A note on necessary conditions for blow-up of energy solutions to the Navier-Stokes equations

G. Seregin

Abstract In the present note, we address the question about behavior of $L_3$-norm of the velocity field as time $t$ approaches blow-up time $T$. It is known that the upper limit of the above norm must be equal to infinity. We show that, for blow-ups of type I, the lower limit of $L_3$-norm equals to infinity as well.

1991 Mathematical subject classification (Amer. Math. Soc.): 35K, 76D.
Key Words: Navier-Stokes equations, Cauchy problem, weak Leray-Hopf solutions, local energy solutions, backward uniqueness.

1 Motivation

Consider the Cauchy problem for the classical 3D-Navier-Stokes system

\[
\begin{align*}
\partial_t v + v \cdot \nabla v - \nu \Delta v &= -\nabla q \\
\text{div } v &= 0 \\
v|_{t=0} &= a \in C^\infty_{0,0}(\mathbb{R}^3),
\end{align*}
\]

in $Q_+$, \hspace{1cm} (1.1)

\[v|_{t=0} = a \in C^\infty_{0,0}(\mathbb{R}^3). \hspace{1cm} (1.2)\]

Here, $v$ and $q$ stand for the velocity field and for the pressure field, respectively, $Q_+ = \mathbb{R}^3 \times ]0, +\infty[$, and

\[C^\infty_{0,0}(\mathbb{R}^3) = \{ a \in C^\infty_0(\mathbb{R}^3) : \text{div } a = 0 \text{ in } \mathbb{R}^3 \}.\]

In what follows, we always assume that $\nu = 1$.

It is well known due to J. Leray, see [5], the Cauchy problem (1.1), (1.2) has at least one solution called the weak Leray-Hopf solution. To give its modern definition, let us introduce standard energy spaces $H$ and $V$. $H$ is
the closure of the set $C_0^\infty(\mathbb{R}^3)$ in $L_2(\mathbb{R}^3)$ and $V$ is the closure of the same set with respect to the norm generated by the Dirichlet integral.

**Definition 1.1** A velocity field $v \in L_\infty(0, +\infty; H) \cap L_2(0, +\infty; V)$ is called a weak Leray-Hopf solution to the Cauchy problem (1.1), (1.2) if the following conditions hold:

$$\int_{Q_+} (v \cdot \partial_t w + v \otimes v : \nabla w - \nabla v : \nabla w) \, dx \, dt = 0$$

for any $w \in C_0^\infty(Q_+)$ with $\text{div} \, w = 0$ in $Q_+$; the function

$$t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot w(x) \, dx$$

is continuous on $[0, +\infty[$ for all $w \in L_2(\mathbb{R}^3)$;

$$\|v(\cdot, t) - a(\cdot)\|_2 \to 0$$

as $t \to +0$;

$$\|v(\cdot, t)\|_2^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla v|^2 \, dx \, dt' \leq \|a\|_2^2$$

for all $t \in [0, +\infty[.$

The definition does not contain the pressure field at all. However, using the linear theory, we can introduce so-called associated pressure $q(\cdot, t)$, which, for all $t > 0$, is the Newtonian potential of $v_{i,j}(\cdot, t) v_{j,i}(\cdot, t)$ and satisfies the pressure equation

$$-\Delta q(\cdot, t) = v_{i,j}(\cdot, t) v_{j,i}(\cdot, t) = \text{div div} \, v(\cdot, t) \otimes v(\cdot, t)$$

in $\mathbb{R}^3$. Since $v$ is known to belong $L_{10}^{10}(Q_+)$, pressure $q$ is in $L_{4}^{4}(Q_+)$. Moreover, the Navier-Stokes system is satisfied in the sense of distributions and even a.e. in $Q_+$. We refer to the paper [3] for details.

Uniqueness of weak Leray-Hopf solutions is still unknown. However, there is a simple but deep connection between smoothness and uniqueness. It has been pointed out by J. Leray in his celebrated paper [5] and reads: any smooth solution to (1.1), (1.2) is unique in the class of weak Leray-Hopf solutions. The problem of smoothness of weak Leray-Hopf solutions is actually one of the seven Millennium problems.
In the paper, we deal with certain necessary conditions for possible blow-ups of solutions to the Cauchy problem \((1.1), (1.2)\). Suppose that \(T > 0\) is the first moment of time when singularities occur. Then, as it has been shown by J. Leray, given \(3 < s \leq +\infty\), there exists a constant \(c_s\) such that

\[
\|v(\cdot, t)\|_{s, \mathbb{R}^3} \geq \frac{c_s}{(T - t)^{\frac{s}{2s}}}
\]  

(1.8)

for \(T/2 \leq t < T\).

However, in the marginal case \(s = 3\), we have a weaker result

\[
\limsup_{t \to T^-} \|v(\cdot, t)\|_3 = +\infty,
\]  

(1.9)

which has been established in [2]. Apparently, a natural question can be raised whether the statement

\[
\lim_{t \to T^-} \|v(\cdot, t)\|_3 = +\infty
\]  

(1.10)

is true or not. In [9], there has been proved a weaker version of (1.10), namely,

\[
\lim_{t \to T^-} \frac{1}{T - t} \int_t^T \|v(\cdot, \tau)\|_3^3 d\tau = +\infty.
\]  

(1.11)

The aim of the present paper is to show validity of (1.10) provided the blow-up of type I takes place, i.e.,

\[
\|v(\cdot, t)\|_\infty \leq \frac{C_\infty}{\sqrt{T - t}}
\]  

(1.12)

for any \(T/2 \leq t < T\) and for some positive constant \(C_\infty\). Our main result can be formulated as follows.

