Cubic threefolds and abelian varieties of dimension five. II

Sebastian Casalaina-Martin

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Abstract

This paper extends joint work with R. Friedman to show that the closure of the locus of intermediate Jacobians of smooth cubic threefolds, in the moduli space of principally polarized abelian varieties (ppavs) of dimension five, is an irreducible component of the locus of ppavs whose theta divisor has a point of multiplicity three or more. This paper also gives a sharp bound on the multiplicity of a point on the theta divisor of an indecomposable ppav of dimension less than or equal to five; for dimensions four and five, this improves the bound due to J. Kollár, R. Smith-R. Varley, and L. Ein-R. Lazarsfeld.

Introduction

The geometric Schottky problem is to identify Jacobian varieties among all principally polarized abelian varieties (ppavs) via geometric conditions on the polarization. For a smooth cubic hypersurface $X \subset \mathbb{P}^4$, which we will simply call a cubic threefold, the intermediate Jacobian $(J_X, \Theta_X)$ is a ppav of dimension five and one can consider the analogous problem, the “geometric Schottky problem for cubic threefolds,” which is to identify these intermediate Jacobians among all ppavs of dimension five via geometric conditions on the polarization.

Since a theorem of Mumford’s [16] states that $\Theta_X$ has a unique singularity, which is of multiplicity three, it is natural to ask to what extent the existence of triple points on the theta divisor of a ppav of dimension five characterizes intermediate Jacobians of cubic threefolds. In fact, in [6], Friedman and the author showed that a ppav of dimension five whose...
The theta divisor has a unique singular point of multiplicity three, and no other singular points, is the intermediate Jacobian of a cubic threefold.

The first goal of this paper is to extend this result:

**Theorem 1** Let $I \subset A_5$ be the locus of intermediate Jacobians of cubic threefolds, let $N_1$ be the locus of ppavs in $A_5$ whose theta divisor has a singular locus of dimension at least one, and let $S_3$ be the locus of ppavs in $A_5$ whose theta divisor has a point of multiplicity three or more.

$$I = S_3 - (N_1 \cap S_3).$$

A more detailed statement is given in Theorem 2.3.1, which provides a complete description of the locus $S_3$. Roughly speaking, $S_3$ consists of three irreducible components, all of dimension ten, one of which is the closure of $I$. Theorem 2.3.1 also gives new information on the boundary of $I$ (cf. Remark 2.3.5).

The techniques of this paper can also be used to give a sharp bound on the multiplicity of a point on the theta divisor of an indecomposable ppav of dimension at most five; for dimensions four and five, this improves the bound due to Kollár [14], Smith-Varley [19], and Ein-Lazarsfeld [11]. To be precise, let $\text{Sing}_k \Theta = \{x \in \Theta : \text{mult}_x \Theta \geq k\}$. A result of Kollár’s [14] shows that if $(A, \Theta) \in A_d$, then $\dim(\text{Sing}_k\Theta) \leq d - k$; generalizing a result of Smith and Varley [19], Ein and Lazarsfeld [11] showed that $\dim(\text{Sing}_k\Theta) = d - k$ only if $(A, \Theta)$ splits as a $k$-fold product. It follows that if $(A, \Theta)$ is indecomposable then

$$\dim(\text{Sing}_k\Theta) \leq d - k - 1,$$

As a special case we see that if $x \in \Theta$, then $\text{mult}_x \Theta \leq d - 1$. For $d \leq 3$ it is easy to see that these bounds are sharp; this paper shows that these bounds are not sharp for $d = 4, 5$.

Motivation for this result comes from the case of Jacobian and Prym varieties. For the Jacobian $(JC, \Theta_C)$ of a smooth curve $C$ of genus $g$, applying the Riemann singularity theorem and Martens’ theorem [13], it follows that $\dim(\text{Sing}_k \Theta_C) \leq g - 2k + 1$, with equality holding only if $C$ is hyperelliptic. Similarly, in the case of an indecomposable Prym variety $(P, \Xi)$ associated to a connected étale double cover of a smooth curve $C$ of genus $g$, the results of [5] show that $\dim(\text{Sing}_k \Xi) \leq (g - 1) - 2k + 1$.

The following theorem extends these results to all ppavs of dimension less than or equal to five.
Theorem 2 Suppose $(A, \Theta) \in A_d$ for $d \leq 5$. For $k$ and $j$ nonnegative integers such that $k \geq 1$ and $k - j \geq 1$, if $\dim(\text{Sing}_k \Theta) = d - k - j$, then $(A, \Theta) = \prod_{i=1}^{k-j} (A_i, \Theta_i)$. Equivalently, if $(A, \Theta)$ is indecomposable, $\dim(\text{Sing}_k \Theta) \leq d - 2k + 1$.

This theorem is obtained from the following:

Theorem 3 Suppose $(A, \Theta) \in A_d$ is indecomposable and either $d \leq 5$ or $(A, \Theta)$ is the Prym variety associated to the double cover of an irreducible stable curve. For any $x \in \Theta$ we have $\text{mult}_x \Theta \leq (d+1)/2$.

