BOUND GEODESICS
FOR THE ATIYAH-HITCHIN METRIC

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ABSTRACT. The Atiyah-Hitchin metric has bounded geodesics which describe bound states of a monopole pair.

Introduction. The dynamics of two nonrelativistic BPS monopoles was described by Manton [1] as the geodesic flow on the space of collective coordinates of the monopoles $M^0_2$ with a special metric found explicitly by Atiyah and Hitchin [2]. Gibbons and Manton [3] studied the asymptotic metric (the Taub-Nut metric) and using its additional symmetry they integrated the equations of geodesics. They found in particular quasiperiodic solutions which describe bound states of a pair of monopoles. It is thus natural to treat the Atiyah-Hitchin metric as a small perturbation of the Taub-Nut metric and to apply the KAM theory to establish the existence of quasiperiodic geodesics. In the present note we sketch an implementation of this idea. The detailed exposition will appear elsewhere.

1. Analytic description of the Atiyah-Hitchin metric on $M^0_2$. The Atiyah-Hitchin metric on the four-dimensional manifold $M^0_2$ admits SO(3) as a symmetry group and the orbits of the action are nondegenerate, i.e., 3-dimensional with only one exception. Hence we can identify the tangent space to the orbit with the Lie algebra $\text{so}(3)$ and write the metric in the form

$$ds^2 = f^2 d\eta^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2,$$

where $\eta$ is a transversal coordinate, and $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the standard one-forms in $\text{so}(3)^*$.

$a, b, c$ and $f$ are functions of $\eta$ which can be described in the following way.

$$a^2 = 4K(K - E)(E - Kk'^2)/E,$$

$$b^2 = 4K(K - E)/E(E - Kk'^2),$$

$$c^2 = 4KE(E - Kk'^2)/(K - E),$$

where

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{-1/2} d\phi, \quad E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{1/2} d\phi$$

are complete elliptic integrals and $k' = \sqrt{1 - k^2}$ is the conjugate modulus.
In the formulas (2) \( k \) is assumed to be a function of \( \eta \) which can be chosen arbitrarily for the price of changing the function \( f \) appropriately. Gibbons and Manton [3] suggested the use of \( \eta = 2K(k), \pi < \eta < +\infty \), which leads to

\[
f^2 = (K - E)E/K(E - Kk^2).
\]

2. The reduced hamiltonian system. The SO(3) symmetry of the Atiyah-Hitchin metric (1) allows the reduction of the geodesic equations to the Euler type equations

\[
\begin{align*}
\frac{dM_1}{dt} &= \left( \frac{1}{b^2} - \frac{1}{c^2} \right) M_2 M_3, \\
\frac{dM_2}{dt} &= \left( \frac{1}{c^2} - \frac{1}{a^2} \right) M_3 M_1, \\
\frac{dM_3}{dt} &= \left( \frac{1}{a^2} - \frac{1}{b^2} \right) M_1 M_2, \\
\frac{d\eta}{dt} &= \frac{\partial H}{\partial p}, \\
\frac{dp}{dt} &= -\frac{\partial H}{\partial \eta},
\end{align*}
\]

where

\[
H = \frac{1}{2} \left( \frac{p^2}{f^2} + \frac{M_1^2}{a^2} + \frac{M_2^2}{b^2} + \frac{M_3^2}{c^2} \right).
\]

Geometrically the reduction means that the geodesic flow factors onto the system (4). In particular the metric (1) has bounded geodesics if and only if the system (4) has solutions bounded in \( \eta \).

