Information Geometry of Entanglement Renormalization for free Quantum Fields

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We provide an explicit connection between the differential generation of entanglement entropy in a tensor network representation of the ground states of two field theories, and a geometric description of these states based on the Fisher information metric. We show how the geometrical description remains invariant despite there is an irreducible gauge freedom in the definition of the tensor network. The results might help to understand how spacetimes may emerge from distributions of quantum states, or more concretely, from the structure of the quantum entanglement concomitant to those distributions.

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INTRODUCTION

Recently, it has been proposed that the structure of spacetime in gravitational theories may inextricably be related with the entanglement structure of some fundamental degrees of freedom [1, 2]. The holographic formula of the entanglement entropy [3], which provides a prescription to quantify the entanglement structure of some fundamental degrees of freedom [1, 2], is an irreducible gauge freedom in the definition of the tensor network. The results might help to understand how spacetimes may emerge from distributions of quantum states, or more concretely, from the structure of the quantum entanglement concomitant to those distributions.

ENTANGLEMENT RENORMALIZATION FOR QFT

Entanglement renormalization (MERA) is a real-space renormalization group formulation on the quantum state (instead of the Wilsonian RG scheme) [4, 5]. MERA represents the wavefunction of the system at each relevant length scale \( u \) of the system. By convention, \( u = 0 \) refers to the state description at short lengths (UV-state \( |\Psi_{UV}\rangle \)). Starting from it, (in principle, this amounts to a highly entangled state), each scale \( u \) of MERA performs a renormalization transformation in which, prior to coarse graining the effective degrees of freedom at that scale, the short range entanglement between them is removed through the action of an unitary transformation called \textit{disentangler}. Thus iteratively, MERA removes the quantum correlations between small adjacent regions of space at each length scale. This RG procedure is applied arbitrarily many times until one reaches the IR-state \( |\Psi_{IR}\rangle \). Namely, the procedure may be run backwards so, starting from \( |\Psi_{IR}\rangle \), one unitarily adds entanglement at each length scale until the correct \( |\Psi_{UV}\rangle \) is generated.

To be precise, let us consider the state \( |\Psi(u)\rangle \) obtained by adding entanglement between modes of momentum \( k \leq \Lambda e^{-u} \) to the unentangled state \( |\Psi_{IR}\rangle \),

\[
|\Psi(u)\rangle = P e^{-i \int_{u}^{u_{IR}} d\hat{u}(\hat{K}(u)+L)} |\Psi_{IR}\rangle,
\]

where \( P \) is a path ordering symbol which allocates operators with bigger \( u \) to the right and \( \Lambda \) is the UV momentum cutoff. The operator \( \hat{K}(\hat{u}) \) generates the entanglement along the cMERA flow from \( u_{IR} \) to a given \( u \). It reads as,

\[
\hat{K}(\hat{u}) = \int d\hat{k} \Gamma(k/\Lambda) g(\hat{u}, k) \mathcal{O}_k,
\]

where \( \Gamma(x) = 1 \) for \( 0 < x < 1 \) and zero otherwise. The function \( g(\hat{u}, k) \) is model/state dependent and gives the strenght of the entangling process at a given scale. The operator \( L \) corresponds to the coarse-graining process [4–6]. In this paper we mainly focus on the entangling process, thereby, to get rid of the \( L \) process in our analysis, we proceed by rescaling the cMERA states as,

\[
|\Phi(u)\rangle = P e^{-i \int_{u}^{u_{IR}} d\hat{u} \tilde{K}(\hat{u})} |\Psi_{IR}\rangle.
\]

Here, the entangler operator is given in the interaction picture \( \tilde{K}(\hat{u}) = e^{-i \hat{u} L} \hat{K}(\hat{u}) e^{i \hat{u} L} \), and reads as,

\[
\tilde{K}(\hat{u}) = \int d\hat{k} \Gamma(k e^{i \hat{u}}/\Lambda) g(\hat{u}, k e^{i \hat{u}}) \tilde{\mathcal{O}}_k,
\]

with \( \tilde{\mathcal{O}}_k = e^{-i \hat{u} L} \mathcal{O}_k e^{i \hat{u} L} \).