**Theorem 1.2** Let \(T\) be a blow-up time and let, for some \(3 < s \leq +\infty\), there exist a positive constant \(C_s\) such that

\[
\|v(\cdot, t)\|_s \leq \frac{C_s}{(T - t)^{\frac{s-1}{2s}}}
\]  

(1.13)

for any \(T/2 \leq t < T\). Then (1.10) holds.
Let us outline our proof of Theorem 1.2. Firstly, we reduce the general case to a particular one showing that if \((1.13)\) is true for some \(3 < s < +\infty\), then it is true for \(s = +\infty\) as well. Secondly, assuming that \((1.10)\) is violated, i.e., a sequence \(t_k\) tending to \(T\) exists such that
\[
\sup_k \|v(\cdot, t_k)\|_3 = M < +\infty,
\]
we may use a blow-up machinery and construct a non-trivial ancient solution defined in \(\mathbb{R}^3 \times ]-\infty, 0]\) with the following properties. It vanishes at time \(t = 0\) and its \(L_3\)-norm is finite say at time \(t = -1\). In order to apply backward uniqueness results, proved in [2], we need to check that the above ancient solution has a certain behavior at infinity with respect to spatial variables. This can be done with the help of the conception of so-called local energy solutions to the Cauchy problem, see [4] and also [7].

Finally, it is interesting to figure out whether condition \((1.14)\) itself implies regularity. It is worthy to note that the important consequence of \((1.14)\) is that
\[
v(\cdot, T) \in L_3(\mathbb{R}^3).
\]

2 Some auxiliary things

In the paper, we are going to use the following notion. \(B(x_0, R)\) stands for a spatial ball centered at a point \(x_0\) and having radius \(R\), \(B(0, R) = B(0, R)\), and \(B = B(1)\). By \(Q(z_0, R)\), where \(z_0 = (x_0, t_0)\) is a space-time point, we denote a parabolic ball \(B(x_0, R) \times ]t_0 - R^2, t_0[\), and \(Q(R) = Q(0, R), Q = Q(1)\). All constants depending on non-essential parameters will be denoted simply by \(c\).

Lemma 2.1 Suppose that \((1.13)\) holds for some \(3 < s < +\infty\). Then it is true for \(s = +\infty\).

Proof From \((1.4)\) and \((1.13)\) it follows that \(q(\cdot, t) \in L_4^2(\mathbb{R}^3)\) for \(T/2 < t < T\).

Fix \(\varepsilon > 0\) and \(z_0 = (x_0, t_0)\) with \(t_0 < T\) arbitrarily. Applying \((1.13)\) and Hölder inequality, we find
\[
\frac{1}{R^2} \int_{Q(z_0, R)} (|v|^3 + |q|^3)dz \leq
\]
\[
\leq c(s) \frac{1}{R^2} \left( \int_{Q(x_0, R)} (|v|^s + |q|^\frac{s}{2}) \, dz \right)^{\frac{s}{2}} R^{5(1 - \frac{s}{2})} \leq
\]

\[
\leq c(s) R^{5(1 - \frac{s}{2}) - 2} \left( \int_{t_0 - R^2}^{t_0} \frac{C_s^s}{(T - t)^{\frac{3}{2}}/s} \, dt \right)^{\frac{3}{2}} \leq
\]

\[
\leq c(s) C_s^{3} R^{3 - \frac{3}{2} s} \frac{1}{(T - t_0)^{\frac{3}{2} s}} \leq c(s) C_s^{3} \left( \frac{R}{\sqrt{T - t_0}} \right)^{\frac{3}{2} s}.
\]

We let \( R = \sqrt{\gamma(T - t_0)} \) and pick up \( 0 < \gamma < 1 \) so that \( c(s) C_s^{3} \gamma^{3 - \frac{3}{2} s} \leq \epsilon/2. \)

Now, we apply the local regularity theory for suitable weak solutions to the Navier-Stokes equations, developed in [1], [6], [3], and [2]. It reads that if \( \epsilon \leq \epsilon_0 \), where \( \epsilon_0 \) is a universal constant, then

\[
|v(z_0)| \leq \frac{c}{R} = \frac{c}{\sqrt{\gamma(T - t_0)}},
\]

for all \( z_0 = (x_0, t_0) \) with \( x_0 \in \mathbb{R}^3 \) and \( T/2 < t_0 < T \) and for some universal constant \( c \). Lemma 2.1 is proved.

So, we need prove Theorem 1.2 in a particular case \( s = +\infty \) only.

3 Ancient Solution

By assumptions of Theorem 1.2, there must be singular points at \( t = T \).

We take any of them say \( x_0 \in \mathbb{R}^3 \). Then local regularity theory gives the following inequality

\[
\frac{1}{R^2} \int_{Q((x_0, T), R)} (|v|^3 + |q|^\frac{3}{2}) \, dz \geq \epsilon_0 > 0 \quad (3.1)
\]

for \( 0 < R < R_0 = \frac{1}{3} \min\{1, \sqrt{T}\} \) with universal constant \( \epsilon_0 \). Without loss of generality, we may assume that \( x_0 = 0 \).

