THE STABLE HYPERELLIPTIC LOCUS IN GENUS 3: AN APPLICATION OF PORTEOUS FORMULA

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ABSTRACT. We compute the class of the closure of the locus of hyperelliptic curves in the moduli space of stable genus-3 curves in terms of the tautological class $\lambda$ and the boundary classes $\delta_0$ and $\delta_1$. The expression of this class is known, but here we compute it directly, by means of Porteous Formula, without resorting to blowups or test curves.

1. Introduction

Porteous Formula gives an expression for the class of a degeneracy scheme of a map of vector bundles on a smooth variety in terms of the Chern classes of the bundles, as long as the degeneracy scheme has no excess, that is, has the expected codimension; see [4], Thm. 14.4, p. 254. In [7], Question 3.134, p. 169, Harris and Morrison posed the question of finding a Porteous-type formula for maps between torsion-free sheaves.

Their question was motivated by the following problem. Let $k$ be an algebraically closed field of characteristic different from 2. Let $\overline{M}_3$ be the moduli space of genus-3 smooth, projective, connected curves, and $\overline{M}_3$ its compactification by (Deligne–Mumford) stable curves over $k$. The vector space $\text{Pic}(\overline{M}_3) \otimes \mathbb{Q}$ is generated by a certain class $\lambda$. This class is the restriction of one in $\text{Pic}(\overline{M}_3) \otimes \mathbb{Q}$, which will be denoted by the same symbol. The latter space is generated by $\lambda$, $\delta_0$ and $\delta_1$, the latter two being boundary classes.

Let $H \subseteq \overline{M}_3$ be the locus parameterizing hyperelliptic curves. It is a closed subvariety of codimension 1. Let $\overline{H}$ be its closure in $\overline{M}_3$. We may ask what are the expressions for the class $[H]$ as a multiple of $\lambda$ and $[\overline{H}]$ as a linear combination of $\lambda$, $\delta_0$ and $\delta_1$. In [7], pp. 162–188, it is shown that

\begin{align*}
[H] &= 9\lambda, \\
\overline{[H]} &= 9\lambda - \delta_0 - 3\delta_1.
\end{align*}

(The first formula above had already appeared as a special case of Mumford’s formula for $[H]_Q$ on [10], p. 314.)

The strategy for obtaining the formula for $[H]$, culminating on page 164 of loc. cit., reviewed in Section 2, was to consider a general family of smooth curves,

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a natural map of vector bundles over its total space whose top degeneracy scheme parameterizes the Weierstrass points of hyperelliptic fibers of $\pi$, apply Porteous Formula to compute the class of this scheme, and then compute the pushforward of this class to the base of the family. However, according to [7], p. 169, even though “trying to extend the application of Porteous’ formula” to a general family of stable curves “is the most obvious approach” to obtaining a formula for $[H]$, the problem is that a certain bundle of jets, namely (2), defined on the smooth locus of the family, “cannot be extended to a vector bundle over the nodes of fibers of the family of curves.” This motivated their question, mentioned above. Not disposing of the asked-for Porteous-type formula, they proposed a less direct approach, by means of the so-called test curves, culminating with the formula for $[H]$ on page 188 of loc. cit..

In [1] Diaz proposes a blowup procedure to obtain a Porteous-type formula for maps between torsion-free sheaves. Though no explicit formula is produced, as an example, the procedure is carried out to obtain a different and more direct proof that $[H] = 9\lambda - \delta_0 - c\delta_1$ for a certain $c \in \mathbb{Z}$, not computable because of excess; see Prop. 1, p. 510 of loc. cit..

In these notes we will see that the bundle of jets which “cannot be extended to a vector bundle” does in fact extend; see Section 3. As observed in [7], p. 169, the obvious extension, namely the sheaf of jets, or principal parts, (3), is not a bundle. However, it becomes so after a pushout construction, (4). So we do get a map of vector bundles over the total space $C$ of a family of stable curves $C/S$, namely (5), to which we can consider applying Porteous Formula. In contrast to Diaz’s approach, no blowups are necessary. But, as in [1], Porteous Formula cannot be directly applied because of excess.

Though the excess could be handled in an ad hoc way, we will see that a simple “twist,” typical of the theory of limit linear series explained in [7], Ch. 5, is enough to produce a map of vector bundles over $C$, namely (7), whose top degeneracy scheme $D$ has the expected codimension by Proposition 2, the class of which can thus be computed by Porteous Formula. Its pushforward to $S$ is given by Proposition 3.

As in [1], $D$ comprises more than the Weierstrass points of the (smooth) hyperelliptic fibers of $\pi$. To get the formula for $[H]$ of Theorem 1, the excess points must be counted out. To remove them, we need to establish their multiplicities in $D$. This is Proposition 4. The three Propositions imply the Theorem.

Though sheaves of jets, or principal parts, for a family of stable curves are not vector bundles, there are vector bundle substitutes that agree with them on the smooth locus of the family. These substitutes have appeared, in various degrees of generality, in [2], [3], [5], [6], [8] and [9]. It may thus well be that a Porteous-type formula for maps between torsion-free sheaves is not necessary for dealing with stable curves. What would certainly be useful instead, is a way of dealing with excessive degeneracy schemes, a phenomenon typical in enumerative questions.