In this paper, we will consider two examples of free fields in (1+1) dimensions, namely the free massive boson and a free massive Dirac fermion. For the free boson theory with action,

\[
S_B = \int dt dx \left[ (\partial_t \phi)^2 + (\partial_x \phi)^2 - m^2 \phi^2 \right],
\]
one has,
\[
\tilde{K}_B(\tilde{u}) = i \int dk \left( g_{k}^B(\tilde{u}) a_k^\dagger a_{-k}^\dagger - g_{k}^B(\tilde{u})^* a_k a_{-k} \right),
\]
where \(g_{k}^B(\tilde{u}) = \Gamma(ke^\tilde{u}/\Lambda) g^B(\tilde{u}, k)\) and \(a_k^\dagger, a_k\) are the creation and annihilation operators of the field mode with momentum \(k\) such that, if \(|\Psi_{UV}\rangle\rangle_B = |0\rangle_B\) is the ground state of the theory then, \(a_k|0\rangle_B = a_{-k}|0\rangle_B = 0\).

The free Dirac fermion theory is given by the action,
\[
S_F = \int dt dx \left[ i\bar{\psi} \left( \gamma^0 \partial_t + \gamma^\alpha \partial_x \right) \psi - m\bar{\psi} \psi \right],
\]
where \(\psi\) is a two component complex fermion with \(\gamma^0 = \sigma_3, \gamma^\alpha = i\sigma_2\) and \(\bar{\psi} = \psi^\dagger \gamma^\dagger\). The entangler operator in this case reads as,
\[
\tilde{K}_F(\tilde{u}) = i \int dk \left( g_{k}^F(\tilde{u}) c_k^\dagger d_k^\dagger + g_{k}^F(\tilde{u})^* c_k d_k \right),
\]
where \(g_{k}^F(\tilde{u}) = (ke^\tilde{u}/\Lambda) \Gamma(ke^\tilde{u}/\Lambda) g^F(\tilde{u}, k)\) and \(c_k, d_k\) are the anihilation operators for field modes of each component (particles and anti-particles) such that \([10]\).

In the following we shall omit the subscripts \((B, F)\) while it will be clear to which case we are referring.

**Coherent State description of cMERA**

In the bosonic theory, the state in eq.\([3]\) can be written as,
\[
|\Phi(u)\rangle = N \exp \left[ \frac{1}{2} \int dk \left( f_k(u) a_k^\dagger a_{-k}^\dagger - f_k^\dagger(u) a_k a_{-k} \right) \right] |\Psi_{IR}\rangle,
\]
where we have defined \(f_k(u) = \int_{-\Lambda} u_{IR} g_k(u) d\tilde{u}\). The path ordering symbol \(P\) is dropped since now the state is normalized by taking \(N = \exp \left[ -\frac{1}{2} \int dk |f_k(u)|^2 \right]\). This state is a Gaussian coherent state annihilated by the operator,
\[
b_k(u) = A_k(u) a_k + B_k(u) a_k^\dagger,
\]
i.e., \(b_k(u)|\Phi(u)\rangle = 0\) with \(|A_k(u)|^2 - |B_k(u)|^2 = 1\). Eq.\([11]\) amounts to a scale-dependent Bogoliubov transformation whose model dependent coefficients are given by \([3]\).

\[
A_k(u) = \cosh f_k(u) a_k - \sinh f_k(u) \beta_k,
B_k(u) = -\sinh f_k(u) a_k + \cosh f_k(u) \beta_k,
\]
with \(\alpha_k \equiv A_k(u_{IR}), \beta_k \equiv B_k(u_{IR})\), i.e.,
\[
(\alpha_k a_k + \beta_k a_k^\dagger)|\Psi_{IR}\rangle = 0.
\]

