most \( N - h - 1 \).

The problem we shall study can be roughly formulated as follows: give a classification of irreducible (possibly smooth) non-degenerate varieties of dimension \( n \) in \( \mathbb{P}^N \) containing a “large-dimensional family” of degenerate subvarieties.

To clarify the meaning to be given to the expression “large-dimensional family”, let us consider some classical examples.

**Example 1.** Let \( n = 2, N = 4 \). It is well known (see for instance [B]) that the only surfaces of \( \mathbb{P}^4 \) containing a 2-dimensional family of conics are the projections of the Veronese surface of \( \mathbb{P}^5 \). There are no irreducible non-degenerate surfaces in \( \mathbb{P}^4 \) with a 3-dimensional family of conics. This result was improved by Corrado Segre ([S1], see also [CS]), who proved that, if an irreducible surface of \( \mathbb{P}^4 \) contains a family of dimension 2 of plane curves of degree \( d \), then \( d \leq 2 \). So either the curves are conics, and we fall in the above case, or they are lines, and the surface is a plane.

**Example 2.** The study of varieties containing “many” linear subspaces is also classical: it corresponds to the case \( N = n + 1 \) in the problem above. In the notes [bS], Beniamino Segre stated the problem of:

1. identifying what integers \( c > n - k \) can be dimension of a family of linear subspaces of dimension \( k \) contained in a variety of dimension \( n \);
2. classifying varieties of dimension \( n \) containing such a family of subspaces.
He proves that the maximal dimension is \( \delta(n, k) = (k+1)(n-k) \), which is achieved only by linear varieties. Moreover some classification results are given, but there is no complete solution to the problem (see §2).

**Example 3.** Let \( n = 2, N = 5 \). In [S1] and [S2] Corrado Segre studied the surfaces \( S \) in \( \mathbb{P}^5 \) with a family \( \mathcal{F} \) of dimension \( c \geq 2 \) of curves of \( \mathbb{P}^3 \). His result may be summarized as follows:

1. if \( c = 3 \), then \( S \) is a rational normal scroll of degree 4;
2. if \( c = 2 \), then there are two cases: either \( S \) is any surface contained in a 3-dimensional rational normal scroll of degree 3 or in a cone over a Veronese surface, or the curves in \( \mathcal{F} \) have degree at most 5 and a finite number of cases is possible.

Now we give a more precise formulation of the general problem.

**Problem.** Let \( 2 \leq n < N, 1 \leq h < n \) be integers. Classify irreducible non-degenerate varieties of dimension \( n \) in \( \mathbb{P}^N \), containing an algebraic family \( \mathcal{F} \) of dimension \( c \geq h + 1 \) of subvarieties of dimension \( n - h \), each one spanning a linear space of dimension at most \( N - h - 1 \).

In this paper, we propose an approach to this problem, which extends the classical one and stems from the notion of *foci* of a family of varieties: a classical notion that was recently rediscovered and used in various situations (geometry of canonical curves [CS], lifting problems [CC], [CCD], [Me]). We assume that \( c = h + 1 \) and that a general variety in \( \mathcal{F} \) generates a linear space of dimension exactly \( N - h - 1 \), and study the foci of the 1\(^{st}\), 2\(^{nd}\),..., order of the family of \( \mathbb{P}^{N-h-1} \)'s spanned by the varieties of \( \mathcal{F} \). We get (§1) that the solutions may be essentially divided in two parts: varieties contained in a higher dimensional variety with similar properties but bigger \( h \), and varieties that we call *of isolated type*, for which there is a bound on the degree of the degenerate subvarieties. So one sees that to completely solve the problem for varieties of codimension \( N - n \), one has to solve before the problem for varieties of codimension \( < N - n \).

Then we discuss the first particular cases of the question. In §2 we collect the results of B. Segre for \( N - n = 1 \) and some recent related results by Lanteri–Palleschi. In §3 we consider the case of varieties of codimension two in \( \mathbb{P}^N \): we treat the cases \( h = n - 1, n - 2 \), corresponding to varieties containing many plane curves or surfaces of \( \mathbb{P}^3 \). For \( n = 3 \), we see that the problem is easy if \( h = 1 \), since it reduces, by cutting with a general hyperplane, to the case \( n = 2 \) which is well known; if \( h = 2 \), the problem for the non–isolated type seems not to be easy. We discuss some examples but we have not uniqueness results. In §4, we consider varieties with \( N - n = 3 \); since the very long proof of the theorem of C. Segre, on surfaces in \( \mathbb{P}^5 \) containing a 2-dimensional family of space curves (see Example 3), seems to us not competely correct, we have tried to rewrite and translate it in a modern language, clarifying some rather oscure points. Then we make some remarks on the next cases.

**Notations.** We work over \( k \), an algebraically closed field of characteristic 0. \( \mathbb{P}^N \) will denote the projective space of dimension \( N \) over \( k \). By *variety* we will always mean an algebraic reduced scheme over \( k \). If \( V \) is a subscheme of \( W \), \( N_{V,W} \) will denote the normal sheaf of \( V \) in \( W \), \( T_W \) the tangent sheaf of \( W \).
1. Foci of families of linear spaces

In this section we will first recall some facts about foci of families of projective varieties (see [S], [CS], [CC]).

Let us fix integers \( N > 0 \) and \( h < N - 1 \). Let \( Z \) be a non–singular quasi–projective variety of dimension \( c \) and \( \Phi \subset Z \times \mathbb{P}^N \) be a flat family of projective irreducible subvarieties of codimension \( h+1 \) in \( \mathbb{P}^N \). Let \( q_1, q_2 \) be the natural projections from \( Z \times \mathbb{P}^N \) to \( Z, \mathbb{P}^N \); \( p_1, p_2 \) their restrictions to \( \Phi \). We will assume that the natural map from \( Z \) to the Hilbert scheme of subvarieties of \( \mathbb{P}^N \) is finite.

The family \( \Phi \) is said to be non–degenerate if the image of the projection \( p_2 \) has dimension \( N - h - 1 + c \), i.e. if \( p_2 \) is generically finite onto its image. A fundamental point for the family \( \Phi \) is by definition a point \( p \) of \( \mathbb{P}^N \) such that \( \dim p_1(p) - \dim p_2(\Phi) > N - h - 1 + c \); in particular, if \( \Phi \) is non–degenerate, then \( p \) is a fundamental point if it belongs to infinitely many varieties of \( \Phi \).

Let us consider now \( \tilde{\Phi} \), a desingularization of \( \Phi \), that we may assume to be flat over \( Z \), with smooth irreducible fibers. Moreover we may assume that the fibers of \( \tilde{\Phi} \) are desingularizations of the fibers of \( \Phi \). Denote by \( u \) the natural map \( \tilde{\Phi} \to \Phi \subset Z \times \mathbb{P}^N \): its differential \( du : T_{\tilde{\Phi}} \to u^* T_{Z \times \mathbb{P}^N} \) is an injective morphism of sheaves, whose cokernel is \( \mathcal{N} \), the normal sheaf to the map \( u \), a non–necessarily torsion–free sheaf. The following exact sequence of sheaves on \( \tilde{\Phi} \) defines a locally free sheaf \( T(p_2) \) of rank \( c \):

\[
0 \to T(p_2) \to u^* T_{Z \times \mathbb{P}^N} \to u^* q_2^* T_{\mathbb{P}^N} \to 0.
\]

We have the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & ker \chi & \to & T(p_2) & \xrightarrow{\chi} & \mathcal{N} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_{\tilde{\Phi}} & \to & u^* T_{Z \times \mathbb{P}^N} & \to & \mathcal{N} & \to & 0 \\
\downarrow d(q_2 u) & & \downarrow & & \downarrow & & \downarrow & & \\
u^* q_2^* T_{\mathbb{P}^N} & = & u^* q_2^* T_{\mathbb{P}^N} & \to & \to & \to & \to & \to & \to
\end{array}
\]

where \( \chi \) is the global characteristic map for \( \tilde{\Phi} \); it is such that

\[
\text{rk } \chi = \dim p_2(\Phi) - (N - h - 1), \quad \text{rk}(\ker \chi) = \dim \Phi - \dim p_2(\Phi).
\]

If \( \Phi \) is non–degenerate, then \( \text{rk } \chi = c \) and \( \text{rk}(\ker \chi) = 0 \).

