SUFFICIENT CONDITIONS FOR STARLIKENESS ASSOCIATED WITH PARABOLIC REGION

V. RAVICHANDRAN, A. GANGADHARAN, and T. N. SHANMUGAM

Received 15 June 2001

An analytic function \( f(z) = z + a_{n+1}z^{n+1} + \cdots \), defined on the unit disk \( \Delta = \{ z : |z| < 1 \} \), is in the class \( S_p \) if \( zf'(z)/f(z) \) is in the parabolic region \( \Re w > |w - 1| \). This class is closely related to the class of uniformly convex functions. Sufficient conditions for function to be in \( S_p \) are obtained. In particular, we find condition on \( \lambda \) such that the function \( f(z) \), satisfying

\[
(1 - \alpha)(f(z)/z)^\mu + \alpha f'(z)(f(z)/z)^\mu - 1 \prec 1 + \lambda z, \quad z \in \Delta.
\]

holds for all \( z \in \Delta \). It can be observed that \( f \in UCV \) if and only if \( zf' \in S_p \). Let \( \Omega = \{ w : |w - 1| < \Re w \} \). It follows that \( f \in UCV \) or \( S_p \) are equivalent to saying that \( 1 + zf''(z)/f'(z) \) or \( zf'(z)/f(z) \) are in \( \Omega \), respectively. Note that \( \Omega \) is a parabolic region symmetric with respect to the real axis and \((1/2,0)\) as its vertex. The function \( k(z) \), with \( k(0) = k'(0) - 1 = 0 \) and

\[
1 + \frac{zk''(z)}{k'(z)} = 1 + \frac{2}{\pi^2} \left[ \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right]^2,
\]

is an example of function in UCV.

1. Introduction. Let \( \mathcal{A}_n \) be the family of analytic functions \( f(z) = z + a_{n+1}z^{n+1} + \cdots \) in the unit disk \( \Delta = \{ z : |z| < 1 \} \), and let \( \mathcal{A}_1 = \mathcal{A} \). For \( 0 \leq \alpha < 1 \), let \( S^*(\alpha) \) and \( C(\alpha) \) denote the subclasses of \( \mathcal{A} \) of starlike functions and convex functions of order \( \alpha \), respectively; for \( \alpha = 0 \), \( S^*(0) = S^* \), the class of starlike functions in \( \Delta \). The function \( f \in \mathcal{A} \) is uniformly convex (starlike) if, for every circular arc \( \gamma \) contained in \( \Delta \) with center \( \zeta \in \Delta \), the image arc \( f(\gamma) \) is convex (starlike with respect to \( f(\zeta) \)). The class of all uniformly convex functions denoted by UCV was introduced by Goodman [1] in 1991. Rønning [5] and Ma and Minda [2] independently proved that \( f \in UCV \) if and only if

\[
\Re \left\{ 1 + zf''(z)/f'(z) \right\} > \left| zf''(z)/f'(z) \right|, \quad z \in \Delta.
\]

Further, Ronning [5] defined the class \( S_p \) of functions \( f \in \mathcal{A} \) for which

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (1.2)
\]

holds for all \( z \in \Delta \). It can be observed that \( f \in UCV \) if and only if \( zf' \in S_p \). Let 

\[ \Omega = \{ w : |w - 1| < \Re w \}. \]

It follows that \( f \in UCV \) or \( S_p \) are equivalent to saying that \( 1 + zf''(z)/f'(z) \) or \( zf'(z)/f(z) \) are in \( \Omega \), respectively. Note that \( \Omega \) is a parabolic region symmetric with respect to the real axis and \((1/2,0)\) as its vertex. The function \( k(z) \), with \( k(0) = k'(0) - 1 = 0 \) and

\[
1 + \frac{zk''(z)}{k'(z)} = 1 + \frac{2}{\pi^2} \left[ \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right]^2,
\]
Ponnusamy and Singh [4] obtained bounds on $\lambda$ such that the Alexander transform of $f \in \mathcal{A}$, satisfying $f' < 1 + \lambda z$, is uniformly convex. We extend their result in two directions. Specifically, we find condition on $\lambda$ such that the function $f(z)$, satisfying

\[(1 - \alpha) \left( \frac{f(z)}{z} \right)^\mu + \alpha f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} < 1 + \lambda z, \quad (1.4)\]

is in $S_p$.

Let $a > 1/2$ and let $R_a = \min \{ |w - a| : |w - 1| = \text{Re} \, w \}$. A simple computation gives

\[
R_a = \begin{cases} 
\frac{a - 1}{2} & \text{if } \frac{1}{2} < a \leq \frac{3}{2}, \\
\sqrt{2a - 2} & \text{if } a \geq \frac{3}{2}.
\end{cases} \quad (1.5)
\]

Now, $D(a, R_a) = \{ w : |w - a| < R_a \}$ is the largest disk centered at $a$ which lies inside $\Omega$. If we restrict the value of $a$ by $3/4 < a < 3$, then the disk will contain the point 1.

