Extraction of $V_{ub}$ from the Decay $B \to \pi l \nu$

Hsiang-nan Li$^1$ and Hoi-Lai Yu$^2$,

$^1$Department of Physics, National Chung-Cheng University,
Chia-Yi, Taiwan, R.O.C.

$^2$Institute of Physics, Academia Sinica, Taipei, Taiwan, R.O.C.

March 7, 2019

PACS numbers: 13.20He, 12.15Hh, 12.38Bx

Abstract

We develop the perturbative QCD formalism including Sudakov effects for semi-leptonic $B$ meson decays. We evaluate the differential decay rate of $B \to \pi l \nu$, and find that the perturbative calculation is reliable for the energy fraction of the pion above 0.3. Combining predictions from the soft pion theorems, we extract the value of the matrix element $|V_{ub}|$ which is roughly $2.7 \times 10^{-3}$. 

Exclusive semi-leptonic meson decays have been intensively studied by several approaches, which provide the information of the mixing angles in the Cabibbo-Kobayashi-Maskawa matrix of the standard model. Chiral symmetry and heavy quark symmetry have been applied to the decays \( K \to \pi l\nu \) [1] and \( B \to D l\nu \) [2], respectively, from which model-independent extraction of the matrix elements \(|V_{us}|\) and \(|V_{cb}|\) is obtained. For the heavy-to-light transition \( B \to \pi l\nu \), which gives reliable estimation of \(|V_{ub}|\) [3], neither of the above theories is appropriate. Recently, this decay has been investigated using the heavy quark effective theory (HQET) [4], in which only the normalization of the relevant form factors is determined in terms of the soft pion relations. An explicit evaluation of the decay \( B \to \pi l\nu \) based on the perturbative QCD (PQCD) formalism has been proposed [5]. However, it leads to results which are too small compared to current experimental data.

In this letter we shall develop a modified PQCD approach to heavy meson decays, which includes Sudakov effects [6, 7]. These effects, arising from the all-order summation of large radiative corrections, suppress contributions from the nonperturbative region, and have been found to extend the applicability of PQCD down to the energy scale of few GeV in the study of elastic hadron form factors [7]. We shall show that the PQCD approach is proper for the decay \( B \to \pi l\nu \), at least when the pion is energetic.

The amplitude of the considered process is written as

\[
A(P_1, P_2) = \frac{G_F}{\sqrt{2}} V_{ub} \bar{u}\gamma_\mu(1 - \gamma_5)l \langle \pi(P_2)|\bar{u}\gamma^\mu b|B(P_1)\rangle ,
\]

where \( G_F \) is the Fermi coupling constant, and \( P_1 \) and \( P_2 \) are the \( B \) meson and pion momenta, respectively. We start with the lowest-order factorization for the matrix element \( M^\mu = \langle \pi(P_2)|\bar{u}\gamma^\mu b|B(P_1)\rangle \), in which the \( b \) quark carries the momentum \( P_1 - k_1 \), and its light partner carries \( k_1 \). These momenta satisfy the on-shell conditions \((P_1 - k_1)^2 = m_b^2, P_2^2 = m_B^2 \) and \( k_1^2 = 0, m_b \) and \( m_B \) being the \( b \) quark and \( B \) meson masses, respectively. We choose the Breit frame such that \( P_1^+ = P_1^- = m_B/\sqrt{2}, P_2^+ = \eta m_B/\sqrt{2} \) and all other components of \( P_1 \)'s vanish, where \( \eta \) is related to the energy fraction of the pion by \( P_2^0 = \eta m_B/2, 0 \leq \eta \leq 1 \). \( k_1 \) has a minus component, defining the momentum fraction \( x_1 = k_1^-/P_1^- \) in the \( B \) meson wave function, and small amount of transverse components \( k_{1T} \). The light valence quark of the \( B \) meson, after absorbing the hard gluon, goes into the pion with the momentum fraction \( x_2 \) and transverse momenta \( k_{2T} \).
We then consider how to group radiative corrections into the basic factorization by locating their leading momentum regions, from which important contributions to loop integrals arise. The important corrections are characterized by large single logarithms, which are either collinear or soft. These two regions may overlap and give double logarithms. It is known that single logarithms can be summed to all orders using renormalization group (RG) methods, while double logarithms must be organized by the resummation technique \[8\], which has been developed in axial gauge \( n \cdot A = 0 \), \( n \) being the gauge vector and \( A \) the gauge field.

