A DETAILED PROOF OF POHST’S INEQUALITY

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Abstract. In 1977 Pohst conjectured a certain inequality for $n$ variables and gave a computer-assisted proof for $n \leq 10$. We give a proof for all $n$ using a combinatorial argument. This inequality yields a better bound for the regulator in terms of the discriminant for totally real number fields.

1. Introduction

In 1952 Remak [Re] proved that the product
\[
\prod_{1 \leq i \leq j \leq n} |1 - \prod_{k=i}^{j} x_k|
\]
is bounded above by \( (n+1)(n+1)/2 \) when the variables $x_i$ are complex numbers with modulus $|x_i| \leq 1$. He used this inequality to obtain a lower bound for the regulator $R_k$ in terms of the discriminant $D_k$ for number fields $k$.

In 1977 Pohst [Po] proved for $n \leq 10$ that the same product is bounded above by \( 2^{\lfloor (n+1)/2 \rfloor} \) when the variables are real with absolute value $|x_i| \leq 1$. This improved the regulator lower bound for totally real number fields. He gave a computer-assisted proof by using some elementary inequalities (see Lemma 1) and factorizing the product for each possible combination of signs for the variables $x_i$. Here we prove it for all $n$.

Theorem. Let $v = (x_1, \ldots, x_n) \in [-1, 1]^n$ and $f_n(v) := \prod_{1 \leq i \leq j \leq n} (1 - \prod_{k=i}^{j} x_k)$. Then $f_n(v) \leq 2^{\lfloor (n+1)/2 \rfloor}$, where $|x|$ denotes the floor of the real number $x$. Moreover, the maximum is attained if and only if the coordinates of $v$ are 0 or $-1$ and have the greatest possible number of $-1$’s without two consecutive $-1$’s.

Pohst’s motivation [Po] was to improve on Remak’s inequality relating discriminant and regulator for primitive totally real fields (fields with no subfields other than itself and $\mathbb{Q}$) [Fr, (3.15)]. Indeed, our theorem improves Remak’s
\[
\log |D_k| \leq m \log(m) + \sqrt{\gamma_{m-1}(m^3 - m)/3} (\sqrt{mR_k})^{1/(m-1)}
\]
to
\[
\log |D_k| \leq \lfloor m/2 \rfloor \log(4) + \sqrt{\gamma_{m-1}(m^3 - m)/3} (\sqrt{mR_k})^{1/(m-1)},
\]
where $m := [k : \mathbb{Q}]$ and $\gamma_{m-1}$ is Hermite’s constant in dimension $m-1$.

The idea of our proof of Pohst’s inequality is to note that the problem is easily solved when each of the variables $x_i$ is negative. We then show that we can always bound the product $f_n(v)$ by this special case, i.e.
\[
f_n(x_1, \ldots, x_n) \leq f_n(-|x_1|, \ldots, -|x_n|) \quad \text{for} \quad (x_1, \ldots, x_n) \in [-1, 1]^n.
\]
To prove this we will use a variation of Pohst’s elementary inequalities (Lemma 2) that will allow us to exchange the signs of certain combinations of terms of the product $f_n(v)$. We then show that for any combination of signs for the variables $x_i$ we can find a factorization of $f_n(v)$ to which the new elementary inequalities apply (Lemma 3).
There is a proof of Pohst's inequality in [Be]. However, the proof is incomplete so we
give a new one here. Battistoni and Molteni [BM] gave a solution to this same problem,
using similar methods. This solutions was not known by the author and not yet published
at the time of submission of this article.

2. Proof of Theorem

Lemma 1. (Pohst)

(1) If \( a \in [-1, 1] \), then \( (1 - a) \leq 2 \).

(2) If \( a \in [0, 1] \) and \( b \in [-1, 0] \), then \( (1 - a)(1 - ab) \leq 1 \).

(3) If \( a, b \in [-1, 1] \), then \( (1 - a)(1 - b)(1 - ab) \leq 2 \).

(4) If \( a \in [0, 1] \) and \( b, c \in [-1, 0] \), then \( (1 - a)(1 - ab)(1 - ac)(1 - abc) \leq 1 \).

In (3) the maximum is attained if and only if \( (a, b) = (0, -1) \) or \( (a, b) = (-1, 0) \).

Proof. See [Pa] p. 468, or use undergraduate calculus.

Lemma 2. For \( a, b, c \in [0, 1] \) the following hold.

(1) \( (1 - a) \leq (1 + a) \).

(2) \( (1 - a)(1 + ab) \leq (1 + a)(1 - ab) \).

(3) \( (1 - a)(1 + ab)(1 - abc) \leq (1 + a)(1 - ab)(1 - ac)(1 + abc) \).

Proof. For (2) note that \( (1 - a) \leq (1 - ab) \) and \( (1 + ab) \leq (1 + a) \). For (3) note that,
since \( x(1 - y) \leq 1 - y \) for \( x \leq 1 \) and \( y \leq 1 \), then \( x + y \leq 1 + xy \). From this we get

\[
(1 + ab)(1 + ac) = 1 + a(b + c) + a^2bc \leq 1 + a(1 + bc) + a^2bc = (1 + a)(1 + abc),
\]

which we multiply with the following inequality,

\[
(1 - a)(1 - abc) = 1 - a(1 + bc) + a^2bc \leq 1 - a(b + c) + a^2bc = (1 - ab)(1 - ac). \]

Let \( v = (x_1, \ldots, x_n) \in [-1, 1]^n \) and \( a_v(i, j) := 1 - \prod_{k=1}^i x_k \). Note that the terms \( a_v(i, j) \)
can be easily ordered in a triangular way, as seen in figures 1, 2 and 3. Next we treat
the case where \( x_k \in [-1, 0] \) for all \( k \).

Lemma 3. If \( v = (x_1, \ldots, x_n) \in [-1, 0]^n \), then \( f_n(v) \leq 2^{\frac{n-2}{2}} \). Moreover, the maximum
is attained if and only if the coordinates of \( v \) are 0 or -1 and \( v \) has the greatest possible
number of -1's without two consecutive -1's.

