On Testing Entropic Inequalities for Superconducting Qudit

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The aim of this work is to verify the new entropic and information inequalities for non-composite systems using experimental $5 \times 5$ density matrix of the qudit state, measured by the tomographic method in a multi-level superconducting circuit. These inequalities are well-known for bipartite and tripartite systems, but have never been tested for superconducting qudits. Entropic inequalities can also be used to evaluate the accuracy of experimental data and the value of mutual information, deduced from them, may characterize correlations between different degrees of freedom in a noncomposite system.

INTRODUCTION

Properties of composite quantum systems, i.e. systems containing subsystems, have been extensively studied during the last few decades, which resulted in numerous practical applications. These systems were also described in terms of classical information theory [1] in the quantum domain [2] and their information and entropic characteristics were investigated, including the von Neumann entropy and quantum mutual information, discord related measures, entropic inequalities, contextuality, causality, subadditivity and strong subadditivity conditions.

On the contrary, the idea of using noncomposite quantum systems for quantum technologies was suggested [3–5] and quantum correlations in such systems have been analyzed only in recent times [6, 7]. The latter opened a way of mapping information and entropic measures for composite quantum systems on the noncomposite quantum systems[6–11].

Along with the development of quantum information theory, tremendous progress has been made in experimental control over quantum systems. In particular, experiments with superconducting circuits, based on Josephson junction devices [12, 13], have been rapidly developing recently [14]. Specifically spectroscopical [15, 16] and time-domain [17] properties of such systems were studied both theoretically and experimentally. With the improvement of coherence time of superconducting qubits it became possible to obtain the density matrices of such systems, using quantum state tomography [18] as well as Wigner tomography [19].

In this work, we aim to verify the entropic and information inequalities using experimental $5 \times 5$ density matrix of the qudit state ($j = 2$), obtained using direct Wigner tomography in a superconducting circuit [19, 20]. The inequalities were obtained using approach [6–10] to get analogs of subadditivity and strong subadditivity conditions, well-known for bipartite and tripartite systems, for a single qudit state.

SUPERCONDUCTING CIRCUITS

Superconducting circuits with Josephson junctions are macroscopic quantum objects, that can be several micrometers wide while still preserving quantum properties. This happens because they are artificially isolated from the environment which leaves them with a single degree of freedom. The intrinsic parameters of these circuits can be engineered as desired and adjusted with an external parameter (for example, a magnetic field). Such superconducting circuits are thereby often called "artificial atoms".

Josephson junction

The Josephson junction in superconducting circuits serves as a non-dissipative nonlinear element (namely, the nonlinear inductance). It consists of two superconductors, separated by a thin insulating layer, through which Cooper-pairs can coherently tunnel. This system was described by Brian Josephson [21], who showed that superconducting current across the junction depends on the phase difference between the superconductors:

$$I = I_c \sin(\phi_2 - \phi_1) = I_c \sin \phi$$

(1)

Here $I_c$ stands for the maximum current, which can flow through the junction without any dissipation, i.e.
the critical current. Josephson also showed that when the voltage is applied across the junction the phase difference changes in time, which leads to the oscillations of the critical current with the angular frequency $\omega$:

$$h\dot{\phi} = h\omega = 2eV$$

(2)

When we substitute this into the time derivative of Eq. (1) and compare it to the Faraday’s law, we obtain the Josephson inductance:

$$L_J(\phi) = \frac{h}{2eI_c \cos \phi} = \frac{\Phi_0}{2\pi(I_c^2 - I^2)^{1/2}}$$

(3)

As the Josephson junction has some intrinsic capacitance $C$ it behaves as a nonlinear oscillator with angular frequency $\omega_p$:

$$\omega_p(I) = \frac{1}{\sqrt{LJC}} = \frac{(2\pi I_c/\Phi_0 C)^{1/2}}{(1 - I^2/I_c^2)^{1/4}}$$

(4)

FIG. 1: The tilted washboard potential and quantized energy levels inside one of the potential wells.

