Limit leaves of a CMC lamination are stable

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Abstract

Suppose $L$ is a lamination of a Riemannian manifold by hypersurfaces with the same constant mean curvature. We prove that every limit leaf of $L$ is stable for the Jacobi operator. A simple but important consequence of this result is that the set of stable leaves of $L$ has the structure of a lamination.

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1 Introduction.

In this paper we prove that given a codimension one lamination $L$ in a Riemannian manifold $N$, whose leaves have a fixed constant mean curvature (minimality is included), then every limit leaf $L$ of $L$ is stable with respect to the Jacobi operator. Our result is motivated by a partial result of Meeks and Rosenberg in Lemma A.1 in [6], where they proved the stability of $L$ under the constraint that the holonomy representation on any compact subdomain $\Delta \subset L$ has subexponential growth (i.e., the covering space $\tilde{\Delta}$ of $\Delta$ corresponding to the kernel of the holonomy representation has subexponential area growth). In general, if we assume stability for a covering space $\tilde{M}$ of a constant mean curvature (CMC) hypersurface $M$ in $N$ and for any connected compact domain $\Delta \subset M$ the related restricted covering $\tilde{\Delta} \to \Delta$ has subexponential area growth, then $M$ is also stable, see Lemma 6.2 in [4] for a proof using cutoff functions. However, if the area growth of the covering is exponential over some compact domain in $M$, then the stability of $\tilde{M}$ does not imply the stability of $M$, as can be seen in the example described in the next paragraph. The existence of this example makes it clear that the application in [6]

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of cutoff functions used to prove the stability of a limit leaf \( L \) with holonomy of subexponential growth cannot be applied to case when the holonomy representation of \( L \) has exponential growth.

Consider a compact surface \( \Sigma \) of genus at least two endowed with a metric \( g \) of constant curvature \(-1\), and a smooth function \( f: \mathbb{R} \to (0,1] \) with \( f(0) = 1 \) and \(-\frac{1}{8} < f''(0) < 0\). Then in the warped product metric \( f^2 g + dt^2 \) on \( \Sigma \times \mathbb{R} \), each slice \( M_c = \Sigma \times \{c\} \) is a CMC surface of mean curvature \(-f'(c)f(c)\) oriented by the unit vector field \( \partial_\tau \), and the stability operator on the totally geodesic (hence minimal) surface \( M_0 = \Sigma \times \{0\} \) is \( L = \Delta + \text{Ric}(\partial_\tau) = \Delta - 2f''(0) \), where \( \Delta \) is the laplacian on \( M_0 \) with respect to the induced metric \( f(0)^2 g = g \) and \( \text{Ric} \) denotes the Ricci curvature of \( f^2 g + dt^2 \). The first eigenvalue of \( L \) in the (compact) surface \( M_0 \) is \( 2f''(0) \), hence \( M_0 \) is unstable as a minimal surface. On the other hand, the universal cover \( \tilde{M}_0 \) of \( M_0 \) is the hyperbolic plane. Since the first eigenvalue of the Dirichlet problem for the laplacian in \( \tilde{M}_0 \) is \( \frac{1}{4} \), we deduce that the first eigenvalue of the Dirichlet problem for the Jacobi operator on \( \tilde{M}_0 \) is \( \frac{1}{4} + 2f''(0) > 0 \). Thus, \( \tilde{M}_0 \) is an immersed stable minimal surface. Similarly, for \( c \) sufficiently small, the CMC surface \( M_c \) is unstable but its related universal cover is stable.

2 The statement and proof of the main theorem.

In order to help understand the results described in this paper, we make the following definitions.

**Definition 1** Let \( M \) be a complete, embedded hypersurface in a manifold \( N \). A point \( p \in N \) is a **limit point** of \( M \) if there exists a sequence \( \{p_n\} \subset M \) which diverges to infinity on \( M \) with respect to the intrinsic Riemannian topology on \( M \) but converges in \( N \) to \( p \) as \( n \to \infty \). Let \( L(M) \) denote the set of all limit points of \( M \) in \( N \). In particular, \( L(M) \) is a closed subset of \( N \) and \( M - M \subset L(M) \), where \( M \) denotes the closure of \( M \).

