PERRON VECTOR ANALYSIS FOR IRREDUCIBLE NONNEGATIVE TENSORS AND ITS APPLICATIONS

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Abstract. In this paper, we analyse the Perron vector of an irreducible nonnegative tensor, and present some lower and upper bounds for the ratio of the smallest and largest entries of a Perron vector based on some new techniques, which always improve the existing ones. Applying these new ratio results, we first refine two-sided bounds for the spectral radius of an irreducible nonnegative tensor. In particular, for the matrix case, the new bounds also improve the corresponding ones. Second, we provide a new Ky Fan type theorem, which improves the existing one. Third, we refine the perturbation bound for the spectral radii of nonnegative tensors, from which one may derive a comparison theorem for spectral radii of nonnegative tensors. Numerical examples are given to show the efficiency of the theoretical results.

1. Introduction. As a generalization of a matrix, the study of a tensor plays an important role in numerical multilinear algebra because of its applications. Analogical to nonnegative matrices, nonnegative tensors also arise in many practical applications, such as the multilinear PageRank, hypergraphs, higher order Markov chains (e.g., see [3, 7, 8, 9, 19]).

Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex fields, respectively. We consider an order $m$ dimension $n$ tensor $\mathcal{A}$ consisting of $n^m$ entries in $\mathbb{R}$:

$$\mathcal{A} = (\mathcal{A}_{i_1,i_2,\cdots,i_m}), \quad \mathcal{A}_{i_1,i_2,\cdots,i_m} \in \mathbb{R}, \quad i_1,i_2,\cdots,i_m \in \langle n \rangle,$$

where $\langle n \rangle = \{1, 2, \cdots, n\}$. A tensor $\mathcal{A}$ is called nonnegative (or, respectively, positive), if all its entries are nonnegative (or, respectively, positive). Based on this definition, we denote the set of all order $m$ dimension $n$ tensors by $\mathbb{R}^{[m,n]}$, and the set of all order $m$ dimension $n$ nonnegative (or, respectively, positive) tensors by $\mathbb{R}_+^{[m,n]}$ (or, respectively, $\mathbb{R}_+^{[m,n]}$).

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For a given vector $x = (x_1, \cdots, x_n)^T$, real or complex, we define a tensor-vector multiplication $\mathbf{A}x^{m-1}$ to be a dimension $n$ vector:

$$\mathbf{A}x^{m-1} := \left( \sum_{i_2, \cdots, i_m=1}^{n} \mathbf{A}_{i_1, i_2, \cdots, i_m} \right)_{1 \leq i \leq n}.$$

With this multiplication, Lim [12] and Qi [17] firstly introduced the following definition of the eigenpair of a tensor, which has brought much attention in recent studies of numerical multilinear algebra.

**Definition 1.1.** Let $\mathbf{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenpair of $\mathbf{A}$ if

$$\mathbf{A}x^{m-1} = \lambda x^{m-1},$$

where $x^{m-1} = (x_1^{m-1}, \cdots, x_n^{m-1})^T$. We call $(\lambda, x)$ an H-eigenpair if both $\lambda$ and $x$ are real.

**Definition 1.2.** Let $\mathbf{A} \in \mathbb{R}^{[m,n]}$. By $\rho(\mathbf{A})$ we denote the spectral radius of $\mathbf{A}$, i.e., the greatest eigenvalue of $\mathbf{A}$ in magnitude.

For the largest eigenvalue of a nonnegative tensor, the theoretical results about the Perron-Frobenius theorem were given by Chang et al. [1], Friedland et al. [5] and Lim [12]. Besides, Yang and Yang [21] presented the weak Perron-Frobenius theorem. In [12], the concept of irreducibility of a matrix has been extended to a tensor.

**Definition 1.3.** [12] A tensor $\mathbf{A} = (\mathbf{A}_{i_1, i_2, \cdots, i_m}) \in \mathbb{R}^{[m,n]}$ is called reducible if there exists a nonempty proper index subset $\mathbb{J} \subset \{1, 2, \cdots, n\}$ such that

$$\mathbf{A}_{i_1, i_2, \cdots, i_m} = 0, \forall i_1 \in \mathbb{J}, i_2, \cdots, i_m \notin \mathbb{J}.$$ 

If $\mathbf{A}$ is not reducible, we call $\mathbf{A}$ irreducible.

**Lemma 1.4.** [1] Let $\mathbf{A} \in \mathbb{R}_{+}^{[m,n]}$. Then there exist $\lambda_0 \geq 0$ and $x \geq 0$ such that $\mathbf{A}x^{m-1} = \lambda_0 x^{m-1}$. In particular, if $\mathbf{A}$ is irreducible, then $\lambda_0 > 0$ and $x > 0$. Moreover, if $\lambda$ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \lambda_0$. If $\lambda$ is an eigenvalue of $\mathbf{A}$, then $|\lambda| \leq \lambda_0$.

Here we call $x$ a Perron vector of a nonnegative tensor corresponding to its largest nonnegative eigenvalue. By Lemma 1.4, we know that the spectral radius of an irreducible nonnegative tensor is also an eigenvalue. Thus some researches have focused on studying this special eigenvalue and the Perron vector (e.g., see [1, 2, 11, 15, 16, 18, 21, 22]). In this paper, we find a new technique to refine the bounds on this Perron vector, from which one gives some sharper bounds on the spectral radius, the Ky Fan type inequality and the variation of the spectral radii of nonnegative tensors.

Let $\mathbf{A} = (\mathbf{A}_{i_1, i_2, \cdots, i_m}) \in \mathbb{R}_{+}^{[m,n]}$ be irreducible, and $x$ be its Perron vector. Then by $r_i(\mathbf{A})$ we denote the sum of the $i$th row, i.e., $r_i(\mathbf{A}) = \sum_{i_2, \cdots, i_m=1}^{n} \mathbf{A}_{i_1, i_2, \cdots, i_m}$, and

$$\eta = \min_{1 \leq i_1, \cdots, i_m \leq n} \mathbf{A}_{i_1, i_2, \cdots, i_m}, \ x_{\min} = \min_{1 \leq i \leq n} x_i \text{ and } x_{\max} = \max_{1 \leq i \leq n} x_i.$$
For the matrix case, i.e., $A = (A_{i,j})$ is an $n \times n$ positive matrix, a classical result for the Perron vector was given as follows (e.g., see [14]):

$$\frac{\min_{1 \leq i,j \leq n} A_{i,j}}{\max_{1 \leq s,t \leq n} A_{s,t}} \leq \frac{x_{\min}}{x_{\max}} \leq \sqrt{\frac{\min_{1 \leq i \leq n} r_i(A) - \eta}{\max_{1 \leq i \leq n} r_i(A) - \eta}} \equiv \sigma. \tag{1}$$

This result was extended to the positive tensor by Wang and Wu [20], i.e.,

$$\kappa_0 \leq \frac{x_{\min}}{x_{\max}} \leq \omega_0 \equiv \eta,$$ \tag{2}

where

$$\kappa_0 = \min_{s,t,i_2,\ldots,i_m} \left( \frac{A_{s,i_2,\ldots,i_m}}{A_{t,i_2,\ldots,i_m}} \right)^{\frac{1}{m-1}}, \quad \omega_0 = \left( \frac{\min_{1 \leq i \leq n} r_i(A) - \eta}{\max_{1 \leq i \leq n} r_i(A) - \eta} \right)^{\frac{1}{2}}.$$

Li and Ng [11] considered an irreducible nonnegative tensor $A \in \mathbb{R}^{[m,n]}_+$, and presented the following bound:

$$\kappa_1 \leq \frac{x_{\min}}{x_{\max}} \leq \omega_1 \equiv \eta,$$ \tag{3}

where

$$\kappa_1 = \max_{2 \leq k,k' \leq m} \min_{1 \leq i_1,i_2,\ldots,i_m \leq n, \text{ except } i_k} \frac{\sum_{i_1,i_2,\ldots,i_m=1} A_{i_1,i_2,\ldots,i_m}}{\sum_{i_1,i_2,\ldots,i_m=1} A_{i_1,i_2,\ldots,i_m}}$$

and

$$\omega_1 = \left( \frac{\min_{1 \leq i \leq n} r_i(A) - \min_{1 \leq i \leq n} A_{i_1,i_2,\ldots,i_m}}{\max_{1 \leq i,j \leq n} A_{i_1,i_2,\ldots,i_m} - \min_{1 \leq i,j \leq n} A_{i_1,i_2,\ldots,i_m}} \right)^{\frac{1}{2}}.$$

By a similar technique, the upper bound of (3) can be further improved as below (see Theorem 8 in [13]):

$$\kappa_2 \leq \frac{x_{\min}}{x_{\max}} \leq \omega_2 \equiv \eta,$$ \tag{4}

where

$$\kappa_2 = \left[ \min_{(i_2,\ldots,i_m) \neq (i_1,\ldots,i)} \frac{A_{i_1,i_2,\ldots,i_m}}{\max_{(i_2,\ldots,i_m) \neq (i_1,\ldots,i)} A_{i_1,i_2,\ldots,i_m}} \right]^{\frac{1}{m-1}}, \quad \omega_2 = \sqrt{\frac{\min_{1 \leq i \leq n} r_i(A) - \min_{1 \leq i \leq n} A_{i_1,i_2,\ldots,i_m}}{\max_{1 \leq i \leq n} r_i(A) - \min_{1 \leq i \leq n} A_{i_1,i_2,\ldots,i_m}}}.$$

and $\delta = \max_{1 \leq i \leq n} A_{i_1,i_2,\ldots,i_m} + \min_{(i_2,\ldots,i_m) \neq (i_1,\ldots,i)} A_{i_1,i_2,\ldots,i_m} - \min_{1 \leq i \leq n} A_{i_1,i_2,\ldots,i_m}$.

