CLASSIFICATION OF DOUBLY DISTRIBUTIVE SKEW HYPERFIELDS AND STRINGENT HYPERGROUPS

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Abstract. A hypergroup is stringent if \( a \boxplus b \) is a singleton whenever \( a \neq -b \). A hyperfield is stringent if the underlying additive hypergroup is. Every doubly distributive skew hyperfield is stringent, but not vice versa. We present a classification of stringent hypergroups, from which a classification of doubly distributive skew hyperfields follows. It follows from our classification that every such hyperfield is a quotient of a field.

1. Introduction

The notion of hyperfield was first introduced by Krasner in [Kra57, Kra83]. It is an algebraic structure similar to a field except that the addition \( \boxplus \) is multivalued. In [Vir10], Viro provided an excellent introduction to and motivation for hyperfields and also gave several good examples, including doubly distributive hyperfields. A hyperfield \( F \) is doubly distributive if \((a \boxplus b)(c \boxplus d) = ac \boxplus ad \boxplus bc \boxplus bd\) for any \( a, b, c, d \in F \). Our interest in doubly distributive hyperfields derive not so much from the actual objects themselves as from the results in tropical geometry and matroid theory based on them.

Fields, the Krasner hyperfield \( K \), the sign hyperfield \( S \), the tropical real hyperfield \( TR \), the ultratriangle hyperfield \( T \triangle \) and the tropical hyperfield \( T \) are all doubly distributive hyperfields.

Viro has illustrated the utility of \( T \) for the foundations of tropical geometry in several interesting papers (cf. [Vir10, Vir11]). \( TR \) and \( T \triangle \) are also very important in tropical geometry.

In [BB17], Baker and Bowler present an algebraic framework which simultaneously generalizes the notion of linear subspaces, matroids, valuated matroids, and oriented matroids, and call the resulting objects matroids over tracts. They also provided two natural notions of matroids over a hyperfield \( F \), weak \( F \)-matroids and strong \( F \)-matroids, and when \( F \) is doubly distributive, the two notions coincide.

Two important special cases of tracts are hyperfields and partial fields. Baker and Bowler defined partial hyperfields, which generalize both of them in a natural way. The property of double distributivity also extends to the partial hyperfields.

The classes of matroids over the doubly distributive hyperfields mentioned before are all important classes of matroids with extra structure. A matroid over a field \( F \) corresponds to a subspace of some \( F^n \). A \( K \)-matroid is just a matroid. An \( S \)-matroid is an oriented matroid. And a \( T \triangle \)-matroid is a valuated matroid, as defined in [DW92].

We will classify the doubly distributive skew hyperfields in Section 5. The classification itself will be described in Section 4 but has the following important consequence:

Definition 1.1. A valuation \( \nu \) of a skew hyperfield \( F \) is a map from \( F \) to \( G \cup \{-\infty\} \), where \((G, <)\) is a linearly ordered group, satisfying

1. \( \nu(x) = -\infty \) if and only if \( x = 0 \).
2. \( \nu(xy) = \nu(x) \cdot \nu(y) \).
3. \( \nu(x) > \nu(y) \) implies \( x \boxplus y = \{x\} \).

Theorem 1.2. For every doubly distributive skew hyperfield \( F \), there is always a valuation \( \nu \) of \( F \) such that \( \nu^{-1}(1_G) \) is either the Krasner hyperfield, or the sign hyperfield, or a skew field.

This compact description is from the paper [BPT19]. In particular, since any nontrivial ordered group is infinite, it follows from our results that the only finite doubly distributive hyperfields are the Krasner hyperfield, the sign hyperfield and the finite fields.

This classification has a number of applications. For example, we use it in Section 7 to show that any doubly distributive skew hyperfield is a quotient of a skew field. Bowler and Pendavingh use it in [BPT19] to
show that any doubly distributive skew hyperfield is perfect and to provide vector axioms for matroids over such skew hyperfields.

Our classification uses a property of the underlying hypergroup which we call stringency. A hyperfield \( F \) is \textit{stringent}, if \( a \uplus b \) is a singleton whenever \( a \neq -b \).

**Theorem 1.6.** Every stringent hypergroup is a wedge sum \( \bigvee_{g \in G} F_g \) where each \( F_g \) is either a copy of the Krasner hypergroup, or a copy of the sign hypergroup, or a group.

This classification of hypergroups is used to derive the classification of doubly distributive skew hyperfields discussed above.

\textbf{Proposition 1.3.} Every doubly distributive skew hyperfield is stringent.

\textbf{Proof.} Let \( F \) be a doubly distributive skew hyperfield. Let \( a, b \in F^x \) be such that \( a \neq -b \). Let \( x, y \in F^x \) be such that \( x, y \in a \uplus b \). By double distributivity, we have

\[(a \uplus b)(x^{-1} \uplus -y^{-1}) = (a \uplus b) \cdot x^{-1} \uplus (a \uplus b) \cdot (-y^{-1}) \supseteq x \cdot x^{-1} \uplus y \cdot (-y^{-1}) = 1 \uplus -1 \ni 0.\]

As \( a \neq -b \), then \( x^{-1} = y^{-1} \), and so \( x = y \). So \( a \uplus b \) is a singleton if \( a \neq -b \). \( \square \)

However, not every stringent skew hyperfield is doubly distributive. The following is a counterexample.

\textbf{Example 1.4.} Let \( F := \mathbb{Z} \cup \{-\infty\} \) be the stringent hyperfield with multiplication given by \( a \circ b = \{a + b\} \) and multiplicative identity 0, and hyperaddition given by

\[a \uplus b = \begin{cases} \{a\} & \text{if } a > b, \\ \{b\} & \text{if } a < b, \\ \{c \mid c < a\} & \text{if } a = b, \end{cases}\]

so that the additive identity is \(-\infty\). Here we use the standard total order on \( \mathbb{Z} \) and set \(-\infty < x \) for all \( x \in \mathbb{Z} \).

\( F \) is not doubly distributive because

\[(0 \uplus 0) \circ (0 \uplus 0) = \{z \mid z < 0\} \circ \{z \mid z < 0\} = \{z \mid z < -1\}, \]

\[0 \uplus 0 \uplus 0 \uplus 0 = \{z \mid z < 0\} \uplus \{z \mid z < 0\} \uplus \{z \mid z < 0\} = \{z \mid z < 0\}.\]

We use our classification of stringent skew hyperfields to derive a classification of stringent skew hyperrings in Section 6. However, this does not give a classification of doubly distributive skew hyperrings, since not every doubly distributive skew hyperring is stringent.

In fact, we classify all stringent hypergroups, and our classification of doubly distributive hyperfields follows from this.

\textbf{Definition 1.5.} Let \((G, <)\) be a totally ordered set, let \((F_g \mid g \in G)\) be a family of hypergroups with a common identity element 0 in each \( F_g \) but otherwise disjoint, and let \( \psi \) be the surjective function from \( \bigcup_{g \in G} F_g^x \) to \( G \) sending \( f \) in \( F_g^x \) to \( g \). We denote the hyperaddition of \( F_g \) by \( \uplus_g \). For any \( g \in G \) we denote by \( g \downarrow \) the set of \( h \in G \) with \( h < g \).

Then the \textbf{wedge sum} \( F = \bigvee_{g \in G} F_g \) is the hypergroup with ground set \( \bigcup_{g \in G} F_g \) and hyperaddition given by

\[x \uplus 0 = 0 \uplus x = \{x\},\]

\[x \uplus y = \begin{cases} \{x\} & \text{if } \psi(x) > \psi(y), \\ \{y\} & \text{if } \psi(x) < \psi(y), \\ (x \uplus_{\psi(x)} y) & \text{if } \psi(x) = \psi(y) \text{ and } 0 \notin x \uplus_{\psi(x)} y, \\ (x \uplus_{\psi(x)} y) \cup \psi^{-1}(\psi(x) \downarrow) & \text{if } \psi(x) = \psi(y) \text{ and } 0 \in x \uplus_{\psi(x)} y. \end{cases}\]

We can also define \( \bigcup_{g \in G} F_g \) up to isomorphism if the \( F_g \)'s don’t have the same identity or aren’t otherwise disjoint, by replacing the \( F_g \)'s with suitably chosen isomorphic copies.

We will show in Section 6 that this construction always yields a hypergroup, and we classify the stringent hypergroups as follows:

\textbf{Theorem 1.6.} Every stringent hypergroup is a wedge sum \( \bigvee_{g \in G} F_g \) where each \( F_g \) is either a copy of the Krasner hypergroup, or a copy of the sign hypergroup, or a group.
The structure of this paper is as follows: After the classification of stringent hypergroups in Section 3, we show in Section 4 that every stringent skew hyperfield arises from a short exact sequence of groups, where the first group in the sequence is the multiplicative group of either the Krasner hyperfield or the sign hyperfield or a skew field, and the last group in the sequence is a totally ordered group. The underlying additive hypergroup is a wedge sum of isomorphic copies of hypergroups. Then we present the classification of doubly distributive skew hyperfields in Section 5 following from the classification of stringent skew hyperfields. We show the surprising result that every stringent skew hyperring is either a skew ring or a stringent skew hyperfield in Section 6. We use our classification to show that every stringent skew hyperfield is a quotient of a skew field in Section 7.

Acknowledgements. We thank Matthew Baker and Laura Anderson (second author’s PhD advisor) for introducing the two authors to each other. We thank Laura Anderson and Tom Zaslavsky, who gave us important comments on early versions of the work. Thanks also to Pascal Gollin for asking whether our classification might hold for all stringent hypergroups.

2. Background

Notation 2.1. Throughout \( G \) and \( H \) denotes groups. For a set \( S \), \( S^\times \) denotes \( S - \{0\} \).

For a function \( f \) from a set \( A \) to a set \( B \), \( \text{supp}(f) \) denotes the set of support of \( f \) (the elements of \( A \) where the function value is not zero).

2.1. Hypergroups, hyperrings and hyperfields.

Definition 2.2. A hyperoperation on a set \( S \) is a map \( \boxplus \) from \( S \times S \) to the collection of non-empty subsets of \( S \).

If \( A, B \) are non-empty subsets of \( S \), we define

\[
A \boxplus B := \bigcup_{a \in A, b \in B} a \boxplus b
\]

and we say that \( \boxplus \) is associative if \( a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c \) for all \( a, b, c \in S \).

All hyperoperations in this paper will be associative.

