Mode Solutions of the Klein-Gordon equation in warped spacetimes

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Abstract

In order to reduce the Klein-Gordon equation (with minimal coupling), we introduce a generalization of the so-called "mode solutions" that are well known in the special case of a Robertson-Walker universe. After separation of the variables, we end up with a partial differential equation in lower dimension. A reduced version of the Gordon current arises and is conserved. When the first factor-manifold is Lorentzian, distinct modes appear as orthogonal in the sense of the sesquilinear form associated with the Gordon current. Moreover, a sesquilinear form is defined on the space of solutions to the reduced equation. Extension of this picture to curvature coupling is possible when the second factor manifold is of constant scalar curvature.

1 Introduction

A warped spacetime is a product manifold \( V_1 \times V_2 \) endowed with (omitting the canonical projections) a metric

\[
(g) = \alpha \oplus (-S)\gamma
\]  

(1)

where \( \alpha \) and \( \gamma \) respectively are metric tensors on \( V_1 \) and \( V_2 \). Either \( \alpha \) or \( \gamma \) is Lorentzian, so that \( (g) \) is normal hyperbolic. \( S \) is a positive function on \( V_1 \), it is convenient to set \( S = e^{2i\Theta} \).

The most simple example of warped spacetime is probably given by the Friedman-Robinson-Walker (FRW) universe. But the warped structure accommodates a large number of metrics of physical interest in General Relativity.

\( ^1 \)The minus sign is dictated by our signature \(+ \ldots -\). See next Section.
In FRW, the wave equation has been studied in details for many decades, in relation with the old problem of defining its positive-frequency solutions. To this end, the general solution is usually developed over the so-called mode solutions, so the Klein-Gordon equation gets reduced to an ordinary differential equation involving the time variable only.

In fact the property of being spatially homogeneous, exhibited by FRW universes, plays no role in this reduction, which permits to follow the same line in generalized FRW spacetimes where the spatial sections have not necessarily a constant curvature. The geometrical properties of these spacetimes have been systematically investigated by M. Sánchez.

Warped structures in general have been widely studied in the literature, the main issue being of course to establish relations between symmetry and curvature properties of the total spacetime and that of its factor manifolds.

In this article we are concerned with interesting features exhibited by the Klein Gordon equation in a warped spacetime of arbitrary dimension and type (with some preference however for the case where $V_1$ is Lorentzian). Assuming the minimal coupling we write

$$\left(\nabla^2 + m^2\right)\Psi = 0$$

for a complex $c$-number-valued wave function.

In spite of well-known limitations, a particle interpretation of (2) remains of interest. In the operator approach to quantum field theory in curved spacetime, one-particle concepts play at least the role of useful tools. For instance, the kernel which projects any solution onto a positive-frequency subspace and determines a definition of the vacuum, is a solution of the KG equation (often referred to as the two-point Wightman function).

Our goal is to show that most results obtained up to now in the context of generalized FRW spacetimes can be systematically extended to any kind of warped spacetime.

We shall take advantage of the warped-product structure in order to carry out a systematic reduction of (2), and shall end up with an equation to be solved for a reduced wave function which depends only on the coordinates running in the first factor manifold $V_1$.

The principle of this procedure consists in developing the general solution of equation (2) over special ones that generalize the mode solutions arising in the customary FRW framework.

The possibility of reducing the KG equation with help of generalized modes can be tracked back to a remarkable feature of classical motion in warped spacetimes. That point is briefly exposed in Section 2, in parallel to a study of the differential operator which describes the quantum motion. Generalized modes are introduced in Section 3, where the separation of variables is carried out.

Together with the KG equation, we analyse the sesquilinear form defined on its solutions. Our motivation is the fact that, for complex (resp. real) solutions sesquilinear (resp. bilinear) forms are fundamental with respect to quantum mechanics; they provide a framework for the construction of a "complex structure positive operator"
ensuring the splitting of any solution into positive and negative-frequency parts.

We shall demonstrate in Sections 4-5 that the sesquilinear form defined for solutions to equation (3) (by conservation of the Gordon current) undergoes some kind of reduction. Indeed, under very general assumptions, the space of complex solutions to the reduced equation can be in turn endowed with a sesquilinear form of its own. Actually, a conserved vector-density defined on the first factor manifold is associated with the reduced wave equation; we display the relationship of this object with the usual Gordon current.

The possibility of extending this study to nonminimal couplings is discussed in Section 6.

Although we have in mind possible applications to quantum mechanics, this article is written from the viewpoint of differential geometry; all functions and tensors are supposed to be smooth, that is $C^\infty$.

Moreover all manifolds considered here are implicitly assumed to be connected.

1.1 Notation

In the product $(V) = V_1 \times V_2$, the factors $V_1$, $V_2$ respectively have dimensions $p$, $q$. We define Type I (resp. Type II) by this property that $(V_1, \alpha)$ (resp. $(V_2, \gamma)$) is Lorentzian.

Throughout this paper we use coordinate charts adapted to the warped structure. The metric takes on the orthogonal form

$$ds^2 = \alpha_{AB} dx^A dx^B - S(x^C)\gamma(x^k)_{ij} dx^i dx^j$$

(3)

where $A, B, C \in I_1$, $i, j, k \in I_2$, $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2$ covers the whole set of integers $0, 1, 2, \ldots, p + q - 1$.

[For instance in 4 dimensions: $I_1 \ni 0$, $I_2 \ni 1, 2, 3$ for FRW spacetimes, and we can take $I_1 \ni 0, 3$ $I_2 \ni 1, 2$ for spherically symmetric universes]

The minus sign in equation (3) is dictated by our signature $+ - - -$. Type I necessarily corresponds to $\gamma$ positive definite. In contrast, Type II implies having the quadratic form $\alpha$ elliptic but negative definite.

g_{AB} = \alpha_{AB}$ and $g_{ij} = -S\gamma_{ij}$.

[Example: In FRW spacetimes, $p = 1$, $q = 3$ and we have $A, B, C = 0$ and $i, j, k = 1, 2, 3$].

