Renormalization of the neutrino mass operators in the multi-Higgs-doublet Standard Model

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Abstract

We derive the renormalization group equations (RGE) for the flavour coupling matrices of the effective dimension-five operators which yield Majorana neutrino masses in the multi-Higgs-doublet Standard Model; in particular, we consider the case where two different scalar doublets occur in those operators. We also write down the RGE for the scalar-potential quartic couplings and for the Yukawa couplings of that model, in the absence of quarks. As an application of the RGE, we consider two models which, based on a $\mu$–$\tau$ interchange symmetry, predict maximal atmospheric neutrino mixing, together with $U_{e3} = 0$, at the seesaw scale. We estimate the change of those predictions due to the evolution of the coupling matrices of the effective mass operators from the seesaw scale down to the electroweak scale. We derive an upper bound on that change, thereby finding that the radiative corrections to those predictions are in general negligible.

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1 Introduction

The Standard Model (SM) in the strict sense, i.e. without right-handed neutrino singlets, forbids neutrino masses. However, it was noticed a long time ago [1] that, if one allows for lepton-number nonconservation, then one can construct, with the SM multiplets, operators of dimension higher than four which give Majorana masses to the neutrinos. The lowest-dimensional such operators have dimension five and contain two left-handed lepton doublets and two Higgs doublets; those operators can be thought of as arising from the seesaw mechanism [2] after one has integrated out the right-handed neutrino singlets.1

Under the assumption that the SM is valid up to the seesaw scale \( m_R \), the renormalization group evolution of the dimension-five neutrino mass operators from \( m_R \) down to the electroweak scale, as represented for instance by the \( Z^0 \) mass \( m_{Z} \), can be determined; the evolution equations have been computed in the SM and in its minimal supersymmetric extension [4, 5, 6, 7] (for a review see [8]). This is an important issue in view of testing mechanisms and symmetries for explaining the neutrino masses and the lepton mixing angles, since such mechanisms and symmetries are usually operative or imposed at the seesaw scale, while the measurements are effected at the electroweak scale. (For the experimental and theoretical status of neutrino masses and lepton mixing see, for instance, [9] and [10], respectively.)

In this paper we extend the existing results for the SM renormalization group equations (RGE) to the case of an arbitrary number of Higgs doublets. In particular, we focus on dimension-five neutrino mass operators which contain two different Higgs doublets; indeed, to our knowledge, that case has not yet been treated in the literature. The reason to consider the multi-Higgs-doublet SM is that, within that framework, several models have been produced in recent years which predict, for instance, lepton mixing angles \( \theta_{13} = 0 \) and \( \theta_{23} = \pi/4 \) [11, 12], or \( \theta_{13} = 0 \) alone [13, 14], or \( \theta_{23} = \pi/4 \) and \( \delta = \pi/2 \) [15].2

(See [9, 10], for instance, for the definition of the lepton mixing angles.) Those predictions usually hold at the seesaw scale and, in order to compare them with experiment, one needs to know the corresponding corrections at the electroweak scale.

In Sect. 2 we display the Lagrangian of the multi-Higgs-doublet SM, without quarks but with dimension-five neutrino mass operators, and present the RGE for the couplings of that Lagrangian. In Sect. 3 we discuss the specific RGE for the models, referred to above, which predict \( \theta_{13} = 0 \) and \( \theta_{23} = \pi/4 \) at the seesaw scale. In Sect. 4 we show explicitly how those seesaw-scale predictions arise, and how they may be changed by the renormalization group evolution. In Sect. 5 we derive an upper bound on the effect of that evolution. A short summary of our main results is provided in Sect. 6. An appendix contains some details of the calculation of the beta functions for the neutrino mass operators.

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1 The effect of the dimension-six operators which also arise from the seesaw mechanism has been studied in [3].

2 The predictions \( \theta_{23} = \pi/4 \) and \( \delta = \pi/2 \) have first been obtained in a supersymmetric extension of the SM [10].
2 General case

2.1 The model

We consider the SM with \( n_H \) Higgs doublets \( \phi_i \) \((i = 1, 2, \ldots, n_H)\) with weak hypercharge 1/2. The \( SU(2) \) gauge coupling constant is denoted \( g \) while the \( U(1) \) gauge coupling constant (with the above normalization for the weak hypercharge) is denoted \( g' \). The scalar potential \( V \) has the form

\[
V = \text{quadratic terms} + \sum_{i,j,k,l=1}^{n_H} \lambda_{ijkl} (\phi_i^\dagger \phi_j)(\phi_k^\dagger \phi_l),
\]

where the dimensionless couplings \( \lambda_{ijkl} \) satisfy

\[
\lambda_{ijkl} = \lambda_{klij} = \lambda_{jilk}^*.
\]

The lepton Yukawa Lagrangian \( \mathcal{L}_{Y\ell} \) is

\[
\mathcal{L}_{Y\ell} = -\sum_{i=1}^{n_H} \left( \bar{\ell}_R \phi_i^\dagger Y_i D_L + D_L Y_i^\dagger \phi_i \ell_R \right),
\]

where \( D_L \) denotes the left-handed lepton doublets and \( \ell_R \) the right-handed charged-lepton singlets. We have defined the dimensionless flavour coupling matrices \( Y_i \) in the same way as [6, 7]. Note that in this paper we do not use the summation convention.

