BI-TRACEABLE GRAPHS, THE INTERSECTION OF THREE LONGEST PATHS AND HIPPCHEN’S CONJECTURE

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Abstract. Let $P, Q$ be longest paths in a simple graph. We analyze the possible connections between the components of $P \cup Q \setminus (V(P) \cap V(Q))$ and introduce the notion of a bi-traceable graph. We use the results for all the possible configurations of the intersection points when $|V(P) \cap V(Q)| \leq 5$ in order to prove that if the intersection of three longest paths $P, Q, R$ is empty, then $|V(P) \cap V(Q)| \geq 6$. We also prove Hippchen’s conjecture for $k \leq 6$: If a graph $G$ is $k$-connected for $k \leq 6$, and $P$ and $Q$ are longest paths in $G$, then $|V(P) \cap V(Q)| \geq 6$.

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Introduction

Different configurations of the intersection points of longest paths in simple graphs have been analyzed by various authors with different purposes. Axenovich in $[1]$ used two configurations $Q_1$ and $Q_2$ that described forbidden connections in $G \setminus V(P) \cap V(Q)$, in order to prove that in outerplanar graphs every three longest paths share a common point. Hence, for outerplanar graphs the following conjecture is true.

Conjecture 1: For every connected graph, any three of its longest paths have a common vertex.

In $[5]$ this conjecture is proven for a broader class of graphs and in $[6]$ an approach to this conjecture using certain distance functions is given. It is known that two longest paths in a

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connected graph have necessarily a common vertex, but there are connected graphs, where seven longest paths do not have a common vertex (See for example [10]). The equivalent question for 3, 4, 5 and 6 longest paths is still open. See [11] for a survey on this and similar problems.

On the other hand Hippchen in [9] analyzed forbidden connections in $G \setminus V(P) \cap V(Q)$ when $|(V(P) \cap V(Q))| = 2$ in order to prove that in 3-connected graphs the intersection of two longest path contains at least 3 vertices. He then stated the following conjecture:

**Conjecture 2:** The intersection of two longest paths in a $k$-connected simple graph has cardinality at least $k$.

This conjecture was an adaptation of a similar conjecture for longest cycles instead of paths, which appeared first in [7, p.188] and was attributed to Scott Smith, and where some partial results are known (See for example [8], [12] and [3]).

In [8] the first author proved Hippchen’s conjecture for $k = 4$, and in [4] Hippchen’s conjecture is proven for $k = 5$. In both cases the possible configurations and forbidden connections of the graph induced by $V(P) \cap V(Q)$ is analyzed.

The configuration of the intersection vertices of longest cycles (instead of paths) has been considered for example in [2], [7] and [12]. Babai [2, Lemma 2] uses a forbidden connection in order to prove that in a 3-connected graph, any two longest cycles have at least three points in common. Groetschel in [7] proved that the complement of the intersection of two longest cycles is not connected when there are at most 5 intersection points, Steward and al. used a computer program in [12] to analyze the different configurations of the intersection points in order to show that this remains true for $k = 6, 7$ (for $k = 8$ there is a counterexample).

In the present paper we will make a complete analysis of the possible configurations of the intersection points of two longest paths $P$ and $Q$, when there are $\ell \leq 5$ intersection points. We will analyze the different possibilities for the sequential order of the intersection vertices in each of the two longest paths $P$ and $Q$, which is given by a permutation of the $\ell$ intersection vertices. Two permutations give the same configuration if they can be transformed into each other by exchanging $P$ and $Q$, or by changing the direction in which we travel through the points in each path.

If the intersection is a single point or has two points, there is only one configuration. If the intersection has three points, there are two configurations, already described in [8]. If the intersection has 4 points, then the 24 different permutations give 7 different configurations, which correspond to 7 different cases described in [4]. When the intersection has 5 points, we find 23 different cases arising from the 120 different permutations.

For each configuration we analyze in detail the possible connections between the components of the graph $(P \cup Q) \setminus (V(P) \cap V(Q))$. We find a large class of graphs $P \cup Q$, such that in each exterior swap unit (see Definition 2.9) $P \setminus (V(P) \cap V(Q))$ cannot be connected with $Q \setminus (V(P) \cap V(Q))$ outside $V(P) \cap V(Q)$ (See Proposition 2.14 and Remark 2.15). In these graphs, $V(P) \cap V(Q)$ is a separator, if we additionally assume that $V(P) \neq V(Q)$. For graphs not in this class we analyze in detail whether two components of $(P \cup Q) \setminus (V(P) \cap V(Q))$ can be connected such that $P$ and $Q$ are still longest paths.

We will use our results in order to approach three different problems on longest paths, the intersection of three longest paths, Hippchen’s conjecture and a path version of Groetschel’s result on the connectedness of the complement of the intersection of two longest paths. One of our main results is the following result on a possible counterexample of Conjecture 1 in Theorem 6.5:

*If the intersection of three longest paths $P, Q, R$ is empty, then $|(V(P) \cap V(Q))| \geq 6$.***
By symmetry, this result implies that in this case we also have \( \#(V(R) \cap V(Q)) \geq 6 \) and \( \#(V(P) \cap V(R)) \geq 6 \). Hence we can rephrase our result in the following way:

Let \( P, Q, R \) be three longest paths in a graph. If one of \( V(P) \cap V(Q), V(P) \cap V(R) \) or \( V(Q) \cap V(R) \) has less than 6 points, then the three paths have a common vertex.

On the other hand, if Conjecture 1 is false and there exist three longest paths in a graph with empty intersection, then with our methods one can carry out a systematic search for a counterexample.

As second main result of the present paper we prove Hippchen’s conjecture for \( k = 6 \) (Corollary 5.3) and give a new proof for \( k = 5 \) (Corollary 5.5). The same methods should be useful in order to prove Hippchen’s conjecture for higher \( k \), or to prove it completely.

The third problem to which we apply our methods is the path-version of [7, Theorem 1.2(a)] (see also [12]). We arrive at the following results (Theorem 5.7 and Corollary 5.8):

\[ [\text{Gr}(\ell \leq 5)] \text{ Assume that } P \text{ and } Q \text{ are two longest paths in a simple graph } G. \text{ If } V(Q) \neq V(P) \text{ and } V(P) \cap V(Q) \text{ has cardinality } \ell \leq 5, \text{ then it is a separator (called an articulation set in } [7]), \text{ which means that the complement is not connected.} \]

\[ [\text{Gr}(n \leq 7)] \text{ If } V(Q) \neq V(P) \text{ and } n = |V(G)| \leq 7 \text{ then } V(Q) \cap V(P) \text{ is a separator.} \]

Open question: Which are the maximal \( \ell \) and \( n \) such that the above results remain true? In an hypotraceable graph we can choose two longest paths \( P \) and \( Q \) that leave out two connected vertices, and so \( V(Q) \cap V(P) \) is a not separator in that case. Since there exists a hypotraceable graph on 34 vertices (see [13]), we know automatically that \( n_{\text{max}} \leq 33 \) and \( \ell_{\text{max}} \leq 31 \).

Now consider the following simple graph with 11 vertices, that has two longest path \( P \) and \( Q \) of length 9, which satisfy \( V(Q) \neq V(P) \) and moreover, the complement of \( V(Q) \cap V(P) \) is connected. Since \( \#(V(P) \cap V(Q)) = 9 \) we have \( n = 11 \) and \( \ell = 9 \).

\[
\text{Simple graph, } n = 11 \text{ vertices} \quad \text{Two longest paths, } \ell = \#(V(P) \cap V(Q)) = 9
\]

Thus \( n_{\text{max}} \leq 10 \) and \( \ell_{\text{max}} \leq 8 \). Moreover, we have verified all cases up to \( n = 10 \) and will write down the lengthy computations in a systematic way in a forthcoming article. This implies that \( n_{\text{max}} = 10 \) and by the above result we already know that \( 5 \leq \ell_{\text{max}} \leq 8 \).

The article is organized as follows. In the first section we prove Theorem 1.2 which illustrates our method in a simple case. The theorem says that if the intersections vertices of two longest path \( P \) and \( Q \) have the same sequential order in \( P \) and in \( Q \), then \( V(P) \cap V(Q) \) is a separator, thus \( P \) and \( Q \) cannot be a counterexample to the Hippchen conjecture.

In the second section we introduce our main new tools: bi-traceable (BT) graphs, the standard representations \( BT(P, Q) \) of these graphs and what we call exterior swap units (ESU) (see Definition 2.1). We also introduce in Definitions 2.2, 2.12 and 2.17 three different conditions on how you can connect, or rather on how you cannot connect, two components of \( (P \cup Q) \backslash (V(P) \cap V(Q)) \). They can be not directly connectable (NDC), non connectable (NC) or locally non connectable (LNC). We also prove that if \( V(P) \neq V(Q) \) and if all ESU’s in \( BT(P, Q) \)
are NC, then $V(P) \cap V(Q)$ is a separator. In the third section we prove that for every graph $BT(P,Q)$ with $\ell \leq 4$ all ESU’s are NC, which reproves the Hippchen conjecture for $k = 5$. In section 4 we analyze the case $\#(V(P) \cap V(Q)) = 5$ and find that in all but three exceptional cases all ESU’s are NC. In section 5 we prove that in these three cases $V(P) \cap V(Q)$ is a separator, proving the Hippchen conjecture for $k = 6$. In the last section we prove that in none of the three cases there can be a third longest path $R$ with $P \cap Q \cap R = \emptyset$, which proves that if the intersection of three longest paths $P, Q, R$ is empty, then $\#(V(P) \cap V(Q)) \geq 6$.

If one wants to continue our analysis for the case where the intersection consists of 6 points, it will be necessary to make a computer assisted analysis of the 720 permutations and the approximately 100 configurations resulting from them, and find the exceptional configurations. The methods developed in the present paper will be useful in order to treat exceptional cases that will arise for $\ell \geq 6$. The complete tables of possible configurations can be useful for other problems involving intersections of longest paths.