Proof. We find a good factorization for the product \( f_n(v) = \prod_{1 \leq i \leq j \leq n} a_v(i, j) \) by means of lemma 1. Figure 4 shows an illustrative example of how the partition is done. The
case \( n = 1 \) is obvious. If \( n \) is even then

\[
f_n(v) = \left( \prod_{k=1}^{n/2} a_v(2k, 2k-1) a_v(2k, 2k) a_v(2k-1, 2k) \right) \left( \prod_{1 \leq k < n/2} a_v(2k, 2k + 2j + 1) a_v(2k-1, 2k + 2j + 1) a_v(2k, 2k + 2j + 2) a_v(2k-1, 2k + 2j + 2) \right).
\]

The first product corresponds to the terms at the base of the triangle while the second
product corresponds to the rectangles in the rest of the triangle. Using lemma 1 (3) for
the first product and lemma 1 (4) for the second one we conclude the even case. If \( n \geq 3 \)
is odd, note that

\[
\prod_{k=1}^{n/2} a_v(2k, 2k-1) a_v(2k, 2k) a_v(2k-1, 2k) \leq \prod_{k=1}^{(n-1)/2} a_v(2k, 2k + 2j + 1) a_v(2k-1, 2k + 2j + 1) a_v(2k, 2k + 2j + 2) a_v(2k-1, 2k + 2j + 2) \leq 2^{\frac{n-2}{2}}.
\]
\[ f_n(v) = f_{n-1}(x_1, \ldots, x_{n-1})a_{v(n,n)} \prod_{k=1}^{(n-1)/2} a_{v(2k,n)}a_{v(2k-1,n)} \]

and use lemma \( \text{(2)} \) in the last product.

\[
\begin{array}{cccccc}
  a_{v(1,n)} & a_{v(2,n)} & a_{v(3,n)} & a_{v(4,n)} & a_{v(5,n)} & a_{v(6,n)} & \cdots & a_{v(n,n)} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{v(1,6)} & a_{v(2,6)} & a_{v(3,6)} & a_{v(4,6)} & a_{v(5,6)} & a_{v(6,6)} & & \\
  a_{v(1,5)} & a_{v(2,5)} & a_{v(3,5)} & a_{v(4,5)} & a_{v(5,5)} & & & \\
  a_{v(1,4)} & a_{v(2,4)} & a_{v(3,4)} & a_{v(4,4)} & & & & \\
  a_{v(1,3)} & a_{v(2,3)} & a_{v(3,3)} & & & & & \\
  a_{v(1,2)} & a_{v(2,2)} & & & & & & \\
  a_{v(1,1)} & & & & & & & \\
\end{array}
\]

**Figure 1.** Factorization of \( f_n(v) \) when \( n \) is odd.

A necessary condition for the maximum to be attained is that each of the terms in the base of the triangle equals 2, i.e. \( a_{v(2k-1,2k-2)}a_{v(2k-2,2k)}a_{v(2k-1,2k)} = 2 \), and \( a_{v(n,n)} = 2 \) if \( n \) is odd. Using the last statement in lemma \( \text{(1)} \) this happens exactly when \( x_{2k-1} = 0 \) and \( x_{2k} = -1 \) or \( x_{2k-1} = -1 \) and \( x_{2k} = 0 \). However, we cannot have \( x_t = x_{t+1} = -1 \) for \( t = 1, \ldots, n - 1 \) since this implies that \( a_{v(t,t+1)} = 0 \) and then \( f_n(v) = 0 \). If the coordinates of \( v \) are 0 or \(-1\) and have the greatest number of \(-1\)’s but without two consecutive \(-1\)’s, then each each of the factors in the base of the triangle equal 2. Since for two consecutive coordinates of \( v \) one of them is 0, each factor \( a_{v(i,j)} \) with \( i \neq j \) is equal to 1. In consequence the product of the factors in the rectangles equals 1.

**Definition 4.** Let \( v = (x_1, \ldots, x_n) \in \{[-1,1] \setminus \{0\}\}^n \). Define the product sign \( s \) of \( a_{v(i,j)} \) by \( s(a_{v(i,j)}) := \text{sign}(1 - a_{v(i,j)}) = \prod_{k=1}^{j} \text{sign}(x_k) \). We say that the term \( a_{v(i,j)} = 1 - \prod_{k=1}^{j} x_k \) is non canonical if \( s(a_{v(i,j)}) = (-1)^{i+j} \). We also define the set \( J_v \) of non canonical indices of \( v \), \( J_v := \{(i,j) : s(a_{v(i,j)}) = (-1)^{i+j}\} \).

**Remark 5.** Note that if we let \( -|v| := (\lceil x_1, \ldots, -x_n \rceil) \in [-1,0]^n \) and if \( a_{v(i,j)} \) is canonical, then \( a_{v(i,j)} = a_{-|v|(i,j)} \) and \( s(a_{-|v|(i,j)}) = s(a_{v(i,j)}) = (-1)^{i+j+1} \).

We will partition \( J_v \) into subsets of 1, 2 or 4 elements so that we can apply to the corresponding products cases (1), (2) or (3) respectively of lemma \( \text{(2)} \). For this we need the following definition.

**Definition 6.** We say that \( J_v \) has a good partition if there is a partition \( \pi_v \) of \( J_v \) such that if \( p \in \pi_v \) then one of the following holds.

