ASYMPTOTICS OF DEGENERATIONS OF MIXED HODGE STRUCTURES

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Abstract. We construct a hermitian metric on the classifying spaces of graded-polarized mixed Hodge structures and prove analogs of the strong distance estimates \[ \text{CKS} \] between an admissible period map and the approximating nilpotent orbit. We also consider the asymptotic behavior of the biextension metric introduced by Hain \[ \text{Ha2} \], analogs of the norm estimates of \[ \text{KN1} \] and the asymptotics of the naive limit Hodge filtration considered in \[ \text{KP2} \].

1. Introduction

Let \( X \) and \( S \) be smooth complex algebraic varieties, and \( f : X \to S \) be a morphism such that \( f \) is smooth, proper and connected. Then \[ G \], a choice of projective embedding of \( X \) determines a polarized variation of Hodge structure \( H \to S \) of weight \( k \) via the \( k \)’th cohomology groups of the fibers. Parallel translation of the data of \( H \) to a fixed reference fiber \( H \) using the Gauss–Manin connection then determines an associated period map

\[ \varphi : S \to \Gamma \backslash D \]

(1.1)

where \( \Gamma \) is monodromy group and \( D \) is a classifying space of pure Hodge structures which are polarized by a non-degenerate bilinear form \( Q \) on \( H_Z \) of parity \( (-1)^k \).

In the case where \( S \) is a curve with smooth compactification \( \bar{S} \), the period integrals which define \[ \text{L} \] have at worst logarithmic singularities at the punctures \( p \in \bar{S} - S \). More generally, let \( \bar{S} \) be a smooth completion of \( S \) such that \( \bar{S} - S \) has only normal crossing singularities. Then, the local monodromy of \( H \) about any singular point \( p \in \bar{S} - S \) is quasi-unipotent. By passage to a finite cover, we can then assume that the local monodromy at \( p \) is unipotent. The analysis of the singularities of the period map can therefore be reduced to the following case: Let \( \Delta^r \subset S \) be a polydisk with local coordinates \( (s_1, \ldots, s_r) \) such that \( S \cap \Delta^r \) is the locus of points where \( s_1 \cdots s_r \neq 0 \). Then, we can construct a commutative diagram

\[ \begin{array}{ccc}
U^r & \xrightarrow{F} & D \\
\downarrow s_j = e^{2 \pi i z_j} & & \downarrow \\
\Delta^r & \xrightarrow{\varphi} & \Gamma \backslash D
\end{array} \]

(1.2)

where \( U^r \) is the product of upper half-planes with coordinates \( z_j \), and

\[ F(z + 1) = T_j F(z) \]

where \( T_j = e^{N_j} \) is the monodromy of \( H \) about \( s_j = 0 \).

Date: March 11, 2014.
The group $G_R = \text{Aut}_R(Q)$ acts transitively on the classifying space $D$. The compact dual $\hat{D}$ is the orbit inside a suitable flag variety of any point in $D$ under the action of $G_C = \text{Aut}_C(Q)$. The monodromy transformations $T_j$ belong to $G_R$, and hence

$$e^{N(z)} = e^{\sum_j z_j N_j} \in G_C$$

Accordingly, $z \mapsto e^{-N(z)}F(z)$ is a holomorphic map from $U^r \to \hat{D}$ which is invariant under the deck transformations $(z_1, \ldots, z_r) \mapsto (z_1, \ldots, z_j + 1, \ldots, z_r)$, and hence descends to a holomorphic map

$$\psi: \Delta^r \to \hat{D}$$

Theorem 1.4. (Nilpotent Orbit Theorem) [S] The map $\psi$ extends to a holomorphic map $\Delta^r \to \hat{D}$. Let $F_\infty = \psi(0)$ denote the limiting Hodge flag. Then,

(a) Each $N_j$ is horizontal with respect to $F_\infty$, i.e. $N_j(F_\infty) \subseteq F_\infty^{-1}$ for all $p$;

(b) There exists a constant $\alpha \geq 0$ such

$$\text{Im}(z_1), \ldots, \text{Im}(z_r) > \alpha \implies \theta(z) = e^{N(z)}F_\infty \in D$$

(c) There exists constants $\beta$ and $K$ such that if $\text{Im}(z_1), \ldots, \text{Im}(z_r) > \alpha$ then

$$d(F(z), \theta(z)) \leq K(\Pi_{j=1}^r \text{Im}(z_j))^{\beta} \sum_{j=1}^r e^{-2\pi \text{Im}(-z_j)}$$

for any $G_R$-invariant metric on $D$.

The basic defect of this distance estimate is that it fails to establish convergence of the period map and the nilpotent orbit when the imaginary parts of $z_1, \ldots, z_r$ diverge at very different rates. In [CKS], the authors present an argument by Deligne which shows that there exist constants $\beta_1, \ldots, \beta_r$ and $K$ such that if $\text{Im}(z_1), \ldots, \text{Im}(z_r) > \alpha$ then

$$d(F(z), \theta(z)) \leq K \sum_{j=1}^r \text{Im}(z_j)^{\beta_j} e^{-2\pi \text{Im}(-z_j)}$$

for any $G_R$-invariant metric on $D$. Accordingly, the period map and nilpotent orbit converge as $\text{Im}(z_1), \ldots, \text{Im}(z_r) \to \infty$.

In particular, by [CKS], variations of pure Hodge structure degenerate to variations of mixed Hodge structure along the boundary strata, and hence one is naturally lead to consider the theory of nilpotent orbits for degenerations of mixed Hodge structure. In analogy with the pure case [U], a variation of mixed Hodge structure $V \to S$ gives rise to a period map

$$\varphi: S \to \Gamma \backslash \mathcal{M}$$

where $\Gamma$ is the monodromy group of the underlying local system of $V$ acting on a fixed reference fiber $V$ of $V$ and $\mathcal{M}$ is a classifying space of mixed Hodge structures on $V$ with a given weight filtration $W$, Hodge numbers and graded-polarizations. In analogy with the pure case, a mixed period map gives rise to a commutative diagram

$$
\begin{array}{ccc}
U^r & \xrightarrow{F} & \mathcal{M} \\
\downarrow s_j=e^{2\pi iz_j} & & \downarrow \varphi \\
\Delta^r & \xrightarrow{\varphi} & \Gamma \backslash \mathcal{M}
\end{array}
$$

(1.5)
where $\mathcal{M}$ is an open subset of a homogeneous space $\hat{\mathcal{M}}$ upon which a complex Lie group acts transitively.

To obtain an analog of Schmid’s distance estimate in the mixed case, one must first endow $\mathcal{M}$ with a hermitian structure. In section 2 of this paper, we describe two different hermitian structures on $\mathcal{M}$ which we call the standard metric and the twisted metric. In section 3, we prove the analog of Schmid’s distance estimate for period maps of admissible variations of mixed Hodge structure and their associated nilpotent orbits with respect to the standard metric.

The fact $\mathcal{M}$ is no longer the homogeneous space of a semisimple Lie group in the mixed case however introduces a distortion factor which seems to prevent one from obtaining Deligne’s stronger distance estimate with respect to the standard metric. The twisted metric contains additional factors designed to compensate for this distortion at the cost of reducing the symmetry of the metric. In section 4, we prove Deligne’s strong estimate for the twisted metric. We also prove Deligne’s strong estimate for unipotent variations of mixed Hodge structure arising in the work of Hain and Morgan on the mixed Hodge theory of the fundamental group of a smooth complex algebraic variety.

More precisely, we recall that as a consequence of the $SL_2$-orbit theorem given any mixed Hodge structure $(F,W)$ there exists an associated mixed Hodge structure $(\hat{F},W)$ which is split over $\mathbb{R}$, and $\epsilon$ is given by a certain universal Lie polynomials [CKS, Lemma (6.60)] in the Hodge components of Deligne’s $\delta$-splitting [CKS, Prop. (2.20)].

Given a point $F \in \mathcal{M}$, we can therefore define

$$
\tau(F) = 1 + \sum_{p,q < 0} \|e^{p,q}\|^{-\frac{1}{p+q}}
$$

where $\sum_{p,q < 0} e^{p,q}$ is the decomposition (see (2.1)) of $\epsilon$ into Hodge components with respect to $(F,W)$ and $\|\ast\|$ is the standard metric with respect to $(F,W)$. The twisted metric is then obtained by rescaling the standard norm of an element of Hodge type $(p,q)$ by $\tau^{\frac{p+q}{2}}$. The resulting metric remains invariant under $G_\mathbb{R}$, but has lower symmetry than the standard metric (cf. Lemma (2.5) and (2.19)).

Remark 1.7. The construction of standard metric appears in [K]. In §12 of [KNU1], the authors consider the twisted metric attached to the function $y_1$ along a period map, but do not appear to consider the case where $\tau$ is a function on the classifying space itself (cf. §12.9 of [KNU1]). In §4 of [KNU2] the authors consider an analog of the twisted metric on the space of $SL_2$-orbits, see the discussion of norm estimates at the end of this section for more details.

Remark 1.8. In the case where the classifying space $\mathcal{M}$ parametrizes mixed Hodge structures with exactly two weight graded quotients which are adjacent, the group $G_\mathbb{R}$ acts transitively on $\mathcal{M}$ by isometries. However, a simple transcription of the argument to prove the strong distance estimate given in [CKS, §1] appears to fail because the group $G_\mathbb{R}$ is no longer semisimple. Moreover, the curvature and other geometric properties of $\mathcal{M}$ differ from the pure case [PP].

Remark 1.9. For future reference we record the following property of Deligne’s $\delta$-splitting: If $g \in GL(V_\mathbb{R})$ preserves $W$ then $(g.F,W)$ is a mixed Hodge structure (not necessarily graded-polarized) and $\delta(g.F,W) = Ad(g) \delta(F,W)$. Since $\epsilon$ is given
by universal Lie polynomials in the Hodge components of $\delta$, the same formula holds for $\epsilon$ as well.

The key technical step in proving these results is a relative compactness result for period maps of degenerations of mixed Hodge structure. To state the result, we recall that Schmid’s $SL_2$-orbit theorem attaches to each nilpotent orbit of pure Hodge structure $e^{zN}F$ an $sl_2$-pair $(N,H)$ such that $H$ acts via a real morphism of type $(0,0)$ of an associated limit mixed Hodge structure of the nilpotent orbit (see §3, [CK]). More generally [CKS, CK], given a nilpotent orbit of pure Hodge structure $e^{z_1N_1+\cdots+z_rN_r}F$ the several variable $SL_2$-orbit theorem gives a commuting family of representations of $sl_2(\mathbb{R})$ with semisimple elements $H_1,\ldots,H_r\in\mathfrak{g}_\mathbb{R}$.

Variations of mixed Hodge structure of geometric origin satisfy a set admissibility conditions [SZ] which ensure that if $V$ is an admissible variation of mixed Hodge structure with weight filtration $W$ and unipotent local monodromy transformations $T_j = e^{N_j}$ then the relative weight filtration of $W$ with respect to $N_j$ exists for each $j$. Via the diagram (1.5), an admissible variation of mixed Hodge structure over $\Delta^r$ determines an admissible nilpotent orbit

$$e^{z_1N_1+\cdots+z_rN_r}F,W$$

in analogy with Schmid’s construction. In particular, the nilpotent orbit (1.10) is a nilpotent orbit of pure Hodge structure on each graded quotient of $W$, and hence determines a corresponding system of semisimple elements $H_j$ acting on $Gr^W$.

A choice of isomorphism (grading) from $Gr^W$ to the reference fiber $V$ can be viewed as a choice of direct sum decomposition

$$V = \bigoplus_j V_j$$

such that $W_k = W_{k-1} \oplus V_k$ for each index $k$.

