Muon anomalous magnetic moment: 
a consistency check for the next-to-leading order hadronic contributions

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Abstract
A model for verifying a consistency of the next-to-leading order hadronic contributions to the muon anomalous magnetic moment with those of the leading order is proposed. A part of the next-to-leading order hadronic contributions related to the vacuum polarization is rather accurately reproduced in the model. I find a new numerical value for the light-by-light hadronic contribution that leads to agreement with recent experimental result for the muon anomalous magnetic moment.
1 Introduction

A numerical value of the muon anomalous magnetic moment (MAMM) measured experimentally with high precision can be used to quantitatively test the theories suggested for describing particle interactions. The experimental result for MAMM presented in ref. [1] reads

\[ a^\text{exp}_\mu = 116 592 023(151) \times 10^{-11} \]  

with the uncertainty \(151 \times 10^{-11}\). The main anomalous effect is due to Schwinger term

\[ a^\text{Schw}_\mu = \frac{\alpha}{2\pi} \]  

where \(\alpha\) is the fine structure constant \(\alpha^{-1} = 137.036\ldots\). The theoretical contributions presently computed in the standard model for the comparison with the experimental value given in eq. (1) are divided into three parts: leptonic (QED), electroweak (EW), and hadronic (had) one. The pure leptonic part is computed in perturbative QED through \(\alpha^5\) order [2, 3]. The numerical value of the QED contribution to the muon anomalous magnetic moment reads (as a review see [1])

\[ a^\text{QED}_\mu = 116 584 705.7(2.9) \times 10^{-11} \]  

The EW corrections are well defined in the perturbation theory framework of the standard model and have been computed with the two-loop accuracy (as a review see [1])

\[ a^\text{EW}_\mu = 152(4) \times 10^{-11} \]  

Numerically this contribution matches the present experimental uncertainty. The EW correction will be noticeable if a goal to reach the planned experimental accuracy \(40 \times 10^{-11}\) is accomplished (as a review see [1]).

The hadronic contribution to MAMM is sensitive to the infrared region and cannot be computed in perturbative QCD with light quarks. The current masses of light quarks are too small to provide a necessary infrared cutoff and explicit models of hadronization are required for the quantitative analysis. This constitutes a main difficulty of the theoretical analysis of MAMM in the standard model. Writing

\[ a^\text{SM}_\mu = a^\text{QED}_\mu + a^\text{EW}_\mu + a^\text{had}_\mu \]  

and assuming

\[ a^\text{exp}_\mu = a^\text{SM}_\mu \]  

one finds a numerical value for the hadronic contribution to MAMM

\[ a^\text{had}_\mu\big|_{\text{th}} = (7165.3 + 151|_{\exp} + 2.9|_{\text{QED}} + 4|_{\text{EW}}) \times 10^{-11} \]
with the experimental error dominating the uncertainty.

Since the hadronic contribution is sensitive to the details of the strong coupling regime of QCD at low energies and cannot be unambiguously computed in perturbation theory framework the theoretical prediction for MAMM in the standard model depends crucially on how this contribution is estimated. In the absence of reliable theoretical tool for computation in this region one turns to experimental data on low-energy hadron interactions for extracting a necessary numerical value. In general terms the hadronic contribution to MAMM is determined by the correlation functions of electromagnetic (EM) currents. As the source for the EM current is readily available for a wide range of energies one tries to extract these functions (or some their characteristics) from experiment. Without an explicit use of QCD the correction $a_\mu^{\text{had}}$ is generated through the EM interaction $e j_\mu^{\text{had}} A^\mu$ with $j_\mu^{\text{had}}$ being the hadronic part of the EM current. At the leading order ($\alpha^2$ in the formal power-counting) only the two-point correlation function of the EM currents emerges in the analysis of hadronic contributions to MAMM

$$\Pi_2 \sim \langle j_\mu^{\text{had}}(x) j_\nu^{\text{had}}(0) \rangle.$$  \hspace{1cm} (8)

At the next-to-leading order ($\alpha^3$) the four-point correlation function appears

$$\Pi_4 \sim \langle j_\mu^{\text{had}}(x) j_\nu^{\text{had}}(y) j_\alpha^{\text{had}}(z) j_\beta^{\text{had}}(0) \rangle.$$ \hspace{1cm} (9)

These correlators are not calculable perturbatively in the region that is essential for the determination of the hadronic contributions to MAMM. The leading contribution to MAMM comes from the two-point correlator eq. (8) referred to as the hadronic vacuum polarization contribution while the four-point function eq. (9) first emerges at the $\alpha^3$ order, most explicitly as the light-by-light scattering. To avoid using QCD in the strong coupling mode one has to extract the necessary contribution to MAMM by studying these two correlation functions experimentally without an explicit realization of the hadronic EM current $j_\mu^{\text{had}}$ in terms of elementary fields. Historically this was a way of studying the EM properties of hadrons before emerging QCD as a fundamental theory of strong interactions (e.g. [4]).

## 2 Hadronic contribution at the leading order

At the leading order in $\alpha$ the hadronic contribution is described by the correlator

$$i \int \langle T j_\mu^{\text{had}}(x) j_\nu^{\text{had}}(0) \rangle e^{iqx} dx = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi^{\text{had}}(q^2)$$ \hspace{1cm} (10)

which reduces to a single function $\Pi^{\text{had}}(q^2)$ of one variable $q^2$. The correlator is transverse due to conservation of the hadronic EM current in the standard model. This function gives a contribution to the muon anomalous magnetic moment (e.g. [5])

$$a_\mu^{\text{had}}(\text{LO}) = 4\pi \left( \frac{\alpha}{\pi} \right)^2 \int_{4m_e^2}^{\infty} \frac{ds}{s} K(s) \text{Im} \, \Pi^{\text{had}}(s)$$ \hspace{1cm} (11)
with the one-loop kernel of the form

$$K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)s/m^2_{\mu}}.$$  \hfill (12)

Here \(\text{Im} \Pi^\text{had}(s) = \text{Im} \{\Pi^\text{had}(q^2)|_{q^2=s+i0}\}\), \(m_{\mu}\) is a muon mass.

