Asymptotic dimension of a hyperbolic space and capacity dimension of its boundary at infinity

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Abstract

We introduce a quasi-symmetry invariant of a metric space $Z$ called the capacity dimension, $cdim\ Z$. Our main result says that for a visual Gromov hyperbolic space $X$ the asymptotic dimension of $X$ is at most the capacity dimension of its boundary at infinity plus 1, $\text{asdim} \ X \leq \text{cdim} \partial_\infty X + 1$.

1 Introduction

The notion of the asymptotic dimension, which is a quasi-isometry invariant of metric spaces, has been introduced in [Gr]. The present paper arose as an attempt to fill in details of a sketch of the proof given in [Gr, 1.E$'$1] that the asymptotic dimension of a negatively pinched Hadamard manifold $X$ is bounded above by $\dim \ X$, $\text{asdim} \ X \leq \dim \ X$. In that way, we came to the notion of the capacity dimension of a metric space, $\text{cdim}$, which should play, as we expect, an important role in many questions.

Recall that for every Gromov hyperbolic space $X$ there is a canonical class of metrics on the boundary at infinity $\partial_\infty X$ called visual metrics, see Sect. 6. Our main result is the following.

Theorem 1.1. Let $X$ be a visual Gromov hyperbolic space. Then

$$\text{asdim} \ X \leq \text{cdim} \partial_\infty X + 1,$$

for any visual metric on $\partial_\infty X$.

The notion of a visual hyperbolic space ([BoS]) is a rough version of the property that given a base point $o \in X$, for every $x \in X$ there is a geodesic ray in $X$ emanating from $o$ and passing through $x$, see Sect. 6.

The inequality of Theorem 1.1 is sharp. It is known that $\text{asdim} \ H^n = n$ for the real hyperbolic space $H^n$, $n \geq 2$. On the other hand, the standard unit sphere metric is a visual metric on the boundary at infinity $\partial_\infty H^n = S^{n-1}$, and $\text{cdim} S^{n-1} = n - 1$, see Corollary 3.5.

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By the definition, the capacity dimension is a bilipschitz invariant. A remarkable fact discovered in [LS] is that the close notion of the Assouad-Nagata dimension is a quasi-symmetry invariant. It turns out that the capacity dimension is also a quasi-symmetry invariant, see Sect. 4, in particular, the right hand side of the inequality from Theorem 1.1 is independent of the choice of a visual metric on \( \partial_\infty X \). This is also compatible with the fact that every quasi-isometry of hyperbolic geodesic spaces induces a quasi-symmetry of their boundaries at infinity.

Now, we briefly describe the structure of the paper. In Sect. 2 we collect notions and facts from the dimension theory needed for the paper. Here we also recall a definition of the asymptotic dimension, see Sect. 2.4. In Sect. 3 we give three definitions of the capacity dimension each of which is useful in different circumstances and prove their equivalence. Here we also compare the capacity dimension with the Assouad-Nagata dimension and obtain monotonicity of the capacity dimension. In Sect. 4 we prove that the capacity dimension is a quasi-symmetry invariant. The proof is based on ideas from [LS]. The core of the paper is Sect. 5 where we recall the notion of the hyperbolic cone over a bounded metric space \( Z \) and prove the relevant estimate for the asymptotic dimension of the cone via the capacity dimension of the base \( Z \). In the last Sect. 6 we discuss some facts from the hyperbolic spaces theory and prove Theorem 1.1.

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2 Preliminaries

Here we collect some (more or less) known notions and facts from the dimension theory needed in what follows.

Let \( Z \) be a metric space. For \( U, U' \subset Z \) we denote by \( \text{dist}(U, U') \) the distance between \( U \) and \( U' \), \( \text{dist}(U, U') = \inf\{|uu'| : u \in U, u' \in U'\} \) where \( |uu'| \) is the distance between \( u, u' \). For \( r > 0 \) we denote by \( B_r(U) \) the open \( r \)-neighborhood of \( U \), \( B_r(U) = \{ z \in Z : \text{dist}(z, U) < r \} \), and by \( \overline{B}_r(U) \) the closed \( r \)-neighborhood of \( U \), \( \overline{B}_r(U) = \{ z \in Z : \text{dist}(z, U) \leq r \} \). We extend these notations over all real \( r \) putting \( B_r(U) = U \) for \( r = 0 \), and defining \( B_r(U) \) for \( r < 0 \) as the complement of the closed \( |r| \)-neighborhood of \( Z \setminus U \), \( B_r(U) = Z \setminus \overline{B}_{|r|}(Z \setminus U) \).

2.1 Coverings

Given a family \( \mathcal{U} \) of subsets in a metric space \( Z \), we put \( \text{mesh}(\mathcal{U}, z) = \sup\{\text{diam} U : z \in U \in \mathcal{U}\} \) for every \( z \in Z \), and \( \text{mesh}(\mathcal{U}) = \sup\{\text{diam} U : U \in \mathcal{U}\} \). Clearly, \( \text{mesh}(\mathcal{U}) = \sup_{z \in Z} \text{mesh}(\mathcal{U}, z) \). In the case \( D = \text{mesh}(\mathcal{U}) < \infty \) we say that \( \mathcal{U} \) is \( D \)-bounded.
The multiplicity of $U$, $m(U)$, is the maximal number of members of $U$ with nonempty intersection. For $r > 0$, the $r$-multiplicity of $U$, $m_r(U)$, is the multiplicity of the family $U_r$ obtained by taking open $r$-neighborhoods of the members of $U$. So $m_r(U) = m(U_r)$. We say that a family $U$ is disjoint if $m(U) = 1$.

A family $U$ is called a covering of $Z$ if $\bigcup\{U : U \in U\} = Z$. A covering $U$ is said to be colored if it is the union of $m \geq 1$ disjoint families, $U = \bigcup_{a \in A} U^a$, $|A| = m$. In this case we also say that $U$ is $m$-colored. Clearly, the multiplicity of a $m$-colored covering is at most $m$.

Let $U$ be an open covering of a metric space $Z$. Given $z \in Z$ we let
\[ L'(U, z) = \sup\{\text{dist}(z, Z \setminus U) : U \in U\}, \]
\[ L(U, z) = \min\{L'(U, z), \text{mesh}(U, z)\} \]
be the Lebesgue number of $U$ at $z$ (the auxiliary $L'(U, z)$ might be larger than $\text{mesh}(U, z)$ and even infinite as e.g. in the case $Z = U$ for some member $U \in U$), $L(U) = \inf_{z \in Z} L(U, z)$ be the Lebesgue number of $U$. We have $L(U, z) \leq \text{mesh}(U, z)$, $L(U) \leq \text{mesh}(U)$ and for every $z \in Z$ the open ball $B_r(z)$ of radius $r = L(U)$ centered at $z$ is contained in some member of the covering $U$.

We shall make use the following

**Lemma 2.1.** Let $U$ be an open covering of $Z$ with $L(U) > 0$. Then for every $s \in (0, L(U))$ the family $U_{-s} = B_{-s}(U)$ is still an open covering of $Z$, and its $s$-multiplicity $m_s(U_{-s}) \leq m(U)$.

**Proof.** For every $z \in Z$ the ball $B_r(z)$, $r = L(U)$, is contained in some $U \in U$. Then $z \in B_{-s}(U)$ since $s < r$, thus $U_{-s}$ is an open covering of $Z$. Furthermore, since $B_s(B_{-s}(U)) \subset U$ for every $U \subset Z$, we have $m_s(U_{-s}) = m(B_s(U_{-s})) \leq m(U)$. \hfill \Box

A covering $U$ is locally finite, if for every $z \in Z$ only finitely many its elements intersect some neighborhood of $z$.

One defines the nerve of $U$ as a simplicial polyhedron whose vertex set is $U$, and a finite collection of vertices spans a simplex iff the corresponding covering elements have a nonempty intersection. Thus its (combinatorial) dimension is $m(U) - 1$.

### 2.2 Uniform polyhedra

Given a index set $J$, we let $R^J$ be the Euclidean space of functions $J \rightarrow \mathbb{R}$ with finite support, i.e., $x \in \mathbb{R}^J$ iff only finitely many coordinates $x_j = x(j)$ are not zero. The distance $|xx'|$ is well defined by
\[ |xx'|^2 = \sum_{j \in J} (x_j - x'_j)^2. \]
Let $\Delta^J \subset \mathbb{R}^J$ be the standard simplex, i.e., $x \in \Delta^J$ iff $x_j \geq 0$ for all $j \in J$ and $\sum_{j \in J} x_j = 1$.

A metric in a simplicial polyhedron $P$ is said to be uniform if $P$ is isometric to a subcomplex of $\Delta^J \subset \mathbb{R}^J$ for some index set $J$. Every simplex $\sigma \subset P$ is then isometric to the standard $k$-simplex $\Delta^k \subset \mathbb{R}^{k+1}$, $k = \dim \sigma$ (so, for a finite $J$, $\dim \Delta^J = |J| - 1$). For every simplicial polyhedron $P$ there is the canonical embedding $u : P \to \Delta^J$, where $J$ is the vertex set of $P$, which is affine on every simplex. Its image $P' = u(P)$ is called the uniformization of $P$, and $u$ is the uniformization map.

For example, the nerve $\mathcal{N} = \mathcal{N}(\mathcal{U})$ of a covering $\mathcal{U} = \{U_j\}_{j \in J}$ can always be considered as subcomplex of $\Delta^J$, $\mathcal{N} \subset \Delta^J$, and therefore as a uniform polyhedron.

### 2.3 Barycentric maps

Let $\mathcal{U} = \{U_j\}_{j \in J}$ be a locally finite open covering of a metric space $Z$ by bounded sets, $\mathcal{N} = \mathcal{N}(\mathcal{U}) \subset \Delta^J$ its nerve. One defines the barycentric map $p : Z \to \mathcal{N}$

associated with $\mathcal{U}$ as follows. Given $j \in J$, we put $q_j : Z \to \mathbb{R}$, $q_j(z) = \min\{\text{diam } Z, \text{dist}(z, Z \setminus U_j)\}$. Since $\mathcal{U}$ is open, $\sum_{j \in J} q_j(z) > 0$ for every $z \in Z$. Since $\mathcal{U}$ is locally finite and its elements are bounded, $\sum_{j \in J} q_j(z) < \infty$ for every $z \in Z$. Now, the map $p$ is defined by its coordinate functions $p_j(z) = q_j(z) / \sum_{j \in J} q_j(z)$, $j \in J$. Clearly, its image lands at the nerve, $p(Z) \subset \mathcal{N}$. Assume in addition that $L(\mathcal{U}) \geq d > 0$ and that the multiplicity $m(\mathcal{U}) = m + 1$ is finite. Then it is easy to check (see for instance [BD], [BS]) that $p$ is Lipschitz with Lipschitz constant

$$\text{Lip}(p) \leq \frac{(m + 2)^2}{d}.$$ 

Furthermore, for each vertex $v \in \mathcal{N}$ the preimage of its open star, $p^{-1}(\text{st}_v) \subset Z$, coincides with the member of the covering $\mathcal{U}$ corresponding to $v$.

