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S-SHAPED BIFURCATION CURVES FOR A COMBUSTION PROBLEM WITH GENERAL ARRHENIUS REACTION-RATE LAWS

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Abstract. We study the bifurcation curve and exact multiplicity of positive solutions of the combustion problem with general Arrhenius reaction-rate laws

\[ u''(x) + \lambda (1 + \epsilon u)^m e^{\frac{u}{1+u}} = 0, \quad -1 < x < 1, \]
\[ u(-1) = u(1) = 0, \]

where the bifurcation parameters \( \lambda, \epsilon > 0 \) and \( -\infty < m < 1 \). We prove that, for \((-4.103 \approx)\) \( \tilde{m} \leq m < 1 \) for some constant \( \tilde{m} \), the bifurcation curve is S-shaped on the \((\lambda, ||u||_\infty)\)-plane if \( 0 < \epsilon \leq \frac{6}{9} \epsilon_{tr}^{Sem}(m) \), where

\[ \epsilon_{tr}^{Sem}(m) = \begin{cases} \left( \frac{1 - \sqrt{1-m}}{m} \right)^2 & \text{for } -\infty < m < 1, \ m \neq 0, \\ \frac{1}{4} & \text{for } m = 0, \end{cases} \]

is the Semenov transitional value for general Arrhenius kinetics. In addition, for \(-\infty < m < 1 \), the bifurcation curve is S-like shaped if \( 0 < \epsilon \leq \frac{8}{9} \epsilon_{tr}^{Sem}(m) \). Our results improve and extend those in Wang (Proc. Roy. Soc. London Sect. A, 454 (1998), 1031–1048.)

1. Introduction

We study the bifurcation curve and exact multiplicity of positive solutions of the problem with the Dirichlet (Frank-Kamenetskii) boundary conditions

\[ \frac{d^2 u}{dx^2} + \lambda (1 + \epsilon u)^m e^{\frac{u}{1+u}} = 0, \quad -1 < x < 1, \]
\[ u(-1) = u(1) = 0, \]

where the bifurcation parameters \( \lambda, \epsilon > 0 \) and \( -\infty < m < 1 \). Problem (1.1) is an one-dimensional case of an equation arising in combustion theory, which governs the steady-state thermal explosions in a material undergoing an \( m \)-th-order exothermic reaction. It can be considered as a special case for an \( n \)-dimensional Dirichlet problem for the infinite slab. In (1.1), \( \lambda \), called the Frank-Kamenetskii parameter, is a dimensionless rate of heat production,
$u$ is the dimensionless temperature, and $\epsilon$ is the reciprocal activation energy parameter or the ambient temperature parameter. The reaction term

$$f(u) \equiv (1 + \epsilon u)^m e^{\frac{u}{1+u}}$$

is the temperature dependence of the $m$th-order reaction rate obeying the general Arrhenius reaction-rate law in which heat flow is purely conductive, see e.g. [1, 2, 3, 12]. Problem (1.1) was studied mainly for physically important range of numerical exponent $m < 1$, and particularly, for $m = -2$ (sensitized reaction rate), $m = 0$ (Arrhenius reaction rate) and $m = 1/2$ (bimolecular reaction rate) (see e.g. [1, 2, 3, 4]). Reaction reports have been given with $m = 2$ in [11]. Reaction reports have also been given with $m = 0$ in [9].

Criticality (bifurcation) persists as long as the reciprocal activation energy is smaller than a transitional value $t_r$. As $\epsilon$ approaches $t_r$, the function $f(u)$ becomes “saturated” and a transition from criticality to continuity results, see [2]. Boddington et al. [4, Section 4] obtained

$$\epsilon_{tr}^{F.K.}(m) < \epsilon_{tr}^{Sem}(m) \equiv \begin{cases} \left(\frac{1-\sqrt{1-m}}{m}\right)^2 & \text{for } -\infty < m < 1, \ m \neq 0, \\ \frac{1}{4} & \text{for } m = 0, \end{cases}$$

where $\epsilon_{tr}^{F.K.}(m)$ is the transitional value for general Arrhenius kinetics under Frank-Kamenetskii boundary conditions ($u(-1) = u(1) = 0$) and $\epsilon_{tr}^{Sem}(m)$ is the Semenov transitional value for general Arrhenius kinetics under Semenov boundary conditions ($u'(-1) = u'(1) = 0$). Thus, a transition to continuity does occur and there is no Frank-Kamenetskii criticality unless $-\infty < m < 1$ and $0 < \epsilon < \epsilon_{tr}^{Sem}(m)$. Accurate transitional values for $\epsilon_{tr}^{F.K.}(m)$ have been calculated by numerical quadrature for the infinite slab; i.e. for (1.1). We note that in the previous numerical work, especially that by Boddington et al. in [1, 2, 3, 4]; they found an S-shaped bifurcation diagram and three solutions for some parameter values.

We define the bifurcation curve of positive solutions of (1.1)

$$S = \{(\lambda, \|u_\lambda\|_\infty) : \lambda \geq 0 \text{ and } u_\lambda \text{ is a solution of (1.1)}\}.$$ 

We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve $S$ is S-shaped if $S$ has exactly two turning points at some points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ where $\lambda_* < \lambda^*$ are two positive numbers such that

(i) $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$;

(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve $S$ turns to the left,

(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the bifurcation curve $S$ turns to the right.

See Figure 1(i) for example. Similarly, we say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve $S$ is S-like shaped if $S$ starts from the origin and initially continues to the right, $S$ tends to infinity as $\lambda \to -\infty$, and $S$ has at least two turning points.
For (1.1) with \(-\infty < m < 1\), it has been a long-standing conjecture that, \(\epsilon_{tr}^{F,K}(m)\) (< \(\epsilon_{tr}^{SEM}(m)\)) is a continuous function such that, on the \((m, \epsilon)\)-plane, the bifurcation curve \(S\) is S-shaped for \(0 < \epsilon < \epsilon_{tr}^{F,K}(m)\), and is monotone increasing for \(\epsilon \geq \epsilon_{tr}^{F,K}(m)\). In particular, when \(\epsilon = \epsilon_{tr}^{F,K}(m)\), there is a unique turning point. See Figure 1(i)–(iii). (This kind of global bifurcation result is useful in understanding the profiles of the solutions to the full exothermic reaction-diffusion system, cf. Mimura & Sakamoto in [10] for details.)

In the case \(m = 0\) (Arrhenius reaction rate), (1.1) becomes to famous perturbed Gelfand problem and \(\epsilon_{tr}^{F,K}(0) < 1/4 = 0.25\). Wang [13, Theorem 1.1] used quadrature method (time-map method) to prove that the bifurcation curve is S-shaped for \(0 < \epsilon < \epsilon_1 \approx 1/4.4967 \approx 0.222\) for some constant \(\epsilon_1\), and hence \((0.222 \approx) \epsilon_1 < \epsilon_{tr}^{F,K}(0) ( < 0.25)\). This lower bound for \(\epsilon_{tr}^{F,K}(0)\) was improved to \(\epsilon_2 \approx 1/4.35 \approx 0.230\) by Korman & Li [8] by applying a bifurcation theorem of Crandall & Rabinowitz [5]. This lower bound for \(\epsilon_{tr}^{F,K}(0)\) was further improved by Hung & Wang [7] to \(\epsilon_3 \approx 1/4.166 \approx 0.240\) for some constant \(\epsilon_3\) defined in Hung & Wang [7, Eq. (3.22)].
Wang [14, Theorems 1.4 and 1.5] proved the following theorem for (1.1) for the shape of the bifurcation curve \( S \) and a lower bound of the Frank-Kamenetskiii transition value \( \epsilon_{tr}^{F.K.}(m) \) with \(-\infty < m < 1, m \neq 0.\)

**Theorem 1.1.** Consider (1.1) with \(-\infty < m < 1\). Then:

(i) For \( 0 < m < 1 \), the bifurcation curve \( S \) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane if

\[
0 < \epsilon < \max \left( \frac{1}{5}, \frac{1}{2} \epsilon_{tr}^{Sem}(m) \right) = \begin{cases} 
\frac{1}{5}, & \text{if } 0 < m \leq \frac{1}{2} (2\sqrt{10} - 5) \approx 0.662, \\
\frac{1}{2} \epsilon_{tr}^{Sem}(m), & \text{if } \frac{1}{2} (2\sqrt{10} - 5) < m < 1.
\end{cases}
\]

In particular, for \( m = 1/2 \), the bifurcation curve \( S \) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane if \( 0 < \epsilon < \frac{1}{2} \epsilon_{tr}^{Sem}(m) = \frac{1}{2} (2\sqrt{10} - 5) \approx 0.330 \).

