ON THE DERIVED FUNCTORS OF DESTABILIZATION AT ODD PRIMES

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Abstract. An explicit chain complex is constructed to calculate the derived functors of destabilization at an odd prime, generalizing constructions of Zarati and of Hung and Sum. These methods are based on the ideas of Singer and also apply at the prime two. These results are applied to give structural results on the derived functors of destabilization.

1. INTRODUCTION

The destabilization functor $D$ from the category $\mathcal{M}$ of modules over the mod $p$ Steenrod algebra $\mathcal{A}$ to the category $\mathcal{U}$ of unstable modules is the left adjoint to the inclusion $\mathcal{U} \hookrightarrow \mathcal{M}$; it is right exact and has non-trivial left derived functors $D_s : \mathcal{M} \to \mathcal{U}$. These derived functors are of considerable interest in homotopy theory: for example Lannes and Zarati [LZ87] used them to prove a weak version of the Segal conjecture; they have recently been used by Hai, Nam and Schwartz [NST09] ($p = 2$) and Hai [Hai10] (for $p$ odd) to prove a generalization of the weak Segal conjecture. The current research was motivated in part by the need to gain a better understanding of the action of Lannes’ $T$-functor on the derived functors of destabilization at odd primes; this will be treated in future work.

At the prime two, Singer constructed functorial chain complexes with homology calculating the derived functors of iterated loop functors [Sin80a]; these can be used to construct chain complexes for calculating the functors $D_s$ (cf. Goerss [Goe86], who works with homology). Lannes and Zarati [LZ87] gave an independent approach at the prime two, calculating the derived functors $D_s(\Sigma^{-t}M)$ where $M$ is an unstable module and $t \leq s$.

Both approaches depend upon the relationship between the Steenrod algebra and the Dickson invariants, the algebra of invariants of the cohomology $H^*(BV_s)$ of a rank $s$ elementary abelian $p$-group $V_s$ under the action of the general linear group $GL_s$ (as $s$ varies). For odd primes, this relationship is slightly more subtle and is explained by the results of Mui [Mui75] in terms of an explicit subalgebra of the invariants $H^*(BV_s)^{\tilde{SL}_s}$, where $\tilde{SL}_s$ is an index two subgroup of $GL_s$ containing the special linear group.

The purpose of this paper is to give a construction of a functorial chain complex to calculate the derived functors of destabilization for odd primes. This unifies and builds upon results in the literature: the derived functors of iterated loop functors at odd primes were studied by Li in his thesis [Lia00] and, in joint work with Singer [LS82], a chain complex for calculating the homology of the Steenrod algebra was given; Hung and Sum [HS95] modified and generalized to odd primes the approach of Singer [Sin83], leading to an invariant-theoretic description of the
Lambda algebra at odd primes. Correspondingly, Zarati generalized the approach of [LZ87] to the odd primary case.

The methods of this paper use a generalization of Zarati’s constructions to all \( A \)-modules, drawing on the observations of Hung and Sum, which are based on ideas going back to Singer and Miller. Writing \( \text{Ch}_{ \geq 0} \mathcal{M} \) for the abelian category of homological chain complexes (in non-negative degrees), the main result is the following:

**Theorem 1.** There is an exact functor

\[
\mathcal{D}_s : \mathcal{M} \to \text{Ch}_{ \geq 0} \mathcal{M}
\]

such that, for \( M \in \text{Ob} \mathcal{M} \) and \( s \in \mathbb{N} \), there is an isomorphism of \( A \)-modules:

\[
\mathcal{D}_s M \cong H_s(\mathcal{D}_s M).
\]

The proof that the chain complex \( \mathcal{D}_s M \) calculates the derived functors of destabilization is inspired by the argument of Singer [Sin80a] for the case of the derived functors of iterated loop functors and avoids explicit calculation by using a connectivity argument for the chain complex (similar to an argument used by Zarati [Zar84]). This recovers the results of Zarati and the method also applies at the prime two.

As is clear from Singer’s work, underlying these methods is the fact that the Steenrod algebra is a quadratic algebra which is non-homogeneous Koszul (in the terminology of Priddy [Pri70]). The destabilization functor appears by exploiting the total Steenrod power, thus going back to the very origins of instability.

Structural results on the derived functors of destabilization can be deduced using these methods. Recall that the algebra of Dickson invariants (the polynomial part of \( H^*(BV)_{GL_n} \)) is the algebra \( F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \) and that, if \( \Phi : \mathcal{M} \to \mathcal{M} \) denotes the Frobenius functor on the category \( \mathcal{M} \), then \( \Phi^k F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \) \((k \in \mathbb{N})\) is the unstable algebra \( F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \).

**Theorem 2.** Let \( M \) be an unstable module and \( s, t, w \) be natural numbers such that \( p^w \geq \left\lceil \frac{s-t+1}{2} \right\rceil \). Then the unstable module \( \mathcal{D}_s(\Sigma^t M) \) is equipped with a natural \( \Phi^w + 1 \mathcal{F}_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-structure in \( \mathcal{U} \).

If \( t \leq s \), then \( \mathcal{D}_s(\Sigma^t M) \) has a natural \( F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-module structure in \( \mathcal{U} \).

The methods of Zarati [Zar84] establish a stronger result in the case \( t < s \). The method of proof is related to that employed in [Pow10] (at the prime two).

**Outline of the proof.** The proof of Theorem 1 is in two steps: the construction of the functorial chain complex and the proof that the homology of the chain complex calculates the derived functors of destabilization.

The construction begins with the definition of an exact functor \( \mathcal{R}_1 : \mathcal{M} \to \mathcal{M} \), given explicitly as a subfunctor of \( M \mapsto H^*(BV)(Q_{1,0}^{-1}) \otimes M \), where \( \otimes \) is a half-completed tensor product. This is achieved by first constructing a larger functor \( \mathcal{R}_1 \) and then restricting using an eigenspace decomposition associated to an action by the group \( \mathbb{Z}/2 \). The functor \( \mathcal{R}_1 \) is a generalization of the functor \( R_1 \) defined by Zarati [Zar84] on the category of unstable modules. There is a technical difficulty introduced by passage to the category of \( A \)-modules, since the functor \( \mathcal{R}_1 \) does not commute with inverse limits.

There are higher functors \( \mathcal{R}_s \), \( s \in \mathbb{N} \), which are constructed by iteration and using a restriction induced by the definition of \( \mathcal{R}_2 \). Namely, an explicit subfunctor \( \mathcal{R}_2 \subset \mathcal{R}_1 \mathcal{R}_1 \) is constructed and, for \( s \geq 2 \), the functor \( \mathcal{R}_s \) can be defined as \( \bigcap_{i+j=s-2} \mathcal{R}_i^j \circ \mathcal{R}_2 \circ \mathcal{R}_1^j \) as subfunctors of \( \mathcal{R}_1^s \). To compare with the work of Zarati
The functors \( R_s \) are identified as subfunctors of \( M \mapsto H^*(BV_s)[Q_{s,0}] \otimes M \); the presence of the half-completed tensor product leads to a number of technicalities, which do not seem to be avoidable.

The differential of the chain complex is induced by Singer’s \( \mathcal{A} \)-linear morphism \( H^*(BV_1)[Q_{1,0}] \rightarrow \Sigma^{-1}F_p \), which induces a natural transformation \( R_1 M \rightarrow \Sigma^{-1}M \). The higher differentials are induced by using the canonical inclusions \( R_s \subset R_{s-1}R_1 \), as the composites

\[
R_s(\Sigma^{s-1}M) \hookrightarrow R_{s-1}R_1\Sigma^{s-1}M \xrightarrow{\partial_s \Sigma^{s-1}M} R_{s-1}\Sigma^{s-1}M,
\]
giving rise to a chain complex

\[
\ldots \rightarrow \Sigma R_s\Sigma^{s-1}M \rightarrow \ldots \rightarrow \Sigma R_2\Sigma M \rightarrow \Sigma R_1\Sigma M \rightarrow M.
\]

The proof that this is a chain complex reduces to showing that the composite \( \Sigma R_2\Sigma M \rightarrow \Sigma R_1\Sigma M \rightarrow M \) is trivial; this is a consequence of the relationship between the Steenrod algebra and invariant theory, highlighted (at the prime 2) by Singer [Sin83].

A formal argument is used to show that the homology of the chain complex calculates the derived functors of destabilization, reflecting the behaviour of the chain complex with respect to suspension. The essential tool is the natural transformation \( \rho_1 : R_1 M \rightarrow \Sigma^{-2}\Phi\Sigma M \), introduced by Zarati in the unstable module setting. This has higher analogues, which fit together to form a morphism of chain complexes. Using this, the proof proceeds by showing vanishing of the higher homology on the projective generators of the category \( \mathcal{M} \). This uses a connectivity argument, reminiscent of that used by Zarati.

An effort has been made to keep the paper reasonably self-contained. The paper can be read independently of Zarati’s unpublished 1984 thèse d’état [Zar84]; references have been given to [Zar84] so as to make clear how the constructions generalize those of Zarati.

**Organization of the Paper.** The paper has two parts; the first covers background and technical foundations and the second presents the constructions outlined above.

Section 2 reviews modules over the Steenrod algebra; Section 3 introduces the notion of a weakly continuous functor and that of connectivity, necessary for analysis of the composition of weakly continuous functors. The half-completed tensor product is reviewed in Section 4, where some basic multiplicative properties are established. The total Steenrod power morphisms \( St_s \) are introduced in Section 5, paying attention to the behaviour of their composition.

The construction commences in Section 6, where the first total Steenrod power \( St_1 \) is used to define the functor \( R_1 \); the differential \( d \) and the morphism \( \rho_1 \) are also introduced. The higher functors \( R_s \) are defined in Section 7, together with the higher morphisms \( \rho_s \); the higher differentials are defined and analysed in Section 8. This material is put together in Section 9 to define the chain complex and to prove Theorem 1. Section 10 considers module structures on the chain complex and gives the proof of Theorem 2.

The appendix A presents a self-contained proof the \( \mathcal{A} \)-stability of \( R_1 M \).

**Contents**

1. Introduction
Part 1. Foundations

2. Preliminaries

The Steenrod algebra over a fixed odd prime $p$ is denoted $\mathcal{A}$. The category $\mathcal{M}$ of (graded) $\mathcal{A}$-modules contains the category $\mathcal{U}$ of unstable modules as a full subcategory; the categories $\mathcal{M}$ and $\mathcal{U}$ are abelian, complete and cocomplete. (See Schwartz’s monograph [Sch94] for details relating to the Steenrod algebra, unstable modules and unstable algebras).

2.1. The destabilization functor. The destabilization functor $D : \mathcal{M} \to \mathcal{U}$ is the left adjoint to the inclusion $\mathcal{U} \hookrightarrow \mathcal{M}$; it is right exact and admits left derived functors $D_s, s \geq 0$. The functor $D$ is constructed explicitly as follows: for $M$ an $\mathcal{A}$-module, the subspace $BM$ generated by the elements of the form $\beta x$ with $\epsilon + 2i > |x|$, $\epsilon \in \{0, 1\}$, is stable under the action of the Steenrod algebra and $DM \cong M/BM$.

There is a natural short exact sequence for calculating the composite of derived functors of destabilization with desuspension. Recall that the loop functor $\Omega : \mathcal{U} \to \mathcal{U}$ is the left adjoint to the suspension functor $\Sigma$ and $\Omega_1$ is the unique non-trivial higher left derived functor of $\Omega$.

Proposition 2.1.1. (Cf. [Zar84]) For $M$ an unstable module and $s$ a positive integer, there is a natural short exact sequence of unstable modules

$$0 \to \Omega D_s M \to D_s \Sigma^{-1} M \to \Omega_1 D_{s-1} M \to 0.$$ 

2.2. Algebras and modules. The categories $\mathcal{M}$ and $\mathcal{U}$ are symmetric monoidal with respect to the tensor product of graded vector spaces and the interchange functor which uses the Koszul sign convention. Unital commutative algebras in $\mathcal{M}$ with respect to this structure are referred to here simply as algebras; an algebra $K$ in $\mathcal{M}$ is unstable if the underlying $\mathcal{A}$-module is unstable and the Cartan condition is satisfied (if $k$ is an element of even degree, $P^{k/2}k = k^p$). A $B$-module in $\mathcal{M}$ is an $\mathcal{A}$-module equipped with a $B$-module structure such that the structure morphism is $\mathcal{A}$-linear.

Notation 2.2.1. Let $B$ be an algebra in $\mathcal{M}$ and $K$ be an unstable algebra.

(1) The category of $B$-modules in $\mathcal{M}$ is denoted by $B\cdot \mathcal{M}$;
(2) the category of $K$-modules in $\mathcal{U}$ is denoted by $K\cdot \mathcal{U}$.

The following result is standard.
Proposition 2.2.2. Let $B$ be an algebra in $\mathcal{M}$ and $K$ be an unstable algebra. The categories $B\cdot \mathcal{M}$ and $K\cdot \mathcal{U}$ are abelian categories for which the forgetful functors

$$
B\cdot \mathcal{M} \to \mathcal{M} \\
K\cdot \mathcal{U} \to \mathcal{U}
$$

are exact. In particular, the forgetful functor $K\cdot \mathcal{U} \to K\cdot \mathcal{M}$ is exact.

2.3. The Frobenius functor, $\Phi$. The exact functor $\Phi : \mathcal{M} \to \mathcal{M}$ is the analogue for odd primes of the familiar doubling functor for $p = 2$. Recall that

$$(\Phi M)^n = \begin{cases} 
M^{2k} & n = 2kp \\
M^{2k+1} & n = 2kp + 2 \\
0 & \text{otherwise.}
\end{cases}$$

In particular, $\Phi M$ is concentrated in even degrees and an element $x \in M^d$ gives rise to $\Phi x \in M^n$, where $n = pd$ if $d$ is even and $n = p(d - 1) + 2$ if $d$ is odd. The action of the Steenrod algebra is given by

$$
P^n(\Phi x) = \Phi(P^n/x) \quad p | i, \quad |x| \equiv 0(2)$$

$$
P^n(\Phi x) = \Phi(\beta P^{i-1}/x) \quad p | i - 1, \quad |x| \equiv 1(2)$$

$$
\beta(\Phi x) = 0.
$$

There is a natural $\mathbb{F}_p$-linear morphism, for $M \in \text{Ob} \mathcal{M}$:

$$
\lambda_M : \Phi M \to M
$$

defined by $\lambda_M(\Phi x) := \beta^e P^i x$, where $|x| = 2i + e$ and $e \in \{0, 1\}$. If $M$ is unstable, then $\lambda_M$ is $A$-linear.

Remark 2.3.1. For an $A$-module $M$, the canonical surjection $M \to DM$ induces a natural surjection $\mathbb{D}\Phi M \to \Phi DM$. For $p$ odd, this is not in general an isomorphism; for example, consider the free unstable module $F(1)$ on a generator of degree one.

2.4. Cohomology of elementary abelian $p$-groups and invariants. For $V$ an elementary abelian $p$-group, $H^*(BV)$ denotes the $\mathbb{F}_p$-cohomology of the classifying space $BV$. For $s$ a natural number, $V_s$ denotes an elementary abelian $p$-group of rank $s$. There is an isomorphism of unstable algebras $H^*(BV_s) \cong \Lambda(u) \otimes \mathbb{F}_p[v]$, where $\Lambda(-)$ denotes the exterior algebra functor, $|u| = 1$, $|v| = 2$ and the generators are linked by the Bockstein operator $\beta u = v$. By the Künneth isomorphism

$$
H^*(BV_s) \cong \Lambda(u_1, \ldots, u_s) \otimes \mathbb{F}_p[v_1, \ldots, v_s]
$$

and $\mathbb{F}_p[v_1, \ldots, v_s] \subset H^*(BV_s)$ is a sub unstable algebra generated in degree two by $H^2(BV_s) \cong V_s^\ast$.

The general linear group $GL_s := GL(V_s)$ acts naturally on $H^*(BV_s)$ by morphisms of unstable algebras and $\mathbb{F}_p[v_1, \ldots, v_s]$ is stable under this action. Hence, for $G \subset GL_s$, the algebras of invariants

$$
\mathbb{F}_p[v_1, \ldots, v_s]^G \to H^*(BV_s)^G \hookrightarrow H^*(BV_s)
$$

are unstable algebras.

The algebra $\mathbb{F}_p[v_1, \ldots, v_s]^{GL_s}$ is the algebra of Dickson invariants, which is isomorphic to $\mathbb{F}_p[Q_{s,0}, Q_{s,1}, \ldots, Q_{s,s-1}]$, where $|Q_{s,i}| = 2(p^s - p^i)$. The generators $Q_{s,i}$ are defined by

$$
f_s(X) := \prod_{v \in V_s^\ast} (X - v) = \sum_{i=0}^s (-1)^{s-i}Q_{s,i} X^i.
$$

In particular, the top Dickson invariant, $Q_{s,0}$, is $(-1)^s \prod_{v \in V_s^\ast \setminus 0} v$.

Definition 2.4.1. (Cf. [MUBS94]) Let $\tilde{SL}_s \subset GL_s$ denote the kernel of the morphism $GL_s \to \mathbb{Z}/2$, $g \mapsto \det(g)^{\pm 1}$. 
The morphism $GL_s \rightarrow \mathbb{Z}/2$ is a split surjection, which leads to the following general eigenspace splitting:

Lemma 2.4.2. Suppose that the group $GL_s$ acts on the unstable algebra $K$ by morphisms of unstable algebras. The unstable algebra $K^{SL_s} \subset K$ inherits a canonical action by the group $\mathbb{Z}/2$. The algebra of invariants $(K^{SL_s})^{\mathbb{Z}/2}$ is isomorphic to $K^{GL_s}$ and there is a canonical splitting in the category $K^{GL_s}$.ι:

$$K^{SL_s} \cong K^{GL_s} \oplus (K^{SL_s})^{-}.$$ 

In the following definition, fix a choice of ordered basis of $V_s^*$, giving an ordered set $\{v_1, \ldots, v_s\}$ of generators of the polynomial part of $H^*(BV_s)$.

Definition 2.4.3. For $s$ a natural number, let $L_s \in H^*(BV_s)$ be the class of degree $2 \left(\frac{p^s - 1}{p-1}\right)$:

$$\begin{vmatrix}
    v_1 & \cdots & v_s \\
    v_1^p & \cdots & v_s^p \\
    \vdots & \cdots & \vdots \\
    v_1^{p^s - 1} & \cdots & v_s^{p^s - 1}
\end{vmatrix}$$

and $\epsilon_s$ be the class $\frac{L_s}{\sqrt{p}}$, which has degree $p^s - 1$.

Lemma 2.4.4. Let $s$ be a positive integer. Then

1. $L_s$ is $SL_s$-invariant;
2. $\epsilon_s^2 = Q_s.0$;
3. $\epsilon_s$ is $SL_s$-invariant.

Example 2.4.5. There are identifications

$$H^*(BV_1)^{GL_1} \cong \Lambda(\tilde{M}_{1,0}) \otimes \mathbb{F}_p[Q_{1,0}] \hookrightarrow \Lambda(M_{1,0}) \otimes \mathbb{F}_p[\epsilon_1] \cong H^*(BV_1)^{\tilde{SL}_1},$$

where $\epsilon_1 = \sqrt{p}$, $\tilde{M}_{1,0} = v_1^p$, $Q_{1,0} = \epsilon_2^p = v_1^{p-1}$. (The notation $\tilde{M}_{1,0}$ is due to Mui [Mui77] and is explained in Section 5.3)

The complementary $H^*(BV_1)^{GL_1}$-module $(H^*(BV_1)^{\tilde{SL}_1})^{-}$ is the free $\mathbb{F}_p[Q_{1,0}]$-module on generators $\tilde{M}_{1,0}$ and $\epsilon_1$.

2.5. Localization. The interest of localizing $H^*(BV_s)$ by inverting the top Dickson invariant $Q_{s,0}$ is well established (see, for example, [Sim83] and [LZ95]).

Notation 2.5.1. For $s$ a natural number, let $\Phi_s$ denote the algebra $H^*(BV_s)[Q_s^{-1}]$, equipped with the canonical algebra structure in $\mathcal{M}$ for which $H^*(BV_s) \hookrightarrow \Phi_s$ is a morphism of algebras in $\mathcal{M}$ (Cf. [W977] [Sim80b]).

Lemma 2.5.2. Let $s$ be a positive integer. There is a commutative diagram of monomorphisms of algebras in $\mathcal{M}$:

$$\begin{array}{ccc}
H^*(BV_1) \otimes H^*(BV_{s-1}) & \cong & H^*(BV_s) \\
\Phi_1 \otimes \Phi_{s-1} & \hookrightarrow & \Phi_s.
\end{array}$$

Proof. Standard. $\square$
Upon passage to algebras of invariants, there are inclusions of algebras in $\mathcal{M}$:

\[ \Phi_{s}^{GL, s} \rightarrow \Phi_{s}^{SL, s} \rightarrow \Phi_{s} \]

\[ (H^{*}(BV_{s})^{GL, s})[Q_{s,0}^{-1}] \rightarrow (H^{*}(BV_{s})^{SL, s})[Q_{s,0}^{-1}]. \]

### 3. Weak continuity and connectivity

The notion of a weakly continuous functor is introduced here as a technical tool for controlling inverse limits; it is applied to the half completed tensor product in Section 3.

#### 3.1. Weakly continuous functors

The category of inverse systems in a category $C$ is written $C^{\text{N}^{\text{op}}}$. 

