First Passage Time Densities through Hölder curves

Jimyeong Lee*

Gran Sasso Science Institute, Viale Francesco Crispi, 7, 67100, L’Aquila, Italy

March 8, 2022

Abstract

We prove that for a standard Brownian motion, there exists a first-passage-time density function through a Hölder curve with exponent greater than 1/2. By using a property of local time of a standard Brownian motion and the theories of partial differential equations in Cannon [2], we find a sufficient condition for existence of the density function. We also show that this density function is proportional to the space derivative of the Green function of the heat equation with Dirichlet boundary condition at the moving boundary.

1 Introduction

In this paper we will study the probability of the hitting time to a moving boundary. To state the main result, we need some notations on a Brownian motion. Let us call \( P_{r,s}, r \in \mathbb{R}, s \geq 0 \), the law on \( C([s, \infty)) \) of the Brownian motion \( B_t, t \geq s \), which starts from \( r \) at time \( s \), i.e. \( B_s = r \). For each \( t > s \) the law of \( B_t \) is absolutely continuous with respect to the Lebesgue measure and has a density \( G_{s,t}(r, \cdot) \) which is the Gaussian \( G(\cdot, t; r, s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(\cdot - r)^2}{2(t-s)}\right) \). We denote by \( E_{r,s} \) the expectation under \( P_{r,s} \). For \( s=0 \), we use abbreviated forms \( P_r, E_r, \tau^X_r, F^X_r(dx) \) instead of \( P_{r,0}, E_{r,0}, \tau^X_{r,0}, F^X_{r,0}(dx) \) respectively whenever it is needed. In addition, for \( t > 0 \), let us call \( d\mu_{t_0}(\cdot, t) \) the positive measure on \((-\infty, X_t)\) such that

\[
\int_{(-\infty, X_t)} d\mu_{t_0}(x, t)f(x) = E_{t_0}[f(B_t); \tau^X_{t_0} > t]
\]

for all \( f \in C_c^\infty(\mathbb{R}) \) with supp \( f \subseteq (-\infty, X_t) \). The main result in the paper is;

\*E-mail: jimyeong.lee@gssi.infn.it
Theorem 1 If $X$ is Hölder continuous on any finite interval in $[0, \infty)$ with exponent $\gamma \in (1/2, 1]$ and $r_0 < X_0$, then

1. \(d\mu_{r_0}(x, t) = G_{0,t}^X(r_0, x)\, dx\) where for all $x < X_t$,

\[
G_{0,t}^X(r_0, x) = G_{0,t}(r_0, x) - \int_{[0,t]} F_{r_0}^X(ds) G_{s,t}(X_s, x).
\] (1.3)

2. $F_{r_0}^X(ds)$ has a density function $p$ on $[0, \infty)$, namely

\[
P_{r_0}^X(ds) = p(s)\, ds.
\]

3. \(p(t) = -\frac{1}{2} \frac{\partial}{\partial x} G_{0,t}^X(r_0, x) \bigg|_{x=X_t} \) for all $t > 0$.

4. $G_{0,t}^X(r_0, x)$ solves

\[
v_t = \frac{1}{2} v_{xx}, \quad -\infty < x < X_t, \ t > 0,
\] (1.4)

\[
\lim_{(x,t) \to (X_s,s)} v(x,t) = 0, \ s > 0,
\] (1.5)

\[
\lim_{(x,t) \to (y,0)} v(x,t) = \delta_{r_0}(y), \ -\infty < y < X_0.
\] (1.6)

Remarks

Item 4 of Theorem 1 states that for any $r_0 < X_0$ the function $G_{0,t}^X(r_0, \cdot)$ given by (1.3) is the Green function of the heat equation with Dirichlet boundary conditions at the moving boundary $X$. In [8], when a moving boundary is infinitely differentiable, it is shown that the space derivative of the Green function of the heat equation at the boundary is proportional to the hitting time density function. Likewise by items 2 and 3 of Theorem 1, the space derivative of $G_{0,t}^X(r_0, \cdot)$ at the moving boundary is proportional to the hitting time density function $p$.

For the case when $X_t = a + bt$, it is well known (see for instance [3]) that for $r < a$, $\tau_r^X$ has a probability density function given by $f(t) = \frac{a-r}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a+bt-r)^2}{2t}\right) 1_{t>0}$. In addition, there is another result in [4] when $X_t = a + bt^p$ for $p \geq 2$, $r < a$, then $\tau_r^X$ has a probability density function. In [6], it is showed that if the boundary behaves as a Lipschitz curve in a local time, then the first passage time density can be expressed explicitly.

In [4], it is proved that for any continuous curve $X_t$ and $r < X_0$, there is a distribution $F_{r_0}^X$ of $\tau_r^X$ which satisfies the following integral equation (called the Master Equation):

\[
\Psi\left(\frac{z-r}{\sqrt{t}}\right) = \int_0^t \Psi\left(\frac{z-X_s}{\sqrt{t-s}}\right) F_{r_0}^X(ds),
\] (1.7)

where $z \geq X_t$, $t > 0$ and $\Psi(z) = \int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$. This can be proved intuitively as follows; the left hand side of (1.7) is the probability that a Brownian motion starts at $r$ at time 0 and reaches $z$ greater or equal to $X_t$ at time $t$. Then it should hit the boundary at least once which implies the right hand side of of (1.7). Moreover, it is showed that if
If $X_t$ is $C^1$, then there exists a continuous density function $f$ of $F^X_r$. There is an extension of this result, in [1], to curves $X$ which are differentiable with $|dX_t/dt| \leq Ct^{-\alpha}$ for some constant $C > 0$ and $\alpha < 1/2$.

