Research article

Certain generalized fractional integral inequalities

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Abstract: The principal aim of this article is to establish certain generalized fractional integral inequalities by utilizing the Marichev-Saigo-Maeda (MSM) fractional integral operator. Some new classes of generalized fractional integral inequalities for a class of \( n \) \((n \in \mathbb{N})\) positive continuous and decreasing functions on \([a,b]\) by using the MSM fractional integral operator also derived.

Keywords: Marichev-Saigo-Maeda fractional integral operator; fractional integral inequalities

Mathematics Subject Classification: 6D10, 26A33, 26D53

1. Introduction

Fractional integral inequalities (FII in short) have made a great impact on scientists and mathematicians because of its potential applications in various fields. This subject plays a vital role in the development of differential equations and related problems in applied mathematics. In recent few decades, a variety of various integral inequalities and their generalizations have been established by utilizing fractional integral, fractional derivative operators and their generalizations are found in \([4–6, 10, 14–16, 19–21, 29, 35]\). Also, the applications of \((k, s)\)-Riemann-Liouville (R-L) fractional integral is found in \([30]\). In the past few years, various researchers have established the generalization of some classical inequalities by using different mathematical techniques. The generalized Hermite-Hadamard type inequalities with fractional integral operators and Hermite-Hadamard type inequalities by using the generalized k-fractional integrals are given in \([34]\) and \([2]\) respectively.
In [1], the authors established FII for a class of $n$ decreasing positive functions where $n \in \mathbb{N}$ by using $(k, s)$-fractional integral operator. Recently, the researchers [17, 18, 22–26] have established certain inequalities by employing some recent type(proportional and conformable) of fractional integrals. Without any doubt one can state that fractional and $k$-fractional calculus have become a very powerful tool for the modern studies, see for example [36, 37]. To move towards our main results, we recall the following definitions [9, 27, 31].

**Definition 1.1.** Let $f(\tau), \tau \geq 0$, real valued function, is said to be in the space $C_{\mu}([a, b]), \mu \in \mathbb{R}$ if there exist $p \in \mathbb{R}$ such that $p > \mu$ and $f(\tau) = \tau^{p}f_{1}(\tau)$ where $f_{1}(\tau) \in C([a, b])$.

**Definition 1.2.** Let $\nu, \nu', \xi, \xi' \in \mathbb{C}$ such that $R(\nu) > 0$ and $x \in \mathbb{R}$. Then MSM fractional integral is defined by

$$\left(\mathcal{I}_{a,x}^{\nu,\nu',\xi,\xi'}^{\eta} f\right)(x) = \frac{x^{-\nu}}{\Gamma(\eta)} \int_{a}^{x} (x-t)^{\nu-1} t^{\nu'} F_{3}\left(\nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t) dt$$

where $F_{3}(\cdot)$ represents the Appell function (or Horn function) which is given in [8] as

$$F_{3}(\nu, \nu', \xi, \xi'; \theta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\nu)_{m}(\nu')_{n}(\xi)_{m}(\xi')_{n}}{(\theta)_{m+n}} x^{m}y^{n} \frac{1}{m!n!}, \max\{|\nu|, |\theta|\} < 1,$n

and $(\nu)_{m} = \nu(\nu+1) \cdots (\nu+m-1)$ is the Pochhammer symbol.

The operator (1.1) is introduced in [13] and extended in [31, 32]. The use of this function in connection with special functions is appeared in many recent papers [3, 11, 12].

**2. Main results**

In this section, we employ the MSM fractional integral operator to establish the generalization of some classical inequalities. Recalling the following Theorem which will be used to establish our main result.

**Theorem 1.** (see [28], Theorem 1) If $\nu, \nu', \xi, \xi' \in \mathbb{R}$ such that $\eta > \max\{\nu, \nu', \xi, \xi'\} > 0$, then the following inequality holds

$$F_{3}\left(\nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) > 0,$$

provided $-1 < (1 - \frac{t}{x}) < 0$ and $0 < (1 - \frac{x}{t}) < \frac{1}{2}$. Also, if $f(x) > 0$, then

$$\left(\mathcal{I}_{a,x}^{\nu,\nu',\xi,\xi'}^{\eta} f\right)(x) > 0.$$

**Theorem 2.** Let $g$ be a positive continuous and decreasing function on the interval $[a, b]$. Let $\nu, \nu', \xi, \xi' \in \mathbb{R}$ such that $\eta > \max\{\nu, \nu', \xi, \xi'\} > 0$, $a < x \leq b$, $\vartheta > 0$ and $\sigma \geq \gamma > 0$. Then for MSM fractional integral operator (1.1), we have

$$\frac{\mathcal{I}_{a,x}^{\nu,\nu',\xi,\xi'}^{\eta} [g^{\vartheta}(x)]}{\mathcal{I}_{a,x}^{\nu,\nu',\xi,\xi'}^{\eta} [g^{\gamma}(x)]} \leq \frac{\mathcal{I}_{a,x}^{\nu,\nu',\xi,\xi'}^{\eta} [(x-a)^{\sigma}g^{\gamma}(x)]}{\mathcal{I}_{a,x}^{\nu,\nu',\xi,\xi'}^{\eta} [(x-a)^{\vartheta}g^{\gamma}(x)]},$$

provided $-1 < (1 - \frac{t}{x}) < 0$ and $0 < (1 - \frac{x}{t}) < \frac{1}{2}$. 

