ON NON-UNIQUENESS OF RECOVERING STURM–LIOUVILLE OPERATORS WITH DELAY AND THE NEUMANN BOUNDARY CONDITION AT ZERO

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Abstract. As is known, for each fixed \( \nu \in \{0,1\} \), the spectra of two operators generated by \(-y''(x) + q(x)y(x-a)\) and the boundary conditions \(y^{(j)}(0) = y^{(j)}(\pi) = 0\), \( j = 0, 1 \), uniquely determine the complex-valued square-integrable potential \(q(x)\) vanishing on \((0,a)\) as soon as \(a \in [2\pi/5, \pi)\). Meanwhile, it actually became the main question of the inverse spectral theory for Sturm–Liouville operators with constant delay whether the uniqueness holds also for smaller values of \(a\). Recently, a negative answer was given by the authors [Appl. Math. Lett. 113 (2021) 106862] for \(a \in [\pi/3, 2\pi/5)\) in the case \(\nu = 0\) by constructing an infinite family of iso-bispectral potentials. Moreover, an essential and dramatic reason was established why this strategy, generally speaking, fails in the remarkable case when \(\nu = 1\). Here we construct a counterexample giving a negative answer for \(\nu = 1\), which is an important subcase of the Robin boundary condition at zero. We also refine the former counterexample for \(\nu = 0\) to \(W_2^1\) -potentials.

Key words: Sturm–Liouville operator with delay, inverse spectral problem, iso-bispectral potentials

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1. Introduction

In recent years, there appeared an interest in inverse spectral problems for Sturm–Liouville-type operators with varying domain, see, e.g., papers [1][19], among which a big part is devoted to operators with delay. For \(j = 0, 1\), denote by \(\{\lambda_{n,j}\}_{n \geq 1}\) the spectrum of the boundary value problem

\[-y''(x) + q(x)y(x-a) = \lambda y(x), \quad 0 < x < \pi,\]

\[U(y) = y^{(j)}(0) = 0\]  \( (1)\)

with delay \(a \in (0, \pi)\) and a complex-valued potential \(q(x) \in L_2(0, \pi)\) such that \(q(x) = 0\) on \((0,a)\), while \(U(y) = y(0)\) or \(U(y) = y'(0) - hy(0)\), \(h \in \mathbb{C}\). Such cases of \(U(y)\) correspond to Dirichlet and Robin boundary conditions at zero, to which we will refer as Case 1 and case Case 2, respectively.

Inverse Problem 1. Given the spectra \(\{\lambda_{n,0}\}_{n \geq 1}\) and \(\{\lambda_{n,1}\}_{n \geq 1}\), find the potential \(q(x)\).

Alternatively, one can consider the case of Robin boundary conditions also at the right end:

\[U(y) = y'(\pi) + H_j y(\pi) = 0, \quad j = 0, 1, \quad H_0, H_1 \in \mathbb{C}, \quad H_0 \neq H_1,\]  \( (3)\)

which, however, can be easily reduced to conditions \(\nu \). Moreover, in the cases of Robin boundary conditions, the coefficients \(h\) and \(H_0\), \(H_1\) are uniquely determined by the two spectra (see [15]).

Various aspects of Inverse Problem 1 were studied in [1][2][4][6][8][14][16][19] and other works. In particular, it is well known that the two spectra uniquely determine the potential as soon as \(a \geq \pi/2\). Moreover, the inverse problem is overdetermined (see [5]). For \(a < \pi/2\), the dependence of the characteristic function of any problem of the form \(\nu \) on the potential is nonlinear. It became actually the main question of the inverse spectral theory for the operators with constant delay whether the uniqueness holds also for small \(a\). Recently, a positive answer when \(a \in [2\pi/3, \pi/2]\) was given in [8] for Case 1 and independently in [9] for Case 2. However, recent authors’ paper [19] gave a negative answer in Case 1 as soon as \(a \in [\pi/3, 2\pi/5)\). Specifically, for each such \(a\), we constructed an infinite family of different iso-bispectral potentials \(q(x)\), i.e. for which both problems consisting of \(\nu \) and \(\nu \) possess one and the same pair of spectra. This appeared quite unexpected because of the inconsistence with Borg’s classical uniqueness result for \(a = 0\) [20], and also in light of recent paper [15] announcing the uniqueness for \(a \in [\pi/3, \pi)\) in Case 2 for boundary conditions [3]. In [19], we also established an essential and dramatic reason why the idea of that counterexample, generally speaking, fails in Case 2 (see Remark 2 in [19]).

