GENERALIZING BROUWER:
ADDING POINTS TO CONFIGURATIONS IN CLOSED BALLS

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Abstract. We answer the question of when a new point can be added in a continuous way to a given configuration of \(n\) distinct points in a closed ball of arbitrary dimension. We show that this is possible given an ordered configuration of \(n\) points if and only if \(n \neq 1\). On the other hand, when the points are not ordered and the dimension of the ball is at least 2, a point can be added continuously if and only if \(n = 2\).

1. Introduction

Let \(B^m\) be the closed ball of dimension \(m\), with \(m \geq 1\). This paper answers the following basic question:

Given \(n\) distinct points in \(B^m\), when can a new point be added in a continuous fashion?

The challenge here is that the new point must be distinct from all of the existing ones, and it is not clear whether such a choice can be made continuously. In the special case of \(n = 1\), the resolution of our question is exactly Brouwer fixed-point theorem \([\text{Bro11}]\).

Example 1 (Case \(n = 1\): Brouwer fixed-point theorem). Given one point in \(B^m\), a continuous choice of a second distinct point can be thought of as a continuous function from the closed ball to itself with no fixed points. By Brouwer fixed-point theorem no such continuous function exists, and therefore introducing a second distinct point continuously is impossible.

Extending Brouwer fixed-point theorem to \(n > 1\), the question diverges into two versions: either the \(n\) points are given with an ordering \((p_1, \ldots, p_n)\), or the points are instead unordered. The first author addressed both versions of this question with respect to point configurations lying in the infinite plane, in the sphere, and in closed surfaces \(S_g\) with \(g \geq 2\) in \([\text{Che17}]\) and her joint work with Salter \([\text{CS17}]\).

It is perhaps surprising that having more points to avoid when adding a point does not necessarily make the task harder. When \(n = 2\), there is in fact a way to introduce a new point in both the ordered and unordered versions of the problem.

Example 2 (Case \(n = 2\): midpoint). Given two distinct points in the closed ball, their midpoint provides a continuous way to introduce a distinct third point.

We now state more formally the problem of adding a point continuously to a configuration of \(n\) points in \(B^m\), which from this point on will be abbreviated to \(\mathbb{B}\). Let \(\text{PConf}_n(\mathbb{B})\) denote the pure configuration space of \(n\) distinct ordered points in \(\mathbb{B}\), topologized as a subspace of \((\mathbb{B})^n\). A continuous map \(\text{PConf}_n(\mathbb{B}) \to \mathbb{B}\) is said to ‘add a point’ if the image of every configuration \((p_1, \ldots, p_n)\) is a point \(p_0\) distinct from all \(p_i\) with \(i \geq 1\). Equivalently, consider the map \(g_{m,n} : \text{PConf}_{n+1}(\mathbb{B}) \to \text{PConf}_n(\mathbb{B})\) that forgets the 0th point \((p_0, p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_n)\). The problem of continuously introducing a new point is precisely the question of finding a section for \(g_{m,n}\).

Next, the symmetric group \(\Sigma_n\) acts on ordered configurations by permuting the indices, and the quotient \(\text{Conf}_n(\mathbb{B}) := \text{PConf}_n(\mathbb{B})/\Sigma_n\) is the configuration space of \(n\) distinct unordered points. The map \(g_{m,n}\) forgetting the point \(p_0\) is \(\Sigma_n\)-equivariant and descends to a map \(\bar{g}_{m,n} : \text{PConf}_{n+1}(\mathbb{B})/\Sigma_n \to \text{Conf}_n(\mathbb{B})\). With
this, our problem of adding a point distinct from a given unordered set of \( n \) points is the problem of finding a section for \( \bar{g}_{m,n} \).

The easy positive result for \( n = 2 \) in Example 2 suggests that the question for larger \( n \) might also be possible. We might be able to use more points to cook up some sections geometrically. However, this is when the two versions of the problem diverge: the ordered version extends the \( n = 2 \) case, while the unordered version reverts back to the behavior at \( n = 1 \).

**Theorem A (Ordered).** For \( m \geq 1 \) and \( n \geq 1 \), the forgetful map \( g_{m,n} \) has a section if and only if \( n \neq 1 \). That is, one can continuously add a new point to an ordered configuration exactly when it consists of \( 2 \) or more ordered points.

