Abstract

We establish the equality of two definitions of an Euler class in algebraic geometry: the first definition is as a "characteristic class" with values in Chow-Witt theory, while the second definition is as an "obstruction class." Along the way, we refine Morel’s relative Hurewicz theorem in $\mathbb{A}^1$-homotopy theory, and show how to define (twisted) Chow-Witt groups for geometric classifying spaces.

1 Introduction

Suppose $k$ is a perfect field having characteristic unequal to 2, $X$ is a $d$-dimensional smooth $k$-scheme, and $\xi : E \to X$ is a rank $r$ vector bundle on $X$. A choice of isomorphism $\theta : \det \xi \to O_X$ is called an orientation of $\xi$. There are (at least) two ways to define an “Euler class” $e(\xi)$ of $\xi$ that provides the primary obstruction to existence of a now-where vanishing section of $\xi$; both definitions are simpler in case $\xi$ is oriented. The goal of this note is to prove equivalence of these definitions. Let us now recall the definitions.

Using the notation of [Fas08], one possible definition is as follows (this is the “characteristic class” approach mentioned in the abstract). If $s_0 : X \to E$ is the zero section of $E$, then there are pullback and Gysin pushforward homomorphisms in Chow-Witt groups. There is a canonical element $\langle 1 \rangle \in \tilde{CH}^0(X)$, and we define

$$e_{cw}(\xi) := (\xi^*)^{-1}(s_0)_*(1) \in \tilde{CH}^r(X, \det(\xi)^\vee)$$

(see [Fas08, Definition 13.2.1]; the definition there is equivalent to forming the product with this class using the ring structure in Chow-Witt groups of [Fas07] by the excess intersection formula of [Fas09]). This definition of Euler class is functorial for pullbacks and coincides with the definition of Euler class given in [BM00, §2.1] for oriented vector bundles.

An alternative definition comes from [Mor12, Remark 8.15], where one constructs an Euler class as the primary obstruction to existence of a non-vanishing section of $\xi$. In that case, if $Gr_\nu$ denotes
the infinite Grassmannian, and $\gamma_r$ is the universal rank $r$ vector bundle on $Gr_r$, the first non-trivial stage of the Moore-Postnikov factorization in $A^1$-homotopy theory of the map $Gr_{r-1} \to Gr_r$ gives rise to a canonical morphism $Gr_r \to K^{G_m}(K_w^{MW})$ (see [Mor12, Appendix B] for a discussion of twisted Eilenberg-Mac Lane spaces in our setting) yielding a canonical (equivariant) cohomology class $o_r \in H_{Nis}^r(Gr_r, K_w^{MW}(\det \gamma_r^\vee))$.

Given any smooth scheme $X$ and an $A^1$-homotopy class of maps $\xi : X \to Gr_r$ “classifying” a vector bundle $\xi$ as above, pullback of $o_r$ along $\xi$ yields a class

$$e_{ob}(\xi) := \xi^*(o_r).$$

This definition of Euler class is evidently functorial for pullbacks as well (note: with this definition $k$ is not required to have characteristic unequal to 2).

Assuming the Milnor conjecture on quadratic forms, now a theorem [OVV07], it follows that both Euler classes live in the same group. Thus, it makes sense to ask if the two classes coincide. In [Mor12, Remark 8.15], Morel asserts that the two definitions of Euler class given above coincide, but provides no proof. The main result of this paper provides justification for Morel’s assertion, and can be viewed as an analog in algebraic geometry of [MS74, Theorem 12.5].

**Theorem 1.** If $k$ is a perfect field having characteristic unequal to 2, and if $\xi : E \to X$ is an oriented rank $r$ vector bundle on a smooth $k$-scheme $X$, then

$$e_{ob}(\xi) = e_{cw}(\xi).$$

The method of proof we propose is classical and is likely the one envisaged by Morel: we establish this result by the method of the universal example. Nevertheless, we felt it useful to provide a complete proof of the above result for at least three other reasons. First, as is perhaps evident from the length of this note, a fair amount of effort is required to develop the technology necessary for the proof, and some of the preliminary results proven will be used elsewhere. Second, as Morel observes in [Mor12, Remark 8.15], in combination with his $A^1$-homotopy classification of vector bundles and the theory of the Euler class [Mor12, Theorems 8.1 and 8.14], the above result completes the verification of the main conjecture of [BM00]. Third, the results of this paper are already used in [AF12a, Lemma 3.3] and therefore implicitly in [AF12b]. The method of this paper can also be used to give another proof of [AF12c, Proposition 5.2] that the two (equivalent) Euler classes defined above are mapped to the top Chern class under the natural map from (twisted) Chow-Witt groups to Chow groups.

