Collocation Method using Quartic B-Splines for Solving the Modified RLW Equation

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Abstract

The Modified Regularized Long Wave (MRLW) equation is solved numerically by giving a new algorithm based on collocation method using quartic B-splines at the mid knots points as element shape. Also, we use the Fourth Runge-Kutta method for solving the system of first order ordinary differential equations instead of finite difference method. Our test problems, including the migration and interaction of solitary waves, are used to validate the algorithm which is found to be accurate and efficient. The three invariants of the motion are evaluated to determine the conservation properties of the algorithm. The temporal evaluation of a Maxwellian initial pulse is then studied.

Keywords: Collocation Method, MRLW Equation, Quartic B-splines, Solitons.

1. Introduction

Solitary waves are wave packets or pulses, which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects these waves retain a stable waveform. The Regularized Long Wave (RLW) equation of the form:

\[ u_t + u_x + uu_x - \delta u_{xxx} = 0 \]  

where \( \delta \) is a positive constant, was originally introduced to describe the behavior of the undular bore by Peregrine. This equation is very important in physics media since it describes phenomena with weak nonlinearity and dispersion waves, including nonlinear transverse waves in shallow water, ion-acoustic and magnetohydrodynamic waves in plasma, and phonon packets in nonlinear crystals. The solutions of this equation are kinds of solitary waves called solitons whose shape is not affected by a collision. RLW equation was solved numerically by various forms of finite element methods such as Galerkin method, Least square method and Collocation Method with quadratic B-splines, cubic B-splines, and recently septic splines. Indeed, the RLW equation is a special case of the Generalized Long Wave (GRLW) equation which has the form:

\[ u_t + uu_x + \mu u^p u_x - \delta u_{xxx} = 0 \]  

Where \( \delta \) and \( \mu \) are positive constants and \( p \) is a positive integer. The GRLW equation is studied by a few authors, L. Zhang with finite difference method for a Cauchy problem, and Dogan Kaya with Adomian decomposition method (ADM). In this paper we design a new technique for solving the MRLW equation. In this paper, we consider another special case of the GRLW which is called the Modified Regularized Long Wave (MRLW) equation. This equation was considered only by Gardner et al using Petrov-Galerkin method with quintic B-splines finite element. Here, we consider a colloca-

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tion method with quartic B-spline finite elements and use the Fourth Runge-Kutta method to solve the system of first order ordinary differential equations instead of the finite difference method which is accurate and efficient. \cite{6,14}

The interaction of solitary waves and other properties of the MRLW equation are also studied.

2. The Governing Equation and Finite Element Solution

The MRLW equation, derived for long waves propagating in the positive x-direction has the form

\[ u_t + u_x + 6u^2u_x - \delta u_{xxt} = 0 \]  

where \( \delta \) is a positive parameter satisfies the boundary conditions \( u \to 0 \) as \( x \to \pm \infty \).\cite{14} In this paper we shall use periodic boundary conditions for a region \( a \leq x \leq b \). The form of the initial pulse will be chosen so that at large distances from the pulse \( |u| \) is extremely small and essentially attains the free space boundary condition \( u=0 \). In the fluid problem \( u \) is related to the vertical displacement of the water surface, while in the plasma application \( u \) is the negative of the electrostatic potential. The MRLW equation as in the form:

\[ u_t + u_x + 6u^2u_x - \delta u_{xxt} = 0 \]  

the exact solution of our equation can be written as in the article:\cite{14}

\[ u(x,t) = \sqrt{c} \sec h(p(x - (c+1)t - x_0)), \]  

where \( p = \frac{c}{\sqrt{\delta(c+1)}} \), and \( x_0, c \) are arbitrary constants.

