Directed Reachability for Infinite-State Systems

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Abstract. Numerous tasks in program analysis and synthesis reduce to deciding reachability in possibly infinite graphs such as those induced by Petri nets. However, the Petri net reachability problem has recently been shown to require non-elementary time, which raises questions about the practical applicability of Petri nets as target models. In this paper, we introduce a novel approach for efficiently semi-deciding the reachability problem for Petri nets in practice. Our key insight is that computationally lightweight over-approximations of Petri nets can be used as distance oracles in classical graph exploration algorithms such as \textit{A}\textsuperscript{*} and greedy best-first search. We provide and evaluate a prototype implementation of our approach that outperforms existing state-of-the-art tools, sometimes by orders of magnitude, and which is also competitive with domain-specific tools on benchmarks coming from program synthesis and concurrent program analysis.

Keywords: Petri nets · reachability · shortest paths · model checking

1 Introduction

Many problems in program analysis, synthesis and verification reduce to deciding reachability of a vertex or a set of vertices in infinite graphs, \textit{e.g.}, when reasoning about concurrent programs with an unbounded number of threads, or when arbitrarily many components can be used in a synthesis task. For automated reasoning tasks, those infinite graphs are finitely represented by some mathematical model. Finding the right such model requires a trade-off between the two conflicting goals of maximal expressive power and computational feasibility of the relevant decision problems. Petri nets are a ubiquitous mathematical model that provides a good compromise between those two goals. They are

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expressive enough to find a plethora of applications in computer science, in particular in the analysis of concurrent processes, yet the reachability problem for Petri nets is decidable [50,43,44,46]. Counter abstraction has evolved as a generic abstraction paradigm that reduces a variety of program analysis tasks to problems in Petri nets or variants thereof such as well-structured transition systems, see e.g. [32,42,64,5]. Due to their generality and versatility, Petri nets and their extensions find numerous applications also in other areas, including the design and analysis of protocols [22], business processes [60], biological systems [36,10] and chemical systems [2]. The goal of this paper is to introduce and evaluate an efficient generic approach to deciding the Petri net reachability problem on instances arising from applications in program verification and synthesis.

A Petri net comprises a finite set of places with a finite number of transitions. Places carry a finite yet unbounded number of tokens and transitions can remove and add tokens to places. A marking specifies how many tokens each place carries. An example of a Petri net is given on the left-hand side of Figure 1, where the two places \( \{p_1, p_2\} \) are depicted as circles and transitions \( \{t_1, t_2, t_3\} \) as squares. Places carry tokens depicted as filled circles; thus \( p_1 \) carries one token and \( p_2 \) carries none. We write this as \([p_1:1, p_2:0]\), or \((1,0)\) if there is a clear ordering on the places. Transition \( t_1 \) can add a single token to place \( p_1 \) at any moment. As soon as a token is present in \( p_1 \), it can be consumed by transition \( t_2 \), which then adds a token to place \( p_2 \) and puts back one token to place \( p_1 \). Finally, transition \( t_3 \) consumes tokens from \( p_1 \) without any adding token at all.

A Petri net induces a possibly infinite directed graph whose vertices are markings, and whose edges are determined by the transitions of the Petri net, cf. the right side of Figure 1. Given two markings, the reachability problem asks whether they are connected in this graph. In Figure 1, the marking \((0,1)\) is reachable from \((0,0)\), e.g., via paths of lengths 3 and 5: \((0,0) \xrightarrow{t_1} (1,0) \xrightarrow{t_2} (1,1) \xrightarrow{t_3} (0,1)\) and \((0,0) \xrightarrow{t_1} (1,0) \xrightarrow{t_2} (2,0) \xrightarrow{t_3} (2,1) \xrightarrow{t_3} (1,1) \xrightarrow{t_3} (0,1)\).

In practice, the Petri net reachability problem is a challenging decision problem due to its horrendous worst-case complexity: an exponential-space lower bound was established in the 1970s [48], and a non-elementary time lower bound.
has only recently been established [12]. One may thus question whether a problem with such high worst-case complexity is of any practical relevance, and whether reducing program analysis tasks to Petri net reachability is anything else than merely an intellectual exercise. We debunk those concerns and present a technique which decides most reachability instances appearing in the wild. When evaluated on large-scale instances involving Petri nets with thousands of places and tens of thousands of transitions, our prototype implementation is most of the time faster, even up to several orders of magnitude on large-scale instances, and solves more instances than existing state-of-the-art tools. Our implementation is also competitive with specialized domain-specific tools. One of the biggest advantages of our approach is that it is extremely simple to describe and implement, and it readily generalizes to many extensions of Petri nets. In fact, it was surprising to us that our approach has not yet been discovered. We now describe the main observations and techniques underlying our approach.

Ever since the early days of research in Petri nets, state-space over-approximations have been studied to attenuate the high computational complexity of their decision problems. One such over-approximation is, informally speaking, to allow places to carry a negative numbers of tokens. Deciding reachability then reduces to solving the so-called state equation, a system of linear equations associated to a Petri net. Another over-approximation are continuous Petri nets, a variant where places carry fractional tokens and “fractions of transitions” can be applied [13]. The benefit is that deciding reachability drops down to polynomial time [26]. While those approximations have been applied for pruning search spaces, see e.g. [23,4,8,31], we make the following simple key observation:

*If a marking $m$ is reachable from an initial marking in an over-approximation, then the length of a shortest witnessing path in the over-approximation lower bounds the length of a shortest path reaching $m$.*

The availability of an oracle providing lower bounds on the length of shortest paths between markings enables us to appeal to classical graph traversal algorithms which have been highly successful in artificial intelligence and require such oracles, namely $A^*$ and greedy best-first search, see e.g. [55]. In particular, determining the length of shortest paths in the over-approximations described above can be phrased as optimization problems in (integer) linear programming and optimization modulo theories, for which efficient off-the-shelf solvers are available [35,7]. Thus, oracle calls can be made at comparably modest computational cost, which is crucial for the applicability of those algorithms. As a result, a large class of existing state-space over-approximations can be applied to obtain a highly efficient forward-analysis semi-decision procedure for the reachability problem. For example, in Figure 1, using the state equation as distance oracle, $A^*$ only explores the four vertices in the blue region and directly reaches the target vertex, whereas a breadth-first search may need to explore all vertices of the figure and a depth-first search may even not terminate.

In theory, our approach could be turned into a decision procedure by applying bounds on the length of shortest paths in Petri nets [47]. However, such
lengths can grow non-elementarily in the number of places [12], and just computing the cut-off length will already be infeasible for any Petri net of practical relevance. It is worth mentioning that, in practice, it has been observed that the over-approximations we employ also often witness non-reachability though, see e.g. [23]. Still, when dealing with finite state spaces, our procedure is complete.

A noteworthy benefit of our approach is that it enables finding shortest paths when \( A^* \) is used as the underlying algorithm. In program analysis, paths usually correspond to traces reaching an erroneous configuration. In this setting, shorter error traces are preferred as they help understanding why a certain error occurs. Furthermore, in program synthesis, paths correspond to synthesis plans. Again, shorter paths are preferred as they yield shorter synthesized programs. In fact, we develop our algorithmic framework for weighted Petri nets in which transitions are weighted with positive integers. Classical Petri nets correspond to the special instance where all weights are equal to one. Weighted Petri nets are useful to reflect cost or preferences in synthesis tasks. For example, there are program synthesis approaches where software projects are mined to determine how often API methods are called to guide a procedure by preferring more frequent methods [28,27,49]. Similarity metrics can also be used to obtain costs estimating the relevance of invoking methods [25]. It has further been argued that weighted Petri nets are a good model for synthesis tasks of chemical reactions as they can reflect costs of various chemical compounds [61]. Finally, weights can be viewed as representing an amount of time it takes to fire a transition, see e.g. [53].

Related work. Our approach falls under the umbrella term directed model checking coined in the early 2000s, which refers to a set of techniques to tackle the state-explosion problem via guided state-space exploration. It primarily targets disproving safety properties by quickly finding a path to an error state without the need to explicitly construct the whole state space. As such, directed model checking is useful for bug-finding since, in the words of Yang and Dill [63], in practice, model checkers are most useful when they find bugs, not when they prove a property. The survey paper [20] gives an overview over various directed model checking techniques for finite-state systems.

