Inverse scattering transform and high-order pole solutions for the NLS equation with quartic terms under vanishing/non-vanishing boundary conditions

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Abstract

The aim of this article is to investigate the inverse scattering transform and high-order pole solutions for the focusing nonlinear Schrödinger equation with quartic terms. Under the vanishing and non-vanishing boundary conditions, the scattering data are obtained respectively for one high-order pole and multi high-order poles. The formulas of soliton solutions are established by Laurent expansion of the Riemann-Hilbert problem. The method used here is different from the usual method of computing residues since the coefficients of higher negative power of $\lambda - \lambda_j$ and $\lambda - \bar{\lambda}_j$ are difficult to be obtained. The determinant formula of high-order pole solution for non-vanishing boundary condition is given. The dynamical properties and characteristics for high-order pole solutions are further analyzed.

Keywords: the focusing nonlinear Schrödinger equation with quartic terms;

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vanishing and non-vanishing boundary conditions; Riemann-Hilbert problem; soliton solution; high-order poles.
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4 Summary
1 Introduction

The nonlinear Schrödinger (NLS) equation can well describe the wave evolution in deep water [1] and optical fibers [2]. For some multiplicity of nonlinear phenomena in plasma waves [3], Bose-Einstein condensates [4, 5] and other physical phenomena, the NLS equation is also a generic model. One of the most important application is the soliton in optical fibers. Hasegawa and Tappert showed soliton propagation in optical fibers remarkable stability via numerical computations [6]. Nonetheless, several phenomena can not be described by low-order dispersion NLS equation. The NLS equation with high-order dispersion is particularly important for describing the more complex phenomena.

A general NLS equation with the high-order dispersion terms was given by [7]

\[ iq_t + S - i\alpha H + \gamma P - i\delta Q = 0, \]  

(1.1)

where the dispersion terms

\[ S = \frac{1}{2} q_{xx} + q|q|^2, \quad H = q_{xxx} + 6|q|^2 q_x, \]  

(1.2a)

\[ P = q_{xxxx} + 8|q|^2 q_{xx} + 6q|q|^4 + 4q|q_x|^2 + 6q^2 q + 2q^2 q_{xx}, \]  

(1.2b)

\[ Q = q_{xxxx} + 10|q|^2 q_{xxxx} + 30|q|^4 q_x + 10(q|q_x|^2)_x + 20q_x q_{xx}. \]  

(1.2c)

The coefficients \( \alpha, \gamma \) and \( \delta \) in equation (1.1) determine some important physical models:

- For \( \alpha = \gamma = \delta = 0 \), the equation (1.1) reduces to the well known NLS equation

\[ iq_t + \frac{1}{2} q_{xx} + |q|^2 q = 0, \]  

(1.3)

which had been extensively investigated [10, 11].

- For \( \gamma = \delta = 0 \), the equation (1.1) reduces to the Hirota equation [12, 13]

\[ iq_t + \frac{1}{2} q_{xx} + |q|^2 q - i\alpha H = 0. \]  

(1.4)

- For \( \alpha = \delta = 0 \), the equation (1.1) reduces to the NLS equation with quartic term (QNLS) [7, 14, 17]

\[ iq_t + \frac{1}{2} q_{xx} + |q|^2 q + \gamma P = 0. \]  

(1.5)
In this article, we investigate the inverse scattering transform and soliton solutions for the QNLS equation (1.5). Soliton solutions as one of the most important consequence in integrable systems, which can be constructed by many different methods. Recently, the Riemann-Hilbert method as another form of inverse scattering method is favorite to study the soliton solutions with vanishing boundary condition [18–21], as well as non-vanishing boundary condition [22–29, 37–41]. The Riemann-Hilbert method, Darboux transform and dressing method were also considered to obtain the higher order pole solutions [30–36].

The aim of this article is to study inverse scattering transform for the equation (1.5) with vanishing/non-vanishing boundary conditions via Riemann-Hilbert method. To derive the high-order pole soliton solutions, the coefficients of $(\lambda - \lambda_j)^{-s}$ and $(\lambda - \bar{\lambda}_j)^{-s}$ with $s > 1$ should be considered, except to the residue conditions. Zhang et al investigated the high-order pole solutions under vanishing boundary conditions [37–39]. We extend this method to the equation (1.5) with vanishing/non-vanishing boundary conditions.

This work is organized as follows. In Section 2, we investigate inverse scattering transform for the equation (1.5) with vanishing boundary conditions. The formula of one high-order pole and multiple high-order pole solutions are further given. The dynamical properties and characteristic for the high-order pole solutions are further analyzed. Section 3, we investigate inverse scattering transform for the equation (1.5) with non-vanishing boundary conditions. Two formulas of one high-order pole and multiple the high-order pole solutions are further obtained. The dynamical properties and characteristic for high-order pole solutions are further analyzed. Finally a conclusion is given in Section 4.

2 Inverse scattering transform for vanishing boundary conditions

In this section, we consider the inverse scattering for the NLS equation with quartic terms (1.5) under the vanishing boundary condition

$$\lim_{|x| \to \infty} q(x, t) = 0.$$ (2.1)
It is known that the equation (1.5) admit the Lax pair

$$
\psi_x = h\psi, \quad \psi_t = I\psi, \quad (2.2)
$$

where

$$
h = i\lambda\sigma_3 + iQ, \quad I = A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0,
$$

$$
A_4 = -8i\gamma\sigma_3, \quad A_3 = -8i\gamma Q,
$$

$$
A_2 = i(1 + 4\gamma Q^2)\sigma_3 - 4\gamma\sigma_3 Q_x, \quad A_1 = 2\gamma[Q_x, Q]\sigma_3 + i(Q + 4\gamma Q^3 + 2\gamma Q_{xx}),
$$

$$
A_0 = -3i\gamma\sigma_3 Q^4 + i\gamma\sigma_3 Q_x^2 - i\gamma\sigma_3(QQ_{xx} + QxxQ) - \frac{i}{2}\sigma_3 Q^2 + \frac{1}{2}\sigma_3 Q_x + 6\gamma\sigma_3 Q^2 Q_x + \gamma\sigma_3 Q_{xxx},
$$

$$
Q = \begin{bmatrix}
0 & \tilde{q}(x, t) \\
q(x, t) & 0
\end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

### 2.1 Spectral analysis

Under the vanishing boundary (2.1), the Lax pair (2.2) becomes

$$
\psi_{\pm, x} = h_{\pm}\psi_{\pm}, \quad \psi_{\pm, t} = I_{\pm}\psi_{\pm}, \quad (2.3)
$$

where

$$
h_{\pm} = i\lambda\sigma_3, \quad I_{\pm} = A_4\lambda^4 + A_2\lambda^2,
$$

$$
A_4 = -8i\gamma\sigma_3, \quad A_2 = i\sigma_3.
$$

To solve the spectral problem (2.3), we introduce the Jost solution $\phi(x, t, \lambda)$ by a gauge transform

$$
\psi(x, t, \lambda) = \phi(x, t, \lambda)e^{i\theta(x, t, \lambda)\sigma_3}, \quad \theta(x, t, \lambda) = \lambda x + (-8\gamma\lambda^4 + \lambda^2)t, \quad (2.4)
$$

then $\phi(x, t, \lambda)$ satisfies

$$
\phi_{\pm}(x, t, \lambda) \sim I, \quad |x| \to \infty, \quad (2.5)
$$

and a new Lax pair

$$
\begin{cases}
\phi_x(x, t, \lambda) = i\lambda[\sigma_3, \phi(x, t, \lambda)] + iQ\phi(x, t, \lambda), \\
\phi_t(x, t, \lambda) = i(-8\gamma\lambda^4 + \lambda^2)[\sigma_3, \phi(x, t, \lambda)] + [A_3\lambda^3 + (A_2 - i\sigma_3)\lambda^2 + A_1\lambda + A_0]\phi(x, t, \lambda).
\end{cases} \quad (2.6)
$$

Consider the asymptotic expansion of $\phi(x, t, \lambda)$

$$
\phi(x, t, \lambda) = \phi[0] + \phi[1]\lambda^{-1} + \phi[2]\lambda^{-2} + \cdots, \quad \lambda \to \infty, \quad (2.7)
$$
where \( \phi[j], (j = 0, 1, 2, \cdots) \) are independent of \( \lambda \). Substituting the asymptotic expansion (2.7) into the system (2.6) and comparing the corresponding coefficients of \( \lambda^{-j} \), one can derive the following equations

\[
i[\sigma_3, \phi[0]] = 0,
\]

\[
\phi[0]_x = i[\sigma_3, \phi[1]] + iQ\phi[0],
\]

from which, we get

\[
\phi[0] = I,
\]

\[
q(x, t) = \lim_{\lambda \to \infty} 2\lambda [\phi(x, t, \lambda)]_{21},
\]

where the subscript ‘21’ represents the element in the second row and the first column of the matrix.

We rewrite the Lax pair (2.6) into a complete differential form

\[
d[e^{-i\theta(x,t,\lambda)\sigma_3} \phi(x, t, \lambda)] = e^{-i\theta(x,t,\lambda)\sigma_3} [W_1 dx + W_2 dt],
\]

where

\[
W_1 = iQ\phi(x, t, \lambda), \quad W_2 = [A_3\lambda^3 + (A_2 - i\sigma_3)\lambda^2 + A_1\lambda + A_0]\phi(x, t, \lambda).
\]

According to its asymptotic, the Jost solution \( \phi(x, t, \lambda) \) can be rewritten as

\[
\phi_-(x, t, \lambda) = I + \int_{-\infty}^{x} e^{i\lambda(x-y)\sigma_3} iQ(y)\phi(y)dy,
\]

\[
\phi_+(x, t, \lambda) = I - \int_{x}^{+\infty} e^{i\lambda(x-y)\sigma_3} iQ(y)\phi(y)dy.
\]

Further we can show the analyticity of two columns \( \phi_{\pm,j}(x, t, \lambda), \ j = 1, 2 \) for \( \phi_{\pm}(x, t, \lambda) \)

\[
\bullet \phi_{+,1}(x, t, \lambda) \quad \phi_{-,2}(x, t, \lambda) \text{ analytic in } D_+,
\]

\[
\bullet \phi_{-,1}(x, t, \lambda) \quad \phi_{+,2}(x, t, \lambda) \text{ analytic in } D_-,
\]

\[
\bullet \left[ \phi_{+,1}(x, t, \lambda) \quad \phi_{-,2}(x, t, \lambda) \right] \to I, \quad \lambda \to \infty,
\]

\[
\bullet \left[ \phi_{-,1}(x, t, \lambda) \quad \phi_{+,2}(x, t, \lambda) \right] \to I, \quad \lambda \to \infty.
\]

where \( D_+ = \{ \lambda \in \mathbb{C} : \text{Im} \lambda > 0 \} \), \( D_- = \{ \lambda \in \mathbb{C} : \text{Im} \lambda < 0 \} \). Figure 1 show that the distribution of the analytic regional.
Figure 1: The distribution of the analytic regional and discrete spectrum.

