POGORELOV TYPE ESTIMATES FOR A CLASS OF HESSIAN QUOTIENT EQUATIONS IN LORENTZ-MINKOWSKI SPACE $\mathbb{R}_{1}^{n+1}$

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Abstract. Let $\Omega$ be a bounded domain (with smooth boundary) on the hyperbolic plane $\mathbb{H}^{n}(1)$, of center at origin and radius 1, in the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{R}_{1}^{n+1}$. In this paper, by using a priori estimates, we can establish Pogorelov type estimates of $k$-convex solutions to a class of Hessian quotient equations defined over $\Omega \subset \mathbb{H}^{n}(1)$ and with the vanishing Dirichlet boundary condition.

Keywords: Hessian quotient equations, $k$-convex, Lorentz-Minkowski space, Dirichlet boundary condition, a priori estimates.

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1. Introduction

As we know, Pogorelov firstly \[32\] obtained Pogorelov’s $C^{2}$ interior estimates for Monge-Ampère equations (see also \[20\]). Later, Chou and Wang \[8,40\] improved this result to a more general setting – they can obtain Pogorelov type estimates for $k$-Hessian equations. More precisely, for a bounded Euclidean domain $\Omega \subset \mathbb{R}^{n}$ with smooth boundary $\partial \Omega$, they considered the following Dirichlet problem of the $k$-Hessian equation

\[
\begin{cases}
\sigma_k(\nabla^2 u) = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.1) with as usual \[1\]

\[
\sigma_k(\nabla^2 u) := \sigma_k(\lambda(\nabla^2 u)) = \sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}
\]

(1.2) the $k$-th elementary symmetric function of eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda(\nabla^2 u)$ of the Hessian matrix $\nabla^2 u$, and successfully proved that if $u \in C^2(\Omega)$ is a $k$-convex solution to the Hessian equation (1.1), then there exists a constant $\beta > 0$ such that

\[
\sup_{\Omega} (-u)^{\beta} |\nabla^2 u| \leq C
\]

for some constant \[2\] $C$. Here, $u$ is said to be $k$-convex if $\lambda(\nabla^2 u)$ belongs to the Garding’s cone (see also \[2,0\])

\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0, \ j = 1, 2, \ldots, k \}.
\]

In some literatures, “$k$-convex” is also called “$k$-admissible”, and we have also used this terminology – see, e.g., \[15,16,17\]. Several years ago, by imposing a stronger assumption, Li-Ren-Wang \[27\] improved Chou-Wang’s above result to a more general setting, that is, if furthermore $f$ in

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1 We make an agreement that in this paper, $\sigma_k(\lambda(\cdot))$ would denote the $k$-th elementary symmetric function of eigenvalues of a given tensor.

2 In the sequel, many constants would appear, and, by abuse of notations and without any potential confusion, we prefer to use the notation $C$ to represent some of them.
the RHS of the first equation in (1.1) depends also on the gradient term $\nabla u$, then they can also establish Pogorelov type estimates for $(k + 1)$-convex solutions to the Dirichlet problem of the $k$-Hessian equation. Besides, there is a special case – when $k = 2$, their $3$-convex assumption can be replaced by $2$-convexity to derive the Pogorelov type estimates. Based on these conclusions, one might ask a natural question as follows:

- **Problem 1.** Whether Pogorelov type estimates are still valid for Hessian quotient equations

$$\frac{\sigma_k(\lambda(\nabla^2 u))}{\sigma_l(\lambda(\nabla^2 u))} = f(x, u, \nabla u)$$

or not?

As far as we know, **Problem 1** is still open.

In 2020, Chu and Jiao [9] considered the Hessian type equation with vanishing Dirichlet boundary condition (DBC for short)

$$(1.3) \begin{cases} \sigma_k(\lambda(\tilde{U}[u])) = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial \Omega$, where $\tilde{U}[u] := (\Delta u) \cdot I - \nabla^2 u$ with $\Delta u$ the Laplacian of $u$ and $I$ the identity matrix, and, as before, $\sigma_k(\lambda(\tilde{U}[u]))$ stands for the $k$-th elementary symmetric function of eigenvalues of the $(0, 2)$-type tensor $\tilde{U}[u]$. They can establish Pogorelov type estimates for $k$-convex solutions to (1.3). Recently, inspired by this work, Chen, Tu and Xiang [7] investigated the Hessian quotient version of (1.3) as follows

$$(1.4) \begin{cases} \frac{\sigma_k(\lambda(\tilde{U}[u]))}{\sigma_l(\lambda(\tilde{U}[u]))} = f(x, u, \nabla u) & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \partial \Omega, \quad l + 2 \leq k \leq n, \quad l > 0, \end{cases}$$

and successfully obtained Pogorelov type estimates for $k$-convex solutions to (1.4), which improves Chu-Jiao’s result mentioned above a lot. When $\tilde{U}[u]$ in the problem (1.3) was replaced by $\tau(\Delta u) \cdot I - \nabla^2 u$, $\tau \geq 1$, Qing Dai have established the similar result in her master’s dissertation [10].

