Analytical Study for Certain Ordinary Differential Equations with Variable Coefficients via $G_\alpha$-Transform

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Abstract. $G_\alpha$-transform, which is a comprehensive and essential form of Laplace-type integral transforms, has both advantages and limitations. The purpose of this study is to consider the applicable range of $G_\alpha$-transform in finding solutions of ordinary differential equations with variable coefficients. Finally, several examples are given to demonstrate the effectiveness of these results.

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1. Introduction

The differential equations have played a central role in every aspect of applied mathematics for a very long time, and their importance has increased further with the advent of computers. Several mathematical methods have been applied by various researchers in various fields of science and engineering to obtain the analytical solutions of differential equations, which appeared in the literature [26, 34, 36]. To solve the differential equations, the integral transforms were extensively used. The Laplace transform is one of many integral transforms in applied mathematics and is often used to solve differential equations.

The Laplace transform reduces a linear differential equation to an algebraic equation, which can then be solved using algebra’s formal rules. After that, the differential equation can then be solved by applying the inverse Laplace transform [33]. The Laplace transform is beneficial for finding the solution of the diffusion equation in transient flow [8, 35, 43]. In

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addition, many researchers mainly had paid attention to study for theory and applications of Laplace transform, see [9–11, 21, 41] for more details.

The Laplace transform is a well-known fact that it converts a function \( f \) of a real variable \( t \) to a function \( F \) of a complex variable \( s \), which is defined by

\[
F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt.
\]

In addition, if \( f(t) \) is a piecewise continuous on \([0, \infty)\) and has an exponential order \( k \), then the Laplace transform \( F(s) = \mathcal{L}\{f(t)\} \) exists for \( s > k \).

For \( s = 1/u \), the Laplace transform \( \mathcal{L}\{f(t)\} \) can be rewritten as

\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-t/u}f(t)\,dt.
\]

In the last two decades, many integral transforms in the class of Laplace-typed integral transform are introduced, such as Sumudu transform, Elzaki transform, natural transform, Aboodh transform, Mohand transform, \( G_\alpha \)-transform, HY-transform, and Kamal transform. These transforms have been used for solving different types of integral equations, ordinary differential equations, partial differential equations, and fractional differential equations, see [1, 3, 15, 22, 37, 42, 45] for more details.

Since the Laplace transform is not suitable for solving some differential equations, in 1993, G. Watugala [44] introduced a new transform, named Sumudu transform, and shown that Sumudu transform has fascinating properties, making it easy to visualize and apply it for finding the solution of ordinary differential equations in control engineering problems. Thus, the Sumudu transform is an ideal transform for control engineering and applied mathematics.

In 2010, H. Eltayeb and A. Kilicman [14] introduced some relationships between Sumudu transform and Laplace transform. They showed that the solution which is given by Laplace transform into a complex domain and given by Sumudu transform into a real domain. Thus, this leads them to consider that if the solution exists by Sumudu transform, then the solution also exists by Laplace transform. Moreover, they showed a strong relationship between Sumudu transform and other integral transforms, see A. Kilicman et al. [13].

Many researchers applied Sumudu transform to solve the system of dynamic equations, partial differential equations with variable coefficient, a semi-infinite string, an integro-differential equation, the fractional neutron transport equation, see [2, 4–6, 12, 20, 23–25, 27, 28] for more details.

The Sumudu transform converts a function \( f \) of a real variable \( t \) to a function of a complex variable \( u \), which is defined by

\[
S\{f(t)\} = \frac{1}{u} \int_0^\infty e^{-t/u}f(t)\,dt.
\]

In addition, if \( f(t) \) is a piecewise continuous on \([0, \infty)\) and has an exponential order \( k \), then the Sumudu transform \( S\{f(t)\} \) exists for \( u < 1/k \).
Elzaki transform is the modified version of Laplace transform and Sumudu transform, which was first introduced by T.M. Elzaki [16] in 2011. Elzaki transform was then presented when Sumudu transform failed to solve some differential equations with variable coefficients [19]. T.M. Elzaki et al. [17, 18] showed that Elzaki transform provides a method for analyzing ordinary differential equations such as linear dynamic systems equation, signals-delay differential equation, and the renewal equation in statistics.

The Elzaki transform converts a function \( f \) of a real variable \( t \) to a function of a complex variable \( u \), which is defined by

\[
E\{f(t)\} = u \int_0^\infty e^{-t/u} f(t)dt.
\]

In particular, if \( f(t) \) is a piecewise continuous on \( t \geq 0 \) and has an exponential order \( k \), then the Elzaki transform \( E\{f(t)\} \) exists for \( u < 1/k \).

Recently, Hj. Kim [29] introduced the intrinsic structure and some properties of \( G_\alpha \)-transform, which is defined by

\[
F(u) = G_\alpha\{f(t)\} = u^\alpha \int_0^\infty e^{-t/u} f(t)dt,
\]

where \( \alpha \in \mathbb{Z} \) and \( u \) is a complex variable. The \( G_\alpha \)-transform can be applied directly to any situation by choosing \( \alpha \) appropriately.