1. \( p = \{(i,j)\} \) with \( s(a_{v(i,j)}) = 1 \).
2. \( p = \{(i,j),(i',j')\} \), \( s(a_{v(i,j)}) = 1 \), \( s(a_{v(i',j')}) = -1 \). Also, \( i' \leq i \) and \( j = j' \), or \( i' = i \) and \( j \leq j' \).
3. \( p = \{(i,j),(i-l,j),(i,j+l),(i-l,j+l)\} \), \( s(a_{v(i,j)}) = s(a_{v(i-l,j)}) = 1 \), \( s(a_{v(i,j+l)}) = s(a_{v(i-l,j+l)}) = -1 \), and \( l, l' \geq 1 \).
Suppose \( p \) is as in case (3). Then

\[
\prod_{(i,j) \in p} a_{v(i,j)} = (1 - \prod_{k=i}^{j} x_k)(1 - \prod_{k=i-l}^{j} x_k)(1 - \prod_{k=1}^{j+l'} x_k)
\]

\[
= (1 - \prod_{k=i}^{j} x_k)(1 + \prod_{k=i-l}^{j} |x_k|)(1 + \prod_{k=i}^{j+l'} |x_k|)
\]

\[
\leq (1 + \prod_{k=i}^{j} x_k)(1 - \prod_{k=i-l}^{j} |x_k|)(1 + \prod_{k=i}^{j+l'} |x_k|)
\]

\[
= \prod_{(i,j) \in p} a_{-|v(i,j)|},
\]

where the inequality follows from lemma 2 (3) on setting \( a := \prod_{k=i}^{j} x_k, b := \prod_{k=i-l}^{j} x_k \) and \( c := \prod_{k=j+1}^{j+l'} x_k \). The other cases are analogous, as we now record.

**Remark 7.** For \( p \) as in definition 2 we have \( \prod_{(i,j) \in p} a_{v(i,j)} \leq \prod_{(i,j) \in p} a_{-|v(i,j)|} \).

The following lemma will be central in our proof. However we will postpone its proof to the next section.

**Lemma 8.** For every \( v = (x_1, \ldots, x_n) \in \left( [-1,1]\setminus \{0\} \right)^n \) the set \( J_v \) has a good partition.

We can conclude the proof of the theorem by induction on \( n \). If \( n = 1 \), we obviously have \( f_1(v) \leq 2 \). If \( n \geq 2 \), let \( v := (x_1, \ldots, x_n) \in [-1,1]^n \). If some of the coordinates \( x_i \) is 0, then we note that for \( v_1 := (x_1, \ldots, x_{i-1}) \) and \( v_2 := (x_{i+1}, \ldots, x_n) \) by induction we obtain

\[
f_n(v) = f_{n-1}(v_1)f_{n-1}(v_2) \leq 2^{\left(\frac{i-1}{2}\right)+1}2^{\left(\frac{n-i}{2}\right)+1} \leq 2^{\frac{n+1}{2}}.
\]

If \( v = (x_1, \ldots, x_n) \in \left( [-1,1]\setminus \{0\} \right)^n \), we separate the product into the canonical elements and the non canonical elements, and we factor the non canonical elements using a partition obtained from lemma 8. Thus,

\[
f_n(x_1, \ldots, x_n) := \prod_{1 \leq i \leq j \leq n} a_{v(i,j)} = \left( \prod_{(i,j) \notin J_v} a_{v(i,j)} \right) \cdot \left( \prod_{(i,j) \in J_v} a_{v(i,j)} \right)
\]

\[
= \left( \prod_{(i,j) \notin J_v} a_{-|v(i,j)|} \right) \cdot \left( \prod_{p \in \pi_v, (i,j) \in p} a_{v(i,j)} \right) \quad \text{(see remark 5)}
\]

\[
\leq \left( \prod_{(i,j) \notin J_v} a_{-|v(i,j)|} \right) \cdot \left( \prod_{p \in \pi_v, (i,j) \in p} a_{-|v(i,j)|} \right) \quad \text{(see remark 7)}
\]

\[
f_n(-|x_1|, \ldots, -|x_n|) \leq 2^{\frac{n+1}{2}} \quad \text{(see lemma 3)}.
\]

As for the final claim in the theorem, we use lemma 2 for each \( p \in \pi_v \) and conclude with lemma 3.

3. **Proof of Lemma 8**

The following remark, whose proof is a straightforward calculation, will be very useful.
Remark 9. If in the set \{(i, j), (i - l, j), (i, j + l'), (i - l, j + l')\}, where \(l, l' \geq 1\), three of the pairs are non canonical, then the last one is also non canonical. Moreover we have the relation \(s(a_{v(i,j)}) s(a_{v(i-l,j+l')}) = s(a_{v(i-l,j)}) s(a_{v(i,j+l')}).

Lemma 10. If \(v = (x_1, \ldots, x_n) \in \{[-1, 1] \setminus \{0\}\}^n\), then \(v\) is even and if the number of \(a_i\) with \(\operatorname{sign}(a_i) = -1\) is odd, then \(b_1(v) = b_{-1}(v)\), where
\[
b_1(v) := \{(i,j) \in J_v : s(a_{v(i,j)}) = l, i = 1 \text{ or } j = n\}.
\]

Proof. Note that we just need to inspect the vertical and horizontal edges of the triangle in Figure 2. We proceed by induction on \(n\), the case \(n = 2\) being trivial. If the sequence of signs of the \(x_k\) is \((-,-,+,-,+,-,-,\ldots)\) or \((+,+,+,+,\ldots,+,+,-)\), in which case necessarily \(n \equiv 2 \pmod{4}\), the lemma is a straightforward calculation that does not even require the inductive assumption. If the \(x_k\) do not have this sign pattern, then the sequence of signs must necessarily contain consecutive ++ or --. The inductive step amounts to the following

Claim. If \(b_1(v) = b_{-1}(v)\) for some \(v = (x_1, \ldots, x_n) \in \{[-1, 1] \setminus \{0\}\}^n\), then \(b_1(v') = b_{-1}(v')\) for \(v' := (x_1, \ldots, x_{k+1}, e_1, e_2, x_{k+1} + 1, \ldots, x_n) \in \{[-1, 1] \setminus \{0\}\}^{n+2}\) where \(\operatorname{sign}(e_1) = \operatorname{sign}(e_2)\) and \(0 \leq k \leq n\).

