Non-linear generalization of the $sl(2)$ algebra *

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Abstract

We present a generalization of the $sl(2)$ algebra where the algebraic relations are constructed with the help of a general function of one of the generators. When this function is linear this algebra is a deformed $sl(2)$ algebra. In the non-linear case, the finite dimensional representations are constructed in two different ways. In the first case, which provides finite dimensional representations only for the non-linear case, these representations come from solutions to a dynamical equation and we show how to construct explicitly these representations for a general quadratic non-linear function. The other type of finite dimensional representation comes from solutions to a cut condition equation. We give examples of solutions of this type in the non-linear case as well.

Keywords: $sl(2)$ algebra; dynamical systems; attractors; quantum algebras; $su_q(2)$ algebra.
1 Introduction

Two years ago, a class of generalized Heisenberg algebras was introduced where, within this class, one finds deformed and non-deformed Heisenberg-type algebras [1, 2]. The representations of these algebras were constructed using concepts of dynamical systems as attractors and their stabilities [2]. A simple example of this class of algebras is the well-known \(q\)-oscillators [3]. It was also shown that this class describes the Heisenberg-type algebraic structure of a family of one-dimensional quantum systems having any two successive energy levels related by \(\epsilon_{n+1} = f(\epsilon_n)\), where \(f\) is a characteristic function of the algebra [4]. As a possible physical consequence of this family of Heisenberg-type algebras, it was shown that it is possible to construct a non-standard quantum field theory based on this algebra [5].

In [6] it was shown that this way of generalizing an algebraic structure can be implemented to the \(sl(2)\) algebra as well. In this case the algebraic relations of the generalized \(sl(2)\) algebra also present a characteristic function of one of the generators of the algebra so that when this function is taken as \(f(x) = x - 1\) the well-known \(sl(2)\) algebra is recovered.

In this letter we discuss the finite dimensional representations of this generalized \(sl(2)\) class of algebras. These representations are constructed by solving an equation that admits two types of solutions. The first type, which happens only in the non-linear generalization of \(sl(2)\), is obtained by solving a dynamical equation and the second type of finite dimensional representation is obtained from the solutions to a cut condition equation. This last type of solution is responsible for all finite dimensional representations of the linear case. We also construct a map connecting the linear case of this generalized \(sl(2)\) algebra with \(sl_q(2)\). Since the generalized \(sl(2)\) class of algebras presented here recovers a deformed \(sl(2)\) algebra for the linear case which is known to be connected with some simple two-dimensional integrable systems, [6] we hope that the non-linear case of this non-linear generalized algebra can be of help in understanding more complex integrable systems.

In section II we introduce the iterative generalized \(sl(2)\) algebra and present the general conditions to obtain finite and infinite dimensional representations. In section III, we establish the connection of the linear case of this algebra with the \(sl_q(2)\) algebraic structure. We also show that it is possible to study the finite dimensional representations taking into account the stability of the fixed points of \(f\). In section IV we study the generalized algebra for a non-linear function \(f\) and show that part of the finite dimen-
sional representations is obtained by finding the cycles of the characteristic function of the algebra, \( f(x) \). We also construct another type of finite dimensional representations in this case, which is associated with the solutions to a cut condition equation.

## 2 Iterative generalized \( sl(2) \) algebra

Let us consider the following algebraic relations among the generators \( J_0 \), \( J_+ \) and \( J_- \):

\[
J_0 J_- = J_- f(J_0) ,
\]

\[
J_+ J_0 = f(J_0) J_+ ,
\]

\[
[J_+, J_-] = J_0(J_0 + 1) - f(J_0)(f(J_0) + 1) ,
\]

where we assume \( J_- = J_+^\dagger \), \( J_0^\dagger = J_0 \) and that \( f(J_0) \) is an analytical function in \( J_0 \).

This algebra satisfies, for all functions \( f \), the Jacobi identity

\[
[J_0, [J_+, J_-]] + [J_-, [J_0, J_+]] + [J_+, [J_-, J_0]] = 0 .
\]

The first term of the L.H.S. is identically null due to eq. (3). To show that the sum of the other two terms is equal to zero it is enough to expand them and use the property, derived from eqs. (1) and (2), that \( [J_0, J_+ J_-] = 0 \).

Using the algebraic relations in eqs. (1-3) it can be shown that the operator

\[
C = 1/2 \{ J_+ J_- + J_- J_+ + J_0(J_0 + 1) + f(J_0)(f(J_0) + 1) \}
\]

satisfies the commutation relations \( [C, J_0] = [C, J_\pm] = 0 \), i.e., \( C \) is a Casimir operator of the algebra.

