NUCLEAR DIMENSION AND SUMS OF COMMUTATORS

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Abstract. The problem of expressing a selfadjoint element that is zero on every bounded trace as a finite sum (or a limit of sums) of commutators is investigated in the setting of C*-algebras of finite nuclear dimension. Upper bounds – in terms of the nuclear dimension of the C*-algebra – are given for the number of commutators needed in these sums. An example is given of a simple, nuclear C*-algebra (of infinite nuclear dimension) with a unique tracial state and with elements that vanish on all bounded traces and yet are “badly” approximated by finite sums of commutators. Finally, the same problem is investigated on (possibly non-nuclear) simple unital C*-algebras assuming suitable regularity properties in their Cuntz semigroups.

1. Introduction

Let $A$ be a C*-algebra. Let $A_0$ denote the intersection of the kernels of all bounded traces on $A$ further intersected with the selfadjoint elements of $A$. A theorem of Cuntz and Pedersen says that $A_0$ agrees with the set of elements of the form $\sum_{i=1}^{\infty} [x_i^* x_i]$, with $x_i \in A$, and where the infinite sum is norm convergent. A line of research pursued by several authors – [Fac82], [Tho93], [Pop02], [Mar06], [KNZ12], [Ng12] – has been to find conditions on $A$ under which the elements of $A_0$ are expressible as finite, rather than infinite, sums of self-commutators (i.e., commutators of the form $[x^*, x]$). In this paper we explore the link between this question and the nuclear dimension of C*-algebras. The latter is a notion of dimension for C*-algebras due to Winter and Zacharias, which extends the covering dimension of topological spaces to the non-commutative setting (see [WZ10]). We prove the following theorem:

Theorem 1.1. Let $A$ be a C*-algebra of nuclear dimension $m \in \mathbb{N}$. If $a \in A_0$ then $a$ is a limit of sums of $m + 1$ commutators of the form $[x^*, x]$, with $\|x\|^2 \leq 2\|a\|$.

In [Fac82], Fack devised a technique – further developed in [Tho93] and [Ng12] – to show that elements of $A_0$, under suitable hypothesis, are finite sums of commutators. We generalize Fack’s technique to prove the following theorem:

Theorem 1.2. Let $m \in \mathbb{N}$. There exists $M = O(m^3)$ such that if $A$ is a unital C*-algebra of nuclear dimension $m \in \mathbb{N}$, with no finite dimensional representations and no simple purely infinite quotients, then each element of $A_0$ is exactly the sum of $M$ self-commutators.

The hypothesis of no simple purely infinite quotients is most likely unnecessary, but our present method for constructing “large” orthogonal positive elements in $A$ requires it.

In [Mar06], Marcoux gives a “reduction argument” for the number of commutators needed for expressing elements of $A_0$. Ng further developed this method in [Ng12] and [Ng13], showing for example that a unital embedding of the Jiang-Su algebra is often sufficient to make Marcoux’s reduction work. Theorem 1.2 above, combined with [Ng13, Theorem 3.2], yields the following corollary: if $A$ is a unital C*-algebra of finite nuclear dimension, and the Jiang-Su algebra $Z$ embeds unitally in $A$, then every element of $A_0$ is the sum of three commutators (see Corollary 4.1). In Section 3 we relax the assumption of having an embedding of the Jiang-Su algebra by asking for an embedding of a dimension drop algebra $Z_{n,n+1}$.

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Theorem 1.3. Let $A$ be a unital $C^*$-algebra of nuclear dimension $m \in \mathbb{N}$, with no finite dimensional representations and no purely infinite quotients. Suppose also that $A$ admits a unital embedding of a dimension drop algebra $\mathbb{Z}_{n,n+1}$ for $n$ large enough (depending only on $m$). Then each element of $A_0$ is the sum of three commutators.

An example of Pedersen and Petersen shows that for each $m \in \mathbb{N}$ there exists a homogeneous $C^*$-algebra $A$ of nuclear dimension $2m$ and $a \in A_0$ such that $a$ is not a limit of sums of $m + 1$ self-commutators (see [PP70] Lemma 3.5 and [BF13] Theorem 2.1). In Section 3 we construct a variety of examples of the same nature. We then use a construction à la Villadsen (of the second type) to prove the following theorem:

Theorem 1.4. There exists a simple unital AH $C^*$-algebra $A$ with a unique tracial state such that for each $m \in \mathbb{N}$ there exists a contraction $a_m \in A_0$ such that for all $x_1, x_2, \ldots, x_m \in A$ we have $\|a_m - \sum_{i=1}^{m} [x_i^*, x_i]\| \geq 1$.

This theorem answers [BF13] Question 2.3.

It was demonstrated in [Mar06] that the property of strict comparison of projections could be used to express elements of $A_0$, with $A$ of real rank zero, as sums of a small number of commutators. Strict comparison of positive elements and a unique tracial state were exploited in [Ng13] for the same purpose. Since such regularity properties continue to exist beyond nuclear $C^*$-algebras, they provide a path for extending the previous results without assuming nuclearity. Almost divisibility in the Cuntz semigroup is another key property, satisfied for example by all $\mathbb{Z}$-stable $C^*$-algebras. In [Win12], Winter calls “pure $C^*$-algebra” one with both strict comparison of positive elements and almost unperforated Cuntz semigroup. In the last section of this paper we prove the following theorem:

Theorem 1.5. Let $A$ be a separable simple, unital, pure $C^*$-algebra of stable rank one. Assume also that bounded 2-quasitraces on $A$ are traces (e.g., $A$ is exact). Then each element of $A_0$ is the sum of three commutators.

