A refined model for spinning dust radiation

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ABSTRACT
We present a comprehensive treatment of the spectrum of electric dipole emission from spinning dust grains, updating the commonly used model of Draine & Lazarian. Grain angular velocity distributions are computed using the Fokker-Planck equation; we revisit the drift and diffusion coefficients for the major torques on the grain, including collisions, grain-plasma interactions, and infrared emission. We use updated grain optical properties and size distributions. The theoretical formalism is implemented in the companion code, SpDust, which is publicly available. The effect of some environmental and grain parameters on the emissivity is shown and analysed.

Key words: dust, extinction – radio continuum: ISM – radiation mechanisms: non-thermal.

1 INTRODUCTION
Observational cosmology has entered an area of high precision, exemplified by the most recent temperature results from sensitive cosmic microwave background (CMB) experiments (Dickinson et al. 2004; Readhead et al. 2004; Kuo et al. 2007; Hinshaw et al. 2008). However, foreground separation and removal remains a major challenge for any CMB measurement (e.g. Eriksen et al. 2008; Leach et al. 2008). In addition to the standard Galactic foregrounds, free-free, synchrotron and thermal dust emission, an unknown “anomalous” dust-correlated emission has been observed over the last decade, in the microwave region of the spectrum. The anomalous emissions was first interpreted as free-free emission from shock-heated gas by Leitch et al. (1997), but Draine & Lazarian (1998a) showed that this would require an extremely high plasma temperature and a corresponding unrealistic energy injection rate. They proposed instead two possible mechanisms to explain the anomalous microwave emission. One of them is the magnetic dipole emission from thermal fluctuations in the magnetisation of interstellar dust grains (Draine & Lazarian 1999). The other possible mechanism, on which the present work focuses, is electric dipole radiation from the smallest carbonaceous grains, described in Draine & Lazarian (1998a), hereafter DL98a. The physical principle is quite straightforward: dust grains are presumably asymmetric, and thus will have a nonzero electric dipole moment. These grains will spin due to interaction with the ambient interstellar medium (ISM) and radiation field, and thus radiate electromagnetic waves due to the rotation of their electric dipole moment. To get the electric dipole radiation spectrum, one thus needs three ingredients: the quantity of small grains, then their dipole moment, and finally their rotation rates.

Although the observational interest in electric dipole radiation from spinning dust grains has only grown in the last decade, there is a long standing history of theoretical work on the subject. Erickson (1957) was the first to consider the possibility that rotating dust grains could be the source of non-thermal radio-noise. Hoyle & Wickramasinghe (1974) showed that this process was dominated by grains with radius \( a \lesssim 10^{-6}\) cm and could lead to radio emission around 10 GHz. Ferrara & Dettman (1994) estimated the spinning dust emissivity for thermally rotating grains. The first to provide a detailed treatment of rotational excitation of small grains were Rouan et al. (1992). They considered the effect of collisions with gas atoms and absorption and emission of radiation. Anderson & Watson (1993) evaluated the effect of collisions with ions and “plasma drag” (torques due to the electric field of passing ions).

DL98b provided the first comprehensive study of the rotational dynamics of small grains, including all the previous effects. They evaluated, as a function of grain radius and environmental conditions, rotational damping and excitation rates through collisions, “plasma drag”, infrared emission, emission of electric dipole radiation, photoelectric emission and formation of H\(_2\) molecules. The spectra they provided are now widely used in interpreting ISM microwave emission (e.g. Finkbeiner 2004; Watson et al. 2005; Casassus et al. 2006, 2007, 2008; Dickinson et al. 2007, 2008; Dobler et al. 2008) and for CMB foreground analyses (e.g. Banday et al. 2003; Davies et al. 2006; Bonaldi et al. 2008).
optical properties and size distribution are used throughout the study of anomalous emission, it is timely to revisit the theory of spinning dust emission, including the approximations made in DL98b. This is the purpose of this paper.

As in DL98b, we concentrate on the rotation rate of the grains; the size distribution has been reconsidered by other authors, and the grain dipole moment distribution should be regarded as a model parameter since one cannot compute it from first principles. We first review and generalize DL98b rotational excitation and damping rates. We modify the rotational excitation and damping rates by collisions with neutral species, such that it respects detailed balance in the case where the evaporation temperature is equal to the gas temperature. We include the electric dipole potential in the case where the evaporation temperature is equal to the gas temperature. We include the electric dipole potential when evaluating the effect of collisions with ions. Full hyperbolic trajectories and rotating grains are used when computing the effect of plasma drag. We correct the infrared emission damping rate which was underestimated for a given infrared spectrum. Finally, we use these excitation and damping rates to calculate the grain rotational distribution function by solving the Fokker Planck equation. Updated grain optical properties and size distribution are used throughout this analysis. An Interactive Data Language (IDL) code implementing the formulas in this paper, SpDust, is available on the web

The paper is organized as follows. In Section 2 we remind the reader of the electric dipole radiation formula and give the resulting expected emissivity. In Section 3 we discuss the size distribution and dipole moments, along with other grain properties. We then turn to the main thrust of this study, which is the computation of the angular velocity distribution function. The theoretical formalism is exposed in Section 4, which presents the Fokker-Planck equation. Sections 5–9 discuss the various rotational damping and excitation processes: collisions with ions and neutral species, plasma drag, infrared emission, photoelectric emission, and random H₂ formation. The reader interested primarily in the predicted emission may wish to proceed directly to Section 10 where we present the resulting emissivity and the effect of various parameters and environment conditions. Our conclusions are given in Section 11. Appendix A exposes the techniques used to numerically evaluate integrals of rapidly oscillating functions involved in the plasma drag calculation. Appendix B presents an alternate, quantum mechanical derivation of the rotational damping rate through infrared emission.

2 ELECTRIC DIPOLE RADIATION

The power radiated by a dust grain spinning with an angular velocity \( \omega \), of electric dipole moment \( \mu \), with component \( \mu_{\perp} \) perpendicular to \( \omega \), is

\[
P = \frac{2}{3} \frac{\mu_{\perp}^2 \omega^4}{c^3}.
\]

This power is emitted at the frequency \( \nu = \omega/2\pi \).

To get the emissivity of electric dipole radiation per H atom, in \( \text{erg s}^{-1} \text{sr}^{-1} \text{H atom}^{-1} \), one needs several ingredients:

- The grain size distribution function: \( n_H^{-1} \frac{d\sigma_{\text{gr}}}{da} \), which gives the number of dust grains per unit size per H atom.
- The electric dipole moments as a function of grain size \( a \): \( \mu(a) \).
- The angular velocity distribution function, \( f_{\omega}(\omega) \), which depends upon the grain radius and environmental conditions. It depends on the angular velocity modulus only in a perfectly isotropic environment, with no strong electromagnetic fields forcing the dipole moments to align in some particular direction.

One then readily gets the emissivity of spinning dust grains per H atom:

\[
\frac{\dot{E}}{n_H} = \frac{1}{4\pi} \int_{a_{\text{min}}}^{a_{\text{max}}} \frac{d\sigma_{\text{gr}}}{da} \frac{4\pi \omega^2 f_{\omega}(\omega)}{2} \frac{2}{3} \frac{\mu_{\perp}^2 \omega^4}{c^3},
\]

where \( \omega = 2\pi\nu \).

3 DUST GRAINS PROPERTIES

3.1 Grain shapes

The grains are characterized by their volume-equivalent radius \( a \), such that the grain volume is \( 4\pi a^3/3 \). The radius \( a \) is in fact a measure of the number of C atoms in the grain, which we assume to be

\[
N_C = \frac{4\pi a^3 \rho_C}{3m_C} \approx 468 \, a_{\perp}^3
\]

where \( \rho_C = 2.24 \, \text{g cm}^{-3} \) is the density of ideal graphite and \( a_{\perp} \equiv a/(10^{-7} \text{ cm}) \).

We follow Draine & Li (2001), hereafter DL01, for the number \( N_H \) of H-atoms in the grains (see their Eq. 8). Following DL98b, we account for the fact that the smallest grains may be sheetlike as expected for polycyclic aromatic hydrocarbons (PAHs). We assume that this is the case for \( a < a_2 = 6 \, \text{ Å} \) (this corresponds to \( N_C \approx 100 \) carbon atoms, the size of a large PAH). We model them as disks of thickness \( d = 3.35 \, \text{ Å} \), the interlayer separation in graphite. In many cases, these grains will be rotating primarily around the axis of largest moment of inertia (Purcell 1979), which is perpendicular to the plane of the grain. When computing various cross sections, we will usually assume a spherical geometry, with a “surface-equivalent” radius \( a_{\perp} \) or a “cylindrical excitation-equivalent” radius \( a_{c_{\perp}} \), defined as :

\[
4\pi a_{\perp}^2 \equiv \int dS \quad \text{and} \quad 4\pi a_{c_{\perp}}^2 = \frac{3}{2} \int \rho^2 dS,
\]

where \( \rho = r \sin \theta \) is the distance to the axis of symmetry and dS is the surface area element.

Although the assumption of cylindrical grains for \( a < a_2 \) is not critical, it does have an effect on the spectrum, which is shown in Fig. 1.

\footnote{DL98b allow for a possible population of linear grains, although they do not actually use them.}

1 http://www.tapir.caltech.edu/~yacine/spdust/spdust.html

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Figure 1. Effect of relaxing the assumption of cylindrical grains on the spectrum, for a fiducial Cold Neutral Medium environment (CNM, defined in Eq. (153)). At equal radius, spherical grains have a smaller moment of inertia than the cylindrical ones, which are rotating primarily about their axis of largest moment of inertia. They will thus radiate at slightly higher frequencies. For the CNM, we find an increase of peak frequency $\Delta v_{\text{peak}}/v_{\text{peak}} \approx 6\%$. The high-frequency tail of the spectrum is due to the smallest dipole moments of the assumed Gaussian distribution for the intrinsic dipole moments (see Section 3.3 and Fig. 10). For a spherical distribution of dipole moments, there are fewer grains with a low dipole than for a planar distribution. This explains the decrease in power at high frequencies. For the CNM, this results in a decrease of total emitted power $\Delta j_{\text{tot}}/j_{\text{tot}} \approx -16\%$.

3.2 Size distribution

Following Weingartner & Draine (2001a), hereafter WD01a, we adopt the following size distribution for carbonaceous dust, for grain radii $a_{\text{min}} = 3.5$ Å < $a < a_{\text{max}} = 100$ Å:

$$\frac{1}{n_{\text{H}}} \frac{dn_{\text{gr}}}{da} = D(a) + C \left( \frac{a}{a_t} \right)^{\alpha} F(a; \beta, a_t)$$

$$\times \begin{cases} 1, & a_{\text{min}} < a < a_t, \\ e^{-[a-a_t]/a_c^2}, & a > a_t, \end{cases}$$

where

$$F(a; \beta, a_t) = \begin{cases} 1 + \beta a/a_t, & \beta \geq 0, \\ (1 - \beta a/a_t)^{-1}, & \beta < 0. \end{cases}$$

The function $D(a)$ describes truncated (at 3.5 Å) lognormal grain populations,

$$D(a) = \sum_{i=1}^{2} B_i \exp \left\{ -\frac{1}{2} \left[ \frac{\ln(a/a_{0,i})}{\sigma} \right]^2 \right\},$$

with the normalizations $B_i$ defined to place a total number $b_{C,i}$ of carbon atoms per hydrogen nucleus in the $i$th lognormal population. Here $b_{C,1} = 0.75b_c$, $b_{C,2} = 0.25b_c$, $b_c$ being the total carbon abundance per hydrogen nucleus in the lognormal populations, $a_{0,1} = 3.5$ Å, $a_{0,2} = 30$ Å, and $\sigma = 0.4$. This size distribution has a total of six adjustable parameters ($b_c, C, a_t, a_c, \alpha, \beta$). For a given choice of $b_c$, the other five parameters can be found in WD01b, Table 1.

We consider only carbonaceous grains in this work. The abundance of small silicate grains in the diffuse phases is limited by the absence of the 10 μm band in emission, as discussed in WD01b. Note, however, that Li & Draine (2001a) found that as much as $\sim 10\%$ of the interstellar silicate mass could be in the form of ultrasmall particles ($a \lesssim 15$ Å) without violating any existing observational constraints. While consistent with observations, our assumption is thus not required by them.

3.3 Dipole moments

Although only the component of the dipole moment perpendicular to $\omega$ is of importance for the electric dipole emission, the total dipole moment will be needed in coming calculations. In case of spherical grains, we assume the dipole moment and rotation axis are randomly oriented. For cylindrical grains, the dipole moment is mainly oriented in the plane of the grain, perpendicular to the rotation axis.

The dipole moments have two components. First, an intrinsic part $\mu_i$, which results from the addition of dipole moments from individual molecular bonds. For a given grain radius, we assume a multivariate Gaussian distribution, with variance proportional to the number of atoms in the grain, $N_{\text{at}} = N_C + N_{\text{H}}$:

$$P(\mu_i) \propto \begin{cases} \mu_i^2 e^{-\mu_i^2/(2\sigma^2)}, & \text{spherical grains} \\ \mu_i e^{-\mu_i^2/(\sigma^2)}, & \text{disklike grains} \end{cases}$$

with

$$\langle \mu_i^2 \rangle_{a=10^{-7}\text{ cm}} = (9.3\text{ debye})^2.$$  

In addition, for grains with charge $Ze$, a displacement $d$ between charge centroid and center of mass (e.g. due to asymmetric grain shape or isotopic substitution) may add another, uncorrelated component. We assume that the displacement is proportional to the excitation equivalent radius: $d = \epsilon a_{\text{ex}}$, where $\epsilon = 0.01$ (DL98b). In most cases this is negligible compared to the intrinsic component, so we model it as a single value for the sake of simplicity. The total dipole moment is thus given by

$$\mu^2 = \mu_i^2 + (Ze a_{\text{ex}})^2,$$

where $q_e$ is the elementary charge.

3.4 Grain charge

The rotational damping and excitation rates will be dependent on the grain charge. DL98b showed that the characteristic timescale for changes in charge is much shorter than the
characteristic rotational damping time. We will therefore average the damping and excitation rates over grain charges, as well as the electric dipole moment when computing the power radiated. We therefore need the charge distribution function of the grains as a function of their radius and environmental conditions, \( f_a(Z) \).

There are three main processes contributing to grain charging: collisional charging by electrons and ions, which rates we denote \( J_i(Z, a) \) and \( J_p(Z, a) \) respectively, and photoelectric emission of electrons caused by the impinging radiation, which rate is \( J_p(Z, a) \). For every grain radius, the steady state charge distribution function is obtained by solving recursively the following equations:

\[
\begin{align*}
J_i(Z, a) + J_p(Z, a) f_a(Z) & = J_i(Z + 1, a) f_a(Z + 1). 
\end{align*}
\] (12)

We use the equations of [Draine & Sutin (1987)] for collisional processes, updated with the [Weingartner & Draine (2001b)] electron sticking coefficients, for \( J_i \) and \( J_p \). The photoelectric emission rate is computed according to [dl98b] electron sticking coefficients, for \( J_p \). As one can see from the Fokker-Planck equation, this implies that the fluctuation-dissipation theorem), to obtain the damping coefficient take up the following form:

\[
D^i(\omega) \equiv - \lim_{\delta t \to 0} \frac{\langle \delta \omega^i \rangle}{\delta t} \quad \text{and} \quad E^{ij}(\omega) \equiv \lim_{\delta t \to 0} \frac{\langle \delta \omega^i \delta \omega^j \rangle}{\delta t}. \] (14)

We assume that the medium is isotropic, and there are no physical processes that allow for a preferred direction, such as a magnetic field. As a consequence, the rotational distribution function only depends upon the magnitude \( \omega \) of \( \omega \).