A careful analysis shows that reducible corrections on the pion side produce double logarithms with soft ones cancelled in the asymptotic region \( b \to 0 \) \[7\], \( b \) being the conjugate variable to \( k_T \). Hence, they are dominated by collinear enhancements, and can be absorbed into the pion wave function. Reducible corrections on the \( B \) meson side also give double logarithms, but the soft ones do not cancel and the collinear ones are suppressed by the \( B \) meson wave function. Therefore, these corrections can be absorbed into the \( B \) meson wave function, which is also dominated by soft dynamics. Irreducible corrections, with an extra gluon connecting the pion and the \( B \) meson, give only soft divergences, which cancel asymptotically. They are then absorbed into a hard scattering amplitude. Hence, the factorization picture holds after radiative corrections are included.

With the above reasoning, the factorization formula for \( M^\mu \) in the transverse configuration space can be written as,

\[
M^\mu = \int_0^1 dx_1 dx_2 \int \frac{d^2b_1}{(2\pi)^2} \frac{d^2b_2}{(2\pi)^2} P_\pi(x_2, b_2, P_2, \mu) \times \tilde{H}^\mu(x_1, x_2, b_1, b_2, m, \mu) P_B(x_1, b_1, P_1, \mu),
\]

with \( P_\pi \) and \( P_B \) the pion and \( B \) meson wave functions, respectively, and \( \tilde{H}^\mu \) the Fourier transform of the hard scattering amplitude to \( b \) space. \( \mu \) is the factorization and renormalization scale. Both \( P_\pi \) and \( P_B \) contain double logarithms, which will be summed up below. The approximation \( m_b \approx m_B = m = 5.28 \text{ GeV} \) has been made to simplify the analysis.

We outline the resummation procedure employed for \( P_\pi \) \[8\]. If the double logarithms are grouped into an exponential \( P \sim \exp[- \ln m \ln(\ln m/\ln b)] \), the problem will become simpler by considering the derivative \( dP/d\ln m = C P \), where the coefficient \( C \) contains only single logarithms, and can be
treated by RG methods. Because of the scale invariance of \( n \) in the gluon propagator,

\[
N^{\mu\nu}(q) = \frac{-i}{q^2} \left( g^{\mu\nu} - \frac{n^\mu q^\nu + q^\mu n^\nu}{n \cdot q} + n^2 \frac{q^\mu q^\nu}{(n \cdot q)^2} \right),
\]

(3)

\( \mathcal{P}_\pi \) depends only on the ratio \( \nu_2^2 = (P_2 \cdot n)^2/n^2 \). It is then possible to relate the derivative \( d\mathcal{P}_\pi/d\ln P_2^2 \) to \( d\mathcal{P}_\pi/dn \), which can be easily computed using the relations \( dN^{\mu\nu}/dn_\alpha = -(N^{\alpha\mu} q^\nu + N^{\mu\alpha} q^\nu)/q \cdot n \). The momentum \( q \) appearing at both ends of the differentiated gluon line hints the application of the Ward identity. After adding together all diagrams with different differentiated gluon lines, we obtain an equation graphically described by fig. 1a, in which the square vertex represents \( gT^a_{\alpha\nu}n^2P_\alpha^2/(P_2^2 \cdot nq \cdot n) \), \( T^a \) being related to the Gell-Mann matrices \( \lambda^a \) by \( T^a = \lambda^a/2 \).

Due to the factor \( 1/q \cdot n \) in the new vertex and the nonvanishing of \( n^2 \), the leading regions of \( q \) are soft and ultraviolet, in which fig. 1a can be factorized according to fig. 1b to lowest order of \( \alpha_s \). The part on the left-hand side of the dashed line is exactly \( \mathcal{P}_\pi \), and that on the right-hand side is assigned to the coefficient \( C \). We introduce a function \( K \) to organize the soft enhancements in the first two diagrams of fig. 1b, and \( G \) for the ultraviolet divergences of the other two diagrams. The soft subtraction in \( G \) is to avoid double counting. We then derive the differential equation,

\[
\frac{d}{d\ln P_2^2} \mathcal{P}_\pi = \left[ 2K(b_2 \mu) + \frac{1}{2} G(x_2 \nu_2/\mu) + \frac{1}{2} G((1-x_2)\nu_2/\mu) \right] \mathcal{P}_\pi ,
\]

(4)

where the functions \( K \) and \( G \) have been calculated using RG methods [6]. Solving eq. (4), we obtain the solution

\[
\mathcal{P}_\pi(x_2, b_2, P_2, \mu) = \exp \left[ - \sum_{\xi=x_2, 1-x_2} s(\xi, b_2, \eta m) \right] \mathcal{P}_\pi(x_2, b_2, \mu). \]

(5)

The explicit expression of the exponent \( s(\xi, b, Q) \) has been obtained in [7], and will not be shown here due to its complexity.