A construction of Deligne presented in [Sch] associates to an admissible nilpotent orbit (1.10) a functorial grading $Gr^W \cong V$. Let $H_1,\ldots,H_r$ be the corresponding semisimple endomorphisms of $V$, and let $Y_0$ be the endomorphism of $V$ which acts as multiplication by $j$ on $V_j$. Let

$$t(y) = y_1^{-Y_0/2}\Pi_{j=1}^r y_j^{-H_j/2}$$

(1.11)

**Remark 1.12.** The construction of $t(y)$ using Deligne systems as described above appears in [BP2]. A different construction of $t(y)$ appears in the introduction to [KNU1] where it is constructed using the work of [CKS] and the limit of a grading of the nilpotent orbit. In the pure case [CK], the factor $y_1^{-Y_0/2}$ acts trivially on the classifying space, and can be omitted.

**Theorem 1.13.** ([§7 BP2]) Let $V \to \Delta^r$ be an admissible variation of mixed Hodge structure with unipotent monodromy with associated period map $F: U^r \to \mathcal{M}$ as in (1.3) and nilpotent orbit (1.10). Let $t(y)$ be the associated family of automorphisms (1.11) and

$$I' = \{(x_1+iy_1,\ldots,x_r+iy_r) \in U^r \mid y_1 \geq y_2 \geq \cdots \geq y_r > 1\}$$
Then, the image of $I'$ under the map
$$z \in U^r \mapsto t^{-1}(y)e^{-\sum_j x_j N_j} F(z)$$
is a relatively compact subset of $M$.

**Remark 1.14.** In the pure case, this is Theorem (4.7) of [CK]. A special case of this result appears in §12 of [KNU1]. A proof of this result is given in [BP2] where it plays a crucial role in the analysis of the asymptotic behavior of normal functions.

**Applications**

**Normal Functions.** Let $X$ be a smooth complex projective variety of dimension $n$. Then, for any pair of homologically trivial algebraic cycles $\alpha$ and $\beta$ on $X$ of dimension $a$ and $b$ on $X$ with disjoint support such that $a + b = n - 1$, there is an associated archimedean height
$$\langle \alpha, \beta \rangle = -\int_{\alpha} G_{\beta}$$
defined by integration of a suitable Green’s current $G_{\beta}$ over $\alpha$. Moreover, as discussed in section 1.1 of [Hai1], the height can be viewed as a period of a subquotient of the mixed Hodge structure on $H^{2a+1}(X - |\beta|, |\alpha|) \times \mathbb{Z}(a)$ with weight graded quotients
$$Gr^W_0 = \mathbb{Z}(0), \quad Gr^W_{-1} = H^{2a+1}(X) \otimes \mathbb{Z}(a), \quad Gr^W_{-2} = \mathbb{Z}(1)$$
In particular [Hai1], as the triple $(X, \alpha, \beta)$ vary in a flat family $(X_s, \alpha_s, \beta_s)$ the pairing (1.15) corresponds to a period of a variation of mixed Hodge structure $V \to S$ with weight graded quotients
$$Gr^W_0 = \mathbb{Z}(0), \quad Gr^W_{-1} = \mathcal{H}, \quad Gr^W_{-2} = \mathbb{Z}(1) \quad (1.16)$$
such that the extension class
$$0 \to \mathcal{H} \to W_0/W_{-2} \to \mathbb{Z}(0) \to 0$$
corresponds to the normal function $\nu_\alpha$ attached to the family of homologically trivial cycles $\alpha_s$ whereas the extension class
$$0 \to \mathbb{Z}(1) \to W_{-1} \to \mathcal{H} \to 0$$
is dual to the normal function $\nu_\beta$ attached to the family $\beta_s$. Once these extension classes are fixed, there is a natural action of $O^*$ on the set of possible variations of mixed Hodge structure with these extension classes. Moreover, as explained in section 5, the resulting line bundle
$$B \to S$$
carries a natural hermitian metric $h$ (see [Hai2]).

Suppose now that $S$ is a Zariski open subset of a complex manifold $\bar{S}$ such that $\bar{S} - S$ is a normal crossing divisor along which the monodromy is unipotent. Then, it is natural to extend $B$ to a line bundle $\bar{B} \to \bar{S}$ by declaring the local extending sections to be admissible variations of mixed Hodge structures with the the given extension data of the admissible normal function $\nu_\alpha$ and $\nu_\beta$ (see [BP3] for details).

In section 5, we will prove the following result:
Theorem 1.17. Let $V \to \Delta^{\ast r}$ be an admissible variation of biextension type with unipotent monodromy and $|V| = e^{-\varphi}$.

Then, $\varphi \in L^1_{\text{loc}}(\Delta^r)$, and hence defines a singular hermitian metric on $\bar{B}$ (see [D]).

In particular, as explained in [BFNP], a triple $(X, L, \zeta)$ consisting of a smooth complex projective variety $X$ of dimension $2n$, a very ample line bundle $L \to X$ and a Hodge class $\zeta \in H^{n,n}(X, \mathbb{Z})$ which is primitive with respect to $L$ determines a normal function $\nu_\zeta : S \to J(H)$ where $S$ is the complement of the dual variety of $X$ in $|L|$ and $H$ is the variation of Hodge structure of weight $2n - 1$ defined by the hyperplane sections of $X$ with respect to $S$. Using $\nu_\zeta$ in place of the normal functions $\nu_\alpha$ and $\nu_\beta$ considered above, we obtain a biextension line bundle $B \to S$. Moreover, as the 2nd author will show in joint work with Patrick Brosnan, the failure of the biextension metric to extend to $\bar{B}$ in this setting is equivalent to the existence of singularities of $\nu_\zeta$ of the type considered by Griffiths and Green in their study of the Hodge conjecture.

Norm Estimates. In §12 of [KNU1], the authors prove analogs of the norm estimates of [CKS] for what is effectively the twisted metric for $\tau(e^{\sum_j y_j N_j} F)$ artificially set equal to $y_1$, i.e. there twisting factor does arise from a function defined globally on the classifying space $M$. On the other hand, by the main theorem of [KNU1] we have

$$e^{\sum_j y_j N_j} F = t(y) e^{\epsilon(y)} e^{r(y)}$$

and

$$e^{\sum_j y_j N_j} r \to 1$$

as $y_j / y_{j+1} \to \infty$, $r \in M$ is split over $\mathbb{R}$ and $\epsilon(y)$ has a finite limit as all $y_j / y_{j+1} \to \infty$. If this limit is non-zero, we say that the original period map/nilpotent orbit is asymptotically non-split. Equivalently, by [KNU1]

$$\lim_{y_j / y_{j+1} \to \infty} t(y) e^{\sum_j z_j N_j} F \to 0$$

is not split over $\mathbb{R}$. By Corollary 12.8 in [KNU1], limit (1.19) has the same limit if replace $e^{\sum_j z_j N_j} F$ by the period map $F(z)$. In this case, it follows directly from the arguments of §12 of [KNU1] and equation (2.12) that the same norm estimates hold for the twisted metric, which is defined on all of $M$.

In §4 of [KNU2], the authors put a metric on a space of $SL_2$-orbits and obtain the norm estimates described above for a metric which appears to be quasi-isometric to the twisted metric studied in this paper near boundary points which correspond to asymptotically non-split orbits. In this respect, our results on the distance between the period map and its nilpotent orbit can be considered as a complement to the results of [KNU2]. Globally however, the metric constructed in [KNU2] does not seem to be globally quasi-isometric to the twisted metric studied in this paper.

Reduced Period Map. Another potential application of the results on relative compactness considered above concerns partial compactifications of period domains of pure Hodge structure. More precisely, we recall (cf. [KU]) that the period domain quotients $\Gamma \backslash \mathcal{D}$ can be partially compactified with respect to horizontal maps by the addition of boundary components which parametrize nilpotent orbits with local monodromy contained in the faces of a fan $\Sigma$ which satisfies certain compatibility conditions relative to $\Gamma$. In the classical case of Hodge structures of weight 1, these
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partial compactifications correspond to toroidal compactification of Ash, Mumford, Rapoport and Tai. The object which corresponds to the limit point in Satake compactification is the reduced limit Hodge filtration

\[ \lim_{m(z) \to \infty} e^{zN}F^r \]

of a nilpotent orbit of weight \( k \). We discuss the relationship between the reduced limit Hodge filtration and the Satake boundary component in §6.2 for more detail.

By Lemma (3.12) in [CKS], this filtration always takes values in the topological boundary \( \partial D \) of \( D \) in the compact dual \( \check{D} \). This boundary is again a union of \( G_{\mathbb{R}} \)-orbits, and the study of such boundary strata is potentially very useful in the study of certain infinite dimensional representations of \( G_{\mathbb{R}} \) (see §5 of [KP2]).

More precisely, Cattani, Kaplan and Schmid show that if \((N,H,N^+)\) is the \( \text{sl}_2 \)-triple attached to the Deligne splitting

\[ (\hat{F}, W(N)[−k]) = (e^{−i\delta}F, W(N)[−k]) \]

of the limit mixed Hodge structure attached to (1.20) then

\[ e^{zN}F = e^{i\delta}e^{(1/z)N^+}\Phi, \quad \Phi^p = \bigoplus_{s \leq k−p} \Gamma_{(\hat{F}, W(N)[−k])}^{r,s} \]

In §6, we consider the several variable analog of this question. Namely, given a nilpotent orbit generated by \((N_1, \ldots, N_r; F)\) then for any \( y_1, \ldots, y_r > 0 \) the data \((N(y), F)\) generates a 1-variable nilpotent orbit where \( N(y) = \sum y_jN_j \). Using the several variable \( SL_2 \)-orbit theorem, we obtain an asymptotic formula for \( N^+(y) \) in the associated \( sl_2 \)-triple. We close §6 with a discussion of the reduced limit map in the case of normal functions in one variable.

Acknowledgments

The authors would like to thank Patrick Brosnan, Phillip Griffiths, Aroldo Kaplan, Zhiqin Lu, Chikara Nakayama, Chris Peters and Colleen Robles for helpful discussion regarding various aspects of this paper. This research was partially supported by Research Fund for International Young Scientists NSFC 11350110209 (Hayama) and NSF grant DMS 1002625 (Pearlstein). We also acknowledge generous support from the IHES, Institut de Mathématiques de Jussieu, National Taiwan University and the Fields Institute which facilitated this work.

Notations and conventions

Let \( V \) be a \( \mathbb{R} \)-vector space and let \( W = \{W_k\}_{k \in \mathbb{Z}} \) be an increasing filtration of \( V \). We use the following notations:

- \( \text{gr}^W_k \) is the quotient space \( W_k/W_{k+1} \);
- \( \text{gr}^W \) is the graded quotient map \( \text{gr}^W : V \to \bigoplus_k \text{gr}^W_k ; \)
- \( \text{End} (V)^W \) (resp. \( \text{Aut} (V)^W \)) is the group of endomorphisms (resp. automorphisms) which preserves \( W \);
- For \( X \in \text{End} (V)^W \) (resp. \( \text{Aut} (V)^W \)), \( \text{gr}^W_k (X) \) is the induced action of \( X \) on \( \text{gr}^W_k ; \)
- For \( X \in \text{End} (V)^W \) and \( \lambda > 0 \), we write \( \lambda^X = \exp (\log (\lambda)X) \in \text{Aut} (V)^W ; \)
For the decreasing filtration $F = \{F_p\}_{p \in \mathbb{Z}}$ on $V_C := V \otimes \mathbb{C}$, $F(\mathfrak{gr}_k^W) = \{F_p(\mathfrak{gr}_k^W)\}_{p \in \mathbb{Z}}$

is the filtration induced by $F$ on $\mathfrak{gr}_k^W \otimes \mathbb{C}$.