An (open, locally finite) covering $\mathcal{U}'$ is inscribed in $\mathcal{U}$ if every its element is contained in some element of $\mathcal{U}$. In this case there is a simplicial map $\rho : \mathcal{N}' \to \mathcal{N}$ of the nerves which associates to every vertex $v' \in \mathcal{N}'$ some vertex $v \in \mathcal{N}$ with $v' \subset v$ (as coverings elements). One easily checks that this rule is compatible with simplicial structures of $\mathcal{N}$, $\mathcal{N}'$, and moreover $\rho \circ p'(z)$ lies in a face of the minimal simplex containing $p(z) \in \mathcal{N}$ for every $z \in Z$.

Note that if $\text{mesh}(\mathcal{U}') < L(\mathcal{U})$ then $\mathcal{U}'$ is inscribed in $\mathcal{U}$.

### 2.4 Asymptotic dimension

The asymptotic dimension is a quasi-isometry invariant of a metric space introduced in [Gr]. There are several equivalent definitions, see [Gr], [BD].
and we shall use the following one. The *asymptotic dimension* of a metric space \( X \), \( \text{asdim} \ X \), is a minimal \( n \) such that for every \( \lambda > 0 \) there is a \( \lambda \)-Lipschitz map \( f : X \to P \) into a uniform simplicial polyhedron \( P \) of dimension \( \leq n \) for which the preimages \( f^{-1}(\sigma) \subset X \) of all simplices \( \sigma \subset P \) are uniformly bounded. We say that \( f \) is *uniformly cobounded* if the last property is satisfied.

### 3 Capacity dimension

We give three equivalent definitions of the capacity dimension. Each of them is useful in appropriate circumstances.

Let \( \mathcal{U} \) be an open covering of a metric space \( Z \). We define the *capacity* of \( \mathcal{U} \) by

\[
\text{cap}(\mathcal{U}) = \frac{L(\mathcal{U})}{\text{mesh}(\mathcal{U})} \in [0, 1],
\]

in the case \( \text{mesh}(\mathcal{U}) = 0 \) or \( L(\mathcal{U}) = \text{mesh}(\mathcal{U}) = \infty \) we put \( \text{cap}(\mathcal{U}) = 1 \) by definition.

#### 3.1 First definition

For \( \tau > 0, \delta \in (0, 1) \) and an integer \( m \geq 0 \) we put

\[
c_1,\tau(Z, m, \delta) = \sup_{\mathcal{U}} \text{cap}(\mathcal{U}),
\]

where the supremum is taken over all open, \((m + 1)\)-colored coverings \( \mathcal{U} \) of \( Z \) with \( \delta \tau \leq \text{mesh}(\mathcal{U}) \leq \tau \).

Next, we put

\[
c_1(Z, m, \delta) = \lim_{\tau \to 0} \inf_{\delta} c_{1,\tau}(Z, m, \delta).
\]

The function \( c_1(Z, m, \delta) \) is monotone in \( \delta \), \( c_1(Z, m, \delta') \geq c_1(Z, m, \delta) \) for \( \delta' < \delta \). Hence, there exists a limit \( c_1(Z, m) = \lim_{\delta \to 0} c_1(Z, m, \delta) \). Now, we define the capacity dimension of \( Z \) as

\[
\text{cdim}_1(Z) = \inf \{ m : c_1(Z, m) > 0 \}.
\]

#### 3.2 Second definition

For \( \tau > 0, \delta \in (0, 1) \) and an integer \( m \geq 0 \) we put

\[
c_2,\tau(Z, m, \delta) = \sup_{\mathcal{U}} \text{cap}(\mathcal{U}),
\]

where the supremum is taken over all open coverings \( \mathcal{U} \) of \( Z \) with multiplicity \( \leq m + 1 \) and \( \delta \tau \leq \text{mesh}(\mathcal{U}) \leq \tau \). Now, we proceed as above putting

\[
c_2(Z, m, \delta) = \lim_{\tau \to 0} \inf_{\delta} c_{2,\tau}(Z, m, \delta),
\]
\[ c_2(Z, m) = \lim_{\delta \to 0} c_2(Z, m, \delta) \] and finally
\[ \text{cdim}_2(Z) = \inf\{ m : c_2(Z, m) > 0 \}. \]

### 3.3 Third definition

Let \( f : Z \to P \) be a map into a \( m \)-dimensional uniform polyhedron \( P \). We define \( \text{mesh}(f) \) as the supremum of \( \text{diam \, f}^{-1}(\text{st}_v) \) over open stars \( \text{st}_v \subset P \) of vertices \( v \in P \). Next, we introduce the capacity of \( f \) as
\[ \text{cap}(f) = (\text{Lip}(f) \cdot \text{mesh}(f))^{-1}, \]
and for \( \tau > 0, \, \delta \in (0, 1) \) and an integer \( m \geq 0 \) we define \( c_{3, \tau}(Z, m, \delta) = \sup_f \text{cap}(f) \), where the supremum is taken over all Lipschitz maps \( f : Z \to P \) into \( m \)-dimensional uniform polyhedra \( P \) with \( \delta \tau \leq \text{mesh}(f) \leq \tau \). Then we define as above \( c_3(Z, m, \delta) \), \( c_3(Z, m) \) and
\[ \text{cdim}_3(Z) = \inf\{ m : c_3(Z, m) > 0 \}. \]

The basic motivation of the capacity dimension is that in some circumstances we need control over the Lebesgue number \( L(U) \) of coverings involved in the definition of a dimension, e.g., that the capacity \( \text{cap}(U) \) stays separated from zero for appropriately chosen \( U \)'s. In general, there is no reason for that. However, if we allow coverings with larger multiplicity, we can typically gain control over \( L(U) \), and it may happen that \( \text{dim} Z < \text{cdim} Z \).

Another feature of the definitions is that they involve the auxiliary variable \( \delta \) and the functions \( c_{i, \tau}(Z, m, \delta) \), \( i = 1, 2, 3 \). This is done for a technical reason to enable “Čech approximations” \( U_k \) for which \( \text{mesh}(U_{k+1}) \) is not extremely small compared with \( \text{mesh}(U_k) \) for every \( k \).

### 3.4 Equivalence of the definitions

The proof that three capacity dimensions coincide is standard, cf. [BD], [BS], [LS].

**Proposition 3.1.** All three capacity dimensions coincide,
\[ \text{cdim}_1 = \text{cdim}_2 = \text{cdim}_3. \]

**Proof.** The multiplicity of every \((m + 1)\)-colored covering is at most \( m + 1 \). Thus \( c_{1, \tau}(Z, m, \delta) \leq c_{2, \tau}(Z, m, \delta) \) for all \( \tau > 0, \, \delta \in (0, 1) \) and integer \( m \geq 0 \), and hence \( \text{cdim}_2(Z) \leq \text{cdim}_1(Z) \).

Given an integer \( m \geq 0 \), every open covering \( U \) of \( Z \) with multiplicity \( \leq m + 1 \) is locally finite. If in addition its Lebesgue number is positive, \( L(U) > 0 \), then the barycentric map \( p : Z \to \mathcal{N} \), \( \mathcal{N} = \mathcal{N}(U) \) is the nerve, \( \text{dim} \mathcal{N} \leq m \), is Lipschitz with \( \text{Lip}(p) \leq (m + 2)^2 / L(U) \). Since \( p^{-1}(\text{st}_v) = U \)
for the vertex \( v \in \mathcal{N} \), corresponding to \( U \in \mathcal{U} \), we have \( \text{mesh}(p) = \text{mesh}(U) \).

Thus for capacities we have \( (m + 2)^2 \text{cap}(p) \geq \text{cap}(U) \), and \( c_2(Z, m, \delta) \leq (m + 2)^2 c_3(Z, m, \delta) \) for all \( \tau > 0 \), \( \delta \in (0, 1) \) and integer \( m \geq 0 \). Hence, \( \text{cdim}_3(Z) \leq \text{cdim}_2(Z) \).

Finally, we can assume that \( m = \text{cdim}_3(Z) < \infty \). Then \( c_0 = \frac{1}{8} c_3(Z, m) > 0 \) and \( c_3(Z, m, \delta) \geq 4c_0 \) for all sufficiently small \( \delta \), \( c_3(Z, m, \delta) \geq 2c_0 \) for all sufficiently small \( \tau > 0 \). Thus there is a Lipschitz map \( f : Z \to P \) into \( m \)-dimensional uniform polyhedron \( P \) with \( \delta \tau \leq \text{mesh}(f) \leq \tau \) and \( \text{cap}(f) \geq c_0 \).

Every \( a \)-dimensional simplex \( \sigma \subset P \) is matched by its barycenter, which is the vertex \( v_\sigma \) in the first barycentric subdivision \( ba \) of \( P \). We let \( A = \{0, \ldots, m\} \) be the color set, and for \( a \in A \) let \( \mathcal{P}^a \) be the family of the open stars \( st_\sigma \) of \( ba \) of all \( v_\sigma \) with \( \text{dim} \sigma = a \), \( \mathcal{P}^a = \{st_\sigma : \sigma \subset P, \; \text{dim} \sigma = a\} \).

The family \( \mathcal{P}^a \) is disjoint, \( st_\sigma \cap st_{\sigma'} = \emptyset \) for \( \sigma \neq \sigma' \), and \( \mathcal{P} = \bigcup_{a \in A} \mathcal{P}^a \) is a covering of the polyhedron \( P \). Thus the open covering \( \mathcal{P} \) is \((m + 1)\)-colored.

Now, \( \mathcal{U} = p^{-1}(\mathcal{P}) \) is a \((m + 1)\)-colored open covering of \( Z \), \( \mathcal{U} = \bigcup_{a \in A} \mathcal{U}^a \), where \( \mathcal{U}^a = p^{-1}(\mathcal{P}^a) \). Since the stars \( st_\sigma \) of \( ba \) are contained in appropriate open stars of \( P \), we have \( \text{mesh}(U) \leq \text{mesh}(f) \leq \tau \). Since the polyhedron \( P \) is uniform, there is a lower bound \( l_m > 0 \) for the Lebesgue number of the covering \( \mathcal{P} \). Therefore, for the Lebesgue number of \( \mathcal{U} \) we have \( L(\mathcal{U}) \geq l_m/\text{Lip}(f) \). This implies \( \text{cap}(\mathcal{U}) \geq l_m \text{cap}(f) \geq l_m c_0 \) and

\[
\text{mesh}(\mathcal{U}) \geq L(\mathcal{U}) \geq l_m \text{cap}(f) \text{mesh}(f) \geq l_m c_0 \delta \tau.
\]

Putting everything together we obtain \( c_{1, \tau}(Z, m, l_m c_0 \delta) \geq c_0 > 0 \) for every sufficiently small positive \( \tau, \delta \). Thus \( \text{cdim}_1(Z) \leq m = \text{cdim}_3(Z) \).