\( M^2 = M_1^2 + M_2^2 + M_3^2 \) and \( H \) are first integrals of the system. Without loss of generality we can set \( M^2 = 1 \). Then the system can be put into an explicitly hamiltonian form by letting

\[
M_1 = \sqrt{1 - M_3^2} \cos \phi, \quad M_2 = \sqrt{1 - M_3^2} \sin \phi,
\]

which leads to the system

\[
\begin{align*}
\frac{d\eta}{dt} &= \frac{\partial H}{\partial p}, \\
\frac{d\phi}{dt} &= \frac{\partial H}{\partial M_3}, \\
\frac{dp}{dt} &= -\frac{\partial H}{\partial \eta}, \\
\frac{dM_3}{dt} &= -\frac{\partial H}{\partial \phi},
\end{align*}
\]

where

\[
H = \frac{1}{2} \left[ \frac{p^2}{f^2} + \frac{(1 - M_3^2) \cos^2 \phi}{a^2} + \frac{(1 - M_3^2) \sin^2 \phi}{b^2} + \frac{M_3^2}{c^2} \right].
\]

Transforming \( H \) further, we get \( H = H_0 + H_1 \), where

\[
H_0 = \frac{1}{2} \left[ \frac{p^2}{f^2} + \frac{1}{2} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) (1 - M_3^2) + \frac{M_3^2}{c^2} \right],
\]

\[
H_1 = \frac{1}{4} \left( \frac{1}{a^2} - \frac{1}{b^2} \right) (1 - M_3^2) \cos 2\phi.
\]

The hamiltonian \( H_0 \) does not depend on \( \phi \) so it defines an integrable hamiltonian system, the other integral being \( M_3 \). Moreover there are bounded quasiperiodic motions in the system. Indeed

\[
H_0 = \frac{1}{2} \frac{p^2}{f^2} + V(\eta, M_3), \quad V(\eta, M_3) = \frac{1}{4} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) (1 - M_3^2) + \frac{M_3^2}{2c^2}
\]
and for a fixed value of $M_3$, $V$ has a global minimum $V_0(M_3)$, at least for small values of $M_3$. To see this note that $1/a^2 + 1/b^2$ is decreasing to zero and $1/c^2$ is increasing to some positive value as $\eta \to +\infty$. The manifold \( \{ H_0 = \text{const}, M_3 = \text{const} \} \) for values of $H_0$ close to $V_0(M_3)$ is compact and hence by the Liouville-Arnold Theorem it must be the torus carrying a quasiperiodic motion.

3. **KAM theory.** We want to treat the hamiltonian system (5) as a perturbation of the integrable system with the hamiltonian $H_0$. To apply the KAM theory we have to find the action-angle variables for $H_0$ and to estimate the perturbation.

Expanding $K$ in the conjugate modulus $k'$ we have

\[
K = -\ln(k'/4)(1 + O(k'^2)) + O(k'^2) \quad \text{as } k' \to 0.
\]

Hence

\[
(6) \quad k'^2 = O(e^{-\eta}) \quad \text{as } \eta \to +\infty.
\]

Also

\[
E = 1 + \ln(k'/4)O(k'^2) + O(k'^2) \quad \text{as } k' \to 0
\]

so that

\[
(7) \quad E = 1 + O(\eta e^{-\eta}) \quad \text{as } \eta \to +\infty.
\]

Applying (6) and (7) to the formulas (2) and (3), we get

\[
\frac{1}{4} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) = \frac{1}{2\eta(\eta - 2)} + O(\eta^{-2}e^{-\eta}),
\]

\[
\frac{1}{2} \frac{1}{c^2} = \frac{1}{8} \frac{\eta - 2}{\eta} + O(\eta e^{-\eta}),
\]

\[
\frac{1}{f^2} = \frac{\eta}{\eta - 2} + O(\eta e^{-\eta}), \quad \text{and}
\]

\[
\frac{1}{a^2} - \frac{1}{b^2} = O(\eta^{-1}e^{-\eta}).
\]

We introduce the integrable hamiltonian

\[
H_{00} = \frac{1}{2} p^2 \frac{\eta}{\eta - 2} + \frac{1}{8} M_3^2 + \frac{1}{4} \left( \frac{1}{\eta - 2} (1 - M_3^2) - \frac{1}{\eta} \right).
\]

By (8) $H_{01} \equiv H_0 - H_{00} = O(\eta e^{-\eta})$ and also $H_1 = O(\eta^{-1}e^{-\eta})$.