It is easy to see that the bounds $\omega_1$ have the order: $\omega_2 \leq \omega_1 \leq \omega_0$. However, the proof technique to get these bounds is similar to the one in the matrix case. In this paper, we provide a new idea in order to get some sharper two-sided bounds for the ratio of the entries of the Perron vector. As a result, all mentioned bounds (1)-(4) are improved.

The paper is organized as follows. In Section 2, we present some sharper lower and upper bounds for the ratio of the entries in a Perron vector, respectively. In
Section 3, we apply the new bounds given in Section 2 to obtain some further results for the spectral radius, Ky Fan type theorem and the perturbation bound for the nonnegative tensor case. Finally, the concluding remarks are given in Section 4.

2. Two-sided bounds for the ratio of the entries in the Perron vector. Let $\mathcal{A} \in \mathbb{R}^{[m,n]}_+$ be an irreducible tensor with a Perron vector $x = (x_1, \cdots, x_n)^T$. In this section, we consider estimating $\frac{x_{\min}}{x_{\max}}$, where the definitions of $x_{\min}$ and $x_{\max}$ are given in Section 1. With these estimates, we may derive some further results such as the two-sided bounds for the spectral radius, the Ky Fan type theorem. A perturbation bound for the spectral radii of nonnegative tensors is also improved below.

2.1. The upper bounds. In this subsection, we consider the upper bound for the ratio of the smallest and largest entries in a Perron vector. The main result is given below.

**Theorem 2.1.** Suppose $\mathcal{A} \in \mathbb{R}^{[m,n]}_+$ is an irreducible tensor with a Perron vector $x$. Then we have

$$
\left(\frac{x_{\min}}{x_{\max}}\right)^{m-1} \leq \omega_3,
$$

where

$$
\omega_3 = \min_{1 \leq i,j \leq n} \left\{ \frac{A_{j,\cdots,j} - A_{i,\cdots,i}}{2(r_i(\mathcal{A}) - A_{i,\cdots,i})} + \sqrt{\left[ \frac{A_{i,\cdots,i} - A_{j,\cdots,j}}{2(r_i(\mathcal{A}) - A_{i,\cdots,i})} \right]^2 + \frac{r_j(\mathcal{A}) - A_{j,\cdots,j}}{r_i(\mathcal{A}) - A_{i,\cdots,i}}} \right\},
$$

and $r_i(\mathcal{A}) = \sum_{i_2,\cdots,i_m=1}^n A_{i,i_2,\cdots,i_m}$.

**Proof.** Since $\mathcal{A}$ is an irreducible nonnegative tensor, by Lemma 1.4 we know that $x > 0$ and

$$
Ax^{m-1} = \rho(\mathcal{A})x^{[m-1]}.
$$

Then it follows that

$$
\rho(\mathcal{A})x_i^{m-1} = \sum_{i_2,\cdots,i_m=1}^n A_{i,i_2,\cdots,i_m}x_{i_2} \cdots x_{i_m}
$$

$$
= \sum_{\alpha \in \Delta_{i,l}^{(k)}} A_{i,\alpha}x_l^{k}x_{u_1}^{m-1-k} + \sum_{\alpha \notin \Delta_{i,l}^{(k)}} A_{i,\alpha}x_{i_2} \cdots x_{i_m},
$$

for $i = 1, 2, \cdots, n$. Let $s = \arg \min x_i$ and $l = \arg \max x_i$. Taking $u = l$, from (6) we have

$$
\rho(\mathcal{A})x_s^{m-1} \geq \rho(\mathcal{A})x_i^{m-1}
$$

$$
\geq \sum_{\alpha \in \Delta_{i,l}^{(k)}} A_{i,\alpha}x_l^{k}x_{u_1}^{m-1-k} + \sum_{\alpha \notin \Delta_{i,l}^{(k)}} A_{i,\alpha}x_{i_2}^{m-1}
$$

$$
= \sum_{\alpha \in \Delta_{i,l}^{(k)}} A_{i,\alpha}x_l^{k}x_{u_1}^{m-1-k} + \left( r_i(\mathcal{A}) - \sum_{\alpha \in \Delta_{i,l}^{(k)}} A_{i,\alpha} \right) x_{s}^{m-1}.
$$
Hence it follows that

\[ \rho(A) \geq \left( r_i(A) - \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha} \right) \left( \frac{x_s}{x_l} \right)^{m-1} + \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha}. \tag{7} \]

For avoiding confusion in following deduction, we replace \( i \) by \( j \) in (6). Then similarly we get

\[ \rho(A)x_j s^{-m-1-k} \leq \rho(A)x_j s^{-m-1} \leq \sum_{\alpha \in \Delta_{j,k}^{(k)}} A_{j,\alpha}x_j s^{-m-1-k} + \sum_{\alpha \notin \Delta_{j,k}^{(k)}} A_{j,\alpha}x_j s^{-m-1} \]

\[ = \sum_{\alpha \in \Delta_{j,k}^{(k)}} A_{j,\alpha}x_j s^{-m-1-k} + \left( r_j(A) - \sum_{\alpha \in \Delta_{j,k}^{(k)}} A_{j,\alpha} \right) x_j s^{-m-1}. \]

Then we have

\[ r_j(A) - \sum_{\alpha \in \Delta_{j,k}^{(k)}} A_{j,\alpha} \]

\[ \rho(A) \leq \frac{\left( \frac{x_s}{x_l} \right)^{m-1}}{\left( \frac{x_s}{x_l} \right)^{m-1}} + \sum_{\alpha \in \Delta_{j,k}^{(k)}} A_{j,\alpha}. \tag{8} \]

Combining (7) and (8), we get

\[ \left( r_i(A) - \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha} \right) \left( \frac{x_s}{x_l} \right)^{m-1} + \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha} \]

\[ r_j(A) - \sum_{\alpha \in \Delta_{j,k}^{(k)}} A_{j,\alpha} \]

\[ \leq \rho(A) \leq \frac{\left( \frac{x_s}{x_l} \right)^{m-1}}{\left( \frac{x_s}{x_l} \right)^{m-1}} + \sum_{\alpha \in \Delta_{j,k}^{(k)}} A_{j,\alpha}. \]

Therefore, here comes a quadratic inequality about \( \left( \frac{x_s}{x_l} \right)^{m-1} \) as follows:

\[ \left( r_i(A) - \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha} \right) \left( \frac{x_s}{x_l} \right)^{2(m-1)} + \left( \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha} - \sum_{\alpha \in \Delta_{j,k}^{(k)}} A_{j,\alpha} \right) \left( \frac{x_s}{x_l} \right)^{m-1} \]

\[ - \left( r_j(A) - \sum_{\alpha \in \Delta_{j,k}^{(k)}} A_{j,\alpha} \right) \leq 0. \]

Since \( r_i(A) - \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha} \geq 0 \) (specially, when \( k = m - 1, r_i(A) - \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha} = r_i(A) - A_{i,...,i} > 0 \) because \( A \) is irreducible), hence for any \( 1 \leq i, j \leq n \) and \( 0 \leq k \leq m - 1, \) when \( r_i(A) - \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha} \neq 0, \) we have

\[ \left( \frac{x_s}{x_l} \right)^{m-1} \leq \frac{\sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha} - \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha}}{2 \left( r_i(A) - \sum_{\alpha \in \Delta_{i,i}^{(k)}} A_{i,\alpha} \right)}. \]
Then it follows from (7) that
\[
\rho(\mathcal{A}) \geq r_l(\mathcal{A}) \left( \frac{x_s}{x_l} \right)^{m-1} + \sum_{\alpha \in \Delta_{i,l}^{(k)}} A_{i,\alpha} \left[ 1 - \left( \frac{x_s}{x_l} \right)^{m-1} \right]
\]
\[
\geq r_l(\mathcal{A}) \left( \frac{x_s}{x_l} \right)^{m-1} + \eta_{(k)} \left[ 1 - \left( \frac{x_s}{x_l} \right)^{m-1} \right]
\]
Specially, when \( k = m - 1 \), the equality of above formula is true. Therefore, we have
\[
\omega_3 = \min_{1 \leq i,j,l \leq n} \left\{ \frac{A_{i,\ldots,j} - A_{i,\ldots,l}}{2 (r_l(\mathcal{A}) - A_{i,\ldots,l})} + \sqrt{\frac{A_{i,\ldots,j} - A_{i,\ldots,l}}{2 (r_l(\mathcal{A}) - A_{i,\ldots,l})}^2 + \frac{r_l(\mathcal{A}) - A_{i,\ldots,l}}{r_l(\mathcal{A}) - A_{i,\ldots,l}}} \right\}
\]
which proves the theorem.

