Self-Dual $\mathcal{N}=8$ Supergravity

as a Closed String Field Theory in Twistor Space

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Abstract

A closed string field theory action is formulated for the $\mathcal{N} = 8$ self-dual supergravity which is off-shell and Lorentz covariant. The bosonic truncation in the quantum field theory limit gives the Plebanski action in the super space.
1. Introduction:

The conventional quantum field theory calculation of gravity/supergravity loops in four dimensions is maligned with ultraviolet divergences and hence it is non-renormalizable [1]. This is easily realized by naive power counting and the presence of higher derivative terms in the Lagrangian. On the contrary it is claimed that $\mathcal{N}=4$ super-Yang Mills theory is a finite theory [3]. By Kawai, Lewellen and Tye (KLT) rule the gravity tree amplitudes can be expressed as the Yang-Mills tree amplitudes [4]. By the method of unitarity cutting rules it is shown that $\mathcal{N}=8$ supergravity has the same ultraviolet behaviour as that of $\mathcal{N}=4$ super-Yang Mills theory and hence it is claimed to be finite [2]. This fact renews ample motivation to study off-shell behaviour of $\mathcal{N}=8$ supergravity in the form of closed string field theory. The usual $\mathcal{N}=8$ supergravity action is obtained from compactifying type II string theory on a six torus or $d=11$ and $\mathcal{N}=1$ supergravity M-theory action. On the contrary we study here the self-dual supergravity as a $\mathcal{N}=2$ closed string field theory in the Attiyah-Ward (AW) space-time. The reason being it is simple and in the linearized level it is same as full non-selfdual theory. The action has the same field content where fields with negative helicities are treated as Lagrangian multiplier to the gauge covariant kinetic terms of the positive helicity states. In the pioneering work of Witten on twistor string theory [5], all the maximally helicity violating (MHV) amplitudes of $\mathcal{N}=4$ super-Yang Mills theory are reproduced as D-instanton corrections to the topological B-type string where a Calabi-Yau supermanifold $CP^{3|4}$ space emerges naturally as the target space of the super-twistor string. The Lagrangian for this B-model is the Open String Field theory Lagrangian of super-holomorphic Chern-Simons theory, which happened to be anti-self-dual Super Yang-Mills theory. Here the D-instanton correlation function plays the vital role in producing the full non-self-dual super Yang-Mills amplitudes. However Berkovits [6] gives an alternative formulation for the same MHV amplitudes as the conventional open string amplitudes of the super-twistor string. There is a natural generalization of this twistor string proposal of $\mathcal{N}=4$ Super-Yang-Mills theory to that of $\mathcal{N}=4$ conformal supergravity [7]. The conformal supergravity is very nice in the classical level however the quantization is not very physically interesting due to the presence of higher derivative terms even in the linearized approximation. Due to this Hilbert space is not positive definite and consequently S-matrix violates Unitarity. So a covariant string formulation of $\mathcal{N}=8$ Einstein Supergravity is more desirable. There are attempts to formulate a super twistor string theory for $\mathcal{N}=8$ supergravity [12]. On the contrary we start from a $\mathcal{N}=2$ critical closed string theory and write a covariant string field theory lagrangian for self-dual $\mathcal{N}=8$ supergravity. Recently a Lorentz covariant Open string field theory
Lagrangian for $\mathcal{N} = 4$ self-dual Super-Yang-Mills theory has been proposed by Lechtenfeld and Popov [8]. In the same spirit we are trying to formulate a closed string field theory for the $\mathcal{N} = 8$ supergravity. However the problems and intricacies are infinitely bigger for this case than the Super-Yang Mills. First of all to show the local gauge invariance of the super-space one needs to generate a target super-space from the world sheet variables. Generating super space from the Neveue-Schwarz variables is extremely complicated may be possible. Thus we followed Siegel’s [11] approach of putting non-holomorphic scalar grassman variables to formulate the super-space.

The organization of the paper is as follows. We give adequate review of the $\mathcal{N} = 2$ critical string theory and basic tools needed for the further construction. First part of section 3. has basic review of Berkovit’s formulation of Closed string and also open string for the Self-dual Yang Mills. In the later part of section 3. we derive our results.

2. The Preliminaries:

The self-dual Yang-Mills and self-dual Gravity theories in $2 + 2$ dimensions are described by $\mathcal{N} = 2$ critical open and closed strings respectively [9]. The perturbative spectrum of $\mathcal{N} = 2$ string has only $p^2 = 0$ as the only ground state. For the four dimensional Euclidean space as the target space there will be each $p_i$ zero so that there will be no dynamics and hence this theory is not at all interesting. For the non-trivial dynamics of the physical ground state, the target space need to have an $SO(2, 2)$ symmetry so that momenta can be complex. Also only non-vanishing S-matrix amplitude is the three point function and all higher n-point functions are zero due to the kinematics of $(2, 2)$ space. So this massless scalar of the ground state of Open/closed string describes the dynamics of self-dual Yang-Mills/ Gravity which is topological in nature. Before proceeding further it is worthwhile to address the space-time supersymmetry of the $\mathcal{N} = 2$ fermionic string. The three point function is only invariant under the subgroup $U(1, 1)$ of the total Lorentz group $SO(2, 2)$ of the target space. This makes the the space-time supersymmetry of $\mathcal{N} = 2$ critical string theory obscure. However the ground states of the fermionic string with $\mathcal{N} = 4$ world-sheet supersymmetry has larger symmetry and is covariant under $SU(1, 1) \times SU(1, 1)$. Since the highest weight states of $\mathcal{N} = 4$ Superconformal algebra are the same as the physical states of $\mathcal{N} = 2$ algebra, it has been argued that both the theories are one and the same [10]. The four world-sheet fermions with the combinations of both periodic and antiperiodic boundary conditions have sixteen bosonic and sixteen fermionic states as the ground states for the closed string and the spectrum is clearly
supersymmetric. To realize explicit Lorentz covariant one needs these extra missing symmetry $SO(2, 2)/SU(1, 1)$ which happened to be $SL(2, R)$ as twistor variables. The Brink and Schwarz [17] action for $\mathcal{N} = 2$ string is:

$$
S = \int d^2 \sigma \sqrt{g} \left\{ \frac{1}{2} g^{ab} \partial_a x^i \partial_b \bar{x}^j + \frac{i}{4} \bar{\psi}^i \gamma^a \overset{\leftrightarrow}{D}_a \psi^i + A_a \bar{\psi}^i \gamma^a \psi^i 
+ (\partial_a \bar{x}^j + \bar{\psi}^i \chi_a^i) \chi_b^a \gamma^a \gamma^b \psi^i + h.c. \right\} \eta_{ii},
$$

(1)

where $g^{ab}$ and $A_a$ with $a=0,1$, are the (real) worldsheet metric and $U(1)$ gauge connection, respectively and $D_a$ is the gravitational covariant derivative for the spinor with spin connection in two dimension and $\gamma_a$ are the worldsheet gamma matrices. Here $x^i$ and $\psi^i$ are complex valued in the target space with $\eta_{ii}$ metric which has a complex structure. In the $\mathcal{N} = 2$ superconformal gauge one has both left and right hand sectors of superconformal constraint algebras denoted as $T, G^+, G^-$ and $J$ and $\tilde{T}, \tilde{G}^+, \tilde{G}^-$ and $\tilde{J}$ respectively. In both the sectors one has usual $(b,c)$ ghosts, a pair of $(\beta_{\pm}, \gamma_{\pm})$ ghosts and a pair of $(u,v)$ ghosts for $U(1)$ gauge fixing. The cancelation of anomaly dictates the central charge of the matter sector to be 6. It is more convenient for our purpose to use $SL(2, R) \times SL(2, R)'$ basis instead of complex coordinate basis for the target space. In $SL(2, R) \times SL(2, R)'$ spinor notation

$$
x_{\alpha \dot{\alpha}} = \sigma_{\mu}^{\alpha \dot{\alpha}} x^\mu = \left( \begin{array}{cc} x^4 + x^2 & x^1 - x^3 \\ x^1 + x^3 & x^4 - x^2 \end{array} \right), \quad \alpha \in \{0, 1\}, \quad \dot{\alpha} \in \{0, 1\},
$$

(2)

where $\sigma_{\mu}^{\alpha \dot{\alpha}}$ describes the chiral $\gamma$ matrices. However in these basis the associated constraints in the left moving sectors are

$$
T = \partial_z x^\alpha \dot{x}_\alpha + \psi^{\dot{\alpha}} \partial_z \psi^\alpha,
$$

(3)

$$
G^{0 \dot{1}} = \psi^{\dot{\alpha}} \partial_z x^0_{\dot{\alpha}}, \quad G^{1 \dot{0}} = \psi^{\dot{\alpha}} \partial_z x^1_{\dot{\alpha}},
$$

(4)

$$
J^{0 \dot{1}} = \psi^{\dot{\alpha}} \psi^\alpha.
$$

(5)

There are similar constraints for the right moving sectors as well. Here $\alpha, \beta, \dot{\alpha}, \dot{\beta}$ are space-time spinor indices and $\bar{\alpha}, \bar{\beta}$ are world-sheet $SL(2, R)'$ internal indices for the $\mathcal{N}=2$ R-symmetry. The operator product expansions for $X$ and $\psi$ are

$$
x_{\alpha \dot{\alpha}}(z, \bar{z}) x_{\beta \dot{\beta}}(w, \bar{w}) \sim \epsilon^{\alpha \beta} \epsilon^{\dot{\alpha} \dot{\beta}} \ln |z - w|^2, \quad \psi^{\dot{\alpha} \beta}(z) \psi^\gamma \delta(w) \sim \frac{2 \epsilon^{\dot{\alpha} \gamma} \epsilon^{\beta \delta}}{z - w}.
$$

(6)

In case we bosonize the fermions, the $U(1)$ current $J^{0 \dot{1}}$ becomes $\partial H$. It is noticed by Berkovits and Vafa [?] that commutation relations among $\exp \left\{ H \right\}$,
exp \((-H)\) and \(\partial H\) form an \(SU(2)\) or \(SU(1,1)\) algebra. Incorporating these as new currents, \(\mathcal{N} = 2\) superconformal algebra is extended to small \(\mathcal{N} = 4\) superconformal algebra as

\[
G_\alpha^{\dot{\beta}} = \psi^{\dot{\alpha}} \partial_z x^\alpha_{\dot{\gamma}} \,, \quad J^{\dot{\alpha} \dot{\beta}} = \psi^{\dot{\alpha}} \dot{\psi}^{\dot{\beta}} \,.
\]  

(7)