**Notation 3.1.1.**

1. Let $\mathcal{M}^{\text{bd}} \subset \mathcal{M}$ denote the full subcategory of bounded-above modules.
2. For $M \in \text{Ob}\, \mathcal{M}$ and $c \in \mathbb{Z}$, let $M^{\geq c}$ denote the sub $A$-module of elements of degree at least $c$ and $M^{< c}$ denote the quotient module $M/M^{\geq c}$.

The following statement resumes the standard properties of these constructions:

**Lemma 3.1.2.** Let $c$ be an integer and $M$ be an $A$-module.

1. The associations $M \mapsto M^{\geq c}$ and $M \mapsto M^{< c}$ define exact functors $(-)^{\geq c}$, $(-)^{< c}$ on $\mathcal{M}$ and there is a canonical short exact sequence in $\mathcal{M}$:

   \[ 0 \rightarrow M^{\geq c} \rightarrow M \rightarrow M^{< c} \rightarrow 0, \]

   which splits canonically in the category of graded vector spaces.
2. The modules $M^{< c}$ define a functorial inverse system of surjections $M^{< c}$ in $\mathcal{M}^{\text{bd}}$:

   \[ \ldots \rightarrow M^{< c+1} \rightarrow M^{< c} \rightarrow \ldots \]

3. The inverse limit induces a functor $\lim_{\leftarrow} : (\mathcal{M}^{\text{bd}})^{\text{N}^{\text{op}}} \rightarrow \mathcal{M}$ and the canonical morphism $M \rightarrow \lim_{\leftarrow} (M^{< c})$ is an isomorphism in $\mathcal{M}$.

**Notation 3.1.3.**

1. For $\alpha : \mathcal{M} \rightarrow \mathcal{M}$ a functor, let $\alpha^{\text{bd}} : \mathcal{M}^{\text{bd}} \rightarrow \mathcal{M}$ denote the restriction of $\alpha$ to $\mathcal{M}^{\text{bd}}$.
2. For $\beta : \mathcal{M}^{\text{bd}} \rightarrow \mathcal{M}$ a functor, let $R_{\text{Kan}}\beta : \mathcal{M} \rightarrow \mathcal{M}$ denote the right Kan extension of $\beta$, given by $R_{\text{Kan}}\beta(M) := \lim_{\leftarrow} (\beta(M^{< c}))$.

**Definition 3.1.4.** A functor $\alpha : \mathcal{M} \rightarrow \mathcal{M}$ is weakly continuous if the canonical natural transformation $\alpha \rightarrow R_{\text{Kan}}(\alpha^{\text{bd}})$ is an isomorphism.

**Remark 3.1.5.**

1. A weakly continuous functor need not be continuous (in the categorical sense [ML98, Chapter IV, Section 6]); the key functors considered in this paper are weakly continuous but not continuous.
2. There is an analogous definition of weak continuity for functors between $\mathbb{Z}$-graded vector spaces.

**Proposition 3.1.6.** Let $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ be functors $\mathcal{M}^{\text{bd}} \rightarrow \mathcal{M}$. Then

1. If $\overline{\alpha}$ is exact, then $R_{\text{Kan}}\overline{\alpha}$ is exact;
2. If $0 \rightarrow \overline{\alpha} \rightarrow \overline{\beta} \rightarrow \overline{\gamma} \rightarrow 0$ is a short exact sequence of exact functors, then

   \[ 0 \rightarrow R_{\text{Kan}}\overline{\alpha} \rightarrow R_{\text{Kan}}\overline{\beta} \rightarrow R_{\text{Kan}}\overline{\gamma} \rightarrow 0 \]

   is a short exact sequence of functors;
(3) there is a bijection of sets of natural transformations:
\[ \text{Nat}(\pi, \beta) \cong \text{Nat}(R_{\text{Kan}}\pi, R_{\text{Kan}}\beta). \]

**Proof.** If \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) is a short exact sequence of \( A \)-modules, then there is an induced short exact sequence
\[ 0 \rightarrow A^<\bullet \rightarrow B^<\bullet \rightarrow C^<\bullet \rightarrow 0 \]
of inverse systems of bounded above \( A \)-modules in which the transition morphisms of the inverse systems are surjective. Applying \( \pi \) gives a short exact sequence of inverse systems
\[ 0 \rightarrow \pi(A^<\bullet) \rightarrow \pi(B^<\bullet) \rightarrow \pi(C^<\bullet) \rightarrow 0 \]
such that the transition morphisms are surjective in each of the inverse systems, since \( \pi \) is exact by hypothesis. Hence, by the Mittag-Leffler condition, the corresponding \( \lim^1 \) terms vanish, so passing to the inverse limit gives the required short exact sequence of \( A \)-modules.

The proof of the second statement is similar and the final statement follows formally from the construction of the right Kan extension. \( \square \)

### 3.2. Stable inverse systems.

Since a weakly continuous functor need not be continuous, it is necessary to place a restriction on the inverse systems which are considered.

**Definition 3.2.1.** An inverse system \( X_\bullet \) of \( M^{\text{op}} \) is stable (or stabilizes) if there exists a map \( \mu : N \rightarrow N \) such that
1. \( \lim_{s \rightarrow \infty} \mu(s) = \infty \).
2. \( X_{s+1}^{<\mu(s)} \rightarrow X_s^{<\mu(s)} \) is an isomorphism, \( \forall s \in N \);

**Lemma 3.2.2.** Let \( \alpha : \mathcal{M} \rightarrow \mathcal{M} \) be a weakly continuous functor, \( X_\bullet \) be a stable inverse system of \( A \)-modules and write \( X_\infty^< \) for \( \lim_{\leftarrow, i} X_i^< \). Then the natural morphism
\[ \alpha(X_\infty^<) \rightarrow \lim_{\leftarrow, i} \alpha(X_i^<) \]
is an isomorphism.

**Proof.** Since \( \alpha \) is weakly continuous, there is an isomorphism \( \alpha(X_i) \cong \lim_{\leftarrow, i} \alpha(X_i^{<c}) \), hence there are isomorphisms
\[ \lim_{\leftarrow, i} \alpha(X_i) \cong \lim_{\leftarrow, i} \lim_{\leftarrow, c} \alpha(X_i^{<c}) \cong \lim_{\leftarrow, c} \lim_{\leftarrow, i} \alpha(X_i^{<c}), \]
where the second isomorphism reverses the order of the inverse limits. The hypothesis that \( X_\bullet \) is stable implies that \( X_i^{<c} \) is isomorphic to \( X_{<\infty}^{<c} \) for \( i \gg 0 \), hence \( \lim_{\leftarrow, i} \alpha(X_i^{<c}) \cong \alpha(X_{<\infty}^{<c}) \). Finally, since \( \alpha \) is weakly continuous, there is a natural isomorphism \( \alpha(X_\infty^<) \cong \lim_{\leftarrow, c} \alpha(X_{<\infty}^{<c}) \). \( \square \)

### 3.3. Weak connectivity for functors.

**Notation 3.3.1.** The connectivity of \( M \in \text{Ob } \mathcal{M} \) is \( |M| := \inf\{d | M_d \neq 0\} \in \mathbb{Z} \cup \{-\infty, \infty\} \).

**Definition 3.3.2.** A functor \( \alpha : \mathcal{M} \rightarrow \mathcal{M} \) is weakly connective if there exists a function \( \kappa : \mathbb{Z} \rightarrow \mathbb{Z} \) such that
1. \( |\alpha(M)| \geq \kappa(|M|) \) if \( |M| \in \mathbb{Z} \);
2. \( \lim_{t \rightarrow -\infty} \kappa(t) = \infty \).

The functor \( \alpha \) will also be said to be \( \kappa \)-connective if the above conditions are satisfied.
Lemma 3.3.3. Let $\alpha : \mathcal{M} \to \mathcal{M}$ be an exact $\kappa$-connective functor. Then the natural morphism

$$\alpha(M)^{<\kappa(c)} \to \alpha(M^{<c})^{<\kappa(c)}$$

is an isomorphism.

Proof. Straightforward. \qed

Proposition 3.3.4. Let $\beta, \gamma$ be weakly continuous functors, such that $\gamma$ is exact and $\kappa$-connective. Then the composite $\beta \circ \gamma$ is weakly continuous. Moreover

1. if $\beta$ is exact, then $\beta \circ \gamma$ is exact;
2. if $\beta$ is $\lambda$-connective, then $\beta \circ \gamma$ is $\lambda \circ \kappa$-connective.

Proof. Consider an $A$-module $M$. The hypothesis that $\gamma$ is weakly continuous implies that $\gamma(M) \cong \lim_{\leftarrow, c} \gamma(M^{<c})$; moreover, the $\kappa$-connectivity and exactness of $\gamma$ implies that the inverse system $\gamma(M^{<c})$ is stable, by Lemma 3.3.3. Now $\beta \circ \gamma(M) \cong \beta(\lim_{\leftarrow, c} \gamma(M^{<c}))$ and, by Lemma 3.2.2, the right hand side is isomorphic to $\lim_{\leftarrow, c} \beta \circ \gamma(M^{<c})$, since $\gamma$ is weakly continuous, which establishes that $\beta \circ \gamma$ is weakly continuous. If $\beta$ is exact, then it follows from Proposition 3.1.6 that the composite $\beta \circ \gamma$ is exact.

The final statement on the weak connectivity is clear. \qed

Examples of connective functors are provided by the derived functors of destabilization (a stronger result is established in Corollary 9.4.5): 

Lemma 3.3.5. [Zar84] For $M \in \text{Ob} \mathcal{M}$ and $s$ a natural number,

$$|D_s M| \geq p(|M| + s - 1) + 1.$$ 

4. Large tensor products and completion

As in the work of Singer (eg. [Sim81]), Lin and Singer [LS82] and Hung and Sum [HS95], it is necessary to use large tensor products when working with arbitrary modules over the Steenrod algebra.

4.1. Large tensor products.

Definition 4.1.1. For $M, N \in \text{Ob} \mathcal{M}$, let $M \otimes N$ define the large tensor product $M \otimes N$ as $\lim_{\leftarrow, c} (M \otimes N^{<c})$.

Remark 4.1.2. An analogous definition applies for graded vector spaces.

Proposition 4.1.3. Let $M, N$ be $A$-modules.

1. There is a canonical embedding $M \otimes N \hookrightarrow M \otimes N$ which is an isomorphism if $N$ is bounded above or if $M$ is bounded below.
2. The association $N \mapsto M \otimes N$ defines an exact functor

$$M \otimes - : \mathcal{M} \to \mathcal{M}$$

which identifies with $R_{\text{Kan}}((M \otimes -)^{\text{bd}})$, in particular is weakly continuous.
3. The association $M \mapsto M \otimes N$ defines an exact functor

$$- \otimes N : \mathcal{M} \to \mathcal{M}.$$ 

4. There is a canonical isomorphism

$$\lim_{d \to -\infty} M \otimes (N_{\geq d}) \cong M \otimes N.$$
Example 4.1.4. The functor $\Phi_1\otimes -$ is weakly continuous but is not continuous.

4.2. Full completion versus large tensor products.

Definition 4.2.1. For $M, N \in \text{Ob} \mathcal{M}$, let $M \hat{\otimes} N$ denote the completed tensor product: $M \hat{\otimes} N := \lim_{\leftarrow} (M < s \otimes N < s)$.

Lemma 4.2.2. Let $M, N$ be $\mathcal{M}$; there is a natural monomorphism in $\mathcal{M}$:

$$M \otimes N \hookrightarrow M \hat{\otimes} N.$$  

Care is required in manipulating the full completed tensor product.

Example 4.2.3. The underlying algebra of $\mathbb{F}_p[v^k, w^k] \otimes \mathbb{F}_p[w^k]$ is not an integral domain. For example, consider the expansion of $\frac{1}{v+w}$ as a power series respectively in $v$ and in $w$. The expansions are obtained by considering the expressions $\frac{1}{v(1+x)}$ and $\frac{1}{w(1+y)}$ respectively. Both are multiplicative inverses for $v + w$, hence $u + v$ is a zero divisor.

4.3. Multiplicative structures. For $M, N \in \text{Ob} \mathcal{M}$, there is a natural morphism in $\mathcal{M}$:

$$(H^*(BV_s) \otimes M) \otimes (H^*(BV_s) \otimes N) \otimes (M \otimes N)$$

where $\tau$ is the interchange of tensor factors and the second morphism is induced by the product structure of $H^*(BV_s)$. This identifies with the projection

$$(H^*(BV_s) \otimes M) \otimes (H^*(BV_s) \otimes N) \rightarrow (H^*(BV_s) \otimes M) \otimes H^*(BV_s) \otimes (H^*(BV_s) \otimes N)$$

provided by the respective $H^*(BV_s)$-module structures.

Proposition 4.3.1. For $M, N \in \text{Ob} \mathcal{M}$ and $s$ a natural number, there is a product morphism

$$(\Phi \otimes M) \otimes (\Phi \otimes N) \rightarrow \Phi \otimes (M \otimes N)$$

in $\mathcal{M}$, which extends the product $(H^*(BV_s) \otimes M) \otimes (H^*(BV_s) \otimes N) \rightarrow H^*(BV_s) \otimes (M \otimes N)$. Hence,

1. $\Phi \otimes M$ is a $\Phi_s$-module in $\mathcal{M}$;
2. if $B$ is an algebra in $\mathcal{M}$, then $\Phi \otimes B$ is a $(\Phi_s \otimes B)$-algebra in $\mathcal{M}$;
3. if, furthermore, $N$ is a $B$-module in $\mathcal{M}$, then $\Phi \otimes N$ is a $\Phi_s \otimes B$-module in $\mathcal{M}$.

Proof. The result is clear with $\otimes$ in place of $\otimes$; it is necessary to verify that this passes to $\otimes$. Using the canonical isomorphism $\lim_{d \rightarrow -\infty} X \otimes (Y_{\geq d}) \cong X \otimes Y$ provided by Proposition 4.1.3, it is sufficient to consider the case where $M, N$ are bounded below.
Proposition 4.3.3. Let \( M \cong M_{\geq m} \), \( N \cong N_{\geq n} \). The surjection \( M \otimes N \twoheadrightarrow (M \otimes N)^{<e} \) factors canonically across \( M \otimes N \twoheadrightarrow M^{<e-n} \otimes N^{<e-m} \). This induces a natural morphism
\[
(\Phi_s \otimes M) \otimes (\Phi_s \otimes N) \to \Phi_s \otimes (M \otimes N)^{<e}
\]
to the inverse system defining \( \Phi_s \otimes (M \otimes N) \), which is the required morphism.

The remaining statements are proved by establishing associativity and graded commutativity, which follow from the uncompleted case, by the construction above.

\[\Box\]

Corollary 4.3.2. Let \( M \) be an \( \mathcal{A} \)-module and \( s \) be a natural number. Then \( \Phi_1 \otimes (\Phi_s \otimes M) \) is naturally an \( \Phi_1 \otimes \Phi_s \)-module in \( \mathcal{M} \).

There is an analogous result for the subalgebra \( F_p[v^{\pm 1}] \) of \( \Phi_1 \). The behaviour exhibited by Example 4.2.3 can be avoided by using the large tensor product.

Proposition 4.3.3. Let \( B \) be an algebra in \( \mathcal{M} \) such that the underlying \( F_p \)-algebra is an integral domain. Then the underlying \( F_p \)-algebra of \( F_p[v^{\pm 1}] \otimes B \) is an integral domain.

Proof. The hypothesis that \( B \) is an integral domain implies that it is concentrated in even degrees. Consider elements \( 0 \neq X = \Sigma_i v_i^{d-i} \otimes b_i, Y = \Sigma_i v_i^{e-i} \otimes b'_i \) of \( \Phi_1 \otimes B \), of degrees \( d, e \) respectively; choose integers \( m, n \) such that \( b_i = 0 \) for \( i < m \) and \( b_{m} \neq 0 \) and \( b'_{j} = 0 \) for \( j < n \).

Suppose that \( XY = 0 \), it is necessary to show that \( Y = 0 \); by induction, it is sufficient to show that \( b'_n = 0 \). The projection of \( B \) (considered as a graded vector space) to \( B_{m+n} \) sends \( XY \) to \( v_i^{d+e-(m+n)} \otimes b_m b'_n \), which is therefore zero. Since \( B \) is an integral domain and \( b_m \) is non-zero by hypothesis, it follows that \( b'_n = 0 \), as required.

\[\Box\]

4.4. Embeddings. An isomorphism of vector spaces \( V_1 \oplus V_s \cong F_p \oplus F_p^{\oplus s} \cong F_p^{s+1} \cong V_{s+1} \) induces an isomorphism of unstable algebras over the Steenrod algebra \( H^*(BV_{s+1}) \cong H^*(BV_1) \otimes H^*(BV_s) \).

Lemma 4.4.1. Let \( s \) be a natural number. An isomorphism \( V_1 \oplus V_s \cong F_p^{s+1} \) induces a unique monomorphism of algebras in \( \mathcal{M} \), \( \Phi_1 \otimes \Phi_s \hookrightarrow \Phi_1 \otimes \Phi_s \), which fits into a commutative diagram of morphisms of algebras in \( \mathcal{M} \):

\[
\begin{array}{ccc}
H^*(BV_{s+1}) & \cong & H^*(BV_1) \otimes H^*(BV_s) \\
\downarrow & & \downarrow \\
\Phi_{s+1} & \hookrightarrow & \Phi_1 \otimes \Phi_s
\end{array}
\]

in which the vertical morphisms are induced by localization.

Proof. There is a composite monomorphism
\[
H^*(BV_{s+1}) \cong H^*(BV_1) \otimes H^*(BV_s) \hookrightarrow \Phi_1 \otimes \Phi_s \hookrightarrow \Phi_1 \otimes \Phi_s
\]
of algebras in \( \mathcal{M} \). By the universality of localization of algebras in \( \mathcal{M} \), it is sufficient to show that the image of the Dickson invariant \( Q_{s+1,0} \) is invertible in \( \Phi_1 \otimes \Phi_s \).

Writing the polynomial part of \( H^*(BV_{s+1}) \) as \( F_p[v_1, \ldots, v_{s+1}] \) and working with respect to the isomorphism \( F_p[v_1, \ldots, v_{s+1}] \cong F_p[v_1] \otimes F_p[v_2, \ldots, v_{s+1}] \), the Dickson invariant can be written as
\[
Q_{s+1,0} = -Q_{s,0} \prod_{\lambda \in F_p^+} (\lambda v_1 + y)
\]
where $Q_{s,0}$ is the top Dickson invariant for $F_p[v_2, \ldots, v_{s+1}]$ and $y$ ranges over the elements of the vector space $\langle v_2, \ldots, v_{s+1} \rangle$. Reindexing and using $\prod_{x \in F_p^*} \lambda = -1$, this gives

$$Q_{s+1,0} = Q_{s,0}v_1^{p(p-1)} \prod_{y} \left(1 + \frac{y}{v_1}\right)^{p-1},$$

which is a well-defined element of $\Phi_1 \otimes \Phi_s$, since $v_1$ is invertible. Moreover, the element $Q_{s,0}$ is invertible, hence it suffices to observe that each element $1 + \frac{y}{v_1}$, for $y \in \langle v_2, \ldots, v_{s+1} \rangle$, is invertible in $\Phi_1 \otimes \Phi_s$. The unique (by Proposition 4.3.3) inverse is given by $\sum_{i \geq 0} (-1)^i \left(\frac{y}{v_1}\right)^i$, which is a well-defined element of $\Phi_1 \otimes \Phi_s$. □

**Proposition 4.4.2.** Let $s$ be a natural number and $M$ be an $A$-module. There is a natural monomorphism of $\Phi_{s+1}$-modules in $\mathcal{M}$

$$\Phi_{s+1} \otimes M \hookrightarrow \Phi_1 \otimes (\Phi_s \otimes M)$$

where the right hand side is considered as a $\Phi_{s+1}$-module by restriction of the $\Phi_1 \otimes \Phi_s$-module structure of Corollary 4.3.2.

**Proof.** There is a canonical embedding $M \hookrightarrow \Phi_1 \otimes (\Phi_s \otimes M)$ induced by the respective units of $\Phi_1$, $\Phi_s$. This extends to an $A$-linear embedding of $\Phi_{s+1}$-modules, by Lemma 4.4.3. □

**Remark 4.4.3.** For a general $M$ in $\mathcal{M}$ and positive integer $s$, the large tensor product $\Phi_{s+1} \otimes M$ does not embed in $\Phi_1 \otimes (\Phi_s \otimes M)$. This stems from the fact that the functor $\Phi_1$ is only weakly continuous and not continuous, together with the fact that $\Phi_s$ is not weakly connective.

For example, consider $M = F_p[v] \subset H^*(BV_1)$ and the subalgebra $F_p[v_1, v_2][Q_{2,0}^{-1}] \subset \Phi_2$. There is an element $z$ of degree zero of $\Phi_3 \otimes F_p[v]$.

$$z := \prod_{t \geq 0} \left(\frac{v_1^{p^2-2} t}{Q_{2,0}}\right)^t \otimes v^t.$$ 

This can be rewritten in the form

$$\prod_{t \geq 0} \left(\frac{v_1^{p^2-2} t}{v_2^{p-1} \prod_{y} (1 + \frac{y}{v_1})^{p-1}}\right)^t \otimes v^t,$$

as in the proof of Lemma 4.4.1 ($y$ is a scalar multiple of $v_2$). For a fixed $t$, the leading term of the power series expansion in $v_1$ is $v_1^{(p-2)t} \otimes v_2^{(p-1)t} \otimes v^t$. Hence, the element $z$ does not belong to the large tensor product $F_p[v_1^{\pm 1}] \otimes (F_p[v_2^{\pm 1}] \otimes F_p[v])$, since the powers of $v_1$ are not bounded above.

