THE $\lambda$-COSINE TRANSFORMS, DIFFERENTIAL OPERATORS, AND FUNK TRANSFORMS ON STIEFEL AND GRASSMANN MANIFOLDS

B. RUBIN

Abstract. We introduce a new family of invariant differential operators associated with $\lambda$-cosine and Funk-Radon transforms on Stiefel and Grassmann manifolds. These operators reduce the order of the $\lambda$-cosine transforms and yield new inversion formulas. Intermediate Funk-cosine transforms corresponding to integration over matrices of lower rank are studied. The main tools are polar decomposition and Fourier analysis on matrix space.

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1. INTRODUCTION

In the present paper we introduce new invariant differential operators that can be used in the study of $\lambda$-cosine, $\lambda$-sine, and Funk transforms on Stiefel and Grassmann manifolds. These operators are generalizations of polynomials of the Beltrami-Laplace operator in the inversion

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formulas for the classical Funk transform on the unit sphere. We recall that the \( \lambda \)-cosine transform of a function \( f \) on the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \) is an integral operator of the form
\[
(C^\lambda f)(u) = \int_{S^{n-1}} f(v) |u \cdot v|^\lambda \, d_* v, \quad u \in S^{n-1},
\]
where integration is performed with respect to the rotation invariant probability Haar measure. The name \textit{cosine transform} for \( \lambda = 1 \) is due to Lutwak \cite[p. 385]{Lutwak1991} and reflects the fact that \( |u \cdot v| \) is the cosine of the smallest angle between the straight lines along \( u \) and \( v \). The associated operator
\[
(F f)(u) = \int_{\{v \in S^{n-1} : u \cdot v = 0\}} f(v) \, d_u v
\]
is called the Funk transform or a spherical Radon transform of \( f \). An extensive bibliography related to operators (1.1), (1.2), their generalizations, and applications can be found in \cite{Burtz, Spherical, Grassmann}.

In recent decades there is an increasing interest to analogues of (1.1) and (1.2), when the lines along \( u \) and \( v \) are replaced by higher dimensional linear subspaces. These generalizations lead to integral operators that take functions on the Grassmannian \( G_{n,m} \) of \( m \)-dimensional linear subspaces of \( \mathbb{R}^n \) to functions on the similar Grassmannian \( G_{n,k} \). In place of the Grassmannians, one can take their orthonormal bases (or frames), which are elements of the Stiefel manifolds \( V_{n,m} \) and \( V_{n,k} \), respectively.

Historically the first publications related to higher rank generalizations of (1.1) and (1.2), were probably short articles by Petrov \cite{Petrov1964} and Matheron \cite{Matheron1965}, though close mathematical objects on matrix spaces were studied before by Gårding \cite{Garding1949}, Gindikin \cite{Gindikin1963}, and some other authors, in particular, in multivariate statistics. The paper \cite{Petrov1964} deals with inversion of Radon transforms on matrices and Grassmannians, while \cite{Matheron1965} contains a famous injectivity conjecture for the higher rank cosine transform with \( \lambda = 1 \). This conjecture was disproved by Goodey and Howard \cite{Goodey1975, Howard1975}. More information and further references can be found in \cite{Burtz, Spherical, Grassmann, Parallel, Spherical, Grassmann, Spherical, Grassmann, Spherical, Grassmann, Spherical, Grassmann, Spherical, Grassmann, Spherical, Grassmann, Spherical, Grassmann, Spherical, Grassmann, Spherical, Grassmann, Spherical, Grassmann, Spherical, Grassmann}. A great deal has been written about Radon transforms on affine Grassmann manifolds. This circle of problems lies beyond the scope of the present paper. Information can be found in \cite{AffineGrassmann, AffineGrassmann, AffineGrassmann, AffineGrassmann, AffineGrassmann, AffineGrassmann, AffineGrassmann}.

Let us describe the contents of the paper and main results.

\textbf{1.} The following question was asked by Alesker \cite{Alesker1996}:
Given a complex number \( \lambda \), what differential operator \( D \) satisfies

\[ D \mathcal{C}^{\lambda+2} = \mathcal{C}^{\lambda} \quad \text{(1.3)} \]

The answer is known for the unit sphere [41, p. 285], where \( D \) is a polynomial of the Beltrami-Laplace operator and the reasoning relies on the spherical harmonic technique. In the Grassmannian set-up, when the lines along \( u \) and \( v \) in (1.1) are replaced by linear subspaces, say \( \xi \in G_{n,m} \) and \( \tau \in G_{n,k} \), the question was studied by Alesker, Gourevitch, and Sahi [4] for \( k = m \). The authors used the tools of the representation theory, in terms of which the operator \( D \) looks pretty complicated.

In the present paper we consider arbitrary \( k \) and \( m \) and use an equivalent language of Stiefel manifolds. This setting of the problem yields a dual pair of \( \lambda \)-cosine transforms, \( \mathcal{C}^{\lambda}_{m,k} \) and \( \mathcal{C}^{\lambda *}_{m,k} \), which coincide when \( k = m \); see (4.1), (4.2). For the dual transform \( \mathcal{C}^{\lambda *}_{m,k} \) we obtain a generalization of (1.3) having the form

\[ D_{\ell} \mathcal{C}^{\lambda+2\ell}_{m,k} = \mathcal{C}^{\lambda *}_{m,k}; \quad \ell = 0, 1, 2, \ldots \quad \text{(1.4)} \]

The operator \( D_{\ell} \) has a simple form; see Theorem 7.3. In the case of the unit sphere, \( D_{\ell} \) boils down to the known polynomial of the Beltrami-Laplace operator. We also obtain an analogue of (1.4) for the \( \lambda \)-sine transforms; see (7.9).

Unlike [4], our method relies on the extension of orthonormal Stiefel matrices by homogeneity onto the ambient space of real rectangular matrices with subsequent implementation of the Fourier transform technique. This method was developed in [40] and used in [42] to prove (1.3) on the unit sphere without spherical harmonics. An analogue of (1.4) for \( \mathcal{C}^{\lambda *}_{m,k} \) (without “*”) is still an open problem if \( k \neq m \).

2. We also study analytic continuations of properly normalized \( \lambda \)-cosine transforms, which include Stiefel analogues of the Funk transform (1.2), as well as their intermediate modifications. We call them the intermediate Funk-cosine transforms and denote \( F^{(j)}_{m,k}; \ j = 0, 1, \ldots, m-1 \).

In the case \( j = 0 \), the operator \( F_{m,k} \equiv F^{(0)}_{m,k} \) is a straightforward generalization of (1.2) and coincides with the latter if \( m = k = 1 \). All these transforms can be written in Grassmannian terms and expressed as convolutions with positive Radon measures. Convolutions (or distributions) of similar nature are well known in Analysis and deal with integration over matrices of lower rank; see, e.g., [7, Chapter VII, Section 2], [39, Section 4].
Intermediate Funk-cosine transforms in the case $k = m$ were considered by Cross [6], who defined them using the group representation tools developed by Ólafsson and Pasquale [33]. Our approach, invoking Stiefel manifolds and zeta integrals, is different in principle. It is straightforward, has simple geometric and group-theoretic interpretation, and covers all admissible $k$ and $m$; see formulas (5.1), (5.9), (5.11), (5.13).

3. We apply (1.4) to inversion of the Funk transforms $F_{m,k}$ and the intermediate Funk-cosine transforms $F^{(j)}_{m,k}$; see Section 8. For the sake of simplicity, the results are formulated in terms of the right $O(m)$-invariant functions on the Stiefel manifold $V_{n,m}$, but the reader can easily reformulate them in the Grassmannian language. We obtain new local inversion formulas and some nonlocal formulas, the structure of which depends on the parity of dimensions and agrees with known results for the unit sphere [21, 41, 42]. Some cases, related to nonlocal inversion formulas, remain open and need new ideas; see Section 9 for the list of open problems that might be of interest.

It should be noted that nonlocal inversion formulas for the Funk transform on Grassmannians are known in terms, which differ from those in the present paper. Such formulas, invoking the Crofton symbol and the kappa operator, can be found in [10] and are pretty involved. An alternative approach in terms of Gårding-Gindikin fractional integrals was suggested in [17] for real Grassmannians and extended in [49] to complex and quaternionic cases. Unlike these works, our goal in the present paper is to find simple differential operators that agree with elegant formulas by Helgason [21, p. 133] and our formulas in [42] for the unit sphere.

Differential operators with determinantal power weights were studied by Sahi and Zhang [44] in the general context of real, complex, and quaternionic matrix spaces. These operators have common features with $D_\ell$ in (1.4), descend to Grassmannians, and can be used to obtain local inversion formulas for the Funk transform. The method of [44] heavily relies on the group representation technique and essentially differs from ours. In contrast with [44], the core of our approach is the classical Fourier analysis. It allows us to obtain not only local but also some nonlocal inversion formulas and covers intermediate Funk-cosine transforms, that were not considered in [44].

4. A distinctive feature of our paper in comparison with other related publications (see, e.g., [3, 4, 16, 24, 34, 44]) is that we think of smooth functions on Stiefel (or Grassmann) manifolds not in terms of
coordinate charts, but using homogeneous continuation of the relevant orthonormal frames onto the space of rectangular matrices, where classical Calculus can be applied. This transition is performed with the aid of the polar decomposition of matrices. As a result, it becomes possible to write the desired differential operators in a simple analytic form.

5. The above approach to the definition of smooth functions entails, however, some extra work. Specifically, we need to show equivalence of our definition and the classical one, as, e.g., in the Lie theory, and carefully justify the $C^\infty \to C^\infty$ action of all operators under consideration\footnote{Such a justification is sometimes skipped in [40], as “obvious.”}. Information about differentiable structures on Stiefel or Grassmann manifolds and related diffeomorphisms is highly scattered and presented in different sources from different points of view. For convenience of the reader, we have written an Appendix, in which this auxiliary material is organized in a unified consistent form. Most of the facts, except probably Lemma A.7, are well known; some of them look folklorish.

In Sections 2-6 we introduce basic objects of our investigation and study their properties (see Contents). The main results are presented in Sections 7,8.

2. Preliminaries

2.1. Notation. Let $\mathfrak{M}_{n,m}$ be the space of real matrices $x = (x_{i,j})$ having $n$ rows and $m$ columns. We associate $\mathfrak{M}_{n,m}$ with the real space $\mathbb{R}^{nm}$ of $nm$-tuples

$$(x_{1,1}, \ldots, x_{n,1}, x_{1,2}, \ldots, x_{n,2}, \ldots, x_{1,m}, \ldots, x_{n,m});$$

$dx = \prod_{i=1}^{n} \prod_{j=1}^{m} dx_{i,j}$; $x'$ is the transpose of $x$; $|x|_m = \det(x'x)^{1/2}$; $I_m$ is the identity $m \times m$ matrix; $0$ stands for the zero entries. In the case $n \geq m$, we denote by $\tilde{\mathfrak{M}}_{n,m}$ the set of all matrices $x \in \mathfrak{M}_{n,m}$ of rank $m$. This set is an open subset of $\mathfrak{M}_{n,m}$ in the standard topology of $\mathbb{R}^{nm}$; $GL(n, \mathbb{R}) = \mathfrak{R}_{n,n}$ is the general linear group of $\mathbb{R}^n$. We write

$$e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$$

for the coordinate unit vectors in $\mathbb{R}^n$.

The notation $L^1(M)$, $C(M)$, $C^\infty(M)$ for the function spaces of Lebesgue integrable, continuous, and infinitely differentiable functions on $M$ is standard. It is assumed that $M$ is equipped with suitable structure. If a group $H$ acts on $M$ from the right, then $L^1(M)^H$, $C(M)^H$, $C^\infty(M)^H$, \ldots, $C^\infty(M)^H$, \ldots, $C^\infty(M)^H$, \ldots.
and \( C^\infty(M)^H \) denote the corresponding spaces of right \( H \)-invariant functions.

We will be dealing with the compact Stiefel manifold \( V_{n,m} = \{ v \in \mathfrak{m}_{n,m} : v^t v = I_m \} \) of orthonormal \( m \)-frames in \( \mathbb{R}^n \) and the Grassmann manifold \( G_{n,m} \) of \( m \)-dimensional linear subspaces of \( \mathbb{R}^n \) equipped with the relevant Haar probability measures. Basic facts about these manifolds are collected in Appendix. If \( v \in V_{n,m} \), then \( \{ v \} = \text{span}(v) \in G_{n,m} \) is a linear subspace spanned by \( v \); \( v^\perp \in G_{n,n-m} \) is a subspace perpendicular to \( v \). If \( m = n \), then \( V_{n,n} = O(n) \) is the group of orthogonal transformations of \( \mathbb{R}^n \). If \( m = 1 \), then \( V_{n,1} = S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \).

