An approach to solve replacement problems under intuitionistic fuzzy nature

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Abstract. Due to impreciseness to solve the day to day problems the researchers use fuzzy sets in their discussions of the replacement problems. The aim of this paper is to solve the replacement theory problems with triangular intuitionistic fuzzy numbers. An effective methodology based on fuzziness index and location index is proposed to determine the optimal solution of the replacement problem. A numerical example is illustrated to validate the proposed method.

1. Introduction

In day to day life problems the replacement theory is used for the decision makers to take decision of replacing used equipment with a substitute; mostly a new equipment of better usage. The replacement of existing items might be necessary due to the deteriorating property of failure or breakdown of particular equipment. The replacement problem is used in the cases like; existing items have outlived or it may not be economical anymore to continue with them or the items might have been destroyed either by accident or otherwise. These situations can be handled mathematically. The main objective of this replacement model is to find the optimum time to replace the existing items. To handle the replacement problem, the parameters of the model should be known precisely. In real world problems due to lack of information or impreciseness it is not possible to get exact data for the decision parameters. Hence the classical mathematical models are inefficient to tackle these problems. Intuitionistic fuzzy set theory introduced by Atanassov [2] is one of the best tool to model and find the optimum period for the replacement of existing items. Most of the authors have studied about optimal period of replacement. Bishwas et.al [3] analyzed the replacement problem using Yeager’s ranking method. Mahdevi [7] discussed the Optimum age replacement policy under reliability based model. Nezhad et.al [8] classified one-stage two-machine replacement strategy using Bayesian inference method. Zhao et.al [11] is analyzed three kinds of replacement models combined with additive and independent damages. Uthra et.al [10] discussed replacement problem with TIFN. Here we used the easiest concept of fuzziness index and location index to find the optimum period of replacement and the minimum average cost.

This paper is organized as follows. In section 2 we discussed the basic concepts of triangular intuitionistic fuzzy numbers, arithmetic operations and ranking of TIFNs. Section 3, the procedure to find the optimum replacement period is discussed. A numerical example is given to validate the proposed method in Section 4. Section 5 concludes this paper.

2. Preliminaries
Definition 2.1
Let $X$ be a universe of discourse. An Intuitionistic Fuzzy Set (IFS) $\tilde{a}_I^{IF}$ in $X$ is given by $\tilde{a}_I^{IF} = \{(x, \mu_{a_I}^{IF}, \gamma_{a_I}^{IF}) / x \in X \}$ where the function $\mu_{a_I}^{IF}(x) \rightarrow [0,1]$ and $\gamma_{a_I}^{IF}(x) \rightarrow [0,1]$ represents the degree of membership and degree of non membership of the element $x \in X$, respectively, and $0 \leq \mu_{a_I}^{IF}(x) + \gamma_{a_I}^{IF}(x) \leq 1$ for every $x \in X$.

Definition 2.2. For every common fuzzy subset $a$ on $X$, Intuitionistic Fuzzy Index of is defined as $\pi_{a_{IF}}(x) = 1 - \mu_{a_{IF}}(x) - \gamma_{a_{IF}}(x), 0 \leq \pi_{a_{IF}}(x) \leq 1$, for every $x \in X$, which is known as degree of hesitancy or degree of uncertainty of the element $x$ in $a_{IF}$.

Definition 2.3. An Intuitionistic Fuzzy Number (IFN) $\tilde{a}_I^{IF}$ is

(i) an intuitionistic fuzzy subset of the real line,

(ii) normal, that is there is any $x \in R$, such that $\mu_{a_I}^{IF}(x) = 1$ and $\gamma_{a_I}^{IF}(x) = 0$.

(iii) Convex for the membership function $\mu_{a_I}^{IF}(x)$ that is,

$$\mu_{a_I}^{IF}(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_{a_I}^{IF}(x_1), \mu_{a_I}^{IF}(x_2))$$

for every $x_1, x_2 \in R, \lambda \in [0,1]$.

(iv) concave for the non-membership function $\gamma_{a_I}^{IF}(x)$ that is,

$$\gamma_{a_I}^{IF}(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\gamma_{a_I}^{IF}(x_1), \gamma_{a_I}^{IF}(x_2))$$

for every $x_1, x_2 \in R, \lambda \in [0,1]$.