If we substitute the specific function \( f(J_0) = J_0 - 1 \) in eqs. (1-3), we reobtain the well-known \( sl(2) \) algebra. Thus, the algebraic relations proposed in eqs. (1-3) contain, as a particular case, the \( sl(2) \) algebra when we choose a specific linear function of \( J_0 \). To discuss this algebra and the role of the function \( f \) in it, we present its respective representation theory.

Under the hypothesis that the function \( f \) and the initial value \( \alpha_j \) satisfy

\[
\alpha_j > f(\alpha_j) > f(f(\alpha_j)) > \ldots > f^m(\alpha_j) > \ldots ,
\]

where \( f^m \) means the \( m \)-th
iterate of $\alpha_j$ through $f$ and $m$ is a positive integer, and that there is a vector, the highest weight vector, such that

$$J_+|\alpha_j, j\rangle = 0,$$  \hspace{1cm} (6)

we obtain for a general $m$ lying between 0 and $2j$ [3]:

$$J_0|\alpha_j, j - m\rangle = \alpha_{j-m}|\alpha_j, j - m\rangle$$  \hspace{1cm} (7)

$$J_+|\alpha_j, j - m\rangle = N_{m-1}|\alpha_j, j - m + 1\rangle$$  \hspace{1cm} (8)

$$J_-|\alpha_j, j - m\rangle = N_m|\alpha_j, j - m - 1\rangle$$  \hspace{1cm} (9)

$$C|\alpha_j, j - m\rangle = \alpha_j(\alpha_j + 1)|\alpha_j, j - m\rangle,$$  \hspace{1cm} (10)

where $N_m^2 = (\alpha_j - \alpha_{j-m-1})(\alpha_j + \alpha_{j-m-1} + 1) = \alpha_j(\alpha_j + 1) - \alpha_{j-m-1}(\alpha_{j-m-1} + 1)$ that can be proved in a similar way to the proof made for the generalized Heisenberg algebra in [3], where the additional equation, eq. (10), is obtained using eqs. (3) and (5). As shown in [6] the allowed values of $\alpha_i$ satisfy

$$\alpha_j \geq \alpha_i \geq \alpha_b = f^{d-1}(\alpha_j)).$$  \hspace{1cm} (11)

If we put $m = d - 1$ in eq. (9), in order to have a $d$-dimensional representation, we must solve

$$N_{d-1} = 0.$$  \hspace{1cm} (12)

This equation is satisfied if $\alpha_j = \alpha_{j-d}$ or if the highest weight of the representation satisfies a cut condition equation

$$\alpha_j + \alpha_{j-d} + 1 = 0.$$  \hspace{1cm} (13)

As it is clear in what follows, the equation $\alpha_j = \alpha_{j-d}$ will have solutions only in the non-linear case (for $d > 1$) while the finite dimensional representations coming from solutions to eq. (13) will have highest weights satisfying

$$\alpha_j > -\alpha_j - 1 \Rightarrow \alpha_j > -1/2,$$  \hspace{1cm} (14)

since $\alpha_j > \alpha_{j-d}$.

3 Linear functions
3.1 \( sl(2) \)

As we have seen, with \( f(J_0) = J_0 - 1 \) we reproduce the commutation relations of the \( sl(2) \) algebra. It is straightforward to verify that for the function under consideration a general eigenvalue \( \alpha_{j-m} \) can be written as:

\[
\alpha_{j-m} = f^m(\alpha_j) = \alpha_j - m.
\]

(15)

Using eq. (15) in eq. (13) and remembering that \( d = 2j + 1 \), we have \( \alpha_j = j \). As \( 2j + 1 \) is an integer, this means that \( j \) and consequently \( \alpha_j \) can be an integer or a semi-integer, as it is well-known. We can also see that the lowest eigenvalue is \( -j \) and the eigenstates can be written as \( |j, j-m\rangle \), where \( m \) goes from zero to \( 2j \). We should also say that the Casimir operator of \( sl(2) \) is obtained from eq. (5).

In general, in order to see the iterations through a graphical analysis of the function \( f \) we graph \( y = f(x) \) together with \( y = x \). Where the lines intersect we have \( x = y = f(x) \), so that the intersections are precisely the fixed points. Now, for a point \( x_0 \), different from the fixed point, in order to follow its path through iterations with the function \( f \), we perform the following steps:

1. move vertically to the graph of \( f(x) \),
2. move horizontally to the graph of \( y = x \), and
3. repeat steps 1, 2, etc.

