Constructions of affine Kac-Moody algebras

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Abstract. I review our recent work on contractions of affine Kac-Moody algebras (KMA) and present new results. We study generalized contractions of KMA with respect to their twisted and untwisted KM subalgebras. As a concrete example, we discuss contraction of $D(4)_4$ and $D(3)_4$, based on $Z_3$-grading. We also describe examples of ‘level-dependent’ contractions, which are based on $Z$-gradings of KMA. Our work generalizes the Inönü-Wigner contraction of P. Majumdar in several directions. We also give an algorithm for constructing Kac-Moody-like algebras $\hat{g}$ for any Lie algebra $g$.

1. A historical introduction

In 1991 I was spending part of my sabbatical at the Instituto de Física Teórica (IFT) in São Paulo - Brazil. There, Prof. Abraham H. Zimerman and his collaborators were working on Kac-Moody construction of Toda field models. When I wanted to read about Kac-Moody algebra (KMA), I was referred to the review article of Goddard and Olive [1]. At that time this review was the standard initiation article to the subject for physicists. I found the section on the twisted KMA quite confusing, and I did not understand why the half integers and the third integers indices are necessary for describing the twisted Kac Moody algebras of the type $g(2)$ and $g(3)$, respectively. Later, I realized that mathematicians use only integer indices for the twisted algebras [2, 3]. And for a very good reason: the twisted KMA are nothing but subalgebras of the affine KMA $\hat{g} \equiv g(1)$.

I gained this insight when I worked with my daughter and her thesis adviser, the late Prof. Peter Slodowy, on identifying the symmetry algebra of the 3-dimensional hydrogen atom. We denoted this algebra by $H_3$, and identified it as a twisted KMA $so(4)^{(2)+}$ or $D(2)^{(2)+} [4]$, where $\hat{g}^+$ denotes the positive subalgebra of $\hat{g}$, as defined in (44) below. However, later, as we generalized our identification to N-dimensional hydrogen atoms [5], we realized that $D(2)^{(2)+}$ can be ‘untwisted’ [1], so that $H_3$ is isomorphic to $A_1^{(1)+} \equiv A_1^{(1)+}$.

Actually, I was not sure whether I should write up our identification of $H_3$, especially since the positive part of KMA, cannot have the central term $\kappa$. However, Dr. Svetlana Nussinov, who had an office next door at Ben Gurion University, encouraged me to write up the article, saying that most physicists are not familiar with twisted KMA, even those who have been working with untwisted KMA for many years. It is possible that physicists were scared off by the unnatural

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1 Based on a talk by J Daboul at the 5th International Symposium on Quantum Theory and Symmetries (QTS5), July 22-28, 2007, Valladolid (Spain)
notation of Goddard and Olive. Moreover, this notation camouflaged the fact that the twisted KMA are subalgebras of the untwisted ones.

When I met in 2005 Marc de Montigny at the symmetry conference in Kiev, he had been working on contractions. So I suggested that we contract KMA, relative to their twisted subalgebras, mainly to emphasize that the twisted KMA are the subalgebras of the untwisted. He knew of a paper by P. Majumdar [6] who studied Inönü-Wigner contraction of KMA. This paper will be reviewed in Sec. 4.2 below.

Our investigation led to several types of general contractions of KMA, which are based on appropriate choice of gradings [8]. In particular, we studied contractions based on $\mathbb{Z}_3$ grading and on infinite $\mathbb{Z}$ gradings, which we call ‘level gradings’.

These contractions yield non-standard ‘Kac-Moody-like’ algebras, which have abelian ideals. To describe these algebras it is necessary to use the general commutation relations of KMA, given (2) below, instead of the more familiar commutation relations in (1). These general commutation relations are defined in terms of the Killing form. Here the expertise of my daughter, who did her Ph.D. on Kac-Moody algebras and contractions [7], became very helpful.

The present paper is organized as follows: In Sec. 2 we give the two definitions of untwisted KMA algebras, and illustrate, by using the example of $\mathbb{Z}_3$ grading, why it is necessary to use the general commutation relations (2). We then describe an algorithm for constructing Kac-Moody-like algebras for any Lie algebra $\mathfrak{g}$. In Sec. 3 we define gradings and the corresponding automorphisms and emphasize the distinction between normal and twist gradings of KMA, and define twisted KMA. In Sec. 4, we give several generalized contractions of KMA, and review the IW contraction of Majumdar. In Sec. 5 we study ‘level contractions’, which by definition depend on the ‘level’, $n$, of the KM basis elements $T_a^n \in \hat{\mathfrak{g}}$. Finally, in Sec. 6 we give concluding remarks.

2. Affine Kac-Moody algebras and Killing form

Consider a finite-dimensional Lie algebra $\mathfrak{g}$, defined by the commutation relations:

$$[T_a, T_b] = c_{ab}^c T_c, \quad a, b, c = 1, 2, \ldots, \dim \mathfrak{g},$$

where summation over repeated indices is implied.

