EMBEDDING UNICRITICAL CONNECTEDNESS LOCI

MALAVIKA MUKUNDAN

Abstract. In this article, for degree \( d \geq 1 \), we construct an embedding \( \Phi_d \) of the connectedness locus \( \mathcal{M}_{d+1} \) of the polynomials \( z^{d+1} + c \) into the connectedness locus of degree \( 2d + 1 \) bicritical odd polynomials.

1. Introduction

Relationships between different families of holomorphic maps have been encountered in various contexts in complex dynamics. In rational dynamics, quadratic polynomials of the form \( z^2 + c \) are the fundamental objects of study, and much of the field involves the study of the Mandelbrot set pioneered by Douady and Hubbard, in \([5, 6, 7, 15]\), and developed by Milnor (\([20]\)), Lyubich and Dudko (\([9]\)) and several others. In general, polynomials with a single critical point, normalized as \( z^d + c \), and their connectedness loci \( \mathcal{M}_d \)—that is, the set of parameters \( c \) for which the filled Julia set is connected, commonly referred to as the Multibrot sets, have also been studied in \([11, 24]\), etc, and the properties of these sets have been used to conjecture and prove several results in both rational and transcendental dynamics.

We establish a relationship between two families—unicritical polynomials and symmetric bicritical polynomials. We take symmetric polynomials to mean polynomials that commute with some affine map \( M \) satisfying \( M^2 = \text{Id} \). A polynomial is bicritical if it has exactly two finite critical points up to multiplicity. Symmetric cubic polynomials are encountered, for example, in the study of core entropy (see \([13]\)).

As we will show in Section 2, any symmetric bicritical polynomial is affine conjugate to

\[
p_{a,d}(z) = a \int_0^z \left( 1 - \frac{w^2}{d} \right)^d dw
\]

for some \( a \in \mathbb{C}^* \) and \( d \geq 1 \). For each \( a \), \( p_{a,d} \) is an odd polynomial—that is, it commutes with \( z \mapsto -z \). For a fixed \( d \), we let \( p_a = p_{a,d} \). We let \( \text{CBO}_d \) denote the set of \( a \in \mathbb{C}^* \) such that \( p_a \) has connected filled Julia set. Using the results of \([4]\), it follows that \( \text{CBO}_d \) is a closed, compact connected subset of \( \mathbb{C} \). The following is the main result of our article.

**Theorem 1.1.** For \( d \geq 1 \), there exists a continuous map \( \Phi_d : \mathcal{M}_{d+1} \to \text{CBO}_d \) that is a homeomorphism onto its image.
Such relationships between other families have been proved by several others. In his study of renormalizable maps in [18], McMullen proved that unicriticals are universal in the sense that there are small copies of the Multibrot sets found in any holomorphic family of rational maps.

Branner and Douady constructed a continuous map from the basilica limb of the Mandelbrot set \( M_2 \) to the rabbit limb (see [2]). This was later extended by Branner and Fagella in [3] into homeomorphisms between various limbs of the Mandelbrot set, and in a different spirit, by Dudko and Schleicher in [10]. In [23], Riedl and Schleicher also construct a homeomorphism from a subset of any \( \frac{p}{nq} \)-limb of the Mandelbrot set to the \( \frac{n}{q} \)-limb.

Our proof is along the lines of Douady and Branner’s use of quasiconformal surgery in [2]. We shall perform a quasiconformal surgery along a \( \beta \)− fixed point and its pre-images. The map \( \Phi_d \) is natural in the following sense: its inverse at all points \( a \in \Phi_d(\mathcal{M}_{d+1}) \), \( a \neq 1 \) can be described by a renormalization operator on a subset of \( CBO_d \). We shall also give a complete description of the image under \( \Phi_d \) (see Section 2.3).

The family \( p_{a,d} \) is interesting in its own right: as \( d \to \infty \), \( p_{a,d}(z) \to a \int_0^z e^{-w^2}dw \) locally uniformly on \( \mathbb{C} \). The limit function is entire, odd, has two asymptotic values \( \pm \frac{a\sqrt{\pi}}{2} \) and no critical points. It is called an error function (see [21] for an introduction). Error functions belong to the larger Speiser class—the family of entire functions with finitely many critical and asymptotic values. This family is studied in [12], [14] and several others.

The simplest of the Speiser class is the family of exponential functions. A lot of the analysis of exponential functions is a direct application of the tools used in the analysis of unicritical polynomials, normalized as \( \lambda(1+\frac{z}{d})^d \) and using the fact that they converge to \( \lambda \exp z \) as \( d \to \infty \). This is a theme that is explored in [1].

Our work in progress aims to carry out a similar analysis for the error functions \( a \int_0^z e^{-w^2}dw \).

We note that the polynomials \( z^{d+1} + c \) are in one-one correspondence with the family \( (1+\frac{z}{d})^d + c \), which approximate the exponentials \( e^z + c \). This paper presents a structural similarity between approximating exponential functions, and the polynomials \( p_{a,d} \) that approximate error functions, and prompts us to make Conjecture 1.2:

**Conjecture 1.2.** Let \( E_c(z) = \exp z + c \), and \( E^a_d(z) = a \int_0^z e^{-w^2}dw \). There exists a continuous map from \( \{ c \in \mathbb{C} | \{ E_c^n(c) \}_{n \geq 0} \text{ is bounded} \} \) to the set \( \{ a \in \mathbb{C}^* | \{ E^a_d^n(a) \}_{n \geq 0} \text{ is bounded} \} \) that is a homeomorphism onto its image.

There is some evidence to show that this is reasonable; work in progress indicates that it may be possible to embed postsingularly finite exponential functions into the collection of postsingularly finite error functions in a dynamically meaningful manner. We do not, however, address error functions in this article.

The paper is organized as follows. In Section 2, we introduce symmetric polynomials and establish some of their basic properties, provide motivation for Theorem 1.1, while laying out our proof strategy, and describe the image of \( \Phi_d \). In Sections 3 and 4 respectively, we define \( \Phi_d \) and prove that it is continuous. We end in Section 5 by constructing a continuous inverse for \( \Phi_d \) on its image.
2. Preliminaries

For an introduction to polynomial dynamics and the Multibrot sets, see [20], [19], [16] and [11].

2.1. Introduction. We call a polynomial \( f \) of degree \( > 1 \) symmetric if it commutes with an affine map \( M \) that satisfies \( M \circ f = f \circ M \). Any such map \( M \) is of the form \( M(z) = -z + b \) for some \( b \in \mathbb{C} \), and we may conjugate \( M \) by a translation \( \tau \) so that \( \tau^{-1} \circ M \circ \tau(z) = -z \). We have

\[
\tau^{-1} \circ f \circ \tau(z) = \tau^{-1} \circ (M \circ f \circ M^{-1}) \circ \tau(z) \\
= (\tau^{-1} \circ M \circ \tau) \circ (\tau^{-1} \circ f \circ \tau) \circ (\tau^{-1} \circ M^{-1} \circ \tau)(z) \\
= - (\tau^{-1} \circ f \circ \tau)(-z).
\]

That is, \( \tau^{-1} \circ f \circ \tau \) is an odd polynomial. Therefore, every symmetric polynomial contains an odd polynomial in its affine conjugacy class. We recall that an odd polynomial has only odd degree terms.

2.2. Bicritical odd polynomials. \( f \) is bicritical if it has, up to multiplicity, exactly two critical points on the plane. Let \( f \) be a bicritical odd polynomial of degree \( 2d + 1 \), with \( d \geq 1 \). The Riemann Hurwitz formula shows that \( f \) has local degree \( d + 1 \) at both critical points. Furthermore, the critical points are of

\[\begin{array}{cc}
\text{(a) } CBO_1 & \text{(b) } CBO_2 \\
\text{(c) } CBO_3 & \text{(d) } CBO_4 \\
\text{(e) } CBO_5
\end{array}\]

Figure 1. The families \( CBO_d \) for \( d = 1, 2, 3, 4, 5 \)
A portion of $CBO_2$. The cut point on the mid-left is $a = 1$. Note the resemblance to $M_3$.

**FIGURE 2.** The figure on the top left is the unicritical locus $M_3$. The figure on the top right is the locus $CBO_2$ of odd bicritical polynomials of degree 5. The figure at the bottom zooms in on the right of the figure on the top right—we will show that this region contains a copy of $M_3$. 
2.3. Monic representatives of polynomials in $\mathbb{C}^*$. Let $\phi(z) = kz$ be such that $\phi(\{x, -x\}) = \{\sqrt{d}, -\sqrt{d}\}$. Then there exists a constant $a \in \mathbb{C}^*$ such that

$$\phi \circ f \circ \phi^{-1}(z) = a \left(1 - \frac{z^2}{d}\right).$$

Therefore,

$$\phi \circ f \circ \phi^{-1}(z) = a \int_0^z \left(1 - \frac{w^2}{d}\right) dw.$$

For $a \in \mathbb{C}^*$, let

$$p_a(z) = a \int_0^z \left(1 - \frac{w^2}{d}\right) dw.$$

We also note that $p_a$ is affine conjugate to $p_{a'}$ if and only if $a = a'$: if there exists an affine map $\psi$ with $\psi \circ p_a \circ \psi^{-1} = p_{a'}$, then we must have $\psi(\{\sqrt{d}, -\sqrt{d}\}) = \{\sqrt{d}, -\sqrt{d}\}$. From this we can infer that $\psi(z) = kz$, where $k = \pm 1$. In both cases, we have

$$a' = \frac{dp_{a'}}{dz}(0) = \frac{dp_a}{dz}(\psi^{-1}(0)) = \frac{dp_a}{dz}(0) = a.$$

To summarize, the space of bicritical odd polynomials modulo conjugation by scaling (or, the space of symmetric bicritical polynomials modulo conjugation by affine conjugation) is the family $a \mapsto p_a$ over $\mathbb{C}^*$. We shall denote this family $\mathcal{CBO}_d$, and let

$$\mathcal{CBO}_d = \{a : K_{p_a} \text{ is connected}\}.$$

Figure 1 illustrates $\mathcal{CBO}_d$ for $d = 1, 2, 3, 4, 5, 19$.

By the techniques in [4] used for cubic polynomials, we can show that $\mathcal{CBO}_d$ is connected and compact, and that $\hat{\mathbb{C}} \setminus \mathcal{CBO}_d$ is isomorphic to $\mathbb{D}$. We consider the part of $\mathcal{CBO}_2$ illustrated in Figure 2a, and present some of the Hubbard trees of postcritically finite polynomials in this region in Figures 3 and 4. The images indicate a relationship between $\mathcal{M}_3$ and $\mathcal{CBO}_2$, and in general, between $\mathcal{M}_{d+1}$ and $\mathcal{CBO}_d$.

2.3. Monic representatives of polynomials in $\mathcal{B}_d$. Any polynomial $p_a$ for $a \in \mathcal{B}_d$ has leading coefficient $T(a) := \frac{(-1)^d a}{d^{(2d+1)}}$ attached to $z^{2d+1}$. Let $w_s(z) = \frac{z}{s}$. We note that $P_s(z) = w_s^{-1} \circ p_a \circ w_s$ is monic if and only if $s^{2d} = T(a)$.