The Fourier transform of a function \( \varphi \in L^1(\mathfrak{m}_{n,m}) \) is defined by

\[
\hat{\varphi}(y) = \int_{\mathfrak{m}_{n,m}} e^{\text{tr}(iy^t x)} \varphi(x) \, dx, \quad y \in \mathfrak{m}_{n,m}.
\]

The corresponding Parseval equality has the form

\[
(\hat{\varphi}, \hat{\omega}) = (2\pi)^{nm}(\varphi, \omega), \quad (\varphi, \omega) = \int_{\mathfrak{m}_{n,m}} \varphi(x) \overline{\omega(x)} \, dx. \tag{2.1}
\]

If \( \omega \) belongs to the Schwartz space \( S(\mathfrak{m}_{n,m}) \) of rapidly decreasing smooth functions and \( \varphi \in S'(\mathfrak{m}_{n,m}) \) is a tempered distribution, the equality (2.1) serves as a definition of the Fourier transform of \( \varphi \).

The Cayley-Laplace operator \( \Delta \) on \( \mathfrak{m}_{n,m} \) is defined by

\[
\Delta = \det(\partial^t \partial), \tag{2.2}
\]

where \( \partial \) is the \( n \times m \) matrix whose entries are partial derivatives \( \partial/\partial x_{i,j} \). In the Fourier transform terms, the action of \( \Delta \) represents a multiplication by \((-1)^m|y|^2 \). It follows that \( \Delta \) is left \( O(n) \)-invariant and right \( O(m) \)-invariant, that is,

\[
\Delta : f(\rho x) \to (\Delta f)(\rho x), \quad f(x\gamma) \to (\Delta f)(x\gamma), \tag{2.3}
\]

for all \( \rho \in O(n), \gamma \in O(m), x \in \mathfrak{m}_{n,m} \). These relations can be easily checked using the Fourier transform. More information about the Cayley-Laplace operator can be found in [27, 39].

In the following, \( \text{Sym}_m \simeq \mathbb{R}^{m(m+1)/2} \) is the space of \( m \times m \) real symmetric matrices \( s = (s_{i,j}); ds = \prod_{i \leq j} ds_{i,j}; \Omega_m \) denotes the cone of positive definite matrices in \( \text{Sym}_m \). The Siegel gamma function of \( \Omega_m \) is defined by

\[
\Gamma_m(\alpha) = \int_{\Omega_m} \det(s)^{\alpha-(m+1)/2} e^{-\text{tr}(s)} \, ds = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\alpha-j/2); \tag{2.4}
\]
This integral is absolutely convergent if \( \Re \alpha > (m - 1)/2 \) and extends a meromorphic function of \( \alpha \) with the polar set 
\[ \{(m - 1 - j)/2 : j = 0, 1, 2, \ldots \} . \]

The abbreviation \( a.c. \) mean analytic continuation. Normalized probability measures will be usually denoted by \( d_s \) followed by the variable of integration. The letter \( c \) (sometimes with subscripts) is used for a constant that can be different at each occurrence.

2.2. Zeta integrals. Suppose that \( n \geq m \geq 2 \) and denote 
\[ Z_{n,m}(f, \lambda) = \int_{M_{n,m}} f(x)|x|^\lambda dx, \quad f \in S(M_{n,m}), \quad \lambda \in \mathbb{C} . \] (2.5)

This expression is called the zeta integral \([23, 46]\) and represents a meromorphic \( S^\ell \)-distribution.

**Lemma 2.1.** ([45, 26], [39, Lemma 4.2]) The integral (2.5) is absolutely convergent if \( \Re \lambda > m - n - 1 \) and extends to \( \Re \lambda \leq m - n - 1 \) as a meromorphic function of \( \lambda \) with the only poles \( m - n - 1, m - n - 2, \ldots \). These poles and their orders are the same as of \( \Gamma_m((\lambda + n)/2) \).

The normalized integral 
\[ \zeta_{n,m}(f, \lambda) = \frac{Z_{n,m}(f, \lambda)}{\Gamma_m((\lambda + n)/2)} \] (2.6)
is an entire function of \( \lambda \).

If \( \Delta \) is the Cayley-Laplace operator (2.2), the following identity of the Bernstein type holds:
\[ \Delta^{\ell}|x|^\lambda dx_{m} = B_{\ell,m,n}(\lambda)|x|^\lambda, \quad \ell = 1, 2, \ldots , \]
\[ B_{\ell,m,n}(\lambda) = \prod_{i=0}^{m-1} \prod_{j=0}^{\ell-1} ((\lambda + n - i + 2j)(\lambda + 2 + 2j + i)); \] (2.7)
see [39, p. 565]. It allows us to represent meromorphic continuation of \( Z(f, \lambda) \) in the form
\[ Z_{n,m}(f, \lambda) = \frac{1}{B_{\ell,m,n}(\lambda)} Z_{n,m}(\Delta^{\ell} f, \lambda + 2\ell), \quad \Re \lambda > m - n - 1 - 2\ell. \] (2.8)

The values
\[ \lambda = -n, 1 - n, \ldots , m - n - 1, \]
for which the corresponding zeta distribution is a positive measure, deserve special mentioning; cf. [7, Theorem VII.3.1].
Lemma 2.2. [39, Theorem 4.4, Lemma 4.7] If \( f \in S(\mathbb{M}_{n,m}) \), then
\[
\zeta_{n,m}(f, \lambda)|_{\lambda = j - n} = \int_{\mathbb{M}_{n,m}} f(x) d\nu_j(x), \quad j = 0, 1, \ldots, m - 1,
\]
where \( \nu_j \) is a Radon measure supported on the set \( \{ x \in \mathbb{M}_{n,m} : \text{rank}(x) \leq j \} \). Specifically, if \( j = 1, 2, \ldots, m - 1 \), then
\[
\zeta_{n,m}(f, j - n) = \pi^{n-1} \left( \frac{m}{2} \right) \int_{O(n)} d\gamma \int_{\mathbb{M}_{j,m}} f \left( \gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right) d\omega.
\]
If \( j = 0 \), then
\[
\zeta_{n,m}(f, -n) = \pi^m \left( \frac{n}{2} \right) f(0).
\]

2.3. Convolutions. Most of the operators in our paper are expressed as convolutions on the group \( O(n) \). In general, let \( G \) be a compact Lie group, \( \mathcal{M}(G) \) be the space of Radon measures on \( G \). The convolution of \( f \in L^1(G) \) with \( \mu \in \mathcal{M}(G) \) is defined by
\[
(f * \mu)(x) = \int_G f(xy^{-1}) d\mu(y)
\]
and belongs to \( L^1(G) \).

Proposition 2.3. If \( f \in C^\infty(G), \mu \in \mathcal{M}(G) \), then \( f * \mu \in C^\infty(G) \).

Proof. This statement can be found in [18, p. 83] without proof. The proof was briefly outlined in [20, p. 147] for the special case of Radon transforms. In the general case it can be proved as follows.

We first note that, by definition of the Lie group, the map
\[
\kappa : G \times G \to G, \quad (x, y) \to xy^{-1},
\]
is smooth. Hence the function \( F = f \circ \kappa \) is smooth on \( G \times G \), as a composition of smooth maps. Fix any coordinate chart \( (U, \varphi) \) for \( G \) and let \( B = \varphi(U) \) be a Euclidean ball in \( \mathbb{R}^d \), \( d = \dim G \). Consider the diffeomorphism \( B \times G \to U \times G \), \( (\xi, y) \to (\varphi^{-1}\xi, y) \), and let \( \tilde{F}(\xi, y) = F(\varphi^{-1}\xi, y) \). If \( x \in U \), the convolution \( f * \mu \) is locally represented as
\[
(f * \mu)(\varphi^{-1}\xi) = \int_G \tilde{F}(\xi, y) d\mu(y), \quad \xi \in B.
\]

The function \( \tilde{F} \) is smooth on \( B \times G \) as a composition of smooth maps. Note also that \( \tilde{F}(\xi, y) \) is smooth as a function of two variables if and only if both “single-variable functions” \( \xi \to \tilde{F}(\xi, y) \) and \( y \to \tilde{F}(\xi, y) \) are
smooth. In particular, if \((\xi, y) \rightarrow \tilde{F}(\xi, y)\) is smooth, then any partial derivative \((\xi, y) \rightarrow \partial_{\xi} \tilde{F}(\xi, y)\) is smooth too. The last observation allows us to differentiate under the sign of integration in (2.9) infinitely many times, and we are done. \(\square\)

3. Funk Transforms on Stiefel and Grassmann Manifolds

Let \(V_{n,k}\) and \(V_{n,m}\) be a pair of Stiefel manifolds; \(1 \leq k, m \leq n - 1\). We consider Funk-type transforms, which are formally defined by

\[
(F_{m,k}f)(u) = \int_{\{v \in V_{n,m} : u'v = 0\}} f(v) \, dv, \quad u \in V_{n,k}, \tag{3.1}
\]

\[
(F^*_{m,k}\varphi)(v) = \int_{\{u \in V_{n,k} : u'v = 0\}} \varphi(u) \, du, \quad v \in V_{n,m}. \tag{3.2}
\]

The condition \(u'v = 0\) means that the subspaces \(\{u\} \in G_{n,k}\) and \(\{v\} \in G_{n,m}\) are mutually orthogonal. Hence, necessarily,

\[k + m \leq n.\]

To give \((F_{m,k}f)(u)\) and \((F^*_{m,k}\varphi)(v)\) precise meaning, we set

\[u_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \in V_{n,k}, \quad v_0 = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in V_{n,m},\]

and let \(g_u\) and \(g_v\) be orthogonal transformations satisfying \(g_u u_0 = u, g_v v_0 = v\). Denote \(f_u(v) = f(g_u v), \varphi_v(u) = \varphi(g_v u)\). Then (3.1) and (3.2) can be explicitly written as

\[
(F_{m,k}f)(u) = \int_{V_{n-k,m}} f_u \left( \begin{bmatrix} \vartheta \\ 0 \end{bmatrix} \right) d_* \vartheta = \int_{O(n-k)} f_u \left( \begin{bmatrix} \gamma \\ 0 \\ I_k \end{bmatrix} \right) v_0 \, d_* \gamma, \tag{3.3}
\]

\[
(F^*_{m,k}\varphi)(v) = \int_{V_{n-m,k}} \varphi_v \left( \begin{bmatrix} 0 \\ \theta \end{bmatrix} \right) d_* \theta = \int_{O(n-m)} \varphi_v \left( \begin{bmatrix} I_m \\ 0 \\ \rho \end{bmatrix} \right) u_0 \, d_* \rho. \tag{3.4}
\]

These expressions are independent of the afore-mentioned choice of \(g_u\) and \(g_v\) and agree with the case \(m = k = 1\) of the unit sphere. Operators \(F_{m,k}\) and \(F^*_{m,k}\) are \(O(n)\)-equivariant, the function \(F_{m,k}f\) is right \(O(k)\)-invariant, and \(F^*_{m,k}\varphi\) is right \(O(m)\)-invariant.

If \(k = m\) we set \(F_m = F_{m,m}\). In this case, (3.1) and (3.2) essentially coincide. If \(k + m = n\), then \(V_{n-k,m} = O(m), V_{n-m,k} = O(k)\), and our
Funk transforms are averages of the form
\[(F_{m,n-m}f)(u) = \int_{O(n-m)} f(u_0 \theta) d\theta, \quad (F_{m,n-m}^* \varphi)(v) = \int_{O(n)} \varphi(u_0 \theta) d\theta.\]

Note also that
\[F_{m,k} f = F_{m,k} f_{\text{ave}}, \quad F_{m,k} \varphi = F_{m,k}^* \varphi_{\text{ave}}, \quad (3.5)\]
where
\[f_{\text{ave}}(v) = \int_{O(n-m)} f(v \beta) d\beta, \quad \varphi_{\text{ave}}(u) = \int_{O(n)} \varphi(u \alpha) d\alpha.\]

By Proposition A.5, the maps \(f \rightarrow f_{\text{ave}}\) and \(\varphi \rightarrow \varphi_{\text{ave}}\) act from \(L^1\) to \(L^1\) and from \(C^\infty\) to \(C^\infty\) on the corresponding Stiefel manifolds. The functions \(f\), for which \(f_{\text{ave}} = 0\), belong to the kernel (the null space) of the operator \(F_{m,k}\) (similarly for \(F_{m,k}^*\)). Thus, in general, \(F_{m,k}\) and \(F_{m,k}^*\) are non-injective.