There is an elegant way to make the theory with vanishing central charge without ghosts by twisting the energy-momentum tensor \(T \rightarrow T' := T + \frac{1}{2} \partial J^{01}\). By this the conformal weight of the spinor \(\psi^{0 \dot{0}}\) has become zero and that of \(\psi^{\dot{0} \dot{1}}\) has become one. Both the fermions are now integer moded and also fermion of zero weight has zero modes on the sphere. Due to the twisting the curvature singularities of the world sheet are taken care of by these twisted fermions rather than the ghosts. In a sense the \(U(1)\) anomaly plays the role of the ghost anomaly and \(\psi's\) play the role of the \((b,c)\) ghosts. Consequently \(G_\alpha^{\dot{0}}\) has spin one with conformal weight one and \(G_\alpha^{\dot{1}}\) has spin two with conformal weight two. Hence \(G_\alpha^{\dot{0}}\) play the role of BRST-type of currents. Similarly \(J^{00}\) and \(J^{11}\) have conformal weight zero and two respectively. This theory has now \(\mathcal{N} = 4\) worldsheet super symmetry and also has \(SL(2,R) \times SL(2,R)'\) space-time symmetry. In order to formulate \(\mathcal{N} = 8\) supergravity we keep one of the \(SL(2,R)\) symmetries rigid and another to extend it to \(O(8 | 2)\) and consequently gauge it. This needs eight fermionic variables to form the superspace. Infact the ground state closed string spectrum has eight such states. However it is quite obscure to use these as super coordinates. Following Siegel [11] we introduce eight fermionic coordinates \(\theta^{\alpha i}\) as super partner of \(x^{\alpha i}\), which are worldsheet scalars with non-holomorphic correlation function. Inorder to maintain the conformal invariance a set of bosonic variables \((b^i, c^j)\) are introduced. The operator product expansions of \(\theta's\) and for \((b^i, c^j)\) are

\[
\theta^{\alpha i}(z, \bar{z}) \theta^{\beta j}(w, \bar{w}) \sim \epsilon^{\alpha \beta} \delta_{ij} \ln |z - w|^2 \quad , \quad b^i(z) c^j(w) \sim \frac{\delta_{ij}}{z - w} \,.
\]  

(8)

The associated stress tensor is

\[
T = \partial_z x^{\alpha \beta} \partial_{\bar{z}} x_{\alpha \beta} + \psi^{\dot{\alpha} \dot{\beta}} \partial_z \psi_{\dot{\alpha} \dot{\beta}} + \frac{1}{2} \partial_z \theta^{\alpha i} \partial_{\bar{z}} \theta_{\alpha i} + b^i \partial_z c^j \,.
\]  

(9)

and also similarly its super partners

\[
G^{\alpha \dot{1}} = \psi^{\dot{\alpha} \dot{i}} \partial_z x^\alpha_{\dot{i}} + c^i \partial_z \theta^{\alpha i} \,, \quad G^{\dot{0} \dot{0}} = \psi^{\dot{\alpha} \dot{0}} \partial_z x^\alpha_{\dot{0}} + c^i \partial_z \theta^{\alpha i} \,,
\]  

(10)

\[
J^{\dot{0} \dot{1}} = \psi^{\dot{\alpha} \dot{0}} \dot{\psi}^{\dot{\alpha} \dot{1}} + c^i b^i \,.
\]  

(11)
However the OPE of these $J^{\tilde{\alpha}\tilde{\beta}}$'s are singular as $\frac{2+\mathcal{N}}{\bar{z}}$ indicating a $U(1)$ anomaly with charge $-(2+\mathcal{N})$ which are 2 due to $\psi^{\tilde{\alpha} \bar{\beta}}$ and $\mathcal{N}$ due to $(b^{i}, \hat{c}^{i})$ systems respectively. In the sequel we will show the proper insertions of these $\psi^{i}$'s and $c^{i}$'s for taking the vacuum expectation values of the vertex operators. (For the closed string all the right moving anti-holomorphic fields are described as hatted variables.)

3. Covariant Closed String Field theory:

The most conventional approach to covariant String field theory is Witten’s Open String Field theory (OSFT) [14], where strings join together at their mid point. The gauge invariance of the OSFT demands only a cubic term in its interaction. This procedure of joining and splitting of interacting strings maintains the length of the string intact. If one extends this approach to closed string and tries to maintain the length of the interacting string fixed then the required prescription would be to overlap one of the closed strings with the other half of another closed string by that the resulting closed string will have the same length as that of the overlapping strings. The gauge invariance of this type of gluing procedure demands the action to be non-polynomial in string field [20]. There is another school of thought for covariant string field theory pursued by Hata et al. [15], who generalized the light-cone string field theory to covariant one by BRST procedure. In this approach the open strings join at their end points and closed string join together at one point only. The gauge invariance of the string fields requires that closed string should have cubic interaction where as open string should have non-polynomial interaction term. However by this the string length cannot be held fixed hence behave as a free parameter in the theory. Although this parameter is easily gauged away in the on-shell limit, the ambiguity remains for the off-shell theory. Despite this ambiguity we follow this formulation for the simplicity due to the cubic interaction.