The polynomial $P_s$ admits a unique Böttcher chart $\varphi_s$ in a neighborhood of $\infty$ that satisfies $\lim_{z \to \infty} \frac{\varphi_s(z)}{z} = 1$, and if $s^{2d} = s_2^{2d}$, then $\varphi_{s_1}(z) = \omega \varphi_{s_2}(z)$ where $\omega = \frac{s_2}{s_1}$ satisfies $\omega^{2d} = 1$. Let $\mathcal{R}_\theta(s)$ denote the ray at angle $\theta$ in the dynamical plane of $P_s$. Then it is easy to see that if $\frac{s_2}{s_1} = e^{\frac{2\pi i j}{2d}}$ for some integer $j$, then for all $\theta \in \mathbb{R}/\mathbb{Z},$

$$\mathcal{R}_\theta(s_2) = \mathcal{R}_{\theta + \frac{j}{2d}}(s_1).$$

Additionally, since $P_s$ is odd, $\varphi_s(z) = \lim_{n \to \infty} P_s^\circ n(z) \frac{1 + (-1)^d n}{2d+1}$ satisfies

$$\varphi_s(-z) = -\varphi_s(z) \implies \mathcal{R}_{\theta + \frac{1}{2d}}(s) = -\mathcal{R}_\theta(s).$$
Figure 3. Postcritically finite polynomials along with their Hubbard trees in the family $p_a(z) = af_0^z(1 - w^2)dw$. $x_0^- = -1$, $x_0^+ = 1$ are the two critical points, with $x_i^\pm = p_{0}^{\pm}(x_0^\pm)$. Terminology: (+,−) type polynomial $p'$ refers to the polynomial in $CBO_1$ that looks like a pair of copies of the polynomial $p = f_c$, where $c \in M_2$. 

(a) A section of $CBO_1$

(b) The (+,−) type $z \mapsto z^2$

(c) The (+,−) type basilica
Figure 4. More examples of Hubbard trees in $CBO_1$
For any $s$, the point $z = 0$ is a repelling or parabolic fixed point of $P_s$ if and only if there exists a subset $\Theta$ of $\{0, 1, \ldots, 2d-1\}$ satisfying $\Theta + \frac{1}{2} = \Theta$, such that the dynamical rays landing at 0 are exactly those with angles in $\Theta$. Moreover, if $s' = e^{2\pi i/s} s$, then the set of angles that land at 0 in the dynamical plane of $P_{s'}$ is $\theta_2 + \Theta$.

Thus for any $a \in \mathcal{CBO}_d$, the point 0 is a repelling or parabolic fixed point of $p_a$ if and only if there exists a monic representative $P_s$ of $p_a$ so that 0 is the landing point of $R_0(s)$ and $R_2(s)$. The union of the rays $R_0(s)$ and $R_2(s)$ separates the plane into two connected components $F_s^L$ and $F_s^R$, named so that $F_s^L$ is the component that contains the critical point $-s\sqrt{d}$, and $F_s^R$ contains $s\sqrt{d}$ ($L$ and $R$ stand for left and right; e.g., see Figure 14).

2.3.1. Polynomials in $\mathcal{CBO}_d$ with separated critical orbits. For all $a \in \mathbb{D}$ with $a \neq 0$, the point $z = 0$ is an attracting fixed point of $p_a$. We note the following.

- The point 0 is a repelling fixed point of $p_a$ if and only if $|a| > 1$.
- The point 0 is a parabolic fixed point of $p_a$ if and only if $a = e^{2\pi it}$ for some $t \in \mathbb{Q}/\mathbb{Z}$. In particular, the point $a = 1$ is on the boundary of $\mathcal{CBO}_d$.

**Proposition 2.1.** There exists $\varepsilon > 0$ such that for all real values $a \in (1, 1 + \varepsilon)$, the polynomial $p_a$ is hyperbolic.

**Proof.** We will show that for real $a$ in some interval of the form $(1, 1 + \varepsilon)$, the polynomial $p_a$ has an attracting fixed point. Given $a \in \mathbb{R}$, with $a > 0, \forall z \in (0, \sqrt{d}]$, then by the Mean Value theorem, there exists $r \in [0, 1]$ such that

$$p_a(z) = a \int_0^z \left(1 - \frac{w^2}{d}\right)^d \, dw = az \int_0^1 \left(1 - \frac{t^2 z^2}{d}\right)^d \, dt \leq az \left(1 - \frac{r^2 z^2}{d}\right)^d.$$ 

Since $\left(1 - \frac{r^2 z^2}{d}\right)^d < 1$ for all $r \in [0, 1]$, we have

$$p_a(z) < az.$$ 

In particular, this implies $p_1(\sqrt{d}) < \sqrt{d}$. Since $a \mapsto p_a(\sqrt{d})$ is continuous, $\exists \varepsilon > 0$ such that $\forall a \in (1, 1 + \varepsilon)$, we have $p_a(\sqrt{d}) < \sqrt{d}$. However, $\forall a \in (1, 1 + \varepsilon)$, since 0 is a repelling fixed point of $p_a$, there exists $r_a > 0$ such that $\forall z \in (0, r_a)$, we have $p_a(z) > z$.

Combining the above two statements, we note that $\forall a \in (1, 1 + \varepsilon)$, there exists $z_a \in (r_a, \sqrt{d})$ such that $p_a(z_a) = z_a$ (assume that $z_a$ is the largest such point in $(r_a, \sqrt{d})$). Clearly, all $z \in (z_a, \sqrt{d})$ are attracted to $z_a$ under iteration of $p_a$, so $z_a$ is either an attracting or parabolic fixed point of $p_a$. However, the set of $a$ such that $p_a$ has a parabolic fixed point is finite: therefore for sufficiently small $\varepsilon$, the point $z_a$ is an attracting fixed point of $p_a$. \qed

Since $a = 1 \in \partial \mathcal{CBO}_d$ and $\hat{\mathcal{C}} \setminus \mathcal{CBO}_d$ is a disk, there exists an arc $\gamma : [0, \infty) \to \hat{\mathbb{C}}$ such that $\gamma(0) = 1$, $\lim_{t \to \infty} \gamma(t) = \infty$ and $\gamma(t) \notin \mathcal{CBO}_d \forall t > 0$. By symmetry, we must also have $-\gamma(t) \notin \mathcal{CBO}_d \forall t > 0$. Then it is easy to see that the hyperbolic component $\{0 < |a| < 1\}$ and the hyperbolic component containing $(1, 1 + \varepsilon)$ (with $0 < \varepsilon << 1$) are in two distinct connected components of $\hat{\mathbb{C}} \setminus (\gamma \cup -\gamma)$. In other words, $a = 1$ disconnects $\mathcal{CBO}_d$ into two connected components.
Let $H$ be the component of $\mathcal{CBO}_d \setminus \{1\}$ that intersects the right half plane. For all $a \in H \cup \{1\}$, the point 0 is either repelling or parabolic for $p_a$. By the discussion at the beginning of the section, on $H \cup \{1\}$, there exists a branch of the function $a \mapsto (T(a))^\frac{i}{2\pi}$, which we shall denote $s(a)$, such that 0 is the landing point of $\mathcal{R}_0(s(a))$ and $\mathcal{R}_1(s(a))$.

For any $s \in \mathbb{C}^*$, the critical points of $P_s$ are $\pm s\sqrt{d}$. Let $\mathcal{O}^R_s$ and $\mathcal{O}^L_s$ be the orbits under $P_s$ of $s\sqrt{d}$ and $-s\sqrt{d}$ respectively. Define

$$\mathcal{CBO}_d^{(+,+)} = \{ a \in \mathcal{CBO}_d | (\mathcal{O}^R_{s(a)} \subset P_{s(a)} \cup \{0\}) \text{ and } (\mathcal{O}^L_{s(a)} \subset P_{s(a)} \cup \{0\}) \}.$$

That is, $\mathcal{CBO}_d^{(+,+)}$ represents the set of polynomials where the dynamical rays at angles $0, \frac{1}{2}$ separate the orbits of the two distinct critical points. It is easy to see that $\mathcal{CBO}_d^{(+,+)} \subset \mathcal{CBO}_d$ (in fact $\mathcal{CBO}_d^{(+,+)} \subset H \cup \{1\}$), and that $\mathcal{CBO}_d^{(+,+)}$ is closed. We have described in detail the dynamics of polynomials in $\mathcal{CBO}_d^{(+,+)}$ in Section \ref{sec:continuous-map}.

The continuous map from $\mathcal{M}_{d+1}$ to $\mathcal{CBO}_d$ that we shall construct in this paper maps $\mathcal{M}_{d+1}$ homeomorphically onto $\mathcal{CBO}_d^{(+,+)}$. Figure \ref{fig:cbos} illustrates this in the case $d = 1$ by pointing out the images of well-known polynomials like the rabbit, co-rabbit, airplane, etc.

### 2.4. Quotienting by $z^2$.

Given $p_a \in \mathcal{BO}_d$, there exists a unique polynomial $\mathcal{P}_a : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ so that the following diagram commutes:

$$\begin{array}{ccc}
\hat{\mathbb{C}} & \xrightarrow{p_a} & \hat{\mathbb{C}} \\
\downarrow_{z \rightarrow z^2} & & \downarrow_{z \rightarrow z^2} \\
\hat{\mathbb{C}} & \xrightarrow{\mathcal{P}_a} & \hat{\mathbb{C}}
\end{array}$$

The critical points of $\mathcal{P}_a$ are $d$ and $\{x_\ell^d\}_{\ell = 1}^d$, where $\pm x_\ell$, $\ell = 1, 2, \ldots, d$, are the pre-images of 0 that are not equal to 0. When $d = 1$, the family $\mathcal{P}_a$ corresponds to the collection of cubic polynomials where one critical point is a pre-image of a $\beta$–fixed point (that is, the landing point of a dynamical ray at angle 0 or $\frac{1}{2}$) and the other is free. This is isomorphic to the collection $\mathcal{F} = \{ (a, b) | Q_{a,b}(a) = -2a \}$, where $Q_{a,b}(z) = z^3 - 3a^2z + b$ discussed in \cite{Douady:1993} Chapters I,II] in the following way: letting $\bar{a} = 9a^2$, we have

$$\mathcal{P}_{-a} = \mathcal{P}_a = \mathcal{M} \circ Q_{\bar{a}, 2\bar{a}^3 - 2\bar{a}} \circ \mathcal{M}^{-1},$$

where $\mathcal{M}$ is some affine map.

Let $F_+ \subset \mathcal{F}$ be the collection of polynomials $Q_{a,b}$ for which the critical point $-a$ maps to the landing point of the dynamical ray at angle 0, and the other critical point $a$ is in the filled Julia set. Douady and Branner show that there exists a homeomorphism $\Phi_B$ from the basilica limb of the Mandelbrot set to $F_+$. The relationship between $\mathcal{CBO}_1^{(+,+)}$ and $F_+$ is as follows:

$$\{(\bar{a}, 2\bar{a}^3 - 2\bar{a}) | a \in \mathcal{CBO}_1^{(+,+)}, \bar{a} = 9a^2 \} \subset F_+. $$
For $d = 1$, the map $\Phi_d$ we construct in this paper exhibits different behaviour from the $\Phi_B$ that the authors construct in [2, Chapter II]. Firstly, it is defined on the whole of the Mandelbrot set, and not just the basilica limb. Secondly, generally, given $c$ in the basilica limb, if $Q_{5,253 \ldots 2a}$ is the polynomial corresponding to $\Phi_B(c)$, $\overline{a}$ does not equal $9\Phi_d(c)^2$. Thirdly, it is evident that our map does not change the combinatorics of critical portraits, whereas $\Phi_B$ does.

2.5. **Proof strategy and tools.** We will use all the theorems listed in this section. Their statements are borrowed from [3].

**Theorem 2.2** (The measurable Riemann mapping theorem). Let $\mu_0$ be the standard complex structure on $\mathbb{C}$. If $\mu$ is a complex structure on a simply connected domain $U \subset \mathbb{C}$ that has bounded dilation with respect to $\mu_0$, then there exists a quasiconformal homeomorphism $\psi: U \to V \subset \hat{\mathbb{C}}$ satisfying

$$f^*\mu_0 = \mu$$

unique up to post composition by a Möbius transformation.

1. Let $\mu_n$ be a sequence of Beltrami forms on a bounded domain $U \subset \mathbb{C}$ such that $\|\mu_n\|_\infty \leq m < 1$ and $\mu_n \to \mu$ in the $L^1$ norm, where $\mu$ is a Beltrami form on $U$ with $\|\mu\|_\infty \leq m$. There exists a sequence of integrating maps $\phi_n$ for $\mu_n$ and an integrating map $\phi$ for $\mu$ such that $\phi_n \to \phi$ uniformly on $U$.