For Petri nets, directed reachability algorithms based on over-approximations as developed in this work have not been described. In [59], it is argued that exploration heuristics, like \( A^* \), can be useful for Petri nets, but they do not consider over-approximations for the underlying heuristic functions. The authors of [39] use Petri nets for scheduling problems and employ the state equation, viewed as a system of linear equations over \( \mathbb{Q} \), in order to explore and prune reachability graphs. This approach is, however, not guaranteed to discover shortest paths. There has been further work on using \( A^* \) for exploring the reachability graph of Petri nets for scheduling problems, see, e.g., [45,51] and the references therein.

2 Preliminaries

Let \( \mathbb{N} := \{0, 1, \ldots\} \). For all \( D \subseteq \mathbb{Q} \) and \( \succ \in \{\geq, >\} \), let \( D_{\succ 0} := \{a \in D : a \succ 0\} \), and for every set \( X \), let \( D^X \) denote the set of vectors \( D^X := \{v \mid v : X \to D\} \).
We naturally extend operations componentwise. In particular, \((u + v)(x) := u(x) + v(x)\) for every \(x \in X\), and \(u \geq v\) iff \(u(x) \geq v(x)\) for every \(x \in X\).

**Graphs.** A \textit{(labeled directed) graph} is a triple \(G = (V, E, A)\), where \(V\) is a set of nodes, \(A\) is a finite set of elements called \textit{actions}, and \(E \subseteq V \times A \times V\) is the set of \textit{edges} labeled by actions. We say that \(G\) has \textit{finite out-degree} if the set of outgoing edges \(\{(w, a, w') \in E : w = v\}\) is finite for every \(v \in V\). Similarly, it has \textit{finite in-degree} if the set of ingoing edges is finite for every \(v \in V\). If \(G\) has both finite out- and in-degree, then we say that \(G\) is \textit{locally finite}. A \textit{path} \(\pi\) is a finite sequence of nodes \((v_i)_{1 \leq i \leq n}\) and actions \((a_i)_{1 \leq i < n}\) such that \((v_i, a_i, v_{i+1}) \in E\) for all \(1 \leq i < n\). We say that \(\pi\) is a \textit{path from} \(v\) \textit{to} \(w\) \((\text{or a} \ v\text{-}w \text{path})\) if \(v = v_1\) and \(w = v_n\), and its \textit{label} is \(a_1 a_2 \cdots a_{n-1}\), where \(\varepsilon\) denotes the empty sequence.

A \textit{weighted} graph is a tuple \(G = (V, E, A, \mu)\) where \((V, E, A)\) is a graph with a \textit{weight function} \(\mu : E \to \mathbb{Q}_{>0}\). The weight of path \(\pi\) is the weight of its edges, i.e., \(\mu(\pi) := \sum_{1 \leq i < n} \mu(v_i, a_i, v_{i+1})\). A shortest path from \(v\) \textit{to} \(w\) is a \(v\text{-}w\) path \(\pi\) minimizing \(\mu(\pi)\). We define \(\text{dist}_G : V \times V \to \mathbb{Q}_{\geq 0} \cup \{\infty\}\) as the \textit{distance function} where \(\text{dist}_G(v, w)\) is the weight of a shortest path from \(v\) \textit{to} \(w\), with \(\text{dist}_G(v, w) = \infty\) if there is none. We assume throughout the paper that weighted graphs have a \textit{minimal weight}, i.e., that \(\min \{\mu(e) : e \in E\}\) exists. For graphs with finite out-degree, this ensures that if a path exists between two nodes, then a shortest one exists.\(^4\) This mild assumption always holds in our setting.

**Petri nets.** A \textit{weighted Petri net} is a tuple \(\mathcal{N} = (P, T, f, \lambda)\) where

- \(P\) is a finite set whose elements are called \textit{places},
- \(T\) is a finite set, disjoint from \(P\), whose elements are called \textit{transitions},
- \(f : (P \times T) \cup (T \times P) \to \mathbb{N}\) is the \textit{flow function} assigning multiplicities to arcs connecting places and transitions, and
- \(\lambda : T \to \mathbb{Q}_{>0}\) is the \textit{weight function} assigning weights to transitions.

A \textit{marking} is a vector \(m \in \mathbb{N}^P\) which indicates that place \(p\) holds \(m(p)\) \textit{tokens}. A weighted Petri net with \(\lambda(t) = 1\) for each \(t \in T\) is called a \textit{Petri net}. For example, Figure 1 depicts a Petri net \(\mathcal{N}\) with \(P = \{p_1, p_2\}\), \(T = \{t_1, t_2, t_3\}\), \(f(p_1, t_3) = f(t_1, t_2) = f(t_1, p_1) = f(t_2, p_1) = f(t_2, p_2) = 1\) (multiplicity omitted on arcs) and \(f(-, -) = 0\) elsewhere (no arc). Moreover, \(\mathcal{N}\) is marked with \([p_1 : 1, p_2 : 0]\).

The \textit{guard} and \textit{effect} of a transition \(t \in T\) are vectors \(g_t \in \mathbb{N}^P\) and \(\Delta_t \in \mathbb{Z}^P\) where \(g_t(p) := f(p, t)\) and \(\Delta_t(p) := f(t, p) - f(p, t)\). We say that \(t\) is \textit{firable} from marking \(m\) if \(m \geq g_t\). If \(t\) is firable from \(m\), then it may be \textit{fired}, which leads to marking \(m' := m + \Delta_t\). We write this as \(m \xrightarrow{\Delta_t \mathcal{N}} m'\). These notions naturally extend to sequences of transitions, i.e., \(\mathcal{N}^\ast\) denotes the identity relation over \(\mathbb{N}^P\), \(\Delta_x := 0\), \(\lambda(e) := 0\), and for every \(t_1, t_2, \ldots, t_k \in T\): \(\Delta_{t_1 t_2 \cdots t_k} := \Delta_{t_1} + \Delta_{t_2} + \cdots + \Delta_{t_k}\), \(\lambda(t_1 t_2 \cdots t_k) := \lambda(t_1) + \lambda(t_2) + \cdots + \lambda(t_k)\), and

\[
\Delta_{t_1 t_2 \cdots t_k} \xrightarrow{\mathcal{N}} \Delta_{t_2} \xrightarrow{\mathcal{N}} \cdots \xrightarrow{\mathcal{N}} \Delta_{t_k} \xrightarrow{\mathcal{N}} \Delta_{t_1} \xrightarrow{\mathcal{N}} \lambda(t_1 t_2 \cdots t_k),
\]

\(^4\) Otherwise, there could be increasingly better paths, e.g., of weights 1, 1/2, 1/4, \ldots.
We say that \( \rightarrow_N \) is the reflexive transitive closure of \( \rightarrow \). Note that the latter is the reflexive transitive closure of \( \rightarrow_N \).

For example, \( m \xrightarrow{t_2 t_3} N m' \) and \( m \xrightarrow{t_1 t_2 t_3} N m'' \) in Figure 1, where \( m := [p_1: 1, p_2: 0] \) and \( m' := [p_1: 0, p_2: 1] \). Moreover, \( t_2 \) is not firable in \( m' \).

Given a sequence \( \sigma \in T^* \), denote by \( |\sigma|_t \in \mathbb{N} \) the number of times transition \( t \) occurs in \( T \). The Parikh image of \( \sigma \) is the vector \( \sigma \in N^T \) that captures the number of occurrences of transitions appearing in \( \sigma \), i.e., \( \sigma(t) := |\sigma|_t \) for all \( t \in T \).

Each weighted Petri net \( \mathcal{N} = (P, T, f, \lambda) \) induces a locally finite weighted graph \( G_N(\mathcal{N}) := (V, E, T, \mu) \), called its reachability graph, where \( V := \mathbb{N}^P \), \( E := \{(m, t, m') : m \xrightarrow{t} N m' \} \) and \( \mu(m, t, m') := \lambda(t) \) for each \( (m, t, m') \in E \). An example of a reachability graph is given on the right of Figure 1. We write \( \text{dist}_\mathcal{N} \) to denote \( \text{dist}_{G_N(\mathcal{N})} \). We have \( \text{dist}_\mathcal{N}(m, m') \neq \infty \) iff \( m \xrightarrow{\gamma} N m' \) for some \( \sigma \in T^* \), and if the latter holds, then \( \text{dist}_\mathcal{N}(m, m') \) is the minimal weight among such firing sequences \( \sigma \). Moreover, for (unweighted) Petri nets, \( \text{dist}_\mathcal{N}(m, m') \) is the minimal number of transitions to fire to reach \( m' \) from \( m \).