The effective dimension-five neutrino mass operators are defined as

\[
\mathcal{O}_{ij} = \sum_{\alpha,\beta=e,\mu,\tau} \sum_{a,b,c,d=1}^{2} \left( D_{\alpha a}^T \kappa_{ij}^{(\alpha\beta)} C^{-1} D_{\beta c} \right) \left( \varepsilon^{ab} \phi_{ib} \right) \left( \varepsilon^{cd} \phi_{jd} \right),
\]

where, contrary to what we had done in [11] and [3], we have made explicit both the flavour and gauge-\( SU(2) \) indices, and the summations thereover. In [4], \( C \) is the Dirac–Pauli charge-conjugation matrix; \( \alpha \) and \( \beta \) are flavour indices; \( a, b, c, \) and \( d \) are \( SU(2) \) indices; and \( \varepsilon \) is the antisymmetric \( 2 \times 2 \) matrix, with \( \varepsilon^{12} = 1 \). The flavour coupling matrices \( \kappa^{(ij)} \) in [11] have dimension \(-1\) and satisfy

\[
\kappa^{(ij)}_{\alpha\beta} = \kappa^{(ji)}_{\beta\alpha}, \text{ i.e. } \kappa^{(ij)} = \kappa^{(ji)T}.
\]

2.2 The RGE

The RGE are first-order differential equations which give the evolution of the couplings of a model relative to \( t = \ln \mu \), where \( \mu \) is the mass parameter used in the regularization of ultraviolet-divergent integrals; the basic equation is

\[
\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2) \left[ (k + p)^2 - m'^2 \right]} = \frac{it}{8\pi^2} + \cdots,
\]

where \( p \) is a typical momentum and \( m, m' \) are typical masses appearing in the loop integral; the dots represent either \( \mu \)-independent terms or terms which disappear in the
limit that regularizes the integral. We have computed, at the one-loop level, the RGE for
the model outlined in the previous subsection. For the RGE of the coupling matrices of
the effective mass operators we have found
\[ 16\pi^2 \frac{d\lambda_{ij}}{dt} = -3g^2\lambda_{ij} + 4 \sum_{k,l=1}^{n_H} \lambda_{kl} \lambda_{ijkl} + \sum_{k=1}^{n_H} \left[ T_{ki} \lambda_{ik} + T_{kj} \lambda_{ki} \right] + \lambda_{ij} P + P^T \lambda_{ij} \]
\[ + 2 \sum_{k,l=1}^{n_H} \left\{ \lambda_{ij} \lambda_{kl} \lambda_{ijkl} + \lambda_{ij} \lambda_{kl} \lambda_{ijkl} + \lambda_{ij} \lambda_{kl} \lambda_{ijkl} + \lambda_{ij} \lambda_{kl} \lambda_{ijkl} \right\}, \]  

where
\[ T_{ij} := \text{tr} \left( Y_i Y_j^\dagger \right), \]
\[ P := \frac{1}{2} \sum_{k=1}^{n_H} Y_k^\dagger Y_k. \]

The third line of (7) is obtained from the second line through the interchange \( i \leftrightarrow j \) together with transposition, in agreement with (5). Our result (7) coincides, when \( i = j \), with the result given in [6, 7]; it generalizes that result for the case \( i \neq j \). Note that, for
the sake of simplicity, in the present paper we dismiss quarks; in general, one would have
to add to (8) analogous trace terms featuring the Yukawa-coupling matrices of the Higgs
doublets \( \phi_i \) and \( \phi_j \) to the up and down quarks, multiplied by a colour factor 3—see, for
instance, [5].

The terms in the second and third lines of (7) arise from diagrams like the one in
figure 1. We dwell on the explicit derivation of those terms in the appendix.

In order to solve the RGE for the effective neutrino mass operators one also needs
the RGE for the other couplings occurring in (7). The general RGE for an arbitrary
renormalizable gauge field theory have been derived in [17, 18] at the one- and two-loop
levels, respectively. It is convenient to have the results of [17] specialized to the case of
the multi-Higgs-doublet SM. We have found that
\[ 16\pi^2 \frac{d\lambda_{ijkl}}{dt} = 4 \sum_{m,n=1}^{n_H} \left( 2 \lambda_{ijmn} \lambda_{mnkl} + \lambda_{ijmn} \lambda_{kmnl} + \lambda_{imnj} \lambda_{mnkl} \right) \]
\[ + \lambda_{imkn} \lambda_{mjnl} + \lambda_{mjkn} \lambda_{imnl} \right) \right) \delta_{ij} \delta_{kl} + \frac{3g^2 g^{'2}}{4} \delta_{ij} \delta_{kl} \right) \]
\[ + \sum_{m=1}^{n_H} \left( T_{mj} \lambda_{imkl} + T_{ml} \lambda_{ijkm} + T_{im} \lambda_{mjkl} + T_{km} \lambda_{ijml} \right) \]
\[ - 2 \text{tr} \left( Y_i Y_j^\dagger Y_k^\dagger Y_l^\dagger \right), \]
\[ 16\pi^2 \frac{dY_i}{dt} = \sum_{k=1}^{n_H} \left( T_{ik} Y_k + Y_k Y_i^\dagger Y_i + \frac{1}{2} Y_i Y_k^\dagger Y_k \right) - \frac{9g^2 + 15g^{'2}}{4} Y_i. \]

It is well known that the RGE for \( g \) and \( g' \) are
\[ 16\pi^2 \frac{dg}{dt} = \left( -\frac{22}{3} + \frac{4N}{3} + \frac{n_H}{6} \right) g^3, \]
Figure 1: A typical vertex correction in the renormalization of the operator $O_{ij}$. The relevant Yukawa-coupling matrices are indicated.

$$16\pi^2 \frac{dg'}{dt} = \left( \frac{20N}{9} + \frac{n_H}{6} \right) g^3,$$

where $N = 3$ is the number of fermion families.