It is clear that

g_{\mu\nu} = \begin{pmatrix} \alpha_{AB} & 0 \\ 0 & -S\gamma_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \alpha^{AB} & 0 \\ 0 & -S^{-1}\gamma^{ij} \end{pmatrix}

where $g^{AB}g_{BC} = \delta^A_C$, $g^{ij}g_{jk} = \delta^i_k$. Notice that $g^{AB} = \alpha^{AB}$

if we define $\alpha^{AB}$ (resp. $\gamma^{ij}$) as the contravariant tensor inverting $\alpha_{AB}$ (resp. $\gamma_{ij}$), that is $\alpha^{AB}\alpha_{BC} = \delta^A_C$, $\gamma^{AB}\gamma_{BC} = \delta^A_C$. We obviously can write $g_{AB} = \alpha_{AB}$.
Notice that
\[ g^{ij} = (-S)^{-1} \gamma^{ij} \]

Caution that the Types I, II defined above must not be confused with the Classes A, B introduced by Carot and da Costa for four-dimensional warped spacetimes [6].

For \( p + q = 4 \), intersection of Classes \( A_1 \), \( A_2 \) and \( B \) with both Types give rise to six possibilities.

In references [6], [5] each class is combined with the Types by choosing a sign \( \pm \) in the generic form of the metric. The contact with the notation of Ref. [6] can be made as follows:

\[ \alpha = -h_1, \quad \gamma = h_2 \]

where \( h_1 \) and \( h_2 \) are the metrics assigned to the factor manifolds in Ref. [6]. Our quadratic form \( ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \) and their have opposite signs.

Geometric objects corresponding to \((V_1, \alpha)\), \((V_2, \gamma)\) are affected by the index 1, 2 respectively. This label will be put on the left for connexions, curvature tensors and their contractions.

Remark
It is a trivial observation that the Minkowski metric is (globally) decomposable in several ways. Similarly, it may happen that a given spacetime can be considered as warped in several ways, so that, for instance, it may be of Class A for one structure whilst it is of Class B for another one [6]. But here, we consider only one structure at one time, so the six possibilities mentioned above are mutually exclusive.

\[ \eta = \sqrt{|g|} \varepsilon \] where \( \varepsilon \) is the Levi-Civita tensor. Useful determinants are as follows: Setting

\[ g = \det g_{\alpha\beta} \quad \gamma = \det \gamma_{ij} \quad \alpha = \det \alpha_{AB} \equiv \det g_{AB} \]

we have

\[ \det g_{ij} = (-S)^q \det \gamma_{ij} \]

and therefore \( g = \alpha (-S)^q \eta \) \( g = \det g_{AB} (-S)^q \det \gamma_{ij} \) thus \( g = \alpha (-S)^q \gamma \). We shall rather use

\[ \sqrt{|g|} = \sqrt{|\alpha| S^{q/2}} \sqrt{|\gamma|} \] (4)

Volume elements: Setting

\[ d^p x = \bigwedge dx^A, \quad d^q x = \bigwedge dx^j, \quad A \in I_1, \quad j \in I_2 \]

we have \( d^{p+q} x = d^p x \wedge d^q x \).

The volume elements of \((V_1, \alpha)\) and \((V_2, \gamma)\) respectively are \( \sqrt{|\alpha|} d^p x \) and \( \sqrt{|\gamma|} d^q x \), whereas naturally the volume form of \((V)\) is \( \sqrt{|g|} d^{p+q} x \).

For Type I, it is convenient to take \( x^A \) running from 0 to \( p - 1 \) and to factorize out the time coordinate by setting

\[ \omega = dx^1 \wedge \ldots \wedge dx^{p-1} \] (5)

so that \( d^p x = dx^0 \wedge \omega \).
With this convention, we have

\[ \omega = \frac{1}{(p-1)!} \varepsilon_{0B_1...B_{p-1}} \, dx^{B_1} \wedge \ldots \wedge dx^{B_{p-1}} \]  

(6)

2 Classical and Quantum Motion in Warped Spacetime

2.1 Geodesic motion

An equation like (6) can be thought of as describing the quantum motion of a test particle, in the approximation where possible particle creation is neglected. It is in order to point out that, in any warped spacetime, the classical motion of a free particle (geodesic motion) already enjoys an interesting property which directly stems from the warping.

Indeed the equations of motion of a test particle in \( V \) are canonically generated by a "Hamiltonian function" \([10]\)

\[ G(x, p) = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta = \frac{1}{2} (\alpha^{AB} p_A p_B - S^{-1} \gamma^{ij} p_i p_j) \]

which is a scalar in the cotangent bundle \( T_*(V) \). According to the canonical symplectic form of this bundle, we have the usual Poisson brackets \( \{ x^\alpha, p_\beta \} = \delta^\alpha_\beta \), etc. Constants of the motion are characterized by a vanishing Poisson bracket with \( G \).

It is easy to verify that

**Proposition 1** In any warped spacetime, with the metric written like in (4), geodesic motion admits the first integral

\[ 2K = \gamma^{ij} p_i p_j \]  

(7)

where \( p_\alpha \) are the momenta.

[ Proof: We see that \( \{ K, \alpha^{AB} p_A p_B \} \) vanishes. This is obvious since \( K \) only depends on \( x^C, \vec{p}_D \). Then we observe that \( \{ p_j, x^A \} = 0 \). It follows that \( \{ K, S \} = 0 \), so finally \( \{ g^{\alpha\beta} p_\alpha p_\beta, K \} = 0 \) and \( K \) is a constant of the motion. ] Indeed we derive with help of the standard Poisson brackets that \( \{ G, K \} = 0 \).

For Type I, the quantity \( K/m \) somehow generalizes the kinetic energy, although its conservation is ensured even if \( (V_2, \gamma) \) fails to admit a group of translations. For instance, when \( V \) is simply \( \mathbb{R} \times \mathbb{R}^3 \) warped with some time-depending scale factor, the conservation of \( K \) can also be derived from the existence of a translation group in \( \mathbb{R}^3 \). And in this case, \( K/m \) is just the kinetic energy in the usual sense. But the point is that this property survives when \( \mathbb{R}^3 \) is replaced by any other three-dimensional manifold.
2.2 Quantum motion

We assume minimal coupling, so we write the KG equation as (2) where $\nabla^2 \Psi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Psi$. But we shall use the well-known formula

$$\nabla^2 \Psi = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \Psi)$$

(8)

