An Application of the Deutsch-Josza Algorithm to Formal Languages and the Word Problem in Groups

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Abstract

We adapt the Deutsch-Josza algorithm to the context of formal language theory. Specifically, we use the algorithm to distinguish between trivial and nontrivial words in groups given by finite presentations, under the promise that a word is of a certain type. This is done by extending the original algorithm to functions of arbitrary length binary output and with the introduction of a more general concept of parity. We provide examples in which properties inherited directly from the original algorithm allow to reduce the number of oracle queries with respect to the deterministic classical case. This has some consequences for the word problem in groups with a particular kind of presentation.

1 The Deutsch-Josza algorithm adapted to formal languages

We apply a direct generalization of the Deutsch-Josza algorithm to the context of formal language theory. More particularly, we adapt the algorithm to distinguish between trivial and nontrivial words in groups given by finite presentations, under the promise that a word is of a certain type. For background information, we refer the reader to [1] and [2].

The Deutsch-Josza algorithm concerns maps \( f : \{0,1\}^n \rightarrow \{0,1\} \), which we may think of as words of length \( n \) in a two-letter alphabet. Instead, let us consider maps \( f : \{0,1\}^n \rightarrow \{0,1\}^k \), where \( k \) does not necessarily depend on \( n \). Once fixed \( k = 2 \), we can identify the letters of the alphabet \( \mathcal{A} = \{a,b,c,d\} \) with the binary strings of \( \{0,1\}^2 \): \( a \leftrightarrow 00, b \leftrightarrow 01, c \leftrightarrow 10 \) and \( d \leftrightarrow 11 \).

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We describe below the simplest possible case: the map $f$ takes a single binary digit as input and gives two binary digits as output. The output corresponds to one of the letters from $A$. Although this example is not general enough to be interesting, it is still useful to see how the “balanced VS. constant” question in the original Deutsch-Jozsa task can be lifted to different parities related to the function. This is essentially the same identical quantum circuit implementing the Deutsch algorithm, but with auxiliary input $|11\rangle$ rather than $|1\rangle$. For the sake of clarity, let us see the steps of the algorithm. After applying the Hadamard gates to the two registers, the state of the system is

$$H \otimes H^{\otimes 2}(|0\rangle \otimes |11\rangle) = |+\rangle \otimes |\overline{-}\rangle^{\otimes 2}. $$

If $z \in \{0,1\}$, the oracle works as follows:

$$U_f (|z\rangle \otimes |\overline{-}\rangle) = |z\rangle \otimes \frac{1}{2} (|00\rangle \oplus f(z)) - |01\rangle \oplus f(z)) - |10\rangle \oplus f(z)) + |11\rangle \oplus f(z))$$

$$= (-1)^{p(f(z))}|z\rangle \otimes (|\overline{-}\rangle)^{\otimes 2}. $$

For a binary string $y$, we denote by $p(y)$ the parity of $y$, that is $p(y) = m(mod 2)$, where $m$ is the Hamming weight of $y$. After querying the oracle $\hat{U}_f$, we obtain the state

$$\frac{(-1)^{p(f(0))}|0\rangle + (-1)^{p(f(1))}|1\rangle}{\sqrt{2}} \otimes |\overline{-}\rangle^{\otimes 2}. $$

Finally, after the last Hadamard gate, the first qubit will be in the state $|0\rangle$ if $p(f(0)) = p(f(1))$ or $|1\rangle$ if $p(f(0)) \neq p(f(1))$. We shall say that $f$ is parity constant if $p(f(0)) = p(f(1))$; parity balanced, otherwise. By measuring the final state, we obtain $|0\rangle$ with probability 1 if $f$ is parity balanced and $|1\rangle$ with probability 1 if $f$ is parity constant. In the same spirit, moving to a larger number of bits, a function $f$ is parity balanced if exactly half of the elements of the image of $f$ have odd parity. We will show that properties of the Deutsch-Jozsa algorithm are inherited when extending the co-domain of $f$ and generalizing the notion of parity in less trivial ways.

Let us now introduce some terminology related to formal languages. Given a word $w : \{0,1\}^n \rightarrow \{a,b,c,d\}$, an anagram of $w$ is a word of the form $w \circ \phi$, where $\phi : \{0,1\}^n \rightarrow \{0,1\}^n$ is a permutation. We write $[w]$ for the set of all anagrams of $w$. More formally, let $F$ denote the free monoid on $\{a,b,c,d\}$ and let $M$ denote the free commutative monoid on $\{a,b,c,d\}$. Let $R$ denote the natural map from $F$ to $M$ and suppose that $w \in M$. Then $R(w) = [w]$, the set of all anagrams of $w$. It is clear that the definition of parity balanced and parity constant extends to the words of $M$. Let $x \in \{01,10,11\}$, we denote the sets of $x$-constant and $x$-balanced words of length $k$ over $A$ by $C_k^x(A)$ and $B_k^x(A)$, respectively. The set of 11-constant words is then a union of sets of anagrams

$$C^{11}_2(a,b,c,d) = [aa] \cup [bb] \cup [ac] \cup [dd] \cup [bc] \cup [ad].$$
Similarly, the set of 11-balanced words is
\[ B_{11}^2(a, b, c, d) = [ab] \cup [ac] \cup [bd] \cup [cd]. \]

