STRICTLY NEF VECTOR BUNDLES AND CHARACTERIZATIONS OF $\mathbb{P}^n$

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ABSTRACT. In this note, we give a brief exposition on the differences and similarities between strictly nef and ample vector bundles, with particular focus on the circle of problems surrounding the geometry of projective manifolds with strictly nef bundles.

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1. INTRODUCTION

Let $X$ be a complex projective manifold. A line bundle $L$ over $X$ is said to be strictly nef if

$$L \cdot C > 0$$

for each irreducible curve $C \subset X$. This notion is also called "numerically positive" in literatures (e.g. [Har70]). The Nakai-Moishezon-Kleiman criterion asserts that $L$ is ample if and only if

$$L^{\dim Y} \cdot Y > 0$$

for every positive-dimensional irreducible subvariety $Y$ in $X$. Hence, ample line bundles are strictly nef. In 1960s, Mumford constructed a number of strictly nef but non-ample line bundles over ruled surfaces (e.g. [Har70]), and they are tautological line bundles of stable vector bundles of degree zero over smooth curves of genus $g \geq 2$. By using the terminology of Harshorne ([Har66]), a vector bundle $E \to X$ is called strictly nef (resp. ample) if its tautological line bundle $\mathcal{O}(E)(1)$ is strictly nef (resp. ample). One can see immediately that the strictly nef vector bundles constructed by Mumford are actually Hermitian-flat. Therefore, some functorial properties for ample bundles ([Har66]) are not valid for strictly nef bundles. In this note, we give a brief exposition on the differences and similarities between strict nefness and ampleness, and survey some recent progress on understanding the geometry of projective manifolds endowed with some strictly nef bundles.

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Starting in the mid 1960s, several mathematicians—notably Grauert, Griffiths and Hartshorne ([Gra62, Gri65, Gri69, Har66])—undertook the task of generalizing to vector bundles the theory of positivity for line bundles. One of the goals was to extend to the higher rank setting as many as possible of the beautiful cohomological and topological properties enjoyed by ample divisors. In the past half-century, a number of fundamental results have been established. For this rich topic, we refer to the books [Laz04a, Laz04b] of Lazarsfeld and the references therein.

1.A. Abstract strictly nef vector bundles. Let’s recall a criterion for strictly nef vector bundles (see [LOY19, Proposition 2.1]) which is analogous to the Barton-Kleiman criterion for nef vector bundles (e.g. [Laz04b, Proposition 6.1.18]). This criterion will be used frequently in the sequel.

**Proposition 1.1.** Let \( E \) be a vector bundle over a projective manifold \( X \). Then the following conditions are equivalent.

1. \( E \) is strictly nef.
2. For any smooth projective curve \( C \) with a finite morphism \( \nu : C \to X \), and for any line bundle quotient \( \nu^* (E) \to L \), one has \( \deg L > 0 \).

Recall that for an ample line bundle \( L \), one has the Kodaira vanishing theorem

\[
H^i (X, L^*) = 0 \quad \text{for} \quad i < \dim X.
\]

For an ample vector bundle \( E \) with rank \( r \geq 2 \), one can only deduce

\[
H^0 (X, E^*) = 0
\]

and the higher cohomology groups \( H^i (X, E^*) (i \geq 1) \) may not vanish. By using Proposition 1.1, we obtain a similar vanishing theorem for strictly nef vector bundles.

**Theorem 1.2.** Let \( E \) be a strictly nef vector bundle over a projective manifold \( X \). Then

\[
H^0 (X, E^*) = 0.
\]

It worths to point out that for a strictly nef vector bundle \( E \) with rank \( r \geq 2 \), the cohomology group \( H^0 (X, \text{Sym}^k E^*) \) may not vanish for \( k \geq 2 \), which is significantly different from properties of ample vector bundles. Indeed, the strict nefness is not closed under tensor product, symmetric product or exterior product of vector bundles. These will be discussed in Section 2.

It is well-known that vector bundles over \( \mathbb{P}^1 \) split into direct sums of line bundles. By using Proposition 1.1 again, one deduces that strictly nef vector bundles over \( \mathbb{P}^1 \) are ample. In [LOY19, Theorem 3.1], the following result is obtained.

**Corollary 1.3.** If \( E \) is a strictly nef vector bundle over an elliptic curve \( C \), then \( E \) is ample.

As we mentioned before, over smooth curves of genus \( g \geq 2 \), there are strictly nef vector bundles which are Hermitian-flat. There also exist strictly nef but non-ample bundles on some rational surfaces ([LR94, Cha20]). It is still a challenge to investigate strictly nef vector bundles over higher dimensional projective manifolds. We propose the following conjecture, which is also the first step to understand such bundles.
Conjecture 1.4. Let $E$ be a strictly nef vector bundle over a projective manifold $X$. If $-K_X$ is nef, then $\det E$ is ample.

Although this conjecture is shown to be a consequence of the "generalized abundance conjecture" (e.g. [LP20a, LP20b]), we still expect some other straightforward solutions. Indeed, we get a partial answer to it.

Theorem 1.5. Let $E$ be a strictly nef vector bundle over a projective manifold $X$. If $-K_X$ is nef and big, then $\det E$ is ample.

We refer to [HMP10, BHN19] for more details on positivity of equivariant vector bundles.

