On the physical significance of $q$-deformation to isovector pairing interactions in nuclei

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Abstract

The quantum deformation concept is applied to a study of isovector pairing correlations in nuclei of the mass $40 \leq A \leq 100$ region. While the non-deformed ($q \to 1$) limit of the theory provides a reasonable global estimate for strength parameters of the pairing interaction, the results show that the $q$-deformation plays a significant role in understanding higher-order effects in the interaction.

1. Introduction — In addition to purely mathematical examinations of quantum algebraic concepts (see e.g. [1]), studies focused on quantum (or $q$-) deformation [2,3,4] in various fields of physics has been the focus of considerable attention in recent years. Studies of interest include applications in string/brane theory, conformal field theory, statistical mechanics, and metal clusters [5,6,7,8]. A feature of any quantum theory is that in the $q \to 1$ limit one recovers the “classical” (non-deformed) results.

The earliest applications of $q$-deformation in nuclear physics were related to a description of rotational bands in axially deformed nuclei [9] using the second order Casimir operator of $SU_q(2)$. Although optimum values for the deformation parameter did not differ much from the “classical” $q \to 1$ limit, an overall improved fit to the experimental excitation energies was achieved with $q \neq 1$. Nevertheless, such applications can contradict the physical interpretation of the generators of the algebra when the generators are associated with fundamental symmetries of the system [10]. In particular, when one $q$-deforms $SO(3) \sim SU(2)$, rotational invariant of the system is compromised. But the situation can be very different if one considers the algebra of a many-body interaction. In such scenarios, the $q$-deformation accounts for non-linear contributions of higher-order interactions [11,12] without affecting physical observables associated with a subset of the generators of the algebra that
remain unchanged by the deformation. This approach also allows one to construct $q$-deformed nuclear Hamiltonians with exact solutions. By considering a $q$-deformed generalization of some nuclear structure models, the role of the deformation can be explored by comparing the “classical” and $q$-deformed results with the experimental data.

It is well-known that effective interactions in nuclei are dominated by pairing and quadrupole terms. The former accounts for the formation of fermion pairs that give rise to a superconducting/pairing gap in nuclear spectra, and the latter is responsible for the strong enhanced electric quadrupole transitions in collective rotational bands. Indeed, within the framework of the harmonic oscillator shell-model with degenerate single-particle energies, these two limiting cases have a very clear algebraic structure in the sense that the spectra exhibit a dynamical symmetry. In the pairing limit (appropriate for near closed-shell nuclei), the Kerman-Klein quasi-spin $SU(2)$ group [13] together with its dual, the unitary symplectic group $Sp(2\Omega)$ [14,15], where $2\Omega$ is equal to the degeneracy of the shell, allows one to introduce the seniority quantum number that can be used to classify the spectra. On the other hand, in the quadrupole limit the symplectic group $Sp(6,\mathbb{R})$ [16] governs a shape-determined dynamics.

The inclusion of a proton-neutron isovector (isospin $T = 1$) pairing interaction enlarges the group structure from the Kerman-Klein $SU(2) \sim Sp(2)$ to $SO(5) \sim Sp(4)$ [17,18,19,20]. And a further extension to the $SO(8)$ group [21] allows one to also take the competing isoscalar ($T = 0$) proton-neutron pairing mode into account [22,23]. In this letter we focus on isovector pairing correlations in nuclei and, in addition, include a proton-neutron isoscalar term in the interaction that is diagonal in an isospin basis. The simple two-body $Sp(4)$ pairing model gives a very reasonable estimate for the nuclear interaction strength, which we assume to be constant for all nuclei within a major shell. The new feature reported on in this article is an extension of the theory to include non-linear deviations from the pairing solution as realized through a $q$-deformation of the underlying $Sp(4)$ symmetry group.

Our results show that the $q$-deformation parameter is decoupled from the interaction strength. In addition, since the $q \neq 1$ results are uniformly superior to those of the non-deformed limit, the results also suggest that the deformation has physical significance over-and-above the simple pairing gap concept, extending to the very nature of the nuclear interaction itself. In short, our results suggest that the $q$-deformation has physical significance beyond what can be achieved by simply tweaking the parameters of a two-body interaction. The results also underscore the need for additional studies to achieve a more comprehensive understanding of $q$-deformation in nuclear physics.

