Neutrino mixing matrix in the 3-3-1 model with heavy leptons and $A_4$ symmetry

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We study the lepton sector in the model based on the local gauge group $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ which do not contain particles with exotic electric charges. The seesaw mechanism and discrete $A_4$ symmetry are introduced into the model to understand why neutrinos are especially light and the observed pattern of neutrino mixing. The model provides a method for obtaining the tri-bimaximal mixing matrix in the leading order. A non-zero mixing angle $V_{e3}$ presents in the modified mixing matrix.

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I. INTRODUCTION

There is convincing evidence for solar and atmospheric neutrino oscillations [1]. And the experimental results of Super-Kamiokande [2], KamLAND [3] and SNO [4] confirm that neutrinos have small but non-zero masses and oscillate. The current experimental data are consistent with so called the tri-bimaximal form [5][6] which, apart from phase redefinitions, is given by,

$$V_{\text{tri-bi}} = \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}$$

(1)

The explanation of the smallness of the neutrino masses and the profile of their mixing as required by recent experiments have been a great puzzle in particle physics. Since in the successful Standard Model(SM), only the massless neutrinos which pair charged leptons in three left-handed flavor generations are considered, it appears obviously that massive neutrinos can be regarded as the definitely signature of new physics beyond SM.

The neutrinos may acquire naturally small Majorana masses through the effective dimension-five operator $O_5$ [7]. For the standard $SU(2)_L \otimes U(1)_Y$ gauge model, the realizations of this operator at tree and one loop level were already investigated in the Ref [8], one of the tree level realizations is of course the canonical seesaw mechanism [9] with one heavy right-handed neutrino $N_R$ for each $\nu_i$, whereas the new particles required in the tree-level realizations are most likely too heavy to be observed experimentally in the near future. In order to reduce the scale of new physics to be only a few TeV and thus be observable at future accelerators, another higgs doublet with a naturally small VEV is introduced into the model [5].

On the other hand, it is an interesting challenge to formulate dynamical principles that can lead to the tri-bimaximal mixing pattern given by Eq. (1) in a completely natural way as a first approximation, and many theoretical efforts have been made to produce such a mixing pattern [10]-[14]. For some years Ma [11] has advocated choosing $A_4$, namely, the symmetry group of the tetrahedron as a family group. In a number of interesting papers with various collaborators, Ma has shown that a broken flavour symmetry based on the non-Abelian discrete group $A_4$ appears to be particularly fit for this purpose [11]-[14]. This non-Abelian discrete finite group admits one three-dimensional representation $3$ as well as three one-dimensional representations $1'$, $1''$, and $1'''$, respectively, or in opposition. The SM singlet right-handed neutrinos $\nu_R$ which transform as $3$ or $(1' \oplus 1'' \oplus 1''')$ under $A_4$ are introduced into the model for obtaining see-saw neutrinos masses and at the same time preserving the $SU(2)_L \otimes U(1)_Y$ gauge symmetry. With these representations, the $A_4$ models can naturally obtain the tri-bimaximal mixing at first approximation [13].

Here we would like to extend the above application to models based on the local gauge group $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ (hereafter 3-3-1 model) with a corresponding enlargement of fermion representations [12]. Because of the two

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important features that the number of family is related by anomaly cancellation to the number of colors, and the third family is treated differently from the first and second families the 3-3-1 model has received much attentions \cite{15-23}. With experimental establishment of the neutrino oscillations a number of paper has been published to discuss the neutrino masses and mixing patterns in the model \cite{16}. In this paper, we introduce the canonical seesaw mechanism and $A_4$ symmetry into the framework of one specific 3-3-1 model and study the neutrinos masses and mixing matrix. We show that in this model, the masses of neutrinos and charged leptons are generated by two separate scalar sectors, and the neutrinos can naturally obtain the small masses. In the leading order the neutrino mixing is just the tri-bimaximal matrix, and after a simple modification, the non-zero mixing parameter $V_{e3}$ is present.

The paper is organized as follows: In Section II we give a brief review of the 3-3-1 models and define the framework of our work. In Section III the $A_4$ symmetry is introduced into the model, and the mass mechanisms and mixing matrix of leptons are represented. Section IV discusses the modified neutrino masses and mixing matrix. The conclusions are given in Section V. Appendix states the basic of $A_4$ symmetry and the potentials which can give the VEV form of the scalars we used in the paper.

II. THE 3-3-1 MODEL WITH THE SINGLET RIGHT-HANDED NEUTRINOS

There are different versions of 3-3-1 model. They are nicely reviewed in ref \cite{24}. Consider the electric charge associated with the unbroken gauge symmetry $U(1)_Q$ which is defined in general as a linear combination of the diagonal generators of the group,

$$Q = \hat{T}_3 + \frac{2}{\sqrt{3}}b\hat{I}_8 + X\hat{I}_3,$$

and then

$$Y = \frac{2}{\sqrt{3}}b\hat{I}_8 + X\hat{I}_3,$$

where $T_a \ (a = 1, 2, \cdots, 8)$ are the eight generators of $SU(3)_L$ and $I$ is the unit matrix.

