Hosoya Polynomial, Wiener Index, Coloring and Planar of Annihilator Graph of $Z_n$

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Received on: 13/04/2020  
Accepted on: 20/05/2020

ABSTRACT

Let $R$ be a commutative ring with identity. We consider $\Gamma_B(R)$ an annihilator graph of the commutative ring $R$. In this paper, we find Hosoya polynomial, Wiener index, Coloring, and Planar annihilator graph of $Z_n$ denote $\Gamma_B(Z_n)$, with $n = p^m$ or $n = p^m q$, where $p, q$ are distinct prime numbers and $m$ is an integer with $m \geq 1$.

Keywords: Annihilator graph of ring, Zero-divisor graph, Hosoya polynomial, Wiener index, coloring graph and planar graph.

1. Introduction

Let $R$ be a commutative ring with identity the annihilator of $R$ is the set of all element $x \in R$ satisfy $ann(R) = \{x \in R: x.y = 0, \forall y \in R\}$ [6], and let $ann(R)$ be the set of all annihilator in $R$. We consider a simple graph $\Gamma_B(R)$ to $R$ with vertices $ann(R)$, for every two distinct vertices $x,y$ are adjacent if and only if $\{x.y = 0: x,y \in ann(R)\}$, and let $Z(R)$ be the set of all zero-divisors in $R$, and $Z(R)^*$ is the set of all non-zero zero-divisors in it. A simple graph $\Gamma(R)$ with vertices $Z(R)^*$, for every two distinct vertices $x,y$ are adjacent if and only if $\{x.y = 0: x,y \in Z(R)^*\}$.
The notion of an annihilator graph of a commutative ring was first introduced in 1988 by Beck [5], where he was interested in colorings, this investigation was then continued by Anderson and Nasser [3] zero-divisor graph of a commutative ring, further that Anderson and Livingston [2]. They denoted that by $\Gamma(R)$. It is clear that from Beck’s definition of annihilator graph of a commutative ring and Anderson’s definition of a zero-divisor graph of a commutative ring can be defined Annihilator graph of a commutative ring can be defined $\Gamma_B(R) = \left( (\Gamma(R) \cup \{ann(R^*) - Z(R)^*\}) + k_1 \right)$. Such that: $\Gamma(R)$ zero-divisor graph of the ring, $ann(R^*)$ set of all vertices in $R$ non-zero, $Z(R)^*$ set of all non-zero zero-divisors in $R$ and $k_1 = 0$.

A graph $G$ is called a connected graph if there is at least one path between any pair of vertices in $G$, otherwise it is called disconnected [7]. For vertices $x,y$ of $G$, let $d(x,y)$ be the length of the shortest path from $x$ to $y$ (and it is called distance between two vertices $x,y$ in $G$). The maximum distance between any two vertices $x,y$ in $G$ is called the diameter graph $G$ [7], that is $diam(G) = \max_{x,y \in V(G)} \{d(x,y)\}$, where $V(G)$ is the set of all vertices of $G$. A graph $G$ is complete if every two of its vertices are adjacent, so the complete graph of order $n$ is denoted by $k_n$. If the vertex set of a graph $G$ can be split into two disjoint sets $A$ and $B$ (such that the induced subgraph that generated by either $A$ or $B$ is null graph), then we said $G$ is a bipartite graph. This graph is also said to be a complete bipartite graph is a bipartite graph in graph if each vertex in the set $A$ has joined to every vertex in the set $B$ with just one edge.

Hosoya polynomial of the graph $G$ is defined by $H(G;x) = \sum_{k=0}^{diam(G)} d(G,k)x^k$, where $d(G,k)$ is the number of pairs of vertices of a graph $G$, that are at distance $k$ apart, for $k = 0,1,2, ..., diam(G)$. The Wiener index of $G$ is defined as the sum of all distances between vertices of the graph $G$, and denoted by $W(G)$, we can also find this index by differentiating Hosoya polynomial with respect to $x$ at $x = 1$, by symbols we can write: $W(G) = \frac{d}{dx}H(G;x) \bigg|_{x=1}$, See [8,12].