3 Application of the RGE to two particular models

3.1 The $\mathbb{Z}_2$ and $D_4$ models

We now apply the general RGE derived in Sect. 2 to the so-called $\mathbb{Z}_2$ and $D_4$ models—for a review see [19]. Those models predict

$$\theta_{13} = 0, \quad \theta_{23} = \frac{\pi}{4}$$

at the seesaw scale and are, in what regards the practical application of the RGE, identical. They both have three Higgs doublets $\phi_1$, $\phi_2$, and $\phi_3$. Below the seesaw scale the structure of both models is dictated by the symmetries

$$\mathbb{Z}_2^{(\text{aux})} : \quad e_R \rightarrow -e_R, \quad \phi_1 \rightarrow -\phi_1,$$

$$\mathbb{Z}_2^{(\text{tr})} : \quad D_{L\mu} \leftrightarrow D_{L\tau}, \quad \mu_R \leftrightarrow \tau_R, \quad \phi_3 \rightarrow -\phi_3.$$
values (VEVs) of $\phi_0^i$ and $\phi_3^0$, respectively, at the low scale. Because of the symmetries in (15) and (16), the Higgs potential is given by

$$V = \text{quadratic terms} + \sum_{i=1}^3 \lambda_i \left( \phi_i^\dagger \phi_i \right)^2 + \lambda_4 \left( \phi_1^\dagger \phi_1 \right) \left( \phi_2^\dagger \phi_2 \right) + \lambda_5 \left( \phi_1^\dagger \phi_1 \right) \left( \phi_3^\dagger \phi_3 \right) + \lambda_6 \left( \phi_2^\dagger \phi_2 \right) \left( \phi_3^\dagger \phi_3 \right) + \lambda_7 \left( \phi_2^\dagger \phi_2 \right) \left( \phi_3^\dagger \phi_3 \right) + \lambda_8 \left( \phi_1^\dagger \phi_1 \right) \left( \phi_3^\dagger \phi_3 \right) + \lambda_9 \left( \phi_2^\dagger \phi_2 \right) \left( \phi_3^\dagger \phi_3 \right) + \lambda_{10} \left( \phi_1^\dagger \phi_2 \right)^2 + \lambda_{11} \left( \phi_1^\dagger \phi_3 \right)^2 + \lambda_{12} \left( \phi_2^\dagger \phi_3 \right)^2 + \text{H.c.},$$

(17)

where $\lambda_{10}$, $\lambda_{11}$, and $\lambda_{12}$ are the only non-real quartic couplings. Comparing (17) with (1) and (2), we arrive at the identifications $\lambda_{iii} = \lambda_i$ (for $i = 1, 2, 3$), $\lambda_{112} = \lambda_{2121} = \lambda_4/2$, $\lambda_{1221} = \lambda_{2112} = \lambda_7/2$, $\lambda_{1212} = \lambda_{2121} = \lambda_{10}$, and so on.

The $\mathbb{Z}_2$ and $D_4$ models have three other symmetries, the family-lepton-number symmetries $L_\alpha$, which are broken at the seesaw scale—softly in the $\mathbb{Z}_2$ model, spontaneously in the $D_4$ model. Because of the symmetries in (15) and (16), and also because of the symmetries $L_\alpha$—which remain valid for the quartic couplings of the light fields below the seesaw scale—the lepton Yukawa Lagrangian is

$$L_{Y\ell} = -y_3 \bar{D}_{L\alpha} e_R \phi_1 - y_4 \left( \bar{D}_{L\mu} \mu_R + \bar{D}_{L\tau} \tau_R \right) \phi_2 - y_5 \left( \bar{D}_{L\mu} \mu_R - \bar{D}_{L\tau} \tau_R \right) \phi_3 + \text{H.c.}$$

(18)

(The coupling constants $y_{1,2}$ occur in the Yukawa interactions of the right-handed neutrino singlets [19] and are thus of no concern here.) Comparing (18) with (3), we see that the Yukawa-coupling matrices are

$$Y_1 = \text{diag} \left( y_{3}^*, \ 0, \ 0 \right),$$

$$Y_2 = \text{diag} \left( 0, \ y_{4}^*, \ y_{4}^* \right),$$

$$Y_3 = \text{diag} \left( 0, \ y_{5}^*, \ -y_{5}^* \right).$$

(19)

Hence, from (8),

$$T_{11} = |y_3|^2,$$

$$T_{22} = 2 |y_4|^2,$$

$$T_{33} = 2 |y_5|^2,$$

(20-22)

and the $T_{ij}$ with $i \neq j$ vanish, a fact which simplifies considerably the RGE in this particular case.

As emphasized before, the symmetries [15] and [16] are broken only at the electroweak scale. The validity of the symmetry $\mathbb{Z}_2^{\text{aux}}$—which changes the sign of $\phi_1$ but does not affect the lepton doublets $D_L$—has an important consequence: the operators $\mathcal{O}_{12}$, $\mathcal{O}_{21}$, $\mathcal{O}_{13}$, and $\mathcal{O}_{31}$ are altogether absent. The symmetry $\mathbb{Z}_2^{(t)}$, on the other hand, changes the sign of $\phi_3$ simultaneously with the interchange of $D_{L\mu}$ with $D_{L\tau}$. This implies that the coupling matrices $\kappa^{(ii)}$ ($i = 1, 2, 3$) must be of the form

$$\begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix},$$

(23)
while the matrices $\kappa^{(23)}$ and $\kappa^{(32)} = \kappa^{(23)^T}$ are of the form

$$
\begin{pmatrix}
0 & p & -p \\
q & s & r \\
-q & -r & -s
\end{pmatrix}.
$$

(24)

