Stabilization of a heat-like equation in rectangle domain

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Abstract. In this paper, we discuss the stabilization of the heat-like equation in rectangle domain with boundary control. We firstly transfer the heat-like equation in 2D into series of heat equations in 1D via eigenfunction expansion, then, by using the normal backstepping transformation, we obtain the existence and estimate for series of kernel functions in transformations. Finally, combined with the states and controls estimate of series of heat equations, we finally obtain the exponential and finite time stabilization of the heat-like equation in 2D.

1. Introduction
The stabilization control problem of PDEs is a basic problem in control theory. There are many methods can be used in stabilization of PDEs, for example, the backstepping method [1], spectral methods [2], the LQR approach [3], the multiplier technique [4], the microlocal analysis [5] and so on. Among all these methods, the backstepping method is a higher efficient tool in designing boundary control for various types of PDEs (see [6-10] and the reference therein). The advantage of backstepping method embody on the efficiency in designing the explicit feedback control.

The main idea of backstepping method is to change the displacement of eigenvalue for corresponding eigenvalue problems via an invertible backstepping transformation, after that, the feedback control of the original system can be easily obtained according to backstepping transformation and target system. Up to now, the backstepping method has been extensively used in stabilization of ODEs and boundary control of PDEs. However, how to use the backstepping method in designing the stabilization control for heat-like equation in 2D is very interesting. In fact, very limited paper had considered the stabilization of higher order PDE by backstepping method, the main reason lies in the kernel equation had not solved if the coefficient function in control system is not analytic but continuous. In this paper, our ideas lies in transferring the heat-like equation in 2D into 1D heat equation, then, by borrowing the known results for 1D heat equation with backstepping method, we expect to obtain the stabilization results of the heat-like equation in 2D. The presented procedure avoid solving the kernel equation in higher dimension case. Meanwhile, the backstepping method can also be used in finite time stabilization and controllability of PDEs (see [11] and [12]).

In this paper, we consider the exponential stabilization and finite time stabilization of a heat-like equation

$$\begin{align*}
    u_t(x, y, t) &= u_{xx}(x, y, t) - u_{yy}(x, y, t) + c(x)u(x, y, t), \\
    u(0, y, t) &= 0, \quad u(1, y, t) = U(y, t), \\
    u(x, 0, t) &= 0, \quad u(x, 1, t) = 0, \\
    u(x, y, 0) &= u_0(x, y),
\end{align*}$$

(1)
where the coefficient function \( c(\cdot) \in C[0,1] \), \( U(y,t) \) is the control input and \( u_0(x,y) \) is the initial data. In fact, without control, system (1) maybe not stable. For example, take \( u_0(x,y) = \sin(\pi x) \sin(\pi y) \), \( c(x) = 1 \), the solution is \( u(t, x, y) = e^{t}\sin(\pi x) \sin(\pi y) \), obviously, it is not stable. Though there are some other methods can that be used to design the stabilization controller of system (1), here, we will obtain the stabilization control via backstepping method via decomposing the original system into series of 1D heat equations.

According to the eigenfunction expansion, we have

\[
    u(x, y, t) = \sum_{n=1}^{\infty} u_n(x, t) \sin(n\pi y)
\]

where \( u_n(x, t) = \int_0^1 u(x, y, t) \sin(n\pi y) dy \).

Hence, we obtain series of heat equations of 1D with state \( u_n(x, t) \) satisfying

\[
    \begin{cases}
        u_{n,t}(x, t) = u_{n,xx}(x, t) + (n^2\pi^2 + c(x))u_n(x, t), \\
        u_n(0,t) = 0, \quad u_n(1, t) = U_n(t), \\
        u_n(x, 0) = u_{n,0}(x),
    \end{cases}
\]

with the initial data \( u_{n,0}(x) = \int_0^1 u_0(x, y) \sin(n\pi y) dy \) and boundary control

\[
    U_n(t) = \int_0^1 u(1, y, t) \sin(n\pi y) dy.
\]

Next, we will choose a different target system for obtaining the exponential and finite time stabilization of the original system. According to (2) and (3), choosing appropriate target systems in different time interval and estimating the controller, we can obtain the stabilization of the original system.