1.B. The geometry of projective manifolds endow with strictly nef bundles.
Since the seminal works of Mori and Siu-Yau ([Mor79], [SY80]) on characterizations of projective spaces, it becomes apparent that the positivity of the tangent bundle of a complex projective manifold carries important geometric information. In the past decades, many remarkable generalizations have been established, and for instances, Mok’s uniformization theorem on compact Kähler manifold with semipositive holomorphic bisectional curvature ([Mok88]) and fundamental works of Campana, Demailly, Peternell and Schneider ([CP91], [DPS94], [Pet96]) on the structure of projective manifolds with nef tangent bundles. For this comprehensive topic, we refer to [Yau74, Mor79, SY80, Mok88, CP91, DPS94, Pet96, DPS01, MOSC+15, Cao16, LOY19, CH19, CH19, CCP19] and the references therein.

As we pointed out before, strict nefness is a notion of positivity weaker than ampleness. Even though there are significant differences between them, we still expect that the strict nefness could play similar roles as ampleness in many situations. The following result is obtained in [LOY19, Theorem 1.4] which extends Mori’s Theorem.

Theorem 1.6. Let $X$ be a projective manifold. If $T_X$ is strictly nef, then $X$ is isomorphic to a projective space.

Therefore, $T_X$ is ample if and only if it is strictly nef. However, this is not valid for cotangent bundles. Indeed, Shepherd-Barron proved in [SB95] that there exists a projective surface whose cotangent bundle is strictly nef but not ample (see e.g. Example 4.3).

Let’s consider manifolds with strictly nef canonical or anti-canonical bundles. Campana and Peternell proposed in [CP91, Problem 11.4] the following conjecture, which is still a major problem along this line.

Conjecture 1.7. Let $X$ be a projective manifold. If $K_X^{-1}$ is strictly nef, then $X$ is Fano.

This conjecture has been verified for projective manifolds of dimension 2 in [Mae93] and dimension 3 in [Ser95] (see also [Ueh00] and the references therein). Recently, some progress has been achieved in [LOY19, Theorem 1.2].

Theorem 1.8. If $K_X^{-1}$ is strictly nef, then $X$ is rationally connected.

Indeed, we show in [LOY21] that if $(X, \Delta)$ is a projective simple normal crossing pair and $-(K_X + \Delta)$ is strictly nef, then $X$ is rationally connected.
The following dual version of Conjecture 1.7 is actually a consequence of the abundance conjecture.

**Conjecture 1.9.** If $K_X$ is strictly nef, then $K_X$ is ample.

As analogous to the Fujita conjecture, Serrano proposed in [Ser95] the following conjecture, which is a generalization of Conjecture 1.7.

**Conjecture 1.10.** Let $X$ be a projective manifold. If $L$ is a strictly nef line bundle, then $K_X \otimes L^m$ is ample for $m \geq \dim X + 2$.

This conjecture has been solved for projective surfaces in [Ser95]. For the progress on projective threefolds and higher dimensional manifolds, we refer to [CCP08, Theorem 0.4] and the references therein.

It is also known that the existence of "positive" subsheaves of the tangent bundle can also characterize the ambient manifold. For instance, Andreatta and Wiśniewski obtained in [AW01, Theorem] the following characterization of projective spaces:

**Theorem 1.11.** Let $X$ be a projective manifold. If the tangent bundle $T_X$ contains a locally free ample subsheaf $\mathcal{F}$, then $X$ is isomorphic to a projective space.

When $\mathcal{F}$ is a line bundle, this result is proved by Wahl in [Wah83], and in [CP98], Campana and Peternell established the cases $r \geq n - 2$. It is also shown that the assumption on the local freeness can be dropped ([AKP08, Liu19]). On the other hand, according to Mumford’s construction (see Example 2.5), Theorem 1.11 does not hold if the subsheaf $\mathcal{F}$ is assumed to be strictly nef. Indeed, we obtained in [LOY20, Theorem 1.12] the following result, which is an extension of Theorem 1.11.

**Theorem 1.12.** Let $X$ be a projective manifold. Assume that the tangent bundle $T_X$ contains a locally free strictly nef subsheaf $\mathcal{F}$ of rank $r$. Then $X$ admits a $\mathbb{P}^d$-bundle structure $\varphi: X \to T$ for some integer $d \geq r$. Furthermore, if $T$ is not a single point, then it is a hyperbolic projective manifold of general type.

Recall that a single point is also considered to be hyperbolic in the sense that every holomorphic map from $C$ to it is constant. We expect a stronger geometric positivity on the cotangent bundle of the base $Y$ in Theorem 1.12 when $\dim Y > 0$. As an application of Theorem 1.12, we obtain in [LOY20, Theorem 1.4] a characterization of projective spaces.

**Theorem 1.13.** Let $X$ be an $n$-dimensional complex projective manifold such that $T_X$ contains a locally free strictly nef subsheaf $\mathcal{F}$. If $\pi_1(X)$ is virtually solvable, then $X$ is isomorphic to $\mathbb{P}^n$, and $\mathcal{F}$ is isomorphic to either $\mathcal{O}_{\mathbb{P}^n}(1)\oplus r$.

There are many other characterizations of projective spaces, and we refer to [KO73, Pet90, CP98, Dru04, Hwa06, Ara06, ADK08, Yan17, Tos17, FLW17, Li18] and the references therein.