**Definition 2** A **codimension one lamination** of a Riemannian manifold \( N^{n+1} \) is the union of a collection of pairwise disjoint, connected, injectively immersed hypersurfaces, with a certain local product structure. More precisely, it is a pair \((\mathcal{L}, \mathcal{A})\) satisfying:

1. \( \mathcal{L} \) is a closed subset of \( N \);
2. \( \mathcal{A} = \{\varphi_\beta: \mathbb{D}^n \times (0,1) \to U_\beta\}_\beta \) is a collection of coordinate charts of \( N \) (here \( \mathbb{D}^n \) is the open unit ball in \( \mathbb{R}^n \), \((0,1)\) the open unit interval and \( U_\beta \) an open subset of \( N \));
3. For each \( \beta \), there exists a closed subset \( C_\beta \) of \((0,1)\) such that \( \varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D}^n \times C_\beta \).

We will simply denote laminations by \( \mathcal{L} \), omitting the charts \( \varphi_\beta \) in \( \mathcal{A} \). A lamination \( \mathcal{L} \) is said to be a foliation of \( N \) if \( \mathcal{L} = N \). Every lamination \( \mathcal{L} \) naturally decomposes into a collection of disjoint connected hypersurfaces, called the leaves of \( \mathcal{L} \). As usual, the regularity of \( \mathcal{L} \) requires the corresponding regularity on the change of coordinate charts. Note that if \( \Delta \subset \mathcal{L} \) is any collection of leaves of \( \mathcal{L} \), then the closure of the union of these leaves has the structure of a lamination within \( \mathcal{L} \), which we will call a **sublamination**.
Definition 3 For $H \in \mathbb{R}$, an $H$-hypersurface $M$ in a Riemannian manifold $N$ is a codimension one submanifold of constant mean curvature $H$. A codimension one $H$-lamination $\mathcal{L}$ of $N$ is a collection of immersed (not necessarily injectively) $H$-hypersurfaces $\{L_\alpha\}_{\alpha \in I}$, called the leaves of $\mathcal{L}$, satisfying the following properties.

1. $\mathcal{L} = \bigcup_{\alpha \in I} \{L_\alpha\}$ is a closed subset of $N$.
2. If $H = 0$, then $\mathcal{L}$ is a lamination of $N$. In this case, we also call $\mathcal{L}$ a minimal lamination.
3. If $H \neq 0$, then given a leaf $L_\alpha$ of $\mathcal{L}$ and given a small disk $\Delta \subset L_\alpha$, there exists an $\varepsilon > 0$ such that if $(q, t)$ denote the normal coordinates for $\exp_q(t\eta)$ (here $\exp$ is the exponential map of $N$ and $\eta$ is the unit normal vector field to $L_\alpha$ pointing to the mean convex side of $L_\alpha$), then:

   (a) The exponential map $\exp: U(\Delta, \varepsilon) = \{(q, t) \mid q \in \text{Int}(\Delta), t \in (-\varepsilon, \varepsilon)\}$ is a submersion.
   (b) The inverse image $\exp^{-1}(\mathcal{L}) \cap \{q \in \text{Int}(\Delta), t \in [0, \varepsilon)\}$ is a lamination of $U(\Delta, \varepsilon)$.

The reader not familiar with the subject of minimal or CMC laminations should think about a geodesic $\gamma$ on a Riemannian surface. If $\gamma$ is complete and embedded (a one-to-one immersion), then its closure is a geodesic lamination $\mathcal{L}$ of the surface. When the geodesic $\gamma$ has no accumulation points, then it is proper. Otherwise, there pass complete embedded geodesics in $\mathcal{L}$ through the accumulation points of $\gamma$ forming the leaves of $\mathcal{L}$. A similar result is true for a complete, embedded $H$-hypersurface of locally bounded second fundamental form (bounded in compact extrinsic balls) in a Riemannian manifold $N$, i.e., the closure of a complete, embedded $H$-hypersurface of locally bounded second fundamental form has the structure of an $H$-lamination of $N$. For the sake of completeness, we now give the proof of this elementary fact in the case $H \neq 0$ (see the beginning of Section 1 in [5] for the proof in the minimal case).