By (1.7), for any continuous curve $X_t$ and $r < X_0$, we have

$$\Psi \left( \frac{X_t - r}{\sqrt{t}} \right) = \int_0^t \Psi \left( \frac{X_t - X_s}{\sqrt{t-s}} \right) F^X_r(ds).$$

which can be regarded as an integral equation for $F^X_r(ds)$. In [7], the equation (1.8) is studied when $X$ is Hölder continuous with exponent greater than $1/2$. It is proved that there exists a unique continuous function $q$ such that

$$\Psi \left( \frac{X_t - r}{\sqrt{t}} \right) = \int_0^t \Psi \left( \frac{X_t - X_s}{\sqrt{t-s}} \right) q(s)ds.$$ (1.9)

To conclude that $F^X_r(ds) = q(s)ds$, one still needs that $F^X_r(ds)$ is absolutely continuous with respect to Lebesgue measure. The analysis in [7] as well as the proof of Theorem 1 uses extensively the work by Cannon [2] on the heat equation with the moving boundary.

## 2 Preliminaries: a weaker form of Theorem 1

We define $D$ as

$$D := \{(x,t) : -\infty < x < X_t, \ t > 0\}$$

and

$$u(x,t) := \int_{-\infty}^0 h(\xi)G^X_{0,t}(\xi,x)d\xi$$

for given $h \in C^\infty_c((-\infty, X_0); \mathbb{R}_+)$ and all $(x,t) \in D$. If $X$ is Hölder continuous with exponent $\gamma \in [1/2, 1]$, then $u$ solves the heat equation in $D$ with the initial condition $h$ and Dirichlet boundary condition. (See Theorem 2 below) However uniqueness fails for the heat equation with the initial function $h$ and Dirichlet boundary conditions (see Remark 1 and 2 below).

To have the uniqueness, first of all, we restrict time variable of $D$ in a finite interval. Thus we fix $T > 0$ and define the parabolic cylinder $D_T$ which is a subset of $D$ as

$$D_T := \{(x,t) : -\infty < x < X_t, \ 0 < t \leq T\}.$$ (2.2)

Consider the following initial-boundary value problem

$$\begin{cases} v \in C(D_T) \cap C^{2,1}(D_T), \\ v_t = \frac{1}{2}v_{xx}, \ (x,t) \in D_T, \\ v(X_t,t) = 0, \ 0 < t \leq T, \\ v(x,0) = h(x), \ -\infty < x < X_0, \\ \lim_{x \to -\infty} \sup_{0 < t < T} |v(x,t)| = 0. \end{cases}$$

We prove the following weaker form of Theorem 1
Theorem 2 Let $X$ be a Hölder continuous curve on any finite interval in $[0, \infty)$ with exponent $\gamma \in [1/2, 1]$ and let $X_0 = 0$.

1. The function $u$ defined in (2.1) is the unique solution of (2.2).

2. If $\gamma \in (1/2, 1]$, then $u$ has the left hand derivative at the boundary $u_x(X_t^-, t)$ which is continuous on $(0, \infty)$. Moreover, for all $t > 0$, $p_h(t) := -\frac{1}{2}u_x(X_t^-, t)$ satisfies

$$p_h(t) = -\int_{-\infty}^0 h(\xi)G_x(X_t,t;\xi,0)d\xi + \int_0^t G_x(X_t,t;X_\tau,\tau)p_h(\tau)d\tau. \quad (2.3)$$

Remark 1 The uniqueness for (2.2) is not guaranteed if we do not assume $v \in C(D_T)$. When $X_t = 0$ for all $t$ and $h$ is identically 0, if $v(x,t)$ is given by $\frac{1}{\sqrt{2\pi t}}\left\{-\frac{x}{t}\right\} \exp\left(-\frac{x^2}{2t}\right)$, then this satisfies the heat equation with the initial data 0 and is also 0 on the boundary, but this is not continuous at $(0, 0)$.

Remark 2 We need the condition $\lim_{x \to -\infty} \sup_{0 < t < T} |v(x,t)| = 0$ to have uniqueness. Indeed, for $X_t = 0$ for all $t \geq 0$, the function

$$v(x,t) = \sum_{n=0}^\infty f(n) \left(\frac{t}{2}\right)^{2n+1} \frac{1}{(2n+1)!},$$

where

$$f(t) = \begin{cases} \exp\left(-\frac{1}{t^2}\right), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

satisfies the heat equation with the initial data 0 and is also 0 on the boundary. Furthermore, $v \in C(D_T) \cap C^{2,1}(D_T)$.

Remark 3 It can be shown $\lim_{t \to 0} p_h(t) = 0$ by (2.3) and Lemma 3 below. Thus $u_x(X_t^-, t)$ is continuous on $[0, \infty)$.

Following [2], we introduce $C_0^\nu((0,T])$, $0 < \nu \leq 1$, as the subspace of $C((0,T])$ that consists of those functions $\varphi$ such that

$$\|\varphi\|_T^{(\nu)} = \sup_{0 < t \leq T} t^{1-\nu}|\varphi(t)| < \infty.$$

Then $C_0^\nu((0,T])$ is a Banach space under the norm $\|\cdot\|_T^{(\nu)}$.

We also introduce the following lemmas from [2] which play an essential role in our analysis.

Lemma 1 (jump relation) For $\varphi \in C_0^\nu((0,T])$, we have

$$\lim_{x \to X_t^+} \frac{\partial w_\varphi}{\partial x}(x,t) = \mp \varphi(t) + \int_0^t G_x(X_t,t;X_\tau,\tau)\varphi(\tau)d\tau, \quad (2.4)$$

where $w_\varphi(x,t) = \int_0^t G(x,t;X_\tau,\tau)\varphi(\tau)d\tau$. 

4
For two continuous curves $s_1, s_2$ such that $s_1(t) < s_2(t), t \in [0, T]$, let us set $E_T := \{(x,t) : s_1(t) < x < s_2(t), 0 < t \leq T\}$ and $B_T := \{(s_i(t), t) : 0 \leq t \leq T, i \in \{1, 2\}\} \cup \{(x,0) : s_1(0) < x < s_2(0)\}$.