Definition 2.3. [Vir10] A hypergroup is a tuple \( (G, \boxplus, 0) \) where \( \boxplus \) is an associative hyperoperation on \( G \) such that:

1. \( 0 \boxplus x = x \boxplus 0 = \{x\} \) for all \( x \in G \).
2. For every \( x \in G \) there is a unique element \( x' \) of \( G \) such that \( 0 \in x \boxplus x' \) and there is a unique element \( x'' \) of \( G \) such that \( 0 \in x'' \boxplus x \). Furthermore, \( x' = x'' \). This element is denoted by \(-x\) and called the hyperinverse of \( x \).
3. (Invertibility of sums) \( x \in y \boxplus z \) if and only if \(-x \in -z \boxplus -y \).

A hypergroup is said to be commutative if \( x \in y \boxplus z \) if and only if \( x \in z \boxplus y \).

Theorem 2.4. [Vir10] In Definition 2.3, the axiom (3) can be replaced by

(Reversibility property) \( x \in y \boxplus z \) implies \( y \in x \boxplus -z \) and \( z \in -y \boxplus x \).

The Reversibility property was introduced by Marshall in [Mar06].

Definition 2.5. A skew hyperring is a tuple \( (R, \odot, \boxplus, 1, 0) \) such that:

1. \( (R, \odot, 1) \) is a monoid.
2. \( (R, \boxplus, 0) \) is a commutative hypergroup.
3. (Absorption rule) \( x \odot 0 = 0 \odot x = 0 \) for all \( x \in R \).
4. (Distributive Law) \( a \odot (x \boxplus y) = (a \odot x) \boxplus (a \odot y) \) and \( (x \boxplus y) \odot a = (x \odot a) \boxplus (y \odot a) \) for all \( a, x, y \in R \).

A hyperring is a skew hyperring with commutative multiplication.

A skew hyperring \( F \) is called a skew hyperfield if \( 0 \neq 1 \) and every non-zero element of \( F \) has a multiplicative inverse.

A hyperfield is then a skew hyperfield with commutative multiplication.
Definition 2.6. Let $F$ and $G$ be skew hyperrings. We may define a skew hyperring $F \times G$ with $(x_1, y_1) \sqoplus (x_2, y_2)$ defined as $(x_1 \sqoplus_F x_2) \times (y_1 \sqoplus_G y_2)$ and multiplication defined pointwise. Its additive identity is $(0_F, 0_G)$ and its multiplicative identity is $(1_F, 1_G)$. We call $F \times G$ the product of $F$ and $G$.

Let $x, y \in F$, we will sometimes write $xy$ instead of $x \odot y$ if there is no risk of confusion.

Example 2.7. In [Vir10], Viro provides a good introduction to hyperfields. Several of the following hyperfields were first introduced there.

(1) If $F$ is a field, then $F$ is a hyperfield with $a \odot b = a \cdot b$ and $a \oplus b = \{a + b\}$, for any $a, b \in F$.

(2) The Krasner hyperfield $K := \{0, 1\}$ has the usual multiplication rule and the hyperaddition is defined by $0 \oplus x = \{x\}$ for $x \in K$ and $1 \oplus -1 = \{0, 1\}$.

(3) The sign hyperfield $S := \{0, 1, -1\}$ has the usual multiplication rule and the hyperaddition is defined by $0 \oplus x = \{x\}, x \oplus x = \{x\}$ for $x \in S$ and $1 \oplus -1 = \{0, 1, -1\}$.

(4) The phase hyperfield $P := \mathbb{S}^1 \cup \{0\}$ has the usual multiplication rule and the hyperaddition is defined by $0 \oplus x = \{x\}, x \oplus -x = \{x, -x, 0\}$ and $x \oplus y = \left\{\frac{ax + by}{|ax + by|} : a, b \in \mathbb{R}_{\geq 0}\right\}$ for $x, y \in S^1$ with $y \neq -x$.

(5) The triangle hyperfield $\triangle := \mathbb{R}_{\geq 0}$ has the usual multiplication rule and the hyperaddition is defined by $x \oplus y = \{z | |z - y| \leq z \leq x + y\}$.

(6) The tropical real hyperfield $TR := \mathbb{R}$ has the usual multiplication rule and the hyperaddition is defined by

$$x \oplus y = \begin{cases} \{x\} & \text{if } |x| > |y|, \\ \{y\} & \text{if } |x| < |y|, \\ \{x\} & \text{if } x = y, \\ \{z | |z| \leq |x|\} & \text{if } x = -y. \end{cases}$$

(7) The tropical hyperfield $T_+ := \mathbb{R} \cup \{-\infty\}$ has the multiplication rule defined by $x \odot y = x + y$ (with $-\infty$ as an absorbing element), for $x, y \in T_+$. The hyperaddition is defined by

$$x \oplus y = \begin{cases} \max(x, y) & \text{if } x \neq y, \\ \{z | z \leq x\} & \text{if } x = y. \end{cases}$$

Here we use the standard total order on $\mathbb{R}$ and set $-\infty < x$ for all $x \in \mathbb{R}$. The additive hyperidentity is $-\infty$ and the multiplicative identity is $0$.

(8) The tropical complex hyperfield $TC := \mathbb{C}$ has the usual multiplication rule and the hyperaddition is defined by

$$x \oplus y = \begin{cases} \{x\} & \text{if } |x| > |y|, \\ \{y\} & \text{if } |x| < |y|, \\ \{x | |x| \leq |y|\} & \text{if } |x| = |y| \text{ and } x \neq -y, \\ \{z : |z| \leq |x|\} & \text{if } x = -y. \end{cases}$$

(9) The ultratriangle hyperfield $T\triangle := \mathbb{R}_{\geq 0}$ has the usual multiplication rule and the hyperaddition is defined by

$$x \oplus y = \begin{cases} \max(x, y) & \text{if } x \neq y, \\ \{z | z \leq x\} & \text{if } x = y. \end{cases}$$

Definition 2.8. [Vir10, BB17] A skew hyperring $R$ is said to be doubly distributive if for any $a, b, c$ and $d$ in $R$, we have $(a \oplus b)(c \oplus d) = ac \oplus ad \oplus bc \oplus bd$.

Example 2.9. Fields, $K, S, TR, T\triangle$ are all doubly distributive, but $P, \triangle$ and $TC$ are not doubly distributive.

Definition 2.10. A hypergroup $F$ is said to be stringent if for any $a, b \in F$ the set $a \oplus b$ is a singleton whenever $a \neq -b$.

A skew hyperring is said to be stringent if the underlying hypergroup $F$ is stringent.

\footnote{Although this is the definition of the phase hyperfield in Viro’s paper, more recent papers have often worked with a variant in which, in this definition, $\mathbb{R}_{\geq 0}$ is replaced by $\mathbb{R}_{>0}$ at the one place where it appears. The confusion on this point is exacerbated by the fact that Viro incorrectly claims that his phase hyperfield is the same as the quotient hyperfield of the complex numbers by the positive real numbers, but this construction actually gives the second variant.}
2.2. Homomorphism.

Definition 2.11. A hypergroup homomorphism is a map \( f : G \to H \) such that \( f(0) = 0 \) and \( f(x \oplus y) \subseteq f(x) \oplus f(y) \) for all \( x, y \in G \).

A hyperring homomorphism is a map \( f : R \to S \) which is a homomorphism of additive hyperrings as well as a homomorphism of multiplicative monoids (i.e., \( f(1) = 1 \) and \( f(x \odot y) = f(x) \odot f(y) \) for \( x, y \in R \)).

A hyperfield homomorphism is a homomorphism of the underlying hyperrings.

A hypergroup (resp. hyperring, hyperfield) isomorphism is a bijection \( f : G \to H \) which is a hypergroup (resp. hyperring, hyperfield) homomorphism and whose inverse is also a hypergroup (resp. hyperring, hyperfield) homomorphism.

Example 2.12. The map \( \exp : \mathbb{T}_+ \to T \Delta \) is a hyperfield isomorphism.

3. Classification of stringent hypergroups

Our aim in this section is to prove Theorem 1.6, the Classification Theorem for stringent hypergroups. We will work with the definition of wedge sums given as Definition 1.4. First we must show that \( F := \bigvee_{g \in G} F_g \) is indeed a hypergroup.

Lemma 3.1. \( F \) is again a hypergroup. If every hypergroup in \((F_g | g \in G)\) is stringent, then so is \( F \). If every hypergroup in \((F_g | g \in G)\) is commutative, then so is \( F \).

Proof. For associativity, suppose we have \( x_1, x_2, x_3 \in F \). If any of them is 0, then associativity is clear, so suppose that each \( x_i \) is in \( H \). If one of the elements \( \psi(x_i) \) of \( G \), say \( \psi(x_{i_0}) \), is bigger than the others, then \( x_i \circ (x_2 \circ x_3) = \{ x_{i_0} \} = (x_1 \circ x_2) \circ x_3 \). If one of the \( \psi(x_i) \) is smaller than the others, then both \( x_1 \circ (x_2 \circ x_3) \) and \( (x_1 \circ x_2) \circ x_3 \) evaluate to the sum of the other two \( x_j \). So we may suppose that all \( \psi(x_i) \) are equal, taking the common value \( g \). If \( 0 \notin x_1 \circ g \circ x_2 \circ g \circ x_3 \), then both \( x_1 \circ (x_2 \circ g) \) and \( (x_1 \circ x_2) \circ g \) evaluate to \( x_1 \circ g \circ x_2 \circ g \), whereas if \( 0 \in x_1 \circ g \circ x_2 \circ g \circ x_3 \), then both evaluate to \( (x_1 \circ g \circ x_2 \circ g) \cup \psi^{-1}(g \downarrow) \). The hyperinverse of 0 is 0 and the hyperinverse of any other \( x \) is its hyperinverse in \( F_\psi(x) \), and 0 is the additive identity.

For invertibility of sums, suppose we have \( x, y, z \in F \). We would like to show that \( x \oplus y \oplus z \equiv x \oplus z \oplus y \oplus z \). It suffices to prove one direction, say if \(-x \in -z \oplus -y \). If \( y \prec \psi(z) \), then \( x \in y \oplus z = \{ z \} \) and \( \psi(-y) \prec \psi(-z) \). So \(-x \oplus -y = \{ -z \} = \{ -x \} \). Similarly, we have if \( y \succ \psi(z) \), then \(-x \oplus -y = \{ -x \} \). If \( \psi(x) = \psi(y) = \psi(z) \), then the statement holds by the reversibility of the hypergroup \((F_\psi(x), \circ_\psi(y), 0)\). Otherwise we have \( \psi(x) \prec \psi(y) = \psi(z) \), and so \( y = -z \). Then \( \psi(-x) \prec \psi(-y) = \psi(-z) \) and so \( -x \in -z \oplus -y \).