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Define a function 

\[(\rho - a)^\theta - (t - a)^\theta)(g^{\sigma - \gamma}(t) - g^{\sigma - \gamma}(\rho)) \geq 0, \quad (2.3)\]

where \(a < t, \rho \leq b, \theta > 0, \sigma \geq \gamma > 0.\)

By (2.3), we have

\[(\rho - a)^\theta g^{\sigma - \gamma}(t) + (t - a)^\theta g^{\sigma - \gamma}(\rho) - (\rho - a)^\theta g^{\sigma - \gamma}(\rho) - (t - a)^\theta g^{\sigma - \gamma}(t) \geq 0. \quad (2.4)\]

Define a function

\[
\tilde{\gamma}(x, t) = (x - t)^{\gamma - 1} t^{-\gamma} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right)
\]

\[
= (x - t)^{\gamma - 1} t^{-\gamma} \left[ 1 + \frac{(\nu t) (t^t)}{t^{-\gamma}} \right) + \frac{(\nu t) (t^t)}{t^{-\gamma}} \right) (1 - \frac{t}{x}) \cdot \ldots \right]. \quad (2.5)
\]

In view of Theorem 1, we observe that the function \(\tilde{\gamma}(x, t)\) remain positive for all \(t \in (a, x), x > a,\) since each term of the above function is positive in view of conditions stated in Theorem 2. Therefore multiplying (2.4) by

\[
\tilde{\gamma}(x, t) g^\gamma(t) = (x - t)^{\gamma - 1} t^{-\gamma} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^\gamma(t), \quad t \in (a, x), a < x \leq b,
\]

we get

\[
\tilde{\gamma}(x, t) \left[ (\rho - a)^\theta g^{\sigma - \gamma}(t) + (t - a)^\theta g^{\sigma - \gamma}(\rho) - (\rho - a)^\theta g^{\sigma - \gamma}(\rho) - (t - a)^\theta g^{\sigma - \gamma}(t) \right] g^\gamma(t)
\]

\[
= (\rho - a)^\theta (x - t)^{\gamma - 1} t^{-\gamma} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^\gamma(t) g^\sigma(t)
\]

\[
+ (t - a)^\theta (x - t)^{\gamma - 1} t^{-\gamma} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^\gamma(t) g^\sigma(t)
\]

\[
- (\rho - a)^\theta (x - t)^{\gamma - 1} t^{-\gamma} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^\gamma(t) g^\sigma(t)
\]

\[
- (t - a)^\theta (x - t)^{\gamma - 1} t^{-\gamma} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^\gamma(t) g^\sigma(t) \geq 0. \quad (2.6)
\]

Integrating (2.6) with respect to \(t\) over \((a, x),\) we have

\[
(\rho - a)^\theta \int_a^x (x - t)^{\gamma - 1} t^{-\gamma} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^\gamma(t) dt
\]

\[
+ g^{\sigma - \gamma}(\rho) \int_a^x (x - t)^{\gamma - 1} t^{-\gamma} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) (t - a)^\theta g^\gamma(t) dt
\]

\[
- (\rho - a)^\theta g^{\sigma - \gamma}(\rho) \int_a^x (x - t)^{\gamma - 1} t^{-\gamma} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^\gamma(t) dt
\]

\[
- \int_a^x (x - t)^{\gamma - 1} t^{-\gamma} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) (t - a)^\theta g^\gamma(t) dt \geq 0. \quad (2.7)
\]

Multiplying (2.7) by \(\frac{t^t}{t^{-\gamma}},\) we get

\[
(\rho - a)^\theta \mathcal{S}_{\nu, x}^{\nu', \xi, \xi'} \left[ g^\sigma(x) \right] + g^{\sigma - \gamma}(\rho) \mathcal{S}_{\nu, x}^{\nu', \xi, \xi'} \left[ x - a \right]^\theta g^\gamma(x)
\]

\textit{Aims Mathematics} Volume 5, Issue 2, 1588–1602.
Multiplying (2.8) by
\[ \frac{x^{-\gamma}}{\Gamma(\eta)} \tilde{\mathcal{F}}(x, \rho) g^\gamma(\rho) = \frac{x^{-\gamma}}{\Gamma(\eta)} (x - \rho)^{-1} \rho^{-\gamma} F_3 \left( \nu', \xi', \xi'; \eta; 1 - \frac{\rho}{x}, 1 - \frac{x}{\rho} \right) g^\gamma(\rho) \]
where \( \tilde{\mathcal{F}}(x, \rho) \) is defined by (2.5) and integrating the resultant identity with respect to \( \rho \) over \((a, x)\), we get
\[ \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\sigma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] - \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\gamma(x)] \geq 0. \]
It follows that
\[ \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\sigma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\gamma(x)] \geq 0. \]
Dividing the above equation by \( \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\gamma(x)] \), we get the desired inequality (2.2).

Remark 2.1. The inequality in Theorem 2 will reverse if \( g \) is an increasing function on the interval \([a, b]\).

Theorem 3. Let \( g \) be a positive continuous and decreasing function on the interval \([a, b]\). Let \( a < x \leq b, \theta > 0, \sigma \geq \gamma > 0 \). Then for the MSM fractional integral (1.1), we have
\[ \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\sigma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] + \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] \geq 1, \]
where \( \alpha, \beta, \xi', \lambda, \nu', \xi, \xi' \in \mathbb{R} \) such that \( \eta > \max\{\nu', \xi', \xi'\} > 0 \) and \( \lambda > \max\{\nu', \xi', \xi'\} > 0 \).