Note the following facts ([CC]):
- if \( \tilde{H} \) is a general fiber of \( \tilde{\Phi} \), then \( \mathcal{N} |_{\tilde{H}} \) is isomorphic to the normal sheaf to the map \( \tilde{H} \to \mathbb{P}^N \) induced by \( \tilde{\Phi} \);
Let $1.3$. Theorem. Let $X$ be an integral smooth variety $\langle N \rangle$, has dimension $n-1$. Hence $p$ is a fundamental point for $\Phi$ and also for $\Phi_1$ and the lemma follows.

1.3. Theorem. Let $X \subset P^N$ be an integral variety of dimension $n$ containing a family $\mathcal{F}$ of dimension $c = h + 1$ of integral subvarieties of dimension $n - h$. Let
Let $Y$ be a general variety of $\mathcal{F}$ and assume that $Y$ spans a $\mathbb{P}^{N-h-1}$. Then one of the following happens:

1. there exists an integer $r$, $1 \leq r < N - n$, such that $X$ is contained in a variety $V_r$ of dimension at most $N - r$ containing $\infty^{h+1}$ varieties of dimension $N - h - r$, each one contained in a linear space of dimension $N - h - 1$;

2. $\deg Y$ is bounded by a function of $h, N - n$.

Proof. Let $\Phi_1$ be the family of linear spaces generated by the varieties of $\mathcal{F}$ and $H = (Y)$ be a general element of $\Phi_1$. Then $F^1_H$, the locus of the $1^\text{st}$ order loci of $\Phi_1$ on $H$, is defined by the vanishing of $\det \chi_H$, where $\chi_H$ is as follows:

$$T_{Z,z} \otimes O_H \xrightarrow{\chi_H} N_{H,\mathbb{P}^N}$$

$$\downarrow \quad \downarrow$$

$$O^*_{H^{+1}} \xrightarrow{\chi_H} O_H(1)^{h+1}.$$ 

If $\det \chi_H \equiv 0$, then $\Phi_1$ is degenerate, so $V_1 := \bigcup_{H \in \Phi_1} H$ is a variety of dimension $N - 1$ containing $X$ which is covered by $\infty^{h+1}$ linear spaces of dimension $N - h - 1$, and (1) holds with $r = 1$. If $\det \chi_H \not\equiv 0$, the condition $\det \chi_H = O$ defines a hypersurface $F^1_H$ of degree $h + 1$ in $H \approx \mathbb{P}^{N-h-1}$ containing $Y$. If $N - n = 1$, only the first case may happen; if $N - n = 2$ and the second case happens, we have $\dim Y = \dim F^1_H$ so $\deg Y \leq h + 1$ and the theorem is proved.

If $N - n > 2$, we consider the family $\{F^1_H\}$. If the general $F^1_H$ is integral, we set $\Phi_2 = \{F^1_H\}$; otherwise we replace $F^1_H$ with one of its irreducible components containing $Y$ with reduced structure. If $\Phi_2$ is degenerate, case (1) happens; if it is non-degenerate, by (1.2) any point $p$ of $X$ is covered by $\infty^{h+1}$ linear spaces of dimension $N - h - 1$.

Therefore $Y \subset \bigcup_{i=1}^{N-n-1} F^i_H$ for $Y$ being integral, it is completely contained in one of them.

If $N - n = 2$ then $\deg Y \leq \max\{\deg \text{Sing } F^1_H, \deg F^2_H\}$; otherwise we proceed in this way, by defining families $\Phi_i = \{F^{i-1}_H\}$ for $i \leq N - n - 1$. If one of these families is degenerate, (1) happens. Otherwise

$$\deg Y \leq \max\{\deg \text{Sing } F^{N-n-2}_H, \deg F^{N-n-1}_H\}$$

To have an estimate of such bound, note that $\dim F^i_H = N - h - 1 - i$; moreover, denoting $d_i = \deg F^i_H(i = 1, ..., N - n - 1), s_i = \deg \text{Sing } F^i_H$, there are the relations (where $k_i$ is the canonical class on $F^{i-1}_H$): 

- $d_i \leq (N + 1)d_{i - 1} + \deg K_{i - 1}$ where $K_{i - 1}$ is the canonical class on $F^{i-1}_H$;
- $\deg K_{i - 1} \leq (1 - N + h + i)d_{i - 1} - 2s_{i - 1} + k_i(k_i - 1)(i - 1) + 2e_i k_i - 2;
- s_{i - 1} \leq \frac{k_i(k_i - 1)(i - 1) + e_i k_i}{i - 1}$

where $k_i = \frac{d_i - 1 - 1}{i - 1}, 0 \leq e_i = d_{i - 1} - 1 - k_i(i - 1) < i - 1$.

So we get the recursive formula:

$$d_i \leq \frac{d_{i - 1}^2}{i - 1} + (h + 2 + i)d_{i - 1} + o(d_{i - 1})$$
where the latter term is independent of $h$; we conclude that $\deg Y$ is bounded by a function of $h$ and $N - n$ having the same order as $d_{N-n-1}$.

1.4. Definition. If a variety $X$ satisfies condition (2) of 1.3 we say that $X$ is of isolated type.

1.5. Remarks.

(i) The hypersurfaces of degree $h + 1$ of $\mathbb{P}^{N-h-1}$ arising as $1^{st}$ order foci are not general; they are in fact linear determinantal varieties. For example in $\mathbb{P}^3$ the general surface of degree $d \geq 4$ is not determinantal (see [H]). If $h = 1$ and $N \geq 5$ they are quadrics of rank at most 4.

(ii) The codimension 2 subvarieties of $\mathbb{P}^{N-h-1}$ which are $2^{nd}$ order foci of $\Phi_1$ are aritmetically Cohen–Macaulay. In fact they are defined by the condition $\text{rk } \chi_1 \leq h$ where $\chi_1 : \mathcal{O}_{\mathcal{F}_H}^{h+1} \to \mathcal{O}_{\mathcal{F}_H}(1)^{h+1} \oplus \mathcal{O}_{\mathcal{F}_H}(h + 1)$. By considering a lifting of $\chi_1$ to $H$, we see that it degenerates along a subvariety of codimension 2 in $H$, of degree

$$\deg c_2(\mathcal{O}_H(1)^{h+1} \oplus \mathcal{O}_H(h + 1)) = (h + 1)(3h + 2)/2.$$  

This variety is linked in a complete intersection of type $(h + 1, 2h + 1)$ to a variety of degree $h(h + 1)/2$ defined by a $h \times (h + 1)$-matrix of linear forms. In particular, if $h = 1$ it is a Castelnuovo variety.

(iii) When considering the $i^{th}$ order foci of $\Phi_1$, $i \geq 3$, we find the normal bundle $\mathcal{N}_{\mathcal{F}_H^{-1}, H}$ which is not decomposable in general, so we cannot lift the map $\chi_1$ to a map between bundles on $H$. For example, if $N = 6, n = 2, h = 1$, the $2^{nd}$ order foci form a Castelnuovo surface $S$ of degree 5 in $\mathbb{P}^4$, whose normal bundle is not decomposable. In this case a bound on the degree of $Y$ is given by $\deg c_1(\mathcal{N}_{S|\mathbb{P}^4}) = 5(\deg S)^2 + 2\pi - 2 - \deg S = 22$ (here $\pi$ denotes the sectional genus of $S$).

(iv) If $h > 1$, by the proof of (1.3) we have that $d_1 \leq h + 1$, and $d_i$ is bounded by a function of order $\frac{d_i}{i-1}$. So we get an upper bound of order

$$h^{2i-1}\frac{1}{(i-1)(i-2)^2(i-3)^2...2^{i-3}}.$$  

(v) Note that if $V_r$ is a variety as in (1), then any variety $X$ of dimension $n$ contained in $V_r$ satisfies the assumption of the Theorem.