The leading order hadronic contribution to MAMM is represented by an integral over the hadron spectrum. No specific information about the function \(\text{Im} \Pi^\text{had}(s)\) is necessary point-wise except its threshold structure in the low-energy region. For the applications at the leading order in \(\alpha\) the function \(\text{Im} \Pi^\text{had}(s)\) can be uniquely identified with data from \(e^+e^-\) annihilation into hadrons. Introducing the experimental \(R^\text{exp}(s)\) ratio

$$R^\text{exp}(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}, \quad s = (p_{e^+} + p_{e^-})^2$$  \hfill (13)

and identifying it with the theoretical quantity \(R^\text{th}|_{LO}(s)\) taken at the leading order in \(\alpha\) as

$$R^\text{th}|_{LO}(s) = 12\pi \text{Im} \Pi^\text{had}(s)$$  \hfill (14)

one finds

$$a_{\mu}^\text{had}(LO) = \frac{1}{3} \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{R^\text{exp}(s)K(s)}{s} ds.$$  \hfill (15)

The contribution to MAMM based on the representation given in eq. (15) is well studied. Several recent determinations are

$$a_{\mu}^\text{had}(LO) = 7011(94) \times 10^{-11} \quad \text{(ref. [6]);} \hfill (16)$$

$$a_{\mu}^\text{had}(LO) = 6924(62) \times 10^{-11} \quad \text{(ref. [7]);} \hfill (17)$$

$$a_{\mu}^\text{had}(LO) = 6988(111) \times 10^{-11} \quad \text{(ref. [8]).} \hfill (18)$$

I use in the further analysis a naive average of these three results (both central values and errors are averaged) which reads

$$a_{\mu}^\text{had}(LO) = 6974(89) \times 10^{-11}. \hfill (19)$$

Writing

$$a_{\mu}^\text{had}|_{th} = a_{\mu}^\text{had}(LO) + a_{\mu}^\text{had}(NLO)$$

and comparing with eq. (1) one has (in units \(10^{-11}\))

$$a_{\mu}^\text{had}(NLO) = 7165 + 151|_{\text{exp}} + 2.9|_{\text{QED}} + 4|_{\text{EW}} - 6974(89)|_{LO}$$

$$= 191 + 151|_{\text{exp}} + 2.9|_{\text{QED}} + 4|_{\text{EW}} + 89|_{\text{had}}. \hfill (20)$$

I use in the further analysis a naive average of these three results (both central values and errors are averaged) which reads

$$a_{\mu}^\text{had}(LO) = 6974(89) \times 10^{-11}.$$  \hfill (19)
Assuming the statistical independence of the uncertainties one finds after adding them in quadratures

\[ a_{\mu}^{\text{had}}(\text{NLO}) = (191 \pm 175) \times 10^{-11} \]  

that does not allow to see higher order hadronic effects clearly. The error comes mainly from the experimental value eq. (1) and the leading order hadronic data eq. (19) the statistical correlation of which is supposed to be small as they come from different sources. Other errors are negligible. For the target experimental error of MAMM at the level of \( 40 \times 10^{-11} \) one finds the uncertainty of the NLO hadronic contribution to become \( 98 \times 10^{-11} \). Assuming that the mean value of \( a_{\mu}^{\text{exp}} \) in the planned experiment will not change one finds a numerical value for the NLO hadronic contribution

\[ a_{\mu}^{\text{had}}(\text{NLO}) = (191 \pm 98) \times 10^{-11} \]  

that makes the NLO hadronic effects noticeable at the level of two standard deviations. If the mean value of \( a_{\mu}^{\text{exp}} \) will change in the range of the present experimental uncertainty \( 151 \times 10^{-11} \) the NLO hadronic effects can be more or less pronounced. From the naive counting in \( \alpha \) a numerical value for the theoretical NLO hadronic contribution about \( 50 \times 10^{-11} \) can be expected. This number is comparable in magnitude with the uncertainty in eq. (22) and should be taken into account.

### 3 Hadronic contribution at next-to-leading order

In NLO there is no such a transparency with determining hadronic contributions as in LO. Basically there are two new features. On the experiment side the interpretation of data to be used in the NLO theoretical calculations is more involved. The problem is to avoid double counting as a part of the hadronic contributions has already been accounted through the use of data at LO. On the theory side a new correlation function \( \Pi_4 \) from eq. (19) which is much more complicated than the two-point correlator enters the game. At present there is no accurate experimental determination of the four-point function in the kinematical range necessary for MAMM computation and one has to rely on phenomenological models used for this function. It is difficult to control the accuracy of such models that introduces an explicit model dependence in the calculation of the NLO hadronic contribution and makes predictions less definite than in LO.

#### 3.1 Interpretation of data at NLO of EM interaction

For applications at NLO in \( \alpha \) the extraction of data is more involved. For instance, one should explicitly take into account the NLO corrections to theoretical factors that emerge in
a description of the process from which a set of data is taken. These “theoretically corrected”
data should be used in NLO applications for computing MAMM. As the sets of data are mainly
extracted from $e^+e^-$ annihilation I discuss this particular process in some detail.