In the fermionic theory, \(|\Phi(u)\rangle\) reads as,
\[
|\Phi(u)\rangle = N \exp \left[ \frac{1}{2} \int dk f_k(u) c_k^\dagger d_k \right] |\Psi_{IR}\rangle,
\]
where again, \(f_k(u) = \int_{-\Lambda} u_{IR} g_k(u) d\tilde{u}\) and the state is normalized by \(N = \exp \left[ -\frac{1}{2} \int dk |f_k(u)|^2 \right]\). Eq.\([14]\) is a *displaced* vacuum coherent state which is annihilated by the operator,
\[
\psi_k(u) = A_k(u) c_k + B_k(u) d_k^\dagger,
\]
i.e., \(\psi_k(u)|\Phi(u)\rangle = 0\) with coefficients,
\[
A_k(u) = \cos f_k(u) \alpha_k + \sin f_k(u) \beta_k,
B_k(u) = -\sin f_k(u) \alpha_k + \cos f_k(u) \beta_k,
\]
such that \(|A_k(u)|^2 + |B_k(u)|^2 = 1\), and
\[
(\alpha_k c_k + \beta_k d_k^\dagger)|\Psi_{IR}\rangle = 0.
\]

In this framework, the entangling operation of \(c\text{MERA}\) in the free theories under consideration amounts to a sequential generation of a set of coherent states \(|\Phi(u)\rangle\) defined through eqs.\([10]\), \([13]\). In both cases, \(|\Phi(u)\rangle\) is an non-entangled vacuum for the Bogoliubov-quasiparticles at that scale, while as displaced vacuum states, they are highly entangled relative to any state defined on a higher scale of cMERA.

**ENTANGLEMENT FLOW IN MERA**

In this section, we quantify the entanglement flow required to generate \(|\Phi(u)\rangle\) starting from \(|\Psi_{IR}\rangle\). Let us first consider the bosonic case by writing the state in eq.\([10]\) as a superposition of Fock states,
\[
|\Phi(u)\rangle = \prod_k \sum_{n=0}^\infty c^n_k |n_k, n_{-k}\rangle,
\]
where \(|n_k, n_{-k}\rangle \sim (a_k^\dagger)^n (a_k^\dagger)^n |\Psi_{IR}\rangle\) and,
\[
N_k = \gamma_k(u)^{n/2} \sqrt{1 - \gamma_k(u)}, \quad \gamma_k(u) = \left[ \frac{B_k(u)}{A_k(u)} \right]^2.
\]
Here, \(A_k(u)\) and \(B_k(u)\) are those in eq.\([12]\). The total amount of entanglement generated between all the modes with opposite momenta \(|k| \leq \Lambda e^{-u}\) when creating \(|\Phi(u)\rangle\) from \(|\Psi_{IR}\rangle\) amounts to the von Neumann entropy of
\[
\rho(u) = \text{Tr}_{\{k\}} \langle|\Phi(u)\rangle \langle\Phi(u)|\rangle = \sum_k \sum_{n=0}^\infty |c^n_k|^2 |n_k\rangle \langle n_k|,
\]
where \(\gamma \equiv \gamma_k(u)\). In a free theory where all modes are decoupled, the entanglement entropy \(S(u)\) can be written as,
\[
S(u) = -\int dk \Gamma(k e^u/\Lambda) \text{Tr} \left[ \rho_k(u) \log \rho_k(u) \right],
\]
FIG. 1: Entropy rate $\partial_u S_k(u)$ (dots) vs $-2g_k(u)$ (continuous line) for three different masses $m = 0.05, 1, 5$ of the free boson. The plot has been created taking $k = 0.005$ and $\Lambda = 100$. $\partial_u S_k(u)$ is computed by numerically differentiating values of $S_k(u)$ obtained through eq. (22). The cMERA scale $u$ runs from the 0 to 9. For each case, the rate $\partial_u S_k(u)$ vanishes at a different scale $u_{IR}$ which increases as $m$ decreases. This scale indicates that the renormalization process has reached the state $|\psi_{IR}\rangle$.

with $\rho_k(u) = \sum_n \gamma^n (1 - \gamma) |n_k\rangle\langle n_k|$. Thus, it is possible to carry out the analysis only focusing on the entanglement generated between two modes with opposite momenta $S_k(u) = -\text{Tr} [\rho_k(u) \log \rho_k(u)]$. This entropy reads as [16],

$$S_k(u) = \gamma \frac{\log \gamma - \log(1 + \gamma)}{1 - \gamma}.$$  (22)