From now on, we denote by \( \text{cdim} Z \) the common value of the capacity dimensions of \( Z \). Clearly, the capacity dimension dominates the topological dimension, \( \text{dim} Z \leq \text{cdim} Z \). The condition for coverings to be open in the first and second definitions is inessential, and one can define \( \text{cdim} Z \) using coverings by arbitrary sets.

The following characterization of the capacity dimension allows to compare it with the Assouad-Nagata dimension, see [AS], [LS].

**Proposition 3.2.** The capacity dimension of a metric space \( Z \) is the infimum of all integers \( m \) with the following property: There exists a constant \( c > 0 \) such that for all sufficiently small \( s > 0 \), \( Z \) has a \( cs \)-bounded covering with \( s \)-multiplicity at most \( m + 1 \).

**Proof.** We have to prove that \( \text{cdim} Z = \text{cdim}' Z \), where \( \text{cdim}' Z \) is defined by the statement of the Proposition,

\[
\text{cdim}' Z + 1 = \lim_{c \to \infty} \lim_{s \to 0} \inf \{ m_s(\mathcal{U}) : \text{mesh}(\mathcal{U}) \leq cs \}.
\]

Let \( m' = \text{cdim}' Z \). Then there are positive \( c, s_0 \) such that for every \( s \in (0, s_0] \) there is a covering \( \mathcal{U} \) of \( Z \) with \( m_s(\mathcal{U}) \leq m' + 1 \) and \( \text{mesh}(\mathcal{U}) \leq cs \). Given
a covering $\mathcal{U}$ of $Z$ with $\text{mesh}(\mathcal{U}) \leq cs$, note that the covering $\mathcal{U}_s = B_s(\mathcal{U})$ is open, $m(\mathcal{U}_s) = m_s(\mathcal{U})$, and

$$s \leq L(\mathcal{U}_s) \leq \text{mesh}(\mathcal{U}_s) \leq \text{mesh}(\mathcal{U}) + 2s \leq (c + 2)s.$$ 

Thus for the capacity of $\mathcal{U}_s$ we have $\text{cap}(\mathcal{U}_s) \geq 1/(c + 2)$. It follows that $c_{2,\tau}(Z, m', \delta) \geq 1/(c + 2)$ for $\tau = (c + 2)s$ and $\delta \in (0, 1/(c + 2))$. Hence, $c_2(Z, m') \geq 1/(c + 2) > 0$ and therefore $\text{cdim} Z \leq m'$.

Conversely, put $m = \text{cdim} Z$. Then $c_0 = \frac{1}{8} c_2(Z, m) > 0$, $c_2(Z, m, \delta) \geq 4c_0$ for all sufficiently small $\delta > 0$, $c_{2,\tau}(Z, m, \delta) \geq 2c_0$ for all $\tau$, $0 < \tau \leq \tau_0$. This means that there is an open covering $\mathcal{U}$ of $Z$ with $\delta \tau \leq \text{mesh}(\mathcal{U}) \leq \tau$ and $L(\mathcal{U}) \geq c_0 \text{mesh}(\mathcal{U})$ having the multiplicity $\leq m + 1$. Then $s = c_0 \delta \tau/2 < L(\mathcal{U})$. By Lemma 2.1 the family $\mathcal{U}_{-s} = B_{-s}(\mathcal{U})$ is a covering of $Z$ with $\text{mesh}(\mathcal{U}_{-s}) \leq \text{mesh}(\mathcal{U}) \leq \frac{2}{c_0 \delta} s$, and its $s$-multiplicity $m_s(\mathcal{U}_{-s}) \leq m(\mathcal{U}) \leq m + 1$. Fixing a sufficiently small $\delta$ as above and taking $\tau \to 0$, we obtain $\text{cdim}' Z \leq m$. $\square$

From this characterization we immediately obtain

**Corollary 3.3.** The capacity dimension is monotone, $\text{cdim} Y \leq \text{cdim} Z$ for every $Y \subset Z$. $\square$

If we omit “sufficiently small” from the statement of Proposition 3.2, then we come up with the Assouad-Nagata dimension of $Z$, $\text{ANdim} Z$. Thus we obtain

**Corollary 3.4.** For every metric space $Z$ we have $\text{cdim} Z \leq \text{ANdim} Z$. $\square$

From [LS] we obtain

**Corollary 3.5.** Every compact Riemannian manifold $M$ satisfies $\text{cdim} M = \dim M$.

Vice versa, the Assouad-Nagata dimension can be characterized by the formula $\text{ANdim} Z = \inf \{m \geq 0 : c'_2(Z, m) > 0\}$, where

$$c'_2(Z, m) = \lim_{\delta \to 0, \tau > 0} \inf c_{2,\tau}(Z, m, \delta).$$

Speaking loosely, the Assouad-Nagata dimension takes into account all scales, while the capacity dimension only all sufficiently small scales as the topological dimension does.

### 4 Quasi-symmetry invariance

The capacity dimension as well as the Assouad-Nagata dimension is obviously a bilipschitz invariant. The striking fact discovered in [LS] is that the Assouad-Nagata dimension is a quasi-symmetry invariant.
A map \( f : X \to Y \) between metric spaces is called \textit{quasi-symmetric}, if it is not constant and there is a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) such that from \( |xa| \leq t |xb| \) it follows that \( |f(x)f(a)| \leq \eta(t) |f(x)f(b)| \) for any \( a, b, x \in X \) and all \( t \geq 0 \). In this case, we say that \( f \) is \( \eta \)-quasi-symmetric. One easily sees that \( f \) is injective and continuous, and \( f^{-1} : f(X) \to X \) is \( \eta^{-1} \)-quasi-symmetric with \( \eta(t) = 1/\eta^{-1}(t^{-1}) \) for \( t > 0 \). Furthermore, if \( f : X \to Y \) and \( g : Y \to Z \) are \( \eta_f \)- and \( \eta_g \)-quasi-symmetric respectively, then \( g \circ f : X \to Z \) is \( (\eta_g \circ \eta_f) \)-quasi-symmetric. A quasi-symmetric homeomorphism is called a \textit{quasi-symmetry}. For more details on quasi-symmetric maps see [He].

For example, the transformation \( d \mapsto d^p \) of a metric \( d \), where \( 0 < p < 1 \), is a quasi-symmetry called a \textit{snow-flake transformation}, see e.g. [BoS]. Such a transformation can be far away from being bilipschitz because nontrivial rectifiable paths w.r.t. the metric \( d \) are nonrectifiable w.r.t. the metric \( d^p \).

**Theorem 4.1.** The capacity dimension is a quasi-symmetry invariant of metric spaces.

Combining with Corollary 3.3 we obtain

**Corollary 4.2.** Assume that there is a quasi-symmetric \( f : X \to Y \). Then \( \operatorname{cdim} X \leq \operatorname{cdim} Y \). \( \square \)

Using Proposition 3.2, one can refer to [LS, Theorem 1.2] for the proof of Theorem 4.1 because actually the same argument works in our case. However, we give a different proof of Theorem 1.1 based on ideas from [LS], as an attempt to understand very nice and concise arguments from [LS].

A key ingredient of the proof of Theorem 1.1 is the existence of a sequence of coverings established in Proposition 1.4.

We say that a family \( \mathcal{U} \) of sets in a space \( X \) is \textit{separated}, if different members \( U, U' \in \mathcal{U} \) are either disjoint, \( U \cap U' = \emptyset \), or one of them is contained in the other. Note that if \( \mathcal{U} \) is separated, then \( B_{-s}(\mathcal{U}) \) is separated for every \( s \geq 0 \).

Let \( \mathcal{U}, \mathcal{U}' \) be families of sets in \( X \). We denote by \( \mathcal{U} \ast \mathcal{U}' \) the family obtained by taking for every \( U \in \mathcal{U} \) the union \( V \) of \( U \) and all members \( U' \in \mathcal{U}' \) which intersect \( U \), \( \mathcal{U} \ast \mathcal{U}' = \{ V : U \in \mathcal{U} \} \). It is straightforward to check that the following is true.

**Lemma 4.3.** Let \( \mathcal{U}, \widehat{\mathcal{U}} \) be separated families in \( X \) such that no member of \( \widehat{\mathcal{U}} \) intersects disjoint members of \( \mathcal{U} \). Then the family \( \mathcal{V} = \mathcal{U} \ast \widehat{\mathcal{U}} \cup \widehat{\mathcal{U}} \) is separated, moreover, if \( \mathcal{U} \) is disjoint then \( \mathcal{U} \ast \widehat{\mathcal{U}} \) is also disjoint. \( \square \)

**Proposition 4.4.** Suppose that \( X \) is a metric space with finite capacity dimension, \( \operatorname{cdim} X \leq n \). Then there are positive constants \( c_0, \delta \) such that for every sufficiently small \( r > 0 \) there exists a sequence of open coverings \( \mathcal{U}_j \) of \( X \), \( j \in \mathbb{N} \), with the following properties

\begin{itemize}
  \item \( \operatorname{diam}(U) \leq r \) for every \( U \in \mathcal{U}_j \).
  \item \( \operatorname{diam}(V) \leq c_0 \delta \) for every \( V \in \mathcal{V}_j \).
  \item \( \mathcal{V}_j \) is \( \eta \)-quasi-symmetric for \( \eta \) as above.
  \item \( \operatorname{cdim} X \leq n \).
\end{itemize}
(i) For every \( j \in \mathbb{N} \) the covering \( \mathcal{U}_j \) is \((n+1)\)-colored by one and the same color set \( A \), \( \mathcal{U}_j = \cup_{a \in A} \mathcal{U}^a_j \), \(|A| = n+1\);

(ii) for every \( j \in \mathbb{N} \) we have
\[
\delta r^j \leq \text{mesh}(\mathcal{U}_j) \leq r^j \quad \text{and} \quad L(\mathcal{U}_j) \geq c_0 \text{mesh}(\mathcal{U}_j);
\]

(iii) for every \( i > j \) the covering \( \mathcal{U}_i \) is inscribed in the covering \( \mathcal{U}_j \);

(iv) for every \( a \in A \) the union \( \mathcal{U}^a = \cup_{j \in \mathbb{N}} \mathcal{U}^a_j \) is separated.