In the present paper, we return to Case 2 and construct a counterexample giving a negative answer for the Neumann boundary condition at zero (i.e. when \(h = 0\), which, unfortunately, refutes the uniqueness

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Theorem in [15] for \( a \in [\pi/3, 2\pi/5] \). For this purpose, we establish Theorem 1 (see the next section) first, which reduces finding a counterexample in Case 2 to constructing a Hermitian integral operator of a special form possessing an eigenfunction with the mean value zero. Even though the existence of such an operator was highly believable, finding its concrete example appeared to be a quite difficult task. After a series of computational experiments we constructed several numerical examples, one of which fortunately admitted a precise elementary implementation (see Proposition 1).

This new non-uniqueness result along with the one in [19] changes the further strategy of studying inverse problems for the operators with delay (see also Remark 1 in [19]). In particular, there appears the relevance of finding various conditions on the class of potentials that would guarantee the uniqueness of recovering \( q(x) \). That is especially important for justifying constructive procedures for solving Inverse Problem 1 when \( a < 2\pi/5 \), otherwise the corresponding algorithms in [11] and [15] become indefinite. By virtue of [13], as such a condition one can impose holomorphy of \( q(x) \) on an appropriate part of \((a, \pi)\).

Finally, we note that both our counterexamples involve discontinuous potentials. So it is also relevant to investigate the possibility of constructing iso-bispectral potentials in \( W^k_2[0, \pi] \) with \( k \in \mathbb{N} \) so large as possible. Here we construct such potentials in Case 1 for \( k = 1 \) and \( a \in (\pi/3, 2\pi/5) \) (see Theorem 2).

2. The main results

For \( \nu, j \in \{0, 1\} \), denote by \( L_{\nu,j}(a, q) \) the eigenvalue problem for equation (1) under the boundary conditions
\[
y^{(\nu)}(0) = y^{(j)}(\pi) = 0.
\]
(4)

Fix \( a \in [\pi/3, 2\pi/5] \). Following the main idea of the work [19], we consider the integral operator
\[
M_h f(x) = \int_{-\frac{\pi}{2}}^{\pi} K_h \left(x + t - \frac{a}{2}\right) f(t) \, dt, \quad \frac{3a}{2} < x < \pi - a,
\]
where \( K_h(x) = \int_{x}^{\pi} h(\tau) \, d\tau \),
(5)

with a nonzero real-valued function \( h(x) \in L_2(5a/2, \pi) \). Thus, \( M_h \) is a nonzero compact Hermitian operator in \( L_2(3a/2, \pi - a) \) and, hence, it has at least one nonzero eigenvalue \( \eta \).

Further, fix \( \nu \in \{0, 1\} \) and put \( h_{\nu}(x) := (-1)^\nu h(x)/\eta \). Then \((-1)^\nu\) is an eigenvalue of the operator \( M_{h_{\nu}} \). Let \( e_{\nu}(x) \) be some related eigenfunction, i.e.
\[
M_{h_{\nu}} e_{\nu}(x) = (-1)^\nu e_{\nu}(x), \quad \frac{3a}{2} < x < \pi - a.
\]
(6)

Consider the one-parametric family of potentials \( B_{\nu} := \{q_{a, \nu}(x)\}_{a \in \mathbb{C}} \) determined by the formula
\[
q_{a, \nu}(x) = \begin{cases}
0, & x \in \left(0, \frac{3a}{2}\right) \cup \left(\pi - a, 2a\right) \cup \left(\pi - \frac{a}{2}, \frac{5a}{2}\right), \\
\alpha e_{\nu}(x), & x \in \left(\frac{3a}{2}, \pi - a\right), \\
-\alpha K_{h_{\nu}} \left(x + \frac{a}{2}\right) \int_{-\frac{\pi}{2}}^{\frac{x-a}{2}} e_{\nu}(t) \, dt, & x \in \left(2a, \frac{\pi - a}{2}\right), \\
h_{\nu}(x), & x \in \left(\frac{5a}{2}, \pi\right).
\end{cases}
\]
(7)

In [19], it was established that, for \( j = 0, 1 \), the spectrum of the problem \( L_{0,j}(a, q_{a,0}) \) is independent of \( a \) for any function \( h(x) \) conditioned above. This means that \( B_0 \) is an iso-bispectral set (of potentials) for these two problems, i.e. the solution of Inverse Problem 1 in Case 1 is not unique.

