Abstract. For a given quadratic irrational \( \alpha \), let us denote by \( D(\alpha) \) the length of the periodic part of the continued fraction expansion of \( \alpha \). We prove that for a positive integer \( d \), which is not a perfect square, the sequence \( (D(n\sqrt{d}))_{n=1}^{\infty} \) has infinitely many limit points.

1. Introduction

Throughout the paper we assume that \( d \) is a positive integer, which is not a perfect square. Every real number \( \alpha \) has a continued fraction expansion of the form:

\[
\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}
\]

for some non-negative integers \( a_1, a_2, \ldots \). We will denote this fraction by \([a_0, a_1, a_2, \ldots]\). Lagrange showed that continued fraction of \( \alpha \) is eventually periodic if and only if \( \alpha \) is a quadratic irrational – that is an irrational root of a quadratic polynomial with integer coefficients. Later Galois proved that for numbers of the form \( \alpha = \sqrt{d} \) this expansion is of the form

\[
\sqrt{d} = [a_0, a_1; a_2, \ldots, a_2, a_1, 2a_0],
\]

where overlined part is periodic and consists of a palindrome with added \( 2a_0 \) at the end. By \( D(\alpha) \) we denote the length of the periodic part of the continued fraction expansion of \( \alpha \).

Continued fraction expansions of the quadratic irrationals and the length of their periodic parts have been widely studied by many authors. In the paper from 1972, Chowla and Chowla [2] asked the following question.

**Question 1.1.** For a given integer \( k \geq 1 \), are there infinitely many integers \( d \geq 1 \) such that \( D(\sqrt{d}) = k \)?

The answers turns out to be positive, which was shown by Friesen in [3]. In fact, Friesen’s approach gives more than that. He proved that for palindromes \((a_1, a_2, \ldots, a_2, a_1)\), satisfying certain quite general condition related to the parity, there exist infinitely many integers \( d \geq 1 \) with \( \sqrt{d} = [a_0, a_1, \ldots, a_1, 2a_0] \). This was later refined by Halter-Koch, who proved that it is possible to impose some additional conditions on \( d \), related to its \( p \)-adic valuations (see Theorem 2 in [4]). The equation \( D(\sqrt{d}) = k \) for small \( k \geq 1 \) was studied in [1] by Balková and Hrušková.

A similar but different line of research has been recently taken by Rada and Starosta in [6]. They studied the behaviour of the length of the continued fraction expansion of a certain transformation of \( \sqrt{d} \). A Möbius transformation is a transformation of the form

\[
h(x) = \frac{ax + b}{cx + d},
\]

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where \( a, b, c, d \) are integers. Rada and Starosta were able to provide some lower and upper bounds on \( D(b(\sqrt{d})) \), in terms of \( D(\sqrt{d}) \). These estimates are expressed with help of the determinant \( |ad - bc| \).

The main goal of the paper is to study a natural question, that falls somewhere in between the two previously mentioned lines of research. A transformation \( x \to nx \), where \( n \geq 1 \) is an integer, is the simplest example of a Möbius transformation. It is therefore natural to consider a variant of the Question 1.1 in the class of the number of the form \( n\sqrt{d} \). We shall state it in a rather general form.

**Question 1.2.** For given integers \( k \geq 1 \) and \( d \geq 1 \), are there infinitely many integers \( n \geq 1 \) such that \( D(n\sqrt{d}) = k \)?

In the full generality, the answer to this question turns out easily to be negative, as opposed to Question 1.1. For example, if \( D(\sqrt{d}) \) is even, then \( D(n\sqrt{d}) \) is also even for every \( n \geq 1 \) (see Lemma 2.3). Therefore, in order to make this question more specific and interesting, we define

\[
A_d = \{ k \in \mathbb{N} : \text{there exist infinitely many } n \text{ for which } D(n\sqrt{d}) = k \}.
\]

In other words, \( A_d \) is the set of limit points of the sequence \( (D(n\sqrt{d}))_{n=1}^\infty \). Our main result goes as follows.

**Theorem 1.3.** Let \( d \) be a positive integer, which is not a perfect square. Then, the set \( A_d \) is infinite. In other words, the sequence \( (D(n\sqrt{d}))_{n=1}^\infty \) has infinitely many different limit points.