Caffarelli-Nirenberg-Spruck [3, 4] and Trudinger [39] considered a class of fully nonlinear elliptic equations with DBC, which covers (1.4) as a special case, and obtained the existence of solutions under suitable assumptions. It is easy to see that if $k = n$ in (1.3), then the corresponding Dirichlet problem degenerates into a Monge-Ampère type equation with vanishing DBC. Harvey-Lawson [25, Example 4.3.2] studied the $p$-convex $(1 \leq p \leq n)$ solutions to this Monge-Ampère type equation with vanishing DBC and solved the Dirichlet problem with $f = 0$ on suitable domains. Besides, they introduced the $(n - 1)$-convexity or the general $p$-convexity $(1 \leq p \leq n)$ for solutions to the Dirichlet problem of nonlinear elliptic equations in a series of papers [21, 23, 24].

Similar to the concept of $(n - 1)$-convexity, for a complex-valued function $u$, it is called $(n - 1)$-plurisubharmonic if $(\Delta u) \cdot I - \nabla^2 u$ is nonnegative definite, and this concept was also introduced by Harvey-Lawson [22, 23]. Clearly, complex Monge-Ampère type equations for

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3 A function $u$ is called $(n - 1)$-convex if the matrix $(\Delta u) \cdot I - \nabla^2 u$ is nonnegative definite.
(n - 1)-plurisubharmonic functions can be defined as follows

\[
\det \left( \sum_{m=1}^{n} \frac{\partial^2 u}{\partial z_m \partial \bar{z}_m} \delta_{ij} - \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right) = f.
\]

For strict pseudo-convex domains in the complex Euclidean \( n \)-space \( \mathbb{C}^n \), the Dirichlet problem of (1.5) with positive \( f \) was solved by Li [25]. On compact complex manifolds, the Eq. (1.5) was investigated by Tosatti-Weinkove [37, 38], and moreover, it was shown that this equation has relation with the Gauduchon conjecture in complex geometry (see [18, 35]). For more progresses on the Eq. (1.5), we refer readers to [11, 12, 36] and references therein.

Elliptic equations of type similar to the one in (1.3) also arise naturally in conformal geometry – see [7, pp. 274-275] for a brief explanation and check [33] for details.

Therefore, from the above introduction, it should be interesting and meaningful to study the Dirichlet problem (1.3) and its more general version (1.4).

In this paper, we study the Lorentz-Minkowski version of the problem (1.4) and try to get the related Pogorelov type estimates. In order to state our main result clearly, we need to give some notions. Throughout this paper, let \( \mathbb{R}^{n+1}_1 \) be the \((n+1)\)-dimensional \((n \geq 2)\) Lorentz-Minkowski space with the following Lorentzian metric

\[
\langle \cdot, \cdot \rangle_L = dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2.
\]

In fact, \( \mathbb{R}^{n+1}_1 \) is an \((n+1)\)-dimensional Lorentz manifold with index 1. Denote by

\[
\mathcal{H}^n(1) = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_1 | x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\},
\]

which is exactly the hyperbolic plane of center \((0, 0, \ldots, 0)\) (i.e., the origin of \( \mathbb{R}^{n+1}_1 \)) and radius 1 in \( \mathbb{R}^{n+1}_1 \). Clearly, \( \mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1 \) is a spacelike hypersurface which is simply-connected Riemannian \( n \)-manifold with constant negative curvature and is geodesically complete. This is the reason why we call \( \mathcal{H}^n(1) \) a hyperbolic plane. Let \( \Omega \subset \mathcal{H}^n(1) \) be a bounded domain with smooth boundary \( \partial \Omega \), and then consider the following Hessian quotient equation with vanishing DBC

\[
\begin{align*}
\sigma_k(\lambda(U[u])) &= f(x, u, \nabla u) & \text{ in } \Omega \subset \mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1, & l + 2 \leq k \leq n, \ l \geq 0, \\
u &= 0 & \text{ on } \partial \Omega,
\end{align*}
\]

where \( u(x) \) is a function defined over \( \Omega \), \( U[u] := \tau(\Delta u) \cdot I - \nabla^2 u \) with \( \tau \geq 1 \), and, with the abuse of notations, \( \nabla, \Delta, \nabla^2 \) are the gradient, the Laplace and the Hessian operators on \( \mathcal{H}^n(1) \) respectively. For the Dirichlet problem (1.3), we can prove:

**Theorem 1.1.** Suppose that \( k > l + 1, \ u \in C^4(\Omega) \cap C^2(\overline{\Omega}) \) is a solution to the Hessian quotient equation (1.6) with \( \lambda(U[u]) \in \Gamma_k \), \( f \) is a positive smooth function. Then there exists a constant \( \beta > 0 \) such that for \( n \geq k \geq l + 2 \), we have

\[
\sup_{x \in \Omega} (-u)^{\beta} |\nabla^2 u|(x) \leq C,
\]

where \( C \) depends on \( n, k, l, \tau, \sup_{\Omega} |u| \) and \( \sup_{\Omega} |\nabla u| \).

