Homological Connectivity of Random $k$-dimensional Complexes

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Abstract

Let $\Delta_{n-1}$ denote the $(n-1)$-dimensional simplex. Let $Y$ be a random $k$-dimensional subcomplex of $\Delta_{n-1}$ obtained by starting with the full $(k-1)$-dimensional skeleton of $\Delta_{n-1}$ and then adding each $k$-simplex independently with probability $p$. Let $H_{k-1}(Y;R)$ denote the $(k-1)$-dimensional reduced homology group of $Y$ with coefficients in a finite abelian group $R$. It is shown that for any fixed $R$ and $k \geq 1$ and for any function $\omega(n)$ that tends to infinity

$$\lim_{n \to \infty} \Pr \left[ H_{k-1}(Y;R) = 0 \right] = \begin{cases} 0 & p = \frac{k \log n - \omega(n)}{n} \\ 1 & p = \frac{k \log n + \omega(n)}{n} \end{cases}$$

1 Introduction

Let $G(n,p)$ denote the probability space of graphs on the vertex set $[n] = \{1,\ldots,n\}$ with independent edge probabilities $p$. Let $\log$ denote the natural logarithm. A classical result of Erdős and Rényi [2] asserts that the threshold probability for connectivity of $G \in G(n,p)$ coincides with the threshold for

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the non-existence of isolated vertices in $G$. In particular, for any function $\omega(n)$ that tends to infinity

$$\lim_{n \to \infty} \Pr \left[ G \in G(n, p) : G \text{ connected} \right] = \begin{cases} 0 & p = \frac{\log n - \omega(n)}{n} \\ 1 & p = \frac{\log n + \omega(n)}{n} \end{cases}.$$ 

A 2-dimensional analogue of the Erdős-Rényi result was considered in [3], where the threshold for homological 1-connectivity of random 2-dimensional complexes was determined (see below). In this paper we study the homological $(k - 1)$-connectivity of random $k$-dimensional complexes for a general fixed $k$.

We recall some topological terminology (see e.g. [4]). Let $X$ be a finite simplicial complex on the vertex set $V$. Let $X^{(k)} = \{ \sigma \in X : \dim \sigma \leq k \}$ denote the $k$-dimensional skeleton of $X$, and let $X(k)$ denote the set of $k$-dimensional simplices in $X$, each taken with an arbitrary but fixed orientation. Denote by $f_k(X) = |X(k)|$ the number of $k$-dimensional simplices in $X$. Let $R$ be a fixed finite abelian group of cardinality $r$. A simplicial $k$-cochain is an $R$-valued skew-symmetric function on all ordered $k$-simplices of $X$. For $k \geq 0$ let $C^k(X)$ denote the group of $k$-cochains on $X$. The $i$-face of an ordered $(k + 1)$-simplex $\sigma = [v_0, \ldots, v_{k+1}]$ is the ordered $k$-simplex $\sigma_i = [v_0, \ldots, \hat{v_i}, \ldots, v_{k+1}]$. The coboundary operator $d_k : C^k(X) \to C^{k+1}(X)$ is given by

$$d_k \phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i).$$

It is convenient to augment the cochain complex $\{C^i(X)\}_{i=0}^{\infty}$ with the $(1)$-degree term $C^{-1}(X) = R$ with the coboundary map $d_{-1} : C^{-1}(X) \to C^0(X)$ given by $d_{-1}a(v) = a$ for $a \in R$, $v \in V$. Let $Z^k(X) = \ker(d_k)$ denote the space of $k$-cocycles and let $B^k(X) = \operatorname{Im}(d_{k-1})$ denote the space of $k$-coboundaries. For $k \geq 0$ let $H^k(X; R) = Z^k(X)/B^k(X)$ denote the $k$-th reduced cohomology group of $X$ with coefficients in $R$. We abbreviate $H^k(X) = H^k(X; R)$.

Let $\Delta_{n-1}$ denote the $(n-1)$-dimensional simplex on the vertex set $V = [n]$. Let $Y_{k}(n, p)$ denote the probability space of complexes $\Delta_{n-1}^{(k-1)} \subset Y \subset \Delta_{n-1}^{(k)}$ with probability measure

$$\Pr(Y) = p^{f_k(Y)}(1 - p)^{\binom{n}{k+1} - f_k(Y)}.$$
A \((k - 1)\)-simplex \(\sigma \in \Delta_{n-1}(k - 1)\) is isolated in \(Y\) if it is not contained in any of the \(k\)-simplices of \(Y\). If \(\sigma\) is isolated then the indicator function of \(\sigma\) is a non-trivial \((k - 1)\)-cocycle of \(Y\), hence \(H^{k-1}(Y) \neq 0\). Our main result is that the threshold probability for the vanishing of \(H^{k-1}(Y)\) coincides with the threshold for the non-existence of isolated \((k - 1)\)-simplices in \(Y\).