Consider a complete, embedded $H$-hypersurface $M$ with locally bounded second fundamental form in a manifold $N$. Choose a limit point $p$ of $M$ (if there are no such limit points, then $M$ is proper and it is an $H$-lamination of $N$ by itself), i.e., $p$ is the limit in $N$ of a sequence of divergent points $p_n$ in $M$. Since $M$ has bounded second fundamental form near $p$ and $M$ is embedded, then for some small $\varepsilon > 0$, a subsequence of the intrinsic $\varepsilon$-balls $B_M(p_n, \varepsilon)$ converges to an embedded $H$-ball $B(p, \varepsilon) \subset N$ of intrinsic radius $\varepsilon$ centered at $p$. Since $M$ is embedded, any two such limit balls, say $B(p, \varepsilon), B'(p, \varepsilon)$, do not intersect transversally. By the maximum principle for $H$-hypersurfaces, we conclude that if a second ball $B'(p, \varepsilon)$ exists, then $B(p, \varepsilon), B'(p, \varepsilon)$ are the only such limit balls and they are oppositely oriented at $p$.

Now consider any sequence of embedded balls $E_n$ of the form $B(q_n, \frac{\varepsilon}{4})$ such that $q_n$ converges to a point in $B(p, \frac{\varepsilon}{2})$ and such that $E_n$ locally lies on the mean convex side of $B(p, \varepsilon)$. For $\varepsilon$ sufficiently small and for $n, m$ large, $E_n$ and $E_m$ must be graphs over domains in $B(p, \varepsilon)$ such that when oriented as graphs, they have the same mean curvature. By the maximum principle, the graphs $E_n$ and $E_m$ are disjoint or equal. It follows that near $p$ and on the mean convex side
of $B(p,\varepsilon)$, $\overline{M}$ has the structure of a lamination with leaves of the same constant mean curvature as $M$. This proves that $\overline{M}$ has the structure of an $H$-lamination of codimension one.

**Definition 4** Let $\mathcal{L}$ be a codimension one $H$-lamination of a manifold $N$ and $L$ be a leaf of $\mathcal{L}$. We say that $L$ is a *limit leaf* if $L$ is contained in the closure of $\mathcal{L} - L$.

We claim that a leaf $L$ of a codimension one $H$-lamination $\mathcal{L}$ is a limit leaf if and only if for any point $p \in L$ and any sufficiently small intrinsic ball $B \subset L$ centered at $p$, there exists a sequence of pairwise disjoint balls $B_n$ in leaves $L_n$ of $\mathcal{L}$ which converges to $B$ in $N$ as $n \to \infty$, such that each $B_n$ is disjoint from $B$. Furthermore, we also claim that the leaves $L_n$ can be chosen different from $L$ for all $n$. The implication where one assumes that $L$ is a limit leaf of $\mathcal{L}$ is clear. For the converse, it suffices to pick a point $p \in L$ and prove that $p$ lies in the closure of $\mathcal{L} - L$. By hypothesis, there exists a small intrinsic ball $B \subset L$ centered at $p$ which is the limit in $N$ of pairwise disjoint balls $B_n$ in leaves $L_n$ of $\mathcal{L}$, as $n \to \infty$. If $L_n \neq L$ for all $n \in \mathbb{N}$, then we have done. Arguing by contradiction and after extracting a subsequence, assume $L_n = L$ for all $n \in \mathbb{N}$. Choosing points $p_n \in B_n$ and repeating the argument above with $p_n$ instead of $p$, one finds pairwise disjoint balls $B_{n,m} \subset L$ which converge in $N$ to $B_n$ as $m \to \infty$. Note that for $(n_1,m_1) \neq (n_2,m_2)$, the related balls $B_{n_1,m_1}, B_{n_2,m_2}$ are disjoint. Iterating this process, we find an uncountable number of such disjoint balls on $L$, which contradicts that $L$ admits a countable basis for its intrinsic topology.

