FERMIONIC EFFECTIVE ACTION AND THE PHASE STRUCTURE
OF NON COMPACT QUANTUM ELECTRODYNAMICS IN 2+1 DIMENSIONS

V. Azcoiti, X.Q. Luo and C.E. Piedrafita
Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza,
50009 Zaragoza (Spain)

G. Di Carlo and A.F. Grillo
Istituto Nazionale di Fisica Nucleare, Laboratori Nazionali di Frascati,
P.O.B. 13 - Frascati (Italy).

A. Galante
L.N.F - I.N.F.N. and Dipartimento di Fisica, Università dell’ Aquila, L’ Aquila 67100, (Italy)

ABSTRACT
We study the phase diagram of non compact $QED_3$ using the microcanonical fermionic average method described elsewhere. We present evidence for a continuous phase transition line in the $\beta, N$ plane, extending down to arbitrarily small flavour number $N$. 
In this Letter we continue a systematic study of Lattice Abelian models with dynamical fermions [1,2,3] that we are making through the use of the microcanonical average method [1] for dealing with fermionic lattice simulations.

Earlier work was related to four dimensional compact [1], and non compact [2,3] models. In particular the non compact abelian model is interesting, being a candidate for a strongly interacting continuum theory: in this case we have presented detailed analysis of the phase structure, both in $(\beta, m)$, including $m = 0$ [2] and $(\beta, N)$ [3] planes.

Lower dimensional models have a particular interest on theoretical grounds. In 1 + 1 dimensions, quantum electrodynamics at zero mass (i.e. the Schwinger model) is analytically solvable, confining and asymptotically free; in this case lattice results can be directly related to exact (continuum) ones.

The 2 + 1 dimensional Abelian model shares some features with Schwinger’s: it confines static charges and is asymptotically free, so its understanding should be relevant to more physical theories in four dimensions. Although not solvable, the model is super renormalizable. It is also potentially interesting in relation to models of high $T_c$ superconductivity [4].

In this paper we describe our study of the phase structure of non compact $QED_3$ in the $(\beta, N)$ plane. We find a critical line that consists of two segments, a first order line in the large-$N$ and a second order one in the small-$N$ region. Our analysis is based on the studies of two quantities, the effective fermionic action and the chiral condensate. An intriguing feature that emerges from these studies is that the phase diagrams obtained using the two quantities differ in the low-$N$ region. The analysis of the fermionic action suggests that the second order critical line terminates on the $N = 0$ axis at finite coupling, $\beta_c = 0.49$, implying the existence of two phases even in the quenched theory. The behavior of the chiral condensate, however, shows no restoration of chiral symmetry for any finite $\beta$, in the investigated range, implying that the nonanalyticities occurring at $\beta_c = 0.49$ are not related to the chiral symmetry breaking, in agreement with the conclusion of previous studies of $QED_3$ [5,6,7] as well as with theoretical expectations.

We mention that extensive amount of work on the subject has been done in the continuum formulation of the model. This has been reported in refs.[8-11]. Also, simulations of the compact variant of $QED_3$, with the use of the method described below, have been presented in [12].

The method we use, is based on the introduction of an effective fermionic action $S_{eff}^F(E, N, m)$ [1-3]. An advantage of this method is that it allows simulations to be performed exactly in the chiral limit. The effective fermionic action, which is a function of the pure gauge energy $E$, fermion mass $m$ and number of flavours $N$, is defined through,
\[ e^{-S^{\text{eff}}_{\text{ferm}}(E,m,N)} = \]

\[
\frac{\int [dA_\mu(x)](\det \Delta(m,A_\mu(x)))^{N/2}\delta\left(\frac{1}{2} \sum_{x,\mu<\nu} F^2_{\mu\nu}(x) - 3VE\right)}{\int [dA_\mu(x)]\delta\left(\frac{1}{2} \sum_{x,\mu<\nu} F^2_{\mu\nu}(x) - 3VE\right)}
\] (1)

where \( \Delta(m,A_\mu(x)) \) is the fermionic matrix (we use 4-components staggered fermions) and \( E \) is the normalized pure gauge energy. The denominator in (1) is the density of states at fixed energy

\[ N(E) = C_G E^\frac{\beta}{2} \] (2)

with \( C_G \) being an unimportant (divergent) constant and \( V \) the lattice volume.

