Ordered Biclique Partitions and Communication Complexity Problems

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Abstract

An ordered biclique partition of the complete graph $K_n$ on $n$ vertices is a collection of bicliques (i.e., complete bipartite graphs) such that (i) every edge of $K_n$ is covered by at least one and at most two bicliques in the collection, and (ii) if an edge $e$ is covered by two bicliques then each endpoint of $e$ is in the first class in one of these bicliques and in the second class in other one. In this note, we give an explicit construction of such a collection of size $n^{1/2+o(1)}$, which improves the $O(n^{2/3})$ bound shown in the previous work [2, Disc. Appl. Math., 2014].

As the immediate consequences of this result, we show (i) a construction of $n \times n$ 0/1 matrices of rank $n^{1/2+o(1)}$ which have a fooling set of size $n$, i.e., the gap between rank and fooling set size can be at least almost quadratic, and (ii) an improved lower bound $(2-o(1)) \log N$ on the nondeterministic communication complexity of the clique vs. independent set problem, which matches the best known lower bound on the deterministic version of the problem shown by Kushilevitz, Linial and Ostrovsky [10, Combinatorica, 1999].

Keywords biclique partition, Boolean matrix, fooling set, rank, complete graphs

1 Introduction

Let $G = (V, E)$ be an undirected graph. For two disjoint subsets $U$ and $W$ of $V$, the complete bipartite graph with edge set $U \times W$ is called a biclique and is denoted by $B(U, W)$. For an integer $k \geq 1$, a collection of bicliques $\{B(U_i, W_i)\}$ is called a $k$-biclique covering of $G$ if every edge in $G$ lies in at least one and at most $k$ bicliques in the collection. The minimum size of a $k$-biclique covering of $G$ is denoted by $bp_k(G)$. In particular, a 1-biclique covering is called a biclique partition and its minimum size $bp_1(G)$ is just denoted by $bp(G)$. Biclique coverings of graphs have been widely investigated in the literature (see e.g., [1] [6] [9]).

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In the preceding work [2], we introduced an “intermediate” notion between the biclique partition and 2-biclique covering, which we call an ordered biclique partition. An ordered biclique partition of $G$ is a 2-biclique covering $\{B(U_i, W_i)\}_i$ with an additional restriction that if an edge $e = \{u, v\}$ is covered by two bicliques, say $B(U_k, W_k)$ and $B(U_\ell, W_\ell)$, then each endpoint of $e$ belongs to a distinct color class in these bicliques, i.e., $w \in U_k \cap W_\ell$ or $w \in U_\ell \cap W_k$ for $w \in \{u, v\}$. The minimum size of such a partition is denoted by $bp_{1.5}(G)$. Recently, in [2], the second author of this note showed $bp_{1.5}(K_n) = O(n^{2/3})$ by giving an explicit construction of such a partition, where $K_n$ denotes the complete graph on $n$ vertices.

In this note, we improve this bound to $bp_{1.5}(K_n) = n^{1/2+o(1)}$, which is the main contribution of this note. This bound is almost tight since $bp_{1.5}(K_n) \geq bp_2(K_n) = \Theta(n^{1/2})$ where the bound on $bp_2(K_n)$ is due to Alon [1].

The original motivation for considering such a parameter is its close connection to the problems related to communication complexity. One of such is the “rank” vs. “fooling set” problem. Let $M$ be an $n \times n$ $0/1$-matrix. The rank of $M$ over the reals is denoted by $\text{rank}(M)$. A set $S \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ of the index set of $M$ is called a fooling set for $M$ if there exists a value $z \in \{0, 1\}$ such that

1. for every $(k, \ell) \in S$, $M_{k, \ell} = z$,
2. for any distinct $(k_1, \ell_1)$ and $(k_2, \ell_2)$ in $S$, $M_{k_1, \ell_2} \neq z$ or $M_{k_2, \ell_1} \neq z$.

The largest size of a fooling set of $M$ is denoted by $\text{fool}(M)$. Analyzing the size of a fooling set is one of the main tools for proving lower bounds on the communication complexity (see e.g., [11]).