Let $(V', W')$ be a pair of $\mathbb{R}$-vector space and an increasing filtration and let $p : V \to V'$ be the $\mathbb{R}$-linear map which preserves the filtration. $\mathfrak{gr}(p)$ (resp. $\mathfrak{gr}_k(p)$) is the linear map $\mathfrak{gr}^W(V) \to \mathfrak{gr}^{W'}(V')$ (resp. $\mathfrak{gr}_k^W \to \mathfrak{gr}_k^{W'}$).

Let $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r$. We denote $\mu(j) = \mu_j$ for $1 \leq j \leq r$.

2. Metrics

2.1. Classifying Spaces. We fix a finite-dimensional $\mathbb{R}$-vector space $V$, an increasing filtration $W$ on $V$, non-degenerate bilinear forms $\langle \cdot, \cdot \rangle_w : \mathfrak{gr}_w^W \times \mathfrak{gr}_w^W \to \mathbb{R}$ for $w \in \mathbb{Z}$, which is symmetric if $w$ is even and anti-symmetric if $w$ is odd, and non-negative integers $h_{p,q}^w$ with $h_{p,q}^w = h_{q,p}^w$ and with $h_{p,q}^w = 0$ unless $p + q = w$ such that $\dim \mathfrak{gr}_w^W = \sum_{p,q} h_{p,q}^w$. Let $\mathcal{M}$ be the set of all decreasing filtrations $F$ on $V_C$ for which $(W, F)$ is a mixed Hodge structure (MHS) such that, for all $w \in \mathbb{Z}$, $(\mathfrak{gr}_w^W, F(\mathfrak{gr}_w^W), \langle \cdot, \cdot \rangle_w)$ is a polarized Hodge structure with Hodge numbers $\{h_{p,q}^w\}$.

Let $\tilde{\mathcal{M}}$ be the set of all decreasing filtration $F$ on $V_C$ satisfying the following two conditions:

- $\dim (F_p(\mathfrak{gr}_w^W)/F_{p+1}(\mathfrak{gr}_w^W)) = h_{p-w,-p}^w$;
- $(F_p(\mathfrak{gr}_w^W), F^q(\mathfrak{gr}_w^W))_w = 0$ for $p + q > w$.

Here $\tilde{\mathcal{M}}$ is a flag manifold including $\mathcal{M}$ as an open subset. $\tilde{\mathcal{M}}$ is the called compact dual.

Let $G_A = \{g \in GL(V_A)^W \mid \mathfrak{gr}_w^W(g) \in Aut(\mathfrak{gr}_w^W, \langle \cdot, \cdot \rangle_w)\}$ for $A = \mathbb{R}, \mathbb{C}$. Then $G_C$ acts on $\tilde{\mathcal{M}}$ transitively ($[U]$ (2.11.1)).

2.2. Standard Metric. By a theorem of Deligne, a mixed Hodge structure $(F, W)$ on $V_C$ determines a unique, functorial bigrading

$$V_C = \bigoplus I^{p,q}$$

such that

- $(a) \quad F^p = \bigoplus_{a \geq p} I^a, b$;
- $(b) \quad W_k = \bigoplus_{a+b=k} I^a, b$;
- $(c) \quad I^{p,q} = \bar{I}^{p,q} \mod \bigoplus_{r<p,s<q} I^{r,s}$

A MHS $(F, W)$ is called $\mathbb{R}$-split if $I^{p,q} = \bar{I}^{p,q}$.

A grading of $W$ corresponds to a semisimple endomorphism $Y$ of $V_C$ such that $W_k$ is the direct sum of $W_{k-1}$ and the $k$-eigenspace of $Y$ for each index $k$. In particular, a mixed Hodge structure $(F, W)$ determines an associated grading $Y_{(F, W)}$ of $W$ which acts as multiplication by $p + q$ on $I^{p,q}$.

**Definition 2.2.** $[K]$ [PT] Let $(F, W)$ be a graded-polarized mixed Hodge structure with underlying complex vector space $V_C$. Then, the associated mixed Hodge metric $h$ is the unique hermitian inner product on $V_C$ which makes the associated bigrading orthogonal and satisfies

$$h(u, v) = i^{p-q} \langle \mathfrak{gr}_{p+q}^W(u), \mathfrak{gr}_p^W(v) \rangle_{p+q}$$

for $u, v \in I^{p,q}$. The associated norm will be denoted $\| \cdot \|$. 
Suppose now that $M$ is a classifying space of graded-polarized mixed Hodge structures with underlying complex vector space $V$. Then, mixed Hodge metric endows the trivial bundle $V \times M \to M$ with a hermitian structure. In particular, using the identification of $M$ with a submanifold of a flag variety, we obtain an associated hermitian metric on $M$ which we shall call the standard metric on $M$.

**Lemma 2.4.** [P1] Let $g_A = \text{Lie } G_A$ ($A = \mathbb{R}, \mathbb{C}$). Then, by the functoriality of the above constructions, a point $F \in M$ induces a mixed Hodge structure on $g_C$ and hence determines an associated decomposition 

$$g_C = \bigoplus_{p,q} g^{p,q}$$

as in (2.1). In particular, the Lie algebra of the stabilizer of $F \in M$ is 

$$g^F_C = \bigoplus_{p \geq 0} g^{p,q}$$

and hence the subalgebra 

$$q_F = \bigoplus_{p<0, p+q \leq 0} g^{p,q}$$

is a vector space complement to $g^F_C$ in $g_C$. Accordingly, we may identify $q_F$ with $T_F(M)$ via the map 

$$v \mapsto \frac{d}{dt} e^{tv} F \bigg|_{t=0}$$

Under this identification, the standard metric on $M$ is given by the formula 

$$h_F(\alpha, \beta) = \text{Tr}(\alpha^* \beta)$$

for $\alpha, \beta \in q_F$ where $\alpha, \beta$ in the right hand side are represented as matrices with respect to the unitary basis for $h_F$.

**Lemma 2.5.** [K, P2] Given a point $F \in M$, let 

$$\Lambda^{-1,-1} = \bigoplus_{r,s<0} g^{r,s}$$

and $g \in G_\mathbb{R} \cup \exp(\Lambda^{-1,-1})$. Then, 

$$L_{g} : T_F(M) \to T_{g.F}(M)$$

is an isometry. In particular, $d$ is a $G_\mathbb{R}$-invariant distance ([P1] Theorem 2.5)).

Let $Y$ be a real grading of $W$. Then $\lambda^{-\frac{1}{2}}Y = \exp\left(-\frac{1}{2} \log(\lambda)Y\right)$ for $\lambda > 0$ is an automorphism of $M$. We then have the following Lemma:

**Lemma 2.6.** ([P1] Lemma 5.7). Let $Y$ be a grading of $W$ defined over $\mathbb{R}$, $y > 0$, $\alpha \in \mathbb{R}$ and $F \in M$. Let $\| \cdot \|$ denote the standard metric (2.3). Then, 

$$\|y^\alpha Y(v)\|_{y^\alpha Y,F} = y^{(p+q)\alpha}\|v\|_F$$

if $v \in I^{p,q}_{(F,W)}$.

**Proof.** Let $v \in I^{p,q}_{(F,W)}$. Then, since $Y$ is defined over $\mathbb{R}$,

$$y^\alpha Y(v) \in I^{p,q}_{(y^\alpha Y,F,W)}.$$
Moreover, since both \( Y \) and \( Y_{(F,W)} \) are gradings of \( W \), we can find \( u \in W_{p+q-1} \) so that \( Y(v + u) = (p + q)(v + u) \). Accordingly,

\[
y^{\alpha Y}(v) = y^{\alpha Y}(v + u - u) = y^{\alpha(p+q)}(v + u) + y^{\alpha Y}(u) \\
y^{\alpha(p+q)}v \mod W_{p+q-1}
\]

and hence by the definition of the standard metric,

\[
\|y^{\alpha Y}(v)\|_{y^{\alpha Y}.F} = y^{\alpha(p+q)}\|v\|_F
\]

since \( y^{\alpha Y}.F \) and \( F \) induce the same Hodge structure on \( Gr^W \).

\[\Box\]

**Remark 2.8.** To see that \( y^{\alpha Y}.F \) remains in \( \mathcal{M} \), observe that it is a mixed Hodge structure by Remark (1.9) since \( y^{\alpha Y} \in GL(V_R) \) preserves \( W \). Moreover, since \( y^{\alpha Y} \) acts by scalar multiplication on each \( Gr^W_k \), it acts trivially on the induced filtrations \( FGr^W_k \).

The length of the weight filtration \( W \) is the difference

\[
L = \min\{k \mid W_k = V\} - \max\{k \mid W_k = 0\} \quad (2.9)
\]

**Corollary 2.10.** Under the hypothesis of Lemma (2.6)

\[
d(y^{\alpha Y}.F, y^{\alpha Y}.F') \leq y^{-\alpha(L-1)}d(F, F')
\]

if \( \alpha < 0 \) and \( y > 1 \).

**Proof.** Let \( \gamma : [0,1] \to \mathcal{M} \) be a smooth curve from \( F \) to \( F' \). Then, \( y^{\alpha Y}.\gamma(u) \) is a curve from \( y^{\alpha Y}.F \) to \( y^{\alpha Y}.F' \). Therefore,

\[
d(y^{\alpha Y}.F, y^{\alpha Y}.F') \leq \int_0^1 \|L_{y^{\alpha Y}}\gamma'(u)\|_{y^{\alpha Y}.\gamma(u)} \, du
\]

Let

\[
\gamma_k(u) = \sum_{p+q=k} \gamma'(u)^{p,q}
\]

according to the decomposition (2.1) along \( \gamma \). Then, by the adjoint form of equation (2.7),

\[
\|L_{y^{\alpha Y}}\gamma_k(u)\|_{y^{\alpha Y}.\gamma(u)} = y^k \alpha \|\gamma_k'(u)\|_{\gamma(u)}
\]

Accordingly, since the subspaces \( \oplus_{p+q=k} \mathfrak{g}^{a,b} \) are orthogonal for different values of \( k \), it then follows that

\[
\|L_{y^{\alpha Y}}\gamma'(u)\|_{y^{\alpha Y}.\gamma(u)} \leq y^{-\alpha(L-1)}\|\gamma'(u)\|_{\gamma(u)}
\]

and hence

\[
d(y^{\alpha Y}.F, y^{\alpha Y}.F') \leq y^{-\alpha(L-1)}\int_0^1 \|\gamma'(u)\|_{\gamma(u)} \, du
\]

Taking the infimum over all such paths \( \gamma \) yields

\[
d(y^{\alpha Y}.F, y^{\alpha Y}.F') \leq y^{-\alpha(L-1)}d(F, F')
\]

as required. \[\Box\]
2.3. **Twisted Metric.** Let $\mathcal{M}$ be a classifying space of graded-polarized mixed Hodge structures, and 

$$\tau : \mathcal{M} \to (0, \infty)$$

be a smooth function. Then, the associated twisted metric $|\ast|$ on $V_C \times \mathcal{M} \to \mathcal{M}$ is obtained by scaling the metric Hodge metric (2.3) on $I_{F,W}^{p,q}$ by $\tau(F)^{(p+q)/2}$.

The reason to introduce the twisted metric is to undo the scaling effect seen in Lemma (2.6). More precisely, as discussed in the introduction, in order to obtain a relative compactness result in the mixed case, we have to twist the period map by the inverse of

$$t(y) = y_1^{-Y_0/2} \prod_{j=1}^{r} y_j^{-H_j/2}$$

where $H_1, \ldots, H_r \in \mathfrak{g}_\mathbb{R}$ are the semisimple elements of commuting representations of $sl_2(\mathbb{R})$ on $V$ and $Y_0$ is a real grading of $W$ which commutes with $H_1, \ldots, H_r$.