**Overview of contents**

Write $Gr_n$ for the infinite Grassmannian of $n$-dimensional subspaces of an infinite dimensional $k$-vector space. Let $\gamma_n$ be the universal vector bundle on $Gr_n$. Since $Gr_n$ is a colimit of smooth schemes, the Chow-Witt Euler class of $\gamma_n$ is *a priori* undefined. In Section 2, we show how to extend the definition of the Chow-Witt Euler class to certain colimits of smooth schemes, including, in particular, vector bundles over geometric classifying spaces of algebraic groups in the sense of [Tot99] or [MV99, §4.2].
The obstruction theoretic Euler class is, by construction, a \( k \)-invariant in the Moore-Postnikov tower of the map \( Gr_{n-1} \rightarrow Gr_n \). Classically, this class is, up to choices of orientation, the transgression of the fundamental class of the fiber. To generalize this to our setting, we need to establish versions of the relative \( \mathbb{A}^1 \)-Hurewicz theorem, which we do in Section 3. Then, we establish that the obstruction theoretic Euler class is transgressive in Section 4.

With these technical results established, since both the Chow-Witt Euler class and obstruction theoretic Euler class are stable under pullbacks (the latter by its very definition), it suffices to check the two classes coincide for the universal rank \( n \) vector bundle over \( Gr_n \); this is the content of Section 5. To do this, requires some results comparing fiber and cofiber sequences.

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Notation and preliminaries

Our notation regarding \( \mathbb{A}^1 \)-homotopy theory, fiber sequences, etc., follows the conventions we laid out in [AF12a, §2]. To keep this paper short, we have used results of [MV99] and [Mor12] rather freely. Any time we refer to the results of [Mor12], the reader should understand that \( k \) is assumed perfect. For the most part, we follow the notation of [Fas08] for Chow-Witt theory. Any time we refer to the results of Chow-Witt theory, the reader should understand that \( k \) has characteristic unequal to 2. Given the truth of the Milnor conjecture on quadratic forms [OVV07], the \( n \)-th Chow-Witt group of a smooth scheme \( X \), can be identified as \( H^n(X, K_n^{MW}) \); see also [Mor12, Theorem 5.47].

2 The Chow-Witt Euler class

The infinite Grassmannian \( Gr_n \) is usually presented as a colimit of smooth schemes. Up to \( \mathbb{A}^1 \)-homotopy, there are many possible models for \( Gr_n \) (see, e.g., [MV99, §4] for a discussion of this point). For example, given any faithful finite dimensional representation of \( GL_n \), it is possible to use Totaro’s finite dimensional approximations [Tot99]. Chow-Witt groups are, a priori defined only for smooth schemes. While it is possible to define Chow-Witt groups for an arbitrary space, we take the “poor man’s approach”: we prove a stabilization result for Chow-Witt groups analogous to that used to define equivariant Chow groups.

Lemma 2.1. If \( X \) is an \( n \)-dimensional smooth scheme over a perfect field \( k \), \( L \) is a line bundle on \( X \) and \( Y \subset X \) is a closed subscheme of codimension \( c \) with open complement \( j : U \rightarrow X \), then the induced map

\[
j^* : \widetilde{CH}^i(X, L) \rightarrow \widetilde{CH}^i(U, j^* L)
\]

is bijective for \( i < c - 1 \).

Proof. The Chow-Witt groups of \( X \) are defined using an explicit Gersten-type complex \( C(X, G^*) \). If \( U \) is an open subscheme, then we have a surjective map of complexes \( C(X, G^*) \rightarrow C(U, G^*) \)
whose kernel is the subcomplex $C_Y(X, G^*) \subset C(X, G^*)$ of cycles supported on $Y$. If $Y$ is of codimension $c$, then the morphism of complexes $C(X, G^*) \to C(U, G^*)$ is thus an isomorphism in degrees $\leq c - 1$. The result follows.