Moreover, in [19], it was noted that acting in the analogous way but for the problems \( L_{1,j}(a, q) \), \( j = 0, 1 \), would lead to the family of potentials \( B_1 \). However, an essential reason was established why \( B_1 \), generally speaking, does not form an iso-bispectral set for these two problems (see also Remark 1 in Section 3). In this paper, we find a concrete example when it is. We begin with the following theorem.

**Theorem 1.** For \( j = 0, 1 \), the spectrum of the problem \( L_{1,j}(a, q_{a,1}) \) is independent of \( a \) as soon as
\[
\int_{-\frac{\pi}{2}}^{\frac{\pi-a}{2}} e_1(x) \, dx = 0.
\]
(8)
Thus, the problem of constructing iso-bispectral potentials for the problems $\mathcal{L}_{1,0}(a, q)$ and $\mathcal{L}_{1,1}(a, q)$ is reduced to the question of finding a function $h(x) \in L_2(5a/2, \pi)$ such that the operator $M_h$ has at least one eigenfunction possessing the zero mean value but related to a nonzero eigenvalue. The answer to this question is given by the following assertion.

**Proposition 1.** Put

$$h_1(x) := \frac{6\pi^2}{(2\pi - 5a)^2} \cos \frac{\pi \sqrt{10(\pi - x)}}{2\pi - 5a}, \quad e_1(x) := \cos \frac{2\pi(2x - 3a)}{2\pi - 5a} + \cos \frac{\pi(2x - 3a)}{2\pi - 5a}. \tag{9}$$

Then relation (3) for $\nu = 1$ as well as equality (3) are fulfilled.

Theorem 1 and Proposition 1 imply that the family $B_1$ constructed by using the functions $h_1(x)$ and $e_1(x)$ that are determined by (9) consists of iso-bispectral potentials for the problems $\mathcal{L}_{1,0}(a, q)$ and $\mathcal{L}_{1,1}(a, q)$. Thus, Inverse Problem 1 is not uniquely solvable also in Case 2.

Finally, we construct iso-bispectral potentials in $W_2^1[0, \pi]$ in Case 1. For this purpose, we consider

$$e_0(x) := \sin \frac{2\pi(2x - 3a)}{2\pi - 5a} + 2\sin \frac{\pi(2x - 3a)}{2\pi - 5a} \tag{10}$$

and introduce the family of potentials $B_0 := \{q_\alpha(x)\}_{\alpha \in \mathbb{C}}$ determined by the formula

$$q_\alpha(x) = \begin{cases} 
0, & x \in \left[0, \frac{3a}{2}\right) \cup \left(\pi - a, 2a\right), \\
\alpha e_0(x), & x \in \left(\frac{3a}{2}, \pi - a\right), \\
-\alpha K_{h_0}\left(x + \frac{a}{2}\right) \int_{\frac{a}{2}}^x e_0(t) \, dt, & x \in \left[2a, \pi - \frac{a}{2}\right), \\
g(x), & x \in \left(\pi - \frac{a}{2}, \frac{5a}{2}\right), \\
h_0(x), & x \in \left(\frac{5a}{2}, \pi\right],
\end{cases} \tag{11}$$

with $h_0(x) = h_1(x)$, where $h_1(x)$ is, in turn, determined in (9), while $g(x)$ is an arbitrary fixed function in $W_2^1[\pi - a/2, 5a/2]$ obeying the boundary conditions

$$g\left(\pi - \frac{a}{2}\right) = 0, \quad g\left(\frac{5a}{2}\right) = \frac{6\pi^2}{(2\pi - 5a)^2} \cos \frac{\pi \sqrt{10}}{2}. \tag{12}$$

**Theorem 2.** For $j = 0, 1$, the spectrum of the boundary value problem $\mathcal{L}_{0,j}(a, q_\alpha)$ is independent of $\alpha$. Moreover, the inclusion $B_0 \subset W_2^1[0, \pi]$ holds as soon as $a \in (\pi/3, 2\pi/5)$.