Let $X(G)$ denote the chromatic number of vertices, i.e., the minimal number of colors, which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors [7]. We let $\bar{X}(G)$ denote the chromatic number of edges, i.e., the minimal number of colors, which can be assigned to the edges of $G$ in such a way that every two adjacent edges have different colors [7]. And last we assumed $f(G)$ denote the chromatic number of faces, i.e., the minimal number of colors, that can be assigned to the faces of planar graph $G$ in such a way that every two adjacent faces have different colors [7]. A planar graph $G$ is a graph that can be drawn in the plane without crossings for any two edges in $G$ [7]. There are many studies in the graph properties and commutative ring. See [1],[4],[10]&[11].

2. Some Properties of Graph $\Gamma_B(Z_p^m)$

We will start this item by a lemma.

Lemma 2.1: The vertex (0) connect with every vertex of the graph $\Gamma_B(Z_n)$.

Proof: Since $0.a = 0$ for $a \in Z_n$, so it is the vertex (0) connect with every vertex of the graph $\Gamma_B(Z_n)$.

Lemma 2.2 [7]: Let $G$ be a connected graph of order $p$, then: $\sum_{k=0}^{diam(G)} d(G,k) = \binom{p+1}{2} = \frac{1}{2} p(p + 1)$. 

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Theorem 2.3: The Hosoya polynomial of graph $\Gamma_B(Z_p^m)$ where $p$ is a prime number and $m$ is an integer with $m \geq 1$.

$$H(\Gamma_B(Z_p^m); x) = p^m + \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right] x$$

$$+ \frac{1}{2} \left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right] x^2.$$  

Proof: From the definition of the graph $\Gamma_B(R)$, since the vertex $(0)$ connect with every vertex of the graph $\Gamma_B(Z_n)$, so the order of the graph $\Gamma_B(Z_n)$ which represents absolute term of Hosoya polynomial of the graph $\Gamma_B(Z_p^m)$.

Now, we find the coefficient of $x$ that represent size of the graph $\Gamma_B(Z_p^m)$ using the definition of the graph $\Gamma_B(R)$ is the sum of $(p^m - 1)$ of the edges (since the vertex $(0)$ connect with every vertex the graph $\Gamma_B(Z_p^m)$ from the Lemma (2.1), with $a_1$ of the graph $\Gamma(Z_p^m)$ [9] where as $a_1 = \frac{1}{2} \left[(m-1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2\right]$ so we get.

$$a_1 + (p^m - 1) = \frac{1}{2} \left[(m-1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor} + 2\right] + (p^m - 1)$$

$$= \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right].$$

Now, we find the coefficient of $x^2$ as the diameter of the graph $\Gamma_B(Z_p^m)$ is two from the Lemma (2.1) and using Lemma (2.2) so we get:

$$\sum_{k=0}^{\text{diam}(\Gamma_B(Z_p^m))} d(\Gamma_B(Z_p^m),k) = \binom{p^m+1}{2}$$

$$\Rightarrow \frac{p^m(p^m+1)}{2} = d(\Gamma_B(Z_p^m),0) + d(\Gamma_B(Z_p^m),1) + d(\Gamma_B(Z_p^m),2)$$

$$d(\Gamma_B(Z_p^m),2) = \frac{p^m(p^m+1)}{2} - d(\Gamma_B(Z_p^m),0) - d(\Gamma_B(Z_p^m),1)$$

$$= \frac{p^m(p^m+1)}{2} - p^m - \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right]$$

$$= \frac{1}{2} \left[p^{2m} + p^m - 2p^m - mp^m - p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right]$$

$$= \frac{1}{2} \left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right]. \blacksquare$$

$$\therefore H(\Gamma_B(Z_p^m); x) = p^m + \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right] x$$

$$+ \frac{1}{2} \left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right] x^2.$$  

Corollary 2.4: The Wiener index of graph $\Gamma_B(Z_p^m)$ where $p$ is prime number and $m$ is an integer with $m \geq 1$.

$$W(\Gamma_B(Z_p^m)) = \frac{1}{2} \left[2p^{2m} - (m+3)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right].$$  