According to the Abel’s theorem, \( \partial_x (\det \phi) = \partial_t (\det \phi) = 0 \). So we have

\[
\det \psi(x, t, \lambda) = 1.
\] (2.13)

Since \( \psi_{\pm}(x, t, \lambda) \) are the linear related solutions of (2.3), so there exists a scattering matrix \( S(\lambda) \) such that

\[
\psi_+(x, t, \lambda) = \psi_-(x, t, \lambda) S(\lambda),
\] (2.14)

where

\[
S(\lambda) = \begin{bmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{bmatrix}.
\]

Then the Jost solutions \( \phi_{\pm}(x, t, \lambda) \) satisfy the linear relation

\[
\phi_+(x, t, \lambda) = \phi_-(x, t, \lambda) e^{i\theta(x, t, \lambda) \sigma_3} S(\lambda) e^{-i\theta(x, t, \lambda) \sigma_3}.
\] (2.15)

Direct calculation shows that

\[
\frac{\phi_{+,1}(x, t, \lambda)}{s_{11}(\lambda)} = \phi_{-,1}(x, t, \lambda) + e^{-2i\theta} \rho(\lambda) \phi_{-,2}(x, t, \lambda),
\] (2.16a)

\[
\frac{\phi_{+,2}(x, t, \lambda)}{s_{22}(\lambda)} = e^{2i\theta} \tilde{\rho}(\lambda) \phi_{-,1}(x, t, \lambda) + \phi_{-,2}(x, t, \lambda),
\] (2.16b)

where

\[
\rho(\lambda) = \frac{s_{21}(\lambda)}{s_{11}(\lambda)}, \quad \tilde{\rho}(\lambda) = \frac{s_{12}(\lambda)}{s_{22}(\lambda)}.
\] (2.17)
And $s_{ij}(\lambda)$ have the following determinant expressions

$$s_{11}(\lambda) = \det \begin{bmatrix} \phi_{+,1} & \phi_{-,2} \end{bmatrix}, \quad s_{12} = \det \begin{bmatrix} \phi_{+,2} & \phi_{-,2} \end{bmatrix} e^{-2i\theta},$$  \hspace{1cm} (2.18a)

$$s_{21} = \det \begin{bmatrix} \phi_{-,1} & \phi_{+,1} \end{bmatrix} e^{2i\theta}, \quad s_{22}(\lambda) = \det \begin{bmatrix} \phi_{-,1} & \phi_{+,2} \end{bmatrix}. \hspace{1cm} (2.18b)$$

It can be shown that the Jost solution $\phi(x, t, \lambda)$ admit the symmetry

$$\phi(x, t, \lambda) = \sigma \phi(\overline{\lambda}) \sigma,$$  \hspace{1cm} (2.19)

where

$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \sigma^2 = -I.$$

$S(\lambda)$ admits the symmetry of matrix

$$s_{11}(\lambda) = \overline{s_{22}(\lambda)}, \quad s_{21}(\lambda) = -\overline{s_{12}(\lambda)},$$  \hspace{1cm} (2.20)

so the scattering matrix $S(\lambda)$ can be rewritten as

$$S(\lambda) = \begin{bmatrix} a(\lambda) & -\overline{b(\lambda)} \\ \overline{b(\lambda)} & a(\lambda) \end{bmatrix},$$

and

$$\tilde{\rho}(\lambda) = -\overline{\rho(\lambda)}. \hspace{1cm} (2.21)$$

The scattering data $a(\lambda)$ and $b(\lambda)$ with the following properties:

- $|a(\lambda)|^2 + |b(\lambda)|^2 = 1$;
- $a(\lambda) \to 1$ and $b(\lambda) \to 0$, $\lambda \to \infty$;
- $a(\lambda)$ analytic and continuous in $D_+$, $b(\lambda)$ only continuous on the real axis.

### 2.2 The Riemann-Hilbert problem

According to equations (2.16), one can construct a meromorphic function $M(x, t, \lambda)$:

$$M(x, t, \lambda) = \begin{cases} \begin{bmatrix} \phi_{+,1}(x, t, \lambda) & \phi_{+,2}(x, t, \lambda) \\ \phi_{-,1}(x, t, \lambda) & \phi_{-,2}(x, t, \lambda) \end{bmatrix}, & \lambda \in D_+, \\ \begin{bmatrix} \phi_{+,1}(x, t, \lambda) & \phi_{-,2}(x, t, \lambda) \\ \phi_{-,1}(x, t, \lambda) & \phi_{+,2}(x, t, \lambda) \end{bmatrix}, & \lambda \in D_-. \end{cases} \hspace{1cm} (2.23)$$

which satisfies the following properties

- $\det M(x, t, \lambda) = 1.$  \hspace{1cm} (2.24a)
\( M(x, t, \lambda) \rightarrow I, \quad \lambda \rightarrow \infty \);

\( M^+(x, t, \lambda) = M^-(x, t, \lambda)J(\lambda), \) where the jump matrix

\[
J(\lambda) = \begin{bmatrix}
1 + \rho(\lambda)\overline{\rho(\lambda)} & \rho(\lambda)e^{2i\theta(x,t,\lambda)} \\
\rho(\lambda)e^{-2i\theta(x,t,\lambda)} & 1
\end{bmatrix}.
\]

Moreover, the solution \((2.9)\) of equation \((1.5)\) can be rewritten by

\[
q(x, t) = \lim_{\lambda \to \infty} 2\lambda[M(x, t, \lambda)]_{21}.
\]

For constructing soliton solutions of equation \((1.5)\), we consider the \( b(\lambda) = 0 \) situation. Now we assume \( a(\lambda) \) with \( N \) high-order zeros \( \lambda_j \) (\( j = 1, 2, \ldots, N \)). This is very different from \( a(\lambda) \) with the simple zeros. If \( a(\lambda) \) with the simple zeros, the Riemann-Hilbert method can be straightly solved by applying the residue conditions and the formula of \( N \)-order zeros can be obtained. But for \( a(\lambda) \) with the high order zeros, except to the residue conditions, the coefficients of \((\lambda - \lambda_j)^{-s}\) and \((\lambda - \overline{\lambda}_j)^{-s}\) should be considered. However, these coefficients are not straightforward derived.

### 2.3 Single high-order pole

In this subsection, we firstly consider \( a(\lambda) \) with one \( N \)-order zero \( \lambda_0 \), that is \( a(\lambda) \) can be expanded as

\[
a(\lambda) = a_0(\lambda)(\lambda - \lambda_0)^N,
\]

and \( a_0(\lambda) \neq 0 \) for all \( \lambda \) (\( \text{Im}\lambda > 0 \)). So \( \rho(\lambda) \) and \( \overline{\rho(\lambda)} \) with one \( N \)-order pole. \( \rho(\lambda) \) and \( \overline{\rho(\lambda)} \) can be expanded as the Laurent expansion

\[
\rho(\lambda) = \rho_0(\lambda) + \sum_{m=1}^{N} \frac{\rho_m}{(\lambda - \lambda_0)^m}, \quad \overline{\rho(\lambda)} = \overline{\rho_0(\lambda)} + \sum_{m=1}^{N} \frac{\overline{\rho_m}}{(\lambda - \overline{\lambda}_0)^m},
\]

where

\[
\rho_m = \lim_{\lambda \to \lambda_0} \frac{1}{(N-m)!} \frac{\partial^{N-m}}{\partial(\lambda - \lambda_0)^{N-m}}[(\lambda - \lambda_0)^N \rho(\lambda)], \quad (m = 1, 2, \ldots, N).
\]

According to the definition of \( M(x, t, \lambda) \), one can obtain that \( \lambda = \lambda_0 \) is the \( N \)-order pole of \( M_1(x, t, \lambda) \) and \( \lambda = \overline{\lambda}_0 \) is the \( N \)-order pole of \( M_2(x, t, \lambda) \).
$M_2(x, t, \lambda)$ is analytic as $\lambda = \lambda_0$ and $M_1(x, t, \lambda)$ is analytic as $\lambda = \lambda_0$. So we have the following expansions

$$M_{21}(\lambda) = \sum_{s=1}^{N} \frac{G_s(x, t)}{(\lambda - \lambda_0)^s}, \quad M_{22}(\lambda) = 1 + \sum_{s=1}^{N} \frac{F_s(x, t)}{(\lambda - \lambda_0)^s}.$$  \hspace{1cm} (2.26)

According to the analyticity one can get the Taylor expansion

$$e^{-2i\theta(\lambda)} = \sum_{l=0}^{+\infty} f_l(x, t)(\lambda - \lambda_0)^l, \quad e^{2i\theta(\lambda)} = \sum_{l=0}^{+\infty} f_l(x, t)(\lambda - \lambda_0)^l, \hspace{1cm} (2.27a)$$

$$M_{21}(\lambda) = \sum_{l=0}^{+\infty} \zeta_l(x, t)(\lambda - \lambda_0)^l, \quad M_{22}(\lambda) = \sum_{l=0}^{+\infty} \mu_l(x, t)(\lambda - \lambda_0)^l, \hspace{1cm} (2.27b)$$

where

$$f_l(x, t) = \lim_{\lambda \to \lambda_0} \frac{1}{l!} \frac{\partial^l}{\partial \lambda^l} e^{-2i\theta(\lambda)}, \hspace{1cm} (2.28a)$$

$$\mu_l(x, t) = \lim_{\lambda \to \lambda_0} \frac{1}{l!} \frac{\partial^l}{\partial \lambda^l} M_{22}(\lambda), \hspace{1cm} (2.28b)$$

$$\zeta_l(x, t) = \lim_{\lambda \to \lambda_0} \frac{1}{l!} \frac{\partial^l}{\partial \lambda^l} M_{21}(\lambda). \hspace{1cm} (2.28c)$$

Then according to the equation (2.15) and the definition of $M(x, t, \lambda)$, comparing the corresponding coefficients of $(\lambda - \lambda_0)^s$ and $(\lambda - \lambda_0)^{-s}$. One can derive the following results

$$F_s(x, t) = -\sum_{j=s}^{N} \sum_{l=0}^{j-s} \bar{\rho}_j f_{j-s-l}(x, t) \zeta_l(x, t), \hspace{1cm} (2.29a)$$

$$G_s(x, t) = \sum_{j=s}^{N} \sum_{l=0}^{j-s} \rho_j f_{j-s-l}(x, t) \mu_l(x, t), \hspace{1cm} (2.29b)$$

where $s = 1, 2, \cdots, N$. For $N = 1$, $F_s(x, t)$ and $G_s(x, t)$ degenerate into the residue conditions. In addition, $\mu_l(x, t)$ and $\zeta_l(x, t)$ can be expressed as $F_s(x, t)$ and $G_s(x, t)$ via direct calculation, that is

$$\zeta_l(x, t) = \sum_{s=1}^{N} \left( \begin{array}{c} s + l - 1 \cr l \end{array} \right) \frac{(-1)^l G_s(x, t)}{(\lambda_0 - \lambda_0)^{l+s}}, \quad l = 0, 1, 2, \cdots, \hspace{1cm} (2.30a)$$

$$\mu_l(x, t) = \begin{cases} 1 + \sum_{s=1}^{N} \frac{F_s(x, t)}{(\lambda_0 - \lambda_0)^s}, & l = 0, \\ \sum_{s=1}^{N} \left( \begin{array}{c} s + l - 1 \cr l \end{array} \right) \frac{(-1)^l F_s(x, t)}{(\lambda_0 - \lambda_0)^{l+s}}, & l = 1, 2, \cdots. \end{cases} \hspace{1cm} (2.30b)$$


Then one can obtain the following system

\[
F_s(x,t) = - \sum_{j=s}^{N} \sum_{l=0}^{s-1} \sum_{p=1}^{N} \left( p + l - 1 \right) \frac{(-1)^p \rho_j f_j(x,t) G_p(x,t)}{(\lambda_0 - \lambda_l)^{l+p}}, \quad (2.31a)
\]

\[
G_s(x,t) = \sum_{j=s}^{N} \rho_j f_j(x,t) + \sum_{j=s}^{N} \sum_{l=0}^{s-1} \sum_{p=1}^{N} \left( p + l - 1 \right) \frac{(-1)^p \rho_j f_j(x,t) G_p(x,t)}{(\lambda_0 - \lambda_l)^{l+p}}.
\]

For convenience, we introduce the following symbols

\[
|F\rangle = [ \begin{array}{cccc} F_1 & F_2 & \cdots & F_N \end{array} ]^T, \quad |G\rangle = [ \begin{array}{cccc} G_1 & G_2 & \cdots & G_N \end{array} ]^T, \quad (2.32a)
\]

\[
\chi = [\chi_{sp}]_{N \times N} = \left( - \sum_{j=s}^{N} \sum_{l=0}^{s-1} \left( p + l - 1 \right) \frac{(-1)^p \rho_j f_j(x,t) G_p(x,t)}{(\lambda_0 - \lambda_l)^{l+p}} \right)_{N \times N},
\]

\[
|\eta\rangle = [ \begin{array}{cccc} \eta_1 & \eta_2 & \cdots & \eta_N \end{array} ]^T, \quad \eta_s(x,t) = \sum_{j=s}^{N} \rho_j f_j(x,t), \quad s, p = 1, 2, \cdots, N.
\]

So the equations (2.31) can be rewritten as

\[
|F\rangle = \chi |G\rangle, \quad |G\rangle = |\eta\rangle - \chi |F\rangle.
\]

And \(|F\rangle\) and \(|G\rangle\) are solved as

\[
|F\rangle = \chi (I + \chi \chi)^{-1} |\eta\rangle, \quad |G\rangle = (I + \chi \chi)^{-1} |\eta\rangle.
\]

