A DIFFERENTIAL ALGEBRA AND THE HOMOTOPY TYPE OF THE COMPLEMENT OF A TORIC ARRANGEMENT

CORRADO DE CONCINI, GIOVANNI GAIFFI

Abstract. We show that the rational homotopy type of the complement of a toric arrangement is completely determined by two sets of combinatorial data. This is obtained by introducing a differential graded algebra over $\mathbb{Q}$ whose minimal model is equivalent to the Sullivan minimal model of $A$.

1. Introduction

Let $T \simeq G_m^n$ be a complex $n$ dimensional algebraic torus and let us denote by $X^*(T) \simeq \mathbb{Z}^n$ its character group.

A layer in $T$ is the subvariety $K_{\Gamma, \phi} = \{ t \in T | \chi(t) = \phi(\chi), \forall \chi \in \Gamma \}$

where $\Gamma$ is a split direct summand of $X^*(T)$ and $\phi : \Gamma \to \mathbb{C}^*$ is a homomorphism.

A toric arrangement $\mathcal{A}$ is given by a finite set of layers $\mathcal{A} = \{K_1, \ldots, K_m\}$ in $T$; if for every $i = 1, \ldots, m$ the layer $K_i$ has codimension 1 the arrangement $\mathcal{A}$ is called divisorial.

We will denote by $M(\mathcal{A})$ the complement $T - \bigcup_i K_i$ of the arrangement. We notice that if we consider the saturation $\check{\mathcal{A}}$ of $\mathcal{A}$, i.e. the arrangement consisting of all the layers which are obtained as connected components of intersections of layers in $\mathcal{A}$, we have $M(\mathcal{A}) = M(\check{\mathcal{A}})$.

The purpose of this note is to show that the rational homotopy type of $M(\mathcal{A})$ is completely determined by

(1) The partially ordered set $\check{\mathcal{A}}$ ordered by reverse inclusion.
(2) The set of lattices $\Gamma \subset X^*(T)$ for $K_{\Gamma, \phi} \in \check{\mathcal{A}}$.

We will call these data the combinatorial data of $\mathcal{A}$.

This is obtained by introducing an object that may be of independent interest: a differential graded algebra over $\mathbb{Q}$, defined using the combinatorial data of $\mathcal{A}$, whose minimal model is equivalent to the Sullivan minimal model of $\mathcal{A}$. In particular we have that its cohomology is isomorphic to the rational cohomology of $M(\mathcal{A})$.

Before giving a sketch of our construction, we recall some previous results on this subject.

As far as we are aware, the results regarding the (rational) homotopy of $M(\mathcal{A})$ have been obtained in the divisorial case. In this case, in [9] De Concini and Procesi determined the generators of the rational cohomology modules of $M(\mathcal{A})$, as well as the ring structure in the case of totally unimodular arrangements. By a rather general approach, Dupont in [10] proved the rational formality of $M(\mathcal{A})$. In turn, in [2], it was shown extending the results in [3, 4] and [19] that the data needed in order to state the presentation of the rational cohomology ring of $M(\mathcal{A})$ is fully encoded in the partially ordered set $\check{\mathcal{A}}$. It follows that the combinatorics of the poset $\check{\mathcal{A}}$ determines the rational homotopy of $M(\mathcal{A})$.

Our approach to the study of the rational homotopy type and of the cohomology of $M(\mathcal{A})$ in full generality involves the construction of projective wonderful models for $M(\mathcal{A})$. 
Results analogous to those in this note were previously obtained in [8] for arrangements of linear subspaces in a projective space using the construction of wonderful models for subspace arrangements and the fundamental results of Morgan in [18]. In particular Morgan introduced, in the case of a compactification \( V \) of a complex algebraic variety \( X \) such that \( V \setminus X = D \) is a divisor with normal crossing, a differential graded algebra (which we are going to call a Morgan differential algebra) whose minimal model is equivalent to the Sullivan minimal model of \( X \).

In the toric case the idea is to do similar considerations using the projective wonderful models of \( M(\mathcal{A}) \) we constructed in [3] and the presentation by generators and relations of the integer cohomology rings of these models and of the strata in their boundary given in [4].

Indeed in [16] for each of these models these presentations were used to describe its Morgan differential algebra which determines the rational homotopy type of \( M(\mathcal{A}) \).

However the projective models described in [5] do not depend only on the combinatorial data of the toric arrangement \( \mathcal{A} \) but also on some extra choices in the construction process and indeed the differential Morgan algebras one obtains also depend on these choices.

To overcome this problem we are going to construct a new differential graded algebra as a direct limit of the differential Morgan algebras of the projective wonderful models described in [5]. This algebra, based on the notion of the ring of conditions of \( T \), is rather large and it is not the Morgan algebra of any compactification of \( M(\mathcal{A}) \).

However we show in Proposition 5.4 that it is quasi isomorphic to any of the Morgan algebras of the projective wonderful models of \( M(\mathcal{A}) \) and it has a simple presentation which depends only on the combinatorial data of \( \mathcal{A} \).

Let us describe more in detail the structure of this paper. First (in Section 2) we briefly provide a self contained presentation of the ring of conditions \( C(T) \) of the torus \( T \). We recall that it was shown by Fulton and Sturmfels in [11] that this ring over \( \mathbb{Q} \) is isomorphic to the McMullen polytope algebra (see [15], and [17] for a similar construction) and is the direct limit of the rational Chow rings of all the compactifications of \( T \). For other descriptions of the ring of conditions of the torus the reader can see for instance [1], [13] (and [7] where rings of conditions appeared in a more general setting).

In Section 2 we first introduce an equivariant version \( B_T(T) \) of the ring of conditions as follows. Let us consider the lattice of one parameter subgroups \( X_*(T) = \text{hom}(X^*(T), \mathbb{Z}) \) and the vector space \( V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \). We denote by \( \Sigma \) the space of the continuous functions \( f \) on \( V \) such that \( f(X_*(T)) \subset \mathbb{Q} \) and there exists a smooth projective fan such that \( f \) restricted to every face of this fan is linear. Then we consider the algebra \( B_T(T) \) of continuous functions \( V \) generated by \( \Sigma \) and finally we obtain \( C(T) \) as the quotient of \( B_T(T) \) modulo the ideal generated by a basis of \( X^*(T) \).

Then in Section 3 we construct a differential graded algebra \( C \) whose degree 0 term is \( C(T) \). This algebra is the direct limit of all the differential graded Morgan algebras associated to the compactifications of \( T \).

So far the toric arrangement \( \mathcal{A} \) has not been taken into account. It appears in Section 4 where we first recall from [5] the construction of the projective wonderful model \( Y(X_F) \) associated to \( \mathcal{A} \) and to a suitable smooth projective fan \( F \). Then we recall from [6] the presentation of the cohomology of \( Y(X_F) \) and of its strata in the boundary and we construct, following Moci and Pagaria (see [16]), the Morgan differential algebra \( N_F \) for \( Y(X_F) \).

Finally, in Section 5 we introduce the differential graded algebra \( N \) as a direct limit of the algebras \( N_F \) and we present it by generators and relations (see Theorem 5.2) starting from \( C \otimes B \), where \( C \) is the limit algebra mentioned above and \( B \) is a quotient of a Weyl
algebra. We immediately obtain Proposition \[ \text{5.4} \] (the minimal model of \( N \) is isomorphic to the minimal model of \( \mathcal{M}(\mathcal{A}) \)) and, since the generators and relations of \( N \) depend only on the combinatorial data of \( \mathcal{A} \), we deduce Theorem \[ \text{5.5} \] on the rational homotopy type.

2. The ring of conditions, recollections

Let \( T \simeq (C^*)^n \) be a complex \( n \) dimensional algebraic torus. Denote by \( X^*(T) \simeq \mathbb{Z}^n \) its character group and by \( X_*(T) = \text{hom}(X^*(T), \mathbb{Z}) \) its lattice of one parameter subgroups. We set \( V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \).

We take a rational smooth projective fan \( \mathcal{F} \) in \( V \) and let \( \Gamma_\mathcal{F} = \{ c_1, \ldots, c_N \} \) be vertices, that is the set of primitive vectors in the one dimensional cones (rays) of \( \mathcal{F} \). Any cone \( C \in \mathcal{F} \) is of the form

\[
C = C(c_1, \ldots, c_k) = \{ v = \sum_{r=1}^{k} a_rc_{r} \mid a_r \geq 0 \}
\]

with \( c_1, \ldots, c_k \) the basis of a split direct summand in \( X_*(T) \).