Then, we would like to show that \( F \) is stringent if every hypergroup \( F_g \) in \((F_g | g \in G)\) is. By definition of \( F \), we just need to show that for any \( x, y \in F \) with \( \psi(x) = \psi(y) \) and \( 0 \notin x \circ_\psi(x) y \), \( x \circ_\psi(x) y \) is a singleton. As \( F_\psi(x) \) is stringent and \( 0 \notin x \circ_\psi(x) y \), then \( x \circ_\psi(x) y \) is a singleton. So \( x \oplus y = x \circ_\psi(x) y \) is also a singleton.

Finally, it is clear that \( \circ_\psi \) is commutative if each \( \circ_g \) is commutative. 

Now we begin the proof of the Classification Theorem. We first introduce a useful lemma. Note that this lemma automatically holds for stringent commutative hypergroups, so readers only interested in that case may skip the proof.

Lemma 3.2. Let \( F \) be a stringent hypergroup. If \( y \in x \oplus y \), then \( y \in y \oplus x \).

Proof. We will divide the proof into four cases.

Case 1: If \( x = y \), this is immediate.

Case 2: If \( x = -y \), then by reversibility we get
\[ y \in x \oplus y \Rightarrow y \in -y \oplus y \Rightarrow y \in y \oplus y \Rightarrow y \in y \oplus -y \Rightarrow y \in y \oplus x. \]

Case 3: If \( y = -y \), then by reversibility we get
\[ y \in x \oplus y \Rightarrow x \in y \oplus -y \Rightarrow x \in -y \oplus y \Rightarrow y \in y \oplus x. \]

Case 4: Now we suppose \( x \notin \{ y, -y \} \) and \( y \neq -y \). Let \( z \in F^x \) be such that \( y \oplus x = \{ z \} \) and let \( t \in F^x \) be such that \(-y \oplus -y = \{ t \} \). Then by associativity we get
\[ z \oplus y \oplus t = (y \oplus x) \oplus y \oplus (-y \oplus -y) = y \oplus (x \oplus y) \oplus -y \oplus -y = y \oplus y \oplus -y \oplus -y \oplus -y \equiv 0. \]
Lemma 3.3. We define a relation \( <_F \) on \( F^\times \) by \( x <_F y \) if \( x \uplus y = y \uplus x = \{ y \} \) but \( x \neq y \).

Definition 3.4. \( <_F \) is a strict partial order on \( F^\times \).

Proof. Irreflexivity is built into the definition, so it remains to check transitivity. Suppose that \( x <_F y <_F z \). Then \( x \uplus y = x \uplus y \uplus z = z \uplus y = x \). By stringency, \( x \neq y \uplus z \). Therefore, \( x \neq y \) and \( x \neq z \). We cannot have \( x = z \), since then \( \{ y \} = y \uplus x = y \uplus z = \{ z \} \), contradicting our assumptions. \( \square \)

Lemma 3.5. If \( x <_F y \), then \( \pm x <_F \pm y \).

Proof. By Lemma 3.3, we have \( \pm x <_F \pm y \). Suppose that \( z \not<_F y \). If \( z \in \{ y \} \), then we have \( x <_F z \). Otherwise, \( y \uplus z = y \uplus y = y \uplus z = \{ z \} \). So by stringency, \( x \uplus z = x \uplus y \uplus z = z \uplus y \uplus z = \{ z \} \). This implies that the relation \( <_F \) is a strict total order on \( \pm F^\times \). If \( x <_F z \), then \( x = z = y \) or \( x <_F z \).

Lemma 3.6. If \( x <_F y \), then for any \( z \in F^\times \) we have either \( x <_F z \) or \( z <_F y \).

Proof. By Lemma 3.5, we have \( \pm x <_F \pm y \). Suppose that \( z \not<_F y \). If \( z \in \{ y \} \), then we have \( x <_F z \). Otherwise, \( y \not\in z \uplus y \) and \( y \not\in y \uplus z \). Then \( 0 \not\in z \uplus y \uplus y = y \uplus z \). So by stringency, \( z \uplus z = z \uplus y \uplus z = \{ z \} \). However, \( x \in y \uplus y \uplus y \) and \( x \in y \uplus y \), since \( x <_F y \). So \( z \uplus x = \{ z \} \) and \( z \uplus x = \{ z \} \). Now if \( z \neq x \) this implies that \( x <_F z \), but if \( z = x \) then we have \( z <_F y \). \( \square \)

Now we define a relation \( \sim_F \) on \( F^\times \) by \( x \sim_F y \) if and only if both \( x \not<_F y \) and \( y \not<_F x \).

Lemma 3.7. \( \sim_F \) is an equivalence relation.

Proof. \( \sim_F \) is clearly reflexive and symmetric. For transitivity, suppose that \( x \sim_F y \) and \( y \sim_F z \). If \( x <_F z \) then either \( x <_F y \), contradicting \( x \sim_F y \), or else \( y <_F z \), contradicting \( y \sim_F z \), so this is impossible. Similarly we have \( z \not<_F x \). So \( x \sim_F z \).

Lemma 3.8. If \( x \sim_F y \), then \( x \not<_F z \) or \( z \not<_F x \).

Proof. Suppose that \( x \sim_F y \) and \( x <_F z \). Then since \( y \not<_F x \) we have \( x <_F z \). The case \( x <_F y \sim_F z \) is similar. \( \square \)

This implies that the relation \( <_F \) lifts to a relation (denoted by \( <'_F \)) on the set \( G \) of \( \sim_F \)-equivalence classes.

Lemma 3.9. \( (G, <'_F) \) is a totally ordered set.

Proof. \( <'_F \) is a strict total order on \( G \) from the definition of \( \sim_F \). \( \square \)

Lemma 3.10. For every \( x \in F^\times \), \( -x \sim_F x \).

Proof. As \( 0 <_F -x \), we have \( x \not<_F -x \) and \( -x \not<_F x \). So \( -x \sim_F x \).

Lemma 3.11. Let \( x, y, z \in F^\times \) with \( x \neq -y \), \( y \neq -z \) and \( z \neq -x \). If \( 0 < x \uplus y \uplus z \), then \( x \sim_F y \sim_F z \).

Proof. If not, then without loss of generality we have \( x <_F y \), so \( -z \in x \uplus y = \{ y \} \), giving \( y = -z \), contradicting our assumptions. \( \square \)

Lemma 3.12. Let \( \{ x_i | i \in I \} \) be a finite family of elements of \( F \), and \( z \in F \) with \( x_i <_F z \) for all \( i \in I \). Then for any \( y \in \biguplus_{i \in I} x_i \), we have \( y \not<_F z \).

Proof. It suffices to prove this when \( I \) has just 2 elements, say \( x_1 \) and \( x_2 \), since the general result then follows by induction. Suppose \( x_1, x_2 <_F z \) and \( y \in x_1 \uplus x_2 \), then we have

\[
y \uplus z \subseteq x_1 \uplus x_2 \uplus z = x_1 \uplus z = \{ z \}\]

and

\[
z \uplus y \subseteq z \uplus x_1 \uplus x_2 = z \uplus x_2 = \{ z \}.
\]

So \( y \uplus z = \{ z \} \) and \( z \uplus y = \{ z \} \). If \( z \in x_1 \uplus x_2 \), then \( -x_1 \in x_2 \uplus -z = \{ -z \} \), contradicting \( x_1 <_F z \). So \( z \not\in x_1 \uplus x_2 \), and so \( z \neq y \). So \( y \not<_F z \). \( \square \)
It follows from the above results that the sum \( x \boxplus y \) is given by \( \{ x \} \) if \( x >_F y \), by \( \{ y \} \) if \( x <_F y \), by \( \{ z \} \) for some \( z \) in the \( \sim_F \)-equivalence class of \( x \) and \( y \) if \( x \sim_F y \) but \( x \neq -y \), and by some subset of that class together with \( \{ t \} \) if \( t <_F x \) or \( t >_F y \). This looks very similar to the hyperaddition given in Definition 1.6.

We now want to consider the structure of the equivalence classes. Let \( g \) be an equivalence class in \( G \) and let \( F_g \) be the set \( g \cup \{ 0 \} \). We can define a multivalued binary operation \( \boxplus_g \) on \( F_g \) by \( x \boxplus_g y = (x \boxplus y) \cap F_g \).

**Lemma 3.13.** For any element \( g \) in \( G \), \( F_g \) is again a hypergroup, with hyperaddition given by \( \boxplus_g \).

**Proof.** For every \( x \in F_g \), we have \( 0 \boxplus_g x = \{ x \} \cap F_g = \{ x \} \).

Suppose \( 0 \in x \boxplus_g y \), then \( 0 \in x \boxplus y \), and so \( y = -x \). Similarly, if \( 0 \in y \boxplus_g x \), then \( y = -x \).

For invertibility of sums, let \( x, y, z \in F_g \) with \( x \in y \boxplus z \). Then we have \( x \in y \boxplus z \). By invertibility of sums of \( F \), \( -x \in -z \boxplus y \). So \( -x \in -z \boxplus_g y \).

For associativity, suppose we have \( x, y, z \in F_g \). We would like to show that

\[
(x \boxplus_g y) \boxplus_g z = x \boxplus_g (y \boxplus_g z).
\]

Let \( t \in F_g \). Let us first show that \( t \in x \boxplus_g (y \boxplus_g z) \) if and only if \( t \in x \boxplus (y \boxplus z) \). It is clear that \( x \boxplus_g (y \boxplus_g z) \subseteq x \boxplus (y \boxplus z) \). So it suffices to prove the other direction. So suppose \( t \in x \boxplus (y \boxplus z) \), so that there exists \( k \in F \) such that \( k \in y \boxplus z \) and \( t \in x \boxplus k \). If \( k \in F_g \), then we are done. If not, we have \( y = -z \) and \( k \in F \). So we also have \( k <_F y \), and so \( t = x \in x \boxplus_g 0 \subseteq x \boxplus_g (y \boxplus_g z) \). Similarly, we can also get \( t \in (x \boxplus_g y) \boxplus_g z \) if and only if \( t \in (x \boxplus y) \boxplus z \). By associativity of \( F \), \( (x \boxplus y) \boxplus z = x \boxplus (y \boxplus z) \). So \((x \boxplus_g y) \boxplus_g z = x \boxplus_g (y \boxplus_g z) \).

**Lemma 3.14.** For any element \( g \) in \( G \), \( F_g \) is either isomorphic to \( \mathbb{K} \) or isomorphic to \( \mathbb{S} \) or is a group.