This paper is organized as follows. In Section 1, we introduce the stabilization control problem. In Section 2, we present the design of stabilized feedback control according to a Volterra integral transformation and do some preliminaries for the proof of main theorem. In Section 3, we prove the exponential stabilization of the original system. In Section 4, we prove the finite time stabilization of the original system. Finally, we summarize the results and present further considering problem.

2. Controller design for the original system

We construct different target systems in different intervals \( t_m < t < t_{m+1} \) for obtaining the appropriate decay rate of target systems and energy estimate of kernels. As for exponential stabilization of the original system, we set \( t_m \rightarrow +\infty \), however, as for finite time stabilization of the original system, we set \( t_m \rightarrow T^- \). For series of systems (3), motivated by [1], we know the integral transformation

\[
    w_n(x, t) = u_n(x, t) - \int_0^x k_{nm}(x,y)u_n(y,t)dy
\]

transferring the systems (3) into exponentially stable target systems

\[
    \begin{cases}
        w_{n,t} = w_{n,xx} - (n^2\pi^2 + \lambda_{mn})w_n, \quad in \ (0,1) \times (t_m, t_{m+1}), \\
        w_n(0,t) = 0, \quad w_n(1, t) = 0, \quad in \ (t_m, t_{m+1}), \\
        w_n(x, 0) = w_{n,0}(x), \quad in \ (0,1),
    \end{cases}
\]

in interval \( (t_m, t_{m+1}) \), where \( \lambda_{mn} \) can be taken different value depending on \( m \) and \( n \) in different case, \( k_{nm}(x,y) \) are undetermined kernels. In [5], we know the kernels \( k_{nm}(x,y) \) satisfy

\[
    \begin{align*}
        &k_{nm,xx}(x,y) - k_{nm,yy}(x,y) = (2n^2\pi^2 + c(y) + \lambda_{mn})k_{nm}(x,y), \\
        &2 \frac{d}{dx}k_{nm}(x,x) + 2n^2\pi^2 + c(x) + \lambda_{mn} = 0, \quad k_{nm}(x,0) = 0,
    \end{align*}
\]
Where the function $c(y)$ is the same function appeared in (1). At the same time, there exists a unique classical solution for (7). After that, we can obtain the boundary feedback controller

$$U_n(t) = \int_0^{t_1} k_{nm}(1,y)u_n(y, t)dy, \quad t \in (t_m, t_{m+1}).$$

(8)

Next, we will recite some known results appeared in [3] for obtaining the estimate of the state and controller.

**Lemma 1.** ([11]) Let $\lambda_{mn} \in \mathbb{R}$, $f \in L^2((0,1)^2)$ and let $c(x,y)$ be a bounded measurable function defined in $(0,1)^2$ such that, for some $\Lambda \geq 1$, $|c(x,y)| \leq \Lambda$ for $(x,y) \in (0,1)^2$. Then, there exists a unique solution $K \in L^2((0,1); H^1_0((0,1)) \cap H^1 ((0,1)^2)$ to the equation

$$K_{xx} - K_{yy} - (2n^2 \pi^2 + c(x,y) + \lambda_{mn})K = f(x,y),$$

(9)

such that

$$K(x,0) = K(x,1) = 0 \text{ for } x \in [0,1]$$

and

$$K(0,y) = K(1,y) = 0 \text{ for } y \in [0,1].$$

(10)

Moreover,

$$\int_0^1 |\nabla K(x,y)|^2 \, dy \leq C \max \{e^{C\text{sign}(2n^2 \pi^2 + \lambda_{mn})\sqrt{|2n^2 \pi^2 + \lambda_{mn}|}}, 1\} ||f||^2_{L^2}.$$  

Using Lemma 1, we can establish the following lemma:

**Lemma 2.** ([11]) Let $\lambda_{mn} \in \mathbb{R}$, $c \in L^2(0,1)$, then, there exists a unique solution $k_{nm} \in H^1(D)$ to the system

$$\begin{cases}
k_{nm,xx}(x,y) - k_{nm,yy}(x,y) = (2n^2 \pi^2 + c(y) + \lambda_{mn})k_{nm}(x,y), \\
2 \frac{d}{dx}k_{nm}(x,x) + 2n^2 \pi^2 + c(x) + \lambda_{mn} = 0, \quad k_{nm}(x,0) = 0,
\end{cases}$$

(11)

Moreover,

$$\int_0^x |\nabla k_{nm}(x,y)|^2 \, dy \leq C \max \{e^{C\text{sign}(2n^2 \pi^2 + \lambda_{mn})\sqrt{|2n^2 \pi^2 + \lambda_{mn}|}}, 1\}(n^2 \pi^2 + \lambda_{mn})^2$$

for some positive constant $C$ independent of given function.

From Lemma 2, we can get the following corollary, which shows the estimate of $k_{nm}$.

**Corollary 1.** ([11]) For given $n$, let $\lambda_{m,0} > 0$. For every $\lambda_{mn} > \lambda_{m,0}$ and the solution $k_{nm} \in H^1$ to the system

$$\begin{cases}
k_{nm,xx}(x,y) - k_{nm,yy}(x,y) = (2n^2 \pi^2 + c(y) + \lambda_{mn})k_{nm}(x,y), \\
2 \frac{d}{dx}k_{nm}(x,x) + (2n^2 \pi^2 + c(x) + \lambda_{mn}) = 0, \quad k_{nm}(x,0) = 0,
\end{cases}$$

(12)

we have the estimate

$$||k_{nm}||_{H^1} \leq e^{C\sqrt{2n^2 \pi^2 + \lambda_{mn}}},$$

(13)

for some positive constant $C$ independent of $\lambda_{mn} \in [\lambda_{m,0}, +\infty)$.

To compute $u_n$ from $w_n$, one searches the kernels such that (5) is written as

$$u_n(x, t) = w_n(x, t) + \int_0^x l_{nm}(x, y)w_n(y, t)dy,$$

(14)

by the equations of $w_n$ and $u_n$, we can obtain the kernels $l_{nm}(x,y)$ satisfy

$$\begin{cases}
l_{nm,xx}(x,y) - l_{nm,yy}(x,y) = (2n^2 \pi^2 + c(y) + \lambda_{mn})l_{nm}(x,y), \\
2 \frac{d}{dx}l_{nm}(x,x) + (2n^2 \pi^2 + c(x) + \lambda_{mn}) = 0, \quad l_{nm}(x,0) = 0.
\end{cases}$$

(15)

Similar to Corollary 1, we obtain the following consequence of Lemma 1.

**Corollary 2.** ([11]) For given $n$, let $\lambda_{m,0} > 0$. For every $\lambda_{mn} > \lambda_{m,0}$, there exists a unique solution $l_{nm} \in H^1$ to the system

$$\begin{cases}
l_{nm,xx}(x,y) - l_{nm,yy}(x,y) = (2n^2 \pi^2 + c(y) + \lambda_{mn})l_{nm}(x,y), \\
2 \frac{d}{dx}l_{nm}(x,x) + (2n^2 \pi^2 + c(x) + \lambda_{mn}) = 0, \quad l_{nm}(x,0) = 0,
\end{cases}$$

(16)

Moreover,
\[ ||l_{nm}||_{H^1} \leq C(2n^2\pi^2 + \lambda_{mn}), \tag{17} \]
for some positive constant \( C \) independent of \( \lambda_{mn} \in [\lambda_{m_0}, +\infty) \).

3. Exponential stabilization of the original system

To establish the exponential stability of the closed-loop system, it needs to show that the transformation (5) is invertible and its inverse transformation is a linear bounded operator. Then, the stability of the closed-loop system is established.

Next we will prove the exponential stability for the closed-loop system.