In this paper we will work over $\mathbb{C}$. The techniques used here are similar to those used in [6] and [5]: we use the fact due to Beauville [2] that a ppav of dimension at most five is the Prym variety associated to an admissible double cover of a stable curve, and then, by studying deformations of line bundles on curves, describe the possible singularities of the theta divisor of a Prym variety. Section 1 recalls the basic setup in [6] and [5]; a more general situation is considered in this paper, and the technical difficulties this introduces are dealt with in Section 1.2, culminating in Propositions 1.2.4 and 1.2.5. In Section 2 we apply the results of Section 1 to the Prym theta divisor, and to description of $S_3 \subseteq A_5$. The main theorems are proven in Sections 2.2 and 2.3.

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1 Preliminaries

1.1 Prym varieties of nodal curves

In this section, following Beauville [2], we recall some basic results about the theta divisors of Prym varieties. Throughout the paper, when discussing Pryms, $\tilde{C}$ will be a connected curve with at worst ordinary double points, equipped with an involution $\tau : \tilde{C} \to \tilde{C}$ satisfying Beauville’s condition (*):

(*) the fixed points of $\tau$ are exactly the singular points of $\tilde{C}$, and at a singular point, the two branches are not exchanged under $\tau$. 

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It is easy to show that $C = \tilde{C}/(\tau)$ has only ordinary double points ([2, Lemma 3.1]), and we will define $\pi : \tilde{C} \rightarrow C$ to be the induced morphism. Observe as well that under these conditions on $\tilde{C}$, $\deg(\omega_{\tilde{C}}|_{\tilde{C}})$ is even for each irreducible component $\tilde{C}_i \subseteq \tilde{C}$, so that there is a well-defined theta divisor for $JC$.

Let $\nu : \tilde{N} \rightarrow \tilde{C}$, and $\nu : N \rightarrow C$ be the normalization of the respective curves, and let $Nm : \text{Pic}(\tilde{C}) \rightarrow \text{Pic}(C)$ be the usual norm for line bundles ([EGA II.6.5]). In analogy with the smooth case, the Prym variety $P \subseteq \ker(Nm) \subseteq JC$ is defined to be the connected component of the identity. Setting $JC^*$ to be the variety of line bundles $L$ on $C$ such that $2c_1(L) = c_1(\omega_C)$, Beauville has shown that $P$ can be identified with the set

$$P^* = \{ L \in JC^* \mid Nm(L) = \omega_C, \ h^0(L) \equiv 0 \ (\text{mod} \ 2) \}.$$

A principal polarization is given on $P$ by a divisor $\Xi$, which can be identified with the set $\Xi^* = \{ L \in P^* \mid h^0(L) \geq 2 \}$, and $\Theta|_P = 2 \cdot \Xi$, where $\Theta$ is the canonical polarization of $JC$. Thus, if $x \in P$ corresponds to a line bundle $L \in P^*$, then $\text{mult}_x \Xi \geq \frac{1}{2}h^0(L)$.

Our starting point for studying $\Xi$ will be the following. Let $L$ be a line bundle over $C \times JC$ of multidegree $\frac{1}{2}c_1(\omega_C)$ with the property that for all $x \in JC$, $L|_{C \times x}$ is isomorphic to the line bundle $M \in JC^*$ corresponding to $x$. The existence of such a line bundle is established in Beauville [2, Lemma 1.3].

Now suppose that $S$ is a smooth curve with $s_0 \in S$. Let $t$ be a local coordinate for $S$ centered at $s_0$ and set $S_k = \text{Spec} \mathbb{C}[t]/(t^{k+1})$. For each $k$ there is a map $S_k \rightarrow S$, so that if we set $\tilde{C}_k = \tilde{C} \times S_k$, there are induced maps $\tilde{C}_k \rightarrow \tilde{C} \times S$. Let $\mathcal{L}_k$ be the pull-back of $\mathcal{L}$ to $\tilde{C}_k$.

The following was proven in [6].

**Lemma 1.1.1** Suppose $S$ is a smooth curve, $f : S \rightarrow P$ is a morphism, and $f(s_0) = x \in P$ corresponds to a line bundle $L \in P^*$. Let $\pi_2 : C \times S \rightarrow S$ be the second projection, let $(\text{id}_C \times f) : C \times S \rightarrow C \times P$ be the induced morphism, and set $\mathcal{L}' = (\text{id}_C \times f)^* \mathcal{L}$. Then

$$\frac{1}{2}h^0(L) \leq \text{mult}_x \Xi \leq \frac{1}{2} \deg_{s_0} \widetilde{\Theta}|_S = \frac{1}{2} \ell((R^1\pi_2_*\mathcal{L}')_{s_0}),$$

where $\ell((R^1\pi_2_*\mathcal{L}')_{s_0})$ is the length of $(R^1\pi_2_*\mathcal{L}')_{s_0}$ as an $\mathcal{O}_{S,s_0}$-module. Moreover, there exist $S$, $f$, and $s_0$ as above such that $\text{mult}_x \Xi = \frac{1}{2} \ell((R^1\pi_2_*\mathcal{L}')_{s_0})$. If $f(S) \not\subseteq \Xi$, there is an $N \in \mathbb{Z}$ such that $\forall k \geq N$,

$$\ell((R^1\pi_2_*\mathcal{L}')_{s_0}) = \ell(H^0(\mathcal{L}_k)).$$
In particular, if for some \( n \in \mathbb{Z} \), \( \ell(H^0(\mathcal{L}_n)) = \ell(H^0(\mathcal{L}_{n+1})) \), we may take \( N = n \).

\[ \square \]

1.2 Line bundles on double covers

The results of [5] Section 2 will be extended to the case of nodal curves. In particular, we will consider the case where the double cover \( \pi : \widetilde{C} \to C \) is a finite degree-two morphism of stable curves, induced by an involution \( \tau : \widetilde{C} \to \widetilde{C} \), such that:

1. the singular points are fixed points, and at the singular points, the two branches are not exchanged under \( \tau \).

