Optimality conditions for the buckling of a clamped plate.

1 Introduction

We consider the following variational problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let

$$\mathcal{R}(u, \Omega) := \frac{\int_{\Omega} \lvert \Delta u \rvert^2 \, dx}{\int_{\Omega} \lvert \nabla u \rvert^2 \, dx}$$

for $u \in H^{2,2}_0(\Omega)$. We set $\mathcal{R}(u, \Omega) = \infty$ if the denominator vanishes. We define

$$\Lambda(\Omega) := \inf \left\{ \mathcal{R}(u, \Omega) : u \in H^{2,2}_0(\Omega) \right\}.$$  

(1.1)
The infimum is attained by the first eigenfunction $u$, which solves the Euler Lagrange equation

\begin{align}
\Delta^2 u + \Lambda(\Omega) \Delta u &= 0 \quad \text{in } \Omega \tag{1.2} \\
u = \partial_\nu u &= 0 \quad \text{in } \partial \Omega. \tag{1.3}
\end{align}

If we normalize $u$ by $\|\nabla u\|_{L^2(\Omega)} = 1$, the first eigenfunction is uniquely determined. Otherwise any multiple of $u$ is an eigenfunction as well. The sign of the first eigenfunction may change depending on $\Omega$.

The quantity $\Lambda(\Omega)$ is called buckling eigenvalue of $\Omega$. It is well known that there is a discrete spectrum of positive eigenvalues of finite multiplicity and their only accumulation point is $\infty$. The corresponding eigenfunctions form an orthonormal basis of $H^2_{0,2}(\Omega)$.

In the sequel, we will assume that $u$ is normalized. If we multiply (1.2) with $x \cdot \nabla u$ and integrate by parts, we obtain

\begin{equation}
\Lambda(\Omega) = \frac{1}{2} \int_{\partial \Omega} |\Delta u|^2 x \cdot \nu \, dS. \tag{1.4}
\end{equation}

In 1951, G. Polya and G. Szegö formulated the following conjecture (see [9]).

*Among all domains $\Omega$ of given volume, the ball minimizes $\Lambda(\Omega)$.\*

This conjecture is still open. However, partial results are known. In [11] Szegö proved the conjecture for all smooth domains under the additional assumption that $u > 0$ in $\Omega$. M.S. Ashbaugh and D. Bucur proved that among simply connected domains of prescribed volume there exists an optimal domain [1]. In [12] H. Weinberg and B. Willms proved the following uniqueness result for $n = 2$. If an optimal plane domain $\Omega$ exists and if $\partial \Omega$ is smooth (at least $C^{2,\alpha}$), then $\Omega$ is a disc.

There also exist bounds for $\Lambda(\Omega)$. We only mention Payne’s inequality (see [13]) which states that

$$\Lambda(\Omega) \geq \lambda_2(\Omega),$$

where $\lambda_2$ denotes the second Dirichlet eigenvalue for the Laplacian. Equality holds if and only if $\Omega$ is a ball.

In this paper, we assume that there exists an optimal domain $\Omega$, which is smooth and simply connected. We will prove that $\Omega$ must be a ball. Thus we generalize the result of H. Weinberg and B. Willms in [12] to higher dimensions.
To consider the second domain variation for $\Lambda(\Omega)$ is motivated by the work of E. Mohr in [6]. He was interested in the clamped plate eigenvalue, where

$$ \mathcal{R}(u, \Omega) = \frac{\int_{\Omega} |\Delta u|^2 \, dx}{\int_{\Omega} u^2 \, dx} $$

and $\Omega$ is a smoothly bounded domain in $\mathbb{R}^2$. For the corresponding eigenvalue he computed the second domain variation. The explicit computation of the kernel of the second domain variation then implies that the disc is a unique minimizer among smooth domains of equal volume.

Our strategy will be as follows. In Chapter 2 we introduce a smooth family $(\Omega_t)_t$ of perturbations of $\Omega$ of equal volume. We denote by $\Lambda(t) := \Lambda(\Omega_t)$ the corresponding first buckling eigenvalue of $\Omega_t$. As a consequence of the optimality of $\Omega$, the eigenfunction $u$ satisfies the overdetermined boundary value problem

$$ \Delta^2 u + \Lambda(\Omega) \Delta u = 0 \text{ in } \Omega $$
$$ u = \partial_{\nu} u = 0 \text{ in } \partial \Omega $$
$$ \Delta u = c_0 \text{ in } \partial \Omega, \text{ where } c_0 = \frac{2\Lambda(\Omega)}{|\Omega|} \text{ by (1.4)}. $$

This follows from the fact that the first domain variation of $\Lambda(\Omega)$ - computed in Chapter 3 - for an optimal domain necessarily vanishes.

In Chapter 4 we compute the second domain variation of $\Lambda(\Omega)$. It turns out that

$$ (1.5) \quad \ddot{\Lambda}(0) = \frac{d^2}{dt^2} \Lambda(t) \bigg|_{t=0} = 2 \int_{\Omega} |\Delta u'|^2 - 2\Lambda(\Omega) \int_{\Omega} |\nabla u'|^2 \, dx, $$

where $u'$ is the so called shape derivative of $u$. It solves

$$ (1.6) \quad \Delta^2 u' + \Lambda(\Omega) \Delta u' = 0 \text{ in } \Omega $$
$$ (1.7) \quad u' = 0 \text{ in } \partial \Omega $$
$$ (1.8) \quad \partial_{\nu} u' = -c_0 v \cdot v \text{ in } \partial \Omega $$

and

$$ (1.9) \quad \int_{\Omega} \nabla u \cdot \nabla u' \, dx = 0. $$

The vector field $v$ is the first order approximation of $\Omega_t$ in the sense that for $y \in \Omega_t$ there exists an $x \in \Omega$ such that

$$ y = x + tv(x) + o(t). $$
Thus, $\bar{\Lambda}(0)$ is equal to a quadratic functional in the shape derivative $u'$ which we denote by $\mathcal{E}(u')$ and $\mathcal{E}(u')$ is given by the right hand side of (1.5). Since we assume the optimality of $\Omega$, we have $\mathcal{E}(u') \geq 0$. It turns out that the kernel of $\mathcal{E}(u')$ contains the directional derivatives $\partial_1 u, \ldots, \partial_n u$ of $u$. Each directional derivative is a shape derivative, which corresponds to a domain perturbation given by translations.

The key idea is to enlarge the class of shape derivatives on which $\mathcal{E}$ is defined. This new class will be denoted by $\mathcal{Z}$ and contains the shape derivatives as a true subset. Nevertheless we can show that $\mathcal{E}$ is still bounded from below and even nonnegative on $\mathcal{Z}$. Moreover $\min_{\mathcal{Z}} \mathcal{E} = 0$ since the directional derivatives of $u$ are in $\mathcal{Z}$. This is done in Chapter 5. In Chapter 6 we construct a function $\psi \in \mathcal{Z}$ for which we will show

$$0 \leq \mathcal{E}(\psi) \leq (\lambda_2(\Omega) - \Lambda(\Omega)) \lambda_2(\Omega).$$

By Payne’s inequality we have equality and this proves that the optimal domain is a ball.

Some of these results were obtained in the Diplom thesis of the first author [5].

## 2 Domain variations

Let $\Omega$ be a bounded smooth (at least $C^{2,\alpha}$) and simply connected domain in $\mathbb{R}^n$. We denote by $\nu$ the unit normal vector field on $\partial \Omega$. Let $\delta$ be the distance function to the boundary, i.e. for $x \in \Omega$ we have

$$\delta(x) := \inf \{|x - z| : z \in \partial \Omega\}.$$  

Then, for smooth $\partial \Omega$, $\nu := \nabla \delta$ defines a smooth extension of $\nu$ into a sufficiently small tubular neighbourhood of $\partial \Omega$. With this the following identities hold.

$$(2.1) \quad \nu \cdot \nu = 1, \quad \nu \cdot D\nu = 0 \quad \text{and} \quad D\nu \cdot \nu = 0$$

on $\partial \Omega$. See e.g. Proposition 5.4.14 in [4] for a proof.