For the bound \( \omega_3 \), we may do some modifications in the proof of Theorem 5 for getting some simple bounds. Let \( \hat{l} = \arg \max r_l(\mathcal{A}) \) and
\[
\eta_{(k)} = \min_{1 \leq i \leq n} \sum_{\alpha \in \Delta_{i,l}^{(k)}} A_{i,\alpha}
\]
Then it follows from (7) that
\[
\omega_3 \leq \left( \frac{x_{\min}}{x_{\max}} \right)^{m-1}
\]
where \( \hat{x} \) of the Perron vector, which always improves the bound \( \omega \).

Similarly, we could modify (8) as follows:

\[
\rho(A) \leq \frac{\max_i \hat{x}_i}{\min_i \hat{x}_i} + \sum_{\alpha \in \Delta_{\hat{s}}^{(k)}} A_{\hat{s},\alpha},
\]

(10)

where \( \hat{s} = \arg \min_i r_i(A) \).

Replacing (7) and (8) by (9) and (10), respectively, we have the following upper bound:

\[
\frac{\max_i \hat{x}_i}{\min_i \hat{x}_i} \leq \omega_4,
\]

where

\[
\omega_4 = \min_{0 \leq k \leq m-1} \max_{1 \leq v \leq n} \left\{ \frac{A_{\hat{s},\ldots,\hat{s}} - \eta^{(k)}_v}{2 (r_i(A) - \eta^{(k)}_v)} + \frac{\left[ \frac{\eta^{(k)}_v - A_{\hat{s},\ldots,\hat{s}}}{2 (r_i(A) - \eta^{(k)}_v)} \right]^2 + \frac{r_s(A) - A_{\hat{s},\ldots,\hat{s}}}{r_i(A) - \eta^{(k)}_v}}{r_i(A) - \eta^{(k)}_v} \right\}.
\]

Now we define a function

\[
f(x) = \frac{A_{\hat{s},\ldots,\hat{s}} - x}{2 (r_i(A) - x)} + \frac{\left[ \frac{x - A_{\hat{s},\ldots,\hat{s}}}{2 (r_i(A) - x)} \right]^2 + r_s(A) - A_{\hat{s},\ldots,\hat{s}}}{r_i(A) - x},
\]

where \( x \in [0, r_i(A)] \). It is noted that its first derivative \( f'(x) \leq 0 \) while \( x \in [0, r_i(A)] \). Thus \( f(x) \) is a monotone decreasing function. In addition, it is noted that

\[
\max_{0 \leq k \leq m-1} \min_{1 \leq \ell \leq n} \eta^{(k)}_\ell = \min_{1 \leq \ell \leq n} A_{i,\ldots,i}.
\]

Hence we have

\[
\omega_4 = \min_{0 \leq k \leq m-1} \left\{ \frac{A_{\hat{s},\ldots,\hat{s}} - \min_{1 \leq \ell \leq n} \eta^{(k)}_\ell}{2 (r_i(A) - \min_{1 \leq \ell \leq n} \eta^{(k)}_\ell)} \right\}
\]

\[
+ \frac{\left[ \min_{1 \leq \ell \leq n} \eta^{(k)}_\ell - A_{\hat{s},\ldots,\hat{s}} \right]^2 + r_s(A) - A_{\hat{s},\ldots,\hat{s}}}{2 (r_i(A) - \min_{1 \leq \ell \leq n} \eta^{(k)}_\ell)} + \frac{r_s(A) - A_{\hat{s},\ldots,\hat{s}}}{r_i(A) - \min_{1 \leq \ell \leq n} \eta^{(k)}_\ell}.
\]

Therefore, we get the following simple upper bound for the ratio of the entries of the Perron vector, which always improves the bound \( \omega_1 \) in (3).
Corollary 1. With the same hypothesis as in Theorem 2.1, we have
\[
\left( \frac{x_{\min}}{x_{\max}} \right)^{m-1} \leq \omega_4,
\]  
(11)

where
\[
\omega_4 \equiv \frac{A_{\hat{s}, \ldots, \hat{s}} - \min_{1 \leq i \leq n} A_i}{2 \left( r_\hat{l}(A) - \min_{1 \leq i \leq n} A_i \right)} + \frac{\left[ \min_{1 \leq i \leq n} A_i - A_{\hat{s}, \ldots, \hat{s}} \right]^2}{\left( r_\hat{l}(A) - \min_{1 \leq i \leq n} A_i \right)} + \frac{r_\hat{s}(A) - A_{\hat{s}, \ldots, \hat{s}}}{r_\hat{l}(A) - \min_{1 \leq i \leq n} A_i},
\]
\[\hat{l} = \arg \max r_i(A) \text{ and } \hat{s} = \arg \min r_i(A).\]

Remark 1. Since \( f(x) \) is a monotone decreasing function and \( \min_{1 \leq i \leq n} A_i \leq A_{\hat{l}} \), it is easy to see that \( \omega_4 \geq \omega_3 \).

On the other hand, for magnifying \( \omega_4 \), we need to analyse another function
\[
g(x) = \frac{x - a}{2 \left( r_\hat{l}(A) - a \right)} + \frac{\left[ \frac{a - x}{2 \left( r_\hat{l}(A) - a \right)} \right]^2 + r_\hat{s}(A) - x}{r_\hat{l}(A) - a},
\]
where \( a = \min_{1 \leq i \leq n} A_i \) and \( x \in [0, r_\hat{s}(A)] \). It is noted that its first order derivative \( g'(x) \leq 0 \) when \( x \in [0, r_\hat{s}(A)] \). Hence \( g(x) \) is also a monotone decreasing function. Since \( A_{\hat{s}, \ldots, \hat{s}} \geq \min_{1 \leq i \leq n} A_i \), we have
\[
\omega_4 \leq g(a) = \frac{r_\hat{s}(A) - \min_{1 \leq i \leq n} A_i}{r_\hat{l}(A) - \min_{1 \leq i \leq n} A_i} = \omega_2 \leq \omega_1.
\]  
(12)

Hence from Remark 1 it is easy to see that
\[
\left( \frac{x_{\min}}{x_{\max}} \right)^{m-1} \leq \omega_3 \leq \omega_4 \leq \omega_2 \leq \omega_1.
\]

Here we give an example to illustrate the differences among these upper bounds (3), (4), (5) and (11).

Example 1. Let \( A \in \mathbb{R}^{[3,3]} \) be a positive tensor given by
\[
A(1, :, :) = \begin{pmatrix} 5 & 4 & 2 \\ 5 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}, A(2, :, :) = \begin{pmatrix} 4 & 10 & 10 \\ 10 & 4 & 2 \\ 10 & 4 & 10 \end{pmatrix}, A(3, :, :) = \begin{pmatrix} 6 & 9 & 3 \\ 3 & 12 & 15 \\ 6 & 15 & 1 \end{pmatrix}.
\]

Then by a simple computation we get the following table:

| \( \omega_1 \) in (3) | \( \omega_2 \) in (4) | \( \omega_3 \) in (5) | \( \omega_4 \) in (11) |
|------------------------|------------------------|------------------------|------------------------|
| 0.5575                 | 0.5307                 | 0.4855                 | 0.5244                 |

Table 1. Comparisons with the upper bounds for the ratio

From Table 1, it is easy to see that the computation result is consistent with the conclusion in Remark 1.
2.2. The lower bounds. In this subsection, we consider to bound the lower bound for the ratio of the entries of the Perron vector. The result will improve the existing ones. The following theorem is to give a lower bound:

**Theorem 2.2.** Let \( \mathcal{A} \in \mathbb{R}^{[m,n]} \) be an irreducible nonnegative tensor with a Perron vector \( \mathbf{x} \). Then we get

\[
\frac{x_{\min}}{x_{\max}} \geq \kappa_3 \equiv \max_{1 \leq k \leq m-1} \kappa_3^{(k)},
\]

where

\[
\kappa_3^{(k)} \equiv \max_{\{j_1,j_2,\cdots,j_k\} \subseteq \{2,3,\cdots,m\}} \min_{\{j_1',j_2',\cdots,j_k'\} \subseteq \{2,3,\cdots,m\}} \min_{1 \leq i_j \leq \min_{1 \leq i \leq n} A_{i_1,\cdots,i}} \left( \frac{\sum_{i_2,\cdots,i_m=1}^n T_{i_1,i_2,\cdots,i_m}}{\sum_{i_2',\cdots,i_m'=1}^n T_{i_1,i_2',\cdots,i_{m'}}} \right)^{\frac{1}{k}}
\]

and

\[
T_{i_1,\cdots,i_m} = \left\{ \begin{array}{ll}
A_{i_1,\cdots,i_m} - t, & i_1 = \cdots = i_m, \\
A_{i_1,\cdots,i_m}, & \text{else}
\end{array} \right.
\]