### 3.2 The RGE for the $\mathbb{Z}_2$ and $D_4$ models

The renormalization group equations for the Yukawa couplings of the $\mathbb{Z}_2$ and $D_4$ models are

$$
16\pi^2 \frac{dy_3}{dt} = \left( \frac{5}{2} |y_3|^2 - \frac{9g^2 + 15g'^2}{4} \right) y_3,
$$

(25)

$$
16\pi^2 \frac{dy_4}{dt} = \left( \frac{7}{2} |y_4|^2 + \frac{3}{2} |y_5|^2 - \frac{9g^2 + 15g'^2}{4} \right) y_4,
$$

(26)

$$
16\pi^2 \frac{dy_5}{dt} = \left( \frac{3}{2} |y_4|^2 + \frac{7}{2} |y_5|^2 - \frac{9g^2 + 15g'^2}{4} \right) y_5.
$$

(27)

The RGE for the scalar-potential couplings are

$$
16\pi^2 \frac{d\lambda_1}{dt} = 24\lambda_1^2 + \lambda_1^4 + (\lambda_4 + \lambda_7)^2 + \lambda_5^2 + (\lambda_5 + \lambda_8)^2 + 4 |\lambda_{10}|^2 + 4 |\lambda_{11}|^2
$$

$$
+ \left( 4 |y_3|^2 - C \right) \lambda_1 + \frac{9g^4}{8} + \frac{3g^2g'^2}{4} + \frac{3g'^4}{8} - 2 |y_3|^4,
$$

(28)

$$
16\pi^2 \frac{d\lambda_2}{dt} = 24\lambda_2^2 + \lambda_2^4 + (\lambda_4 + \lambda_7)^2 + \lambda_5^2 + (\lambda_5 + \lambda_8)^2 + 4 |\lambda_{10}|^2 + 4 |\lambda_{12}|^2
$$

$$
+ \left( 8 |y_4|^2 - C \right) \lambda_2 + \frac{9g^4}{8} + \frac{3g^2g'^2}{4} + \frac{3g'^4}{8} - 4 |y_4|^4,
$$

(29)

$$
16\pi^2 \frac{d\lambda_3}{dt} = 24\lambda_3^2 + \lambda_3^4 + (\lambda_5 + \lambda_8)^2 + \lambda_5^2 + (\lambda_6 + \lambda_9)^2 + 4 |\lambda_{11}|^2 + 4 |\lambda_{12}|^2
$$

$$
+ \left( 8 |y_5|^2 - C \right) \lambda_3 + \frac{9g^4}{8} + \frac{3g^2g'^2}{4} + \frac{3g'^4}{8} - 4 |y_5|^4,
$$

(30)

$$
16\pi^2 \frac{d\lambda_4}{dt} = (\lambda_1 + \lambda_2) (12\lambda_1 + 4\lambda_7) + 4\lambda_4^2 + 2\lambda_7^2 + 4\lambda_5\lambda_6 + 2 (\lambda_5\lambda_9 + \lambda_6\lambda_8) + 8 |\lambda_{10}|^2
$$

$$
+ \left( 2 |y_3|^2 + 4 |y_4|^2 - C \right) \lambda_4 + \frac{9g^4}{4} - \frac{3g^2g'^2}{2} + \frac{3g'^4}{4},
$$

(31)

$$
16\pi^2 \frac{d\lambda_5}{dt} = (\lambda_1 + \lambda_3) (12\lambda_1 + 4\lambda_8) + 4\lambda_5^2 + 2\lambda_8^2 + 4\lambda_4\lambda_6 + 2 (\lambda_4\lambda_9 + \lambda_6\lambda_7) + 8 |\lambda_{11}|^2
$$

$$
+ \left( 2 |y_3|^2 + 4 |y_5|^2 - C \right) \lambda_5 + \frac{9g^4}{4} - \frac{3g^2g'^2}{2} + \frac{3g'^4}{4},
$$

(32)

$$
16\pi^2 \frac{d\lambda_6}{dt} = (\lambda_2 + \lambda_3) (12\lambda_2 + 4\lambda_9) + 4\lambda_6^2 + 2\lambda_9^2 + 4\lambda_4\lambda_5 + 2 (\lambda_4\lambda_8 + \lambda_5\lambda_7) + 8 |\lambda_{12}|^2
$$

$$
+ \left( 4 |y_4|^2 + 4 |y_5|^2 - C \right) \lambda_6 + \frac{9g^4}{4} - \frac{3g^2g'^2}{2} + \frac{3g'^4}{4} - 8 |y_4y_5|^2,
$$

(33)

$$
16\pi^2 \frac{d\lambda_7}{dt} = (4\lambda_1 + 4\lambda_2 + 8\lambda_4 + 4\lambda_7 + 2 |y_3|^2 + 4 |y_4|^2 - C) \lambda_7
$$
The reason why no fourth-order terms in the Yukawa couplings and in the gauge couplings appear in the RGE for \( \lambda_{10} \) and \( \lambda_{11} \) is that the condition \( \lambda_{10} = \lambda_{11} = 0 \) may be enforced through an additional \( U(1) \) symmetry: \( \phi_1 \to e^{i\alpha} \phi_1, \epsilon_R \to e^{-i\alpha} \epsilon_R \), where \( \alpha \in \mathbb{R} \).