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2. Basic properties and examples

In this section, we investigate basic properties of strictly nef bundles and discuss some examples. As we mentioned before, Mumford constructed a strictly nef vector bundle which is not ample (see [Har70, Chapter I, Example 10.6]). We shall describe this example in details. Let $E$ be a rank 2 vector bundle over a smooth curve $C$ of genus $g \geq 2$, $X = \mathbb{P}(E)$ be the projectivized bundle and $\pi : \mathbb{P}(E) \to C$ be the projection. Let $\mathcal{O}_E(1)$ be the tautological line bundle of $\mathbb{P}(E)$ and $D$ be the corresponding divisor over $X$.

Lemma 2.1. [Har70, Chapter I, Proposition 10.8] For any $m > 0$, there is a one-to-one correspondence between

1. effective curves $Y$ on $X$, having no fibers as components, of degree $m$ over $C$; and
2. sub-line bundles $L$ of $\text{Sym}^m \otimes E$.

Moreover, under this correspondence, one has

$$D \cdot Y = m \deg(E) - \deg(L).$$

For any effective curve $Y$ on $X$, we denote by $m(Y)$ the degree of $Y$ over $C$. Then there is an exact sequence

$$0 \to \text{Pic}(C) \xrightarrow{\pi^*} \text{Pic}(X) \xrightarrow{\pi_*} \mathbb{Z} \to 0.$$

It follows that the the divisors on $X$, modulo numerical equivalence, form a free abelian group of rank 2, generated by $D$ and $F$ where $F$ is any fiber of $\mathbb{P}(E)$.

Lemma 2.2. [Har70, Chapter I, Theorem 10.5] Let $C$ be a smooth curve of genus $g \geq 2$.

1. If $E$ is a stable vector bundle, then every symmetric power $\text{Sym}^k \otimes E$ is semi-stable.
2. For any $r > 0$ and $d \in \mathbb{Z}$, there exists a stable vector bundle with rank $r$ and degree $d$ such that all symmetric powers $\text{Sym}^k \otimes E$ are stable.

Theorem 2.3. [Har70, Chapter I, Example 10.6] Let $E$ be a rank 2 vector bundle over a smooth curve of genus $g \geq 2$. If $\deg(E) = 0$ and all symmetric powers $\text{Sym}^k \otimes E$ are stable, then $E$ is a strictly nef vector bundle, i.e. $\mathcal{O}_E(1)$ is a strictly nef line bundle. Moreover $E$ is not ample.

Proof. Let $Y$ be an arbitrary irreducible curve on $X$. If $Y$ is a fiber, then $D \cdot Y = 1$. If $Y$ is an irreducible curve of degree $m > 0$ over $C$, then by Lemma 2.1, $Y$ is corresponding to a subline bundle $L$ of $\text{Sym}^m \otimes E$. Note that since $\text{Sym}^m \otimes E$ are stable and of degree zero for all $m \geq 1$, we have

$$\deg(L) < \frac{\deg(\text{Sym}^m \otimes E)}{\text{rank}(\text{Sym}^m \otimes E)} = 0.$$

Therefore, by formula (2.1)

$$D \cdot Y = m \deg(E) - \deg(L) = - \deg(L) > 0.$$

Hence, the line bundle $\mathcal{O}_E(1)$ of the divisor $D$ is strictly nef, i.e. $E$ is a strictly nef vector bundle. Since $\deg(E) = 0$, $E$ can not be ample. □

Corollary 2.4. Let $E$ be a vector bundle over a smooth curve $C$. If $E$ is stable and $\deg(E) = 0$, then $E$ admits a Hermitian-flat metric.
Proof. Since $E$ is stable over $C$, there exists a Hermitian-Einstein metric $h$ on $E$ (e.g. [UY86]), i.e. $g^{-1} \cdot R_{\alpha \bar{\beta}} = c \cdot h_{\alpha \bar{\beta}}$ for some constant $c$ where $g$ is a smooth metric on $C$. Since $\text{deg}(E) = 0$, we deduce $c = 0$, i.e. $(E, h)$ is Hermitian-flat. □

We summarize Mumford’s example as following.

**Example 2.5.** Let $C$ be a smooth curve of genus $g \geq 2$. There exists a rank 2 vector bundle $E \rightarrow C$ satisfying the following properties:

1. $\text{deg}(E) = 0$;
2. $\text{Sym}^{\otimes k}E$ are stable for all $k \geq 1$;
3. $E$ is strictly nef but not ample; $E^*$ is strictly nef but not ample;
4. $E$ admits a Hermitian-flat metric.
5. Let $X = \mathbb{P}(E)$, $\pi : X \rightarrow C$ be the projection and $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the tautological line bundle. Then $T_{X/C} = K_{X/C}^{-1} \cong \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^* \text{det} E^*$ is strictly nef.

Although the strict nefness is not closed under tensor products and wedge products, we still have the following properties by using the Barton-Kleiman type criterion (Proposition 1.1).