The Funk transforms \(F_{m,k} f\) and \(F_{m,k}^* \varphi\) can be thought of as convolutions on the group \(G = O(n)\) with delta measures \(\mu_U\) and \(\mu_V\) associated with stabilizers
\[U = \{ g \in G : g = \begin{bmatrix} \gamma & 0 \\ 0 & I_k \end{bmatrix}, \quad \gamma \in O(n-k) \}, \quad V = \{ g \in G : g = \begin{bmatrix} I_m & 0 \\ 0 & \rho \end{bmatrix}, \quad \rho \in O(n-m) \}\]
of \(u_0\) and \(v_0\), respectively. These measures are defined by
\[\int_G \omega(g) d\mu_U(g) = \int_U \omega(g) dU g, \quad \int_G \omega(g) d\mu_V(g) = \int_V \omega(g) dV g, \quad \omega \in C(G),\]
where \(dU g\) and \(dV g\) are the relevant Haar probability measures. We denote
\[\varphi_0(\alpha) = \varphi(\alpha u_0), \quad f_0(\beta) = f(\beta v_0); \quad \alpha, \beta \in G.\]
Then, by (3.3) and (3.4),
\[(F_{m,k} f)(\alpha u_0) = \int_G f_0(\alpha g^{-1}) d\mu_U(g), \quad (3.6)\]
\[(F_{m,k}^* \varphi)(\beta v_0) = \int_G \varphi_0(\beta g^{-1}) d\mu_V(g). \quad (3.7)\]
Lemma 3.1. Let $1 \leq k, m \leq n - 1; \ k + m \leq n$. The operators $F_{m,k}$ and $F_{m,k}^*$ act from $L^1$ to $L^1$ and from $C^\infty$ to $C^\infty$ on the corresponding Stiefel manifolds. Moreover,
\[
\int_{V_{n,k}} (F_{m,k}f)(u) \varphi(u) \, dsu = \int_{V_{n,m}} f(v) (F_{m,k}^* \varphi)(v) \, dv,
\]
provided that at least one of these integrals is finite when $f$ and $\varphi$ are replaced by $|f|$ and $|\varphi|$, respectively.

Proof. The first statement follows from (3.6) and (3.7), taking into account the properties of convolutions on compact Lie groups (use, e.g., Propositions A.4 and 2.3). The duality (3.8) agrees with Helgason’s double fibration scheme [20, p. 144]. A straightforward proof of (3.8) can be found in [40, Lemma 3.2]; see also (5.10) for the more general statement.

There is an obvious relationship between the Funk transforms (3.1) and (3.2) and Radon type transforms on Grassmannians, defined by
\[
(R_{p,q} \hat{f})(\eta) = \int_{\xi \subset \eta} \hat{f}(\xi) \, d\eta \xi, \quad (R_{p,q} \hat{\varphi})(\xi) = \int_{\eta \supseteq \xi} \varphi(\eta) \, d\xi \eta,
\]
where $\xi \in G_{n,p}$, $\eta \in G_{n,q}$, $1 \leq p \leq q \leq n - 1$, $d\eta \xi$ and $d\xi \eta$ being the relevant probability measures. Specifically, suppose that $f$ is a right $O(m)$-invariant function on $V_{n,m}$, $\varphi$ is a right $O(k)$-invariant function on $V_{n,k}$, and set $p = m$, $q = n - k$. If we define $\hat{f}$ on $G_{n,m}$ and $\hat{\varphi}$ on $G_{n,n-k}$ by
\[
\hat{f}(\{v\}) = f(v), \quad \hat{\varphi}(\{v\}) = \varphi(u),
\]
then
\[
(F_{m,k}f)(u) = (R_{m,n-k} \hat{f})(u^\perp), \quad (F_{m,k} \varphi)(v) = (R_{m,n-k} \hat{\varphi})(\{v\}).
\]
In the case $p = q$, both expressions in (3.9) represent the identity maps.

Lemma 3.2. $1 \leq p \leq q \leq n - 1$. The operators $R_{p,q}$ and $R_{p,q}^*$ act from $L^1$ to $L^1$ and from $C^\infty$ to $C^\infty$ on the corresponding Grassmannians. Moreover,
\[
\int_{G_{n,q}} (R_{p,q} \hat{\varphi})(\eta) \, d\eta \eta = \int_{G_{n,p}} \hat{\varphi}(\eta) \, d\eta \eta,
\]
provided that at least one of these integrals is finite when $\hat{f}$ and $\hat{\varphi}$ are replaced by $|\hat{f}|$ and $|\hat{\varphi}|$, respectively.
This well known statement follows from Lemma 3.1 and Remark A.13. Just as Lemma 3.1, it also falls into the scope of Helgason’s double fibration theory.

4. The $\lambda$-Cosine Transforms

In this section we follow our paper [40], however, the notation for some parameters has been changed for the sake of consistency with [41, 42].

4.1. Preparations. Let $1 \leq m, k \leq n - 1$. The non-normalized $\lambda$-cosine transform and its dual are defined by

$$
(C^\lambda_{m,k} f)(u) = \int_{V_{n,m}} f(v) |u'v|_m^\lambda d_* v, \quad u \in V_{n,k}, \quad (4.1)
$$

$$
(\star C^\lambda_{m,k} \phi)(v) = \int_{V_{n,k}} \phi(u) |u'v|_m^\lambda d_* u, \quad v \in V_{n,m}. \quad (4.2)
$$

In the self-adjoint case $m = k$ we set

$$(C^\lambda_{m} f)(u) = (C^\lambda_{m,m} f)(u) \equiv \int_{V_{n,m}} f(v) |\det(u'v)|^\lambda d_* v. \quad (4.3)$$

Recall that $|u'v|_m = \det(v'uu'v)^{1/2}$, where $v'uu'v$ is a positive semi-definite $m \times m$ matrix. We restrict our consideration to $m \leq k$, because, otherwise, $|u'v|_m = 0$ for all $v \in V_{n,m}$ and $u \in V_{n,k}$.

The functions $C^\lambda_{m,k} f$ and $\star C^\lambda_{m,k} \phi$ are a right $O(k)$-invariant and right $O(m)$-invariant, respectively. Moreover,

$$
C^\lambda_{m,k} f = C^\lambda_{m,k} f_{\mathrm{ave}}, \quad \star C^\lambda_{m,k} \phi = \star C^\lambda_{m,k} \phi_{\mathrm{ave}}, \quad (4.3)
$$

as in (3.5).

Because the quantity $|u'v|_m$ is invariant under change of variables $u \to u\alpha, \alpha \in O(k)$, and $v \to v\beta, \beta \in O(m)$, it is actually a function of Grassmannian variables

$$
\xi = \{v\} \in G_{n,m} \quad \text{and} \quad \tau = \{u\} \in G_{n,k}.
$$

We denote this function by $|\mathrm{Cos}(\xi, \tau)|$, taking into account that if $k = m = 1$, then $|u'v|_m$ is exactly the cosine of the smallest angle between the lines $\xi$ and $\tau$. Thus we define

$$
|\mathrm{Cos}(\xi, \tau)| \equiv |\mathrm{Cos}(\{v\}, \{u\})| = |u'v|_m. \quad (4.4)
$$

This definition does not depend on the choice of the orthonormal bases $v$ in $\xi$ and $u$ in $\tau$. Geometrically, $|\mathrm{Cos}(\xi, \tau)|$ is the $m$-volume of the orthogonal projection onto $\tau$ of a generic set of unit volume in $\xi$. 

Setting
\[ \tilde{f}(\xi) \equiv \tilde{f}(\{v\}) = f(v), \quad \tilde{\varphi}(\tau) \equiv \tilde{\varphi}(\{u\}) = \varphi(u), \]
\[ (T_{m,k}^{\lambda} \tilde{f})(\tau) \equiv (T_{m,k}^{\lambda} \tilde{f})(\{u\}) = (C_{m,k}^{\lambda} f)(u), \quad (4.5) \]
\[ (T_{m,k}^{\lambda} \tilde{\varphi})(\xi) \equiv (T_{m,k}^{\lambda} \tilde{\varphi})(\{v\}) = (C_{m,k}^{\lambda} \varphi)(v), \quad (4.6) \]
we can write (4.1) and (4.2) in the Grassmannian language as
\[ (T_{m,k}^{\lambda} \tilde{f})(\tau) = \int_{G_{n,m}} \tilde{f}(\xi) |\cos(\xi, \tau)|^{\lambda} d*\xi, \quad \tau \in G_{n,k}, \quad (4.7) \]
\[ (T_{m,k}^{\lambda} \tilde{\varphi})(\xi) = \int_{G_{n,k}} \tilde{\varphi}(\tau) |\cos(\xi, \tau)|^{\lambda} d*\tau, \quad \xi \in G_{n,m}, \quad (4.8) \]
and reformulate all our results in these terms. However, for the sake of convenience (especially in proofs), we prefer the Stiefel terminology.

Note that, unlike (4.1) and (4.2), the operators \( T_{m,k}^{\lambda} \) and \( T_{m,k}^{\ast \lambda} \) are \( GL(n, \mathbb{R}) \)-equivariant, because
\[ |\cos(g \xi, g \tau)| = |\cos(\xi, \tau)| \quad \text{for all} \quad g \in GL(n, \mathbb{R}). \quad (4.9) \]
The latter can be easily checked if we write \( g \) in polar coordinates \( g = \omega r^{1/2} \), where \( \omega \in O(n) \) and \( r = g'g \) is a positive definite \( n \times n \) matrix. Specifically, if \( \xi = \{v\} \), then
\[
\begin{align*}
g \xi &= g \{v\} = \{g v\} = \{\omega r^{1/2} v\} \\
&= \omega \{r^{1/2} v\} = \omega \{v\} = \{\omega v\}
\end{align*}
\]
(similarly for \( g \tau \)). Hence
\[
|\cos(g \xi, g \tau)| = |\cos(\{\omega v\}, \{\omega u\})| = |(\omega u)'(\omega v)|_m \\
&= |u' v|_m = |\cos(\xi, \tau)|.
\]

**Lemma 4.1.** Let \( 1 \leq m \leq k \leq n - 1 \). If \( \text{Re} \lambda > m - k - 1 \), then the operators \( C_{m,k}^{\lambda}, \hat{C}_{m,k}^{\lambda}, T_{m,k}^{\lambda}, \) and \( T_{m,k}^{\ast \lambda} \) act from \( L^1 \) to \( L^1 \) and from \( C^\infty \) to \( C^\infty \) on the corresponding Stiefel or Grassmann manifolds.

**Proof.** We can write \( (C_{m,k}^{\lambda} f)(u) \) and \( (C_{m,k}^{\lambda} \varphi)(v) \) as convolutions on the group \( G = O(n) \). Specifically, let
\[
\begin{align*}
u_0 = \begin{bmatrix} 0 \\
I_k \end{bmatrix} &\in V_{n,k}, \quad v_0 = \begin{bmatrix} I_m \\
0 \end{bmatrix} \in V_{n,m}, \quad u = \alpha u_0, \quad v = \beta v_0.
\end{align*}
\]

\( ^2\text{Although} \ GL(n, \mathbb{R}) \text{does not act directly on} \ V_{n,m}, \text{one can consider representations of this group on the spaces of right} \ O(m) \text{-invariant functions on} \ V_{n,m}; \text{see} \ [34], [40, Section 7.4.3], \text{and references therein.} \)
Setting \( f_0(\beta) = f(\beta v_0), \varphi_0(\alpha) = \varphi(\alpha u_0) \), we obtain
\[
(C^{\lambda}_{m,k}f)(\alpha u_0) = \int \limits_{G} f_0(\alpha \gamma^{-1}) h(\gamma) \, d_\gamma, \\
(C^{\lambda}_{m,k}\varphi)(\beta v_0) = \int \limits_{G} \varphi_0(\beta \gamma^{-1}) h^*(\gamma) \, d_\gamma,
\]
where \( h(\gamma) = |u'_0 v_0|^\lambda_m, \quad h^*(\gamma) = |u'_0 v_0|^\lambda_m \) (recall that \( \gamma^{-1} = \gamma' \)). If \( \Re \lambda > m - k - 1 \), then \( h \) and \( h^* \) are integrable on \( G \). The latter follows from the equality
\[
\int \limits_{G} h(\gamma) \, d_\gamma = \int \limits_{V_{n,m}} |u'_0 v_0|^\lambda_m \, d_\gamma = \frac{\Gamma_m(n/2) \Gamma_m((\lambda + k)/2)}{\Gamma_m(k/2) \Gamma_m((\lambda + n)/2)}; \tag{4.10}
\]
see [40, formula (A.16)]. Now the \( L^1 \) action is obvious and the smoothness result holds by Proposition 2.3. The corresponding statements for \( T^{\lambda}_{m,k} \) and \( T^{\lambda}_{m,k} \) then follow from (4.5) and (4.6) by Remark A.13 and Proposition A.15. \( \square \)

Remark 4.2. The condition \( \Re \lambda > m - k - 1 \) in Lemma 4.1 is sharp because \( (C^{\lambda}_{m,k}1)(u) = (C^{\lambda}_{m,k}1)(v) \) coincide with (4.10). If \( \Re \lambda \leq m - k - 1 \), then the gamma function \( \Gamma_m((\lambda + k)/2) \) in this expression represents a divergent integral.

4.2. Connection Between the \( \lambda \)-Cosine Transform and Its Dual.
Given \( u \in V_{n,k} \) and \( v \in V_{n,m} \), we denote by \( \tilde{u} \in V_{n,n-k} \) and \( \tilde{v} \in V_{n,n-m} \) arbitrary frames, which are orthogonal to the subspaces \( \{u\} = \text{span}(u) \) and \( \{v\} = \text{span}(v) \), respectively. By Proposition A.14 there is a one-to-one correspondence \( f \simeq f_* \) between the right \( O(m) \)-invariant functions \( f \) on \( V_{n,m} \) and right \( O(n-m) \)-invariant functions \( f_* \) on \( V_{n,n-m} \) (similarly \( \varphi \simeq \varphi_* \)). Clearly,
\[
\int \limits_{V_{n,k}} \varphi(u) \, d_u u = \int \limits_{V_{n,n-k}} \varphi_*(\tilde{u}) \, d_\tilde{u} \tilde{u}, \quad \int \limits_{V_{n,m}} f(v) \, d_v v = \int \limits_{V_{n,n-m}} f_*(\tilde{v}) \, d_\tilde{v} \tilde{v},
\]
which follows from (A.9).