Moreover, in a local orthonormal frame \((\hat{e}_\omega, \hat{e}_\theta, \hat{e}_\phi)\), where \( \omega, \theta \) and \( \phi \) are the usual spherical polar coordinates defining \( \omega \), the excitation coefficient take up the following form:

\[
E^{\omega \omega} = E^{\|}(\omega) 
\] (15)
accounts for fluctuations along \( \omega \), and

\[
E^{\omega \theta} = E^{\theta \phi} = E^{\perp}(\omega) 
\] (16)
accounts for fluctuations perpendicular to \( \omega \). The components in the coordinate basis are thus:

\[
E^{\omega \omega} = E^{\|}(\omega), \quad E^{\omega \theta} = \frac{E^{\|}(\omega)}{\omega^2}, \quad E^{\omega \phi} = \frac{E^{\perp}(\omega)}{\omega^2 \sin^2 \theta}. \] (17)

Moreover, we assume there are no systematic torques, so the damping coefficient is directed along \( \omega \) and we have

\[
D(\omega) = D(\omega) \hat{e}_\omega. \] (18)

In the spherical polar coordinate basis, the Fokker-Planck equation then becomes:

\[
\frac{1}{\omega^2} \frac{d}{d\omega} \left[ \omega^2 D(\omega) f_a(\omega) \right] + \frac{1}{2\omega^2} \frac{d^2}{d\omega^2} \left[ \omega^2 E^{\|}(\omega) f_a(\omega) \right] - \frac{1}{\omega^2} \frac{d}{d\omega} \left[ \omega E^{\perp}(\omega) f_a(\omega) \right] = 0. \] (19)

Integrating once, we get the following first order differential equation:

\[
\frac{df_a}{d\omega} + \frac{D}{E^{\|}} f_a = 0, \] (20)
where

\[
\tilde{D} \equiv D + \frac{1}{\omega} (E^{\|}_j - E^{\perp}_j) + \frac{1}{2} \frac{dE^{\|}_j}{d\omega}. \] (21)

Note that \( \tilde{D} \) is simply equal to \( D \) if the fluctuations are isotropic and independent of \( \omega \).

The coefficients \( D, E^{\|}_j, E^{\perp}_j \), and therefore \( \tilde{D} \) from various independent rotational damping and excitation processes are additive.

A given process is said to respect detailed balance, when, if that process were the only one taking place, the grain would rotate thermally, i.e. \( f_a(\omega) \propto \exp(-I\omega^2/2kT) \). As one can see from the Fokker-Planck equation, this implies that this process must satisfy:

\[
\tilde{D} = \frac{I}{2kT} E^{\|}_j. \] (22)

Excitation rates are often easier to calculate than damping rates, since they are positive definite and do not rely on near-cancellation of processes that increase versus decreasing \( \omega \). Thus in some cases, we will make use of detailed balance (i.e. the fluctuation-dissipation theorem), to obtain the damping rate, knowing the excitation rate.
4.2 Normalized damping and excitation coefficients

We will see in the next section that for collisions with neutral H atoms, at a temperature $T$, for a spherical dust grain at the same temperature $T$, the damping and parallel excitation coefficients have the following form:

$$\dot{D}_H = \frac{\omega}{\tau_H} \quad \text{and} \quad E_{\parallel, H} = E_{\perp, H} = \frac{2kT}{I\tau_H},$$

where

$$\tau_H \equiv \left[ n_H m_H \left( \frac{2kT}{\pi m_H} \right)^{1/2} \frac{4\pi a^2_x}{3I} \right]^{-1}$$

is the characteristic rotational damping timescale for collisions with neutral H atoms. Taking detailed balance condition, we define, following DL98b:

$$F_X(\omega) \equiv \frac{\tau_H}{\omega} \tilde{D}_X$$

and

$$G_X(\omega) \equiv \frac{I\tau_H}{2kT} E_{\parallel, X}(\omega)$$

A special case is made of the rotational damping through electric dipole radiation (subscript $ed$), because of its specific $\omega^3$ dependence:

$$\frac{d}{dt} \left( \frac{1}{2} I \omega^2 \right)_{ed} = \frac{2\mu^2}{3} \frac{\omega^4}{c^3},$$

so

$$\frac{d\omega}{dt}_{ed} = -D_{ed}(\omega) = -\frac{2\mu^2}{3} \frac{\omega^3}{kT} = -\frac{I\omega^3}{3kT} \frac{1}{\tau_{ed}}.$$}

Here we define, following DL98b:

$$\tau_{ed} \equiv \frac{I^2 c^3}{2kT \mu^2}$$

Using Eqs. (23), (24) and (28) in Eq. (20), the final equation for the distribution function is

$$\frac{df}{d\omega} + \left[ \frac{I \omega}{kT G} + \frac{\tau_H}{\tau_{ed}} \frac{I^2 \omega^3}{3G(kT)^2} \right] f_{\omega} = 0,$$

where

$$F \equiv \sum_X F_X \quad \text{and} \quad G \equiv \sum_X G_X.$$}

One can see that the conditions to get a thermal, Maxwellian distribution $f_{\omega}(\omega) \propto \exp(-I\omega^2/2kT)$ are:

$$F = G = \text{constant} \quad \text{and} \quad \frac{\tau_H}{\tau_{ed}} \to 0.$$}

Otherwise, the general solution to this equation is:

$$f_{\omega}(\omega) \propto \exp \left[ -\int_{0}^{\omega} d\omega' \left\{ -\frac{I \omega'}{kT G(\omega')} + \frac{\tau_H}{3\tau_{ed} G(\omega')} \frac{I^2 \omega'^3}{(kT)^2} \right\} \right].$$

If all $F_X$’s and $G_X$’s are constant, this has a simple form:

$$f_{\omega}(\omega) \propto \exp \left[ -\frac{F I \omega^2}{G 2kT} - \frac{\tau_H}{\tau_{ed}} \frac{1}{3G} \frac{I^2 \omega^3}{(kT)^2} \right].$$

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Note that the damping through electric dipole radiation causes the distribution to be non-Maxwellian.

In the general case, some $F_X$’s and $G_X$’s may depend upon $\omega$ and one has to compute numerically the resulting distribution function, using Eq. (33).

We now turn to the calculation of the various damping and excitation coefficients, due to collisions, plasma drag, infrared emission, photoelectron emission, and random H$_2$ formation. In the following microphysics sections that form the heart of the paper, we compute excitation and damping coefficients as a function of grain radius and environmental conditions. We evaluate them numerically for a fiducial Cold Neutral Medium (CNM) environment, defined explicitly in Eq. (102).

5 COLLISIONAL DAMPING AND EXCITATION

In this section we correct the results of DL98b, Appendix B, which did not take into account the fact that not all neutrals escape the grain surface when computing the damping rate. The microphysics of collisions is complex and beyond the scope of this study (for a discussion of the physics and chemistry of PAHs and their relation with the interstellar gas see for example [Montn 1984]). We therefore use the following simplifying assumptions:

- The grain is in a stationary state: the rate at which species collide with it is equal to the rate at which they leave its surface.
- We assume that all species (neutrals and ions) colliding with the grain stick and that they depart the grain as neutrals. In extremely dense environments, the colliding species may bounce off the grain surface instead of sticking. This case is discussed at the end of Section 5.1.4.
- Even if the impacting species may not collide equiprobably everywhere on the grain’s surface (e.g. if the grain is non-spherical or if it has a dipole moment), we assume they somehow get re-distributed on the grain surface and leave it equiprobably from any point.
- We assume, as in DL98b, that neutrals leave the grain surface with a thermal velocity distribution in the grain’s frame, with a temperature $T_{ev}$ of the order of the infrared emission characteristic temperature. Unlike DL98b, we estimate $T_{ev}$ as a function of grain radius and ambient radiation field (see section 5.1.4).

Using those assumptions, one can compute the rate of collisional damping and excitation. We will perform the calculations for a spherical grain in the general case. To find the relevant equivalent radius to use for a cylindrical grain, we will carry out the explicit calculation in the case of collisions of a neutral grain with neutral H atoms. Note that as pointed in DL98b, the rotational excitation in case of collisions has two origins: the random excitation by incoming particles (superscript (in)), as well as the random excitation by “evaporating” neutrals (superscript (ev)).
5.1 General considerations: spherical grain

We use the usual spherical polar coordinates around the spherical grain, taking the rotation axis as a reference. The local phase-space density at the grain surface is:

\[ f_{ev}(v, \theta) = K(\theta) \exp \left[ - \frac{m(v - v_0)^2}{2kT_{ev}} \right] \]

(35)

with the local velocity

\[ v_0 \equiv \omega \times r = a \omega \sin \theta \hat{e}_\varphi. \]

(36)

The normalization constant \( K(\theta) \) is found by imposing that, locally, the flux of evaporating (and escaping) particles is equal to the flux of colliding particles. Except for the case of ions interacting with the electric dipole of the grain, the flux of colliding particles will be homogenous on the grain surface. If it is not the case, we approximate the local flux by the total rate of collisions \( dN_{coll}/dt \) divided by the grain area:

\[ \frac{1}{4\pi a^2} \frac{dN_{coll}}{dt} = K(\theta) \int v_r \exp \left[ - \frac{m(v - v_0)^2}{2kT_{ev}} \right] P_{esc} \, d^3v, \]

(37)

where \( P_{esc} = 1 \) for velocities at the grain surface leading to escape, and 0 otherwise.

All particles evaporating from the grain are neutrals. They interact with the grain through the induced dipole potential (we neglect the dipole-induced dipole interaction with the dipole moment of the grain):

\[ U(r) = -\frac{1}{2} \left( \frac{Z_0^2 e^2}{r^4} \right). \]

(38)

where \( \alpha \) is the polarizability of the escaping neutral and \( q_\varphi \) is the elementary charge. The polarizability of hydrogen is a standard result in nonrelativistic quantum mechanics and is \( \frac{a_0^3}{\epsilon_0} = 0.67 \, \text{Å}^3 \) where \( a_0 \) is the Bohr radius (Landau & Lifshitz 1965). We also take \( \alpha = 0.20 \, \text{Å}^3 \) for helium (Thomas & Humbertson 1972), and \( \alpha = 1.54 \, \text{Å}^3 \) for carbon (Miller & Kelly 1972), which is important since \( C^+ \) is often the dominant ion if the hydrogen is self-shielded. For molecular hydrogen \( \text{H}_2 \), we take \( \alpha = 0.79 \, \text{Å}^3 \) (Marlow 1965).

5.1.1 Computation of \( P_{esc} \)

The radial coordinate of the escaping neutral is the solution of the following equation:

\[ r^2 + V_{eff}(r) = r^2 + \frac{a^2 v_\|^2}{r^2} - \frac{a^4}{r^4} v_\|^2 = \frac{2E}{m}, \]

(39)

where \( v_\| \) is the modulus of the tangential velocity at the grain surface and

\[ v_\|^2 = \frac{Z_0^2 e^2 a_\varphi \alpha}{m a^3}. \]

(40)

The effective potential has a maximum at the radius

\[ r_a = \sqrt{2} \frac{a_\varphi}{a_\parallel}, \quad V_{eff}(r_a) = \frac{v_\|^4}{4a^2}. \]

(41)

To escape, a neutral needs to have either \( a > r_a \) or \( 2E/m > V_{eff}(r_a) \). These two conditions can be combined to get:

\[ P_{esc} = 1 \text{ if } \begin{cases} v_\| > v_\| \quad \text{or} \quad 0 < v_\parallel < v_\| \quad \text{and} \quad \sqrt{v_\|(v - v_\|)} > 2v_\|(v - v_\|) \end{cases} \]

(42)

where \( v_\| \) is the radial velocity at the grain surface.

5.1.2 Computation of \( K(\theta) \)

Following DL98b, we define \( \epsilon_e^2 = m v_\|^2/2kT_{ev} \), which describes whether the typical evaporating atom has enough energy to overcome the induced dipole attraction to the grain (\( \epsilon_e < 1 \)) or not (\( \epsilon_e > 1 \)). We also define the ratio of rotational velocity to thermal velocity at the grain surface, which is small compared to unity:

\[ \Omega \equiv a_\varphi \sqrt{\frac{m}{kT_{ev}}} \sim \left( \frac{m_{\text{grain}}}{T_{ev}} \right)^{1/2} \ll 1. \]

(43)

In terms of those dimensionless quantities, we can find the normalization constant \( K \). The right-hand side of Eq. (34) can be expanded using the substitution

\[ (v_\|, v_\|, v_\|) = \sqrt{\frac{2kT_{ev}}{m}} (u_\|, u \cos \psi, u \sin \psi) \]

(44)

to yield

\[ \frac{1}{4\pi a^2} \frac{dN_{coll}}{dt} = K \left( \frac{2kT_{ev}}{m} \right)^2 \frac{2}{\pi} \left[ e^{-\epsilon_e^2} \right. \]

\[ \left. + e^{-(\Omega \sin \theta)^2} \int_0^{\epsilon_e} 2u_\| du_\| \, e^{-u_\|^2} \right. \]

\[ \left. \times \int_{\sqrt{u_\|}}^\infty \! 2u_\| du \, e^{-u_\|^2} I_0(2u_\| \Omega \sin \theta) \right], \]

(45)

where

\[ I_0(X) = \frac{1}{2\pi} \int_0^{2\pi} e^{X \sin \psi} d\psi = 1 + \frac{1}{4} X^2 + ... \]

(46)

is a modified Bessel function of the first kind.

Expanding to second order in \( \Omega \), we get:

\[ K = \left( \frac{2kT_{ev}}{m} \right)^2 \frac{2}{\pi} \frac{e^{\epsilon_e^2}}{e^{\epsilon_e^2} + \sqrt{\pi} \epsilon_e \epsilon_\psi \text{erf} (\epsilon_e)} \frac{1}{4\pi a^2} \frac{dN_{coll}}{dt} \]

(47)

up to corrections of order \( O(\Omega^2) \).

5.1.3 Damping and excitation rates

Each escaping neutral particle takes away an angular momentum

\[ L = ma(v_\| \hat{e}_\| - v_\| \hat{e}_\|). \]

(48)

As \( P_{esc} \) is an even function of \( v_\| \), the average of \( v_\| \) vanishes.