The previous theorem allows us to deal with the case of separable simple unital exact $\mathbb{Z}$-stable $C^*$-algebra completely: Rørdam’s dichotomy [Rør04, Theorem 6.7] says that a simple unital $\mathbb{Z}$-stable $C^*$-algebra is either purely infinite or of stable rank one. In the purely infinite case it is known that two commutators suffice [Pop02]. On the other hand, the previous theorem is applicable in the stable rank one case. Indeed, $\mathbb{Z}$-stability implies pureness (by [Rør04]) and a well known theorem of Haagerup says that all bounded 2-quasitraces on an exact $C^*$-algebra are traces.

2. Preliminaries

Let $A$ be a $C^*$-algebra. A commutator of $A$ is an element of the form $[x, y] := xy - yx$ for some $x, y \in A$. We call self-commutator a commutator of the form $[z^*, z]$. By [Mar06] Theorem 2.4], if a selfadjoint element is a sum of $m$ commutators, then it is a sum of $2m$ self-commutators.

We will make use of the arithmetic of Cuntz equivalence classes of positive elements. Let us recall it briefly (see [APT11] for more). Let $A_+$ denote the set of positive elements of $A$. Given $a, b \in A_+$, $a$ is said to be Cuntz smaller than $b$ if there exist $d_n \in A$ such that $d_n b d_n \to b$; $a$ is Cuntz equivalent to $b$ if $a$ is Cuntz smaller than $b$ and $b$ is Cuntz smaller than $a$. These relations we will be denoted by $a \lesssim b$ and $a \sim b$ respectively. We will make use of the following reformulation of Cuntz comparison: $a \lesssim b$ if and only if for each $\varepsilon > 0$ there exists $x \in A$ such that $(a - \varepsilon)_+ = x^* x$ and $xx^* \in \text{her}(b)$. Here and throughout the paper, $\text{her}(b)$ denotes the hereditary subalgebra generated by $b$, i.e., $bA_b$ and $(t - \varepsilon)_+ = \max(t - \varepsilon, 0)$ for $t \geq 0$. 
The Cuntz semigroup of $A$, denoted by $\text{Cu}(A)$, is defined as the set of Cuntz equivalence classes of positive elements in $A \otimes K$ (where $K$ is the C*-algebra of compact operators on a separable Hilbert space). Given a positive element $a \in A \otimes K$ we will denote its Cuntz equivalence class by $[a]$. The order on $\text{Cu}(A)$ is defined as $[a] \leq [b]$ if $a \preceq b$ and addition is defined as $[a] + [b] = [a' + b']$, where $a \sim a'$, $b \sim b'$ and $a' \perp b'$ (it is always possible to find such $a', b'$ in $A \otimes K$).

Let us now recall the definition of nuclear dimension, as introduced by Winter and Zacharias in [WZ10]. We start by defining order zero maps: A completely positive contractive (c.p.c.) map between C*-algebras $\phi: A \to B$ is said to have order zero if it preserves orthogonality, i.e., $ab = 0$ implies $\phi(a)\phi(b) = 0$ for all $a, b \in A$. By [WZ10 Corollary 3.1], in this case there exists a homomorphism $\pi^\phi: A \otimes C_0(0, 1) \to B$ such that $\pi^\phi(a) = \pi(a \otimes t)$, where $t \in C_0(0, 1)$ is the identity function on $(0, 1]$. Observe, for later use, that this implies that $\phi([x^*, x]) = [y^*, y]$, with $y = \pi^\phi(x \otimes 1/2)$, and that $\phi$ maps $A_0$ to $B_0$.

The C*-algebra $A$ is said to have nuclear dimension at most $m \in \mathbb{N}$ if for each $k = 0, 1, \ldots, m$ there exist nets of c.p.c. maps $\psi^k: A \to \prod_{\lambda} C^k_{\lambda}$ and $\phi^k: \prod_{\lambda} C^k_{\lambda} \to A$, where $C^k_{\lambda}$ is a finite dimensional C*-algebra and $\phi^k$ has order zero for all $\lambda \in \Lambda$, such that

$$\sum_{k=0}^{m} \phi^k \psi^k(a) \to_A a$$

for all $a \in A$. It is a useful fact that the maps $\psi^k$ may be chosen to be asymptotically of order zero. That is, such that $\psi^k: A \to \prod_{\lambda} C^k_{\lambda}/\bigoplus_{\lambda} C^k_{\lambda}$ is an order zero map. We then have that

$$A^\iota = \sum_{k=0}^{m} \phi^k \psi^k \to_A A^\infty$$

where $\iota$ is the diagonal inclusion, $N^k = \prod_{\lambda} C^k_{\lambda}/\bigoplus_{\lambda} C^k_{\lambda}$, $A^\infty = \prod_{\lambda} A/\bigoplus_{\lambda} A$, and the maps $\phi^k: N^k \to A^\infty$ are defined entrywise using the maps $\phi^k_{\lambda}$, $\lambda \in \Lambda$.

3. Proof of Theorem 1.1

The following lemma is well known:

Lemma 3.1. Let $A$ be either a finite dimensional C*-algebra, a product $\prod_{i=1}^{\infty} A_i$, or the quotient $\prod_{i=1}^{\infty} A_i/\bigoplus_{i=1}^{\infty} A_i$, where each $A_i$ is a finite dimensional C*-algebra. If $a \in A_0$ then there exists $x \in A$ such that $a = [x^*, x]$ and $\|x\|^2 \leq 2\|a\|$.