1 Preliminaries and an illustrative example

Along the present paper $G$ is a simple graph, and $P, Q$ are longest paths. By definition, a path cannot pass twice through the same vertex, and the length of a path is the number of its edges. We will construct a large class of graphs $P \cup Q$, such that in each exterior swap unit (see Definition 2.3) $P \setminus (V(P) \cap V(Q))$ cannot be connected with $Q \setminus (V(P) \cap V(Q))$ outside $V(P) \cap V(Q)$ (See Proposition 3.1 and Remark 3.15). As usual, removing the vertices also removes the incident edges. In these graphs, $V(P) \cap V(Q)$ is a separator, if we additionally assume that $V(P) \neq V(Q)$.

We define an operation $\cup''$ on two paths $P, Q$, which is defined only if $P$ and $Q$ share exactly one endpoint, and $P + Q$ is simply the union of both paths. If we write $P + Q + R$, then $P$ and $Q$ share one endpoint, and the other endpoint of $Q$ is also an endpoint of $R$, and $P + Q + R$ is the union of the three paths.

**Notation 1.1.** Along this work we assume that $\#(V(P) \cap V(Q)) = \ell$ for some $\ell \geq 1$. If we write $P = P_0 + P_1 + \cdots + P_\ell$ such that $P_i \cap P_{i-1} = \{a_i\}$, and $\{a_1, \ldots, a_\ell\} = V(P) \cap V(Q)$, and we write $Q = Q_0 + Q_1 + \cdots + Q_\ell$ such that $Q_i \cap Q_{i-1} = \{b_i\}$, and $\{b_1, \ldots, b_\ell\} = V(P) \cap V(Q)$, then the two paths $P, Q$ determine a permutation $\sigma$ on $\{1, 2, \ldots, \ell\}$ given by $b_j = a_{\sigma(j)}$.

Write $Q'_i := Q_i \setminus (V(P) \cap V(Q))$ and $P'_i := P_i \setminus (V(P) \cap V(Q))$. These are the components of $P \cup Q \setminus (V(P) \cap V(Q))$. Note that $P'_0 = \emptyset$ if and only if $P_0 = \{a_1\}$, i.e., $L(P_0) = 0$. The same holds for $Q'_0, P'_i$ and $Q'_i$. On the other hand if $i \notin \{0, \ell\}$, then $P'_i = \emptyset$ if and only if $L(P'_i) = 1$ and the same holds for $Q'_i$.

Let

$$X, Y \in \{P'_i, Q'_i : P'_i \neq \emptyset, Q'_i \neq \emptyset\}.$$  

A direct connection between $X$ and $Y$ is a path $R$ from $X$ to $Y$ internally disjoint from $V(P) \cup V(Q)$. We write $X \sim R Y$ (Note that this is not an equivalence relation). Clearly such a path $R$ cannot be part of a graph where $P$ and $Q$ are longest paths, if there is a path in $P \cup Q \cup R$, which is longer than $P$ (or $Q$).

The case in which $\sigma = Id$, is the prototype of the large class of graphs mentioned above. Our principal ideas and methods are already present in this case, so we will give a detailed proof and describe an example with $\ell = 7$.

**Theorem 1.2.** Let $G$ be a simple graph, and let $P, Q$ be longest paths in $G$. Assume that $a_i = b_i$ for all $i$ (which means that $\sigma = Id$) and that $V(P) \neq V(Q)$. Then the complement of $V(P) \cap V(Q)$ cannot be connected.
Proof. Assume by contradiction that it is connected. Since \( V(P) \neq V(Q) \), there exists \( i_0 \) such that \( P_{i_0}' \neq \emptyset \), and it follows that \( Q_{i_0}' \neq \emptyset \), since \( L(P_{i_0}) = L(Q_{i_0}) \). Consequently there exist
\[
X_1, \ldots, X_r \in \{P_i', Q_i' : P_i' \neq \emptyset, Q_i' \neq \emptyset\},
\]
such that \( X_i \sim X_{i+1} \) and \( X_1 = P_{i_0}' \) and \( X_r = Q_{i_0}' \).

Let \( \hat{X}_i = \begin{cases} P_j, & \text{if } X_i = Q_j \\ Q_j, & \text{if } X_i = P_j \end{cases} \). Set
\[
j_0 = \min\{j > 1, \hat{X}_j \in \{X_1, \ldots, X_{j-1}\}\}.
\]

We can assume that \( X_j \subset P \) for \( j < j_0 \). In fact, if \( X_j = Q_j \), we redefine \( P \) as \( P_{<i} + Q_{j} + P_{>i} \) and \( Q \) as \( Q_{<i} + P_{j} + Q_{>i} \).

There exists \( i_1 \) such that \( X_{j_0} = Q_{i_1} \) and so there is a path \( R \) from \( P_{i_1} \) to \( Q_{i_1} \) internally disjoint from \( Q \) and from \( P_{i_1} \). The endpoints of \( R \) split the subpaths \( Q_{i_1} \) and \( P_{i_1} \) into two subpaths each, which we name \( P_{i_1,1}, P_{i_1,2}, Q_{i_1,1}, Q_{i_1,2} \). The two paths
\[
Q_{<i_1} + Q_{i_1,1} + R + P_{i_1,2} + Q_{>i_1} \quad \text{and} \quad Q_{<i_1} + P_{i_1,1} + R + Q_{i_1,2} + Q_{>i_1}
\]
have lengths that sum \( 2L(Q) + 2L(R) \), a contradiction that concludes the proof. \( \square \)

Example 1.3. We will illustrate the proof of the theorem in an example with \( \ell = 7 \). The green path is \( P \), the red path is \( Q \) and the blue paths are in \( G \setminus (P \cup Q) \), with endpoints in \( P \cup Q \setminus (V(P) \cap V(Q)) \).
In this example $i_0 = 2$, since the additional blue paths connect $P_2$ with $Q_2$. Moreover
\[ X_1 = P'_2, \quad X_2 = P'_4, \quad X_3 = P'_3, \quad X_4 = P'_5, \quad X_5 = Q'_6, \quad X_6 = Q'_4, \quad X_7 = Q'_2, \]
and so
\[ \hat{X}_1 = Q'_2, \quad \hat{X}_2 = Q'_4, \quad \hat{X}_3 = Q'_3, \quad \hat{X}_4 = Q'_5, \quad \hat{X}_5 = P'_6, \quad \hat{X}_6 = P'_4, \quad \hat{X}_7 = P'_2. \]
We have $j_0 = 6$, since
\[ \hat{X}_2 \notin \{X_1\}, \quad \hat{X}_3 \notin \{X_1, X_2\}, \quad \hat{X}_4 \notin \{X_1, X_2, X_3\}, \quad \hat{X}_5 \notin \{X_1, X_2, X_3, X_4\}, \]
but
\[ \hat{X}_6 = P'_4 = X_2 \in \{X_1, X_2, X_3, X_4, X_5\}. \]
Note that we also have $\hat{X}_7 = P'_2 = X_1 \in \{X_1, X_2, X_3, X_4, X_5, X_6\}$. Since $X_5 = Q'_6$, we redefine $P$ and $Q$, swapping $Q_6$ with $P_6$.

Then $i_1 = 4$, since $X_{j_0} = X_6 = Q'_4$, and we arrive at the following situation, where $P'_{i_1} = P'_4$ is connected with $Q'_{i_1} = Q'_4$ via a path $R$ that is internally disjoint from $Q$ and from $P_4 \cup Q_4$.

As in the proof of the theorem, now we can construct the two paths, whose length sum $2L(Q) + 2L(R)$.

We want to generalize the theorem in order to apply to other permutations, and not only for $\sigma = Id$. Note that in the proof of the theorem we use the following properties of the pair $\{P_i, Q_i\}$:

- $L(P_i) = L(Q_i)$,
- They share both endpoints, when $1 < i < \ell$, and one endpoint when $i \in \{0, \ell\}$.
- Thus they can be interchanged, i.e., $P_{<i} + Q_i + P_{>i}$ and $Q_{<i} + P_i + Q_{>i}$ are two longest paths having the same intersection and the same union as $P$ and $Q$.
- An internally disjoint path $R$ between $P'_{i_1}$ and $Q'_{i_1}$ when $0 < i < \ell$, enables to construct the two paths $P_{i_1} + R + Q_{i_2}$ and $Q_{i_1} + R + P_{i_2}$ such that both paths connect the endpoints of $P_1$. Moreover, the lengths of the new paths sum $2L(P_i) + 2R > L(Q_i) + L(P_i)$. 

\[ R \]
2 Bi-traceable graphs

We want to consider the graph which is the union of the two longest paths $P$ and $Q$. In this section we assume the same notations as in Notation 1.1.

Definition 2.1. 
(1) A bi-traceable graph (BT-graph) is a simple graph that has two longest paths such that the union is the whole graph.

(2) The representation of a bi-traceable graph associated to $P$ and $Q$ is a two colored graph $BT(P, Q)$, obtained from the BT-graph $P \cup Q$ by the following process. First we assign to all the edges of $P$ one color. Then the edges of $Q$ are colored with the other color. If the paths share an edge, then this edge is duplicated, and each copy is colored with one of the colors. The vertices are not colored.

Note that the resulting graph is no longer simple, but $P$ and $Q$ are still longest paths in it. We also have $BT(P, Q) = BT(Q, P)$.

(3) The subpaths $P_i$, $Q_i$ are called the completed components of $BT(P, Q)$, or simply the completed components, if $P$ and $Q$ are clear from the context. Note that each of the extremal completed components $P_0$, $Q_0$, $P_\ell$ and $Q_\ell$ may be only a point.

Note that some components $P'_i$ or $Q'_i$ might be empty.

Definition 2.2. Two components $X$, $Y$ are said to be not directly connectable (NDC), if one of them is empty, or if $X \sim_R Y$ implies that there exist two paths $\hat{P}$ and $\hat{Q}$ in $BT(P, Q) \cup R$ such that

$$L(\hat{P}) + L(\hat{Q}) = 2L(R) + L(P) + L(Q) = 2L(P) + 2L(R).$$

The following result is one of the main ingredients in the proof of the Hippchen conjecture for $k = 4$ in [8], and the idea was already present in [9, Lemma 2.2.3].

Lemma 2.3 ([8, Lemma 4.1]). If two completed components of different colors are adjacent (which means that they have a common vertex), then they are NDC.

Proof. See the diagram, or see also [8, Lemma 4.1].