It does not seem possible to deduce Theorem 1.3 from any of the previously mentioned results. The proof is based on establishing a lower bound on \( D(n\sqrt{d}) \) for some specific sequence \( (n_i)_{i=1}^\infty \). In order to do this, we explore some connections between the continued fraction expansion, the Euclidean algorithm and the Pell equation.

The paper is organized as follows. In Section 2 we prove some preparatory results. Theorem 1.3 is proved in Section 3 Section 4 concludes the paper with some further open questions.

## 2. Auxiliary results

For each continued fraction \( [a_0, a_1, a_2, a_3, \ldots] \) we define sequences \( (p_i)_{i=0}^\infty \), \( (q_i)_{i=0}^\infty \) as follows:

\[
\begin{align*}
p_0 &= a_0 \\
p_1 &= a_0a_1 + 1 \\
p_k &= a_kp_{k-1} + p_{k-2} \\
q_0 &= 1 \\
q_1 &= a_1 \\
q_k &= a_kq_{k-1} + q_{k-2}
\end{align*}
\]

Let us recall (see Chapter 1.5 and 1.6 in [5]) that these sequences satisfy the equations

\[
\begin{align*}
p_kq_{k-1} - p_{k-1}q_k &= (-1)^{k-1} \\
\frac{p_k}{q_k} &= [a_0, a_1, \ldots, a_k]
\end{align*}
\]

for all \( k \geq 1 \). The reader is referred to [5] for some basics on the continued fraction expansion.

We start our investigation with some preparatory lemmas related to the Pell equation. Let \( (x_n, y_n)_{n=1}^\infty \) be the sequence of the solutions of the Pell equation \( x^2 - dy^2 = 1 \). It is easy to see that for every integer \( a \) the sequence \( x_n \mod a \) is periodic \( \mod a \). By \( m_d(a) \) we denote the period length of this sequence modulo \( a \). We recall a folklore result.

**Lemma 2.1.** For any prime \( p > 2 \) we have \( m_d(p) | p^2 - 1 \).

In the next two lemmas we relate the Pell equation to \( D(n\sqrt{d}) \).

**Lemma 2.2.** For every positive integer \( d \), there exists a positive integer \( n \) such that \( D(n\sqrt{d}) = 2 \).
Proof. We will show that, if a pair \((a, b)\) fulfills a Pell equation
\[ a^2 - db^2 = 1, \]
then
\[ b\sqrt{d} = [a - 1, 1, 2(a - 1)]. \]

Let \(\beta = [1, 2(a - 1)]\). We have
\[ \beta = [1, 2(a - 1), \beta] = \frac{p_1\beta + p_0}{q_1\beta + q_0} = \frac{(2a - 1)\beta + 1}{2(a - 1)\beta + 1}. \]

Solving this for \(\beta\) we obtain:
\[ \beta = \frac{(a - 1) + \sqrt{a^2 - 1}}{2(a - 1)}, \]
as we can exclude the second negative root. Therefore
\[ a - 1 + \frac{1}{\beta} = a - 1 + \frac{2(a - 1)}{(a - 1) + \sqrt{a^2 - 1}} = \frac{a^2 - 1 + (a - 1)\sqrt{a^2 - 1}}{(a - 1) + \sqrt{a^2 - 1}} = \sqrt{a^2 - 1} = b\sqrt{d}, \]
which concludes the proof. \(\square\)

Lemma 2.3. Suppose that \(D(\sqrt{d})\) is even. Then \(D(n\sqrt{d})\) is even for every \(n \geq 1\).