The function \( \mathcal{P}_\pi \) still contains single logarithms from ultraviolet divergences, which need to be summed using RG methods [6]. The large-\( b \) behavior of \( \mathcal{P}_\pi \) is then written as

\[
\mathcal{P}_\pi = \exp \left[ - \sum_{\xi=x_2, 1-x_2} s(\xi, b_2, \eta m) - 2 \int_{1/b_2}^{\mu} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(g(\bar{\mu})) \right] \phi_\pi(x_2, b_2), \]

(6)
\( \gamma_q = -\alpha_s/\pi \) being the quark anomalous dimension in axial gauge. The \( b_2 \) dependence in \( \phi_\pi \), which corresponds to the intrinsic transverse momentum dependence of the pion wave function \([3]\), will be neglected.

As to \( P_B \), the resummation of the double logarithms is subtler. The self-energy correction to the massive \( b \) quark, giving only soft single logarithms, should be excluded. On the other hand, \( P_B \) involves the invariants such as \( P_1^2 \), which can not be related to \( n \), so that the technique of replacing \( d/dm \) by \( d/dn \) fails. However, the above difficulties can be removed by applying the eikonal approximation to the heavy quark line. In the collinear region with the loop momentum \( q \) parallel to \( k_1 \) and in the soft region, the \( b \) quark line can be replaced by an eikonal line:

\[
\frac{(P_1 - k_1 + \mu + m)\gamma^\alpha}{(P_1 - k_1 + q)^2 - m^2} \approx \frac{(P_1 - k_1)^\alpha}{(P_1 - k_1) \cdot q} + R ,
\]

where the remaining part \( R \) is less important. The involved physics is that a soft gluon or a gluon moving parallelly to \( k_1 \) can not explore the details of the \( b \) quark, and its dynamics can be factorized.

The first difficulty is then resolved, because self-energy diagrams of an eikonal line are excluded by definition \([10]\). With the scale invariance of \( P_1 - k_1 \) as shown in eq. \((4)\), which is equivalent to the flavor symmetry in HQET, \( P_1 - k_1 \) does not lead to a large scale, and the remaining large scale is only \( k_1^- \). Furthermore, an explicit lowest-order investigation shows that \( P_B \) depends only on the single ratio \( \nu_1^2 = (k_1 \cdot n)^2/n^2 \), and thus \( d/dk_1^- \) can be replaced by \( d/dn \) now.

Following the similar procedures to those for the pion, we obtain the differential equation,

\[
\frac{d}{d \ln k_1^-} P_B = \left[ \mathcal{K}(b_1 \mu) + \frac{1}{2} \mathcal{G}(\nu_1/\mu) \right] P_B .
\]

It can be shown that the functions \( \mathcal{K} \) and \( \mathcal{G} \) for the \( B \) meson are exactly the same as those in eq. \((4)\). It is then straightforward to derive the solution

\[
P_B = \exp \left[ -s(x_1, b_1, m) - 2 \int_{1/b_1}^{\mu} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(g(\bar{\mu})) \right] \phi_B(x_1, b_1) .
\]

At last, the RG analysis of \( \tilde{H}^\mu \) gives

\[
\tilde{H}^\mu(x_i, b_i, m, \mu) = \exp \left[ -4 \int_{\mu}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(g(\bar{\mu})) \right] \tilde{H}^\mu(x_i, b_i, m, t) ,
\]
where $t$ is taken as the largest mass scale associated with the hard gluon, $t = \max(\sqrt{x_1 x_2 \eta m}, 1/b_1, 1/b_2)$. Having factorized all the large logarithms into the exponents, we can then compute $\tilde{H}^\mu$ to $O(\alpha_s)$.

