Encoding an arbitrary state in a [7,1,3] quantum error correction code

Sidney D. Buchbinder · Channing L. Huang · Yaakov S. Weinstein

Received: 26 January 2012 / Accepted: 20 April 2012 / Published online: 4 May 2012
© Springer Science+Business Media, LLC 2012

Abstract We calculate the fidelity with which an arbitrary state can be encoded into a [7, 1, 3] Calderbank-Shor-Steane quantum error correction code in a non-equiprobable Pauli operator error environment with the goal of determining whether this encoding can be used for practical implementations of quantum computation. The determination of usability is accomplished by applying ideal error correction to the encoded state which demonstrates the correctability of errors that occurred during the encoding process. We also apply single-qubit Clifford gates to the encoded state and determine the accuracy with which these gates can be implemented. Finally, fault tolerant noisy error correction is applied to the encoded states allowing us to compare noisy (realistic) and perfect error correction implementations. We find the encoding to be usable for the states $|0\rangle$, $|1\rangle$, and $|\pm\rangle = |0\rangle \pm |1\rangle$. These results have implications for when non-fault tolerant procedures may be used in practical quantum computation and whether quantum error correction must be applied at every step in a quantum protocol.

Keywords Quantum fault tolerance · Quantum error correction · Pauli operator error environment

S. D. Buchbinder · C. L. Huang · Y. S. Weinstein
Quantum Information Science Group, MITRE, 260 Industrial Way West, Eatontown, NJ 07224, USA
e-mail: weinstein@mitre.org

Present Address:
S. D. Buchbinder
California Institute of Technology, Pasadena, CA 91125, USA

Present Address:
C. L. Huang
Princeton University, Princeton, NJ 08544, USA
1 Introduction

Quantum fault tolerance is a framework designed to allow accurate implementations of quantum algorithms despite inevitable errors [1–4]. While the construction of this framework is important and necessary to demonstrate the possibility of successful quantum computation, the realization of a quantum computer which utilizes the complete edifice of quantum fault tolerance requires huge numbers of qubits and is a monumental task. Thus, it is worthwhile to explore whether it is possible to achieve successful quantum computation without using the full toolbox of quantum fault tolerance techniques. Initial studies along these lines have been done for logical encoding of the state zero into the Steane [7,1,3] quantum error correction code [5]. In this paper we extend these results by determining the accuracy with which an arbitrary state can be encoded into the Steane code in a non-equiprobable Pauli operator error environment. Direct encoding of an arbitrary state in the Steane code cannot be done in a fault tolerant manner [1], and we explore whether the accuracy achieved for a non-fault-tolerant encoding is sufficient for use in a realistic quantum computation. We then apply (noisy) single qubit Clifford gates and error correction to this encoded state, again concentrating on the accuracy of the implementation.

Encoding quantum information into the logical states of a quantum error correction (QEC) code is the first step of any fault tolerant quantum computation [6,7]. Here we encode into the [7,1,3] Steane QEC code [8–10] which protects one qubit of quantum information by encoding it into seven physical qubits. Encoding into the Steane code can be done via the gate sequence originally designed in [10]. However, this method is not fault tolerant because an error on a given qubit may spread to other qubits. A fault tolerant method does exist for encoding into the logical zero and one states [1]. In this study we determine whether the arbitrary encoding sequence is at all 'usable' for practical implementations of quantum computation.

We define a protocol (in this case the encoding sequence) as usable if perfect error correction applied after the protocol can correct errors occurring during the protocol to first order. We believe this definition is appropriate since the [7,1,3] QEC code only corrects first order errors. We apply perfect error correction after the encoding sequence and look at the fidelity of the resulting state with respect to the perfectly encoded state. If there are no first order error probability terms remaining in the fidelity we will say the encoding sequence is usable for practical quantum computation.

We assume a non-equiprobable error model [11,12] in which qubits taking part in any gate are subject to a $\sigma_x$ error with probability $p_x$, a $\sigma_y$ error with probability $p_y$, and a $\sigma_z$ error with probability $p_z$. Thus, an attempted implementation of a single qubit transformation $T$ on qubit $j$ described by density matrix $\rho_0$ would produce:

$$\rho = \sum_{a=0,x,y,z} p_a \sigma_a^j T^j \rho_0 T^j \sigma_a^{j^+},$$

where $\sigma_0$ is the identity matrix, and $p_0 = 1 - \sum_{i=x,y,z} p_i$. A two-qubit controlled-NOT (CNOT) gate with control qubit $j$ and target qubit $k$ ($c_j \text{NOT}_k$) would cause the following evolution of an initial two-qubit density matrix $\rho_0$: 

$\text{CNOT} \rho \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rho \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
\[ \rho = \sum_{a,b=0,x,y,z} p_a p_b \sigma_d^j \sigma_b^k C_j \text{NOT}_k \rho_0 \text{NOT}_k \sigma_b^\dagger \sigma_d^\dagger. \]  

(2)

After noisy encoding of an arbitrary state, we apply ideal error correction as a means of determining whether the errors that occurred during encoding can, at least in principle, be corrected, thus making the encoded state usable for practical instantiations of quantum computation. In addition, we apply each of three logical Clifford gates (Hadamard, NOT, phase) to the noisy encoded state, and measure the accuracy of their implementation in the non-equiprobable error environment. We apply ideal error correction after each of these gates. Finally, we apply noisy QEC to the encoded states, to model the reliability with which data can be encoded and maintained in a realistic noisy environment.

In this work, we utilize three different measures of fidelity to evaluate accuracy. The first fidelity quantifies the accuracy of the seven-qubit state after the intended operations have been performed compared to the desired seven-qubit state. The second fidelity measures the accuracy of the logical qubit, the single qubit of quantum information stored in the QEC code. To obtain this measure, we perfectly decode the seven-qubit state and partial trace over all but the logical qubit. The final fidelity measure compares the state of the seven qubits after application of perfect error correction to the state that has undergone perfect implementation of the desired transformations. This fidelity reveals whether the errors that occur in a noisy encoding process can be corrected and thus whether the protocol is usable for practical quantum computation.

2 General state encoding

Our first step is to encode an arbitrary pure state \(|\psi(\alpha, \beta)\rangle = \cos \alpha |0\rangle + e^{i\beta} \sin \alpha |1\rangle\) into a [7,1,3] QEC code using the non-fault tolerant gate sequence shown in Fig. 1. Ideally, this gate sequence would produce the state \(|\psi_L(\alpha, \beta)\rangle = \cos \alpha |0_L\rangle + e^{i\beta} \sin \alpha |1_L\rangle\) where:

\[
|0_L\rangle = |0000000\rangle + |0001111\rangle + |0110011\rangle + |0111100\rangle + |1010101\rangle + |1011010\rangle + |1100110\rangle + |1101001\rangle \\
|1_L\rangle = |1111111\rangle + |1110000\rangle + |1001100\rangle + |1000011\rangle + |0101010\rangle + |0100101\rangle + |0011001\rangle + |0010110\rangle 
\]

(3)

However, in the non-equiprobable error environment, as each gate is applied the qubits taking part in the gate probabilistically undergo Pauli errors, as in Eqs. 1 and 2. The resulting state of the noisy, error-prone [7,1,3] encoding, \(\rho_{\text{EncN}}^{(7)}\), depends on the state being encoded (parameterized by \(\alpha\) and \(\beta\)) and the probabilities of error \((p_x, p_y, \text{and } p_z)\). We determine the accuracy of the encoding process by calculating the fidelity as compared to the ideally (non-error-prone) encoded state. This seven-qubit fidelity, \(\langle \psi_L^{(7)}(\alpha, \beta) | \rho_{\text{EncN}}^{(7)} | \psi_L^{(7)}(\alpha, \beta) \rangle\), reflects the accuracy with which the general state encoding process is achieved and is given (to second order) in the Appendix, Eq. 4. We find that no first order error terms are dependent on \(\beta\), and the only first order term
dependent on the state to be encoded is $p_z$. This indicates a relatively small dependence on the initial state in general. Encoding with an initial state of $\alpha = 0$ or $\alpha = \frac{\pi}{2}$ results in the highest fidelity, while encoding an initial state with $\alpha = \frac{\pi}{4}$ results in a lower fidelity. We further note that there is no dominant error term, in that the magnitudes of the coefficients of the first order terms are similar.