The corresponding **affine Kac-Moody algebras (KMA)** are often defined by the following commutation relations: [1, 6]

$$[T_a^m, T_b^n] = c_{ab}^c T_c^{m+n} + m \kappa \delta_{m+n,0} \delta_{a,b},$$

$$[\kappa, T_a^m] = 0,$$

$$[d, T_a^m] = m T_a^m.$$  \hfill (1)

The $\kappa$ in (1) is called the **central term or central charge**. Henceforth, we shall omit the **derivation operator $d$**, since it is irrelevant for our purposes.

The commutation relations in (1) hold for totally antisymmetric structure constants $c_{ab}^c$. But for many gradings (see Sec. 3) these commutation relations are inconsistent with the choice of totally antisymmetric $c_{ab}^c$. For example, let $\mathfrak{g}$ have $\mathbb{Z}_3$-grading, such that

$$T_a, T_b \in \mathfrak{g}_1, \quad \text{with} \quad c_{a,b}^c \neq 0 \quad \text{for some} \quad T_c \in \mathfrak{g}_2.$$  

Since by assumption $c_{a,b}^c = c_{a,b}^c \neq 0$, it follows that $[T_c, T_a]$ must have at least one nonzero element $c_{c,a}^b T_b$ in $\mathfrak{g}_1$. But this contradicts the grading assumption, namely $[T_c, T_a] \in \mathfrak{g}_0$. 


The commutation relations for using general basis are obtained by replacing the \( \delta_{ab} \) in (1) by the corresponding Killing form \( K(T_a, T_b) \). We obtain
\[
\begin{align*}
[T^m_a, T^n_b] &= \epsilon^c_{ab} T^{m+n+c} + m \kappa \delta_{m+n,0} K(T_a, T_b), \\
[\kappa, T^m_a] &= 0, \quad \text{for } -\infty < m, n < \infty,
\end{align*}
\]
where
\[
K(x, y) := \frac{1}{2} \text{Tr}(\text{ad}(x)\text{ad}(y)), \quad \forall \ x, y \in g,
\]
and \( \text{ad}(x) \) is the adjoint representation of \( x \in g \).

2.1. Algorithm for constructing Kac-Moody-like algebras from any finite Lie algebras

The ‘standard affine KMA’ \( \tilde{g}^{(1)} \) are usually constructed for semisimple finite-dimensional Lie algebras \( g \), by starting with the Cartan matrices \( [2, 3] \).

However, we can reproduce the standard KMA and in addition construct similar ‘Kac-Moody-like’ algebras from any Lie algebra \( g \) by using the following algorithm:

For every basis element \( T_a \in g \) define formally an infinite set of elements, \( \{ T^n_a \mid -\infty < n < \infty \} \), by adding an index \( n \), which we call level. Among these \( \{ T^n_a \} \) we formally define commutation relations, exactly as in (2). We now give a simple proof that the above algorithm yields Lie algebras, which we shall also call ‘Kac-Moody algebras’ and denote them by \( \tilde{g} \equiv \tilde{g}^{(1)} \)

Proof: The commutators in (2) are clearly antisymmetric. We now prove the Jacobi identity: Writing (2) in a basis-independent form, we obtain
\[
[x^m, y^n] = [x, y]^{m+n} + m \kappa \delta_{m+n,0} \ (x|y) ,
\]
where \( (x|y) \equiv K(x, y) \). It follows that
\[
[x^s, [y^m, z^n]] = [x^s, [y, z]^{m+n}] \\
= [x, [y, z]]^{s+m+n} + s \kappa \delta_{m+n+s,0} \ (x|[y, z]) .
\]
Hence, the Jacobi identity is equivalent to
\[
[x^s, [y^m, z^n]] + \text{cyclic} = \kappa \delta_{m+n+s,0} \ (s(x|[y, z]) + n(z|[x, y]) + m(y|[z, x])) \\
= \kappa \delta_{m+n+s,0} (s + n + m)(x|[y, z]) = 0
\]
where we used the symmetry of the Killing form: \( (x|[y, z]) = (z|[x, y]) = (y|[z, x]) \).

3. Gradings of affine and twisted Kac-Moody algebras

3.1. Gradings of finite-dimensional and KM Lie algebras

We begin this section by describing some gradings of KMA, since they are useful for classifying and contracting KMA. A \( \mathbb{Z}_M \)-grading is a decomposition of a Lie algebra \( g \),
\[
g = g_0 \oplus g_1 \oplus \cdots \oplus g_{M-1},
\]
which satisfies
\[
[g_i, g_j] \subseteq g_{i+j}, \quad i, j, i + j \ (M),
\]
where \( (M) \) denotes ‘modulo \( M \)’. If \( M \) in (8) is equal to \( \infty \), then we speak of \( \mathbb{Z}^+ \)-grading. On the other hand, a \( \mathbb{Z} \)-grading is defined as in (9) below.
Note that $g_0$ is a subalgebra of $g$. All the other sets $g_i$, $i \neq 0$, however, are not Lie subalgebras, but they constitute representation spaces of $g_0$.

Affine KMA have an inherent $\mathbb{Z}$-grading:

$$\hat{g} = \bigoplus_{n=-\infty}^{\infty} \hat{g}^{(n)},$$

where

$$\hat{g}^{(n)} := \{T_a^n \mid T_a \in g\} \oplus (\kappa \oplus d) \delta_{n,0} , \quad n \in \mathbb{Z} .$$

Since we refer to the upper index $n$ of $T_a^n$ as its level, the above $\mathbb{Z}$-grading is called level grading.