In addition, if \( f(t) \) is a piecewise continuous on \( t \geq 0 \) and has an exponential order \( k \), then the \( G_\alpha \)-transform \( G_\alpha\{f(t)\} \) exists for \( u < 1/k \).

The \( G_\alpha \)-transform is a Laplace-type integral transform can be reduced to the Laplace transform, Sumudu transform, and Elzaki transform for \( \alpha = 0, -1, 1 \), respectively.

Moreover, we know that the Laplace transform has a strong point in the transforms of derivatives. If we set \( \alpha = -2 \), then we obtain a simple tool for transforms of integral, which can be rewritten as

\[
G_{-2}\{f(t)\} = \frac{1}{u^2} \int_0^\infty e^{-t/u} f(t)dt,
\]

see [30]. Further, Hj. Kim [31] also solved Laguerre’s equation by the \( G_{-2} \)-transform.

In 2019, S. Sattaso et al. [39] studied the properties of \( G_\alpha \)-transform and presented an example that cannot be solved by the Sumudu and Elzaki transforms, but it can be solved by the \( G_\alpha \)-transform.

Furthermore, Hj. Kim et al. [38] considered an application of \( G_\alpha \)-transform in partial differential equations by using the \( n \)-th partial derivatives, and Hj. Kim [7, 32] also considered a proof concerning the Laplace transform of the \( n \)-th derivative of any order by mathematical induction and considered a variant of \( G_\alpha \)-transform represented by a logarithmic function. The connection of this transform to the convolutional neural network can be found in [40].

In this paper, we give some conditions of certain ordinary differential equations that can be solved by \( G_\alpha \)-transform. Furthermore, we include examples to demonstrate the effectiveness of these results.
2. Preliminaries

In this section, we give some basic properties of the $G_\alpha$-transform, which would appear in this study quite frequently. The proofs of the following properties are given in [29, 39].

**Lemma 1.** [29] ($G_\alpha$-transform of derivatives) If $f(t), f'(t), \ldots, f^{(m-1)}(t)$ are continuous and $f^{(m)}(t)$ is a piecewise continuous function on $[0, \infty)$ for $m \in \mathbb{N} \cup \{0\}$ and has an exponential order $k$ for $u < 1/k$, then the following properties hold:

(i) $G_\alpha\{f'(t)\} = \frac{F(u)}{u} - u^\alpha f(0);

(ii) $G_\alpha\{f''(t)\} = \frac{F(u)}{u^2} - u^{\alpha-1} f(0) - u^\alpha f'(0);

(iii) $G_\alpha\{f^{(m)}(t)\} = \frac{F(u)}{u^m} - \sum_{k=0}^{m-1} u^{\alpha-m+(k+1)} f^{(k)}(0),$

where $F(u) = G_\alpha\{f(t)\}$.

**Lemma 2.** [39] ($G_\alpha$-transform of multiplication by power of $t$) If $f(t)$ is a piecewise continuous function on $[0, \infty)$ and has an exponential order $k$ for $u < 1/k$, then the following properties hold:

(i) $G_\alpha\{tf(t)\} = u^2 F'(u) - \alpha u F(u);

(ii) $G_\alpha\{t^2f(t)\} = u^4 F''(u) - 2(\alpha - 1) u^3 F'(u) + (\alpha - 1) \alpha u^2 F(u);

(iii) $G_\alpha\{t^nf(t)\} = u^{2n} F^{(n)}(u) - \binom{n}{1} (\alpha - (n - 1)) u^{2n-1} F^{(n-1)}$

\[+ \sum_{k=1}^{n} \binom{n}{k} (\alpha - (n - 1)) (\alpha - (n - 2)) \cdots (\alpha - 1) u^{n+1} F'(u)
\]

\[+ (\alpha - (n - 1)) (\alpha - (n - 2)) \cdots \alpha u^n F(u),
\]

where $F(u) = G_\alpha\{f(t)\}$.

**Lemma 3.** [39] If $f^{(m)}(t)$ is a piecewise continuous function on $[0, \infty)$ for $m \in \mathbb{N} \cup \{0\}$ and has an exponential order $k$ for $u < 1/k$, then

\[G_\alpha\{t^nf^{(m)}(t)\} = u^{2n} \frac{d^n G_\alpha\{f^{(m)}(t)\}}{du^n} - \binom{n}{1} (\alpha - (n - 1)) u^{2n-1} \frac{d^{n-1} G_\alpha\{f^{(m)}(t)\}}{du^{n-1}}
\]

\[+ \sum_{k=1}^{n} \binom{n}{k} (\alpha - (n - 1)) (\alpha - (n - 2)) \cdots (\alpha - 1) u^{n+1} \frac{d G_\alpha\{f^{(m)}(t)\}}{du}
\]

\[+ (\alpha - (n - 1)) (\alpha - (n - 2)) \cdots \alpha u^n G_\alpha\{f^{(m)}(t)\}.\]

**Lemma 4.** [29] If $f(t) = t^n$ for $n \in \mathbb{N} \cup \{0\}$, then

\[G_\alpha\{t^n\} = n! u^{n+\alpha+1}.\]
Remark 1. By using Lemma 3, substituting \( n = 1, 2, \) and \( 3 \) in (1) and derivatives, after some simplification, we obtain