The value of the $b$ parameter determines the fermion assignment and it is customary to use this number to classify the different 3-3-1 models. Taking $b = \pm 3/2$, for example, we obtain the original Frampton, Pisano and Pleitez models \cite{15}. In this version the charge conjugation of the right-handed charged lepton for each generation is combined with the usual $SU(2)_L$ doublet left-handed leptons components to form an $SU(3)$ triplet $(\nu, e, e^c)_L$. In the sense that no extra leptons are needed the version can be considered as minimal. There is no right-handed neutrino in this minimal version but there are quarks with exotic charges $-4/3$ and $5/3$. As it shown in refs. \cite{16, 23}, if we accommodate extra leptons are needed the version can be considered as minimal. There is no right-handed neutrino in this minimal version \cite{24-25} or the version with the right-handed neutrino \cite{26} where all lepton degrees of freedom, i.e., $e_L$, $(e_R)^c$, or $\nu_L$, $(\nu_R)^c$ belong to the same triplet. Therefore the present model has an extra global $U(1)$ symmetry and we can assign a lepton number for every fields. The introduction of right-handed neutral Weyl states $N_{iR}$ is optional. The version with no neutral Weyl states has been studied in refs. \cite{23, 27}. They are introduced here is for the tree level realization of the canonical see-saw mechanism and obviously it does not change the anomaly cancellation.

In the quark sector, we have

$$Q_{aL} = \begin{pmatrix} d_a \\ u_a \\ U_a \end{pmatrix}_L \sim (3, 3^*, \frac{1}{3}), \quad U_{aR} \sim (3, 1, \frac{2}{3}) \quad a = 1, 2,$$
\[ Q_{3L} = \begin{pmatrix} \frac{u_3}{d_3} \\ \frac{d_3}{D_3} \end{pmatrix}_L \sim (3, 3, 0), \quad D_{3R} \sim (3, 1, -\frac{1}{3}) \]
\[ u_{iR} \sim (3, 1, -\frac{2}{3}), \quad d_{iR} \sim (3, 1, -\frac{1}{3}) \quad i = 1, 2, 3. \]  

Note here the five quarks \( u_{iR} \) and \( u_{iR} \) have the same quantum number and so are the four quarks \( d_{iR} \) and \( D_{3R} \).

One can see that the third generation is treated differently from the first two generations here as we mentioned in the introduction.

We now consider the most general set of scalars that can Yukawa couple to the above leptons and quarks through either lepton bilinears, quark bilinears, or quark-lepton bilinears. Then all the possible scalar representations under \( SU(3)_c \otimes SU(3)_L \otimes U(1)_X \) are: \( (1, 1, 0), (1, 1, -1), (1, 1, -2), (1, 3, -\frac{2}{3}), (1, 3, \frac{1}{3}), (1, 6, -\frac{4}{3}), (3, 1, -\frac{1}{3}), (3, 1, \frac{2}{3}), (3, 1, -\frac{4}{3}), (3, 3, 0), (3, 3, 1), (3, 3^*, -\frac{2}{3}), (3, 3^*, \frac{1}{3}), (3, 3^*, 0), (3, 6, -\frac{4}{3}), (3, 6, -\frac{2}{3}), (3, 6, \frac{2}{3}), (3, 8, \frac{3}{2}), (6, 1, \frac{1}{2}), (6, 1, \frac{3}{2}), (6, 3, \frac{1}{2}), (6, 3^*, 0), (6, 6, 0), (6, 8, \frac{1}{2}), (8, 3^*, -\frac{4}{3}), (8, 3^*, \frac{3}{2}), \) and their complex conjugates. As for us, in this work, we choose the scalars to break the symmetry following the pattern,

\[ SU(3)_c \otimes SU(3)_L \otimes U(1)_X \longrightarrow SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \longrightarrow SU(3)_c \otimes U(1)_Q \]

and give, at the same time, masses to the fermion fields in the model. Then the minimally required scalars are:

\[ \chi = \begin{pmatrix} \chi^+ \chi^0 \\ \chi^0 \chi^0 \end{pmatrix} \sim (1, 3, 1, 3), \quad \rho = \begin{pmatrix} \rho^+ \rho^0 \\ \rho^0 \rho^0 \end{pmatrix} \sim (1, 3, 1, 3), \quad \eta = \begin{pmatrix} \eta^0 \eta^0 \\ \eta^0 \eta^0 \end{pmatrix} \sim (1, 3, -\frac{2}{3}, 3), \]

and their complex conjugates. And there may be multiple scalars of each type, in particular when \( A_4 \) symmetry is introduced later. The necessary VEVs are:

\[ \langle \chi \rangle = \begin{pmatrix} 0 \\ 0 \\ V \end{pmatrix}, \quad \langle \rho \rangle = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}, \quad \langle \eta \rangle = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}. \]