3.1 Preliminaries of closed string field theory action:

The basic string field $\Phi$ is a functional of the worldsheet string coordinates $\{x^{\alpha\beta}, \psi^{\tilde{\alpha}\tilde{\beta}}\psi^{\tilde{\alpha}\bar{\beta}}, \theta^{\alpha i}, b^{i}, \hat{b}^{i}, c^{i}, \hat{c}^{i}\}$. The basic ingredients needed for the gluing and splitting of the closed strings are prescription dependent and these are presented in the appendix. For illustration we present here the Berkovits $\mathcal{N} = 2$ closed string theory [16]. For the gluing of string fields, a curvature is created at the joining points which is related to the $U(1)$ anomaly as explained in the appendix. The back-ground charge due to the $U(1)$ current which is equi-
lent to ghost current of \((b, c)\) system is \(-\frac{D}{2}\) which happens to be -2 in this case. One needs insertion of zero modes of \(\psi^{\bar{\alpha}0}\) to compensate this charge in each vertex operator. Taking this into account Berkovits [16] has proposed a closed string field theory action for \(\mathcal{N} = 2\) string field theory which kinetic term is given as:

\[
S = \int \langle \Phi (J^{\bar{0}0} + \hat{J}^{\bar{0}0}) (G^{0\bar{0}} + \hat{G}^{0\bar{0}}) (G^{1\bar{0}} + \hat{G}^{1\bar{0}}) \Phi \rangle \quad (12)
\]

Here subscript zero denotes the zero modes of the operators. More explicitly \(G^{0\bar{0}} = \psi^{\bar{\alpha}0} \frac{\partial}{\partial x^{\bar{\alpha}}}\) and similarly \(\hat{G}^{0\bar{0}} = \hat{\psi}^{\bar{\alpha}0} \frac{\partial}{\partial x^{\bar{\alpha}}}\) and using the contraction \(\langle \psi^{\bar{\alpha}0} \psi_{\alpha}^{\bar{0}} \rangle\) one gets the correct kinetic term as

\[
K.E = \epsilon^{\alpha\beta} \epsilon^{\bar{\alpha}\bar{\beta}} \frac{\partial}{\partial x^{\alpha}} \Phi \frac{\partial}{\partial x^{\beta}} \Phi. \quad (13)
\]

The gluing of three string \(A, B\) and \(C\) satisfy \((A(BC)) = ((BA)C)+(AC)B\). The interaction term for covariant closed string is given as

\[
S_{int} = \int [(\hat{G}^{\bar{0}0} - \hat{G}^{1\bar{0}}) \Phi, (G^{0\bar{0}} + \hat{G}^{0\bar{0}})(G^{1\bar{0}} - \hat{G}^{1\bar{0}}) \Phi] (G^{0\bar{0}} + \hat{G}^{0\bar{0}}) \Phi \quad (14)
\]

where \([,]\) is anti-symmetric product. After a lengthy calculation taking into account all sorts of contractions and permutations we arrive at the required result which gives the Plebanshki action as:

\[
S_{int} = \int \epsilon^{\alpha\beta} \epsilon^{\bar{\gamma}\bar{\delta}} \frac{\partial}{\partial x^{\alpha}} \Phi \frac{\partial}{\partial x^{\beta}} \Phi \frac{\partial}{\partial x^{\bar{\gamma}}} \Phi \frac{\partial}{\partial x^{\bar{\delta}}} \Phi \quad (15)
\]

where \(\Phi\) is the pre-potential.

### 3.2 Berkovits-Siegel covariant string field theory for self-dual Yang-Mills:

Berkovits and Siegel [17] have proposed a manifestly Lorentz covariant string field theory action for self-dual Yang-Mills theory where string fields are the gauge connection \(A_\alpha\) which makes the BRST operator \(Q_\alpha\) gauge invariant and the auxiliary field \(g_{\alpha\beta}\) which enforces the self-dual constraint. To illustrate further – the gauge covariant derivative \(\nabla_\alpha = G_\alpha^{\bar{\alpha}} + A_\alpha\) and under the gauge transformation

\[
\nabla'_\alpha = \Omega \nabla_\alpha \Omega^{-1}, \quad g'_{\alpha\beta} = \Omega g_{\alpha\beta} \Omega^{-1} + \nabla^\gamma K_{\gamma\alpha\beta} \quad (16)
\]
where $\Omega$ and $K_{\gamma\alpha\beta}$ are gauge parameters of the internal symmetry of the Yang-Mills gauge group such that the Lorentz invariant string field theory action remains gauge invariant.

$$S = \int <\text{tr} g_{\alpha\beta} f^{\alpha\beta}>.$$  \hspace{1cm} (17)

Here $f$ is for integral over zero modes of $x$ space and $\langle,\rangle$ is the expectation value of string field operator products where massive unphysical modes contribute in the off-shell, and

$$f_{\alpha\beta} = \{\nabla_\alpha, \nabla_\beta\} = G^0_{\alpha\beta} A_\beta + G^0_{\beta\alpha} A_\alpha + \{A_\alpha, A_\beta\}.$$  \hspace{1cm} (18)

The $\{,\}$ denotes the symmetric product of the string fields and each product is the Witten’s OSFT star product. Also

$$(G^0_{\alpha\beta}(z)) = \oint_z \frac{dw}{2\pi i} G^0_{\alpha\beta}(w) A_\beta(z).$$  \hspace{1cm} (19)

The equation of motion for the string fields from (13) gives

$$f_{\alpha\beta} = 0, \quad G^0_{\alpha\beta} g^{\alpha\beta} + [A_\alpha, g^{\alpha\beta}] = 0.$$  \hspace{1cm} (20)

Berkovits and Siegel [17] have also generalized this action for the $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory. More recently Lechtenfeld and Popov [8] have given a covariant cubic string field theory action for the self-dual $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory in the supertwistor space.