The loss of angular momentum along the z-direction per unit time per unit area is given by

\[ \frac{dL_z}{dt} = -ma \sin \theta K \int v_\| v_\psi \exp \left[ - \frac{m(v - v_0)^2}{2kT_{ev}} \right] \]

\[ \times P_{esc} \, dv_\| \, dv_\psi \, dv_\| \, dv_\psi. \]

(49)

Here we differ from DL98b as we take into account the fact that not all particles escape from the grain. Expanding in \( \Omega \)
and using the expression for $K$ we get, up to corrections of order $O(\Omega^2)$:

$$\frac{dL_\nu}{dt} = -\frac{1}{4\pi} m \sin^2 \theta \frac{e^{-\epsilon^2} + 2\epsilon^2 e^{-\epsilon^2}}{e^{-\epsilon^2} + 2\epsilon^2 e^{-\epsilon^2}} \frac{dN_{\text{coll}}}{dt}. \quad (50)$$

Integrating over the whole grain surface, we find the damping rate

$$D(\omega) = \frac{1}{T} \frac{dL_\nu}{dt} = \frac{e^{-\epsilon^2} + 2\epsilon^2 e^{-\epsilon^2}}{e^{-\epsilon^2} + 2\epsilon^2 e^{-\epsilon^2}} \frac{2ma^2}{3I} \frac{dN_{\text{coll}}}{dt}. \quad (51)$$

A similar calculation leads to the excitation rate through evaporating particles:

$$E^{(ev)}(\omega) = \frac{1}{T^2} \frac{d(\Delta L_j^2)}{dt}$$

$$= \frac{e^{-\epsilon^2} + 2\epsilon^2 e^{-\epsilon^2}}{e^{-\epsilon^2} + 2\epsilon^2 e^{-\epsilon^2}} \frac{2ma^2}{3I} \frac{dN_{\text{coll}}}{dt} kT_{ev}$$

$$= \frac{kT_{ev}}{I} \frac{1}{\omega} D(\omega), \quad (52)$$

up to terms quadratic in $\Omega$.

This implies the remarkable relation

$$C_{\text{coll}}^{(ev)} = \frac{T_{ev}}{2T} C_{\text{coll}}. \quad (53)$$

Physically, this occurs because if $T_{ev} = T$ then the collisions with neutrals satisfy detailed balance, Eq. (22). The factor of 2 arises since in this case there is an equal contribution to the excitation from incoming and evaporating particles.

We derive a stronger damping rate due to evaporating atoms than DL98b: for $\epsilon_e \ll 1$ this results in no change, but for $\epsilon_e \gg 1$ we find much stronger damping. The physical origin of this is that atoms that evaporate with prograde velocities relative to the local grain surface ($v_0 > v_{\text{th}}$) typically have more angular momentum than atoms that evaporate with retrograde velocities. Therefore the centrifugal potential helps them to escape the grain. DL98b neglected this effect, but for $\epsilon_e \gg 1$ it is dominant.

The excitation rate through incoming particles will be calculated for each case.

5.1.4 Evaporation temperature $T_{ev}$

DL98b assume that the evaporating temperature is a constant, independent of grain size. This accurately describes the largest grains, for which the temperature may be approximated as a constant, obtained from equating the absorbed and emitted energy (DL98b):

$$T_e = \frac{hc}{k} \left[ \frac{\langle Q \rangle_{v*} u_*}{8\pi h c Q_0 \lambda_0^4 \Gamma(\alpha + 4)} \right]^{1/(\alpha + 4)} \quad (54)$$

where in the infrared the grain absorption efficiency is assumed to be a power-law

$$Q_\nu = Q_0 \left( \frac{\nu}{\nu_0} \right) ^\alpha, \quad \lambda_0 = \frac{c}{\nu_0} \quad (55)$$

with typically $\alpha = 2$ and $\langle Q \rangle_{v*} u_* \equiv \int dv Q_\nu v_\nu$. Note that we have $T_e \propto \nu_0^{1/6}$ and a weak dependence on grain radius as the absorption efficiencies cancel out.

However, the smallest grains undergo sudden thermal spikes after each photon absorption, followed by long intervals during which the grain drops to its vibrational ground state. The neutrals or ions that have stuck to the grain after a collision cannot be thermally ejected from a grain in the ground state so we assume ejection during thermal spikes. A simple assumption is that in this case ejection occurs after a photon absorption and thermalization of the photon’s energy. We take

$$E_\gamma = \frac{\int Q_\nu u_\nu dv}{\int Q_\nu \frac{u_\nu}{h} dv} \quad (56)$$

as the typical energy of an absorbed photon. Typically, $E_\gamma \approx 5$ eV. We then calculate the corresponding grain temperature following DL01: we solve for $T_q$ such that $\bar{E}(T_q) = E_\gamma$, where

$$\bar{E}(T) = \sum_{j=1}^{N_m} \text{exp}(\hbar \omega_j/kT) - 1 \quad (57)$$

is the expectation value of the energy of the grain, and the sum runs over its $N_m$ vibrational degrees of freedom. We take $T_{ev} = \max(T_e, T_q)$ as the evaporation temperature. The result is shown in Fig. 2. One can see that we obtain much higher evaporation temperatures than the ones used by DL98b. The effect may be significant on the final spectrum, as can be seen from Fig. 3.

High density, low radiation field case

The previous treatment is valid only if the rate of photon absorption is high enough to eject all stuck species before all available sites on the grain are occupied. We approximate the number of available sites on the grain by the number of superflcial C-atoms:

5 The mechanism we describe for atomic ejection from grains is called photo-thermo-dissociation (PDT). Rouan et al. (1992) also mention another possible mechanism, photo-dissociation (PD), which is an atomic ejection following the direct interaction of a UV photon with a given C-H bond. PD may lead to even higher ejection temperatures, of order 10 000 K.
where $N$ is the peak frequency of the spectrum. The damping rate (see discussion at the end of section 5.1.3) and an in-
spinning dust spectrum for the Cold Neutral Medium (CNM, Eq. a constant $T = 20$ K for all grain sizes. This leads to a decreased
temperature for the smallest grains, compared to DL98b, who assume
Figure 3. Effect of the evaporation temperature model on the
spinning dust spectrum for the Cold Neutral Medium (CNM, Eq. (102)). Our prescription results in a much higher evaporation temperature
for the smallest grains, compared to DL98b, who assume a constant $T_{ev} = 20$ K for all grain sizes. This leads to a decreased
damping rate (see discussion at the end of section 5.1.3) and an in-
creased excitation rate through collisions, and therefore increases the peak frequency of the spectrum.

$$N_{\text{sites}} = \begin{cases} 
N_C(a) & \text{for cylindrical grains} \\
\frac{4d}{a}N_C(a) & \text{for spherical grains}
\end{cases}, \quad (58)
$$

where $N_C(a)$ was defined in Eq. (3) and $d = 3.35\text{Å}$ is the interlayer separation in graphite. The ratio of collision rate to photon absorption rate is given by:

$$R_{\text{coll/abs}} = \frac{n_H \sqrt{8kT/\pi m_H}}{Q_v \frac{\nu}{c_0} d\nu c} \approx 0.1 \times \frac{n_H}{30 \text{cm}^{-3}} T^{1/2} \chi^{-1} a^{-1}. \quad (59)$$

In most environments, $R_{\text{coll/abs}} \ll N_{\text{sites}}$ so there is no accumulation of stuck species. In very dense and dark clouds however, the rate of collisions may become so high compared to the rate of photon absorption that all the sites are occupied. In that case, the assumption that incoming species stick to the grain is no more valid. They will instead
bounce off the irregular grain surface. From the fluctuation-
dissipation theorem, one expects that, for collisions with neutral species, $F_n = G_n$. Thus, we set the effective evaporation temperature equal to the gas temperature in that case (see Eq. (73) and discussion below):

$$T_{ev} = T \quad \text{if} \quad R_{\text{coll/abs}} > N_{\text{sites}}. \quad (60)$$

The actual transition from sticking to elastic collisions should of course be smooth, unlike the discontinuous step we assume here. Our treatment should approximately reflect the physics of collisions except near the transition regime $R_{\text{coll/abs}} \sim N_{\text{sites}}$.

5.2 Collision with neutral H atoms: neutral grain, general grain shape

We assume that the grain is neutral, and has no dipole moment, so there is no interaction whatsoever between the grain and the neutral H atoms (purely geometric cross-section). The phase-space density of incoming H atoms at the grain surface is simply

$$f_{in}(\nu) = n_H \left( \frac{m_H}{2\pi kT} \right)^{3/2} e^{-m_H \nu^2/2kT}, \quad (61)$$

from which one can easily get the excitation rate through incoming particles:

$$\frac{d\Delta L^2}{dt} = \int v_n (m_H c^2) f_{in}(\nu) d^3\nu \quad (62)$$

where $v_n$ is the component of the velocity normal to the grain surface. This evaluates to

$$\frac{d\Delta L^2}{dt} = n_H m_H^2 \pi^2 \left( \frac{2kT}{\pi m_H} \right)^{3/2}. \quad (63)$$

Integrating over the grain surface, we get

$$\frac{d\Delta L^2}{dt} = kT n_H m_H \left( \frac{2kT}{\pi m_H} \right)^{1/2} \frac{4\pi a_{ex}^4}{3}, \quad (64)$$

where $a_{ex}$ was defined in equation (1). For a spherical grain, $a_{ex} = a$. For a disklike grain of thickness $d$ and radius $b$, spinning around its axis of symmetry, we have

$$a_{ex} = \left[ \frac{3}{8} b^2 (2d + b) \right]^{1/4}. \quad (65)$$

We can write the excitation rate by incoming H atoms as

$$E_{\nu, H} = \frac{kt}{I_{\nu H}} \quad (66)$$

where $I_{\nu H}$ was defined in equation (22).

The case of evaporating particles is very similar. Assuming the grain surface is at the same temperature $T$ as the gas, the phase-space density of evaporating particles is

$$f_{ev}(\nu) = n_H \left( \frac{m_H}{2\pi kT} \right)^{3/2} \exp \left( -\frac{m_H (\nu - v_0)^2}{2kT} \right). \quad (67)$$

In that case $P_{esc} = 1$ for all outgoing particles. The same calculation therefore leads to

$$E_{\nu, H} = \frac{kt}{I_{\nu H}} \quad (68)$$

up to terms of order $O(\Omega^2)$, which comes from the fact that we did not take into account the slight change of $\omega$ after the particle has collided (we assumed the same $\omega$ for the incoming and the outgoing particle). Detailed balance ensures that

$$D_{\nu} = \frac{\omega}{\tau_{\nu H}}. \quad (69)$$

Therefore, for non spherical grains, we will compute collision rates assuming a spherical geometry with radius $a_{ex}$. We just showed that this is an exact result for collisions with neutral H atoms. The collision rates are indeed proportional to the area of the grain, but the angular momentum gained depends on $a_{ex}$, so $a_{ex}$ will approximately reflect both dependencies.

5.3 Collisions with neutral atoms: charged grains

In that case, the incoming neutrals interact with the same potential as the outgoing particles:

$$U(r) = -\frac{1}{2} \frac{Z^2 e^2}{r^2}. \quad (70)$$

We use the same notation as DL98b and define
\[ \epsilon_n \equiv \sqrt{\frac{m_n v_a^2}{2kT}} \quad \text{and} \quad b_0(v) \equiv a \sqrt{\frac{2m_n}{v^3}}, \]  
\[ (71) \]

where \( v_a \) was defined in Eq. (40).

We recall, from DL98b, that a trajectory with impact parameter \( b \) and velocity at infinity \( v \) leads to a collision if

\[ b \leq b_{\max}(v) = \left\{ \begin{array}{ll} b_0(v) & \text{if } v \leq v_a \\ a\sqrt{1 + v_a^2/v^2} & \text{if } v \geq v_a. \end{array} \right. \]  
\[ (72) \]

We compute the collision rate

\[ \frac{dN_{\text{coll}}}{dt} = n_v \int_0^{\infty} dv 4\pi v^3 n_{\text{max}}^2(v) \left( \frac{m_n}{2\pi kT} \right)^{3/2} e^{-m_n^2/2kT} \]  
\[ = n_v 2\pi a^2 \left( \frac{2kT}{\pi m} \right)^{1/2} \left[ e^{-t^2} + \sqrt{\pi} \epsilon_n \text{erf} \epsilon_n \right]. \]  
\[ (73) \]

We can now get the normalized damping and excitation rates for collisions with neutrals:

\[ F_n = \frac{n_v}{n_H} \sqrt{\frac{m_n}{m_H}} e^{-t^2/2T} + \sqrt{\pi} \epsilon_n \text{erf} \epsilon_n \left( e^{-t^2} + 2\epsilon_n \right), \]

\[ G_n^{(\text{ev})} = \frac{T_e}{2T} F_n, \quad \text{and} \]

\[ G_n^{(\text{in})} = \frac{n_v}{2n_m} \sqrt{\frac{m_n}{m_H}} e^{-t^2/2T} + 2\epsilon_n \left( e^{-t^2} + 2\epsilon_n \right). \]  
\[ (74) \]

where the result for \( G_n^{(\text{in})} \) is identical to that of DL98b. Note that when \( T = T_e \), \( G_n^{(\text{ev})} = G_n^{(\text{in})} = F_n/2 \) so the principle of detailed balance holds. Moreover, in the case of a neutral grain, if the only rotational excitation and damping process were collisions with neutral species, then the rotational distribution function would be a Maxwellian. In that case the rotational temperature would be given by \( T_{rot} = G_n/F_n \times T = \frac{1}{2} \left( T + T_{rot} \right) \), the arithmetic mean of the gas and evaporation temperatures, as was already shown by Purcell (1979).

This is the contribution of an individual neutral, for a given grain charge. To get the total contribution, one must average over all grain charges (DL98b showed that the charging timescale is much shorter than the collision timescale) and sum over all neutral species, which we take to be atomic and molecular hydrogen, and helium with \( n_{He}/n_H = 1/12 \).

### 5.4 Collisions with ions: charged grains

The ion interacts with the grain through the Coulomb, electric dipole, and “image charge” potentials. The latter dominates over the Coulomb potential only in the immediate vicinity of the grain surface, so we will neglect it for charged grains. Properly accounting for it would result in a slight increase in both damping and excitation rates as this is an attractive potential. The general solution for this problem, with a rotating electric dipole moment, is still not analytical. Thus, for simplicity, we will only consider the case where the electric dipole moment can be considered as non-rotating, i.e. when the timescale of the collision is short compared to the rotation period of the grain. This is justified as, when the ion reaches the vicinity of the grain surface, the ratio of collision timescale to rotation timescale is approximately \( \omega_s/\nu \sim \sqrt{m_i/m_{gr}} \ll 1 \). We will assume that the grain is spherical, so that the electric dipole moment is randomly oriented relative to the rotation axis (for cylindrical grains this is not the case but we will assume so for simplicity). Note that when the grain rotates rapidly, the component of the dipole moment perpendicular to the rotation axis averages out, but not the parallel component. Although this problem will be different in nature as this alignment creates anisotropic excitation by collisions, the magnitude of the non rotating part of the dipole moment will remain of the same order (a factor \( 1/\sqrt{3} \) smaller only), so our approximation should give a decent idea of what the effect of the dipole moment is on the trajectory.