3.2. $\mathbb{Z}_M$ normal and twist gradings of affine KMA and their automorphisms

For any $\mathbb{Z}_M$-grading in Eq. (7) one can define the following automorphism:

$$\tau(T_a) = (\lambda_M)^k T_a, \quad \text{for} \quad T_a \in g_k, \quad \text{and} \quad k = 0, 1, \ldots, M - 1 , \quad (11)$$

where $\tau^M = 1$ and

$$\lambda_M := \exp(2\pi i / M).$$

We see from (11) that the subspaces $g_k$ become eigenspaces of $\tau$. Conversely, every automorphism $\tau$ as in Eq. (11) yields a $\mathbb{Z}_M$-grading.

We can extend the automorphism $\tau$ for $g$ to the corresponding affine KM algebra $\hat{g}$ in two ways, as follows:

$$\tau_0(T_a^n) = \lambda_M^k T_a^n, \quad \text{for} \quad T_a \in g_k, \quad k = 0, 1, \ldots, M - 1 , \quad (12)$$

$$\tau_s(T_a^n) = \lambda_M^{k-n} T_a^n, \quad \text{and} \quad -\infty \leq n \leq \infty ,$$

The $\tau_0$ ignores the level index $n$. In contrast, $\tau_s$ gives the level index $n$ the same importance as to the $k$ index. We shall call $\tau_0$ and $\tau_s$ normal and twist automorphisms, respectively.

By defining the following eigenspaces of $\tau_0$ and $\tau_s$

$$\hat{g}_j := \bigoplus_{n=-\infty}^{\infty} \{T_a^n \mid T_a \in g_j\} \oplus \kappa \delta_{j,0} , \quad 0 \leq j < M,$$

$$\hat{g}_{[j]} := \bigoplus_{k=0}^{M-1} \left( \bigoplus_{n=-\infty}^{\infty} \{T_a^{(nM+k-j)} \mid T_a \in g_k\} \oplus \kappa \delta_{j,0} \right) \oplus \kappa, \quad 0 \leq j < M ,$$

we obtain two $\mathbb{Z}_M$ gradings, which we shall call ‘normal and twist gradings’:

$$\hat{g} = \bigoplus_{j=0}^{M-1} \hat{g}_j , \quad \text{with} \quad \tau_0(\hat{g}_j) = \lambda_M^j \hat{g}_j ,$$

(14)

$$\hat{g} = \bigoplus_{j=0}^{M-1} \hat{g}_{[j]} , \quad \text{with} \quad \tau_s(\hat{g}_{[j]}) = \lambda_M^j \hat{g}_{[j]} .$$

(15)

The $\hat{g}_0 = (\hat{g})_0$ in (14) is an affine KM subalgebra of $\hat{g}$, which is equal to $(g_0) \equiv g_0^{(1)}$.

The subspace $\hat{g}_{[0]}$ of the twist grading (15) is a subalgebra of $\hat{g}$, which is called ‘twisted KMA’ and denoted as $g^{(M)}$:

$$g^{(M)} \equiv \hat{g}_{[0]} := \bigoplus_{k=0}^{M-1} \left( \bigoplus_{n=-\infty}^{\infty} \{T_a^{(nM+k)} \mid T_a \in g_k\} \oplus \kappa \right) .$$

(16)
3.3. Conditions on \( K(T_a, T_b) \) by using automorphisms

The Killing form is invariant under all automorphisms \( \tau \) of \( g \) [9], i.e.

\[
K(\tau(x), \tau(y)) = K(x, y), \quad \forall \ x, y \in g.
\] (17)

This invariance follows from the symmetry property of the trace, and the relation \( \text{ad}(\tau(x)) = \tau \circ \text{ad} \circ \tau^{-1} \), which in turn follows from

\[
\text{ad}(\tau(x))(y) := [\tau(x), y] = \tau( [x, \tau^{-1}(y)] ).
\]

By applying this invariance equality to a Lie algebra \( g \) which has a \( Z_M \)-grading, we obtain

\[
K(x, y) = 0 \text{ if } j + k \neq 0 (M), \quad \text{for } x \in g_j \text{ and } y \in g_k .
\] (18)

This condition is important: it tells us which commutators of KMA cannot have a central term \( \kappa \).

3.4. Example: Normal and twist \( Z_3 \)-gradings of \( D_{4(1)} \)

It is important to distinguish between normal and twist gradings of affine KMA. As an instructive example, let us study the twisted \( Z_3 \) grading of \( D_{4(1)} \) in some details, because \( D_{4(3)} \) is the only genuinely twisted KMA \( g^{(M)} \) with \( M \neq 2 \).