**Proof.** The most important property is (iv). Existence of a sequence of coverings \( \hat{\mathcal{U}}_j \), \( j \in \mathbb{N} \), of \( X \) possessing (i)–(iii) is an easy consequence of the condition \( \text{cdim} X \leq n \) and the definition of \( \text{cdim} \). Namely, as in the proof of Proposition 3.2 we have \( c'_0 = \frac{1}{s} c_1(X, n) > 0 \), \( c_1(X, n, \delta') \geq 4c'_0 \) for all sufficiently small \( \delta' > 0 \). We fix such a \( \delta' \), and note that \( c_1, e(X, n, \delta') \geq 2c'_0 \) for all \( \tau, 0 < \tau \leq \tau_0 \). This means that for every \( \tau \in (0, \tau_0) \) there is an open \((n+1)\)-colored covering \( \mathcal{U}_\tau \) of \( X \) with \( \delta' \tau \leq \text{mesh}(\mathcal{U}_\tau) \leq \tau \) and the capacity arbitrarily close to \( c_1, e(X, n, \delta') \), in particular, \( L(\mathcal{U}_\tau) \geq c'_0 \text{mesh}(\mathcal{U}_\tau) \).

Take a positive \( \tau < \min\{c'_0 \delta'/4, \tau_0\} \) and for every \( j \in \mathbb{N} \) consider the covering \( \hat{\mathcal{U}}_j = \mathcal{U}_{j, \tau} \), where \( \tau_j = \tau^j \). Then the sequence \( \hat{\mathcal{U}}_j \), \( j \in \mathbb{N} \), of open coverings of \( X \) satisfies the conditions (i), (ii) (with \( \delta = \delta' \) and \( c_0 = c'_0 \)). Since \( L(\hat{\mathcal{U}}_j) \geq c'_0 \delta' \tau^j > \tau^{j+1} \geq \text{mesh}(\hat{\mathcal{U}}_{j+1}) \), the covering \( \hat{\mathcal{U}}_{j+1} \) is inscribed in \( \hat{\mathcal{U}}_j \), thus (iii) is also satisfied.

Now, we modify the sequence \( \{\hat{\mathcal{U}}_j\} \) in a way to keep properties (i)–(iii) preserved and to obtain (iv). This can be done, e.g. as in [23, Proposition 4.1]. We take another track to show a different possibility.

For \( k \in \mathbb{N} \) we put \( s_k = c'_0 \delta' r^k/4 \). Fix a color \( a \in A \) and define \( \mathcal{V}_1^a = \mathcal{U}_{1, \tau}^a := \hat{\mathcal{U}}_1^a \). Assume that for \( k \geq 1 \) the family \( \mathcal{V}_k^a \) is already defined, it is separated and \( \mathcal{V}_k^a = \cup_{j=1}^k \mathcal{U}_{j,k}^a \) where each family \( \mathcal{U}_{j,k}^a \) is disjoint. Now, we define
\[
\mathcal{V}_{k+1}^a := B_{-s_k} (\mathcal{V}_k^a) \ast \hat{\mathcal{U}}_{k+1}^a \cap \hat{\mathcal{U}}_{k+1}^a.
\]
Then \( \mathcal{V}_{k+1}^a + 1 = \cup_{j=1}^k \mathcal{U}_{j+1,k}^a \), where \( \mathcal{U}_{j+1,k}^a = B_{-s_k} (\mathcal{U}_{j,k}^a) \ast \hat{\mathcal{U}}_{k+1}^a \) for \( 1 \leq j \leq k \) and \( \mathcal{U}_{k+1,k+1}^a = \hat{\mathcal{U}}_{k+1}^a \). Since \( \text{mesh}(\hat{\mathcal{U}}_{k+1}) \leq \text{mesh}(\mathcal{U}_{k+1}) \leq r^{k+1} < s_k \), no member of \( \mathcal{V}_{k+1}^a \) intersects disjoint members of \( B_{-s_k} (\mathcal{V}_k^a) \). Then by Lemma 4.3 the family \( \mathcal{V}_{k+1}^a \) is separated and the family \( \mathcal{U}_{j,k+1}^a \) is disjoint for every \( 1 \leq j \leq k + 1 \).

It follows from the definition of \( \mathcal{U}_{j,k+1}^a = B_{-s_k} (\mathcal{U}_{j,k}^a) \ast \hat{\mathcal{U}}_{k+1}^a \) and the fact that \( r^{k+1} < s_k \), that for every \( k \geq 1 \), \( 1 \leq j \leq k \), every member \( U \in \mathcal{U}_{j,k+1}^a \) is contained in a unique member \( U' \in \mathcal{U}_{j,k}^a \). In this sense, for every \( j \in \mathbb{N} \) the sequence of families \( \mathcal{U}_{j,k}^a \), \( k \geq j \), is monotone, \( \mathcal{U}_{j,k}^a \supseteq \mathcal{U}_{j,k+1}^a \), and the intersection \( \cap_{k \geq j} \mathcal{U}_{j,k}^a \) is defined in the obvious sense.

We put \( \tilde{s}_j = \sum_{k \geq j} s_k = c'_0 \delta' r^j/4(1 - r) \). Then \( \tilde{s}_j < c'_0 \delta' r^j \leq L(\mathcal{U}_j) \) for every \( j \geq 1 \). By Lemma 2.1 the family \( \hat{\mathcal{U}}_j = B_{-\tilde{s}_j} (\mathcal{U}_j) \) is still an
open covering of $X$, and $B_{-\bar{s}_j}(\hat{U}_j^a) \subset \bigcap_{k \geq j} U_{j,k}^a$ for every $a \in A, j \in \mathbb{N}$. Now, for the interior $U_j^a = \text{Int} \left( \bigcap_{k \geq j} U_{j,k}^a \right)$ the family $U_j = \bigcup_{a \in A} U_j^a$ is an open $(n+1)$-colored covering of $X$ inscribed in $\hat{U}_j$ for every $j \in \mathbb{N}$. Then $\text{mesh}(U_j) \leq \text{mesh}(\hat{U}_j) \leq r^j$.

Since $r < 1/2$, we have $\bar{s}_j = c_0 \delta^j r^j/(1-r) \leq c_0 \delta^j r^j/2$ for every $j \in \mathbb{N}$. The covering $\hat{U}_j$ is inscribed in $U_j$, thus $L(U_j) \geq L(\hat{U}_j) \geq c_0 \delta^j r^j - \bar{s}_j \geq c_0 \delta^j r^j/2$. We put $\delta := c_0 \delta^j/2 =: c_0$. Then $\text{mesh}(U_j) \geq L(U_j) \geq \delta r^j$ and $L(U_j) \geq c_0 \text{mesh}(U_j)$. Since $L(U_j) > r^{j+1} \geq \text{mesh}(U_{j+1})$, the covering $U_{j+1}$ is inscribed in $U_j$ for every $j \in \mathbb{N}$. Therefore, the sequence of open coverings $U_j, j \in \mathbb{N}$, satisfies (i)-(iii). For every $a \in A, k \geq 1$, the family $V_k^a$ is separated. It follows that the family $U^a = \bigcup_{j \in \mathbb{N}} U_j^a$ is also separated, hence (iv).

We define the local capacity of an open covering $U$ of a metric space $Z$ by

$$\text{cap}_{\text{loc}}(U) = \inf_{z \in Z} \frac{L(U, z)}{\text{mesh}(U, z)}.$$ 

Clearly, $1 \geq \text{cap}_{\text{loc}}(U) \geq \text{cap}(U)$. The advantage of the local capacity over the capacity is that its positivity is preserved under quasi-symmetries quantitatively, see Lemma 4.5. This implies that a dimension defined exactly as the capacity dimension replacing the capacity of coverings by the local capacity is a quasi-symmetry invariant. However, that invariant is not as good as the capacity dimension for applications. We use the local capacity of coverings only as an auxiliary tool to prove quasi-symmetry invariance of the capacity dimension.

**Lemma 4.5.** Let $U$ be an open covering of a metric space $Z$, $f : Z \to Z'$ be an $\eta$-quasi-symmetry, $U' = f(U)$ be the image of $U$. Then for the local capacities of $U$ and $U'$ we have

$$\frac{1}{\text{cap}_{\text{loc}}(U')} \leq 16\eta \left( \frac{2}{\text{cap}_{\text{loc}}(U)} \right).$$

**Proof.** We can assume that $\text{cap}_{\text{loc}}(U) > 0$. We fix $z \in Z$ and consider $U \subset U$ for which $z \in U$ and $\text{dist}(z, Z \setminus U) \geq L(U, z)/2$. For $z' = f(z)$ and $U' = f(U)$ there is $a' \in Z' \setminus U'$ with $|z'a'| \leq 2 \text{dist}(z', Z' \setminus U')$. Then $|z'a'| \leq 2L(U', z')$, and for $a = f^{-1}(a')$ we have $|za| \geq \text{dist}(z, Z \setminus U) \geq L(U, z)/2$.

Similarly, consider $V' \in U'$ with $z' \in V'$ and $\text{diam} V' \geq \text{mesh}(U', z')/2$. There is $b' \in V'$ with $|z'b'| \geq \text{diam} V'/4$. Then $|z'b'| \geq \text{mesh}(U', z')/8$, and for $b := f^{-1}(b')$ we have $|zb| \leq \text{mesh}(U, z)$. Therefore, we have

$$\text{cap}_{\text{loc}}(U) \leq \frac{L(U, z)}{\text{mesh}(U, z)} \leq \frac{2|za|}{|zb|} \quad \text{and} \quad |zb| \leq t|za|$$

for some $t \geq 1$. This completes the proof.
with \( t = 2/\operatorname{cap}_{\text{loc}}(\mathcal{U}) \). It follows \( |z'b'| \leq \eta(t)|z'a'| \) and

\[
\frac{L(\mathcal{U}', z')}{\operatorname{mesh}(\mathcal{U}', z')} \geq \frac{|z'a'|}{16|z'b'|} \geq (16\eta(t))^{-1}
\]

for every \( z' \in Z' \). Then \( \operatorname{cap}_{\text{loc}}(\mathcal{U}') \geq \left(16\eta \left(\frac{2}{\operatorname{cap}_{\text{loc}}(\mathcal{U})}\right)\right)^{-1}. \)

\( \square \)

A covering \( \mathcal{U} \) of \( Z \) is said to be \( c \)-balanced, \( c > 0 \), if \( \inf\{\operatorname{diam}(U) : U \in \mathcal{U}\} \geq c \cdot \operatorname{mesh}(\mathcal{U}) \). The notion of a balanced covering combined with the local capacity allows to estimate from below the capacity of a covering as follows.