**Theorem 1.1.** Let \(k \geq 1\) and \(R\) be fixed, and let \(\omega(n)\) be any function which satisfies \(\omega(n) \to \infty\) then

\[
\lim_{n \to \infty} \Pr \left[ Y \in Y_k(n, p) : H^{k-1}(Y; R) = 0 \right] = \begin{cases} 
0 & p = \frac{k \log n - \omega(n)}{n} \\
1 & p = \frac{k \log n + \omega(n)}{n} 
\end{cases} \quad (1)
\]

**Remarks:**
1. Theorem 1.1 remains true when \(H^{k-1}(Y)\) is replaced by the \((k - 1)\)-th reduced homology group \(H_{k-1}(Y) = H^{k-1}(Y; R)\). This follows from the universal coefficient theorem since \(H_{k-2}(Y) = 0\) for \(Y \in Y_k(n, p)\).
2. The \(k = 1\) case of Theorem 1.1 is the Erdős-Rényi result. For \(k = 2\) and \(R = \mathbb{Z}_2\) the theorem was proved in [3]. Our approach to the general case combines the method of [3] with some additional new ideas.

The case \(p = \frac{k \log n - \omega(n)}{n}\) of Theorem 1.1 is straightforward: Let \(g(Y)\) denote the number of isolated \((k - 1)\)-simplices of \(Y\). Then

\[E[g] = \binom{n}{k} (1 - p)^{n-k} = \Omega(\exp(\omega(n)))\ .\]

A standard second moment argument then shows that

\[
\Pr[H^{k-1}(Y) = 0] \leq \Pr[g = 0] = o(1) 
\]

The case \(p = \frac{k \log n + \omega(n)}{n}\) is more involved. For a \(\phi \in C^{k-1}(\Delta_{n-1})\) denote by \([\phi]\) the image of \(\phi\) in \(H^{k-1}(\Delta_{n-1}^{(k-1)})\). Let

\[b(\phi) = |\{\tau \in \Delta_{n-1}(k) : d_{k-1}\phi(\tau) \neq 0\}| .\]

For any complex \(Y \supset \Delta_{n-1}^{(k-1)}\) we identify \(H^{k-1}(Y)\) with its image under the natural injection \(H^{k-1}(Y) \to H^{k-1}(\Delta_{n-1}^{(k-1)})\). It follows that for \(\phi \in C^{k-1}(\Delta_{n-1})\)

\[
\Pr[\ [\phi] \in H^{k-1}(Y) ] = (1 - p)^{b(\phi)} .
\]
For $\phi \in C^{k-1}(\Delta_{n-1})$ let $\operatorname{supp}(\phi) = \{ \sigma \in \Delta_{n-1}(k-1) : \phi(\sigma) \neq 0 \}$. The weight of such $\phi$ is defined by

$$w(\phi) = \min \{ |\operatorname{supp}(\phi')| : \phi' \in C^{k-1}(\Delta_{n-1}) , [\phi'] = [\phi] \} = \min \{ |\operatorname{Supp}(\phi + d_{k-2}\psi)| : \psi \in C^{k-2}(\Delta_{n-1}) \}.$$ 

A $k$-uniform hypergraph $F \subset \binom{[n]}{k}$ is connected if for any $\sigma, \tau \in F$ there exists a sequence $\sigma = \sigma_0, \ldots, \sigma_t = \tau \in F$ such that $|\sigma_i \cap \sigma_{i-1}| = k-1$ for all $1 \leq i \leq t$. Let

$$G_n = \{ 0 \neq \phi \in C^{k-1}(\Delta_{n-1}) : \operatorname{supp}(\phi) \text{ is connected} , w(\phi) = |\operatorname{supp}(\phi)| \}.$$ 

If $H^{k-1}(Y) \neq 0$ and $\phi \in C^{k-1}(\Delta_{n-1})$ is a cochain of minimum support size such that $0 \neq [\phi] \in H^{k-1}(Y)$, then $\phi \in G_n$. Therefore

$$\Pr[H^{k-1}(Y) \neq 0] \leq \sum_{\phi \in G_n} \Pr[ [\phi] \in H^{k-1}(Y) ] = \sum_{\phi \in G_n} (1 - p)^{b(\phi)}.$$ 