Lemma 2.4. Two extremal components of different colors are NDC. The extremal components are $P'_0$, $P'_\ell$, $Q'_0$, $Q'_\ell$.

Proof. Assume for example $P'_0 \sim_R Q'_\ell$. Let $P_{0,1} + P_{0,2} = P_0$ such that $P_{0,1} \cap P_{0,2}$ is one endpoint of $R$ and such that $P_{0,1}$ and $P$ share one endpoint. Similarly define $Q_{\ell,1}$ and $Q_{\ell,2}$, with $Q_{\ell,2}$ having a common endpoint with $Q$. Then

$$\hat{P} = Q_{\ell,2} + R + P_{0,2} + P_1 + \cdots + P_\ell$$

and

$$\hat{Q} = P_{0,1} + R + Q_{\ell,1} + Q_{\ell-1} + \cdots + Q_1 + Q_0.$$
Let $G$ be a graph, $P$ and $Q$ be longest paths. Assume that $BT(P, Q)$ has at least two non-empty components of different colors. If all pairs of different colors in $BT(P, Q)$ are NDC, then $V(P) \cap V(Q)$ is a separator of $G$. Verifying this condition for all pairs of different colors in small graphs is a manageable task, but in order to obtain results generalizing Theorem 1.2 we need a notion of “connectable” with a broader scope. For this we formalize the process of swapping colors of some completed components in the proof of Theorem 1.2 and we analyze the block structure of $BT(P, Q)$.

**Definition 2.5.** A **swap unit in** $BT(P, Q)$ is a set of completed components of two colors, such that if we swap the colors in the given set of completed components, then we obtain a new representation $BT(\tilde{P}, \tilde{Q})$ of the original $BT$-graph.

**Definition 2.6.** An **internal building block (IBB)** of $BT(P, Q)$ is a 2-connected subgraph which is the union of two subpaths $\tilde{P}$ of $P$ and $\tilde{Q}$ of $Q$, such that $\tilde{P}$ and $\tilde{Q}$ have the same endpoints, and don’t intersect other subpaths of $P$ and $Q$ other than in these endpoints, which are called the endpoints of the IBB.

An **elementary IBB** is the union of two completed components that share both endpoints.

For example, if the $P$ and $Q$ share one edge, then the duplicated edge is an elementary IBB.

Note that every IBB is a swap unit. In fact, write $P = P_1 + P_2 + P_3$ and $Q = Q_1 + Q_2 + Q_3$, where $P_2$ and $Q_2$ are the subpaths spanning the building block. Swapping the colors generates two longest paths

$$\tilde{P} = P_1 + Q_2 + P_3 \quad \text{and} \quad \tilde{Q} = Q_1 + P_2 + Q_3,$$

such that the union is still $BT(P, Q)$ and so we obtain a new representation $BT(\tilde{P}, \tilde{Q})$. Hence, if $B$ is an IBB, the subpath of $P$ and the subpath of $Q$ have the same length, which we call the length of the block $L(B)$.

**Remark 2.7.** An IBB with endpoints $a$ and $b$, can be defined independently of $BT(P, Q)$ as the representation of a bitraceable graph with fixed endpoints $a$ and $b$, where such a graph is
generated by two paths from $a$ to $b$ that have the same length, and such that there is no longer path from $a$ to $b$.

If the intersection of two IBB’s is one common endpoint, then the concatenation of the IBB’s is their union, and similarly we can concatenate three or more IBB’s.

**Definition 2.8.** An extremal building unit of $BT(P,Q)$ is a subgraph which is the union of two extremal subpaths $\tilde{P}$ of $P$ and $\tilde{Q}$ of $Q$, such that $\tilde{P}$ and $\tilde{Q}$ have exactly one endpoint in common, and this common endpoint is neither an endpoint of $P$ nor of $Q$. We also require that $\tilde{P}$ and $\tilde{Q}$ don’t intersect other subpaths of $P$ and $Q$.

An extremal building block (EBB) of $BT(P,Q)$, is a minimal extremal building unit. This means that it is an extremal building unit, which is not the concatenation of one or more IBB’s with another extremal building unit.

An elementary EBB is an EBB, which is the union of two extremal completed components that share one endpoint.

A building block (BB) is an IBB, an EBB or all of $BT(P,Q)$, if $BT(P,Q)$ is not the concatenation of IBB’s and/or EBB’s, and $P$ and $Q$ have no common endpoints.

![EBB with two embedded IBB’s](image1)

Elementary EBB

$BT(P,Q)$ is a BB

Note that an IBB can be contained in another BB, and that BB’s are also swap units. Note also that in the case $\sigma = Id$, for each $0 < i < \ell$ the union of $P_i$ and $Q_i$ is an IBB, and the union of $P_0$ and $Q_0$ and the union of $P_\ell$ and $Q_\ell$ are the EBB’s.

**Definition 2.9.** Given a building block, the exterior swap unit (ESU) associated with the building block, is the union of all completed components in the building block, that are not contained in any embedded IBBs.

Note that the ESU of an elementary IBB is the whole IBB and the same holds for an elementary EBB.

The following representation of a BT-graph is the concatenation of three IBB’s, two of them are elementary IBB’s and the middle one contains five elementary IBBs. Hence $BT(P,Q)$ has seven ESU’s of two completed components each and one ESU consisting of 8 completed components (it has ten edges).
Note that swapping the color of the edges in a swap unit doesn’t change the intersection vertices, since each intersection vertex has two incident edges of each color. After the swapping it must still have two incident edges of each color, since otherwise one of the paths would visit this vertex twice.

**Lemma 2.10.** In a swap unit, the sum of the lengths of the completed components of one color equals the sum of the lengths of the completed components of the other color. Moreover, the number of components of each color coincide.

**Proof.** The sum of the lengths coincide, since otherwise one of the new paths resulting from the swap would be longer. The same holds for the number of components, since swapping the color of the edges doesn’t change the intersection vertices, and so the number of components of each longest path remains constant.

**Remark 2.11.** Note that the union of the ESU’s in $BT(P, Q)$ is $BT(P, Q)$ and that no completed component is in two ESU’s at the same time.

**Definition 2.12.** Let $X$ and $Y$ be components of different colors in an ESU. The pair $X, Y$ is **non connectable (NC)**, if a path $R$ that connects $X$ with $Y$, satisfying $R \cap V(P) \cap (Q) = \emptyset$ and such that $R$ is internally disjoint from the given ESU and in each of the other ESU’s touches at most one color, allows to construct two paths $\hat{P}$ and $\hat{Q}$ in $BT(P, Q) \cup R$ such that

$$L(\hat{P}) + L(\hat{Q}) = 2L(R) + L(P) + L(Q) = 2L(P) + 2L(R).$$

The ESU is called NC, if all the pairs of component of the ESU of different colors are NC.

**Proposition 2.13.** The components of an elementary IBB are NC.

**Proof.** Let $R$ be a path that connects one component of one color in the ESU with one component of another color in the ESU, such that $R$ is internally disjoint from the given ESU and in each of the other ESU’s touches at most one color.
Then $R$ is internally disjoint from $\tilde{Q}$, which is the path obtained from $Q$ by swapping the colors in the ESU’s where $R$ touches only the color of $Q$.

Write $\tilde{Q}$ as $\tilde{Q} = \tilde{Q}_1 + Q_i + \tilde{Q}_2$, where $Q_i$ is the subpath of $Q$ in the given ESU, and assume that $P_j$ is the subpath of $P$ in the given ESU. The endpoints of $R$ split the subpaths $Q_i$ and $P_j$ into two subpaths each, which we name $P_{j,1}, P_{j,2}, Q_{i,1}, Q_{i,2}$. The two paths

$\hat{Q} = \tilde{Q}_1 + Q_{i,1} + R + P_{j,2} + \tilde{Q}_2$ and $\hat{P} = \tilde{Q}_1 + P_{j,1} + R + Q_{i,2} + \tilde{Q}_2$

$\hat{Q} = \tilde{Q}_1 + Q_{i,1} + R + P_{j,2} + \tilde{Q}_2$ in red

$\hat{P} = \tilde{Q}_1 + P_{j,1} + R + Q_{i,2} + \tilde{Q}_2$ in green
have lengths that sum $2L(\bar{Q}) + 2L(R) = 2L(P) + 2L(R)$, as desired. □

**Proposition 2.14.** Let $G$ be a graph and $P, Q$ longest paths. If all the ESU’s in $BT(P, Q)$ are NC, then there can be no path in $G \setminus (V(P) \cap V(Q))$ from a component of one color in one ESU to a component of the other color in the same ESU.

*Proof.* Assume by contradiction that such a path from $P'_p$ to $Q'_q$ exists. Then there exist $X_1, \ldots, X_r \in \{P'_p, Q'_q : P'_p \neq \emptyset, Q'_q \neq \emptyset\}$, such that $X_i \sim X_{i+1}$ and $X_i = P'_p$ and $X_r = Q'_q$.

Let $i_0 < j_0$ be such that $X_{i_0}$ and $X_{i_0}$ are in the same ESU and have different colors, and such that

$$j_0 - i_0 = \min\{j - i : i < j, X_j \text{ and } X_i \text{ are in the same ESU and have different colors}\}.

Then the subpath $R'$ of $R$ which connects successively $X_{i_0}, X_{i_0+1}, \ldots, X_{j_0}$, is internally disjoint from the ESU of $X_{i_0}$ and $X_{i_0}$ and in each of the other ESU’s touches at most one color. Since the ESU of $X_{j_0}$ and $X_{i_0}$ is NC, by Definition 2.12 there exists a path in $BT(P, Q) \cup R$ that is longer than $P$. This path determines a path of the same length in $G$, a contradiction that concludes the proof. □

**Remark 2.15.** Proposition 2.14 characterizes the large class of graphs $BT(P, Q)$, in which all the ESU’s are NC. When $\#(V(P) \cap V(Q))$ is small, in most cases $BT(P, Q)$ is in this class, in particular we will show that this class includes all $BT(P, Q)$ such that $\#(V(P) \cap V(Q)) \leq 2$.

**Proposition 2.16.** Let $G$ be a graph and $P, Q$ longest paths. If $V(P) \neq V(Q)$ and all the ESU’s in $BT(P, Q)$ are NC, then $V(P) \cap V(Q)$ is a separator of $G$.