We now present in fig. (1) a curious graphical representation of a finite dimensional representation of the \( sl(2) \) algebra where the iterative aspect of this algebra is emphasized. In fig. (1) we plot the function \( f(\alpha) = \alpha - 1 \) versus \( \alpha \) for \( \alpha_j = j = 2 \). We also plot the vertical line representing the cut condition given by eq. (13), i.e., \( \alpha_j - d = -\alpha_j - 1 = -j - 1 = -3 \).

We see that the iterations of \( j \) through \( f \) reach exactly the intersection of the vertical line given by the cut condition and the function itself. If the starting point \( \alpha_j \) is not an integer or semi-integer, the future iterations of this value will never reach the intersection with the cut line and the iterations will evolve forever. There is no lower bound. The dimension of this representation is infinite. Note, also, that there is no fixed point in this case. In this way, a graphical analysis of the function \( f \) gives us quick and useful information about the representation of the algebra, without the need
Figure 1: Five dimensional representation of the $sl(2)$ algebra for $\alpha_j = 2$. The dashed line ($\alpha_j - d = -3$) is the cut condition. The iterations are indicated by the arrows. The thinner line is the function $f(\alpha) = \alpha$ and the thicker one is the function $f(\alpha) = \alpha - 1$.

to realize an extensive calculation. It is also easy to see in this graph that the eigenvalues always decrease in absolute value when the iterations of the function $f$ increase.

3.2 Linear deformations of $sl(2)$

Let us consider the function $f = rJ_0 - s$, where $r$ and $s$ are real numbers. The eqs. (1-3) can be written as:

\[
\begin{align*}
[J_0, J_-]_r &= -sJ_- \\
[J_0, J_+]_{r^{-1}} &= (s/r)J_+ \\
[J_+, J_-] &= (1 - r^2)J_0^2 + (1 + 2rs - r)J_0 + s(1 - s),
\end{align*}
\]

where $[A, B]_r = AB - rBA$ is the $r$-deformed commutation of two operators $A$ and $B$.

There are three cases of interest to be analysed: (I) $r = 1$ and $s > 0$; $s \neq 1$, (II) $r > 1$ and (III) $|r| < 1$. In the first case we consider only positive (unlike $s = 1$) values of $s$ because negative values of $s$ will not satisfy the condition given by eq. (11). The $r$-commutator in eqs. (16-17) turns out to be standard commutator and the eigenvalues of $J_0$ are $\alpha_{j-m} = \alpha_j - ms$. Once the dimension $(d)$ of the representation is chosen, the cut condition
given by eq. (13) leads to the following value of $\alpha_j$:

$$\alpha_j = \frac{sd - 1}{2}.$$  \hfill (19)

For a fixed value of $s$ and for each different dimension of the representation we want, we have a different initial value allowed. As $s$ can be any positive real number ($\neq 1$), this implies that $\alpha_j$ can also be a real number. Note that the transformed operators $\tilde{J}_\pm = J_\pm / s$ and $\tilde{J}_0 = J_0 / s + (1 - s)/(2s)$ obey the $sl(2)$ algebra, while $J_\pm$ and $J_0$ satisfy the algebra given by eqs. (14-18) for $r = 1$ and $s > 0; \ s \neq 1$. The graphical representation of this case is similar to the $sl(2)$ case, with constant spacing $s$ between the eigenvalues.

In case (II), $r > 1$ and thus exists a fixed point $\alpha^* = s/(r - 1)$, where the function $f$ crosses the diagonal $y = \alpha$. This fixed point is unstable, $\left(\partial f / \partial \alpha\right)_{\alpha^*} = r > 1$, showing that only points below $\alpha^*$ are allowed if we obey the condition given by eq. (11). Also, eq. (14) shows that $\alpha_j > -1/2$, a necessary condition to have $\alpha_j > \alpha_{j-1}$. Then, $s$ and $r$ should satisfy the inequality $\alpha^* > -1/2$ and $\alpha_j$ lies within $-1/2 < \alpha_j < \alpha^*$. But even in this interval, only those values of $\alpha_j$ that satisfy eq. (13) are allowed. The eigenvalues of $J_0$ can be written as:

$$\alpha_{j-m} = f^m(\alpha_j) = r^m \alpha_j - s [m]_r,$$ \hfill (20)

where $[m]_r \equiv (r^m - 1)/(r - 1)$ is the Gauss number. The cut condition given by eq. (13) and the eq. (20) yield us an expression for $\alpha_j$ once the function $f$ has been given (i.e., that $r$ and $s$ are given) and the dimension of the representation is chosen. We have for $\alpha_j$:

$$\alpha_j = (s[d]_r - 1)/(r^d + 1).$$ \hfill (21)

For each dimension we want, we have a different starting point, generally a real number. In fig. (2) we show an example of this case.