This is a slightly weaker condition than (§), which required that the fixed points be exactly the singular points.

We will begin by studying how points \( p, \tau(p) \in \widetilde{C} \) impose conditions on global sections of line bundles on \( \widetilde{C} \). The difficulty here is due to the fact that \( \widetilde{C} \) may be reducible, and hence nonzero sections may vanish identically on irreducible components. We will fix the following notation: let \( \nu : \widetilde{N} \to \widetilde{C} \) and \( N \to C \) be the normalizations, let \( \widetilde{C} = \bigcup_{i=1}^\mu \widetilde{C}_i \), where the \( \widetilde{C}_i \) are the irreducible components of \( \widetilde{C} \), let \( \text{Sing} \widetilde{C} = \{x_1, \ldots, x_\delta\} \), let \( \nu^{-1}(x_i) = \{r_i, r'_i\} \), and let \( \hat{R} = \sum_{i=1}^\delta (r_i + r'_i) \). Recall that we are allowing the covering to be ramified at some smooth points, say \( \{t_1, \ldots, t_a\} \) of \( \widetilde{C} \); let \( \hat{T} = \sum_{i=1}^a t_i \).

The following definition generalizes the notation used in [6] and [5] to this setting: for a line bundle \( L \) on \( \widetilde{C} \), a decomposition of \( L \) is an isomorphism

\[
\phi : \nu^* L \to \pi^* M \otimes O_{\widetilde{N}}(R + T + B),
\]

where \( M \) is a line bundle on \( N \) such that \( h^0(N, M) > 0 \), and \( B, T \) and \( R \) are effective divisors on \( \widetilde{N} \) satisfying the properties \( B \cap \tau^* B = \emptyset, T \leq \hat{T}, R \leq \hat{R} \) and \( r_i \in \text{supp}(R) \) if and only if \( r'_i \in \text{supp}(R) \).

Given a decomposition of \( L \), we will let \( r \in H^0(\widetilde{N}, R) \), (resp. \( t \in H^0(\widetilde{N}, T) \), \( b \in H^0(\widetilde{N}, B) \)), be a section vanishing on \( R \), (resp. \( T, B \)). We can consider \( H^0(\widetilde{C}, L) \subseteq H^0(\widetilde{N}, \nu^* L) \) as a linear subspace; a priori, the sections of \( \pi^* H^0(N, M) \cdot r \cdot t \cdot b \) are not necessarily sections of \( H^0(\widetilde{C}, L) \). Define \( \mathfrak{M} \subseteq H^0(N, M) \) to be the maximal subspace with the property that \( \pi^* \mathfrak{M} \cdot r \cdot t \cdot b \subseteq H^0(\widetilde{C}, L) \). We say that the decomposition satisfies (†) if \( \mathfrak{M}|_{C_i} \neq 0 \) for all irreducible curves \( C_i \subseteq C \).

We have the following lemma:
Lemma 1.2.1 If \( H^0(L)|_{\tilde{C}} \neq 0 \) for all \( 1 \leq i \leq \mu \), there is a decomposition of \( L \) satisfying (†). Moreover, given a decomposition of \( L \) satisfying (†) with \( B = 0 \), there is an isomorphism \( \tau^*(\nu^*L) \cong \nu^*L \) such that

\[
H^0(\tilde{N}, \nu^*L)^+ = \pi^*H^0(N, M) \cdot r \cdot t,
\]

where \( H^0(\tilde{N}, \nu^*L)^+ \) refers to the positive eigen-space of the induced action of \( \tau \) on \( H^0(\tilde{N}, \nu^*L) \). \( \square \)

For irreducible curves, there are the following two results, which closely reflect those in [5] for smooth curves. The proofs are similar.

Corollary 1.2.2 ([5], Corollary 2.1.3) If \( \tilde{C} \) is irreducible there exists a decomposition of \( L \) satisfying (†). Moreover, given such a decomposition, let \( n_1 = \dim(\mathfrak{M}) \), let \( n_2 = h^0(L) - \dim(\mathfrak{M}) \), and let \( p_1, \ldots, p_k \) be general points of \( \tilde{C} \). Set \( D_k = \sum_{i=1}^{k} (p_i + \tau(p_i)) \) and \( \bar{p}_i = \pi(p_i) \).

(a) If \( \dim(\mathfrak{M}) > h^0(L)/2 \) and \( k \leq n_2 \), \( h^0(L(-D_k)) = h^0(L) - 2k \).

(b) If \( \dim(\mathfrak{M}) > h^0(L)/2 \) and \( n_2 \leq k \leq n_1 \), then \( h^0(L(-D_k)) = h^0(L) - n_2 - k \). In this case, \( H^0(L(-D_k)) = \pi^*\mathfrak{M}(-\sum_{i=1}^{k} \bar{p}_i) \cdot r \cdot t \cdot b \).