The proofs of Theorems 1 and 2 as well as Propositions 1 are provided in the next section.

### 3. The proofs

Denote by $y_0(x, \lambda)$ and $y_1(x, \lambda)$ the unique solutions of equation (11) under the initial conditions $y^{(j)}_\nu(0, \lambda) = \delta_{\nu,j}$, $\nu, j = 0, 1$, where $\delta_{\nu,j}$ is the Kronecker delta. For any pair of $\nu, j \in \{0, 1\}$, eigenvalues of the boundary value problem $\mathcal{L}_{\nu,j}(a, q)$ with account of multiplicity coincide with zeros of the entire function $\Delta_{\nu,j}(\lambda) = y^{(j)}_{\nu,\nu}(\pi, \lambda)$, which is called characteristic function of the problem $\mathcal{L}_{\nu,j}(a, q)$. Thus, the spectrum of any problem $\mathcal{L}_{\nu,j}(a, q)$ does not depend on $q(x) \in B$ for some subset $B \subset L_2(0, \pi)$ as soon as neither does the corresponding characteristic function $\Delta_{\nu,j}(\lambda)$. Put $\rho^2 = \lambda$ and denote

$$\omega := \int_a^\pi q(x) \, dx. \tag{13}$$

Before proving Theorem 1, we provide necessary information from [19]. For $\nu, j = 0, 1$, we have

$$\Delta_{\nu,\nu}(\lambda) = (-\lambda)^\nu \left(\sin \frac{\rho \pi}{\rho} - \cos \frac{\rho(\pi - a)}{2\lambda} \int_a^\pi w_\nu(x) \cos \rho(\pi - 2x + a) \, dx\right), \tag{14}$$

$$\Delta_{\nu,j}(\lambda) = \cos \rho \pi + \omega \frac{\sin \rho(\pi - a)}{2\rho} + \frac{(\rho^2)\nu}{2\rho} \int_a^\pi w_\nu(x) \sin \rho(\pi - 2x + a) \, dx, \quad \nu \neq j, \tag{14}$$
where the functions \( w_\nu(x) \) are determined by the formula

\[
w_\nu(x) = \begin{cases} 
q(x), & x \in (a, \frac{3a}{2}) \cup \left( \pi - \frac{a}{2}, \pi \right), \\
q(x) + Q_\nu(x), & x \in \left( \frac{3a}{2}, \pi - \frac{a}{2} \right),
\end{cases}
\]

(15)

while

\[
Q_\nu(x) = \int_a^{x - \frac{a}{2}} q(t) dt \int_{x + \frac{a}{2}}^\pi q(\tau) d\tau - (-1)^\nu \int_a^{x - \frac{a}{2}} q(t) dt \int_{x + \frac{a}{2}}^\pi q(\tau) d\tau.
\]

(16)

**Remark 1.** As was established [19], the difference between the cases \( \nu = 0 \) and \( \nu = 1 \) is as follows. Since the functions \( \Delta_{\nu,\lambda}(\lambda) \) are entire in \( \lambda \), the first representation in (14) for \( \nu = 0 \) implies

\[
\omega = \int_0^\pi w_0(x) dx,
\]

(17)

which can also be checked directly using (15) and (16) for \( \nu = 0 \). Thus, for \( \nu = 0 \), the iso-bispectrality of \( B \) requires only \( w_0(x) \)'s independence of \( q(x) \). However, for \( \nu = 1 \), there is no relation analogous to (17). In other words, the constant \( \omega \) is not determined by \( w_1(x) \). Thus, both functions \( \Delta_{1,0}(\lambda) \) and \( \Delta_{1,1}(\lambda) \) may depend on \( q(x) \in B \) even when \( w_1(x) \) does not.

