Coverings of small categories and nerves

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Abstract

We prove a certain proposition which states a relationship between coverings of small categories and nerves. As its application, we prove that for a covering \( P : E \to B \) of finite categories, the zeta function of \( E \) is the zeta function of \( B \) to the number of sheet of \( P \). Moreover, we prove the formula \( \chi(E) = \chi(F)\chi(B) \) for Euler characteristic of categories and coverings.

1 Introduction

A covering space is very useful and interesting tools in geometry. For instance, it is used for calculations of fundamental groups and it has an analogy of Galois theory [Hat02]. The notion of coverings is also defined for small categories. Many people have studied about it, for example [BH99], [CM] and [Tan]. In this paper, we show the following proposition.

Proposition 1.1. Let \( P : E \to B \) be a covering of small categories and let \( b \) be an object of \( B \). For any \( n \geq 0 \), \( N_n(E) \) is bijective to \( \prod_{x \in P^{-1}(b)} N_n(B) \) where

\[
N_n(E) = \{ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} x_n) \ \text{in} \ E \}
\]

and \( N_n(B) \) is also defined in the same way.

As its application, we prove that for a covering \( P : E \to B \) of finite categories, the zeta function of \( E \) is the zeta function of \( B \) to the number of sheet of \( P \). Moreover, we prove the formula \( \chi(E) = \chi(F)\chi(B) \) for Euler characteristic of categories and coverings.

First, we give a historical background of the zeta function of a finite category and coverings of finite categories. In [NogA], the zeta function of a finite category \( I \) was defined by

\[
\zeta_I(z) = \exp \left( \sum_{n=1}^{\infty} \frac{\#N_n(I)}{n} z^n \right)
\]

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and it was shown that the zeta function of a connected finite groupoid is a rational function and for a covering of finite groupoids $P : E \to B$, the inverse zeta function of $B$ divides the inverse zeta function of $E$ (Proposition 2.3 and Proposition 2.7 of [NogA]).

In this paper, we generalize Proposition 2.7 of [NogA] as follows.

**Main Theorem.** Suppose $P : E \to B$ is a covering of finite categories and let $b$ be an object of $B$. Then, the zeta function of $E$ is the zeta function of $B$ to the number of sheet of $P$, that is,

$$\zeta_E(z) = (\zeta_B(z))^\#P^{-1}(b).$$

Note that the number $\#P^{-1}(b)$ does not depend on the choice of $b$ (Proposition 2.1). It is an analogue of Corollary 1 of §2 of [ST96].

Next, we recall Euler characteristic of categories.

The Euler characteristic of a finite category $\chi_L$ was defined by Leinster [Lei08]. After this paper, various definitions of Euler characteristics for categories were defined, series Euler characteristic $\chi_{\Sigma}$ by Berger-Leinster [BL08], $L^2$-Euler characteristic $\chi^{(2)}$ by Fiore-Lück-Sauer [FLS11], extended $L^2$-Euler characteristic $\chi_{\Sigma}^{(2)}$ [Nog] and Euler characteristic of $\mathbb{N}$-filtered acyclic categories $\chi_{\mathrm{fil}}$ [Nog11] by the author. For a finite acyclic category, four Euler characteristics $\chi_L, \chi_{\Sigma}, \chi^{(2)}$ and $\chi_{\mathrm{fil}}$ coincide (see, for example, Introduction of [Nog]). A small category is acyclic if every endomorphism and every isomorphism is an identity morphism. For a finite groupoid $G$, three Euler characteristic $\chi_L, \chi_{\Sigma}$ and $\chi^{(2)}$ coincide and the value is

$$\sum_{x \in \text{Ob}(G)/\sim} \frac{1}{\#\text{Aut}(x)}$$

where this sum runs over all isomorphism classes of objects of $G$ (Example 2.7 of [Lei08], Theorem 3.2 of [BL08] and Example 5.12 of [FLS11]).

A topological fibration $F \hookrightarrow E \to B$ under certain suitable hypothesis satisfies the equation

$$\chi(E) = \chi(F)\chi(B).$$

An analogue of this formula for Euler characteristic of categories and category fibrations was considered in [Lei08] and [FLS11]. In [Lei08], such formula was found for the Grothendieck construction (Proposition 2.8 of [Lei08]). In [FLS11], such formula for coverings of connected finite groupoids and isofibrations of connected finite groupoids were found (Theorem 5.30 and Theorem 5.37 of [FLS11]).