Proceeding in the same way as in [10], we can find that condition (1.13) implies the following bound

\[
\sup_{0 < R \leq R_0} \left\{ \frac{1}{R^2} \int_{Q((0, T), R)} (|v|^3 + |q|^\frac{3}{2}) \, dz + \frac{1}{R} \int_{Q((0, T), R)} |\nabla v|^2 \, dz + \right. \]

\[
\int_{Q((0, T), R)} |v|^2 \, dz \right\}
\]

\[
\leq c(s) C_s^{3} \left( \frac{R}{\sqrt{T - t_0}} \right)^{\frac{3}{2} s}.
\]
Next, we may scale our functions $v$ and $q$ essentially in the same way as it has been done in [9], namely, 

$$u^{(k)}(y, s) = R_k v(R_k y, T + R_k^2 s), \quad p^{(k)}(y, s) = R_k^2 q(R_k y, T + R_k^2 s)$$

for $y \in B(R_0/R_k)$ and for $s \in \left] - (R_0/R_k)^2, 0 \right]$, where $R_k = \sqrt{T - t_k}$. 

Now, let us see what happens if $k \to +\infty$. This is more or less well-understood procedure and the reader can find details in [2], [9]–[11]. As a result, we have two measurable functions $u$ and $p$ defined on $Q_- = \mathbb{R}^3 \times \left] -\infty, 0 \right]$ with the following properties:

$$u^{(k)} \to u \quad \text{in } L_3(Q(a)),$$

$$\nabla u^{(k)} \to \nabla u \quad \text{in } L_2(Q(a)),$$

$$p^{(k)} \to p \quad \text{in } L_{\frac{3}{2}}(Q(a)),$$

$$u^{(k)} \to u \quad \text{in } C([-a^2, 0]; L_\frac{3}{2}(B(a)))$$

for any $a > 0$. The pair $u$ and $p$ satisfies the Navier-Stokes equations in $Q_-$ in the sense distributions. We call it an ancient solution to the Navier-Stokes equations. Moreover, since inequalities (1.12) and (3.2) are invariant with respect to the Navier-Stokes scaling, we can show that

$$\sup_{0 < a < +\infty} \left\{ \frac{1}{a^2} \int_{Q(a)} (|u|^3 + |p|^2)de + \frac{1}{a} \int_{Q(a)} |\nabla u|^2de + \right.$$ 

$$\left. + \frac{1}{a} \sup_{-a^2 < s < 0} \int_{B(a)} |u(y, s)|^2dy \right\} \leq M_1 < +\infty$$

(3.4)

and

$$|u(y, s)| \leq \frac{C_{\infty}}{\sqrt{-s}}$$

(3.5)

for all $e = (y, s) \in Q_-$. 

The important consequence of (1.15) and the last line in (3.3), is the following fact

$$u(\cdot, 0) = 0$$

(3.6)
in $\mathbb{R}^3$, see [9] in a similar situation.

Now, our goal is to show that the above ancient solution is non-trivial. Unfortunately, we cannot get this by direct passing to the limit in the formula

$$\frac{1}{a^2} \int_{Q(a)} (|u^{(k)}|^3 + |p^{(k)}|^{\frac{3}{2}}) \, de =$$

$$= \frac{1}{a^2 R_k^2} \int_{Q(aR_k)} (|u|^3 + |q|^{\frac{3}{2}}) \, dz \geq \varepsilon_0 > 0 \quad (3.7)$$

for $aR_k < 3/4$. The reason is simple: there is no hope to prove strong convergence of the pressure. However, we still have local strong convergence of $u^{(k)}$ so that

$$\frac{1}{a^2} \int_{Q(a)} |u^{(k)}|^3 \, de \to \frac{1}{a^2} \int_{Q(a)} |u|^3 \, de \quad (3.8)$$

for any $0 < a \leq 3/4$.

To prove that our ancient solution is non-trivial, let us first note that according to (3.2)

$$\frac{1}{a^2} \int_{Q(a)} (|u^{(k)}|^3 + |p^{(k)}|^{\frac{3}{2}}) \, de \leq M_1 \quad (3.9)$$

for sufficient large $k$ and for all $a \in [0, 3/4]$.

The second observation is quite typical when treating the pressure. In the ball $B(3/4)$, the pressure can be split into two parts

$$p^{(k)} = p_1^{(k)} + p_2^{(k)},$$

where the first term is defined by the variational identity

$$\int_{B(3/4)} p_1^{(k)}(y, s) \Delta \varphi(y) \, dy = - \int_{B(3/4)} u^{(k)}(y, s) \otimes u^{(k)}(y, s) : \nabla^2 \varphi(y) \, dy$$

being valid for any $\varphi \in W^2_2(B(3/4))$ with $\varphi = 0$ on $\partial B(3/4)$. It is not difficult to show that the first counter-part of the pressure satisfies the estimate

$$\|p_1^{(k)}(\cdot, s)\|_{L^\infty,B(3/4)} \leq c\|u^{(k)}(\cdot, s)\|_{L^3,B(3/4)}^2 \quad (3.10)$$
for all $-\infty < s < 0$ while the second one is a harmonic function in $B(3/4)$ for the same $s$. Since $p_2^{(k)}(\cdot, s)$ is harmonic, we have

$$\sup_{y \in B(1/2)} |p_2^{(k)}(y, s)|^{3/2} \leq c \int_{B(3/4)} |p_2^{(k)}(y, s)|^{3/2} dy \leq c \int_{B(3/4)} |p^{(k)}(y, s)|^{3/2} dy + c \int_{B(3/4)} |u^{(k)}(y, s)|^3 dy. \quad (3.11)$$