In the next sections we will treat the first particular cases of the question, for $N - n = 1, 2, 3$. As we will see, for fixed codimension, the situation becomes simpler as $h$ decreases.

2. The case $N - n = 1$

In this section we collect the known results about the codimension 1 case, i.e. the case of a variety $X$ of dimension $n$ embedded in $\mathbb{P}^{n+1}$ and containing a family of dimension $c \geq h + 1$ of $\mathbb{P}^k$, $k = n - h$.

In the above quoted papers [bS] B. Segre, who does not make restrictions on the dimension of the projective space containing $X$, first states some existence results.
Assume $\delta = \delta(n, k) = (k+1)(n-k)$, the dimension of the Grassmannian of $k$–planes in $\mathbb{P}^n \ G(k, n)$. He proves that:

- if $k = 1$, then $c$ may assume all values from $n$ to $\delta(n, 1)$;
- if $k = 2$, then $c$ may assume all values from $n - 1$ to $\delta(n, 2)$, except $\delta - 1$;
- we say that $c = \delta - 1$ is a gap;
- if $k > 2$, then the following $k - 1$ numbers are gaps for $c$: $\delta - k + 1, ..., \delta - 1$; $\delta - k$ is not a gap; then there are $k - 2$ gaps: $\delta - 2k + 2, ..., \delta - k - 1$.

As for uniqueness results, he proves:

1. if $c = \delta$, then $X$ is linear;
2. if $c = \delta - k$, then $X$ is a scroll in $\mathbb{P}^{n-1}$'s, or a quadric if $k = 1$;
3. if $c = \delta - 2k + 1$, then $X$ is a quadric.

There is one more case he treats, i.e. $k = 1, c = \delta - 2$. He quotes results by Togliatti and Bompiani ([T], [Bo]) claiming that in this case $X$ has to be a scroll in $\mathbb{P}^{n-2}$'s, or a quadric bundle ($n \geq 4$), or a section of $\mathbb{G}(1, 4)$ with a $\mathbb{P}^7$ (the latter variety is not a hypersurface). Recently, this claim has been proved by Lanteri–Palleschi ([LP]) under smoothness assumption, by adjunction–theoretic techniques. Unfortunately, it is not clear if this result is true without assuming that $X$ is smooth. There is work in progress on this subject by E.Rogora.

3. The case $N - n = 2$

In this section $X$ is a codimension 2 subvariety of $\mathbb{P}^N$ containing $\infty^{h+1}$ varieties of dimension $n - h$ each one generating a $\mathbb{P}^{N-h-1}$. Then, by (1.3), there are the following possibilities:

1. $X$ is contained in a hypersurface $V_1$ containing all the linear spaces of the family $\Phi_1$;
2. $\deg Y \leq h + 1$.

Let us study the first particular cases.

\begin{enumerate}
\item[a)] $h = n - 1$
\end{enumerate}

The varieties $Y$ are plane curves. There are two subcases:

a1) $X$ is contained in a variety $V_1$ of dimension $n + 1$ containing at least $\infty^n$ planes. By §2 these varieties are classified only for $n \leq 3$: if $n = 2$ they are linear, which is impossible in our case because $X$ is non–degenerate; if $n = 3$, then $V_1$ is a quadric or a scroll in $\mathbb{P}^3$'s;

a2) the varieties $Y$ are plane curves of degree $d \leq n, d > 1$.

If $n = 2$, we find again the classical case of surfaces of $\mathbb{P}^4$ containing $\infty^2$ conics (see the Introduction).

If $n = 3$, the curves are conics or cubics. Looking at the list of the known smooth threefolds of $\mathbb{P}^5$, we find two examples, precisely the Bordiga and Palatini scrolls (see [O]). The Bordiga scroll is an arithmetically Cohen–Macaulay 3–fold in $\mathbb{P}^5$ which is defined by the $(3 \times 3)$–minors of a general $(4 \times 3)$–matrix with linear entries. Its general hyperplane section is a Bordiga surface of $\mathbb{P}^4$, a rational surface $S$ of degree 6, which can be realized as the blowing–up of $\mathbb{P}^2$ in 10 points, embedded by the complete linear series of quartics through these points. Then any line through 2 of the 10 points corresponds to a conic on $S$ and any cubic through 9 of them corresponds to an elliptic cubic; so $S$ contains 45 conics and 10 plane cubics. Hence the Bordiga scroll contains $\infty^3$ conics and $\infty^3$ plane cubics.
The Palatini scroll $X$ is an arithmetically Buchsbaum 3–fold whose ideal has the following $\Omega$–resolution (see [Ch]):

\[(3.1) \quad 0 \to 4\mathcal{O}_P^5 \to \Omega^1_{P^5}(2) \to \mathcal{I}_X(4) \to 0;\]

i.e. $X$ is the degeneration locus of a bundle map defined by 4 independent sections of $\Omega^1_{P^5}(2)$. A general hyperplane section of $X$ is a rational surface $S$ of degree 7 and sectional genus 4 (see [Ok]); the map that gives the embedding in $\mathbb{P}^4$ is associated to the linear system of the sextics plane curves with 6 double points and 5 simple points in assigned general position. Any line through two of the 6 double points corresponds to a conic on $S$ and any cubic through the 6 double points and 3 of the simple points corresponds to an elliptic cubic. So also the Palatini scroll contains $\infty^3$ conics and $\infty^4$ plane cubics.

Let us give another construction of such plane curves on $X$ by means of a geometrical interpretation of the exact sequence (3.1); it is the direct generalization of a construction by Castelnuovo which gives the Veronese variety in $\mathbb{P}^4$ (see [Ca], [O]). Let us define $V = H^0(\mathcal{O}_{P^5}(1))^*; \,$ by the Euler’s sequence, $H^0(\Omega^1(2)) \simeq \Lambda^2 V^*$, so any section of $\Omega^1(2)$ may be seen as a bilinear antisymmetric form on $V^*$, associated to an antisymmetric $(6 \times 6)$–matrix $A = (a_{ij})$, $i,j = 0, \ldots, 5$, with constant entries. It defines a null correlation $\Phi : \mathbb{P}^5 \to \mathbb{P}^5$, a rational linear map which associates to a point $p$ a hyperplane through $p$. $\Phi$ associates to a line $l$ a $\mathbb{P}^3$, the intersection of the hyperplanes corresponding to the points of $l$. Let us consider the set of lines $l$ such that $l \subset \Phi(l)$: it is classical that this is a linear complex $\Gamma$ of lines in $\mathbb{P}^5$, i.e. a hyperplane section of the Grassmannian $G(1, 5)$, which has equation precisely $\sum_{i,j} a_{ij} p_{ij} = 0$ ($p_{ij}$ coordinates of a line in the Plücker embedding of $G(1, 5)$).

Four sections of $\Omega^1(2)$ define four matrices $A_1, \ldots, A_4$, $A_k = (a_{ij}^k)$, four null correlations $\Phi_1, \ldots, \Phi_4$, four linear complexes $\Gamma_1, \ldots, \Gamma_4$; for a general point $p$, the four corresponding hyperplanes via $\Phi_1, \ldots, \Phi_4$ intersect along a line: by definition $X$ is the set of points $p$ such that they intersect along a plane $\pi$, i.e. the matrix

$$M := \begin{pmatrix} \sum a_{01}^1 x_i & \cdots & \sum a_{01}^4 x_i \\ \cdots & \cdots & \cdots \\ \sum a_{04}^1 x_i & \cdots & \sum a_{04}^4 x_i \end{pmatrix}$$

has rank $< 4$. So $X$ is defined by a system of equations of degree 4.