There is also a marginal two dimensional representation for $r = -1$. The case $r < -1$ is not allowed because it is not possible to obtain a highest weight representation.

In case (III), there is also a fixed point with the same formal expression for $\alpha^*$, but in this case $|r| < 1$. This fixed point is stable $\left(\partial f / \partial \alpha\right)_{\alpha^*} = r; \ |r| < 1$, indicating that only the region with $\alpha_j > \alpha^*$ is allowed (since $\alpha_j$ is the highest value). The formal expressions given by eqs. (20) and (21) are still valid here, but with $|r| < 1$. However, as in this case we are only considering
Figure 2: Case II; 4-dimensional representation for \( r = 2, s = 1, \alpha_j = 14/17 \) and \( \alpha^* = 1 \). The dashed line is the cut condition \( \alpha_j - \alpha = -31/17 \). The thinner line is the function \( f(\alpha) = \alpha \) and the thicker one is the function \( f(\alpha) = 2\alpha - 1 \).

\( \alpha_j > \alpha^* \), the iterations of \( f \) will approach the fixed point. Yet, note that in order to have a finite dimensional representation we must have \( \alpha^* < -\alpha_j - 1 \), otherwise the dimension of the representation will be infinite. Also, under the restriction given by eq. (14) yields, in this case, \( \alpha^* < -1/2 \). In this way, the cut condition given by eq. (13) should be obeyed by the allowed values of \( \alpha_j \). For a fixed function, there are infinitely countable possible values of \( \alpha_j \), one for each respective dimension. A graphical representation of this case can be seen in fig. (3).

3.3 Connection with \( sl_q(2) \)

In this section we are going to show the connection of the generalized \( sl(2) \) algebra for linear \( f(J_0) \), eqs. (1-3), with the deformation of \( sl(2) \) found in the literature as \( sl_q(2) \) [7]. The \( sl_q(2) \) generators, let us call them \( S_3 \) and \( S_{\pm} \), have the following commutation relations among them [7]:

\[
[S_3, S_{\pm}] = \pm S_{\pm}, \quad (22)
\]

\[
[S_+, S_-] = [2S_3], \quad (23)
\]

where \([x] \equiv (q^x - q^{-x})/(q - q^{-1})\). The parameter \( q \) is a real number and is called the deformation parameter of the algebra. When \( q \to 1 \) the above commutation relations recover the \( sl(2) \) relations. A simple transformation shows that \([x] = q^{-x+1}[x]_{q^2}\), where \([x]_{q^2} = (q^{2x} - 1)/(q^2 - 1)\). The action of
these generators to the states of an irreducible representation of the $sl_q(2)$ algebra, dimension of which is $2j + 1$, can be written as $\text{[7]}$:

\[
S_\pm |j, j - m\rangle = \sqrt{q^{-2j + 1} \left[ j \mp (j - m) \right] q^2 \left[ j \pm (j - m) + 1 \right] q^2 } |j, j - m \pm 1\rangle
\]

\[
S_3 |j, j - m\rangle = (j - m) |j, j - m\rangle
\]

(24)

where $2j + 1$ is a positive integer and $m = 0, 1, 2, \ldots, 2j$.

In general, explicit functionals that map generators of a specific algebra to another one can be found $\text{[8]}$. In particular, there is a specific map that converts the $sl(2)$ generators into the generators of $sl_q(2)$ $\text{[8]}$. The relevance of these maps is that they may provide information on an unknown co-algebra structure based on a known co-algebra.