*Proof.* Since $V(P) \neq V(Q)$, there exists $i_0$ such that $P'_i \neq \emptyset$. Every $P'_i$ is part of an ESU in $BT(P, Q)$, and by Lemma 2.10 there is a $Q'_i \neq \emptyset$ in that ESU. By Proposition 2.14 these components cannot be connected in $G \setminus (V(P) \cap V(Q))$, so $V(P) \cap V(Q)$ is a separator of $G$, as desired. □

**Definition 2.17.** a) Let $B$ be an BB in $BT(P, Q)$ and let $X, Y$ be components of different colors of the corresponding ESU. The pair $X, Y$ is called **locally non connectable (LNC)**, if for any $x \in X$ and $y \in Y$, there exist two pairs of disjoint paths in $B$: one pair of disjoint paths, $X_1$ from $x$ to one endpoint of $B$, and $Y_1$ from $y$ to another endpoint of $B$; and another pair of disjoint paths, $X_2$ from $x$ to one endpoint of $B$, and $Y_2$ from $y$ to another endpoint of $B$, such that

$$X_1 \cup Y_1 \cup X_2 \cup Y_2 = B,$$

and such that the intersection of each of $X_1, X_2, Y_1, Y_2$ with an embedded ESU is either empty, or is equal to the intersection of the embedded ESU with $P$ or with $Q$.

\[\begin{align*}
&X_1 \text{ and } Y_1 \text{ thickened} \\
&X_2 \text{ and } Y_2 \text{ in blue}
\end{align*}\]

b) The ESU of a building block $B$ is called LNC, if all the pairs of component of the ESU of different colors are LNC. The building block $B$ is called LNC, if its ESU and the ESU’s of all embedded IBB’s are LNC.
Remark 2.18. Note that if one of the paths of the pair \( X_1, Y_1 \) has an edge contained in one of the embedded IBB’s, then it goes from one endpoint of the IBB to the other, since by definition an IBB can touch the rest of \( BT(P, Q) \) only at its endpoints. Consequently, the other path in the pair cannot touch this embedded IBB. The same holds for the pair \( X_2, Y_2 \). Since the union is the whole block, each embedded IBB has to be travelled twice, and so in each of the two pairs one of the paths has to go through the given embedded IBB.

Proposition 2.19. If a pair of completed components of different colors in an ESU is LNC, then it is NC. Consequently, if an ESU is LNC, then it is NC.

Proof. Take a pair of components of different colors \( X = P_i \) and \( Y = Q_j \) of a given ESU. Assume that there exists a path \( R \) that connects \( X \) and \( Y \), such that \( R \) is internally disjoint from the given ESU and in each of the other ESU’s touches at most one color. Then \( R \) is internally disjoint from \( Q \), which is the path obtained from \( Q \) by swapping the colors in the ESU’s where \( R \) touches only the color of \( Q \). Let \( x, y \) be the endpoints of \( R \) in \( P_i \) and \( Q_j \), respectively, and let \( X_1, X_2, Y_1, Y_2 \) be as in Definition 2.17(a).

Let \( \tilde{X}_1 \) be the path obtained from \( X_1 \) by swapping the ESU’s of the embedded IBB’s, where \( R \) touches \( X_1 \). This means that if \( R \) touches \( X_1 \) in the ESU of an embedded block in a certain color, then, since all the completed components of \( X_1 \) in this ESU have this one color, we can replace these completed components of \( X_1 \) with the completed components of the ESU of the other color, which are not touched by \( R \), and we have still a path. Similarly we define \( \tilde{X}_2, \tilde{Y}_1 \) and \( \tilde{Y}_2 \). By Remark 2.18 these are paths that have the same endpoints and the same length as the original ones, \( \tilde{X}_1 \) and \( \tilde{Y}_1 \) are disjoint, and \( \tilde{X}_2 \) and \( \tilde{Y}_2 \) are disjoint. Hence

\[
\hat{P} = \tilde{X}_1 + R + \tilde{Y}_1 \quad \text{and} \quad \hat{Q} = \tilde{Y}_2 + R + \tilde{X}_2,
\]

are two paths such that

\[
L(\hat{P}) + L(\hat{Q}) = 2L(B) + 2L(R),
\]

and such that the set of the endpoints of the paths coincides with the set of the endpoints of \( B \). If the building block \( B \) is all of \( BT(P, Q) \), then this finishes the proof, since then \( L(P) = L(B) \). If \( B \) is an EBB or an IBB, then we extend \( \hat{P} \) and \( \hat{Q} \) using \( Q \). For this we write \( \hat{Q} = \hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 \), where \( \hat{Q}_2 \) is the intersection of \( \hat{Q} \) with the given block. Then we set

\[
\hat{\hat{P}} := \hat{Q}_1 + \hat{P} + \hat{Q}_3 \quad \text{and} \quad \hat{\hat{Q}} := \hat{Q}_1 + \hat{Q} + \hat{Q}_3,
\]

in order to obtain two paths \( \hat{\hat{P}} \) and \( \hat{\hat{Q}} \) such that

\[
L(\hat{\hat{P}}) + L(\hat{\hat{Q}}) = 2L(P) + 2L(R),
\]

which concludes the proof. \( \square \)

Theorem 2.20. Let \( G \) be a graph and let \( P, Q \) be two longest paths. If \( V(P) \neq V(Q) \) and all the ESU’s in \( BT(P, Q) \) are LNC, then \( V(P) \cap V(Q) \) is a separator of \( G \).

Proof. By Propositions 2.16 and 2.19. \( \square \)

Our next goal is to construct recursively new LNC building blocks out of some given LNC building blocks. For this we first generalize Lemmas 2.3 and 2.4.

Proposition 2.21. If two completed components \( X, Y \) of different colors in an ESU are adjacent, then they are LNC.
Proof. Let one of the subpaths spanning the block be \( \tilde{P}_1 + X + \tilde{P}_2 \) and the other \( \tilde{Q}_1 + Y + \tilde{Q}_2 \), where one common vertex is \( u = X \cap \tilde{P}_2 = Y \cap \tilde{Q}_1 \).

Let \( x \in X \) and \( y \in Y \) be internal vertices, and write \( X = \tilde{X}_1 + \tilde{X}_2 \), with \( x = \tilde{X}_1 \cap \tilde{X}_2 \) and similarly \( Y = \tilde{Y}_1 + \tilde{Y}_2 \). Then

\[
X_1 = \tilde{P}_1 + \tilde{X}_1, \quad X_2 = \tilde{Q}_1 + \tilde{X}_2, \quad Y_1 = \tilde{P}_2 + \tilde{Y}_1 \quad \text{and} \quad Y_2 = \tilde{Q}_2 + \tilde{Y}_2.
\]

\( X = \tilde{X}_1 + \tilde{X}_2 \) and \( Y = \tilde{Y}_1 + \tilde{Y}_2 \) are adjacent satisfy the conditions of Definition 2.17. Note that if the common vertex is an end point of the building block, then \( \tilde{P}_2 \) and \( \tilde{Q}_1 \) have length zero.

\( \square \)

Proposition 2.22. If two extremal components of \( P \) and \( Q \) are disjoint and in the same BB, then they are LNC.

Proof. The proof is similar to the proof of Lemma 2.4. Let

\[
\tilde{P} + \tilde{X}_1 + \tilde{X}_2 \quad \text{and} \quad \tilde{Q} + \tilde{Y}_1 + \tilde{Y}_2
\]

be two paths spanning the BB, such that \( X = \tilde{X}_1 + \tilde{X}_2 \) and \( Y = \tilde{Y}_1 + \tilde{Y}_2 \) are the disjoint extremal components. Then

\[
X_1 = \tilde{P} + \tilde{X}_1, \quad X_2 = \tilde{X}_2, \quad Y_1 = \tilde{Y}_1 \quad \text{and} \quad Y_2 = \tilde{Q} + \tilde{Y}_2
\]

satisfy the conditions of Definition 2.17. \( \square \)

Definition 2.23. A locally non connectable internal building unit (LNC IBU) is a LNC IBB, or the concatenation of LNC IBB’s.

Proposition 2.24. The following four constructions yield LNC blocks, starting from some given LNC IBU’s.

1. Let \( B' \) be a LNC IBU with endpoints \( d \) and \( e \), as represented in the figure. We can embed \( B' \) as in the figure and obtain a LNC IBB \( B \) with endpoints \( a \) and \( b \). The four new subpaths \( W \) from \( a \) to \( e \), \( X \) from \( d \) to \( b \), \( Y \) from \( a \) to \( d \) and \( Z \) from \( e \) to \( b \) with the given colors are the ESU of the new building block. Moreover,

\[
L(W) + L(X) = L(Y) + L(Z).
\]
The LNC IBU $B'$

(2) Let $B'$ be a LNC IBU with endpoints $d$ and $e$, as represented in the figure. We can embed $B'$ as in the figure and obtain a LNC EBB $B$ with common endpoint $c$ and with two other endpoints $a$ and $b$. The four new subpaths with the given colors are the ESU of the new building block.

The LNC IBU $B'$

The new EBB $B$

(3) Let $B'$ be an a LNC IBU with endpoints $d$ and $e$, as represented in the figure. We can embed $B'$ as in the figure and obtain a LNC IBB $B$ with endpoints $a$ and $b$. The six new subpaths with the given colors are the ESU of the new building block.

The LNC IBU $B'$

The new IBB $B$

(4) Let $B'$ be an a LNC IBU with endpoints $d$ and $e$, as represented in the figure. We can embed $B'$ as in the figure and obtain a LNC EBB $B$ with common endpoint $c$ and with two other endpoints $a$ and $b$. The six new subpaths with the given colors are the ESU of the new building block.
Proof. (1) By assumption all the ESU’s in $B'$ are LNC. All the pairs of different colors in the new ESU are adjacent, so Proposition 2.21 concludes the proof in this case.

(2) By assumption, all the ESU’s in $B'$ are LNC. On the other hand all the pairs of different colors of the new components are either adjacent or both extremal, so Propositions 2.21 and 2.22 conclude the proof in this case.

(3) By assumption, all the ESU’s in $B'$ are LNC. On the other hand all the pairs of the new components of different color except one, are adjacent, so Propositions 2.21 proves that they are LNC. The remaining pair is proven to be LNC by the paths $X_1, X_2, Y_1, Y_2$ in the following diagrams, that satisfy the conditions of Definition 2.17.