**Proposition 2.6.** Let $E$ and $F$ be two vector bundles on a projective manifold $X$.

1. $E$ is a strictly nef vector bundle if and only if for every smooth curve $C$ and for any non-constant morphism $f : C \rightarrow X$, $f^*E$ is strictly nef.
2. If $E$ is strictly nef, then any non-zero quotient bundle $Q$ of $E$ is strictly nef.
3. $E \oplus F$ is strictly nef if and only if both $E$ and $F$ are strictly nef.
4. If the symmetric power $\text{Sym}^{\otimes k}E$ is strictly nef for some $k \geq 1$, then $E$ is strictly nef.
5. Let $f : Y \rightarrow X$ be a finite morphism such that $Y$ is a smooth projective variety. If $E$ is strictly nef, then so is $f^*E$.
6. Let $f : Y \rightarrow X$ be a surjective morphism such that $Y$ is a smooth projective variety. If $f^*E$ is strictly nef, then $E$ is strictly nef.

**Example 2.7.** Let $E$ be the strictly nef vector bundle in Example 2.5.

1. $\Lambda^2 E = \text{det} E$ is numerically trivial and it is not strictly nef;
2. the tensor product $E \otimes E^*$ is not strictly nef since $H^0(C, E \otimes E^*) \cong C$;
3. $E \oplus E^*$ is strictly nef, but $\text{Sym}^{\otimes 2}(E \oplus E^*)$ is not strictly nef.

The following result is well-known and it will be used frequently in the sequel.

**Lemma 2.8.** If $L$ is a strictly nef and semi-ample line bundle, then it is ample.

Proof. Since $L$ is semi-ample, $mL$ is globally generated for some large $m$. Let $\phi_m : X \rightarrow Y$ be the morphism defined by $|mL|$. Since $L$ is strictly nef, it is easy to see that $\phi_m$ is finite and so $L$ is ample. □

**Corollary 2.9.** Let $L$ be a strictly nef line bundle. Then $H^0(X, L^*) = 0$. Moreover, for any line bundle $F$, there exists a positive integer $m_0 = m_0(L, F)$ such that for $m \geq m_0$

$$H^0(X, (L^*)^m \otimes F) = 0. \quad (2.2)$$

Proof. Suppose $H^0(X, L^*) \neq 0$, then $L^*$ is effective. Since $L$ is nef, we deduce that $L$ is trivial and this is a contradiction. Hence $H^0(X, L^*) = 0$ and $L^*$ is not pseudo-effective. By [Yan19], the vanishing theorem (2.2) holds. □
Recall that for an ample line bundle $L$, one has the Kodaira vanishing theorem. For an ample vector bundle $E$, one can only deduce $H^0(X, E^*) = 0$ and the higher cohomology groups $H^i(X, E^*)$ ($i \geq 1$) may not vanish. For instance, when $X = \mathbb{P}^n$ with $n \geq 2$ and $E = TP^n$, one has

$$H^1(X, E^*) \cong H^{1,1}(X, C) \cong C \neq 0.$$ 

For strictly nef vector bundles, we have a similar vanishing theorem.

**Theorem 2.10.** Let $E$ be a strictly nef vector bundle over a projective manifold $X$. Then

$$H^0(X, E^*) = 0.$$ 

**Proof.** Suppose $\sigma \in H^0(X, E^*)$ is a nonzero section. Then by [DPS94, Proposition 1.16], $\sigma$ does not vanish anywhere. This section gives a trivial subbundle of $E^*$ and so a trivial quotient bundle of the strictly nef vector bundle $E$. This contradicts to (2) of Proposition 2.6. \hfill \square

Note that the vanishing theorem in (2.2) does not hold for higher rank vector bundles. Indeed, let $E$ be the strictly nef vector bundle in Example 2.5, $X = \mathbb{P}(E)$ and $\pi : X \rightarrow C$ be the projection. Let $F = K^{\otimes m}_C$ for some sufficiently large $m$. Then

$$H^0(C, \text{Sym}^k F) \neq 0$$

for all $k > 0$.

**Remark 2.11.** For a strictly nef vector bundle $E$ with rank $r \geq 2$, in general, $H^0(X, \text{Sym}^k E^*) = 0$ does not hold for $k \geq 2$.

We give more examples on strictly nef vector bundle over higher dimensional projective manifolds (see [LOY20, Section 5] for details). A line bundle $L$ over a projective variety $X$ of dimension $n$ is called $k$-strictly nef if

$$L \cdot \text{dim} Y : Y > 0$$

for every irreducible subvariety $Y$ in $X$ with $0 < \text{dim} Y \leq k$. Hence, 1-strictly nef is exactly strictly nef, and an $n$-strictly nef line bundle is ample.

**Theorem 2.12.** [Sub89, Lemma 3.2 and Theorem 6.1] Let $C$ be a smooth curve of genus $g \geq 2$. Then for any $r \geq 2$, there exists a Hermitian flat vector bundle $E$ of rank $r$ such that the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is $(r - 1)$-strictly nef. In particular, $E$ is strictly nef.

Fix a smooth curve $C$ of genus $g \geq 2$. Let $r \geq 2$ and $E$ be a vector bundle of rank $r$ provided in Theorem 2.12.

**Example 2.13.** Let $X = \mathbb{P}(E)$. Then we have the following relative Euler exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow p^* E^* \otimes \mathcal{O}_{\mathbb{P}(E)}(1) \rightarrow T_{X/C} \rightarrow 0,$$

where $p : X = \mathbb{P}(E) \rightarrow C$ is the natural projection. It is shown in [LOY20, Example 5.9] that $p^* E^* \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$ is strictly nef.