**Remark 1.1.** (1) Obviously, by Theorem 1.1 one knows that if \( C^0, C^1 \) estimates could be obtained for the \( k \)-convex solutions to (1.6), then the interior \( C^2 \) estimates follows directly.

(2) In the Dirichlet problem (1.6), if the function \( u \) was replaced by a spacelike graphic function \( \tilde{u} \), the \((0, 2)\)-type tensor \( \tilde{U} \) was replaced by the second fundamental form of the spacelike graphic hypersurface determined by \( \tilde{u} \), and the zero DBC was replaced by an affine function defined over \( \partial \Omega \), then (1.6) would become a prescribed curvature problem (PCP for short) recently
considered by the corresponding author, Prof. J. Mao, and his collaborators. They successfully obtained the a priori estimates for $k$-admissible solutions to the PCP, and, together with the method of continuity, showed the existence and uniqueness of $2$-admissible solution to the PCP – see [16, Theorem 1.4] for details. Of course, in [16], the $C^2$ interior estimates for the $k$-admissible solutions to the PCP are important to get the existence. This research experience, together with the continuous study of the Dirichlet problem (1.3) and its more general version (1.4) in different settings, is exactly our motivation of considering the problem (1.6) in this paper.

(3) Inspired by the work [27], it should be interesting to know whether the Pogorelov type estimates in Theorem 1.1 only for the case $\Omega \subset \mathbb{R}^n(1)$.

(4) Our Theorem 1.1 here and [7, Theorem 1.1] somehow reveal the reasonability of considering Problem 1 and show the hope of possibly solving it.

(5) Inspired by Theorem 1.1 here, it is natural and feasible to try to improve the existing Pogorelov type estimates for solutions to some prescribed elliptic PDEs (with DBC) defined over bounded domains (with boundary) in the Euclidean space or Riemannian manifolds to our setting – investigating elliptic PDEs of the same type defined over bounded domains (with boundary) in the spacelike hypersurface $\mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1$. We have already obtained an interesting result and now we prefer to leave this attempt to readers who are interested in this topic.

(6) In fact, one can consider the problem (1.6) defined over more general bounded domains (with smooth boundary) in Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$, and similar conclusion can be obtained – see Remark 3.1 for details.

(7) Since in our previous works [15, 16], the PCPs therein were considered for spacelike graphic functions defined over bounded domains (with boundary) in the hyperbolic plane $\mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1$, in order to embody the continuity of our study on geometric elliptic PDEs in $\mathcal{H}^n(1)$, we insist on giving the Pogorelov-type estimates in Theorem 1.1 only for the case $\Omega \subset \mathcal{H}^n(1)$.

The paper is organized as follows. In Section 2, some useful formulae (including the structure equations for spacelike hypersurfaces in $\mathbb{R}^{n+1}_1$, some basic properties of elementary symmetric functions, etc) will be listed. The proof of Theorem 1.1 will be shown in Section 3.

2. SOME USEFUL FORMULAE

Let $g_{\mathcal{H}^n(1)}$ be the Riemannian metric on $\mathcal{H}^n(1)$ induced by the Lorentzian metric $(\cdot, \cdot)_L$ of $\mathbb{R}^{n+1}_1$. For a $(s, r)$-tensor field $\alpha$ on $\mathcal{H}^n(1)$, its covariant derivative $\nabla \alpha$ is a $(s, r+1)$-tensor field given by

$$\nabla \alpha(Y_1, \cdots, Y^s, X_1, \cdots, X_r, X) = \nabla_X \alpha(Y_1, \cdots, Y^s, X_1, \cdots, X_r) = X(\alpha(Y_1, \cdots, Y^s, X_1, \cdots, X_r)) - \alpha(\nabla_X Y^1, \cdots, Y^s, X_1, \cdots, X_r) - \cdots - \alpha(Y^1, \cdots, Y^s, X_1, \cdots, \nabla_X X_r),$$

and its components in local coordinates are denoted by

$$\nabla^2 \alpha_{l_1 \cdots l_s}^{r_1 \cdots r_1} = \alpha_{l_1 \cdots l_s}^{r_1 \cdots r_1},$$

where $1 \leq l_i, k_j \leq n$ with $i = 1, 2, \cdots, s$ and $j = 1, 2, \cdots, r+1$. Here the comma “,” in subscript of a given tensor means doing covariant derivatives. Besides, we make an agreement that, for simplicity, in the sequel the comma “,” in subscripts will be omitted unless necessary. One can continue to define the second covariant derivative of $\alpha$ as follows:

$$\nabla^2 \alpha(Y_1, \cdots, Y^s, X_1, \cdots, X_r, X, Y) = (\nabla_Y (\nabla \alpha))(Y^1, \cdots, Y^s, X_1, \cdots, X_r, X).$$
and then its components in local coordinates are denoted by
\[ \alpha_{\ell_1 \cdots \ell_s}^{k_1 \cdots k_r, k_{r+1} \cdots k_{r+2}}, \]
where \( 1 \leq \ell_i, k_j \leq n \) with \( i = 1, 2, \cdots, s \) and \( j = 1, 2, \cdots, r + 2 \). Similarly, the higher order covariant derivatives of \( \alpha \) are given as follows
\[ \nabla^3 \alpha = \nabla(\nabla^2 \alpha), \quad \nabla^4 \alpha = \nabla(\nabla^3 \alpha), \cdots, \]
and so on.

On \( \mathcal{H}^n(1) \), for any tangent vector fields \( X, Y, Z \), the Riemannian curvature \((1,3)\)-tensor \( R \) w.r.t. \( g_{\mathcal{H}^n(1)} \) is defined as
\[ R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, \]
where, as usual, \([\cdot,\cdot]\) denotes the Lie bracket. In a local coordinate chart \( \{\xi^i\}_{i=1}^n \) of \( \mathcal{H}^n(1) \), the component of the curvature tensor \( R \) is defined by
\[ R \left( \frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j} \right) \frac{\partial}{\partial \xi^k} = R^l_{kij} \frac{\partial}{\partial \xi^l}, \]
where \( R^m_{ijkl} = \sigma_{jm} R^m_{ki\ell} \) and \( \sigma_{lm} := g_{\mathcal{H}^n(1)} \left( \frac{\partial}{\partial \xi^l}, \frac{\partial}{\partial \xi^m} \right) \). Then, we have the standard commutation formulas (i.e., Ricci identities)
\[ \alpha_{\ell_1 \cdots \ell_s}^{k_1 \cdots k_r, j} - \alpha_{\ell_1 \cdots \ell_s}^{k_1 \cdots k_r, i,j} = -\sum_{a=1}^r R^m_{ijkl} \alpha_{a \ell_1 \cdots \ell_s}^{k_1 \cdots k_a \cdots k_{r+1} \cdots k_{r+s}} - \sum_{b=1}^s R^m_{ijb} \alpha_{k_1 \cdots k_r l_{b+1} \cdots l_s}^{l_1 \cdots l_b-1 \cdots l_s}. \]
Denote by \( X \) the position vector field of the spacelike hypersurface \( \mathcal{H}^n(1) \subset \mathbb{R}^{n+1}_1 \). Clearly, for any \( x \in \mathcal{H}^n(1) \), \( X(x) \) is a one-to-one correspondence w.r.t. \( x \). Let \( \nu \) be the future-directed timelike unit normal vector field and \( h_{ij} \) be coefficient components of the second fundamental form of the hypersurface \( \mathcal{H}^n(1) \) w.r.t. \( \nu \), that is,
\[ h_{ij} = -\langle X_{,ij}, \nu \rangle_L. \]
Recall the following identities
\[ X_{,ij} = h_{ij} \nu, \quad \text{(Gauss formula)} \]
\[ \nu_{,i} = h_{ij} X^j, \quad \text{(Weingarten formula)} \]
where \( X^j = \sigma^{ij} X_i \). Moreover, \( \mathcal{H}^n(1) \) has constant sectional curvature \(-1\) w.r.t. \( g_{\mathcal{H}^n(1)} \) and satisfies the following Gauss equation
\[ R_{ijkl} = \overline{R}_{ijkl} - \langle h_{ik} h_{jl} - h_{il} h_{jk} \rangle, \quad 1 \leq i, j, k, l \leq n, \]
which together with the fact \( \overline{R} = 0 \) implies
\[ R_{ijkl} = -\langle h_{ik} h_{jl} - h_{il} h_{jk} \rangle, \quad 1 \leq i, j, k, l \leq n, \]
where \( \overline{R} \) stands for the curvature tensor of \( \mathbb{R}^{n+1}_1 \). For a brief introduction to the structure equations of spacelike hypersurfaces in \( \mathbb{R}^{n+1}_1 \), one can also check some of the corresponding author’s previous works, e.g., [13, 14, 15, 16]. We refer readers to an interesting book [1] about systematical knowledge on submanifolds in pseudo-Riemannian geometry.