(4) By assumption, the ESU’s in $B'$ are LNC. On the other hand all the pairs of the new components of different color except one, are adjacent or both extremal, so Propositions 2.21 and 2.22 prove that they are LNC. The remaining pair is proven to be LNC by the paths in the following diagrams.

3 Bi-traceable graphs with $\#(V(P) \cap V(Q)) \leq 4$

In this section we will prove in Theorem 3.4 that if $G$ is a simple graph, $P, Q$ are longest paths and $\#(V(P) \cap V(Q)) \leq 4$, then $BT(P, Q)$ is either a concatenation of LNC blocks, or it is totally disconnected (TD), according to the following definition.
Definition 3.1. $BT(P, Q)$ is totally disconnected (TD), if all pairs of components are NDC.

Remark 3.2. If $BT(P, Q)$ is TD and $V(P) \neq V(Q)$, then $V(P) \cap V(Q)$ is a separator.

Clearly, if $\ell = \#(V(P) \cap V(Q)) = 1$, then $BT(P, Q)$ is TD. When $\ell = 2$, then the representation of the resulting BT-graph is a concatenation of LNC blocks.

When $\ell = 3$, then there are two different representation types for $BT(P, Q)$. Either it is the concatenation of two elementary IBB’s and eventually up to two elementary EBB’s, or it is the concatenation of an elementary EBB and an EBB as in Proposition 2.24 (2). For this we write $(i, j, k)$ for the permutation $\sigma$ with $\sigma(1) = i$, $\sigma(2) = j$ and $\sigma(3) = k$, and similarly for $\ell > 3$. Consider the equivalence relation generated by $\sigma \sim \sigma^{-1}$ and $\sigma \sim \sigma^{\perp}$, where $\sigma^{\perp}(j) = \sigma(\ell - j)$.

We will draw generic representations for each equivalence class of permutations for $\ell = 3, 4, 5$. This means that in each completed component the number of vertices is left open. Note that two equivalent permutations $(i_1, j_1, k_1) \sim (i_2, j_2, k_2)$ have the same generic representation. For the identity permutation $(1, 2, 3)$ we have

$$(1, 2, 3) \sim (3, 2, 1)$$
and the (generic) representation of $BT(P, Q)$ is 

For the remaining permutations we have

$$(1, 3, 2) \sim (2, 1, 3) \sim (2, 3, 1) \sim (3, 1, 2)$$
and the representation of the BT-graph is 

The case $\ell = 4$

Note that if $\ell = 4$, then $\sigma^\perp = (4, 3, 2, 1) \circ \sigma$, where $(4, 3, 2, 1)$ corresponds to $(14)(23)$ in the standard form, so our equivalence relation is the same as in [4]. We have seven classes, which coincide with the cases in [4], and are listed in the following table.

| Case | Permutations | Representation | Conn. |
|------|--------------|----------------|------|
| 1.   | $(1, 2, 3, 4), (4, 3, 2, 1)$ | LNC            |
| 2.   | $(1, 2, 4, 3), (3, 4, 2, 1), (4, 3, 1, 2), (2, 1, 3, 4)$ | LNC            |
| 3.   | $(1, 3, 2, 4), (4, 2, 3, 1)$ | LNC            |
| 4.   | $(1, 3, 4, 2), (1, 4, 2, 3), (2, 3, 1, 4), (2, 4, 3, 1), (3, 1, 2, 4), (3, 2, 4, 1), (4, 1, 3, 2), (4, 2, 1, 3)$ | LNC            |
| 5.   | $(1, 4, 3, 2), (2, 3, 4, 1), (4, 1, 2, 3), (3, 2, 1, 4)$ | LNC            |
| 6.   | $(2, 1, 4, 3), (3, 4, 1, 2)$ | LNC            |
| 7.   | $(2, 4, 1, 3), (3, 1, 4, 2)$ | TD             |

Theorem 3.3. If $G$ is a simple graph, $P, Q$ are longest paths and $\#V(P) \cap V(Q) \leq 4$, then $BT(P, Q)$ is either a concatenation of LNC blocks, or it is TD.
Proof. For $\ell = 1$ and $\ell = 2$, we know by the discussion after Definition 3.1 that $BT(P, Q)$ is TD. For $\ell = 3$ and for the first five subcases of the case $\ell = 4$, $BT(P, Q)$ is the concatenation of LNC blocks. In fact, the first case for $\ell = 3$ and the first case for $\ell = 4$ is a concatenation of elementary building blocks which are LNC (for example by Proposition 2.21). The second case for $\ell = 3$ and the cases 2. and 5. for $\ell = 4$ have additional blocks corresponding to Proposition 2.24(2). The cases 3. and 4. for $\ell = 4$ have additional blocks corresponding to items (1) and (4) of Proposition 2.24. So it remains to prove that for $\ell = 4$ in Case 6. $BT(P, Q)$ is LNC and in Case 7. it is TD. In the sixth case there is one BB, with two embedded elementary IBB’s and an ESU with six completed components. By Proposition 2.21 the two elementary ESU’s are LNC.

In the ESU with six completed components all the pairs of the components of different color except one, are adjacent or both extremal, so by Propositions 2.21 and 2.22 they are LNC. The remaining pair is proven to be LNC by the paths in the second diagram.

Finally we prove that $BT(P, Q)$ in Case 7 is TD. We claim that adjacent components are NDC, and that any pair of extremal components is also NDC. In fact, since the graph is symmetric, we can assume that these pairs are of different colors and so Lemmas 2.3 and 2.4 prove the claim. Again by symmetry, we are left with two cases:

- a) Either one component is extremal and the other is not adjacent and internal, or
- b) both components are internal and not adjacent.

The following diagrams show that in both cases the pairs are NDC.

This concludes Case 7 and thus finishes the proof. □

Corollary 3.4. If $G$ is a simple graph, $P, Q$ are longest paths, $V(P) \neq V(Q)$ and $\#V(P) \cap V(Q) \leq 4$, then $V(P) \cap V(Q)$ is a separator.
Proof. By Theorem 3.3 and Proposition 2.19 in the first six cases all the ESU’s in $BT(P, Q)$ are NC. Hence, Proposition 2.16 implies that $V(P) \cap V(Q)$ is a separator, as desired. Case 7 follows from Remark 3.2.

Corollary 3.5. Assume that $P$ and $Q$ are two longest paths in a 5-connected simple graph $G$. Then $\#(V(P) \cap V(Q)) \geq 5$.

Proof. We know that $\#V(P) \geq 5$, so we can assume that $V(P) \neq V(Q)$. Since a 5-connected graph is also 4-connected, by [8, Theorem 4.2] we know that $\#(V(P) \cap V(Q)) \geq 4$. Assume by contradiction that $\#(V(P) \cap V(Q)) < 5$, i.e., that $\#(V(P) \cap V(Q)) = 4$. Since $G$ is 5-connected, the complement of $V(P) \cap V(Q)$ is connected, which contradicts the fact that by Corollary 3.4 $V(P) \cap V(Q)$ is a separator. This contradiction concludes the proof.

Proposition 3.6. Let $P, Q, R$ be three longest paths in a graph $G$. If all the ESU’s in $BT(P, Q)$ are NC, or $BT(P, Q)$ is TD, then the intersection of the three longest paths $V(P) \cap V(Q) \cap V(R)$ is not empty.

Proof. On one hand, $R$ must touch in at least one ESU the components of different colors, since otherwise we could swap the colors conveniently and obtain a longest path $\tilde{Q}$ such that $\tilde{Q} \cap R = \emptyset$, which is impossible. On the other hand, it is impossible that $R$ touches two components of different colors in an ESU. In fact, if all ESU’s are NC, then this follows from Proposition 2.14, and if $BT(P, Q)$ is TD, then $R$ can touch only one component of $BT(P, Q)$.

Corollary 3.7. If the intersection of three longest paths $P, Q, R$ is empty, then $\#(V(P) \cap V(Q)) \geq 5$.

Proof. If $\#(V(P) \cap V(Q)) < 5$, then by Theorem 3.3 we know that in $BT(P, Q)$ all ESU’s are NC, or $BT(P, Q)$ is TD. Proposition 3.6 concludes the proof.

Remark 3.8. The following representation of a BT-graph with $\ell = V(P) \cap V(Q) = 5$ shows that the method for $\ell \leq 4$ cannot be carried over to this case, since the highlighted ESU is not NC. In fact, the blue path $R$ connects two components of different colors in the same highlighted ESU, but cannot be completed to two paths whose lengths sum $2L(R) + 2L(P)$, contradicting Definition 2.12.

However, we will prove in Corollary 5.3 that this graph is not a counterexample to the Hippchen conjecture. We will even prove that in this graph $V(P) \cap V(Q)$ is a separator in Theorem 5.7.