**Theorem 1.** Taking \( \lambda_{mn} = m^2\pi^2, t_0 = 0, t_{m+1} - t_m = 1 \) and series of controller (8), then, we have the estimate for series of states and controllers

\[ \int_0^{+\infty} |u_n(t)|^2 dt \leq e^{-2n^2\pi^2} \sum_{m=0}^{\infty} e^{-(2m^3)/3} ||u_n(\cdot, 0)||_{L^2}^2, \tag{18} \]

and

\[ \int_0^{+\infty} |U_n(t)|^2 dt \leq Ce^{-n^2\pi^2} ||u_n(\cdot, 0)||_{L^2}^2. \tag{19} \]

Meanwhile, the state and controller of the original system (1) satisfying the estimate

\[ ||u(\cdot, t)||^2_{L^2((0,1)^2)} \leq Ce^{-t} ||u_0(\cdot, \cdot)||^2_{L^2((0,1)^2)} \tag{20} \]

and

\[ \int_0^{+\infty} |U(t)|^2 dt \leq \sum_{n=1}^{\infty} Ce^{-n^2\pi^2} ||u_n(\cdot, 0)||_{L^2}^2. \tag{21} \]

**Proof.** Firstly, choose the target system

\[
\begin{aligned}
&\{ w_{nt} = w_{n,xx} - (n^2\pi^2 + \lambda_{mn}) w_n, \quad \text{in} \ (0,1) \times (t_m, t_{m+1}), \\
&w_n(0, t) = 0, \ w_n(1, t) = 0, \quad \text{in} \ (t_m, t_{m+1}), \\
&w_n(x, 0) = w_n(x, 0), \quad \text{in} \ (0,1),
\end{aligned}
\tag{22}
\]

let \( t_m \) be an increasing sequence satisfying \( t_0 = 0, t_{m+1} - t_m = 1 \), which shows \( t_m \) converges \( +\infty \) as \( m \to +\infty \). Notice that we have dividing \((0, +\infty)\) into series of intervals \((t_m, t_{m+1})\), according to corollary 1 and 2, we obtain

\[ ||k_{nm}||_{H^1} \leq e^{C\sqrt{2n^2\pi^2 + \lambda_{mn}}}, \ ||l_{nm}||_{H^1} \leq C(2n^2\pi^2 + \lambda_{mn}). \]

Then, for \( t_m \leq t < t_{m+1} \), we have

\[ ||w_n(\cdot, t)||_{L^2}^2 \leq e^{C\sqrt{2n^2\pi^2 + \lambda_{mn}}} ||u_n(\cdot, t)||_{L^2}^2, \tag{23} \]

\[ ||u_n(\cdot, t)||_{L^2}^2 \leq C(2n^2\pi^2 + \lambda_{mn})^2 ||w_n(\cdot, t)||_{L^2}^2, \tag{24} \]

and for \( t_m \leq t < t_{m+1} \),

\[ ||w_n(\cdot, t_m)||_{L^2}^2 \leq e^{-2(2n^2\pi^2 + \lambda_{mn})(t_2 - t_1)} ||w_n(\cdot, t_1)||_{L^2}^2, \tag{25} \]

A combination of (23) and (25) yields

\[ ||u_n(\cdot, t_m)||_{L^2}^2 \leq Ce^{-2(2n^2\pi^2 + \lambda_{mn})} ||u_n(\cdot, t_m)||_{L^2}^2. \tag{26} \]

Then,

\[ ||u_n(\cdot, t_{m+1})||_{L^2}^2 \leq C^m e^{-2n^2\pi^2 m^3} e^{-(\lambda_{mn} + \lambda_{m-1} + \cdots + \lambda_{m_0})} ||u_n(\cdot, t_0)||_{L^2}^2. \tag{27} \]

Hence, for \( t_m \leq t < t_{m+1} \), \( ||u_n(\cdot, t)||_{L^2}^2 \leq e^{-2n^2\pi^2 e^{Cm-(m^2+(m-1)^2+\cdots+1)^2}} ||u_n(\cdot, 0)||_{L^2}^2. \]

