Impact of gravity on vacuum stability

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Abstract – In a pioneering paper on the role of gravity on false vacuum decay, Coleman and De Luccia showed that a strong gravitational field can stabilize the false vacuum, suppressing the formation of true vacuum bubbles. This result is obtained for the case in which the energy density difference between the two vacua is small, the so-called thin-wall regime, but is considered of more general validity. Here we show that when this condition does not hold, however, the effect of gravity on the bubble nucleation rate amounts to a small correction to the flat space-time result even in a strong gravity regime. This result suggests that the impact of gravity on the stability analysis of the electroweak vacuum could be essentially negligible.

Introduction. – The search for new physics beyond the Standard Model (BSM) is one of the most important and challenging questions of present theoretical and experimental physics. From the theoretical side, crucial for our understanding and construction of BSM theories is the stability analysis of the electroweak (EW) vacuum [1–7]. The discovery of the Higgs boson boosted new interest on this question, making it one of the hottest topics in theoretical particle physics [8–15].

Coleman and Callan studied the decay of the false vacuum in a flat space-time background [16]. Later, Coleman and De Luccia extended this analysis to include the impact of gravity [17]. They found that the probability of materialization of a true vacuum bubble decreases for increasing values of its size. In a flat space-time background, this probability turns out to be always finite, no matter how large the bubble size. In a curved background, things are different. For the transition from a false Minkowski vacuum to a true AdS vacuum it was found [17] that, when the gravitational effects are weak, the probability of materialization of such a bubble is close to the flat space-time result, while in a strong gravitational regime the presence of gravity stabilizes the false vacuum, preventing the materialization of a true vacuum bubble. This result was obtained under the thin-wall condition, namely for the case in which \( V(\phi_{\text{false}}) - V(\phi_{\text{true}}) \ll V(\phi_{\text{top}}) - V(\phi_{\text{false}}) \), where \( \phi_{\text{false}} \), \( \phi_{\text{true}} \), and \( \phi_{\text{top}} \) respectively, are the locations of the false vacuum, the true vacuum, and the maximum of the potential barrier.

Going back to the SM, it is known that due to the top loop corrections the Higgs potential \( V(\phi) \) bends down for values of \( \phi > v \sim 246 \text{ GeV} \) (the location of the EW minimum), and develops a second minimum, much deeper than the EW one and at a much larger value of the field, \( \phi_{\text{true}}^{(2)} \gg v \). Clearly, the conditions under which the Coleman-De Luccia result is derived are not fulfilled. However, it is still expected that in the presence of Planckian physics, the bubble nucleation rate vanishes [18–20]. More technically, it is expected that the bounce solution to the Euclidean equation of motion disappears.

Motivated by the above considerations, in this letter we focus on the decay of a false vacuum with zero energy density to a true vacuum with negative energy density, i.e., the decay from a Minkowski false vacuum state to an AdS true vacuum.

To study the impact of gravity on the nucleation rate of a bubble of true vacuum, we consider a potential [21] that allows to explore all possible cases, from the thin-wall regime to the case in which the difference in height between the two minima of the potential is large. For this latter case we find that the suppression of the Minkowski false vacuum decay (if still present) is pushed to very high-energy regimes, even much higher than the Planck scale, so that the presence of a Coleman-De Luccia pole becomes physically irrelevant (this is similar to what happens for the Landau pole in QED, where the latter occurs at such a high energy scale that the theory has already lost its significance several orders of magnitudes below...
that scale). At energies below the pole scale, but yet Planckian or trans-Planckian, we observe that for the case considered the decay probability is non-vanishing and very close to the flat space-time result, i.e., the result obtained ignoring the presence of gravity.

In our opinion these results represent a significant indication for current studies and for model building of BSM physics, where we are often interested in considering new physics at Planckian and/or trans-Planckian scales.