Proof. It is known (see for example [7, Theorem 7.26]), that a negative Pell equation
\[ x^2 - dy^2 = -1, \]
has a solution (and thus infinitely many) if and only if, the number \(D(\sqrt{d})\) is odd. Therefore, if \(D(n\sqrt{d})\) is odd, then the equation
\[ x^2 - (dn^2)y^2 = -1 \]
has an integer solution \((a, b)\). But then, the equation
\[ u^2 - dz^2 = -1 \]
has an integer solution \((a, nb)\). This implies that \(D(\sqrt{d})\) is odd and gives us a contradiction. \(\square\)

Now we turn our attention to the Euclidean algorithm. For given positive integers \(x, y\), by \(L(x, y)\) we denote the length of the Euclidean algorithm for \(x\) and \(y\). Here we assume that for every positive integer \(n \geq 1\) we have \(L(n, 1) = 1\). So, for example, \(L(25, 7) = 4\), because the Euclidean algorithm in this case goes as follows: \((25, 7) \rightarrow (7, 4) \rightarrow (4, 3) \rightarrow (3, 1)\). We suppose that for the Fibonacci sequence \((F_n)_{n=0}^\infty\) we have \(F_0 = 0, F_1 = F_2 = 1\) and \(\varphi\) denotes the golden ratio. A following lemma gives a lower bound on \(L(a, b)\) for \(a\) and \(b\) with ratio close to \(\varphi\).

Lemma 2.4. Let \(a > b\) be positive integers such that \(|\frac{a}{b} - \varphi| < |\frac{F_{k+1}}{F_k} - \varphi|\). Then \(L(a, b) \geq k\).

Proof. Let \(x_0 = a, x_1 = b\) and for \(n \geq 2\) let \(x_n\) be the smaller of the numbers obtained in the \((n-1)\)-th step of the Euclidean algorithm for \(a\) and \(b\), i.e. \(x_n\) is the remainder of \(x_{n-1}\) modulo \(x_{n-1}\). We shall prove inductively that \(x_n = (-1)^naF_{n-1} + (-1)^{n+1}bF_n\) for \(1 \leq n \leq k\). The statement is clearly true for \(n = 1\), so let us assume that it is true for some \(1 \leq n \leq k - 1\). We will show that \(2x_n > x_{n-1}\). Indeed, by the definition of \(x_n\), it is equivalent to
\[ 2(-1)^n F_{n-1}a + 2(-1)^{n+1} F_n b > (-1)^{n+1} F_{n-2}a + (-1)^n F_{n-1} b \]
or
\[ (-1)^n(2F_{n-1} + F_{n-2})a > (-1)^n(2F_n + F_{n-1})b. \]

Simplifying we get
\[ (-1)^n F_{n+1}a > (-1)^n F_{n+2}b, \]
which is finally equivalent to
\[ (-1)^n \frac{a}{b} > (-1)^n \frac{F_{n+2}}{F_{n+1}} \]
and this clearly follows from our assumption and the monotonic convergence of \( \frac{F_{n+2}}{F_{n+1}} \) to \( \varphi \). In particular, \( x_k > 2x_{k-1} \) and therefore \( x_k \neq 0 \). Hence \( L(a, b) \geq k \) and conclusion follows. \( \square \)

In the last lemma of this section we establish some relations concerning the middle of part of the palindrome with its right end.

**Lemma 2.5.** Suppose that \( k = 2l = D(\sqrt{d}) \) is even. Then, the following properties are true:
\[
q_{k-1} = q_{l-1}(q_l + q_{l-2}) = q_{l-1}(aq_{l-1} + 2q_{l-2}), \quad (3)
\]
\[
p_{k-1} = a_0q_{k-1} + q_{k-2}, \quad (4)
\]
\[
q_{l-1}|q_{k-2} + (-1)^{l-1}, \quad (5)
\]
\[
a_{lq_{l-1} + 2q_{l-2}|q_{k-2} + (-1)^l}. \quad (6)
\]

**Proof.** We will use an equivalent matrix definition of \( (p_i)_{i=0}^{\infty}, (q_i)_{i=0}^{\infty} \), which follows directly from construction of these sequences (see [8] for details). For
\[
D = [a_0, a_1, \ldots, a_{l-1}, a_l, a_{l-1}, \ldots, a_1, 2a_0]
\]
we have
\[
\begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{l-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_l & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{l-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} =
\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_l & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{l-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}.
\]
Thus
\[
\begin{pmatrix} p_l & p_{l-1} \\ q_l & q_{l-1} \end{pmatrix} = \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Finally, we obtain:
\[
\begin{pmatrix} p_{l-1}(p_l + p_{l-2}) & p_{l-1}(p_{l-1} + p_{l-1}q_{l-2}) \\ q_{l-1}(q_l + q_{l-2}) + (-1)^{l-1} + q_l + q_{l-2} + (-1)^{l-2} & (-1)^l = q_{l-1}(p_l + p_{l-2} + a_0q_{l-1} + q_{k-2}).
\]