In this paper, we consider such formula for $\chi_{\Sigma}$ and $\chi_{\mathrm{fil}}$ and coverings.

**Main Theorem.**

1. Let $P : E \to B$ be a covering of finite categories and let $b$ be an object of $B$. Then, $E$ has series Euler characteristic if and only if $B$ has series Euler characteristic. In this case, we have

$$\chi_{\Sigma}(E) = \chi_{\Sigma}(P^{-1}(b))\chi_{\Sigma}(B).$$

2. Suppose $(A, \mu_A)$ and $(B, \mu_B)$ are $\mathbb{N}$-filtered acyclic categories and $b$ is an object of $B$ and $P : A \to B$ be a covering whose fiber is finite satisfying
the equation $\mu_A(x) = \mu_B(P(x))$ for any object $x$ of $A$. Then, $(A, \mu_A)$ has Euler characteristic $\chi_{fil}(A, \mu_A)$ if and only if $B$ has Euler characteristic $\chi_{fil}(B, \mu_B)$. In this case, we have

$$\chi_{fil}(A, \mu_A) = \chi_{fil}(P^{-1}(b), \mu)\chi_{fil}(B, \mu_B)$$

for any $\mathbb{N}$-filtration $\mu$ of $P^{-1}(b)$.

This paper is organized as follows.

In section 2 we investigate relationships between coverings of small categories and nerves.

In section 3 we prove our main theorem.

In section 4 we give some examples of coverings of small categories.

2 Coverings and Nerves

In this section, we investigate relationships between coverings of small categories and nerves.

Here, let us recall a covering of small categories [BH99].

Let $C$ be a small category. Then, $C$ is connected if there exists a zig-zag sequence of morphisms in $C$

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} y$$

for any objects $x$ and $y$ of $C$. We do not have to care about the direction of the last morphism $f_n$ since we can insert an identity morphism to the sequence. For an object $x$ of $C$, let $S(x)$ is the set of morphisms of $C$ whose source is $x$ $S(x) = \{f : x \to * \in \text{Mor}(C)\}$

and $T(x)$ is the set of morphisms of $C$ whose target is $x$ $T(x) = \{g : * \to x \in \text{Mor}(C)\}$.

Suppose $E$ and $B$ are small categories and $B$ is connected. Then, a functor $P : E \to B$ is a covering if the following two restrictions of $P$

$$P : S(x) \longrightarrow S(P(x))$$

$$P : T(x) \longrightarrow T(P(x))$$

are bijections for any object $x$ of $E$. This condition is an analogue of the condition of an unramified covering of graphs (see [ST96]). For an object $b$ of $B$, the inverse image $P^{-1}(b)$ of the restriction of $P$ with respect to objects $P^{-1}(b) = \{x \in \text{Ob}(E) \mid P(x) = b\}$

is called the fiber of $b$. The cardinality of $P^{-1}(b)$ is called the number of sheet of $P$ and it does not depend on the choice of $b$ since the base category $B$ is connected (see Proposition 2.1). In particular, a covering of groupoids was studied in [May99]. Applying the classifying space functor $B$ to a covering $P : E \to B$, we have the covering space $BP$ in the topological sense (see [Tan]) and examples are given in Section 4.

The following proposition is briefly introduced in [BH99] with no proof, but this proposition is very important in this paper. So we give a proof of it to make sure.
Proposition 2.1. Let $P : E \to B$ be a covering of small categories. Then, for any objects $b$ and $b'$ of $B$, $P^{-1}(b)$ is bijective to $P^{-1}(b')$.

Proof. It suffices to show that $P^{-1}(b)$ is bijective to $P^{-1}(b')$ if there exists a morphism $f : b \to b'$. Indeed, if it is proven, then we have for any objects $b$ and $b'$, we have a zig-zag sequence

$$b \longrightarrow b_1 \longrightarrow b_2 \longrightarrow \ldots \longrightarrow b'$$

so that we obtain

$$P^{-1}(b) \cong P^{-1}(b_1) \cong \ldots \cong P^{-1}(b').$$

Suppose there exists a morphism $f : b \to b'$. For any $x$ of $P^{-1}(b)$, we have the bijection $P : S(x) \to S(b)$. Since $f$ belongs to $S(b)$, there exists a unique morphism $f_x : x \to y_x$ of $S(x)$ such that $P(f_x) = f$. Define a map $F_f : P^{-1}(b) \to P^{-1}(b')$ by $F_f(x) = y_x$. Then, $F_f$ is a bijection. We first show it is injective. If $x \neq x'$, then $y_x \neq y_{x'}$. Indeed, then if we assume $y_x = y_{x'}$

