Gabriel-Quillen embedding for $n$-exact categories

Ramin Ebrahimi$^{a,b}$

$^a$Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, Isfahan, Iran; $^b$School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran

ABSTRACT
Our first aim is to provide an analog of the Gabriel-Quillen embedding theorem for $n$-exact categories. Also we give an example of an $n$-exact category that is not an $n$-cluster tilting subcategory, and we suggest two possible ways for realizing $n$-exact categories as $n$-cluster tilting subcategory.

1. Introduction
Higher Auslander-Reiten theory was introduced by Iyama in [4, 5]. It deals with $n$-cluster tilting subcategories of abelian and exact categories. Recently, Jasso [7] introduced $n$-abelian and $n$-exact categories as a higher-dimensional analogs of abelian and exact categories, they are axiomatization of $n$-cluster tilting subcategories. Jasso proved that each $n$-cluster tilting subcategory of an abelian (res, exact) category is $n$-abelian (res, $n$-exact).

In [2] and [9], independently it has been shown that any small $n$-abelian category is equivalent to an $n$-cluster tilting subcategory of an abelian category. This note is an attempt to generalize this result for $n$-exact categories. We give an example of an $n$-exact category that is not equivalent to an $n$-cluster tilting subcategory, so we have to use a different strategy for realizing $n$-exact categories as $n$-cluster tilting subcategories.

Let $\mathcal{M}$ be a small $n$-exact category. We denote by $\text{Mod}\mathcal{M}$ the category of all additive contravariant functors from $\mathcal{M}$ to the category of all abelian groups. Let $\text{Eff}(\mathcal{M})$ be the subcategory of weakly effaceable functors, parallel to the proof of Gabriel-Quillen embedding theorem we will show that composition of the Yoneda functor with the localization functor

$$\mathcal{M} \xrightarrow{Y} \text{Mod}\mathcal{M} \xrightarrow{\mathcal{q}} \frac{\text{Mod}\mathcal{M}}{\text{Eff}(\mathcal{M})}$$

sends $n$-exact sequences in $\mathcal{M}$ to exact sequences in $\mathcal{A} = \frac{\text{Mod}\mathcal{M}}{\text{Eff}(\mathcal{M})}$. Furthermore we will show that the essential image is $n$-rigid in $\mathcal{A}$. In the end we suggest two possible ways for realizing $n$-exact categories as $n$-cluster tilting subcategory.
In section 2 we recall the definitions of $n$-exact categories, $n$-cluster tilting subcategories and some of their basic properties. And we give an example of an $n$-exact category that is not an $n$-cluster tilting subcategory. In section 3 after recalling some results from localization theory of abelian categories, we construct the embedding $\mathcal{M} \hookrightarrow \mathcal{A} = \text{Mod} \mathcal{M} / \text{Eff} \mathcal{M}$ with desired properties. We end with a question that by results of this paper it make sense to has positive answer.

1.1. Notation

Throughout this paper, unless otherwise stated, $n$ always denotes a fixed positive integer and $\mathcal{M}$ is a fixed small $n$-exact category.

2. Preliminaries

In this section we recall the definition of $n$-exact category and $n$-cluster tilting subcategory. And we give an example of an $n$-exact category that can’t be an $n$-cluster tilting subcategory of an exact category.

2.1. $n$-Exact categories

Let $\mathcal{M}$ be an additive category and $f : A \to B$ a morphism in $\mathcal{M}$. A weak cokernel of $f$ is a morphism $g : B \to C$ such that for all $C' \in \mathcal{M}$ the sequence of abelian groups

$$\text{Hom}(C, C') \xrightarrow{(g, C')} \text{Hom}(B, C') \xrightarrow{(f, C')} \text{Hom}(A, C')$$

is exact. The concept of a weak kernel is defined dually.

Let $d^0 : X^0 \to X^1$ be a morphism in $\mathcal{M}$. An $n$-cokernel of $d^0$ is a sequence

$$(d^1, \ldots, d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots \xrightarrow{d^n} X^n \xrightarrow{d^n} X^{n+1}$$

of objects and morphisms in $\mathcal{M}$ such that for each $Y \in \mathcal{M}$ the induced sequence of abelian groups

$$0 \to \text{Hom}(X^{n+1}, Y) \to \text{Hom}(X^n, Y) \to \cdots \to \text{Hom}(X^1, Y) \to \text{Hom}(X^0, Y)$$

is exact. Equivalently, the sequence $(d^1, \ldots, d^n)$ is an $n$-cokernel of $d^0$ if for all $1 \leq k \leq n - 1$ the morphism $d^k$ is a weak cokernel of $d^{k-1}$, and $d^n$ is moreover a cokernel of $d^{n-1}$ [7, Definition 2.2]. The concept of an $n$-kernel of a morphism is defined dually.