(i) \( G_\alpha \{ t f^{(m)}(t) \} = \frac{F'(u)}{u^{m-2}} - (m + \alpha) \frac{F(u)}{u^{m-1}} - \sum_{k=0}^{m-1} (1 + k - m) u^{2+k+\alpha-m} f^{(k)}(0); \)

(ii) \( G_\alpha \{ t^2 f^{(m)}(t) \} = \frac{F''(u)}{u^{m-4}} - 2(m + \alpha - 1) \frac{F'(u)}{u^{m-3}} + [m(m + 1) + 2(\alpha - 1)m + (\alpha - 1)\alpha]
\times \frac{F(u)}{u^{m-2}} - \sum_{k=0}^{m-1} [(\alpha - m + k + 1)(\alpha - m + k) - 2(\alpha - 1)(\alpha - m + k + 1)
+ (\alpha - 1)\alpha] u^{3+k+\alpha-m} f^{(k)}(0); \)

(iii) \( G_\alpha \{ t^3 f^{(m)}(t) \} = \frac{F'''(u)}{u^{m-6}} - 3(m + \alpha - 2) \frac{F''(u)}{u^{m-5}} + [3m(m + 1) + 6(\alpha - 2)m
+ 3(\alpha - 2)(\alpha - 1)\alpha] \frac{F'(u)}{u^{m-4}} - [m(m + 1)(m + 2) + 3(\alpha - 2)m(m + 1)
+ 3(\alpha - 2)(\alpha - 1)m + (\alpha - 2)(\alpha - 1)\alpha] \frac{F(u)}{u^{m-3}}
- \sum_{k=0}^{m-1} [(\alpha - m + k + 1)(\alpha - m + k) - 3(\alpha - 2)(\alpha - m + k + 1)
- (\alpha - 2)(\alpha - 1)\alpha] u^{4+k+\alpha-m} f^{(k)}(0), \)

where \( F(u) = G_\alpha \{ f(t) \}. \)

3. Main Results

In this section, we show some conditions of certain ordinary differential equations to ensure that those ordinary differential equations can be solved by \( G_\alpha \)-transform.

Theorem 1. Consider the \( m \)-th order ordinary differential equation of the form

\[
(a_m t^2 + b_m t + c_m) y^{(m)}(t) + (a_{m-1} t^2 + b_{m-1} t + c_{m-1}) y^{(m-1)}(t) \\
+ \cdots + (a_0 t^2 + b_0 t + c_0) y(t) = g(t),
\]

where \( a_j, b_j, c_j \) are constants, \( j = 0, 1, 2, \ldots, m \) and \( g(t) \) is an unknown function. The \( G_\alpha \)-transform is a suitable method for solving (2), if the following conditions are satisfies

\[
c_m = b_m = c_{m-1} = (\alpha - 1)\alpha a_0 = 2(\alpha - 1)a_0 = 0,
\]

\[
[2 + 2(\alpha - 1) + (\alpha - 1)\alpha] a_1 - \alpha b_0 = b_{i-1} - 2(\alpha + i - 1)a_i = 0
\]

for \( i = 1, 2, 3, \ldots, m, \) and

\[
[i(i + 1) + 2(\alpha - 1)i + (\alpha - 1)\alpha] a_i - (i + \alpha - 1)b_{i-1} + c_{i-2} = 0
\]

for \( i = 2, 3, 4, \ldots, m. \)
Proof. By using Remark 1(1-2) and taking $G_\alpha$-transform of both sides to (2), we obtain

$$
\left[ \frac{a_m}{u^{m-1}} + \frac{a_{m-1}}{u^{m-2}} + \ldots + \frac{a_1}{u^2} + \frac{a_0}{u} \right] F''(u) + \left[ -2(\alpha + m - 1) \frac{a_m}{u^{m-3}} - 2(\alpha + m - 2) \frac{a_{m-1}}{u^{m-4}} - \ldots - 2\alpha \frac{a_1}{u^2} - 2(\alpha - 1) \frac{a_0}{u} + \frac{b_m}{u^{m-2}} + \frac{b_{m-1}}{u^{m-3}} + \ldots + \frac{b_1}{u-1} + \frac{b_0}{u-2} \right] F'(u)
$$

$$+ \left[ (m(m + 1) + 2(\alpha - 1)m + (\alpha - 1)\alpha) \frac{a_m}{u^{m-2}} + ((m - 1)m + 2(\alpha - 1)(m - 1) + (\alpha - 1)\alpha) \frac{a_{m-1}}{u^{m-3}} + \ldots + (2 + 2(\alpha - 1) + (\alpha - 1)\alpha) \frac{a_1}{u} + (\alpha - 1)\alpha \frac{b_m}{u^{m-1}} - (\alpha + m - 1) \frac{b_{m-1}}{u^{m-2}} - \ldots - (\alpha + 1) \frac{b_1}{u} - \alpha \frac{b_0}{u-1} + \frac{c_m}{u^{m-1}} + \frac{c_{m-1}}{u} + \ldots + \frac{c_1}{u} + c_0 \right] F(u)
$$

$$= G_\alpha \{g(t)\} - q(u), \quad (3)
$$

where $q(u)$ be contained in some expressions that are started by summation and do not influence the proof steps.