To prove this, note that if \(j \leq k\) then \(a_{v'}(i,j) = a_{v}(i,j)\). Also, if \(i \geq k + 3\) then \(a_{v'}(i,j) = a_{v}(i-2,j-2)\). Thus in both cases \(s(a_{v'}(i,j)) = s(a_{v}(i,j))\) and \(s(a_{v'}(i,j)) = s(a_{v}(i-2,j-2))\). Moreover, if \(i \leq k\) and \(j \geq k + 3\), then \(s(a_{v'}(i,j)) = s(a_{v}(i,j-2))\). So we just need to inspect the elements \(a_{v'}(1,k+2), a_{v'}(1,k+2), a_{v'}(k+1,n+2)\) and \(a_{v'}(k+2,n+2)\).

If \(\operatorname{sign}(e_1) = 1\), then \(s(a_{v'}(1,k+1)) = s(a_{v'}(1,k+2))\) and therefore \(a_{v'}(1,k+1) \in J_v\) if and only if \(a_{v'}(1,k+2) \notin J_v\). The same statement holds for \(a_{v'}(k+1,n+2)\) and \(a_{v'}(k+2,n+2)\). Also, since the number of \(\{x_i\}_{i=1}^n\) with \(\operatorname{sign}(x_i) = -1\) is odd, then \(\operatorname{sign}(\prod_{i=1}^n x_i) = -\operatorname{sign}(\prod_{i=k+1}^n x_i)\). This implies that \(s(a_{v'}(1,k+1)) \neq s(a_{v'}(1,k+2))\), so we have \(b_1(v') = b_{-1}(v')\).

If \(\operatorname{sign}(e_1) = -1\), then \(s(a_{v'}(1,k+1)) \neq s(a_{v'}(1,k+2))\) and therefore \(a_{v'}(1,k+1) \in J_v\) if and only if \(a_{v'}(1,k+2) \notin J_v\). The same statement holds for \(a_{v'}(k+1,n+2)\) and \(a_{v'}(k+2,n+2)\). So we have \(b_1(v') = b_{-1}(v')\). \(\square\)
We will construct the partition of \( J_v \) inductively. Let us define a total order over \( \{(i, j) : i \leq j\} \) as in Figure 3 (i, j) \( \prec (i', j') \) if and only if \( j' > j \), or \( j' = j \) and \( i' < i \).

\[
(1, n) \prec (2, n) \prec (3, n) \prec (4, n) \prec (5, n) \prec (6, n) \prec \cdots \prec (n, n)
\]

\[
(1, 6) \prec (2, 6) \prec (3, 6) \prec (4, 6) \prec (5, 6) \prec (6, 6)
\]

\[
(1, 5) \prec (2, 5) \prec (3, 5) \prec (4, 5) \prec (5, 5)
\]

\[
(1, 4) \prec (2, 4) \prec (3, 4) \prec (4, 4)
\]

\[
(1, 3) \prec (2, 3) \prec (3, 3)
\]

\[
(1, 2) \prec (2, 2)
\]

\[
(1, 1)
\]

**Figure 3.** Order used to construct the partition of \( J_v \).

We define \( \pi_0 \subset \mathcal{P}(J_v) \), a subset of the power set of \( J_v \), as

\[
\pi_0 := \{ \{(i, j)\} \mid s(a_v(i, j)) = 1 = (-1)^{i+j}, 1 \leq i \leq j \leq n \}.
\]

Let \( N \geq 0 \) be the number of pairs \((i, j) \in J_v\) with \( s(a_v(i, j)) = -1 \). If \( N = 0 \), then \( \pi_0 \) is already a good partition. If \( N > 0 \), for \( 1 \leq k \leq N \) we will add inductively to \( \pi_{k-1} \) the \( k \)-th \((i, j)\) such that \( a_v(i, j) \) has negative product sign. Thus (some element of) \( \pi_k \) contains the first \( k \) pairs \((i, j)\) in the order \( \prec \). We will show that \( \pi_N \) is the partition of \( J_v \) claimed in lemma 8.

If \((i, j)\) is the \( k \)-th element of \( J_v \) for which \( s(a_v(i, j)) = -1 \) we will choose one of two operations to apply to \( \pi_{k-1} \) to produce \( \pi_k \).

**Operation 1.** If \( \{(i', j')\} \in \pi_{k-1} \) with \( i' = i \) and \( i \leq j' < j \), or \( j' = j \) and \( i < i' \leq j \), where \( a_v(i', j') \) has positive product sign, then

\[
\pi_k := \left( \pi_{k-1} \setminus \{\{(i', j')\}\} \right) \cup \{\{(i', j'), (i, j)\}\}.
\]

**Operation 2.** If \( \{(i, l), (r, l)\} \in \pi_{k-1} \) and \( \{(r, j)\} \in \pi_{k-1} \) with \( i \leq l < j \) and \( 1 \leq r < i \), where \( a_v(i, l) \) and \( a_v(r, j) \) have positive product sign and \( a_v(r, l) \) has negative product sign, then

\[
\pi_k := \left( \pi_{k-1} \setminus \{\{(i, l), (r, l)\}, \{(r, j)\}\} \right) \cup \{\{(i, l), (r, l), (i, j), (r, j)\}\}.
\]

Thus operation 1 removes a singleton from \( \pi_{k-1} \) and inserts a doubleton containing this singleton. Operation 2 removes a doubleton and a singleton from \( \pi_{k-1} \) and inserts a quadruplet containing the removed elements and forming the vertices of a rectangle. It is useful to visualize the effect of the operations as in Figure 3, where the subindex gives the sign of the corresponding product.

**Remark 11.** After either operation the new set \( \pi_k \) only contains sets \( p \) that correspond to one of the cases in definition 2 and \( \sum_{p \in \pi_k} \|p\| = 1 + \sum_{p \in \pi_{k-1}} \|p\| \). Furthermore, all \((l, t) \in J_v\) with \((l, t) \prec (i, j)\) are already in \( \pi_{k-1} \) (regardless of the product sign of \( a_v(l, t) \)), but \((l, t)\) is not contained in \( \pi_{k-1} \) if \((i, j) \preceq (l, t)\) and \( s(a_v(l, t)) = -1 \).
For simplicity we are going to say that a non-canonical pair \((i, j)\) is positive (respectively negative) if the corresponding term \(a_{v(i, j)}\) has positive (respectively negative) product sign. If we already know the product sign of a pair we will add it as a sub-index. The following definition will also be useful.

