From the Cooper problem to canted supersolids in Bose-Fermi mixtures

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We calculate the phase diagram of the Bose-Fermi Hubbard model on the 3d cubic lattice at fermionic half filling and bosonic unit filling by means of single-site dynamical mean-field theory (DMFT). For fast bosons, this is equivalent to the Cooper problem in which the bosons can induce s-wave pairing between the fermions. We also find miscible superfluid and canted supersolid phases depending on the interspecies coupling strength. In contrast, slow bosons favor fermionic charge density wave structures for attractive fermionic interactions. These competing instabilities lead to a rich phase diagram within reach of cold gas experiments.

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Interactions between bosons and fermions play a crucial role in various physics contexts. Examples include the atomic nucleus, quarks exchanging gluons via the strong force, electrons dressed by lattice vibrations forming polarons, conventional superconductors where phonons induce an attraction between the electrons at the Fermi energy, and the phase separation between 3He and 4He mixtures. Beyond mean-field, these systems are notoriously difficult to describe. Cold atom experiments can be used to simulate this physics, thanks to the experimental control over the coupling strength between fermions and bosons, effectively performing quantum simulation of superconductors.

The first experiments investigated the influence of fermions on the bosonic Mott insulator, and found that the bosonic visibility always decreases when adding fermions attractively interacting with the bosons [1, 2]. This has been explained by self-trapping [3], corrections to higher bands [4, 5], or by adiabatic heating [1, 2, 8]. At weaker inter-species interactions, symmetry between repulsion and attraction was found [6]. In a dynamics experiment the strength of the potential terms has been measured with astonishing precision [7]. However, many more exotic phases such as supersolids [10] and pair superfluids [11] have been predicted [12–14], though not yet realized in experiment. Such may become possible though thanks to the recent discovery of multiple Feshbach resonances between 23Na and 40K at MIT [15].

In this Letter, we revisit the Cooper problem of conventional superconductors in a cold atom setup, that is we study the conditions under which bosons induce s-wave pairing between spin-1/2 fermions [21, 23]. We will see that a bosonic condensate leads to a strong static enhancement of s-wave pairing. Our formalism also allows us to explore physics in the strong Bose-Fermi coupling regime as well as bosons that are slow compared to the Fermi velocity. In such cases, instabilities favoring density waves compete against pairing, leading to a rich and unexpected phase diagram.

Our model consists of spinless bosons and spin-1/2 fermions on a cubic lattice with Hamiltonian

\[ H = -t_\ell \sum_{\langle ij \rangle} c_{i\sigma}^\dagger c_{j\sigma} - t_b \sum_{\langle ij \rangle} b_i^\dagger b_j - \mu_b \sum_i n_i^b - \mu_f \sum_i n_i^f + U_{bf} \sum_i n_i^f n_i^b + \frac{U_{bb}}{2} \sum_i n_i^b (2n_i^b - 1) + U_{bd} \sum_i n_i^b n_i^d, \]

where \( b_i^\dagger \) and \( c_i^\dagger \) (\( c_i^\dagger \) and \( c_i \)) are the bosonic (fermionic) creation and annihilation operators at site \( i \) with spin \( \sigma \) and \( n_i^b \) (\( n_i^f \)) denote the corresponding number operator. Particles can hop between neighboring sites via the hopping amplitude \( t_{b(f)} \) and the particle number is adjusted through the chemical potential \( \mu_{b(f)} \). The particles can interact via an onsite interaction, where \( U_{bb}, U_{ff} \) and \( U_{bd} \) denotes the boson-boson, fermion-fermion and boson-fermion interaction, respectively. We will work at unit filling for the bosons and half filling for the fermions, in which case the sign of \( U_{bb} \) is irrelevant. This model is a direct extension of the previous cold atom experiments with spin-polarized fermions. We restrict the discussion to the case where the spin-up and spin-down fermions interact equally strongly with the bosons.