Proof. If $A$ is finite dimensional, this is [Par82 Lemma 3.5]. The result for $\prod_{i=1}^{\infty} A_i$ follows at once applying it to each entry. Finally, let $A = \prod_{i=1}^{\infty} A_i/\bigoplus_{i=1}^{\infty} A_i$ and let $a \in A_0$. Then $a = \sum_{n=1}^{\infty} [x^*_n, x_n]$. Let us choose lifts $(a_i)_{i=1}^{\infty}, (x_{i,n})_{i=1}^{\infty} \in \prod_{i=1}^{\infty} A_i$. For each $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $\|a - \sum_{i=1}^{N} [x^*_i, x_i]\| < \varepsilon$. Then $\|a_i - \sum_{i=1}^{N} [x_{i,n}^*, x_{i,n}]\| < \varepsilon$ for all sufficiently large $i$. Thus, for all such $i$, we find $y_i$ such that $\|a_i - [y_i^*, y_i]\| < \varepsilon$, with $\|y_i\|^2 \leq 2(\|a_i\|^2 + \varepsilon)^2$. The result follows letting $\varepsilon = 1/2, 1/3, \ldots$ and using a standard diagonal sequence argument.

Proof of Theorem 1.1. Let $a \in A_0$, and assume that $a = \sum_{i=1}^{\infty} [z^*_i, z_i]$. By [WZ10 Proposition 2.6], we may assume that $A$ is separable and $z_i \in A$ for all $i$. Let $\psi_k$ and $\phi_k$ be maps as in (2.1), with $k = 0, 1, \ldots, m$ (and with $\Lambda = \mathbb{N}$). Then $\psi_k(a) \in (N^k)_0$, since $\psi_k$ is a c.p.c. map of order zero. By Lemma 3.1 applied to the C*-algebra $N^k$, $\psi_k(a)$ is expressible as a single self-commutator $[x^*_k, x_k]$, with $\|x_k\|^2 \leq 2\|\psi_k(a)\|$. Since $\phi_k$ is also a c.p.c. map of order zero,
each self-commutator $[x_k^*, x_k]$ gets mapped by $\phi^k$ to a self-commutator $[y_k^*, y_k] \in A_\infty$, with $y_k = \pi^k (x_k \otimes 1^{1/2})$ (whence $\|y_k\| \leq \|x_k\|$). Thus,

$$\nu(a) = \sum_{k=0}^{m} \phi^k \psi^k (a) = \sum_{k=0}^{m} [y_k^*, y_k],$$

from which the result clearly follows. □

**Remark 3.2.** If in Theorem 1.1 we relax the requirement of using self-commutators, and simply seek to express all the commutators to satisfy $\|x\| \cdot \|y\| \leq \|a\|$. Let us show this: By the method used to prove Theorem 1.1, this boils down to showing that for $a \in A_0$ with $A$ a matrix algebra, we can find $x, y \in A$ with $a = [x, y]$ and $\|x\| \cdot \|y\| \leq \|a\|$. To prove this we may assume that $a$ is a diagonal matrix. Furthermore, conjugating by a suitable permutation matrix, we can arrange for the partial sums of the diagonal entries of $a$ to be in $[-\|a\|, \|a\|]$. We can then use a commutator formula similar to equation (5.1) below.

### 4. Finitely Many Commutators

In this section we prove Theorem 1.2.

**Lemma 4.1.** Let $a, b \in A_+$ be such that

$$L[a] \leq (L - 1)[(a - \varepsilon)_+] + K[b] \tag{4.1}$$

for some $L, K \in \mathbb{N}$ and $\varepsilon > 0$. Then for each $x \in \text{her}((a - \varepsilon)_+)\text{we have}$

$$x = \sum_{k=1}^{L(L+K-1)} [x_k, y_k] + z,$$

for some $z \in \text{her}(b)$ such that $\|z\| \leq K\|x\|$ and $x_k, y_k \in A$ such that $\|x_k\| \cdot \|y_k\| \leq \|x\|$ for all $k$.

**Proof.** We may assume without loss of generality that $a$ and $b$ are contractions. Let us set

$$c := (a - \varepsilon)_+ \otimes 1_{L-1} \oplus (b \otimes 1_K) \in M_{L+K-1}(A).$$

The relation (4.1) implies that there exists an $L \times (L + K - 1)$ matrix $V$ with entries in $A$ such that $g_{\varepsilon/2}(a) \otimes 1_L = V^*V$ and $VV^* \in \text{her}(c)$. Adding over the main diagonal of $V^*V$ we get

$$L \cdot g_{\varepsilon/2}(a) = \sum_{j=1}^{L} \sum_{i=1}^{L+K-1} v_{i,j}^* v_{i,j}.$$

On the other hand, using that $VV^* \in \text{her}(b)$, and adding over the first $L - 1$ diagonal terms of $VV^*$ we get that

$$\sum_{i=1}^{L-1} \sum_{j=1}^{L} v_{i,j} v_{i,j}^* \leq (L - 1) \cdot g_{\varepsilon/2}(a). \tag{4.2}$$

Finally, looking at the remaining $K$ diagonal terms of $VV^*$ we get $v_{i,j} v_{i,j}^* \in \text{her}(b)$ for all $i \geq L$ and all $j$.