**Lemma 2 (The Weak Maximum(Minimum) Principle)** For a solution $u$ of $u_t = \frac{1}{2}u_{xx}$ in $E_T$, which is continuous in $E_T \cup B_T$,

\[
\max_{E_T \cup B_T} u = \max_{E_T \cup B_T} u. \quad \left(\min_{E_T \cup B_T} u = \min_{E_T \cup B_T} u.\right)
\]  

(2.5)

Before going to the proof of Theorem 2, we need the following proposition.

**Proposition 1** Let $X$ be a Hölder continuous curve on any finite interval in $[0, \infty)$ with exponent $\gamma \in [1/2, 1]$. If the starting point of the Brownian motion is close to $X$, the first hitting time converges to 0. Precisely, \(\lim_{x \to X_0} P_\xi [\tau^X_\xi > s] = 0\) for all $s > 0$.

**Proof** Without loss of generality, we may reduce this problem as the case for Brownian motion starting at 0 and $X_0 = \epsilon > 0$ and let $\epsilon$ go to 0. For $s > 0$, let $m := \sup_{0 \leq t_2 < t_1 \leq s} \frac{|X_{t_2} - X_{t_1}|}{|t_2 - t_1|^\gamma}$. Since $\limsup_{t \to 0} \frac{B_t}{\sqrt{t}} = \infty$ a.e., for $M > m$, we have a sequence $t_k \downarrow 0$ such that $M \sqrt{t_k} \leq B_{t_k}$ a.e. and $M \sqrt{t_k} - mt_k^\gamma \downarrow 0$ for all $k$. Thus, for $0 < t \leq s$, we deduce that for all $t, \epsilon > 0$,

\[
\sup_{0 \leq t \leq s} \{B_t - (\epsilon + mt_k^\gamma)\} \geq \sup_{t_k \leq t} \{B_{t_k} - (\epsilon + mt_k^\gamma)\} \geq \sup_{t_k \leq t} \{M \sqrt{t_k} - (\epsilon + mt_k^\gamma)\}.
\]

Therefore,

\[
P_0 \left(\sup_{0 \leq t \leq s} \{B_t - X_t\} < 0\right) \leq P_0 \left(\sup_{t_k \leq t} \{M \sqrt{t_k} - mt_k^\gamma\} < \epsilon\right) .
\]

For each sufficiently small $\epsilon > 0$, there is a greatest $k(\epsilon)$ such that $M \sqrt{t_k(\epsilon)} - mt_k^\gamma \geq \epsilon$. Thus we obtain that $\tau^X_0 \leq t_k(\epsilon)$ a.e. for all sufficiently small $\epsilon > 0$. Since $k(\epsilon)$ is an increasing function as $\epsilon$ decreases and $\lim_{\epsilon \to 0} k(\epsilon) = \infty$, the proposition follows.

**Proof of Theorem 2**

Let us define $X'_{t} := X_{t-\tau}$ for all $0 \leq \tau \leq t$. Using the invariance of the law of the Brownian motion under time reversal, we have

\[
u(x,t) = \int_{-\infty}^{0} h(r')G_{0,t}(r',x)dr' = E_x[h(B_t); \tau^X_x > t].
\]  

(2.6)

Using this equality, we also have

\[
|u(x,t)| = \left|E_x[h(B_t); \tau^X_x > t]\right| \leq \|h\|_\infty P_x[\tau^X_x > t].
\]  

(2.7)

For $s > 0$, let us choose $0 < s^* < s$. Then for all $(x,t)$ sufficiently close to $(X_s, s)$, we obtain

\[
P_x[\tau^X_x > t] \leq P_x[\tau^X_x > s^*]
\]  

(2.8)
which vanishes when \((x, t) \to (X_s, s)\) by **Proposition 1** so that \(\lim_{(x, t) \to (X_s, s)} u(x, t) = 0\). In addition, we have

\[
|u(x, t)| = \left| E_x[h(B_t); \tau^X_x > t] \right| \leq E_x[|h(B_t)|] = \int_{-\infty}^{0} |h(\xi)| G_{0,t}(x, \xi) d\xi \tag{2.9}
\]

which also vanishes when \((x, t) \to (0, 0)\), since the support of \(h\) is strictly away from 0. To prove that \(u\) satisfies the initial data \(h\), we write \(y_t = \min_{s \in [0, t]} X'_s\). For \(x < 0\) and any positive \(\lambda > 0\),

\[
P_x\left[\tau^X_x \leq t \right] \leq P_x \left[ \max_{s \in [0, t]} B_s \geq y_t \right] = P_x \left[ \max_{s \in [0, t]} \exp(\lambda B_s) \geq \exp(\lambda y_t) \right]. \tag{2.10}
\]

Since the exponential of Brownian motion is a positive submartingale, we can apply Doob’s inequality, then

\[
P_x \left[ \max_{s \in [0, t]} \exp(\lambda B_s) \geq \exp(\lambda y_t) \right] \leq \frac{E_x[\exp(\lambda B_t)]}{\exp(\lambda y_t)} = \exp \left( \frac{1}{2} \lambda^2 t - \lambda (y_t - x) \right). \tag{2.11}
\]

From (2.10) and (2.11), we get

\[
\lim_{(x, t) \to (y, 0)} P_x \left[ \tau^X_x \leq t \right] \leq \exp(-\lambda (X_0 - y)) = \exp(\lambda y) \tag{2.12}
\]

so that the left hand side vanishes since \(\lambda > 0\) is arbitrary and we are assuming \(y < X_0 = 0\). Thus we obtain that

\[
\lim_{(x, t) \to (y, 0)} \int_{-\infty}^{0} h(\xi) G_{0,t}(\xi, x) d\xi = \lim_{(x, t) \to (y, 0)} E_x[h(B_t)] = h(y). \tag{2.13}
\]

By the properties of the Gaussian kernel, we deduce that \(u\) solves (2.2). We now prove uniqueness. If \(v_1, v_2\) satisfy all the above conditions, then \(v_1 - v_2 \in C(D_T) \cap C^{2,1}(D_T)\) satisfies the heat equation with the initial data 0 and is also 0 on the boundary. Moreover, \(\lim_{x \to -\infty,0 < t < T} |v_1(x, t) - v_2(x, t)| = 0\) so that by the weak maximum(minimum) principle, \(v_1 - v_2\) is all 0 in \(D_T\). Therefore, \(v_1 = v_2\) so that item 1 is proved.