Note that both the terms of the alphabet in the bracket have the same parity with the notation \( a \leftrightarrow 00, \ b \leftrightarrow 01, \ c \leftrightarrow 10 \) and \( d \leftrightarrow 11. \)

Suppose not to input \( |11 \rangle \) into the auxiliary workspace, but rather some arbitrary string of length two. How does this affect the sets of words we can distinguish between? It is interesting to observe that we may define as follows a more general type of parity. The set \( \{00, 01, 10, 11\} \) is considered in natural way as the vector space \( (\mathbb{Z}_2)^2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2. \) Define \( p^x(y) \) to be equal to 0, if \( y \) is in the subspace \( \langle x \rangle = \{00, x\} \) and equal to 1, otherwise. With this notation, \( p^{11}(y) = p(y), \) the usual parity function. A similar circuit, taking the auxiliary input \( \neg(x) \), that is the binary complement of \( x, \) will distinguish between whether the word is \( x \)-constant or \( x \)-balanced. Again, measurement of the state will yield this information with certainty. It is clear that if \( x = 00 \) then the output of the circuit is independent of \( f, \) and so this is of no use. Let us now suppose that \( x = 01. \) Then \( x \)-constant means that the outputs of \( f \) are in the same coset of the subgroup \( \{00, x\} \) in \( (\mathbb{Z}_2)^2 \) and \( x \)-balanced means that \( f(0) \) and \( f(1) \) are in different cosets, or, in other words, both in or out the subspace \( \langle x \rangle = \{00, x\}. \) The set of 01-constant words is
\[ C_{01}^2(a, b, c, d) = [aa] \cup [bb] \cup [cc] \cup [dd] \cup [ac] \cup [cd] \]
and the set of 01-balanced words is
\[ B_{01}^2(a, b, c, d) = [ab] \cup [ac] \cup [bd] \cup [cd]. \]

With the same notation,
\[ C_{10}^2(a, b, c, d) = [aa] \cup [bb] \cup [cc] \cup [dd] \cup [ac] \cup [bd] \]
and
\[ B_{10}^2(a, b, c, d) = [ab] \cup [ad] \cup [bc] \cup [cd]. \]

As before, the first term and the second term in the bracket represent the first output and the second output of the function, respectively. Also the parity is the same as described before. Note that when the set is parity constant both terms are in or out the subspace \( \langle x \rangle, \) while in the parity balanced case one term is in the subspace and the other one is out.

We write
\[ \mathcal{F}_k^x(A) = C_k^x(A) \cup B_k^x(A) \]
and call this the set of \( x \)-feasible words of length \( k. \) Notice that
\[ A^k = \bigcup_x \mathcal{F}_k^x(A). \]

The following fact is central in the context of our discussion.

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Theorem 1 Fixed an $x$-parity, we can decide if a function $f$ is $x$-constant or $x$-balanced with a single quantum query. Equivalently, we can determine if the output of $f$ is a language in $C^x_k$ or $B^x_k$, with a single quantum query.

Already in the seminal work [3], it was pointed out that a classical randomized algorithm solves the Deutsch-Josza task with three classical queries on average, whereas the quantum approach solves it with probability 1 using one single query (see also [4]). Here the output of the function $f$ is no more a single bit but a bit string. If the number of letters of the alphabet is $d$ then the output of the function is an $n$-bit string, where $n = \log_2 d$. A word is given by $k$ repeated random output of the function, where $k$ is the length of the word. In other terms, a word is like a sequence obtained by tossing a dice with $d$ faces. It is easy to see that the probability of being constant over all possible anagrams, interpreting the output binary string of $k$ queries as anagrams of $k$ letters, is higher than in the balanced case. The difference decreases while increasing the number of queries. As long as any possible parity function partitions into two classes the function co-domain, the number of quantum queries required to distinguish between the parity constant and parity balanced cases remains constant. This is due to the fact that binary strings always form a bipartition with respect to the Hamming weight. The method described allows us to extend the Deutsch-Josza algorithm to functions with output of any dimension, $f : \{0,1\}^n \rightarrow \{0,1\}^k$. Defining appropriate parities, based on subgroups or code membership problems, could give arise to potentially interesting applications.