\*\*This fact will be shown clearly in Section 3.\*\*
At the end of this section, we prefer to give some properties of elementary symmetric functions. Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), and then for \( 1 \leq k \leq n \), the \( k \)-th elementary symmetric function of \( \lambda \) can be defined as follows

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.
\]

We also set \( \sigma_0 = 1 \) and \( \sigma_k = 0 \) for \( k > n \) or \( k < 0 \). Recall that the Garding’s cone is defined by

\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n \mid \sigma_i(\lambda) > 0, \forall 1 \leq i \leq k \}.
\]

Denote by \( \sigma_{k-1}(\lambda|i) = \frac{\partial \sigma_k}{\partial \lambda_i} \). By, e.g., [4, 19, Lemma 2.2.19], [26, 29 Chapter XV], one has the following facts:

**Lemma 2.1.** Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) and \( 1 \leq k \leq n \), then we have

1. \( \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n \);
2. \( \sigma_{k-1}(\lambda|i) > 0 \) for \( \lambda \in \Gamma_k \) and \( 1 \leq i \leq n \);
3. \( \sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i) \) for \( 1 \leq i \leq n \);
4. \( \left[ \frac{\sigma_k}{\sigma_i} \right]^{\frac{1}{k-1}} \) are concave and elliptic in \( \Gamma_k \) for \( 0 \leq l < k \);
5. If \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), then \( \sigma_{k-1}(\lambda|1) \leq \sigma_{k-1}(\lambda|2) \leq \cdots \leq \sigma_{k-1}(\lambda|n) \) for \( \lambda \in \Gamma_k \);
6. \( \sum_{i=1}^{n} \sigma_{k-1}(\lambda|i) = (n-k+1)\sigma_{k-1}(\lambda) \).

The following Newton-MacLaurin inequality will be used as well (see, e.g., [30, 31]).

**Lemma 2.2.** Let \( \lambda \in \mathbb{R}^n \). For \( 0 \leq l \leq k \leq n \), \( r > s \geq 0 \), \( k \geq r \), \( l \geq s \), we have

\[
k(n-l+1)\sigma_{l-1}(\lambda)\sigma_k(\lambda) \leq l(n-k+1)\sigma_l(\lambda)\sigma_{k-1}(\lambda)
\]

and

\[
\left[ \frac{\sigma_k(\lambda)/C_n^k}{\sigma_l(\lambda)/C_n^l} \right]^{\frac{1}{k-l}} \leq \left[ \frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s} \right]^{\frac{1}{r-s}}, \quad \text{for } \lambda \in \Gamma_k.
\]

For convenience, we introduce the following notations

\[
F(U) = \left[ \frac{\sigma_k(U)}{\sigma_l(U)} \right]^{\frac{1}{k-l}}, \quad T(D^2 u) = F(\tau \Delta u I - D^2 u),
\]

\[
(2.7) \quad F^{ij} = \frac{\partial F}{\partial U_{ij}}, \quad F^{ij,rs} = \frac{\partial^2 F}{\partial U_{ij} \partial U_{rs}}, \quad T^{ii} = \tau \sum_{j=1}^{n} F^{ij} - F^{ii}.
\]

Thus,

\[
F^{ii} = \frac{1}{k-l} \left( \frac{\sigma_k(U)}{\sigma_l(U)} \right)^{\frac{1}{k-l} - 1} \frac{\sigma_{k-1}(U|i)\sigma_l(U) - \sigma_k(U)\sigma_{l-1}(U|i)}{\sigma_l^2(U)}.
\]

To handle the ellipticity of the equation (1.4), we need the following important proposition and its proof is almost the same as that of [5, Proposition 2.2.3].

**Lemma 2.3.** Let \( \lambda(U[u]) \in \Gamma_k \) and \( 0 \leq l \leq k-1 \). Then the operator

\[
(2.8) \quad F(U(u)) = \left( \frac{\sigma_k(\lambda(U))}{\sigma_l(\lambda(U))} \right)^{\frac{1}{k-l}}
\]
is elliptic and concave with respect to $U(u)$. Moreover we have

$$\sum_{i=1}^{n} F^{ii} \geq \left( \frac{C_k^i}{C_k^n} \right)^{\frac{1}{k-l}}.$$  

We also need the following well-known result.

**Lemma 2.4.** Let $U$ be a diagonal matrix with $\lambda(U[u]) \in \Gamma_k$, $0 \leq l \leq k-1$ and $k \geq 3$. Then

$$-F^{ii,ii}(U) = \frac{F^{11} - F^{ii}}{U_{ii} - U_{11}}$$  

for $i \geq 2$. Moreover, if $U_{11} \geq U_{22} \geq \cdots \geq U_{nn}$, we have

$$F^{11} \leq F^{22} \leq \cdots \leq F^{nn}.$$  

**Proof.** See [4, Proposition 2.3] for the proof of (2.10). The proof of (2.11) is almost the same with that of [2] Lemma 2.2 and we prefer to omit here. \hfill $\Box$

3. PROOF OF THEOREM 1.1

In this section, we will use an idea (which is similar to that in [7, 9]) to give the proof of Theorem 1.1.

Assume that $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is a solution of the problem (1.6) with $\lambda(U) \in \Gamma_k$. Without loss of generality, we assume $u < 0$ in $\Omega$ by the maximum principle.