The entanglement flows in the process amounts to quantify how much entanglement is added at each infinitesimal cMERA layer. Differentiating eq. (22) wrt $u$ and noticing that $\partial_u f_k(u) = g_k(u)$, yields,

$$\partial_u S_k(u) = \left[ \frac{2\sqrt{\gamma}}{(1 - \gamma)} \log \gamma \right] g_k(u),$$  (23)

which explicitly relates the rate of entanglement generation with the strength of the entangling operation $g_k(u)$. For $\gamma \sim 1$, the factor $2\sqrt{\gamma}/(1 - \gamma) \log \gamma \approx -(1 + \gamma) \approx -2$. This allows to write,

$$g_k(u) \approx -\frac{1}{2} \partial_u S_k(u).$$  (24)

Figure 1 illustrates this relation for the ground state of a free scalar theory with mass $m$. In this case, by variationally minimizing the energy density $E = \langle \Psi_{IR} | H(u_{IR}) | \Psi_{IR} \rangle$ for $k < \Lambda e^{-u}$, one obtains [8,10],

$$g_k(u) = g(u) = -\frac{1}{2} \frac{e^{-2u}}{e^{-2u} + m^2/\Lambda^2}. $$  (25)

where $H$ is the Hamiltonian of the system.

Next, we address the fermionic case. As $\psi_k(u)\langle \Phi(u) | = 0$, and taking into account eq. (13), the state in eq. (14) can be written as,

$$|\Phi(u)\rangle = \mathcal{N} \prod_k \left( 1 + \gamma_k^{1/2}(u) c_k^{\dagger} d_k \right) |\Psi_{IR}\rangle$$

$$= \mathcal{N} \prod_k \left( |0_c, 0_d\rangle_k + \gamma_k^{1/2}(u) |1_c, 1_d\rangle_k \right)$$

$$= \mathcal{N} \prod_k |\Psi_k(u)\rangle.$$  (26)

Here, $\gamma_k(u) = |B_k(u)/A_k(u)|^2$ with $A_k(u)$ and $B_k(u)$ given by eq. (16). The states $|0_c, 0_d\rangle$ and $|1_c, 1_d\rangle$ refer to fermionic Fock states with no $c$-particles and $d$-antiparticles of momentum $k$ and one $c$-particle and one $d$-antiparticle of momentum $k$ respectively; $|\Psi_{IR}\rangle \equiv \prod_k |0_c, 0_d\rangle_k$ and

$$|\Psi_k(u)\rangle = \frac{1}{\sqrt{1 + \gamma_k(u)}} \left( |0_c, 0_d\rangle_k + \gamma_k(u)^{1/2} |1_c, 1_d\rangle_k \right).$$  (27)

In likeness manner as before, we proceed by focusing on the entanglement of the state $|\Psi_k(u)\rangle$. This amounts to the entanglement between a $c$-mode and a $d$-mode given by the entropy $S_k(u) = -\text{Tr} [\rho_k(u) \log \rho_k(u)]$, where the reduced density matrix $\rho_k(u)$ reads as,

$$\rho_k(u) = \text{Tr}_{d} \left[ |\Psi_k(u)\rangle \langle \Psi_k(u)| \right]$$

$$= \begin{pmatrix} 1/(1 + \gamma_k(u)) & 0 \\ 0 & \gamma_k(u)/(1 + \gamma_k(u)) \end{pmatrix}. $$  (28)

Then, a straightforward calculation yields,

$$S_k(u) = \log(1 + \gamma_k(u)) - \frac{\gamma_k(u)}{1 + \gamma_k(u)} \log \gamma_k(u).$$  (29)

To obtain the rate of entanglement generation along the cMERA flow in the free fermion theory, one simply differentiates eq. (29). The entanglement flow, as in the bosonic case, results proportional to the strength of the entangling operation and can be written as,

$$\partial_u S_k(u) = \left[ \frac{2\sqrt{\gamma_k(u)}}{(1 + \gamma_k(u))} \log \gamma_k(u) \right] g_k(u).$$  (30)

Both eq. (24) and eq. (30) are major results of this work and, as it will be shown below, they shall allow to write explicit formulas linking the rate of entanglement generation in cMERA flows with the geometric descriptions of the process proposed in [10].