2. The $q$-deformed pairing model — The $sp_q(4)$ deformed algebra [24,25,26] is constructed in terms of $q$-deformed creation and annihilation operators, $\alpha_{jm\sigma}^+$ and $\alpha_{jm\sigma}$, each of which creates and annihilates a nucleon with isospin $\sigma$ ($+\frac{1}{2}$ for proton, $-\frac{1}{2}$ for neutron) in a single-particle state of total angular momentum $j$ (half-integer) with projection $m$ along the $z$-axis. The $q$-
The generators of $Sp_N\sigma$ particle levels, the dimension of the space for given $J\sigma$erators coupled to total angular momentum and parity non-deformed (“classical”) version of the theory is obtained in the limit $AB^\pm$ in addition to the proton (neutron) number operators $N$. Deformed fermion operators are defined through their anticommutation relations, $\{\alpha_{jm\sigma}, \alpha^\dagger_{jm\sigma}\} = q^{\pm N^\pi} \delta_{mn}$, $\{\alpha_{jm\sigma}, \alpha^\dagger_{k\nu\tau}\} = 0 \ (j \neq k, \ \sigma \neq \tau)$, and $\{\alpha^\dagger_{jm\sigma}, \alpha^\dagger_{k\nu\tau}\} = \{\alpha_{jm\sigma}, \alpha_{k\nu\tau}\} = 0$ [26], where the $q$-anticommutator is $\{A, B\}_q = AB + q^qBA$ and $N_{2\sigma} = (\sum_{jm} \alpha^\dagger_{jm\sigma} \alpha_{jm\sigma})_{q=1}$. In a model with degenerate single-particle levels, the dimension of the space for given $\sigma$ is $2\Omega = \sum_j (2j + 1)$. The non-deformed (“classical”) version of the theory is obtained in the limit $q \to 1$.

The generators of $Sp_q(4)$ are constructed as a bilinear product of fermion operators coupled to total angular momentum and parity $J\pi = 0^+$,

\[
T_\pm = \frac{1}{\sqrt{2\Omega}} \sum_{jm} \alpha^\dagger_{jm, \pm \frac{1}{2}} \alpha_{jm, \pm \frac{1}{2}},
\]

\[
A_\mu^\dagger = \frac{1}{\sqrt{2\Omega(1 + \delta_{\sigma\tau})}} \sum_{jm} (-1)^{j-m} \alpha^\dagger_{jm, \sigma} \alpha^\dagger_{j, -m, \tau},
\]

\[
A_\mu = (A^\dagger_\mu)^\dagger, \ \mu = \sigma + \tau
\]

in addition to the proton (neutron) number operators $N_{\pm 1}$, which remain undeformed. These generators of the Cartan subalgebra can also be realized in terms of the isospin projection operator $T_0 = \frac{1}{2}(N_+ - N_-)$ and the total nucleon number operator $N = N_+ + N_-$. The $Sp_q(4)$ algebra contains four distinct $su_q(2)$ $q$-deformed subalgebras (see table 1). The commutation relations between the generators are symmetric with respect to the exchange $q \leftrightarrow q^{-1}$ [26].

As for the microscopic “classical” approach [27], the most general Hamiltonian of a system with $Sp_q(4)$ dynamical symmetry and conserved proton and neutron particle numbers can be expressed in terms of the generators (1) [28]

\[
H_q = -\varepsilon_q N - F_q(A^\dagger_{+1} A_{+1} + A^\dagger_{-1} A_{-1}) - G_q A^\dagger_{0} A_0
- 2C_q([\frac{X}{2}]^2 - [\Omega]^2) - D_q [T_0]^2 - E_q \sum_{\mu} \{T^\mu, T^-\} - \frac{N}{2M_1},
\]

where $[X]^2 = \Omega \left[\frac{X}{2M_1}\right] \left([X + 1]\frac{1}{2M_1} + [X - 1]\frac{1}{2M_1}\right)$, and by definition $[X]_\kappa = \frac{\kappa X - q^{-\kappa} X}{q^{\kappa} - q^{-\kappa}} \to X$. In principle, the parameters $\gamma_q = \{\varepsilon_q, F_q, G_q, C_q, D_q, E_q\}$ can be different from their non-deformed counterparts $\gamma = \{\varepsilon, F, G, C, D, E\}$. The model describes the motion of $N_+$ valence protons and $N_-$ valence neutrons in the mean-field of a doubly-magic nuclear core. The basis states are constructed by the action of the pair-creation operators $A^\dagger_{0, \pm 1}$, on the vacuum. The quantum numbers used to specify the basis set $|n_1, n_0, n_{-1}\rangle$ count the number of $pp$, $pn$ and $nn$ pairs, respectively, and are associated with the corresponding limits of $Sp_q(4)$ (see table 1). The basis vectors model $0^+$ states with dominant isovector pair correlations. In the mass $40 \leq A \leq 100$ region they can be used to describe $0^+$ ground states of even-$A$ nuclei and the lowest isobaric analog $0^+$ state in the odd-odd nuclei with a $J \neq 0$ ground state.