We next write down the RGE for the coupling matrices of the effective neutrino mass operators. They are

\[
16\pi^2 \frac{d\lambda_8}{dt} = +2\lambda_8 \lambda_9 + 32 |\lambda_{10}|^2 + 3g^2 g'^2, \tag{34}
\]

\[
16\pi^2 \frac{d\lambda_9}{dt} = \left( 4\lambda_1 + 4\lambda_3 + 8\lambda_5 + 4\lambda_8 + 2 |y_3|^2 + 4 |y_5|^2 - C \right) \lambda_8 + 2\lambda_7 \lambda_9 + 32 |\lambda_{11}|^2 + 3g^2 g'^2, \tag{35}
\]

\[
16\pi^2 \frac{d\lambda_10}{dt} = \left( 4\lambda_1 + 4\lambda_2 + 8\lambda_4 + 4\lambda_9 + 4 |y_4|^2 + 4 |y_5|^2 - C \right) \lambda_9 + 2\lambda_7 \lambda_8 + 32 |\lambda_{12}|^2 + 3g^2 g'^2 - 8 |y_4 y_5|^2, \tag{36}
\]

\[
16\pi^2 \frac{d\lambda_11}{dt} = \left( 4\lambda_1 + 4\lambda_2 + 8\lambda_4 + 12\lambda_7 + 2 |y_3|^2 + 4 |y_4|^2 - C \right) \lambda_10 + 4\lambda_{11} \lambda_{12}^*, \tag{37}
\]

\[
16\pi^2 \frac{d\lambda_12}{dt} = \left( 4\lambda_2 + 4\lambda_3 + 8\lambda_6 + 12\lambda_9 + 4 |y_4|^2 + 4 |y_5|^2 - C \right) \lambda_{11} + 4\lambda_{10} \lambda_{12}, \tag{38}
\]

\[
16\pi^2 \frac{d\lambda_13}{dt} = \left( 4\lambda_2 + 4\lambda_3 + 8\lambda_6 + 12\lambda_9 + 4 |y_4|^2 + 4 |y_5|^2 - C \right) \lambda_{12} + 4\lambda_{10} \lambda_{11} - 4y_4^2 y_5^2, \tag{39}
\]

where

\[
C := 9g^2 + 3g'^2. \tag{40}
\]
where \(\{R, S\}\) and \([R, S]\) denote the anticommutator and the commutator, respectively, of the matrices \(R\) and \(S\). Moreover, we have defined

\[
P_1 := \text{diag} \left( |y_3|^2, 0, 0 \right), 
\]

\[
P_2 := \text{diag} \left( 0, |y_4|^2, |y_4|^2 \right), 
\]

\[
P_3 := \text{diag} \left( 0, |y_5|^2, |y_5|^2 \right), 
\]

\[
P_{23} := \text{diag} \left( 0, y_4 y_5^*, -y_4 y_5^* \right), 
\]

\[
P_{32} := \text{diag} \left( 0, y_4^* y_5, -y_4^* y_5 \right). 
\]

Notice that the matrix \(P\) in (9) is equal to \((P_1 + P_2 + P_3)/2\).

4 Predictions of the \(\mathbb{Z}_2\) and \(D_4\) models

The Lagrangian of neutrino Majorana masses is

\[
\mathcal{L}_{\text{Majorana}} = \frac{1}{2} \sum_{\alpha,\beta = e,\mu,\tau} \nu_{L\alpha}^T C^{-1} (\mathcal{M}_\nu)_{\alpha\beta} \nu_{L\beta} + \text{H.c.}, 
\]

where \(\mathcal{M}_\nu = \mathcal{M}_\nu^T\). Taking \(b = d = 2\) in (4), it is clear that

\[
\mathcal{O}_{ij} = \sum_{\alpha,\beta = e,\mu,\tau} \kappa_{i\beta}^{(ij)} \phi_\alpha^0 \phi_j^0 \nu_{L\alpha}^T C^{-1} \nu_{L\beta} + \cdots. 
\]

Therefore, if we denote the VEV of \(\phi_i^0\) by \(v_i\), then the neutrino Majorana mass matrix \(\mathcal{M}_\nu\) is given by

\[
\frac{1}{2} \mathcal{M}_\nu = \sum_{i=1}^3 v_i^2 \kappa_{i(i)} + v_2 v_3 \left[ \kappa^{(23)} + \kappa^{(32)} \right], 
\]

since \(\kappa^{(12)} = \kappa^{(21)} = \kappa^{(13)} = \kappa^{(31)} = 0\) in the \(\mathbb{Z}_2\) and \(D_4\) models. This is valid at all scales \(t\).

In general one may write [20]

\[
\mathcal{M}_\nu = \begin{pmatrix} X & A (1 + \epsilon) & A (1 - \epsilon) \\
A (1 + \epsilon) & B (1 + \epsilon') & C \\
A (1 - \epsilon) & C & B (1 - \epsilon') \end{pmatrix}. 
\]

We already know that the form of the flavour coupling matrices \(\kappa_{i(i)}\) is described by [23], while \(\kappa^{(23)} = \kappa^{(32)}\) is described by [24]. Therefore,

\[
\sum_{i=1}^3 2v_i^2 \kappa_{i(i)} = \begin{pmatrix} X & A & A \\
A & B & C \\
A & C & B \end{pmatrix}, 
\]

while

\[
\kappa^{(e)} := \kappa^{(23)} + \kappa^{(32)} = \begin{pmatrix} 0 & c_1 & -c_1 \\
c_1 & c_2 & 0 \\
-c_1 & 0 & -c_2 \end{pmatrix}. 
\]
\[ \epsilon A = 2v_2v_3c_1, \quad (57) \]
\[ \epsilon' B = 2v_2v_3c_2. \quad (58) \]

Once again, all this is valid at any scale \( t \).