**Example 2.14.** We consider the following extension of vector bundles

$$0 \rightarrow Q \rightarrow G \rightarrow E^* \rightarrow 0,$$

where $Q$ is a nef vector bundle of positive rank. Since $E^*$ is Hermitian flat, it is numerically flat. In particular, $E^*$ is nef and so is $G$ ([DPS94, Proposition 1.15]).
Let \( X = \mathbb{P}(G) \) and \( p : X = \mathbb{P}(G) \to C \) the natural projection. Then we have the following relative Euler sequence
\[
0 \to \mathcal{O}_X \to p^*G^* \otimes \mathcal{O}_{\mathbb{P}(G)}(1) \to T_{X/C} \to 0.
\]
Since \( E \) is a subbundle of \( G^* \), it follows that \( p^*E \otimes \mathcal{O}_{\mathbb{P}(G)}(1) \) is a subbundle of \( p^*G^* \otimes \mathcal{O}_{\mathbb{P}(G)}(1) \). We proved in [LOY20, Example 5.10] that \( p^*E \otimes \mathcal{O}_{\mathbb{P}(G)}(1) \) is strictly nef and the restriction of \( F \) to fibers of \( p \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^1}(1)^{r+1} \). In particular, \( F \) is not a subbundle of \( T_{X/C} \).

3. Strictly nef vector bundles

In this section, we consider strictly nef vector bundles over higher dimensional projective manifolds.

**Theorem 3.1.** Let \( E \) be a strictly nef vector bundle over a projective manifold \( X \). If \(-K_X\) is nef and big, then \( \det E \) is ample.

**Proof.** If \( E \) is a strictly nef line bundle, then \( E - K_X \) is nef and big. By Kawamata-Reid-Shokurov base point free theorem, \( E \) is semi-ample. Thanks to Lemma 2.8, \( E \) is ample. If \( E \) has rank \( r \geq 2 \), we consider the projective bundle \( Y = \mathbb{P}(E) \).

Let \( \mathcal{O}_E(1) \) be the tautological line bundle of the projection \( \pi : Y \to X \). By the projection formula, we have
\[
-K_Y = \mathcal{O}_E(r) \otimes \pi^*(-K_X) \otimes \pi^*(\det E^*).
\]
For any \( m > 0 \), the line bundle \( L = \mathcal{O}_E(m) \otimes \pi^*(\det E) \) is strictly nef. Since
\[
L - K_Y = \mathcal{O}_E(m + r) \otimes \pi^*(-K_X),
\]
we deduce \( L - K_Y \) is strictly nef. On the other hand, \( L - K_Y \) is big. Indeed, since both \( \mathcal{O}_E(1) \) and \(-K_X\) are nef, the top intersection number
\[
(L - K_Y)^{n+r-1} = (\mathcal{O}_E(m + r) \otimes \pi^*(-K_X))^{n+r-1} \geq (\mathcal{O}_E(m + r))^{r-1} \cdot (\pi^*(-K_X))^n > 0.
\]

Therefore, by the base point free theorem again, \( L \) is semi-ample and so \( L \) is ample. By the positivity of direct image sheaves ([Mou97]), we deduce that \( \pi_*((K_{Y/X} \otimes L^\otimes k)) \) is ample for \( k \) large enough. By using the projection formula, one can see that the ample vector bundle \( \pi_*((K_{Y/X} \otimes L^\otimes k)) \) is of the form \( \text{Sym}^{\otimes k_0} E \otimes (\det E)^{\otimes k_1} \) where \( k_0 \) and \( k_1 \) are some positive integers. In particular, \( \det E \) is ample. \( \square \)

More generally, we propose the following conjecture.

**Conjecture 3.2.** Let \( E \) be a strictly nef vector bundle over a projective manifold \( X \). If \(-K_X\) is nef, then \( \det E \) is ample.

It is known that every strictly nef line bundle over an abelian variety is ample ([Ser95, Proposition 1.4]), and Chaudhuri proved in [Cha20] that every strictly nef homogeneous bundle on a complex flag variety is ample. We observe that Conjecture 3.2 can be implied by Serrano’s Conjecture 1.10.

**Proposition 3.3.** Conjecture 1.10 implies Conjecture 3.2.
Proof. Suppose Conjecture 1.10 is valid. Let $Y = \mathbb{P}(E), L = \mathcal{O}_{\mathbb{P}(E)}(1)$ and $\pi : Y \to X$ be the projection. For large $m$, $K_Y \otimes L^m$ is ample. Since $-K_X$ is nef, $K_{Y/X} \otimes L^m$ is ample. We know $\text{det} \pi_*(K_{Y/X} \otimes (K_{Y/X} \otimes L^m))$ is ample and so is $\text{det} E$. □

The following result is proved in [LOY19, Section 3].

**Proposition 3.4.** Let $E$ be a strictly nef vector bundle over a projective manifold $X$. If either of the following holds
1. the Kodaira dimension $\kappa(X)$ satisfies $0 \leq \kappa(X) < \dim X$,
2. $-K_X$ is pseudo-effective,
then $\text{det} E$ is not numerically trivial.

4. **Geometry of projective manifold with strictly nef bundles**

In this section, we describe the geometry related to strictly nef and ample bundles. As we mentioned before, the following result is obtained in [LOY19, Theorem 1.3]:

**Theorem 4.1.** Let $X$ be a projective manifold. If $T_X$ is strictly nef, then $X$ is isomorphic to a projective space.

