G-Family Polynomials

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Abstract

We introduce two notions of quandle polynomials for G-families of quandles: the quandle polynomial of the associated quandle and a G-family polynomial with coefficients in the group ring of G. As an application we define image subquandle polynomial enhancements of the G-family counting invariant for trivalent spatial graphs and handlebody-links. We provide examples to show that the new enhancements are proper.

Keywords: Quandle polynomials, G-families of quandles, enhancements of counting invariants, handlebody-links, trivalent spatial graphs

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1 Introduction

In [7] and [8] algebraic structures known as quandles were introduced, with axioms derivable from the Reidemeister moves in knot theory. To every knot is associated a fundamental quandle, also called the knot quandle, whose isomorphism class determines the knot up to (possibly orientation-reversing) homeomorphism, making it an extremely powerful knot invariant. Direct comparison of knot quandles given by presentations is difficult, but given any finite quandle X the set of quandle homomorphisms from the knot quandle to X provides a number of easily computable knot and link invariants. In particular, cardinality of the homset is an integer-valued invariant known as the quandle counting invariant $\Phi^X_\mathbb{Z}(K)$, and stronger invariants which determine the counting invariant are known as enhancements. For more detail and further references, see [2].

In [4], the notion of a G-family of quandles for a group G and set X was introduced and applied to define invariants of handlebody-links and trivalent spatial graphs. A G-family of quandles induces a quandle structure on the product $G \times X$ known as the associated quandle, which raises an interesting question: given a quandle Q, for which G-families is Q the associated quandle?

In [9], the second listed author introduced the notion of quandle polynomials, two-variable polynomial invariants of finite quandles which reflect the distribution of trivial action throughout the quandle. These quandle invariants were then used to enhance the quandle counting invariants, providing a set of new knot and link invariants. Generalizations of the quandle polynomial were studied in [1, 10] and elsewhere.

In this paper we extend the quandle polynomial idea to the case of G-families of quandles. As an application we obtain new invariants of handlebody-links and trivalent spatial graphs. The paper is organized as follows. In Section 2 we review the basics of quandles and quandle polynomials. In Section 3 we review the basics of G-families of quandles and their handlebody-link and trivalent spatial graph invariants. In Section 4 we introduce our G-family polynomials and provide examples. As an application, in Section 5 we introduce new enhanced invariants of handlebody-links and trivalent spatial graphs using G-family polynomials. We end with some questions for future research in Section 6.

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2 Quandles and Quandle Polynomials

We begin with a definition; see [7, 8, 2] etc. for more.

**Definition 1.** Let $X$ be a set. A **quandle operation** or **quandle structure** on $X$ is a binary operation $\triangleright : X \times X \to X$ such that

(i) For all $x \in X$, we have $x \triangleright x = x$,

(ii) For all $x, y \in X$ there is a unique $z \in X$ such that $x = z \triangleright y$, and

(iii) For all $x, y, z \in X$, we have $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

These properties are known respectively as **idempotence**, **right-invertibility** and **right self-distributivity**. Axiom (ii) is equivalent to

(ii') There is a binary operation $\triangleright^{-1}$ on $X$ such that for all $x, y \in X$ we have $(x \triangleright y) \triangleright^{-1} y = x$ and $(x \triangleright^{-1} y) \triangleright y = x$.

A set $X$ with a choice of quandle operation is a **quandle**. If the $\triangleright^{-1}$ operation is the same as the $\triangleright$ operation, i.e. if for all $x, y \in X$ we have $(x \triangleright y) \triangleright^{-1} y = x$, we say $X$ is an **involutory quandle** or **kei**.

**Example 1.** Standard examples of quandles include

- The empty set $\emptyset$ is a quandle since the quandle axioms are not existentially quantified,
- Any singleton set $\{x\}$ is a quandle with $x \triangleright x = x$,
- More generally, any set $X$ is a quandle with the **trivial quandle operation** $x \triangleright y = x$ for all $x, y \in X$,
- Any group $G$ is a quandle with the **core quandle operation** $x \triangleright y = y^{-1}x^y$ for all $x, y \in X$ for a choice of $n \in \mathbb{Z}$,
- Any $\mathbb{Z}[t^{\pm 1}]$-module $M$ is a quandle with the **Alexander quandle operation** $x \triangleright y = ty + (1-t)y$ for all $x, y \in X$ for a choice of $n \in \mathbb{Z}$,
- Any vector space over a field of characteristic other than 2 with symplectic form $[,]$ is a quandle with the **symplectic quandle operation** $x \triangleright y = x + [x, y]y$ for all $x, y \in X$.