Analysis. – Let us consider the (Euclidean) action of a scalar field in curved space-time together with the Einstein term:

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) - \frac{R}{16\pi G} \right],
\]

where \( R \) is the Ricci scalar, \( G \) the Newton constant, and \( V(\phi) \) the potential [21]:

\[
V(\phi) = \frac{\rho^2}{4} \left( V(\phi) - a^2 \right)^2 + \frac{4b}{3} (a \phi^2 - 3a^3 \phi + 2a^4),
\]

(2)

where \( a < 0 \). \( \phi \) has two non-degenerate minima at \( \phi = \pm a \), \( \phi = a \) being the absolute minimum, and a potential barrier at \( \phi = -b a \). Note that for \( b = 0 \) this is just the double-well potential with two degenerate minima at \( \phi = \pm a \), while for \( b = 1 \) the minimum at \( \phi = -a \) becomes an inflection point and the barrier disappears.

This potential is perfectly suited for our scopes. By varying the dimensionless parameter \( b \), we control the difference \( V(-a) - V(a) \) and the height of the potential barrier \( V(-ba) - V(-a) \). By varying the dimensional parameter \( a \) (the only mass scale of our potential), and pushing it toward the Planck scale and beyond, we will test the impact of strong gravitational physics on the decay of the false vacuum.

To calculate the bubble nucleation rate \( \Gamma \) we need the bounce solution to the (Euclidean) equations of motion derived by varying the action in eq. (1). The saddle point approximation to the transition rate \( \Gamma \) per unit volume is given by the ratio between the exponential of minus the (Euclidean) action evaluated at the bounce and the action calculated for the false vacuum solution [17]:

\[
\Gamma / V = e^{-(S_{bounce} - S_{false})} \equiv e^{-B}.
\]

Following [17], we consider the most general \( O(4) \)-symmetric Euclidean metric, \((ds)^2 = (dr)^2 + \rho(r)^2 (d\Omega_3)^2\), where \( r \) is the radial coordinate along a radial curve, \( d\Omega_3 \) is the element of distance on a unit three-sphere, and \( \rho(r) \) the radius of curvature of each three-sphere.

With this metric, the Einstein equation \( G_{r r} = -\kappa T_{r r} \) reduces to the following equation for \( \rho(r) \) (the \( \kappa \) denotes derivation with respect to \( r \)):

\[
\rho^2 = 1 + \frac{\kappa}{3} \rho^2 \left( \frac{1}{2} \phi'^2 - V(\phi) \right),
\]

(3)

where \( \kappa \equiv 8\pi G \equiv 8\pi / M_P^2 \), and \( M_P \) is the Planck scale. Equation (3) is coupled to the scalar field equation derived from (1),

\[
\phi'' + \frac{3\rho'}{\rho} \phi' = \frac{dV}{d\phi}.
\]

(4)

The false vacuum solution to these coupled equations consists of the flat space-time metric, \( \rho_{false}^2(r) = 1 \), together with the constant solution for the profile, \( \phi_{false}(r) = -a \).

The Ricci scalar, \( R \equiv 2\Lambda \), is the element of distance on a unit three-sphere, and \( \Lambda \) the Newton constant, and

\[
\rho = \frac{\rho_0}{\rho_0} \left( \frac{r_0}{r} \right)^{-\frac{2}{3}},
\]

(2)

where the right-hand side is obtained by making use of eqs. (3) and (4), and the boundary terms B.T. appear after integrating by parts \( \rho^2 \phi'' \) on the first line of eq. (5).

The B.T. are infinite. However, the computation of \( \Gamma \) only involves the difference \( B \) between the action evaluated at the bounce and at the false vacuum solutions. These two solutions coincide at infinity, and the two actions provide the same divergent contribution, that is then cancelled out in the difference \( B = S_{bounce} - S_{false} \).

It is worth noting that, due to the particular dependence of the potential \( V(\phi) \) in (2) on the coupling \( g \), it is possible to re-express eqs. (3) and (4) in terms of the rescaled quantities \( \tau = gr \), \( \rho = g \rho \) and \( V = V/g^2 \), so that no explicit dependence on \( g \) is left in eqs. (3), (4). Accordingly, the dependence on \( g \) of the action can be factored out: \( S = S/g^2 \) and \( B = B/g^2 \).

Let us solve now eqs. (3) and (4) with the help of a numerical shooting procedure. The typical bounces are decreasing profiles \( \phi_0(r) \), such that \( \phi_0(r \rightarrow \infty) \rightarrow -a \), and can be parametrized by their size \( \tau \), the phase of the radial coordinate where the height of the profile is decreased by one half. For \( r \gg \tau \), the solution \( \rho(r) \) approaches the flat space-time solution, \( \rho'(r) = 1 \), while for \( r \ll \tau \), i.e., inside the bubble of true vacuum, the presence of a negative energy density creates a distortion of this solution.