3.3 Supertwistor space:

As mentioned in the section 2, the perturbative string spectrum has only manifest $SU(1, 1)$ instead of space-time $SO(2, 2)$ Lorentz symmetry. This missing $SO(2, 2)/SU(1, 1)$ symmetry is treated as twistor transformation, which will be introduced to rotate all the complex structure of the $SO(2, 2)$ and finally integration over this parameter space will ensure the Lorentz covariance. This geometric space $\mathcal{H}^2 = SO(2, 2)/SU(1, 1)$ is a two sheeted hyperboloid. For the Euclidean space $R^4$ the quotient space $SO(4)/U(2)$ parametrizes all the complex structure happened to be the Riemann sphere $CP^1$. Atiyah, Hitchin and Singer [13] describe the twistor space of $R^4$ as a complex manifold $\mathcal{P}_E = R^4 \times CP^1$, which is a rank 2 vector bundle over $CP^1$. Similarly the twistor space of $R^{2,2}$ is a complex manifold $\mathcal{P}_H$ which is $R^{2,2} \times \mathcal{H}^2$. This space is covered by two (acyclic i.e. in one of the patches $\lambda^0$ is zero and in another patch $\lambda^1$ is zero.
coordinate patches $U_{\pm}$ with complex coordinates $(\omega^\alpha_{\pm}, \lambda^\alpha_{\pm})$ on each patches where

$$
(\lambda^\pm_a) = \begin{pmatrix} 1 \\ \lambda^\pm \end{pmatrix}, \quad (\lambda^\alpha_{\pm}) = \begin{pmatrix} -\lambda^\pm \\ 1 \end{pmatrix}, \quad (\bar{\lambda}\,^\beta_{\pm}) = \begin{pmatrix} 1 \\ -\bar{\lambda}\,^\pm \end{pmatrix}
$$

with the normalization

$$
\nu_+ = (1 - \lambda^\pm \bar{\lambda}\,^\pm)^{-1}, \quad \nu_- = -(1 - \lambda_- \bar{\lambda}\,_-)^{-1}
$$

and $\omega^\pm_a = x^{\alpha\bar{\alpha}} \lambda^\pm_a$. On the overlap $U_- \cap U_+$ it is $\lambda_+ = \lambda_-$. Using the above normalization the inverse transformation of $x^{\alpha\bar{\alpha}}$ in terms of $\omega^\pm_a$ is easily found. This shows the reparametrization of the twistor space. The transformations of anti-holomorphic vector fields and coordinate vector fields are

$$
\frac{\partial}{\partial \bar{w}^\beta_{\pm}} = \nu^\pm_{\alpha\beta} \frac{\partial}{\partial x^\alpha}, \quad \frac{\partial}{\partial \bar{w}^\beta_{\pm}} = \frac{\partial}{\partial \lambda^\pm} - x^{\alpha\bar{\alpha}} \lambda^\alpha \frac{\partial}{\partial x^\alpha}. \tag{23}
$$

Thus the anti-holomorphic vector fields are

$$
\bar{v}^\pm_a = \lambda^\alpha_{\pm} \frac{\partial}{\partial x^\alpha}, \quad \bar{v}^\pm_i = \frac{\partial}{\partial \lambda^\pm}. \tag{24}
$$

Similarly one can construct holomorphic vector fields. Let us extend this approach to superspace geometry. For the graded space $\{x^{\alpha\bar{\alpha}}, \theta^{\alpha i}\}$, one has a graded twistor space $PS_H$ which is $\mathbb{R}^{(2,2)} | N \times H^2$. The local coordinates on patches $U_{S\pm}$ are $(\omega^A_{\pm}, \lambda^\alpha_{\pm})$ where $\omega^A_{\pm} = [x^{\alpha\bar{\alpha}} \lambda^\pm_a \lambda^\pm \theta^{ai}]$ The anti holomorphic vector fields are

$$
\bar{v}^\pm_a = \lambda^\alpha_{\pm} \frac{\partial}{\partial x^\alpha}, \quad \bar{v}^\pm_i = \lambda^\alpha_{\pm} \frac{\partial}{\partial \theta^ai}. \tag{25}
$$

### 3.4 BRST formulation in Supertwistor space:

Here we treat the twistorial parameter $\lambda^\alpha$ as a constant complex parameter unlike the usual twistor string theory where the super-twistor space is the target space and also constrained to be a superconformal theory. In this sense $\lambda^\alpha$ can be treated as a zero mode. (However we can always treat $\lambda^\alpha$ and its canonical conjugate as a conformal theory with additional ghosts for conformal invariance.) As mentioned in section 2. $G^{\alpha\bar{\alpha}}$ has spin one and is treated as the BRST current for the left movers and similarly $\hat{G}^{\alpha\bar{\alpha}}$ for the right movers. So

$$
Q_{BRS}^\alpha = \int dz \ G^{\alpha\bar{\alpha}}(z), \quad \hat{Q}_{BRS}^\alpha = \int \bar{d}z \ \hat{G}^{\alpha\bar{\alpha}}(\bar{z}). \tag{26}
$$
For the closed string theory let us define

\[ Q = \lambda_\alpha \left( Q^\alpha_{BRS} + \hat{Q}^\alpha_{BRS} \right) \]

which gives

\[ Q = \lambda_\alpha \left[ \left( \psi^{\hat{\alpha}0} + \hat{\psi}^{\hat{\alpha}0} \right) \frac{\partial}{\partial \psi^{\hat{\alpha}0}} + \left( c^i + \hat{c}^\ell \right) \frac{\partial}{\partial \theta^i} \right] \]