(c) If \( \dim(\mathfrak{M}) > h^0(L)/2 \), \( n_2 \leq k \leq n_1 \), and \( 1 \leq k_1 \leq k \),

\[
h^0(L(-D_k - D_{k_1})) = h^0(L(-D_k - \sum_{i=1}^{k_1} p_i)) = \max(h^0(L) - n_2 - k - k_1, 0).
\]

In this case, \( H^0(L(-D_k - D_{k_1})) = \pi^*\mathfrak{M}(-\sum_{i=1}^{k} \bar{p}_i - \sum_{i=1}^{k_1} \bar{p}_i) \cdot r \cdot t \cdot b. \) \( \square \)

Corollary 1.2.3 ([5], Lemma 2.1.4) Suppose \( \tilde{C} \) is irreducible, \( h^0(L) = 2n > 0 \), and for general points \( p_1, \ldots, p_n \) of \( \tilde{C} \), \( h^0(L(-D_n)) > 0 \), where \( D_n = \sum_{i=1}^{n} (p_i + \tau(p_i)) \). Then \( L \) has a decomposition satisfying (†) such that \( \dim(\mathfrak{M}) > h^0(L)/2 \). \( \square \)

In general, for reducible curves it is hard to understand how points impose conditions on sections of line bundles. For curves of low genus we can say a little more. The proof of the following proposition is similar to the arguments in [6] on pages 27-30. Recall the parity of a connected étale double cover of a plane quintic is that of \( h^0(O_{\mathbb{P}^2}(1)|_{C} \otimes \eta) \), where \( \eta \) is the two-torsion line bundle associated to the double cover.
**Proposition 1.2.4** Suppose $p_a(C) \leq 6$. If $h^0(L) = 4$, and there exists a decomposition of $L$ satisfying (†), then $h^0(N, M) \leq 3$. If $\dim(\mathfrak{M}) = 3$, then

(a) if $p_a(C) = 5$, then $C$ is hyperelliptic;

(b) if $p_a(C) = 6$, then $C$ is either hyperelliptic, or obtained from a hyperelliptic curve by identifying two points, or trigonal, or is a plane quintic, in which case $\pi$ is an odd double cover.

The following proposition complements Proposition 1.2.4. The proof is similar to those in [5].

**Proposition 1.2.5** Suppose $p_a(C) \leq 6$, $h^0(L) = 4$, there exists a decomposition of $L$ satisfying (†), and for every such decomposition $\dim(M) < 3$.

There exist smooth points $p_1, p_2 \in \tilde{C}$ such that $H^0(L(-\sum_{i=1}^2 (p_i + \tau(p_i)))) = 0$. □

### 2 Prym theta divisors

The results of Section 2 will allow us to determine the multiplicity of a point on the Prym theta divisor in certain cases.

#### 2.1 General statements

The following result was proven in [6]:

**Proposition 2.1.1** ([6], Lemma 2.1) Suppose $x \in P$ corresponds to a line bundle $L \in P^*$ such that $h^0(L) = 2n$, and there exist $n$ distinct points $p_1, \ldots, p_n$ of $\tilde{C}$ such that $H^0(L(-D_n)) = 0$, where $D_n = \sum_{i=1}^n (p_i + \tau(p_i))$. Then $\text{mult}_x \Xi = h^0(L)/2$. □

The next proposition rephrases a result from [6] in the notation used in Section 2.

**Proposition 2.1.2** ([6], Theorem 2.3) If $x \in P$ corresponds to a line bundle $L \in P^*$ which has a decomposition satisfying (†), $\text{mult}_x \Xi \geq \dim(\mathfrak{M})$.

*Proof.* The proof follows directly from [6] Theorem 2.3, and Lemma 1.2.1. □

We also have the following:
Theorem 2.1.3 Suppose $x$ is a point of $\Xi$, corresponding to a line bundle $L \in P\ast$ such that $h^0(L) = 2$. If there exists a general point $p$ on $\tilde{C}$ such that $h^0(L(-p-\tau(p))) = 0$, then $x$ is a smooth point of $\Xi$. Otherwise, $\text{mult}_x\Xi = 2$.

The first statement is a special case of Proposition 2.1.1. The second statement is a consequence of the following:

Proposition 2.1.4 In the notation above, suppose $\tilde{C} = \bigcup_{i=1}^d \tilde{C}_i$, where the $\tilde{C}_i$ are connected. If $h^0(L) = n \leq d$, $\dim(H^0(L)|_{\tilde{C}_i}) = 1$ for all $i \in \{1, \ldots, n\}$, and $\dim(H^0(L)|_{\tilde{C}_j}) = 0$ for all $j \in \{n+1, \ldots, d\}$, then $\text{mult}_x\Xi = h^0(L)$.