If the $G_\alpha$-transform is suitable method for solving (2), then the coefficient of $F(u)$ and $F'(u)$ in (3) should be equal to zero. Thus, if the coefficient of $F(u) = 0$, then

$$
u^m \to c_m = 0;
$$
$$uv^{m-1} \to c_{m-1} - (m + \alpha) b_m = 0;
$$
$$uv^{m-2} \to c_{m-2} - (m + \alpha - 1) b_{m-1} + (m(m + 1) + 2(\alpha - 1)m + (\alpha - 1)\alpha) a_m = 0;
$$
$$
\vdots
$$
$$nu^0 \to c_0 - (\alpha + 1) b_1 + (6 + 4(\alpha - 1) + (\alpha - 1)\alpha) a_2 = 0;
$$
$$nu^{-1} \to -\alpha c_0 + (2 + 2(\alpha - 1) + (\alpha - 1)\alpha) a_1 = 0;
$$
$$nu^{-2} \to (\alpha - 1)a a_0 = 0.
$$

And if the coefficient of $F'(u) = 0$, then

$$
u^{m-2} \to b_m = 0
$$
$$nu^{m-3} \to b_{m-1} - 2(m + \alpha - 1) a_m = 0
$$
$$nu^{m-4} \to b_{m-2} - 2(m + \alpha - 2) a_{m-1} = 0
$$
$$\vdots
$$
$$nu^{-1} \to b_1 - 2(\alpha + 1) a_2 = 0
$$
$$nu^{-2} \to b_0 - 2a a_1 = 0
$$
$$nu^{-3} \to 2(\alpha - 1) a_0 = 0.
$$

In general, we can show that

$$c_m = b_m = c_{m-1} = (\alpha - 1)a a_0 = 2(\alpha - 1)a_0 = 0,$$
for $i = 1, 2, 3, \ldots, m$, and

$$[i(i + 1) + 2(\alpha - 1)i + (\alpha - 1)\alpha]a_i - (i + \alpha - 1)b_{i-1} + c_{i-2} = 0$$

for $i = 2, 3, 4, \ldots, m$. This completes the proof.

**Remark 2.** From Theorem 1, if $g(t) = 0$, we can just set the coefficient of $F(u)$ equal to zero to reduce conditions. Therefore, the $G_\alpha$-transform is a suitable method for solving equation (2), if

$$c_m = c_{m-1} - (m + \alpha)b_m = [2 + 2(\alpha - 1) + (\alpha - 1)\alpha]a_1 - \alpha b_0 = (\alpha - 1)\alpha a_0 = 0,$$

and

$$[i(i + 1) + 2(\alpha - 1)i + (\alpha - 1)\alpha]a_i - (i + \alpha - 1)b_{i-1} + c_{i-2} = 0$$

for $i = 2, 3, 4, \ldots, m$.

**Theorem 2.** Consider the $m$-th order ordinary differential equation of the form

$$ (a_mt^3 + b_mt^2 + c_m t + d_m) y^{(m)}(t) + (a_{m-1}t^3 + b_{m-1}t^2 + c_{m-1}t + d_{m-1}) y^{(m-1)}(t) + \cdots + (a_0t^3 + b_0t^2 + c_0t + d_0) y(t) = g(t),$$

(4)

where $a_j, b_j, c_j, d_j$ are constants, $j = 0, 1, 2, \ldots, m$ and $g(t)$ is an unknown function. The $G_\alpha$-transform is a suitable method for solving (4), if the following conditions are satisfied

$$d_m = c_m = b_m = d_{m-1} = c_{m-1} = d_{m-2} = 0,$$

$$[(\alpha - 2)(\alpha - 1)\alpha a_0 = 3(\alpha - 2)(\alpha - 1)a_0 = 3(\alpha - 2)a_0 = 0,$$

$$\alpha c_0 - [2 + 2(\alpha - 1) + (\alpha - 1)\alpha]b_1$$

$$+ [24 + 18(\alpha - 2) + 6(\alpha - 2)(\alpha - 1) + (\alpha - 2)(\alpha - 1)\alpha]a_2 = 0,$$

$$(\alpha - 1)\alpha b_0 - [\alpha c_0 - [2 + 2(\alpha - 1) + (\alpha - 1)\alpha]b_1$$

$$+ [24 + 18(\alpha - 2) + 6(\alpha - 2)(\alpha - 1) + (\alpha - 2)(\alpha - 1)\alpha]a_2 = 0,$$

$$2(\alpha - 1)b_0 - [\alpha c_0 - [2 + 2(\alpha - 1) + (\alpha - 1)\alpha]b_1$$

$$+ [24 + 18(\alpha - 2) + 6(\alpha - 2)(\alpha - 1) + (\alpha - 2)(\alpha - 1)\alpha]a_2 = 0,$$