Let \( g \) have a \( Z_3 \)-grading

\[
g = g_0 \oplus g_1 \oplus g_2 := \{x_a\} \oplus \{y_a\} \oplus \{z_a\}.
\]

For example, \( D_4 \) and \( G_2 \) have the following \( Z_3 \)-gradings:

\[
D_4 = G_2 + 7 + \tilde{7} \quad \text{and} \quad G_2 = su(3) + 3 + \tilde{3}.
\] (19)

The corresponding affine KMA \( \widehat{g} = g_{0(1)} \) have two types of \( Z_3 \)-gradings:

(i) normal grading: defined by \( \widehat{g}_i = \{T_a^i | T_a \in g_i\} \), so that

\[
\widehat{g} = \widehat{g}_0 \oplus \widehat{g}_1 \oplus \widehat{g}_2 = \{x_a^0\} \oplus \{y_a^0\} \oplus \{z_a^0\}.
\] (20)

(ii) twist grading: defined by \( \widehat{g} = \widehat{g}_{[0]} \oplus \widehat{g}_{[1]} \oplus \widehat{g}_{[2]} \), where

\[
\begin{align*}
\widehat{g}_{[0]} &= \{X_a^n\} := \{x_{3a}^{3n}, y_{a}^{3n+1}, z_{a}^{3n+2}\}, \\
\widehat{g}_{[1]} &= \{Y_a^n\} := \{x_{3a}^{3n+2}, y_{a}^{3n}, z_{a}^{3n+1}\}, \\
\widehat{g}_{[2]} &= \{Z_a^n\} := \{x_{a}^{3n+1}, y_{3a}^{3n+2}, z_{3a}\}.
\end{align*}
\]

The normal \( Z_3 \) grading in (20) can be used to define IW or generalized contractions with respect to affine KMA \( \widehat{g}_0 \equiv g_{0(1)} \). For example, \( D_{4(1)} \) can be contracted with respect to \( G_{2(1)} \).

Similarly, the twist \( Z_3 \) grading as in (21) is necessary to contract \( D_{4(1)} \), with respect to its twisted subalgebra \( D_{4(3)} \).
4. Constructions of Kac-Moody algebras

A (generalized) **contraction** is a limit process through which a Lie algebra $g$ is transformed into a non-isomorphic Lie algebra $g'$, with the same dimension. The commutation relations of the 'contracted Lie algebra' $g'$ are defined by [10]

$$[x, y]' = \lim_{\varepsilon \to 0} U_\varepsilon^{-1}([U_\varepsilon(x), U_\varepsilon(y)]),$$  \hspace{1cm} (21)

where $U_\varepsilon$ is an $\varepsilon$-dependent linear transformation of $g$.

The simplest contraction [11, 12] is called Inönü-Wigner (IW) contraction. It is defined by decomposing $g$ into two parts:

$$g := \{T_\alpha\} = g_R \oplus g_C =: \{T_\alpha\} \oplus \{T_j\},$$ \hspace{1cm} (22)

where $g_R$ is a Lie subalgebra of $g$. The complement $g_C$ is simply a vector subspace. Henceforth, we employ the three types of indices defined in (22) to distinguish the generators encountered in IW contractions, which are defined by the following map

$$U_\varepsilon(T_\alpha) = T_\alpha, \quad U_\varepsilon(T_j) = \varepsilon T_j.$$ \hspace{1cm} (23)

The resulting commutation relations are

$$[T_\alpha, T_\beta]_\varepsilon = c_{\alpha\beta}^\gamma T_\gamma,$$  \hspace{1cm} (24)

$$[T_\alpha, T_j]_\varepsilon = \varepsilon c_{\alpha j}^\gamma T_\gamma + c_{\alpha j}^k T_k \Rightarrow c_{\alpha j}^k T_k,$$  \hspace{1cm} (25)

$$[T_i, T_j]_\varepsilon = \varepsilon^2 c_{ij}^\gamma T_\gamma + \varepsilon c_{ijk}^k T_k \Rightarrow 0.$$  \hspace{1cm} (26)

The commutation relations in (24) are those of the invariant subalgebra $g_R$. Eq. (26) show that the subspace $g_C$ becomes the following abelian subalgebra of $g'$:

$$g'_C = \text{abel}(g_C) := \{\tilde{T}_j\}, \quad \text{with} \quad [\tilde{T}_j, \tilde{T}_k] = 0.$$ \hspace{1cm} (27)

Moreover, the commutation relations in (25) tell us that $g'_C$ is an ideal of the contracted algebra $g'$. Hence, the above IW contraction yields

$$g \searrow g' = g_R \oplus g'_C = g_R \oplus \text{abel}(g_C).$$ \hspace{1cm} (28)

where we use $g \searrow g'$ to represent the statement 'the Lie algebra $g$ is contracted to the Lie algebra $g'$'.

4.1. IW contraction of KMA relative to any subalgebra

We can apply IW contraction to any KMA $\hat{g}$, if it has a subalgebra $\hat{g}_R$. Using the familiar decomposition

$$\hat{g} := \hat{g}_R \oplus \hat{g}_C,$$ \hspace{1cm} (29)

and the map

$$U_\varepsilon(\hat{g}_R) = \hat{g}_R, \quad U_\varepsilon(\kappa) = \kappa, \quad \text{and} \quad U_\varepsilon(\hat{g}_C) = \varepsilon \hat{g}_C,$$

we obtain the following commutation relations in the limit $\varepsilon \to 0$:

$$[T_\alpha^m, T_\beta^n]_0 = c_{\alpha\beta}^\gamma T_\gamma^{m+n} + m\kappa \delta_{m+n,0} K(T_\alpha, T_\beta),$$  \hspace{1cm} (30)

$$[T_\alpha^m, T_j^n]_0 = c_{\alpha j}^k T_k^{m+n},$$  \hspace{1cm} (31)

$$[T_i^m, T_j^n]_0 = 0.$$  \hspace{1cm} (32)
The above IW contraction for $\hat{g}$ yields a semidirect sum,
\[ \hat{g} \setminus \hat{g}' = \hat{g}_R \oplus \hat{g}_C := \hat{g}_R \oplus \mathfrak{abel}(\hat{g}_C), \tag{33} \]
effectively as the IW contraction in (28) of finite-dimensional Lie algebras $g$.

