A Characterization of Inner Product Spaces Related to the Skew-Angular Distance

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Abstract—A new refinement of the triangle inequality is presented in normed linear spaces. Moreover, a simple characterization of inner product spaces is obtained by using the skew-angular distance.

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1. INTRODUCTION

In 2006, Maligranda [1, Theorem 1] (also see [2]) introduced the following strengthening of the triangle inequality and its reverse: For any nonzero vectors $x$ and $y$ in a real normed linear space $X = (X, \| \cdot \|)$ it is true that

$$
\| x + y \| \leq \| x \| + \| y \| - \left( 2 - \frac{\| x \| + \| y \|}{\| x \| + \| y \|} \right) \min\{\| x \|, \| y \|\},
$$

(1.1)

$$
\| x + y \| \geq \| x \| + \| y \| - \left( 2 - \frac{\| x \| + \| y \|}{\| x \| + \| y \|} \right) \max\{\| x \|, \| y \|\}.
$$

(1.2)

Also, the author used (1.1) and (1.2) for the following estimation of the angular distance

$$
\alpha[x, y] = \left\| \frac{x}{\| x \|} - \frac{y}{\| y \|} \right\|
$$

between two nonzero elements $x$ and $y$ in $X$ which was defined by Clarkson in [3]:

$$
\frac{\| x - y \| - \| x \| - \| y \|}{\min\{\| x \|, \| y \|\}} \leq \alpha[x, y] \leq \frac{\| x - y \| + \| x \| - \| y \|}{\max\{\| x \|, \| y \|\}}.
$$

(1.3)

The right-hand of estimate (1.3) is a refinement of the Massera–Schaffer inequality proved in 1958 (see [4, Lemma 5.1]): for nonzero vectors $x$ and $y$ in $X$ we have

$$
\alpha[x, y] \leq \frac{2\| x - y \|}{\max\{\| x \|, \| y \|\}},
$$

which is stronger than the Dunkl–Williams inequality

$$
\alpha[x, y] \leq \frac{4\| x - y \|}{\| x \| + \| y \|}
$$

proved in [5]. In the same paper, Dunkl and Williams proved that the constant 4 can be replaced by 2 if and only if $X$ is an inner product space.

The main aim of this paper is to obtain a new and simple characterization of inner product spaces. To proceed in this direction, we first present a refinement of the triangle inequality in normed linear spaces and introduce the notion of skew-angular distance. Next, we compare the angular distance and skew-angular distance with each other.

*The text was submitted by the author in English.

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2. A REFINED STATEMENT OF THE TRIANGLE INEQUALITY

We start with the following strengthening of the triangle inequality.

**Theorem 2.1.** For nonzero vectors $x$ and $y$ in a real normed linear space $X = (X, \| \cdot \|)$,
\[
\|x + y\| \leq \|x\| + \|y\| - \left(\frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \frac{x}{\|y\|} + \frac{y}{\|x\|}\right) \min\{\|x\|, \|y\|\}, \tag{2.1}
\]
\[
\|x + y\| \geq \|x\| + \|y\| - \left(\frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \frac{x}{\|y\|} + \frac{y}{\|x\|}\right) \max\{\|x\|, \|y\|\}. \tag{2.2}
\]

**Proof.** Without loss of generality, we may assume that $\|x\| \leq \|y\|$. Then, by the triangle inequality,
\[
\|x + y\| = \left\| \frac{\|x\|}{\|y\|} x + \frac{\|y\|}{\|x\|} y + \left(1 - \frac{\|y\|}{\|x\|}\right) x \right\|
\]
\[
\leq \|x\| \left(\frac{x}{\|y\|} + \frac{y}{\|x\|}\right) + \|x\| - \frac{x}{\|y\|} = \|x\| + \|y\| - \left(\frac{x}{\|y\|} + \frac{y}{\|x\|}\right)
\]
\[
= \|x\| + \|y\| + \|x\| \left(\frac{x}{\|y\|} + \frac{y}{\|x\|}\right) - \|x\| - \|y\|,
\]
which establishes estimate (2.1). Similarly, the computation
\[
\|x + y\| = \left\| \frac{\|y\|}{\|x\|} x + \frac{\|y\|}{\|x\|} y + \left(1 - \frac{\|y\|}{\|x\|}\right) y \right\|
\]
\[
\geq \|y\| \left(\frac{x}{\|y\|} + \frac{y}{\|x\|}\right) - \|y\| = \|y\| + \|y\| - \left(\frac{x}{\|y\|} + \frac{y}{\|x\|}\right)
\]
\[
= \|x\| + \|y\| + \|y\| \left(\frac{x}{\|y\|} + \frac{y}{\|x\|}\right) - \|x\| - \|y\|
\]
gives inequality (2.2). □

The following examples show that neither our refinement nor Maligranda’s refinement of the triangle inequality is always better.