$$2(\alpha - 1)b_0 - [\alpha c_0 - [2 + 2(\alpha - 1) + (\alpha - 1)\alpha]b_1$$

$$+ [24 + 18(\alpha - 2) + 6(\alpha - 2)(\alpha - 1) + (\alpha - 2)(\alpha - 1)\alpha]a_2 = 0,$$

$$(\alpha - 1)c_2 - (\alpha + i - 2)c_{i-2} + ([i(i - 1)i + 2(\alpha - 1)(i - 1) + (\alpha - 1)\alpha]b_{i-1}$$

$$- [i(i + 1)(i + 2) + 3(\alpha - 2)i(i + 1) + 3(\alpha - 2)(\alpha - 1)i + (\alpha - 2)(\alpha - 1)\alpha]a_i = 0$$

for $i = 3, 4, 5, \ldots, m$,

$$c_{i-2} - 2(\alpha + i - 2)b_{i-1} + [3i(i + 1) + 6(\alpha - 2)i + 3(\alpha - 2)(\alpha - 1)]a_i = 0$$

for $i = 2, 3, 4, \ldots, m$, and $b_{i-1} - 3(\alpha + i - 2)a_i = 0$ for $i = 1, 2, 3, \ldots, m$.

**Proof.** By using Remark 1 and taking $G_\alpha$-transform of both sides to (4), we obtain

$$\left[\frac{a_m}{u^{m-6}} + \frac{a_{m-1}}{u^{m-7}} + \cdots + \frac{a_1}{u^3} + \frac{a_0}{u^2}\right] F''''(u) + \left[-3(m + \alpha - 2)\frac{a_m}{u^{m-6}} - 3(m + \alpha - 3)\frac{a_{m-1}}{u^{m-6}}\right]$$
\[ - \cdots - 3(\alpha - 1) \frac{a_1}{u^4} - 3(\alpha - 2) \frac{a_0}{u^5} + \frac{b_m}{u^{m-4}} + \frac{b_{m-1}}{u^{m-5}} + \cdots + \frac{b_1}{u^3} + \frac{b_0}{u^4} \] \[ F''(u) \]
\[ + \left[ (3m(m+1) + 6(\alpha - 2)m + 3(\alpha - 2)(\alpha - 1)) \frac{a_m}{u^{m-4}} + (3(m-1)m + 6(\alpha - 2)(m-1) \frac{a_1}{u^3} + 3(\alpha - 2)(m-1) \right] \]
\[ \times \frac{a_0}{u^4} - 2(\alpha + m - 1) \frac{b_m}{u^{m-3}} - 2(\alpha + m - 2) \frac{b_{m-1}}{u^{m-4}} - \cdots - 2\alpha \frac{b_1}{u^2} - 2(\alpha - 1) \frac{b_0}{u^3} + \frac{c_m}{u^{m-2}} \]
\[ + \frac{c_{m-1}}{u^{m-3}} + \cdots + \frac{c_1}{u-1} + \frac{c_0}{u-2} \] \[ F'(u) + \left[ - (m(m+1)(m+2) + 3(\alpha - 2)m(m+1) \right] \]
\[ + 3(\alpha - 2)(\alpha - 1)m + (\alpha - 2)(\alpha - 1) \alpha \frac{a_m}{u^{m-3}} - ((m-1)m(m+1) + 3(\alpha - 2)(m-1)m \]
\[ + 3(\alpha - 2)(\alpha - 1)(m-1) + (\alpha - 2)(\alpha - 1) \alpha \frac{a_1}{u^3} - (\alpha - 2)(\alpha - 1) \alpha \frac{a_0}{u^4} - (m(m+1) \]
\[ + 2(\alpha - 1)m + (\alpha - 1) \alpha \frac{b_m}{u^{m-2}} + (m-1)m + 2(\alpha - 1)(m+1) + (\alpha - 1) \alpha \frac{b_{m-1}}{u^{m-3}} \]
\[ + \cdots + (2 + 2(\alpha - 1) + (\alpha - 1) \alpha \frac{b_1}{u^2} + (\alpha - 1) \alpha \frac{b_0}{u^3} - (m + \alpha) \frac{c_m}{u^{m-1}} \]
\[ - (m + \alpha - 1) \frac{c_{m-1}}{u^{m-2}} - \cdots - (\alpha + 1) \frac{c_1}{u^0} - \alpha \frac{c_0}{u^1} + \frac{d_m}{u^m} + \frac{d_{m-1}}{u^{m-1}} + \cdots + \frac{d_1}{u^1} + \frac{d_0}{u^0} \] \[ F(u) \]
\[ = G_\alpha \{g(t)\} - r(u), \]

where \( r(u) \) be contained in some expressions that are started by summation and do not influence the proof steps.

