New representations of Taylor coefficients of the Weierstrass $\sigma$-function

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Abstract

We provide two kinds of representations for the Taylor coefficients of the Weierstrass $\sigma$-function $\sigma(\cdot; \Gamma)$, where $\Gamma$ is an arbitrary lattice in $\mathbb{C}$. The first one in terms of Hermite-Gauss series over $\Gamma$ and the second one in terms of Hermite-Gauss integrals over $\mathbb{C}$. As applications, we derive identities of arithmetic type relating some modular forms to such Hermite-Gaussian series. This will be possible by comparing our obtained representations to the one elaborated by Weierstrass himself in 1882.

1 Introduction and statement of main result

Let $\sigma(z; \Gamma)$ be the standard Weierstrass $\sigma$-function associated to a given two-dimensional lattice $\Gamma \subset \mathbb{C}$. It is an odd entire function on $\mathbb{C}$ given by the convergent infinite product

$$\sigma(z; \Gamma) = z \prod_{\gamma \in \Gamma \setminus \{0\}} \left(1 - \frac{z}{\gamma}\right) e^{z/\gamma + \frac{1}{2}(z/\gamma)^2}. \quad (1.1)$$

Its Taylor series expansion at $z = 0$ can then be written as

$$\sigma(z; \Gamma) = \sum_{r=0}^{\infty} W_r(\Gamma) \frac{z^{2r+1}}{(2r+1)!}. \quad (1.2)$$

The first few terms of $\sigma(z; \Gamma)$ are known to be given explicitly by [8, page 391]

$$\sigma(z; \Gamma) = z - \frac{g_2}{24 \cdot 3 \cdot 5} z^5 - \frac{g_3}{23 \cdot 3 \cdot 5 \cdot 7} z^7 - \frac{g_2^2}{2^9 \cdot 3^2 \cdot 5 \cdot 7} z^9 - \frac{g_2 g_3}{2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11} z^{11} + \frac{(23g_2^3 - 576g_3^2)}{2^{10} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13} z^{13} + \cdots. \quad (1.3)$$
where the parameters $g_2$ and $g_3$ are essentially the Eisenstein series of weight 4 and 6 respectively given by

$$g_2 = 60 \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{\gamma^4} \quad \text{and} \quad g_3 = 140 \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{\gamma^6}. \quad (1.4)$$

More precisely, a representation of the involved Taylor coefficients $\mathcal{H}_r(\Gamma)$ has been obtained in 1882 by K. Weierstrass [14], to wit

$$\mathcal{H}_r(\Gamma) := \sum_{2m + 3n = r \atop m, n \geq 0} a_{m,n}(g_2/2)^m(2g_3)^n \quad (1.5)$$

with $a_{0,0} = 1$ and $a_{m,n} = 0$ whenever $m < 0$ or $n < 0$. Otherwise the coefficients $a_{m,n}$ are connected by the recursion formula

$$a_{m,n} = 3(m+1)a_{m+1,n-1} + \frac{16}{3}(n+1)a_{m-2,n+1} - \frac{1}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n}. \quad (1.6)$$

In the present paper, we give new representations of the Taylor coefficients $\mathcal{H}_r(\Gamma)$ either as series over $\Gamma$ of Hermite-Gauss type, or as integrals over $\mathcal{C}$ of the $\sigma$-function. Thus, let $\Gamma$ be a lattice in $\mathcal{C}$ and set

$$\nu = \nu(\Gamma) = \frac{\pi}{S(\Gamma)} = \frac{\pi}{\Im(\omega_1\omega_2)} = \frac{2i\pi}{\omega_1\omega_2 - \omega_1\bar{\omega}_2} \in \mathbb{R} \quad (1.7)$$

and

$$\mu = \mu(\Gamma) = \frac{i}{S}(\zeta(\omega_1/2; \Gamma)\bar{\omega}_2 - \zeta(\omega_2/2; \Gamma)\bar{\omega}_1) \in \mathcal{C}, \quad (1.8)$$

where $S = S(\Gamma)$ is the area of a fundamental cell $\Lambda(\Gamma)$ and $\{\omega_1, \omega_2\}$ is an arbitrary oriented $\mathbb{R}$-basis $\Gamma = Z\omega_1 + Z\omega_2$ such that $\Im(\omega_2/\omega_1) > 0$. Above $\zeta(z; \Gamma) = \sigma'(z; \Gamma)/\sigma(z; \Gamma)$ denotes the Weierstrass zeta-function (not to be confused with the Riemann zeta-function). Also, let $\chi_w$ be the "Weierstrass pseudo-character" defined on $\Gamma$ by

$$\chi_w(\gamma) = \begin{cases} +1 & \text{if } \gamma/2 \in \Gamma \\ -1 & \text{if } \gamma/2 \not\in \Gamma \end{cases} \quad (1.9)$$

Fix $\nu$ (1.7), $\mu$ (1.8) and $\chi_w$ (1.9), and consider for ever positive integer $r$ the series

$$\mathcal{H}_r(\Gamma) = \nu \frac{(2r+1)!}{2^r \cdot r!} \sum_{\gamma \in \Gamma} \chi_w(\gamma)|\gamma|^2 \mu^r F\left(-r; \frac{3}{2}; -\frac{\nu^2 r^2}{2\mu}\right) e^{-\frac{\nu \gamma}{2}} |\gamma|^2. \quad (1.10)$$

In above (1.10), $|\gamma|^2$ denotes the square of the euclidian length of the vector $\gamma \in \mathbb{C} = \mathbb{R}^2$ and $F(-r; \frac{3}{2}; -\frac{\nu^2 r^2}{2\mu})$ is the usual confluent hypergeometric function

$$F(a; c; x) = 1 + \frac{ax}{c} + \frac{a(a+1)x^2}{c(c+1)2!} + \frac{a(a+1)(a+2)x^3}{c(c+1)(c+2)3!} + \cdots.$$
Noting that the Hermite polynomial (see [8, page 252])

\[ H_{2r+1}(z) = (-1)^r \frac{(2r+1)!}{r!} 2z \binom{3}{2} F\left(-r; \frac{3}{2}; z^2\right) \]

with \( z = \sqrt{\frac{x}{2\mu}} v^2 \) and the Gaussian \( e^{-\frac{v^2}{2}|\gamma|^2} \) occur in the quantities \( \mathcal{H}_r(\Gamma) \). We may call them here Hermite-Gauss series over \( \Gamma \).