Proof. To begin, one can easily check that for any deformation $L$ which lies in $P$, all sections of $L$ lift to first order. In other words, in the notation of Section 1 and Lemma 1.1.1, taking $L$ general and setting $\tilde{L}_k$ to be the truncation of $L$ to order $k$, $\ell(H^0(L_1)) = 2h^0(L)$, and for some $k > 0$,

\begin{equation}
(2.1.5) 
\frac{1}{2} \ell(H^0(L_1)) \leq \frac{1}{2} \ell(H^0(L_k)) = \text{mult}_x\Xi.
\end{equation}

We now show that for a particular deformation, there are no nontrivial sections lifting to second order. For $i \in \{1, \ldots, n\}$ let $p_i \in \tilde{C}_i$ be a general point. Let $S \subset \mathbb{C}$ be a neighborhood of the origin, and let $q_i : S \to U_i \subset \tilde{C}$ be an isomorphism centered at $p_i$. There is a deformation $\mathcal{L}$ of $L$ given by

$$
L \otimes \mathcal{O}_{\tilde{C}}(p_1 - \tau(p_1) - q_1 + \tau(q_1)) \otimes \cdots \otimes \mathcal{O}_{\tilde{C}}(p_n - \tau(p_n) - q_n + \tau(q_n))
$$

corresponding to a morphism $f : S \to P$ sending 0 to $x$. For more details on these deformations see [5] and [6]. Let $t$ be a local coordinate on $S$ centered at 0, and in the notation of Section 1, define $\tilde{L}_k$ on $\tilde{C}_k = \tilde{C} \times S_k$ to be the truncation of $L$ to order $k$. Locally, for an open set $U \times S_k \subset \tilde{C}_k$, a section of $H^0(\tilde{L}_k)$ can be written as $\sum_{i=0}^k \sigma^{(i)} t^i$ for some functions $\sigma^{(i)} \in \mathcal{O}_{\tilde{C}}(U)$.

Observe that a section $s \in H^0(L)$ lifts to second order only if $s|_{\tilde{C}_i} \in H^0(L|_{\tilde{C}_i})$ lifts to second order. Now let $\tilde{C}_v = \bigcup_{i \neq 1} \tilde{C}_i$, and suppose that $s \in H^0(L)$ is such that $s|_{\tilde{C}_i} \neq 0$, and $s|_{\tilde{C}_v} \equiv 0$. Let $\tilde{C}_1 \cap \tilde{C}_v = \{x_1, \ldots, x_m\}$, let $L' = L|_{\tilde{C}_1} \otimes \mathcal{O}_{\tilde{C}_1}(-\sum_{i=1}^m x_i)$, and consider the deformation $\mathcal{L}'$ of $L'$ induced by $\mathcal{L}$, which in the notation above is given by

$$
L' \otimes \mathcal{O}_{\tilde{C}_1}(p_1 - \tau(p_1) - q_1 + \tau(q_1)).
$$

One can prove inductively that if $\sum_{i=0}^k \sigma^{(i)} t^i \in H^0(\tilde{L}_k)$, then $\sigma^{(i)}(x_j) = 0$ for $1 \leq i \leq k, 1 \leq j \leq m$. In other words, the section $s$ lifts to second order
as a section of \( L^*_2 \) only if \( s|_{\tilde{C}_1} \in H^0(L') \) lifts to second order as a section of \( L'_2 \). Since \( H^0(L') = H^0(L)|_{\tilde{C}_1} \), it follows that \( h^0(L') = 1 \). One can check that \( h^0(\tilde{C}_1, L'(p_1 + \tau(p_1))) = 2 \), and that \( h^0(\tilde{C}_1, L'(p_1 + 2\tau(p_1))) = 2 \), and thus the natural inclusion \( H^0(L'(p_1 + \tau(p_1))) \subseteq H^0(L'(p_1 + 2\tau(p_1))) \) is an equality. It follows from \[5\] Lemma 1.3.4 that only the trivial section of \( L' \) lifts to second order.

We now complete the proof. Let \( s \in H^0(L) \) be such that \( s|_{\tilde{C}_1} \neq 0 \). Then \( s \) lifts to order two only if \( s|_{\tilde{C}_1} \) lifts to order two as a section of \( L'_2 \). We have seen that only the trivial section lifts to second order as a section of \( L'_2 \). By symmetry, only the zero section of \( L \) lifts to second order; in other words \( H^0(L_2) = H^0(L_1) \). From Lemma \[1.1.1\] and the inequality \( (2.1.3) \), it follows that \( h^0(L) \leq \text{mult}_x \Xi \leq \frac{1}{2} \ell(H^0(L_1)) = h^0(L) \). □

Proof of Theorem 2.1.3. If there exist smooth points \( p, \tau(p) \) such that \( H^0(L(-p - \tau(p))) = 0 \), then \( \text{mult}_x \Xi = 1 \), by Proposition 2.1.1. So suppose that there do not exist such points. In the case where there is no component \( \tilde{C}_j \subseteq \tilde{C} \) such that \( \dim(H^0(L)|_{\tilde{C}_j}) = 2 \), the theorem follows from Proposition 2.1.4. In the case where there is a component \( \tilde{C}_j \subseteq \tilde{C} \) such that \( \dim(H^0(L)|_{\tilde{C}_j}) = 2 \), it is straightforward to adapt the proof of \[5\] Lemma 4.1.1 and \[5\] Theorem 2 to show that \( \text{mult}_x \Xi = 2 \). □

The following theorem extends a theorem of Smith and Varley’s \[20\], and a theorem of the author’s \[5\], to the case of an irreducible curve with nodes. We continue to use the notation of Section 1.