The seven-qubit fidelity quantifies the accuracy with which the seven physical qubits are in the desired state. However, certain errors may lower the fidelity but not impact the single logical qubit of quantum information stored in the QEC code. We would like to determine the accuracy of that single logical qubit of information. To do this the seven-qubit system is perfectly decoded, and a partial trace is applied to qubits 2 through 7. The resulting one-qubit state $\rho^{(1)}_{EncN}$ is compared to the initial state of the logical qubit. This fidelity is then given by: $\langle \psi(\alpha, \beta) | \rho^{(1)}_{EncN} | \psi(\alpha, \beta) \rangle$, and is given (to second order) in Eq. 5. As opposed to the seven qubit fidelity we find that this single qubit fidelity is highly dependent on the initial state. All terms, including the first order terms, are dependent on $\alpha$. In addition, dependence on $\beta$ appears in all terms but $p_z$ and $p_z^2$. As above, no single type of error dominates the loss of fidelity, as indicated by the similar coefficients of the first order probability terms. The initial states that result in the highest fidelity occur at $\alpha = 0$ and $\alpha = \frac{\pi}{2}$, while the state that results in the lowest fidelity occurs at $\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{4}$. Contour plots of the seven qubit and single qubit fidelities are shown in Fig. 2.

3 Application of Clifford gates

We now apply the one logical-qubit Clifford gates to the encoded arbitrary state in the presence of the non-equiprobable error environment described above, and determine the accuracy of their implementation. CSS codes in general, and the Steane code in particular, allow logical Clifford gates to be implemented bitwise, so that only single (physical) qubit gates are required. The Clifford gates applied in this study are the Hadamard (H), NOT (X), and phase (P) gates:
Fig. 2 Contour plots for the fidelity of an arbitrary state noisily encoded in the [7, 1, 3] QEC code as a function of initial state parameterized by \( \alpha \) (\( \beta \) is set to 0), and error probabilities \( p_x \), and \( p_z \) (\( p_y \) is set to zero). Left The seven qubit fidelity for the general encoding procedure. Right The single logical qubit fidelity for the general encoding procedure. The contours for the seven qubit fidelity values are (from top to bottom) 0.85, 0.90, 0.95, and 0.99. The contours for the single qubit fidelities are (from top to bottom) 0.97, 0.985, and 0.995.

The logical Hadamard gate is implemented by applying a single qubit Hadamard gate on each of the seven qubits in the encoding. The logical NOT gate is implemented by applying single qubit NOT gates to the first three qubits of the encoded state. The logical phase gate is implemented by applying an inverse phase gate to each of the seven qubits. Each of these gates is applied to the noisily encoded arbitrary state of the last section in the non-equiprobable error environment such that any qubit acted upon by a gate evolves via Eq. 1. We quantify the accuracy of the gate implementation by calculating the fidelities as a comparison of the state after application of the Clifford gate with the state of a perfectly encoded arbitrary state that undergoes perfect application of the Clifford transformation via the seven qubit fidelity, \( \langle \psi(\alpha, \beta)_{C}^{(7)} | \rho_{ENC_{CN}}^{(7)} | \psi(\alpha, \beta)_{C}^{(7)} \rangle \), and the one qubit fidelity \( \langle \psi(\alpha, \beta)_{C}^{(1)} | \rho_{ENC_{CN}}^{(1)} | \psi(\alpha, \beta)_{C}^{(1)} \rangle \) where \( C \) represents \( H \), \( X \), or \( P \) applied to the encoded (seven qubit) or unencoded (one qubit) states. Plots of these fidelities are given in Fig. 3, and the expressions of these fidelities are given in the Appendix, Eqs. 6–11.

The encoded state (seven qubit) fidelities for the three Clifford gates follow several general trends. The fidelities of the Hadamard and phase gates, Eqs. 6 and 10, are nearly identical and slightly lower than the fidelity of the NOT gate, Eq. 8. This is likely due to the fact that the same number of gates are applied to implement the logical Hadamard and phase gates which is more than the number needed to implement the logical NOT gate. In addition, all three fidelities have similar dependence on the initial state as parameterized by \( \alpha \) and \( \beta \) (though dependence on \( \beta \) is small). While
Fig. 3 Contour plots of fidelity for application of the single qubit Clifford gates on an arbitrary state encoded in the [7, 1, 3] code as a function of the initial state parameterized by $\alpha$ ($\beta$ is set to 0), and error probabilities $p_x$ and $p_z$ ($p_y$ is set to zero). Left Seven qubit fidelities, Right single logical qubit fidelities for the Hadamard, NOT, and phase gates (top–bottom). The contours for the seven qubit fidelity values are (from top–bottom) 0.85, 0.90, 0.95, and 0.99. The contours for the single qubit fidelities are (from top to bottom) 0.97, 0.985, and 0.995. Note that the contour plots of the seven qubit fidelities for the Hadamard and phase gates (top-left and bottom-left) are nearly indistinguishable.
the coefficient of the first order \( p_x \) term is larger than the coefficients of the first order \( p_x \) and \( p_z \) terms for all three fidelities, the difference is only slight. No specific error dominates the loss in fidelity.

Different trends are apparent in the fidelities of the single logical qubit. First, the fidelity of the state after the Hadamard, Eq. 7, is lower than the fidelities after the NOT and phase gates, Eqs. 9 and 11. Furthermore, all three fidelities exhibit relatively large dependence on initial state, in that \( \alpha \) appears in every first order term, and \( \beta \) appears in nearly all first order terms. The fidelities after application of the NOT gate and Hadamard gate exhibit similar magnitude dependencies on \( \alpha \), while the phase gate fidelity changes more significantly as the initial state varies. For all gates, the most stable initial states are \( \alpha = 0, \alpha = \frac{\pi}{2} \), in that a higher fidelity occurs at the same \( p_x, p_y, \) and \( p_z \) values when compared with fidelities of other initial states. We note as well that the single logical qubit fidelities of the NOT gate and phase gate become independent of \( \sigma_z \) errors for these values of \( \alpha \). In both fidelity measures there are first order terms which would ideally be suppressed to second order through the application of quantum error correction.

4 Quantum error correction

The purpose of encoding a state via the [7, 1, 3] code is to protect the data from being corrupted by errors. While errors may have occurred during the encoding itself, these can be corrected by subsequent application of QEC if the encoding is done in a fault tolerant fashion thus ensuring that at most one data qubit will be corrupted. In this paper we have encoded an arbitrary state in a necessarily non-fault tolerant fashion. Nevertheless, we would like to know if QEC applied after encoding can suppress errors to the same degree as if the encoding had been done fault tolerantly.