Remark 2.2. Note that the restriction map $\widetilde{CH}^i(X) \to \widetilde{CH}^i(U)$ attached to an open immersion $U \to X$ is not surjective in general.

Suppose $G$ is a linear algebraic group over a perfect field $k$, $(\rho, V)$ is a faithful finite dimensional $k$-rational representation of $G$, $V_n \subset V^\oplus n + \dim V$ is the (maximal) open subscheme on which $G$ acts (scheme-theoretically) freely, and $X$ is a smooth $G$-scheme such that the quotients $X \times^G V_n$ exist as smooth schemes. The $G$-equivariant open immersion $V_n \hookrightarrow V_{n+1}$ which arise from the inclusion of $V^\oplus n + \dim V \hookrightarrow V^\oplus n + 1 + \dim V$ as the first $n + \dim V$-summands yields bonding maps $b_n : X \times^G V_n \to X \times^G V_{n+1}$. Write $X_G(\rho) := \text{colim}_n b_n$ and $BG(\rho)$ in the case when $X = \text{Spec } k$; the spaces $X \times^G V_n$ will be called finite-dimensional approximations to $X_G(\rho)$.

The space $X_G(\rho) := \text{colim}_n b_n$ has $\mathbb{A}^1$-homotopy type independent of $\rho$; when $X = \text{Spec } k$ this independence result is established in [MV99, §4.2, esp. Remark 4.2.7]. In general, this independence statement can be established by the “Bogomolov double fibration trick;” see, e.g., [Tot99, p. 5] or [EG98, Definition Proposition 1]. If $\rho$ and $\rho'$ are two faithful representations on vector spaces $V$ and $V'$, we can consider $\rho \oplus \rho'$. There are induced maps $X_G(\rho \oplus \rho') \to X_G(\rho)$ that one can check are $\mathbb{A}^1$-weak equivalences. In any case, $\text{Pic}(X_G(\rho))$ is well-defined, independent of the choice of $\rho$. Then, if $L$ is a line bundle on $X_G(\rho)$ whose pullback to $X \times^G V_n$ is denoted $L_n$, then for any integer $i$ the groups $\lim_n \widetilde{CH}^i(X \times^G V_n, L_n)$ are defined.

Theorem 2.3. With $\rho, V_n$ and $L_n$ as described in the preceding paragraphs, the groups $\widetilde{CH}^i(X \times^G V_n, L_n)$ stabilize, i.e., are independent of $n$ for $n$ sufficiently large.

Proof. In the situation above, the inclusion $V_n \to V_{n+1}$ can be factored as $V_n \to V \oplus V_n \to V_{n+1}$. This factorization induces maps

$$X \times^G V_n \to X \times^G (V \oplus V_n) \to X \times^G V_{n+1}.$$

The projection map $V \oplus V_n \to V_n$ descends to the projection map of a vector bundle $X \times^G (V \oplus V_n) \to X \times^G V_n$, and the first map in the above displayed sequence is the zero section. The second map is a open immersion and it is known that the codimension of the complement tends to $\infty$ as $n \to \infty$. In particular, by Lemma 2.1 the induced maps on Chow-Witt groups stabilize.

Because of the Theorem 2.3, the following notation makes sense.

Notation 2.4. We set $\widetilde{CH}^i(X_G(\rho), L) := \lim_n \widetilde{CH}^i(X \times^G V_n, L_n)$.

Remark 2.5. Applying the above result with $G = GL_n$, $X = \text{Spec } k$ and $(V, \rho)$ the standard $n$-dimensional representation of $GL_n$ we see that the theorem applies to $Gr_n := \text{colim}_N Gr_{n,n+N}$ and any line bundle on $Gr_n$. More generally, the above results show that it makes sense to talk about (Borel style, twisted) equivariant Chow-Witt groups.

Corollary 2.6. If $X = Gr_n$ and $\gamma_n : V_n \to Gr_n$ is the universal rank $n$ vector bundle on $Gr_n$, then there is a well defined element $e_{cw}(\gamma_n) \in \widetilde{CH}^i(Gr_n, \det(\gamma_n)^\vee)$.
3 A refined relative Hurewicz theorem in $\mathbb{A}^1$-homotopy theory

In this section, we prove a refined version of F. Morel’s relative $\mathbb{A}^1$-Hurewicz theorem [Mor12, Theorem 6.56].