By formal analogy with (7) it is natural to consider that the quantum mechanical analog of $K$ is $K_{\text{quant}} = -\frac{1}{2} \Delta_2$ where $\Delta_2$ is the Laplace-Beltrami operator in $(V_2, \gamma)$. Indeed we have in obvious notations $\Delta_2 = \gamma^{ij} (\nabla_i \partial_j) \psi$. But we rather use the formula

$$\Delta_2 \Psi = \frac{1}{\sqrt{|\gamma|}} \partial_i (\sqrt{|\gamma|} \gamma^{ij} \partial_j \Psi)$$

Developing formula (8) we get

$$\sqrt{|g|} \nabla^2 \Psi = \partial_A (\sqrt{|g|} g^{AB} \partial_B \Psi) + \partial_t (\sqrt{|g|} g^{ij} \partial_j \Psi)$$

but $g^{AB} = \alpha^{AB}$ and $g^{ij} = -S^{-1} \gamma^{ij}$ thus

$$\sqrt{|g|} \nabla^2 \Psi = \partial_A (\sqrt{|g|} g^{AB} \partial_B \Psi) - \partial_t (\sqrt{|g|} S^{-1} \gamma^{ij} \partial_j \Psi)$$

where $\partial_t S = 0$. But in view of (4) we have

$$\sqrt{|g|} \nabla^2 \Psi = \partial_A (\sqrt{|g|} g^{AB} \partial_B \Psi) - S^{-1} S^{q/2} \partial_t (\sqrt{|\alpha|} \sqrt{|\gamma|} \gamma^{ij} \partial_j \Psi)$$

Since $\alpha$ only depends on $x^A$ we obtain

$$\sqrt{|g|} \nabla^2 \Psi = \partial_A (\sqrt{|g|} g^{AB} \partial_B \Psi) - S^{-1} S^{q/2} \sqrt{|\alpha|} \partial_t (\sqrt{|\gamma|} \gamma^{ij} \partial_j \Psi)$$

Again develop $\sqrt{|g|}$ and remember that $\partial_A \gamma = 0$. We get

$$\sqrt{|g|} \nabla^2 \Psi = \sqrt{|\gamma|} \partial_A (\sqrt{|\alpha|} S^{q/2} g^{AB} \partial_B \Psi) - S^{-1} S^{q/2} \sqrt{|\alpha|} \partial_t (\sqrt{|\gamma|} \gamma^{ij} \partial_j \Psi)$$

Again develop $\sqrt{|g|}$ hence

$$\nabla^2 \Psi = \frac{1}{\sqrt{|\alpha|} S^{q/2}} \partial_A (\sqrt{|\alpha|} S^{q/2} g^{AB} \partial_B \Psi) - S^{-1} \sqrt{|\gamma|} \partial_t (\sqrt{|\gamma|} \gamma^{ij} \partial_j \Psi)$$

It is convenient to define

$$D \Psi = \frac{1}{\sqrt{|\alpha|} S^{q/2}} \partial_A (\sqrt{|\alpha|} S^{q/2} g^{AB} \partial_B \Psi)$$

irrespective of whether $\Psi$ is a solution to (2) or not.

Indeed the second order differential operator $D$ only affects quantities depending on the $x^A$ variables. So we can write

$$\nabla^2 \Psi = D \Psi - S^{-1} \frac{1}{\sqrt{|\gamma|}} \partial_t (\sqrt{|\gamma|} \gamma^{ij} \partial_j \Psi)$$
In other words we have the identity
\[ \nabla^2 \Psi = D\Psi - S^{-1} \Delta_2 \Psi \] (9)
where \( \Delta_2 \) is the \( q \)-dimensional Laplace-Beltrami operator, associated with the manifold \((V_2, \gamma)\). As an operator extended to functions on \( V \), it does not affect the quantities depending on \( x^A \) only.

It is clear that \( \Delta_2 \) commutes with \( D \), because these operators act on separate sets of variables. As \( S \) does not depend on the \( x^j \) coordinates, it is clear that \( \Delta_2 \) (or equivalently \( K_{\text{quant}} \)) commutes with \( \nabla^2 \). In the classical limit (\( \hbar \to 0 \)) this property reduces to the conservation of \( K \).

Let us re-arrange \( D \) in order to simplify the expression of \( \nabla^2 \Psi \).

Let us provisionally use coordinates where \( |\alpha| = 1 \). We obtain
\[ D\Psi = S^{-q/2} \partial_A (S^{q/2} g^{AB} \partial_B \Psi) \]
\[ D\Psi = \partial_A (g^{AB} \partial_B \Psi) + \frac{q}{2} (\partial_A \log S) g^{AB} \partial_B \Psi \]
But \( g^{AB} = \alpha^{AB} \) and \( |\alpha| = 1 \), thus
\[ D\Psi = \Delta_1 \Psi + \frac{q}{2} \alpha^{AB} (\partial_A \log S) \partial_B \Psi \] (10)
which is valid in all coordinates and for arbitrary \( \Psi \). This expression, where the coordinates \( x^j \) are ignorable, is to be inserted into equation (9).

## 3 Mode Solutions, Product Solutions

Since \([\Delta_2, \nabla^2] \) vanishes, it is clear that some solutions of the KG equation are also eigenstates of \( \Delta_2 \). If \( \Phi \) is such a solution we have
\[ \Delta_2 \Phi = -\lambda \Phi \] (11)
for some constant number \( \lambda \in \text{Spec}(V_2) \). We shall generalize the terminology which is commonly used when \( V \) is FRW universe and shall call \( \Phi \) a Mode Solution to the KG equation. For a solution in mode \( \lambda \), the KG equation reduces to
\[ (D + \lambda S^{-1} + m^2)\Phi = 0 \] (12)
where the differential operator \( D \) acting on \( \Phi \) affects the \( x^A \)'s only.

Finally, \( x^A \) and \( x^j \) are respectively ignorable in equations (12) and (11), which realizes separation of the variables.

Leaving aside the solving of (12), the original KG equation in \( (V) \) has been reduced to a (linear) partial differential equation in \( p \) dimensions.

From now on, we look for solutions to (12) in the form of a superposition of various mode solutions corresponding to all possible values taken by \( \lambda \) in the spectrum of \( V_2 \).
In the special case where \( p = 1 \), equation (12) is an ordinary 2nd order equation and its solutions form a two-dimensional vector space.

Otherwise, the space of solutions still has infinitely many dimensions.

Some special solutions of the wave equation (2) have the form of a product of functions compatible with the product structure of spacetime, namely

\[
\Phi = f(x^A) \ F(x^k)
\]

(13)

We observe that

**Proposition 2** Any product solution \( fF \) to the KG equation is a mode solution, and \( F \) is eigenfunction of \( \Delta_2 \).