4 Bi-traceable graphs with $\#(V(P) \cap V(Q)) = 5$

In this section we will prove in Theorem 4.2 that if $G$ is a simple graph, $P, Q$ are longest paths and $\#V(P) \cap V(Q) = 5$, then $BT(P, Q)$ is either a concatenation of LNC building blocks, or it is TD, or it is one of the three cases 6., 13. or 14. in the following table.
| Case | Permutations | Representation | Conn. |
|------|--------------|----------------|-------|
| 1.   | (1, 2, 3, 4, 5), (5, 4, 3, 2, 1) | [Image] | LNC |
| 2.   | (1, 2, 3, 5, 4), (4, 5, 3, 2, 1), (2, 1, 3, 4, 5), (5, 4, 3, 1, 2) | [Image] | LNC |
| 3.   | (1, 2, 4, 3, 5), (5, 3, 4, 2, 1), (1, 3, 2, 4, 5), (5, 4, 2, 3, 1) | [Image] | LNC |
| 4.   | (1, 2, 4, 5, 3), (3, 5, 4, 2, 1), (1, 2, 5, 3, 4), (4, 3, 5, 2, 1), (3, 1, 2, 4, 5), (5, 4, 2, 1, 3), (2, 3, 1, 4, 5), (5, 4, 1, 3, 2) | [Image] | LNC |
| 5.   | (1, 2, 5, 4, 3), (3, 4, 5, 2, 1), (3, 2, 1, 4, 5), (5, 4, 1, 2, 3) | [Image] | LNC |
| 6.   | (1, 3, 2, 5, 4), (4, 5, 2, 3, 1), (2, 1, 4, 3, 5), (5, 3, 4, 1, 2) | [Image] | – |
| 7.   | (1, 3, 4, 2, 5), (5, 2, 4, 3, 1), (1, 4, 2, 3, 5), (5, 3, 2, 4, 1) | [Image] | LNC |
| 8.   | (1, 3, 4, 5, 2), (2, 5, 4, 3, 1), (4, 1, 2, 3, 5), (5, 3, 2, 1, 4), (1, 5, 2, 3, 4), (4, 3, 2, 5, 1), (2, 3, 4, 1, 5), (5, 1, 4, 3, 2) | [Image] | LNC |
| 9.   | (1, 3, 5, 2, 4), (4, 2, 5, 3, 1), (2, 4, 1, 3, 5), (5, 3, 1, 4, 2), (1, 4, 2, 5, 3), (3, 5, 2, 4, 1), (3, 1, 4, 2, 5), (5, 2, 4, 1, 3) | [Image] | LNC |
| 10.  | (1, 3, 5, 4, 2), (2, 4, 5, 3, 1), (4, 2, 1, 3, 5), (5, 3, 1, 2, 4), (1, 5, 2, 4, 3), (3, 4, 2, 5, 1), (3, 2, 4, 1, 5), (5, 1, 4, 2, 3) | [Image] | LNC |
| 11.  | (1, 4, 3, 2, 5), (5, 2, 3, 4, 1) | [Image] | LNC |
| 12.  | (1, 4, 3, 5, 2), (2, 5, 3, 4, 1), (4, 1, 3, 2, 5), (5, 2, 3, 1, 4), (1, 5, 3, 2, 4), (4, 2, 3, 5, 1), (2, 4, 3, 1, 5), (5, 1, 3, 4, 2) | [Image] | LNC |
| 13.  | (1, 4, 5, 2, 3), (3, 2, 5, 4, 1), (3, 4, 1, 2, 5), (5, 2, 1, 4, 3) | [Image] | – |
| 14.  | (1, 4, 5, 3, 2), (2, 3, 5, 4, 1), (4, 3, 1, 2, 5), (5, 2, 1, 3, 4), (1, 5, 4, 2, 3), (3, 2, 4, 5, 1), (3, 4, 2, 1, 5), (5, 1, 2, 4, 3) | [Image] | – |
| 15.  | (1, 5, 3, 4, 2), (2, 4, 3, 5, 1), (4, 2, 3, 1, 5), (5, 1, 3, 2, 4) | [Image] | LNC |
| Case | Permutations | Representation | Con. |
|------|--------------|----------------|-----|
| 16.  | (1, 5, 4, 3, 2), (2, 3, 4, 5, 1), (4, 3, 2, 1, 5), (5, 1, 2, 3, 4) | ![Representation](image1) | LNC |
| 17.  | (2, 1, 3, 5, 4), (4, 5, 3, 1, 2) | ![Representation](image2) | LNC |
| 18.  | (2, 1, 4, 5, 3), (3, 5, 4, 1, 2), (3, 1, 2, 5, 4), (4, 5, 2, 1, 3), (2, 1, 5, 3, 4), (4, 3, 5, 1, 2), (2, 3, 1, 5, 4), (4, 5, 1, 3, 2) | ![Representation](image3) | LNC |
| 19.  | (2, 1, 5, 4, 3), (3, 4, 5, 1, 2), (3, 2, 1, 5, 4), (4, 5, 1, 2, 3) | ![Representation](image4) | LNC |
| 20.  | (2, 3, 5, 1, 4), (4, 1, 5, 3, 2), (2, 5, 1, 3, 4), (4, 3, 1, 5, 2), (4, 1, 2, 5, 3), (3, 5, 2, 1, 4), (3, 1, 4, 5, 2), (2, 5, 4, 1, 3) | ![Representation](image5) | LNC |
| 21.  | (2, 4, 1, 5, 3), (3, 5, 1, 4, 2), (3, 1, 5, 2, 4), (4, 2, 5, 1, 3) | ![Representation](image6) | TD |
| 22.  | (2, 4, 5, 1, 3), (3, 1, 5, 4, 2), (3, 5, 1, 2, 4), (4, 2, 1, 5, 3), (4, 1, 5, 2, 3), (3, 2, 5, 1, 4), (3, 4, 1, 5, 2), (2, 5, 1, 4, 3) | ![Representation](image7) | LNC |
| 23.  | (2, 5, 3, 1, 4), (4, 1, 3, 5, 2) | ![Representation](image8) | TD |

Now we will verify that for all cases in the table $BT(P, Q)$ is made of LNC blocks, when the last entry is LNC, and that it is TD, when the last entry is TD. We know by Proposition 2.13 that elementary blocks are LNC, so Case 1. is clear. Cases 2., 5., 15., 16. and 17. follow from Proposition 2.24(2) with different embedded LNC IBU’s, Cases 3. and 11. from Proposition 2.24(1), Cases 4. and 8. from Proposition 2.24(4), and Case 7. from Proposition 2.24(3).

**Case 9.:** In this case we have one elementary EBB and one EBB with 10 components and no embedded IBB.

![ESU with 10 components](image9)\(\text{ESU with 10 components}\)

![The case 1b](image10)\(\text{The case 1b}\)

![The case 1e](image11)\(\text{The case 1e}\)

One verifies directly that in this ESU all the pairs of components of different colors except seven, are adjacent or both extremal, so Propositions 2.21 and 2.22 prove that they are LNC. The two pairs 1b and 1e are proven to be LNC by the paths in the second and third diagrams above, where $X_1$ and $Y_1$ are in black and $X_2$ and $Y_2$ are in blue. The remaining pairs 2e, 3c, 4α, 5α, 5d are proven to be LNC by the paths in the following diagrams, where $X_1$ and $Y_1$ are in black and $X_2$ and $Y_2$ are in blue.
Remark 4.1. From now on, when we identify in a graph $X_1, X_2, Y_1, Y_2$ and $x, y$ satisfying Definition 2.17, then we will draw the paths $X_1$ and $Y_1$ in black, the paths $X_2$ and $Y_2$ in blue and $x$ and $y$ in red. We don’t need to write the names in the graph and we won’t do it.

Case 10.: In this case we have one elementary EBB, and one bigger EBB, that has one embedded IBB and that has an ESU with 8 components:

One verifies directly that in this ESU all the pairs of components of different colors except four, are adjacent or both extremal, so Propositions 2.21 and 2.22 prove that they are LNC. The remaining pairs $1d, 2a, 3d, 4c$ are proven to be LNC by the paths in the following diagrams. The paths $X_1$ and $Y_1$ are in black and the paths $X_2$ and $Y_2$ are in blue.
**Case 12.** In this case we have one elementary EBB, and one bigger EBB, that has one embedded IBB and that has an ESU with 8 components:

![ESU with 8 components](image)

One verifies directly that in this ESU all the pairs of components of different colors except four, are adjacent or both extremal, so Propositions 2.21 and 2.22 prove that they are LNC. The remaining pairs $1b, 2a, 3d, 4a$ are proven to be LNC by the paths in the following diagrams. The paths $X_1$ and $Y_1$ are in black and the paths $X_2$ and $Y_2$ are in blue.

**Case 18.** In this case we have one BB with two embedded elementary IBB’s and an ESU with 8 components:

![ESU with 8 components](image)

One verifies directly that in this ESU all the pairs of components of different colors except four, are adjacent or both extremal, so Propositions 2.21 and 2.22 prove that they are LNC. The remaining pairs $1c, 2b, 2d, 3a$ are proven to be LNC by the paths in the following diagrams. The paths $X_1$ and $Y_1$ are in black and the paths $X_2$ and $Y_2$ are in blue.

**Case 19.** In this case we have one BB with three embedded elementary IBB’s and an ESU with 6 components:
One verifies directly that in this ESU all the pairs of components of different colors except one, are adjacent or both extremal, so Propositions 2.21 and 2.22 prove that they are LNC. The remaining pair is proven to be LNC by the paths in the second diagram, where the paths $X_1$ and $Y_1$ are in black and the paths $X_2$ and $Y_2$ are in blue.

**Case 20.** In this case we have one BB with one embedded elementary IBB and an ESU with 10 components:

One verifies directly that in this ESU all the pairs of components of different colors except seven, are adjacent or both extremal, so Propositions 2.21 and 2.22 prove that they are LNC. The two pairs $1b$ and $2b$ are proven to be LNC by the paths in the second and third diagrams above, where $X_1$ and $Y_1$ are in black and $X_2$ and $Y_2$ are in blue. The remaining pairs $2c, 3a, 3c, 4a, 5d$ are proven to be LNC by the paths in the following diagrams, where $X_1$ and $Y_1$ are in black and $X_2$ and $Y_2$ are in blue.

**Case 21.** and **Case 23.** In both cases we have the same bi-traceable graph, which is TD. In order to see this, we first use Lemmas 2.3 and 2.4 and the symmetry of the graph, in order to verify that adjacent pairs and pairs of extremal components are NDC. Using again the symmetry of the
graph, in order to check the remaining pairs, it suffices to prove that the five pairs of components $ab$, $ac$, $ad$, $be$ and $ce$ are NDC, which follows from the paths in the diagrams below, where $X_1$ and $Y_1$ are in black and $X_2$ and $Y_2$ are in blue. In fact, the paths $\hat{P}$ and $\hat{Q}$ of Definition 2.2 are given in each case by

$$\hat{P} = X_1 + R + Y_1 \quad \text{and} \quad \hat{Q} = X_2 + R + Y_2.$$ 

Case 22: In this case we have one BB with one embedded elementary IBB and an ESU with 10 components:

One verifies directly that in this ESU all the pairs of components of different colors except seven, are adjacent or both extremal, so Propositions 2.21 and 2.22 prove that they are LNC. The two pairs $1b$ and $2e$ are proven to be LNC by the paths in the second and third diagrams above, where $X_1$ and $Y_1$ are in black and $X_2$ and $Y_2$ are in blue. The remaining pairs $3c, 4a, 4c, 5b, 5d$ are proven to be LNC by the paths in the following diagrams, where $X_1$ and $Y_1$ are in black and $X_2$ and $Y_2$ are in blue.
The case 5b

The case 5d

This finishes all the cases in the table, and thus we have proven the following theorem.