2. Let $\Lambda$ be an open set in $\mathbb{C}^n$ and $(\mu_\lambda)_{\lambda \in \Lambda}$ be a family of Beltrami forms on $U$. Suppose $\lambda \mapsto \mu_\lambda(z)$ is holomorphic for almost every $z \in U$, and that there exists a constant $m < 1$ such that $\|\mu_\lambda\|_\infty < m$ for each $\lambda$. For each $\lambda$, extend $\mu_\lambda$ to $\hat{\mathbb{C}}$ by $\mu_\lambda = 0$ on $\mathbb{C} \setminus U$, and let $f_\lambda: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the unique quasi-conformal homeomorphism such that $f_\lambda^*\mu_0 = \mu_\lambda$, and $\frac{f_\lambda(z)}{z} \to 1$ when $|z| \to \infty$. Then $(\lambda, z) \mapsto (\lambda, f_\lambda(z))$ is a homeomorphism of $\Lambda \times \mathbb{C}$ onto itself, and for each $z \in \mathbb{C}$ the map $\lambda \mapsto f_\lambda(z)$ is holomorphic.

**Definition 2.3** (Polynomial-like maps). Given Jordan domains $U, V \subset \mathbb{C}$ with $\overline{U} \subset V$, a polynomial-like map $f: U \to V$ is an analytic proper map of finite degree $d$.

The filled Julia set of $f$ is the set

$$K_f = \bigcap_{n \geq 0} f^{\circ n}(U).$$

Given a polynomial $p: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d$, we can always find suitable domains $U, V$ such that $\overline{U} \subset V$ and $p|_{U}: U \to V$ is polynomial-like of degree $d$.

**Definition 2.4** (Hybrid equivalence). Given two polynomial like maps $f: U \to V$ and $g: U' \to V'$, we say that $f$ and $g$ are hybrid equivalent if there exists a quasiconformal homeomorphism $\psi: (V, U) \to (V', U')$ satisfying $g \circ \psi = \psi \circ f$, with zero dilation on $K_f$.

Theorem 2.5 is due to Douady and Hubbard, and we shall be using it several times.

**Theorem 2.5** (The straightening theorem for polynomial-like maps). Every polynomial-like map of degree $d$ is hybrid equivalent to a polynomial of degree $d$. 

Our strategy for constructing $\Phi_d$ follows the general layout in [2]. Given $c \in M_d$, we will perform a topological surgery in the dynamical plane using the dynamics around one of the $\beta$-fixed points. At the end of this surgery, we will construct a quasiregular map $g_c$ from a simply-connected Riemann surface $X_1$ to a simply connected Riemann surface $X$ with $X_1 \subset X$.

Next, we will show that $g_c$ has an invariant complex structure, and is therefore equivalent to a polynomial-like map of degree $2d + 1$. We will finally show that this map is hybrid equivalent to an odd polynomial $p_a$ with $a \in \mathcal{CB}_d$. The strategy for constructing an inverse for $\Phi_d$ is similar.

All the figures in this paper are illustrations of the case $d = 2$.

### 3. Construction of $\Phi_d$

As mentioned before, we proceed along the lines of holomorphic surgery as outlined in [2]. Let $f_c(z) = z^{d+1} + c$, and $\varphi_c$ be a Böttcher chart at $\infty$ that satisfies $\lim_{z \to \infty} \frac{\varphi_c(z)}{z} = 1$. In the absence of the latter condition, $\varphi_c$ is unique only up to multiplication by a $d$th root of unity. By including the condition, we fix a choice of $\varphi_c$ for every $c$ that makes it continuous in the following sense: given $c \in \mathbb{C}$ and $z \in \mathbb{C}$ such that $\varphi_c(z)$ is well-defined for $\bar{c}$ in a neighborhood of $c$, $\bar{c} \mapsto \varphi_c(z)$ is continuous in $\bar{c}$.

Now fix $c \in M_{d+1}$. The dynamical ray $R_0$ at angle 0 lands at a fixed point $\beta$ on the dynamical plane of $c$. For a fixed $r > 0$, choose $q, \eta$ such that $q \eta < r$. We will explain how to choose $r$ in the following passages. Let $G_c$ denote the Green’s escape rate function.

Let

$$W_0 = \{ z : G_c(z) < \eta \}$$

and for $i \in \mathbb{Z}$ with $i \geq 1$, let $W_i = f_c^{-i}(W_0)$. We note that

$$W_i = \left\{ z : G_c(z) < \frac{\eta}{(d+1)^i} \right\}.$$  

Also define

$$\tilde{S}(0) = \{ \varphi_c^{-1}(e^{s+2\pi i t}) : s \in (0, \eta), |t| < qs \}. $$

We call $\tilde{S}(0)$ a sector based at $\beta$. Its inverse image under $f_c$ is a union of similar sectors, each based at a pre-image of $\beta$. More precisely, for $\ell \in \{1, 2, \ldots, d\}$, let

$$\tilde{S} \left( \frac{\ell}{d+1} \right) = \left\{ \varphi_c^{-1}(e^{s+2\pi i t} e^{2\pi i \ell/d}) : s \in (0, \eta), |t| < qs \right\} \subset W_0.$$  

Then

$$f_c^{-1}(\tilde{S}(0)) = \bigcup_{\ell=0}^{d} \left( W_1 \cap \tilde{S} \left( \frac{\ell}{d+1} \right) \right).$$

When imposing the condition $q \eta < r$, we choose $r$ small enough so that the sectors $\tilde{S}(\frac{\ell}{d+1})$, $\ell = 0, 1, \ldots, d$, are pairwise disjoint (see Figure [5]).
Figure 5. The dynamical plane of $z^3 + c$

Figure 6. Cutting the dynamical plane of $z^3 + c$

Figure 7. The Riemann Surface $\tilde{X}$
Additionally, for \( i \in \mathbb{Z} \) with \( i \geq 1 \), form open subsets \( \tilde{S}_i(0) = W_i \cap \tilde{S}(0) \) of \( \tilde{S}(0) \). All points in \( \tilde{S}_i(0) \) have escape rates in the interval \( \left( 0, \frac{\eta}{d+1} \right) \), and \( f_c \) maps \( \tilde{S}_i(0) \) conformally onto \( \tilde{S}_{i-1}(0) \). Then there exists a branch of the log function that satisfies
\[
\log \circ \varphi_c(\tilde{S}(0)) = \{ z : Re(z) > 0 \text{ and } |Im(z)| < 2\pi q Re(z) \}.
\]

### 3.1. Steps in the definition of \( \Phi_d(c) \)

As in \([2] \) Chapter II], we shall follow this sequence of steps.

1. First, we cut along \( \mathcal{R}_0 \) and glue together two copies of \( W_0 \), one rotated by \( 180^\circ \), to get a quotient Riemann surface \( \tilde{X} \).
2. We then construct a function \( f \) on an open subset of \( \tilde{X} \) that
   - is analytic and acts like \( f_c \) away from the sectors \( \tilde{S}(\frac{\ell}{d+1}) \), \( \ell \in \{0, 1, 2, \ldots, d\} \) on both copies of \( W_0 \) and
   - has lines of discontinuities at the two copies of \( \mathcal{R}_{\frac{\ell}{d+1}} \) for \( \ell \in \{1, 2, \ldots, d\} \).
3. We show that by changing \( f \) in sectors around these rays, and by modifying the boundary of these sectors, we may construct a quasiregular map \( g : X_1 \to X \) where \( \overline{X_1} \subset X \subset \tilde{X} \) with \( X_1 \), \( X \) simply connected, and an almost complex structure \( \sigma \) on \( X \) that is \( g \) invariant. By Theorem \([2, 2] \), there exists a quasiconformal map \( \psi \) such that \( \psi \circ g \circ \psi^{-1} \) is analytic.
4. Finally, we will apply Theorem \([2, 5] \) to obtain a unique polynomial \( p_\alpha \) hybrid equivalent to \( \psi \circ g \circ \psi^{-1} \).

We will now implement these steps one by one.

### 3.1.1. Cutting along \( \mathcal{R}_0 \)

Let us cut along \( \mathcal{R}_0 \). In this slit disk, \( \tilde{S}(0) \) is now split into two components \( \tilde{S}_A \) and \( \tilde{S}_B \); we will denote as \( \mathcal{R}_0^{(A)} \) the copy of \( \mathcal{R}_0 \) bounding \( \tilde{S}_A \), and the other copy of \( \mathcal{R}_0 \) as \( \mathcal{R}_0^{(B)} \). Every \( x \in \mathcal{R}_0 \) now has two copies \( x^{(A)} \in \mathcal{R}_0^{(A)} \) and \( x^{(B)} \in \mathcal{R}_0^{(B)} \) (see Figure \([4] \)).

Consider a second copy of this slit \( W_0 \), and rotate it by \( \pi \). We will accent all objects in this (slit) second copy with a \( ^- \) superscript, and all objects in the original copy with a \( ^+ \) superscript. Glue the slit copies \( W_0^+, W_0^- \) together using the following rule:

\[
\forall x \in \mathcal{R}_0, \quad x^{(A+)} \sim x^{(B-)}, \quad x^{(B+)} \sim x^{(A-)}.
\]

This gives a quotient map
\[
\pi : W_0^+ \sqcup W_0^- \to W_0^+ \sqcup W_0^- / \sim.
\]

This quotient surface, denoted \( \tilde{X} \), can be endowed with a Riemann surface structure that makes \( \pi \) analytic away from \( \beta^\pm \). We can think of \( \tilde{X} \) as an open subset of the branched cover over \( W_0 \) corresponding to \( w \mapsto w^2 + \beta \), and \( \pi \) as a branch of \( \sqrt{z - \beta} \) on each of the slit copies \( W_0^+ \) and \( W_0^- \). We name this Riemann surface \( \tilde{X} \), and note that \( \tilde{X} \) has a smooth boundary. Let \( \tilde{X}_1 = \pi(W_1^+ \sqcup W_1^-) \). Then \( \tilde{X}_1 \) is an open
subset of $X$. We also define the sectors $\tilde{A}$ and $\tilde{B}$ as follows:

$$\tilde{A} = \pi(\tilde{S}_A^- \cup \mathcal{R}_0^{(A^-)} \cup \tilde{S}_B^+),$$
$$\tilde{B} = \pi(\tilde{S}_B^- \cup \mathcal{R}_0^{(B^-)} \cup \tilde{S}_A^+).$$

See Figure 7 for an illustration.

Remark 3.1. We could have performed our cut and paste surgery by cutting along $\mathcal{R}_j^0$ for any $j \in \{0, 1, \ldots, d - 1\}$ (these rays land at distinct fixed points of $f_c$). To get a continuous embedding $\Phi_d$ of $\mathcal{M}_{d+1}$, however, we will use the same $j$ for all $c \in \mathcal{M}_{d+1}$.

3.1.2. Constructing a map $f$ on a subset of $\tilde{X}_1$. For $z \in W_1^+$, define

$$f(\pi(z)) = \begin{cases} 
\pi(f_c(z)) & z \not\in \mathcal{R}_0^{(A^\pm)} \cup \mathcal{R}_0^{(B^\pm)}, \\
\pi(f_c(z^{(A^+)}) = \pi(f_c(z^{(B^{-})))) & z \in \mathcal{R}_0^{(A^+)} \cup \mathcal{R}_0^{(B^-)}, \\
\pi(f_c(z^{(B^+)}) = \pi(f_c(z^{(A^{-})))) & z \in \mathcal{R}_0^{(B^+)} \cup \mathcal{R}_0^{(A^-)}, \\
\pi(\beta^+) = \beta^- & z \in \{\beta, \beta^\pm\}.
\end{cases}$$

For $\ell \in \{1, 2, \ldots, d\}$, $f$ is not well defined on $\pi(\mathcal{R}_j^\pm(\frac{\ell}{d+1}))$ and we cannot extend it over any of these rays continuously since one component of the complement of such a ray in $\pi(\tilde{S}_{\ell-1}^\pm(\frac{\ell}{d+1}))$ is mapped to $\tilde{A}$, and the other is mapped to $\tilde{B}$.