**Lemma 1.1** [6]. Let $f \in \mathcal{A}$. If, for any $a$, $3/4 < a < 3$,

\[\frac{zf'(z)}{f(z)} - a < R_a, \quad z \in U, \quad (1.6)\]

then $f \in S_p$.

Also, we need the following result.

**Lemma 1.2** [3]. Let $h(z)$ be convex and $\gamma \neq 0$, $\text{Re} \, \gamma \geq 0$. If $p(z) = a + p_n z^n + \cdots$, $n \geq 2$, is analytic in $\triangle$ and

\[p(z) + \frac{z p'(z)}{\gamma} < h(z), \quad h(0) = p(0), \quad (1.7)\]

then

\[p(z) < \frac{\gamma}{n} z^{-\gamma/n} \int_0^z h(t) t^{\gamma/n - 1} \, dt. \quad (1.8)\]

**2. Main results.** We begin with proving the following result.

**Theorem 2.1.** Let $\mu > 0$, $\alpha \geq 0$, and $0 \leq \beta < 1$. Let $f \in \mathcal{A}$ and

\[0 < \lambda \leq \frac{\alpha(\mu + \alpha n)(a - \beta - |1 - a|)}{\mu(1 + (a - \beta) \alpha + |(a - 1) \alpha + 1|) + \alpha n}. \quad (2.1)\]

Then, for $a > (1 + \beta)/2$,

\[(1 - \alpha) \left( \frac{f(z)}{z} \right)^\mu + \alpha f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} < 1 + \lambda z, \quad (2.2)\]

implies

\[
\left| \frac{zf'(z)}{f(z)} - a \right| \leq \frac{\lambda |\mu + \alpha n + \mu| (a - 1) \alpha + 1| + \alpha |1 - a| (\mu + \alpha n)}{\alpha(\mu + \alpha n - \lambda \mu)} \leq a - \beta, \quad (2.3)
\]

and $f \in S^*(\beta)$. 
PROOF. Define the functions $Q(z)$ and $w(z)$ by

$$Q(z) = \left( \frac{f(z)}{z} \right)^\mu, \quad w(z) = \frac{zf'(z)}{f(z)} - a, \quad z \in \Delta. \quad (2.4)$$

Then, $Q(z)$ and $w(z)$ are analytic in $\Delta$, and $w(0) = 1 - a$. Clearly,

$$(1 - \alpha)Q(z) + \alpha[w(z) + a]Q(z) = (1 - \alpha)\left( \frac{f(z)}{z} \right)^\mu + \alpha \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^\mu < 1 + \lambda z,$$

$$\frac{1}{\mu} \frac{zQ'(z)}{Q(z)} + 1 = w(z) + a. \quad (2.5)$$

This shows that

$$Q(z) + \alpha \frac{zQ'(z)}{Q(z)} < 1 + \lambda z,$$

and hence, by Lemma 1.2, we have

$$Q(z) < 1 + \frac{\lambda \mu}{\mu + \alpha n} z, \quad z \in \Delta. \quad (2.7)$$

Since

$$\lambda \leq \frac{\alpha(\mu + \alpha n)(a - \beta - |1 - a|)}{\mu[1 + (a - \beta)\alpha + |(a - 1)\alpha + 1|] + \alpha n} \leq \frac{\mu + \alpha n}{\mu} \quad (2.8)$$

and $a \geq (1 + \beta)/2$, we see that $\mu + \alpha n - \lambda \mu > 0$.

Since

$$\frac{zf'(z)}{f(z)} - a = w(z)$$

$$= \frac{[(1 - \alpha)Q(z) + \alpha Q(z)(w(z) + a) - 1] - (Q(z) - 1)[(a - 1)\alpha + 1] + \alpha(1 - a)}{\alpha Q(z)}$$

we have

$$\left| \frac{zf'(z)}{f(z)} - a \right| \leq \frac{\lambda + (\lambda \mu/(\mu + \alpha n)) |(a - 1)\alpha + 1| + \alpha |1 - a|}{\alpha(1 - \lambda \mu/(\mu + \alpha n))}$$

$$\leq \frac{\lambda[\mu + \alpha n + \mu |(a - 1)\alpha + 1|] + \alpha |1 - a|/(\mu + \alpha n)}{\alpha(\mu + \alpha n - \lambda \mu)} \quad (2.10)$$

provided condition (2.1) is satisfied. This shows that $\text{Re} \, zf'(z)/f(z) > \beta$ and $f(z)$ is starlike of order $\beta$.