Proposition 4.3. Let \( 1 \leq m \leq k \leq n - 1 \), \( \Re \lambda > m - k - 1 \). If \( \varphi \in L^1(V_{n,k})^{O(k)} \), then
\[
(C^{\lambda}_{m,k}\varphi)(v) = (C^{\lambda}_{n-k,n-m} \varphi_*)(\tilde{v}). \tag{4.11}
\]
Proof. Note that \( \det(v'u'u'v) = \det(\tilde{u}'\tilde{v}'\tilde{u}) \). The latter can be proved using Sylvester’s equality

\[
\det(I_m - ab) = \det(I_n - ba); \quad a \in \mathcal{M}_{m,n}, \ b \in \mathcal{M}_{n,m}.
\]

Indeed,

\[
det(v'u'u'v) = \det(I_m - v'\tilde{u}'\tilde{u}) = \det(I_{n-k} - \tilde{u}'v'\tilde{u}) = \det(\tilde{u}'\tilde{v}'\tilde{u}).
\]

Hence, by (A.9),

\[
(\mathcal{C}_{m,k}^\lambda \varphi)(v) = \int_{V_{n,k}} \varphi(u) |u'|^\lambda d_* u = \int_{V_{n,n-k}} \varphi_*(\tilde{u}) |\tilde{v}'\tilde{u}|^\lambda_{n-k} d_* \tilde{u} = (\mathcal{C}_{n-k,n-m}^\lambda \varphi_*)(\tilde{v}).
\]

\[\blacksquare\]

4.3. Analytic Continuation. Given a frame \( u \in V_{n,k} \), we denote by \( g_u \) an orthogonal transformation that takes \( u_0 = \begin{bmatrix} 0 & I_k \end{bmatrix} \in V_{n,k} \) to \( u \) and set \( f_u(v) = f(g_u v), \ v \in V_{n,m} \).

**Theorem 4.4.** Let \( 1 \leq m \leq k \leq n - 1 \). If \( f \in C^\infty(V_{n,m}) \), then the function

\[
\lambda \mapsto (\mathcal{C}_{m,k}^\lambda f)(u), \quad \text{Re} \lambda > m - k - 1,
\]

extends meromorphically to \( \text{Re} \lambda \leq m - k - 1 \). The polar set of the extended function consists of the poles \( m - k - 1, m - k - 2, \ldots \) of \( \Gamma_m((\lambda + k)/2) \). The normalized integral

\[
I_f(\lambda, u) = \frac{(\mathcal{C}_{m,k}^\lambda f)(u)}{\Gamma_m((\lambda + k)/2)}
\]

is an entire function of \( \lambda \). Moreover, if

\[
m - n \leq j - k \leq m - k - 1, \quad j \geq 0,
\]

then

\[
I_f(j - k, u) \equiv a.c. I_f(\lambda, u) \quad (4.13)
\]

\[
= c_j \int_{O(k)} d\gamma \int_{V_{n-k+j,m}} f_u \left( \begin{bmatrix} I_{n-k} & 0 \\ 0 & \gamma \end{bmatrix} \left[ \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right] \right) d_* \omega,
\]

\[
c_j = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2) \Gamma_m((n - k + j)/2)}.
\]
In particular, if \( j = 0 \), \( n - k \geq m \), then
\[
\begin{align*}
\text{a.c. } & \quad \lambda = -k \\
(C_{m,k}^\lambda f)(u) = c_0 (F_{m,k} f)(u),
\end{align*}
\]
where \( F_{m,k} f \) is the Funk transform (3.3). If \( j - k < m - n \), then \( I_f(j - k, u) \equiv 0 \).

**Proof.** The reasoning below is a generalization of [40, Subsection 7.1]. By invariance, it suffices to assume \( u = u_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \). Let
\[
F(\lambda) = \int_{\mathfrak{M}_{n,m}} f(x(x'x)^{-1/2}) |u_0' x_m^\lambda \psi(x'x) e^{-tr(x'x)} dx,
\]
where \( \psi \) is a nonnegative \( C^\infty \) function on the cone \( \Omega_m \) (see Notation) with compact support away from the boundary of \( \Omega_m \). The function
\[
\varphi(x) \equiv f(x(x'x)^{-1/2}) \psi(x'x) e^{-tr(x'x)}
\]
belongs to \( S(\mathfrak{M}_{n,m}) \) and is supported away from the surface \( \text{det}(x'x) = 0 \). Passing to polar coordinates \( x = wr^{1/2}, w \in V_{n,m}, r \in \Omega_m \) (see Lemma A.6), we obtain
\[
F(\lambda) = \kappa(\lambda) (C_{m,k}^\lambda f)(u_0),
\]
where
\[
\kappa(\lambda) = 2^{-m} \sigma_{n,m} \int_{\Omega_m} \text{det}(r)^{(\lambda+n-m-1)/2} \psi(r) e^{-tr(r)} dr.
\]
Because \( \kappa(\lambda) \) and its reciprocal are entire functions, the analyticity of \( \lambda \to (C_{m,k}^\lambda f)(u_0) \) is equivalent to that of \( F(\lambda) \) and the poles of both functions are the same and have the same order.

The integral (4.15) can be represented as
\[
F(\lambda) = \int_{\mathfrak{M}_{n,m}} \varphi(x) |u_0' x_m^\lambda dx = \int_{\mathfrak{M}_{k,m}} \hat{\varphi}(y) |y_m^\lambda dy = \mathcal{Z}_{k,m}(\hat{\varphi}, \lambda)
\]
(cf. (2.5)), where the function
\[
\hat{\varphi}(y) = \int_{\mathfrak{M}_{n-k,m}} \varphi \left( \begin{bmatrix} \eta \\ y \end{bmatrix} \right) d\eta
\]
belongs to \( S(\mathfrak{M}_{k,m}) \). Thus,
\[
(C_{m,k}^\lambda f)(u_0) = \kappa(\lambda)^{-1} \mathcal{Z}_{k,m}(\hat{\varphi}, \lambda)
\]
and
\[
I_f(\lambda, u_0) = \frac{\mathcal{Z}_{k,m}(\tilde{\varphi}, \lambda)}{\varphi(\lambda) \Gamma_m((\lambda + k)/2)} = \frac{\zeta_{k,m}(\tilde{\varphi}, \lambda)}{\varphi(\lambda)}; \quad (4.19)
\]
cf. (2.6).

Analytic properties of \(\mathcal{Z}_{k,m}(\tilde{\varphi}, \lambda)\) and \(\zeta_{k,m}(\tilde{\varphi}, \lambda)\) are described in Lemmas 2.1 and 2.2 (with \(n\) replaced by \(k\)). In particular, the integral \(\mathcal{Z}_{k,m}(\tilde{\varphi}, \lambda)\) converges absolutely if \(\text{Re} \lambda > m - k - 1\) and extends to \(\text{Re} \lambda \leq m - k - 1\) as a meromorphic function of \(\lambda\). The polar set of the extended function is a subset of the set of poles \(m - k - 1, m - k - 2, \ldots\) of \(\Gamma_m((\lambda + k)/2)\). The normalized integral \(\zeta_{k,m}(\tilde{\varphi}, \lambda)\) is an entire function of \(\lambda\). Since the left-hand side of (4.19) is independent of the choice of \(\psi\), the analytic continuation of the right-hand side (in which \(\psi\) is hidden) is independent of \(\psi\) too, thanks to the uniqueness property of analytic functions.

If \(0 \leq j \leq m - 1\), then for \(\lambda = j - k \leq m - k - 1\) we have
\[
\zeta_{k,m}(\tilde{\varphi}, j - k) = \frac{\pi (k-j)m/2}{\Gamma_m(k/2)} \int_{O(k)} d\gamma \int_{\mathfrak{M}_{j,m}} \tilde{\varphi} \left( \gamma \left[ \begin{array}{c} \omega \\ 0 \end{array} \right] \right) d\omega. \quad (4.20)
\]
In particular,
\[
\zeta_{k,m}(\tilde{\varphi}, -k) = \frac{\pi km/2}{\Gamma_m(k/2)} \tilde{\varphi}(0). \quad (4.21)
\]
Combining (4.20) and (4.21) with (4.17) and (4.16), we obtain
\[
I_f(j-k, u_0) = \frac{\pi (k-j)m/2}{\Gamma_m(k/2) \varphi(j-k)} \int_{O(k)} d\gamma \int_{\mathfrak{M}_{n-k+j,m}} \varphi \left( \tilde{\gamma} \left[ \begin{array}{c} \xi \\ 0 \end{array} \right] \right) d\xi,
\]
where
\[
\tilde{\gamma} = \left[ \begin{array}{cc} I_{n-k} & 0 \\ 0 & \gamma \end{array} \right] \in O(n),
\]
\[
\varphi \left( \tilde{\gamma} \left[ \begin{array}{c} \xi \\ 0 \end{array} \right] \right) = f \left( \tilde{\gamma} \left[ \begin{array}{c} \xi \xi' \end{array} \right] \right) (\xi'\xi)^{-1/2} \psi(\xi'\xi)e^{-\text{tr}(\xi'\xi)}.
\]
If \(n-k+j \geq m\), that is, \(m-n \leq j-k\) (cf. (4.12)), then, passing to polar coordinates in \(\mathfrak{M}_{n-k+j,m}\) (see Lemma A.6), we get \(I_f(j-k, u_0) = c I_1 I_2\), where
\[
I_1 = \int_{O(k)} d\gamma \int_{\mathfrak{M}_{n-k+j,m}} f \left( \tilde{\gamma} \left[ \begin{array}{c} \omega \\ 0 \end{array} \right] \right) d_\omega,
\]
\[
I_2 = \int_{\Omega_m} \det(r)(n-k+j-m-1)/2 \psi(r) e^{-\text{tr}(r)} dr,
\]
\[
c = \frac{\pi (k-j)m/2 \sigma_{n-k+j,m}}{\sigma_{n,m} \Gamma_m(k/2) I_2} = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2) \Gamma_m((n-k+j)/2) I_2}.
\]
Hence, if \( \max(m+k-n,0) \leq j \leq m-1 \), then

\[
I_f(j-k,u_0) = c_j \int_{O(k)} d\gamma \int_{V_{n-k+j,m}} f \left( \begin{bmatrix} I_{n-k} & 0 \\ 0 & \gamma \end{bmatrix} \right) \omega d\omega,
\]

\[
c_j = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2) \Gamma_m((n-k+j)/2)}.
\]

If \( j = 0 \) and \( n-k \geq m \), the expression for \( I_f(j-k,u) \) has a simpler form

\[
I_f(-k,u_0) = c_0 \int_{V_{n-k,m}} f \left( \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right) d\omega = c_0 (F_{m,k,f})(u_0);
\]

cf. (3.3). If \( j-k < m-n \), then the rank of \( \xi \) is less than \( m \), \( \xi' \xi \) is the boundary point of the cone \( \Omega_m \), and therefore, by the definition of \( \varphi \) in (4.16), we have \( \varphi \equiv 0 \). This gives \( I_f(j-k,u_0) \equiv 0 \).

Remark 4.5. The function \( I_f(\lambda,u) = (C_{m,k,f}^\lambda(u))/\Gamma_m((\lambda + k)/2) \) may have zeros at some \( \lambda \). For example, if \( f \equiv 1 \), then, by (4.10),

\[
I_f(\lambda,u) = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2) \Gamma_m((\lambda + n)/2)} = 0 \quad \forall \lambda = m-n-1, m-n-2, \ldots.
\]

Thus \( C_{m,k,f}^\lambda \) and \( \Gamma_m((\lambda + k)/2) \) may have poles of different order. 3

Lemma 4.6. Let \( 1 \leq m \leq k \leq n-1 \). If \( f \in C^\infty(V_{n,m}) \), then the function

\[
u \to a.c. \frac{(C_{m,k,f}^\lambda)(u)}{\Gamma_m((\lambda + k)/2)}, \quad u \in V_{n,k}, \tag{4.22}
\]

is infinitely differentiable for every complex \( \lambda \).

