Tidal Love numbers and moment-Love relations of polytropic stars

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ABSTRACT
The physical significance of tidal deformation in astronomical systems has long been known. The recently discovered universal I-Love-Q relations, which connect moment of inertia, quadrupole tidal Love number, and spin-induced quadrupole moment of compact stars, also underscore the special role of tidal deformation in gravitational wave astronomy. Motivated by the observation that such relations also prevail in Newtonian stars and crucially depend on the stiffness of a star, we consider the tidal Love numbers of Newtonian polytropic stars whose stiffness is characterised by a polytropic index $n$. We first perturbatively solve the Lane-Emden equation governing the profile of polytropic stars through the application of the scaled delta expansion method and then formulate perturbation series for the multipolar tidal Love number about the two exactly solvable cases with $n = 0$ and $n = 1$, respectively. Making use of these two series to form a two-point Padé approximant, we find an approximate expression of the quadrupole tidal Love number, whose error is less than $2.5 \times 10^{-5}$ per cent ($0.39$ per cent) for $n \in [0, 1]$ ($n \in [0, 3]$). Similarly, we also determine the mass moments for polytropic stars accurately. Based on these findings, we are able to show that the I-Love-Q relations are in general stationary about the incompressible limit irrespective of the equation of state (EOS) of a star. Moreover, for the I-Love-Q relations, there is a secondary stationary point near $n \approx 0.4444$, thus showing the insensitivity to $n$ for $n \in [0, 1]$. Our investigation clearly tracks the universality of the I-Love-Q relations from their validity for stiff stars such as neutron stars to their breakdown for soft stars.

Key words: methods: analytical - stars: neutron - stars: white dwarfs - stars: interiors - stars: oscillations - hydrodynamics

1 INTRODUCTION

The phenomenon of tidal deformation is ubiquitous as well as important in many astronomical systems such as coalescing binary stars (see, e.g., Lai & Wu 2006, and references therein), our own solar system and other planetary systems (see, e.g., Beuthe 2015, and references therein). In order to quantify and analyse the tidal effects on a star (or a planet) due to other companion stars (or planets), the concept of tidal Love numbers was introduced (Love 1909), which is a measure of the ratio of the induced multipole moment to an external multipole field applied on the star (or planet) under consideration in the linear response regime. The tidal Love numbers of a star (or a planet) could depend sensitively on its physical characteristics (e.g., mass, radius and elasticity) and its equation of state (EOS), (see, e.g., Brooker & Olle 1955; Damour & Nagar 2009; Postnikov et al. 2010; Beuthe 2015, and references therein). As a result, some proposals have been put forward to infer the physical characteristics and the EOS of neutron stars (NSs) and quark stars (QSs) from their tidal Love numbers (see, e.g., Damour & Nagar 2009; Postnikov et al. 2010).

Although the study of tidal Love numbers was first started more than a century ago (Love 1909), it has regained proper attention in this new millennium for its special role in gravitational wave astronomy, which has been realized recently (Abbott et al. 2016). Flanagan & Hinderer (2008) pointed out that tidal Love numbers of NSs could be inferred from the associated gravitational wave signals emitted in the inspiral stage of neutron star-black hole (NS-BH) and neutron star-neutron star (NS-NS) binaries. In addition, Yagi & Yunes (2013a,b) discovered the universal I-Love-Q relations among the moment of inertia, the quadrupole tidal Love number (deformability) and the spin-induced quadrupole moment of NSs and QSs are almost independent of the EOS of these stars. Such relations are remarkable in the sense that a measurement of one of these three quantities could provide estimates of the other two with high precision, and it is claimed that such relations could facilitate the analysis of gravitational wave signals detected from mergers of compact star binaries (see Yagi & Yunes 2017, and references therein).

The physical origin of these universal relations has attracted a lot of interest since their discovery. Yagi & Yunes (2013a) attempted to attribute these universal relations of compact stars to the no-hair theorem, which states that black holes do not have interior structure, in the black hole limit. On the other hand, several
studies (Sham et al. 2015; Chan et al. 2014, 2015, 2016) indicate that the I-Love-Q relations indeed originate from the facts that (i) compact stars (including both NSs and QSs) are stiff enough to be considered as incompressible during tidal deformation, and (ii) the I-Love-Q relations are stationary and remarkably flat around the incompressible limit. As a matter of fact, the I-Love-Q relations still prevail in the Newtonian limit and their accuracies were found to worsen towards the high compactness regime (Sham et al. 2015; Chan et al. 2015, 2016).

Motivated by the findings of Sham et al. (2015); Chan et al. (2015, 2016), we study analytically in the present paper how the I-Love-Q relations depend on the stiffness of respective stars under Newtonian gravity by considering polytropic stars whose stiffness can be varied continuously and is characterised by the polytropic index \( n \). As the spin-induced quadrupole moment is completely specified by the quadrupole tidal Love number and the moment of inertia for Newtonian stars, we focus our attention on the I-Love relation as well as its multipolar generalisation in the present paper.

Our main objective here is to obtain accurate approximants for the I-Love relation of stable polytropic stars as a function of the polytropic index \( n \). Firstly, we make use of the scaled delta expansion method proposed recently by Yip et al. (2017) to find the density profile of polytropic stars. Secondly, a perturbative expansion for the tidal Love numbers in terms of moments of the mass density distribution for spherical Newtonian stars is derived. When such an expansion is applied to polytropic stars whose density profile can be found approximately from the scaled delta expansion method, we succeed in finding perturbative corrections to the tidal Love numbers of degree \( l = 2 \) and \( l = 3 \), up to the third order about the incompressible limit \( n = 0 \), and up to the first order about the linear model \( n = 1 \). The perturbation formulae yield accurate two-point Padé analytical expressions of the tidal Love numbers of degree \( l = 2 \) and \( l = 3 \), with the errors within \( 2.5 \times 10^{-5} \) per cent and \( 7.5 \times 10^{-5} \) per cent respectively over the range of polytropic indices \( n \in [0, 1] \). Overall, the entire physical interval of stability, where \( n \in [0, 3] \), the errors are still less than 1 per cent (see Table 3).

On the other hand, the scaled moment of inertia (or its multipolar counterpart) of polytropic stars can also be found directly from integrating the density profile resulting from the application of the scaled delta expansion method. We see that the I-Love relation vary less than 0.4 per cent for \( n \in [0, 1] \). However, the I-Love relation and likewise the multipolar moment-Love relations demonstrate more obvious dependence on \( n \) for \( n > 1 \) (see Table 5). Therefore, these relations can be considered as universal only for sufficiently stiff stars. Thus, our analytical form of the I-Love relation (or the multipolar moment-Love relations) further supports the proposal reported in Sham et al. (2015); Chan et al. (2015, 2016).

In addition to the analytical formulae of the I-Love relation and the multipolar moment-Love relation of polytropic stars in the Newtonian limit, we further supplement the universality of these relations by analysing the perturbative response of the tidal Love numbers to arbitrary density variations. We find that the stationarity of the I-Love relation, as well as the multipolar moment-Love relation, at the incompressible limit is actually independent of the exact form of the density variation. In fact, the stationarity of these relations is due to the cancellation of the respective variations in the mass, the mass moment, and the tidal Love number.

The structure and the major findings of the paper are summarized as follows. In Section 2, we briefly review the I-Love-Q relations. In Section 3, we introduce the polytropic EOS and the Lane-Emden equation (LEE) governing the hydrostatic equilibrium of polytropic stars. In Section 4, we outline the scaled delta expansion method proposed by Yip et al. (2017) to perturbatively solve the Lane-Emden equation from the incompressible limit \( n = 0 \) and the case \( n = 1 \). Such results will be applied to the perturbation calculations of the tidal Love numbers and the moments of polytropic stars. In Section 5, we review the formulation of tidal Love numbers in Newtonian gravity, and analyse the effects of a jump-discontinuity in the density profile on the gravitational potential. Through balancing the singularities, we define a modified potential whose spatial derivative remains continuous in the presence of density discontinuities. In Section 6, we derive a recursive formula for the perturbative corrections to the tidal Love numbers from an arbitrary solvable system. In particular, we express the perturbative corrections from the incompressible limit in terms of the moments and the overlap integrals of the density perturbations up to the third order. In Section 7, we apply the results obtained in the preceding section to the polytropic EOS. The perturbative corrections to the tidal Love numbers of degree \( l = 2 \) and \( l = 3 \) are derived, up to the third order about the incompressible limit \( n = 0 \), and up to the first order about the case \( n = 1 \). Accurate two-point Padé approximants for the tidal Love numbers are formed from these perturbative results. In Section 8, we apply the perturbation results developed about the incompressible limit in Section 6 to examine the validity of the I-Love relation and the multipolar moment-Love relation. We show that these relations are stationary with respect to any density variations away from the incompressible limit. In Section 9, we study the I-Love relation and the multipolar moment-Love relation of polytropic stars in greater detail. We find that our perturbative results can nicely capture the behaviour of these relations, demonstrating the robustness of the perturbative approach developed in the present paper. In Section 10, we present the conclusion of our paper and further discuss the physical origin of the universality of these moment-Love relations. For reference, we list the scaled delta expansion method results to the Lane-Emden equation in Appendix A, and argue in Appendix B that the tidal Love numbers behaves as \((n−5^2)\) in the limit \( n \rightarrow 5 \). In Appendix C, accurate approximants of the radius, tidal Love numbers (deformabilities), mass and moment of inertia are listed in a self-contained manner.

## 2 I-LOVE-Q RELATIONS

The I-Love-Q universal relations first discovered by Yagi & Yunes (2013a,b) connect three dimensionless (in geometric units where \( G = c = 1 \)) physical quantities of compact stars, namely, the scaled moment of inertia \( I \equiv I/M^3 \), the dimensionless tidal deformability \( \lambda \equiv (2k_2/3)(R/M)^3 \) and the scaled spin-induced quadrupole moment \( Q \equiv Q/M, P^2 Q^2 \), where \( M \), \( R \), \( I \), \( k_2 \), \( Q \), and \( \Omega \) are the mass, radius, moment of inertia, (quadrupole) tidal Love number, spin-induced quadrupole moment and the spin (angular) frequency of a star. Standard procedures evaluating these quantities in the Newtonian and Einstein gravity have been widely discussed in the literature (see, e.g., Tassoul 1978; Hartle 1967; Hartle & Thorne 1968;Damour & Nagar 2009; Postnikov et al. 2010; Yagi & Yunes 2013a).

The I-Love-Q relations are universal, i.e., EOS-insensitive, for compact stars. As shown in Fig. 1 of Yagi & Yunes (2013a), where \( I \) and \( Q \) are plotted against \( \lambda \), the graphs for compact stars constructed with different EOSs models are close to each other and the difference between them is usually less than one per cent level. However, it is worth noting that the EOSs considered in Fig. 1 of Yagi & Yunes (2013a) are either realistic EOSs of nuclear matter or polytropic EOSs with polytropic indices less than unity. These
popular EOSs for compact stars are all stiff in the sense that their adiabatic indices (see Section 3 for the definition of adiabatic index) are larger than 2 (at least in the high density region). To further investigate the dependence of the I-Love-Q relations on the stiffness of stellar matter, Sham et al. (2015) showed in Fig. 3 of their paper, that the ratio of the fractional differences is larger than 2 (at least in the high density region). They found that both and increase if the adiabatic index (the compactness) of a star decreases (increases). As a result, the I-Love-Q universal relations hold nicely for stiff stars in Newtonian gravity.

Given the discovery reported in Sham et al. (2015), we aim to study in the present paper the I-Love-Q universal relations for polytropic stars, whose stiffness can be adjusted continuously, in Newtonian gravity. Analytic perturbative methods will be developed to derive accurate approximate formulae, which can work for stars with a wide range of stiffness, for the I-Love-Q relations. In particular, we note that in Newtonian gravity (see, e.g., Tassoul 1978; Yagi & Yunes 2013a). Within the scope of the present paper, analytical formulae of and will suffice to examine the physical nature of the I-Love-Q relations. Therefore, we shall concentrate on the I-Love relation in the majority of our paper and show in Section 10 how the scaled spin-induced quadrupole moment also acquires similar universal behaviour.

3 POLYTROPIC STARS

The EOS of polytropic stars is given by:

\[ P(r) = K \rho^{1 + 1/n}, \]  

where \( K > 0 \) and the polytropic index \( n \geq 0 \) are given parameters. The stiffness of a polytropic star is usually characterised by the adiabatic index \( \Gamma = (p/P)(dP/\rho) = 1 + 1/n \). For example, polytropic stars with \( n = \infty \) (i.e., \( n = 0 \)) are indeed incompressible.