### 3 Directed search algorithms

Our approach relies on classical pathfinding procedures guided by node selection strategies. Their generic scheme is described in Algorithm 1. Its termination with a value \( d \neq \infty \) indicates that the weighted graph \( G = (V, E, A, \mu) \) has a path from \( s \) to \( t \) of weight \( d \), whereas termination with \( d = \infty \) signals that \( \text{dist}_G(s, t) = \infty \).

Algorithm 1 maintains a set of frontier nodes \( C \) and a mapping \( g : V \rightarrow \mathbb{Q}_{\geq 0} \cup \{\infty\} \) such that \( g(w) \) is the weight of the best known path from \( s \) to \( w \).

In Line 4, a selection strategy \( S \) determines which node \( v \) to expand next. Starting from Line 6, a successor \( w \) of \( v \) is added to the frontier if its distance improves.

Let \( h : V \rightarrow \mathbb{Q}_{\geq 0} \cup \{\infty\} \) estimate the distance from all nodes to a target \( t \in V \). The selection strategies sending \((g, v)\) respectively to \( g(v) \), \( g(v) + h(v) \) or \( h(v) \) yield the classical Dijkstra's, \( A^* \) and greedy best-first search (GBFS) algorithms.

When instantiating \( S \) with Dijkstra's selection strategy, a return value \( d \neq \infty \) is guaranteed to equal \( \text{dist}_G(s, t) \). This is not true for \( A^* \) and GBFS. However, if \( h \) fulfills the following consistency properties, then \( A^* \) also has this guarantee: \( h(t) = 0 \) and \( h(v) \leq \mu(v, a, w) + h(w) \) for every \((v, a, w) \in E \) (see, e.g., [55]).

In the setting of infinite graphs, unlike GBFS, \( A^* \) and Dijkstra's selection strategies guarantee termination if \( \text{dist}_G(s, t) \neq \infty \). Yet, we introduce unbounded heuristics for which termination is also guaranteed for GBFS. Note that these
guarantees would vanish in the presence of zero weights. An infinite path $\pi$ is a sequence of nodes $(v_i)_{i \in \mathbb{N}}$ and actions $(a_i)_{i \in \mathbb{N}}$ such that $(v_i, a_i, v_{i+1}) \in E$ for all $i \in \mathbb{N}$. We say that $\pi$ is bounded w.r.t. $h$ if its nodes are pairwise distinct and there exists $b \in \mathbb{Q} \geq 0$ with $h(v_i) \leq b$ for all $i \geq 0$. We say that $h$ is unbounded if it admits no bounded sequence. The following technical lemma enables to prove termination of GFBS in the presence of unbounded heuristics.

**Lemma 1.** If $G$ is locally finite and $h$ is unbounded, then the following holds:

1. The set of paths of weight at most $c \in \mathbb{Q} \geq 0$ starting from node $s$ is finite.
2. Let $W \subseteq V$. The set $\text{dist}_G(W, t) := \{ \text{dist}_G(w, t) : w \in W \}$ has a minimum.
3. No node is expanded infinitely often by Algorithm 1.

**Theorem 1.** Algorithm 1 with the greedy best-first search selection strategy always finds reachable targets for locally finite graphs and unbounded heuristics.

**Proof.** First observe that Algorithm 1 satisfies this invariant:

if $g(v) \neq \infty$, then $g(v)$ is the weight of a path from $s$ to $v$ in $G$
whose nodes were all expanded, except possibly $v$. (*)

Assume dist$_G(s, t) \neq \infty$. For the sake of contradiction, suppose $t$ is never expanded. Let $K_i$ be the subgraph of $G$ induced by nodes expanded at least once within the first $i$ iterations of the while loop. In particular, $K_1$ is the graph made only of node $s$. Let $K = K_1 \cup K_2 \cup \cdots$. By Lemma 1 (3), no node is expanded infinitely often, hence $K$ is infinite. Moreover, $K$ has finite out-degree, and each node of $K$ is reachable from $s$ in $K$ by (*). Thus, by König’s lemma, $K$ contains an infinite path $v_0, v_1, \ldots \in V$ of pairwise distinct nodes.

Let $w$ be a node of $K$ minimizing dist$_G(w, t)$. It is well-defined by Lemma 1 (2).
We have dist$_G(w, t) \neq \infty$ as $t$ is reachable from $s$ and the latter belongs to $K_1 \subseteq K$. By minimality of $w \neq t$, there exists an edge $(w, a, w')$ of $G$ such that dist$_G(w', t) < \text{dist}_G(w, t)$ and $w'$ does not appear in $K$. Note that $w'$ is added to $C$ at some point, but is never expanded as it would otherwise belong to $K$. Let $i$ be the smallest index such that $w$ belongs to $K_i$. Since $h$ is unbounded, there exists $j$ such that $h(v_j) > h(w')$ and $v_j$ is expanded after iteration $i$ of the while loop. This is a contradiction as $w'$ would have been expanded instead of $v_j$. □

### 4 Directed reachability

In this section, we explain how to instantiate Algorithm 1 for finding short(est) firing sequences witnessing reachability in weighted Petri nets. Since Dijkstra’s selection strategy does not require any heuristic, we focus on A* and greedy best-first search which require consistent and unbounded heuristics. More precisely, we introduce distance under-approximations (Section 4.1); present relevant concrete distance under-approximations (Section 4.2); and put everything together into our framework (Section 4.3).
4.1 Distance under-approximations

A distance under-approximation of a weighted Petri net \( \mathcal{N} = (P, T, f, \lambda) \) is a function \( d: \mathbb{N}^P \times \mathbb{N}^P \to \mathbb{Q}_{\geq 0} \cup \{\infty\} \) such that for all \( m, m', m'' \in \mathbb{N}^P \):

- \( d(m, m') \leq \text{dist}_\mathcal{N}(m, m') \),
- \( d(m, m'') \leq d(m, m') + d(m', m'') \) (triangle inequality), and
- \( d \) is effective, i.e. there is an algorithm that evaluates \( d \) on all inputs.

We naturally obtain a heuristic from \( d \) for a directed search towards marking \( m_{\text{target}} \). Indeed, let \( h: \mathbb{N}^P \to \mathbb{Q}_{\geq 0} \cup \{\infty\} \) be defined by \( h(m) := d(m, m_{\text{target}}) \).

The following proposition shows that \( h \) is a suitable heuristic for \( A^* \):

**Proposition 1.** Mapping \( h \) is a consistent heuristic.

**Proof.** Let \( m, m' \in \mathbb{N}^P \) and \( t \in T \) be such that \( m \xrightarrow{t}N m' \). We have:

\[
\begin{align*}
    h(m) &= d(m, m_{\text{target}}) \quad \text{(by def. of } h) \\
    &\leq d(m, m') + d(m', m_{\text{target}}) \quad \text{(by the triangle inequality)} \\
    &\leq \text{dist}_\mathcal{N}(m, m_{\text{target}}) + d(m', m_{\text{target}}) \quad \text{(by distance under-approximation)} \\
    &= \lambda(t) + h(m') \quad \text{(since } m \xrightarrow{t}N m' ).
\end{align*}
\]

Moreover, \( h(m_{\text{target}}) = d(m_{\text{target}}, m_{\text{target}}) \leq \text{dist}_\mathcal{N}(m_{\text{target}}, m_{\text{target}}) = 0 \), where the last equality follows from the fact that weights are positive. \( \square \)

4.2 From Petri net relaxations to distance under-approximations

We now introduce classical relaxations of Petri nets which over-approximate reachability and consequently give rise to distance under-approximations. The main source of hardness of the reachability problem stems from the fact that places are required to hold a non-negative number of tokens. If we relax this requirement and allow negative numbers of tokens, we obtain a more tractable relation. More precisely, we write \( m \xrightarrow{t}Z m' \) iff \( m' = m + \Delta t \). Note that transitions are always firable under this semantics. Moreover, they may lead to “markings” with negative components.