### 3.1.1 One-photon mediated $e^+e^-$ annihilation

The main object studied experimentally in this sector is the full photon propagator $D(q^2)$

$$D(q^2) = \frac{1}{-q^2 1 + e^2 \Pi(q^2)} \tag{23}$$

with $\Pi(q^2) = \Pi^{\text{lept}}(q^2) + \Pi^{\text{had}}(q^2)$ being a one-particle irreducible block, $e^2 = 4\pi\alpha$. Note that in higher orders the one-particle irreducible block does not split into a sum of pure leptonic
and pure hadronic contributions. It happens first at NNLO which is far beyond the practical
interest though. I discuss only NLO or $\alpha^3$ terms in the formal $\alpha$ power-counting. Since the
data are collected at low energies the EW sector can be excluded. With these restrictions the
cross section of $e^+e^-$ annihilation through the one-photon exchange at NLO without vertex
corrections to initial states is proportional to

$$\text{Im} \left\{ q^2 D(q^2) \right|_{q^2 = s + i0} = \frac{e^2 \text{Im}\Pi(s)}{(1 + e^2 \text{Re}\Pi(s))^2 + e^4 \text{Im}\Pi(s)^2}$$

$$= \frac{e^2 (\text{Im}\Pi^{\text{lept}}(s) + \text{Im}\Pi^{\text{had}}(s))}{(1 + e^2 \text{Re}\Pi(s))^2 + e^4 \text{Im}\Pi(s)^2}. \tag{24}$$

The theoretical expression for the $R$ ratio at NLO reads

$$R_{NLO}^{\text{th}}(s) = \frac{\text{Im}\Pi^{\text{had}}(s)}{\text{Im}\Pi^{\mu\mu}(s)}. \tag{25}$$

If $R_{NLO}^{\text{th}}(s)$ is identified with $R_{\text{exp}}(s)$ from eq. (13) then $\text{Im}\Pi^{\text{had}}(s)$ can be restored by using
a theoretically calculated $\text{Im}\Pi^{\mu\mu}(s)$. For $s \gg m_\mu^2$ one finds with NLO accuracy

$$12\pi \text{Im}\Pi^{\text{had}}(s) = R_{\text{exp}}^{\text{exp}}(s) \left(1 + \frac{3\alpha}{4\pi}\right). \tag{26}$$

In some analyses the cross section $\sigma(e^+e^- \rightarrow \text{hadrons})$ divided by the normalization factor

$$\sigma_0 = \frac{4\pi\alpha^2}{3s} \tag{27}$$

is used as a data set [1]. Then the relation

$$12\pi \text{Im}\Pi^{\text{had}}(s) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma_0 + O(\alpha) \tag{28}$$

is valid only at the leading order in $\alpha$. One of the differences with the $R$ ratio at NLO from
eq (24) is the term $\text{Re}\Pi^{\text{had}}(s)$ from the denominator in eq. (24). The quantity $\text{Re}\Pi^{\text{had}}(s)$
can be found by reiterating the leading order term $\text{Im}\Pi^{\text{had}}(s)$ through the dispersion relation that gives a relative error of $\alpha^2$ order. The NLO contribution in the denominator is related to the running of the EM coupling constant and can partly be taken into account through the renormalization group technique for the energies far from the resonances [9]. Another difference is the corrections to the production vertex that should properly be taken into account as they enter the cross section. Extracting $\text{Im}\Pi^{\text{had}}(s)$ from the cross section requires to subtract these corrections from the data in order to avoid double counting in the NLO analysis of MAMM if a theoretical NLO kernel for averaging the two-point correlator is used.

The use of the $R$-ratio is preferable from the theoretical point of view as it relates data to the imaginary part of the two-point hadronic correlator $\text{Im}\Pi^{\text{had}}(s)$ in a simple way. It is also preferable from the experimental point of view since the total normalization of the data is fixed that helps to eliminate systematic errors. In this respect a $\tau$-data set that can be used to determine that part of the two-point function that is generated by the isovector part of the hadronic EM current in the limit of exact isotopic invariance has a different normalization at NLO and should be corrected by an explicit account for contributions of the relative $\alpha$ order. Note that the NLO correction emerging from the interpretation of data can be controlled theoretically within counting in $\alpha$ while corrections due to the violation of isotopic invariance for the data obtained from the $\tau$ and $e^+e^-$ channels can only be estimated in models. The problem of different normalization remains also for heavy hadrons if their contribution to the cross section is calculated from their leptonic branchings. The NLO contribution of heavy flavors is not essential though because it is small.

3.1.2 Two-photon mediated $e^+e^-$ annihilation

The NLO cross section of $e^+e^-$ annihilation contains a contribution of the two-photon annihilation with one hadronic insertion into the photon propagator. This contribution requires a special treatment before the data set is related to the hadronic two-point function with the NLO accuracy. For instance, the NLO kernel for the MAMM diagram with a vertex correction integrates the part of data emerging through the double photon scattering channel in $e^+e^-$ annihilation. This can lead to double counting at NLO for MAMM.