Definition 2.1. Let $\mathcal{M}$ be an additive category. A left $n$-exact sequence in $\mathcal{M}$ is a complex

$$X^0 \xrightarrow{d^0} X^1 \to \cdots \to X^n \xrightarrow{d^n} X^{n+1}$$

such that $(d^0, \ldots, d^{n-1})$ is an $n$-kernel of $d^n$. The concept of right $n$-exact sequence is defined dually. An $n$-exact sequence is a sequence which is both a right $n$-exact sequence and a left $n$-exact sequence.

Let

$$\begin{array}{cccccccc}
X & \xrightarrow{d^0_x} & X^1 & \xrightarrow{d^1_x} & \cdots & \xrightarrow{d^{n-2}_x} & X^{n-1} & \xrightarrow{d^{n-1}_x} & X^n \\
Y & \xrightarrow{d^0_y} & Y^1 & \xrightarrow{d^1_y} & \cdots & \xrightarrow{d^{n-2}_y} & Y^{n-1} & \xrightarrow{d^{n-1}_y} & Y^n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^0 & \xrightarrow{f^0} & X^1 & \xrightarrow{f^1} & \cdots & \xrightarrow{f^{n-2}} & X^{n-1} & \xrightarrow{f^{n-1}} & X^n \\
Y^0 & \xrightarrow{f^0} & Y^1 & \xrightarrow{f^1} & \cdots & \xrightarrow{f^{n-2}} & Y^{n-1} & \xrightarrow{f^{n-1}} & Y^n 
\end{array}$$
be a morphism of complexes in an additive category. The mapping cone \( C = C(f) \) is the complex
\[
\begin{align*}
X^0 &\xrightarrow{d_C^1} X^1 \oplus Y^0 \xrightarrow{d_C^0} X^2 \oplus Y^1 \xrightarrow{d_C^{n-2}} \cdots \xrightarrow{d_C^1} X^n \oplus Y^{n-1} \xrightarrow{d_C^{n-1}} Y^n,
\end{align*}
\]
where
\[
d_C^k := \begin{pmatrix}
-d_X^{k+1} & 0 \\
-f_Y^{k+1} & d_Y^k
\end{pmatrix} : X^{k+1} \oplus Y^k \rightarrow X^{k+2} \oplus Y^{k+1}
\]
for each \( k \in \{-1, 0, \ldots, n-1\} \). In particular \( d_C^{n-1} = (f^n,d_Y^{n-1}) \).

- We say that the above diagram is an \( n \)-pull back of \( Y \) along \( f^n \) if (2.1) is a left \( n \)-exact sequence.
- We say that the above diagram is an \( n \)-push out of \( X \) along \( f^0 \) if (2.1) is a right \( n \)-exact sequence.

Let \( A \) be an additive category and \( B \) be a full subcategory of \( A \). \( B \) is called covariantly finite in \( A \) if for every \( A \in A \) there exists an object \( B \in B \) and a morphism \( f : A \rightarrow B \) such that, for all \( B' \in B \), the sequence of abelian groups \( \text{Hom}_A(B',B') \rightarrow \text{Hom}_A(A,B') \rightarrow 0 \) is exact. Such a morphism \( f \) is called a left \( B \)-approximation of \( A \). The notions of contravariantly finite subcategory of \( A \) and right \( B \)-approximation are defined dually. A functorially finite subcategory of \( A \) is a subcategory which is both covariantly and contravariantly finite in \( A \).

Let \( X \) and \( Y \) be two \( n \)-exact sequences. We remained that a morphism \( f : X \rightarrow Y \) of \( n \)-exact sequences is a morphism of complexes. We say that a morphism \( f : X \rightarrow Y \) of \( n \)-exact sequences is a weak isomorphism if \( f^k \) and \( f^{k+1} \) are isomorphisms for some \( k \in \{0,1,\ldots,n+1\} \) with \( n + 2 := 0 \).