Moreover, a characterization of quadrics is established in [LOY19, Theorem 1.5], which is analogous to classical results of Cho-Miyaoka-Shepherd-Barron ([CMSB02]) and Dedieu-Hoering ([DH17]).

**Theorem 4.2.** Let $X$ be a projective manifold of dimension $n \geq 3$. Suppose that $\Lambda^2 T_X$ is strictly nef, then $X$ is isomorphic to $\mathbb{P}^n$ or a quadric $Q^n$.

Gachet studied in [Gac19] the case when $\Lambda^3 T_X$ is strictly nef. We have established that the tangent bundle $T_X$ is strictly nef if and only if it is ample. However, it is not valid for cotangent bundles.

**Example 4.3.** Let $X$ be a bidisk quotient, $\Delta \times \Delta / \Gamma$, with $\Gamma$ an irreducible torsion-free cocompact lattice. Let $E = T^*_X$ and $L$ be its tautological line bundle. It is proved in [SB95] that $L$ is strictly nef and big, but it is not semi-ample.

We propose the following problem which is analogous to the class result of Kobayashi that projective manifolds with ample cotangent bundle are hyperbolic.

**Problem 4.4.** Let $X$ be a projective manifold. If $T^*_X$ is strictly nef, is $X$ hyperbolic?

Let’s consider manifolds with strictly nef canonical or anti-canonical bundles by recalling the Conjecture 1.7 of Campana and Peternell.

**Conjecture 4.5.** Let $X$ be a projective manifold. If $K^{-1}_X$ is strictly nef, then $X$ is Fano.

Recently, some evidences are established in [LOY19, Theorem 1.2].

**Theorem 4.6.** Let $X$ be a projective manifold. If $\Lambda^r T_X$ is strictly nef, then $X$ is rationally connected. In particular, if $K^{-1}_X$ is strictly nef, then $X$ is rationally connected.

Let $f : X \to Y$ be a smooth morphism between two projective manifolds. It is well-known that if $K^{-1}_X$ is ample, then so is $K^{-1}_Y$ ([KMM92], see also [BC16] for semi-ampileness). It is natural to propose the following conjecture.
Conjecture 4.7. Let $f : X \to Y$ be a smooth morphism between two projective manifolds. If $K_X^{-1}$ is strictly nef, then so is $K_Y^{-1}$.

Indeed, this conjecture can be regarded as a consequence of Conjecture 4.5. Thanks to Theorem 4.6, one obtains a partial answer to Conjecture 4.7.

Corollary 4.8. Let $f : X \to Y$ be a smooth morphism between two projective manifolds. If $K_X^{-1}$ is strictly nef, then $Y$ is rationally connected.

Example 4.9. Let $f : X \to Y$ be a smooth morphism between two projective manifolds. It is well-known that $K_X^{-1}$ can not be ample ([KMM92]). However, it can be strictly nef.

We also propose the following general conjecture concerning strictly nef bundles.

Conjecture 4.10. Let $X$ be a projective manifold.

1. If $\Lambda^r T_X$ is strictly nef for some $r > 0$, then $K_X^{-1}$ is ample;
2. If $\Lambda^r T_X^*$ is strictly nef for some $r > 0$, then $K_X$ is ample.

When $T_X$ is strictly nef, it is of particular interest and it is also related to the Kobayashi-Lang conjecture on hyperbolicity.

Let’s consider the geometry of projective manifolds whose tangent bundle contains a "positive" subsheaf. Recall that, Andreatta and Wiśniewski obtained in [AW01, Theorem] the following characterization of projective spaces.

Theorem 4.11. Let $X$ be an $n$-dimensional projective manifold. Assume that the tangent bundle $T_X$ contains a locally free ample subsheaf $\mathcal{F}$ of rank $r$. Then $X \cong \mathbb{P}^n$ and either $\mathcal{F} \cong T_{\mathbb{P}^n}$ or $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$.

According to Example 2.5, Theorem 4.11 does not hold if the subsheaf $\mathcal{F}$ is assumed to be strictly nef. Indeed, we obtained in [LOY20, Theorem 1.3] the following structure theorem for projective manifolds whose tangent bundle contains a strictly nef subsheaf.

Theorem 4.12. Let $X$ be a projective manifold. Assume that the tangent bundle $T_X$ contains a locally free strictly nef subsheaf $\mathcal{F}$ of rank $r$. Then $X$ admits a $\mathbb{P}^d$-bundle structure $\varphi : X \to T$ for some $d \geq r$. Moreover, $T$ is a hyperbolic projective manifold.

Actually, we obtained in [LOY20, Theorem 8.1] a concrete description on the structure of the subsheaf $\mathcal{F}$ and it is exactly one of the following:

1. $\mathcal{F} \cong T_{X/T}$ and $X$ is isomorphic to a flat projective bundle over $T$;
2. $\mathcal{F}$ is a numerically projectively flat vector bundle and its restriction on every fiber of $\varphi$ is isomorphic to $\mathcal{O}_{\mathbb{P}^d}(1)^{\oplus r}$.

When $\dim T = 0$, we obtained in [LOY20, Theorem 1.4] a new characterization of projective spaces.

Corollary 4.13. Let $X$ be an $n$-dimensional projective manifold such that $T_X$ contains a locally free strictly nef subsheaf $\mathcal{F}$. If $\pi_1(X)$ is virtually abelian, then $X$ is isomorphic to $\mathbb{P}^n$, and $\mathcal{F}$ is isomorphic to either $T_{\mathbb{P}^n}$ or $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$.