Moreover, the mean curvature of $\partial \Omega$ is bounded since $\Omega$ is smooth, i.e. for each $x \in \partial \Omega$ there holds

$$(2.2) \quad |H_{\partial \Omega}(x)| \leq \max_{\partial \Omega} |H_{\partial \Omega}| < \infty.$$  

We will frequently use integration by parts on $\partial \Omega$. Let $f \in C^1(\partial \Omega)$ and $v \in C^{0,1}(\partial \Omega, \mathbb{R}^n)$. The next formula is often called the Gauss theorem on surfaces.

$$(2.3) \quad \oint_{\partial \Omega} f \text{ div } \partial \Omega v \, dS = -\oint_{\partial \Omega} v \cdot \nabla f \, dS + (n - 1) \oint_{\partial \Omega} f(v \cdot \nu) H_{\partial \Omega} \, dS,$$
where
\begin{equation}
\nabla^\tau f = \nabla f - (\nabla f \cdot \nu)\nu
\end{equation}
denotes the tangential gradient of $f$.

In this chapter, we describe the class of admissible variations for the domain functional $\Lambda(\Omega)$. For given $t_0 > 0$ and $t \in (-t_0, t_0)$ let $(\Omega_t)_t$ be a family of perturbations of the domain $\Omega \subset \mathbb{R}^n$ of the form
\[ \Omega_t = \Phi_t(\Omega) \]
where
\[ \Phi_t : \overline{\Omega} \to \mathbb{R}^n \]
is a diffeomorphism which is smooth in $t$ and $x$. Thus we may write
\[ \Omega_t := \{ y = x + tv(x) + \frac{t^2}{2} w(x) + o(t^2) : x \in \Omega, \ t \text{ small} \} \]
where
\[ v = (v_1(x), v_2(x), \ldots, v_n(x)) = \partial_t \Phi_t(x)|_{t=0} \]
and
\[ w = (w_1(x), w_2(x), \ldots, w_n(x)) = \partial^2_t \Phi_t(x)|_{t=0} \]
are smooth vector fields and where $o(t^2)$ collects terms such that $\frac{o(t^2)}{t^2} \to 0$ as $t \to 0$. For small $t_0$ the sets $\Omega_t$ and $\Omega$ are diffeomorphic. We will frequently use the notation $y := \Phi_t(x)$. Consider the functional
\[ \Lambda(\Omega_t) := \inf \left\{ R(u, \Omega_t) : u \in H^{2,0}_0(\Omega_t) \right\} , \]
which only depends on $\Omega_t$. Let $u(t, y) \in H^{2,0}_0(\Omega_t)$ be the minimizer. For short we will write
\begin{equation}
\tilde{u}(t) := u(t, y).
\end{equation}
Then $\tilde{u}(t)$ solves
\begin{equation}
\Delta^2 \tilde{u}(t) + \Lambda(\Omega_t) \Delta \tilde{u}(t) = 0 \text{ in } \Omega_t
\end{equation}
\begin{equation}
\tilde{u}(t) = |\nabla \tilde{u}(t)| = 0 \text{ in } \partial \Omega_t.
\end{equation}
for each $t \in (-t_0, t_0)$. With this notation we define
\[ \Lambda(t) := R(\tilde{u}(t), \Omega_t) . \]
Since we assume smoothness of $\Omega$ and $\Phi_t$ the eigenfunction $\tilde{u}$ is also smooth in $t$ and $x$. This has several consequences which we list as remarks.
Remark 1 Since $\partial \Omega_t$ is smooth and since $\tilde{u}(t) = 0$ on $\partial \Omega_t$, then necessarily
\begin{equation}
\Delta \tilde{u} = \partial_t^2 \tilde{u} + (n - 1) \partial_t \tilde{u} H_{\partial \Omega_t} \quad \text{in } \partial \Omega_t,
\end{equation}
where $H_{\partial \Omega_t}$ denotes the mean curvature of $\partial \Omega_t$. Clearly, if $\tilde{u} = |\nabla \tilde{u}| = 0$ on $\partial \Omega_t$, then necessarily
\begin{equation}
\Delta \tilde{u} = \partial_t^2 \tilde{u} \quad \text{in } \partial \Omega_t.
\end{equation}

Remark 2 Since (2.7) holds for all $t \in (-t_0, t_0)$, we also have
\begin{equation}
\dot{\tilde{u}}(t) = |\nabla \tilde{u}(t)| = 0 \quad \text{in } \partial \Omega_t
\end{equation}
for all $t \in (-t_0, t_0)$.

Remark 3 Straightforward computations yield
\begin{equation*}
\dot{\tilde{u}}(t) = \frac{d}{dt} u(t, y) = \partial_t u(t, \Phi_t(\Phi_t^{-1}(y)) + \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nabla u(t, y)
\end{equation*}
for all $t \in (-t_0, t_0)$. Let $y \in \partial \Omega_t$. Then (2.10) and (2.7) imply
\begin{equation}
0 = \dot{\tilde{u}}(t) = \partial_t u(t, y) \quad \text{for } y \in \partial \Omega_t
\end{equation}
for all $t \in (-t_0, t_0)$.

In particular for $t = 0$ we compute $\tilde{u}(0) = u(x)$ and
\begin{align*}
\dot{\tilde{u}}(0) &= \partial_t u(0, x) + v(x) \cdot Du(0, x) \\
\tilde{u}(0) &= \partial_t^2 u(0, x) + 2v(x) \cdot D\partial_t u(0, x) + w(x) \cdot Du(0, x) + v(x) \cdot D(v(x) \cdot Du(0, x)).
\end{align*}

We will use the notation
\begin{equation*}
u'(x) := \partial_t u(0, x) \quad \text{and} \quad u''(x) := \partial_t^2 u(0, x).
\end{equation*}

Hence,
\begin{align*}
\dot{\tilde{u}}(0) &= u'(x) + v(x) \cdot Du(x) \\
\tilde{u}(0) &= u''(x) + 2v(x) \cdot Du'(x) + w(x) \cdot Du(x) + v(x) \cdot D(v(x) \cdot Du(x)).
\end{align*}

Note that all these quantities are defined for $x \in \overline{\Omega}$. For $x \in \partial \Omega$ we thus get
\begin{equation*}
0 = \dot{\tilde{u}}(0) = u'(x) \quad \text{and} \quad 0 = \nabla \tilde{u}(0) = \nabla u'(x) + v(x) \cdot D^2 u(x),
\end{equation*}
where $(v(x) \cdot D^2 u(x))_j = \sum_{i=1}^n v_i(x) \partial_i \partial_j u(x)$ for $j = 1, \ldots, n$. Thus, we get the following boundary conditions for $u'$.
\begin{equation}
\begin{aligned}
u'(x) &= 0 \quad \text{and} \quad \partial_n u'(x) = -v(x) \cdot D^2 u(x) \cdot \nu(x) \quad \text{for } x \in \partial \Omega.
\end{aligned}
\end{equation}
Here we used the notation $v(x) \cdot D^2u(x) \cdot \nu(x) = \sum_{i,j=1}^{n} v_i(x) \partial_i \partial_j u(x) v_j(x)$.

Let $\nu(t,y)$ be the unit normal vector in $y \in \partial \Omega_t$. We also write this as
\begin{equation}
\nu_t(y) = \nu(t, \Phi_t(x)) \quad \forall t \in (-t_0, t_0) \quad x \in \partial \Omega.
\end{equation}

Then we have
\begin{equation}
\nu' = -\nabla^\tau (v \cdot \nu), \quad \nu \cdot \nu' = 0.
\end{equation}

This follows from direct calculations (see e.g. (5.64) in [4]).