**Definition 2.2.** ([7, Definition 4.2]) Let \( \mathcal{M} \) be an additive category. An \( n \)-exact structure on \( \mathcal{M} \) is a class \( \mathcal{X} \) of \( n \)-exact sequences in \( \mathcal{M} \), closed under weak isomorphisms of \( n \)-exact sequences, and which satisfies the following axioms:

\begin{itemize}
\item[(E0)] The sequence \( 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \) is an \( \mathcal{X} \)-admissible \( n \)-exact sequence.
\item[(E1)] The class of \( \mathcal{X} \)-admissible monomorphisms is closed under composition.
\item[(E1\textsuperscript{op})] The class of \( \mathcal{X} \)-admissible epimorphisms is closed under composition.
\item[(E2)] For each \( \mathcal{X} \)-admissible \( n \)-exact sequence \( X \) and each morphism \( f : X^0 \rightarrow Y^0 \), there exists an \( n \)-pushout diagram of \( (d_X^0, \ldots, d_X^{n-1}) \) along \( f \) such that \( d_Y^n \) is an \( \mathcal{X} \)-admissible monomorphism. The situation is illustrated in the following commutative diagram:
\[
\begin{array}{ccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
\downarrow f & & \downarrow d_Y^0 & & \downarrow d_Y^1 & & \cdots & & \downarrow d_Y^{n-1} & \downarrow \ \\
Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & \cdots
\end{array}
\]
\item[(E2\textsuperscript{op})] For each \( \mathcal{X} \)-admissible \( n \)-exact sequence \( Y \) and each morphism \( g : X^{n+1} \rightarrow Y^{n+1} \), there exists an \( n \)-pull back diagram of \( (d_Y^1, \ldots, d_Y^n) \) along \( g \) such that \( d_Y^n \) is an \( \mathcal{X} \)-admissible epimorphism. The situation is illustrated in the following commutative diagram:
\[
\begin{array}{ccccccc}
X^n & \xrightarrow{d_Y^n} & Y^n & \xrightarrow{d_Y^0} & \cdots & \xrightarrow{d_Y^{n-2}} & Y^1 & \xrightarrow{d_Y^1} & Y^0 \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}
\]
\end{itemize}
An $n$-exact category is a pair $(\mathcal{M}, \mathcal{X})$ where $\mathcal{M}$ is an additive category and $\mathcal{X}$ is an $n$-exact structure on $\mathcal{M}$. If the class $\mathcal{X}$ is clear from the context, we identify $\mathcal{M}$ with the pair $(\mathcal{M}, \mathcal{X})$.

The members of $\mathcal{X}$ are called $\mathcal{X}$-admissible $n$-exact sequences, or simply admissible $n$-exact sequences when $\mathcal{X}$ is clear from the context. Furthermore, if

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^n} X^n \xrightarrow{d^{n+1}} X^{n+1}$$

is an admissible $n$-exact sequence, $d^0$ is called admissible monomorphism and $d^n$ is called admissible epimorphism.

**Definition 2.3.** (cf. [7, Definition 4.13]) Let $(\mathcal{E}, \mathcal{X})$ be an exact category and $\mathcal{M}$ a subcategory of $\mathcal{E}$. $\mathcal{M}$ is called an $n$-cluster tilting subcategory of $(\mathcal{E}, \mathcal{X})$ if the following conditions are satisfied.

(i) Every object $E \in \mathcal{E}$ has a left $\mathcal{M}$-approximation by an $\mathcal{X}$-admissible monomorphism $E \rightarrowtail M$.

(ii) Every object $E \in \mathcal{E}$ has a right $\mathcal{M}$-approximation by an $\mathcal{X}$-admissible epimorphism $M \twoheadrightarrow E$.

(iii) We have

$$\mathcal{M} = \{ E \in \mathcal{E} \mid \forall i \in \{1, \ldots, n-1\}, \text{Ext}^i_{\mathcal{E}}(E, \mathcal{M}) = 0 \}$$

$$= \{ E \in \mathcal{E} \mid \forall i \in \{1, \ldots, n-1\}, \text{Ext}^i_{\mathcal{E}}(\mathcal{M}, E) = 0 \}.$$

Note that $\mathcal{E}$ itself is the unique 1-cluster tilting subcategory of $\mathcal{E}$.

A full subcategory $\mathcal{M}$ of an exact or abelian category $\mathcal{E}$ is called $n$-rigid, if for every two objects $M, N \in \mathcal{M}$ and for every $k \in \{1, \ldots, n-1\}$, we have $\text{Ext}^k_{\mathcal{E}}(\mathcal{M}, \mathcal{M}) = 0$. Any $n$-cluster tilting subcategory $\mathcal{M}$ of an exact category $\mathcal{E}$ is $n$-rigid.