(ii) For \(-\infty < m < 0\), the bifurcation curve \( S \) is S-like shaped on the \((\lambda, \|u\|_\infty)\)-plane if \( 0 < \epsilon < \frac{1}{2} \epsilon_{tr}^{Sem}(m) \). In particular, the bifurcation curve \( S \) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane for

\[
m = -1 \quad \text{and} \quad 0 < \epsilon < \frac{1}{2} \epsilon_{tr}^{Sem}(m) = \frac{1}{2} (3 - 2\sqrt{2}) \approx 0.0858,
m = -2 \quad \text{and} \quad 0 < \epsilon < \frac{1}{2} \epsilon_{tr}^{Sem}(m) = \frac{1}{4} (2 - \sqrt{3}) \approx 0.0670.
\]

Notice that Theorem 1.1(i) implies that

\[
\max \left( \frac{1}{5}, \frac{1}{2} \epsilon_{tr}^{Sem}(m) \right) \leq \epsilon_{tr}^{F.K.}(m) \quad \text{for } 0 < m < 1.
\]

In this section, we finally note that problem (1.1) is also of mathematical interest for the case \( m \geq 1 \). Du [6, Theorems 3.3 and 3.4] proved that, if \( m > 1 \), the bifurcation curve \( S \) is \( \triangledown \)-shaped on the \((\lambda, \|u\|_\infty)\)-plane for \( \epsilon > 0 \). In addition, if \( m = 1 \), the bifurcation curve \( S \) is \( \triangledown \)-shaped for \( 0 < \epsilon < 1 \) and is a monotone curve for \( \epsilon \geq 1 \). See Figure 1(iv)–(vi). Note that the results of Du [6] cover not only dimension 1 for problem (1.1) but also dimension 2 when the domain is a unit open ball.

2. **Main result**

The main result in this paper is next Theorem 2.1 which improves and extends Theorem 1.1. In Theorem 2.1(i), we prove that, for \( \tilde{m} \leq m < 1 \), where \( \tilde{m} \approx -4.103 \) is a negative constant defined in (6.2), the bifurcation curve \( S \) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane if \( 0 < \epsilon < \frac{1}{7} s_{tr}^{Sem}(m) \). Moreover, we prove that \( \max \left( \frac{1}{5}, \frac{1}{2} \epsilon_{tr}^{Sem}(m) \right) < \frac{8}{9} s_{tr}^{Sem}(m) < \epsilon_{tr}^{F.K.}(m) \) for \( \tilde{m} \leq m < 1 \). This improves and extends (1.3). In particular, for \( m = 1/2 \) (bimolecular reaction rate), we give a better lower bound of the Frank-Kamenetskiii transition value \( \epsilon_{tr}^{F.K.}(m = 1/2) \). In Theorem 2.1(ii), we prove that, for \(-\infty < m < 1\), the bifurcation curve \( S \) is S-like shaped on the \((\lambda, \|u\|_\infty)\)-plane if \( 0 < \epsilon < \frac{8}{9} s_{tr}^{Sem}(m) \); our result improves and extends Theorem 1.1(ii).
Theorem 2.1. (See Figure 2.) Consider (1.1) with $-\infty < m < 1$. Then:

(i) For $(-4.103 \approx \bar{m} \leq m < 1$, the bifurcation curve $S$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane if

$$0 < \epsilon \leq \frac{6}{7} \epsilon_{S_{tr}}^{Sem}(m).$$

In particular, the bifurcation curve $S$ is S-shaped for

- $m = 1/2$ and $0 < \epsilon \leq \epsilon_4 \approx 0.328$, where $\epsilon_4$ is a constant defined in (3.45),
- $m = -1$ and $0 < \epsilon \leq \frac{6}{7} \epsilon_{S_{tr}}^{Sem}(m) = \frac{6}{7}(3 - 2\sqrt{2}) \approx 0.147$,
- $m = -2$ and $0 < \epsilon \leq \frac{6}{7} \epsilon_{S_{tr}}^{Sem}(m) = \frac{6}{7}(2 - \sqrt{3}) \approx 0.114$.

(ii) For $-\infty < m < 1$, the bifurcation curve $S$ is S-like shaped on the $(\lambda, \|u\|_\infty)$-plane if $0 < \epsilon \leq \frac{8}{9} \epsilon_{S_{tr}}^{Sem}(m)$.

3. Lemmas

To prove our main result, we modify the time-map techniques developed recently in Hung & Wang [7]. The time map formula which we apply to study (1.1) takes the form as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha [F(\alpha) - F(u)]^{-1/2} du \equiv T(\alpha) \quad \text{for} \quad 0 < \alpha < \infty, \quad (3.1)$$

where $F(u) \equiv \int_0^u f(t)dt$. So positive solutions $u$ of (1.1) correspond to

$$\|u\|_\infty = \alpha \quad \text{and} \quad T(\alpha) = \sqrt{\lambda}. \quad (3.2)$$
Thus, studying of the exact number of positive solutions of (1.1) is equivalent to studying the shape of the time map \( T(\alpha) \) on \((0, \infty)\). Also, proving that the bifurcation curve \( S \) is S-shaped (resp. S-like shaped) on the \((\lambda, ||u||_{\infty})\)-plane is equivalent to proving that \( T(\alpha) \) has exactly two (resp. at least two) critical points, a local maximum at some \( \alpha_* \) and a local minimum at some \( \alpha^* > \alpha_* \), on \((0, \infty)\). See Figure 1(i).

The following lemma contains some basic properties of the time map \( T(\alpha) \), which follows from Wang [14, Proposition 1.2].

**Lemma 3.1.** Consider (3.1) with \(-\infty < m < 1\). If \( 0 < \epsilon < \epsilon_{tr}^\text{Sem}(m) \), then

\[
\lim_{\alpha \to 0^+} T(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} T(\alpha) = \infty. \tag{3.3}
\]

In the following Lemma 3.2, we study the concavity of \( f(u) \), which is used to study the shape of the bifurcation curve \( S \). Lemma 3.2 follows from Wang [14, Lemmas 2.1 and 2.2].