2 Distinguishing between languages

In this section we construct languages given by intersecting the images of binary maps. We show that acceptance of a word of length $k$ in one of these languages can be determined with $k$ quantum queries. This can be easily done on the basis of the discussion carried on in the previous section. The problem defined is artificial, but nonetheless indicates a way to use repeated applications of the modified Deutsch-Josza algorithm, with special reference to formal languages. We define languages constructed by intersecting the images of functions promised to be $x$-constant or $x$-balanced with respect to different subspaces. If $X \subset \{01,10,11\}$, let us write

$$F^X_k(A) := \bigcap_{x \in X} F^x_k(A).$$

We have
For these four languages the word is. This is an improvement over the classical deterministic setting, where we need at least $2^n - 1$ queries for each function. The remaining possibilities for $x$ are

$$C_2^{11}(a, b, c, d) = [aaaa] \cup [bbbb] \cup [cccc] \cup [dddd] \cup [aadd] \cup [aacc] \cup [accd] \cup [abbd] \cup [bccc] \cup [bcdd] \cup [abcd],$$

$$B_2^{11}(a, b, c, d) = [aabb] \cup [aacc] \cup [bbdd] \cup [ccdd] \cup [bbcc] \cup [bcdd] \cup [addd],$$

$$C_2^{01}(a, b, c, d) = [aaaa] \cup [bbbb] \cup [cccc] \cup [dddd] \cup [aaab],$$

$$B_2^{01}(a, b, c, d) = [aacc] \cup [aadd] \cup [bbcc] \cup [bbdd] \cup [ccdd] \cup [abcd],$$

$$F_2^{01,11} = [aaaa] \cup [bbbb] \cup [cccc] \cup [dddd] \cup [aacc] \cup [aacc] \cup [bbcc] \cup [bbdd] \cup [ccdd] \cup [abcd].$$

We also have

$$B_2^{11}(a, b, c, d) \cap B_2^{01}(a, b, c, d) = [abcd] \cup [aacc] \cup [bbdd],$$

$$C_2^{11}(a, b, c, d) \cap B_2^{01}(a, b, c, d) = [aadd] \cup [bbcc],$$

$$B_2^{11}(a, b, c, d) \cap C_2^{01}(a, b, c, d) = [aabb] \cup [ccdd],$$

$$C_2^{11}(a, b, c, d) \cap C_2^{01}(a, b, c, d) = [aaaa] \cup [bbbb] \cup [cccc] \cup [dddd].$$

Therefore, given a word in $F_2^{01,11}$, we can decide with two quantum queries in which of these four languages the word is. This is an improvement over the classical deterministic setting, where we need at least $2^n - 1$ queries for each function. The remaining possibilities for $x$ are

$$C_2^{10}(a, b, c, d) = [aaaa] \cup [bbbb] \cup [cccc] \cup [dddd] \cup [aacc] \cup [aacc] \cup [bbbd] \cup [bcdd] \cup [bdad],$$

$$B_2^{10}(a, b, c, d) = [aabb] \cup [aadd] \cup [bbcc] \cup [ccdd] \cup [aadd] \cup [bbcc] \cup [abcc] \cup [acdd] \cup [abcd],$$

$$F_2^{10,11} = [aaaa] \cup [bbbb] \cup [cccc] \cup [dddd] \cup [aacc] \cup [aacc] \cup [bbcc] \cup [bbdd] \cup [ccdd] \cup [abcd].$$

We then have

$$F_2^{01,11} = F_2^{10,11}.$$

It can be checked that this is also equal to $F_2^{01,10}$. However, the three possibilities $X = \{01,11\}, \{10,11\}$ and $\{01,10\}$ all distinguish between different languages, since we
have

\[ B_{2}^{11}(a, b, c, d) \cap B_{2}^{10}(a, b, c, d) = [abcd] \cup [aabb] \cup [ccdd], \]
\[ C_{2}^{11}(a, b, c, d) \cap B_{2}^{10}(a, b, c, d) = [aadd] \cup [bbcc], \]
\[ B_{2}^{11}(a, b, c, d) \cap C_{2}^{10}(a, b, c, d) = [aacc] \cup [bbdd], \]
\[ C_{2}^{11}(a, b, c, d) \cap C_{2}^{10}(a, b, c, d) = [aaaa] \cup [bbbb] \cup [cccd] \cup [dddd], \]
\[ B_{0}^{11}(a, b, c, d) \cap B_{10}^{2}(a, b, c, d) = [abcd] \cup [aadd] \cup [bbcc], \]
\[ C_{0}^{11}(a, b, c, d) \cap B_{10}^{2}(a, b, c, d) = [aacc] \cup [bbdd], \]
\[ B_{0}^{11}(a, b, c, d) \cap C_{10}^{2}(a, b, c, d) = [aacc] \cup [bbdd], \]
\[ C_{0}^{11}(a, b, c, d) \cap C_{10}^{2}(a, b, c, d) = [aaaa] \cup [bbbb] \cup [cccd] \cup [dddd]. \]

We have then seen that

\[ F^{11} = F^{10} = F^{10,11} = F^{11,10}. \]

This fact will be useful later, when dealing with the word problem.

