SOME SUPPORTS OF FOURIER TRANSFORMS OF SINGULAR MEASURES ARE NOT RAJCHMAN

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Abstract. The notion of Riesz sets tells us that a support of Fourier transform of a measure with non-trivial singular part has to be large. The notion of Rajchman sets tells us that if the Fourier transform tends to zero at infinity outside a small set, then it tends to zero even on the small set. Here we present a new angle of an old question: Whether every Rajchman set should be Riesz.

1. Introduction

The consideration of the properties of measures and their Fourier transforms is a classical area of Harmonic Analysis. In particular the following is well known.

Theorem 1.1 (Rajchman, 1929 [4]). If for a finite measure μ on the unit circle T holds \( \hat{\mu}(n) \to 0 \) when \( n \to -\infty \), then it holds also that \( \hat{\mu}(n) \to 0 \) when \( n \to +\infty \).

This motivates the following.

Definition 1.2. We say that Λ ⊂ \( \mathbb{Z} \) is a Rajchman set if as soon as \( \hat{\mu}(n) \to 0 \) when \( |n| \to +\infty, n \in \mathbb{Z} \setminus \Lambda \), then \( \hat{\mu}(n) \to 0 \) when \( |n| \to +\infty, n \in \Lambda \).

With this definition the Rajchman theorem says that the non-negative integers is a Rajchman set.

Now, given a (signed) Radon measure μ on the unit circle T, we can present it as \( \mu = f \cdot m + \mu_s \), where m is the Lebesgue measure and \( \mu_s \) is the singular with respect to Lebesgue measure part of the measure μ. We known the following.

Theorem 1.3 (F. and M. Riesz’s, 1916, [5]). If a finite measure μ has the property \( \hat{\mu}(-n) = 0 \) for \( n = 1, \ldots \), then the measure is absolutely continuous with respect to Lebesgue measure, i.e. \( \mu = f \cdot m \), where \( f \in L^1(T) \).

This result motivates the following definition.

Definition 1.4. We say that a subset Λ ⊂ \( \mathbb{Z} \) is a Riesz set if it has the property, that if \( \text{supp}(\hat{\mu}) \subset \Lambda \) then μ has no singular part.

With this definition the F. and M. Riesz theorem says that the non-negative integers is a Riesz set.

Theorem 1.5 (Host, Parreau, 1978 [1]). A set Λ ⊂ \( \mathbb{Z} \) is a Rajchman set iff it doesn’t contain any shift of the Fourier support of a Riesz product, i.e. any set Ω((\( n_j \))) = \{ \sum \epsilon_j n_j : \epsilon_j = -1, 0, 1; \sum |\epsilon_j| < \infty \}, where (\( n_j \)) is an infinite sequence.

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1It is actually proven in [1] not only for T but for any compact group.
Thus, any set which is not Rajchman, contains the support of the Fourier transform of a singular measure, and thus is not Riesz (or, without negations, that every Riesz set is a Rajchman set).

A natural question is following: Is every Rajchman set a Riesz set? (I.e. Do the classes of Riesz and Rajchman sets coincide?) As far to the author’s knowledge, this question was first raised by Figno, 1978 [3].

As we are unable to answer the question, we want to diversify it:

**Definition 1.6.** We say that a closed set $E \subset \mathbb{T}$ is a *parisian set* if for every non absolutely continuous measure $\mu \in M(E)$, the support of it’s Fourier transform is not a Rajchman set.

The original question thus becomes: Is $\mathbb{T}$ a parisian set?

While we are not able to answer the question above, we can show that some parisian sets do exist. As any subset of a parisian set is parisian, it is clear that a positive answer on the original question would imply all the results we prove here. Yet, there are good chances that the answer is negative and a negative answer would give the study of the parisian sets some interest.

It is natural to expect that the parisian sets should be "small". Thus we try to construct a "big" parisian set.

**Main Theorem A.** For any $\alpha < 1$ there exists a closed parisian set $E$, such that $\dim_H(E) \geq \alpha$, where $\dim_H(E)$ means the Hausdorff dimension of $E$.

**Main Theorem B.** For any $\alpha < 1$ there exists a Borel parisian set $E$ such that it is an additive subgroup of $\mathbb{T}$ and $\dim_H(E) \geq \alpha$.

**Notations.** In what follows we identify $\mathbb{T}$ with $(-1, 1]$, so that the Fourier coefficients are $\hat{\mu}(n) = \frac{1}{2} \int e^{i\pi nx} d\mu(x)$.

### 2. Construction of a big parisian set

Let us first introduce a test to establish that a set is parisian.