**Lemma 3.2.** Consider (1.1) with \(-\infty < m < 1\) and \( m \neq 0 \). If \( 0 < \epsilon < \epsilon_{tr}^\text{Sem}(m) \), then:

(i) If \( 0 < m < 1 \), then

\[
f''(u) = \frac{f(u)}{(1 + \epsilon u)^4} [m(m-1)\epsilon^4 u^2 + 2\epsilon^2(-1 + m - \epsilon m + \epsilon^2 m^2)u + (1 - 2\epsilon + 2\epsilon m + \epsilon^2 m - \epsilon^2 m^2)]
\]

\[
\begin{align*}
&> 0 \quad \text{on } (0,C), \\
&= 0 \quad \text{when } u = C, \\
&< 0 \quad \text{on } (C, \infty)
\end{align*}
\]

where

\[
C = \frac{-1 + \sqrt{1-m + m - \epsilon m + \epsilon m^2}}{\epsilon^2(1-m)m}. \tag{3.5}
\]

(ii) If \(-\infty < m < 0\), then

\[
f''(u) \begin{cases} 
> 0 & \text{on } (0,C) \cup (D, \infty), \\
= 0 & \text{when } u = C, D, \\
< 0 & \text{on } (C, D)
\end{cases}
\]

where

\[
C = \frac{-1 + \sqrt{1-m + m - \epsilon m + \epsilon m^2}}{\epsilon^2(1-m)m} < D = \frac{-1 - \sqrt{1-m + m - \epsilon m + \epsilon m^2}}{\epsilon^2(1-m)m}. \tag{3.6}
\]

For \( 0 < m < 1 \) and \( 0 < \epsilon < \epsilon_{tr}^\text{Sem}(m) \), thus by Lemma 3.2(i), it is easy to check that \( f(u) = (1 + \epsilon u)^m e^{\frac{\epsilon u}{1+m}} \) satisfies the following properties (P1)–(P3):

(P1) \( f(0) > 0 \) (positone) and \( f(u) > 0 \) on \((0, \infty)\).
(P2) $f$ is convex-concave on $(0, \infty)$; that is, $f$ has exactly one positive inflection point at $C = \frac{-1 + \sqrt{1 - m + 4m - 2em + m^2}}{e^2(1 - m)m}$ such that
\[
f''(u) \begin{cases} > 0 & \text{on } [0, C), \\ = 0 & \text{when } u = C, \\ < 0 & \text{on } (C, \infty). \end{cases}
\]

(P3) $f$ is asymptotic sublinear at $\infty$; that is, $\lim_{u \to \infty} (f(u)/u) = 0$.

Next we define the following function
\[
H(u) = 3 \int_0^u tf(t)dt - u^2 f(u) \quad \text{for } u \geq 0. \tag{3.7}
\]

For $0 < m < 1$, since $f$ satisfies (P1)-(P3), thus our main result in Theorem 2.1 in the case $0 < m < 1$ is based upon the following Lemma 3.3 proved by Hung & Wang [7, Theorem 2.1].

**Lemma 3.3.** Consider (1.1) with $0 < m < 1$. If $0 < \epsilon < \epsilon_{\text{tr}}(m)$, then:

(i) If $H(C) \leq 0$, then the bifurcation curve $S$ is S-shaped on the $(\lambda, \|u\|_{\infty})$-plane.

(ii) If $H(u_0) \leq 0$ for some $u_0 > 0$, then the bifurcation curve $S$ is S-like shaped on the $(\lambda, \|u\|_{\infty})$-plane.

For $T(\alpha)$ in (3.1), we compute that
\[
T'(\alpha) = \frac{1}{2\sqrt{2\alpha}} \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{[F(\alpha) - F(u)]^{3/2}} du, \tag{3.8}
\]
where
\[
\theta(u) \equiv 2F(u) - uf(u) = 2 \int_0^u f(t)dt - uf(u). \tag{3.9}
\]

Thus by (3.9), we compute that, there exist positive numbers $A < C < B$ such that
\[
\theta'(u) = f(u) - uf'(u) = \frac{f(u)}{(1 + \epsilon u)^2} \left[ (1 - m)\epsilon^2 u^2 + (-1 + 2\epsilon - m\epsilon)u + 1 \right]
\begin{cases} > 0 & \text{on } [0, A) \cup (B, \infty), \\ = 0 & \text{when } u = A, B, \\ < 0 & \text{on } (A, B). \end{cases} \tag{3.10}
\]

where
\[
A = \frac{1 - 2\epsilon + cm - \sqrt{1 - 4\epsilon + 2cm + \epsilon^2m^2}}{2\epsilon^2(1 - m)} < B = \frac{1 - 2\epsilon + cm + \sqrt{1 - 4\epsilon + 2cm + \epsilon^2m^2}}{2\epsilon^2(1 - m)}, \tag{3.11}
\]
and we also compute that
\[
\theta''(u) = -uf''(u). \tag{3.12}
\]

By (3.12) and Lemma 3.2(i)–(ii), we notice that:
(i) If $0 < m < 1$ and $0 < \epsilon < \epsilon_{tr}^{Sem}(m)$, then

$$\theta''(u) = -uf''(u) \begin{cases} < 0 & \text{on } (0,C), \\ = 0 & \text{when } u = C, \\ > 0 & \text{on } (C,\infty). \end{cases}$$ (3.13)

(ii) If $-\infty < m < 0$ and $0 < \epsilon < \epsilon_{tr}^{Sem}(m)$, then

$$\theta''(u) = -uf''(u) \begin{cases} < 0 & \text{on } (0,C) \cup (D,\infty), \\ = 0 & \text{when } u = C, D, \\ > 0 & \text{on } (C,D). \end{cases}$$ (3.14)

For $H(u)$ in (3.7), we compute that

$$H'(u) = uf(u) - u^2f'(u) = u\theta'(u) \quad \text{for } u \geq 0,$$

$$H''(u) = \theta'(u) + u\theta''(u) \quad \text{for } u \geq 0.$$ (3.15) (3.16)

Moreover, by (3.15) and (3.10),

$$H'(u) = u\theta'(u) \begin{cases} > 0 & \text{on } (0,A) \cup (B,\infty), \\ = 0 & \text{when } u = A, B, \\ < 0 & \text{on } (A,B). \end{cases}$$ (3.17)

The following lemma contains some properties of $\theta(u)$, which basically follow from Wang [14, Lemmas 3.3, 3.6, 4.2 and 4.5] after some slight modification; we omit the proof. Lemma 3.4 is useful in studying the auxiliary function $H(u)$.

**Lemma 3.4.** Consider (1.1) with $-\infty < m < 1$ and $m \neq 0$. If $0 < \epsilon < \epsilon_{tr}^{Sem}(m)$, then:

(i) For fixed $m < 1$ and $m \neq 0$, $\theta'(C)$ is a strictly increasing function of $\epsilon \in (0,\epsilon_{tr}^{Sem}(m))$; that is,

$$\frac{\partial \theta'(C)}{\partial \epsilon} > 0.$$ (3.18)

(ii) If $0 < m < 1$, then

$$\theta''(C) > 0,$$ (3.19)

and $\theta''(u)$ changes sign exactly once on $(0,C)$ and changes sign exactly once on $[C,\infty)$.

(iii) If $-\infty < m < 0$, then

$$\theta''(C) > 0, \quad \theta''(D) < 0,$$ (3.20)

and $\theta''(u)$ changes sign exactly once on $(0,C)$ and changes sign exactly once on $[C,D)$.

**Lemma 3.5.** Consider (1.1) with fixed $m < 1$ and $m \neq 0$. If $0 < \epsilon \leq \frac{6}{l} \epsilon_{tr}^{Sem}(m)$, then $-H'(C) > \frac{C}{2}$.
Proof of Lemma 3.5. Let $-\infty < m < 1$ and $m \neq 0$. By (3.10) and (3.5), we compute and obtain that

$$\theta'(C)|_{\epsilon = \frac{6}{7} \epsilon_{l_{tr}}^{S_{sem}}(m)} = \frac{(1 - m) \left[ \frac{m(m-1+\sqrt{1-m})}{(1+\sqrt{1-m})^2(1-m)} \right]^m \left[ 2 - 2\sqrt{1-m} + (-2 + \sqrt{1-m})m \right]}{6^n \sqrt{n-m(m-1+\sqrt{1-m})^2}}$$

is an increasing function of $m < 1$, see Figure 3.

![Graph of $\theta'(C)$](image)

Figure 3. Graph of $\theta'(C)$ when $\epsilon = \frac{6}{7} \epsilon_{l_{tr}}^{S_{sem}}(m)$ for $-\infty < m < 1$.