Proof.

\[
\nabla^2 \Phi = (Df)F - S^{-1} f \ \Delta_2 F
\]

\[
(\nabla^2 + m^2) \Phi = F(D + m^2)f - S^{-1} f \ \Delta_2 F
\]

According to KG equation this quantity vanishes.

\[
F(D + m^2)f = S^{-1} f \ \Delta_2 F
\]

Discarding a trivial case, neither \( f \) nor \( F \) can identically vanish. When \( f \) and \( F \) are not zero, we divide by \( fF \) and multiply by \( S \). We get

\[
S \frac{(D + m^2)f}{f} = \frac{\Delta_2 F}{F}
\]

The l.h.s. of this equation depends on \( x^A \) only while the r.h.s. only depends on \( x^k \). Both are thus necessarily constants, so there exists some \( \lambda \) such that

\[
\Delta_2 F = -\lambda F
\]

(14)

Let \( \mathcal{E}[\lambda] \) be the space of smooth functions on \( V_2 \) satisfying (14) for a given value of \( \lambda \). For \( V_2 \) compact, \( \Delta_2 \) has a discrete spectrum which is the infinite sequence

\[
\text{Spec}(V_2, \gamma) = \{\lambda_0 = 0, < \lambda_1, \ldots < \lambda_n \ldots\}
\]

In this case \( \mathcal{E}_n \) denotes \( \mathcal{E}[\lambda_n] \) and we know that its dimension \( d(n) \) is finite [?].

The converse of the previous Proposition is not true, but

**Proposition 3** In a warped spacetime of Type I with compact \( (V_2) \), any mode solution corresponding to a given \( \lambda \) in \( \text{Spec}(V_2, \gamma) \) is a finite sum of product solutions.
Let \( f_u(x^A) \) be the coefficients of \( \Phi \) in a development over a basis \( F_1(x^j), \ldots \).

\[
\Phi = \sum_{u=1}^{d(n)} f_u(x^A) \, F_u(x^j)
\]  

(15)

\[
(\nabla^2 + m^2)\Phi = \sum F_u(x^j) \left[ D + S^{-1}\lambda_n + m^2 \right] f_u(x^A)
\]  

(16)

This expression must vanish. Since \( F_1 \ldots F_d \) form a basis, it is clear that each \( f_u \) must be a solution to the equation

\[
(D + \lambda_n S^{-1} + m^2)f = 0
\]  

(17)

It follows that each \( f_u F_u \) is a product solution.

Let \( S[\lambda] \) be the space of smooth functions on \( V_1 \) satisfying (17) for a given value of \( \lambda \).

Except in the very special case where \( p = 1 \), (17) is a partial differential equation and has infinitely many linearly independent solutions.

Developing \( D \) with help of (10) we obtain after simplification

\[
\Delta_1 f + \frac{q}{2} \alpha^{AB} (\partial_A \log S) \partial_B f + (\lambda S^{-1} + m^2)f = 0
\]  

(18)

which is a \( p \)-dimensional problem only, formulated in terms of the metric \( \alpha \). The \( x^j \) do not arise in this equation.

To summarize, equation (2) has been reduced to a pair of equations involving separate sets of variables. These equations are (14) and (18). For Type I (resp. Type II) the Laplacian in the former is elliptic (resp. hyperbolic) whereas the partial differential operator in the latter is hyperbolic (resp. elliptic). Notice that (14) involves only the geometry of \( V_2 \), whereas (18) involves not only the geometry of \( V_1 \) but also the shape of the warping function \( S \).

For Type I with compact \( V_2 \), we write \( S_n \) for \( S[\lambda_n] \). The \( n \)th mode will be noted

\[
\mathcal{H}_n = S_n \otimes \mathcal{E}_n
\]

Now, a comparison of (18) with (2) is in order: Equation (2) simply involves the \( p+q \)-dimensional Laplacian, which implies that the Gordon current in \( p+q \) dimensions is conservative. In contrast, (18) not only involves the \( p \)-dimensional Laplacian \( \Delta_1 \) and an innocent multiplicative operator \( \lambda S^{-1} + m^2 \), but also first order partial differentiation. As a result, the \( p \)-dimensional sesquilinear field constructed with a couple of solutions \( f, h \) to (18), that is to say

\[
I^A = -i(f^* \nabla_1^A h - h \nabla_1^A f^*)
\]  

fails to be divergence-free.
For the same reason, the second order linear differential operator involved in (18) is not symmetric with respect to the scalar product
\[ < f, h >_1 = \int_{V_1} f^* h \sqrt{|\alpha|} d^p x \]
defined with help of the \( p \)-dimensional volume element \( \sqrt{|\alpha|} d^p x \) determined by the metric \( \alpha \) in \( V_1 \).

This situation is more specially unpleasant in Type I spacetimes, where (18) includes the dynamical aspects of the original equation KG equation. In this case, a scalar product for any couple of solutions to (18) would be of interest for quantum mechanics. So, it is desirable to construct some conserved \( p \)-dimensional current, sesquilinear with respect to the couple \( f, h \). But the presence of \( \partial f \) in (18) is an obstacle.

Before we focus our attention on Type I, let us first replace the vector field of formula (19) by a better candidate in order to make up a conservation law.

[N.B. Statements about the divergence of a vector necessarily refer to a metric. In contradistinction, the divergence of a vector-density is intrinsic.]

The difficulty associated with the presence of first derivatives in (18) can be circumvented by two manners.

- Use another metric (conformal to \( \alpha_{AB} \)) on the manifold \( V_1 \) and re-write (18) in terms of it. This procedure amounts to consider another differential operator that is symmetric in the sense of this new metric.
- Keep the metric of \( V_1 \) unaltered, but make a conformal change of function, say \( f = S^r \hat{f} \).

### 4 Conserved Currents

#### 4.1 First Method

Let us consider in \( V_1 \) a new metric \( \bar{\alpha}_{AB} \) such that
\[ \alpha_{AB} = U(x^C)\bar{\alpha}_{AB} \]
for some conformal factor \( U(x^C) \) which must be suitably chosen. It is clear that \( \alpha^{AB} = U^{-1}\bar{\alpha}^{AB} \), if we call \( \bar{\alpha}^{AB} \) the contravariant tensor inverting \( \alpha_{AB} \). We set \( \bar{\nabla}^A = \bar{\alpha}^{AB}\partial_B \).