### 3 Larger alphabets

A similar approach can be taken for larger alphabets:

\[ \{a, b, c, d, e, f, g, h\} \rightarrow \{000, 001, 010, 011, 100, 101, 110, 111\}. \]

The previous treatment applies in a straightforward manner. It is in fact still possible to define a parity, based on the even number of 1s, like \( p^{11} \). This is equivalent to determine if a word \( w \) is in the subspace \( \{000,011,101,110\} \), also denoted \( p^{adfg} \). In this case, the set of parity constant and parity balanced words can be obtained using the auxiliary input \( |111\rangle \) in the circuit described before:

\[
U_f \left( |z\rangle \otimes |-\rangle^{\otimes 3} \right) = |z\rangle \otimes \frac{1}{2} \left( |000 \oplus f(z)\rangle - |001 \oplus f(z)\rangle - |010 \oplus f(z)\rangle + |011 \oplus f(z)\rangle \right) \\
- |100 \oplus f(z)\rangle + |101 \oplus f(z)\rangle + |110 \oplus f(z)\rangle - |111 \oplus f(z)\rangle \right) \\
= (-1)^{p(f(z))} |z\rangle \otimes (-|\rangle{\rangle}^{\otimes 3} .
\]

Then \( U_f \) gives the following set:

\[
C_{2}^{adfg}(a, b, c, d, e, f, g, h) = [aa] \cup [bb] \cup [cc] \cup [dd] \cup [ee] \cup [ff] \cup [gg] \cup [hh] \cup [ad] \cup [af] \\
\cup [ag] \cup [df] \cup [dg] \cup [fg] \cup [bc] \cup [be] \cup [bh] \cup [ce] \cup [ch] \cup [eh].
\]
Similarly, the set of parity balanced words is
\[
\mathcal{B}_2^{abfg}(a, b, c, d, e, f, g, h) = \left[ ab \cup [ac] \cup [bd] \cup [cd] \cup [ae] \cup [ah] \cup [de] \cup [dh] \right. \\
\left. \cup [bf] \cup [bg] \cup [cf] \cup [cg] \cup [fe] \cup [fh] \cup [ge] \cup [gh] \right]
\]

Other parities can be defined considering different set of vectors. For our purposes it is sufficient to define a set composed by the elements \( p_{abcd} = \{000, 001, 010, 011\} \). This plays the same role as \( p_0^1 \). In this case, the set of parity constant word can be obtained by using \( |100\rangle \) as auxiliary input. The circuit has the following output:

\[
U_f (|z\rangle \otimes |\rangle \otimes |+\rangle^{\otimes 2}) \\
= |z\rangle \otimes \frac{1}{2} (|000 \oplus f(z)\rangle + |001 \oplus f(z)\rangle + |010 \oplus f(z)\rangle + |011 \oplus f(z)\rangle \\
- |100 \oplus f(z)\rangle - |101 \oplus f(z)\rangle - |110 \oplus f(z)\rangle - |111 \oplus f(z)\rangle) \\
= (-1)^{p(f(z))}|z\rangle \otimes (-|\rangle \otimes |+\rangle^{\otimes 2}.
\]

As we have said before, this procedure gives
\[
\mathcal{C}_2^{abcd}(a, b, c, d, e, f, g, h) = \left[ aa \cup [bb] \cup [cc] \cup [dd] \cup [ee] \cup [ff] \cup [gg] \cup [hh] \cup [ab] \cup [ac] \\
\cup [ad] \cup [bc] \cup [bd] \cup [cd] \cup [ef] \cup [eg] \cup [eh] \cup [fg] \cup [fh] \cup [gh] \right]
\]
and
\[
\mathcal{B}_2^{abcd}(a, b, c, d, e, f, g) = \left[ ae \cup [af] \cup [ag] \cup [ah] \cup [be] \cup [bf] \cup [bg] \cup [bh] \\
\cup [ce] \cup [cf] \cup [cg] \cup [ch] \cup [de] \cup [df] \cup [dg] \cup [dh] \right].
\]

For reasons that will be clear later, it is important to define also the parity, based on the subspace \( p_{adch} = \{000, 011, 101, 111\} \), for which the set of parity constant words is obtained by setting as auxiliary input the state \( |011\rangle \):

\[
U_f (|z\rangle \otimes |+\rangle \otimes |\rangle^{\otimes 2}) \\
= |z\rangle \otimes \frac{1}{2} (|000 \oplus f(z)\rangle - |001 \oplus f(z)\rangle - |010 \oplus f(z)\rangle + |011 \oplus f(z)\rangle + \\
- |100 \oplus f(z)\rangle + |101 \oplus f(z)\rangle - |110 \oplus f(z)\rangle + |111 \oplus f(z)\rangle) \\
= (-1)^{p(f(z))}|z\rangle \otimes (-|\rangle \otimes |+\rangle^{\otimes 2} \otimes |+\rangle).
\]