Definition 2.4. A Triangular Intuitionistic Fuzzy Number (TIFN) with parameters $a'_1 \leq a_1 \leq a_2 \leq a_3 \leq a'_3$ is denoted by $\tilde{a}_I^{IF} = < a_1, a_2, a_3, a'_1, a'_2, a'_3 >$ with membership and non-membership functions are

$$\mu_{a_I}^{IF} = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2 \\ 1, & x = a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x < a_3 \\ 0, & x < a_2 \text{ and } x > a_3 \end{cases}$$

and

$$\gamma_{a_I}^{IF} = \begin{cases} \frac{a_2-x}{a_2-a'_1}, & a'_1 \leq x < a_2 \\ 0, & x = a_2 \\ \frac{x-a_2}{a_3-a'_2}, & a_2 \leq x < a_3' \\ 1, & x < a_1' \text{ and } x > a_3' \end{cases}$$

Let $F(R)$ denote the set of all TIFNs, $m = a_2$ be the midpoint and $\alpha_1 = (a_2 - a_1)$, $\beta_1 = (a_3 - a_2)$ represent the left and right spread of membership function and $\alpha_2 = (a_2 - a'_1)$, $\beta_2 = (a'_3 - a_2)$ the left and right spread of non-membership function.
Figure 1  Triangular Intuitionistic Fuzzy Number (TIFN)

Note.1. In particular, if $a'_1 = a_1$ and $a'_3 = a_3$, then $\tilde{a}^IF$ is a triangular fuzzy number.

Note.2. In particular, if $a'_1 = a_1 = a'_3 = a_3 = m$, then $\tilde{a}^IF$ is a real number $m$.

**Definition 2.5.** The triangular intuitionistic fuzzy number $\tilde{a}^IF \in \mathbb{F}(\mathbb{R})$ can also be represented as $\tilde{a}^IF = (a, \bar{a}; a', \bar{a}')$ of functions $a(r), \bar{a}(r), a'(r)$ and $\bar{a}'(r)$ for $0 \leq r \leq 1$ which satisfies the following requirements:

(i) $a(r)$ is a bounded monotonic increasing left continuous function for membership function

(ii) $\bar{a}(r)$ is a bounded monotonic decreasing left continuous function for membership function

(iii) $a'(r) \leq \bar{a}(r), \ 0 \leq r \leq 1$.

(iv) $\bar{a}'(r)$ is a bounded monotonic decreasing left continuous function for non-membership function

(v) $a'(r)$ is a bounded monotonic increasing left continuous function for non-membership function

(vi) $a'(r) \leq \bar{a}'(r), \ 0 \leq r \leq 1$.

**Definition 2.6.** For an arbitrary TIFN $\tilde{a}^IF = (a, \bar{a}; a', \bar{a}')$, the number $a_0 = \frac{a(1) + \bar{a}(1)}{2}$ or $a_0 = \frac{a'(1) + \bar{a}'(1)}{2}$ are said to be location index number of membership and non-membership functions. The non-decreasing left continuous functions $a_* = (a_0 - a)$, $a' * = (\bar{a} - a_0)$ are called the left fuzziness index function and right fuzziness index function for membership function and non decreasing left continuous functions $a_*' = (a_0 - a')$, $a'* = (\bar{a}' - a_0)$ are called the are called the left fuzziness index function and right fuzziness index function for non-membership functions respectively. Therefore every TIFN $\tilde{a}^IF = <a_1, a'_2, a'_3; a'_1, a_2, a_3>$ can be represented in parametric form as $\tilde{a}^IF = <a_0, a_*; a_0, a'_*, a'*>$. 
2.1. Arithmetic Operations on Triangular Intuitionistic Fuzzy Numbers

We propose a new fuzzy arithmetic on triangular intuitionistic fuzzy numbers based upon both location index and fuzziness index functions for membership and non-membership functions. The location index number is taken in the ordinary arithmetic, whereas the fuzziness index functions are considered to follow the lattice rule which is least upper bound in the lattice \( L \). That is for \( a, b \in L \), we define
\[
a \lor b = \max\{a, b\} \quad \text{and} \quad a \land b = \min\{a, b\}
\]
In particular for any two triangular intuitionistic fuzzy numbers \( \bar{a}^{IF} = < a_0, a_+, a^-; a_0', a_+', a^-'> \) and \( \bar{b}^{IF} = < b_0, b_+, b^-; b_0', b_+', b^-'> \) the arithmetic operations between them are