From a “classical” perspective, the deformation introduces higher-order, many-body terms into a theory that starts with only one-body and two-body inter-
actions, the latter including an isovector pairing interaction (parameters $F$, $G$) together with a proton-neutron isoscalar force diagonal in an isospin basis. The way in which the higher-order effects enter into the theory is governed by the $[X]$ form. Since $\kappa$ and $q$ are related to one another, $q=e^\kappa$, everything is tied to the deformation with $[X]=\frac{\sinh(\kappa X)}{\sinh(\kappa)}=X(1+\frac{\kappa^2 X^2-1}{6}+\kappa^4\frac{3X^4-10X^2+7}{360}+\ldots)\to X$ in the $\kappa\to0$ ($q\to1$) limit. The deformation is applied to the same region of nuclei where the “classical” model has already proven to provide for a reasonable description of the $0^+$ states under consideration. Thus the $q$-deformation does not remedy the non-deformed model but complementarily can improve it.

3. Novel properties of $q$-deformation — To explore the physics of $q$-deformation, we fit the eigenvalues of the deformed Hamiltonian to the relevant experimental $0^+$ state energies for groups of 36 and 100 nuclei. This was carried out in two steps. First we determined a set of $\gamma$ parameters for the non-deformed limit ($q = 1$) of the theory that yielded a best overall fit to the data. Then another global fit was made with the $\gamma_q$ and $q$ parameters allowed to vary. The $\gamma_q$ set that was found differed very little from that of the non-deformed case. In short, varying the deformation parameter affected the pairing strengths very little. This observation underscores the fact that the deformation represents something fundamentally different, a feature that cannot be “mocked up” by allowing the strengths of the non-deformed interaction to absorb its effect.

Once the general $\gamma$ parameters were determined, the deviation of the predicted $0^+$ state energy from the corresponding experimental number, $|\langle H_q \rangle - E_{\text{exp}}|^2$, was minimized with respect to $q$ for each nucleus. This procedure yielded either two symmetric solutions for $\kappa$ ($\langle H_q \rangle = E_{\text{exp}}$), that is one physical solution $|\kappa|$, or one value of $q$ at the minimum of the $q$-deformed energy $\langle H_q \rangle$ (see Fig. 1). In the second case the minimum occurs at the “classical” energy ($\kappa = 0$) and its difference from the experimental value can be attributed to the presence of other types of interactions that are not in the model.

The higher-order terms, which correspond to many-body interactions, can be recognized through the expansion of the eigenvalues of the $q$-deformed Hamiltonian (2). As an example, in the limit of identical pairing interactions, i.e. $\mathfrak{su}_q^\pm(2)$ subalgebra (table 1), and in the $pn$ pairing limit, i.e. $\mathfrak{su}_q^0(2)$ limit, the ratio between the $q$-deformed $E_q^{\pm}$ ($E_q^{pn}$) and “classical” $E^{\pm}$ ($E^{pn}$) energies is

$$R_{pp}^{\langle nn \rangle} = \frac{E_q^{\pm}}{E^{\pm}} = 1 + \frac{\kappa^2}{6} \left( \frac{8n_1^2 - 5 + 4\Omega^2}{8\Omega^2} + \left( \frac{E_{q^{\pm}}}{E_{n^{\pm}}} \right)^2 \right) + \ldots,$$

$$R_{pn}^{\langle pn \rangle} = \frac{E_q^{pn}}{E^{pn}} = 1 + \frac{\kappa^2}{6} \left( \frac{n_0^2 - 1 + 4\Omega^2}{4\Omega^2} + \left( \frac{E_{q^{pn}}}{E_{n^{pn}}} \right)^2 \right) + \ldots,$$