**Proof.** By Lemma 1.4, we know that \( \mathbf{x} > 0 \) and

\[\mathcal{A} \mathbf{x}^{m-1} = \rho(\mathcal{A}) \mathbf{x}^{[m-1]} \]

Let \( t \) be a parameter such that \( t \leq \min_{1 \leq i \leq n} A_{i_1,\cdots,i} \). Hence we get

\[\mathcal{T} \mathbf{x}^{m-1} = (\rho(\mathcal{A}) - t) \mathbf{x}^{[m-1]} \]

It is noted that \( \rho(\mathcal{A}) > \min_{1 \leq i \leq n} A_{i_1,\cdots,i} \geq t \) and \( \mathcal{T} \) is also an irreducible nonnegative tensor. So \( \rho(\mathcal{T}) = \rho(\mathcal{A}) - t \) is an eigenvalue of \( \mathcal{T} \) with a positive eigenvector \( \mathbf{x} \). Let \( s = \arg \min x_i \) and \( l = \arg \max x_i \). Then we have

\[
\rho(\mathcal{T}) \mathbf{x}_s^{m-1} = \sum_{i_2,\cdots,i_m=1}^n T_{s,i_2,\cdots,i_m} x_{i_2} \cdots x_{i_m} 
\]

\[
\geq \left( \sum_{i_2,\cdots,i_m=1}^n T_{s,i_2,\cdots,i_m} x_{i_{j_1}} \cdots x_{i_{j_k}} \right) x_s^{m-1-k},
\]

where \( \{j_1,j_2,\cdots,j_k\} \subseteq \{2,3,\cdots,m\} \). Hence it follows that

\[
\rho(\mathcal{T}) \mathbf{x}_s^k \geq \sum_{i_2,\cdots,i_m=1}^n T_{s,i_2,\cdots,i_m} x_{i_{j_1}} \cdots x_{i_{j_k}},
\]

Similarly, for \( x_l \), we have

\[
\rho(\mathcal{T}) \mathbf{x}_l^k \leq \sum_{i_2,\cdots,i_m=1}^n T_{l,i_2,\cdots,i_m} x_{i_{j_1}} \cdots x_{i_{j_k}}.
\]
Combining (14) and (15), we get

\[
\left( \frac{x_s}{x_l} \right)^k \geq \frac{\sum_{\substack{i_1, \ldots, i_m = 1 \atop i_1', \ldots, i_m' = 1 \atop \text{except } i_{j_1}, \ldots, i_{j_k}}}^n \mathcal{T}_{s, i_1', \ldots, i_m'} x_{i_1 j_1} \cdots x_{i_k j_k}}{\sum_{\substack{i_1, \ldots, i_m = 1 \atop i_1', \ldots, i_m' = 1 \atop \text{except } i_{j_1}, \ldots, i_{j_k}}}^n \mathcal{T}_{s, i_1', \ldots, i_m'} x_{i_1 j_1} \cdots x_{i_k j_k}} = \frac{\sum_{\substack{i_1, \ldots, i_m = 1 \atop i_1', \ldots, i_m' = 1 \atop \text{except } i_{j_1}, \ldots, i_{j_k}}}^n \mathcal{T}_{s, i_1', \ldots, i_m'} x_{i_1 j_1} \cdots x_{i_k j_k}}{\sum_{\substack{i_1, \ldots, i_m = 1 \atop i_1', \ldots, i_m' = 1 \atop \text{except } i_{j_1}, \ldots, i_{j_k}}}^n \mathcal{T}_{s, i_1', \ldots, i_m'} x_{i_1 j_1} \cdots x_{i_k j_k}} \geq \min_{1 \leq j_s = j' \leq n} \min_{1 \leq s \leq k} \sum_{\substack{i_1, \ldots, i_m = 1 \atop i_1', \ldots, i_m' = 1 \atop \text{except } i_{j_1}, \ldots, i_{j_k}}}^n \mathcal{T}_{s, i_1', \ldots, i_m'} x_{i_1 j_1} \cdots x_{i_k j_k}
\]

which proves the theorem. \(\square\)

**Remark 2.** It is noted that, for a given positive integer \(k\) and sequences \(\{j_1, j_2, \ldots, j_k\}\) and \(\{j'_1, j'_2, \ldots, j'_k\}\), there are only \(2n^2 - n\) fractions containing the diagonal entry of \(A\). Thus the choice of parameter \(t\) is only based on these \(2n^2 - n\) formulas. Hence \(t\) can be found by solving a corresponding simple optimization problem.

**Remark 3.** Since \(\kappa_3\) in (13) is too difficult to calculate, we present some simple bounds below:

- For \(\kappa_3^{(m-1)}\), taking \(t = 0\) and \(j_s = j'_s\) for \(1 \leq s \leq m - 1\), the bound reduces to \(\kappa_0\) in (2). Hence we have \(\kappa_3^{(m-1)} \geq \kappa_0\).
- For \(\kappa_3^{(1)}\), taking \(t = 0\), the bound becomes \(\kappa_1\) in (3).
- For \(\kappa_3^{(m-1)}\), taking \(t = \min_{1 \leq i \leq n} A_{\overline{i}, i} - \min_{(i_1', \ldots, i_m') \not= (i, \ldots, i)} A_{i_1', \ldots, i_m'}\) \(j_s = j'_s\) and \(1 \leq s \leq m - 1\) gives a bound \(\kappa'_3 \leq \kappa_3^{(m-1)}\). It is easy to see that \(\kappa'_3 \geq \kappa_2\).
- When \(m \geq 3\), \(\kappa_3^{(2)}\) gives a simple lower bound as follows:

\[
\frac{x_{\min}}{x_{\max}} \geq \kappa_3^{(2)}
\]
where

\[ \kappa_3^{(2)} \equiv \max_{2 \leq j \neq k \neq k' \leq m} \min_{i \leq i_1, \ldots, i_m \leq n} \left( \sum_{t_1, \ldots, t_m = 1}^{n} \mathcal{T}_{i_1, i_2, \ldots, i_m} \right) \sqrt{\sum_{t_1, \ldots, t_m = 1}^{n} \mathcal{T}_{i_1', i_2', \ldots, i_m'}}, \]

except \( i_j = i_{j'}, i_k = i_{k'} \).

By the above analysis, we obtain that \( \kappa_3 \geq \kappa_i, \ i = 0, 1, 2 \). So the bound (13) improves the existing bounds.

Next we give examples to show the differences among these lower bounds.

**Example 2.** Let \( A \in \mathbb{R}^{[4,2]} \) be a positive tensor, where

\[
\begin{align*}
A(;;1,1) &= \begin{pmatrix} 3 & 4 \\ 4 & 1 \end{pmatrix}, \ A(;;2,1) = \begin{pmatrix} 4 & 1 \\ 5 & 4 \end{pmatrix}, \\
A(;;1,2) &= \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix}, \ A(;;2,2) = \begin{pmatrix} 4 & 4 \\ 5 & 3 \end{pmatrix}.
\end{align*}
\]

**Example 3.** Let \( A \in \mathbb{R}^{[4,2]} \) be a positive tensor, where

\[
\begin{align*}
A(;;1,1) &= \begin{pmatrix} 1 & 3 \\ 16 & 12 \end{pmatrix}, \ A(;;2,1) = \begin{pmatrix} 3 & 3 \\ 12 & 8 \end{pmatrix}, \\
A(;;1,2) &= \begin{pmatrix} 3 & 3 \\ 12 & 16 \end{pmatrix}, \ A(;;2,2) = \begin{pmatrix} 2 & 3 \\ 12 & 4 \end{pmatrix}.
\end{align*}
\]

**Example 4.** Let \( A \in \mathbb{R}^{[4,2]} \) be a positive tensor, where

\[
\begin{align*}
A(;;1,1) &= \begin{pmatrix} 3 & 1 \\ 10 & 4 \end{pmatrix}, \ A(;;2,1) = \begin{pmatrix} 1 & 3 \\ 4 & 8 \end{pmatrix}, \\
A(;;1,2) &= \begin{pmatrix} 4 & 2 \\ 10 & 6 \end{pmatrix}, \ A(;;2,2) = \begin{pmatrix} 1 & 3 \\ 8 & 8 \end{pmatrix}.
\end{align*}
\]