Proof: Since wiener index is the first derivative polynomial of Hosoya after compensation for a value $x = 1$ so we get:

$$\therefore W(\Gamma_B(Z_p^m)) = \frac{d}{dx} H(\Gamma_B(Z_p^m); x) \bigg|_{x=1}$$

$$\therefore W(\Gamma_B(Z_p^m)) = \frac{d}{dx} \left[p^m + \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right] x$$

$$+ \frac{1}{2} \left[p^{2m} - (m+2)p^m + mp^{m-1} + p^{\lfloor \frac{m}{2} \rfloor}\right] x^2\right] \bigg|_{x=1}$$

$$= 0 + \frac{1}{2} \left[(m+1)p^m - mp^{m-1} - p^{\lfloor \frac{m}{2} \rfloor}\right]$$

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Example 1: The Hosoya polynomial and wiener index of graph $\Gamma_B(Z_{16})$.

The graph is clear $\Gamma_B(Z_{16})$ of formula $\Gamma_B(Z_{p^m})$, where $p = 2$ and $m = 4$.

$\therefore H(\Gamma_B(Z_{p^m}); x) = p^m + \frac{1}{2} \left[(m + 1)p^m - mp^{m-1} - p^{\left\lfloor \frac{m}{2} \right\rfloor} \right]x$

$\therefore W(\Gamma_B(Z_{p^m})) = \frac{1}{2} \left[2p^{2m} - (m + 3)p^m + mp^{m-1} + p^{\left\lfloor \frac{m}{2} \right\rfloor} \right].$

Theorem 2.5: (Coloring of graph $\Gamma_B(Z_{p^m})$).

A- Chromatic number of vertices of the graph $\Gamma_B(Z_{p^m}) = \begin{cases} p^{\frac{m-1}{2}} + 1 & , m \text{ is an odd.} \\ p^{\frac{m}{2}} & , m \text{ is an even.} \end{cases}$

B- Chromatic number of edges of the graph $\Gamma_B(Z_{p^m})$ is $p^m - 1$.

Proof: A- Case 1: if $m$ is an odd.

Since the multiplication of the number $p^{\frac{m+1}{2}}$ by one of its complication

$\left(2p^{\frac{m+1}{2}}, 3p^{\frac{m+1}{2}}, ..., p^{\frac{m-1}{2}}p^{\frac{m+1}{2}} = 0\right)$, that the product is one of its complications of the number $p^m$ which is equal to (0) in the ring $Z_{p^m}$. Or multiplication one complication of the number $p^{\frac{m+1}{2}}$ in another complication of the number $p^{\frac{m+1}{2}}$ that the product is one of the complications of the number $p^m$ which is equal to (0) in the ring $Z_{p^m}$ as in the Figure (2.1).
Clearly the complete graph $k \frac{m-1}{p^{\frac{m}{2}}}$ be subgraph from the graph $\Gamma_B(Z_{p^m})$ (when $m$ is an odd), and also the multiplication of the number $p^{\frac{m-1}{2}}$ or one of its complications $\left(2p^{\frac{m-1}{2}}, 3p^{\frac{m-1}{2}}, \ldots, p^{\frac{m+1}{2}} \cdot p^{\frac{m-1}{2}} = 0\right)$ by the number $p^{\frac{m}{2}}$ or one of its complications is the product $p^m$ or one of its complications which is equal to (0), in the ring $Z_{p^m}$. And thus the complete graph $k \frac{m-1}{p^{\frac{m}{2}}+1}$ is the largest complete subgraph that exist in the graph $\Gamma_B(Z_{p^m})$. And since the chromatic number of vertices of a complete graph $k \frac{m-1}{p^{\frac{m}{2}}+1}$ is $(p^{\frac{m}{2}} + 1)$ [7], so it is the chromatic number of the graph $\Gamma_B(Z_{p^m})$ is $(p^{\frac{m}{2}} + 1)$ and also of vertices (when $m$ is an odd).