By comparing matrix entries we get:
- Property [3]: by using the equality \( q_i = a_iq_{i-1} + q_{i-2} \).
- Property [4]: by comparing matrix entries we have \( p_{k-1} = p_l q_{l-1} + p_{l-1}q_{l-2} \). Then, using [4] we can write it as
\[
q_{l-1}(p_l + p_{l-2} + (-1)^{l-1} + q_l + q_{l-2} + (-1)^{l-2} = q_{l-1}q_l + p_{l-1}q_{l-2} + a_0q_{l-1} + q_{k-2}.
\]
- Property [5]: from previous properties we have
\[
p_{k-1} = p_l q_{l-1} + p_{l-1}q_{l-2} = a_0q_{l-1} + q_{k-2}.
\]
From [3] we get:
\[
a_0q_{k-1} + q_{k-2} = a_0q_{l-1}(a_lq_{l-1} + 2q_{l-2}) + q_{k-2}.
\]
Comparing this two equalities we obtain:
\[
a_0q_{l-1}(a_lq_{l-1} + 2q_{l-2}) + q_{k-2} = p_{l-1}q_{l-1} + p_{l-1}q_{l-2} = q_{l-1}q_l + q_{l-1}q_{l-2} + (-1)^{l-2}.
\]
Therefore
\[
q_{k-2} + (-1)^{l-1} = q_{l-1}(p_l + p_{l-2} - a_0(a_lq_{l-1} + 2q_{l-2})).
\]
• Property (5) similarly to property (3):
\[
 a_0q_l-1(a_0q_l-1 + 2q_l-2) + q_k-2 = pq_l-1 + p_{l-1}q_l-2 = q_l p_{l-1} + (-1)^{l-1} + p_{l-1}q_l-2
\]
\[
 q_k-2 + (-1)^{l} = (aq_l-1 + 2q_l-2)(p_l-1 - a_0q_l-1),
\]
where we used the relation \( q_l + q_{l-2} = a_0q_l-1 + 2q_l-2. \)

It is worth mentioning that property (4) was used also by Friesen [3] in the proof of the affirmative answer to Question 1.11

3. Proof of Theorem 1.3

Before proving our main result, we need one more lemma, which connects the number \( D(n\sqrt{d}) \), a corresponding Pell equation and the length of the Euclidean algorithm (as promised in the Introduction). In fact, Theorem 1.3 is an easy consequence of this lemma.

Lemma 3.1. Let \( p,q \) be odd numbers. If \( 2|D(\sqrt{d}) \) and \((x_a,y_a)\) is a solution of a Pell equation \( x^2 − dy^2 = 1 \), satisfying the conditions:
- \( 2pq|y_a \),
- \( p|x_a \pm 1 \),
- \( q|x_a \mp 1 \),
then there exist infinitely many positive integers \( n \) such that
\[
 D(n\sqrt{d}) \in \{2L(p,q), 2L(p,q) + 1\}. \]