Let us recall that a linear complex $\Gamma$ of lines of $\mathbb{P}^5$ is said to be non-special if the associated matrix $A$ is non–degenerate; special if it is degenerate. The rank of $A$, $\rho(A)$, is always an even number; if $\rho(A) = 4$, there is a line, the center of $\Gamma$, which is met by all lines in $\Gamma$; if $\rho(A) = 2$, there is a $\mathbb{P}^3$ which is the center of the complex. Given four linear complexes $\Gamma_1, \ldots, \Gamma_4$, with associated matrices $A_1, \ldots, A_4$, they define a linear system of complexes; the special complexes of the system correspond to the zeros $(\lambda_1, \ldots, \lambda_4)$ of the pfaffian of the matrix $\lambda_1 A_1 + \ldots + \lambda_4 A_4$, so they are parametrized by a cubic surface of $\mathbb{P}^3$. The union of the lines, centers of these complexes, is the degeneracy locus of the bundle map

$$4\mathcal{O}_P^5 \to \Omega^1_{P^5}(2)$$

given by $A_1, \ldots, A_4$, i.e. the Palatini scroll. The intersection of $\Gamma_1, \ldots, \Gamma_4$ is a family of lines of dimension 4, $\Sigma_4$, such that there is one line of $\Sigma_4$ through a general point
of \( \mathbb{P}^5 \); but if \( p \in X \), there is a pencil of lines of \( \Sigma_4 \) through \( p \), spanning a plane \( \pi \). One can easily see, by exactly the same argument as in [Ca], that \( X \cap \pi \) is the union of the point \( p \) and a plane cubic, generally not containing \( p \). So we get a family of plane cubics on \( X \) parametrized by the points of \( X \).

Assume now that one of the matrices \( A_i \), \( 1 \leq i \leq 4 \), has rank 2; in particular, we may suppose that \( A_1 \) has the following canonical form:

\[
A_1 = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
-1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix};
\]

in this case the center of the complex \( \Gamma_1 \) is the space \( H : x_0 = x_1 = 0 \). Note that \( M \) takes the form

\[
M = \begin{pmatrix}
x_1 & -x_0 & 0 & 0 & 0 \\
P & N
\end{pmatrix}.
\]

It easily follows that \( X \) splits; in fact its components are \( H \) and the variety \( Y \) defined by the maximal minors of the \( 3 \times 4 \) matrix \( N \); hence \( Y \) is a Bordiga scroll.

So this construction gives both the known examples of smooth threefolds with \( \infty^3 \) plane curves, of isolated type.

The above construction may be generalized to higher dimension, giving examples of (singular) varieties of isolated type with \( N - n = 2 \), \( h = n - 1 \), for any \( n \geq 3 \).

\[ b) \ h = n - 2 \]

The varieties \( Y \) are surfaces of \( \mathbb{P}^3 \).

\[ b1) \ X \text{ is contained in a variety of dimension } n + 1, \ V_1, \text{ containing } \infty^{n-1}\mathbb{P}^3. \] If we cut \( V_1 \) with a hyperplane, we find a variety of dimension \( n \) containing \( \infty^{n-1} \) planes, the same situation as in \( a1) \): such varieties exist if \( n \geq 4 \) and are classified only for \( n = 4 \).

\[ b2) \ \deg Y \leq n - 1. \] If \( n = 3 \), then \( X \) is a \( 3 \)-fold of \( \mathbb{P}^5 \) containing \( \infty^2 \) quadric surfaces; by cutting with a hyperplane we see that either \( X \) is a cone over a Veronese surface, or a rational normal scroll.

If \( n = 4 \), then \( X \) is a \( 4 \)-fold of \( \mathbb{P}^6 \) containing a family \( \mathcal{F} \) of dimension 3 of cubic or quadric surfaces; also in this case, by cutting with a general hyperplane, we reduce to the case \( a2) \).

If the surfaces are quadrics, we are able to say a little bit more: since any quadric contains a family of dimension 1 of lines, then either \( X \) contains a family of dimension 4 of lines, or \( X \) contains a family of lines of dimension 3 such that through the general one there are \( \infty^1 \) quadrics of \( \mathcal{F} \). In the former case, by the results quoted in \( \S 2 \), if \( X \) is smooth, it is either a scroll in \( \mathbb{P}^2 \)'s over a surface or a quadric bundle over a smooth curve: it is easy to see that both possibilities cannot happen. In the second case, there are \( \infty^1 \) quadrics and a finite number of lines through a general point \( p \) of \( X \): if the quadrics were smooth, for any \( p \) all the quadrics through \( p \) should be tangent at \( p \), which is impossible; so the quadrics of the family are cones.
4. $N - n = 3$

In this section $X \subset \mathbb{P}^{n+3}$ is a variety containing a family $\mathcal{F}$ of dimension $h+1$ of subvarieties $Y$ of dimension $n - h$, spanning a $\mathbb{P}^{n-h+2}$. By (1.3) there are the following three possibilities:

1. $X \subset V_1$, with $\dim V_1 = n + 2$ and $V_1$ contains a family of dimension $h+1$ of $\mathbb{P}^{n-h+2}$;
2. $X \subset V_2$, with $\dim V_2 = n + 1$ and $V_2$ contains a family of dimension $h+1$ of subvarieties $Z$ of dimension $n - h + 1$ and degree $h+1$ each one generating a $\mathbb{P}^{n-h+2}$;
3. $\deg Y$ is bounded by a function of $h$; by 1.5, (ii) $\deg Y \leq (h+1)(3h+2)/2$.

a) $h = n - 1$

The varieties $Y$ are curves of $\mathbb{P}^3$.

a1) In case (1) the discussion is analogous to a1) of §3; there are no varieties $V_1$ if $n \leq 2$, they are classified if $n = 3$.

a2) In case (2) $V_2$ contains $\infty^n$ surfaces of $\mathbb{P}^3$; this is case b2) of §3.

a3) In case (3) $\deg Y \leq n(3n-1)/2$.

Let $n = 2$: $X$ is a surface of $\mathbb{P}^5$ containing $\infty^2$ curves of $\mathbb{P}^3$; case a1) is impossible, so the first order foci on a fixed space $H$ of the family $\Phi_1$ form a quadric; it can be thought of as the union of the focal lines, i.e. of the intersections of $H$ with the spaces of $\Phi_1$ which are “infinitely near” to $H$. In case a2) $X$ either is contained in a cone over a Veronese surface or in a rational normal scroll. In case a3), $\deg Y \leq 5$; the second order foci $F^2_H$ on a general space $H$ of the family $\Phi_1$ form a curve of degree 5 and genus 2 which is linked to a line on the focal quadric.

In the papers [S1] and [S2] C.Segre gave a classification of the surfaces $X$ as in a3); but there are some gaps and some rather obscure points in his proof. Here we give a new proof of his result which follows as much as possible the original one.

4.1. **Theorem (C.Segre).** Let $S \subset \mathbb{P}^5$ be a surface containing a family $\mathcal{F}$ of dimension 2 of irreducible non-degenerate curves $Y$ of $\mathbb{P}^3$, parametrized by an irreducible surface $Z$. If $S$ is of isolated type and not a cone, then one of the following happens:

1. $Y$ is a rational cubic and $S$ is a rational normal scroll of degree 4;
2. $Y$ is an elliptic quartic and $S$ is an elliptic normal scroll of degree 6;
3. $S$ is a rational surface, isomorphic to a blowing-up of $\mathbb{P}^2$, embedded in $\mathbb{P}^5$ by a linear system of cubics;
4. $S$ is a rational surface isomorphic to a blowing-up of $\mathbb{P}^2$ embedded in $\mathbb{P}^5$ by a linear system of quartics, such that the images of the lines are rational quartics of $\mathbb{P}^3$.

**Proof.** We give the proof assuming first that $S$ is smooth. At the end we will indicate how the argument can be modified if $S$ is singular.

Let us remark first that there are no base points for the family $\mathcal{F}$; otherwise, by projecting from a base point $q$, we find a surface $S'$ of $\mathbb{P}^4$ containing $\infty^2$ plane curves: $S'$ is a Veronese surface or a rational normal scroll of degree 3; so, since $S$ lies on the cone of vertex $p$ over $S'$, it is not of isolated type.