Assume that a point on $H^n(1)$ is described by local coordinates $\xi^1, \ldots, \xi^n$, that is, $x = x(\xi^1, \ldots, \xi^n)$. For convenience, let $\partial_i := \partial/\partial \xi^i$ be the corresponding coordinate vector fields on $H^n(1)$ and then $\sigma_{ij} = g_{\mathcal{H}^n(1)}(\partial_i, \partial_j)$ be the Riemannian metric on $H^n(1)$. For spacelike graphic hypersurfaces $G := \{(x, \tilde{u}(x))|x \in \mathcal{H}^n(1)\}$ in $\mathbb{R}^{n+1}_1$, one knows that its induced metric from the Lorentzian metric $\langle \cdot, \cdot \rangle_L$ of $\mathbb{R}^{n+1}_1$ has the form $g := u^2g_{\mathcal{H}^n(1)} - dr^2$. By [13, Lemma 3.1] (see also [14, 15, 16]), it is easy to know that for $G$, the tangent vectors, the future-directed timelike unit normal vector and the second fundamental form are given by

$$X_i = \partial_i + \tilde{u}_i \partial_r, \quad i = 1, 2, \cdots, n,$$

$$\nu = \frac{1}{v} \left( \partial_r + \frac{1}{u^2} \tilde{u}^j \partial_j \right),$$

with $\tilde{u}^j := \sigma^{ij}_{\tilde{u}}$, $v := \sqrt{1 - \tilde{u}^{-2} |\nabla \tilde{u}|^2}$, and

$$h_{ij} = -\frac{1}{v} \left( \frac{2}{u} \tilde{u}_i \tilde{u}_j - \tilde{u}_{ij} - \tilde{u} \sigma_{ij} \right),$$

Since $\mathcal{H}^n(1)$ can be seen as the special case of $G$ with $\tilde{u} \equiv 1$, one has

$$\nu = \partial_r, \quad h_{ij} = \sigma_{ij},$$

which implies further the Gauss equation (2.25) has the following form

$$R_{ijkl} = -(\sigma_{ik} \sigma_{jl} - \sigma_{il} \sigma_{jk}), \quad 1 \leq i, j, k, l \leq n$$

for the spacelike hypersurface $\mathcal{H}^n(1)$. Then by Schur’s theorem (see, e.g., [4, Chapter 4]), one knows.\footnote{We prefer to mention readers that the computation $R_{ijkl} = \sigma_{ij} R^{\alpha}_{\alpha kl}$ (for components of curvature tensor) used in Section 2 has an opposite sign with the corresponding one used in [4, Chapter 4]. However, there is no essential difference between two settings – when using them to calculate sectional (or Ricci, scalar) curvatures, they coincide with each other. The reason why one meets two settings for $R_{ijkl}$ is that opposite orientations have been chosen for the unit normal vector when computing components of the second fundamental form.}
• $\mathcal{H}^n(1)$ has constant sectional curvature $-1$.

Obviously, when $\mathcal{G}$ degenerates into $\mathcal{H}^n(1)$, its metric $g$ becomes $g_{\mathcal{H}^n(1)}$ directly. Now, we would like to give more information about the induced metric $g_{\mathcal{H}^n(1)}$ of $\mathcal{H}^n(1) \subset \mathbb{R}^{n+1}$ with components $\sigma_{ij} = g_{\mathcal{H}^n(1)}(\partial_i, \partial_j)$ for any $1 \leq i, j, k, l \leq n$. As we know, for any point on $\mathcal{H}^n(1) \subset \mathbb{R}^{n+1}$, its global Lorentz-Minkowski coordinates $(x^1, x^2, \cdots, x^{n+1})$ can be reparameterized as follows

\[
\begin{align*}
  x^1 &= \cos \xi^1 \cos \xi^2 \cdots \cos \xi^{n-1} \sinh \xi^n \\
  x^2 &= \cos \xi^1 \cos \xi^2 \cdots \sin \xi^{n-1} \sinh \xi^n \\
  &\quad \cdot \cdots \cdot \\
  x^{n-1} &= \cos \xi^1 \sin^2 \sinh \xi^n \\
  x^n &= \sin \xi^1 \sinh \xi^n \\
  x^{n+1} &= \cosh \xi^n,
\end{align*}
\]

which implies

\[
g_{\mathcal{H}^n(1)} = \sinh^2 \xi^n (d\xi^1)^2 + \sinh^2 \xi^n \cos^2 \xi^1 (d\xi^2)^2 + \cdots + \sinh^2 \xi^n \cos^2 \xi^1 \cdots \cos^2 \xi^{n-1} (d\xi^n)^2 + (d\xi^n)^2,
\]

and then

\[
\sigma_{ij} = \delta_{ij} \sinh^2 \xi^n \cos^2 \xi^1 \cdots \cos^2 \xi^{i-1}, \quad \sigma_{11} = \sinh^2 \xi^n, \quad \sigma_{nn} = 1, \quad \sigma_{1i} = \sigma_{i1} = \sigma_{nn} = \sigma_{nn} = 0,
\]

for any $2 \leq i, j \leq n - 1$, with the inverse

\[
\begin{align*}
  \sigma^{ij} &= \delta^{ij} \sinh^{-2} \xi^n \cos^{-2} \xi^1 \cdots \cos^{-2} \xi^{i-1}, \\
  \sigma^{11} &= \sinh^{-2} \xi^n, \quad \sigma^{nn} = 1, \quad \sigma^{1i} = \sigma^{i1} = \sigma^{nn} = \sigma^{nn} = 0.
\end{align*}
\]