Note that to prove (2.10) it is enough to assume that $0 < \lambda \leq (\mu + \alpha n)/\mu$. \qed
Corollary 2.2. If \( f(z) = z + a_{n+1}z^{n+1} + \cdots \) is analytic in \( \Delta \) and if
\[
\left| f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} - 1 \right| < \lambda, \quad z \in \Delta,
\]
then, for \( a > 1/2 \), we have
\[
\left| zf'(z) \left( \frac{f(z)}{z} \right)^{\mu} - a \right| \leq \lambda \left[ \mu(a+1) + (\mu+n) |1-a| \right] \left( \mu + n - \lambda \mu \right) \]
provided \( \mu > 0 \) and
\[
0 < \lambda \leq \frac{(\mu+n)(a-|1-a|)}{\mu(1+2a) + n}.
\]

When \( \mu = 1 \), Corollary 2.2 reduces to the result by Ponnusamy and Singh [4].

Theorem 2.3. Let \( \lambda \) be defined by
\[
\lambda = \begin{cases} 
\frac{\alpha(\mu+\alpha n)(4a-3)}{\mu[2+(2a-1)\alpha+2|a-1\alpha+1|]+2\alpha n} & \left( \frac{3}{4} < a \leq 1 \right), \\
\frac{\alpha(\mu+\alpha n)}{\mu[\alpha(4a-1)+2]+2\alpha n} & \left( 1 \leq a \leq \frac{3}{2} \right), \\
\frac{\alpha(\mu+\alpha n)(1-a+\sqrt{2a-2})}{\alpha n + \mu[2+\alpha(a-1)+\alpha\sqrt{2a-2}]} & \left( \frac{3}{2} \leq a < 3 \right).
\end{cases}
\]
If \( f \in A_n \) satisfies
\[
(1-\alpha) \left( \frac{f(z)}{z} \right)^{\mu} + \alpha f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} < 1 + \lambda z,
\]
then \( f \in S_p \).

It should be noted that if \( 3/4 < a \leq 3/2 \), then the condition on \( \lambda \) in Theorem 2.3 reduces to the condition in Theorem 2.1. Hence, with the same condition as in Theorem 2.1 (with \( \beta = 1/2 \)), we get a stronger conclusion that \( f \in S_p \).

Proof. We first verify that \( \lambda \) defined in Theorem 2.3 satisfies the condition \( 0 < \lambda \leq (\mu+\alpha n)/\mu \). This condition is equivalent to
\[
0 \leq \begin{cases} 
\mu[\alpha(2a-1)+1] & \left( 1 \leq a \leq \frac{3}{2} \right), \\
2\mu + \alpha n & \left( \frac{3}{4} \leq a \leq 1, (a-1)\alpha+1 \geq 0 \right), \\
2\mu\alpha(1-a) + \alpha n & \left( \frac{3}{4} \leq a \leq 1, (a-1)\alpha+1 \leq 0 \right), \\
\alpha n + 2\mu[\alpha(a-1)+1] & \left( \frac{3}{2} \leq a < 3 \right).
\end{cases}
\]
The above inequality is obviously correct. Let
\[
R_a = \frac{\lambda[\mu+\alpha n + \mu|a-1\alpha+1| + \alpha|1-a|(\mu+\alpha n)]}{\alpha(\mu+\alpha n - \lambda \mu)}.
\]
Then, a computation shows that
\[
R_a = \begin{cases} 
\frac{a - \frac{1}{2}}{2} & \left(\frac{3}{4} < a \leq \frac{3}{2}\right), \\
\sqrt{2a - 2} & \left(\frac{3}{2} \leq a < 3\right).
\end{cases}
\] (2.18)

Then, from the proof of Theorem 2.1, we have
\[
\left|zf'(z) - f(z) - a\right| \leq R_a.
\] (2.19)

Using Lemma 1.1, we have the desired result.

This result for \(\mu = 1\) and \(\alpha = 1\) is obtained in Ponnusamy and Singh [4].

**Theorem 2.4.** Suppose \(\alpha \in \mathbb{C}, a > 1/2,\) and \(\lambda \in \mathbb{R}\) satisfy
\[
0 < \lambda \leq \frac{|1 + n\alpha|(\mu + n)(a - |1 - a|)}{\mu(1 + 2a) + n}.
\] (2.20)

If
\[
\left(\frac{f(z)}{z}\right)^{\mu-1} \left\{\alpha(\mu - 1) \frac{zf''(z)}{f(z)^2} + \alpha f''(z) + (1 + (1 - \mu)\alpha)f'(z)\right\} < 1 + \lambda z, \quad z \in \Delta,
\] (2.21)

then
\[
\left|zf'(z) - f(z) - a\right| \leq \frac{\lambda_1[n + \mu(a + 1)] + (\mu + n)|1 - a|}{\mu(1 - \lambda_1) + n},
\] (2.22)

where \(\lambda_1 = \lambda/|1 + n\alpha|\).