To numerically study the above model we use DMFT where the full many body problem is mapped onto a self-consistent determination of an impurity model. In the Nambu notation the kinetic impurity action for sublattice \( s \) is given by

\[ S_{s}^{\text{kin}} = -\frac{1}{2} \int_0^\beta d\tau d\tau' (b_s^\dagger(\tau) - \Phi_s^\dagger) \Delta_{b,s}(\tau - \tau')(b_s(\tau') - \Phi_s) - zt \Phi_s^\dagger \int_0^\beta d\tau b_s(\tau) - \int_0^\beta d\tau d\tau' c_s^\dagger(\tau) \Delta_{c,s}(\tau - \tau') c_s(\tau') \]

where \( \Delta_{b(s)} \) is the matrix hybridization function of the bosons (fermions) and the corresponding creation and destruction operators are given by \( b_s^\dagger(\tau) = (b_s^\dagger(\tau), b_s(\tau)) \).
and \( c_{l}^{\dag}(\tau) = (c_{l,s}^{\dag}(\tau), c_{l,A}(\tau)) \), and \( s = A, B \) denote the two sublattices. \( \Phi_{-s} = (\mathbf{b}_{-s})_s = (\Phi_{-s}, \Phi_{-s}) \) is the time independent condensate order parameter of the bosons determined selfconsistently on the other sublattice as denoted by the subscript \(-s\). For a cubic lattice, the coordination number is \( z = 2d = 6 \). The fermionic hybridization function is determined by the following form of the inverse lattice Green function \[ \mathcal{G}_{-s}^{-1}(k, i\omega_n) = \begin{bmatrix} \zeta - \Sigma_A & -\Sigma_A & -\epsilon_k & 0 \\ -\Sigma_A & -\zeta^{\ast} + \Sigma_A & 0 & \epsilon_k \\ -\epsilon_k & 0 & \zeta - \Sigma_B & -\Sigma_B \\ 0 & \epsilon_k & -\Sigma_B & -\zeta^{\ast} + \Sigma_B \end{bmatrix} \]

(with \( \zeta = i\omega_n + \mu, \epsilon_k = 2\hbar \sum_{j=1}^{d} \cos(k_j) \), and standard notation for the normal and anomalous selfenergies on the respective sublattices) such that (charge) density wave ordering and s-wave pairing are allowed, and can occur independently of each other. The nature of the density-density coupling between bosons and fermions implies that a density wave ordering for fermions immediately creates density wave ordering for the bosons, and vice versa. The possible symmetry breakings in the spin sector are expected to be the same as in the pure fermionic model. The (local) potential energy terms are absorbed in the potential part of the impurity action \( S_{\text{pot}} = \int_{0}^{\beta} dt H_{\text{loc}}(\tau) \).

As impurity solver we use a continuous-time Monte Carlo method based on an expansion of the partition function in powers of the impurity-bath hybridization \( \Delta_{\text{bf}(l)} \) and the condensate order parameter \( \Phi_{l} \). The method is a direct extension of the fermionic \( \Phi \) and bosonic \( \mathcal{G}_{-s}^{-1} \) impurity solvers. This method allows for the first time to study Bose-Fermi mixtures within the full DMFT formalism (see however Refs. \[23, 24\] regarding the broken symmetry in the action). An illustration of a possible Monte Carlo configuration is shown in Fig. 1. Details of the algorithm will be presented elsewhere \[23\].

At half filling the pure fermionic system \( U_{\text{bf}} = 0 \) exhibits particle-hole symmetry: The superfluid phase transition on the attractive side \( U_{\text{hf}} < 0 \) is mirror reflected around \( U_{\text{hf}} = 0 \) into an anti-ferromagnetic transition on the repulsive side, \( U_{\text{hf}} > 0 \), as is shown in Fig. 2 (although both have SU(2) character, we already use the terminology appropriate for \( U_{\text{bf}} \neq 0 \)). The DMFT results interpolate between the Weiss mean-field result \( T_{\text{MF}} = 6t^{2}/|U_{\text{hf}}| \) valid at strong coupling and the T-matrix/BCS result at weak coupling \[26\]. We first study how the superfluid and anti-ferromagnetic phase transition are affected by the presence of strongly condensed bosons with a speed of sound exceeding the Fermi velocity (referred to as fast bosons), and focus on the s-wave pairing transition. The bosons can then be treated in the Bogoliubov approximation \[23\] and the effective interaction between the fermions is given by

\[
U_{\text{eff}}^{\text{ff}} (k, \omega) = U_{\text{ff}} + U_{\text{bf}}^{2}\chi_{0}(k, \omega) = U_{\text{ff}} + \frac{U_{\text{bf}}^{2}2n_{b}(zt_{b} + \epsilon_{b})}{\omega^{2} - (zt_{b} + \epsilon_{b})^{2}((zt_{b} + \epsilon_{b})^{2} + 2n_{b}U_{\text{bb}})},
\]