**Definition 5** A minimal hypersurface $M \subset N$ of dimension $n$ is said to be *stable* if for every compactly supported normal variation of $M$, the second variation of area is non-negative. If $M$ has constant mean curvature $H$, then $M$ is said to be *stable* if the same variational property holds for the functional $A - nHV$, where $A$ denotes area and $V$ stands for oriented volume. A *Jacobi function* $f: M \to \mathbb{R}$ is a solution of the equation $\Delta f + |A|^2 f + \text{Ric}(\eta)f = 0$ on $M$; if $M$ is two-sided, then the stability of $M$ is equivalent to the existence of a positive Jacobi function on $M$ (see Fischer-Colbrie [1]).

The proof of the next theorem is motivated by a well-known application of the divergence theorem to prove that every compact domain in a leaf of an oriented, codimension one minimal foliation in a Riemannian manifold is area-minimizing in its relative $\mathbb{Z}$-homology class. For other related applications of the divergence theorem, see [8].

**Theorem 1** The limit leaves of a codimension one $H$-lamination of a Riemannian manifold are stable.

**Proof.** We will assume that the dimension of the ambient manifold $N$ is three in this proof; the arguments below can be easily adapted to the $n$-dimensional setting. The first step in the proof is the following result.
Assertion 1  Suppose $D(p,r)$ is a compact, embedded $H$-disk in $N$ with constant mean curvature $H$ (possibly negative), intrinsic diameter $r > 0$ and center $p$, such that there exist global normal coordinates $(q,t)$ based at points $q \in D(p,r)$, with $t \in [0,\varepsilon]$. Suppose that $T \subset [0,\varepsilon]$ is a closed disconnected set with zero as a limit point and for each $t \in T$, there exists a function $f_t : D(p,r) \to [0,\varepsilon]$ such that the normal graphs $q \mapsto \exp_q(f_t(q)\eta(q))$ define pairwise disjoint $H$-surfaces with $f_t(p) = t$, where $\eta$ stands for the oriented unit normal vector field to $D(p,r)$.

For each component $(t_\alpha, s_\alpha)$ of $[0,\varepsilon] - T$ with $s_\alpha < \varepsilon$, consider the interpolating graphs $q \mapsto \exp_q(f_t(q)\eta(q))$, $t \in [t_\alpha, s_\alpha]$, where

$$f_t = f_{t_\alpha} + (t - t_\alpha)\frac{f_{s_\alpha} - f_{t_\alpha}}{s_\alpha - t_\alpha}.$$

(See Figure 1). Then, the mean curvature functions $H_t$ of the graphs of $f_t$ satisfy

$$\lim_{t \to 0^+} \frac{H_t(q) - H}{t} = 0 \quad \text{for all } q \in D(p,\varepsilon/2).$$

Proof of Assertion 1. Reasoning by contradiction, suppose there exists a sequence $t_n \in [0,\varepsilon] - T$, $t_n \searrow 0$, and points $q_n \in D(p,r/2)$, such that $|H_{t_n}(q_n) - H| > C t_n$ for some constant $C > 0$. Let $(t_{\alpha_n}, s_{\alpha_n})$ be the component of $[0,\varepsilon] - T$ which contains $t_n$. Then, we can rewrite $f_{t_n}$ as

$$f_{t_n} = t_n \left[ \frac{t_{\alpha_n} f_{t_{\alpha_n}}}{t_n} \right] + \left( 1 - \frac{t_{\alpha_n}}{t_n} \right) \frac{f_{s_{\alpha_n}} - f_{t_{\alpha_n}}}{s_{\alpha_n} - t_{\alpha_n}}.$$