**Theorem B (Unordered).** For \( m \geq 2 \) and \( n \geq 1 \), the forgetful map \( \bar{g}_{m,n} \) has a section if and only if \( n = 2 \). That is, one cannot continuously add a new point to an unordered configuration unless it consists of precisely \( 2 \) unordered points.

Note that the case of points on a line segment (\( m = 1 \)) is excluded from Theorem B. In this case, the unordered version coincides with the ordered one, as the points in an unordered configuration are nevertheless forced into a linear order. We may therefore add a point continuously to a configuration of \( n \) points in \( \mathbb{B}^1 \) as long as \( n \neq 1 \).

Recasting Theorem B as a direct generalization of Brouwer fixed-point theorem, we claim that except for when \( n = 2 \), every continuous map \( f : \text{Conf}_n(\mathbb{B}) \to \mathbb{B} \) has a ‘fixed point’. This is meant in the sense that there must exist some configuration \( S = \{p_1, \ldots, p_n\} \) whose image under \( f \) lies inside \( S \).

The negative results in Theorem B contrasts with the fact that the problem of introducing a new point to an unordered configuration of points in \( \mathbb{R}^m \) (or equivalently, on the open ball) always has a solution: adding a point ‘from infinity’, i.e. placing it very far away from all the others. This construction is of course not possible on the closed ball. However, since the configuration spaces of the open and closed balls are homotopy equivalent, there is no purely homotopy theoretic obstruction to finding a section in the case of a closed ball. This means that a different approach is needed. Even more, many standard tools of algebraic topology fail in our context since the forgetful map \( g_{m,n} \) is not a fibration: it has fibers of distinct homotopy types, depending on how many points lie on the boundary of the ball.

The main argument of our proof of Theorem B proceeds by contradiction. Any section of \( \bar{g}_{m,n} \) induces an \( \Sigma_n \)-equivariant section of \( g_{m,n} : \text{PConf}_{n+1}(\mathbb{B}) \to \text{PConf}_n(\mathbb{B}) \). We shall give a cohomological classification of such sections, and then arrive at a contradiction by pulling back a cohomology class in two different ways.

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### 2. Case of ordered configurations

In this section we prove Theorem A. We use a generalization of the midpoint construction of Example 2 to build a section of \( g_{m,n} \) for all \( n \geq 2 \).

**Proof of Theorem A.** Let \( m \geq 1 \). The \( n = 1 \) case is covered by Example 1. Otherwise, fix \( n \geq 2 \). For any pair \((i, j)\) with \( 1 \leq i, j \leq n \) and \( i \neq j \), the line segment \([p_i, p_j]\) is contained within \( \mathbb{B} \). Let \( v_{ij} := p_j - p_i \) be the vector pointing from \( p_i \) to \( p_j \). Further, let \( d_i := \min \{ \|p_k - p_i\| : k \neq i\} \) be the minimal distance between \( p_i \) and any other point in the configuration. We may then introduce a new point “close” to \( p_i \), lying at a distance \( d_i/2 \) from \( p_i \) along the interval \([p_i, p_j]\): that is, \( p_0 := p_i + \frac{d_i}{2\|v_{ij}\|}v_{ij} \). The added point \( p_0 \) is clearly distinct from all other points, and it depends on the configuration continuously since \( d_i \) and \( v_{ij} \) do. \( \square \)
For notational convenience, denote \( a \) cohomological classification of sections of the forgetful map \( \text{PConf}_{n+1}(\mathbb{R}^m) \to \text{PConf}_n(\mathbb{R}^m) \) that are equivariant under the action of the symmetric group \( \Sigma_n \).

The cohomology ring \( H^*(\text{Conf}_n(\mathbb{R}^m); \mathbb{Z}) \) is generated by classes \( G_{ab} \in H^{m-1}(\text{Conf}_n(\mathbb{R}^m); \mathbb{Z}) \) for distinct \( 1 \leq a, b \leq n \). Explicitly, the generator \( G_{ab} \) measures the winding of point \( a \) around point \( b \). These generators are subject to the following relations (see, e.g., Totaro \cite{Tot96} and the references therein):

\[
G_{ab} = (-1)^m G_{ba} \quad (1) \\
G_{ab}^2 = 0 \quad (2) \\
G_{ab}G_{ac} + G_{bc}G_{ba} + G_{ca}G_{cb} = 0 \quad \text{for all distinct } a, b \text{ and } c. \quad (3)
\]