We assume a spherical geometry with radius \( a_{ex} \). Taking \( \mu \) as the polar axis for spherical polar coordinates, the interaction potential of the ion in the Coulomb and dipole field of the grain is given by

\[ V(r, \theta) = \frac{Z_e q_i q_e^2}{r} + \frac{Z_e q_i q_e \cos \theta}{r^2}. \]  
\[ (75) \]

The motion in this potential has two obvious constants: the energy \( E \) and the angular momentum along the \( z \)-axis (along \( \mu \)), \( L_z \). For this special potential, however, there exists a third constant of the motion. The torque \( \mathbf{L} \) exerted on the ion comes entirely from the second term in the potential and is

\[ \mathbf{L} = -\mathbf{r} \times \nabla V(r) = \frac{Z_e q_i q_e \sin \theta}{r^2} \mathbf{e}_\varphi. \]  
\[ (76) \]

Since the azimuthal component of angular momentum is \( \mathbf{L} \cdot \mathbf{e}_\varphi = m_r \mathbf{r} \cdot \mathbf{\hat{e}}_\varphi \), we can then determine the overall rate of change of the angular momentum,

\[ \frac{d}{dt} (L_z^2) = 2m_i Z_i q_i q_e \sin \theta \frac{d \theta}{dt} = -2m_i Z_i q_i q_e \cos \theta \frac{d \theta}{dt}. \]  
\[ (77) \]

Therefore we find the constant of the motion\(^6\)

\[ \mathcal{A} \equiv L_z^2 + 2m_i Z_i q_i q_e \cos \theta. \]  
\[ (78) \]

Its value can be determined by taking the incoming trajectory at infinity with approach angle \( \theta_\infty \),

\[ \mathcal{A} = (m_i \nu_e)^2 + 2m_i Z_i q_i q_e \cos \theta_\infty. \]  
\[ (79) \]

The energy of the trajectory is

\[ \frac{1}{2} m_i \nu_e^2 + V_{eff}(r) = E \]  
\[ (80) \]

where \( V_{eff}(r) \) is the sum of the potential \( V(r, \theta) \) and the tangential kinetic energy \( L_z^2/2m_i \nu_e^2 \):

\[ V_{eff}(r) \equiv \frac{Z_e q_i q_e^2}{r} + \frac{m_i (\nu_e)^2}{2} + 2Z_e q_i q_e \cos \theta_\infty. \]  
\[ (81) \]

It is easier to work with the following dimensionless parameters

\[ \psi \equiv \frac{Z_e q_i q_e^2}{a_{ex} kT}, \quad \tilde{\mu} \equiv \frac{Z_e q_i q_e}{a_{ex} kT}. \]  
\[ (82) \]

\(^6\) Collisions with neutral helium have little effect on the spectrum: the helium contribution dominates \( F_n \) and \( G_n \) only in the case where the medium is strongly ionized, i.e. when the dominant rotational excitation and damping mechanisms are rather collisions with ions or plasma drag. We include them for completeness.

\(^7\) This may also be derived by the Hamilton-Jacobi method in spherical polar coordinates.
Their physical meaning is as follows: $|\psi| \ll 1$ when the thermal energy of the ion dominates over the electrostatic interaction energy, and $|\psi| \gg 1$ when the electrostatic interaction dominates. The sign determines whether the interaction is attractive ($\psi < 0$) or repulsive ($\psi > 0$). $\tilde{\mu}$ is the equivalent quantity for the dipole interaction. Note that we consider only positively charged ions so $\tilde{\mu} > 0$. We also work with the dimensionless variables

\[ e = \frac{b}{a_{ex}}, \quad u = \sqrt{\frac{m_i}{2kT}v}. \]  

(83)

The effective potential can now be written

\[ V_{\text{eff}}(r) = kT \left[ \psi \frac{a_{ex}}{r} + \left( u^2 c^2 + \tilde{\mu} \cos \theta_{\infty} \right) \left( \frac{a_{ex}}{r} \right)^2 \right]. \]  

(84)

A study of this potential leads to the following condition for collision:

\[ \cos \theta_{\infty} < X_{\text{max}}(c, u) \equiv \tilde{\mu}^{-1} \left( u^2 - u^2 c^2 - \psi \right) \]  

(85)

Note that if $X_{\text{max}} < -1$, then there is never collision, for any angle. If $X_{\text{max}} > 1$, then all angles lead to a collision. We define

\[ X(c, u) \equiv \max \{ -1, \min \{ 1, X_{\text{max}}(c, u) \} \}. \]  

(86)

Now, we can compute the collision rate

\[ \frac{dN_{\text{coll}}}{dt} = n_i \int 2\pi^3 dv \pi db \left( \frac{m_i}{2kT} \right)^{3/2} \times e^{-m_i v^2/2kT} \]  

\[ \times \left( u^2 + \tilde{\mu} \psi \right) \frac{a_{ex}}{r^2} \]  

\[ \times \int 2 \psi e^{-u^2} du 2c dc \left( X + \frac{1}{2} \right). \]  

(87)

We can also get the excitation rate by incoming

\[ \frac{d\Delta L^2}{dt} = n_i \int \frac{(mbv)^2}{3} 2\pi^3 dv \pi db \left( \frac{m_i}{2kT} \right)^{3/2} \]  

\[ \times e^{-m_i v^2/2kT} \]  

\[ \times \left( u^2 + \tilde{\mu} \psi \right) \frac{a_{ex}}{r^2} \]  

\[ \times \int u^2 e^{-u^2} du 4c^3 dc \left( X + \frac{1}{2} \right). \]  

(88)

These integrals can be evaluated explicitly and one then gets, for the charged grains $Z_0 \neq 0$

\[ F_i(Z_0 \neq 0) = n_i \int \frac{m_i}{m_i} m_i e^{-v^2/2kT} + 2k^2 \]  

\[ \times \psi \frac{a_{ex}}{r^2} \int 2 \psi e^{-u^2} du 2c dc \left( X + \frac{1}{2} \right). \]  

(89)

where we have defined $g_1(\psi, \tilde{\mu}) = \left\{ \begin{array}{ll} 1 - \psi & \psi < 0 \\ e^{-\psi} \sinh \tilde{\mu}/\tilde{\mu} & \psi > 0 \end{array} \right.$, $\tilde{\mu} \leq |\psi|$

\[ 1 - e^{-(\psi + \tilde{\mu})} + \tilde{\mu} - \psi + \frac{1}{2}(\tilde{\mu} - \psi)^2 \]  

\[ \frac{2\tilde{\mu}}{2\tilde{\mu}}, \quad \tilde{\mu} > |\psi|, \]  

(90)

and $g_2(\psi, \tilde{\mu}) = \left\{ \begin{array}{ll} 1 - \psi + \psi^2/2 + \tilde{\mu}^2/6 & \psi < 0 \\ e^{-\psi} \sinh \tilde{\mu}/\tilde{\mu} & \psi > 0 \end{array} \right.$, $\tilde{\mu} \leq |\psi|$

\[ 1 - e^{-(\psi + \tilde{\mu})} + \tilde{\mu} - \psi + \frac{1}{2}(\tilde{\mu} - \psi)^2 + \frac{1}{2}(\tilde{\mu} - \psi)^3 \]  

\[ \frac{2\tilde{\mu}}{2\tilde{\mu}}, \quad \tilde{\mu} > |\psi|. \]  

(91)

Note that these functions coincide with the functions $g_1(\psi)$, $g_2(\psi)$ defined in DL98b for $\tilde{\mu} = 0$. We also defined $\epsilon_i^2 \equiv Z_i q_i^2 \alpha_i/2n_i kT$, here $\alpha_i$ is the polarizability of species $i$ after it neutralizes on the grain surface, e.g. when considering collisions with the C$^+$ ion we take the polarizability of the neutral C atom). Note that even when $T_{\text{co}} = T$, $F_i \neq G_i$, as the incoming and outgoing particles are in different ionization states; detailed balance does not apply since realistic ISM phases are not in Saha equilibrium. Numerically, one has (with $T_2 \equiv T/100K$)

\[ \psi \approx 170 \frac{Z_o a_{ex}^{-2} T_2^{-1}}{9.3 \text{ debye}} \]  

(92)

\[ \tilde{\mu} \approx 30 \frac{(\mu^2)^{1/2} 10^{-7} \text{ cm}}{a_{ex}^{-1/2} T_2^{-1}} \]  

(93)

From these values, one can see that in general the effect of the dipole moment cannot be neglected a priori, as $\tilde{\mu}$ is not small compared to unity. However, in general $\tilde{\mu} < |\psi|$. This implies that, for negatively charged grains, the dipole moment has little or no effect on the excitation and damping rate. For positively charged grains, the damping and excitation rate are both increased by the huge factor sinh $\tilde{\mu}/\tilde{\mu}$, but still remain extremely small due to the coulomb repulsion, which shows in the factor $e^{-\psi}$.

We therefore conclude that DL98b approximation of neglecting the effect of the electric dipole moment on the trajectory of ions, is essentially valid in the case of collisions with charged grains. It only has a significant effect for positively charged grains, for which the coulomb repulsion implies an extremely small rate of collisions with ions anyway. We still account for the electric dipole moment for the sake of completeness.

5.5 Collisions with ions, neutral grain

In this case the Coulomb potential vanishes, and the “image charge” potential has to be taken into account. We carry the calculation using the same assumptions as in the previous section: slowly rotating spherical grain, with radius $a_{ex}$. Taking $\mu$ as the polar axis for spherical polar coordinates, the interaction potential of the ion in the dipole and induced dipole field of the grain is given by

\[ V(r, \theta) = -\frac{Z_i q_i a_{ex}}{2r^2(r^2 - a_{ex}^2)} + \frac{Z_i q_i \mu \cos \theta}{r^2}. \]  

(94)

The considerations that lead to the third constant of motion $A$ hold again. The energy of the trajectory is

\[ \frac{1}{2}m_i r^2 + V_{\text{eff}}(r) = E \]  

(95)

where $V_{\text{eff}}(r)$ is given by

\[ V_{\text{eff}}(r) \equiv -\frac{Z_i q_i a_{ex}^3}{2r^2(r^2 - a_{ex}^2)} + \frac{m_i b^2}{2r^2} + 2Z_i q_i \mu \cos \theta_{\infty}. \]  

(96)

Following DL98b, we define the dimensionless parameter

\[ \phi^2 \equiv \frac{2Z_i q_i^2}{a_{ex} kT}. \]  

(97)
which describes whether the image charge attraction dominates over the thermal energy (\(\phi \gg 1\)) or the thermal energy dominates (\(\phi \ll 1\)). The effective potential can be written
\[
V_{\text{eff}}(r) = kT \left[ -\frac{\phi^2 a_{\infty}}{4r^2} + \left( u^2 c^2 + \mu \cos \theta_{\infty} \right) \left( \frac{a_{\infty}}{r} \right)^2 \right]
\]
where \(\mu\), \(c\), and \(u\) were defined in equations (82) and (83). A study of this potential leads to the following condition for collision:
\[
\cos \theta_{\infty} < X_{\text{max}}(c, u) \equiv \tilde{\mu}^{-1} \left( u^2 - u_0^2 \right) + \phi u_0
\]
(99)
The collision and excitation rates are obtained as in Eqs. (86), (87) and (88). One can then obtain the normalized damping and excitation rates for collisions of ions with a neutral grain:
\[
F_i(Z_0 = 0) = \frac{n_i}{n_{\text{H}}} \frac{m_i}{m_{\text{H}}} h_1(\phi, \tilde{\mu})
\]
(100)
\[
G_i^{(\alpha)}(Z_0 = 0) = \frac{T_e}{2T} F_i(Z_0 = 0)
\]
(101)
\[
G_i^{(\alpha)}(Z_0 = 0) = \frac{n_i}{2n_{\text{H}}} \frac{m_i}{m_{\text{H}}} h_2(\phi, \tilde{\mu})
\]
(102)
where we have defined
\[
h_1(\phi, \tilde{\mu}) \equiv \frac{1}{2} + \frac{\tilde{\mu}}{4} + \frac{2 + \phi^2}{4\tilde{\mu}} \left( 1 - e^{-u_0^2} \right) - \frac{\phi u_0}{4\tilde{\mu}} e^{-u_0^2}
\]
\[
+ \frac{\pi^{1/2}}{2} \phi \left( 1 + \frac{3 - 2\tilde{\mu}}{4\tilde{\mu}} \right) \text{erf } u_0
\]
(103)
\[
h_2(\phi, \tilde{\mu}) \equiv \frac{1}{2} + \frac{3\phi^{1/2}}{4} + \frac{\phi^2}{12} + \frac{\tilde{\mu}}{4}
\]
\[
+ \frac{1 + \phi^2}{2\tilde{\mu}} \left( 1 - e^{-u_0^2} \right) - \frac{2\phi u_0}{16\tilde{\mu}} e^{-u_0^2}
\]
\[
+ \frac{\pi^{1/2}}{32\mu} \left( 4\mu^2 - 12\tilde{\mu} + 15 + 2\phi^2 \right) \text{erf } u_0
\]
(104)
\[
u_0 \equiv -\phi + \sqrt{\phi^2 + 4\tilde{\mu}}
\]
(105)
Note that in the limit \(\tilde{\mu} \to 0\) we recover DL98b result, as
\[
h_1(\phi, \tilde{\mu} \to 0) = 1 + \frac{\pi^{1/2}}{2} \phi + O(\tilde{\mu}^2)
\]
(106)
\[
h_2(\phi, \tilde{\mu} \to 0) = 1 + \frac{3\pi^{1/2}}{4} \phi + \frac{\phi^2}{2} + O(\tilde{\mu}^2)
\]
(107)
However, the parameter \(\tilde{\mu}\) is not small in general, as we saw in Eq. (83), so the effect of the dipole moment on the trajectory cannot be neglected. Note that we also have \(\phi \approx 18 \sigma_{\infty}^{-1/2} T_{\infty}^{-1/2}\). The net effect of the dipole moment is to increase the collision and excitation rates, as can be seen from Fig. 4 In contrast to the case of charged grains, the electric dipole moment does have a significant effect and cannot be discarded.

The effect of the dipole moment is always to increase both the collision and excitation rates, for both charged and neutral grains. This can be understood as follows. When the dipole moment vanishes, ions with a given velocity at infinity collide with the grain if their impact parameter is such that \(b < b_{\text{max}}(v)\). The effect of the dipole moment is to make a smooth transition from non-colliding to colliding trajectories: all ions with impact parameter \(b < b_{i}(v)\) collide with the grain, a fraction \((X(b, v) + 1)/2\) of those for which \(b_{i}(v) < b < b_{2}(v)\) do collide, and none of the ions with \(b > b_{2}(v)\) collide. \(b_{1}\) and \(b_{2}\) are such that \(b_{1} < b_{\text{max}} < b_{2}\). As a result, a fraction of trajectories for which \(b_{1}(v) < b < b_{\text{max}}(v)\) do not lead to collision anymore (compared to the vanishing dipole case), and a fraction of trajectories \(b_{\text{max}}(v) < b < b_{2}(v)\) now lead to collision. The suppressed colliding trajectories have a lower rate of collision and angular momentum than the added colliding trajectories. Thus the net effect of the dipole moment is to increase the collision and rotational excitation rates.