For a polytropic star with central density \( \rho_c = \rho(r = 0) \), a length scale \( a \) is conventionally introduced:

\[ a = \sqrt[3]{\frac{K(n+1)}{4\pi G}} \rho_c^{-1} \left[ \frac{1}{n} \right]^{(1-n)/(2n)}, \]

(2)

to define a dimensionless radius \( x = r/a \) and the polytropic function \( \theta(x) \):

\[ [\theta(x)]^n = \frac{\rho(x)}{\rho_c}. \]

(3)

Under Newtonian gravity and hydrostatic equilibrium condition, the polytropic function \( \theta(x) \) satisfies the Lane-Emden equation (LEE) (see, e.g., Chandrasekhar 1958; Cox 1980; Binney & Tremaine 2011):

\[ \frac{1}{x^2} \frac{d}{dx} \left[ x^2 \frac{d\theta(x)}{dx} \right] + [\theta(x)]^n = 0, \]

(4)

as well as the conventional initial conditions \( \theta(0) = 1 \) and \( \theta'(0) = 0 \). Physically speaking, \([\theta(x)]^n\) is a measure of the density distribution. Besides, it is readily shown that \( \theta(x) \) is, up to an additive constant, proportional to the gravitational potential.

To our knowledge, the LEE admits exact analytical solution only at \( n = 0, n = 1 \) and \( n = 5 \) under the conventional initial conditions mentioned above. These closed form solutions can be verified by direct substitution into the LEE (see Seidov 2004, equations (3)- (5)), which are given explicitly as follows:

\[ \begin{align*}
    n &= 0, \quad \theta(x) = 1 - \frac{1}{6} x^2, \quad \xi = \sqrt{6}, \\
    n &= 1, \quad \theta(x) = \frac{\sin(x)}{x}, \quad \xi = \pi, \\
    n &= 5, \quad \theta(x) = \frac{1}{\sqrt{1 + 2x^2/\sqrt{3}}}, \quad \xi = \infty,
\end{align*} \]

(5-7)

where \( \xi \) denotes the first zero of the solution \( \theta(x) \), and \( \xi = \infty \) means that the solution has no root on the positive real line. These solutions can be used as the starting point of perturbation analysis, and are good checks of numerical calculations as well.

The total mass \( M \) of a polytropic star is given by:

\[ M = \rho_c \int_0^\infty \frac{d^3}{d\xi} \left[ \frac{K(n+1)}{4\pi G} \right]^{3/2} \theta(x)^n 4\pi x^2 dx. \]

(8)

Considering the change of the total mass with respect to the variation of the central density, we get:

\[ \frac{dM}{d\rho_c} = \frac{3 - n}{2n}, \]

(9)

which is positive when \( n < 3 \) and negative when \( n > 3 \). It implies that, polytropes are unstable against radial oscillations when \( n > 3 \) (see, e.g., Shapiro & Teukolsky 1983, for the criterion on the stability of stars). In the present paper we are mainly interested in stable polytropic stars with \( 0 < n < 3 \).

For polytropic indices other than \( n = 0, n = 1 \) and \( n = 5 \), there is no known closed form solution to the LEE. Instead, different analytical approximate methods to solve the LEE, including power series expansion methods (see, e.g., Seidov & Kuzakhmedov 1977; Hunter 2001; Iacono & De Felice 2015), resummation of power series solutions (see, e.g., Iacono & De Felice 2015; Pascual 1977) and the delta expansion method (DEM) (see, e.g., Bender et al. 1989; Seidov 2004; Yip et al. 2017) have been proposed. In order to evaluate the tidal Love number as well as the moment of inertia (or other mass moments) of polytropic stars, we adopt the scaled delta expansion method (Yip et al. 2017), which is a variant of the DEM originally proposed by Bender et al. (1989) to solve a wide range of non-linear problems through expanding the non-linear term in a series of its power, to solve the LEE and hence the density profile of polytropic stars. We shall develop proper perturbation schemes to determine the tidal Love number and the moment of inertia (or other mass moments) from the density profile obtained from the SDEM in the later part of this paper.

4 SCALED DELTA EXPANSION METHOD

Recently, Yip et al. (2017) have proposed the SDEM based on the DEM (Bender et al. 1989; Seidov 2004) to solve the LEE perturbatively around \( n = 0 \) and \( n = 1 \), where the LEE admits closed form solutions in terms of elementary functions. In this section we briefly review the main idea and the results of the SDEM to make our paper self-contained.

The spirit of the SDEM is to scale the dimensionless radial distance \( x \) in the LEE (4) in a way that polytropes of different polytropic indices \( n \) share a common scaled radius at \( z = \pi \) through the introduction of a scaling factor \( S(n) \) as follows:

\[ x = S(n)^{(n-1)/2} z. \]

(10)

As a result, the scaled polytropic function \( \Theta(z) = \theta(S^{(1-n)/2} z) \) attains its first zero at \( z = \pi \) in the new length scale. Thus, we seek
for a solution to the following scaled LEE (SLEE):

$$\frac{1}{z} \frac{d}{dz} \left[ z^4 \frac{dS(n)(z)}{dz} + |S(n)(z)|^2 \right] = 0,$$

(11)

under the three boundary conditions (1) \( \Theta(z = 0) = 1 \), (2) \( \Theta'(z = 0) = 0 \), and (3) \( \Theta(z = \pi) = 0 \). The SLEE (11) is nicer than the original LEE (4) because \( \Theta(z) > 0 \) for \( 0 \leq z < \pi \). As a result, the nonlinear term \( \Theta(z)^n \) can then be expanded as a power series in \( n \), which is free of singularities in the whole physical range \( 0 \leq z < \pi \) and in turn leads to better accuracy of the ensuing perturbative results.

In general, the SLEE (11) cannot simultaneously satisfy the three boundary conditions mentioned above. For each polytropic index \( n \), the additional constraint allows us to determine the scale factor \( S(n) \). In other words, the LEE is recast into an eigenvalue problem, with the scale factor \( S \) being the eigenvalue and the scaled polytrope function \( \Theta(z) \) being the eigenfunction.

As the SLEE admits closed form solution of finite radius at \( n = 0 \) and \( n = 1 \), the scale factor \( S \) and the polytrope function \( \Theta(z) \) are solved at these points by expanding them into perturbation series in \( n \) and \( n - 1 \), respectively, as follows:

\[
\Theta(z) = \Theta_0^{(0)}(z) + (n - p)\Theta_0^{(1)}(z) + (n - p)^2\Theta_0^{(2)}(z) + (n - p)^3\Theta_0^{(3)}(z) + O[(n - p)^4],
\]

(12)

\[
S(n) = S_0^{(0)} + (n - p)S_0^{(1)} + (n - p)^2S_0^{(2)} + (n - p)^3S_0^{(3)} + O[(n - p)^4],
\]

(13)

where \( p = 0 \) or \( 1 \) denotes the perturbation centre, \( n - p \) is considered as the perturbation parameter, and \( \Theta_0^{(l)} \) and \( S_0^{(l)} \) respectively denote the \( l \)th order perturbation correction of the scaled polytrope function \( \Theta(z) \) and scale factor \( S \) about the perturbation centre \( p \). The perturbative corrections to the scaled polytrope function \( \Theta(z) \) have been solved up to the second order at \( n = 0 \) and the first order at \( n = 1 \) (Yip et al. 2017), and are summarized in Appendix A.

The SDEM approach is particularly useful in our present context. Firstly, as will be shown in Section 5, the tidal Love numbers are independent of the length scale. In the SDEM approach, finite polytropes of different physical radii are scaled to the same radius at \( \pi \) in the scaled length variable \( z = \pi \) via the scale factor \( S(n) \). In the determination of the tidal Love number, the length scale factor \( S(n) \) associated with the \( n \)-dependence of the physical radius completely disappears, so there is no need to look for the solution to \( S(n) \) in equation (13). Only the density scaled to a common radius at \( z = \pi \) has to be considered. Secondly, the SDEM approach determines the corrections of the scaled polytrope function perturbatively from the exact solutions of the SLEE available at \( n = 0 \) (i.e., the incompressible limit) and \( n = 1 \). These perturbative corrections to the scaled polytrope function straightforwardly determine the density perturbation \( \rho_1(r) \) (see equation (34)) and consequently form the input to the perturbative analyses developed in the following sections.

## 5 TIDAL LOVE NUMBER

Traditionally, multipolar tidal Love numbers can be obtained by solving the Poisson’s equation that governs the spatial variation of the gravitation potential inside a star. In general, whenever the density of a star encounters a jump discontinuity (e.g., the density discontinuity at the surface of a QS), there is an associated jump discontinuity in the derivative of the gravitation potential, which could render perturbative expansion inapplicable. To remedy this problem, in the subsequent discussion a modified potential, which is continuous everywhere inside the star, is introduced to replace the traditional gravitation potential. We show that the multipolar tidal Love numbers can be found by solving the governing equation of such a modified potential.

### 5.1 Definition

Consider a spherically symmetric star of radius \( r \), density profile \( \rho(r) \) and pressure profile \( P(r) \) under the influence of an axially symmetric external multipole tidal field \( H_{\text{ext}} \equiv A_l r l P_l(\cos \theta) \), where \( A_l \) is a constant, \( P_l(\cos \theta) \) is the standard Legendre polynomial of order \( l \), and \( l = 2, 3, 4, \ldots \). Inside the star, the Eulerian change of the Newtonian gravitational potential \( \Pi(r) \) satisfies the standard Poisson’s equation (see, e.g., Brooker & Olle 1955; Beuthe 2015; Postnikov et al. 2010; Damour & Nagar 2009, and references therein):

\[
H''(r) + \frac{2}{r} H'(r) - \left[ \frac{l(l+1)}{r^2} - 4\pi G \rho(r) \frac{dP}{d\rho} \right] H(r) = 0,
\]

(14)

where \( G \) is the constant of universal gravitation and the term \( \rho(dP/d\rho)H \) is the Eulerian change in the mass density in response to the tidal field. On the other hand, outside the star \( H(r) = H_{\text{ext}} + q r^{-l-1} P_l(\cos \theta) \), and the multipole moment \( q_l \) of the star induced by the tidal field is defined accordingly. Equation (14) is solved under the condition that \( H(r) \) is continuous at \( r = R \).

The tidal Love number \( k_l \) of degree \( l \) \( (l = 2, 3, 4, \ldots) \), also known as the constants of apsidal motion, is given by half of the limiting ratio of the gravitational potential due to the induced multipole moment \( q_l \) to the applied gravitational potential evaluated at surface of the star (see, e.g., Brooker & Olle 1955; Postnikov et al. 2010; Damour & Nagar 2009; Beuthe 2015), i.e.,

\[
k_l \equiv \frac{q_l}{2A_l R^{2l+1}}.
\]

(15)

In the I-Love-Q relations, the quadrupolar tidal Love number \( k_2 \) is considered specifically (Yagi & Yusnes 2013a,b).

It follows directly from (14) that \( k_l \) can be found from the ratio \( H'(R^+) / H(R) \) as follows:

\[
k_l = \frac{l - [H'(R^+)/H(R)]}{2(l + 1 + [H'(R^+)/H(R)]).}
\]

(16)

where, in general, \( r^\pm \equiv r \pm \delta \) in the limit of \( \delta \to 0^+ \). It is straightforward to show from equation (14) that as the result of a density jump of \( \Delta \rho \equiv \rho(r^+) - \rho(r^-) \) across \( r = r_* \), \( H(r) \) correspondingly develops a discontinuity of \( 4\pi r_*^2 \Delta \rho(r_*)/m(r_*) \) (see Section III of Postnikov et al. 2010, for a discussion on this issue), where \( m(r) \) is the mass enclosed within the radius \( r \):

\[
m(r) = \int_0^r \rho(r)4\pi r^2 dr.
\]

(17)

Therefore, for stars with non-vanishing surface mass density \( \rho(R^-) \), e.g., incompressible stars and QSs, equation (16) has to be modified as:

\[
k_l = \frac{l - [H'(R^-)/H(R) - 4\pi R^3 \rho(R^-)/m(R)]}{2(l + 1 + [H'(R^-)/H(R) - 4\pi R^3 \rho(R^-)/m(R)])},
\]

(18)

if the ratio \( H'(R^-)/H(R) \), instead of \( H'(R^+)/H(R) \), is used to evaluate \( k_l \) (see, e.g., Postnikov et al. 2010).

On the other hand, as the tidal Love numbers \( k_l \) depends only on the ratio \( H'(R)/H(R) \), it is customary to recast equation (14)
into a first-order nonlinear Ricatti differential equation (see, e.g., Postnikov et al. 2010):

$$Y'(r) + \frac{1}{r}Y(r) + \frac{1}{r}Y^2(r) = \frac{l(l+1)}{r} + \frac{4\pi r^2}{m(r)} \frac{d\rho}{dr},$$

(19)

where

$$Y(r) \equiv \frac{d\ln H}{d\ln r} = \frac{H'(r)}{H(r)}$$

(20)

is the logarithmic derivative of the potential $H(r)$ (LDP) satisfying the initial condition $Y(0) = l$.