The induced \( \Sigma_n \)-action is given by \( \sigma^*(G_{ab}) = G_{\sigma(a)\sigma(b)} \). In particular, we have that the \( \Sigma_n \)-representation \( H^{m-1}(\text{Conf}_n(\mathbb{R}^m); \mathbb{Z}) = \langle G_{ab} \mid a < b \rangle \) is

- \( A^2(\mathbb{Z}^n) \cong V_{(n-1,1)} \oplus V_{(n-2,1,1)} \) when \( m \) is odd, and
- \( \text{Sym}^2(\mathbb{Z}^n)/\mathbb{Z}^n \cong V_{(n)} \oplus V_{(n-1,1)} \oplus V_{(n-2,2)} \) when \( m \) is even.

For notational convenience, denote \( G_i := \sum_{a \neq i} G_{ia} \) and let \( G := \sum_{a \neq b} G_{ab} \). Note that \( \langle G_1, \ldots, G_n \rangle \) is the standard subrepresentation \( V_{(n-1,1)} \) along with a trivial subrepresentation \( \langle G \rangle \) when \( m \) is even. When \( m \) is odd we have \( G = 0 \). By abuse, we also use \( G_{ij} \) and \( G_i \) to denote the corresponding classes in \( H^{m-1}(\text{PConf}_{n+1}(\mathbb{R}^m); \mathbb{Z}) \), where now the indexes range over \( 0 \leq i, j \leq n \). In particular \( G_0 := \sum_{a > 0} G_{0a} \).

When \( m \) is even we shall use the notation \( \overline{G}_i := \frac{1}{2} G - G_i = \sum_{a < b \neq i} G_{ab} \) for the class that has no instances of \( i \).

If \( s : \text{PConf}_n(\mathbb{R}^m) \to \text{PConf}_{n+1}(\mathbb{R}^m) \) is a section of the forgetful map \( g_{m,n} \), then it follows that \( s^*(G_{ab}) = G_{ab} \) for all \( 1 \leq a, b \leq n \). Thus the map that \( s \) induced on cohomology is determined by its values on the classes \( G_{0a} \) for \( 1 \leq a \).

**Lemma 3.1 (Cohomological classification of equivariant sections).** For \( m \geq 2 \) and \( n \geq 3 \), let \( s : \text{PConf}_n(\mathbb{R}^m) \to \text{PConf}_{n+1}(\mathbb{R}^m) \) be a \( \Sigma_n \)-equivariant section of the forgetful map \( g_{m,n} \). Then at the level of cohomology we have the following equations for all \( i \geq 1 \).

- If \( m \) is odd, then
  \[
s^*(G_{0i}) = 0.
  \]
- If \( m \) is even, there exists an integer \( \mu \in \mathbb{Z} \) such that either
  \[
s^*(G_{0i}) = \frac{\mu}{2} G = \mu(G_i + \overline{G}_i)
  \]
  or otherwise, \( \mu = 2k + 1 \) is odd and
  \[
s^*(G_{0i}) = (-k)G_i + \mu\overline{G}_i
  \]
  Moreover, if \( n \geq 4 \) then the latter case has \( \mu = -1 \), i.e. \( s^*(G_{0i}) = G_i - \overline{G}_i \).
Proof. As a representation of $\Sigma_n$, the classes $G_{01}, \ldots, G_{0n}$ generate the permutation representation $\mathbb{Z}^n \cong V(a) \oplus V(n-1,1)$, with invariant class $G_0$. Since there are no non-zero homomorphisms between non-isomorphic irreducible representations, it follows that the section

$$s^* : \langle G_{01}, \ldots, G_{0n} \rangle \to \langle G_{ab} \mid a < b \rangle$$

must land in the subrepresentation $(G_1, \ldots, G_n, \frac{1}{2}G)$.

Next, since any endomorphism of an irreducible representation over an algebraically closed field is scalar, it follows that $s^*(G_{0i}) = \lambda G_i + \nu G$ for some scalars $\lambda, \nu \in \overline{\mathbb{Q}}$. But since $s^*$ is defined integrally, it follows that the coefficients of $G_{ia}$ and $G_{ab}$ in $s^*(G_{0i})$ are integers. From this it follows that $\lambda_i(1+(-1)^m)\nu \in \mathbb{Z}$. When $m$ is even this implies $\nu \in \frac{1}{2}\mathbb{Z}$. Let us remark that the same characterization of $s^*(G_{0i})$ can be derived in an elementary way by comparing the coefficient of $G_{ab}G_{cd}$ in the equalities $s^*(G_{0\sigma(i)}) = \sigma \cdot s^*(G_{0i})$ for various $\sigma \in \Sigma_n$.