However, on the complement in $\tilde{X}_1$ of the rays above, $f$ is analytic.

3.1.3. A new map on some sectors. By our definition of sectors, note that $\tilde{S}(\frac{\ell}{d+1}) = \omega^{\ell}\tilde{S}(0)$.

We will produce a quasiregular map $g$ that agrees with $f$ on the complement of the sets $\pi\left(\tilde{S}_i^\pm(\frac{\ell}{d+1})\right)$ for $\ell = 1, 2, \ldots, d$. We let

$$\tilde{S}_i(\frac{\ell}{d+1}) = \tilde{S}\left(\frac{\ell}{d+1}\right) \cap W_i.$$  

$f_c$ maps $\tilde{S}_i(\frac{\ell}{d+1})$ conformally to $\tilde{S}_{i-1}(0)$.

Choose $p, q'$ such that $0 < p < q' < q$, and consider the set $\Delta_q$ in log Böttcher coordinates as illustrated in Figure 8. Its boundary is defined so that it is smooth away from the points $0, \log(\frac{p}{d+1})(1 \pm 2\pi q)$, and such that it coincides with an arc of the circle $x^2 + y^2 = \frac{y^2}{(d+1)^2}$ on the two connected regions bounded by $y = \pm 2\pi px$ and $y = \pm 2\pi q'x$. We will also require the boundary of $\Delta_q$ to be symmetric about the $x-$axis in Figure 8. Additionally, let

$$\Delta_{q'} = \Delta_q \cap \{|y| < 2\pi q' x\},$$
$$\Delta_p = \Delta_q \cap \{|y| < 2\pi px\}. $$
Figure 8. The set $\Delta_q \subset \log \circ \phi_c(S_1(0))$. We will eventually define a map that is conformal on the white and lightly shaded regions, and quasiconformal on the darkly shaded region.

For $\ell \in \{0, 1, \ldots, d\}$, define

\begin{align*}
S_1\left(\frac{\ell}{d+1}\right) &= \omega^\ell \varphi_c^{-1} \circ \exp(\Delta_q), \\
Q_1\left(\frac{\ell}{d+1}\right) &= \omega^\ell \varphi_c^{-1} \circ \exp(\Delta_q'), \\
T_1\left(\frac{\ell}{d+1}\right) &= \omega^\ell \varphi_c^{-1} \circ \exp(\Delta_p), \\
Y_1\left(\frac{\ell}{d+1}\right) &= \omega^\ell \varphi_c^{-1} \circ \exp(\Delta_q \setminus \Delta_q') = S_1\left(\frac{\ell}{d+1}\right) \setminus Q_1\left(\frac{\ell}{d+1}\right).
\end{align*}

Clearly,

$$T_1\left(\frac{\ell}{d+1}\right) \subset Q_1\left(\frac{\ell}{d+1}\right) \subset S_1\left(\frac{\ell}{d+1}\right) \subset \bar{S}_1\left(\frac{\ell}{d+1}\right).$$

We additionally define a subset $V_0$ of the slit disk $W_0 \setminus R_0$ as follows:

\begin{equation}
V_0 = f_c\left(W_1 \setminus \bigcup_{\ell=0}^{d} S_1\left(\frac{\ell}{d+1}\right)\right) \cup f_c(Y_1(0)).
\end{equation}

See Figure 9 for details. Let $z_1, z_2, \tilde{z}_1$ and $\tilde{z}_2$ be points defined as in Figure 9. By the Riemann mapping theorem, there exist analytic maps $\tilde{k} : \mathbb{D} \to V_0$ and $\tilde{m} : T_1(0) \to \mathbb{D}$ such that

$$\tilde{k}(\tilde{m}(\beta)) = \beta,$$

$$\tilde{k}(\tilde{m}(z_1)) = \tilde{z}_1,$$

$$\tilde{k}(\tilde{m}(z_2)) = \tilde{z}_2.$$
Figure 9. The Riemann map $\tilde{h}$ maps the shaded region $T_1(0)$ in the top image to the shaded region $V_0$ in the bottom image.

Figure 10. The darkly shaded region above indicates the set $\Delta$-the image under $\hat{h}$ of a connected component of $\Delta_{q'} \setminus \Delta_p$. 
Let \( \tilde{h} = \tilde{k} \circ \tilde{m} \). The map \( \tilde{h} : T_1(0) \rightarrow V_0 \) is the unique analytic function that sends the triple \((\beta, z_1, z_2)\) to \((\beta, \tilde{z}_1, \tilde{z}_2)\) (see Figure 9). \( \partial V_0, \partial T_1(0) \) are quasi-circles. Therefore, \( \tilde{k}, \tilde{m} \) extend to quasisymmetric maps on the boundaries of their respective domains. Furthermore, by [22, Chapter 3.4, Exercise 1],

\[
\begin{align*}
\tilde{k}(z) &= \beta + a_k(z - 1)^{2-4q'} + O(|z - 1|^{2-4q' + (2-4q')\gamma_k}), \\
\tilde{m}^{-1}(z) &= \beta + a_m(z - 1)^{4p} + O(|z - 1|^{4p + 4p \gamma_k})
\end{align*}
\]

for some \( \gamma_k, \gamma_m \in (0, 1), a_k, a_m \in \mathbb{C} \setminus \{0\}. \)

It then follows that

\[
\tilde{h}(z) = \beta + a_h(z - \beta)^{\frac{1-2q'}{2p}} + O\left(|z - \beta|^{\frac{1-2q'}{2p} + (\frac{1-2q'}{2p})\gamma_h}\right)
\]

for some \( \gamma_h \in (0, 1), a_h \in \mathbb{C} \setminus \{0\} \). Since \( \varphi_c \) does not distort angles, conjugating \( \tilde{h} \) by \( \varphi_c \) does not change the exponents in this equation, and we have the following:

**Proposition 3.2.** For all \( z \in \exp(\Delta_p) = \varphi_c(T_1(0)) \),

\[
(3.6) \quad \varphi_c \circ \tilde{h} \circ \varphi_c^{-1}(z) = 1 + a'(z - 1)^{\frac{1-2q'}{2p}} + O\left(|z - 1|^{\frac{1-2q'}{2p} + (\frac{1-2q'}{2p})\gamma}\right)
\]

for some \( \gamma \in (0, 1), a' \in \mathbb{C} \setminus \{0\} \).

Let \( G \) be a connected component of \( \Delta_q \setminus \Delta_p \), say the one bounded by \( y = 2\pi q'x \) and \( y = 2\pi px \). The polynomial \( f_c \) induces the map \( \mu_{d+1}(z) = (d+1)z \) on the part of its boundary where \( y = 2\pi q'x \). The map \( \tilde{h} = (\log \circ \varphi_c) \circ \tilde{h} \circ (\varphi_c^{-1} \circ \exp) \) extends to a continuous map on the part of the boundary where \( y = 2\pi px \).

Let \( \Delta \) be the set \( \Delta_q \cap \{x^2 + y^2 < \eta^2\} \) (see Figure 10 for an illustration of \( \Delta \)). The set \( S(0) := \varphi_c^{-1} \circ \exp(\Delta) \) is an open subset of \( \tilde{S}(0) \). For \( \ell \in \{1, 2, \ldots, d\} \), let \( S(\frac{\ell}{d+1}) = \omega^\ell S(0) \). The sector \( S(\frac{\ell}{d+1}) \) contains \( \tilde{S}(\frac{\ell}{d+1}) \).

The following is a crucial lemma.

**Lemma 3.3.** There exists a quasiconformal map from \( G \) to \( \Delta \) that restricts to \( \mu_{d+1} \) on one boundary, and to \( \tilde{h} \) on the other boundary.

**Proof.** \( G \) has a positive angle at the vertex 0. In log coordinates, \( G' = \log G \) is a half-infinite horizontal strip with \( \mu_{d+1}(z) = z + \log(d+1) \) induced by \( \mu_{d+1} \) on the part of the boundary where \( y = \arctan(2\pi q') \), and \( H(z) = \log \tilde{h}(e^z) \) on the part of the boundary where \( y = \arctan(2\pi p) \).

We will interpolate between \( \mu_{d+1} \) and \( H \) by mapping vertical lines in \( G' \) to lines joined by the images of the endpoints. If we can show that these image lines have uniformly bounded slope, the resulting map will be quasiconformal. We prove this in detail below.

Set \( \theta_p = \arctan(2\pi p), \theta_q' = \arctan(2\pi q') \). We will define \( H \) on \( \overline{G'} \) by extending along vertical lines:

\[
H \left( x + i((1-t)\theta_p + t\theta_q') \right) = (1-t)H(x + i\theta_p) + t\mu_{d+1}(x + i\theta_q').
\]
Proposition 3.4. There exists $R > 0$ such that for all $z \in \{ \text{Im}(z) = \arctan(2\pi p) \} \cap \mathbb{C}^*$,
\begin{equation}
|H(z) - z| \leq R.
\end{equation}

Proof. We will prove this by showing that both $z \mapsto z - H(z)$ and $z \mapsto H(z) - z$ are bounded above.

Suppose $z - H(z)$ is not bounded above, then for each natural number $n$, there exists $z_n$ such that
\[
z_n - H(z_n) > n
\]
and up to a subsequence, the $z_n$ tend to $-\infty$. But this implies that
\[
|\hat{h}(u_n)| < \left| \frac{u_n}{e^n} \right|,
\]
where $u_n = \exp z_n$. Furthermore,
\[
\text{Re}(\hat{h}(u_n)) \leq |\hat{h}(u_n)|
\leq \left| \frac{u_n}{e^n} \right| = \frac{1}{e^n} \sqrt{\text{Re}(u_n)^2 (1 + 4\pi^2 p^2)}
\leq C \frac{\text{Re}(u_n)}{e^n} = C \frac{\text{Re}(u_n)}{e^n}
\]
for some constant $C > 0$.

Set $w_n = \exp u_n$, and note that $\exp \hat{h}(u_n) = \varphi_c \circ \tilde{h} \circ \varphi_c^{-1}(w_n)$. Thus
\begin{equation}
|\varphi_c \circ \tilde{h} \circ \varphi_c^{-1}(w_n)| < b|w_n|^{\frac{1}{2m}}
\end{equation}
for some $b > 0$.

But equation (3.8) implies that $|\varphi \circ \tilde{h} \circ \varphi_c^{-1}(w_n)|$ converges much faster to 1 than allowed by equation (3.6) and forms a contradiction. This proves that $z - H(z)$ is bounded above.

Similarly, suppose $H(z) - z$ is not bounded above as $z \to -\infty$, there exists a sequence $z_n \to -\infty$ such that
\[
H(z_n) - z_n > n.
\]
Thus
\[
|\hat{h}(u_n)| > |u_n| e^n.
\]
Consequently, we have
\[
\text{Re}(\hat{h}(u_n)) = \frac{|\hat{h}(u_n)|}{\sqrt{1 + 4\pi^2 q^2}}
\geq \frac{|u_n| e^n}{\sqrt{1 + 4\pi^2 q^2}}
\geq \frac{\sqrt{1 + 4\pi^2 p^2}}{\sqrt{1 + 4\pi^2 q^2}} \text{Re}(u_n) e^n.
\]
But this gives us
\[
|\varphi_c \circ \tilde{h} \circ \varphi_c^{-1}(w_n)| > \epsilon|w_n| e^n
\]
for some constant $\epsilon > 0$, which also contradicts equation (3.6).

This proves Proposition 3.4. □
**Figure 11.** The map $H$ is defined by mapping vertical lines to lines joining the images of their endpoints.