**Lemma 1** With the notation from above the following equality holds.
\begin{equation}
\nu_t \cdot \nabla (\partial_t u(t,y)) = -\Delta u(t,y) \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \in \partial \Omega_t
\end{equation}
for all $t \in (-t_0, t_0)$. Alternatively, we write this for all $t \in (-t_0, t_0)$ and $x \in \partial \Omega$ as
\begin{equation}
\nu(t, \Phi_t(x)) \cdot \nabla \{\partial_t u(t, \Phi_t(x))\} = -\Delta u(t, \Phi_t(x)) \nu(t, \Phi_t(x)) \cdot \partial_t \Phi_t(x).
\end{equation}

**Proof** Since $\nabla u(t, \Phi_t(x)) = 0$ for all $|t| < t_0$ and all $x \in \partial \Omega$ we have
\begin{equation*}
0 = \frac{d}{dt} \nabla u(t, \Phi_t(x)) = \nabla \partial_t u(t, \Phi_t(x)) + D^2 u(t, \Phi_t(x)) \cdot \partial_t \Phi_t(x).
\end{equation*}
This implies
\begin{equation*}
0 = \nu_t \cdot \nabla (\partial_t u(t,y)) + \nu_t \cdot D^2 u(t,y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \in \partial \Omega_t
\end{equation*}
for all $t \in (-t_0, t_0)$. Here we used the notation
\begin{equation*}

\nu_t \cdot D^2 u(t,y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) = \sum_{i,j=1}^{n} \nu_{t,i} \partial_i \partial_j u(t,y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) j.
\end{equation*}

Since $\nabla \bar{u}(t) = 0$ in $\partial \Omega_t$, we get
\begin{equation*}
\nu_t \cdot D^2 u(t,y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) = \nu_t \cdot D^2 u(t,y) \cdot \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)).
\end{equation*}
Thus,
\begin{equation*}
\nu_t \cdot \nabla (\partial_t u(t,y)) = -\nu_t \cdot D^2 u(t,y) \cdot \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \in \partial \Omega_t.
\end{equation*}
Formula (2.9) simplifies to
\begin{equation*}
\nu_t \cdot \nabla (\partial_t u(t,y)) = -\Delta u(t,y) \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \in \partial \Omega_t.
\end{equation*}
This proves the lemma. \qed
The first derivative of $\Lambda(t)$ with respect to the parameter $t$ is called the first domain variation and the second derivative is called the second domain variation.

Our domain variations will be chosen within the class of volume preserving perturbations up to order 2. Hence, they are chosen such that

\begin{equation}
L^n(\Omega_t) = L^n(\Omega) + o(t^2)
\end{equation}

holds. This puts constraints on the vector fields $v$ and $w$. They were discussed e.g. in [2], formula (2.13) and Lemma 1.

**Lemma 2** Let $v, w \in C^{0,1}(\Omega, \mathbb{R}^n)$ be such that (2.19) holds. Then

\begin{equation}
\int_{\Omega} \text{div} v \, dx = 0
\end{equation}

and

\begin{equation}
\int_{\Omega} ((\text{div} v)^2 - Dv : Dv + \text{div} w) \, dx = 0,
\end{equation}

where $Dv : Dv = \sum_{i,j=1}^{n} \partial_i v_j \partial_j v_i$. The second equality is equivalent to

\begin{equation}
\int_{\partial \Omega} (v \cdot \nu) \text{div} v \, dS - \int_{\partial \Omega} v \cdot Dv \cdot \nu \, dS + \int_{\partial \Omega} (w \cdot \nu) \, dS = 0.
\end{equation}

Note that rotations do not satisfy these conditions (see e.g. Remark 1 in [2]).

3 The first domain variation

We will use the following formula for the computations of the first domain variation of $\Lambda$. It is well known as Reynolds transport theorem and is analyzed in detail in Chapter 5.2.3 in [4].

**Theorem 1** Let $t \in (-t_0, t_0)$ for some $t_0 > 0$. Let $\Phi_t \in C^{0,1}(\mathbb{R}^n)$ be differentiable in $t$ and let $t \rightarrow f(t) \in L^1(\mathbb{R}^n)$ be a function which is differentiable in $t$. Moreover, let $f(t) \in W^{1,1}(\mathbb{R}^n)$. Then $t \rightarrow I(t) := \int_{\Omega_t} f(t) \, dy$ is differentiable in $t$. Moreover, we have the formula

\begin{equation}
\dot{I}(t) = \int_{\Omega_t} \partial_t f(t) + \text{div } (f(t) \partial_t \Phi_t(\Phi_t^{-1}(y))) \, dy.
\end{equation}

If $\partial \Omega$ is sufficiently smooth (at least Lipschitz continuous), this is equivalent to

\begin{equation}
\dot{I}(t) = \int_{\Omega_t} \partial_t f(t) \, dy + \int_{\partial \Omega_t} f(t) \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu(y) \, dS(y).
\end{equation}
In particular, for \( t = 0 \) we get

\[
\dot{I}(0) = \int_{\Omega} \partial_t f(t)|_{t=0} + \text{div} \left( f(0) \, v(x) \right) \, dx.
\]

Again, if \( \partial \Omega \) is sufficiently smooth, this is equivalent to

\[
\dot{I}(0) = \int_{\Omega} \partial_t f(t) \, dx + \int_{\partial \Omega} f(0) \, v(x) \cdot \nu(x) \, dS(x).
\]

We apply this formula to \( \Lambda(t) = \frac{D(t)}{N(t)} \) where

\[
D(t) := \int_{\Omega_t} |\Delta \tilde{u}(t)|^2 \, dy \quad \text{and} \quad N(t) := \int_{\Omega_t} |\nabla \tilde{u}(t)|^2 \, dy
\]

and we assume the normalization

\[
(3.1) \quad N(t) = \int_{\Omega_t} |\nabla \tilde{u}(t)|^2 \, dy = 1 \quad \forall \, t \in (-t_0, t_0).
\]

We then obtain

\[
\dot{\Lambda}(t) = 2 \int_{\Omega_t} \Delta \tilde{u}(t) \Delta \partial_t \tilde{u}(t) \, dy - 2 \Lambda(t) \int_{\Omega_t} \nabla \tilde{u}(t) \cdot \nabla \partial_t \tilde{u}(t) \, dy
\]

\[
+ \int_{\partial \Omega_t} |\Delta \tilde{u}(t)|^2 \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) \, dS(y),
\]

where \( \nu_t(y) \) denotes the unit normal vector in \( y \in \partial \Omega_t \). We integrate by parts and use (2.10). Then

\[
\dot{\Lambda}(t) = 2 \int_{\Omega_t} \left\{ \Delta^2 \tilde{u}(t) + \Lambda(t) \Delta \tilde{u}(t) \right\} \partial_t \tilde{u}(t) \, dy + 2 \int_{\partial \Omega_t} \Delta \tilde{u}(t) \partial_{\nu_t} \partial_t \tilde{u}(t) \, dS(y)
\]

\[
-2 \int_{\partial \Omega_t} \partial_{\nu_t} \Delta \tilde{u}(t) \partial_t \tilde{u}(t) \, dS(y) + \int_{\partial \Omega_t} |\Delta \tilde{u}(t)|^2 \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) \, dS(y).
\]

The first integral vanishes since \( \tilde{u}(t) \) solves (2.6). The third integral vanishes since (2.11) holds. Finally we use (2.17). This proves the following lemma.

**Lemma 3** Let \( \tilde{u}(t) \) be an eigenfunction (i.e. a solution of (2.6) - (2.7)) and assume (3.1) holds. Let

\[
\Lambda(t) = \int_{\Omega_t} |\Delta \tilde{u}(t)|^2 \, dy.
\]
Then
\[ \dot{\Lambda}(t) = - \int_{\partial \Omega} |\Delta \tilde{u}(t)|^2 \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) \, dS(y). \]

**Remark 4** Note that if \( \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) > 0 \), this implies \( \mathcal{L}^n(\Omega_t) > \mathcal{L}^n(\Omega) \) for small \( t \). Thus, \( \dot{\Lambda}(t) \) is negative in this case. From this we conclude that the first buckling eigenvalue is decreasing under set inclusion.