We then compute that

$$\lim_{m \to 1^-} \theta'(C)|_{\epsilon = \frac{6}{7} \epsilon_{l_{tr}}^{S_{sem}}(m)} = -\frac{e^{7/6}}{6} \approx -0.535.$$ 

Thus

$$\theta'(C)|_{\epsilon = \frac{6}{7} \epsilon_{l_{tr}}^{S_{sem}}(m)} < -\frac{1}{2}, \quad \text{for } -\infty < m < 1,$$ 

and hence by (3.18) and (3.21), for $0 < \epsilon \leq \frac{6}{7} \epsilon_{l_{tr}}^{S_{sem}}(m)$, we have that

$$\theta'(C) < -\frac{1}{2}. \quad (3.22)$$

Then by (3.15) and (3.22), we obtain that $-H'(C) = -C\theta'(C) > \frac{C}{2}$. The proof of Lemma 3.5 is complete.

Lemma 3.6. Consider (1.1) with $-\infty < m < 1$ and $m \neq 0$. If $0 < \epsilon \leq \frac{6}{7} \epsilon_{l_{tr}}^{S_{sem}}(m)$, then $C > \frac{13}{5} A$.

Proof of Lemma 3.6. For $-\infty < m < 1$ and $m \neq 0$, we compute and obtain that

$$C - \frac{13}{5} A = \frac{m(-10 + 10\sqrt{1-m} - 3m + 16m\epsilon - 3m^2\epsilon) + 13m^2\sqrt{1 - 4\epsilon + 2m\epsilon + \epsilon^2m^2}}{10(1-m)m^2\epsilon^2} > 0$$

$$ \frac{6}{7} \epsilon_{l_{tr}}^{S_{sem}}(m)$$
if
\[ K \equiv (13m^2\sqrt{1 - 4\epsilon + 2m\epsilon + \epsilon^2m^2})^2 - \left[ m(-10 + 10\sqrt{1 - m} - 3m + 16m\epsilon - 3m^2\epsilon) \right]^2 > 0. \] (3.23)

Notice that, if \(-\infty < m < 1\) and \(m \neq -\frac{8}{5}\), then we obtain that
\[ K = 4m^4(-64 + 24m + 40m^2)(\epsilon - \hat{\epsilon})(\epsilon - \check{\epsilon}) \]

where
\[ \hat{\epsilon} = \frac{-m(80 - 80\sqrt{1 - m} - 160m + 15m\sqrt{1 - m} + 80m^2)}{2m^2(-64 + 24m + 40m^2)} + \sqrt{5mF^\frac{1}{2}} > 0, \]
\[ \check{\epsilon} = \frac{-m(80 - 80\sqrt{1 - m} - 160m + 15m\sqrt{1 - m} + 80m^2)}{2m^2(-64 + 24m + 40m^2)} - \sqrt{5mF^\frac{1}{2}} < 0, \]

(see Figure 4) and
\[ F = -1280 + 1280(1 - m) - 3648m + 5408\sqrt{(1 - m)m} - 480(1 - m)m + 11136m^2 - 5408\sqrt{(1 - m)m^2} + 45(1 - m)m^2 - 6208m^3 > 0. \]

Observe that
\[ -64 + 24m + 40m^2 \begin{cases} < 0 & \text{on } (-\frac{8}{5}, 1), \\ > 0 & \text{on } (-\infty, -\frac{8}{5}). \end{cases} \] (3.24)

Thus by (3.23) and (3.24), we obtain the following results (i)–(iii), see Figure 4.

(i) For \(m \in (-\frac{8}{5}, 1)\) and \(\hat{\epsilon} < 0 < \epsilon \leq \frac{6}{7}e_{Sem}^t(m) < \check{\epsilon}\), we have that \(K = 4m^4(-64 + 24m + 40m^2)(\epsilon - \hat{\epsilon})(\epsilon - \check{\epsilon}) > 0\). This implies \(C > \frac{13}{5}A\) by (3.23).
(ii) For \( m = -\frac{8}{5} \) and \( 0 < \epsilon \leq \frac{6}{7} \epsilon_{tr}^\text{Sem}(m) = \frac{6}{7} \left( \frac{45}{32} - \frac{5}{32} \sqrt{65} \right) \approx 0.125 \), we compute that

\[
K = \frac{46592}{125} + \frac{6656 \sqrt{13}}{25} + \left( \frac{212992}{125} \sqrt{\frac{13}{5}} - \frac{5537792}{625} \right) \epsilon
\]

\[
> \frac{46592}{125} + \frac{6656 \sqrt{13}}{25} + \left( \frac{212992}{125} \sqrt{\frac{13}{5}} - \frac{5537792}{625} \right) \frac{6}{7} \left( \frac{45}{32} - \frac{5}{32} \sqrt{65} \right)
\]

(note that \( \frac{212992}{125} \approx 6113.0 < 0 \))

\[
\approx 34.306 > 0.
\]

This implies \( C > \frac{13}{5} A \) by (3.23).

(iii) For \( m \in (-\infty, -\frac{8}{5}) \) and \( 0 < \epsilon \leq \frac{6}{7} \epsilon_{tr}^\text{Sem}(m) \approx \frac{45}{32} - \frac{5}{32} \sqrt{65} \), we have that \( K = 4m^4(-64 + 24m + 40m^2)(\epsilon - \hat{\epsilon})(\epsilon - \hat{\epsilon}) > 0 \). This implies \( C > \frac{13}{5} A \) by (3.23). The proof of Lemma 3.6 is complete.

**Lemma 3.7.** Consider (1.1) with \(-\infty < m < 1 \) and \( m \neq 0 \). If \( 0 < \epsilon < \epsilon_{tr}^\text{Sem}(m) \), then:

(i) If \( 0 < m < 1 \), then \( H^m(C) > 0 \) and there exists a number \( J \in (0, C) \) such that

\[
H^m(u) \begin{cases}
< 0 & \text{on } (0, J), \\
= 0 & \text{when } u = J, \\
> 0 & \text{on } (J, \infty).
\end{cases}
\]

(ii) If \(-\infty < m < 0 \), then \( H^m(C) > 0 \) and \( H^m(D) < 0 \), and there exist two numbers \( J_1 \in (0, C) \) and \( J_2 \in (C, D) \) such that

\[
H^m(u) \begin{cases}
< 0 & \text{on } (0, J_1) \cup (J_2, D], \\
= 0 & \text{when } u = J_1, J_2, \\
> 0 & \text{on } (J_1, J_2).
\end{cases}
\]

**Proof of Lemma 3.7.** First by (3.16) and (3.4), we compute that

\[
H^m(u) = 2\theta^m(u) + u\theta^m(u)
\]

\[
= ue^{\frac{\nu}{1+\epsilon}}(1 + eu)^{m-6} \left\{ m \left( 1 - m^2 \right) \epsilon^6 u^4 + 3m \left( 1 - m \right) \left( 2\epsilon + m\epsilon + 1 \right) \epsilon^4 u^3 + 3 \left( 1 - m \right) \left( m^2 \epsilon^2 + 4m\epsilon^2 + 2m\epsilon + 1 \right) \epsilon^2 u^2
\right.
\]

\[
+ \left[ \left( 10 - 9m - m^2 \right) me^3 + \left( 12 - 9m - 3m^2 \right) \epsilon^2 - 3m\epsilon - 1 \right] u
\]

\[
+ 3m \left( 1 - m \right) \epsilon^2 + 6 \left( 1 - m \right) \epsilon - 3 \left\} \right.
\]

\[
≡ ue^{\frac{\nu}{1+\epsilon}}(1 + eu)^{m-6} Q(u)
\]
where
\[
Q(u) = m \left(1 - m^2\right) \epsilon^6 u^4 + 3m (1 - m) (2\epsilon + m\epsilon + 1) \epsilon^4 u^3 \\
+ 3 (1 - m) \left(m^2 \epsilon^2 + 4m\epsilon^2 + 2m\epsilon + 2\epsilon + 1\right) \epsilon^2 u^2 \\
+ \left[10 - 9m - m^2\right] m\epsilon^3 + \left(12 - 9m - 3m^2\right) \epsilon^2 - 3m\epsilon - 1\right] u \\
+ 3m (1 - m) \epsilon^2 + 6 (1 - m) \epsilon - 3.
\]