Also, the MRLW equation has three invariants in the form as in the reported article:\cite{14}

\[ I_1 = \int u dx, \quad I_2 = \int (u^2 + \delta u_x^2) dx, \quad I_3 = \int (u^4 - \delta u_x^4) dx, \]  

the approximation solution to the solution \( u(x,t) \) is given by

\[ u^n(x,t) = \sum C_i(t)B_i(x), \]  

where \( C_i(t) \) is time dependent parameters to be determined at each time level, and \( B_i(x) \) are the quartic B-splines which have the form:

\[ B_i(x) = \begin{cases} 
(x - x_{i-2})^4, & x_{i-2} \leq x \leq x_{i-1} \\
(x - x_{i-1})^4 - 5(x - x_{i-1})^4, & x_{i-1} \leq x \leq x_i \\
(-x + x_{i+3})^4 - 5(-x + x_{i+3})^4 + 10(-x + x_{i+1})^4, & x_i \leq x \leq x_{i+1} \\
(-x + x_{i+3})^4 - 5(-x + x_{i+2})^4, & x_{i+1} \leq x \leq x_{i+2} \\
(-x + x_{i+3})^4, & x_{i+2} \leq x \leq x_{i+3} \\
0, & \text{otherwise}
\end{cases} \]  

(8)
In this section we use the collocation method using the quartic B-splines as in the following table:

**Table 1.** Quartic B-spline at mid points

| $X_{i-3}$ | $X_{i-2}$ | $X_{i-1}$ | $X_i$ | $X_{i+1}$ | $X_{i+2}$ | $X_{i+3}$ |
|-----------|-----------|-----------|-------|-----------|-----------|-----------|
| $B_i$     | 0         | $1/16$    | $19/4$ | $115/8$   | $1/16$    | 0         |
| $B'_i$    | 0         | $1/2h$    | $11/h$ | 0         | $-11/h$   | $-1/2h$  |
| $B''_i$   | 0         | $3/h^2$   | $12/h^2$ | $-30/h^2$ | $12/h^2$ | $3/h^2$  |

Then, the discretized equations for the space derivative are derived as

$$\sum (B_i(x) - \delta B'_i(x)) \hat{C}_i(t) = -(1+6(\sum C_i(t) B_i(x))) \sum C_i(t) B'_i(x),$$  \hspace{1cm} (9)

where $x$ takes the values at the selected collocation mid points. From these equations the following system of first order ordinary differential equations can be obtained.

$$\hat{C}(t) = f(C(t)).$$  \hspace{1cm} (10)

In other studies, the first order ordinary differential system (10) can be solved by using the central difference approximation for , but in the present study we solve the system (10) by using the Fourth Runge-Kutta method to obtain a numerical solution for the last system. We use the fourth order Runge-Kutta method in the following way:

$$k_1 = \Delta t f(C^n)$$
$$k_2 = \Delta t f(C^n + 0.5k_1)$$
$$k_3 = \Delta t f(C^n + 0.5k_2)$$
$$k_4 = \Delta t f(C^n + k_3)$$

and hence calculate

$$C^{n+1} = C^n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$  \hspace{1cm} (12)

the time evolution of the approximate solution $u^N(x,t)$ is determined from that of the vector $C^n$ which is found by repeatedly applying the above procedure once the starting vector $C^n$ has been computed from the initial condition.

In section (3) we first study the single soliton, also the interaction of two and three solitons with different amplitudes and study the Maxwellian initial condition.

### 3. Numerical Results

In this section, we present some numerical experiments to assign the numerical solution of single soliton waves. In addition, we determine the solution of two and three solitons interaction at different time levels; also we show the development of Maxwellian initial condition into solitary waves.

#### 3.1 The Motion of Single Solitary Waves

To examine the validity and the efficiency of our scheme, we consider two cases in our numerical work, since $L_\infty$ -norm and $L_2$ -norm are used to compare our numerical solutions with the exact solution (5). Also, the quantities $I_1$, $I_2$, and $I_3$ are evaluated to measure the conservation properties of the collocation scheme. The analytical values of these invariants can be found as in [14]: $I_1 = \frac{\pi \sqrt{c}}{2}$, $I_2 = c + \frac{4\delta c}{3}$ and $I_3 = \frac{2c^2}{3} - \frac{4\delta c}{3}$. In the first case, the parameters $c=0.25$, $k=0.2$, $\delta=1$ and $x_0 = 40$ with range [0,100] are chosen to coincide with the scheme in the reported article, then solitary wave has amplitude 0.5, and the simulations are done up to $t=10$. The analytical values for the invariants are $I_1=3.512403$, $I_2=1.192569$ and $I_3=0.1118034$. The invariants $I_1$, $I_2$, and $I_3$ change from their initial values by less than $3 \times 10^{-5}$, $3 \times 10^{-6}$ and $8 \times 10^{-7}$, respectively, during the time of running. Thus satisfactory quantities are obtained, also the error norms $L_\infty$ and $L_2$ are satisfactorily small at all times, where $L_\infty$ -error $=5.513430E-06$ and $L_2$ -error $=1.302204E-05$ at $t=10$. All results for the first case are documented in Table 2. Moreover, Table 3 represents the values of the invariants and error norms of the present method at time 10 against the recorded results of Gardner et al., we find that our scheme provides good results than others. The motion of solitary wave using our collocation scheme is plotted at different time levels in Figure 1.
3.2 Interaction of Two Solitary Waves