Let $\tau$ be the twisting function defined in (1.6) and $Y$ be a real grading of $W$. Then, by Remark (1.9)

$$\tau(t(y)(v)) = 1 + \sum_{p,q<0} y^p \|\epsilon^{p,q}(F,W)\|_{p^{-1/2},F}^{-2} = 1 + \sum_{p,q<0} y^p \|\epsilon^{p,q}(F,W)\|_{p^{-1/2},F}^{-2}$$

for any $F \in \mathcal{M}$. Likewise, by the $G_\mathbb{R}$-invariance of the standard metric and Remark (1.9), it follows that

$$\tau(t(y).F) = 1 + \sum_{p,q<0} y^p \|\epsilon^{p,q}(F,W)\|_{p^{-1/2},F}^{-2} = 1 + \tau(F) - 1$$

(2.11)

In particular, as long as $\epsilon(F,W)$ is non-zero (i.e. $(F,W)$ is not split over $\mathbb{R}$), it follows that

$$\tau(t(y).F) = 1 + Cy_1$$

(2.12)

for some non-zero constant $C$. Inspection of the right hand side of equation (2.11) shows that (2.12) holds for variable $F$ which remains in a compact set which does not intersect the locus of points $\mathcal{M}_R \subseteq \mathcal{M}$ for which the associated mixed

**Remark 2.13.** The series of computations outlined above hold for Deligne’s $\delta$-splitting in place of $\epsilon$, and hence the conclusions of this paper for the twisted metric also hold for the corresponding metric obtained by replacing $\epsilon^{p,q}$ by $\delta^{p,q}$ in the definition of the twisted metric.

**Corollary 2.14.** Let $Y$ be a grading of $W$ defined over $\mathbb{R}$ and $F \in \mathcal{M}$. Then, the twisted metric satisfies

$$|\frac{t(y)(v)}{|v|_F}|_{t(y).F} = \left(\frac{y_1^{-1} + \tau(F) - 1}{\tau(F)}\right)^{\frac{p+q}{2}}$$

(2.15)
if \( v \in I_{p,q}^{p,q}(F,W) \).

**Proof.** By (2.7) and the previous calculations, it follows from the definition of the twisted metric that

\[
|t(y)(v)|_{t(y),F} = (y_1^{-1} + \tau(F) - 1)^{\frac{p+q}{2}} \|v\|_F
\]

whereas \( |v|_F = \tau(F)^{\frac{p+q}{2}} \). \( \square \)

Before stating out next result, we recall the following from [C, pg 334]

**Definition 2.16.** Let \( M \) a Riemannian manifold. A subset \( X \subseteq M \) is geodeically convex if any of its points are joined by a unique shortest geodesic in \( M \), and this geodesic lies in \( X \).

**Theorem 2.17.** (Whitehead) Let \( W \subset M \) be open and \( x \in W \). Then, there is a geodesically convex open neighborhood \( U \subset W \) of \( x \).

**Theorem 2.18.** Let \( d_{\tau} \) denote the Riemannian distance associated to the twisted metric. Assume that for \( y_1, \ldots, y_r \) sufficiently large the points \( t^{-1}(y).F_1 \) and \( t^{-1}(y).F_2 \) are contained in a geodesically convex set \( U \) that is contained in a compact set which does not intersect the split locus \( M_R \). Then, there exists a constant \( K \) such that

\[
d_{\tau}(F_1, F_2) \leq K d_{\tau}(t^{-1}(y).F_1, t^{-1}(y).F_2)
\]

**Proof.** Let \( \gamma : [0, a] \to U \) be the geodesic from \( t^{-1}(y).F_1 \) to \( t^{-1}(y).F_2 \). Then,

\[
d_{\tau}(F_1, F_2) \leq \int_0^a |L_{t(y)} \gamma'(u)|_{t(y),\gamma(u)} \, du
\]

where \( |*| \) denotes the twisted metric. Let

\[
\gamma_k(u) = \bigoplus_{p+q=k} \gamma'(u)^{p,q}
\]

according to the decomposition (2.1) along the curve \( \gamma \). Then, by Corollary (2.14),

\[
|L_{t(y)} \gamma_k(u)|_{t(y),\gamma(u)} = \left( \frac{y_1^{-1} + \tau(\gamma(u)) - 1}{\tau(\gamma(u))} \right)^{\frac{k}{2}} |\gamma'(u)|_{\gamma(u)}
\]

In particular, as long as \( \gamma(u) \) stays in a compact set which does not intersect the split locus (where \( \tau = 1 \)), the distortion factor in the previous equation can be uniformly bounded away from zero. \( \square \)

Of course, twisting the metric in this way reduces its symmetry.

**Lemma 2.19.** The twisted metric \( h_{\tau} \) is \( G_R \)-invariant, i.e. for any \( g \in G_R \) and \( F \in \mathcal{M} \)

\[
L_g : T_F(M) \to T_{g.F}(M)
\]

is an isometry.

**Proof.** \( \epsilon(g.F, W) = g.\epsilon(F, W) \), and hence

\[
\|e^{p,q}(g.F, W)\|_F = \|e^{p,q}(F, W)\|_F
\]

The rest follows from the \( G_R \)-invariance of the standard metric. \( \square \)
2.4. Variation of mixed Hodge structure. The definition of the notion of an admissible variation of mixed Hodge structure is due to Steenbrink and Zucker in the 1-variable case [SZ] and to Kashiwara [Ka] in the several variable case. In particular, locally, a variation of mixed Hodge structure is given by a holomorphic map into a classifying space of graded-polarized mixed Hodge structure $M$ which satisfies Griffiths horizontality condition

$$\frac{\partial}{\partial s_j} F^p(s) \subseteq F^{p-1}(s)$$

As alluded to in the introduction, the admissibility of a variation of mixed Hodge structure $V \to \Delta^r$ implies that it has a limit Hodge filtration in analogy with the pure case. The second part of admissibility is the existence of the relative weight filtration $M(N_j,W)$ of $W$ and each of the local monodromy logarithms. By a theorem of Kashiwara, the limit Hodge filtration $F_\infty \in \hat{M}$ pairs with the relative weight filtration $M = M(N_1 + \cdots + N_r, W)$ to yield a mixed Hodge structure for which each $N_j$ is $(-1,-1)$-morphism. Moreover,

$$(e^{\sum_j z_j N_j}, F_\infty, W)$$

is an admissible nilpotent orbit.

Two basic references for period maps and nilpotent orbits in the mixed case are [KNU1] and [BP2]. Unless otherwise noted, the conventions in this paper conform to §2 of [BP2], and we refer the reader to these two papers for additional background information on mixed period maps.

3. Distance estimate

Let $V$ be an admissible variation of mixed Hodge structure over $\Delta^r$ with period map $F : U^r \to M$ and associated nilpotent orbit $e^{\sum_j z_j N_j} F_\infty$ (3.1) In this section, we prove the Schmid’s original estimate between $F(z)$ and $e^{\sum_j z_j N_j} F_\infty$ holds for the standard metric on the classifying space $M$. More precisely, we prove that if $L$ is the length (2.9) of the weight filtration then

$$d(F(z), F_{nilp}(z)) \leq K y_1^{(L-1)/2} \sum_j y_j^{b_j} e^{-2\pi y_j}$$

(3.2)

The two key steps are the derivation of equation (3.11) and Lemma (3.15) below, which both ultimately depend on Griffiths’ horizontality.

To begin, fix $(z_{j+1}, \ldots, z_r) \in U^{r-j}$. Then, the map

$$(z_1, \ldots, z_j) \mapsto F(z_1, \ldots, z_r)$$

arises from an admissible variation of mixed Hodge structure over $\Delta^{*j}$ with monodromy logarithms $N_1, \ldots, N_j$. Let $F_\infty(z_{j+1}, \ldots, z_r)$ be the limit Hodge filtration of this period map and

$$F_j(z_1, \ldots, z_r) = e^{\sum_{k \leq j} z_k N_k} F_\infty(z_{j+1}, \ldots, z_r)$$

be the associated nilpotent orbit.

Following [CK], the partial period map $F_j$ admits the following description: The Deligne bigrading

$$V_C = \bigoplus_{p,q} H^p_q(F_\infty, M)$$
of the limit mixed Hodge structure \((F_{\infty}, M)\) of \(\mathcal{V}\) induces a bigrading
\[
\mathfrak{g}_C = \bigoplus_{a, b} \mathfrak{g}^{a, b}
\]
of \(\mathfrak{g}_C \subset V_C \otimes V_C^*\) such that
\[
\mathfrak{g}^{a, b}(I_{p, q}(F_{\infty}, M)) \subseteq I_{p+a, q}(F_{\infty}, M)
\]
By equation (6.11) in \([P1]\), the period map has a local normal form
\[
F(z) = e^{\sum_j z_j N_j e^{\Gamma(s)}}, F_{\infty}
\]
where
\[
\Gamma : \Delta^r \to \mathfrak{q}, \quad \Gamma(0) = 0
\]
is a holomorphic function taking values in the complement
\[
\mathfrak{q} = \bigoplus_{a<0, b} \mathfrak{g}^{a, b}
\]
to the Lie algebra of the stabilizer of \(F_{\infty}\) in \(G_C\). For future reference, we record that \(\mathfrak{q}\) has a natural grading
\[
\mathfrak{q} = \bigoplus_{a<0} \mathfrak{q}_a, \quad \mathfrak{q}_a = \bigoplus_{b} \mathfrak{g}^{a, b}
\]
To simplify notation, let us define \(N(z) = \sum_{j=1}^r z_j N_j\) and set
\[
\Gamma_j(s) = \Gamma(0, \ldots, 0, s_{j+1}, \ldots, s_n)
\]
Then, tracing through the above definition, it follows that
\[
F_{\infty}(z_{j+1}, \ldots, z_r) = e^{\sum_{k>j} z_k N_k e^{\Gamma_j(s)}} F_{\infty}
\]
and hence \(F_j(z) = e^{N(z)} e^{\Gamma_j(s)} F_{\infty}\).

**Remark 3.5.** The local normal form is only valid in some relative neighborhood of 0 in \(\Delta^*\). Moreover, since \(F_j(z)\) is constructed as a nilpotent orbit, technically we must further shrink \(\Delta^*\) in order get an admissible variation, i.e. \(F_j(z)\) only takes values in \(\mathcal{M}\) when
\[
\text{Im}(z_1), \ldots, \text{Im}(z_r) > \alpha
\]
for some constant \(\alpha\). However, since we are only interested in asymptotics, for simplicity of notation we shall leave (3.6) as an implicit assumption in the remainder of this section.

**Lemma 3.7 ([CK] Proposition 2.6).** For \(1 \leq k \leq j \leq r\), \([N_k, \Gamma_j] = 0\).

**Proof.** By the horizontality condition, \(\frac{\partial}{\partial z_j} \log (e^{N(z)} e^{\Gamma(s)})\) is a holomorphic function on \(\Delta^r\) and the right hand side of (3.8) is a complex vector space, the equation holds on all of \(\Delta^r\). Setting
We have $e^{-\text{ad} \Gamma(s)} N_j \in \varphi_{-1}$. Since $N_j \in \varphi_{-1}$ and $\Gamma(s) \in q_F$, this is possibly only if $[N_j, \Gamma|_{s_j=0}] = 0$. □

The basic strategy of the proof of Schmid’s distance estimate in the mixed case is observe that $F_0 = F(z)$ whereas $F_r(z)$ is the associated nilpotent orbit. Therefore, we have the triangle inequality

$$d(F_0, F_r) \leq d(F_0, F_1) + \cdots + d(F_{r-1}, F_r)$$

(3.9)

where $F_j(z)$ and $F_{j+1}(z)$ differ by

$$e^{\Gamma(z)} = e^{\Gamma_{j-1}(z)} e^{-\Gamma_j(z)}$$

where $\Gamma_j(z)$ is a holomorphic function which is divisible by $s_j$.