**Theorem 2.1.6** Suppose \( \tilde{C} \) is irreducible. In the notation above, if \( x \in \Xi \) corresponds to a line bundle \( L \in P^* \), and \( C_x\tilde{\Theta} \) is the tangent cone to \( \tilde{\Theta} \) at \( x \), then the following are equivalent:

(a) \( T_xP \subseteq C_x\tilde{\Theta} \);

(b) \( \text{mult}_x \Xi > h^0(L)/2 \);

(c) \( L \) has a decomposition satisfying (†), with \( \dim(\mathfrak{M}) > h^0(L)/2 \).

Moreover, if (c) holds, \( \text{mult}_x \Xi = \dim(\mathfrak{M}) \).

Proof. The proof of the equivalence follows directly from Corollary \[1.2.3\] and Proposition \[2.1.2\]. In the case where (c) holds, using Corollary \[1.2.2\] it is straightforward to adapt the proof of \[5\] Lemma 4.1.1, and \[5\] Theorem 2 to show \( \text{mult}_x \Xi = \dim(\mathfrak{M}) \). □
Corollary 2.1.7 Suppose \((P, \Xi) \in A_{g-1}\) is an indecomposable Prym variety associated to an admissible double cover of an irreducible curve. For \(x \in \Xi\), \(\text{mult}_x \Xi \leq g/2 = (\dim(P) + 1)/2\). □

2.2 Proof of Theorem 2, and Theorem 3

In this section we will only be concerned with indecomposable ppavs, and so we will always assume that the double cover satisfies \((\ast)\), and \((\#)\) if there is a decomposition \(\bar{C} = \bar{C}_1 \cup \bar{C}_2\), with \(\bar{C}_1 \cap \bar{C}_2\) finite, then \(#(\bar{C}_1 \cap \bar{C}_2)\) is even and \(\geq 4\).

Otherwise, by [2], Theorem 5.4, and Lemma 4.11, \((P, \Xi)\) is either reducible or a Jacobian.

Theorem 2.2.1 Suppose \((P, \Xi)\) is indecomposable, and \(x \in \Xi\). If \(p_a(C) = 5\), then \(\text{mult}_x \Xi \leq 2\).

Proof. Suppose \(x\) corresponds to the line bundle \(L \in P^*\). By Lemmas 3.15 and 3.16 of [6], we may assume \(h^0(L) < 6\). If \(h^0(L) = 2\), then by Theorem 2.1.3 \(\text{mult}_x \Xi \leq 2\). If \(h^0(L) = 4\), then again, by Lemmas 3.15 and 3.16 of [6], \(H^0(L)\) is nonzero on every irreducible component of \(\bar{C}\), and so \(L\) has a decomposition satisfying \((\dag)\). If there does not exist such a decomposition with \(\dim(\mathfrak{M}) \geq 3\), it follows from Proposition 1.2.5 that for general points \(p_1\) and \(p_2\) of \(C\), \(H^0(L(-p_1 - \tau(p_1) - p_2 - \tau(p_2))) = 0\); Proposition 2.1.1 implies that in this case \(\text{mult}_x \Xi = 2\). If on the other hand there is a decomposition of \(L\) with \(\dim(\mathfrak{M}) \geq 3\), then by virtue of Proposition 1.2.4 \(C\) is hyperelliptic. A result of Mumford’s implies that \((P, \Xi)\) is the Jacobian of a curve of genus four. The Riemann singularity theorem implies that \(\text{mult}_x \Xi \leq 2\). □

Remark 2.2.2 It has been shown by Beauville in [2] that if \(\text{mult}_x \Xi = 2\), then \((P, \Xi)\) is the Jacobian of a curve, or has a vanishing theta null.

Theorem 2.2.3 Suppose \((P, \Xi)\) is indecomposable, and \(x \in \Xi\). If \(p_a(C) = 6\), then \(\text{mult}_x \Xi \leq 3\).

Moreover, if equality holds then \(C\) is either hyperelliptic, or obtained from a hyperelliptic curve by identifying two points, or trigonal, or a plane quintic, in which case \(\pi : \bar{C} \to C\) is the double cover associated to an odd theta characteristic. It follows that \((P, \Xi)\) is either the Jacobian of a hyperelliptic curve, or the intermediate Jacobian of a smooth cubic threefold in \(\mathbb{P}^4\).
Proof. Suppose $x$ corresponds to the line bundle $L \in P^*$, and $\text{mult}_x \Xi \geq 3$. Again, by Lemmas 3.15 and 3.16 of [6], we may assume $h^0(L) \leq 6$, and equality holds only if $C$ is hyperelliptic, and hence $(P, \Xi)$ is the Jacobian of a hyperelliptic curve. If $h^0(L) = 4$, then by Theorem 2.1.3 $\text{mult}_x \Xi \leq 2$, which is a contradiction. If $h^0(L) = 4$, then $H^0(L)$ is nonzero on every irreducible component of $\bar{C}$, and so $L$ has a decomposition satisfying (i). We claim that there exists such a decomposition of $L$ with $\dim(\mathfrak{M}) \geq 3$. Indeed, if not, it would follow from Proposition 1.2.5 that for general points $p_1$ and $p_2$ of $C$, $H^0(L(-p_1 - \tau(p_1) - p_2 - \tau(p_2))) = 0$; Proposition 2.1.1 would then imply that $\text{mult}_x \Xi = 2$, which is a contradiction. Thus there is a decomposition of $L$ with $\dim(\mathfrak{M}) \geq 3$, and the claims on $C$ follow directly from Proposition 1.2.3. Results of Mumford [16], Shokurov [18], Beauville [2] and [4], [cf. 6], Theorem 4.1, then imply that $(P, \Xi)$ is either the Jacobian of a curve, or the intermediate Jacobian of a cubic threefold. It follows that $\text{mult}_x \Xi \leq 3$, and equality holds only if $(P, \Xi)$ is the Jacobian of a hyperelliptic curve, or the intermediate Jacobian of a cubic threefold.