A couple of remarks are as follows. Firstly, equation (19) is invariant under length (or mass) scale transformation. Hence, tidal Love numbers are independent of the size and the density scale of the stellar object, i.e., two stars of density profiles $\rho(r)$ and $\alpha \rho(\beta r)$ have the same tidal Love numbers, for arbitrary $\alpha > 0$ and $\beta > 0$. As a result, there is no need for us to explicitly determine the scaling factor $S(n)$ introduced in the SDEM because $k_l$ is independent of $S(n)$.

Secondly, it is easy to see that both equations (14) and (19) could contain divergent terms in some situations. For example, for polytropes of polytropic index $n \neq 0$, it is well known that near the stellar surface, the pressure $P(r)$, the density $\rho(r)$ and $d\rho(r)/dr$ behave as $(R - r)^{n+1}$, $(R - r)^n$ and $(R - r)^{n-1}$, respectively. As mentioned previously, in equation (14), the induced change in the mass density is proportional to $\rho(d\rho/dr) \propto (R - r)^{n+1}$. Similarly, on the RHS of equation (19), there is a term proportional to $d\rho(r)/dr \propto (R - r)^{n-1}$. These terms are obviously divergent as $r$ approaches $R$ for polytropes with $n \in [0, 1)$. On the other hand, they are also problematic for stars with density discontinuities or rapidly varying density profile. In fact, $d\rho(r)/dr$ is actually proportional to a delta-function at a density discontinuity. These diverging terms in equations (14) and (19) are likely to hamper both analytical perturbative as well as numerical calculations (see the relevant discussion in Postnikov et al. 2010). In order to get rid of these troublesome terms, we propose in the following a modified potential whose governing equation and the associated Ricatti equation are free of divergence.

### 5.2 Modified potential

To remedy the problem of divergence in equations (14) and (19) as mentioned above, we consider the modified potential $h(r)$ defined by:

$$h(r) = H(r)/m(r),$$

(21)

and obtain the differential equation governing $h(r)$:

$$h''(r) + \left[ \frac{2}{r} + \frac{8\pi r^2 \rho(r)}{m(r)} \right] h'(r) + \left[ \frac{16\pi \rho(r)}{m(r)} - \frac{l(l+1)}{r^2} \right] h(r) = 0.$$

(22)

The proper boundary condition for this modified potential is $h(r) \sim r^{-1}$ as $r \to 0$. It is obvious that this new field equation is divergence-free (except at the origin). As a result, while $H'(r)$ is discontinuous across density discontinuities where $d\rho/dr$ is infinite, $h'(r)$ is continuous across such points.

Moreover, it can be shown that the logarithmic derivative of the modified potential (LDMP), $y(r) \equiv d\ln h/d\ln r = rH'(r)/h(r)$, is related to the LDP $Y(r)$, through the following equation:

$$y(r) = Y(r) - \frac{4\pi r^2 \rho(r)}{m(r)},$$

(23)

and the governing equation of $y(r)$ can be obtained from equation (22):

$$y'(r) + \left[ \frac{1}{r} + \frac{8\pi r^2 \rho(r)}{m(r)} \right] y(r) + \frac{1}{r} y^2(r) = \frac{l(l+1)}{r} - \frac{16\pi r^2 \rho(r)}{m(r)}.$$

(24)

which is to be solved under the initial condition $y(0) = l - 3$. It follows directly from equations (18) and (23) that the tidal Love numbers, $k_l$, depends only on the LDMP, $y(r)$:

$$k_l = \frac{l - y(R)}{2l + 1 + y(R)}.$$

(25)

As the derivative of the density profile no longer appears explicitly in the governing equation of $y(r)$, $y(r)$ is amenable to perturbative calculations even for cases with $n \in [0, 1)$. We shall see in the subsequent discussion equation (24) indeed opens up the possibility of perturbative analysis from the incompressible limit (i.e., $n = 0$). Otherwise, for a slight perturbation of the density profile from the incompressible limit, the derivative of the change in the density profile appearing in (19), in general, dominates the derivative of the original constant density profile, and precludes the perturbative analysis of $Y(r)$.

The delicate balance between the discontinuities in the two terms $Y(r)$ and $4\pi r^2 \rho(r)/m(r)$ appearing in the definition of $y(r)$ echoes the relationship between the electric field and electric displacement field in the theory of electrodynamics in dielectric media. While the electric field is discontinuous across the surface of a polarized material, the electric displacement takes into account of the contributions due to polarization in a way that the electric displacement is continuous (see, e.g., Jackson 1999).

It is interesting to note that the Ricatti equation (24) originally derived from the consideration of the Eulerian change in the gravitational potential $H$ is analogous to the Clairaut-Radau equation in the traditional theory of tides (see, e.g., Brooker & Olle 1955; Tassoul 1978), which was obtained by monitoring the deformation of the equipotential surfaces inside a star upon the influence of a tidal field. In fact, the solutions of these two equations are related through the linear transformation $y(r) \rightarrow y(r) - l + 3$.

### 5.3 Exactly solvable models

For the case of polytropic stars, equation (24) can be exactly solved for two different cases, namely, $n = 0$ and $n = 1$. For the case $n = 0$ (i.e., incompressible stars), the governing equation (24) reduces to, for $r < R$:

$$y'(r) + \frac{7}{r} y(r) + \frac{1}{r} y^2(r) = \frac{l^2 - l - 12}{r},$$

(26)

which is solved under the initial condition $y(0) = l - 3$. By inspection, one can see that, for $r < R$:

$$y(r) = l - 3.$$  

(27)

As $y$ is continuous at the stellar surface, the tidal Love number $k_l$ of an incompressible star can be obtained from equation (25):

$$k_l = \frac{3}{4(l - 1)}.$$  

(28)

For the case with $n = 1$, for $r < R$, the tidal field equation $H(r)$ is the standard spherical Bessel equation:

$$H''(r) + \frac{2}{r} H'(r) - \left( \frac{l(l+1)}{r^2} - \frac{1}{a^2} \right) H(r) = 0.$$  

(29)
The solution that is non-singular at the origin is the standard spherical Bessel function,
\[ H(r) \propto j_l(r/a), \tag{30} \]
for \( r < R \). Besides, we also have:
\[ \rho(r) = \rho_0 \frac{\sin(r/a)}{r/a}. \tag{31} \]

Therefore, we can determine the \( y_l(r) \) by equation (23):
\[ y_l(r) = -\frac{r \sin(r/a)}{a j_l(r/a)} \frac{r^2 \sin(r/a)}{a \cos(r/a) - a^2 \sin(r/a)}. \tag{32} \]

Hence, evaluating at the stellar surface at \( R = a \pi \), by equation (25), we have:
\[ k_l = \frac{\pi}{2} \left( 2l + 1 \right) j_{l+1}(\pi) - \pi j_{l+1}(\pi). \tag{33} \]

Although the LEE also admits exact solution at \( n = 5 \), neither \( H(r) \) nor \( h(r) \) can be found analytically for \( n = 5 \), at least to our knowledge. However, it can be shown that \( k_l \) is proportional to \((n-5)^2\) for \( n \) close to 5. Hence, \( k_l \) vanishes identically for polytropes with \( n = 5 \) as a consequence of their infinite spatial extent. These results are useful in improving the accuracy of the perturbative formula (to be developed in the subsequent discussion) for \( k_l \).

In order not to distract the readers’ attention from the main theme of the present paper, instead of working out a detailed proof for any asymptotic behaviour here, we move on to develop a perturbative theory for the tidal Love number in the following section. However, we will briefly outline the proof in Appendix B.

## 6 Perturbative Expansion of Tidal Love Numbers

The removal of the explicit dependence on the derivative of density profile in the governing equation (22) allows perturbative analysis of the tidal Love numbers. We first derive a recursion formula for the perturbative corrections of the LDMP, \( y(r) \), due to variation of the density profile from an arbitrary exactly solvable state. Then, we consider an incompressible star as the unperturbed system and find that the perturbative corrections to the tidal Love numbers are expressible in terms of the moments of the density perturbations.

### 6.1 General cases

We derive the effects on the modified potential due to a density perturbation \( \varepsilon \rho_1(r) \), where \( \varepsilon \) is a parameter measuring the strength of perturbation, on an exactly solvable star with a given density distribution \( \rho_0(r) \) and correspondingly a known solution \( y_0(r) \) to (24). As a result, the density profile \( \rho(r) \) and the LDMP, \( y(r) \), can be expressed respectively as:
\[
\rho(r) = \rho_0(r) + \varepsilon \rho_1(r), \tag{34}
\]
\[
y(r) = y_0(r) + \varepsilon y_1(r) + \varepsilon^2 y_2(r) + \ldots, \tag{35}
\]
where \( \varepsilon y_j(r) \) is the \( j \)th order perturbative correction to \( y(r) \) due to the density perturbation \( \varepsilon \rho_1(r) \). For notational convenience, we write:
\[
m(r) = m_0(r) + \varepsilon m_1(r) = \int_0^r \rho_0(t) 4\pi t^2 dt + \varepsilon \int_0^r \rho_1(t) 4\pi t^2 dt. \tag{36}
\]

The zeroth order and the higher order equations for \( j \geq 1 \) read:
\[
\frac{dy_0}{dr} + \frac{1}{r} y_0 + \frac{1}{r^2} y_0^2 + \frac{8\pi^2 \rho_0}{m_0} y_0 = \frac{l(l+1)}{r} - 16\pi^2 \rho_0/m_0, \tag{37}
\]
\[
\frac{dy_j}{dr} + \left( \frac{2y_0 + 1}{r} + \frac{8\pi^2 \rho_0}{m_0} \right) y_j = 8\pi^2 \left( \frac{\rho_1}{m_1} - \frac{\rho_0}{m_0} \right) (y_0 + 2) + g_j, \tag{38}
\]
where \( g_1(r) = 0, \) and for \( j \geq 2 \):
\[
g_j(r) = \int_0^r \left( \frac{8\pi^2}{m_0(s)^2} \left( \frac{\rho_1(s)}{m_1(s)} - \frac{\rho_0(s)}{m_0(s)} \right) y_0(s) + 2 + g_j(s) \right) ds. \tag{39}
\]

### 6.2 Incompressible stars

As mentioned previously, the case of incompressible stars is one of the few exactly solvable models whose tidal Love number can be found analytically. Here we consider an unperturbed star of constant density \( \rho_0 \). Then, we have, for \( r < R \):
\[
m_0(r) = \frac{4\pi}{3} \rho_0 r^3, \tag{41}
\]
\[ y_0(r) = l - 3. \tag{42}\]

In this case, equation (40) could be simplified, and we shall see that the perturbative corrections \( y_j \) are expressible in terms of the moments of the density perturbation \( \rho_1(r) \). For notational convenience, we denote the dimensionless \( k \)th moment \( \mu_k(r) \) of the dimensionless density perturbation by:
\[
\mu_k(r) = \frac{1}{\rho_0} \int_0^r \rho_1(t) t^{k+2} dt. \tag{43}\]

Solving equation (40) recursively for \( y_0^{(0)}(r), y_1^{(1)}(r), y_2^{(2)}(r), \) and \( y_0^{(0)}(r) \), we obtain:
\[
y(r) = l - 3 + \varepsilon \left[ 9\mu_0 - 3(2l+1)\mu_{2l-2} \right]
+ \varepsilon^2 \left\{ -27\mu_0^2 + 9(2l+1)(\mu_{2l-2})^2 + 18(l-1)y_0^{(0)} \right\}
+ \varepsilon^3 \left\{ 81\mu_0^3 - 27(2l+1)(\mu_{2l-2})^3 - 108(l-1)\mu_{2l-2} \right\}
+ \varepsilon^4 \left\{ 27(2l+1)(\mu_{2l-2})^4 - 162\mu_0 \mu_{2l-2} \right\}
+ O(\varepsilon^5), \tag{44}\]
where \( \varepsilon \) denotes the dimensionless overlap integral with \( \rho_1(t) t^{2l-2} \):
\[
\varepsilon [f](r) = \frac{1}{r^{2l+1}} \int_0^r f(t) \frac{\rho_1(t)}{\rho_0} t^{2l-2} dt. \tag{45}\]

MNRAS 000, 1–17 (2017)
and the \( r \)-dependence of \( \mu_0, \mu_{2l-2}, \mathcal{C}[(\mu_0)^2], \mathcal{C}[\mu_0\mu_{2l-2}], \) and \( \mathcal{C}'[\mu_{2l-2}]^2 \) are suppressed. By equation (25), the tidal Love number of degree \( l \) is evaluated perturbatively from the incompressible limit and given up to the third-order in \( \varepsilon \) by:

\[
k_l = \frac{3}{4(l-1)} + \frac{3\varepsilon(2l+1)}{8(1-l)^2} \left[ -3\mu_0 + (2l+1)\mu_{2l-2} \right] + \frac{9\varepsilon^2(2l+1)}{16(l-1)^3} \left[ 3(2l+1)\left(\mu_0 - \mu_{2l-2}\right)^2 - 4(l-1)^2\mathcal{C}[\mu_0] \right] + \frac{27\varepsilon^3(2l+1)}{32(l-1)^4} \left[ 3(2l+1)\left(\mu_0 - \mu_{2l-2}\right)^2 \right.
\]
\[
\times \left. \left\{ (2l+1)\mu_0 + 3\mu_{2l-2} \right\} + 24(l-1)^2\mathcal{C}'[\mu_0] \left(\mu_0 - \mu_{2l-2}\right) + 4(l-1)^2 \left\{ (2l-5)\mathcal{C}'[(\mu_0)^2] + 6\mathcal{C}'[\mu_0\mu_{2l-2}] \right\} \right] + O(\varepsilon^4),
\]

(46)

where all the moments and the overlap integrals appearing in RHS of equation (46) are evaluated at the stellar surface \( r = R \).