From Lemma 3 we get in particular
\[ \dot{\Lambda}(0) = - \int_{\partial \Omega} |\Delta u|^2 \, v(x) \cdot \nu(x) \, dS(x). \]

From Lemma 2 and (2.20) we deduce \( |\Delta u| = \text{const.} \) if \( \Omega \) is a critical point of \( \Lambda(t) \). Due to formula (1.4), this constant is equal to
\[ c_0 := \frac{2\Lambda(0)}{|\Omega|}. \]

We denote this result as a theorem.

**Theorem 2** Let \( \Omega_t \) be a family of volume preserving perturbations of \( \Omega \) as described in Chapter 2. Then \( \Omega \) is a critical point of the energy \( \Lambda(t) \), i.e. \( \dot{\Lambda}(0) = 0 \), if and only if
\[ \Delta u = c_0 \quad \text{on} \quad \partial \Omega. \]

In particular, \( u \) is a solution of the overdetermined boundary value problem
\[
\begin{align*}
\Delta^2 u + \Lambda(\Omega) \Delta u & = 0 \quad \text{in} \ \Omega \\
u_t \cdot \nabla u & = 0 \quad \text{in} \ \partial \Omega. \\
\Delta u & = c_0 > 0 \quad \text{in} \ \partial \Omega.
\end{align*}
\]

Note that if we set \( U := \Delta u + \Lambda(\Omega)u \) (3.5) - (3.7) implies
\[ \Delta U = 0 \text{ in } \Omega \text{ and } U = c_0 \text{ in } \partial \Omega. \]

Hence,
\[ U = \Delta u + \Lambda(\Omega)u = c_0 \quad \text{in } \overline{\Omega}. \]

From [12] we know that for \( n = 2 \) this implies that \( \Omega \) is a ball. In particular,
\[ \partial_t \Delta u = 0 \quad \text{in} \ \partial \Omega. \]
4 The second domain variation

Throughout this chapter we assume that $\Omega$ is an optimal domain, i.e. $\dot{\Lambda}(0) = 0$ and $\ddot{\Lambda}(0) \geq 0$. This implies that $u$ solves (3.5) - (3.7) and (3.8). As a consequence (2.14) reads as

$$(4.1) \quad u'(x) = 0 \quad \text{and} \quad \partial_\nu u'(x) = -c_0 v(x) \cdot \nu(x) \quad \text{for} \quad x \in \partial \Omega.$$

Note that if we differentiate (2.6) - (2.7) in $t = 0$ and use the fact that $\dot{\Lambda}(0) = 0$, we obtain an equation for $u'$:

$$(4.2) \quad \Delta^2 u'(x) + \Lambda(\Omega) \Delta u'(x) = 0 \quad \text{in} \quad \Omega.$$

The boundary conditions for $u'$ are given by (4.1). Furthermore, the normalization (3.1) implies

$$(4.3) \quad \int_{\Omega} \nabla u \cdot \nabla u' \, dx = 0.$$

We recall formula (3.2). Before we differentiate with respect to $t$ again we state the following consequence of Reynold’s theorem (see e.g. Chapter 5.4.2 in [4]).

**Theorem 3** Let $\Omega$ be a bounded smooth domain of class $C^3$. Let $t \in (-t_0, t_0)$ and let $\Phi_t \in C^{0,1}(\mathbb{R}^n)$ be differentiable in $t$. Let $t \to g(t) \in L^1(\mathbb{R}^n)$ be a function which is differentiable in $t$. Moreover, let $g(t) \in W^{1,1}(\mathbb{R}^n)$. Then $t \to J(t) := \int_{\partial \Omega} g(t) \, dS(y)$ is differentiable in $t$. Moreover, for $t = 0$ we have the formula

$$\dot{J}(0) = \int_{\partial \Omega} \partial_t g(0) + (v(x) \cdot \nu(x)) \{ \partial_\nu g(0) + (n - 1)g(0) H_{\partial \Omega}(x) \} \, dS(x),$$

where $H_{\partial \Omega}$ denotes the mean curvature of $\partial \Omega$ in $x$.

We apply this theorem to (3.2). It is convenient to apply (2.17) and to rewrite (3.2) as

$$\dot{\Lambda}(t) = \int_{\partial \Omega} \Delta \tilde{u}(t) \nu_t \cdot \nabla(\partial_t u(t, y)) \, dS(y).$$

Let

$$g(t) := \Delta \tilde{u}(t) \nu_t \cdot \nabla(\partial_t u(t, y)).$$

An application of Theorem 3 yields

$$(4.4) \quad \ddot{\Lambda}(0) = \int_{\partial \Omega} \Delta u' \partial_\nu u' \, dS + \int_{\partial \Omega} \Delta u \nu' \cdot \nabla u' \, dS + \int_{\partial \Omega} \Delta u \partial_\nu u'' \, dS$$