In our case, if we remember the action of the $J_0$ generator in a space of dimension $2j + 1$, eq. (7), and use the expression of $\alpha_{j-m}$ given by eq. (20) with $r \equiv q^2$, we see that expressing the generator $J_0$ as:

\[
J_0 = q^{2(j-S_3)} \alpha_j - s [j - S_3] q^2,
\]

(25)

this generator acts to the $2j + 1$ states of the representation of the $sl_q(2)$ algebra exactly as it does on its own space of same dimension.
If we identify
\[ J_+ = \sqrt{(Q_1 \alpha_j - Q_2 [j - S_3 + 1] q^2)(Q_3 \alpha_j + 1 + Q_2 [j - S_3 + 1] q^2)} S_+ , \]
where \( Q_1 \equiv (q^2 - 2)/(q^2 - 1) \), \( Q_2 \equiv (q^2 - 1) \alpha_j - s \) and \( Q_3 \equiv q^2/(q^2 - 1) \), this generator also acts to the \( 2j + 1 \) states of the \( sl_q(2) \) algebra exactly the same way it does to its own \( 2j + 1 \) space of states as given by eqs. (25) and (26).

As \( J_- = J_+^\dagger \), the transformations given by eqs. (25, 26) connect the \( sl_q(2) \) algebra with the \( r \neq 1 \) linear case of our formalism.

Applying the same procedure just described above, we can compute the inverse map, i.e., to express the \( sl_q(2) \) generators in terms of \( J(\pm, 0) \), that could be used to obtain some information on the co-algebra structure of the generalized \( sl(2) \) algebra given by eqs. (16-18).

Therefore, we have shown that this linear case is connected to the \( sl_q(2) \) algebra. Moreover, this formalism allows generalizations of \( sl(2) \) to more complex algebras obtained by considering non-linear functions \( f \) in eqs. (1-3).

These algebras, depending on the function \( f \), will not simply be deformations of \( sl(2) \).

4 Non-linear functions

In this section we consider some aspects of the representation theory of the algebra defined by eqs. (1-3) for \( f(x) = t x^2 + r x - s \). In this case the algebra becomes
\[
\begin{align*}
[J_0, J_+]_{r-1} &= -r^{-1} (t J_0^2 - s) J_+ , \\
[J_0, J_-]_r &= J_- (t J_0^2 - s) , \\
[J_+, J_-] &= -t^2 J_0^4 - 2t r J_0^3 + (1 - s) t - r^2 + s t ) J_0^2 \\
&+ (1 - r(1 - 2s)) J_0 + s(1 - s) .
\end{align*}
\]

When \( t = 0 \) we recover the linear (or \( r \)-deformed) \( sl(2) \) algebra given in eqs. (10-18). For \( t = 0 \) and \( r = s = 1 \) we recover the standard \( sl(2) \) algebra.

We focus now on the analysis of the finite dimensional representations of the above quadratic \( sl(2) \) algebra [3]. To this aim we have to look for

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\[^2\text{From now one we are considering } t > 0; \text{ the analysis of negative values of } t \text{ is similar.}\]
the finite dimensional representation solutions to eqs. (7-10). Since we are starting from a highest weight vector and in order to have a finite dimensional representation we must find the conditions where the eq. (9) is identically null. This is obtained by analysing the zeros of the equation

\[ N_{d-1}^2 = (\alpha_j - \alpha_{j-d})(\alpha_j + \alpha_{j-d} + 1) = 0. \]  

(30)

In this case, we can find solutions to eq. (30) in two different ways. In the first manner, as in the linear case, we find the solutions to \( N_{d-1} = 0 \) that satisfy the cut condition \( \alpha_j + \alpha_{j-d} + 1 = 0 \). In the non-linear case we can also find the zeroes of \( N_{d-1} = 0 \) coming from \( \alpha_j = \alpha_{j-d} \). The zeros of the term \( \alpha_j - \alpha_{j-d} \) can be obtained through the analysis and the stability of the fixed points of \( f(x) = tx^2 + rx - s \) and their composed functions [2].

In order to analyse the stability of the fixed points of \( f(x) \) it is convenient to sort this analysis out in three cases: (I) \( \Delta < 0 \), (II) \( \Delta = 0 \) and (III) \( \Delta > 0 \), for \( \Delta = (r-1)^2 + 4ts \). In the first case there is no fixed point and we see, by a graphical analysis similar to that discussed in subsection 3.1, that there is no finite dimensional representation coming from \( \alpha_j = \alpha_{j-d} \); in case (II), we have one fixed point given by \( \alpha_\ast = (1-r)/2t \). This fixed point corresponds to a trivial one-dimensional representation of the algebra for \( \alpha_j = \alpha_\ast \) since \( N_0 = 0 \) \( (\alpha_{j-1} = \alpha_j = \alpha_\ast) \).