$$x \xleftarrow{f_x} \leftarrow \ x' \xrightarrow{f_{x'}}$$

$f_x$ and $f_{x'}$ belong to $T(y_x)$ and they are different morphisms, but the map $P : T(y_x) \to T(b')$ is bijective and $P(f_x) = P(f_{x'}) = f$. This contradiction implies $y_x \neq y_{x'}$. Next, we show $F_f$ is a surjection. For any $y$ of $P^{-1}(b')$, there exists a unique morphism $g : x \to y$ such that $P(g) = f$ since the map $P : T(y) \to T(b')$ is bijective. Then, $P(x) = b$, so $x$ belongs to $P^{-1}(b)$. Hence, $F_f(1) = y$.

The following lemma is clear, but this formulation will make the proof of Proposition 2.1 easier to understand.

Lemma 2.2. Suppose $f : X \to Y$ is a bijection and $X = \coprod_{\lambda \in A} A_\lambda$ and $Y = \coprod_{\lambda \in B} B_\lambda$ and for each restriction $f|_{A_\lambda}$, its image belongs to $B_\lambda$. Then, each restriction $f|_{A_\lambda}$ is a bijection.

Let $C$ be a small category and $x$ be an object of $C$. Then, let $N_n(C)_x$ be the set of chains of morphisms in $C$ of length $n$ whose target is $x$

$$N_n(C)_x = \{ (x_0 \overset{f_1}{\longrightarrow} x_1 \overset{f_2}{\longrightarrow} \ldots \overset{f_n}{\longrightarrow} x_n) \mid x_i = x \}.$$

Proposition 2.3. Let $P : E \to B$ be a covering of small categories. Then, for any object $b$ of $B$ and any $x$ of $P^{-1}(b)$ and $n \geq 0$, $N_n(E)_x$ is bijective to $N_n(B)_b$.
Proof. We prove this proposition by induction on $n$. If $n = 0$, we have

$$N_0(E)_x = \{1_x\} \cong \{1_b\} = N_0(B)_b.$$  

Suppose it is true for $n$. Then we have

$$N_{n+1}(E)_x = \prod_{y \in \text{Ob}(E)} N_n(E)_y \times \text{Hom}_E(y, x)$$

$$\cong \prod_{b_i \in \text{Ob}(B)} \prod_{y_i \in \text{P}^{-1}(b_i)} N_n(B)_{b_i} \times \text{Hom}_E(y_i, x)$$

$$\cong \prod_{b_i \in \text{Ob}(B)} N_n(B)_{b_i} \times \left( \prod_{y_i \in \text{P}^{-1}(b_i)} \text{Hom}_E(y_i, x) \right). \tag{1}$$

We have the following diagram

$$T(x) = \prod_{b_i \in \text{Ob}(B)} \left( \prod_{y_i \in \text{P}^{-1}(b_i)} \text{Hom}_E(y_i, x) \right) \quad \xrightarrow{P} \quad \prod_{y_i \in \text{P}^{-1}(b_i)} \text{Hom}_E(y_i, x)$$

$$T(b) = \prod_{b_i \in \text{Ob}(B)} \text{Hom}_B(b_i, b) \quad \xleftarrow{P} \quad \text{Hom}_B(b_i, b)$$

Lemma 2.2 implies

$$P : \prod_{y_i \in \text{P}^{-1}(b_i)} \text{Hom}_E(y_i, x) \rightarrow \text{Hom}_B(b_i, b)$$

is a bijection since for each $f : y_i \rightarrow x$, $P(f) : b_i \rightarrow b$ belongs to $\text{Hom}_B(b_i, b)$. Hence, the equation (1) is

$$\prod_{b_i \in \text{Ob}(B)} N_n(B)_{b_i} \times \text{Hom}_B(b_i, b) \cong \prod_{b_i \in \text{Ob}(B)} N_n(B)_{b_i} \times \text{Hom}_B(b_i, b)$$

$$= N_{n+1}(B)_b.$$  

\[\Box\]

**Proposition 2.4.** Let $P : E \rightarrow B$ be a covering of small categories and let $b$ be an object of $B$. For any $n \geq 0$, $N_n(E)$ is bijective to $\prod_{x \in \text{P}^{-1}(b)} N_n(B)$.