The following theorem gives the main source of $n$-exact categories.

**Theorem 2.4.** (cf. [7, Theorem 4.14]) Let $(\mathcal{E}, \mathcal{X})$ be an exact category and $\mathcal{M}$ be an $n$-cluster tilting subcategory of $(\mathcal{E}, \mathcal{X})$. Let $\mathcal{Y} = \mathcal{Y}(\mathcal{M}, \mathcal{X})$ be the class of all $\mathcal{X}$-acyclic complexes

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^n} X^n \xrightarrow{d^{n+1}} X^{n+1}$$

such that for all $k \in \{0, 1, \ldots, n+1\}$ we have $X^k \in \mathcal{M}$. Then $(\mathcal{M}, \mathcal{Y})$ is an $n$-exact category.

Let $\mathcal{M}$ be an additive category and $M$ be an object of $\mathcal{M}$. A morphism $e \in \mathcal{M}(M, M)$ is called idempotent if $e^2 = e$. $\mathcal{M}$ is called idempotent complete if for every idempotent $e \in \mathcal{M}(M, M)$ there exist an object $N$ and morphisms $f \in \mathcal{M}(M, N)$ and $g \in \mathcal{M}(N, M)$ such that $gf = e$ and $fg = 1_N$. Assume that $r : M \rightarrow M'$ is a retraction with section $s : M' \rightarrow M$. Then $sr : M \rightarrow M$ is an idempotent. It is well known that if $r : M \rightarrow M'$ has a kernel $k : K \rightarrow M$, this idempotent splits and there is a canonical isomorphism $M \cong K \oplus M'$ [1].

In abelian categories all retractions have kernels, but in exact categories this does not happen in general. An exact category where all retractions have kernels are called weakly idempotent.
complete [1]. But it is obvious that any admissible epimorphism in an exact category, that is a retraction has a kernel.

Let \( \mathcal{M} \) be an \( n \)-cluster tilting subcategory of an exact category \( (\mathcal{E}, \mathcal{X}) \), and \( \mathcal{Y} \) be the class of all \( \mathcal{X} \)-acyclic complexes

\[
X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \ldots \xrightarrow{d^n} X^{n+1}
\]
such that for all \( k \in \{0, 1, \ldots, n + 1\} \) we have \( X^k \in \mathcal{M} \). By Theorem 2.4, \( (\mathcal{M}, \mathcal{Y}) \) is an \( n \)-exact category. If \( M, N \in \mathcal{M} \), a morphism \( f : M \to N \) is \( \mathcal{Y} \)-admissible epimorphism if and only if it is \( \mathcal{X} \)-admissible epimorphism [7]. Thus if \( f : M \to N \) is an \( \mathcal{Y} \)-admissible epimorphism that is a retraction with section \( g : N \to M \), the idempotent \( gf : M \to M \) splits and \( M \cong N \oplus \text{Ker}(f) \). By the definition of \( n \)-cluster tilting subcategory, \( \text{Ker}(f) \in \mathcal{M} \).

**Example 2.5.** ([7, Theorem 3.5]) Let \( n \geq 2 \) and \( K \) be a field. Consider the full subcategory \( \mathcal{V} \) of \( \text{mod} \ K \) given by the finite dimensional \( K \)-vector spaces of dimension different from 1. Then it has been shown in [6, Example 3.5] that \( \mathcal{V} \) is not idempotent complete, but it satisfies other axioms of \( n \)-abelian category. By a similar argument the class of all exact sequences with \( n + 2 \) term is an \( n \)-exact structure on \( \mathcal{V} \). But there exist an admissible epimorphism \( K^3 \to K^2 \) which is a retraction, that doesn’t give a splitting of \( K^3 \). Thus \( \mathcal{V} \) can’t be an \( n \)-cluster tilting subcategory. Note that we can consider \( \mathcal{V} \) as an \( n \)-cluster tilting subcategory of itself, but in this case the induced \( n \)-exact structure is different than the class of all exact sequences in \( \text{mod} \ K \).

### 3. Embeddings into abelian categories

Let \( \mathcal{M} \) be a small \( n \)-exact category. In this section we find an abelian category \( \mathcal{A} \) and an embedding \( H : \mathcal{M} \to \mathcal{A} \), such that \( H \) sends \( n \)-exact sequences in \( \mathcal{M} \) to exact sequences in \( \mathcal{A} \). Furthermore we will show that the essential image of \( H \) is \( n \)-rigid in \( \mathcal{A} \).