III. DISCRETE SYMMETRY \( A_4 \) AND LEPTONS MASSES

As discussed in section II the non-abelian discrete group \( A_4 \) provides interesting examples of leading to the tri-bimaximal mixing matrix. The group consists of 12 elements and has 4 irreducible representations (see refs. [30, 31] or the appendix). Depending on what representations of \( A_4 \) we choose for the various fermion and scalar fields there are different schemes. Here we follow the discussion on the \( A_4 \) model of the standard \( SU(2)_L \times U(1)_Y \) theory in refs. [30] and try the following \( A_4 \) assignment to the leptons of the 3-3-1 model defined in the last section,

\[ \varphi_{3L} = (\nu_i, e_i, E_i)_L \sim \frac{3}{2}, \quad e_{1R} + e_{2R} + e_{3R} \sim (1 \oplus \frac{1}{2} \oplus 1'), \quad N_{iR} \sim \frac{3}{2}, \quad E^c_{3L} \sim \frac{3}{2}. \]

Notice that the SM right-handed charged fermions are assigned to a \( \frac{3}{2} \oplus \frac{1}{2} \oplus 1' \) structure whereas the right-handed neutrinos and heavy right-handed charged fermions are each given to the \( \frac{3}{2} \) representation.

Then \( A_4 \) and \( SU(3)_C \otimes SU(3)_L \otimes U(1)_X \) invariant Yukawa interactions require to enlarge the scalar sector correspondingly. Most general scalars now can be:
• scalars $\rho_i$ ($i = 1, 2, 3$) which transform as $\mathbf{3}$ representation of $A_4$ and Yukawa couple to the lepton bilinears $\varphi_{iL}$ and $e_iR$;

• scalars $\chi$, $\chi'\nu$, $\chi''\nu$ and $\chi_i$ which transform as $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, and $\mathbf{3}$ representations of $A_4$ respectively are needed to Yukawa couple to the lepton bilinears $\varphi_{iL}$ and $E_iR$;

• scalars $\eta$, $\eta'$, $\eta''$ and $\eta_i$ which transform as $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and $\mathbf{3}$ representations respectively and Yukawa couple to the lepton bilinears $\varphi_{iL}$ and $N_{ir}$.

and finally we need $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ singlet scalar $\xi$ to generate tree-level Majorana mass term $M_N(N_R)^cCN_R$.

In practice we find it is sufficient to consider scalars $\rho_i$, $\chi$, $\eta$, $\eta_i$ and $\xi$. Then the $A_4$ and $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ invariant Yukawa interactions in the lepton sector read,

\begin{equation}
L_Y = \lambda_1 (\varphi_{iL}\rho_j) e_1R + \lambda_2 (\varphi_{iL}\rho_j)^{e2R} + \lambda_3 (\varphi_{iL}\rho_j)^{e3R} + \lambda_4 (\varphi_{iL}E_3R)\chi + \lambda_5 (\varphi_{iL}\rho_j E_{kr}) + M_N(N_R)^cCN_R + h_1(\varphi_{iL}N_{jr})\eta + h_2(\varphi_{iL}N_{jr}\eta_i) + h.c. + \cdots
\end{equation}

(10)

Here, $\mathbf{3}(33)$ transforms as $\mathbf{1}$, $\mathbf{3}(33)'$ transforms as $\mathbf{1}'$, $\mathbf{3}(33)''$ transforms as $\mathbf{1}''$, $(\mathbf{3}33)$ transforms as $\mathbf{1}$ under $A_4$ symmetry. We can see that this interaction actually has a quite simple structure.

From the above Yukawa interaction, the $6 \times 6$ mass matrix of charged lepton ($e_i$, $E_i$) is fund to have the form,

\begin{equation}
m_{eE} = \begin{pmatrix}
\lambda_{11}v_1 & \lambda_{21}v_1 & \lambda_{31}v_1 & 0 & \lambda_{51}v_1 & 0 \\
\lambda_{12}v_2 & \lambda_{22}v_2 & \lambda_{32}v_2 & \lambda_{52}v_2 & 0 & 0 \\
\lambda_{13}v_2 & \lambda_{23}v_3 & \lambda_{33}v_3 & \lambda_{53}v_3 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 v & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_4 v & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_4 v
\end{pmatrix}
\end{equation}

(11)

where $V$ and $v_i$ are the VEVs for $\chi$ and $\rho_i$ respectively. $v_i$ are taken to be relatively real, and the numerical subscripts 1, 2, 3 of $v$ denote the $A_4$ components, as in the appendix.

For simplicity we take,

\begin{equation}
v_1 = v_2 = v_3 = v
\end{equation}

(12)

Then the mass matrix can be diagonalized by using the unitary transformations from the weak interaction eigenstates to mass eigenstates,

\begin{equation}
\begin{pmatrix}
E_i\\e_i
\end{pmatrix}_L^w = U_L^\dagger \begin{pmatrix}
E_i\\e_i
\end{pmatrix}_L^m, \quad \begin{pmatrix}
E_i\\e_i
\end{pmatrix}_R^w = U_R^\dagger \begin{pmatrix}
E_i\\e_i
\end{pmatrix}_R^m
\end{equation}