A- Case 2: if $m$ is an even:

Since the multiplication of the number $p^{\frac{m}{2}}$ by one of its complications $\left(2p^{\frac{m}{2}}, 3p^{\frac{m}{2}}, \ldots, p^{\frac{m}{2}} \cdot p^{\frac{m}{2}} = 0\right)$, that the product is one of its complications of the number $p^m$ which is equal to (0) in the ring $Z_{p^m}$. Or multiplying one complication of the number $p^{\frac{m}{2}}$ in another complication of the number $p^{\frac{m}{2}}$, that the product is one of its complications of the number $p^m$, which is equal to (0) in the ring $Z_{p^m}$ as in the Figure (2.2).

Clearly the complete graph $k \frac{m}{p^{\frac{m}{2}}}$ is the largest complete subgraph that exist in the graph $\Gamma_B(Z_{p^m})$ (when $m$ is an even). And since the chromatic number of vertices of a complete graph $k \frac{m}{p^{\frac{m}{2}}}$ is $p^{\frac{m}{2}}$ [7], so it is the chromatic number of the graph $\Gamma_B(Z_{p^m})$ is $p^{\frac{m}{2}}$ and also of vertices (when $m$ is an even).

B- From the Lemma (2.1) the vertex (0) connect with every vertex in the graph $\Gamma_B(Z_{p^m})$ then the degree of the vertex (0) is $(p^m - 1)$ so the chromatic number of the edges is $(p^m - 1)$.

**Theorem 2.6** [7]: (kuratowski’s Theorem), The graph $G$ is planar if and only if it does not contain $G$ on subgraph that is homeomorphic to $k_5$ or $k_{3,3}$. 

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Theorem 2.7:

a- The graph $\Gamma_B(Z_{p^m})$ contains a subgraph that is homeomorphic to $k \frac{m-1}{p^{\frac{m+1}{m}}} + 1$ and $k \left( \frac{m+1}{p^{\frac{m+1}{2}}} - \frac{m-1}{p^{\frac{m-1}{2}}} \right) \frac{m-1}{p^{\frac{m-1}{2}}}$ (when $m$ is an odd).

b- The graph $\Gamma_B(Z_{p^m})$ contains a subgraph that is homeomorphic to $k \frac{m}{p^{\frac{m+1}{m}}} + 1$ and $k \left( \frac{m}{p^{\frac{m+1}{2}}} - \frac{m-2}{p^{\frac{m-2}{2}}} \right) \frac{m-2}{p^{\frac{m-2}{2}}}$ (when $m$ is an even).

Proof:

a- From the Theorem (2.5-A-1) we get the first part of the Theorem directly.

Since the multiplication of the number $p^{\frac{m+1}{2}}$ or one of its complications $\left( 2p^{\frac{m+1}{2}}, 3p^{\frac{m+1}{2}}, \ldots, p^{\frac{m+1}{2}} \cdot p^{\frac{m+1}{2}} = 0 \right)$ by number $p^{\frac{m-1}{2}}$ or one of its complications $\left( 2p^{\frac{m-1}{2}}, 3p^{\frac{m-1}{2}}, \ldots, p^{\frac{m-1}{2}} \cdot p^{\frac{m-1}{2}} = 0 \right)$ be the product $p^m$ or one of its complications of the number $p^m$ which is equal to (0) in the ring $Z_{p^m}$. Thus, the graph $\Gamma_B(Z_{p^m})$ contains a subgraph homeomorphic with complete bipartite graph $k \left( \frac{m+1}{p^{\frac{m+1}{2}}} - \frac{m-1}{p^{\frac{m-1}{2}}} \right) \frac{m-1}{p^{\frac{m-1}{2}}}$ it is the largest complete bipartite graph there is in the graph $\Gamma_B(Z_{p^m})$ as in the Figure (2.3).

\[ \text{The graph } \Gamma_B(Z_{p^m}) \]
\[ \text{Fig (2.3)} \]

b- From the Theorem (2.5-A-2) we get the first part of the Theorem directly.