Proof. By multiplying both sides of (2.8) by
\[ \frac{x^{-\gamma}}{\Gamma(\lambda)} \tilde{\mathcal{F}}(x, \rho) g^\gamma(\rho) = \frac{x^{-\gamma}}{\Gamma(\lambda)} (x - \rho)^{-1} \rho^{-\gamma} F_3 \left( \alpha, \beta, \xi', \lambda; 1 - \frac{\rho}{x}, 1 - \frac{x}{\rho} \right) g^\gamma(\rho) \]
where \( \tilde{\mathcal{F}}(x, \rho) \) is defined by (2.5) and integrating the resultant identity with respect to \( \rho \) over \((a, x)\), we have
\[ \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\sigma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] \]
\[ + \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\gamma(x)] \]
\[ - \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\gamma(x)] \geq 0. \]
Hence, dividing (2.10) by
\[ \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\gamma(x)] \]
\[ + \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [(x - a)^\theta g^\gamma(x)] \mathcal{Y}_{a, x}^{\nu', \xi', \xi'} [g^\gamma(x)], \]
we get the required results. □
Remark 2.2. Applying Theorem 3 for \( \alpha = \nu, \beta = \nu', \zeta = \xi, \zeta' = \xi', \lambda = \eta \), we get Theorem 2.

Theorem 4. Let \( g \) and \( h \) be positive continuous functions on the interval \([a, b]\) such that \( h \) is increasing and \( g \) be decreasing functions on the interval \([a, b]\). Let \( a < x \leq b, \theta > 0, \sigma \geq \gamma > 0 \). Then for the MSM fractional integral (1.1), we have

\[
\frac{\mathcal{S}_{\nu, \xi, \xi'}^{\nu', \xi', \eta} \left[ g^{\sigma}(x) \right]}{\mathcal{S}_{\nu, \xi, \xi'}^{\nu', \xi', \eta} \left[ h^{\rho}(x) g^{\sigma}(x) \right]} \geq 1, 
\]

(2.11)

where \( \nu, \nu, \xi, \xi' \in \mathbb{R} \) such that \( \eta > \max\{\nu, \nu', \xi, \xi'\} > 0 \).

Proof. Under the conditions stated in Theorem 4, we can write

\[
\left( h^{\rho}(p) - h^{\rho}(t) \right) \left( g^{\sigma-\gamma}(t) - g^{\sigma-\gamma}(\rho) \right) \geq 0 \tag{2.12}
\]

where \( a < x \leq b, \theta > 0, \sigma \geq \gamma > 0 \).

From (2.12), we have

\[
h^{\rho}(\rho) g^{\sigma-\gamma}(\rho) + h^{\rho}(t) g^{\sigma-\gamma}(\rho) - h^{\rho}(\rho) g^{\sigma-\gamma}(\rho) - h^{\rho}(t) g^{\sigma-\gamma}(t) \geq 0. \tag{2.13}
\]

Multiplying both sides of (2.13)

\[
\mathfrak{S}(x, t) g^{\gamma}(t) = (x-t)^{\gamma-1} t^{\gamma'} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^{\gamma}(t), \quad t \in (a, x), a < x \leq b,
\]

where \( \mathfrak{S}(x, t) \) is defined by (2.5), we get

\[
\mathfrak{S}(x, t) g^{\gamma}(t) \left[ h^{\rho}(\rho) g^{\sigma-\gamma}(\rho) + h^{\rho}(t) g^{\sigma-\gamma}(\rho) - h^{\rho}(\rho) g^{\sigma-\gamma}(\rho) - h^{\rho}(t) g^{\sigma-\gamma}(t) \right] \\
= h^{\rho}(\rho) (x-t)^{\gamma-1} t^{\gamma'} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^{\gamma}(t) \\
+ h^{\rho}(t) (x-t)^{\gamma-1} t^{\gamma'} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^{\sigma-\gamma}(\rho) g^{\gamma}(t) \\
- h^{\rho}(\rho) (x-t)^{\gamma-1} t^{\gamma'} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^{\sigma-\gamma}(\rho) g^{\gamma}(t) \\
- h^{\rho}(t) (x-t)^{\gamma-1} t^{\gamma'} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^{\gamma}(t) \geq 0. \tag{2.14}
\]

Integrating (2.14) with respect to \( t \) over \((a, x)\), we have

\[
h^{\rho}(\rho) \int_a^x (x-t)^{\gamma-1} t^{\gamma'} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^{\gamma}(t) dt \\
+ g^{\sigma-\gamma}(\rho) \int_a^x (x-t)^{\gamma-1} t^{\gamma'} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) h^{\rho}(t) g^{\gamma}(t) dt \\
- h^{\rho}(\rho) g^{\sigma-\gamma}(\rho) \int_a^x (x-t)^{\gamma-1} t^{\gamma'} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) g^{\gamma}(t) dt \\
- \int_a^x (x-t)^{\gamma-1} t^{\gamma'} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) h^{\rho}(t) g^{\gamma}(t) dt \geq 0. \tag{2.15}
\]
Multiplying (2.15) by $\frac{\psi}{\Gamma(\eta)}$, we get
\[
\begin{align*}
\alpha^{\beta, \zeta, \lambda} [g^\sigma(x)] + \beta^{\gamma, \nu, \xi} [h^\nu(x)]g^\gamma(x) \\
-\alpha^{\beta, \zeta, \lambda} [h^\nu(x)]g^\gamma(x) \geq 0.
\end{align*}
\]
(2.16)