It is clear that $H$ interpolates between the maps on the two horizontal boundaries, and that $H(G' \cap \{Re(z) = \log(\frac{3}{2})\}) = \{\log \eta + iy : |y| < \arctan(2\pi q')\}$. Furthermore, $H$ is a quasiconformal map whose dilatation is bounded above by some $M \geq 1$ (see Figure 11): this is because vertical lines in the domain are mapped by $H$ to lines whose slopes are bounded below by some uniform constant, by equation 3.7. We conjugate $H$ by the exponential map to obtain a quasiconformal map $\hat{h}$ from $G$ that satisfies the properties in the statement of Lemma 3.3.

$G$ could be taken to be either connected component of $\Delta_{q'} \setminus \Delta_p$. On the dynamical plane, it corresponds to a component $G$ of $Q_1(0) \setminus T_1(0)$. We shall henceforth denote the copy of $S(0)$ in $\hat{A}$ as $\hat{A}$, and the copy in $\hat{B}$ as $\hat{B}$. With this in mind, we will take $h_{G,A}$ to mean the map $(\omega^\ell \circ \varphi^{-1}_c \circ \exp) \circ h \circ (\log \circ \varphi_c \circ \omega^{-\ell})$ from the component $G$ of $Q_1(\frac{\ell}{\pi+1}) \setminus T_1(\frac{\ell}{\pi+1})$ to $\hat{A}$, and $h_{G,B}$ to mean the same map, but from $G$ to $B$. We will use the same names for the extended maps from $\hat{G}$.

### 3.1.4. Constructing a quasiregular map $g$. Let $S$ be a sector of the form $\pi\left(S_1^+\left(\frac{\ell}{\pi+1}\right)\right)$, where $\ell \in \{1, 2, \ldots, d+1\}$. The map $f$ defined in Section 3.1.2 has a line of discontinuities in $S$ along the ray $\pi\left(\mathcal{R}_{\frac{\ell}{\pi+1}}^+\right)$—on one side of this ray, $f$ maps into $\hat{A}$ and approaches $\pi(\mathcal{R}_0^{(A^-)})$, whereas on the other side, $f$ maps into $\hat{B}$ and approaches $\pi(\mathcal{R}_0^{(B^-)})$. Define a map $h$ on $\bar{S}$ as follows.

- On $\pi\left(T_1^+\left(\frac{\ell}{\pi+1}\right)\right)$, let $h(\pi(z^+)) = \pi(h(\omega^{-\ell}z^-)) \in \pi(V_0^-)$.
- On the connected component $G$ of $\pi\left(Q_1^+\left(\frac{\ell}{\pi+1}\right) \setminus T_1^+\left(\frac{\ell}{\pi+1}\right)\right)$ with $\ell = 1, 2, \ldots, d$ part of whose boundary $f$ maps into $\partial\hat{A}$, let $h(\pi(z^+)) = h_{G,A}(z)$. 


Figure 12. An illustration of $h$ on $\pi(S_1(\frac{1}{3})^+)$; the dark components in the top image are quasiconformally mapped to the dark components in the bottom image, the lightly shaded region maps by the Riemann map $\omega^{-1}h$ to the lightly shaded region below, and the white region maps by $f$ to the white region below.

- On the connected component $G$ of $\pi\left(Q_1^+\left(\frac{\ell}{d+1}\right) \setminus T_1^+\left(\frac{\ell}{d+1}\right)\right)$ with $\ell = 1, 2, \ldots, d$ part of whose boundary $f$ maps into $\partial B$, let $h(\pi(z^+)) = h_{G,B}(z)$.
- On $\pi\left(Y_1^+\left(\frac{\ell}{d+1}\right)\right)$, let $h(\pi(z^+)) = f(\pi(z^+))$, where $f$ is defined as in Section 3.1.2.

The map $h$ so defined is a quasiconformal homeomorphism from $S$ to $\pi(V_0^-) \cup A \cup B$ (see Figure 12 for an illustration of $h$ on $\pi(S(\frac{1}{3})^+)$ when $d = 2$), and restricts to an analytic map on $\pi\left(T_1\left(\frac{\ell}{d+1}\right)\right)$.

Furthermore, the latter set has smooth boundary at the points $\pi(\tilde{z}_1^-)$ and $\pi(\tilde{z}_2^-)$ in Figure 12, consider $\pi(\tilde{z}_1^-)$ for instance. In Böttcher coordinates, the boundary in a neighborhood of $\tilde{z}_1$ looks like $f_c(\varphi^{-1}_c \circ \exp(\gamma))$, where $\gamma$ is a neighborhood of the boundary of $\Delta_q$ at the point $\log \circ \varphi_c(z_1)$; $\gamma$ is clearly smooth.
On $\pi\left(S^{-}\left(\frac{\ell}{\pi+1}\right)\right)$, we define $h$ the same way, except with the following change: on $\pi\left(T_{1}^{-}\left(\frac{\ell}{\pi+1}\right)\right)$, let $h(\pi(z^{-})) = \pi(\tilde{h}(\omega^{-\ell}z^{+})) \in \pi(V^{+}_{0})$. This $h$ is a quasiconformal homeomorphism from $S$ to $\pi(V^{+}_{0}) \cup A \cup B$. Finally, we construct a quasiregular map on newly defined subsets of $\tilde{X}_{1}, \tilde{X}$. Let

$$X = \pi(V^{+}_{0}) \cup A \cup B \cup \pi(V^{-}_{0}).$$

Also let $X_{1}$ be the subset of $\tilde{X}_{1}$ where all sectors of the form $\pi\left(S^{\pm}_{1}\left(\frac{\ell}{\pi+1}\right)\right)$, $\pi\left(S_{1}^{-}\left(\frac{\ell}{\pi+1}\right)\right)$ are replaced by $\pi\left(S^{\pm}_{1}\left(\frac{\ell}{\pi+1}\right)\right)$ for $\ell = 1, 2, \ldots, d$, and let $X$ be the open subset of $\tilde{X}$ where $\tilde{A}$ and $\tilde{B}$ are replaced by $A$, $B$ respectively. See Figure 13b for details. Clearly, $X_{1}$ is an open subset of the Riemann surface $X$ compactly
contained in \( X \). We define

\[
g : X_1 \rightarrow X,
\]

\[
g(\pi(z)) = \begin{cases} f(\pi(z)) & z \in \left(W_1^+ \cup \bigcup_{\ell=1}^{d} S_1^+ \left(\frac{\ell}{\tau} \right)\right) \cup \left(W_1^- \cup \bigcup_{\ell=1}^{d} S_1^- \left(\frac{\ell}{\tau} \right)\right), \\
h(\pi(z)) & z \in S_1^+ \left(\frac{\ell}{\tau} \right), \ell = 1, 2, \ldots, d. \end{cases}
\]

See Figure 13.11 for an illustration of \( g \).

\( g \) is quasiregular. Furthermore, any \( g \) orbit visits \( \pi \left(Q_1^+ \left(\frac{\ell}{\tau} \right) \right) \setminus T_1^+ \left(\frac{\ell}{\tau} \right) \) or \( \pi \left(Q_1^- \left(\frac{\ell}{\tau} \right) \right) \setminus T_1^- \left(\frac{\ell}{\tau} \right) \) at most once, and these are the only regions where \( g \) is not analytic. We will use this fact to define a new complex structure \( \sigma \) (given by an ellipse field \( E_x \) for \( x \in X \)) by setting

- \( E_x = S^1 \) if \( x \in X \setminus X_1 \) or if the orbit of \( x \) never visits \( \pi \left(Q_1^+ \left(\frac{\ell}{\tau} \right) \right) \setminus T_1^+ \left(\frac{\ell}{\tau} \right) \)
  or \( \pi \left(Q_1^- \left(\frac{\ell}{\tau} \right) \right) \setminus T_1^- \left(\frac{\ell}{\tau} \right) \),
- \( E_x = (T_x g)^{-1}(S^1) \) for \( x \in \pi \left(Q_1^+ \left(\frac{\ell}{\tau} \right) \right) \setminus T_1^+ \left(\frac{\ell}{\tau} \right) \) or \( x \in \pi \left(Q_1^- \left(\frac{\ell}{\tau} \right) \right) \setminus T_1^- \left(\frac{\ell}{\tau} \right) \) for some \( \ell \in \{1, 2, \ldots, d\} \), and
- \( E_x = (T_x g^o)^{-1}(E_g^o(x)) \) if \( g^o(x) \) is the first point in the \( g \)-orbit of \( x \) that is in one of the regions above.

The complex structure \( \sigma \) thus defined has bounded dilatation, and \( g^* \sigma = \sigma \).

### 3.1.5. Obtaining a polynomial

Define the map \( \tau : X \rightarrow Y \) by sending \( \pi(z^+) \) to \( \pi(z^-) \). \( \tau \) satisfies \( \tau^* \sigma = \sigma \), \( \tau(X_1) = X_1 \), and \( \tau^o \sigma = \tau \sigma \). We note that

\[
g \circ \tau = \tau \circ g.
\]

By Theorem 2.2 there exists an integrating map \( \psi \) for \( \sigma \) mapping \( \pi(\beta^+) \) to 0, and satisfying \( \frac{\psi(z)}{z} \rightarrow 1 \) as \( z \rightarrow \infty \). The map \( g = \psi \circ g \circ \psi^{-1} : U' \rightarrow U \) is polynomial-like, and has two critical points with local degree \( d + 1 \). The map \( \kappa = \psi \circ \tau \circ \psi^{-1} \) is an analytic involution on \( U \), and commutes with the map \( g \). We can further conjugate by a Riemann map taking the pair \((U, 0)\) to \((\mathbb{D}, 0)\). By the Schwarz lemma, we can assume without loss of generality that \( \kappa \vert_U(z) = -z \); in particular, \( \kappa \) has a global extension.

By Theorem 2.5 \( g \) is hybrid equivalent to a degree \( 2d + 1 \) polynomial \( p \) with two critical points. We may choose this hybrid equivalence \( h : \mathbb{C} \rightarrow \mathbb{C} \) such that \( h(0) = 0 \). Then \( \delta = h \circ \kappa \circ h^{-1} \) is an affine map of \( \mathbb{C} \) with \( \delta(0) = 0, \delta^o = \text{id}, \delta \neq \text{id} \). Therefore, \( \delta(z) = -z \). \( p \) commutes with \( \delta \), and can now be normalized to the form

\[ a \int_0^z \left(1 - \frac{w^a}{d}\right)^d dw \text{ for a unique } a \in \mathbb{C}^+ \]

The choice of \( a \) does not depend on our initial choice of \( q, \eta, \psi \) (\( h \) is determined by \( c, p, q, q', \eta \))—we can show that different choices give rise to hybrid equivalent polynomials.

### 3.2. The image of \( \Phi_d \)

Clearly, \( \Phi_d(M_{d+1}) \subseteq \text{CBO}_d \). Let \( a = \Phi_d(c) \). By our construction, 0 is a fixed point of \( p_a \) belonging to the Julia set, and it disconnects the Julia set into two components.

Under our surgery, the original dynamical ray \( R_0(s(a)) \) landing at \( \beta \) gets transformed into an arc \( \Gamma \) from 0 to \( \infty \) in the dynamical plane of \( p_{\Phi_d(c)} \) whose interior is contained in the escaping set. In the monic representation \( P_{s(a)} = p_a \), \( \Gamma \) has the same access as \( R_0(s(a)) \). The union \( \Gamma \cup -\Gamma \), and indeed \( R_0(s(a)) \cup R_\frac{1}{2}(s(a)) \),
separates the orbits of the two critical points of $P_{s(a)}$. From Section 2.3.1 we recall the definition

$$CBO_d^{(+,-)} = \{ a \in CBO_d | O^{L}_{s(a)} \subset F^{L}_{s(a)} \cup \{0\} \text{ and } O^{R}_{s(a)} \subset F^{R}_{s(a)} \cup \{0\} \}.$$ 

The above discussion shows that $\Phi_d(M_{d+1}) \subseteq CBO_d^{(+,-)}$. We will show in Section 5 that $\Phi_d(M_{d+1}) = CBO_d^{(+,-)}$.