Then, for any $0 < a < 1/2$,

$$\varepsilon_0 \leq \frac{1}{a^2} \int_{Q(a)} (|u^{(k)}|^3 + |p^{(k)}|^{3/2}) de \leq c \frac{1}{a^2} \int_{Q(a)} (|u^{(k)}|^3 + |p_1^{(k)}|^{3/2} + |p_2^{(k)}|^{3/2}) de$$

$$\leq c \frac{1}{a^2} \int_{Q(a)} (|u^{(k)}|^3 + |p_1^{(k)}|^{3/2}) de + c a^3 \frac{1}{a^2} \int_{-a^2}^0 \sup_{y \in B(1/2)} |p_2^{(k)}(y, s)|^{3/2} ds.$$

Combining $\eqref{3.9}$-\eqref{3.11}, we find

$$\varepsilon_0 \leq c \frac{1}{a^2} \int_{Q(3/4)} |u^{(k)}|^3 de + ca \int_{-a^2}^0 \sup_{y \in B(3/4)} \left( |p^{(k)}(y, s)|^{3/2} + |u^{(k)}(y, s)|^3 \right) dy \leq c \frac{1}{a^2} \int_{Q(3/4)} |u^{(k)}|^3 de + ca \int_{Q(3/4)} \left( |p^{(k)}|^{3/2} + |u^{(k)}|^3 \right) de \leq c \frac{1}{a^2} \int_{Q(3/4)} |u^{(k)}|^3 de + cM_1 a$$
for the same $\alpha$. Passing to the limit and choosing sufficiently small $\alpha$, we show that
\[ 0 < c_0 \alpha^2 \leq \int_{Q(3/4)} |u|^3 \, de \] (3.12)
for some positive $0 < \alpha < 1/2$. So, our ancient solution $u$ is non-trivial.

If we show that for some positive $R_*$
\[ |u| + |\nabla u| \in L_\infty((\mathbb{R}^3 \setminus B(R_*)) \times] - (5/6)^2, 0[), \]
we could use arguments from [2] and conclude that, by (3.6), $\nabla \cdot u \equiv 0$ in $\mathbb{R}^3 \times] - (3/4)^2, 0[$. This, together with the incompressibility condition, means that $u(\cdot, t)$ is harmonic in $\mathbb{R}^3$. And it is bounded there. So, $u$ must be a function of $t$ only. But estimate (3.4) says that such a function must be zero in $] - (3/4)^2, 0[$. The latter contradicts (3.12).

### 4 Spatial decay for ancient solutions

We know that
\[ \|u^{(k)}(\cdot, -1)\|_3 \leq M \]
and thus by (3.3)
\[ \|u(\cdot, -1)\|_3 \leq M. \] (4.1)

Now, let us consider the following Cauchy problem
\[ \begin{align*}
\partial_t w + w \cdot \nabla w - \Delta w &= -\nabla r \\
\text{div } w &= 0 \\
\end{align*} \]
\[ \text{in } \bar{Q} = \mathbb{R}^3 \times] - 1, 1[, \] (4.2)
\[ w(\cdot, -1) = u(\cdot, -1). \] (4.3)

We would like to construct a solution to problem (4.2), (4.3) satisfying the local energy inequality. To this end, let us recall notation and some facts from [7].

\[ L_{m, \text{unif}} = \{ u \in L_{m, \text{loc}} : \|u\|_{L_{m, \text{unif}}} = \sup_{x_0 \in \mathbb{R}^3} \left( \int_{B(x_0, 1)} |u(x)|^m \, dx \right)^{1/m} < +\infty \}, \]
\[ E_m = \{ u \in L_{m, \text{unif}} : \int_{B(x_0,1)} |u(x)|^m dx \to 0 \text{ as } |x_0| \to +\infty \}, \]

\[ \overset{\circ}{E}_m = \{ u \in E_m : \text{div} u = 0 \text{ in } \mathbb{R}^3 \}. \]

Apparently,

\[ u(\cdot, -1) \in \overset{\circ}{E}_2. \] (4.4)