The BRST operator is equivalent to the \((0,1)\) vector fields as described in eq.(25). If we assume that the string field also depends on the twistor variable \(\lambda\), then inorder that the gauge invariance in the entire twistor space is maintained we have to add to the \(Q\) another antiholomorphic differential \(Q = d\bar{\lambda} H_{\bar{\lambda}}\). One can easily show that \(Q^2 = 0\), \(Q_{\bar{\lambda}}^2 = 0\) and also the anticommutator of \(\{Q, Q_{\bar{\lambda}}\} = 0\) due to the Berezinian nature of \(\psi, \theta\) and \(\lambda\). The desired vertex operator in this BRST cohomology is

\[ V = \lambda^\alpha \left[ \left( \psi^{\hat{\alpha}0} + \hat{\psi}^{\hat{\alpha}0} \right) H_{\alpha \alpha} + \left( c^i + \hat{c}^\ell \right) H_{\alpha i} \right] + d\bar{\lambda} H_{\bar{\lambda}}. \]

From eq.(9) and eq.(11) we show that \(\psi^{\hat{\alpha}0}\) has conformal spin zero and charge one and also for \(c^i\)’s. (Henceforth we will use \(\psi^{\hat{\alpha}0}\) as \(\psi^{\hat{\alpha}}\) for convenience). The vacuum expectation of these zero modes are

\[ \langle \psi^{\hat{\alpha}} \psi^{\hat{\beta}} \Pi_{i=1..N} c_0^i \rangle = \epsilon^{\hat{\alpha} \hat{\beta}} \]

and also similarly for right moving hatted variables. Let us define \(\theta^{\alpha i} = c_0^j \theta^{\alpha i}\) just to make \(\theta^{\alpha i}\) as fermionic coordinates of charge one. (Here any one of the zero modes of \(c^i\) will do. We can work on either of the coordinate patches say on \(U_-\) and drop the - sign from \(\lambda^\alpha\)). For the open string and \(N = 4\), we have

\[ H_{\alpha \alpha} = A_{\alpha \alpha} + \theta^j_\alpha \bar{\xi}^{\alpha j} + \theta^j_\alpha \theta^{\beta k} \partial_{\beta \alpha} \phi^{lm} \epsilon_{jklm} \]

\[ + \theta^j_\alpha \theta^{\beta k} \theta^{\gamma l} \partial_{\beta \alpha} \xi^{\gamma l} \epsilon_{jklm} + (\theta^4_\alpha)^2 \theta^i_\alpha \theta^j_\alpha G_{\gamma \delta} \]

\[ H_{\alpha i} = (\theta^j_\alpha \phi^{kl} + \theta^j_\alpha \theta^k_\beta \xi^{\gamma l} + \theta^j_\alpha \theta^k_\beta \theta^\ell_\delta G^{\alpha \delta}) \epsilon_{ijkl}, \]

and

\[ H_{\lambda} = \left( \frac{1}{2} \nu^2 \lambda^\alpha \lambda^\beta \theta^i_\alpha \theta^j_\beta \phi^{kl} + \frac{\nu^3}{3!} \lambda^\alpha \lambda^\beta \lambda^\gamma \theta^i_\alpha \theta^j_\beta \theta^k_\gamma \lambda^\ell \xi^{\alpha l} \right) \]

\[ + \frac{\nu^4}{4!} \lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta \theta^i_\alpha \theta^j_\beta \theta^k_\gamma \theta^\ell_\delta \lambda^\kappa \lambda^\lambda G^{\sigma \kappa} \epsilon_{ijkl} \]
Witten's OSFT action is given as

\[ S = \int Tr \langle VQV + \frac{2}{3}V^3 \rangle. \]  

(34)

where \( f \) denotes integration over all the zero modes namely \( \int d\lambda d^4x d^4\theta \) and \( Tr \) is the Trace over the Chan-Paton factors. Using vacuum expectation values (eq.(30)) and integrating over the fermionic variables and \( d^2\lambda \) one gets

\[ S = \int d^4x \left[ G^{\alpha\beta} (\partial^\alpha A_{\beta}^\dagger + \partial_{\beta} A_{\alpha}^\dagger + [A_{\alpha\beta}^\dagger, A_{\alpha\beta}]) \right. 
\]

\[ + \xi^\alpha j \nabla_{aa} \tilde{\xi}_{\alpha j} + \nabla_{aa} A_{\alpha j} \right], \]  

(35)

where \( \nabla_{aa} = \partial_{aa} + A_{aa} \). This is the \( \mathcal{N} = 4 \) self-dual super Yang-Mills action.

3.5 Closed String field theory:

The usual closed string field theory action [15] is given as

\[ S = \frac{1}{2} V \cdot QV + \frac{g}{3} V \cdot V \ast V, \]  

(36)

where \( \cdot \) and \( \ast \) are explained in the appendix. Also invariant under the BRS transformation

\[ \delta_B V = QV + gV \ast V. \]  

(37)

Conventional bosonic closed string theory is quite subtle due to the gost number conservation. However our case here is further complicated due to the unusual closed string vacuum which we chose has charge -20. So gauge invariance is unusual which we will discuss in the sequel.