When $m$ is odd one gets $s^*(G_{0i}) = \lambda G_i$ with $\lambda \in \mathbb{Z}$, since in this case we have $G = 0$. To see that $\lambda$ must in fact vanish in this case, consider the relation $G_{0i}G_{0j} + G_{ij}G_{i0} + G_{j0}G_{ji} = 0$ for $i \neq j$. The map $s^*$ is a ring homomorphism, the $G_{ab}$’s commute and $G_{ab} = -G_{ba}$, so we have

$$0 = s^*(G_{0i}G_{0j} + G_{ij}G_{i0} + G_{j0}G_{ji}) = \lambda^2 G_i G_j - \lambda G_{ij}(G_i + G_j)$$

Let us read off some of the coefficients of this expression. Since $n \geq 3$ one can take $k \neq i, j$ and observe:

- the coefficient of $G_{ik}G_{jk}$ is $-\lambda^2 - \lambda$,
- the coefficient of $G_{ji}G_{j2}$ is $-\lambda^2 + \lambda$, and
- the coefficient of $G_{ki}G_{kj}$ is $\lambda^2$.

The only way that the sum of these terms can be zero in cohomology is via Relation $[3]$, i.e. their coefficients must be equal. It therefore follows that $\lambda = 0$, as claimed.

When $m$ is even set $\mu := 2\nu \in \mathbb{Z}$. If $\lambda = 0$ then $s^*(G_{0i}) = \frac{\mu}{2}G$ as claimed. Otherwise, assume $\lambda \neq 0$ and again consider the relation $G_{0i}G_{0j} + G_{ij}G_{i0} + G_{j0}G_{ji} = 0$. Since the $G_{ab}$’s anti-commute and $G_{ab} = -G_{ba}$,

$$0 = s^*(G_{0i}G_{0j} + G_{ij}G_{i0} + G_{j0}G_{ji}) = (\lambda G_i + \nu G)(\lambda G_j + \nu G) + G_{ij}(\lambda G_i + \nu G - \lambda G_j - \nu G) \quad (4)$$

$$= \lambda^2 G_i G_j + \lambda(G_{ij} - \nu G)(G_i - G_j) \quad (5)$$

As above, we read off the coefficients of various terms $G_{ab}G_{cd}$. Since $n \geq 3$, find $k \neq i, j$ and observe:

- the coefficient of $G_{ij}G_{ik}$ in $[5]$ is $-\lambda^2 + \lambda - 2\mu\lambda$,
- the coefficient of $G_{ij}G_{jk}$ in $[6]$ is $-\lambda^2 + \lambda - 2\mu\lambda$, and lastly,
- the coefficient of $G_{ij}G_{ki}$ in $[5]$ is $\lambda^2 + 4\mu\lambda$.

As in the above case, the only way that the sum of these terms can be zero in cohomology is if their coefficients are equal. It therefore follows that

$$-\lambda^2 - 2\mu\lambda + \lambda = \lambda^2 + 4\mu\lambda$$

which simplifies to

$$1 = 2\lambda + 6\mu = 2\lambda + 3\mu. \quad (6)$$

In particular, $\mu$ must be odd. Next, using the notation $\overline{G}_i = \frac{1}{2}G - G_i$ we expand

$$s^*(G_{0i}) = \lambda G_i + \nu G = (\lambda + 2\nu)G_i + 2\nu\overline{G}_i = (\lambda + \mu)G_i + \mu\overline{G}_i.$$
In the case where \( n \geq 4 \) it is possible to choose distinct indices \( k, \ell \not\in \{i, j\} \). The coefficient of \( G_{ik} G_{j \ell} \) in \((\text{II})\) is \( \lambda^2 + 4 \mu \nu \). But since classes of this form satisfy no linear relations, their coefficient must be zero. From this it follows that
\[
0 = \lambda + 4 \mu = \lambda + 2 \mu
\] (7)
which along with \((\text{II})\) is only solved by \( \mu = -1 \).