**Definition 4.1** A pair of functions \( w \) and \( r \) defined in the space-time cylinder \( \tilde{Q} \) is called a local energy weak Leray-Hopf solution or simply local energy solution to the Cauchy problem (4.2), (4.3) if the following conditions are satisfied:

\[ w \in L_\infty(-1, 1; L_{2, \text{unif}}), \quad \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{1} \int_{B(x_0,1)} |\nabla w|^2 dz < +\infty, \]

\[ r \in L_\frac{3}{2}(-1, 1; L_{\frac{3}{2}, \text{loc}}(\mathbb{R}^3)); \] (4.5)

\[ w \text{ and } r \text{ meet (4.2) in the sense of distributions}; \] (4.6)

\[ \text{the function } t \mapsto \int_{\mathbb{R}^3} w(x, t) \cdot \tilde{w}(x) \, dx \text{ is continuous on } [-1, 1] \] (4.7)

for any compactly supported function \( \tilde{w} \in L_2(\mathbb{R}^3); \)

for any compact \( K, \)

\[ ||w(\cdot, t) - u(\cdot, -1)||_{L_2(K)} \to 0 \text{ as } t \to -1 + 0; \] (4.8)

\[ \int_{\mathbb{R}^3} \varphi|w(x,t)|^2 \, dx + 2 \int_{-1}^{t} \int_{\mathbb{R}^3} \varphi|\nabla w|^2 \, dx dt \leq \int_{-1}^{t} \int_{\mathbb{R}^3} \left(|w|^2 (\partial_t \varphi + \Delta \varphi) + \\
+ w \cdot \nabla \varphi |w|^2 + 2r) \right) \, dx dt \] (4.9)

for a.a. \( t \in ]-1, 1[ \) and for all nonnegative functions \( \varphi \in C_0^\infty(\mathbb{R}^3 \times ]-1, 2[); \)
for any \( x_0 \in \mathbb{R}^3 \), there exists a function \( c_{x_0} \in L^2_\mathbb{T}(-1,1) \) such that
\[
r_{x_0}(x,t) \equiv r(x,t) - c_{x_0}(t) = r^1_{x_0}(x,t) + r^2_{x_0}(x,t),
\]
(4.10)
for \((x,t) \in B(x_0,3/2) \times ]-1,1[\), where
\[
r^1_{x_0}(x,t) = -\frac{1}{3} |w(x,t)|^2 + \frac{1}{4\pi} \int_{B(x_0,2)} K(x-y) : w(y,t) \otimes w(y,t) \, dy,
\]
\[
r^2_{x_0}(x,t) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(x_0,2)} (K(x-y) - K(x_0-y)) : w(y,t) \otimes w(y,t) \, dy
\]
and \( K(x) = \nabla^2(1/|x|) \).

We have, see [4] and also [7].

**Proposition 4.2** Under assumption (4.4), there exists at least one local energy solution to problem (4.2), (4.3).

To describe spatial decay of local energy solution, we need additional notation
\[
\alpha_w(t) = \|w(\cdot,t)\|^2_{L^2_{\text{uni}}}, \quad \beta_w(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0,1)} |\nabla w|^2 \, dx \, dt',
\]
\[
\gamma_w(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0,1)} |w|^3 \, dx \, dt', \quad \delta_r(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0,3/2)} |r_{x_0}|^2 \, dx \, dt'.
\]

One of the most important properties of local energy solutions is a kind of uniform local boundedness of the energy, i.e.,
\[
\sup_{-1 \leq t \leq 1} \alpha_w(t) + \beta_w(1) + \gamma_w^2(1) \leq A < +\infty.
\]
(4.11)

Next, fix a smooth cut-off function \( \chi \) so that \( \chi(x) = 0 \) if \( x \in B \), \( \chi(x) = 1 \) if \( x \notin B(2) \), and then let \( \chi_R(x) = \chi(x/R) \). Hence, one can define
\[
\alpha^R_w(t) = \|\chi_R w(\cdot,t)\|^2_{L^2_{\text{uni}}}, \quad \beta^R_w(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0,1)} |\chi_R \nabla w|^2 \, dx \, dt',
\]
\[ \gamma_w^R(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0,1)} \left| \chi_{\mathbb{R}^w} \right|^3 dx dt', \quad \delta_r^R(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0,3/2)} \left| \chi_{\mathbb{R}^r} \right|^2 dx dt'. \]

As it was shown in [7], the following decay estimate is true.

**Lemma 4.3** Assume that the pair \( w \) and \( r \) is a local energy solution to (4.2), (4.3). Then

\[
\sup_{-1 \leq t \leq \alpha} \gamma_w^R(t) + \beta_w^R(1) + \left( \gamma_w^R \right)^2(1) \leq C(A) \left[ \left\| \chi_{\mathbb{R}^u} (\cdot, -1) \right\|_{L^2(\text{unif})}^2 + 1/R^2 \right]. \tag{4.12}
\]

Since any local energy solution to the Cauchy problem (4.2), (4.3) is also a suitable weak solution to the Navier-Stokes equations, one can apply the local regularity theory to them and deduce from Lemma 4.3 that there exists a positive number \( R^* \) such that

\[
|w(z)| + |\nabla w(z)| \leq A_1 \tag{4.13}
\]

for all \( z = (x, t) \in \left( \mathbb{R}^3 \setminus B(R^*) \right) \times [-(5/6)^2, 1] \).

If we would show that

\[
u \equiv w \tag{4.14}
\]
on \( \mathbb{R}^3 \times [0, 1[, \) this would make it possible to apply backward uniqueness results (actually, to vorticity equations) and conclude that \( u = 0 \) on \( \mathbb{R}^3 \times [-\left(3/4\right)^2, 0] \) which contradicts (3.12). So, the rest of the paper is devoted to a proof of (4.14).