**Proof.** For any \( y \) and any \( x \) with \( x \in y \boxplus -y \), we have \( y \in x \boxplus y \) and \( x <_F y \) unless \( x \in \{ -y, 0, y \} \). So for any \( y \in F_g \) we have \( y \boxplus_g -y \subseteq \{ -y, 0, y \} \). Now suppose that there is some \( y \in F_g \) with \( y \boxplus_g -y \neq \{ 0 \} \). Then \( y \) is nonzero and \( y, -y \in -y \boxplus_g y \). Suppose for a contradiction that there is some \( z \in F_g \setminus \{ -y, 0, y \} \), and let \( t \) be the unique element of \(-y \boxplus z \). Then by Lemma 3.5, \( t \notin \{ y, -y \} \), since \( z \notin_F y \). So \( y \boxplus t = \{ t \} \). Thus

\[
y \in y \boxplus_g 0 \subseteq (-y \boxplus_g y) \boxplus_g (t \boxplus_g -t) = -y \boxplus_g (y \boxplus_g t) \boxplus_g -t = -y \boxplus_g z \boxplus_g -t = t \boxplus_g -t,
\]

and so \( y \notin \{ -t, 0, t \} \), which is the desired contradiction.

So if there is any \( y \) with \( y \boxplus_g -y \neq \{ 0 \} \), then \( F_g = \{ -y, 0, y \} \). It is now not hard to check that in this case if \( y = -y \) then \( F_g \cong \mathbb{K} \), and if \( y \neq -y \) then \( F_g \cong \mathbb{S} \). On the other hand, if there is no such \( y \) then the hyperaddition on \( F_g \) is single-valued, and so \( F_g \) is a group.

We can finally prove the Classification Theorem.

**Proof of Theorem 1.6.** Let \( H \) be \( F^\times \), let \( G \) be given as above and let \( \psi \) be the map sending an element \( h \) of \( H \) to its equivalence class in \( G \). For any \( x \) and \( y \) in \( H \), if \( \psi(x) >_F \psi(y) \) then \( x >_F y \) and so \( x \boxplus y = \{ x \} \). Similarly if \( \psi(x) <^F \psi(y) \) then \( x \boxplus y = \{ y \} \). If \( \psi(x) = \psi(y) \) then \( x \boxplus_{\psi(x)} y = (x \boxplus y) \cap [x]_{\sim_F} \). So by the remarks following Lemma 6.12 we have that the hyperaddition of \( F \) agrees with that of \( \bigcup_{g \in G} F_g \) in this case as well.

4. Classification of stringent skew hyperfields

**Definition 4.1.** Let \( F \) be a skew hyperfield and let \( G \) be a totally ordered group. Suppose that we have a short exact sequence of groups

\[
1 \to F^\times \overset{\varphi}{\to} H \overset{\psi}{\to} G \to 1.
\]

Since \( \varphi \) is injective, by replacing \( H \) with an isomorphic copy if necessary we may (and shall) suppose that \( \varphi \) is the identity. As usual, we define \( x^h \) to be \( h^{-1} \cdot x \cdot h \) for \( x, h \in H \). We extend this operation by setting \( 0^h := 0 \).

We say that the short exact sequence has stable sums if for each \( h \in H \) the operation \( x \mapsto x^h \) is an automorphism of \( F \). Since this operation clearly preserves the multiplicative structure, this is equivalent to the condition that it is always an automorphism of the underlying additive hypergroup. Furthermore, any short exact sequence as above with \( H \) abelian automatically has stable sums.

Suppose now that we have a short exact sequence with stable sums as above. Then we may define a hyperfield with multiplicative group \( H \) as follows. We begin by choosing some object not in \( H \) to serve as the additive identity, and we denote this object by \( 0 \). For each \( g \) in \( G \), let \( A_g \) be \( \psi^{-1}(g) \cup \{ 0 \} \). For any \( h \)
in $\psi^{-1}(g)$ there is a bijection $\lambda_h$ from $F$ to $A_g$ sending $0_F$ to 0 and $x$ to $h \cdot x$ for $x \in F^\times$, and so there is a unique hypergroup structure on $A_g$ making $\lambda_h$ an isomorphism of hypergroups. Furthermore, this structure is independent of the choice of $h$ since for $h, h' \in \psi^{-1}(g)$ the map $\lambda_h^{-1} \cdot \lambda_{h'}$ is just left multiplication by $h^{-1} \cdot h'$, which is an automorphism of the additive hypergroup of $F$. In this way we obtain a well defined hypergroup structure on $A_g$, whose hyperaddition we denote by $\boxplus_{g}$.

Then the $G$-layering $F \rtimes_{H, \psi} G$ of $F$ along this short exact sequence has as ground set $H \cup \{0\}$. The multiplication is given by $x \cdot y = 0$ if $x$ or $y$ is 0 and by the multiplication of $H$ otherwise. $H \cup \{0\}$ is the underlying set of the hypergroup $\bigvee_{g \in G} A_g$, and we take the hyperaddition of $F \rtimes_{H, \psi} G$ to be given by that of this hypergroup. Explicitly: the hyperaddition is given by taking 0 to be the additive identity and setting

$$x \boxplus y = \begin{cases} \{x\} & \text{if } \psi(x) > \psi(y) \\
\{y\} & \text{if } \psi(x) < \psi(y) \\
x \boxplus_{\psi(x)} y & \text{if } \psi(x) = \psi(y) \text{ and } 0 \notin x \boxplus_{\psi(x)} y \\
(x \boxplus_{\psi(x)} y) \cup \psi^{-1}(\psi(x) \downarrow) & \text{if } \psi(x) = \psi(y) \text{ and } 0 \in x \boxplus_{\psi(x)} y
\end{cases}$$

**Lemma 4.2.** $F \rtimes_{H, \psi} G$ is again a skew hyperfield. If $F$ is stringent, then so is $F \rtimes_{H, \psi} G$.

**Proof.** As shown in Lemma 3.1, $\bigvee_{g \in G} A_g$ is a commutative hypergroup. So it suffices to show that $\cdot$ distributes over $\boxplus$. For left distributivity, we must prove an equation of the form $x_1 \cdot (x_2 \boxplus x_3) = x_1 \cdot x_2 \boxplus x_1 \cdot x_3$. As usual, if any of the $x_i$ is 0, then this is trivial, so we suppose that each $x_i$ is in $H$. If $\psi(x_2) > \psi(x_3)$, then both sides are equal to $x_1 \cdot x_2$. If $\psi(x_2) < \psi(x_3)$, then both sides are equal to $x_1 \cdot x_3$. So we may assume that $\psi(x_2) = \psi(x_3)$ and we call their common value $g$. Then $x_2 \boxplus_g x_3 = \lambda_{x_2}(1 \boxplus_R x_2^{-1} \cdot x_3)$ and $(x_1 \cdot x_2) \boxplus_{\psi(x_1) \cdot g} (x_1 \cdot x_3) = \lambda_{x_1 \cdot x_2}(1 \boxplus_R x_2^{-1} \cdot x_3)$. So if $0 \notin x_2 \boxplus_g x_3$, then also $0 \notin (x_1 \cdot x_2) \boxplus_{\psi(x_1) \cdot g} (x_1 \cdot x_3)$, and so both sides of the equation are equal to $x_1 \cdot (x_2 \boxplus_g x_3)$. If $0 \in x_2 \boxplus_g x_3$, then also $0 \in (x_1 \cdot x_2) \boxplus_{\psi(x_1) \cdot g} (x_1 \cdot x_3)$, and so both sides of the equation are equal to $x_1 \cdot (x_2 \boxplus_g x_3) \cup x_1 \cdot \psi^{-1}(g \downarrow)$.

For the right distributivity, we need to consider bijections $\lambda'_h : F \to A_{\psi(h)}$ similar to the $\lambda_h$. We take $\lambda'_h(x)$ to be $x \cdot h$ for $x \in F^\times$ and to be 0 for $x = 0_F$. Then since $\lambda'_h(x) = \lambda_h(x)^h$ for any $x$ and the short exact sequence has stable sums, the $\lambda'_h$ are also hyperfield isomorphisms. So we may argue as above but with the $\lambda'_h$ in place of the $\lambda_h$.

Finally, we must show that $F \rtimes_{H, \psi} G$ is stringent if $F$ is. By definition of $F \rtimes_{H, \psi} G$, we just need to show that for $x, y \in F \rtimes_{H, \psi} G$ with $\psi(x) = \psi(y)$ and $0 \notin x \boxplus_{\psi(x)} y$, $x \boxplus y$ is a singleton. As $F$ is stringent and $0 \notin x \boxplus_{\psi(x)} y$, then $x \boxplus_{\psi(x)} y$ is a singleton. So $x \boxplus y = x \boxplus_{\psi(x)} y$ is also a singleton. \hfill $\Box$

**Example 4.3.** If $F$ is the Krasner hyperfield, $G$ and $H$ are both the group of real numbers, and $\psi$ is the identity, then $F \rtimes_{H, \psi} G$ is the tropical hyperfield.

**Example 4.4.** $F := \mathbb{Z} \cup \{ -\infty \}$ in Example 1.3 can arise from the short exact sequence of groups

$$0 \to F_2^\times \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \to 0.$$

**Example 4.5.** In [AD18], Anderson and Davis draw a diagram encoding many popular and important hyperfields and homomorphisms. Our classification gives a good relationship between the hyperfields in each column and we can construct each hyperfield in the third row from the corresponding element of the second row and the ordered group $\mathbb{R}_{>0}$.

1. From the short exact sequence of groups

$$1 \to S^\times \to \mathbb{R}^\times \to \mathbb{R}_{>0} \to 1,$$

we can get the tropical real hyperfield $\text{TR} = \mathbb{S} \times \mathbb{R}_{>0}$.

2. From the short exact sequence of groups

$$1 \to \mathbb{P}^\times \to \mathbb{C}^\times \to \mathbb{R}_{>0} \to 1,$$

we can get the tropical complex hyperfield $\text{TC} = \mathbb{P} \times \mathbb{R}_{>0}$.

3. From the short exact sequence of groups

$$1 \to \mathbb{K}^\times \to \mathbb{R}_{>0} \to \mathbb{R}_{>0} \to 1,$$

we can get the ultratriangle hyperfield $\mathbb{T} \triangle = \mathbb{K} \times \mathbb{R}_{>0}$.
Since in each column the second element is obtained as a quotient of the first by a \( \mathbb{R}_{>0} \)–subgroup, this operation of putting back the factor of \( \mathbb{R}_{>0} \) yields a hyperfield on the same ground set as the top element of the column.

Our aim is to show that every stringent skew hyperfield is of the form \( F \times_{H, \psi} G \) with \( F \) either the Krasner hyperfield or the sign hyperfield or a skew field.