Proof. Let first \( \text{Re} \lambda > m-k-1 \). We replace \( f \) in (4.18) by \( f_{\gamma} = f \circ \gamma \), \( \gamma \in G = O(n) \), to get

\[
(C_{m,k,f}^\lambda)(\gamma u_0) = \kappa(\lambda)^{-1} \mathcal{Z}_{k,m}(\tilde{\varphi}_{\gamma}, \lambda),
\]

\[
\tilde{\varphi}_{\gamma}(y) = \int_{\mathfrak{M}_{n-k,m}} \varphi_{\gamma} \left( \begin{bmatrix} \eta \\ y \end{bmatrix} \right) d\eta, \quad y \in \mathfrak{M}_{k,m},
\]

\[
\varphi_{\gamma}(x) = \omega(x)f(\gamma v)|_{v=x(x'x)^{-1/2}}, \quad \omega(x) = \psi(x'x) e^{-tr(x'x)} \in C^\infty(\mathfrak{M}_{n,m}),
\]

where \( \kappa(\lambda) \) and its reciprocal are entire functions of \( \lambda \); cf. (4.16), (4.17). The maps

\[
w : G \times V_{n,m} \to V_{n,m}, \quad (\gamma, v) \to \gamma v,
\]

\[
\sigma : \mathfrak{M}_{n,m} \to V_{n,m}, \quad x \to x(x'x)^{-1/2},
\]

3The converse statement in [40, Theorem 7.1(i)] should be corrected.
are smooth; see Proposition A.1 and the proof of Lemma A.7. Hence the function

\[ F(\gamma, x) = \varphi_\gamma(x) = \omega(x) f(\gamma v) \big|_{v=x(x')^{-1/2}} = \omega(x)(f \circ w)(\gamma, \sigma(x)) \]

is smooth on \( G \times \tilde{M}_{n,m} \), and therefore

\[ F_1(\gamma, \eta, y) = \varphi_\gamma \left( \begin{bmatrix} \eta \\ y \end{bmatrix} \right) = F \left( \gamma, \begin{bmatrix} \eta \\ y \end{bmatrix} \right) \]

is a smooth function on \( G \times \tilde{M}_{n-k,m} \times \tilde{M}_{k,m} \). It follows that \( \tilde{\varphi}_\gamma(y) \) is a smooth function of \((\gamma, y) \in G \times \tilde{M}_{k,m}\). Using the meromorphic continuation formula (2.8) for zeta integrals, we obtain

\[
\text{a.c.} \left( C_{m,k}^\lambda f \right)(\gamma u_0) = \frac{1}{\kappa(\lambda) B_{t,m,k}(\lambda)} Z_{k,m}(\Delta^\ell \tilde{\varphi}_\gamma, \lambda + 2\ell) = \frac{1}{\kappa(\lambda) B_{t,m,k}(\lambda)} \int_{\tilde{M}_{k,m}} (\Delta^\ell \tilde{\varphi}_\gamma)(y) |y|^{\lambda+2\ell} \, dy,
\]

\( \Re \lambda > m-k-1-2\ell, \quad \ell = 1, 2, \ldots, \)

where \( B_{t,m,k}(\lambda) \) is the Bernstein polynomial (2.7) (with \( n \) replaced by \( k \)). By above, \( \Delta^\ell \tilde{\varphi}_\gamma(y) \) is a smooth function of \((\gamma, y) \), and therefore, \( \text{a.c.} \left( C_{m,k}^\lambda f \right)(\gamma u_0) \) is a smooth function of \( \gamma \in G \). Now the smoothness of the normalized function (4.22) follows from Proposition A.4. \( \Box \)

The next statement is an analogue of Theorem 4.4 and Lemma 4.6 for the dual transform \( \hat{C}_{m,k}^\lambda \varphi \).

**Theorem 4.7.** Let \( 1 \leq m \leq k \leq n-1 \).

(i) If \( \varphi \in C^\infty(V_{n,k}) \), then the function \( \lambda \mapsto (\hat{C}_{m,k}^\lambda \varphi)(v) \) extends meromorphically to \( \Re \lambda \leq m-k-1 \) for every \( v \in V_{n,m} \). The polar set of the extended function consists of the poles \( m-k-1, m-k-2, \ldots \) of \( \Gamma_{n-k}((\lambda + n - m)/2) \).

(ii) The normalized function

\[ \lambda \mapsto \frac{(\hat{C}_{m,k}^\lambda \varphi)(v)}{\Gamma_n((\lambda + n - m)/2)} \]

is an entire function of \( \lambda \) belonging to \( C^\infty(V_{n,m})^{O(m)} \) in the \( v \)-variable.

(iii) An alternative normalized function

\[ \lambda \mapsto \frac{(\hat{C}_{m,k}^\lambda \varphi)(v)}{\Gamma_m((\lambda + k)/2)} \]
extends meromorphically with the only possible poles
\[-k - 1, -k - 2, \ldots .\]

**Proof.** By (4.3), it suffices to consider right \(O(k)\)-invariant functions \(\varphi\). The statements (i) and (ii) follow from Theorem 4.4, Propositions 4.6, 4.3, and A.14. To prove (iii), we observe that by (2.4),

\[
\Gamma_{n-k}((\lambda + n - m)/2) = c(\lambda) \Gamma_m((\lambda + k)/2),
\]

where \(c(\lambda) = \pi^{(n-k-m)/2} \Gamma_{n-k-m}((\lambda + n - m)/2)\) is a meromorphic function with the polar set \([-k - 1, -k - 2, \ldots ]\). □

### 5. Intermediate Funk-Cosine Transforms

Theorem 4.4 leads to new Radon-like transforms

\[
(F^{(j)}_{m,k}f)(u) = \int_{O(k)} d_* \gamma \int_{V_{n-k+j,m}} f_u \left( \begin{bmatrix} I_{n-k} & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \omega \\ 0 \end{bmatrix} \right) d_* \omega, \quad (5.1)
\]

which take functions on \(V_{n,m}\) to functions on \(V_{n,k}\). Following this theorem, we assume

\[
1 \leq m \leq k \leq n - 1, \quad n - k + j \geq m, \quad 0 \leq j \leq m - 1. \quad (5.2)
\]

Recall that

\[
f_u(v) = f(g_u v), \quad g_u \in G = O(n), \quad g_u u_0 = u, \quad u_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \in V_{n,k}.
\]

One can formally write (5.1) as

\[
(F^{(j)}_{m,k}f)(u) = \int_{\{v \in V_{n,m} : \text{rank}(u'v) \leq j\}} f(v) d_u,j(v). \quad (5.3)
\]

The case \(j = 0\) agrees with the usual Funk transform \(F_{m,k}\). By (4.13),

\[
\frac{C_{m,k}^\lambda}{\lambda = j-k \Gamma_m((\lambda + k)/2)} = c_j (F^{(j)}_{m,k}f)(u), \quad (5.4)
\]

\[
c_j = \frac{\Gamma_m(n/2)}{\Gamma_m(k/2) \Gamma_m((n - k + j)/2)}. \quad (5.5)
\]

The integral transform (5.1) has a nice geometric interpretation in the Grassmannian language (3.9). Specifically, suppose first \(u = u_0\). Given a right \(O(m)\)-invariant function \(f\) on \(V_{n,m}\), we define the associated function \(\tilde{f}\) on \(G_{n,m}\) by \(\tilde{f}\{v\} = f(v)\). Denote

\[
\eta_0 = \text{span}(e_1, \ldots, e_{n-k}) = \mathbb{R}^{n-k}, \quad \zeta_0 = \text{span}(e_1, \ldots, e_{n-k+j}) = \mathbb{R}^{n-k+j}.
\]
Then the inner integral in (5.1) can be written as
\[ \int_{\{\xi \in G_{n,m} : \xi \subset \tilde{\gamma} \zeta_0\}} \hat{f}(\xi) \, d\tilde{\gamma} \xi = (R_{m,n-k+j} \hat{f})(\tilde{\gamma} \zeta_0), \quad \tilde{\gamma} = \begin{bmatrix} I_{n-k} & 0 \\ 0 & \gamma \end{bmatrix}. \tag{5.6} \]

Let
\[ G_{n,n-k+j}(\eta) = \{\zeta \in G_{n,n-k+j} : \zeta \supset \eta\}, \quad \eta \in G_{n,n-k}. \]

Integrating (5.6) over \( \gamma \in O(k) \), and noting that \( \tilde{\gamma} \) leaves \( \eta_0 \) fixed, we obtain
\[ (F_{m,k}^{(j)} f)(u_0) = \int_{G_{n,n-k+j}(\eta_0)} (R_{m,n-k+j} \hat{f})(\zeta) \, d\zeta \]
\[ = (R_{n-k,n-k+j} R_{m,n-k+j} \hat{f})(\eta_0). \tag{5.7} \]

Hence, by rotation invariance,
\[ (F_{m,k}^{(j)} f)(u) = (R_{n-k,n-k+j} R_{m,n-k+j} \hat{f})(\eta), \quad \eta = u^\perp. \tag{5.8} \]

We denote
\[ (R_{m,n-k+j} \hat{f})(\eta) = (R_{n-k,n-k+j} R_{m,n-k+j} \hat{f})(\eta), \quad \eta \in G_{n,n-k}. \tag{5.9} \]

This expression is the Grassmannian model of the intermediate Funk-cosine transform \( F_{m,k}^{(j)} \). If \( j = 0 \), it boils down to the usual Radon transform \( R_{m,n-k+j} \hat{f} \), as in (3.9).

The dual intermediate Funk-cosine transform \( \hat{F}_{m,k}^{(j)} \varphi \) is naturally defined as an integral operator satisfying
\[ \int_{V_{n,m}} (F_{m,k}^{(j)} f)(u) \varphi(u) \, d_*u = \int_{V_{n,m}} f(v) \left( \hat{F}_{m,k}^{(j)} \varphi \right)(v) \, d_*v. \tag{5.10} \]

To obtain an explicit formula for \( \hat{F}_{m,k}^{(j)} \varphi \), we set
\[ u_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix}, \quad v_0 = \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \]
\[ \tilde{\gamma} = \begin{bmatrix} I_{n-k} & 0 \\ 0 & \gamma \end{bmatrix}, \quad \tilde{a} = \begin{bmatrix} a & 0 \\ 0 & I_{k-j} \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} I_m & 0 \\ 0 & b \end{bmatrix}, \]
where \( \gamma \in O(k) \), \( a \in O(n-k+j) \), \( b \in O(n-m) \). Then, by (5.1),
\[ (F_{m,k}^{(j)} f)(u) = \int_{O(k)} d_\gamma \int_{O(n-k+j)} f_u(\tilde{\gamma} \tilde{a} b v_0) \, d_\gamma a 
\]
\[ = \int_{O(n-m)} d_\gamma b \int_{O(k)} \int_{O(n-k+j)} f_u(\tilde{\gamma} \tilde{a} b v_0) \, d_\gamma a. \]
Thus we can set
\[ (F_{m,k}^{(j)} \varphi)(v) = \int_{O(n)} d_{-m} \int_{O(n-k+j)} d_{+a} \int_{O(k)} \varphi_v(b \tilde{a} \gamma u_0) d_{+a} \] (note that \( \gamma u_0 = u_0 \gamma \)). If \( j = 0 \), the above formula gives the usual dual Funk transform (3.2).

To obtain a Grassmannian analogue of (5.11), we define a function \( \varphi_v \) on \( G_{n,n-k} \) by the formula \( \varphi_v(u) = \varphi(g_v u) \), \( g_v \in O(n) \), \( g_v v_0 = v \). If \( \varphi \) is right \( O(k) \)-invariant, then
\[ (F_{m,k}^{* (j)} \varphi)(v) = \int_{O(n-m)} d_{-m} \int_{O(n-k+j)} d_{+a} \int_{O(k)} \varphi_v(b \tilde{a} \gamma u_0) d_{+a} \] (5.11)
and, by rotation invariance,
\[ (F_{m,k}^{* (j)} \varphi)(v) = (R_{n,n-k+j} R_{n-k,n-k+j} \varphi_0)(\{v_0\}) \] (5.12)
Thus the Grassmannian modification of the dual intermediate Funk-cosine transform is
\[ (R_{m,n-k}^{* (j)} \varphi_0)(\xi) = (R_{m,n-k+j} R_{n-k,n-k+j} \varphi_0)(\xi), \quad \xi \in G_{n,m}. \] (5.13)

**Lemma 5.1.** If \( j, k, m, n \) satisfy (5.2), then the operators \( F_{m,k}^{(j)} \), \( F_{m,k}^{* (j)} \), \( R_{m,n-k}^{(j)} \), and \( R_{m,n-k}^{* (j)} \) act from \( L^1 \) to \( L^1 \) and from \( C^\infty \) to \( C^\infty \) on the corresponding Stiefel or Grassmann manifolds.