Choose a fixed point $q \in \mathcal{H}^n(1)$ which is outside the bounded domain $\Omega \subset \mathcal{H}^n(1)$, and define a function $\rho$ on $\Omega$ as follows

\[
\rho := \rho(q, x), \quad \forall x \in \Omega,
\]

where $\rho(q, x)$ measures the Riemannian distance between $q$ and $x$. Let $\gamma(t)$ be the minimizing geodesic connecting $q$ and $x$, with $\gamma(0) = q$, $\gamma(\rho) = x$. Let $\{\frac{d\gamma(t)}{dt}|_x, Y_1, \ldots, Y_{n-1}\}$ be an orthonormal basis of the tangent space $T_{\gamma(\rho)}\mathcal{H}^n(1)$. Parallel transport vectors $Y_1, \ldots, Y_{n-1}$ along the geodesic $\gamma(t)$ yields orthogonal vector fields $\tilde{Y}_i$, $i = 1, 2, \cdots, n - 1$. Clearly, each $\tilde{Y}_i$ is the Jaccobi field along $\gamma(t)$ satisfying $\tilde{Y}_i(\rho) = Y_i$. Since $\mathcal{H}^n(1)$ has constant sectional curvature $-1$, one has

\[
\tilde{Y}_i(t) = f(t)Y_i(t)
\]
with \( f(t) = \frac{1}{\sinh \rho} \sinh t \), and then

\[
\Delta \rho = \sum_{i=1}^{n-1} \text{Hess}(\rho)(Y_i, Y_i)
\]

\[
= \sum_{i=1}^{n-1} \int_0^\rho \left( |\nabla \beta i|_Y^2 + \langle \beta Y_i, \nabla \beta Y_i \rangle_{g,\mathcal{H}^n(1)} \right) dt
\]

\[
= (n-1) \int_0^\rho \left( \left| \frac{df(t)}{dt} \right|^2 + f^2(t) \right) dt
\]

\[
= (n-1) \int_0^\rho \frac{1}{\sinh^2 \rho} \sinh^2 t + \cosh^2 t dt
\]

\[
= (n-1) \cot \rho,
\]

where Hess is the Hessian operator on \( \mathcal{H}^n(1) \), and \( \langle \cdot, \cdot \rangle_{g,\mathcal{H}^n(1)} \) is the inner product w.r.t. the metric \( g,\mathcal{H}^n(1) \). Since \( \Omega \) is bounded and complete, from the definition (3.2), it is easy to know that \( \rho = \rho(q, x) \) has infimum \( c^- > 0 \) and supremum \( c^+ \) simultaneously. Therefore, we have

\[
(3.3) \quad (n-1) \coth(c^+) \leq \Delta \rho(q, x) \leq (n-1) \coth(c^-), \quad \forall x \in \Omega.
\]

On \( \Omega \), consider the following test function

\[
\tilde{P}(x) = \beta \log(-u) + \log \lambda_{\text{max}}(x) + \frac{a}{2} |\nabla u|^2 + A \cdot \rho(q, x),
\]

where \( \rho(q, x) \) is defined as (3.2), \( \lambda_{\text{max}}(x) \) is the biggest eigenvalue of the Hessian matrix \( u_{ij} \), and \( \beta, a, A \) are positive constants which will be determined later. Suppose that \( \tilde{P} \) attains its maximum value in \( \Omega \) at \( x_0 \). Choosing a local orthonormal frame field at \( x_0 \) such that \( g_{ij}(x_0) = \delta_{ij}(x_0) \). This can always be assured – e.g., choosing local coordinates \( \xi^1, \xi^2, \ldots, \xi^{n-1}, \sinh^{-1} \xi^n \) around \( x_0 \), where \( \xi^i, i = 1, 2, \ldots, n \) are determined by (3.1). Rotating further the coordinate axes, we can diagonal the matrix \( \nabla^2 u = (u_{ij}) \),

\[
u_{ij}(x_0) = u_{ii}(x_0) \delta_{ij}, \quad u_{11}(x_0) \geq u_{22}(x_0) \geq \cdots \geq u_{nn}(x_0),
\]

and then

\[
U_{11}(x_0) \leq U_{22}(x_0) \leq \cdots \leq U_{nn}(x_0).
\]