To prove item 2, assuming that \(\gamma \in (1/2, 1]\), we define, \((x, t) \in D_T,\)

\[
v(x, t) := \int_{-\infty}^{0} h(\xi) G(x, t; \xi, 0) d\xi + \int_{0}^{t} G(x, t; X_\tau, \tau) \varphi(\tau) d\tau, \tag{2.14}
\]

where \(\varphi \in C_{(\gamma)} ((0, T])\) satisfies

\[
0 = \int_{-\infty}^{0} h(\xi) G(x_t, t; \xi, 0) d\xi + \int_{0}^{t} G(x_t, t; X_\tau, \tau) \varphi(\tau) d\tau. \tag{2.15}
\]

If we apply the Abel inverse operator \(A\) defined by

\[
(AF)(t) = \frac{1}{\pi} \frac{d}{dt} \int_{0}^{t} \frac{F(\eta)}{(t - \eta)^{\frac{1}{2}}} d\eta
\]
on both sides of (2.15), then from Chapter 14 in [2] we have a equivalent Volterra integral equation of the second kind:

\[ \varphi(t) = \psi(t) + \int_0^t H(t, \tau) \varphi(\tau) d\tau, \]  

(2.16)

where \( \psi \in C^0(\gamma)((0, T]) \) and \(|H(t, \tau)| \leq C(t - \tau)^{2\gamma - 2} \). The existence of \( \varphi \in C^0(\gamma)((0, T]) \) which satisfies (2.16) can be proved similarly to the proof of Proposition 3 that we will show later. Then \( v \) is well-defined and solves (2.12) so that \( v = u \) since \( u \) is the unique solution of (2.2). By the jump relation, we have

\[ u_{x}(X_{t}^{-}, t) = \int_{-\infty}^{0} h(\xi) G_{x}(X_{t}, t; \xi, 0) d\xi + \varphi(t) + \int_{0}^{t} G_{x}(X_{t}, t; X_{\tau}, \tau) \varphi(\tau) d\tau \]  

(2.17)

so that \( u_{x}(X_{t}^{-}, t) \in C^0(\gamma)((0, T]). \) Since \( T \) is arbitrary, we deduce \( u_{x}(X_{t}^{-}, t) \) is continuous on \((0, \infty)\).

To show (2.3), let us fix \((x, t) \in D_{T}\) and let us define \( D_{x,r}^{(t)} := \{(\xi, \tau) : r < \xi < X_{\tau} - \epsilon, \epsilon < \tau < t - \epsilon \} \) for each \( \epsilon > 0 \) and \( r \in \mathbb{R} \). By the Green’s identity, we have

\[ \frac{1}{2}(u_{x}G - uG_{x})_{\xi} - (uG)_{\tau} = 0 \quad \implies \quad \oint_{\partial D_{x,r}^{(t)}} \frac{1}{2}(u_{x}G - uG_{x}) d\tau + (uG)d\xi = 0. \]  

(2.18)

It can be also shown that \( \lim_{x \to -\infty} \sup_{0 < t < T} |u_{x}(x, t)| = 0 \) by the properties of the Gaussian kernel. Hence we obtain another representation of \( u \) by letting \( \epsilon \to 0, r \to -\infty, \)

\[ u(x, t) = \int_{-\infty}^{0} h(\xi) G(x, t; \xi, 0) d\xi + \frac{1}{2} \int_{0}^{t} G(x, t; X_{\tau}, \tau) u_{x}(X_{\tau}^{-}, \tau) d\tau. \]  

(2.19)

Differentiating both sides of (2.19) with respect to \( x \) and applying the jump relation, we get

\[ \frac{1}{2} u_{x}(X_{t}^{-}, t) = \int_{-\infty}^{0} h(\xi) G_{x}(X_{t}, t; \xi, 0) d\xi + \frac{1}{2} \int_{0}^{t} G_{x}(X_{t}, t; X_{\tau}, \tau) u_{x}(X_{\tau}^{-}, \tau) d\tau. \]  

(2.20)

which implies (2.3).

\[ \bullet \]

3 Proof of of Theorem 1

To prove item 1 let \( X \) be a continuous curve defined on \([0, \infty)\) and let \( r_{0} < X_{0}. \) Using the strong Markov property, we have, for \( f \in C_{c}^{\infty}(\mathbb{R}) \) with \( \text{supp} f \in (-\infty, X_{t}) \) and \( 0 \leq s \leq t, \)

\[ E_{r_{0}}[f(B_{t})|\tau_{r_{0}}^{X} = s] = E_{X_{s},s}[f(B_{t})]. \]  

(3.21)

Thus we get

\[ E_{r_{0}}[f(B_{t}); \tau_{r_{0}}^{X} \leq t] = \int_{[0,t]} F_{r_{0}}^{X}(ds)E_{X_{s},s}[f(B_{t})]. \]  

(3.22)

It follows that item 1 holds with \( G_{0,t}^{X}(r_{0}, x) \) given by (1.3).

To prove item 2 from now on, \( X \) is a Hölder continuous curve defined on \([0, \infty)\) with
exponent $\gamma \in (1/2, 1]$ and $X_0 = 0$. Comparing the definition (2.1) of $u$ and (2.19), using (1.3), we see the following equality:

$$
\int_{[0,t]} G_{t,t}(X_\tau, x) \int_{-\infty}^{0} h(\xi) F^X_\xi(d\tau) d\xi = -\frac{1}{2} \int_0^t G(x, t; X_\tau, \tau) u_x(X^-_\tau, \tau) d\tau.
$$

(3.23)

Set $F^X_h(d\tau) := \int_{-\infty}^{0} h(\xi) F^X_\xi(d\tau) d\xi$.

