GLOBAL SURFACES OF SECTION FOR DYNAMICALLY
CONVEX REEB FLOWS ON LENS SPACES

A. SCHNEIDER

ABSTRACT. We show that a dynamically convex Reeb flow on a lens space
\( L(p, 1), p > 1 \), admits a \( p \)-unknotted closed Reeb orbit \( P \) which is the binding
of a rational open book decomposition with disk-like pages. Each page is a
rational global surface of section for the Reeb flow and the Conley-Zehnder
index of the \( p \)-th iterate of \( P \) is 3. This result applies to the Hénon-Heiles
Hamiltonian whose energy level presents \( \mathbb{Z}_3 \)-symmetric and for all energies <
1/6 the flow restricted to the sphere-like component descends to a dynamically
convex Reeb flow on \( L(3, 1) \). Due to a \( \mathbb{Z}_4 \)-symmetry the result also applies to
Hill’s lunar problem.

1. Introduction

1.1. The existence of the binding orbit

1.2. Basic concepts and main result

1.3. Some applications

1.4. The Hénon-Heiles Hamiltonian

1.5. The regularized Hill’s lunar problem

2. Preliminaries

2.1. The Conley-Zehnder index

2.2. The rational self-linking number

2.3. Pseudo-holomorphic curves in symplectizations

3. Proof of main result

3.1. The non-degenerate case

3.2. The degenerate case

Appendix A. Proof of Lemma 3.7

Appendix B. Proof of lemma 3.8

Appendix C. Proof of Lemma 3.10

References

1. Introduction

It is a classical problem in conservative dynamics to investigate the existence
of periodic motions of Hamiltonian systems restricted to energy levels. Poincaré
and Birkhoff used global surfaces of section to study periodic trajectories in the
restricted three body problem and for geodesic flows in the 2-sphere. In such
systems, the energy levels are diffeomorphic to the real projective 3-space and,

The author was partially supported by CAPES and CNPq.
under some good conditions, the flow is reduced to a surface map. Tools from discrete dynamics then come into play and one can derive the existence of other periodic trajectories.

The systems considered by Poincaré and Birkhoff are particular cases of Reeb flows on the universally tight lens space $L(2, 1) \simeq \mathbb{R}P^3$, as shown in [2]. In order to study such systems, one usually lifts the flow to the tight 3-sphere via a suitable double covering map. In some cases, it is even possible to show that the Reeb flow is dynamically convex [1].

Hofer, Wysocki and Zehnder [20] used pseudoholomorphic curves in symplectizations to study dynamically convex Reeb flows on the tight 3-sphere. They found a special periodic orbit, which is unknotted and has Conley-Zehnder index 3, bounding a disk-like global surface of section. One of the consequences of this global section and a result of Franks [12] is that the Reeb flow admits either 2 or infinitely many periodic orbits. This result applies to Hamiltonian dynamics on strictly convex hypersurfaces in $\mathbb{R}^4$.

Using the same methods in [20], Hryniewicz and Salomão [31] proved a similar result for Reeb flows on $\mathbb{R}P^3 = S^3/\mathbb{Z}_2 = L(2, 1)$, equipped with the universally tight contact structure: if the Reeb flow is dynamically convex then it admits a rational open book decomposition whose binding orbit is elliptic and 2-unknotted. Each page is a 2-disk for the binding and constitutes a rational global surface of section for the Reeb flow. The main motivation of this result was to study the circular planar restricted three body problem directly on the regularized $\mathbb{R}P^3$, without considering the usual lift to the tight 3-sphere.

Here we generalize results in [20] and [31] to lens spaces $L(p, 1)$, $p > 1$, equipped with the universally tight contact structure. We use the theory of pseudoholomorphic curves in symplectizations in order to show that if the Reeb flow on such contact manifold is dynamically convex then it admits a rational open book decomposition whose binding orbit is elliptic and $p$-unknotted and bounds a rational disk-like global surface of section. The Conley-Zehnder index of the $p$-th iterate of $P$ is 3 when computed with respect to a $p$-disk for $P$. In fact, $P$ is the binding of a rational open book decomposition whose pages are rational global surfaces of section.

We apply our results to some Hamiltonian systems with two-degree-of-freedom whose sphere-like components of the corresponding energy levels present $\mathbb{Z}_p$-symmetry and the flow descends to a Reeb flow on the universally tight lens space $L(p, 1)$. This is the case of the Hénon-Heiles Hamiltonian for energies $< 1/6$. The flow on the sphere-like component of the energy level descends to a Reeb flow on $L(3, 1)$. Dynamical convexity follows from strict convexity which is directly checked on the corresponding Hill’s region using a criterium found in [37].

We also apply our results to the regularized Hill’s lunar problem whose flow on a low energy level descends to a Reeb flow on the universally tight $L(4, 1)$. In this case, dynamical convexity follows from the simple fact that the an energy level sufficiently close to a nondegenerate minimum point of the Hamiltonian is strictly convex.

1.1. The existence of the binding orbit. Finding necessary and sufficient conditions for a closed Reeb orbit to be the boundary of a disk-like global surface of section constitutes an important question in Reeb dynamics. The importance of this problem relies on the fact that, under some good circumstances, this closed
Reeb orbit characterizes the contact manifold. Relevant research in this direction is found in \[18, 20, 21, 25, 26\].

In \[18\] the authors characterize the tight 3-sphere. If a nondegenerate and dynamically convex contact form on a co-orientable contact 3-manifold \((M,\xi)\) admits an unknotted closed Reeb orbit \(P_0\) with self-linking number \(-1\) and Conley-Zehnder index 3 then \((M,\xi)\) is contactomorphic to the tight 3-sphere. The proof is based on the construction of an open book decomposition with binding \(P_0\) whose pages are disk-like global surfaces of section for the Reeb flow. The dynamical convexity hypothesis used in the construction of the open book can be dropped and one may only require that the Reeb orbits with Conley-Zehnder index 2 are linked to \(P_0\), see \[29\]. In fact, although such unlinked closed Reeb orbits with Conley-Zehnder index 2 represent an obstruction to the existence of an open book decomposition adapted to the Reeb flow, other types of transverse foliations might be considered, see \[10, 11\] for the existence of \(3-2-3\) foliations adapted to Reeb flows on the tight 3-sphere.

Recently, Hryniewicz, Licata and Salomão \[26\] characterized the universally tight lens spaces. If \((M,\xi)\) is a closed co-orientable contact manifold admitting a Reeb flow with a \(p\)-unknotted closed Reeb orbit \(P_0\), whose self-linking number is \(-1/p\) and the Conley-Zehnder index of its \(p\)-th iterate is 3 then a suitable necessary and sufficient condition on the closed Reeb orbits which are contractible in \(M \setminus P_0\) implies that \((M,\xi)\) is a lens space \(L(p,q)\), for some \(1 \leq q \leq p\), equipped with the universally tight contact structure. The proof is also based on the construction of a rational open book decomposition with binding \(P_0\) whose pages are rational disk-like global surfaces of section for the Reeb flow.

The results mentioned above show that in order to construct a disk-like global surface of section, the first work to be done is to prove the existence of the special closed Reeb orbit which will be the binding of an open book decomposition. Hence, it is of central interest to provide sufficient conditions on the contact form which assure the existence of such a closed Reeb orbit. The following question is not answered in its full generality.

- Is dynamically convexity a sufficient condition for a contact form on the universally tight lens space \(L(p,q)\) to admit a \(p\)-unknotted closed Reeb orbit with self-linking number \(-1/p\) and whose \(p\)-th iterate has Conley-Zehnder index 3?

Here we give an affirmative answer to this question in the case \(q = 1\). Together with results in \[26\] and \[31\], we show that \(P_0\) is the binding orbit of a rational open book decomposition with disk-like pages and each page is a global surface of section.

Recall that a rational global surface of section forces the existence of a second closed Reeb orbit \(P_1\) associated to a fixed point of the first return map. The link formed by \(P_0\) and \(P_1\) is called a Hopf link. A non-resonance condition on the rotation numbers of \(P_0\) and \(P_1\) corresponds to a twist condition on the first return map and forces the existence of infinitely many closed Reeb orbits, see also \[27\]. Alternatively, a result of Franks \[12\] gives infinitely many periodic orbits in case a third one exists. In particular, all such Reeb flows admitting disk-like global surfaces of section have 2 or infinitely many closed Reeb orbits. It is still an open question whether every 3-dimensional Reeb flow admits either 2 or infinitely many closed Reeb orbits. See \[13\] for a partial answer to this question.
1.2. Basic concepts and main result. Let $M$ be a smooth oriented 3-manifold. A contact structure on $M$ is a smooth hyperplane distribution $\xi \subset TM$, locally defined by the kernel of a 1-form $\lambda$ satisfying $\lambda \wedge d\lambda \neq 0$. The pair $(M, \xi)$ is a contact manifold. We say that $(M, \xi)$ is co-orientable if there exists a globally defined 1-form $\lambda$ on $M$ satisfying $\xi = \ker \lambda$. If $f : M \to \mathbb{R} \setminus \{0\}$ is smooth then $\lambda$ and $f\lambda$ define the same contact structure. Furthermore, since $(f\lambda) \wedge d(f\lambda) = f^2\lambda \wedge d\lambda$, any contact form defining $\xi$ induces the same orientation on $M$ and we say $\xi$ is positive if its induced orientation coincides with the orientation of $M$.

A contact structure $\xi$ on $M$ is called overtwisted if there exists an embedded disk $D \hookrightarrow M$ such that $T_z(\partial D) \subset \xi_z$ and $T_zD \neq \xi_z$ for all $z \in \partial D$. If such a disk does not exist then the contact structure is called tight.

Let $\lambda$ be a contact form which defines the contact structure $\xi$ on $M$. The vector field on $M$ uniquely determined by
\[ i_{X_\lambda}d\lambda = 0 \quad \text{and} \quad i_{X_\lambda}\lambda = 1, \]
is called the Reeb vector field of $\lambda$. Its flow $\{\varphi_t, t \in \mathbb{R}\}$ is the Reeb flow of $\lambda$. Let $P = (x, T)$ be a periodic orbit of the Reeb flow of $\lambda$, that is $x : \mathbb{R} \to M$ is periodic, it satisfies $x(t) = \varphi_t(x(0)) \forall t$, and $T > 0$ is a period of $x$. $P$ is also called a closed Reeb orbit of $\lambda$. We say that $P$ is simple if $T$ is the least positive period of $x$. We denote by $P(\lambda)$ the set of equivalence classes of periodic orbits of $\lambda$ with the identification
\[ P = (x, T) \sim Q = (y, T') \iff T = T' \text{ and } x(\mathbb{R}) = y(\mathbb{R}). \]
We say that $P = (x, T)$ is contractible if the loop
\[ (1.1) \quad x_T : \mathbb{R}/\mathbb{Z} \to M : t \in \mathbb{R}/\mathbb{Z} \mapsto x(Tt), \]
is contractible on $M$. If the first Chern class $c_1(\xi)$ vanishes on $\pi_2(M)$ then every contractible periodic orbit $P \in P(\lambda)$ has a well-defined Conley-Zehnder index $\mu_{CZ}(P) \in \mathbb{Z}$.

The dynamical convexity condition on a contact form was introduced in [20].

**Definition 1.1** (Hofer, Wysocki and Zehnder). A contact form $\lambda$ on a smooth closed 3-manifold $M$ is called dynamically convex if $c_1(\ker \lambda)$ vanishes on $\pi_2(M)$ and every contractible periodic orbit $P \in P(\lambda)$ satisfies $\mu_{CZ}(P) \geq 3$.

Dynamical convexity imposes contact-topological obstructions as the following theorem shows.

**Theorem 1.2** (Hofer, Wysocki and Zehnder [21]). If $\lambda$ is a dynamically convex contact form on a closed 3-manifold $M$ then $\pi_2(M)$ vanishes and the contact structure $\ker \lambda$ is tight.

A knot $K \hookrightarrow M$ is called $k$-unknotted, for some $k \in \mathbb{N}$, if there exists an immersion $u : \mathbb{D} \to M$, so that $u|_{\mathbb{D}\setminus \partial \mathbb{D}}$ is an embedding and $u|_{\partial \mathbb{D}} : \partial \mathbb{D} \to K$ is a $k$-covering map. The map $u$ is called a $k$-disk for $K$. If $K$ is oriented then we say that $u$ induces the same orientation as $K$ if $u|_{\partial \mathbb{D}}$ preserves orientation, where $\partial \mathbb{D}$ has the counter-clockwise orientation. If the $k$-unknotted $K$ is transverse to $\xi$ then $K$ is oriented by $\lambda$ and there exists a well-defined rational self-linking number $sl(K, u) \in \mathbb{Q}$, computed with respect to a $k$-disk $u$ for $K$, see [4]. If the first Chern class $c_1(\xi)$ vanishes on $\pi_2(M)$ then $sl(K) = sl(K, u)$ does not depend on the choice of $u$. 

Definition 1.3. Let $\lambda$ be a defining contact form for a closed contact 3-manifold $(M, \xi)$. Let $K \rightarrow M$ be a $k$-unknotted closed Reeb orbit of $\lambda$. A rational disk-like global surface of section bounded by $K$ is a $k$-disk $u : \mathbb{D} \rightarrow M$ for $K = u(\partial \mathbb{D})$ so that $u(\mathbb{D} \setminus \partial \mathbb{D})$ is transverse to $X_\lambda$, and every Reeb trajectory in $M \setminus K$ hits $u(\mathbb{D} \setminus \partial \mathbb{D})$ infinitely many times in the past and in the future. In particular, the Reeb flow of $\lambda$ is encoded in the corresponding first return map $\psi : u(\mathbb{D} \setminus \partial \mathbb{D}) \rightarrow u(\mathbb{D} \setminus \partial \mathbb{D})$.

Let $(z_1, z_2)$ be coordinates in $\mathbb{C}^2$ and consider the 3-sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$ 

The Liouville form

$$\lambda_0 = \frac{1}{4\pi}(\bar{z}_1 dz_1 - z_1 d\bar{z}_1 + \bar{z}_2 dz_2 - z_2 d\bar{z}_2),$$

is a contact form when restricted to $S^3$. Its contact structure $\xi_0 = \ker \lambda_0$ is called standard. It is well known that it is the unique tight contact structure on $S^3$ up to diffeomorphisms. Given relatively prime integers $p \geq q \geq 1$, there exists a free action of $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ on $(S^3, \xi_0)$ generated by the contactomorphism $g_{p,q} : S^3 \rightarrow S^3$,

$$(1.2) \quad g_{p,q}(z_1, z_2) = \left(e^{2\pi i/p}z_1, e^{2\pi i/q}z_2\right).$$

The orbit space

$$L(p, q) := S^3/\mathbb{Z}_p,$$

is called a lens space. The contact structure $\xi_0$ descends to a tight contact form on $L(p, q)$, still denoted $\xi_0$. It is called the universally tight contact structure on $L(p, q)$. On $L(p, 1)$ this is the unique tight contact structure which lifts to $(S^3, \xi_0)$ up to diffeomorphisms.