A refined relative $\mathbb{A}^1$-Hurewicz theorem

Fix a field $k$. Suppose $f : X \to Y$ is a morphism of pointed spaces. If we write $F := \text{hofib}(f)$ (throughout, we mean $\mathbb{A}^1$-homotopy fiber), then there is a diagram of the form

$$
\begin{array}{ccc}
F & \to & * \\
\downarrow & & \downarrow \\
X & \to & Y.
\end{array}
$$

By functoriality of homotopy colimits, there is an induced map:

$$
\Sigma^1_s F \to \text{hocofib}(f).
$$

The refined relative Hurewicz theorem we refer to gives information about the $\mathbb{A}^1$-connectivity of this morphism assuming connectivity hypotheses on the $\mathbb{A}^1$-homotopy fiber.

**Theorem 3.1.** Suppose $k$ is a perfect field. Assume $(X, x)$ and $(Y, y)$ are pointed $k$-spaces with $X$ assumed $\mathbb{A}^1$-1-connected. If $f : X \to Y$ is a pointed $\mathbb{A}^1$-m-connected morphism, then the induced morphism $\Sigma^1_s F \to \text{hocofib}(f)$ is $\mathbb{A}^1$-$(m + 2)$-connected. If $Y$ is $\mathbb{A}^1$-n-connected for some $n \geq 2$ and $f$ is $\mathbb{A}^1$-m-connected for some $m \geq 2$, then the map $\Sigma^1_s F \to \text{hocofib}(f)$ is $\mathbb{A}^1$-$(m + n + 1)$-connected.

**Proof.** By using functorial fibrant replacements, we can assume that $f : X \to Y$ is an $\mathbb{A}^1$-fibration of $\mathbb{A}^1$-fibrant spaces. The $\mathbb{A}^1$-homotopy fiber $F$ of $f$ is then precisely the point-set fiber of this...
morphism and, by assumption, it is simplicially $n$-connected. In particular, the stalks of $F$ are $n$-connected simplicial sets, and the stalks of $X$ are simply connected simplicial sets.

Now, using functorial cofibrant replacements, we can up to simplicial weak equivalence replace $f$ by a cofibration (i.e., replace $Y$ by the mapping cylinder of $f$). (Note: forming the mapping cylinder only changes $Y$ and therefore does not change the actual fiber $F$ of $f$.) In that case, the homotopy colimit coincides with the ordinary colimit (i.e., it is the quotient sheaf). Since $A^1$-localization is a left adjoint, it follows that $A^1$-localization commutes with formation of colimits. In other words, we can assume that $L_{A^1}(\mathrm{hocofib}(f)) \cong \mathrm{hocofib}(L_{A^1}f)$.

Taking stalks commutes with formation of finite limits or arbitrary colimits. Since $\Sigma^1_\ast F$ is $S^1_\ast \times F/(S^1_\ast \vee F)$, the stalk of the suspension at a point coincides with the suspension of the stalk. One can conclude that the stalks of the morphism $\Sigma^1_\ast F \to \mathrm{hocofib}(f)$ are simplicially $(m + 2)$-connected by appealing to [GJ09, Theorem 3.11]. Likewise, under the second set of hypotheses of the theorem statement, the stalks are simplicially $m + n + 1$-connected by [Mat76, Theorem 50].

We can then apply Morel’s relative $A^1$-Hurewicz theorem [Mor12, Theorem 6.57] to conclude that the map $L_{A^1}\Sigma^1_\ast F \to L_{A^1}\mathrm{hocofib}(f)$ is simplicially $(n + 2)$-connected (resp. $A^1$-$(m + n + 1)$ connected under the second set of hypotheses). The latter is isomorphic to $\mathrm{hocofib}(L_{A^1}f)$ as we saw before, and this is the desired conclusion.

4 Transgression and $k$-invariants

The goal of this section is to give a nice representative of the $k$-invariant that defines the obstruction theoretic Euler class; we show that the obstruction theoretic Euler class can be described in terms of a “fundamental class” under transgression (see Lemma 4.1 and Examples 4.2 and 4.3). This is the analog of a classical fact relating Moore-Postnikov $k$-invariants and transgressions of cohomology of the fiber (see [Tho66, Chapter III p. 12] for a general statement $k$-invariants or [Har02, p. 237] for the corresponding topological statement). In the setting in which we work (simplicial sheaves), it is more or less a question of unwinding definitions; we build on the theory of [GJ09, VI.5] in the setting of simplicial sets.