$$+ \int_{\partial \Omega} (v \cdot \nu) \partial_\nu (\Delta u \partial_\nu u') \, dS + (n - 1) \int_{\partial \Omega} (v \cdot \nu) \Delta u \partial_\nu u' H_{\partial \Omega} \, dS.$$
Note that
\[ \nu_t \cdot \nu_t = 1 \text{ in } \partial \Omega_t \implies \nu \cdot \nu' = 0 \text{ in } \partial \Omega, \]
where
\[ \nu'(x) = \partial_t \nu(t, \Phi_t(x))|_{t=0} \text{ for } x \in \partial \Omega. \]
Since (4.1) implies \( \nabla u' = \partial_{\nu'} u \), this implies
\[ \int_{\partial \Omega} \Delta u \nu' \cdot \nabla u' \, dS = 0. \]
For the fourth integral we apply (3.4) and (3.9).
\[
\int_{\partial \Omega} (v \cdot \nu) \partial_{\nu'} (\Delta u \partial_{\nu'} u') \, dS = \int_{\partial \Omega} (v \cdot \nu) \partial_{\nu'} \Delta u \, dS + \int_{\partial \Omega} (v \cdot \nu) \Delta u \partial_{\nu''} u' \, dS
\]
\[= 0 + c_0 \int_{\partial \Omega} (v \cdot \nu) \partial_{\nu''} u' \, dS. \]
With the help of (4.1) and (2.8) we write
\[ \partial_{\nu''} u' = \Delta u' - (n - 1) \partial_{\nu'} u' H_{\partial \Omega}. \]
Hence,
\[ \int_{\partial \Omega} (v \cdot \nu) \partial_{\nu'} (\Delta u \partial_{\nu'} u') \, dS = c_0 \int_{\partial \Omega} (v \cdot \nu) \Delta u' \, dS - c_0 (n - 1) \int_{\partial \Omega} (v \cdot \nu) \partial_{\nu'} u' H_{\partial \Omega} \, dS. \]
Our computations yield a first simplification of (4.4):
\[ \ddot{\Lambda}(0) = \int_{\partial \Omega} \Delta u' \partial_{\nu'} u' \, dS + \int_{\partial \Omega} \Delta u \partial_{\nu''} u'' \, dS + c_0 \int_{\partial \Omega} (v \cdot \nu) \Delta u' \, dS. \]
In the first integral on the right hand side we use (4.1) again. Thus, we get
\[ (4.5) \quad \ddot{\Lambda}(0) = c_0 \int_{\partial \Omega} \partial_{\nu'} u'' \, dS. \]
In order to find a lower bound for \( \ddot{\Lambda}(0) \), we analyze the integral in (4.5). Recall (2.18). We differentiate this equation with respect to \( t \) in \( t = 0 \). Then (3.9) and (3.4) yield
\[ \nu' \cdot \nabla u' + v \cdot D \nu \cdot \nabla u' + \partial_{\nu'} u'' + \nu' \cdot \partial_{\nu} D^2 u' \cdot v =
\]
\[ -\Delta u' (v \cdot \nu) - c_0 (v \cdot v') - c_0 v \cdot D \nu \cdot v - c_0 (w \cdot v). \]
As before, \(u' \cdot \nabla u' = 0\) on \(\partial \Omega\). Moreover, by (4.1)
\[
v \cdot Dv \cdot \nabla u' = -c_0 v \cdot Dv \cdot (v \cdot \nu) = 0,
\]
where the last equality follows from (2.1). Thus,
\[
(4.6) \quad \ddot{\Lambda}(0) = -c_0 \int_{\partial \Omega} (v \cdot \nu) \Delta u' \, dS - c_0 \int_{\partial \Omega} v \cdot D^2 u' \cdot v \, dS
\]
\[
- c_0^2 \int_{\partial \Omega} (v \cdot v') \, dS - c_0^2 \int_{\partial \Omega} v \cdot Dv \cdot v \, dS - c_0^2 \int_{\partial \Omega} (w \cdot \nu) \, dS.
\]
For the first integral we use (4.1) and we observe that Gauß theorem, partial integration and equation (4.2) for \(u'\) gives
\[
(4.7) \quad -c_0 \int_{\partial \Omega} (v \cdot \nu) \Delta u' \, dS = \int_{\partial \Omega} \Delta u' \partial_{\nu} u' \, dS = \int_{\Omega} |\Delta u'|^2 \, dx - \Lambda(\Omega) \int_{\Omega} |\nabla u'|^2 \, dx.
\]
The second integral is slightly more involved. We set \(v^\tau = v - (v \cdot \nu)\nu\). Since \(\nabla u' = (\partial_{\nu} u')\nu\) and since (2.8) can be applied to \(u'\), we get
\[
-c_0 \int_{\partial \Omega} v \cdot D^2 u' \cdot v \, dS = -c_0 \int_{\partial \Omega} v^\tau \cdot D^2 u' \cdot v \, dS - c_0 \int_{\partial \Omega} (v \cdot \nu) (\Delta u' - (n-1)\partial_{\nu} u' H_{\partial \Omega}) \, dS
\]
\[
= -c_0 \int_{\partial \Omega} v^\tau \cdot D (\partial_{\nu} u' \nu) \cdot v \, dS - c_0 \int_{\partial \Omega} (v \cdot \nu) \Delta u' \, dS
\]
\[
- c_0^2 (n-1) \int_{\partial \Omega} (v \cdot \nu)^2 H_{\partial \Omega} \, dS.
\]
For the last equality we also used
\[
v^\tau \cdot Dv \cdot \nu = v^\tau \cdot D^\tau v \cdot \nu = 0 \quad \text{in} \ \partial \Omega.
\]
Next we note that with (4.1) we have
\[
-c_0 \int_{\partial \Omega} v^\tau \cdot D (\partial_{\nu} u' \nu) \cdot v \, dS = -c_0 \int_{\partial \Omega} v^\tau \cdot D^\tau (\partial_{\nu} u' \nu) \cdot v \, dS = c_0^2 \int_{\partial \Omega} v^\tau \cdot D^\tau ((v \cdot \nu) \nu) \cdot v \, dS
\]
\[
= c_0^2 \int_{\partial \Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) \, dS,
\]
where the last equality uses (2.1).
For the third integral in (4.6) we apply formula (2.16):
\[
-c_0^2 \int_{\partial \Omega} (v \cdot v') \, dS = c_0^2 \int_{\partial \Omega} v \cdot \nabla^\tau (v \cdot \nu) \, dS = c_0^2 \int_{\partial \Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) \, dS.
\]
These computations simplify (4.6) and we obtain

\[ \ddot{\Lambda}(0) = 2 \int_{\partial \Omega} \partial_{u'} \Delta u' \, dS + 2c_0^2 \int_{\partial \Omega} v^{\tau} \cdot \nabla^{\tau} (v \cdot \nu) \, dS - c_0^2 (n - 1) \int_{\partial \Omega} (v \cdot \nu)^2 H_{\partial \Omega} \, dS \]

\[ -c_0^2 \int_{\partial \Omega} v \cdot D\nu \cdot v \, dS - c_0^2 \int_{\partial \Omega} (w' \cdot \nu) \, dS. \]

Next we use the volume constraint (2.21).

\[ -c_0^2 \int_{\partial \Omega} (w' \cdot \nu) \, dS = c_0^2 \int_{\partial \Omega} (v^{\tau} \cdot D^{\tau} (v \cdot \nu)) \, dS - c_0^2 \int_{\partial \Omega} v^{\tau} \cdot D^{\tau} v \cdot \nu \, dS. \]

We integrate by parts in the first integral (see formula (2.3) and (2.4)).

\[ -c_0^2 \int_{\partial \Omega} (w' \cdot \nu) \, dS = -c_0^2 \int_{\partial \Omega} v^{\tau} \cdot D^{\tau} (v \cdot \nu) \, dS + c_0^2 (n - 1) \int_{\partial \Omega} (v \cdot \nu)^2 H_{\partial \Omega} \, dS \]

\[ -c_0^2 \int_{\partial \Omega} v^{\tau} \cdot D^{\tau} v \cdot \nu \, dS. \]

Thus, (4.8) becomes

\[ \ddot{\Lambda}(0) = 2 \int_{\partial \Omega} \partial_{u'} \Delta u' \, dS + c_0^2 \int_{\partial \Omega} v^{\tau} \cdot \nabla^{\tau} (v \cdot \nu) \, dS - c_0^2 \int_{\partial \Omega} v^{\tau} \cdot D^{\tau} v \cdot \nu \, dS \]

\[ -c_0^2 \int_{\partial \Omega} v \cdot D\nu \cdot v \, dS. \]

An application of (2.1) and (2.16) yields

\[ v^{\tau} \cdot \nabla^{\tau} (v \cdot \nu) - v^{\tau} \cdot D^{\tau} v \cdot \nu - v \cdot D\nu \cdot v = v^{\tau} \cdot D^{\tau} v \cdot v - v \cdot D\nu \cdot v \]

\[ = -(v \cdot \nu) v \cdot D\nu \cdot v = 0. \]

Thus, with (4.8) we proved the following lemma.

**Lemma 4** Let \( u' \) be the shape derivative of \( u \) resulting from a volume preserving perturbation of \( \Omega \). Then there holds

\[ \frac{\ddot{\Lambda}(0)}{2} = \mathcal{E}(u'), \]

where

\[ \mathcal{E}(u') = \int_{\Omega} |\Delta u'|^2 \, dx - \Lambda(\Omega) \int_{\Omega} |\nabla u'|^2 \, dx. \]
5 Minimization of the second domain variation

In this chapter we consider the quadratic functional

\[ E(\varphi) := \int_{\Omega} |\Delta \varphi|^2 \, dx - \Lambda(\Omega) \int_{\Omega} |\nabla \varphi|^2 \, dx \]

for \( \varphi \in H^{1,2}_0 \cap H^{2,2}(\Omega) \). It will be convenient to work with an alternative representation of \( E \).

For \( \varphi \in H^{1,2}_0 \cap H^{2,2}(\Omega) \) there holds

\[ E(\varphi) = \int_{\Omega} |D^2 \varphi|^2 \, dx - \Lambda(\Omega)|\nabla \varphi|^2 \, dx + \int_{\partial \Omega} \Delta \varphi \partial_{\nu} \varphi - \varphi \cdot D^2 \varphi \cdot \nu \, dS. \]

We apply (2.8) and (2.1).

\[ \Delta \varphi \partial_{\nu} \varphi - \varphi \cdot D^2 \varphi \cdot \nu = \partial^2_{\nu} \varphi \partial_{\nu} \varphi + (n-1)(\partial_{\nu} \varphi)^2 H_{\partial \Omega} - \varphi \cdot D^2 \varphi \cdot \nu \]

\[ = \nu \cdot D^2 \varphi \cdot \nu (\nu \cdot \nabla \varphi) + (n-1)(\partial_{\nu} \varphi)^2 H_{\partial \Omega} - \varphi \cdot D^2 \varphi \cdot \nu \]

\[ = (n-1)(\partial_{\nu} \varphi)^2 H_{\partial \Omega}. \]

Consequently, we get

\[ E(\varphi) = \int_{\Omega} |D^2 \varphi|^2 \, dx - \Lambda(\Omega)|\nabla \varphi|^2 \, dx + \int_{\partial \Omega} (\partial_{\nu} \varphi)^2 H_{\partial \Omega} \, dS. \]

(5.2)

Remark 5 The functional \( E \) is lower semicontinuous with respect to weak convergence in \( H^{1,2}_0 \cap H^{2,2}(\Omega) \).