**Lemma 2.1.** If there exist $\delta > 0$ and a sequence $(N_j)_{j=1}^{\infty}$ such that for every $j$ the set $E$ is a subset of $\frac{2}{N_j^2} \mathbb{Z} + [-1/2N_j^{1+\delta}, 1/2N_j^{1+\delta}]$, then the set $E$ is parisian.

**Proof.** Let us fix $\mu \in M_s(E)$. We want to show that $\text{supp}(\hat{\mu})$ contains a shift of a set $\Omega((n_j))$. Up to a shift of the Fourier transform we may assume without loss of generality that $\hat{\mu}(0) \neq 0$.

Here we construct the sequence $(n_j)$ as a subsequence of $(N_j)$ inductively. Assume that $(k - 1)$ first terms of the sequence $(n_j)$ are chosen. This means that for all combinations of $\epsilon_j$ the sum $\sum_{0}^{k-1} \epsilon_j n_j \in \text{supp}(\hat{\mu})$. Thus, we know that

$$\int e^{i\pi \sum_{j=1}^{k-1} \epsilon_j n_j x} d\mu(x) \neq 0,$$

for all combinations $(\epsilon_j = -1, 0, 1)_{j=1}^{k-1}$. We can take $\gamma_{k-1}$ to be the minimum of the absolute value of the $3^{k-1}$ non-zero numbers, so that $|\int e^{i\pi \sum_{j=1}^{k-1} \epsilon_j n_j x} d\mu(x)| \geq \gamma_{k-1}$. We want to show that for some sufficiently large $n_k = N_{j_k}$ for all combinations of $\epsilon_j$ holds $\int e^{i\pi \sum_{j=1}^{k} \epsilon_j n_j x} d\mu(x) \neq 0$. 

Indeed, as $E \subset 2\mathbb{Z}/N_m + [-1/N_m^{1+\delta}, 1/N_m^{1+\delta}]$, we know that $|e^{i\pi(x \pm N_m x)} - 1| \leq \frac{2}{N_m^{1+\delta}}$, when $x \in E$. Now we see that

$$|\int_E e^{i\pi \sum_{j=1}^k \varepsilon_j n_j x} \, d\mu(x) - \int_E e^{i\pi \sum_{j=1}^{k-1} \varepsilon_j n_j x} \, d\mu(x)| \leq \int_E |d\mu| |e^{i\pi x \pm N_m x} - 1| \leq \frac{\|\mu\|}{N_m^{1+\delta}}.$$ 

Thus, for sufficiently large $m$ we can be sure that the later is less than $\frac{1}{2} \gamma_{k-1}$. Now, we see that by the triangle inequality $|\int_E e^{i\pi \sum_{j=1}^k \varepsilon_j n_j x} \, d\mu(x)| \geq \frac{1}{2} \gamma_{k-1} > 0$ for all the combinations of $\varepsilon_j = -1, 0, 1$, with $j = 1, \ldots, k$, and $n_k = N_m$. \hfill \Box

A slight modification of the proof gives us the following.

**Lemma 2.2.** For an increasing sequence $(N_j) \subset \mathbb{N}$ and $\delta > 0$ the set $E = \{x \in \mathbb{T} : \sup_j (\text{dist}(x, 2\mathbb{Z}/N_j)/N_j^{1+\delta}) < \infty\}$ is a paraisan set.

**Proof.** We start from observing that $E = \bigcup_{t \in \mathbb{N}} E_t$, where

$$E_t = \{x \in \mathbb{T} : \sup_j (\text{dist}(x, 2\mathbb{Z}/N_j)/N_j^{1+\delta}) \leq t\}$$

is an increasing sequence of closed sets.

Now, we start the proof exactly as the previous one, but after the choice of $\gamma_{k-1}$ and before the choice of $n_k$ we do one more step: We pick $t_k$ large enough that $\mu_k = \mu\big|_{E_t}$ satisfies $\|\mu - \mu_k\| < \frac{1}{2} \gamma_{k-1}$. Then we see that $|\int e^{i\pi \sum_{j=1}^{k-1} \varepsilon_j n_j x} \, d\mu_k(x)| \geq \frac{1}{2} \gamma_{k-1}$. We proceed in the same way as before with $\mu_k$ in place of $\mu$, and find $n_k = N_{m_k}$ such that $|\int e^{i\pi \sum_{j=1}^k \varepsilon_j n_j x} \, d\mu_k(x)| \geq \frac{1}{2} \gamma_{k-1} > 0$. Then, $|\int e^{i\pi \sum_{j=1}^k \varepsilon_j n_j x} \, d\mu(x)| \geq \frac{1}{2} \gamma_{k-1} > 0$. \hfill \Box

**Remark 2.3.** The set $E$ is obviously an additive subgroup of $\mathbb{T}$ and thus either finite or dense in $\mathbb{T}$.