where the expansions include higher-order terms that may not be negligible and the like-particles limit (3) can be compared with the earlier studies [11,12]. While the quadratic coefficient in $R_{pp}^{\langle nn \rangle}$ is positive, the one in $R_{pn}^{\langle pn \rangle}$ is negative. This leads to a decrease of the binding energy of the $pn$ pairs as $|\kappa|$ increases from zero. As the deformation parameter increases from the “classical” limit, the like-particle pairing is strengthen, yielding a larger pairing gap.
4. Analysis of the role of the $q$-deformation — The analysis yields solutions for the deformation parameter $|\kappa|$ that fall on a smooth curve that tracks with the energy of the lowest $2^+$ states (see Fig. 2). These energies are largest near closed shells where the pairing effect is essential for determining the low-lying spectrum and decrease with increasing collectivity and shape deformation. Similar properties are suggested for the $q$-deformation and this result, even though qualitative, gives some insight into the understanding of the nature of the $q$-deformation. The observed smooth behavior of the deformation parameter reveals its functional dependence on the model quantum numbers.

The many-body nature of the interaction is most important around closed shells and the regions with $N_+ \approx N_-$. For these nuclei the $q$-parameter has significant values and the experimental energies can be reproduced exactly. An interesting point is that $q$ tends to peak for even-even nuclei when $N_+ = N_-$ where strong pairing correlations are expected (see Fig. 2).

“Classical” values of the $q$-deformation parameter ($q \approx 1$) are found in nuclei with only one or two particle/hole pairs from a closed shell. This is an expected result since the number of particles is insufficient to sample the effect of higher-order terms in a deformed interaction. For these nuclei the non-deformed limit gives a good description. Around mid-shell ($N \approx 2\Omega$) the deformation adds little improvement to the theory with the experimental values remaining close to the “classical” limit. This suggests that for these nuclei the pairing interaction is not sufficient for their description. The results imply that even though the $q$-parameter gives additional freedom for all the nuclei, it only improves the model around regions of dominant pairing correlations.

A $q$-deformed extension of the $Sp(4)$ model, which is the underlying symmetry for describing isovector ($T = 1$) pairing correlations in atomic nuclei, has been investigated. When compared to experimental data the theory shows a smooth functional dependence of the deformation parameter $q$ on the proton and neutron numbers, which resembles the behavior of the lowest $2^+$ state energies. Since a $q$-deformation of $Sp_q(4)$ introduces higher-order, non-linear terms in the $np$, $pp$ and $nn$ pairing interactions into the nuclear Hamiltonian, the outcome suggests the presence and importance of higher-order pairing correlations in nuclei, especially for nuclei just beyond closed shells and with $N_+ \approx N_-$. The results also show that the deformation is decoupled from parameters that are used to characterize the two-body interaction itself, which means the latter can be assigned best-fit global values for the model space under consideration without compromising overall quality of the theory. Moreover, the specific features of the nuclear structure can be investigated through the use of a local $q$ value that varies smoothly with nuclear mass number. In summary, the concept of quantum deformation has been linked to the smooth behavior of physical phenomena in atomic nuclei.

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Table 1
Realizations of various $su_q(2) \subset sp_q(4)$.  

| symmetry ($\mu$) | $su_q^\mu(2)$ | $u^\mu(1)$ |
|------------------|---------------|-------------|
| $pp$ pairs (+)   | $A^+_1, \frac{1}{2}(N_1 - \Omega), A_{+1}$ | $N_{-1}$ |
| $nn$ pairs (−)   | $A^-_{1}, \frac{1}{2}(N_{-1} - \Omega), A_{-1}$ | $N_{+1}$ |
| $pn$ pairs (0)   | $A^+_0, \frac{1}{2}N - \Omega, A_{0}$ | $T_0$ |
| isospin ($T$)    | $T_+, T_0, T_-$ | $N$ |

Fig. 1. The difference between the theoretical and the corresponding experimental energies as a function of the deformation parameter $\kappa$ for a typical near-closed shell nucleus (solid line) and for a mid-shell nucleus (dashed line).

Fig. 2. Deformation parameter $\kappa$ (symbol ■) as a function of neutron numbers ($N_-$) for various isotopes with $^{56}$Ni as a core. The solid line is the excitation energies of the $2^+_1$ level measured in MeV. The arrows indicate $N_{+} = N_{-}$ with the value of $q = e^{\kappa}$. The global parameters are $\varepsilon = 13.851$ MeV, $F/\Omega = 0.296$ MeV, $G/\Omega = 0.352$ MeV, $C = 0.190$ MeV, $D = -0.796$ MeV, $E/(2\Omega) = -0.489$ MeV.