Let $\Phi : \text{her}((a - \varepsilon)_+) \to \text{her}((a - \varepsilon)_+)$ be defined by

$$\Phi(x) = \frac{1}{L} \sum_{i=1}^{L-1} \sum_{j=1}^{L} v_{i,j} y v_{i,j}^*.$$
From (4.2) we deduce that \( \|\Phi\| \leq \frac{L-1}{L} \). Thus, \( \text{Id} - \Phi \) is invertible with \( \|(1 - \Phi)^{-1}\| \leq L \). Let \( y \in \text{her}((a - \varepsilon)_+) \) be such that \( x = y - \Phi(y) \), with \( \|y\| \leq L\|x\| \). Notice that \( g_{\varepsilon/2}(a)y = y \). Then
\[
x = y - \Phi(y) = g_{\varepsilon/2}(a)y - \frac{1}{L} \sum_{i=1}^{L-1} \sum_{j=1}^{L} v_{i,j}y v_{i,j}^* \]
\[
= \frac{1}{L} \sum_{i=1}^{L+K-1} \sum_{j=1}^{L} v_{i,j}^* v_{i,j} y - \frac{1}{L} \sum_{i=1}^{L-1} \sum_{j=1}^{L} v_{i,j} y v_{i,j}^* \]
\[
= \sum_{i=1}^{L+K-1} \sum_{j=1}^{L} \frac{1}{L}[v_{i,j}^*, v_{i,j} y] + z,
\]
with
\[
z = \frac{1}{L} \sum_{i=1}^{L+K-1} \sum_{j=1}^{L} v_{i,j} y v_{i,j}^* \in \text{her}(b).
\]
Observe that there are \( L(L + K - 1) \) commutators \( \frac{1}{L}[v_{i,j}^*, v_{i,j} y] \), that \( \frac{1}{L} \|v_{i,j}^*\| \cdot \|v_{i,j} y\| \leq \|x\| \) for all \( i, j \), and that
\[
\|z\| \leq \left\| \sum_{i=1}^{L+K-1} \sum_{j=1}^{L} v_{i,j} v_{i,j}^* \right\| \|x\| \leq K\|x\|,
\]
as desired. \( \square \)

The following proposition is an adaptation of Fack’s method from [Fac82] that uses positive elements instead of projections. (We will only use it below with \( L = 1 \).)

**Proposition 4.2.** Let \( A \) be a \( C^\ast \)-algebra. Suppose that

(i) there exists a sequence of positive elements \( (e_i)_{i=0}^\infty \) in \( A \) and \( \varepsilon_i > 0 \) such that \( e_i \perp e_j \) for all \( i, j \geq 1 \),

(4.3) \[
L[e_i] \leq (L - 1)[(e_i - \varepsilon_i)_+] + K[(e_{i+1} - \varepsilon_{i+1})_+] \]

for all \( i \geq 0 \), and

(ii) there exist \( M, \overline{M} \geq 1 \) such that for each hereditary subalgebra \( B \subseteq A \) if \( b \in B_0 \) then \( b \) is limit of elements of the form \( \sum_{i=1}^{M} x_i, y_i \), with \( x_i, y_i \in B \) and \( \|x_i\| \cdot \|y_i\| \leq \overline{M}\|b\| \) for all \( i = 1, 2, \ldots, M \).

Then each element \( z \in \text{her}((e_0 - \varepsilon_0)_+) \cap A_0 \) is the sum of \( L(L + K - 1) + \max(M, L(L + K - 1)) \) commutators \( [x, y] \), with \( \|x\| \cdot \|y\| \leq \overline{M}\|z\| \).

**Proof.** Let \( N = L(L + K - 1) \) and \( B^{(i)} = \text{her}(e_i - \varepsilon_i) \) for \( i \geq 0 \). Note that, by assumption, \( B^{(i)} \perp B^{(j)} \) for all \( i, j \geq 1 \) with \( i \neq j \).

Let us choose a sequence \( \delta_1, \delta_2, \ldots \) of positive real numbers tending to 0.

Let \( z \in B^{(0)} \cap A_0 \). By Lemma 4.1 and (4.3), we can find elements \( x_k^{(1)}, y_k^{(1)} \in A \), with \( k = 1, \ldots, N \), such that \( z = \sum_{k=1}^{N} [x_k^{(1)}, y_k^{(1)}] + z_1 \), where \( z_1 \in B^{(1)} \). Applying the assumption (ii) in the hereditary subalgebra \( B^{(1)} \), we get \( z_1 = \sum_{k=1}^{M} [\tilde{x}_k^{(1)}, \tilde{y}_k^{(1)}] + \tilde{z}_1 \), where the elements \( \tilde{x}_k^{(1)}, \tilde{y}_k^{(1)} \in B^{(1)} \) are such that \( \|x_k^{(1)}\| \cdot \|y_k^{(1)}\| \leq \overline{M}\|z_1\| \) for all \( k \), and where \( \tilde{z}_1 \in B^{(1)} \) is such that \( \|\tilde{z}_1\| < \delta_1 \). Now let us apply Lemma 4.1 to \( \tilde{z}_1 \), and get \( \tilde{z}_1 = \sum_{k=1}^{M} [x_k^{(2)}, y_k^{(2)}] + z_2 \), with
Lemma 4.3. Let $A$ be a C*-algebra of nuclear dimension $m \in \mathbb{N}$ such that every representation of $A$ has dimension at least $2m+3$ and $A$ has no simple purely infinite quotients. Let $c \in A_+$ be a strictly positive element and $\varepsilon > 0$. Then there exist orthogonal positive elements $d_0, d_1 \in A$ such that
\[
[(c - \varepsilon)] \leq (2m+3)[d_0], \quad [c] \leq (2m+2)(2m+3)[d_1].
\]

Proof. The construction of $d_0$ and $d_1$ with these properties is given in the proof of [RT13, Lemma 3.4].

Proposition 4.4. Let $A$ be a C*-algebra of nuclear dimension $m$, with no finite dimensional representations and no simple purely infinite quotients. Then for each strictly positive element $c \in A_+$ and $\varepsilon > 0$, there exist positive elements $e_0, e_1, \ldots, e_m$ and positive numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$, with $e_0 = (c - \varepsilon)_+$, $e_i \perp e_j$ for all $i, j \geq 1$, and $|e_i| \leq K[(e_{i+1} - \varepsilon_{i+1})_+]$ for all $i \geq 0$, where $K = O(m^3)$.