So far we did not specify the subalgebra $\hat{g}_R$. We consider four choices. The first two are:

(i) $\hat{g}_R$ is the 0-level subalgebra $\hat{g}_R = \{ T^0_a | T_a \in g \}$ of $\hat{g}$.

(ii) $\hat{g}_R$ is the positive subalgebra of $\hat{g}$, i.e. $\hat{g}_R = \hat{g}^+ := \{ T^a_n \mid T_a \in g, n \geq 0 \}$.

The other two subalgebras will be considered in the next two subsections.

4.2. IW Contraction of affine KMA relative to their affine subalgebras: the IW Contraction by Majumdar

Let $g = g_R \oplus g_C$ is a decomposition as in (29), where $g_R$ is a subalgebra. The ‘affinization’ of $g$ yields
\[ \hat{g} = \{ T^0_a \} \oplus \kappa = (\{ T^0_a \} \oplus \kappa) \oplus \{ T^a_j \} =: \hat{g}_R \oplus \hat{g}_C, \tag{34} \]
Majumdar [6] applied IW contraction to $g$ and $\hat{g}$, and showed by using the Jacobi identity, that
\[ \hat{g} \setminus \hat{g}' = \hat{g}_R \oplus \mathfrak{abel}(\hat{g}_C) \]
\[ = \hat{g}' \equiv (g'(1)) = (g_R \oplus \mathfrak{abel}(g_C))^{(1)}. \tag{35} \]

In particular, he proved that the affinization of $g'$ yields $[(\hat{T}_j)^m, (\hat{T}_k)^n] = 0$, in accordance with $[T^m_i, T^n_i]_0 = 0$ in (32); i.e. he showed that the commutators between the $(\hat{T}_j)^n$ are not proportional to the central term $\kappa$. This result follows automatically from our algorithm in Sec. 2.1, since $K(\hat{T}_j, \hat{T}_k) = 0$.

4.3. Contractions of affine KMA relative to their twisted KM subalgebras

The subalgebra $\hat{g}_R$ of the KMA $\hat{g}$ in Sec. 4.1 may be a twisted KM subalgebra $g^{(M)}$. We now illustrate the simplest case $M = 2$ explicitly:
\[ \hat{g} = \hat{g}_R \oplus \hat{g}_C := \{ \{ T^2n_i, T^2n+1_i \} \oplus \kappa \} \oplus \{ T_{2n+1}^a, T_{2n+2}^a \}, \tag{36} \]
where we used the index notation in Eq. (22). The IW contraction yields
\[ \hat{g} = g^{(1)} \setminus g' = g^{(1)'} = g^{(2)} \oplus \mathfrak{abel}(\hat{g}_C). \]

Note that unlike the IW contractions of Majumdar, the contractions of $g^{(1)}$ relative to twisted subalgebras, $g^{(M)}, M \geq 2$, cannot be reproduced by first contracting $\hat{g}$ to some $g'$ and then affinizng $g'$. This is because there is no finite algebra $g'$, such that $\hat{g}' = g^{(M)} \oplus \mathfrak{abel}(\hat{g}_C)$.

4.4. Contracting twisted KMA relative to their affine subalgebras

Twisted algebras $\hat{g}^{(M)}$ admit a natural $Z_M$-grading given by (see Eq. (16)):
\[ g^{(M)} = \bigoplus_{k=0}^{M-1} g_k^{(M)}, \quad \text{where} \quad g_k^{(M)} := \bigoplus_{n=-\infty}^{\infty} \{ T^a_{nM+k} \mid T_a \in g_k \} \oplus \kappa \delta_{k0}. \]
We can use this grading to contract \( \mathfrak{g}^{(M)} \) relative to its KM subalgebra \( \mathfrak{g}_0^{(M)} \). For instance, applying an IW contraction to the twisted KMA \( D_4^{(3)} \), relative to its untwisted subalgebra \( G_2^{(1)} \), we obtain

\[
D_4^{(3)} = (D_4)_0^{(3)} ⊕ ((D_4)_1^{(3)} ⊕ (D_4)_2^{(3)}) =: \mathfrak{g}_R^{(3)} ⊕ \mathfrak{g}_C^{(3)}
\]

where we used

\[
(D_4)_0^{(3)} = \bigoplus_{n=\infty}^{-\infty} \{ T_a^{(3n)} | T_a ∈ (D_4)_0 ≃ G_2 \} ⊕ \kappa ≃ G_2^{(1)}.
\]