When $\dim T > 0$, we established in [LOY20, Corollary 1.5] the existence of non-zero symmetric differentials.
Corollary 4.14. Let $X$ be a projective manifold whose tangent bundle contains a locally free strictly nef subsheaf. If $X$ is not isomorphic to a projective space, then $X$ has a non-zero symmetric differential, i.e. $H^0(X, \text{Sym}^i \Omega_X) \neq 0$ for some $i > 0$.

REFERENCE

[ADK08] Carolina Araujo, Stéphane Druel, and Sándor J. Kovács. Cohomological characterizations of projective spaces and hyperquadrics. *Invent. Math.*, 174(2):233–253, 2008.

[AKP08] Marian Aprodu, Stefan Kebekus, and Thomas Peternell. Galois coverings and endomorphisms of projective varieties. *Math. Z.*, 260(2):431–449, 2008.

[Ara06] Carolina Araujo. Rational curves of minimal degree and characterizations of projective spaces. *Math. Ann.*, 335(4):937–951, 2006.

[AW01] Marco Andreatta and Jarosław A. Wiśniewski. On manifolds whose tangent bundle contains an ample subbundle. *Invent. Math.*, 146(1):209–217, 2001.

[BC16] Caucher Birkar and Yifei Chen. Images of manifolds with semi-ample anti-canonical divisor. *J. Algebraic Geom.*, 25(2):273–287, 2016.

[BHN19] Indranil Biswas, Krishna Hanumanthu, and Donihakkal S. Nagaraj. Positivity of vector bundles on homogeneous varieties. *ArXiv preprint*, 1904.09310, 2019.

[Cao16] Junyan Cao. Albanese maps of projective manifolds with nef anticanonical bundles. *ArXiv preprint*, 1612.05921, 2016.

[CCP08] Frédéric Campana, Jungkai A. Chen, and Thomas Peternell. Strictly nef divisors. *Math. Ann.*, 342(3):565–585, 2008.

[CCP19] Frédéric Campana, Junyan Cao, and Mihai Păun. Subharmonicity of direct images and applications. *ArXiv preprint* arXiv:1906.11317, 2019.

[CH19] Junyan Cao and Andreas Höring. A decomposition theorem for projective manifolds with nef anticanonical bundle. *Journal of Algebraic Geometry*, 28(3):567–597, 2019.

[Cha20] Priyankur Chaudhuri. Strictly nef divisors and some remarks on a conjecture of Serrano. *ArXiv preprint*, 2008.05009, 2020.

[CMSB02] Koji Cho, Yoichi Miyaoka, and Nicholas I. Shepherd-Barron. Characterizations of projective space and applications to complex symplectic manifolds. In *Higher dimensional birational geometry (Kyoto, 1997)*, volume 35 of *Adv. Stud. Pure Math.*, pages 1–88. Math. Soc. Japan, Tokyo, 2002.

[CP91] Frédéric Campana and Thomas Peternell. Projective manifolds whose tangent bundles are numerically effective. *Math. Ann.*, 289(1):169–187, 1991.

[CP98] Frédéric Campana and Thomas Peternell. Rational curves and ampleness properties of the tangent bundle of algebraic varieties. *Manuscripta Math.*, 97(1):59–74, 1998.

[DH17] Thomas Dedieu and Andreas Höring. Numerical characterisation of quadrics. *Algebr. Geom.*, 4(1):120–135, 2017.

[DPS94] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider. Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geom.*, 3(2):295–345, 1994.

[DPS01] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider. Pseudo-effective line bundles on compact Kähler manifolds. *Internat. J. Math.*, 12(6):689–741, 2001.

[Dru04] Stéphane Druel. Caractérisation de l’espace projectif. *Manuscripta Math.*, 115(1):19–30, 2004.

[FLW17] Huitao Feng, Kefeng Liu, and Xueyuan Wan. Compact Kähler manifolds with positive orthogonal bisectional curvature. *Math. Res. Lett.*, 24(3):767–780, 2017.

[Gac19] CÃ©cile Gachet. PositivitÃ© du fibrÃ© tangent et de ses puissances extÃ©rieures. Master’s thesis, Paris 6, 2019.

[Gra62] Hans Grauert. Über Modifikationen und exzeptionelle analytische Mengen. *Math. Ann.*, 146:331–368, 1962.

[Gri65] Phillip A. Griffiths. Hermitian differential geometry and the theory of positive and ample holomorphic vector bundles. *J. Math. Mech.*, 14:117–140, 1965.

[Gri69] Phillip A. Griffiths. Hermitian differential geometry, Chern classes, and positive vector bundles. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 185–251. Univ. Tokyo Press, Tokyo, 1969.

[Har66] Robin Hartshorne. Ample vector bundles. *Inst. Hautes Études Sci. Publ. Math.*, (29):63–94, 1966.
[Har70] Robin Hartshorne. *Ample subvarieties of algebraic varieties*. Lecture Notes in Mathematics, Vol. 156. Springer-Verlag, Berlin-New York, 1970. Notes written in collaboration with C. Musili.

[HMP10] Milena Hering, Mircea Mustaţă, and Sam Payne. Positivity properties of toric vector bundles. *Ann. Inst. Fourier (Grenoble)*, 60(2):607–640, 2010.