After extracting a subsequence, we may assume that as $n \to \infty$, the sequence of numbers $\frac{t_{\alpha_n}}{t_n}$ converges to some $A \in [0,1]$, and the sequences of functions $\frac{f_{t_{\alpha_n}}}{t_{\alpha_n}}, \frac{f_{s_{\alpha_n}} - f_{t_{\alpha_n}}}{s_{\alpha_n} - t_{\alpha_n}}$ converge smoothly.
to Jacobi functions $F_1, F_2$ on $\overline{D}(p, r/2)$, respectively. Now consider the normal variation of $D(p, r/2)$ given by

$$\tilde{\psi_t}(q) = \exp_q (t[A F_1 + (1-A) F_2](q) \eta(q)),$$

for $t > 0$ small. Since $AF_1 + (1-A) F_2$ is a Jacobi function, the mean curvature $\bar{H}_t$ of $\tilde{\psi}_t$ is $\bar{H}_t = H + O(t^2)$, where $O(t^2)$ stands for a function satisfying $t O(t^2) \to 0$ as $t \to 0^+$. On the other hand, the normal graphs of $f_{t_n}$ and of $t_n(A F_1 + (1-A) F_2)$ over $\overline{D}(p, r/2)$ can be taken arbitrarily close in the $C^4$-norm for $n$ large enough, which implies that their mean curvatures $H_{t_n}, \bar{H}_{t_n}$ are $C^2$-close. This is a contradiction with the assumed decay of $H_{t_n}$ at $q_0$. \hfill $\square$

We now continue the proof of the theorem. Let $L$ be a limit leaf of an $H$-lamination $\mathcal{L}$ of a manifold $N$ by hypersurfaces. If $L$ is one-sided, then we consider the two-sided $2:1$ cover $\tilde{L} \to L$ and pullback the $H$-lamination $\mathcal{L}$ to a small neighborhood of the zero section $\tilde{L}_0$ of the normal bundle $\tilde{L}^\perp$ to $\tilde{L}$ ($\tilde{L}_0$ can be identified with $\tilde{L}$ itself). In this case, we will prove that $\tilde{L}_0$ is stable, which in particular implies stability for $L$, see Remark \[1\]. Hence, in the sequel we will assume $L$ is two-sided.

Arguing by contradiction, suppose there exists an unstable compact subdomain $\Delta \subset L$ with non-empty smooth boundary $\partial \Delta$. Given a subset $A \subset \Delta$ and $\varepsilon > 0$ sufficiently small, we define

$$A_{\varepsilon} = \{\exp_q(t \eta(q)) \mid q \in A, t \in [0, \varepsilon]\}$$

to be the one-sided vertical $\varepsilon$-neighborhood of $A$, written in normal coordinates $(q,t)$ (here we have picked the unit normal $\eta$ to $L$ such that $L$ is a limit of leaves of $\mathcal{L}$ at the side $\eta$ points into). Since $\mathcal{L}$ is a lamination and $\Delta$ is compact, there exists $\delta \in (0, \varepsilon)$ such that the following property holds:

(**) *Given an intrinsic disk $D(p, \delta) \subset L$ centered at a point $p \in \Delta$ with radius $\delta$, and given a point $x \in L$ which lies in $D(p, \delta)^{\perp, \varepsilon/2}$, then there passes a disk $D_x \subset L$ through $x$, which is entirely contained in $D(p, \delta)^{\perp, \varepsilon}$, and $D_x$ is a normal graph over $D(p, \delta)$.*

Fix a point $p \in \Delta$ and let $x \in L \cap \{p\}^{\perp, \varepsilon/2}$ be the point above $p$ with greatest $t$-coordinate. Consider the disk $D_x$ given by property (**), which is the normal graph of a function $f_x$ over $D(p, \delta)$. Since $\Delta$ is compact, $\varepsilon$ can be assumed to be small enough so that the closed region given in normal coordinates by $U(p, \delta) = \{(q,t) \mid q \in D(p, \delta), 0 \leq t \leq f_x(q)\}$ intersects $\mathcal{L}$ in a closed collection of disks $\{D(t) \mid t \in T\}$, each of which is the normal graph over $D(p, \delta)$ of a function $f_t: D(p, \delta) \to [0, \varepsilon)$ with $f_t(p) = t$, and $T$ is a closed subset of $[0, \varepsilon/2]$, see Figure \[2\].