In both the \( \mathbb{Z}_2 \) and \( D_4 \) models, the symmetry \( \mathbb{Z}_2^{(\text{aux})} \) inverts the signs of the right-handed-neutrino fields which are present above the seesaw scale. Hence, only the doublet \( \phi_1 \) has Yukawa couplings to those fields, above the high scale. This implies that \( \mathcal{M}_\nu(t_0) \), where \( t_0 := \ln m_R \), originates solely from the VEV of \( \phi_1^0 \). Therefore, at the seesaw scale [11, 12]
\[ \kappa^{(11)}(t_0) = \mathcal{M}_\nu(t_0)/(2v_1^2), \]
\[ \kappa^{(ij)}(t_0) = 0 \quad \text{for all other } (ij). \quad (59) \]

We conclude that \( \mathcal{M}_\nu(t_0) \) has the same form as \( \kappa^{(11)} \), i.e. \( \mathcal{M}_\nu(t_0) \) is of the form [23]. Clearly then, \( (0, 1, -1)^T \) is an eigenvector of \( \mathcal{M}_\nu(t_0) \) and therefore, at the seesaw scale, the predictions [14] hold.

At any other scale, though, the matrix \( \kappa^{(\epsilon)} \) in [60] is not zero. Thus, for any \( t < t_0 \), \( \mathcal{M}_\nu(t) \) is not \( \mu \leftrightarrow \tau \)-symmetric. This fact renders the predictions [14] inexact for any scale other than the seesaw scale. In [20] it has been shown that, if one assumes the parameters \( \epsilon \) and \( \epsilon' \) in [62] to be small, then, to first order in those parameters, one has, instead of [14],
\[ U_{e3} = \begin{pmatrix} s_{12} c_{12} & \frac{s_{12} c_{12}}{m_3^2 - m_1^2} \left( \bar{\epsilon} s_{12} \hat{m}_2^* + \bar{\epsilon'} s_{12} \hat{m}_3 - \bar{\epsilon'} \hat{m}_2^* - \bar{\epsilon} \hat{m}_3 \right) + \frac{s_{12} c_{12}}{m_3^2 - m_1^2} \left( \bar{\epsilon} c_{12} \hat{m}_1^* + \bar{\epsilon'} c_{12} \hat{m}_3 + \bar{\epsilon'} \hat{m}_1 + \bar{\epsilon} \hat{m}_3 \right) \end{pmatrix}, \quad (60) \]
\[ \cos 2\theta_{23} = \text{Re} \left[ \frac{2c_{12}}{m_3^2 - m_1^2} \left( \bar{\epsilon} s_{12} - \bar{\epsilon'} \right) (\hat{m}_2 + m_3) \right] - \frac{2s_{12}^2}{m_3^2 - m_1^2} \left( \bar{\epsilon} c_{12} + \bar{\epsilon'} \right) (\hat{m}_1 + m_3), \quad (61) \]

where
\[ \bar{\epsilon} := (\hat{m}_1 - \hat{m}_2) \epsilon, \quad (62) \]
\[ \bar{\epsilon'} := \frac{\hat{m}_1 s_{12}^2 + \hat{m}_2 c_{12}^2 + m_3}{2} \epsilon', \quad (63) \]

and
\[ \hat{m}_1 = m_1 e^{-2i\rho}, \quad (64) \]
\[ \hat{m}_2 = m_2 e^{-2i\sigma}. \quad (65) \]

Here, \( m_1, m_2, \) and \( m_3 \) are the (real, non-negative) neutrino masses, while \( \rho \) and \( \sigma \) are Majorana phases. In [60], [63], \( s_{12} \) and \( c_{12} \) are the sine and cosine, respectively, of the solar mixing angle. We are using the standard parametrization [9, 10] for the lepton mixing matrix \( U \); that parametrization fixes the significance of the phase of [60].
We see in (50) and (51) that the deviation from the predictions (14) is numerically of the same order as the (small) parameters $\epsilon$ and $\epsilon'$, except in the case of quasi-degenerate neutrinos with common mass $m_0$; in that case, an enhancement factor $m_0^2/\Delta m^2_{\text{atm}}$ appears, where $\Delta m^2_{\text{atm}} = |m_3^2 - m_2^2| \approx |m_3^2 - m_1^2|$ is the atmospheric mass-squared difference.

5 An approximate solution of the RGE

In this section we assume the Yukawa couplings $y_3$, $y_4$, and $y_5$ to be very small. This assumption is reasonable when one considers the $\mathbb{Z}_2$ or $D_4$ models with their minimal content of three Higgs doublets, since in that case $\sum_{i=1}^3 |v_i|^2$ must be equal to $(174 \text{ GeV})^2$, and therefore all the $|v_i|$ are in principle much larger than the charged-lepton masses. We estimate, to first order in the Yukawa couplings squared, the deviation from the predictions (14) is numerically of the same order as the (small) parameters $\epsilon$ and $\epsilon'$, except in the case of quasi-degenerate neutrinos with common mass $m_0$; in that case, an enhancement factor $m_0^2/\Delta m^2_{\text{atm}}$ appears, where $\Delta m^2_{\text{atm}} = |m_3^2 - m_2^2| \approx |m_3^2 - m_1^2|$ is the atmospheric mass-squared difference.

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and taking into account that \( c_1 (t_0) = c_2 (t_0) = 0 \), one has
\[
c_1 (t) = - \frac{1}{8\pi^2} S_c (t) \int_{t_0}^{t} dt' S_c^{-1} (t') \left[ y_{15}^* y_5 \kappa_{12}^{(22)} + y_{45}^* \kappa_{12}^{(33)} \right] (t')
\]
\[
= - \frac{\kappa_{12}^{(11)} (t_0)}{8\pi^2} S_c (t) \int_{t_0}^{t} dt' S_c^{-1} (t') \left[ \nu (t') T (t', t_0)_{21} + \nu^* (t') T (t', t_0)_{31} \right],
\]
(75)
\[
c_2 (t) = - \frac{1}{4\pi^2} S_c (t) \int_{t_0}^{t} dt' S_c^{-1} (t') \left[ y_{15}^* y_5 \kappa_{22}^{(22)} + y_{45}^* \kappa_{22}^{(33)} \right] (t')
\]
\[
= - \frac{\kappa_{22}^{(11)} (t_0)}{4\pi^2} S_c (t) \int_{t_0}^{t} dt' S_c^{-1} (t') \left[ \nu (t') T (t', t_0)_{21} + \nu^* (t') T (t', t_0)_{31} \right],
\]
(76)
where we have used (73) and defined \( \nu := y_5^* y_5 \).