(13)

where the $6 \times 6$ unitary matrices $U_L^\dagger, U_R^\dagger$ can be written as $\mathbf{32}$,

\begin{equation}
U_L^\dagger = \begin{pmatrix}
A_L & B_L \\
F_L & G_L
\end{pmatrix}, \quad U_R^\dagger = \begin{pmatrix}
A_R & B_R \\
F_R & G_R
\end{pmatrix}
\end{equation}

(14)

with

\begin{align*}
A_L &= U(\omega) \cdot \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix}, \quad B_L = U(\omega) \cdot \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix} \\
F_L &= U(\omega) \cdot \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix}, \quad G_L = U(\omega) \cdot \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix} \\
A_R &= I_{3 \times 3}, \quad B_R = F_R = 0, \quad G_R = U(\omega)
\end{align*}

\begin{equation}
U(\omega) = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}
\end{equation}

(15)
Where \( c_i = \cos \theta_i \) represents the mixing of \( e_{iL} \) with the heavy left-handed charged leptons \( E_{iL} \) with,

\[
tg \theta_1 = \frac{2 \lambda_{4v}}{\lambda_5 V}, \quad tg \theta_2 = -\frac{\lambda_{4v}}{\lambda_5 V}, \quad tg \theta_3 = -\frac{\lambda_{4v}}{\lambda_5 V}.
\]

The charged lepton masses are given by,

\[
m_{e_i} = \sqrt{3} \lambda_i c_i v, \quad m_{E_i} = \lambda_4 c_i V + \Delta m_i.
\]

Please note that one also can add another symmetry (such as \( U(1) \) or \( Z_2 \)) into the model to let the gauge invariant term \( \overline{\nu}_{iL} \rho_j E_{kR} \) absent from the Lagrangian and \( c_i = 1 \). Then the mass matrix and transform matrix will be more simple.

Now let us consider the neutrino mass matrix. The right-handed neutrino bare Majorana mass term is trivial, which is \( M_N \) times the identity. The Yukawa term \( \overline{\nu}_{iL} \nu_N \eta \) also contributes trivially to the Dirac mass matrix a term proportional to the \( 3 \times 3 \) identity matrix, i.e. \( h_1 u \) times the identity. The only non-trivial structure is from contribution to the Dirac mass matrix supplied by the Yukawa coupling to \( \eta_i \), which is,

\[
\begin{pmatrix}
0 & h_2 < \eta_3 > & h_2 < \eta_2 > \\
h_2 < \eta_3 > & 0 & h_2 < \eta_1 > \\
h_2 < \eta_2 > & h_2 < \eta_1 > & 0
\end{pmatrix}.
\]  

(16)

After making the assumption about \( A_4 \) breaking as,

\[
< \eta_1 >= u', \quad < \eta_2 >= < \eta_3 >= 0
\]

(17)

We obtain the full \( 6 \times 6 \) neutrino mass matrix on the weak interaction eigenstates,

\[
m_\nu N = \begin{pmatrix}
0 & 0 & 0 & h_1 u & 0 & 0 \\
0 & 0 & 0 & h_1 u & h_2 u' & 0 \\
h_1 u & 0 & 0 & M_N & 0 & 0 \\
0 & h_1 u & h_2 u' & 0 & M_N & 0 \\
0 & h_2 u' & h_1 u & 0 & 0 & M_N
\end{pmatrix} = \begin{pmatrix}
0 & m_D \\
m_D & m_S
\end{pmatrix}
\]

(18)

So the see-saw mass matrix for \( (\nu_i) \) is,

\[
m_\nu = -m_D m_S^{-1} m_D^T = -\frac{1}{M_N} \begin{pmatrix}
(h_1 u)^2 & 0 & 0 \\
0 & (h_1 u)^2 + (h_2 u')^2 & 0 \\
0 & 2 h_1 h_2 u u' & (h_1 u)^2 + (h_2 u')^2
\end{pmatrix},
\]

(19)

which can be written as a simple form,

\[
m_\nu = -\frac{1}{M_N} \begin{pmatrix}
\gamma & 0 & 0 \\
0 & \alpha & \beta \\
0 & \beta & \alpha
\end{pmatrix}.
\]

(20)

This mass matrix can be diagonalized by the transformation,

\[
V_L' = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & \sqrt{2} & 0 \\
1 & 0 & -1 \\
1 & 0 & 1
\end{pmatrix}.
\]

(21)

And the mass matrix is given by,

\[
-\frac{1}{M_N} \begin{pmatrix}
(h_1 u + h_2 u')^2 & 0 & 0 \\
0 & (h_1 u)^2 & 0 \\
0 & 0 & (h_1 u - h_2 u')^2
\end{pmatrix}.
\]

(22)

Then we can choose the free parameters \( h_1, M_N \) and \( u \) to get the small masses of neutrinos. If \( h_i \) is of order 1 and \( M_N \) is of order \( \sim TeV \), \( u, u' \sim MeV \), The neutrinos will have the masses of order eV.
Using the definition for the observed neutrino mixing matrix \( V = A^iv_L \), we obtain the mixing matrix in this order,

\[
V = A^iv_L = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{2}} \end{pmatrix} = P_c V_{tri-bi} P_\phi
\]  

(23)

where \( V_{tri-bi} \) is defined in the Eq. (11), the phase matrices \( P_c \) and \( P_\phi \) are both diagonal and with the diagonal elements \( c_i \) and the diagonal elements 1, 1 and \( i \) respectively.