**Lemma 4.6.** If an open covering \( \mathcal{U} \) of a metric space \( Z \) is \( c_1 \)-balanced and its local capacity satisfies \( \operatorname{cap}_{\text{loc}}(\mathcal{U}) \geq c_0 \), then \( \operatorname{cap}(\mathcal{U}) \geq c_0 \cdot c_1. \)

**Proof.** Since \( \mathcal{U} \) is \( c_1 \)-balanced, we have \( \operatorname{mesh}(\mathcal{U}, z) \geq c_1 \cdot \operatorname{mesh}(\mathcal{U}) \) for every \( z \in Z \). Since \( \operatorname{cap}_{\text{loc}}(\mathcal{U}) \geq c_0 \), we have \( L(\mathcal{U}, z) \geq c_0 \cdot \operatorname{mesh}(\mathcal{U}, z) \) for every \( z \in Z. \) Therefore \( L(\mathcal{U}) \geq c_0 c_1 \cdot \operatorname{mesh}(\mathcal{U}). \)

Let \( f : X \to Y \) be a quasi-symmetry. To prove Theorem 4.1, it suffices to show that \( \operatorname{cdim} Y \leq \operatorname{cdim} X \). The idea is to construct out of a sequence \( \mathcal{U}_j \), \( j \in \mathbb{N} \), of coverings of \( X \) as in Proposition 4.1 a covering \( \mathcal{V} \) of \( X \) with local capacity uniformly separated from \( 0 \), see Lemma 4.9, such that its image \( f(\mathcal{V}) \) has an arbitrarily small mesh and is balanced, see Lemma 4.10. Then by Lemma 4.6 combined with Lemma 4.5, the capacity of the covering \( f(\mathcal{V}) \) of \( Y \) is positive quantitatively, which implies \( \operatorname{cdim} Y \leq \operatorname{cdim} X \).

Fix a sufficiently small \( r > 0 \) and consider the sequence of open coverings \( \mathcal{U}_j \) of \( X \) as in Proposition 4.1. We can assume additionally that \( \operatorname{diam} U \geq L(\mathcal{U}_j) \) for every \( U \in \mathcal{U}_j \), since if \( \operatorname{diam} U < L(\mathcal{U}_j) \) then \( U \) is contained in another member \( U' \in \mathcal{U}_j \) and thus it can be deleted from \( \mathcal{U}_j \) without destroying any property from (i)-(iv).

Following [LS], we put \( \mathcal{U} = \bigcup_{j \in \mathbb{N}} \mathcal{U}_j \) and for \( s > 0 \) consider the family \( \mathcal{U}(s) = \{U \in \mathcal{U} : \operatorname{diam} f(U) \leq s\} \).

**Lemma 4.7.** For every \( s > 0 \) the family \( \mathcal{U}(s) \) is a covering of \( X \).

**Proof.** We fix \( x \in X \), consider \( x' \in X \) different from \( x \) and put \( y = f(x), y' = f(x') \). For every \( j \in \mathbb{N} \) there is \( U_j \in \mathcal{U}_j \) containing \( x \). Take \( y_j \in f(U_j) \) with \( \operatorname{diam} f(U_j) \leq 4|yy_j| \) and consider \( x_j = f^{-1}(y_j) \). Then \( |xx_j| \leq t_j |xx'| \) with \( t_j \to 0 \) as \( j \to \infty \), since \( \operatorname{diam} U_j \leq r_j \to 0 \). Therefore, \( \operatorname{diam} f(U_j) \leq 4|yy_j| \leq 4\eta(t_j)|yy'| \leq s \) for sufficiently large \( j \). Hence, \( U_j \in \mathcal{U}(s) \).

A family \( \mathcal{V} \subset \mathcal{U}(s) \) is minimal if every \( U \in \mathcal{U}(s) \) is contained in some \( V \in \mathcal{V} \) and neither of different \( V, V' \in \mathcal{V} \) is contained in the other.

**Lemma 4.8.** For every \( s > 0 \) there is a minimal family \( \mathcal{V} \subset \mathcal{U}(s) \). Every minimal family \( \mathcal{V} \subset \mathcal{U}(s) \) is a \((n+1)\)-colored covering of \( X \).
Proof. Given $s > 0$ we construct a family $\mathcal{V} \subset \mathcal{U}(s)$ deleting every $U \in \mathcal{U}(s)$ which is contained in some other $U' \in \mathcal{U}(s)$. Now, $\mathcal{V}$ is what remains. One needs only to check that for every $U \in \mathcal{U}(s)$ there is a maximal $U' \in \mathcal{U}(s)$ with $U \subset U'$. It follows from Proposition 4.10 that for every $j \in \mathbb{N}$ there are only finitely many $U' \in \mathcal{U}_j$ containing $U$ (since all of them must have different colors). Since $\text{mesh}(\mathcal{U}_j) \to 0$ as $j \to \infty$, there are only finitely many $U' \in \mathcal{U}(s)$ containing $U$ and hence there is a maximal $U' \in \mathcal{U}(s)$ among them.

Let $\mathcal{V} \subset \mathcal{U}(s)$ be a minimal family. By Lemma 4.7 the family $\mathcal{U}(s)$ is a covering of $X$, and it follows from the definition of a minimal family that $\mathcal{V}$ is also a covering of $X$. It follows from Proposition 4.4(iv) that different $V$, $V' \in \mathcal{V}$ having one and the same color are disjoint. Thus $\mathcal{V}$ is $(n + 1)$-colored. \qed

Lemma 4.9. There is a constant $\nu > 0$ depending only on $c_0, \delta, r$ and $\eta$ such that for every $s > 0$ every minimal covering $\mathcal{V} \subset \mathcal{U}(s)$ has the local capacity $\text{cap}_{\text{loc}}(\mathcal{V}) \geq \nu$.

Proof. Let $\mathcal{V} \subset \mathcal{U}(s)$ be a minimal family. Given $x \in X$ we put $j = j(x) = \min\{i \in \mathbb{N} : x \in V \in \mathcal{V} \cap \mathcal{U}_i\}$. Then $\text{mesh}(\mathcal{V}, x) \leq r^j$. We fix $V \in \mathcal{V} \cap \mathcal{U}_j$ containing $x$, $v \in V$ with $4|xv| \geq \text{diam} V$ and note that $\text{diam} V \geq L(\mathcal{U}_j) \geq c_0 \delta r^j$ by our assumptions.

Furthermore, we fix $\mu > 0$ with $4\eta(4\mu/c_0\delta) \leq 1$. Now we check that for $i \in \mathbb{N}$ with $r^{i-j} \leq \mu$ every $U \in \mathcal{U}_i$ containing $x$ is a member of $\mathcal{U}(s)$. There is $u \in U$ with $4|f(x)f(u)| \geq \text{diam} f(U)$. We have $|xv| \leq t|xv|$ for some $t \leq 4\text{diam} U / \text{diam} V \leq 4r^{i-j}/c_0\delta \leq 4\mu/c_0\delta$. Then $\text{diam} f(U) \leq 4|f(x)f(u)| \leq 4\eta(4\mu/c_0\delta)|f(x)f(v)| \leq \text{diam} V \leq s$, thus $U \in \mathcal{U}(s)$.

Therefore, $L(\mathcal{V}, x) \geq L(\mathcal{U}_j) \geq c_0 \delta r^j$. Assuming that $i$ is taken minimal with $r^{i-j} \leq \mu$, we obtain $L(\mathcal{V}, x) \geq c_0 \delta r^{j+1}$. Thus $\frac{L(\mathcal{V}, x)}{\text{mesh}(\mathcal{V}, x)} \geq \nu = c_0 \delta \mu r$ for every $x \in X$ and $\text{cap}_{\text{loc}}(\mathcal{V}) \geq \nu$. \qed

Lemma 4.10. Given a minimal family $\mathcal{V} \subset \mathcal{U}(s)$, the $(n + 1)$-colored covering $W = f(\mathcal{V})$ of $Y$ satisfies $\text{diam} W \geq s/4\eta(t)$ for every $W \in \mathcal{W}$, where $t = 4/c_0\delta r$. In particular, $\text{mesh}(W) \geq s/4\eta(t)$ and $W$ is $c$-balanced with $c \geq 1/4\eta(t)$.

Proof. Note that $\text{mesh}(W) \leq s$ by the definition of $\mathcal{U}(s)$. Take any $W \in \mathcal{W}$ and consider $V = f^{-1}(W)$. We can assume that $V \in \mathcal{U}_j$ for some $j \in \mathbb{N}$. Then $\text{diam} V \geq L(\mathcal{U}_j) \geq c_0 \delta r^j$ by our assumption on the sequence $\{\mathcal{U}_j\}$.

For any $U \in \mathcal{U}$ with $V \subset U$ we have $\text{diam} f(U) \geq s$, since the family $\mathcal{V}$ is minimal. The covering $\mathcal{U}_j$ is inscribed in $\mathcal{U}_{j-1}$, thus there is $U \in \mathcal{U}_{j-1}$ containing $V$, in particular, $\text{diam} f(U) \geq s$.

Take $y \in W \subset f(U)$. There is $y' \in f(U)$ with $|yy'| \geq \text{diam} f(U)/4 > s/4$. For $x = f^{-1}(y)$, $x' = f^{-1}(y')$ we have $|xx'| \leq \text{diam} U \leq \text{mesh}(\mathcal{U}_{j-1}) \leq r^{j-1}$. There is $v \in V$ with $|xv| \geq \text{diam} V/4 \geq c_0 \delta r^{j}/4$. Thus $|xx'| \leq r^{j-1} \leq
$t|xv|$ for $t = 4/c_0 \delta r$. For $w = f(v) \in W$ we obtain $|yy'| \leq \eta(t)|yw| \leq \eta(t) \text{diam } W$. Hence, $\text{diam } W \geq s/4\eta(t)$. \hfill \Box

Proof of Theorem 4.4. Let $f : X \to Y$ be an $\eta$-quasi-symmetry. We show that $\text{cdim } Y \leq n$ for every $n \geq \text{cdim } X$. Fix a sufficiently small $r > 0$ and consider a sequence $U_j$, $j \in \mathbb{N}$, of coverings of $X$ as in Proposition 4.4 with positive $c_0$, $\delta$. Then by Lemmas 4.8 and 4.10 for every $s > 0$ we have an open $(n + 1)$-colored covering $W$ of $Y$ with $s/4\eta(t) \leq \text{mesh}(W) \leq s$, which is $c$-balanced, $c \geq 1/4\eta(t)$, where $t = 4/c_0 \delta r$. Moreover, by Lemmas 4.9 and 4.5 its local capacity $\text{cap}_{\text{loc}}(W) \geq d$, where the constant $d > 0$ depends only on $\eta$, $c_0$, $\delta$, $r$. Then by Lemma 4.6 we have $\text{cap}(W) \geq c \cdot d$ independently of $s$.