It is known that $\text{fool}(M) \leq (\text{rank}(M) + 1)^2$ (see Dietzfelbinger, Hromkovič and Schnitger [4]). The open question is whether this quadratic gap can be improved or not (see e.g., [1] Open Problem 2). M. Hühne (described in [4], and see also [2] [14]) constructed a matrix $M$ such that $\text{fool}(M) \geq \text{rank}(M)^{\log_4 6} = \text{rank}(M)^{1.292\ldots}$. This was improved to $\text{fool}(M) \geq \Omega(\text{rank}(M)^{1.5})$ in the previous work of the second author of this note [2]. A biclique partition presented in this note immediately gives a new separation $\text{fool}(M) \geq \text{rank}(M)^{2-o(1)}$, which is almost tight. Note that recently Friezen and Theis [4] proved that the exponent 2 on the rank is tight if we take the rank in a field of characteristic two. See also [7] for a recent development on a related problem.

Our new partition also gives an improved bound on the nondeterministic communication complexity of the clique vs. independent set problem (see e.g., a textbook [11] for the background and definition of the problem). It was shown that finding a graph $H$ with $\chi(H) \geq f(bp_{1.5}(H))$ for some function $f(\cdot)$ is essentially equivalent to proving $\log_2 f(N)$ lower bound on the nondeterministic communication complexity for the problem for an explicit graph on $N$ vertices, where $\chi(H)$ denotes the chromatic number of $H$ [2]. (See also [3] [13] for this equivalence. In these papers, $bp_{1.5}(\cdot)$ is denoted by $bp_{\text{or}}(\cdot).$) Combining this with our biclique partition yields that the nondeterministic communication complexity of the problem is at least $(2 - o(1)) \log_2 N$, which improves the previously known bounds of $1.5 \log_2 N$ in [2] and $1.2 \log_2 N$ in [8] and matches...
2 Ordered Biclique Partition of Complete Graphs

Let \([n]\) denote the set \(\{1, 2, \ldots, n\}\). As defined in the Introduction, for an undirected graph \(G\), \(bp_{1,5}(G)\) is the minimum size of an ordered biclique partition of \(G\). The following is an example of such a partition \(\{B(U_i, W_i)\}_{i=1}^{4}\) of size four for \(K_6\) on the vertex set \([6]\):

\[
\begin{align*}
U_1 &= \{1, 2\}, & W_1 &= \{4, 6\}, \\
U_2 &= \{1, 3\}, & W_2 &= \{2, 5\}, \\
U_3 &= \{3, 6\}, & W_3 &= \{1, 4\}, \\
U_4 &= \{2, 4, 6\}, & W_4 &= \{3, 5\}.
\end{align*}
\]

The edges \(\{1, 6\}\), \(\{2, 3\}\) and \(\{3, 4\}\) are covered twice. It can be checked that \((1, 6) \in (U_1 \times W_1) \cap (W_3 \times U_3), (2, 3) \in (U_4 \times W_4) \cap (W_2 \times U_2)\) and \((3, 4) \in (U_3 \times W_3) \cap (W_4 \times U_4)\). An easy case analysis verifies that \(bp_{1,5}(K_6) = 4\).

In the previous work [2], we showed \(bp_{1,5}(K_n) = O(n^{2/3})\). The following theorem improves this result when we put \(k \geq 3\).

**Theorem 1** \(bp_{1,5}(K_{n^{2k-1}}) = O(kn^k)\).

**Proof.** The theorem is obvious for \(k = 1\). For \(k \geq 2\), we consider the complete graph \(K_{n^{2k-1}}\) on the vertex set \(V = [n]^{2k-1} = \{(x_1, x_2, \ldots, x_{2k-1}) \mid x_i \in [n]\}\). Define three types of subsets of the edge set of \(K_{n^{2k-1}}\):