(1)

with \( \chi_{0}(k, \omega) \) the density-density response function. With a strong condensate, the zero temperature expression can be used since the Bose condensation temperature is much higher than the BCS temperature. When the bosonic sound velocity \( s_{b} = (2n_{b}U_{\text{bb}}t_{b})^{1/2} \) is much higher than the Fermi velocity, retardation effects are negligible \[23\] and the induced interaction is always attractive on the Fermi sphere. The induced interaction is then \( U_{\text{ind}}^{\text{ff}} (k) = -\frac{U_{\text{bf}}^{2}}{U_{\text{bb}}^{2}} c_{1}(k) = -\frac{U_{\text{bf}}^{2}2n_{b}(zt_{b} + \epsilon_{b})}{t_{b}^{2}1 + \xi^{2}(z - \sum_{j=1}^{d} \cos(k_{j}a))} \)

(with \( \xi = \sqrt{t_{b}/2n_{b}U_{\text{bb}}} \) the healing length), and an on-site effective interaction \( U_{\text{eff}}^{\text{ff}} = U_{\text{ff}} - \frac{U_{\text{bf}}^{2}}{U_{\text{bb}}^{2}} \sum_{k} c_{1}(k) \) is found. The effective hopping follows from a mean-field decoupling of the nearest neighbor interaction and is

![FIG. 1. (Color online) Illustration of a typical Monte Carlo configuration. The full (empty) circles denote creation (annihilation) operators in the imaginary time interval \([0, \beta]\) for bosons (black), and spin-up (blue) and spin-down (red) fermions. a) The local contribution to the weight of the operator sequence is determined by the length of the segments and the overlap between segments of different particles (segments mark time intervals in which a particle resides on the impurity). b) A possible configuration of bosonic hybridization functions and source fields with density \( n_{b} = 0 \) at imaginary time \( \tau = 0 \). c) All possible combinations to connect the fermionic creation and annihilation operators.](image-url)
with quasi-particle weight \[28\], far outside the perturbative regime. An analysis of the transition lines can be collapsed onto each other; \(U\) forms purely fermionic system. On the basis of the perturbation was disabled confirmed this picture (not shown): Zero density-density correlation function in Eq. \[\text{1}\] changes dramatically in the absence of a condensate. It may change sign when \(\omega\) cannot be set to zero thereby suppressing pairing. This motivates us to numerically investigate the dependence of the phase transition on the bosonic hopping \(t_b\), shown in Fig. \[\text{3}\] for strong interactions \(U_{\text{ff}}/t_t = -10\). We see that the system undergoes a sharp first order transition around \(t_b/t_t = 0.75\) between a fermionic superfluid (corresponding to spin singlets in the fermionic spin sector) and a (molecular) charge density wave (corresponding to Neel ordering in the fermionic spin sector). The bosons remain strongly condensed at this point \((n_0 \approx 0.6)\), but pick up charge density wave order. The transition temperature varies remarkably little over the different phases, reflecting the underlying \(SU(2) \times SU(2)\) symmetry of the pure fermionic model. At very low hoppings \((t_b/t_t < 0.2)\), the bosons become insulating and are very ineffective in influencing the fermions. The fermions can undergo a simultaneous a pairing and molecular charge order transition, which couples back to the bosons and generates bosonic charge order. We also observed that, except in the close vicinity of a bosonic superfluid-insulator phase transition, bosonic static mean field approximation provides quantitatively correct results in our DMFT scheme, which may be useful for future cluster extensions of this work.