The total current flow through the junction can be written as $J = I_c \sin \phi + CV$. Substituting $\dot{V} = (h/2e)\dot{\phi}$ from Eq. (2) we obtain:

$$\frac{hC}{2e} \dot{\phi} = J - I_c \sin \phi - \frac{2e}{h} \frac{\partial U}{\partial \phi}$$

(5)

where $U = -\Phi_0 \frac{\partial U}{\partial \phi}$ is a tilted washboard potential for a particle with mass $hC/2e$, shown on the Fig. 1(a).

Superconducting qudit

A closer look at one of the wells in the tilted washboard potential in Fig.1(b) with the quantized energy levels gives us a perfectly suitable d-level system (qudit). Varying the potential by an external magnetic field, we can achieve a desired number of energy levels in the well. The physical implementation of this system is called the Josephson phase circuit [22, 23] and is shown in Fig. 2.

The quantum state of the Josephson phase circuit is controlled via DC and microwave pulses of bias current. The measurement of the state employs the escape from the potential well via tunneling. For example, to measure the occupation probability of state $|1\rangle$ one can pump microwaves at frequency $\omega_{11}$, which will induce a $|1\rangle \rightarrow |4\rangle$ transition. Then the state will rapidly tunnel due to the large tunneling rate $\Gamma_4$. When the tunneling occurs, a voltage appears across the junction, which can be measured directly by an on-chip SQUID.

In this paper we utilize the results, obtained in the experiment by Shalibo et al. [19, 20], in which the Wigner distribution of the Josephson phase circuit was directly measured using simple tomography pulses.

ENTROPIC INEQUALITIES

Quantum states are generally described by the density matrix operator $\hat{\rho}$, which has the following properties:

$$\text{Tr}(\hat{\rho}) = 1, \quad \hat{\rho} = \hat{\rho}^\dagger, \quad \hat{\rho} \geq 0$$

(6)

We consider a $5 \times 5$ density matrix for a qudit with $j = 2$:

$$\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} \\
\rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55}
\end{pmatrix}$$

(7)

We can rewrite this as a $6 \times 6$ matrix, by adding one more zero row and zero column:

$$\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & 0 \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} & 0 \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} & 0 \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} & 0 \\
\rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

(8)

While looking at this system one can realize that it can be viewed as tensor product of two subsystems - a qubit and a qutrit. So, using an invertible mapping of indices $1 \leftrightarrow -1 \leftrightarrow -1/2; 2 \leftrightarrow -1 1/2; 3 \leftrightarrow 0 \leftrightarrow -1/2; 4 \leftrightarrow 0 1/2; 5 \leftrightarrow 1 \leftrightarrow -1/2; 6 \leftrightarrow 1 1/2$, we obtain the density matrix, which describes the bipartite qubit-qutrit state. The density matrices of the subsystems are generally derived by taking the partial trace over the corresponding sample.
indices. We propose a simplified approach by dividing the density matrix into several blocks with fewer dimensions:

\[
\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} \\
\rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55}
\end{pmatrix} = \begin{pmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{pmatrix}
\]

(9)

Then the density matrices of the subsystems are:

\[
\rho_1 = \frac{\text{Tr} R_{11} \text{Tr} R_{12}}{\text{Tr} R_{21} \text{Tr} R_{22}} = \begin{pmatrix}
\rho_{11} + \rho_{22} + \rho_{33} & \rho_{14} + \rho_{25} \\
\rho_{41} + \rho_{52} & \rho_{44} + \rho_{55}
\end{pmatrix}
\]

(10)

\[
\rho_2 = (R_{11} + R_{22}) = \begin{pmatrix}
\rho_{11} + \rho_{44} & \rho_{12} + \rho_{45} & \rho_{13} \\
\rho_{21} + \rho_{54} & \rho_{22} + \rho_{55} & \rho_{23} \\
\rho_{31} & \rho_{32} & \rho_{33}
\end{pmatrix}
\]

(11)