We now foliate the region $U(p, \delta) - \bigcup_{t \in T} D(t)$ by interpolating the graphing functions as we did in Assertion \[1\]. Consider the union of all these locally defined foliations $\mathcal{F}_p$ with $p$ varying in $\Delta$. Since $\Delta$ is compact, we find $\varepsilon_1 \in (0, \varepsilon/2)$ such that the one-sided normal neighborhood $\Delta^{\perp, \varepsilon_1} \subset \bigcup_{p \in \Delta} \mathcal{F}_p$ of $\Delta$ is foliated by surfaces which are portions of disks in the locally defined foliations $\mathcal{F}_p$. Let $\mathcal{F}(\varepsilon_1)$ denote this foliation of $\Delta^{\perp, \varepsilon_1}$. By Assertion \[1\] the mean curvature function of the foliation $\mathcal{F}(\varepsilon_1)$ viewed locally as a function $H(p, t)$ with $p \in \Delta$ and $t \in [0, \varepsilon_1]$,
Figure 2: The shaded region between $D_x$ and $D(p, \delta)$ corresponds to $U(p, \delta)$.

satisfies

$$\lim_{t \to 0^+} \frac{H(p, t) - H}{t} = 0, \quad \text{for all } p \in \Delta. \quad (1)$$

On the other hand since $\Delta$ is unstable, the first eigenvalue $l_1$ of the Jacobi operator $J$ for the Dirichlet problem on $\Delta$, is negative. Consider a positive eigenfunction $h$ of $J$ on $\Delta$ (note that $h = 0$ on $\partial \Delta$). For $t \geq 0$ small and $q \in \Delta$, $\exp_q(th(q)\eta(q))$ defines a family of surfaces $\{\Delta(t)\}_t$ with $\Delta(t) \subset \Delta^{1, \epsilon}$ and the mean curvature $\widehat{H}_t$ of $\Delta(t)$ satisfies

$$\frac{d}{dt} \bigg|_{t=0} \widehat{H}_t = Jh = -l_1 h > 0 \quad \text{on the interior of } \Delta. \quad (2)$$

Let $\Omega(t)$ be the compact region of $N$ bounded by $\Delta \cup \Delta(t)$ and foliated away from $\partial \Delta$ by the surfaces $\Delta(s)$, $0 \leq s \leq t$. Consider the smooth unit vector field $V$ defined at any point $x \in \Omega(t) - \partial \Delta$ to be the unit normal vector to the unique leaf $\Delta(s)$ which passes through $x$, see Figure 3. Since the divergence of $V$ at $x \in \Delta(s) \subset \Omega(t)$ equals $-2\widehat{H}_s$ where $\widehat{H}_s$ is the mean curvature of $\Delta(s)$ at $x$, then (2) gives

$$\text{div}(V) = -2\widehat{H}_s = -2H + 2l_1 sh + O(s^2) \quad \text{on } \Delta(s)$$

for $s > 0$ small. It follows that there exists a positive constant $C$ such that for $t$ small,

$$\int_{\Omega(t)} \text{div}(V) = -2H \text{Vol}(\Omega(t)) + 2l_1 \int_{\Omega(t)} sh + O(t^2) < -2H \text{Vol}(\Omega(t)) - Ct. \quad (3)$$
Figure 3: The divergence theorem is applied in the shaded region $\Omega(t)$ between $\Delta$ and $\Delta(t)$.