Moreover, we computed that
\[
Q(0) = 3m (1 - m) \epsilon^2 + 6 (1 - m) \epsilon - 3 = 3m (1 - m) (\epsilon - \mu)(\epsilon - \nu) \quad (3.29)
\]

where
\[
\mu = \frac{m - 1 + \sqrt{1 - m}}{m(1 - m)} \quad \text{and} \quad \nu = \frac{m - 1 - \sqrt{1 - m}}{m(1 - m)}.
\]

(i) Let \(0 < m < 1\) and \(0 < \epsilon < \epsilon_{\text{tr}}^{\text{Sem}}(m)\). By (3.27), (3.13) and (3.19), we have that
\[
H'''(C) = 2\theta'''(C) + C\theta'''(C) = C\theta'''(C) > 0,
\]
and hence by (3.28),
\[
Q(C) > 0. \quad (3.30)
\]

It is easy to check that \(\nu < 0 < \epsilon_{\text{tr}}^{\text{Sem}}(m) < \mu\). Thus by (3.29), we have that
\[
Q(0) < 0. \quad (3.31)
\]

We compute that
\[
Q''(u) = m \left(1 - m^2\right) \epsilon^6 u^4 + 3m (1 - m) (2\epsilon + m\epsilon + 1) \epsilon^4 u^3 \\
+ 3 (1 - m) \left(m^2 \epsilon^2 + 4m\epsilon^2 + 2m\epsilon + 2\epsilon + 1\right) \epsilon^2 u^2 \\
> 0 \quad (3.32)
\]
since it is easy to see that all coefficients of terms \(u^4, u^3\) and \(u^2\) are all positive. Hence \(Q(u)\) is always concave up on \((0, \infty)\). So by (3.30)–(3.32), there exists a number \(J \in (0, C)\) such that
\[
Q(u) \begin{cases} 
< 0 & \text{on } (0, J), \\
= 0 & \text{when } u = J, \\
> 0 & \text{on } (J, \infty).
\end{cases}
\]

Thus by (3.28), (3.25) holds.

(ii) Let \(-\infty < m < 0\) and \(0 < \epsilon < \epsilon_{\text{tr}}^{\text{Sem}}(m)\). By (3.27), (3.14) and (3.20), we have that
\[
H'''(C) = 2\theta'''(C) + C\theta'''(C) = C\theta'''(C) > 0,
\]
\[
H'''(D) = 2\theta'''(D) + D\theta'''(D) = D\theta'''(D) < 0.
\]

Thus by (3.28), we obtain that
\[
Q(D) < 0 < Q(C). \quad (3.33)
\]

It is easy to check that \(\epsilon_{\text{tr}}^{\text{Sem}}(m) < \mu < \nu\). Thus by (3.29), we have that
\[
Q(0) < 0. \quad (3.34)
\]
In addition, we compute that
\[ Q\left(-\frac{1}{\epsilon}\right) = \frac{1}{\epsilon} > 0. \quad (3.35) \]

We consider the following three cases as follows:

Case (i). \(-1 < m < 0\). In this case, \(Q(u)\) is a quartic polynomial with negative leading coefficient \(m(1-m^2)\epsilon^6\). Thus by (3.34) and (3.35), \(Q(u)\) has (exactly) two distinct negative zeros. In addition, by (3.33) and (3.34), there exist two numbers \(J_1 \in (0, C)\) and \(J_2 \in (C, D)\) such that
\[
Q(u) \begin{cases} < 0 & \text{on } (0, J_1) \cup (J_2, D], \\ = 0 & \text{when } u = J_1, J_2, \\ > 0 & \text{on } (J_1, J_2). \end{cases} \quad (3.36)
\]

Case (ii). \(m = -1\). In this case, \(Q(u)\) is a cubic polynomial with negative leading coefficient \(-6(\epsilon + 1)\epsilon^4\). Thus by (3.33) and (3.34), \(Q(u)\) has (exactly) one negative zeros. In addition, by (3.33) and (3.34), there exist two numbers \(J_1 \in (0, C)\) and \(J_2 \in (C, D)\) such that the result in (3.36) holds.

Case (iii). \(-\infty < m < -1\). In this case, \(Q(u)\) is a quartic polynomial with positive leading coefficient \(m(1-m^2)\epsilon^6\). Thus by (3.33) and (3.34), \(Q(u)\) has (exactly) one negative zeros. In addition, by (3.33) and (3.34), there exist two numbers \(J_1 \in (0, C)\) and \(J_2 \in (C, D)\) such that the result in (3.36) holds.

Hence by (3.28), for Cases (i)–(iii), (3.26) always holds. The proof of Lemma 3.7 is complete.

**Lemma 3.8.** Consider (1.1) with \(-\infty < m < 1\) and \(m \neq 0\). If \(0 < \epsilon \leq \frac{6}{7} \epsilon_{tr}^{Sem}(m)\), then \(H(C) \leq 0\).

**Proof of Lemma 3.8.** Let \(-\infty < m < 1\) and \(m \neq 0\). By Lemma 3.7 and (3.17), for \(0 < \epsilon \leq \frac{6}{7} \epsilon_{tr}^{Sem}(m)\), there exists a number \(M \in (0, C)\) such that the function \(H'\) satisfies:
\[
(H')''(u) < 0 \text{ for } u \in (0, M), \quad (H')''(u) > 0 \text{ for } u \in (M, C), \\
H'(0) = 0, \quad (H')'(0) > 0, \quad H'(C) < 0, \quad (H')'(C) < 0. \quad (3.37)
\]

Let \(U = (A, 0), P = (C, 0), Q = (C, H'(C))\). Then the tangent line of \(y = H'(u)\) at \(U = (A, 0)\) intersects the line through the points \(P\) and \(Q\) at some point \(V = (C, C_0)\). There are four cases to be considered as follows:
Figure 5. Four possible graphs of $H'(u)$ on $[0, C]$. (i) $H'''(A) < 0$ and $H'(C) \leq C_0$. (ii) $H'''(A) < 0$ and $H'(C) > C_0$. (iii) $H'''(A) = 0$. (iv) $H'''(A) > 0$.

Case (i). $H'''(A) < 0$ and $H'(C) \leq C_0$. By Lemma 3.6, $C > \frac{13}{5} A > 2A$. In addition, by (3.37) the convexity of $H'(u)$ on $(0, C)$, it is easy to see that

$$0 < H(A) = \int_0^A H'(t)dt < -\int_A^{2A} H'(t)dt < -\int_A^C H'(t)dt = H(A) - H(C).$$

So $H(C) < 0$.

Case (ii). $H'''(A) < 0$ and $H'(C) > C_0$. By (3.15), (3.10), (3.13) and (3.14), we obtain that

$$H'(u) = u\theta'(u) \leq A\theta'(u) \leq A\theta'(0) = Af(0) = A \text{ for } 0 \leq u \leq A.$$  \hspace{1cm} (3.38)

By Figure 5(ii) and Lemma 3.5, we have that

$$\text{area}(\Delta U PQ) = \frac{(C - A) [-H'(C)]}{2} > \frac{C^2 - AC}{4}.$$ \hspace{1cm} (3.39)

Thus by (3.38), (3.39) and Figure 5(ii),

$$\int_0^A H'(t)dt < A^2 \quad \text{and} \quad -\int_A^C H'(t)dt > \text{area}(\Delta U PQ) > \frac{C^2 - AC}{4}.$$

Moreover, by Lemma 3.6, we obtain that $A < \frac{5}{13} C$ and

$$H(C) = \int_0^A H'(t)dt + \int_A^C H'(t)dt < A^2 + \frac{AC - C^2}{4} = \frac{4A^2 + AC - C^2}{4} < \frac{1}{169} C^2 < 0.$$
Case (iii). $H'''(A) = 0$. By (3.37), we obtain that $H'(C) > C_0$. Applying the same arguments in Case (ii), we are able to obtain that $H(C) < 0$.