Now we can take a look at correlations in our system. One of the most important correlation characteristics is entropy. In this work we deal with the von Neumann entropy [24]:

\[
S_N = -\text{Tr} \rho \ln \rho
\]

(12)

For the von Neumann entropy of the bipartite system one can write the subadditivity condition:

\[-\text{Tr} \rho \ln \rho \leq -\text{Tr} \rho_1 \ln \rho_1 - \text{Tr} \rho_2 \ln \rho_2\]

(13)

Now we can repeat this process for another partition of the 6 \times 6 density matrix:

\[
\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} \\
\rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55}
\end{pmatrix} = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} \\
\rho_{21} & \rho_{22} & \rho_{23} \\
\rho_{31} & \rho_{32} & \rho_{33} \\
\rho_{41} & \rho_{42} & \rho_{43} \\
\rho_{51} & \rho_{52} & \rho_{53}
\end{pmatrix}
\]

(14)

and get the density matrices of the subsystems:

\[
\hat{\rho}_1 = \frac{\text{Tr} \rho_{11} \text{Tr} \rho_{12} \text{Tr} \rho_{13}}{\text{Tr} \rho_{21} \text{Tr} \rho_{22} \text{Tr} \rho_{23}} = \begin{pmatrix}
\rho_{11} + \rho_{22} + \rho_{33} & \rho_{14} + \rho_{25} \\
\rho_{31} + \rho_{42} & \rho_{34} + \rho_{45} \\
\rho_{51} & \rho_{52}
\end{pmatrix}
\]

(15)

\[
\hat{\rho}_2 = (\rho_{11} + \rho_{22} + \rho_{33}) = \begin{pmatrix}
\rho_{11} + \rho_{33} & \rho_{13} \\
\rho_{21} + \rho_{33} & \rho_{23}
\end{pmatrix}
\]

(16)

So the subadditivity condition takes the form:

\[-\text{Tr} \rho \ln \rho \leq -\text{Tr} \hat{\rho}_1 \ln \hat{\rho}_1 - \text{Tr} \hat{\rho}_2 \ln \hat{\rho}_2\]

(17)

Next we add two more zero rows and columns to this matrix to get an 8 \times 8 matrix. The system, described by this density matrix, can be divided into three subsystems (represented by 2 \times 2 matrices) by the following mapping of indices:

\[
1 \leftrightarrow -1/2 -1/2 -1/2; \quad 2 \leftrightarrow -1/2 -1/2 1/2; \\
3 \leftrightarrow -1/2 1/2 -1/2; \quad 4 \leftrightarrow -1/2 1/2 1/2; \\
5 \leftrightarrow 1/2 -1/2 -1/2; \quad 6 \leftrightarrow 1/2 -1/2 1/2; \\
7 \leftrightarrow 1/2 1/2 -1/2; \quad 8 \leftrightarrow 1/2 1/2 1/2.
\]

Here, we use the same approach of dividing the matrix into blocks to calculate the partial traces and get the matrices for the subsystems.

\[
\rho = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_{11} & \rho_{12} & \rho_{13} & 0 & \rho_{14} & \rho_{15} & 0 \\
0 & \rho_{21} & \rho_{22} & \rho_{23} & 0 & \rho_{24} & \rho_{25} & 0 \\
0 & \rho_{31} & \rho_{32} & \rho_{33} & 0 & \rho_{34} & \rho_{35} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(18)

\[
\rho_{12} = \begin{pmatrix}
\rho_{11} & \rho_{13} & \rho_{14} & 0 \\
\rho_{31} & \rho_{33} & \rho_{34} & \rho_{35} & \rho_{43} & \rho_{45} & 0 \\
\rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55}
\end{pmatrix},
\]

(19)

\[
\rho_{23} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \rho_{11} + \rho_{14} + \rho_{13} + \rho_{43} + \rho_{23} + \rho_{24} + \rho_{25} + \rho_{45} + \rho_{33} & \rho_{13} + \rho_{33} & \rho_{14} + \rho_{34} & \rho_{24} + \rho_{25} + \rho_{45} + \rho_{33} & \rho_{43}
\end{pmatrix},
\]