Let us take variables \( x_c, c \in \Gamma_\mathcal{F} \), and in the polynomial ring \( \mathbb{Q}[x_c]_{c \in \Gamma_\mathcal{F}} \) take the ideal \( I_\mathcal{F} \) generated by the monomials \( m_J = \prod_{c \in J} x_c \) for all subsets of \( J \subseteq \Gamma_\mathcal{F} \) for which \( C(c_j)_{j \in J} \) is not a cone in \( \mathcal{F} \). Take the algebra \( A_\mathcal{F} := \mathbb{Q}[x_{c_1}, \ldots, x_{c_N}]/I_\mathcal{F} \). \( A_\mathcal{F} \) is the Stanley-Reisner ring of \( \mathcal{F} \) and it is the equivariant cohomology ring of the toric variety \( X_\mathcal{F} \) corresponding to \( \mathcal{F} \) (see \[ \text{1} \] Corollary 1.3 and Proposition 2.2). The algebra \( A_\mathcal{F} \) inherits a grading from the grading of \( \mathbb{Q}[x_{c_1}, \ldots, x_{c_N}] \) in which \( \deg x_c = 2 \) for all \( c \in \Gamma_\mathcal{F} \). The degree 2 part is spanned by the classes of the elements \( x_c \) and we denote it by \( S_\mathcal{F} \). We may associate to each \( x_c \) the function \( s_c \) on \( V \) defined as follows. We set \( s_c(d) = \delta_{c,d} \). Then for \( v \in V \) there exist a unique cone \( C = C(c_1, \ldots, c_k) \in \mathcal{F} \) such that \( v \) lies in the relative interior of \( C \), so \( v = \sum_{r=1}^{k} a_rc_{r}, a_r > 0 \). We then set

\[
s_c(v) = \begin{cases} 
0 & \text{if } c \neq c_r \forall r \\
\delta_{c,r} & \text{if } c = c_r 
\end{cases}
\]

Notice that \( s_c \) is continuous so that sending \( x_c \) to \( s_c \) we get a homomorphism

\[
\rho_\mathcal{F} : \mathbb{Q}[x_{c_1}, \ldots, x_{c_N}] \to C(V),
\]

where \( C(V) \) is the algebra of continuous functions on \( V \).

The functions \( s_c \) are linearly independent in \( C(V) \). Their span will be identified with \( S_\mathcal{F} \) and denoted by the same letter.

The following result is well known and we prove it for completeness.

**Proposition 2.1.**  
(1) The space \( S_\mathcal{F} \subseteq C(V) \) is the space of continuous functions on \( V \) with the property that their restriction to each cone of \( \mathcal{F} \) is linear. 

(2) The ideal \( I_\mathcal{F} \) is the kernel of \( \rho_\mathcal{F} \). In particular we obtain an inclusion

\[
\mu_\mathcal{F} : A_\mathcal{F} \to C(V).
\]

(3) Let \( \mathcal{G} \) be a smooth refinement of \( \mathcal{F} \), that is every cone in \( \mathcal{F} \) is subdivided by cones in \( \mathcal{G} \). We know that there is a map

\[
\gamma_\mathcal{G}^\mathcal{F} : A_\mathcal{F} \to A_\mathcal{G}.
\]

Then

\[ \mu_\mathcal{F} = \mu_\mathcal{G} \gamma_\mathcal{G}^\mathcal{F} \]
Proof. The first claim is clear since it is immediate to check that any function \( f \in C(V) \) whose restriction to each cone of \( \mathcal{F} \) is linear can written as
\[
f = \sum_c f(c)s_c.
\]

To see the second claim recall that for any cone \( C \in \mathcal{F} \), its star \( S(C) \) consists of cones in \( \mathcal{F} \) having \( C \) as a face. The function \( s_c \) is clearly supported on \( S(c) \) (for brevity we write \( S(c) \) instead than \( S(S(c)) \)). From this it follows that for any monomial \( m = x_{c_1}^{h_1} \cdots x_{c_n}^{h_n} \) the support of \( \rho_{\mathcal{F}}(m) = s_{c_1}^{h_1} \cdots s_{c_n}^{h_n} \) is \( S(C) \) if \( C = (c_1, \ldots, c_n) \in \mathcal{F} \) while \( \rho_{\mathcal{F}}(m) = 0 \) otherwise. We deduce that \( I_{\mathcal{F}} \subset \ker(\rho_{\mathcal{F}}) \). In particular we obtain a homomorphism \( \mu_{\mathcal{F}} : A_{\mathcal{F}} \to C(V) \).

Thus for a monomial \( m = x_{c_1}^{h_1} \cdots x_{c_n}^{h_n} \) with \( C = (c_1, \ldots, c_n) \in \mathcal{F} \), if \( C' = (c_1, \ldots, c_n) \) is a cone of maximal dimension, \( \mu_{\mathcal{F}}(mx_{c_1} \cdots x_{c_n}) \) is supported on \( C' \) if \( C \) is a face of \( C' \), while \( \mu_{\mathcal{F}}(mx_{c_1} \cdots x_{c_n}) = 0 \) otherwise.

Take now a polynomial \( P(x_{c_1}, \ldots, x_{c_n}) \). The restriction of \( \mu_{\mathcal{F}}(P(x_{c_1}, \ldots, x_{c_n})) \) to \( C' \) is just the evaluation of \( P(x_{c_1}, \ldots, x_{c_n}) \) hence it is zero if and only if \( P(x_{c_1}, \ldots, x_{c_n}) \equiv 0 \).

Take \( a \in \ker \mu_{\mathcal{F}} \). If \( a \neq 0 \) there is an \( n \)-dimensional cone \( \overline{C} = (c_1, \ldots, c_n) \) such that \( b = ax_{c_1} \cdots x_{c_n} \neq 0 \). Then \( \mu_{\mathcal{F}}(b) \) is the restriction of a polynomial \( P(x_{c_1}, \ldots, x_{c_n}) \) to \( \overline{C} \). Hence it is zero if and only if \( b = 0 \). A contradiction.

The last statement follows since, if we denote by \( y_d \) the variable corresponding to a vertex \( d \) of \( \mathcal{G} \), for any vertex \( c \) of \( \mathcal{F} \)
\[
(2.2) \quad \gamma^{\overline{G}}(x_c) = \sum_{\text{vertex of } \mathcal{G}} (\mu_{\mathcal{F}}(x_c)(d))y_d.
\]
as the reader can easily verify. \( \square \)

We now take a suitable algebra of continuous functions on \( V \)

**Definition 2.1.**

(1) The space \( \Sigma \) consists of the functions \( f \in C(V) \) such that
(a) If \( \lambda \in X_*(T) \), \( f(\lambda) \in \mathbb{Q} \).
(b) There exists a rational smooth projective fan \( \mathcal{F} \) such that for any \( C \in \mathcal{F} \) the restriction of \( f \) to \( C \) is linear.

(2) The equivariant ring of conditions \( B_T(T) \) is the \( \mathbb{Q} \) subalgebra of the ring of continuous functions on \( V \) generated by \( \Sigma \).

In this paper we will always consider rational fans, so from now on the adjective ‘rational’ will be omitted. Since any two smooth projective fans admit a common refinement which is still smooth and projective, it is clear that \( \Sigma \) is a \( \mathbb{Q} \)-vector space.

Let us order the set of \( \mathcal{S} \) of smooth projective fans using refinement. We get a directed system \( (A_{\mathcal{F}}, \gamma^{\overline{G}}_{\mathcal{F}}) \). By Proposition 2.1 we deduce that \( B_T(T) \) is the union of the images of the homomorphisms \( \mu_{\mathcal{F}} \) and we deduce (see [11]):

**Proposition 2.2.**

\[
B_T(T) = \lim_{\mathcal{F}} A_{\mathcal{F}}.
\]

Each element \( \ell \in X^*(T) \) is a linear function on \( V \) taking integral values on \( X_*(T) \) which, in terms of classes of the elements \( x_{c_i} \), is the class of
\[
\sum_{i=1}^{N} (\ell, c_i)x_{c_i}.
\]
(we are taking into account the identification of \( S_F \) with \( \rho_F(S_F) \)). In this way if we take a basis \( \xi_1, \ldots, \xi_n \) of \( X^*(T) \) and set \( R := \mathbb{Q}[\xi_1, \ldots, \xi_n] \), \( A_F \) is a free \( R \) module (see [1] Corollary 1.3 and Proposition 2.2.) and we may consider the quotient algebra
\[
B_F := A_F / (\xi_1, \ldots, \xi_n) \simeq H^*(X_F, \mathbb{Q}).
\]
It is clear that the \( \gamma_F \) induces an algebra homomorphism \( \overline{\gamma_F} : B_F \to B_G \).