Definition 1.4. Let $\lambda$ be a contact form on $L(p, q)$ inducing the universally tight contact structure $\xi_0$. A rational open book decomposition with disk-like pages and binding orbit $K$ is a pair $(\pi, K)$ formed by a $p$-unknotted closed Reeb orbit $K \rightarrow L(p, q)$ and a smooth fibration $\pi : M \setminus K \rightarrow S^1$ so that the closure of each fiber $\pi^{-1}(t)$ is the image of a rational disk-like global surface of section bounded by $K$.

Finally we can state our main result.

Theorem 1.5. Let $\lambda$ be a dynamically convex contact form on $L(p, 1)$. Then its Reeb flow admits a $p$-unknotted closed Reeb orbit with self-linking number $sl(P) = -1/p$. The Conley-Zehnder index of the $p$-th iterate of $P$ is 3. Moreover, $P$ is the binding of a rational open book decomposition whose pages are rational disk-like global surfaces of section. Moreover, every $p$-unknotted closed Reeb orbit with self-linking number $-1/p$ is the binding of a rational open book decomposition whose pages are rational disk-like global surfaces of section.

1.3. Some applications. A Hamiltonian function $H : \mathbb{R}^4 \rightarrow \mathbb{R}$ determines the vector field $X_H$ by

$$\iota_{X_H} \omega_0 = -dH,$$

where $\omega_0 = \sum_{i=1}^2 dy_i \wedge dx_i$ is the standard symplectic form on $\mathbb{R}^4$. Furthermore, the Hamiltonian flow $\phi_t, t \in \mathbb{R}$, preserves each energy level $S_E = H^{-1}(E)$, $E \in \mathbb{R}$. Assume that the energy level has contact type. This means that there exists a Liouville vector field $Y$ defined on a neighborhood of $S_E$ which is transverse to
The Hamiltonian flow is equivalent to the Reeb flow on $S_E$ associated to the contact form

\begin{equation}
\lambda := i_Y\omega_0|_{S_E}
\end{equation}

Assume also that $S_E$ is diffeomorphic to the 3-sphere and presents the following $\mathbb{Z}_p$-symmetry: there exists a $\mathbb{Z}_p$-action on $\mathbb{R}^4$ which makes $S_E$ invariant and so that $S_E/\mathbb{Z}_p \simeq L(p,1)$. Assume, moreover, that the contact form defined in (1.3) is preserved by the $\mathbb{Z}_p$-action. Then this quotient induces a contact form $\tilde{\lambda}$ on $L(p,1)$. If $\lambda$ is dynamically convex then, by Theorem 1.2, $\lambda$ is tight. Hence $\tilde{\lambda}$ is universally tight. By a classification of tight contact structures on lens spaces due to Honda, see [24], there exists a unique universally tight contact structure on $L(p,1)$ up to diffeomorphisms. Theorem 1.5 gives

**Theorem 1.6.** If $\lambda$ is a dynamically convex contact form on $S_E$ then the Hamiltonian flow on $S_E$ admits a $\mathbb{Z}_p$-symmetric unknotted periodic orbit $P$ with Conley-Zehnder index 3 which is the binding of an open book decomposition whose pages are disk-like global surfaces of section for $P$. Moreover, this open book is the lift to $S_E$ of a rational open book decomposition of $S_E/\mathbb{Z}_p \simeq L(p,1)$.

1.4. **The Hénon-Heiles Hamiltonian.** Hénon and Heiles [14] proposed in 1964 a galactic model to describe the dynamical behavior of a star. This model is a two-degree-of-freedom Hamiltonian system with Hamiltonian function

\begin{equation}
H(x_1, x_2, y_1, y_2) = \frac{y_1^2 + y_2^2}{2} + V(x_1, x_2),
\end{equation}

and time independent potential

\[ V(x_1, x_2) = \frac{x_1^2 + x_2^2}{2} + \frac{x_1^3 x_2 - x_2^3}{3}. \]

In [34] Lunsford and Ford show that the behavior of some generic systems is equivalent to the behavior of the Hénon-Heiles system if we disregard cubic superior order terms in the Taylor series expansion of the potential. In [9] the authors present a list of problems related to the Hénon-Heiles system which ranges from the planar three body problem to stationary plasm systems.

The Hamiltonian $H$ is invariant under the $\mathbb{Z}_3$-action on $\mathbb{R}^4$ generated by $\tilde{g}_{3,1}: \mathbb{R}^4 \to \mathbb{R}^4$

\[ (w_1 := x_1 + ix_2, w_2 := y_1 + iy_2) \mapsto \tilde{g}_{3,1}(w_1, w_2) = \left( e^{\frac{2\pi i}{3}} w_1, e^{\frac{2\pi i}{3}} w_2 \right). \]

A direct computation shows that the Liouville form on $\mathbb{R}^4$

\[ \lambda_0 = \frac{1}{2} \sum_{i=1}^{4} (x_i dy_i - y_i dx_i). \]

is $\mathbb{Z}_3$-symmetric, that is $\tilde{g}_{3,1}^* \lambda_0 = \lambda_0$. Note that $d\lambda_0 = -\omega_0$.

The energy level $H^{-1}(E)$, $0 < E < \frac{1}{3}$, contains a component $S_E$ diffeomorphic to $S^3$. As we shall see below, $S_E$ is a strictly convex hypersurface and hence the Hamiltonian flow restricted to $S_E$ is dynamically convex. Due to its $\mathbb{Z}_3$-symmetry the Hamiltonian flow on $S_E$ descends to a dynamically convex Reeb flow on $L(3,1)$ and Theorem 1.6 applies. Hence, for each $0 < E < \frac{1}{6}$, we obtain an unknotted periodic orbit $P_{3,E} \subset S_E$, which is $\mathbb{Z}_3$-symmetric, has Conley-Zehnder index 3 and
is the binding of an open book decomposition whose pages are order 3 disk-like
global surfaces of section for the Hamiltonian flow restricted to $S_E$.

The critical level $H^{-1}(\frac{1}{6})$ contains a subset $S_1$ homeomorphic to $S^3$ which is
the $C^0$-limit of $S_E$ as $E \to \frac{1}{6}^-$, see Figure 1.1 for the boundaries of Hill’s regions
projected to the $x_1x_2$-plane. The subset $S_0$ contains precisely three critical points
which are $Z_3$-symmetric and correspond to saddle-center equilibrium points for the
Hamiltonian flow of $H$.

Despite the simplicity of the Hamiltonian function, its dynamics is very complex,
having multiple horseshoes and even chaotic behavior [8, 36]. Arioli and Zgliczyski
[3] show that the critical level has a rich symbolic structure which guarantees in-
finitely many periodic orbits for energy levels slightly above and slightly below the
critical one.

At this moment we focus on the sphere-like component $S_E \subset H^{-1}(E)$ for energies
below the critical value, that is, $0 < E < \frac{1}{6}$. Next we show that $S_E$ is strictly convex
and, in particular, is star-shaped with respect to $0 \in \mathbb{R}^4$. Hence the Liouville form
(1.3) restricts to a $Z_3$-symmetric dynamically convex contact form on $S_E$.

In [37] Salomão presents a criterium to check strict convexity of $S_E$.

**Theorem 1.7** (Salomão [37]). Let $H = \frac{x_1^2 + x_2^2}{2} + V(x_1, x_2)$ be a Hamiltonian function
on $\mathbb{R}^4$ with smooth potential $V : \mathbb{R}^2 \to \mathbb{R}$. Let $S_E \subset H^{-1}(E)$ be a sphere-like
regular component of its energy level. Denote by $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ the natural projection
to the $x_1x_2$-plane. Let $B_E = \pi(S_E)$. Then $S_E$ is strictly convex if and only if the function

$$G_E := 2(E - V)(x_1V_{x_1}x_1V_{x_2}x_2 - V_{x_1}^2x_2 + V_{x_2}x_1V_{x_2}x_2 - 2V_{x_1}V_{x_2}x_1x_2)$$

is positive on $B_E$.

We use Theorem 1.7 to prove that $S_E$ is strictly convex.

**Theorem 1.8.** For every $0 < E < \frac{1}{6}$, $S_E$ is a strictly convex hypersurface.

**Proof.** A direct computation shows that

$$G_E = 2(E - V)(1 - 4x_1^2 - 4x_2^2) + I,$$
Theorem 1.9. For every \( 0 < E < \frac{1}{6} \), \( B_E \) is contained in the interior of the triangle \( B_{\pi} = \pi(S_1) \), whose vertices are the points
\[
\left( -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right), \ (0,1) \text{ and } \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right),
\]
see Figure 1.1. In particular, every point \( (x_1,x_2) \in B_E \) satisfies
\[
x_1^2 + x_2^2 < 1.
\]
Since \( 6V = 3x_1^2 + 3x_2^2 + 6x_1^2x_2 - 2x_2^3 \), we have
\[
x_1^2(3 + 6x_2) = 6V - 3x_2^2 + 2x_2^3.
\]
This implies
\[
I := x_1^2 - 3x_1^4 + 2x_2^5 - 2x_2^3(1 - 6x_2^2 - 4x_2^3 + 6V - 3x_2^2 - 2x_2^3) \geq x_1^2 - 3x_1^4 + 2x_2^5 - 2x_2^3(1 - 6V - 3x_2^2 - 6x_2^3) = x_2^2 - 3x_1^4 + 2x_2^5 + x_2^7(1 - 6V) - 6Vx_2^2 + 3x_2^4 - 2x_2^5 = (1 - 6V)(x_1^2 + x_2^2).
\]
Using (1.6) and the inequality \( 1 > 6E \) we obtain
\[
G_E = 2(E - V)(1 - 4x_1^2 - 4x_2^2) + (1 - 6V)(x_1^2 + x_2^2) \geq 2(E - V)(1 - 4x_1^2 - 4x_2^2) + (6E - 6V)(x_1^2 + x_2^2) = 2(E - V)(1 - x_1^2 - x_2^2) \geq 0,
\]
Observe that the first inequality above is strict if \( (x_1, x_2) \neq (0, 0) \). Since \( V(0, 0) = 0 < E \), the last inequality above is strict if \( (x_1, x_2) = (0, 0) \). We conclude that \( G_E > 0 \) on \( B_E \), as desired. An application of Theorem 1.7 finishes the proof of this theorem. \( \square \)

Theorems 1.6 and 1.8 imply the existence of an open book decomposition adapted to the flow on \( S_E \) for every \( 0 < E < \frac{1}{6} \).

Theorem 1.9. For every \( 0 < E < \frac{1}{6} \), there exist two 3-unknotted periodic orbits \( P_1, P_2 \subset S_E/\mathbb{Z}_3 \simeq L(3,1) \) which form a Hopf link and so that each \( P_i \) is the binding of a rational open book decompositions adapted to the flow. Each page is a rational disk-like global surface of section. Moreover, the Conley-Zehnder index of \( P_1^3 \) is 3.

In [6, 7] the authors prove the existence of at least eight periodic orbits in \( S_E \) for every \( 0 < E < \frac{1}{6} \). Combining Theorem 1.9 and a result of J. Franks in [12] we obtain the following corollary.

Corollary 1.10. For every \( 0 < E < \frac{1}{6} \), \( S_E \) admits infinitely many periodic orbits.
1.5. **The regularized Hill’s lunar problem.** A very interesting problem in Hamiltonian dynamics is the well-known circular planar restricted three body problem (CPRTBP). Roughly speaking it consists of two primaries \( Q_1 \) and \( Q_2 \) following the Keplerian circular motion about their center of mass and one studies the motion of a massless particle \( R \) under Newtonian forces produced by the the primaries. Due to its difficulty, the CPRTBP is commonly studied under some simplifications. In [15] Hill studies the motion of the Moon. One can think of \( Q_1 \) and \( Q_2 \) as the Earth and the Sun, respectively, whose mass ratio is taken to be 0. After suitably re-scaling the motion near \( Q_1 \) the Hamiltonian function in the rotating frame takes the form

\[
H_{\text{HLP}} = \frac{y_1^2 + y_2^2}{2} + x_2y_1 - x_1y_2 - \frac{1}{\sqrt{x_1^2 + x_2^2}} - x_1^2 + \frac{1}{2}x_2^2.
\]

We refer to [35] for a precise exposition of Hill’s lunar equation and its relation to the CPRTBP.

In [32] Lee proves that energy levels of \( H_{\text{HLP}} \) below the first critical value are fiberwise convex. This implies that the Conley Zehnder index of every periodic orbit is nonnegative. However, it is not known whether these energy levels are dynamically convex.

Simó and Stuchi [40] study Hill’s lunar problem after Levi-Civita regularization for which the new Hamiltonian is

\[
H = \frac{y_1^2 + y_2^2}{2} + \frac{x_1^2 + x_2^2}{2} + 2(x_1^2 + x_2^2)(x_2y_1 - x_1y_2) - 4(x_1^6 - 3x_1^4x_2^2 - 3x_1^2x_2^4 + x_2^6).
\]

In [33] the authors prove the existence of at least two periodic orbits on each positive energy level of \( H \).

The point \( 0 \in \mathbb{R}^4 \) is a nondegenerate local minimum point of \( H \) since \( H = 0 \), \( dH(0) = 0 \) and the order 2 term in the Taylor expansion of \( H \) at 0 is \( H_2 = \frac{y_1^2 + y_2^2}{2} + \frac{x_1^2 + x_2^2}{2} \). Observe that \( H_2 \) is the Hamiltonian of two decoupled harmonic oscillators and is degenerate as a Hamiltonian flow. However, for every \( E > 0 \) sufficiently small, \( H^{-1}(E) \) contains a sphere-like subset \( S_E \) close to 0, which is strictly convex due to the nondegeneracy of 0 as a critical point of \( H \). See Figure 1.2 for the boundaries of Hill’s regions.

![Figure 1.2](image-url)
The regularized Hamiltonian $H$ is invariant under the $\mathbb{Z}_4$-action generated by $\tilde{g}_{4,1} : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$(w_1 := x_1 + ix_2, w_2 := y_1 + iy_2) \mapsto \tilde{g}_{4,1}(w_1, w_2) = \left( e^\frac{\pi i}{2} w_1, e^\frac{\pi i}{2} w_2 \right).$$

The Liouville form on $\mathbb{R}^4$ is preserved by $\tilde{g}_{4,1}$. It follows that the Hamiltonian flow on $S_E$ descends to a dynamically convex Reeb flow on $L(4, 1)$. Applying Theorem 1.6 we obtain the existence of an open book decomposition adapted to the flow.

**Theorem 1.11.** For every $E > 0$ sufficiently small, there exist two 4-unknotted periodic orbits $P_1, P_2 \subset S_E/\mathbb{Z}_4 \simeq L(4, 1)$, forming a Hopf link, so that each $P_i$ is the binding of a rational open book decomposition adapted to the flow. Each page is an rational disk-like global surface of section. Moreover, the Conley-Zehnder index of $P_1$ is 3.

**Acknowledgements.** I would like to thank my PhD advisor Pedro Salomão for proposing this problem to me and for all his support and motivation. Without his help this work would probably not be done. AS was partially supported by CAPES grant 1526852 and CNPq grant 142059/2016-1.