With Examples 2, 3 and 4, we have the following results:

| Actual value of \( \frac{\hat{x}_{\min}}{\hat{x}_{\max}} \) | Example 2 | Example 3 | Example 4 |
|--------------------------------------------------------|------------|------------|------------|
| \( \kappa_0 \) in (2)                                  | 0.9873     | 0.6402     | 0.6794     |
| \( \kappa_1 \) in (3)                                  | 0.7857     | 0.2083     | 0.3077     |
| \( \kappa_2 \) in (4)                                  | 0.5848     | 0.5000     | 0.4642     |
| \( \kappa_3 \) in (13)                                 | **0.9662** | 0.2808     | 0.3445     |
| \( \kappa_3 \) in (13) (t = -5.5602)                    |            | 0.5000     | 0.5539     |
| \( \kappa_3 \) in (13) (t = -5.1168)                    | 0.9258     |            |            |
| \( \kappa_3 \) in (13) (t = -5.2956)                    |            | 0.5724     |            |
| \( \kappa_3 \) in (13) (t = -5.0250)                    |            |            | 0.5539     |

Table 2. Comparisons with the lower bounds for ratio.
It is easy to see that $\kappa_3$ in (13) is the best one among all the bounds. For a given parameter $k$, the bound $\kappa^{(1)}_3$ in (13) is always better than $\kappa_1$ in (3), and the bound $\kappa^{(2)}_3$ in (13) is always sharper than $\kappa_0$ in (2) and $\kappa_2$ in (4). Besides, for the bound $\kappa^{(k)}_3$ in (13) with different $k$, the examples show that each bound could be the best one for some cases.

3. Applications. Based on the new estimates for the Perron vector given in Section 2, we may provide some further spectrum analysis in the following various aspects.

3.1. The bounds for the spectral radius. The estimation of the spectral radius for nonnegative matrices is of important significance. For a nonnegative matrix $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$, some classical bounds for the spectral radius are given as follows (see Theorem 8.1.22 in [6]):

$$\min_{1 \leq i \leq n} \sum_{j=1}^{n} a_{i,j} \leq \rho(A) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{i,j},$$

(17)

and

$$\min_{1 \leq j \leq n} \sum_{i=1}^{n} a_{i,j} \leq \rho(A) \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{i,j}. \tag{18}$$

Ostrowski improved the bound (17) for a positive square matrix (see Theorem 2.1.4 in [14]):

$$\min_{1 \leq i \leq n} r_i(A) + \eta\left(\frac{1}{\sigma} - 1\right) \leq \rho(A) \leq \max_{1 \leq i \leq n} r_i(A) - \eta(1 - \sigma), \tag{19}$$

where $r_i(A) = \sum_{1 \leq j \leq n} a_{i,j}$, $\eta = \min_{1 \leq i \leq n} a_{i,j}$, and $\sigma$ is given by (1).

Brauer also presented a two-sided bound for positive matrix case (see Theorem 2.1.5 in [14]) as follows:

$$r + \eta(h - 1) \leq \rho(A) \leq R - \eta(1 - 1/g), \tag{20}$$

where $R = \max_{1 \leq i \leq n} r_i(A)$, $r = \min_{1 \leq i \leq n} r_i(A)$, $r_i(A)$ and $\eta$ are given in (19),

$$g = \frac{R - 2\eta + \sqrt{R^2 - 4\eta(R - r)}}{2(r - \eta)} \quad \text{and} \quad h = \frac{-r + 2\eta + \sqrt{r^2 + 4\eta(R - r)}}{2\eta}.$$ 

It is noted that the bound (20) is sharper than (19). For the tensor case, Yang and Yang [21] extended the bound (17) to the nonnegative tensors:

Theorem 3.1. [21] Let $A = (a_{i_1,i_2,\ldots,i_m}) \in \mathbb{R}_{+}^{[m,n]}$. Then

$$\min_{1 \leq i \leq n} r_i(A) \leq \rho(A) \leq \max_{1 \leq i \leq n} r_i(A), \tag{21}$$

where $r_i(A) = \sum_{i_2,\ldots,i_m = 1}^{n} a_{i_1,i_2,\ldots,i_m}$.

Further, Wang and Wu [20] extended the bound (19) to the positive tensor case:

$$r_{\hat{s}}(A) + \eta\left(\frac{1}{\omega_0} - 1\right) \leq \rho(A) \leq r_i(A) - \eta(1 - \omega_0), \tag{22}$$

where $\eta$ and $r_i(A)$ are given in Section 1, $r_{\hat{s}}(A) = \min_{1 \leq i \leq n} r_{i}(A)$, $r_i(A) = \max_{1 \leq i \leq n} r_i(A)$ and $\omega_0$ is given in (2). Li and Ng [11] considered an irreducible
nonnegative tensor $A \in \mathbb{R}^{[m,n]}_+$, based on the bound (3) they gave the following bound for the spectral radius of an irreducible nonnegative tensor:

$$\min_{1 \leq i,j \leq n} \{ r_i(A) + A_{i,j} \omega^{-1} - 1 \} \leq \rho(A) \leq \max_{1 \leq i,j \leq n} \{ r_i(A) - A_{i,j} \omega^{-1} + 1 \}. \quad (23)$$

Clearly, the bound (23) improves (21) and (22).

Recently, Liu, Li and Zhang (see Theorem 8 in [13]) got a spectral radius bound for a positive tensor by (4):

$$r^\hat{s}(A) + \eta(\omega - 1) \leq \rho(A) \leq r^\hat{l}(A) - \eta(1 - \omega). \quad (24)$$

This bound improves (22). In the above formulas, the spectral radius bounds can be derived based on the improvement of the upper bound for the ratio of entries in the Perron vector. With this observation, we may employ (5) to get the spectral radius bounds.

Next, we present the two-sided bound for the spectral radius of an irreducible nonnegative tensor, which always improves all the mentioned bounds in (21)-(24).

Let $\omega$ be a parameter so that for the Perron vector $x$ of $A$, $x_{\min} \leq \omega x_{\max}$. From the sections 1 and 2, it is known that $\omega$ can be taken as $\omega_i, i = 0, 1, 2, 3$.

Recall that the definition of $\Delta_{i,j}^{(k)}$ is given in Section 2.

**Theorem 3.2.** Let $A \in \mathbb{R}^{[m,n]}_+$ be an irreducible tensor. Then we have

$$\min_{0 \leq k \leq m-1} \min_{1 \leq i,j \leq n} \left[ \omega^{\frac{m-1-k}{m-1}} - 1 \right] \sum_{\alpha \in \Delta_{i,j}^{(k)}} A_{i,\alpha} + r_i(A) \leq \rho(A) \leq \max_{0 \leq k \leq m-1} \max_{1 \leq i,j \leq n} \left[ \omega^{\frac{m-1-k}{m-1}} - 1 \right] \sum_{\alpha \in \Delta_{i,j}^{(k)}} A_{i,\alpha} + r_i(A). \quad (25)$$

**Proof.** Since $A$ is nonnegative and irreducible, there exists a positive eigenvector $x$ corresponding to $\rho(A)$. Let $x_s = \min x_i$ and $x_l = \max x_i$. By (6), we get

$$\rho(A)x_{s}^{m-1} \geq \sum_{\alpha \in \Delta_{i,j}^{(k)}} A_{s,\alpha} x_s x_l^{m-1-k} + \sum_{\alpha \notin \Delta_{i,j}^{(k)}} A_{s,\alpha} x_s^{m-1}. \quad (26)$$

Hence it follows that

$$\rho(A) \geq \sum_{\alpha \in \Delta_{i,j}^{(k)}} A_{s,\alpha} x_l^{m-1-k} + \sum_{\alpha \notin \Delta_{i,j}^{(k)}} A_{s,\alpha},$$

$$\geq \left( \omega^{\frac{m-1-k}{m-1}} - 1 \right) \sum_{\alpha \in \Delta_{i,j}^{(k)}} A_{s,\alpha} + r_s(A)$$

$$\geq \min_{1 \leq i,j \leq n} \left[ \omega^{\frac{m-1-k}{m-1}} - 1 \right] \sum_{\alpha \in \Delta_{i,j}^{(k)}} A_{s,\alpha} + r_i(A).$$
Similarly, for the upper bound, by (6) we have
\[ \rho(A)x_l^{m-1} \leq \sum_{\alpha \in \Delta_{l,i}^{(k)}} A_{i,\alpha} x_i^k x_{l}^{m-1-k} + \sum_{\alpha \notin \Delta_{i,j}^{(k)}} A_{i,\alpha} x_l^{m-1}. \]

Then we get
\[ \rho(A) \leq \sum_{\alpha \in \Delta_{l,i}^{(k)}} A_{i,\alpha} \left( \frac{x_i}{x_l} \right)^{m-1-k} + \sum_{\alpha \notin \Delta_{i,j}^{(k)}} A_{i,\alpha} \leq (\omega^{\frac{m-1-k}{m-1}} - 1) \sum_{\alpha \in \Delta_{i,j}^{(k)}} A_{i,\alpha} + r_l(A) \]
\[ \leq \max_{1 \leq i,j \leq n} \left( (\omega^{\frac{m-1-k}{m-1}} - 1) \sum_{\alpha \in \Delta_{i,j}^{(k)}} A_{i,\alpha} + r_l(A) \right). \]

This completes the proof of theorem. \( \square \)

**Remark 4.** By taking different \( k \) we may get some simple bounds.
- When we take \( k = m - 1 \), the two-sided bound (25) reduces to (21).
- When we take \( k = 0 \) and \( \omega = \omega_1 \), the bound (25) reduces to (23).
- When we take \( k = 0 \) and \( \omega = \omega_3 \), then we have
\[ \min_{1 \leq i,j \leq n} \{ r_i(A) + A_{i,j} (\omega_3^{-1} - 1) \} \leq \rho(A) \leq \max_{1 \leq i,j \leq n} \{ r_i(A) - A_{i,j} (1 - \omega_3) \}. \]
\[ (26) \]

It is easy to see that the bound (26) always improves each one among (21)-(24).