**Proposition 2** Let $F^X_h(d\tau) := -\frac{1}{2} u_x(X^-_\tau, \tau) d\tau$.

For the proof of Proposition 2, we introduce the mass lost $\Delta^X_I(u)$, $I = [t_1, t_2] \subset [0, T]$, $t_1 \leq t_2$, is defined by

$$
\Delta^X_I(u) = \int_{-\infty}^{X_{t_1}} u(r, t_1) dr - \int_{-\infty}^{X_{t_2}} u(r, t_2) dr.
$$

(3.24)

If we see the right hand side of (2.19), we can extend $u$ to $\bar{u}$ defined in $\{(x, t) : x \in \mathbb{R}, 0 < t \leq T\}$ as

$$
\bar{u}(x, t) = \int_{-\infty}^{0} h(\xi) G(x, t; \xi, 0) d\xi + \frac{1}{2} \int_0^t G(x, t; X_\tau, \tau) u_x(X^-_\tau, \tau) d\tau.
$$

(3.25)

Then this satisfies the heat equation with $\lim_{(x,t) \to (y,0)} \bar{u}(x, t) = 0$ for all $y \geq 0$ and also satisfies $\bar{u}(X_t, t) = 0$ for all $0 < t \leq T$. Moreover, by the properties of Gaussian kernel, we have

$$
\lim_{x \to \infty} \sup_{0 < t < T} |\bar{u}(x, t)| = 0.
$$

(3.26)

It follows that $\bar{u}(x, t) = 0$ in $\{(x, t) : x \geq X_t, 0 < t \leq T\}$ by the weak maximum(minimum) principle. Thus we assume that $u$ is defined $\{(x, t) : x \in \mathbb{R}, 0 \leq t \leq T\}$ such that it is 0 in $\{(x, t) : x \geq X_t, 0 \leq t \leq T\}$.

Heuristically

$$
\Delta^X_I(u) = -\int_{t_1}^{t_2} u_t(x, t) dt dx = -\int_{t_1}^{t_2} \frac{1}{2} u_{xx}(x, t) dx dt = -\frac{1}{2} \int_{t_1}^{t_2} u_x(X^-_t, t) dt.
$$

Since we do not control $u_{xx}$ at the moving boundary, we cannot make this argument rigorously. Thus we use a different approach.

**Proof of Proposition 2**

It suffices to show

$$
-\frac{1}{2} \int_0^t u_x(X^-_\tau, t) dt = \Delta^X_I(u) = F^X_h(I).
$$

If we integrate both sides of (2.19), then

$$
\int_{-\infty}^{\infty} u(x, t) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{0} h(\xi) G(x, t; \xi, 0) d\xi dx + \int_{-\infty}^{\infty} \frac{1}{2} \int_0^t G(x, t; X_\tau, \tau) u_x(X^-_\tau, \tau) d\tau dx.
$$
Applying Fubini’s theorem, we get
\[
\int_{\infty}^{X_t} u(x, t)dx = \int_{\infty}^{0} u(x, t)dx = \int_{\infty}^{0} h(\xi)d\xi + \frac{1}{2} \int_{0}^{t} u_x(X_{\tau}^{-}, \tau)d\tau. \quad (3.27)
\]
Thus we get the first equality of the proposition,
\[
\Delta^X_{I_{t_1, t_2}}(u) = -\frac{1}{2} \int_{t_1}^{t_2} u_x(X_{\tau}^{-}, \tau)d\tau. \quad (3.28)
\]
From \((1.2)\) and item \((1)\) of \textbf{Theorem 1} we get
\[
P_\xi[\tau^X_\xi > t] = \int_{\infty}^{X_t} G^X_{0, t}(\xi, x)dx. \quad (3.29)
\]
For \(0 = t_1 < t_2\), using Fubini’s theorem again, we get
\[
\Delta^X_I(u) = \int_{\infty}^{0} h(\xi)d\xi - \int_{\infty}^{X_{t_2}} \int_{\infty}^{0} h(\xi)G^X_{0, t_2}(\xi, x)d\xi dx
\]
\[
= \int_{\infty}^{0} h(\xi)d\xi - \int_{\infty}^{0} h(\xi)P_\xi[\tau^X_\xi > t_2]d\xi = \int_{\infty}^{0} h(\xi)P_\xi[0 \leq \tau^X_\xi \leq t_2]d\xi.
\]
For \(0 < t_1 < t_2\), similarly,
\[
\Delta^X_I(u) = \int_{\infty}^{X_{t_1}} \int_{\infty}^{0} h(\xi)G^X_{0, t_1}(\xi, x)d\xi dx - \int_{\infty}^{X_{t_2}} \int_{\infty}^{0} h(\xi)G^X_{0, t_2}(\xi, x)d\xi dx
\]
\[
= \int_{\infty}^{0} h(\xi)P_\xi[t_1 < \tau^X_\xi \leq t_2]d\xi.
\]
Then for \(I_\epsilon = [t_1 - \epsilon, t_1]\), we get
\[
\lim_{\epsilon \to 0} \Delta^X_I(u) = \lim_{\epsilon \to 0} -\frac{1}{2} \int_{I_\epsilon} u_x(X_{\tau}^{-}, t)dt = 0 = \lim_{\epsilon \to 0} \int_{\infty}^{0} h(\xi)P_\xi[t_1 - \epsilon < \tau^X_\xi \leq t_1]d\xi
\]
\[
= \int_{\infty}^{0} h(\xi)P_\xi[\tau^X_\xi = t_1]d\xi.
\]
Finally we conclude that
\[
\Delta^X_I(u) = \int_{\infty}^{0} h(\xi)P_\xi[t_1 \leq \tau^X_\xi \leq t_2]d\xi = F^X_h(I). \quad (3.30)
\]
By approximating the initial delta measure of \textbf{Theorem 1} we prove the proposition below.