6 PLASMA DRAG

DL98b computed the effect of torques from passing ions on the electric dipole moments of the dust grains, which they named “plasma drag”. They computed this effect for straight-line trajectories (the “Born approximation”). Here we include the full hyperbolic trajectory in the case of charged grains. We also account for the rotation of the grain explicitly. Moreover, we do not include trajectories leading to collisions, as they will give away their entire angular momentum through collision, which we already accounted for. A precise calculation is important because plasma drag is one of the major excitation processes in some environments. Note that treatments of the plasma drag effect that treat the plasma as a linear dielectric medium [Ragot 2002] with the drag due to the imaginary part of the dielectric constant \(\Delta\epsilon(\omega, k)\) implicitly assume the Born approximation and do not capture the effects considered here.

We will find that the straight-line approximation usually overestimates the plasma drag. In the case of positively charged grains, there is a range of impact parameters where the ion trajectory is deflected away from the grain, thereby suppressing angular momentum transfer. For negatively charged grains, ions can be focused by electrostatic attraction. [Anderson & Watson 1993] argued that this is not a significant correction because the increased torque during close approach balances the shorter interaction time since the ion gains kinetic energy as it is attracted to the grain; however, we will see that in these cases there is a cancellation of angular momentum transfer in different parts of the trajectory that leads to reduced drag. For very special cases, the grain can corotate with the ion during close approach leading to an enhancement of the plasma drag, but this occurs for only a narrow range of impact parameters and does not compensate for the reduction of plasma drag that we find in other regimes.

As in DL98b, we find it easiest to directly compute the plasma excitation \(G_p\) and use the fluctuation-dissipation theorem to infer \(F_p = G_p\).

6.1 Charged grain

We consider the trajectories of positively charged ion (charge \(Z_i > 0\) in the electric potential of a charged dust grain (charge \(Z_g \neq 0\)). The trajectories are not strictly hyperbolic due to the presence of the electric dipole potential (see Section 5.3). However, we saw that it has little influence on collisions and we will neglect its effect on the trajectory here, assuming they are hyperbolic and determined by the Coulomb potential only. The eccentricity of the hyperbolic
trajectory of the ion will be denoted $e$ (as opposed to the elementary charge $q_e$).

Let the ion trajectory (a hyperbola) be in the $(\hat{e}_y, \hat{e}_z)$ plane, symmetric about the $\hat{e}_y$ axis. The position is given by

$$r = r\hat{e}_r = r(\alpha) (\cos \alpha \hat{e}_y + \sin \alpha \hat{e}_z).$$

The hyperbolic trajectory of impact parameter $b$ and velocity at infinity $v$ can be described in polar coordinates as

$$r(\alpha) = \frac{p}{e \cos \alpha - 1} \quad \alpha \in (-\alpha_e, \alpha_e) \quad (Z_g > 0),$$

$$r(\alpha) = \frac{p}{1 - e \cos \alpha} \quad \alpha \in (\alpha_e, 2\pi - \alpha_e) \quad (Z_g < 0);$$

the eccentricity and semilatus rectum $p$ of the trajectory are

$$e = \sqrt{1 + \left(\frac{m_ibv^2}{Z_gq_e^2}\right)^2} \quad \text{and} \quad p = b\sqrt{e^2 - 1}. \quad (108)$$

The range of longitudes $\alpha$ of the trajectory are determined by the limiting case

$$\alpha_e \equiv \arccos \frac{1}{e}. \quad (111)$$

The longitude can be related to the true anomaly $f$ familiar from planetary dynamics by $\alpha = f$ for repulsive ($Z_g > 0$) cases and $\alpha = \pi + f$ for attractive ($Z_g < 0$) cases. We will need the following expression for the time $t(\alpha)$, valid in both cases (for the case of an attractive potential, see e.g. Geving & Westerman (1971), Eq. (2.4.12)):

$$t(\alpha) = \frac{b}{v \sqrt{e + 1}} \left[ \sqrt{e + 1 + \ln \left| \frac{\tan \frac{\alpha}{2} + \sqrt{\tan^2 \frac{\alpha}{2} + \frac{1}{e + 1}}} \tan \frac{\alpha}{2} - \sqrt{\tan^2 \frac{\alpha}{2} + \frac{1}{e + 1}} \right|} \right] - \frac{2v \tan \frac{\alpha}{2}}{\tan^2 \frac{\alpha}{2} - \frac{1}{e + 1}}. \quad (112)$$

In order to characterize the torque on the grain, we must first take the unit vector in the direction of grain rotation,

$$\hat{e}_\omega = \sin \theta \cos \phi \hat{e}_x + \sin \theta \sin \phi \hat{e}_y + \cos \theta \hat{e}_z,$$

so that $\omega = \omega\hat{e}_\omega$. We use $(\theta, \phi)$ to parameterize the (general) direction of rotation. We define the other two axes:

$$\hat{e}_\theta = \cos \theta \cos \phi \hat{e}_x + \cos \theta \sin \phi \hat{e}_y - \sin \theta \hat{e}_z \quad \text{and} \quad \hat{e}_\phi = -\sin \phi \hat{e}_x + \cos \phi \hat{e}_y.$$ 

In this system the electric dipole moment of the grain is

$$\mu = \mu_\parallel \hat{e}_\omega + \mu_\perp [\cos(\omega t + \chi)\hat{e}_\theta + \sin(\omega t + \chi)\hat{e}_\phi],$$

where $t = 0$ is taken to be the time when the ion is at the closest approach (i.e. $r \parallel \hat{e}_y$) and $\chi \in [0, 2\pi)$ is the random angle that $\mu_\perp$ makes with $\hat{e}_\phi$ at that time.

The ion electric field exerts a torque on the grain dipole moment:

$$\int \frac{d\omega}{dt} = \mu \times E = -\frac{Z_gq_e}{r^2} \mu \times \hat{e}_r. \quad (116)$$

Using the conservation of angular momentum, $r^2 \dot{\alpha} = bv$, we can rewrite:

$$\frac{d\omega}{dt} = -\frac{Z_gq_e}{r^2} \mu \times \hat{e}_r. \quad (117)$$

We project that along the direction of $\hat{e}_\omega$:

$$\frac{d\omega}{\dot{\alpha}} = -\frac{Z_gq_e}{r^2} \frac{\mu_\perp}{\mu_\parallel} \left[ \cos(\omega t + \chi) \cos \alpha \cos \phi \right.$$

$$\sin(\omega t + \chi) \left( \cos \alpha \cos \phi \cos \phi - \sin \alpha \sin \theta \right).$$

Expanding the sines and cosines, we integrate over the trajectory. We keep only the parts of the integral for which the inbound and outbound parts do not cancel, i.e. those which are even under $\alpha \to -\alpha$ ($Z_g > 0$) or $\alpha \to 2\pi - \alpha$ ($Z_g < 0$); note that $t(\alpha)$ is even. We are then left with

$$\delta \omega_\parallel = \frac{Z_gq_e}{r^2} \frac{\mu_\perp}{\mu_\parallel} \left[ \sin \chi \cos \theta \sin \phi - \cos \chi \cos \phi \right.$$

$$\times \cos \omega t \cos \phi \cos \alpha \sin \alpha \cos \phi \left. - \cos \chi \sin \theta \int \sin \omega t \sin \phi \cos \alpha \right].$$

---

Note: The positive and negative cases of Eq. (109) could have been unified by taking the negative branches of the square root in Eq. (110), however we have not taken this route.
In order to find the plasma excitation coefficient, we need to sum \( \delta \omega^2 \) over collisions. We begin by averaging \( \delta \omega^2 \) over solid angles for \((\theta, \phi)\) and over angles for \(\chi\). The result is

\[
\langle \delta \omega^2 \rangle = \frac{1}{3} \left( \frac{2Z_i e b v}{I b v} \right)^2 \mathcal{I} \left( \frac{\omega b}{v}, e, Z_g \right).
\]

We have defined the integral

\[
\mathcal{I} \left( \frac{\omega b}{v}, e, Z_g \right) \equiv \left( \int \cos \omega t \cos \alpha \, d\alpha \right)^2 + \left( \int \sin \omega t \sin \alpha \, d\alpha \right)^2,
\]

where the integration limits are given by \(0 < \alpha < \alpha_c\) \((Z_g > 0)\) or \(\alpha_c < \alpha < \pi\) \((Z_g < 0)\). Note that \(\mathcal{I}\) only integrates over the inbound part of the trajectory; the outbound part is equal by symmetry.

The excitation rate due to plasma drag is then given by:

\[
\frac{d\Delta \omega^2}{dt} = \int_0^\infty \frac{d\omega}{\omega} \int_{b_{\max}(v)}^{\infty} \frac{2n_i b v}{\pi \lambda_D^2} \left( \frac{m_i}{2\pi kT} \right)^{3/2} e^{-m_i v^2/2kT} \left( \frac{2Z_i e b v}{I b v} \right)^2 \mathcal{I} \left( \frac{\omega b}{v}, e, Z_g \right),
\]

where \(b_{\max}(v)\), the maximum impact parameter for collision to occur, is defined as

\[
b_{\max}(v) = \left\{ \begin{array}{ll}
0 & \text{if } a_{ex} \sqrt{1 - \left( \frac{2kT}{mv^2} \right)^2} \leq \psi

\left( \frac{m_i}{m_v} \right) \left( \frac{a_{ex} kT}{a_c} \right) & \text{if } \frac{mv^2}{2kT} \leq \psi
\end{array} \right.
\]

where \(\psi = Z_g Z_i a_{ex}^2/a_{ex} kT\).

Note that technically the integration over impact parameters should stop at the Debye length

\[
\lambda_D = \sqrt{\frac{kT}{4\pi n_i e^2}} \approx 398 \left( \frac{T_2}{n_e/0.03 \text{cm}^{-3}} \right)^{1/2}
\]

However, we will see below that the integrand vanishes exponentially for

\[
b > v/\omega \approx 4.5 \times 10^{-5} a_{ex} \frac{v}{\sqrt{m_i/v}} \frac{\omega b}{\omega} \text{ cm}
\]

which is much smaller than the Debye length.

Converting this into an excitation coefficient, we find

\[
G_p = \frac{n_i}{n_H} \sqrt{\frac{m_i}{m_H}} \left( \frac{Z_i e b v}{a_{ex} kT} \right)^2 \mathcal{G}_p \left( \frac{\psi}{\sqrt{\frac{m_i a_{ex}^2}{2kT}}}, \omega \right)
\]

where

\[
\mathcal{G}_p \left( \frac{\psi}{\sqrt{\frac{m_i a_{ex}^2}{2kT}}}, \omega \right) \equiv \int_0^\infty 2\omega e^{-u^2} du \int_{b_{\max}}^{\infty} \frac{dc}{c} \mathcal{I} \left( \frac{\Omega c}{u}, e, Z_g \right),
\]

where the eccentricity is given by

\[
e = \sqrt{1 + \left( \frac{2a_{ex}^2}{\psi} \right)^2}.
\]

Note that we recover DL98b result \(^9\) in the limit \(I = 1\).

This expression has to be averaged over the grain charge and summed over all present ions.

**Straight line limit for \(I\)**

In the limit \(e \to \infty\), it is easier to express the integrals as a function of time, using

\[
\cos \alpha = \frac{y}{\sqrt{y^2 + z^2}} = \frac{b}{\sqrt{b^2 + (vt)^2}}, \quad \sin \alpha = \frac{z}{\sqrt{b^2 + (vt)^2}}, \quad \text{and} \quad \frac{d\tau}{dt} = \frac{1}{1 + \left( \frac{vt}{b} \right)^2}.
\]

In this case, the first integral for \(I\) reduces to

\[
\int \cos \omega t \cos \alpha \, d\alpha = \frac{v}{b} \int_0^\infty \frac{\cos \omega t \, dt}{1 + \left( \frac{vt}{b} \right)^2} = \frac{\omega b}{v} K_1 \left( \frac{\omega b}{v} \right),
\]

where \(K_1\) is a modified Bessel function of the second kind.

[Here we used Eq. (9.6.25) of Abramowitz & Stegun (1972)](https://dl.acm.org/) with \(v = 1\), \(z = 1\), and \(x = \omega b/v\).] The other integral is

\[
\int \sin \omega t \sin \alpha \, d\alpha = \frac{v}{b} \int_0^\infty \frac{vt}{b} \frac{\sin \omega t \, dt}{1 + \left( \frac{vt}{b} \right)^2} = \int_0^\infty \frac{\tau \sin \tau \, d\tau}{\sqrt{1 + \tau^2}},
\]

where \(x = \omega b/v\). Since \(\tau(1 + \tau^2)^{-3/2} = (1 + \tau^2)^{-1/2}\), we can integrate by parts and find

\[
\frac{-\sin \tau \tau}{\sqrt{1 + \tau^2}} \bigg|_0^\infty + \int_0^\infty \frac{x \cos \tau \tau \, d\tau}{\sqrt{1 + \tau^2}}.
\]

The boundary terms evaluate to zero, and the second integral can again be evaluated to \(x K_0(x)\) using Eq. (9.6.25) of [Abramowitz & Stegun (1972)](https://dl.acm.org/) with \(v = 0\) and \(z = 1\). Thus we have

\[
I = x^2 \left[ K_0^2(x) + K_1^2(x) \right], \quad x = \frac{\omega b}{v}
\]

Note that when \(\omega \to 0\) we recover DL98b result, i.e.

\[
I \left( \frac{\omega b}{v}, 0, e \to \infty, Z_g \right) = 1.
\]

We moreover have an exact functional shape for the cutoff at large rotation rates.

**Non rotating grain limit for \(I\)**

It is straightforward to show that

\[
I \left( \frac{\omega b}{v}, 0, e \right) = 1 - \frac{1}{e^2}
\]

for both positively and negatively charged grains. Thus, the nearly parabolic trajectories \(e - 1 \ll 1\) are suppressed by a factor \(~2(e - 1)\).

The numerical calculation of \(I\) in the general case is tricky because it involves integrating an oscillating function which frequency goes to infinity at one limit of the integral, as \(f(\alpha \to \alpha_c) \to \infty\). We refer the reader to Appendix A.
for the description of the method used for numerical computation. Fig. 3 shows the resulting dimensionless torques. An important feature is that for negatively charged grains, ions with nearly parabolic trajectories may corotate with the grain which results in an enhanced torque.

### 6.2 Neutral Grain

The exact calculation of the trajectory in the electric dipole potential and the “image charge potential” is untractable analytically, and would require a heavy numerical calculation. Therefore, we will make the following simplifications. First, we neglect the effect of the electric dipole moment on the trajectory. Therefore, we will make the following simplifications.

An important feature is that for negatively charged grains, ions with nearly parabolic trajectories may corotate with the grain which results in an enhanced torque.

### 7 Infrared Emission

A dust grain absorbs light and reemits it in the infrared. A rotating grain will also radiate angular momentum, which damps its rotation.