We have the following consequence.

**Corollary 2.2.4** An indecomposable ppav of dimension five whose theta divisor has a triple point is either the intermediate Jacobian of a smooth cubic threefold in $\mathbb{P}^4$, or the Jacobian of a hyperelliptic curve.

In [6] the same statement was proven for ppavs of dimension five with a unique triple point.

**Proof of Corollary 2.2.4 and Theorem 3** The statements are now a direct consequence of Theorem 2.2.3, Theorem 2.2.1, Corollary 2.1.7.

Theorems 2 will follow easily from these results, together with the lemma below, which is essentially proven in Ein and Lazarsfeld [11].

**Lemma 2.2.5** Let $N$ and $k$ be a positive integers. The following statements are equivalent:

(a) For a ppav $(A, \Theta)$ of dimension $d \leq N$, $\dim(\text{Sing}_k \Theta) > d - 2k + 1$ implies that $(A, \Theta)$ is reducible.

(b) For a ppav $(A, \Theta)$ of dimension $d \leq N$, and all nonnegative integers $j$ such that $k - j \geq 1$, if $\dim(\text{Sing}_k \Theta) = d - k - j$, then $(A, \Theta) = \prod_{i=1}^{k-j}(A_i, \Theta_i)$. □
Proof of Theorem 2.2. By virtue of the lemma, we need only show that for $d \leq 5$, if $(A, \Theta) \in \mathcal{A}_5$, and $\Theta$ is irreducible, then $\dim(S_{\Theta}) \leq d - 2k + 1$. By Theorems 2.2.1 and 2.2.3 the pertinent cases are $d = 5$, $k \leq 3$, and $d = 4$, $k \leq 2$. If $d = 5$ and $k = 3$, then the result follows from Theorem 2.2.3 if $d = 5$ and $k = 2$, or $d = 4$ and $k = 3$, then the result follows from Ein and Lazarsfeld [11], Corollary 2.

**Remark 2.2.6** For the Jacobian of a curve, Martens’ theorem implies that $\dim(S_{\Theta}) = g - 2k + 1$ only if the curve is hyperelliptic. It is a result of Beauville [2] that if $(A, \Theta)$ is an indecomposable Prym variety, and $\dim(S_{\Theta}) \geq d - 4 + 1$, then $(A, \Theta)$ is a hyperelliptic Jacobian. Thus at least in dimension five or less, any indecomposable ppav whose theta divisor has double points in codimension three is a hyperelliptic Jacobian. In regards to these results, it was asked in [5] to what extent $k$-fold points in codimension $2k - 1$ on an indecomposable ppav characterize hyperelliptic Jacobians. Theorem 2.2.1 implies that in dimension less than or equal to five, $k$-fold points in codimension $2k - 1$ imply that $(A, \Theta)$ is either a hyperelliptic Jacobian, or the intermediate Jacobian of a smooth cubic threefold.

2.3 The triple point locus

Recall the definition of the Andreotti-Mayer loci $N_k \subseteq \mathcal{A}_g$:

$$N_k = \{(A, \Theta) \in \mathcal{A}_g : \dim(\text{Sing}_{k}\Theta) \geq k\}.$$ 

These are closed subschemes, and similarly, it is easy to see that the subloci $S_k = \{(A, \Theta) \in \mathcal{A}_g : \text{Sing}_k \Theta \neq \emptyset\}$ are closed as well.

It is a result of Andreotti and Mayer [1] that $\bar{J}_g$ is an irreducible component of $N_{g-4}$. In [2], Beauville showed that $N_0 \subseteq \mathcal{A}_4$ consists of two irreducible components of dimension nine; to be exact, $N_0 = \bar{J}_4 \cup \theta_{\text{null}}$, where $\theta_{\text{null}}$ is the locus of ppavs with a vanishing theta null. We will prove a similar statement for $S_3 \subseteq \mathcal{A}_5$.

Let $I \subseteq \mathcal{A}_5$ denote the locus of intermediate Jacobians of smooth cubic threefolds in $\mathbb{P}^4$, and let $J_{g_1, \ldots, g_n} \subseteq \mathcal{A}_{\sum g_i}$ be the locus of ppavs which are the product of $n$ Jacobians of hyperelliptic curves of genera $g_1, \ldots, g_n$. For a sublocus $V \subseteq \mathcal{A}_g$, let $\mathcal{A}_g^V \subseteq \mathcal{A}_{g'+g}$ be the locus of ppavs which are the product of a ppav of dimension $g'$ with a ppav in $V$.