We hint that the perturbation solution offers an analytical understanding of the I-Love relation. The first-order perturbative response of the tidal Love number is proportional to the zeroth moment, and the \( (2 - l) \)th moment of the density perturbation. In the special case when \( l = 2 \), the moments correspond to the variations of the mass and the moment of inertia of the stellar structure. We shall see in Section 8 that the tidal Love number, the mass and the moment of inertia are combined in the I-Love relation, so that the perturbation responses cancel exactly in the first order.

7 TIDAL LOVE NUMBERS OF POLYTROPIC STARS

In this section we shall use the SDEM results of the SLEE (Yip et al. 2017) and the perturbative expansion of the tidal Love number derived in the preceding section to find the perturbative series of the polytropic tidal Love numbers of degree \( l = 2 \) and \( l = 3 \), up to the third order about the incompressible limit \( n = 0 \), and up to the first order about the case \( n = 1 \). With these two series derived respectively at \( n = 0 \) and \( n = 1 \), we form a two point Padé approximants of \( k_2 \) and \( k_3 \), whose errors are within 0.39 per cent and 0.93 per cent respectively for stable polytropes for \( n \in [0, 3] \). Between the two perturbation centres \( n \in [0, 1] \), the errors are \( 2.5 \times 10^{-5} \) per cent and \( 7.5 \times 10^{-5} \) per cent respectively (see Table 3). In the following we provide the technical details of our calculations.

7.1 \( n \approx 0 \) case

The density of a polytrope is proportional to \( \Theta^\rho \), so the density perturbation away from the incompressible limit is given up to the third order in \( n \) by (Yip et al. 2017):

\[
\varepsilon \rho_1(\zeta; n = 0) \approx n\rho_e \left[ \ln \Theta_0 - n \left( \frac{3}{2} \ln^2 \Theta_0 + \Theta_1^{(1)} - \Theta_0^{(0)} \right) \right] + n^2 \left[ \frac{1}{6} \ln^3 \Theta_0 + n \Theta_1^{(2)} - \Theta_0^{(0)} \right] \Theta_0^{(0)} + O(n^3),
\]

(47)

where the perturbation corrections \( \Theta_0^{(j)} \) for \( j = 0, 1, \) or 2 are obtained by Yip et al. (2017) and listed in Appendix A for reference.

In this case we shall let \( n \) be the perturbation parameter \( \varepsilon \) from the incompressible limit and use equation (46) to evaluate \( k_2 \) and \( k_3 \), yielding the results:

\[
k_2 = \frac{3}{4} - \frac{3}{4}n \left[ \frac{766}{75} - \pi^2 \right] n^2 + n^3 \left[ -\frac{471797}{1875} + \frac{173}{25} \pi^2 \right. \]
\[
\left. + \left( -\frac{1087}{75} + \frac{2\pi^2}{5} \right) \ln 2 + \frac{216}{25} \ln^2 2 - \frac{12}{5} \ln^3 2 + 156\zeta(3) \right) - \frac{135}{8} J(2,2;4) + \frac{405}{4} J(2,4;4) - \frac{675}{8} J(4,4;4) + O(n^4) \right] \]
\[
\approx 0.75 - 0.75n + 0.343728932n^2 - 0.10794273n^3 + O(n^4),
\]

(48)

\[
k_3 = \frac{3}{8} - \frac{9}{20}n^2 + \left( \frac{37531}{4900} - \frac{3\pi^2}{4} \right)n^3 + n^4 \left[ -\frac{1114120459}{5402250} \right. \]
\[
\left. + \frac{11114}{1960} \pi^2 + \left( \frac{289253}{25725} + \frac{2}{7} \pi^2 \right) \ln 2 + \frac{1608}{245} \ln^2 2 - \frac{12}{7} \ln^3 2 + 132\zeta(3) \right) - \frac{189}{32} J(2,2;6) + \frac{675}{16} J(2,6;6) - \frac{1323}{32} J(6,6;6) + O(n^4) \right] \]
\[
\approx 0.375 - 0.45n + 0.257184454n^2 - 0.103284215n^3 + O(n^4),
\]

(49)

where \( \zeta(s) \) is the Riemann zeta function defined by the infinite series (see, e.g., Olver et al. 2010):

\[
\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}
\]

(50)

for \( s > 1 \), and \( J(p,q;r) \) is a third-order interaction integral defined to be:

\[
J(p,q;r) = \int_0^1 \left[ \frac{1}{r^{p+1}} \int_0^r \ln(1-s^2)^q ds \right] \frac{1}{r^{q+1}} \int_0^r \ln(1-s^2)^q ds \times \ln(1-r^2) \, dr.
\]

(51)

We note that \( J(p,q;r) = J(q,p;r) \), and these integrals are caused by the third order interaction integrals \( \mathcal{C}'[\mu_0]^2, \mathcal{C}'[\mu_0\mu_{2l-2}] \) and \( \mathcal{C}'[\mu_{2l-2}]^2 \).

7.2 \( n \approx 1 \) case

Next, we apply the SDEM results developed about \( n = 1 \) to derive the approximate expressions for \( k_2 \) and \( k_3 \) that are valid for \( n \approx 1 \). Polytropic stars with \( n = 1 \) are exactly solvable with \( \rho_0(r) \) and \( \Theta_0(r) \) given by equations (31) and (32). On the other hand, for polytropic stars with \( n \approx 1 \), the density perturbation is given up to the first-order in \( n - 1 \):

\[
\varepsilon \rho_1(\zeta; n = 1) \approx (n-1)\rho_e \left[ \Theta_1^{(1)} + \Theta_0^{(0)} \ln \Theta_0^{(0)} \right] + O((n-1)^2),
\]

(52)

where the explicit forms of the perturbation corrections \( \Theta_0^{(j)} \) for \( j = 0, 1 \) can be found in Appendix A (see also Yip et al. 2017).

Considering \( n - 1 \) as the perturbative parameter \( \varepsilon \) in equation
and their numerical values are $40$, $53$, $1$, $54$, $54$, $1016$.

We form two-point Padé approximants of the tidal Love numbers $k_2$ and $k_3$, which are valid for stable polytropic stars with $n \in [0, 3]$. Such Padé approximants, denoted by $k_{2,\mathrm{tp}}$ and $k_{3,\mathrm{tp}}$, are in the form of:

$$k_{l,\mathrm{tp}}(n) = (5 - n)^3 \frac{a_1 + na_2 + n^2a_3 + n^3a_4}{1000 + n^3a_6}$$

where $l = 2$ or $3$, the constants $a_1, a_2, \ldots, a_6$ are determined analytically from the perturbation solutions in equations $(48)$, $(49)$, $(53)$ and $(54)$ by applying the two-point Padé approximation scheme (Baker & Graves Morris 1996), and their numerical values are shown in Table 1. In the above expression, we have also incorporated an empirical $(5 - n)^3$ factor to match the fact that both $k_2$ and $k_3$ vanish in the limit $n \to 5$. Readers may refer to Appendix B for more discussion of the empirical factor $(5 - n)^3$.

### Table 1. The numerical values of the constants $a_1, a_2, \ldots, a_6$ appearing in $(55)$.

| $l$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|-----|-------|-------|-------|-------|-------|-------|
| 2   | 6     | 0.0701474999044 | -0.0540432029285 | -0.292566896945 | 0.0248516990271 |
| 3   | | 0.0269208110373 | 0.0346247187561 | 411.691249849 | 581.98559024 |
|     | 17.6102741235 | 60.5271512307 |

(40), and using equations $(25)$ and $(40)$, we obtain:

$$k_2 = \frac{15 - \pi^2}{2\pi^2} - \frac{45}{2\pi^2}y_1(z = \pi; l = 2)(n - 1)$$
$$\approx 0.259908877 - 0.295642768(n - 1) + O[(n - 1)^2]$$

$$k_3 = \frac{105 - 10\pi^2}{6\pi^2} - \frac{7}{18\pi^4}y_1(z = \pi; l = 3)(n - 1)$$
$$\approx 0.106454047 - 0.142830875(n - 1) + O[(n - 1)^2]$$

where $y_1(z; l)$ for $l = 2$ and $l = 3$ are defined in equation $(40)$.

### 7.3 Two-point approximations

We form two-point Padé approximants of the tidal Love numbers $k_2$ and $k_3$, which are valid for stable polytropic stars with $n \in [0, 3]$. Such Padé approximants, denoted by $k_{2,\mathrm{tp}}$ and $k_{3,\mathrm{tp}}$, are in the form of:

$$k_{l,\mathrm{tp}}(n) = (5 - n)^3 \frac{a_1 + na_2 + n^2a_3 + n^3a_4}{1000 + n^3a_6}$$

where $l = 2$ or $3$, the constants $a_1, a_2, \ldots, a_6$ are determined analytically from the perturbation solutions in equations $(48)$, $(49)$, $(53)$ and $(54)$ by applying the two-point Padé approximation scheme (Baker & Graves Morris 1996), and their numerical values are shown in Table 1. In the above expression, we have also incorporated an empirical $(5 - n)^3$ factor to match the fact that both $k_2$ and $k_3$ vanish in the limit $n \to 5$. Readers may refer to Appendix B for more discussion of the empirical factor $(5 - n)^3$.

### 7.4 Numerical results

Next, we gauge the accuracy of the perturbation series of $k_2$ and $k_3$ shown respectively in equations $(48)$, $(49)$, $(53)$, $(54)$, as well as the two-point approximants $k_{2,\mathrm{tp}}$ and $k_{3,\mathrm{tp}}$. To illustrate the convergence of the perturbation series of $k_2$ and $k_3$, we introduce the symbol $[k_j]_p^i$ to denote the $i$th order partial sum of the perturbation series of the tidal Love number of degree $l = 2$ or $3$ about the perturbation centre $p = 0$ or $1$. As could be seen in Figures 1 and 2, the accuracy improves by two orders of magnitude for each increment of the perturbation order near the perturbation centre. The two-point Padé approximants are formed by merging the third-order perturbation results at $n = 0$ and the first-order results at $n = 1$. The percentage errors of the two-point Padé approximants $k_{2,\mathrm{tp}}$ and $k_{3,\mathrm{tp}}$ are within $0.39$ per cent and $0.93$ per cent respectively over the range $n \in [0, 3]$. In particular, the percentage errors of $k_{2,\mathrm{tp}}$ and $k_{3,\mathrm{tp}}$ within the range of polytropic index $n \in [0, 1]$ between the two perturbation centres $n = 0$ and $n = 1$, are $2.5 \times 10^{-3}$ per cent and $7.5 \times 10^{-3}$ per cent respectively (see also Table 3).

These accurate approximations of the tidal Love numbers are useful to astrophysicists in understanding the tidal responses of polytropic stars, and lay the foundation of an analytical understanding of the stationarity of the moment-Love relations. We observe that the tidal Love number decreases monotonically with increasing $n$ from the incompressible limit. Since the density tail of polytropes lengthens with increasing polytropic index $n$, our observation implies that the tidal Love number declines with a more extended density distribution.

### 8 STATIONARITY OF MULTIPOLAR MOMENT-LOVE RELATIONS IN THE INCOMPRESSIBLE LIMIT

As reviewed in Section 2, the mass, the moment of inertia and the tidal Love number of degree $l = 2$, $k_2$, is plotted for the numerically exact solution $k_{2,\mathrm{exact}}$ (black line), the two-point Padé approximant $k_{2,\mathrm{tp}}$ (red cross) in equation $(55)$, the approximants about $n = 0$, $[k_2]_0^0$ in equation $(48)$ for $j = 1$ (blue triangle), $j = 2$ (orange triangle) and $j = 3$ (grey square), and the first-order approximation about $n = 1$, $[k_2]_1^1$ in equation $(53)$ (purple diamond). The associated error plot is shown in the lower panel, where $\Delta k_2 = |k_2 - k_{2,\mathrm{exact}}|/k_{2,\mathrm{exact}} \times 100\%$ represents the percentage error of the approximant.