To continue, we observe that if we view $F_j(z)$ as a period map in the variables $(z_1, \ldots, z_r)$ then it has the same associated nilpotent orbit (3.1) as $F(z)$, and hence the same associated family of semisimple operators

$$t(y) = y_1^{-Y_0/2} \prod_j y_j^{-H_j/2}$$

Let $N(x) = \sum_j x_j N_j$ where $z_j = x_j + \sqrt{-1} y_j$.

Lemma 3.10. Define

$$\tilde{F}_j(z) = t^{-1}(y) e^{N(y)} e^{\Gamma_j(z)} F_\infty.$$

Then, $d(F_{j-1}(z), F_j(z)) \leq y_1^{(L-1)/2} d(\tilde{F}_{j-1}(z), \tilde{F}_j(z))$.

Proof. We have

\begin{align*}
    d(F_{j-1}(z), F_j(z)) &= d(e^{N(z)} e^{i N(y)} e^{\Gamma_{j-1}(z)} F_\infty, e^{N(z)} e^{i N(y)} e^{\Gamma_j(z)} F_\infty) \\
    &= d(e^{i N(y)} e^{\Gamma_{j-1}(z)} F_\infty, e^{i N(y)} e^{\Gamma_j(z)} F_\infty) \\
    &= d(t(y) t^{-1}(y) e^{i N(y)} e^{\Gamma_{j-1}(z)} F_\infty, t(y) t^{-1}(y) e^{i N(y)} e^{\Gamma_j(z)} F_\infty) \\
    &= d(t(y) \tilde{F}_{j-1}(z), t(y) \tilde{F}_j(z)) \\
    &= d(y_1^{-Y_0/2} \prod_j y_j^{-H_j/2} \tilde{F}_{j-1}(z), y_1^{-Y_0/2} \prod_j y_j^{-H_j/2} \tilde{F}_j(z)) \\
    &= d(y_1^{-Y_0/2} \tilde{F}_{j-1}(z), y_1^{-Y_0/2} \tilde{F}_j(z)) \\
    &\leq y_1^{(L-1)/2} d(\tilde{F}_{j-1}(z), \tilde{F}_j(z))
\end{align*}

where the last step is justified by Corollary (2.10). All other transformations are either based on substitutions or the fact that the group $G_R$ acts by isometries with respect to the standard metric. □

Combining the previous corollary with (3.9), it follows that

$$d(F_0(z), F_r(z)) \leq y_1^{(L-1)/2} \sum_j d(\tilde{F}_j(z), \tilde{F}_{j-1}(z))$$

(3.11)

and hence it remains to prove that

$$d(\tilde{F}_{j-1}(z), \tilde{F}_j(z)) \leq K y_j^{b_j} e^{-2\pi y_j}$$

for suitable constants $K$ and $b_j$. For this we state two preliminary results:
Lemma 3.12. ([BP2, §6, KNU, §10, CK, §4]) Let \((e^{\sum z_jN_j}, F, W)\) be an admissible nilpotent orbit. Then

\[
\text{Ad}(t^{-1}(y))e^{N(y)} = e^{P(t)}
\]

where \(P(t)\) is a polynomial in non-negative half-integer powers of \(t_j = y_{j+1}/y_j\) where we formally set \(y_{r+1} = 1\).

A short calculation shows that

\[
t(y) = \prod_{j=1}^r t_j^{Y_j}
\]

where \(t_j = y_{j+1}/y_j\) where formally \(y_{r+1} = 1\), and

\[
Y_1 = Y_0 + H_1, \ldots, Y_r = Y_0 + H_1 + \cdots + H_r
\]

are commuting semisimple endomorphisms. Let

\[
\Gamma_j = \sum_{\mu \in \mathbb{Z}^n} \Gamma_j^{[\mu]}\]

relative to \(Y^1, \ldots, Y^r\), i.e. \(\text{Ad}(t(y)^{-1})\) acts on \(\Gamma_j^{[\mu]}\) as multiplication by

\[
\prod_{j=1}^r t_j^{\mu(j)/2}.
\]

Lemma 3.13. \(\Gamma_j^{[\mu]} = 0\) if \(\mu(k) > 0\) for some \(1 \leq k \leq j - 1\).

Proof. This follows from relative compactness of \(\tilde{F}_j\) since we can make the variables \(z_1, \ldots, z_{j-1}\) arbitrarily large, independent of \((s_j, \ldots, s_r)\). For details, see equation (7.32) in [BP2].

Remark 3.14. Another way of looking at the previous Lemma is that since we know that \(\Gamma_j\) commutes with \(N_k\) for \(k \leq j\), we expect \(\Gamma_j\) to have only non-positive weights with respect to \(H_1, \ldots, H_{j-1}\).

Lemma 3.15. There exist constants \(K\) and \(b_j\) such that

\[
d(\tilde{F}_{j-1}(z), \tilde{F}_j(z)) \leq Ky_j^{b_j}e^{-2\pi y_j}
\]

for \(\text{Im}(z_1), \ldots, \text{Im}(z_r)\) sufficiently large.

Proof. Since \(e^{\Gamma_j^{-1}} = e^{\Gamma_j}e^{\Gamma_j^{-1}}\), we have

\[
\tilde{F}_j(z) = t(y)^{-1}e^{iN(y)}e^{\Gamma_j^{-1}(s)}F_{\infty} = (t(y)^{-1}e^{iN(y)}e^{\Gamma_j(s)}e^{\Gamma_j(s)}F_{\infty} = (t(y)^{-1}e^{-iN(y)}e^{\Gamma_j(s)}e^{-iN(y)}e^{\Gamma_j(s)}F_{\infty} = \text{Ad}(t(y)^{-1})(e^{iN(y)}\text{Ad}(t(y)^{-1})(e^{\Gamma_j(s)}\text{Ad}(t(y)^{-1})(e^{-iN(y)}\tilde{F}_j(z).

Accordingly,

\[
\tilde{F}_{j-1}(z) = \exp(\text{Ad}(e^{P(y)}w_j(z)))\tilde{F}_j(z)
\]

where \(w_j(z) = \text{Ad}(t(y)^{-1})(\Gamma_j(s))\). In particular, by Theorem 3.13, \(\tilde{F}_j(z)\) assumes values in a relatively compact subset of \(M\). Likewise, by Lemma 4.12, under these hypothesis, \(e^{P(y)}\) takes values in a compact subset of \(G_C\). Finally, by Lemma 3.13 and the fact that \(s_j|\Gamma_j\), it follows that \(w_j(z)\) satisfies a bound of the form

\[
|w_j(z)| \leq c_jy_j|s_j|
\]
for some fixed norm) since \( y_j \geq y_{j+1} \geq \cdots \geq y_r \) on \( I' \). Combining these observations, it then follows that
\[
d(\tilde{F}_{j-1}(z), \tilde{F}_j(z)) \leq Ky_j^b e^{-2\pi y_j}
\]
□

**Remark 3.16.** In Schmid’s estimate the factor \( \prod_j \Im(z_j) \) appears instead of a leading power of \( y_1 \). However, since we can always reorder to obtain \( y_1 \geq y_2 \geq \cdots \geq y_r \), this is really just a fully symmetric version of our result.

### 4. Strong estimates

A variation of mixed Hodge structure is said to satisfy the strong distance estimate with respect to a metric on \( \mathcal{M} \) if there is a constant \( K \) such
\[
d(F(z), F_{\text{nilp}}(z)) \leq K \sum_j y_j^b e^{-2\pi y_j}
\]for all \( z \in I' \) with the imaginary part of \( z \) sufficiently large.

#### 4.1. Pure Case

For variations of pure Hodge structure, \( L = 1 \), and we recover the strong distance estimate of [CKS] from \([32]\).

#### 4.2. Twisted Metric

The distance between the period map and the nilpotent orbit satisfies the strong distance estimate with respect to the twisted metric, provided that we also restrict the real parts of \( z_1, \ldots, z_r \) to a compact set, and assume that the associated nilpotent orbit is asymptotically non-split in the sense of \([18]\). To this end, we note that since \([19]\) has the same limit when we replace the nilpotent orbit by the period map, the twists of \( F_{j-1}(z) \) and \( F_j(z) \) are asymptotically non-split and take values in a geodesically convex neighborhood as \( y_j/y_{j+1} \to \infty \).

The proof proceeds as in the previous section up to Lemma \([310]\). At this juncture, use Theorem \([2.18] \) to write
\[
d_\tau(F_{j-1}(z), F_j(z)) \leq d_\tau(t^{-1}(y)F_{j-1}(z), t^{-1}(y)F_j(z))
\]
However, this is not exactly what appears in our relative compactness result, since
\[
t^{-1}(y)F_j(z) = (\Ad(t^{-1}(y)) e^{N(x)}) \tilde{F}_j(z)
\]
but as discussed in the proof of Lemma \((7.26)\) in \([BP2]\),
\[
\Ad(t^{-1}(y)) e^{N(x)} = e^{Q(x,t)}
\]
where \( Q(x,t) \) is a polynomial in non-negative half-integral powers of \( t_j = y_{j+1}/y_j \) with coefficients which are polynomials in \( x_1, \ldots, x_r \). Accordingly, when the variables \( x_j \) are restricted to a compact set, we still have a relative compactness result.

The rest of the proof proceeds as in the previous section. Consequently, we obtain the strong distance estimate since we never multiply by \( y_1^{(L-1)/2} \).
4.3. **Unipotent Variations.** Let $X$ be a smooth complex algebraic variety and $J_x$ denote the augmentation ideal of the group ring $\mathbb{Z}\pi_1(X,x)$, i.e.

$$J_x = \left\{ \sum_{g \in \pi_1} a_g g \mid \sum_{g \in \pi_1} a_g = 0 \right\}$$

Then, the quotients $\mathbb{Z}\pi_1(X,x)/J_x^\ell$ carry functorial mixed Hodge structures which patch together to form an admissible variation of mixed Hodge structure $\mathcal{J}$ over $X$ with monodromy representation given by the natural action of $\pi_1(X,x)$ on $\mathbb{Z}\pi_1(X,x)/J_x^\ell$. This is called a tautological variation in [HZ]. The weight graded-quotients of $\mathcal{J}$ are locally constant.

A variation of mixed Hodge structure $\mathcal{V}$ over $X$ is said to be **unipotent** if the induced variations of pure Hodge structure on the weight graded-quotients are constant. By Theorem 1.6 of [HZ], there is an equivalence of categories between unipotent variations of mixed Hodge structure and mixed Hodge theoretic representations of $\mathbb{Z}\pi_1(X,x)/J_x^{\ell+1}$ and unipotent variations of mixed Hodge structure with index of unipotency $\leq \ell$.

Suppose now that we have an (admissible) unipotent variation $\mathcal{V}$ over the punctured disk with monodromy operator $T$. Then, since $\text{gr}^W$ is constant, and $T$ is quasi-unipotent in the usual sense, it follows that $T = \text{id}$ on $\text{gr}^W$, and hence $T = e^N$ where $N = 0$ on $\text{gr}^W$. Admissibility then forces:

(a) $N(W_k) \subseteq W_{k-2}$;

(b) $M(N,W) = W$;

(see the paragraph above [HZ Theorem 1.6]). Thus, for a unipotent variation the Hodge filtration may degenerate, but the weight filtration does not.

**Theorem 4.2.** If $\mathcal{V} \to \Delta^*$ is a unipotent variation of mixed Hodge structure then period map and the nilpotent orbit satisfy the strong distance estimate (4.1) with respect to the standard metric.

We start with the estimate

$$d(F_0(z), F_1(z)) \leq K y_1^{b_1} e^{-2\pi y_1}$$

which we prove as in the previous section. It remains therefore to bound

$$d(F_k(z), F_{k+1}(z))$$

for $k = 1, \ldots, n - 1$.