We note that disjoint unions of quandles acting trivially on each other form quandles and that every quandle can be decomposed as a disjoint union of orbit subquandles acting on each other (not necessarily trivially).

**Definition 2.** Let $X$ and $Y$ be quandles. A map $f : X \to Y$ is a **quandle homomorphism** if for all $x, x' \in X$ we have

$$f(x \triangleright x') = f(x) \triangleright f(x').$$

For finite quandles, a bijective quandle homomorphism is a **quandle isomorphism**, and we say two quandles $X, Y$ are **isomorphic** if there exists an isomorphism $f : X \to Y$.

Next, we review the notion of quandle polynomials. For more, see [2] or [9].

**Definition 3.** Let $X$ be a finite quandle. For each $x \in X$, define quantities $c(x)$ and $r(x)$ by

$$c(x) = \{|y \in X | y \triangleright x = y\}|$$

$$r(x) = \{|y \in X | x \triangleright y = x\}.$$

Then the **quandle polynomial** of $X$ is the two-variable polynomial

$$\phi(X) = \sum_{x \in X} t^{c(x)} s^{r(x)}.$$
Example 2. Let \( X = \{1, 2, 3\} \) have the quandle structure given by the operation table

\[
\begin{array}{c|ccc}
\triangleright & 1 & 2 & 3 \\
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 \\
3 & 3 & 3 & 3 \\
\end{array}
\]

To compute \( \phi(X) \), for each element of \( X \) we count the number of times the row number appears in the row and column corresponding to the element. In this case, looking in row 1 we see the row number 1 twice and

\[
\phi(X) = ts^3 + 2t^3s^2.
\]

In [9] we have the following result:

**Theorem 1.** If \( X \) and \( Y \) are isomorphic, then \( \phi(X) = \phi(Y) \).

In [9] it was shown by direct calculation that all quandles of order up to five were distinguished from each other by their quandle polynomials, raising the hope that the quandle polynomial might determine the quandle up to isomorphism class; however, in [1], examples were found of nonisomorphic quandles of order six with the same quandle polynomial. Quandle polynomials were generalized to rack polynomials in [1] and biquandle polynomials in [10]. In the remainder of this paper, we will extend quandle polynomials to the case of \( G \)-families of quandles.

3 \quad G\text{-Families of Quandles and Spatial Graphs}

We begin this section with a definition from [4].

**Definition 4.** Let \( G \) be a group and \( X \) a set. A \( G \)-family of quandles is a choice of quandle operation \( \triangleright^g \) on \( X \) for each element of \( G \) such that

(iv) For all \( x \in X \) and \( g \in G \), \( x \triangleright^g x = x \),

(v) For all \( x, y \in X \) and \( g, h \in G \),

\[
x \triangleright^{gh} y = (x \triangleright^g y) \triangleright^h y \quad \text{and} \quad x \triangleright^1 y = x
\]

where \( 1 \in G \) is the identity, and

(vi) For all \( x, y, z \in G \),

\[
(x \triangleright^g y) \triangleright^h z = (x \triangleright^h z) \triangleright^{h^{-1}gh} (y \triangleright^h z).
\]

**Example 3.** Any group \( G \) can be regarded as a \( G \)-family of singleton quandles \( X = \{x\} \) with \( x \triangleright^g x = x \) for all \( g \in G \).

**Example 4.** Any group \( G \) and set \( X \) defines a \( G \)-family of quandles by setting \( x \triangleright^g y = x \) for all \( x, y \in X \) and \( g \in G \); this is the trivial \( G \)-family on \( X \).