Our first observation in this direction is that \( u \) is \( C^\infty \)- function in \( Q_- \). This follows from [3.5]. Detail discussion on differentiability properties of bounded ancient solutions can be found in [8] and [10]. In addition, the pressure \( p(\cdot, t) \) is in addition, the BMO-solution to the pressure equations

\[-\Delta p(\cdot, t) = \text{div} u(\cdot, t) \otimes u(\cdot, t) \]
in \( \mathbb{R}^3 \).

Using a suitable cut-off function in time and differentiability properties of \( w \) and \( u \), we can get the following three relations:

\[
\int_{\mathbb{R}^3} \varphi(x) w(x, \tau) \cdot u(\cdot, t) dx \bigg|_{\tau = t}^{\tau = -1} =
\]
\[
\int_{-1}^{t} \int_{\mathbb{R}^3} (w \otimes w - \nabla w) : (\varphi \nabla u + u \otimes \nabla \varphi) dx \, d\tau + \\
\int_{-1}^{t} \int_{\mathbb{R}^3} ru \cdot \nabla \varphi dx \, d\tau + \int_{-1}^{t} \int_{\mathbb{R}^3} \phi \cdot \partial_t u dx \, d\tau;
\]

\[
\int_{\mathbb{R}^3} \varphi(x) |w(x, \tau)|^2 dx \bigg|_{\tau = t}^{\tau = -1} + 2 \int_{-1}^{t} \int_{\mathbb{R}^3} \varphi |\nabla w|^2 dx \, d\tau \leq \\
\int_{-1}^{t} \int_{\mathbb{R}^3} \left( |w|^2 \Delta \varphi + w \cdot \nabla \varphi (|w|^2 + 2r) \right) dx \, d\tau;
\]

\[
\int_{\mathbb{R}^3} \varphi(x) |u(x, \tau)|^2 dx \bigg|_{\tau = t}^{\tau = -1} + 2 \int_{-1}^{t} \int_{\mathbb{R}^3} \varphi |\nabla u|^2 dx \, d\tau = \\
= \int_{-1}^{t} \int_{\mathbb{R}^3} \left( |u|^2 \Delta \varphi + u \cdot \nabla \varphi (|u|^2 + 2p) \right) dx \, d\tau
\]

for any \(0 \leq \varphi \in C_0^\infty(\mathbb{R}^3)\). Letting \(\overline{u} = w - u\) and \(\overline{p} = r - p\), we can find from them the main inequality

\[
\int_{\mathbb{R}^3} \varphi(x) |\overline{u}(x, t)|^2 dx + 2 \int_{-1}^{t} \int_{\mathbb{R}^3} \varphi |\nabla \overline{u}|^2 dx \, d\tau \leq \\
\leq \int_{-1}^{t} \int_{\mathbb{R}^3} \left( |\overline{u}|^2 \Delta \varphi + \overline{u} \cdot \nabla \varphi (|\overline{u}|^2 + 2|\overline{p}|) + u \cdot \nabla \varphi |\overline{u}|^2 + \right. \\
\left. -2\varphi \nabla u : (\overline{u} \otimes \overline{u}) \right) dx \, d\tau
\]

(4.15)

for a.a. \(t \) in \([-1, 0]\).

Next, for \(u\) and \(\overline{u}\), we may introduce the analogous quantities

\[
\alpha_u(t) = \|u(\cdot, t)\|^2_{L^2_{\text{unif}}} , \quad \alpha_{\overline{u}}(t) = \|\overline{u}(\cdot, t)\|^2_{L^2_{\text{unif}}}
\]
\[
\beta_u(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0, 1)} |\nabla u|^2 dx dt', \quad \beta_{\overline{\pi}}(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0, 1)} |\nabla \overline{\pi}|^2 dx dt',
\]

\[
\gamma_u(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0, 1)} |u|^3 dx dt', \quad \gamma_{\overline{\pi}}(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0, 1)} |\overline{\pi}|^3 dx dt',
\]

\[
\delta_p(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0, 3/2)} |p_{x_0}|^{1/2} dx dt', \quad \delta_{\overline{p}}(t) = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t} \int_{B(x_0, 3/2)} |\overline{p}_{x_0}|^{1/2} dx dt',
\]

where

\[
p_{x_0}(x, t) \equiv p(x, t) - p_{x_0}^0(t) = p_{x_0}^1(x, t) + p_{x_0}^2(x, t),
\]

\[
p_{x_0}^1(x, t) = -\frac{1}{3} |u(x, t)|^2 + \frac{1}{4\pi} \int_{B(x_0, 2)} K(x - y) : u(y, t) \otimes u(y, t) dy,
\]

\[
p_{x_0}^2(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(x_0, 2)} (K(x - y) - K(x_0 - y)) : u(y, t) \otimes u(y, t) dy,
\]

\[
\overline{p}_{x_0}(x, t) \equiv \overline{p}(x, t) - \overline{p}_{x_0}^0(t) = \overline{p}_{x_0}^1(x, t) + \overline{p}_{x_0}^2(x, t),
\]

\[
\overline{p}_{x_0}^1(x, t) = r_{x_0}^1(x, t) - p_{x_0}^1(x, t), \quad \overline{p}_{x_0}^2(x, t) = r_{x_0}^2(x, t) - p_{x_0}^2(x, t).
\]