4.1 Perfect error correction

We apply perfect (non error-prone) error correction as shown in Fig. 4 to the noisily encoded arbitrary state and to the encoded states after one of the three Clifford gates has been applied. Performing ideal error correction on noisy states enables us to determine if there is any feasible recovery from the errors that occur during the noisy encoding process and noisy gate application. Perfectly applied error correction should suppress the probability of error terms in the fidelity to at least second order for the gate sequence encoding method to be usable in practical quantum computing. If the first order error probability terms are not suppressed under ideal error correction conditions, error correction that occurs in a noisy environment (which is more realistic) will certainly not be able to reliably preserve encoded data.

The fidelity resulting from implementing perfect error correction will depend on the measured error syndrome. For encoding purposes one can afford to utilize only encodings which upon application of error correction yield desirable error syndromes (i.e. error syndromes corresponding to states with high fidelity). The most desirable syndrome consists of all zeros and will occur with probability \( 1 - 22p_x - 25p_y - 21p_z + \mathcal{O}(p_i^2) \). The fidelity of the encoded arbitrary state when the syndrome is
measured to be all zeros is given by Eq. 12. While perfect error correction on the encoded arbitrary state does improve the fidelity, there is a remaining first order term $p_z$ (the errors associated with the first order $p_x$ and $p_y$ have been suppressed by perfect error correction). This would suggest that the gate sequence encoding scheme is not appropriate for practical quantum computation. However, for certain initial states $\alpha = 0, \frac{\pi}{2}$ the first order error term drops out and thus this encoding protocol is usable in creating logical $|0\rangle$ and $|1\rangle$ states.

When the error syndrome is not all zeros a recovery operation (also assumed to be perfectly implemented) is necessary to complete the error correction procedure. Table 1 lists the possible error syndromes that can occur with probability first order in the error probabilities, $p_j$, when applying QEC after the encoding protocol, along with the probability that they occur and the fidelity of the resulting error corrected state. For some of these error syndromes the corrected state will also have no first order error terms when the state to be encoded has $\alpha = 0, \frac{\pi}{2}$. For other error syndromes the first order error terms will not drop out despite the fact that the syndrome can occur to first order in error probability. This is likely due to the fact that a single error on one qubit during the gate sequence encoding can spread to other qubits because the encoding circuit is not fault tolerant. The encoded states that arise when these syndromes are measured should be discarded and a new encoding should be performed. We stress that even a fault tolerant encoding into the $[7,1,3]$ code requires some attempts be abandoned (for example due to failed Shor state verification or the inability to get the same syndrome measurement twice in a row). Thus, the fact that certain error syndromes will not lead to acceptable encoded states using the gate sequence encoding should not, in and of itself, be perceived as a negative of this encoding method.

We observe similar trends when perfect QEC is performed on the states that have undergone Clifford transformations, Eqs. 13, 14, and 15. In all three cases, when the
Table 1  Probability of each error syndrome measurement and fidelity of the resulting seven qubit state to first order in error probabilities for the application of a noisy encoding sequence followed by perfect error correction and recovery