We now make use of (55) and (57) to write
\[
\epsilon (t_1) = \frac{v_2 v_3 c_1 (t_1)}{\sum_{i=1}^{3} v_i^2 \kappa_{12}^{(11)} (t_1)} = \frac{v_2 v_3 c_1 (t_1)}{\sum_{i=1}^{3} v_i^2 T (t_1, t_0)_{i1} \kappa_{12}^{(11)} (t_0)},
\]
(77)

Similarly,
\[
\epsilon' (t_1) = \frac{v_2 v_3 c_2 (t_1)}{\sum_{i=1}^{3} v_i^2 T (t_1, t_0)_{i1} \kappa_{22}^{(11)} (t_0)},
\]
(78)

Putting (75)–(78) together, we conclude that
\[
\epsilon (t_1) = - \frac{v_2 v_3}{8\pi^2 \sum_{i=1}^{3} v_i^2 T (t_1, t_0)_{i1}} S_c (t_1) \int_{t_0}^{t_1} dt' S_c^{-1} (t') \left[ \nu (t') T (t', t_0)_{21} + \nu^* (t') T (t', t_0)_{31} \right]
\]
(79)

while \( \epsilon' (t_1) = 2 \epsilon (t_1) \). From (60) and (61) one can derive that
\[
\epsilon' = 2 \epsilon \Rightarrow \begin{cases} 
U_{e3} = 2 m_3 c_{12} s_{12} \left( \frac{\hat{m}_1 + m_3}{m_3 - m_1} + \frac{\hat{m}_3 + m_3}{m_2 - m_3} \right) \text{Re} \epsilon, \\
\cos 2\theta_{e3} = 2 \left( s_{12}^2 \left| \hat{m}_1 + m_3 \right|^2 + c_{12}^2 \left| \hat{m}_2 + m_3 \right|^2 \right) \text{Re} \epsilon.
\end{cases}
\]
(80)

Now we want to estimate the maximum possible order of magnitude of \( \epsilon (t_1) \) by using (79). The length of the integration interval of \( t' = t_0 - t_1 = \ln (m_R / m_Z) \sim 10 \). The functions \( S_c (t) \) and \( S_c^{-1} (t') \) are of order 1 since, in (74), \( \lambda_6 / (16\pi^2) \) and \( \lambda_9 / (16\pi^2) \) are necessarily small. The functions \( T (t', t_0)_{21} \) and \( T (t', t_0)_{31} \) are governed by \( \lambda_{10} \) and \( \lambda_{11} \); in any case, \( T (t', t_0)_{21} / T (t_1, t_0)_{i1} \) and \( T (t', t_0)_{31} / T (t_1, t_0)_{i1} \) should be \( \lesssim 1 \). Similarly, \( v_2 v_3 / v_i^2 \) should not be larger than 1. We conclude that
\[
|\epsilon (t_1)| \sim \frac{10 |\nu|}{8\pi^2} \sim \frac{|y_4 y_5|}{10}.
\]
(81)

Even if we allow for rather small VEVs, \( |y_{4,5}| \) cannot be larger than 0.1. We thus have the generous upper bound \( |\epsilon (t_1)| \lesssim 10^{-3} \).  

12
Equation (80) tells us that the only chance to have a non-negligible $U_{e3}$ is in the case of a degenerate neutrino spectrum. Let us consider the extreme case of a common mass $m_0 = 0.3\text{ eV}$ \cite{21}. Since $\Delta m^2_{\text{atm}} \simeq 2 \times 10^{-3} \text{ eV}^2$, in that case we have $|U_{e3}| \simeq 100 |\text{Re} \epsilon| \lesssim 0.1$. Here we have used Majorana phases $\rho = \sigma = 0$ for simplicity. That choice of $m_0$ is indeed extreme; if take $m_0 = 0.1\text{ eV}$ instead, then the upper bound becomes one order of magnitude smaller, due to the quadratic dependence of $|U_{e3}|$ on $m_0$ in the case of a degenerate neutrino spectrum. In any case, we expect $|U_{e3}|$ and $|\cos 2\theta_{23}|$ to be no larger than 0.1 in our model, but most likely they are two or more orders of magnitude smaller.

6 Summary

In this paper we have computed the RGE for the dimension-five neutrino mass operators in the multi-Higgs-doublet SM. Thus, the main result of this paper is (7), which describes the evolution of the coupling matrices of the mass operators in the SM with an arbitrary number of Higgs doublets. We have argued in favour of the usefulness of (7) by citing models for lepton mixing which have been constructed in the framework of the multi-Higgs-doublet SM.