Let \( R \) be a stringent skew hyperring. By Theorem 1.6 we can classify \( R \) to be the wedge sum \( \bigvee_{g \in G} R_g \) with a surjective mapping \( \psi \) from \( R^\times \) to the set \( G \) defined in last section and an ordering \( \prec'_R \) on \( G \) by \( \psi(x) \prec'_R \psi(y) \) if and only if \( x \oplus y = \{ y \} \) but \( x \neq y \), where the hypergroup \( R_g \) is either isomorphic to \( \mathbb{K} \) or isomorphic to \( S \) or is an abelian group. Thus by distributivity of \( R \), \( \psi(x) \prec'_R \psi(y) \) if and only if \( \psi(ax) \prec'_R \psi(ay) \) if and only if \( \psi(xa) \prec'_R \psi(ya) \) for \( a \in R^\times \). So the multiplication of \( R \) lifts to a multiplication on \( G \) respecting the ordering, with identity \( \psi(1) := 1_G \).

**Lemma 4.6.** \((G, \cdot, \prec'_R)\) is a totally ordered monoid. If \( R \) is a skew hyperfield, then \( G \) is a totally ordered group.

**Proof.** This is obvious by Lemma 3.9. \( \square \)

Now we want to consider the structure of \( R_g \).

**Lemma 4.7.** \( R_{1_G} \) is again a skew hyperring, with hyperaddition given by \( \boxplus_{1_G} \) and multiplication by that of \( R \).

**Proof.** By Lemma 3.13 it suffices to check the distributivity. To prove left distributivity we must show that any element \( t \in R_{1_G} \) of \( x \cdot (y \oplus z) \) is also an element of the same expression evaluated in \( R_{1_G} \). So let \( w \) be an element of \( y \oplus z \) with \( x \cdot w = t \). This second equation implies that the equivalence class of \( \psi(w) \) is \( 1_{G} \), as desired. The right distributivity is similar. \( \square \)

**Lemma 4.8.** If \( R \) is a skew hyperfield, \( R_{1_G} \) is either the Krasner hyperfield or the sign hyperfield or a skew field.

**Proof.** By Lemma 3.14 and Lemma 4.7 we can get \( R_{1_G} \) is either the Krasner hyperfield or the sign hyperfield or a skew ring.

Since \( \sim_R \) respects the multiplication, the multiplicative inverse of anything equivalent to \( 1_R \) is again equivalent to \( 1_R \), so that \( R_{1_G} \) is a skew field if it is a skew ring. \( \square \)

**Lemma 4.9.** For every \( g \in G \), the hypergroup of \( R_g \) is isomorphic to the hypergroup of \( R_{1_G} \).

**Proof.** Let \( a \in R^\times_g \). Define \( f : R_g \to R_{1_G} \) by sending 0 to 0 and \( x \) in \( R^\times_g \) to \( a^{-1} \cdot x \). Since \( f \) has an inverse operation, namely left multiplication by \( a \), this is a bijection. Now we would like to show \( f(x \oplus_y y) = f(x) \boxplus_{1_G} f(y) \).

\[
\begin{align*}
f(x \oplus_y y) &= a^{-1} \cdot (x \oplus_y y) \\
&= a^{-1} \cdot ((x \oplus y) \cap R_g) \\
&= (a^{-1} \cdot (x \oplus y)) \cap (a^{-1} \cdot R_g) \\
&= ((a^{-1} \cdot x) \oplus (a^{-1} \cdot y)) \cap R_{1_G} \\
&= (a^{-1} \cdot x) \boxplus_{1_G} (a^{-1} \cdot y) \\
&= f(x) \boxplus_{1_G} f(y)
\end{align*}
\]

Now we can classify the stringent skew hyperfield as follows.

**Theorem 4.10.** Any stringent skew hyperfield \( R \) has the form \( F \times_{H, \psi} G \), where \( F \) is either the Krasner hyperfield or the sign hyperfield or a skew field.

**Proof.** Let \( F \) be \( R_{1_G} \), let \( H \) be \( R^\times \) and let \( G \) be given as above. Let \( \varphi \) be the injection of \( F^\times \) as a subgroup of \( H \) and let \( \psi \) be the map sending an element \( h \) of \( H \) to its equivalence class in \( G \). Then

\[
1 \to F^\times \xrightarrow{\varphi} H \xrightarrow{\psi} G \to 1
\]
is a short exact sequence. For any $x$ and $y$ in $H$, if $\psi(x) \succ_R \psi(y)$, then $x \succ_R y$ and so $x \uplus y = \{x\}$. Similarly if $\psi(x) \prec_R \psi(y)$, then $x \downarrow y = \{y\}$. If $\psi(x) = \psi(y)$, then $x \uplus \psi(x)y = x \cdot ((1 \uplus x^{-1} \cdot y) \cap [1]_{\sim_R}) = (x \uplus y) \cap [x]_{\sim_R}$.

So by the remarks following Lemma 3.12 we have that the hyperaddition of $R$ agrees with that of $F \times H, \psi G$ in this case as well.

Using results of Marshall’s paper [Mar06], we can show that the structure is even more constrained if the multiplication of $F$ is commutative (so that $F$ is a stringent hyperfield) and $F_{1_G}$ is the Krasner or the sign hyperfield.

**Proposition 4.11.** Let $R$ be a stringent skew hyperfield with $R_{1_G} = \mathbb{S}$ and let $a \in R^\times - \{1,-1\}$. Then $a^2 \notin \{1,-1\}$.

**Proof.** As $a \notin \{1,-1\}$, then $a \not\sim_R 1$. So $\psi(a) \neq 1$. Then $\psi(a^2) = (\psi(a))^2 \neq 1$, since $G$ is a totally ordered group. So $a^2 \not\sim_R 1$. That is $a^2 \notin \{1,-1\}$. □

Following are some useful Lemmas in Marshall’s paper (cf. section 3, [Mar06]).

**Definition 4.12.** [Mar06] Let $F$ be a hyperfield. A subset $P$ of $F$ is called an ordering if

$$P \uplus P \subseteq P, P \cap P \subseteq P, P \cup -P = F \text{ and } P \cap -P = \{0\}.$$  

**Definition 4.13.** [Mar06] A hyperfield $F$ is said to be real if $-1 \notin F^\times \uplus F^\times$ where $F^\times := \{a^2 \mid a \in F\}$.

**Lemma 4.14.** [Mar06] Lemma 3.3 Let $F$ be a hyperfield. $F$ has an ordering if and only if $F$ is real.

**Lemma 4.15.** [Mar06] Lemma 3.2, 3.3 Let $F$ be a hyperfield with $1 \neq -1$. If $F$ has an ordering $P$, then $-1 \notin P$.

Based on above lemmas, we get the following.

**Proposition 4.16.** If $F$ is a stringent hyperfield with $F_{1\{1\}} = \mathbb{S}$, then $F$ has an ordering.

**Proof.** By Lemma 4.14 we just need to show that $F$ is real.

Suppose that $-1 \in F^\times \uplus F^\times$. Then there exist $a, b \in F$ such that $-1 \in a^2 \uplus b^2$. By Proposition 4.11 $a^2 \neq -1$ and $b^2 \neq -1$. Thus $a \neq 0$ and $b \neq 0$. And by reversibility, $-b^2 \in 1 \uplus a^2$. As $a^2 \neq -1$, then $1 \uplus a^2 \subseteq \{1,a^2\}$. Thus $-b^2 = a^2$. Then $-1 = a^2 b^{-2} = (ab^{-1})^2$, a contradiction to Proposition 4.11.

So $F$ is real, and therefore has an ordering. □

**Theorem 4.17.** If $F$ is a stringent hyperfield with $F_{1\{1\}} \in \{\mathbb{K},\mathbb{S}\}$, then $F$ arises from a short exact sequence

$$1 \rightarrow F_{1\{1\}}^\times \xrightarrow{\psi} F_{1\{1\}}^\times \times G \xrightarrow{\psi \times 1} G \rightarrow 1.$$  

**Proof.** If $F_{1\{1\}} = \mathbb{K}$, this is trivial.

If $F_{1\{1\}} = \mathbb{S}$, by Theorem 4.10 we may suppose $F = \mathbb{S} \times H, \psi G = H \cup \{0\}$ with a short exact sequence of groups

$$1 \rightarrow \mathbb{S}^\times \xrightarrow{\psi} H \xrightarrow{\psi \times 1} G \rightarrow 1.$$  

By Proposition 4.16 we know that $F$ has an ordering $P$. Let $O = P - \{0\}$. As $P \cup -P = F$ and $P \cap -P = \{0\}$, then $F = O \cup -O \cup \{0\}$. By Lemma 4.15 $-1 \notin O$. Then $1 \notin O$. And as $P \cap P \subseteq P$, then $O \cap O \subseteq O$. For any $a \in O$, $a^{-1} \in O$. Otherwise $-a^{-1} \in O$. Then $a \cap -a^{-1} = -1 \in O$. Contradiction. So $O$ is a multiplicative group with $1 \in O$ and $F = O \cup -O \cup \{0\}$, and $\psi \mid O$ is an isomorphism from $O$ to $G$.

Now we can identify $x \in H$ with $(1,\psi(x))$ if $x \in O$, and with $(-1,\psi(x))$ if $x \notin O$, giving a bijection from $H$ to $\mathbb{S}^\times \times G$.

So $F \cong (\mathbb{S}^\times \times G) \cup \{0\}$. □

It is not clear whether this result extends to stringent skew hyperfields.
5. Classification of Doubly Distributive Skew Hyperfields

**Proposition 5.1.** The doubly distributive skew hyperfields are precisely those of the form \( F \times_{H, \psi} G \) of exactly one of the following types:

1. \( F \) is the Krasner hyperfield,
2. \( F \) is the sign hyperfield,
3. \( F \) is a skew field and \( G \) satisfies

\[
\{ab \mid a, b < 1_G\} = \{c \mid c < 1_G\}.
\]

Before showing the proof, we'll first introduce a useful lemma.

**Lemma 5.2.** Let \( R \) be a stringent skew hyperfield. \( R \) is doubly distributive if and only if

\[
(1 \boxplus -1)(1 \boxplus -1) = 1 \boxplus -1 \boxplus 1 \boxplus -1.
\]

**Proof.** By Definition 2.8, \( R \) is doubly distributive if and only if \((a \boxplus b)(c \boxplus d) = ac \boxplus ad \boxplus bc \boxplus bd\), for any \( a, b, c, d \in R \).

As \( R \) is stringent, we have \( u \boxplus v \) is a singleton if \( u \neq -v \). So if either \( a \neq -b \) or \( c \neq -d \), then the equation above is just about distributivity. It already holds.