Proof. If there exists at least one solution satisfying above conditions, then there exist infinitely many of them, as the sequence of solutions of Pell’s equation is periodic modulo \( 2pq \). Let us take one of these solutions \((x_m, y_m)\), such that \( x_m, y_m > 4p^2q^2 \). We can consider another Pell equation of the form
\[
 w^2 − d\left(\frac{y_m}{2pq}\right)^2 z^2 = 1
\]
Clearly \((w,z) = (x_m,2pq)\) is a solution of this equation. In fact, it is the fundamental solution. Indeed, let \((w_1,z_1)\) be the fundamental solution and \((w_2,z_2)\) be the next one. Assume that \( z_1 < 2pq \). Then
\[
 2pq ≥ z_2 ≥ 2w_1z_1 ≥ 2w_1 > \frac{y_m}{2pq},
\]
which contradicts our choice of \( m \). Therefore \((w_1,z_1) = (x_m,2pq)\). Let \( n = \frac{y_m}{2pq} \) and \( k = D(n\sqrt{d}) \). By Lemma 2.3 we know that \( k \) is even, so let us write \( k = 2l \). From [5] (see Chapter 4.8) we know that the fundamental solution \((w_1,z_1)\) is a pair \((p_{k-1},q_{k-1})\). Hence
\[
 (x_m,2pq) = (p_{k-1},q_{k-1}) \tag{7}
\]
First part of Lemma 2.3 yields the equality
\[
 2pq = q_{l-1}(aq_{l-1} + 2q_{l-2}) \tag{8}
\]
Also from Lemma 2.3 we easily get the following congruences
\[
 p_{k-1} \equiv (-1)^{l} \pmod{q_{l-1}},
\]
\[
 p_{k-1} \equiv (-1)^{l-1} \pmod{aq_{l-1} + 2q_{l-2}}.
\]
Using (7) we can write this in the form:
\[
 q_{l-1}|x_m + (-1)^{l},
\]
\[
 aq_{l-1} + 2q_{l-2}|x_m + (-1)^{l-1}.
\]
Since \( p | x_m \pm 1 \), we have that \( \gcd(p, a_i q_{i-1} + 2q_{i-2}) = 1 \) or \( \gcd(p, q_{i-1}) = 1 \) (as \( p \) is odd). Similar property holds also for \( q \). As \( p \) and \( q \) divide \( x_m + (-1)^\alpha \) with different parity of \( \alpha \), the two inequalities: \( \gcd(p, a_i q_{i-1} + 2q_{i-2}) > 1 \) and \( \gcd(q, a_i q_{i-1} + 2q_{i-2}) > 1 \) can not be true at the same time. Without loss of generality, we assume that
\[
\gcd(p, a_i q_{i-1} + 2q_{i-2}) = 1.
\]
Then, from [5] it follows that \( p | q_{i-1} \). Furthermore, if \( 2 | q_{i-1} \), then \( 2 | a_i q_{i-1} + 2q_{i-2} \), so again by [5] we have \( 4 | 2pq \), which is false. In consequence, \( 2 | a_i q_{i-1} + 2q_{i-2} \), which implies that \( 2 | a_i \) and
\[
p = q_{i-1},
\]
\[
q = \frac{a_i}{2} q_{i-1} + q_{i-2}.
\]
Now, we consider the number \( L(p, q) \). We have that
\[
L(p, q) = L\left(q_{i-1}, \frac{a_i}{2} q_{i-1} + q_{i-2}\right) = L(q_{i-1}, q_{i-2}) + 1 = L(a_{i-1} q_{i-2} + q_{i-3}, q_{i-2}) + 1 = L(q_{i-2}, q_{i-3}) + 2
\]
\[
= L(a_{i-2} q_{i-3} + q_{i-4}, q_{i-3}) + 2 = L(q_{i-3}, q_{i-4}) + 3 = \ldots = L(q_2, q_1) + l - 2.
\]
It follows that for \( q_1 = 1 \) we have \( L(p, q) = l - 1 \) and for \( q_1 > 1 \) we have \( L(p, q) = l \). So we proved that for \( n = \frac{2m}{2pq} \) we have
\[
D(n\sqrt{d}) \in \{2L(p, q), 2(L(p, q) + 1)\}.
\]
Since we have infinitely many options for choosing sufficiently large \( m \), this finishes the proof. \[ \square \]

Now we are ready to prove our main result.

**Proof of Theorem 1.3** By Lemma 2.4 it is enough to consider the case \( D(\sqrt{d}) = 2 \) – every limit point of the sequence \( D(nc\sqrt{d}) \) for a fixed \( c \), is also a limit point of the sequence \( D(n\sqrt{d}) \).

Let \( r \) be any odd positive integer. Suppose that there exists an odd prime \( p \) not dividing \( d \), but dividing \( x_{8r} + 1 \) where \( (x_{8r}, y_{8r}) \) is the \( 8r \)-th solution of the Pell equation
\[
x^2 - dy^2 = 1.
\]
Let \( t \) be the positive integer such that \( p \in (\phi^2, \phi^{2(t+1)}) \).