So the expansions of \(M_{21}(x,t,\lambda)\) and \(M_{22}(x,t,\lambda)\) can be given as

\[
M_{21}(x,t,\lambda) = \sum_{s=1}^{N} \frac{G_s(x,t)}{(\lambda - \lambda_0)^s} = \frac{\det (I + \chi \chi + |\eta\rangle \langle Y(\lambda)\chi|)}{\det (I + \chi \chi)} - 1, \quad (2.35a)
\]

\[
M_{22}(x,t,\lambda) = 1 + \sum_{s=1}^{N} \frac{F_s(x,t)}{(\lambda - \lambda_0)^s} = \frac{\det (I + \chi \chi + |\eta\rangle \langle Y(\lambda)\chi|)}{\det (I + \chi \chi)}, \quad (2.35b)
\]

where

\[
\langle Y(\lambda)\rangle = \left[ \begin{array}{cccc} \frac{1}{\lambda - \lambda_0} & \frac{1}{(\lambda - \lambda_0)^2} & \cdots & \frac{1}{(\lambda - \lambda_0)^N} \end{array} \right].
\]

According to the symmetric property, one can derive

\[
M_{12}(x,t,\lambda) = 1 - \frac{\det (I + \chi \chi + |\eta\rangle \langle Y(\lambda)\chi|)}{\det (I + \chi \chi)}, \quad (2.37a)
\]
\[ M_{11}(x, t, \lambda) = \frac{\det (I + \chi \chi + |\eta\rangle \langle Y(\lambda)|\chi)}{\det (I + \chi \chi)}. \] (2.37b)

So one can derive the solutions of equation (1.5) with one N-order pole

\[ q(x, t) = 2 \left( \frac{\det (I + \chi \chi + |\eta\rangle \langle Y_0|)}{\det (I + \chi \chi)} - 1 \right), \] (2.38)

where

\[ \langle Y_0| = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{1 \times N}. \] (2.39)

### 2.3.1 Single 1\textsuperscript{nd}-order pole solution

Set \( \lambda_0 \) is 1-order zero of \( a(\lambda) \), so \( \rho(\lambda) \) can be expanded as

\[ \rho(\lambda) = \rho_0(\lambda) + \frac{\rho_1}{\lambda - \lambda_0}. \] (2.40)

Now the coefficient \( \rho_1 \) reduces to \( \text{Res}[\rho(\lambda), \lambda_0] \). The matrices \( \chi \), \( |\eta\rangle \) and \( \langle Y_0| \) in (2.38) are reduced as

\[ \chi = -\frac{\rho_1 f_0}{\lambda_0 - \lambda_0}, \quad |\eta\rangle = \rho_1 f_0, \quad \langle Y_0| = 1. \] (2.41)

According to (2.38) the one soliton solution of the QNLS equation can be derived. The patterns of soliton are shown in Figure 2-5 with different parameters. The 2-dimensional figure show the propagate of soliton at different time: orange full line is \( t = 0.5 \), red dashed line is \( t = 0 \) and green dashed line is \( t = -0.5 \).

![Figure 2](image)

Figure 2: \( \gamma = 0, \lambda_0 = \frac{1}{2} + i \frac{\sqrt{2}}{2} \).

### 2.3.2 Single 2\textsuperscript{nd}-order pole solution

Set \( \lambda_0 \) is 2-order zero of \( a(\lambda) \), reflection coefficient \( \rho(\lambda) \) can be expanded as

\[ \rho(\lambda) = \rho_0(\lambda) + \frac{\rho_1}{\lambda - \lambda_0} + \frac{\rho_2}{(\lambda - \lambda_0)^2}, \]
Figure 3: $\gamma = 1, \lambda_0 = \frac{1}{2} + i \frac{\sqrt{2}}{2}$.

Figure 4: $\gamma = 1, \lambda_0 = -\frac{1}{10} + i \frac{\sqrt{2}}{2}$.

Figure 5: $\gamma = 1, \lambda_0 = i \frac{\sqrt{2}}{2}$. 
\( \chi \) is defined as

\[
\chi = \begin{bmatrix}
\chi_{11} & \chi_{12} \\
\chi_{21} & \chi_{22}
\end{bmatrix},
\]

where

\[
\chi_{sp} = -\sum_{j=s}^{j-s} \sum_{l=0}^{l-s} \left( p + l - 1 \right) \frac{(-1)^l \rho_j f_{j-s-l}(x, t) G_p(x, t)}{(\lambda_0 - \lambda_0)^{l+p}}, \quad s, p = 1, 2
\]

|\( \eta \rangle = [ \eta_1 \eta_2 ]^T \) is defined as

\[
\eta_s(x, t) = \sum_{j=s}^{j-s} \rho_j f_{j-s}(x, t),
\]

and \( \langle Y_0 | = [ 1 \ 0 ] \).

According to (2.38), the 2-soliton solution of the QNLS equation can be derived. The interactions of soliton with different parameters are shown in Figure 6-9. The 2-dimensional figures show the propagate at different time: orange full line is \( t = 0.5 \), red dashed line is \( t = 0 \) and green dashed line is \( t = -0.5 \).

Figure 6: \( \gamma = 0, \rho_1 = 1, \rho_2 = 2, \lambda_0 = \frac{1}{2} + i \sqrt{2} \).

Figure 7: \( \gamma = 1, \rho_1 = 1, \rho_2 = 2, \lambda_0 = \frac{1}{3} + i \sqrt{3} \).
2.4 Multiple high-order pole solutions

In this subsection, we consider the general case: \( a(\lambda) \) with \( N \) high-order zeros \( \lambda_1, \lambda_2, \cdots, \lambda_N \), and the powers are \( n_1, n_2, \cdots, n_N \), respectively. So \( a(\lambda) \) can be expanded as

\[
a(\lambda) = a_0(\lambda)(\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_N)^{n_N}.
\]

Similar to the one high-order pole, \( \rho(\lambda) \) can be expanded as the Laurent series

\[
\rho(\lambda) = \rho_{j,0}(\lambda) + \sum_{m_j=1}^{n_j} \frac{\rho_{j,m_j}}{(\lambda - \lambda_j)^{m_j}} \quad \text{and} \quad \overline{\rho(\lambda)} = \overline{\rho_{j,0}(\lambda) + \sum_{m_j=1}^{n_j} \frac{\overline{\rho_{j,m_j}}}{(\lambda - \overline{\lambda_j})^{m_j}}},
\]

where

\[
\rho_{j,m_j} = \lim_{\lambda \to \lambda_j} \frac{1}{(\lambda - \lambda_j)^{m_j}} \frac{\partial^{n_j-m_j}}{(n_j-m_j)!} \left[ (\lambda - \lambda_j)^{n_j} \rho(\lambda) \right], \quad j = 1, \cdots, N,
\]

and \( \rho_0(\lambda) \) is analytic for all \( \lambda \) (\( \text{Im} \lambda > 0 \)). So, in the similar method, one can derive the soliton solution formula with \( N \) high-order poles.
Theorem 2.1. For the vanishing boundary condition, if \( a(\lambda) \) with \( N \) high-order zeros, then the soliton solutions of equation \((1.3)\) are given by the following formula

\[
q(x,t) = 2 \left( \frac{\det (I + \chi \chi^T + |\eta \rangle \langle Y_0|)}{\det (I + \chi \chi^T)} - 1 \right), \tag{2.43}
\]

where

\[
|\eta| = [ \eta_1 \eta_2 \cdots \eta_N ]^T, \quad (Y_0) = [ Y_0^1 \ Y_0^2 \cdots \ Y_0^N ],
\]

\[
Y_j^0 = [ 1 \ 0 \ \cdots \ 0 ]_{1 \times n_j}, \quad \eta_j = [ \eta_{j,1} \ \eta_{j,2} \ \cdots \ \eta_{j,n_j} ],
\]

\[
\eta_{j,l} = \sum_{m_j=m}^{n_j} \rho_{j,m_j}f_{j,m_j-l}(x,t),
\]

\[
\chi = \begin{bmatrix}
[\varpi_{11}] & [\varpi_{12}] & \cdots & [\varpi_{1N}]
[\varpi_{21}] & [\varpi_{22}] & \cdots & [\varpi_{2N}]
\vdots & \vdots & \ddots & \vdots
[\varpi_{N1}] & [\varpi_{N2}] & \cdots & [\varpi_{NN}]
\end{bmatrix},
\]

and \([\varpi_{j,l}]\) \((j,l = 1, 2, \cdots, N)\) are \(n_j \times n_l\) matrices

\[
[\varpi_{j,l}] = ([\varpi_{j,l}]_{p,q})_{n_j \times n_l} = \left( -\sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \left( \frac{q + s_j - 1}{s_j} \right) \frac{(-1)^{s_j} p_{j,m_j} f_{j,m_j-p-s_j}(x,t)}{(\lambda_j - \lambda_l)^{s_j+q}} \right)_{n_j \times n_l}.
\]

2.4.1 Double 2\(^{nd}\)-order pole solutions

Let \( \lambda_1 \) and \( \lambda_2 \) are 2\(^{nd}\)-order zeros of \( a(\lambda) \), The reflection coefficient \( \rho(\lambda) \) can be expand as Laurent series

\[
\rho(\lambda) = \rho_{j,0}(\lambda) + \sum_{m_j=1}^{n_j} \frac{\rho_{j,m_j}}{(\lambda - \lambda_j)^{m_j}},
\]

Now \( \chi \) in \((2.43)\) be defined as

\[
\chi = \begin{bmatrix}
[\varpi_{11}] & [\varpi_{12}] & \cdots & [\varpi_{1N}]
[\varpi_{21}] & [\varpi_{22}] & \cdots & [\varpi_{2N}]
\vdots & \vdots & \ddots & \vdots
[\varpi_{N1}] & [\varpi_{N2}] & \cdots & [\varpi_{NN}]
\end{bmatrix}_{4 \times 4},
\]

where

\[
[\varpi_{j,l}] = ([\varpi_{j,l}]_{p,q})_{n_j \times n_l} = \left( -\sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \left( \frac{q + s_j - 1}{s_j} \right) \frac{(-1)^{s_j} p_{j,m_j} f_{j,m_j-p-s_j}(x,t)}{(\lambda_j - \lambda_l)^{s_j+q}} \right)_{n_j \times n_l}.
\]

and \( j, l = 1, 2, n_1 = n_2 = 2 \). \((Y_0) = [ 1 \ 0 \ 1 \ 0 ] \) and \(|\eta|\) in \((2.43)\) is defined as a column vector \(|\eta| = [ \eta_1 \ \eta_2 ]^T\), where
According to Theorem 2.1 the 4-soliton solution of the QNLS equation can be derived. The interactions of soliton with different parameters are shown in Figure 10-11. The 2-dimensional figure show the propagate at different time: orange full line is \( t = 0.5 \), red dashed line is \( t = 0 \) and green dashed line is \( t = -0.5 \).

\[
\eta_{j,l} = \sum_{m_j=l}^{n_j} \rho_{j,m_j} f_{j,m_j-l}(x,t),
\]

\( \gamma = 0, \rho_{1,1} = 1, \rho_{1,2} = 2, \rho_{2,1} = 2, \rho_{2,2} = 1, \lambda_1 = \frac{1}{2} + i \frac{\sqrt{2}}{2}, \lambda_2 = -\frac{1}{2} + i \frac{\sqrt{3}}{4}, \)

\[
\gamma = 1, \rho_{1,1} = 1, \rho_{1,2} = 2, \rho_{2,1} = 2, \rho_{2,2} = 1, \lambda_1 = -\frac{\sqrt{3}}{2} + i \frac{\sqrt{2}}{2}, \lambda_2 = \frac{\sqrt{3}}{2} + i \frac{\sqrt{2}}{2}.
\]

### 2.4.2 Mixed 2\(^{nd}\)- and 1\(^{nd}\)-order pole solution

Let \( \lambda_1 \) is a 2-order zero point of \( a(\lambda) \) and \( \lambda_2 \) is a 1-order zero point of \( a(\lambda) \). The reflection coefficient \( \rho(\lambda) \) can be expand as Laurent series

\[
\rho(\lambda) = \rho_{j,0}(\lambda) + \sum_{m_j}^{n_j} \frac{\rho_{j,m_j}}{(\lambda - \lambda_j)^{m_j}},
\]
Now $\chi$ in (2.43) be defined as

$$
\chi = \left[ \begin{array}{cc}
[\varpi_{11}] & [\varpi_{12}] \\
[\varpi_{21}] & [\varpi_{22}]
\end{array} \right]_{3 \times 3},
$$

where

$$
[\varpi_{j,l}] = \left( \begin{array}{cc}
\rho_{j,m} & f_{j,m} \end{array} \right)_{n_j \times n_l} = 
\sum_{m_j+p, s_j=0}^{n_j} \left( q + s_j - 1 \right) \frac{(-1)^{s_j}}{\lambda_j - \lambda_l}^{s_j+q}.
$$

$j, l = 1, 2$, $n_1 = 2$ and $n_2 = 1$. $\langle Y_0 \rangle = [1 \ 0 \ 1]$ and $|\eta\rangle$ in (2.43) is defined as a column vector $|\eta\rangle = [\eta_1 \ \eta_2]^T$, where

$$
\eta_{j,l} = \sum_{m_j=m_l}^{n_j} \rho_{j,m} f_{j,m-j}(x,t),
$$

According to Theorem 2.1 the 3-soliton solution of the QNLS equation can be derived. The interactions of soliton with different parameters are shown in Figure 12. The 2-dimensional figure show the propagate at different time: orange full line is $t = 0.5$, red dashed line is $t = 0$ and green dashed line is $t = -0.5$.