For convenience let us rescale each \( \theta \) variable with the zero modes of the \( c_0^i \). Let us define

\[ \theta^i = c_0^i \hat{c}_0^i \lambda_\alpha \theta^{\alpha i} \]  

(38)

where \( \lambda \) is chosen on either of the coordinate patches of the twistor space. Now each \( \theta^i \) has charge +2. To avoid clutter let us define

\[ \theta^i = \theta^i, \quad \theta^{ij} = \frac{1}{2!} \theta^i \theta^j, \quad \theta^{ijk} = \frac{1}{3!} \theta^i \theta^j \theta^k, \]

\[ \theta^{ijkl} = \frac{1}{4!} \theta^i \theta^j \theta^k \theta^l, \quad \theta_{ijklm} = \frac{1}{5!} \epsilon_{ijklmnp} \theta^m \theta^n \theta^p \theta^q, \]
\[
\theta_{ij} = \frac{1}{6!} \epsilon_{ijklmnop} \theta^k \theta^l \theta^m \theta^n \theta^o \theta^p, \quad \theta_i = \frac{1}{7!} \epsilon_{ijklmnop} \theta^j \theta^l \theta^m \theta^n \theta^o \theta^p,
\]

\[
\theta^8 = \frac{1}{8!} \epsilon_{ijklmnop} \theta^i \theta^j \theta^k \theta^l \theta^m \theta^n \theta^o \theta^p.
\]

The \(2^N\) component fields will appear in the expansion of \(H_{aa}^\ast\), \(H_{ai}\) and \(H_{\bar{\lambda}}\) in terms \(\theta^i\) s.

\[
H_{aa}^\ast = h_{aa} + \theta^i \tilde{\lambda}_a \xi_{ai} + \theta^{ij} A_{ai} + \theta^{ij} \tilde{\lambda}_a \chi_{aij} + \theta^{ijkl} \partial_{ai} \cdot \phi_{ijkl}
\]

\[
+ \theta_{ijkl} \tilde{\lambda}_a \chi_{ij} \chi_{kl} + \theta_{ijkl} \tilde{\lambda}_a \tilde{\lambda}_b \tilde{\lambda}_c \tilde{\lambda}_d G^\beta \gamma ij + \theta_i \tilde{\lambda}_a \partial_{\alpha} \cdot \tilde{\xi}_i \Omega_{\gamma\delta}
\]

(39)

Similarly

\[
H_{\bar{\lambda}} = \theta^{ijkl} \phi_{ijkl} + \theta_{ijkl} \tilde{\lambda}_a \tilde{\lambda}_b \tilde{\lambda}_c \tilde{\lambda}_d G^\alpha \Omega_{\gamma\delta}
\]

\[
+ \theta_i \tilde{\lambda}_a \tilde{\xi}_i + \theta^8 \tilde{\lambda}_a \tilde{\lambda}_b \Omega_{\alpha\beta}
\]

(40)

Let us first fix the Kinetic energy part of the string field theory action. As explained in the Appendix the dot product for the conventional bosonic string fields satisfies

\[
\Psi \cdot \Phi = \langle R(1, 2) | b_0^{-2} | \Psi_2 \rangle |\Phi_1 > (41)
\]

due to that the kinetic part of the action \(\frac{1}{2} V \cdot Q V\) is defined as \(\int \langle V | (c_0 - \hat{c}_0) Q V \rangle\).

When one imposes the linear gauge invariance \(\delta V = Q \Lambda\) this will be invariant only when \((b_0 - \hat{b}_0)V = (b_0 - \hat{b}_0) \Lambda = 0\). This condition is imposed as the gauge condition on string field. In this case for the vacuum charge to be maintained we prescribe (as has done in section 3.1)

\[
\int \langle V | (\tilde{J}_0^{\tilde{0} \tilde{0}} + \tilde{J}_0^{\tilde{0} \tilde{0}}) Q V \rangle.
\]

(42)

Demanding the gauge invariance \(\delta V = Q \Lambda\) we can verify that the gauge condition happens to be either \((\tilde{J}_0^{\tilde{0} \tilde{0}} - \tilde{J}_0^{\tilde{0} \tilde{0}}) Q V = 0\) or in otherwords

\[
(G^\alpha \tilde{\gamma} - \tilde{G}^\alpha \tilde{\gamma}) V = 0
\]

(43)

The \(G^\alpha \tilde{\gamma}\) are weight 2 operators which is equivalent to \(b_0\). \(H_{\bar{\lambda}}\) part of the field does nothave \(\psi^{\alpha}\) part and contains all the auxiliary fields. After contractions
of $\psi^\alpha$ we get

$$S_{K.E} = \int d^4x \, d^8\theta d^2\lambda \left\langle \lambda^\alpha \lambda^\beta \epsilon^\alpha \epsilon^\beta \bar{H}_\lambda \left( \partial_{\hat{\alpha}} H_{\beta\hat{\beta}} \right) \right\rangle$$

(44)

The fermionic integration will give the Kinetic energy part of all the fields of positive helicity. For the bosonic truncation let us take only $\theta^8$ term of $H_\lambda$ which gives

$$\int d^4x \, \epsilon^{\alpha\beta} \Omega_{\alpha\beta} \partial_{\hat{\alpha}} h_{\beta \hat{\beta}}$$

(45)

The interaction part of the action is $\frac{g}{3} V \cdot V \ast V$ where $\ast$ operation and its combinatorics are defined in the Appendix. As mentioned earlier three string combinatorics are $A(BC) = ((BA)C) + ((AC)B)$. Since $H_{\alpha\alpha}$ is fermionic and $H_{\lambda}$ is bosonic we have the interaction term which will be glued are in the form be $[H_{\alpha\alpha} H_{\beta\beta}, - H_{\beta\beta} H_{\alpha\alpha}] \cdot H_{\lambda}$. Demanding the gauge condition

$$\left( G^{\alpha\dot{\alpha}} - \hat{G}^{\alpha\dot{\alpha}} \right) \left[ H_{\alpha\alpha} H_{\beta\beta} - H_{\beta\beta} H_{\alpha\alpha} \right] = 0 ,$$