![Figure 1](image)

The upper panel, the value of the tidal Love number of degree $l = 2$, $k_2$, is plotted for the numerically exact solution $k_{2,\mathrm{exact}}$ (black line), the two-point Padé approximant $k_{2,\mathrm{tp}}$ (red cross) in equation $(55)$, the approximants about $n = 0$, $[k_2]_0^0$ in equation $(48)$ for $j = 1$ (blue triangle), $j = 2$ (orange triangle) and $j = 3$ (grey square), and the first-order approximation about $n = 1$, $[k_2]_1^1$ in equation $(53)$ (purple diamond). The associated error plot is shown in the lower panel, where $\Delta k_2 = |k_2 - k_{2,\mathrm{exact}}|/k_{2,\mathrm{exact}} \times 100\%$ represents the percentage error of the approximant.

As reviewed in Section 2, the mass, the moment of inertia and the tidal Love number of degree $l = 2$, $k_2$, is plotted for the numerically exact solution $k_{2,\mathrm{exact}}$ (black line), the two-point Padé approximant $k_{2,\mathrm{tp}}$ (red cross) in equation $(55)$, the approximants about $n = 0$, $[k_2]_0^0$ in equation $(48)$ for $j = 1$ (blue triangle), $j = 2$ (orange triangle) and $j = 3$ (grey square), and the first-order approximation about $n = 1$, $[k_2]_1^1$ in equation $(53)$ (purple diamond). The associated error plot is shown in the lower panel, where $\Delta k_2 = |k_2 - k_{2,\mathrm{exact}}|/k_{2,\mathrm{exact}} \times 100\%$ represents the percentage error of the approximant.
We hint that, in Newtonian gravity, to analyse the universality of the I-Love-Q relations, it suffices to study the constancy of $U_2$ (see Sections 2 and 10).

While such independency has been checked numerically and analytically for several model EOSs, including polytropic stars with $n = 0$ and 1 for the case $l = 2$ (Yagi & Yunes (2013a) and the modified Tolman model for the case $l > 2$ (Chan et al. 2014; Sham et al. 2015), to our knowledge, as yet there is no generally valid theoretical justification.

### 8.2 Stationarity of multipolar moment-Love relations

Our perturbative analysis (46) of the tidal Love number from the incompressible limit is applicable to arbitrary EOSs. We shall use it to show the stationarity of the moment-Love relation in an EOS-independent manner. The variation of $U_l$ in the moment-Love relation (59) against an arbitrary perturbation $\varepsilon \rho_1(r)$ in the incompressible limit is given by the sum of three terms:

$$\begin{align*}
\frac{1}{U_l} \frac{dU_l}{d\varepsilon} |_{\varepsilon = 0} &= \frac{d\ln U_l}{d\varepsilon} |_{\varepsilon = 0} \\
&= \left[ \frac{d\ln k_l}{d\varepsilon} + 2l + 1 \frac{d\ln \mathcal{M}_0}{d\varepsilon} - \frac{2l + 1}{2l - 2} \frac{d\ln \mathcal{M}_{2l-2}}{d\varepsilon} \right] |_{\varepsilon = 0}.
\end{align*}$$

From the incompressible limit, the $l$th moment of the density profile, $\mathcal{M}_k$, is related to the $k$th moment of the density perturbation, $\mu_k$, by:

$$\mathcal{M}_k = \frac{1}{k + 3} \left[ 1 + \varepsilon (k + 3) \mu_k(R) \right].$$

By equations (46) and (64), we find that:

$$\begin{align*}
\frac{d\ln k_l}{d\varepsilon} |_{\varepsilon = 0} &= - \frac{3(2l + 1)}{2l - 2} \mu_0(R) + \frac{(2l + 1)^2}{2l - 2} \mu_{2l-2}(R), \\
\frac{2l + 1}{2l - 2} \frac{d\ln \mathcal{M}_0}{d\varepsilon} |_{\varepsilon = 0} &= \frac{3(2l + 1)}{2l - 2} \mu_0(R), \\
\frac{2l + 1}{2l - 2} \frac{d\ln \mathcal{M}_{2l-2}}{d\varepsilon} |_{\varepsilon = 0} &= \frac{(2l + 1)^2}{2l - 2} \mu_{2l-2}(R).
\end{align*}$$
Hence, we conclude that $U_l$ is stationary against an arbitrary perturbation $\varepsilon \rho_1(r)$ in the incompressible limit:

$$\frac{dU_l}{d\varepsilon}|_{\varepsilon=0} = 0,$$

and at the incompressible limit:

$$U_l(\varepsilon = 0) = \frac{3}{4(l-1)} \left(\frac{2l+1}{3}\right)^{(2l+1)/(2l-2)}.$$

We stress that the stationarity of the multipolar moment-Love relation in the incompressible limit is general, because it does not depend specifically on the functional form of the density perturbation $\rho_1$. Hence, the multipolar moment-Love relation is insensitive to the equation of state near the incompressible limit. The stationarity of $U_l$ and the universality of the multipolar moment-Love relation are remarkable, in the sense that two seemingly unrelated physical quantities, the tidal Love number and the mass moments, are related in an EOS-independent manner from the incompressible limit. Intuitively speaking, to the first order, the tidal Love numbers $k_l$ respond to the density variation $\rho_1(r)$ through its zeroth moment $\mu_0(R)$ and the $(2l-2)$th moment $\mu_{2l-2}(R)$. While the former is purely a mass effect due to the density variation, the latter further measures the spatial extent of $\rho_1(r)$. However, these two changes are exactly cancelled by the corresponding responses in the moments $\mathcal{M}_l$ and $\mathcal{M}_{l-2}$, which both appear in the multipolar moment-Love relations. This observation is indeed an independent corroboration of the correctness of the form of such EOS-insensitive relations.

9. MOMENT-LOVE RELATIONS OF POLYTROPIC STARS

The perturbative formulae for the tidal Love numbers $k_2$ and $k_3$ of polytropes obtained in Section 7 allow us to examine the 1-Love relation (or the multipolar moment-Love relation) of polytropic stars analytically. The analytical results, which are also consistent with the numerical ones, show that $U_2$ in the 1-Love relation is, within 0.4 per cent, a constant near the incompressible limit where $n \in [0, 1]$. The deviation grows sharply to 6 per cent when $n = 2$, and reaches 25 per cent at $n = 3$ (see Table 3). We shall show that the constancy of $U_2$ near the incompressible limit where $n \in [0, 1]$ is actually attributable to the universal stationary point at $n = 0$ and, in addition, the presence of a secondary stationary point at $n \approx 0.4444$. Numerical and analytical results of the moment-Love relation for $l = 3$ are also presented. Compared to the 1-Love relation with $l = 2$, the value of $U_3$ appearing in the moment-Love relation for $l = 3$ varies more significantly than that of $U_2$. While $U_3$ is stationary at $n = 0$, there is no secondary stationary point for $n \in [0, 1]$.

9.1 1-Love relation of polytropic stars

We first focus on the 1-Love relation of polytropic stars, where $l = 2$, and $U_2 = k_2 \mathcal{M}_2/\mathcal{M}_0 = 5^{3/2}$. As an accurate analytical approximation to the tidal Love number $k_2$, the two-point Padé approximant in equation (55), has been derived, it remains to determine the moment ratio $\mathcal{M}_2/\mathcal{M}_0$. Physically, $\mathcal{M}_2$ and $\mathcal{M}_0$ are proportional to the mass $M$ and the moment of inertia $I$ respectively, by $M = 4\pi \rho R^3 \mathcal{M}_0$ and $I = (8\pi/3) \rho R^5 \mathcal{M}_2$. Hence, the fraction of the dimensionless moments, $\mathcal{M}_2/\mathcal{M}_0$, is equal to $3I/(2MR^2)$.

In fact, we find that the perturbation series of the fraction $\mathcal{M}_{l+2}/\mathcal{M}_l$ is more accurate than individual moments because for each singularity in $\mathcal{M}_l$ inherited from the density profile, there is a singularity of the same kind in $\mathcal{M}_{l+2}$. Following directly from the solution of the LEE with $n = 0$, the leading order approximation of $\mathcal{M}_l(n)$ from the incompressible limit is given by:

$$\mathcal{M}_l(n) = \frac{1}{\pi^3/5} \left(1 - \frac{\pi^2}{3}\right)^{l+1} \frac{d^2}{dx^2} \left[4\pi^3/5 \left(\frac{\pi}{3}\right)^{l+2} + O(n^2)\right]$$

for $n > -1$. As could be seen, the $k$th moment contains simple poles at $n = -1, -2, -3, \ldots$ due to the term $\Gamma(1+n)$. Such singularities are caused by the density profile $(1-z^2/3)^n$, which is integrable over the domain of interest for $n > -1$. As both the zeroth moment $\mathcal{M}_0$ and the $(2l-2)$th moment $\mathcal{M}_{l-2}$ are directly proportional to the density profile, such singularities due to the $\Gamma(1+n)$ cancel out in the fraction $\mathcal{M}_{l+2}/\mathcal{M}_l$. While the $n$-series approximation of $\mathcal{M}_0$ and $\mathcal{M}_2$ have a common radius of convergence of 1 due to the nearest singularity at $n = -1$, the fractions $\mathcal{M}_2/\mathcal{M}_0$ has a larger interval of validity.

Therefore, it is more advantageous for us to determine the perturbation series for the fractions $\mathcal{M}_l/\mathcal{M}_0$ instead of the moments themselves, about $n = 0$ and $n = 1$, respectively, and then derive a two-point Padé approximant to connect these series. Using the results of SDEM at $n = 0$ and $n = 1$ and going through some standard but a bit tedious calculations, we arrive at the following results: (i) an expansion of $\mathcal{M}_2/\mathcal{M}_0$ about $n = 0$,

$$\mathcal{M}_2/\mathcal{M}_0 = \frac{3}{5} \frac{6n}{25} \left[\frac{1496}{375} + \frac{2\pi^2}{5}\right] n^2$$

$$+ \left[\frac{63504}{625} + \frac{202\pi^2}{75} + \frac{312}{5} \pi(3)\right] n^3 + O(n^4)$$

$$\approx \frac{3}{5} - 6n/25 + 0.041491573n^2 - 0.015914723n^3 + O(n^4),$$

(ii) an expansion of $\mathcal{M}_3/\mathcal{M}_0$ about $n = 1$,

$$\mathcal{M}_3/\mathcal{M}_0 = 1 - \frac{6}{\pi^2} \frac{3(n-1)}{\pi^2} \left[-5\pi + 4\pi \ln 2 + 2\text{Si}(\pi) + \text{Si}(2\pi)\right]$$

$$+ O((n-1)^2)$$

$$\approx 0.39207290 - 0.18147228(n-1) + O((n-1)^2),$$

and (iii) a two-point Padé approximant obtained by merging the results shown in equations (71) and (72),

$$\frac{\mathcal{M}_{l+2}/\mathcal{M}_l}{[\mathcal{M}_2/\mathcal{M}_0]}_{[p]} = \left(5 - n\right) \frac{b_1 + nb_2 + n^2b_3 + n^3b_4}{1000 + nb_5 + n^2b_6},$$

where $\text{Si}(\pi)$ is the integral of sine function (see, e.g., Olver et al. 2010, and equation (A7)), and the constants $b_1, b_2, \ldots, b_6$ for the case $l = 2$ are determined from equations (71) and (72) and tabulated in Table 2. In equation (73) we have incorporated an empirical factor of $(5-n)^3$, which is found to greatly enhance the accuracy of the two-point approximant. The motivation for the form of such an empirical factor can be found in Appendix B.

We gauge the accuracy of the perturbation series of $\mathcal{M}_2/\mathcal{M}_0$ shown respectively in equations (71) and (72), as well as the two-point approximant $[\mathcal{M}_2/\mathcal{M}_0]_{[p]}$ in equation (73). To illustrate the convergence of the perturbation series of $\mathcal{M}_2/\mathcal{M}_0$, we introduce the symbol $[\mathcal{M}_2/\mathcal{M}_0]_{[p]}$ to denote the $j$th order partial sum of the perturbation series of the moment fraction about $p = 0$ or 1. As could be seen in Figure 3, the accuracy improves by an order of magnitude for each increment of the perturbation order near the
The range of polytropic index near the incompressible limit for $n$ two-point Padé approximant $\left[\frac{M_{2}}{M_{0}}\right]$ in equation (73), the approximants about $n = 0$, $\left[\frac{M_{2}}{M_{0}}\right]_{\text{exact}}$ in equation (71) for $j = 1$ (blue triangle), $j = 2$ (orange triangle) and $j = 3$ (grey square), and the first-order approximation about $n = 1$, $\left[\frac{M_{2}}{M_{0}}\right]$ in equation (72) (purple diamond). The associated error plot is shown in the lower panel, where $\Delta_{\frac{M_{2}}{M_{0}} = \left[\frac{M_{2}}{M_{0}}\right]} = \left[\frac{M_{2}}{M_{0}}\right] - \left[\frac{M_{2}}{M_{0}}\right]_{\text{exact}} - 1 \times 100\%$ represents the percentage error of the approximant.