\[
\begin{align*}
C_i &= \{\{u, v\} \mid u_{k+i} \neq v_{k+i} \text{ and } u_{i+\ell} = v_{i+\ell} \ (1 \leq \ell \leq k - 1)\} \text{ for } 0 \leq i \leq k - 1, \\
D_j &= \{\{u, v\} \mid u_j \neq v_j \text{ and } u_{k+j+\ell} = v_{k+j+\ell} \ (0 \leq \ell \leq k - 2)\} \text{ for } 1 \leq j \leq k - 1, \\
E_{i,j} &= \{\{u, v\} \mid u_j \neq v_j, u_{k+i} \neq v_{k+i} \text{ and } u_\ell = v_\ell \ (1 \leq \ell \leq j - 1, \ k + j \leq \ell \leq k + i - 1)\} \text{ for } 1 \leq i \leq k - 1, \ 1 \leq j \leq i.
\end{align*}
\]

Here the index “\(k + j + \ell\)” in the definition of \(D_j\) is modulo \(2k - 1\).

For example, for \(k = 4\), we define \(C_i\)’s for \(i \in \{0, 1, 2, 3\}\), \(D_j\)’s for \(j \in \{1, 2, 3\}\) and \(E_{i,j}\)’s for \((i, j) \in \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}\) as shown in Table 1. In this table, ‘○’, ‘×’ and ‘-’ denote \(u_i = v_i, u_i \neq v_i\) and don’t care, respectively.

We first see that the union of these subsets covers all edges in \(K_{n^{2k-1}}\). This can easily be verified by checking

\[
\bigcup_i C_i \supset \{\{u, v\} \mid u \neq v \text{ and } u_\ell = v_\ell \ (1 \leq \ell \leq k - 1)\},
\]

the best known lower bound on the deterministic version of the problem shown by Kushilevitz, Linial and Ostrovsky shown in [10] (see also [12]).
Table 1: $C_i$, $D_j$ and $E_{i,j}$ in the case $k = 4$. For example, $C_0$ is the set of edges such that the first three coordinates (out of $2k - 1 = 7$ in total) of its two endpoints are identical and the fourth coordinates of them are different, which is represented by "$\bigcirc \bigcirc \bigcirc \times \bigcirc \bigcirc \bigcirc$".

|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|-------|-------|-------|-------|-------|-------|-------|
| $C_0$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\times$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ |
| $C_1$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\times$ | $\bigcirc$ |
| $C_2$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\times$ |
| $C_3$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ |
| $D_1$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ |
| $D_2$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ |
| $D_3$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ |

and, for each $1 \leq j \leq k - 1$,

$$D_j \cup \bigcup_{i} E_{i,j} = \{ \{u, v\} \mid u_j \neq v_j \text{ and } u_\ell = v_\ell \ (1 \leq \ell \leq j - 1) \}.$$  

A key property of the collection of these subsets is that among all pairwise intersections of subsets in the collection, only $C_i \cap E_{i,j}$ is nonempty, while the others are empty; namely,

$$C_i \cap E_{i,j} = \{ \{u, v\} \mid u_j \neq v_j, u_{k+i} \neq v_{k+i} \text{ and } u_\ell = v_\ell \ (1 \leq \ell \leq j - 1, \ i+1 \leq \ell \leq k+i-1) \} \ (1)$$

for $1 \leq i \leq k - 1, \ 1 \leq j \leq i$.

In order to construct an ordered biclique partition of $K_{n^{2k-1}}$, we design a biclique partition of graph $G_{E} = (V, E)$ for each $E \in \{C_i, D_j, E_{i,j}\}_{i,j}$ separately. We can observe that $G_{C_i}$ (and also $G_{D_j}$) is the $n^{k-1}$-blowup of $n^{k-1}$ independent copies of $K_n$. Here an $m$-blowup of a “base” graph $H$ is obtained by replacing every vertex of $H$ by a group of $m$ vertices and every edge of $H$ by an $m \times m$ biclique between the corresponding groups of vertices. We also observe that $G_{E_{i,j}}$ is the $n^{2k-i-2}$-blowup of $n^{i-1}$ independent copies of the complement of $n \times n$ grid graph $\overline{G}_{n,n}$. Let $\tilde{G}_E$ denote the base graph of $G_E$, i.e., $\tilde{G}_{C_i}$ and $\tilde{G}_{D_j}$ are the $n^{k-1}$ independent copies of $K_n$ and $\tilde{G}_{E_{i,j}}$ is the $n^{i-1}$ independent copies of $\overline{G}_{n,n}$.