Since \( \Omega \) is optimal, we know from Lemma 4 that

\[ E(\varphi) \geq 0 \]

for all \( \varphi \) which are shape derivatives of \( u \). Recall that \( \varphi \) is a shape derivative, if it solves (1.6) - (1.9) for some vector field \( v \) in the class described in Chapter 1 (Lemma 2).

The following remark shows a property of shape derivatives we have not yet mentioned.

Remark 6 Let \( \varphi \) be a shape derivative and assume that \( \partial_{\nu} \varphi \equiv 0 \) in \( \partial \Omega \). Then \( \varphi \in H^{2,2}_0(\Omega) \) and, since \( \varphi \) satisfies equation (4.2), \( \varphi \) is a buckling eigenfunction in \( \Omega \). Thus by uniqueness of \( u \) we get \( \varphi = \alpha u \) for any \( \alpha \in \mathbb{R} \). Then formula (1.4) yields

\[ \Lambda(\Omega) = \int_{\partial \Omega} |\Delta \varphi|^2 x \cdot \nu \, dS = \alpha^2 c_0^2 \int_{\partial \Omega} x \cdot \nu \, dS = \alpha^2 \int_{\partial \Omega} |\Delta u|^2 x \cdot \nu \, dS = \alpha^2 \Lambda(\Omega). \]
Thus, \( \alpha^2 = 1 \) and there holds

\[ \left| \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx \right| = 1. \]

This is contradictory to (4.3) and thus \( \partial_{\nu} \varphi \) cannot vanish identically on \( \partial \Omega \).

This motivates the following definition.

\[ Z := \left\{ \varphi \in H^{1,2}_0(\Omega) \cap H^{2,2}(\Omega) : \int_{\partial \Omega} \partial_{\nu} \varphi \, dS = 0, \int_{\partial \Omega} (\partial_{\nu} \varphi)^2 \, dS > 0, \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = 0 \right\}. \]

Note that \( Z \) contains elements which are not shape derivatives. Nevertheless we will show that

\[ \mathcal{E}|_Z \geq 0. \]

The next lemma ensures that \( Z \) is not empty and that at least for a specific shape derivative \( \mathcal{E} \) is equal to zero.

**Lemma 5** For each \( 1 \leq k \leq n \) the directional derivative \( \partial_k u \) satisfies \( \partial_k u \in Z \). Furthermore, \( \mathcal{E}(\partial_k u) = 0 \).

**Proof** Let \( 1 \leq k \leq n \). Due to (1.2) and (1.3) \( \partial_k u \) satisfies

\[ \Delta^2 \partial_k u + \Lambda(\Omega) \Delta \partial_k u = 0 \quad \text{in } \Omega \]
\[ \partial_k u = 0 \quad \text{in } \partial \Omega. \]

According to (2.9) there holds \( \partial_{\nu} \partial_k u = c_0 \nu_k \) on \( \partial \Omega \). Hence,

\[ \int_{\partial \Omega} \partial_{\nu} \partial_k u \, dS = c_0 \int_{\partial \Omega} \nu_k \, dS = 0. \]

In addition, we find that

\[ \int_{\Omega} \nabla u \cdot \nabla \partial_k u \, dx = \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 \nu_k \, dS = 0. \]

Following the idea of Remark 6, we obtain that \( \partial_{\nu} \partial_k u \) does not vanish identically on \( \partial \Omega \). Thus, \( \partial_k u \in Z \). Moreover, (3.9) and (5.3) imply

\[ \mathcal{E}(\partial_k u) = \int_{\Omega} (\Delta^2 \partial_k u + \Lambda(\Omega) \Delta \partial_k u) \partial_k u \, dx + \int_{\partial \Omega} \partial_k \Delta u \partial_{\nu} \partial_k u \, dS = 0. \]

This proves the lemma. \( \square \)

Note that each directional derivative of \( u \) is a shape derivative resulting from translations of \( \Omega \).
Theorem 4 The infimum of the functional $E$ in $Z$ is finite.

Proof We argue by contradiction. Let us assume that $\inf_Z E = -\infty$ and consider a sequence $(\hat{w}_k)_k \subset Z$ such that

$$\lim_{k \to \infty} E(\hat{w}_k) = -\infty.$$ 

For this sequence there either holds

$$\int_{\partial \Omega} (\partial_\nu \hat{w}_k)^2 dS \xrightarrow{k \to \infty} 0 \quad \text{or} \quad \int_{\partial \Omega} (\partial_\nu \hat{w}_k)^2 dS \xrightarrow{k \to \infty} 0.$$ 

If the second case holds true, we normalize the sequence $(\hat{w}_k)_k$ such that $||\partial_\nu \hat{w}_k||_{L^2(\partial \Omega)} = 1$. Hence, in either case, for each $\hat{w}_k$ there holds $||\partial_\nu \hat{w}_k||_{L^2(\partial \Omega)} \leq 1$. Thus, (2.2) gives

$$\left| \int_{\partial \Omega} H_{\partial \Omega} (\partial_\nu \hat{w}_k)^2 dS \right| \leq \max_{\partial \Omega} |H_{\partial \Omega}| < \infty.$$ 

We use (5.2) and obtain

(5.4) \hfill $E(\hat{w}_k) \geq -\Lambda(0) \int_\Omega |\nabla \hat{w}_k| \, dx - (n - 1) \max_{\partial \Omega} |H_{\partial \Omega}|.$ \hfill 

The assumption $\lim_{k \to \infty} E(\hat{w}_k) = -\infty$ implies

$$\int_\Omega |\nabla \hat{w}_k|^2 \, dx \xrightarrow{k \to \infty} \infty.$$ 

We define

$$w_k := \frac{1}{||\nabla \hat{w}_k||_{L^2(\Omega)}} \hat{w}_k.$$ 

Then there holds

(5.5) \hfill $||\nabla w_k||_{L^2(\Omega)} = 1$ and $\int_{\partial \Omega} (\partial_\nu w_k)^2 \, dS \xrightarrow{k \to \infty} 0.$ \hfill 

Moreover, for each $k \in \mathbb{N}$ estimate (5.4) implies

$$E(w_k) \geq -\Lambda(0) - C$$ 

and the infimum of $E$ in $M := \{ w_k : k \in \mathbb{N} \}$ is finite. Therefore, we can choose a subsequence of $(w_k)_k$, denote by $(w_k)_k$ as well, such that

$$\lim_{k \to \infty} E(w_k) = \inf_M E.$$
Now Poincaré’s inequality and the previous estimates imply
\[
||w_k||^2_{H^2,2(\Omega)} = \int_\Omega |D^2 w_k|^2 + |\nabla w_k|^2 + w_k^2 \, dx \\
\leq \mathcal{E}(w_k) + C \int_\Omega |\nabla w_k|^2 \, dx + (n-1) \int_{\partial\Omega} |H_{\partial\Omega}(\partial_\nu w_k)|^2 \, dS \\
\leq C.
\]
Thus, the sequence \((w_k)_k\) is uniformly bounded in \(H^{2,2}(\Omega)\) and there exists a \(w \in H^{2,2}(\Omega)\) such that \((w_k)_k\) weakly converges to \(w\). In view of (5.5), the limit function \(w\) satisfies \(||\nabla w||_{L^2(\Omega)} = 1\) and \(\partial_\nu w = 0\) on \(\partial\Omega\). Since \(w_k = 0\) in \(\partial\Omega\) for each \(k \in \mathbb{N}\), we conclude that \(w \in H_0^{2,2}(\Omega)\).