**Example 5.** A kei \( X \) can be completed to a \( \mathbb{Z}_2 \)-family of quandles where \( \mathbb{Z}_2 = \{1, t \mid t^2 = 1\} \) is the cyclic group of order 2 by including a trivial quandle on the same set. Let us define \( x \triangleright^1 y = x \) and \( x \triangleright^t y = x \triangleright y \) where \( \triangleright \) is the quandle operation of \( X \). Then we verify:

(iv) For all \( x \in X \), \( x \triangleright^1 x = x \) by definition of \( \triangleright^1 \) and \( x \triangleright^t x = x \) since \( X \) is a quandle,
(v) Let \( x, y \in X \). There are four cases to verify:

\[
\begin{align*}
\triangledown^{1(1)} y &= x \triangledown^{1} y = (x \triangledown^{1} y) \triangledown^{1} y, \\
\triangledown^{1(1)} y &= x \triangledown^{1} y = (x \triangledown^{1} y) \triangledown^{1} y, \\
\triangledown^{1(1)} y &= x \triangledown^{y} y = (x \triangledown^{1} y) \triangledown^{1} y \text{ and} \\
\triangledown^{1(1)} y &= x \triangledown^{1} y = x = (x \triangledown^{1} y) \triangledown^{1} y
\end{align*}
\]

where the last condition holds since \( X \) is a kei.

Thus, the notion of \( G \)-families of quandles can be considered a generalization of kei to larger groups \( G \).

**Example 6.** We can specify a finite \( G \)-family of quandles with operation tables for the group \( G \) and the operations \( \triangledown^{g} \). For example, the data

| \cdot | 1 | 2 | 3 | 4 | \triangledown^{1} | 1 | 2 | 3 | 4 | \triangledown^{2} | 1 | 2 | 3 | 4 | \triangledown^{3} | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|-----------|---|---|---|---|-----------|---|---|---|---|-----------|---|---|---|---|
| 1   | 1 | 1 | 1 | 1 | 1         | 2 | 3 | 4 | 1 |           | 2 | 3 | 4 | 1 |           | 1 | 1 | 3 | 4 | 2
| 2   | 2 | 2 | 2 | 2 | 2         | 3 | 2 | 4 | 1 |           | 4 | 2 | 1 | 3 |           | 2 | 4 | 3 | 1 |
| 3   | 3 | 3 | 3 | 3 | 3         | 4 | 1 | 3 | 2 |           | 4 | 2 | 3 | 1 |           | 3 | 1 | 2 | 4 |

defines a \( \mathbb{Z}_{3} \)-family of quandles.

As noted in [4], if \((G, X)\) is a \( G \)-family of quandles, then the set \( G \times X \) is a quandle under the operation

\[
(g, x) \triangledown^{g} (h, y) = (h^{-1} gh, x \triangledown^{h} y).
\]

This quandle structure is known as the associated quandle of the \( G \)-family \((G, X)\).

**Example 7.** Let \( G \) be a group and \( X = \{x\} \) a singleton quandle. Then there is an isomorphism of quandles between the associated quandle of the \( G \)-family \((G, X)\) and the conjugation quandle of \( G \) given by \( f(g, x) = g \):

\[
f((g, x) \triangledown^{g} (h, x)) = f(h^{-1} gh, x \triangledown^{h} x) = h^{-1} gh = g \triangledown^{h} h = f(g, x) \triangledown^{g} f(h, x).
\]

We note that not every quandle is the associated quandle of a \( G \)-family of quandles. For example, the quandle structure on the set \( X = \{1, 2, 3\} \) given by the operation table

\[
\begin{array}{cccc}
\triangledown & 1 & 2 & 3 \\
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 \\
3 & 3 & 3 & 3 \\
\end{array}
\]

is a quandle of cardinality 3, so we would need either \( G = \{1\} \) and \(|X| = 3 \) or \( G = \mathbb{Z}_{3} \) and \( X = \{x\} \) a singleton quandle. In the former case, the requirement of axiom (v) that \( x \triangledown^{1} y = x \) for all \( x, y \in X \) is contradicted and in the latter case, the condition

\[
(g, x) \triangledown^{g} (h, y) = (h^{-1} gh, x \triangledown^{h} y)
\]

says

\[
(g, x) \triangledown^{g} (h, x) = (\neg h + g + h, x) = (g, x)
\]

and the associated quandle is trivial.

**Example 8.** The \( \mathbb{Z}_{2} \)-family of quandles given by

\[
\begin{array}{cccc|cccc|cccc}
\cdot & 1 & 2 & \triangledown^{1} & 1 & 2 & 3 & \triangledown^{2} & 1 & 2 & 3 \\
1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 3 & 2 & 2 \\
2 & 2 & 1 & 2 & 2 & 2 & 2 & 3 & 2 & 1 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 1 & 3 & 1 \\
\end{array}
\]
Definition 5. Let \((G, X)\) be a \(G\)-family of quandles and \(Y \subset X\) a subset. We say \((G, Y)\) is a \(G\)-subfamily of quandles of \((G, X)\) if \((G, Y)\) is a \(G\)-family of quandles under the operations \(\triangleright^g\) inherited from \((G, X)\).