(46)

shows that the two form $[H, H]$ is a constant and hence has a poisson structure. More explicitly this will be satisfied if the antisymmetric bracket will be taken with each form operated by its conjugate $\left( G^{\alpha\dot{\alpha}} - \hat{G}^{\alpha\dot{\alpha}} \right)$. After performing the vacuum contractions of $\psi$ we get

$$S_{int} = \int d^4x \, d^8\theta \, d^2\lambda \, \lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} \bar{H}_\lambda \left( \partial_{\hat{\alpha}} H_{\beta\beta} \partial_{\gamma\gamma} H_{\delta\delta} \right)$$

(47)

So far we have not taken the gauge part of the fermionic coordinate. Similarly we shall expand $H_{\alpha\alpha}$ as in the previous case for the $N = 4$ super Yang-Mills. Going through the same exercise we shall get interaction term in the super space as

$$S_{int} = \int d^4x \, d^8\theta \, d^2\lambda \, \lambda^\alpha \lambda^\beta \lambda^\gamma W^{AB} W^{CD} H_{\lambda} \left( \partial_{\alpha A} H_{\beta C} \partial_{\gamma B} H_{\delta D} \right)$$

(48)

where $W^{AB}$ is the metric in the super-space as $\{ \epsilon^{\alpha\beta} , \delta_{ij} \}$. This is the main result which resembles the Plebanski equation in the super-space. One can make the bosonic truncation after taking only the $\theta^8$ component of the $H_\lambda$ and get the usual Plebanski equation. One also get the Prepotential equation of Karanas and Ketov [21] once one identifies $H_{\alpha\alpha\alpha\alpha d} = \partial_{\alpha \alpha} \Phi$ where $\Phi$ is the prepotential.

4. The Conclusion:
We have given here a closed string field theory formulation of $\mathcal{N} = 8$ self-dual supergravity. This is completely off-shell and Lorentz covariant. The field theory limit gives the correct action which was earlier formulated by Karanas and Ketov\cite{21} and recently by Mason and Wolf \cite{18} in the super twistor space. Unlike the conventional twistor string formulation of recent times \cite{5, 7, 19} and \cite{7, 19} where twistor space is the target space we treat here the twistor variable as a complex parameter of mixing the world sheet super conformal generators. We will calculate the N point MHV amplitudes using the Witten and Nair’s \cite{5, 19} principle which enables us to calculate the higher point function in the twistor space although without the twistor space more than 3 point function vanishes.

Appendix

In this appendix we present very basic properties of Closed string field theories. For any generic string functionals $\Phi_i$ $(i = 1, 2, 3)$, the dot and the star products have the following properties:

$$
\Phi_1 \cdot \Phi_2 = (-)^{|1||2|} \Phi_2 \cdot \Phi_1,
$$

$$
\Phi_1 \star \Phi_2 = -(-)^{|1||2|} \Phi_2 \star \Phi_1,
$$

$$
\Phi_1 \cdot (\Phi_2 \star \Phi_3) = (-)^{(|1|(|2|+|3|))} \Phi_2 \cdot (\Phi_2 \star \Phi_1),
$$

$$
\Phi_1 \cdot Q_{BRS} \Phi_2 = -(-)^{|1|}(Q_{BRS} \Phi_1) \cdot \Phi_2,
$$

where $|i|$ $(i = 1, 2, 3)$ is 0 (1) if $\Phi_i$ is Grassmann- even or odd.

A1. The Reflector:

Let $| \Phi_i >$ is expanded as a basis in a Hilbert space and let $< \Phi_i |$ be its dual then $< \Phi_i | \Phi^j > = \delta_{ij}$. Thus the dot product is defined as

$$
\Psi \cdot \Phi = < R(1, 2) | \Psi_2 > | \Phi_1 >
$$

The reflector maps the state $| O >$ to its BPZ conjugate state. More explicitly if two string states are described by two punctured spheres with a uniformizing coordinate say $z$. The local coordinate $z_1 = z = 0$ and another with local coordinate $z_2 = \frac{1}{z} = 0$ So

$$
< R(1, 2) | \Psi_i(1) > | \Phi_j(2) > = < \Psi_i(z_2 = 0) \Phi_j(z_1 = 0) > = G_{ij}
$$

This will be non-vanishing of only when the total ghost number of the matrix element vanishes. The closed string vacuum has ghost number - 6. Since any
physical bosonic closed string has ghost number 2, inorder that the dot product will be non-vanishing one defines that

\[ \Psi \cdot \Phi = \langle R(1, 2)b_0^{-2} | \Psi_2 | \Phi_1 \rangle \]  \hspace{1cm} (55)

where \( b_0^{-} = b_0 - \hat{b}_0 \).

A2. The \( \star \) Product:

The \( \star \) product is defined as mapping from two string fields to one string field such as

\[ | \Phi \rangle \langle \Psi | = | \Phi \star \Psi \rangle \]  \hspace{1cm} (56)

For the three point vertex in the language of conformal field theory [22], which is written as three conformal mapping of \( \{ h_r | r = 1, 2, 3 \} \) from a unit disk to sphere with coordinates \( z \) as

\[ \langle \mathcal{V}_{123} || \phi_1 > | \phi_2 > | \phi_3 > \rangle = \langle h_1(\phi_1)(z_1)h_2(\phi_2)(z_2)h_3(\phi_3)(z_3) \rangle_{S^2} \]  \hspace{1cm} (57)

Each map \( h_r \) is a conformal transformation of the vertex operator at the origin on the disk to one on the sphere.
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