5. The total Steenrod power

The total Steenrod power is fundamental to the constructions of this paper, as in the work of Zarati [Zar84] and the work of Hung and Sum [HS95], who use a stable version. The precise relationship between the total Steenrod power and algebras of invariants was established by Mu [Min86].

5.1. The morphism $St_1$. By convention, all morphisms of graded vector spaces considered here are degree preserving; the following degree-multiplying functor is required in order to describe the morphism $St_1$.
Notation 5.1.1. For $M$ a graded $\mathbb{F}_p$-vector space, let $\Psi M$ denote the graded vector space
\[(\Psi M)^n = \begin{cases} M^n & n = ip \\ 0 & \text{otherwise}. \end{cases}\]

Notation 5.1.2. Write $w$ for the element $\frac{1}{e}$ of $\Phi_1 \cong \Lambda(u) \otimes \mathbb{F}_p[v^{\pm 1}]$ of degree $-1$.

Definition 5.1.3. [Zar84, Section 2.4.1] For $M \in \text{Ob } \mathcal{M}$, let $\text{St}_1 : \Psi M \to \Phi_1 \otimes M$ be the morphism of graded $\mathbb{F}_p$-vector spaces defined on $x \in M^{2k+\delta}$, $\delta \in \{0, 1\}$ $k = \lfloor \frac{x}{2} \rfloor$, by
\[
\text{St}_1(x) := \sum_{k \geq t > -\infty, e \in \{0, 1\}} (-1)^{k+e} e_1^{e+2t} w^e \otimes \beta^e p^{k-1}(x).
\]

Remark 5.1.4. The definition of the morphism $\text{St}_1$ depends on the choice of the generator $u$ of $H^1(BV_1)$; whilst the class $w$ is $GL_1$-invariant, the class $e_1$ is only determined up to sign (cf. Lemma 5.1.4).

The morphism $\text{St}_1$ is compared in Lemma 5.1.9 below with the following stabilized linear morphism:

Definition 5.1.5. (Cf. [HS95, Definition 2.4].) For $M \in \text{Ob } \mathcal{M}$, let $S_1 : M \to \Phi_1 \otimes M$ be the linear morphism defined by
\[
S_1(x) = \sum_{i \geq 0, e \in \{0, 1\}} (-1)^{i+e} w^e Q_{1,0}^{-1} \otimes \beta^e p^i(x).
\]

Lemma 5.1.6. For $M \in \text{Ob } \mathcal{M}$ and $x \in M^{2k+\delta}$ ($\delta \in \{0, 1\}$), there is an identity
\[
\text{St}_1(x) = (-1)^k \epsilon_1^{k| \epsilon_1} S_1(x).
\]

Proof. Reindex the sum defining $\text{St}_1$ by $i = k - l$ and compare the two expressions, using the identity $\epsilon_1^2 = Q_{1,0}$. \hfill \Box

Remark 5.1.7. By [HS95, Lemma 2.7], this makes the relation between $\text{St}_1$ and the total Steenrod power $d_1^i P_1$ explicit (using the notation of [Mui86, HS95]). Namely, for $x \in M^{2k+\delta}$ ($\delta \in \{0, 1\}$)
\[
d_1^i P_1(x) = (-1)^k \mu(|x|) \text{St}_1(x),
\]
where the normalization factor $\mu(q)$ is given by $\mu(q) := (h!)^q (-1)^h q^{-1}/2$, for $h = \frac{q-1}{2}$.

Notation 5.1.8. For $M$ a $\mathbb{Z}$-graded $\mathbb{F}_p$-vector space, let $M^{\text{ev}}$ (respectively $M^{\text{odd}}$) be the graded sub vector space of even (resp. odd) degree elements. If $M \in \text{Ob } \mathcal{M}$, this is applied to the underlying graded vector space.

Notation 5.1.9. Let $(\Phi_1^{SL_1})^-$ be defined by the eigenspace decomposition $\Phi_1^{SL_1} \cong \Phi_1^{GL_1} \oplus (\Phi_1^{SL_1})^-$ induced by the $\mathbb{Z}/2$-action on $\Phi_1^{SL_1}$ (cf. Lemma 2.1.2).

Lemma 5.1.10. Let $M \in \text{Ob } \mathcal{M}$. The linear morphism $\text{St}_1$ induces morphisms:
\[
\Psi M^{\text{ev}} \to \Phi_1^{GL_1} \otimes M \\
\Psi M^{\text{odd}} \to (\Phi_1^{SL_1})^- \otimes M.
\]

Proof. Clear. \hfill \Box

There is a canonical isomorphism of $\mathcal{A}$-modules $\Sigma \Phi_1 \otimes M \cong \Phi_1 \otimes \Sigma M$; this corresponds to the isomorphisms
\[
(\mathbb{F}_p \otimes \Phi_1) \otimes M \cong (\Phi_1 \otimes \mathbb{F}_p) \otimes M \cong \Phi_1 \otimes (\mathbb{F}_p \otimes M)
\]
where the first isomorphism is the interchange of tensor factors.

The behaviour of $\text{St}_1$ with respect to suspension is described by the following result:

**Lemma 5.1.11.** (Cf. [Zar84] Lemma 4.2.1.) For $x$ an element of $M \in \text{Ob } \mathcal{M}$,

$$\text{St}_1(\Sigma x) = (-1)^{|x|} \Sigma \epsilon_1 \text{St}_1(x).$$

**Proof.** Straightforward. □

Using the product introduced in Proposition 4.3.1, the multiplicativity of $S_1$ (Cf. [HS95] Proposition 2.6) corresponds to the following result:

**Lemma 5.1.12.** (Cf. [Zar84] Lemme 2.4.2). For $M_1, M_2 \in \text{Ob } \mathcal{M}$ and elements $x_1 \in M_1, x_2 \in M_2$,

$$\text{St}_1(x_1 \otimes x_2) = (-1)^{|x_1||x_2|} \text{St}_1(x_1) \ast \text{St}_1(x_2).$$

**Proof.** Straightforward. □

This result implies the multiplicativity (up to sign) of the morphism $\text{St}_1$, in the presence of product structures.

**Lemma 5.1.13.** Let $\mathcal{M}$ be an unstable algebra in $\mathcal{M}$ and $M$ be a $B$-module in $\mathcal{M}$ then, with respect to the product structures given by Proposition 4.3.1, for elements $b_1, b_2 \in B$ and $m \in M$:

$$\text{St}_1(b_1 b_2) = (-1)^{|b_1||b_2|} \text{St}_1(b_1) \text{St}_1(b_2),$$

$$\text{St}_1(b_1 m) = (-1)^{|b_1||m|} \text{St}_1(b_1) \text{St}_1(m).$$

**Proof.** The multiplicative structures are provided by Proposition 4.3.1 and the behaviour on products of elements follows from Lemma 5.1.12. □

**Example 5.1.14.** Consider the unstable algebra $H^*(BV_1) \cong \Lambda(x) \otimes F_p[y]$. Then,

$$\text{St}_1(x) = \epsilon_1 \otimes x - M_{1,0} \otimes y,$$

$$\text{St}_1(y) = -Q_{1,0} \otimes y + 1 \otimes y^p.$$  

This determines $\text{St}_1$ on $H^*(BV_1)$ by multiplicativity.

5.2. The morphisms $\text{St}_s$ on unstable modules, for $s \geq 0$. By convention, for any $M \in \text{Ob } \mathcal{M}$, the morphism $\text{St}_0$ is the identity morphism of $M$. The morphism $\text{St}_1$ has higher analogues, $\text{St}_s$, for $s \geq 1$,

$$\text{St}_s : \Psi^s M \to \Phi^s \otimes M.$$

The definition for unstable modules is more elementary, and is treated in this section; the general case is postponed until the technical underpinnings have been introduced.

**Definition 5.2.1.** [Zar84] Section 2.4.1 For $M$ an unstable module and $s$ a positive integer, define the natural linear morphism

$$\text{St}_s : \Psi^s M \to H^*(BV_s) \otimes M$$

recursively by $\text{St}_s = \text{St}_1 \circ \text{St}_{s-1}$.

**Remark 5.2.2.** A priori, the construction of $\text{St}_s$ depends on a choice of ordered basis of $H^1(BV_s)$. The relationship with the stabilized morphism $S_s$, (see Definition 5.4.1 for unstable modules and Definition 5.5.1 for the general case), together with the invariance property of $S_s$ (established in Proposition 5.5.3), show that the linear morphism depends only upon the associated class $\epsilon_s$.  

Lemma 5.2.3. Let $s$ be a positive integer, then for $H^*(BV_1) \cong \Lambda(x) \otimes F_p[y]$:

$$\text{St}_s(y) = f_s(y) = \prod_{v \in V_s^*} (y - v),$$

in $H^*(BV_1)$ and $H^*(BV_1)$. Hence $\text{St}_s(y) = \sum_{i=0}^{s} (-1)^{s-i} Q_{s,i} y^i$.

Proof. Proof by induction, using the fact that $\text{St}_1$ is multiplicative, by Lemma 5.1.13.

Lemma 5.2.4. Let $s$ be a positive integer, then

$$Q_{s+1,0} = Q_{1,0} \text{St}_1(Q_{s,0})$$

$$Q_{s+1,i} = Q_{1,0} \text{St}_1(Q_{s,i}) + \text{St}_1(Q_{s+1,i-1}), \quad 1 \leq i \leq s$$

in $H^*(BV_1) \otimes H^*(BV_1) \cong \Lambda(u) \otimes F_p[v] \otimes H^*(BV_1) \cong H^*(BV_{s+1}),$ where $Q_{1,0}$ denotes the Dickson invariant $v^{p-1}$ of $\Lambda(u) \otimes F_p[v]$.

Similarly,

$$L_{s+1} = v \text{St}_1(L_s)$$

$$\epsilon_{s+1} = \epsilon_1 \text{St}_1(\epsilon_s),$$

where $\epsilon_1 = v^{\frac{p-1}{2}}$.

Proof. The first statement follows from Lemma 5.2.3

$$\text{St}_{s+1}(y) = \text{St}_1 \left( \sum_{i=0}^{s} (-1)^{s-i} Q_{s,i} y^i \right)$$

$$= \sum_{i=0}^{s} (-1)^{s-i} \text{St}_1(Q_{s,i}) \text{St}_1(y)^i.$$ 

Now $\text{St}_1(y) = -Q_{1,0} y^2 + y^p$, from which the result follows.

The result for $L_s$ follows by a similar calculation (cf. Lemma 5.3.1); the key point is to identify the normalization factor. The calculation for $\epsilon_s$ follows by multiplicativity.

Remark 5.2.5. Zarati uses a different sign convention in defining the generators $Q_{s,i}$; this leads to a sign appearing in [Zar84, Section 4.3.2].

Lemma 5.2.6. Let $M \in \text{Ob } \mathcal{V}$ and $x$ be an element of $M$, then

$$\text{St}_s(\Sigma x) = (-1)^{s|x|} \Sigma \epsilon_s \text{St}_s(x).$$

Proof. (Cf. [Zar84, Lemma 4.2.1].) Use an induction upon $s$, starting with the case $s = 1$, given by Lemma 5.1.11. For the inductive step, apply $\text{St}_1$ to the equation $\text{St}_s(\Sigma x) = (-1)^{s|x|} \Sigma \epsilon_s \text{St}_s(x)$ and use the identity $\epsilon_{s+1} = \epsilon_1 \text{St}_1(\epsilon_s)$ (note that the parity of $|\epsilon_s \text{St}_s(x)|$ is equal to the parity of $|x|$).

Lemma 5.1.12 generalizes to the following:

Lemma 5.2.7. (Cf. [Zar84, Lemma 2.4.2].) Let $s$ be a positive integer. For $M_1, M_2 \in \text{Ob } \mathcal{V}$ and elements $x_1 \in M_1$, $x_2 \in M_2$,

$$\text{St}_s(x_1 \otimes x_2) = (-1)^{s|x_1||x_2|} \text{St}_s(x_1) \bullet \text{St}_s(x_2).$$

Proof. Induction upon $s$. 

□
5.3. Mui invariants. This section reviews the calculation of \( S_t(x) \) (\( s \) a positive integer), where \( x \) is the degree one generator of \( H^*(BV_1) \cong \Lambda(x) \otimes \mathbb{F}_p[y] \); this relies on the ideas of Mui [Mui86] (cf. [Zar84]).

Consider the unstable algebra \( \bigotimes_{i=1}^s \Lambda(u_i) \otimes \mathbb{F}_p[v_i], \) isomorphic to \( H^*(BV_s) \), together with a choice of generators. There is a morphism of algebras \( T : \bigotimes_{i=1}^s \Lambda(u_i) \otimes \mathbb{F}_p[v_i] \to \bigotimes_{i=1}^s \Lambda(u_i) \otimes \mathbb{F}_p[v_i] \otimes \Lambda(u) \otimes \mathbb{F}_p[v^{+1}], \) defined by

\[
T : \left\{ \begin{array}{l}
u_i \mapsto u_i - \frac{v_i}{v^{p^i}}, \\
v_i \mapsto v_i - \frac{1}{v^{p^i}},
\end{array} \right.
\]

which is modelled on the total operation \( S_1 \).

**Lemma 5.3.1.** Let \( T \) be the multiplicative operator defined above, then

\[
\begin{vmatrix}
  u_1 & \cdots & u_s \\
  v_1 & \cdots & v_s \\
  \vdots & \vdots & \vdots \\
  v_1^{p^i-1} & \cdots & v_s^{p^i-1}
\end{vmatrix}
= \frac{1}{v^{p^s}}
\begin{vmatrix}
  u_1 & \cdots & u_s & u \\
  v_1 & \cdots & v_s & v \\
  \vdots & \vdots & \vdots & \vdots \\
  v_1^{p^i} & \cdots & v_s^{p^i} & v^{p^i}
\end{vmatrix}
\]

**Proof.** The result follows by developing the determinant after applying the transformation \( T \). Writing \( \lambda \) for \( \frac{u}{v} \) and \( \mu \) for \( \frac{1}{v^{p^s}} \), a straightforward calculation shows that the left hand side is equal to

\[
\begin{vmatrix}
u_1 & \cdots & u_s & \lambda \mu_0 \\
v_1 & \cdots & v_s & \mu_0 \\
\vdots & \vdots & \vdots & \vdots \\
v_1^{p^i} & \cdots & v_s^{p^i} & \mu_s
\end{vmatrix}
\]

where \( \mu_i := \prod_{j=1}^{i-1} \mu_j \) (for \( i < s \)) and \( \mu_s = 1 \). To complete the proof, substitute the values for \( \lambda \) and \( \mu \), using the identification \( \mu_i = \mu^{s-i} \frac{v_i^{p^i-1}}{v^{p^i-1}} \).

This result is the essential ingredient in the proof of the following:

**Proposition 5.3.2.** (Cf. [Mui86] Proposition 2.6.) Let \( s \) be a positive integer and consider \( M = \Lambda(x) \otimes \mathbb{F}_p[y] \cong H^*(BV_1) \)

\[
\begin{vmatrix}
u_1 & \cdots & u_s & x \\
v_1 & \cdots & v_s & y \\
\vdots & \vdots & \vdots & \vdots \\
v_1^{p^i} & \cdots & v_s^{p^i} & y^{p^i}
\end{vmatrix}
\]

**Proof.** By Lemma 5.3.1 it is straightforward to show by induction upon \( s \) that, up to a normalization factor, \( S_t(x) \) is equal to the determinant. The normalization factor is determined by considering the coefficient of \( x \) in \( S_t(x) \), which is the Euler class \( \mathfrak{c}_s \).

**Definition 5.3.3.** [Mui86] Section 2] Let \( s \) be a positive integer. Using the choice of generators \( \{u_i, v_i \mid 1 \leq i \leq s \} \) for the algebra \( H^*(BV_s) \), let:

1. \( M_{s, t} \) (\( 0 \leq t < s \)) be the determinant

\[
\begin{vmatrix}
u_1 & \cdots & u_s \\
v_1 & \cdots & v_s \\
\vdots & \vdots & \vdots \\
v_1^{p^i} & \cdots & v_s^{p^i}
\end{vmatrix}
\]
in which the row \((v'_1 \cdots v'_e)\) is omitted from the array;

(2) \(\tilde{M}_{s,i}\) be the element
\[
\tilde{M}_{s,i} := M_{s,i}L_s^{\frac{p^s-1}{2}}
\]
so that \(|\tilde{M}_{s,i}| = p^s - 2p^i\);

(3) \(R_{s,i}\) be the element \(R_{s,i} := \tilde{M}_{s,i}e_s\) of degree \(|R_{s,i}| = 2(p^s - p^i) - 1\); (this notation is consistent with that used by [HS95]).

**Proposition 5.3.4.** Let \(s\) be a positive integer. Then

1. \(e_s \in H^*(BV_1)\)
2. \(\tilde{M}_{s,i} \in H^*(BV_1)\)
3. \(R_{s,i} \in H^*(BV_1)\)

for \(0 \leq i < s\).

**Proof.** (Cf. [Mùi86, Section 2].) The \(\tilde{L}_s\) invariance of \(e_s\) is stated in Lemma 2.4.4 and the invariance of the elements \(\tilde{M}_{s,i}\) follows from the definition. The \(GL_s\)-invariance of \(R_{s,i}\) follows.

**Corollary 5.3.5.** Let \(s\) be a positive integer, then
\[
\text{St}_s(x) = e_s \otimes x + \sum_{i=0}^{s-1} (-1)^{i+1} \tilde{M}_{s,i} \otimes y^{p^i}.
\]
Hence:
\[
\tilde{M}_{s+1,0} = -Q_1,0\text{St}_1(\tilde{M}_{s,0}) + \tilde{M}_{1,0}\text{St}_1(e_s)
\]
\[
\tilde{M}_{s+1,i} = -Q_i,0\text{St}_1(\tilde{M}_{s,i}) - \text{St}_1(\tilde{M}_{s-1,i-1}), \ 0 < i < s + 1.
\]

**Proof.** A consequence of Proposition 5.3.4. The calculation of the identities follows, as in Lemma 5.2.7.

5.4. The morphisms \(S_s\) for unstable modules. For the purposes of this section, define the stable version of the total Steenrod power on unstable modules by:

**Definition 5.4.1.** For \(M \in \text{Ob } \mathcal{U}\), let \(S_s : M \rightarrow \Phi_s \otimes M\) be the natural linear morphism given by
\[
S_s(x) := (-1)^{|x|\frac{|x|}{2}} e_s - |x|\text{St}_s(x).
\]

This extends to a linear morphism defined on \(\mathcal{M}\), by using a completed tensor product in the image; the delicate point is to show that it takes values in \(\Phi_s \otimes M\).

**Lemma 5.4.2.** (Cf. [HS95, Proposition 2.6].) Let \(s\) be a positive integer; for \(M_1, M_2 \in \text{Ob } \mathcal{U}\) and elements \(x_1 \in M_1, x_2 \in M_2\),
\[
S_s(x_1 \otimes x_2) = S_s(x_1) \bullet S_s(x_2).
\]

**Proof.** A consequence of Lemma 5.2.7.

**Corollary 5.4.3.** Let \(s\) be a positive integer and consider the unstable algebra \(H^*(BV_1) \cong \Lambda(x) \otimes \mathbb{F}_p[y]\) Then
\[
S_s(x) = x + \sum_{i=0}^{s-1} (-1)^{i+1} \frac{R_{s,i}}{Q_{s,0}} \otimes y^{p^i},
\]
\[
S_s(y) = \sum_{i=0}^{s} (-1)^{i} \frac{Q_{s,i}}{Q_{s,0}} \otimes y^{p^i}
\]
in \(\Phi_s \otimes M\).
Proof. Follows from Corollary 5.3.3 and Lemma 5.2.3. □

Recall that the dual Steenrod algebra \( A^* \) is isomorphic as an algebra to \( \Lambda(\tau_j | j \geq 0) \otimes \mathbb{F}_p[\xi_i | i \geq 1] \) (for \( p \) an odd prime) \[Mil58\]. In the following, use the convention that \( Q_{s,s} = 1 \) and \( Q_{s,i} = 0 \) for \( i > s \); similarly, \( R_{s,j} = 0 \) for \( j \geq s \).

**Definition 5.4.4.** For \( s \) a positive integer, let \( \theta_s : A^* \to \Phi_s \) denote the morphism of algebras

\[
\begin{align*}
\xi_i & \mapsto (-1)^i \frac{Q_{s,i}}{Q_{s,0}} \\
\tau_j & \mapsto (-1)^{j+1} \frac{R_{s,j}}{Q_{s,0}}.
\end{align*}
\]

If \( M \) is a module over the Steenrod algebra, then \( M \) can be considered as a comodule over \( A^* \) (with respect to a completed tensor product), with coaction \( M \to M \hat{\otimes} A^* \). By transposition of tensor factors, this defines a natural morphism \( \psi_M : M \to A^* \otimes M \).

**Remark 5.4.5.** The switch of tensor factors is required to allow the large tensor product to be used in place of the completed tensor product.

**Proposition 5.4.6.** Let \( M \) be an unstable module and \( s \) be a positive integer. The following diagram commutes

\[
\begin{array}{ccc}
M & \xrightarrow{\psi_M} & A^* \otimes M \\
\downarrow{S_s} & & \downarrow{\theta_s \otimes M} \\
\Phi_s \otimes M & \xleftarrow{} & \Phi_s \otimes M.
\end{array}
\]

**Proof.** By the naturality of the construction of \( S_s \), it suffices to prove this result on the fundamental class of \( M = \mathcal{F}(n) \), the free unstable module on a generator of degree \( n \), for each \( n \). Since \( \mathcal{F}(n) \) embeds in \( H^*(BV_n) \) and the functor \( \Phi_s \otimes - \) is exact, it suffices to establish the result for each \( H^*(BV_n) \). For \( n = 0 \), this is trivial.