Example 9. In the \(\mathbb{Z}_2\)-family given by the operation tables

\[
\begin{array}{c|ccccccc}
\triangleright & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1 & 1 & 1 & 1 & 3 & 2 \\
2 & 2 & 2 & 2 & 3 & 2 & 1 \\
3 & 3 & 3 & 3 & 2 & 1 & 3 \\
4 & 4 & 4 & 4 & 4 & 6 & 5 \\
5 & 5 & 5 & 4 & 4 & 6 & 5 \\
6 & 6 & 6 & 6 & 5 & 4 & 6 \\
\end{array}
\]

we have \(G\)-subfamilies with \(X = \{1, 2\}\), \(X = \{1\}\), \(X = \{2\}\) and \(X = \{3\}\).

\(G\)-families of quandles were used in [4] for distinguishing \(Y\)-oriented trivalent spatial graphs and their quotient objects, handlebody-links. More precisely, let \(K\) be a diagram consisting of oriented classical crossings and trivalent vertices where we disallow sources and sinks.

Two trivalent spatial graph diagrams represent ambient isotopic spatial graphs if they are related by a sequence of the following moves:

Additionally allowing the \(IH\)-move

\[
\begin{array}{c|c|c}
& \text{IH} & \\
\hline
\text{IH} & \text{IH' } & \text{IH''}
\end{array}
\]

give us handlebody-links as a quotient of trivalent spatial graphs. See [4] for more.

Then given a \(G\)-family of quandles \((G, X)\), a \((G, X)\)-coloring of \(K\) is an assignment of a pair \((g, x)\) to
each arc in $K$ such that at crossings and vertices, the following conditions are satisfied:

Then as shown in [4], each $(G, X)$-coloring of $K$ before a Reidemeister move corresponds to a unique $(G, X)$-coloring of $K$ after the move, and we have:

**Theorem 2.** Let $(G, X)$ be a $G$-family of quandles. The cardinality of the set $C((G, X), K)$ of $(G, X)$-colorings of a $Y$-oriented trivalent spatial graph $K$ is an invariant of $Y$-oriented trivalent spatial graphs and $Y$-oriented handlebody-knots under ambient isotopy. This is known as the $G$-family counting invariant, denoted

$$\Phi^Z_{(G, X)}(K) = |C((G, X), K)|.$$ 

**Example 10.** Consider the theta graph, one of two unknotted trivalent spatial graphs with two vertices, and let $(G, X)$ be the $\mathbb{Z}_2$-family given by

| $G$ | 1 | 2 | $\triangleright^1$ | 1 | 2 | 3 | $\triangleright^2$ | 1 | 2 | 3 |
|-----|---|---|---------------|---|---|---|---------------|---|---|---|
| 1   | 1 | 2 | 1            | 1 | 1 | 1 | 1            | 3 | 3 | 3 |
| 2   | 2 | 1 | 2            | 2 | 2 | 2 | 2            | 3 | 2 | 1 |

Then there are 12 $X$-colorings of this graph as shown:

Thus, the $G$-family counting invariant for this theta graph with respect to $X$ is $\Phi^Z_{(G, X)}(\theta) = 12$. 
Recall that the *image subquandle* of a quandle homomorphism \( f : X \to Y \) is the set of elements \( y = f(x) \in Y \) which are the images of elements \( x \in X \) under \( f \).

**Definition 6.** The \( G \)-subfamily generated by the set of pairs \( (g, x) \in (G, X) \) appearing in a particular \((G, X)\)-coloring \( f \in C((G, X), K)\), i.e., the smallest \( G \)-subfamily of \((G, X)\) containing all the pairs \((g, x)\) in the coloring, is the *image \( G \)-subfamily* of the coloring, denoted \( \text{Im}(f) \).

**Example 11.** The image \( G \)-subfamilies for the colorings in Example 10 are \( \{(1, 1), (2, 1)\} \) for the four colorings in the top row, \( \{(1, 2), (2, 2)\} \) for the four colorings in the middle row and \( \{(1, 3), (2, 3)\} \) for the four colorings in the last row.