Thus, one concludes that at NLO a strict correlation between sets of data and theoretical expressions for the NLO kernels emerges. This leads to additional contributions of the relative $\alpha$ order that numerically amounts to about 1% of the leading order contribution which is the precision one is trying to reach for the comparison with the experimental result for MAMM.
3.2 Four-point correlator

At NLO a new correlation function of hadronic EM currents emerges. This correlation function leads to a new effect which is known as light-by-light scattering. Besides this explicit effect a less pronounced mixed effect also emerges. The four-point function gives a contribution to the full photon propagator of the form

\[ \int dxdy D_{\mu\nu}(x-y) \langle T j_{\mu}^{\text{had}}(x) j_{\nu}^{\text{had}}(y) j_{\alpha}^{\text{had}}(z) j_{\beta}^{\text{had}}(0) \rangle \]  

(29)

where \( D_{\mu\nu}(x) \) is a free photon propagator with a scalar amplitude \( D(x) \sim 1/x^2 \). In other words a projection of the four-point function of the form

\[ \int dxdy \frac{(x-y)^2}{2} \langle T j_{\mu}^{\text{had}}(x) j_{\nu}^{\text{had}}(y) j_{\alpha}^{\text{had}}(z) j_{\beta}^{\text{had}}(0) \rangle \]  

(30)

is present in the two-photon Green function. In QCD and other models where the EM current is explicitly expressed through the elementary fields this contribution is interpreted as an EM correction to the one-particle irreducible block. There is an option to include this contribution to the two-point function. I do not consider this option since the picture of the local interaction of the photon with the hadronic EM current is lost in such case.

Thus, an accurate account of NLO hadronic contributions to MAMM from general principles is rather a challenging task both experimentally and theoretically. As a first approach to it one can use an effective theory with few free parameters providing a unique framework for calculations at LO and NLO. In such an approach the LO information is used to obtain numerical values for the model parameters. The NLO results are then computed theoretically. This approach can also serve as a base for verifying a consistence of the estimates for the NLO hadronic contributions made in different phenomenological models.

4 A model for hadronic contributions

In this section I describe a model to check a consistence of the NLO hadronic contributions and especially the light-by-light contribution with the results of LO analysis for MAMM. The simplest version of the model contains three light quarks with QCD quantum numbers and the mass \( m_q \) which is the only model parameter. The numerical value of \( m_q \) is fixed from the LO hadronic contribution and then used to find the NLO result. Heavy quarks enter the model with their standard masses. In this model the calculations are explicit and can be performed analytically that is an advantage. Indeed, the model differs from the leptonic sector only by the QCD group factors and the numerical values of fermion masses.
4.1 Fixing $m_q$ from the LO hadronic contribution

A fermion with mass $m_q$ without QCD group factors (as a lepton) gives the LO contribution to MAMM of the form

$$a_{\mu}^{\text{form}}(\text{LO}) = I(m_q) \left( \frac{\alpha}{\pi} \right)^2$$

with

$$I(m_q) = \int_{4m_q^2}^{\infty} \frac{\rho_q(s)K(s)}{s} ds$$

and

$$\rho_q(s) = \frac{1}{3} \sqrt{1 - \frac{4m_q^2}{s}} \left( 1 + \frac{2m_q^2}{s} \right).$$

Explicit integration over $s$ with the kernel $K(s)$ from eq. (12) gives

$$I(m_q) = \int_0^1 dx (1 - x)[-\pi(x, m_q)]$$

where

$$\pi(x, m_q) = \left( \frac{1}{3z} - 1 \right) \varphi(z) - \frac{1}{9}$$

and

$$\varphi(z) = \frac{1}{\sqrt{z}} \text{ArcTanh}(\sqrt{z}) - 1, \quad z = \frac{m_{\mu}^2x^2}{4m_q^2(1 - x) + m_{\mu}^2x^2}.$$  

An analytical expression for the function $I(m_q)$ is known, however, the integral representation given in eq. (34) is sufficient for practical applications.

Contributions of the heavy $c$ and $b$ quarks can directly be computed in QCD perturbation theory independently of the model. In the present calculation I use only free quark approximation for simplicity. For the $c$ quark with the mass $m_c = 1.6$ GeV [10] and the charge $e_c = 2/3$ one finds from eq. (31) multiplied by the group factor $3e_c^2 = 4/3$

$$a_{\mu}^{\text{mod}}(\text{LO}; c) = 69.3 \times 10^{-11}.$$  

The $b$-quark contribution for $m_b = 4.8$ GeV [11] and $e_b = -1/3$ is small and reads

$$a_{\mu}^{\text{mod}}(\text{LO}; b) = 1.9 \times 10^{-11}.$$  

Thus the contribution of light hadronic modes that is represented in our model by light fermions with the mass $m_q$ amounts to

$$a_{\mu}^{\text{mod}}(\text{LO}; uds) = (6974.3 - 69.3 - 1.9) \times 10^{-11} = 6903(89) \times 10^{-11}.$$  

I assume that this result directly corresponds to the contribution of the two-point correlator at the leading order as given in eq. (11). It means that a real data set is properly corrected to
extract \( r m I m \Pi^{\text{had}}(s) \). As was discussed above the extraction of \( r m I m \Pi^{\text{had}}(s) \) with the NLO accuracy requires a careful interpretation of data which is assumed to be done.

From eqs. (31-36) and eq. (39) one obtains a numerical value for the single model parameter \( m_q \)

\[
m_q = 179 \pm 1 \text{ MeV}.
\]  

(40)

This value is rather close to that of the charged pion mass, \( m_\pi = 139.6 \text{ MeV} \), which is expected as the LO contribution is mainly sensitive to the first derivative at \( q^2 = 0 \) of the two-point function \( \Pi^{\text{had}}(q^2) \) given in eq. (11). For definiteness I give the LO contribution of the light hadronic modes within the model obtained literally with the number from eq. (40)

\[
a_{\mu}^{\text{mod}}(\text{LO; uds})(m_q = 179 \pm 1 \text{ MeV}) = (6920 \pm 70) \times 10^{-11}.
\]  

(41)

Since in the framework of the model the NLO hadronic corrections to MAMM are determined by the single parameter \( m_q \) with the numerical value from eq. (40) they can readily be found.