4.5. \( \mathbb{Z}_M \)-graded contraction of affine KMA relative to affine KMA

Let \( \mathfrak{g} \) be a finite-dimensional algebra with a \( \mathbb{Z}_M \)-grading. Then we can define (generalized) \( \mathbb{Z}_M \)-graded contractions of the corresponding affine KMA \( \hat{\mathfrak{g}} \), by using the following contraction map

\[
\hat{\mathfrak{g}}_i → \varepsilon^{n_i} \hat{\mathfrak{g}}_i, \quad \text{for } i = 0, 1, \ldots, M - 1.
\]

The commutation relations of the contracted KMA \( \hat{\mathfrak{g}}' \) are obtained by taking the limit of the following commutation relations, as \( \varepsilon → 0 \):

\[
[T_a^m, T_b^n] = \varepsilon^{n(a) + n(b) - n(c)} e_{ab} T_c^m + \varepsilon^{n(a) + n(b)} m \kappa \delta_{m + n, 0},
\]

where

\[
n(a) = n_i ≥ 0, \quad \text{if } T_a ∈ \mathfrak{g}_i, \quad \text{etc.}
\]

In order for \( \hat{\mathfrak{g}}_0 \) to be the invariant subalgebra under the above contraction, we define \( n_0 = 0 \). Moreover, in order for the contraction factors \( \varepsilon^{n(a) + n(b) - n(c)} \) in (38) to be finite for \( \varepsilon → 0 \), we must require that

\[
n_i + n_j - n_k ≥ 0, \quad \forall \quad 0 ≤ i, j, k ≤ M - 1, \quad i + j ≤ k (M).
\]

Clearly, different choices of the \( \{ n_i \} \) yield different contractions.

In particular, by choosing \( n_0 = 0 \) and \( n_i = 1 \) for \( 1 ≤ i ≤ M - 1 \) we obtain IW contraction.

4.6. The ‘modulo’ contraction

Another obvious possibility is \( n_i = i \) for \( 0 ≤ i ≤ M - 1 \). For the commutator of the contracted algebra we get for \( x_i ∈ \mathfrak{g}_i \) and \( y_j ∈ \mathfrak{g}_j \) with \( i, j < M \):

\[
[x_i, y_j]' = [x_i, y_j], \quad \text{if } i + j < M \quad (39)
\]

\[
[x_i, y_j]' = 0, \quad \text{if } i + j ≥ M. \quad (40)
\]

Thus the subspace

\[
n := \mathfrak{g}_1 ⊕ \cdots ⊕ \mathfrak{g}_{M-1}
\]

becomes a nilpotent ideal of the contracted Lie algebra \( \mathfrak{g}' \), which becomes a semidirect sum of \( \mathfrak{g}_0 \) and \( n \):

\[
\mathfrak{g}' = \mathfrak{g}_0 ⊕ n.
\]

Note the difference to Inönü-Wigner contraction, with respect to \( \mathfrak{g}_0 \): It lies only in the structure of the ideal, which is abelian for the IW contraction, but nilpotent, and in general not abelian, for our contraction.
We call the above contraction the ‘modulo contraction’, because it takes advantage of the ‘(M) := modulo M’, when calculating the powers of $\varepsilon$.

For example, if $M = 5$, then the modulo map (37) would yield

$$[U_\varepsilon(x_3), U_\varepsilon(x_4)] = \varepsilon^{3+4-2} [x_3, x_4] \Rightarrow 0, \quad \text{for } \varepsilon \to 0, \ x_i \in g_i.$$ 

The modulo contraction can be applied to finite Lie algebras $g$ and also to KMA $g^{(M)}$.

### 4.7. Example: Modulo contraction of KMA with a $Z_3$-twist gradings

As an example of the contractions discussed in the previous subsection, we now carry out a twist contraction explicitly, for the simple but important special case of $M = 3$. The map $U_\varepsilon(\hat{g}) = \varepsilon^{i} \hat{g}(i)$, $i = 0, 1, 2$ yields the following contraction:

$$
\begin{align*}
[X_a^{m}, X_b^{n}] &= \varepsilon^{c} c^{(m+n)} + m \kappa \delta_{m+n,0} K(X_a, X_b), \\
[X_a^{m}, Y_b^{n}] &= \varepsilon^{c} c^{(m+n)}, \\
[X_a^{m}, Z_b^{n}] &= \varepsilon^{c} c^{(m+n)}, \\
[Y_a^{m}, Y_b^{n}] &= \varepsilon^{c} c^{(m+n)}, \\
[Y_a^{m}, Z_b^{n}] &= \varepsilon^{3} c^{(m+n)} + \varepsilon^{3} m \kappa \delta_{m+n,0} K(Y_a, Z_b) \Rightarrow 0, \\
[Z_a^{m}, Z_b^{n}] &= \varepsilon^{3} c^{(m+n)} \Rightarrow 0,
\end{align*}
$$

where we noted the condition (18) when we added the central term.

We see that the vector space $\mathfrak{n} := \hat{g}[1] ' \oplus \hat{g}[2] ' = \{Y_a^{n}\} \oplus \text{abel}(Z_a^{n})$.

is a nilpotent ideal of $\hat{g}'$. Hence,

$$\hat{g} \setminus \hat{g}' = \hat{g}[0] \oplus \mathfrak{n} = \hat{g}^{(3)} \oplus \mathfrak{n}.$$ 

This contraction can be applied to $D^{(3)}_4$ to obtain $D^{(3)}_4$. Of course, $D^{(3)}_4$ can also be obtained by IW contraction.