Case (III) is less trivial. In this case it is also possible to have attractors of period 1, 2, 4, \( \cdots \) and even a chaotic region in the space of parameters \( (t, r, s, \alpha_0) \) where, as it is well known, there are cycles associated with all integer numbers.

For \( 0 < \Delta < 4 \) there is only trivial one-dimensional representations associated to the fixed points \( \alpha_\ast = f(\alpha_\ast) \), with highest weight:

\[ \alpha_\ast^\pm = \frac{1-r \pm \sqrt{\Delta}}{2t}. \]  

(31)

At \( \Delta = 4 \) the one-cycle looses stability and a stable two-cycle, solution to \( \beta_\ast^\prime = f^2(\beta_\ast) \) \( (f^2(x) \equiv f(f(x))) \), appears. The solutions to the two-cycle equation that are not attractors of period 1 (fixed point of \( f \)) are

\[ \beta_\ast^\pm = \frac{-1-r \pm \sqrt{\Delta_1}}{2t}, \]  

(32)

where \( \Delta_1 = -3 - 2r + r^2 + 4ts \). In this case, \( \Delta > 4 \), we have a two-dimensional representation of the algebra simply by choosing the highest
weight, \( \alpha_j \), as the highest element of the two cycle, i.e., \( \beta^*_\downarrow \). Note that in eq. (30) the term \( \alpha_j - \alpha_{j-d} \) becomes, in this two-dimensional case, \( \beta^*_\downarrow - f^2(\beta^*_\downarrow) \), that is identically zero. The matrix representation is given by

\[
J_0 = \begin{pmatrix} \beta^*_\downarrow & 0 \\ 0 & \beta^*_\uparrow \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & 0 \\ N_0 & 0 \end{pmatrix}, \quad J_- = J_+^\dagger,
\]  

(33)

where \( N_0 \) is computed for \( \Delta > 4 \) and \( \alpha_j = \beta^*_\downarrow \).

For \( \Delta > 6 \) we will have other cycles, of length 4, 8, \ldots, \( 2^k \ldots \), entering the chaotic region where all cycles will be present. In general, for a \( d \)-cycle we have a \( d \)-dimensional representation where the highest weight of the representation is the largest element of the cycle. The term \( \alpha_j - \alpha_{j-d} \) of eq. (30) is identically null for this cycle.

There are also finite dimensional representations coming from the zeroes of the cut condition \( \alpha_j + \alpha_{j-d} + 1 = 0 \) in the expression of \( N_{d-1} \). For example, the regions associated with the possible highest weight vector solutions for the first cycles are better understood by studying the corresponding \( \Delta \) intervals. For the one-cycle (\( 0 < \Delta < 4 \)), the following region

\[
\alpha^*_\downarrow < \alpha_j < \alpha^*_\uparrow,
\]  

(34)
is the only region in the \( \alpha \) real axis where it is possible to find highest weight vectors since iterations of \( \alpha \) in this region give lower values than the initial one. In order to select \( d \)-dimensional representations we must pick up the points in this interval that satisfy the cut condition \( \alpha_j + \alpha_{j-d} + 1 = 0 \). For example, \( r = s = 1, t = 0.1 (\Delta = 0.4) \) with highest weight \( \alpha_j = 0.476105 \) is a possible solution to the cut condition corresponding to a two-dimensional representation. Note also that the highest weight of this two-dimensional representation is within the \( \alpha \)-region for the one-cycle given in eq. (34) since \( \alpha^*_\downarrow = -3.16228 = -\alpha^*_\uparrow \) for the values of the parameters under consideration.

For the two-cycle (\( 4 < \Delta < 6 \)), the following region

\[
\beta^*_\downarrow < \alpha_j < \alpha^*_\uparrow,
\]  

(35)
is the region in the \( \alpha \) real axis where it is possible to find highest weight vectors for \( d \)-dimensional representation for any finite value of \( d \), apart of the two-dimensional representation (\( \alpha_j = \beta^*_\downarrow \)). In order to seek for finite dimensional representations we have to find the points in this interval that satisfy the cut condition. For example, in this region we have for the parameters...
$r = t = 1, s = 1.1 \ (\Delta = 4.4)$, giving $\beta^*_+ = -0.683772$ and $\alpha^*_+ = 1.04881$, two possible two-dimensional representations with highest weights given by $\alpha_j = \pm 0.316228$.

For higher cycles the analysis is similar. In all cycles the allowed regions to get possible finite dimensional representations range from the largest element of the cycle up to $\alpha^*_+$. 

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