### 4 G-Family Polynomials

Let us now turn to the question of quandle polynomials for \( G \)-families of quandles. The simplest approach is to use the *associated quandle*. As observed in [4], given a \( G \)-family of quandles, there is a quandle structure on \( G \times X \) given by

\[
(g, x) \triangleright (g', x') = (g^{-1} g g', x \triangleright g x')
\]

known as the *associated quandle* of the \( G \)-family. We will denote this quandle structure by \( A(G, X) \). Since \( A(G, X) \) is a quandle, it has a quandle polynomial. Thus, we have:

**Definition 7.** Let \((G, X)\) be a \( G \)-family of quandles. The *associated quandle polynomial* of \((G, X)\), denoted \( \phi_{A(G, X)} \), is the quandle polynomial of \( A(G, X) \), i.e.,

\[
\sum_{(g, x) \in A(G, X)} t^{c(g,x)} s^{r(g,x)}.
\]

**Example 12.** The \( \mathbb{Z}_2 \)-family of quandles given by

\[
\begin{array}{c|cccc}
\cdot & 1 & 2 & \triangleright_2 & \triangleright_2 \\
1 & 1 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
\end{array}
\begin{array}{c|cccc}
| & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

has associated quandle

\[
\begin{array}{c|cccccccc}
\triangleright & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 7 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\end{array}
\]

and hence associated quandle polynomial \( \phi_{A(G, X)} = 3t^8s^8 + 4t^8s^7 + t^4s^8 \).

A second approach to quandle polynomials for \( G \)-families makes use of the \( G \)-family structure to get a two-variable polynomial with coefficients in the group ring \( \mathbb{Z}[G] \) over \( G \).

**Definition 8.** Let \((G, X)\) be a \( G \)-family of quandles. For each \( g \in G \), we have a quandle structure on \( X \) given by \( \triangleright^g \); let

\[
\begin{align*}
c_g(x) &= |\{y \in x \mid y \triangleright^g x = y\}| \\
r_g(x) &= |\{y \in x \mid x \triangleright^g y = x\}|.
\end{align*}
\]
The \( G \)-family polynomial, denoted \( \phi_{(G,X)} \), is given by
\[
\sum_{(g,x) \in (G,X)} g t^{[c_g(x)]} s^{[r_g(x)]}.
\]

**Remark 1.** For \( G \)-families of quandles presented via operation tables, we will specify elements of \( \mathbb{Z}[G,t,s] \) as \(|G|\)-tuples of polynomials in \( \mathbb{Z}[s,t] \). For example, if \( G = \{ g_1, g_2, g_3 \} \) then the polynomial
\[
\phi = 3g_1t^2s + 6g_2ts^2 + 2g_2ts + 4g_3
\]
will be expressed as
\[
[3t^2s, 6ts^2 + 2ts, 4].
\]

**Example 13.** Consider the \( \mathbb{Z}_3 \)-family of quandles with \( X = \{ 1, 2, 3, 4 \} \) given by
\[
\begin{array}{c|cccc}
\cdot & 1 & 2 & 3 & \\hline
1 & 1 & 2 & 3 & \\hline
2 & 2 & 3 & 1 & \\hline
3 & 3 & 1 & 2 & \\hline
\end{array}
\]
\[
\begin{array}{cccc|cccc}
| & b^1 & 1 & 2 & 3 & 4 & \\hline
1 & 1 & 1 & 1 & 1 & \\hline
2 & 2 & 2 & 2 & 2 & \\hline
3 & 3 & 3 & 3 & 3 & \\hline
\end{array}
\quad
\begin{array}{cccc|cccc}
| & b^2 & 1 & 2 & 3 & 4 & \\hline
1 & 1 & 3 & 4 & 2 & \\hline
2 & 2 & 4 & 1 & 3 & \\hline
3 & 3 & 2 & 4 & 3 & \\hline
\end{array}
\quad
\begin{array}{cccc|cccc}
| & b^3 & 1 & 2 & 3 & 4 & \\hline
1 & 1 & 4 & 2 & 3 & \\hline
2 & 2 & 3 & 4 & 1 & \\hline
3 & 3 & 1 & 3 & 2 & \\hline
\end{array}
\]
The element \( 1 \in X \) contributes \( g_1t^4s^4 + g_2ts + g_3t^4s^4 \) to the \( G \)-family polynomial; repeating for the other elements of \( X \), we have
\[
\phi = [4t^4s^4, 4ts, 4s4t].
\]