The determinants are related by
\[ \det \alpha_{AB} = U^p \det(\bar{\alpha}_{AB}) \]  
(20)

In view of this formula, we find
\[ \Delta_1 f = U^{-1} \bar{\Delta}_1 f + (\partial_A \log U^p/2) \alpha^{AB}\partial_B f \]
This is to be inserted into \( Df \) which is given by (18). We get
\[ Df = U^{-1} \bar{\Delta}_1 f + (\partial_A \log U^p/2) \alpha^{AB}\partial_B f + \frac{q}{2}(\partial_A \log S) \alpha^{AB}\partial_B f \]  
(21)
The first derivatives of $f$ are eliminated from $Df$ provided that

$$U^{p/2-1} = \text{const. } S^{-q/2}$$

which is possible (for nontrivial $S$) under the condition that $p \neq 2$. In this case it is sufficient to take $U^{p-2} = S^{-q}$, which leads to

$$U = S^{\frac{q}{2-p}}$$

Thus (20) entails

$$\sqrt{|\alpha|} = S^{\frac{q}{2-p}} \sqrt{|\tilde{\alpha}|}$$

(23)

With the choice (22) we simply have

$$Df = U^{-1} \tilde{\Delta}_1 f$$

to be inserted into equation (17) for mode $\lambda$.

We end up with an equation of the form

$$\tilde{\Delta}_1 f + Qf = 0$$

(24)

where $Q = U(\lambda S^{-1} + m^2)$ but equation (22) implies that

$$Q = S^{\frac{q}{2-p}} (\lambda S^{-1} + m^2)$$

Notice that $\tilde{\Delta}_1 + Q$ in equation (24) is symmetric with respect to the scalar product $<\tilde{f}, \tilde{h}>_1$, defined with help of the $p$-dimensional volume $\sqrt{|\alpha|} \, d_1^p \, x$. Moreover, the $p$-dimensional current

$$\tilde{I}^A(f, h) = -i(f^* \tilde{\nabla}_1^A \tilde{h} - \tilde{h} \tilde{\nabla}_1^A \tilde{f}^*)$$

(25)

is divergence-free in $(V_1, \tilde{\alpha})$ for any couple of solutions $f, h$ to the same equation (24). We mean the same $\lambda$ in (24) for $f$ and $h$ and of course $\tilde{\nabla}_1^A = \tilde{\alpha}^{AB} \partial_B$. In other words we have

$$\partial_A(\sqrt{|\tilde{\alpha}|} \, \tilde{I}^A) = 0$$

Remark: When $p = 1$ then $\tilde{I}^A$ has a single component only, so the above conservation law reduces to the well-known constancy of the Wronskian.

So, provided $p \neq 2$, it is possible to eliminate $\alpha^{AB} \partial_B \log S$ from (21) by chosing $U$ as in (22).

It is noteworthy that in four-dimensional warped spacetime, we precisely have $p = 2$ in the case of Class $B$. Cases with $p = 2$ will be handled by the second method (next section).
4.2 Second Method

Let us try a conformal change of function involving a suitable power of $S$. So we introduce $\hat{f}$ by setting $f = S^r \hat{f}$, for some $r$ to be determined below. Our goal is to eliminate $\partial \hat{f}$.

Define in any dimension the Laplace-Beltrami operator on scalars $\Delta u = \nabla \cdot \nabla u$.
Well known that
$$\Delta (uv) = (\Delta u)v + u\Delta v + 2\nabla u \cdot \nabla v$$
We apply this formula in manifold $V_1$, where the Laplacian operator is $\Delta_1$, to the product $S^r \hat{f}$. In the present case
$$\Delta_1 f = (\Delta_1 S^r) \hat{f} + S^r \Delta_1 \hat{f} + 2\alpha^{AB}(\partial_A S^r) \partial_B \hat{f}$$
But (10) tells that
$$Df = \Delta_1 f + q/2 \alpha^{AB}(\partial_A \log S) \partial_B f$$
where $\partial_B f = (\partial_B S^r) \hat{f} + S^r \partial_B \hat{f}$. Thus
$$Df = \Delta_1 f + q/2 \alpha^{AB}(\partial_A \log S)(\partial_B S^r) \hat{f} + q/2 \alpha^{AB}(\partial_A \log S)(\partial_B \log S) \hat{f}$$
First order derivatives of $\hat{f}$ get cancelled from (18) provided that
$$2\partial_A S^r + q/2 S^r \partial_A \log S = 0.$$ Thus
$$r = -q/4 \quad (26)$$
We are left with
$$Df = (\Delta_1 S^r) \hat{f} + S^r \Delta_1 \hat{f} + q/2 \alpha^{AB}(\partial_A \log S)(\partial_B S^r) \hat{f}$$
$$S^{-r}Df = S^{-r}(\Delta_1 S^r) \hat{f} + \Delta_1 \hat{f} + qr/2 \alpha^{AB}(\partial_A \log S)(\partial_B \log S) \hat{f}$$
In view of (26) we conclude that
$$S^{q/4}Df = \Delta_1 \hat{f} + (S^{q/4} \Delta_1 S^{-q/4}) \hat{f} - q^2/8 \alpha^{AB}(\partial_A \log S)(\partial_B \log S) \hat{f} \quad (27)$$
Equation (12) yields $(D + \lambda S^{-1} + m^2) f = 0$. Multiply by $S^{q/4}$, and use $f = S^{-q/4}\hat{f}$. We get
$$S^{q/4}Df + (\lambda S^{-1} + m^2) \hat{f} = 0$$
where the first term is to be developed in terms of $\hat{f}$ as in (27).
We end up with a reduced equation of the form $\Delta_1 \hat{f} + \Xi \hat{f} = 0$.
If $\hat{f}$ and $\hat{h}$ are two solutions to this equation (for the same $\lambda$) they correspond to $f$ and $h$ through the formulas
$$f = S^r \hat{f}, \quad h = S^r \hat{h} \quad (28)$$
and the $p$-dimensional current
$$J^A(f, h) = I^A(\hat{f}, \hat{h}) = -i(\hat{f}^* \nabla^A \hat{h} - \hat{h} \nabla^A \hat{f}^*) \quad (29)$$
is conserved
\[ \partial_A \sqrt{|\alpha| J^A} = 0 \]