**Definition 2.2.** The ring of conditions for \( T \) is the algebra
\[
\mathbb{C}(T) = \lim_{\leftarrow F} B_F = B_T(T) / (\xi_1, \ldots, \xi_n).
\]

### 3. Two differential graded algebras

We now want to define some differential graded algebras (DGA). Again we take a projective smooth fan \( F \) in \( V \).

We start with the algebra \( \mathbb{Q}[x_c] \otimes \bigwedge (\tau_c), c \in \Gamma_F \). We define a bigrading on this algebra by setting \( \deg x_c = (2,0), \deg \tau_c = (0,1) \). In general all the differential graded algebras we are going to consider will be easily seen to be bigraded, so we will often omit to specify how their bigrading is defined.

**Definition 3.1.**

1. The algebra \( D_F \) is the quotient of the algebra \( \mathbb{Q}[x_c] \otimes \bigwedge (\tau_c) \) modulo the bigraded ideal \( J_F \) generated by “the square free monomials”
\[
x_{c_1} \cdots x_{c_h} \tau_{c_{j_1}} \cdots \tau_{c_{j_k}}
\]
for each sequence of vertices \( c_{i_1}, \ldots, c_{i_m}, c_{j_1}, \ldots, c_{j_k} \) not spanning a cone in \( F \).

2. The differential \( d \) is the unique derivation on \( D_F \) defined by
\[
d(x_c) = 0, \quad d(\tau_c) = x_c.
\]

Remark that \( d \) preserves the relations in \( D_F \) and hence it is well defined and of degree 1. It is easily seen that if for any cone \( C = C(c_{i_1}, \ldots, c_{i_k}) \in F \), we take the algebra
\[
A_{C,F} := \mathbb{Q}[x_{c_{i_1}} \cdots x_{c_N}] / [I_F : (x_{c_{i_1}} \cdots x_{c_{i_k}})],
\]
we have
\[
D_F = \bigoplus_{C \in F} A_{C,F} \tau_{c_{i_1}} \cdots \tau_{c_{i_k}},
\]
and setting for each \( m = 0, \ldots, n \)
\[
D_{m,F} = \bigoplus_{C = C(c_{i_1}, \ldots, c_{i_m})} A_{C,F} \tau_{c_{i_1}} \cdots \tau_{c_{i_m}},
\]
the decomposition
\[
D_F = \bigoplus_{m=0}^n D_{m,F}.
\]

**Proposition 3.1.**

\[
H^i(D_F, d) = \begin{cases} 
\mathbb{Q} & \text{if } i = 0 \\
0 & \text{if } i > 0.
\end{cases}
\]

**Proof.** Let us consider the complex \( (D_F^+, d) \) of elements of positive degree. We need to prove that this complex is exact. Let us define define a map of degree \(-1\)
\[
S : D_F^+ \to D_F^+
\]
and show that \( Sd + dS \) is the identity. This will give our claim.

For this let fix a total order \( c_1, \ldots, c_N \) of the vertices of \( F \).
Below, for brevity, we will write $x_{i_1}, \tau_{i_1}$ instead of $x_{c_{i_1}}, \tau_{c_{i_1}}$.

Let $m = x_{i_1}^{h_1} \cdots x_{i_s}^{h_s} \tau_{j_1} \cdots \tau_{j_r} \in D^+_F$, with $j_1 < j_2 \cdots < j_r$ and if $s > 0, h_i > 0$. If $s = 0$ (resp. $r = 0$) we set $i_1 = \infty$ (resp. $j_1 = \infty$). Notice that $r + s > 0$ and set $f = \min(i_1, j_1)$. We define

$$S(m) = \begin{cases} 0 & \text{if } f = j_1 \\ x_{i_1}^{h_1-1} \cdots x_{i_s}^{h_s} \tau_{j_1} \cdots \tau_{j_r} & \text{if } f = i_1 < j_1 \end{cases}$$

Now let us compute $(dS + Sd)(m)$. If $f = j_1$ we have $dS(m) = 0$ and

$$Sd(m) = S(x_{j_1} x_{i_1}^{h_1} \cdots x_{i_s}^{h_s} \tau_{j_1} \cdots \tau_{j_r} + \sum_{\ell=2}^r (-1)^{\ell+1} x_{j_\ell} x_{i_1}^{h_1} \cdots x_{i_s}^{h_s} \tau_{j_1} \cdots \tau_{j_{\ell-1}} \tau_{j_\ell} \cdots \tau_{j_r}) = m.$$ 

If $f = i_1 < j_1$, one easily sees that $dS(m) = m - \sum_{\ell=1}^r (-1)^{\ell+1} x_{j_\ell} x_{i_1}^{h_1-1} x_{i_2}^{h_2} \cdots x_{i_s}^{h_s} \tau_{j_1} \cdots \tau_{j_{\ell-1}} \tau_{j_\ell} \cdots \tau_{j_r} = m - Sd(m)$

and everything follows. \(\square\)

When the fan $G$ is a (smooth, projective) refinement of $F$, we want to compare the algebras $D_F$ and $D_G$.

As before in order to avoid confusion for $d \in \Gamma_G$ we denote by $y_d$ and $v_d$ the corresponding even and odd variables. We define a homomorphism

$$\xi^F_G : \mathbb{Q}[x_c] \otimes \bigwedge(\tau_c) \to \mathbb{Q}[y_d] \otimes \bigwedge(v_d)$$

by setting

$$\xi^F_G(x_c) = \sum_{\text{d vertex of } G} (\rho_F(x_c)(d))y_d.$$ \hspace{1cm} (3.1)

$$\xi^F_G(\tau_c) = \sum_{\text{d vertex of } G} (\rho_F(x_c)(d))v_d.$$ \hspace{1cm} (3.2)

We then set

$$\bar{\xi}^F_G = q \circ \xi^F_G : \mathbb{Q}[x_c] \otimes \bigwedge(\tau_c) \to D_G,$$

$q$ being the quotient modulo $J_G$. We then have

**Proposition 3.2.** $\bar{\xi}^F_G(J_F) = 0$. It follows that $\bar{\xi}^F_G$ factors through a homomorphism of differential graded algebras

$$\zeta^F_G : D_F \to D_G.$$

**Proof.** Let us take a monomial $x_{c_{i_1}} \cdots x_{c_{i_h}} \in J_F$ that is $c_{i_1}, \ldots, c_{i_h}$ do not span a cone in $F$.

We know by Proposition 2.1 that $\bar{\xi}^F_G(x_{c_{i_1}} \cdots x_{c_{i_h}}) = 0$.

Now notice that $\xi^F_G(x_c)$ is a linear combination with non negative coefficients of the $y_d$. We deduce that $\xi^F_G(x_{c_{i_1}} \cdots x_{c_{i_h}})$ is a linear combination with non negative coefficients of monomials in the $y_d$. Thus each monomial appearing with non zero coefficient has to lie in $J_G$.

Necessarily if in any such monomial we substitute some of the $y_d$’s with the corresponding $v_d$’s we also get a relation in $D_G$. 

This immediately implies that for any \( h > 0 \), \( \xi^F_G(x_{c_1} \cdots x_{c_{i+1}} \cdots x_{c_{i+l}}) \) is well defined. We just have to show that if the function \( s \) has the required properties, such that each \( s \) is non-zero, we can map \( s \) into \( \Lambda(\Sigma) \). Passing to the limit and recalling that \( \xi^F_G \) is clearly a quasi isomorphism, we deduce that if we write \( D = \bigoplus_m D_m \) we get that

\[
H^i(D, d) = \begin{cases} 
\mathbb{Q} & \text{if } i = 0 \\
0 & \text{if } i > 0.
\end{cases}
\]

Finally, let us remark that \( \xi^F_G(D_{m,F}) \subset D_{m,G} \) for each \( m = 0, \ldots, n \). It follows that, taking the limit, \( D_m = \lim_{\longrightarrow} D_{m,F} \) we get a direct sum decomposition

\[
D = \bigoplus_m D_m.
\]

In particular for \( m = 0 \), \( D_0 = B_T(T) \). We denote by \( \nu_F : D_F \to D \) the natural morphism.