**Definition 12.** Let \(0 \leq k \leq N\). If \((i, j)_+ \in J_v\) is positive, then we will say that

1. \((i, j)_+\) is in a singleton if \(\{(i, j)_+\} \in \pi_k\). We call it a \([\oplus]\)-configuration.
2. \((i, j)_+\) is in an \(h\)-doubleton if \((\{(i, j)_+, (i-l, j)_-\}) \in \pi_k\), \(i-l < i\). We call it a \([\ominus\oplus]\)-configuration.
3. \((i, j)_+\) is in a \(v\)-doubleton if \(\{(i, j)_+, (i, j+l)_-\} \in \pi_k\), \(j < j+l\). We call it a \([\ominus\ominus]\)-configuration.
4. \((i, j)_+\) is in an \(i\)-quadruplet if \(\{(i, j)_+, (i-l, j)_-, (i, j+l)_-, (i-l_1, j+l_2)_+\} \in \pi_k\), \(l_1, l_2 \geq 1\). We call it a \([\ominus\ominus\ominus\ominus]\)-configuration.
5. \((i, j)_+\) is in a \(k\)-quadruplet if \(\{(i, j)_+, (i+l, j)_-, (i, j-l)_-, (i+l_1, j-l_2)_+\} \in \pi_k\), \(l_1, l_2 \geq 1\) and \(i+l_1 \leq j-l_2\). We call it a \([\ominus\ominus\ominus\ominus]\)-configuration.

If \((i, j)_-\) is negative and is contained in some subset of \(\pi_k\), then we will say that

1. \((i, j)_-\) is in an \(h\)-doubleton if \(\{(i, j)_-, (i+l, j)_+\} \subset \pi_k\), \(i < i+l \leq j\). We call it a \([\ominus\oplus]\)-configuration.
2. \((i, j)_-\) is in a \(v\)-doubleton if \(\{(i, j)_-, (i, j-l)_+\} \subset \pi_k\), \(i \leq j-l < j\). We call it a \([\ominus\ominus]\)-configuration.

Finally, given \((i, j)_-\) and \((i, l)_+\), \(i \leq l < j\), in \([\ominus\ominus]\)-configuration with \((i', l')_-, 1 \leq i' \leq l\). We say that we can apply operation 2 when the pair \((i', j)_+\) is in \([\ominus\ominus]\)-configuration.

Now we show the method used to obtain \(\pi_k\) from \(\pi_{k-1}\) for each \(k, 1 \leq k \leq N\). Suppose \((i, j)_-\) is the \(k\)-th negative pair and fix the horizontal list

\[
 l_{(i,j)} := \{ (i,j),(i+1,j),\ldots,(j,j) \} \cap J_v,
\]

we do one of the following

- **Case 1.** If there is some positive pair \((i, j)_+\) in the list \(l_{(i,j)}\) in a \([\oplus]\)-configuration then we will use operation 1 with the maximal (with the order \(<\)) positive pair \((i+l, j)_+\), \(i < i+l \leq j\), contained in the list \(l_{(i,j)}\) that is in a \([\oplus]\)-configuration.

- **Case 2.** If \((i, j)_-\) is the minimal negative pair on the list \(l_{(i,j)}\) for which we couldn’t apply Case 1. We will prove (Lemma 10) that there is a unique positive pair in the vertical list \(s_{(i,j)} := \{ (i,i),\ldots,(i,j-1),(i,j) \} \cap J_v\) that is contained in a \([\ominus\ominus]\)-configuration or in a \([\ominus\ominus]\)-configuration such that we can apply operation 2. In the first instance we use operation 1 while in the second instance we use operation 2.

- **Case 3.** If we couldn’t apply Case 1 nor Case 2, we consider the minimal negative pair \((i_1, j)_-\) on the list \(l_{(i,j)}\) for which we couldn’t apply Case 1.
From Case 2, \((i_1, j_1)_-\) is contained in a \([\pi_+^{\oplus}]-configuration\) with some positive pair \((i_1, j_1)_+\). We will prove (Lemma \ref{lem:positive_pairs}) that the positive pair \((i, j)_+\) is contained in a \([\pi_+^{\oplus}]-configuration\) or in a \([-\pi_+^{\oplus}]-configuration\) such that we can apply operation \ref{case:2}. In the first instance we use operation \ref{case:1} while in the second instance we use operation \ref{case:2}.

Proving lemmas \ref{lem:impossible_configurations} and \ref{lem:positive_pairs} will conclude our proof of lemma \ref{lem:impossible_pairs}.

**Remark 13.** Let \(0 \leq k \leq N\), suppose that we are able to construct \(\pi_k\) following the method above. The following five configurations are impossible in \(\pi_k\).

1. \((i, j)_-\) is in a \([\oplus_+]-configuration\) with \((i_l, j)_+\), \((i', j)_-\) is in a \([\oplus_+]-configuration\) with \((i'+l', j)_+\) and \(i < i' < i + l < i' + l'\).
2. \((i, j)_-\) is in a \([\oplus_+]-configuration\) with \((i_l, j)_+\), \((i', j)_-\) is in a \([\pi_+^{\oplus}]-configuration\) with \((i', j - l')_+\) and \(i < i' < i + l\).
3. \((i, j)_-\) is not in a \([\oplus_+]-configuration\), \((i', j)_+\) is in a \([\pi_+^{\oplus}]-configuration\) with \((i', j + l)_-\) and \(i < i'\).
4. \((i, j)_-\) is in a \([\pi_+^{\oplus}]-configuration\) with \((i, j - l)_+\), \((i', j)_-\) is in a \([\pi_+^{\oplus}]-configuration\) with \((i', j - l')_+\), \(i < i'\) and \(l_1 \neq l_2\).
5. \((i, j)_-\) is in a \([\pi_+^{\oplus}]-configuration\) with \((i, j - l)_+\), \((i', j)_-\) is in a \([\pi_+^{\oplus}]-configuration\) with \((i', j - l')_+\), \(i < i'\) and \((i, j)_-\) is the minimal negative pair on the list \(l_{(i,j)}\) for which we couldn't apply Case 1.