(i) \( \bar{a}^{IF} + \bar{b}^{IF} = < a_0, a_+, a^-; a_0', a_+', a^-'> + < b_0, b_+, b^-; b_0', b_+', b^-'> = < a_0 + b_0, \max(a_+, b_+), \max(a^-, b^-); a_0 + b_0', \max(a'_+, b'_+), \max(a'_-, b'_-) > \)

(ii) \( \bar{a}^{IF} \times \bar{b}^{IF} = < a_0, a_+, a^-; a_0', a_+', a^-'> \times < b_0, b_+, b^-; b_0', b_+', b^-'> = < (a_0 \times b_0), \max(a_+, b_+), \max(a^-, b^-); (a_0 \times b_0'), \max(a'_+, b'_+), \max(a'_-, b'_-) > \)

(iii) \( \bar{a}^{IF} - \bar{b}^{IF} = < a_0, a_+, a^-; a_0', a_+', a^-'> - < b_0, b_+, b^-; b_0', b_+', b^-'> = < (a_0 - b_0), \min(a_+, b_+), \min(a^-, b^-); (a_0 - b_0'), \min(a'_+, b'_+), \min(a'_-, b'_-) > \)

(iv) \( \bar{a}^{IF} \div \bar{b}^{IF} = < a_0, a_+, a^-; a_0', a_+', a^-'> \div < b_0, b_+, b^-; b_0', b_+', b^-'> = < (a_0 \div b_0), \min(a_+, b_+), \min(a^-, b^-); (a_0 \div b_0'), \min(a'_+, b'_+), \min(a'_-, b'_-) > \)

2.2. Ranking of Triangular Intuitionistic Fuzzy Numbers

Prabakaran et al.\[9\] proposed a new ranking method for triangular intuitionistic fuzzy numbers based on the left and the right spreads at some \( \alpha \)-levels of fuzzy numbers.

For an arbitrary triangular intuitionistic fuzzy number \( \bar{a}^{IF} = < a_0, a_+, a^-; a_0', a_+', a^-'> \) with parametric form \( \bar{a}^{IF} = (\bar{a}, \bar{a}'; \bar{a}'') \), we define the magnitude of the triangular intuitionistic fuzzy number \( \bar{a}^{IF} \) by
\[
\text{Mag}(\bar{a}^{IF}) = \frac{1}{2} \int_0^1 \left( \bar{a} + \bar{a}' + 2a_0 + \bar{a}'' + \bar{a}''' \right) f(r) \, dr = \frac{1}{2} \int_0^1 \left( a'' + a'' + 6a_0 - a_0 - a_0'' \right) f(r) \, dr
\]
where the \( f(r) \) is a non-negative and increasing function on \([0,1]\) with \( f(0)=0, f(1)=1 \) and \( \int_0^1 f(r) \, dr = \frac{1}{3} \). In real life applications, the decision maker can choose \( f(r) \) according to their situation.

For our convenience, we have taken \( f(r) = r^2 \). Hence
\[
\text{Mag}(\bar{a}^{IF}) = \frac{1}{6} \left( \bar{a} + \bar{a}' + 2a_0 + \bar{a}'' + \bar{a}''' \right) = \frac{1}{6} \left( \bar{a} + \bar{a}' + 2a_0 + \bar{a}'' + \bar{a}''' \right)
\]
Which is used to rank the TIFNs, the larger value of \( \text{Mag}(\bar{a}^{IF}) \) is the larger TIFN.