We want to give a more explicit description of the algebra \( D \). In order to do so let us take the exterior algebra \( \Lambda(\Sigma) \). Define the algebra \( E = B_T(T) \otimes \Lambda(\Sigma)/H \), where \( H \) is the ideal generated by the elements \( x_1 \cdots x_t \otimes \sigma_1 \cdots \sigma_t \), with \( x_i, \sigma_j \in \Sigma \), such that \( x_1 \cdots x_t \sigma_1 \cdots \sigma_t = 0 \) in \( B_T(T) \). Notice that the natural differential \( d \) on \( B_T(T) \otimes \Lambda(\Sigma) \) defined by setting \( d(a \otimes s) = as \otimes 1 \in B_T(T) \) and extended as an algebra derivation, clearly preserves \( H \). It follow that we get a differential on \( E \). We claim, Proposition 3.3. \( E \simeq D \) as differential graded algebras.

Proof. For our usual smooth projective fan \( F \), we have already defined a map \( \mu_F : A_F \to B_T(T) \), with the property that \( \mu_F(x_c) = s_c \) for any ray \( c \) of \( F \), which gives an inclusion of \( S_F \) into \( \Sigma \) and hence a map \( \Lambda(S_F) \to \Lambda(\Sigma) \). Tensoring, we obtain a map

\[
A_F \otimes \Lambda(S_F) \to B_T(T) \otimes \Lambda(\Sigma)
\]

and composing with the quotient, a map

\[
A_F \otimes \Lambda(S_F) \to E
\]

By the very definition of \( D_F \) we deduce that this map factors through a map

\[
\nu_F : D_F \to E
\]

Passing to the limit and recalling that \( \Sigma \) is spanned by the functions \( s_c \) for some ray in a suitable fan, we get a surjective map

\[
\mu : D \to E
\]

On the other hand, if we take an element \( s_1 \cdots s_t \otimes \sigma_1 \cdots \sigma_r \), we can find a fan \( F \) with the required properties, such that each \( s_i = \mu_F(x_c) \) and each \( \sigma_j = \mu_F(x_c') \) with \( x_i, \tau_j \in S_F \setminus \{0\} \). Thus we can map \( s_1 \cdots s_t \otimes \sigma_1 \cdots \sigma_r \) to \( \nu_F(x_1 \cdots x_t \otimes \tau_1 \cdots \tau_r) \). In order to see that this map is well defined we just have to show that if the function \( s_1 \cdots s_t \sigma_1 \cdots \sigma_r = 0 \) in \( C(V) \), the element \( x_1 \cdots x_t \tau_1 \cdots \tau_r = 0 \) in \( A_F \).

If we write each \( x_i \) and each \( \tau_j \) as a linear combination of the basis elements \( x_c, c \) a ray of \( F \), of \( S_F \) we get that \( x_1 \cdots x_t \tau_1 \cdots \tau_r \) is the image of a polynomial \( P(x_c) \in \mathbb{Q}[x_c] \) which is a product of non zero linear functions and hence non zero. We deduce that if we write \( P(x_c) \)
as a linear combination of monomials and we compute it as a function on $V$, we get 0 if and only if each monomial appearing with non zero coefficient in $P(x_c)$ is zero in $A_F$ hence the element $x_1 \cdots x_t \otimes \tau_1 \cdots \tau_s = 0$ in $A_F$.

It follows that we get a map $E \to D$ and it is immediate to check that this map is the inverse of $\mu$. □

For every cone $C \in F$, $A_{C,F}$ is a $R$-module. Again by Corollary 1.3 and Proposition 2.2. in [1], $A_{C,F}$ is free of rank equal to the number of $n$ dimensional cones in the star $S(C)$ of $C$. Furthermore $A_{C,F}$ is isomorphic to the $T$-equivariant cohomology of the closure $X_{C,F}$ of the $T$-orbit associated to the cone $C \in F$ and the quotient algebra

$$B_{C,F} := A_F/(\xi_1, \ldots, \xi_n) \simeq H^*(X_F, \mathbb{Q}).$$

From this we deduce in particular that $D_F$ is a free $R$ module and, setting by abuse of notation, $\xi_i := \xi_i \otimes 1$, for each $i = 1, \ldots, n$, we may consider the quotient algebra

$$C_F := D_F/(\xi_1, \ldots, \xi_n).$$

Since each element $\xi_j$ is a cocycle, we deduce that the ideal $(\xi_1, \ldots, \xi_n)$ is preserved by the differential $d$ and we have an induced differential on $C_F$ which we shall denote by the same letter.

Notice that if we set in $D_F$

$$\psi_j = \sum_{i=1}^{N} \langle \ell_j, c_i \rangle \tau_{c_i},$$

in $C_F$ we get that $d(\psi_j) = 0$ so that we obtain an inclusion of the exterior algebra $\wedge(\psi_1, \ldots, \psi_n)$ into the subalgebra $Z(C_F)$ of cocycles and a degree preserving homomorphism

$$j_F : H^*(T) \to H^*(C_F),$$

defined by setting $j_F(\ell_j) = \psi_j$.

We have

**Proposition 3.4.** The homomorphism $j_F$ is an isomorphism.

*Proof.* We shall deduce this by induction from a slightly more general fact. For any $0 \leq h \leq n$ consider

$$C_F^{(h)} = \begin{cases} D_F & \text{if } h = 0 \\ D_F/(\xi_1, \ldots, \xi_h) & \text{if } h > 0 \end{cases}.$$

Our claim is that for every $h$, $H^*(C_F^{(h)}) \simeq \wedge(\psi_1, \ldots, \psi_h)$.

For $h = 0$, $C_F^{(0)} = D_F$ and our claim is Proposition 3.1.

We proceed by induction on $h$ and assume the claim proved for $h - 1$. By reasoning as above we deduce that $\psi_h$ is a cocycle in $Z(C_F^{(h)})$ and hence gives a class in $H^1(C_F^{(h)})$.

Since clearly $\xi_h$ is a non zero divisor in $C_F^{(h-1)}$ we take the exact sequence

$$0 \to C_F^{(h-1)}[-2] \to C_F^{(h-1)} \to C_F^{(h)} \to 0. \tag{3.3}$$

the corresponding long exact sequence in cohomology and using the $\xi_h$ being a coboundary induces the trivial homomorphism in cohomology, we deduce the exact sequence

$$0 \to H^*(C_F^{(h-1)}) \to H^*(C_F^{(h)}) \to H^{*-1}(C_F^{(h)}) \psi_h \to 0. \tag{3.4}$$

Since by induction $H^*(C_F^{(h-1)}) \simeq \wedge(\psi_1, \ldots, \psi_{h-1})$, this implies our claim. □
We finish by remarking that clearly the map \( \zeta^F_G \) is a map of \( R \)-modules, so it induces a map

\[
\chi^F_G : C_F \to C_G
\]

and we may also consider

\( C = D/(\xi_1, \ldots, \xi_n) = \lim_{\mathbb{F}} C_F. \)

It is immediate to see that \( C \) inherits a direct sum decomposition

\[
C = \bigoplus_m C_m,
\]

with \( C_m = \lim_{\mathbb{F}} C_{m,F} = \lim_{\mathbb{F}} D_{m,F}/(\xi_1, \ldots, \xi_n) \). In particular for \( m = 0 \)

\[
C_0 = C(T) = \mathcal{B}_F(T)/(\xi_1, \ldots, \xi_n)
\]

is the ring of conditions of the torus \( T \).

Furthermore the \( \chi^F_G \) are quasi isomorphism and we deduce that also for \( C \) we have

\[
H^*(C, d) \simeq \bigwedge (\psi_1, \ldots, \psi_n) \simeq H^*(T).
\]

4. Toric arrangements

Let us now recall from the Introduction the definition of a toric arrangement. A layer in \( T \) is the subvariety

\[
K_{\Gamma, \phi} = \{ t \in T | \chi(t) = \phi(\chi), \forall \chi \in \Gamma \}
\]

where \( \Gamma \) is a split direct summand of \( X^*(T) \cong \mathbb{Z}^n \) and \( \phi : \Gamma \to \mathbb{C}^* \) is a homomorphism.