**Lemma 4.3.** $N_j$ is a $(-1, -1)$-morphism of $(F_k(z), W)$ for $1 \leq j \leq k$.

**Proof.** By Lemma 3.7 we know that

$$[N_j, F_k(s)] = 0$$

for $j = 1, \ldots, k$, and hence

$$N_j F_k^p(z) = N_j(\sum z_i N_i e^{\Gamma_k(s)} F_k^p) = e^{\sum z_i N_i e^{\Gamma_k(s)} N_j F_k^p} \subset e^{\sum z_i N_i e^{\Gamma_k(s)} F_k^{p-1}} = F_k^{p-1}(z)$$

On the other hand, $N_j$ is real and $N_j(W_{\ell}) \subseteq W_{\ell-2}$. Taken together, these three facts imply that if $j \leq k$ then $N_j$ is a $(-1, -1)$-morphism of $(F_k(z), W)$. \qed
Regarding the local normal form of $F(z)$, a short argument shows that since the graded variations of Hodge structure are constant, and the monodromy logarithms act trivially on $Gr^W$ so do the functions $\Gamma_j$ constructed in the previous section. In particular, if $A$ acts trivially on $Gr^W$ then $e^A : \mathcal{M} \to \mathcal{M}$.

To compactify notation, write
\[ z = (u, v), \quad u = (z_1, \ldots, z_k), \quad v = (z_{k+1}, \ldots, z_r). \]

and observe that $F_k(v) = e^{N(0,v)}e^{\Gamma_k(0,v)}F_{\infty} \in \mathcal{M}$ by the remarks of the previous paragraph. Let $\gamma(t,v)$ be the path
\[ \gamma(t,v) = e^{N(0,v)}e^{t\Gamma_k}e^{\Gamma_{k+1}}F_{\infty} \]
from $e^{-N(u,0)}F_{k+1}(z)$ to $e^{-N(u,0)}F_k(z)$.

**Lemma 4.5.** $N_1, \ldots, N_k$ are $(-1, -1)$-morphisms along $\gamma(t,v)$.

**Proof.** Since both $\Gamma_k$ and $\Gamma_{k+1}$ commute with $N_j$ for $j = 1, \ldots, k$ so does $\Gamma_k$. Arguing as in equation (4.4) it then follows that $N_1, \ldots, N_k$ are $(-1, -1)$-morphisms along $\gamma(t,v)$. \qed

**Lemma 4.6.** Given $k > 0$ let $L(v)$ denote the length of the curve $\gamma(t,v)$ with respect to the standard metric. Then, $d(F_k(z), F_{k+1}(z)) \leq L(v)$.

**Proof.** By definition, $d(F_k(z), F_{k+1}(z))$ is less than or equal to the length $\tilde{L}$ of the path $e^{N(u,0)}\gamma(t,v)$ from $F_k(z)$ to $F_{k+1}(z)$. Now we have
\[
\tilde{L} = \int_0^1 \left\| \frac{d}{dt} e^{N(u)} \gamma(t,v) \right\|_{e^{N(u,0)}\gamma(t,v)} dt = \int_0^1 \left\| e^{N(u,0)} \frac{d}{dt} \gamma(t,v) \right\|_{e^{N(u,0)}\gamma(t,v)} dt.
\]

By Lemma (2.5), the fact that $N(u,0)$ is a $(-1, -1)$-morphism along $\gamma(t,v)$ implies that
\[
\tilde{L} = \int_0^1 \left\| e^{N(u)} \frac{d}{dt} \gamma(t,v) \right\|_{e^{N(u)}\gamma(t,v)} dt = \int_0^1 \left\| \frac{d}{dt} \gamma(t,v) \right\|_{\gamma(t,v)} dt = L(v)
\]
and hence $d(F_k(z), F_{k+1}(z)) \leq L(v)$ as required. \qed

It remains to estimate $L(v)$:

**Lemma 4.7.** $L(v) \leq Ky_k^{b+1}e^{-2\pi y_{k+1}}$ for suitable non-negative constants $K$ and $b$.

**Proof.** We begin with the observation that
\[ F(0,v) = e^{\sum_{j>k} z_jN_j}e^{\Gamma_{k+1}}F_{\infty} \]
is the period map of a variation of mixed Hodge structure over $\Delta^{r-k}$ in the variables $(s_{k+1}, \ldots, s_r)$. Let
\[ F_{\text{nilp}}(0,v) = e^{\sum_{j>k} z_jN_j}F_{\infty} \]
be the associated nilpotent orbit. Let $t(y)$ $(y = (y_k, \ldots, y_r))$ be as in (1.11) constructed from $F_{\text{nilp}}(0,v)$. Then, by Theorem (1.13), the image of the map
\[ \tilde{F}(0,v) = t^{-1}(y).F(0,v) \]
will be a relative compact subset of \( M \) for \( v \) in an appropriate \( I' \) (with the real component of \( v \) restricted to a compact set). Therefore,

\[
L(v) = \int_{0}^{1} \left| \frac{d}{dw} \gamma(w, v) \right|_{\gamma(w,v)} dw \\
= \int_{0}^{1} \left| \frac{d}{dw} e^{N(0,v)}e^{w\Gamma_{k+1}}.F_{\infty} \right|_{e^{N(0,v)}e^{w\Gamma_{k+1}}.F_{\infty}} dw \\
= \int_{0}^{1} \left| \frac{d}{dw} (Ad(e^{N(0,v)})e^{w\Gamma_{k}}).F(0,v) \right|_{(Ad(e^{N(0,v)})e^{w\Gamma_{k}}).F(0,v)} dw
\]

To continue, let

\[
A(w, v) = Ad(t^{-1}(y)e^{N(0,v)})e^{w\Gamma_{k}}
\]

Then, by the above computations,

\[
L(v) = \int_{0}^{1} \left| t(y) \frac{d}{dw} A(w, v).\tilde{F}(0,v) \right|_{t(y)A(w,v).\tilde{F}(0,v)} dw
\]

Finally, since \( s_{k+1} \) divides \( \Gamma_{k} \) and \( t(y) \) and \( e^{N(v)} \) can be bounded by polynomials in half-integral powers of \( y_{k+1} \) as in \( \S 3 \), it follows that

\[
|\frac{d}{dw} A(w, v)|\tilde{F}(0,v) \leq Cy_{k+1}^{b}e^{-2\pi y_{k+1}}
\]

for suitable constants \( C \) and \( b \). In particular, since \( \tilde{F}(0, v) \) takes values in a relatively compact set, and the distortion introduced by \( t(y) \) is again at worst a non-negative, half-integral power of \( y_{k+1} \), it follows that

\[
L(v) \leq Cy_{k+1}^{b}e^{-2\pi y_{k+1}}
\]

\[
\square
\]

5. Archimedean Heights

In this section, we prove Theorem [137]. We begin with a discussion of the biextension metric.

Let \((F,W)\) be a mixed Hodge structure on \( V \) with weight graded quotients

\[
Gr_{-2}^{W} \cong \mathbb{Z}(1), \quad Gr_{-1}^{W} = H, \quad Gr_{0}^{W} \cong \mathbb{Z}(0)
\]

(5.1)

Assume that \( V \) has lattice \( V_{\mathbb{Z}} \) such that \( W_{a}V_{\mathbb{Z}}/W_{b}V_{\mathbb{Z}} \) is torsion free \( (b < a) \) and \( H = Gr_{-1}^{W} \) is a polarizable Hodge structure of weight \(-1\).

For \((F,W)\) as in (5.1), fix the isomorphisms \( Gr_{-2}^{W} \cong \mathbb{Z}(j) \) and the extension classes

\[
0 \to W_{-2} \to W_{-1} \to Gr_{-1}^{W} \to 0 \\
0 \to Gr_{-1}^{W} \to W_{0}/W_{-2} \to Gr_{0}^{W} \to 0
\]

(5.2)

Let \( 1^{V} \) the generators of \( \mathbb{Z}(0) \) and \( \mathbb{Z}(1) \), and \( \mu : V \to W_{-2} \) be the linear map defined by the following properties:

(i) \( \mu \) annihilates \( W_{-1} \);
(ii) \( \mu(v) = 1^{V} \) for any lifting \( v \in V \) of \( 1 \in Gr_{0}^{W} \);
(iii) \( \mu \) is in the center of \( g_{C} \);
Let $\tilde{B}$ be the set of all mixed Hodge structures $(\tilde{F}, W)$ on $V$ with extension data $(5.2)$, and $B$ denote the quotient of $\tilde{B}$ by the equivalence relation

$$(\tilde{F}, W) \equiv (e^{a\mu} \cdot \tilde{F}, W), \quad a \in \mathbb{Z}$$

Then, $\mathbb{C}^*$ acts simply transitively on $B$ via the rule

$$[t \cdot F] = [e^{\frac{1}{2\pi} \log(t) \mu} \cdot F] \quad (5.3)$$

Let $Y_{(F,W)}$ be the semisimple endomorphism of $V_{\mathbb{C}}$ which acts as multiplication by $p + q$ on $P_{(F,W)}^{p,q}$. Then,

$$\delta_{(F,W)} \in W_{-2} \text{End}(V_{\mathbb{C}})$$

is defined (see Prop. (2.20) in [CKS]) by the equation

$$\tilde{Y}_{(F,W)} = Y_{(F,W)} - 4i\delta_{(F,W)}$$

(due to the short length of $W$), and hence

$$\delta_{(F,W)} = \frac{1}{4i}(Y_{(F,W)} - \tilde{Y}_{(F,W)}) \quad (5.4)$$

Moreover, since $\delta$ depends only on the isomorphism class of the underlying $\mathbb{R}$-MHS, $\delta$ descends to $B$.

Regarding the $\mathbb{C}^*$ action, $(5.3)$, we have:

$$\delta_{(t \cdot F,W)} = \frac{1}{4i}(Y_{(t \cdot F,W)} - \tilde{Y}_{(t \cdot F,W)})$$

$$= \frac{1}{4i}((Y_{(F,W)} + \frac{1}{\pi i} \log(t) \mu) - (Y_{(F,W)} + \frac{1}{\pi i} \log(t) \mu))$$

$$= \delta_{(F,W)} - \frac{1}{2\pi} \log |t| \mu$$

Accordingly, if for any equivalence class $[F] \in B$, we define

$$||F|| = e^{-2\pi \delta_{(F,W)}/\mu} \quad (5.5)$$

it follows that

$$|t \cdot [F]| = e^{-2\pi \delta_{(F,W)}}/\mu - \frac{1}{2\pi} \log |t| \mu \mu = e^{-2\pi \delta_{(F,W)}/\mu + \log |t|} = |t||[F]|$$

Suppose now that $S$ is a Zariski open subset of a complex manifold $\tilde{S}$. Let $\mathcal{V}_\mathbb{Z}$ be a local system of torsion free $\mathbb{Z}$-modules of finite rank equipped with a weight filtration $W$ which satisfy the analog of $(5.1)$, e.g. $Gr_{n,x}^{W_{-2j}} \cong \mathbb{Z}(j)$ and $\mathcal{H} = Gr_{n,x}^{W_{-1}}$ is a polarizable variation of Hodge structure of weight $-1$. Let $\nu_A$ and $\nu_B$ be admissible normal functions on $S$ which correspond to extensions

$$0 \to \mathbb{Z}(1) \to W_{-1} \to \mathcal{H} \to 0$$

$$0 \to \mathcal{H} \to W_0/W_{-2} \to \mathbb{Z}(0) \to 0 \quad (5.6)$$

as in $(5.2)$.

**Lemma 5.7.** For any sufficiently small open set $U$ of $S$ in the analytic topology, there exists an admissible variation of mixed Hodge structure $\mathcal{V}$ over $U$ with underlying weight filtration $W, \mathcal{V}_\mathbb{Z}$ and extension data $(5.0)$. 