First we recall localization theory of abelian categories, for reader can find proof in textbooks or Gabriel thesis [3]. Let \( \mathcal{A} \) be an abelian category. A subcategory \( \mathcal{C} \) of \( \mathcal{A} \) is called a **Serre subcategory** if for any exact sequence

\[
0 \to A_1 \to A_2 \to A_3 \to 0
\]

we have that \( A_2 \in \mathcal{C} \) if and only if \( A_1 \in \mathcal{C} \) and \( A_3 \in \mathcal{C} \). In this case we have the quotient category \( \mathcal{A}/\mathcal{C} \) that is by definition localization of \( \mathcal{A} \) with respect to the class of all morphisms \( f : X \to Y \) such that \( \text{Ker}(f), \text{Coker}(f) \in \mathcal{C} \).

**Theorem 3.1.** Let \( \mathcal{C} \) be a Serre subcategory of \( \mathcal{A} \), and let \( q : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) denote the canonical functor to the localization. The following statements hold:

(i) \( \mathcal{A}/\mathcal{C} \) is an abelian category and \( q \) is an exact functor.

(ii) \( q(C) = 0 \) for all \( C \in \mathcal{C} \), and any exact functor \( F : \mathcal{A} \to \mathcal{D} \) annihilating \( \mathcal{C} \) where \( \mathcal{D} \) is abelian must factor uniquely through \( q \).

A Serre subcategory \( \mathcal{C} \subseteq \mathcal{A} \) is called a **localizing subcategory** if the canonical functor \( q : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) admits a right adjoint \( r : \mathcal{A}/\mathcal{C} \to \mathcal{A} \). The right adjoint \( r \) is called the **section functor**, which always is fully faithful. Note that a localizing subcategory is closed under all coproducts which exist in \( \mathcal{A} \). The converse is true for Grothendieck categories, indeed we have the following result.

**Theorem 3.2.** Let \( \mathcal{C} \) be a Serre subcategory of a Grothendieck category \( \mathcal{A} \). The following statements hold:

(i) \( \mathcal{C} \) is a localizing subcategory if and only if it is closed under coproducts.

(ii) In this case the quotient category \( \mathcal{A}/\mathcal{C} \) is a Grothendieck category.
Let $C$ be a Serre subcategory of an abelian category $A$. Recall that an object $A \in A$ is called $C$-closed if for every morphism $f : X \to Y$ with $\text{Ker}(f) \in C$ and $\text{Coker}(f) \in C$ we have that $\text{Hom}_A(f, A)$ is bijective. Denote by $C^\perp$ the full subcategory of all $C$-closed objects. The following result is well known.

**Theorem 3.3.** Let $C$ be a Serre subcategory of an abelian category $A$. The following statements hold:

(i) We have

$$C^\perp = \{ A \in A \mid \text{Hom}(C, A) = 0 = \text{Ext}^1(C, A) \}.$$

(ii) For $A \in A$ and $B \in C^\perp$, the natural homomorphism $q_{A,B} : \text{Hom}_A(A, B) \to \text{Hom}_A(q(A), q(B))$ is an isomorphism.

(iii) If $C$ is a localizing subcategory, the restriction $q : C^\perp \to \underline{\text{C}}$ is an equivalence of categories.

(iv) If $C$ is localizing and $A$ has injective envelopes, then $C^\perp$ has injective envelopes and the inclusion functor $C^\perp \to A$ preserves injective envelopes.

We also need the following technical lemma.

**Lemma 3.4.** Let $0 \to A \to L \to M \to 0$ be an exact sequence in $A$ with $L \in C^\perp$, then $A \in C^\perp$ if and only if $\text{Hom}(C, M) = 0$.

Now we want to apply the above general results to $\text{Mod}M$, where $M$ is a small $n$-exact category. Recall that $\text{Mod}M$ is the category of all additive contravariant functors from $M$ to the category of all abelian groups. It is an abelian category with all limits and colimits, which are defined point-wise. Also by the Yoneda’s lemma, representable functors are projective and the direct sum of all representable functors $\Sigma_{X \in M} \text{Hom}(\cdot, X)$, is a generator for $\text{Mod}M$. Thus $\text{Mod}M$ is a Grothendieck category.