Theorem 1.1 will thus follow from

**Theorem 1.2.** For $p = \frac{k \log n + \omega(n)}{n}$

$$\sum_{\phi \in G_n} (1 - p)^{b(\phi)} = o(1).$$

The main ingredients in the proof of Theorem 1.2 are a lower bound on $b(\phi)$ given in Section 2, and an estimate for the number of $\phi \in G_n$ with prescribed values of $b(\phi)$ given in Section 3. In Section 4 we combine these results to derive Theorem 1.2. The group $R$ and the dimension $k$ are fixed throughout the paper. We use $c_i = c_i(r, k)$ to denote constants depending on $r$ and $k$ alone.

## 2 A lower bound on $b(\phi)$

We bound $b(\phi)$ in terms of the weight $w(\phi)$.

**Proposition 2.1.** For $\phi \in C^{k-1}(\Delta_{n-1})$

$$b(\phi) \geq \frac{nw(\phi)}{k+1}.$$
**Proof:** For an ordered simplex \( \tau = [v_0, \ldots, v_\ell] \) and a vertex \( v \notin \tau \), let \( v\tau = [v, v_0, \ldots, v_\ell] \). For \( u \in V \) define \( \phi_u \in C^{k-2}(\Delta_{n-1}) \) by

\[
\phi_u(\tau) = \begin{cases} 
\phi(u\tau) & u \notin \tau \\
0 & u \in \tau 
\end{cases}
\]

(4)

Let \( \sigma \in \Delta_{n-1}(k-1) \) and \( u \in V \). Then

\[
\phi(\sigma) - d_{k-2}\phi_u(\sigma) = \begin{cases} 
d_{k-1}\phi(u\sigma) & u \notin \sigma \\
0 & u \in \sigma
\end{cases}
\]

It follows that

\[
(k + 1)|\text{supp}(d_{k-1}\phi)| = |\{(\tau, u) : u \in \tau \in \text{supp}(d_{k-1}\phi)\}| = |\{(\sigma, u) \in \Delta_{n-1}(k-1) \times V : \sigma \in \text{supp}(\phi - d_{k-2}\phi_u)\}| = \\
\sum_{u \in V} |\text{supp}(\phi - d_{k-2}\phi_u)| \geq nw(\phi) .
\]

\( \square \)

**Remark:** The following example shows that equality can be attained in (3). Let \( n \) be divisible by \( k + 1 \), and let \( [n] = \cup_{i=0}^{k} V_i \) be a partition of \([n]\) with \( |V_i| = \frac{n}{k+1} \). Consider the unique cochain \( \phi \in C^{k-1}(\Delta_{n-1}) \) that satisfies

\[
\phi([v_0, \ldots, v_{k-1}]) = \begin{cases} 
1 & v_i \in V_i \text{ for all } 0 \leq i \leq k-1 \\
0 & |\{v_0, \ldots, v_{k-1}\} \cap V_i| \neq 1 \text{ for some } 0 \leq i \leq k-1.
\end{cases}
\]

Then \( b(\phi) = (\frac{n}{k+1})^{k+1} \), and it can be shown that \( w(\phi) = |\text{supp}(\phi)| = (\frac{n}{k+1})^k \).

### 3 The number of \( \phi \) with prescribed \( b(\phi) \)

Let

\[
\mathcal{G}_n(m) = \{ \phi \in \mathcal{G}_n : |\text{supp}(\phi)| = m \}
\]

and for \( 0 \leq \theta \leq 1 \) let

\[
\mathcal{G}_n(m, \theta) = \{ \phi \in \mathcal{G}_n(m) : b(\phi) = (1 - \theta)mn \} .
\]

Write \( g_n(m) = |\mathcal{G}_n(m)| \) and \( g_n(m, \theta) = |\mathcal{G}_n(m, \theta)| \). Proposition 2.1 implies that \( g_n(m, \theta) = 0 \) for \( \theta > \frac{k}{k+1} \). Our main estimate is the following
Proposition 3.1. There exists a constant $c_1 = c_1(r,k)$ such that for any $n \geq 10k^2$, $m \geq \frac{n}{2k}$, and $\theta \geq \frac{1}{2k}$

$$g_n(m, \theta) \leq \left(c_1 \cdot n^{(k-1)(1-\theta(1-\frac{1}{2k}))}\right)^m.$$ (5)

The proof of Proposition 3.1 depends on a certain partial domination property of hypergraphs. Let $F \subset \left(\left[\frac{n}{k}\right]\right)$ be a $k$-uniform hypergraph of cardinality $|F| = m$. For $\sigma \in F$ let

$$\beta_F(\sigma) = |\{ \tau \in \left(\left[\frac{n}{k}\right]\right) : (\tau \cap F) = \{\sigma\}\}|$$

and let $\beta(F) = \sum_{\sigma \in F} \beta_F(\sigma)$. Clearly $\beta_F(\sigma) \leq n - k$ and $\beta(F) \leq m(n - k)$.