In this section, we study the interaction of two MRLW solitary waves which having different amplitudes and traveling in the same direction. We consider the MRLW equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes:

\[ u(x,0) = A_1 \text{sech}(p_1(x-x_1)) + A_2 \text{sech}(p_2(x-x_2)), \quad (13) \]

where \( A_i = \sqrt{c_i} \), \( p_i = \frac{c_i}{\sqrt{\delta(c_i + 1)}} \), \( i = 1, 2 \), \( x_1 \) and \( c_i \) are arbitrary constants. The analytical values of the conservation laws of this case can be found as

\[
I_1 = \frac{\pi \sqrt{c_1}}{p_1} + \frac{\pi \sqrt{c_2}}{p_2}, \quad I_2 = \frac{2c_1}{p_1} + \frac{2c_2}{p_2} + \frac{2\delta p_1 c_1}{3} + \frac{2\delta p_2 c_2}{3},
\]

and

\[
I_3 = \frac{4c_1^2}{3p_1} + \frac{4c_2^2}{3p_2} - \frac{2\delta p_1 c_1}{3} - \frac{2\delta p_2 c_2}{3}.
\]

In our computational work, we choose \( c_1 = 4, c_2 = 1, x_1 = 25, x_2 = 55, \delta = 1, h = 0.2, k = 0.025 \), with interval \([0,250]\), then the amplitudes are in ratio 2:1, where \( A_1 = 2A_2 \). The analytical values for the invariants of this case are \( I_1 = 11.467698, I_2 = 14.629243 \) and \( I_3 = 22.880466 \), and the changes in \( I_1, I_2 \) and \( I_3 \) as seen in Table 3 are small. Also, Figure 2 shows the computer plot of the interaction of these solitary waves at different time levels, where the simulation is done up to \( t=15 \).

### Table 3. Invariants of interaction of two solitary waves of MRLW equation

| Time | \( I_1 \) | \( I_2 \) | \( I_3 \) |
|------|----------|----------|----------|
| 0    | 11.467700 | 14.629210 | 22.880500 |
| 1    | 11.467700 | 14.553450 | 22.578430 |
| 2    | 11.467700 | 14.480940 | 22.291290 |
| 4    | 11.467700 | 14.411420 | 22.017820 |
| 6    | 11.467700 | 14.347230 | 21.764590 |
| 8    | 11.466230 | 14.321610 | 21.662270 |

Figure 1. Single solitary waves from \( t=0 \), up to \( t=10 \).
3.3 Interaction of Three Solitary Waves

The interaction of three MRLW solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the MRLW equation with initial conditions given by the linear sum of three well separated solitary waves of various amplitudes:

\[ u(x,0) = A_1 \sec h(p_1(x-x_1)) + A_2 \sec h(p_2(x-x_2)) + A_3 \sec h(p_3(x-x_3)), \]

(14)

Where \( A_i = \sqrt{c_i} \), \( p_i = \frac{c_i}{\sqrt{\delta (c_i + 1)}} \), and \( c_i \) are arbitrary constants. The analytical values of the conservation laws of this case can be found as:

\[ I_1 = \frac{\pi \sqrt{c_1}}{p_1} + \frac{\pi \sqrt{c_2}}{p_2} + \frac{\pi \sqrt{c_3}}{p_3}, \]

\[ I_2 = \frac{c_1}{p_1} + \frac{2c_2}{p_2} + \frac{2c_3}{p_3} + \frac{2\delta p_1c_1}{3p_3} + \frac{2\delta p_2c_2}{3p_3} + \frac{2\delta p_3c_3}{3p_3} \]

and

\[ I_3 = \frac{4c_1^2}{3p_1^2} + \frac{4c_2^2}{3p_2^2} + \frac{4c_3^2}{3p_3^2} - \frac{2\delta p_1c_1}{3p_3} - \frac{2\delta p_2c_2}{3p_3} - \frac{2\delta p_3c_3}{3p_3}. \]