Case (iv). $H'''(A) > 0$. By (3.37), we obtain that $H'(C) > C_0$. Applying the same arguments in Case (ii), we are able to obtain that $H(C) < 0$.

In all Cases (i)–(iv), we obtain that $H(C) < 0$. The proof of Lemma 3.8 is complete.

The following lemma follows immediately by slight modification of the proof of Hung & Wang [7, Theorem 2.1]; we omit the proof.

**Lemma 3.9.** Consider (1.1) with $-\infty < m < 0$. If $0 < \epsilon \leq \frac{6}{7} \epsilon^{Sem}(m)$, the bifurcation curve $S$ is S-shaped if

$$\theta(D) > \theta(A) (> 0). \quad (3.40)$$

**Lemma 3.10.** Consider (1.1) with $(-4.103 \approx \tilde{m} \leq m < 0$. If $0 < \epsilon \leq \frac{8}{9} \epsilon^{Sem}(m)$, then $\theta(D) > \theta(A) (> 0)$.

The proof of Lemma 3.10 is put in Appendix.

**Lemma 3.11.** Consider (1.1) with fixed $m < 1$ and $m \neq 0$. If $0 < \epsilon \leq \frac{8}{9} \epsilon^{Sem}(m)$, then $-H'(C) > \frac{9}{25} C$.

The proof of Lemma 3.11 follow by the similar arguments in the proof of Lemma 3.5; we omit the proof.

**Lemma 3.12.** Consider (1.1) with $-\infty < m < 1$ and $m \neq 0$. If $0 < \epsilon \leq \frac{8}{9} \epsilon^{Sem}(m)$, then $C > \frac{12}{5} A$ and $B > 4A$.

The proof of Lemma 3.12 follow by the same arguments in the proof of Lemma 3.6; we omit the proof.

**Lemma 3.13.** Consider (1.1) with $-\infty < m < 1$ and $m \neq 0$. If $0 < \epsilon \leq \frac{8}{9} \epsilon^{Sem}(m)$, then:

(i) $H'''(B) > 0$,

(ii) $H(B) \leq 0$.

**Proof of Lemma 3.13.** (i) By (3.27), (3.14) and (3.4), we compute that

$$H'''(B) = 2\theta''(B) + \theta''(B) > \theta''(B) \quad (3.41)$$

and

$$\theta''(u) = e^{\frac{u}{1 + \epsilon u}}(1 + \epsilon u)^{m-6}[-(1 + 2\epsilon - 2\epsilon m + \epsilon^2 m - \epsilon^2 m^2)
- (1 - 4\epsilon + 3\epsilon m - 3\epsilon^2 m - 2\epsilon^3 m + 3\epsilon^2 m^2 + \epsilon^3 m^2 + \epsilon^3 m^3) u
+ \epsilon^2(5 - 6\epsilon - 3m + 12\epsilon m + 6\epsilon m^2 + 3\epsilon^2 m^2 - 3\epsilon^2 m^3) u^2
- \epsilon^4(4 - 7m + 2\epsilon m + 3m^2 - 5\epsilon m^2 + 5\epsilon m^3) u^3 - \epsilon^5 m(1 - m^2) u^4]
\equiv e^{\frac{u}{1 + \epsilon u}}(1 + \epsilon u)^{m-6}R(u) \quad (3.42)$$
where
\[
R(u) = (-1 + 2\epsilon - 2\epsilon m + \epsilon^2 m - \epsilon^2 m^2) \\
\quad + (1 - 4\epsilon + 3\epsilon m - 2\epsilon^2 m + 3\epsilon m^2 - \epsilon^2 m^2 + \epsilon^3 m^3)u \\
\quad + \epsilon^2(5 - 6\epsilon - 3m + 12\epsilon m - 6\epsilon m^2 + 3\epsilon^2 m^2 - 3\epsilon^2 m^3)u^2 \\
\quad - \epsilon^4(4 - 7m + 2\epsilon m + 3m^2 - 5\epsilon m^2 + 5\epsilon m^3)u^3 - \epsilon^6 m(1 - m)^2 u^4.
\]

By (3.11), we compute that
\[
R(A + B) = \frac{1}{(m - 1)^2\epsilon}[4 - 13\epsilon + 10\epsilon^2 + (3 - 2\epsilon - 3\epsilon^2)\epsilon m + \epsilon^3 m^2]
\]
\[
= \frac{\epsilon^2}{(m - 1)^2(m - m_1)(m - m_2)}
\]  
(3.43)

where
\[
m_1 = \frac{-3 + 2\epsilon + 3\epsilon^2 + (\epsilon - 1)\sqrt{9 - 10\epsilon + 9\epsilon^2}}{2\epsilon^2},
\]
\[
m_2 = \frac{-3 + 2\epsilon + 3\epsilon^2 - (\epsilon - 1)\sqrt{9 - 10\epsilon + 9\epsilon^2}}{2\epsilon^2}.
\]

Notice that \(\frac{s_{\operatorname{sem}}}{e_{\operatorname{tr}}}(\epsilon)\) is a strictly increasing function of \(m < 1\), and we can solve \(\epsilon = \frac{s_{\operatorname{sem}}}{e_{\operatorname{tr}}}(\epsilon)\) as \(m = m_0 = \frac{4(-2+3\sqrt{2})}{9\epsilon_{\operatorname{tr}}^2}\). We thus find that \(m_0 > m_1 > m_2\) for \(0 < \epsilon \leq \frac{8}{9}\), see Figure 6. Thus for \(0 < \epsilon \leq \frac{8}{9}\), we have that \(R(A + B) > 0\) by (3.43). Thus by (3.42),
\[
\theta''(A + B) > 0.
\]  
(3.44)

![Figure 6. Graphs of \(m_1\) and \(m_2\) for \(0 < \epsilon \leq 1\), and of \(\epsilon = \frac{s_{\operatorname{sem}}}{e_{\operatorname{tr}}}(\epsilon)\) for \(0 < \epsilon \leq 8/9\).](image)

Case (I). If \(0 < m < 1\), we obtain \(\theta''(B) > 0\) immediately by Lemma 3.4(ii) and (3.44). Thus by (3.41), \(H''(B) > 0\).

Case (II). If \(-\infty < m < 0\), it can be checked that \(C < B < A + B < D\). Thus we obtain \(\theta''(B) > 0\) immediately by Lemma 3.4(iii) and (3.44). So by (3.41), \(H''(B) > 0\).
(ii) The proof of $H(B) \leq 0$ for $0 < \epsilon \leq \frac{8}{9} \epsilon_{tr}^{Sem}(m)$ follows by Lemma 3.13(i) and the similar arguments in the proof of $H(C) \leq 0$ for $0 < \epsilon \leq \frac{6}{7} \epsilon_{tr}^{Sem}(m)$ in Lemma 3.8; we omit the proof. The proof of Lemma 3.13 is complete.

The following lemma follows by the same arguments of Hung & Wang [7, Lemma 3.5]; we omit the proof.

**Lemma 3.14.** Consider (1.1) with $m = 1/2$. Then there exists a positive constant $\epsilon_4 < \epsilon_{tr}^{Sem}(m) = 6 - 4\sqrt{2} \approx 0.343$ satisfying

$$H(C(\epsilon_4)) = 0 \text{ and } H(C(\epsilon)) < 0 \text{ for all } 0 < \epsilon < \epsilon_4. \quad (3.45)$$

(Numerical simulation shows that $\epsilon_4 \approx 0.328$).