Proof. When $n = 0$, Proposition 2.1 implies

$$N_0(E) = \prod_{b \in \text{Ob}(B)} \text{P}^{-1}(b)$$

$$\cong \text{P}^{-1}(b) \times N_0(B)$$

$$\cong \prod_{x \in \text{P}^{-1}(b)} N_0(B).$$
For $n \geq 1$, Proposition 2.3 implies

$$N_n(E) = \prod_{x \in \text{Ob}(E)} N_{n-1}(E) x \times S(x)$$

$$\cong \prod_{b \in \text{Ob}(B)} \prod_{x \in P^{-1}(b)} N_{n-1}(B) b \times S(b)$$

Since the number of sheet of $P$ does not depend on the choice of $b$ (Proposition 2.1), we have

$$\cong \prod_{x \in P^{-1}(b)} N_{n-1}(B) b \times S(b)$$

(2)

The two propositions above hold when nerves are non-degenerate, it means that we do not use identity morphisms. Let $C$ be a small category and let $x$ and $y$ be objects of $C$. We define the following symbols by

$$S(x) = S(x) \setminus \{1_x\}, T(x) = T(x) \setminus \{1_x\}$$

$$\text{Hom}_C(x, y) = \begin{cases} \text{Hom}_C(x, y) \setminus \{1_x\} & \text{if } x = y \\ \text{Hom}_C(x, y) & \text{if } x \neq y \end{cases}$$

$$\overline{N}_n(C) = \{ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} x_n) \text{ in } C \mid f_i \neq 1 \}$$

$$\overline{N}_n(C)_x = \{ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} x_n) \text{ in } C \mid f_i \neq 1, x_n = x \}.$$ 

Note that $\overline{N}_0(C) = \overline{N}_0(C)$.

**Proposition 2.5.** Let $P : E \to B$ be a covering of small categories. Then, for any object $b$ of $B$ and any $x$ of $P^{-1}(b)$ and $n \geq 0$, $N_n(E)_x$ is bijective to $\overline{N}_n(B)_b$.

**Proof.** If we replace the symbols in the proof of Proposition 2.3 by the symbols with bars above, we can use the same proof. \qed

**Proposition 2.6.** Let $P : E \to B$ be a covering of small categories and $b$ be an object of $B$. For any $n \geq 0$, $N_n(E)$ is bijective to $\prod_{x \in P^{-1}(b)} \overline{N}_n(B)$.

### 3 Applications

In this section, we prove our main theorem.

The following is an analogue of Corollary 1 of §2 of [ST96].

**Theorem 3.1.** Let $P : E \to B$ be a covering of finite categories and let $b$ be an object of $B$. Then, we have

$$\zeta_E(z) = (\zeta_B(z))^{#P^{-1}(b)}.$$
Proof. The definition of the zeta function of a finite category and Proposition 2.4 directly imply this fact, that is,

\[ \zeta_E(z) = \exp \left( \sum_{n=1}^{\infty} \frac{\#N_n(E)}{n} z^n \right) \]
\[ = \exp \left( \sum_{n=1}^{\infty} \frac{\#P^{-1}(b)\#N_n(B)}{n} z^n \right) \]
\[ = (\zeta_B(z))^{\#P^{-1}(b)}. \]

Let \( I \) be a finite category. Then, \( I \) has series Euler characteristic [BL08] if and only if the rational function

\[ f_I(t) = \frac{\text{sum(adj}(E - (A_I - E)t))}{\det(E - (A_I - E)t)} \]

can be substituted \(-1\) to \( t \) where \( A_I \) is an \( n \times n \)-matrix, called adjacency matrix, whose \((i, j)\)-entry is the number of morphisms from \( x_i \) to \( x_j \) when

\[ \text{Ob}(I) = \{x_1, x_2, \ldots, x_n\}. \]

If \( I \) has series Euler characteristic, then the series Euler characteristic \( \chi_{\Sigma}(I) \) of \( I \) is defined by \( f_I(-1) \). This rational function is the rational expression of the power series \( \sum_{n=0}^{\infty} \#N_n(I)t^n \) (Theorem 2.2 of [BL08]).

A discrete category consists of only objects and identity morphisms. For a covering \( P : E \to B \), its fiber is a discrete category when we regard it as a category.

**Theorem 3.2.** Let \( P : E \to B \) be a covering of finite categories and let \( b \) be an object of \( B \). Then, \( E \) has series Euler characteristic if and only if \( B \) has series Euler characteristic. In this case, we have

\[ \chi_{\Sigma}(E) = \chi_{\Sigma}(P^{-1}(b))\chi_{\Sigma}(B). \]

Proof. We give two types of proofs.