**Remark 5.** Let \( A \in \mathbb{R}^{n \times n} \) be an irreducible nonnegative matrix, i.e., \( m = 2 \). Then the bound (25) reduces to the following one:
\[ \min_{1 \leq i,j \leq n} \left[ (\omega^{-1} - 1) A_{i,j} + r_i(A) \right] \leq \rho(A) \leq \max_{1 \leq i,j \leq n} \left[ (\omega - 1) A_{i,j} + r_i(A) \right], \]
\[ (27) \]
where \( r_i(A) = \sum_{1 \leq j \leq n} A_{i,j} \) and
\[ \omega = \min_{1 \leq i,j \leq n} \left\{ \frac{A_{i,j} - A_{i,i}}{2 (r_i(A) - A_{i,i})} + \sqrt{\frac{A_{i,i} - A_{j,j}}{2 (r_i(A) - A_{i,i})}^2 + \frac{r_j(A) - A_{j,j}}{r_i(A) - A_{i,i}}} \right\}. \]

Since \( 0 \leq \omega \leq 1 \), it is easy to see that the two-sided bound (27) is better than the one in (17). Furthermore, from Remark 1, it is noted that
\[ \omega \leq \sqrt{\frac{r_i(A) - \min_{1 \leq i,j \leq n} A_{i,j}}{r_i(A) - \min_{1 \leq i,j \leq n} A_{i,j}}} = \sigma, \]
where \( \sigma \) is given by (1). And thus the bound (27) always improves the one in (19).

However, we could not compare the bound (27) with (20) theoretically. Hence we generate some nonnegative matrices randomly by the uniform distribution on \((0, 1)\), and test two bounds in order to see which one has a better performance. In the following experiment, the sizes of nonnegative matrices are taken as \( n = 5, n = 10, n = 15 \) and \( n = 20 \), respectively, whose entries are generated randomly by uniform distribution on \((0, 1)\). For each dimension, we run 10000 random cases. In the following table, we show the percentages of results that the bound in (27) is
sharper. And it is easy to see that the bound (27) has better performance in the great majority cases when the size of the matrix is large.

| Dimension | n = 5 | n = 10 | n = 15 | n = 20 |
|-----------|------|-------|-------|-------|
| Lower bound | 42.86% | 64.02% | 75.64% | 81.37% |
| Upper bound | 91.76% | 94.50% | 95.77% | 96.83% |

**Table 3.** Comparisons between (20) and (27)

### 3.2. The Ky Fan type theorem

Let $A = (A_{i_1, \cdots, i_m})$ be a tensor. By $|A|$ we denote a tensor whose the $(i_1, \cdots, i_m)$-entry $(|A|)_{i_1, \cdots, i_m} = |A_{i_1, \cdots, i_m}|$. In [21], Yang and Yang presented the following Ky Fan type theorem for tensors.

**Theorem 3.3.** [21] Let $A, B \in \mathbb{R}^{[m,n]}$ with $B \geq |A|$. Then for any eigenvalue $\lambda$ of $A$, there exists $i$ such that

$$|\lambda - A_{i_1, \cdots, i_m}| \leq \rho(B) - B_{i_1, \cdots, i_m}. \quad (28)$$

For improving the inequality (28), Li and Ng [11] gave the following theorem.

**Theorem 3.4.** [11] Let $A, B \in \mathbb{R}^{[m,n]}$ with $m \geq 3$. If $B$ is irreducible and $B \geq |A|$, then for any eigenvalue $\lambda$ of $A$, there exists $i$ such that

$$|\lambda - A_{i_1, \cdots, i_m}| \leq \rho(B) - B_{i_1, \cdots, i_m} - (\kappa_1(B))^{m-1}\hat{r}_i(B - |A|), \quad (29)$$

where $r_i(C) = \sum_{i_2, \cdots, i_m=1}^{n} C_{i_2, \cdots, i_m}$ for a tensor $C \in \mathbb{R}^{[m,n]}$, $\hat{r}_i(C) = r_i(C) - C_{i_1, \cdots, i_m}$ and $\kappa_1$ is given in (3).

In this subsection, we consider improving the inequality (29). In order to derive the new Ky Fan type theorem, it needs to study the diagonal scaling on tensor $A$. For any given positive diagonal matrix $D = \text{diag}(d_1, \cdots, d_n)$, Li and Ng [11] defined a tensor $A_D$ as follows:

$$A_D \equiv A \times_1 D^{1-m} \times_2 D \times_3 \cdots \times_m D, \quad (30)$$

where $\times_k$ is mode-$k$ tensor-matrix multiplication between $A$ and $D$ [4]. Hence the entries of $A_D$ are given by

$$(A_D)_{i_1, i_2, \cdots, i_m} = A_{i_1, i_2, \cdots, i_m} d_{i_1}^{1-m} d_{i_2} \cdots d_{i_m}, 1 \leq i_1, i_2, \cdots, i_m \leq n.$$ 

And thus we get the following lemma.

**Lemma 3.5.** [22] The tensor $A$ and $A_D$ have the same set of eigenvalues.

From the lower bound for the ratio of the entries of Perron vector, we can derive the following Ky Fan type theorem.

**Theorem 3.6.** Let $A, B \in \mathbb{R}^{[m,n]}$. If $B$ is irreducible and $B \geq |A|$, then for any eigenvalue $\lambda$ of $A$, there exist $i \neq j$ such that

$$|\lambda - A_{i_1, \cdots, i_m}|^{m-1}|\lambda - A_{j_1, \cdots, j_m}| \leq \left[\rho(B) - B_{i_1, \cdots, i_m} - (\kappa(B))^{m-1}\hat{r}_i(\mathcal{E})\right]^{m-1} \left[\rho(B) - B_{j_1, \cdots, j_m} - (\kappa(B))^{m-1}\hat{r}_j(\mathcal{E})\right], \quad (31)$$

where $\mathcal{E} = B - |A|$, $r_i(\mathcal{E}) = \sum_{i_2, \cdots, i_m=1}^{n} \mathcal{E}_{i_2, \cdots, i_m}$, $\hat{r}_i(\mathcal{E}) = r_i(\mathcal{E}) - \mathcal{E}_{i_1, \cdots, i_m}$ and $\kappa(B)$ is given by $\kappa_3$ in (13).
Proof. Since \( B \) is nonnegative and irreducible, by Lemma 1.4, there exists a positive vector \( y \) such that
\[
\rho(B) y_i^{m-1} = \sum_{i_2,\ldots,i_m=1}^{n} B_{i,i_2,\ldots,i_m} y_{i_2} \cdots y_{i_m}.
\]

Let \( D = \text{diag}(y_1, \ldots, y_m) \) and define \( A_D \) and \( B_D \) as in (30). Then we get
\[
\rho(B) = \sum_{i_2,\ldots,i_m=1}^{n} (B_D)_{i,i_2,\ldots,i_m}
\]
and
\[
\lambda(A) = \lambda(A_D),
\]
where the latter one is given by Lemma 3.5. Let \( x \) be an eigenvector of \( A_D \) corresponding to \( \lambda \), and \( E = B - |A| \). Hence \( E \geq 0 \), and for any \( 1 \leq i \leq n \)
\[
\lambda^m x_i^{m-1} = \sum_{i_2,\ldots,i_m=1}^{n} (A_D)_{i,i_2,\ldots,i_m} x_{i_2} \cdots x_{i_m}.
\]