\textbf{Proposition 3} Let \(X\) be a Hölder continuous curve on any finite interval in \([0, \infty)\) with exponent \(\gamma \in (1/2, 1]\) and let \(X_0 = 0\). We fix \(r_0 < X_0 = 0\) and choose a sequence \(\{h_n\} \subset C^\infty_c(\mathbb{R}; \mathbb{R}^+)\) with \(\text{supp } h_n = [r_0 - \frac{1}{n}, r_0 + \frac{1}{n}] \subset (-\infty, 0)\) and \(\|h_n\|_1 = 1\). For each \(h_n\), there exists a corresponding solution \(u_n\) with \(-\frac{1}{2} \frac{\partial u_n}{\partial x} \bigg|_{(X^-_n, t)} =: p_n(t)\) in the sense of \textbf{Theorem 2}. Then we have the following statements:
1. There is a unique \( p \in C([0, \infty)) \) with \( p(0) = 0 \) such that
\[
p(t) = -G_x(X_t, t; r_0, 0) + \int_0^t G_x(X_t, t; X_\tau, \tau)p(\tau)d\tau \quad \text{for all } t > 0. \tag{3.31}
\]

2. \( p_n \) converges to \( p \) in \( C^0_{(\eta)}((0, T]) \) for all \( 0 < \eta < 1/2 \).

We use the following lemma extensively to prove Proposition 3.

**Lemma 3** \[
\int_0^t \tau^{\alpha_1}(t - \tau)^{\alpha_2}d\tau = \frac{\Gamma(1 + \alpha_1)\Gamma(1 + \alpha_2)}{\Gamma(2 + \alpha_1 + \alpha_2)}t^{1+\alpha_1+\alpha_2} \quad \text{for } \alpha_1, \alpha_2 > -1, \text{ where } \Gamma \text{ is the gamma function.}
\]

**Proof of Proposition 3**

Let \( T_s > 0 \). Since \(|G_x(X_t, t; X_\tau, \tau)| = \left| \frac{1}{\sqrt{2\pi(t - \tau)}} \left\{ -\frac{X_t - X_\tau}{t - \tau} \right\} \exp \left( -\frac{(X_t - X_\tau)^2}{2(t - \tau)} \right) \right| \leq \frac{C_0}{(t - \tau)^{\frac{3-\gamma}{2}}}, \tag{3.32}
\]
degree that for \( q_1, q_2 \in C([0, T_s]), 0 < t \leq T_s, \)
\[
\left\| \int_0^t G_x(X_t, t; X_\tau, \tau)(q_1(\tau) - q_2(\tau))d\tau \right\| \leq \int_0^t \frac{C_0\|q_1 - q_2\|_{T_s}}{(t - \tau)^{\frac{3-\gamma}{2}}}d\tau \]
\[
= C_1t^{\gamma - \frac{3}{2}}\|q_1 - q_2\|_{T_s} \leq C_1T_s^{\gamma - \frac{3}{2}}\|q_1 - q_2\|_{T_s}.
\]

We define \( F : C([0, T_s]) \to C([0, T_s]) \) as, for \( q \in C([0, T_s]) \),
\[
(Fq)(t) = -G_x(X_t, t; r_0, 0) + \int_0^t G_x(X_t, t; X_\tau, \tau)q(\tau)d\tau \quad \text{for } t > 0,
\]
and \( (Fq)(0) = 0 \). Since \( \lim_{t \to 0} G_x(X_t, t; r_0, 0) = 0 \), it is well-defined. If we choose \( T_s \) such that \( C_1T_s^{\gamma - \frac{3}{2}} < 1 \), then \( F \) is a contraction mapping so that \( F \) has a unique fixed point since \( C([0, T_s]) \) is a Banach space. Let’s call this \( p_{T_s} \).

Now we assume that we have \( p_{T_s} \) for some \( T_s > 0 \). For \( T^* > T_s \), we define \( H : C([T_s, T^*]) \to C([T_s, T^*]) \) as, for \( q \in C([T_s, T^*]) \),
\[
(Hq)(t) = -G_x(X_t, t; r_0, 0) + \int_0^{T_s} G_x(X_t, t; X_\tau, \tau)p_{T_s}(\tau)d\tau + \int_{T_s}^t G_x(X_t, t; X_\tau, \tau)q(\tau)d\tau.
\]

Then for \( q_1, q_2 \in C([T_s, T^*]) \), we have
\[
\|Hq_1 - Hq_2\|_{\infty} \leq C_2(t - T_s)^{\gamma - \frac{3}{2}}\|q_1 - q_2\|_{\infty} \leq C_2(T^* - T_s)^{\gamma - \frac{3}{2}}\|q_1 - q_2\|_{\infty}. \tag{3.33}
\]

Similarly, if we choose \( T^* \) such that \( C_2(T^* - T_s)^{\gamma - \frac{3}{2}} < 1 \), then \( H \) is a contraction mapping so that \( H \) has a unique fixed point since \( C([T_s, T^*]) \) is a Banach space.