Let \( q(x) = 0 \) on \( (a, 3a/2) \). Hence, formulae (15) and (16) give

\[
w_\nu(x) = \begin{cases} 
0, & x \in \left( a, \frac{3a}{2} \right), \\
q(x) - (-1)^\nu M_h q(x), & x \in \left( \frac{3a}{2}, \pi - a \right), \\
q(x), & x \in \left( \pi - a, 2a \right), \\
q(x) + K_h \left( x + \frac{a}{2} \right) \int_{x + \frac{a}{2}}^{-\frac{a}{2}} q(t) dt, & x \in \left( 2a, \pi - \frac{a}{2} \right), \\
q(x), & x \in \left( \pi - \frac{a}{2}, \frac{5a}{2} \right), \\
h(x), & x \in \left( \frac{5a}{2}, \pi \right),
\end{cases}
\]

(18)

where \( h(x) = q(x) \) on \( (5a/2, \pi) \), while \( M_h \) and \( K_h(x) \) are determined by (15).

**Proof of Theorem 1.** Substituting \( q(x) = q_{a,1}(x) \) into (18) for \( \nu = 1 \), where \( q_{a,1}(x) \) is determined by (7) with \( \nu = 1 \), and taking (3) for \( \nu = 1 \) into account, we arrive at

\[
w_1(x) = 0, \quad a < x < \frac{5a}{2}, \quad w_1(x) = h_1(x), \quad \frac{5a}{2} < x < \pi.
\]

Thus, the function \( w_1(x) \) is independent of \( \alpha \). Hence, it remains to prove that so is also the value \( \omega \) determined by formula (13) with \( q(x) = q_{a,1}(x) \). Integrating the third line in (7) for \( \nu = 1 \), we get

\[
\mathcal{I} := \int_{2a}^{\pi - \frac{a}{2}} K_{h_1} \left( x + \frac{a}{2} \right) dx \int_{\frac{\pi}{2}}^{\pi - \frac{a}{2}} e_1(t) dt = \int_{2a}^{\pi - \frac{a}{2}} K_{h_1} \left( x + \frac{a}{2} \right) dx \int_{\frac{\pi}{2}}^{\pi - \frac{a}{2}} e_1(x + a - t) dt.
\]

Changing the order of integration and then the internal integration variable, we calculate

\[
\mathcal{I} = \int_{\frac{\pi}{2}}^{\pi - \frac{a}{2}} dx \int_{x + \frac{\pi}{2}}^{\pi - \frac{a}{2}} K_{h_1} \left( t + \frac{a}{2} \right) e_1(t + a - x) dt = \int_{\frac{\pi}{2}}^{\pi - a} dx \int_{\frac{\pi}{2}}^{\pi - \frac{a}{2}} K_{h_1} \left( t + \frac{a}{2} \right) e_1(t) dt,
\]

which along with the first equality in (5) as well as (6) for \( \nu = 1 \) and (8) implies

\[
\mathcal{I} = \int_{\frac{\pi}{2}}^{\pi - a} M_{h_1} e_1(x) dx = - \int_{\frac{\pi}{2}}^{\pi - a} e_1(x) dx = 0.
\]
Thus, according to (7) for $\nu = 1$, the assumption (8) of the theorem gives
\[
\omega = \int_{a}^{\pi} q_{\alpha,1}(x) \, dx = \int_{\pi}^{\frac{\pi}{5}} h_{1}(x) \, dx,
\]
i.e. the value $\omega$ does not depend on $\alpha$, which finishes the proof.

For shortening the remaining proofs, we provide the following auxiliary assertion.