We repeated this calculation for different values of \(U_{\text{ff}}\) for a temperature \(T/t_t = 0.2\) close to the ground state resulting in the phase diagram in the \((U_{\text{ff}}, t_b)\) plane, shown in Fig. \[\text{4}\]. For large values of \(U_{\text{ff}}\) we find the same phases as in Fig. \[\text{4}\]: a double superfluid, a CDW with a bosonic density wave on the attractive side. Collapse of the curves with different \(U_{\text{ff}}\) is observed provided the on-site repulsions, hoppings and \(Z_{\text{sp}}\) factors are rescaled.

The density-density correlation function in Eq. \[\text{1}\] is:

\[\rho_{\alpha\beta}(r) = \frac{1}{\Omega} \int \rho_{\alpha\beta}(\mathbf{r}, \mathbf{k}) \cos(kz) \, dk_z\]

\(Z_{\text{sp}}\) factors are rescaled.

### FIG. 2
(Color online) S-wave superfluid (left) and antiferromagnetic \((U_{bf} = 0)\) and \(|U_{bf}|/t_t = 8\) on the right) phase transition of the Bose-Fermi Hubbard model on the 3d cubic lattice at filling \(n_b = 1\) and \(n_\uparrow = n_\downarrow = 1/2\) and with \(U_{bb}/t_t = 20\) and \(t_b/t_t = 1\) for different boson-fermion interactions \(U_{bf}\). The DMFT results interpolate between the Weiss mean-field result and the T-matrix/BCS result (‘modified HF’) (see text). Inset: Critical temperature for pairing for non-interacting fermions \((U_{bf} = 0)\) as a function of the boson-fermion interaction \(U_{bf}\). The transition temperature is exponentially suppressed at low \(U_{\text{ff}}\) for all \(U_{bf}\) (not shown).

### FIG. 3
(Color online) The phase diagrams of Fig. 2 can be collapsed onto the phase diagram of a pure fermionic model with renormalized hoppings \(t_{\text{eff}} = t_t - c_j^2 U_{bf}^2/U_{bb}\) and on-site repulsions \(U_{\text{ff}} = U_t - c_j^2 U_{bf}^2/U_{bb}\), with \(c_j^2\) fitting constants. Error bars are of the order of the symbol size and omitted for clarity.
superfluid, and a CDW with a fermionic superfluid.

However, for rather low values of \( U_{bf} \) and sufficiently large bosonic hoppings we find a supersolid phase, in which bosons and fermions have both types of orderings. In this supersolid, the gaps for pairing and charge order are not equal; this supersolid is a realization of the canted supersolids put forward in Refs. [32, 33]. The RG study of Ref. [34] finds that a \( d \)-wave superfluid develops for certain parameters in this regime, which may compete with the supersolid. However, seeing such a phase is not possible with single site DMFT. We expect \( d \)-wave only to be feasible for low values of \( U_{bf} \) and \( U_{bf} \) while for large values of \( U_{bf} \) and \( U_{bf} \) the supersolid is most likely stable. The transition temperature of the supersolid phase for \( |U_{bf}| = 2 \) and \( t_0 = 0 \) is \( T_c \approx 0.48 t_0 \), rendering an experimental observation with cold gases realistic. This is the same transition temperature as for a supersolid in a bosonic model on a triangular lattice [33], and 50\% higher than the one of an anti-ferromagnet in the 3d Hubbard model [38]. One example of a mixture with promising scattering properties for the supersolid phase is \(^{6}\text{Li}^{7}\text{Li} \) [37]. For low values of \( U_{bf} \) the structure of the phase diagram is identical to the one shown in Fig. 4A, from which we conclude that the BCS-BEC crossover is not a driving force for the Bose-Fermi Hubbard model at half filling.

In conclusion, we developed a single-site DMFT formalism for the Bose-Fermi-Hubbard model allowing for \( s \)-wave pairing and charge density wave ordering. We computed changes to the pure fermionic phase diagram at fermionic half filling induced by the commensurate bosons, focusing on attractive \( U_{bf} \). While fast bosons favor \( s \)-wave pairing, slow bosons favor charge density order. These different type of instabilities compete, leading to some unexpected phases such as a canted supersolid and the CDW+SF\(_b\) phase shown in the phase diagram of Fig. 4.

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