**Proof.** Let \(p(z) = f'(z)(f(z)/z)^{\mu-1}, \) \(z \in \Delta.\) Then, \(p(z)\) is analytic in \(\Delta\) and
\[
zp'(z) = \left(\frac{f(z)}{z}\right)^{\mu-1} \left\{zf''(z) + (\mu - 1) \left(\frac{zf'(z)}{f(z)} - 1\right)f'(z)\right\}.
\] (2.23)

This shows that
\[
p(z) + \alpha z p'(z) < 1 + \lambda z,
\] (2.24)

and hence, by Lemma 1.2,
\[
p(z) < 1 + \frac{\lambda}{1 + n\alpha} z < 1 + \frac{\lambda}{|1 + n\alpha|} z.
\] (2.25)

The result now follows from Theorem 2.1 where \(\lambda_1 = \lambda/|1 + n\alpha|\).

**Corollary 2.5.** Suppose that \(\alpha \in \mathbb{C}, a > 1/2,\) and \(\lambda \in \mathbb{R}\) satisfy
\[
0 < \lambda \leq \frac{|1 + n\alpha|(1 + n)(a - |1 - a|)}{1 + 2a + n}.
\] (2.26)

If \(f \in \mathcal{A}\) satisfies
\[
|\alpha z f''(z) + f'(z) - 1| < \lambda, \quad z \in \Delta,
\] (2.27)

then
\[
\left|zf'(z) - f(z) - a\right| \leq \frac{\lambda_1[n + a + 1] + (1 + n)|1 - a|}{1 - \lambda_1 + n},
\] (2.28)

where \(\lambda_1 = \lambda/|1 + n\alpha|\).
The result follows from Theorem 2.4 when $a = n = 1$ and is obtained in [4].

**Theorem 2.6.** Let $\lambda$ be defined by

$$
\lambda = \begin{cases}
\frac{|1 + n\alpha|(1 + n)(4a - 3)}{4a + 2n + 1} & (\frac{3}{4} < a \leq 1), \\
\frac{|1 + n\alpha|(1 + n)}{4a + 2n + 1} & (1 < a \leq \frac{3}{2}), \\
\frac{|1 + n\alpha|(1 + n)(\sqrt{2a - 2} + 1 - a)}{n + 1 + a + \sqrt{2a - 2}} & (\frac{3}{2} \leq a < 3).
\end{cases}
$$

(2.29)

If $|\alpha zf''(z) + f'(z) - 1| < \lambda$, then $f \in S_p$.

**Proof.** From the definition of $\lambda$, it is clear that

$$
\lambda_1 = \frac{[n + a + 1] + (1 + n)[1 - a]}{1 - \lambda_1 + n} = \begin{cases}
\frac{a - \frac{1}{2}}{\sqrt{2a - 2}} & (\frac{3}{4} < a \leq \frac{3}{2}), \\
\frac{3}{2} & (\frac{3}{2} \leq a < 3),
\end{cases}
$$

(2.30)

where $\lambda_1 = \lambda / |1 + n\alpha|$. Since

$$
0 < \lambda \leq \frac{|1 + n\alpha|(1 + n)(a - |1 - a|)}{2a + n + 1},
$$

(2.31)

the result follows from Corollary 2.5.

\[\square\]

**References**

[1] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. 56 (1991), no. 1, 87–92.

[2] W. C. Ma and D. Minda, *Uniformly convex functions*, Ann. Polon. Math. 57 (1992), no. 2, 165–175.

[3] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. 28 (1981), no. 2, 157-172.

[4] S. Ponnusamy and V. Singh, *Criteria for strongly starlike functions*, Complex Variables Theory Appl. 34 (1997), no. 3, 267–291.

[5] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. 118 (1993), no. 1, 189–196.

[6] T. N. Shanmugam and V. Ravichandran, *Certain properties of uniformly convex functions*, Computational Methods and Function Theory (Penang, Malaysia, 1994) (R. M. Ali, St. Ruscheweyh, and E. B. Saff, eds.), Ser. Approx. Decompos., vol. 5, World Scientific Publishing, New Jersey, 1995, pp. 319–324.
Mathematical Problems in Engineering

Special Issue on
Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

| Event                  | Date       |
|------------------------|------------|
| Manuscript Due         | December 1, 2008 |
| First Round of Reviews | March 1, 2009  |
| Publication Date       | June 1, 2009  |

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob’evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru

Hindawi Publishing Corporation
http://www.hindawi.com