![Figure 12](image)

Figure 12: $\gamma = 0$, $\rho_{1,1} = 1$, $\rho_{1,2} = 2$, $\rho_{2,1} = 3$, $\lambda_1 = -\frac{1}{3} + i\frac{2\sqrt{2}}{3}$, $\lambda_2 = \frac{1}{2} + i\frac{\sqrt{2}}{2}$.

### 3 Inverse scattering transform for non-vanishing boundary conditions

In this section we consider the soliton solution of the equation (1.5) with the non-vanishing boundary conditions. For convenient, we directly discuss the following equation:

$$
i q_{t} + \frac{1}{2} q_{xx} + (|q|^2 - q_0^2)q + \gamma [q_{xxxx} + 6(|q|^4 - q_0^4)q + 4q \bar{q}_x q_x + 2q^2 \bar{q}_x^2 + 8|q|^2 q_{xx}] = 0.
$$

(3.1)
This equation can be transformed as the general QNLS equation (1.5) under the gauge transformation

\[ q(x, t) = \tilde{q}(x, t)e^{-i(q_0^2 + 6\gamma q_0^4)t}. \]  

(3.2)

### 3.1 Spectral analysis

Under the non-vanishing asymptotic condition

\[ \lim_{x \to \pm \infty} q(x, t) = q_\pm = q_0 e^{i\theta \pm}, \quad |q_\pm| = q_0, \]  

(3.3)

the Lax pair (2.2) admits the following asymptotic spectral problem

\[ \phi_{\pm, x} = h_{\pm} \phi_{\pm}, \quad \phi_{\pm, t} = l_{\pm} \phi_{\pm}, \]  

(3.4)
where
\[ h_\pm = \lim_{x \to \pm \infty} h = i \lambda \sigma_3 + i Q_\pm, \]
\[ l_\pm = \lim_{x \to \pm \infty} l = \hat{A}_4 \lambda^4 + \hat{A}_3 \lambda^3 + \hat{A}_2 \lambda^2 + \hat{A}_1 \lambda + \hat{A}_0, \]
\[ \hat{A}_4 = -8i \gamma \sigma_3, \quad \hat{A}_3 = -8i \gamma Q_\pm, \]
\[ \hat{A}_2 = i(1 + 4 \gamma Q_\pm^2) \sigma_3, \quad \hat{A}_1 = i(Q_\pm + 4 \gamma Q_\pm^3), \]
\[ \hat{A}_0 = -3i \gamma \sigma_3 Q_\pm^4 - \frac{i}{2} \sigma_3 Q_\pm^2, \]
\[ Q_\pm = \begin{bmatrix} 0 & \frac{q_\pm}{q_0} \\ q_\pm & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

It is easily verified via direct calculation \([h_\pm, l_\pm] = 0\). The eigenvalues of \(h_\pm\) and \(l_\pm\) are \(\pm i \sqrt{\lambda^2 + q_0^2}\) and \(\pm i \sqrt{\lambda^2 + q_0^2}(\lambda - 8 \gamma Q_\pm^3 + 4 \gamma Q_\pm^3)\), respectively. Moreover, \(h_\pm\) and \(l_\pm\) share the same eigenvector
\[ \Omega_\pm = \begin{bmatrix} 1 \\ q_\pm \\ \lambda \pm \sqrt{\lambda^2 + q_0^2} \\ 1 \end{bmatrix}. \]

Since the eigenvalues are doubly branched. One can transform the \(\lambda\)-plane to a two-sheet Riemann surface by
\[ k^2 = \lambda^2 + q_0^2. \]

So \(k(\lambda)\) is a single value function on the Riemann surface, and \(\lambda = \pm iq_0\) are the branch point. Let
\[ \lambda + iq_0 = \varepsilon_1 e^{i \theta_1}, \quad \lambda - iq_0 = \varepsilon_2 e^{i \theta_2}, \quad -\frac{\pi}{2} \leq \theta_1, \theta_2 \leq \frac{3\pi}{2}. \]

So \(k\) on each sheet as
\[ k_1 = \sqrt{\varepsilon_1 \varepsilon_2} e^{i \frac{\theta_1 + \theta_2}{2}}, \quad k_11 = -k_1 = \sqrt{\varepsilon_1 \varepsilon_2} e^{i \left( i \frac{\theta_1 + \theta_2}{2} \right)}. \]

And the branch cut of the Riemann surface is the segment \([-iq_0, iq_0]\) in the complex \(k\)-plane. We introduce the uniformization variable
\[ z = \lambda + k, \]
and we note that
\[ k = \frac{1}{2}(z + \frac{q_0^2}{z}), \quad \lambda = \frac{1}{2}(z - \frac{q_0^2}{z}). \]
Table 1: Conformal mapping from $k_j$-plane ($j = I, II$) to $z$-plane ($q_j^k$ or $q_j^*$ ($j = 1, 2, 3, 4$) mark the four quadrants excluding the boundary of $k$-plane or $z$-plane).

| $k_I$-plane | $\rightarrow$ | $z$-plane | $k_{II}$-plane | $\rightarrow$ | $z$-plane |
|------------|-------------|-----------|----------------|-------------|---------|
| $q_j^k$ ($j = 1, 2, 3, 4$) | $\rightarrow$ | $q_j^k \cap \{|z| > q_0\}, (j = 1, 2, 3, 4)$ | $q_j^k$ ($j = 1, 2, 3, 4$) | $\rightarrow$ | $q_j^k \cap \{|z| < q_0\}, (j = 1, 2, 3, 4)$ |
| $[0, iq_0]$ | $\rightarrow$ | $\{z = q_0 e^{i\theta}, \theta \in [0, \frac{\pi}{2}]\}$ | $[0, iq_0]$ | $\rightarrow$ | $\{z = q_0 e^{i\theta}, \theta \in [-\pi, -\frac{\pi}{2}]\}$ |
| $[-iq_0, 0]$ | $\rightarrow$ | $\{z = q_0 e^{i\theta}, \theta \in [\pi, \frac{3\pi}{2}]\}$ | $[-iq_0, 0]$ | $\rightarrow$ | $\{z = q_0 e^{i\theta}, \theta \in [-\frac{\pi}{2}, 0]\}$ |
| $[\pm iq_0, \pm \infty)$ | $\rightarrow$ | $[\pm iq_0, \pm \infty)$ | $[\pm iq_0, \pm \infty)$ | $\rightarrow$ | $[\pm iq_0, \pm \infty)$ |
| $[0, \pm \infty)$ | $\rightarrow$ | $[\pm iq_0, \pm \infty)$ | $[0, \pm \infty)$ | $\rightarrow$ | $[\pm iq_0, 0]$ |

Finally, the two-sheet Riemann surfaces $k_I$-plane and $k_{II}$-plane are transformed as $z$-plane, the Table 1 describes the changes in detail. So $\text{Im}k \nless 0$ replaced by $D_{\pm}$ and the complex $z$-plane with new version

$$z \in \mathbb{C} = \begin{cases} (|z|^2 - q_0^2) \text{Im}z > 0, & z \in D_+, \\ (|z|^2 - q_0^2) \text{Im}z < 0, & z \in D_-, \\ \mathbb{R} \cup C_0, & \text{otherwise,} \end{cases}$$

(3.5)

where $C_0$ is the circle of radius $q_0$ in the $z$-plane.

The eigenvalues and eigenvectors of $h_{\pm}$ and $l_{\pm}$ can be rewritten as

$$h_{\pm} \Omega_{\pm} = \Omega_{\pm} H_1, \quad l_{\pm} \Omega_{\pm} = \Omega_{\pm} H_2.$$
where
\[
\Omega_\pm = \begin{bmatrix}
\frac{1}{\bar{q}_0} & -\bar{q}_0 \\
\frac{\bar{q}_0}{z} & 1
\end{bmatrix} = I - \frac{1}{z}\sigma_3 Q_\pm,
\]
\[
H_1 = i\frac{1}{2}(z + \frac{q_0^2}{z})\sigma_3,
\]
\[
H_2 = i\frac{1}{2}(z + \frac{q_0^2}{z})\left[\frac{1}{2}(z - \frac{q_0^2}{z}) - \gamma(z - \frac{q_0^2}{z})^3 + 2\gamma q_0^3(z - \frac{q_0^2}{z})\right]\sigma_3.
\]
And we also note that
\[
\det \Omega_\pm(z) = 1 + \frac{q_0^2}{z^2} = \omega,
\]
\[
\Omega_\pm^{-1}(z) = \frac{1}{\omega}(I + \frac{1}{z}\sigma_3 Q_\pm).
\]
The asymptotic spectral problem \([3.4]\) satisfy the following fundamental matrix solution
\[
\phi_\pm(x, t, z) = \Omega_\pm(z)e^{i\theta(x, t, z)\sigma_3}.
\]

3.1.1 Jost solutions

The Lax pair \([2.2]\) can be rewritten as
\[
\phi_x = (h_\pm + \Delta Q_\pm)\phi,
\]
\[
\phi_t = (l_\pm + \Delta \hat{Q}_\pm)\phi,
\]
where
\[
\Delta Q_\pm = iQ - iQ_\pm, \quad \Delta \hat{Q}_\pm = \hat{Q} - \hat{Q}_\pm,
\]
\[
\hat{Q} = A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0,
\]
\[
\hat{Q}_\pm = \hat{A}_3\lambda^3 + \hat{A}_2\lambda^2 + \hat{A}_1\lambda + \hat{A}_0.
\]

Now, we introduce the Jost solutions as the simultaneous solutions of Lax pair \([3.7]\) such that
\[
\Psi_\pm(x, t, z) \sim \Omega_\pm(z)e^{i\theta(x, t, z)\sigma_3}, \quad z \in \Sigma, \quad x \to \pm\infty.
\]
For factoring the asymptotic exponential oscillations, we define a modified Jost solutions
\[
\mu_\pm(x, t, z) = \Psi_\pm(x, t, z)e^{-i\theta(x, t, z)\sigma_3}.
\]
The modified Jost solutions \( \mu_\pm(x, t, z) \) satisfy the asymptotic properties

\[
\lim_{|x| \to \infty} \mu_\pm(x, t, z) = \Omega_\pm(z).
\]  

(3.10)

One can derive the following satisfactory conclusion via take some calculations for equation (3.9)

\[
\begin{aligned}
(\Omega_\pm^{-1} \mu_\pm(x, t, z))_x &= -ik \left[ \sigma_3, \Omega_\pm^{-1}(z) \mu_\pm(x, t, z) \right] + \Omega_\pm^{-1}(z) \nabla \mu_\pm(x, t, z), \\
(\Omega_\pm^{-1} \mu_\pm(x, t, z))_t &= -ik(\lambda - 8\gamma \lambda^3 + 4\gamma \lambda u_0^3) \left[ \sigma_3, \Omega_\pm^{-1}(z) \mu_\pm(x, t, z) \right] + \Omega_\pm^{-1}(z) \nabla \hat{Q}_\pm \mu_\pm(x, t, z).
\end{aligned}
\]  

(3.11a)

(3.11b)