| Syndrome | Probability | Fidelity |
|----------|-------------|----------|
| 000000   | 1 – 22px – 25py – 21pz | 1 – 2pz(1 – Cos[4α]) |
| 000001   | 2pz         | \(\frac{1}{4}(3 + Cos[4α]) = (\frac{7p_x}{8} + \frac{7p_y}{4} + \frac{3p_xp_y}{8p_z}) (1 – Cos[4α])\) |
| 000010   | py + 7pz    | \(\frac{1}{4}(6 + Cos[4α]) = (\frac{p_y}{99} + \frac{79p_x}{345} – \frac{p_xp_y}{p_z}) (1 – Cos[4α]) – \frac{p_y}{5} (\frac{115 + 81Cos[4α]}{9}) – Cos[2β]Sin[2α]²\) |
| 000011   | py + 5pz    | \(\frac{1}{10}(9 + Cos[4α]) = (\frac{p_y}{50} + \frac{53p_x}{250} + \frac{101p_xp_y}{50p_z}) (1 – Cos[4α]) – \frac{p_y}{5p_z} (\frac{74 + 26Cos[4α]}{25} – Cos[2β]Sin[2α]²\) |
| 000100   | py + 4pz    | 1 – \(\frac{1}{4}(\frac{3}{4}p_y + \frac{p_y}{16}) (1 – Cos[4α]) – \frac{p_y}{8p_z} (\frac{3 + Cos[4α]}{2} – Cos[2β]Sin[2α]²\) |
| 000101   | pz          | 1 – \(\frac{3p_y}{2} + 20p_z = (1 – Cos[4α])\) |
| 000111   |pz           | 1 – \(\frac{9p_x}{2} + 13p_z = (1 – Cos[4α]) – \frac{p_y}{2p_z} \frac{3 + 3Cos[4α]}{2} – Cos[2β]Sin[2α]²\) |
| 001000   | 3px + py    | \(\frac{1}{2}(\frac{1 + Cos[4α]}{4}) – (\frac{27p_x}{4} + \frac{3p_y}{2p_z} + 2p_x + \frac{21p_xp_y}{p_z}) (1 – Cos[4α]) + p_y(1 – 3Cos[4α]) + (p_y – 6p_x)Cos[2β]Sin[2α]²\) |
| 001001   | py          | 1 – \(\frac{5p_x}{4} + 3p_y = (3 + Cos[4α]) – (3p_x + \frac{1}{2}p_xp_y + (1 + p_y) – \frac{p_x}{2p_z} Cos[2β]Sin[2α]²\) |
| 001010   | py          | \(\frac{1}{2}(\frac{1 + Cos[4α]}{4}) + (\frac{5 – 3Cos[4α]}{4})p_x + \frac{1}{2}(\frac{1 – 3Cos[4α]}{4})p_y + \frac{7p_x}{2} + \frac{1}{2}p_y \frac{15p_xp_y}{2p_z}) (1 – Cos[4α]) + \frac{p_y}{2} + p_z = Cos[2β]Sin[2α]²\) |
| 010000   | 2px         | \(\frac{5}{3} \frac{p_x(1 + 5Cos[4α])}{10p_y} – \frac{p_x}{2p_z} + \frac{1}{2}p_y = (\frac{1}{10} – \frac{p_y}{p_z} – \frac{p_y}{25p_z} – \frac{1}{50p_z} \frac{26 + p_y}{25p_z} + \frac{1}{10p_z} + \frac{44p_y}{5p_z} + \frac{3}{4p_y})\) |
| 010010   | py          | 1 = (\(\frac{3}{4} + 3Cos[4α])p_x – 6p_y – \(\frac{17 – 3Cos[4α]}{2}p_y = (1 + 5Cos[4α])p_x + \frac{p_y}{2p_z} \frac{25p_x}{2} – 3p_y + \frac{3p_x}{2p_z} p_y = Cos[2β]Sin[2α]²\) |
| 010011   | py          | \(\frac{1}{4} (1 – Cos[4α] + 2Cos[2β]Sin[2α]² + \frac{3 + Cos[4α]}{2} + (\frac{1 + 5Cos[4α]}{2})p_x + \frac{p_y}{2} (1 + Cos[4α])) + \frac{1}{(\frac{1 + 3Cos[4α]}{2})p_y} + \frac{p_z}{p_y} (\frac{1 + 5Cos[4α]}{2})p_x + \frac{p_y}{2} + (7p_x + 4p_y + 5p_z + \frac{7p_x}{2p_z}) Cos[2β]Sin[2α]²\) |
| 011000   | 3px         | \(\frac{23 – 7Cos[4α] + 10Cos[2β]}{2p_z} Sin[2α]² = (\frac{1}{8} – \frac{71p_x}{6p_y} – \frac{21p_x}{16p_z} + \frac{5p_y}{5712p_z} – \frac{21619}{2p_y} + \frac{3549}{2p_z} + \frac{1}{p_z} (\frac{1171}{2} + (5295 – 216p_z) p_y)\) + \(\frac{1}{2p_z} (\frac{51p_x}{169p_y} – \frac{23p_x}{6p_y} + \frac{5p_y}{5712p_z} – \frac{5265}{2p_y} + \frac{1}{p_z} (\frac{1287}{2} + (6999 – \frac{5p_y}{2p_z} + \frac{504}{p_z}) p_y)\) Cos[4α] + \(\frac{1}{2p_z} (\frac{214p_x}{169p_y} – \frac{43p_y}{338p_z} + \frac{p_z}{2p_z} \frac{2585p_z}{p_y} – \frac{35399}{2p_y} + \frac{1859}{2p_z} + \frac{819}{2} + (6749 – \frac{576}{p_z} p_y)\) Cos[2β]Sin[2α]² |
| Syndrome | Probability | Fidelity |
|----------|-------------|----------|
| 011010   | $p_y$       | \[\frac{1}{4}(1 - \cos(4\alpha_1) + 2\cos(2\beta_1)\sin(2\alpha_1)^2) \rightleftharpoons (p_x + p_z)(1 + \cos(4\alpha_1) + \frac{(1 + 3\cos(4\alpha_1))p_x +}{4p_y}) - \frac{8p_x}{9p_y} + \frac{6p_z}{2p_y} + \frac{15p_x p_z}{2p_y} \cos(2\beta_1)\sin(2\alpha_1)^2)}{4p_y} \rightleftharpoons (p_x + p_z)(1 + \cos(4\alpha_1) + \frac{(1 + 3\cos(4\alpha_1))p_x +}{4p_y}) - \frac{8p_x}{9p_y} + \frac{6p_z}{2p_y} + \frac{15p_x p_z}{2p_y} \cos(2\beta_1)\sin(2\alpha_1)^2} | |
| 011011   | $p_y$       | $1 - \frac{(9 + 3\cos(4\alpha_1))p_x}{4p_y} - \frac{(17 - 5\cos(4\alpha_1))p_x}{4p_y} - \frac{(8 + 2\cos(4\alpha_1)p_x p_z}{p_y} + \cos(2\beta_1)(\frac{2p_x - \frac{5p_x}{p_y} - \frac{p_x}{p_y}}{p_y}) \sin(2\alpha_1)^2} | |
| 011100   | $p_y$       | \[\frac{1}{4}(1 - \cos(4\alpha_1) + 2\cos(2\beta_1)\sin(2\alpha_1)^2) \rightleftharpoons \frac{(\cos(4\alpha_1)p_x p_z}{p_y} + \cos(2\beta_1)\sin(2\alpha_1)^2} \rightleftharpoons (p_x + p_z)(1 + \cos(4\alpha_1) + \frac{(1 + 3\cos(4\alpha_1))p_x +}{4p_y}) - \frac{8p_x}{9p_y} + \frac{6p_z}{2p_y} + \frac{15p_x p_z}{2p_y} \cos(2\beta_1)\sin(2\alpha_1)^2) | |
| 100000   | $p_x$       | $1 - \frac{2p_x}{4p_y} - \frac{5p_x^2}{3p_x} + \frac{5p_x}{12p_x} + \frac{197p_x p_y}{6p_y} - 61p_x p_y}{6p_y} - \frac{15p_x p_y}{4p_y} + \frac{2p_x}{3p_y} - \frac{5p_x p_y}{9p_y} + \frac{5p_x}{36p_y} + \frac{229p_x p_y}{18p_y} - \frac{77p_x}{18p_y} | |
| 100100   | $p_y$       | $1 - \frac{(3 + \cos(4\alpha_1))p_x}{4p_y} - \frac{(17 - 5\cos(4\alpha_1)p_x}{4p_y} + \frac{(2p_x - \frac{5p_x}{2p_y}) \cos(2\beta_1)\sin(2\alpha_1)^2} | |
| 100100   | $p_x$       | $\frac{17 + 3\cos(4\alpha_1)p_x}{20p_y} - \frac{3p_x}{2p_y} + \frac{3p_x}{250p_y} - \frac{63p_x p_y}{2000p_y} - \frac{9p_x}{200p_y} + \frac{p_y}{200p_y} | |
| 100100   | $p_y$       | $\frac{1}{4}(1 + \cos(4\alpha_1)) - \frac{(1 + 3\cos(4\alpha_1))p_x}{2p_y} - \frac{(1 + 3\cos(4\alpha_1))p_x}{2p_y} | |
### Table 1 continued

| Syndrome    | Probability | Fidelity                                                                 |
|-------------|-------------|--------------------------------------------------------------------------|
| 110110      | $p_y$       | $1 - \frac{(15 - 3\cos[4\alpha])p_x}{4} - \frac{5p_y}{2} (1 - \cos[4\alpha]) - \frac{3p_y}{2}\cos[2\beta]\sin[2\alpha]^2$ |
| 111000      | $4p_x + 2p_y$ | $\frac{1}{8}(3 + \cos[4\alpha] - 2\cos[2\beta]\sin[2\alpha]^2) - \frac{(69 - 25\cos[4\alpha])p_x}{16} + \frac{(7 - 11\cos[4\alpha])p_x}{16} + p_z + \frac{(7 - 47p_x - (3 - 27p_x)\cos[4\alpha])p_x}{8p_y} + \left(\frac{-57p_x + 15p_x}{8} + 2p_z + \frac{(3 - 27p_x)p_x}{4p_y}\right)\cos[2\beta]\sin[2\alpha]^2$ |
| 111011      | $p_y$       | $\frac{1}{2}(1 + \cos[4\alpha]) + \frac{(17 - 29\cos[4\alpha])p_x}{4} - \frac{(3 + 9\cos[4\alpha])p_x}{4} + \frac{(5 - 25\cos[4\alpha])p_x}{4} + \frac{(19 - 39\cos[4\alpha])p_xp_z}{4p_y} + \frac{(3p_y + 2p_z)}{4p_y}\cos[2\beta]\sin[2\alpha]^2$ |
| 111111      | $p_y$       | $1 - \frac{(1 + \cos[4\alpha])p_x}{4} - \frac{(13 - 9\cos[4\alpha])p_x}{4} - \frac{(3 + \cos[4\alpha])p_xp_z}{4p_y} - \left(\frac{3p_y}{2} + \frac{p_z}{2} - \frac{p_xp_z}{2p_y}\right)\cos[2\beta]\sin[2\alpha]^2$ |

Error syndromes appearing with probability second order (or higher) in the error probabilities $p_i$ are not given. Note that in some of the fidelity terms error probabilities appear in the denominator. This is a result of keeping only first order terms.
syndrome measurement is all zeros, the error probabilities in the fidelities are suppressed to second order in $p_x$ and $p_y$, while $p_z$ is suppressed to second order only for the initial states $|0\rangle$ and $|1\rangle$. Application of the NOT and phase gates to $|0\rangle$ or $|1\rangle$ does not provide any advantage. However, the ability to apply a Hadamard leads to usable encoding of the $|+\rangle$ and $|-\rangle$ states as there are no first order error terms. This cannot be directly done by encoding with fault-tolerant error correction.