Since the foliation $\mathcal{F}(\varepsilon_1)$ has smooth leaves with uniformly bounded second fundamental form, then the unit normal vector field $W$ to the leaves of $\mathcal{F}(\varepsilon_1)$ is Lipschitz on $\Delta^{\perp,\varepsilon_1}$ and hence, it is Lipschitz on $\Omega(t)$. Since $W$ is Lipschitz, its divergence is defined almost everywhere in $\Omega(t)$ and the divergence theorem holds in this setting. Note that the divergence of $W$ is smooth in the regions of the form $U(p, \delta) - \bigcup_{t \in T} D(t)$ where it is equal to $-2$ times the mean curvature of the leaves of $\mathcal{F}_p$. Also, the mean curvature function of the foliation is continuous on $\mathcal{F}(\varepsilon_1)$ (see Assertion 1). Hence, the divergence of $W$ can be seen to be a continuous function on $\Omega(t)$ which equals $-2H$ on the leaves $D(t)$, and by Assertion 1, $\text{div}(W)$ converges to the constant $-2H$ as $t \to 0$ to first order. Hence,

$$\int_{\Omega(t)} \text{div}(W) > -2H \text{Vol}(\Omega(t)) - Ct, \quad (4)$$

for any $t > 0$ sufficiently small.

Applying the divergence theorem to $V$ and $W$ in $\Omega(t)$ (note that $W = V$ on $\Delta$), we obtain the following two inequalities:

$$\int_{\Omega(t)} \text{div}(V) = \int_{\Delta(t)} \langle V, \eta(t) \rangle - \int_{\Delta} \langle V, \eta \rangle = \text{Area}(\Delta(t)) - \text{Area}(\Delta),$$

$$\int_{\Omega(t)} \text{div}(W) = \int_{\Delta(t)} \langle W, \eta(t) \rangle - \int_{\Delta} \langle V, \eta \rangle < \text{Area}(\Delta(t)) - \text{Area}(\Delta),$$
where $\eta(t)$ is the exterior unit vector field to $\Omega(t)$ on $\Delta(t)$. Hence, $\int_{\Omega(t)} \text{div}(W) < \int_{\Omega(t)} \text{div}(V)$. On the other hand, choosing $t$ sufficiently small such that both inequalities (3) and (4) hold, we have $\int_{\Omega(t)} \text{div}(W) > \int_{\Omega(t)} \text{div}(V)$. This contradiction completes the proof of the theorem. \qed

**Remark 1** The proof of the theorem shows that given any two-sided cover $\tilde{L}$ of a limit leaf $L$ of $\mathcal{L}$ as described in the statement of the theorem, then $\tilde{L}$ is stable. This follows by lifting $\mathcal{L}$ to a neighborhood $U(\tilde{L})$ of $\tilde{L}$ in its normal bundle, considered to be the zero section in $U(\tilde{L})$. In the case of non-zero constant mean curvature hypersurfaces, $L$ is already two-sided and then stability is equivalent to the existence of a positive Jacobi function. However, in the minimal case where a hypersurface $L$ may be one-sided, this observation concerning stability of $L$ is generally a stronger property; for example, the projective plane contained in projective three-space is a totally geodesic surface which is area minimizing in its $\mathbb{Z}_2$-homology class but its oriented two-sided cover is unstable, see Ross [9] and also Ritoré and Ros [7].

Next we give a useful and immediate consequence of Theorem 1. Let $L$ be a codimension one $H$-lamination of a manifold $N$. We will denote by $\text{Stab}(\mathcal{L})$, $\text{Lim}(\mathcal{L})$ the collections of stable leaves and limit leaves of $\mathcal{L}$, respectively. Note that $\text{Lim}(\mathcal{L})$ is a closed set of leaves and so, it is a sublamination of $\mathcal{L}$.

**Corollary 1** Suppose that $N$ is a not necessarily complete Riemannian manifold and $\mathcal{L}$ is an $H$-lamination of $N$ with leaves of codimension one. Then, the closure of any collection of its stable leaves has the structure of a sublamination of $\mathcal{L}$, all of whose leaves are stable. Hence, $\text{Stab}(\mathcal{L})$ has the structure of a minimal lamination of $N$ and $\text{Lim}(\mathcal{L}) \subset \text{Stab}(\mathcal{L})$ is a sublamination.

**Remark 2** Theorem 1 and Corollary 1 have many useful applications to the geometry of embedded minimal and constant mean curvature hypersurfaces in Riemannian manifolds. We refer the interested reader to the survey [2] by the first two authors and to our joint paper in [3] for some of these applications.

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