The layout of these notes is the following: In Section 2 we review the approach in [7] for computing $[H]$. In Section 3, we show how to produce a map of vector
bundled over the total space $C$ of a general family of stable curves $\pi: C \to S$, whose top degeneracy scheme $D$ has the expected codimension, and contains the eight Weierstrass points on each (smooth) hyperelliptic fiber, among others on singular fibers; see Proposition 2. In Section 4, we apply Porteous Formula to compute the pushforward of the class of $D$ to $S$; see Proposition 3. To conclude the proof of Theorem 1 in Section 5, we compute the multiplicity with which the points on singular fibers appear in $D$, our Proposition 4, thereby finishing the computation of $[H]$.

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2. Smooth curves

The formulas in Theorem 1 can be obtained by considering the Picard group of the functor, or that of the associated stack. Given a flat, projective map $\pi: C \to S$ with smooth, connected fibers of dimension 1 and genus 3, the class $[H]$ corresponds to a class $h^\pi$ on $S$, and $\lambda$ to a class $\lambda^\pi$, the first Chern class of the rank-3 locally free sheaf $\pi_*\Omega^1_{C/S}$. To show that $[H] = 9\lambda^\pi$, it is equivalent to show that $h^\pi = 9\lambda^\pi$ for every such $\pi$.

Likewise, to show that $[H] = 9\lambda - \delta^\pi_0 - 3\delta^\pi_1$, we may consider the corresponding classes $h^\pi$, $\lambda^\pi$, $\delta^\pi_0$ and $\delta^\pi_1$ on the target of flat, projective maps $\pi: C \to S$ whose fibers are stable curves of (arithmetic) genus 3, and show that

$$[H] = 9\lambda - \delta^\pi_0 - 3\delta^\pi_1. \quad (1)$$

Since $\overline{M}_3$ is stackwise smooth and complete, we need only consider one-dimensional smooth projective targets $S$, and show the above formula holds for a general such map $\pi$. Alternatively, we may show that the degrees on both sides of the formula are equal for three special such maps $\pi$, yielding three linearly independent triples $(\deg(\lambda^\pi), \deg(\delta^\pi_0), \deg(\delta^\pi_1))$. The latter is known as the method of test curves, employed in [7].

The strategy to compute $[H]$ in [7] is as follows. Let $\pi: C \to S$ be a projective, flat map between schemes of finite type over the algebraically closed field $k$. Assume the (geometric) fibers of $\pi$ are smooth, connected curves of genus 3. Given a (closed) point $s \in S$, we will let $C_s := \pi^{-1}(s)$.

Let $\Omega^1_{C/S}$ be the relative cotangent sheaf. Set $E := \pi^*\pi_*\Omega^1_{C/S}$. At a (closed) point $P$ of $C$, letting $s := \pi(P)$, we have

$$E|_P = H^0(C_s, \Omega^1_{C_s/k}).$$

Since the fibers of $\pi$ have genus three, $E$ is locally free of rank 3.
Let $F$ be the relative bundle of first-order jets, or principal parts, of $\Omega^1_{C/S}$. In other words,

\[
F := p_1^*(p_2^*\Omega^1_{C/S} \otimes \frac{\mathcal{O}_{C \times_S C}}{I^2_{C \times_S C}}),
\]

where $p_i : C \times_S C \to C$ is the projection onto the indicated factor, for $i = 1, 2$, and $\Delta \subset C \times_S C$ is the diagonal. At a point $P \in C$, letting $s := \pi(P)$, we have

\[
F|_P := H^0(C_s, \Omega^1_{C_s}|_k \big|_{\text{Spec}(\mathcal{O}_{C_s,P}/m_{C_s,P})}),
\]

where $m_{C_s,P}$ is the maximal ideal of the local ring $\mathcal{O}_{C_s,P}$. Since $C_s$ is smooth, $F$ is locally free of rank 2.

Since $\pi^*\pi_*\Omega^1_{C/S} = p_1^*p_2^*\Omega^1_{C/S}$, there is a natural map $\nu : \mathcal{E} \to \mathcal{F}$. At a given $P \in C$, the map is an evaluation map: Given a local parameter $t$ of $C_s$ at $P$, where $s := \pi(P)$, using $dt$ to trivialize $\Omega^1_{C_s}|_k$ at $P$, the map $\nu|_P$ assigns to a global differential form, whose germ at $P$ can be written as $f(t)dt$ for $f \in \mathcal{O}_{C_s,P}$, the class $(f(0) + f'(0)t)dt \mod t^2dt$, where $f' := df/dt$. Thus, $\nu|_P$ fails to be surjective if and only if $H^0(C_s, \Omega^1_{C_s}|_k(-2P))$ fails to have codimension 2 in $H^0(C_s, \Omega^1_{C_s}|_k)$, that is, if and only if $C_s$ is hyperelliptic, and $P$ is a Weierstrass point of $C_s$.