A toric arrangement \( \mathcal{A} \) is given by a finite set of layers \( \mathcal{A} = \{ K_1, \ldots, K_m \} \) in \( T \).

In [5] it is shown how to construct projective wonderful models for the complement \( \mathcal{M}(\mathcal{A}) = T - \bigcup_i K_i \).

A projective wonderful model is a smooth projective variety containing \( \mathcal{M}(\mathcal{A}) \) as an open set and such that the complement of \( \mathcal{M}(\mathcal{A}) \) is a divisor with normal crossings and smooth irreducible components. As we mentioned in the Introduction, we have \( \mathcal{M}(\mathcal{A}) = \mathcal{M}(\tilde{\mathcal{A}}) \), where \( \tilde{\mathcal{A}} \) is the saturation of \( \mathcal{A} \), i.e. the arrangement consisting of all the layers which are obtained as connected components of intersections of layers in \( \tilde{\mathcal{A}} \).

Therefore from now on, for brevity of notation, we are going to assume \( \mathcal{A} = \tilde{\mathcal{A}} \).

Let us put \( V = \text{hom}_{\mathbb{Z}}(X^*(T), \mathbb{R}) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \). A layer \( K_{\Gamma, \phi} \) is a coset with respect to the torus \( T_\Gamma = \cap_{\chi \in \Gamma} K e r(e^{2\pi i \chi}) \), and we can consider the subspace

\[
V_\Gamma = \{ v \in V | \langle \chi, v \rangle = 0, \forall \chi \in \Gamma \}.
\]

Since \( X^*(T_\Gamma) = X^*(T)/\Gamma \), \( V_\Gamma \) is naturally isomorphic to \( \text{hom}_{\mathbb{Z}}(X^*(T_\Gamma), \mathbb{R}) = X_*(T_\Gamma) \otimes_{\mathbb{Z}} \mathbb{R} \).

Definition 4.1. Let \( F \) be a fan in \( V \). A finite set \( \{ \chi_1, \ldots, \chi_s \} \) of vectors in \( X^*(T) \) is said to have equal sign with respect to \( F \) if for each \( i = 1, \ldots, s \) and each cone \( C \in F \), the function \( \langle \chi_i, - \rangle \) has constant sign on \( C \), i.e. it is either non negative or non positive on \( C \).

In [5] (see Proposition 6.1) it was shown how to construct a projective smooth \( T \)-embedding \( X_F \) whose fan \( F \) in \( V \) has the following property. For every \( \Gamma_i \) there is an integral basis of \( \Gamma_i, \chi_1, \ldots, \chi_s \), which has equal sign with respect to \( F \). The basis \( \chi_1, \ldots, \chi_s \) is called an equal sign basis for \( \Gamma_i \).

In fact by the same proof one can even show that one can construct \( F \) such that for any pair of layers \( K_{\Gamma, \phi} \subset K_{\Gamma', \psi} \in \mathcal{A} \), there is an equal sign basis for \( \Gamma \) whose intersection with \( \Gamma' \) is an equal sign basis for \( \Gamma' \).
In view of this we define

**Definition 4.2.** Let \( \mathcal{A} \) be a toric toric arrangement. A smooth projective fan \( \mathcal{F} \) is compatible with \( \mathcal{A} \) if for any pair of layers \( \mathcal{K}_{\Gamma,\phi} \subset \mathcal{K}_{\Gamma',\psi} \in \mathcal{A} \), there is an equal sign basis for \( \Gamma \) whose intersection with \( \Gamma' \) is an equal sign basis for \( \Gamma' \).

In what follows we are always going to consider fans \( \mathcal{F} \) compatible with \( \mathcal{A} \). Once such \( \mathcal{F} \) has been constructed, the strategy used in \([5]\) is to first embed the torus \( T \) in \( X_{\mathcal{F}} \).

In such a toric variety \( X_{\mathcal{F}} \) consider the closure \( \overline{\mathcal{K}_{\Gamma,\phi}} \) of a layer. This closure turns out to be a toric variety, whose explicit description is provided by \([5]\).

**Theorem 4.1** (Proposition 3.1 and Theorem 3.1 in \([5]\)). For every layer \( \mathcal{K}_{\Gamma,\phi} \), let \( T_{\Gamma} \) be the corresponding subtorus and let \( V_{\Gamma} = \{ v \in V | \langle \chi, v \rangle = 0, \forall \chi \in \Gamma \} \). Then,

1. For every cone \( C \in \mathcal{F} \), its relative interior is either entirely contained in \( V_{\Gamma} \) or disjoint from \( V_{\Gamma} \).
2. The collection of cones \( C \in \mathcal{F} \) which are contained in \( V_{\Gamma} \) is a smooth fan \( \mathcal{F}_{\Gamma} \).
3. \( \overline{\mathcal{K}_{\Gamma,\phi}} \) is a smooth \( T_{\Gamma} \)-variety whose fan is \( \mathcal{F}_{\Gamma} \).
4. Let \( O \) be a \( T \) orbit in \( X := X_{\mathcal{F}} \) and let \( C_{O} \in \mathcal{F} \) be the corresponding cone. Then
   - (a) If \( C_{O} \) is not contained in \( V_{\Gamma} \), \( \overline{O \cap \mathcal{K}_{\Gamma,\phi}} = \emptyset \).
   - (b) If \( C_{O} \subset V_{\Gamma} \), \( O \cap \mathcal{K}_{\Gamma,\phi} \) is the \( T_{\Gamma} \) orbit in \( \mathcal{K}_{\Gamma,\phi} \) corresponding to \( C_{O} \in \mathcal{F}_{\Gamma} \).

Let us denote by \( Q' \) (resp. \( Q \)) the set whose elements are the subvarieties \( \overline{\mathcal{K}_{\Gamma_i,\phi_i}} \) of \( X_{\mathcal{F}} \) (resp. the subvarieties \( \mathcal{K}_{\Gamma_i,\phi_i} \) and the irreducible components of the complement \( X_{\mathcal{F}} - T \)). We then denote by \( \mathcal{L}' \) (resp. \( \mathcal{L} \)) the poset made by all the connected components of all the intersections of some of the elements of \( Q' \) (resp. \( Q \)). In \([5]\) (Theorem 7.1) we have shown that the family \( \mathcal{L} \) is an arrangement of subvarieties in \( X_{\mathcal{F}} \) in the sense of Li’s paper \([14]\). As a consequence also \( \mathcal{L}' \), being contained in \( \mathcal{L} \) and closed under intersection, is an arrangement of subvarieties.

Let \( \mathcal{L}' = \{ G_1, \ldots, G_m \} \), ordered in such a way that if \( G_i \subseteq G_j \) then \( i < j \). Thus for each \( i = 1, \ldots, m \) we have \( G_i = \overline{\mathcal{K}_{\Gamma_i,\phi_i}} \) for a suitable pair \( (\Gamma_i, \phi_i) \).

A this point, following Li’s construction for \( \mathcal{L}' \) we construct the variety \( Y(X_{\mathcal{F}}) \), which is a projective wonderful model for \( M(\mathcal{A}) = X_{\mathcal{F}} - \bigcup_{A \in \mathcal{L}} A \). This means that \( Y(X_{\mathcal{F}}) \) contains \( M(\mathcal{A}) \) as a dense open set whose complement is a divisor with smooth irreducible components having transversal intersections.

More in detail we choose \( \mathcal{L}' \) as a *building set* (see Definition 2.5 in \([6]\)). Then we obtain \( Y(X_{\mathcal{F}}) \) starting from \( X_{\mathcal{F}} \) and blowing up the elements of \( \mathcal{L}' \) (after the first step, their transforms) in any order such that if \( G_{i_1} \subset G_{i_2} \) we blow up (the transform of ) \( G_{i_1} \) before (the transform of ) \( G_{i_2} \). In particular we notice that the ordering we chose in \( \mathcal{L}' \) is one of the admissible orderings to perform these blowups.

In Proposition 5.2 of \([6]\) we observed that \( Y(X_{\mathcal{F}}) \) is isomorphic to the variety \( Y^+(X_{\mathcal{F}}) \) obtained by choosing as a building set the set \( \mathcal{L}^+ = \mathcal{L}' \cup \{ D_{c_i} \}_{i=1,\ldots,N} \) where for every vertex \( c_i \) of \( \mathcal{F} \), \( D_{c_i} \) is the associated irreducible divisor in the boundary of \( X_{\mathcal{F}} \). The isomorphism is an immediate consequence of the fact that the \( D_{c_i} \)'s (and hence their transforms in \( X_{\mathcal{F}} \)) are divisors.