The sets produced are
\[
\mathcal{C}_2^{adch}(a, b, c, d, e, f, g, h) = \left[ aa \cup [bb] \cup [cc] \cup [dd] \cup [ee] \cup [ff] \cup [gg] \cup [hh] \cup [ad] \cup [ae] \\
\cup [ah] \cup [de] \cup [dh] \cup [eh] \cup [bc] \cup [bf] \cup [bg] \cup [cf] \cup [cg] \cup [fg] \\
\cup [ce] \cup [cf] \cup [eg] \cup [hb] \cup [hc] \cup [hf] \cup [hg] \right];
\]
for the balanced case, we have
\[
\mathcal{B}_2^{adch}(a, b, c, d, e, f, g, h) = \left[ ab \cup [ac] \cup [af] \cup [ag] \cup [db] \cup [dc] \cup [df] \cup [dg] \\
\cup [eb] \cup [ec] \cup [ef] \cup [eg] \cup [hb] \cup [hc] \cup [hf] \cup [hg] \right].
\]
It is indeed possible to generalize the circuit for an arbitrary length binary function co-domain. In particular, the length of the output binary string will be determined by the logarithm of the cardinality of the alphabet considered (for example, two bits for a 4-elements alphabet). Moreover, to each parity function subspace corresponds a unique input to be fed into the circuit shown before. The Hadamard gate transforms each qubit of the input binary string into the state $|+\rangle$ or $|-\rangle$ depending on the value of the qubit. For the generic input $|0\ldots1\rangle$, we have

$$U_f(|z\rangle \otimes |+\rangle \otimes \ldots \otimes |+\rangle \otimes |-\rangle) = (-1)^{p(f(x))}|z\rangle \otimes |+\rangle^\otimes n+1.$$  

If $k = 4$, for $p^{adj}$, the set of parity balanced words is

$$C^{adj}_4(a, b, c, d, e, f, g, h) = [aaaa] \cup [bbbb] \cup [cccc] \cup [dddd] \cup [eeee] \cup [ffff] \cup [gggg] \cup [hhhh] \cup [aaad] \cup [aadd] \cup [dddd] \cup [aaa] \cup [aaaf] \cup [afff] \cup [aaag] \cup [aggg] \cup [dddf] \cup [ddff] \cup [ffgg] \cup [fsgs] \cup [bbcc] \cup [bcce] \cup [bbbe] \cup [bbee] \cup [beee] \cup [ceee] \cup [eeee] \cup [bbbb] \cup [bbhh] \cup [bhhh] \cup [cccc] \cup [chhh] \cup [ehhh] \cup [eenn] \cup [eeph] \cup [eeehh] \cup [eehh] \cup [ehee] \cup [eehh] \cup [eenh] \cup [ehhh];$$

while the set of parity balanced words is

$$B^{adj}_4(a, b, c, d, e, f, g, h) = [aabb] \cup [aacc] \cup [aeee] \cup [aahh] \cup [ddbb] \cup [ddcc] \cup [ddeee] \cup [ddhh] \cup [ffbb] \cup [ffcc] \cup [ffee] \cup [ffhh] \cup [ggbb] \cup [ggcc] \cup [ggee] \cup [ghhh] \cup [adbc] \cup [afce] \cup [agbe] \cup [agce] \cup [adce] \cup [adbc] \cup [adhe] \cup [agch] \cup [afce] \cup [afch] \cup [adbe] \cup [adbb] \cup [afbc] \cup [afbb] \cup [agbh] \cup [ageh] \cup [afeh] \cup [afbe].$$
For $p^{abcd}$, we have

$$C_4^{abcd}(a, b, c, d, e, f, g, h) = [aaaa] \cup [bbbb] \cup [cccc] \cup [dddd] \cup [eeee]$$
$$\cup [ffff] \cup [gggg] \cup [hhhh] \cup [aaab] \cup [aabb]$$
$$\cup [abbb] \cup [aaac] \cup [aacc] \cup [accc] \cup [aaad]$$
$$\cup [aadd] \cup [addd] \cup [addd]$$

and

$$B_4^{abcd}(a, b, c, d, e, f, g, h) = [aeee] \cup [aaef] \cup [aagg] \cup [aahh] \cup [bbee]$$
$$\cup [bbff] \cup [bbgg] \cup [bbhh] \cup [cccc] \cup [fccc]$$
$$\cup [ccgg] \cup [cchh] \cup [ddce] \cup [ddf] \cup [ddgg]$$
$$\cup [ddhh] \cup [abef] \cup [abeg] \cup [abeh] \cup [acef]$$
$$\cup [aceg] \cup [aceh] \cup [adef] \cup [adeg] \cup [adeh]$$