Let $D$ be the top degeneracy scheme of the map $\nu$, supported on the set of points $P \in C$ where $\nu|_P$ fails to be surjective. Then, assuming the general fiber of $\pi$ is nonhyperelliptic, $D$ has codimension at least 2. This is however the expected codimension, thus the actual codimension. Then, assuming $S$ is smooth, or at least Cohen–Macaulay, Porteous Formula ([4], Thm. 14.4, p. 254 or [7], Thm. 3.114, p. 161) gives an expression for $[D]$: \n
\[ [D] = c_2(\mathcal{E}^* - \mathcal{F}^*) \cap [C]. \]

There are 8 Weierstrass points on a hyperelliptic curve of genus 3. Thus \n
\[ 8h^3 = \pi_*[D] = \pi_*(c_2(\mathcal{E}^* - \mathcal{F}^*) \cap [C]). \]

If we compute the right-hand side of the formula above we will get $72\lambda^3$. We will not do this here, as in Sections 3, 4 and 5 we will use the same procedure to prove the more general ([1]).

3. Stable curves

In [7], the expression for $[H]$ is computed by the method of test curves. Instead, we will use a “twist” of the same method used to compute $[H]$.

Let $\pi : C \to S$ be a projective, flat map between schemes of finite type over the algebraically closed field $k$. Assume the fibers of $\pi$ are stable curves of genus 3.

As pointed out in [7], the relative cotangent sheaf $\Omega^1_{C/S}$ is not locally free, but can be replaced by the (invertible) relative dualizing sheaf $\omega_{C/S}$, equal to the cotangent sheaf away from the nodes of the fibers. The restriction $\omega_{C_s}$ of $\omega_{C/S}$ to a fiber $C_s$ is Rosenlicht’s sheaf of regular differential forms, those being the meromorphic forms
regular everywhere, except over a node, where the form must have at most simple pole at each branch with zero residue sum.

Set \( \mathcal{E} := \pi^* \pi_* \omega_{C/S} \). As in Section 2, the sheaf \( \mathcal{E} \) is locally free of rank 3. On the other hand, [7] asserts that the restriction to the smooth locus of \( C/S \) of

\[
p_1^* \left( p_2^* \omega_{C/S} \otimes \frac{\mathcal{O}_{C \times_S} I^2_{\Delta(C \times_S C)}}{I^2_{\Delta(C \times_S C)}} \right),
\]

where, as before, \( p_i : C \times_S C \to C \) is the projection onto the indicated factor, for \( i = 1, 2 \), and \( \Delta \subset C \times_S C \) is the diagonal, does not extend to a locally free sheaf on the whole \( C \). This is false.

It is true, as pointed out in [7], that (3), the sheaf of first-order principal parts of \( \omega_{C/S} \) is not locally free, but there is a locally free substitute that coincides with that sheaf away from the nodes. In the case at hand, it is easy to produce the substitute, by considering the pushout for the following diagram of maps:

\[
\begin{array}{ccc}
0 & \longrightarrow & \omega_{C/S} \otimes \Omega^1_{C/S} \\
\downarrow & & \downarrow \\
\omega_{C/S}^{\otimes 2} & \longrightarrow & F \longrightarrow \omega_{C/S} \longrightarrow 0
\end{array}
\]

where the exact sequence is obtained from the natural exact sequence

\[
0 \longrightarrow \frac{I_{\Delta(C \times_S C)}}{I^2_{\Delta(C \times_S C)}} \longrightarrow \frac{\mathcal{O}_{C \times_S C}}{I^2_{\Delta(C \times_S C)}} \longrightarrow \frac{\mathcal{O}_{C \times_S C}}{I^2_{\Delta(C \times_S C)}} \longrightarrow 0,
\]

and the vertical map is obtained from the "canonical class" \( \Omega^1_{C/S} \to \omega_{C/S} \) by tensoring with \( \omega_{C/S} \). The pushout construction completes the above diagram to a map of short exact sequences:

\[
\begin{array}{ccc}
0 & \longrightarrow & \omega_{C/S} \otimes \Omega^1_{C/S} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \omega_{C/S}^{\otimes 2} \longrightarrow F \longrightarrow \omega_{C/S} \longrightarrow 0
\end{array}
\]

Of course, \( F \) is locally free of rank 2.