Then, observe that a general curve $Y$ is smooth; otherwise, if there is a point $p$ which is singular for all curves of $\mathcal{F}$, by projecting $S$ from $p$ we would find as above
that $S$ is not of isolated type. If the singular points of the curves of $F$ are variable, let $p$ be a variable singular point of $Y$: it should lie on all the focal lines of the space $H$ which is spanned by $Y$, so for any $Y$ the focal quadric on $H$ should be a quadric cone and $p$ it vertex. There are only two possible cases: either $Y$ is a quartic with a double point or it is a quintic with a triple point; in both cases $Y$ is rational so the surface $S$ is rational too. Since the singularities of $Y$ are variable, then $Y$ varies in a linear system of dimension at least 3 whose general curve is smooth.

The intersection number $j = Y^2$ of two curves of $F$ is constant; since it may be computed by intersecting $Y$ with a curve $Y'$ of an “infinitely near” space of $\Phi_1$, by monodromy we get that it is equal to the number of second order foci of $Y$ on a focal line, i.e. to the number of variable intersections of $Y$ with the focal lines.

We have the following possibilities:

(a) $j = 1$, $p_a(Y) = 0$, deg$Y = 4$ or 3;
(b) $j = 2$, $p_a(Y) = 2$, deg$Y = 5$;
(c) $j = 2$, $p_a(Y) = 1$, deg$Y = 4$;
(d) $j = 2$, $p_a(Y) = 0$, deg$Y = 3$.

In case (a), the family $F$ is linear (a homaloidal net); it defines a morphism $\phi: S \to \mathbb{P}^2$ which sends the curves of $F$ to the lines, so (3) or (4) happens.

Let us assume now that $j = 2$.

Claim (4.2). Let $p, q$ be two general points of $S$; then there are at least two curves of $F$ through them. If $p_a(Y) > 0$, there are exactly two.

Proof. Assume that the claim is not true; let $p', q'$ be the intersection of two general curves of $F$: then there are infinitely many curves of $F$ through $p'$ and $q'$, so they are fundamental points of $F$ and by consequence of $\Phi_1$: the infinitely many linear spaces containing $p'$ and $q'$ contain the whole line $\langle p', q' \rangle$, so it consists of fundamental points of $\Phi_1$ and is focal of first and second order, against the assumption that $S$ is of isolated type.

Assume now that there are $r > 2$ curves of $F$ through two general points of $S$. Let us fix a general point $p$ on $S$ and denote by $F_p$ the subfamily of $F$ of curves through $p$; it is parametrized by a curve $C_p \subset Z$. $F_p$ is a family of curves of dimension 1, degree 1 and index $r$ (let us recall that the degree of $F_p$ is the number of variable intersections of two curves of $F_p$, and the index of $F_p$ is the number of curves of $F_p$ passing through a general point of $S$); in fact, two general curves of $F_p$ intersect at a point $q$ different from $p$ and there are $r$ curves of $F_p$ through a general $q$. So, if $Y \in F_p$, we may define a rational map $g_Y$ from $C_p$ to $Y$ sending a curve $C$ to $\{Y \cap C \} - p$. The fibers $\{ \tilde{g}_Y^{-1}(q) \}_{q \in Y}$ form a pencil $\Pi_Y$ of divisors on $C_p$. There is an algebraic family of such pencils, one for any $Y$ in $F_p$; note that if $Y \neq Y'$ are curves of $F_p$, then obviously $\Pi_Y \neq \Pi_{Y'}$. The classical theorem of Castelnuovo–Humbert asserts that there cannot be an algebraic family of irrational pencils on a curve (see [M] for a modern version); it follows that the curves of $F$ are rational. This concludes the proof of Claim (4.2).

Let $S$ be as in (d). The curves of $F$ are skew cubics with selfintersection 2. Observe that in this case $S$ is rational. Let us consider the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(Y) \to \mathcal{O}_Y(Y) \to 0;$$

we have $h^0(\mathcal{O}_S(Y)) = 4$ which means that $S$ contains in fact a linear family of dimension 3 of skew cubics. In particular there are $\infty^2$ skew cubics through any
point $p$ of $S$; if we project $S$ from $p$, we get a surface $S'$ in $\mathbb{P}^4$ with a 2–dimensional family of conics, all intersecting a line, the image of (the tangent plane at) $p$. So it is clear that both $S'$ and $S$ are rational normal scrolls.

Let $S$ be as in (c): fix $q$, a general point on $S$, and $Y$ in $\mathcal{F}_q$: there is a birational map $Y \dashrightarrow C_q$ (the curve in $Z$ parametrizing $\mathcal{F}_q$), which associates to $p \in Y$ the point of $C_q$ corresponding to the unique curve of $\mathcal{F}_q$ through $p$. So $C_q$ is an elliptic curve with the same modulus as all the curves in $\mathcal{F}$.

We will construct now some rational curves on $S$: let us fix a $g^1_2$, i.e. a linear series of degree 2 and dimension 1, on $C_q$; it gives a rational family of pairs of curves of $\mathcal{F}$ through $q$. The variable intersections of such pairs of curves form a rational curve $L$ on $S$; as the $g^1_2$ varies on $C_q$, we get a family $\mathcal{L}$ of rational curves, such that there is exactly one curve of $\mathcal{L}$ through a general point of $S$. Note that $\mathcal{L}$ is an elliptic pencil, with the same modulus as any $Y$ of $\mathcal{F}$; in fact we may construct a rational map associating to $p \in Y (Y \in \mathcal{F}_q)$ the curve of $\mathcal{L}$ through $p$, which does not intersect $Y$ elsewhere. This implies also that $Y \cdot L = 1$ because for $L$ general the intersection $Y \cap L$ is transversal. Since there is a curve $Y$ of $\mathcal{F}$ through any two points of $L$, such a $Y$ must split: $Y = L \cup \Lambda$; so there is a subfamily of dimension 1 of $\mathcal{F}$ whose curves are of the form $L \cup \Lambda$, where $L$ is rational, $\Lambda$ is elliptic, $L \cdot \Lambda = 1$ and $\Lambda^2 = 0$. Since $\deg Y = 4$, then $\Lambda$ is a plane cubic and $L$ is a line. Therefore $S$ is an elliptic scroll, having elliptic quartics pairwise meeting in two points as unisecant curves. We conclude that $S$ is an elliptic normal scroll of degree 6.

It remains to exclude the case (b). Note first that in this case $S$ is birational to the symmetric product $Y^{(2)}$ of $Y$, where $Y$ is a general curve of $\mathcal{F}$; in fact, as in case (c), fixed $p \in S$, the system $\mathcal{F}_p$ is parametrized by $Y$ and we may associate to a point $q$ of $S$ the pair of curves of $\mathcal{F}_p$ passing through $q$. So $S$ is birational to the Jacobian of a curve of genus 2.

Claim (4.3). $S$ does not contain any rational curve; in particular, if it is smooth, it is minimal.

Proof. Note first that if $D$ is any irreducible curve on $S$, then $D \cdot Y > 0$: in fact if $D \cdot Y = 0$, then the curves of $\mathcal{F}_q$, $q \in D$, split as $D + \Delta$, where $\Delta$ varies in a 1–dimensional family; since $S$ does not contain any algebraic family of rational or elliptic curves, then $\Delta$ is a plane curve of degree 4. So we have a 1–dimensional family of planes; each of them intersects the spaces of $\Phi_1$ along a line; these lines generate a rational normal scroll which contains $S$, against the assumption that $S$ is of isolated type.