## 2. Preliminaries

In this section we remind some definitions in contact and symplectic geometry such as Conley-Zehnder index, transverse rotation number and self-linking number. After that we introduce pseudo-holomorphic curves in symplectizations and some related algebraic invariants.

### 2.1. The Conley-Zehnder index

Let $S : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2 \times 2}$ be a smooth path of $2 \times 2$ symmetric matrices. Identify $\mathbb{R}^2 \simeq \mathbb{C}$, $\mathbb{R}^{2 \times 2} \simeq L_\mathbb{R}(\mathbb{C})$ and consider the unbounded self-adjoint operator

$$(2.1) \quad L_S = -i\partial_t - S(t),$$

defined on the Hilbert space $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, with the structure induced by the euclidian inner product in $\mathbb{C}$. In [17], Hofer, Wysocky and Zehnder present some important properties of the operator (2.1). Its spectrum $\sigma(L_S)$ consists of countably many real eigenvalues which accumulate only at $\pm \infty$.

For each $\eta \in \sigma(L_S)$, every non-trivial $\eta$-eigensection $e : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ never vanishes and so it has a well-defined winding number

$$\text{wind}(\eta) = \frac{1}{2\pi}(\theta(1) - \theta(0)) \in \mathbb{Z},$$

where $\theta : [0, 1] \to \mathbb{R}$ is any continuous function satisfying $e(t) \in \mathbb{R}^+ e^{i\theta(t)}$. It can be shown that wind($\eta$) does not depend on $e$. If $\eta_1 \leq \eta_2 \in \sigma(L_S)$, then wind($\eta_1$) $\leq$ wind($\eta_2$), and for each $k \in \mathbb{Z}$, there exist precisely two eigenvalues (multiplicities counted) whose winding number is $k$.

Let Sp(2) be the group of $2 \times 2$ symplectic matrices and consider the set $\Sigma$ of smooth maps $\varphi : \mathbb{R} \to \text{Sp}(2)$ starting at the identity and satisfying

$$\varphi(t + 1) = \varphi(t)\varphi(1), \quad \forall t.$$ 

It follows that $S(t) := -i\varphi(t)\varphi(t)^{-1}$ is a smooth path of 1-periodic symmetric matrices. Therefore one can associate to the path $\varphi$ an unbounded self-adjoint operator $L_S$ as in (2.1).
Consider the special eigenvalues
\[ \eta^{<0} := \max \{ \eta \in \sigma(L_S) | \eta < 0 \}, \]
\[ \eta^{\geq0} := \min \{ \eta \in \sigma(L_S) | \eta \geq 0 \}. \]

**Definition 2.1.** The *Conley-Zehnder index* of the path \( \varphi \) is defined as
\[ \mu_{CZ}(\varphi) := \text{wind}(\eta^{<0}) + \text{wind}(\eta^{\geq0}). \]

Let \( \Pi : E \to \mathbb{R}/\mathbb{Z} \) be an oriented real vector bundle satisfying \( \text{rank}_\mathbb{R}(E) = 2 \). Denote by \( \Omega_E^+ \) the set of homotopy classes of oriented trivializations of \( E \). For any continuous non-vanishing section \( t \in \mathbb{R}/\mathbb{Z} \mapsto Z(t) \in \Pi^{-1}(t) \), there exists another continuous non-vanishing section \( Z' \), such that \( \{ Z(t), Z'(t) \} \) is an oriented basis of \( \Pi^{-1}(t) \forall t \). The frame \( \{ Z, Z' \} \) determines an oriented trivialization and its homotopy class \( \beta \in \Omega_E^+ \) depends only on \( Z \), up to homotopy through non-vanishing sections; it is called the homotopy class induced by \( Z \), see [26]. Given a continuous non-vanishing section \( W \) of \( E \), it follows that \( W(t) = a(t)Z(t) + b(t)Z'(t) \) for unique continuous functions \( a, b \) and
\[ \text{(2.2)} \]
\[ \text{wind}(W, Z) := \frac{1}{2\pi} (\theta(1) - \theta(0)) \in \mathbb{Z}, \]
where \( \theta : [0, 1] \to \mathbb{R} \) is a continuous function satisfying \( a(t) + ib(t) \in \mathbb{R}^+ e^{i\theta(t)} \). This integer depends only on the homotopy classes of non-vanishing sections \( Z \) and \( W \).

Denoting by \( \beta' \in \Omega_E^+ \) the homotopy class of oriented trivializations induced by \( W \), we may write \( \text{wind}(\beta, \beta') = \text{wind}(W, Z) \).

Now let \( P = (x, T) \in \mathcal{P}(\lambda) \) be a closed Reeb orbit of the Reeb flow of a contact \( \lambda \) on a closed 3-manifold \( M \). The contact structure \( \xi = \ker \lambda \) is preserved by the flow. Consider the map \( x_T : \mathbb{R}/\mathbb{Z} \to M \) defined by \( x_T := x(T \cdot) \). Then the bundle \( x_T^* \xi \to \mathbb{R}/\mathbb{Z} \) is oriented by \( d\lambda \). Choose a \( d\lambda \)-symplectic trivialization \( \Psi : x_T^* \xi \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \) representing a class \( \beta \in \Omega^+_{x_T^* \xi} \). Define the path of symplectic matrices \( \varphi \in \Sigma \) by
\[ \text{(2.3)} \]
\[ \varphi(t) := \Psi_t \circ d\phi_T \circ \Psi_0^{-1}, \]
where \( \Psi_t \) is the restriction of \( \Psi \) to the fiber over \( t \). The *Conley-Zehnder index* of \( (P, \beta) \) is defined as
\[ \text{(2.4)} \]
\[ \mu_{CZ}(P, \beta) := \mu_{CZ}(\varphi) \in \mathbb{Z}. \]
It does not depend on the choice of \( \Psi \) in class \( \beta \). Moreover,
\[ \text{(2.5)} \]
\[ \mu_{CZ}(P, \beta) = \mu_{CZ}(P, \beta') + 2\text{wind}(\beta', \beta), \]
for every \( \beta, \beta' \in \Omega^+_{x_T^* \xi} \).

If \( P \subset M \) is contractible and \( c_1(\xi)|_{\tau_2(M)} \equiv 0 \) then there exists a special class \( \beta_{\text{disk}} \in \Omega^+_{x_T^* \xi} \) induced by any trivialization of \( x_T^* \xi \) which extends to a trivialization of \( u^* \xi \), where \( u : D \to M \) is a capping disk for \( P \), i.e., \( u \) is continuous and \( u_t(e^{2\pi it}) = x_T(t)\cdot u \). We may denote \( \mu_{CZ}(P, \beta_{\text{disk}}) \) simply by \( \mu_{CZ}(P) \).

Let \( \varphi \) be the path of \( 2 \times 2 \) symplectic matrices defined as in (2.3), which is associated to the closed Reeb orbit \( P \in \mathcal{P}(\lambda) \) and the homotopy class \( \beta \in \Omega^+_{x_T^* \xi} \). In [23] the authors present a more geometrical definition of \( \mu_{CZ}(P) \). Given \( 0 \neq v \in \mathbb{R}^2 \), choose a continuous argument \( \theta_v(t) \) of \( \varphi_v(t) \). Let \( \Delta(v) := (\theta_v(1) - \theta_v(0))/2\pi \) and...
consider the interval $J := \{ \Delta(v) : 0 \neq v \in \mathbb{R}^2 \}$. It is possible to show that $J$ has length $< 1/2$. Moreover, for every $\epsilon > 0$ sufficiently small, we have

$$
\mu_{CZ}(P, \beta) = \begin{cases} 
2k, & \text{if } k \in J, \\
2k - 1, & \text{if } J \subset (k - 1, k),
\end{cases}
$$

where $J := J - \epsilon$.

As before, it does not depend on $\Psi$ is class $\beta$ degenerate, then following relation is verified

$$\rho(P, \beta) = \lim_{t \to \infty} \frac{\theta_i(t)}{2\pi t}.
$$

The limit above does not depend on $v$. It satisfies

$$\rho(P, \beta) = \rho(P, \beta') + \text{wind}(\beta', \beta),$$

for every $\beta, \beta' \in \Omega^+_{x_2\xi}$.

Given $n \in \mathbb{N}$, denote by $P^n := (x, nT)$ the $n$-th iterate of $P$. It follows that

$$\rho(P^n, \beta^n) = n\rho(P, \beta),$$

where $\beta^n \in \Omega^+_{x_2^n\xi}$ is the homotopy class induced by $\beta \in \Omega^+_{x_2\xi}$. If $P^n$ is non-degenerate, then following relation is verified

$$\mu_{CZ}(P^n, \beta^n) = \begin{cases} 
2n\rho(P, \beta), & \text{if } P^n \text{ is hyperbolic}, \\
2\lfloor n\rho(P, \beta) \rfloor + 1, & \text{if } P^n \text{ is elliptic}.
\end{cases}
$$

Moreover, $\mu_{CZ}(P, \beta) = 2$ if and only if $P$ is hyperbolic and $\rho(P, \beta) = 1$.

A $d\lambda$-compatible complex structure $J$ on $x_2^\perp\xi$ determines the inner product

$$\int_{\mathbb{R}/\mathbb{Z}} d\lambda_{x_2^\perp\xi}(t)(Z(t), J_t \cdot W(t))dt, \quad \forall Z, W \in L^2(\mathbb{R}/\mathbb{Z}).$$

Here $J_t$ is the restriction of $J$ to the fiber over $t$. Using an auxiliary symmetric connection $\nabla$ on $TM$ we define the so called asymptotic operator $A_P$ as the unbounded self-adjoint operator

$$A_P : \eta \mapsto -J_t \cdot (\nabla_t \eta - T\nabla_{\eta}X_\lambda),$$

defined on $W^{1,2}(x_2^\perp\xi) \subset L^2(x_2^\perp\xi)$, where $X_\lambda$ is the Reeb vector field of $\lambda$ and $\nabla_t$ is the covariant derivative along $x_T$. The operator $A_P$ does not depend on $\nabla$. Choosing a $d\lambda$-symplectic trivialization $\Psi$ of $x_2^\perp\xi$, the operator $A_P$ is represented as $-J(t) \partial_t - S(t)$, for smooth curves $t \in \mathbb{R}/\mathbb{Z} \mapsto J(t), S(t) \in \mathcal{L}_R(\mathbb{C})$, satisfying $J(t)^2 = -I$ and $\det J(t) = 1$. If $\Psi$ is unitary then $J(t) \equiv i$ and $S(t)^T = S(t)\overline{t}$. Therefore the operator $A_P$ takes the form $L_S$ as in (2.1). Thus given $\eta \in \sigma(A_P) = \sigma(L_S)$, a non-vanishing $\eta$-eigensection is represented in this trivialization by an eigenvector $e \in \ker(L_S - \eta I)$ which has a winding number $\text{wind}(\zeta, \beta) := \text{wind}(\eta)$.

Defining

$$\text{wind}^{<0}(A_P, \beta) := \text{wind}^{<0}(L_S),$$

$$\text{wind}^{\geq0}(A_P, \beta) := \text{wind}^{\geq0}(L_S),$$

it follows that

$$\mu_{CZ}(P, \beta) = \text{wind}^{<0}(A_P, \beta) + \text{wind}^{\geq0}(A_P, \beta).$$

As before, it does not depend on $\Psi$ is class $\beta$. 

2.2. The rational self-linking number. Let \((M, \xi = \ker \lambda)\) be a co-oriented contact manifold and let \(K \subset M\) be a \(k\)-unknot transverse to \(\xi\) oriented by \(\lambda\). Let \(u : \mathbb{D} \to M\) be a \(k\)-disk for \(K\) inducing the same orientation and let \(Z\) be a smooth non-vanishing section of \(u^*\xi\). Given \(\epsilon > 0\) small, consider the loop \(Z_\epsilon : \mathbb{R}/\mathbb{Z} \to M \setminus K\) defined by \(Z_\epsilon(t) := \exp_{u(e^{2\pi \epsilon t})}(\epsilon Z(e^{2\pi it}))\), for some exponential map exp.

**Definition 2.2.** The rational self-linking number \(sl(K, u) \in \mathbb{Q}\) is defined as the oriented intersection number

\[
sl(K, u) = \frac{1}{k^2} Z_\epsilon \cdot u,
\]

where \(u\) has the orientation induced by \(\mathbb{D}\) and \(Z_\epsilon\) inherits the orientation of \(Z\). The ambient manifold \(M\) is oriented by \(\lambda \wedge d\lambda\).

The integer \(sl(K, u)\) does not depend on \(\epsilon, \exp\) and \(Z\). If \(c_1(\xi)|_{\pi_2(M)} \equiv 0\) then it does not depend on the \(k\)-disk \(u\) as well. In this case, it is simply denoted by \(sl(K)\).

2.3. Pseudo-holomorphic curves in symplectizations. Let \((M, \xi = \ker \lambda)\) be a co-oriented contact 3-manifold. Its symplectization is the symplectic manifold \((\mathbb{R} \times M, \omega + \lambda^2)\), where \(\lambda\) is the \(\mathbb{R}\)-coordinate. Denote by \(\pi_\lambda : TM \to \xi\) the projection along the Reeb direction \(X_\lambda\), i.e., \(\pi_\lambda(v) = v - \lambda(v)X_\lambda\). \(\forall v \in TM\).

Let \(J_+\) be the set of \(D\lambda\)-compatible complex structures on \(\xi\). Given \(J \in J_+\) consider the almost-complex structure \(\tilde{J}\) on \(\mathbb{R} \times M\) defined by

\[
\tilde{J}_a \cdot \partial_a = X_\lambda \quad \text{and} \quad \tilde{J}|_{\xi} \equiv J.
\]

Let \((S, j)\) be a closed Riemann surface and let \(\Gamma \subset S\) be a finite set. A non-constant map \(\tilde{u} = (a, u) : S \setminus \Gamma \to \mathbb{R} \times M\) is called a finite energy \(\tilde{J}\)-holomorphic curve if it satisfies

\[
\partial\tilde{J}(\tilde{u}) := \frac{1}{2}(d\tilde{u} + \tilde{J}(\tilde{u}) \circ d\tilde{u} \circ j) = 0,
\]

and has finite Hofer’s energy

\[
E(\tilde{u}) := \sup_{\phi \in \Lambda} \int_{S \setminus \Gamma} \tilde{u}^* d(\phi \lambda) < \infty,
\]

where \(\Lambda := \{ \phi \in C^\infty(\mathbb{R}, [0, 1]), \phi' \geq 0 \}\).

**Definition 2.3.** The map \(\tilde{u} : S \setminus \Gamma \to \mathbb{R} \times M\) is called somewhere injective if there exists \(z_0 \in S \setminus \Gamma\) so that \(d\tilde{u}(z_0) \neq 0\) and \(\tilde{u}^{-1}(\tilde{u}(z_0)) = \{z_0\}\).

The elements in \(\Gamma\) are called punctures. A puncture \(z \in \Gamma\) is called positive if \(a(\zeta) \to +\infty\) as \(\zeta \to z\). It is called negative if \(a(\zeta) \to -\infty\) as \(\zeta \to z\). It is called removable if \(a\) is bounded near \(z\). If \(z\) is removable then \(\tilde{u}\) can be smoothly extended over \(z\) \([16]\). We always assume that no puncture is removable. In this case a puncture is either positive or negative. By Stokes’ theorem there always exists a positive puncture.