On $k$-invariants in Moore-Postnikov towers

The Moore-Postnikov tower of a morphism of spaces is constructed in the simplicial homotopy category by sheafifying the classical construction in simplicial homotopy theory [GJ09, VI.2]. To perform the same construction in $A^1$-homotopy theory, one applies the construction in the simplicial homotopy category to a fibration of fibrant and $A^1$-local spaces [Mor12, Appendix B]. The description of the $k$-invariants in the Moore-Postnikov factorization is then an appropriately sheafified version of the classical construction. The first result is an analog of [GJ09, Lemma VI.5.4] in the context of $A^1$-homotopy theory.

Lemma 4.1. Suppose $f : X \to Y$ is a morphism of pointed $A^1$-1-connected spaces and write $F$ for the $A^1$-homotopy fiber of $f$. If $f$ is an $A^1$-$(n - 1)$-equivalence for some $n \geq 2$, then for any strictly $A^1$-invariant sheaf $A$ there are isomorphisms

$$f^* : H^i(Y, A) \xrightarrow{\sim} H^i(X, A) \text{ if } i < n,$$
and an exact sequence of the form

\[ 0 \rightarrow H^n(\mathcal{Y}, A) \xrightarrow{f^*} H^n(X, A) \rightarrow Hom(\pi_n^A(\mathcal{F})), A) \xrightarrow{d} H^{n+1}(\mathcal{Y}, A) \rightarrow H^{n+1}(X, A). \]

Moreover the sequence above is natural in morphisms \( f \) satisfying the above hypotheses.

**Proof.** If \( C \) is the homotopy cofiber of \( f \), then there is a cofiber sequence

\[ \mathcal{X} \rightarrow \mathcal{Y} \rightarrow C \rightarrow \Sigma_1 \mathcal{X} \rightarrow \cdots. \]

By assumption \( \mathcal{F} \) is \( \mathbb{A}^1 \)-\((n-1)\)-connected for some \( n \geq 2 \). By the \( \mathbb{A}^1 \)-Freudenthal suspension theorem, \( \Sigma_1 \mathcal{F} \) is at least \( \mathbb{A}^1 \)-\( n \)-connected. By the relative Hurewicz theorem 3.1, we know that \( \Sigma_1 \mathcal{F} \rightarrow C \) is an \( \mathbb{A}^1 \)-(\( n+1 \))-equivalence, so \( C \) is at least \( \mathbb{A}^1 \)-\( n \)-connected.

Since \( C \) is at \( \mathbb{A}^1 \)-\( n \)-connected, it follows that \( H^n(C, A) = Hom(\pi_n^A(C), A) \) for \( i \leq n \) by, e.g., [AD09, Theorem 3.30]. The first statement then follows from the long exact sequence in cohomology associated with the above cofiber sequence. For the second statement, observe that there are canonical isomorphisms

\[ \pi_n^A(\mathcal{F}) \xrightarrow{\sim} \pi_{n+1}(\Sigma_1 \mathcal{F}) \xrightarrow{\sim} \pi_{n+1}(C) \]

by Morel’s \( \mathbb{A}^1 \)-Freudenthal suspension theorem [Mor12, Theorem 6.61] and the relative Hurewicz theorem 3.1. Another application of [AD09, Theorem 3.30] then implies that

\[ H^{n+1}(C, A) \cong Hom(\pi_{n+1}^A(C), A), \]

which then yields the identification

\[ H^{n+1}(C, A) \cong Hom(\pi_{n+1}^A(\mathcal{F}), A) \]

by the isomorphisms stated above. The functoriality statement is a consequence of the functoriality of the various construction involved. \( \square \)