Since the multiplication of the number $p^{\frac{m+2}{2}}$ or one of its complications $\left( 2p^{\frac{m+2}{2}}, 3p^{\frac{m+2}{2}}, \ldots, p^{\frac{m+2}{2}} \cdot p^{\frac{m+2}{2}} = 0 \right)$ by number $p^{\frac{m-2}{2}}$ or one of its complications $\left( 2p^{\frac{m-2}{2}}, 3p^{\frac{m-2}{2}}, \ldots, p^{\frac{m-2}{2}} \cdot p^{\frac{m-2}{2}} = 0 \right)$ be the product $p^m$ or one of its complications of the number $p^m$ which is equal to (0) in the ring $Z_{p^m}$. Thus, the graph $\Gamma_B(Z_{p^m})$ contains a subgraph homeomorphic complete bipartite graph $k \left( \frac{m+2}{p^{\frac{m+2}{2}}} - \frac{m-2}{p^{\frac{m-2}{2}}} \right) \frac{m-2}{p^{\frac{m-2}{2}}}$ it is the largest complete bipartite graph there is in the graph $\Gamma_B(Z_{p^m})$ as in the Figure (2.4).
Remarks:
1. From the Theorem (2.7-a), the only graphs $\Gamma_B(Z_p)$ and $\Gamma_B(Z_8)$ from the formula $\Gamma_B(Z_p^m)$ when $m$ is an odd it does not contain subgraph homeomorphic to $k_5$ or $k_{3,3}$ therefore it is planar graphs by kuratowski’s Theorem.
2. From the Theorem (2.7-b), the only graphs $\Gamma_B(Z_4), \Gamma_B(Z_9)$ and $\Gamma_B(Z_{16})$ from the formula $\Gamma_B(Z_p^m)$ when $m$ is an even it does not contain subgraph homeomorphic to $k_5$ or $k_{3,3}$ therefore it is planar graphs by kuratowski’s Theorem.
3. The only graphs $\Gamma_B(Z_4), \Gamma_B(Z_9), \Gamma_B(Z_{16})$ and $\Gamma_B(Z_p)$ they are colorable for faces.

Example 2: The chromatic number of the graphs $\Gamma_B(Z_{16})$ and $\Gamma_B(Z_{27})$.
The graph is clear $\Gamma_B(Z_{16})$ of formula $\Gamma_B(Z_p^m)$, where $p = 2$ and $m = 4$ and the graph is clear $\Gamma_B(Z_{27})$ of formula $\Gamma_B(Z_p^m)$, where $p = 3$ and $m = 3$.

The chromatic number of vertices the graph $\Gamma_B(Z_p^m)$ is $p^\frac{m}{2}$ (when $m$ is an even).
\[ \therefore \chi(\Gamma_B(Z_{16})) = 4. \]

The chromatic number of edges the graph $\Gamma_B(Z_p^m)$ is $p^m - 1$.
\[ \therefore \tilde{\chi}(\Gamma_B(Z_{16})) = 15. \]
From Theorem (2.7-b) we get the graph $\Gamma_B(Z_{16})$ contains a subgraph that is homeomorphic to $k_4$ and $k_{6,2}$ then the graph $\Gamma_B(Z_{16})$ it is planar by kuratowski’s Theorem.
\[ \therefore f(\Gamma_B(Z_{16})) = 3. \]

The chromatic number of vertices the graph $\Gamma_B(Z_p^m)$ is $p^\frac{m-1}{2} + 1$ (when $m$ is an odd).
\[ \therefore \chi(\Gamma_B(Z_{27})) = 4. \]

The chromatic number of edges the graph $\Gamma_B(Z_p^m)$ is $p^m - 1$.
\[ \therefore \tilde{\chi}(\Gamma_B(Z_{27})) = 26. \]
From Theorem (2.7-a) we get the graph $\Gamma_B(Z_{27})$ contains a subgraph that is homeomorphic to $k_{6,3}$ then the graph $\Gamma_B(Z_{27})$ it is not planar by kuratowski’s Theorem.
3. Some Properties of graph $\Gamma_B(Z_{p^m}q)$.

Theorem 3.1: The Hosoya polynomial of graph $\Gamma_B(Z_{p^m}q)$ where $p, q$ are distinct prime numbers and $m$ is an integer with $m \geq 1$.

$$H(\Gamma_B(Z_{p^m}q); x) = p^m q + \left(\frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{[\frac{m}{2}]}ight) x$$

$$+ \left(\frac{1}{2} [q(p^{m+1}q - 3p - 2mp + 2m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{[\frac{m}{2}]}ight) x^2.$$  

Proof: From the definition of the graph $\Gamma_B(R)$ since the vertex (0) connect with every vertex the graph $\Gamma_B(Z_{p^m}q)$ so the order of the graph $\Gamma_B(Z_{p^m}q)$ which represents absolute term Hosoya polynomial of graph $\Gamma_B(Z_{p^m}q)$.