**Relative Entropy**

A measure of distinguishability between the quantum probability distributions defined by $\tilde{\rho} \equiv \rho_k(u + du, \{\bar{e}\})$ and $\rho \equiv \rho_k(u, \{e\})$ has been computed in terms of the relative entropy between them [17],

$$S(\tilde{\rho} || \rho) = \text{Tr} [\tilde{\rho} \log \tilde{\rho} - \tilde{\rho} \log \rho],$$  (31)
where it has been assumed that \( \bar{\gamma} \equiv \gamma(u + du) \approx \gamma(u) + \partial_u \gamma(u) du \). In the bosonic theory, the computation yields,

\[
S(\bar{\rho} \parallel \rho) = \frac{\bar{\gamma}}{(1 - \bar{\gamma})} \log \frac{\bar{\gamma}}{\gamma} + \log \frac{(1 - \bar{\gamma})}{(1 - \gamma)} \quad (32)
\]

\[
= 4 g_k(u)^2 du^2.
\]

In like manner, for the fermionic case one obtains,

\[
S(\bar{\rho} \parallel \rho) = \frac{\bar{\gamma}}{(1 + \bar{\gamma})} \log \frac{\bar{\gamma}}{\gamma} - \log \frac{(1 + \bar{\gamma})}{(1 + \gamma)} \quad (33)
\]

\[
= 4 g_k(u)^2 du^2.
\]

In information geometry, the Fisher information metric \([18]\) is a Riemannian metric defined on a smooth statistical manifold, i.e., a smooth manifold whose points are probability distributions defined on a common probability space. The metric measures the informational difference between those points (distributions) and amounts to the infinitesimal form of the relative entropy. The computations above, addressed the case in which these probability distributions correspond to the reduced density matrices \( \bar{\rho} \) and \( \rho \) of two infinitesimally displaced cMERA quantum states. Thus, the infinitesimal information distance between these distributions, has been related with the strength of the disentangling operation \( g_k(u) \) and, through eqs. (24) and eq. (30), with the differential generation of entanglement entropy along the renormalization group flow.

**EMERGENT GEOMETRY AND ENTANGLEMENT**

In \([9]\), it has been conjectured that, from the entanglement structure of an static (1+1) wavefunction represented by an entanglement renormalization tensor network, one may define a higher dimensional geometry in which, apart from the coordinate \( x \), it is reasonable to define a "radial" coordinate \( u \) which accounts for the hierarchy of scales. The conjecture has been qualitatively confirmed by comparing how entanglement entropy is computed in MERA tensor networks and in the AdS/CFT correspondence \([3]\). The geometry emerging at the critical point is the hyperbolic AdS spacetime. For more generic static cMERA states, it is hypothesized that the metric must be an asymptotically AdS geometry given by,

\[
ds^2 = g_{uu} du^2 + \Lambda^2 e^{-2u} dx^2.
\]

In \([10]\), authors provide a method to obtain an Euclidean geometric description of the cMERA process. In the AdS/CFT, it is widely accepted that the holographic radial dimension corresponds to the length scale of the renormalization group flow, whereupon it is natural to identify the length scale \( u \) with the radial direction of a dual geometric description of the quantum state \([2]\). This geometric construction amounts to the Fisher information metric between the \{\( |\Phi(u)\rangle / u \in [0, u_{IR}] \}\} states, once a suitable distance measure \( D[\Phi(u), \Phi(u + du)] \), such as the Hilbert-Schmidt distance

\[
D_{HS}^2 [\Phi(u), \Phi(u + du)] = 1 - |\langle \Phi(u) | \Phi(u + du) \rangle|^2, \quad (35)
\]

has been chosen. Then, the proposal for the \( g_{uu} \) component of the metric is given by,

\[
g_{uu} du^2 = V^{-1} D_{HS}^2 [\Phi(u), \Phi(u + du)], \quad (36)
\]

with \( V \) a normalization constant.