This shows that $c_1,s(Y, n, \delta') \geq c \cdot d$ for every $s > 0$, where $\delta' = 1/4\eta(t)$. Hence, $c_1(Y, n) \geq c \cdot d > 0$ and $\text{cdim } Y \leq n$. \hfill \Box

5 Asymptotic dimension of a hyperbolic cone

Let $Z$ be a bounded metric space. Assuming that $\text{diam } Z > 0$ we put $\mu = \pi/\text{diam } Z$ and note that $\mu|zz'| \in [0, \pi]$ for every $z, z' \in Z$. Recall that the hyperbolic cone $\text{Co}(Z)$ over $Z$ is the space $Z \times [0, \infty)/Z \times \{0\}$ with metric defined as follows. Given $x = (z, t)$, $x' = (z', t') \in \text{Co}(Z)$ we consider a triangle $\bar{\sigma}x\bar{x}' \subset \mathbb{H}^2$ with $|\bar{\sigma}x| = t$, $|\bar{\sigma}x'| = t'$ and the angle $\angle_{\bar{\sigma}}(x, x') = \mu|zz'|$. Now, we put $|xx'| := |\bar{\sigma}x|$. In the degenerate case $Z = \{pt\}$ we define $\text{Co}(Z) = \{pt\} \times [0, \infty)$ as the metric product. The point $o = Z \times \{0\} \in \text{Co}(Z)$ is called the vertex of $\text{Co}(Z)$.

Theorem 5.1. For every bounded metric space $Z$ we have

$$\text{asdim } \text{Co}(Z) \leq \text{cdim } Z + 1.$$ 

The proof occupies sect. 5.1. In sect. 5.3 our arguments are close to those from [BD, §2].

5.1 Some estimates from hyperbolic geometry

We denote by $Z_t$ the metric sphere of radius $t > 0$ around $o$ in $\text{Co}(Z)$. There are natural polar coordinates $x = (z, t)$, $z \in Z$, $t \geq 0$, in $\text{Co}(Z)$. Then $Z_t = \{(z, t) : z \in Z\}$ is the copy of $Z$ at the level $t$. For $t > 0$ we denote by $\pi_t : Z_t \to Z$ the canonical homeomorphism, $\pi_t(z, t) = z$.

Let $\mathcal{U}$ be an open covering of $Z$ with multiplicity $m + 1$ and positive Lebesgue number $L(\mathcal{U})$. Let $\mathcal{N} = \mathcal{N}(\mathcal{U})$ be the nerve of $\mathcal{U}$, $p : Z \to \mathcal{N}$ the barycentric map. Then $\text{Lip}(p) \leq \frac{(m + 2)^2}{L(\mathcal{U})}$, see sect. 2.3. For every $t > 0$ we have the induced covering $\mathcal{U}_t = \pi_t^{-1}(\mathcal{U})$ of $Z_t$ whose nerve is canonically isomorphic to $\mathcal{N}$, and the corresponding barycentric map $p_t : \mathcal{U}_t \to \mathcal{N}$. 14
Given \( \lambda > 0 \) we want to find \( t > 0 \) and conditions for \( U \) such that \( \text{Lip}(p_t) \leq \lambda \) and still to get \( p_t \) uniformly cobounded w.r.t. the metric induced from \( \text{Co}(Z) \). To this end, we first recall the hyperbolic cosine law. For \( t > 0 \), \( \alpha \in [0, \pi] \) we define \( a = a(t, \alpha) \) by

\[
\cosh a = \cosh^2(t) - \sinh^2(t) \cos \alpha,
\]

i.e., \( a \) is the length of the base opposite to the vertex \( o \) with angle \( \alpha \) of a isosceles triangle in \( H^2 \) with sides \( t \). Then for \( \alpha \) sufficiently small we have

\[
\cosh a = 1 + \frac{1}{2} \sinh^2(t) \alpha^2 + \sinh^2(t) \cdot o(\alpha^3).
\]

Assume that small \( \lambda, \sigma > 0 \) are fixed so that \( d := \frac{(m+2)^2}{\lambda} - \ln \frac{1}{\sigma} = \frac{(m+2)^2}{2\lambda} \), and

\[
\sigma \tau \leq \mu L(U) \leq \mu \text{mesh}(U) \leq \tau
\]

for sufficiently small \( \tau \). We put

\[
t_\tau = \ln \frac{2}{\tau} + \frac{2(m+2)^2}{\lambda}.
\]

Then \( t_\tau - 2d = \ln \frac{2}{\sigma \tau} \), and for \( t_\tau - 2d \leq t \leq t_\tau \) we have

\[
\sinh^2(t)(\sigma \tau)^2 \approx \frac{1}{4} e^{2t(\sigma \tau)^2} \geq \frac{1}{\sigma^2} = \exp\left(\frac{(m+2)^2}{\lambda}\right) \gg 1,
\]

while

\[
\sinh^2(t) \cdot o(\tau^3) \leq o(\tau) \cdot \exp\left(\frac{4(m+2)^2}{\lambda}\right) \ll 1.
\]

Noting that \( L(U_t) = a(t, \mu L(U)) \) we obtain

\[
\cosh(L(U_t)) \geq 1 + \frac{1}{2} \sinh^2(t)(\sigma \tau)^2 \geq 1 + \frac{1}{2} \exp\left(\frac{(m+2)^2}{\lambda}\right)
\]

up to a negligible error. Hence \( L(U_t) > \frac{(m+2)^2}{\lambda} \) and \( \text{Lip}(p_t) \leq \frac{(m+2)^2}{L(U_t)} < \lambda \).

Similarly, \( \text{mesh}(U_t) = a(t, \mu \text{mesh}(U)) \), and for \( t_\tau - 2d \leq t \leq t_\tau \), we obtain as above

\[
\cosh(\text{mesh}(U_t)) \leq 1 + \frac{1}{2} \sinh^2(t) \tau^2 \simeq 1 + \frac{1}{8} e^{2t} \tau^2 \leq 1 + \frac{1}{2} \exp\left(\frac{4(m+2)^2}{\lambda}\right),
\]

which gives an upper bound for \( \text{mesh}(U_t) \) depending only on \( \lambda \).
5.2 Čech approximation

Here we construct a sequence of coverings \( \{U_k\} \) and associated barycentric maps which will be used in the proof of Theorem 5.1. We can assume that the capacity dimension of \( Z \) is finite, \( m = \text{cdim} Z < \infty \). Then \( c_0 = \frac{1}{4} c_2(Z, m) > 0 \), and \( c_2(Z, m, \delta) \geq 4c_0 \) for all sufficiently small \( \delta > 0 \), see sect. 3. Given \( \lambda > 0 \) we take \( \delta > 0 \) so that

\[
d := \frac{(m + 2)^2}{\lambda} + \ln(c_0\delta) = \frac{(m + 2)^2}{2\lambda},
\]

assuming that \( \lambda \) is sufficiently small to satisfy \( c_2(Z, m, \delta) \geq 4c_0 \). Then \( c_2,\tau(Z, m, \delta) \geq 2c_0 \) for all sufficiently small \( \delta > 0 \), see sect. 3. Given \( \lambda > 0 \) we take \( \delta > 0 \) so that

\[
d := \frac{(m + 2)^2}{\lambda} + \ln(c_0\delta) = \frac{(m + 2)^2}{2\lambda},
\]

assuming that \( \lambda \) is sufficiently small to satisfy \( c_2(Z, m, \delta) \geq 4c_0 \). Then \( c_2,\tau(Z, m, \delta) \geq 2c_0 \) for all \( \tau, 0 < \tau \leq \tau_0 \). Consider the sequence \( \tau_k, k \geq 0 \), recursively defined by \( \tau_{k+1} = e^{-2d}\tau_k \). Recall that \( \mu = \pi / \text{diam} Z \). Then it follows from the second definition of \( \text{cdim} Z \) that for every \( k \geq 0 \) there is an open covering \( \hat{U}_k \) of \( Z \) such that

(i) \( m(\hat{U}_k) \leq m + 1 \);

(ii) \( \delta \tau_k \leq \mu \text{mesh}(\hat{U}_k) \leq \tau_k \) and \( L(\hat{U}_k) \geq c_0 \text{mesh}(\hat{U}_k) \).

Since \( \mu L(\hat{U}_k) \geq c_0 \delta \tau_k > \tau_{k+1} \geq \mu \text{mesh}(\hat{U}_{k+1}) \), we additionally have

(iii) \( \hat{U}_{k+1} \) is inscribed in \( \hat{U}_k \) for every \( k \).

We put \( t_k = \ln \frac{2}{\tau_k} + \frac{2(m + 2)^2}{\lambda} \).

\( Z_k = Z_{t_k} \subset \text{Co}(Z) \) and for every \( t > 0 \) and every integer \( k \geq 0 \) consider the covering \( U_{t,k} = \pi^{-1}_t(\hat{U}_k) \) of \( Z_t \). Note that its nerve is independent of \( t > 0 \) and can be identified with \( \mathcal{N}_k = \mathcal{N}(\hat{U}_k) \). Then \( t_k - t_{k+1} = 2d \), and using the estimates from sect. 3.1 with \( \tau = \tau_k \) and \( \sigma = c_0\delta \) we obtain for the barycentric map \( p_{t,k} : Z_t \to \mathcal{N}_k \) associated with the covering \( U_{t,k} \) that \( \text{Lip}(p_{t,k}) < \lambda \) for all \( t_{k-1} \leq t \leq t_k \) and all \( k \geq 1 \). We put \( U_k = U_{t_k,k} \) and \( p_k = p_{t_k,k} : Z_k \to \mathcal{N}_k \). Then \( \text{Lip}(p_k) < \lambda \). Furthermore, \( \text{mesh}(U_k) \) is bounded above by a constant depending only on \( \lambda, \text{mesh}(\hat{U}_k) \leq \text{const}(\lambda) \), for all \( k \). Hence, preimages of all simplices from \( \mathcal{N}_k \) under \( p_k \) have uniformly bounded diameter \( \leq \text{const}(\lambda) \) independently of \( k \).

5.3 Homotopy between \( p_k \) and \( \rho_k \circ p_{k+1} \)

Due to (iii), for every \( k \) there is a simplicial map \( \rho_k : \mathcal{N}_{k+1} \to \mathcal{N}_k \) such that \( \rho_k \circ p_{k+1}(z, t_{k+1}) \) lies in a face of the minimal simplex containing \( p_k(z, t_k) \in \mathcal{N}_k \) for every \( z \in Z \).
**Lemma 5.2.** For every $k$, the map $\rho_k : N_{k+1} \to N_k$ is $c_1$-Lipschitz with $c_1 = c_1(m)$ depending only on $m$.