**Definition 9.** Let \((G,X)\) be a \( G \)-family of quandles and \((G,Y)\) a \( G \)-subfamily. Then:
- The associated subquandle polynomial of \((G,Y)\) is the subquandle polynomial \( \phi_{A(G,Y) \subset A(G,X)} \) of the associated quandle of \((G,Y)\) considered as a subquandle of the associated quandle of \((G,X)\), and
- The \( G \)-subfamily polynomial of \((G,Y)\), denoted \( \phi_{(G,Y) \subset (G,X)} \), is the sum of contributions of elements of \((G,Y)\) to \( \phi_{(G,X)} \), i.e.
\[
\phi_{(G,Y) \subset (G,X)} = \sum_{(g,x) \in (G,Y)} g t^{[c_g(x)]} s^{[r_g(x)]}.
\]

5 \hspace{1cm} \textbf{G-family Polynomial Enhancements}

We can now apply our definitions to enhance the \( G \)-family counting invariant.

**Definition 10.** Let \((G,X)\) be a \( G \)-family of finite quandles and let \( K \) be a \( Y \)-oriented trivalent spatial graph. Then we define the
- **Associated subquandle polynomial invariant** of \( K \) with respect to \((G,X)\) to be the multiset of associated subquandle polynomials of the image subquandles of each coloring, i.e.
\[
\Phi^A_{(G,X)}(K) = \{ \phi_{A(G,\text{Im}(f)) \subset A(G,X)} \mid f \in \mathcal{C}((G,X), K) \}
\]
and the
- **\( G \)-family subquandle polynomial invariant** of \( K \) with respect to \((G,X)\) to be the multiset of \( G \)-family subquandle polynomials of the image subquandles of each coloring, i.e.
\[
\Phi_{(G,X)}(K) = \{ \phi_{(G,\text{Im}(f)) \subset (G,X)} \mid f \in \mathcal{C}((G,X), K) \}
\]
We then state our main result:
Proposition 3. Let \((G,X)\) be a \(G\)-family of quandles. If two handlebody-link diagrams \(L\) and \(L'\) are related by Reidemeister moves, then \(\Phi_{(G,X)}(L) = \Phi_{(G,X)}(L')\) and \(\Phi^A_{(G,X)}(L) = \Phi^A_{(G,X)}(L')\).

Proof. It suffices to observe that the Reidemeister moves do not change the image \(G\)-subfamily and the image subquandle of the associated quandle for a \((G,X)\)-coloring of a handlebody-link diagram. 

Example 14. To illustrate the invariants, let us consider the trivalent spatial graph below representing handlebody-knot 4_1 with the \(\mathbb{Z}_2\)-family of quandles \(X\) given by 

\[
\begin{array}{cccccc}
1 & 1 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 3 & 3 & 1 \\
3 & 3 & 3 & 3 & 2 & 1 \\
\end{array}
\]

There are eighteen \((G,X)\)-colorings, including for example the two depicted here:

The coloring on the right is a trivial coloring, since all arcs have the same color; the coloring on the left is nontrivial.

The image subfamily of the coloring on the left is the entire \(G\)-family, while the image subfamily on the right is the subfamily \{\(1, 2\), \(2, 2\)\}. Thus the left coloring contributes \(3g_1t^3s^3 + 3g_2ts\) to the \(G\)-family subquandle polynomial invariant and the coloring on the right contributes \(g_1t^3s^3 + g_2ts\). Repeating for the other colorings, we obtain invariant value 

\[
\Phi^G_{(G,X)}(4_1) = \{12 \times [t^3s^3, ts], 6 \times [3t^3s^3, 3ts]\}.
\]

Example 15. Repeating the computation in Example 14 with the associated quandle 

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 \\
3 & 3 & 3 & 3 & 2 & 3 \\
4 & 4 & 4 & 4 & 4 & 6 \\
5 & 5 & 5 & 5 & 6 & 5 \\
6 & 6 & 6 & 6 & 5 & 4 \\
\end{array}
\]

of the \(G\)-family we obtain invariant value 

\[
\Phi^A_{(G,X)}(4_1) = \{3 \times t^6s^4, 9 \times (t^6s^4 + t^2s^4), 6 \times (3t^6s^4 + 3t^2s^4)\}.
\]
Example 16. Using our python code, we compute that the two handlebody-links