**Theorem 2.3.1** The locus $S_3 \subseteq \mathcal{A}_5$ has three irreducible components, $\bar{I}$, $\bar{A}_1\bar{J}_4$ and $\bar{A}_1\theta_{\text{null}}$, each of which is ten-dimensional.
Proof. Let \((A, \Theta) \in S_3\), and suppose first that \((A, \Theta) = (A_1, \Theta_1) \times (A_2, \Theta_2)\). We may as well assume that \(\dim(A_1) < \dim(A_2)\). If \(\dim(A_1) = 1\), then \(\Theta_2\) must have a double point, and so by Beauville’s result on \(N_0\), \((A, \Theta) \in A_1J_4 \cup A_1\theta_{null}\). If \(\dim(A_1) = 2\), then we may assume that \((A_1, \Theta_1)\) and \((A_2, \Theta_2)\) are indecomposable, since otherwise we would be in the previous case. It follows that \(\Theta_2\) has a double point, and so \((A_2, \Theta_2)\) is the Jacobian of a hyperelliptic curve. Thus, \((A, \Theta) \in J_{2,3}^h\). A result of Collino’s [8] (cf. the remark below) states that \(J_{5}^h \cup J_{1,4}^h \cup J_{2,3}^h \subseteq \partial I\); thus \((A, \Theta) \in \bar{I}\). Finally, if \((A, \Theta)\) is indecomposable, then it follows from Corollary 2.2.4 that \((A, \Theta) \in \bar{I}\). The fact that \(A_{1}\theta_{null}\) is irreducible and of dimension ten follows from the fact (cf. Beauville [2]) that \(\theta_{null}\) is irreducible of dimension nine. □

Remark 2.3.2 Samuel Grushevsky has pointed out the following observation: for a ppav \((A, \Theta) \in A_g\) and a point \(x \in A\) such that \(x + x = 0\) and \(\text{mult}_x \Theta = 3\), the gradient of the theta function at \(x\) vanishes, giving \(g\) conditions in \(A_g\) cutting out \(S_3\). The fact that in \(A_5, S_3\) is equidimensional and has codimension five is equivalent to the independence of the conditions arising from the vanishing gradient. Note also that in \(A_4, S_3 = J_{1,3}^h\), which has codimension four, and in \(A_3, S_3 = J_{1,1,1}^h\) which has codimension three. Hence the conditions arising from the vanishing gradient are independent in these cases as well. It would be interesting to know if this were the case for all \(S_3 \subset A_g\) with \(g \geq 3\).

Corollary 2.3.3 (Theorem 1) In the notation above, \(I = S_3 - (N_1 \cap S_3)\).

Proof. Theorems 2.2.3 and 2.3.1 imply that the ppavs in \(S_3\) which are not in \(I\) are either reducible, or are Jacobians of hyperelliptic curves, and therefore the dimension of the singular locus of their theta divisor must be at least one (in fact at least two). □

Remark 2.3.4 The fact that \(J_{5}^h \cup J_{1,4}^h \cup J_{2,3}^h \subseteq \partial I\) was proven by Collino in [8] by considering the secant variety to the rational normal quartic. It is interesting to point out that this result can be obtained as well by combining some arguments used in Beauville [3], Section 2.9, with a result of Griffin’s [12] on degenerations of plane quintics, and a result of Mumford’s [16] on Prym varieties of hyperelliptic curves.

Remark 2.3.5 On the other hand, it is still not clear exactly which ppavs
lie in $\partial I$. The results of this paper tell us that the only indecomposable ppavs in $\partial I$ are the hyperelliptic Jacobians. In general, the description of $S_3$ above certainly narrows the list of possibilities, and one may be able to use similar techniques to give a complete description.

One can also consider the boundary of $I$ in the Satake compactification, and certain special cases are known; e.g. in [7], Clemens and Griffiths show that the intermediate Jacobian of a cubic hypersurface in $\mathbb{P}^4$ with a unique ordinary double point is the extension by a $\mathbb{C}^*$ of the Jacobian of a curve of genus 4. Other treatments and examples are considered by Collino and Murre [9], and Gwena [13].

**Remark 2.3.6** In this paper, we study the locus of intermediate Jacobians of cubic threefolds by classifying those ppavs of dimension five whose theta divisor has a triple point. It should be pointed out that there are other properties of the theta divisor which have been conjectured to classify these intermediate Jacobians as well.

For example, for a cubic threefold $X$, the Fano surface $F$ of lines on $X$ embeds in $JX$ as a nondegenerate surface, and has class $[F] = [\Theta]^3/3! \in H^6(JX, \mathbb{Z})$. In [10], Debarre showed that $\bar{I}$ is an irreducible component of the locus of ppavs of dimension five for which there exists a surface $S \subseteq X$ such that $[S] = [\Theta]^3/3!$. For more discussion in this vein, we refer the reader to Debarre [10].

In another direction, Pareschi and Popa [17] have considered a notion of regularity for subvarieties of abelian varieties which may also give a criterion for a ppav of dimension five to be the intermediate Jacobian of a cubic threefold (cf. Pareschi and Popa [17] Conjecture 2.2).

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Harvard University
Department of Mathematics
One Oxford Street
Cambridge, MA 02138

casa@math.harvard.edu

Department of Mathematics
SUNY Stony Brook
Stony Brook, NY 11794-3651

casa@math.sunysb.edu