In order to compute \( J^A \) we use (28) and write
\[ \nabla^A \hat{f}^* = \alpha^{AB} S^{-r} \partial_B f^* + \alpha^{AB} (\partial_B S^{-r}) f^* \quad (30) \]
\[ \nabla^A \hat{h} = \alpha^{AB} S^{-r} \partial_B h + \alpha^{AB} (\partial_B S^{-r}) h \quad (31) \]

Inserting into (29) yields
\[ i J^A = \hat{f}^* S^{-r} \nabla^A h - \hat{h} S^{-r} \nabla^A f^* \quad (32) \]

After simplification we are left with
\[ i J^A = \hat{f}^* S^{-r} \nabla^A h - \hat{h} S^{-r} \nabla^A f^* \]

After cancellation of two terms we obtain
\[ i J^A = S^{-2r} (f^* \alpha^{AB} \partial_B h - h \alpha^{AB} \partial_B f^*) \quad (33) \]
\[ J^A = S^{q/2} I^A \quad (34) \]

Now, under the assumption that (22) and (26) hold true we can assert

**Proposition 4** If \( p \neq 2 \) then the vector fields \( \tilde{I} \) on \((V_1, \tilde{\alpha})\) and \( J \) on \((V_1, \alpha)\) correspond to the same (conserved) vector-density, in other words
\[ \sqrt{|\alpha|} J^A = \sqrt{|\tilde{\alpha}|} \tilde{I}^A \]

**Proof**
Start from (29), that is
\[ i J^A = \hat{f}^* \nabla^A \hat{h} - \hat{h} \nabla^A \hat{f}^* \]

use
\[ \hat{f}^* = S^{-r} f^*, \quad \hat{h} = S^{-r} h \]

Multiply equation (32) by \( \sqrt{|\alpha|} \) which is given by (23). Remember (26), hence
\[ \frac{pq}{4 - 2p} - 2r = \frac{q}{2 - p} \]

Thus we find
\[ i \sqrt{|\alpha|} J^A = S^{q/2} (f^* \nabla^A h - h \nabla^A f^*) \sqrt{|\alpha|} \]

At this stage it is convenient to notice that for all scalar function \( u(x^A) \) we have
\[ \nabla^A u = S^{q/2} \nabla^A u \quad (35) \]

Taking this formula into account, we can write
\[ i \sqrt{|\alpha|} J^A = (f^* \nabla^A h - h \nabla^A f^*) \sqrt{|\alpha|} \]

which proves our assertion, according to (25).
4.2.1 Comparison of both methods

The first method is widely employed in the literature concerning FRW universes, and we also used it in the framework of generalized FRW spacetimes [4]. As we have just checked, the first method and the second one are equivalent when both can be carried out. But the former cannot be carried out when \( p = 2 \). So we are forced to consider the latter as more fundamental, since it is not affected by any dimensional exception. In particular it remains the only one available in the important case of a Class B four-dimensional spacetime. For a unified approach that encompasses all cases, we are led to a systematic use of the second method.

5 Sesquilinear Forms for Type I

In this section we specialize to warped spacetimes of Type I, with \( V_2 \) compact. The special case \( p = 1, q = 3 \) concerns generalized FRW spacetimes; it has been treated in [4] and yields results similar to those of the case \( p > 1 \) considered below, where the quantity defined by formula (37) is replaced by \(-iW(f^*, h)\) in terms of the Wronskian. Extension to arbitrary \( q \) is straightforward.

When \( p > 1 \) we assume in addition that \( V_1 \approx \mathbb{R} \times \text{compact} \).

If \( L \) is imbedded in \((V_1, \alpha)\) as a \((p - 1)\)-dimensional spacelike surface, let \( dL_A \) be the \((p - 1)\)-dimensional surface element, that is

\[
dL_A = \frac{1}{(p-1)!} \sqrt{|\alpha|} \varepsilon_{AB_1 \cdots B_{p-1}} \, dx^{B_1} \wedge \cdots \wedge dx^{B_{p-1}} \tag{36}
\]

Conservation of \( J^A \) with respect to \((V_1, \alpha)\) implies that

\[
(f; h)_1 = \int_L J^A(f, h) \, dL_A \tag{37}
\]

does not depend on the choice of \( L \). This expression defines a sesquilinear form on \( S_n \).

5.1 The Gordon Current

Remember usual formulas for \( N \)-dimensional spacetime, with \( \alpha, \beta = 0, 1 \ldots N - 1 \). The space of arbitrary solutions to the KG equation is endowed with a sesquilinear form

\[
(\Phi; \Omega) = \int j^\nu \, d\Sigma_\nu \tag{38}
\]

\[
j^\nu(\Phi, \Omega) = -i(\Phi^* \nabla^\nu \Omega - \Omega \nabla^\nu \Phi^*)
\]

Integration is performed over an \( N - 1 \) dimensional spacelike surface \( \Sigma \). Notation \( d^{N-1}x = dx^1 \wedge dx^2 \cdots \wedge dx^{N-1} \). Be cautious that \( d^{N-1}x \) should not be confused with \( dx^{N-1} \).
Provided \((\Sigma)\) is defined by \(x^0 = \text{const}\), we can write

\[
d\Sigma_0 = \frac{1}{(N-1)!} \eta_{01 \cdots N-1} \, dx^1 \wedge \cdots \wedge dx^{N-1} = \sqrt{|g|} \, d^{N-1}x
\]

For the space components we first have

\[
d\Sigma_1 = \frac{1}{(N-1)!} \eta_{\alpha_1 \cdots \alpha_{N-1}} \, dx^\alpha_1 \cdots \wedge dx^{\alpha_{N-1}}
\]

Here the indices \(\alpha_1, \cdots, \alpha_{N-1} \neq 1\), thus one of them all must be 0 (otherwise they would not be all different, so \(\eta_{\alpha_1 \cdots \alpha_{N-1}}\) would vanish). Thus \(d\Sigma_1\) has \(dx^0\) as a factor, say

\[
d\Sigma_1 = (\cdots) \wedge dx^0.
\]

But \(dx^0\) is zero on \((\Sigma)\). Thus finally \(d\Sigma_1\) vanishes on \((\Sigma)\). By a similar argument we check that \(d\Sigma_2, \ldots, d\Sigma_{N-1}\) also vanish on \((\Sigma)\).