If both \( a = -b \) and \( c = -d \), then

\[
(a \boxplus b)(c \boxplus d) = (a \boxplus -a)(c \boxplus -c) = a(1 \boxplus -1)(1 \boxplus -1)c
\]

and

\[
ac \boxplus ad \boxplus bc \boxplus bd = ac \boxplus -ac \boxplus -ac \boxplus ac = a(1 \boxplus -1 \boxplus 1 \boxplus -1)c.
\]

So \( R \) is doubly distributive if and only if

\[
(1 \boxplus -1)(1 \boxplus -1) = 1 \boxplus -1 \boxplus 1 \boxplus -1.
\]

\( \square \)

**Proof of Proposition 5.1.** By Proposition 1.3 and Theorem 4.10, we may suppose that a doubly distributive skew hyperfield \( R = F \times_{H, \psi} G \) arises from a short exact sequence of groups

\[
1 \to F^\times \xrightarrow{\xi} H \xrightarrow{\psi} G \to 1,
\]

where \( F \) is either the Krasner hyperfield or the sign hyperfield or a skew field.

**Case 1:** When \( F = \mathbb{K} = \{1, 0\} \), the hyperaddition is defined by

\[
x \boxplus 0 = \{x\},
\]

\[
x \boxplus y = \begin{cases} 
\{x\} & \text{if } \psi(x) > \psi(y), \\
\{y\} & \text{if } \psi(x) < \psi(y), \\
\{z \mid \psi(z) \leq \psi(x)\} \cup \{0\} & \text{if } \psi(x) = \psi(y), \text{ that is } x = y.
\end{cases}
\]

By Lemma 5.2, \( R \) is doubly distributive if and only if

\[
(1 \boxplus 1)(1 \boxplus 1) = 1 \boxplus 1 \boxplus 1 \boxplus 1.
\]

\[
(1 \boxplus 1)(1 \boxplus 1) = ((\{z \mid \psi(z) \leq 1\} \cup \{0\}) \boxplus ((\{z \mid \psi(z) \leq 1\} \cup \{0\})) = \{z \mid \psi(z) \leq 1\} \cup \{0\},
\]

and

\[
1 \boxplus 1 \boxplus 1 \boxplus 1 = ((\{z \mid \psi(z) \leq 1\} \cup \{0\}) \boxplus ((\{z \mid \psi(z) \leq 1\} \cup \{0\})) = \{z \mid \psi(z) \leq 1\} \cup \{0\}.
\]

So \( R \) is doubly distributive when \( F = \mathbb{K} \).

**Case 2:** When \( F = S = \{-1, 0, 1\} \), the hyperaddition is defined by

\[
x \boxplus 0 = \{x\},
\]

\[
x \boxplus y = \begin{cases} 
\{x\} & \text{if } \psi(x) > \psi(y), \\
\{y\} & \text{if } \psi(x) < \psi(y), \\
\{x\} & \text{if } x = y, \\
\{z \mid \psi(z) \leq \psi(x)\} \cup \{0\} & \text{if } x = -y.
\end{cases}
\]
By Lemma 5.2, $R$ is doubly distributive if and only if

$$(1 ⊕ 1)(1 ⊕ 1) = 1 ⊕ 1 ⊕ 1 ⊕ 1.$$  

and

$$(1 ⊕ 1)(1 ⊕ 1) = (\{z \in \psi(z) \leq 1\} \cup \{0\}) \cdot (\{z \in \psi(z) \leq 1\} \cup \{0\}) = \{z \in \psi(z) \leq 1\} \cup \{0\},$$

and

$$1 ⊕ 1 ⊕ 1 ⊕ 1 = (\{z \in \psi(z) \leq 1\} \cup \{0\}) \cdot (\{z \in \psi(z) \leq 1\} \cup \{0\}) = \{z \in \psi(z) \leq 1\} \cup \{0\}.$$  

So $R$ is doubly distributive when $F = \mathbb{S}$.

Case 3: When $F$ is a skew field, the hyperaddition is defined by

$$x ⊕ y = \begin{cases} \{x\} & \text{if } \psi(x) > \psi(y), \\ \{y\} & \text{if } \psi(x) < \psi(y), \\ x ⊕ \psi(x) y & \text{if } \psi(x) = \psi(y) \text{ and } 0 \not\in x ⊕ \psi(x) y. \\ \{z \in \psi(z) < \psi(x)\} \cup \{0\} & \text{if } \psi(x) = \psi(y) \text{ and } 0 \in x ⊕ \psi(x) y. \end{cases}$$

By Lemma 5.2, $R$ is doubly distributive if and only if

$$(1 ⊕ 1)(1 ⊕ 1) = 1 ⊕ 1 ⊕ 1 ⊕ 1.$$  

and

$$(1 ⊕ 1)(1 ⊕ 1) = (\{z \in \psi(z) \leq 1\} \cup \{0\}) \cdot (\{z \in \psi(z) \leq 1\} \cup \{0\}) = \{xy \mid \psi(x), \psi(y) < 1\} \cup \{0\},$$

and

$$1 ⊕ 1 ⊕ 1 ⊕ 1 = (\{z \in \psi(z) < 1\} \cup \{0\}) \cdot (\{z \in \psi(z) < 1\} \cup \{0\}) = (\{z \in \psi(z) < 1\} \cup \{0\}) \cup \{0\}.$$  

So $R$ is doubly distributive if and only if

$$\{xy \mid \psi(x), \psi(y) < 1\} \cup \{0\} = \{z \mid \psi(z) < 1\} \cup \{0\}.$$  

We claim that

$$\{xy \mid \psi(x), \psi(y) < 1\} \cup \{0\} = \psi^{-1}(\psi(1) \downarrow) \cup \{0\} = \{z \mid \psi(z) < 1\} \cup \{0\}$$

if and only if

$$\{ab \mid a, b < 1_G\} = \{c \mid c < 1_G\}.$$  

$(\Rightarrow)$: If $\{xy \mid \psi(x), \psi(y) < 1\} \cup \{0\} = \{z \mid \psi(z) < 1\} \cup \{0\}$, the direction $\subseteq$ is clear. We just need to consider the other direction. Let $c \in G$ be such that $c < 1_G$ and let $z \in \psi^{-1}(c)$. Then there exist $x, y \in H$ such that $z = xy$ and $\psi(x), \psi(y) < 1$ by our assumption. So $c = \psi(z) = \psi(xy) = \psi(x)\psi(y)$. We have $c \in \{ab \mid a, b < 1_G\}$.

$(\Leftarrow)$: If $\{ab \mid a, b < 1_G\} = \{c \mid c < 1_G\}$, the direction $\subseteq$ is also clear. We just need to consider the other direction. Let $z \in H$ be such that $\psi(z) < 1$ and let $c = \psi(z)$. Then there exist $a, b \in G$ such that $c = ab$ and $a, b < 1_G$ by our assumption. Let $x \in H$ be such that $\psi(x) = a < 1_G$ and let $y = x^{-1}z$. We have $\psi(y) = \psi(x^{-1}z) = a^{-1}c = b < 1_G$ and $z = xy$. So $z \in \{xy \mid \psi(x), \psi(y) < 1\}$.

So $R$ is doubly distributive if and only if

$$\{ab \mid a, b < 1_G\} = \{c \mid c < 1_G\}.$$

6. Reduction of stringent skew hyperrings to hyperfields

In this section, we will show that stringent skew hyperrings are very restricted.

Theorem 6.1. Every stringent skew hyperring is either a skew ring or a stringent skew hyperfield.
They show that this notion is reasonably well behaved if and only if 

\[ \text{Definition 7.1.} \]

Let \( R \) with multiplication given by 

\[ 1 \in x \cdot s \oplus x \cdot s = x \cdot (s \oplus s) , \]
\[ 1 \in t \cdot x \oplus t \cdot x = (t \oplus t) \cdot x . \]

So there exists \( y \in s \oplus s \) and \( z \in t \oplus t \) such that 

\[ 1 = x \cdot y = z \cdot x . \]

Thus \( y = (z \cdot x) \cdot y = z \cdot (x \cdot y) = z \).

So \( x \) has a multiplicative inverse \( y \) in \( R \). Then \( x \) is a unit of \( R \).

So every stringent skew hyperring is either a skew ring or a stringent skew hyperfield.

We cannot classify doubly distributive hyperring using our classification because not every doubly distributive hyperring is stringent. For example, the hyperring \( K \times K \) that is the square of the Krasner hyperring is doubly distributive but not stringent.

7. Every Stringent Skew Hyperfield is a Quotient of a Skew Field

In [DG73], Diller and Grenzdörffer try to unify the treatment of various notions of convexity in projective spaces over a commutative field \( K \) by introducing for any subgroup \( U \leq K^x \) the notion of \( U \)-convexity. They show that this notion is reasonably well behaved if and only if \( U \) is as follows.

**Definition 7.1.** [DG73] Let \( K \) be a field and let \( U \leq K^x \). \( U \) is called \( U \)-‘hüllenbildend’ if \( U \) satisfies

\[ x , y \in K , x + y - xy \in U \rightarrow x \in U \text{ or } y \in U . \]

In [Dre86], Dress presents a simple complete classification of such subgroups \( U \).

**Theorem 7.2.** Let \( U \leq K^x \) satisfy (7) and let \( S_U = \{ x \in K \mid x \notin U \text{ and } x + U \subseteq U \} \). Then \( S_U \) is the maximal ideal of a valuation ring \( R = R_U \) (\( \{ x \in K \mid x \cdot S_U \subseteq S_U \} \)) in \( K \), \( U \) is contained in \( R \), \( \overline{U} = \{ \overline{x} \in K_U = R_U / S_U \mid x \in U \} \) is either a domain of positivity in \( K_U \) (if \( -1 \notin U \), \( 2 \in U \) or \( U = \{ 1 \} \) or \( \overline{U} = K_U \) and, in any case, \( U = \{ x \in R_U \mid \overline{x} \in \overline{U} \} \).

**Example 7.3.** Let \( \Gamma \) be a totally ordered abelian group and let \( k \) be a residue field. Define \( K = k((\Gamma)) \) to be the ring of formal power series whose powers come from \( \Gamma \), that is, the elements of \( K \) are functions as power series

\[ \sum_{a \in \Gamma} p(a)x^a \]

It is trivial to show that \( K \) is a field. The \( p \) with \( \min(\supp(p)) \geq 0 \) (along with 0 in \( K \)), form a subring \( R \) of \( K \) that is a valuation ring. The maximal ideal \( S \) of \( R \) in \( K \) is

\[ S = \{ p \in K^\times \mid \min(\supp(p)) > 0 \} . \]

Let \( U \leq K^x \) be \( U \)-‘hüllenbildend’. By Theorem 7.2, \( U \) is either

(1) \( \{ p \in K^\times \mid \min(\supp(p)) = 0 \} \) and \( p(0_\Gamma) > 0 \) if \( k \) is an ordered field, or

(2) \( \{ p \in K^\times \mid \min(\supp(p)) = \min(\supp(p)) = 0 \} \) if \( p(0_\Gamma) = 1 \), or

(3) \( \{ p \in K^\times \mid \min(\supp(p)) = 0 \} \).