By (3.5) and by our definitions,

\[
\alpha_u(t) \leq \frac{c}{(-t)}, \quad \beta_u(t) \leq \frac{c}{(-t)^2}, \quad \gamma_u(t) + \delta_p(t) \leq \frac{c}{(-t)^{3/2}} \quad (4.16)
\]

for all \(-1 \leq t < 0\). Indeed, the first bound follows directly from (3.5). To get the second one, we need to use (3.5), BMO-estimate of the pressure via velocity field, and then local regularity theory in the same way as in the proof of Lemma 2.1. It is useful to note that the above arguments imply the estimate \(|\nabla u(x, t)| \leq c/(-t)|\) for all \((x, t) \in Q_-\). As to the third bound, the second term is estimated with the help of the singular integral theory, (3.5), and definitions of \(p_{x_0}^1\) and \(p_{x_0}^2\).
We fix \( x_0 \in \mathbb{R}^3 \) and a smooth non-negative function \( \varphi \) such that \( \varphi \equiv 1 \) in \( B \) and \( \text{spt} \varphi \subset B(3/2) \) and let \( \varphi_{x_0}(x) = \varphi(x - x_0) \). Considering (4.15) with such a cut-off function \( \varphi_{x_0} \), taking into account (4.11) and (4.16), and arguing for example as in [7], we can find the inequality

\[
\alpha(t_0) + \beta(t_0) \leq c \left[ \int_{-1}^{t_0} \alpha(t)dt + \gamma(t_0) + \right.
\]

\[
+ \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t_0} \int_{B(x_0,3/2)} |p_{x_0}||\pi|dxdt + \right.
\]

\[
\left. \frac{c}{\sqrt{-t_0}} \int_{-1}^{t_0} \alpha(t)dt + \frac{c}{(-t_0)} \int_{-1}^{t_0} \alpha(t)dt \right] \]

(4.17)

for all \(-1 \leq t_0 < 0\).

Next, we can re-write the well-known (in the theory of the Navier-Stokes equations) multiplicative inequality in terms of quantities introduced above

\[
\gamma(t_0) \leq c \left( \int_{-1}^{t_0} \alpha(t)dt \right)^{\frac{3}{4}} \left( \beta(t_0) + \int_{-1}^{t_0} \alpha(t)dt \right)^{\frac{1}{4}}.
\]

To simplify the latter, we first make use of (4.11) and (4.16) in the following way

\[
\alpha(t_0) \leq c(\alpha_w(t_0) + \alpha_u(t_0)) \leq c \left( A + \frac{c}{(-t_0)} \right) \leq \frac{C(A)}{(-t_0)}
\]

for all \(-1 \leq t_0 < 0\). And thus

\[
\gamma(t_0) \leq \frac{C(A)}{\sqrt{-t_0}} \left( \int_{-1}^{t_0} \alpha(t)dt \right)^{\frac{3}{4}} \left( \beta(t_0) + \int_{-1}^{t_0} \alpha(t)dt \right)^{\frac{1}{4}} \quad (4.18)
\]

It remains to estimate the third term on the right hand side of (4.17)

\[
I = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t_0} \int_{B(x_0,3/2)} |\pi_{x_0}||\pi|dxdt \leq I_1 + I_2, \quad (4.19)
\]
where

\[ I_1 = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t_0} \int_{B(x_0, 3/2)} |\mathcal{P}_1 x_0| |\mathcal{U}| dx dt \leq I'_1 + I''_1, \]

\[ I_2 = \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t_0} \int_{B(x_0, 3/2)} |\mathcal{P}_2 x_0| |\mathcal{U}| dx dt, \]

and

\[ I'_1 = c \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t_0} \int_{B(x_0, 3/2)} \left| |w| - |u| \right| |\mathcal{U}| dx dt, \]

\[ I''_1 = c \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t_0} \int_{B(x_0, 3/2)} \int_{B(x_0, 2)} K(x - y) : \left( w(y, t) \otimes w(y, t) - u(y, t) \otimes u(y, t) \right) dy \left| \mathcal{U}(x, t) \right| dx dt. \]

\( I'_1 \) is evaluated easily, namely,

\[ I'_1 \leq c \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t_0} \int_{B(x_0, 3/2)} |\mathcal{U}|^2 \left( |\mathcal{U}| + 2|u| \right) dx dt \leq c \gamma |\mathcal{U}(t_0)| + \frac{c}{\sqrt{-t_0}} \int_{-1}^{t_0} \alpha |\mathcal{U}(t)| dt. \] (4.20)

To estimate \( I''_1 \), we exploit the same idea and \( L_{3/2} \) and \( L_2 \)-estimates for singular integrals

\[ I''_1 \leq c \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t_0} \int_{B(x_0, 3/2)} |\mathcal{U}(x, t)| \left\{ \left| \int_{B(x_0, 2)} K(x - y) : \mathcal{U}(y, t) \otimes \mathcal{U}(y, t) dy \right| + \left| \int_{B(x_0, 2)} K(x - y) : \left( \mathcal{U}(y, t) \otimes u(y, t) + u(y, t) \otimes \mathcal{U}(y, t) \right) dy \right| \right\} dx dt \leq \]
\[
\leq c \gamma(t_0) + \frac{c}{\sqrt{-t_0}} \int_{-1}^{t_0} \alpha(t) dt.
\]

So, by (4.20), we have

\[
I_1 \leq c \gamma(t_0) + \frac{c}{\sqrt{-t_0}} \int_{-1}^{t_0} \alpha(t) dt. \tag{4.21}
\]