The result holds for \( M = H^*(BV_1) \) by Corollary 5.4.3 by using the multiplicity of \( S_s \) (Lemma 5.4.2). Again, by the multiplicity of \( S_s \), the result holds for \( H^*(BV_n) \), for any natural number \( n \). □

**Remark 5.4.7.** Compare [HS95, Theorem 2.9]: the definition of \( \theta_s \) above incorporates the signs which appear in loc. cit.. The above result can be considered as a reformulation of Mui’s [Mùi86, Theorem 4.6].

5.5. The morphisms \( S_t, S_s \) for arbitrary modules.

**Definition 5.5.1.** For \( s \) a positive integer and \( M \) an \( A \) module, let \( S_s : M \to \Phi_s \otimes M \) be the morphism defined as the composite

\[
\begin{array}{ccc}
M & \xrightarrow{\psi_M} & A^* \otimes M \\
\downarrow{S_s} & & \downarrow{\theta_s \otimes M} \\
\Phi_s \otimes M & \xleftarrow{} & \Phi_s \otimes M.
\end{array}
\]

and let \( S_s : \Psi^* M \to \Phi_s \otimes M \) be the linear morphism defined by

\[
S_s(x) := (-1)^{|x|} \frac{Q_{s,|x|}}{Q_{s,0}} S(x).
\]

**Remark 5.5.2.** By Proposition 5.4.6 this definition is compatible with that given for \( M \) unstable, where it is not necessary to use the large tensor product.
Proposition 5.5.3. Let $s$ be a positive integer and $M$ an $A$-module. Then

1. $S_t$s factorizes naturally as
   \[
   \Psi^s M \rightarrow \Phi_{s}^{\mathbb{Z}/s} \otimes M \hookrightarrow \Phi_{s} \otimes M
   \]
   and $S_s$s factorizes naturally as $M \rightarrow \Phi_{s}^{\mathbb{Z}/s} \otimes M \hookrightarrow \Phi_{s} \otimes M$;

2. the linear morphism $S_t$s factorizes naturally as
   \[
   M \rightarrow S_t \rightarrow \Phi_{s}^{\mathbb{Z}/t} \otimes \mathbb{F}_p \left[ Q_{s,i} \right] \otimes M \rightarrow \Phi_{s} \otimes M,
   \]
   where the vertical morphism is induced by the natural inclusion
   \[
   \Lambda \left( \frac{R_{s,i}}{Q_{s,j}} \right) \otimes \mathbb{F}_p \left[ Q_{s,i} \right] \otimes M \rightarrow \Phi_{s} \otimes M,
   \]
   of algebras.

Proof. The statement about the factorization across the invariants is a consequence of Proposition 5.3.4, together with the definition of $S_t$s and $S_s$s. The final statement follows from the definition of $\theta$.

5.6. Relating $S_s$s and $S_s^{-1}$. Proposition 5.6.4 below gives a precise sense in which $S_t$s can be regarded as the composition $S_t \circ S_s^{-1}$, generalizing the tautological relationship for unstable modules (cf. Proposition 5.4.6). (This technical material is required in order to analyse the functors $R_s$ introduced in Section 7.) It is sufficient to consider the analogous property for the stabilized total operations $S_s$. The relationship between $S_s$ and $S_1 \circ S_{s-1}$ is clear for a module $M$ which is bounded above, by using the embedding $\Phi_{s} \otimes M^{c} \hookrightarrow \Phi_{1} \otimes (\Phi_{s}^{-1} \otimes M^{c})$, provided by Proposition 4.4.2. The functor $M \mapsto \Phi_{s}^{-1} \otimes M$ is not weakly continuous, hence this is not sufficient to establish a general result, since $M \rightarrow \Phi_{s} \otimes (\Phi_{s-1} \otimes M)$ is not weakly continuous for $s > 1$. This technical difficulty is avoided by appealing to Proposition 5.6.3.

Notation 5.6.1. If $X$ is a $\mathbb{Z}$-graded vector space and $C$ is an integer, write $X^{\leq C}$$ for the subspace of elements of degree at most $C$.

Lemma 5.6.2. Let $Y_*$ be an inverse system of $\mathbb{Z}$-graded vector spaces, $C$ be an integer and $X$ be a graded vector space of finite type. Then the canonical morphism

\[
X^{\leq C} \otimes \lim_{\rightarrow t} Y_* \cong \lim_{\rightarrow t} \left( X^{\leq C} \otimes Y_* \right)
\]

is an isomorphism of graded vector spaces.

Proof. There is an isomorphism

\[
\lim_{\rightarrow t} \lim_{\rightarrow s} \left( X^{\leq C} \otimes Y_* \right) \cong \lim_{\rightarrow s} \lim_{\rightarrow t} \left( X^{\leq C} \otimes Y_* \right),
\]

reversing the order of the inverse limits.

In a fixed degree $k$ and for fixed $t$, \(X^{\leq C} \otimes Y_*^{c t})^k\) is a finite sum

\[
\bigoplus_{i+j=k} X^i \otimes (Y_*^{c t})^j,
\]

using the fact that $X^{\leq C}$ is bounded above. Moreover, since $X$ is of finite type, each $X^i$ is of finite dimension. It follows that the inverse limit $\lim_{\rightarrow t} Y_*$ commutes
with the formation of the tensor product. The result follows, since \( \lim_{e^{-t}} (Y^s) \cong \lim_{e^{-t}} Y^s \).

As observed in Remark 3.1.5, there is a notion of weak connectivity for functors defined on the category of graded vector spaces. The previous Lemma implies the following key result:

**Proposition 5.6.3.** Let \( C \) be an integer and \( s \) be a positive integer. The functor

\[
M \mapsto \Phi^C \otimes (\Phi_{s-1} \otimes M)
\]

is weakly continuous.

**Proof.** By definition, \( \Phi_{s-1} \otimes M \cong \lim_{e^{-t}} \Phi_{s-1} \otimes M^t \cong \lim_{e^{-t}} \Phi_{s-1} \otimes M^t \). The result follows from Lemma 5.6.2. \( \square \)

**Proposition 5.6.4.** Let \( s \) be a positive integer and \( M \) be an \( A \)-module.

1. There is a natural monomorphism of \( \Lambda^j \otimes \mathbb{F}_p(Q_{s-1}) \)-modules:

\[
\left( \Lambda^j \otimes \mathbb{F}_p(Q_{s-1}) \right) \otimes M \hookrightarrow \Phi^j \otimes (\Phi_{s-1} \otimes M),
\]

which makes the following diagram commute

\[
\begin{array}{ccc}
\Lambda^j \otimes \mathbb{F}_p(Q_{s-1}) & \otimes M^C & \Phi^j \otimes M \\
\downarrow & \downarrow & \downarrow \\
\Lambda^j \otimes \mathbb{F}_p(Q_{s-1}) & \otimes M^C & \Phi^j \otimes (\Phi_{s-1} \otimes M),
\end{array}
\]

in which the vertical morphism is the embedding of Proposition 5.4.2.

2. There is a natural commutative diagram of linear morphisms

\[
\begin{array}{ccc}
M & \xrightarrow{S_{s-1}} & \Phi_{s-1} \otimes M \\
\downarrow & & \downarrow \\
S_{s-1} & \xrightarrow{S_s} & \Phi_{s} \otimes (\Phi_{s-1} \otimes M).
\end{array}
\]

**Proof.** For the first statement, by Lemma 4.3.1 there is an inclusion of algebras \( \Phi^j \otimes \Phi_{s-1} \) which is induced by \( V^j \cong \mathbb{F}_p \oplus \mathbb{F}_p^{s-1} \). By restriction, this gives a morphism of algebras

\[
\Lambda^j \otimes \mathbb{F}_p(Q_{s-1}) \hookrightarrow \Phi^j \otimes \Phi_{s-1}.
\]

Since \( \Lambda^j \otimes \mathbb{F}_p(Q_{s-1}) \) is a subalgebra of \( \Phi^j \), this morphism is independent of the choice of isomorphism \( V^j \cong \mathbb{F}_p \oplus \mathbb{F}_p^{s-1} \).

The Dickson invariant \( Q_{s,i} \) can be written, up to sign, as

\[
\sum_{W \leq V^*, \dim W = 1} \left( \prod_{v \in V^*_W} v \right).
\]

It follows that

\[
\frac{Q_{s,i}}{Q_{s,0}} = \sum_{W \leq V^*, \dim W = 1} \left\{ \prod_{w \in W \setminus \{0\}} w \right\}^{-1}.
\]

By considering the power series expansion of this element, it follows that \( \frac{Q_{s,i}}{Q_{s,0}} \in \Phi^j \otimes \Phi_{s-1} \). Namely, if \( w \in W \setminus \{0\} \), either \( w \) corresponds to a class of degree two of
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$H^*(BV_{s-1})$, which is therefore an invertible element of $\Phi_{s-1}$, or $w$ can be written as $\lambda(v_1 + w')$, where $v_1$ is the polynomial generator of $H^*(BV_1)$, $\lambda \in \mathbb{F}_p^\times$ and $w'$ corresponds to a class of degree two of $H^*(BV_{s-1})$. By rewriting this element as $\lambda v_1 (1 + w''_n)$, it is clear that the inverse of $w$ in $\Phi_1 \otimes \Phi_{s-1}$ belongs to the subalgebra $\Phi_1^{s \leq 0} \otimes \Phi_{s-1}$ in either case. The result for $\frac{Q_s}{Q_{s,0}}$ follows.

It follows that, for any integer $c$, the induced morphism of $\Lambda \frac{R_{s,i}}{Q_{s,0}} \otimes \mathbb{F}_p \frac{Q_i}{Q_{s,0}}$ modules

\[
\left( M \frac{R_{s,i}}{Q_{s,0}} \otimes \mathbb{F}_p \frac{Q_i}{Q_{s,0}} \right) \otimes M^{<c} \rightarrow \Phi_1 \otimes \left( \Phi_{s-1} \otimes M^{<c} \right)
\]

has image which lies in the subspace $\Phi_1^{s \leq D} \otimes (\Phi_{s-1} \otimes M^{<c})$ of $\Phi_1 \otimes (\Phi_{s-1} \otimes M^{<c})$, for some fixed integer $D$ (the precise value is unimportant; it can be calculated by studying the images of the exterior generators $\frac{R_{s,i}}{Q_{s,0}}$ as for the case $\frac{Q_s}{Q_{s,0}}$). To complete the proof of the first statement, it suffices to pass to the inverse limit, using the weak continuity established in Proposition 5.6.3 and then composing with the monomorphism

$$\Phi_1^{s \leq D} \otimes (\Phi_{s-1} \otimes M) \hookrightarrow \Phi_1 \otimes (\Phi_{s-1} \otimes M).$$

The composite morphism has the required properties.

The second statement is known for $M$ unstable (by comparison with $\text{St}_s$). Consider the case of a bounded above module ($M \cong M^{<c}$ for some $c \in \mathbb{Z}$). An element $x$ of $M$ belongs to the bounded submodule $M_{\geq |x|}$. The morphism $\text{St}_s$ is stable (commutes with suspension); hence by replacing $M$ by a suitable suspension, we may assume that $x$ belongs to an unstable submodule of $M$, so the result follows in this case.

The general case follows by passage to the inverse limit using weak continuity, by considering the image in $\Phi_1^{s \leq D} \otimes (\Phi_{s-1} \otimes M)$ and appealing to Proposition 5.6.3 as above.

5.7. Freeness of submodules. The following general result will be applied to the analysis of the functor $\mathcal{A}_s$ in Section 7.2.

**Proposition 5.7.1.** Let $s$ be a natural number, $K \subset H^*(BV_s)$ be a graded subalgebra and $M$ be an $A$-module. The morphism $\text{St}_s$ induces an isomorphism of $K$-modules

$$K \otimes \Psi^s M \cong K\text{St}_s(M) \subset \Phi_s \otimes M.$$ 

**Proof.** (Cf. the argument of [Zar84, Proposition 3.3.4].) The morphism $\text{St}_s$ induces a morphism of $K$-modules:

$$K \otimes \Psi^s M \rightarrow \Phi_s \otimes M,$$

with image $K\text{St}_s(M)$, by construction.

It suffices to show that the elements $\{\text{St}_s(m_i)|i \in I\}$ are linearly independent over $K$, where $\{m_i|i \in I\}$ is a homogeneous $\mathbb{F}_p$-basis of $M$. Suppose that there exists a relation $\sum j k_j \text{St}_s(m_j) = 0$ (finite sum), with $k_j \in K$; let $d$ be inf $\{|m_j| \mid k_j \neq 0\}$ and consider the projection of graded vector spaces $M \rightarrow \Sigma^d M^d$, which induces a surjection $\Phi_s \otimes M \rightarrow \Phi_s \otimes \Sigma^d M^d$. Under this morphism, $\sum j k_j \text{St}_s(m_j)$ maps to $\pm d^{|s|} \sum l k_l \otimes m_l$ indexed over the subset of indices $l$ such that $|m_l| = d$. It follows that $k_l = 0$, for each $l$, since the element $\epsilon_s$ is invertible in $\Phi_s$. Iterating this argument establishes that $k_l = 0$, for all $l$, as required. 

\[\square\]
Part 2. The construction

6. THE FUNCTORS $\mathcal{A}_1$ AND $\mathcal{A}_1$

This section introduces the functor $\mathcal{A}_1$ on the category $\mathcal{M}$, which generalizes the functor $R_1$ used by Zarati [Zar84]. This is achieved by first introducing the functor $\mathcal{A}_1$ and then restricting to $GL_1$-invariants. The non-trivial step in the construction is provided by Proposition 6.1.3 which establishes invariance under the action of the Steenrod algebra.

6.1. Introducing the functor $\mathcal{A}_1$. Let $K_1$ denote the unstable algebra $H^*(BV_1)^{\tilde{S}L_1}$; by Lemma 5.1.10 for $M \in \text{Ob } \mathcal{M}$, $\text{St}_1$ takes values in $\Phi_1^{\tilde{S}L_1 \otimes M}$.

Definition 6.1.1. (Cf. [Zar84]) For $M \in \text{Ob } \mathcal{M}$, let $\tilde{\mathcal{A}}_1 M$ denote the submodule $\tilde{\mathcal{A}}_1 M$ of $\Phi_1^{\tilde{S}L_1 \otimes M}$ generated by $\text{St}_1(M)$.

Lemma 6.1.2. For $M \in \text{Ob } \mathcal{M}$, the submodule $\tilde{\mathcal{A}}_1 M$ of $\Phi_1^{\tilde{S}L_1 \otimes M}$ is independent of the choice of basis of $H^1(BV_1)$ used in defining $\text{St}_1$.

Proof. Clear.

Proposition 6.1.3. The submodule $\tilde{\mathcal{A}}_1 M \subset \Phi_1^{\tilde{S}L_1 \otimes M}$ is stable under the action of the Steenrod algebra $A$. In particular, $\tilde{\mathcal{A}}_1 M$ is a $K_1$-module in $\mathcal{M}$.

Proof of Proposition 6.1.3. It is sufficient to show that, for any element $x$ of $M$, the elements $\beta \text{St}_1(x)$ and $P^i \text{St}_1(x)$ ($i \in \mathbb{N}$) belong to $\tilde{\mathcal{A}}_1 M \subset \Phi_1^{\tilde{S}L_1 \otimes M}$. A straightforward calculation shows that $\beta \text{St}_1(x) = 0$, hence it suffices to consider the case of the reduced powers.

Lemma 5.1.10 shows that $\text{St}_1(x) = \pm e_1^{[x]} S_1(x)$. By the Cartan formula, it is straightforward to see that it is sufficient to show that $e_1^{[x]} P^i S_1(x)$ belongs to $\tilde{\mathcal{A}}_1 M$ for any $i \in \mathbb{N}$. Proposition 4.10 of [HS95] implies that

$$P^i(S_1(x)) = \sum_{\varepsilon \in \{0,1\}, t \geq 0} \frac{-(p-1)t - \varepsilon}{i - pt - \varepsilon} w^\varepsilon e_1^{2(i-t)} S_1(\beta^\varepsilon P^i x).$$

Writing $w$ as $\tilde{M}_{1,0} e_1^{-1}$, this gives

$$e_1^{[x]} P^i(S_1(x)) = \sum_{\varepsilon \in \{0,1\}, t \geq 0} \frac{-(p-1)t - \varepsilon}{i - pt - \varepsilon} \tilde{M}_{1,0} e_1^{2(i-t)+|x| - \varepsilon} S_1(\beta^\varepsilon P^i x).$$

To show that the right hand side belongs to $\tilde{\mathcal{A}}_1 M$, it is sufficient to show that the terms with

$$2(i - t) + |x| - \varepsilon < |\beta^\varepsilon P^i x|$$

have trivial coefficient. The above condition is equivalent to $i - pt - \varepsilon < 0$, which implies the vanishing of the binomial coefficient, as required. \qed

Remark 6.1.4. A direct proof is given in Appendix A.

Proposition 6.1.5.

1. Let $M \in \text{Ob } \mathcal{M}$, then $\text{St}_1$ induces an isomorphism of $K_1$-modules

$$K_1 \otimes \Psi M \xrightarrow{\sim} \tilde{\mathcal{A}}_1 M.$$

2. (Cf. [Zar84] Corollaire 3.3.5.) The functor $\tilde{\mathcal{A}}_1 : \mathcal{M} \rightarrow K_1 \mathcal{M}$ is exact and commutes with colimits.

3. The functor $\tilde{\mathcal{A}}_1 : \mathcal{M} \rightarrow \mathcal{M}$ is weakly continuous and $\kappa_1$-connective, where $\kappa_1 : n \mapsto pn$. 

The morphisms of $K\text{-}\text{inverse limit}$ can be considered in this category. Let $M$ be an $\mathcal{A}\text{-module}$ and consider the morphisms of $K\text{-}\mathcal{A}$:

$$\mathcal{R}_1 M \to \mathcal{R}_1 M \leftarrow \Phi_1 \otimes M.$$ 

Passing to the inverse limit, as $c \to \infty$, this gives morphisms of $K\text{-}\mathcal{A}$:

$$\mathcal{R}_1 M \to \lim_{\leftarrow c} \mathcal{R}_1 M \leftarrow \Phi_1 \otimes M$$

using the Mittag-Leffler condition to deduce that $\lim_{\leftarrow c}$ preserves the surjection. It is clear (essentially by the definition of $\mathcal{R}_1$) that the composite coincides with the inclusion $\mathcal{R}_1 M \hookrightarrow \Phi_1 \otimes M$. It follows, that the morphism $\mathcal{R}_1 M \to \lim_{\leftarrow c} \mathcal{R}_1 M \leftarrow \Phi_1 \otimes M$ is an isomorphism. □

The functor $\mathcal{R}_1$ commutes with tensor products and, in particular, behaves well with respect to suspension.

**Proposition 6.1.6.** Let $M_1, M_2$ be $\mathcal{A}\text{-modules}$. There is a natural isomorphism

$$\mathcal{R}_1 (M_1 \otimes M_2) \cong \mathcal{R}_1 (M_1) \otimes_{K_1} \mathcal{R}_1 (M_2)$$

in $K\text{-}\mathcal{A}$.

**Proof.** A consequence of Proposition 6.1.5 using the compatibility of $\text{St}_1$ with tensor products provided by Lemma 5.1.12 and the fact that the functor $\Psi$ commutes with tensor products. □

6.2. **Eigenspace splitting.** By Lemma 2.4.2, there is a splitting in unstable modules $K_1 \cong K_1^+ \oplus K_1^-$, where $K_1^+$ is the algebra $H^* (BV_1)^{GL_1}$; moreover, the splitting is defined in $K_1^+\text{-}\mathcal{A}$. Recall that $\Phi_1^{GL_1} \cong K_1 [Q_{11}]$, hence this induces a splitting $\Phi_1^{GL_1} \cong \Phi_1^{GL_1} \oplus (\Phi_1^{GL_1})^-$ of $\Phi_1^{GL_1}\text{-modules}$ in $\mathcal{A}$.

**Notation 6.2.1.** For $M \in \text{Ob} \mathcal{A}$, write $\text{St}_1 (M^{ev})$ (resp. $\text{St}_1 (M^{odd})$) for the images of $\Psi M^{ev}$ (resp. $\Psi M^{odd}$) under the linear morphism $\text{St}_1$.

**Proposition 6.2.2.** For $M \in \text{Ob} \mathcal{A}$, there is a natural splitting in the category $K_1^+\text{-}\mathcal{A}$:

$$\mathcal{R}_1 M \cong \mathcal{R}_1 M \oplus \mathcal{R}_1 M^c,$$

where $\mathcal{R}_1 M := \mathcal{R}_1 M \cap \left( \Phi_1^{GL_1} \otimes M \right)$ and $\mathcal{R}_1 M^c := \mathcal{R}_1 M \cap \left( (\Phi_1^{GL_1})^- \otimes M \right)$. In particular $\mathcal{R}_1 M$ and $\mathcal{R}_1 M^c$ are sub $\mathcal{A}\text{-modules}$ of $\mathcal{R}_1 M$.

There are natural isomorphisms of $K_1^+\text{-modules}$:

$$\mathcal{R}_1 M \cong K_1^+ \text{St}_1 (M^{ev}) \oplus K_1^- \text{St}_1 (M^{odd})$$

$$\mathcal{R}_1 M^c \cong K_1^- \text{St}_1 (M^{ev}) \oplus K_1^+ \text{St}_1 (M^{odd})$$

**Proof.** A straightforward consequence of Lemma 5.1.10. □

**Remark 6.2.3.** When $M$ is an unstable module, $\mathcal{R}_1 M$ coincides with the object $R_1 M$ defined by Zarati in [Zar84].