**Proof.** The result for \( R_{m,n-k}^{(j)} \) and \( R_{m,n-k}^{* (j)} \) follows immediately from the composition formulas (5.9) and (5.13) according to Lemma 3.2. By (5.8) and (5.12), these operators are expressed through \( F_{m,k}^{(j)} \) and \( F_{m,k}^{* (j)} \).
Hence the result for $F_{m,k}^{(j)}$ and $\hat{F}_{m,k}^{(j)}$ follows from Remark A.13 and Proposition A.15. Alternatively, the result for arbitrary $L^1$ or smooth functions on the Stiefel manifolds can be obtained if we represent our operators as convolutions with Radon measures on $O(n)$, as we did in the proof of Lemma 3.1.

\[\text{Proposition 5.2.} \quad \text{Let} \quad 1 \leq m \leq k \leq n-1, \quad \varphi \in C^\infty(V_{n,k}). \quad \text{If} \quad j \geq 0 \quad \text{satisfies} \quad m-n \leq j-k \leq m-k-1, \quad \text{then} \]
\[
\left(\text{a.c.} \lambda = \lambda \right)_{\lambda=j-k} (\mathcal{C}^\lambda_{m,k} \varphi)(v) = c_j (\mathcal{F}_m^{(j)} F_{m,k} \varphi)(v), \tag{5.14}
\]
\[c_j \text{ being the constant (5.5).}\]

\[\text{Proof. Denote} \]
\[A_{\lambda}(v) \equiv \frac{(\mathcal{C}^\lambda_{m,k} \varphi)(v)}{\Gamma_m((\lambda+k)/2)} = \frac{c(\lambda)}{\Gamma_{n-k}((\lambda+n-m)/2)}, \quad \text{Re} \lambda > m-k-1.\]

By Theorem 4.7 (ii), this function extends analytically to $\text{Re} \lambda > -k-1$ and the analytic continuation belongs to $C^\infty(V_{n,m})$. Clearly, $j-k > -k-1$, because $j \geq 0$. Hence, for any test function $w \in C^\infty(V_{n,m})$, owing to (5.4) and (5.10), we have
\[
\left(\text{a.c.} \lambda = \lambda \right)_{\lambda=j-k} (A_{\lambda}, w) = \left(\varphi, \frac{\mathcal{C}^\lambda_{m,k} w}{\Gamma_m((\lambda+k)/2)}\right) = \left(\varphi, \frac{\mathcal{C}^\lambda_{m,k} w}{\Gamma_m((\lambda+k)/2)}\right)
\]
\[= c_j (\varphi, F_{m,k}^{(j)} w) = c_j (\hat{F}_{m,k}^{(j)} \varphi, w), \]
and (5.14) follows. \qed

6. Normalized $\lambda$-Cosine and $\lambda$-Sine Transforms

6.1. Normalized $\lambda$-Cosine Transforms. Let $1 \leq m \leq k \leq n-1$. We introduce the following normalized modifications of the cosine transforms (4.1) and (4.2):
\[
(\mathcal{C}_{m,k}^\lambda f)(u) = \gamma_{m,k}(\lambda) \int_{V_{n,m}} f(v) \left| u' v \right|^\lambda_m d_* v, \quad u \in V_{n,k}, \tag{6.1}
\]
\[
(\mathcal{C}_{m,k}^\lambda \varphi)(v) = \gamma_{m,k}(\lambda) \int_{V_{n,k}} \varphi(u) \left| u' v \right|^\lambda_m d_* u, \quad v \in V_{n,m}; \tag{6.2}
\]
\[
\gamma_{m,k}(\lambda) = \frac{\Gamma_m(m/2)}{\Gamma_m(n/2)} \frac{\Gamma_m(-\lambda/2)}{\Gamma_m((\lambda+k)/2)}, \quad \lambda \neq 1 - m, 2 - m, \ldots
\]

Such a normalization makes our operators consistent with those in the case \( m = 1 \) (cf. \([41, 42]\)) and simplifies many formulas in the sequel. Excluded values of \( \lambda \) belong to the polar set of \( \Gamma_m(-\lambda/2) \).

Both integrals exist in the Lebesgue sense if \( \text{Re} \lambda > m - k - 1 \). If \( k = m \), we set \( C^\lambda_m f = C^\lambda_{m,m} f \).

**Theorem 6.1.** Let \( f \in C^\infty(V_{n,m}) \), \( 1 \leq m \leq k \leq n - 1 \).

(i) The function \( \lambda \to C^\lambda_{m,k} f \) extends meromorphically with the only poles \( \lambda = 1 - m, 2 - m, \ldots \). The extended function belongs to \( C^\infty(V_{n,k})^O(k) \).

(ii) If

\[
m - n \leq j - k \leq \min(-m, m - k - 1), \quad j \geq 0,
\]

then

\[
a.c. \lambda = j - k \quad C^\lambda_{m,k} f = \tilde{c}_j F^{(j)}_{m,k} f, \quad \tilde{c}_j = \frac{\Gamma_m(m/2) \Gamma_m((k - j)/2)}{\Gamma_m(k/2) \Gamma_m((n - k + j)/2)}.
\]

In particular, for \( j = 0 \), \( m + k \leq n \),

\[
a.c. \lambda = -k \quad C^\lambda_{m,k} f = \tilde{c}_0 F_{m,k} f, \quad \tilde{c}_0 = \frac{\Gamma_m(m/2)}{\Gamma_m((n - k)/2)}.
\]

**Proof.** The statement (i) follows from Theorem 4.4 and Proposition 4.6. The restrictions \( m - n \leq j - k \leq m - k - 1, \ j \geq 0 \), are inherited from (4.12). The inequality \( j - k \leq -m \) means that \( \lambda = j - k \) does not belong to the polar set \( \{1 - m, 2 - m, \ldots\} \). The statement (ii) follows from (5.4). \( \square \)

**Remark 6.2.** The inequality \( m - n \leq j - k \leq -m \) in (6.3) implies \( 2m \leq n \). The additional restriction \( j - k \leq -m \) is not imposed in the definition (5.1). The case

\[
m - n \leq 1 - m \leq j - k \leq m - k - 1,
\]

when \( F^{(j)}_{m,k} f \) is well defined, but the left-hand side of (6.4) may be infinite, is not included in Theorem 6.1.

**Conjecture 6.3.** In the case (6.6), the operator \( F^{(j)}_{m,k} \) is non-injective on \( C^\infty(V_{n,m})^O(m) \).

For the dual transform \( \check{C}^\lambda_{m,k} \varphi \), the following result is a consequence of Theorem 4.7 and Proposition 5.2.
Theorem 6.4. Let $\varphi \in C^\infty(V_{n,k}),$

\[ 1 \leq m \leq k \leq n - 1, \quad n - k + j \geq m, \quad 0 \leq j \leq m - 1. \]

(i) The function $\lambda \to {}^\star \mathcal{C}_{m,k}^\lambda \varphi$ extends meromorphically with the only poles

\[ \lambda \in \{-k - 1, -k - 2, \ldots\} \cup \{1 - m, 2 - m, \ldots\}. \]

The extended function belongs to $C^\infty(V_{n,m})_O^m.$

(ii) If, moreover, $j - k \leq \min(-m, m - k - 1),$ then

\[ \text{a.c.} \quad \lambda = j - k \quad \mathcal{C}_{m,k}^\lambda \varphi = \tilde{c}_j \mathcal{F}_{m,k}^{(j)} \varphi, \]

where $\tilde{c}_j$ is defined by (6.5). In particular, for $j = 0, m + k \leq n,$

\[ \text{a.c.} \quad \lambda = -k \quad \mathcal{C}_{m,k}^\lambda \varphi = \tilde{c}_0 \mathcal{F}_{m,k} \varphi, \quad \tilde{c}_0 = \frac{\Gamma_m(m/2)}{\Gamma_m((n - k)/2)}. \]

6.2. Normalized $\lambda$-Sine Transforms. The normalized $\lambda$-sine transform is defined by

\[ (S^\lambda_m f)(u) = \delta_m(\lambda) \int_{V_{n,m}} \det(I_m - v'uu'v)^{\lambda/2} f(v) dv, \quad u \in V_{n,m}, \quad (6.8) \]

\[ \delta_m(\lambda) = \frac{\Gamma_m(m/2)}{\Gamma_m((n - m)/2)}, \quad 2m \leq n; \quad \lambda + m \neq 1, 2, \ldots. \]

More general sine transforms acting from $V_{n,m}$ to $V_{n,k}$ were introduced in [40, Sections 4,6]. If $f \in L^1(V_{n,m}),$ the integral (6.8) is absolutely convergent provided $Re \lambda > 2m - 1 - n.$

Note that

\[ \int_{V_{n,m}} \det(I_m - v'uu'v)^{\lambda/2} dv = \frac{\Gamma_m(n/2) \Gamma_m((\lambda + n - m)/2)}{\Gamma_m((n - m)/2) \Gamma_m((\lambda + n)/2)} \quad (6.9) \]

(cf. [40, Remark 4.4]). This formula shows that the restriction $Re \lambda > 2m - 1 - n$ is sharp.

The function $S^\lambda_m f$ is right $O(m)$-invariant and

\[ S^\lambda_m f = S^\lambda_m f_{\text{ave}}, \quad f_{\text{ave}}(v) = \int_{O(m)} f(v\gamma) dv. \]

\[ \text{Here and on, when dealing with the normalized } \lambda \text{-sine transforms, we use the formula (6.6) from [40], in which } \alpha + m - n \text{ should be replaced by } \lambda. \text{ Similarly,}\]

\[ \text{when dealing with the normalized } \lambda \text{-cosine transforms, we use formulas (6.1)-(6.2) from [40] in which } \alpha - k \text{ is replaced by } \lambda. \]
for every $f \in L^1(V_{n,m})$. Hence, in many occurrences, when dealing with $S^\lambda_m f$, it suffices to assume $f$ to be right $O(m)$-invariant.

If $\Pr_{\{u\}}$ and $\Pr_{\{u\}^\perp}$ stand for the orthogonal projections onto the subspaces $\{u\}$ and $\{u\}^\perp$, then
\[
\det(I_m - v'u'u'v) = \det(I_m - v'\Pr_{\{u\}}v) = \det(v'\Pr_{\{u\}^\perp}v = \det(v'\tilde{u}'v),
\]
where $\tilde{u}$ is an $(n - m)$-frame orthogonal to $\{u\}$. It follows that
\[
(S^\lambda_m f)(u) = (\mathscr{C}^\lambda_{m,n,m} f)(\tilde{u}), \quad 2m \leq n. \tag{6.10}
\]

The assumption $2m \leq n$ is natural because otherwise,
\[
\det(I_m - v'u'u'v) = \det(v'\tilde{u}'v) = 0.
\]

The equality (6.10) combined with Proposition A.14 and Theorem 6.1 yields the following result (see [40, Theorem 7.2] for details).

**Lemma 6.5.** If $f \in C^\infty(V_{n,m})^{O(m)}$, then for each $u \in V_{n,m}$, $(S^\lambda_m f)(u)$ extends meromorphically to all complex $\lambda$ with the only poles
\[
\lambda = 1 - m, 2 - m, \ldots,
\]
so that a.c. $(S^\lambda_m f)(u) \in C^\infty(V_{n,m})^{O(m)}$ in the $u$-variable. Moreover,
\[
a.c. \quad (S^\lambda_m f)(u) = f(u), \quad 2m \leq n. \tag{6.11}
\]

The following statement establishes remarkable connection between the sine transforms, cosine transforms, and Funk transforms.

**Lemma 6.6.** (cf. [40, Theorems 4.5, 4.8]) Let $1 \leq m \leq k \leq n - m$, $f \in L^1(V_{n,m})$. If
\[
\Re \lambda > 2m - 1 - n, \quad \lambda \neq 1 - m, 2 - m, \ldots
\]
then
\[
S_m^\lambda f = \tilde{\delta} \mathscr{C}^\lambda_{m,k} F_m k f = \tilde{\delta} F_m k \mathscr{C}^\lambda_{m,k} f, \quad \tilde{\delta} = \frac{\Gamma_m(k/2)}{\Gamma_m((n-m)/2)}. \tag{6.12}
\]
In particular, if $2m \leq n - k$, then
\[
S_m^{-k} f = \delta_0 \mathscr{C}^\lambda_{m,k} F_m k f, \quad \delta_0 = \frac{\Gamma_m(k/2) \Gamma_m(m/2)}{\Gamma_m((n-k)/2) \Gamma_m((n-m)/2)}. \tag{6.13}
\]

The proof of these formulas is actually an application of Fubini’s theorem.

Lemma 6.5 implies the following

**Corollary 6.7.** If $f \in C^\infty(V_{n,m})^{O(m)}$, $1 \leq m \leq k \leq n - m$, then (6.12) extends to all complex $\lambda \neq 1 - m, 2 - m, \ldots$. In particular, analytic continuations of $\mathscr{C}^\lambda_{m,k} F_m k f$ and $\tilde{F}_m k \mathscr{C}^\lambda_{m,k} f$ belong to $C^\infty(V_{n,m})^{O(m)}$. 

The next result extends (6.13) to intermediate Funk-cosine transforms.

**Corollary 6.8.** If \( f \in C^\infty(V_{n,m})^{O(m)}, \) \( 1 \leq m \leq k \leq n - m, \)

\[
m - n \leq j - k \leq \min(-m, m - k - 1), \quad j \geq 0,
\]

then

\[
S_{m}^{j-k} f = \delta_j F_{m,k}^*(F_{m,k} f) = \delta_j F_{m,k}^{(j)} f,
\]

\[
\delta_j = \frac{\Gamma_m((k - j)/2) \Gamma_m(m/2)}{\Gamma_m((n - k + j)/2) \Gamma_m((n - m)/2)}.
\]

This statement follows from (6.4), (6.7), and (6.12). The assumption (6.14) mimics those in Theorems 6.1 and 6.4.