Again, multiplying (2.16) by
\[
\frac{X^\gamma}{\Gamma(\eta)} g^\gamma(x) = \frac{X^\gamma}{\Gamma(\eta)} (x - \rho \gamma) \gamma_{-1} \rho \gamma F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{\rho}{x}, 1 - \frac{X}{\rho} \right) g^\gamma(\rho)
\]
and integrating the resultant identity with respect to $\rho$ over $(a, x)$, we get
\[
\alpha^{\beta, \zeta, \lambda} [g^\sigma(x)] \beta^{\gamma, \nu, \xi} [h^\nu(x)]g^\gamma(x) \\
-\beta^{\gamma, \nu, \xi} [h^\nu(x)]g^\gamma(x) \geq 0
\]
which completes the desired inequality (2.11) of Theorem 4.

\[\Box\]

**Theorem 5.** Let $g$ and $h$ be positive continuous functions on the interval $[a, b]$ such that $h$ is increasing and $g$ be decreasing functions on the interval $[a, b]$. Let $a < x \leq b$, $\theta > 0$, $\sigma \geq \gamma > 0$. Then for the MSM fractional integral (1.1), we have
\[
\frac{\alpha^{\beta, \zeta, \lambda} [g^\sigma(x)] \beta^{\gamma, \nu, \xi} [h^\nu(x)]g^\gamma(x) + \beta^{\gamma, \nu, \xi} [h^\nu(x)]g^\gamma(x)}{\beta^{\gamma, \nu, \xi} [g^\gamma(x)] + \beta^{\gamma, \nu, \xi} [h^\nu(x)]g^\gamma(x)} \geq 1,
\]
(2.17)
where $\alpha, \beta, \zeta, \lambda, \nu, \nu', \xi, \xi', \eta \in \mathbb{R}$ such that $\eta > \max\{\nu, \nu', \xi, \xi'\} > 0$ and $\lambda > \max\{\nu, \nu', \xi, \xi'\} > 0$.

**Proof.** Multiplying (2.16) by
\[
\frac{X^\gamma}{\Gamma(\eta)} g^\gamma(x) = \frac{X^\gamma}{\Gamma(\eta)} (x - \rho \gamma) \gamma_{-1} \rho \gamma F_3 \left( \alpha, \beta, \zeta, \lambda; 1 - \frac{\rho}{x}, 1 - \frac{X}{\rho} \right) g^\gamma(\rho)
\]
(where $\gamma(x, \rho)$ is defined by (2.5)) and integrating the resultant identity with respect to $\rho$ over $(a, x)$, we get
\[
\alpha^{\beta, \zeta, \lambda} [g^\sigma(x)] \beta^{\gamma, \nu, \xi} [h^\nu(x)]g^\gamma(x) \\
-\beta^{\gamma, \nu, \xi} [h^\nu(x)]g^\gamma(x) \geq 0.
\]

It follows that
\[
\alpha^{\beta, \zeta, \lambda} [g^\sigma(x)] \beta^{\gamma, \nu, \xi} [h^\nu(x)]g^\gamma(x) \\
\geq \beta^{\gamma, \nu, \xi} [h^\nu(x)]g^\gamma(x)
\]

\[\Box\]
Therefore multiplying both sides by
\[ + \mathcal{F}_{\alpha,x}^{\nu',\xi',\eta}(h^{\varphi}(x)) \mathcal{F}_{\alpha,x}^{\nu',\xi',\eta}(g^{\gamma}(x)) \]
which gives the desired inequality (2.32).

\[ \square \]

Remark 2.3. Applying Theorem 5 for \( \alpha = \nu, \beta = \nu', \xi = \xi', \zeta' = \zeta', \lambda = \eta \), we get Theorem 4.

Now, we use the MSM fractional integral fractional integral operator to present some inequalities for a class of \( n \)-decreasing positive functions.

Theorem 6. Let \((g_i)_{i=1,2,3,\ldots,n}\) be \( n \) positive continuous and decreasing functions on the interval \([a, b]\). Let \( a < x < b, \vartheta > 0, \sigma > \gamma_p > 0 \) for any fixed \( p \in \{1, 2, 3, \ldots, n\} \). Then for MSM fractional integral operator (1.1), we have
\[ \begin{align*}
\frac{\mathcal{F}_{\alpha,x}^{\nu',\xi',\eta}(h^{\varphi}(x))}{\mathcal{F}_{\alpha,x}^{\nu',\xi',\eta}(\prod_{i=1}^{n} g_i^{\gamma_p}(x))} & \geq \frac{\mathcal{F}_{\alpha,x}^{\nu',\xi',\eta}(h^{\varphi}(x))}{\mathcal{F}_{\alpha,x}^{\nu',\xi',\eta}(\prod_{i=1}^{n} g_i^{\gamma_p}(x))}, \\
& \geq \frac{(x-a)^{\varphi} \prod_{i=p}^{n} g_i^{\gamma_p}(x)}{\prod_{i=1}^{n} g_i^{\gamma_p}(x)}, \quad (2.18)
\end{align*} \]
where \( \nu, \nu', \xi, \xi' \in \mathbb{R} \) such that \( \eta > \max\{\nu, \nu', \xi, \xi'\} > 0 \).