One can note that the system (3.11) admit the full derivative form

\[
d \left( e^{-i\theta_\pm \sigma_3} \Omega_\pm^{-1} \mu_\pm \right) = e^{-i\theta_\pm \sigma_3} \left( \Omega_\pm^{-1} \nabla \mu_\pm dx + \Omega_\pm^{-1} \nabla \hat{Q}_\pm dt \right). 
\]  

(3.12)

Then one can derive the modified Jost solution

\[
\begin{aligned}
\mu_-(x, t, z) &= \Omega_-(z) + \int_{-\infty}^x \Omega_-(z) e^{ik(x-y)\sigma_3} \Omega_\pm^{-1} \nabla \mu_-(y, t, z) dy, \\
\mu_+(x, t, z) &= \Omega_+(z) - \int_{x}^{+\infty} \Omega_+(z) e^{ik(x-y)\sigma_3} \Omega_\pm^{-1} \nabla \mu_+(y, t, z) dy.
\end{aligned}
\]  

(3.13a)

(3.13b)

Now we analyze the analyticity of \( \mu_\pm(x, t, z) \) in \( D_\pm \). It is well known that

\[
e^{ik(x-y)\sigma_3} A = \begin{bmatrix} a_{11} & a_{12} e^{2ik(x-y)} \\ a_{21} e^{-2ik(x-y)} & a_{22} \end{bmatrix}.
\]

Then the analyticities of \( \mu_\pm \) are decided by \( e^{\pm 2ik(x-y)} \). We make the expand for \( e^{\pm 2ik(x-y)} \)

\[
e^{-2i(\Re k + i\Im k)(x-y)} = e^{-2i(x-y)\Re k} e^{2(x-y)\Im k},
\]

\[
e^{2i(\Re k + i\Im k)(x-y)} = e^{2i(x-y)\Re k} e^{-2(x-y)\Im k}.
\]

So the analyticity decided by \( \Im k \) and \( x-y \). Through further analysis, one can obtain the analyticities of \( \mu_{\pm,j}(x, t, z), (j = 1, 2) \)

\[
\begin{align*}
\mu_{+,1}(x, t, z) \text{ and } \mu_{-,2}(x, t, z) \text{ is analytic in } D_+, \\
\mu_{-,1}(x, t, z) \text{ and } \mu_{+,2}(x, t, z) \text{ is analytic in } D_-
\end{align*}
\]

where the subscripts ‘1’ and ‘2’ identify the columns of matrix.
3.1.2 Scattering matrix

In this subsection, we will consider the scattering matrix. It is easy to check that $\text{tr} \Psi = \text{tr} \ell = 0$, and the Abel formula implies that

$$(\det \Psi)_{x} = (\det \Psi)_{t} = 0.$$  

And for $z \in \Sigma$,

$$\lim_{|x| \to \infty} \Psi(x, t, z)e^{-i\theta(z)\sigma_{3}} = \Omega_{\pm},$$

one can obtain

$$\det \Psi_{\pm}(x, t, z) = \det \Omega_{\pm} = \omega. \quad (3.16)$$

Since $\Psi_{+}(x, t, z)$ and $\Psi_{-}(x, t, z)$ are the fundamental solutions of the spectral problem (2.2), so there exists a constant matrix $S(z)$ which satisfies the following linear relationship

$$\mu_{+}(x, t, z) = \mu_{-}(x, t, z)e^{i\theta(z)\sigma_{3}}S(z)e^{-i\theta(z)\sigma_{3}}, \quad z \in \Sigma \setminus \{\pm iq_{0}\}, \quad (3.17)$$

where $S(z) = (s_{ij}(z))_{2 \times 2}$ is called scattering matrix. Moreover, (3.16) and (3.17) imply that $\det S(z) = 1$. The line relationship (3.17) can be expressed as

$$\mu_{\pm,j}(x, t, z) = \mu_{\pm,1}(x, t, z) + \rho(z)e^{-2i\theta(z)}\mu_{-2}(x, t, z), \quad (3.18a)$$

$$\mu_{\pm,2}(x, t, z) = \tilde{\rho}(z)e^{2i\theta(z)}\mu_{-1}(x, t, z) + \mu_{-2}(x, t, z), \quad (3.18b)$$

where the reflection coefficients are defined as

$$\rho(z) = \frac{s_{21}(z)}{s_{11}(z)}, \quad \tilde{\rho}(z) = \frac{s_{12}(z)}{s_{22}(z)}. \quad (3.19)$$

Solving the equation (3.18) by Cramer’s rule, the scattering coefficients have the following Wronskian representations

$$s_{11}(z) = \frac{\det [\mu_{+1}(x, t, z), \mu_{-2}(x, t, z)]}{\omega}, \quad s_{12}(z) = \frac{\det [\mu_{+2}(x, t, z), \mu_{-2}(x, t, z)] e^{-2i\theta(z)}}{\omega}, \quad (3.20a)$$

$$s_{21}(z) = \frac{\det [\mu_{-1}(x, t, z), \mu_{+1}(x, t, z)] e^{2i\theta(z)}}{\omega}, \quad s_{22}(z) = \frac{\det [\mu_{-1}(x, t, z), \mu_{+2}(x, t, z)]}{\omega}. \quad (3.20b)$$
Take advantage of the analyticities of \( \mu_{\pm,i}(x,t,z) \) and further consider equation (3.20), one can obtain \( s_{11}(z) \) is analytic in \( D_+ \) and \( s_{22}(z) \) is analytic in \( D_- \). However, \( s_{12}(z) \) and \( s_{22}(z) \) are just continuous on \( \Sigma \).

### 3.1.3 Symmetries

As usual, we consider the symmetries of Jost solution and scattering matrix. For the equation (1.5) with nonzero boundary conditions, there exist two kinds of symmetries for the Jost solution \( \Psi_{\pm}(z) \) and scattering matrix \( S(z) \) in \( z \)-plane: \( z \to \bar{z} \) and \( z \to -\frac{q_0^2}{z} \). For this purpose, we firstly introduce a constant matrix \( \sigma \)

\[
\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \sigma^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

And the matrix functions \( h \) and \( l \) in Lax pair (2.2) satisfy the following reduction conditions

\[
h(z) = -\sigma h(\bar{z}) \sigma, \quad l(z) = -\sigma l(\bar{z}) \sigma, \quad \Omega(\pm)(z) = -\sigma \Omega(\pm)(\bar{z}) \sigma, \quad \theta(z) = \overline{\theta(\bar{z})},
\]

(3.21a)

\[
h(z) = h(-\frac{q_0^2}{z}), \quad l(z) = l(-\frac{q_0^2}{z}), \quad \Omega(\pm)(z) = I - zQ^{-1} \sigma_3, \quad \theta(z) = -\theta(-\frac{q_0^2}{z}).
\]

(3.21b)

Now using the above results, one can derive the symmetries

1. The symmetries of up-half and low-half of \( z \)-plane \( (z \mapsto \bar{z}) \)

\[
\bullet \quad \Psi_{\pm}(z) = -\sigma \Psi_{\mp}(\bar{z}) \sigma, \quad \Psi_{\pm,j}(z) = (-1)^{j-1}\sigma \Psi_{\pm,(3-j)}(\bar{z}).
\]

(3.22a)

\[
\bullet \quad S(z) = -\sigma S(\bar{z}) \sigma, \quad s_{11}(z) = \overline{s_{22}(\bar{z})}, \quad s_{12}(z) = -\overline{s_{21}(\bar{z})}.
\]

(3.22b)

\[
\bullet \quad \rho(z) = -\overline{\rho(\bar{z})}.
\]

(3.22c)

2. The symmetries of outside and inside of \( C_0 \) \( (z \mapsto -\frac{q_0^2}{z}) \)

\[
\bullet \quad \Psi_{\pm}(z) = -\frac{1}{z} \Psi_{\pm,\mp}(\frac{q_0^2}{z}) \sigma_3 Q_{\pm}, \quad \Psi_{\pm,j}(z) = (-1)^{j-1} \frac{-1}{z} q_{\pm} \Psi_{\pm,(3-j)}\left(-\frac{q_0^2}{z}\right),
\]

(3.23a)

\[
\bullet \quad S(z) = Q^{-1} \sigma_3 S\left(-\frac{q_0^2}{z}\right) \sigma_3 Q_{+}, \quad s_{11}(z) = \frac{q_+}{q_-} s_{22}(\frac{q_0^2}{z}), \quad s_{12}(z) = -\frac{q_+}{q_-} s_{21}\left(-\frac{q_0^2}{z}\right).
\]

(3.23b)

\[
\bullet \quad \rho(-\frac{q_0^2}{z}) = -\frac{q_-}{q_+} \rho(z).
\]

(3.23c)
According to the above symmetries, the scattering matrix $S(z)$ can be rewritten as a new version

$$S(z) = \begin{bmatrix} a(z) & -\overline{b(z)} \\ b(z) & \overline{a(z)} \end{bmatrix}.$$ 

### 3.1.4 Asymptotic of $\mu_{\pm}(x, t, z)$ and $S(z)$

In this section, we consider the asymptotic of Jost solution and scattering matrix. For convenient, we introduce the following notations

$$A_d = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, \quad A_o = \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} \quad (3.24)$$

Consider the asymptotic expansion of $\mu_{\pm}(x, t, z)$ in equation (3.13a)

$$\mu_{\pm}(x, t, z) = \sum_{n=0}^{\infty} \mu_{\pm}^{(n)}(x, t, z) \quad (3.25)$$

where

$$\mu_{\pm}^{(0)}(x, t, z) = \Omega_{\pm}(z) = I - \frac{1}{z} \sigma_3 Q_{\pm}, \quad (3.26a)$$

$$\mu_{\pm}^{(n+1)}(x, t, z) = \int_{-\infty}^{x} \Omega_{\pm}(z) e^{ik(x-y)\sigma_3} \Omega_{\pm}^{-1}(z) \Delta Q_{\pm}(y, t) \mu_{\pm}^{(n)}(y, t, z) dy. \quad (3.26b)$$

So

$$\mu_{\pm,d}^{(n+1)} = \int_{-\infty}^{x} \left( \Omega_{\pm,o}^{-1} \Delta Q_{\pm,d} \mu_{\pm,d}^{(n)} + \Omega_{\pm,d}^{-1} \Delta Q_{\pm,o} \mu_{\pm,o}^{(n)} \right) dy$$

$$+ \int_{-\infty}^{x} \Omega_{\pm,o} e^{ik(x-y)\sigma_3} \left( \Omega_{\pm,d}^{-1} \Delta Q_{\pm,d} \mu_{\pm,d}^{(n)} + \Omega_{\pm,o}^{-1} \Delta Q_{\pm,o} \mu_{\pm,o}^{(n)} \right) e^{-ik(x-y)\sigma_3} dy,$$

$$\mu_{\pm,o}^{(n+1)} = \int_{-\infty}^{x} \left( \Omega_{\pm,o}^{-1} \Delta Q_{\pm,o} \mu_{\pm,d}^{(n)} + \Omega_{\pm,d}^{-1} \Delta Q_{\pm,o} \mu_{\pm,o}^{(n)} \right) dy$$

$$+ \int_{-\infty}^{x} e^{ik(x-y)\sigma_3} \left( \Omega_{\pm,o}^{-1} \Delta Q_{\pm,d} \mu_{\pm,d}^{(n)} + \Omega_{\pm,d}^{-1} \Delta Q_{\pm,o} \mu_{\pm,o}^{(n)} \right) e^{-ik(x-y)\sigma_3} dy,$$