Now let us consider the quotient hyperfield \( K/U = \{ [g] = gU \mid g \in K \} \), introduced by Krasner in [Kra83] with multiplication given by \( [g] \cdot [h] = [gh] \), for \( [g] , [h] \in K/U \). The hyperaddition is given by \( [g] \ominus [0] = [g] \) and \( [g] \oplus [h] = \{ [f] \subseteq K/U \mid f \in gU + hU \} \), for \( [g] , [h] \in (K/U)^x \).

We would like to show that every stringent skew hyperfield is a quotient of a skew field by some subgroup \( U \) which is ‘hüllenbildend’. By Theorem 4.10, we may suppose that a stringent skew hyperfield \( F = M \rtimes_{\varphi, \psi} G \) arises from a short exact sequence of groups

\[ 1 \rightarrow M^x \xrightarrow{\varphi} H \xrightarrow{\psi} G \rightarrow 1 , \]

where \( G \) is a totally ordered group equipped with a total order \( \leq \) and \( M \) is either \( \mathbb{K} \), or \( \mathbb{S} \), or a skew field. Now let us define an order \( \leq' \) on \( G \) such that \( x \leq' y \) if and only if \( y \leq x \). So \( \leq' \) is also a total order on \( G \).

Let \( k \) be any field if \( M \) is \( \mathbb{K} \) and let \( k \) be the field \( \mathbb{R} \) of real numbers (or any other ordered field) if \( M \) is \( \mathbb{S} \). We can construct a field \( K = k((\Gamma)) \) as in Example 7.3.

**Proof.** If \( G \) is trivial, then \( R = R_U \). So \( R \) is either \( \mathbb{K} \), or \( \mathbb{S} \), or a skew ring.

If \( G \) is nontrivial, we would like to show that every element \( x \in R^x \) is a unit. Now let \( s \) and \( t \) in \( R^x \) be such that \( x \cdot s >_R 1 \) and \( t \cdot x >_R 1 \). Then by the remarks after Lemma 3.12 we have

\[ 1 \in x \cdot s \oplus x \cdot s = x \cdot (s \oplus s) , \]
\[ 1 \in t \cdot x \oplus t \cdot x = (t \oplus t) \cdot x . \]
If $M$ is a skew field, let $k = M$. Define $K = k[[G]]$ to be the set of formal sums of elements of $H$ all from different layers such that the support is well-ordered, that is, an element of $K$ is a function $p$ from $G$ to $H$ such that for any $g$ in $G$, $p(g) \in \psi^{-1}(g) \cup \{0\} = A_g$ and the support of each function is a well-ordered subset of $G$. As $M$ is a skew field and each $\lambda_g$ is an isomorphism of hypergroups, then $(A_g, \square_g, 0)$ is always an abelian group. We claim that $K$ is a field, viewing functions as power series
\[
\sum_{a \in G} p(a)x^a,
\]
with addition $+K$ given by
\[
\sum_{a \in G} p(a)x^a + _K \sum_{a \in G} q(a)x^a = \sum_{a \in G} (p(a) \square_a q(a))x^a,
\]
and the additive identity is $\sum_{a \in G} 0x^a$. The multiplication $\cdot_K$ is given by
\[
\left( \sum_{a \in G} p(a)x^a \right) \cdot_K \left( \sum_{a \in G} q(a)x^a \right) = \sum_{s \in G} \left( \bigoplus_{g \in \text{supp}(p), h \in \text{supp}(q), g \cdot h = s} p(g) \cdot_H q(h) \right)x^s,
\]
and the multiplicative identity is $1x^{1_G}$. Since the proof that this really gives a skew field is a long calculation very similar to that for $k((\Gamma))$, we do not give it here but in appendix A.

A valuation ring $R$ of the field $K$ is $R = \{ p \in K^\times | \min(\text{supp}(p)) \geq 1_G \}$. The maximal ideal $S$ of $R$ is $S = \{ p \in K^\times | \min(\text{supp}(p)) > 1_G \}$. For $p \in K^\times$, let us denote $\min(\text{supp}(p))$ by $m_p$.

(1) If $M$ is $\mathbb{K}$, then let $U = \{ p \in K^\times | m_p = 1_G \}$. Then the quotient hyperfield $K/U = \{ [q] = qU | q \in K \}$ has
\[
[q] = \{ p \in K^\times | m_p = m_q \},
\]
\[0 = \{ 0_K \}.
\]
So we can identify $[q]$ in $(K/U)^\times$ with $m_q$ in $G$ and identify $[0]$ in $K/U$ with 0. So we have $K/U \cong (G \cup \{ 0 \}, \boxplus, \cdot)$ with multiplication given by
\[0 \cdot g = 0,
\]
\[g \cdot h = g \cdot_G h,
\]
where $g, h \in G$. And the hyperaddition is given by
\[g \boxplus 0 = \{ g \},
\]
\[g \boxplus h = \begin{cases} \{ g \} & \text{if } g <' h, \text{ that is } g > h, \\ \{ h \} & \text{if } g >' h, \text{ that is } g < h, \\ \{ f \in G | f \geq g \} \cup \{ 0 \} & \text{if } g = h, \end{cases}
\]
where $g, h \in G$.

Now it is clear to see that $K/U \cong (G \cup \{ 0 \}, \boxplus, \cdot) \cong \mathbb{K} \rtimes_{H, \psi} G = F$.

(2) If $M$ is $\mathbb{S}$, $k = \mathbb{R}$ (or any other ordered field) and $K = k((G))$, then let $U = \{ p \in K^\times | m_p = 1_G \text{ and } p(1_G) > 0 \}$. Then the quotient hyperfield $K/U = \{ [q] = qU | q \in K \}$ has
\[\begin{align*}
[q] &= \{ p \in K^\times | m_p = m_q \text{ and } p(m_p) > 0 \} & \text{if } q(m_q) > 0, \\
[q] &= \{ p \in K^\times | m_p = m_q \text{ and } p(m_p) < 0 \} & \text{if } q(m_q) < 0, \\
[0] &= \{ 0_K \}.
\end{align*}
\]
We can identify $[q]$ in $(K/U)^\times$ with $(1, m_q)$ if $q(m_q) > 0$, identify $[q]$ in $(K/U)^\times$ with $(-1, m_q)$ if $q(m_q) < 0$, and identify $[0]$ with 0. So we have $K/U \cong (\mathbb{S}^\times \times G \cup \{ 0 \}, \boxplus, \cdot)$ with multiplication given by
\[(r, g) \cdot 0 = 0,
\]
\[(r_1, g_1) \cdot (r_2, g_2) = (r_1 \cdot_G r_2, g_1 \cdot_G g_2),
\]
where $r, r_1, r_2 \in \mathbb{S}^\times \text{ and } g, g_1, g_2 \in G$. And the hyperaddition is given by
\[(r, g) \boxplus 0 = \{ (r, g) \}.
\]
and d hyperfield, the sign hyperfield or a field. We divide into cases according to the value of g₁

\( g₁ \trianglelefteq g₂ \), that is \( g₁ > g₂ \),

\( g₁ \triangleright g₂ \), that is \( g₁ < g₂ \),

if \( g₁ = g₂ \) and \( r₁ = r₂ \)

if \( g₁ = g₂ \) and \( r₁ = -r₂ \),

\[(r₁, g₁) \otimes (r₂, g₂) = \begin{cases} \{(r₁, g₁); r₁ \in G, g₁ \in \mathbb{S} \times (\mathbb{S} \times G) \cup \{0\}, \exists, -\} \cong \mathbb{S} \times H, \varphi G = F. \end{cases}\]

(3) If \( M \) is a skew field, \( k = M \) and \( K = k[[G]] \), then let \( U = \{ p \in K^X \mid m_p = 1_G \text{ and } p(1_G) = 1 \} \). Then the quotient hyperfield \( K/U = \{ [q] = qU \mid q \in K \} \) has

\[ [q] = \{ p \in K^X \mid m_p = m_q \text{ and } p(m_p) = q(m_q) \}, \]

\[ [0] = 0_K. \]

We can identify \([q]\) in \((K/U)^X\) with \(q(m_q)\) in \(H\) (clearly \(\psi(q(m_q)) = m_q\)) and identify \([0]\) with \(0_F\). So we have \(K/U \cong F\) with multiplication given by

\[ [q] \cdot [h] = \{ p \in K^X \mid m_p = m_q \cdot_G m_h \text{ and } p(m_p) = q(m_q) \cdot_H h(m_h) \} = [p] \]

The hyperaddition is given by

\[ [q] \oplus 0 = [q], \]

\[ [q] \oplus [h] = \begin{cases} \{ [q] \} & \text{if } m_q \lessdot m_h, \text{ that is } m_q > m_h, \\ \{ [h] \} & \text{if } m_q \succ m_h, \text{ that is } m_q < m_h, \\ \{ [p] \} = \{ p \in K^X \mid m_p = m_q \text{ and } p(m_p) = q(m_q) \oplus m_q h(m_h) \} & \text{if } m_q = m_h \text{ and } 0 \notin q(m_q) \oplus m_q h(m_h), \\ \{ [p] \} \in (K/U)^X \mid m_p < m_q \} \cup \{0\} & \text{if } m_q = m_h \text{ and } 0 \in q(m_q) \oplus m_q h(m_h), \end{cases} \]

where \([q], [h] \in (K/U)^X\).

So \(K/U \cong M \times_{H, \varphi} G = F.\)

**Theorem 7.4.** Every stringent skew hyperfield is a quotient of a skew field.

**Corollary 7.5.** Every doubly distributive skew hyperfield is a quotient of a skew field.

It follows from the construction that the same statements with all instances of the word ‘skew’ removed also hold.

8. The semirings associated to doubly distributive hyperfields

For any doubly distributive hyperfield \( H \) we can define binary operations \( \oplus \) and \( \odot \) on \( \mathcal{P}H \) by setting \( A \oplus B := \bigcup_{a \in A, b \in B} a \circ b \) (this is just the extension of \( \circ \) to subsets of \( H \) from Definition 2.2) and \( A \odot B := \{ a b : a \in A, b \in B \} \). Let \( \langle H \rangle \) be the substructure of \( \langle \mathcal{P}H, \oplus, \odot \rangle \) generated from the singletons of elements of \( H \). It is clear that \( \oplus \) is commutative and associative with identity \( \{ 0_H \} \) and that \( \odot \) is associative with identity \( \{ 1_H \} \). Furthermore, multiplication by \( \{ 0_H \} \) annihilates \( \langle H \rangle \). Nevertheless, \( \langle H \rangle \) is not necessarily a semiring, since \( \odot \) may not distribute over \( \oplus \). Indeed, if \( H \) is not doubly distributive then there are \( a, b, c \) and \( d \) in \( H \) with \( (a \oplus b)(c \oplus d) \neq ac \oplus ad \oplus bc \oplus bd \), which in \( \langle H \rangle \) means that \( \{ a \} \oplus \{ b \} \odot \{ c \} \odot \{ d \} \neq \{ (a \odot (c \odot d)) \} \oplus \{ (b \odot (c \odot d)) \} \).