**Proof.** Recall that the nerve $N_k$ is a uniform polyhedron, and that $\rho_k$ is affine on every simplex sending it either to an isometric copy in $N_k$ or to a face of it. In either case, $\rho_k$ is $c_1$-Lipschitz on every simplex. If $x, x' \in N_{k+1}$ are from disjoint simplices, then $|x - x'| \geq \frac{\sqrt{2}}{c_1}$ for some $c_1 = c_1(m) > 0$, and $|\rho_k(x) - \rho_k(x')| \leq \frac{\sqrt{2}}{c_1}$. The remaining case, when $x, x'$ are sitting in different simplices in $N_{k+1}$ having a common face, we leave to the reader as an exercise.

Consider the annulus $A_k \subset \text{Co}(Z)$ between $Z_k$ and $Z_{t_k+d}$, $A_k = Z \times [t_k, t_k + d]$ in the polar coordinates. We put $s = s_k(t) = \frac{1}{d}(t - t_k)$ for $t_k \leq t \leq t_k + d$ and define the homotopy $h_k : A_k \to N_k \times [0,1]$ between $p_k$ and $\rho_k \circ p_{k+1}$ by

$$h_k(z, t) = ((1 - s)p_k(z, t_k) + s\rho_k \circ p_{k+1}(z, t_k+1), s).$$

This is well defined because the points $p_k(z, t_k)$ and $\rho_k \circ p_{k+1}(z, t_k+1)$, $z \in Z$, can be joined by the segment in the appropriate simplex.

**Lemma 5.3.** The map $h_k$ is $c_2\lambda$-Lipschitz with respect to the product metric on $N_k \times [0,1]$, for some constant $c_2 = c_2(m) > 0$ depending only on $m$.

**Proof.** By convexity of the distance function in Euclidean space, the distance between $h_k(z, t)$, $h_k(z', t) \in N_k \times \{s\}$ is bounded above by the maximum of distances between the end points of the vertical segments $z \times [0,1]$, $z' \times [0,1]$ containing them, thus

$$|h_k(x_t) - h_k(x'_t)| \leq \max\{|p_k(x_t) - p_k(x'_t)|, |\rho_k \circ p_{k+1}(x_{t+1}) - \rho_k \circ p_{k+1}(x'_{t+1})|\},$$

where $t_k \leq t \leq t_k + d$, $x_t = (z, t)$, $x'_t = (z', t)$, $x_k = x_{t_k}$, $x'_k = x'_{t_k}$. On the other hand,

$$|p_k(x_k) - p_k(x'_k)| \leq \text{Lip}(p_k)|x_kx'_k| \leq \lambda|x_kx'_k|,$$

and by Lemma 5.2

$$|\rho_k \circ p_{k+1}(x_{t+1}) - \rho_k \circ p_{k+1}(x'_{t+1})| = |\rho_k \circ p_{t,k+1}(x_t) - \rho_k \circ p_{t,k+1}(x'_t)| \leq c_1 \text{Lip}(p_{t,k+1})|x_tx'_t| \leq c_1 \lambda|x_tx'_t|$$

for $t_k \leq t \leq t_k + d$. Furthermore, since every edge of any standard simplex has length $\sqrt{2}$, we have

$$|h_k(z', t) - h_k(z', t')| \leq \sqrt{3}|s - s'| = \frac{\sqrt{3}}{d}|(z', t)(z', t')|,$$

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where \( s' = s_k(t') \). Taking into account that
\[
|(z, t)(z', t')| \geq \max\{ |x_{i}x'_{i}|, |(z', t)(z', t')| \}
\]
(assuming \( t' \geq t \)), and that \( d = \frac{(m+2)^2}{2\lambda} \), we obtain from all of these that \( \text{Lip}(h_k) \leq c_2\lambda \). \( \square \)

### 5.4 Simplicial mapping cylinder of \( \rho_k \)

Consider the annulus \( B_k \subset \text{Co}(Z) \) between \( Z_{t_k+d} \) and \( Z_{k+1} \), \( B_k = Z \times [t_k + d, t_{k+1}] \) in the polar coordinates, and define \( g_k : B_k \to N_{k+1} \times [0, 1] \) as follows \( g_k(z, t_k + d) = (p_{k+1}(z, t_{k+1}), 0) \), \( g_k(z, t_{k+1}) = (p_{k+1}(z, t_{k+1}), 1) \) and \( g_k \) is affine on every segment \( z \times [t_k + d, t_{k+1}] \subset B_k, z \in Z \).

Since \( \lambda > 0 \), it immediately follows from the estimates of sect. 5.2 that \( g_k \) is \( \lambda \)-Lipschitz (with respect to the product metric on \( N_{k+1} \times [0, 1] \)).

Next, recall the notion of the simplicial mapping cylinder for a simplicial map \( \rho : K \to L \) of simplicial complexes (see, e.g., [Sp]). Assuming that the vertices of \( K \) are linearly ordered, we define the mapping cylinder \( C_\rho \) of \( \rho \) as a simplicial complex whose vertex set is the union of those of \( K \) and \( L \), and simplices are the simplices of \( K \) and \( L \) and all subsets of the sets \( \{v_0, \ldots, v_k, \rho(v_k), \ldots, \rho(v_p)\} \), where \( \{v_0 < \cdots < v_p\} \) is a simplex in \( K \).

Now, assuming that a linear order on \( N_{k+1} \times \{0\} \) is fixed, one triangulates \( N_{k+1} \times \{0\} \) as the mapping cylinder of the identity map, and defines the canonical simplicial map \( \varphi_k : N_{k+1} \times \{0\} \to C_k = C_{\rho_k} \), which sends \( N_{k+1} \times \{0\} \) onto the subcomplex \( \rho_k(N_{k+1}) \subset C_k \) by \( \rho_k \), and \( N_{k+1} \times \{1\} \) onto \( N_{k+1} \subset C_k \) identically.

By [BD], Proposition 3, \( \varphi_k \) is \( c(m) \)-Lipschitz for some constant \( c(m) > 0 \) depending only on \( m \geq \dim N_{k+1} \), where the cylinder \( C_k \) is given the uniform metric (there is a minor inaccuracy in argument there claiming that a simplicial map between uniform complexes is always \( 1 \)-Lipschitz, which is not true as easily seen for \( \Delta^m \to \Delta^1 \) with \( m \geq 2 \); it is only \( c(m) \)-Lipschitz, see Lemma 5.2). In conclusion, the composition \( \varphi_k \circ g_k : B_k \to C_k \) is \( c_3 \lambda \)-Lipschitz with \( c_3 = c_3(m) \). Note that \( h_k \) and \( \varphi_k \circ g_k \) coincide on \( Z_{t_k+d} = A_k \cap B_k \subset \text{Co}(Z) \) if one identifies \( \rho_k(N_{k+1}) \subset C_k \) with subcomplex in \( N_k \times \{1\} \).

### 5.5 Proof of Theorem 5.1

We have to find for every sufficiently small \( \lambda > 0 \) a uniform polyhedron \( P \) with \( \dim P \leq \text{cdim} Z + 1 \) and a uniformly cobounded, \( \lambda \)-Lipschitz map \( f : \text{Co}(Z) \to P \).

Given \( \lambda > 0 \) we take \( \delta > 0 \) as in sect. 5.2. Then we generate sequences \( \{\tau_k\} \) of positive reals, \( \{U_k\} \) of open coverings of \( Z, k \geq 0 \), and all the machinery around them from sects. 5.2, 5.4.
Having that we define \( P \) as the uniformization of the union \( P' = P_{-1} \cup_{k \geq 0} P_k \), where \( P_k \) is constructed out of the uniformization of \( N^*_k \times [0, 1] \) (triangulated by fixing a linear order on \( N^*_k \)) and the simplicial mapping cylinder \( C_k \) by attaching them along the common subcomplex \( \rho_k(N^*_{k+1}) \subset (N_k \times (1)) \cap C_k \). Furthermore, \( P_{k+1} \) is attached to \( P_k \) along the common subcomplex \( N^*_{k+1} \) for every \( k \geq 0 \). The polyhedron \( P_{-1} \) is the cone over \( N^*_0 \) attached to \( P_0 \) along the base. Then \( \text{dim } P \leq m + 1 \) for \( m = \text{cdim } Z \).

The map \( f : \text{Co}(Z) \to P \) is obtained by composing the map \( f' : \text{Co}(Z) \to P' \) with the uniformization of \( P' \), where \( f' \) coincides with \( h_k \) on \( A_k \) and with \( \varphi_k \circ g_k \) on \( B_k \) for every \( k \geq 0 \). Finally, \( f' \) is affine on every segment \( z \times [0,t_0] \), \( z \in Z \), sending \( o = Z \times \{0\} \) to the vertex of \( P_{-1} \). It follows from Lemma 5.3 and sect. 5.4 that \( f \) is \( c \lambda \)-Lipschitz for some \( c = c(m) > 0 \) on every \( A_k, B_k, k \geq 0 \), and on \( Z \times [0,t_0] \subset \text{Co}(Z) \).

Since \( \text{diam } P \leq \sqrt{2} \) and \( \frac{1}{2} < \lambda \), the \( c \lambda \)-Lipschitz condition is certainly satisfied for points \( (z,t), (z',t') \in \text{Co}(Z) \) separated by some annulus \( A_k \) or \( B_k \). Thus we assume that \( (z,t), (z',t') \) are sitting in adjacent annuli. Unfortunately, we cannot directly apply the argument from [BD Proposition 4] which would be well adapted to our situation if \( \text{Co}(Z) \) is geodesic. In general, this is not the case, and we slightly modify it as follows.

Assume that \( t' > t \). We take \( t'' \in (t,t') \) for which \( (z',t'') \) is common for the annuli, and note that

\[
|(z,t)(z',t')| \geq \max\{|(z,t)(z',t)|, |t-t'| = |t-t''| + |t''-t'|\}
\]

by geometry of \( \text{Co}(Z) \). Now, the required Lipschitz condition for the pair \((z,t) \) and \((z',t') \) follows in the obvious way from those for three pairs \((z,t) \) and \((z',t) \) and \((z',t'') \) and \((z',t') \), each of which belong to some annulus.

It remains to check that \( f \) is uniformly cobounded. For every simplex \( \sigma \subset P_k, k \geq 0 \), the preimage \( f'^{-1}(\sigma) \subset A_k \cup B_k \) is contained in \( Z_\sigma \times [t_k,t_{k+1}] \subset \text{Co}(Z) \), where \( \text{diam}(Z_\sigma \times \{t_k\}) \leq \text{const}(\lambda) \) by estimates of sect. 5.3. Thus \( \text{diam } f'^{-1}(\sigma) \leq 4d + \text{diam}(Z_\sigma \times \{t_k\}) \leq \text{const}(\lambda), \) and \( f \) is uniformly cobounded. This completes the proof of Theorem 5.1.