Integrate \( t \) from \( t_m \leq t < t_{m+1} \), we obtain

\[ \int_{t_m}^{t_{m+1}} |u_n(\cdot, t)|^2 dt \leq e^{-2n^2\pi^2 e^{Cm-(m^2+\cdots+1)^2}} ||u_n(\cdot, 0)||_{L^2}^2, \]
After some simple computations, we have
\[
\int_0^{+\infty} |u_n(t, \cdot)|^2_{L^2} dt \leq \sum_{m=0}^{+\infty} \int_{t_m}^{t_{m+1}} |u_n(t, \cdot)|^2_{L^2} dt \\
\leq \sum_{m=0}^{+\infty} \int_{t_m}^{t_{m+1}} e^{-2n^2\pi^2} e^{Cm-(m^2+\cdots+1)\pi^2} |u_n(\cdot,0)|^2_{L^2} dt \\
\leq e^{-2n^2\pi^2} \sum_{m=0}^{+\infty} e^{-(2m^3)/3} ||u_n(\cdot,0)||^2_{L^2}.
\]

Because \(\sum_{m=0}^{+\infty} e^{-m^3/9}\) is convergent, we obtain \(\int_0^{+\infty} |U_n(t)|^2 dt \leq Ce^{-n^2\pi^2} ||u_n(\cdot,0)||^2_{L^2(0,1)}\). Therefore, the controller \(U_n(\cdot) \in L^2(0, +\infty)\). According to (3), we obtain the controller for the original system satisfying the estimate
\[
\int_0^{+\infty} ||U(t)||^2_{L^2} dt \leq \sum_{n=1}^{+\infty} C e^{-n^2\pi^2} ||u_n(\cdot,0)||^2_{L^2}.
\]

Finally, we will prove the exponential stabilization for the original system. For any \(t\) large enough, we can find \(t_m\) and \(t_{m+1}\) such that \(t_m \leq t < t_{m+1}\), then,
\[
||u_n(t, \cdot)||^2_{L^2} \leq e^{-2n^2\pi^2} e^{Cm-(m^2+\cdots+1)\pi^2} ||u_n(\cdot,0)||^2_{L^2} \leq e^{-2n^2\pi^2} e^{-t} ||u_n(\cdot,0)||^2_{L^2}.
\]
Hence,
\[
||u(t, \cdot)||^2_{L^2((0,1)^2)} = \sum_{n=1}^{+\infty} ||u_n(t, \cdot)||^2_{L^2(0,1)} \leq Ce^{-t} ||u_0(\cdot,0)||^2_{L^2((0,1)^2)},
\]
which shows the exponential stabilization of the original system and finishes the proof of Theorem 1.

4. Finite time stabilization of the original system

In this section, we will choose special \(\ell_m\) and \(\lambda_{mn}\) for obtaining the finite time stabilization of the heat-like equation in 2D, we state the main theorem of this section.

**Theorem 2.** Let \(T > 0\), for given \(n, (\lambda_{mn})_{m \geq 0}\) be an increasing sequence of positive numbers converging to infinity, \(\lambda_m = \lambda_{mn} + n^2\pi^2\) (only depend on \(m\)), let \((\ell_m)_{m \geq 0}\) be an increasing sequence which converges to \(T\) with \(\ell_0 = 0\). For \(t_m \leq t < t_{m+1}\), \(U_n(t)\) is given by (8). Set \(s_0 = 0\) and \(s_m = \sum_{i=0}^{m-1} \sqrt{\lambda_i (t_{i+1} - t_i)}\) for \(m \geq 1\), there exists a positive constant \(\gamma\), such that if, for large \(m\),
\[
(\ell_{m+1} - \ell_m) \lambda_m \geq \gamma \sqrt{\lambda_m + 1}
\]
then, for \(t_m \leq t < t_{m+1}\),
\[
||u_n(t, \cdot)||_{L^2} \leq Ce^{-s_m/4+C(m-1)} ||u_n(\cdot,0)||_{L^2},
\]
\[
|U_n(t)| \leq Ce^{-s_m/4+C(m-1)+C\sqrt{\lambda_m}} ||u_n(\cdot,0)||_{L^2}.
\]
In particular, if
\[
\lim_{m \to +\infty} \frac{s_m}{m + \ell_{m+1}} = +\infty,
\]
we have
\[
\lim_{t \to T^-} ||u_n(t, \cdot)||_{L^2((0,1)^2)} = 0, \quad \lim_{t \to T^-} U_n(t) = 0
\]
and
\[
\lim_{t \to T^-} ||u(t, \cdot)||_{L^2((0,1)^2)} = 0, \quad \lim_{t \to T^-} ||U(t)||_{L^2} = 0.
\]