Therefore, if we have \( p \) defined \([0, T_s], p_{T_s} \), then we can extend this to time \( T_s + C_3 \) where \( C_3 \) is an independent constant. Thus if we repeat this step inductively, we have \( p \) defined on \([0, \infty)\) which satisfies (3.31).
Let us choose $T$ for all sufficiently large $n$. For $0 < t < T$, we have

$$p_n(t) = - \int_{-\infty}^{0} h_n(\xi) G_x(X_t, \xi, t; \xi, 0) d\xi + \int_{0}^{t} G_x(X_t, t; X_\tau, \tau) p_n(\tau) d\tau. \quad (3.34)$$

For $0 < t \leq T$, taking the difference between (3.31) and (3.34), we get

$$t^{1-\eta} |p_n(t) - p(t)| \leq t^{1-\eta} \left| \int_{-\infty}^{0} h_n(\xi) [G_x(X_t, \xi, t; \xi, 0) - G_x(X_t, \xi, t; r_0, 0)] d\xi \right| + t^{1-\eta} \left| \int_{0}^{t} G_x(X_t, t; X_\tau, \tau) (p_n(\tau) - p(\tau)) d\tau \right|.$$

For the second term of the right hand side, we have

$$t^{1-\eta} \left| \int_{0}^{t} G_x(X_t, t; X_\tau, \tau) (p_n(\tau) - p(\tau)) d\tau \right| \leq C_0 t^{1-\eta} \int_{0}^{t} \frac{\|p_n - p\|_{T_s}^{(\eta)}}{(t - \tau)^{\frac{3}{2} - \gamma} t^{1-\eta}} d\tau = C_1 t^{\gamma - \frac{3}{2}} \|p_n - p\|_{T_s}^{(\eta)} \leq C_1 T_s^{-\gamma} \|p_n - p\|_{T_s}^{(\eta)}.$$

Let us choose $T > 0$ such that $C_1 T_s^{\gamma - \frac{3}{2}} < 1$. Then

$$(1 - C_1 T_s^{\gamma - \frac{3}{2}}) \|p_n - p\|_{T_s}^{(\eta)} \leq t^{1-\eta} \left| \int_{-\infty}^{0} h_n(\xi) [G_x(X_t, \xi, t; \xi, 0) - G_x(X_t, \xi, t; r_0, 0)] d\xi \right| \leq \int_{-\infty}^{0} h_n(\xi) \sup_{0 < t \leq T_s} \left| \int_{0}^{t\leq \xi} t^{1-\eta} |G_x(X_t, \xi, t; \xi, 0) - G_x(X_t, t; r_0, 0)| d\xi. \right.$$

For all sufficiently large $n$, $0 < t \leq \frac{1}{n}$ and $r_0 - \frac{1}{n} \leq \xi \leq r_0 + \frac{1}{n}$, there exists $C_2$ such that

$$|G_x(X_t, \xi, 0)| = \left| \frac{1}{\sqrt{2\pi}} \left\{ - \frac{X_t - \xi}{\left( \frac{2}{t} \right)^{\frac{3}{2}}} \right\} \exp \left( -\frac{(X_t - \xi)^2}{2t} \right) \right| \leq C_2,$$

Thus

$$\sup_{0 < t \leq \frac{1}{n}, \xi \in [r_0 - \frac{1}{n}, r_0 + \frac{1}{n}]} t^{1-\eta} |G_x(X_t, t; \xi, 0) - G_x(X_t, t; r_0, 0)| \leq \sup_{0 < t \leq \frac{1}{n}} 2t^{1-\eta} C_2 \leq \frac{2C_2}{n^{1-\eta}}.$$

For all sufficiently large $n$, $\frac{1}{n} < t \leq T_s$ and $r_0 - \frac{1}{n} \leq \xi \leq r_0 + \frac{1}{n}$, since $|G_{xx}(\cdot, t; \cdot, 0)| \leq \frac{C_3}{t^{\frac{3}{2}}}$, we have

$$|G_x(X_t, t; \xi, 0) - G_x(X_t, t; r_0, 0)| \leq \frac{C_3 |\xi - r_0|}{t^{\frac{3}{2}}} \leq \frac{C_3}{nt^{\frac{3}{2}}},$$

Thus

$$\sup_{\frac{1}{n} < t \leq T_s, \xi \in [r_0 - \frac{1}{n}, r_0 + \frac{1}{n}]} t^{1-\eta} |G_x(X_t, t; \xi, 0) - G_x(X_t, t; r_0, 0)| \leq \sup_{\frac{1}{n} < t \leq T_s} \frac{C_3}{nt^{\frac{3}{2}+\eta}} \leq \frac{C_3}{n^{\frac{3}{2}-\eta}}.$$
Therefore, we conclude that $p_n$ converges to $p$ in $C_{(0)}^0((0, T_s])$ for all sufficiently small $T_s > 0$.

To extend from $T_s$ to $T^*$, assuming that $p_n$ converges to $p$ in $C_{(0)}^0((0, T_s])$ for some $T_s > 0$ and writing $\|p_n - p\|_{[T_s, T^*]} = \sup_{\tau \in [T_s, T^*]} |p_n(\tau) - p(\tau)|$, for $T_s \leq t \leq T^*$, we deduce that

\[
|p_n(t) - p(t)| \leq \int_{-\infty}^{0} h_n(\xi)[G_x(X_t, t; \xi, 0) - G_x(X_t, t; r_0, 0)]d\xi + \int_{0}^{T_s} G_x(X_t, t; X_\tau, \tau)(p_n(\tau) - p(\tau))d\tau + \int_{T_s}^{t} G_x(X_t, t; X_\tau, \tau)(p_n(\tau) - p(\tau))d\tau
\]

\[
\leq C_3 \int_{-\infty}^{0} h_n(\xi)\left|\xi - r_0\right|d\xi + C_0 \int_{0}^{T_s} \frac{\|p_n - p\|_{T_s}^{(0)}}{(t - \tau)^{1-\gamma}}d\tau + C_0 \int_{T_s}^{t} \frac{\|p_n - p\|_{[T_s, T^*]}^{(0)}}{(t - \tau)^{1-\gamma}}d\tau
\]

\[
\leq \frac{C_3}{nt_s^2} + C_0 \int_{0}^{T_s} \frac{\|p_n - p\|_{T_s}^{(0)}}{(T_s - \tau)^{1-\gamma}}d\tau + C_4 \|p_n - p\|_{[T_s, T^*]}(t - T_s)^{\gamma - \frac{1}{2}}
\]

\[
\leq \frac{C_3}{nt_s^2} + C_1 T_s^{-\frac{\gamma}{2} + \eta} \|p_n - p\|_{T_s}^{(0)} + C_4 \|p_n - p\|_{[T_s, T^*]}(T^* - T_s)^{\gamma - \frac{1}{2}}.
\]