The first three items are consequences of Case 1 of the construction given above, item (4) is a consequence of Case 3 and item (5) is a consequence of Case 2. See Figure 5.

\[
\begin{align*}
(i, j)_- & \quad \ldots \quad (i', j)_- \quad \ldots \quad (i + l, j)_+ \\
& \vdots \\
(i, j)_- & \quad \ldots \quad (i', j)_- \quad \ldots \quad (i + l, j)_+ \quad \ldots \quad (i' + l', j)_+ \quad (i', j - l')_+ \\

(a) \text{ Impossible configuration (1).} & & (b) \text{ Impossible configuration (2).} \\
(i', j + l)_- & \quad \ldots \quad (i, j)_- \quad \ldots \quad (i', j)_- \\
& \vdots \\
(i, j)_- & \quad \ldots \quad (i', j)_+ & (i, j - l)_+ & (i', j - l')_+ \\
(c) \text{ Impossible configuration (3).} & & (d) \text{ Impossible configurations (4) and (5).}
\end{align*}
\]

**Figure 5.** Configurations from remark \ref{lem:impossible_configurations}.

In fact our construction makes a variety of configurations impossible.

**Lemma 14.** Let \(0 \leq k \leq N\), suppose that we are able to construct \(\pi_k\) following the method above and consider a horizontal list \(L := \{(i, j), \ldots, (i + l', j)\} \cap J_v\).

1. If \(L\) contains strictly more positive pairs than negative pairs, then there exist some positive pair \((i + l, j)_+ \in L\) contained in a partition \(p\) such that \(p \cap L = \{(i + l, j)_+\} \).
(2) If \( L \) contains strictly more negative pairs than positive pairs and every negative pair in \( L \) is contained in a partition, then there exist some negative pair \((i + l, j)_-\) in \( L \) contained in a partition \( p \) such that \( p \cap L = \{(i + l, j)_-\} \), in particular \((i + l, j)_-\) must be in a \([-\oplus]_+\)-configuration.

We have analogues statements for vertical lists.

**Proof.** Suppose that for every positive pair \((i + l, j)_+ \in L \) there exist a partition \( p \) such that \( p \cap L \neq \{(i + l, j)_+\} \). By inspecting the different cases of definition \([12]\) we conclude that there exist some \( b_i \) such that \( p \cap L = \{(i + l, j)_+, (i + b_i, j)_-\} \). Since the partitions are disjoint we are able to find a negative pair for each positive pair, which contradicts the hypothesis. The proof of the second statement is analogous.

**Lemma 15.** Let \( 0 \leq k \leq N \), suppose that we are able to construct \( \pi_k \) following the method above. If there is some non canonical negative pair \((i, j)_-\) not in a \([\oplus+\])-configuration, every pair \((a, b) \prec (i, j)_-\) is contained in a partition of \( \pi_k \) and there is a negative pair \((i, j - l_2)\) in a \([\oplus+\])-configuration with the positive pair \((i + l_1, j - l_2)_+\), then in the list \( L := \{(i, j)_-, (i + 1, j), \ldots, (i + l_1, j)_+\} \cap J \), there is at least one positive pair \((i + l, j)_+\) in a \([-\ominus]-configuration or in a \([-\ominus]_+\)-configuration. Moreover, if \((i + l, j)_+\) is in a \([-\ominus]_+\)-configuration with \((i + l + l'_1, j)_-\), \((i + l, j - l'_2)_-\) and \((i + l + l'_1, j - l'_2)_+\), then \( i + l < i + l + l'_1 \) (See figure \([10]\)).

**Proof.** We will prove that in the list \( L_{\text{sub}} := \{(i, j - l_2)_-, (i + 1, j - l_2), \ldots, (i + l_1, j - l_2)_+, \} \cap J \), there are more positive pairs than negative pairs. Suppose that there is some negative pair \((i, j - l_2)_-\) in \( L_{\text{sub}} \), then by remark \([13]\) (1) and (2) it must be in a \([\ominus+\])-configuration with some \((i + l', j - l_2)_+\) such that \( i + l'' < i + l_1 \). So for each negative pair in \( L_{\text{sub}} \) there is a unique corresponding positive pair in \( L_{\text{sub}} \).

Remark \([3]\) implies that \( s(a_{v(i + r, j)}) = s(a_{v(i + r, j - l_2)}) \) for \( 0 \leq r \leq l_1 \), thus there are strictly more positive elements than negative elements in the list \( L' := L - \{(i, j)_-\} \). By lemma \([14]\) there must be at least one positive pair \((i + l, j)_+ \in L \) contained in a partition \( p \in \pi_k \), without a negative pair in the list \( L' \), i.e. \( p \cap L' = \{(i + l, j)_+\} \). We conclude inspecting the possible configurations for \((i + l, j)_+\) of definition \([12]\) \([-\ominus]-configuration, \([-\oplus]_+\)-configuration and \([-\ominus]_+\)-configuration form impossible configurations (2) and (3) of remark \([13]\) and if \((i + l, j)_+\) is in a \([-\ominus]_+\)-configuration with \((i + l, j - l'_2)_-\) and \((i + l + l'_1, j - l'_2)_+\), then \((i + l + l'_1, j)_- \not\in L' \), which is equivalent to the last statement of the lemma.

\[
\begin{align*}
(i, j)_- & \quad \ldots \quad (i + l, j)_+ \quad \ldots \quad (i + l_1, j)_+ \quad \ldots \quad (i + l + l'_1, j)_- \\
\vdots & \\
(i, j - l_2)_- & \quad \ldots \quad (i + l_1, j - l_2)_+
\end{align*}
\]

**Figure 6.** Lemma \([15]\) when \((i + l, j)_+\) is in a \([-\ominus]_+\)-configuration.

In the rest of the paper we let \( 1 \leq k \leq N \), suppose that we are able to construct \( \pi_{k-1} \) following the method above and we fix \((i, j)_-\) the \( k \)-th negative pair.
Lemma 16. Suppose \((i, j)_-\) is the minimal negative pair on the list \(l_{(i,j)}\) for which we couldn’t apply Case 1, then in the vertical list \(s_{(i,j)}\) there is a unique positive pair \((i, j)_+\) for which we can apply operation 1 or operation 2. Specifically we will prove that \((i, j)_+\) is in a \([ \overline{a} ]\)-configuration (So we can apply operation 1) or in a \([- \overline{a} ]\)-configuration with \((i', j')_-\) and \((i', j)_+\) is in a \([ \overline{a} ]\)-configuration (So we can apply operation 2).