**Example 4.2.** In the notation of Lemma 4.1, since \( \mathcal{F} \) is \( \mathbb{A}^1 \)-\((n-1)\)-connected, it follows that if \( A \) is any strictly \( \mathbb{A}^1 \)-invariant sheaf, then \( H^n(\mathcal{F}, A) = Hom(\pi_n^A(\mathcal{F}), A) \), again by [AD09, Theorem 3.30]. In particular, taking \( A = \pi_n^A(\mathcal{F}) \), the identity morphism on \( \pi_n^A(\mathcal{F}) \) gives a canonical element \( 1_{\mathcal{F}} \in H^n(\mathcal{F}, \pi_n^A(\mathcal{F})) \) that we refer to as the “fundamental class of the \( \mathbb{A}^1 \)-homotopy fiber”. Then, \( d \) determines a morphism

\[ H^n(\mathcal{F}, \pi_n^A(\mathcal{F})) \cong H^{n+1}(\mathcal{Y}, \pi_n^A(\mathcal{F})) \xrightarrow{d} H^{n+1}(\mathcal{Y}, \pi_n^A(\mathcal{F})). \]

The image of \( 1_{\mathcal{F}} \) under the above morphism yields a canonical element of \( d(1_{\mathcal{F}}) \in H^{n+1}(\mathcal{Y}, \pi_n^A(\mathcal{F})) \) that we refer to as the *transgression of the fundamental class of the \( \mathbb{A}^1 \)-homotopy fiber*. The hypotheses of the previous result apply to the Moore-Postnikov factorization of a morphism of \( \mathbb{A}^1 \)-connected spaces. In that case the element \( d(1_{\mathcal{F}}) \) is, by definition, the \( k \)-invariant at the relevant stage of the tower [GJ09, VI.5.5-6].
The obstruction theoretic Euler class

Consider the map $Gr_{n-1} \to Gr_n$, and assume $n \geq 3$ (we will treat $n = 2$ shortly). If $V$ is the standard $n$-dimensional representation of $GL_n$ and $GL_{n-1} \to GL_n$ is the usual inclusion as the upper left $(n-1) \times (n-1)$-block, then we can consider $V$ as a representation of $GL_{n-1}$. Since that representation is still faithful, we can form the space $V_n/GL_{n-1}$ (notation as in the previous section). Observe that the map $V_n/GL_{n-1} \to V_n/GL_n$ induced by the inclusion is Zariski locally trivial with fibers isomorphic to $GL_n/GL_{n-1}$. In fact, up to $\mathbb{A}^1$-homotopy, this construction identifies $Gr_{n-1}$ as the complement of the zero section of the universal vector bundle over $Gr_n$. Observe therefore that, in addition to the $\mathbb{A}^1$-fiber sequence

$$\cdots \to \mathbb{A}^n \setminus 0 \to Gr_{n-1} \to Gr_n,$$

we obtain the cofiber sequence

$$Gr_{n-1} \to Gr_n \to Th(\gamma_n) \to \cdots.$$

Now, the space $Gr_n$ is not $\mathbb{A}^1$-connected, which complicates our discussion, but the $\mathbb{A}^1$-universal cover in the sense of [Mor12, §7.1] is not hard to describe.

Indeed, the space $BSL_n$, which we identified in Remark 2.7 as the total space as a $G_m$-torsor over $Gr_n$, is the $\mathbb{A}^1$-universal cover of $Gr_n$. There is an $\mathbb{A}^1$-fiber sequence

$$\mathbb{A}^n \setminus 0 \to BSL_{n-1} \to BSL_n,$$

that, since $BSL_i$ is $\mathbb{A}^1$-connected, corresponds to passing to $\mathbb{A}^1$-connected covers in the $\mathbb{A}^1$-fiber sequence $\mathbb{A}^n \setminus 0 \to Gr_{n-1} \to Gr_n$.

Example 4.3. Since the space $BSL_n$ is $\mathbb{A}^1$-connected, the map $BSL_{n-1} \to BSL_n$ satisfies the hypotheses of Lemma 4.1. In this case, $C = Th(\gamma_n)$ and $\mathcal{F} = \mathbb{A}^n \setminus 0$, and $\pi^n_{n-1}(\mathbb{A}^n \setminus 0) \cong K_n^{MW}$. Thus, Example 4.2 gives a canonical class in $\omega_n \in H^n(BSL_n, K_n^{MW})$ as the image of a “fundamental class” in $H^{n-1}(\mathbb{A}^n \setminus 0, K_n^{MW})$; by definition, this class is $e_{ob}(\gamma_n)$ from the introduction.