Two basic facts are needed to prove this theorem. First, for any graph $H$ on the vertex set $\{v_1, \ldots, v_m\}$, $\text{bp}(H) \leq m - 1$. This is because the collection of $(m - 1)$ stars $\{\mathcal{B}(\{v_i\}, N(v_i) \cap \{v_{i+1}, \ldots, v_m\})\}_{i=1}^{m-1}$ forms a biclique partition of $H$, where $N(v_i)$ denotes the set of neighbors of $v_i$. Second, if $\tilde{H}$ is a blowup of $\tilde{H}$, then $\text{bp}(H) \leq \text{bp}(\tilde{H})$. The reason is that the blowup of a biclique is a biclique itself; the blowup of all the biclique in a partition of $\tilde{H}$ is a biclique.
partition of $H$. Because of these facts, we have

\[
\begin{align*}
\text{bp}(G_{C_i}) & \leq \text{bp}(\tilde{G}_{C_i}) \leq n^{k-1} \cdot \text{bp}(K_n) \leq n^{k-1}(n-1) \text{ for } 0 \leq i \leq k-1, \\
\text{bp}(G_{D_j}) & \leq \text{bp}(\tilde{G}_{D_j}) \leq n^{k-1} \cdot \text{bp}(K_n) \leq n^{k-1}(n-1) \text{ for } 1 \leq j \leq k-1, \\
\text{bp}(G_{E_{i,j}}) & \leq \text{bp}(\tilde{G}_{E_{i,j}}) \leq n^{i-1} \cdot \text{bp}(\overline{G}_{n,n}) \leq n^{i-1}(n^2-1) \text{ for } 1 \leq i \leq k-1, 1 \leq j \leq i.
\end{align*}
\]

It would be worth noting that we can slightly improve the upper bound on $\text{bp}(\overline{G}_{n,n})$ although it affects only a lower order term. If we place the vertices of $\overline{G}_{n,n}$ in an $n \times n$ square grid and the roots of the stars are picked in row-major order then the last row can be skipped. Thus, $n(n-1)$ stars are enough to cover all edges instead of a trivial bound of $n^2-1$, i.e., $\text{bp}(G_{E_{i,j}}) \leq n^i(n-1)$.

Consequently, we obtain a collection of $(2k-1) \cdot n^{k-1}(n-1) + \sum_{i=1}^{k-1} in^i(n-1)$ bicliques that covers all edges in $K_{n^{2k-1}}$.

To complete the proof, we should notice that every edge $e \in C_i \cap E_{i,j}$ for $1 \leq i \leq k-1$, $1 \leq j \leq i$ is covered by exactly two bicliques in the collection (by recalling Eq. (11)). Therefore, in order to satisfy the definition of the ordered biclique partition, each endpoint of an edge $e \in C_i \cap E_{i,j}$ must be in different color classes in two bicliques that cover $e$. For this purpose, we pay attention to the ordering of the roots of the stars in making the partitions of $\tilde{G}_{C_i}$ and $\tilde{G}_{E_{i,j}}$.

For $\tilde{G}_{C_i}$, we pick the root $u$ of the stars in the lexicographic order on the $n$-ary string $(u_{k+i}u_{k+i-1}\cdots u_{i+1})$; whereas for $\tilde{G}_{E_{i,j}}$, we pick them in the reverse on the $n$-ary string $(u_{k+i}u_{k+i-1}\cdots u_{k+j}u_{j-1}\cdots u_1)$. In fact, we should only ensure that the $(k+i)$-th coordinate is the most significant. This guarantees that, for every edge $\{u, v\} \in C_i \cap E_{i,j}$ with $u_{k+i} < v_{k+i}$, $u$ is in the first class of a biclique in the collection for $G_{C_i}$ and in the second class of a biclique in the collection for $G_{E_{i,j}}$. In this way, we have

\[
\text{bp}_{1.5}(K_{n^{2k-1}}) \leq (2k-1) \cdot n^{k-1}(n-1) + \sum_{i=1}^{k-1} in^i(n-1) = O(kn^k).
\]

By putting $k := n$ in Theorem 1, the following is immediate.