4.2 Noisy error correction

Realistic error correction will be noisy and thus cannot be implemented as in Fig. 4. Rather, we apply a fault tolerant error correction scheme in the non-equiprobable error environment to the arbitrary encoded state to determine what might occur in a more realistic quantum computation. In adhering to the rules of fault tolerance we utilize Shor states as syndrome qubits (Fig. 5). The Shor states themselves are prepared in a noisy environment such that the Hadamard evolution is properly described by Eq. 1, and the evolution of the CNOT gates is described by Eq. 2. We verify the Shor state by performing two parity checks (also done in the non-equiprobable error environment). If the state is not suitable, it is discarded and a new Shor state is prepared. To implement error correction the necessary CNOT gates are applied between the data and ancilla (Shor state) qubits, and the ancilla qubits are measured. The parity of the four measurements determines the syndrome value. The phase flip syndrome measurements are performed in a similar manner (Fig. 5). We analyze the case where all four ancilla qubits are measured as zero for each of the syndrome measurements. Other even parity measurement outcomes would yield similar results, while odd parity outcomes would signal the presence of an error and require applying the appropriate recovery operation. We apply each syndrome check twice to confirm the correct parity measurement, as errors may occur while implementing the syndrome measurements themselves.

If error correction is to perform as desired and maintain an encoded state with high reliability, the fidelity of the post error-prone QEC state should be higher than

![Fig. 5](image-url) Left Preparation of the four qubit Shor state. To construct the Shor state, we first prepare a GHZ state. To ensure fault tolerant construction, two parity checks (between qubits 1 and 4 and qubits 1 and 2) are performed. If the qubit measurement gives an odd parity, the state is discarded and a new Shor state is prepared. Application of Hadamard rotations to the four qubits completes the preparation of the Shor states. Right The fault tolerant Steane code utilizing Shor states to perform the syndrome measurements. To check for bit flips, three sets of CNOTs are performed with the Shor state qubits as the target. The parity of the ancilla qubits determines the syndrome value. To perform phase flip checks, we utilize the equality shown in the Box of Fig. 4. We apply Hadamard rotations on the ancilla bits, cancelling the Hadamards at the end of the Shor state construction, and flip the direction of the CNOTs. Measuring in the $x$-basis allows us to eliminate the final Hadamards on the ancilla qubits.
the fidelity of the noisily encoded state with no error correction applied. The fidelity should furthermore be greater than the fidelity expected for unencoded states, which would indicate that going through the $[7,1,3]$ encoding process and QEC procedure effectively maintains information more reliably than undergoing no encoding at all. The fidelity for the (noisily) error corrected arbitrary encoded state is given in Eq. 16.

We find that the fidelity after an error-prone error correction resulting in an error syndrome of all zeros is quite comparable to the fidelity of the encoded state prior to error correction. This fidelity still includes first order terms in $p_x$, $p_y$, and $p_z$. There is little dependence on the initial state, as $\alpha$ only appears in the $p_z$ term, though the initial states with $\alpha = 0$ and $\alpha = \frac{\pi}{2}$ result in slightly higher fidelities than other initial states. We note that the $p_x$ term is now dominant. Furthermore, while the $p_x$ term in the fidelity of the error-prone QEC state is higher than the corresponding term in the fidelity of the pre-QEC encoded state, Eq. 4, the $p_y$ and $p_z$ terms of the QEC state are both significantly lower than the corresponding terms of the encoded state fidelity. Thus, in cases where $p_x$ is lower than $p_y$ and $p_z$, the noisy error correction scheme does indeed improve fidelity. However, when the $p_x$ is high, the noisy error correction can significantly lower the fidelity of the encoded state. Figure 6 displays a contour plot of the seven qubit fidelity with noisy error correction, and a plot showing the region in probability-space for which the post-QEC states attain a higher fidelity than the noisily encoded states (prior to error correction).

The fidelity of the single logical qubit for the QEC code is given in Eq. 17. We find that the single logical qubit fidelity after error-prone QEC is comparable to the fidelity of the single logical qubit before QEC was applied. This fidelity contains first order terms in $p_x$, $p_y$, and $p_z$. Furthermore, the fidelity is highly dependent on initial

![Figure 6](image)

*Fig. 6  Left Contour plots of seven qubit fidelity for noisy QEC on an arbitrary state encoded in the $[7,1,3]$ code as a function of the initial state parameterized by $\alpha$ ($\beta$ is set to 0) and error probabilities $p_x$ and $p_z$ ($p_y$ is set to zero). The contours for the fidelity values are (from top to bottom) 0.85, 0.90, 0.95, and 0.99. Right The shaded region indicates where in probability space the seven qubit fidelity of a noisily encoded arbitrary initial state increases after applying QEC. We find that the choice of initial state does not significantly affect this region. In this plot $\alpha$ and $\beta$ are both set to 0.*
state, as $\alpha$ appears in all first order terms. The initial states $\alpha = 0$ and $\alpha = \frac{\pi}{2}$ result in higher fidelities than other initial states. We note that in this measure of fidelity, the $p_x$ error is dominant as well.

When noisy error correction is implemented a second time on the system, the fidelities remain nearly the same, as seen in Eqs. 18 and 19.

The seven-qubit fidelity of the state after applying noisy error correction resulting in an error syndrome of all zeros to a noisy arbitrary encoded state that has undergone a logical NOT gate is given by Eq. 20. This fidelity features similar characteristics to the fidelity when noisy error correction is applied to the general encoded state, Eq. 16. It contains first order terms, with the $p_x$ term again being dominant. The single logical qubit fidelity of this state is given by Eq. 21, and bears strong resemblance to the post QEC single qubit fidelity given by Eq. 17. These results indicate that it may not be necessary or beneficial to perform error correction after every step in a quantum procedure, but rather that one should apply QEC only at specific intervals or after specific gate sequences.

5 Conclusion

In conclusion, we have explicitly evaluated the accuracy with which an arbitrary state can be encoded into the Steane $[7, 1, 3]$ QEC code. The calculated fidelities for the direct gate encoding method are only slightly worse than those calculated for the fault tolerant encoding of the zero state [5]. However, subsequent application of ideal error correction is able to suppress only some of the first-order error probability terms in the fidelity demonstrating that, in general, this encoding sequence is not usable for practical quantum computation. For the zero and one state, however, all first-order error probability terms are suppressed and, for these states, the encoding protocol is usable. We then applied logical Clifford gates to the encoded arbitrary state. The fidelity of the state after the application of one of these gates is primarily dependent on the number of gates applied. Once again we find the encoding plus Clifford gate usable only for encoding of state zero and one, however, because the Hadamard is usable for these states, the encoding of the $|+\rangle$ and $|-\rangle$ are also usable.