4.2 Hadronic contributions at NLO

The first check is to use the model for computing the higher order hadronic corrections due to vacuum polarization graphs. The data-based analysis gives for the NLO effects of this type \[12\]

\[
a_{\mu}^{\text{had}}(\text{vac; NLO}) = -101(6) \times 10^{-11}.
\]  

(42)

This number is about 1.5\% of the leading term as expected. As was discussed above at this level of precision the numerical value for the NLO contribution depends strongly on the data sets used in the analysis. For different data sets the different expressions of the NLO kernel should be used to avoid double counting. For example, if the \( R \)-ratio is used in the one-loop computation then the LO result should first be divided by the factor (cf. eq. (26))

\[
12\pi \text{Im}\Pi^{\mu\mu}(s)|_{s \gg m_\mu^2} = \left(1 + \frac{3\alpha}{4\pi} + O\left(\frac{m_\mu^4}{s^2}\right)\right)
\]  

(43)

before being used in the NLO analysis that changes the LO result by \( 12 \times 10^{-11} \) that exceeds the uncertainty quoted in eq. (42). In fact, even mass suppressed terms can be important at this level of precision and the entire function \( \text{Im}\Pi^{\mu\mu}(s) \) should be integrated since the mass terms from the leading order can partly cancel the NLO corrections in \( \alpha \). For other types of data (\( \tau \) data especially) the change can be larger. This uncertainty is a reflection of the mixture of contributions at NLO.

In the proposed model the analysis is unambiguous and straightforward. I present different contributions separately for a detailed comparison with the results from ref. [12].

For the vertex type contributions I use the explicit analytical formulae in the leading order of the mass expansion as they are given in ref. [13]. The exact expressions are presented in
The analytical expression for the contribution of a fermion with mass $m_q$ without any group factors reads

$$a_{\mu}^{\text{ferm}}(\text{ver}; m_q) = -\frac{2}{3} \left( \frac{m_\mu}{m_q} \right)^2 \left( -\frac{2689}{5400} + \frac{\pi^2}{15} + \frac{23}{90} \ln \frac{m_q}{m_\mu} \right) \left( \frac{\alpha}{\pi} \right)^3$$

that leads to a numerical result for the light mode contribution in the model

$$a_{\mu}^{\text{mod}}(\text{ver; NLO; uds}) = -172 \times 10^{-11}.$$  

A more accurate evaluation (using numerical integration with the kernel given up to the third order in the mass expansion from ref. [12]) gives for the contribution of light modes

$$a_{\mu}^{\text{mod}}(\text{ver; NLO; uds}) = -188 \times 10^{-11}.$$  

The difference with the result obtained by using only the first term of mass expansion given in eq. (45) is of order 10%. It is smaller than one could expect from the numerical value of the expansion parameter $(m_\mu/m_q)^2 = (0.106/0.179)^2 = 0.36$. For the $c$-quark contribution one finds

$$a_{\mu}^{\text{mod}}(\text{ver; NLO; c}) = -4 \times 10^{-11}$$

while the $b$-quark contribution is small

$$a_{\mu}^{\text{mod}}(\text{ver; NLO; b}) = -0.2 \times 10^{-11}.$$  

The total vertex contribution computed in the model

$$a_{\mu}^{\text{mod}}(\text{ver; NLO}) = -192 \times 10^{-11}$$

should be compared with the result of the data-based analysis from ref. [12]

$$a_{\mu}^{\text{ref} [12]}(\text{ver; NLO}) = -211(5) \times 10^{-11}.$$  

Next check of the model is done for a mixed contribution of the lepton-hadron type. This contribution contains the electron and $\tau$-lepton loops and depends on three masses $m_\mu$, $m_q$, and $m_e$ or $m_\tau$. For fermions without group factors the contribution is given by the integral representation

$$a_{\mu}^{\text{ferm}}(\text{db; } f_1 \& f_2) = \left( \frac{\alpha}{\pi} \right)^3 \int_0^1 dx \left( 1 - x \right) \pi(x, m_{f_1}) \pi(x, m_{f_2}) .$$

For the combined contribution of light modes with the electron loop one has

$$a_{\mu}^{\text{mod}}(\text{db; NLO; e\&uds}) = 105 \times 10^{-11}$$

and with the $\tau$-lepton loop

$$a_{\mu}^{\text{mod}}(\text{db; NLO; } \tau \& \text{uds}) = 0.05 \times 10^{-11}.$$
The contribution of heavy modes is only visible for the combined insertion of the $c$-quark loop and the electron loop

$$a^{\text{mod}}_\mu(\text{db}; \text{NLO}; e\&c) = 1.1 \times 10^{-11}.$$  (54)

The results given in eqs. (52-54) are in good agreement with the data-based estimate

$$a^{\text{ref}}_\mu[12](\text{lept\&had}; \text{NLO}) = 107(2) \times 10^{-11}.$$  (55)

Next comes the contribution from the reiteration of hadronic insertions. The light modes give

$$a^{\text{mod}}_\mu(\text{db}; \text{NLO}; u\&d\&s) = 3 \times 10^{-11}.$$  (56)

The combination of the $c$-quark insertion with the light mode loops gives

$$a^{\text{mod}}_\mu(\text{db}; \text{NLO}; c\&uds) = 0.1 \times 10^{-11}$$  (57)

while the contribution of the two $c$-quark insertions is negligible. The results of the model from eqs. (56,57) are in agreement with the data-based estimates

$$a^{\text{ref}}_\mu[12](\text{had\&had}; \text{NLO}) = 2.7 \times 10^{-11}.$$  (58)