IV. MODIFIED NEUTRINOS MASSES AND MIXING MATRIX

More generally, according to the low energy effective field theory analysis \([7]\) neutrino masses are generated by the unique effective dimension-five operator \( O_5 \) which has been studied by many papers in the standard SU(2)\(_L\otimes U(1)_Y\) model \([7,8]\) and \( A_4 \) model \([12,33]\) and has the form as,

\[
O_5 = \frac{\lambda_{ij}(H^c)^T (H^c)^j}{\Lambda} + h.c.
\]  

(24)

Note here the charge conjugation matrix \( C \) between the lepton fields has been omitted, and in our notation, \( H \) and \( H^c \) denote \( \eta \) or \( \eta_i \), \( \Lambda \) denote \( M_N \), \( \varphi_L \) is the SU(3) lepton triplet and \( \Lambda \) is a matrix in flavour space. Then there has three types of this \( O_5 \) operators, i.e., \( (\eta \varphi_L)^2 \), \( (\eta \varphi_L)(\eta \varphi_L)\) and \( (\eta \varphi_L)^2 \). The operator \( (\eta \varphi_L)^2 \) which has the form of \( 3 \times 3 \) contributes a term proportional to the identity matrix. Next, \( (\eta \varphi_L)(\eta \varphi_L) \), which is formed by \( 3 \times 3 \times 3 \), has the form \( \eta \varphi_L = \eta \varphi_L + \eta \varphi_L + \eta \varphi_L \) and its analogous form. Thus, the operator \( (\eta \varphi_L)(\eta \varphi_L) \) contributes the term denoted by \( \beta \) in \([20]\). Finally, the operator \( (\eta \varphi_L)^2 \) actually denotes schematically 4 different operators since it is formed by \( 3 \times 3 \times (3 \times 3) \) and this contains \( 1 \times 1 \), \( 1 \times 1 \), \( 3 \times 3 \times 3 \) and \( 3 \times 3 \), corresponding respectively to the operators \( (\eta \varphi_L + \eta \varphi_L + \eta \varphi_L)^2 \), \( (\eta \varphi_L + \eta \varphi_L + \eta \varphi_L)(\eta \varphi_L + \eta \varphi_L + \eta \varphi_L) \), \( (\eta \varphi_L + \eta \varphi_L + \eta \varphi_L)(\eta \varphi_L + \eta \varphi_L + \eta \varphi_L) \) and \( (\eta \varphi_L + \eta \varphi_L + \eta \varphi_L)(\eta \varphi_L + \eta \varphi_L + \eta \varphi_L) \).

Where \( \eta \) and \( \eta_i \) acquire the VEVs of \( < \eta > = u \) and \( < \eta_i > = (u^*,0,0) \).

Then we obtain a more general form of the neutrino mass matrix \( m_\nu \), as

\[
m_\nu = -\frac{1}{M_N} \begin{pmatrix} \gamma & 0 & 0 \\
0 & \alpha - \varepsilon & \beta \\
0 & \beta & \alpha + \varepsilon \end{pmatrix},
\]  

(25)

rather than the mass matrix \( m_\nu \) in Eq. \([20]\).

At this point, it could only suppose that \( \varepsilon \) is small compared to \( \beta \), in which case \( U_\nu \) is perturbed from the desired \( V^\nu_L \) in Eq. \([21]\) to

\[
V^\nu_L = \begin{pmatrix} 0 & 1 & 0 \\
\cos \theta & 0 & -\sin \theta \\
\sin \theta & 0 & \cos \theta \end{pmatrix}.
\]  

(26)

where \( \theta = \frac{\pi}{4} + \delta \) ( \( \delta \ll 1 \) ), \( \sin \theta \simeq \frac{1}{\sqrt{2}}(1 + \frac{\pi}{16}) \), \( \cos \theta \simeq \frac{1}{\sqrt{2}}(1 - \frac{\pi}{16}) \). In this order, the neutrino masses come out to be \( \alpha + \beta^2 + \varepsilon^2, \gamma \), and \( \alpha - \sqrt{\beta^2 + \varepsilon^2} \).

Then the matrix \( V \) becomes,

\[
V = A^iv_L^\nu = \frac{1}{\sqrt{3}} \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} \cos \theta + \sin \theta & 1 & \cos \theta - \sin \theta \\
\omega (\sin \theta + \omega \cos \theta) & 1 & \omega (\cos \theta - \omega \sin \theta) \\
\omega (\cos \theta + \omega \sin \theta) & 1 & \omega (-\sin \theta + \omega \cos \theta) \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & -\beta \\
-1 + \frac{\pi}{2} i \sqrt{2} & \sqrt{3} i + \frac{\pi}{2} \\
-1 - \frac{\pi}{2} i \sqrt{2} & -\sqrt{3} i + \frac{\pi}{2} \end{pmatrix}
\]  