**Proposition 2.** For each fixed $\nu \in \{0, 1\}$, relation (10) is equivalent to the relation
\[
m_{\chi_{\nu}}(x) = (-1)^{\nu} m_{\chi}(x), \quad 0 < \xi < 1,
\]
as soon as
\[
e_{\nu}(\xi) = e_{\nu}\left(\frac{3a}{2} + \left(\frac{\pi}{2} - \frac{5a}{2}\right)\xi\right), \quad \chi_{\nu}(\xi) = \left(\frac{\pi}{2} - \frac{5a}{2}\right)^{2} \nu \left(\frac{\pi}{2} - \frac{5a}{2}\right) \xi.
\]

**Proof.** Making in (11) the change of variable $\xi := (2x - 3a)/A$, where $A = 2\pi - 5a$, we obtain
\[
M_{h_{\nu}}\left(\frac{3a}{2} + \left(\frac{\pi}{2} - \frac{5a}{2}\right)\xi\right) = (-1)^{\nu} e_{\nu}(\xi), \quad 0 < \xi < 1.
\]
Using (5) and successively making the changes $\eta := (2t - 3a)/A$ and $\theta := (2\pi - 5a)/A$, we get
\[
(-1)^{\nu} e_{\nu}(\xi) = \left(\frac{\pi}{2} - \frac{5a}{2}\right) \int_{0}^{1} e_{\nu}(\eta) \, d\eta \int_{\pi}^{\pi} h_{\nu}(\tau) \, d\tau = \int_{0}^{1} e_{\nu}(\eta) \, d\eta \int_{\pi}^{1} \chi_{\nu}(\theta) \, d\theta,
\]
which coincides with (19).

**Proof of Proposition 1.** Let us start with (8), which can be checked by the direct substitution:
\[
\int_{\frac{\pi}{5}}^{\pi-a} e_{1}(x) \, dx = \frac{2\pi - 5a}{2\pi} \left(\frac{1}{2} \sin 2\pi\left(\frac{2x - 3a}{2\pi - 5a}\right) + \sin \left(\frac{2\pi - 5a}{2\pi - 5a}\right)\right)\bigg|_{x=\frac{\pi}{5}}^{\pi-a} = 0.
\]
According to Proposition 2, it remains to prove relation (19) for $\nu = 1$ with the functions
\[
\epsilon_{1}(\xi) = \cos \pi \xi + \cos 2\pi \xi, \quad \chi_{1}(\theta) = \frac{3\pi^{2}}{2} \cos \frac{\pi\sqrt{10}(1 - \theta)}{2},
\]
which are determined by (9) and (20) for $\nu = 1$. Indeed, it is easy to calculate
\[
m_{\chi_{1}}(\xi) = \frac{3\pi}{2\sqrt{10}}(A_{1} + A_{2}), \quad A_{j} = 2 \int_{0}^{1} \cos \pi j \eta \cdot \sin \frac{\pi\sqrt{10}(1 - \xi - \eta)}{2} \, d\eta
\]
\[
= \frac{1}{\sqrt{10}} \int_{0}^{1} \sin \frac{\pi\sqrt{10}(1 - \xi)}{2} - \pi \left(\frac{\sqrt{10}}{2} + (-1)^{j}j\right) \eta \, d\eta = \frac{1}{\sqrt{10}} \int_{0}^{1} \frac{5}{2} \eta \, d\eta
\]
where $j = 1, 2$, which leads to (19) for $\nu = 1$.

**Proof of Theorem 2.** Using Proposition 2 as in the preceding proof, one can establish (6) for $\nu = 0$ with $h_{0}(x) = h_{1}(x)$ determined in (9) and $e_{0}(x)$ determined by (10). Further, substituting $q(x) = q_{\alpha}(x)$ into (12) for $\nu = 0$, where $q_{\alpha}(x)$ is determined by (11), and taking (6) for $\nu = 0$ into account, we get
\[
w_{0}(x) = 0, \quad a < x < \pi - \frac{a}{2}, \quad w_{0}(x) = g(x), \quad \pi - \frac{a}{2} < x < \frac{5a}{2}, \quad w_{0}(x) = h_{0}(x), \quad \frac{5a}{2} < x < \pi.
\]
Thus, the function $w_{0}(x)$ is independent of $\alpha$. Moreover, according to (11) and (17), for $j = 0, 1$, the characteristic function $\delta_{0,j}(\lambda)$ of the problem $L_{0,j}(a, \alpha)$ is independent of $\alpha$.

Finally, we note that, for any $\alpha \in \mathbb{C}$ and $a \in (\pi/3, 2\pi/5)$, the inclusion $q_{\alpha}(x) \in W_{2}^{1}[0, \pi]$ follows from (9)–(12) along with the last equality in (5).
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