Within the above notations, the main result of the present paper can be stated as follows:

**Main Theorem 1.1.** Let \( \Gamma \) be a lattice in \( \mathbb{C} \), and keep \( \nu = \nu(\Gamma), \mu = \mu(\Gamma), \chi = \chi_w \) and \( \mathcal{H}_r(\Gamma) \) as above. Then, we have

i) An expansion series over \( \Gamma \) of the Weierstrass \( \sigma \)-function \( \sigma(z; \Gamma) \):

\[ \sigma(z; \Gamma) = \frac{1}{\mathcal{H}_0(\Gamma)} e^{\frac{\mu}{2} z^2} \sum_{\gamma \in \Gamma} \chi_w(\gamma) \gamma e^{-\frac{v^2}{2}|\gamma|^2 + vz\bar{\gamma}}. \tag{1.11} \]

ii) A reproducing integral formula over \( \mathbb{C} \) for \( \sigma(z; \Gamma) \):

\[ \sigma(z; \Gamma) = \frac{\nu}{\pi} e^{\frac{\mu}{2} z^2} \int_{\mathbb{C}} e^{\nu w x - \frac{\mu}{2} |w|^2} \sigma(w; \Gamma) e^{-v|w|^2} dm(w), \tag{1.12} \]

where \( dm \) denotes the usual Lebesgue measure on \( \mathbb{C} \).

iii) A representation of the Taylor coefficients \( \mathcal{W}_r(\Gamma) \) in terms of Hermite-Gauss series over \( \Gamma \) as

\[ \mathcal{W}_r(\Gamma) = \frac{\mathcal{H}_r(\Gamma)}{\mathcal{H}_0(\Gamma)}. \tag{1.13} \]

iv) A representation of the coefficients \( \mathcal{W}_r(\Gamma) \) in terms of Hermite-Gauss integrals over \( \mathbb{C} \) as

\[ \mathcal{W}_r(\Gamma) = \frac{\nu^2}{\pi} \frac{(2r+1)!}{2^r \cdot r!} \int_{\mathbb{C}} w^{\mu r} F\left(-r; \frac{3}{2} - \frac{v^2 w^2}{2\mu}\right) e^{-\frac{v^2}{2}|w|^2} \sigma(w; \Gamma) e^{-v|w|^2} dm(w). \tag{1.14} \]

**Remark 1.2.** In Section 4, we prove in an indirect way that

\[ \mathcal{H}_0(\Gamma) = \nu \sum_{\gamma \in \Gamma} \chi_w(\gamma) |\gamma|^2 e^{-\frac{v^2}{2}|\gamma|^2} \]

is different from zero. However, the proof by direct methods is evidently a nontrivial problem.

**Remark 1.3.** The approach we will be using to obtain the representations (1.11) and (1.13) of the Taylor coefficients \( \mathcal{W}_r(\Gamma) \), is not based on the partial differential equations satisfied by \( \sigma(z; \Gamma) \) involving \( g_2 \) and \( g_3 \) as elaborated by Weierstrass in [14], nor on the bilinear operators as done in [2]. In fact, we will derive them from a concrete description of the functional space of \((\Gamma, \chi)\)-theta functions of magnitude \( \nu \) on \( \mathbb{C} \), i.e., the space \( \mathcal{O}_{\Gamma, \chi}(\mathbb{C}) \) of entire functions \( f \) on \( \mathbb{C} \) satisfying the functional equation

\[ f(z + \gamma) = \chi(\gamma) e^{\frac{v^2}{2}|\gamma|^2 + vz\bar{\gamma}} f(z). \tag{1.15} \]
Some applications of Theorem 1.1 are discussed in Section 5. Indeed, highly non-trivial identities of arithmetic type are obtained and relate some modular forms to Hermite-Gauss series. Moreover, a full series of identities containing those obtained by Perelomov [10] page 1 are also deduced. Representation of the Eisenstein series $G_{2n}(\Gamma) = \sum' 1/\gamma^{2n}$ as rational fraction of the successive derivatives of the Weierstrass-theta series.

The paper is outlined as follows. In Section 2, we review some basic properties of the Hilbert space $O^\nu_{\Gamma,\chi}(\mathbb{C})$ of $(\Gamma,\chi)$-theta functions. In Section 3, we introduce both the modified Weierstrass $\sigma$-function and the Weierstrass $(\Gamma,\chi)$-theta series, which verify the important property of being generators of $O^\nu_{\Gamma,\chi}(\mathbb{C})$. Section 4 deals with the proof of the main result, while Section 5 is devoted to discuss some direct applications.

2 Background on $(\Gamma,\chi)$-theta entire functions

In this section, we review some basic properties of the space $O^\nu_{\Gamma,\chi}(\mathbb{C})$ of $(\Gamma,\chi)$-theta entire functions of magnitude $\nu > 0$ on $\mathbb{C}$, i.e., the functional space

$$O^\nu_{\Gamma,\chi}(\mathbb{C}) = \left\{ f \text{ entire}; \quad f(z + \gamma) = \chi(\gamma)e^{z^2|\gamma|^2 + \nu z\overline{\gamma}}f(z), \, z \in \mathbb{C}, \, \gamma \in \Gamma \right\},$$

(2.1)

where $\Gamma$ is a lattice in $\mathbb{C}$ and $\chi$ a given map on $\Gamma$ satisfying the (RDQ)-condition

$$\chi(\gamma + \gamma') = \chi(\gamma)\chi(\gamma')e^{z^2(\overline{\gamma}^\prime - \overline{\gamma}^\prime)}, \quad \gamma, \gamma' \in \Gamma.$$  

(RDQ)

In fact, this is a necessary and sufficient condition to $O^\nu_{\Gamma,\chi}(\mathbb{C})$ be nontrivial [4]. Whence, the space $O^\nu_{\Gamma,\chi}(\mathbb{C})$ can be viewed as the space of holomorphic sections of the holomorphic line bundle $L = (\mathbb{C} \times \mathbb{C})/\Gamma$ over the complex torus $\mathbb{C}/\Gamma$, where $L$ is constructed as the quotient of the trivial bundle over $\mathbb{C}$ by the action of $\Gamma$ on $\mathbb{C} \times \mathbb{C}$ given by

$$\gamma(z; v) := (z + \gamma; \chi(\gamma)e^{z^2|\gamma|^2 + \nu z\overline{\gamma}.v}).$$

The dimension of $O^\nu_{\Gamma,\chi}(\mathbb{C})$ is then known to be given by $\sqrt{\det E}$, the Pfaffian of the associated skew-symmetric form $E$ (see [3, 7, 10, 11]). In our case, $E(z, w) := (\nu / \pi) \Im(z\overline{w})$ and therefore we have the following

**Proposition 2.1.** The functional space $O^\nu_{\Gamma,\chi}(\mathbb{C})$ is a finite dimensional space and its dimension is given explicitly by

$$\dim O^\nu_{\Gamma,\chi}(\mathbb{C}) = \frac{\nu}{\pi} S(\Gamma),$$

where $S(\Gamma)$ is the area of a fundamental cell of $\Gamma$.