By using the previous results, which similar to the Theorem 1, we know that the coefficients of \( F(u), F'(u) \) and \( F''(u) \) should be equal to zero, by the same process as Theorem 1, we can show that

\[ d_m = c_m = b_m = d_{m-1} = c_{m-1} = d_{m-2} = 0, \]
\[ (\alpha - 2)(\alpha - 1)aa_0 = 3(\alpha - 2)(\alpha - 1)a_0 = 3(\alpha - 2)a_0 = 0, \]
\[ \alpha c_0 - [2 + 2(\alpha - 1) + (\alpha - 1)\alpha]b_1 \]
\[ + [24 + 18(\alpha - 2) + 6(\alpha - 2)(\alpha - 1) + (\alpha - 2)(\alpha - 1)\alpha]a_2 = 0, \]
\[ (\alpha - 1)ab_0 - [6 + 6(\alpha - 2) + 3(\alpha - 2)(\alpha - 1) + (\alpha - 2)(\alpha - 1)\alpha]a_1 = 0, \]
\[ 2(\alpha - 1)b_0 - [6 + 6(\alpha - 2) + 3(\alpha - 2)(\alpha - 1)\alpha]a_1 = 0, \]
\[ d_{i-3} - (\alpha + i - 2)c_{i-2} + [(i - 1)i + 2(\alpha - 1)(i - 1) + (\alpha - 1)\alpha]b_{i-1} \]
\[ - [i(i + 1)(i + 2) + 3(\alpha - 2)i(i + 1) + 3(\alpha - 2)(\alpha - 1)i + (\alpha - 2)(\alpha - 1)\alpha]a_i = 0 \]

for \( i = 3, 4, 5, \ldots, m, \)
\[ c_{i-2} - 2(\alpha + i - 2)b_{i-1} + [3i(i + 1) + 6(\alpha - 2)i + 3(\alpha - 2)(\alpha - 1)]a_i = 0 \]

for \( i = 2, 3, 4, \ldots, m, \) and \( b_{i-1} - 3(\alpha + i - 2)a_i = 0 \) for \( i = 1, 2, 3, \ldots, m. \) The proof is completed.
Remark 3. From Theorem 2, if \( g(t) = 0 \), we can just set the coefficient of \( F(u) \) equal to zero and \( F'(u) \) equal to zero to reduce conditions. Therefore, the \( G_\alpha \)-transform is a suitable method for solving equation (4), if

\[
d_m = c_m = d_{m-1} = c_{m-1} - 2(\alpha + m - 1)b_m = 0, \\
d_{m-2} - (\alpha + m - 1)c_{m-1} + [m(m + 1) + 2(\alpha - 1)m + (\alpha - 1)\alpha]b_m = 0, \\
(\alpha - 2)(\alpha - 1)a_0 = 3(\alpha - 2)(\alpha - 1)a_0 = 0, \\
\alpha c_0 - [2 + 2(\alpha - 1) + (\alpha - 1)\alpha]b_1 + \\
[24 + 18(\alpha - 2) + 6(\alpha - 2)(\alpha - 1) + (\alpha - 2)(\alpha - 1)\alpha]a_2 = 0, \\
(\alpha - 1)\alpha b_0 - [6 + 6(\alpha - 2) + 3(\alpha - 2)(\alpha - 1) + (\alpha - 2)(\alpha - 1)\alpha]a_1 = 0, \\
2(\alpha - 1)b_0 - [6 + 6(\alpha - 2) + 3(\alpha - 2)(\alpha - 1)]a_1 = 0, \\
d_{i-3} - (\alpha + i - 2)c_{i-2} + [(i - 1)i + 2(\alpha - 1)(i - 1) + (\alpha - 1)\alpha]b_{i-1} + \\
- [i(i + 1)(i + 2) + 3(\alpha - 2)i(i + 1) + 3(\alpha - 2)(\alpha - 1)i + (\alpha - 2)(\alpha - 1)\alpha]a_i = 0
\]

for \( i = 3, 4, 5, \ldots, m \), and

\[
c_{i-2} - 2(\alpha + i - 2)b_{i-1} + [3i(i + 1) + 6(\alpha - 2)i + 3(\alpha - 2)(\alpha - 1)]a_i = 0
\]

for \( i = 2, 3, 4, \ldots, m \).

4. Examples

In this section, we show the usage of \( G_\alpha \)-transform for solving the ordinary differential equations with variable coefficients that according to Theorem 1 and Theorem 2 via some examples.

Example 1. Consider the ordinary differential equation with variable coefficients of the form

\[
t^2y''(t) + 4ty'(t) + 2y(t) = t^3.
\]  

(5)

From (2) and (5), we have

\[
a_2 = 1, \quad b_1 = 4, \quad c_0 = 2, \quad a_0 = a_1 = 0, \quad b_0 = b_2 = 0, \quad c_1 = c_2 = 0,
\]

and we define \( \alpha = 1 \) to satisfy with the conditions of Theorem 1, so using the \( G_1 \)-transform leads to find the solution of (5). By applying the \( G_1 \)-transform to (5) and using Lemma 3, we obtain

\[
G_1\{t^2y''(t)\} + G_1\{4ty'(t)\} + G_1\{2y(t)\} = G_1\{t^3\}
\]

\[
u^2F''(u) - 4uF'(u) + 6F(u) + 4uF'(u) - 8F(u) + 2F(u) = 6u^5
\]

\[
F''(u) = 6u^5.
\]

Then, we have

\[
F(u) = \frac{3}{10}u^5 + c_1 u + c_2.
\]
where $c_1$ and $c_2$ are constants. Letting $c_1 = c_2 = 0$, we get $F(u) = \frac{3}{10}u^5$. By using Lemma 4 and the inverse $G_1$-transform, thus the inverse of $u^5$ is $\frac{t^3}{6}$, we obtain $y(t) = \frac{1}{20}t^3$ as a solution of (5). It is not difficult to show that $y(t) = \frac{1}{20}t^3$ satisfies (5).