Remark 4.4. The case $n = 2$ is slightly anomalous. In that case, note that, since $\pi_2^{A^1}(BSL_2) = K_2^{MW}$, we have $BSL_2 = K(K_2^{MW}, 2)$. The composite map $BSL_2 \to BSL_2 \to K(K_2^{MW}, 2)$ defines the universal obstruction class in this case. The reason for this discrepancy is that $BSL_1 = *$. We have the model $H^{\infty}$ as a model for $BSL_2$ by the results of [PW10]. The inclusion $H^{\infty} \to H^{\infty}$ gives, up to $\mathbb{A}^1$-homotopy, a map $\mathbb{P}^{1\wedge 2} \to BSL_2$ that factors the map $\mathbb{P}^{1\wedge 2} \to Th(\gamma_2)$.

By definition, for any smooth $k$-scheme $X$ and an oriented rank $n$ vector bundle $\xi : E \to X$, $e_{ob}(\xi)$ is the pullback of $e_{ob}(\gamma_n)$ along a map $X \to BSL_n$ classifying $\xi$.

Remark 4.5 (Orientation classes). By [AF12a, Lemma 4.2], we know that $H^{n-1}(\mathbb{A}^n \setminus 0, K_n^{MW}) \cong K_0^{MW}(k)$. However, the isomorphism $H^{n-1}(\mathbb{A}^n \setminus 0, K_n^{MW}) \cong K_0^{MW}(k)$ is non-canonical (see [AF12a, Remark 4.3]): it depends on a choice of an “orientation class,” which corresponds to a choice of basis of the tangent space of $0 \in \mathbb{A}^n$. The explanation of how such a choice affects the isomorphism is described in [AF12a, Example 4.4]. If $k$ has characteristic unequal to 2, then there is a canonical isomorphism $K_0^{MW}(k) \cong GW(k)$. To make the orientation class explicit, we will write $H^{n-1}(\mathbb{A}^n \setminus 0, K_n^{MW}) \cong K_0^{MW}(k)\xi$ for $\xi$ the basis of the tangent space corresponding to the standard coordinates $(x_1, \ldots, x_n)$ on $\mathbb{A}^n$. 

4 Transgression and $k$-invariants
5 Proof of Theorem 1

Proof. By Corollary 2.6 and Remark 2.7, it makes sense to speak of the Chow-Witt Euler class of the universal vector bundle over the (oriented) Grassmannian $BSL_n$. Therefore, to prove the main result, since both the Chow-Witt Euler class and Morel’s obstruction theoretic Euler class are functorial under pullbacks, it suffices to check that the two classes coincide in the case of the universal vector bundle over the Grassmannian $BSL_n$.

Note that, by the relative Hurewicz theorem 3.1, we know that $Th(\gamma_n)$ is $A^1-(n-1)$-connected, and $\pi_n(Th(\gamma_n)) \cong K_n^{MW}$. In particular, $H^n(Th(\gamma_n), K_n^{MW}) \cong \text{Hom}(K_n^{MW}, K_n^{MW})$ by [AD09, Theorem 3.30]. There is a non-canonical isomorphism $\text{Hom}(K_n^{MW}, K_n^{MW}) \cong (K_n^{MW})_{-n} \cong K_0^{MW}$, and we can choose the orientation class of Remark 4.5 as a generator of this group. The Chow-Witt theoretic Euler class is then precisely the image of $1 \in K_0^{MW}(\mathbb{k}) = H^n(Th(\gamma_n), K_n^{MW})$ in $H^n(BSL_n, K_n^{MW})$ under the connecting homomorphism in the cofiber sequence

$$BSL_{n-1} \longrightarrow BSL_n \longrightarrow Th(\gamma_n).$$

We observe that the Chow-Witt Euler class is then the $k$-invariant of Section 4, and thus coincides with the obstruction theoretic Euler class.

Remark 5.1. With more work, we expect it is possible to establish the comparison result in the introduction for vector bundles that are not necessarily oriented, i.e., to check that the Euler classes twisted by the dual of the determinant coincide. Describing the $k$-invariant explicitly in this setting is more involved because one has to keep track of $G_m$-equivariance; the corresponding result in the setting of simplicial sets is [GJ09, Lemma VI.5.4]. We have avoided pursuing this generalization because in all cases we know where one wants to actually compute a twisted Euler class one uses the Chow-Witt definition.

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