The first one is a proof by Proposition 2.6, Proposition 2.6, and Theorem 2.2 of [BL08] imply

\[ \sum_{n=0}^{\infty} \#N_n(E)t^n = \#P^{-1}(b) \sum_{n=0}^{\infty} \#N_n(B)t^n \]
\[ = \#P^{-1}(b) \frac{\text{sum(adj}(E - (A_B - E)t))}{\det(E - (A_B - E)t)}. \]

So \( E \) has series Euler characteristic if and only if \(-1\) can be substituted to

\[ \frac{\text{sum(adj}(E - (A_B - E)t))}{\det(E - (A_B - E)t)} \]

if and only if \(-1\) can be substituted to

\[ \frac{\text{sum(adj}(E - (A_B - E)t))}{\det(E - (A_B - E)t)} \]
if and only if $B$ has series Euler characteristic. Hence, we prove the first claims. If $E$ has series Euler characteristic, then we have

$$\chi_{\Sigma}(E) = \#P^{-1}(b)\chi_{\Sigma}(B) = \chi_{\Sigma}(P^{-1}(b)\chi_{\Sigma}(B)).$$

The second proof is a proof by the zeta function of a finite category. Suppose

$$\zeta_B(z) = \prod_{k=1}^{n} \frac{1}{(1 - a_k z)^{b_k,0}} \exp \left( \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} \frac{b_{k,j} z^j}{j(1 - a_k z)^j} \right)$$

for some complex numbers $a_k$ and $b_{k,j}$ and natural numbers $n$ and $e_k$ and a polynomial $Q(z)$ with $\mathbb{Z}$-coefficients whose constant term is 0 (Theorem 3.1 of [NogC]). Then, the uniqueness of the analytic continuation and Theorem 3.3 imply

$$\zeta_E(z) = \prod_{k=1}^{n} \frac{1}{(1 - a_k z)^{\#P^{-1}(b)b_k,0}} \exp \left( \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} \frac{b_{k,j} z^j}{j(1 - a_k z)^j} \right).$$

We have $Q(z) = 0$ if and only if $\#P^{-1}(b)Q(z) = 0$, so that by Lemma 3.3 the first claim is proven. If $B$ has series Euler characteristic, then we have

$$\zeta_B(z) = \prod_{k=1}^{n} \frac{1}{(1 - a_k z)^{\#P^{-1}(b)b_k,0}} \exp \left( \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} \frac{b_{k,j} z^j}{j(1 - a_k z)^j} \right).$$

Theorem 3.3 of [NogC] implies

$$\chi_{\Sigma}(E) = \sum_{k=1}^{n} \sum_{j=0}^{e_k-1} (-1)^j \#P^{-1}(b)b_{k,j} a_k^{j+1}$$

$$= \#P^{-1}(b) \sum_{k=1}^{n} \sum_{j=0}^{e_k-1} (-1)^j b_{k,j} a_k^{j+1}$$

$$= \chi_{\Sigma}(P^{-1}(b)\chi_{\Sigma}(B)).$$

Lemma 3.3. Suppose $I$ is a finite category and the zeta function of $I$ is

$$\zeta_I(z) = \prod_{k=1}^{n} \frac{1}{(1 - a_k z)^{b_k,0}} \exp \left( Q(z) + \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} \frac{b_{k,j} z^j}{j(1 - a_k z)^j} \right)$$

for some complex numbers $a_k$ and $b_{k,j}$ and natural numbers $n$ and $e_k$ and a polynomial $Q(z)$ with $\mathbb{Z}$-coefficients whose constant term is 0. Then, $I$ has series Euler characteristic if and only if $Q(z) = 0$. 

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Proof. Theorem 3.1 of [NogA] implies the zeta function of $I$ is of the form of (4).

If $I$ has series Euler characteristic, Theorem 3.3 of [NogC] implies $Q(z) = 0$. Conversely, let $Q(z) = 0$. Here, let us recall what $Q(z)$ is (Theorem 3.1 of [NogC]). Indeed, $Q(z) = \int q(z)dz$ and $q(z)$ is a polynomial with coefficients in $\mathbb{Z}$, moreover

$$\text{sum}(\text{adj}(E - A_I z)A_I) = q(z)|E - A_I z| + r(z)$$

where

$$\text{deg}(r(z)) < \text{deg}|E - A_I z|.$$  

We have $\int q(z)dz = 0$ implies $q(z) = 0$. Hence,

$$\text{sum}(\text{adj}(E - A_I z)A_I) = r(z).$$

Lemma 2.3 of [NogC] implies $I$ has series Euler characteristic.