**Theorem 4.2.** If $G$ is a simple graph, $P, Q$ are longest paths and $\#V(P) \cap V(Q) = 5$, then $BT(P, Q)$ is either a concatenation of LNC blocks, or it is TD, or it is one of the three cases 6., 13. or 14. in the table above.

5 In the three exceptional cases $V(P) \cap V(Q)$ is a separator

In this section we will prove that none of the three exceptional cases of the previous section is a counterexample to the Hippchen conjecture, which proves the Hippchen conjecture for $k = 6$. Then we prove that in the three cases $V(P) \cap V(Q)$ is a separator.

**Lemma 5.1.** Assume $G$ is $\ell + 1$-connected and let $a_1, b_1, P_0', Q_0'$ be as in Notation 1.1, in particular we assume that $\#(V(P) \cap V(Q)) = \ell$. If $a_1 = b_1$, then $P_0' \neq \emptyset$ and $Q_0' \neq \emptyset$.

**Proof.** Since $a_1 = b_1$, we can interchange $P_0$ and $Q_0$, and $P_0' = \emptyset$ if and only if $Q_0' = \emptyset$. Assume by contradiction that $P_0' = Q_0' = \emptyset$, and so $a_1 = b_1$ is an endpoint of $P$ and of $Q$. Since $G$ is $\ell + 1$-connected, there is an edge connecting $a_1$ with a point $t \not\in \{a_2, \ldots, a_\ell\} = (V(P) \cap V(Q))$, which we call $a_1 t$. If $t \not\in V(P)$, then $L(P + a_1 t) > L(P)$ which contradicts the fact that $P$ is a longest path; and if $t \not\in V(Q)$, then $L(Q + a_1 t) > L(Q)$, which contradicts the fact that $Q$ is a longest path, concluding the proof. \hfill $\Box$

**Proposition 5.2.** In the three cases 6. 13. and 14., we can assume $a_1 = b_1$. In any of the three cases, if $P_0' \neq \emptyset$ and $Q_0' \neq \emptyset$, then $V(P) \cap V(Q)$ is a separator.

**Proof.** Note that in the three exceptional cases can assume $a_1 = b_1$, changing the directions of $P$ and/or $Q$, if necessary.

**Case 6.:** The following diagrams show that $P_0'$ can connect directly only with $Q_3'$ and $P_3'$, and that $Q_3'$ cannot be connected directly with $P_3'$.

In fact, the blue and black path show that the connections shown in the diagrams are impossible. By symmetry of the elementary ESU’s, $P_0'$ cannot connect with each of the other components of the elementary ESU’s. Moreover, the adjacent components cannot be connected with $P_0'$ by Lemma 2.3 and the extremal components cannot be connected by Lemma 2.4. So $Q_3'$ and $P_3'$ are the only left. By symmetry $Q_3'$ can connect directly only with $Q_3'$ and $P_3'$. But $P_0'$ and $Q_0'$ form an NC pair, so they cannot connect to the same component $X \in \{P_3', Q_3'\}$. On the other hand, $Q_3'$ can connect only with $P_0'$ and $Q_0'$, since any other pair that contains $Q_3'$ but not $P_3'$, is either adjacent, or can be discarded by the paths in the following diagrams.
The symmetric argument shows that $P'_3$ can connect only with $P'_0$ and $Q'_0$. Hence $P'_{0}$ and $Q'_{0}$ cannot be connected in $G \setminus (V(P) \cap V(Q))$, which shows that $V(P) \cap V(Q)$ is a separator, as desired.

Case 13.: The following diagrams show that $P'_{0}$ cannot be connected to any other component.

In fact, the first two diagrams show paths that prevent $P'_{0}$ to be connected to two components, and by symmetry also to the other component of the elementary ESU. The third diagram shows that the graph is symmetric, and so we can discard three more components. The remaining components are either adjacent to $P'_{0}$ or are extremal, so they cannot be connected to $P'_{0}$ by Lemmas 2.3 and 2.4. Hence there cannot be a path from $P'_{0}$ to $Q'_{0}$ in $G \setminus (V(P) \cap V(Q))$, and so $V(P) \cap V(Q)$ is a separator, as desired.

Case 14.: The following diagrams show that $P'_{0}$ cannot be connected to any other component.

In fact, the diagrams show paths that prevent $P'_{0}$ to be connected to four components, and by symmetry also to the other component of the elementary ESU’s. The remaining components are either adjacent to $P'_{0}$ or are extremal, so they cannot be connected to $P'_{0}$ by Lemmas 2.3 and 2.4. Hence there cannot be a path from $P'_{0}$ to $Q'_{0}$ in $G \setminus (V(P) \cap V(Q))$, and so $V(P) \cap V(Q)$ is a separator, as desired. This finishes the three cases and concludes the proof. □

Corollary 5.3. Assume that $P$ and $Q$ are two longest paths in a 6-connected simple graph $G$. Then $\#(V(P) \cap V(Q)) \geq 6$.

Proof. We know that $\#V(P) \geq 6$, so we can assume that $V(P) \neq V(Q)$. Since a 6-connected graph is also 5-connected, by Corollary 6.5 we know that $\#(V(P) \cap V(Q)) \geq 5$. Assume by contradiction that $\#(V(P) \cap V(Q)) < 6$, i.e., that $\#(V(P) \cap V(Q)) = 5$. Since $G$ is 6-connected, the complement of $V(P) \cap V(Q)$ is connected, which contradicts the fact that by Lemma 5.1 and Proposition 5.2 we know that $V(P) \cap V(Q)$ is a separator. This contradiction concludes the proof. □

Now we prove that in the three exceptional cases $V(P) \cap V(Q)$ is always a separator.

Lemma 5.4. Let $P'_0$, $Q'_0$, $P'_\ell$ and $Q'_\ell$ be as in Notation 1.1 and assume that one of $P'_0$, $Q'_0$, $P'_\ell$ or $Q'_\ell$ is empty. Then the two adjacent partial paths of the other longest path have length 1.
Proof. Clear, since otherwise one can extend $P$ (respectively $Q$) using the first part of the adjacent partial path. □

We will show that in each of the three cases, if $P'_0 = Q'_0 = \emptyset$, then $BT(P, Q)$ is weakly disconnected (WD), according to the following definition.

**Definition 5.5.** The graph $BT(P, Q)$ is called **weakly disconnected** (WD) if each pair $X, Y$ of components of different colors is either NDC, or is **weakly non directly connectable** (WNDC), which means that for every path $R$ that connects $X$ with $Y$, such that $R$ is internally disjoint from $BT(P, Q)$, we can construct a path $\hat{P}$ and a cycle $C$ in $BT(P, Q) \cup R$ such that

$$L(\hat{P}) + L(C) = 2L(R) + L(P) + L(Q) = 2L(P) + 2L(R).$$

Clearly in a WD graph with $V(P) \neq V(Q)$, the set $V(P) \cap V(Q)$ is a separator, since the length of $\hat{P}$ and the length of the opened cycle $\tilde{C}$ sum $L(\hat{P}) + L(C) - 1 = 2L(P) + 2L(R) - 1 > 2L(P)$, which shows that no pair of components of different colors can be connected.

**Proposition 5.6.** Consider the three cases 6. 13. and 14. and assume $a_1 = b_1$. If $P'_0 = Q'_0 = \emptyset$, then $V(P) \cap V(Q)$ is a separator.

Proof. **Case 6.** The paths in the following diagrams show that the graph is TD.

In fact, the two first diagrams show that one cannot connect $P'_5$ with any other component. Some components are either adjacent, or extremal, $P'_1$ and $Q'_1$ are empty, and for the three remaining components, the two diagrams show that they cannot be connected. Note that two of them are components in an elementary ESU, so by symmetry it suffices to show it for one of them. By symmetry the same holds for $Q'_5$, and the remaining pairs that are not adjacent are shown to be NDC in the last two diagrams. Note that by symmetry it suffices to prove it for one horizontal connection.

**Case 13.** The paths and cycles in the following diagrams show that the graph is WD.

In fact, the two first diagrams show that one cannot connect $P'_5$ with any other component. Some components are either adjacent, or extremal, $P'_1$ and $Q'_1$ are empty, and for the three remaining components, the two diagrams show that they cannot be connected. Note that two of them are components in an elementary ESU, so by symmetry it suffices to show it for one of them. By symmetry the same holds for $Q'_5$, and the remaining pairs that are not adjacent are shown to be WNDC in the last two diagrams. Note that by symmetry it suffices to prove it for one pair of components of different elementary ESU’s.

**Case 14.** The paths and cycles in the following six diagrams show that the graph is WD.
In fact, one uses the fact that $P_1'$ and $Q_1'$ are empty, discard all adjacent pairs and pairs with two extremal components, and the remaining pairs are connected in the six diagrams above, where by symmetry we consider the connection with only one of the components of each elementary ESU.

Thus in all three exceptional cases $V(P) \cap V(Q)$ is a separator, as desired. \hfill \square

**Theorem 5.7.** If $G$ is a simple graph, $P, Q$ are longest paths, $V(P) \neq V(Q)$ and $|V(P) \cap V(Q)| \leq 5$, then $V(P) \cap V(Q)$ is a separator.

**Proof.** If $|V(P) \cap V(Q)| \leq 4$, then it is true by Corollary 3.3. If $|V(P) \cap V(Q)| = 5$ and $BT(P, Q)$ is none of the exceptional cases, then by Theorem 4.2 the graph $BT(P, Q)$ is a concatenation of LNC BB, or it is TD. Hence all ESU’s are NC, and so by Proposition 2.16 and Theorem 2.20 the set $V(P) \cap V(Q)$ is a separator. Finally, if $BT(P, Q)$ is in one of the exceptional cases, then we can assume $a_1 = b_1$. If $P_0' \neq \emptyset$ and $Q_0' \neq \emptyset$, then Proposition 5.2 yields the result. Otherwise necessarily $P_0' = Q_0' = \emptyset$, and then Proposition 5.6 concludes the proof. \hfill \square

Note that Theorem 5.7 implies Corollary 5.3.