After the definition of the fermionic effective action, the partition function of this model can be written as a one-dimensional integral

\[ Z = \int dEN(E)e^{-3\beta VE-S^{\text{eff}}_{\text{ferm}}(E,N,m)} \] (3)

from which we can define an effective full action per unit volume as

\[ \bar{S}^{\text{eff}}_{\text{ferm}}(E,\beta,N,m) = -\ln E + 3\beta E + \bar{S}^{\text{F}}_{\text{eff}}(E,N,m) \] (4)

\( \bar{S}^{\text{F}}_{\text{eff}}(E,N,m) \) in (4) is the fermionic effective action (1) normalized to the lattice volume.

Once the effective action, which entirely characterizes the fermionic contribution, is defined, the qualitative features of the phase structure of the system can be studied analytically. The integrand in (3) is a strongly peaked function of \( E \). Since the effective full action diverges linearly with the lattice volume \( V \), one can evaluate the free energy and its derivatives by saddle point. The mean plaquette energy \( <E_p>=E_0(m,\beta,N) \) will be given by the solution of the saddle point equation [2]

\[
\frac{1}{3E} - \beta - \frac{1}{3} \frac{\partial}{\partial E} \bar{S}^{\text{F}}_{\text{eff}}(E,N,m) = 0
\] (5)

satisfying the minimum condition

\[
\frac{1}{E^2} + \frac{\partial^2}{\partial E^2} \bar{S}^{\text{F}}_{\text{eff}}(E,N,m) > 0
\] (6)

By differentiating equation (5) respect to \( \beta \) we get the specific heat

\[
C_\beta = \frac{\partial}{\partial \beta} <E_p> = -\left\{ \frac{1}{3E^2_0(m,\beta,N)} + \frac{1}{3} \frac{\partial^2}{\partial E^2} \bar{S}^{\text{F}}_{\text{eff}}(E,N,m) \right\}_{E_0(m,\beta,N)}^{-1}
\] (7)

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The effective action must be continuous even in the thermodynamical limit; however its derivatives may be discontinuous in this limit. A discontinuity in the first energy derivative implies an analogous behaviour for the average energy, hence a first order phase transition. If the first energy derivative is continuous but some higher order derivative is discontinuous, then the average energy will be continuous, the specific heat or some of its derivatives being discontinuous, i.e. the system will undergo a continuous phase transition [2].

This is not the only way in which a phase transition is generated, however, since this can be obtained also through a cancellation of the two terms of the denominator in (7). In order that this happen, the second energy derivative of the effective action should be negative. Then, for a range of energies and $N$ large enough, the denominator will be negative and there will be no solution of the saddle point equation, i.e. there will be a range of energies not accessible to the system, indicating again a first order transition.

Finally, if the effective action is non analytic in the flavour number $N$ or the fermion mass $m$, this will cause again phase transitions in the $(\beta, N)$, $(\beta, m)$ planes.

To summarize, the phase structure of the theory can be entirely described in terms of the behaviour of the effective action as a function of the energy and the bare parameters. If it is non analytic in the energy, then a phase transition will appear. On the other hand, if the effective action is analytic in $E$, $N$ and $m$, then the only other mechanism for producing a phase transition, if any, will generate a first order transition line ending in a second order point. Obviously these mechanisms can coexist, as in $QED_4$ [3].

We stress that this characterization of the phase structure of the theory (more transparent in the non compact model since the density of states is known analytically), is rather independent on the details of the numerical evaluation of the effective action. Also, it does not depend on the evaluation and extrapolation of the chiral condensate, which, as for past experience, is very delicate especially in small lattices and in the three dimensional theories [6]. In fact, the above characterization of the phase structure is entirely independent from those existing in the literature for these models. The phase structure of the four-dimensional theory, obtained in this way [2] is in very good agreement with the one obtained in [13].

We now present our results for the effective action. For the fermionic effective action we can write down an expansion in cumulants [2],

$$-S_{eff}^F(E, N, m) = \frac{N}{2} < \ln \det \Delta(m, A_\mu(x)) >_E$$

$$+ \frac{N^2}{8} \left\{ < (\ln \det \Delta)^2 >_E - < \ln \det \Delta >_E^2 \right\} + ... \quad (8)$$
where \( <O>_E \) means the mean value of the operator \( O(A_\mu(x)) \) computed with the probability distribution \( dA_\mu(x) \delta(\frac{1}{2}\sum_{x,\mu<\nu} F_{\mu\nu}^2(x) - 3VE)/N(E) \). Expression (8) is a \( N \) expansion of the fermionic effective action.