4. Continuity of $\Phi_d$

To show continuity of $\Phi_d$, we will follow the strategy laid out in [2, Chapter II.8], and prove it separately when $c$ is on the boundary, or in the interior of $M_{d+1}$. Throughout this section, we shall index all sets and functions in Section 3 in constructing $\Phi_d(c)$ by the subscript $c$. For example, the projecton $\pi$ is referred to as $\pi_c$, the quasiregular map $g$ as $g_c$, the domain of $g_c$ as $(X_1)_c$ and so on.

Lemma 4.1. If $p_a, p_{a'}$ with $a, a' \in CBO_d$ are hybrid equivalent, then they are affine conjugate.

Proof. This follows from [8, Chapter I.6, Corollary 2].

4.1. The interior case. If $c \in M^0_{d+1}$, then proof is based on the proof of [8, Chapter II.5, Proposition 12].

Definition 4.2. Given $f_\lambda : U'_\lambda \rightarrow U_\lambda$, for $\lambda \in \Lambda$ and $\overline{U'_\lambda} \subset U_\lambda$, let

$$U' = \{(\lambda, z) | z \in U'_\lambda\},$$

$$U = \{(\lambda, z) | z \in U_\lambda\}$$

and define $f : U' \rightarrow U$ as $f(\lambda, z) = f_\lambda(z)$. If

1. $U', U$ are homeomorphic over $\Lambda$ to $\Lambda \times \mathbb{D}$,
2. the projection of the closure of $U'$ in $U$ to $\Lambda$ is proper, and
3. $f$ is holomorphic and proper,

then $f_\lambda$ is called an analytic family.

Let us go back to the construction of $\Phi_d(c)$ from $f_c$. We first construct a quasiregular map $g_c : (X_1)_c \rightarrow X_c$. This map is built from $f_c$ away from certain escaping sectors, and from the Riemann map $h_c$ on other sectors. Then we find an invariant complex structure $\sigma_c$ for $g_c$ and find integrating maps $\tilde{\psi}_c$. This gives us the polynomial-like family $\tilde{g}_c : U'_c \rightarrow U_c$.

Proposition 4.3. On a connected component $\Lambda$ of $M^0_{d+1}$, $(c, z) \mapsto (c, \tilde{g}_c(z))$ is an analytic family.

Proof. We show that $\tilde{g}_c$ satisfies the three properties of Definition 4.2.

1. $U'_c, U_c$ are homeomorphic to $\mathbb{D}$ and $c' \mapsto U'_c, c \mapsto U_c$ are both continuous maps in the Hausdorff topology.
2. Let $\Pi$ be the projection $(c, z) \mapsto c$. Given any compact set $K$ in $\Lambda$, and a sequence $(c_n, z_n) \in \Pi^{-1}(K)$, up to a subsequence, $c_n \rightarrow c \in K$. We note that $z_n \in \overline{U'_c}$. The sets $\overline{U'_c} \rightarrow \overline{U_c}$ in the Hausdorff topology, and hence, there exists a sequence $\tilde{z}_n \in \overline{U'_c}$ such that $|z_n - \tilde{z}_n| \rightarrow 0$. Since $\overline{U'_c}$ is compact, up to a subsequence, $\tilde{z}_n \rightarrow z \in \overline{U'_c}$. So $z_n \rightarrow z$ up to the same subsequence. This shows that $\Pi^{-1}(K)$ is compact.
(3) By [17], every parameter $c \in \Lambda$ is structurally stable. More particularly, given $c \in \Lambda$, there exists a holomorphic motion $L: \Lambda \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that $L_c = id$, and for all $\bar{c} \in \Lambda$, $L_{\bar{c}}$ is quasiconformal and satisfies $L_{\bar{c}} \circ f_c \circ L_{\bar{c}}^{-1} = f_{\bar{c}}$. But this also means that $h_{\bar{c}} = L_{\bar{c}} \circ h_c \circ L_{\bar{c}}^{-1}$ on $\varphi_{\bar{c}}^{-1} \circ \exp(\Delta_p)$, and by definition of the quasiconformal extension, $h_{\bar{c}} = L_{\bar{c}} \circ h_c \circ L_{\bar{c}}^{-1}$ on the sector $S_1 \left( \frac{\ell}{d+1} \right)$ in the dynamical plane of $f_{\bar{c}}$. Therefore, $g_c$, and consequently $\sigma_c$, depend analytically on $c$. By Theorem 2.2, the integrating map $\psi_c$ depends holomorphically on $c$.

For a fixed $z \in U'_c$, when $\bar{c}$ is close to $c$, $\bar{g}_{\bar{c}}(z)$ is well-defined, and $\bar{c} \mapsto \bar{g}_{\bar{c}}(z) = \psi_{\bar{c}} \circ g_{\bar{c}} \circ \psi_{\bar{c}}^{-1}(z)$ is holomorphic in $\bar{c}$. For a fixed $c$, $z \mapsto \bar{g}_c(z)$ is holomorphic by definition. Thus $\bar{g}_c(z)$ is holomorphic in both $c$ and $z$; by Hartog’s theorem, it is holomorphic as a function of $(c, z)$. Proof that $\bar{g}_c(z)$ is proper is similar to point (2).

\[ \square \]

In Proposition 4.3, we showed that $(c, z) \mapsto \bar{g}_c(z)$ is an analytic family over every connected component $\Lambda$ of $M_{d+1}^0$. Given $c \in \Lambda$, let us pick the hybrid equivalence $k_c$ conjugating $\bar{g}_c$ to a polynomial such that $k_c(0) = 0$ and $k_c(z)/z \rightarrow 1$. Then, by [8, Chapter II.5, Proposition 12], the polynomials $k_c \circ \bar{g}_c \circ k_c^{-1}$ form a continuous family over $\Lambda$. As proved in Section 3, these are affine conjugate to bicritical odd polynomials, and their critical points vary continuously with respect to $c$. Hence there exists a continuous family of scaling maps $M_c$ that map these critical points to $\pm \sqrt{d}$. But then

$$p_{\Phi_{d}\left(\bar{c}\right)} = M_{\bar{c}} \circ k_{\bar{c}} \circ \bar{g}_{\bar{c}} \circ k_{\bar{c}}^{-1} \circ M_{\bar{c}}^{-1}$$

is clearly continuous in $\bar{c}$.

4.2. The boundary case. Lemma 4.4 and its proof are similar to [2, Chapter II.8, Lemma 3].

**Lemma 4.4.** If $p_a$ and $p_{a'}$ are quasiconformally equivalent via $\psi$, with $a \in \partial \mathbb{C} \mathbb{B} \mathbb{O}_d$, such that $\psi$ satisfies the conditions

$$\psi(0) = 0,$$

$$\psi(\sqrt{d}) = \sqrt{d},$$

$$\lim_{z \rightarrow \infty} \frac{\psi(z)}{z} = 1,$$

then $a = a'$.

**Proof.** We first note that any $\psi$ as above also satisfies $\psi(-\sqrt{d}) = -\sqrt{d}$. If $K_{p_a}$ has measure 0, then $\psi$ is a hybrid equivalence and the result follows.

Otherwise, our strategy is to build a hybrid equivalence between the two polynomials, similar to [8, Chapter I.6, Corollary 2] and use Lemma 4.1. Consider the Beltrami form $\mu = \frac{\partial \psi}{\partial \bar{c}}$ and let $\mu_0$ be the form that agrees with $\mu$ on $K_{p_a}$ and equals 0 on $\mathbb{C} \setminus K_{p_a}$. Set $k = \|\mu_0\|_{\infty}$. 


We note that \( k < 1 \). By Theorem 2.2, for every \( t \in \mathbb{D}_k^+ \), there exists a unique quasiconformal homeomorphism \( \psi_t : \mathbb{C} \to \mathbb{C} \) such that

\[
\frac{\partial \psi_t}{\partial \psi_t} = t \mu_0,
\psi_t(0) = 0,
\lim_{z \to \infty} \frac{\psi_t(z)}{z} = 1.
\]

We note that \( \kappa_t(z) = \psi_t(-\psi_t^{-1}(z)) \) is a family of affine maps that satisfy

\[
\kappa_t(0) = 0,
\lim_{z \to \infty} \frac{\kappa_t(z)}{z} = -1.
\]

Therefore, \( \kappa_t(z) = -z \).

\( \psi_t \circ p_a \circ \psi_t^{-1} \) is a polynomial with exactly two critical points that commutes with \( \kappa_t \). It has the form \( \tilde{a}(t) \int_0^z \left(1 - \frac{w}{z(t)} \right)^d \left(1 + \frac{w}{z(t)} \right)^d dw \). The functions \( \tilde{a}, x : \mathbb{D}_k^+ \to \mathbb{C} \) are holomorphic, with \( x(0) = \sqrt{d}, \tilde{a}(0) = a \). These polynomials are odd, therefore by conjugating them by \( h_t(z) = \frac{z\sqrt{d}}{z(t)} \), we obtain polynomials of the form

\[
a(t) \int_0^z \left(1 - \frac{w^2}{d} \right)^d dw,
\]

where \( a : \mathbb{D}_k^+ \to \mathbb{C} \) is holomorphic, with \( a(t) \in \text{CBO}_d \), and \( a(0) = a \in \partial\text{CBO}_d \). By construction, for all \( t \in \mathbb{D}_k^+ \), the polynomials \( p_a(t) \) and \( p_a(0) \) are quasiconformally conjugate. However, for any \( t \) with \( a(t) \notin \text{CBO}_d \), the Julia set \( J_{p_a(t)} \) has a totally disconnected Julia set, and thus, it is not possible for \( p_a(t) \) to be quasiconformally conjugate to \( p_a(0) \). So we must have \( a(t) \in \text{CBO}_d \) for all \( t \in \mathbb{D}_k^+ \).

This implies that the function \( t \mapsto a(t) \) is constant, and \( \psi \circ \psi_t^{-1} \circ h_t^{-1} \) is a hybrid equivalence between \( p_a \) and \( p_a' \). Lemma 4.4 then implies \( a = a' \).

To show continuity of \( \Phi_d \) at \( c \in \partial\mathcal{M}_{d+1} \), it suffices to show that its graph is closed, that is, if \( c_n \in \mathcal{M}_{d+1} \) converge to \( c \) and \( a_n = \Phi_d(c_n) \to \tilde{a} \), then \( \tilde{a} = \Phi_d(c) \).

Let

\[
a = \Phi_d(c),
g = g_c,
\sigma = \sigma_c,
\psi = \psi_c,
\tilde{g} = \tilde{g}_c = \psi \circ g \circ \psi^{-1},
\varphi = \varphi_c,
V_n = (V_0)_{c_n},
Q_n(\ell) = \left(Q_1\left(\frac{\ell}{d+1}\right)\right)_{c_n},
\]

\[
T_n(\ell) = \left(T_1\left(\frac{\ell}{d+1}\right)\right)_{c_n}.
\]

**Proposition 4.5.** The sequence of quasiregular maps \( g_n \) converge to \( g \).

**Proof.** On both the + and − copies of \( (W_0)_{c_n} \), \( g_n \) coincides with \( f_n \) away from the sectors \( Q_1(\ell) \), for \( \ell = 1, 2, \ldots, d \). On each of these sectors, \( g_n \) has as its
components a conformal map \( \tilde{h}_n : T_n(\ell) \to V_n \), chosen uniquely so that the triple \(((z_1)_n, (z_2)_n, \omega^f \beta_n)\) is mapped to the triple \(((\tilde{z}_1)_n, (\tilde{z}_2)_n, \beta_n)\), and a quasiconformal extension to \( Q_n(\ell) \setminus T_n(\ell) \). Similarly, on \( T(\ell) \), \( g \) agrees with an analytic map \( \tilde{h} : T(\ell) \to V \) chosen so that the triple \((z_1, z_2, \omega^f \beta)\) is mapped to \((\tilde{z}_1, \tilde{z}_2, \beta)\). Fix an \( \ell \in \{1, 2, \ldots, d\} \). We will first show that the \( \tilde{h}_n \) converge to \( \tilde{h} \).