Let \(z \in \Gamma\). Take a holomorphic chart \(\varphi : (\mathbb{D}, 0) \to (U \subset S, z)\) on a neighborhood \(U\) of \(z\). Then we have positive cylindrical coordinates \((s, t) \in [0, +\infty) \times \mathbb{R}/\mathbb{Z}\) near \(z\) given by \((s, t) \simeq \varphi(e^{-2\pi(s+it)})\) and negative cylindrical coordinates \((s, t) \in (-\infty, 0) \times \mathbb{R}/\mathbb{Z}\) near \(z\) given by \((s, t) \simeq \varphi(e^{2\pi(s+it)})\).

The following result due to Hofer establishes a deep connection between finite energy \(\tilde{J}\)-holomorphic curves and closed Reeb orbits.
Theorem 2.4 (Hofer [16]). Let \((s, t) \in [0, +\infty) \times \mathbb{R}/\mathbb{Z}\) be positive cylindrical coordinates near a puncture \(z \in \Gamma\). For each sequence \(s_\eta \to +\infty\), there exists a subsequence, still denoted \(s_\eta\), and a closed Reeb orbit \(P = (x, T)\) so that
\[
u(s_\eta, \cdot) \to x(\epsilon_\eta T \cdot) \text{ in } C^{\infty}_{\text{loc}}(\mathbb{R}/\mathbb{Z}, M) \text{ as } n \to +\infty.
\]
Here \(\epsilon_\eta = +1 \) (\(\epsilon_\eta = -1\)) if \(z\) is a positive (negative) puncture.

Definition 2.5. Let \(z \in \Gamma\) be a puncture of a finite energy \(\bar{J}\)-holomorphic curve \(\bar{u}\). Let \(\epsilon_\eta \in \{-1, 1\}\) be its sign and let \((s, t) \in [0, +\infty) \times \mathbb{R}/\mathbb{Z}\) be positive cylindrical coordinates near \(z\). We say that \(z\) is non-degenerate if the following hold:

- There exists a closed Reeb orbit \(P = (x, T) \in \mathcal{P}(\lambda)\) and \(c \in \mathbb{R}\) so that \(u(s, t) \to x(\epsilon_\eta T s - T c)\) and \(|a(s, t) - \epsilon_\eta T s - c| \to 0\), uniformly in \(t\), as \(s \to +\infty\).
- If \(\zeta(s, t) \in \xi_{x(T s)}\) is defined by \(u(s, t) = \exp_{x(T s)}(\zeta(s, t))\) for \(s \gg 1\) then there exists \(b > 0\) so that \(e^{bs}|\zeta(s, t)| \to 0\), uniformly in \(t\), as \(s \to +\infty\).
- If \(\pi \circ du\) does not vanish identically then it does not vanish for \(s\) large enough.

The closed Reeb orbit above is called the asymptotic limit of \(\bar{u}\) at \(z\). If \(\lambda\) is non-degenerate then every puncture of \(\bar{u}\) is non-degenerate [19].

Definition 2.6. Let \(P(x, T) \in \mathcal{P}(\lambda)\) be a closed Reeb orbit. Denote by \(T_{\text{min}} > 0\) the least period of \(x\). A Martinet tube for \(P\) is a pair \((U, \Psi)\), where \(U\) is a neighborhood of \(x(\mathbb{R})\) in \(M\) and \(\Psi : U \to \mathbb{R}/\mathbb{Z} \times B\) is a diffeomorphism \((B \subset \mathbb{R}^2\) is a ball centered at the origin and \(\mathbb{R}/\mathbb{Z} \times B\) is provided with coordinates \((\theta, x_1, x_2)\)) satisfying
\[
\Psi^* (f \cdot (d\theta + x_1 dx_2)) = \lambda, \text{ where } f : \mathbb{R}/\mathbb{Z} \times B \to \mathbb{R}^+ \text{ is smooth and satisfies}\]
\[
\int_{\mathbb{R}/\mathbb{Z} \times \{0\}} f = T_{\text{min}} \text{ and } df|_{\mathbb{R}/\mathbb{Z} \times \{0\}} \equiv 0.
\]
\[
\Psi(x(T_{\text{min}})) = (t, 0, 0)\forall t.
\]

Let \(\bar{u} = (a, u) : (S \setminus \Gamma, J) \to (\mathbb{R} \times M, \bar{J})\) be a finite energy \(\bar{J}\)-holomorphic curve. Let \(z_0 \in \Gamma\) be a puncture of \(\bar{u}\) with sign \(c \in \{-1, 1\}\). Consider positive cylindrical coordinates \((s, t) \in [0, +\infty) \times \mathbb{R}/\mathbb{Z}\) near \(z_0\) if \(z_0\) is positive or consider negative cylindrical coordinates near \(z_0\) if \(z_0\) is negative. Assume that \(\lambda\) is nondegenerate and let \(P = (x, T) \in \mathcal{P}(\lambda)\) be the asymptotic limit of \(\bar{u}\) at \(z_0\). Choose a Martinet tube \((U, \Psi)\) for \(P\). For \(e s \gg 1\) there are well defined functions \(\theta(s, t) \in \mathbb{R}/\mathbb{Z}\), \(x_1(s, t), x_2(s, t) \in \mathbb{R}\), so that
\[
\Psi \circ u(s, t) = (\theta(s, t), z(s, t)) \in \mathbb{R}/\mathbb{Z} \times B \text{ with } z(s, t) := (x_1(s, t), x_2(s, t)).
\]
We still denote by \(\theta(s, t)\) its lift to \(\mathbb{R}\). Hence it satisfies \(\theta(s, t + 1) = \theta(s, t) + k\), \(\forall(s, t)\), where \(k \in \mathbb{N}\) is the covering multiplicity of \(P\) determined by \(T = k T_{\text{min}}\). We may assume that \(\theta(s, 0) \to 0\) as \(s \to +\infty\). We also assume that \(\pi \circ du\) does not vanish near \(z_0\).

Theorem 2.7 (Hofer, Wysocki and Zehnder [19], Siefring [38]). There exist an eigenvalue \(\eta\) of \(A_P\) satisfying \(e \eta < 0\), a non-vanishing \(\eta\)-eigensection \(v\), a function \(R(s, t) \in \mathbb{R}^2\) defined for \(es \gg 1\), and constants \(c \in \mathbb{R}, r > 0\), such that the following hold:
\[
\lim_{s \to +\infty} e^{rs} ||\partial_s^2 \partial_t^2 [a(s, t) - Ts - c](s, \cdot)||_{L^\infty(\mathbb{R}/\mathbb{Z})} = 0
\]
\[
\lim_{s \to +\infty} e^{rs} ||\partial_s^2 \partial_t^2 [\theta(s, t) - kT](s, \cdot)||_{L^\infty(\mathbb{R}/\mathbb{Z})} = 0,
\]
\( \forall (\beta_1, \beta_2) \in \mathbb{N} \times \mathbb{N}. \) Moreover, if \( t \in \mathbb{R}/\mathbb{Z} \mapsto e(t) \in \mathbb{R}^2 \) is the representation of \( v \) in the frame \( \{ \partial_{x_1}, \partial_{x_2} \} \) along \( x(T) \) then

\[
(2.10) \quad z(s, t) = e^{\psi}(e(t) + R(s, t)) \quad \text{and} \quad \lim_{\epsilon_s \to +\infty} ||\partial_x^2 \partial_t R(s, \cdot)||_{L^\infty(\mathbb{R}/\mathbb{Z})} = 0.
\]

The eigenvalue \( \eta \) and the eigensection \( v \) are called the asymptotic eigenvalue and asymptotic eigensection of \( \tilde{u} \) at \( z_0 \), respectively. Theorem 2.7 remains valid if we only require \( z_0 \) to be a nondegenerate puncture.

Let \( \beta \in \Omega^+(T_{\min}) \) be a homotopy class of symplectic trivializations of \( \xi \) along the prime closed Reeb orbit \( P_{\min} = (x, T_{\min}) \). Recall that \( k \) is the covering number of \( P \) over \( P_{\min} \). Let \( \beta^k \) be the homotopy class of symplectic trivializations of \( \xi \) along \( P \) induced by \( \beta \). Let \( \text{wind}(v, \beta^k) \) be the winding number of the asymptotic eigensection \( v \) of \( \tilde{u} \) at \( z_0 \) with respect to \( \beta^k \).

**Definition 2.8 (Relatively prime puncture, [31]).** The nondegenerate puncture \( z_0 \in \Gamma \) is called relatively prime if \( \gcd(\text{wind}(v, \beta^k), k) = 1 \).

Observe that Definition 2.8 does not depend on \( \beta \). The following lemma will be very useful in the proof of Theorem 1.5.

**Lemma 2.9 (Hryniewicz and Salomão [31]).** Let \( z_0 \in \Gamma \) be a nondegenerate and relatively prime puncture of a finite energy \( \tilde{J} \)-holomorphic curve \( \tilde{u} = (a, u) : S \setminus \Gamma \to \mathbb{R} \times M \). Then there exists a neighborhood \( \mathcal{U} \) of \( z \) in \( S \) so that \( u|_{\mathcal{U} \setminus \{z\}} \) is an embedding.

We finish this section introducing some algebraic invariants of finite energy curves as in [17]. Let \( \tilde{u} = (a, u) : (S \setminus \Gamma, j) \to (\mathbb{R} \times M, \tilde{J}) \) be a finite energy \( \tilde{J} \)-holomorphic curve, where \( S \) is closed and connected. Assume that every puncture of \( \tilde{u} \) is nondegenerate. If \( u^*d\lambda \) does not vanish identically then \( \pi \circ du \) has only finitely many zeros. Indeed, Carleman’s similarity principle implies that the zeros of \( \pi \circ du \) are isolated and contribute positively. Moreover, since every \( z \in \Gamma \) is nondegenerate, \( \pi \circ du \) does not vanish close to \( z \).

Hofer, Wysocky and Zehnder define

\[
\text{wind}_v(\tilde{u}) := \text{algebraic count of zeros of } \pi \circ du.
\]

We have

\[
\text{wind}_v(\tilde{u}) = 0 \Leftrightarrow u \text{ is an immersion}.
\]

Now take a \( d\lambda \)-symplectic trivialization \( \sigma \) of \( u^*\xi \). For each \( z \in \Gamma \), \( \sigma \) induces a homotopy class \( \beta_z \) of \( d\lambda \)-symplectic trivializations of \( x_z(T_z)^*\xi \), where \( P_z = (x_z, T_z) \) is the asymptotic limit of \( \tilde{u} \) at \( z \). Set \( \text{wind}(\tilde{u}, z, \sigma) := \text{wind}(v_z, \beta_z) \), where \( v_z \) is the asymptotic eigensection of \( \tilde{u} \) at \( z \). Then define

\[
\text{wind}_\infty(\tilde{u}) := \sum_{z \in \Gamma^+} \text{wind}(\tilde{u}, z, \sigma) - \sum_{z \in \Gamma^-} \text{wind}(\tilde{u}, z, \sigma),
\]

where \( \Gamma^+ \subset \Gamma \) is the set of positive punctures and \( \Gamma^- \subset \Gamma \) is the set of negative punctures of \( \tilde{u} \). The above sum does not depend on \( \sigma \). In [17] the authors prove that

\[
(2.11) \quad 0 \leq \text{wind}_v(\tilde{u}) = \text{wind}_\infty(\tilde{u}) - \chi(S) + \#\Gamma.
\]

**Definition 2.10.** A finite energy plane \( \tilde{u} : \mathbb{C} \to \mathbb{R} \times M \) is called fast if the puncture at \( \infty \) is nondegenerate and \( \text{wind}_v(\tilde{u}) = 0 \) \((\text{wind}_\infty(\tilde{u}) = 1)\).
3. Proof of main result

The proof of Theorem 1.5 follows the same ideas in [20] and [31] where the cases 
$p = 1$ and $p = 2$ are treated, respectively. Here we assume $p > 2$.

Let $\pi_p : S^3 \to L(p, 1)$ be the natural projection. Since $\lambda$ is dynamically convex, 
$\pi_p^* \lambda$ is dynamically convex as well and hence tight, see [21]. It follows that $\xi = \ker \lambda$ 
is universally tight on $L(p, 1)$ and, we may assume $\xi = \xi_0$ and $\lambda = f\lambda_0$ for some smooth function $f : L(p, 1) \to (0, +\infty)$.

Let $H : \mathbb{R}^4 \to \mathbb{R}$ be the Hamiltonian function

$$H(x_1, x_2, y_1, y_2) = \frac{x_1^2 + y_1^2}{r_1^2} + \frac{x_2^2 + y_2^2}{r_2^2},$$

where $r_1 < r_2 \in \mathbb{R}$. Let $E := H^{-1}(1) \in \mathbb{R}^4$ be the ellipsoid associated to $r_1$ and $r_2$. 

It induces a contact form $\lambda_E = fE\lambda_0$ on $S^3$ by pulling back the Liouville form via 
the radial map $S^3 \to E$. Since $\lambda_E$ is preserved under $g_{p,1}$, it descends to a contact 
form on $L(p, 1)$, still denoted $\lambda_E$.

If $(r_1/r_2)^2 \in \mathbb{R} \setminus \mathbb{Q}$ then the Reeb flow of $\lambda_E$ on $L(p, 1)$ has precisely two simple 
nondegenerate closed Reeb orbits $P_1 = \pi_{p,1}(S^1 \times \{0\})$ and $P_2 = \pi_{p,1}(\{0\} \times S^1)$, 
forming a Hopf link. Their periods are $T_1 = \pi r_1^2/p$ and $T_2 = \pi r_2^2/p$, respectively. 
The closed Reeb orbits $P_1$ and $P_2$ are $p$-unknotted and have self-linking number 
$-1/p$ [26]. Moreover, the Conley-Zehnder indices of the contractible closed Reeb 
orbits $P_{1}^p$ and $P_{2}^p$ are

$$\mu_{CZ}(P_{1}^p) = 3 \quad \text{and} \quad \mu_{CZ}(P_{2}^p) = 2k + 1,$$

where $k \in \mathbb{N}$ is such that $r_1^2/r_2^2 \in (k - 1, k)$. It is always possible to choose 
$0 < r_1 < r_2$ large enough so that $f < f_E$ on $L(p, 1)$, see [28].

3.1. The non-degenerate case. In this section we prove Theorem 1.5 in the non-degenerate case. 
The degenerate case will be treated as a limiting case.

**Proposition 3.1.** Let $\lambda = f\lambda_0$ be a contact form on $L(p, 1)$, where $f : L(p, 1) \to (0, +\infty)$ is smooth. Choose $0 < r_1 < r_2$ and $r_1^2/r_2^2 \in \mathbb{R} \setminus \mathbb{Q}$ so that $f < f_E$ pointwise. 
Assume that every closed Reeb orbit $P \in \mathcal{P}(\lambda)$ with period $\leq \pi r_1^2$ satisfies:

(i) $P$ is nondegenerate.