In the thin-wall regime, which for our potential is the case of small values of the parameter \( b \), it was shown in [17] that \( \tau_0 = T_0 [1 - (T_0/2\Lambda)]^{-1} \) and \( B = B_0 [1 - (T_0/2\Lambda)]^{-2} \), where \( \tau \equiv \rho(\tau) \), while \( B_0 \) and \( T_0 \) are the values of \( B \) and \( \tau \) derived in the flat space-time case (needless to say, in this latter case \( \rho(r) \) coincides with \( r \)). These equations show that \( \tau_0 \) and \( B \) have a pole at \( T_0/2 = \Lambda \equiv \sqrt{3/(8\pi G |V(a)|)} \). The corresponding curve for \( B/B_0 \) is reported in figs. 1, 2 as a solid line having an asymptote at \( T_0/2 = 1 \).

Our numerical approach allows to check the accuracy of the thin-wall predictions and before moving to the cases of particular interest to us, we calculate \( B/B_0 \) for some small values of \( b \) (thin-wall regime). To facilitate the comparison with [17], we present our results by plotting \( B/B_0 \).
against $\tau_0/(2\Lambda)$, the latter being an increasing function of $a$ for each fixed value of $b$. The parameter $a$ gives the location of the minima, and increasing values of $a$ toward the Planck scale (and beyond) mark the transition from the weak to the strong gravitational regime. The results are plotted in figs. 1 and 2. Note that the rescaling to the hatted quantities discussed above shows that all variables on the $x$ and $y$ axes of the figures do not depend on the coupling $g$.

For the smallest value considered for $b$, $b = 0.001$, the points practically sit on the thin-wall curve (blue solid curve), and are not reported in fig. 1. At $b = 0.01$ (red circles) we start to observe a small deviation from the thin-wall curve, that becomes more sizable for $b = 0.13$ (green squares). Figure 1 shows that for small values of $b$, when the two minima of the potential are almost degenerate (thin-wall regime), starting from $\tau_0/(2\Lambda) \sim 0.5$ the effect of gravity grows rapidly for increasing values of $\tau_0/(2\Lambda)$.

We now consider larger values of $b$, thus leaving the thin-wall regime. Some of the curves for $B/B_0$ vs. $\tau_0/(2\Lambda)$ resulting from our numerical analysis, namely those obtained for $b = 0.4, 0.7, 0.85$, are reported in fig. 2. This figure clearly shows that for increasing values of $b$, i.e., when the difference in height between the two minima increases, $B/B_0$ gets flattened, a similar trend being also observed in [22]. This flattening related to the increase of $b$, whose origin is discussed and explained below, is the crucial effect we were looking for. In fact, when values of $b$ sufficiently close to 1 are considered, we actually explore the region of physical interest in the parameter space, while smaller values of $b$, where $B/B_0$ significantly increases, do not belong to the region of physical interest (see below).

For $b \to 1$, $B/B_0$ gets closer and closer to the horizontal line $B/B_0 = 1$. We have not added other curves with larger $b$ in fig. 2 as they would unnecessarily clutter this figure. For the same reason, we have also limited the range of values of $\tau_0/(2\Lambda)$ to $0 < \tau_0/(2\Lambda) < 1.5$. In fact, the Coleman-De Luccia pole (of the thin-wall case) occurs at $\tau_0/(2\Lambda) = 1$, and fig. 2 clearly shows that this point is not a pole for large values of $b$. Finally, for $b = 1$ this curve reaches the horizontal line $B/B_0 = 1$ for the whole range of values $0 < \tau_0/(2\Lambda) < \infty$.

This result is easily understood if we consider the bounce profile function $\phi_b(r)$, that connects the center of the bounce, $\phi_b(0)$, with the false vacuum state at $r \to \infty$, $\phi_b(\infty) = -a$, and look at the numerical solution of eq. (4). For small values of $b$, $\phi_b(0)$ is close to $a$, the true vacuum, while for increasing values of $b$, $\phi_b(0)$ decreases toward $\phi_b(0) \sim -a$. In the limit $b \to 1$, $\phi_b(0)$ reaches this latter value, so that: $\phi_b(r) \to -a$ in the whole range of $r$. In this limit, eq. (3) becomes $\dot{\rho}(r) = 1$, and the flat space-time result is recovered. It is worth noting that this limiting solution obtained by considering the $b \to 1$ limit coincides with the one that is obtained by inserting $b = 1$ from the beginning in eqs. (3) and (4).