The same reasoning carried on for a four-letter alphabet can be applied to form the set of words

$$\mathcal{F}_k^x(A) = C_k^x(A) \cup B_k^x(A)$$

and the relative intersections. A potential generalization could arise in the context of error correcting codes. This could be based on introducing an encoding in which the letters of the alphabet are associated to the codewords of a subspace quantum error correcting code. A form of parity could be defined by considering the remaining subspaces.

4 Applications to the word problem in groups

Let $\{a, b, c = B, d = A\}$ be a paired alphabet, where $A$ represents $a^{-1}$ and $B$ represents $b^{-1}$. We first consider words of length 2. Parity constant words are “character constant”, i.e. consist of only one letter, whether it be lower or upper case. Parity balanced words are “character balanced”. The words corresponding to the parity constant case are $aa, aA, bb, bB, BB, Aa, AA$. Those corresponding to the parity balanced case are $ab, aB, ba, bA, Ba, BA, Ab, AB$. The words $w$ in the first list all satisfy $w \in \langle a \rangle \cup \langle b \rangle$ (in fact
we have \( w \in \langle a^2 \rangle \cup \langle b^2 \rangle \), whereas those \( w \) in the second list all satisfy \( w \notin \langle a \rangle \cup \langle b \rangle \). Thus, for words of length 2, we can determine with a single measurement whether or not \( w \in \langle a \rangle \cup \langle b \rangle \).

If \( x = 01 \) then the \( x \)-constant words are \( aa, ab, ba, bb, BA, BB, AA, AB \) and the \( x \)-balanced words are \( aB, aA, bB, bA, Aa, Ab, Ba, Bb \). So, \( 01 \)-constant and \( 01 \)-balanced may be thought of as “case constant” and “case balanced” where the case can be upper or lower. For example, a commutator word (reduced or not) is always case balanced. “Case constant” and “case balanced” are properties of \( \overline{w} \), rather than \( w \). This is not the case for “parity constant” and “parity balanced”.

If \( x = 10 \) then the \( x \)-constant words are \( aa, aB, bb, Ba, BB, Ab, AA \) and the \( x \)-balanced words are \( ab, aA, ba, bB, Bb, BA, AB, Aa \). This does not seem to have any nice interpretation. The 10-balanced corresponds to the cyclic subgroup generated by \( ab \) and the 11-constant set solves a problem of union of subgroup membership for \( \langle a \rangle \cup \langle b \rangle \). The elements represented by these words are depicted on the following Cayley graph portions:

Note that \( w \) is 11-constant but not 01-constant; also \( w \) is 11-constant and not 10-constant. Then \( w = \mathcal{F}_{11}^2 \). This gives a method of solving the word problem for words of length 2 using two quantum queries.

For \( k = 2 \), if we are promised that \( w \) is \( x \)-feasible then the quantum query complexity of the property “is \( w \) trivial?” seems to be 2. But this is not a reduction in complexity from the classical case. However, there is hope that an analogous method might be an improvement in quantum query complexity for longer words. We have the following:

**Proposition 2** For all \( n \), if we are promised that the word \( w \) of length \( 2^n \) is 11-feasible then the quantum query complexity of the property “Does \( w \) represent an element of \( \langle a \rangle \cup \langle b \rangle \)?” is 1.

This is directly analogous to the Deutsch-Josza algorithm, and the proof is the same. It is unclear how to extend this approach for the word problem beyond two letters. Here are examples of two groups where we require different promises:
Proposition 3  Consider the free abelian group $G = \langle a, b \mid ab = ba \rangle$. Let $w$ be a four-letter word in $A$ which is in $F_{2}^{11} \cap F_{2}^{01} \cap F_{2}^{10}$. Then the quantum query complexity of the question “Does $w$ represent the trivial element of $G$?” is at most 3.