(ii) If $P$ is contractible then $\mu_{CZ}(P, \beta_{\text{disk}}) \geq 3$.

Let $J \in \mathcal{J}_+(\xi)$ and let $\tilde{J}$ be the cylindrical almost complex structure on $\mathbb{R} \times L(p, 1)$ 
induced by $\lambda$ and $J$. Then there exists a $p$-unknotted closed Reeb orbit $P_0 = (x_0, T_0) \in \mathcal{P}(\lambda)$ 
and a finite energy $\tilde{J}$-holomorphic plane $\tilde{u} : \mathbb{C} \to \mathbb{R} \times L(p, 1)$ satisfying

- The asymptotic limit of $\tilde{u}$ at $\infty$ is the nondegenerate closed Reeb orbit $P_0^p = (x_0, pT_0)$ whose Conley-Zehnder index is $\mu_{CZ}(P_0^p, \beta_{\text{disk}}) = 3$.
- $\tilde{u}$ is embedded and $E(\tilde{u}) = pT_0 \leq \pi r_1^2$.
- $\tilde{u}$ is an embedding transverse to $X_\lambda$ and $u(\mathbb{C}) \cap x_0(\mathbb{R}) = \emptyset$.
- the rational self-linking number of $P_0$ is $-1/p$.

The first step in the proof of Proposition 3.1 consists of constructing a symplectic cobordism between $\lambda$ and $\lambda_E$, see [20, 31] for more details. Take a smooth function $h : \mathbb{R} \times L(p, 1) \to \mathbb{R}^+$ satisfying

- $h(a, \cdot) = f$, if $a \leq -2$.
- $h(a, \cdot) = f_E$, if $a \geq 2$.
- $\frac{\partial h}{\partial a} \geq 0$.
The family of contact forms \( \lambda_a := h(a, \cdot)\lambda_0, a \in \mathbb{R} \), interpolates \( \lambda \) and \( \lambda_E \) and the contact structure \( \xi = \text{ker} \lambda_a \) does not depend on \( a \). Choose \( J_E \in \mathcal{J}_+(\lambda_E) \) and consider a family \( J_a \in \mathcal{J}_+(\lambda_a), a \in \mathbb{R} \), such that \( J_a = J \) if \( a \leq -2 \) and \( J_a = J_E \) if \( a \geq 2 \). Consider a smooth almost complex structure \( \bar{J} \) on \( \mathbb{R} \times L(p, 1) \) with the following properties:

- \( \bar{J} = \bar{J}_a \) on \( (\mathbb{R} \setminus [-1, 1]) \times L(p, 1) \), where \( \bar{J}_a \) satisfies \( \bar{J}_a \cdot \partial_a = X_{\lambda_a} \) and \( \bar{J}_a \mid_\xi = J_a \).
- \( \bar{J} \) is compatible with the symplectic form \( d(h\lambda_0) \) on \([-1, 1] \times L(p, 1) \).

The space of such almost complex structures \( \bar{J} \) is non-empty and contractible in the \( C^\infty \)-topology and is denoted by \( \mathcal{F}(\lambda_E, J_E, \lambda, J) \).

Let \( \Gamma \in \mathbb{C} \) be a finite set. Consider non-constant maps

\[
\tilde{u} : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times L(p, 1),
\]

which satisfy

\[
d\tilde{u} \circ i = \bar{J}(\tilde{u}) \circ d\tilde{u},
\]

for some \( \bar{J} \in \mathcal{F}(\lambda_E, J_E, \lambda, J) \), with finite Hofer’s energy \( 0 < E(\tilde{u}) < \infty \). The energy \( E(\tilde{u}) \) is defined as follows: let \( \Lambda \) be the space of smooth functions \( \phi : \mathbb{R} \to [0, 1] \) satisfying \( \phi' \geq 0 \) and \( \phi = 1/2 \) on \([-1, 1]\). Then

\[
E(\tilde{u}) := \sup_{\phi \in \Lambda} \int_{\mathbb{C} \setminus \Gamma} \tilde{u}^* d(\phi(a)\lambda_a),
\]

where \( \lambda_a \) is seen as 1-form on \( \mathbb{R} \times L(p, 1) \). These maps are called generalized finite energy spheres and if \( \Gamma = \emptyset \) then they are called generalized finite energy planes.

As in the case of cylindrical almost complex structures, the set of non-removable punctures \( \Gamma \) of a generalized finite energy sphere \( \tilde{u} = (a, u) \) is non-empty and if \( z_0 \in \Gamma \) then either \( a(z) \to +\infty \) or \( a(z) \to -\infty \) as \( z \to z_0 \in \Gamma \). Therefore, it makes sense to talk about positive and negative punctures. Moreover, if \( \tilde{u} \) is a generalized finite energy sphere then \( \tilde{u} \) has at least one positive puncture due to the exact nature of the symplectic cobordism.

**Proposition 3.2 (Hofer-Wysocki-Zehnder [20]).** If \( \tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times L(p, 1) \) is a generalized finite energy plane then:

- \( a(z) \to +\infty \) as \( |z| \to +\infty \).
- \( u(Re^{2\pi i \cdot}) \to x(T) \) in \( C^\infty(\mathbb{R}/\mathbb{Z}) \) as \( R \to +\infty \).

Here \( P = (x, T) \) is a contractible closed Reeb orbit of \( \lambda_E \) whose period satisfies \( T = E(\tilde{u}) \).

Fix \( \bar{J} \in \mathcal{J}(\lambda, J, \lambda_E, J_E) \). Denote by \( \Theta \) the space of \( \bar{J} \)-holomorphic generalized finite energy planes asymptotic to \( P_1^p \), modulo holomorphic re-parameterizations. Recall that \( P_1 \subset L(p, 1) \) is an elliptic non-contractible closed Reeb orbit of \( \lambda_E \) with period \( \pi r_1^2/p \). Moreover, the Conley-Zehnder index of its \( p \)-th iterate is \( \mu CZ(P_1^p) = 3 \). The energy of each \( \tilde{u} \in \Theta \) is \( E(\tilde{u}) = \pi r_1^2 \).

Following [20] and [31], \( \Theta \) has the structure of a 2-dimensional manifold. Moreover, using the intersection theory of holomorphic curves developed by Siefring in [38, 39], there exists a connected component \( \Theta' \subset \Theta \) whose planes are embedded and do not intersect each other.
3.1.1. Bubbling-off tree. At this moment we are interested in the compactness properties of the component $\Theta' \subset \Theta$. In order to do that we introduce a bubbling-off tree of pseudo-holomorphic curves. Consider an oriented, rooted and finite tree $T$, and a finite set $U$ of finite energy $J$-holomorphic spheres so that the following properties are satisfied:

- There exists a bijective correspondence between vertices $q \in T$ and finite energy spheres $\tilde{u}_q \in U$. Every sphere $\tilde{u}_q : C \setminus \Gamma_q \to \mathbb{R} \times L(p,1)$ is pseudo-holomorphic with respect to $J_E$, $\bar{J}$ or $\tilde{J}$. Moreover, every oriented path $(q_1,\ldots,q_N)$ from the root $q_1 = r$ to a leaf $q_N \in T$ has at most one vertex $q_i$ for which $\tilde{u}_{q_i}$ is $\bar{J}$-holomorphic. In that case, $\tilde{u}_{q_j}$ is $J_E$-holomorphic for $1 \leq j < i$ and $\tilde{u}_{q_i}$ is $\bar{J}$-holomorphic for $i < j \leq N$.

- Every sphere $\tilde{u}_q$ has precisely one positive puncture at $\infty$ and $0 \leq \# \Gamma_q < +\infty$ negative punctures.

- If the vertex $q$ is not a root, then $q$ has an income edge $e$ that comes from a vertex $q'$ and $\# \Gamma_q$ outgoing edges $f_1,\ldots,f_{\# \Gamma_q}$ that go to vertices $p_1,\ldots,p_{\# \Gamma_q} \in T$, respectively. The edge $e$ is associated to the positive puncture of $\tilde{u}_q$ and the edges $f_1,\ldots,f_{\# \Gamma_q}$ are associated to the $\# \Gamma_q$ negative punctures of $\tilde{u}_q$. The asymptotic limit of $\tilde{u}_q$ at its positive puncture coincides with the asymptotic limit of $\tilde{u}_{q'}$ at its negative puncture associated to $e$. In the same way, the asymptotic limit of $\tilde{u}_q$ at its negative puncture corresponding to $f_i$ coincides with the asymptotic limit of $\tilde{u}_p$ at its unique positive puncture. If $\tilde{u}_q$ is $J_E$-holomorphic then $\tilde{u}_p$ is either $J_E$ or $\bar{J}$-holomorphic. If $\tilde{u}_q$ is $\bar{J}$ or $\bar{J}$-holomorphic then $\tilde{u}_p$ is necessarily $\bar{J}$-holomorphic.

- If $\tilde{u}_q$ is $J_E$ or $\bar{J}$-holomorphic and its contact area vanishes then $\# \Gamma_q \geq 2$.

Denote by $B = (T,U)$ the bubbling-off tree, where $T$ and $U$ satisfies the above properties. Take a sequence $\tilde{u}_n = (a_n,u_n)$ of generalized finite energy planes representing elements of $\Theta'$. It follows that $E(\tilde{u}_n) = \pi r^2, \forall n$. Hence the STF compactness theorem in [5] implies the following theorem.

**Theorem 3.3.** There exists a bubbling-off tree $B = (T,U)$ so that, up to a subsequence of $\tilde{u}_n$, the following hold:

- Given a vertex $q \in T$ there exist sequences $z^n_q$, $\delta^n_q \in \mathbb{C}$ and $c^n_q \in \mathbb{R}$ such that
  \begin{equation}
  \tilde{u}_n(z^n_q + \delta^n_q) + c^n_q \to \tilde{u}_q \text{ in } C^\infty_{\text{loc}}(C \setminus \Gamma_q) \text{ as } n \to +\infty.
  \end{equation}

Here, $\tilde{u} + c := (a + c, u)$.

- The curve $\tilde{u}_r$ associated to the root $r \in T$ is asymptotic to $P^\lambda_1$ at $\infty$. Moreover, every asymptotic limit of every $\tilde{u}_r$ is a contractible closed Reeb orbit of $\lambda_E$ or $\lambda$ and its period is $\leq \pi r^2$.

The bubbling-off tree $B = (T,U)$ given in Theorem 3.3 is called the SFT-limit of $\tilde{u}_n$. Due to a deep analysis found in [20] and [31] in the cases $p = 1$ and $p = 2$, respectively, it is possible to obtain a much better description of the bubbling-off tree $B$. Given sequences $z^n_q$, $\delta^n_q \in \mathbb{C}$ and $c^n_q \in \mathbb{R}$ as in (3.2) is verified, precisely one of the following alternatives holds:

1. $c^n_q$ is bounded, $a_n(z^n_q + \delta^n_q) \in C^0_{\text{loc}}(C \setminus \Gamma_q)$-bounded in $n$ and $\tilde{u}_q$ is $J$-holomorphic.

2. $c^n_q \to -\infty$, $a_n(z^n_q + \delta^n_q) \to +\infty$ in $C^0_{\text{loc}}(C \setminus \Gamma_q)$ as $n \to +\infty$, and $\tilde{u}_q$ is $J_E$-holomorphic.
Proposition 3.4 \([20], [31]\). There exists a sequence \(\tilde{u}_n = (a_n, u_n) \in \Theta'\) so that
\[
\min_{z \in \mathbb{C}} a_n(z) \to -\infty \quad \text{as} \quad n \to +\infty.
\]

The proof of Proposition 3.4 essentially follows from the SFT compactness theorem and the Fredholm theory developed for embedded holomorphic curves in symplectizations \([22]\).

Results from Hofer, Wysocki e Zehnder in \([22]\) guarantee the following.

Theorem 3.5. There exists a dense subset \(\mathcal{J}_{reg} \subset \mathcal{J}(\lambda_E, J_E, \lambda, J)\) so that if \(\check{u} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times L(p, 1)\) is a somewhere injective generalized finite energy \(\check{J}\)-holomorphic sphere, with \(J \in \mathcal{J}_{reg}\), then
\[
\text{Fred}(\check{u}) := \mu_{CZ}(P_\infty) - \sum_{z \in \Gamma} \mu_{CZ}(P_z) + \#\Gamma - 1 \geq 0.
\]

The Conley-Zehnder indices above are computed using a trivialization which extends over \(u^*\xi\).

We assume from now on that \(\check{J} \in \mathcal{J}_{reg}\) and so (3.4) holds.

Theorem 3.6. The bubbling-off tree \(B = (\mathcal{T}, \mathcal{U})\) obtained as the STF limit of a sequence \(\tilde{u}_n = (a_n, u_n) \in \Theta'\) satisfying (3.3) contains only two vertices \(r\) and \(q\). The root \(r\) corresponds to a finite energy \(\check{J}\)-holomorphic cylinder \(\tilde{u}_r : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times L(p, 1)\) which is asymptotic to \(P_1^r\) at its positive puncture \(\infty\) and to a closed orbit \(P \in \mathcal{P}(\lambda)\) at its negative puncture \(0\). The Conley-Zehnder index of \(P\) is 3. The vertex \(q\) corresponds to an embedded finite energy \(\check{J}\)-holomorphic plane \(\tilde{u}_q = (a_q, u_q) : \mathbb{C} \to \mathbb{R} \times L(p, 1)\) asymptotic to \(P\) at \(\infty\). There exists a \(p\)-unknotted closed Reeb orbit \(P' \in \mathcal{P}(\lambda)\) so that \(P = (P')^p\) and \(u_q : \mathbb{D} \to L(p, 1)\) is a \(p\)-disk for \(P'\).

Proof. Let \(\tilde{u}_r = (a_r, u_r) : \mathbb{C} \setminus \Gamma_r \to \mathbb{R} \times L(p, 1)\) be the finite energy sphere associated to the root \(r \in \mathcal{T}\). Then \(P_1^p\) is the asymptotic limit of \(\tilde{u}_r\) at its positive puncture. Next we show that \(\tilde{u}_r\) is \(\check{J}\)-holomorphic.

There exist sequences \(z_{\Gamma_r}^r, \delta_{\Gamma_r}^r \in \mathbb{C}\) and \(c_{\Gamma_r}^r \in \mathbb{R}\) so that (3.2) holds for \(\tilde{u}_r\). We know that either (I) or (II) holds. Suppose that (II) holds. Then \(\tilde{u}_r\) is \(\check{J}\)-holomorphic. If the \(d\lambda_E\)-area of \(\tilde{u}_r\) vanishes then \(\#\Gamma_r \geq 2\). Since every asymptotic limit at a puncture in \(\Gamma_r\) is contractible, its action is \(\geq \pi r_1^2\). This implies \(\int_{\mathbb{C} \setminus \Gamma_r} u_r^* d\lambda_E \leq \pi r_1^2 - \#\Gamma_r \pi r_1^2 < 0\), a contradiction. If the \(d\lambda_E\)-area of \(\tilde{u}_r\) does not vanish then we claim that \(\Gamma_r = \emptyset\). If this is not the case then the action of the asymptotic limit \(P_r\) at a negative puncture is strictly less than \(\pi r_1^2\). Since \(P_r\) must be contractible, we get a contradiction. Indeed, \(\pi r_1^2\) is the smallest action of a contractible closed Reeb orbit of \(\lambda_E\).