In order to integrate over \((\Sigma)\) we evaluate the differential for \(m_j \cdot d\Sigma\) on this surface; we can write

\[
j \cdot d\Sigma = j^0 \, d\Sigma_0
\]

\[
j^0 = -i(\Phi^* \nabla^0 \Omega - \Omega \nabla^0 \Phi^*)
\]

Naturally \(\nabla^0 \Phi = g^{0A} \partial_A \Phi\).

So far the formulas of this subsection are general. When we specify to warped spacetimes of Type I, then \(N = p + q\) and \(V_1\) is Lorentzian; \(V_2\) is Riemannian so the label \(k\) cannot be 0. In coordinates adapted to the warped product structure, we know that \(g^{Ak}\) all vanish; in particular \(g^{0k}\) vanishes. We are left with

\[
\nabla^0 \Phi = g^{0A} \partial_A \Phi
\]

Similarly

\[
\nabla^0 \Omega = g^{0A} \partial_A \Omega
\]

We thus have

\[
j \cdot d\Sigma = -i \sqrt{|g|} \, d^{n-1}x \left( \Phi^* g^{0A} \partial_A \Omega - \Omega g^{0A} \partial_A \Phi^* \right)
\]

(39)

Cf. (2.2) in ref. [4].

Let us now consider product solutions.

So we assume that \(\Phi\) and \(\Omega\) are solutions to (2) in the form

\[
\Phi = f(x^A) \, F(x^j) \quad \Omega = h(x^B) \, H(x^k)
\]

but not necessarily on the same mode, say \(\Phi \in \mathcal{H}_n, \ \Omega \in \mathcal{H}_l\) including the possibility that \(n \neq l\). We have

\[
g^{0A} \partial_A \Omega = H g^{0A} \partial_A h
\]

\[
g^{0A} \partial_A \Phi^* = F^* g^{0A} \partial_A f^*
\]

Inserting into (39) yields

\[
j \cdot d\Sigma = -i \left( \Phi^* H g^{0A} \partial_A h - \Omega F^* g^{0A} \partial_A f^* \right) \sqrt{|g|} \, d^{n-1}x
\]

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Develop $\sqrt{|g|}$ according to (4) where $S$ and $\gamma$ are positive, so

$$\sqrt{|g|} = \sqrt{|\alpha|} \ S^{q/2} \sqrt{\gamma}$$

According to (3) we can write $d^{N-1}x = \omega \wedge dq^2 x$, therefore

$$j \cdot d\Sigma = -i(F^* f^A h^0 \partial_A h - hH F^* g^0 A f^*) \sqrt{|\alpha|} \ S^{q/2} \sqrt{\gamma} \omega \wedge dq^2 x$$

$$j \cdot d\Sigma = -i(f^* g^0 A \partial_A h - hg^0 A f^*) \ F^* H \sqrt{|\alpha|} S^{q/2} \sqrt{\gamma} \omega \wedge dq^2 x$$

We now turn to integration.

If $p = 1$ we can take $\Sigma = \{t_0\} \times V_2$ where $t_0$ is a fixed value of the time coordinate $x^0$. This case has been investigated in detail in (4).

If $p > 1$, let us take $\Sigma = L \times V_2$ where $L \subset V_1$ is the submanifold defined by $x^0 = \text{const.}$ Indeed this choice is compatible with the assumption made above that the equation of $\Sigma$ in $V_1 \times V_2$ is just $x^0 = \text{const.}$

Integrate the above formula; we obtain

$$(\Phi; \Omega) = -i \left( \int_{V_2} F^* H \sqrt{\gamma} \ dq^2 x \right) \int_L (f^* g^0 A \partial_A h - hg^0 A f^*) \ S^{q/2} \sqrt{|\alpha|} \omega \right) \ (40)$$

$$(\Phi; \Omega) = -i < F, H >_2 \int_L (f^* g^0 A \partial_A h - hg^0 A f^*) \ S^{q/2} \sqrt{|\alpha|} \omega \right) \ (41)$$

where we have factorized out the scalar product in $(V_2, \gamma)$

$$< F, H >_2 = \int_{V_2} \sqrt{\gamma} F^* H \ dq^2 x \ (42)$$

well-defined and positive for arbitrary couple of functions on $V_2$.

When $F$ and $H$ belong to $\mathcal{E}_n$ and $\mathcal{E}_l$ with $n \neq l$ then $< F, H >_2$ vanishes. It follows that if two product solutions $\Phi$, $\Omega$ belong to distinct modes, $(\Phi; \Omega)$ vanishes.

Now using (15) and a similar development

$$\Omega = \sum_{s=1}^{d(l)} h_s H_s$$

over a basis $H_1, \ldots$ of $\mathcal{E}_l$, we easily check that this property holds for any couple of mode solutions, in other words

**Proposition 5** For Type I under our assumptions, two different modes are orthogonal in the sense of the sesquilinear form defined by the Gordon current.
When \( \Phi \) and \( \Omega \) belong to the same mode we give to (41) a more compact formulation. A look at (33) (34) enables one to write

\[
j \cdot d\Sigma = j^0 \cdot d\Sigma_0 = J^0 \sqrt{|\alpha|} \omega \wedge (F^*H\sqrt{\gamma} \ d^2 x) \quad (43)
\]

We obtain by integration

\[
\int_{\Sigma} j \cdot d\Sigma = \langle F, H \rangle_2 \int_{L} J^0 \sqrt{|\alpha|} \omega
\]

with At this stage it is convenient to remember that \( \omega \) is given by (5) and to observe that

\[
dL_0 = \sqrt{|\alpha|} \omega
\]

When \( A \neq 0 \), if \( B_1 \cdots B_{p-1} \) are all \( \neq A \), one of them must be 0, thus \( dx^0 \) is a factor in \( dL_A \).

On (L), we can write \( dx^0 = 0 \), thus \( dL_A \) vanishes for \( A \neq 0 \). Thus we have on this manifold \( J^A \ dL_A = J^0 \ dL_0 \). Hence

\[
\int_{\Sigma} j \cdot d\Sigma = \langle F, H \rangle_2 \int_{L} J^A \ dL_A
\]

A glance at (37) enables us to state

**Theorem 1** In Type I warped spacetime, for product solutions, the sesquilinear map defined through the usual Gordon current gets factorized according to the formula

\[
(\Phi; \Omega) = (f; h)_1 \ < F, H >_2
\]

where \((f; h)_1 \) is defined by (37).