### 5. Level-dependent contractions of Kac-Moody algebras

In this section, we turn our attention to a class of contractions which can be defined for infinite-dimensional algebras only. We shall refer to them as level contractions. They are defined by the map

$$T^n_a \to \varepsilon^{f(a, n)} T^n_a \quad \text{and} \quad \kappa \to \kappa,$$

in which the power function $f(a, n)$ depends explicitly on the level $n$ of the generators $T^n_a \in g^{(M)}$.

The map (41) yields the following commutation relations:

$$
\begin{align*}
[T^m_a, T^n_b]_\varepsilon &= \varepsilon^{f(a,m)+f(b,n)-f(c,m+n)} c^{(m+n)}_{ab} T^m_a + \varepsilon^{f(a,m)+f(b,n)} m \kappa \delta_{m+n,0} K(T_a, T_b). \quad (42)
\end{align*}
$$

In order to keep the right-hand side of (42) finite in the contraction limit $\varepsilon \to 0$, the power function $f(a, n)$ must satisfy the following two ‘power conditions’:

$$
\begin{align*}
f(a, m) + f(b, n) - f(c, m + n) &\geq 0, \quad \text{for } c^{(m+n)}_{ab} \neq 0, \quad \text{and} \\
f(a, m) + f(b, -m) &\geq 0, \quad \text{for } K(T_a, T_b) \neq 0. \quad (43)
\end{align*}
$$
In the following sections, we shall carry out unusual contractions of various affine KMA and their subalgebras by using different forms of \( f(a,n) \). Such level contractions have not been studied before, because contractions have dealt mostly with finite Lie algebras.

The most interesting level contractions can be defined only for positive (resp. negative) subalgebras \( g^{(M)+} \) (resp. \( g^{(M)-} \)) of KMA \(^7\). These subalgebras contain positive levels only, i.e.

\[
g^{(M)+} := \{ T^n_a \mid T^n_a \in g^{(M)}, n \geq 0 \}.
\]

Note that positive (or negative) KMA do not have a central element \( \kappa \), because of the factor \( m \delta_{m+n,0} \) in the definition of KMA (see Eq. (2)). Hence, positive KM subalgebras can be regarded simply as loop algebras \(^1, 2, 4\).

5.1. Level contractions of \( g^{(M)+} \) by the power function \( f_s(n) \)
In this section, we contract positive KMA \( g^{(M)+} \) by using the following interesting power function for \( s \in \mathbb{N}^+ \):

\[
f_s(n) := \begin{cases} 
  n \quad & \text{for } 0 \leq n \leq s, \quad s \geq 1, \\
  \sqrt{n} \quad & \text{for } s + 1 \leq n.
\end{cases}
\]

(45)

Since \( f_s(0) = 0 \), contracting \( g^{(M)+} \) for \( s \geq 1 \) with \( f_s(n) \) yields the following commutation relations in the limit \( \varepsilon \to 0 \):

\[
[T^0_a, T^n_b]_0 = c^n_{ab} T^0_c \\
[T^m_a, T^n_b]_0 = c^n_{ab} T^{m+n}_c \\
[T^n_a, T^n_b]_0 = 0,
\]

for \( n \geq 0 \), for \( m + n \leq s \), otherwise.

The contracted algebra \( g^{(M)+'} \) can be written as

\[
(g^{(M)+'}) = \{ T^n_a \mid n \geq 1 \},
\]

(46)

\[
= (g^{(M)+}[s]) \subset \text{abel}(\{ T^n_a \mid n \geq s + 1 \}),
\]

(47)

where

\[
(g^{(M)+}[s]) = \{ T^n_a \mid 0 \leq n \leq s \}.
\]

(48)

is a finite-dimensional nonabelian subalgebra of the contracted algebra \( g^{(M)+} \). For \( s \geq 1 \) the \( g^{(M)+'} \) is not a subalgebra of the original algebra \( g^{(M)+} \). Hence it is not an invariant algebra, except for \( s = 1 \); the invariant subalgebras for all \( s \geq 1 \) is the 0-level subalgebra of \( g^{(M)+} \), as we emphasized in (46).

In (47) we write \( (g^{(M)+'}) \) as a semidirect sum of the subalgebra \( (g^{(M)+}[s]) \) and an infinite-dimensional abelian ideal.

For untwisted algebras, \( g^{(1)+} \), i.e. for \( M = 1 \), we use for simplicity the notation \( g^{[s]} \) instead of \( (g^{(1)+}[s]) \). Its dimension is given by

\[
|g^{[s]}| = (s + 1)|g| \quad \text{for } s \geq 1,
\]

(49)

since for these algebras all \( n \)-levels \( n \leq s \) have the same number of elements, namely \(|g|\).

Thus, as \( s \) increases, the subalgebras \( (g^{(M)+}[s]) \) in Eq. (48) become larger. In particular, for \( s = \infty \) we obtain \( (g^{(M)+}[\infty]) = g^{(M)+} \), so that \( f_\infty(n) := n \) yields no contraction whatsoever.