Now, we find the coefficient of $x$ that represent size of the graph $\Gamma_B(Z_{p^m}q)$ using the definition of the graph $\Gamma_B(R)$ is the sum of $(Z_{p^m}q - 1)$ of the edges (since the vertex (0) connect with every vertex the graph $\Gamma_B(Z_{p^m}q)$ from the Lemma (2.1), with $a_1$ of the graph $\Gamma(Z_{p^m}q)$ [9] where as $a_1 = \frac{1}{2} [2m(q - 1) - (m + 1)p + m] p^{m-1} - p^{[\frac{m}{2}]} + 1$ so we get.

$$a_1 + (p^m q - 1) = \frac{1}{2} [2mq(p - 1) - (m + 1)p + m] p^{m-1} - p^{[\frac{m}{2}]} + 1 + p^m q - 1$$

$$= \frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{[\frac{m}{2}]}.$$

Now, we find the coefficient of $x^2$ as the diameter of the graph $\Gamma_B(Z_{p^m}q)$ is two from the Lemma (2.1) and using Lemma (2.2) so we get.

$$\sum_{k=0}^{\text{diam}(f_B(Z_{p^m}q))} d(\Gamma_B(Z_{p^m}q), k) = \left(\frac{p^m q + 1}{2}\right)$$

$$\Rightarrow p^m q(p^{m+1}q+1) = d(\Gamma_B(Z_{p^m}q), 0) + d(\Gamma_B(Z_{p^m}q), 1) + d(\Gamma_B(Z_{p^m}q), 2)$$

$$d(\Gamma_B(Z_{p^m}q), 2) = \frac{p^m q(p^{m+1}q+1)}{2} - d(\Gamma_B(Z_{p^m}q), 0) - d(\Gamma_B(Z_{p^m}q), 1)$$

$$= \frac{p^m q - \left(\frac{1}{2} [2q(mp - m + p) + m] p^{m-1} - \frac{1}{2} p^{[\frac{m}{2}]}ight)}{2}.$$ 

$$= \frac{1}{2} [q(p^{m+1}q - 3p - 2mp + 2m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{[\frac{m}{2}]}.$$

$$\therefore H(\Gamma_B(Z_{p^m}q); x) = p^m q + \left(\frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{[\frac{m}{2}]}ight) x$$

$$+ \left(\frac{1}{2} [q(p^{m+1}q - 3p - 2mp + 2m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{[\frac{m}{2}]}ight) x^2.$$ 

Corollary 3.2: The Wiener index of $\Gamma_B(Z_{p^m}q)$ where $p, q$ are distinct prime numbers and $m$ is an integer with $m \geq 1$.

$$W\left(\Gamma_B(Z_{p^m}q)\right) = \frac{1}{2} [2q(p^{m+1}q - mp - 2p + m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{[\frac{m}{2}]}$$

Proof: Since wiener index is the first derivative polynomial of Hosoya after compensation for a value $x = 1$ so we get:

$$\therefore W(\Gamma_B(Z_{p^m}q)) = \frac{d}{dx} H(\Gamma_B(Z_{p^m}q); x) \bigg|_{x=1}$$

$$\therefore W(\Gamma_B(Z_{p^m}q)) = \frac{d}{dx} (p^m q + \left(\frac{1}{2} [2q(mp - m + p) - (m + 1)p + m] p^{m-1} - \frac{1}{2} p^{[\frac{m}{2}]}ight) x$$

$$+ \left(\frac{1}{2} [q(p^{m+1}q - 3p - 2mp + 2m) + (m + 1)p - m] p^{m-1} + \frac{1}{2} p^{[\frac{m}{2}]}ight) x^2) \bigg|_{x=1}$$
\[
= \left( 0 + \left( \frac{1}{2} [2q(mp - m + p) - (m + 1)p + m]p^{m-1} - \frac{1}{2} \frac{m}{2} \right) \right) + \left( [q(p^{m+1}q - 3p - 2mp + 2m) + (m + 1)p - m]p^{m-1} + \frac{1}{2} \frac{m}{2} \right) x \right]_{x = 1}
\]

\[
= \frac{1}{2} [2q(p^{m+1}q - mp - 2p + m) + (m + 1)p - m]p^{m-1} + \frac{1}{2} \frac{m}{2} \right].\]