Proof. Let us apply the previous lemma with $c$ and $\frac{c}{2}$, and let $d_0$ and $d_1$ be the resulting positive orthogonal elements. From $[(c - \frac{c}{2})] \leq (2m+3)[d_0]$ we get that
\[
[(c - \varepsilon)_+] \leq (2m+3)((d_0 - \delta)_+)
\]
for some $\delta > 0$. Setting $e_1 = (d_0 - \frac{d_0}{2})_+$ and $\varepsilon_1 = \frac{\delta}{2}$ we get $[(c - \varepsilon)_+] \leq [(e_1 - \varepsilon_1)_+]$, as desired. The C*-algebra $h_{d_1}(A)$ does not have finite dimensional representations (since $[c] \leq (2m+2)(2m+3)[d_1]$). Thus, we can apply the previous lemma to it. But first, let us choose $\delta_1 > 0$ such that
\[
[(d_0 - \frac{\delta}{2})_+] \leq (2m+2)(2m+3)[d_1].
\]
Now let us apply the previous lemma to $\text{her}(d_1)$, with strictly positive element $d_1 \in \text{her}(d_1)_+$ and $\frac{\delta}{2}$. We get two positive orthogonal elements, $d_{1,0}, d_{1,1} \in \text{her}(d_1)$, with $[(d_1 - \frac{\delta}{2})_+] \leq (2m + 3)[d_{1,0}]$. Thus, for some $\delta_2 > 0$, we have that $[(d_1 - \delta_1)_+] \leq (2m + 3)[(d_{1,0} - \delta_2)_+]$, and so

$$[(d_0 - \frac{\delta}{2})_+] \leq (2m + 2)(2m + 3)^2[(d_{1,0} - \delta_2)_+] .$$

Setting $e_2 = (d_{1,0} - \frac{\delta}{2})_+$ and $\varepsilon_2 = \frac{\delta}{2}$, this is simply $[e_1] \leq [(e_2 - \varepsilon_2)_+]$. Furthermore, as before with $\text{her}(d_1)$, $\text{her}(d_{1,1})$ has no finite dimensional representations, so we can continue applying this algorithm to get the desired sequence.

Proof of Theorem 5.2. This follows at once from Theorem 1.1, Proposition 4.2, and Proposition 4.4 (applied with $c = 1$).

5. Small number of commutators

Theorem 1.1 combined with [Ng13, Theorem 3.2] yields at once the following

Corollary 5.1. Let $A$ be a unital $C^*$-algebra of nuclear dimension $m$. Suppose that $A$ admits a unital embedding of the Jiang-Su algebra $\mathcal{Z}$. If $a \in A_0$ then there exist $x_i, y_i \in A$, with $i = 1, 2, 3$, such that $a = [x_1, y_1] + [x_2, y_2] + [x_3, y_3]$.

Let us now prove Theorem 5.3. We follow [Ng12] and [Ng13] closely.

Lemma 5.2. Let $A$ be a $C^*$-algebra, $c \in A_+$ and $\varepsilon > 0$. Let $b \in M_n(\text{her}((c - \varepsilon)_+))$ be such that

$$\sum_{i=1}^n b_{i,i} = \sum_{i=1}^n [x_i, y_i],$$

with $x_i, y_i \in \text{her}((c - \varepsilon)_+)$ for all $i$. Then $b$ is the sum of two commutators in $M_n(A)$.

Proof. Let us write $b = b' + b''$, where

$$b' = \begin{pmatrix} b_{1,1} - [x_1, y_1] & b_{2,1} - [x_2, y_1] & \cdots & b_{n,1} - [x_n, y_1] \\ b_{2,2} - [x_2, y_2] & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{n,n} - [x_n, y_n] \\ 0 & \cdots & 0 & b_{n,n} - [x_n, y_n] \end{pmatrix},$$

Then $b'$ and $b''$ are commutators by [Ng12, Lemma 2.7] and [Ng12, Lemma 2.8] respectively. The commutator formula for $b'$ is simply

$$b' = \left[ \begin{pmatrix} 0 & s_1 \\ \vdots & \ddots \\ 0 & s_{n-1} \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right],$$

where $s_i = \sum_{j=1}^i (b_{j,j} - [x_j, y_j])$ and $e \in C^*(c)_+$ is chosen such that $e(c - \varepsilon)_+ = (c - \varepsilon)_+$.

Proof of Theorem 5.3. We will make use of the following relation among positive elements: $a \preceq_s b$ if $a = v^*v$ and $vv^* \in \text{her}(b)$ for some $v \in A$ (equivalently $\overline{aA}$ embeds in $bA$ as a Hilbert $A$-module).

Let $n \geq 3$ (how much larger will be specified later). Since $\mathcal{Z}_{n,n+1}$ embeds unitally in $A$, we can find positive elements $e_i \in A_+$, with $i = 1, 2, \ldots, n$ and $d \in A_+$ such that

1. $e_i \perp e_j$ for all $i \neq j$,
2. there exist $x_i$ such that $e_1 = x_i^* x_i, x_i x_i^* = e_i$, for $i = 2, \ldots, n$,
3. $1 = d + \sum_{i=1}^n (e_i - \frac{1}{2})_+$, and $1 - d \preceq_s (e_1 - \frac{1}{2})_+$. 

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In fact, these elements may be found in $Z_{n,n+1}$ and then moved to $A$ (see the proof of [Rør04, Lemma 4.2]).