We recall the Euler characteristic of an $\mathbb{N}$-filtered acyclic category [Nog11].

**Definition 3.4.** A small category $A$ is acyclic if every endomorphism and every isomorphism is an identity morphism.

**Remark 3.5.** This is the same as a skeletal scwol [BH99].

Define an order on the set $\text{Ob}(A)$ of objects of $A$ by $x \leq y$ if there exists a morphism $x \to y$. Then, $\text{Ob}(A)$ is a poset.

**Definition 3.6.** Let $A$ be an acyclic category. A functor $\mu : A \to \mathbb{N}$ satisfying $\mu(x) < \mu(y)$ for $x < y$ in $\text{Ob}(A)$ is called an $\mathbb{N}$-filtration of $A$. A pair $(A, \mu)$ is called an $\mathbb{N}$-filtered acyclic category.

**Definition 3.7.** Let $(A, \mu)$ be an $\mathbb{N}$-filtered acyclic category. Then, define $\chi_{fil}(A, \mu)$ as follows.

For natural numbers $i$ and $n$, let

$$\overline{N}_n(A)_i = \{ f \in N_n(A) \mid \mu(t(f)) = i \}$$

where $t(f) = x_n$ if

$$f = (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \ldots x_n \xrightarrow{f_n}) .$$

Suppose each $\overline{N}_n(A)_i$ is finite. Define the formal power series $f_{\chi}(A, \mu)(t)$ over $\mathbb{Z}$ by

$$f_{\chi}(A, \mu)(t) = \sum_{i=0}^{\infty} (-1)^i \left( \sum_{n=0}^{i} (-1)^n \# \overline{N}_n(A)_i \right) t^i .$$

Then, define

$$\chi_{fil}(A, \mu) = f_{\chi}(A, \mu)|_{t=-1}$$

if $f_{\chi}(A, \mu)(t)$ is rational and has a non-vanishing denominator at $t = -1$. 

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**Theorem 3.8.** Suppose \((A, \mu_A)\) and \((B, \mu_B)\) are \(\mathbb{N}\)-filtered acyclic categories and \(b\) is an object of \(B\) and \(P : A \to B\) be a covering whose fiber is finite satisfying the equation \(\mu_A(x) = \mu_B(P(x))\) for any object \(x\) of \(A\). Then, \((A, \mu_A)\) has Euler characteristic \(\overline{\chi}(A, \mu_A)\) if and only if \(B\) has Euler characteristic \(\overline{\chi}(B, \mu_B)\). In this case, we have

\[
\overline{\chi}(A, \mu_A) = \chi(B, \mu_B)
\]

for any \(\mathbb{N}\)-filtration \(\mu\) of \(P^{-1}(b)\).

**Proof.** For any object \(b\) of \(B\) and \(x\) of \(P^{-1}(b)\), we have

\[
\mu^{-1}_A(i) = \prod_{b \in \mu^{-1}_B(i)} P^{-1}(b)
\]

for any \(i \geq 0\). Indeed, for any \(x\) of \(\mu^{-1}_A(i)\), \(\mu_B(P(x)) = \mu_A(x) = i\). Hence, \(P(x)\) belongs to \(\mu^{-1}_B(i)\). Moreover, \(x\) belongs to \(P^{-1}(P(x))\), so that \(x\) belongs to \(\prod_{b \in \mu^{-1}_B(i)} P^{-1}(b)\). Conversely, for any \(b\) of \(\mu^{-1}_B(i)\) and \(y\) of \(P^{-1}(b)\), \(y\) belongs to \(\mu^{-1}_A(i)\) since

\[
\mu_A(y) = \mu_B(P(y)) = \mu_B(b) = i.
\]

Hence, \(y\) belongs to \(\mu^{-1}_A(i)\). Proposition 25 implies

\[
\overline{N}_n(A)_i = \prod_{b \in \mu^{-1}_B(i)} \overline{N}_n(A)_x
\]

\[
= \prod_{b \in \mu^{-1}_B(i)} \overline{N}_n(A)_x
\]

\[
\cong \prod_{b \in \mu^{-1}_B(i)} \prod_{x \in P^{-1}(b)} \overline{N}_n(A)_x
\]

\[
\cong \prod_{b \in \mu^{-1}_B(i)} P^{-1}(b) \times \overline{N}_n(B)_b
\]

\[
\cong P^{-1}(b) \times \prod_{b \in \mu^{-1}_B(i)} \overline{N}_n(B)_b
\]

\[
= P^{-1}(b) \times \overline{N}_n(B)_i.
\]