For $S \subset F$ let

$$\Gamma(S) = \{ \eta \in F : |\eta \cap \sigma| = k - 1 \text{ for some } \sigma \in S \}.$$ 

Claim 3.2. Let $0 < \epsilon \leq \frac{1}{2}$ and $n > 2 \log \frac{1}{\epsilon} + k$. Suppose that

$$\beta(F) \leq (1 - \theta)m(n - k)$$

for some $0 < \theta \leq 1$. Then there exists a subfamily $S \subset F$ such that

$$|\Gamma(S)| \geq (1 - \epsilon)\theta m$$

and

$$|S| < (20 \log \frac{1}{\epsilon}) \cdot \frac{m}{n - k} + 2 \log \frac{1}{\epsilon \theta}.$$ 

proof: Let $c(\epsilon) = 2 \log \frac{1}{\epsilon}$. Choose a random subfamily $S \subset F$ by picking each $\sigma \in F$ independently with probability $\frac{c(\epsilon)}{n - k}$. For any $\sigma \in F$ there exist distinct $v_1, \ldots, v_{n-k-\beta_F(\sigma)} \in [n] - \sigma$ and $\tau_1, \ldots, \tau_{n-k-\beta_F(\sigma)} \in \left(\left[\frac{n}{k}\right]\right)^{\left(k-1\right)}$ such that $\tau_i \cup \{v_i\} \in F$ for all $i$. In particular

$$\Pr[ \sigma \notin \Gamma(S) ] \leq \left(1 - \frac{c(\epsilon)}{n - k}\right)^{n-k-\beta_F(\sigma)},$$

hence

$$E[ |F - \Gamma(S)| ] \leq \sum_{\sigma \in F} \left(1 - \frac{c(\epsilon)}{n - k}\right)^{n-k-\beta_F(\sigma)}.$$ (6)
Since
\[ \sum_{\sigma \in F} (n-k - \beta_\mathcal{F}(\sigma)) = m(n-k) - \beta(\mathcal{F}) \geq \theta m(n-k) \]
it follows by convexity from (6) that
\[ E[|\mathcal{F} - \Gamma(S)|] \leq (1 - \theta)m + \theta m \left( 1 - \frac{c(\epsilon)}{n-k} \right)^{n-k} \leq \]
\[ (1 - \theta)m + \theta m e^{-c(\epsilon)} = (1 - \theta)m + \theta m e^2 . \]
Therefore
\[ E[|\Gamma(S)|] \geq (1 - e^2)\theta m . \]
Hence, since $|\Gamma(S)| \leq |\mathcal{F}| = m$, it follows that
\[ \Pr[|\Gamma(S)| \geq (1 - \epsilon)\theta m ] > \epsilon(1 - \epsilon)\theta . \tag{7} \]
On the other hand
\[ E[|S|] = \frac{c(\epsilon)m}{n-k} \]
and by the large deviation inequality (see e.g. Theorem A.1.12 in [1])
\[ \Pr[|S| > \lambda \frac{c(\epsilon)m}{n-k}] < \left( \frac{e^{\lambda}}{\lambda} \right)^{\frac{c(\epsilon)m}{n-k}} \tag{8} \]
for all $\lambda \geq 1$. Let
\[ \lambda = 10 + \frac{n-k}{m} \left( \frac{\log \frac{1}{\theta}}{\log \frac{1}{\epsilon}} + 1 \right) \]
then
\[ \epsilon(1 - \epsilon)\theta > \left( \frac{e}{\lambda} \right)^{\frac{c(\epsilon)m}{n-k}} . \]
Hence by (7) and (8) there exists an $S \subset \mathcal{F}$ such that $|\Gamma(S)| \geq (1 - \epsilon)\theta m$ and
\[ |S| \leq \lambda \frac{c(\epsilon)m}{n-k} = (20 \log \frac{1}{\epsilon}) \cdot \frac{m}{n-k} + 2 \log \frac{1}{\epsilon \theta} . \]