(27)

\[
= P_c V_{tri-bi} \begin{pmatrix} \cos \delta & 0 & -\sin \delta \\
0 & 1 & 0 \\
isin \delta & 0 & icos \delta \end{pmatrix}
\]  

(29)

Compared with Eq. \([23]\), the middle column is uncorrected at this level. At the same time a nonzero \( V_{e3} \) element is generated, and there are other small deviations from exact tri-bimaximal mixing.
V. CONCLUSION

The non-abelian discrete symmetry $A_4$ appears to be particularly fit for the purpose to produce the neutrino tri-bimaximal mixing pattern in the standard $SU(2)_L \times U(1)_Y$ theory. In this paper we have generalized the $A_4$ study to the 3-3-1 model. In the version we consider here there are negative charged leptons and right handed neutrinos in addition to the ordinary SM leptons. By combining the $A_4$ symmetry and canonical see saw mechanism we have reproduced the observed neutrino tri-bimaximal mixing matrix. The smallness of neutrino masses can be explained naturally without introducing too heavy neutral leptons. Our main results are Eq. (23) and Eq. (29). Numerically they are consistent with the present experimental constraints \[34,35,36\]. A small but non-zero mixing angle $V_{e3}$ present in Eq. (29). This angle has been assumed to be zero in tri-bimaximal form but it is only required experimentally to be small i.e. $|V_{e3}| < 0.16^{37}$, and maybe is measured more accurately by the daya bay reactor neutrino experiments \[38\].

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Appendix: Basic $A_4$ properties and the potential

The model is based on the discrete group $A_4$ following refs. \[30,31\], where its structure and representations are described in detail. It is appropriate to recall briefly some relevant features of it. $A_4$ symmetry is the discrete symmetry group of the rotations that leave a tetractetraedron invariant, or the group of the even permutations of 4 objects. It has 12 elements and 4 inequivalent irreducible representations denoted $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and $\mathbf{3}$ in terms of their respective dimensions. Introducing $\omega$, the cubic root of unity, $\omega = \exp i \frac{2\pi}{3}$, so that $1 + \omega + \omega^2 = 0$, the three one-dimensional representations are obtained by dividing the 12 elements of $A_4$ in three classes, which are determined by the multiplication rule, and assigning to (class 1, class 2, class 3) a factor $(1,1,1)$ for $\mathbf{1}$, or $(1,\omega,\omega^2)$ for $\mathbf{1}'$, or $(1,\omega^2,\omega)$ for $\mathbf{1}''$. The product of two $\mathbf{3}$ gives $\mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{1}' + \mathbf{1}'' + \mathbf{3}' + \mathbf{3}^\prime$. Also $\mathbf{1}' \times \mathbf{1}' = \mathbf{1}'$, $\mathbf{1}' \times \mathbf{1}'' = \mathbf{1}'$, $\mathbf{1}'' \times \mathbf{1}'' = \mathbf{1}'$ etc. For $\mathbf{3}$ ~ $(a_1, a_2, a_3)$ and $\mathbf{3}$ ~ $(b_1, b_2, b_3)$, the irreducible representations obtained from their tensor products,

$$
3 \times 3 = 1(a_1 b_1 + a_2 b_2 + a_3 b_3) + 1'(a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3) + 1''(a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3) + 2(a_2 b_3, a_3 b_1, a_1 b_2) + 2(a_3 b_2, a_1 b_3, a_2 b_1)
$$

As required in section \[11\] there are need a mechanism such that the scalar fields develop a VEV along the directions,

$$
< \chi > = V,
$$

$$
< \rho_i > = (v, v, v),
$$

$$
< \eta > = u,
$$

$$
< \eta'_i > = (u', 0, 0).
$$

Here for the convenience of depiction, we replace $\eta_i$ with $\eta'_i$ in this section.

When we study the Higgs potential for $SU(3) \times U(1)$ scalars, we should consider all various representations for $A_4$. For the sake of simplicity, previous to give the whole potential, we restrict to the Higgs potential for a single $SU(3) \times U(1)$ scalar triplet $\phi$, which transform $\mathbf{3}$ under $A_4$. The multiplication $3 \times 3 = 1 + 1' + 1'' + 3 + 3'$ shows that there is only one quadratic invariant. Since $(\mathbf{3} \times \mathbf{3}) \times (\mathbf{3} \times \mathbf{3})$ contains $\mathbf{4}$ five times, corresponding to $\mathbf{1} \times \mathbf{1}$, $\mathbf{1}' \times \mathbf{1}'$, $\mathbf{3} \times \mathbf{3}$, $\mathbf{3} \times \mathbf{3}$, and $\mathbf{3} \times \mathbf{3}$, there should have 5 quartic invariants and the last two terms $\mathbf{3} \times \mathbf{3}$ and $\mathbf{3} \times \mathbf{3}$ are the complex conjugate with each other. Then the Higgs potential of the single $SU(3) \times U(1)$ scalar triplet $\phi$, which transform $\mathbf{3}$ under $A_4$ is given by,

$$
V(\phi) = \mu_0^2 \sum_i \phi_i^\dagger \phi_i + \frac{1}{2} \lambda_1 \left( \sum_i \phi_i^\dagger \phi_i \right)^2 + \frac{1}{2} \lambda_2 \sum_{i,j} (3 \delta_{i,j} - 1)(\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j)
$$

$$
+ \frac{1}{2} \lambda_3 \sum_{i \neq j} (\phi_i^\dagger \phi_j)(\phi_i^\dagger \phi_j) + \frac{1}{2} \lambda_4 \sum_{i \neq j} (\phi_i^\dagger \phi_i)^2.
$$

(32)
Note that for the sake of simplicity, we have taken $\lambda$’s and $v_1$’s to be real, since our focus here is not on $CP$ violation.