A functor $F \in \text{Mod}M$ is called weakly effaceable, if for each object $X \in M$ and $x \in F(X)$ there exists an admissible epimorphism $f : Y \to X$ such that $F(f)(x) = 0$. We denote by $\text{Eff}(M)$ the full subcategory of all weakly effaceable functors. For each $k \in \{1, \ldots, n\}$ we denote by $L_k(M)$ the full subcategory of $\text{Mod}M$ consist of all functors like $F$ such that for every $n$-exact sequence

$$X^0 \to X^1 \to \cdots \to X^n \to X^{n+1}$$

the sequence of abelian groups

$$0 \to F(X^{n+1}) \to F(X^n) \to \cdots \to F(X^{n-k})$$

is exact. Also for a Serre subcategory $C$ of an abelian category $A$ we set $C^{\perp_1} = \{ A \in A \mid \text{Ext}^{0,\ldots,k}(C, A) = 0 \}$. Note that $C^{\perp_1} = C^\perp$ by **Theorem 3.3**.

**Proposition 3.5.**

(i) $\text{Eff}(M)$ is a localizing subcategory of $\text{Mod}M$.

(ii) $\text{Eff}(M)^\perp = L_1(M)$.

**Proof.**

(i) We need to show that $\text{Eff}(M)$ is a Serre subcategory closed under coproducts, because $\text{Mod}M$ is a Grothendieck category. The proof is similar to the classical case of exact categories. We only prove that $\text{Eff}(M)$ is closed under extensions. Let

$$0 \to F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_3 \to 0$$

be a short exact sequence in $\text{Mod}M$ and $F_1, F_3 \in \text{Eff}(M)$. We want to show that $F_2 \in \text{Eff}(M)$. Let $X \in M$ and $x_2 \in F_2(X)$. Set $x_3 = \beta(x_2) \in F_3(X)$. By assumption there exist an admissible epimorphism $f : Y \to X$ such that $F_3(f)(x_3) = 0$. 

Using the above commutative diagram $F_2(f)(x_2) \in \text{Ker}(\beta_Y) = \text{Im}(\alpha_Y)$. Thus there exists $y_1 \in F_1(Y)$ such that $\alpha_Y(y_1) = F_2(f)(x_2)$. Again by assumption there exist an admissible epimorphism $g : Z \to Y$ such that $F_1(g)(y_1) = 0$.

Using the above commutative diagram $F_2(gf)(x_2) = F_2(g)F_2(f)(x_2) = F_2(g)\alpha_Y(y_1) = \alpha_ZF_1(g)(y_1) = 0$. Since $gf$ is an admissible epimorphism, $F_2 \in \text{Eff}(\mathcal{M})$.

(ii) Let $L \in \mathcal{L}_1(\mathcal{M})$, consider the exact sequence $0 \to L \to I \to M \to 0$ where $I$ is injective envelope of $L$. First note that for every $n$-exact sequence $X^0 \to X^1 \to \cdots \to X^n \to X^{n+1}$ by definition

$$0 \to (-, X^0) \to (-, X^1) \to \cdots \to (-, X^n) \to (-, X^{n+1})$$

is exact, applying the exact functor $(-, I)$ to this sequence we obtain that $I(X^{n+1}) \to I(X^n) \to \cdots \to I(X^1) \to I(X^0) \to 0$ is exact. Also because $L \in \mathcal{L}_1(\mathcal{M})$ it doesn’t have any nonzero weakly effaceable subobject, so $I$ doesn’t have any nonzero weakly effaceable subobject because it is an injective envelope if $L$. This means that $I$ is an $n$-exact functor, i.e.

$$0 \to I(X^{n+1}) \to I(X^n) \to \cdots \to I(X^1) \to I(X^0) \to 0$$

is exact for all $n$-exact sequences in $\mathcal{M}$. Consider the following commutative diagram.
All rows are exact by assumption, and the left-hand and middle columns are exact, now long exact sequence theorem [10, Theorem 1.3.1] tells that $0 \to M(X^{n+1}) \to M(X^n)$ is exact. Thus $\text{Hom}(\text{Eff}(M), M) = 0$. Now by Lemma 3.4 $L \in \text{Eff}(M)^{i-1}$. For the converse inclusion $\text{Eff}(M)^{i} \subseteq \mathcal{L}_1(M)$, let $L \in \text{Eff}(M)^{i}$ and consider the short exact sequence $0 \to L \to I \to M \to 0$ where $I$ is an injective envelope of $L$. Thus by Lemma 3.4 $\text{Hom}(\text{Eff}(M), M) = 0$, that means $0 \to M(X^{n+1}) \to M(X^n)$ is exact. Again by long exact sequence theorem, the left-hand column is exact.