Accordingly, there exists a unique \( \mathbb{Q} \)-line bundle \( \mathcal{B} \) over \( S \) have such VMHS as local generators. Indeed, the fibers of \( \mathcal{B} \) are just the sets \( B \) defined above. Since the \( \mathbb{C}^* \)-action (5.3) is simply transitive and the Hodge filtration of a VMHS is holomorphic with respect to the flat structure \( \mathcal{V}_\mathbb{Z} \) it follows that if \( U_i \) and \( U_j \) are open sets which support VMHS \( \mathcal{V}_i \) and \( \mathcal{V}_j \), which satisfy the hypothesis of the Lemma then \( \mathcal{V}_i = f_{ij} \mathcal{V}_j \) for some element \( f_{ij} \in O^*(U_i \cap U_j) \). Furthermore, since (5.3) is a simply transitive group action, \( \{ f_{ij} \} \) is a cocycle.

**Theorem 5.8.** Let \( \mathcal{V} \to \Delta^r \) be an admissible variation of mixed Hodge structure of biextension type and

\[
|\mathcal{V}| = e^{-\phi}
\]

Then, \( \phi \in L^1_{\text{loc}}(\Delta^r) \).

**Proof.** Let \( F : U^r \to \mathcal{M} \) denote the lift of the period map of \( \mathcal{V} \) to the product of upper half-planes, and \( t(y) \) be the associated family of semisimple automorphisms (1.11). Then, since \( \mu \) is central in \( g_\mathbb{C} \) and

\[
F(z_1 + a_1, \ldots, z_r + a_r) = e^{\sum_j a_j N_j} F(z)
\]

for \( a_1, \ldots, a_r \in \mathbb{Z} \) and \( \mu \) is central in \( g_\mathbb{C} \) it follows from (5.4) that

\[
\delta(F(z_1 + a_1, \ldots, z_r + a_r), W) = \delta(F(z), W)
\]

Likewise, \( \delta(e^{-N(x)} F(z), W) = \delta(F(z), W) \) since \( N(x) \) is real. By equation (5.5), it then follows that

\[
|\mathcal{V}|_i = e^{-2\pi \delta(F(z), W)/\mu}
\]

for any lifting of \( s \in \Delta^r \) to \( z \in U^r \), i.e. \( \phi = 2\pi \delta(F(z), W)/\mu \). Returning to (5.4), we see that since \( t(y) \) is real and preserves the weight filtration,

\[
\delta(F(z), W) = \delta(e^{-N(x)} F(z), W)
\]

\[
= \delta(t(y) t^{-1}(y) e^{-N(x)} F(z), W) = t(y) \delta(t^{-1}(y) e^{-N(x)} F(z), W)
\]

As above, the factors \( y_j^{-H_j/2} \) appearing \( t(y) \) act trivially on \( \delta \), and hence

\[
\delta(F(z), W) = y_1^{-Y_1/2} \delta(t^{-1}(y) e^{-N(x)} F(z), W)
\]

\[
= y_1 \delta(t^{-1}(y) e^{-N(x)} F(z), W) = -\frac{1}{2\pi} \log |s_1| \delta(t^{-1}(y) e^{-N(x)} F(z), W)
\]

since \( \delta \) lowers the weight filtration by 2, which is the maximum possible. Combining the above, it then follows that

\[
\phi = -\log |s_1| \delta(t^{-1}(y) e^{-N(x)} F(z), W)/\mu
\]

which is locally integrable since \( t^{-1}(y) e^{-N(x)} F(z) \) takes values in a relatively compact subset of \( \mathcal{M} \), so the corresponding \( \delta \)-values are bounded, and \( \log |s_1| \) is locally integrable.

6. **Reduced Limit Filtrations**

In this section, we considered the reduced limit filtration in the case of a several variable degeneration of Hodge structure \( \mathcal{V} \to \Delta^r \), and show this filtration remains the same even along sequences \( s(m) = (s_1(m), \ldots, s_r(m)) \) for which the components \( s_j(m) \) go to zero at radically different rates.
6.1. Complement to SL(2)-orbit theorem. Let \((N_1, \ldots, N_r, F)\) be a pure nilpotent orbit of weight \(w\), and let \((\rho, \phi)\) be the associated \(SL_2\)-orbit. We denote by \(\hat{N}_j^\pm\) (resp. \(H_j\)) the image of the \(j\)-th \(n_\pm\) (resp. \(h\)) where \((n_-, h, n_+)\) are the standard generator of \(sl_2(\mathbb{R})\), and we write \(H_r = H_1 + \cdots + H_r\). By the bracket relations of the \(sl_2\)-triples, we have

\[
\left[ \sum_{j=1}^{r} \frac{1}{y_j} \hat{N}_j^+, \sum_{j=1}^{r} y_j \hat{N}_j^- \right] = H_r, \quad \left[ H_r, \sum_{j=1}^{r} \hat{N}_j^\pm \right] = \pm 2 \sum_{j=1}^{r} \hat{N}_j^\pm.
\]

and hence \((\sum_{j=1}^{r} \frac{y_j}{y_j}, H_r, \sum_{j=1}^{r} \frac{1}{y_j} \hat{N}_j^+)\) is also an \(sl_2\)-triple.

Let \(g_j(y_1/y_j+1, \ldots, y_j/y_j+1)\) for \(1 \leq j \leq r-1\) be the \(G_2\)-valued function defined in the \(SL_2\)-orbit theorem. We define

\[
g(y) = \prod_{j=1}^{r-1} g_j(y_1/y_j+1, \ldots, y_j/y_j+1)
\]

where \(y_1/y_2, \ldots, y_r/y_{r+1}\) are sufficiently large. We write \(N(y) = \sum_j y_j N_j\). By the \(SL_2\)-orbit theorem, \(g(y)\) commutes with \(H_r\) and \(N(y) = Ad(g(y)) \hat{N}^-(y)\). Therefore,

\[
Ad\left(g(y)\right) (\sum_{j=1}^{r} y_j \hat{N}_j^-, H_r, \sum_{j=1}^{r} \frac{1}{y_j} \hat{N}_j^+) = (N(y), H_r, N^+(y)) \quad (6.1)
\]

is \(sl_2\)-triple where \(N^+(y) = Ad(g(y)) \sum_{j=1}^{r} \frac{1}{y_j} \hat{N}_j^+\).

Now \((N(y), F)\) generates a one variable nilpotent orbit with associated \(SL_2\)-orbit defined by \((N(y), \hat{F})\) where

\[
(\hat{F}, W(N)[-w]) = (e^{-r} F, W(N)[-w])
\]

is the \(sl_2\)-splitting of \((F, W(N)[-w])\), for \(N\) any element in the monodromy cone of positive linear combinations of \(N_1, \ldots, N_r\). The corresponding semisimple element \(H\) of this \(SL_2\)-orbit coincides with \(H_r\) above, and hence the associated \(sl_2\)-triple is \((N(y), H_r, N^+(y))\).

As in the proof of Lemma (3.12) [CKS, p. 478], it then follows that

\[
\exp(z N(y)) \hat{F} = \exp\left(\frac{1}{z} N^+(y)\right) \Phi, \quad \Phi^p = \sum_{b \leq w-p} \int_{(F, W(N)[-w])} I_{(F, W(N)[-w])}^0
\]

Substituting \(z = i\), we have

\[
\exp(i N(y)) \hat{F} = \exp(-i N^+(y)) \Phi \quad (6.2)
\]

Therefore, along any sequence \(y(m)\) to which the \(SL_2\)-orbit theorem applies (i.e. \(y_j(m)/y_{j+1}(m)\) sufficiently large as \(m \to \infty\)),

\[
e^{i N(y(m))} \hat{F} = e^{\epsilon^i N(y(m))} \hat{F} = e^{\epsilon^i N^+(y(m))} \Phi
\]

and hence the limit as \(m \to \infty\) is again \(\Phi\).
6.2. **Relationship with the Satake Boundary Components.** As noted in the introduction, by the work of Kato and Usui, the theory of degenerations of Hodge structure can be used to construct generalizations of the toroidal compactifications of [AMRT] by adjoining spaces of nilpotent orbits as boundary components. In analogy with this classical theory, it is natural to consider the reduced limit period map as providing a generalization of the Satake construction.

To obtain an explicit connection with the standard Satake construction, let $D$ be a classifying space of effective Hodge structures of weight $2k - 1$ upon which the Lie group $G_R$ acts transitively. Let $V \subset G_R$ be the stabilizer of a reference Hodge structure $F \in D$ and $K$ be the stabilizer of

$$S = H^{2k-1,0} \oplus H^{2k-3,2} \oplus \cdots$$

Then, $K$ is a maximal compact subgroup of $G_R$ containing $V$ and the quotient $G_R/K$ is a Siegel space $H$ parametrizing the Hodge structures $S \otimes S$ polarized by $Q$. Via a Tate-twist, we now renormalize $D$ and $H$ to have weight $−1$ and let $p : D \to H$ denote the quotient map $G_R/V \to G_R/K$, i.e.

$$F^0p(H^{•,•}) = \bigoplus_{p \text{ even}} H^{p,-p-1}$$

Let $(N_1, \ldots, N_r, F)$ be a nilpotent orbit with nilpotent cone $\sigma = \sum_{j=1}^r \mathbb{R}_{\geq 0} N_j$ of even-type as defined in [Hay1, Definition 2.6] or [Hay2, Lemma 2.4]. Then we have the Satake boundary component

$$B_S(\sigma) = \{F \in cl(H) \mid F^0 \cap F^0 = W^{-2,\mathcal{C}}\}$$

contained in the closure $cl(H)$ of the compact dual $\mathcal{H}$ of $H$ where $W = W(\sigma)[1]$ is the monodromy weight filtration defined by $\sigma$.

On the other hand, we have the Kato-Usui boundary component $B(\sigma)$ of $D$ which is the set of $\sigma$-nilpotent orbits modulo $\exp(\sigma_{\mathcal{C}})$. In [Hay1, Corollary 2.8], the first author defined a map $p_\sigma : B(\sigma) \to B_S(\sigma)$, where $\Psi := p_\sigma(\sigma, F)$ is given by

$$\Psi^0 = \left( \bigoplus_{p \text{ even}} I_{p,-p-1}^{F^0(F,W^\mathcal{C})} \right) \oplus W^{-2,\mathcal{C}}$$ \hspace{1cm} (6.3)$$

**n.b.** $p_\sigma$ is well defined since $e^N \Psi = \Psi$ for all $N \in \sigma_{\mathcal{C}}$.

Let $(\sigma, \tilde{F})$ generate a nilpotent orbit of even type with limit mixed Hodge structure split over $\mathbb{R}$ and

$$\tilde{F}^0 = \left( \bigoplus_{p \text{ even}} I_{p,-p-1} \right) \oplus \left( \bigoplus_{p} I_{p,-p} \right)$$ \hspace{1cm} (6.4)$$

Then, by Proposition (2.10) of [Hay1] it follows that $(\sigma, \tilde{F})$ generates a nilpotent orbit $e^{\sum_j z_j N_j} \tilde{F}$ with limit split over $\mathbb{R}$ such that

$$p(e^{\sum_j z_j N_j} \tilde{F}) = e^{\sum_j z_j N_j} \tilde{F}$$

By the results of §6.1, the orbit $e^{\sum_j z_j N_j} \tilde{F}$ has a reduced limit $\tilde{\Phi}$. Comparing the defining equations (1.20), (6.3) and (6.4) it follows that

$$\tilde{\Phi} = \Psi$$

and hence $p_\sigma(\sigma, \tilde{F})$ is equal to the reduced limit of the orbit $(\sigma, \tilde{F})$. The same conclusion holds for odd type $\sigma$ defined in [Hay1].
6.3. Complement to [CK]. In this subsection, we prove the following consequence of equation (3.8).