[Hwa06] Jun-Muk Hwang. Rigidity of rational homogeneous spaces. In *International Congress of Mathematicians. Vol. II*, pages 613–626. Eur. Math. Soc., Zürich, 2006.

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational connectedness and boundedness of Fano manifolds. *J. Differential Geom.*, 36(3):765–779, 1992.

[KO73] Shosichi Kobayashi and Takushiro Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973.

[Laz04a] Robert Lazarsfeld. *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, volume 48 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2004.

[Laz04b] Robert Lazarsfeld. *Positivity in algebraic geometry II: Positivity for vector bundles, and multiplier ideals.*, volume 49 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2004.

[Li18] Ping Li. The spectral rigidity of complex projective spaces, revisited. *Math. Z.*, 290(3-4):1115–1143, 2018.

[Liu19] Jie Liu. Characterization of projective spaces and $\mathbb{P}^n$ -bundles as ample divisors. *Nagoya Math. J.*, 233:155–169, 2019.

[LOY19] Duo Li, Wenhao Ou, and Xiaokui Yang. On projective varieties with strictly nef tangent bundles. *J. Math. Pures Appl. (9)*, 128:140–151, 2019.

[LOY20] Jie Liu, Wenhao Ou, and Xiaokui Yang. Projective manifolds whose tangent bundle contains a strictly nef subsheaf. *ArXiv preprint*, 2004.08507, 2020.

[LOY21] Jie Liu, Wenhao Ou, and Xiaokui Yang. Projective manifolds with a strictly nef pair. *preprint*, 2021.

[LP20a] Vladimir Lazic and Thomas Peternell. On Generalised Abundance, I. *Publ. Res. Inst. Math. Sci.*, 56(2):353–389, 2020.

[LP20b] Vladimir Lazic and Thomas Peternell. On Generalised Abundance, II. *Peking Math. J.*, 3(1):1–46, 2020.

[LR94] Antonio Lanteri and Barbara Rondena. Numerically positive divisors on algebraic surfaces. *Geom. Dedicata*, 53(2):145–154, 1994.

[Mae93] Hidetoshi Maeda. A criterion for a smooth surface to be Del Pezzo. *Math. Proc. Cambridge Philos. Soc.*, 113(1):1–3, 1993.

[Mok88] Ngaiming Mok. The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature. *J. Differential Geom.*, 27(2):179–214, 1988.

[Mor79] Shigefumi Mori. Projective manifolds with ample tangent bundles. *Ann. of Math. (2)*, 110(3):593–606, 1979.

[MOSC+15] Roberto Muñoz, Gianluca Occhetta, Luis Eduardo Solà Conde, Kiwamu Watanabe, and Jaroslaw A. Wiśniewski. A survey on the Campana-Peternell conjecture. *Rend. Istit. Mat. Univ. Trieste*, 47:127–185, 2015.

[Mou97] Christophe Mourougane. Images directes de fibrés en droites adjoints. *Publ. Res. Inst. Math. Sci.*, 33(6):893–916, 1997.

[Pet90] Thomas Peternell. A characterization of $\mathbb{P}_n$ by vector bundles. *Math. Z.*, 205(3):487–490, 1990.

[Pet96] Thomas Peternell. Manifolds of semi-positive curvature. In *Transcendental methods in algebraic geometry (Cetraro, 1994)*, volume 1646 of Lecture Notes in Math., pages 98–142. Springer, Berlin, 1996.

[SB95] N. I. Shepherd-Barron. Infinite generation for rings of symmetric tensors. *Math. Res. Lett.*, 2(2):125–128, 1995.

[Ser95] Fernando Serrano. Strictly nef divisors and Fano threefolds. *J. Reine Angew. Math.*, 464:187–206, 1995.

[Sub89] Swaminathan Subramanian. Mumford’s example and a general construction. *Proc. Indian Acad. Sci. Math. Sci.*, 99(3):197–208, 1989.
[SY80] Yum Tong Siu and Shing Tung Yau. Compact Kähler manifolds of positive bisectional curvature. *Invent. Math.*, 59(2):189–204, 1980.

[Tos17] Valentino Tosatti. Uniqueness of $\mathbb{C}P^n$. *Expo. Math.*, 35(1):1–12, 2017.

[Ueh00] Hokuto Uehara. On the canonical threefolds with strictly nef anticanonical divisors. *J. Reine Angew. Math.*, 522:81–91, 2000.

[UY86] Karen Keskuilla Uhlenbeck and Shing-Tung Yau. On the existence of Hermitian-Yang-Mills connections in stable vector bundles. volume 39, pages S257–S293. 1986. Frontiers of the mathematical sciences: 1985 (New York, 1985).

[Wah83] Jonathan M. Wahl. A cohomological characterization of $\mathbb{P}^n$. *Invent. Math.*, 72(2):315–322, 1983.

[Yan17] Xiaokui Yang. Big vector bundles and complex manifolds with semi-positive tangent bundles. *Math. Ann.*, 367(1-2):251–282, 2017.

[Yan19] Xiaokui Yang. A partial converse to the Andreotti-Grauert theorem. *Compos. Math.*, 155(1):89–99, 2019.

[Yau74] Shing Tung Yau. On the curvature of compact Hermitian manifolds. *Invent. Math.*, 25:213–239, 1974.

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