Let now $D \subset S$ be a rational curve, with $D \cdot Y = m > 0$: we have a map $\phi : C_p \to D^{(m)}$, the $m^{th}$ symmetric power of $D$, such that $\phi(Y) = Y \cap D$. Since the family $\mathcal{F}_p$ has index two, then the curve $\phi(C_p)$ is rational; but, if $m > 1$, $\phi$ should be injective, because $j = 2$, so $m = 1$. If we take two points of $D$, there is a curve $Y$ of $\mathcal{F}$ through them, so $Y$ splits: $Y = D + \Delta$ where $Y \cdot \Delta = 1$ and $\Delta$ is irreducible. There are the maps $\Phi_D : C_p \to D$, $\Phi_\Delta : C_p \to \Delta$ defined by the intersection; $\Phi_D$ is 2 : 1 so its fibers form the unique $g^1_2$ on $C_p$; if $\Delta$ is rational, also $\Phi_\Delta$ is 2 : 1, therefore $\Phi_D = \Phi_\Delta$, $D = \Delta$ and $Y = 2D$: this is impossible because $\deg Y = 5$, so $p_a(\Delta) > 0$. Since $\Delta$ cannot have genus one because the family $\mathcal{F}$ has constant moduli, then $p_a(\Delta) = 2$ and $\Delta$ is a plane quartic with $D \cdot \Delta = 1$. This implies that $D$ is a line with $D^2 = 0$: the claim follows because $S$ is not ruled.

As a first consequence, note that, by degree reasons, all the curves $Y$ in $\mathcal{F}$ are irreducible. Moreover, any $Y$ is an ample divisor on $S$, by the Nakai–Moishezon
criterion; in fact $Y^2 = 2 > 0$ and, for any irreducible curve $C$ on $S$, either $C \cdot Y > 0$ or $C$ is a component of $Y$, i.e. $C = Y$. In particular, we may apply the Kodaira vanishing theorem, getting the relations: $h^1(\mathcal{O}_S(Y)) = 0$, $h^0(\mathcal{O}_S(Y)) = 1$.

Claim (4.4). $S$ is linearly normal.

Proof. If $S$ were not linearly normal, then it would be the isomorphic projection of a surface $F$, $F \subset \mathbb{P}^6$; $F$ too should contain $\infty^2$ quintics of genus 2, generating spaces that intersect two by two along a line. Since these spaces cannot have a common point, they should intersect a fixed $\mathbb{P}^3$ along planes and the lines of intersection should lie in this $\mathbb{P}^3$. By projecting in $\mathbb{P}^5$ we would find that the spaces of $\Phi_1$ intersect two by two along lines of a fixed $\mathbb{P}^3$; in particular, all the focal quadrics and, by consequence, the surface $S$ should lie in this $\mathbb{P}^3$: a contradiction.

For any $Y$ we consider now the linear system of the curves $L$ which are residual to $Y$ in a hyperplane section of $S$: it is a complete linear system of dimension 1.

In fact, by the cohomology of the exact sequence

$$0 \rightarrow \mathcal{I}_S(1) \rightarrow \mathcal{I}_Y(1) \rightarrow \mathcal{I}_{Y|S}(1) \rightarrow 0,$$

we get $h^0(\mathcal{I}_Y(1)) = h^0(\mathcal{I}_{Y|S}(1)) = 2$, because $h^1(\mathcal{I}_S(1)) = 0$ by (4.4).

Let $\mathcal{G}$ be the family described by the curves $L$ as $Y$ varies in $\mathcal{F}$.

Claim (4.5). $\dim \mathcal{G} = 3$.

Proof. By the exact sequence

$$0 \rightarrow \mathcal{I}_S(1) \rightarrow \mathcal{I}_L(1) \rightarrow \mathcal{I}_{L|S}(1) \rightarrow 0,$$

we get $h^0(\mathcal{I}_L(1)) = h^0(\mathcal{I}_{L|S}(1)) = 1$, because $h^0(\mathcal{O}_S(Y)) = 1$. Let us consider in $\mathcal{F} \times \mathcal{G}$ the correspondence

$$\{(Y, L)|Y + L \in \mathbb{P}(\mathcal{O}_S(1))\} \xrightarrow{p_1} \mathcal{F} \xrightarrow{p_2} \mathcal{G}$$

where $p_1$ and $p_2$ are the projections. Since the fibers of $p_1$ have dimension 1 and the fibers of $p_2$ have dimension 0, then $\dim \mathcal{G} = 3$.

Let us compute the intersection number $Y \cdot L$. Fix $Y' \in \mathcal{F}$, $H$ a hyperplane containing $Y'$ and let $S \cap H = Y' \cup L$. Then $Y \cdot (S \cap H) = 5 = Y \cdot L + Y \cdot Y'$, so $Y \cdot L = 3$.

All the curves of $\mathcal{G}$ are irreducible: if some $L$ is reducible, then there is an irreducible component $\Lambda$ of $L$ such that $\Lambda \cdot Y < 2$, so $\Lambda$ is a component of any curve $Y$ of $\mathcal{F}$ passing through two of its points; i.e. $\Lambda = Y$; but this cannot happen because $Y^2 = 2$. Moreover $L$ is ample; in fact we will prove that $L^2 > 0$ and $L$ is strictly numerically effective. $S$ being abelian, we have $\mathcal{O}_S \simeq \omega_S$, $\mathcal{O}_L(L) \simeq \omega_L$, so, by the exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(L) \rightarrow \mathcal{O}_L(L) \rightarrow 0,$$

we get $1 \leq p_a(L) \leq 3$. Note that $\deg L = L \cdot (Y + L) = 3 + L^2$; if $L^2 = 0$, then $L$ is a plane cubic, but in this case $S$ should contain a family of dimension 3 of plane cubics,
which is impossible. Let $D$ be an irreducible curve; since all $L$ are irreducible, then $D \cdot L > 0$, because the curves of $\mathcal{G}$ fill up $S$. By the Nakai–Moishezon criterion, we conclude that $L$ is ample.

By Kodaira vanishing we get: $p_a(L) = 3$, $L^2 = 4$, $\deg L = 7$, $\deg S = 12$.

Let us consider now the intersection $Y \cap L$, where $Y \cup L = S \cap H$ is a hyperplane section of $S$.

**Claim (4.6).** $Y \cap L$ consists of three points on a line.

**Proof.** Let $H$ be general in the set of the hyperplanes containing the curves of $\mathcal{F}$. We know that $Y \cdot L = 3$; the points of $Y \cap L$ are singular for $S \cap H$, so either they are singular points of $S$ or $H$ is tangent to $S$ at these points. The former case is impossible. In the second case, we have a family of dimension 3, $\mathcal{H}$, of hyperplanes which are tangent to $S$; the set of foci of $\mathcal{H}$ on a general $H$ is a line $l$ (the “characteristic line”), because it is the degeneracy locus of a map

$$O^3_H \simeq T_{H,H} \otimes O_H \rightarrow O_H(1) \simeq N_{H|\mathbb{P}^5}.$$  

It is well known ([S]) that the tangency points lie on the characteristic line, so we have the claim.

Hence, for any hyperplane containing $Y$, we have a trisecant line of $Y$ (which is a line of the focal quadric).

As a consequence of (4.6), we have that, for fixed $L$ in $\mathcal{G}$, the intersection $L' \cap L$ is fixed if $L' \in L$. In fact, the curves of $|L|$ are all residual to the same $Y$: let $\pi_Y = \{Y\}$; since $L$ is contained in a hyperplane containing $Y$ then the intersection $\pi_Y \cap L$ consists of 7 points; moreover $S \cap \pi_Y = (L \cup Y) \cap \pi_Y = (L \cap \pi_Y) \cup Y$, so $L \cap \pi_Y$ depends only on $Y$, and is equal to $L \cap L'$ for any $L' \in |L|$. None of these points lies on $Y$; otherwise all the curves of $|L|$ should have the same intersection with $Y$ (for any point on $Y$ there is only one trisecant), so a curve in $|L|$ passing through another point of $Y$ should have $Y$ as a component, which has been excluded.