Another source of hardness comes from the fact that markings are discrete. Hence, we can further relax \( \toZ \) into \( \toQ \) where transitions may be scaled down:

\[
m \xrightarrow{t}Q m' \iff m' = m + \delta \cdot \Delta t, \text{ for some } 0 < \delta \leq 1.
\]

One gets a less crude relaxation from considering nonnegative “markings” only:

\[
m \xrightarrow{t}Q_{\geq 0} m' \iff (m \geq \delta \cdot g_t) \text{ and } (m' = m + \delta \cdot \Delta t), \text{ for some } 0 < \delta \leq 1.
\]

Under these, we obtain “markings” from \( \mathbb{Q}^P \) and \( \mathbb{Q}^P_{\geq 0} \) respectively. Petri nets equipped with relation \( \toQ_{\geq 0} \) are known as *continuous Petri nets* [13,14].
To unify all three relaxations, we sometimes write \( m \xrightarrow{\delta t_G} G m' \) to emphasize the scaling factor \( \delta \), where \( \delta = 1 \) whenever \( G = Z \). Let \( d_G : \mathbb{N}^P \times \mathbb{N}^P \rightarrow Q_{\geq 0} \cup \{ \infty \} \) be defined as 
\[
d_G(m,m') := \infty \quad \text{if} \quad m \xrightarrow{\notin \delta t_G} G m', \quad \text{and otherwise:}
\]
\[
d_G(m,m') := \min \left\{ \sum_{i=1}^{n} \delta_i \cdot \lambda(t_i) : m \xrightarrow{\delta t_1 \cdots \delta t_n} G m' \right\}.
\]

In words, \( d_G(m,m') \) is the weight of a shortest path from \( m \) to \( m' \) in the graph induced by the relaxed step relation \( \rightarrow_G \), where weights are scaled accordingly.

We now show that any \( d_G \), which we call the \( G \)-distance, is a distance under-approximation, and first show effectiveness of all \( d_G \). It is well-known and readily seen that reachability over \( G \in \{ Z, Q \} \) is characterized by the following state equation, since transitions are always firable due to the absence of guards:
\[
m \xrightarrow{\sigma} G m' \iff \exists \sigma \in G_{\geq 0}^T : m' = m + \sum_{t \in T} \sigma(t) \cdot \Delta t.
\]

Here, \( \sigma \) can be seen as the Parikh image of a sequence \( \sigma \) leading from \( m \) to \( m' \).

**Proposition 2.** The functions \( d_Z \), \( d_Q \), \( d_{Q_{\geq 0}} \) are effective.

**Proof.** By the state equation, we have:
\[
d_G(m,m') = \min \left\{ \sum_{t \in T} \lambda(t) \cdot \sigma(t) : \sigma \in G_{\geq 0}^T, m' = m + \sum_{t \in T} \sigma(t) \cdot \Delta t \right\}
\]

Therefore, \( d_Q(m,m') \) (resp. \( d_Z(m,m') \)) are computable by (resp. integer) linear programming, which is is complete for \( P \) (resp. \( NP \)), in its variant where one must check whether the minimal solution is at most some bound.

For \( d_{Q_{\geq 0}} \), note that the reachability relation of a continuous Petri net can be expressed in the existential fragment of linear real arithmetic [8]. Hence, effectiveness follows from the decidability of linear real arithmetic.

Altogether, we conclude that \( d_G \) is a distance under-approximation. Furthermore, we can show that \( d_G \) yields unbounded heuristics, which, by Theorem 1, ensure termination of GBFS on reachable instances:

**Theorem 2.** Let \( G \in \{ Z, Q, Q_{\geq 0} \} \), then \( d_G \) is a distance under-approximation. Moreover, the heuristics arising from it are unbounded.

**Proof.** Let \( N = (P,T,f,\lambda) \) be a weighted Petri net. Effectiveness of \( d_G \) follows from Proposition 2. By definitions and a simple induction, \( \sigma_N \subseteq \sigma_G \) for any sequence \( \sigma \in T^* \), with weights left unchanged for unscaled transitions. This implies that \( d_G(m,m') \leq \text{dist}_N(m,m') \) for every \( m, m' \in G' \). Moreover, the triangle inequality holds since for every \( m, m', m'' \in G' \) and sequences \( \sigma, \sigma' \):
\[
m \xrightarrow{\sigma} G m' \xrightarrow{\sigma'} G m'' \text{ implies } m \xrightarrow{\sigma' G} m''.
\]
Let us sketch the proof of the second part. Let \( \mathbf{m}_{\text{target}} \) be a marking and let \( h_G \) be the heuristic obtained from \( d_G \) for \( \mathbf{m}_{\text{target}} \). Since \( h_Q(m) \leq h_G(m) \) for all \( m \) and \( G \in \{ \mathbb{Z}, Q_{\geq 0} \} \), it suffices to prove that \( d_Q \) is unbounded. Suppose it is not. There exist \( b \in Q_{\geq 0} \) and pairwise distinct markings \( m_0, m_1, \ldots \) each with \( h_Q(m_i) \leq b \). Let \( x_t \) be a solution to the state equation that gives \( h_G(m_i) \). By well-quasi-ordering and pairwise distinctness, there is a subsequence such that \( m_{i_0}(p) < m_{i_1}(p) < \cdots \) for some \( p \in P \). Thus, \( \lim_{j \to \infty} m_{\text{target}}(p) - m_i(p) = -\infty \), and hence \( \lim_{j \to \infty} x_{i_j}(s) = \infty \) for some \( s \in T \) with \( \Delta_i(p) < 0 \). This means that \( b \geq h_Q(m_{i_j}) = \sum_{t \in T} \lambda(t) \cdot x_{i_j}(t) > b \) for a sufficiently large \( j \). \( \square \)

4.3 Directed reachability based on distance under-approximations

We have all the ingredients to use Algorithm 1 for answering reachability queries.

A distance under-approximation scheme is a mapping \( D \) that associates a distance under-approximation \( D(N) \) to each weighted Petri net \( N \). Let \( h_D(N), \mathbf{m}_{\text{target}} \) be the heuristic obtained from \( D(N) \) for marking \( \mathbf{m}_{\text{target}} \). By instantiating Algorithm 1 with this heuristic, we can search for a short(est) firing sequence witnessing that \( \mathbf{m}_{\text{target}} \) is reachable. Of course, constructing the reachability graph of \( N \) would be at least as difficult as answering this query, or impossible if it is infinite. Hence, we provide \( G_0(N) \) symbolically through \( N \) and let Algorithm 1 explore it on-the-fly by progressively firing its transitions.

For each \( G \in \{ \mathbb{Z}, Q, Q_{\geq 0} \} \), the function \( D_G \) mapping a weighted Petri net \( N \) to its \( G \)-distance \( d_G \) is a distance under-approximation scheme with consistent and unbounded heuristics by Proposition 1, Theorem 1 and Theorem 2. Although Algorithm 1 is geared towards finding paths, it can prove non-reachability even for infinite reachability graphs. Indeed, at some point, every candidate marking \( m \in C \) may be such that \( h_D(N), \mathbf{m}_{\text{target}}(m) = \infty \), which halts with \( \infty \). There is no guarantee that this happens, but, as reported e.g. by [23,8], the \( G \)-distance for domains \( G \in \{ \mathbb{Z}, Q, Q_{\geq 0} \} \) does well for witnessing non-reachability in practice, often from the very first marking \( \mathbf{m}_{\text{init}} \).

An example. We illustrate our approach with a toy example and \( D_Q \) (the scheme based on the state equation over \( Q_{\geq 0} \)). Consider the Petri net \( N \) illustrated on the left of Figure 1, but marked with \( \mathbf{m}_{\text{init}} := [p_1: 0, p_2: 0] \). Suppose we wish to determine whether \( \mathbf{m}_{\text{init}} \) can reach marking \( \mathbf{m}_{\text{target}} := [p_1: 0, p_2: 1] \) in \( N \).