Let us take the free boson theory as an example. In this case, \( V = \int dk \Gamma(k e^{u}/\Lambda). \) The Hilbert space of the theory consists of a direct product of sectors, each with fixed momentum \( k \). Thus, for \( k \leq \Lambda e^{-u}, f_k(u) = f(u) \in \mathbb{R} \) and henceforth, the overlap between the two coherent states \( |\Phi(u + du)\rangle \) and \( |\Phi(u)\rangle \) defined through eq. (6) (assuming that \( f(u) \) smoothly changes as \( u \) varies), reads as,

\[
|\langle \Phi(u) | \Phi(u + du)\rangle|^2 = \exp \left[ -V (\partial_u f(u))^2 du^2 \right]. \quad (37)
\]

Then, making use of eq. (24) one obtains,

\[
D_{HS}^2 [\Phi(u), \Phi(u + du)] \approx \frac{V}{4} |\partial_u S_k(u)\rangle^2 du^2. \quad (38)
\]

Substituting this result into eq. (36) yields,

\[
g_{uu}(u) = \frac{1}{4} |\partial_u S_k(u)\rangle (\partial_u S_k(u))\rangle, \quad (39)
\]

which explicitly connects the \( g_{uu} \) component of the cMERA metric with the entanglement generated at each step of the process.

An interesting corollary may be obtained from this result. Let us consider an observer (whose density matrix \( \rho(u) \) is given by eq. (20)), wishing to estimate the value of \( u \) through a measurement of the position operator \( X \) such that \( \langle X \rangle = \text{Tr} (\rho(u) X) \). An important result in information theory known as the Cramér-Rao bound, establishes that the lowest bound for \( \langle (\delta u)^2 \rangle = \text{Tr} (\rho(u) (X - u)^2) \) is given by,

\[
\langle (\delta u)^2 \rangle \geq \frac{1}{4 g_{uu}}. \quad (40)
\]

This states that the larger are the changes in the probability distributions along the \( u \)-coordinate (measured by the Fisher metric), the better are the estimations of the value of this coordinate. For the bosonic theory, the bound reads as,

\[
\langle (\delta u)^2 \rangle \geq \left| \partial_u S_k(u)\right|^{-2}. \quad (41)
\]

In the massless limit of this theory, both \( S_k(u) \) and \( \partial_u S_k(u) \) must be proportional to the central charge \( C \) of the theory so, \( \langle (\delta u)^2 \rangle \sim C^{-2}. \) Thus, if one conjectures that for the cMERA construction of theories with large \( C \), still holds that the Fisher information metric \( g_{uu} \propto |\partial_u S_k(u)|^{-2} \) (and hence eq. (41)), then, as a result, one gets that the estimation error \( \langle (\delta u)^2 \rangle \) would become largely suppressed for those theories. This seems to conform to the emergence of classical geometries in the large \( C \) limit, as stated by the AdS/CFT correspondence \([15]\).
cMERA gauge invariance and Fisher metric

Finally, we show how certain gauge invariance of the cMERA flow directly reflects on the invariance of \( g_{uu} \). To this end, let us first note that the cMERA evolution operator \( U(u_s|u_{IR}) = \exp \left( -i \int_{u_{IR}}^{u_s} \tilde{K}(\bar{u}) \, d\bar{u} \right) \) in equation (43), can be written as,

\[
U(u_s|u_{IR}) = U(u_s|u_s + \delta_u) \cdots (42)
\]

\[
U(u - \delta_u|u) \, U(u|u + \delta_u) \cdots \, U(u_{IR} - \delta_u|u_{IR}),
\]

where \( \delta_u = du \) and \( U(u|u + \delta_u) = \exp \left( -i \tilde{K}(u) \, \delta_u \right) \) corresponds to an infinitesimal layer of cMERA. It happens that the operator \( U(u_s|u_{IR}) \) remains invariant if one inserts the product \( G^\dagger(u) \, G(u) \) of a unitary scale-dependent gauge transformation \( G(u) \) and its inverse \( G^\dagger(u) \) in between any two layers,

\[
U(u - \delta_u|u) \, G^\dagger(u) \, G(u) \, U(u|u + \delta_u). \quad (43)
\]