Proof. Since \((g_i)_{i=1,2,3,\ldots,n}\) be \( n \) positive continuous and decreasing functions on the interval \([a, b]\). Therefore, we have
\[ \begin{align*}
\left( (\rho-a)^{\varphi} - (t-a)^{\varphi} \right) (g^{\sigma-\gamma_p}(t) - g^{\sigma-\gamma_p}(\rho)) & \geq 0 \quad \text{(2.19)}
\end{align*} \]
where \( a < x < b, \vartheta > 0, \sigma > \gamma_p > 0 \) and for any fixed \( p \in \{1, 2, 3, \ldots, n\} \).

By (2.19), we have
\[ \begin{align*}
(\rho-a)^{\varphi} g^{\sigma-\gamma_p}(t) + (t-a)^{\varphi} g^{\sigma-\gamma_p}(\rho) - (\rho-a)^{\varphi} g^{\sigma-\gamma_p}(\rho) - (t-a)^{\varphi} g^{\sigma-\gamma_p}(t) & \geq 0 \quad \text{(2.20)}
\end{align*} \]
Therefore multiplying both sides of (2.20)
\[ \begin{align*}
\mathcal{F}(x, t) \prod_{i=1}^{n} g_i^{\gamma_p}(t) & = (x-t)^{\varphi-1} t^{\gamma_p} F_3 \left( t, \gamma_p, \xi, \xi' ; \eta, 1 - \frac{t}{x}, 1 - \frac{\gamma_p}{x} \right) \prod_{i=1}^{n} g_i^{\gamma_p}(t), t \in (a, x), \ a < x \leq b,
\end{align*} \]
where \( \mathcal{F}(x, t) \) is defined by (2.5), we have
\[ \begin{align*}
\mathcal{F}(x, t) \left[ (\rho-a)^{\varphi} g^{\sigma-\gamma}(t) + (t-a)^{\varphi} g^{\sigma-\gamma}(\rho) - (\rho-a)^{\varphi} g^{\sigma-\gamma}(\rho) - (t-a)^{\varphi} g^{\sigma-\gamma}(t) \right] \prod_{i=1}^{n} g_i^{\gamma_p}(t) \\
= (\rho-a)^{\varphi} (x-t)^{\varphi-1} t^{\gamma_p} F_3 \left( t, \gamma_p, \xi, \xi' ; \eta, 1 - \frac{t}{x}, 1 - \frac{\gamma_p}{x} \right) \prod_{i=1}^{n} g_i^{\gamma_p}(t) g^{\sigma-\gamma_p}(\rho) \\
+ (t-a)^{\varphi} (x-t)^{\varphi-1} t^{\gamma_p} F_3 \left( t, \gamma_p, \xi, \xi' ; \eta, 1 - \frac{t}{x}, 1 - \frac{\gamma_p}{x} \right) \prod_{i=1}^{n} g_i^{\gamma_p}(t) g^{\sigma-\gamma_p}(\rho)
\end{align*} \]
The inequality in Theorem 6 will reverse if Remark 2.4.

which completes the desired inequality (2.18). □

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Volume 5, Issue 2, 1588–1602.

Integrating (2.21) with respect to $t$ over $(a, x)$, we have

$$
(\rho - a)^{\theta} \int_{a}^{x} (x - t)^{\nu - 1} t^{-\nu} F_{3}(v, v', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t}) \prod_{i=1}^{n} g_{i}^{\gamma}(t) g_{p}^{\sigma - \gamma}(t) dt \\
+ g_{p}^{\sigma - \gamma}(\rho) \int_{a}^{x} (x - t)^{\nu - 1} t^{-\nu} F_{3}(v, v', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t}) \prod_{i=1}^{n} g_{i}^{\gamma}(t) dt \\
- (\rho - a)^{\theta} g_{p}^{\sigma - \gamma}(\rho) \int_{a}^{x} (x - t)^{\nu - 1} t^{-\nu} F_{3}(v, v', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t}) \prod_{i=1}^{n} g_{i}^{\gamma}(t) dt \\
- \int_{a}^{x} (x - t)^{\nu - 1} t^{-\nu} F_{3}(v, v', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t}) (t - a)^{\theta} \prod_{i=1}^{n} g_{i}^{\gamma}(t) g_{p}^{\sigma - \gamma}(t) dt \geq 0. \quad (2.22)
$$

Multiplying (2.22) by $\frac{x^{\nu}}{\Gamma(\eta)}$, we get

$$
(\rho - a)^{\theta} \Xi_{a,x}^{\nu, \nu', \xi, \xi'; \eta} \prod_{i=1}^{n} g_{i}^{\gamma}(x) + g_{p}^{\sigma - \gamma}(\rho) \Xi_{a,x}^{\nu, \nu', \xi, \xi'; \eta} \prod_{i=1}^{n} g_{i}^{\gamma}(x) \\
- (\rho - a)^{\theta} g_{p}^{\sigma - \gamma}(\rho) \Xi_{a,x}^{\nu, \nu', \xi, \xi'; \eta} \prod_{i=1}^{n} g_{i}^{\gamma}(x) - \Xi_{a,x}^{\nu, \nu', \xi, \xi'; \eta} \prod_{i=1}^{n} g_{i}^{\gamma}(x) \geq 0. \quad (2.23)
$$