To conclude the proof, we will finally consider the family of the trisecants of $S$. Let $t$ be such a line and $Y$ be a curve of $\mathcal{F}$ passing through two of the intersection points. Then $t \subset \pi_Y$, so either the $3^{rd}$ intersection point lies on $Y$ and $t$ is a trisecant of $Y$, or it is one of the base points of $|L|$. Trisecants of the first type form a family $\mathcal{T}$ of trisecants of $S$ of dimension 3; those of the second type a family of dimension 2: from now on, we will consider only the trisecants of $\mathcal{T}$.

Fix $p \in S$: the trisecants through $p$ are parametrized by $C_p$, because there is exactly one trisecant for any curve of $\mathcal{F}$ through $p$, so they form a cone $K_p$ over a curve of genus 2, whose degree is at least 4. If $\deg K_p = 4$, then $K_p$ is a cone over a plane quartic, so $\langle K_p \rangle$ has dimension 3; the intersection points of the trisecants of $K_p$ generate a curve contained in $S$ into this 3–space, whose degree is at least 8. Since curves corresponding to different points of $S$ are different, $S$ should contain also $\infty^2$ curves of $\mathbb{P}^3$ of degree at least 8: a contradiction. Hence, $\deg K_p \geq 5$; in particular the number of trisecants through $p$ contained in a hyperplane, counted with multiplicities, is at least 5.

Let us fix now a curve $Y$ of $\mathcal{F}$ not containing $p$ and consider $H = \langle Y, p \rangle$, $H \cap S = Y \cup L$, $L \in \mathcal{G}$, $p \in L$. We will see now that there are at most 4 trisecants of $S$ contained in $H$ and passing through $p$: this gives a contradiction that concludes the proof of the theorem.

Let $t$ be such a trisecant; since $p \notin \langle Y \rangle$, then $\deg(t \cap Y) \leq 1$, otherwise $t \subset \langle Y \rangle$. So either $t$ is a trisecant of $L$ or it is a secant of $L$ intersecting $Y$. We prove that
if \( p \) is a general point of \( L \), there is at most one trisecant of \( L \) through \( p \). If there are two trisecants through \( p \), let us consider the projection from \( t_i, \pi_i \), \( i = 1, 2 \). If it is birational, then \( \pi_i(L) \) is a plane curve of degree 4 with a node, hence of genus 2; which is impossible. So \( \text{deg} \, \pi_i = 2 \). Note that \( L \) is hyperelliptic: the \( g_2^1 \) is cut out by the \( \mathbb{P}^3 \)'s through the plane \( \langle t_1, t_2 \rangle \); the pairs of points having the same image via \( \pi_i \) form the unique \( g_2^1 \) since they are contained both in planes through \( t_1 \) and through \( t_2 \), then they are contained in lines passing through \( p \). It is clear that this cannot happen for a general point of \( L \).

On the other hand, to compute the number (with multiplicity) of secants of \( L \) through \( p \) intersecting \( Y \), let us assume that \( p \) is one of the 4 base points of \( | L | \) belonging to \( \pi_Y - Y \). In this case, such a trisecant \( t \) lies inside \( \pi_Y \) and there are the following possibilities:

(i) \( t \) is one of the 3 lines joining \( p \) with a point of \( L \cap Y \);

(ii) \( t \) passes through 2 base points of \( | L | \) and meets \( Y \).

In case (i), \( t \) has to be counted with multiplicity one, if \( L \) is general, otherwise it should meet \( Y \) elsewhere; but in the linear system \( | L | \) there is only a finite number of curves passing through the intersection points of the secants of \( Y \) through \( p \). Possibility (ii) does not happen, if \( Y \) is general in \( \mathcal{F} \); in fact, suppose that, for any \( Y \) of \( \mathcal{F} \), a line through 2 of the base points of \( | L | \) intersects \( Y \) and let \( x_Y \) be the intersection point. Let \( \bar{L} \) be the curve of \( | L | \) through \( x_Y \); the other points \( x_1, x_2 \) of \( Y \cup \bar{L} \) are on a line through \( x_Y \), so there are 2 trisecants through \( x_Y \). Arguing as above (see [S2]), we get that \( x_Y \) lies on all lines generated by the pairs of points of the \( g_2^1 \) of \( L \); so \( x_Y \) is the vertex of a cone of trisecants \( S \) having degree 3. Since, as \( Y \) varies in \( \mathcal{F} \), the points \( x_Y \) cover \( S \), we have the required contradiction.

If \( S \) is singular, we may consider \( \tilde{S} \), a minimal desingularization of \( S \) and repeat the above argument on \( \tilde{S} \). There are two points where one must be careful. Precisely, concerning the proof of Claim (4.3), if \( D \) is an irreducible curve on \( \tilde{S} \) such that \( D \cdot \tilde{Y} = 0 \) (where \( \tilde{Y} \), varying in \( \mathcal{F} \), denotes the preimage on \( \tilde{S} \) of a curve \( Y \) of \( \mathcal{F} \), then there is a subfamily of \( \mathcal{F} \) of curves of the form \( D + \Delta \), where \( \Delta \) varies in a 1-dimensional family. In addition to the above discussed case, it may happen that \( D \) is a line which goes to a singular point of \( S \) and \( \Delta \) is a curve of genus 2 whose image on \( S \) is a quintic of the family \( \mathcal{F} \). But in this case \( D \cdot \Delta = D \cdot \tilde{Y} = 1 \) against the assumption. So \( \tilde{S} \) may contain some rational curves, precisely lines \( D \) with \( D \cdot \tilde{Y} = 1 \), but they contract to isolated singular points on \( \tilde{S} \). Moreover, in the proof of Claim (4.6), we have to exclude that, for \( H \) general, some of the 3 points of \( Y \cap L \) lies on the double curve \( D' \) of \( S \). If \( D' \) is a double line, keeping a hyperplane containing \( Y \) and \( D' \), we would find a non-integral residual to \( Y \), which is impossible. So \( D' \) should have degree 2 or 3 and its points of intersection with \( Y \) should be base points for the linear system \( | L | \) of the residuals. If there are 3 points, by imposing the passage through a 4th point of \( Y \), we would find a reducible \( L \); if there are 2, then the third intersection of the curves of \( | L | \) with \( Y \) would give a rational parametrization of \( Y \); therefore both cases are excluded.

4.7. Remarks.

(i) Let \( S \) be a surface of type (4) in (4.1); to get such a surface one has to find a linear system \( \delta \) of dimension 5 of plane quartics such that, for any line \( l \), the curves of \( \delta \) containing \( l \) form a pencil. If one wants a linearly normal surface, then \( \delta \) has the form \( \delta = | 4l - \sum_{i=1}^c P_i | \), where some of the points \( P_i \) may coincide or be infinitely near; so the cubic component of the quartics containing a line \( l \) varies.
in the linear system $\delta(-l) = |3l - \sum_{i=1}^{r} P_i|$. We sketch here how one can proceed provided the points $P_i$ are pairwise distinct. Note that $r \leq 9$ distinct points impose dependent conditions to the quartics if and only if 6 of them lie on a line. We exclude this case, because then the line is a fixed component of $\delta$, and we fall in case (c).

Let $r = 9$: $S$ is a surface of degree 7; $\dim \delta(-l) = 1$ if and only if $P_1, \ldots, P_9$ impose less than 9 conditions to the cubics, i.e. either they are the complete intersection of two cubics, or 8 of them lie on a conic. In the first case, as $l$ varies $\delta(-l)$ is a fixed pencil of cubics, so $\delta = \mathbb{P}(V)$, where $V \subset H^0(\mathcal{O}_{\mathbb{P}^2}(4))$ has the form $V = (x_0, x_1, x_2)(F, G)$ ($x_0, x_1, x_2$ homogeneous coordinates in the plane, $F, G$ polynomials of degree 3 defining two cubics of the pencil). This means that $S \subset \mathbb{P}^4 \times \mathbb{P}^2$, i.e. it is not of isolated type. In the second case, $\delta(-l)$ is also a fixed pencil of cubics, having a common conic $\Gamma$; $\delta$ does not separate the points of $\Gamma$, so the surface $S$ is singular: it has a quadruple point $p$ image of $\Gamma$. If we project $S$ in $\mathbb{P}^4$ from $p$, we get a surface $S'$ of degree 3, i.e. a rational normal scroll. Since $S$ is contained in the cone of vertex $p$ over $S'$ it is not of isolated type.