Let us choose $T^* > T_s$ such that $C_4(T^* - T_s)^{\gamma - \frac{1}{2}} < 1$, then we have

\[
(1 - C_4(T^* - T_s)^{\gamma - \frac{1}{2}})\|p_n - p\|_{[T_s, T^*]} \leq \frac{C_3}{nt_s^2} + C_1 T_s^{-\frac{\gamma}{2} + \eta} \|p_n - p\|_{T_s}^{(0)}.
\]

The right term vanishes when $n$ goes to $\infty$ so that $p_n$ converges to $p$ in $C_{(0)}^0((0, T_s + C_5])$ for some independent constant $C_5 > 0$. By repeating this argument inductively, it follows that $p_n$ converges to $p$ in $C_{(0)}^0((0, T])$.

Now we can prove $p$ from Proposition 3 is the density function of $F_{r_0}^X$.

Hence we have

\[
\lim_{n \to \infty} \int_{-\infty}^{0} h_n(\xi)P_{\xi}^X(\xi) = \lim_{n \to \infty} \int_{I} p_n(t)dt = \int_{I} p(t)dt.
\]

For $I = [0, t] \subset [0, T]$, $P_{\xi}^X(\xi) = I$ is an increasing function of $\xi$, so if we choose $h_n$ so that

\[
\lim_{n \to \infty} \int_{0}^{r_0 + \frac{1}{n}} h_n(\xi)d\xi = 1,
\]

we get

\[
\lim_{\xi \to r_0^+} P_{\xi}^X(\xi) \leq \lim_{n \to \infty} \int_{r_0}^{r_0 + \frac{1}{n}} h_n(\xi)P_{\xi}^X(\xi)d\xi = \lim_{n \to \infty} \int_{-\infty}^{0} h_n(\xi)P_{\xi}^X(\xi)d\xi(3.35)
\]

Similarly, if we choose $h_n$ so that $\lim_{n \to \infty} \int_{r_0 - \frac{1}{n}}^{0} h_n(\xi)d\xi = 1$, we get

\[
\lim_{n \to \infty} \int_{-\infty}^{0} h_n(\xi)P_{\xi}^X(\xi)d\xi = \lim_{n \to \infty} \int_{r_0 - \frac{1}{n}}^{0} h_n(\xi)P_{\xi}^X(\xi)d\xi \leq \lim_{\xi \to r_0^-} P_{\xi}^X(\xi) \leq \lim_{\xi \to r_0^+} P_{\xi}^X(\xi) \leq \lim_{\xi \to r_0^-} P_{\xi}^X(\xi). (3.36)
\]

Therefore, we obtain that

\[
F_{r_0}^X([0, t]) = P_{r_0}(\tau_{r_0}^X \leq t) = \int_{0}^{t} p(\tau)d\tau, \quad (3.37)
\]
which implies that $p$ is the density function of $F^X_{r_0}$, thus item 2 is proved. By Theorem 2 and the properties of the Gaussian kernel, $G^X_{0,t}(r_0, x)$ solves (1.2), (1.5) and (1.6). Hence $G^X$ is the Green function of the heat equation with Dirichlet boundary condition which implies item 1. Furthermore, $G^X_{0,t}(r_0, x)$ can be written as

$$G^X_{0,t}(r_0, x) = G_{0,t}(r_0, x) - \int_0^t G_{\tau,t}(X_\tau, x)p(\tau) d\tau.$$  

(3.38)

Differentiating with respect to $x$, applying the jump relation and (3.31), we have

$$\frac{\partial}{\partial x} G^X_{0,t}(r_0, x) \bigg|_{x=x^-} = \frac{\partial}{\partial x} G_{0,t}(r_0, X_t) - p(t) - \int_0^t \frac{\partial}{\partial x} G_{\tau,t}(X_\tau, X_t)p(\tau) d\tau$$  

(3.39)

Thus $p(t) = -\frac{1}{2} \frac{\partial}{\partial x} G^X_{0,t}(r_0, x) \bigg|_{x=x^-}$, so it proves item 3 of Theorem 1.

Acknowledgement

I am thankful to Errico Presutti, Anna De Masi for helpful suggestions and Victor Perez Abreu for useful comments.

References

[1] L. M. Ricciardi; L. Sacerdote; S. Sato: On an Integral Equation for First-Passage-Time Probability Densities, Journal of Applied Probability, Vol. 21, No. 2, 302–314 (1984).

[2] J.R. Cannon: The One-Dimensional Heat Equation, Addison-Wesley Publishing Company (1984).

[3] Ioannis Karatzas, Steve E.Shreve: Brownian Motion and Stochastic Calculus, Springer (1991).

[4] G. Peskir, A. Shiryaev: Optimal Stopping and Free-Boundary Problems, Birkhäuser (2006).

[5] A. Fasano, Mathematical models of some diffusive processes with free boundaries, SIMAI e-Lecture Notes (2008).

[6] K. Borovkov, A.N. Downes: On boundary crossing probabilities for diffusion processes, Stochastic Processes and their Applications, Volume 120, Issue 2, February 2010, Pages 105–129.

[7] T. Taillefumier, M. Magnasco: A Transition to Sharp Timing in Stochastic Leaky Integrate-and-Fire Neurons Driven by Frozen Noisy Input, Neural Computation Vol. 26, No. 5, 819–859 (2014).
[8] L. Alili, P. Patie: *Boundary crossing identities for Brownian motion and some non-linear ode’s*, Proc. Amer. Math. Soc. **142**, 3811–3824 (2014).

[9] G. Carinci, A. De Masi, C. Giardinà, E. Presutti: *Free Boundary Problems in PDEs and Particle Systems*, Springer (2016).