**Remark 2.2.** Note that for given $f, g \in O^\nu_{\Gamma,\chi}(\mathbb{C})$, the function $z \mapsto f(z)\overline{g(z)}e^{-\nu|z|^2}$ is $\Gamma$-periodic. Thus, one can equip $O^\nu_{\Gamma,\chi}(\mathbb{C})$ with the hermitian inner product

$$\langle \langle f, g \rangle \rangle_{\Gamma} = \int_{\Lambda(\Gamma)} f(z)\overline{g(z)}e^{-\nu|z|^2} \, dm(z).$$

(2.2)
Consider the kernel function \( K_{\Gamma, \lambda}^\nu(z, w) \) on \( \mathbb{C} \times \mathbb{C} \) given as series by
\[
K_{\Gamma, \lambda}^\nu(z, w) := \frac{v}{\pi} \sum_{\gamma \in \Gamma} \chi(\gamma)e^{-\frac{v}{2}|\gamma|^2 + v(z\gamma - \omega\gamma + z\omega)}.
\] (2.3)

Then \( K_{\Gamma, \lambda}^\nu(z, w) \) is well defined, holomorphic in \( z \) and antiholomorphic in \( w \). Moreover, the kernel function \( K_{\Gamma, \lambda}^\nu(z, w) \) possesses the properties summarized in the following

**Theorem 2.3** ([4]). Let \( K_{\Gamma, \lambda}^\nu(z, w) \) be as in (2.3). Then, we have

i) Let \( \Lambda(\Gamma) \) be an arbitrary fundamental cell of the lattice \( \Gamma \) in \( \mathbb{C} \). Then
\[
\int_{\Lambda(\Gamma)} K_{\Gamma, \lambda}^\nu(z, z)e^{-v|z|^2} dm(z) = \frac{v}{\pi} S(\Gamma) \neq 0.
\] (2.4)

ii) The function \( K_{\Gamma, \lambda}^\nu(z, w) \) satisfies the \( \Gamma \)-bi-invariance property
\[
K_{\Gamma, \lambda}^\nu(z + \gamma, w + \gamma') = \chi(\gamma)e^\frac{v}{2}|\gamma|^2 + vzw \nu K_{\Gamma, \lambda}^\nu(z, w) \overline{\chi(\gamma')}e^\frac{v}{2}|\gamma'|^2 + v\omega\gamma'
\] for every \( z, w \in \mathbb{C} \) and \( \gamma, \gamma' \in \Gamma \).

iii) Every \( f \in \mathcal{O}_{\Gamma, \lambda}^\nu(\mathbb{C}) \) can be reproduced either as
\[
f(z) = \frac{v}{\pi} \int_{\mathbb{C}} e^{vzw}f(w)e^{-v|w|^2} dm(w)
\] or also as
\[
f(z) = \int_{\Lambda(\Gamma)} K_{\Gamma, \lambda}^\nu(z, w)f(w)e^{-v|w|^2} dm(w).
\] (2.7)

For the sake of self-countenance, we reproduce here a sketched proof of such theorem. See [4] for more details and precisions.

**Proof.**  

i) follows by direct computation of \( \int_{\Lambda(\Gamma)} K_{\Gamma, \lambda}^\nu(z, z)e^{-v|z|^2} dm(z) \) using 2.3 and taking into account the fact that \( \int_{\Lambda(\Gamma)} e^{v(z\gamma - \omega\gamma)} dm(z) = 0 \), for every \( \gamma \in \Gamma \) with \( \gamma \neq 0 \), under the (RDQ) condition.

ii) By writing 2.3 for \( K_{\Gamma, \lambda}^\nu(z + \gamma, w) \) and next using a change of the variable \( \gamma \) together with the (RDQ) condition, we obtain
\[
k_{\Gamma, \lambda}^\nu(z + \gamma, w) = \chi(\gamma)e^\frac{v}{2}|\gamma|^2 + vzw \nu k_{\Gamma, \lambda}^\nu(z, w).
\]

This gives rise to ii) making use of \( K_{\Gamma, \lambda}^\nu(z, w) = K_{\Gamma, \lambda}^\nu(w, z) \).

iii) For given \( f \in \mathcal{O}_{\Gamma, \lambda}^\nu(\mathbb{C}) \), there exists certain constant \( C > 0 \) such that \( |f(z)| < Ce^\frac{v}{2}|z|^2 \). Then it is easy to check that the holomorphic functions \( f_\varepsilon(z) := f(\varepsilon z); 0 < \varepsilon < 1 \), satisfy the growth condition \( |f_\varepsilon(z)| < Ce^\frac{v}{2}|z|^2 \) and then belong to the Bargmann-Fock space \( B^{2,v}(\mathbb{C}) = \mathcal{O}(\mathbb{C}) \cap L^2(\mathbb{C}; e^{-v|z|^2} dm(z)) \), whose the reproducing kernel function is \( (v/\pi)e^{vzw} \). Finally, by tending \( \varepsilon \) to 1 in the reproducing formula
\[
g_\varepsilon(z) = \frac{v}{\pi} \int_{\mathbb{C}} e^{vzw}g_\varepsilon(w)e^{-v|w|^2} dm(w)
\]
and applying the dominated convergence theorem, we get \(2.6\). Thus \(2.7\) can be obtained from \(2.6\) by writing \(C\) as disjoint union of \(\gamma + \Lambda(\Gamma); \gamma \in \Gamma\), and using the fact that \(f\) belongs to \(O_{\Gamma,\chi}^\nu(C)\).

Corollary 2.4. There exists \(w_0 \in \mathbb{C}\) such that \(z \mapsto K_{\Gamma,\chi}^\nu(z, w_0)\), belonging to \(O_{\Gamma,\chi}^\nu(C)\), is not identically zero.

Proof. Clearly \(z \mapsto K_{\Gamma,\chi}^\nu(z, w); w \in \mathbb{C}\), in view of ii) of the previous theorem. The fact that \(z \mapsto K_{\Gamma,\chi}^\nu(z, w_0)\) is not identically zero for certain \(w_0 \in \mathbb{C}\) follows from i) of such theorem. \(\square\)

3 Generators of \(O_{\Gamma,\chi}^\nu(C)\) associated to Weierstrass pseudo-character

We start with the following result (whose the proof is left to the reader):

Lemma 3.1. Let \(\chi = \chi_w\) be the Weierstrass pseudo-character as in \(1.9\) and \(\nu = \nu(\Gamma) = \pi / S(\Gamma)\). Then \(\chi = \chi_w\) satisfies the \((RDQ)\) condition

\[
\chi(\gamma + \gamma') = \chi(\gamma)\chi(\gamma')e^{\nu(\gamma \bar{\gamma}' - \bar{\gamma} \gamma')}; \quad \gamma, \gamma' \in \Gamma.
\]

Hence, it follows from Theorem 2.3, that the corresponding \(O_{\Gamma,\chi}^\nu(C)\) is of dimension 1. Our concern below, is to exhibit some of its generators. From now on, the triplet \((\Gamma, \chi, \nu)\) is the one considered in the previous lemma.

3.1 The modified Weierstrass \(\sigma\)-function.

The main result of this section is the following

Theorem 3.2. There exist unique real number \(\nu > 0\) and complex number \(\mu\) such that the entire function

\[
\tilde{\sigma}_\mu(z; \Gamma) := e^{-\frac{1}{2} \bar{\mu}z^2} \sigma(z; \Gamma); \quad z \in \mathbb{C},
\]

is a generator of \(O_{\Gamma,\chi}^\nu(C)\). Precisely, \(\nu\) and \(\mu\) are given explicitly by \(3.7\) below.