Now let us recall that \(\mathcal{E}(\hat{w}_k)\) converges to \(-\infty\). Thus there exists a \(k_0 \in \mathbb{N}\) such that
\[
\mathcal{E}(\hat{w}_k) < 0 \quad \text{for all} \quad k \geq k_0.
\]
Since the functional \(\mathcal{E}\) is lower semicontinuous with respect to weak convergence in \(H^{2,2}(\Omega)\), we find that \(\mathcal{E}(w) < 0\). According to the definition of \(\mathcal{E}\) in (5.1), this immediately leads to
\[
\frac{\int_\Omega |\Delta w|^2 \, dx}{\int_\Omega |\nabla w|^2 \, dx} < \Lambda(\Omega).
\]
Since \(w \in H_0^{2,2}(\Omega)\) this is contradictory to the minimum property of \(\Lambda(\Omega)\). \(\square\)

As mentioned in the previous proof, a minimizing sequence \((\varphi_k)_k \subset \mathcal{Z}\) satisfies one of the following two conditions
\[
i) \int_{\partial\Omega} (\partial_\nu \varphi_k)^2 \, dS \xrightarrow{k \to \infty} 0 \\
ii) \int_{\partial\Omega} (\partial_\nu \varphi_k)^2 \, dS \xrightarrow{k \to \infty} 0.
\]
In the sequel, we show that the case i) implies that the minimizing sequence \((\varphi_k)_k\) converges to zero. For this purpose, let \((\varphi_k)_k \subset \mathcal{Z}\) be a minimizing sequence which satisfies condition i).

From (5.2) we get
\[
||\varphi_k||^2_{H^2,2(\Omega)} \leq C,
\]
and thus there exists a \(\varphi \in H^{2,2}(\Omega)\) such that \(\varphi_k\) weakly converges to \(\varphi\) in \(H^{2,2}(\Omega)\) and \(\mathcal{E}(\varphi) = \inf_\mathcal{Z} \mathcal{E}\). Furthermore, condition i) implies \(\varphi \in H_0^{2,2}(\Omega)\). From Lemma 5 we obtain
\[
\inf_\mathcal{Z} \mathcal{E} = \mathcal{E}(\varphi) \leq \mathcal{E}(\partial_\nu u) = 0 \quad \text{for any} \quad 1 \leq l \leq n.
\]
Hence,
\[ \int_{\Omega} |\Delta \varphi|^2 \, dx \leq \int_{\Omega} |\nabla \varphi|^2 \, dx \leq \Lambda(\Omega). \]

Thus \( \varphi \) is necessarily an eigenfunction corresponding to \( \Lambda(\Omega) \). Since the eigenvalue is simple we have \( \varphi = \alpha u \) for \( \alpha \in \mathbb{R} \). Now let us recall that \( \varphi_k \in \mathcal{Z} \) and, therefore,
\[ \alpha = \alpha \int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \lim_{k \to \infty} \int_{\Omega} \nabla u \cdot \nabla \varphi_k \, dx = 0. \]

Consequently, \( \alpha = 0 \) and \( \varphi \equiv 0 \) in \( \Omega \). Hence \( \varphi \not\in \mathcal{Z} \). Since we are interested to find minimizers of \( \mathcal{E} \) in \( \mathcal{Z} \), we restrict ourselves to minimizing sequences which satisfy the condition ii). Thus we consider the functional
\[ \tilde{\mathcal{E}}(\varphi) := \frac{\mathcal{E}(\varphi)}{\int_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS}, \]
where \( \varphi \in \mathcal{Z} \) and we set \( \tilde{\mathcal{E}} = \infty \) if \( \int_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS = 0 \).

**Remark 7** Suppose \( (\varphi_k)_k \subset \mathcal{Z} \) is a minimizing sequence for \( \tilde{\mathcal{E}} \) in \( \mathcal{Z} \). Then there exists a constant \( C > 0 \) such that \( ||\nabla \varphi_k||_{L^2(\Omega)} \leq C \) for every \( k \in \mathbb{N} \). This follows by contradiction. Otherwise we may assume that \( ||\nabla \varphi_k||_{L^2(\Omega)} \) tends to infinity as \( k \to \infty \), we define \( \varphi_k^* := ||\nabla \varphi_k||_{L^2(\Omega)}^{-1} \varphi_k \). Then \( (\varphi_k^*)_k \) is uniformly bounded in \( H^{2,2}(\Omega) \) and
\[ \int_{\partial\Omega} (\partial_\nu \varphi_k^*)^2 \, dS \xrightarrow{k \to \infty} 0. \]

Thus, \( (\varphi_k^*)_k \) converges weakly to a function \( \varphi \in H^{2,2}_0(\Omega) \) and for every \( 1 \leq l \leq n \) there holds
\[ \inf_{\mathcal{Z}} \tilde{\mathcal{E}} = \tilde{\mathcal{E}}(\varphi) \leq \tilde{\mathcal{E}}(\partial_l u) = 0. \]

As the previous considerations have shown, this implies \( \varphi \equiv 0 \) in \( \Omega \). Thus, our assumption cannot be true.

We now consider a minimizing sequence \( (\varphi_k)_k \subset \mathcal{Z} \) which satisfies
\[ (5.6) \quad \int_{\partial\Omega} (\partial_\nu \varphi_k)^2 \, dS = 1 \]
for all \( k \in \mathbb{N} \). As before we obtain the inequality
\[ ||\varphi_k||_{H^{2,2}(\Omega)}^2 \leq \mathcal{E}(\varphi_k) + C \int_{\Omega} |\nabla \varphi_k|^2 \, dx. \]
Thus, \((\varphi_k)_k\) is uniformly bounded in \(H^{2,2}(\Omega)\) and \(\varphi_k\) converges weakly to a \(\varphi^* \in H^{2,2}(\Omega)\). We find that \(\varphi^* \in Z\) and \(\mathcal{E}(\varphi^*) = \inf_Z E\). In addition, there holds
\[
\int_{\partial\Omega} (\partial_\nu \varphi^*)^2 dS = 1.
\]
Hence, \(\varphi^*\) minimizes \(\tilde{E}\) in \(Z\). Suppose \(\theta \in Z\), then the minimality of \(\varphi^*\) implies
\[
\frac{d}{dt} \left. \frac{\mathcal{E}(\varphi^* + t\theta)}{\int_{\partial\Omega} (\partial_\nu (\varphi^* + t\theta))^2 dS} \right|_{t=0} = 0
\]
and we obtain
\[
\int_\Omega [\Delta^2 \varphi^* + \Lambda(\Omega) \Delta \varphi^*] \theta \, dx - \int_{\partial\Omega} [\Delta \varphi^* + \rho \partial_\nu \varphi^*] \partial_\nu \theta \, dS = 0.
\]
Since \(\theta \in Z\) was chosen arbitrary, \(\varphi^*\) satisfies the Euler-Lagrange equalities
\[
\begin{align*}
\Delta^2 \varphi^* + \Lambda(\Omega) \Delta \varphi^* &= 0 \quad \text{in} \ \Omega \\
\Delta \varphi^* + \rho \partial_\nu \varphi^* &= \text{const.} \quad \text{in} \ \partial\Omega,
\end{align*}
\]
where \(\rho := \min_Z \tilde{E}\). The following theorem collects the previous results.

**Theorem 5** There exists a function \(\varphi^* \in Z\) such that \(\tilde{E}(\varphi^*) = \min_Z \tilde{E}\). Furthermore, any minimizer \(\varphi^* \in Z\) satisfies
\[
\begin{align*}
(5.7) \quad &\Delta^2 \varphi^* + \Lambda(\Omega) \Delta \varphi^* = 0 \quad \text{in} \ \Omega \\
(5.8) \quad &\Delta \varphi^* + \rho \partial_\nu \varphi^* = \text{const.} \quad \text{in} \ \partial\Omega \\
&\varphi^* = 0 \quad \text{in} \ \partial\Omega,
\end{align*}
\]
where \(\rho := \min_Z \tilde{E}\).

The next theorem shows that in fact \(\rho = 0\).