Let $a \in A_0$. Then
\[ a = dz + (1 - d)zd + (1 - d)z(1 - d). \]

Let $a' = dz + (1 - d)zd$ and $f = d + ad^2a + (1 - d)ad^2a(1 - d)$. Then $a' \in \text{her}(f)$. Also,
\[ d \lesssim_s (e_1 - \frac{1}{2})_+, \quad ad^2a \lesssim_s d \lesssim_s (e_2 - \frac{1}{2})_+, \quad \text{and} \quad (1 - d)ad^2a(1 - d) \lesssim_s d \lesssim_s (e_3 - \frac{1}{2})_+. \]

Since the $(e_i - \frac{1}{2})_+$ are pairwise orthogonal (and $n \geq 3$), we get that $f \lesssim_s (\sum_{i=1}^n e_i - \frac{1}{2})_+$. Thus, by [Ng13, Lemma 2.5], $a' = [x, y] + a''$, with $z'' \in \text{her}((\sum_{i=1}^n e_i - \frac{1}{2})_+)$. Hence, $a = [x, y] + b$, with $b \in \text{her}(1 - d) = \text{her}((\sum_{i=1}^n e_i - \frac{1}{2})_+)$. Let us show that for $n$ large enough Lemma 5.2 can be applied to get that $b$ is the sum of two commutators. From the properties (1)-(3) of the $e_i$s, we have that $\text{her}(\sum_{i=1}^n e_i) \cong M_n(\text{her}(e_1))$. Thus, in order to be able to apply Lemma 5.2 we only need to show that $\sum_{i=1}^n b_{i,i}$ is a sum of $n$ commutators, where $b = (b_{i,j})_{i,j}$ is regarded as an element of $M_n(\text{her}((e_1 - \frac{1}{2})_+))$. Let us prove this. Since $[1] \leq (n + 1)[e_1]$, the C*-algebra $\text{her}(e_1)$ has no finite dimensional representations and no simple purely infinite quotients. Also, since $\text{her}(e_1)$ is a full hereditary subalgebra of $A$, every bounded trace on $\text{her}(e_1)$ extends uniquely to a bounded trace on $A$. Hence, $b \in A_0$ implies that $\sum_{i=1}^n b_{i,i} \in \text{her}(e_1)_0$. Now, in the same way that we proved Theorem 1.2, we can deduce from Proposition 4.4 and Proposition 4.2 that $\sum_{i=1}^n b_{i,i}$ is a sum of $n$ commutators inside $\text{her}((e_1 - \frac{1}{2})_+)$ for $n = O(m^3)$ depending solely on $m$. Then, by Lemma 5.2 $b$ is the sum of two commutators in $A$. \[ \square \]

Remark 5.3. The hypotheses “no finite dimensional representations and no simple purely infinite quotients” in Theorem 1.3 are only needed to ensure the existence of an infinite sequence of sufficiently large orthogonal positive elements inside $\text{her}(e_1)$, with $e_1$ as in the preceding proof. If instead we assume that $Z$ embeds unitally in $A$, then the existence of such a sequence of orthogonal elements can be obtained by more direct means. Indeed, in this case an embedding $Z_{n,n+1} \hookrightarrow A$ may be chosen that factors through the embedding $Z_{n,n+1} \overset{a \mapsto [a]}{\rightarrow} Z \otimes Z \cong Z$. In this way, we can arrange for an infinite sequence of positive orthogonal elements in her($e_1$) satisfying (1.3) (with $L = 1$ and small $K$), which makes Proposition 1.2 applicable in her($e_1$). This line of reasoning gives an alternative proof to Corollary 5.1.

Remark 5.4. In spite of the number of commutators being reduced – say, from Theorem 1.3 to Theorem 1.4 – this is achieved at the expense of increasing the norms of the elements making up the commutators (see for example the commutator formula in the proof of Lemma 5.2). Although explicit estimates for the norms of these elements may be obtained, it does not seem that if $A$ is a C*-algebra of nuclear dimension $m$ the quantity
\[ \sup_{a \in A_0} \sup_{\varepsilon > 0} \sup_{\|a\| \leq 1} \inf \left\{ \sum_{i=0}^m \|x_i\| \cdot \|y_i\| \mid \|a - \sum_{i=0}^m [x_i, y_i]\| < \varepsilon, x_i, y_i \in A \right\} \]

is reduced by this method. (We know, by Remark 5.2 that it is at most $m + 1$.)

6. LARGE NUMBER OF COMMUTATORS

In this section we prove Theorem 1.4.

Let $X$ be a compact Hausdorff space. It will be helpful in the sequel to bear in mind the correspondences between projections in $C(X, K)$, finitely generated projective modules over $C(X)$, and vector bundles on $X$. For example, given a projection $p \in C(X, K)$ then
$E_p = \{(x,v) \mid v \in \ell_2, p(x)v = v\}$ is a vector bundle over $X$ whose continuous sections $P_p = \{s \in C(X, \ell_2) \mid p(x)s(x) = s(x)\}$ form a finitely generated projective module over $C(X)$.

Let us fix an embedding of the Cuntz algebra $\mathcal{O}_2$ in $B(\ell_2)$ and use it to define direct sums of elements in $C(X, \mathcal{K})$: if $f, g \in C(X, \mathcal{K})$ then

$$f \oplus g := v_1f v_1^* + v_2gv_2^*,$$

with $v_1, v_2 \in B(\ell_2)$ isometries that generate $\mathcal{O}_2$.

In the sequel $1_X \in C(X, \mathcal{K})$ denotes the rank one projection constantly equal to $e_{1,1} \in \mathcal{K}$.

**Lemma 6.1.** Let $p, q \in C(X, \mathcal{K})$ be projections. Suppose that $[p] \leq n[1_X]$ and that there exists $v \in q(C(X, \mathcal{K}))p$ that is full (i.e., $v(x) \neq 0$ for all $x \in X$). Then $[1_X] \leq n[q]$.