Thus one sees a good agreement of the model results with calculations based on data. However, in the model there is a contribution which is missing in the explicit calculations based on data as it is related to the internal structure of the hadronic block. In the data-based calculation this contribution is hidden in data while in the model it can explicitly be resolved as a correction to the one-particle irreducible hadronic block. At the leading order of the mass ratio the analytical expression for this contribution without group factors reads

$$a^{\text{ferm}}_\mu(4; \text{NLO}; m_q) = \frac{41}{486} \left(\frac{m_\mu}{m_q}\right)^2 \left(\frac{\alpha}{\pi}\right)^3.$$  (59)

The result for the light modes of the model is

$$a^{\text{mod}}_\mu(4; \text{NLO}; u\&d) = 25 \times 10^{-11}$$  (60)

while the $c$-quark contribution is small

$$a^{\text{mod}}_\mu(4; \text{NLO}; c) = 0.3 \times 10^{-11}.$$  (61)

The result for the total NLO hadronic contribution of the vacuum polarization type is

$$a^{\text{mod}}_\mu(\text{vac}; \text{NLO}) = -58 \times 10^{-11}.$$  (62)

The difference with eq. (12) comes mainly from two sources: the vertex type contributions and the new term related to the one-particle irreducible hadronic block. Both contributions are
of the \((m_\mu/m_q)^2\) order that explains the magnitude of the difference. All remarks about the
double counting in the data-based approach apply here. Only first few terms of expansions in
the mass ratio \((m_\mu/m_q)^2\) were used for numerical estimates that provided a sufficient accuracy.
Thus, the model reproduces rather accurately the results for the NLO hadronic contributions
found in the data-based analysis for the graphs related to vacuum polarization. This has been
expected as these results are obtained by the integration of the two-point function with the
NLO kernel.
The next try for the model is the computation of the light-by-light contribution which is
given by the four-point correlator. The analytical expression for a contribution of the fermion
without group factors through the \((m_\mu/m_q)^4\) order reads [15]

\[
a_{\mu}^{\text{ferm}}(\text{lbl; NLO}; m_q) = \left(\frac{\alpha}{\pi}\right)^3 \left\{ \left(\frac{m_\mu}{m_q}\right)^2 \left(\frac{3}{2} \zeta(3) - \frac{19}{16}\right) \right.
\]
\[
+ \left(\frac{m_\mu}{m_q}\right)^4 \left(\frac{-161}{810} \ln^2 \left(\frac{m_q}{m_\mu}\right) - \frac{16189}{48600} \ln \left(\frac{m_q}{m_\mu}\right) + \frac{13}{18} \zeta(3) - \frac{161}{9720} \pi^2 - \frac{831931}{972000}\right) \right\}. \tag{63}
\]

With this formula one finds the value for the light modes

\[
a_{\mu}^{\text{mod}}(\text{lbl; NLO; uds}) = 140.5 \times 10^{-11} \tag{64}
\]

and one for the \(c\) quark

\[
a_{\mu}^{\text{mod}}(\text{lbl; NLO; c}) = 2 \times 10^{-11}. \tag{65}
\]

The total light-by-light contribution predicted by the model

\[
a_{\mu}^{\text{mod}}(\text{lbl; NLO}) = 140.5 + 2 = 143 \times 10^{-11} \tag{66}
\]
is different from the number used in the literature [1]

\[
a_{\mu}^{\text{had}}(\text{lbl; standard}) = -85(25) \times 10^{-11}. \tag{67}
\]

I postpone a discussion of this point till sect. [5].
Thus, the NLO hadronic contribution obtained in the model reads

\[
a_{\mu}^{\text{mod}}(\text{NLO}) = (-58 + 143) \times 10^{-11} = 85 \times 10^{-11}. \tag{68}
\]

It agrees with the present experimental result from eq. [21] which we repeat here

\[
a_{\mu}^{\text{had}} = (191 \pm 175) \times 10^{-11}.
\]

The agreement with the future experimental result for MAMM depends on a possible change
of the mean value of \(a_{\mu}^{\text{exp}}\) as one sees from eq. [22].
The prediction of the NLO hadronic contribution obtained in the model is fairly sensitive to the numerical value of the mass parameter for the light modes. This numerical value is however strictly determined by the LO result. To check how sensitive to the details of the model the results are I introduce a mass difference between $s$ and $u, d$ quarks ($SU(3)$ flavor violation in the mass sector in the approximation of exact isotopic invariance). I write $m_s = m_q + 0.18$ GeV with $0.18$ GeV being the value of the running mass for the strange quark. Then one finds

$$m_q = 166 \pm 1 \text{ GeV}$$

with

$$a_\mu^{\text{mod1}}(\text{LO}; \text{uds})(m_q = 166 \pm 1 \text{ MeV}) = (6928 \pm 71) \times 10^{-11}.$$  

The prediction of the NLO contribution in this case as compared to the $SU(3)$ symmetric one is

$$a_\mu^{\text{mod1}}(\text{NLO}) - a_\mu^{\text{mod1}}(\text{NLO}) = (2 + 4 + 14) \times 10^{-11}$$

where the first term comes from vertex corrections, the second term comes from insertions into the photon propagator and the last term comes from the light-by-light graphs. The result is fairly stable. Finally, the model with $SU(3)$ flavor violation in the mass sector gives the NLO hadronic contribution to MAMM

$$a_\mu^{\text{mod1}}(\text{NLO}) = 105 \times 10^{-11}$$

which is rather close to the prediction of the model with $SU(3)$ symmetric mass arrangement from eq. (68).