**Corollary 1** $\text{bp}_{1.5}(K_n) = n^{1/2+o(1)}$. 

Proof. Let $N = n^{2k-1}$ and $k = n$. A simple calculation shows

$$n = \frac{\log N}{2\log n} + \frac{1}{2}$$

$$= \frac{\log N}{2\log(\frac{N}{2\log n} + \frac{1}{2})} + \frac{1}{2}$$

$$= \frac{\log N}{2\log(\log N + \log n) - 2\log(2\log n)} + \frac{1}{2}$$

$$= \Theta\left(\frac{\log N}{\log \log N}\right).$$

By Theorem 1, we have

$$bp_{1.5}(K_N) = O(kn^k) = O(n^{n+1}) = O(N^{\frac{n+1}{n^2+1}}) = O(N^{\frac{1}{2} + o(1)})$$

$$= N^{\frac{1}{2} + \Theta\left(\frac{\log \log N}{\log N}\right)}$$

$$= N^{\frac{1}{2} + o(1)}.$$

□

An almost quadratic separation between rank and fooling set size for 0/1—matrices is immediately follows from Theorem 1.

**Theorem 2** There is a 0/1 matrix $M$ such that $\text{fool}(M) \geq \text{rank}(M)^{2-o(1)}$.

**Proof.** Let $k := n$ and $N := n^k$. Let $\{\mathcal{B}(U_i, W_i)\}_{i=1}^m$ be an ordered biclique partition of $K_N$ constructed in Theorem 1. Let $A_i (1 \leq i \leq m)$ be an $N \times N$ 0/1-matrix whose $(k, \ell)$-entry is 1 iff $k \in U_i$ and $\ell \in W_i$ and $M$ be the component-wise sum of all $A_i$’s. Obviously, $M$ is a 0/1-matrix of rank at most $m = N^{1/2+o(1)}$ since the rank of $A_i$ is 1 for all $i$. In addition, the set of all the diagonal entries of $M$ forms a fooling set of $M$ since all the diagonal entries of $M$ are zero and, for every $k \neq \ell \in [N]$, at least one of $M_{k,\ell}$ or $M_{\ell,k}$ is one. This completes the proof of the theorem. □

Indeed, we constructed an $N \times N$ matrix having 0-fooling set of size $N$ such that all one entries can be covered by $N^{1/2+o(1)}$ disjoint 1-monochromatic rectangles. As noted in [2] Section 2.2, this also yields a separation between the deterministic and unambiguous nondeterministic communication complexities introduced by Yannakakis [15]. See [2] for more details.

By an equivalence between the problem to finding an ordered biclique partition and the one to obtaining a lower bound on the non-deterministic communication complexity for the clique vs. independent set problem described in Introduction, Theorem 1 also implies the following:

**Theorem 3** There exists an infinite family of graphs $G = (V(G), E(G))$ such that the non-deterministic communication complexity of the clique vs. independent set problem is at least $(2 - o(1)) \log_2 |V(G)|$. □
3 Concluding Remarks

In this note, we established an almost tight bound on $\text{bp}_{1.5}(K_n)$. It is now known that

$$\Theta(n^{1/2}) = \text{bp}_2(K_n)\leq \text{bp}_{1.5}(K_n) = n^{1/2+o(1)}.$$ 

It would be interesting to see whether $o(1)$ term in the exponent can be removed or not. A table of $\text{bp}_2(K_n)$ and $\text{bp}_{1.5}(K_n)$ for small values of $n$ ($n \leq 11$) was shown in [2, Section 3].

More challenging problem is to find a graph that has a larger (than quadratic) gap between its chromatic number and ordered biclique partition size. A superpolynomial gap on them gives $\omega(\log |V(G)|)$ lower bounds on the nondeterministic communication complexity of the clique vs. independent set problem, which would resolve a long standing open problem.

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