Following the general method described in [1,2], we have done simulations in \( 6^3, 10^3, 14^3, 18^3 \) and \( 20^3 \) lattices. In Fig. 1 we present the Fermionic Effective Action for two massless flavours on an \( 14^3 \) lattice, computed using the first two contributions to the cumulant expansion. The third one has been found compatible with zero within errors. The result shown in Fig. 1 is qualitatively completely analogous to the one found in the four dimensional theory, so we can repeat the same analysis as in [2,3]. The important point here is that our data strongly indicate a continuous phase transition; in fact the Fermionic Effective Action is linear for small energies and clearly not linear for larger energies, thus suggesting a discontinuity of the second or higher order derivatives in the thermodynamical limit at some critical energy \( E_c \). The first derivative is continuous, as follows from the analysis of the numerical data.

This behaviour is dramatically evident if one plots the Fermionic Effective Action minus the fit to its linear part, Fig. 2. Once the critical energy \( E_c \) is determined by fitting the results reported in Fig.2 with a power law function \( C(E - E_c)^\rho \), the critical coupling is computed from saddle point equations. Notice that at these lattice sizes, the results from the saddle point approximation are undistinguishable from the numerical ones.

In Table I we present the critical values of \( \beta \) at various values of \( N \). The transition line continues down to arbitrarily small values of \( N \), including zero flavour (the quenched theory). The critical values of \( \beta \) and \( E \) in the quenched limit, \( \beta_c = 0.49(1) \), \( E_c = 0.68(1) \), show no variations with the lattice size, in disagreement with expectations for a phase transition with divergent correlation length, and might be interpreted as indicating that we are indeed observing a transition with finite correlation length. However the similarities of these results with those obtained in the four-dimensional non compact model [2,3], cast doubts on the previous interpretation. In fact our determination of the critical couplings \( \beta_c \) in the four-dimensional model with two and four dynamical flavours, were in very good agreement with those reported in [13], the last obtained in much larger lattices. But there is no doubt that the correlation length diverges at the critical point of noncompact QED in four dimensions.

Concerning the phase structure at large number of flavours, here the discussion in [3] also applies: at the critical values \( N = 6.10 \), \( E_c = 1.58 \) the denominator in (6) is zero and for larger values of \( N \) it becomes negative in some energy interval. In such an energy interval, the saddle point equation has no solution, producing a first order transition line. As described in [3] the continuous transition line merges into the first order one, since the energy at which the effective action becomes non analytic falls into the
energy interval not accessible to the system, which widens with the energy. Notice that the critical energy obtained from our simulations seems to be independent on the flavour number at small $N$, as in the four dimensional case [2,3]. In Fig. 3 we present the complete $(\beta, N)$ phase diagram of the model, at $m = 0$. This phase diagram has been obtained using only the first term in the expansion (8).

The only results on the phase structure for the three-dimensional non compact case are those in [5,6,7], suggesting a continuous, chiral symmetry restoring transition ending at $N \approx 3 - 4, \beta = \infty$. On the other hand, the results reported in [6] show unambiguously that chiral symmetry is spontaneously broken in the quenched model for $\beta$ values larger than our $\beta_c = 0.49$, thus suggesting that the phase transition we observe does not restore chiral symmetry, contrary to what happens in the four dimensional non compact model. This is not surprising since quenched $QED_3$ confines static charges for any finite $\beta$ and there are general arguments suggesting that confining forces make the chiral symmetric vacuum unstable [14]. In fact our numerical results for the chiral condensate in the quenched model support well this scenario. In Fig. 4 we plot the inverse logarithm of the chiral condensate against the inverse logarithm of the fermion mass for several values of $\beta$ in the $14^3$ and $18^3$ lattices. This kind of plot was proposed in [13] as a very efficient way to get the critical $\beta$ and the value of the $\delta$ exponent in a continuous chiral restoring transition. The results of Fig. 4 show unambiguously that the "critical $\beta$" obtained in this way moves significantly towards larger values when the lattice size changes from $14^3$ to $18^3$, indicating that the observation of a "critical $\beta$" (which in this plot corresponds to a straight line which passes through the origin) is a pure finite size effect, the critical coupling being pushed towards $\infty$ in the thermodynamical limit.