Table 2. The numerical values of the constants $b_1, b_2, \ldots, b_6$ appearing in (73).

| $l = 2$ | $l = 3$ |
|---------|---------|
| $b_1$ | $24/5$ | $24/7$ |
| $b_2$ | $3.5452950061777$ | $1.605541075759$ |
| $b_3$ | $0.08417922198458$ | $0.5058524906146$ |
| $b_4$ | $0.01816712663676$ | $0.02106526231457$ |
| $b_5$ | $519.3654962035$ | $553.9970994773$ |
| $b_6$ | $-155.4882028232$ | $-137.1355972892$ |

Figure 3. In the upper panel, the value of the moment fraction, $\frac{M_{2}}{M_{0}}$, is plotted for the numerically exact solution $\left[\frac{M_{2}}{M_{0}}\right]_{\text{exact}}$ (black line), the two-point Padé approximant $\left[\frac{M_{2}}{M_{0}}\right]_{\text{exact}}$ (red cross) in equation (73), the approximants about $n = 0$, $\left[\frac{M_{2}}{M_{0}}\right]_{\text{exact}}$ in equation (71) for $j = 1$ (blue triangle), $j = 2$ (orange triangle) and $j = 3$ (grey square), and the first-order approximation about $n = 1$, $\left[\frac{M_{2}}{M_{0}}\right]$ in equation (72) (purple diamond). The associated error plot is shown in the lower panel, where $\Delta_{\frac{M_{2}}{M_{0}} = \left[\frac{M_{2}}{M_{0}}\right]} = \left[\frac{M_{2}}{M_{0}}\right] - \left[\frac{M_{2}}{M_{0}}\right]_{\text{exact}} - 1 \times 100\%$ represents the percentage error of the approximant.

Figure 4. In the upper panel, the quadrupole tidal Love number $k_2$ is plotted over the range of stable polytropes for $n \in [0, 3]$, for the numerically exact solution $k_{2,\text{exact}}$ (black line), and the two-point Padé approximant $k_{2,\text{tp}}$ (red cross) in equation (55). In the middle panel, the moment fraction raised to the power $S/2$, $\left[\frac{M_{2}}{M_{0}}\right]^{S/2}$ is plotted over the same range for $n \in [0, 3]$, for the numerically exact solution $\left[\frac{M_{2}}{M_{0}}\right]^{S/2}$ (black line), and the two-point Padé approximant $\left[\frac{M_{2}}{M_{0}}\right]^{S/2}$ (red cross) in equation (73). In the lower panel, the percentage variation of the I-Love relation $U_2$ over the range of stable polytropes for $n \in [0, 3]$ is plotted for the numerically exact solution $D_{2,\text{exact}} = (U_{2,\text{exact}}/U_2(0) - 1) \times 100\%$ (black line) and the two-point approximant $D_{2,\text{tp}} = (U_{2,\text{tp}}/U_2(0) - 1) \times 100\%$ (red cross).

We define the analytical two-point approximant of $U_2$ by $U_{2,\text{tp}} = k_{2,\text{tp}}^{S/2}$ and first analyse the I-Love relation near the incompressible limit for $n \in [0, 1]$. In Figure 4, the numerical results and the analytical calculations consistently show that the I-Love relation $U_2$ is remarkably flat near the incompressible limit for $n \in [0, 1]$, and the variation is within $0.4444$. In addition, the constancy of the I-Love relation is indeed attributable to the existence of multiple stationary points for $n \in [0, 1]$. Numerically, $U_2$ has stationary points at $n = 0$ and $n \approx 0.4444$. Our analytical calculations can capture such features as well. Demanding $dU_{2,\text{tp}}/dn = 0$, we find stationary points $n = 0$ and $n \approx 0.444364$, which agrees nicely with the numerical value $n \approx 0.4444$.

Alternatively, we can use equations (48) and (71) to find an
expansion for \( U_2 \) about the incompressible limit with \( n = 0 \):

\[
U_2(n) = \sqrt{\frac{5}{3}} \frac{25}{12} n^2 + n^3 \left( \frac{25n^2}{36} - \frac{1487}{216} \right) + n^4 \left( \frac{59093}{300} \right) - \frac{599n^2}{108} + \frac{325}{4} \zeta(3) + \left( -\frac{1087}{27} + \frac{10n^2}{9} \right) \ln 2 + 24\ln^2 2 - \frac{20}{3} \ln^2 2 - \frac{375}{8} J(2, 2; 4) + \frac{1125}{4} J(2, 4; 4) - \frac{1875}{8} J(4, 4; 4) \right] 
\]

\[
\approx 2.6895718 - 0.03920433n^2 + 0.084829962n^3 + O(n^4) 
\]

(74)

which is normalised by the value at the incompressible limit \( n = 0 \):

\[
\frac{U_2(n)}{U_2(n = 0)} = 1 - 0.014576311n^2 + 0.031540323n^3 + O(n^4) 
\]

(75)

The first order variation with respect to \( n \) vanishes at \( n = 0 \) as expected. Moreover, the existence of another stationary point at \( n \approx 0.3081 \) close to the first one where \( n = 0 \) is well-predicted by the third order perturbation result shown in equation (74) or (75). We note that the coefficients of the \( n^2 \) and \( n^3 \) terms in the Taylor series expansion of \( U_2(n)/U_2(n = 0) \) are much smaller than unity, and the second order term has an opposite sign from the zeroth and the third order terms. These, again, hint at an extended interval of validity of the constancy of the I-Love relation from the incompressible limit.

In Figure 4, from \( n = 0 \) to \( n = 1 \), while both the quadrupole Love number \( k_2 \) and the moment fraction \( [\mathcal{M}_2/\mathcal{M}_0]^{7/4} \) roughly drop by a factor of 3, the value of \( U_2(n) \) defined in the I-Love relation is remarkably stable with the variation being within 0.4 per cent. In addition, over the extended range of stable polytropes from \( n = 0 \) to \( n = 3 \), while the quadrupole Love number \( k_2 \) and the moment fraction \( [\mathcal{M}_2/\mathcal{M}_0]^{7/4} \) simultaneously decrease by two orders of magnitude, \( U_2(n) \) only changes by 25 per cent.

Next, we analyse the breakdown of the I-Love relation. In Figure 4, both the numerical and the analytical results show that the \( U_2(n) \) in the I-Love relation significantly deviates from its incompressible counterpart, \( U_2(0) \), for polytropes of larger polytropic indices. As a matter of fact, \( D_2 \equiv \left[ U_2(n)/U_2(0) - 1 \right] \times 100\% \) grows from 0.4 per cent via 6.0 per cent to 25 per cent, when \( n \) increases from 1 via 2 to 3. It illustrates the breakdown of universality when the EOS is soft, or the density distribution is more extended.

It is remarkable that, as shown in Figure 4, the two-point approximants derived in the present paper can accurately capture the variation of the quadrupole tidal Love number and the scaled moment \( [\mathcal{M}_2/\mathcal{M}_0] \). Through these two approximants, we can demonstrate the validity of the I-Love relation as well as its limitation, which clearly pinpoints the physical origin of such universality.

9.2 Octupolar moment-Love relation of polytropic stars

Next, we repeat the calculations for the moment-Love relation for \( l = 3 \). From equation (59), we see that \( U_3 = k_3(\mathcal{M}_3/\mathcal{M}_0)^{7/4} \). The approximation of \( k_3 \) has been determined in Section 7. To analytically analyse the the moment-Love relation \( U_3 \), it remains to evaluate the moment fraction \( \mathcal{M}_3/\mathcal{M}_0 \). We determine the perturbation expansion of \( \mathcal{M}_3/\mathcal{M}_0 \) about the incompressible limit \( n = 0 \):

\[
\frac{\mathcal{M}_3}{\mathcal{M}_0} = \frac{3}{7} \frac{72n^2}{245} + n^3 \left( \frac{147382}{25725} - \frac{4\pi^2}{7} \right) + n^4 \left( \frac{432846317}{2701125} + \frac{908n^2}{245} + \frac{720\zeta(3)}{7} \right) + O(n^5) 
\]

\[
\approx \frac{3}{7} \frac{72n^2}{245} + 0.089361139n^2 - 0.028580246n^3 + O(n^4) 
\]

(76)

as well as the case \( n = 1 \),

\[
\frac{\mathcal{M}_3}{\mathcal{M}_0} = 1 + \frac{120}{\pi^4} \frac{20}{\pi^5} + \frac{10(n-1)}{\pi^6} \left[ \frac{94\pi - 5\pi^3}{\pi^6} - 4\pi(12 - \pi^2)\ln 2 - 248\text{Si}(\pi) - (12 + \pi^2)\text{Si}(2\pi) \right] - 21\pi^2 \zeta(3) + O((n-1)^2) 
\]

\[
\approx 0.20549420 - 0.16501233(n-1) + O((n-1)^2) 
\]

(77)

and merge these two expansions together to obtain a two-point Padé approximant, \([\mathcal{M}_3/\mathcal{M}_0]_{\text{TP}}\), whose explicit form is given in equation (73) with the constants \( b_1, b_2, \ldots, b_6 \) listed in Table 2. Then, a two-point approximant for the \( l = 3 \) moment-Love relation can be formed, namely, \( U_{3,\text{TP}} = k_{3,\text{TP}}[\mathcal{M}_3/\mathcal{M}_0]_{\text{TP}} \).

The accuracy of the perturbation series in equations (76) and (77), and the two-point approximant in equation (73) is studied in Figure 5. As introduced previously in the case with \( l = 2 \), the symbol \([\mathcal{M}_l/\mathcal{M}_0]_j\) denotes the \( j \)-th order partial sum of the perturbation series of the the moment fraction about \( p = 0 \) or 1. Figure 5 clearly shows that the accuracy of \([\mathcal{M}_l/\mathcal{M}_0] \) improves by an order of magnitude for each increment of the perturbation order near the perturbation centre \( p = 0 \) or \( p = 1 \). As a matter of fact, the percentage error of \([\mathcal{M}_l/\mathcal{M}_0] \) between the two perturbation centres \( n = 0 \) and \( n = 1 \) is less than \( 1.6 \times 10^{-4} \) per cent. Even throughout the entire physical range \( n \in [0, 3] \), the percentage error of such a two-point Padé approximant is still bounded by 0.65 per cent, which is impressive.