Multiplying (2.23) by

$$
\frac{x^{-\nu}}{\Gamma(\eta)} \tilde{g}(x, \rho) \prod_{i=1}^{n} g_{i}^{\gamma}(\rho) = \frac{x^{-\nu}}{\Gamma(\eta)} (x - \rho)^{\nu - 1} \rho^{-\nu} F_{3}(v, v', \xi, \xi'; \eta; 1 - \frac{\rho}{x}, 1 - \frac{x}{\rho}) \prod_{i=1}^{n} g_{i}^{\gamma}(\rho)
$$

(where $\tilde{g}(x, \rho)$ is defined by (2.5)) and integrating the resultant identity with respect to $\rho$ over $(a, x)$, we get

$$
\Xi_{a,x}^{\nu, \nu', \xi, \xi'; \eta} \left( \prod_{i=1}^{n} g_{i}^{\gamma}(x) \right) \Xi_{a,x}^{\nu, \nu', \xi, \xi'; \eta} \left( \prod_{i=1}^{n} g_{i}^{\gamma}(x) \right) \\
- (x - a)^{\theta} \prod_{i=1}^{n} g_{i}^{\gamma}(x) \Xi_{a,x}^{\nu, \nu', \xi, \xi'; \eta} \left( \prod_{i=1}^{n} g_{i}^{\gamma}(x) \right) \Xi_{a,x}^{\nu, \nu', \xi, \xi'; \eta} \left( \prod_{i=1}^{n} g_{i}^{\gamma}(x) \right) \geq 0
$$

which completes the desired inequality (2.18). □

**Remark 2.4.** The inequality in Theorem 6 will reverse if $(g_{i})_{i=1,2,3,\ldots,n}$ are increasing functions on the interval $[a, b]$.
Theorem 7. Let \((g_i)_{i=1,2,3,\ldots,n}\) be \(n\) positive continuous and decreasing functions on the interval \([a, b]\). Let \(a < x \leq b\), \(\vartheta > 0\), \(\sigma \geq \gamma_p > 0\) for any fixed \(p \in \{1, 2, 3, \ldots, n\}\). Then for MSM fractional integral (1.1), we have

\[
\begin{align*}
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ (x-a)^{\theta} \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \\
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ (x-a)^{\theta} \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \\
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ (x-a)^{\theta} \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \\
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ (x-a)^{\theta} \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \geq 1,
\end{align*}
\]

(2.24)

where \(\alpha, \beta, \xi, \beta, \gamma, \eta \in \mathbb{R}\) such that \(\eta = \max\{\nu, \nu, \xi, \xi, \lambda\} > 0\) and \(\lambda = \max\{\nu, \nu, \xi, \xi\} > 0\).

Proof. By multiplying both sides of (2.23) by

\[
\frac{x^{-\sigma}}{\Gamma(\alpha)} g_i^{\gamma}(\rho) \prod_{i=1}^{n} g_i^{\gamma}(\rho) = \frac{x^{-\sigma}}{\Gamma(\alpha)} (x-\rho)^{1-\rho} F_3 \left( \alpha, \beta, \xi, \beta; \gamma; 1 - \frac{\rho}{x}, 1 - \frac{x}{\rho} \right) \prod_{i=1}^{n} g_i^{\gamma}(\rho)
\]

(where \(\tilde{g}(x, \rho)\) is defined by (2.5)) and integrating the resultant identity with respect to \(\rho\) over \((a, x)\), we have

\[
\begin{align*}
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ (x-a)^{\theta} \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \\
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ (x-a)^{\theta} \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \\
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ (x-a)^{\theta} \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \\
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ (x-a)^{\theta} \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \geq 0.
\end{align*}
\]

(2.25)

Hence, dividing (2.25) by

\[
\begin{align*}
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ (x-a)^{\theta} \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \\
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ (x-a)^{\theta} \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right]
\end{align*}
\]

which completes the desired proof.

Remark 2.5. Applying Theorem 7 for \(\alpha = \nu, \beta = \nu', \xi = \xi', \zeta' = \xi', \lambda = \eta\), we get Theorem 6.

Theorem 8. Let \((g_i)_{i=1,2,3,\ldots,n}\) and \(h\) be positive continuous functions on the interval \([a, b]\) such that \(h\) is increasing and \((g_i)_{i=1,2,3,\ldots,n}\) be decreasing functions on the interval \([a, b]\). Let \(a < x \leq b\), \(\vartheta > 0\), \(\sigma \geq \gamma_p > 0\) for any fixed \(p \in \{1, 2, 3, \ldots, n\}\). Then for the MSM fractional integral (1.1), we have

\[
\begin{align*}
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ h^{\theta}(x) \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \\
\varGamma_{a,x}^{\varphi,\xi,\zeta} & \left[ h^{\theta}(x) \prod_{i \neq p} g_i^{\sigma} g_p^{\sigma}(x) \right] \varGamma_{a,x}^{\varphi,\xi,\zeta} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \geq 1,
\end{align*}
\]

(2.26)
where \( \nu, \nu', \xi, \xi' \in \mathbb{R} \) such that \( \eta > \max\{\nu, \nu', \xi, \xi'\} > 0 \).

**Proof.** Under the conditions stated in Theorem 8, we can write

\[
(h^0(\rho) - h^0(t)) \left( g^\sigma_{\rho} - g^\sigma_{\rho}(\rho) \right) \geq 0
\]  

where \( a < x \leq b, \theta > 0, \sigma \geq \gamma_p > 0 \) and for any fixed \( p \in \{1, 2, 3, \ldots, n\} \).