Proof. The first query asks whether $w \in C_{2}^{01}$ or $w \in B_{2}^{01}$. If the former is true then $w$ is not trivial so stop. If $w \in B_{2}^{01}$ then proceed to the second query, which is whether $w \in C_{2}^{11}$ or $w \in B_{2}^{11}$. If the former is true then $w$ is trivial so stop. Otherwise we know that $w \in B_{2}^{11} \cap B_{2}^{01}$ and we may proceed to the third query. There are two possibilities. The first possibility is that we have a word with two $A$s and two $B$s or a word with two $a$s and two $b$s. That is, $w$ is a cyclic rotation of $(AAbb)^{\pm 1}$. The second possibility is that we have one each of $A, b, a$ and $B$. In the first case, $w$ is nontrivial and in $C_{2}^{10}$; in the second case, $w$ is trivial and in $B_{2}^{10}$. So our third query is whether $w \in C_{2}^{10}$ or $w \in B_{2}^{10}$; this solves the word problem provided $w$ is as promised. ■

It is indeed possible to generalize this theorem to the 8-letters alphabet introduced earlier, by considering the four-paired alphabet $\{a, b, c, d, e = D, f = C, g = B, h = A\}$, where the upper-case $A, B, C, D$ letters represent respectively $a^{-1}, b^{-1}, c^{-1}, d^{-1}$. In particular, we have the following statement:

Proposition 4  Consider the free group $G = \langle a, b, c, d \mid abcd = dcba \rangle$. Let $w$ be a 8-letter word in $A$ which is in $F_{3}^{abcd} \cap F_{3}^{bed} \cap F_{3}^{adeh}$. Then the quantum query complexity of the question “Does $w$ represent the trivial element of $G$?” is at most 3.

Proof. The first query asks whether $w \in C_{3}^{abcd}$ or $w \in B_{3}^{abcd}$. If the former is true then $w$ is not trivial so stop. If $w \in B_{3}^{abcd}$ then proceed to the second query, which is whether $w \in C_{3}^{adf}g$ or $w \in B_{3}^{adf}g$. If the former is true then $w$ is trivial so stop. Otherwise we know that $w \in B_{3}^{adf}g \cap B_{3}^{bed}$ and we may proceed to the third query. There are two possibilities. The first possibility is that we have a word with two $A$s two $D$s and two $a$s and two $d$s or a word with two $C$s two $B$s, two $c$s and two $b$s. That is, $w$ is a cyclic rotation of $(AADDaadd)^{\pm 1}$ or $(BBCCbbcc)^{\pm 1}$. The second possibility is that we have one each of $A, b, a, B, C, d, c$, and $D$. In the first case, $w$ is nontrivial and in $C_{3}^{adeh}$; in the second, $w$ is trivial and in $B_{3}^{abcd}$. Our third query is whether $w \in C_{3}^{adeh}$ or $w \in B_{3}^{abcd}$; this solves the word problem provided $w$ is as promised. ■

Looking at the first two queries it seems possible to generalize this result for every paired alphabet of dimension $2n-1$ and words of length $2^{n}$, by defining parities based on the even number of ones, like $p^{abcd}$. This is always possible because of the equipartition of the binary strings with respect to Hamming weight. The last parity required is the one used to identify words that are cyclic permutations of elements of the alphabet, for example, $p^{adeh}$. It does not seem easy to distinguish between trivial and nontrivial four-letter words in the free group of rank 2 using less than 4 quantum queries. However, the first indication
that classical query complexity can be improved upon in a nonabelian finitely presented group is the following:

**Proposition 5** Consider the group presented by $G = \langle a, b \mid a^2 = b^2 \rangle$. Suppose we are given a word $w$ of length $4$ in $A$ such that $w \in F_2^{11} \cap F_2^{01}$. Then the quantum query complexity of the question “Does $w$ represent the trivial element of $G$?” is at most $3$.

**Proof.** The first two queries are as in the proof of the last proposition. So we can assume that if we do not already know whether or not $w$ is trivial, $w \in B_2^{11} \cap B_2^{01}$ and we may proceed to the third query. For this, we construct a “syllable function”

$$f : \{0, 1\} \rightarrow \{aa, ab, aA, aB, bB, Ba, Ba, Ab, BA, AA, Ab, AB, AA\}.$$

It maps $AA, BB, Aa, aA, Ab, AB, ab, aB, bB, Ba, Ba, aa, bb$ to $0$ and $Bb, bB, BA, Ba, Ba, ba, bb$ to $1$. Note that, since $w \in B_2^{11} \cap B_2^{01}$, $w$ is either a cyclic rotation of $(AAbb)^{\pm 1}$ or $w$ is an anagram of $AaBb$. Words in the first case are all trivial, because $a^2 = b^2$ is a relation in $G$, and these words are all balanced under the syllable function. Words in the second case are nontrivial if and only if they are nontrivial commutators. Commutators are constant under the syllable function. Words in the second case which are trivial (i.e., not commutators) are all balanced under the syllable function. Thus a third query of “is $w$ syllable-balanced or syllable-constant” will complete the solution of the word problem. The following table lists all 0-syllabs and 1-syllabs:

| 0-syllabs | 1-syllabs |
|-----------|-----------|
| $AA$      | $aa$      |
| $BB$      | $bb$      |
| $Aa$      | $Bb$      |
| $aA$      | $bB$      |
| $Ab$      | $bA$      |
| $AB$      | $BA$      |
| $ab$      | $ba$      |
| $aB$      | $Ba$      |

While the group $G$ in the last proposition is nonabelian, it can be shown to have a free abelian subgroup of rank $2$ and index $4$; it is an extension of $\mathbb{Z} \oplus \mathbb{Z}$ by the Klein 4-group.