**Proof.** Without loss of generality, we may assume that $p \leq 1^{(n)}_X$, with $1^{(n)}_X(x) := 1_n$ (the $n \times n$ identity matrix) for all $x \in X$. Hence, we can reduce to the case that $p = 1^{(n)}_X$. Let $v_1, v_2, \ldots, v_n$ denote the columns of $v$. They can be interpreted as sections of the vector bundle associated to $q$. Since they do not all simultaneously vanish, we get that $[1_X] \leq n[q]$. □

**Proposition 6.2.** Let $p, q \in C(X, \mathcal{K})$ be projections such that $[p] \leq n[1_X]$ and $[1_X] \not\leq nm[q]$. Let $a \in \text{her}(p \oplus q)$ be of the form

$$(p \quad *) \quad \quad (*) \quad \quad (*)$$

Then $a$ cannot be within a distance less than 1 of a sum of $m$ self-commutators in $\text{her}(p \oplus q)$.

**Proof.** Suppose that we have elements

$$x_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix},$$

with $a_i \in \text{her}(p)$, $b_i \in p(C(X, \mathcal{K}))q$, $c_i \in q(C(X, \mathcal{K})p)$ and $d_i \in \text{her}(q)$ for all $i = 1, 2, \ldots, m$, such that $\|a - \sum_{i=1}^m x_i^*x_i\| < 1$. Then

$$\|p - \sum_{i=1}^m [a_i^*, a_i] - \sum_{i=1}^m (c_i^*c_i - b_ib_i^*)\| < 1.$$  (6.1)

Let $v = (c_1^* - b_1) \oplus (c_2^* - b_2) \oplus \cdots \oplus (c_n^* - b_n)$. Then $v \in q^{\oplus m}C(X, \mathcal{K})p$. By the previous lemma, $v$ is not full, i.e., it vanishes at some point $x \in X$. Evaluating at $x$ in (6.1) we get

$$\|p(x) - \sum_{i=1}^m [a_i^*(x), a_i(x)]\| < 1,$$

with $a_i(x) \in \text{her}(p(x))$ for all $i$. This is impossible, since the unit of a C*-algebra with a tracial state (namely, $\text{her}(p(x))$) cannot be within a distance less than 1 from a sum of commutators. □

**Example 6.3.** The previous proposition gives us a way of constructing examples of $a \in A_0$ not well approximated by small sums of commutators. For example, let $P \in M_2(C(S^2))$ denote a Bott projection over the 2-dimensional sphere (i.e., a rank one projection whose associated line bundle is the tautological line bundle of $\text{CP}(1) \cong S^2$). Let $X = \prod_{i=1}^m S^2$ and $p = P^{\oplus m} \in M_{2^m}(C(X)) \subset C(X, \mathcal{K})$. If we identify $H^*(X)$ with $\mathbb{Z}[\alpha_1, \ldots, \alpha_m]/(\alpha_i^2 = 0 \mid i = 1, \ldots, m)$, then the Euler class of the vector bundle associated to $p^{\oplus m}$ is $\prod_{i=1}^m \alpha_i \neq 0$. Hence, $[1_X] \not\leq m[p]$. So the element $a \in \text{her}(1_X \oplus p)$ given by

$$a = \begin{pmatrix} 1_X & 0 \\ 0 & -p \end{pmatrix}$$

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is in $\text{her}(1_X \oplus p)_0$ but not within a distance less than 1 of a sum of $m$ commutators.

**Proof of Theorem 6.4.** Let $(k_n)_{n=1}^{\infty}$ be an increasing sequence of natural numbers. We will use this sequence to construct the $C^*$-algebra $A$ as an inductive. Then, by letting $(k_n)_{n=1}^{\infty}$ grow sufficiently fast, will show that $A$ has the desired properties.

For each $i \in \mathbb{N}$, let $X_i = (S^2)^{k_i}$ and consider the rank one projection $P_i = P \otimes k_i \in C(X_i, K)$, where $P \in M_2(C(S^2))$ is a Bott projection. Let us now form the product $Y_n = \prod_{i=1}^{n} X_i$ and consider, over this space, the projection $p_n(y) = P_1(x_1)^{\otimes l_1} \oplus P_2(x_2)^{\otimes l_2} \oplus \cdots \oplus P_n(x_n)^{\otimes l_n},$

where $y = (x_1, x_2, \ldots, x_n) \in Y_n$. Let us set the numbers $l_n$ recursively such that $l_1 = 1$ and $l_{n+1} = \text{rank}(p_n)$ for $n \geq 1$ (in fact, $l_n = 2^{n-1}$ for $n \geq 2$). Let $\phi_n : \text{her}(p_n) \to \text{her}(p_{n+1})$ be defined as follows:

$$\phi_n(f)(y_n, x_{n+1}) = f(y_n) \oplus (f(p_n) \otimes p_{n+1}(x_{n+1})),$$

where $c_n \in Y_n$. In this formula, the term $f(p_n) \otimes p_{n+1}$ is regarded as an element in $\text{her}(P_{n+1})$ via the identifications her$(P_{n+1}) \cong M_{l_{n+1}}(C) \otimes \text{her}(P_{n+1})$ and her$(p_n(c_n)) \cong M_{l_{n+1}}(C)$ (recall that $p_n$ has rank $l_{n+1}$).

It is known that choosing the points $c_n \in Y_n$ suitably, one can arrange for the inductive limit $C^*$-algebra $A := \lim \text{her}(p_n)$ to be simple and have a unique tracial state (see [Vi99]: the uniqueness of tracial state comes from the fact that if $\sigma$ is a tracial state on $\text{her}(p_{m+n})$ then $\sigma \circ \phi_{m,n} \circ \phi_{m,n}$ is an average of $\sigma$ and $2^n - 1$ traces that don’t depend on $\sigma$, coming from point evaluations).