Indeed, it is not hard to check, either directly or using our classification, that \( \langle H \rangle \) is a semiring if and only if \( H \) is doubly distributive. We will refer \( \langle H \rangle \) as the associated semiring to \( H \). Using our classification, we can easily determine all such associated semirings. Surprisingly, some of the basic examples have already been intensively studied and play an important role in the foundations of tropical geometry. In each case, we find that \( \langle H \rangle \) contains only few elements in addition to the singletons of elements of \( H \).

We have seen that any doubly distributive hyperfield has the form \( F \times_{H, \varphi} G \), where \( F \) is the Krasner hyperfield, the sign hyperfield or a field. We divide into cases according the value of \( F \).
The commutativity and associativity of \( K \) is a skew field. Let \( \psi \) where groups \( H \) of \( x \) elements of \( H \) and the sets \( g^\nu := \{ h \in G \mid h \leq g \} \cup \{ 0 \} \). To simplify the definition of the addition we define an operation \( \nu \) on \( \langle H \rangle \setminus \{ \{ 0 \} \} \) by \( \nu(\{ g \}) = g^\nu = g^\nu \) and we transfer the total order of \( G \) to the \( g^\nu \) in the obvious way. Then the addition is given by \( x \oplus \{ 0 \} = x \) for any \( x \) and otherwise by

\[
\begin{cases}
  x & \text{if } \nu(x) > \nu(y) \\
  y & \text{if } \nu(x) < \nu(y) \\
  \nu(x) & \text{if } \nu(x) = \nu(y).
\end{cases}
\]

The multiplication is given by \( x \odot \{ 0 \} = \{ 0 \} \), by \( \{ g \} \odot \{ h \} = \{ g \cdot h \} \), by \( \{ g \} \odot h^\nu = (g \cdot h)^\nu \) and by \( g^\nu \odot h^\nu = (g \cdot h)^\nu \). In the case that \( G \) is the ordered group of real numbers, this is simply the supertropical semiring introduced by Izhakian in [Izh09]. So it would be reasonable to call such semirings in general supertropical semirings.

8.2. Symmetrised \((max, +)-semirings\). If \( F \) is the sign hyperfield then by Theorem 4.17 without loss of generality it arises from a short exact sequence

\[
1 \to \mathbb{S}^x \to \mathbb{S}^x \times G \to G \to 1.
\]

The elements of \( \langle H \rangle \) then have the form \( 0 := \{ 0_H \} \), \( \ominus g := \{ (1, g) \} \), \( \ominus g := \{ (-1, g) \} \), or \( g^\circ := \{ (i, h) \mid i \in \mathbb{S}^x, h \leq g \} \cup \{ 0_H \} \). There is an obvious projection map \( \pi \) from \( \langle H \rangle \setminus \{ 0 \} \) to \( G \). Then the addition is given by \( x \ominus 0 = x \) for any \( x \), by \( x \ominus y = x \) if \( \pi(x) > \pi(y) \), by \( x \ominus g^\circ = g^\circ \) if \( \pi(x) = g \), by \( (\ominus g) \ominus (\ominus g) = \ominus g \) and by \( (\ominus g) \ominus (\ominus g) = g^\circ \). The multiplication is given by \( x \ominus 0 = 0 \) for any \( x \), by \( x \ominus g^\circ = (\pi(x) \cdot g) \), by \( (\ominus g) \ominus (\ominus h) = g^\circ \cdot h \), by \( (\ominus g) \ominus (\ominus h) = g^\circ \cdot h \) and by \( (\ominus g) \ominus (\ominus h) = (g \cdot h) \).

In the case that \( G \) is the ordered group of real numbers, this is simply the symmetrised \((max, +)-semiring\) introduced by Akian et al in [ACG+99]. So it would be reasonable to call such semirings in general symmetrised \((max, +)-semirings\).

8.3. Linearised \((max, +)-semirings\). If \( F \) is a field, then the elements of \( \langle H \rangle \) are the singletons of elements of \( H \) (which are in canonical bijection with \( H \)) and the sets \( \psi^{-1}(g \downarrow) \cup \{ 0 \} \) (which are in canonical bijection with \( G \)). So \( \langle H \rangle \) is isomorphic to the semiring on \( H \cup G \) with \( x \ominus y \) for \( x, y \in H \) given by the unique element of \( x \ominus y \) if this is a singleton and by \( \psi(x) \) otherwise, with \( x \ominus g \) for \( x \in H \) and \( g \in G \) given by \( x \psi(x) \geq g \) and by \( g \) otherwise, and with \( g \ominus h \) for \( g, h \in G \) given by \( \max(g, h) \). For the multiplication, \( x \ominus y = x \cdot y \) for \( x, y \in H \) and \( x \ominus g = \psi(x) \cdot g \) for \( x \in H \) and \( y \in G \) and finally \( g \ominus h = g \cdot h \) for \( g, h \in G \).

By analogy to the previous construction, we could refer to such semirings as linearised \((max, +)-semirings\).

So far as we know, such semirings have not yet been seriously investigated.

Appendix A.

Lemma A.1. Let \( F = M \rtimes_{H, \psi} G \) be a stringent skew hyperfield arising from a short exact sequence of groups

\[
1 \to \mathbb{M}^\times \xrightarrow{\varphi} H \xrightarrow{\psi} G \to 1,
\]

where \( G \) is a totally ordered group and \( M \) is a skew field. Define \( K = k[[G]] \) as we did in section 2. Then \( K \) is a skew field.

Proof. The commutativity and associativity of \( (K, +_K, \sum_{a \in G} 0^a) \) follow from those of \( (H \cup \{ 0 \}, \oplus, 0) \). So we only need to show the associativity of \( (K, \cdot_K, 1^x) \), the existence of a multiplicative inverse for every element and the distributivity.

An important principle which we will need again and again as we go along is a kind of distributivity of the composition of \( H \) over the various additions \( \boxplus_g \). To express it cleanly, we begin by extending \( \cdot_H \) to \( H \cup \{ 0 \} \) by setting \( x \cdot 0 = 0 \cdot x = 0 \) for all \( x \in H \cup \{ 0 \} \). Suppose that we have elements \( x \) and \( y_1, y_2, \ldots, y_n \) of \( H \) with \( \psi(y_i) = s \) for all \( i \), so that \( \boxplus_i y_i \) is defined. Let \( t = \psi(x) \). Then \( x \mapsto x \cdot_H z \) is a bijection from \( A_x \) to \( A_{t \cdot x} \) whose composition with \( \lambda_{y_1} \) is \( \lambda_{x \cdot y_1} \), so it must also be an isomorphism of hypergroups. Thus \( x \cdot_H (\boxplus_i y_i) = \boxplus_i (x \cdot_H y_i) \).
A similar argument using the $\chi'_h$ shows

$$\left( \bigoplus_{i \leq k} y_i \right) \cdot_H x = \bigoplus_{i \leq k} x_i y_i \cdot_H x.$$ 

To show the associativity of $(K, \cdot_K, 1x_1^G)$, we calculate as follows:

$$(p \cdot_K q) \cdot_K w) (s) = \bigoplus_{c \in \text{supp}(w)} \left( \bigoplus_{a \in \text{supp}(p)} p(a) \cdot_H q(b) \cdot_H w(c) \right)$$

$$= \bigoplus_{a \in \text{supp}(p)} \left( \bigoplus_{b \in \text{supp}(q)} p(a) \cdot_H q(b) \cdot_H w(c) \right)$$

$$= \bigoplus_{a \in \text{supp}(p)} \left( \bigoplus_{b \in \text{supp}(q)} p(a) \cdot_H q(b) \cdot_H w(c) \right)$$

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So $(p \cdot_K q) \cdot_K w = p \cdot_K (q \cdot_K w)$.

Next we will show that all elements of $K$ have multiplicative inverses. We do this first for those $p \in K = k[[G]]$ such that $m_p = 1_G$ and $p(m_p) = 1$. Let $S$ be the set of finite sums of elements of $\text{supp}(p)$. $S$ is well founded.

Define $q \in K = k[[G]]$ such that $q(1_G) := 1$, $q(s) := 0$ for $s \notin S$ and, for $s \in S$, define $q(s)$ recursively by

$$q(s) := -\left( \bigoplus_{g \in \text{supp}(p)-\{1_G\}} p(g) \cdot_H q(h) \right).$$

So

$$p \cdot_K q(1_G) = 1,$$

$$p \cdot_K q(s) = 0 \quad \text{if } s \notin S,$$

$$p \cdot_K q(s) = \bigoplus_{g \in \text{supp}(p)} p(g) \cdot_H q(h) \quad \text{if } s \in S - \{1_G\}.$$

So $p \cdot_K q$ is the identity. Therefore, $q$ is the multiplicative inverse of $p$.

Next we consider elements of $K$ with only a single summand, that is, those of the form $ax^g$. It is clear that each such element also has a multiplicative inverse, namely $a^{-1}x^{g^{-1}}$. 
Now every element of $K$ can be expressed as a product $p_1 \cdot p_2$, with $m_{p_1} = 1_G$ and $p_1(m_{p_2}) = 1$ and such that $p_2$ has only a single summand. As seen above, each of $p_1$ and $p_2$ has a multiplicative inverse, and hence $p_1 \cdot p_2$ also has one, namely $p_2^{-1} \cdot p_1^{-1}$.

For distributivity, we would like to show that $p \cdot K(q + K w) = p \cdot K q + p \cdot K w$. For $s \in G$,

\[
(p \cdot K(q + K w))(s) = \bigoplus_{g \in h = s} p(g) \cdot H(q(h)) \bigoplus_{h \in w(h)}
\]

\[
= \bigoplus_{g \in h = s} (p(g) \cdot H(q(h)) \bigoplus_{h \in w(h)} p(g) \cdot H w(h))
\]

\[
= \left( \bigoplus_{g \in h = s} p(g) \cdot H(q(h)) \bigoplus_{h \in w(h)} \right) (s)
\]

\[
= (p \cdot K q + K p \cdot K w)(s).
\]

So $p \cdot K(q + K w) = p \cdot K q + K p \cdot K w$. A similar calculation shows that $(p + K q) \cdot K w = p \cdot K w + K q \cdot K w$.

So $K = k[[G]]$ is a skew field. □

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