Now suppose that \(\tilde{u}_r\) is a finite energy \(\check{J}_E\)-holomorphic plane. In this case, due to Lemma 3.13 in \([31]\) and the absence of bubbling-off points for the sequence \(\tilde{u}_n\), given \(M > 0\) large there exists \(R > 0\) so that \(a_n(z_n^r + \delta_n^r z^r) + c_n^r > M\) if \(|z| > R_0\).
and \( a_n(z_n^r + \delta_n^r z) > 0 \) if \(|z| \leq R_0\), for all large \( n \). Since \( c_n^r \to -\infty \), we obtain 
\[
\inf_n \min_{z \in \mathbb{C}} a_n(z) \geq 0,
\]
contradicting (3.3). We conclude that (I) holds for \( \tilde{u}_r \).

Now we check that \( \tilde{u}_r \) has at least one negative puncture. In fact, if \( \Gamma_r = \emptyset \) then \( \tilde{u}_r \) is a finite energy \( J \)-holomorphic plane. Since \( c_n^r \) is bounded, it follows from Lemma 3.13 in [31] that there exists \( R_0 > 0 \) so that \( a_n(z_n^r + \delta_n^r z) \in [0, +\infty) \) if \(|z| > R_0\), for all large \( n \). However, \( a_n(z_n^r + \delta_n^r z) \) is uniformly bounded on compact subsets of \( \mathbb{C} \), in particular, on \( \{|z| \leq R_0\} \). This contradicts (3.3). Hence \( \Gamma_r \neq \emptyset \).

So far we know that \( \tilde{u}_r \) is \( J \)-holomorphic and has at least one negative puncture. In the following we consider two different cases. Suppose first that \( \tilde{u}_r \) is not somewhere injective, the other case will be treated later. Then we find a somewhere injective \( J \)-holomorphic curve \( \tilde{v}_r : \mathbb{C} \setminus \Gamma' \to \mathbb{R} \times L(p, 1), 1 \leq \#\Gamma' \), and a polynomial \( Q : \mathbb{C} \to \mathbb{C} \) with degree \( \deg(Q) \geq 2 \) satisfying \( \tilde{u}_r = \tilde{v}_r \circ Q \) and \( Q^{-1}(\Gamma') = \Gamma_r \). All punctures in \( \Gamma' \) are negative and the corresponding asymptotic limits are closed Reeb orbits of \( \lambda \). Furthermore, the asymptotic limit of \( \tilde{v}_r \) at \( \infty \) is \( P_k^1 \), where \( p = k \cdot \deg(Q) \).

Denote by \( \Gamma' = \{w_1, \ldots, w_n\} \subset \mathbb{C} \) the set of negative punctures of \( \tilde{v}_r \), and let \( P_{w_i} = (x_{w_i}, T_{w_i}) \) be the asymptotic limit of \( \tilde{v}_r \) at \( w_i, i = 1, \ldots, n \). Denote by \( P_{\infty} = (x_\infty, T_\infty) = P_k^1 = (x_1, kT_{\min}) \) the asymptotic limit of \( \tilde{v}_r \) at \( \infty \). We shall need the following lemma whose proof is found in Appendix A.

Lemma 3.7. The curve \( \tilde{v}_r \) contains precisely one puncture whose asymptotic limit is non-contractible.

Assume that \( w_1 \) is the puncture given in Lemma 3.7. In particular, \( P_{w_1} \) is non-contractible and \( P_{w_2}, \ldots, P_{w_n} \) are contractible, see Figure 3.1. For each \( i = 2, \ldots, n \), we choose a symplectic trivialization of \( x_{w_i}(T_{w_i})^*\xi \) which extends over a capping disk \( f_i : \mathbb{D} \to L(p, 1) \) for \( P_{w_i} \). Since \( \tilde{v}_r \# f_2 \# \ldots \# f_n \) is topologically a cylinder, each class \( \beta \) of symplectic trivializations of \( x_\infty(T_{\infty})^*\xi \) induces a class \( \beta' \) of symplectic trivializations of \( x_{w_1}(T_{w_1})^*\xi \).

Lemma 3.8. There exists a class \( \beta \) of \( d\lambda_E \)-symplectic trivializations of \( x_1(T_{\min})^*\xi \) such that \( \mu_{CZ}(P_1^j, \beta^j) = 2j - 1 \) for every \( j = 1, \ldots, p \). Moreover, \( \text{wind}(\beta^p, \beta_{\text{disk}}) = 2 - p \).
The proof of Lemma 3.8 is found in Appendix B. Due to Lemma 3.8 we find a class $\beta^k$ of symplectic trivializations of $x_{\infty}(T_{\infty})^*\xi$ so that

$$
\mu_{CZ}(P_{\infty}, \beta^k) = \mu_{CZ}(P^k_{1}, \beta^k) = 2k - 1.
$$

The class $\beta^k$ induces a class $\beta'$ of symplectic trivializations of $x_{w_1}(T_{w_1})^*\xi$.

**Proposition 3.9.** $\mu_{CZ}(P_{w_1}, \beta') = 2k - 1$ and $n = 1$.

**Proof.** Using the class $\beta$ as in Lemma 3.8 and the dynamical convexity of $\lambda$ we obtain

$$
0 \leq \mu_{CZ}(P_{\infty}, \beta^k) - \mu_{CZ}(P_{w_1}, \beta') - \sum_{i=2}^{n} \mu_{CZ}(P_{w_1}, \beta_{disk}) + \# \Gamma' - 1
$$

$$
= \mu_{CZ}(P^k_{1}, \beta^k) - \mu_{CZ}(P_{w_1}, \beta') - \sum_{i=2}^{n} \mu_{CZ}(P_{w_1}, \beta_{disk}) + \# \Gamma' - 1
$$

$$
\leq 2k - 1 - \mu_{CZ}(P_{w_1}, \beta') - 3(n - 1) + n - 1
$$

$$
= -\mu_{CZ}(P_{w_1}, \beta') - 2(n - 1) + 2k - 1,
$$

from where we conclude that

$$
(3.5) \quad \mu_{CZ}(P_{w_1}, \beta') \leq 2k - 1 - 2(n - 1) \leq 2k - 1.
$$

Now the class $\beta^k$ induces the class $(\beta^k)^{p/k} = \beta^p$ of symplectic trivializations of $x_{\infty}(\frac{p}{k}(kT_{\min}))^*\xi = x_{\infty}(pT_{\min})^*\xi$ such that $\mu_{CZ}((P_{\infty})^{p/k}, \beta^p) = 2p - 1$, see Lemma 3.8. Furthermore, wind$((\beta^k)^{p/k}, \beta_{disk}) = 2 - p$. By the hypothesis on $\lambda$, we know that $\mu_{CZ}(P_{w_1}^{p/k}, \beta_{disk}) \geq 3$. Therefore, it follows from (2.5) that

$$
\mu_{CZ}(P_{w_1}^{p/k}, (\beta')^{p/k}) = \mu_{CZ}(P_{w_1}^{p/k}, \beta_{disk}) + 2\text{wind}(\beta_{disk}, (\beta')^{p/k})
$$

$$
\geq 3 - 2\text{wind}((\beta')^{p/k}, \beta_{disk})
$$

$$
= 3 - 2(2 - p) = 2p - 1.
$$

The periodic orbit $P_{w_1}$ is non-degenerate and hence it is either hyperbolic or elliptic. Assume that $P_{w_1}$ is hyperbolic and suppose by contradiction that

$$
\mu_{CZ}(P_{w_1}, \beta') < 2k - 1.
$$

From (2.8), it follows that

$$
\mu_{CZ}(P_{w_1}^{p/k}, (\beta')^{p/k}) = \frac{p}{k} \mu_{CZ}(P_{w_1}, \beta') < \frac{p}{k}(2k - 1) \leq 2p - 1,
$$

which is in contradiction with (3.6).

Assume now that $P_{w_1}$ is elliptic. In view of (3.6) we have

$$
\rho(P_{w_1}^{p/k}, (\beta')^{p/k}) \geq p - 1
$$

and hence

$$
(3.7) \quad \rho(P_{w_1}, \beta') \geq k \frac{p - 1}{p} \geq k - 1.
$$

Therefore, it follows from (2.8) and (3.7) that

$$
\mu_{CZ}(P_{w_1}, \beta') = 2[\rho(P_{w_1}, \beta')] + 1 \geq 2k - 1.
$$

Using (3.5) we conclude that $\mu_{CZ}(P_{w_1}, \beta') = 2k - 1$ and $n = 1$. \qed
Proposition 3.9 implies $\Gamma' = \{w_1\}$. Furthermore, we know that $\mu_{\text{CZ}}(P_{w_1}, \beta') = 2k - 1$. If $P_{w_1}$ is hyperbolic then (2.8) implies that $\rho(P_{w_1}, \beta') = \mu_{\text{CZ}}(P_{w_1}, \beta')/2 = k - 1/2$. It follows that
\[
\mu_{\text{CZ}}(P_{w_1}^{p/k}, (\beta')^{p/k}) = 2 \rho(P_{w_1}^{p/k}, (\beta')^{p/k}) = \frac{2p}{k} \rho(P_{w_1}, \beta') = \frac{2p}{k} \left( k - \frac{1}{2} \right) = 2p - \frac{p}{k} \leq 2p - 1.
\]
Now if $P_{w_1}$ is elliptic then (2.8) implies that $\rho(P_{w_1}, \beta') = \mu_{\text{CZ}}(P_{w_1}, \beta') - 1/2 = k - 1$ and hence $\rho(P_{w_1}, \beta') < k$. Hence
\[
\mu_{\text{CZ}}(P_{w_1}^{p/k}, (\beta')^{p/k}) = 2 \rho((P_{w_1})^{p/k}, (\beta')^{p/k}) + 1 = 2p - 1 + 1 = 2p - 1.
\]
Using (3.6) we conclude that $\mu_{\text{CZ}}(P_{w_1}^{p/k}, (\beta')^{p/k}) = 2p - 1$. Lemma 3.8 and (2.5) imply the following crucial fact
(3.8) $\mu_{\text{CZ}}(P_{w_1}^{p/k}, \beta_{\text{disk}}) = 2p - 1 + 2(2 - p) = 3$.

Recall that $\deg(Q) = p/k$. We claim that the polynomial $Q$ has the form
(3.9) $Q(z) = w_1 + a(z - z_1)^{p/k}$, for some $a \in \mathbb{C}^*$ and $z_1 \in \mathbb{C}$. If this is not the case, then we write $Q(z) = w_1 + b \prod_{i=1}^l (z - z_i)^{m_i}$ for some $b \in \mathbb{C}^*$, $m_i \in \mathbb{N}^*$, $z_i \neq z_j \in \mathbb{C}, \forall i \neq j$, and $l \geq 2$. Observe that $\Gamma_r = \{z_1, \ldots, z_l\}$ is the set of punctures of $\tilde{u}_r$. We have $\sum_{i=1}^l m_i = p/k$ and $\sum_{i=1}^l (m_i - 1) = p/k - l \leq p/k - 2 < \deg(Q')$. Hence $Q$ must have a critical point $z^* \notin \Gamma_r$. This implies that $z^*$ is a critical point of $\tilde{u}_r$ which is a contradiction since $\tilde{u}_r$ is the $C_{\text{loc}}^\infty$-limit of $\tilde{u}_n$ and none of $\tilde{u}_n$ has a critical point. We conclude that $\Gamma_r = \{z_1\}$ and $Q$ has the form (3.9). Moreover, the asymptotic limit of $\tilde{u}_r$ at $z_1$ is the contractible closed Reeb orbit $P := P_{w_1}^{p/k}$ which, in view of (3.8), has Conley-Zehnder index 3.

Now we deal with the case $\tilde{u}_r$ is somewhere injective. The asymptotic limit $P_r$ of $\tilde{u}_r$ at a negative puncture $z \in \Gamma_r$ is a contractible periodic orbit, its period is $\leq \pi r_1^2$ and, therefore, $\mu_{\text{CZ}}(P_r, \beta_{\text{disk}}) \geq 3$. Using that $\mu_{\text{CZ}}(P_{w_1}^{p/k}, \beta_{\text{disk}}) = 3$ and Theorem 3.5 we obtain
\[
0 \leq \mu_{\text{CZ}}(P_{w_1}^{p/k}, \beta_{\text{disk}}) - \sum_{z \in \Gamma_r} \mu_{\text{CZ}}(P_z, \beta_{\text{disk}}) + \# \Gamma_r - 1
\]
\[
= 2 - \sum_{z \in \Gamma_r} \mu_{\text{CZ}}(P_z, \beta_{\text{disk}}) + \# \Gamma_r
\]
\[
\leq 2 - 3\# \Gamma_r + \# \Gamma_r = 2(1 - \# \Gamma_r).
\]
now since $\# \Gamma_r \geq 1$, we conclude that $\# \Gamma_r = 1$. Hence $\tilde{u}_r$ has only one puncture whose asymptotic limit $P$ is a contractible periodic orbit with $\mu_{\text{CZ}}(P, \beta_{\text{disk}}) = 3$.

We have concluded that $\tilde{u}_r$ is a $J$-holomorphic curve with precisely one negative puncture whose asymptotic limit $P = (x, T)$ is contractible and has Conley-Zehnder index 3.

Now we want to prove that the vertex $q$, immediately below the root $r$, is associated to an embedded $J$-holomorphic plane. In particular, $r$ and $q$ are the only vertices of the bubbling-off tree $B$. Since $\tilde{u}_r$ is $J$-holomorphic, we conclude that the finite energy sphere $\hat{u}_q = (a_p, u_q) : \mathbb{C} \setminus \Gamma_q \to \mathbb{R} \times L(p, 1)$ is $\tilde{J}$-holomorphic.
We know that \( \tilde{u}_q \) is asymptotic to \( P = (x,T) \) at \( \infty \), which is a contractible closed Reeb orbit of \( \lambda \) with Conley-Zehnder index 3. Every asymptotic limit of \( \tilde{u}_q \) at a negative puncture is contractible and non-degenerate, with period \( \leq \pi r_1^2 \) and Conley-Zehnder index \( \geq 3 \).

Suppose that \( \Gamma_q \neq \emptyset \) and denote by \( P_z \) the asymptotic limit of \( \tilde{u}_q \) at \( z \in \Gamma_q \). If the \( d\lambda \)-area vanishes then \( \#\Gamma_q \geq 2 \). Theorem 6.11 in [17] implies that \( \tilde{u}_q(C \setminus \Gamma_q) = \mathbb{R} \times P \). Denote by \( P' \) the prime closed Reeb orbit of \( P \), i.e., \( P' \) is simple and \( P = (P')^{m_z} \) for some \( m_z \in \mathbb{N}^* \). Then \( P_z = (P')^{m_z} \) for some \( m_z \in \mathbb{N}^* \), \( \forall z \in \Gamma_q \).

We have \( m_\infty = \sum m_z \geq 2 \) since \( \#\Gamma_q \geq 2 \). In particular, \( P \) is not simple.