For $n = 0$,

$$\mu_{\pm,d}^{(0)}(x, t, z) = I = O\left(\frac{1}{z^0}\right), \quad \mu_{\pm,o}^{(0)}(x, t, z) = -\frac{1}{z} \sigma_3 Q_{\pm} = O\left(\frac{1}{z}\right); \quad (3.27)$$
For $n = 1,$

\[
\begin{align*}
\mu_{\pm,d}^{(1)} &= \int_{-\infty}^{x} \frac{1}{\omega} \left( \frac{1}{z} \sigma_3 Q_{\pm} \Delta Q_{\pm} \mu_{\pm,d}^{(0)} + \Delta Q_{\pm} \mu_{\pm,o}^{(0)} \right) dy \\
&+ \int_{-\infty}^{x} \frac{1}{\omega} \left( -\frac{1}{z} \sigma_3 Q_{\pm} \right) e^{ik(x-y)\sigma_3} \left( \Delta Q_{\pm} \mu_{\pm,d}^{(0)} + \left( \frac{1}{z} \sigma_3 Q_{\pm} \right) \Delta Q_{\pm} \mu_{\pm,o}^{(0)} \right) e^{-ik(x-y)\sigma_3} dy,
\end{align*}
\]

\[
\mu_{\pm,o}^{(1)} = \int_{-\infty}^{x} \frac{1}{\omega} \left( -\frac{1}{z} \sigma_3 Q_{\pm} \right) \left( \frac{1}{z} \sigma_3 Q_{\pm} \Delta Q_{\pm} \mu_{\pm,d}^{(0)} + \Delta Q_{\pm} \mu_{\pm,o}^{(0)} \right) dy \\
&+ \int_{-\infty}^{x} e^{ik(x-y)\sigma_3} \frac{1}{\omega} \left( \Delta Q_{\pm} \mu_{\pm,d}^{(0)} + \left( \frac{1}{z} \sigma_3 Q_{\pm} \right) \Delta Q_{\pm} \mu_{\pm,o}^{(0)} \right) e^{-ik(x-y)\sigma_3} dy.
\]

If $z \to \pm \infty,$ we note the fact that

\[
k = \frac{1}{2} \left( z + \frac{u_0^2}{z} \right), \quad \frac{1}{\omega} = 1 + \frac{u_0^2}{z^2} + \cdots.
\]

(3.28)

And taking the integration by part for the last two terms of $\mu_{\pm,d}^{(1)}$ and $\mu_{\pm,o}^{(1)},$ one can derive the following results

\[
\begin{align*}
\mu_{\pm,d}^{(1)} &= O\left( \frac{\mu_{\pm,d}^{(0)}}{z} \right) + O(\mu_{\pm,o}^{(0)}) + O\left( \frac{\mu_{\pm,d}^{(0)}}{z^2} \right) + O\left( \frac{\mu_{\pm,o}^{(0)}}{z^3} \right) = O\left( \frac{1}{z} \right), \\
\mu_{\pm,o}^{(1)} &= O\left( \frac{\mu_{\pm,d}^{(0)}}{z^2} \right) + O\left( \frac{\mu_{\pm,d}^{(0)}}{z} \right) + O\left( \frac{\mu_{\pm,o}^{(0)}}{z} \right) + O\left( \frac{\mu_{\pm,o}^{(0)}}{z^2} \right) = O\left( \frac{1}{z} \right).
\end{align*}
\]

(3.29a, b)

Iterate in the same method, one can obtain the asymptotic

\[
\begin{align*}
\mu_{\pm,d}^{(2n)} &= O\left( \frac{1}{z^{2n}} \right), \quad \mu_{\pm,o}^{(2n)} = O\left( \frac{1}{z^{2n+2}} \right), \quad \mu_{\pm,d}^{(2n+1)} = O\left( \frac{1}{z^{2n+3}} \right), \quad \mu_{\pm,o}^{(2n+1)} = O\left( \frac{1}{z^{2n+4}} \right).
\end{align*}
\]

(3.30)

If $z \to 0,$ one can obtain the asymptotics

\[
\begin{align*}
\mu_{\pm,d}^{(2n)} &= O(z^{n}), \quad \mu_{\pm,o}^{(2n)} = O(z^{n-1}), \quad \mu_{\pm,d}^{(2n+1)} = O(z^{n}), \quad \mu_{\pm,o}^{(2n+1)} = O(z^{n+1}).
\end{align*}
\]

(3.31)

So we obtain the following asymptotic by some calculations for the equation (3.25)

\[
\begin{align*}
\bullet \mu_{\pm}(x,t,z) &= I + O\left( \frac{1}{z} \right), \quad z \to \pm \infty. \\
\bullet \mu_{\pm}(x,t,z) &= -\frac{1}{z} \sigma_3 Q_{\pm} + O(1), \quad z \to 0.
\end{align*}
\]

(3.32a, b)

And one can derive the potential function

\[
q(x,t) = \lim_{z \to \infty} z |\mu_{\pm}|_{21}.
\]

(3.33)
By substituting the asymptotic of $\mu_\pm$ into (3.20), one can obtain the asymptotic behaviors of the scattering matrix $S(z)$ are given as follows

- $S(z) = I + O(\frac{1}{z})$, \quad z \to \pm\infty, \quad (3.34a)$
- $S(z) = \frac{q^+}{q^-} I + O(z)$, \quad z \to 0. \quad (3.34b)$

3.2 The Riemann-Hilbert problem

The analytic region is divided into $D_\pm$. Now we establish the connection between $D_+$ and $D_-$. And based on the analyticity and asymptotic of eigenfunctions $\mu_\pm$ and $S(z)$, the solutions can be derived by Riemann-Hilbert problem and Plemelj’s formula. So we introduce the Riemann-Hilbert problem. Firstly, we define a sectionally meromorphic matrix $M(x,t,z)$:

**Proposition 4.** The sectionally meromorphic matrix defined as

$$
M(x,t,z) = \begin{cases}
\begin{bmatrix}
\mu_{+,1}(x,t,z) & \mu_{+,2}(x,t,z) \\
\frac{a(z)}{\rho} & \mu_{-,2}(x,t,z)
\end{bmatrix}, & z \in D_+,
\begin{bmatrix}
\mu_{-,1}(x,t,z) & \mu_{-,2}(x,t,z)
\end{bmatrix}, & z \in D_-,
\end{cases}
$$

and $M(x,t,z)$ satisfying the following Riemann-Hilbert problem:

- **Analyticity:** $M(x,t,z)$ is meromorphic in $D_+ \cup D_-$.  
- **Jump condition:**
  $$M^+(x,t,z) = M^-(x,t,z) \left( I - J(x,t,z) \right), \quad z \in \Sigma,$$
  where
  $$J(x,t,z) = e^{i\theta} \begin{bmatrix}
\rho & \hat{\rho} \\
-\hat{\rho} & 0
\end{bmatrix} e^{-i\theta} \sigma_3.\$$
- **Asymptotic behavior:**
  $$M(x,t,z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty.$$  
  $$M(x,t,z) = -\frac{1}{z} \sigma_3 Q_- + O(1), \quad z \to 0.$$  

According to the asymptotic of $\mu_\pm$ and $S(z)$, the solution (3.47) can be rewritten by $M(x,t,z)$

$$q(x,t) = \lim_{z \to \infty} z[M(x,t,z)]_{21}. \quad (3.36)$$
Figure 16: The distribution of discrete spectral.

Similar to vanishing boundary condition, for \( a(z) \) with \( N \) simple zeros, the Riemann-Hilbert method can be straightly solved by applying the residue conditions and the formula of soliton solutions can be obtained. But for \( a(z) \) with \( N \) high order zeros, not only the residue conditions are useful but the coefficients of negative power should be considered. However, these coefficients are not straightforward derived. Moreover, if \( z_n \) is the \( N \)-order zero point of \( a(z) \), so is \( -\frac{q_n^2}{z_0} \). This situation can be equivalent to the zero points of \( a(z) \) always paired in \( D_+ \).

### 3.3 Single high-order pole solutions

We assume \( a(z) \) with one \( N \)-order zero \( z_0 \in \{D_+ \cap \text{Im}z_0 > 0\} \). According to the symmetry properties of \( S(z) \), one can derived \( -\frac{q_n^2}{z_0} \) also is the zero point of \( a(z) \). For convenient, we make the following notations

\[
\begin{align*}
\nu_1 & \doteq z_0, \quad \nu_2 \doteq -\frac{q_0^2}{z_0}, \quad \nu_1, \nu_2 \in D_+, \\
\overline{\nu}_1 & \doteq \overline{z}_0, \quad \overline{\nu}_2 \doteq -\frac{q_0^2}{\overline{z}_0}, \quad \overline{\nu}_1, \overline{\nu}_2 \in D_-.
\end{align*}
\]

The distribution of discrete spectrum is shown in following Figure 16. So \( a(z) \) can be expanded as the Taylor series

\[
a(z) = a_0(z) \prod_{j=1}^{2} (z - \nu_j)^N, \quad (j = 1, 2),
\]
and \( a_0(z) \neq 0 \) for all \( z \in D_+ \). Reflection coefficient \( \rho(z) \) and \( \rho(\overline{z}) \) can be expanded as the Laurent expansion

\[
\rho(z) = \rho_{j,0}(z) + \sum_{m_j=1}^{N} \frac{\rho_{j,m_j}}{(z - \nu_j)^{m_j}},
\]

\[
\overline{\rho(\overline{z})} = \rho_{j,0}(\overline{z}) + \sum_{m_j=1}^{N} \frac{\rho_{j,m_j}}{(z - \overline{\nu}_j)^{m_j}}, \quad j = 1, 2,
\]

where

\[
\rho_{j,m_j} = \lim_{z \to \nu_j} \frac{1}{(N-m_j)!} \frac{\partial^{N-m_j}}{\partial z^{N-m_j}} [(z - \nu_j)^N \rho(z)], \quad m_j = 1, 2, \ldots, N,
\]

\( \rho_0(z) \) is analytic for all \( z \in D_+ \). According to the definition of \( M(x,t,z) \), one can obtain that \( z = \nu_j \) are the \( N \)-order poles of \( M_1(x,t,z) \) and \( z = \overline{\nu}_j \) are the \( N \)-order poles of \( M_2(x,t,z) \). \( M_2(x,t,z) \) analytic as \( z = \nu_j \) and \( M_1(x,t,z) \) analytic as \( z = \nu_j \). So we have the following expand

\[
M_{21}(z) = q \frac{z}{z} + 2 \sum_{j=1}^{N} \sum_{p=1}^{N} \frac{G_{j,p}(x,t)}{(z - \nu_j)^p}, \quad M_{22}(z) = 1 + 2 \sum_{j=1}^{N} \sum_{p=1}^{N} \frac{F_{j,p}(x,t)}{(z - \overline{\nu}_j)^p}. \tag{3.38}
\]

According to the analyticity one can get the Taylor expansion

\[
e^{-2i\theta(z)} = \sum_{s_j=0}^{+\infty} f_{j,s_j}(x,t)(z - \nu_j)^{s_j}, \quad e^{2i\theta(z)} = \sum_{s_j=0}^{+\infty} f_{j,s_j}(x,t)(z - \nu_j)^{s_j},
\]

\[
M_{21}(z) = \sum_{s_j=0}^{+\infty} \zeta_{j,s_j}(x,t)(z - \nu_j)^{s_j}, \quad M_{22}(z) = \sum_{s_j=0}^{+\infty} \mu_{j,s_j}(x,t)(z - \nu_j)^{s_j}, \tag{3.39a}
\]

where

\[
f_{j,s_j}(x,t) = \lim_{z \to \nu_j} \frac{1}{s_j!} \frac{\partial^{s_j}}{\partial z^{s_j}} e^{-2i\theta(z)}, \tag{3.40a}
\]

\[
\mu_{j,s_j}(x,t) = \lim_{z \to \nu_j} \frac{1}{s_j!} \frac{\partial^{s_j}}{\partial z^{s_j}} M_{22}(z), \tag{3.40b}
\]

\[
\zeta_{j,s_j}(x,t) = \lim_{z \to \nu_j} \frac{1}{s_j!} \frac{\partial^{s_j}}{\partial z^{s_j}} M_{21}(z). \tag{3.40c}
\]

Then according to the equation (3.18) and the definition of \( M(x,t,z) \), comparing the corresponding coefficients of \((z - \nu_j)^p\) and \((z - \overline{\nu}_j)^{-p}\). One can derived the
following results

\[
F_{j,p}(x,t) = - \sum_{m_j = p}^{N} \sum_{s_j = 0}^{m_j - p} \rho_{j,m_j} \tilde{F}_{j,m_j - p - s_j}(x,t) \zeta_{j,s_j}(x,t),
\]

(3.41a)

\[
G_{j,p}(x,t) = \sum_{m_j = p}^{N} \sum_{s_j = 0}^{m_j - p} \rho_{j,m_j} f_{j,m_j - p - s_j}(x,t) \mu_{j,s_j}(x,t),
\]