\[ \square \]
Proof of Proposition 3.1: Define

\[ F_n(m, \theta) = \{ \mathcal{F} \subset \binom{[n]}{k} : |\mathcal{F}| = m, \beta(\mathcal{F}) \leq (1 - \theta)mn \} \]

and let \( f_n(m, \theta) = |F_n(m, \theta)| \). If \( \phi \in \mathcal{G}_n(m, \theta) \), then \( \mathcal{F} = \text{Supp}(\phi) \in F_n(m, \theta) \). Indeed, if \( \tau \in \binom{[n]}{k+1} \) satisfies \( \binom{\tau}{k} \cap \mathcal{F} = \{ \sigma \} \), then \( d_{k-1}\phi(\tau) = \pm \phi(\sigma) \neq 0 \), hence \( \beta(\mathcal{F}) \leq b(\phi) = (1 - \theta)mn \). Therefore

\[
g_n(m, \theta) \leq (r - 1)^m f_n(m, \theta) .
\]

We next estimate \( f_n(m, \theta) \). Let \( \mathcal{F} \in F_n(m, \theta) \), then

\[
\beta(\mathcal{F}) \leq (1 - \theta)mn = (1 - \frac{\theta n - k}{n - k})m(n - k) .
\]

Applying Claim 3.2 with \( \theta' = \frac{\theta n - k}{n - k} \) and \( \epsilon = \frac{1}{2k^2} \), it follows that there exists an \( S \subset \mathcal{F} \) of cardinality \(|S| \leq \frac{c_2m}{n} \) with \( c_2 = c_2(k) \), such that \(|\Gamma(S)| \geq (1 - \frac{1}{2k^2})\theta'm \). The injectivity of the mapping

\[
\mathcal{F} \rightarrow (S, \Gamma(S), \mathcal{F} - \Gamma(S))
\]

implies that

\[
f_n(m, \theta) \leq \sum_{i=0}^{c_2m/n} \binom{\binom{n}{k}}{i} \cdot 2^{(\frac{c_2m}{n})kn} \cdot \sum_{j=0}^{(1 - \theta'(1 - \frac{1}{2k^2}))m} \binom{n}{j} \leq \]

\[
c_3^m \binom{n}{k} \left(1 - \theta'(1 - \frac{1}{2k^2})m\right) \leq c_4^m \left(\frac{n}{m}\right)^{(1 - \theta'(1 - \frac{1}{2k^2}))m} .
\]

Therefore

\[
g_n(m, \theta) \leq (r - 1)^m f_n(m, \theta) \leq (r - 1)^mc_4^m \left(\frac{n}{m}\right)^{(1 - \theta'(1 - \frac{1}{2k^2}))m} \leq \left(c_1 \cdot n^{(k-1)(1 - \theta'(1 - \frac{1}{2k^2}))}\right)^m .
\]

\(\square\)
4 Proof of Theorem 1.2

Proof of Theorem 1.2: Let $\omega(n) \to \infty$ and let $p = \frac{k \log n + \omega(n)}{n}$. We have to show that

$$\sum_{m \geq 1} \sum_{\phi \in G_n(m)} (1 - p)^{b(\phi)} = o(1).$$

(9)

We deal separately with two intervals of $m$:

(i) $1 \leq m \leq \frac{n}{2k}$. If $\phi \in G_n(m)$ then $\text{supp}(\phi) \subset \binom{[n]}{k}$ is a connected $k$-uniform hypergraph, hence there exists a subset $S \subset [n]$ of cardinality $|S| \leq m + k - 1$ such that $\text{supp}(\phi) \subset \binom{S}{k}$. Since $d_{k-1}(u\sigma) = \phi(\sigma) \neq 0$ for any $\sigma \in \text{supp}(\phi)$ and $u \not\in S$, it follows that $b(\phi) \geq m(n - m - k + 1)$. The trivial estimate

$$g_n(m) \leq (r - 1)^m \binom{n}{m} \leq c_5^m \left( \frac{n^k}{m} \right)^m$$

implies that for $n \geq 6k$

$$g_n(m)(1 - p)^{m(n - m - k + 1)} \leq$$

$$c_5^m \left( \frac{n^k}{m} \right)^m \left( 1 - \frac{k \log n + \omega(n)}{n} \right)^{m(n - m - k + 1)} \leq$$