![Handlebody Links](image)

both have 324 \((G, X)\)-colorings by the \(S_3\)-family of quandles given by the operation tables

\[
\begin{array}{cccccc}
\cdot & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 2 & 3 & 4 & 5 \\
2 & 2 & 1 & 5 & 6 & 3 \\
3 & 3 & 6 & 1 & 5 & 4 \\
4 & 4 & 5 & 6 & 1 & 2 \\
5 & 5 & 4 & 2 & 3 & 6 \\
6 & 6 & 3 & 4 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\circ^1 & 1 & 2 & 3 & \circ^2 & 1 & 2 & 3 & \circ^3 & 1 & 2 & 3 & \circ^4 & 1 & 2 & 3 & \circ^5 & 1 & 2 & 3 & \circ^6 & 1 & 2 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

but are distinguished by their \(\Phi_{(G, X)}\)-values

\[
\begin{align*}
\Phi_{(X,G)}(L) &= \{162 \times [t^3t^3, ts, ts, t^3t^3, t^3t^3, 162 \times [3t^3s^3, 3ts, 3ts, 3ts, 3ts, 3ts, 3ts, 3ts, 3ts] \} \\
\Phi_{(X,G)}(L') &= \{108 \times [t^3s^3, ts, ts, t^3s^3, t^3s^3, 216 \times [3t^3s^3, 3ts, 3ts, 3ts, 3ts, 3ts, 3ts, 3ts, 3ts] \} \\
\end{align*}
\]

as well as by their \(\Phi^4_{(G, X)}\)-values

\[
\begin{align*}
\Phi^4_{(X,G)}(L) &= \{3 \times t^{18}s^{12}, 27 \times (t^{18}s^{12} + t^2s^4), 18 \times (t^{18}s^{12} + t^3s^3), 18 \times (t^{18}s^{12} + 3t^2s^4), \\
&\quad 36 \times (t^{18}s^{12} + 2t^9s^9 + 3t^2s^4), 54 \times (3t^{18}s^{12} + 3t^2s^4), 54 \times (2t^9s^9 + 3t^2s^4), \\
&\quad 108 \times (6t^9s^9 + 9t^2s^4), 6 \times (t^{18}s^{12} + 2t^3s^3) \} \\
\Phi^4_{(X,G)}(L') &= \{3 \times t^{18}s^{12}, 6 \times 2t^{18}s^{12}, 27 \times (t^{18}s^{12} + t^2s^4), 30 \times (t^{18}s^{12} + t^3s^3), \\
&\quad 18 \times (2t^{18}s^{12} + t^2s^4), 18 \times (3t^{18}s^{12} + t^3s^3), 18 \times (t^{18}s^{12} + 2t^3s^3), \\
&\quad 72 \times (t^{18}s^{12} + 2t^9s^9 + 3t^2s^4), 54 \times (3t^{18}s^{12} + 3t^2s^4), \\
&\quad 12 \times (2t^{18}s^{12} + 2t^9s^9), 30 \times (t^{18}s^{12} + 2t^3s^3), \\
&\quad 36 \times (2t^{18}s^{12} + 2t^9s^9 + 3t^2s^4) \} \\
\end{align*}
\]

In particular, this example shows that both \(\Phi_{(X,G)}\) and \(\Phi^4_{(X,G)}\) are not determined by the the number of \((G, X)\)-colorings and hence are proper enhancements.
Example 17. We selected an $S_3$-family of quandles with group and quandle tables given by

\[
\begin{array}{cccccc}
\cdot & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 2 & 1 & 5 & 6 & 3 & 4 \\
3 & 3 & 6 & 1 & 5 & 4 & 2 \\
4 & 4 & 5 & 6 & 1 & 2 & 3 \\
5 & 5 & 4 & 2 & 3 & 6 & 1 \\
6 & 6 & 3 & 4 & 2 & 1 & 5 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\vartriangleright^1 & 1 & 2 & 3 & 4 & 5 & 6 \\
\vartriangleright^2 & 1 & 1 & 1 & 1 & 1 & 2 \\
\vartriangleright^3 & 1 & 1 & 3 & 1 & 4 & 4 \\
\vartriangleright^4 & 1 & 1 & 4 & 3 & 2 & 3 \\
\vartriangleright^5 & 1 & 1 & 4 & 2 & 1 & 3 \\
\vartriangleright^6 & 1 & 1 & 4 & 2 & 1 & 3 \\
\end{array}
\]

and computed the $G$-family polynomial enhancement for each of the genus 2 handlebody-knots in the table in [6]. The results are collected in the tables.