**Example 3:** The Hosoya polynomial and Wiener index of graph \( \Gamma_B(Z_{18}) \).

The graph is clear \( \Gamma_B(Z_{18}) \) of formula \( \Gamma_B(Z_{p^m q}) \), where \( p = 3 \), \( q = 2 \) and \( m = 2 \).

\[
\Gamma_B(Z_{18}) \cdot x = p^m q + \left( \frac{1}{2} [2q(mp - m + p) - (m + 1)p + m]p^{m-1} - \frac{1}{2} \frac{m}{2} \right) x + \left( \frac{1}{2} [2q(p^{m+1}q - mp - 2p + m) + (m + 1)p - m]p^{m-1} + \frac{1}{2} \frac{m}{2} \right) x^2
\]

\[
\Gamma_B(Z_{18}) \cdot x = 18 + 30x + 123x^2.
\]

\[
W(\Gamma_B(Z_{p^m q})) = \frac{1}{2} [2q(p^{m+1}q - mp - 2p + m) + (m + 1)p - m]p^{m-1} + \frac{1}{2} \frac{m}{2}.
\]

\[
W(\Gamma_B(Z_{18})) = 276.
\]

**Theorem 3.3:** (Coloring of graph \( \Gamma_B(Z_{p^m q}) \)).

A- Chromatic number of vertices of the graph \( \Gamma_B(Z_{p^m q}) \):

\[
\Gamma_B(Z_{p^m q}) = \begin{cases} p^{m-1}+2, & \text{m is an odd.} \\ p^m, & \text{m is an even.} \end{cases}
\]

B- Chromatic number of edges of the graph \( \Gamma_B(Z_{p^m q}) \) is \( p^m q - 1 \).

**Proof:**

A- Case 1: if \( m \) is an even:

From the Theorem (2.5-A-1). Since the subgraph \( k \frac{m-1}{p^m+1} \) is the largest complete subgraph exist in the graph \( \Gamma_B(Z_{p^m}) \) (when \( m \) is an odd). It is also clear that the number \( p^m q \) product of multiplication the number \( p^m \) or one of its complications in the number \( q \) or one of its complications thus a new vertex will be added to the complete graph \( k \frac{m-1}{p^m+1} \) so we have the complete graph \( k \frac{m-1}{p^m+1} \) is the largest complete subgraph exist in the graph \( \Gamma_B(Z_{p^m q}) \) hence the chromatic number of the graph \( \Gamma_B(Z_{p^m q}) \) is \( \left( p^{m-1} + 2 \right) \) [7].

A- Case 2: if \( m \) is an even:

From the Theorem (2.5-A-2). Since the subgraph \( k \frac{m}{p^m+1} \) is the largest complete subgraph, exist in the graph \( \Gamma_B(Z_{p^m}) \) (when \( m \) is an even). It is also clear that the number \( p^m q \) product of multiplication the number \( p^m \) or one of its complications in the number \( q \) or one of its complications thus a new vertex will be added to the complete graph \( k \frac{m}{p^m+1} \) so we have the complete graph \( k \frac{m}{p^m+1} \) is the largest complete subgraph exist in the graph \( \Gamma_B(Z_{p^m q}) \) hence the chromatic number of the graph \( \Gamma_B(Z_{p^m q}) \) is \( \left( \frac{m}{p^m+1} \right) \) [7].
B- From the Lemma (2.1) so it is the vertex (0) connect with every vertex the graph \( \Gamma_B(Z_{p^m q}) \) then the degree of the vertex (0) is \((p^m q - 1)\) so it is the chromatic number of the edges is \((p^m q - 1)\).