$$c_5^m \left( \frac{n^k}{m} \right)^m e^{-\frac{w(n)(n - m - k + 1)}{n}} \leq$$

$$c_6^m \left( \frac{n^k}{m} \right)^m e^{-\frac{w(n)}{3m}} =$$

$$\left( c_6 \frac{n^k}{m} e^{-\frac{w(n)}{3}} \right)^m.$$

Since

$$\frac{n^k}{m} \leq \begin{cases} n^{k-1/3} & m \leq n^{2/3} \\ n^{-1/6} & n^{2/3} \leq m \leq \frac{n}{2k} \\ \end{cases}$$

it follows that there exists a $c_7 = c_7(r, k)$ such that for $m \leq \frac{n}{2k}$ and $n \geq 6k$

$$g_n(m)(1 - p)^{m(n - m - k + 1)} \leq \left( c_7 e^{-\frac{w(n)}{3}} \right)^m.$$
Therefore
\[
\sum_{m=1}^{n/2k} \sum_{\phi \in \mathcal{G}_n(m)} (1-p)^{b(\phi)} \leq \sum_{m=1}^{n/2k} g_n(m)(1-p)^{m(n-m-k+1)} \leq \\
\sum_{m=1}^{n/2k} \left( c_7 e^{-\frac{w(n)}{m}} \right)^m = O(e^{-\frac{w(n)}{m}}) = o(1) .
\] (10)

(ii) \( m \geq \frac{n}{2k} \). Then
\[
\sum_{m \geq n/2k} \sum_{\theta \leq 1/2k} \sum_{\phi \in \mathcal{G}_n(m,\theta)} (1-p)^{b(\phi)} = \\
\sum_{m \geq n/2k} \sum_{\theta \leq 1/2k} g_n(m,\theta)(1-p)^{(1-\theta)mn} \leq \\
\sum_{m \geq n/2k} g_n(m)(1-p)^{(1-\frac{1}{2k})mn} \leq \\
\sum_{m \geq n/2k} \left( \frac{c_5 n^k}{m} \right)^m n^{-(1-\frac{1}{2k})km} \leq \\
\sum_{m \geq n/2k} (2kc_5 n^{-1})^m n^{-(1-\frac{1}{2k})km} = \\
\sum_{m \geq n/2k} \left( 2kc_5 n^{-1/2} \right)^m = n^{-\Omega(n)} .
\] (11)

Next note that by Proposition 2.1, \( g_n(m,\theta) = 0 \) for \( \theta > \frac{k}{k+1} \). Hence, by Proposition 3.1
\[
\sum_{m \geq n/2k} \sum_{\theta \geq 1/2k} \sum_{\phi \in \mathcal{G}_n(m,\theta)} (1-p)^{b(\phi)} = \\
\sum_{m \geq n/2k} \sum_{\theta \geq 1/2k} g_n(m,\theta)(1-p)^{(1-\theta)mn} \leq \\
\sum_{m \geq n/2k} \sum_{\theta \geq 1/2k, g_n(m,\theta) \neq 0} \left( c_1 \cdot n^{(k-1)(1-\theta(1-\frac{1}{2k}))} \right)^m \cdot n^{-(1-\theta)km} = \\
10
\[
\sum_{m \geq n/2k} \sum_{\theta \geq 1/2k} \left( c_1 \cdot n^{\theta(1 + \frac{k-1}{2k}) - 1} \right)^m \leq n^{k+1} \sum_{m \geq n/2k} \left( c_1 \cdot n^{1/4 \left(1 + \frac{k-1}{2k}\right)-1} \right)^m = \\
n^{k+1} \sum_{m \geq n/2k} \left( c_1 n^{-1/2k} \right)^m = n^{-\Omega(n)} \quad (12)
\]

Finally (9) follows from (10), (11) and (12).

\[\blacksquare\]

5 Concluding Remarks

We have shown that in the model \( Y_k(n, p) \) of random \( k \)-complexes on \( n \) vertices, the threshold for the vanishing of \( H^{k-1}(Y; R) \) occurs at \( p = k \log \frac{n}{n} \), provided that both \( k \) and the finite coefficient group \( R \) are fixed. One natural concrete question is whether \( p = \frac{k \log n}{n} \) is also the threshold for the vanishing of \( H^{k-1}(Y; \mathbb{Z}) \).

More generally, in view of the detailed understanding of the evolution of random graphs (see e.g. [1]), it would be interesting to formulate and prove analogous statements concerning the topology of random complexes. For example, what is the higher dimensional counterpart of the remarkable double-jump phenomenon that occurs in random graphs?

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