**Corollary 6.2.4.** The associations $M \mapsto \mathcal{R}_1 M$ and $M \mapsto \mathcal{R}_1^c M$ define functors

$$\mathcal{R}_1, \mathcal{R}_1^c : \mathcal{A} \to K_1^+\text{-}\mathcal{A}$$

which are exact, commute with colimits, are weakly continuous and weakly connective.

**Proof.** The result follows from Proposition 6.2.2 and Proposition 6.1.5. □
Proposition 6.2.5. For \( M \in \text{Ob } \mathcal{M} \), there is a natural isomorphism in \( K_1^+\mathcal{M} \):

\[
\widetilde{\mathcal{R}}_1(\Sigma M) \cong \Sigma e_1 \widetilde{\mathcal{R}}_1 M.
\]

This restricts to natural isomorphisms of \( K_1^+\mathcal{M} \):

\[
\widetilde{\mathcal{R}}_1(\Sigma M) \cong \Sigma e_1 \widetilde{\mathcal{R}}_1^+ M \quad \text{and} \quad \widetilde{\mathcal{R}}_1^- (\Sigma M) \cong \Sigma e_1 \widetilde{\mathcal{R}}_1^- M.
\]

In particular, \( \widetilde{\mathcal{R}}_1(\Sigma M) \) is a sub \( A \)-module of \( \Sigma \mathcal{R}_1 M \).

Proof. Straightforward. \( \Box \)

6.3. Multiplicative properties. The following result is a straightforward consequence of the multiplicativity of \( St_1 \) together with the multiplicative structures introduced in Proposition 4.3.1.

Proposition 6.3.1. Let \( B \) be an algebra in \( \mathcal{M} \) and \( M \) be a \( B \)-module in \( \mathcal{M} \). Then

1. \( \widetilde{\mathcal{R}}_1 B \) has a natural \( K_1 \)-algebra structure in \( \mathcal{M} \) and \( \widetilde{\mathcal{R}}_1 M \) has a natural \( \widetilde{\mathcal{R}}_1 B \)-module structure in \( \mathcal{M} \);
2. \( R_1 B \) has a natural \( K_1^+ \)-algebra structure in \( \mathcal{M} \) and \( R_1 M \) has a natural \( \widetilde{\mathcal{R}}_1 B \)-module structure in \( \mathcal{M} \).

Let \( K \) be an unstable algebra and \( N \) be an object of \( K-\mathcal{U} \). Then

1. \( \widetilde{\mathcal{R}}_1 K \) has a natural \( K_1 \)-algebra structure in unstable algebras and \( \widetilde{\mathcal{R}}_1 N \) is naturally an object of \( \widetilde{\mathcal{R}}_1 K-\mathcal{U} \);
2. \( R_1 K \) has a natural \( K_1^+ \)-algebra structure in unstable algebras and \( R_1 N \) is naturally an object of \( R_1 K-\mathcal{U} \).

Proof. The first statement is proved by verifying that \( \widetilde{\mathcal{R}}_1 B \hookrightarrow \Phi_1 \otimes B \) is a monomorphism in \( K_1-\mathcal{M} \) and that \( \widetilde{\mathcal{R}}_1 M \hookrightarrow \Phi_1 \otimes M \) is a monomorphism of \( \widetilde{\mathcal{R}}_1 B \)-modules, with respect to the products introduced in Proposition 4.3.1; this depends upon the multiplicative properties of \( St_1 \). The second statement follows on restricting to \( GL_1 \)-invariants. The versions for unstable algebras and modules over them are straightforward consequences. \( \Box \)

The following result is used in the analysis of the functors \( \widetilde{\mathcal{R}}_s \) in Section 7.

Lemma 6.3.2. Let \( K \) be an unstable algebra and \( M \) be a \( K \)-module in \( \mathcal{M} \). If \( V \leq M \) is a sub graded vector space such that the induced morphism of \( K \)-modules

\[
K \otimes V \to M
\]

is surjective, then

1. \( \widetilde{\mathcal{R}}_1 M \) is a \( \widetilde{\mathcal{R}}_1 K \)-module in \( \mathcal{M} \);
2. \( St_1 V \) is a sub graded vector spaces of \( \widetilde{\mathcal{R}}_1 M \) and the induced morphism of \( \widetilde{\mathcal{R}}_1 K \)-modules

\[
\widetilde{\mathcal{R}}_1 K \otimes St_1 V \to \widetilde{\mathcal{R}}_1 M
\]

is surjective.

Proof. The \( \widetilde{\mathcal{R}}_1 K \)-module structure is given by Proposition 6.3.1. The verification that \( St_1 V \) is a sub graded vector space of \( St_1 M \) is elementary and the surjectivity of the induced morphism follows from the multiplicative properties of \( St_1 \). \( \Box \)
6.4. The transformation $\rho_1$ and the fundamental short exact sequence.

For $M \in \text{Ob } \mathcal{M}$, there is a natural transformation in $K^+_1 - \mathcal{M}$:

$$\Sigma^{-1} \mathcal{H}_1(\Sigma M) \rightarrow \mathcal{H}_1 M.$$ 

The cokernel identifies, as a graded vector space, with $\text{St}_{1}(\text{ev}) \oplus \tilde{M}_{1,0} \text{St}_{1}(\text{odd})$, by Proposition 6.2.3. This cokernel can be analysed in $\mathcal{M}$ by using the natural transformation $\rho_1$, generalizing the transformation defined by Zarati [Zar84, Définition-Proposition 3.3.7] for unstable modules.

**Definition 6.4.1.** Let $\rho_1 : \mathcal{H}_1 M \rightarrow \Sigma^{-2} \Phi \Sigma M$ be the linear natural transformation which factors over the cokernel of $\Sigma^{-1} \mathcal{H}_1(\Sigma M) \rightarrow \mathcal{H}_1 M$, defined on generators by

$$\text{St}_{1}(m) \mapsto -\Sigma^{-2} \Phi(\Sigma m),$$

$$\tilde{M}_{1,0} \text{St}_{1}(n) \mapsto \Sigma^{-2} \Phi(\Sigma n),$$

where $m \in \text{St}_{1}(\text{ev})$ and $n \in \text{St}_{1}(\text{odd})$.

**Remark 6.4.2.** A sign change has been introduced here which accords better with the application of this morphism.

This allows the introduction of the fundamental short exact sequence for the functor $\mathcal{H}_1$.

**Proposition 6.4.3.** For $M \in \text{Ob } \mathcal{M}$, there is a natural short exact sequence in $\mathcal{M}$:

$$0 \rightarrow \Sigma^{-1} \mathcal{H}_1(\Sigma M) \rightarrow \mathcal{H}_1 M \overset{\rho_1}{\rightarrow} \Sigma^{-2} \Phi \Sigma M \rightarrow 0.$$ (1)

**Proof.** The surjectivity of $\rho_1$ is clear. The main point of the result is to show that $\rho_1$ is $\mathcal{A}$-linear; this is a straightforward calculation, using the method of proof of Proposition 6.1.3 derived from [HS95, Proposition 4.2.10]. Namely, (using the notation of the proof of Proposition 6.1.3) the only terms of $P^j \text{St}_{1}(x)$ which are non-zero correspond to the cases:

(1) $\epsilon = 0$, $|x|$ even and $i = pt$;

(2) $\epsilon = 1$, $|x|$ odd and $i = pt + 1$.

□

**Remark 6.4.4.** The proof of the $\mathcal{A}$-linearity can also be reduced to the case $M$ unstable, which is established in [Zar84], by a different argument.

6.5. The Singer evaluation. Whilst the material introduced above generalizes the constructions of Zarati [Zar84] to the case of general $\mathcal{A}$-modules, the differential introduced below is invisible in the unstable module case.

Writing $H^*(BV_1) \cong \Lambda(u) \otimes F_p[v]$, recall that $\Phi_1$ is the algebra $\Lambda(w) \otimes F_p[v^{\pm 1}]$, where $w = \frac{u^2}{v}$ has degree $-1$. There is a linear morphism

$$\partial : \Phi_1 \rightarrow \Sigma^{-1} F_p$$

sending $w$ to the canonical generator, and it is a fundamental result of Singer (cf. [LS82, Proposition 2.2] for $p$ odd) that $\partial$ is $\mathcal{A}$-linear.

**Definition 6.5.1.** For $M \in \text{Ob } \mathcal{M}$, let $d_M : \mathcal{H}_1 M \rightarrow \Sigma^{-1} M$ be the natural transformation defined by the composite:

$$\mathcal{H}_1 M \mapsto \Phi_1 \otimes M \overset{\partial \otimes M}{\Rightarrow} \Sigma^{-1} M.$$ 

Recall from Section 2.1 the explicit construction of $\text{DM}$ for $M \in \text{Ob } \mathcal{M}$ as the quotient $\text{DM} \cong M/\sim M$. 


Proposition 6.5.2. For \( M \in \text{Ob} \, \mathcal{M} \), the cokernel of 
\[ d_M : \mathcal{E}_1 M \rightarrow \Sigma^{-1} M \]
is \( \Sigma^{-1} D M \).

Proof. \( \text{St}_1(x) \) can be written as 
\[ \text{St}_1(x) = \sum \pm w^i t_i^{x-2i} \otimes \beta^i P^i(x). \]

Consider a basis of \( \mathcal{E}_1 M \), which consists of elements of the form
1. \( |x| \) even: \( Q^n_{1\theta} \text{St}_1(x) \) or \( WQ^n_{1\theta+1} \text{St}_1(x) \)
2. \( |x| \) odd: \( \xi^{2n+1}_1 \text{St}_1(x) \) or \( W\xi^{2n+1}_1 \text{St}_1(x) \),
where \( n \geq 0 \) and \( x \) denotes a homogeneous element of \( M \).

Calculation gives:
1. \( |x| \) even, \( d(Q^n_{1\theta} \text{St}_1(x)) = (-1)^{n+1} \Sigma^{-1} P^{x+2n} \beta P^{x+2n+1} \)
2. \( |x| \) even, \( d(WQ^n_{1\theta+1} \text{St}_1(x)) = (-1)^{n+1} \Sigma^{-1} P^{x+2n+1} \beta P^{x+2n+2} \)
3. \( |x| \) odd, \( d(\xi^{2n+1}_1 \text{St}_1(x)) = (-1)^{n+1} \Sigma^{-1} P^{x+2n+1} \beta P^{x+2n+2} \)
4. \( |x| \) odd, \( d(W\xi^{2n+1}_1 \text{St}_1(x)) = (-1)^{n+1} \Sigma^{-1} P^{x+2n+1} \beta P^{x+2n+2} \).

It follows that the image of \( d \) is equal to \( \Sigma^{-1} BM \), whence the result. \( \square \)

6.6. The tail of the chain complex. The morphism \( \Sigma d_M : \Sigma \mathcal{E}_1 M \rightarrow M \) is the tail of the chain complex introduced in Section 9 for which the short exact sequence \( \xcancel{(1)} \) of Proposition 6.4.3 becomes a short exact sequence of chain complexes. A fundamental point is to understand the connecting morphism induced in homology.

Proposition 6.6.1. For \( M \in \text{Ob} \, \mathcal{M} \), there is a natural commutative diagram in \( \mathcal{M} \)
\[
\begin{array}{ccc}
\mathcal{E}_1 \Sigma M & \xrightarrow{\Sigma \text{St}_1} & \Sigma \mathcal{E}_1 M & \xrightarrow{\Sigma \text{P}_1} & \Sigma^{-1} \Phi \Sigma M \\
\downarrow{d_{\Sigma M}} & & \downarrow{\Sigma d_M} & & \downarrow{\Sigma^{-1} D(\Sigma M)} \\
M & = & M & & \\
\end{array}
\]
in which the three-term sequences are exact.

The induced connecting morphism
\[ \Sigma^{-1} \Phi \Sigma M \rightarrow \Sigma^{-1} D(\Sigma M) \]
is induced by \( \Sigma^{-1} \lambda_{\Sigma M} : \Sigma^{-1} \Phi(\Sigma M) \rightarrow M \).

Proof. Consider an element \( \Sigma^{-1} \Phi \Sigma m \) of \( \Sigma^{-1} \Phi \Sigma M \); if \( m \) is of even degree then, by the definition of \( \text{P}_1 \), \( \Sigma \text{St}_1(m) \in \Sigma \mathcal{E}_1 M \) is a lift; under the differential \( \Sigma d_M \), this maps to \( \beta P^{m+2}/2 \).
(As in the proof of Proposition 6.5.2.)

Similarly, if \( m \) is of odd degree, \( \Sigma \text{P}_1 \text{St}_1(m) \) is a lift; under the differential \( \Sigma d_M \), this maps to \( P^{(m+1)/2} \).

Now \( \Sigma m \in \Sigma M \) is of degree \( |m|+1 \). On passing to the cokernel, the morphism is well-defined and, by inspection, is induced by the morphism \( \Sigma^{-1} \lambda_{\Sigma M} \), as required. \( \square \)

7. The functors \( \mathcal{E}_s \) and \( \widetilde{\mathcal{E}}_s \)

The functors \( \mathcal{E}_1 \) and \( \widetilde{\mathcal{E}}_1 \) have analogues \( \mathcal{E}_s \) and \( \widetilde{\mathcal{E}}_s \) for all natural numbers, \( s \), so that \( \mathcal{E}_s \) is the appropriate generalization to \( \mathcal{M} \) of the functor \( R_s \) used by Zarati in [Zar81].
7.1. A general result. The following result is the key to the inductive understanding of the functors $\mathcal{A}_s$.

**Proposition 7.1.1.** Let $s$ be a natural number and $K \subset H^*(BV)$ a sub unstable algebra such that, for any $M \in \text{Ob } \mathcal{M}$, the sub $K$-module

$$KS_t M \subset \Phi \overline{\otimes} M$$

is $A$-stable.

Then,

1. the association $M \mapsto KS_t M$ defines an exact, weakly continuous, $k_s$-connective functor $\mathcal{M} \to K\mathcal{M}$, where $\kappa_s(n) = p^n n$;
2. the functor $\mathcal{A}_1(KSt_s(-))$ defines an exact, weakly continuous, $\kappa_{s+1}$-connective functor $\mathcal{M} \to \mathcal{A}_1 K\mathcal{M}$;
3. there is a natural inclusion $\mathcal{A}_1(KSt_s(-)) \hookrightarrow \Phi_{s+1} \overline{\otimes} M$ of functors with values in $\mathcal{A}_1 K\mathcal{M}$, which induces an identification

$$\mathcal{A}_1(KSt_s(M)) \cong (\mathcal{A}_1 K)St_{s+1} M.$$

**Proof.** The first statement is an elementary verification, using Proposition 5.7.1 to show that the functor is exact, since the underlying $K$-module is isomorphic to $K \otimes \Psi^s M$, which also makes the properties of this functor transparent.

The functor $\mathcal{A}_1$ is exact, weakly continuous and $\kappa_s$-connective. The properties of the composite functor $\mathcal{A}_1(KSt_s(-))$, considered as taking values in $\mathcal{M}$, follow from Proposition 7.3.4. The fact that the composite functor takes values in $\mathcal{A}_1 K\mathcal{M}$ follows from Proposition 7.3.1.

The functors $\mathcal{A}_1(KSt_s(-)), \Phi_{s+1} \overline{\otimes} M$ are both weakly continuous hence, to construct the natural transformation, it is sufficient to consider modules which are bounded above.

Consider the following diagram for an integer $c$

$$\xymatrix{ & \Phi_{s+1} \otimes M^{<c} \ar[d] & \Phi \overline{\otimes} (\Phi_s \otimes M^{<c}) \ar[d] \\
\mathcal{A}_1(KSt_s(M^{<c})) & \Phi_{s+1} \otimes M^{<c} 
}$$

in which the vertical morphism is induced by $KSt_s(M^{<c}) \subset \Phi_s \otimes M^{<c}$ and the horizontal morphism is provided by Proposition 14.1.2. Both morphisms are $\mathcal{A}$-linear and are morphisms of $\mathcal{A}_1 K$-modules (the horizontal one, by restriction along $\mathcal{A}_1 K \hookrightarrow H^*(BV) \hookrightarrow \Phi_{s+1} \hookrightarrow \Phi \overline{\otimes} \Phi$).

Consider the pullback $P$ of this diagram in the category of $\mathcal{A}_1 K$-modules; by Lemma 5.3.2 the image of $\mathcal{A}_1(KSt_s(M^{<c}))$ in $\Phi \overline{\otimes} (\Phi_s \otimes M^{<c})$ is generated as an $\mathcal{A}_1 K$-module by the subspace $St_s M^{<c}$. This subspace is the image of the subspace $St_{s+1} M^{<c}$ of $\Phi_{s+1} \otimes M^{<c}$. It follows that the pullback $P$ is isomorphic to $\mathcal{A}_1 K \otimes \Psi^{s+1} M^{<c}$ as an $\mathcal{A}_1 K$-module, using Proposition 5.7.1 to deduce the freeness; moreover, the morphism $P \to \mathcal{A}_1(KSt_s(M^{<c}))$ is an isomorphism.

The diagram shows that the image of $P$ in $\Phi_{s+1} \otimes M^{<c}$ is $A$-stable, since the horizontal morphism is $\mathcal{A}$-linear, and is sent bijectively to the image of $\mathcal{A}_1(KSt_s(M^{<c}))$ in $\Phi \overline{\otimes} (\Phi_s \otimes M^{<c})$. $\square$

7.2. The functors $\mathcal{A}_s$. Fix a choice of basis for $V$, giving an isomorphism $H^*(BV) \otimes \ldots \otimes H^*(BV) \cong H^*(BV)$ and hence a choice of monomorphism of unstable algebras:

$$\mathcal{A}_1^{\otimes \mathbb{P}_p} \hookrightarrow H^*(BV).$$
Proposition 7.2.1. Let $s$ be a natural number.

1. The functor $\tilde{\mathcal{A}}_{s}^{\circ} : \mathcal{A} \to \mathcal{A}$ is exact, weakly continuous and $\kappa_{s}$-connective, where $\kappa_{s}(n) = p^{n}$.
2. The functor $\tilde{\mathcal{A}}_{s}^{\circ} : \mathcal{A} \to \mathcal{A}$ takes values in the category $\tilde{\mathcal{A}}_{1}^{s}\mathbb{F}_{p}$-$\mathcal{A}$.
3. There is a natural transformation $\tilde{\mathcal{A}}_{s}^{\circ}M \hookrightarrow (\Phi \otimes -)^{s}M$ of functors from $\mathcal{A}$ to $\tilde{\mathcal{A}}_{1}^{s}\mathbb{F}_{p}$-$\mathcal{A}$.
4. There is a natural transformation $\tilde{\mathcal{A}}_{s}^{\circ}M \hookrightarrow \Phi \circ M$

of functors $\mathcal{A} \to \tilde{\mathcal{A}}_{1}^{s}\mathbb{F}_{p}$-$\mathcal{A}$, where the right hand side is an $\tilde{\mathcal{A}}_{1}^{s}\mathbb{F}_{p}$-module via restriction of the natural $\Phi_{s}$-module structure along the composite $\tilde{\mathcal{A}}_{1}^{s}\mathbb{F}_{p} \to H^{*}(BV_{s}) \to \Phi_{s}$.
5. The morphism $\text{St}_{s}$ induces an isomorphism of $\tilde{\mathcal{A}}_{1}^{s}\mathbb{F}_{p}$-modules

as a submodule of $\Phi \circ M$.

Proof. The result follows by induction on $s$ by applying Proposition 7.2.1 the case $s = 0$ being clear. The freeness statement follows from Proposition 5.7.1. □

Definition 7.2.2. For $s$ a natural number and $M \in \text{Ob} \mathcal{A}$, let

1. $K_{s}$ denote the unstable algebra

$K_{s} = \tilde{\mathcal{A}}_{1}^{s}\mathbb{F}_{p} \cap H^{*}(BV_{s})^{SL_{s}}$;

2. $\tilde{\mathcal{A}}_{s}M$ denote the sub $K_{s}$-module $\tilde{\mathcal{A}}_{s}^{\circ}M \cap \Phi \circ M$.

Remark 7.2.3.

1. It is straightforward to show that, for $s > 0$, $K_{s} \cong \tilde{\mathcal{A}}_{1}K_{s-1} \cap H^{*}(BV_{s})^{SL_{s}}$ and the subalgebra $K_{s}$ of $H^{*}(BV_{s})$ is independent of the basis of $H^{1}(BV_{s})$ used in the construction.
2. By convention, the functor $\tilde{\mathcal{A}}_{0}$ is the identity.

Proposition 7.2.4. Let $s$ be a natural number.

1. The construction $M \mapsto \tilde{\mathcal{A}}_{s}M$ defines a functor $\tilde{\mathcal{A}}_{s} : \mathcal{A} \to K_{s}$-$\mathcal{A}$, equipped with a natural transformation $\tilde{\mathcal{A}}_{s}M \hookrightarrow \Phi \circ M$ of functors with values in $K_{s}$-$\mathcal{A}$.
2. The underlying $K_{s}$-module of $\tilde{\mathcal{A}}_{s}M$ is isomorphic to $K_{s} \otimes \Psi^{s}M$.
3. The functor $\tilde{\mathcal{A}}_{s} : \mathcal{A} \to \mathcal{A}$ is exact, commutes with colimits, is weakly continuous and is $\kappa_{s}$-connective, where $\kappa_{s}(n) = p^{n}$.