From Theorem 1.2 in \([14]\) it follows that \( Y^+(X_{\mathcal{F}}) \backslash M(\mathcal{A}) \) is a divisor with normal crossings whose irreducible components are smooth and indexed by \( \mathcal{L}^+ \). For \( j = 1, \ldots, m \), we denote by \( D_{G_j} \), the component of \( Y^+(X_{\mathcal{F}}) \backslash M(\mathcal{A}) \) corresponding to \( G_j \) and, by abuse of notation, we still denote by \( D_{c_i} \) its transform in \( Y^+(X_{\mathcal{F}}) \), so that

\[
Y^+(X_{\mathcal{F}}) \backslash M(\mathcal{A}) = \left( \bigcup_{j=1}^{m} D_{G_j} \right) \cup \left( \bigcup_{c_i} D_{c_i} \right).
\]
It follows from the theory of torus embeddings that for a collection of rays \( c_{i_1}, \ldots, c_{i_t} \) the intersection \( \cap_{h=1}^t D_{c_{i_h}} \) is non empty if and only if \( C = C(c_{i_1}, \ldots, c_{i_t}) \) is a cone in \( \mathcal{F} \).

Furthermore from the general definition of nested set (see Definition 5.6 of [14] and also Definition 2.7 in [6]), one can easily check that in our special situation, if we take a subset \( \mathcal{G} = \{G_{j_1}, G_{j_2}, \ldots, G_{j_s}\} \) of \( \mathcal{L}' \) and a cone \( C = C(c_{i_1}, \ldots, c_{i_t}) \), the intersection

\[
Y_{(\mathcal{G}, C)} := (\cap_{h=1}^t D_{G_{j_h}}) \cap (\cap_{h=1}^t D_{c_{i_h}})
\]

is non empty if and only if \( G_{j_1} \subseteq G_{j_2} \subseteq \cdots \subseteq G_{j_s} \) and \( C \subseteq V_{G_{j_1}} \). In this case \( Y_{(\mathcal{G}, C)} \) is smooth and irreducible.

**Remark 4.2.** From now on we will identify \( Y(X_{\mathcal{F}}) \) and \( Y^+(X_{\mathcal{F}}) \).

In [6] we have described the cohomology ring \( H^*(Y(X_{\mathcal{F}}), \mathbb{Z}) \) by generators and relations in a greater generality. Here we shall illustrate this result under our assumption, leaving the straightforward translation to the reader. We refer to [6] for the geometric explanation of our relations.

To simplify notation we are going to add to \( \mathcal{L}' \) the element \( G_{m+1} := X_{\mathcal{F}} \). We need to introduce certain polynomials in \( B_{\mathcal{F}}[t_1, \ldots, t_m] \).

Take a pair \((i, j)\) with \( i \in \{1, \ldots, m\} \), and \( j \in \{1, \ldots, m+1\} \) in such a way that \( G_i \subseteq G_j \). Consider the set \( B_i := \{h : G_h \subseteq G_i\} \).

Take an equal sign basis \( \chi \) of \( \Gamma_i \) whose intersection with \( \Gamma_j \) (if \( j = m+1, \Gamma_{m+1} = \{0\} \)) is a basis of \( \Gamma_j \). We then set

\[
P_{G_i}^{G_j}(t) := \prod_{\chi \in \chi \setminus (\chi \cap \Gamma_j)} (t - \chi_{\mathcal{F}}) \in B_{\mathcal{F}}[t]
\]

with

\[
\chi_{\mathcal{F}} = \sum_{c \text{ ray}} \min(0, (\chi, c))x_c,
\]

and

\[
F(i, j) = P_{G_i}^{G_j}(\sum_{h \in B_i} -t_h) t_j,
\]

with \( t_{m+1} := 1 \).

From [6] we easily get

**Theorem 4.3** (Proposition 6.3 and Theorem 7.1 in [6]). Let \( I \) be the ideal in \( B_{\mathcal{F}}[t_1, \ldots, t_m] \) generated by

1. the products \( t_i x_c \) for every ray \( c \in \mathcal{F} \) that does not belong to \( V_{\Gamma_i} \).
2. the products \( t_s t_r \) if \( G_s \) and \( G_r \) are not comparable.
3. the polynomials \( F(i, j) \), for \( G_i \subseteq G_j \).

Then

\( (i) \) \( I \) does not depend on the choice of the polynomials \( F(i, j) \).

\( (ii) \) The cohomology ring \( H^*(Y(X_{\mathcal{F}}), \mathbb{Q}) \) is isomorphic to \( B_{\mathcal{F}}[t_1, \ldots, t_m]/I \).

More generally one can compute the cohomology algebra of every stratum \( Y_{(\mathcal{G}, C)} \) of \( Y(X_{\mathcal{F}}) \) as follows (in fact in Theorem 9.1 of [6] one of the relations, the relation (1), was stated in an incorrect way; this was corrected in Theorem 4.3 of [16]).

First of all one shows that the restriction map

\[
r_{(\mathcal{G}, C)} : H^*(Y(X_{\mathcal{F}}), \mathbb{Q}) \rightarrow H^*(Y_{(\mathcal{G}, C)}, \mathbb{Q})
\]
is surjective.

If a ray $c$ is such that $c \notin V_{T_1}$,

$$r_{(G,C)}(x_c) = 0. \quad (4.1)$$

If $G_i$ is such that $C$ is not contained in $V_{T_1}$ or $G \cup \{G_i\}$ cannot be reordered into a flag we have

$$r_{(G,C)}(t_i) = 0. \quad (4.2)$$

Let us take a pair $(i,j)$ with $i \in \{1, \ldots, m\}$, and $j \in \{1, \ldots, m + 1\}$ in such a way that $G_i \subseteq G_j$ and set $S_i = \{s|G_{i_s} \in \mathcal{G}, G_{i_s} \supseteq G_i\}$.

Let us start with a pair $(i,m+1)$. If $S_i = \emptyset$ one has the relation

$$F(i,m+1) = P_{G_i}^{G_{m+1}} \left( \sum_{h \in B_i} -t_h \right).$$

which already holds in $H^*(Y(X_\mathcal{F}))$.

Otherwise, set $k = \min(s| s \in S_i)$. One has the relation

$$F_{G_i}(i,m+1) = P_{G_i}^{G_{i_k}} \left( \sum_{h \in B_i} -t_h \right) \quad (4.3)$$

in $H^*(Y(G,C), \mathbb{Q})$.

Now let us consider the case of a pair $(i,j)$ with $j \leq m$. If $G \cup \{G_j\}$ cannot be reordered into a flag we already know from the relation (1.2) that $r_{(G,C)}(t_j) = 0$.

Assume now that $G \cup \{G_j\}$ can be reordered in a flag. Then also $S_i \cup \{G_j\}$ is a flag and let $H$ be its smallest element.

If $H = G_j$ and $G_j \notin S_i$, we get the relation

$$F(i,j) = P_{G_i}^{G_{i_j}} \left( \sum_{h \in B_i} -t_h \right) t_j.$$

which already holds in $H^*(Y(X_\mathcal{F}))$.

If $H = G_{i_k} \in S_i$ we get the relation $F_{G_i}(i,j) = P_{G_i}^{G_{i_k}} \left( \sum_{h \in B_i} -t_h \right) t_j$, which is a consequence of (4.3).

**Theorem 4.4.** For any pair $(G,C)$ with $C \subset V_{G_{i_1}}$, the cohomology ring $H^*(Y(G,C), \mathbb{Q})$ is the quotient of the polynomial ring $B_{G,C,F}[t_1, \ldots, t_m]$ modulo the ideal generated by

1. the image of the ideal $I$ modulo the quotient homomorphism

$$\pi : B_F[t_1, \ldots, t_m] \rightarrow B_{G,C,F}[t_1, \ldots, t_m].$$

2. The relations (4.1), (4.2) and (4.3).

We can now apply this to give a presentation of the differential graded algebra associated to $Y(X_\mathcal{F})$ and the divisor with normal crossings $Y(X_\mathcal{F}) \setminus \mathcal{M}(\mathcal{A})$ following [18]. Recall that in our case this algebra is the direct sum

$$M_F = \oplus_{(G,C)} H^*(Y(G,C), \mathbb{Q})[-n_{(G,C)}]$$

with $n_{(G,C)}$ equal to $\dim C + |G|$ which is the codimension of $Y(G,C)$.