The following observation is interesting and is our motivation for Question 3.9.

**Proposition 3.6.** For every $k \in \{1, ..., n\}$, $\text{Eff}(M)^{i-k} = \mathcal{L}_k(M)$.

**Proof.** We want to prove by induction that for all $1 \leq k \leq n$, $\text{Eff}(M)^{i-k} = \mathcal{L}_k(M)$. By Proposition 3.5 $\text{Eff}(M)^{i-1} = \mathcal{L}_1(M)$. Let $k \geq 2$, $L \in \text{Eff}(M)^{i-1} = \mathcal{L}_1(M)$ and $X : X^0 \to X^1 \to \cdots \to X^n \to X^{n+1}$ be an arbitrary $n$-exact sequence. Consider the exact sequence $0 \to L \to I \to M \to 0$ where $I$ is injective envelope of $L$. Note that as we see in the proof of Proposition 3.5

$$0 \to I(X^{n+1}) \to I(X^n) \to \cdots \to I(X^1) \to I(X^0) \to 0$$

is exact. By dimension shifting $L \in \text{Eff}(M)^{i-k}$ if and only if $M \in \text{Eff}(M)^{i-k-1} = \mathcal{L}_{k-1}(M)$.

Applying the long exact sequence theorem [10, Theorem 1.3.1] to the following short exact sequence of complexes.

$$0 \to L(X) \to I(X) \to M(X) \to 0$$

Because the middle column is exact we obtain that $M \in \mathcal{L}_{k-1}(M)$ if and only if $L \in \mathcal{L}_k(M)$.

We denote by $H : \mathcal{M} \to \mathcal{L}_1(\mathcal{M})$ the composition of the Yoneda functor $\mathcal{M} \to \text{Mod}\mathcal{M}$ with the localization functor $\text{Mod}\mathcal{M} \to \text{Mod}\mathcal{M} \simeq \text{Eff}(\mathcal{M})^\perp = \mathcal{L}_1(\mathcal{M})$. Thus $H(X) = (-, X) : \mathcal{M}^{\text{op}} \to \text{Ab}$. For simplicity we denote $(-, X)$ by $H_X$.

**Proposition 3.7.**

(i) **For every $n$-exact sequence** $X^0 \to X^1 \to \cdots \to X^n \to X^{n+1}$ in $\mathcal{M}$,

$$0 \to H_{X^0} \to H_{X^1} \to \cdots \to H_{X^n} \to H_{X^{n+1}} \to 0$$

**is exact in $\mathcal{L}_1(\mathcal{M})$.**

(ii) **The essential image of** $H : \mathcal{M} \to \mathcal{L}_1(\mathcal{M})$ **is $n$-rigid.**

**Proof.** Because the cokernel of $H_{X^n} \to H_{X^{n+1}}$ is weakly effaceable, (i) follows.

Let $X, Y \in \mathcal{M}$ and $H_Y \to I^0$ be the injective envelope of $H_Y$ in $\text{Mod}\mathcal{M}$. Because $H_Y = (-, Y) \in \mathcal{L}_n(\mathcal{M})$, by the proofs of Proposition 3.5 and Proposition 3.6 $I^0 \in \mathcal{L}_n(\mathcal{M})$ and in the short exact sequence

$$0 \to H_Y \to I^0 \to \Omega^{-1}H_Y \to 0$$

of functors in $\text{Mod}\mathcal{M}$ we have that $\Omega^{-1}H_Y \in \text{Eff}(\mathcal{M})^{i-n-1} = \mathcal{L}_{n-1}(\mathcal{M})$, so $I^1$ that is the injective envelope of $\Omega^{-1}H_Y$ belongs to $\mathcal{L}_n(\mathcal{M})$ by the proof of Proposition 3.5. By repeating this argument, in the minimal injective coresolution

$$0 \to H(Y) \to I^0 \to I^1 \to \cdots \to I^n$$

(3.1)

for $H_Y$ in $\text{Mod}\mathcal{M}$ we have $I^0, ..., I^{n-1} \in \mathcal{L}_n(\mathcal{M})$ and $\Omega^{-1}H_Y, ..., \Omega^{-n+1}H_Y \in \mathcal{L}_1(\mathcal{M})$. In the last step applying $\text{Hom}(E, -)$ for an arbitrary weakly effaceable functor $E$ to the short exact sequence of functor
in \( \text{Mod} \mathcal{M} \) we have the following exact sequence of abelian groups.