### 6 Extension to further couplings.

Up to now we have focussed on the minimal coupling of a free particle to gravity. Extending our results to an equation of the form

\[
(\nabla^2 + m^2 + a(x)) \ \Psi = 0 \quad (44)
\]

is straightforward provided that the additional term \( a \) does not depend on the \( x^j \)'s. Otherwize, \( \Delta_2 \) would not commute any more with \( \nabla^2 + a(x) \), and the mode decomposition would be impossible.

Assuming that \( a = a(x^A) \), the differential operator in (14) still commutes with \( \Delta_2 \). Equation (12) is replaced by

\[
(D + \lambda S^{-1} + m^2 + a(x^A)) \ \Phi = 0
\]
Any product solution remains a mode. In $m^2$ must be replaced by $m^2 + a(x^A)$ but the variables $x^j$ remain ignorable. Since $a$ acts on $f$ as a multiplicative operator, the status of the first derivatives $\partial f$ in equation (18) is not modified; therefore condition (20) still avoid the occurrence of them. Moreover formula (27) which determines $Df$ in terms of $f$ is an identity valid for arbitrary $f$. Again we end up with an equation of the form $\Delta_1 \hat{f} + \Xi \hat{f} = 0$ where $\Xi$ now involves the additional term $a$. As well-known, this still ensures conservation of $J^A$, given without modification by (29). Finally the results of Section 4 remain valid.

These remarks permit to treat the case where the KG equation includes an external potential of the special (and rather artificial) form $a(x^A)$. It is more interesting to notice that our study could remain valid for curvature coupling, characterized by the addition to equation (2) of a curvature term $\xi R^\alpha_{\alpha}$ where $\xi$ is a constant and $R^\alpha_{\alpha}$ is the scalar curvature of $(V, g)$. But the condition which legitimates this extension is that the scalar curvature of spacetime depends only on the $x^A$'s.

We are thus led to investigate what kind of warped spacetime supports a scalar curvature of the form $R(x^A)$.

The following lemma will be useful.

**Lemma 1** Let $u(x^A)$ be a function on $V$ satisfying $\partial_j u = 0$. Then the following quantities

$$\nabla^A \partial_B u, \quad \nabla^A u, \quad \nabla^A \nabla^A u$$

are independent from the coordinates $x^k$.

**Proof:**

$$\nabla_A \partial_B u = \partial_A \partial_B u - \Gamma^\mu_{AB} \partial_\mu u.$$ The first term is obviously independent from $x^k$'s. Since $\partial_j u$ is zero, only the coefficients $\Gamma^C_{AB}$ give a contribution to the second term. But it is known that $\Gamma^C_{AB} = 1 \Gamma^C_{AB}$, where $1 \Gamma$ is the connexion for $(V_1, \alpha)$ (see [1]); in 4 dimensions see formula (26) in [3]. Thus the second term has the same property.

The statement about $\nabla^A u$ is obvious, if we remember that $g^{AB} = \alpha^{AB}, \quad g^{A0} = 0$.

Finally, we have $\nabla^A \nabla_A u = \alpha^{AB} \nabla_B \partial_A u$ which is independent of $x^k$ because of the first statement.

The Ricci tensor of a warped product can be expressed in terms of the warping function and of the geometry of the factor manifolds (See O'Neill [1], especially corollary 43, p. 211. When $p + q = 4$, see Carot and da Costa [3], eqs (25), for a more transparent notation).

We are concerned with

$$R = \alpha^{AB} R_{AB} - S^{-1} \gamma^{ij} R_{ij}$$

(45)

In the first term, we know that $R_{AB}$ differs from $1 R_{AB}$ only by a function of $e^\Theta$ and $\nabla_B \nabla_A (e^\Theta)$, which cannot depend on $x^C$ according to the previous result. Therefore $R_{AB}$ does not depend on the $x^k$'s.

$$\partial_j R_{AB} = 0$$

\[\text{2For dimensional reasons, } \xi \text{ necessarily vanishes in the limit where } \hbar \to 0, \text{ thus curvature coupling cannot affect the geodesic motion}\]
This result, in turn, entails that $R$ might depend on these variables only through the quantity $\gamma^{ij} R_{ij}$.

According to the literature cited, $R_{ij}$ takes on the form

$$R_{ij} = 2R_{ij} + N\gamma_{ij}$$

where $N$ is a function of $\Theta$, $\nabla^A \Theta$, $\partial_A \Theta$ and $\nabla^A \nabla_A \Theta$. By the above lemma, we see that

$$\partial_j N = 0$$

By contraction we obtain

$$\gamma^{ij} R_{ij} = \gamma^{ij} 2R_{ij} + qN$$

It is now clear that $\gamma^{ij} R_{ij}$, and therefore equally $R$, depends on the $x^k$'s except when $2R$ is a constant; thus

**Theorem 2** (Only) if $(V_2, \gamma)$ has constant scalar curvature $\gamma^{ij} 2R_{ij}$, the scalar curvature $R$ of the warped product can be regarded as a function on $V_1$.

An well-known instance of this situation is the case of conventional FRW spacetime, where the space sections are homogeneous.

## 7 Concluding remarks

We have separated the variables in the KG equation with help of a generalized mode decomposition which is possible essentially because the motion of a test particle in a warped product spacetime admits a remarkable first integral.

At least for Type I, these modes are actually "normal", that is orthogonal, in the sense of the usual sesquilinear form associated with the Gordon current. This form itself has been analysed in terms of the vector field $J$ defined on the Lorentzian factor manifold and, under very large assumptions, a sesquilinear form has been defined on the solutions to the reduced wave equation.

Type I is perhaps physically the most interesting; in 4 dimensions it encompasses not only generalized FRW spacetimes, but also all kind of spherically symmetric spacetime, including the nonstationary ones. However further investigations of Type II might be of interest.

It is noteworthy that the whole picture remains valid in the presence of a curvature coupling term $\xi R^\alpha_\alpha$ only under the condition that $(V_2, \gamma)$ has a constant scalar curvature. For FRW, abandoning spatial homogeneity would not permit to carry out our mode decomposition when a curvature coupling term is introduced into the KG equation. In contrast, the mode decomposition remains possible with such a term for all kind of spherically symmetric spacetime.
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