Definition 3.3. The function \(\tilde{\sigma}(z; \Gamma) = \tilde{\sigma}_\mu(z; \Gamma)\) defined by \(3.2\) and associated to the special value of \(\mu\) given through \(3.7\) will be called the modified Weierstrass \(\sigma\)-function.

Proof. We have to prove only that the function \(\tilde{\sigma}_\mu(z; \Gamma)\), as given by \(3.2\), satisfies the functional equation

\[
\tilde{\sigma}_\mu(z + \gamma; \Gamma) := \chi(\gamma) e^{\frac{1}{2} |\gamma|^2 + \nu \bar{\gamma} \sigma}(z; \Gamma); \quad z \in \mathbb{C}, \gamma \in \Gamma.
\]

For this, we begin by recalling the pseudo-periodicity \([13, 7, 6]\), satisfied by the standard Weierstrass \(\sigma\)-function,

\[
\sigma(z + \gamma; \Gamma) = \chi_w(\gamma)e^{(z + \gamma/2)\eta} \sigma(z; \Gamma),
\]
where \( \eta(\gamma) \) is defined by \( \eta(m\omega_1 + n\omega_2) = m\eta_1 + n\eta_2 \), where \( \{\omega_1,\omega_2\} \) is a given oriented \( \mathbb{R} \)-basis of the lattice \( \Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \) and the \( \eta_j; j = 1, 2 \), are related to the Weierstrass zeta-function by \( \eta_j = 2\zeta(\omega_j/2) \). Therefore, \( \tilde{\sigma}_\mu(z; \Gamma) \) satisfies (3.3) if and only if
\[
e^{(\mu\gamma + \nu\bar{\gamma} - \eta)(\gamma + \bar{\gamma}/2)} = 1 \tag{3.4}
\] for every \( z \in \mathbb{C} \) and every \( \gamma \in \Gamma \). From Equation (3.4), it follows that the numbers \( \nu \) and \( \mu \) verify linear system
\[
\begin{align*}
\nu\bar{\omega}_1 + \mu\omega_1 &= \eta_1 \\
\nu\bar{\omega}_2 + \mu\omega_2 &= \eta_2
\end{align*} \tag{3.5}
\]
Hence, the numbers \( \nu \) and \( \mu \) exist and are unique for the determinant of (3.5) is
\[
\omega_1\omega_2 - \omega_1\omega_2 = 2iS \neq 0, \quad \text{where} \quad S = S(\Gamma) \quad \text{denotes the cell area of the lattice} \quad \Gamma.
\]
Solving (3.5) and making use of the Legendre’s relation [6, page 102],
\[
\eta_1\omega_2 - \eta_2\omega_1 = 2i\pi, \tag{3.6}
\]
we see that \( \nu \) and \( \mu \) are given by
\[
\nu = \frac{\pi}{S} \quad \text{and} \quad \mu = \frac{i}{S} \left( \zeta(\omega_1/2)\bar{\omega}_2 - \zeta(\omega_2/2)\bar{\omega}_1 \right), \tag{3.7}
\]
respectively. Therefore, for such values, the nonzero function \( \tilde{\sigma}_\mu(z; \Gamma) \) belongs to the one dimensional space \( \mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C}) \) and is then a generator of it. \( \square \)

Remark 3.4. From the obtained explicit expression, we see that \( \nu \) depends only on the lattice \( \Gamma \) for the cell area \( S = \Im(\omega_1\bar{\omega}_2) \) being independent of the choice of basis \( \omega_1,\omega_2 \). Therefore, uniqueness of \( \mu \), and so the independence of the choice of oriented basis of the lattice, follows by applying Liouville theorem to the entire \( \Gamma \)-periodic function \( \tilde{\sigma}_{\mu_1}(z; \Gamma) / \tilde{\sigma}_{\mu_2}(z; \Gamma) = e^{-\frac{1}{2}(\mu_1 - \mu_2)z^2} \).

3.2 The Weierstrass-theta series.

Below, we will introduce an other generator of \( \mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C}) \) as series over \( \Gamma \). Namely, we assert the following

Theorem 3.5. Keep \( \nu = \pi / S \) and \( \chi = \chi_W \) as above. Then, the entire function \( \theta_W(z; \Gamma) \) defined by
\[
\theta_W(z; \Gamma) = \sum_{\gamma \in \Gamma} \chi(\gamma)\gamma e^{-\frac{\pi}{2}|\gamma|^2 + \nu \bar{\gamma}} \tag{3.8}
\]
is a generator of the space \( \mathcal{O}^\nu_{\Gamma,\chi}(\mathbb{C}) \).

Definition 3.6. The odd entire function \( \theta_W(z; \Gamma) \) given by (3.8) will be called here Weierstrass \( (\Gamma, \chi) \)-theta series.

To prove 3.5, we make use of the following lemma.
Lemma 3.7. Keep $\nu = \pi / S$ and $\chi = \chi_w$ as above. Let $f$ be a holomorphic function on $\mathbb{C}$ such that $|f(z)| \leq Ce^{\alpha |z|^\beta}$ for certain constant $C \geq 0$ and given real numbers $\alpha \geq 0$ and $\beta < 2$. Define

$$[\mathcal{D}_V^{\nu}(f)](z) := \sum_{\gamma \in \Gamma} \chi(\gamma)e^{-\frac{\nu}{2}||\gamma||^2 + \nu z\gamma} f(z - \gamma)$$

(3.9)

to be the periodization à la Poincaré of the function $f$. Then,

i) $\mathcal{D}_V^{\nu}(f)$ belongs to $\mathcal{O}_V^{\nu}(\mathbb{C})$.

ii) If $f$ is in addition an even function, then $\mathcal{D}_V^{\nu}(f)$ is also even and consequently is identically zero. vanishes on $\mathbb{C}$ whenever $f$ is an even function.

Proof of Theorem 3.5. By specifying $f(z) = 1$ and appealing ii) of Lemma 3.7, we get easily the following identity,

$$\sum_{\gamma \in \Gamma} \chi(\gamma)e^{-\frac{\nu}{2}||\gamma||^2 + \nu z\gamma} = 0 \quad \text{for every } z \in \mathbb{C}. \quad (3.10)$$

Hence, it follows

$$\theta_W(z; \Gamma) \overset{(3.10)}{=} -\sum_{\gamma \in \Gamma} \chi(\gamma)(z - \gamma)e^{-\frac{\nu}{2}||\gamma||^2 + \nu z\gamma} = -\mathcal{D}_V^{\nu}(z \rightarrow z),$$

and therefore $\theta_W(z; \Gamma)$ belongs to $\mathcal{O}_V^{\nu}(\mathbb{C})$, according to i) of Lemma 3.7. The hard part is to prove that $\theta_W(z; \Gamma)$ is not identically zero on $\mathbb{C}$. For this, recall from Corollary 2.4, that the holomorphic function

$$z \mapsto K_V^{\nu}(z, w_0) := \frac{\nu}{\pi} \sum_{\gamma \in \Gamma} \chi(\gamma)e^{-\frac{\nu}{2}||\gamma||^2 + \nu z(\gamma + \nu w_0)} \in \mathcal{O}_V^{\nu}(\mathbb{C}),$$

is not identically zero on $\mathbb{C}$ for certain $w_0$. This means that $z \mapsto K_V^{\nu}(z, w_0)$ is a generator of the one dimensional space $\mathcal{O}_V^{\nu}(\mathbb{C})$ generated also by the modified Weierstrass function $\sigma(z; \Gamma)$ (Theorem 3.2). This implies the existence of a constant $C_{w_0} \neq 0$ such that $K_V^{\nu}(z, w_0) = C_{w_0}\sigma(z; \Gamma)$. More explicitly,

$$\frac{\nu}{\pi} \sum_{\gamma \in \Gamma} \chi(\gamma)e^{-\frac{\nu}{2}||\gamma||^2 + \nu z(\gamma + w_0)} e^{-\nu |w_0\gamma|} = C_{w_0} e^{-\frac{\nu}{2}z\gamma}$$