Proof. A straightforward consequence of Proposition 7.2.1. □

Corollary 7.2.5. Let $s$ be a natural number and $M$ be an $A$-module. The subobject $\tilde{\mathcal{A}}_{s}M$ of $\Phi \circ M$ is independent of the choice of basis used in defining the embedding $\tilde{\mathcal{A}}_{s}^{\circ}M \hookrightarrow \Phi \circ M$.

Proof. By Proposition 7.2.1, $\tilde{\mathcal{A}}_{s}^{\circ}M$ is generated as an $\tilde{\mathcal{A}}_{1}^{s}\mathbb{F}_{p}$-$\mathcal{A}$-module by $\text{St}_{s}M$. The morphism $\text{St}_{s}$ only depends on $\varepsilon_{s}$, which is well-defined up to sign, and the subalgebra $K_{s} = \tilde{\mathcal{A}}_{1}^{s}\mathbb{F}_{p} \cap H^{*}(BV_{s})^{SL_{s}}$ is independent of choices, the result follows. □

Corollary 7.2.6. Let $s, t$ be natural numbers.

1. The functor $\tilde{\mathcal{A}}_{s} \circ \tilde{\mathcal{A}}_{t}$ is exact, weakly continuous and $\kappa_{s+t}$-connective.
(2) There is a natural monomorphism $\mathcal{R}_{s+t} \hookrightarrow \mathcal{R}_s \circ \mathcal{R}_t$.

Proof. The first statement follows as in Proposition [2.2.3]. For the second, it suffices to check that the defining monomorphism $\mathcal{R}_{s+t} \hookrightarrow \mathcal{R}_s \circ \mathcal{R}_t$ admits a factorization as indicated. This is straightforward. □

Proposition 7.2.7. Let $M, M_1, M_2 \in \text{Ob } \mathcal{M}$ be $\mathcal{A}$-modules. There are natural isomorphisms

$$\mathcal{R}_s(M_1 \otimes M_2) \cong \mathcal{R}_s(M_1) \otimes_{K_s} \mathcal{R}_s(M_2)$$

$$\mathcal{R}_s(\Sigma M) \cong \Sigma \mathcal{R}_s M.$$

in $K_s$-$\mathcal{M}$.

Proof. Consequence of Proposition [7.2.6] (compare the proof of Proposition [6.1.0]). □

7.3. The functors $\mathcal{R}_s$. As in the case $s = 1$, the functor $\mathcal{R}_s$ is constructed by an eigenspace splitting. The action of $\mathbb{Z}/2$ on $H^*(BV_s)^{GL_s}$ induces an action of $\mathbb{Z}/2$ on $K_s$ by morphisms of unstable algebras, hence an eigenspace decomposition

$$K_s \cong K^+_s \oplus K^-_s$$

in the category $K^+_s$-$\mathcal{M}$ (compare Lemma [2.4.2]). Here, $K^+_s$ can be identified with the unstable algebra $\mathcal{R}_1K_{s-1} \cap H^*(BV_s)^{GL_s}$.

Notation 7.3.1. Write $(\Phi^{SL_s})^-$ for the sub $\Phi^{GL_s}$-module $K^-_s[Q_{s,0}^{-1}]$ of $\Phi^{SL_s}$.

Definition 7.3.2. Let $s$ be a natural number. For $M \in \text{Ob } \mathcal{M}$, let

1. $\mathcal{R}^+_s M$ denote the sub $K^+_s$-module $\mathcal{R}_s M \cap \Phi^{GL_s}_s \circ M$;
2. $\mathcal{R}^-_s M$ denote the sub $K^-_s$-module $\mathcal{R}_s M \cap (\Phi^{SL_s})^- \circ M$.

Remark 7.3.3. The functor $\mathcal{R}_0$ is the identity and $\mathcal{R}_0^-$ is the zero functor.

Proposition 7.3.4. Let $s$ be a natural number.

1. The associations $M \mapsto \mathcal{R}^+_s M, M \mapsto \mathcal{R}^-_s M$ define functors $\mathcal{M} \to K^+_s$-$\mathcal{M}$;
2. There is a canonical natural monomorphism of functors $\mathcal{M} \to K^+_s$-$\mathcal{M}$:

$$\mathcal{R}_s(-) \hookrightarrow \Phi^{GL_s}_s \circ (-).$$

Proof. The first statement is clear; the second is a consequence of the restriction to $GL_s$-invariants, which implies that the constructions are independent of choices. □

Proposition 7.3.5. Let $s$ be a natural number and $M$ be an $\mathcal{A}$-module. There is a natural direct sum decomposition:

$$\mathcal{R}_s M \cong \mathcal{R}^+_s M \oplus \mathcal{R}^-_s M$$

in $K^+_s$-$\mathcal{M}$. Moreover, as modules over $K^+_s$,

$$\mathcal{R}^+_s M \cong K^+_s \text{St}_s(M^{ev}) \oplus K^-_s \text{St}_s(M^{odd})$$

$$\mathcal{R}^-_s M \cong K^-_s \text{St}_s(M^{ev}) \oplus K^+_s \text{St}_s(M^{odd}).$$

Proof. Straightforward. □

Remark 7.3.6. This recovers, in the case $M$ unstable, the definition of the functor $R_s$, given by Zarati [Zar84, Définition 2.4.5], once the unstable algebra $K_s$ has been identified.

Corollary 7.3.7. For $s$ a natural number, the functor $\mathcal{R}_s$ is exact, commutes with colimits, is weakly continuous and is $K_s$-connective.
Proof. Follows from Proposition 7.2.4 and Proposition 7.3.5.

**Proposition 7.3.8.** For $M \in \mathcal{M}$, there are natural isomorphisms of $K_+^s$-modules:

$$\mathcal{R}_s(\Sigma M) \cong \Sigma \mathcal{R}_s M$$
$$\mathcal{R}^+_s(\Sigma M) \cong \Sigma \mathcal{R}^+_s M.$$  

In particular, $\mathcal{R}_s(\Sigma M)$ is a sub $A$-module of $\Sigma \mathcal{R}^+_s M$.

Proof. Straightforward.

**Proposition 7.3.9.** Let $M \in \text{Ob } \mathcal{M}$.

1. Let $s, t$ be natural numbers. There is a canonical embedding

$$\mathcal{R}_s tM \hookrightarrow \mathcal{R}_s M,$$

hence a canonical embedding $\mathcal{R}_s M \hookrightarrow \mathcal{R}_s^t M$.

2. Let $s \geq 2$ be an integer; as submodules of $\mathcal{R}_s^t M$, there is an isomorphism

$$\mathcal{R}_s M \cong \bigcap_{i+j=s-2} \mathcal{R}_i \mathcal{R}_j \mathcal{R}_1 M.$$

Proof. There is a natural embedding $\tilde{\mathcal{R}}_{s+t} M \hookrightarrow \tilde{\mathcal{R}}_{s+t} M$; the right hand side decomposes as

$$\mathcal{R}_m \mathcal{R}_n M \oplus \tilde{\mathcal{R}}_m \mathcal{R}_n M \oplus \tilde{\mathcal{R}}_m \mathcal{R}_n M \oplus \mathcal{R}_m \mathcal{R}_n M$$

and it is clear that the restriction of the embedding to $\mathcal{R}_m+n M$ maps canonically to the first factor. By induction, there is a canonical embedding $\mathcal{R}_s M \hookrightarrow \mathcal{R}_s^t M$.

By Proposition 7.2.1, $\mathcal{R}_s^t M$ is a submodule of $\Phi \oplus M$. The second statement is a consequence of the corresponding result for $GL_s$-invariants:

$$H^*(BV_s)^{GL_s} \cong \bigcap_{i+j=s-2} H^*(BV_s)^{GL_s^{i+j}}$$

where, with respect to a choice of basis of $V_s$, $GL_2 \cong GL_2^{i+j} \subset GL_s$ is induced by the direct sum decomposition corresponding to the basis $V_s \cong F_p^{i+j} \oplus F_p^{i+j} \oplus F_p^{i+j}$.

□

The following result is required in studying the properties of the chain complex introduced in Section 9.

**Corollary 7.3.10.** Let $s, t$ be natural numbers and $M \in \text{Ob } \mathcal{M}$, then

1. there is a canonical natural monomorphism $\mathcal{R}_s \hookrightarrow \mathcal{R}_s^t$ of functors taking values in $K_+^s$-$\mathcal{M}$;

2. for $M \in \text{Ob } \mathcal{M}$, the following diagram of natural inclusions commutes:

$$\begin{array}{ccc}
\mathcal{R}_{s+2} M & \longrightarrow & \mathcal{R}_s \mathcal{R}_{s+1} M \\
\downarrow & & \downarrow \\
\mathcal{R}_{s+1} \mathcal{R}_1 M & \longrightarrow & \mathcal{R}_s \mathcal{R}_1 \mathcal{R}_1 M
\end{array}$$

and is cartesian.

Proof. The first statement strengthens the assertion of Proposition 7.3.9 to the statement that the canonical morphism $\mathcal{R}_s \hookrightarrow \mathcal{R}_s^t$ is a morphism of $K_+^s$-$\mathcal{M}$. This follows from the fact that the morphism $K_+^s \hookrightarrow H^*(BV_s)$ is basis-independent, since $K_+^s$ is a subalgebra of $H^*(BV_s)^{GL_s}$.

The second statement follows from the second part of Proposition 7.3.9, which implies directly that the square is cartesian. □
7.4. Identifying the unstable algebras $K_s$. Recall that the regular representation $V_s \hookrightarrow \mathfrak{S}_p$ factors canonically across the inclusion $\mathfrak{A}_p^+ \subset \mathfrak{S}_p$ of the alternating group in the symmetric group.

**Theorem 7.4.1.** ([Mùi75], I.4.14 and [Mùi86],) Let $s$ be a positive integer. The elements $\{\tilde{M}_{s,i}|0 \leq i \leq s - 1\} \subset H^*(BV_s)$ are $SL_s$-invariant and, together with the elements $\{e_s, Q_{s,j}|0 \leq j \leq s - 1\}$ generate a free graded commutative algebra

$$\Lambda(\tilde{M}_{s,i}|0 \leq i \leq s - 1) \otimes \mathbb{F}_p[e_s, Q_{s,j}|1 \leq j \leq s - 1] \subset H^*(BV_s)$$

which is a sub unstable algebra isomorphic to the image of the restriction morphism

$$H^*(B\mathfrak{A}_p^+) \rightarrow H^*(BV_s).$$

Moreover, after localization, there is an isomorphism

$$\Phi^s_{SL_s} \cong \Lambda(\tilde{M}_{s,i}|0 \leq i \leq s - 1) \otimes \mathbb{F}_p[e_s, Q_{s,j}|1 \leq j \leq s - 1][Q_{s,0}^{-1}].$$

**Proposition 7.4.2.** (Cf. [Zar84] Proposition 4.1.2,.) Let $s$ be a natural number. As sub unstable algebras of $H_*(BV_s)$:

$$K_s \cong \Lambda(\tilde{M}_{s,i}|0 \leq i \leq s - 1) \otimes \mathbb{F}_p[e_s, Q_{s,j}|1 \leq j \leq s - 1]$$

is a consequence of [Mùi86] Lemma 3.11, as follows.

For the purposes of this proof, define a functor $\tilde{\mathcal{A}}_1 : \mathcal{M} \rightarrow H^*(B V_1).\mathcal{M}$ by extension of scalars

$$\tilde{\mathcal{A}}_1 M := H^*(B V_1) \otimes_{K_1} \tilde{\mathcal{A}}_1 M.$$ 

It is clear that $\tilde{\mathcal{A}}_1$ restricts to an endofunctor of $\mathcal{M}$ which has good multiplicative properties. In particular, if $K$ is an unstable algebra, then $\tilde{\mathcal{A}}_1 K$ is an unstable algebra and there is an inclusion of unstable algebras $\tilde{\mathcal{A}}_1 K \hookrightarrow \tilde{\mathcal{A}}_1 K$.

In [Mùi86, Theorem 3.9], Mui defines an algebra $\mathcal{M}_p(s)$ (for $s$ a natural number) which is a free graded algebra

$$\mathcal{M}_p(s) \cong \Lambda(U_1, \ldots, U_s) \otimes \mathbb{F}_p[V_1, \ldots, V_s]$$

on explicit generators defined in [Mùi86, Section 2]. Combining [Mùi86] Proposition 2.6 with [Mùi86, Theorem 3.8], one can show that $\mathcal{M}_p(s)$ is isomorphic to $(\tilde{\mathcal{A}}_1^s)^{op}\mathbb{F}_p$ (hence is an unstable subalgebra of $H^*(BV_s)$). In particular, there is an inclusion of unstable algebras $\tilde{\mathcal{A}}_1^s \mathbb{F}_p \hookrightarrow \mathcal{M}_p(s)$. By [Mùi86, Lemma 3.11]

$$\mathcal{M}_p(s) \cap H^*(BV_s)_{SL_s} \cong \Lambda(\tilde{M}_{s,i}|0 \leq i \leq s - 1) \otimes \mathbb{F}_p[e_s, Q_{s,j}|1 \leq j \leq s - 1],$$

hence this establishes the required inclusion.

**Remark 7.4.3.** The result can be proved directly by using the techniques of Mui, as in the proof of [Mùi86] Lemma 3.11.

The following identification of the algebra $K_s^+$ is required below.

**Notation 7.4.4.** Let $s$ be a positive integer.
In particular, there is a natural inclusion of \( K \). Straightforward.

\( \rho \) of unstable algebras: (1) For \( I \subset \{0, \ldots, s-1\} \), let \( \tilde{M}_{s,I} \) denote the monomial \( \prod_{j \in I} \tilde{M}_{s,j} \) (the factors in the product are ordered by the natural order on \( I \)).

Lemma 7.4.5. Let \( s \) be a natural number. Then

(1) \( K^+ \) is a free \( F \) module on \( \{\epsilon_s \tilde{M}_{s,f^{-ad}}, \tilde{M}_{s,f^{-ev}}\} \);

(2) \( K^- \) is a free module on \( \{\epsilon_s \tilde{M}_{s,f^{-ev}}, \tilde{M}_{s,f^{-ad}}\} \).

Proof. Straightforward.

7.5. The higher natural transformation \( \rho_s \).

Definition 7.5.1. For \( M \in \text{Ob } \mathcal{M} \) and \( s \) a positive integer, let \( \rho_s : \mathcal{R}_s M \to \Sigma^{-2}\Phi \Sigma \mathcal{R}_{s-1}M \) denote the natural transformation in \( \mathcal{M} \) given by the composite:

\[
\mathcal{R}_s M \hookrightarrow \mathcal{R}_s \mathcal{R}_{s-1}M \rightarrow \Sigma^{-2}\Phi \Sigma \mathcal{R}_{s-1}M.
\]

The following result allows module structures to be used in studying the morphism \( \rho_s \).

Lemma 7.5.2. Let \( K \rightarrow \) an unstable algebra concentrated in even degrees and \( M \) a module over \( K \) in \( \mathcal{M} \). Then \( \Sigma^{-2}\Phi \Sigma M \) has a natural \( \Phi K \)-module structure defined by

\[(\Phi k)(\Sigma^{-2}\Phi \Sigma m) = \Sigma^{-2}\Phi \Sigma(km).
\]

In particular, there is a natural inclusion of \( K \)-modules \( \Phi K \hookrightarrow \Sigma^{-2}\Phi \Sigma K \).

Proof. Straightforward.

The algebra of Dickson invariants \( F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \) is a sub unstable algebra of \( K^+ \subset H^*(BV_\omega) \). The following is well-known:

Lemma 7.5.3. Let \( s \) be a positive integer and \( i : V_{s-1} \hookrightarrow V_s \) be an inclusion of elementary abelian \( p \)-groups. There is a commutative diagram of morphisms of unstable algebras:

\[
\begin{array}{ccc}
F_p[Q_{s,0}, \ldots, Q_{s,s-1}] & \xrightarrow{\varphi_s} & F_p[Q_{s-1,0}, \ldots, Q_{s-1,s-2}] \\
\downarrow & & \downarrow \\
H^*(BV_\omega) & \xrightarrow{\iota} & H^*(BV_{s-1})
\end{array}
\]

where

\[
\varphi_s Q_{s,j} = \begin{cases} 0 & \text{if } j = 0 \\ Q_{s-1,j-1} & \text{if } j > 0. \end{cases}
\]

In particular, \( \varphi_s \) factorizes as:

\[
F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \rightarrow \Phi F_p[Q_{s-1,0}, \ldots, Q_{s-1,s-2}] \hookrightarrow F_p[Q_{s-1,0}, \ldots, Q_{s-1,s-2}].
\]

Recall that, if \( N \) is a \( K^+ \)-module in \( \mathcal{M} \), then \( \mathcal{R}_1 N \) is naturally a \( \mathcal{R}_1 K^+ \)-module in \( \mathcal{M} \), by Proposition 5.3.1.

Lemma 7.5.4. Let \( N \) be an object of \( K^+ \). Then \( \rho_1 : \mathcal{R}_1 N \to \Sigma^{-2}\Phi \Sigma N \) is a morphism of \( F_p[Q_{s+1,i}] \)-modules, where \( \mathcal{R}_1 N \) is a \( F_p[Q_{s+1,i}] \)-module by restriction along \( F_p[Q_{s+1,i}] \to \mathcal{R}_1 K^+ \) and \( \Sigma^{-2}\Phi \Sigma N \) is a module by restriction of the \( \Phi F_p[Q_{s,j}] \)-module structure provided by Lemma 7.5.3 along the projection \( F_p[Q_{s+1,i}] \to F_p[Q_{s,j}] \) of Lemma 7.5.3.
Proof. The result follows by analysing the $\mathbb{F}_p[Q_{s+1,i}]$-module structure on $\mathcal{R}_1N$ by using the explicit form of the inclusion $\mathbb{F}_p[Q_{s+1,i}] \subset \mathcal{R}_1K^+$, which is given by Lemma 5.2.3. Since the ideal generated by $Q_{1,0}$ is killed by the morphism $\rho_1$, the result follows by inspection. □

The following result can be compared with [Zar84, Sections 4.3.2 and 4.6].

**Lemma 7.5.5.** Let $\rho_{s+1} : \mathcal{R}_{s+1}M \to \Sigma^{-2}\Phi\Sigma\mathcal{R}_sM$

is a morphism of $\mathbb{F}_p[Q_{s+1,i}|0 \leq i \leq s]$-modules.

Proof. Apply the previous result to the case $N = \mathcal{R}_sM$. □

**Proposition 7.5.6.** (Cf. [Zar84, Définition-Proposition 4.5.1].) For $M \in \text{Ob} \mathcal{M}$ and $s$ a positive integer, there is a natural short exact sequence of $\mathcal{A}$-modules:

$$0 \to \Sigma^{-1}\mathcal{R}_s(\Sigma M) \to \mathcal{R}_sM \to \Sigma^{-2}\Phi\Sigma\mathcal{R}_{s-1}M \to 0.$$  

Proof. There is a commutative diagram of exact sequences,

$$0 \to \Sigma^{-1}\mathcal{R}_s(\Sigma M) \to \mathcal{R}_sM \xrightarrow{\rho_s} \Sigma^{-2}\Phi\Sigma\mathcal{R}_{s-1}M \to 0,$$

where the top row is exact since the left hand square is a pullback (this can be checked directly). Hence it remains to show that the morphism $\rho_s$ is surjective; for this, recall the identification of $K_s^+$ and $K_s^-$ given in Lemma 7.4.3.

The cokernel of $\Sigma^{-1}\mathcal{R}_sM \to \mathcal{R}_sM$ is a free $\mathbb{F}_p[Q_{s,j}|1 \leq j \leq s-1]$-module on elements

$$\tilde{M}_{s,j-1}\text{St}_s(m^e)$$

$$\tilde{M}_{s,j-1}\text{St}_s(m^{od}),$$

as $m$ runs through a basis of $M$.

The elements $\tilde{M}_{s,j}$ can be written respectively (according to the parity of $|J|$) in one of the following forms: $\tilde{M}_{s,0}\tilde{M}_{s,j-odd}$ or $\tilde{M}_{s,j-odd}$, where $J \subset \{1, \ldots, s-1\}$ (respectively $\tilde{M}_{s,0}\tilde{M}_{s,j-even}$ or $\tilde{M}_{s,j-even}$). For $J \subset \{1, \ldots, s-1\}$, write $\tilde{J}$ for the subset of $\{0, \ldots, (s-1)-1\}$ which is induced by the order preserving surjection $\{1, \ldots, s-1\} \to \{0, \ldots, (s-1)-1\}$.

The multiplicitivity of $\text{St}_s$ together with the formulae of Corollary 5.3.5 give:

$$\tilde{M}_{s,0}\tilde{M}_{s,j-odd}\text{St}_s(m^{even}) \Rightarrow \pm \Sigma^{-2}\Phi\Sigma\mathcal{R}_{s-1-1, j-odd}\text{St}_{s-1}(m^{even})$$

$$\tilde{M}_{s,j-even}\text{St}_s(m^{even}) \Rightarrow \pm \Sigma^{-2}\Phi\Sigma\tilde{M}_{s-1-1, j-even}\text{St}_{s-1}(m^{even})$$

$$\tilde{M}_{s,0}\tilde{M}_{s,j-odd}\text{St}_s(m^{odd}) \Rightarrow \pm \Sigma^{-2}\Phi\Sigma\mathcal{R}_{s-1-1, j-odd}\text{St}_{s-1}(m^{odd})$$

$$\tilde{M}_{s,j-even}\text{St}_s(m^{odd}) \Rightarrow \pm \Sigma^{-2}\Phi\Sigma\tilde{M}_{s-1-1, j-even}\text{St}_{s-1}(m^{odd}).$$

These identifications imply the surjectivity of $\rho_s$, by using the module structures provided by Lemma 7.5.5. □

8. The differential

8.1. **Defining the differential.** A similar approach to the definition of $\rho_s$ from $\rho_1$ gives a natural transformation which generalizes $d : \mathcal{R}_1\Sigma M \to M$.  