**Lemma 3.10.** Let \( P = (x,T) \in \mathcal{P}(\lambda) \) be a nondegenerate contractible closed Reeb orbit satisfying \( \mu_{CZ}(P, \beta_{disk}) = 3 \) and whose period satisfies \( T \leq \pi r_1^2 \). Assume that \( P \) is not simple. Then there exists a simple non-contractible closed Reeb orbit \( P' \in \mathcal{P}(\lambda) \) so that \( P = (P')^k \) for some \( k \geq 2 \). Moreover, \( k \) divides \( p \) and there exists no \( k' \in \{1, \ldots, k - 1\} \) such that \( (P')^{k'} \) is contractible.

Its proof is given in Appendix C. Applying the above lemma to \( P \) we conclude that \( P' \) is non-contractible and \( m_\infty \) is the least positive integer \( k \) so that \( (P')^k \) is contractible. Since \( P_z = (P')^{m_z} \) is contractible and \( \#\Gamma_q \geq 2 \) we obtain \( m_z < m_\infty \forall z \in \Gamma_q \). This is a contradiction and we conclude that the \( d\lambda \)-area of \( \tilde{u}_q \) is positive.

Now if the \( d\lambda \)-area of \( \tilde{u}_q \) is positive then the invariants \( \text{wind}_{\infty}(\tilde{u}_q) \) and \( \text{wind}_\pi(\tilde{u}_q) \) are well-defined, see section 2.

Let \( \sigma \) be a symplectic trivialization of \( \tilde{u}_q^*\xi \). Since every asymptotic limit of \( \tilde{u}_q \) is contractible we can assume that \( \sigma \) extends over every asymptotic limit as a symplectic trivialization of \( \xi \) in class \( \beta_{disk} \). Since \( \mu_{CZ}(P, \beta_{disk}) = 3 \) and \( \mu_{CZ}(P_z, \beta_{disk}) \geq 3 \forall z \in \Gamma_q \), we have

\[
\text{wind}_{\infty}(\tilde{u}_q, \infty, \sigma) = 1 \quad \text{and} \quad \text{wind}_{\infty}(\tilde{u}_q, z, \sigma) \geq 2, \forall z \in \Gamma_q.
\]

Inequalities above and (2.11) give

\[
0 \leq \text{wind}_\pi(\tilde{u}_q) = \text{wind}_{\infty}(\tilde{u}_q) - 1 + \#\Gamma_q
\]
\[
= \text{wind}_{\infty}(\tilde{u}_q, \infty, \sigma) - \sum_{z \in \Gamma_q} \text{wind}_{\infty}(\tilde{u}_q, z, \sigma) - 1 + \#\Gamma_q
\]
\[
\leq 1 - 2\#\Gamma_q - 1 + \#\Gamma_q = -\#\Gamma_q,
\]

and, therefore, \( \Gamma_q = \emptyset \). Thus \( \tilde{u}_q \) is a \( J \)-holomorphic finite energy plane and

\[
(3.10) \quad \text{wind}_\pi(\tilde{u}_q) = \text{wind}_{\infty}(\tilde{u}_q) - 1 = 0.
\]

In particular, \( u_q \) is an immersion transverse to the Reeb vector field \( X_\lambda \). We claim that \( \tilde{u}_q \) is somewhere injective. Indeed, otherwise we write \( \tilde{u}_q = \tilde{v}_q \circ Q \), for a somewhere injective finite energy \( J \)-holomorphic plane and a polynomial \( Q : \mathbb{C} \to \mathbb{C} \) with degree \( \deg(Q) \geq 2 \). This forces \( \tilde{u}_q \) to have at least one critical point which is impossible since \( u_q \) is an immersion.

Now we claim that \( \tilde{u}_q \) is an embedding. Arguing indirectly, if this is not the case then

\[
D := \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \setminus \{\text{diagonal}\} : \tilde{u}_q(z_1) = \tilde{u}_q(z_2)\}
\]

is non-empty. Furthermore, \( D \) is discrete, since \( \tilde{u}_q \) is somewhere injective. This follows from the similarity principle. Therefore, self-intersections of \( \tilde{u}_q \) are isolated. By the positivity and stability of intersections of pseudo-holomorphic curves, it
follows that $\tilde{u}_n$ also has self-intersections for large $n$. This is a contradiction since every $\tilde{u}_n$ is embedded.

We have concluded that the bubbling-off tree has exactly two vertices, the root $r$ and a single vertex $q$ below it. The curve $\tilde{u}_q$ is an embedded finite energy $J$-holomorphic plane whose asymptotic limit at $\infty$ is a closed Reeb orbit $P = (x, T) \in \mathcal{P}(\lambda)$ whose Conley-Zehnder index is $3$. Moreover, $\text{wind}_\tau(\tilde{u}_q) = 0$ and hence $u_q$ is an immersion transverse to the Reeb vector field $X_\lambda$. The winding number of the asymptotic eigensection $e_\infty$ of $A_P$ at $\infty$ satisfies $\text{wind}_\infty(e_\infty, \beta_{\text{disk}}) = 1$.

At this moment we want to prove that
\begin{equation}
(3.11) \quad u|_{C \setminus B_R(0)} \text{ is an embedding for } R \gg 1.
\end{equation}

Since $P$ is contractible and $\mu_{\text{CZ}}(P, \beta_{\text{disk}}) = 3$, Lemma 3.10 implies that either $P$ is simple or $P = (P')^k$, for a simple non-contractible periodic orbit $P'$, where $k \geq 2$ is an integer which divides $p$. If $P$ is simple, (3.11) follows directly. In the following we assume that $P$ is not simple.

The class $\beta$ of $d\lambda_k$-symplectic trivializations of the contact structure along $P_1$, as given in Lemma 3.8, induces a class $\bar{\beta} \simeq \beta$ of $d\lambda$-symplectic trivializations of the contact structure along closed curves in class $1 \in \mathbb{Z}_p \cong \pi_1(L(p, 1))$. We know that $|P'| = q \in \{1, 2, ..., p - 1\}$ and that there exists $n \in \mathbb{N}^*$ such that
\begin{equation}
(3.12) \quad qk = np.
\end{equation}

Take a $d\lambda$-symplectic trivialization of the contact structure along the contractible closed curve $P = (P')^k$ in class $\bar{\beta}^p$. Since $\text{wind}(\bar{\beta}_{\text{disk}}, \bar{\beta}^p) = p - 2$, it follows that
\begin{align*}
\mu_{\text{CZ}}((P')^k, \bar{\beta}^p) &= \mu_{\text{CZ}}((P')^k, \beta_{\text{disk}}) + 2\text{wind}(\beta_{\text{disk}}, \bar{\beta}^p) \\
&= 3 + 2(p - 2) = 2p - 1.
\end{align*}

In particular,
\begin{equation}
(3.13) \quad \text{wind}(e_\infty, \bar{\beta}^p) = p - 1.
\end{equation}

Now choose a $d\lambda$-symplectic trivialization of the contact structure along $P'$ in class $\bar{\beta}^q$. From (3.12) and (3.13) we have
\begin{align*}
\text{wind}(e_\infty, (\bar{\beta}^q)^k) &= \text{wind}(e_\infty, (\bar{\beta})^q)^k = \text{wind}(e_\infty, (\bar{\beta})^{np}) \\
&= \text{wind}(e_\infty, (\bar{\beta}^p)^n) = n\text{wind}(e_\infty, \bar{\beta}^p) \\
&= n(p - 1).
\end{align*}

We claim that $n(p - 1)$ and $k$ are relatively prime. To show that observe first that $k$ divides $p$, see Lemma 3.10. Hence $k$ and $p - 1$ are relatively prime. If $n$ and $k$ are not relatively prime, then $n = ln_1$ and $k = lk_1$, for some $2 \leq l$, $n_1$, $k_1 \in \mathbb{N}^*$. Together with (3.12) this implies that $qk_1 = nl$ and thus the periodic orbit $(P')^{k_1}$ is contractible. Hence $\mu((P')^{k_1}, \beta_{\text{disk}}) \geq 3$ and
\begin{equation}
(3.14) \quad \mu((P')^{k_1}, \beta_{\text{disk}}) = \mu((P')^{k_1})^l, \beta_{\text{disk}}) > 3,
\end{equation}
a contradiction. We conclude that $n(p - 1)$ and $k$ are relatively prime numbers. It follows that the puncture of $\tilde{u}_q$ at $\infty$ is relatively prime and (3.11) is verified in view of Lemma 2.9. Thus $\tilde{u}$ is an embedded fast plane and hence $u(C) \cap x(\mathbb{R}) = \emptyset$.

For a proof of these facts, see [31, Section 3.1.8].

Finally, due to [17, Theorem 2.3], $u : \mathbb{C} \to L(p, 1) \setminus x(\mathbb{R})$ is injective. Since $u$ is an immersion it follows that $u$ defines an oriented $k$-disk for $x(\mathbb{R})$, where $k$
is the number of times that $P$ covers $x(\mathbb{R})$. Applying [26, Lemma 3.10] we have $\text{sl}(P) = -\frac{1}{k}$, where $k$ divides $p$. By [26, Lemma 7.3] the Reeb flow of $\lambda$ admits a rational open book decomposition with disk-like pages of order $k$ whose binding is $P$. Applying the characterization theorem of contact lens spaces in [26], due to Hryniewicz, Licata and Salomão, we obtain that $L(p, 1)$ is diffeomorphic to $L(k, 1)$. Hence $k$ is necessarily equal to $p$. This concludes the proof of Theorem 3.6. Proposition 3.1 also follows. □

3.2. The degenerate case. Since $\ker \lambda$ is universally tight we can assume that $\lambda = f\lambda_0$ for a smooth function $f : L(p, 1) \to (0, +\infty)$. Choose $r_1 < r_2$, with $r_1^2/r_2^2 \in \mathbb{R} \setminus \mathbb{Q}$, so that $f < f_E$ pointwise. By [20, Proposition 6.1] we can choose a sequence $f_n \to f$ in $C^\infty$ such that $\lambda_n := f_n\lambda_0$ is non-degenerate for all $n$ and, moreover, $f_n < f_E$ pointwise for all large $n$. Now we show that for all large $n$, $\lambda_n$ satisfies the hypotheses of Proposition 3.1. If this is not the case then there exists a subsequence, still denoted by $\lambda_n$, so that $\lambda_n$ admits a contractible periodic orbit $Q_n$ with period $0 < T_n \leq \pi r_1^2$ and $\mu_{CZ}(Q_n) \leq 2$. By the Arzelà-Ascoli theorem, up to a subsequence, $Q_n$ converges in $C^\infty$ as $n \to \infty$ to a contractible periodic orbit $Q = (w, S) \in \mathcal{P}(\lambda)$ with period $S \leq \pi r_1^2$. Due to the lower semi-continuity of the generalized Conley-Zehnder index, it follows that $\mu_{CZ}(Q) \leq 2$, which is a contradiction, since $\lambda$ is dynamically convex. Thus, by Proposition 3.1, $\lambda_n$ admits a $p$-unknotted periodic orbit $P_n = (x_n, T_n)$, with self-linking number $-\frac{1}{p}$ and $\mu_{CZ}(P_n^p) = 3$, $\forall n$. Moreover, the period $T_n$ is uniformly bounded by $\pi r_1^2/p$ in $n$. By the Arzelà-Ascoli theorem, up to a subsequence, $P_n \to P$ in $C^\infty$ as $n \to \infty$, where $P$ is a periodic orbit of $\lambda$ with period $\leq \pi r_1^2/p$.

Since $P_n$ is $p$-unknotted, it follows that $P_n^p$ is contractible and therefore $P^p$ is contractible as well. Due to the lower semi-continuity of the generalized Conley-Zehnder index, we have $\mu_{CZ}(P^p) \leq 3$. Since $\lambda$ is dynamically convex it follows that $\mu_{CZ}(P^p) = 3$. Now suppose by contradiction that $P$ is not simple. Then $P = (P')^k$ for some $k \geq 2$, where $P'$ is simple. Since $\pi_1(L(p, 1)) \cong \mathbb{Z}_p$, we have that $(P')^p$ is contractible and therefore $\mu_{CZ}((P')^p, \beta_{\text{disk}}) \geq 3$. From $P^p = ((P')^p)^k = ((P')^{k})^k$, $k \geq 2$, it follows that $\mu_{CZ}(P^p, \beta_{\text{disk}}) \geq 5$, which is a contradiction. Hence $P$ is simple and $P_n \to P$ as $n \to \infty$. The simple periodic orbit $P$ is transversely isotopic to each $P_n$ for all large $n$. Hence $P$ is $p$-unknotted and has self-linking number $-\frac{1}{p}$.

Applying [31, Corollary 1.8], the periodic orbit $P$ bounds a $p$-disk which is a global surface of section for the Reeb flow. Moreover, this $p$-disk is a page of a rational open book decomposition of $L(p, 1)$ with binding $P$ such that all pages are disk-like global surfaces of section of order $p$. The proof of Theorem 1.5 is now complete.

APPENDIX A. PROOF OF LEMMA 3.7

Observe first that $P_k^\infty$ is non-contractible since $1 \leq k < p$. Hence there exists a puncture $z_1 \in \Gamma_r$ of $\tilde{u}_r$ so that $Q(z_1) = w_1 \in \Gamma'$ and $Q'(z_1) = 0$. In fact, if such a puncture does not exist then $Q$ is a local bi-holomorphism near each puncture of $\Gamma_r$ and hence every asymptotic limit of $\tilde{u}_r$ at a puncture in $\Gamma'$ is contractible. This forces $P_k^\infty$ to be contractible, a contradiction.