We will find the action variables $I, J$ for the hamiltonain $H_{00}$ in the region of the phase space where the motion is bounded. Let $H_{00} = \frac{1}{2} p^2 \eta/((\eta - 2) + W(\eta, M_3)$ and let $\eta_0(M_3)$ be the value of $\eta$ at which $W$ attains its minimum, i.e., $(\partial W/\partial \eta)(\eta_0, M_3) = 0$. We will use $\varepsilon = 1/\eta_0$ as a small parameter. We have

\[
\varepsilon - \varepsilon^2 = \frac{1}{4} M_3^2.
\]

We choose the basic cycles for the torus \( \{ M_3 = \text{const}, H_{00} = \text{const} \} \) to be

\[
\gamma_1 = \{ M_3 = \text{const}, p = \text{const}, \eta = \text{const}, 0 \leq \phi \leq 2\pi \} \quad \text{and} \quad \gamma_2 = \{ M_3 = \text{const},
\]

\[
\eta = \text{const}, 0 \leq \phi \leq 2\pi \}.
\]
\[ \phi = \text{const}, \ H_{00} = \text{const} \} \text{ and then} \]
\[ I = \frac{1}{2\pi} \int_{\gamma_1} p \, d\eta + M_3 \, d\phi = M_3, \]
\[ J = \frac{1}{2\pi} \int_{\gamma_2} p \, d\eta + M_3 \, d\phi = \frac{1}{2\pi} \int_{\gamma_2} p \, d\eta. \]

To evaluate the last integral we make for a fixed \( M_3 \) the change of variables \( \eta = (1 + \tilde{\eta})/\epsilon, \ p = \epsilon \tilde{p} \). We have
\[ H_{00} = \frac{1}{2} \epsilon - \epsilon^2 + \frac{1}{2} \epsilon^2 \left[ \left( \frac{\tilde{\eta}}{\tilde{\eta} + 1} \right)^2 + \tilde{p}^2 \right] \left( 1 + \frac{2\epsilon}{\tilde{\eta} + 1 - 2\epsilon} \right) \]
and \( J = (1/2\pi) \int_{\gamma_2} \tilde{p} \, d\tilde{\eta}. \)

By straightforward integration
\[ J = (1 - c)^{-1/2} - 1 + O(\epsilon), \]
where
\[ c = \left[ \left( \frac{\tilde{\eta}}{\tilde{\eta} + 1} \right)^2 + \tilde{p}^2 \right] \left( 1 + \frac{2\epsilon}{\tilde{\eta} + 1 - 2\epsilon} \right). \]

Finally
\[ H_{00} = \frac{1}{2} (\epsilon - \epsilon^2) - \frac{1}{2} \epsilon^2 \frac{1}{(J + 1)^2} + O(\epsilon^3). \]

Switching back to \( M_3 = I \), we get
\[ H_{00} = \frac{1}{8} I^2 - \frac{1}{32} \frac{I^4}{(J + 1)^2} + O(I^6). \]

For small \( I \) where the perturbation \( H_{01} + H_1 \) is small the hamiltonian \( H_{00} \) is degenerate in the sense that the hessian \( \det(\partial^2 H/\partial (I, J)^2) \) is also small, so that the standard KAM theory does not work in our case. The appropriate version of the KAM theory was actually developed by Arnold \([4]\) and we can conclude that for sufficiently small \( I_0 \), for most initial conditions in the domain \( 0 < I \leq I_0 \) (in the sense of Lebesgue measure) the motion is quasiperiodic and close for all \( -\infty < t < +\infty \) to the motion
\[ \dot{i} = 0, \quad \dot{j} = 0, \quad \dot{\phi}_I = \frac{\partial H_{00}}{\partial I}, \quad \dot{\phi}_J = \frac{\partial H_{00}}{\partial J} \]
with appropriate initial conditions. \( \phi_I \) and \( \phi_J \) are the angle variables conjugate to \( I \) and \( J \) respectively.

**Conclusion.** The Atiyah-Hitchin metric has many bounded geodesics. Their union in the phase space has positive Lebesgue measure.

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