In order to do so, we take the algebra $B = \mathbb{Q}[t_1, \ldots, t_m] \otimes \Lambda(\kappa_1, \ldots, \kappa_m)/K$ where $K$ is the ideal generated by the products $t_i t_j, t_i \kappa_j, \kappa_i \kappa_j$ whenever $G_i$ and $G_j$ are not comparable.
We grade $\mathcal{B}$ by setting $\deg t_j = 2$ and $\deg \kappa_j = 1$ and we remark that the usual differential on $\mathcal{B} = \mathbb{Q}[t_1, \ldots, t_m] \otimes \Lambda(\kappa_1, \ldots, \kappa_m)$ given by $d(\kappa_j) = t_j$ preserves $K$ so that $\mathcal{B}$ inherits a degree 1 differential $d_{\mathcal{B}}$.

We can then consider the algebra $C_{\mathcal{F}} \otimes \mathcal{B}$ with differential $d_{C_{\mathcal{F}}} \otimes 1 + 1 \otimes d_{\mathcal{B}}$.

Remark that $B_{\mathcal{F}}$ is the subalgebra of $C_{\mathcal{F}}$ consisting of element of bidegree $(2n,0)$, $n \geq 0$ and so the polynomial ring $B_{\mathcal{F}}[t_1, \ldots, t_m]$ is a subalgebra of $C_{\mathcal{F}} \otimes \mathcal{B}$. Using this remark we can take the ideal $\Theta_{\mathcal{F}}$ in $C_{\mathcal{F}} \otimes \mathcal{B}$ generated by the elements

1. $x_c t_j$, $\tau_c t_j$, $x_c \kappa_j$, $\tau_c \kappa_j$, $c \notin V_{G_j}$,
2. $F(i,j)$, for $G_j \supseteq G_i$, $i = 1, \ldots, m$, $j = 1, \ldots, m + 1$.
3. $P_{G_i}^c(\sum_{h \in B_i} -t_h)\kappa_j$, with $i \in \{1, \ldots, m\}$, and $j \in \{1, \ldots, m\}$ in such a way that $G_i \subseteq G_j$.

Observe that $\Theta_{\mathcal{F}}$ is preserved by the differential $d_{C_{\mathcal{F}}} \otimes 1 + 1 \otimes d_{\mathcal{B}}$. It follows that we get an induced differential $d_{N_{\mathcal{F}}}$ on the algebra $N_{\mathcal{F}} = C_{\mathcal{F}} \otimes \mathcal{B}/\Theta_{\mathcal{F}}$.

We know that $\Theta_{\mathcal{F}}$ is a graded ideal so that $N_{\mathcal{F}}$ is also graded and the differential $d_{N_{\mathcal{F}}}$ is of degree 1.

**Theorem 4.5.** The differential graded algebra $(N_{\mathcal{F}}, d_{N_{\mathcal{F}}})$ is isomorphic to the Morgan algebra $(M_{\mathcal{F}}, d_{M_{\mathcal{F}}})$.

**Proof.** The proof given in [16] can be applied verbatim in this more special case. \hfill \Box

5. A LIMIT AND THE RATIONAL HOMOTOPY TYPE OF $M(\mathcal{A})$

Before we start, let us briefly discuss the generators of $\Theta_{\mathcal{F}}$. Remark that $c \notin V_{\tau_j}$ if and only if the function $s_c = \rho_{\mathcal{F}}(x_c)$ vanishes on $V_{\tau_j}$. Indeed the support of $s_c$ is the interior of $S(c)$ the star of $c$ and such interior intersects $V_{\tau_j}$, if and only if $c \in V_{\tau_j}$.

Thus the first relations can be written as

\[(5.1) \quad xt_j, \quad \tau t_j, \quad x\kappa_j, \quad \tau\kappa_j,\]

for $x, \tau \in \{f \in S_{\mathcal{F}} \mid \rho_{\mathcal{F}}(f) \equiv 0 \text{ on } V_{\tau_j}\}$.

As for the elements

\[\chi^- = \sum_{\text{ray}} \min(0, (\chi, c)) x_c,\]

appearing in the definition of $P_{G_i}^c(t) := \prod_{\chi \in \Lambda(\chi^\mathcal{F})} (t - \chi^-)^{-1}$, we remark that $\rho_{\mathcal{F}}(\chi^-) = \chi^-$ where

\[\chi^-(v) = \min(0, \chi(v))\]

for each $v \in V$.

Notice that if $\mathcal{G}$ is a refinement of $\mathcal{F}$ an equal sign linear function relative to $\mathcal{F}$ is also equal sign relative to $\mathcal{G}$, so if $\mathcal{F}$ is compatible with $\mathcal{A}$ also $\mathcal{G}$ is compatible with $\mathcal{A}$.

The first consequence of this fact is

**Proposition 5.1.** Let $\mathcal{F}$ be a fan compatible with $\mathcal{A}$. Let $\mathcal{G}$ be a refinement of $\mathcal{F}$ and $\psi_{\mathcal{G}}^\mathcal{F} : X_{\mathcal{G}} \rightarrow X_{\mathcal{F}}$ the unique $T$-equivariant projective morphism extending the identity on $T$.

Then, for each layer $K_{\Gamma,\phi}$ of $\mathcal{A}$, the preimage of the closure of $K_{\Gamma,\phi}$ in $X_{\mathcal{F}}$ is the closure of $K_{\Gamma,\phi}$ in $X_{\mathcal{G}}$.

**Proof.** Fix the layer $K_{\Gamma,\phi}$ of $\mathcal{A}$. Clearly we can identify $K_{\Gamma,\phi}$ with the torus $T_{\Gamma} = \cap_{\chi \in \Gamma} \ker e^{2\pi i \chi}$ and the restriction $\psi_{\mathcal{G}}^\mathcal{F}$ to $K_{\Gamma,\phi}$ is the identity.

We know by the compatibility of $\mathcal{F}$ and $\mathcal{G}$ with $\mathcal{A}$, that a cone $C$ in $\mathcal{F}$ or in $\mathcal{G}$ has either relative interior disjoint from $V_{\tau}$ or it is contained in $V_{\tau}$. 

The cones contained in $V_{\Gamma}$ define smooth projective fans $\mathcal{F}_{\Gamma}$ and $\mathcal{G}_{\Gamma}$ respectively and $\mathcal{G}_{\Gamma}$ is a refinement of $\mathcal{F}_{\Gamma}$.

We can then identify the closure of $Z_{\mathcal{F}}$ (resp. $Z_{\mathcal{G}}$) of $K_{\mathcal{F},\phi}$ in $X_{\mathcal{F}}$ (resp. $X_{\mathcal{G}}$) with the $T_{\Gamma}$-variety associated to the fan $\mathcal{F}_{\Gamma}$ (resp. $\mathcal{G}_{\Gamma}$) and the restriction of $\psi_{\mathcal{G}}^{\mathcal{F}}$ to $Z_{\mathcal{G}}$ with the unique $T_{\Gamma}$-equivariant morphism extending the identity on $K_{\mathcal{F},\phi}$ (identified with $T_{\Gamma}$).

We need to prove that $(\psi_{\mathcal{G}}^{\mathcal{F}})^{-1}(Z_{\mathcal{F}}) = Z_{\mathcal{G}}$. In order to see this let us take a cone $C \in \mathcal{F}$. Consider the new $T$-orbit $\mathcal{O}_{C} \subset X_{\mathcal{F}}$ corresponding to $C$. We know that $(\psi_{\mathcal{G}}^{\mathcal{F}})^{-1}(\mathcal{O}_{C})$ is the union of the $T$-orbits $\mathcal{O}_{C'} \subset X_{\mathcal{G}}$ corresponding to the cones $C' \in \mathcal{G}$ whose relative interior is contained in the relative interior of $C$.

Also as a $\mathcal{O}_{C} \simeq T/T_{\Gamma_{C}}$, were

$$T_{\Gamma_{C}} = \cap_{\chi \in X^* (T), \langle \chi, C \rangle = 0} \ker e^{2\pi i \chi},$$

the restriction of the map $\psi_{\mathcal{G}}^{\mathcal{F}}$ to a $T$-orbit $\mathcal{O}_{C'} \subset X_{\mathcal{G}}$ corresponding to a cone $C' \in \mathcal{G}$ whose relative interior is contained in the relative interior of $C$ can be then identified with the projection $T/T_{\Gamma_{C'}} \rightarrow T/T_{\Gamma_{C}}$ whose fiber is the torus $T_{\Gamma_{C}} / T_{\Gamma_{C'}}$.