In contrast, \( f_1(n) = \sqrt{n} \) yields the ‘maximal contraction’, since \( (g^{(M)+}[1]) = \{ T^0_a \} \).
5.2. Constructing ‘finite-dimensional Kac-Moody-like’ algebras via nilpotent operators
The above non-abelian subalgebras \( g^{[s]} \) can be represented by using an abelian nilpotent operator \( p \), as follows
\[
g^{[s]} \simeq \{ T_a^n \simeq p^n \otimes T_a \mid T_a \in g \}, \quad \text{where} \quad p^{s+1} = 0 .
\] (50)
In fact, we can apply the above definition to any Lie algebra \( g \). In this way, we obtain an infinite number of ‘finite-dimensional Kac-Moody-like’ Lie algebras, \( g^{[s]} \), one for every \( s \geq 1 \).

One can also represent the subalgebras \( (g^{(M)})^{[s]} \) for \( M \geq 2 \) by nilpotent operators (see [2]).

We may think of these \( g^{[s]} \) as ‘finite Loop algebras’. They may become useful for constructing some cutoff procedures.

5.3. Level contractions with \( f(a,n) \neq f(n) \)
In this section we turn to contractions in which the power functions \( f(a,n) \) depend explicitly on both \( a \) and \( n \). First, we consider a positive KMA with a \( \mathbb{Z}_2 \times \mathbb{Z} \) grading:
\[
\hat{g}^+ := \hat{g}_0 + \hat{g}_1 := \{ X_a^m \} + \{ Y_i^n \}
\] (51)
On \( \hat{g}^+ \) we define the following change of basis,
\[
X^m \rightarrow X^m, \quad Y^n \rightarrow \varepsilon^{f(n)}Y^n, \quad \text{with} \quad 0 < f(n + 1) < f(n) ,
\]
This map yields the following commutation relations :
\[
[X_a^m, X_b^n]_\varepsilon = c_{ab}^c X_c^{m+n} ,
\] (52)
\[
[X_a^m, Y_b^n]_\varepsilon = \varepsilon^{f(n)−f(n+m)}c_{ab}^c Y_c^{m+n} ,
\] (53)
\[
\Rightarrow \begin{cases} 
  c_{ab}^c Y_c^n , & \text{for } m = 0 , \\
  0 , & \text{for } m > 0 ,
\end{cases}
\] (54)
\[
[Y_a^m, Y_b^n]_\varepsilon = \varepsilon^{f(n)+f(m)}c_{ab}^c X_c^{m+n} \Rightarrow 0 .
\] (55)
From (53) we see that the right-hand side vanishes, in the limit \( \varepsilon \rightarrow 0 \), if \( m \geq 1 \) and if \( f(n) \) is a strictly monotonically decreasing function of \( n \geq 0 \). A simple example of such function is \( f(n) = 1/(n+c) \), \( c > 0 \). The above commutation relations show that the contracted (infinite-dimensional) algebras \( \hat{g}' \) contain several interesting subalgebras:

(i) The affine KMA \( \hat{g}_0 = \{ X_a^m \} \), defined in Eq. (52), is an invariant subalgebra of \( \hat{g}^+ \).

(ii) The abelian subalgebra \( T_Y := \{ Y_a^n \mid n \geq 0 \} \) is an ideal of \( (\hat{g}^+)' \). Thus, the contracted algebra \( (\hat{g}^+)' \) has the familiar structure of being a semidirect sum of an invariant subalgebra and an abelian ideal.
\[
(\hat{g}^+)' = \hat{g}_0 \oplus \text{abel}(Y_a^n) .
\] (56)

(iii) The level 0 of the contracted algebra in (56) yields a contraction of the underlying algebra \( g = \hat{g}_0 + \hat{g}_1 : \)
\[
g' = \hat{g}_0 \oplus \text{abel}(\hat{g}_1) = \{ X_a^0 \} \oplus \text{abel}(Y_a^0) .
\]
6. Concluding remarks
We contracted affine KMA relative to affine and twisted KMA by using IW contractions and generalized contractions based on \( \mathbb{Z}_M \) and \( \mathbb{Z}_M \times \mathbb{Z} \) gradings. These contractions may become useful in the study of representations of KMA.

There is another aspect which we also find interesting: in the case of finite-dimensional Lie algebras, one can obtain different limits of these algebras and their representations by applying
two contractions in different order. This was done, for example, by first ‘deforming’ a harmonic-oscillator Hamiltonian $H_1(k)$, by adding to it a constant force term $-fx$. The symmetry algebra of the resulting two-parameter Hamiltonian, $H_2(k, f) := H_1(k) - fx$, is $so(3)$. This algebra was contracted by taking the limits $(k, f) \to 0$ in different orders: (1) taking the limits $f \to 0$ followed by $k \to 0$ yields the Euclidean algebra $e(2)$, whereas (2) taking the limits $k \to 0$ followed by $f \to 0$ yields the Heisenberg-Weyl algebra $w_1$. This phenomenon was called the Deformation-Contraction (DC) hysteresis [13, 14, 15]. It should be interesting if one can construct a model which yields a DC hysteresis for affine KM algebras.

Many more contractions of KMA can be constructed by various choices of gradings. In [7] graded contractions were studied which made use of gradings of Kac Moody algebras based on their root lattices.

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