(20)

\[
R_{2} = \begin{pmatrix}
\rho_{11} + \rho_{14} + \rho_{13} + \rho_{33} & \rho_{13} + \rho_{33} & \rho_{14} + \rho_{34} & \rho_{24} + \rho_{25} + \rho_{45} + \rho_{33} & \rho_{43}
\end{pmatrix},
\]

(21)

For this kind of tripartite system one can write the strong subadditivity condition [25]:

\[-\text{Tr} \rho \ln \rho - \text{Tr} R_1 \ln R_1 - \text{Tr} R_2 \ln R_2 \leq -\text{Tr} \rho_{12} \ln \rho_{12} - \text{Tr} \rho_{23} \ln \rho_{23}\]

(22)

**VERIFYING EXPERIMENTAL DATA**

Next, we calculate the density matrices of the subsystems from the experimentally obtained 5 \times 5 density matrix. This density matrix corresponds to the qudit, mentioned in section Superconducting qudit, and was measured in [19, 20, 26]. One can also find this matrix in the Supplementary material.

The density matrices in Eq. 10 and 11 are as follows:

\[
\rho_1 = \begin{pmatrix}
0.985 & 8.3 \cdot 10^{-5} - 2.7 \cdot 10^{-4} i \\
8.3 \cdot 10^{-5} + 2.7 \cdot 10^{-4} i & 0.006
\end{pmatrix}
\]

\[
\rho_2 = \begin{pmatrix}
0.96 & 8.8 \cdot 10^{-4} + 0.011 \\
8.8 \cdot 10^{-4} - 0.001 & 0.004
\end{pmatrix}
\]

\[
\begin{pmatrix}
0.088 + 0.018 i & -7.6 \cdot 10^{-4} + 2.9 \cdot 10^{-4} i \\
-7.6 \cdot 10^{-4} - 2.9 \cdot 10^{-4} i & 0.026
\end{pmatrix}
\]
Then, we calculate the subadditivity condition (Eq. 13), which gives the following result:

$$0.1583 \leq 0.1996$$

and the mutual information equals: $$I = S_\rho - S_{\tilde{\rho}_1} - S_{\tilde{\rho}_2} = 0.1996 - 0.1583 = 0.0413$$

Analogously, for the other way of dividing the system into subsystems (Eq. 15 and 16) we obtain the following density matrices:

$$\hat{\rho}_1 = \begin{pmatrix} 0.96 & 0.008 - 0.018i & 0.005 - 0.0071i \\ 0.008 + 0.018i & 0.028 & 0.012 + 0.0191i \\ -0.006 + 8.6 \times 10^{-4}i & 0.005 + 0.0071i & 0.004 \end{pmatrix},$$

$$\hat{\rho}_2 = \begin{pmatrix} 0.99 & 0.005 - 0.0021i \\ 0.005 + 0.0021i & 0.002 \end{pmatrix}$$

So the subadditivity condition (Eq. 17) for this system with subsystems reads:

$$0.1583 \leq 0.1768$$

and the mutual information: $$I = S_\rho - S_{\hat{\rho}_1} - S_{\hat{\rho}_2} = 0.1768 - 0.1583 = 0.0185$$

Finally, we calculate the density matrices for the tripartite system (Eq. 21, 19, 20):

$$R_2 = \begin{pmatrix} 0.961 & 0.012 - 0.0191i \\ 0.012 + 0.0191i & 0.030 \end{pmatrix},$$

$$\rho_{12} = \begin{pmatrix} 0.005 + 0.0064i \\ 0.006 - 0.004i \\ 0.0064 - 0.006i \end{pmatrix},$$

$$\rho_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.012 + 0.0191i \\ 0 & 0.004 & 0.004 + 0.0064i \end{pmatrix}.$$
information, deduced from entropic inequalities, may characterize correlations between different degrees of freedom in a noncomposite system. There also exist other inequalities for the von-Neumann and q-entropy, which will be checked in future publications.

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