These are the central results of the present letter. In fact, the lesson from [17] (see fig. 1) is that when we move from weak to strong gravity regime, the impact of gravity becomes enormous, to the point that, for the critical value $\tau_0/(2\Lambda) = 1, B$ diverges and the corresponding transition rate $\Gamma$ vanishes. As stressed above, this is expected to hold even out of the thin-wall regime. Models of BSM physics and conclusions on the vacuum stability analysis are often based on this expectation [18–20].

However, we have seen that when the energy density difference between the two vacua of the potential is increased by choosing larger values of $b$, so that the conditions of [17] are no longer verified, the gravitational contribution to $\Gamma$ at fixed $\tau_0/(2\Lambda)$ becomes smaller and smaller. In addition, when this difference becomes sufficiently large, which for our model occurs when $b \to 1$, the quenching of the false vacuum decay (if still present) is pushed to very high scales. We will see below that this is the case of phenomenological interest, relevant to the EW vacuum decay. In such a case the pole of $B/B_0$ is pushed well above the Planck mass, then its presence is physically irrelevant as it occurs at scales where the theory has already lost its validity.
that are orders of magnitude above $M_{\text{Pl}}$. Values of $V_{\text{eff}}(\phi)$ calculated for such values are very close to 1, which in turn implies that curves of the kind of those shown in fig. 2 for potentials that simulate as much as possible the Higgs effective potential from eq. (2) a potential that mimics as much as possible the decay of the EW vacuum. In order to get at least qualitative indications of the effect of gravity on the vacuum decay probability could be inferred from our model potential in eq. (2) when $a$ and $b$ are chosen as explained above.

In order to carry on with our analysis we now consider the results derived in [25] and in particular we focus on the critical line displayed in fig. 10 of [25] that connects the location of the $B/B_0$ pole with the energy scale at which it is found. Actually, the $y$-axis of this figure is $y = 4\sqrt{2\pi a_{cr}}/M_P$, where $a_{cr}$ is the location of the pole, and the $x$-axis is $x = V(\phi_{\text{top}})/\epsilon$, where $V(\phi_{\text{top}})$ is the height of the barrier and $\epsilon$ is the difference in height between the two minima, i.e., $\epsilon = V(-a) - V(a)$. According to [25], close to the $y$-axis the critical line has a power law behaviour given by $\epsilon = 1.22 x^{-0.33}$.

With the help of this last equation, a straightforward computation with the potential of eq. (2) then shows that when $(1 - b) \approx 10^{-8}$ the pole of $B/B_0$ occurs at $a_{cr} \approx 10^7 M_P$. This is the searched result: for the choice of the parameters $a$ and $b$ made above in potential (2), the pole lies well above the Planck scale. Note also that for the largest value of $b$ considered in fig. 2, $b = 0.85$, which as clarified above is already far from the region of phenomenological interest, the pole is located at a scale slightly above the Planck mass: $a_{cr} = 1.2 M_P$. It is then clear that in order to observe quenching of the false vacuum decay at sub-Planckian scales, it is necessary to lower $b$ down to values that correspond to potentials which do not resemble $V_{\text{eff}}(\phi)$, even at a rough qualitative level.

At this point it is worth noting that after the appearance of the first version of this paper, a calculation with a bona fide SM Higgs effective potential was done in [26]. In particular, it is found that the difference in the bounce action calculated with and without gravity (see their fig. 3 with $\zeta = 0$) is $(S_{\text{grav}} - S_{\text{flat}})/S_{\text{flat}} \lesssim 0.2\%$. This tiny change is essentially in line with our indications, thus supporting our expectation that the findings obtained with the potential (2) are not overturned in a phenomenologically relevant context.