The next example will show that if the conditions do not satisfy Theorem 1, then it is not suitable to solve by this method as the following.

**Example 2.** Consider the Legendre differential equation of the form

$$(1 - t^2)y''(t) - 2ty'(t) = t. \quad (6)$$

From (2) and (6), we have

$$a_2 = -1, \quad c_2 = 1, \quad b_1 = -2, \quad a_0 = a_1 = 0, \quad b_0 = b_2 = 0, \quad c_0 = c_1 = 0,$$

and with respect to the conditions in Theorem 1, $c_2$ should be equal to 0, while $c_2$ is equal to 1. Therefore, the conditions of Theorem 1 are not satisfied. If we take $G_\alpha$-transform both sides of (6), we obtain

$$G_\alpha\{(1 - t^2)y''(t)\} - G_\alpha\{2ty'(t)\} = G_\alpha\{t\}$$

$$-u^2F''(u) + 2\alpha uF'(u) + \left((\alpha - 3)\alpha + \frac{1}{u^2}\right)F(u) = u^{\alpha + 2}. \quad (u)$$

Observe that (6) changed into a second-order ordinary differential equation with variable coefficients. Thus, using $G_\alpha$-transform did not lead to finding the solution of (6).

**Example 3.** Consider the ordinary differential equation with variable coefficients of the form

$$t^2y''(t) + 2ty'(t) - 2y(t) = 0. \quad (7)$$

From (2) and (7), we have

$$a_2 = 1, \quad b_1 = 2, \quad c_0 = -2, \quad a_0 = a_1 = 0, \quad b_0 = b_2 = 0, \quad c_1 = c_2 = 0,$$

and we define $\alpha = 1$ to satisfy with the conditions of Remark 2, so using the $G_1$-transform leads to find the solution of (7). By applying the $G_1$-transform to (7) and using Lemma 3, we obtain

$$G_1\{t^2y''(t)\} + G_1\{2ty'(t)\} - G_1\{2y(t)\} = 0$$

$$u^2F''(u) - 4uF'(u) + 6F(u) + 2uF'(u) - 2F(u) - 2F(u) - 2F(u) = 0$$

$$u^2F''(u) - 2uF'(u) = 0.$$

Then, we have $\frac{F''(u)}{F'(u)} = \frac{2}{u}$. By integration both sides, we obtain

$$\ln F'(u) = \ln c_1 u^2 \text{ or } F'(u) = c_1 u^2,$$
and hence 

\[ F(u) = \frac{c_1}{3} u^3 + c_2, \]

where \( c_1 \) and \( c_2 \) are constants. Letting \( c_2 = 0 \), we get \( F(u) = \frac{c_1}{3} u^3 \). By using Lemma 4 and the inverse \( G_1 \)-transform, thus the inverse of \( u^3 \) is \( t \), we obtain \( y(t) = \frac{c_1}{3} t \) as a solution of (7). It is not difficult to show that \( y(t) = \frac{c_1}{3} t \) satisfies (7).

**Example 4.** Consider the ordinary differential equation with variable coefficients of the form

\[ t^3 y'''(t) + 9t^2 y''(t) + 18ty'(t) + 6y(t) = t. \] (8)

From (4) and (8), we have

\[
\begin{align*}
a_3 &= 1, & b_2 &= 9, & c_3 &= 1, \\
c_1 &= 8, & a_0 &= a_1 = a_2 = 0, & b_0 &= b_1 = b_3 = 0, & c_0 &= c_2 = c_3 = 0, \\
d_0 &= d_1 = d_2 = d_3 = 0,
\end{align*}
\]

and we define \( \alpha = 2 \) to satisfy with the conditions of Theorem 2, so using the \( G_2 \)-transform leads to find the solution of (8). By applying the \( G_2 \)-transform to (8) and using Lemma 3, we obtain

\[
G_2\{t^3 y'''(t)\} + G_2\{9t^2 y''(t)\} + G_2\{18ty'(t)\} + G_2\{6y(t)\} = G_2\{t\}
\]

\[
\begin{align*}
u^3 F'''(u) - 9u^2 F''(u) + 36u F'(u) - 60F(u) + 9u^2 F''(u) - 54u F'(u) + 108F(u) \\
+ 18u F'(u) - 54F(u) + 6F(u) = u^4.
\end{align*}
\]

Then, we have \( F'''(u) = u \). By integration both sides, we obtain

\[ F(u) = \frac{1}{24} u^4 + \frac{c_1}{2} u^2 + c_2 u + c_3, \]

where \( c_1, c_2, \) and \( c_3 \) are constants. Letting \( c_1 = c_2 = c_3 = 0 \), we get \( F(u) = \frac{1}{24} u^4 \). By using Lemma 4 and the inverse \( G_2 \)-transform, thus the inverse of \( u^4 \) is \( t \), we obtain \( y(t) = \frac{1}{24} t \) as a solution of (8).