Coming to the non–linearly normal surfaces, $S$ is projection of a surface $F$ of $\mathbb{P}^n$, $n \geq 6$, which is also embedded by a linear system of quartics. There are two cases: either the rational quartics of $F$ are already in $\mathbb{P}^3$ and the projection is general, or they are rational normal quartics and one has to find a suitable projection. In the first case, $F$ is embedded in $\mathbb{P}^n$ by a linear system $\delta = |4l - \sum_{i=1}^{14-n} P_i|$ such that $\delta(-l) = |3l - \sum_{i=1}^{14-n} P_i|$ has dimension $n-4$. As before we exclude that the points $P_i$ lie on a line. So the only possibility is $n = 6$: then $P_1, \ldots, P_8$ are on a conic $\Gamma$ and $\delta$ does not separate the points of $\Gamma$. $F$ has a quadruple point $P$ image of $\Gamma$; projecting from $P$, we find the Veronese surface of $\mathbb{P}^5$. Also in this case $S$ is not of isolated type, because it projects on the Veronese surface in $\mathbb{P}^4$.

The second case is more complicated: $\delta = |4l - \sum_{i=1}^{14-n} P_i|$ is a projective space of dimension $n$: for any line $l$ in the plane, $\delta(-l)$ is a linear subspace of dimension $n-5$ of $\delta$. We have to find a subspace $\delta' \subset \delta$, of dimension 5, such that $\dim(\delta' \cap \delta(-l)) = 1$ for any line $l$.

If $n = 6$, there are 8 base points, the only possible choice for $\delta'$ is imposing the passing through the 9th base point of the pencil of cubics through $P_1, \ldots, P_8$: we fall in one of the previous cases.

Assume $n = 7$: for any line $l$, $\delta(-l)$ is a plane into $\delta$. Let us consider $X = \cup \delta(-l)$: it is a variety of dimension 4 in $\mathbb{P}^7$, ruled by planes. We may identify the net of cubics through $P_1, \ldots, P_7$ with $\mathbb{P}^2; \mathbb{P}^2 \times \mathbb{P}^2$, the set of pairs $(l, \Gamma)$, $l$ line of $\mathbb{P}^2$, $\Gamma$ cubic through $P_1, \ldots, P_7$, is embedded in $\mathbb{P}^8$ via the Segre map, as a variety of degree 6, and $X$ is a projection of its, so $\deg X \leq 6$. We would like to find a subvariety $Y \subset X$, ruled by lines, of dimension 3, generating a $\mathbb{P}^5$, so $Y$ should have degree at most 5. For any hyperplane $H$ containing $Y$, we would find a variety $Y'$ of dimension 3, residual to $Y$ in $H \cap X$, and such varieties $Y'$ should describe a linear system of dimension 1 inside $X$. If $\deg Y = 5$, then the $Y'$ should be linear spaces, which is impossible. If $\deg Y = 3$, $Y$ is the rational normal scroll $\mathbb{P}^1 \times \mathbb{P}^2$; this means that the pencil of cubics is fixed, i.e. the projection is centered in two points of $F$ and we find again the above cases. If $\deg Y = 4$, then its general hyperplane section is a surface of degree 4 in $\mathbb{P}^4$, therefore a Veronese surface, or a Del Pezzo surface, complete intersection of two quadrics, or a cone, or a rational non–normal scroll. In none of the former three cases $Y$ can be a scroll in lines; in the last case,
Y is a rational non-normal 3-fold ruled by planes: these planes should belong to one of the rulings of \( X \), which is impossible.

It is clear that, if \( n > 7 \), a similar analysis becomes more and more complicated and we were not able to conclude it.

(ii) It is easy to see that the rational normal scroll of degree 4 (case (1) of the Theorem) is the unique case of a surface of \( \mathbb{P}^5 \) with a 3-dimensional family of curves of \( \mathbb{P}^3 \). In fact, given such a surface \( S \) and a point \( p \) on \( S \), the curves of the family through \( p \) form a subfamily of dimension 2; if we project \( S \) from \( p \), we get a surface \( S' \) in \( \mathbb{P}^4 \) with a 2-dimensional family of plane curves, all intersecting a line and of degree 2. So it is clear that \( S' \) is a rational normal scroll.

(iii) If there exists a surface \( S \) of isolated type containing \( \infty^2 \) rational quartics, then for any point \( p \) of \( S \) and any curve \( Y \) of \( \mathcal{F} \) through \( p \), there is a finite number of trisecants to \( Y \); so there is a cone of trisecants \( S \) at any point \( p \). Hence such a surface would provide an example of a surface of \( \mathbb{P}^5 \) with a family of dimension 3 of trisecant lines, filling up a variety of dimension 4. Up to now, the only other known example is a non-general Enriques surface of degree 10 (see [CV]).

\[ b) h = n - 2 \]

The varieties \( Y \) are surfaces of \( \mathbb{P}^4 \). If \( n = 3 \), then \( h = 1 \) and \( X \) is a threefold of \( \mathbb{P}^6 \) containing a family of dimension 2 of surfaces of \( \mathbb{P}^4 \). There are as usual 3 possibilities:

- **b1)** \( X \) is contained in a variety of dimension 5 containing \( \infty^2 \) spaces of dimension 4: but this cannot happen for \( X \) non–degenerate by §2;

- **b2)** \( X \) is contained in a cone of dimension 4 over a Veronese surface or in a rational normal scroll of dimension 4;

- **b3)** The second order foci are Castelnuovo surfaces. In this case a general hyperplane section of \( X \), \( X \cap H = S \), is a surface of \( \mathbb{P}^5 \) of isolated type containing a family of dimension 2 of surfaces of \( \mathbb{P}^3 \), hence a surface as in Theorem 4.1. If \( S \) is a rational normal scroll, then \( X \) exists and is a rational normal scroll of degree 4 too.

If \( S \) is as in (4), then \( X \) has to contain a family of surfaces of degree 4 with rational sections, i.e. of Veronese surfaces, or cones, or rational non-normal scrolls (with a node). Each of them must lie in the variety of 1-st order foci, which is a quadric, so the case of the Veronese surfaces cannot happen. If the surfaces are all singular with a fixed singular point \( p \), either \( X \) is a cone or the projection from \( p \) gives a 3-fold \( X' \) in \( \mathbb{P}^5 \) containing \( \infty^2 \) surfaces of \( \mathbb{P}^3 \) of degree at most 2; so \( X' \) is not of isolated type: it should be contained in a 4-fold of \( \mathbb{P}^5 \) containing a 2-dimensional family of \( \mathbb{P}^3 \)'s, but this is impossible (case b1)). If \( p \) is a variable singular point of the surfaces, arguing as in 4.1, we see that it is the vertex of the focal quadric cone and that this vertex is fixed: a contradiction.

For discussing the other two cases, let us restrict to smooth threefolds. If \( S \) is an elliptic normal scroll as in (2), then \( X \) has to be linearly normal in \( \mathbb{P}^6 \) of degree 6: by Ionescu classification (see [I]), it does not exist. Finally, if \( S \) is as in (3), it is a Del Pezzo surface of degree 5 linearly normal in \( \mathbb{P}^5 \), or a projection of a Del Pezzo surface of \( \mathbb{P}^n \), \( 5 < n \leq 9 \). \( X \) should have sectional genus 1, hence it should be either a threefold ruled by planes over an elliptic curve (which cannot happen in our case) or a rational threefold. Such rational threefolds exist (see [Sc]) of degree
$n$, for $n = 5, \ldots, 8$; they may be realized as images of $\mathbb{P}^3$ via some maps given by linear systems of cubic surfaces having a base curve of degree $9 - n$.

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