**Theorem 6** Suppose \(\varphi^* \in Z\) is a minimizer of \(\tilde{E}\). Then there holds \(\tilde{E}(\varphi^*) = 0\). In particular, \(\mathcal{E} \geq 0\) in \(Z\).

**Proof** Let \(\varphi^* \in Z\) be a minimizer of \(\tilde{E}\). Since \(\varphi^*\) satisfies equation (5.7) and \(\partial\Omega\) is smooth, \(\varphi^*\) is a smooth function on \(\Omega\). Hence, we may define a volume preserving perturbation \(\Phi_t\) of \(\Omega\) such that
\[
\partial_\nu u'(x) = \partial_\nu \varphi^*(x) \quad \text{for} \quad x \in \partial\Omega.
\]
Note that this can be achieved by setting \(v = c_0^{-1} \nabla \varphi^*\) in \(\partial\Omega\). In this way, each minimizer \(\varphi^*\) implies the existence of vector fields \(v\) and \(w\) in the sense of Section 2. We define \(\psi := u' - \varphi^*\), then \(\psi \in H^{2,2}_0(\Omega)\) and
\[
\Delta^2 \psi + \Lambda(\Omega) \Delta \psi = 0 \quad \text{in} \ \Omega.
\]
The uniqueness of \( u \) implies \( \psi = \alpha u \) for an \( \alpha \in \mathbb{R} \). Since \( \varphi^* \in \mathcal{Z} \), equation (4.3) yields

\[
0 = \int_\Omega \nabla u \cdot \nabla u' \, dx - \int_\Omega \nabla u \cdot \nabla \varphi^* \, dx = \int_\Omega \nabla u \cdot \nabla \psi \, dx = \alpha.
\]

Consequently, \( u' \equiv \varphi^* \). Thus \( \varphi^* \) is a shape derivative. Since \( \Omega \) is optimal \( \tilde{\mathcal{E}}(\varphi^*) \geq 0 \). Finally we apply Lemma 5. This gives

\[
0 \leq \tilde{\mathcal{E}}(\varphi^*) = \min_{\mathcal{Z}} \tilde{\mathcal{E}} \leq \tilde{\mathcal{E}}(\partial_k u) = 0.
\]

\[\square\]

6 The optimal domain is a ball

We will use an inequality due to L.E. Payne to show that the optimal domain \( \Omega \) is a ball. Payne’s inequality (see [13]) states that for each domain \( G \) there holds

\[
\lambda_2(G) \leq \Lambda(G)
\]

and equality only holds if and only if \( G \) is a ball. Thereby \( \lambda_2 \) denotes the second Dirichlet eigenfunction of the Laplacian. In the sequel, we construct a suitable function \( \psi \in \mathcal{Z} \) such that the condition \( \mathcal{E}(\psi) \geq 0 \) (due to Theorem 6) will imply that the optimal domain \( \Omega \) is a ball. For this purpose, we denote by \( u_1 \) and \( u_2 \) the first and the second Dirichlet eigenfunction for the Laplacian in \( \Omega \). Thus, for \( 1 \leq k \leq 2 \) there holds

\[
\Delta u_k + \lambda_k(\Omega) u_k = 0 \quad \text{in } \Omega
\]

\[
u_k = 0 \quad \text{in } \partial \Omega,
\]

where \( \lambda_k(\Omega) \) is the \( k \)-th Dirichlet eigenvalue for the Laplacian in \( \Omega \). Note that \( 0 < \lambda_1(\Omega) < \lambda_2(\Omega) \). For the sake of brevity, we will write \( \lambda_k \) instead of \( \lambda_k(\Omega) \) and \( \Lambda \) instead of \( \Lambda(\Omega) \). In addition, we assume \( ||u_k||_{L^2(\Omega)} = 1 \) and

\[
\int_\Omega u_1 u_2 \, dx = 0.
\]

Without loss of generality, we may assume that

\[
\int_\Omega u_1 \, dx > 0 \quad \text{and} \quad \int_\Omega u_2 \, dx \leq 0.
\]

Consequently, there exists a \( t \in (0, 1] \) such that

\[
(6.1) \quad \int_\Omega (1 - t) \lambda_1 u_1 + t \lambda_2 u_2 \, dx = 0.
\]
This fixes \( t \). Next we define
\[
\psi(x) := (1 - t) u_1(x) + t u_2(x) + c u(x) \quad \text{for } x \in \Omega,
\]
where \( u \) is the first buckling eigenfunction in \( \Omega \). The constant \( c \) is given by
\[
c := -\frac{1}{\Lambda} \int_{\Omega} (1 - t) \lambda_1 \nabla u. \nabla u_1 + t \lambda_2 \nabla u. \nabla u_2 \, dx.
\]

In a first step we show that \( \psi \in Z \). Note that \( \psi \in H^{1,2} \cap H^{2,2}(\Omega) \). Moreover the definition of \( \psi \), the fact that \( \partial \nu \nabla u = 0 \) on \( \partial \Omega \), the equations for \( u_1 \) and \( u_2 \), and (6.1) imply
\[
\int_{\partial \Omega} \partial \nu \psi \, dS = \int_{\Omega} (1 - t) \Delta u_1 + t \Delta u_2 \, dx = -\int_{\Omega} (1 - t) \lambda_1 u_1 + t \lambda_2 u_2 \, dx = 0.
\]

By the unique continuation principle \( \partial \nu \psi \) does not vanish identically in \( \partial \Omega \). Thus, to show that \( \psi \in Z \), it remains to prove that
\[
\int_{\Omega} \nabla u. \nabla \psi \, dx = 0.
\]

We recall that \( \Delta u = c_0 \) in \( \partial \Omega \). Hence
\[
0 = \int_{\Omega} (\Delta^2 u + \Lambda \Delta u) \psi \, dx = \int_{\Omega} \Delta u \Delta \psi \, dx - \Lambda \int_{\Omega} \nabla u. \nabla \psi \, dx
\]
\[
= -\int_{\Omega} [(1 - t) \lambda_1 u_1 + t \lambda_2 u_2] \Delta u \, dx + c \int_{\Omega} |\Delta u|^2 \, dx - \Lambda \int_{\Omega} \nabla u. \nabla \psi \, dx.
\]

Since \( ||\nabla u||_{L^2(\Omega)} = 1 \), the second integral is equal to \( \Lambda \). Thus, the definition of \( c \) implies (6.2).

Note that \( \psi \) is not a shape derivative since it fails to satisfy (4.2) - unless \( t = 1 \) and \( \Omega \) equals a ball. However, \( \psi \in Z \) and, according to Theorem 6, there holds \( \mathcal{E}(\psi) \geq 0 \). Consequently, \( \mathcal{E}(\psi) \geq 0 \). Thus
\[
\mathcal{E}(\psi) = \int_{\Omega} |\Delta \psi|^2 - \Lambda |\nabla \psi|^2 \, dx
\]
\[
= (1 - t)^2 \lambda_1 (\lambda_1 - \Lambda) + t^2 \lambda_2 (\lambda_2 - \Lambda) + 2 c c_0 \int_{\Omega} (1 - t) \lambda_1 u_1 + t \lambda_2 u_2 \, dx
\]
\[
\overset{(6.1)}{=} (1 - t)^2 \lambda_1 (\lambda_1 - \Lambda) + t^2 \lambda_2 (\lambda_2 - \Lambda) \geq 0.
\]

Since \( \lambda_1 - \Lambda < 0 \) and \( \lambda_2 - \Lambda \leq 0 \), both summands in \( \mathcal{E}(\psi) \) have to vanish. Consequently \( t = 1 \) and \( \lambda_2(\Omega) = \Lambda(\Omega) \). Payne’s inequality implies that \( \Omega \) is a ball. This proves the main theorem of the paper.

**Theorem 7** Let \( \Omega \) be a bounded, smooth and simply connected domain in \( \mathbb{R}^n \), which minimizes the first buckling eigenvalue among all bounded, smooth and simply connected domains in \( \mathbb{R}^n \) with given measure. Then \( \Omega \) is a ball.
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