(3.11)

Differentiating both sides of (3.11) at $z = 0$ and using the fact $\sigma'(0; \Gamma) = 1$, we find

$$\frac{\nu^2}{\pi} \sum_{\gamma \in \Gamma} \chi(\gamma)(w_0 + \gamma)e^{-\frac{\nu}{2}||\gamma||^2 - \nu |w_0\gamma|} = C_{w_0} \neq 0.$$

Now, making the conjugate and changing $\gamma$ to $-\gamma$ keeping in mind that the Weierstrass pseudo-character $\chi$ is real and even, it follows

$$\frac{\nu^2}{\pi} \sum_{\gamma \in \Gamma} \chi(\gamma)(w_0 - \gamma)e^{-\frac{\nu}{2}||\gamma||^2 - \nu |w_0\gamma|} = \overline{C_{w_0}} \neq 0.$$
Finally, from definition of $\theta_w(\cdot; \Gamma)$, we deduce that $\theta_w(w_0; \Gamma) = -\frac{\pi}{\nu} \overline{C_{w_0}} \neq 0$. This completes the proof of Theorem 3.5.

**Sketched proof of Lemma 3.7.** The growth condition $|f(z)| \leq Ce^{\alpha|z|^\beta}$, $\alpha \geq 0$, $\beta < 2$, assumed to be satisfied by the holomorphic function $f$, ensures that the Poincaré series (3.9) converges absolutely and uniformly on compact subsets of $\mathbb{C}$ and then is holomorphic. To conclude for i), note that for every given $\gamma_0 \in \Gamma$, we have

$$[\mathcal{P}_{\Gamma, \chi}^\nu(f)](z + \gamma_0) := \sum_{\gamma \in \Gamma} \chi(\gamma)e^{-\frac{\nu}{2}|\gamma|^2 + \nu(z + \gamma_0)^2} f(z + \gamma_0 - \gamma).$$

Using the change of variable $\gamma = \gamma_0 + \gamma'$ together with the (RDQ) condition (3.1),

$$\chi(\gamma_0 + \gamma') = \chi(\gamma_0)\chi(\gamma')e^{\nu|\gamma_0 - \gamma_0| - \nu|\gamma'|},$$

the above summation over $\Gamma$ reduces further to the following

$$[\mathcal{P}_{\Gamma, \chi}^\nu(f)](z + \gamma_0) = \chi(\gamma_0)e^{\nu|\gamma_0|^2 + \nu z \bar{\gamma}_0} \sum_{\gamma' \in \Gamma} \chi(\gamma')e^{\nu|\gamma_0 - \gamma_0| - \nu|\gamma'|} e^{-\frac{\nu}{2}|\gamma'|^2 + \nu z \bar{\gamma}_0} f(z - \gamma').$$

Now, since $e^{\nu|\gamma_0 - \gamma_0|} = 1$, which is an immediate consequence of the (RDQ) condition, one deduces

$$[\mathcal{P}_{\Gamma, \chi}^\nu(f)](z + \gamma_0) = \chi(\gamma_0)e^{\nu|\gamma_0|^2 + \nu z \bar{\gamma}_0} \sum_{\gamma' \in \Gamma} \chi(\gamma')e^{-\frac{\nu}{2}|\gamma'|^2 + \nu z \bar{\gamma}_0} f(z - \gamma')$$

$$= \chi(\gamma_0)e^{\nu|\gamma_0|^2 + \nu z \bar{\gamma}_0}[\mathcal{P}_{\Gamma, \chi}^\nu(f)](z).$$

To get ii), let $f$ be an even function satisfying the hypothesis of Lemma 3.7. Then, for $\chi$ being even, it is easy to show that $\mathcal{P}_{\Gamma, \chi}^\nu(f)$ is also an even function belonging to the one dimensional space $\mathcal{O}_{\Gamma, \chi}^\nu(\mathbb{C})$ which is generated by the odd function $\tilde{\sigma}(z; \Gamma)$. So necessary $\mathcal{P}_{\Gamma, \chi}^\nu(f) \equiv 0$ on $\mathbb{C}$.

We conclude this section by the following remark related to Lemma 3.7.

**Remark 3.8.**

The assertion ii) of Lemma 3.7 can be used to derive a full series of identities by specifying the function $f$. Here we provide some examples.

i) Consider the function $f(z) = \cos(\lambda z)$ with $\lambda \in \mathbb{C}$, one gets

$$\sum_{\gamma \in \Gamma} \chi(\gamma) \cos(\lambda(z - \gamma)) e^{-\frac{\nu}{2}|\gamma|^2 + \nu z \bar{\gamma}} = 0.$$  

For $z = 0$, they reduce to the following

$$\sum_{\gamma \in \Gamma} \chi(\gamma) \cos(\lambda \gamma) e^{-\frac{\nu}{2}|\gamma|^2} = 0.$$
ii) Other interesting identities involve even polynomials. Indeed, for \( f(z) = z^{2p}, \ p = 0, 1, 2, \cdots \), we obtain
\[
\sum_{\gamma \in \Gamma} \chi(\gamma)(z - \gamma)^{2p}e^{-\frac{1}{2}|\gamma|^2 + vz\gamma} = 0. \tag{3.12}
\]

In particular
\[
\sum_{\gamma \in \Gamma} \chi(\gamma)\gamma^{2p}e^{-\frac{1}{2}|\gamma|^2} = 0.
\]

More generally, for every positive integer \( k = 0, 1, 2, \cdots \), we have
\[
\sum_{\gamma \in \Gamma} \chi(\gamma)\gamma^{k}e^{-\frac{1}{2}|\gamma|^2} = 0, \quad \nu = \pi / S. \tag{3.13}
\]

Remark 3.9. The obtained identities (3.13), are exactly those obtained by Perelomov [10, Equation (47)], for regular lattice with cell area \( S = \pi \) (so \( \nu = 1 \)), when dealing with the completeness of the coherent state system. The only proof the simplest case \( p = 0 \) is given there and done by means of the transformation formulas involving theta functions.

In the next section, we proceed towards a proof of the main result of this paper.