As before, there is a natural homomorphism from \( \mathcal{E} \) to (3), which can be composed to a homomorphism

\[
\nu : \mathcal{E} \to F.
\]

It restricts, over the open subset of points of \( S \) parameterizing smooth fibers, to the map of locally free sheaves considered in Section 2. In fact, away from the nodes, the description is the same: If \( P \in C \) is not a node of \( C_s \), where \( s := \pi(P) \), then, considering a local parameter \( t \) of \( C_s \) at \( P \), and using \( dt \) to trivialize \( \Omega^1_{C_s/k} \) at \( P \), the map \( \nu|_P \) assigns to a regular differential form, whose germ at \( P \) can be written as \( f dt \) for \( f \in \mathcal{O}_{C_s,P} \), the class \( (f(0) + f'(0)t)dt \mod t^2 dt \), where \( f' := df/dt \).
Assume now that $S$ is one-dimensional and $\pi$ is “general.” More precisely, assume $C$ is nonsingular, the general fiber of $\pi$ is smooth, and the (finitely many) singular fibers have only one singularity. There are thus two types of singular fibers: an irreducible curve $Z$ with a node whose two branches on the normalization $\tilde{Z}$ are in general position with respect to the canonical system; and the union of a genus-1 curve $X$ with a genus-2 curve $Y$ meeting transversally at a point $N$ which is not a Weierstrass point of $Y$. From the arguments we will use, we may harmlessly assume that there are just two singular fibers, $Z$ and $X \cup Y$.

The top degeneracy scheme of $\nu$ has dimension 1. Indeed, it contains the curve $X$. This is because $\omega_{C/S}|_X = \mathcal{O}_X(N)$, whose space of global sections has dimension 1. Since the dimension of the degeneracy scheme is not the minimum possible, the so-called expected dimension, which in this case is zero, we say that we have excess. To deal with excess in Intersection Theory is generally hard. In our case, the excess could be handled in an ad hoc way. We will nonetheless avoid it.

What we will do is replace $\omega_{C/S}$ by a “twisted” sheaf:

$$L := \omega_{C/S}(-X) := \omega_{C/S} \otimes \mathcal{O}_C(-X).$$

We will set $\mathcal{E}' := \pi^* \pi_* L$, and let $\mathcal{F}'$ be the sheaf obtained by the pushout construction, part of the map of short exact sequences:

$$0 \longrightarrow L \otimes \Omega^1_{C/S} \longrightarrow p_1^*(p_2^* L \otimes \frac{\mathcal{O}_{C \times S \setminus \Delta}}{\mathcal{I}_{\Delta}(C \times S)}) \longrightarrow L \longrightarrow 0$$

As before, there is a natural map

$$\nu' : \mathcal{E}' \to \mathcal{F}'.$$

**Proposition 2.** The sheaves $\mathcal{E}'$ and $\mathcal{F}'$ are locally free of ranks 3 and 2, respectively. In addition, the top degeneracy scheme of $\nu'$ is zero-dimensional, and consists of the following points:

1. The 8 Weierstrass points of each smooth hyperelliptic fiber of $\pi$;
2. The node of $Z$;
3. The node $N$ of $X \cup Y$;
4. The 3 points $A \in X - \{N\}$ such that $2A \equiv 2N$;
5. The 6 Weierstrass points of $Y$.

**Proof.** That $\mathcal{F}'$ is locally free of rank 2 follows immediately from it being the middle sheaf in the bottom exact sequence of Diagram (6). On the other hand, to show that $\mathcal{E}'$ is locally free of rank 3, we need to show that 3 is the dimension of the space of global sections of $L$ restricted to each fiber. This is clear except for $L|_{X \cup Y}$. Now, there are two exact sequences associated to $L|_{X \cup Y}$:

$$0 \rightarrow L|_X(-N) \rightarrow L|_{X \cup Y} \rightarrow L|_Y \rightarrow 0,$$

$$0 \rightarrow L|_Y(-N) \rightarrow L|_{X \cup Y} \rightarrow L|_X \rightarrow 0.$$
The first yields the exact sequence

\[(9) \quad 0 \to \mathcal{O}_X(N) \to L|_{X \cup Y} \to \omega_Y \to 0,\]

where \(\omega_Y\) is the canonical sheaf of \(Y\). It follows from the long exact sequence in cohomology associated to \([9]\) that \(h^0(X \cup Y, L|_{X \cup Y}) = 3\). So, \(\mathcal{E}'\) is locally free of rank 3.

Let \(D\) denote the top degeneracy scheme of \(\nu'\). Since \(h^0(X \cup Y, L|_{X \cup Y}) = 3\), it follows from the base-change theorem and the exactness of \([9]\) that the two maps in the composition below are surjective:

\[
H^0(C, L) \longrightarrow H^0(X \cup Y, L|_{X \cup Y}) \longrightarrow H^0(Y, L|_Y).
\]

Since \(L|_Y \cong \omega_Y\), and since \(H^0(Y, \omega_Y(-2A)) \neq 0\) if and only if \(A\) is a Weierstrass point of \(Y\), it follows that \(D \cap Y - \{N\}\) consists of the six Weierstrass points of \(Y\).

The second exact sequence in \([8]\) yields the exact sequence:

\[
0 \longrightarrow \omega_Y(-N) \longrightarrow L|_{X \cup Y} \longrightarrow \mathcal{O}_X(2N) \longrightarrow 0.
\]

Since \(h^0(Y, \omega_Y(-N)) = 1\) and \(h^0(X, \mathcal{O}_X(2N)) = 2\), it follows as before that the two maps in the composition below are surjective:

\[
H^0(C, L) \longrightarrow H^0(X \cup Y, L|_{X \cup Y}) \longrightarrow H^0(X, L|_X).
\]

Thus, since \(L|_X \cong \mathcal{O}_X(2N)\), it follows that \(D \cap X - \{N\}\) consists of the three points \(A \in X - \{N\}\) such that \(2A \equiv 2N\).