DL98b compute this damping rate by modelling the grain as composed of six rotating dipoles. We give here a more accurate calculation, using the correlation functions of the dipole moment in the grain frame. Our result is a factor of two greater than that of DL98b. We present a classical calculation in this section; the quantum calculation is presented in Appendix B and gives the same result.

The normalized excitation rate for plasma drag for Cold Neutral Medium conditions, Eq. (162), is shown in Fig. 6. The normalized excitation rate due to plasma drag $G_p(\omega_{th})$ for a neutral grain, a positively charged grain, and a negatively charged grain in CNM conditions (Eq. (162)), evaluated at the “thermal rotation rate” $\omega_{th} = (2kT/I)^{1/2}$. All of them are lower than estimated by DL98b. It is clear that the positively charged grains are much less excited than the neutral and negatively charged grains. The kink at 6Å is due to the change of grain shape.

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rotating around the z-axis, without precession, with angular frequency $\omega$. We have

$$
\begin{align*}
p_x' &= \cos \omega t \ p_x - \sin \omega t \ p_y, \\
p_y' &= \sin \omega t \ p_x + \cos \omega t \ p_y, \quad \text{and} \\
p_z &= p_z. \\
\end{align*}
$$

A straightforward calculation leads to the following expressions in the lab frame:

$$
\begin{align*}
p_x' &= p_x^2 + p_y^2 + p_z^2 + 4\omega (p_x p_y - p_y p_x) \\
&\quad + \omega^2 \left[4 \left(p_x^2 + p_y^2\right) - 2 \left(p_x p_y + p_y p_x\right)\right] \\
&\quad + 4\omega^3 \left(p_x p_y - p_y p_x\right) + \omega^4 \left(p_x^2 + p_y^2\right) \\
(p \times \dot{p})_z &= p_x p_y - p_y p_x + \omega^2 \left(2 \left(p_x^2 + p_y^2\right) - (p_x p_y + p_y p_x)\right) \\
&\quad + 3\omega^2 \left(p_x p_y - p_y p_x\right) + \omega^3 \left(p_x^2 + p_y^2\right).
\end{align*}
$$

Since we are interested in the statistical properties of the emission, we define the unequal-time dipole moment correlation function in grain coordinates,

$$
C_{ij}(\tau) \equiv \langle \left(p_i(t) - \langle p_i \rangle \right) \left(p_j(t + \tau) - \langle p_j \rangle \right) \rangle, \\
$$

where $\langle p_i \rangle = \mu_i$ is just the constant dipole moment of the grain. We further assume statistical spherical symmetry of the dipole moment in the grain coordinates, i.e. $C_{ij} = C_{\delta ij}$.

(For a planar grain, the values of the correlation functions depend on the in-plane or out-of-plane character of the vibrational modes and may be anisotropic. However if the infrared emission arises during thermal spikes when the grain is not rotating around its axis of greatest angular momentum, we expect the isotropic analysis to be a good approximation.) The average values of the previous formulae then become

$$
\begin{align*}
\langle p_1 \rangle &= \mu_1, \\
\langle p_2 \rangle &= \mu_2, \\
\langle p_3 \rangle &= \mu_3, \\
\langle p_4 \rangle &= \mu_4, \\
\langle p_5 \rangle &= \mu_5, \\
\langle p_6 \rangle &= \mu_6.
\end{align*}
$$

10 Expectation values of derivatives such as $\langle \dot{p}_i^2 \rangle$ can be expressed

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Figure 5. Contour levels of $I(\omega b/v, e, Z_\circ > 0)$ (left panel) and $I(\omega b/v, e, Z_\circ < 0)$ (right panel). Both show that $I$ goes to unity for slowly rotating grain, straightline trajectories, and vanishes for rapidly rotating grains or nearly parabolic trajectory. In the case of negatively charged grains, though, there is a visible corotation regime, where $e - 1 \ll 1$ and $\omega b/v(e - 1) \sim 1$, for which the ion and the grain approximately corotate, enhancing the torque given to the grain.

\[
\langle \hat{p}^2 \rangle = 3C'''(0) - 12\omega^2 C''(0) + 2\omega^4 C(0) \tag{144}
\]

and

\[
\langle \hat{p} \times \hat{p} \rangle_2 = -6\omega C''(0) + 2\omega^3 C(0), \tag{145}
\]

where $'$ denotes the derivative of the correlation function with respect to $\tau$.

The Wiener-Khintchine Theorem relates the correlation functions to the spectral density $S_\nu$, $C(\tau) = \int_0^\infty S_\nu \cos(2\pi\nu\tau) d\nu$. Plugging back into Eqs. (144) and (145), we get

\[
\langle \hat{p}^2 \rangle = \int_0^\infty \left[ 3(2\pi\nu)^4 + 12\omega^2(2\pi\nu^2 + 2\omega^4) \right] S_\nu d\nu \tag{146}
\]

and

\[
\langle \hat{p} \times \hat{p} \rangle_2 = \int_0^\infty \left[ 6\omega(2\pi\nu^2)^2 + 2\omega^3 \right] S_\nu d\nu. \tag{147}
\]

Now making use of the assumption that the grain rotates slowly, i.e. that $\nu_{rot} \equiv \omega/2\pi \ll \nu_0 \equiv$ typical frequency of emission, in the infrared, we get, at the lowest order in $\nu_{rot}/\nu_0$, the average total power and average total rate of radiation of angular momentum:

\[
\left\langle \frac{dE}{dt} \right\rangle = \frac{2}{3c^3} \langle \hat{p}^2 \rangle = \frac{2}{c^3} \int_0^\infty (2\pi\nu)^4 S_\nu d\nu \tag{148}
\]

and

\[
\left\langle \frac{dL_z}{dt} \right\rangle = \frac{2}{3c^3} \langle \hat{p} \times \hat{p} \rangle_2 = \frac{4\omega}{c^3} \int_0^\infty (2\pi\nu)^3 S_\nu d\nu. \tag{149}
\]

If one knows the infrared power radiated per steradian per frequency interval $F_\nu$, such that

\[
\left\langle \frac{dE}{dt} \right\rangle = 4\pi \int_0^\infty F_\nu d\nu,
\]

one can deduce the rate of angular momentum loss through infrared emission:

\[
\left\langle \frac{dL_\nu}{dt} \right\rangle = \frac{2\omega}{\pi} \int_0^\infty F_\nu d\nu. \tag{151}
\]

This result is twice as big as the one given in DL98b. [The difference occurs because DL98b modeled the dipole fluctuations with six uncorrelated rotating dipoles, one rotating each direction in the $xy$, $yz$, and $xz$ planes. They counted the radiated power from all six of these, but only considered the angular momentum loss from two of them (in the $xy$ plane). The dipoles rotating in the $xz$ and $yz$ planes containing the rotation axis also emit net angular momentum however, and if they are considered one recovers the factor of 2.]

This classical treatment does not predict the rotational excitation from the recoil given by individual photons, which is a quantum effect. As in DL98b, we set

\[
\left\langle \frac{d\Delta l^2}{dt} \right\rangle = \frac{dN_{phot}}{dt} \hbar^2 = \hbar \int_0^\infty F_\nu d\nu. \tag{152}
\]

The normalized damping and excitation rates are then

\[
F_{IR} = \frac{2\tau_{TL}}{\pi kT} \int_0^\infty F_\nu d\nu \quad \text{and} \quad \quad \quad G_{IR} = \frac{\hbar}{\pi kT} \int_0^\infty F_\nu d\nu. \tag{153}
\]

We calculate the infrared emissivity of PAHs and small carbonaceous grains using the ”thermal continuous” approximation, described in DL01. They indeed show that this treatment leads to spectra very close to those predicted by the exact statistical treatment, and has the advantage of being computationally much faster. We obtain the steady-state energy distribution function and then get the infrared emissivity, as explained in DL01. We checked numerically that we recover DL98 result for low values of the radiation field intensity : $F_{IR}$, $G_{IR} \propto \chi$. However, their result for high values of the radiation field ($F_{IR} \propto \chi^{2/3}$, $G_{IR} \propto \chi^{5/6}$) relies on the fact that the absorption efficiency $Q_{\nu} \propto \nu^2$ at the characteristic frequencies of infrared emission. This is not valid anymore for high radiation fields, which offset the emission spectrum to higher
8 PHOTOELECTRIC EMISSION

An electron ejected from the grain carries away an angular momentum along the rotation axis (z-axis) equal to:
\[ \Delta L_z = m_e \rho (v^\prime - \rho \omega), \]
where \( v^\prime \) is its tangential velocity in the grain frame. From this we deduce that
\[ F_{pe} = \frac{m_e}{m_H} \frac{J_{pe}}{2 \pi a^2 n_H \sqrt{2kT/\pi m_H}}, \]
where \( J_{pe} \) is the photoemission rate and was described in section 4.3. The excitation rate can be obtained by first noticing that the rotational velocity is much smaller than the velocity of ejected electrons:
\[ a \omega \ll v^\prime \]
so that we have, up to small corrections
\[ \Delta L_z^2 = m_e v^\prime_z v^\prime^2. \]
We assume a cosine-law directional distribution for escaping electrons, so that \( \langle v^\prime_z^2 \rangle = \frac{1}{3} v^\prime_z^2 \), where we denote \( v_z \) the average velocity of the electron at the grain surface. The latter satisfies
\[ \frac{1}{2} m_e v^2 = \frac{(Z_a + 1) \omega^2}{a}, \]
where \( E_{pe} \) is the average energy at infinity of the photoejected electron. We finally get
\[ \langle \Delta L_z^2 \rangle = m_e \frac{2}{3} a^2 \frac{1}{a^2} \frac{1}{4} \langle v_z^2 \rangle = \frac{m_e}{3} \frac{a^4}{a^2} \frac{E_{pe}}{a} \left[ \frac{(Z_a + 1) \omega^2}{a} \right]. \]
So the normalized excitation rate is
\[ G_{pe} = \frac{m_e}{4 \pi n_H (8 \pi m_H k T)^{1/2}} \left[ \frac{1}{4} (Z_a + 1) \frac{a^2}{a} \frac{J_{pe}}{a^2} \right], \]
where \( G_{pe} \) is the heating rate due to photoemission of electrons, obtained from WD01b.

9 RANDOM H₂ FORMATION

DL98b showed that the random formation of H₂ molecules on the grain surface does not make a major contribution to rotational excitation. We use their result:
\[ G_{H_2} = \frac{2}{4} (1 - y) \frac{E_f}{kT} \left[ 1 + \frac{(J(J + 1))h^2}{2m_H E_f a^2} \right], \]
where \( y \) is the efficiency of H₂ formation, \( y = 2n_H / n_H \), \( E_f \approx 0.2eV \) is the average translational kinetic energy of the nascent H₂, and \( (J(J + 1)) \approx 10^3 \) gives its average angular momentum.

10 Resulting emissivity and effect of various parameters

Throughout this section and unless otherwise stated, we will take as a fiducial environment the CNM parameters specified frequencies, where the absorption efficiency has not a simple dependence on frequency anymore. We show the resulting infrared emission and damping coefficients in Fig. 7.

![Figure 8](image_url)

**Figure 8.** Rotational distribution function for a grain radius \( a = 7 \) Å, in CNM conditions, for a single value of the dipole moment \( \mu(10^{-7} \text{cm}) = 9.3 \) debye. The plot compares the solution of the Fokker-Planck equation with the DL98b Maxwellian approximation (DL98b Eq. (57) used with our \( F, G \)). Note that DL98b prescription gives \( \langle \omega^2 \rangle \approx 2 \times 10.7 \text{GHz} \), which is in excellent agreement with the value we get, \( \langle \omega^2 \rangle \approx 2 \times 10^9 \text{GHz} \). However, the shape of the distribution function is significantly different.

by
\[ n_H = 30 \text{ cm}^{-3}, \quad T = 100 \text{ K} \]
\[ x_H = n(\text{H}^+)/n_H \approx 10^{-3}, \quad x_C \equiv n(\text{C}^+)/n_H \approx 3 \times 10^{-4} \]
\[ x \equiv u/u_{\text{SRF}} = 1, \quad \gamma = 0. \]
We also take the rms intrinsic dipole moment to be
\[ \langle \mu_i^2 \rangle^{1/2} (\alpha = 10^{-7} \text{cm}) = 9.3 \text{ debye}. \]

For the size distribution parameters, we use those given by WD01a for a ratio of visual extinction to reddening \( R_V = 3.1 \), and a carbon abundance in the log-normal distributions \( b_C = 6 \times 10^{-5} \).

This section is intended to give some intuition into the effect of various parameters on the spinning dust spectrum. However, the reader should keep in mind that environment parameter space is many-dimensional, and changing several parameters at once may lead to modifications that are not superpositions of the effects described here.

10.1 General shape of the rotational distribution function

The rotational distribution function is obtained as described in section 4. We remind the reader that the Fokker-Planck equation is not strictly valid for the smallest grains, for which impulsive torques are important. It however still describes their rotational distribution function with more accuracy than a simple Maxwellian. Moreover, DL98 showed that impulsive torques may be neglected for grain radii \( a \geq 7 \text{Å} \). In Fig. 8 we show that the rotational distribution function obtained by the Fokker-Planck equation differs significantly from a Maxwellian. It has a sharper cutoff at high frequencies due to the proper accounting for rotational damping through electric dipole radiation.
frequency is given by

\[ \nu = \frac{p}{\nu_{\text{max}}} \left( \frac{\nu}{\nu_{\text{peak}}} \right)^{1/3} \]

where we defined the parameter

\[ \xi \equiv \frac{8G \gamma_{\text{th}}}{F^2 \tau_{\text{ed}}} \]

which denotes the non-Maxwellian character of the distribution function.

For \( \xi \ll 1 \), the distribution is nearly Maxwellian.

\[ f_a(\omega) \propto \exp \left( -\frac{F I \omega^2}{G 2kT} \right) \]

and the peak frequency is given by

\[ \nu_{\text{peak}} \approx \left( \frac{G}{F} \right)^{1/2} \frac{1}{2\pi} \sqrt{\frac{6kT}{I}} (\xi \ll 1). \]

Moreover, the total power emitted by a single grain

\[ j_a \propto \mu^2 \int \omega^2 f_a(\omega)d\omega \]

has the following dependence:

\[ j_a \propto \mu^2 \left( \frac{G}{F} \right)^2 T^2 (\xi \ll 1). \]

For \( \xi \gg 1 \) the distribution is strongly non-Maxwellian:

\[ f_a(\omega) \propto \exp \left( -\frac{\gamma_{\text{th}} I \omega^2}{3G^2 \tau_{\text{ed}}^2} \right) \]

and the peak frequency is given by

\[ \nu_{\text{peak}} \approx \left( \frac{G \tau_{\text{ed}}}{2\gamma_{\text{th}}} \right)^{1/4} \frac{1}{2\pi} \sqrt{\frac{6kT}{I}} (\xi \gg 1). \]

The total power is then given by

\[ j_a \propto \mu^2 \frac{G \tau_{\text{ed}}}{\gamma_{\text{th}}} T^2 (\xi \gg 1). \]

In Fig. 7 we show the rms rotation rate \( \omega^2 \gg 1/2 \) as a function of grain radius. As can be expected, the smallest grains are rotating with the highest angular velocity, as they have the smallest moment of inertia. Consequently, they radiate at the highest frequencies, and constitute the peak of the spectrum. Therefore, we will use Eqs. (161) to (163) for a grain of radius \( a_{\text{min}} = 3.5 \) Å to evaluate the effect of various parameters on the emissivity.