**Theorem 6.5.** For any choice of branch cuts for \( z_j = \frac{1}{2\pi i} \log(s_j) \),

\[
\lim_{s \to 0} \text{Ad}(e^{\sum_j z_j N_j}) \Gamma(s) = 0
\]

Define the length of \( q \) to be the smallest non-negative integer \( \lambda \) such that \( \wp - a = 0 \) for \( a > \lambda \). Recall that

(i) \( q = \oplus_{a<0} \wp_a \);
(ii) \( [\wp_a, \wp_b] \subseteq \wp_{a+b} \);
(iii) \( N_j \in \wp_{-1} \).

Given a multi-index \( J = (a_1, \ldots, a_r) \) with non-negative entries define

\[
\deg_j(J) = a_j, \quad (\log s)^J = \Pi_j (\log s_j)^{\deg_j(J)}, \quad \deg(J) = \sum_j \deg_j(J)
\]

By the commutativity of the \( N_j \)'s, and properties (i)–(iii) above, we have

\[
\text{Ad}(e^{\sum_j z_j N_j}) \Gamma(s) = \left( \sum_{\deg(J) < \lambda} (\log s)^J A_J \right) \Gamma(s) \quad (6.6)
\]

where \( A_J = b_J \Pi(\text{ad} N_j)^{\deg_j(J)} \) for an appropriate constant \( b_J > 0 \) and \( \lambda \) is the length of \( q \).

Given a holomorphic function \( f(s_1, \ldots, s_r) \) which vanishes at \( s = 0 \) let

\[
f(s) = \sum_{\deg(K) > 0} c_K s^K f_K \quad (6.7)
\]

denote the Taylor expansion of \( f \) about \( s = 0 \) where \( s^K = \Pi s_j^{\deg_j(K)} \) and

\[
f_K = \left. \frac{\partial^{\deg(K)} f}{\partial s_1^{\deg_1(K)} \cdots \partial s_r^{\deg_r(K)}} \right|_0
\]

Combining (6.6) and (6.7) and using the fact that \( \lim_{t \to 0} |t|^{\ell} \log |t| = 0 \) for \( \ell > 0 \), it then follows that in order to show that (6.6) converges to zero, it is sufficient to prove [see Lemma (6.9)] that

\[
\deg_j(K) = 0 \implies [N_j, \Gamma_K] = 0 \quad (6.8)
\]

To verify (6.8), we note that \( \deg_j(K) = 0 \) implies that

\[
\Gamma_K = \left. \frac{\partial K}{\partial s^K} \Gamma_j \right|_0
\]

Therefore, by (iv) and the fact that Lie brackets commute with derivatives, we have \([N_j, \Gamma_K] = 0\). To complete the proof of Theorem (6.5) it remains to establish:

**Lemma 6.9.** \((\log s)^J A_J \Gamma(s) \to 0\) for any multi-index \( J \) with non-negative entries.

**Proof.** We can assume that \( \deg(J) < \lambda \). Otherwise \( A_J \Gamma(s) = 0 \). For a pair of multi-indices \( J \) and \( K \) define

\[
J \cdot K = \max(\deg_j(J) | \deg_j(K) = 0)
\]
Then, $J \cdot K > 0$ if and only if there is some index $j$ such that $\deg_j(J) > 0$ and $\deg_j(K) = 0$. Therefore, by (6.8) and the commutativity of $N_i$'s:

$$J \cdot K > 0 \implies (A_j \Gamma)_K = b_j \Pi (\text{ad } N_i)^{\deg_j(J)} \Gamma_K = 0$$

As such,

$$A_j \Gamma(s) = \sum_{\deg(K) > 0, J \cdot K = 0} c_K s^K (A_j \Gamma)_K$$

(6.10)

To continue, let $|J| = \{ j \mid \deg_j(J) > 0 \}$ and $s^{[J]} = \Pi_{j \in |J|} s_j$. Observe that $J \cdot K = 0$ implies that $s^{[J]} s^K$ in $\mathbb{C}[s_1, \ldots, s_r]$ and hence by (6.10),

$$s^{[J]} |A_j \Gamma(s)| \rightarrow (6.11)$$

in $\mathcal{O}$. Therefore,

$$(\log s)^{J} A_j \Gamma(s) = ((\log s)^{J} s^{[J]})(A_j \Gamma(s)/s^{[J]})$$

where the first factor $((\log s)^{J} s^{[J]}) \rightarrow 0$ as $s \rightarrow 0$ by elementary calculus and the second factor $A_j \Gamma(s)/s^{[J]}$ is bounded near $s = 0$ by (6.11).

6.4. Convergence of Reduced Limit. By equation (6.2) it follows that

$$e^{\sum_j z_j N_j e^\Gamma(s)} F = \left( e^{\sum_j z_j N_j e^\Gamma(s)} e^{-\sum_j z_j N_j} \right) \left( e^{\epsilon^\epsilon e \sum_j x_j N_j} \right) \left( \exp(-i N^+(y)) \Phi \right)$$

where the rightmost parenthesized quantity

$$\left( \exp(-i N^+(y)) \Phi \right) = (1 + O(1/y)) \Phi$$

by §6.1. The next parenthesized quantity $\left( e^{\epsilon^\epsilon e \sum_j x_j N_j} \right)$ fixes $\Phi$, and so

$$\left( e^{\epsilon^\epsilon e \sum_j x_j N_j} \right) \left( \exp(-i N^+(y)) \Phi \right) \rightarrow \Phi$$

Finally, by §6.2,

$$\left( e^{\sum_j z_j N_j e^\Gamma(s)} e^{-\sum_j z_j N_j} \right) \rightarrow 1$$

Combining the above, it follows that

$$\lim_{y_j/y_{j+1} \rightarrow \infty} F(z) = \Phi$$

so long as $x_1, \ldots, x_r$ remain bounded. In the case where $y_r \rightarrow \infty$ but some other ratios $y_j/y_{j+1}$ remain bounded, one can group the resulting $N_j$'s together and use the fact that the $SL_2$-orbit theorem remains valid for variable $N_{\alpha}$ taken from a compact set. See [BP2] for details on this type of argument.

6.5. Extension to Normal Functions. In this paragraph, we consider the extension of the reduced limit map to the case of normal functions in 1-variable. To this end, we recall that a mixed Hodge structure $(F, W)$ is determined by $FGr W$ and the grading $Y_{(F, W)}$ of $W$ which acts as multiplication by $(p + q)$ on $P_{(F, W)}$. By Theorem (3.9) of [BP1], given a lifting

$$F : U \rightarrow \mathcal{M}$$

of an admissible normal function $\varphi : \Delta^* \rightarrow \Gamma \backslash \mathcal{M}$ to the upper half-plane, we have a well defined limit

$$\lim_{\text{Im}(z) \rightarrow \infty} Y_{(F(z), W)}$$
provided that Re(z) remains in a compact set. By the previous paragraphs (see also §5 of [KP2]) \( F(z)G_W \) also has a well defined limit as \( \text{Im}(z) \to \infty \). Combining these two results it follows that \( F(z) \) has a well defined limit as \( \text{Im}(z) \to \infty \) with \( \text{Re}(z) \) confined to a compact set.

References

[AMRT] A. Ash, D. Mumford, M. Rapoport, Y. S. Tai, Smooth compactifications of locally symmetric varieties, Math. Sci. Press, Brookline, 1975.

[BFNP] P. Brosnan, H. Fang, Z. Nie, G. Pearlstein, Singularities of admissible normal functions, Invent. Math. 177 (2009) 599-629.

[BP1] P. Brosnan and G. Pearlstein, The zero locus of an admissible normal function, Annals of Math., 170 (2009), 883–893.

[BP2] P. Brosnan and G. Pearlstein, On the algebraicity of the zero locus of an admissible normal function, To Appear, Compositio Math.

[BP3] P. Brosnan and G. Pearlstein, Jumps in the Arakelov Height, preprint.

[CKS] E. Cattani, A. Kaplan and W. Schmid, Degeneration of Hodge structures, Ann. of Math. 123 (1986), 457–535.

[CK] E. Cattani and A. Kaplan, Degenerating variations of Hodge structure, Actes du Colloque de Théorie de Hodge (Luminy, 1987). Astérisque 179-180 (1989), 67–96.

[C] L. Conlon, Differentiable Manifolds, Birkhäuser, 2001.

[D] J. Demailly, Singular Hermitian metrics on positive line bundles, Complex algebraic varieties (Bayreuth, 1990) 87 – 104, Lecture Notes in Math, 1507, Springer (1992).

[G] Griffiths P., Periods of integrals on algebraic manifolds III Publ. Math. I.H.E.S. 38 (1970) 125–180.

[Hai1] R. Hain, Birextensions and heights associated to curves of odd genus, Duke Math. Journ., 61 (1990) 859–898.

[Hai2] R. Hain, Normal functions and the geometry of the moduli space of curves, in Handbook of Moduli, edited by Gavril Farkas, Ian Morrison, vol. I (March, 2013), pp. 527-578, International Press.

[HZ] R. Hain and S. Zucker, Unipotent variations of mixed Hodge structure, Invent. Math. 88 (1987), 83–124.

[Hay1] T. Hayama, Boundaries of cycle spaces and degenerating Hodge structures, to appear in Asian J. Math.

[Hay2] T. Hayama, Kato-Usui compactifications over the toroidal compactifications, Contemp. Math. 608 (2014), AMS, 143–155.

[K] A. Kaplan, Notes on the moduli spaces of Hodge structures, preprint, 1995.

[Ka] M. Kashiwara, A study of variation of mixed Hodge structure, Publ. Res. Inst. Math. Sci. 22 (1986), no. 5, 991–1024.

[KU] K. Kato, S. Usui, Classifying space of degenerating polarized Hodge structures, Annals of Mathematics Studies, 169, Princeton University Press, Princeton, NJ, 2009.

[KNU1] K. Kato, C. Nakayama and S. Usui, SL(2)-orbit theorem for degeneration of mixed Hodge structure. J. Algebraic Geom. 17 (2008), 401–479.

[KNU2] K. Kato, C. Nakayama and S. Usui, Classifying spaces of degenerating mixed Hodge structures, III: Spaces of nilpotent orbits. arXiv:1011.4353.

[KP1] M. Kerr, G. Pearlstein, Boundary components of Mumford-Tate domains, arxiv:1210.5301.

[KP2] M. Kerr, G. Pearlstein, Naive boundary strata and nilpotent orbits, arxiv:1307.7945.

[L1] Z. Lu, On the geometry of classifying spaces and horizontal slices, Amer. J. Math., Vol.121 (1999), pp 177-198.

[L2] Z. Lu, Private letter, dated November 16, 2013.

[P0] G. Pearlstein, Variations of mixed Hodge structure, Higgs fields and quantum cohomology, Manuscripta Math., 102 (2000), 269–310

[P1] G. Pearlstein, Degenerations of mixed Hodge structure., Duke Math. J. 110 (2001), 217–251.

[P2] G. Pearlstein, SL₂-Orbits and degenerations of mixed Hodge structure, Jour. Diff. Geom., 74 (2006), 1–67.

[PP] G. Pearlstein, C. Peters, Curvature for mixed Period Domains, preprint.
[S] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*. Invent. Math. 22 (1973), 211–319.

[Sch] C. Schwarz, *Relative monodromy weight filtrations*, Math. Z., 236 (2001), 11-21.

[SZ] J. Steenbrink and S. Zucker, *Variation of mixed Hodge structure. I*. Invent. Math. 80 (1985), no. 3, 489–542.

[U] S. Usui, *Variation of mixed Hodge structure arising from family of logarithmic deformations. II. Classifying space*. Duke Math. J. 51 (1984), 851–875.

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