In order to find upper bound for \(I_2\), we simply repeat arguments of Lemma 2.1 in [7] with \(R = 1\) there. This gives us the following estimate

\[
|p_{x_0}^2(x, t)| \leq c \int_{\mathbb{R}^3 \setminus B(x_0, 2)} |K(x - y) - K(x_0 - y)| w(y, t) \otimes w(y, t) - u(y, t) \otimes u(y, t) dx \leq c \|w(\cdot, t) \otimes u(\cdot, t) - u(\cdot, t) \otimes u(\cdot, t)\|_{L^1,unif}
\]

being valid for any \(x \in B(x_0, 3/2)\) and thus

\[
I_2 \leq c \sup_{x_0 \in \mathbb{R}^3} \int_{-1}^{t_0} \|w(\cdot, t) \otimes u(\cdot, t) - u(\cdot, t) \otimes u(\cdot, t)\|_{L^1,unif} \int_{B(x_0, 3/2)} |\bar{u}(x, t)| dx.
\]

Furthermore, by (4.11),

\[
\|w(\cdot, t) \otimes u(\cdot, t) - u(\cdot, t) \otimes u(\cdot, t)\|_{L^1,unif} =
\]

\[
= \sup_{x_0 \in \mathbb{R}^3} \int_{B(x_0, 1)} |\bar{u}(y, t) \otimes w(y, t) + u(y, t) \otimes \bar{u}(y, t)| dx \leq \alpha_\tilde{\tau}(t) \alpha_\tilde{n}(t) + \alpha_\tilde{\tau}(t) \alpha_\tilde{n}(t) \leq \frac{C(A)}{\sqrt{-t}} \alpha_\tilde{n}(t).
\]

So,

\[
I_2 \leq \frac{C(A)}{\sqrt{-t_0}} \int_{-1}^{t_0} \alpha(t) dt
\]
and, by (4.18) and (4.21), we have

\[ I \leq c\gamma_u(t_0) + \frac{C(A)}{\sqrt{-t_0}} \int_{-1}^{t_0} \alpha_\pi(t) dt \leq \]

\[ \leq \frac{C(A)}{\sqrt{-t_0}} \left[ \left( \int_{-1}^{t_0} \alpha_\pi(t) dt \right)^{\frac{1}{4}} \left( \beta_\pi(t_0) + \int_{-1}^{t_0} \alpha_\pi(t) dt \right)^{\frac{3}{4}} + \int_{-1}^{t_0} \alpha_\pi(t) dt \right]. \quad (4.22) \]

Combining (4.17) and (4.22) and applying Young inequality, we arrive at the final estimate

\[ \alpha_\pi(t_0) \leq C(A, \delta) \int_{-1}^{t_0} \alpha_\pi(t) dt \]

which is valid for all \(-1 \leq t_0 \leq \delta < 0\). The latter says that \(\alpha_\pi(t) = 0\) in \([-1, 0]\) and, hence, \(u(\cdot, t) = w(\cdot, t)\) for the same \(t\). This completes the proof of Theorem 1.2.

Acknowledgement This work was partially supported by the RFFI grant 08-01-00372-a.

References

[1] Caffarelli, L., Kohn, R.-V., Nirenberg, L., Partial regularity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure Appl. Math., Vol. XXXV (1982), pp. 771–831.

[2] Escauriaza, L., Seregin, G., Šverák, V., \(L^{3,\infty}\)-Solutions to the Navier-Stokes Equations and Backward Uniqueness, Russian Mathematical Surveys, 58(2003)2, pp. 211-250.

[3] Ladyzhenskaya, O. A., Seregin, G. A., On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations, J. math. fluid mech., 1(1999), pp. 356-387.

[4] Lemarie-Riesset, P. G., Recent developemnets in the Navier-Stokes problem, Chapman&Hall/CRC resarch notes in mathematics series, 431.

[5] Leray, J., Sur le mouvement d’un liquide visqueux emplissant l’espace, Acta Math. 63(1934), pp. 193–248.
[6] Lin, F.-H., A new proof of the Caffarely-Kohn-Nirenberg theorem, Comm. Pure Appl. Math., 51(1998), no.3, pp. 241–257.

[7] Kikuchi, N., Seregin, G., Weak solutions to the Cauchy problem for the Navier-Stokes equations satisfying the local energy inequality, AMS translations, Series 2, Volume 220, pp. 141-164.

[8] Koch, G., Nadirashvili, N., Seregin, G., Šverák, V., Liouville theorems for Navier-Stokes equations and applications, to appear in Acta Mathematica.

[9] Seregin, G.A., Navier-Stokes equations: almost $L_{3,\infty}$-cases, Journal of mathematical fluid mechanics, 9(2007), pp. 34-43.

[10] Seregin, G., Šverák, V., On Type I singularities of the local axisymmetric solutions of the Navier-Stokes equations, Communications in PDE's, 34(2009), pp. 171-201.

[11] Seregin, G., Zajaczkowski, W., A sufficient condition of regularity for axially symmetric solutions to the Navier-Stokes equations, SIMA J. Math. Anal., (39)2007, pp. 669–685.

G. Seregin
Center for Nonlinear PDE's,
Mathematical Institute, University of Oxford, UK