\[
\begin{align*}
0 & \to \Omega^{-n+1}H_Y \to I^{n-1} \to \Omega^{-n}H_Y \to 0
\end{align*}
\]

Thus \( \text{Hom}(\text{Eff}(\mathcal{M}), \Omega^{-n}H_Y) = 0 \), and because \( I^n \) is an essential extension of \( \Omega^{-n}H_Y \) and \( \text{Eff}(\mathcal{M}) \subseteq \text{Mod} \mathcal{M} \) is a Serre subcategory we have that \( \text{Hom}(\text{Eff}(\mathcal{M}), I^n) = 0 \). Therefore by the proof of Proposition 3.5 \( I^n \) belongs to \( \mathcal{L}_n(\mathcal{M}) \). Thus we constructed an injective coresolution (3.1) for \( HY \) with \( I^0, \ldots, I^n \in \mathcal{L}_1(\mathcal{M}) \). Since the inclusion functor \( L_1(\mathcal{M}) \to \text{Mod} \mathcal{M} \) preserve monomorphisms \( I^0, \ldots, I^n \) are injective objects in the abelian category \( \mathcal{L}_1(\mathcal{M}) \). Thus we have

\[
\begin{align*}
\text{Ext}^i_{\mathcal{L}_1(\mathcal{M})}(H_X, H_Y) & \cong \text{Ext}^i_{\text{Mod} \mathcal{M}}(H_X, H_Y) = 0,
\end{align*}
\]

for every \( 1 \leq i \leq n - 1 \), because representable functors are projective objects in \( \text{Mod} \mathcal{M} \). \( \square \)

**Remark 3.8.** By the Example 2.5 there are \( n \)-exact categories that aren’t equivalent to \( n \)-cluster tilting subcategories. Motivated by the above proposition one can try to prove the following.

Let \( \mathcal{M} \) be a small \( n \)-exact subcategory, is there an exact category \( \mathcal{E} \) and an embedding \( \mathcal{M} \hookrightarrow \mathcal{E} \) such that the additive closure \( \text{add}(\mathcal{M}) \) is an \( n \)-cluster tilting subcategory of \( \mathcal{E} \)?

By Example 2.5 there are \( n \)-exact categories that are not \( n \)-cluster tilting. Every \( n \)-abelian category has a natural structure of \( n \)-exact category [7, Theorem 4.4]. The positive answer to the following question tells that every \( n \)-exact category can be viewed as a nice subcategory of an \( n \)-abelian category.

**Question 3.9.** Let \( \mathcal{M} \) be a small \( n \)-exact category. Is \( \text{Eff}(\mathcal{M}) \) \( \mathcal{L}_n(\mathcal{M}) \) an \( n \)-cluster tilting subcategory of the abelian category \( \text{Eff}(\mathcal{M}) \)?

**Remark 3.10.** Note that positive answer to Question 3.9 complete the following table in a natural way. Recall that for an additive category \( B, \text{mod} B \) is the full subcategory of \( \text{Mod} \mathcal{M} \) consist of all finitely presented functors, and \( \text{eff}(B) = \text{Eff}(B) \cap \text{mod} B \). The first equivalence is called” Auslander’s formula”. The second equivalence is called” Gabriel-Quillen embedding theorem” (see [8, Appendix A]). And the third equivalence recently was proved in [2, 9].

| \( \mathcal{A} \) is a small abelian category. | \( \text{mod} \mathcal{A} \simeq \text{eff}(\mathcal{A})^{\perp} \simeq \mathcal{A} \) |
| --- | --- |
| \( \mathcal{E} \) is a small exact category. | \( \text{Mod} \mathcal{E} \simeq \text{Eff}(\mathcal{E})^{\perp} \simeq \text{Lex}(\mathcal{E}) \), and \( \mathcal{E} \) is an extension-closed subcategory of it. |
| \( \mathcal{M} \) is a small \( n \)-abelian category. | \( \text{mod} \mathcal{M} \simeq \text{eff}(\mathcal{M})^{\perp} \) has an \( n \)-cluster tilting subcategory equivalent to \( \mathcal{M} \). |
| \( \mathcal{M} \) is a small \( n \)-exact category. | \( \text{Mod} \mathcal{M} \simeq \text{Eff}(\mathcal{M})^{\perp} \simeq \mathcal{L}_1(\mathcal{M}) \) has an \( n \)-cluster tilting subcategory \( \text{Eff}(\mathcal{M})^{\perp} \) that \( \mathcal{M} \) nicely embed in it. |

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