**Example 2.2.** Let $X$ be the normed space $\mathbb{R}$ with the norm $\|x\| = |x|$. Then for $x = 1$ and $y = -2$ we have
\[
2 - \left(\frac{1}{2}, \frac{2}{2}\right) = 2 > 1 = \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \frac{x}{\|y\|} + \frac{y}{\|x\|}.
\]

**Example 2.3.** Let $X = \mathbb{R}^2$ with the norm of $x = (a, b)$ be given by $\|x\| = |a| + |b|$. Take $x = (3/4, 3/4)$ and $y = (-1, 0)$, then $\|x\| = 3/2$ and $\|y\| = 1$. Therefore,
\[
\frac{x}{\|x\|} = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \frac{y}{\|y\|} = (-1, 0), \quad \frac{x}{\|y\|} = \left(\frac{3}{4}, \frac{3}{4}\right), \quad \frac{y}{\|x\|} = \left(-\frac{2}{3}, 0\right),
\]
\[
2 - \left(\frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|}\right) = 1 < \frac{8}{6} = \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \frac{x}{\|y\|} + \frac{y}{\|x\|}.
\]

We can gather estimates (2.1) and (2.2) together as
\[
\|x + y\| + \left(\frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \frac{x}{\|y\|} + \frac{y}{\|x\|}\right) \min\{\|x\|, \|y\|\}
\]
\[
\leq \|x\| + \|y\| \leq \|x + y\| + \left(\frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} - \frac{x}{\|y\|} + \frac{y}{\|x\|}\right) \max\{\|x\|, \|y\|\}.
\]

Also, we use them as the estimates for a distance in normed linear spaces which we call **skew-angular distance**.
Definition 2.4. For two nonzero elements $x$ and $y$ in a real normed linear space $X = (X, \| \cdot \|)$, the distance
\[ \beta[x, y] = \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \]
(2.3)
is called the skew-angular distance between $x$ and $y$.

Corollary 2.5. For any nonzero elements $x$ and $y$ in a real normed linear space $X = (X, \| \cdot \|)$,
\[ \beta[x, y] \leq \frac{\|x - y\|}{\max\{\|x\|, \|y\|\}} + \frac{\||x| - \|y||}{\min\{\|x\|, \|y\|\}}, \]
(2.4)
\[ \beta[x, y] \geq \frac{\|x - y\|}{\min\{\|x\|, \|y\|\}} - \frac{\||x| - \|y||}{\max\{\|x\|, \|y\|\}}. \]
(2.5)

Proof. Estimate (2.2) implies that
\[ \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \max\{\|x\|, \|y\|\} \leq \|x - y\| - \|x\| - \|y\| + \left( \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} \right) \max\{\|x\|, \|y\|\}. \]
Without loss of generality, we may assume that $\|x\| \leq \|y\|$. Then
\[ \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \|y\| \leq \|x - y\| + \frac{\|y\|}{\|x\|}(\|y\| - \|x\|), \]
and so
\[ \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \leq \frac{\|x - y\|}{\|y\|} + \frac{\|y\| - \|x\|}{\|x\|}. \]
Similarly, inequality (2.1) implies that
\[ \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \|x\| \geq \|x - y\| - \frac{\|x\|}{\|y\|}(\|y\| - \|x\|), \]
and so
\[ \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\| \geq \frac{\|x - y\|}{\|x\|} - \frac{\|y\| - \|x\|}{\|y\|}, \]
which completes the proof. \hfill \Box