We try here to discuss on the reliability of our results concerning the continuous phase transition line at small $N$. A first criticism might be that the method we use forces for some reason a phase transition through a (non physical) non analiticity of the effective action. We argue that this is extremely unlikely: on one hand, our determination of the phase diagram of $QED_4$ through the Effective Action is in very good agreement with those obtained using traditional methods (using the behaviour of the chiral condensate); this agreement extends to all the physical observables measured [2]. On the other hand we have obtained within this approach preliminary results for massless $QED_2$ (the Schwinger model), showing a good analytical behaviour of $S_{eff}$, i.e. the absence of phase transitions at finite $\beta$ in the one flavour model, with a scaling behaviour in agreement with simple dimensional counting.

In Fig. 5 we plot the mean value of the normalized singular part of the fermionic action in the quenched limit against $\beta$ in a $18^3$ lattice. This singular part is defined as
where $a_0 = 0.145$, $a_1 = -0.256$ are the zero energy intercept and the slope in the small energy region of the first cumulant contribution to the fermionic effective action (8). The existence of two phases is evident in this figure. From a numerical point of view, we must note that our critical coupling at $N = 0$, $\beta_c = 0.49$, corresponds to a region where the results for the chiral condensate reported in [5,6] show a very rapid change. Unfortunately this $\beta$ region has not been explored intensively in [5,6], so no definite conclusions can be extracted from their results.

The phase structure of the model in the $(\beta, N)$ plane (Fig. 3) shows the existence of two completely separated phases. However, chiral symmetry should be spontaneously broken in both phases since it is broken for small $N$ including the quenched limit. We have also explored if the continuous phase transition line is related to the percolation of topological structures associated to the lattice regularization. However, the monopole [15] and string densities at the critical values of $\beta$ are too small to produce percolation of these objects. Therefore we have at the moment no compelling evidence for this interpretation. We want to notice here that this continuous transition was not observed in the simulations of the compact version of this model [12] thus suggesting again important qualitative differences between the compact and non-compact regularizations, like in the four dimensional case.

Conversely the first order line is clearly produced by pure fermionic effects, like in the four dimensional non compact model. We would like to remark that our quantitative results for large $N$ could change when including all the cumulants in the expansion of the effective action. What can be analytically proved is that the effective action is linear with $N$ in the large $N$ limit [3] so higher order terms in the cumulant expansion conspire between them in order to give this linear behaviour. However the main qualitative features like the fact that the first order line does not intercept the $\beta = 0$ axis [3] remain unchanged.

Concerning the possibility to have a chiral restoring phase transition, we have not seen evidence for such a transition. However we can not exclude a non analyticity of the effective fermionic action as a function of $N$ which could originate this transition. Indeed our approach is based in an expansion of the fermionic effective action in powers of $N$ and the implicit assumption that the convergence radius of this expansion is $\infty$.

Several interesting issues emerging out of this study and which are left open at this moment are: the physical origin of the continuous transition, qualitative differences between the strong and weak coupling phases, correlation length and critical exponents associated with the continuous transition, possibility to define a non superrenormalizable, but
renormalizable field theory, etc.. A more detailed study of these issues is underway.

The numerical simulations quoted above have been done using the Transputer Networks of the Theoretical Group of the Frascati National Laboratories, of the University of L’ Aquila and the Reconfigurable Transputer Network (RTN), a 64 Transputers array of the University of Zaragoza.

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REFERENCES

1. V. Azcoiti, G. Di Carlo and A.F. Grillo, Phys. Rev. Lett. 65 (1990) 2239; V. Azcoiti, A. Cruz, G. Di Carlo, A.F. Grillo and A. Vladikas, Phys. Rev. D43 (1991) 3487; V. Azcoiti et al. "The microcanonical fermionic average method for Monte Carlo simulations of lattice gauge theories with dynamical fermions", INFN preprint (1992) LNF-93/004(P) (1993), to appear in Phys. Rev. D.

2. V. Azcoiti, G. Di Carlo and A.F. Grillo, Mod. Phys. Lett. A7 (1992) 3561; V. Azcoiti, G. Di Carlo and A.F. Grillo, "A New Approach to Non Compact Lattice QED with Light Fermions ", DFTUZ 91.34 (1992), to appear in Int. Jour. Mod. Phys. A.