Let \( |x_i| = \max_{1 \leq i \leq n} |x_i| \), \( \Delta_i^{(k)} \) be a set of all sequences \( (i_2, \ldots, i_m) \), each of which exactly consists of \( k \) ‘i’s, and \( \bar{\Delta}_i^{(k)} \) be the complement of \( \Delta_i^{(k)} \). Therefore it follows from (32) and (33) that
\[
|\lambda - (A_D)_{l,\ldots,l}| \leq \sum_{(i_2,\ldots,i_m) \in \Delta_i^{(m-1)}} |(A_D)_{l,i_2,\ldots,i_m}| \frac{|x_{i_2}| \cdots |x_{i_m}|}{|x_l|^{m-1}}
\]
\[
\leq \sum_{(i_2,\ldots,i_m) \in \Delta_i^{(m-1)}} |(A_D)_{l,i_2,\ldots,i_m}| \frac{|x_l|}{|x_l|}
\]
\[
= \left[ \sum_{(i_2,\ldots,i_m) \in \Delta_i^{(m-1)}} (B_D)_{l,i_2,\ldots,i_m} - \sum_{(i_2,\ldots,i_m) \in \Delta_i^{(m-1)}} (E_D)_{l,i_2,\ldots,i_m} \right] \frac{|x_l|}{|x_l|}
\]
\[
= \left[ \rho(B) - (B_D)_{l,\ldots,l} - \sum_{(i_2,\ldots,i_m) \in \Delta_i^{(m-1)}} (E_D)_{l,i_2,\ldots,i_m} \right] \frac{|x_l|}{|x_l|}
\]
\[
\leq \left[ \rho(B) - (B_D)_{l,\ldots,l} - \tilde{\gamma}(E) \left( \frac{y_{\min}}{y_{\max}} \right)^{m-1} \right] \frac{|x_l|}{|x_l|}
\]
\[
\leq \left[ \rho(B) - (B_D)_{l,\ldots,l} - (\kappa(B))^{m-1} \tilde{\gamma}(E) \right] \frac{|x_l|}{|x_l|},
\]
where \( |x_l| = \max_{1 \leq l \leq n} |x_l| \) \( (i \neq l) \), \( y_{\min} = \min_{1 \leq l \leq n} y_l \), \( y_{\max} = \max_{1 \leq l \leq n} y_l \) and \( \kappa(B) \) is a lower bound for \( \frac{y_{\min}}{y_{\max}} \). Similarly, by (32) and (33) again, for index \( t \) we have
\[
|\lambda - (A_D)_{t,\ldots,t}| |x_t|^{m-1} \leq \sum_{(i_2,\ldots,i_m) \in \Delta_i^{(m-1)}} |(A_D)_{t,i_2,\ldots,i_m}| |x_{i_2}| \cdots |x_{i_m}|
\]
\[ \leq \left[ \rho(B) - B_{i_i \cdots i_l} - (\kappa(B))^{m-1} \check{r}_i(E) \right] |x|^{m-1}. \]  

Combining (34) and (35), we get

\[ |\lambda - A_{i_i \cdots i_l}|^{m-1} |\lambda - A_{j_j \cdots j_l}| \]
\[ \leq \left( \rho(B) - B_{i_i \cdots i_l} - (\kappa(B))^{m-1} \check{r}_i(E) \right)^{m-1} \left[ \rho(B) - B_{j_j \cdots j_l} - (\kappa(B))^{m-1} \check{r}_j(E) \right], \]

which proves the desired inequality.

**Remark 6.** It is noted that the disks in (31) are contained in the union of the ones in (29). For any complex point \( z \) satisfied (31), it must be true that

\[ |z - A_{i_i \cdots i_l}| \leq \rho(B) - B_{i_i \cdots i_l} - (\kappa(B))^{m-1} \check{r}_i(E) \]

or

\[ |z - A_{j_j \cdots j_l}| \leq \rho(B) - B_{j_j \cdots j_l} - (\kappa(B))^{m-1} \check{r}_j(E). \]

Otherwise, if \( z \) does not satisfy both the above inequalities, then

\[ |z - A_{i_i \cdots i_l}|^{m-1} |z - A_{j_j \cdots j_l}| \]
\[ > \left[ \rho(B) - B_{i_i \cdots i_l} - (\kappa(B))^{m-1} \check{r}_i(E) \right]^{m-1} \left[ \rho(B) - B_{j_j \cdots j_l} - (\kappa(B))^{m-1} \check{r}_j(E) \right], \]

which contradicts to (31). This proves the assertion. Hence the bound in (31) always improves the corresponding one in (29).

Next we give an example to reveal the difference of (29) and (31).

**Example 5.** Let \( A \in \mathbb{R}_{+}^{4,2} \): \( A_{1,1,1,1} = 1/2, A_{2,2,2,2} = 3 \), and \( A_{i_i,j,j,k,l} = 1/3 \) elsewhere, and \( B \in \mathbb{R}_{+}^{4,2} \): \( B_{i_i,j,j,k,l} = 3 \) for any \( 1 \leq i_i, j, k, l \leq 2 \). Thus \( B \) is a positive tensor and \( B \geq |A| \). Then we have the following figure.

---

**Figure 1.** Comparison between two Ky Fan type Theorems

In Figure 1, the green area is the union of two disks in (31). The two red circles are the boundaries of two disks in (29). And the blue star symbols are the eigenvalues of \( A \). Hence it is easy to see that the green area is contained in the union of two disks with red borders, which shows the theoretical result.
3.3. Perturbation bounds for a spectral radius. In [10], Li and Ng obtained the following perturbation bound for the spectral radii of nonnegative tensors.

**Theorem 3.7.** Suppose \( A \in \mathbb{R}^{[m,n]}_+ \), and \( \tilde{A} = A + \mathcal{E} \) is the perturbed nonnegative tensor of \( A \). Then

\[
|\rho(\tilde{A}) - \rho(A)| \leq \tau(A) \cdot \|\mathcal{E}\|_{\infty} \tag{36}
\]

provided that \( \tau(A) > 0 \), where \( \tau(A) = 1/(\kappa_1(A))^{n-1} \), \( \kappa_1(A) \) is given in (3) and \( \|\mathcal{E}\|_{\infty} \equiv \max_{1 \leq i \leq n} \sum_{i_2, \ldots, i_m=1}^{n} |\mathcal{E}_{i_1, i_2, \ldots, i_m}| \).

In this subsection, we consider to present a new perturbation bound for two nonnegative tensors \( A \) and \( \tilde{A} \). First we state some preliminary results.

**Lemma 3.8.** [10] Suppose \( A \in \mathbb{R}^{[m,n]}_+ \) with a positive Perron vector \( x \), and \( \tilde{A} = A + \mathcal{E} \) is the perturbed nonnegative tensor of \( A \). Then

\[
\min_{1 \leq i \leq n} r_i(\mathcal{E}_D) \leq \rho(\tilde{A}) - \rho(A) \leq \max_{1 \leq i \leq n} r_i(\mathcal{E}_D),
\]

where \( r_i(\mathcal{E}_D) = \sum_{i_2, \ldots, i_m=1}^{n} \mathcal{E}_{i,i_2, \ldots, i_m} \), \( D = \text{diag}(x_1, \ldots, x_n) \) and \( \mathcal{E}_D \) is defined as in (30).

**Lemma 3.9.** [21, 10] Let \( A \in \mathbb{R}^{[m,n]}_+ \), and \( A_t = A + (1/t)\mathcal{J} \), where \( \mathcal{J} \) is a tensor with all entries being 1. Then \( \rho(A_t) = \lim_{t \to \infty} \rho(A_t) \).

The perturbation bound for the spectral radius of a positive tensor \( A \) is given below:

**Theorem 3.10.** Let \( A \in \mathbb{R}^{[m,n]}_+ \) be a positive tensor, and \( \tilde{A} = A + \mathcal{E} \) be the perturbed nonnegative tensor of \( A \). Then

\[
|\rho(\tilde{A}) - \rho(A)| \leq \max_{1 \leq i \leq n} \left\{ |\mathcal{E}_{i_1, \ldots, i}| + \frac{1}{(\kappa(A))^{n-1}} \hat{r}_i(|\mathcal{E}|) \right\} \tag{37}
\]

provided that \( \kappa(A) > 0 \), where \( r_i(\mathcal{E}) = \sum_{i_2, \ldots, i_m=1}^{n} \mathcal{E}_{i,i_2, \ldots, i_m} \), \( \hat{r}_i(\mathcal{E}) = r_i(\mathcal{E}) - \mathcal{E}_{i_1, \ldots, i} \) and \( \kappa(A) \) is given by \( \kappa_3 \) in (13).