Proof. Existence. We will use lemma 10. Since \(s(a_{(i,j)}) = -1\), there is an odd number of negative \(x_r\)'s with \(i \leq r \leq j\). As we are also assuming that \(a_{(i,j)}\) is non canonical, \(j - i\) is odd. Thus the hypotheses of lemma 10 with the vector \(v' = (x_i, x_{i+1}, \ldots, x_j)\) are satisfied. Hence, one of the following holds.

• Subcase 1. List \(A_1 := s_{(i,j)} - \{(i, j)\} = \{(i, i), (i, i+1), \ldots, (i, j-1)\} \cap J_v\) has strictly more positive pairs than negative pairs.

• Subcase 2. List \(A_2 := l_{(i,j)} - \{(i, j)\} = \{(i+1, j), \ldots, (j-1, j), (j, j)\} \cap J_v\) has strictly more positive pairs than negative pairs.

Subcase 2 is impossible, if \(A_2\) has strictly more positive pairs than negative pairs then by lemma 14 there must be some positive pair in the list \(l_{(i,j)}\) in \([ \overline{a} ]\)-configuration or in \([ \overline{a} ]\)-configuration. The first possibility is a contradiction with the fact that we couldn’t apply Case 1 of the construction and the second possibility is impossible since we would obtain impossible configuration \((4)\) of remark 13.

So \(A_1\) has strictly more positive pairs than negative pairs, by lemma 14 it must have a positive pair \((i', j')_+\) in \([ \overline{a} ]\)-configuration or in \([- \overline{a} ]\)-configuration with \((i', j')_-\), for \(i' < i\). If \((i', j')_+\) is in \([ \overline{a} ]\)-configuration we can apply operation 1. If \((i', j')_+\) is in \([- \overline{a} ]\)-configuration we have to prove that \((i', j)_+\) is in \([ \overline{a} ]\)-configuration so that we can apply operation 2. Inspecting the different configurations for the positive pair \((i', j)_+\), by remark 11 it can’t be in a \([- \overline{a} ]\)-configuration, \([- \overline{a} ]\)-configuration or \([- \overline{a} ]\)-configuration. Suppose it is in a \([ \overline{a} ]\)-configuration (See figure 7) with some pairs \((i'', j)_-\), \((i'', j''_+\) and \((i', j'')_-\), then by remark 11 \(i < i''\). Thus \((i'', j)_-\) is in a \([ \overline{a} ]\)-configuration with \((i'', j'')_+\). Since \(i < i''\) we would obtain impossible configuration \((5)\) of remark 13. We conclude that \((i', j)_+\) must be in \([ \overline{a} ]\)-configuration and we can apply operation 2.

\[
(i', j)_+ \ldots (i, j)_- \ldots (i'', j)_- \\
:\:\:\:
(i', j')_+ \ldots (i, j')_+
\]

**Figure 7.** If \((i', j)_+\) is in \([\overline{a}]\)-configuration.

\[
(i, j)_- \ldots (i, j)_+ \ldots (i', j')_+ \\
:\:\:\:
(i, j')_- \ldots (i, j')_+
\]

**Figure 8.** If \((i', j)_+\) is not unique.

Uniqueness. We already proved that in \(A_2\) there are more negative pairs than positive pairs. However, if there are strictly more negative pairs than positive pairs, then by lemma 14 we can find some negative pair in \(A_2\) in a \([ \overline{a} ]\)-configuration and would obtain impossible configuration \((5)\) of remark 13. So in the list \(A_2\) there is the same

---

\(^1\) To apply lemma 10 to \(v'\), note that \(a_{v(t, l)} = a_{v'((l-1)+1, t-1+1)} (i \leq l \leq t \leq j)\), so that \((l, t)\) is canonical with respect to \(v\) if and only if \((l-1+1, t-1+1)\) is canonical with respect to \(v'\).
number of positive and negative pairs. Now lemma 11 guarantees that \( s_{(i,j)} \) has exactly the same number of positive and negative pairs.

Suppose there are two positive pairs in \([\oplus]\)-configuration or in \([\ominus\oplus]\)-configuration in the list \( A_1 \). If every negative pair in the list \( A_1 \) is in \([\oplus\ominus]\)-configuration, since the partitions are disjoint, we would obtain strictly more positive pairs than negative pairs in \( s_{(i,j)} \).

So there is at least one negative pair \((i,j)_-\) in \([\ominus\oplus]\)-configuration with \((i,l,j')_+\) in the list \( A_1 \) (See figure 9). Applying lemma 15 to \((i,j)_-\) and \((i,j')_-\) in \([\ominus\oplus]\)-configuration with \((i+l,j')_+\) we obtain a positive pair in the list \( l_{(i,j)} \) in a \([\oplus]\)-configuration or in a \([\ominus\oplus\ominus]\)-configuration. The first possibility is a contradiction with the fact that we couldn’t apply Case 1 of the construction and the second possibility is impossible since we would obtain impossible configuration (5) of remark 13.