Let \( \rho_n \) be the Riemann map that sends \( \mathbb{D} \) to \( V_n \), with \( \rho_n(0) = 0 \) and \( \rho_n'(0) > 0 \). Observe that \( V_n \) converges to \( V \) with respect to the point 0 in the sense of kernel convergence (see \cite{22} Section 1.4). By Carathéodory’s kernel convergence theorem (\cite{22} Theorem 1.8), \( \rho_n \to \rho \) uniformly in \( \mathbb{D} \), where \( \rho : \mathbb{D} \to V \) is a conformal map that sends 0 to 0 and satisfies \( \rho'(0) > 0 \). Since the boundaries of \( V_n, V \) are quasicircles, the \( \rho_n \) extend to \( \partial \mathbb{D} \) and these boundary maps converge uniformly to the boundary extension of \( \rho \). Thus, the triples \((s_n, t_n, w_n)\) in \( S^1 \) that under \( \rho_n \) to \(((\tilde{z}_1)_n, (\tilde{z}_2)_n, \beta_n)\) converge to the triple \((s, t, w)\) in \( S^1 \) that maps under \( \rho \) to \((\tilde{z}_1, \tilde{z}_2, \beta)\).

Let \( M_n : \mathbb{D} \to \mathbb{D} \) be a sequence of automorphisms that send \((1, i, -1)\) to \((s_n, t_n, w_n)\), and let \( M \) be the automorphism of \( \mathbb{D} \) that sends \((1, i, -1)\) to \((s, t, w)\). Then \( M_n \to M \) on \( \mathbb{D} \).

Lastly, for a given \( \ell \), note that \( \varphi_n(T_n(\ell)) \) is the same domain \( D = \exp(\Delta_\beta) = \varphi(T(\ell)) \) for each \( n \), and furthermore, \( (\varphi_n((z_1)_n), \varphi_n((z_2)_n), \varphi_n(\omega^f \beta_n)) = (\varphi(z_1), \varphi(z_2), \varphi(\omega^f \beta)) \) (we note that \( \omega^f \beta \) and \( \omega^f \beta' \) are tips, ie. unique dynamical rays at each of these points, so evaluating the Böttcher chart at these points makes sense). Let \( e : D \to \mathbb{D} \) be the Riemann map that takes the triple \((\varphi(z_1), \varphi(z_2), \varphi(\omega^f \beta))\) in \( \partial D \) to \((1, i, -1)\). Then

\[
\tilde{h}_n = \rho_n \circ M_n \circ e \circ \varphi_n,
\]

\[
\tilde{h} = \rho \circ M \circ e \circ \varphi.
\]

It is clear by our discussion that \( \tilde{h}_n \to \tilde{h} \).

Therefore, on the sectors \( T_n(\ell) \), the sequence \( g_n \) converges to \( g \). But note that the quasiconformal extension to \( Q_n(\ell) \) is done in the same way for each \( n \). Therefore, \( g_n \to g \).

By definition of \( \sigma_n \) and \( \sigma \), we must have \( \sigma_n \to \sigma \), and consequently, by Theorem 2.2, \( \psi_n \to \psi \).

This discussion tells us that

\[
\tilde{g}_n = \psi_n \circ g_n \circ \psi_n^{-1} \to \psi \circ g \circ \psi^{-1} = \tilde{g}.
\]

Now consider the hybrid equivalences \( k_n \) that conjugate \( \tilde{g}_n \) to \( p_{a_n} \). These have bounded dilatation ratio and map 0 to 0, and hence form an equicontinuous family. Upto a subsequence, \( k_n \) converge to a quasiconformal map \( \tilde{k} \). Thus, \( k_n \circ \tilde{g}_n \circ k_n^{-1} \to \tilde{k} \circ \tilde{g} \circ k^{-1} \). We will call the latter map \( \tilde{g} \).

Using \cite{8} Chapter II.7, Lemma, p.313], \( \tilde{g} \) is quasiconformally equivalent to \( p_{\tilde{a}} \) (not necessarily hybrid equivalent), but it is also quasiconformally equivalent to \( \tilde{k} \circ \tilde{g} \circ k^{-1} \), which in turn is hybrid equivalent to \( p_{\tilde{a}} \).

This shows that \( p_{\tilde{a}} \) is quasiconformally equivalent to \( p_{\tilde{a}} \). We can choose the equivalence so that the conditions of Lemma 4.4 are satisfied. But in order to use this lemma, we also need to show that \( a \in \partial CBO_d \).
Consider a sequence $c_n^*$ of Misiurewicz parameters tending to $c$, and let $a_n^* = \Phi_d(c_n^*)$. Then $a_n^*$ is Misiurewicz, and there exists a subsequence $a_n^* \to a^* \in \partial \mathcal{BO}_d$. By the paragraphs above, $a^* = a$, and hence, $a \in \partial \mathcal{BO}_d$. Now we apply Lemma 4.4 again to get $a = a'$.

5. Injectivity of $\Phi_d$

We recall once more the following definition:

$$\mathcal{BO}_d^{(+, -)} = \{a \in \mathcal{BO}_d \mid \mathcal{O}_{s(a)}^L \subset F_{s(a)}^L \cup \{0\} \text{ and } \mathcal{O}_{s(a)}^R \subset F_{s(a)}^R \cup \{0\}\}.$$ 

In this section will construct an inverse $\Psi_d : \mathcal{BO}_d^{(+, -)} \to \mathcal{M}_{d+1}$ of $\Phi_d$.

5.1. Dynamics of maps in $\mathcal{BO}_d^{(+, -)}$. Given $a \in \mathcal{BO}_d^{(+, -)}$, let $P_{s(a)}$ be the monic representative of $p_a$ as defined in Section 2. As in the construction of $\Phi_d$, for $\theta = 0, \frac{1}{2}$, let $S_{\theta}$ be invariant sectors at 0 with same slope. That is,

$$S_0 = \{\varphi_{s(a)}^{-1}(e^{s+2\pi it}) : s \in (0, \eta), |t| < qs\},$$

$$S_{\frac{1}{2}} = \{\varphi_{s(a)}^{-1}(-e^{s+2\pi it}) : s \in (0, \eta), |t| < qs\}.$$ 

Note that $S_{\frac{1}{2}} = -S_0$.

**Figure 14.** Dynamics of $P_{s(a)}$ for $a \in \mathcal{BO}_d^{(+, -)}$
We choose \( q \) to be small enough so that \( S_0 \cap S_{\frac{1}{2}} = \{0\} \), and the inverse image of each \( S_\theta \) under \( P_s(a) \) consists of exactly \( 2d + 1 \) components. The point 0 has pre-images \( \{0 = x_0, x_1, x_2, \ldots, x_{2d}\} \) under \( P_s(a) \), of which \( d \)—say \( x_1, x_2, \ldots, x_d \), are in \( F_s^L \), and \( d \) are in \( F_s^R \). Let \( S_\theta(x_\ell) \) be the inverse image of \( S_\theta \) based at \( x_\ell \) for \( \ell \neq 0 \).

Let \( W \) be the region bounded by an equipotential \( \{z | G_s(z) = \eta\} \) and define \( W_i = P_s^{-a}(W) \). For a given \( \ell \in \{1, 2, \ldots, 2d\} \), let \( S \) be the connected component of \( W_1 \setminus (S_{0}(x_\ell) \cup S_{\frac{1}{2}}(x_\ell)) \) that does not contain 0. Then \( P_s(a) \) maps \( S \) to \( F_s^L \) if \( S \subset F_s^L \), and to \( F_s^R \) if \( S \subset F_s^R \).

We have illustrated this in Figure 13.

5.2. Definition of \( \Psi_d \).

5.2.1. Using renormalization. For the sake of intuition, we give a first definition of \( \Psi_d(a) \) for \( a \in \mathcal{CB}_d^{(+,-)} \) with \( a \neq 1 \), by choosing a renormalization of \( P_s(a) \).

Consider a neighborhood \( \mathcal{E} \) of 0, in which \( P_s(a) \) is conjugate to \( z \mapsto rz \) for some \( r \in \mathbb{C} \) with \( |r| > 1 \), small enough so that \( \overline{\mathcal{E}} \) does not contain any critical points, and satisfying

\[
\mathcal{E} \cap S_0 = S_0 \cap W_i, \\
\mathcal{E} \cap S_{\frac{1}{2}} = S_{\frac{1}{2}} \cap W_i.
\]

Let \( \mathcal{V} \) be an open set defined the union of \( W_i \cap F_s^L \) and \( \mathcal{E} \). Then, there exists a connected component \( \mathcal{V}' \) of \( P_s^{-1}(\mathcal{V}) \) such that \( \overline{\mathcal{V}'} \subset \mathcal{V} \), and \( P_s(a) \big|_{\mathcal{V}'} : \mathcal{V}' \rightarrow \mathcal{V} \) is polynomial-like of degree \( d + 1 \) (see Figure 15). This polynomial-like map has a unique critical point at \(-\sqrt{d}\), and by Theorem 2.5 it is hybrid equivalent to a unicritical degree \( d + 1 \) polynomial \( z^{d+1} + c \). We let \( \Psi_d(a) = c \).

5.2.2. Another construction of \( \Psi_d \). Given \( a \in \mathcal{CB}_d^{(+,-)} \),

construct the Riemann surface \( Y \) as follows: let \( Y_0 = W \cap F_s^L \), and identify the boundaries \( Y_0 \cap \mathcal{R}_0(s(a)) \) and \( Y_0 \cap \mathcal{R}_1(s(a)) \) by identifying points on either ray with same speed of escape. Additionally, if necessary, smooth the boundary of \( Y_0 \) at the point \( w \) as shown in Figure 16, \( S_0 \cap F_s(a) \) and \( S_{\frac{1}{2}} \cap F_s(a) \) with this boundary identification become a single sector which we shall call \( \tilde{S} \). We let \( Y_1 = P_s^{-1}(Y_0) \)

with this boundary identification. Clearly, \( \overline{Y_1} \subset Y_0 \).

Given \( \ell \in \{1, 2, \ldots, d\} \), let \( S \) be the connected component of \( Y_1 \setminus (S_0(x_\ell) \cup S_{\frac{1}{2}}(x_\ell)) \) that does not contain 0, and let \( S' \) be the component that does. Let \( S_\ell = S \cup S_0(x_\ell) \cup S_{\frac{1}{2}}(x_\ell) \) (see Figure 16). Pick a quasiconformal homeomorphism \( e_\ell : S_\ell \rightarrow \tilde{S} \) that extends to a homeomorphism from \( \partial S_\ell \) to \( \partial \tilde{S} \), and coincides with \( P_s(a) \) on \( \partial S_\ell \cap \partial S' \). For example, this can be constructed in a manner similar to \( g_c \big|_{\pi_c(S_\ell(\frac{x}{\pi(c)}) \big)} \) in Section 3. Define

\[
F : Y_1 \rightarrow Y_0, \\
F(z) = \begin{cases} 
P_s(a)(z) & z \in Y_1 \setminus \bigcup_{\ell=1}^d S_\ell, \\
e_\ell(z) & z \in S_\ell \text{ for some } \ell \in \{1, 2, \ldots, d\}.
\end{cases}
\]
Figure 15. Alternative construction of $\Psi_d(a)$ by renormalization; the ‘∗’ marks the critical point $-\sqrt{d}s(a)$ of $P_{s(a)}$.

Figure 16. Cut and paste surgery on $P_{s(a)}$

$F$ is clearly a quasiregular map of degree $d + 1$ with a single critical point. We define an $F$-invariant complex structure $\sigma$ on $Y_0$ as

- $E_z = S^1$ if $z \in Y_0 \setminus Y_1$ or the $F-$ orbit of $z$ does not intersect $S_\ell$ for any $\theta, \ell$, and
- $E_z = (DF^n)^{-1}(S^1)$ if $F^n(z)$ is the first point in the orbit of $z$ that is in $S_\ell$.