Substituting eqs. (6), (9) and (10) into (2), we obtain the factorization formula for $M^\mu = f_1 P^\mu_1 + f_2 P^\mu_2$, where the form factors $f_1$ and $f_2$ are given by

$$f_1 = -16\pi C_F m^2 \int_0^1 dx_1 dx_2 \int_0^\infty b_1 db_1 b_2 db_2 \phi_B(x_1, b_1) \phi_\pi(x_2) \times x_1 \eta h(x_1, x_2, b_1, b_2, m) \exp[-S(x_i, m)] ,$$  

and

$$f_2 = 16\pi C_F m^2 \int_0^1 dx_1 dx_2 \int_0^\infty b_1 db_1 b_2 db_2 \phi_B(x_1, b_1) \phi_\pi(x_2) \times \left[ x_1 h(x_1, x_2, b_1, b_2, m) + (3 - x_2 \eta) h(x_2, x_1, b_1, b_2, m) \right] \times \exp[-S(x_i, m)] ,$$  

respectively, with

$$h(x_1, x_2, b_1, b_2, m) = \alpha_s(t) K_0(\sqrt{x_1 x_2 \eta m b_2}) \times \left[ \theta(b_1 - b_2) K_0(\sqrt{x_1 \eta m b_1}) I_0(\sqrt{x_1 \eta m b_2}) + \theta(b_2 - b_1) K_0(\sqrt{x_1 \eta m b_2}) I_0(\sqrt{x_1 \eta m b_1}) \right] .$$  

$C_F$ is the color factor defined by $\text{tr}(T^a T^a) = N_c C_F$, $N_c$ being the number of colors, and $K_0$ and $I_0$ are the modified Bessel functions of order zero. The complete Sudakov exponent $S$ is given by

$$S(x_i, m) = s(x_1, b_1, m) + s(x_2, b_2, \eta m) + s(1 - x_2, b_2, \eta m) - \frac{1}{\beta} \left[ \ln \frac{\ln(t/\Lambda)}{\ln(1/b_1 \Lambda)} + \ln \frac{\ln(t/\Lambda)}{\ln(1/b_2 \Lambda)} \right]$$  

with $\beta = (33 - 2n_f)/12$, $n_f = 4$ being the number of quark flavors, and $\Lambda \equiv \Lambda_{QCD} = 100$ MeV here. The Sudakov factor $\exp(-S)$ decreases quickly in the large $b_i$ region and vanishes as $b_i > 1/\Lambda$. We have kept the intrinsic transverse momentum dependence in $\phi_B$. Since the $B$ meson wave function is dominated by soft dynamics, this dependence is more important than that in the pion.
\( \phi_\pi \) is chosen as the Chernyak-Zhitnitsky model \(^{[11]}\),
\[
\phi_\pi(x) = 5\sqrt{3}f_\pi x(1-x)(1-2x)^2
\]
with \( f_\pi = 93 \) MeV the pion decay constant. For the \( B \) meson wave function we consider \(^{[12]}\)
\[
\Phi_B(x, k_T) = N \left[ C + \frac{m^2}{1-x} + \frac{k_T^2}{x(1-x)} \right]^{-2},
\]
with \( N = 1.232 \) GeV\(^3\) and \( C = -0.99993m^2 \), which are obtained from the normalizations \( \int dx \int d^2k_T \Phi_B = f_B/2\sqrt{3} \) and \( \int dx \int d^2k_T \Phi_B^2 = 1/2 \), \( f_B = 160 \) MeV being the \( B \) meson decay constant \(^{[13]}\). The Fourier transform of \( \Phi_B \) gives
\[
\phi_B(x, b) = \frac{\pi Nb^2(1-x)^2}{\sqrt{m^2x + Cx(1-x)}} K_1 \left( \sqrt{m^2x + Cx(1-x)b} \right),
\]
\( K_1 \) being the modified Bessel function of order one.

Results of \( f_1 + f_2 \) with \( b_1 \) and \( b_2 \) integrated up to the same cutoff \( b_c \) are shown in fig. 2a. We observe that at \( \eta = 0.3 \) approximately 50\% of the contribution to \( f_1 + f_2 \) comes from the region with \( \alpha_s(1/b_c) < 1 \), or equivalently, \( b_c < 0.5/\Lambda \). At \( \eta = 0.4 \), 55\% of the contribution is accumulated in this perturbative region. As \( \eta = 1 \), perturbative contribution has reached 75\%. It implies that the modified PQCD analysis of the decay \( B \to \pi l\nu \) in the range of \( \eta > 0.3 \) is relatively reliable according to the criteria given in \(^{[7]}\).

The differential decay rate for the specific case \( B^0 \to \pi^0 l^+ l^- \nu \) with vanishing lepton masses is given by
\[
\frac{d\Gamma}{d\eta} \equiv |V_{ub}|^2 R(\eta) = |V_{ub}|^2 \frac{G_F^2m^5\eta^3}{768\pi^4} |f_1 + f_2|^2.
\]
Substituting the results of \( f_i \) into eq. \(^{[18]}\), we derive the behavior of \( R(\eta) \) as in fig. 2b, which shows a slow decrease with \( \eta \).