Definition 8.1.1. Let \( d_s : R_s \Sigma M \rightarrow R_{s-1} M \) denote the composite natural transformation
\[
R_s \Sigma M \rightarrow R_{s-1} (R_1 \Sigma M) \xrightarrow{d_{s \Sigma M}} R_{s-1}M,
\]
which is a morphism of \( A \)-modules.

Remark 8.1.2. A choice has been made in the above definition, using \( d_{\Sigma M} \) rather than \( \Sigma d_M \). Upon restricting to \( GL_s \)-invariants, the same morphism is obtained (for \( s > 1 \)).

8.2. Coproduct morphisms for \( \Phi_{sGL}^* \). The relationship between the dual Steenrod algebra and the algebras \( \Phi_{sGL}^* \) via the morphisms \( \theta_s \) introduced in Definition 5.4.4 gives the natural way to introduce coproduct morphisms on the localized invariant rings; these are important in the work of Singer [Sin83] and [HS95] and are used to give the explicit identification of the differential.

Notation 8.2.1. For a natural number, let \( \Gamma_s \) denote the algebra \( \Phi_{sGL}^* \), which is isomorphic to \( \Lambda(R_s, [0 \leq i < s]) \otimes F_p[q_s, 0, q_s, 1, \ldots, q_s, s-1] \).

Proposition 8.2.2. Let \( s, t \) be positive integers. There exists a unique morphism of algebras \( \psi_{s,t} : \Gamma_{s+t} \rightarrow \Gamma_s \otimes \Gamma_t \) such that
\[
\begin{aligned}
A^* &\xrightarrow{\Delta} A^* \otimes A^* \\
\theta_{s+t} &\xrightarrow{\psi_{s,t}} \theta_s \otimes \theta_t \\
\Gamma_{s+t} &\xrightarrow{\Gamma_s \otimes \Gamma_t}
\end{aligned}
\]
where \( \Delta \) is the coproduct of the dual Steenrod algebra.

The morphism \( \psi_{s,t} \) is given explicitly by:
\[
\begin{aligned}
Q_{s+t,i} &\mapsto \sum_j Q_{s+t-j}^{p^i} Q_{s,j}^{p^j} \otimes Q_{t,j} \\
R_{s+t,i} &\mapsto R_{s+t,0} Q_{s+t,0}^{p^i-1} \otimes Q_{t,0} + \sum_{j} Q_{s+t-j}^{p^j} Q_{s,t-j}^{p^i} \otimes R_{t,j}.
\end{aligned}
\]

Proof. Straightforward: the morphism \( \psi_{s,t} \) is determined uniquely by the coproduct \( \Delta \). (Compare [HS95, Proposition 3.3].)

Corollary 8.2.3. The structure \((\Gamma_*, \psi_*, *)\) defines an augmented graded coalgebra, where the elements of \( \Gamma_s \) have grading \( s \).

Proof. The result follows from the coassociativity of the diagonal \( \Delta \) of the dual Steenrod algebra \( A^* \).

Remark 8.2.4. The above result should be compared with [HS95, Corollary 3.4]. Moreover, \((\Gamma_*, \psi_*, *)\) has the structure of a graded bialgebra equipped with an internal grading (cf. [Sin05]).

The following morphisms arise naturally in considering the composite linear morphisms \( S_s \circ S_t \).

Definition 8.2.5. For \( s, t \) positive integers, let \( \tilde{\psi}_{s,t} : \Gamma_{s+t} \rightarrow \Gamma_s \otimes \Gamma_t \) denote the composite morphism of algebras
\[
\begin{aligned}
\Gamma_{s+t} &\xrightarrow{\psi_{s,t}} \Gamma_s \otimes \Gamma_t \\
&\xrightarrow{S_s} \Gamma_s \otimes \Gamma_t
\end{aligned}
\]
where \( S_s \) denotes the morphism of \( \Gamma_s \)-algebras induced by \( S_s : \Gamma_t \rightarrow \Gamma_s \otimes \Gamma_t \).
8.3. A general framework. The identification of the differential in terms of the above morphisms $\tilde{\psi}_{s,t}$ can be considered in a general framework; this is presented here in a simplified setting (without large tensor products). Throughout we work in the category of graded $\mathbb{F}_p$-vector spaces, considered as a symmetric monoidal category with respect to the graded tensor product and braiding $\tau$ involving Koszul signs. Commutativity is understood within this symmetric monoidal structure.

Let $H$ be a commutative bialgebra and let $\{A_s | s \in \mathbb{N}\}$ be a set of commutative algebras equipped with structure morphisms $\psi_{s,t} : A_{s+t} \to A_s \otimes A_t$ which satisfy the evident notion of coassociativity. Suppose furthermore that there are algebra morphisms $\{H \theta_s : A_s | s \in \mathbb{N}\}$ such that the diagrams

$$
\begin{array}{ccc}
H & \longrightarrow & H \otimes H \\
\downarrow & & \downarrow \\
A_{s+t} & \longrightarrow & A_s \otimes A_t
\end{array}
$$

commute for all $(s, t) \in \mathbb{N}^2$.

The constructions below are based on the category of right $H$-comodules, using the following hypothesis:

Hypothesis 8.3.1. For each $s \in \mathbb{N}$, the algebra $A_s$ is a right $H$-comodule algebra (a right $H$-comodule such that the structure morphism is a morphism of algebras).

Definition 8.3.2. For $M$ a right $H$-comodule and $s$ a natural number, let $S_s : M \to A_s \otimes M$ denote the linear morphism defined as the composite

$$
M \longrightarrow M \otimes H \xrightarrow{M \otimes \theta_s} M \otimes A_s \xrightarrow{\tau} A_s \otimes M,
$$

where the first morphism is the comodule structure morphism.

Recall that, if $M, N$ are right $H$-comodules, then $M \otimes N$ has a natural right $H$-comodule structure, induced by the multiplication of $H$. This has the following counterpart.

Lemma 8.3.3. Let $M, N$ be right $H$-comodules and $s$ a natural number, then the morphism $S_s : M \otimes N \to A_s \otimes M \otimes N$ identifies with the composite:

$$
M \otimes N \xrightarrow{S_s \otimes S_s} A_s \otimes M \otimes A_s \otimes N \xrightarrow{\mu \otimes \mu} A_s \otimes A_s \otimes M \otimes N.
$$

Proof. A straightforward verification relying on the symmetric monoidal structure. $\square$

Since $A_s$ is a right $H$-comodule algebra, by hypothesis, for each pair of natural numbers $(s, t)$, there is an algebra morphism $S_t : A_s \to A_t \otimes A_s$.

Definition 8.3.4. For a pair of natural numbers $(s, t)$, let $\tilde{\psi}_{t,s}$ denote the composite morphism of algebras:

$$
A_{s+t} \xrightarrow{\psi_{t,s}} A_t \otimes A_s \xrightarrow{\mu \otimes A_s} A_t \otimes A_t \otimes A_s \xrightarrow{\mu \otimes A_s} A_t \otimes A_s.
$$

Proposition 8.3.5. Let $M$ be a right $H$-comodule and $s, t$ be a pair of natural numbers. The following diagram commutes:

$$
\begin{array}{ccc}
M & \longrightarrow & A_s \otimes M \\
S_{s+t} \downarrow & & \downarrow S_t \\
A_{s+t} \otimes M & \longrightarrow & A_t \otimes A_s \otimes M.
\end{array}
$$
Proof. (Indications.) The proof is a standard verification, based upon the coassociativity of the comodule structure of $M$, using Lemma 8.3.3 to analyse the morphism $S_t$ on $A_s \otimes M$. (Note that, although the braiding has been used to transfer the coproduct to the left hand side, the coproduct which arises is still derived from the coproduct $\psi_{s,t}$.) A final subtlety is that the commutativity of the product of $A_s$ is required in analysing the morphism $S_t$ on $A_s \otimes M$, relating to the order in which the tensor factors occur. □

Remark 8.3.6. The above (simplified) framework is inspired by the case where $H$ is the dual Steenrod algebra $A^*$, and $A_s$ is the algebra $\Gamma_s$.

8.4. Identifying the differential. Using the general framework developed in the previous section, the differential can be identified in terms of the natural inclusions $\mathcal{R}_s M \hookrightarrow \Gamma_s \otimes M$, where $\Gamma_s$ denotes the algebra $\Phi^G L_s$.

Definition 8.4.1. For $s$ a positive integer, let $\partial_s : \Gamma_s \to \Gamma_{s-1} \otimes \Sigma^{-1}F_p$ denote the composite

$$\Gamma_s \xrightarrow{\psi_{s-1,1}} \Gamma_{s-1} \otimes \Gamma_1 \xrightarrow{\Gamma_{s-1} \otimes \partial_1} \Gamma_{s-1}.$$  

Lemma 8.4.2. Let $s$ be a positive integer and $M$ be an $A$-module. The morphism $\partial_s$ induces a natural morphism

$$\partial_s \otimes \Sigma M : \Gamma_s \otimes \Sigma M \to \Gamma_{s-1} \otimes M.$$

Proof. The functors are weakly continuous, hence it again suffices to restrict to the case of modules which are bounded above, where it is defined by forming the usual tensor product. □

Proposition 8.4.3. For $M \in \text{Ob} \mathcal{M}$ and $s$ a positive integer, there is a natural commutative diagram

$$
\begin{array}{c}
\mathcal{R}_s \Sigma M \\
\downarrow d_s \\
\Gamma_s \otimes \Sigma M \\
\downarrow \partial_s \otimes \Sigma M \\
\Gamma_{s-1} \otimes M.
\end{array}
$$

Proof. All functors considered are weakly continuous, hence it again suffices to restrict to the case of modules which are bounded above. If $M$ is bounded above, then the morphism $\psi_{s-1,1} : \Gamma_s \to \Gamma_{s-1} \otimes \Gamma_1$ induces a natural morphism

$$\Gamma_s \otimes \Sigma M \to \Gamma_{s-1} \otimes (\Gamma_1 \otimes \Sigma M).$$

Moreover, the abstract framework of the previous section (in particular, Proposition 8.3.3) extends to prove the commutativity of the left hand square in the following diagram:

$$
\begin{array}{c}
\mathcal{R}_s \Sigma M \\
\downarrow \\
\Gamma_s \otimes \Sigma M \\
\downarrow \psi_{s-1,1} \\
\Gamma_{s-1} \otimes (\Gamma_1 \otimes \Sigma M) \\
\downarrow \\
\Gamma_{s-1} \otimes M.
\end{array}
$$

where the right hand square is commutative by the definition of $d_{\Sigma M}$.

The bottom horizontal composite is induced by the composite

$$\Gamma_s \xrightarrow{\bar{\psi}_{s-1,1}} \Gamma_{s-1} \otimes \Gamma_1 \xrightarrow{\Gamma_{s-1} \otimes \partial_1} \Gamma_{s-1} \otimes \Sigma^{-1}F_p.$$  

It is sufficient to show that this morphism identifies with $\partial_s$. This is proved in Proposition 8.4.7 below, after some preliminary results. □
Lemma 8.4.4. Let $k$ be a positive integer. Then the binomial coefficient
\[
\binom{-pk + k - 1}{k} \equiv 0 \mod p.
\]

Proof. Use the identity
\[
\binom{-pk + k - 1}{k} = -\frac{pk}{k} \binom{-pk + k - 1}{k - 1}.
\]

To simplify the induction argument below, for $s$ a natural number, $S_s$ is considered to have range $\Phi_s \otimes M$.

Lemma 8.4.5. The following diagram commutes:
\[
\begin{array}{ccc}
\Phi_1 & \xrightarrow{S_1} & \Phi_1 \otimes \Phi_1 \\
\downarrow \phi & & \downarrow \phi \\
\Sigma^{-1}F_p & \xrightarrow{\eta} & \Phi_1 \otimes \Sigma^{-1}F_p,
\end{array}
\]
where $\eta: \Sigma^{-1}F_p \to \Phi_1 \otimes \Sigma^{-1}F_p$ is induced by the unit of $\Phi_1$.

Proof. Using the notation of Corollary 5.4.3, the morphism $S_1$ is determined by
\[
S_1(x) = x - \frac{M_{1,0}}{Q_{1,0}} \otimes y,
\]
\[
S_1(y) = y - \frac{1}{Q_{1,0}} \otimes y^p.
\]
Consider the image of $xy^n$, for $n$ an integer, passing around the top of the diagram. Terms involving $\frac{x}{y}$ can only arise in the expression $x(y - \frac{1}{Q_{1,0}} y^p)^n$. Hence, it suffices to show that
\[
\binom{n}{i} y^n \left(-\frac{1}{Q_{1,0}} y^{p-1}\right)^i
\]
is a non-trivial scalar multiple of $\frac{y}{i}$ (for $i$ a natural number) if and only if $n = -1$ and $i = 0$.

The term is a scalar multiple of $y^{-1}$ if and only if $n + i(p - 1) = -1$, when $n = -(p - 1)i - 1 = -pi + i - 1$. By Lemma 8.4.4 if $i > 0$, the binomial coefficient is zero. This completes the proof.

Lemma 8.4.6. Let $s$ be a positive integer. Then the following diagram commutes
\[
\begin{array}{ccc}
\Phi_1 & \xrightarrow{S_s} & \Phi_s \otimes \Phi_1 \\
\downarrow \phi & & \downarrow \phi \\
\Sigma^{-1}F_p & \xrightarrow{\eta_s} & \Phi_s \otimes \Sigma^{-1}F_p,
\end{array}
\]
where $\eta_s: \Sigma^{-1}F_p \to \Phi_s \otimes \Sigma^{-1}F_p$ is induced by the unit of $\Phi_1$.

Proof. The result follows by induction on $s \geq 1$ using the identity $S_s = S_1 \circ S_{s-1}$ and Lemma 8.4.3 for the initial step. (Here, the morphisms factorize naturally across the quotient $\Phi_1 \to \Phi_1'$, hence the large tensor product can be avoided and Proposition 4.4.2 can be used in the inductive step.)
Proposition 8.4.7. Let s be a positive integer. The morphism $\partial_s : \Gamma_s \to \Gamma_{s-1} \otimes \Sigma^{-1}F_p$ identifies with

$$
\Gamma_s \xrightarrow{\partial_{s-1}} \Gamma_{s-1} \otimes \Gamma_1 \xrightarrow{\Gamma_{s-1} \otimes \partial} \Gamma_{s-1} \otimes \Sigma^{-1}F_p.
$$

Proof. The result follows from Lemma 8.4.6 by considering the definition of the morphism $\tilde{\psi}_{s-1,1}$. \qed

Remark 8.4.8. Proposition 8.4.3 establishes the relation between the complex studied by Hung and Sun [HS95] and the complex introduced here. (Cf. [HS95, Definition 3.5] and [HS95, Proposition 3.6].)

8.5. Vanishing of $d^2$. The identification of $d_s$ given above is used to deduce that $d^2$ is trivial, so that the morphisms $d_s$ do define a differential.

Proposition 8.5.1. Let $s \geq 2$ be an integer. For $M \in \text{Ob} \mathcal{M}$, the composite morphism

$$
\mathcal{R}_s \Sigma^2 M \to \mathcal{R}_{s-1} \Sigma M \to \mathcal{R}_{s-2} M
$$

is trivial.

Proof. By a standard argument (cf. [HS95]), it is sufficient to prove the result for $s = 2$. Moreover, by Proposition 8.4.3, it is sufficient to show that the morphism $\partial_1 \circ \partial_2$ is trivial. This follows from [HS95, Proposition 3.6 (ii)]. \qed

9. The chain complex and its homology

The functors $\mathcal{R}_s$ and the differentials $d_s : \mathcal{R}_s \Sigma \to \mathcal{R}_{s-1}$ allow the construction of a natural chain complex, $\mathcal{D}_s$, which is the key to analysing the derived functors of destabilization. This is analogous to the approach taken by Singer [Sin80a] to the study of the derived functors of iterated loop functors. This leads to Theorem 9.4.3, the main result of the paper.

9.1. Defining the chain complex.

Definition 9.1.1. For $M \in \text{Ob} \mathcal{M}$, let $\mathcal{D}_s M$ denote the chain complex of $A$-modules given by

$$
\mathcal{D}_s M := \Sigma \mathcal{R}_s \Sigma^{s-1} M,
$$

with differential $\mathcal{D}_s M \to \mathcal{D}_{s-1} M$ induced by $d_s : \mathcal{R}_s \Sigma \to \mathcal{R}_{s-1}$.

Remark 9.1.2. That this is a chain complex follows from Proposition 8.5.1. Moreover, it is clear that this construction is functorial in $M$.

The chain complex $\mathcal{D}_s M$ has the form

$$
\cdots \to \Sigma \mathcal{D}_3 (\Sigma^2 M) \to \Sigma \mathcal{D}_2 (\Sigma M) \to \Sigma \mathcal{D}_1 M \to M \to 0.
$$

Proposition 9.1.3. The functor $\mathcal{D}_s$ from $\mathcal{M}$ to chain complexes in $\mathcal{M}$ is exact.

Proof. Follows from the exactness of $\mathcal{R}_s$ (see Corollary 7.3.7). \qed

9.2. The short exact sequence of chain complexes.

Theorem 9.2.1. For $M \in \text{Ob} \mathcal{M}$, there is a natural short exact sequence of chain complexes in $\mathcal{M}$:

$$
0 \to \Sigma^{-1} \mathcal{D}_s \Sigma M \to \mathcal{D}_s M \xrightarrow{\delta} \Sigma^{-1} \Phi \mathcal{D}_{s-1} \Sigma M \to 0,
$$

which, in homological degree $s$, is the suspension of the natural short exact sequence

$$
0 \to \Sigma^{-1} \mathcal{R}_s (\Sigma^s M) \to \mathcal{R}_s (\Sigma^{s-1} M) \xrightarrow{\delta} \Sigma^{-2} \Phi \Sigma \mathcal{R}_{s-1} (\Sigma^{s-1} M) \to 0.
$$
Proof. It is sufficient to show that the short exact sequences, for $s \geq 0$, induce morphisms of chain complexes. It suffices to show that the morphisms $\rho_s$ induce a morphism of chain complexes.

Fix $s \geq 2$ and set $N := \Sigma^{s-1}M$. There is a natural commutative diagram in $\mathcal{M}$:

\[
\begin{array}{c}
\mathcal{R}_s N \downarrow \mathcal{R}_1 \mathcal{R}_{s-1} N \\
\mathcal{R}_s \mathcal{R}_1 N \downarrow \mathcal{R}_s \mathcal{R}_{s-2} N \\
\mathcal{R}_s \mathcal{R}_1 \Sigma^{-1} N \\
\end{array}
\]

in which the lower vertical arrows are induced by $\mathcal{R}_1 N \xrightarrow{d} \Sigma^{-1}N$ and the upper part of the diagram corresponds to embedding the functor $\mathcal{R}_s$ in composite functors (see Corollary 7.3.10). The commutativity of the right hand side of the diagram follows from the naturality of $\rho_1$.

The top and bottom horizontal composites correspond respectively to $\rho_s$ and $\rho_{s-1}$, whereas the the left and right hand vertical composites are (up to suspension) the differentials in the respective chain complexes. It follows (noting that the case $s = 1$ is trivial), that $\rho_\bullet$ defines a morphism of chain complexes. □

9.3. First properties of the chain complex. For the purposes of this section, introduce the following notation:

**Notation 9.3.1.** For $M \in \text{Ob } \mathcal{M}$ and $s \geq 0$ an integer, let $\mathcal{D}_s M$ denote the homology $H_s(\mathcal{D}_s M)$.

**Lemma 9.3.2.** For a natural number $s$, $\mathcal{D}_s$ defines an additive functor $\mathcal{M} \to \mathcal{M}$ and, for a short exact sequence $0 \to K \to M \to Q \to 0$ of $\mathcal{M}$, there is a long exact sequence in $\mathcal{M}$:

\[
\ldots \to \mathcal{D}_s K \to \mathcal{D}_s M \to \mathcal{D}_s Q \to \mathcal{D}_s-1 K \to \ldots
\]

**Proof.** Standard. □

**Lemma 9.3.3.** For $M \in \text{Ob } \mathcal{M}$, the natural short exact sequence of chain complexes

\[
0 \to \Sigma^{-1} \mathcal{D}_s \Sigma M \to \mathcal{D}_s M \xrightarrow{\rho_s} \Sigma^{-1} \mathcal{D}_s-1 \Sigma M \to 0
\]

induces a natural long exact sequence

\[
\ldots \to \Sigma^{-1} \mathcal{D}_s \Sigma M \to \mathcal{D}_s M \to \Sigma^{-1} \mathcal{D}_s-1 \Sigma M \xrightarrow{\lambda_{s-1}} \Sigma^{-1} \mathcal{D}_{s-1} \Sigma M \to \ldots
\]

in $\mathcal{M}$.

**Proof.** Straightforward. □

**Remark 9.3.4.** It is not necessary to identify the connecting morphisms $\lambda_{s-1}$ explicitly, for $s > 1$; the case $s = 1$ is treated by Proposition [6.6.1].

The complex $\mathcal{D}_\bullet$ has the following connectivity property:

**Lemma 9.3.5.** Let $s \geq 0$ be an integer and $M \in \text{Ob } \mathcal{M}$, then

\[
|\mathcal{D}_s M| \geq 1 + p^s(|M| + s - 1).
\]

**Proof.** A consequence of the $\kappa_s$-connectivity of the functor $\mathcal{R}_s$ (see Corollary 7.3.7). □

By construction of the chain complex, we have:
Proposition 9.3.6. For $M \in \text{Ob } \mathcal{A}$, there is a natural isomorphism $D_0 M \cong D M$.