4 Proof of main Theorem 1.1

Proof of i): We have to prove the following representation series of the Weierstrass \( \sigma \)-function
\[
\sigma(z; \Gamma) = \frac{1}{\mathcal{K}_0(\Gamma)} e^\frac{\eta^2}{2} \sum_{\gamma \in \Gamma} \chi(\gamma)\gamma e^{-\frac{1}{2}|\gamma|^2 + vz\gamma}, \tag{4.1}
\]

where \( \mathcal{K}_0(\Gamma) = \nu \sum_{\gamma \in \Gamma} \chi(\gamma)|\gamma|^2 e^{-\frac{1}{2}|\gamma|^2} \). Indeed, since \( \theta_W(z; \Gamma) \) and \( \tilde{\sigma}(z; \Gamma) \) are both generators of the one dimensional space \( \mathcal{O}_{1,\chi}(\mathbb{C}) \) (see Theorems 3.5 and 3.2), there exists a constant \( C \neq 0 \) such that \( \theta_W(z; \Gamma) = C\tilde{\sigma}(z; \Gamma) \). This reads explicitly as
\[
\sum_{\gamma \in \Gamma} \chi(\gamma)\gamma e^{-\frac{1}{2}|\gamma|^2 + vz\gamma} = C\tilde{\sigma}(z; \Gamma) = Ce^{-\frac{1}{2}|\mu|^2} \sigma(z; \Gamma); \quad C \neq 0. \tag{4.2}
\]

Differentiation of both sides of (4.2) and next taking \( z = 0 \) yield
\[
v \sum_{\gamma \in \Gamma} \chi(\gamma)|\gamma|^2 e^{-\frac{1}{2}|\gamma|^2} = C\sigma'(0; \Gamma) = C \neq 0. \tag{4.3}
\]

Note that the right hand side in (4.3) is exactly \( \mathcal{K}_0(\Gamma) \) and therefore \( \mathcal{K}_0(\Gamma) = C \neq 0 \). Inserting this in (4.2) yields the representation series (4.1). \( \square \)
Proof of ii): By applying iii) of Theorem 2.3 to the modified Weierstrass $\sigma$-function
\[ \tilde{\sigma}(z; \Gamma) := e^{-\frac{1}{2}z^2} \sigma(z; \Gamma) \] which belongs to $\mathcal{O}_\nu^\infty(\mathbb{C})$ (Theorem 3.2), we get the following
\[ \tilde{\sigma}(z; \Gamma) = \frac{V}{\pi} \int_{\mathbb{C}} e^{\nu z \pi z} \tilde{\sigma}(w; \Gamma) e^{-\nu |w|^2} dm(w) \] (4.4)
and therefore, we obtain the reproducing integral formula over $\mathbb{C}$ for the Weierstrass $\sigma$-function
\[ \sigma(z; \Gamma) = \frac{V}{\pi} e^{\nu z^2} \int_{\mathbb{C}} e^{\nu z \pi \frac{b}{a} z^2} \sigma(w; \Gamma) e^{-\nu |w|^2} dm(w). \] (4.5)
This completes the proof of ii). \[ \square \]

Proof of iii): The expression provided for the Taylor coefficients of the Weierstrass $\sigma$-function $\sigma(z; \Gamma)$ can be checked easily by making use of the obtained representation series (4.1) combined with the following technical lemma.

Lemma 4.1. For arbitrary complex numbers $a, b$, we have the following expansion in power series of the exponential function $e^{az^2 + bz}$, to wit
\[ e^{az^2 + bz} = \sum_{r \geq 0} \frac{a^r}{r!} F(-r; \frac{1}{2}, \frac{b^2}{4a}) z^{2r} + b \sum_{r \geq 0} \frac{a^r}{r!} F(-r; \frac{3}{2}, \frac{b^2}{4a}) z^{2r+1}, \] (4.6)
where $F(a; c; x)$ is the usual confluent hypergeometric function.

In fact, substitution of (4.6) into the representation series (4.1), with $a = \mu/2$ and $b = \nu \gamma$, infers
\[ \sigma(z; \Gamma) = \frac{1}{\mathcal{H}_0(\Gamma)} \sum_{r \geq 0} \frac{(\mu/2)^r}{r!} \left( \sum_{\gamma \in \Gamma} \chi(\gamma) \gamma F(-r; \frac{1}{2}; -\frac{\mu^2 \gamma^2}{2\mu}) e^{-\frac{\gamma}{2} |\gamma|^2} \right) z^{2r} \]
\[ + \frac{\nu}{\mathcal{H}_0(\Gamma)} \sum_{r \geq 0} \frac{(\mu/2)^r}{r!} \left( \sum_{\gamma \in \Gamma} \chi(\gamma) |\gamma|^2 F(-r; \frac{3}{2}; -\frac{\mu^2 \gamma^2}{2\mu}) e^{-\frac{\gamma}{2} |\gamma|^2} \right) z^{2r+1}. \]

Next, changing $\gamma$ by $-\gamma$ in the first sum of the right hand side, using the fact that $\chi(-\gamma) = \chi(\gamma)$, we see easily that
\[ \sum_{\gamma \in \Gamma} \chi(\gamma) \gamma F(-r; \frac{1}{2}; -\frac{\mu^2 \gamma^2}{2\mu}) e^{-\frac{\gamma}{2} |\gamma|^2} = 0 \]
and therefore the above expression of $\sigma(z; \Gamma)$ reduces further to the following one
\[ \sigma(z; \Gamma) = \sum_{r \geq 0} \frac{\mathcal{H}_r(\Gamma)}{\mathcal{H}_0(\Gamma) (2r+1)!} z^{2r+1} \]
where we have set
\[ \mathcal{H}_r(\Gamma) = \nu \frac{(2r+1)!}{r!} \left( \frac{\mu}{2} \right)^r \sum_{\gamma \in \Gamma} \chi(\gamma) |\gamma|^2 F(-r; \frac{3}{2}; -\frac{\mu^2 \gamma^2}{2\mu}) e^{-\frac{\gamma}{2} |\gamma|^2}. \]
This completes the proof of iii) of Theorem 1.1. □

**Proof of iv):** This is an immediate consequence of the reproducing integral formula (4.5) (i.e., ii) of Theorem 1.1. Indeed, by replacing there the involved exponential function $e^{\mu z^2 + \nu zw}$ by its expansion in power series as given by (4.6) (with $a = \mu/2$ and $b = \nu w$), and comparing the obtained expression to $\sigma(z; \Gamma) = \sum_{r=0}^{\infty} \mathcal{W}_r(\Gamma) \frac{z^{2r+1}}{(2r+1)!}$, one deduces easily the representation of the coefficients $\mathcal{W}_r(\Gamma)$ in terms of the Gauss-Hermite integrals. □

The proof of Theorem 1.1 is completed.