**Proposition 6** Consider the group presented by $G = \langle a, b, c, d \mid a^2b^2 = b^2a^2 \rangle$. Suppose we are given a word $w$ of length $8$ in $A$ such that $w \in F_3^{adj} \cap F_3^{bcd}$. Then the quantum query complexity of the question “Does $w$ represent the trivial element of $G$?” is at most $3$.

**Proof.** The first two queries are as in the proof of the last proposition. So we can assume that if we do not already know whether or not $w$ is trivial, $w \in B_3^{adj} \cap B_3^{bcd}$ and we may
proceed to the third query. For this, we construct an extended syllable function whose output has a cardinality of $2^n - 1$. Some of the elements are listed below:

$$f : \{0, 1\} \rightarrow \{aaaa, bbbb, BBBBB, AAAA, aaab, aabb, abbb, aaaaB, aaBB, abBB, baBB, baaa, bAAA, aBBB, bBBB, bBBb, BBba, bbAb, baaA, bBBa, BBaa, Baaa, BbbB, BbBA, BBAA, BAAA, AAAAa, AAaa, Aaaa, AABb, AAbb, AAAB, ABaa, ABBa, ABAB, AABB, ABBA, ...\}$$

Examples of this map are

AAAA, BBBB, Abb, Aaaa, aBBB, aaBB, aaaaB, abAB, ABab, ABab, AAB, ABBB, ABB, ABA, ... to 0

and

aaaa, bbbb, Bbbb, BBbb, bBBB, bAAA, BAAA, Aaaa, baaa, bAAA, abbb, bbaa, abbb, bbba, bbaa ... to 1.

Note that since $w \in B_3^{\text{gdfg}} \cap B_3^{\text{abcd}}$, $w$ is either a cyclic rotation of $(AABBaabb)^{\pm 1}$ or $w$ is an anagram of $AAaaBBbb$. Words in the first case are all trivial, because $a^2b^2 = b^2a^2$ is a relation in $G$, and these words are all balanced under the syllable function. Words in the second case are nontrivial if and only if are nontrivial sequence of letters, that is not commutator-like sequence with respect to the presentation. Words in the second case which are trivial (i.e., not trivial sequence) are all balanced under the extended syllable function. Thus a third query of “is $w$ syllable-balanced or syllable-constant” will complete the solution of the word problem.

The same considerations can be made by looking at different sets of generators or relations like $G = \langle a, b, c, d \mid c^2d^2 = d^2c^2 \rangle$ and $G = \langle a, b, c, d \mid b^2c^2 = c^2b^2 \rangle$. It is important to notice that all the alternate sets of relations five groups isomorphic to the group considered in Proposition 5. To see this, it is sufficient to relabel the generators. The relation in $G$ is in fact very general and it is possible to obtain the same result with a whole family of similar relations. This can be done by varying the parity function used for the queries, choosing the presentation accordingly. Moreover such a group is a free group of rank 2 with $G = \langle a, b, c, d \mid a^2b^2 = b^2a^2 \rangle$. It is simple to see that since the other two generators, $c$ and $d$, are not involved in the proof, it is possible to take the free product of $G$ with any free group and get to the same conclusion. In particular it is possible to extend the free product with any group and see the invariance of those three quantum queries under free products.

Notice that the choice of some particular kind of relations and an higher number of generators in the setting of the problem may increase the number of queries required. The reason of this is the exponential growth in the number of permutations, in particular, in those cases where splitting the words in parity balanced and parity constant does not help. Generalize to other different sets of generators and possibly for free products, and limiting to commutator words might give interesting promises.
5 Conclusions

We have extended the original Deutsch-Josza algorithm to functions of arbitrary length binary output, and we have introduced a more general concept of parity. The setting described allows us to consider maps between binary strings and alphabet of various length. In the quantum regime, some instances of the word problem for small alphabets and free groups, can be solved in a reduced number of queries with respect to the deterministic classical case. Extensions to more general groups and presentations may give interesting promises. It is not clear that the success of procedures similar to the ones discussed here depends or not on the group considered. We have seen that the $X$-parity of a function, for some fixed set of binary strings $X$, can be determined with the Deutsch-Jozsa procedure, when $X$ consists of an appropriate subgroup (in our examples, a subgroup of index two). It has to be verified that the toy problems considered here can be re-interpreted as instances of the Abelian Hidden Subgroup Problem. In such a case, the problems could be solved with a slightly different technique, but with essentially the same number of oracle queries.

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