From (2.27), we have

\[
(h^0(\rho)g^\sigma_{\rho} - h^0(t)g^\sigma_{\rho}(\rho) - h^0(\rho)g^\sigma_{\rho}(\rho) - h^0(t)g^\sigma_{\rho}(t) \geq 0.
\]  

Multiplying both sides of (2.28)

\[
\tilde{\gamma}(x, t) \prod_{i=1}^{n} g^\gamma_i(t) = (x - t)^{\nu - 1} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) \prod_{i=1}^{n} g^\gamma_i(t)
\]

(where \( \tilde{\gamma}(x, \rho) \) is defined by (2.5)), we get

\[
\begin{align*}
&h^0(\rho)(x - t)^{\nu - 1} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) \prod_{i=1}^{n} g^\gamma_i(t) g^\sigma_{\rho}(t) \\
&+ h^0(t)(x - t)^{\nu - 1} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) \prod_{i=1}^{n} g^\gamma_i(t) g^\sigma_{\rho}(\rho) \\
&- h^0(\rho)(x - t)^{\nu - 1} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) \prod_{i=1}^{n} g^\gamma_i(t) g^\sigma_{\rho}(\rho) \\
&- h^0(t)(x - t)^{\nu - 1} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) \prod_{i=1}^{n} g^\gamma_i(t) g^\sigma_{\rho}(t) \geq 0.
\end{align*}
\]  

Integrating (2.29) with respect to \( t \) over \( (a, b) \), we have

\[
\begin{align*}
&h^0(\rho) \int_a^x (x - t)^{\nu - 1} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) \prod_{i=1}^{n} g^\gamma_i(t) g^\sigma_{\rho}(t) dt \\
&+ g^\sigma_{\rho}(\rho) \int_a^x (x - t)^{\nu - 1} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) h^0(t) \prod_{i=1}^{n} g^\gamma_i(t) dt \\
&- h^0(\rho) g^\sigma_{\rho}(\rho) \int_a^x (x - t)^{\nu - 1} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) \prod_{i=1}^{n} g^\gamma_i(t) dt \\
&- \int_a^x (x - t)^{\nu - 1} F_3 \left( \nu, \nu', \xi, \xi'; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) h^0(t) \prod_{i=1}^{n} g^\gamma_i(t) g^\sigma_{\rho}(t) dt \geq 0.
\end{align*}
\]  

Multiplying (2.30) by \( \frac{x - t}{t} \), we get

\[
\begin{align*}
&h^0(\rho) \tilde{\gamma}_{a, x}^{\nu, \nu', \xi, \xi'; \eta} \left[ \prod_{i=1}^{n} g^\gamma_i g^\sigma_{\rho} \right] + g^\sigma_{\rho}(\rho) \tilde{\gamma}_{a, x}^{\nu, \nu', \xi, \xi'; \eta} \left[ h^0(\rho) \prod_{i=1}^{n} g^\gamma_i \right]
\end{align*}
\]
\[ -h^\theta(p)g_p^{\sigma-\rho}(\rho) \left[ \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \sum_{i=1}^{n} g_i^{\gamma}(x) \right] - \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ h^\theta(x) \sum_{i=1}^{n} g_i^{\gamma}(x) \right] \geq 0. \]  

(2.31)

Again, multiplying (2.31) by

\[ \frac{x^{-\nu}}{\Gamma(\eta)} \tilde{g}(x,\rho) \prod_{i=1}^{n} g_i^{\gamma}(\rho) = \frac{x^{-\nu}}{\Gamma(\eta)} (x-\rho)^{\eta-1} \rho^{-\nu} F_3 \left( \nu,\nu',\xi,\xi'; \eta; 1 - x \frac{\rho}{x} \right) \prod_{i=1}^{n} g_i^{\gamma}(\rho) \]

(where \( \tilde{g}(x,\rho) \) is defined by (2.5)) and integrating the resultant identity with respect to \( \rho \) over \( (a, x) \),

we get

\[ \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ h^\theta(x) \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \]

\[ - \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ h^\theta(x) \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \geq 0 \]

which completes the desired inequality (2.26) of Theorem 8. \( \square \)

**Theorem 9.** Let \((g_i)_{i=1,2,3,\ldots,n}\) and \(h\) be positive continuous functions on the interval \([a, b]\) such that \(h\) is increasing and \((g_i)_{i=1,2,3,\ldots,n}\) be decreasing functions on the interval \([a, b]\). Let \(a < x \leq b\), \(\theta > 0\), \(\sigma \geq \gamma_p > 0\) for any fixed \(p \in \{1, 2, 3, \ldots, n\}\). Then for MSM fractional integral (1.1), we have

\[ \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \mathcal{A}^{\alpha,\beta,\xi,\eta}_{a,x} \left[ h^\theta(x) \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \]

\[ \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ h^\theta(x) \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \mathcal{A}^{\alpha,\beta,\xi,\eta}_{a,x} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \]

\[ \mathcal{A}^{\alpha,\beta,\xi,\eta}_{a,x} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ h^\theta(x) \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \]

\[ \mathcal{A}^{\alpha,\beta,\xi,\eta}_{a,x} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \geq 1, \]  

(2.32)

where \(\alpha, \beta, \xi, \eta, \nu, \eta, \xi, \xi' \in \mathbb{R}\) such that \(\eta > \max(\nu, \nu', \xi, \xi') > 0\) and \(\lambda > \max(\nu, \nu', \xi, \xi') > 0\).