This potential has minimum at $v_1 = v_2 = v_3 = v = \sqrt{-\frac{\mu^2}{3\lambda_1+2\lambda_2+2\lambda_3}}$ or at $v_1 = \sqrt{\frac{\mu^2}{\lambda_1+2\lambda_2}}$, $v_2 = v_3 = 0$.

Consider all the scalar triplets in the model and note that under $A_4$ the $\frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = 1$ is possible, i.e. 1 2 3 + permutations. So the $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes A_4$ invariant higgs potential of the model can be written as,

$$V(\chi) = \mu_\chi^2 (\chi^\dagger \chi) + \frac{1}{2} \lambda_{\chi}^2 (\chi^\dagger \chi)^2$$ (33)

$$V(\rho) = \mu_\rho^2 \sum_i \rho_i \rho_i + \frac{1}{2} \lambda_{\rho}^2 \left[ (\sum_i \rho_i \rho_i)^2 + \frac{1}{2} \lambda_{\rho}^2 \sum_{i,j} (3\delta_{ij} - 1)(\rho_i^\dagger \rho_i)(\rho_j^\dagger \rho_j) \right]$$

$$+ \frac{1}{2} \lambda_{\rho}^2 \sum_{i \neq j} (\rho_i^\dagger \rho_j)(\rho_i^\dagger \rho_j) + \frac{1}{2} \lambda_{\rho}^2 \sum_{i \neq j} (\rho_i^\dagger \rho_j)^2.$$ (34)

$$V(\eta) = \mu_\eta^2 (\eta^\dagger \eta) + \frac{1}{2} \lambda_{\eta}^2 (\eta^\dagger \eta)^2$$ (35)

$$V(\eta') = \mu_{\eta'}^2 \sum_i \eta_i^\dagger \eta_i + \frac{1}{2} \lambda_{\eta'}^2 \sum_i (\eta_i^\dagger \eta_i)^2 + \frac{1}{2} \lambda_{\eta'}^2 \sum_{i,j} (3\delta_{ij} - 1)(\eta_i^\dagger \eta_i)(\eta_j^\dagger \eta_j)$$

$$+ \frac{1}{2} \lambda_{\eta'}^2 \sum_{i \neq j} (\eta_i^\dagger \eta_j)(\eta_i^\dagger \eta_j) + \frac{1}{2} \lambda_{\eta'}^2 \sum_i (\eta_i^\dagger \eta_j)^2.$$ (36)

$$V(\chi, \rho) = \lambda_{\chi \rho}^2 \sum_i \rho_i \rho_i + \lambda_{\chi \rho} ( \chi^\dagger \chi)(\rho_i^\dagger \rho_i) + \frac{1}{2} \lambda_{\chi \rho}^2 \sum_i [(\chi^\dagger \chi)(\rho_i^\dagger \rho_i) + h.c]$$

$$+ \lambda_{\chi \rho} \varepsilon_{ijk} [(\chi_i^\dagger \rho_j)(\rho_j^\dagger \rho_k) + h.c.]$$ (37)

$$V(\chi, \eta) = \lambda_{\chi \eta}^2 (\chi^\dagger \chi)(\eta^\dagger \eta) + \lambda_{\chi \eta}^2 (\chi^\dagger \chi)(\eta^\dagger \eta) + \mu_{\chi \eta} \chi \chi \eta \eta$$ (38)

$$V(\chi, \eta') = \lambda_{\chi \eta'}^2 (\chi^\dagger \chi)(\eta_i^\dagger \eta_i) + \lambda_{\chi \eta'}^2 (\chi^\dagger \chi)(\eta_i^\dagger \eta_i)$$ (39)

$$V(\rho, \eta) = \lambda_{\rho \eta} ( \rho_i^\dagger \rho_i)(\eta^\dagger \eta) + \lambda_{\rho \eta} ( \rho_i^\dagger \rho_i)(\eta^\dagger \eta) + \mu_{\rho \eta} \sum_i (\rho_i^\dagger \rho_i)$$ (40)

$$V(\rho, \eta') = \lambda_{\rho \eta'}^2 ( \rho_i^\dagger \rho_i)(\eta_i^\dagger \eta_i) + \lambda_{\rho \eta'}^2 ( \rho_i^\dagger \rho_i)(\eta_i^\dagger \eta_i)$$ (41)