**Theorem 3.4:** The graph \( \Gamma_B(Z_{p^m q}) \) contains a subgraph that is homeomorphic to \( k_{(p^m - 1),q} \) and \( k_{(p,q-p^m-1),p^m-1} \).

**Proof:** The first part, since the multiplying the number \( p^m \) or one of its complications \((2p^m,3p^m,...,(q-1)p^m,p^m q = 0)\) by number \( q \) or one of its complications \((2q,3q,...,(p^m - 1),q,p^m q = 0)\) be the product \( p^m q \) or one a complications of the number \( p^m q \) which is equal to \((0)\) in the ring \( Z_{p^m q} \). Thus the graph \( \Gamma_B(Z_{p^m q}) \) contains a subgraph homeomorphic complete bipartite graph \( k_{(p^m - 1),q} \), as in the Figure (3.1).

![The graph \( \Gamma_B(Z_{p^m q}) \)](image)

Fig (3.1)

The second part, since the multiplying the number \( pq \) or one of its complications \((2pq,3pq,...,(p^m - 1)pq,p^m q = 0)\) by number \( p^m - 1 \) or one of its complications \((2p^m - 1,3p^m - 1,...,(pq - 1)p^m - 1,p^m q = 0)\) be the product \( p^m q \) or one a complications of the number \( p^m q \) which is equal to \((0)\) in the ring \( Z_{p^m q} \). Thus the graph \( \Gamma_B(Z_{p^m q}) \) contains a subgraph homeomorphic complete bipartite graph \( k_{(pq-p^m-1),p^m-1} \), as in the Figure (3.2).

![The graph \( \Gamma_B(Z_{p^m q}) \)](image)

Fig (3.2)

**Remark:**

From the Theorem (3.4), the only graphs of the formula \( \Gamma_B(Z_{p^m q}) \) when \( q = 2 \) and \( m = 1 \) does not contain a subgraph homeomorphic \( k_{3,3} \) or \( k_5 \) therefore it is planar and colorable for faces. Otherwise, the graphs of the formula \( \Gamma_B(Z_{p^m q}) \) contain a subgraph homeomorphic \( k_{3,3} \) or \( k_5 \) therefore it is not planar graphs by kuratowski’s Theorem.

**Example 4:** The chromatic number of the graphs \( \Gamma_B(Z_{18}) \) and \( \Gamma_B(Z_{22}) \)
The graph is clear $\Gamma_B(Z_{18})$ of formula $\Gamma_B(Z_{p^m})$, where $p = 3$, $q = 2$ and $m = 2$ and the graph is clear $\Gamma_B(Z_{22})$ of formula $\Gamma_B(Z_{p^m})$, where $p = 11$, $q = 2$ and $m = 1$.

The chromatic number of vertices of the graph $\Gamma_B(Z_{p^m})$ is $p^\frac{m-1}{2} + 1$ (when $m$ is an even).

∴ $\chi(\Gamma_B(Z_{18})) = 4$.

The chromatic number of edges the graph $\Gamma_B(Z_{p^m})$ is $p^m q - 1$.

∴ $\bar{\chi}(\Gamma_B(Z_{18})) = 17$.

From Theorem (3.4) we get the graph $\Gamma_B(Z_{18})$ contains a subgraph that is homeomorphic to $k_{3,3}$ then the graph $\Gamma_B(Z_{18})$ it is not planar by kuratowski’s Theorem.

The chromatic number of vertices the graph $\Gamma_B(Z_{p^m})$ is $p^\frac{m-1}{2} + 2$ (when $m$ is an odd).

∴ $\chi(\Gamma_B(Z_{22})) = 3$.

The chromatic number of edges the graph $\Gamma_B(Z_{p^m})$ is $p^m q - 1$.

∴ $\bar{\chi}(\Gamma_B(Z_{22})) = 21$.

From Theorem (3.4) we get the graph $\Gamma_B(Z_{22})$ contains a subgraph that is homeomorphic to $k_{10,2}$ it is the largest complete bipartite graph there is in the graph $\Gamma_B(Z_{22})$ then the graph $\Gamma_B(Z_{22})$ it is planar by kuratowski’s Theorem.

∴ $f(\Gamma_B(Z_{22})) = 3$. 
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