We consider the case where Algorithm 1 follows a greedy best-first search, but the markings would be expanded in the same way with \( \mathbb{N} \). Let us abbreviate a marking \( [p_1: x, p_2: y] \) as \( (x, y) \). Since \( \Delta_{t_2} = (0, 1) \), the heuristic considers that \( \mathbf{m}_{\text{init}} \) can reach \( \mathbf{m}_{\text{target}} \) in a single step using transition \( t_2 \) (it is unaware of the guard). Marking \((1, 0)\) is expanded and its heuristic value increases to 2 as the state equation considers that both \( t_2 \) and \( t_3 \) must be fired (in some unknown order). Markings \((2, 0)\) and \((1, 1)\) are both discovered with respective heuristic values 3 and 1. The latter is more promising, so it is expanded and target \((0, 1)\) is discovered. Since its heuristic value is 0, it is immediately expanded and the correct distance \( \text{dist}_N(\mathbf{m}_{\text{init}}, \mathbf{m}_{\text{target}}) = 3 \) is returned. Note that, in this example, the only markings expanded are precisely those occurring on the shortest path.
Handling multiple targets. Algorithm 1 can be adapted to search for some marking from a given target set \( X \subseteq N^P \). The idea consists simply in using a heuristic \( h_X : N^P \to Q_{\geq 0} \cup \{\infty\} \) estimating the weight of a shortest path to any target:

\[
h_X(m) := \min \{ h_{D(N)}(m), m_{\text{target}}(m) : m_{\text{target}} \in X \}.
\]

This is convenient for partial reachability instances occurring in practice, i.e.

\[
X := \{ m_{\text{target}} \in N^P : m_{\text{target}}(p) \sim_p c(p) \} \text{ where } c \in N^P \text{ and each } \sim_p \in \{=, \geq\}.
\]

5 Experimental results

We implemented Algorithm 1 in a prototype, called FastForward, which supports all selection strategies and distance under-approximations presented in the paper. We evaluate FastForward empirically with three main goals in mind. First, we show that our approach is competitive with established tools and can even vastly outperform them, and we also give insights on its performance w.r.t. its parameterizations. Second, we compare the length of the witnesses reported by the different tools. Third, we briefly discuss the quality of the heuristics.

Technical details. Our tool is written in C# and uses Gurobi [35], a state-of-the-art MILP solver, for distance under-approximations. We performed our benchmarks on a machine with an 8-Core Intel® Core™ i7-7700 CPU @ 3.60GHz running on Ubuntu 18.04 with memory constrained to \( \sim 8GB \). We used a timeout of 60 seconds per instance, and all tools were invoked from a Python script using the time module for time measurements.

A minor challenge arises from the fact that many instances specify an upward-closed set of initial markings rather than a single one. For example, \( m_{\text{init}}(p) \geq 1 \) to specify, e.g., an arbitrary number of threads. We handle this by setting \( m_{\text{init}}(p) = 1 \) and adding a transition \( t_p \) producing a token into \( p \).

As a preprocessing step, we implemented sign analysis [31]. It is a general pruning technique that has been shown beneficial for reducing the size of the state-space of Petri nets. Initially, places that carry tokens are viewed as marked. For each transition whose input places are marked, the output places also become marked. When a fixpoint is reached, places left unmarked cannot carry tokens in any reachable marking, so they are discarded.

Benchmarks. Due to the lack of tools handling reachability for unbounded state spaces, benchmarks arising in the literature are primarily coverability instances\(^5\), i.e. reachability towards an upward closed set of target markings. We gathered 61 positive and 115 negative coverability instances originating from five suites \([42,29,6,38,18]\) previously used for benchmarking \([23,8,31]\). They arise from the analysis of multi-threaded C programs with shared-memory; mutual

\(^5\) The Model Checking Contest focuses on reachability for finite state spaces.
exclusion algorithms; communication protocols; provenance analysis in the context of a medical messaging and a bug-tracking system; and the verification of ERLANG concurrent programs. We further extracted the sypet suite made of 30 positive (standard) reachability instances arising from queries encountered in type-directed program synthesis [25]. The overall goal of this work is to enable a vast range of untapped applications requiring reachability over unbounded state-spaces, rather than just coverability. To obtain further (positive) instances of the Petri net reachability problem, we performed random walks on the Petri nets from the aforementioned coverability benchmarks. To this end, we used the largest quarter of distinct Petri nets from each coverability suite, for a total of 33. We performed one random walk each of lengths 20, 25, 30, 35, 40, 50, 60, 75, 90 and 100, and we saved the resulting marking as the target. For nets with an upward-closed initial marking, we randomly chose to start with a number of tokens between 1 and 20\% of the length of the walk. It is important to note that even with long random walks, instances can (and in fact tend to) have short witnesses. To remove trivial instances and only keep the most challenging ones, we removed those instances where FastForward or LoLA reported a witness of length at most 20, disregarding the transitions used to generate the initial marking. This leaves us with 127 challenging instances on which the shortest witness is either unknown or has length more than 20. Moreover, this yields real-world Petri nets with no bias towards any specific kind of targets.

This table summarizes the characteristics of the various benchmarks:

| Suite          | Size | Number of places min. | med. | mean | max. | Number of transitions min. | med. | mean | max. |
|----------------|------|-----------------------|------|------|------|---------------------------|------|------|------|
| Coverability   | 61   | 16                    | 87   | 226  | 2826 | 14                        | 181  | 1519 | 27370|
| Sypet          | 30   | 65                    | 251  | 320  | 1199 | 537                       | 2307 | 2646 | 8340 |
| Random walks   | 127  | 52                    | 306  | 531  | 2826 | 60                        | 3137 | 5885 | 27370|

Tool comparison. To evaluate our approach on reachability instances, we compare FastForward to LoLA [56], a tool developed for two decades that wins several categories of the Model Checking Contest every year. LoLA is geared towards model checking of finite state spaces, but it implements semi-decision procedures for the unbounded case. We further compare the three selection strategies of Algorithm 1: A*, GBFS and Dijkstra; the two first with the distance under-approximation scheme $D_Q$, which provides the best trade-off between estimate quality and efficiency. We also considered comparing with KReach [17], a tool showcased at TACAS’20 that implements an exact non-elementary algorithm. However, it timed out on all instances, even with larger time limits.

Figure 2 depicts the number of reachability instances decided by the tools within the time limit. As shown, all approaches outperform LoLA, with GBFS as the clear winner on the Random-Walk suite and A* slightly better on the Sypet suite. Note that Dijkstra’s selection strategy sometimes competes due to its locally very cheap computational cost (no heuristic evaluation), but its performance generally decreases as the distance increases.
To demonstrate the versatility of our approach, we also benchmarked FastForward on the original coverability instances. Recall that coverability is an \textsc{ExpSpace}-complete problem that reduces to reachability in linear time \cite{Kesten2007,Ye2012}. While its complexity exceeds the \textsc{PSpace}-completeness of reachability for finite state-spaces \cite{Schewe2012,Leino2003}, it is much more tame than the non-elementary complexity of (unbounded) reachability. We compare FastForward to four tools implementing algorithms tailored specifically to the coverability problem: LoLA, Bfc \cite{Czarnecki2012}, ICover \cite{Colom2005} and the backward algorithm (based on \cite{Bjornsson1992}) of \textsc{Mist} \cite{Czarnecki2010}. We did not test \textsc{Petrinizer} \cite{Colom2007} since it only handles negative instances, while we focus on positive ones; likewise for \textsc{QCover} \cite{Colom2006} since it is superseded by ICover.

Figure 3 illustrates the number of coverability instances decided within the time limit. The left side corresponds to an evaluation on the original instances where FastForward performs pruning (included in its runtime). On the right hand right side the pruned instances are the input for all tools, and the time for this pruning is not included for any tool. As a caveat, ICover performs its
own preprocessing which includes pruning among techniques specific to coverability. This preprocessing is enabled (and its time is included) even when pruning is already done. Using \texttt{FastForward}(A^*, D_Q), we decide more instances than all tools on unpruned Petri nets, and one less than Bfc for pre-pruned instances. It is worth mentioning that with a time limit of 10 minutes per instance, \texttt{FastForward}(A^*, D_Q) is the only tool to decide all 61 instances.

![Fig. 4. Runtime comparison against FF(A^*, D_Q) (left) and FF(GBFS, D_Q) (right), in seconds, for individual instances without pre-pruning. Tools on the first column of each side include coverability and reachability instances, while those on the second column of each side include coverability only. Marks on the gray lines denote timeouts (60 s).]

We also compared the running time of A^* and GBFS with D_Q to the other tools and approaches. For each tool, we considered the type of instances it can handle: either reachability and coverability, or coverability only. Figure 4 depicts this comparison, where the base approach is faster for data points that lie in the upper-left half of the graph. The axes start at 0.1 second to avoid a comparison based on technical aspects such as the programming language. Yet, LoLA, Bfc and mist regularly solve instances faster than this, which speaks to their level of optimization. We can see that \texttt{FastForward} outperforms ICover, LoLA and mist overall. We cannot compete with Bfc in execution time as it is a highly optimized tool specifically tailored to only the coverability problem that can employ optimization techniques such as Karp-Miller trees that do not work for reachability queries.

**Length of the witnesses.** Since our approach is also geared towards the identification of short(est) reachability witnesses, we compared the different tools with respect to length of the reported one, depicted in Figure 5. Positive values on the y-axis mean the witness was not minimal, while y = 0 means it was.
Note that the points for Bfc must be taken with a grain of salt: it uses a different file format, and its translation utility can introduce additional transitions. This means that even if Bfc found a shortest witness, it could be longer than a shortest one of the original instance.