In parallel to the \( l = 2 \) case, we proceed to study the moment-Love relation for the \( l = 3 \) case with the two-point two-point Padé approximants for the tidal Love number and the moment ratio. We first consider the relation in the range near the incompressible limit for \( n \in [0, 1] \), and then extend our investigation to the entire range of stable polytropes for \( n \in [0, 3] \). In Figure 6, the numerical and the analytical results consistently show that the variation of \( U_3 \) is within 2.8 per cent near the incompressible limit for \( n \in [0, 1] \). However, over the extended range of stable polytropes from the incompressible limit at \( n = 0 \) to the verge of stability at \( n = 3 \), while both the Love number of degree \( l = 3 \), \( k_3 \), and the moment fraction, \([\mathcal{M}_3/\mathcal{M}_0] \), decrease by two orders of magnitude, the maximum change of \( U_3(n) \) the moment-Love relation is relatively mild at 53 per cent. Compared to \( U_2 \) in the I-Love relation, the variation of \( U_3 \) is much larger. Besides, there is only one stationary point located at \( n = 0 \), while the stationary point at \( n = 0 \) is expected analytically by our analysis in Section 8.2, throughout the whole physical range where \( n \in [0, 3] \). To better understand the absence of the secondary stationary point, we can find the Taylor series of \( U_3(n) \) about the incompressible limit \( n = 0 \) up to the third order in
Using the third order perturbation of $\Delta$, is plotted for the numerically exact solution $[M_{0}/M_{0}]_{\text{exact}}$ (black line), the two-point Padé approximant $[M_{0}/M_{0}]_{\text{tp}}$ (red cross) in equation (73), the approximants about $n = 0$, $[M_{0}/M_{0}]_{0}$ in equation (76) for $j = 1$ (blue triangle), $j = 2$ (orange triangle) and $j = 3$ (grey square), and the first-order approximation about $n = 1$, $[M_{0}/M_{0}]_{1}$ in equation (72) (purple diamond). The associated error plot is shown in the lower panel, where $\Delta[M_{0}/M_{0}]_{\text{exact}}/\Delta[M_{0}/M_{0}]_{\text{tp}}$ (black line), the two-point Padé approximant $[M_{0}/M_{0}]_{\text{tp}}$ (red cross) in equation (75). In the middle panel, the moment fraction raised to the power $7/4$, $[M_{0}/M_{0}]_{7/4}$ is plotted over the same range for $n \in [0, 3]$, for the numerically exact solution $[M_{0}/M_{0}]_{7/4}^{\text{exact}}$ (black line), and the two-point Padé approximant $[M_{0}/M_{0}]_{7/4}^{\text{tp}}$ (red cross) in equation (73). In the lower panel, the percentage variation of the third moment-Love relation $U_{3}$ over the range of stable polytropes for $n \in [0, 3]$ is plotted for the numerically exact solution $U_{3,\text{exact}}$ (black line) and the two-point Padé approximant $U_{3,\text{tp}}$ (red cross) in equation (74). In Figure 6, both the numerical and the analytical results show that $U_{3}(n)$ in such a relation varies significantly for polytropes of larger polytropic indices. In particular, our numerical results show that the percentage change of $U_{3}(n)$, $D_{3,\text{exact}} = [U_{3,\text{exact}}/U_{3}(0) - 1] \times 100\%$, are 2.8 per cent, 15.4 per cent and 52.7 per cent at $n = 1$, $n = 2$ and $n = 3$, respectively. We, again, observe the breakdown of the universality for soft polytropic stars.

$$n^3$$ from equations (49) and (76):

$$U_{3}(n) = \left(\frac{7}{3}\right)^{3/4} \left\{ \frac{7}{8} + n^2 \left(\frac{7\pi^2}{24} - \frac{72269}{25200}\right) + n^4 \left(\frac{320866909}{37044000}\right) \right\} - 19\pi^2 \frac{28}{28} + \left(\frac{289253}{11025} + \frac{2\pi^2}{3}\right) \ln 2 + \frac{536}{35} \ln^2 2 - \frac{3087}{32} J(6, 6; 6) \right\} + O[n^4]$$

which is normalised by the value at the incompressible limit $n = 0$:

$$\frac{U_{3}(n)}{U_{3}(n = 0)} = 1 + 0.2136264652n^2 + 0.026387963n^4 + O[n^6].$$

Using the third order perturbation of $U_{3}(n)$ above and setting $dU_{3}/dn = 0$, we obtain two solutions $n = 0$ and $n = -0.3153$. The latter solution lies outside the physical domain of interest, and our analytical results thus consistently predict the absence of a secondary stationary point. In stark contrast to the case where $l = 2$, the second and the third order terms carry the same sign as the zeroth order term, suggesting a more pronounced variation around the incompressible limit and the absence of a secondary stationary point.

Next, we study the octupole moment-Love relation over an extended range of polytropic index for $n \in [0, 3]$. In Figure 6, both the numerical and the analytical results show that $U_{3}(n)$ in such a relation varies significantly for polytropes of larger polytropic indices. In particular, our numerical results show that the percentage change of $U_{3}(n)$, $D_{3,\text{exact}} = [U_{3,\text{exact}}/U_{3}(0) - 1] \times 100\%$, are 2.8 per cent, 15.4 per cent and 52.7 per cent at $n = 1$, $n = 2$ and $n = 3$, respectively. We, again, observe the breakdown of the universality for soft polytropic stars.
10 DISCUSSION AND CONCLUSION

The I-Love relation has been a hot topic of scholarly account to date (see Yagi & Yunes 2017, for a review on this topic). Various compact stellar objects, including both NSs and QoSs, appear to follow the same relation between the moment of inertia and the tidal Love number. Its physical origin is certainly an interesting issue. It has been observed and proved in the cases of the modified Tolman model and self-bound stars, which both possess non-zero surface mass density, that the I-Love relation is stationary in the incompressible limit (Chan et al. 2014; Sham et al. 2015; Chan et al. 2015, 2016). Along a similar line of thought, using our perturbation solutions to the tidal Love numbers and the mass moments, we have established in the present paper that the I-Love relation is stationary with respect to arbitrary density changes in the incompressible limit, and generalised the stationarity to cases where \( l \geq 2 \) (see Section 8.2). Our proof is generally valid in the sense that there is no need to confine the variation to a certain class (e.g., from a modified Tolman model (self-bound) star to another modified Tolman model (self-bound) star as in the previous studies mentioned above. In particular, we have pinpointed the physical nature of such an EOS-independent stationarity of the I-Love relation, which is attributable to the cancellation of the moments of density variation to the first order in the incompressible limit.

In Newtonian gravity, to study the universality of the I-Love-Q relation, it suffices to understand the constancy of \( U_2 = k_2 (\mathcal{M}_2/\mathcal{M}_0)^{-3/2} \). By the relation \( \lambda_2 \sim \mathcal{Q}^2 \) and equation (62), we obtain the following constitutive relations, in terms of the scaled quadrupole moment \( \mathcal{Q} \) and the scaled moment of inertia \( \mathcal{I} \):

\[
U_2 = \left( \frac{2}{7} \right)^{3/2} \mathcal{Q}^{-1/2},
\]

and in terms of the scaled tidal deformability \( \lambda_2 \) and the scaled quadrupole moment \( \mathcal{Q} \):

\[
U_2 = \left( \frac{2}{7} \right)^{3/2} \lambda_2^{1/4} \mathcal{Q}^{5/4}.
\]

Therefore, the EOS-insensitivity of \( U_2 \), the I-Love relation studied in depth in the present paper, readily leads to the universality of the I-Love-Q relations.

One of the major breakthroughs achieved in the present paper is the establishment of a perturbation expansion for the tidal Love number, which is divergence-free and enables us to evaluate the tidal Love number of an arbitrary star with zero/non-zero surface density from that of an incompressible star. Through balancing the singularities of the tidal field and density profile, we have proposed the modified potential \( \mathcal{h}(r) = H(r)/m(r) \). The governing equations of the modified potential and its associated logarithmic derivative, (22) and (24), are singularity-free, upon which we have developed the perturbation expansion. As a result, we can express the perturbative corrections to the tidal Love number in terms of the moments and the overlap integrals of the density variation (see equation (46)) up to the third order when the unperturbed star is incompressible. Likewise, we have derived a recursive formula to find the corrections to the tidal Love numbers due to an arbitrary density perturbation on a general density profile (see equation (40)).

Using the density profile obtained by applying the SDEM to polytropic stars (Yip et al. 2017), and the perturbative analyses for the tidal Love numbers and the multipole moments developed in Sections 6 and 9 respectively, we have determined perturbation series about the centres \( n = 0, 1 \) as well as two-point Padé approximants for various physical quantities, including \( k_2, \mathcal{M}_2/\mathcal{M}_0, U_2, \)

| Quantity | Two-point approximant | \( 0 \leq n \leq 1 \) | \( 0 \leq n \leq 3 \) |
|----------|-----------------------|----------------|----------------|
| \( k_{2p} \) | \( k_{2p} \) | 2.5 \times 10^{-5\%} | 0.39\% |
| \( \mathcal{M}_{2p}/\mathcal{M}_0 \) | \( \mathcal{M}_{2p}/\mathcal{M}_0 \) | 6.1 \times 10^{-3\%} | 0.21\% |
| \( U_{2p} \) | \( U_{2p} \) | 1.3 \times 10^{-4\%} | 0.93\% |
| \( \lambda_{2p} \) | \( \lambda_{2p} \) | 7.5 \times 10^{-9\%} | 0.93\% |
| \( \mathcal{M}_{4p}/\mathcal{M}_0 \) | \( \mathcal{M}_{4p}/\mathcal{M}_0 \) | 1.6 \times 10^{-4\%} | 0.65\% |
| \( \lambda_{4p} \) | \( \lambda_{4p} \) | 3.5 \times 10^{-4\%} | 2.1\% |
| \( R/a \) | \( \xi_l \) | 8.1 \times 10^{-7\%} | 0.022\% |
| \( M/(\rho a^3) \) | \( m_k \) | 8.5 \times 10^{-5\%} | 0.091\% |

In particular, as shown in Table 3, the two-point Padé approximants for these quantities are very accurate. Throughout the entire physical range where \( 0 \leq n \leq 3 \), the maximum errors for the two-point Padé approximants listed in the table is less than 1 per cent, except for the case \( U_{3p} \) with a maximum error slightly greater than 2 per cent. As a matter of fact, for stiff polytropic stars with \( 0 \leq n \leq 1 \), these two-point Padé approximants are nearly exact. To make Table 3 self-contained, we have also included the results of the corresponding approximants for the dimensionless radius in equation (C2) (see Yip et al. 2017, equation (83)), and the dimensionless mass in equation (C6) (see Yip et al. 2017, equation (86)). One can therefore apply these simple approximants to accurately evaluate various physical quantities, including mass, radius, moment of inertia, and tidal Love numbers (deformabilities), of stable polytropic stars. For ease of reference, a self-contained summary of the accurate two-point approximants are given in Appendix C.

Furthermore, as could be seen the accuracy of the values of \( U_2 \) and \( U_3 \) shown in Figures 4 and 6, and Table 3, the I-Love relation (or multipolar moment-Love relations) of polytropic stars is also accurately reproduced by our perturbative analysis. Consequently, we can study these relations analytically. Both the analytical and numerical results show that the variation of the I-Love relation (i.e., \( U_2 \)) is within 0.4 per cent near the incompressible limit where \( n \in [0, 1] \), growing sharply to 6 per cent in softer EOS when \( n = 2 \), and reaching 25 per cent in the stability limit when \( n = 3 \). Moreover, the universality of the I-Love relation around the incompressible limit is found to be attributable to the existence of the universal stationary point at the incompressible limit \( n = 0 \), and a secondary stationary point around \( n = 0.4444 \). The existence of such a secondary stationary point is in good agreement with the Taylor expansion of \( U_2 \) (see equation (74)). The octupole moment-Love relation of polytropes \( (l = 3) \) is analysed in the same fashion, and it is observed to vary more significantly for the absence of a secondary stationary point. This signifies that the universality of the multipolar moment-Love relations would degrade as the angular momentum index \( l \) increases.

The form of the multipolar moment-Love relation (59) is also worthy of discussion. As \( U_l \) is universal, it is expected to be independent of the spatial size and the mass of a star. It is obvious that \( U_l \) is dimensionless and equation (59) is dimensionally correct.
However, one can similarly construct other dimensionally correct formulae, such as
\[ V_l \equiv k_l \left( \frac{\mathcal{M}_{l+2}}{\mathcal{M}_0} \right)^{-\chi}, \]  
where \( V_l \) and \( \chi \) are two dimensionless constants. How can one determine the value of \( \chi \) such that the above formula is EOS-insensitive?

The answer to this question lies in the physical properties of \( k_l \) and \( \mathcal{M}_l \), which are defined by equations (15) and (60), respectively. In order to make the quantity \( V_l \) EOS-insensitive, the RHS of (82) should remain unchanged if a mass shell of zero (or low) density is artificially added on a specific star. However, the radius of such a composite star increases from \( R \) to \( R + \Delta R \), where \( \Delta R \) is the thickness of the mass shell. On the other hand, such a massless shell cannot change the values of \( q_l \) and \( l_k \) appearing in equations (15) and (60), respectively. Physically speaking, both \( q_l \) and \( l_k \) are global dynamical variables of the star and should not depend on an artificial increment in the stellar radius due to a variation in the extremely low density portion of the relevant EOS. As \( k_l \equiv q_l/R^{l+1} + \mathcal{M}_l \) and \( \mathcal{M}_l \equiv k_l/R^{l+1} \), it is obvious that the value of \( V_l \) will be affected by this zero-density mass shell unless \( \chi = (2l + 1)/(2l - 2) \). Thus, the correct form of the I-Love relation (or the multipolar moment-Love relation), (59), can be obtained.