In our computational work, we choose \( c_1 = 4, c_2 = 1, c_3 = 0.25, x_1 = 15, x_2 = 45, x_3 = 60, \delta = 1 \) with interval \([0,250]\), then the amplitudes are in ratio 4:2:1, where \( A_1 = 2A_2 = 4A_3 \). The analytical values for the invariants of this case are \( I_1 = 14.9801, I_2 = 15.8218 \) and \( I_3 = 22.9923 \) and we find from our numerical scheme that the invariants \( I_1, I_2, I_3 \) for interaction of these solitary waves are sensible constants, comparing with their big amplitudes, the changes are \( 1.02 \times 10^{-3}, 3 \times 10^{-4} \) and \( 2.41 \times 10^{-4} \) percent, respectively for the computer run and the results are recorded in Table 4. Figure 3 shows details of interaction of these solitary waves at different time levels, and the simulation is done up to \( t = 20 \).

Table 4. Invariants of interaction of three solitary waves of MRLW equation

| Time | \( I_1 \) | \( I_2 \) | \( I_3 \) |
|------|----------|----------|----------|
| 0    | 14.980090| 15.837450| 23.008200|
| 1    | 14.980110| 15.799140| 22.855150|
| 2    | 14.980100| 15.761700| 22.706120|
| 3    | 14.980120| 15.725070| 22.560780|
| 4    | 14.980110| 15.689200| 22.418970|
| 5    | 14.979780| 15.654060| 22.280570|

3.4 The Maxwellian Initial Condition

In final series of numerical experiments, the development of the Maxwellian initial condition:

\[ u(x,0) = e^{-x-\delta^2 - (x-\delta^2)^2}, \]

(15)

into a train of solitary waves is examined. As it is known, with the Maxwellian (15), the behavior of the solution depends on the values of \( \delta \). For \( \delta > \delta_c \), where \( \delta_c \) is some critical value, the Maxwellian does not break up into solutions but exhibits rapidly oscillating wave packets. When \( \delta \approx \delta_c \), a mixed type of solutions is found which consists of a leading soliton and an oscillating tail. For \( \delta < \delta_c \), the Maxwellian breaks up into a number of solitons according to the value of \( \delta \). The recorded values of the invariants \( I_1, I_2, I_3 \) are given in Table 5. The conservation properties are all good.
Table 5. Computed values $I_1$, $I_2$, $I_3$ for Maxwellian initial condition when $\delta=0.1, \kappa=0.1$, [0,100]

| $\delta$ | Time | $I_1$   | $I_2$   | $I_3$   |
|---------|------|---------|---------|---------|
| 1       | 0    | 1.772454| 2.506624| -0.3670824|
|         | 1    | 1.772452| 2.506620| -0.3670842|
|         | 2    | 1.772451| 2.506617| -0.3670933|
|         | 3    | 1.772450| 2.506616| -0.3670944|
|         | 4    | 1.772449| 2.506614| -0.3670933|
|         | 5    | 1.772449| 2.506616| -0.3670935|
| 0.1     | 1    | 1.772458| 1.370170| -1.120874|
|         | 2    | 1.772459| 1.358382| -1.183677|
|         | 3    | 1.772460| 1.347160| -1.177421|
|         | 4    | 1.772462| 1.336518| -1.168438|
|         | 5    | 1.772462| 1.326405| -1.160032|
| 0.05    | 1    | 1.772462| 1.248352| -2.471305|
|         | 2    | 1.772452| 1.168798| -2.490366|
|         | 3    | 1.772453| 1.113770| -2.346704|
|         | 4    | 1.772457| 1.072956| -2.224114|
|         | 5    | 1.772461| 1.040952| -2.124501|

Figure 4. Maxwellian initial condition, $h=0.1, \kappa=0.1$, [0,100].

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