4. **Proof of Theorem 2.1**

(i) For $0 < m < 1$, if $0 < \epsilon \leq \frac{6}{7} \epsilon_{tr}^{Sem}(m)$, then we obtain that $H(C) \leq 0$ by Lemma 3.8. Thus by Lemma 3.3(i), the bifurcation curve $S$ is S-shaped on the $(\lambda, \|u\|_{\infty})$-plane. For $(-4.103 \approx \tilde{m} \leq m < 0$, if $0 < \epsilon \leq \frac{6}{7} \epsilon_{tr}^{Sem}(m)$, we obtain that $\theta(D) > \theta(A)$ by Lemma 3.10. Thus by Lemma 3.9, the bifurcation curve $S$ is S-shaped on the $(\lambda, \|u\|_{\infty})$-plane. In particular, for $m = 1/2$, the result in Theorem 2.1(i) follows by Lemma 3.14. Hence Theorem 2.1(i) holds.

(ii) For $0 < m < 1$, if $0 < \epsilon \leq \frac{8}{9} \epsilon_{tr}^{Sem}(m)$, then we obtain that $H(B) \leq 0$ by Lemma 3.13(ii). Thus by Lemma 3.3(ii), the bifurcation curve $S$ is S-like shaped on the $(\lambda, \|u\|_{\infty})$-plane immediately. For $-\infty < m < 0$, if $0 < \epsilon \leq \frac{8}{9} \epsilon_{tr}^{Sem}(m)$, we obtain that $H(B) \leq 0$ by Lemma 3.13(ii). Recall $T'(\alpha)$ defined in (3.8). Applying the same arguments used in Hung & Wang [7, Theorem 2.2], it can be proved that $T'(B) < 0$. So by (3.3) obtained by Lemma 3.1, the time map $T(\alpha)$ has at least two critical points on $(0, \infty)$. So the bifurcation curve $S$ is S-like shaped on the $(\lambda, \|u\|_{\infty})$-plane. Hence Theorem 2.1(ii) holds.

The proof of Theorem 2.1 is complete.

5. **Final Remarks**

We finish this paper by giving next two remarks.

**Remark 5.1.** For each fixed $m \in (0, 1)$, by applying similar arguments as we did in Lemma 3.14; that is, Hung & Wang [7, Lemma 3.5], it can be shown that the following assertion (i) and (ii) hold:

(i) There exists a constant $\epsilon^*_m \in (\frac{8}{9} \epsilon_{tr}^{Sem}(m), \epsilon_{tr}^{Sem}(m))$ satisfying $H(C(\epsilon^*_m)) = 0$ and $H(C(\epsilon)) < 0$ for all $0 < \epsilon < \epsilon^*_m$.

(ii) There exists a constant $\epsilon^{**} \in (\epsilon^*_m, \epsilon_{tr}^{Sem}(m))$ satisfying $H(B(\epsilon^{**})) = 0$ and $H(B(\epsilon)) < 0$ for all $0 < \epsilon < \epsilon^{**}_m$.

Thus by Lemma 3.3, for $0 < m < 1$, on the $(\lambda, \|u\|_{\infty})$-plane, the bifurcation curve $S$ is S-shaped if $0 < \epsilon \leq \epsilon^*_m$, and is S-like shaped if $\epsilon^*_m < \epsilon \leq \epsilon^{**}$.
Remark 5.2. For Lemma 3.9, we observe that:

(i) For \(-20 < m < 0\) and \(0 < \epsilon \leq \frac{6}{7} \epsilon_0^{\text{semi}}(m)\), numerical simulation shows that assertion (3.40) holds, and hence our arguments are able to show that the bifurcation curve \(S\) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane.

(ii) However, for \(m < 0\) negatively large enough, assertion (3.40) do not hold. For example, let \(m = -25\) and \(0 < \epsilon < 0.001\), then numerical simulation shows that \(\theta(D(\epsilon)) < 0\). Thus our arguments does not apply to show that the bifurcation curve \(S\) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane. Further investigation is needed.

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6. Appendix

Proof of Lemma 3.10.

We divide the proof into the following four steps 1–4:
Step 1. \( f(D/2) \geq f(D) \).
Step 2. \( D > 4 \).
Step 3. \( \theta(D) > 2 \).
Step 4. \( 2 > \theta(A) > 0 \).

Thus we obtain \( \theta(D) > 2 > \theta(A) > 0 \) and the proof of Lemma 3.10 is complete by Steps 3 and 4.

**Proof of Step 1.** By (1.2) and (3.6), we compute that

\[
\frac{\partial}{\partial \epsilon} \frac{f(D/2)}{f(D)} = m^3(1-m)^2 \left[ \frac{1-m+\sqrt{1-m}}{1+\sqrt{1-m}+m^2(1+\epsilon)} \right]^{-m} \frac{e^{(m-1)(1-m)+m-m^2\epsilon-m(1+\epsilon)}}{2^m \left[ 1+\sqrt{1-m+m^2\epsilon-m(1+\epsilon)} \right]^2} \left[ \epsilon - \frac{m-1+\sqrt{1-m}}{m(1-m)} \right] > 0 \text{ for } m < 0 \text{ and } 0 < \epsilon \leq \frac{6}{7} \epsilon^{\text{Sem}} \left( m \right) < \frac{m-1+\sqrt{1-m}}{m(1-m)}. \quad (6.1)
\]

Thus \( \frac{f(D/2)}{f(D)} \) is a strictly increasing function of \( \epsilon \in (0, \frac{6}{7} \epsilon^{\text{Sem}} \left( m \right)] \) for fixed \( m < 0 \). Setting \( \epsilon = 0 \) into \( \frac{f(D/2)}{f(D)} \), we obtain that

\[
\left. \frac{f(D/2)}{f(D)} \right|_{\epsilon=0} = 2^{-m} e^{\frac{m(1-m)(1-m+\sqrt{1-m})}{(1-m+\sqrt{1-m})^2}} \begin{cases} > 1 & \text{on } (\tilde{m}, 0), \\ = 1 & \text{when } m = \tilde{m}, \\ < 1 & \text{on } (-\infty, \tilde{m}), \end{cases} \quad (6.2)
\]

where constant \( \tilde{m} \approx -4.103 \). Thus by (6.1) and (6.2), for \( \tilde{m} \leq m < 0 \) and \( 0 < \epsilon \leq \frac{6}{7} \epsilon^{\text{Sem}} \left( m \right) \), we have that \( f(D/2) \geq f(D) \).

**Proof of Step 2.** Let \( \tilde{m} \leq m < 0 \) and \( 0 < \epsilon \leq \frac{6}{7} \epsilon^{\text{Sem}} \left( m \right) \). By (3.6), we compute that

\[
D - 4 = -4m \sqrt{1-m} e^2 + m \sqrt{1-m} + \sqrt{1-m} + 1 \quad = -4 e^2 (\epsilon - \xi)(\epsilon - \eta) \quad (6.3)
\]

where

\[
\xi = \frac{-m \sqrt{1-m} + \sqrt{-16m - 16m \sqrt{1-m} + 17m^2 - m^3}}{8m \sqrt{-m + 1}} > 0,
\]
\[
\eta = \frac{-m \sqrt{1-m} - \sqrt{-16m - 16m \sqrt{1-m} + 17m^2 - m^3}}{8m \sqrt{-m + 1}} < 0.
\]
Since \(0 < \epsilon \leq \frac{6}{7} \epsilon_{tr}^e(m)\), we obtain that \(\eta < 0 < \epsilon \leq \frac{6}{7} \epsilon_{tr}^e(m) < \xi\) for \(\tilde{m} \leq m < 0\), see Figure 7. Thus by (6.3), we have that \(D > 4\).