Before ending this paper, we have to say some words on previous work [27] on the role of gravity on the EW vacuum decay, where a perturbative expansion of $\phi_b(r)$ and $\rho(r)$ in powers of $\kappa$ around the flat space-time solution was attempted. Actually, this analytical method seems to have a serious drawback, as the boundary conditions (see above), essential for the classical solution to be a bounce (and accordingly for $\Gamma$ to be interpreted as the bubble nucleation rate) are not respected already at first order in $\kappa$. Therefore, the output of this analysis cannot be trusted and a fortiori cannot be used for comparison. A detailed study of this issue goes beyond the scope of the present paper and will be presented elsewhere.

Conclusions. – We studied the decay of a false Minkowski vacuum to a true AdS vacuum by taking into account the impact of gravity. We find that when the energy density difference between the false and the true

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Table 1: Set of values of $g \tau_0 a$ and $g^2 B_0$ computed at different $b$.

| $b$   | $g \tau_0 a$            | $g^2 B_0$          |
|-------|-------------------------|--------------------|
| 0.001 | 2121.32                 | $4.44132 \times 10^{10}$ |
| 0.01  | 212.113                 | $4.4094 \times 10^{7}$  |
| 0.13  | 16.2379                 | 19921.0            |
| 0.4   | 3.93908                 | 276.288            |
| 0.7   | 2.73633                 | 70.9009            |
| 0.85  | 2.51328                 | 22.2602            |

In [23] and [24] arguments were given in favour of the existence of such a pole. After the appearance in the archives of the first version of this paper, the question was also addressed in [25], and the same conclusions of [23] and [24] were reached. However, we stress that these results are not in contrast with our findings. In fact the pole that they find lies at very high scales when $b \to 1$; below we provide numerical examples.

For the reader’s convenience, in table 1 some values of $g \tau_0 a$ and $g^2 B_0$ for different values of $b$ are reported. These are useful parameters to reconstruct physical quantities relevant to the analysis that we presented and to better appreciate the results reported in the figures (see below). These quantities are obtained in the flat space-time case, and are both $g$ and $a$ independent. Therefore, they are univocally determined once $b$ is given.

In particular, the values of $g \tau_0 a$ in table 1 are useful to establish the relation between $\tau_0/(2\Lambda)$, that appears on the $x$-axis of the figures, and the ratio $a/M_P$ for each value of $b$, that is for each single curve in the figures. In fact, by recalling the definition of $\Lambda$, one obtains the following relation: $\tau_0/(2\Lambda) = (2/3)\sqrt{2\pi b} (g \tau_0 a)/(a/M_P)$. Then, for instance, the point $\tau_0/(2\Lambda) = 1$ corresponds for $b = 0.001$ to $a \sim M_P/100$, and for $b = 0.85$ to $a \sim M_P/4$. It is then evident from the lowest curve of fig. 2 that when $b = 0.85$ even at energy scales as high as $a \sim M_P/10$ the effect of gravity on the false vacuum decay probability only amounts to a few percent correction.

We now explore to what extent these results can give some hint on the case of phenomenological interest, namely the decay of the EW vacuum. In order to get from eq. (2) a potential that mimics as much as possible the Higgs effective potential $V_{\text{eff}}(\phi)$, we require that the location of its maximum does not exceed $10^{11}$ GeV and that the location of the true vacuum is above the Planck scale $M_P$. This leads to the following upper bound for $b$: $(1 - b) < 10^{-8}$ which indicates that in our case $b$ must be very close to 1, which in turn implies that curves of the same kind of those shown in fig. 2 calculated for such values of $b$ have a significant deviation from 1 only at scales that are orders of magnitude above $M_P$.

It must be remarked that even for $b \sim 1$ our potential does not reproduce all of the features of $V_{\text{eff}}(\phi)$, and in order to get firmer conclusions a complete analysis involving $V_{\text{eff}}(\phi)$ is needed. Yet, it is reasonable to expect that

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vacuum is significantly larger than the height of the potential barrier, the effect of gravity on the transition rate amounts to a small correction to the flat space-time result even in a strong gravity regime.

In particular, this result holds when the characteristic scales of the potential under investigation are tuned on those of the Higgs effective potential and, if confirmed by a thorough analysis involving the latter potential, it would be of considerable interest for the stability analysis of the EW vacuum, as well as for the construction of BSM models where the role of gravity needs to be carefully understood.

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