Still, the graph shows that reported witnesses can be far from minimal. For example, on one instance LoLA returns a witness that is 53 transitions longer than the one of FastForward(A∗, DQ). Still, LoLA returns a shortest witness on 28 out of 43 instances. Similarly, FastForward(GBFS, DQ) finds a shortest path on 60 out of 83 instances. In contrast, MIST finds a shortest witness on all instances since its backward algorithm is guaranteed to do so on unweighted Petri nets, which constitute all of our instances. Again, this approach is tailored to coverability and cannot be lifted to reachability.

**Heuristics and pruning.** We briefly discuss the quality of the heuristics and the impact of pruning. The left-hand side of Figure 6 compares the exact distance to the estimated distance from the initial marking.\(^7\) It shows that it is incredibly accurate for all \(G\)-distances, but even more so for \(G = Q \geq 0\). We experimented with this distance using the logical translation of [8] and Z3 [52] as the optimization modulo theories solver. At present, it appears that the gain in estimate quality does not compensate for the extra computational cost.

As depicted on the right-hand side of Figure 6, pruning can make some instances trivial, but in general, many challenging instances remain so. On average, around 50% of places and 40% of transitions were pruned.

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\(^6\) These numbers disregard instances where the tool did not finish or where a shortest witness is not known, i.e. no method guaranteeing one finished in time.

\(^7\) Z3 reported two non optimal solutions which explains the two points above the line.
6 Conclusion

We presented an efficient approach to the Petri net reachability problem that uses state-space over-approximations as distance oracles in the classical graph traversal algorithms $A^*$ and greedy best-first search. Our experiments have shown that using the state equation over $Q^S_T \geq 0$ provides the best trade-off between computational feasibility and the accuracy of the oracle. However, we expect that further advances in optimization modulo theories solvers may enable employing stronger over-approximations such as continuous Petri nets in the future.

Moreover, non-algebraic distance under-approximations also fit naturally in our framework, e.g. the syntactic distance of [58] and “$\alpha$-graphs” of [25]. These are crude approximations with low computational cost. Our preliminary tests show that, although they could not compete with our distances, they can provide early speed-ups on instances with large branching factors. An interesting line of research consists in identifying cheap approximations with better estimates.

We wish to emphasize that our approach to the reachability problem has the potential to also be naturally used for semi-deciding reachability in extensions of Petri nets with a recursively enumerable reachability problem, such as Petri nets with resets and transfers [3,19] as well as colored Petri nets [40]. These extensions have, for instance, been used for the generation of program loop invariants [57], the validation of business processes [62] and the verification of multi-threaded C and JAVA program skeletons with communication primitives [15,42]. Linear rational and integer arithmetic over-approximations for such extended Petri nets exist [11,9,37,34] and could smoothly be used inside our framework.

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A primer on applications of Petri net reachability

This section provides two representative examples from the literature that illustrate the important role of Petri net reachability. They allow us to underpin our claim that it is desirable to find shortest paths witnessing reachability. Our examples come from program synthesis and concurrent program analysis. We conclude this section with a brief discussion on further applications.

Program synthesis. The authors of [25] and [33] have recently employed the Petri net reachability problem for automated program synthesis. In their setting, one is given an API containing hundreds or thousands of functions, together with a type signature and a number of test cases. The goal is to automatically synthesize a loop-free program using functions from the API that respects the specified type signature and satisfies the given test cases.

Let us illustrate the approach with an example from [25]. Suppose we have access to library `java.awt.geom`, and we wish to synthesize a function `rotate` with type signature

\[
\text{Area rotate(Area object, Point2D point, double angle)}.
\]

Naturally, the function should rotate the supplied `Area` around `point` by `angle` degrees. We assume the `java.awt.geom` library is sufficient for this task in that it contains the functions needed to synthesize the method. Figure 7 presents an excerpt of functions contained in the API.

![Java API excerpt](image)

**Fig. 7.** A small sample of methods from library `java.awt.geom`.

The authors of [25] suggest to view an API as a Petri net whose places correspond to types and transitions correspond to API functions which, informally speaking, consume input types and produce an output type. Figure 8 illustrates the Petri net corresponding to the excerpt of API functions listed in Figure 7. To synthesize the `rotate` function above, we start with tokens in the places corresponding to the input parameters of our function. Thus, in Figure 8 we have one token in each of the places corresponding to `Area`, `Point2D` and `double`. The goal is then to reach a marking with a single token in the place corresponding to the return type. In our example, we aim for one token in `Area`, and no token in
any other place. This corresponds to invoking a sequence of functions that “use up” all input parameters, and finally return the correct type. To allow reuse of variables, additional “copy” transitions are introduced for each place; they take one token from a place and put two tokens back. If the target marking is reachable, then the witnessing path corresponds to a partial sketch of a program.

For example, the path

\[
\text{copyPoint2D} \rightarrow \text{GetY} \rightarrow \text{GetX} \rightarrow \text{new AffineTransformation} \rightarrow \\
\text{copyAffineTransformation} \rightarrow \text{setToRotation} \rightarrow \text{createTransformedArea}
\]

tells us which functions to apply, and in which order to apply them. Since Petri nets do not store information about the identity of tokens, when we have multiple objects of the same type, we do not know which to supply as an argument to which function. This can be figured out by a separate process involving SAT solving (see [25] for more details).

As discussed in [25], finding short paths of the Petri net is a natural goal. Indeed, since short programs are easier to test, there are fewer possibilities for the arguments of each function, and it is easier for humans to verify that the synthesized program has the desired functionality.

Concurrent program analysis. Perhaps most prominently, Petri nets have been used in order to model and analyze concurrent processes. Let us begin with a simple example illustrating how the Petri net reachability problem can be used in order to detect race conditions in concurrent programs. Consider function \texttt{fun()} of Figure 9 in which \texttt{s} is a global shared Boolean variable. If there is a single thread running \texttt{fun()}, then the condition of the if-statement in Line 3 never evaluates to true and an error cannot occur. However, if there are two independently interleaved threads running \texttt{fun()}, it is possible that one thread reaches Line 3 whilst \texttt{s} is set to 1, which means an error could occur.

In more technical terms, we consider non-recursive Boolean programs in which an unbounded number of identical programs run in parallel. The authors
of [32] showed that verifying safety properties of such concurrent programs can be reduced to the coverability problem for Petri nets using a technique called counter abstraction. The coverability problem is a weaker version of the reachability problem. Given a target marking, the coverability problem asks whether it is possible to reach a marking in which every place carries at least as many tokens as specified by the target marking. The Petri net obtained by applying the approach of [32] to the program from Figure 9 is depicted in Figure 10. The places on the top of the Petri net correspond to the program locations of Figure 9. Tokens in each of the places on the top count the number of threads which are currently at the respective program location, which is a form of counter abstraction. At any time, transition `fun()` can add tokens to `loc_1`, reflecting that a new thread executing `fun()` can be spawned at any point in time arbitrarily often. The two places on the bottom encode the state of the Boolean variable `s` which is updated whenever a transition moves tokens from `loc_1` to `loc_2`, or from `loc_2` to `loc_3`. Determining whether an error can occur then reduces to deciding whether the marking `[Err: 1]` is coverable, i.e., whether there is an interleaving in which at least one thread produces an error.

In stark contrast to the reachability problem, it was shown in [54] that the coverability problem belongs to \textsc{ExpSpace}. There is a natural reduction from the coverability problem to the reachability problem: by introducing additional transitions that can non-deterministically remove tokens from every place corresponding to program lines, a target marking in the original Petri net is coverable iff it is reachable in the Petri net with the additional transitions. Alternatively,
deciding coverability can be rephrased as the problem of determining whether an upward-closed set of markings is reachable in the directed graph induced by a given Petri net, which is the approach that we take.

Further applications The authors of [24] show how proofs involving counting arguments, which can, for instance, naturally prove properties of concurrent programs with recursive procedures, can automatically be synthesized by a reduction to the Petri net reachability problem. The authors of [30] propose a model for reasoning about finite-data asynchronous programs. They show that proving liveness properties of such programs in their model is inter-reducible with the Petri net reachability problem. In a broader context, it was shown that various verification problems for population protocols, a formal model of sensor networks, reduce to the Petri net reachability problem [22]. The authors of [16] develop a method that allows for verifying rich models of data-driven workflows by a reduction to the coverability problem for Petri nets. See also survey [53] for further classical application areas of Petri nets and their extensions.