As explained in Section 2, \(D\) does not intersect nonhyperelliptic fibers, and intersects each hyperelliptic fiber in its 8 Weierstrass points.

As for the fiber \(Z\), consider the following composition of natural maps:

\[
H^0(C, L) \longrightarrow H^0(Z, L|_Z) \longrightarrow H^0(\tilde{Z}, L|_{\tilde{Z}}).
\]

The first map above is surjective by the base-change theorem. The second is the pullback map to the normalization, which is injective, and so an isomorphism, since both source and target have dimension 3. Here we used the Riemann–Roch Theorem and the fact that \(L|_{\tilde{Z}} = \omega_{\tilde{Z}}(M_1 + M_2)\), where \(\omega_{\tilde{Z}}\) is the canonical sheaf of \(\tilde{Z}\), and \(M_1\) and \(M_2\) are the two points over the node \(M\) of \(Z\). Assuming \(M_1\) and \(M_2\) are in general position, \(M_1 + M_2\) does not move, and thus, by the Riemann–Roch Theorem, \(h^0(\tilde{Z}, \omega_{\tilde{Z}}(M_1 + M_2 - 2B)) = 1\) for every \(B \in \tilde{Z}\) distinct from \(M_1\) and \(M_2\). Hence \(D\) intersects \(Z\) at most at its node.

It remains only to show that the nodes of \(Z\) and of \(X \cup Y\) belong to \(D\), but this will be done in Proposition 3. What is important for what follows is that we have already shown \(D\) to be finite. \[\square\]

4. An application of Porteous Formula

**Proposition 3.** Let \(D\) be the top degeneracy scheme of \(\nu'\): \(\mathcal{E}' \to \mathcal{F}'\). Then

\[
\pi_*[D] = 72\lambda^7 - 7\delta_0^7 - 7\delta_1^7.
\]
Proof. Since, by Proposition [2] \( D \) has the right codimension, we may compute its class in \( C \) by Porteous Formula ([1], Thm. 14.4, p. 254 or [7], Thm. 3.114, p. 161):

\[
[D] = c_2(\mathcal{E}' - \mathcal{F}'') \cap [C].
\]

Expanding,

\[
[D] = \left[ c(\mathcal{E}'') \right]_2 \cap [C]
\]

\[
= \left[ \frac{1 - c_1(\mathcal{E}') + c_2(\mathcal{E}')} {1 - c_1(\mathcal{F}') + c_2(\mathcal{F}')} \right] \cap [C]
\]

\[
=(1 - c_1(\mathcal{E}'))(1 + c_1(\mathcal{F}'') - c_2(\mathcal{F}')) \cap [C]
\]

\[
=(c_2(\mathcal{E}') - c_1(\mathcal{E}')(1 + c_2(\mathcal{F}'')) + 1 - c_2(\mathcal{F}'')) \cap [C].
\]

Now, \( \mathcal{F}' \) sits in the middle of an exact sequence of the form:

\[
0 \longrightarrow L \otimes \omega_{C/S} \longrightarrow \mathcal{F}' \longrightarrow L \longrightarrow 0.
\]

By the Whitney Sum Formula ([1], Thm. 3.2(e), p. 50), letting \( K := c_1(\omega_{C/S}) \cap [C] \),

\[
c_1(\mathcal{F}') \cap [C] = (2c_1(L) + c_1(\omega_{C/S})) \cap [C] = 3K - 2[X],
\]

\[
c_2(\mathcal{F}') \cap [C] = c_1(L)c_1(\omega_{C/S}) \cap [C] = (K - [X])(2K - [X]).
\]

On the other hand, from the exact sequence

\[
0 \longrightarrow \omega_{C/S}(-X) \overset{\cdot X} \longrightarrow \omega_{C/S} \longrightarrow \omega_{C/S}|_X \longrightarrow 0,
\]

since \( \omega_{C/S}|_X = \mathcal{O}_X(N) \) and \( H^1(X, \mathcal{O}_X(N)) = 0 \), we get the long exact sequence

\[
0 \longrightarrow \pi_*(\omega_{C/S}(-X)) \longrightarrow \pi_*\omega_{C/S} \overset{\beta} \longrightarrow \pi_*\mathcal{O}_X(N) \overset{\gamma} \longrightarrow R^1\pi_*(\omega_{C/S}) \longrightarrow 0.
\]

As we have seen in the proof of Proposition [2] \( \pi_*(\omega_{C/S}(-X)) \) is a locally free sheaf of rank 3 with formation commuting with base change, whence \( R^1\pi_*(\omega_{C/S}(-X)) \) is invertible. Since so is \( R^1\pi_*(\omega_{C/S}) \), it follows that \( \gamma \) is an isomorphism. So \( \beta \) is surjective. Since \( h^0(X, \mathcal{O}_X(N)) = 1 \), it follows that

\[
c_1(\pi_*(\omega_{C/S}(-X))) = c_1(\pi_*\omega_{C/S}) - \delta = \lambda^\pi - \delta^\pi.
\]