Hence, we have

\[
f_X(A, \mu_A)(t) = \sum_{i=0}^{\infty} (-1)^i \left( \sum_{n=0}^{i} (-1)^n \overline{N}_n(A)_i \right) t^i
\]

\[
= \sum_{i=0}^{\infty} (-1)^i \left( \sum_{n=0}^{i} (-1)^n \overline{N}_n(B)_i \right) t^i
\]

\[
= \#P^{-1}(b) f_X(B, \mu_B)(t).
\]

Hence, there exists \(\overline{\chi}(A, \mu_A)\) if and only if the power series \(f_X(A, \mu_A)(t)\) is rational and \(-1\) can be substituted to the rational function if and only if the power series \(f_X(B, \mu_B)(t)\) is rational and \(-1\) can be substituted to the rational
function if and only if there exists \( \chi_{\text{fil}}(B, \mu_B) \). So the first claim is proven. If there exists \( \chi_{\text{fil}}(A, \mu_A) \), then we have

\[
\chi_{\text{fil}}(A, \mu_A) = \#P^{-1}(b)\chi_{\text{fil}}(B, \mu_B) = \chi_{\text{fil}}(P^{-1}(b), \mu)\chi_{\text{fil}}(B, \mu_B).
\]

It is clear that for any \( \mathbb{N} \)-filtration \( \mu \), \( \chi_{\text{fil}}(P^{-1}(b), \mu) = \#P^{-1}(b) \). We can give a filtration to \( P^{-1}(b) \), for example, define \( \mu : P^{-1}(b) \to \mathbb{N} \) by \( \mu(x) = 0 \) for any \( x \) of \( P^{-1}(b) \).

\[ \square \]

4 Examples

In this section, we give three examples of coverings of small categories.

**Example 4.1.** Let

\[
\Gamma = \begin{array}{c}
\text{x} \\
\downarrow \\
\text{y}
\end{array} \quad \begin{array}{c}
\text{f} \\
\downarrow \\
\text{f}^{-1}
\end{array}
\]

and \( B = \mathbb{Z}_2 = \{1, -1\} \). A group can be regarded as a category whose object is just one object \( * \) and morphisms are elements of \( G \) and composition is the operation of \( G \). Define \( P : \Gamma \to B \) by \( P(f) = P(f^{-1}) = -1 \). Then, \( P \) is a covering and this covering was studied in Example 5.33 of [FLSI11]. Since \( \Gamma \) and \( B \) are finite groupoids, Proposition 2.3 of [NogA] implies

\[
\zeta_{\Gamma}(z) = \frac{1}{(1 - 2z)^2}, \quad \zeta_{B}(z) = \frac{1}{1 - 2z}.
\]

The number of sheet of \( P \) is 2. We have \( \zeta_{\Gamma}(z) = \zeta_{B}(z)^2 \).

Applying the classifying space functor \( B \) to \( P \), we obtain the famous covering \( p_\infty : S^\infty \to \mathbb{R}P^\infty \)

\[
\begin{array}{c}
B\Gamma \\
\downarrow \\
BP
\end{array} \quad \begin{array}{c}
S^\infty \\
\downarrow \\
\mathbb{R}P^\infty
\end{array}
\]

For a finite groupoid \( G \), three Euler characteristic \( \chi_L \), \( \chi_\Sigma \) and \( \chi^{(2)} \) coincide and the value is

\[
\sum_{x \in \text{Ob}(G)/\approx} \frac{1}{\#\text{Aut}(x)}
\]

where this sum runs over all the isomorphism classes of objects of \( G \) (Example 2.7 of [Lei08], Theorem 3.2 of [BL08] and Example 5.12 of [FLSI11]). Hence, we have

\[
\chi_\Sigma(\Gamma) = 1, \quad \chi_\Sigma(B) = \frac{1}{2}, \quad \chi_\Sigma(P^{-1}(*) = 2
\]

and

\[
\chi_\Sigma(\Gamma) = \chi_\Sigma(P^{-1}(*))\chi_\Sigma(B).
\]
Example 4.2. Let

\[ A = y_1 y_2 y_3 \ldots y_n \]

and

\[ B = a \xrightarrow{h_1} b . \]

Define a functor \( P : A \to B \) by \( P(x_i) = a, P(y_i) = b, P(f_i) = h_1 \) and \( P(g_i) = h_2 \) for any \( i \). Then, \( P \) is a covering. By Proposition 2.9 of \([\text{NogA}]\), we have

\[ \zeta_A(z) = \frac{1}{(1-z)^2} \exp \left( \frac{2nz}{1-z} \right), \quad \zeta_B(z) = \frac{1}{(1-z)^2} \exp \left( \frac{2z}{1-z} \right) . \]

The number of sheet of \( P \) is \( n \). We have \( \zeta_A(z) = \zeta_B(z)^n \).