Write

\begin{equation}
Q(z) = w_1 + a(z - z_1)^{k_1}(z - z_2)^{k_2} \cdots (z - z_n)^{k_n},
\end{equation}
where $0 \neq a, z_i \in \mathbb{C}$, $k_i \in \mathbb{N}^*$ and $z_i \neq z_j, \forall i \neq j$. Notice that $Q(z_i) = w_1, \forall i = 1 \ldots n$. We claim that

$$k_i \geq 2, \ \forall i = 1, \ldots, n. \tag{A.2}$$

The inequality above follows from the fact that the asymptotic limit of $\tilde{v}_r$ at $w_1$ is non-contractible. In fact, if $k_i = 1$ for some $i \in \{1, \ldots, n\}$ then $Q$ is a local bi-holomorphism near $z_i$. Since the asymptotic limit $P_{z_i}$ of $\tilde{u}_r$ at $z_i$ is contractible, this implies that $P_{w_1}$ is contractible as well, a contradiction. Hence (A.2) holds and, in particular,

$$n \leq \frac{\deg(Q)}{2}. \tag{A.3}$$

Moreover, each $z_i$ is a critical point of $Q$ with multiplicity $k_i - 1 \geq 1$. Hence the total contribution of $z_i$ to the critical points of $Q$ is given by

$$\sum_{i=1}^{n} (k_i - 1) = \deg(Q) - n \geq \frac{\deg(Q)}{2}. \tag{A.4}$$

Now suppose that there exists a puncture $w_2 \neq w_1 \in \Gamma'$ of $\tilde{v}_r$ whose asymptotic limit is also non-contractible. As before, write

$$Q(z) = w_2 + b(z - \tilde{z}_1)^{l_1}(z - \tilde{z}_2)^{l_2} \ldots (z - \tilde{z}_m)^{l_m}, \tag{A.5}$$

where $0 \neq b, \tilde{z}_i \in \mathbb{C}$ is a constant, $l_i \geq 2, \forall j = 1, \ldots, m$, and

$$m \leq \frac{\deg(Q)}{2}. \tag{A.6}$$

Since $w_2 \neq w_1$, we clearly have $\tilde{z}_j \neq z_i, \forall i, j$. Hence the total contribution of $\tilde{z}_i$ to the critical points of $Q$ is given by

$$\sum_{i=1}^{m} (l_i - 1) = \deg(Q) - m \geq \frac{\deg(Q)}{2}. \tag{A.7}$$

Summing up (A.4) and (A.7) we conclude that $Q$ has at least $\deg(Q)$ critical points, a contradiction. We conclude that $\tilde{v}_r$ has precisely one puncture whose asymptotic limit is non-contractible. \hfill \qed

**Appendix B. Proof of Lemma 3.8**

Let us consider the $\lambda_E$-closed Reeb orbit $\tilde{P}_1 = (x_1, \tilde{T}_1 = \pi r_1^2) \subset S^3$, where $x_1(t) = (e^{2\pi it}/T_1, 0) \in \mathbb{C}^2$, $t \in \mathbb{R}/T_1 \mathbb{Z}$. The contact structure $\xi_0$ along $P_1$ is spanned by $\{\partial_x, \partial_y\}$ in coordinates $(x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2$. Consider the non-vanishing section $\tilde{Z}_1$ of $x_1(\tilde{T}_1)^*\xi_0$ given by $\tilde{Z}_1(t) = -e^{-2\pi it}$, $t \in \mathbb{R}/\mathbb{Z}$, where $a + ib \equiv a\partial_x + b\partial_y$. Defining $\tilde{Z}_2 = i\tilde{Z}_1$ we see that $\{\tilde{Z}_1, \tilde{Z}_2\}$ induces the class $\beta_{\text{disk}}$ of $d\lambda_E$-symplectic trivializations of $x_1(\tilde{T}_1)^*\xi_0$, see [30].

Now consider the following non-vanishing section of $x_1(\tilde{T}_1)^*\xi_0$

$$\tilde{Z}(t) := -e^{(p-1)(-2\pi it)}, \ t \in \mathbb{R}/\mathbb{Z}. \tag{B.1}$$

A direct computation shows that $\text{wind}(\tilde{Z}, \tilde{Z}_1) = 2 - p$. Hence, denoting by $\tilde{\beta}$ the homotopy class of the $d\lambda_E$-symplectic trivializations of $x_1(\tilde{T}_1)^*\xi_0$ induced by $\tilde{Z}$, we obtain

$$\text{wind}(\tilde{\beta}, \beta_{\text{disk}}) = 2 - p. \tag{B.2}$$
Furthermore, the section $\tilde{Z}$ presents the following $\mathbb{Z}_p$-symmetry
\begin{equation}
\tilde{Z}(t + 1/p) = e^{2\pi i \frac{t}{p}} \tilde{Z}(t), \forall t \in \mathbb{R}/\mathbb{Z},
\end{equation}
see Figure B.1. Hence, using the projection $\pi_{p,1} : S^3 \rightarrow L(p,1)$, $\tilde{Z}$ descends to a non-vanishing section $\tilde{Z}$ of the contact structure $\xi_0$ along the simple closed Reeb orbit $P_1 = (x_1, T_1 = \tilde{T}_1/p) \subset L(p,1)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure.png}
\caption{The non-vanishing section $\tilde{Z}$ is $\mathbb{Z}_p$-symmetric.}
\end{figure}

Denote by $\beta$ the homotopy class of $d\lambda_E$-symplectic trivializations of $x_1(T_1)\ast \xi_0$ induced by $\tilde{Z}$. It follows from (B.2) that
\begin{equation}
\text{wind}(\beta^p, \beta_{\text{disk}}) = 2 - p.
\end{equation}
We know that $\mu_{CZ}(P_1^p, \beta_{\text{disk}}) = 3$ and hence, by (2.5) and (B.4), we obtain
\begin{equation}
\mu_{CZ}(P_1^p, \beta^p) = \mu_{CZ}(P_1^p, \beta_{\text{disk}}) - 2\text{wind}(\beta^p, \beta_{\text{disk}}) = 3 - 2(2-p) = 2p - 1.
\end{equation}
Since $P_1$ is elliptic and nondegenerate, we know from (2.8) that
\begin{equation}
2p - 1 = \mu_{CZ}(P_1^p, \beta^p) = 2 \lfloor p \rho(P_1, \beta) \rfloor + 1,
\end{equation}
which implies that
\begin{equation}
\frac{p-1}{p} < \rho(P_1, \beta) < 1.
\end{equation}
Now given $1 \leq k \leq p$, $k \in \mathbb{N}$, we obtain from the inequalities above that
\begin{equation}
k - 1 \leq k \frac{(p-1)}{p} < k \rho(P_1, \beta) < k,
\end{equation}
and hence $\lfloor k \rho(P_1, \beta) \rfloor = k - 1$. This implies that $\mu_{CZ}(P_1^k, \beta^k) = 2k - 1$. \hfill $\square$

**Appendix C. Proof of Lemma 3.10**

Let $P'$ be a contractible periodic orbit so that $P = (P')^k$. Since $P$ is nondegenerate, $P'$ is nondegenerate as well. Using (2.7), (2.8) and that $\mu_{CZ}(P, \beta_{\text{disk}}) = 3$ we obtain
\begin{equation}
1 < \rho(P, \beta_{\text{disk}}) < 2 \Rightarrow \frac{1}{k} < \rho(P', \beta_{\text{disk}}) < \frac{2}{k}.
\end{equation}
Since the period of $P'$ is $T/k \leq \frac{\pi r_1^2}{k}$, the assumptions on $\lambda$ imply that
\begin{equation}
\mu_{CZ}(P', \beta_{\text{disk}}) \geq 3.
\end{equation}
Hence $\rho(P', \beta_{\text{disk}}) > 1$ which, in view of (C.1), gives $k = 1$.

We conclude that if $P$ is not simple then there exist $k \geq 2$ and a simple non-contractible closed orbit $P' \in P(\lambda)$ so that $(P')^k = P$. Next we prove that such $k$ divides $p$. Since $\pi_1(L(p,1)) \simeq \mathbb{Z}_p$ and $P'$ is non-contractible, we have

$$[P'] = q \in \{1, 2, \ldots, p-1\}.$$ 

Since $P$ is contractible, $qk = np$ for some $n \in \mathbb{N}^*$.

If $k$ does not divide $p$ then $k = k_1k_2$ for some integers $k_1, k_2$ where $k_1 \geq 2$ divides $n$. Then $(P')^{k_2}$ is contractible since $qk_2 = \frac{n}{k_1}p$ is a multiple of $p$. Using the above argument we must have $k_1 = 1$ since $((P')^{k_2})^{k_1} = P$. This is a contradiction and we conclude that $k$ divides $p$.

Now assume that there exists $k' \in \{1, \ldots, k-1\}$ so that $(P')^{k'}$ is contractible. Take the least $k'$ with this property. We claim that $k'$ divides $k$. If this is not the case then let $1 \leq r < k'$ be the remainder of the division $k/k'$ and let $m \in \mathbb{N}^*$ be such that $k = mk' + r$. Since $(P')^{k'}$ is contractible we have $qk' = np$ for some $n' \in \mathbb{N}^*$. Then $qr = qk - qmk' = np - mnp$ is a multiple of $p$, which implies that $(P')^{r'}$ is contractible. This contradicts the minimality of $k'$. We conclude that $k'$ divides $k$. Since $(P')^{k'}$ is contractible and $((P')^{k'})^{k'/k'} = P$ we obtain by the previous argument that $k/k' = 1$, again a contradiction. We conclude that there exists no $k' \in \{1, \ldots, k-1\}$ so that $(P')^{k'}$ is contractible. \hfill \Box

References

1. P. Albers, J. W. Fish, U. Frauenfelder, H. Hofer, and O. van Koert, *Global surfaces of section in the planar restricted 3-body problem*, Archive for Rational Mechanics and Analysis 204 (2012), no. 1, 273–284.

2. P. Albers, U. Frauenfelder, O. Van Koert, and G. P. Paternain, *Contact geometry of the restricted three-body problem*, Communications on Pure and Applied Mathematics 65 (2012), no. 2, 229–263.

3. G. Arioli and P. Zgliczyński, *Symbolic dynamics for the Hénon–Heiles Hamiltonian on the critical level*, Journal of Differential Equations 171 (2001), no. 1, 173–202.

4. K. Baker and J. Etnyre, *Rational linking and contact geometry*, Perspectives in analysis, geometry, and topology, Springer, 2012, pp. 19–37.

5. F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, *Compactness results in symplectic field theory*, Geometry & Topology 7 (2003), no. 2, 799–888.

6. R. C. Churchill, M. Kummer, and D. L Rod, *On averaging, reduction, and symmetry in Hamiltonian systems*, Journal of differential equations 49 (1983), no. 3, 359–414.

7. R. C. Churchill, G. Pecelli, and D. L. Rod, *A survey of the Hénon-Heiles Hamiltonian with applications to related examples*, Stochastic behavior in classical and quantum Hamiltonian systems (1979), 76–136.

8. R. C. Churchill and D. L. Rod, *Pathology in dynamical systems. III. Analytic Hamiltonians*, Journal of Differential Equations 37 (1980), no. 1, 23–38.

9. J. C. B. de Figueiredo, C. G. Ragazzo, and C. P. Malta, *Two important numbers in the Hénon-Heiles dynamics*, Physics Letters A 241 (1998), no. 1, 35–40.

10. N. de Paulo and Pedro A. S. Salomão, *On the multiplicity of periodic orbits and homoclinics near critical energy levels of Hamiltonian systems in $\mathbb{R}^4$*, arXiv (2018).

11. , *Systems of transversal sections near critical energy levels of Hamiltonian systems in $\mathbb{R}^4$*, Memoirs of the AMS - to appear (2018).

12. J. Franks, *Geodesics on $S^2$ and periodic points of annulus homeomorphisms*, Inventiones mathematicae 108 (1992), no. 1, 403–418.

13. D. C. Gardiner, M. Hutchings, and D. Pomerleano, *Torsion contact forms in three dimensions have two or infinitely many Reeb orbits*, arXiv:1701.02262 (2017).

14. M. Henon and C. Heiles, *The applicability of the third integral of motion: some numerical experiments*, The Astronomical Journal 69 (1964), 73.
15. G. W. Hill, *Researches in the lunar theory*, American Journal of Mathematics 1 (1878), no. 1, 5–26.
16. H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Inventiones mathematicae 114 (1993), no. 1, 515–563.
17. H. Hofer, K. Wysocki, and E. Zehnder, *Properties of pseudo-holomorphic curves in symplectisations II: Embedding controls and algebraic invariants*, Geometries in Interaction, Springer, 1995, pp. 270–328.
18. \text{same author}, *A characterisation of the tight three-sphere*, Duke J. Math 81 (1996), no. 1, 159–226.
19. \text{same author}, *Properties of pseudoholomorphic curves in symplectisations I: asymptotics*, Annales de l'I.H.P. Analyse non linéaire 13 (1996), no. 3, 337–379 (eng).
20. \text{same author}, *The dynamics on three-dimensional strictly convex energy surfaces*, Annals of Mathematics (1998), 197–289.
21. \text{same author}, *A characterization of the tight three sphere II*, Commun. Pure Appl. Anal 55 (1999), 1139–1177.
22. \text{same author}, *Properties of Pseudoholomorphic Curves in Symplectizations III: Fredholm Theory*, pp. 381–475, Birkhäuser Basel, Basel, 1999.
23. \text{same author}, *Finite energy foliations of tight three-spheres and Hamiltonian dynamics*, Annals of Mathematics (2003), 125–255.
24. K. Honda, *On the classification of tight contact structures I*, Geometry & Topology 4 (2000), no. 2000, 309–368.
25. U. L. Hryniewicz, *Fast finite-energy planes in symplectizations and applications*, Transactions of the American Mathematical Society 364 (2012), no. 4, 1859–1931.
26. U. L. Hryniewicz, J. E. Licata, and P. A. S. Salomão, *A dynamical characterization of universally tight lens spaces*, Proceedings of the London Mathematical Society 110 (2015), no. 1, 215–269.
27. U. L. Hryniewicz, A. Momin, and P. A. S. Salomão, *A Poincaré–Birkhoff theorem for tight Reeb flows on \( S^3 \)*, Inventiones mathematicae 199 (2015), no. 2, 333–422.
28. U. L. Hryniewicz and P. A. S. Salomão, *Uma introdução à geometria de contato e aplicações à dinâmica Hamiltoniana*, Publicações Matemáticas do IMPA, 2009.
29. \text{same author}, *On the existence of disk-like global sections for Reeb flows on the tight 3-sphere*, Duke J. Math 160 (2011), no. 3, 415–465.
30. \text{same author}, *Introdução à Geometria Finsler*, 1 ed., Publicações Matemáticas do IMPA, 2013.
31. \text{same author}, *Elliptic bindings for dynamically convex Reeb flows on the real projective three-space*, Calculus of Variations and Partial Differential Equations 55 (2016), no. 2, 1–57.
32. J. Lee, *Fiberwise convexity of Hill’s lunar problem*, Journal of Topology and Analysis (2014), 1–60.
33. J. Llibre and L. A. Roberto, *On the periodic orbits and the integrability of the regularized Hill lunar problem*.
34. G. H. Lunsford and J. Ford, *On the stability of periodic orbits for nonlinear oscillator systems in regions exhibiting stochastic behavior*, Journal of Mathematical Physics 13 (1972), no. 5, 700–705.
35. K. R. Meyer and D. S. Schmidt, *Hill’s lunar equations and the three-body problem*, Journal of Differential Equations 44 (1982), no. 2, 263–272.
36. C. G. Ragazzo, *Nonintegrability of some Hamiltonian systems, scattering and analytic continuation*, Communications in Mathematical Physics 166 (1994), no. 2, 255–277.
37. P. A. S. Salomão, *Convex energy levels of Hamiltonian systems*, Qualitative Theory of Dynamical Systems 4 (2004), no. 2, 439–454.
38. R. Siefring, *Relative asymptotic behavior of pseudoholomorphic half-cylinders*, Communications on Pure and Applied Mathematics 61 (2008), no. 12, 1631–1684.
39. \text{same author}, *Intersection theory of punctured pseudoholomorphic curves*, Geometry & Topology 15 (2011), no. 4, 2351–2457.
40. C. Simó and T. de J. Stuchi, *Central stable/unstable manifolds and the destruction of KAM tori in the planar Hill problem*, Physica D: Nonlinear Phenomena 140 (2000), no. 1, 1–32.

Universidade Estadual do Centro-Oeste, Rua Camargo Varela de Sá, 3, Guarapuava PR, Brazil, 85040-080
E-mail address: alexsandro@unicentro.br