**Proof of Step 3.** Let \(\tilde{m} \leq m < 0\) and \(0 < \epsilon \leq \frac{6}{7} \epsilon_{tr}^e(m)\). First we compute that \(f(0) = 1\) and

\[
f'(u) = \frac{f(u)}{(1 + \epsilon u)^2} \left(me^2u + m\epsilon + 1\right) \begin{cases} > 0 & \text{on } [0, \rho), \\ = 0 & \text{when } u = \rho = \frac{-(1 + me^2)}{me^2}, \\ < 0 & \text{on } (\rho, \infty). \end{cases}
\]

(6.4)

It is easy to show that \(D/2 < \rho < D\); we omit the proof. Thus by (6.4), we have that

\[
f(u) > 1 \quad \text{on } (0, D/2)
\]

(6.5)

and

\[
f'(D/2) > 0, \quad f'(D) < 0.
\]

(6.6)

By Step 1, (6.5) and (6.6), we obtain the following Figure 8 on two possible graphs of \(f(u)\) with \(f(D/2) \geq f(D)\).

Figure 8. Two possible graphs of \(f(u)\) with \(f(D/2) \geq f(D)\). (i) \(f(D) \leq 1\). (ii) \(f(D) > 1\).
Case (i). Suppose \( f(D) \leq 1 \) (see Figure 8(i)). Then by (3.9), (6.5) and Step 2, we obtain that

\[
\theta(D) = 2 \int_0^D f(t)dt - Df(D)
\]

\[
= \int_0^D f(t)dt + \int_0^D [f(t) - f(D)] dt + \int_0^D f(D)dt - Df(D)
\]

\[
= \int_0^D f(t)dt + \int_0^D [f(t) - f(D)] dt
\]

\[
> \int_0^D f(t)dt > \int_0^{D/2} f(t)dt > D/2 > 2.
\]

Case (ii). Suppose \( f(D) > 1 \) (see Figure 8(ii)). Then by (6.4) and (6.6), there exists a positive number \( \tilde{D} \leq D/2 \) such that \( f(\tilde{D}) = f(D) \). Thus by (3.9), (6.5) and Step 2, we obtain that

\[
\theta(D) = 2 \int_0^D f(t)dt - Df(D)
\]

\[
= 2\left( \int_0^{\tilde{D}} f(t)dt + \int_{\tilde{D}}^{D/2} f(D)dt + \int_{D/2}^{D} f(D)dt + \int_{D/2}^{\tilde{D}} [f(t) - f(D)] dt \right)
\]

\[
- \left( \int_0^{D/2} f(D)dt + \int_{D/2}^{D} f(D)dt \right)
\]

\[
= 2\left( \int_0^{\tilde{D}} f(t)dt + \int_{\tilde{D}}^{D/2} f(D)dt + \int_{D/2}^{\tilde{D}} [f(t) - f(D)] dt \right)
\]

\[
> \int_0^{\tilde{D}} f(t)dt + \int_{\tilde{D}}^{D/2} f(D)dt > \int_0^{\tilde{D}} 1dt + \int_{\tilde{D}}^{D/2} 1dt = D/2 > 2.
\]

Proof of Step 4. Let \( \tilde{m} \leq m < 0 \) and \( 0 < \epsilon \leq \frac{6}{7} \epsilon_{\text{crit}}(m) \). First we show that \( \frac{\partial}{\partial \epsilon} \theta(A(\epsilon)) > 0 \). By (1.2), (3.9) and (3.10), we have that \( \theta(A(\epsilon)) = \theta(\epsilon, A(\epsilon)) \) and \( \frac{\partial}{\partial \epsilon} \theta(\epsilon, A(\epsilon)) = 0 \). In
addition, we compute that
\[
\frac{\partial}{\partial \epsilon} \theta(A(\epsilon)) = \frac{\partial}{\partial \epsilon} \theta(\epsilon, A(\epsilon)) + \frac{\partial}{\partial u} \theta(\epsilon, A(\epsilon)) \frac{\partial A(\epsilon)}{\partial \epsilon}
\]
\[
= \frac{\partial}{\partial \epsilon} \theta(\epsilon, A(\epsilon))
\]
\[
= \left[ \frac{\partial}{\partial \epsilon} \left( 2 \int_0^u f(t) dt - uf(u) \right) \right]_{u=A(\epsilon)}
\]
\[
= \left[ 2 \int_0^u f(t) \frac{t(m-t+\epsilon m t)}{(1+\epsilon t)^2} dt - uf(u) \frac{u(m-u+\epsilon m u)}{(1+\epsilon u)^2} \right]_{u=A(\epsilon)}
\]
\[
= \left[ 2 \int_0^u \tilde{f}(t) dt - u \tilde{f}(u) \right]_{u=A(\epsilon)}
\]
\[
= \tilde{\theta}(A(\epsilon))
\]
(6.7)

where
\[
\tilde{\theta}(u) \equiv 2 \int_0^u \tilde{f}(t) dt - u \tilde{f}(u)
\]
\[
\tilde{f}(u) \equiv f(u) \frac{u(m-u+\epsilon m u)}{(1+\epsilon u)^2}.
\]

We compute that \( \tilde{\theta}(0) = 0 \) and
\[
\tilde{\theta}'(u) = \tilde{f}(u) - uf'(u)
\]
\[
= \frac{u^2 f(u)}{(1+\epsilon u)^4} e^2 (-1 + m + m\epsilon - m^2\epsilon) (u-u_1)(u-u_2)
\]
\[
> 0 \quad \text{for } 0 < u < u_2 \quad \text{(and since } m < 0, \quad -1 + m + m\epsilon - m^2\epsilon < 0) \]

where
\[
u_2 \equiv \frac{1 + 2m^2 - 2m^2 \epsilon^2 + \sqrt{1 + 4\epsilon^2 - 4m^2 \epsilon^2}}{2\epsilon^2(1 - m - m\epsilon + m^2\epsilon)} > 0,
\]
\[
u_1 \equiv \frac{1 + 2m^2 - 2m^2 \epsilon^2 - \sqrt{1 + 4\epsilon^2 - 4m^2 \epsilon^2}}{2\epsilon^2(1 - m - m\epsilon + m^2\epsilon)} < 0
\]

for \( \tilde{m} \leq m < 0 \) and \( 0 < \epsilon \leq \frac{6}{7} \epsilon_{tr} \)\( \text{Sem} \); we omit the proof. So
\[
\tilde{\theta}(u) > 0 \quad \text{for } 0 < u < u_2.
\]
(6.8)

It is easy to check that
\[
0 < A(\epsilon) = \frac{1 - 2\epsilon + m\epsilon - \sqrt{1 - 4\epsilon + 2m\epsilon + m^2 \epsilon^2}}{2\epsilon^2(1 - m)} < u_2
\]
(6.9)

for \( \tilde{m} \leq m < 0 \) and \( 0 < \epsilon \leq \frac{6}{7} \epsilon_{tr} \)\( \text{Sem} \); we omit the proof. Thus by (6.7)–(6.9),
\[
\frac{\partial}{\partial \epsilon} \theta(A(\epsilon)) = \tilde{\theta}(A(\epsilon)) > 0,
\]
(6.10)

and hence \( \theta(A(\epsilon)) \) is a strictly increasing function of \( \epsilon \in (0, \frac{6}{7} \epsilon_{tr} \)\( \text{Sem} \)] for fixed \( m < 0 \).
We find that

$$2 > \theta(A)_{\epsilon = \frac{6}{7} \epsilon_{t_t}^{\text{Sem}}(m)} \quad \text{for} \quad \bar{m} \leq m < 0,$$

see Figure 9. (Notice that $2 > \theta(A)_{\epsilon = \frac{6}{7} \epsilon_{t_t}^{\text{Sem}}(m)} \approx 1.566$ for $m = \bar{m}$.) Thus by (6.10) and (6.11), we obtain that $2 > \theta(A)$. Moreover, we compute that $\theta(0) = 0$. Thus by (3.10), we obtain that $\theta(A) > 0$. The proof of Lemma 3.10 is complete.

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