Thus

\[
c_1(\mathcal{E}') \cap [C] = \pi^*(\lambda^\pi - \delta^\pi),
\]

\[
c_2(\mathcal{E}') \cap [C] = 0.
\]

Replacing (11) and (12) in (10), we get

\[
[D] = \pi^*(\delta^\pi - \lambda^\pi)(3K - 2[X]) + (3K - 2[X])^2 - (K - [X])(2K - [X])
\]
Thus,
\[
\pi_*[D] = (\delta_1^\pi - \lambda^\pi) \pi_* (3K - 2[X]) + \pi_* ((3K - 2[X])^2) - \pi_* ((K - [X])(2K - [X]))
\]
\[
= 12(\delta_1^\pi - \lambda^\pi) + \pi_* (9K^2 - 16[N] + 4[X] \pi^* \delta_1^\pi) - \pi_* (2K^2 - 4[N] + [X] \pi^* \delta_1^\pi)
\]
\[
= 12(\delta_1^\pi - \lambda^\pi) + 9\kappa^\pi - 16\delta_1^\pi - 2\kappa^\pi + 4\delta_1^\pi
\]
\[
= 7\kappa^\pi - 12\lambda^\pi,
\]
where \(\kappa^\pi := \pi^* K^2\). In the first equality above we used the projection formula. In the second, we used that \(\pi\) collapses \(X\) to a point, that \(\omega_{C/S}\big|_X \sim 0\), that \([X] = \pi^* \delta_1^\pi - [Y]\), that \(\pi_* K = 4[S]\), and that \([X][Y] = [N]\). In the third, we used again the projection formula and the fact that \(\pi\) collapses \(X\) to a point.

Finally, using that (7), (3.110), p. 158
\[
\kappa^\pi = 12\lambda^\pi - \delta_0^\pi - \delta_1^\pi,
\]
we get
\[
\pi_*[D] = 7(12\lambda^\pi - \delta_0^\pi - \delta_1^\pi) - 12\lambda^\pi,
\]
which yields the stated formula. \(\square\)

5. Multiplicities

Once we remove the contribution of the points of items (2) to (5) of Proposition\(^2\) from the expression of \(\pi_*[D]\) in Proposition\(^3\) we get \(8\tilde{h}^\pi\), and thus (1). This means that the node of \(Z\) should appear in \(D\) with multiplicity 1, and the remaining points, items (3) to (5), should count to 17, with multiplicities. Indeed:

**Proposition 4.** The scheme \(D\) consists of:

1. the 8 Weierstrass points of each smooth hyperelliptic fiber of \(\pi\), each with multiplicity 1;
2. the node of \(Z\), with multiplicity 1;
3. the node \(N\) of \(X \cup Y\), with multiplicity 2;
4. the 3 points \(A \in X - \{N\}\) such that \(2A \equiv 2N\), each with multiplicity 1;
5. the 6 Weierstrass points of \(Y\), each with multiplicity 2.

**Proof.** We have seen in Proposition\(^2\) that \(D\) consists at most of the points listed above. The multiplicity of a Weierstrass point of a smooth hyperelliptic fiber of \(\pi\) has been established in [7], Ex. 3.116, p. 164. The multiplicity of the node of \(Z\) can be computed in essentially the same way as the multiplicity computation in [1]; we will do it at the end for the sake of completeness.

We will first establish the multiplicities at smooth points, starting with (1). Let \(A \in X - N\) such that \(2A \equiv 2N\). Let \(t\) be a local parameter of \(O_{S,s}\), where \(s := \pi(A)\). Since \(\pi\) is smooth at \(A\), there is \(u \in O_{C,A}\) such that \(t, u\) form a regular system of parameters for \(O_{C,A}\). Set \(L' := L(-X)\). The exactness of the natural sequence
\[
0 \to L' \to L \to L|_X \to 0,
\]
coupled with the surjectivity of the restriction map $H^0(C, L) \rightarrow H^0(X, L|_X)$ shown in the proof of Proposition 2, yields the exactness of

$$0 \rightarrow H^0(C, L') \rightarrow H^0(C, L) \rightarrow H^0(X, L|_X) \rightarrow 0.$$ 