We finally remind the reader with the dependencies of characteristic timescales:

\[ \xi \ll 1 \]

Interestingly, Erickson (1957) had already obtained a result similar to Eq. (171) with a Fokker-Planck equation.
\[
\tau_{\text{H}} \propto n_{\text{H}}^{-1} T^{-1/2} , \quad \tau_{\text{ed}} \propto \mu^{-2} T^{-1}
\]  
(174)

10.2 Emissivity

Once the rotational distribution function is known, as a function of the intrinsic electric dipole moment, \( f_a(\omega; \mu_i) \), one can get the power radiated by a grain of radius \( a \) by averaging over the intrinsic dipole moments gaussian distribution \( P(\mu_i) \) defined in equation \( \ref{eq:gaussian} \). One gets :

\[
P_v(a) = \int d\mu_i P(\mu_i) \frac{\mu_i^2 \omega^6}{c^2} 2\pi f_a(\omega; \mu_i)
\]  
(175)

where \( \mu_i^2 = \frac{2}{3} \mu^2 \) for spherical grains, and \( \mu_i^2 = \mu^2 \) for cylindrical grains.

The overall effect of averaging over the dipole moments distribution is to broaden the spectrum, as can be seen in Fig. 10. The peak frequency remains approximately equal to that of \( P_v(\mu_i = (\mu_i^2)^{1/2}) \). We will discuss the effect of the rms intrinsic dipole moment in section 10.3.

The emissivity per H atom is then obtained by integrating the power radiated by each grain over the grain size distribution function, described in section 3.2. The emissivity for the CNM environment is shown in figure 11. Note that the grain size distribution directly weights the spectrum, and thus needs to be known with accuracy, which is not quite the case yet for the very small grains.

10.3 Effect of the rms intrinsic dipole moment \( (\mu_i^2)^{1/2} \)

Varying the rms intrinsic dipole moment affects the spectrum in three main ways. First, it affects the total power radiated, as \( P_v \propto \mu^2 \). Then, it affects the non-Maxwellian character of the distribution function, as \( \tau_{\text{ed}} \propto \mu^2 \). Finally, it affects the rotational damping and excitation rates essentially through plasma drag, which has \( G_p \propto \mu^2 \) (the effect on \( G_i \) is not as important).

Throughout the range of values considered, the total power has a weak dependence on \( \mu_i \), for low
values of the intrinsic dipole moment, but is not strictly independent of it, which comes from the multiple approximations made in this analysis (neglecting the charge displacement-induced dipole moment, and using Eq. (173) for the total power, after integration over the size distribution, instead of the total power radiated by a single grain).

**High dipole moment limit**

For high values of the electric dipole moment, plasma drag dominates both rotational damping and excitation. Therefore, \( G \propto G_p \propto \mu^2 \) so we get

\[
\nu_{\text{peak}} (\mu \to \infty) \to \text{constant} \quad (179)
\]

\[
j/n_H (\mu \to \infty) \propto \mu^2, \quad (180)
\]

which describe approximately the behavior observed in Fig. 12.

**10.4 Effect of the number density \( n_H \)**

The main effects of the number density are:

- Changing the relative contribution of gas-induced and radiation-induced rotational damping and excitation. For very low number densities, \( f_{\text{IR}} \) and \( G_{\text{IR}} \propto n_H \propto n_H^{-1} \) dominate over other rotational damping and excitation rates. For high densities, plasma drag and collisions are dominant. Note that the charge distribution is also modified as the higher the density, the more important is collisional charging compared to photoemission. As a consequence, the grains are positively charged at low densities, and tend to be negatively charged at high densities, due to the higher rate of collisions with electrons.

- Influencing the non-Maxwellian character of the rotational distribution function. The higher the number density, the closer is the distribution function to a Maxwellian. Numerical calculation shows that starting from CNM conditions and varying only \( n_H \), we transition to the Maxwellian regime (\( \xi \ll 1 \)) if \( n_H \gtrsim 10^5 \text{ cm}^{-3} \).

**Low density limit**

For very low number densities, the distribution is highly non-Maxwellian and we can use Eqs. (172) and (173), with \( G = G_{\text{IR}} \), to estimate the peak frequency and total power. As \( G_{\text{IR}}/n_H \) is independent of \( n_H \), both the number density and total power should asymptote to a constant value. We can estimate numerically the peak frequency in CNM conditions and get:

\[
\nu_{\text{peak}} (n_H \to 0) \approx 13 \text{ GHz} \quad (181)
\]

\[
j/n_H (n_H \to 0) \to \text{constant} \quad (182)
\]

which is in good agreement with Fig. 13.

**Intermediate densities**

Over the range \( 10^2 \text{ cm}^{-3} \lesssim n_H \lesssim 10^4 \text{ cm}^{-3} \), gas processes are dominant over infrared emission, so \( F_{\text{IR}}, G \) are roughly independent of \( n_H \). In addition, the distribution is still strongly non-Maxwellian. Using Eqs. (172) and (173), we thus find

\[
\nu_{\text{peak}} (10^2 \text{ cm}^{-3} \lesssim n_H \lesssim 10^4 \text{ cm}^{-3}) \propto n_H^{1/4} \quad (183)
\]

\[
j/n_H (10^2 \text{ cm}^{-3} \lesssim n_H \lesssim 10^4 \text{ cm}^{-3}) \propto n_H \quad (184)
\]

**10.5 Effect of the gas temperature \( T \)**

Temperature has a less obvious effect on the spectrum and we need to analyse in detail every damping and excitation process. It turns out the charge distribution of the smallest grains varies very little over the range of temperature considered \( 1 \text{ K} < T < 10^5 \text{ K} \) and they remain mostly neutral throughout this interval. The distribution remains strongly non-Maxwellian for \( T \) greater than a few K.

**Low temperature limit**

At very low temperatures, the dominant excitation process is collisions with ions. Indeed, the grains being mostly neutral, the ions interact strongly with the electric dipole potential. As \( \tilde{\mu} \propto T^{-1} \) and \( \phi \propto T^{-1/2} \), one can see from Eq. (112) and (113) that \( G_i \propto T^{-2} \). Plasma drag has also \( G_p \propto T^{-2} \) in principle but this becomes a shallower power law at low temperatures as the interaction timescale becomes longer than the rotation timescale. We find numerically, though, that roughly \( G \propto T^{-1.5} \) as \( G \) is not strictly equal to \( G_i \) (collisions with neutrals are also significant at low temperatures). Using Eqs. (169), (170), and (173), we find

\[
\nu_{\text{peak}} (3 \text{ K} \lesssim T \lesssim 10^2 \text{ K}) \approx 35 \text{ GHz} \quad (187)
\]

\[
j/n_H (3 \text{ K} \lesssim T \lesssim 10^2 \text{ K}) \approx \text{constant} \quad (188)
\]

Note that for extremely low temperatures, the distribution would become Maxwellian, and one would get, according to Eqs. (172), (173),

\[
\nu_{\text{peak}} (T \to 0) \propto T^{1/2} \quad (189)
\]

\[
j/n_H (T \to 0) \propto T^2 \quad (190)
\]

which can be guessed at the extreme low temperature end of Fig. 13. Temperatures below \( \sim 3 \text{ K} \) are of course unphysical, but for other environmental conditions than those of Eq. (102), the behaviour discussed above could take place at higher, observed temperatures.

**High temperature limit**

At very high temperatures, collisions with neutrals are the dominant damping and excitation process. The CNM environment being mostly neutral, \( F_{\text{IR}} \to 1 \) and \( G_{\text{IR}} \to 1/2 \) at high temperatures \( (G_{\text{IR}}^{(\infty)} \propto T_{\text{es}}/T \to 0) \). Moreover, the distribution becomes strongly non-Maxwellian, as \( \xi \propto T^{1/2} \). We therefore obtain

\[
\nu_{\text{peak}} (T \to \infty) \approx 200 T_3^{3/8} \text{ GHz} \quad (191)
\]
increases the total power radiated. Increasing the electric dipole moment decreases the peak frequency and increases the total power radiated.

\[ j/n_H (T \to \infty) \propto T^{3/2} \]  
(192)

Fig. 13 shows that these power laws describe the behavior of the peak frequency and total power with very good accuracy.

### 10.6 Effect of the radiation field intensity \( \chi \)

The radiation field affects the spectrum through only two ways. First of all, it changes the charge distribution of the grains as an increased radiation implies a higher photoemission rate. Second of all, it affects the rate of damping and excitation through infrared emission. (and photoelectric emission, but this is subdominant).

**Low radiation intensity limit**

In a low radiation field, \( F_{\text{rad}} \) and \( G_{\text{rad}} \) become negligible. The photoemission charging rate becomes insignificant compared with collisional charging, and the charge distribution function depends only on other environment parameters. Thus, one expects the spectrum to reach an asymptotic shape for very low radiation fields. The distribution is strongly non-Maxwellian, and the dominant excitation mechanism is collisions with ions, whereas the dominant damping mechanisms are plasma drag and collisions with neutrals. Thus, we find

\[ \nu_{\text{peak}}(\chi \to 0) \approx 35 \text{ GHz} \]  
(193)
\[ j/n_H (\chi \to 0) \to \text{constant} \]  
(194)

The kink around \( \chi \approx 2 \times 10^{-2} \) is due to our discontinuous treatment of the evaporation temperature for low intensities of the radiation field.

Around \( \chi \approx 1 - 10 \), the grain becoming more and more positively charged, the collisions with ions start being less efficient, although still the dominant excitation mechanism. This results in a slight decrease in both \( \nu_{\text{peak}} \) and \( j/n_H \).

**High radiation intensity limit**

In a high radiation field, \( F \approx F_{\text{IR}} \), and \( G \approx G_{\text{IR}} \). Both \( F_{\text{IR}} \) and \( G_{\text{IR}} \) are approximately (although not strictly) linear in \( \chi \), as shown in DL98b for the thermal spikes limit (see their equations (31) and (44)). Thus, \( \xi \propto \chi^{-1} \) so the distribution becomes Maxwellian. The peak frequency and total emitted power are then given by Eqs. (109) and (120), which imply that

\[ \nu_{\text{peak}}(\chi \to \infty) \approx \text{constant} \]  
(195)
\[ j/n_H (\chi \to \infty) \approx \text{constant} \]  
(196)

These asymptotic forms are not strictly valid because \( F_{\text{IR}} \) and \( G_{\text{IR}} \) are not strictly linear in \( \chi \), and do not have a simple dependence on that parameter.

### 10.7 Effect of the ionization fraction \( x_H \)

The Hydrogen ionization fraction affects the charge distribution by modifying the contribution from collisions with protons. It also changes the contribution of collisions with ions, neutrals and plasma drag. Characteristic timescales are left invariant, and \( \xi \gg 1 \) for any ionization fraction in otherwise CNM conditions.

**Low ionization fraction limit**

In that limit the rotational distribution function reaches an asymptotic form, and the collisions with ions and plasma drag can be neglected. However, there are still \( C^+ \) ions in the gas so collisions with ions and plasma drag may still be important, although the dominant excitation process is collisions with neutrals. We find

\[ \nu_{\text{peak}}(x_H \to 0) \approx 30\text{GHz} \]  
(197)
\[ j/n_H (x_H \to 0) \to \text{constant} \]  
(198)

**High ionization fraction limit**

In that case collisions with ions are the dominant excitation process. Using Eqs. (172) and (173) along with \( G \approx G_i \propto x_H \), we find

\[ \nu_{\text{peak}}(x_H \to 1) \approx 90(x_H/0.1)^{1/4}\text{GHz} \]  
(199)
\[ j/n_H (x_H \to 1) \propto x_H \]  
(200)

### 10.8 Concluding remarks

We remind the reader that all the estimates in the previous section were given by assuming that the peak frequency of the spinning dust spectrum is determined by that of the
Figure 13. Effect of various environmental parameters on the peak frequency and the total spinning dust emission. When one parameter is varied, other environment parameters are set to CNM conditions defined in Eq. (162). See sections 10.4 to 10.7 for a detailed description.
smallest grains, and that the total power follows the same
dependence upon environmental parameters as the power
emitted by the smallest grains. Therefore they should be
taken as an aid to understand the physics of spinning dust,
but not as an accurate description, which requires numerical
calculations.

The overall conclusion of this section is that varying a
single environmental parameter may change the peak fre-
quency by up to an order of magnitude, and the total emit-
ted power by several orders of magnitude. There is therefore
a very large range of possible peak frequencies and total pow-
ers that can be produced by spinning dust radiation. Mul-
tiphase environments, in particular, could emit very broad
spinning dust spectra. Deducing the environment parama-
ters from an observed spectrum could therefore be a difficult
task.

We show the spinning dust spectrum for various envi-
ronments and compare them to DL09b results in Fig. 14.

11 CONCLUSION

We have presented a detailed analysis of the rotational exci-
tation and damping of small carbonaceous grains. We have
refined DL09b results in the case of collisions, accounting
properly for the centrifugal potential which increases the
net damping rate. In the case of collisions with ions, we ac-
counted for the effect of the electric dipole potential on the
collision cross section. We found that this is a small effect
in the case of charged grains, but that it may significantly
increase the damping and excitation rates in the case of neu-
tral grains. We evaluated the contribution of “plasma drag”
by considering hyperbolic trajectories and rotating grains in
the case of charged grains, and straightlines in the case of
neutral grains. We corrected DL09b results for the damp-
ing through infrared emission. Finally, we calculated the ro-
tational distribution function by solving the Fokker-Planck
equation.

We believe our model provides a much more accurate
description of the spinning dust spectrum than previous
work. However, we would like to remind the reader of its
uncertainties and limitations. First, our model only com-
putes the total intensity of the emitted radiation and not
the polarization, which would require an additional study
of the alignment mechanisms for PAHs. Secondly, the dust
grains properties are poorly known:

• The size distribution and abundance of the smallest
  grains is uncertain, and in particular the nature of the cutoff
  at small grain sizes \( a \sim a_{\text{min}} \) can have a large effect on the
  spectrum.

• The permanent electric dipole moments of dust grains
  are not directly constrained by other dust observables. Given
  that it cannot be computed from first principles, one may
  regard it as a free parameter (or parameters) of the spinning
dust model.

Thirdly, we made some simplified calculations in some cases,
as an accurate calculation would have been intractable nu-
merically or substantially complicated the code:

• We used the Fokker-Planck approximation, which starts
to break down for our smallest grains because a single colli-
sion suffices to change the rotational state. We expect that

the main consequence of a full treatment would be a tail
in the emission spectrum extending to high frequencies, be-
cause impulsive collisions would be able to increase the ro-
tation velocities of the grains to \( > 2v_{\text{peak}} \) before dissipative
forces had time to act (an effect missed by the Fokker-Planck
treatment). Therefore one should not place too much confi-
dence in the many order-of-magnitude falloff at \( \sim 100 \text{ GHz} \)
seen in most of our models. (In many cases this will be unim-
portant observationally since at high frequencies the vibra-
tional dust contribution is dominant.)