One could consider an even more sophisticated model including a violation of the isotopic invariance by using different masses for $u$ and $d$ quarks. An additional uncertainty emerges from the errors in the numerical value for the $c$-quark mass. By using the $\overline{\text{MS}}$ mass around 1.3 GeV for the $c$-quark one could enhance its LO contribution by about 50% (a leading order rescaling factor is $(m_c(\text{pole})/m_c(\overline{\text{MS}}))^2 = (1.6/1.3)^2 = 1.5$). Within the proposed model the use of the pole mass of the heavy quark looks more natural while an account of the difference between the numerical values for the pole and $\overline{\text{MS}}$ masses is beyond the accuracy of the approximation used for heavy quarks. It can readily be done since the contribution of heavy quarks is perturbative and corrections in the strong coupling constant can reliably be found.

5 Discussion

The underlying idea of the presented analysis is to introduce a framework for computing the NLO hadronic contributions to MAMM using the LO information. Presently the results for the light-by-light contribution that is the most interesting term at NLO are available analytically for
fermions that dictates the choice of the model from the technical point of view almost uniquely. Thus, a model of massive quarks with the EM interaction emerges as a suitable candidate. It is not an approximation for QCD as a gauge model with constituent quarks. It is just a bridge from LO to NLO results for hadronic contributions to a particular observable. Note that for another important parameter of the standard model -- the running EM coupling constant at the scale of the Z-boson mass -- there is no possibility to use such kind of a model as there is no important next-to-leading order terms to compute. Calculations for the infrared (IR) sensitive observables using constituent quarks with masses around 300–500 MeV as the only IR scales are unjustified in pQCD in general since the higher order corrections in the strong coupling constant cannot be found. Also the introduction of finite masses for the light quarks explicitly violates chiral invariance which is a well established symmetry of the light hadronic sector. In this sense the approximation for QCD with constituent quarks cannot be considered as a reasonable general framework. In the high energy limit the massless approximation for strong interactions is perturbative and quite precise. This means that high energy contributions to MAMM can be represented by almost any model that satisfies the duality constraints. In this sense the fermionic model fits the standard approximation for large energies. However, the main contribution to MAMM comes from the IR region where there is no sensible approximation for strong interactions deduced from QCD. Therefore the necessary characteristics of the strong interaction amplitudes relevant for the computation of MAMM has to be extracted from data. The first amplitude that emerges is the two-point correlator that is given by a single function of one variable with simple analytic properties (see eqs. (8,10)). For computing MAMM one need not know the point-wise behavior of the spectrum but only the integral over all energies with some enhancement of the threshold region. A model of massive fermions is then well suitable to fit this integral over data. When hadrons are introduced into the threshold IR region to fit experiment the effective masses of quarks increase. Therefore an account of low-energy hadronization for the two-point function entering MAMM is achieved by introducing an explicit cut in energy in the sum over the states. In practice, at the leading order the hadron contributions are represented by the pion with an EM interaction of the form \( e j_\mu^\pi A^\mu \) where at the leading order \( j_\mu^\pi = i(\pi^+ \overrightarrow{\partial}_\mu \pi^-) \). The inclusion of pions leads to the scalar type of the spectrum near the threshold

\[
\rho_\pi(s) = \frac{1}{12} \sqrt{1 - \frac{4m_\pi^2}{s}} \left(1 - \frac{4m_\pi^2}{s}\right)
\]

(73)

instead of the fermionic form given in eq. (33). Furthermore, the fermionic contributions can be moved to higher energies by using vector mesons. In the vector meson dominance model one identifies the EM current with the canonically normalized elementary \( \rho \)-meson field \( \rho_\mu \) through the relation \( j_\mu^{\text{had}} = f_\rho \rho_\mu \). Here \( f_\rho \) gives a form factor related to the leptonic width of the \( \rho \)-meson. Because of the nature of the MAMM observable this contribution can well be
represented by fermions already as it resides at a rather large scale. This hadronization picture is transparent for the two-point function which is sufficient for the LO analysis. At NLO an hadronization procedure for the four-point function is necessary. Within a hadron picture of the low-energy spectrum the most important contribution to MAMM comes from pions. To quantitatively handle contributions from the four-point function a quantum field model for pions given by the Lagrangian

\[ L_{\text{lowenergy}} = |D_\mu \pi|^2 - m_\pi^2 \pi^2, \quad D_\mu = \partial_\mu - ieA_\mu \]  

(74)
is introduced. This model generates vertices that allow to compute the pion contribution to the four-point hadronic EM current correlator that enters the light-by-light diagram explicitly. The high energy contribution of this model should then be replaced by the standard quark contributions. In the pure fermionic model with the small effective mass this replacement is effectively made at rather low energies that makes the separate contribution of pions small or even vanishing. Thus, the hadronization procedure of the model is realized through light massive quarks rather than real hadrons. Note that the hadronization picture need not be universal for all strong interaction processes but can specially be tailored for a given observable.

The results for MAMM related to the two-point function which have been obtained in data-based analysis are well reproduced by the model with the mass of the light fermion around the pion mass. Using the model prediction for the light-by-light graph I find an agreement of the NLO hadron contribution to MAMM with experiment. The results for the light-by-light graph in the pion model are available numerically. In the absence of analytical expressions for the light-by-light contributions in the pion model I could not quantitatively check how fermionic contributions replace the pion ones when the effective fermion mass decreases. However, it seems probable that the explicit inclusion of the pion contributions in the framework of the present model will shift the effective mass of light quarks larger.