Since $L|_X \cong \mathcal{O}_X(2N)$, we may choose a $k[[t]]$-basis $s_1, s_2, s_3$ of $H^0(C, L)$ such that $\psi := s_1$ generates $L_A$ and

$$s_2 \in (u^2 + (t, u^3))\psi,$$

$$s_3 \in (t)\psi.$$

Now, since $L' = L(-X)$, it follows that $ts_1, ts_2, s_3 \in H^0(C, L')$. Moreover, as $h^0(X, L|_X) = 2$, they form a $k[[t]]$-basis of $H^0(C, L')$. But $L'|_X \cong \mathcal{O}_X(3N)$ and, since $H^0(Y, \omega_Y(-2N)) = 0$, the restriction map $H^0(C, L') \rightarrow H^0(X, L'|_X)$ is an isomorphism. Since $3A \neq 3N$, the vanishing orders of sections of $\mathcal{O}_X(3N)$ at $A$ are $0, 1, 2$. Since $L'_A = \mathcal{O}_{C,A}t\psi$, the vanishing orders of $ts_1|_X$ and $ts_2|_X$ at $A$ are 0 and 2, respectively. Thus, replacing $s_3$ by $s_3 - (s_3/ts_1)(A)ts_1$, we may assume that the vanishing order of $s_3|_X$ at $A$ is 1. So, we may assume

$$s_1 \in \{\psi\},$$

$$s_2 \in (u^2 + (t, u^3))\psi,$$

$$s_3 \in (u + (t, u^2))t\psi.$$

Thus, $D$ is the zero scheme given by the maximal minors of a matrix whose entries are in the corresponding entries of

$$\begin{bmatrix}
    \{1\} & u^2 + (t, u^3) & tu + (t^2, tu^2) \\
    \{0\} & 2u + (t, u^2) & t + (t^2, tu)
\end{bmatrix}$$

Since the characteristic of the ground field $k$ is assumed different from 2, it follows that the multiplicity of $D$ at $A$ is 1.

We will establish the multiplicity in (13) now. Let $A$ be a Weierstrass point of $Y$. Let $t$ be a local parameter of $\mathcal{O}_{S,A}$, where $s := \pi(A)$. Since $\pi$ is smooth at $A$, there is $u \in \mathcal{O}_{C,A}$ such that $t, u$ form a regular system of parameters for $\mathcal{O}_{C,A}$.

Let $L' := L(-Y)$ and $L'' := L(-2Y)$. Then $L'_A = tL_A$ and $L''_A = t^2L_A$. As we have seen in the proof of Proposition 2, the restriction map

$$(13) \quad H^0(C, L) \rightarrow H^0(Y, L|_Y)$$

is surjective. Since $L|_Y \cong \omega_Y$, and $A$ is a Weierstrass point of $Y$, there is a basis $s_1, s_2, s_3$ of $H^0(C, L)$ such that $s_1(A) \neq 0$, $s_2|_Y$ vanishes at $A$ with multiplicity 2, and $s_3|_Y = 0$. Let $\psi$ be the germ of $s_1$ at $A$. So, in $L_A$, we may assume

$$s_1 \in \{\psi\},$$

$$s_2 \in (u^2 + (t, u^3))\psi,$$

$$s_3 \in (t)\psi.$$

Since (13) is surjective, and since $h^0(Y, L|_Y) = 2$, the sections $ts_1, ts_2, s_3$ of $H^0(C, L')$ form a basis. Since $L'|_X \cong \mathcal{O}_X(N)$ and $L'|_Y \cong \omega_Y(N)$, and since
$h^0(X, \mathcal{O}_X(N)) = 1$ and $h^0(Y, \omega_Y(N)) = 2$, it follows that the restriction map

$$H^0(C, L') \rightarrow H^0(Y, L'|_Y)$$

is surjective, and $h^0(Y, L'|_Y) = 2$. Thus there is $s' \in H^0(C, L')$ forming a $k[[t]]$-basis of $H^0(C, L')$ together with $ts_1, ts_2$ such that $s'|_Y = 0$. Up to replacing $s_3$ by $s_3 - fts_1 - gts_2$ for $f, g \in k[[t]]$, we may assume $s' = s_3$. So $s_3 \in (t^2)^\psi$.

In addition, it follows that the sections $t^2s_1, t^2s_2, s_3$ of $H^0(C, L''')$ form a basis. But $L''|_Y \cong \omega_Y(2N)$. Since $A$ is a Weierstrass point of $Y$, and $Y$ has genus 2,

$$h^0(Y, \omega_Y(2N - 3A)) = h^0(Y, \mathcal{O}_Y(2N - A)) = h^1(Y, \mathcal{O}_Y(2N - A))$$

where the last equality follows from the fact that $N$ is not a Weierstrass point of $Y$. Then there must be a section $s''$ of $L''$, forming a $k[[t]]$-basis with $t^2s_1, t^2s_2$,

whose restriction to $Y$ vanishes at $A$ with multiplicity 1. Up to replacing $s_3$ by

$s_3 - (s_3/t^2s_1)(A)t^2s_1$, we may assume that $s'' = s_3$. Thus we may assume

$$s_3 \in t^2(u + (t, u^2))\psi.$$

It follows that $D$ is given at $A$ by the maximal minors of a matrix whose entries belong to the corresponding entries of the matrix below:

$$\begin{bmatrix}
1 & u^2 + (t, u^3) & t^2(u + (t, u^2)) \\
0 & 2u + (t, u^2) & t^2 + t^2(t, u)
\end{bmatrix}.$$  

Then the minors belong to

$$2u + (t, u^2), \quad t^2 + (t, u)^3, \quad (t, u)^3.$$  

Since the characteristic of $k$ is not 2, it follows that $D$ has multiplicity 2 at $A$.