Having recalled these facts let us examine the preimage of $Z_{\mathcal{F}}$. Fix a cone $C \in \mathcal{F}$.

If $C$ is not contained in $V_{\Gamma}$ then $\mathcal{O}_{C} \cap Z_{\mathcal{F}} = \emptyset$, so no orbit $\mathcal{O}_{C'} \subset X_{\mathcal{G}}$ corresponding to a cone $C' \in \mathcal{G}$ whose relative interior is contained in the relative interior of $C$ intersects $(\psi_{\mathcal{G}}^{\mathcal{F}})^{-1}(Z_{\mathcal{F}})$. If $C$ is contained in $V_{\Gamma}$ then also every cone $C' \in \mathcal{G}$ whose relative interior is contained in the relative interior of $C$ is contained in $V_{\Gamma}$. Furthermore we have inclusions $T_{\Gamma_{C'}} \subset T_{\Gamma_{C}} \subset T_{\Gamma}$. From this and the description of the restriction of the map $\psi_{\mathcal{G}}^{\mathcal{F}}$ to $\mathcal{O}_{C'}$, it follows that $(\psi_{\mathcal{G}}^{\mathcal{F}})^{-1}(Z_{\mathcal{F}}) \cap \mathcal{O}_{C'} \subset Z_{\mathcal{G}}$, proving our claim. 

At this point we observe that the universal property of Blowing up (see, \[12\], pp.164-165) implies that we get a morphism

$$\nu_{\mathcal{G}}^{\mathcal{F}} : Y(X_{\mathcal{G}}) \rightarrow Y(X_{\mathcal{F}})$$

extending the identity of $\mathcal{M}(A)$.

The map $\nu_{\mathcal{G}}^{\mathcal{F}}$ then induces a homomorphism

$$\Phi_{\mathcal{G}}^{\mathcal{F}} : N_{\mathcal{F}} \rightarrow N_{\mathcal{G}}$$

of differential graded algebras having the following properties: $\Phi_{\mathcal{G}}^{\mathcal{F}}$ coincides with $\chi_{\mathcal{G}}^{\mathcal{F}}$ on $\mathcal{C}_{\mathcal{F}}$ and one has $\Phi_{\mathcal{G}}^{\mathcal{F}}(t_{i}) = t_{i}$, $\Phi_{\mathcal{G}}^{\mathcal{F}}(\kappa_{i}) = \kappa_{i}$ for every $i = 1, \ldots, m$.

Let us now remark that the function $\chi^{-}$ depends only on $\chi$ and not on the choice of any particular fan. This allows us to define the polynomial $P_{G_{i}}^{C_{j}}(t) := \prod_{\chi \in \mathcal{C}_{\mathcal{F}} \setminus \mathcal{C}_{\mathcal{G}}}(t - \chi^{-}) \in \mathcal{C}[t]$.

We can then consider the algebra $\mathcal{C} \otimes \mathcal{B}$ with differential $d_{\mathcal{C}} \otimes 1 + 1 \otimes d_{\mathcal{G}}$.

For any $\Gamma_{j}$ we consider the subspace $S_{\Gamma_{j}}$, which is the image modulo $V^{*}$ of the space of functions whose restriction to $V_{\Gamma_{j}}$ is linear.

Now remark that the space $\Sigma$ surjects on both $\mathcal{C}_{(2,0)}$ and $\mathcal{C}_{(0,1)}$. So, given $\nu \in \Sigma$, we may take the corresponding elements $s_{\nu} \in \mathcal{C}_{(2,0)}$ and $\sigma_{\nu} \in \mathcal{C}_{(0,1)}$.

In $\mathcal{C} \otimes \mathcal{B}$ we then take the ideal $\Theta$ generated by the elements

1. $s_{\nu} t_{j}$, $\sigma_{\nu} t_{j}$, $s_{\nu} \kappa_{j}$, $\sigma_{\nu} \kappa_{j}$, for $v \in S_{\Gamma_{j}}$.
2. $F(i, j)$, for $G_{i} \supseteq G_{j}$, $i = 1, \ldots, m$, $j = 1, \ldots, m + 1$.
3. $P_{G_{i}}^{C_{j}}(\sum_{h \in B_{i}} -t_{h}) \kappa_{j}$, with $i \in \{1, \ldots, m\}$, and $j \in \{1, \ldots, m\}$ in such a way that $G_{i} \subsetneq G_{j}$, and set $N = \mathcal{C} \otimes \mathcal{B} / \Theta$.

Observe that $\Theta$ is preserved by the differential $d_{\mathcal{C}} \otimes 1 + 1 \otimes d_{\mathcal{G}}$. It follows that we get an induced differential $d_{N}$ on $N$. 

We know that $\Theta$ is a graded ideal so that $\mathcal{N}$ is also graded and the differential $d_{\mathcal{N}}$ is of degree 1.

From (3.3) and the above observations we then deduce

**Theorem 5.2.** $(N, d_N) = \lim_{\rightarrow} (N_F, d_{N_F}).$

We now want to remark a few facts about the algebras $(N_F, d_{N_F})$ and $(N, d_N)$. First of all we have seen that although the set of generators of the defining ideal of $N_F$ depends on the choice of equal sign bases, by Theorem 4.3(i) the algebra itself is independent on the choice of these equal sign bases.

Assume now that we have given two smooth projective fans $\mathcal{F}$ and $\mathcal{F}'$ each equipped with a choice of equal sign bases for our arrangement. We know that we can find a common refinement $\mathcal{G}$ of $\mathcal{F}$ and $\mathcal{F}'$. We have already remarked that the two sets of equal sign bases are both choices of equal sign bases for $\mathcal{G}$, so that the algebra $(N_G, d_{N_G})$ is independent from these choices.

We deduce

**Proposition 5.3.** The differential graded algebra $(N, d_N)$ is independent from any choice of equal sign bases.

Let us now recall that according to [18], for smooth projective fans $\mathcal{F}$ as above the minimal model of the differential graded algebra $(N_F, d_{N_F})$ is isomorphic to the minimal model of $\mathcal{M}(\mathcal{A})$ so it determines the rational homotopy type of $\mathcal{M}(\mathcal{A})$ and in particular its cohomology is the cohomology ring $H^*(\mathcal{M}(\mathcal{A}), \mathbb{Q})$. Furthermore as we have already seen the homomorphism $\Phi_{\mathcal{F}}$ is induced by the morphism $\nu_{\mathcal{F}} : Y(X_G) \to Y(X_F)$ which is the identity when restricted to $\mathcal{M}(\mathcal{A})$, so $\Phi_{\mathcal{F}}$ is a quasi isomorphism of differential graded algebras.

Since, taking homology of a chain complex is an exact functor we deduce that also the cohomology of the differential graded algebra $(N, d_N)$ is $H^*(\mathcal{M}(\mathcal{A}), \mathbb{Q})$ and that the natural maps $(N_F, d_{N_F}) \to (N, d_N)$ are quasi isomorphisms.

We deduce

**Proposition 5.4.** The minimal model of the differential graded algebra $(N, d_N)$ is isomorphic to the minimal model of $\mathcal{M}(\mathcal{A})$.

To state our final result in its more general form, let us now consider an arbitrary toric arrangement $\mathcal{A}$ (i.e. we drop the assumption $\mathcal{A} = \tilde{\mathcal{A}}$). We recall that, according to the definition in the Introduction, the combinatorial data of $\mathcal{A}$ are provided by the following two sets:

1. The partially ordered set $\tilde{\mathcal{A}}$ ordered by reverse inclusion.
2. The set of lattices $\Gamma$ for $K_{\Gamma, \phi} \in \tilde{\mathcal{A}}$.

We have:

**Theorem 5.5.** The rational homotopy type of the complement $\mathcal{M}(\mathcal{A})$ depends only on the combinatorial data of $\mathcal{A}$.

**Proof.** By Proposition 5.4 the rational homotopy type of the complement $\mathcal{M}(\mathcal{A})$ depends only by the differential graded algebra $(N, d_N)$. In turn we have that this algebra is defined only in terms of the combinatorial data of $\mathcal{A}$. Hence our claim follows. $\Box$
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