(3.41b)

where \( p = 1, 2, \cdots, N \). For \( N = 1 \), \( F_{j,p}(x,t) \) and \( G_{j,p}(x,t) \) degenerate into the residue conditions. In addition, \( \mu_{j,s_j}(x,t) \) and \( \zeta_{j,s_j}(x,t) \) can be expressed as \( F_{j,p}(x,t) \) and \( G_{j,p}(x,t) \) via direct calculation, that is

\[
\zeta_{j,s_j}(x,t) = \frac{(-1)^s q_-}{p_j^{s_j+1}} + \sum_{p = 1}^{N} \left( \frac{p + s_j - 1}{s_j} \right) \frac{(-1)^s G_{j,p}(x,t)}{(p_j - \nu_p)^{s_j+p}}, \quad s_j = 0, 1, 2, \cdots
\]

\[
\mu_{j,s_j}(x,t) = \begin{cases} 
1 + \sum_{p = 1}^{N} \frac{F_{j,p}(x,t)}{(\nu_j - \nu_p)^p}, & s_j = 0, \\
\sum_{p = 1}^{N} \left( \frac{p + s_j - 1}{s_j} \right) \frac{(-1)^s F_{j,p}(x,t)}{(p_j - \nu_p)^{s_j+p}}, & s_j = 1, 2, \cdots.
\end{cases}
\]

Then one can obtain the following system

\[
F_{j,p}(x,t) = - \sum_{m_j = p}^{N} \sum_{s_j = 0}^{m_j - p} \left( -1 \right)^s \rho_{j,m_j} \tilde{F}_{j,m_j - p - s_j}(x,t) q_- \frac{1}{p_j^{s_j+1}}
\]

\[- \sum_{m_j = p}^{N} \sum_{s_j = 0}^{m_j - p} \sum_{q = 1}^{N} \left( q + s_j - 1 \right) \frac{(-1)^s \rho_{j,m_j} \tilde{F}_{j,m_j - p - s_j}(x,t) G_{j,q}(x,t)}{(p_j - \nu_q)^{s_j+q} + q},
\]

\[
G_{j,p}(x,t) = \sum_{m_j = p}^{N} \rho_{j,m_j} f_{j,m_j - p}(x,t)
\]

\[+ \sum_{m_j = p}^{N} \sum_{s_j = 0}^{m_j - p} \sum_{q = 1}^{N} \left( q + s_j - 1 \right) \frac{(-1)^s \rho_{j,m_j} f_{j,m_j - p - s_j}(x,t) F_{j,q}(x,t)}{(\nu_j - p_j)^{s_j+q}},
\]

For convenient, we introduce the following symbols \((s, p = 1, 2, \cdots, N)\):

\[
|F| = |F_1 F_2 \cdots F_N|\!, \quad |G| = |G_1 G_2 \cdots G_N|\!,
\]

\[
|\beta| = |\beta_1 \beta_2 \cdots \beta_N|\!, \quad \beta_j = |\beta_{j,1} \beta_{j,2} \cdots \beta_{j,N}|,
\]

\[
|\eta| = |\eta_1 \eta_2 \cdots \eta_N|\!, \quad \eta_j = |\eta_{j,1} \eta_{j,2} \cdots \eta_{j,N}|,
\]

\[
\beta_{j,l} = - \sum_{m_j = l}^{N} \sum_{s_j = 0}^{m_j - l} \left( -1 \right)^s \frac{\rho_{j,m_j} \tilde{F}_{j,m_j - l - s_j}(x,t) q_-}{p_j^{s_j + 1}}.
\]
\[ \eta_{j,l}(x,t) = \sum_{m_j=l}^{N} \rho_{j,m_j} f_{j,m_j - l}(x,t), \]

\[ \chi = \begin{bmatrix} \varpi_{11} & \varpi_{12} \\ \varpi_{21} & \varpi_{22} \end{bmatrix}, \]

\[ [\varpi_{j,l}] = ([\varpi_{j,l}]_{p,q})_{N \times N} = \left( -\sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} \left( q + s_j - 1 \right) \frac{(-1)^{s_j} \bar{\rho}_{j,m_j} \bar{f}_{j,m_j - p - s_j}(x,t)}{\varpi_{j} - \nu_{l})^{s_j + q}} \right)_{N \times N}. \]

So the equations (2.31) can be rewritten as

\[ |F\rangle = |\beta\rangle + \chi |G\rangle, \quad |G\rangle = |\eta\rangle - \chi |F\rangle. \tag{3.44} \]

|F\rangle and |G\rangle are solved as

\[ |F\rangle = \chi (I + \chi \chi)^{-1} |\eta\rangle + (I - \chi (I + \chi \chi)^{-1} \chi)^{-1} |\beta\rangle, \quad |G\rangle = (I + \chi \chi)^{-1} (|\eta\rangle - \chi |\beta\rangle), \tag{3.45} \]

So the expansions of \( M_{21}(x,t,\lambda) \) can be given as

\[ M_{21}(x,t,z) = \frac{q_0}{z} + 2 \sum_{j=1}^{N} \sum_{s=1}^{N} G_{j,s}(x,t) \left( \frac{z - \nu_j}{z - \nu_j} \right)^{s} = \frac{q_0}{z} + \frac{\det (I + \chi \chi + (|\eta\rangle - \chi |\beta\rangle) (Y(\lambda))]}{\det (I + \chi \chi)} - 1. \tag{3.46} \]

where

\[ \{ Y(z) \} = \begin{bmatrix} \frac{1}{z - \nu_1} & \frac{1}{z - \nu_1^2} & \cdots & \frac{1}{(z - \nu_1)^N} \\ \frac{1}{z - \nu_2} & \frac{1}{z - \nu_2^2} & \cdots & \frac{1}{(z - \nu_2)^N} \end{bmatrix}. \]

In the similar way as vanishing boundary condition, one can derive the solutions of equation (3.1) with one \( N \)-order pole

\[ q(x,t) = q_0 + \left( \frac{\det (I + \chi \chi + (|\eta\rangle - \chi |\beta\rangle) (Y_0)}{\det (I + \chi \chi)} - 1 \right), \tag{3.47} \]

where

\[ |\eta\rangle = [ \eta_1 \quad \eta_2 ]^T, \quad |\beta\rangle = [ \beta_1 \quad \beta_2 ]^T, \quad (Y_0) = [ Y_0^0 \quad Y_2^0 ], \]

\[ \eta_j = [ \eta_j,1 \quad \eta_j,2 \quad \cdots \quad \eta_j,N ], \quad \eta_j,t = \sum_{m_j=t}^{N} \rho_{j,m_j} f_{j,m_j - t}(x,t), \]

\[ \beta_j = [ \beta_j,1 \quad \beta_j,2 \quad \cdots \quad \beta_j,N ], \quad (Y_0) = [ 1 \quad 0 \quad \cdots \quad 0 ]_{1 \times N}, \]

\[ \beta_j,t = -\sum_{m_j=t}^{N} \sum_{s_j=0}^{m_j-t} \frac{(-1)^{s_j} \bar{\rho}_{j,m_j} \bar{f}_{j,m_j - t - s_j}(x,t) q_0}{\varpi_{j}^{s_j + t}}, \]
χ = \left[ \begin{array}{cc} \varpi_{11} & \varpi_{12} \\ \varpi_{21} & \varpi_{22} \end{array} \right],

and \([\varpi_{j,l}]_{(j,l = 1,2)}\) are \(N \times N\) matrix

\[
[\varpi_{j,l}] = ( [\varpi_{j,l}]_{p,q} )_{N \times N} = \left( - \sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} \left( \frac{q + s_j - 1}{s_j} \right) \frac{(-1)^{s_j} \rho_{j,m_j, f_{j,m_j-p-s_j}(x,t)}}{(\varpi_j - \nu_j)^{s_j+q}} \right)_{N \times N}.
\]

### 3.3.1 Single 1\(^{st}\)-order pole solution

Let \(\nu_1\) is 1-order zero of \(a(z)\), so is \(\nu_2\). The reflection coefficient \(\rho(z)\) can be expand as Laurent series

\[
\rho(z) = \rho_{j,0}(z) + \frac{\rho_{j,1}}{z - \nu_j}.
\]

Now \(\chi\) in (3.47) is defined as

\[
\chi = \left[ \begin{array}{cc} \varpi_{11} & \varpi_{12} \\ \varpi_{21} & \varpi_{22} \end{array} \right]_{2 \times 2},
\]

where

\[
[\varpi_{j,l}] = [\varpi_{j,l}]_{p,q} = - \sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} \left( \frac{q + s_j - 1}{s_j} \right) \frac{(-1)^{s_j} \rho_{j,m_j, f_{j,m_j-p-s_j}(x,t)}}{(\varpi_j - \nu_j)^{s_j+q}}.
\]

\((Y_0) = [ 1 \ 1 ],\)

\(|\eta\rangle = [ \eta_1 \ \eta_2 ]^T, \ |\beta\rangle = [ \beta_1 \ \beta_2 ]^T,\)

\[
\eta_{j,l} = \sum_{m_j=l}^{N} \rho_{j,m_j,f_{j,m_j-l}(x,t)},
\]

\[
\beta_{j,l} = - \sum_{m_j=l}^{N} \sum_{s_j=0}^{m_j-l} \frac{(-1)^{s_j} \rho_{j,m_j, f_{j,m_j-l-s_j}(x,t)q}}{\varpi_j^{s_j+1}}, \quad j, l = 1, 2, \quad N = 1
\]

According to (3.47) the soliton solution of the QNLS equation can be derived. The patterns of soliton with different parameters and \(q_\pm = 1\) are shown in Figure 17-20. Figure 18 and 20 show that the soliton as \(|\nu_j| \to 1\). The 2-dimensional figure show the propagate at different time: orange full line is \(t = 0.5\), red dashed line is \(t = 0\) and green dashed line is \(t = -0.5\).
3.3.2 Single 2nd-order pole solutions

Let \( \nu_1 \) is 2-order zero of \( a(z) \), so is \( \nu_2 \). The reflection coefficient \( \rho(z) \) can be expand as Laurent series

\[
\rho(z) = \rho_{j,0}(z) + \sum_{m_j=1}^{2} \frac{\rho_{j,m_j}}{(z - \nu_j)^{m_j}}.
\]

Now \( \chi \) in (3.47) is defined as

\[
\chi = \begin{bmatrix} \varpi_{11} & \varpi_{12} \\ \varpi_{21} & \varpi_{22} \end{bmatrix},
\]

where

\[
(\varpi_{j,l}) = [\varpi_{j,l}]_{p,q} = -\sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j} \left( q + s_j - 1 \right) \frac{(-1)^{q+s_j} \rho_{j,m_j} f_{j,m_j-p-s_j}(x,t)}{(\nu_j - \nu_l)^{s_j+q}},
\]

\[
(Y_0) = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix},
\]

\[
|\eta\rangle = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix}^T, \quad |\beta\rangle = \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}^T,
\]

\[
\eta_{j,l} = \sum_{m_j=l}^{N} \rho_{j,m_j} f_{j,m_j-l}(x,t),
\]

\[
\text{Figure 17: } \gamma = 0, \rho_{1,1} = 1, \rho_{2,1} = 1, \nu_1 = 3 + 2i, \nu_2 = -\frac{1}{3-2i}.
\]

\[
\text{Figure 18: } \gamma = 0, \rho_{1,1} = 1, \rho_{2,1} = 1, \nu_1 = \frac{1}{100} + i, \nu_2 = -\frac{1}{100-i}.
\]
According to (3.47) the soliton solution of the QNLS equation can be derived.