• In the plasma drag calculation, we neglected the electric
dipole potential when evaluating the trajectory of ions, tak-
ing the straight-line (neutral grain) or hyperbolic (charged
grain) approximation. Relying on the study of collisions with
ions, we may expect the dipole moment to have a small effect
in the case of a charged grain. On the other hand, its effect
in the case of a neutral grain may be more important, as in
that case the electric dipole potential provides the dominant
interaction.

• We assumed the evaporation temperature for the small-
est grains was the “temperature” of the grain just after it has
absorbed a UV photon. This is a physically motivated as-
sumption but its validity is not established. The evaporation
temperature can have a significant effect on the spectrum,
as can be seen from Fig. 13 and one should be aware of the
uncertainty in this parameter. Also, we assumed that collisi-
sions transition from being sticking to elastic, as the density
exceeds a given threshold. Our model is therefore inaccurate
in the transition region.

• When calculating the infrared emission spectrum of the
grains, we used DL01 “thermal continuous” approximation,
which is not very accurate to describe the low energy part
of the spectrum. Whereas these uncertainties are not impor-
tant if one only wants the spectrum \( F_{\nu} \) in the mid-infrared,
they may lead to significant errors when calculating the corre-
spending damping and excitation rates, which are propor-
tional to \( \int \nu^{-2}F_{\nu}d\nu \) and \( \int \nu^{-1}F_{\nu}d\nu \) respectively.

• We ignored systematic torques, although this may not
be a major omission for the smallest dust grains.

Despite these uncertainties, we believe that this model
is the most complete thus far, and will be a useful tool for
comparison to observations and testing the spinning dust
hypothesis for anomalous microwave emission in various ISM
phases.

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Figure 14. Spinning dust spectra for several environment conditions: Cold Neutral Medium (CNM), Warm Neutral Medium (WNM), Warm Ionized Medium (WIM), Molecular Cloud (MC), Dark Cloud (DC) and Reflection Nebula (RN). The environments are defined in DL98, Table 1. The thin solid line is the result of our calculation, the dotted line being DL98 prediction, and the dashed line is the free-free emission (the free-free gaunt factor were taken from Sutherland (1998)). The parameters for the grain size distribution are: $R_V = 3.1$, $b_C = 6 \times 10^{-5}$ for the diffuse CNM, WNM and WIM environments, and $R_V = 5.5$, $b_C = 3 \times 10^{-5}$ for the dense MC, DC and RN environments. The apparent systematic increase of power around the peak frequency for our result is mainly due to the grain size distribution we use, which has an increased number of small grains compared to that used by DL98 (compare e.g. Fig. 2 from WD01a with Fig. 8 from DL98b). Note that for the DC environment, for which rotational excitation and damping is dominated by collisions with neutral species (mainly H$_2$ molecules), DL98 prediction largely overestimates the peak frequency and total power because they underestimate the damping rate (see Eq. 53 and subsequent discussion).

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We first make the change of variable
\[ A_1 \]
Positively charged grains:
integrating an oscillating function whose frequency goes to infinity at one limit of the integral, as \( t \to \infty \) at one limit of the integral, as 
\[ e^{i\omega t} \] (A3)

The functions inside the integrals are analytical on the complex plane, deprived from the branch cut \([-1, 1]\) on the real axis and the two poles \( \pm i\sqrt{\gamma} \). The integrands are at least \( O(z^{-2}) \) as \( |z| \to \infty \). Moreover, for \( y \to 0^+ \),
\[ \Re[\omega t(1 - iy)] \approx \Re\left[ i \ln \left( -1 + \frac{2i y}{y} \right) - \frac{e^{i\pi y}}{y} \right] < 0 \] (A4)

Thus, using the fact that the integral over the lower right part of the complex plane vanishes, we can replace our integrals by integrals over the axis
\[ z = 1 - iy, \quad 0 < y < +\infty. \] (A5)

Note that for \( e \to 1, I = \mathcal{O}(e^{-1}) \), as one may expect from almost parabolic trajectories if the grain repels the ion. Also, in the limit \( \omega b/v \to 0, I \to (e^2 - 1)/e^2 \).

A2 Negatively charged grains: \( Z_q < 0 \)
This time we make the change of variable
\[ z = \frac{1}{\sqrt{\gamma}} \tan \frac{\alpha}{2} \] (A6)
The expression for the time is now
\[ \omega(t) = \frac{\omega b}{v} \frac{1}{\sqrt{e^{2z}} - 1} \left( \ln \frac{z + 1}{z - 1} + 2e^{-\frac{z}{2}} - 1 \right) \]. (A7)

And we have:
\[ I = 4\gamma \left[ \Re \int_1^{\infty} e^{i\omega t(z)} \frac{1 - \gamma z^2}{(1 + \gamma z^2)^2} dz \right]^2 + 16\gamma^2 \left[ \Re \int_1^{\infty} e^{i\omega t(z)} \frac{z}{(1 + \gamma z^2)^2} dz \right]^2. \] (A8)

The functions inside the integrals are analytical on the complex plane, deprived from the branch cut \([-1, 1]\) on the real axis and the two poles \( \pm i\sqrt{\gamma} \). This time \( \Re(\omega t) \) is negative for \( z \) close to 1 when \( 3z > 1 \). Moreover, the two poles tend to infinity when \( e \to 1 \) so to avoid integrating too close to the poles, we integrate over the line
\[ z = 1 + e^{i\pi/4} y, \quad 0 < y < +\infty. \] (A9)

In that case, the integrals are not simply bounded anymore for nearly parabolic trajectories. One can show, by making the previous change of variables, that
\[ I \left( \frac{\omega b}{v}, e, Z_q < 0 \right) = \exp \left( \frac{2\pi \omega b}{v \sqrt{e^{2z}} - 1} I \left( \frac{\omega b}{v}, e, Z_q > 0 \right) \right). \] (A10)

This expression is ill behaved for nearly parabolic trajectories, as the exponential factor diverges whereas the \( I \)-integral vanishes. In order to avoid numerical problems, in the case of nearly parabolic trajectories, we make the change of variables
\[ u = \left( \tan \frac{\alpha}{2} \right)^{-1} \] (A11)
The expression for the \( I \)-integral is then, for \( e - 1 \ll 1 \):
\[ I \approx 4 \left\{ \int_{\sqrt{2}}^{\infty} \cos \left( \frac{\omega b}{v} (e - 1)(u + \frac{u^3}{3}) \right) \frac{u^2 - 1}{(u^2 + 1)^2} du \right\}^2 + 16 \left\{ \int_{\sqrt{2}}^{\infty} \sin \left( \frac{\omega b}{v} (e - 1)(u + \frac{u^3}{3}) \right) \frac{u}{(u^2 + 1)^2} du \right\}^2. \] (A12)
Note that in terms of the true anomaly \( f \) we have \( u = -\tan f/2 \) and the expression for the time can be found in Gevling & Westerman (1971), Eq. (2.3.9).

Here again we integrate along \( u = e^{i\nu/y} \), \( 0 < y < \infty \), which cancels the \( O(u^3) \) real part of the time and maximizes its positive imaginary part at infinity. Notice that for very small eccentricities, this is mainly a function of \((\omega b/v)(e - 1)\).

**APPENDIX B: QUANTUM TREATMENT OF INFRARED EMISSION**

In Section 7, we computed the net angular momentum loss due to infrared emission using classical electrodynamics. Here we reconsider the effect with a quantum calculation. We assume a spherically symmetric grain for simplicity, and neglect vibration-rotation interaction. We will recover the classical result in the limit \( J \gg 1 \), which is applicable to the dust grains considered in this paper.

The Hilbert space of the grain is characterized by the vibrational quantum numbers (generically denoted \( v \)) and the three rotation quantum numbers \( J, K, M \), where \( K \) is the projection of angular momentum onto the grain \( z \)-axis. The energy levels are given by

\[
E_{J,K,v} = E_v^0 + \frac{\hbar^2 J(J + 1)}{2I},
\]

where \( I \) is the grain moment of inertia and \( E_v^0 \) is the vibrational energy. The rotational wave functions are

\[
\Psi_{J,K}(\chi) = \sqrt{\frac{2J + 1}{8\pi^2}} D_{J,K}^{\pm}(\chi),
\]

where \( \chi = (\theta, \phi, v) \in SO(3) \) is the set of Euler angles, \( 8\pi^2 \) is the volume of \( SO(3) \), and \( D_J^{\pm} \) is the passive rotation matrix in the spin-\( J \) representation, i.e. \( D_M^{J1, \text{grain}} = D_M^{J2, \text{lab}} \).

Spontaneous infrared vibrational transitions are possible from vibrational state \( v \) to \( v' \); their rate is given by

\[
A_{J,K,v \rightarrow J',K',v'} = \frac{4(E_{J,v}^0 - E_{J',v'}^0)^3}{3\hbar^2 c^3} \times |\langle J', K', v' | \mu | J, K, M, v \rangle|^2,
\]

where \( \mu \) is the electric dipole moment operator. In the absence of vibration-rotation interaction, we may take the operator \( \mu \) to depend only on the vibrational quantum numbers and on the rotation matrix \( R(\chi) \) that converts grain-fixed to lab-fixed coordinates:

\[
\langle J', K', M', v' | \mu | J, K, M, v \rangle = \langle J', K', M' | R(\chi) | J, K, M \rangle \langle v' | \mu^{(g)} | v \rangle.
\]

Here \( \mu^{(g)} \) is the dipole moment in grain coordinates.

The transition rates can be determined by writing \( \mu^{(g)} \) in the polar basis,

\[
\mu^{(g)}_0 = \mu_{\pm}^{(g)} = \frac{\mp \mu_x^{(g)} \pm i \mu_y^{(g)}}{\sqrt{2}},
\]

in which \( \{\mu_{m}^{(g)}\}_{m=-1} \) transform in the \( L = 1 \) representation of \( SO(3) \). Written in this basis, the rotation matrix \( R(\chi) \) is the inverse of \( D_J^{\pm}(\chi) \), which for unitary \( D_J^{\pm} \) is the same as the Hermitian conjugate:

\[
\langle J', K', M', v' | \mu_{m}^{(g)} | J, K, M, v \rangle = \langle J', K', M' | D_m^{\pm}(\chi) | J, K, M \rangle \langle v' | \mu^{(g)}_m | v \rangle.
\]

The first matrix element can be evaluated by the rotation matrix integral,

\[
\langle J', K', M' | D_m^{\pm}(\chi) | J, K, M \rangle = \sqrt{\frac{(2J + 1)(2J + 1)}{8\pi^2}} \times \int \frac{D_{J,K,M}^{\pm}(\chi)}{D_{J',K',M'}^{\pm}(\chi)} \frac{1}{\pi^2} \chi.
\]

This transforms the spontaneous decay rate, Eq. (B3), into

\[
\frac{4(E_{J,v}^0 - E_{J,v'}^0)^3}{3\hbar^2 c^3} \times \frac{1}{m} \left| \langle v' | \mu^{(g)}_m | v \rangle \right|^2.
\]

We would now like to find the net decay rates to states of different \( J' \). To do this, we assume the grain is randomly oriented, i.e. we average over initial projections \( K \), and sum over final projections \( K' \). Using the 3j symbol orthonormality relations, one obtains

\[
A_{J,v \rightarrow J',M',v'} = \frac{4(E_{J,v}^0 - E_{J,v'}^0)^3}{3\hbar^2 c^3} \times \frac{1}{m} \left| \langle v' | \mu^{(g)}_m | v \rangle \right|^2.
\]

(The terms mixing different values of \( m' \) are eliminated by orthogonality relations.) The summation over \( m \) of course has at most one term, with \( m = \Delta M \pm M' - M \).

We are interested in the net angular momentum loss, which is most easily obtained by taking an initial state with \( M = J \). There are then six possible values of \( \Delta J \) and \( \Delta M \), constrained by selection rules \( \Delta J, \Delta M = -1, 0, +1 \) and the restriction \( \Delta M \leq \Delta J \). The branching ratios are constrained by (i) the energy difference factors in Eq. (B9), (ii) the factor of \( 2J' + 1 \), and (iii) the 3j symbol. We consider each.

The energy factors do not depend on \( \Delta M \). If we take natural frequency \( \nu = (E_v^0 - E_v'^0)/\hbar \), then the energies differences are given by

\[
E_v^0 - E_v'^0 = h\nu - \frac{\hbar^2}{2J} (2J + 1 + \Delta J) \Delta J.
\]

The classical grain rotation rate is \( \omega = \hbar J/\nu \). In the limit of \( J \gg 1 \) and \( \omega \ll \nu \), the energy difference is proportional to \( 1 - \nu \Delta J/2\nu \); so the cube of the energy difference is proportional to \( 1 - 3\nu \Delta J/2\nu \).
The square of the $3j$ symbol, multiplied by $2J' + 1$, can be directly evaluated for the six cases of interest. It is:

\[
\begin{align*}
\frac{2J - 1}{2J + 1} & \quad \Delta J = -1, \Delta M = -1, \\
\frac{1}{J + 1} & \quad \Delta J = 0, \Delta M = -1, \\
\frac{J}{J + 1} & \quad \Delta J = 0, \Delta M = 0, \\
\frac{1}{(J + 1)(2J + 1)} & \quad \Delta J = +1, \Delta M = -1, \\
\frac{1}{J + 1} & \quad \Delta J = +1, \Delta M = 0, \text{ and} \\
1 & \quad \Delta J = +1, \Delta M = +1. 
\end{align*}
\] (B11)

By multiplying these relative probabilities by $1 - 3\omega \Delta J / 2\pi \nu$, it is easily seen that the average $\langle \Delta M \rangle$ is exactly zero if $\omega = 0$. Therefore the leading contribution to $\langle \Delta M \rangle$ can be obtained by taking the large-$J$ limit of the $3j$ symbols. Transitions with $\Delta J \neq \Delta M$ are suppressed by powers of $J$ in Eq. (B11), so one has three available transitions: $\Delta M = -1, 0, +1, \Delta J = \Delta M$. Since the factors in Eq. (B11) go to unity, the branching ratio for these three transitions is determined entirely by the energy factor:

\[
P(\Delta M) = \frac{1}{3} \cdot \frac{\omega}{2\pi \nu} \Delta M.
\] (B12)

This implies an average loss of $z$-component of angular momentum

\[
\langle \Delta M \rangle = -\frac{\omega}{\pi \nu}.
\] (B13)

In particular, we may find the ratio of angular momentum loss to energy loss ($h\nu$), which is

\[
\frac{\dot{L}_z}{E} = \frac{h\omega / \pi \nu}{h \nu} = \frac{\omega}{2\pi^2 \nu^2}.
\] (B14)

With the normalization of Eq. (150) and this ratio, one recovers Eq. (151).