**Proof.** Multiplying (2.31) by

\[ \frac{x^{-\alpha}}{\Gamma(\lambda)} \tilde{g}(x,\rho) \prod_{i=1}^{n} g_i^{\gamma}(\rho) = \frac{x^{-\alpha}}{\Gamma(\lambda)} (x-\rho)^{\lambda-1} \rho^{-\alpha} F_3 \left( \alpha,\beta,\xi,\zeta', \lambda; \eta, \frac{\rho}{x}, 1 - \frac{x}{\rho} \right) \prod_{i=1}^{n} g_i^{\gamma}(\rho) \]

(where \( \tilde{g}(x,\rho) \) is defined by (2.5)) and integrating the resultant identity with respect to \( \rho \) over \( (a, x) \),

we get

\[ \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \mathcal{A}^{\alpha,\beta,\xi,\eta}_{a,x} \left[ h^\theta(x) \prod_{i=1}^{n} g_i^{\gamma}(x) \right] + \]

\[ + \mathcal{A}^{\alpha,\beta,\xi,\eta}_{a,x} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ h^\theta(x) \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \]

\[ - \mathcal{A}^{\nu,\xi,\varepsilon,\eta}_{a,x} \left[ h^\theta(x) \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \mathcal{A}^{\alpha,\beta,\xi,\eta}_{a,x} \left[ \prod_{i=1}^{n} g_i^{\gamma}(x) \right] \]
It follows that

\[
\begin{align*}
\sum_{a,x}^{\nu,\nu,\xi,\xi,\lambda,\lambda} \left[ h^\nu(x) \prod_{i \neq p} g_i^\nu g_p^\nu(x) \right] \sum_{a,x}^{\alpha,\beta,\lambda,\lambda} \left[ h^\nu(x) \prod_{i = 1}^n g_i^\nu(x) \right] \\
+ \sum_{a,x}^{\alpha,\beta,\lambda,\lambda} \left[ h^\nu(x) \prod_{i \neq p} g_i^\nu g_p^\nu(x) \right] \sum_{a,x}^{\nu,\nu,\xi,\xi,\lambda,\lambda} \left[ h^\nu(x) \prod_{i = 1}^n g_i^\nu(x) \right] \\
\geq \sum_{a,x}^{\nu,\nu,\xi,\xi} \left[ h^\nu(x) \prod_{i \neq p} g_i^\nu g_p^\nu(x) \right] \sum_{a,x}^{\alpha,\beta,\lambda,\lambda} \left[ h^\nu(x) \prod_{i = 1}^n g_i^\nu(x) \right] \\
+ \sum_{a,x}^{\alpha,\beta,\lambda,\lambda} \left[ h^\nu(x) \prod_{i \neq p} g_i^\nu g_p^\nu(x) \right] \sum_{a,x}^{\nu,\nu,\xi,\xi} \left[ h^\nu(x) \prod_{i = 1}^n g_i^\nu(x) \right].
\end{align*}
\]

Dividing both sides by

\[
\begin{align*}
\sum_{a,x}^{\nu,\nu,\xi,\xi} \left[ h^\nu(x) \prod_{i \neq p} g_i^\nu g_p^\nu(x) \right] \sum_{a,x}^{\alpha,\beta,\lambda,\lambda} \left[ h^\nu(x) \prod_{i = 1}^n g_i^\nu(x) \right] \\
+ \sum_{a,x}^{\alpha,\beta,\lambda,\lambda} \left[ h^\nu(x) \prod_{i \neq p} g_i^\nu g_p^\nu(x) \right] \sum_{a,x}^{\nu,\nu,\xi,\xi} \left[ h^\nu(x) \prod_{i = 1}^n g_i^\nu(x) \right],
\end{align*}
\]

which gives the desired inequality (2.32). \(\square\)

**Remark 2.6.** Applying Theorem 9 for \(\alpha = \nu, \beta = \nu', \xi = \xi, \zeta = \xi', \lambda = \eta,\) we get Theorem 8.

**Remark 2.7.** The results presented in this paper generalize some previous works cited therein.

3. **Concluding remarks**

In this present paper, the we introduced certain inequalities by employing the MSM fractional integral operator. Also, they presented some inequalities for a class of \(n\) positive continuous and decreasing functions on the interval \([a, b]\). The inequalities obtained in this present paper are more general than the classical inequalities available in the literature. The MSM operator defined by (1.1) was introduced by [13] as Mellin type convolution operator with a special function \(F_3(\cdot)\) in the kernel. This MSM operator was re-discovered by Saigo [31] which is the generalized form of Saigo fractional integral operator [11]. The MSM operator (1.1) will led to the Saigo fractional integral operator [31] due to the following relation \(\sum_{a,x}^{\nu,\nu,\xi,\xi}(x) = \sum_{a,x}^{\nu,\nu,\xi,\xi}(x), (\gamma \in \mathbb{C}).\) Thus, the inequalities obtained in this paper will reduce to the inequalities integral inequalities involving Saigo fractional integral operators recently defined by Houas [7].
Acknowledgment

The author K.S. Nisar thanks to Prince Sattam bin Abdulaziz University, Saudi Arabia for providing facilities and support.

Conflict of interest

All authors declare no conflicts of interest.

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