The patterns of soliton with different parameters and \( q \pm = 1 \) are shown in Figure 21-24. Figure 23-24 show that the soliton as \(|\nu_j|\to 1\). The 2-dimensional figure show the propagate at different time: orange full line is \( t = 1 \), red dashed line is \( t = 0 \) and green dashed line is \( t = -1 \) in Figure 21. Orange full line is \( t = 0 \), red dashed line is \( t = 0 \) and green dashed line is \( t = -0.2 \) in Figure 22.
3.4 Multiple high-order pole solutions

In this subsection, we consider the more general case: \(a(z)\) with \(N\) high-order zeros \(z_1, z_2, \ldots, z_N\), and the powers are \(n_1, n_2, \ldots, n_N\), respectively. Then \(-\frac{q_0}{z_1}, -\frac{q_0}{z_2}, \ldots, -\frac{q_0}{z_N}\) also are the zeros of \(a(z)\). And the powers are \(n_1, n_2, \ldots, n_N\), too. We also make the following notations

\[
\begin{align*}
\nu_j &= z_i, \\
\nu_j |_{j=N+i} &= \frac{q_0^2}{z_i}, \\
\nu_j &= z_i, \\
\nu_j |_{j=N+i} &= -\frac{q_0^2}{z_i},
\end{align*}
\]

So \(a(z)\) can be expanded as

\[
a(z) = a_0(z)(z - \nu_1)^{n_1} \cdots (z - \nu_N)^{n_N}(z - \nu_{N+1})^{n_{N+1}} \cdots (z - \nu_{2N})^{n_{2N}},
\]

Similar to the one high-order pole, \(\rho(z)\) can be expanded as the Laurent series

\[
\rho(z) = \rho_0(z) + \sum_{m_j=1}^{n_j} \frac{\rho_{j,m_j}}{(z - \nu_j)^{m_j}},
\]

Figure 22: \(\gamma = 1, \rho_{1,1} = 1, \rho_{1,2} = 1, \rho_{2,1} = 1, \rho_{2,2} = 1, \nu_1 = 1 + i, \nu_2 = -\frac{1}{1+i}\).

Figure 23: \(\gamma = 0, \rho_{1,1} = 1, \rho_{1,2} = 1, \rho_{2,1} = 1, \rho_{2,2} = 1, \nu_1 = \frac{1}{20} + i, \nu_2 = -\frac{1}{20-i}\).
\[ \gamma = 1, \rho_{1,1} = 1, \rho_{1,2} = 1, \rho_{2,1} = 1, \rho_{2,2} = 1, \nu_1 = \frac{1}{10} + i, \nu_2 = -\frac{1}{10}. \]

\[
\rho(z) = \rho(z) = \rho_{j,0}(z) + \sum_{m_j=1}^{n_j} \frac{\rho_{j,m_j}}{(z - \nu_j)^{m_j}},
\]

where
\[
\rho_{j,m_j} = \lim_{z \to \nu_j} \frac{1}{(n_j - m_j)!} \frac{\partial^{n_j-m_j}}{\partial (z - \nu_j)^{n_j-m_j}} [ (z - \nu_j)^{n_j} \rho(z)],
\]

and \( \rho_{j,0}(z) \) is analytic for all \( z \in D_+ \) and \( j = 1, \ldots, 2N \). So, in the similar method, one can derive the soliton solution formula with \( N \) high-order poles.

**Theorem 3.1.** For the non-vanishing boundary condition, if \( a(z) \) with \( N \) high-order zeros, then the soliton solutions of equation (3.1) with the following formula
\[
q(x,t) = q_+ + \left( \frac{\det(I + \chi Y + (|\eta| - \chi |\beta|) \langle Y_0 \rangle)}{\det(I + \chi Y)} - 1 \right), \tag{3.54}
\]

where
\[
|\eta\rangle = [ \eta_1 \ \eta_2 \ \cdots \ \eta_{2N} ]^T, \quad |\beta\rangle = [ \beta_1 \ \beta_2 \ \cdots \ \beta_{2N} ]^T,
\]
\[
\langle Y_0 \rangle = [ \ Y_1^0 \ Y_2^0 \ \cdots \ Y_{2N}^0 \ ], \quad Y_j^0 = [ \ 1 \ 0 \ \cdots \ 0 \ ]_{1 \times n_j},
\]
\[
\eta_j = [ \eta_{j,1} \ \eta_{j,2} \ \cdots \ \eta_{j,n_j} ], \quad \beta_j = [ \beta_{j,1} \ \beta_{j,2} \ \cdots \ \beta_{j,n_j} ],
\]
\[
\eta_{j,t} = \sum_{m_j=t}^{n_j} \rho_{j,m_j} f_{j,m_j-t}(x,t),
\]
\[
\beta_{j,t} = -\sum_{m_j=t}^{n_j} \sum_{s_j=0}^{m_j-t} (-1)^{s_j} \bar{\rho}_{j,m_j} \bar{f}_{j,m_j-t-s_j}(x,t) q_+,\]

\[
\chi = \begin{bmatrix}
\varphi_{11} & \varphi_{12} & \cdots & \varphi_{1(2N)} \\
\varphi_{21} & \varphi_{22} & \cdots & \varphi_{2(2N)} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_{(2N)1} & \varphi_{(2N)2} & \cdots & \varphi_{(2N)(2N)}
\end{bmatrix},
\]
and \([\varpi_{j,l}(j, l = 1, 2, \cdots, 2N)]\) are \(n_j \times n_l\) matrix

\[
[\varpi_{j,l}] = ([\varpi_{j,l}]_{p,q})_{n_j \times n_l} = \left( -\sum_{m_j=p}^{m_j-p} \sum_{s_j=0}^{s_j} \binom{q + s_j - 1}{s_j} \frac{(-1)^{s_j} \varpi_{j,m_j} \varpi_{j,m_j-p-s_j}(x,t)}{(\varpi_j - \nu_l)^{s_j+q}} \right)_{n_j \times n_l}.
\]

### 3.4.1 Double 1\(^{st}\)-order pole solution

Let \(\nu_1\) and \(\nu_2\) are the 1-order zeros point of \(a(z)\), so is \(\nu_3\) and \(\nu_4\). The reflection coefficient \(\rho(z)\) can be expand as Laurent series

\[
\rho(z) = \rho_{j,0}(z) + \frac{\rho_{j,1}}{z - \nu_j}.
\]

Now \(\chi\) in (3.54) is defined as

\[
\chi = \begin{bmatrix}
[\varpi_{11}] & [\varpi_{12}] & [\varpi_{13}] & [\varpi_{14}]

[\varpi_{21}] & [\varpi_{22}] & [\varpi_{23}] & [\varpi_{24}]

[\varpi_{31}] & [\varpi_{32}] & [\varpi_{33}] & [\varpi_{34}]

[\varpi_{41}] & [\varpi_{42}] & [\varpi_{43}] & [\varpi_{44}]
\end{bmatrix}_{4 \times 4},
\]

where

\[
[\varpi_{j,l}] = ([\varpi_{j,l}]_{p,q}) = -\sum_{m_j=p}^{m_j-p} \sum_{s_j=0}^{s_j} \binom{q + s_j - 1}{s_j} \frac{(-1)^{s_j} \varpi_{j,m_j} \varpi_{j,m_j-p-s_j}(x,t)}{(\varpi_j - \nu_l)^{s_j+q}}.
\]

\(\eta_0 = [1 \ 1 \ 1 \ 1]^T\),

\(\eta = [\eta_1 \ \eta_2 \ \eta_3 \ \eta_4]^T\),

\(\beta = [\beta_1 \ \beta_2 \ \beta_3 \ \beta_4]^T\),

\[
\eta_{j,l} = \sum_{m_j=l}^{n_j} \rho_{j,m_j} f_{j,m_j-l}(x,t),
\]

\[
\beta_{j,l} = -\sum_{m_j=l}^{n_j} \sum_{s_j=0}^{s_j} \frac{(-1)^{s_j} \varpi_{j,m_j} \varpi_{j,m_j-l-s_j}(x,t) q_{j,l-s_j}}{(\varpi_j)^{s_j+1}},
\]

\(j, l = 1, 2, 3, 4\), \(n_1 = n_2 = n_3 = n_4 = 1\).

According to Theorem 3.1, the soliton solution of the QNLS equation with two 1-order discrete spectral can be derived. The patterns of soliton with different parameters and \(q_{\pm} = 1\) are shown in Figure 25–28. Figure 27–28 show that the soliton as \(|\nu_j| \to 1\). The 2d figure show the propagate at different time: orange full line is \(t = 1\), red dashed line is \(t = 0\) and green dashed line is \(t = -1\) in Figure 25. Orange full line is \(t = 0.2\), red dashed line is \(t = 0\) and green dashed line is \(t = -0.2\) in Figure 26.
\[ \chi \text{ in (3.54) is defined as} \]

\[
\chi = \begin{bmatrix}
\omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \\
\omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \\
\omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} \\
\omega_{41} & \omega_{42} & \omega_{43} & \omega_{44}
\end{bmatrix}_{6 \times 6},
\]

where

\[ [\omega_{j,l}] = [\omega_{j,l}]_{p,q} = -\sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \left( q + s_j - 1 \right) \frac{(-1)^{s_j} \rho_{j,m_j} f_{j,m_j-s} (x, t)}{(\nu_j - \nu_l)^{s_j+q}}. \]

\[ \langle Y_0 \rangle = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix}. \]
Figure 27: $\gamma = 0, \rho_{1,1} = 1, \rho_{2,1} = 2, \rho_{3,1} = 1, \rho_{4,1} = 2, \nu_1 = \frac{1}{10} + i, \nu_2 = -\frac{1}{10} + \frac{11}{100}, \nu_3 = -\frac{1}{10} - \frac{1}{100}, \nu_4 = -\frac{1}{10} - \frac{1}{100}$.

Figure 28: $\gamma = 0, \rho_{1,1} = 1, \rho_{2,1} = 2, \rho_{3,1} = 1, \rho_{4,1} = 2, \nu_1 = \frac{1}{10} + i, \nu_2 = -\frac{1}{10} + \frac{11}{100}, \nu_3 = -\frac{1}{10} - \frac{1}{100}, \nu_4 = -\frac{1}{10} - \frac{1}{100}$.

$$|\eta\rangle = [\eta_1 \eta_2 \eta_3 \eta_4]^T, \quad |\beta\rangle = [\beta_1 \beta_2 \beta_3 \beta_4]^T,$$

$$\eta_{j,l} = \sum_{m_j=l}^{n_j} \rho_{j,m_j} f_{j,m_j-l}(x,t),$$

$$\beta_{j,l} = \sum_{m_j=l}^{n_j} \sum_{s_j=0}^{m_j-l} (-1)^{s_j} \rho_{j,m_j} f_{j,m_j-l-s_j}(x,t) q_{s_j}^{\pm},$$

$j, l = 1, 2, 3, 4, \quad n_1 = n_3 = 2, \quad n_2 = n_4 = 1$.

According to Theorem 3.1 the soliton solution of the QNLS equation with one 2-order and one 1-order discrete spectral can be derived. The patterns of soliton are shown in Figure 29-30 with different parameters and $q_{\pm} = 1$.

### 4 Summary

In this article, we investigated the soliton solution for the equation (1.5) with two different boundary conditions and solved the initial problem:
Figure 29: $\gamma = 0$, $\rho_{1,1} = \rho_{1,2} = \rho_{2,1} = \rho_{3,1} = \rho_{3,2} = \rho_{4,1} = 1$, $\nu_1 = 1 + i$, $\nu_2 = -1 + \frac{i}{2}$, $\nu_3 = -\frac{1}{1+i}$, $\nu_4 = -\frac{1}{1-i}$.

Figure 30: $\gamma = 1$, $\rho_{1,1} = \rho_{1,2} = \rho_{2,1} = \rho_{3,1} = \rho_{3,2} = \rho_{4,1} = 1$, $\nu_1 = 1 + i$, $\nu_2 = -1 + i$, $\nu_3 = -\frac{1}{1-i}$, $\nu_4 = -\frac{1}{1+i}$.

- For deriving the soliton solution with $N$ high-order poles via the method in [22] is not very convenient. Although, the residue conditions of $N$ high-order poles can be obtained [42], the coefficients of $(\lambda - \lambda_j)^{-s}$ and $(\lambda - \overline{\lambda}_j)^{-s}$ should be considered. However, these coefficients are not very easily obtained.

- For solving the first problem, we expand reflection coefficient $\rho$ and matrix $M$ as Laurent series and compare the coefficient of negative powers.

- The analytic region of the non-vanishing boundary condition more complex than vanishing boundary condition: the discrete spectral always paired. In this situation, it would be well if we consider the paired discrete spectral as two different discrete spectral with the same power. So we reform the method in [37] [39] and extend this method for non-vanishing boundary condition with high-order poles. Under the non-vanishing boundary condition, the zeros of scattering coefficient always
paired in $D_+$.  

- It is possible for the formula of soliton solution to be constructed as a determinant form. For vanishing boundary condition it is $N$-order determinant, but for non-vanishing boundary condition it is $2N$-order determinant.

In this article, we just show a small part soliton solutions. According the formula more situation can be derived. In addition, the order of determinant in formula always even for non-vanishing boundary condition which is different with vanishing boundary condition.

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