**Corollary 5.8.** Assume that $P$ and $Q$ are two longest paths in a simple graph $G$. If $V(Q) \neq V(P)$ and $n = |V(G)| \leq 7$ then $V(Q) \cap V(P)$ is a separator.

**Proof.** Since $V(Q) \neq V(P)$ and $|V(G)| \leq 7$, it follows that $|V(P) \cap V(Q)| \leq 5$, and the result follows from the previous theorem. \hfill \square

6 Three longest paths in the exceptional cases

In this section we will show that in none of the three exceptional cases 6. 13. and 14., we can have three disjoint longest paths. In order to do this, we will analyze the number of times and the sequential order in which a third longest path $R$ intersects the components of $BT(P, Q)$. We will show that in the three cases there is only one pair of components of different colors in $BT(P, Q)$ connected directly by $R$, and in the next lemma we prove that this is impossible.

**Lemma 6.1.** Assume $P, Q, R$ are disjoint longest paths. Then it is impossible that there is only one pair of components of different colors in $BT(P, Q)$ connected directly by $R$.

**Proof.** We will prove the following three statements.
(1) Given two components of different colors in $BT(P, Q)$, there cannot be two disjoint subpaths of $R$ joining directly one component with the other.

(2) If there is only one pair of components of different colors that are connected directly by $R$, then there is only one subpath of $R$ connecting directly $P$ and $Q$.

(3) It is impossible that there is only one subpath of $R$ connecting directly $P$ and $Q$.

Clearly the lemma follows from statements (2) and (3).

(1) The corresponding result for cycles is well known (See for example [3, p.145, Claim 1]). Assume by contradiction that $R_1$ has one endpoint $x_1$ in $P$ and the other endpoint $y_1$ in $Q$, and $R_2$ has one endpoint $x_2$ in $P$ and the other endpoint $y_2$ in $Q$. Then $R_1 + Q_{[y_1, y_2]} + R_2$ is internally disjoint from $P$ and $R_1 + P_{[x_1, x_2]} + R_2$ is internally disjoint from $Q$. Then

\[ \hat{P} = P_{\le x_1} + R_1 + Q_{[y_1, y_2]} + R_2 + P_{\ge x_2} \quad \text{and} \quad \hat{Q} = Q_{\le y_1} + R_1 + P_{[x_1, x_2]} + R_2 + Q_{\ge y_2} \]

are two paths that have lengths that sum $L(P) + L(Q) + 2L(R_1) + 2L(R_2)$, a contradiction that proves the first statement.

(2) By item (1) the only other possibility is that two consecutive subpaths of $R$ go back and forth between the components of the only pair connected by $R$. But then $R$ intersects one of the paths $P$ or $Q$ in only one point, which is impossible, for example by Corollary 3.7 applied to $V(R) \cap V(P)$ or $V(R) \cap V(Q)$.

(3) Assume there is only one subpath $R_2$ of $R$ connecting directly $P$ and $Q$. Let the endpoint of that subpath of $R$ in $P$ be $a$ and the other endpoint in $Q$ be $b$. Then $a$ and $b$ partition $R$ into three subpaths $R_1, R_2, R_3$, the point $a$ partitions the path $P$ into $P_1, P_2$ and $b$ partitions the path $Q$ into $Q_1, Q_2$. Interchanging if necessary $P$ with $Q$, and $Q_1$ with $Q_2$, we can assume without loss of generality that $L(R_1) \ge L(R_3)$ and that $L(Q_1) \ge L(Q_2)$. But then the path $R_1 + R_2 + Q_1$ has length

\[ L(R_1) + L(R_2) + L(Q_1) > L(R)/2 + L(Q)/2 = L(Q) = L(R), \]
contradicting that $R$ is a longest path and thus proving statement (3).

□

**Proposition 6.2.** If $P, Q, R$ are disjoint longest paths, then $BT(P, Q)$ cannot have the configuration of Case 6 in the table of section 4.

**Proof.** We can assume $a_1 = b_1$.

From the proof of Proposition 5.2 we know that if $R$ touches $P'_0$, then it can touch only $P'_3$ and $Q'_3$, and that $P'_3$ cannot be connected directly with $Q'_3$, and that $Q'_3$ and $P'_3$ can be connected only with $P'_0$ or $Q'_0$. Since $P'_0$ and $Q'_0$ are NC, if $R$ touches one of $P'_0, Q'_0, P'_3$ or $Q'_3$, then the only possibilities for the set of components touched by $R$ are four sets with two elements and two sets with three elements, which are connected in the following way:

$$P'_0 \sim P'_3, \quad Q'_0 \sim Q'_3, \quad Q'_0 \sim Q'_3, \quad P'_0 \sim Q'_3, \quad P'_3 \sim Q'_0 \sim Q'_3 \quad \text{or} \quad P'_3 \sim P'_0 \sim Q'_3.$$ 

In all cases there is at most one pair of components of different colors connected by $R$, which is impossible by Lemma 6.1. Hence $R$ cannot touch $P'_0, Q'_0, P'_3$ nor $Q'_3$.

We are searching for pairs of components that can be connected directly by a subpath of $R$. We have discarded all pairs that contain $P'_0, Q'_0, P'_3$ or $Q'_3$, and we can also discard all pairs that are adjacent and have different colors, and all pairs of extremal components of different colors. Using the paths in the following 3 diagrams,

and using that by symmetry, if you cannot connect one component of an elementary ESU, then you cannot connect the other; we can discard all pairs except the following four.
The blue and the black paths in the following diagram, whose lengths sum more than $2L(P)$, show that $R$ cannot connect simultaneously $P'_1$ with $Q'_5$ and $Q'_1$ with $P'_5$.

We cannot have $P'_1 \sim Q'_5$ and $Q'_1 \sim P'_5$.

Thus there is only one pair of components of different colors in $BT(P, Q)$ connected by $R$, which contradicts Lemma 6.1 and concludes the proof. \[\square\]

**Proposition 6.3.** If $P, Q, R$ are disjoint longest paths, then $B(T(P, Q))$ cannot have the configuration of Case 13 in the table of section 4.

**Proof.** We can assume $a_1 = b_1$.

As in the proof of Proposition 5.2, we know that $P'_0$ cannot be connected with any other component, and by symmetry the same holds for $Q'_0$. The paths in the following three diagrams show that the components in the elementary ESU’s cannot be connected to any other component.
Note that we use the symmetry of the graph. We discard the adjacent pairs and the pair with two extremal components, and are left with 9 possible pairs that $R$ can connect directly. There are six pairs with the same color:

\[ P'_1 \sim P'_3, \quad P'_1 \sim P'_5, \quad P'_5 \sim P'_3, \quad Q'_1 \sim Q'_3, \quad Q'_1 \sim Q'_5, \quad Q'_5 \sim Q'_3, \]

and three pairs with different colors:

\[ P'_1 \sim Q'_5, \quad P'_3 \sim Q'_3, \quad P'_5 \sim Q'_1. \]

The paths in the following diagrams show that $R$ can connect only one pair of different colors.

Thus Lemma 6.1 concludes the proof. \[ \square \]

**Proposition 6.4.** If $P, Q, R$ are disjoint longest paths, then $BT(P, Q)$ cannot have the configuration of Case 14 in the table of section 4.

**Proof.** We can assume $a_1 = b_1$.

As in the proof of Proposition 5.2 we know that $P'_0$ cannot be connected with any other component, and by symmetry the same holds for $Q'_0$. We discard also the pairs of adjacent components and if both components are extremal. There are 14 possible connections left, (where we count the connection to the two components of an elementary ESU only once), and 8 of them are discarded by the paths in the following diagrams.
So we are left with 6 possible connections.

Since \( P_2 \) and \( Q_4 \) are the components of an elementary ESU, we also have the connections \( P'_2 \sim Q'_4 \) and \( Q'_4 \sim P_5 \). So we have two possibilities: either \( R \) touches some of \( P'_2, P'_3, Q'_1, Q'_4 \), or \( R \) touches some of \( P'_1, P'_3, Q'_3, Q'_5 \). In the first case, since \( P'_2 \) and \( Q'_4 \) are NC, they cannot be connected simultaneously by \( R \) to the same component, thus we have two possibilities

\[
Q'_1 \sim P'_2 \sim P'_5 \quad \text{or} \quad Q'_1 \sim Q'_4 \sim P'_5.
\]

But both cases are impossible by Lemma 6.1.

So \( R \) must touch some of \( P'_1, P'_3, Q'_3, Q'_5 \). The paths in the following diagram show that \( R \) cannot connect simultaneously \( P'_1 \sim Q'_3 \) and \( P'_3 \sim Q'_5 \).

Since \( P_3 \) and \( Q_5 \) are adjacent and \( P_3 \) and \( Q_3 \) are also adjacent, there is only one pair of components of different colors in \( BT(P, Q) \) connected by \( R \), which contradicts Lemma 6.1 and concludes the proof.

**Theorem 6.5.** If the intersection of three longest paths \( P, Q, R \) is empty, then \( \#(V(P) \cap V(Q)) \geq 6 \).

**Proof.** Assume that the intersection of three longest paths \( P, Q, R \) is empty, then \( \#(V(P) \cap V(Q)) \geq 5 \) by Corollary 3.7. If \( \#(V(P) \cap V(Q)) = 5 \), then we know by Theorem 4.2 that in \( BT(P, Q) \) all the ESU’s are NC or \( BT(P, Q) \) is TD, or it has the representation type of one of the cases 6., 13 or 14. If all the ESU’s are NC or \( BT(P, Q) \) is TD, then Proposition 3.6 yields a contradiction. Else Propositions 6.2, 6.3 and 6.4 lead to contradictions and thus finish the proof. □
Remark 6.6. In each of the cases that couldn’t be discarded constructing longer paths in the three exceptional cases, we proved that $R$ connects only one pair of components of different colors of $BT(P,Q)$, which is impossible by Lemma [7.1].

It would be interesting to characterize the configurations in which this strategy works. One could start trying to find all configurations (in which not all ESU’s are NC), in which $R$ can connect only one pair of components of different colors of $BT(P,Q)$.

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