One must also impose that \( G(u_{IR})|\Psi_{IR} \rangle = |\Psi_{IR} \rangle \) to guarantee that \( \langle \Phi(u) \rangle \) also remains invariant. The gauge transformed layer operator \( \tilde{U}(u|u + du) \) up to first order in \( du \) reads as,

\[
\tilde{U}(u|u + du) = \exp \left( -i du \left( G \, \tilde{K} \, G^\dagger + i \, G \, \partial_u \, G^\dagger \right) \right), \quad (44)
\]

where \( G = G(u) \), \( G^\dagger = G^\dagger(u) \), \( \tilde{K} = \tilde{K}(u) \). Thus, under a gauge transformation \( G(u) \), the entanglement generator of the cMERA flow \( \tilde{K}(u) \) transforms as,

\[
\tilde{K}'(u) = G(u) \, \tilde{K}(u) \, G^\dagger(u) + i \, G(u) \, \partial_u \, G^\dagger(u). \quad (45)
\]

Here we are interested in the class of gauge transformations which leaves the Fisher information metric \( g_{uu}(u) \) in (39) invariant. Formally, the Fisher metric is defined through,

\[
g_{uu} = \langle \partial_u \Phi | \partial_u \Phi \rangle - \langle \partial_u \Phi | \Phi \rangle \langle \Phi | \partial_u \Phi \rangle, \quad (46)
\]

where we have used the compressed notation \( |\Phi \rangle \equiv |\Phi(u) \rangle \) and \( |\Phi | \equiv \partial_u \Phi | \langle \Phi(u) \rangle = -i \tilde{K}(u) |\Phi(u) \rangle \). With this, \( g_{uu} \) can be written as the variance of \( \tilde{K}(u) \),

\[
g_{uu} = \langle \Phi(u) | \tilde{K}(u) |\Phi(u) \rangle - \langle \Phi(u) | \tilde{K}(u) |\Phi(u) \rangle^2. \quad (47)
\]

Now, regarding eq. (43), it is clear that \( g_{uu} \) will remain invariant under a gauge transformation \( G(u) \) of the cMERA flow provided that \( \tilde{K}(u) = \tilde{K}'(u) \). In this sense, let us focus on the gauge transformations \( G(u) = \exp \left( i \epsilon \, \mathcal{O}(u) \right) \) generated by a self-adjoint scale-dependent local operator \( \mathcal{O}(u) \), with \( \epsilon \) being a small parameter. Under these assumptions, the transformation in (43) can be written as,

\[
\tilde{K}'(u) = \tilde{K}(u) + \epsilon \, \nabla_u \mathcal{O}(u), \quad (48)
\]

\[
\nabla_u \mathcal{O}(u) = \partial_u \mathcal{O}(u) - i \left[ \tilde{K}(u), \mathcal{O}(u) \right]. \quad (49)
\]

We note that the equation which defines the cMERA flow for an operator such as \( \mathcal{O}(u) \), is similar to the equation of motion of an operator in the Heisenberg picture with respect to the \( u \)-dependent Hamiltonian \( \tilde{K}(u) \) [8]. This reads as,

\[
\partial_u \mathcal{O}(u) = i \left[ \tilde{K}(u), \mathcal{O}(u) \right], \text{ i.e., } \quad \nabla_u \mathcal{O}(u) = 0, \quad (50)
\]

which immediately implies both that, \( \tilde{K}'(u) = \tilde{K}(u) \) in (48) and, regarding eq. (47), the invariance of \( g_{uu} \) for this class of gauge transformations on the cMERA flow.

CONCLUSIONS

We have explicitly shown how the information metric emerging from a static cMERA state amounts to the differential generation of entanglement entropy along the renormalization group flow. We also characterized a class of gauge transformations of this flow which leaves the metric invariant. Further investigations should address non-stationary settings such as quantum quenches. Entanglement renormalization deals with time dependent states [10] but tackling space and time on quite different grounds. It is worth to investigate if the analysis of the entanglement flow in these states could provide some light in order to formulate a time-dependent cMERA in a covariant way. Finally, it would be also worth to clarify if it is possible to generate cMERA states/cMERA entanglement flows, compatible with information metrics which extremize a (gravitational-like) action functional.

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The state $|\Psi_{IR}\rangle$ has no real space entanglement, i.e., is a completely unentangled state in case of massive theories. When considering a massless CFT, this state coincides with the vacuum $|0\rangle$ of the theory which is an entangled state.