4. Case of unordered configurations

In this section we prove Theorem \((\text{III})\) which says that for \( m \geq 2 \) there is no section of \( \tilde{g}_{m,n} \) except when \( n = 2 \). We begin with a few preliminary observations. First, our cohomological classification was for equivariant sections of configurations in \( \mathbb{R}^m \) rather than in \( \mathbb{B}^m \). But since \( \mathbb{R}^m \) is homeomorphic to the open unit ball \( U \), their configuration spaces are also equivariantly homeomorphic. Then scaling by \( 0 < t < 1 \) gives a sequence of inclusions
\[
t U \subset t B \subset U \subset \mathbb{B}
\]
with compositions isotopic to the identity. It follows that \( U \) and \( \mathbb{B} \) have equivariantly homotopy equivalent configuration spaces. In particular our classification of equivariant sections applies to \( \mathbb{B} \) just the same.

Second, observe that a section \( \mathcal{s} \) of \( \tilde{g}_{m,n} \) for the unordered configuration spaces lifts to a \( \Sigma_n \)-equivariant section \( s \) of \( g_{m,n} \) for the ordered spaces. This follows from the lifting criterion for connected coverings.

Third, we leverage the fact that this equivariant section \( s \) provides a solution to the corresponding section problem subject to the added restriction that certain points are constrained to the boundary sphere. This will give us more maps, whose pullback on cohomology we compute in two contradictory ways. Let \( U \subset \mathbb{B} \) denote the interior of the closed ball and consider the subspace of \( \text{PConf}_n(\mathbb{B}) \) in which only the \( i \)-th point lies on the boundary sphere \( S^{m-1} \):

\[
\mathcal{B}_n^i := \{(p_1, \ldots, p_n) \in \text{PConf}_n(\mathbb{B}) \mid p_i \in \partial \mathbb{B}, (p_1, \ldots, \hat{p}_i, \ldots, p_n) \in \text{PConf}_n(U)\} \cong \text{PConf}_{n-1}(U) \times S^{m-1}
\]

Define \( \mathcal{B}_{n+1}^i \subseteq \text{PConf}_{n+1}(\mathbb{B}) \) similarly. Lastly, let \( E_{n+1}^i \) denote the preimage \( g_{m,n}^{-1}(\mathcal{B}_n^i) \) and consider the inclusion \( \mathcal{B}_{n+1}^i \hookrightarrow E_{n+1}^i \). The difference between these two spaces is that in the larger space the additional point \( p_0 \) may lie on the boundary \( \partial \mathbb{B} \), while in the smaller space this is not allowed. Despite this apparent difference, we have the following.

**Lemma 4.1.** The inclusion \( \mathcal{B}_{n+1}^i \hookrightarrow E_{n+1}^i \) is a homotopy equivalence.

**Proof.** Choosing an inward-pointing vector field that vanishes on the point \( p_i \in \partial \mathbb{B} \), one can push any other point into the interior of \( \mathbb{B} \). Explicitly, the vector field \( -(p - p_i) \) on \( \mathbb{B} \) gives rise to a smooth vector field on \( \text{PConf}_{n+1}(\mathbb{B}) \). Its flow produces an isotopy \( \Phi_t^i \) such that for all \( t > 0 \),
\[
\Phi_t^i(B_{n+1}^i) \subset \Phi_1^i(E_{n+1}^i) \subset B_{n+1}^i \subset E_{n+1}^i
\]
thus establishing the homotopy equivalence. \(\square\)

Consider the compatible projections onto the \( i \)-th coordinate
\[
\begin{array}{ccc}
\mathcal{B}_{n+1}^i & \xrightarrow{g_{m,n}} & \mathcal{B}_n^i \\
& \searrow \downarrow \swarrow & \downarrow \nearrow \searrow \\
& S^{m-1} & \searrow \downarrow \swarrow \\
\end{array}
\]
Pulling back an orientation class \([S^{m-1}] \in H^{m-1}(S^{m-1})\), we get a class \( 0 \neq X_i \in H^{m-1}(\mathcal{B}_n^i) \) whose pullback will also be abusively denoted by \( X_i \in H^{m-1}(B_{n+1}^i) \). These classes measure how many times the point \( p_i \) wraps around the boundary sphere.
Lemma 4.2. Under the inclusion \( i : B^{i}_{n+1} \hookrightarrow \text{PConf}_{n+1}(\mathbb{B}) \) we have

\[ i^*(G_{ia}) = X_i \text{ for all } a \neq i \]

and the class \( G_{ab} \) for \( a, b \neq i \) pulls back to \( G_{ab} \in H^{m-1}(\text{PConf}_n(U)) \). The same is true for \( B^i_n \subseteq \text{PConf}_n(\mathbb{B}) \).