Applying noisy quantum error correction to the noisy encoded state maintains the fidelity of the state at a level comparable to that prior to QEC application, though the relative values of $p_x$, $p_y$, and $p_z$ can cause significant increases or decreases in this fidelity. These results indicate that QEC need not be applied after every step in a quantum protocol. We believe that simulations of this sort will help determine what quantum gates will be acceptable in order to successfully implement quantum computation.

Appendix: Fidelities

In this Appendix we present the fidelities discussed in the main part of the paper. In the equations, $\alpha$ and $\beta$ parameterize the initial state of the qubit to be encoded, $|\psi(\alpha, \beta)\rangle = \cos \alpha |0\rangle + e^{i\beta} \sin \alpha |1\rangle$, $p_x$ represents the probability of a $\sigma_x$ error, $p_y$...
represents the probability of a $\sigma_y$ error, and $p_z$ represents the probability of a $\sigma_z$ error. The superscript label of each equation indicates the number of qubits in the state for which the fidelity is calculated: the entire seven physical qubit system or the single logical qubit. All fidelities are given to second order in the error probabilities and are thus good approximations only for $p_x, p_y, p_z \ll 1$. To emphasize this we write the below equations using $\approx$ rather than $=$. 

Equation 4 is the fidelity of the seven qubit encoding of an arbitrary state:

$$F^{(7)}_{\text{enc}} \approx 1 - 22p_x - 25p_y - (23 - 2\cos(4\alpha))p_z + \left( \frac{515}{2} - \frac{3}{2}\cos(4\alpha) + 3\cos(2\beta)\sin(2\alpha)^2 \right) p_x^2$$
$$+ \left( 532 + 2\cos(4\alpha) - 2\cos(2\beta)\sin(2\alpha)^2 \right) p_x p_y$$
$$+ \left( \frac{1205}{4} - \frac{1}{4}\cos(4\alpha) + \frac{1}{2}\cos(2\beta)\sin(2\alpha)^2 \right) p_y^2$$
$$+ (486 - 42\cos(4\alpha))p_x p_z + \left( \frac{1127}{2} - \frac{93}{2}\cos(4\alpha) \right) p_y p_z$$
$$+ \left( \frac{577}{2} - \frac{81}{2}\cos(4\alpha) \right) p_z^2. \tag{4}$$

Equation 5 is the fidelity of the single logical qubit of an encoded arbitrary state that has undergone perfect (non error-prone) decoding:

$$F^{(1)}_{\text{enc}} \approx 1 - \left( \frac{9}{2} + \frac{3}{2}\cos(4\alpha) - 3\cos(2\beta)\sin(2\alpha)^2 \right) p_x$$
$$- \left( \frac{15}{2} - \frac{3}{2}\cos(4\alpha) + \cos(2\beta)\sin(2\alpha)^2 \right) p_y - (5 - 5\cos(4\alpha))p_z$$
$$+ \left( \frac{45}{2} + \frac{15}{2}\cos(4\alpha) - 15\cos(2\beta)\sin(2\alpha)^2 \right) p_x^2$$
$$+ (53 + 7\cos(4\alpha) - 46\cos(2\beta)\sin(2\alpha)^2) p_x p_y$$
$$+ \left( \frac{103}{2} - \frac{43}{2}\cos(4\alpha) + 17\cos(2\beta)\sin(2\alpha)^2 \right) p_y^2$$
$$+ (28 - 28\cos(4\alpha) - 56\cos(2\beta)\sin(2\alpha)^2) p_x p_z + (82 - 82\cos(4\alpha)$$
$$+ 16\cos(2\beta)\sin(2\alpha)^2) p_y p_z + (45 - 45\cos(4\alpha)) p_z^2. \tag{5}$$

Equations 6–11 are the fidelities of the state after application of a noisy Clifford gate compared to the perfectly encoded arbitrary state with ideal application of the Clifford gate. Equations 6, 8, and 10 represent the fidelities of the encoded seven-qubit system after a logical Hadamard, NOT, and phase gate has been applied, respectively. Equations 7, 9, and 11 represent the fidelities of the single logical qubit after a logical Hadamard, NOT, and phase gate has been applied, respectively:
\[ F_H^{(7)} \approx 1 - 29 p_x - 32 p_y - (30 - 2\cos[4\alpha]) p_z \]
\[ + \left( \frac{865}{2} - \frac{3}{2} \cos[4\alpha] + 3\cos[2\beta] \sin[2\alpha]^2 \right) p_x^2 \]
\[ + \left( 906 + 2\cos[4\alpha] - 2\cos[2\beta] \sin[2\alpha]^2 \right) p_x p_y \]
\[ + \left( \frac{2017}{4} - \frac{1}{4} \cos[4\alpha] + \frac{1}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_y^2 \]
\[ + \left( 881 - 55\cos[4\alpha] + 2\cos[2\beta] \sin[2\alpha]^2 \right) p_x p_z \]
\[ + \left( \frac{3773}{4} - \frac{237}{4} \cos[4\alpha] - \frac{1}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_y p_z \]
\[ + \left( \frac{941}{2} - \frac{109}{2} \cos[4\alpha] \right) p_z^2 \]
\[ (6) \]
\[ F_H^{(1)} \approx 1 - \left( \frac{21}{2} + \frac{3}{2} \cos[4\alpha] - 6\cos[2\beta] \sin[2\alpha]^2 \right) p_x \]
\[ - \left( \frac{65}{4} - \frac{5}{4} \cos[4\alpha] - \frac{1}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_y \]
\[ - \left( \frac{37}{4} - \frac{25}{4} \cos[4\alpha] - \frac{3}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_z + (120 + 12\cos[4\alpha]) \]
\[ - 102\cos[2\beta] \sin[2\alpha]^2 p_x^2 + (323 + 13\cos[4\alpha] - 214\cos[2\beta] \sin[2\alpha]^2) p_x p_y \]
\[ + \left( \frac{499}{2} - \frac{79}{2} \cos[4\alpha] - 17\cos[2\beta] \sin[2\alpha]^2 \right) p_y^2 \]
\[ + (181 - 109\cos[4\alpha] - 206\cos[2\beta] \sin[2\alpha]^2) p_x p_z \]
\[ + (302 - 72\cos[4\alpha] - 72\cos[2\beta] \sin[2\alpha]^2) p_y p_z \]
\[ + \left( \frac{233}{2} - \frac{221}{2} \cos[4\alpha] - 45\cos[2\beta] \sin[2\alpha]^2 \right) p_z^2 \]
\[ (7) \]
\[ F_X^{(7)} \approx 1 - 25 p_x - 28 p_y - (26 - 2\cos[4\alpha]) p_z \]
\[ + \left( \frac{1329}{4} - \frac{9}{4} \cos[4\alpha] + \frac{9}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_x^2 \]
\[ + \left( \frac{1359}{2} + \frac{5}{2} \cos[4\alpha] - 2\cos[2\beta] \sin[2\alpha]^2 \right) p_x p_y \]
\[ + \left( \frac{1529}{4} - \frac{1}{4} \cos[4\alpha] + \frac{1}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_y^2 \]
\[ + (627 - 48\cos[4\alpha]) p_x p_z + \left( \frac{1431}{2} - \frac{105}{2} \cos[4\alpha] \right) p_y p_z \]
\[ + \left( \frac{745}{2} - \frac{89}{2} \cos[4\alpha] \right) p_z^2 \]
\[ (8) \]
\[ F_X^{(1)} \approx 1 - \left( \frac{27}{4} + \frac{9}{4} \cos[4\alpha] - \frac{9}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_x \]
The fidelity of an arbitrary state that has undergone error-prone encoding, followed by a perfect (non error-prone) quantum error correction scheme that resulted in an error syndrome of all zeros. Note that the only first order error probability term is \( p_z \) and this term falls out for \( \alpha = 0, \frac{\pi}{2} \). The fidelities of the state that would result upon measurement of different error syndromes are given in Table 1.