3. V. Azcoiti, G. Di Carlo and A.F. Grillo, Phys. Lett. 305B (1993) 275.

4. E. Dagotto, E. Fradkin and A. Moreo, Phys. Rev. B38 (1988) 2926.

5. E. Dagotto, J.B. Kogut and A. Kocic, Phys. Rev. Lett. 62 (1989) 1083; E. Dagotto, A. Kocic and J.B. Kogut Nucl. Phys. B334 (1990) 279.

6. S. Hands and J.B. Kogut, Nucl. Phys. B335 (1990) 455.

7. J.B. Kogut and J.-F. Lagae Nucl. Phys. B (Proc. Suppl.) 30 (1993) 737.

8. R. Pisarski, Phys. Rev. D29 (1984) 2423; Phys. Rev. D44 (1991) 1866.

9. T. Appelquist, M. Bowick, E. Choler and L.C.R. Wijewardhana, Phys. Rev. Lett. 55 (1985) 1715; T. Appelquist, M. Bowick, D. Karabali and L.C.R. Wijewardhana, Phys. Rev. D33 (1986) 3704; T. Appelquist, D. Nash and L.C.R. Wijewardhana, Phys. Rev. Lett. 60 (1988) 2575; D. Nash, Phys. Rev. Lett. 62 (1989) 3024.

10. M.R. Pennington and D. Walsh, Phys. Lett. 253B (1991) 246; D.C. Curtis, M.R. Pennington and D. Walsh, Phys. Lett. 295B (1992) 313.

11. K.I. Kondo and H. Nakatani, Progr. Theor. Phys. 87 (1992) 193.

12. V. Azcoiti and X.Q. Luo, Nucl. Phys. B (Proc. Suppl.) 30 (1993) 741; V. Azcoiti and X.Q. Luo DFTUZ.92/25 (1992).

13. S.J. Hands, A. Kocic, J.B. Kogut, R.L. Renken, D.K. Sinclair and K.C. Wang, "Spectroscopy, Equation of State and monopole percolation in lattice QED with two flavours", CERN-TH.6609/92 (1992); A. Kocic, J.B. Kogut and K.C. Wang, " Monopole Percolation and the Universality Class of the Chiral Transition in Four flavour non compact Lattice QED." ILL-TH-92-17 (1992).

14. R. Brout, F. Englert and J.M. Frere, Nucl. Phys. B134 (1978) 327; A. Casher, Phys. Lett. B83 (1979) 395; A. Amer, A. LeYaouanc, L.
Oliver, O. Pene and J.C. Raynal, Phys. Rev. Lett. 50 (1983) 87.
15. H.R. Fiebig and R.M. Woloshyn, Phys. Rev. D42 (1990) 3520.
FIGURE CAPTIONS

1) Normalized fermionic effective action as a function of the pure gauge energy on a $14^3$ lattice, $m = 0.0$ and $N = 2$.
2) The order parameter, obtained from the results of Fig. 1
3) Phase diagram on the $(\beta, N)$ plane.
4) Inverse logarithm of the chiral condensate against the inverse fermion mass logarithm for several values of $\beta$ in the $14^3$ (4a) and $18^3$ (4b) lattices (quenched case).
5) Singular part of the mean value of the effective action normalized by the lattice volume $V$ against $\beta$ in a $18^3$ lattice (quenched case).
TABLE CAPTION

I) Critical values of $\beta$ at several values of $N$ on the $18^3$ lattice.
Figure 2

$S_{\text{sing}}$ vs $E$
\(-1 / \log \psi \overline{\psi}\)
\[-\frac{1}{\log \psi \bar{\psi}}\]
| $n_f$ | $\beta_c$     | transition       |
|------|---------------|------------------|
| 0    | 0.490         | continuous       |
| 1    | 0.442         | continuous       |
| 2    | 0.394         | continuous       |
| 3    | 0.345         | continuous       |
| 4    | 0.297         | continuous       |
| 6    | 0.201         | continuous       |
| 7    | 0.106, 0.153  | discontinuous,   |
|      |               | continuous       |
| 8    | 0.091, 0.105  | discontinuous,   |
|      |               | continuous       |
| 10   | 0.069         | discontinuous    |

Table I