Via the homotopy equivalence \( B^i_{n+1} \simeq E^i_{n+1} \), we consider Lemma 4.2 as a statement about \( E^i_{n+1} \) as well. In particular we shall keep the notation \( X_i \) for the corresponding class in \( H^{m-1}(E^i_{n+1}) \).

Proof of Lemma 4.2 These facts are geometrically obvious: recall that \( G_{ab} \) is pulled back from \( S^{m-1} \) under the ‘Gauss map’, sending a configuration to the direction vector from \( p_a \) to \( p_b \). When \( a, b \neq i \), this Gauss map factors through the projection \( B^i_{n+1} \to \text{PConf}_n(U) \), as claimed.

Otherwise, if \( a \neq i \) then since \( p_i \) lies on the boundary and \( p_a \) is internal, the Gauss map is homotopic to a map in which \( p_a \) is fixed at the origin. But when \( p_a = 0 \) the Gauss map coincides with the projection which records only \( p_i \).

With these facts in hand, we can now complete our proof.

Proof of Theorem 4.3 Let \( m \geq 2 \). If \( n = 1 \), we have no section by Example 1. If \( n = 2 \), we have a section by Example 2. Otherwise, let \( n \geq 3 \). Assume that \( \bar{s} \) is a section of \( \bar{g}_{m,n} \) so that we have an \( \Sigma_n \)-equivariant section \( s \) of \( g_{m,n} \). The assumption that \( s \) is a section forces \( s(B^i_n) \subseteq E^i_{n+1} \), thus it restricts to a section \( s' : B^i_n \to E^i_{n+1} \) of \( g_{m,n} \). From this one observes that \( (s')^*X_i = X_i \). Let us show that this contradicts the cohomological classification of Lemma 3.1.

Indeed, we have a commutative diagram

\[
\begin{array}{ccc}
E^i_{n+1} & \longrightarrow & \text{PConf}_{n+1}(\mathbb{B}) \\
\downarrow s' & & \downarrow s \\
B^i_n & \longrightarrow & \text{PConf}_n(\mathbb{B})
\end{array}
\]

and so the restriction of the class \( G_{0i} \in H^{m-1}(\text{PConf}_{n+1}) \) along the two different paths must agree. Pulling it back through the top-left corner,

\[ G_{0i} \longrightarrow X_i \xrightarrow{(s')^*} X_i. \]

But pulling back through the bottom right corner, one first applies \( s^* \), which falls under our classification. If \( m \) is odd then \( s^*(G_{0i}) = 0 \), restricting to 0 in \( B^i_n \) and contradicting the fact that \( X_i \neq 0 \). Otherwise, if \( m \) is even then either \( s^*(G_{0i}) = \mu(G_i + \overline{G}^i) \) or \( (-k)G_i + (2k+1)\overline{G}^i \). Let us consider both cases at once by denoting \( s^*(G_{0i}) = \delta G_i + \mu \overline{G}^i \).

Since the restriction of \( G_{ab} \) with \( a, b \neq i \) to \( B^i_n \) gives the class \( G_{ab} \) again, it follows that \( \overline{G}^i \) restricts to itself. In particular, it is linearly independent from \( X_i \). At the same time, since each \( G_{ia} \) restricts to \( X_i \), the restriction of their sum \( G_i \) is \( (n-1)X_i \).

Therefore the restriction of \( s^*(G_{0i}) \) to \( B^i_n \) is \( \delta(n-1)X_i + \mu \overline{G}^i \). But this must coincide with \( X_i \) by pullback along the top-left corner. We therefore deduce that \( \mu = 0 \) and is thus even, eliminating one possibility from the classification. But if \( \mu = 0 \), this now brings us to a situation analogous to the odd \( m \) case: it follows that \( s^*(G_{0i}) = 0 \) and it restricts to 0 \( \neq X_i \), which is a contradiction. Therefore no section \( \bar{s} \) of \( \bar{g}_{m,n} \) exists for \( n \geq 3 \), as claimed. \( \square \)
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