\[
F_p^{(7)} \approx 1 - 29\, p_x - 32\, p_y - (30 - 2\cos[4\alpha])\, p_z \\
+ \left( \frac{865}{2} - \frac{3}{2} \cos[4\alpha] + 3\cos[2\beta]\sin[2\alpha]^2 \right) p_x^2 + (929 + \cos[4\alpha]) \, p_x \, p_y \\
+ (499 + \cos[4\alpha]) \, p_y^2 + (843 - 56\cos[4\alpha]) \, p_x \, p_z \\
+ \left( \frac{1889}{2} - \frac{121}{2} \cos[4\alpha] \right) \, p_y \, p_z + \left( \frac{979}{2} - \frac{105}{2} \cos[4\alpha] \right) \, p_z^2 \tag{10}
\]

\[
F_p^{(1)} \approx 1 - \left( \frac{45}{4} - \frac{9}{4} \cos[4\alpha] + \frac{3}{2} \cos[2\beta]\sin[2\alpha]^2 \right) \, p_x \\
+ \left( -\frac{37}{4} + \frac{1}{4} \cos[4\alpha] - \frac{5}{2} \cos[2\beta]\sin[2\alpha]^2 \right) \, p_y - \left( \frac{13}{2} - \frac{13}{2} \cos[4\alpha] \right) \, p_z \\
+ \left( \frac{243}{2} - \frac{99}{2} \cos[4\alpha] + 39\cos[2\beta]\sin[2\alpha]^2 \right) \, p_z^2 \, d \\
+ \left( 199 - 55\cos[4\alpha] + 92\cos[2\beta]\sin[2\alpha]^2 \right) \, p_x \, p_y \\
+ \left( \frac{159}{2} - \frac{15}{2} \cos[4\alpha] + 45\cos[2\beta]\sin[2\alpha]^2 \right) \, p_y^2 \\
+ \left( 167 - 167\cos[4\alpha] + 40\cos[2\beta]\sin[2\alpha]^2 \right) \, p_x \, p_z \\
+ \left( 119 - 119\cos[4\alpha] + 64\cos[2\beta]\sin[2\alpha]^2 \right) \, p_y \, p_z + (78 - 78\cos[4\alpha]) \, p_z^2 . \tag{11}
\]
\[- \left( 6 - 2\cos[4\alpha] + 2\cos[2\beta]\sin[2\alpha]^2 \right) p_x p_y \]
\[- \left( \frac{3}{4} + \frac{1}{4}\cos[4\alpha] - \frac{1}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_y^2 \]
\[- (2 - 2\cos[4\alpha]) p_x p_z - \left( \frac{7}{2} - \frac{7}{2}\cos[4\alpha] \right) p_y p_z \]
\[- \left( \frac{3}{2} - \frac{3}{2}\cos[4\alpha] \right) p_z^2. \] (12)

Equations 13, 14, and 15 represent the fidelities of an arbitrary state that has undergone error-prone encoding and error-prone application of the logical single qubit Clifford gates (Hadamard, NOT, and phase, respectively), followed by perfect quantum error correction resulting in an error syndrome of all zeros. Note that the only first order term in all three equations is \( p_z \) (and this term falls out for \( \alpha = 0, \frac{\pi}{2} \)):

\[ F_{P-QEC,H}^{(7)} \approx 1 - (2 - 2\cos[4\alpha]) p_z - \left( \frac{9}{2} + \frac{3}{2}\cos[4\alpha] - 3\cos[2\beta]\sin[2\alpha]^2 \right) p_x^2 \]
\[- \left( 6 - 2\cos[4\alpha] + 2\cos[2\beta]\sin[2\alpha]^2 \right) p_x p_y \]
\[- \left( \frac{3}{4} + \frac{1}{4}\cos[4\alpha] - \frac{1}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_y^2 \]
\[- (2 - 2\cos[4\alpha]) p_x p_z - \left( \frac{7}{2} - \frac{7}{2}\cos[4\alpha] \right) p_y p_z \]
\[- \left( \frac{3}{2} - \frac{3}{2}\cos[4\alpha] \right) p_z^2. \] (13)

\[ F_{P-QEC,X}^{(7)} \approx 1 - (2 - 2\cos[4\alpha]) p_z - \left( \frac{27}{4} + \frac{9}{4}\cos[4\alpha] - \frac{9}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_x^2 \]
\[- \left( \frac{13}{2} - \frac{5}{2}\cos[4\alpha] + 2\cos[2\beta]\sin[2\alpha]^2 \right) p_x p_y \]
\[- \left( \frac{3}{4} + \frac{1}{4}\cos[4\alpha] - \frac{1}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_y^2 \]
\[- (2 - 2\cos[4\alpha]) p_x p_z - \left( \frac{7}{2} - \frac{7}{2}\cos[4\alpha] \right) p_y p_z \]
\[- \left( \frac{7}{2} - \frac{7}{2}\cos[4\alpha] \right) p_z^2. \] (14)

\[ F_{P-QEC,P}^{(7)} \approx 1 - (2 - 2\cos[4\alpha]) p_z - \left( \frac{9}{2} + \frac{3}{2}\cos[4\alpha] - 3\cos[2\beta]\sin[2\alpha]^2 \right) p_x^2 \]
\[- (9 - \cos[4\alpha]) p_x p_y - \left( 3 - \cos[4\alpha] \right) p_y^2 - (2 - 2\cos[4\alpha]) p_x p_z \]
\[- \left( \frac{7}{2} - \frac{7}{2}\cos[4\alpha] \right) p_y p_z - \left( \frac{7}{2} - \frac{7}{2}\cos[4\alpha] \right) p_z^2. \] (15)
Equation 16 represents the fidelity of an arbitrary state that has been encoded via the error-prone $[7, 1, 3]$ code and has then undergone error-prone quantum error correction resulting in an error syndrome of all zeros. Equation 17 represents the fidelity of the single logical qubit of this state:

$$F^{(7)}_{QEC} \approx 1 - 55p_x - 7p_y - (9 - 2\cos[4\alpha])p_z$$
$$+ \left( \frac{3111}{2} - \frac{3}{2}\cos[4\alpha] + 3\cos[2\beta]\sin[2\alpha]^2 \right) p_x^2$$
$$+ \left( 226 + 2\cos[4\alpha] - 2\cos[2\beta]\sin[2\alpha]^2 \right) p_x p_y$$
$$- \left( \frac{1079}{4} + \frac{1}{4}\cos[4\alpha] - \frac{1}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_y^2$$
$$+ (330 - 103\cos[4\alpha]) p_x p_z - \left( \frac{1293}{2} + \frac{21}{2}\cos[4\alpha] \right) p_y p_z$$
$$- \left( \frac{297}{2} + \frac{5}{2}\cos[4\alpha] \right) p_z^2.$$  