B Missing proofs of Section 3

Recall the following invariant satisfied by Algorithm 1:

if $g(v) \neq \infty$, then $g(v)$ is the weight of a path from $s$ to $v$ in $G$
whose nodes were all expanded, except possibly $v$.  \((*)\)

We prove this lemma from the main text:

Lemma 1. If $G$ is locally finite and $h$ is unbounded, then the following holds:
1. The set of paths of weight at most $c \in \mathbb{Q}_{\geq 0}$ starting from node $s$ is finite.
2. Let $W \subseteq V$. The set $\text{dist}_G(W, t) := \{\text{dist}_G(w, t) : w \in W\}$ has a minimum.
3. No node is expanded infinitely often by Algorithm 1.

Proof. Let $d := \min\{\mu(e) : e \in E\}$.

1. Any path of weight at most $c$ traverses at most $k := \lceil c/d \rceil$ edges. Since the graph has finite out-degree, the number of paths from $s$ using at most $k$ edges is finite.
2. Suppose the claim false. We have $\text{dist}_G(v_0, t) > \text{dist}_G(v_1, t) > \cdots$ for some $v_0, v_1, \ldots \in W$. Let $k := \lceil \text{dist}_G(v_0, t)/d \rceil$. Let $V_{\leq k}$ be the set of nodes that can reach $t$ by traversing at most $k$ edges. Since $G$ has finite in-degree, $V_{\leq k}$ is finite. Moreover, any node $v \in V \setminus V_{\leq k}$ is such that $\text{dist}_G(v, t) > k \cdot d \geq \text{dist}_G(v_0, t)$. Hence, $\{v_0, v_1, \ldots\} \subseteq V_{\leq k}$ is finite, which is a contradiction.
3. For the sake of contradiction, assume a node $v$ is expanded infinitely often. Each time node $v$ is expanded, it is removed from $C$. Hence, it is reinserted infinitely often in $C$. Moreover, each time this happens, value $g(v)$ is decreased. Let $q_0, q_1, \ldots \in \mathbb{Q}_{\geq 0}$ denote these increasingly smaller values. By \((*)\), there is a path $\pi_i$ from $s$ to $v$ of weight $q_i$ in $G$. By (1), $\{\pi_i : i \in \mathbb{N}\}$ is finite as the weight of these paths is at most $q_0$. This contradicts $q_0 > q_1 > \cdots$. \(\square\)
C  Missing proofs of Section 4.2

Proposition 2. The functions $d_Z$, $d_Q$, $d_{Q > 0}$ are effective.

Proof. Let us prove the case of $d_{Q > 0}$, which was only sketched in the main text. The reachability relation of a continuous Petri net can be expressed in the existential fragment of linear real arithmetic, i.e. $\mathrm{FO}(\mathbb{Q}, +, \cdot)$, the first-order theory of the rationals with addition and order [8]. More precisely, there exists a linear-time computable formula $\psi \in \exists \mathrm{FO}(\mathbb{Q}, +, <)$ such that $\psi(m, x, m')$ holds if

there exists a sequence $\sigma \in (0, 1] \times T$ s.t. $m \xrightarrow{\sigma} m'$ and $\sigma = x$.

Let $\mathcal{F} = (P, T, f, \lambda)$ be a weighted Petri net, let $m_{\text{target}}$ be a target marking, and let $h_{d_{Q}}$ be the heuristic obtained from $d_{Q}$ for $m_{\text{target}}$. Observe that $h_{d_{Q}}(m) \leq h_{d}(m)$ for every marking $m$ and every $d_{Q} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Q}_{\geq 0}\}$. Hence, if $h_{d}$ is unbounded, so are all three heuristics. Thus, it suffices to prove the case $d_{Q} = Q$.

For the sake of contradiction, suppose $h_{Q}$ is not unbounded. There exists $b \in Q_{\geq 0}$ and an infinite sequence of pairwise distinct markings $m_0, m_1, \ldots \in \mathbb{N}^P$ with $h_{Q}(m_i) \leq b$ for every $i \geq 0$. Let $x_i \in Q_{\geq 0}^T$ be a solution to the state equation over $Q_{\geq 0}$ that yields $h_{Q}(m_i)$, i.e. such that $h_{Q}(m_i) = \sum_{t \in T} \lambda(t) \cdot x_i(t)$ is minimized subject to

$$m_{\text{target}} = m_i + \sum_{t \in T} x_i(t) \cdot \Delta_i. \tag{1}$$

Since $\mathbb{N}^P$ is well-quasi-ordered, there exist indices $i_0 < i_1 < \cdots$ such that $m_{i_0} \leq m_{i_1} \leq \cdots$. Since these markings are pairwise distinct, we may assume w.l.o.g. the existence of a place $p \in P$ such that $m_{i_0}(p) < m_{i_1}(p) < \cdots$ (otherwise, we could extract such a subsequence).

Let us define the following constants:

$$c := \min \{\lambda(t) : t \in T\} \quad \text{and} \quad d := \frac{b \cdot |T| \cdot \max \{|\Delta_i(p)| : t \in T\}}{c}.$$

Let $j \geq 0$ be such that $m_{\text{target}}(p) - m_{i_j}(p) < -d$. Such an index $j$ exists as $p$ takes arbitrarily large values along our infinite sequence. By (1), we have:

$$\sum_{t \in T} x_{i_j}(t) \cdot \Delta_i(p) = m_{\text{target}}(p) - m_{i_j}(p) < -d.$$
Thus, there exists $s \in T$ such that $\Delta_s(p) < 0$ and $x_{ij}(s) > b/c$. Indeed, if it was not the case, it would be impossible to obtain a negative value smaller than $-d$.

We are done since we obtain the following contradiction:

$$h_Q(m_{ij}) = \sum_{t \in T} \lambda(t) \cdot x_{ij}(t) \quad \text{(by definition)}$$

$$\geq \lambda(s) \cdot x_{ij}(s) \quad \text{(by } \lambda(t) > 0 \text{ and } x_{ij}(t) \geq 0 \text{ for each } t \in T)$$

$$\geq \lambda(s) \cdot (b/c) \quad \text{(by } \lambda(s) > 0 \text{ and } x_{ij}(s) > b/c)$$

$$\geq \lambda(s) \cdot (b/\lambda(s)) \quad \text{(by } \lambda(s) \geq c)$$

$$= b$$

$$\geq h_Q(m_{ij}) \quad \text{(by boundedness).}$$

\[\square\]

### D Experimental results

Figure 11 depicts an evaluation on reachability instances where all tools were given the pruned Petri nets (preprocessing time not included for any tool). The results are essentially the same as those of Figure 2.

![Cumulative number of reachability instances decided over time](image)

**Fig. 11.** Cumulative number of reachability instances decided over time (on pre-pruned instances). **Left:** SYPET suite (semi-log scale). **Right:** RANDOM-WALK suite (log scale).

### E Structural distance

All three $G$-distances presented in the main text have an algebraic flavor. While their complexity is significantly lower than the non-elementary time complexity of Petri net reachability, they involve solving optimization problems. An alternative avenue, mentioned in the conclusion, consists in constructing less precise but more efficient distance under-approximation based on structural properties.