Proof. Follows from Proposition 6.5.2. □

9.4. The main results. The following is the key result:

Proposition 9.4.1. For any integer $t \in \mathbb{Z}$ and integer $s \geq 1$, $D_s(\Sigma^t A) = 0$.

Proof. The proof is by induction on $s$; start with the initial case $s = 1$.

The long exact sequence of Lemma 9.3.3 in low degrees is

$\Sigma^{-1} D_1 \Sigma M \to D_1 M \to \Sigma^{-1} \Phi D \Sigma M \to \Sigma^{-1} \lambda \Sigma^{-1} D \Sigma M$

using Proposition 9.3.6 to identify $D_0$ with $D$ and Proposition 6.6.1 to identify the connecting morphism.

Now, if $M = \Sigma^t A$, the morphism $\Sigma^{-1} \lambda$ is the desuspension of $\lambda : \Phi F(t + 1) \to F(t + 1)$, since $D \Sigma(\Sigma^t A)$ is the free unstable module $F(t + 1)$. This morphism is injective, hence, for any $t \in \mathbb{Z}$, there is a surjection

$\Sigma^{-1} D_1 (\Sigma^t A) \to D_1 (\Sigma^t A)$,

by Lemma 9.3.3. It follows using $\kappa_1$-connectivity from Lemma 9.3.3 that $D_1 (\Sigma^t A) = 0$, for all integers $t$.

The inductive step uses a similar argument: by induction, we may suppose that $D_{s-1} \Sigma^t A$ is zero for all integers $t$. Hence, by Lemma 9.3.3 there is a surjection

$\Sigma^{-1} D_s (\Sigma^{t+1} A) \to D_s (\Sigma^t A)$,

for all integers $t$. Again, it follows from Lemma 9.3.3 that $D_s (\Sigma^t A) = 0$, for all integers $t$. □

Remark 9.4.2. The reader should compare this argument with the proof of [Sin80a, Propositions 6.4 and 6.5], which concerns the derived functors of iterated loop functors (for $p = 2$).

The main result of the paper follows, as for the proof of [Sin80a, Theorem 6.6].

Theorem 9.4.3. For $M \in \text{Ob } \mathcal{A}$ and $s$ a natural number, there is a natural isomorphism

$D_s M \cong D_s M$.

That is, the homology of the chain complex $D_\bullet M$ calculates the derived functors of destabilization.

Proof. Formal. □

Remark 9.4.4. The theorem shows, in particular, that the homology $H_s(D_\bullet M)$ is an unstable module, which is not immediately transparent from the construction, for $s > 0$.

Along the way, we have established the fact that the functors $D_s$ satisfy the following connectivity property (Cf. Lemma 3.3.5).

Corollary 9.4.5. Let $s \geq 0$ be an integer and $M \in \text{Ob } \mathcal{A}$, then

$|D_s M| \geq 1 + p^s(|M| + s - 1)$.

Theorem 9.4.3 recovers the main result of [Zar84]:

Corollary 9.4.6. [Zar84, Théorème 2.5] Let $M$ be an unstable module and $s \geq 0$ be an integer, then there is a natural isomorphism

$D_s(\Sigma^{1-s} M) \cong \Sigma D_s M$. 
Proof. The relevant portion of the chain complex $\mathfrak{D}_s^{\Sigma^{1-s}}M$ is
\[
\Sigma \mathfrak{R}_{s+1}^\Sigma M \to \Sigma \mathfrak{R}_s^\Sigma M \to \Sigma \mathfrak{R}_{s-1}^\Sigma^{-1}M;
\]
the fact that $M$ is unstable implies that the two differentials are trivial. \qed

9.5. Further results. The above techniques give some further information on the structure of the unstable module $\mathfrak{D}_s^{\Sigma^{-s}}M$, for $s$ a positive integer and $M$ an unstable module. The following result follows from Corollary 9.4.6 and Proposition 9.5.1.

Proposition 9.5.1. (Cf. [Zar84] Section 4.8.) Let $M$ be an unstable module and $s$ be a positive integer, then there is a natural short exact sequence of unstable modules
\[
0 \to \mathfrak{R}_s^\Sigma M \to \mathfrak{D}_s^{\Sigma^{-s}}M \to \Omega_1\mathfrak{D}_{s-1}^{\Sigma^{1-s}}M \to 0.
\]
However, as observed by Zarati in [Zar84] Remarque 4.8], even when the natural morphism $\lambda : \Phi M \to M$ is injective, the morphism $\mathfrak{R}_s^\Sigma M \to \mathfrak{D}_s^{\Sigma^{-s}}M$ is not in general an isomorphism. In particular, the morphism $\mathfrak{R}_s^\Sigma \mathfrak{F} \to \mathfrak{D}_s^{\Sigma^{-s}}\mathfrak{F}$ is not an isomorphism for $s \gg 0$.

The following result provides a complement to Proposition 9.5.1.

Corollary 9.5.2. Let $M$ be an unstable module and $s$ be a positive integer. Then $\mathfrak{D}_s^{\Sigma^{-s}}M$ is the sub $\mathcal{A}$-module of $\Sigma \mathfrak{R}_s^{\Sigma^{-1}}M$ given as the kernel of
\[
\Sigma \mathfrak{R}_s^{\Sigma^{-1}}M \xrightarrow{d} \Sigma \mathfrak{R}_{s-1}^{\Sigma^{-2}}M.
\]
In particular, $\mathfrak{D}_s^{\Sigma^{-s}}M$ embeds in the largest unstable submodule of $\Sigma \mathfrak{R}_s^{\Sigma^{-1}}M$.

Proof. Generalization of the proof of Corollary 9.4.6 \qed

10. Module structures

For a natural number $s$, $K^+_s$ is a sub-unstable algebra of $H^*(BV_s)$ and, for any $\mathcal{A}$-module $N$, $\mathfrak{R}_sN$ is naturally an object of $K^+_s\mathcal{A}$. The purpose of this section is to show to what extent this structure passes to the derived functors $\mathfrak{D}_s$. The main result of the section is Theorem 10.2.1.

10.1. Linearity results. Recall that $\Gamma_s$ denotes the $\mathcal{A}$-algebra $\Phi^{GL_s}$, which contains the algebra of Dickson invariants $F_p[Q_{s,0}, Q_{s,1}, \ldots, Q_{s,s-1}]$.

Remark 10.1.1. We are interested in the linearity properties of the differential $d_s$ of $\mathfrak{D}_sN$
\[
d_s : \Sigma^{s-1}N \to \Sigma^{s-2}N.
\]
with respect to the Dickson algebra.

The first term is naturally a module over $\mathbb{F}_p[Q_{s,0}, \ldots, Q_{s,s-1}]$ and the second over $\mathbb{F}_p[Q_{s-1,0}, \ldots, Q_{s-1,s-2}]$ and, hence, over $\mathbb{F}_p[Q_{s,0}, \ldots, Q_{s,s-1}]$, via the morphism $\varphi_s$ of Lemma 7.5.3. In general, the differential is not $\mathbb{F}_p[Q_{s,0}, \ldots, Q_{s,s-1}]$-linear; the aim is to show linearity when $N$ is an iterated desuspension of an unstable module, after restricting to subalgebras of the form $\Phi^k\mathbb{F}_p[Q_{s,0}, \ldots, Q_{s,s-1}]$, for suitably large $k$.

The differential of the complex $\mathfrak{D}_sN$ is determined by the morphisms $\partial_s : \Gamma_{s-1} \to \Gamma_{s-1} \otimes \Sigma^{-1}\mathbb{F}_p$, by Proposition 8.3.3. In the case of interest, Proposition 8.2.2 gives the following:
Proposition 10.1.2. Let \( s \) be a positive integer.

\[
\psi_{s-1,1} Q_{s,j} = \begin{cases} 
Q_{s-1,0}^p \otimes Q_{1,0} & j = 0 \\
Q_{s-1,0}^p Q_{s-1,j} \otimes Q_{1,0} + Q_{s-1,j}^p \otimes 1 & j > 0;
\end{cases}
\]

\[
\psi_{s-1,1} R_{s,j} = \begin{cases} 
Q_{s-1,0}^p R_{s-1,j} \otimes Q_{1,0} + Q_{s-1,j}^p Q_{s-1,0} \otimes R_{1,0} & 0 \leq j < s - 1 \\
Q_{s-1,0}^p \otimes R_{1,0} & j = s - 1.
\end{cases}
\]

The key to the linearity results is provided by the following Lemma.

Lemma 10.1.3. Let \( M \) be an unstable module, \( t \) a natural number and set \( u := \left[\frac{t+1}{2}\right] \). Then the submodule \( Q_{u,0}^u R_s(\Sigma^{-t} M) \) of \( R_s(\Sigma^{-t} M) \subset \Gamma_s \otimes \Sigma^{-t} M \) is contained in \( H^*(BV_q)^{GL_s} \otimes \Sigma^{-t} M \subset \Gamma_s \otimes \Sigma^{-t} M \).

Proof. By choice of \( u \), \( \Sigma^{2u}\Sigma^{-t} M \) is unstable and hence \( R_s(\Sigma^{2u}\Sigma^{-t} M) \) is an unstable submodule of \( H^*(BV_q)^{GL_s} \otimes \Sigma^{2u}\Sigma^{-t} M \).

There is a natural isomorphism

\[ R_s(\Sigma^{2u}\Sigma^{-t} M) \cong \Sigma^{2u} Q_{u,0}^u R_s(\Sigma^{-t} M), \]
considered as a submodule of \( \Sigma^{2u} R_s(\Sigma^{-t} M) \), by Proposition 7.3.8. The result follows.

The differential of the complex is induced by the natural transformation \( d_s : R_s(\Sigma^{-t} M) \to R_{s-1}(\Sigma^{-(t+1)} M) \).

By Remark 10.1.1, both of the modules above have natural \( \mathbb{F}_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-module structures in the category \( \mathcal{M} \).

Proposition 10.1.4. Let \( M \) be an unstable module, \( t \) a natural number and set \( u := \left[\frac{t+1}{2}\right] \). For \( w \) a natural number such that \( p^w \geq u \), the morphism

\[ d_s : R_s(\Sigma^{-t} M) \to R_{s-1}(\Sigma^{-(t+1)} M). \]

is \( \Phi^w \mathbb{F}_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-linear.

Proof. The morphism \( d_s \) appears in a commutative diagram

\[
\begin{array}{ccc}
\mathcal{M} \Sigma^{-t} M & \xrightarrow{d_s} & R_{s-1}(\Sigma^{-(t+1)} M) \\
\Gamma_s \otimes \Sigma^{-t} M & \xrightarrow{d_s} & \Gamma_{s-1} \otimes \Sigma^{-(t+1)} M \\
\end{array}
\]

by Proposition 8.4.3 where the lower horizontal arrows are induced by \( \psi_{s-1,1} \) and \( \partial_t \) respectively. (The fact that \( M \) is unstable implies that it is unnecessary to use large tensor products here.)

By construction, the vertical inclusions are morphisms of \( \mathbb{F}_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-modules and \( F_p[Q_{s-1,0}, \ldots, Q_{s-1,s-2}] \)-modules respectively, thus can be considered as morphisms of \( \mathbb{F}_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-modules as above. Thus, to prove the linearity result, it suffices to consider the composite in this diagram.

Lemma 10.1.3 together with Proposition 10.1.2 implies that the image of \( R_s(\Sigma^{-t} M) \) in \( \Gamma_{s-1} \otimes \Sigma^{-(t+1)} M \) lies in the submodule

\[ (Q_{s-1,0}^p \otimes Q_{1,0})^{-u} (H^*(BV_{s-1})^{GL_{s-1}} \otimes H^*(BV_1)^{GL_1} \otimes \Sigma^{-t} M). \]

After multiplying by an element of \( H^*(BV_{s-1})^{GL_{s-1}} \otimes H^*(BV_1)^{GL_1} \) of the form \( \alpha \otimes Q_{1,0}^p \), where \( n \geq u \), this lies in \( \Gamma_{s-1} \otimes H^*(BV_1)^{GL_1} \otimes \Sigma^{-t} M \), which is contained in the kernel of the morphism induced by \( \partial_t \).
By Proposition 10.1.2
\[ \psi_{s-1,1}Q_{s,j}^w = \begin{cases} Q_{s-1,0}^{p+1} \otimes Q_{s,j}^w & j = 0 \\ Q_{s-1,0}^{p(w+1)} \otimes Q_{s,j}^w + Q_{s-1,j}^{p+1} \otimes 1 & j > 0. \end{cases} \]

The hypothesis that \( p^w \geq w \) allows the previous remark to be applied, so that the terms involving \( Q_{p+1}^w \) can be discarded. The result follows.

\[ \square \]

10.2. Module structures on derived functors of destabilization.

**Theorem 10.2.1.** Let \( M \) be an unstable module and \( s, t, w \) be natural numbers such that \( p^w \geq \left\lfloor \frac{w}{2} \right\rfloor \). Then the unstable module \( D_s(\Sigma^{-t}M) \) is equipped with a natural \( \Phi^{w+1}F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-structure in \( \mathcal{M} \).

If \( t \leq s \), then \( D_s(\Sigma^{-t}M) \) has a natural \( F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-module structure in \( \mathcal{M} \).

**Proof.** By Theorem 9.4.3 \( D_s(\Sigma^{-t}M) \) is calculated as the homology of:
\[ \Sigma \mathcal{A}_{s+1}(\Sigma^{-s-t}M) \xrightarrow{d_{s+1}} \Sigma \mathcal{A}_s(\Sigma^{-s-t-1}M) \xrightarrow{d_s} \Sigma \mathcal{A}_{s-1}(\Sigma^{-s-t-2}M). \]

The morphism \( d_s \) is \( \Phi^wF_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-linear, by Proposition 10.1.1; hence the kernel has a natural \( \Phi^wF_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-module structure in \( \mathcal{M} \), which restricts to a natural \( \Phi^{w+1}F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-module structure in \( \mathcal{M} \).

Similarly, the morphism \( d_{s+1} \) is \( \Phi^wF_p[Q_{s+1,0}, \ldots, Q_{s+1,s}] \)-linear; hence the image is naturally a sub \( \Phi^wF_p[Q_{s+1,0}, \ldots, Q_{s+1,s}] \)-module of \( \Sigma \mathcal{A}_s(\Sigma^{-s-t-1}M) \) in \( \mathcal{M} \). Since the \( F_p[Q_{s+1,0}, \ldots, Q_{s+1,s}] \)-structure on \( \Sigma \mathcal{A}_s(\Sigma^{-s-t-1}M) \) is induced by restriction along \( \varphi_{s+1}: F_p[Q_{s+1,0}, \ldots, Q_{s+1,s}] \to F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \), which surjects onto the subalgebra \( \Phi^wF_p[Q_{s,0}, \ldots, Q_{s,s-1}] \) by Lemma 7.5.3 it follows that the image of \( d_{s+1} \) is a sub \( \Phi^{w+1}F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-module in \( \mathcal{M} \). It follows that the homology has the natural structure of a \( \Phi^{w+1}F_p[Q_{s,0}, \ldots, Q_{s,s-1}] \)-module, as required.

In the case \( t \leq s \), one can take \( w = 0 \); the differential \( d_{s+1} \) is trivial, hence the additional Frobenius twist is unnecessary.

**Remark 10.2.2.** In the case \( t < s \), this result is immediate from Zarati’s result (see Corollary 9.4.6), which provides a \( \Kw^+ \)-module structure.

**Appendix A.**

\[ \text{A.1. A direct proof of the } A\text{-stability of } \mathcal{A}_1(M). \] The following exploits the (anti)symmetry in calculating \( S_1 \circ S_1 \). For an \( A \)-module \( M \), consider \( S_1 \circ S_1(x) \) as belonging to
\[ \Phi_1 \otimes (\Phi_1 \otimes M) \cong (\Lambda(u') \otimes F_p[v']) \otimes (\Lambda(u) \otimes F_p[v] \otimes M). \]

To avoid ambiguity, write \( S_1^{u', v'} \) for the linear morphism \( S_1^{u'} : M \to (\Lambda(u') \otimes F_p[v']) \otimes M \) with respect to the variables \( u', v' \) (respectively \( S_1^{u, v} \) with respect to the variables \( u, v \)).

**Proof of Proposition 6.1.3** It is sufficient to show that the elements \( \beta \text{St}_1(x) \) and \( P^i \text{St}_1(x) \) (\( i \in \mathbb{N} \)) belong to \( \mathcal{A}_1(M \subset \Phi_1 \otimes M) \), for any element \( x \) of \( M \). By Lemma 5.4.6 \( \text{St}_1(x) = \pm \epsilon_1^{x|} \text{S}_1(x) \); a straightforward calculation with the Cartan formula shows that it is sufficient to establish that \( \epsilon_1^{x|} \beta^i P^i \text{S}_1(x) \) belongs to \( \mathcal{A}_1(M \subset \Phi_1 \otimes M) \), for all operations \( \beta^i P^i \).
Recall that $S_{1}^{\mu',\nu'}(z) = \sum \pm \left( \frac{u'}{v'} \right)^{\varepsilon} (v')^{-i(p-1)} \beta \varepsilon \Phi(z)$. Hence
\[
S_{1}^{\mu',\nu'} \circ S_{1}^{\mu,\nu}(x) = \sum \pm \left( \frac{u'}{v'} \right)^{\varepsilon} (v')^{-i(p-1)} \beta \varepsilon \Phi(S_{1}^{\mu,\nu}(x))
\]
\[
= \sum \pm (u')^{\varepsilon} (v')^{-i(p-1)-\varepsilon} \beta \varepsilon \Phi(S_{1}^{\mu,\nu}(x)),
\]
considered as an element of $\Phi_{1} \otimes (\Phi_1 \otimes M)$. This expression encodes the elements $(u')^{\varepsilon} (v')^{-i(p-1)-\varepsilon}$ of interest: up to sign, we require to calculate the coefficient of $\beta \varepsilon \Phi(S_{1}(x))$. As in the proof presented in Section 5.6.4, one can reduce to the case $\varepsilon = 0$.

The element $S_{1}^{\mu',\nu'} \circ S_{1}^{\mu,\nu}(x)$ identifies with $S_{2}(x) \in \Phi_{2} \otimes \Phi_1 \otimes M$, by Proposition 5.6.4 (strictly speaking, one should consider the element $S_{2}(x)$, as in loc. cit.). This implies that the element $S_{1}^{\mu',\nu'} \circ S_{1}^{\mu,\nu}(x)$ is invariant under the action of the subgroup of $GL_2$ which is generated by
\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]
(This invariance has to be interpreted suitably, since it involves reversing the roles of the variables $u, v$ and $u', v'$ in the expression $\Phi_{1} \otimes (\Phi_1 \otimes M)$). Hence, the term $\Phi_{1}(S_{1}^{\mu',\nu'}(x))$ is equal (up to sign) to the coefficient of $v^{-i(p-1)}$ in the expression $S_{1}^{\mu',\nu'} \circ S_{1}^{\mu,\nu}(x)$, considered as an element of $(\Lambda(u) \otimes \mathbb{F}_p[v]) \otimes \left((\Lambda(u') \otimes \mathbb{F}_p[v']) \otimes M\right)$.

Write $S_{1}^{\mu',\nu'}(x) = \sum \pm u^{\eta} v^{-t(p-1)-\eta} \beta \eta \Phi^t(x)$; since the morphism $S_{1}^{\mu',\nu'}$ is multiplicative, this gives:
\[
S_{1}^{\mu',\nu'} \circ S_{1}^{\mu,\nu}(x) = \sum \pm S_{1}^{\mu',\nu'}(u^{\eta} v^{-t(p-1)-\eta}) S_{1}^{\mu',\nu'}(\beta \eta \Phi^t(x))
\]
The linear morphism $S_{1}^{\mu',\nu'}$ on $\Lambda(u) \otimes \mathbb{F}_p[v]$ is determined by:
\[
u \mapsto \left( 1 - \left( \frac{v'}{v} \right)^{p-1} \right)
\]
To form the power series expansion in $(\Lambda(u) \otimes \mathbb{F}_p[v]) \otimes ((\Lambda(u') \otimes \mathbb{F}_p[v']) \otimes M)$, the expression $S_{1}(v^{-1})$ has to be written in the form
\[

v^{-1} \mapsto \left( \frac{u'}{v} \right)^{p-1} \left[ 1 - \left( \frac{v'}{v} \right)^{p-1} \right]^{-1}.
\]
For fixed $\eta, t$, we require to calculate (up to sign) the coefficient (as an expression of $u', v'$) of $v^{-i(p-1)}$ in
\[

(\eta - \eta \left( \frac{v'}{v} \right) v^{-1} \left[ 1 - \left( \frac{v'}{v} \right)^{p-1} \right]^{-1})^{t(p-1)+\eta}.
\]
By inspection, this coefficient is trivial if
\[
i(p-1) > pt + \eta,
\]
which is equivalent to $i < pt + \eta$.

Up to (possibly zero) scalar multiple, the coefficient is:
\[
(u')^{\eta}(v')^{i-t(p-1)-\eta} = (\tilde{M}_{1,0})^{\eta}(\epsilon_1)^{2(i-t+1/2)},
\]
where $\tilde{M}_{1,0}$ and $\epsilon_1$ are defined in terms of $u', v'$. As in the proof presented in Section 5.6.4, it suffices to show that the scalar coefficient is trivial if
\[
2(i-t+1/2) < (\eta + 2t(p-1)).
\]
This inequality is equivalent to $i < pt + \eta$, which is the condition for triviality exhibited above.

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