$$V(\rho, \eta') = \lambda_{\rho \eta'}^2 ( \rho_i^\dagger \rho_i)(\eta_i^\dagger \eta_i) + \lambda_{\rho \eta'}^2 ( \rho_i^\dagger \rho_i)(\eta_i^\dagger \eta_i)$$ (42)

$$V(\chi, \rho) = \lambda_{\chi \rho}^2 \varepsilon_{ijk} [(\chi_i \rho_j)(\rho_j \rho_k) + h.c.] + \lambda_{\chi \rho} \varepsilon_{ijk} [(\chi_i \eta_i)(\eta_i \eta_i) + h.c.] + \mu_{\chi \rho} \sum_i (\rho_i^\dagger \rho_i$$ (43)

$$V(\chi, \rho') = \lambda_{\chi \rho'}^2 \varepsilon_{ijk} [(\chi_i \rho_j)(\rho_j \rho_k) + h.c.] + \lambda_{\chi \rho'} \varepsilon_{ijk} [(\chi_i \eta_i)(\eta_i \eta_i) + h.c.] + \mu_{\chi \rho'} \sum_i (\rho_i^\dagger \rho_i$$ (44)

Note that, in the order we discussed, the charged lepton mass matrix and the neutrino mass matrix are related by two separate scalar sectors $\chi$, $\rho_i$ and $\eta$, $\eta_i$, respectively. If there is no communication between the two scalar sectors, the VEVs of $\rho$ and $\eta_i$ given in Eq. (31) will break the $A_4$ symmetry to $Z_2$ symmetry in charged sector and $Z_2$ symmetry in neutrino sector and these residual symmetries will be maintained. In general, $\chi$, $\rho_i$ and $\eta$, $\eta_i$ mix in the potential and it is not possible to keep the VEVs structure for $\rho$ and $\eta_i$ as Eq. (31). One needs to separate them from communicating in the scalar potential and therefore to simplify the vacuum alignment problem. Suppose that, at least at some level, the interchange between the fields $\chi$, $\rho_i$ and $\eta$, $\eta_i$ to produce the desired mass matrices in the charged and neutrino lepton sectors. We can determine the minima of two scalar potentials $V_c$ and $V_0$, depending
only, respectively, on $\chi$, $\rho_i$ and $\eta, \eta'_i$. There are whole regions of the parameter space where $V_c(\chi, \rho)$ and $V_0(\eta, \eta')$ have the minima VEVs given in Eq. (31).

First consider the scalar potential $V_c(\chi, \rho)$,

$$V_c(\chi, \rho) = V(\chi) + V(\rho) + V(\chi \rho). \quad (45)$$

Analyzing the field configuration as,

$$< \chi > = V, \quad < \rho > = (v, v, v), \quad (46)$$

then the minimum conditions are,

$$\frac{\partial V_c}{\partial \chi} = 2V(\mu_0^2 + \lambda_1^\rho V^2 + 3\lambda_1^\rho v^2) \quad (47)$$

$$\frac{\partial V_c}{\partial \rho} = 2v(\mu_0^2 + 3\lambda_0^\rho v^2 + 2\lambda_0^\rho v^2 + 2\lambda_0^\rho v^2 + \lambda_3^\rho V^2) \quad (48)$$

Therefore the $< \chi > = V$ and $< \rho > = (v, v, v)$ can be local minima of $V_c$ depending on the parameters. Take into account $A_4$ symmetry there are four degenerate minima, i.e., $< \rho_i > = (v, v, v)$, $< \rho_i > = (v, v, v)$, $< \rho_i > = (v, v, -v)$ and $< \rho_i > = (v, v, -v)$ in this region.

Then consider the scalar potential $V_0(\eta, \eta')$,

$$V_0(\eta, \eta') = V(\eta) + V(\eta') + V(\eta \eta'). \quad (49)$$

Search for the minimum conditions at $< \eta > = u$, $< \eta'_i > = (u', 0, 0)$, then

$$\frac{\partial V_0}{\partial \eta} = 2u(\mu_0^2 + \lambda_1^\eta u^2 + \lambda_1^\eta' u^2 + \lambda_2^\eta u^2 + \lambda_3^\eta u^2) \quad (50)$$

$$\frac{\partial V_0}{\partial \eta'_i} = 2u'(\mu_0^2 + \lambda_1^\eta u^2 + 2\lambda_2^\eta u^2 + \lambda_3^\eta u^2 + \lambda_3^\eta u^2) \quad (51)$$

In this case $(\partial V_0/\partial \eta_{2,3}) = 0$ are automatically satisfied. There a large portion of the parameter space where the minimum is. In this region, there are six degenerate minima, i.e., $< \eta'_i > = (\pm u', 0, 0)$, $< \eta'_i > = (0, \pm u', 0)$ and $< \eta'_i > = (0, 0, \pm u')$, related by $A_4$ symmetry. Putting together the minima of $V_c(\chi, \rho)$ and $V_0(\eta, \eta')$ there are 24 degenerate minima of the potential energy, differing for signs or ordering. It can be shown that these 24 minima produce exactly the same mass pattern discussed in section III up to field and parameter redefinitions. Therefore it is not restrictive to choose one of them.

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