6 Hyperbolic spaces

6.1 Basics of hyperbolic spaces

We briefly recall necessary facts from the hyperbolic spaces theory. For more details the reader may consult e.g. [BoS].

Let \( X \) be a metric space. Fix a base point \( o \in X \) and for \( x, x' \in X \) put \( (x|x')_o = \frac{1}{2}(|x|_o + |x'|_o - |x|x'|) \). The number \( (x|x')_o \) is nonnegative by the triangle inequality, and it is called the Gromov product of \( x, x' \) w.r.t. \( o \). A \( \delta \)-triple is a triple of three real numbers \( a, b, c \) with the property that the two smallest of these numbers differ by at most \( \delta \).
A metric space $X$ is (Gromov) hyperbolic if for some $\delta \geq 0$, some base point $o \in X$ and all $x, x', x'' \in X$ the numbers $(x|x')_o$, $(x'|x'')_o$, $(x|x'')_o$ form a $\delta$-triple. This condition is equivalent to the $\delta$-inequality

$$(x|x'')_o \geq \min\{ (x|x')_o, (x'|x'')_o \} - \delta.$$

Let $X$ be a hyperbolic space and $o \in X$ be a base point. A sequence of points $\{x_i\} \subset X$ converges to infinity, if

$$\lim_{i,j \to \infty} (x_i|x_j)_o = \infty.$$ 

Two sequences $\{x_i\}, \{x'_i\}$ that converge to infinity are equivalent if

$$\lim_{i \to \infty} (x_i|x'_i)_o = \infty.$$

The boundary at infinity $\partial_\infty X$ of $X$ is defined as the set of equivalence classes of sequences converging to infinity. The Gromov product extends to $X \cup \partial_\infty X$ as follows. For points $\xi, \xi' \in \partial_\infty X$ the Gromov product is defined by

$$(\xi|\xi')_o = \inf \liminf_{i \to \infty} (x_i|x'_i)_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{x'_i\} \in \xi'$. Note that $(\xi|\xi')_o$ takes values in $[0, \infty]$, and that $(\xi|\xi')_o = \infty$ if and only if $\xi = \xi'$. Furthermore, for $\xi, \xi', \xi'' \in \partial_\infty X$ the following holds, see [BoS, Sect.3]

(1) for sequences $\{y_i\} \in \xi$, $\{y'_i\} \in \xi'$ we have

$$(\xi|\xi')_o \leq \liminf_{i \to \infty} (y_i|y'_i)_o \leq \limsup_{i \to \infty} (y_i|y'_i)_o \leq (\xi|\xi')_o + 2\delta$$

(2) $(\xi|\xi')_o$, $(\xi'|\xi'')_o$, $(\xi''|\xi')_o$ is a $\delta$-triple.

Similarly, the Gromov product

$$(x|\xi)_o = \inf \liminf_{i \to \infty} (x|x_i)_o$$

is defined for any $x \in X$, $\xi \in \partial_\infty X$, where the infimum is taken over all sequences $\{x_i\} \in \xi$, and the $\delta$-inequality holds for any three points from $X \cup \partial_\infty X$.

A metric $d$ on the boundary at infinity $\partial_\infty X$ of $X$ is said to be visual, if there are $o \in X$, $a > 1$ and positive constants $c_1, c_2$, such that

$$c_1 a^{-(\xi|\xi')_o} \leq d(\xi, \xi') \leq c_2 a^{-(\xi|\xi')_o}$$

for all $\xi, \xi' \in \partial_\infty X$. In this case we say that $d$ is the visual metric w.r.t. the base point $o$ and the parameter $a$. 

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6.2 The hyperbolic cone

Proposition 6.1. Let $Z$ be a bounded metric space. Then the hyperbolic cone $Y = \text{Co}(Z)$ is a $\delta$-hyperbolic space with $\delta = \delta(H^2)$, there is a canonical inclusion $Z \subset \partial_\infty Y$, and the metric of $Z$ is visual. If in addition $Z$ is complete then $\partial_\infty Y = Z$.

Proof. We can assume that $(y|y')_o \leq (y|y'')_o \leq (y''|y')_o$ for $y, y', y'' \in Y$. We show that $(y|y'')_o \leq (y|y')_o + \delta$. To this end, consider triangles $\overrightarrow{\sigma y y''}$ and $\overrightarrow{\sigma y y'}$ in $H^2$ with common side $\overrightarrow{\sigma y}$ separating them such that $|\overrightarrow{\sigma y}| = |\overrightarrow{oy}|$, $|\overrightarrow{\sigma y'}| = |\overrightarrow{oy'}|$, and $|\overrightarrow{\sigma y''}| = |\overrightarrow{oy''}|$, $|\overrightarrow{y y'}| = |y''y'|$. Then $|yy'| \leq |\overrightarrow{y y'}|$ by the triangle inequality in $Z$. It follows that $(\overrightarrow{y y'})_\pi = (y|y')_o$, $(\overrightarrow{y y'})_\pi = (y|y'')_o$ and $(\overrightarrow{y y'})_\pi \leq (y|y'')_o$. Therefore, $(y|y'')_o - (y|y')_o \leq (\overrightarrow{y y'})_\pi - (\overrightarrow{y y'})_\pi \leq \delta$ since $H^2$ is $\delta$-hyperbolic.

For every $z \in Z$ the ray $\{z\} \times [0, \infty) \subset Y$ represents a point from $\partial_\infty Y$ which we identify with $z$. This yields the inclusion $Z \subset \partial_\infty Y$. The last assertion of the proposition is easy to check.

It remains to show that the metric of $Z$ is visual. Given $z, z' \in Z$, consider the geodesic rays $\gamma (t) = (z, t), \gamma (t) = (z', t)$ in $\text{Co}(Z)$. Then $\gamma \in Z$, $\gamma' \in z'$ viewed as points at infinity, and for $(\gamma|\gamma')_o = \lim_{t \to \infty} (\gamma(t)|\gamma'(t))_o$ (it is easy to see that the Gromov product $(\gamma(t)|\gamma'(t))_o$ is monotone) we have

$$(z|z')_o \leq (\gamma|\gamma')_o \leq (z|z')_o + 2\delta.$$

For comparison geodesic rays $\overrightarrow{\gamma}, \overrightarrow{\gamma'} \subset H^2$ with common vertex $\overrightarrow{\gamma}$ and

$$\angle_\pi(\overrightarrow{\gamma}, \overrightarrow{\gamma'}) = \mu |zz'|$$

(recall $\mu = \pi / \text{diam} Z$) we have $(\overrightarrow{\gamma|\gamma'})_\pi = (\gamma|\gamma')_o$ and $(\overrightarrow{\gamma|\gamma'})_\pi \leq d \leq (\overrightarrow{\gamma|\gamma'})_\pi + \delta$, where $d = \text{dist}(\overrightarrow{\gamma}, \overrightarrow{\gamma'})$ and $\overrightarrow{\gamma}, \overrightarrow{\gamma'} \subset H^2$ is the infinite geodesic with the end points at infinity $\overrightarrow{\gamma} = \gamma(\infty), \overrightarrow{\gamma'} = \gamma'(\infty)$. By the angle of parallelism formula from geometry of $H^2$ we have $\tan \frac{\mu |zz'|}{4} = e^{-d}$, therefore, we conclude that

$$e^{-3\delta} e^{-(z|z')_o} \leq \tan \frac{\mu |zz'|}{4} \leq e^{-(z|z')_o}$$

for every $z, z' \in Z$. The function $s \mapsto \frac{1}{8} \tan \frac{\mu s}{T}$ is uniformly bounded and separated from zero on $[0, \text{diam} Z]$. It follows that the metric of $Z \subset \partial_\infty Y$ is visual w.r.t. the vertex $o \in Y$ and the parameter $a = e$.

Let $X$ be a hyperbolic space, $x_0 \in X$ a base point. For $x \in X$ we denote $|x| = |xx_0|$. We also omit the subscript $x_0$ from the notations of Gromov products w.r.t. $x_0$. The space $X$ is called visual, if for some base point $x_0 \in X$ there is a positive constant $D$ such that for every $x \in X$ there is $\xi \in \partial_\infty X$ with $|x| \leq (x|\xi) + D$ (one easily sees that this property is independent of the choice of $x_0$). This definition is due to V. Schroeder, cf. [BoS Sec. 5].

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Proposition 6.2. Every visual hyperbolic space $X$ is roughly similar to a subspace of the hyperbolic cone over the boundary at infinity, $\text{Co}(\partial_\infty X)$, where $\partial_\infty X$ is taken with a visual metric.

Proof. We fix a visual metric on $\partial_\infty X$ w.r.t. $x_0 \in X$ and a parameter $a > 1$. Replacing $X$ by $\lambda X$ with $\lambda = 1/\ln a$ we can assume that $a = e$. Since $X$ is visual, there is a constant $D > 0$ such that for every $x \in X$ there is $\xi = \xi(x) \in \partial_\infty X$ with $|x| \leq (x|\xi) + D$. We define $F : X \to Y$, $Y = \text{Co}(\partial_\infty X)$ by $F(x) = (\xi(x), |x|) \in Y$. Note that $F(x_0) = o$.

It follows from Proposition 6.1 that the Gromov product $(\xi|\xi')$ in $X$ coincides with the Gromov product $(\xi|\xi')_o$ in $Y$ up to a uniformly bounded error for every $\xi, \xi' \in \partial_\infty X$. Since $|F(x)| = |x|$ for every $x \in X$, by [BoS, Lemma 5.1] we have

$$|xx'| \equiv |x| + |x'| - 2\min\{(|\xi(x)||\xi(x')|), |x|, |x'|\} \equiv |F(x)F(x')|$$

up uniformly bounded error for every $x, x' \in X$. Hence $F$ is roughly isometric and $X$ is roughly similar to a subspace of $Y$.

6.3 Proof of Theorem 1.1

The asymptotic dimension is a quasi-isometry invariant, thus $\text{asdim} X \leq \text{asdim} \text{Co}(\partial_\infty X)$ by Proposition 6.2. By Theorem 5.1 we obtain $\text{asdim} X \leq \text{cdim}(\partial_\infty X) + 1$.

Remark 6.3. Coming back to the Gromov argument (see Introduction) that $\text{asdim} X \leq n$ for every negatively pinched Hadamard manifold $X$ of dimension $n$, in my opinion to complete the proof one needs to show that $\text{cdim} \partial_\infty X = n - 1$ where $\partial_\infty X = S^{n-1}$ is considered with a visual metric. However, at the moment this is only known for $X = \mathbb{H}^n$.

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