In order to have the full spectrum in \( \eta \), we approximate \( d\Gamma/d\eta \) in the range of \( \eta < 0.3 \) by the soft pion limits of \( f_i \) \(^{[4]}\):
\[
\lim_{\eta \to 0} R(\eta) = \frac{G_F^2m^5\eta}{192\pi^3} \frac{f_{\pi^0}^2}{f_\pi^2} g_{BB^*\pi}^2,
\]
where \( \phi_\pi \) is chosen as the Chernyak-Zhitnitsky model \(^{[11]}\).
which shows a linear relation with $\eta$. Here $f_{B^*} \approx 1.1 f_B$ is the decay constant of the $B^*$ meson, and $g_{BB^*\pi} \approx 0.8$ is the $BB^*\pi$ coupling constant. We extrapolate eq. (19) to $\eta = 0.3$ as shown in fig. 2b, and a good match between the soft pion and PQCD predictions is observed. Certainly, this extrapolation of the soft pion limit may not be reliable, but the match of the two different approaches does justify our calculation to some extent.

It is then possible to estimate the total decay rate $\Gamma$ by combining eq. (19) for $\eta < 0.3$ with the PQCD predictions for $\eta > 0.3$. We obtain $\Gamma \approx 2.4 \times 10^{-11} |V_{ub}|^2$ GeV, which corresponds to a branching ratio $0.47 \times 10^2 |V_{ub}|^2$ for the total width $(0.51 \pm 0.02) \times 10^{-9}$ MeV of the $B^0$ meson. Current experimental limit on the branching ratio of $B^0 \rightarrow \pi^- l^+ \nu$ is $3.3 \times 10^{-4}$. We then extract the matrix element $|V_{ub}| \approx 2.7 \times 10^{-3}$, close to the value $0.003$ given in the literature.

We thank G.L. Lin, M. Neubert, G. Sterman and Y.P. Yao for helpful discussions. This work was supported by the National Science Council of R.O.C. under Grant Nos. NSC84-2112-M194-006 and NSC84-211-M001-034.
References

[1] H. Leutwyler and M. Roos, Z. phys. C25, 91 (1984).

[2] M. Neubert, Phys. Lett. B264, 455 (1991).

[3] N. Isgur and M.B. Wise, Phys. Rev. D42, 2388 (1990).

[4] G. Burdman, Z. Ligeti, M. Neubert and Y. Nir, Phys. Rev D49, 2331 (1994).

[5] R. Akhoury, G. Sterman and Y.-P Yao, Phys. Rev. D50, 358 (1994).

[6] J. Botts and G. Sterman, Nucl. Phys. B325, 62 (1989).

[7] H.-n. Li and G. Sterman, Nucl. Phys. B381, 129 (1992); H.-n. Li, Phys. Rev. D48, 4243 (1993).

[8] J.C. Collin and D.E. Soper, Nucl. Phys. B193, 381 (1981).

[9] R. Jakob and P. Kroll, Phys. Lett. B315, 463 (1993); J. Bolz, R. Jakob, P. Kroll, M. Bergmann and N.G. Stefanis, Wuppertal Preprint WU-B-94-09.

[10] J.C. Collins, in Perturbative Quantum Chromodynamics, ed. A.H. Mueller (World Scientific, Singapore, 1989).

[11] V.L. Chernyak and A.R. Zhitnitsky, Nucl. Phys. B201, 492 (1982); Phys. Rep. 112, 173 (1984).

[12] F. Schlumpf, SLAC-PUB-6335.

[13] C. Bernard, J. Labrenz and A. Soni, Nucl. Phys. B(Proc.)30, 465 (1993).

[14] M. Neubert, Phys. Rev. D46, 1076 (1992).

[15] T.M. Yan et al., Phys. Rev. D46, 1148 (1992).

[16] Review of Particle Properties, Phys. Rev. D45 (1992).

[17] B. Ong et al. (CLEO collaboration), Phys. Rev. Lett. 70, 18 (1993).
Figure Captions

Fig. 1. Graphic representation of eq. (4).

Fig. 2. (a) Dependence of $f_1 + f_2$ on the cutoff $b_c$ for (1) $\eta = 0.3$, (2) $\eta = 0.4$, and (3) $\eta = 1.0$. (b) Dependence of $R(\eta)$ on $\eta$ derived from the modified PQCD formalism (solid line) and from the soft pion theorems (dashed line).
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9409313v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9409313v1