Let us show that for a suitable choice of the sequence $(k_n)_{n=1}^{\infty}$ the inductive limit $A$ has the desired properties. It suffices, for each $m \in \mathbb{N}$, to find an element $a_m$ of norm 1 in $\text{her}(p_m)_0$ for some $n$, that not only it is not within a distance less than 1 from any sum of $m$ commutators, but also this property is not destroyed by moving the element along the inductive limit.

Fix $m \in \mathbb{N}$. Let us assume that the numbers $k_1, \ldots, k_m$ have been chosen. Let us find $M_m \in \mathbb{N}$ such that $[p_m] \leq M_m[1_{Y_m}]$. Now let us choose $k_{m+1} \in \mathbb{N}$ such that $k_{m+1} \geq mM_ml_m$. An Euler class computation (as in Example 6.3) then shows that $[1_{Y_{m+1}}] \not\leq mM_ml_{m}[P_{m+1}(x_{m+1})]$ in her$(p_{m+1})$. It follows by Proposition 6.2 that the element $a_m \in \text{her}(p_{m+1}) = \text{her}(p_m \oplus P_{m+1}^{\otimes l_m}(x_{m+1}))$ given by

$$a(y, x_{m+1}) = \begin{pmatrix} p_m(y) \\ -P_{m+1}^{\otimes l_m}(x_{m+1}) \end{pmatrix},$$

with $(y, x_{m+1}) \in Y_m \times X_{m+1} = Y_{m+1}$, is not within a distance less than 1 of a sum of $m$ commutators in her$(p_{m+1})$. Observe on the other hand that $a \in \text{her}(p_{m+1})_0$. Let us continue choosing the numbers $k_{m+2}, k_{m+3}, \ldots$ in this way. Let us now consider the image of $a \in \text{her}(p_{m+1})$, as defined above, by the connecting map $\phi_{m+1,m+n} : \text{her}(p_{m+1}) \to \text{her}(p_{m+n})$. Then

$$\phi_{m+1,m+n}(a) = \begin{pmatrix} p_m \\ * \end{pmatrix},$$

where we regard her$(p_{m+n})$ as her$(p_m \oplus q_{m,n})$, with

$$q_{m,n}(y) = P_{m+1}(x_{m+1})^{\otimes l_{m+1}} \oplus \cdots \oplus P_{m+n}(x_{m+n})^{\otimes l_{m+n}},$$

for $y \in Y_{m+n}$. Again a routine Euler class computation shows that $[1_{Y_{m+n}}] \not\leq mM_m[q]$. Hence, $\phi_{m,m+n}(a)$ is not within a distance less than 1 of a sum of $m$ commutators in her$(p_{m+n})$, by Proposition 6.2.
7. Beyond nuclearity

Here we prove Theorem 1.5. Let us first recall the properties of almost unperforation and almost divisibility in the Cuntz semigroup of a C*-algebra: The Cuntz semigroup \( \text{Cu}(A) \) is said to have almost divisibility if for each \( k \in \mathbb{N} \), \( [a] \in \text{Cu}(A) \), and \( \varepsilon > 0 \), there exists \( [b] \in \text{Cu}(A) \) such that \( k[b] \leq [a] \) and \( [(a - \varepsilon)_{+}] \leq [(k + 1)[b]] \). The property of strict comparison of positive elements of \( A \otimes \mathcal{K} \) is equivalent to almost unperforation in the Cuntz semigroup. \( \text{Cu}(A) \) is said to be almost unperforated and almost divisible, it can be computed to be

\[
\text{Cu}(A) = V(A) \sqcup \text{SAff}(T(A)).
\]

This computation is obtained in [BT07, Theorem 3.6] and also in [ABP11, Theorem 5.6] (in both references exactness is only used to guarantee that bounded 2-quasitraces are traces). The additive structure of the ordered semigroup on the right hand side encodes the pairing \( \lambda \) between tracial states \( \tau \in T(A) \) and Murray von-Neumann classes of projections \( [p] \in V(A) \), so that \( \text{Cu}(A) \) is in this case functorially equivalent to \( (V(A), T(A), \lambda) \) (see [ADPS11]). On the other hand, Elliott has shown in [Ell96, Theorem 2.2] that such data is exhausted by inductive limits of 1-dimensional NCCW complexes with trivial \( K_1 \)-group. Therefore, there exists a simple unital C*-algebra \( B \), expressible as an inductive limit of 1-dimensional NCCW complexes with trivial \( K_1 \)-group, such that \( \text{Cu}(B) \cong \text{Cu}(A) \) with \([1] \mapsto [1]\). It follows by the classification theorem of [Rob12] that there exists a unital embedding of \( B \) in \( A \) inducing an isomorphism at the level of the Cuntz semigroups.

Let \( a \in A_0 \). By the previous lemma, there exists \( a' \in B \) unitarily equivalent to \( a \). In particular, \( \tau(a') = 0 \) for all bounded traces on \( A \). But the fact that the inclusion of \( B \) in \( A \) induces an isomorphism at the level of their Cuntz semigroups implies that the restriction map \( \tau \mapsto \tau|_B \) is a bijection from \( T(A) \) to \( T(B) \). Thus, \( a' \in B_0 \). Since the nuclear dimension of \( B \) is at most 1, we get that \( a' \) is the limit of sums of two self-commutators \([x_0^*, x_0] + [x_i^*, x_i] \), with \( \|x_i\|^2 \leq 2\|a'\| \) for \( i = 0, 1 \). Thus, the same holds for \( a \). Furthermore, \( Z \) embeds unitally in \( B \) (which is in fact \( Z \)-stable), whence also in \( A \). Thus, the theorem follows from [Ng13, Theorem 3.2] (alternatively, see Remark 5.3).

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