**Proof.** Firstly, according to corollary 1 and 2, for \(t_m \leq t < t_{m+1}\), we have,
\[
||w_n(t, \cdot)||_{L^2} \leq e^{C\sqrt{\ell_m}} ||u_n(t, \cdot)||_{L^2},
\]

and for $t_m \leq t < t_{m+1}$,
\[
||u_n(\cdot, t)||_{L^2}^2 \leq C\lambda_m^2||w_n(\cdot, t)||_{L^2}^2.
\]  
Moreover, for $t_m \leq \eta_1 < \eta_2 < t_{m+1}$,
\[
||w_n(\cdot, \eta_1)||_{L^2}^2 \leq e^{-2\lambda_m(\eta_2-\eta_1)}||w_n(\cdot, \eta_1)||_{L^2}^2.
\]  
A combination of (31), (32) and (33) yields, $t_m \leq t < t_{m+1}$,
\[
||u_n(\cdot, t_{m+1})||_{L^2}^2 \leq C\lambda_m e^{C\sqrt{\lambda_m-2\lambda_m(t_{m+1}-t_m)}}||u_n(\cdot, t_m)||_{L^2}^2.
\]  
We derive from (28) and (34) that, when $m$ large enough,
\[
||u_n(\cdot, t_{m+1})||_{L^2}^2 \leq Ce^{-\lambda_m(t_{m+1}-t_m)}||u_n(\cdot, t_m)||_{L^2}^2.
\]  
This, together with the definition of $s_m$, implies
\[
||u_n(\cdot, t_{m+1})||_{L^2}^2 \leq e^{-s_{m+1}+cm}||u_n(\cdot, 0)||_{L^2}^2.
\]  
Therefore, for $t_m \leq t < t_{m+1}$, we have
\[
||u_n(\cdot, t)||_{L^2}^2 \leq C\lambda_m^2||w_n(\cdot, t)||_{L^2}^2 \leq C\lambda_m^2 e^{-2\lambda_m(t-t_m)}||w_n(\cdot, t_m)||_{L^2}^2 \\
\leq Ce^{-s_{m-1/2}+c(m-2)}||u_n(\cdot, 0)||_{L^2}^2.
\]  
and $|U_n(t)| \leq Ce^{-s_{m-1/2}+c(m-2)}+\sqrt{\lambda_m}||u_n(\cdot, 0)||_{L^2}^2$. Taking $m \to +\infty$, we have $\lim_{t \to -}\n ||u_n(\cdot, t)||_{L^2}^2 = 0$, $\lim_{t \to -} U_n(t) = 0$ and $\lim_{t \to -}\n ||u(\cdot, t)||_{L^2(0,1)}^2 = 0$, $\lim_{t \to -}\n ||U(\cdot, t)||_{L^2(0,1)}^2 = 0$, which finish the proof of Theorem 2.

5. Conclusions
In this paper, we have studied the exponential and finite time stabilization of a heat-like equation in 2D rectangle domain with boundary control via decomposing the original system into series of 1D heat equations with boundary control. Then, combined with some estimates of series of states and controllers, we obtained the stabilization results. However, for more general function $c(x,y)$, the procedure of this paper does not work, moreover, for some other kind of 2D domain, maybe the corresponding original control problem cannot be transferred into 1D heat equations with boundary control, these problems will be further considered by using other methods in the future.

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