The fermionic model gives a smooth spectrum at low energies. An important question is whether such a smooth spectrum is a reasonable approximation for computing MAMM. In the two-point correlator there are no resonances in the relevant region. The contribution of the \(\rho\)-meson is located at relatively large scales. In the axial channel, for instance, the situation is different because of the presence of the pion resonance and probably the model with massive quarks would not fit. Note, however, that as soon as the pion is considered to be massive (not a pure Goldstone mode) the chiral invariance is explicitly broken that makes quarks massive as well (or vice versa). Note also that a model can be suited for a description of a specific observable and need not give a universal approximation of any Green function. For instance, in the axial-vector two-point correlator the projection related to spin-one particles contains only massive resonances and the spectrum can well be approximated by a fermionic model without the pion pole. For the four-point functions the situation is more complicated though. In the literature there are models where the four-point function at low energies is represented through
the elementary fields of neutral pseudoscalar bosons in order to compute contributions of the light-by-light graphs to MAMM. The representation employs the neutral pion contribution to the four-point function through the iteration of an effective Lagrangian for the interaction of the neutral pion $\pi^0$ with photons due to the Abelian anomaly in axial current (as a review see ref. [16]). The result for the light-by-light contribution obtained using the neutral pion dominance eq. (57) is different from one obtained in the present model eq. (56). In general, the reduction of the four-point amplitude of the hadronic EM currents to a two-point correlator of axial currents uses the operator product expansion at small distances

$$iT_{\mu} j_{\nu}^{had}(x) j_{-\mu}^{had}(-x)|_{x \to 0} = \varepsilon_{\mu\nu\lambda\omega} x^\lambda \cdot \cdot \cdot$$

(75)

where $C(x^2)$ is a coefficient function of the local operator $j_{5}^{\lambda}(0)$ which has quantum numbers of axial current (e.g. [17]). In other words the combination of two hadronic EM currents of the form

$$\varepsilon_{\mu\nu\lambda} \xi^\lambda T_{\mu} j_{\nu}^{had}(x + \xi) j_{\nu}^{had}(x - \xi) F(\xi^2)$$

(76)

taken at small $\xi$ with some form factor $F(\xi^2)$ may act in some applications as a local axial current that can serve as an interpolation field for the pion. Thus, this combination can be replaced by a fundamental pion field in a hadronization procedure. This kind of factorization for the four-point amplitude is valid for $\gamma\gamma \rightarrow \gamma\gamma$ scattering in a specific region of the phase space in kinematic variables where all three external momenta are essential. In other regions of the phase space the saturation of the scattering amplitude with the pion pole contribution is invalid. The projection of the four-point function that emerges in the light-by-light graphs for the MAMM calculation has the form

$$\int dx \delta \langle T_{\mu} j_{\nu}^{had}(x) j_{\alpha}^{had}(y) j_{\beta}^{had}(z) j_{5}^{had}(0) \rangle.$$

(77)

In momentum space this projection depends on two external momenta only as the third momentum is set to zero after differentiation according to the definition of MAMM. In the neutral pseudoscalar model the projection of the four-point function given in eq. (77) is saturated by the contribution of the neutral pion that seems to be invalid in the kinematical region relevant for MAMM computation. In the absence of the neutral pion pole contribution in the hadronization picture for the light-by-light graph the fermionic model can be used for its computation on the same footing as it was used for vacuum polarization graphs. In fact, the neutral pion contribution gives the major difference with the present analysis based on the fermionic model. However, the corresponding contribution of the neutral pion to the projection of the four-point function emerging in the photon propagator is usually not considered. In other words, the neutral pion approximation for the four-point function should also be taken into account in eq. (30) as it is accounted for in eq. (77). If the neutral pion contribution to eq. (30) does not vanish by some symmetry considerations it can lead to a cut starting from the pion mass
that seems to contradict the threshold behavior of the spectrum known in $e^+e^-$ annihilation. This calls for a necessity to evaluate the validity of the neutral pion dominance model for the four-point function in the kinematical region relevant for computing the NLO hadronic contribution to MAMM.

Despite the fact that the fermionic model with the mass $m_q = 179$ MeV predicts the value for the NLO hadronic contribution to MAMM in agreement with experiment there remains a disturbing feeling that this prediction is obtained within an unrealistic approximation for strong interactions and, therefore, cannot be taken seriously. A historic reminiscence may be appropriate here. A century ago thinking about the light as existing in the form of discrete portions – photon quanta – was rather disturbing for classical physics. However, the quantum representation allowed for the quantitative explanation of experimental facts on photoeffect and black body radiation. It did not change the description of electromagnetic phenomena insensitive to the quantum nature of the light. It may happen that the muon anomalous magnetic moment is sensitive to the contribution of all hadrons in a way it would be sensitive to that of free fermions with an appropriate mass which is a standard realization of duality concept. The direct application of this concept to a particular case of MAMM looks suspicious because the IR region is explicitly involved in the analysis and the results depend strongly on the numerical value of the effective quark mass which happens to be rather small. The model, however, is only designed for computing the NLO hadronic contribution to MAMM using the LO result as input. This does not mean that this model approximation suited for computing MAMM is in any sense a universal limit of QCD automatically applicable to other observables.

6 Conclusion

A model for describing the NLO hadronic contributions to the muon anomalous magnetic moment is proposed. The model contains a single parameter that is fixed from the experimental result for the LO hadronic contribution to MAMM. The model describes the NLO hadronic contributions of the vacuum polarization type in agreement with existing estimates. However, it predicts a numerical value for the light-by-light contribution which is different from one used in the literature that considerably changes the prediction of the total NLO hadronic contribution to MAMM. The prediction of the model agrees with the present experimental value for MAMM. A resolution of the contradiction between the estimates for the NLO hadronic contribution, or rather for the light-by-light contribution, obtained in the present model and existing in the literature could help in verifying the validity of the standard model.

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