\[
\begin{array}{c|c}
K & \Phi_{(G,X)}(K) \\
\hline
41 & \{216 \times [t^4s^4, t^2s^4, t^2s^2, t^2s^2, ts, ts], 96 \times [4t^4s^4, 4t^2s^2, 4t^2s^2, 4ts, 4ts]\} \\
51 & \{24 \times [t^4s^4, t^2s^4, t^2s^2, t^2s^2, ts, ts]\} \\
52 & \{216 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts], 96 \times [4t^4s^4, 4t^2s^2, 4t^2s^2, 4ts, 4ts]\} \\
53 & \{48 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
54 & \{144 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
61 & \{144 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts], 48 \times [4t^4s^4, 4t^2s^2, 4t^2s^2, 4ts, 4ts]\}, \\
62 & \{120 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts], 24 \times [4t^4s^4, 4t^2s^2, 4t^2s^2, 4ts, 4ts]\} \\
63 & \{24 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
64 & \{72 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
65 & \{24 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
66 & \{24 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
67 & \{24 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
68 & \{24 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
69 & \{72 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
610 & \{216 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts], 72 \times [4t^4s^4, 4t^2s^2, 4t^2s^2, 4ts, 4ts]\} \\
611 & \{48 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
612 & \{24 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
613 & \{144 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts]\} \\
614 & \{144 \times [t^4s^4, t^2s^2, t^2s^2, t^2s^2, ts, ts], 72 \times [4t^4s^4, 4t^2s^2, 4t^2s^2, 4ts, 4ts]\}. \\
\end{array}
\]

In particular, $\Phi_{(G,X)}$ distinguishes the handlebody-knot $6_2$ from $6_{13}$ and $5_3$ despite the counting invariants being equal. We further note that with this $S_3$-family of quandles, the associated quandle version of our invariant further distinguishes handlebody-knots $4_1$ and $5_2$ despite both having the same number of colorings.

\[
\begin{array}{c|c}
K & \Phi_{(G,X)}^3(K) \\
\hline
41 & \{4 \times t^{24}s^{12}, 36 \times (t^{24}s^{12} + t^4s^6), 24 \times (t^{24}s^{12} + t^4s^6), 24 \times (t^{24}s^{12} + t^4s^6), 48 \times (t^{24}s^{12} + 3t^4s^6), 24 \times (t^{24}s^{12} + 3t^4s^6), 48 \times (3t^{24}s^{12} + 3t^4s^6), 48 \times (4t^{24}s^{12} + 12t^4s^6 + 8t^3s^6), 72 \times (3t^4s^6 + 2t^3s^6), 8 \times (t^{24}s^{12} + 2t^3s^6), 24 \times (4t^{24}s^{12} + 4t^3s^6)\} \\
52 & \{4 \times t^{24}s^{12}, 32 \times (t^{24}s^{12} + t^4s^6), 24 \times (t^{24}s^{12} + 3t^4s^6), 16 \times (t^{24}s^{12} + t^3s^6), 24 \times (t^{24}s^{12} + 3t^4s^6), 120 \times (3t^4s^6 + 2t^3s^6), 48 \times (12t^4s^6 + 8t^3s^6), 8 \times (t^{24}s^{12} + 2t^3s^6), 8 \times 2t^3s^6, 24 \times (8t^3s^6)\} \\
\end{array}
\]

6 Questions

We conclude with some questions for future research.
• $G$-families of quandles have a number of generalizations such as multiple conjugation quandles\cite{4} and multiple conjugation biquandles\cite{5}. $G$-family polynomials should be likewise extendable to these structures.

• What additional enhancements of the $G$-family counting invariant are possible, either alone or in combination with $G$-family polynomials?

• What further refinements can be made to the $G$-family polynomial definition, perhaps allowing for subgroups for $G$?

• In\cite{4} the notion of a $Q$-family of quandles for a quandle $Q$ is also introduced, suggesting an obvious notion of $Q$-family polynomials; these should give invariants analogous to the ones in this paper.

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