We describe such a distance under-approximation adapted from the syntactic distance of [58] and related to the “$\alpha$-graphs” used by [25]. Let $N = (P, T, f, \lambda)$ be a weighted Petri net. The structural abstraction of $N$ is a weighted graph
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$G_{\text{struct}}(\mathcal{N})$ with places as nodes with an edge $(p, t, q)$ iff transition $t$ consumes tokens from $p$ and produces tokens into $q$. Since some transitions may consume or produce no token, we imagine these as consuming from, or producing to, an artificial “sink place” $\perp$. Intuitively, if $m$ can reach $m'$, then each token of $m$ must either make its way to $m'$ or disappear. Of course, tokens cannot move independently and freely in $\mathcal{N}$. However, paths in $G_{\text{struct}}(\mathcal{N})$ yield a lower bound on an actual path from $m$ to $m'$. A structural abstraction is given in Figure 12.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure12.png}
\caption{Left: A Petri net $\mathcal{N}$. Right: Its structural abstraction $G_{\text{struct}}(\mathcal{N})$.}
\end{figure}

Formally, let $\text{in}(t) := \{p \in P : f(p, t) > 0\}$ be the set of input places of $t$ if it is nonempty, and $\text{in}(t) := \{\perp\}$ otherwise; and let $\text{out}(t) := \{p \in P : f(p, t) > 0\}$ be the set of output places of $t$ if it is nonempty, and $\text{out}(t) := \{\perp\}$ otherwise. We define $G_{\text{struct}}(\mathcal{N}) := (V, E, T, \mu)$ with $V := P \cup \{\perp\}$, $\mu(p, t, q) := \lambda(t)$ and

$$E := \{(p, t, q) : p \neq q, t \in T, p \in \text{in}(t) \text{ and } q \in \text{out}(t)\}.$$ 

We obtain the structural distance $d_{\text{struct}} : \mathbb{N}^P \times \mathbb{N}^P \to \mathbb{Q}_{\geq 0} \cup \{\infty\}$ defined as follows. For every marking $m$, let $[m] := \{p \in P : m(p) > 0\}$ be the places marked in $m$ together with $\perp$ (considered permanently marked). Let:

$$d_{\text{struct}}(m, m') := \max \{\kappa_{m'}(p) : p \in [m]\},$$

where

$$\kappa_{m'}(p) := \min \{\text{dist}_{G_{\text{struct}}}(p, q) : q \in [m']\}.$$ 

Informally, $\kappa_{m'}(p)$ is the distance required to freely move a token from place $p$ to a place marked in $m'$, or to destroy it. Since every token of $m$ must achieve this task, $d_{\text{struct}}$ maximizes $\kappa_{m'}(p)$ among all places marked in $m$. Consider the Petri net of Figure 12 with $m := [p_1: 0, p_2: 1, p_3: 1]$ and $m' := [p_1: 1, p_2: 0, p_3: 0]$. We have $d_{\text{struct}}(m, m') = 2$ since $\kappa_{m'}(p_2) = 2$ and $\kappa_{m'}(p_3) = 1$.

We show that $d_{\text{struct}}$ is an under-approximation by first proving a lemma:

**Lemma 2.** If $m \xrightarrow{\sigma} m'$, then for every $p \in [m]$ there exists a path of weight at most $\lambda(\sigma)$ from $p$ to some $q \in [m']$ in $G_{\text{struct}}(\mathcal{N})$.

**Proof.** We proceed by induction on $|\sigma|$. If $|\sigma| = 0$, then the claim follows immediately with the empty path. Assume $\sigma = t\tau$ with $t \in T$ and $\tau \in T^*$. There is some marking $m''$ such that $m \xrightarrow{t} m'' \xrightarrow{\tau} m'$. By induction hypothesis, for every $r \in [m'']$, there exists a path $\pi_r$ of weight at most $\lambda(\tau)$ from $r$ to some $q \in [m']$ in $G_{\text{struct}}(\mathcal{N})$. Let $p \in [m]$. We must exhibit a path from $p$.

If $p \in [m']$, then we are done as path $\pi_p$ satisfies $\mu(\pi_p) \leq \lambda(\tau) \leq \lambda(\sigma)$. So, assume $p \not\in [m']$. By definition of $E$, we have $e := (p, t, r) \in E$ for some
Proposition 3. It is the case that $d_{\text{struct}}$ is a distance under-approximation.

Proof. Let $m, m', m'' \in \mathbb{N}^P$ be markings. We prove admissibility by establishing each property.

Distance under-approximation. We must show that $d_{\text{struct}}(m, m') \leq \text{dist}_N(m, m')$. Assume the latter differs from $\infty$, as we are otherwise done. Let $\sigma \in T^*$ be a shortest firing sequence such that

$$m \xrightarrow{\sigma} \mathbb{N} m'. $$

Let $p \in \llbracket m \rrbracket$ maximize $\kappa_{m'}(p)$. By Lemma 2, $G_{\text{struct}}(N)$ has a path $\pi$ of weight at most $\lambda(\sigma)$ from $p$ to some $q \in \llbracket m \rrbracket$. Thus, $d_{\text{struct}}(m, m') = \kappa_{m'}(p) \leq \text{dist}_{G_{\text{struct}}(N)}(p, q) \leq \lambda(\sigma) = \text{dist}_N(m, m')$.

Triangle inequality. We show $d_{\text{struct}}(m, m'') \leq d_{\text{struct}}(m, m') + d_{\text{struct}}(m', m'')$.

Assume the right-hand side does not equal $\infty$ as we are otherwise done. Let $p, p' \in \llbracket m \rrbracket$ and $q \in \llbracket m' \rrbracket$ respectively maximize $\kappa_{m'}(p)$, $\kappa_{m''}(p')$ and $\kappa_{m''}(q)$.

Let $q' \in \llbracket m' \rrbracket$ and $r \in \llbracket m'' \rrbracket$ be such that $\kappa_{m''}(p') = \text{dist}_{G_{\text{struct}}}(p', q')$ and $\kappa_{m''}(q') = \text{dist}_{G_{\text{struct}}}(q', r)$. Note that they are well-defined by $\kappa_{m'}(p) \neq \infty$ and $\kappa_{m''}(q) \neq \infty$.

We have:

\[
d_{\text{struct}}(m, m'') = \kappa_{m''}(p') \quad \text{(by def. of } d_{\text{struct}}) \\
\leq \text{dist}_{G_{\text{struct}}}(p', r) \quad \text{(by } r \in \llbracket m'' \rrbracket \text{ and min. of } \kappa_{m''}(p')) \\
\leq \text{dist}_{G_{\text{struct}}}(p', q') + \text{dist}_{G_{\text{struct}}}(q', r) \quad \text{(by the triangle inequality)} \\
= \kappa_{m''}(p') + \kappa_{m''}(q') \\
\leq \kappa_{m''}(p') + \kappa_{m''}(q') \quad \text{(by } q' \in \llbracket m' \rrbracket \text{ and max. of } q) \\
\leq \kappa_{m'}(p) + \kappa_{m''}(q) \quad \text{(by } p' \in \llbracket m' \rrbracket \text{ and max. of } p) \\
= d_{\text{struct}}(m, m') + d_{\text{struct}}(m', m'') \quad \text{(by def. of } d_{\text{struct}}). \\
\]

Effectiveness. The structural abstraction $G_{\text{struct}}(N)$ can be precomputed in linear time from $N$, and $\text{dist}_{G_{\text{struct}}(N)}(p, q)$ can then be precomputed in polynomial time using e.g. Dijkstra’s algorithm. After these steps, $d_{\text{struct}}(m, m')$ can be evaluated in time $O(\|\llbracket m \rrbracket\| \cdot \|\llbracket m' \rrbracket\|)$. ☐

Let us stress that $d_{\text{struct}}(m, m')$ yields a crude estimation of $\text{dist}_N(m, m')$. Indeed, its value is always upper bounded by $|P| \cdot \max\{\lambda(t) : t \in T\}$, while the actual distance could be arbitrarily large in $m$ and $m'$. Nevertheless, it is lightweight since it enables pre-computations. This makes it useful in particular for reachability graphs with short paths but large branching factors.

For example, instances from the SYPET suite have a large branching factor. They have between 23 and 187 unguarded transitions. Most markings tend to
enable some guarded transitions as well, so the average branching factor is larger. In particular, the branching factor of initial markings ranges from 30 to 300.\footnote{Chess and Go respectively have an average branching factor of \(~35\) and \(~350\) \cite{55}.}

Let $D_{\text{struct}}$ be distance under-approximation scheme obtained from the structural distance. This scheme is not unbounded, but can still be used with GBFS without termination guarantee. Figure 13 compares the performance of FastForward using $A^\ast$ with $D_Q$ and using GBFS with $D_{\text{struct}}$ on a time limit of 600 seconds. The former is faster on most instances, but it is vastly outperformed by $A^\ast$ on a few instances. An explanation is provided by the large branching factor and short paths, and how these emphasize the characteristics of the different approaches. Note that the structural abstraction can be precomputed. On the other hand, $A^\ast$ requires computing the heuristic on each successor before the next node is chosen for expansion. It thus is at a slight disadvantage on instances where a shortest witness is so short that it is found rather quickly even with the coarse structural distance. Its advantage is on the instances where the length of a shortest witness is at the upper end of the range. There, the large branching factor fully comes into play and a search algorithm must more aggressively discard parts of the search space.

![Figure 13. Results on the spet suite with a time limit of 600 seconds. Left: Cumulative number of instances shown reachable. Right: Performance comparison per instance of FastForward with two different schemes. Marks on the gray lines denote timeouts.](image-url)