**Remark 4.2.** The technical Lemma 4.1 is basically an alternative appropriate form of $e^{-z^2 + 2tz}$ which is the generating function of the Hermite polynomials $H_n(t)$. Indeed, using this and next splitting the obtained sum by collecting terms in $z^{2r}$ and those in $z^{2r+1}$ together with the transformations [8, page 252]

$$H_{2r}(z) = (-1)^r \frac{(2r)!}{r!} F(-r; \frac{1}{2}; z^2) \quad \text{and} \quad H_{2r+1}(z) = (-1)^r \frac{(2r+1)!}{r!} 2z F(-r; \frac{3}{2}; z^2)$$

yield the desired result as stated in Lemma 4.1.
5 Some applications

In this section, we keep \( \nu = \nu(\Gamma) = \pi / S \), \( \mu = \mu(\Gamma) \) and \( \chi = \chi_w \) as in the introduction and we discuss some direct applications of Theorem [1.1]

5.1 Some identities of arithmetic type.

Direct comparison of the obtained representations (1.13) and (1.14) to the Weierstrass representation (1.5) yields a full series of free highly nontrivial identities of arithmetic type on our Hermite-Gauss series. Thus, owing to the fact \( \psi_1(\Gamma) = 0 \), we can deduce an intrinsic expression of \( \mu = \mu(\Gamma) \) defined by (1.8). Namely

**Corollary 5.1.** The complex number \( \mu = \mu(\Gamma) \) is independent of the choose of the oriented basis of the lattice \( \Gamma \) and is given as series over \( \Gamma \) by

\[
\mu = -\frac{\nu^2}{3} \left( \frac{\sum_{\gamma \in \Gamma} \chi(\gamma) \gamma^2 |\gamma|^2 e^{-\frac{\nu}{2} |\gamma|^2}}{\sum_{\gamma \in \Gamma} \chi(\gamma) |\gamma|^2 e^{-\frac{\nu}{2} |\gamma|^2}} \right). \tag{5.1}
\]

**Remark 5.2.** As immediate consequence, the \( \eta \) function on \( \Gamma \) in the proof of Theorem 3.2 can also be defined intrinsically by

\[
\eta(\gamma) = \nu \bar{\gamma} + \mu \gamma; \quad \gamma \in \Gamma.
\]

Now, according to the facts \( \psi_2(\Gamma) = -(1/2)g_2 \) and \( \psi_3(\Gamma) = -2 \cdot 3g_3 \), we can deduce easily the following

**Corollary 5.3.** The modular forms \( g_2(\Gamma) = 60 \sum \gamma^4 \) and \( g_3(\Gamma) = 140 \sum \gamma^6 \) can be expressed as series over \( \Gamma \) as

\[
g_2(\Gamma) = \left( \frac{-30}{\sum_{\gamma \in \Gamma} \chi(\gamma) \gamma^2 e^{-\frac{\nu}{2} |\gamma|^2}} \right) \sum_{\gamma \in \Gamma} \chi(\gamma) |\gamma|^2 \mu^2 F(-2; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e^{-\frac{\nu}{2} |\gamma|^2} \tag{5.2}
\]

\[
g_3(\Gamma) = \left( \frac{-35}{2 \sum_{\gamma \in \Gamma} \chi(\gamma) \gamma^2 e^{-\frac{\nu}{2} |\gamma|^2}} \right) \sum_{\gamma \in \Gamma} \chi(\gamma) |\gamma|^2 \mu^3 F(-3; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e^{-\frac{\nu}{2} |\gamma|^2}. \tag{5.3}
\]

or also as integrals on \( \mathbb{C} \) as

\[
g_2(\Gamma) = -\frac{30
u^2}{\pi} \int_{\mathbb{C}} \bar{w} \mu^2 F(-2; \frac{3}{2}; -\frac{v^2 \bar{w}^2}{2\mu}) e^{-\frac{\nu}{2} |\bar{w}|^2} \sigma(w; \Gamma) e^{\nu |w|^2} \ dm(w) \tag{5.4}
\]

\[
g_3(\Gamma) = -\frac{35
u^2}{2\pi} \int_{\mathbb{C}} \bar{w} \mu^3 F(-3; \frac{3}{2}; -\frac{v^2 \bar{w}^2}{2\mu}) e^{-\frac{\nu}{2} |\bar{w}|^2} \sigma(w; \Gamma) e^{\nu |w|^2} \ dm(w) \tag{5.5}
\]

A byproduct of (5.2) and (5.3) with the help of the Weierstrass representation gives rise to a unlimited list of identities on our Hermite-Gauss series. As particular examples, one can use the facts \( \psi_4(\Gamma) = -[1/(2^9 \cdot 3^2 \cdot 5 \cdot 7)]g_2^2 \) and \( \psi_5(\Gamma) = [1/(2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11)]g_2g_3 \) to get the following ones
Corollary 5.4. Let $e_{\chi}^{\nu}(\gamma)$ be the mapping defined on $\Gamma$ by $e_{\chi}^{\nu}(\gamma) := |\gamma|^2 \chi(\gamma) e^{-\frac{1}{2} |\gamma|^2}$. Then the following identities hold

$$\left( \sum_{\gamma \in \Gamma} \mu^2 F(-2; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e_{\chi}^{\nu}(\gamma) \right)^2 = \frac{7}{15} \sum_{\gamma \in \Gamma} e_{\chi}^{\nu}(\gamma) \times \sum_{\gamma \in \Gamma} \mu^4 F(-4; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e_{\chi}^{\nu}(\gamma)$$ \hspace{1cm} (5.6)

$$\sum_{\gamma \in \Gamma} \mu^2 F(-2; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e_{\chi}^{\nu}(\gamma) \times \sum_{\gamma \in \Gamma} \mu^3 F(-3; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e_{\chi}^{\nu}(\gamma)$$

$$= -\frac{11}{10} \sum_{\gamma \in \Gamma} e_{\chi}^{\nu}(\gamma) \times \sum_{\gamma \in \Gamma} \mu^5 F(-5; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e_{\chi}^{\nu}(\gamma).$$ \hspace{1cm} (5.7)

Consequently, from (5.6) and (5.7) together with

$$\mathcal{W}_S(\Gamma) = \frac{23g_2^3 - 576g_3^2}{2^{10} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13},$$

we get

Corollary 5.5.

$$\sum_{\gamma \in \Gamma} \mu^3 F(-3; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e_{\chi}^{\nu}(\gamma) \times \sum_{\gamma \in \Gamma} \mu^4 F(-4; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e_{\chi}^{\nu}(\gamma)$$

$$= -\frac{33}{14} \sum_{\gamma \in \Gamma} \mu^2 F(-2; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e_{\chi}^{\nu}(\gamma) \times \sum_{\gamma \in \Gamma} \mu^5 F(-5; \frac{3}{2}; -\frac{v^2 \gamma^2}{2\mu}) e_{\chi}^{\nu}(\gamma).$$ \hspace{1cm} (5.8)

For the particular lattices $\Gamma$ such that $\mu(\Gamma) = 0$, like the standard lattices of Gauss numbers $\Gamma = \mathbb{Z}[i]$ and of Jacobi-Eisenstein numbers $\Gamma = \mathbb{Z}[j]$, one obtains similar identities involving the quantities

$$\sum_{\gamma \in \Gamma} |\gamma|^2 \chi(\gamma) e^{-\frac{1}{2} |\gamma|^2} = \sum_{\gamma \in \Gamma} e_{\chi}^{\nu}(\gamma); \hspace{1cm} k = 0, 1, 2, \ldots$$