**Proof.** Since \( A \) is a positive tensor, by Lemma 1.4, we know that \( A \) has a positive Perron vector \( x \). Let \( D = \text{diag}(x_1, \ldots, x_n) \). By Lemma 3.8, we have

\[
|\rho(\tilde{A}) - \rho(A)| \leq \max_{1 \leq i \leq n} |r_i(\mathcal{E}_D)|
\]

\[
\leq \max_{1 \leq i \leq n} \left\{ \sum_{i_2, \ldots, i_m=1}^{n} |\mathcal{E}_{i,i_2, \ldots, i_m}| x_i^{-(m-1)} x_{i_2} \cdots x_{i_m} \right\}.
\]

Let \( s = \arg \min x_i \), \( l = \arg \max x_i \), and \( \Delta^{(k)}_{i,j} \) be defined as in Section 2. Then we have

\[
|\rho(\tilde{A}) - \rho(A)| \leq \max_{1 \leq i \leq n} \left\{ \sum_{a \in \Delta^{(k)}_{i,j}} |\mathcal{E}_{i,a}| \left( \frac{x_i}{x_a} \right)^{m-1-k} + \sum_{(i_2, \ldots, i_m) \notin \Delta^{(k)}_{i,j}} |\mathcal{E}_{i,i_2, \ldots, i_m}| x_i^{-(m-1)} x_{i_2} \cdots x_{i_m} \right\}
\]
Theorem 3.11. Let 

\[ \text{Corollary 2.} \quad \text{Let} \] 

\[ \text{Therefore, when} \] 

\[ \text{which completes the proof of theorem.} \] 

Based on Theorem 3.10, we give the following perturbation bound:

**Corollary 2.** Let \( \mathcal{A} \in \mathbb{R}^{[m,n]}_+ \) and \( \tilde{\mathcal{A}} = \mathcal{A} + \mathcal{E} \) be the perturbed nonnegative tensor of \( \mathcal{A} \). Then

\[ |\rho(\tilde{\mathcal{A}}) - \rho(\mathcal{A})| \leq \max_{1 \leq i \leq n} \left\{ \sum_{\alpha \in \Delta_i} |\mathcal{E}_{i,\alpha}| + \frac{1}{(\kappa(\mathcal{A}))^{m-1}} \sum_{(i_2, \ldots, i_m) \in \Delta_i^{(k)}} |\mathcal{E}_{i,i_2,\ldots,i_m}| \right\} \]

(38)

provided that \( \kappa(\mathcal{A}) > 0 \), where \( r_i(\mathcal{E}) = \sum_{i_2, \ldots, i_m=1}^{n} \mathcal{E}_{i,i_2,\ldots,i_m} \), \( \hat{r}_i(\mathcal{E}) = r_i(\mathcal{E}) - \mathcal{E}_{i,\ldots,i} \) and \( \kappa(\mathcal{A}) \) is given by \( \kappa_3 \) in (13).

**Proof.** Let \( \mathcal{A}_t = \mathcal{A} + (1/t)\mathcal{J} \) and \( \tilde{\mathcal{A}}_t = \mathcal{A}_t + \mathcal{E} \), where \( \mathcal{J} \) is a tensor with all entries equal to 1. Then \( \mathcal{A}_t \) and \( \tilde{\mathcal{A}}_t \) are positive tensors. Hence by Theorem 3.10 we get

\[ |\rho(\tilde{\mathcal{A}}_t) - \rho(\mathcal{A}_t)| = \max_{1 \leq i \leq n} \left\{ |\mathcal{E}_{i,\ldots,i}| + \frac{1}{(\kappa(\mathcal{A}_t))^{m-1}} \hat{r}_i(|\mathcal{E}|) \right\} \]

Therefore, when \( t \to \infty \), by Lemma 3.9, we get the desired inequality. \( \square \)

By an analogous proof as Corollary 2, we may have a new comparison theorem for the nonnegative tensor case as follows.

**Theorem 3.11.** Let \( \mathcal{A}, \mathcal{B} \in \mathbb{R}^{[m,n]}_+ \). If \( \mathcal{B} \leq \mathcal{A} \) and \( \mathcal{A} \) is irreducible, then

\[ \min_{1 \leq i \leq n} \left\{ \mathcal{E}_{i,\ldots,i} + (\kappa(\mathcal{A}))^{m-1} \hat{r}_i(\mathcal{E}) \right\} \leq \rho(\mathcal{A}) - \rho(\mathcal{B}) \leq \max_{1 \leq i \leq n} \left\{ \mathcal{E}_{i,\ldots,i} + \frac{1}{(\kappa(\mathcal{A}))^{m-1}} \hat{r}_i(\mathcal{E}) \right\} \]

(39)

where \( \mathcal{E} = \mathcal{A} - \mathcal{B} \), \( r_i(\mathcal{E}) = \sum_{i_2, \ldots, i_m=1}^{n} \mathcal{E}_{i,i_2,\ldots,i_m} \), \( \hat{r}_i(\mathcal{E}) = r_i(\mathcal{E}) - \mathcal{E}_{i,\ldots,i} \) and \( \kappa(\mathcal{A}) \) is given by \( \kappa_3 \) in (13).

**Proof.** From the proof of Theorem 3.10, it is easy to note that

\[ \rho(\mathcal{A}) - \rho(\mathcal{B}) \leq \max_{1 \leq i \leq n} \left\{ \mathcal{E}_{i,\ldots,i} + \frac{1}{(\kappa(\mathcal{A}))^{m-1}} \hat{r}_i(\mathcal{E}) \right\} \]

(40)
Similarly, for the lower bound, by Lemma 3.8 we have
\[ \rho(A) - \rho(B) \geq \min_{1 \leq i \leq n} \frac{1}{\kappa(A)} \sum_{i=2}^{m} E_{i} x_{i}^{1-m} x_{i} \cdot x_{i} \]
\[ \geq \min_{1 \leq i \leq n} \left\{ \frac{x_{\min}}{x_{\max}} \right\} - \frac{1}{\kappa(A)} \left( \sum_{i=1}^{n} r_{i}(E) \right) \]
\[ \geq \min_{1 \leq i \leq n} \left\{ \frac{x_{\min}}{x_{\max}} - \frac{1}{\kappa(A)} \sum_{i=1}^{n} r_{i}(E) \right\}, \] (41)

where \( x \) is a Perron vector of \( A \), \( x_{\min} = \min_{1 \leq i \leq n} x_{i} \) and \( x_{\max} = \max_{1 \leq i \leq n} x_{i} \).

Hence we may prove the desired formula by (40) and (41).

**Remark 7.** Since \( \kappa(A) \leq 1 \), it is easy to note that
\[ \max_{1 \leq i \leq n} \left\{ \frac{x_{\min}}{x_{\max}} - \frac{1}{\kappa(A)} \sum_{i=1}^{n} r_{i}(E) \right\} \leq \frac{1}{\kappa(A)} \sum_{i=1}^{n} r_{i}(E) \leq \tau(A) \cdot \|E\|_{\infty}. \]

Hence the bound (38) always improves the corresponding one in (36) and the bound (41) in [11].

Next we do numerical experiment to demonstrate the differences of the perturbation bounds (36) and (38). First, we compare the bounds by some tensors generated randomly. We construct the positive tensors \( A, \tilde{A} \in \mathbb{R}^{[4,5]} \), whose entries are generated by uniform distribution on \((0,1)\). Their difference is denoted by symbol:

- The bound in (36) — the bound in (38): plus symbol in red color.

In Figure 2, we plot the numerical results by 100 generated tensors. It is easy to see that the symbols are above the \( x \)-axis, i.e., the bound (38) is better than that in (36).

Second, we conduct another experiment, which is similar to Example 18 in [10], to verify the perturbation bounds. We construct a positive tensor \( B \) randomly, whose entries are generated by uniform distribution on \((0,1)\). And let \( D = \text{diag}(1,2,\cdots,n) \) be a diagonal matrix. Then we suppose \( A = B \times_{1} D. \) In addition, \( A \) is perturbed to a positive tensor \( \tilde{A} \) with perturbation \( \varepsilon E \), i.e., \( \tilde{A} = A + \varepsilon E \), where \( \varepsilon \) is a positive number and \( E \) is a positive tensor generated by uniform distribution on \((0,1)\). In this experiment, we get the absolute difference...
$|\rho(A) - \rho(\tilde{A})|$ and the perturbation bounds (36) and (38). Some symbols are denoted by

- $|\rho(A) - \rho(\tilde{A})|$ (the actual difference): cross symbol in blue color;
- the bound in (36): circle symbol in red color;
- the bound in (38): plus symbol in magenta color.

Figure 3. The results of perturbation bounds (left) $n = 5$ and (right) $n = 10$

We plot Figure 3 for $n = 5, 10$ and $m = 4$, respectively, in which we report the numerical results of 100 cases, and give the average values of $|\rho(A) - \rho(\tilde{A})|$ and the bounds (36) and (38). The $x$-axis refers to the value of coefficient $\varepsilon$: 0.01, 0.005, 0.001, 0.0005, 0.0001 and 0.00005. We see from Figure 3 that the average values (in logarithm scale) depend linearly on $\varepsilon$ (in logarithm scale), and the bound (38) is closer to the actual difference, which shows the theoretical results.

4. Conclusion. In this paper, we have derived some new lower and upper bounds for the ratio of the smallest and largest entries of the Perron vector. Based on these new bounds, we obtain the two-sided bound for the spectral radius of an irreducible nonnegative tensor, which always improves any ones among (21)-(24). In particular, the spectral radius bound also improves the corresponding ones in the nonnegative matrix case. In addition, we present a new Ky Fan type theorem and a perturbation bound for the largest eigenvalue of a nonnegative tensor, respectively. Each of these new bounds is always better than the existing one. Numerical examples are given to show the validity of the theoretical bounds.

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