Applying the classifying space functor \( B \) to \( P \), we obtain the famous covering \( p^n : \mathbb{S}^1 \to \mathbb{S}^1 \) where

\[ \mathbb{S}^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \]

and \( p^n \) is the \( n \)-th power mapping. The map \( p^n \) is a covering (see \([\text{Hat02}]\), for example)

\[
\begin{array}{c}
BA \xrightarrow{\cong} \mathbb{S}^1 \\
\downarrow BP \\
BB \xrightarrow{\cong} \mathbb{S}^1.
\end{array}
\]

The two categories \( A \) and \( B \) are finite acyclic categories. For a finite acyclic category, four Euler characteristics \( \chi_L, \chi_\Sigma, \chi^{(2)} \) and \( \chi_{fil} \) coincide (see, for example, Introduction of \([\text{Nog}]\)). Furthermore, they coincide the Euler characteristic for cell complexes of the classifying space of an acyclic category (Proposition 2.11 of \([\text{Lei08}]\)). We have

\[ \chi_\Sigma(A) = 0, \chi_\Sigma(B) = 0, \chi_\Sigma(P^{-1}(a)) = 2 \]

and

\[ \chi_\Sigma(A) = \chi_\Sigma(P^{-1}(a))\chi_\Sigma(B) . \]

We introduce an example of a covering of infinite categories.

Example 4.3. Suppose

\[ A = x_0 \xrightarrow{x_1} x_1 \xrightarrow{x_2} x_2 \ldots \]

and

\[ B = b_0 \xrightarrow{b_1} b_1 \xrightarrow{b_2} b_2 \ldots \]
Where $A$ is a poset, that is, $A$ acyclic and each hom-set has at most exactly one morphism and for $n < m$ and $b_n$ and $b_m$, define

$$\Hom_B(b_n, b_m) = \{\phi^0_{n,m}, \phi^1_{n,m}\}$$

and a composition of $B$ is defined by $\phi^k_{m,t} \circ \phi^l_{n,m} = \phi^k_{n,t}$ where $k \equiv i + j \mod 2$ for $n < m < t$. Define $P : A \to B$ by $P(x_i) = P(y_i) = b_i$ and $P((x_n, x_m)) = P((y_n, y_m)) = \phi^0_{n,m}$ and $P((y_n, x_m)) = \phi^1_{n,m}$ for $n < m$. Then, $P$ is a covering. The indexes of objects of $A$ and $B$ give $\mathbb{N}$-filtrations $\mu_A$ and $\mu_B$ to $A$ and $B$, respectively. We have

$$f_X(A, \mu_A)(t) = \sum_{i=0}^{\infty} \left( \sum_{n=0}^{i} (-1)^{n+1} \binom{i}{n} \right) t^i$$

$$= 2 \sum_{i=0}^{\infty} t^i$$

$$= \frac{2}{1-t},$$

so that $\chi_{\text{fil}}(A, \mu_A) = 1$. We have

$$f_X(B, \mu_B)(t) = \sum_{i=0}^{\infty} \left( \sum_{n=0}^{i} (-1)^{n} 2^n \binom{i}{n} \right) t^i$$

$$= \sum_{i=0}^{\infty} t^i$$

$$= \frac{1}{1-t},$$

so that $\chi_{\text{fil}}(B, \mu_B) = \frac{1}{2}$. In fact, the categories $A$ and $B$ are the barycentric subdivision of $\Gamma$ of Example 4.1 and $\mathbb{Z}_2$ (see [Nog11] and [Nog]). Hence, Theorem 4.9 of [Nog11] and Example 4.1 imply their Euler characteristic $\chi_{\text{fil}}(A, \mu_A)$ and $\chi_{\text{fil}}(B, \mu_B)$. We obtain

$$\chi_{\text{fil}}(A, \mu_A) = \chi_{\text{fil}}(P^{-1}(b_0), \mu) \chi_{\text{fil}}(B, \mu_B)$$

for any $\mathbb{N}$-filtration $\mu$ of $P^{-1}(b)$.

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