### 7. The Fourier Transform and Differential Operators

Now, after we are done with all preparations, we can proceed to the main topic of the paper. We first introduce an auxiliary integral operator

\[
(A_{k,m}\varphi)(v) = \int_{V_{n,m,k,m}} \varphi \left( g_v \left[ \begin{array}{c} a \\ 0 \\ I_m \end{array} \right] \right) d_* a, \quad v \in V_{n,m},
\]

where \( 1 \leq m \leq k \leq n - 1, \) \( g_v \left[ \begin{array}{c} 0 \\ I_m \end{array} \right] = v, \) \( g_v \in O(n). \) If \( k = m, \) \( A_{k,m} \) is the identity operator. One can show \([40, \text{Lemma 5.2}]\) that \( A_{k,m} \) is a linear bounded operator from \( L^1(V_{n,k}) \) to \( L^1(V_{n,m}). \) Given a function \( f \) on \( V_{n,m}, \) using polar decomposition (A.6), we set

\[
(E_\lambda f)(x) = |x|^\lambda f(x(x')^{-1/2}), \quad x \in \tilde{M}_{n,m}.
\]

**Theorem 7.1.** \([40, \text{Corollary 5.5}]\) Let \( \varphi \in L^1(V_{n,k})^{O(k)}, \) \( \omega \in S(\mathfrak{m}_{n,m}), \) \( 1 \leq m \leq k \leq n - 1. \) Then for every \( \lambda \in \mathbb{C}, \)

\[
\left( \frac{E_\lambda \mathcal{C}^\lambda_{m,k}\varphi}{\Gamma_m((\lambda + k)/2)}, \hat{\omega} \right) = c \left( \frac{E_{-\lambda-n}A_{k,m}\varphi}{\Gamma_m(-\lambda/2)}, \omega \right),
\]

\[
c = \frac{2^{m(n+\lambda)} \pi^{nm/2} \Gamma_m(n/2)}{\Gamma_m(m/2)},
\]

where both sides are understood in the sense of analytic continuation.

For the normalized transform \( \mathcal{C}^\lambda_{m,k}\varphi, \) (7.3) yields

\[
(E_\lambda \mathcal{C}^\lambda_{m,k}\varphi, \hat{\omega}) = c_{m,\lambda} (E_{-\lambda-n}A_{k,m}\varphi, \omega),
\]

(7.4)
where $\Delta$ is the Cayley-Laplace operator (2.2). If $m \neq 1$, then $\Delta$ is the usual Laplace operator and $\Delta_\lambda$ expresses through the Beltrami-Laplace operator $\Delta_S$ on the unit sphere as

$$\Delta_\lambda f = -\frac{1}{4} [\Delta f + (\lambda + 2)(n + \lambda)f].$$

The latter can be easily checked using the product formula

$$\Delta(\varphi \psi) = \varphi \Delta \psi + 2 (\text{grad } \varphi) \cdot (\text{grad } \psi) + \psi \Delta \varphi;$$

cf. [42, Proposition 2.2].

**Lemma 7.2.** If $f$ is a right $O(m)$-invariant function on $V_{n,m}$, then $\Delta_\lambda f$ is a right $O(m)$-invariant function on $V_{n,m}$ too, and therefore $\Delta_\lambda$ can be viewed as a differential operator on the Grassmannian $G_{n,m}$.

**Proof.** We need to show that $(\Delta_\lambda f)(v \gamma) = (\Delta_\lambda f)(v)$ for all $\gamma \in O(m)$, $v \in V_{n,m}$. Let

$$F(x) = |x|^{\lambda+2/n} f(x(x')^{-1/2}), \quad x \in \mathfrak{M}_{n,m}.$$ 

Then

$$(\Delta_\lambda f)(v \gamma) = \left( -\frac{1}{4} \right)^m \frac{1}{4} \left( \Delta E_{\lambda+2} f \right)(x) \bigg|_{x = v \gamma} = \left( -\frac{1}{4} \right)^m \frac{1}{4} (\Delta F)(x) \bigg|_{x = v \gamma}.$$ 

By (2.3),

$$(\Delta F)(x) \bigg|_{x = v \gamma} = (\Delta F)(x \gamma) \bigg|_{x = v} = (\Delta F)(x) \bigg|_{x = v}.$$ 

Here $F_\gamma(x) = F(x \gamma) = |x|^{\lambda+2/n} f(x \gamma(\gamma' x' x \gamma)^{-1/2})$. Using polar decomposition $x = vr^{1/2}$, $v \in V_{n,m}$, $r = x' x \in \Omega_m$, we write

$$f(x \gamma(\gamma' x' x \gamma)^{-1/2}) = f(v r^{1/2} \gamma(\gamma' r \gamma)^{-1/2}).$$ 

The $m \times m$ matrix $r^{1/2} \gamma$ can be written in the polar form as $r^{1/2} \gamma = \theta s^{1/2}$, $\theta \in O(m)$, $s \in \Omega_m$. Hence we continue:

$$f(v r^{1/2} \gamma(\gamma' r \gamma)^{-1/2}) = f(v \theta s^{1/2} s^{-1/2}) = f(v \theta) = f(v),$$

because $f$ is right $O(m)$-invariant. Noting that $|x \gamma|_m = |x|_m$, we obtain $F(x \gamma) = |x|^{\lambda+2/n} f(x(x')^{-1/2})$, which means that $(\Delta_\lambda f)(v \gamma) = (\Delta_\lambda f)(v).$
An analogue of Lemma 7.2 still holds if we replace the Cayley-
Laplace operator $\Delta$ by its power and set
\[
(\Delta_{\lambda,\ell} f)(v) = \left(-\frac{1}{4}\right)^{m\ell} \left.(\Delta^\ell E_{\lambda+2\ell} f)(x)\right|_{x=v}; \quad \ell = 1, 2, \ldots \tag{7.6}
\]

**Theorem 7.3.** Let $1 \leq m \leq k \leq n - 1$, $\varphi \in C^\infty(V_{n,k})^{O(k)}$. Then for all complex $\lambda$,
\[
\Delta_{\lambda,\ell} \mathcal{C}_{m,k}^{\lambda+2\ell} \varphi = \mathcal{C}_{m,k}^{\lambda} \varphi, \tag{7.7}
\]
provided that both sides of this equality are meaningful and smooth.

**Proof.** Let first $\ell = 1$. Suppose $\omega \in S(\mathfrak{g}_{n,m})$, and let $\hat{\omega}(y)$ be the Fourier transform of $\omega$. Then, by (7.4),
\[
(E_{\lambda} \mathcal{C}_{m,k}^{\lambda} \varphi, \hat{\omega}) = c_{m,\lambda} (E_{-\lambda-n A_{k,m}} \varphi, \omega), \quad c_{m,\lambda} = 2^{m(n+\lambda)} \pi^{nn/2}, \tag{7.8}
\]
for all complex $\lambda \notin \{1-m, 2-m, \ldots\}$. Setting $\omega_1(x) = |x|^2_{m} \omega(x)$ and using (7.8) repeatedly, we obtain
\[
\left(\Delta [E_{\lambda+2} \mathcal{C}_{m,k}^{\lambda+2} \varphi], \hat{\omega}\right) = (-1)^m \left(E_{\lambda+2} \mathcal{C}_{m,k}^{\lambda+2} \varphi, \hat{\omega}_1\right)
= (-1)^m c_{m,\lambda+2} (E_{-\lambda-2-n A_{k,m}} \varphi, \omega_1)
= (-1)^m c_{m,\lambda+2} (E_{-\lambda-n A_{k,m}} \varphi, \omega)
= (-1)^m \frac{c_{m,\lambda+2}}{c_{m,\lambda}} \left(E_{\lambda} \mathcal{C}_{m,k}^{\lambda} \varphi, \hat{\omega}\right)
= (-4)^m \left(E_{\lambda} \mathcal{C}_{m,k}^{\lambda} \varphi, \hat{\omega}\right).
\]

Since $\mathcal{C}_{m,k}^{\lambda+2} \varphi$ and $\mathcal{C}_{m,k}^{\lambda} \varphi$ are smooth on $V_{n,m}$, the functions
\[
\Delta [E_{\lambda+2} \mathcal{C}_{m,k}^{\lambda+2} \varphi] \quad \text{and} \quad E_{\lambda} \mathcal{C}_{m,k}^{\lambda} \varphi
\]
are smooth on $\mathfrak{g}_{n,m}$, and therefore
\[
(\Delta [E_{\lambda+2} \mathcal{C}_{m,k}^{\lambda+2} \varphi])(x) = (-4)^m (E_{\lambda} \mathcal{C}_{m,k}^{\lambda} \varphi)(x)
\]
for all $x \in \mathfrak{g}_{n,m}$. Setting $x = v \in V_{n,m}$, we obtain the result. In the general case the proof is similar: just set $\omega_1(x) = |x|^{2\ell}_{m} \omega(x)$ to obtain
\[
(\Delta^\ell E_{\lambda+2\ell} \mathcal{C}_{m,k}^{\lambda+2\ell} \varphi)(x) = (-4)^{m\ell} (E_{\lambda} \mathcal{C}_{m,k}^{\lambda} \varphi)(x), \quad x \in \mathfrak{g}_{n,m}.
\]
This completes the proof. \[\square\]
Remark 7.4. By Theorem 6.4 (i), the conditions of Theorem 7.3 are satisfied if
\[ \lambda, \lambda + 2\ell \notin \{-k - 1, -k - 2, \ldots\} \cup \{1 - m, 2 - m, \ldots\}. \]
If, moreover, \( k \leq n - m \), then, by Corollary 6.7, both \( \mathcal{C}_m^{\lambda + 2\ell} \varphi \) and \( \mathcal{C}_m^{\lambda,k} \varphi \) are smooth for \( \lambda + 2\ell \neq 1 - m, 2 - m, \ldots \) provided that \( \varphi \) lies in the range of the Funk transform, i.e., \( \varphi = F_{m,k} f, f \in C^\infty(V_{n,m}) \).

Combining the formula \( S_m^\lambda f = \tilde{\delta} \mathcal{C}_m^{\lambda,k} F_{m,k} f \) (see (6.12)) with (7.7), we obtain
\[ \Delta_{\lambda,\ell} S_m^{\lambda + 2\ell} f = \mathcal{S}_m^\lambda f, \quad \lambda \in \mathbb{C}, \quad \lambda + 2\ell \neq 1 - m, 2 - m, \ldots, \quad (7.9) \]
where \( 1 \leq m \leq n - m \) and \( \Delta_{\lambda,\ell} \) is defined by (7.6). Regarding applicability of this reasoning, we recall that whenever \( 1 \leq m \leq k \leq n - m \), \( F_{m,k} \) acts from \( C^\infty(V_{n,m}) \) to \( C^\infty(V_{n,k})^{O(k)} \) and \( \mathcal{C}_m^{\lambda,k} \) acts from the range \( F_{m,k}(C^\infty(V_{n,m})) \) to \( C^\infty(V_{n,m})^{O(m)} \) for all \( \lambda \in \mathbb{C} \), except poles.

8. Inversion Formulas

As usual in the Radon transform theory, we distinguish the local inversion formulas and the nonlocal ones, depending on the parity of dimensions involved. In the case of the Funk-cosine transform \( F_{m,k}^{(j)} \), the formulas of the fist kind correspond to \( n - m + j - k \) even, while the most difficult case, when \( n - m + j - k \) is odd, deals with formulas of the second kind. For the sake of simplicity and consistency with the previous text, all inversion formulas in this section are presented for functions on the Stiefel manifold \( V_{n,m} \). Because functions on the Grassmannian \( G_{n,m} \) can be viewed as right \( O(m) \)-invariant functions on \( V_{n,m} \), the reader can easily reformulate the results in the Grassmannian terms.

8.1. Local Inversion.

Theorem 8.1. Let \( \varphi = F_{m,k}^{(j)} f, f \in C^\infty(V_{n,m})^{O(m)}, \)
\[ 1 \leq m \leq k \leq n - m, \quad m - n \leq j - k \leq \min(-m, m - k - 1), \quad j \geq 0. \]
If \( n - m + j - k \) is even and \( \ell = (n - m + j - k)/2 \geq 0 \), then
\[ f = \delta_j \Delta_{m-n,\ell} \psi, \quad \psi = F_{m,k} \varphi, \]
where
\[ \delta_j = \frac{\Gamma_m((k-j)/2) \Gamma_m(m/2)}{\Gamma_m((n-k+j)/2) \Gamma_m((n-m)/2)}. \]
(Δ_{m-n,\ell} \psi)(v) = \left(-\frac{1}{4}\right)^{m\ell} (\Delta^\ell E_{m-n+2\ell} \psi)(x) \bigg|_{x=v}, \quad v \in V_{n,m}.