So, by (2.11) we can obtain

\[
F^{11}(x_0) \geq F^{22}(x_0) \geq \cdots \geq F^{nn}(x_0) > 0,
\]

\[
0 < T^{11}(x_0) \leq T^{22}(x_0) \leq \cdots \leq T^{nn}(x_0).
\]

On \( \Omega \), define a new function

\[
P(x) = \beta \log(-u) + \log u_{11}(x) + \frac{a}{2} |\nabla u|^2 + A \cdot \rho(q, x),
\]

which also attains a local maximum at \( x_0 \). Differentiating \( P \) at \( x_0 \) once yields

\[
\frac{\beta u_{ii}}{u} + \frac{u_{11} u_{ii}}{u_{11}} + au_{ii} u_{ii} + A \cdot \rho_i(q, x) = 0,
\]

and differentiating \( P \) at \( x_0 \) twice results in

\[
\frac{\beta u_{ii}}{u} - \frac{\beta u_{ii}^2}{u_2} + \frac{u_{11} u_{ii}}{u_{11}} - \frac{u_{11}^2}{u_{11}^2} + a \sum_{p=1}^n u_{p} u_{pp} + a u_{ii}^2 + A \cdot \rho_{ii}(q, x) \leq 0.
\]
Thus, at \( x_0 \),

\[
0 \geq T^{ii} P_{ii} \\
= \frac{T^{ii} \beta u_{ii}}{u} - \frac{\beta T^{ii} u^2_i}{u^2} + \frac{T^{ii} u_{11ii}}{u_{11}} - \frac{T^{ii} u^2_{11i}}{u^2_{11}} \\
+ a \sum_{p=1}^{n} u_p T^{ii} u_{pii} + aT^{ii} u^2_i + AT^{ii} p_{ii}
\]

\[
(3.4)
\]

where the Einstein summation convention has been used – repeated superscripts and subscripts should be made summation. The second inequality in (3.4) holds since \( \Delta \rho \) is globally defined and is independent of the choice of local coordinates. The last inequality in (3.4) holds because of the estimate (3.3). If one goes back to the proof of [7, Theorem 1.1], he or she would find that for calculations done at a point, the above inequality holds as well. If one goes back to the proof of [7, Theorem 1.1], he or she would find that for calculations done at a point, the above inequality holds as well. If one goes back to the proof of [7, Theorem 1.1], he or she would find that for calculations done at a point, the above inequality holds as well. If one goes back to the proof of [7, Theorem 1.1], he or she would find that for calculations done at a point, the above inequality holds as well. If one goes back to the proof of [7, Theorem 1.1], he or she would find that for calculations done at a point, the above inequality holds as well. If one goes back to the proof of [7, Theorem 1.1], he or she would find that for calculations done at a point, the above inequality holds as well. If one goes back to the proof of [7, Theorem 1.1], he or she would find that for calculations done at a point, the above inequality holds as well. If one goes back to the proof of [7, Theorem 1.1], he or she would find that for calculations done at a point, the above inequality holds as well. If one goes back to the proof of [7, Theorem 1.1], he or she would find that for calculations done at a point, the above inequality holds as well.
comparison theorem (see, e.g., [34]), one has
\[ \frac{n-1}{\rho} \leq \Delta \rho \leq \frac{n-1}{\rho} \sqrt{K} \rho \coth(\sqrt{K} \rho) \leq \frac{n-1}{\rho} \left( 1 + \sqrt{K} \rho \right). \]
Together with the fact \( 0 < c^{-} \leq \rho \leq c^{+} \), it is easy to know that
\[ 0 < \frac{n-1}{c^{+}} \leq \Delta \rho \leq \sqrt{K} + \frac{n-1}{c^{-}}. \]

Inspired by this observation and the construction of auxiliary functions in our proof of Theorem 1.1, it is not hard to know that our Pogorelov type estimates in Theorem 1.1 would still be valid if the domain \( \Omega \subset \mathcal{M}^{n}(1) \) in the problem (1.6) was replaced by \( \Omega \subset M \) mentioned as above. Of course, in this setting, \( M \) must be noncompact since \( M \) is simply connected and \( \text{Sec}(M) \leq 0 \) (using Cartan-Hadamard theorem directly, see, e.g., [6, Chapter 5]).

(2) One might see that Dai’s main conclusion in [10] can be improved to bounded domains (with smooth boundary) of simply connected complete Riemannian manifolds whose sectional curvature is non-positive and whose Ricci curvature is bounded from below by non-positive constant. What about bounded domains on complete manifolds with positive curvature? If one checks our proof here carefully, one might see that the key point is how to find a strictly positive lower bound for \( \Delta \rho \). Based on this reason, maybe complete manifolds with suitable pinching assumption for positive curvature could be expected to get similar conclusion as well. Readers who have interest can try this improvement. So far, we do not have this interest to finish this, since what we are caring about is the Lorentz-Minkowski situation.

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