The distance between two consecutive numbers of the form \( \frac{b}{p} \) is equal to \( \frac{1}{p} \). Thus, we can choose a positive integer \( b \) such that
\[
\left| \frac{b}{p} - \phi \right| < \frac{1}{p} < \phi^{-2t}.
\]
We note also that
\[
\left| \frac{F_{t+1}}{F_t} - \phi \right| = \frac{\phi - \phi^{-1}}{\phi^{2t} + (-1)^t} > \phi^{-2t}.
\]
Therefore, by Lemma 2.4 we have
\[
L(b, p) \geq t - 1.
\]

Now, let \( q \) be any prime such that:

a) \( 2r \) and \( \frac{q^2 - 1}{8} \) are relatively prime
b) \( q \equiv b \) (mod \( p \)).

Since \( 8 | q^2 - 1 \) the system of congruences
\[
m \equiv 8r \quad (\text{mod } 16r), \quad m \equiv 0 \quad (\text{mod } q^2 - 1)
\]
has a solution \( m \). Let us recall that \( p | x_{8r} + 1 \) and \( p \) does not divide \( d \). Hence, \( p \) divides \( y_{8r} \). Thus, looking modulo \( p \) the pair \( (x_{8r}, y_{8r}) \) is congruent to the pair \( (-1, 0) \) and therefore the pair \( (x_{16r}, y_{16r}) \) is congruent to \( (1, 0) \). This shows that \( m_d(p) | 16r \). Hence, from the first congruence it follows that \( p | x_m + 1 \). From the second congruence and Lemma 2.3 we get that \( m_d(q) | m \) and thus
q|x_m − 1. This shows that p and q satisfy conditions of Lemma 3.1. It follows that for infinitely many positive integers n we have:

$$D(n\sqrt{d}) \in \{2L(p,q), 2(L(p,q) + 1)\}.$$ 

From \(q \equiv b \pmod{p}\) we obtain

$$L(p,q) = L(b,p) \geq t - 1 = \left\lfloor \frac{\log \phi p}{2} \right\rfloor - 1.$$ 

On the other hand, we have

$$L(p,q) \leq \log \phi p + 1.$$ 

This shows that for infinitely many n we have

$$D(n\sqrt{d}) \in (\log \phi p - 3, 2 \log \phi p + 4).$$ 

In particular, we get a limit point in each interval \((\log \phi p - 3, 2 \log \phi p + 4)\).

To finish the proof, we are left with proving that we can choose infinitely many primes p for a variable r. As d has only finitely many divisors, it is enough to prove that there are infinitely many primes p such that \(p|x_r + 1\) for some \(r \geq 1\). Let us consider r prime. In this case, we have \(m_d(p)|16r\). If r does not divide \(m_d(p)\), then \(m_d(p)|16\) which gives us \(p|y_{16}\) and therefore it is satisfied by only finitely many primes p. If \(r|m_d(p)\), then by Lemma 2.1 we have also that \(r|p^2 - 1\). This clearly yields the desired infinite number of possibilities for p and the proof is finished. \(\square\)

4. Concluding remarks

It is natural to ask, if something more can be said in general about the set \(A_d\) of the limit points of the sequence \((D(n\sqrt{d}))_{n=1}^{\infty}\), besides the fact that it is of infinite cardinality. More specifically, we pose the following question.

**Question 4.1.** Is it true, that for every non-square integer \(d \geq 1\) and every \(k \geq 1\) at least one of the numbers \(k, k + 1\) belongs to \(A_d\)?

Such a conjecture may seem to be quite strong at first glance, but there is a motivation behind it. We recall that by \(L(x, y)\) we denoted the length of the Euclidean algorithm applied for \(x\) and \(y\). Let us state perhaps somewhat more natural question, that is related directly to \(L(x, y)\).

**Question 4.2.** Is it true, that for every integer \(k \geq 1\), there exists an integer \(N \geq 1\), such that for every \(n > N\) and \(1 \leq i \leq k\) there exists \(1 \leq m \leq n\) such that \(L(m, n) = i\)?

It turns out, that a slight modification of the argument used in the proof of Theorem 1.3 shows that an affirmative answer to Question 4.2 would directly imply an affirmative answer to Question 4.1. We do not know if any of these questions has a positive answer, but we believe that some further properties of \(A_d\) could be established in the full generality.

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