Every $F-$ orbit visits $S_\theta(x_i)$ at most once. So, $\sigma$ has bounded dilation. Note that $F^*\sigma = \sigma$, and thus, $F$ is quasiconformally equivalent to a polynomial-like map $y : V_1 \to V$ with degree $d + 1$ and a single critical point. The map $y$ is hybrid equivalent to a polynomial of the form $f_c(z) = z^{d+1} + c$. Note that $y$ only determines the affine equivalence class of $f_c$, and thus $c$ is not unique if $d > 1$; however, we impose the condition that the identified rays $R_0(s(a))$ and $R_\frac{1}{2}(s(a))$ are eventually mapped to the same access as the dynamical ray at angle 0 to $f_c$ (with respect to the Böttcher chart where $\frac{\phi_c(z)}{z} \to 1$ as $z \to \infty$). This determines $c$ uniquely. It is clear that $c \in M_{d+1}$; we therefore define $\Psi_d(a) = c$.

Remark 5.1. When $a \neq 1$, we can show that for an appropriate choice of domains, the map $P_{s(a)}|_{\mathcal{Y}}$ from Section 5.2.1 and the map $F$ from Section 5.2.2 are hybrid equivalent, therefore the definitions of $\Psi_d(a)$ from these two sections coincide.

We may use the same methods as in Section 4 to show that $\Psi_d$ is continuous.
5.3. Ψd is the inverse of Φd. Given \( c \in \mathcal{M}_{d+1} \), let \( c' = \Psi_d \circ \Phi_d(c) \). We will follow the construction to show that \( f_{c'} \) and \( f_c \) are hybrid equivalent, and thus, \( c' = c \).

Let \( a = \Phi_d(c) \). The construction \( a \rightarrow \Psi_d(a) \) involves picking the sectors \( S_0 \) and \( S_\frac{1}{2} \) in the dynamical plane of \( P_{s(a)} \), constructing a Riemann surface \( Y \), a quasiregular map \( F_{s(a)} \), and lastly, a polynomial-like map \( y_{s(a)} \).

On the other hand, the construction \( c \rightarrow \Phi_d(c) \) goes through the steps \( f_c \rightarrow g_c \rightarrow \tilde{g}_c \rightarrow P_{s(a)} \). We will only be working with the \( '−1 ' \) copies of \( S(\frac{\ell}{\ell+1}), K_{f_c}, \) etc., and so we shall drop the \( '−1 ' \) superscript. The first step in the construction of \( \Phi_d(c) \) uses the quotient map \( \pi_c \), and we have

\[
g_c(\pi_c(z)) = \pi_c(f_c(z)) \text{ away from sectors } \pi_c\left(S\left(\frac{\ell}{d+1}\right)\right),
\]

\[
\tilde{g}_c = \psi_c \circ g_c \circ \psi_c^{-1},
\]

\[
P_{s(a)} = k_c \circ \tilde{g}_c \circ k_c^{-1},
\]

where \( \psi_c \) is quasiconformal and \( k_c \) is a hybrid equivalence. In the dynamical plane of \( P_{s(a)} \), for \( \ell \in \{1, 2, \ldots, d\} \), define

\[
\tilde{S}_0(x_\ell) = \left\{ \varphi^{-1}_{s(a)}(e^{\theta + 2\pi it}) : s \in (0, \eta), \left| t - \frac{\ell}{2d}\right| < qs \right\},
\]

\[
\tilde{S}_{\frac{1}{2}}(x_\ell) = \left\{ \varphi^{-1}_{s(a)}(-e^{\theta + 2\pi it}) : s \in (0, \eta), \left| t - \frac{\ell}{2d}\right| < qs \right\},
\]

and let \( \tilde{S}_\ell \) be the union of \( \tilde{S}_0(x_\ell), \tilde{S}_{\frac{1}{2}}(x_\ell) \) and the connected component of \( Y_0 \setminus \tilde{S}_0(x_\ell) \cup \tilde{S}_{\frac{1}{2}}(x_\ell) \) that contains \( S_\ell \), as defined in Section 5.2. See Figure 16 for an illustration of \( \tilde{S}_\ell \). Let \( \tilde{\phi} = k_c \circ \psi_c \circ \pi_c \).

In the dynamical plane of \( f_c \), let \( S_1(\frac{\ell}{\ell+1}), \ell = 0, 1, \ldots, d \), be as defined in equations 3.1 to 3.2 (the equipotential \( \eta \) and the slope factor \( q \) may be different from the ones used for \( P_{s(a)} \)). There are two copies of \( S(0) = f_c(S_1(0)) \) in the dynamical plane of \( \tilde{g}_c \), but we will pick the copy that eventually gets mapped to a sector that intersects \( S_0 \). More generally, for a suitable choice of equipotential and slope factor in the \( f_c \)-plane, we may assume that the open sets \( S(\frac{\ell}{\ell+1}) = \omega^j S(0) \) are eventually mapped into \( \tilde{S}_\ell \), and that \( S(0) \) is eventually mapped to \( S_0 \) (or to \( S_{\frac{1}{2}} \)). That is,

\[
S_\ell(c) = \tilde{\phi}\left(S\left(\frac{\ell}{d+1}\right)\right) \subset \tilde{S}_\ell \text{ for } \ell = 0, 1, \ldots, d,
\]

\[
\tilde{\phi}(V_0) = Y_0 \setminus S_0(c),
\]

where the domain \( V_0 \) is as defined in equation 3.5.

Our strategy will be to set up a quasiconformal map \( \phi : V_0 \cup S(0) \rightarrow Y_0 \) that agrees with \( \tilde{\phi} \) away from certain sectors, and conjugates \( f_c \) and \( F_{s(a)} \). Let

\[
V = \tilde{\phi}^{-1}(Y_0 \setminus \tilde{S}),
\]

\[
V_1 = f_c^{-1}(V),
\]

\[
S = V_0 \cup S(0) \setminus V,
\]

\[
S\left(\frac{\ell}{d+1}\right) = \tilde{\phi}^{-1}(\tilde{S}_\ell) \text{ for } \ell = 1, 2, \ldots, d.
\]
Figure 17. Building a conjugacy between $f_c$ and $F_s(a)$. The wavy shaded region in the top figure is $S(0)$; it is contained in $S$ and its two copies map under $k_c \circ \psi_c \circ \pi_c$ to the sectors $S_0$ and $S_1$ respectively, which we cut to make $\tilde{S}$. We define $\hat{\phi}$ on the darkly shaded region on the top to the darkly shaded region at the bottom.

See Figure 17 for details. For all $z \in V_1 \setminus \left( S \cup \bigcup_{\ell} S\left( \frac{\ell}{d+1} \right) \right)$,

$$F_{s(a)} \circ \tilde{\phi}(z) = \tilde{\phi} \circ f_c(z).$$

Furthermore, with degree one,

$$f_c\left( S\left( \frac{\ell}{d+1} \right) \cap V_1 \right) = S \text{ for } \ell = 1, 2, \ldots, d,$$

$$f_c(S \cap V_1) = S.$$

For $z \in S$, $z = f_c(w)$ for $d$ distinct $w \in V_1$. We can assume that $F_{s(a)} \circ \tilde{\phi}(w)$ does not depend on the choice of preimage $w$, since $\tilde{\phi}(w) \in S_\ell$ and $F_{s(a)}|_{S_\ell}$ depends on the homeomorphisms $e_\ell$ defined as in Section 5.2 which we have freedom in choosing. So we set

$$\hat{\phi}(z) = F_{s(a)} \circ \tilde{\phi}(w).$$
Define
\[
\phi : V_0 \cup S(0) \longrightarrow Y_0, \\
\phi(z) = \begin{cases} \\
\tilde{\phi}(z) & z \not\in S, \\
\hat{\phi}(z) & z \in S.
\end{cases}
\]

By the discussion above, for all \( z \in f_c^{-1}(V_0 \cup S(0)) \),
\[
F_s(a) \circ \phi(z) = \phi \circ f_c(z).
\]

We note that \( \pi_c \) changes the angle at \( \beta_c \) from \( 2\pi \) to \( \pi \), and has zero dilation on \( K_{f_c} \setminus \{ \beta_c \} \). Also note that \( \psi_c \) has zero dilation on \( \pi_c(K_{f_c}) \).

On the other hand, the cutting procedure in Section 5.2 changes the angle \( \pi \) made by the boundary of \( F^{L}_{s(a)} \) at 0 to the angle \( 2\pi \) in the plane of \( F_{s(a)} \). Lastly, note that \( k_c \) has zero dilation on \( \psi_c \circ \pi_c(K_{f_c}) \setminus \{ 0 \} \).

Combined, this information tells us that we have constructed a quasiconformal map \( \phi : V_0 \cup S(0) \longrightarrow Y \) that has zero dilation on \( K_c \), and conjugates \( f_c \) to \( F_{s(a)} \).

Now, if \( z \in \phi(K_c) \), a point \( F^{\infty}_{s(a)} \) in the orbit of \( z \) cannot be in the interior of \( S_\ell \) for any \( \ell \). Therefore, the quasiconformal map that conjugates \( F_{s(a)} \) to \( y_{s(a)} \) has zero dilation on \( \phi(K_c) \). That is, \( y_{s(a)} \) and \( f_c \) are hybrid equivalent. Thus, \( f_c \) and \( f_c \) are hybrid equivalent, implying \( c = c' \).

In a similar manner, we can show that \( p_a \), where \( \tilde{a} = \Phi_d \circ \Psi_d(a), \) is hybrid equivalent to \( p_a \). That is, \( \Phi_d \circ \Psi_d(a) = a \).

This finishes the proof of Theorem 1.1.

We will end with a discussion of how the image under \( \Phi_d \) fits inside \( \mathcal{CBO}_d \).

**Lemma 5.2.** \( \mathcal{CBO}_d^{(+,-)} \) disconnects \( \mathcal{CBO}_d \) into infinitely many components.

**Proof.** Let \( f_c(z) = z^{d+1} + c \) be a polynomial where the orbit of \( c \) contains the \( \beta \)-fixed point where the dynamical ray at angle 0 lands. There are infinitely many values of \( c \) in \( \mathcal{M}_{d+1} \) that satisfy this condition—these are precisely the landing points of parameter rays at angles \( \frac{1}{d^n} \) for \( n \geq 1 \) and \( 0 < i < d^n \). These are included in the set of tips of \( \mathcal{M}_{d+1} \).

Given such a \( c \), let \( a = \Phi_d(c) \). Then the orbit of both critical points \( \pm \sqrt{d} \) of \( p_a \) eventually lands on 0—that is, there exists \( k \) such that \( P^k_a(\pm \sqrt{d}) = 0 \).

In the dynamical plane of the monic representative \( P_{s(a)} \), the dynamical rays at angles 0, \( \frac{1}{2} \) land at 0. Thus there exist two angles \( \theta_1, \theta_2 \) such that \( (2d+1)^{k-1} \theta_1 \equiv 0 \) and \( (2d+1)^{k-1} \theta_2 \equiv \frac{1}{2} \), which both land at the critical value \( P_{s(a)}(-s(a)\sqrt{d}) \). In the parameter plane of \( \mathcal{MBO}_d \), the parameter rays at angle \( \theta_1, \theta_2 \) both land at \( s(a) \)—which means that \( s(a) \) is a cut-point of \( \mathcal{MBO}_d \), which is equivalent to saying that \( a \) is a cut-point of \( \mathcal{CBO}_d \).

Another way to show this is to see that exists \( a' \in \mathcal{CBO}_d \) close to \( a \) such that \( P_{s(a')}^k(\sqrt{d}) \in F^{L}_{s(a')} \) and \( P_{s(a')}^k(-\sqrt{d}) \in F^{R}_{s(a')} \). That is, the orbits of both critical points eventually “cross over” to the other side. So \( a' \notin \mathcal{CBO}_d^{(+,-)} \).
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Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Current address: Department of Mathematics and Statistics, Boston University, Boston, Massachusetts 02215

Email address: mmukunda@bu.edu