Let us establish the multiplicity in (3) now. Let $t$ be a local parameter of $\mathcal{O}_{S,s}$, where $s := \pi(N)$. Let $x$ (resp. $y$) be a local equation for $Y$ (resp. $X$) at $N$. We may choose them such that $t = xy$ in the local ring $\mathcal{O}_{C,N}$. Since the restriction maps

$$H^0(C, L) \rightarrow H^0(X, \mathcal{L}|_X) \quad \text{and} \quad H^0(C, L) \rightarrow H^0(Y, \mathcal{L}|_Y)$$

are surjective, $\mathcal{L}|_X \cong \mathcal{O}_X(2N)$ and $\mathcal{L}|_Y \cong \omega_Y$, there is a basis $s_1, s_2, s_3$ of $H^0(C, L)$ as a $k[[t]]$-module such that $s_1(N) \neq 0$, that $s_2|_Y = 0$ and $s_2|_X$ vanishes at $N$, necessarily to order 2, and $s_3|_X = 0$ and $s_3|_Y$ vanishes at $N$ to order 1. Thus, letting $\psi$ be the germ of $s_1$ at $N$, we may assume that

$$s_1 \in \{\psi\},$$

$$s_2 \in x(x + (y, x^2))\psi,$$

$$s_3 \in y(1 + (x, y))\psi.$$  

Now, $\omega_{C/S}$ is generated at $N$ by the meromorphic differential $\tau := dx/x = -dy/y$. This means that the canonical derivation $\partial$ on $L_N$ induced by the composition of the universal derivation $\mathcal{O}_C \rightarrow \Omega^1_{C/S}$ with the canonical class $\Omega^1_{C/S} \rightarrow \omega_{C/S}$ satisfies $\partial(x) = x\tau$ and $\partial(y) = -y\tau$.  

Thus, $D$ is given at $N$ by the maximal minors of a matrix whose entries belong to the corresponding entries of the matrix below:

$$
\begin{bmatrix}
\{1\} & x(x + (y, x^2)) & y(1 + (x, y)) \\
\{0\} & 2x^2 + x(y, x^2) & -y + y(x, y)
\end{bmatrix}.
$$

Then the minors belong to

$$
2x^2 + kxy + (x, y)^3, \quad -y + (x, y)^2, \quad (x, y)^3.
$$

Since the characteristic of $k$ is not 2, it follows that $D$ has multiplicity 2 at $N$.

Finally, let us establish the multiplicity in (2). Let $M$ denote the node of $Z$. Let $t$ be a local parameter of $S$, $s := \pi(M)$. Since $Z$ is a node of the special fiber, there are local parameters $x$ and $y$ at $N$ such that $t \equiv xy \mod (x, y)^3$. Let $\tilde{Z}$ be the normalization of $Z$ and $M_1, M_2 \in \tilde{Z}$ the points above $M$. The normalization map induces a canonical isomorphism

$$
H^0(Z, \omega_Z) \longrightarrow H^0(\tilde{Z}, \omega_{\tilde{Z}}(M_1 + M_2)).
$$

Since the restriction map $H^0(C, \omega_{C/S}) \to H^0(Z, \omega_Z)$ is surjective, there is a basis $s_1, s_2, s_3$ of $H^0(C, L)$ as a $k[[t]]$-module such that $s_1(M) \neq 0$ and the pullbacks $\tilde{s}_2$ and $\tilde{s}_3$ of $s_2$ and $s_3$ to $\tilde{Z}$ are such that both vanish at $M_1$ and $M_2$ but $\tilde{s}_2$ vanishes at $M_1$ to order 2 and $\tilde{s}_3$ vanishes at $M_2$ to order 2. Thus, letting $\psi$ be the germ of $s_1$ at $M$, we may assume

$$
s_1 \in \{\psi\},
$$

$$
s_2 \in (x + (x, y)^2)\psi,
$$

$$
s_3 \in (y + (x, y)^2)\psi.
$$

Let $\tau$ be a generator of $\omega_{C/S}$ at $M$, and let $\partial$ be the canonical derivation of $\omega_{C/S, M}$, induced by the composition of the universal derivation $\mathcal{O}_C \to \Omega^1_{C/S}$ with the canonical class $\Omega^1_{C/S} \to \omega_{C/S}$. Choosing $\tau$ appropriately, we have $\partial(x) \equiv x \mod (x, y)^2$ and $\partial(y) \equiv -y \mod (x, y)^2$.

Thus, $D$ is given at $M$ by the maximal minors of a matrix whose entries belong to the corresponding entries of the matrix below:

$$
\begin{bmatrix}
\{1\} & x + (x, y)^2 & y + (x, y)^2 \\
\{0\} & x + (x, y)^2 & -y + (x, y)^2
\end{bmatrix}.
$$

Then the minors belong to

$$
x + (x, y)^2, \quad -y + (x, y)^2, \quad (x, y)^2,
$$

and thus $D$ has multiplicity 1 at $M$. \square

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