The next example will show that if the conditions do not satisfy Theorem 2, then it is not suitable to solve by this method as the following.

**Example 5.** Consider the ordinary differential equation with variable coefficients of the form

\[ (t^3 + t)y'''(t) + 6t^2 y''(t) + 6ty'(t) = t^2. \] (9)

From (4) and (9), we have

\[
\begin{align*}
a_3 &= 1, & c_3 &= 1, & b_2 &= 6, & c_1 &= 6, & a_0 &= a_1 = a_2 = 0, & b_0 &= b_1 = b_3 = 0, & c_0 &= c_2 = 0, \\
d_0 &= d_1 = d_2 = d_3 = 0,
\end{align*}
\]

and hence

\[ F(u) = \frac{c_1}{3} u^3 + c_2, \]
and with respect to the conditions in Theorem 2, $c_3$ should be equal to 0, while $c_3$ is equal to 1. Therefore, the conditions of Theorem 2 are not satisfied. If we take $G_\alpha$-transform both sides of (9), we obtain

$$
G_\alpha\{(t^3 + t)y'''(t)\} + G_\alpha\{6t^2y''(t)\} + G_\alpha\{6ty'(t)\} = G_\alpha\{t^2\}
$$

$$
u^3F'''(u) - [3 + 3(\alpha - 2)]u^2F''(u) + \left[18 - 18(\alpha - 2) + 3(\alpha - 2)(\alpha - 1) - 12(\alpha - 1) + \frac{1}{u^2}\right]uF'(u) - \left[24 + 36(\alpha - 2) + 9(\alpha - 2)(\alpha - 1) + \alpha - 2)(\alpha - 1) + (\alpha - 1)(\alpha + 3)\frac{1}{u^2} - 24(\alpha - 1) + 6(\alpha + 1)\right]F(u) = 2u^{\alpha+3}.
$$

Observe that (9) changed into a third order ordinary differential equation with variable coefficients. Thus, by using $G_\alpha$-transform did not lead to find the solution of (9).

**Example 6.** Consider the ordinary differential equation with variable coefficients of the form

$$
t^3y'''(t) + 4t^2y''(t) - 2ty'(t) - 4y(t) = 0. \tag{10}
$$

From (4) and (10), we have

$$
a_3 = 1, \quad b_2 = 4, \quad c_1 = -2, \quad d_0 = -4, \quad a_0 = a_1 = a_2 = 0, \quad b_0 = b_1 = b_3 = 0,
\quad c_0 = c_2 = c_3 = 0, \quad d_1 = d_2 = d_3 = 0,
$$

and we define $\alpha = 1$ to satisfy with the conditions of Remark 3, so using the $G_1$-transform leads to find the solution of (10). By applying the $G_1$-transform to (10) and using Lemma 3, we obtain

$$
G_1\{t^3y'''(t)\} + G_1\{4t^2y''(t)\} - G_1\{2ty'(t)\} - G_1\{4y(t)\} = 0
$$

$$
u^3F'''(u) - 9u^2F''(u) + 36uF'(u) - 60F(u) + 3u^2F''(u) - 18uF'(u) + 36F(u)
+ 4u^2F''(u) - 16uF'(u) + 24F(u) - 2uF'(u) + 2F(u) + 2F(u) - 4F(u) = 0.
$$

Then, we have $\frac{F'''(u)}{F''(u)} = \frac{2}{u}$, By integration both sides, we obtain

$$
\ln F''(u) = \ln c_1u^2 \quad \text{or} \quad F''(u) = c_1u^2,
$$

and hence

$$
F(u) = \frac{c_1}{12}u^4 + c_2u + c_3,
$$

where $c_1$, $c_2$, and $c_3$ are constants. Letting $c_2 = c_3 = 0$, we get $F(u) = \frac{c_1}{12}u^4$. By using Lemma 4 and the inverse $G_1$-transform, thus the inverse of $u^4$ is $\frac{t^2}{2}$, we obtain $y(t) = \frac{c_1}{24}t^2$ as a solution of (10).
**Remark 4.** We can see that Example 1, 3, and 6 can be solved by $G_1$-transform, and Example 4 can be solved by $G_2$-transform, it is clear that Sumudu transform cannot be solved for these ordinary differential equations.

**Remark 5.** If we choose the suitable value for $\alpha$ and the problem is consistent with the conditions of Theorem 1 or Theorem 2, then we can easily find the solution of the ordinary differential equation. But if the problem is not consistent with the conditions of Theorem 1 or Theorem 2, it will be difficult to find the solution of the ordinary differential equation.

5. **Conclusions**

We obtained some conditions of certain ordinary differential equations to ensure that it can be solved by $G_\alpha$-transform. In this regard, we observed that $G_\alpha$-transform more appropriate than other Laplace-typed integral transforms to solve the ordinary differential equations with variable coefficients by choosing the suitable value for $\alpha$.

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