For the TIFNs \( \bar{a}^{IF} = < a_0, a_+, a^-; a_0', a_+', a^-'> \) and \( \bar{b}^{IF} = < b_0, b_+, b^-; b_0', b_+', b^-'> \) in \( F(R) \),
We define the ranking as

(i) $\tilde{a}^F \succeq \tilde{b}^F$ if and only if $\text{Mag}(\tilde{a}^F) \geq \text{Mag}(\tilde{b}^F)$

(ii) $\tilde{a}^F \preceq \tilde{b}^F$ if and only if $\text{Mag}(\tilde{a}^F) \leq \text{Mag}(\tilde{b}^F)$

(iii) $\tilde{a}^F = \tilde{b}^F$ if and only if $\text{Mag}(\tilde{a}^F) = \text{Mag}(\tilde{b}^F)$

3. **Replacement problems with triangular intuitionistic fuzzy numbers**

The aim of this paper is to determine the optimum replacement year of an item whose maintenance (running) cost increases with time, the money value remains static during that period and the time is a discrete variable.

Here, $\tilde{R}^F_n$ - Intuitionistic Fuzzy Running cost for $n^{th}$ year

$\tilde{C}^F$ - Intuitionistic Fuzzy capital cost of the item

$\tilde{S}^F_n$ - Intuitionistic Fuzzy scrap value of the item in the $n^{th}$ year

Annual cost for the $n^{th}$ year = $\tilde{R}^F_n + \tilde{C}^F - \tilde{S}^F_n$

The total cost incurred on the item is $\tilde{P}^F = \sum_{i=1}^{n} \tilde{R}^F_i + \tilde{C}^F - \tilde{S}^F_n$

The average total cost is $\bar{T}^F(n) = \frac{\tilde{P}^F(n)}{n} = \frac{\sum_{i=1}^{n} \tilde{R}^F_i + \tilde{C}^F - \tilde{S}^F_n}{n}$

$\bar{T}^F(n)$ is minimum if $\Delta\bar{T}^F(n-1) < 0 < \Delta\bar{T}^F(n)$ is satisfied.

$\Delta\bar{T}^F(n) = \bar{T}^F(n+1) - \bar{T}^F(n)$

For minimum $\bar{T}^F(n)$, $\Delta\bar{T}^F(n-1) < 0 < \Delta\bar{T}^F(n)$

$\Rightarrow \tilde{R}^F_n > \frac{\sum_{i=1}^{n} \tilde{R}^F_i + \tilde{C}^F - \{(n+1)\tilde{S}^F_n - n\tilde{S}^F_{n+1}\}}{n}$

$\Rightarrow \tilde{R}^F_n > \frac{\sum_{i=1}^{n} \tilde{R}^F_i + \tilde{C}^F - \{(n+1)\tilde{S}^F_n - n\tilde{S}^F_{n+1}\}}{n}$

$\Rightarrow \tilde{R}^F_n > \frac{\sum_{i=1}^{n} \tilde{R}^F_i + \tilde{C}^F - \tilde{S}^F_n}{n}$

Similarly $\tilde{R}^F_n < \frac{\tilde{P}^F_n}{n}$.
Thus, \( \bar{R}_{n+1} > \frac{F^n}{n} > \bar{R}_n \) is used to determine the optimum replacement period.

4. Numerical example

Consider the Problem discussed in Uthra et.al [10], the intuitionistic fuzzy cost of a machine in hundred is \( \tilde{C}^{IF} < 50,60,65;45,50,70 > \). The intuitionistic fuzzy running cost for the year and salvage value at the end of the year (in hundreds) are given in the following table.

| Year | Intuitionistic fuzzy Maintenance cost | Intuitionistic fuzzy Resale value |
|------|-------------------------------------|---------------------------------|
| 1    | < 8,10,12 ; 6,10,14 >              | < 25,30,32 ; 22,30,34 >        |
| 2    | < 10,12,13 ; 8,12,15 >              | < 14,15,17 ; 12,15,18 >        |
| 3    | < 13,14,15 ; 12,14,16 >             | < 7.7,5,7.8 ; 6.6,7.5,8 >      |
| 4    | < 16,18,19 ; 15,18,20 >             | < 2.5,3.5,4.5 ; 2,3.5,4.8 >    |
| 5    | < 22,23,24 ; 20,23,26 >             | < 1.5,2,2.2 ; 1.2,2,2.5 >      |
| 6    | < 27,28,29 ; 26,28,30 >             | < 1.4,2,2.3 ; 1.1,2,2.6 >      |

Using fuzziness index and location index, the parametric form of the intuitionistic fuzzy running cost for the year and salvage value at the end of the year are given in the following table.