**Lemma 17.** Suppose \((i,j)_-\) is not the minimal negative pair on the list \( l_{(i,j)} \) for which we couldn’t apply Case 1, that \((i_1,j)_-\) is the minimal negative pair on the list \( l_{(i,j)} \) for which we couldn’t apply Case 1 and that \((i_1,j)_-\) is contained in a \([\ominus\ominus]\)-configuration with some positive pair \((i_1,j_1)_+\), then it is possible to apply operation 1 or operation 2 with the positive pair \((i,j_1)_+\).

Specifically we will prove that \((i,j_1)_+\) is in a \([\ominus\ominus]\)-configuration (So we can apply operation 1) or in a \([\ominus\ominus\ominus]\)-configuration with \((i-l,j_1)_-\) and \((i-l,j)_+\) is in a \([\ominus\ominus]\)-configuration (So we can apply operation 2).

**Proof.** We will study separately each possible configuration for the pair \((i,j_1)_+\).

\[
(i - l, j)_+ \quad \ldots \quad (i, j)_- \quad \ldots \quad (i', j)_- \\
\vdots \quad \vdots \quad \vdots \\
(i - l, j_1)_- \quad \ldots \quad (i, j_1)_+ \quad \ldots \quad (i', j')_+ \\
\]

(a) If \((i, j_1)_+\) is in a \([\ominus\ominus]\)-configuration.

\[
(i, j)_- \quad \ldots \quad (i_1, j)_- \quad \ldots \quad (i', j)_+ \quad (i, j)_- \quad \ldots \quad (i_1, j)_- \quad \ldots \quad (i', j)_+ \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
(i, j')_- \quad \ldots \quad (i_1, j')_- \quad \ldots \quad (i', j')_+ \quad (i, j_1)_+ \quad \ldots \quad (i_1, j_1)_+ \quad \ldots \quad (i', j)_- \quad \ldots \quad (i', j')_+ \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
(i, j_1)_+ \quad \ldots \quad (i_1, j_1)_+ \quad (i, j')_- \quad \ldots \quad (i_1, j')_- \quad \ldots \quad (i', j')_+ \\
\]

(b) If \((i,j_1)_+\) is in a \([\ominus\ominus]\)-configuration or in a \([\ominus\ominus\ominus]\)-configuration.

(c) If \((i,j_1)_+\) is in a \([\ominus\ominus\ominus]\)-configuration.

**Figure 9.** Figures of lemma 17.

- If \((i,j_1)_+\) is in a \([\ominus\ominus]\)-configuration we can directly apply operation 1.
- If \((i,j_1)_+\) is in a \([\ominus\ominus]\)-configuration with \((i - l, j)_-\) we have to prove that \((i - l, j)_+\) is a \([\ominus\ominus]\)-configuration, so we inspect the different configurations for the pair \((i - l, j)_+\) (See figure 9a). By remark 11 it can’t be in a \([\ominus\ominus\ominus]\)-configuration, \([\ominus\ominus\ominus]\)-configuration or \([\ominus\ominus\ominus]\)-configuration. Suppose it is in a \([\ominus\ominus\ominus]\)-configuration with...
some pairs \((i', j)_-, (i-l, j')_-, (i', j')_+,\) then to avoid impossible configuration (4) of remark 13 necessarily \(j' = j_1.\) So the pair \((i-l, j_1)_-\) would be contained in two different partitions, which is impossible.

- If \((i, j)_+\) is in a \([\bigcirc-\bigcirc]\)-configuration or in a \([\bigcirc-\bigcirc\bigcirc]\)-configuration (See figure 9c), then there is some \((i, j')_+\) in a \([\bigcirc\bigcirc]\)-configuration with \((i, j)_+\) and remark 9 implies that the pair \((i_1, j')_+\) is negative.

  If \((i_1, j')_-\) was in \([\bigcirc\bigcirc\bigcirc]\)-configuration with \((i_1, j_{i''})_+\), since \((i, j')_-\) in a \([\bigcirc\bigcirc]\)-configuration with \((i, j_1)_+\), to avoid impossible configuration (4) of remark 13 necessarily \(j'' = j_1\) and so the pair \((i_1, j_1)_-\) would be contained in two different partitions, which is impossible. Thus \((i_1, j')_+\) is in a \([\bigcirc+\bigcirc\bigcirc]\)-configuration with some pair \((i', j')_+\).

  We finally apply lemma 15 to the negative pair \((i_1, j)_-\) and to \((i_1, j'_1)_-\) in a \([\bigcirc+\bigcirc\bigcirc]\)-configuration with \((i', j')_+\). We conclude that there must be some positive pair \((i'', j')_+\) with \(i_1 < i''\) in a \([\bigcirc+\bigcirc\bigcirc]\)-configuration or in a \([-\bigcirc\bigcirc\bigcirc]\)-configuration, however both possibilities are impossible. The first possibility is a contradiction with the fact that we couldn’t apply Case 1 of the construction and the second possibility is impossible since we would obtain impossible configuration (5) of remark 13.

- If \((i, j)_-\) is in a \([\bigcirc+\bigcirc\bigcirc]\)-configuration with \((i', j)_-, (i', j')_-, (i', j')_+\) (See figure 9b). Remark 13 (2) implies \(i_1 < i'\). By remark 9 the pair \((i_1, j')_-\) is negative and by remark 13 (2) the pair \((i_1, j')_-\) must be in \([\bigcirc+\bigcirc\bigcirc\bigcirc]\)-configuration with some positive pair \((i'', j')_+\). We apply lemma 14 to \((i_1, j')_-\) and the pair \((i_1, j')_-\) in \([\bigcirc+\bigcirc\bigcirc\bigcirc]\)-configuration with \((i'', j')_+\) to obtain a positive pair \((i''', j')_+\) with \(i_1 < i''\) in a \([\bigcirc\bigcirc\bigcirc\bigcirc]\)-configuration or in a \([\bigcirc+\bigcirc\bigcirc\bigcirc]\)-configuration, however both possibilities are impossible. The first possibility is a contradiction with the fact that we couldn’t apply Case 1 of the construction and the second possibility is impossible since we would obtain impossible configuration (5) of remark 13.

\(\Box\)

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