Equation 18 represents the fidelity of an arbitrary state that has been encoded via the error-prone $[7, 1, 3]$ code and has undergone error-prone quantum error correction two times successively each resulting in an error syndrome of all zeros. Equation 19 represents the fidelity of the single logical qubit of this procedure:

$$F^{(1)}_{QEC} \approx 1 - \left( \frac{57}{4} + \frac{19}{4}\cos[4\alpha] - \frac{19}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_x$$
$$- \left( \frac{13}{4} - \frac{1}{4}\cos[4\alpha] - \frac{1}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_y$$
$$- \left( \frac{7}{2} - \frac{7}{2}\cos[4\alpha] \right) p_z + (210 + 70\cos[4\alpha] - 140\cos[2\beta]\sin[2\alpha]^2) p_x^2$$
$$+ \left( \frac{169}{4} + \frac{43}{4}\cos[4\alpha] - \frac{109}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_x p_y$$
$$- \left( \frac{175}{2} + \frac{37}{2}\cos[4\alpha] - 34\cos[2\beta]\sin[2\alpha]^2 \right) p_y^2$$
$$+ \left( \frac{51}{4} - \frac{271}{4}\cos[4\alpha] - \frac{209}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_x p_z$$
$$- \left( \frac{829}{4} + \frac{127}{4}\cos[4\alpha] - \frac{137}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_y p_z$$
$$- (42 - 42\cos[4\alpha]) p_z^2.$$  

Equation 18 represents the fidelity of an arbitrary state that has been encoded via the error-prone $[7, 1, 3]$ code and has undergone error-prone quantum error correction two times successively each resulting in an error syndrome of all zeros. Equation 19 represents the fidelity of the single logical qubit of this procedure:

$$F^{(7)}_{2xQEC} \approx 1 - 55p_x - 7p_y - (9 - 2\cos[4\alpha])p_z$$
$$+ \left( \frac{3081}{2} - \frac{3}{2}\cos[4\alpha] + 3\cos[2\beta]\sin[2\alpha]^2 \right) p_x^2.$$
\[
F_{2^{(1)}_{QEC}} \approx 1 - \left( \frac{57}{4} + \frac{19}{4} \cos[4\alpha] - \frac{19}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_x
\]
\[
- \left( \frac{13}{4} - \frac{1}{4} \cos[4\alpha] - \frac{1}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_y
\]
\[
- \left( \frac{7}{2} - \frac{7}{2} \cos[4\alpha] \right) p_z + (198 + 66 \cos[4\alpha] - 132 \cos[2\beta] \sin[2\alpha]^2) p_x^2
\]
\[
+ \left( \frac{177}{4} + \frac{43}{4} \cos[4\alpha] - \frac{107}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_x p_y
\]
\[
- \left( 85 + 19 \cos[4\alpha] - 33 \cos[2\beta] \sin[2\alpha]^2 \right) p_y^2
\]
\[
+ \left( \frac{51}{4} - \frac{271}{4} \cos[4\alpha] - \frac{209}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_x p_z
\]
\[
- \left( \frac{773}{4} + \frac{135}{4} \cos[4\alpha] - \frac{133}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_y p_z
\]
\[
- \left( 38 - 38 \cos[4\alpha] \right) p_z^2
\]

Equation 20 represents the fidelity of an arbitrary state that has been encoded via the error-prone [7, 1, 3] code, and has then undergone noisy application of the logical single qubit NOT gate, followed by error-prone quantum error correction resulting in an error syndrome of all zeros. Equation 21 represents the fidelity of the single logical qubit of this procedure:

\[
F^{(7)}_{QEC,X} \approx 1 - 55 p_x - 7 p_y - (9 - 2 \cos[4\alpha]) p_z
\]
\[
+ \left( \frac{6193}{4} - \frac{9}{4} \cos[4\alpha] + \frac{9}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_x^2
\]
\[
+ \left( \frac{451}{2} + \frac{5}{2} \cos[4\alpha] - 2 \cos[2\beta] \sin[2\alpha]^2 \right) p_x p_y
\]
\[
- \left( \frac{1079}{4} + \frac{1}{4} \cos[4\alpha] - \frac{1}{2} \cos[2\beta] \sin[2\alpha]^2 \right) p_y^2
\]
\[
+ (330 - 103 \cos[4\alpha]) p_x p_z
\]
\[
- \left( \frac{1309}{2} + \frac{21}{2} \cos[4\alpha] \right) p_y p_z - \left( \frac{311}{2} + \frac{1}{2} \cos[4\alpha] \right) p_z^2
\]
\[
F_{QEC,X}^{(1)} \approx 1 - \left( \frac{57}{4} + \frac{19}{4}\cos[4\alpha] - \frac{19}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_x \\
- \left( \frac{13}{4} - \frac{1}{4}\cos[4\alpha] - \frac{1}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_y - \left( \frac{7}{2} - \frac{7}{2}\cos[4\alpha] \right) p_z \\
+ \left( \frac{819}{4} + \frac{273}{4}\cos[4\alpha] - \frac{273}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_x^2 \\
+ \left( \frac{167}{4} + \frac{45}{4}\cos[4\alpha] - \frac{109}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_x p_y \\
- \left( \frac{175}{2} + \frac{37}{2}\cos[4\alpha] - 34\cos[2\beta]\sin[2\alpha]^2 \right) \frac{1}{2} p_y^2 \\
+ \left( \frac{51}{4} - \frac{271}{4}\cos[4\alpha] - \frac{209}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_x p_z \\
- \left( \frac{847}{4} + \frac{125}{4}\cos[4\alpha] - \frac{139}{2}\cos[2\beta]\sin[2\alpha]^2 \right) p_y p_z \\
- \left( 46 - 46\cos[4\alpha] \right) p_z^2 \tag{21}
\]

References

1. Preskill, J.: Reliable quantum computers. Proc. Roy. Soc. Lond. A 454, 385–410 (1998)
2. Shor, P.W.: Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. In: Proceedings of the 35th Annual Symposium on Fundamentals of Computer Science. IEEE Press, Los Alamitos (1996)
3. Gottesman, D.: Theory of fault-tolerant quantum computation. Phys. Rev. A 57, 127–137 (1998)
4. Aleferis, P., Gottesman, D., Preskill, J.: Quant. Inf. Comput. 6, 97 (2006)
5. Weinstein, Y.S.: Fidelity of an encoded \([7,1,3]\) logical zero. Phys. Rev. A 84, 012323 (2011)
6. Nielsen, M., Chuang, I.: Quantum Information and Computation. Cambridge University Press, Cambridge (2000)
7. Shor, P.W.: Scheme for reducing decoherence in quantum computer memory. Phys. Rev. A 52, R2493–R2496 (1995)
8. Calderbank, A.R., Shor, P.W.: Good quantum error-correcting codes exist. Phys. Rev. A 54, 1098–1105 (1996)
9. Steane, A.M.: Error correcting codes in quantum theory. Phys. Rev. Lett. 77, 793–797 (1996)
10. Steane, A.: Multiple particle interference and quantum error correction. Proc. Roy. Soc. Lond. A 452, 2551–2577 (1996)
11. Aggarwal, V., Calderbank, A.R., Gilbert, G., Weinstein, Y.S.: Volume thresholds for quantum fault tolerance. Quant. Inf. Proc. 9, 541–549 (2010)
12. Aliferis, P., Preskill, J.: Fault-tolerant quantum computation against biased noise. Phys. Rev. A 78, 052331 (2008)