| Year | Intuitionistic fuzzy Maintenance cost | Intuitionistic fuzzy Resale value |
|------|-------------------------------------|---------------------------------|
| 1    | < 10,2-2r,2-2r ; 10,4-4r,4-4r >    | < 30,5-5r,5-5r ; 30,8-8r,4-4r > |
| 2    | < 12,2-2r,1-r ; 12,4-4r,3-3r >     | < 15,1-r,2-2r ; 15,3-3r,3-3r > |
| 3    | < 14,1-r,1-r ; 14,2-2r,2-2r >      | < 7.5,0.5-0.5r,0.3-0.3r ; 7.5,0.9-0.9r,0.5-0.5r > |
| 4    | < 18,2-2r,1-r ; 18,3-3r,2-2r >     | < 3.5,1-r,1-r ; 3.5,1.5-1.5r,1.3-1.3r > |
| 5    | < 23,1-r,1-r ; 23,3-3r,3-3r >      | < 2.0,5-0.5r,0.2-0.2r ; 2.0,8-0.8r,0.5-0.5r > |
| 6    | < 28,1-r,1-r ; 28,2-2r,2-2r >      | < 2.0,6-0.6r,0.3-0.3r ; 2.0,9-0.9r,0.6-0.6r > |
The total running cost and the Total cost are calculated as follows

\[
\text{Total running cost } = \sum_{i=1}^{n} \overline{R}_i^m
\]
\[
\text{Total cost } = \sum_{i=1}^{n} \overline{R}_i^m + (\overline{C}_m^u - \overline{S}_m^n)
\]

| Year (n) | Total running cost                                                                 | Total cost                                                                 |
|---------|-----------------------------------------------------------------------------------|---------------------------------------------------------------------------|
| 1       | $< 10,2-2r; 10,4-4r,4-4r >$                                                      | $< 40,5-5r,4-4r >$                                                       |
| 2       | $< 22,2-2r; 22,4-4r >$                                                          | $< 67,2-2r,4-4r >$                                                       |
| 3       | $< 36,2-2r; 36,4-4r >$                                                           | $< 88,2-2r; 88,4-4r >$                                                   |
| 4       | $< 54,2-2r; 54,4-4r >$                                                           | $< 110,5-5r,4-4r >$                                                      |
| 5       | $< 77,2-2r; 77,4-4r >$                                                           | $< 135,2-2r; 135,4-4r >$                                                |
| 6       | $< 105,2-2r; 105,4-4r >$                                                         | $< 163,2-2r; 163,4-4r >$                                                |

The following table gives the triangular intuitionistic fuzzy average cost and its rank using \( \text{Mag}(\tilde{a}^n) \).

| Year (n) | Average cost                                                                 | Rank |
|---------|------------------------------------------------------------------------------|------|
| 1       | $< 40,5-5r,2-2r; 40,4-4r,4-4r >$                                             | 39.5+3r |
| 2       | $< 33,5-5r,2-2r; 33,4-4r,4-4r >$                                             | 33.5 |
| 3       | $< 29,5-5r,2-2r; 29,4-4r,4-4r >$                                             | 29.5 |
| 4       | $< 27,6-5r,2-2r; 27,4-4r,4-4r >$                                             | 27.6 |
| 5       | $< 27,5-5r,2-2r; 27,4-4r,4-4r >$                                             | 27** |
| 6       | $< 27,2,5-5r,2-2r; 27,2-4r,4-4r >$                                           | 27.2 |

Therefore the minimum average cost = $< 27, 5-5r, 2-2r; 27, 4-4r, 4-4r >$

\[
= < 25+2r, 27, 29-2r; 23+4r, 27, 31-4r >.
\]

Which occurs in the 5th year and is better than $< 23,4,27,29.3; 20.7,27,32 >$ obtained by Uthra et.al [10].

5. Conclusion

While we model a replacement problem it is observed that the parameters of the problem are not exactly known one. To handle these kinds of situations we proposed intuitionistic fuzzy replacement problem when the capital cost, scrap value, maintenance or running cost are TIFNs. The Proposed method is easy to follow and effective in finding the optimum period of replacement. The magnitude of the TIFN used to
compare the TIF average cost to find the optimum period of replacement. This work can be extended to find the optimum period of replacement using trapezoidal intuitionistic fuzzy number.

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