As an application of our RGE we have considered the $\mathbb{Z}_2$ model of \cite{11} and the $D_4$ model of \cite{12}, which—from the field-theoretical point of view—are identical below the seesaw scale. The predictions (14) of those models hold at the seesaw scale and we have used the RGE to estimate the corrections to those predictions which appear due to the evolution of the coupling matrices of the dimension-five neutrino mass operators down to the electroweak scale. We have found that those corrections are in general negligible, with the possible exception of a degenerate neutrino mass spectrum with a rather large common mass $m_0 \gtrsim 0.2\text{ eV}$. In that case, $s_{13}^2$ could be as large as 0.01 and be within the sensitivity of the planned long-baseline neutrino experiments \cite{9}. On the other hand, even in the degenerate neutrino case the deviation of $\theta_{23}$ from $\pi/4$ will be hard to uncover in those experiments \cite{22}, since they will be sensitive to the parameter $\sin^2 2\theta_{23} = 1 - \cos^2 2\theta_{23}$ and we have estimated $\cos^2 2\theta_{23} \lesssim 0.01$ in our models. Thus, with respect to the experiments presently envisaged the models discussed here have the following properties: should a non-zero $s_{13}^2$ be discovered, then the neutrino mass spectrum must be degenerate; while deviations from $\sin^2 2\theta_{23} = 1$ should be invisible.

A Vertex corrections to the neutrino mass operator

The last two lines of (7) originate in vertex corrections of the type displayed in figure 1. In this appendix we show how we have arrived at those two lines. Perturbation theory yields the expression

$$
\frac{(-i)^2}{2!} T \left[ \sum_{m,n=1}^{n_H} \sum_{a,b,c,d=1}^{2} \left( D_{La}^{T} \phi_{mn}^{(mn)} \right) C^{-1} D_{Lc} \varepsilon^{ab} \phi_{mb} \varepsilon^{cd} \phi_{nd} \right] x
\times 2 \int d^4 x_1 \sum_{k=1}^{n_H} \sum_{\ell=1}^{2} \left( D_{La} Y_k^{\dagger} \phi_{\ell R} \right) x_1 \int d^4 x_2 \sum_{i=1}^{n_H} \sum_{f=1}^{2} \left( \bar{\ell}_R \phi_{if}^{\dagger} Y_i D_{Lf} \right) x_2, \tag{A1}
$$

13
where $d$ is the dimension of space–time and $x, x_1, x_2$ are space–time points. We have left out the flavour indices in (A1). The gauge-$SU(2)$ indices are $a, b, \ldots, f$. The symbol $\mathbf{T}$ denotes time ordering. When computing (A1), the field $\ell_R$ must be contracted with $\bar{\ell}_R$. As for $\bar{D}_{Le}$, it may be contracted either with $D_{La}$ or with $D_{Le}$; it is easy to see that both possibilities yield the same contribution to the RGE of $\kappa^{(ij)}$—this fact explains the factor 2 in the second line of (7). In the following we compute explicitly the case where $\bar{D}_{Le}$ is contracted with $D_{Le}$.

For the contraction of $\phi^\dagger_{lf}$ there are also two possibilities: one may contract it either with $\phi_{mb}$ or with $\phi_{nd}$. We consider the second possibility first. We use dimensional regularization with minimal subtraction. In the evaluation of (A1) we only need the pole terms in $\epsilon = 4 - d$. The computation is straightforward and we arrive at

$$
\frac{1}{16\pi^2\epsilon} \sum_{k,m,n=1}^{n_H} \sum_{a,b,c,d=1}^{2} D_{La}^T C^{-1} \left[ \kappa^{(mn)} Y_k^\dagger Y_n \right] D_{Ld} \varepsilon^{ab} \phi_{mb} \varepsilon^{cd} \phi_{kc},
$$

(A2)

where all the fields are now meant to be at the same space–time point. The minus sign from the factor $(-i)^2$ in (A1) has been removed through the interchange of the indices $c$ and $d$ in $\epsilon^{cd}$. If, instead, we contract $\phi^\dagger_{lf}$ with $\phi_{mb}$, then we get

$$
- \frac{1}{16\pi^2\epsilon} \sum_{k,m,n=1}^{n_H} \sum_{a,b,c,d=1}^{2} D_{La}^T C^{-1} \left[ \kappa^{(mn)} Y_k^\dagger Y_m \right] D_{Lb} \varepsilon^{ab} \varepsilon^{cd} \phi_{kc} \phi_{nd}.
$$

(A3)

The $SU(2)$ structure of (A3) is different from the one in (4). Therefore, we need to apply the identity

$$
\varepsilon^{ab} \varepsilon^{cd} + \varepsilon^{ac} \varepsilon^{db} + \varepsilon^{ad} \varepsilon^{bc} = 0
$$

(A4)

to (A3), obtaining

$$
\frac{1}{16\pi^2\epsilon} \sum_{k,m,n=1}^{n_H} \sum_{a,b,c,d=1}^{2} D_{La}^T C^{-1} \left[ \kappa^{(mn)} Y_k^\dagger Y_m \right] D_{Lb} \left( -\varepsilon^{ac} \phi_{kc} \varepsilon^{bd} \phi_{nd} + \varepsilon^{ad} \phi_{nd} \varepsilon^{bc} \phi_{kc} \right).
$$

(A5)

The terms in the last two lines of (7) are obtained from (A2) and (A5) as follows. Firstly, one substitutes the factor $1/\epsilon$ by $-1/\epsilon$. Secondly, the indices $k$ and $m$—in (A2)—or $k$ and $n$—in (A5)—of the scalar doublets must be replaced by $i$ and $j$; the contribution to the beta function of $\kappa^{(ij)}$ is then given by the flavour matrix in between the lepton doublets. If, in each expression, the first Higgs doublet is labeled $i$ and the second one is labeled $j$, then one obtains the terms in the second line of (7): $-\sum_n \kappa^{(in)} Y_i^\dagger Y_n$ from (A2), $\sum_m (\kappa^{(mj)} Y_j^\dagger Y_m - \sum_m \kappa^{(mi)} Y_i^\dagger Y_m)$ from (A5). Reversing the role of $i$ and $j$ leads to the terms in the last line of (7).

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