Let us now construct a set $E$ of large Hausdorff dimension which satisfies the hypothesis of the Lemma 2.1 and is thus paraisan. As the constructed set is a subset of $E$ it will also give us the estimate on the Hausdorff dimension of $E$. Fix $\alpha \in (0, 1)$, and choose $\delta > 0$ so that $\delta = 1 - \alpha$. We will construct a rapidly increasing sequence $\{N_j\}$, and related sequence of closed sets $C_j \subset (-1, 1)$, such that the sets $C_j$ is the union of the closed intervals with centrums in $2\mathbb{Z}/N_j$, of length $1/N_j^{1+\delta}$ which are entirely contained in $\bigcap_{k=1}^{j-1} C_k$. We will let then the set $E = \bigcap_j C_j$, which is obviously closed. The set constructed in such a way is a Cantor-type set, and we show that provided the sequence $N_j$ grows quickly enough the dimension of such a set is at least $\alpha$.

**Lemma 2.4.** $\dim_H(E) \geq \alpha$.

**Proof.** In order to prove that the Hausdorff dimension of $E$ is at least $\alpha$ we will show that it is at least $s$ for any $0 < s < \alpha$, and to do so we construct a finite measure $\mu$ supported on $E$ such that $\mu(I) \leq c_s |I|^s$ for any interval $I$ (it is a standard fact of

\textsuperscript{2}This estimate is well known, but we give the proof for the sake of completeness.
Geometric Measure Theory that a measure satisfying such an estimate should have support of Hausdorff dimension at least \( s \), see for example [2]).

Let us take a subset \( D_k \) of \( \bigcap_{j=1}^{k} C_j \), which is a collection of intervals of length \( 1/N_k^{1+\delta} \). This collection is defined inductively: we know that every interval of length \( 1/N_k^{1+\delta} \) contains at least \( N_k/2N_k^{1+\delta} - 1 \) points of \( 2\mathbb{Z}/N_k \). Thus, every interval of \( D_{k-1} \) contains (entirely) at least \( M_k = N_k/2N_k^{1+\delta} - 3 \) intervals with centrum in \( 2\mathbb{Z}/N_k \) and length \( 1/N_k^{1+\delta} \). (To make the estimates more simple we assume \( (N_k) \) to grow so rapidly that \( M_k \geq N_k/4N_k^{1+\delta} \))

We pick from each interval of \( D_{k-1} \) exactly \( M_k \) such intervals. All together we will have picked \( M_k \prod_{j=1}^{k-1} M_j \) intervals of length \( \frac{1}{N_k^{1+\delta}} \). Then we take the probability measure \( \mu_k \) equally distributed on the \( \prod_{j=1}^{k} M_j \) intervals of \( D_k \). We introduce \( \mu \) as a weak limit point of \( \mu_k \) (which has to be a probability measure supported by \( E = \cap C_j \)).

Let us estimate \( \mu(I) \) where \( 1/N_{k-1} > |I| \geq 1/N_k \). The interval can intersect at most \( N_k|I|/2 + 3 \) intervals of \( D_k \) (as \( N_k|I| \geq 1 \), we may use that it is at most \( 4N_k|I| \) intervals). As the measure of each interval of \( D_k \) is \( 1/\prod_{j=1}^{k} M_j \) we see that \( \mu(I) \leq 4|I|/N_k/\prod_{j=1}^{k} M_k \) where

\[
\prod_{j=1}^{k} M_k \geq (N_k/N_1)/(4^{k-1}(\prod_{j=1}^{k-1} N_j^\delta)).
\]

Thus, \( \mu(I) \leq N_1 4^k (\prod_{j=1}^{k-1} N_j^\delta)|I| = N_1 4^k (\prod_{j=1}^{k-1} N_j^\delta)|I|^{1-s}|I|^s \).

Our task is fulfilled if we show that \( c_{k,s} = N_1 4^k (\prod_{j=1}^{k-1} N_j^\delta)|I|^{1-s} \) is bounded above independently from \( k \). We know that \( |I| < 1/N_{k-1} \), and, as \( \delta = 1 - \alpha \), we see that \( c_{k,s} \leq N_1 4^k (\prod_{j=1}^{k-2} N_j^\delta)/N_k^{\alpha-s} \). It remains to take the sequence \( (N_k) \) such that \( (N_1 4^{k+2}(\prod_{j=1}^{k} N_j^\delta)|I|^{\alpha-s} < N_{k+1} \). For any fixed \( s \) the sequence \( c_{k,s} \) tends to zero, and so is bounded. (Notice that the bound \( c_s = \sup_{k} \{c_{k,s} \} \) grows as \( s \rightarrow \alpha \), but we only need it to be finite.)

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