Indeed, they read simply as

$$\left( \sum_{\gamma \in \Gamma} \gamma e_{\chi}^{\nu}(\gamma) \right)^2 = \frac{1}{11} \left( \sum_{\gamma \in \Gamma} e_{\chi}^{\nu}(\gamma) \right) \left( \sum_{\gamma \in \Gamma} \gamma^8 e_{\chi}^{\nu}(\gamma) \right)$$ \hspace{1cm} (5.9)

$$\left( \sum_{\gamma \in \Gamma} \gamma^4 e_{\chi}^{\nu}(\gamma) \right) \left( \sum_{\gamma \in \Gamma} \gamma^6 e_{\chi}^{\nu}(\gamma) \right) = -\frac{1}{6} \left( \sum_{\gamma \in \Gamma} e_{\chi}^{\nu}(\gamma) \right) \left( \sum_{\gamma \in \Gamma} \gamma^{10} e_{\chi}^{\nu}(\gamma) \right)$$ \hspace{1cm} (5.10)

$$\left( \sum_{\gamma \in \Gamma} \gamma^6 e_{\chi}^{\nu}(\gamma) \right) \left( \sum_{\gamma \in \Gamma} \gamma^8 e_{\chi}^{\nu}(\gamma) \right) = -\frac{3}{2} \left( \sum_{\gamma \in \Gamma} \gamma^4 e_{\chi}^{\nu}(\gamma) \right) \left( \sum_{\gamma \in \Gamma} \gamma^{10} e_{\chi}^{\nu}(\gamma) \right).$$ \hspace{1cm} (5.11)
5.2 Eisenstein series $G_{2n}(\Gamma)$ as rational fraction of $\theta^{2j+1}_W(z; \Gamma)|_{z=0}$.

Our concern in this subsection is to give representation of the Eisenstein series

$$G_{2n}(\Gamma) = \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{\gamma^{2n}}$$

as rational fraction of the successive derivatives of the Weierstrass $(\Gamma, \chi)$-theta series

$$\theta_W(z; \Gamma) := \sum_{\gamma \in \Gamma} \gamma \chi(\gamma) e^{-\frac{1}{2} |\gamma|^2 + v z} = \sum_{j \geq 0} \theta_W^{(2j+1)}(0; \Gamma) \frac{z^{2j+1}}{(2j+1)!}$$

where for every $j = 0, 1, 2, \cdots$, the coefficient $\theta^{(2j+1)}_W(0; \Gamma)$ stands for

$$\theta^{(2j+1)}_W(0; \Gamma) = \theta^{(2j+1)}_W(z; \Gamma)|_{z=0} = v^{2j+1} \sum_{\gamma \in \Gamma} \gamma^{2j} |\gamma|^2 \chi(\gamma) e^{-\frac{1}{2} |\gamma|^2}.$$ 

For this, let recall that $\theta_W(z; \Gamma)$ is connected to the Weierstrass $\sigma$-function by (see i) of Theorem 1.1

$$\theta_W(z; \Gamma) = \mathcal{K}_0(\Gamma) e^{-\frac{v}{2} z^2} \sigma(z; \Gamma).$$

Therefore, the Weierstrass zeta-function $\zeta(z; \Gamma) = \sigma'(z; \Gamma) / \sigma(z; \Gamma)$ can be rewritten as follows

$$\zeta(z; \Gamma) = \mu z + \frac{\theta'_W(z; \Gamma)}{\theta_W(z; \Gamma)} = \mu z + \frac{1}{z} \sum_{j \geq 0} \frac{(2j + 1) X_j z^{2j}}{\sum_{j \geq 0} X_j z^{2j}},$$

(5.12)

where we have set

$$X_j = X_j(\Gamma) := \frac{1}{(2j+1)!} \frac{\theta^{(2j+1)}_W(z; \Gamma)|_{z=0}}{\theta_W(z; \Gamma)|_{z=0}}$$

(5.13)

$$= \frac{v^{2j}}{(2j+1)!} \left( \sum_{\gamma \in \Gamma} \gamma^{2j} |\gamma|^2 \chi(\gamma) e^{-\frac{1}{2} |\gamma|^2} \right)$$

(5.14)

so that $X_0 = 1$. Thus, keeping in mind that $z \mapsto \zeta(z; \Gamma)$ is an odd function, we can write down the Taylor expansion at $z = 0$ of $z \mapsto \left( \sum_{j \geq 0} (2j + 1) X_j z^{2j} \right) / \left( \sum_{j \geq 0} X_j z^{2j} \right)$ as

$$\sum_{j \geq 0} \frac{(2j + 1) X_j z^{2j}}{\sum_{j \geq 0} X_j z^{2j}} = \sum_{j \geq 0} Y_j z^{2j}.$$ 

(5.15)

It follows that the coefficients $Y_j$ can be obtained by induction through the formula

$$Y_j = 2j X_j - \sum_{k=1}^{j-1} Y_k X_{j-k}.$$ 

(5.16)
for every \( j \geq 1 \). By comparing this to the well known Laurent expansion of \( \zeta(z; \Gamma) \) about zero \([6]\)

\[
\zeta(z; \Gamma) = \frac{1}{z} - \sum_{n=2}^{\infty} G_{2n}(\Gamma)z^{2n-1}
\]

one sees easily that \( Y_0 = 1, \ Y_1 = -\mu \) and \( Y_j = -G_{2j} \) for every \( j \geq 2 \). Moreover, we assert the following

**Proposition 5.6.** For every positive integer \( n \) there exists a polynomial \( P_n \) in the variables \( X_1, X_2, \ldots, X_n \) with integer coefficients such that

\[
G_{2n}(\Gamma) = P_n(X_1, X_2, \ldots, X_n).
\]

Particularly, the few first expressions of \( G_{2n} \) up to \( n \leq 6 \) are given explicitly by

\[
\begin{align*}
0 &= \mu + 2X_1 \quad \text{(5.16)} \\
G_4 &= 2(X_1^2 - 2X_2) \quad \text{(5.17)} \\
G_6 &= -2(X_1^3 - 3X_1X_2 + 3X_3) \quad \text{(5.18)} \\
G_8 &= 2(X_1^4 - 4X_1^2X_2 + 4X_1X_3 + 2X_2^2 - 4X_4) \quad \text{(5.19)} \\
G_{10} &= -2(X_1^5 - zX_1^3X_2 + 5X_1^2X_3 + 5X_1X_2^2 - 5X_1X_4 - 5X_2X_3 + 5X_5) \quad \text{(5.20)} \\
G_{12} &= 2(X_1^6 - 3X_1^4X_2 + 6X_1^3X_3 + 9X_1^2X_2^2 - 6X_1^2X_4 - 12X_1X_2X_3 + 6X_1X_5 \\
&\quad - 2X_2^3 + 6X_2X_4 + 3X_3^2 - 6X_6) \quad \text{(5.21)}
\end{align*}
\]

Whence, one can rederive again from (5.16) the intrinsic expression (1.8) of the complex number \( \mu \).

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