On the other hand, one could also suggest another possible form of the universal relation:
\[ V_{l,m} \equiv k_{l,m} \left( \frac{\mathcal{M}_{2m-2}}{\mathcal{M}_0} \right)^{-(2l+1)/(2m-2)}, \]  
where \( m \) is a positive integer other than \( l \). The above formula seems to be correct because (i) it is dimensionally correct, and (ii) remains invariant upon the introduction of a massless shell as mentioned above. In fact, this form has been considered by Chan et al. (2014) and was coined as the off-diagonal relation to signify that \( l \neq m \). However, it follows directly from the derivation developed in Section 8 that the so defined quantity \( V_{l,m} \) is no longer stationary about the incompressible limit where \( n = 0 \). As shown in equations (65) and (67), in the limit \( \varepsilon = 0 \), \( d \ln k_l/d\varepsilon \) consists of two contributions, namely, \( \mu_R(R) \) and \( \mu_{2m-2}(R) \), while \( d \ln \mathcal{M}_{2m-2}/d\varepsilon \) is proportional to \( \mu_{2m-2}(R) \). It is then obvious that, in general, \( dV_{l,m}/d\varepsilon \) does not vanish unless \( m = 1 \). Therefore, the relation defined in (83) is not stationary about the incompressible limit. We conclude that such a relation would not hold universally for compact stars with good precision. As a matter of fact, our conclusion is consistent with the numerical result obtained by Chan et al. (2014).

The scope of the present paper is confined to Newtonian gravity. However, the approximants of the tidal Love numbers and mass moments of polytropic stars are accurate over a wide range of polytropic indices. They are useful to astrophysicists in their own right. In addition, they also provide a physically transparent picture for the I-Love-Q relations, at least in the Newtonian regime. We expect that similar investigation could also be performed in Einstein gravity in due course.

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APPENDIX A: PERTURBATIVE SOLUTION TO THE SCALED POLYTROPE FUNCTION IN SDM

The perturbative solution to the scaled polytrope function at the perturbation centres \( n = 0 \) and \( n = 1 \) are summarized below. For
details, readers may refer to a recent article proposing the SDEM in LEE (Yip et al. 2017).

\[
\Theta_0^{(0)}(z) = 1 - \frac{z^2}{\pi^2}, \tag{A1}
\]

\[
\Theta_0^{(1)}(z) = -4 + \frac{4z^2}{\pi^2} (1 - \ln 2) + \left(3 - 2\frac{\pi}{z} - \frac{z^2}{\pi^2} \right) \ln \left(1 - \frac{z}{\pi} \right) + \left(3 + \frac{2\pi}{z} - \frac{z^2}{\pi^2} \right) \ln \left(1 + \frac{z}{\pi} \right), \tag{A2}
\]

\[
\Theta_0^{(2)}(z) = 40 + \frac{7\pi^2}{3} + 8\ln 2 - 14\ln^2 2 \\
+ \left[ \frac{7\pi^2}{3} - 40 + 32\ln 2 - 8\ln^2 2 \right] \frac{z^2}{\pi^2} \\
+ \left[-23 + \frac{20\pi}{z} + \frac{3z^2}{2\pi^2} + (14 + \frac{10\pi}{z} - \frac{z^2}{2\pi^2}) \ln 2 \right] \ln \left(1 - \frac{z}{\pi} \right) \\
+ \left[-23 - \frac{20\pi}{z} + \frac{3z^2}{2\pi^2} + (14 + \frac{10\pi}{z} - \frac{z^2}{2\pi^2}) \ln 2 \right] \ln \left(1 + \frac{z}{\pi} \right) \\
+ \left(1 - \frac{z^2}{\pi^2} \right) \ln \left(1 - \frac{z}{\pi} \right) \ln \left(1 + \frac{z}{\pi} \right) \\
+ \left(14\frac{\pi}{z} - 14 \right) L_2 \left( \frac{\pi - z}{2\pi} \right) \\
+ \left(14\frac{\pi}{z} - 14 \right) L_2 \left( \frac{\pi + z}{2\pi} \right), \tag{A3}
\]

\[
\Theta_1^{(0)}(z) = \frac{\sin z}{z}, \tag{A4}
\]

\[
\Theta_1^{(1)}(z) = \frac{\sin z}{z} \left[ 1 - \ln (2\pi) - \frac{\sin (2\pi)}{2\pi} \right] + \frac{\sin (2\pi)}{4\pi} \frac{(2\pi)}{4} - \frac{1}{2} \ln \sin z + \frac{\cos \frac{1}{2} \ln (2\pi)}{2} + \frac{1}{2} \int_0^\pi \ln \sin t \, dt. \tag{A5}
\]

Here \(L_2(z)\), called the dilogarithm (or polylogarithm of order 2), is defined by (see, e.g., Olver et al. 2010; Lewin 1991):

\[
L_2(z) = \int_0^\infty \frac{1}{t} \ln (1 - t) \, dt, \tag{A6}
\]

\[S_i(x)\text{ and }C_i(x)\text{ are integrals of sine and cosine defined respectively by (see, e.g., Olver et al. 2010):}
\]

\[S_i(x) = \int_0^x \sin t \, dt, \tag{A7}\]

\[C_i(x) = \int_0^x \cos t \, dt. \tag{A8}\]

**APPENDIX B: ASYMPTOTIC BEHAVIOUR OF TIDAL LOVE NUMBER AND MOMENTS OF SOFT POLYTROPES**

Although we are primarily interested in stable polytropic stars with \(n \in [0, 3]\) in the present paper, we find that the asymptotic behaviour of tidal Love numbers and moments around the case \(n = 5\), whose LEE is also exactly solvable, is noteworthy. In both the two-point approximants (55) and (73), we have included an empirical factor \((n - 5)^3\) to improve their accuracy. Such a factor can be argued from the above-mentioned asymptotic behaviour and extrapolation. In the following we briefly sketch our argument.

The LEE for polytropes with \(n = 5\) can be solved exactly with \(\Theta(x) = (1 + x^2/3)^{1/2}\). Hence, for soft polytropes with \(n \approx 5\) and \(x \gg 1\), we have the following approximate dependence: (i) \(\Theta(x) \approx 1/x\), (ii) \(P \approx 1/x^2\), and (iii) \(P \approx 1/x^4\). As a result, the induced mass density in the Poisson’s equation (14) is proportional to \(1/x^4\). On the other hand, the potential at a large distance is dominated by the external tidal field, which goes as \(x^4\). Using standard Green’s function method and the multipole expansion of \(1/|r - r'|\) (see, e.g., Jackson 1999),

\[
\frac{1}{|r - r'|} = \sum_{l=0}^\infty \frac{r^{l+1}}{r'^{l+1}} P_l(\cos \gamma), \tag{B1}
\]

where \(r\) and \(r'\) are two arbitrary 3-dimensional vectors, \(\gamma\) is the angle formed between these two vectors, \(r_{max} = \max(|r|, |r'|)\) and \(r_{min} = \min(|r|, |r'|)\), it is straightforward to show that the induced multipole potential outside the star is given by a term proportional to \(R^{2l+1}/r^{l+1}\), implying that the induced multipole moment \(q_l\) is proportional to \(R^{2l+1}\). As the tidal Love number \(k_l\) is given by \(q_l/(2R^{2l+1})\), we see that \(k_l \propto R^2\).

Given that \(R \propto \sqrt{1/(n - 5)}\) as \(n \to 5\) (Buchdahl 1978), we can show that \(k_l \approx (n - 5)^2\) under the same limit. We have confirmed this analytic result by comparing it with our numerical results. However, we are interested in stable stars with \(n \in [0, 3]\). We found numerically that \(k_l\) is approximately proportional \((n - 5)^2\) for \(n \approx 3\). Therefore, we have included an empirical factor \((n - 5)^2\) instead of \((n - 5)^3\), in the construction of the two-point approximant (55).

On the other hand, it follows directly from equation (60) that the leading dependence of \(M_{2\ell}\) and \(M_{4\ell}\) are given by \(R^{-2} \ln R \sim R^{-2} \sim R^{-2}\), respectively, as \(R \to \infty\). To take care of this behaviour for soft yet stable polytropic stars with \(n \approx 3\), we similarly add an empirical factor \((n - 5)^2\) in the two-point approximant (73). Once again we see from the results discussed in Section 9 that the introduction this factor can indeed significantly improves the accuracy of the approximant for soft polytropic stars.

**APPENDIX C: LIST OF USEFUL FORMULAE**

For ease of reference, in the following we quote the approximate expressions of the radius, the tidal Love numbers of degree \(l = 2\) and \(l = 3\), the mass, the moment of inertia and the octupole moment of polytropes, as a function of the polytropic index \(n\).

**C1 Radius**

For a polytrope of central density \(\rho_c\) under the polytropic equation of state \(P(r) = K \rho_r^n\), its radius is equal to (Yip et al. 2017):

\[
R = \sqrt{\frac{K(n + 1)}{4\pi G}} \rho_c^{(1-n)/(2n)} \xi(n) = a_n^2(n), \tag{C1}
\]

where \(\xi(n)\) is the first zero of the normalised polytrope function, approximately given by the following approximant:

\[
\xi(n) \approx \frac{\pi}{1 + h_2 n + h_3 n^2 + h_4 n^3} \left[ \frac{2}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}} \right]^{(n-1)/2},
\]

\(h_2, h_3, h_4\) are polytropic constants.
The error of $\hat{\xi}_n(n)$ is within $8.1 \times 10^{-7}$ per cent near the incompressible limit for $n \in [0, 1]$, and 0.022 per cent throughout the physical range of stable polytropes for $n \in [0, 3]$ (see Yip et al. 2017, Figure 6).

### C2 Tidal deformabilities and tidal Love numbers

The tidal love numbers $k_l$ and the tidal deformabilities $\lambda_l$ are related by:

$$\lambda_l = \frac{2}{(2l-1)} R^{2l+1} k_l.$$  

(C3)

For the polytropic equation of state, $k_2$ and $k_3$ are approximated by:

$$k_{l,\text{tp}}(n) = (5-n) \frac{a_1 + a_2 + n_2 a_3}{1000 + na_5 + n^2 a_6}$$  

(C4)

where are constants are listed in Table C1. The percentage errors are, respectively, within $2.5 \times 10^{-5}$ per cent and $7.5 \times 10^{-5}$ per cent near the incompressible limit for $n \in [0, 1]$, and 0.39 per cent and 0.93 per cent for stable polytropes over the range $n \in [0, 3]$. The details could be found in Figures 1 and 2.

### C3 Mass

The physical mass $M$ of the same polytrope is given by (Yip et al. 2017):

$$M = \int_0^R \rho(r) 4\pi r^2 dr = \frac{(n+1)K}{4\pi G} \rho_0^{(3-n)/(2n)} \rho_0(n),$$  

(C5)

where $\rho$ is the dimensionless mass approximately equal to $m_b(n)$:

$$m_b(n) = j_0(5-n)^{(15-3n)/4} n^{8(n-1)/8} + 4\pi \left[ \frac{h_1 + h_2 n + h_3 n^2 + h_4 n^3 + h_5 n^4}{\sqrt{5-n(1+h_5 n^2)}} \right]^{3(n-1)/2} \times (5-n)^3 j_1 + n j_2 + n^2 j_3 + n^3 j_4 + 1 + n j_5 + n^2 j_6.$$  

(C6)

where the constants $h_i$ for $i = 1, 2, \ldots, 6$ are given above and the constants $j_i$ for $i = 0, 1, 2, \ldots, 6$ are determined by the SDEM:

$$j_0 \approx -3.42086751650 \times 10^{-10},$$

$$j_1 \approx 0.0826834044808,$$

$$j_2 \approx 0.0570927374428,$$

$$j_3 \approx -0.00213715241113,$$

$$j_4 \approx -0.000633277094516,$$

$$j_5 \approx 1.37086609691,$$

$$j_6 \approx 0.415498502167.$$  

The error is within $8.5 \times 10^{-5}$ per cent for $n \in [0, 1]$, and 0.091 per cent throughout the range of stable polytropes for $n \in [0, 3]$ (see Yip et al. 2017, Figure 7).

### C4 Moment of inertia and octupole moment

Define the 4th moment of the density profile by:

$$I_4 = \int_0^R \rho(r) r^4 dr,$$  

(C7)

where $r^2$ factor accounts for the spherical volume element. Note that the total mass $M = 4\pi I_0$, and the moment of inertia $I = (8\pi/3) I_2$. For $l = 2$ and $l = 3$, accurate two-point Padé approximant is determined by the SDEM:

$$\frac{I_2}{I_0 R^{2l-2}} = (5-n)^{-1} j_1 + n j_2 + n^2 j_3 + n^3 j_4 + \frac{1 + n j_5 + n^2 j_6}{1000 + n b_5 + n^2 b_6},$$  

(C8)

where the constants $b_i$ for $i = 1, 2, \ldots, 6$ are listed in Table C2. The percentage errors of the moment fractions for $l = 2$ and $l = 3$ are, respectively, within $6.1 \times 10^{-5}$ and $1.6 \times 10^{-4}$ per cent near the incompressible limit for $n \in [0, 1]$, and 0.21 per cent and 0.65 per cent for stable polytropes over the range $n \in [0, 3]$. The details could be found in Figures 3 and 5. Together with the aforementioned approximation to the radius and the mass, the expressions yield accurate approximation to the moment of inertia and the octupole moment.

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