Estimates (2.1) and (2.2) for the skew-angular distance mean that
\[ \frac{\|x - y\|}{\min\{\|x\|, \|y\|\}} - \frac{\||x| - \|y||}{\max\{\|x\|, \|y\|\}} \leq \beta[x, y] \leq \frac{\|x - y\|}{\max\{\|x\|, \|y\|\}} + \frac{\||x| - \|y||}{\min\{\|x\|, \|y\|\}}. \]
Since $\|\|x\| - \|y|| \leq \|x - y\|$, we obtain the estimate
\[ \beta[x, y] \leq \left( \frac{1}{\max\{\|x\|, \|y\|\}} + \frac{1}{\min\{\|x\|, \|y\|\}} \right) \|x - y\| = \left( \frac{1}{\|x\|} + \frac{1}{\|y\|} \right) \|x - y\|. \]
(2.6)
The constant 1 in the estimate (2.6) is the best possible even for an inner product space. In fact, consider $X = \mathbb{R}$ with the norm of $x$ given by $\|x\| = |x|$. Take $x = -1$ and $y = \epsilon$, where $\epsilon > 0$ is small. Then
\[ \beta[x, y] = \epsilon + \frac{1}{\epsilon} \quad \text{and} \quad \left( \frac{1}{\|x\|} + \frac{1}{\|y\|} \right) \|x - y\| = \left( 1 + \frac{1}{\epsilon} \right) (1 + \epsilon). \]
Hence
\[ \beta[x, y] \leq \frac{\|x\| \|y\|}{\|x\| + \|y\|} \|x - y\| = \frac{1 + \epsilon^2}{(1 + \epsilon)^2} \rightarrow 1 \]
as $\epsilon \rightarrow 0^+$. 

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3. CHARACTERIZATION OF INNER PRODUCT SPACES

In this section we compare the norm-angular distance $\alpha[x, y]$ with the skew-angular distance $\beta[x, y]$. The next theorem due to Lorch will be useful in what follows.

**Theorem 3.1** (see [6]). Let $(X, \| \cdot \|)$ be a real normed linear space. Then the following statements are mutually equivalent:

(i) for each $x, y \in X$, if $\|x\| = \|y\|$, then $\|x + y\| \leq \|\gamma x + \gamma^{-1}y\|$ (for all $\gamma \neq 0$);

(ii) for each $x, y \in X$, if $\|x + y\| \leq \|\gamma x + \gamma^{-1}y\|$ (for all $\gamma \neq 0$), then $\|x\| = \|y\|$;

(iii) $(X, \| \cdot \|)$ is an inner product space.

**Proof.** Let $X = (X, \langle \cdot, \cdot \rangle)$ be an inner product space, $x, y \in X$, and $x, y \neq 0$. We have

$$\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|^2 - \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|^2 = \langle \frac{x}{\|y\|} - \frac{y}{\|x\|}, \frac{x}{\|y\|} - \frac{y}{\|x\|} \rangle - \langle \frac{x}{\|y\|} - \frac{y}{\|x\|}, \frac{x}{\|y\|} - \frac{y}{\|x\|} \rangle = \frac{\|x\|^2}{\|y\|^2} + \frac{\|y\|^2}{\|x\|^2} - 2\langle \frac{x}{\|x\|} - \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|} \rangle = 2 \left( \frac{\|x\|^2}{\|y\|^2} - \frac{\|y\|^2}{\|x\|^2} - \frac{2\langle x, y \rangle}{\|x\|\|y\|} \right) = \frac{\|x\|^2}{\|y\|^2} + \frac{\|y\|^2}{\|x\|^2} - 2 = \left( \frac{\|x\|}{\|y\|} - \frac{\|y\|}{\|x\|} \right)^2 \geq 0,$$

which proves the necessity.

To prove the sufficiency, let $x, y \in X$, $\|x\| = \|y\|$, and $\gamma \neq 0$. From Theorem 3.1 it is enough to prove that

$$\|x + y\| \leq \|\gamma x + \gamma^{-1}y\|.$$

If $x = 0$ or $y = 0$, then the proof is clear. Let $x \neq 0$, $y \neq 0$, and $\gamma > 0$. Applying inequality (3.1) to $\gamma^{1/2}x$ and $-\gamma^{-1/2}y$, instead of $x$ and $y$, respectively, we obtain

$$\left\| \frac{\gamma^{1/2}x}{\|x\|} + \frac{\gamma^{-1/2}y}{\|y\|} \right\| \leq \left\| \frac{\gamma^{1/2}x}{\|x\|} + \frac{\gamma^{-1/2}y}{\|y\|} \right\| \leq \left\| \gamma^{1/2}x \right\| + \gamma^{-1/2}y \right\|.$$

Thus,

$$\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \leq \left\| \frac{x}{\|y\|} + \gamma^{-1}y \right\|.$$

Since $\|x\| = \|y\| \neq 0$, we have

$$\|x + y\| \leq \|\gamma x + \gamma^{-1}y\|.$$

Now, let $\gamma$ be negative. Put $\mu = -\gamma > 0$. From the positive case, we obtain

$$\|x + y\| \leq \|\mu x + \mu^{-1}y\| = \|\gamma x + \gamma^{-1}y\|,$$

which completes the proof. \qed

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