Proof. By (6.11), \( f = S_m^{m-n} f \), where, by (7.9) with \( \lambda = m-n \), we have

\[ S_m^{m-n} f = \Delta_m^{m-n,\ell} S_m^{j-k} f. \]

Because \( S_m^{j-k} f = \delta_j F_m^{(j)} F_m^{(j)} \) (see (6.15)), the result follows. \( \square \)

The case \( j = 0 \), when \( \varphi = F_m^{(j)} \) is the usual Funk transform, deserves special mentioning. If \( n - m - k \) is even, the above theorem yields

\[ f = \delta_0 \Delta_m^{m-n,\ell} F_m^{(1)} \varphi, \quad (8.1) \]

where

\[ \delta_0 = \frac{\Gamma_m(k/2) \Gamma_m(m/2)}{\Gamma_m((n-k)/2) \Gamma_m((n-m)/2)}, \quad \ell = \frac{n - m - k}{2} \geq 0. \quad (8.2) \]

In the case \( m = k = 1 \), (8.1) agrees with Helgason’s formula in [21, Theorem 1.17, p. 133] and the corresponding formula in [42]. In the general case, our formula can be used as a substitute for known local inversion formulas in [10, 16, 17, 24, 44].

8.2. Nonlocal Inversion of \( F_m^{(k)} \) in the Case \( m < k \). If \( n - k - m \) is odd, a local inversion formula, like (8.1), is not available. However, if \( m < k \), the following result for the Funk transform \( F_m^{(k)} \) (i.e., \( j = 0 \)) can be obtained by making use of the intermediate dual Funk-cosine transform \( F_m^{(1)} \).

**Theorem 8.2.** Let \( \varphi = F_m^{(k)} f, f \in C^\infty(\mathbb{V}_{n,m})^{O(m)}, 1 \leq m < k \leq n-m. \) Suppose that \( n - k - m \) is odd. Then

\[ f = c \Delta_m^{m-n,\ell} F_m^{(1)} \varphi, \]

where

\[ c = \frac{\Gamma_m(m/2) \Gamma_m((k-1)/2)}{\Gamma_m((n-m)/2) \Gamma_m((n-k+1)/2)}, \quad \ell = \frac{n - k - m + 1}{2}. \]

Proof. By (6.12) and Corollary 6.7 (with \( \lambda = 1-k \)) we have

\[ S_m^{1-k} f = \tilde{\delta} \tilde{\mathcal{C}}_m^{1-k} \varphi, \quad \tilde{\delta} = \frac{\Gamma_m(k/2)}{\Gamma_m((n-m)/2)}, \]

where, by (6.7),

\[ \tilde{\mathcal{C}}_m^{1-k} \varphi = \tilde{c}_1 F_m^{(1)} \varphi, \quad \tilde{c}_1 = \frac{\Gamma_m(m/2) \Gamma_m((k-1)/2)}{\Gamma_m(k/2) \Gamma_m((n-k+1)/2)}. \]
Hence, by (7.9),
\[ f = S_{m-n}^m f = \Delta_{m-n,\ell} S_{m-n}^m f = \Delta_{m-n,\ell} S_{m-n}^m f \]
\[ f = \delta_0 \Delta_{m-n,\ell} \psi_{m,k} \varphi = \delta_1 \Delta_{m-n,\ell} F_{m,k}^{(1)} \varphi, \]
which gives the result. \(\square\)

8.3. The case \(k = m, j = 0\). In this case the dual Funk transform intertwines the differential operator in the local inversion formula, and, as a result, we have two different inversion formulas. The proof relies on the previous formulas for \(F_{m,k}^{(j)} f\), according to which the restriction \(k = m\) implies \(j = 0\). It means that our reasoning works only for the Funk transform \(F_m\) acting from \(C^\infty(V_{n,m})^{O(m)}\) into itself.

**Theorem 8.3.** Let \(\varphi = F_m f\), \(f \in C^\infty(V_{n,m})^{O(m)}, 1 \leq m \leq n - m\). Suppose that \(n\) is even and \(\ell = (n-2m)/2\). If
\[ D = c \Delta_{m-n,\ell}, \quad c = \left( \frac{\Gamma_m(m/2)}{\Gamma_m((n-m)/2)} \right)^2, \]
then
\[ f = D F_m \varphi = F_m D \varphi. \] (8.3)

**Proof.** In view of Theorem 8.1, it remains to prove the second equality in (8.3). As above, \(f = S_{m-n}^m f\), where by (6.12) and Corollary 6.7 (with \(\lambda = m - n\)),
\[ S_{m-n}^m f = \delta F_m \psi_{m-n}^m f, \quad \delta = \frac{\Gamma_m(m/2)}{\Gamma_m((n-m)/2)}. \]
Using (7.7) and (6.4), we can write
\[ \psi_{m-n}^m f = \Delta_{m-n,\ell} \psi_{m-n}^m f = \Delta_{m-n,\ell} \psi_{m-n}^m f = \delta_0 \Delta_{m-n,\ell} F_m f, \]
where \(\delta_0 = \delta\). Hence \(f = \delta^2 F_m \Delta_{m-n,\ell} F_m f = F_m D \varphi. \) \(\square\)

Theorem 8.3 agrees with the known case \(m = 1\) for the sphere; see [42, Theorem 2.6 (i)].

9. Conclusion and Open Problems

In the present paper we introduced a new family of differential operators on Stiefel (or Grassmann) manifolds and applied these operators to the study of \(\lambda\)-cosine transforms, Funk transforms, and their intermediate modifications. Our main objective was inversion formulas on smooth functions. The main tool was the classical Fourier analysis on matrix space. Of course, many problems are still open. Below we list
some of them with the hope that the reader will be inspired to make further progress.

1. A nonlocal inversion formula for the Funk transform $F_m = F_{n,m}$ in terms of the differential operator $\Delta_{\lambda,\ell}$ and a suitable back-projection operator, when $n$ is odd, $1 \leq m < n/2$.

Theorems 8.1, 8.2, do not cover this case. Nonlocal inversion formulas for the Funk transform on Grassmannians are known in different terms [10, 17, 49].

2. A nonlocal inversion formula for the intermediate Funk-cosine transform $F^{(j)}_{m,k}$ when $j > 0$ and $n - m + j - k$ is odd; cf. Theorem 8.1, where $n - m + j - k$ is even.

3. An analogue of the equality $\Delta_{\lambda,\ell} \ast C^\lambda_{m,k+2\ell} \varphi = \ast C^\lambda_{m,k} \varphi$ for $C^\lambda_{m,k} f$ and $C^\lambda_{m,k} f$, $k > m$; cf. Theorem 7.3.

4. Intertwining formulas for $k > m$ and $j > 0$, generalizing (8.3). Formulas of this kind are well known in the Radon transform theory; cf. Theorems 3.1 and 3.8 in [21, Chapter 1, Section 3] for the hyperplane Radon transform on $\mathbb{R}^n$.

5. Non-injectivity of $F_{m,k}$ (on right $O(m)$-invariant functions) when $m > k$ or, equivalently, of $R_{p,q}$ when $p + q > n$. Is it possible to give a relatively simple counterexample?

Note that if $k + m \leq n$, then, by (A.1), $\dim V_{n,m} > \dim V_{n,k}$ if and only if $m > k$. Similarly, if $p < q$, then $p + q > n$ if and only if $\text{rank } G_{n,p} > \text{rank } G_{n,q}$. An outline of the proof, that $R_{p,q}$ is non-injective when $p + q > n$, was communicated by T. Kakehi [25] in group representation terms.

6. An analogue of Problem 5 for intermediate Funk-cosine transforms and their Grassmannian modifications; see Conjecture 6.3.

This conjecture resembles the known fact that the non-normalized $\lambda$-cosine transform is non-injective, when $\lambda$ is a positive integer; see [15, 35, 36].

APPENDIX. SMOOTH FUNCTIONS ON STIEFEL AND GRASSMANN MANIFOLDS

Below we present necessary information about Stiefel and Grassmann manifolds in a consistent way and prove some auxiliary statements. Our main objective is characterization of smooth functions on these manifolds. Most of this material is scattered in numerous books and
papers, according to needs and taste of the authors; see, e.g., [1, 5, 29, 51]. Lemma A.7 is new. We refer to [29] for terminology.

A.1. **Stiefel manifolds.** Recall that $\mathfrak{m}_{n,m}$, $n \geq m$, is the real vector space of matrices $x = (x_{i,j})$ having $n$ rows and $m$ columns. We equip $\mathfrak{m}_{n,m}$ with a natural linear manifold structure. A chart of this manifold is given by a map

$$\mathfrak{m}_{n,m} \rightarrow \mathbb{R}^{nm}, \quad x \rightarrow (x_{1,1}, \ldots, x_{n,1}, x_{1,2}, \ldots, x_{n,2}, \ldots, x_{1,m}, \ldots, x_{n,m}),$$

that stacks the columns of $x$ below one another. Let $\tilde{\mathfrak{m}}_{n,m}$ be the set of all matrices $x \in \mathfrak{m}_{n,m}$ of rank $m$, which is an open subset of $\mathfrak{m}_{n,m}$. We consider $\tilde{\mathfrak{m}}_{n,m}$ as a smooth manifold with the differentiable structure inherited from $\mathfrak{m}_{n,m}$.

Let $V_{n,m} = \{v \in \tilde{\mathfrak{m}}_{n,m} : v'v = I_m\}$ be the set of all orthonormal $m$-frames in $\mathbb{R}^n$. Because $v'v = I_m$ gives $m(m+1)/2$ functionally independent polynomial conditions on the $nm$ entries $v_{i,j}$ of $v$, $V_{n,m}$ is an algebraic variety of dimension

$$d_m = nm - m(m+1)/2. \quad (A.1)$$

It is also a closed subset of the sphere of radius $\sqrt{m}$ in $\mathbb{R}^{nm}$.

The subset $V_{n,m} \subset \tilde{\mathfrak{m}}_{n,m}$ can be regarded as an embedded submanifold of $\tilde{\mathfrak{m}}_{n,m}$. To prove the latter, consider the polynomial map

$$F : \tilde{\mathfrak{m}}_{n,m} \rightarrow \mathfrak{m}_{m,m}, \quad F(x) = x'x.$$

The differential of $F$ is given by

$$DF(x)y = x'y + y'x, \quad y \in \mathfrak{m}_{n,m}.$$ 

It follows that $F$ has full rank; see, e.g., [1, p. 26] for details. Then, by the inverse function theorem [51, Theorem 1.38], $V_{n,m} = F^{-1}(I_m)$ is an embedded submanifold of $\tilde{\mathfrak{m}}_{n,m}$ with unique differentiable structure inherited from $\tilde{\mathfrak{m}}_{n,m}$. With this manifold structure, the set $V_{n,m}$ is known as the **Stiefel manifold**. Important particular cases are $m = 1$ and $m = n$, when $V_{n,1} = S^{n-1}$ and $V_{n,n} = O(n)$.

**Proposition A.1.** The maps

$$O(n) \times V_{n,m} \rightarrow V_{n,m}, \quad (g, v) \rightarrow gv,$$

$$V_{n,m} \times O(m) \rightarrow V_{n,m}, \quad (v, \gamma) \rightarrow v\gamma,$$

are smooth.

**Proof.** The statement follows, e.g., from [29, Corollary 8.25], since, by the above definition, $V_{n,m}$ is an embedded submanifold of $\tilde{\mathfrak{m}}_{n,m}$ and the maps

$$F_1, F_2 : V_{n,m} \rightarrow \tilde{\mathfrak{m}}_{n,m}, \quad F_1 : v \rightarrow gv, \quad F_2 : v \rightarrow v\gamma,$$
are smooth. The smoothness of $F_1$ and $F_2$ is obvious because matrix entries of $gv$ and $v\gamma$ depend polynomially on the matrix entries of $v$. □

The Stiefel manifold $V_{n,m}$ has several diffeomorphic realizations. The next one is especially important. Let us fix a unit frame $v_0 = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in V_{n,m}$. The isotropy subgroup of $O(n)$ at $v_0$, that can be identified with $O(n - m)$, is a closed embedded Lie subgroup of $O(n)$. Hence the left coset space $O(n)/O(n - m)$ has a unique smooth manifold structure such that the quotient map
\[ \pi : O(n) \to O(n)/O(n - m) \]  
(A.2)
is a smooth submersion and the map
\[ F : O(n)/O(n - m) \to V_{n,m}, \quad F(gO(n - m)) = gv_0, \]  
(A.3)
is an $O(n)$-equivariant diffeomorphism; see [29, Lemma 9.23 and Theorems 9.22, 9.24]. Thus we have the following

**Proposition A.2.** The Stiefel manifold $V_{n,m}$ is diffeomorphic to the quotient manifold $O(n)/O(n - m)$.

We will need the following general statement.

**Proposition A.3.** [29, Proposition 7.17] Suppose $M$, $N$, and $P$ are smooth manifolds, $\sigma : M \to N$ is a surjective submersion, and $f : N \to P$ is any map. Then $f$ is smooth if and only if $f \circ \sigma$ is smooth:

\[ \begin{array}{ccc}
M & \xrightarrow{\sigma} & N \\
\downarrow f & & \downarrow f \\
N & \xrightarrow{f \circ \sigma} & P
\end{array} \]

**Proposition A.4.** Let $v_0 = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in V_{n,m}$. A function $f$ on $V_{n,m}$ is smooth if and only if a function $f_0$ on $O(n)$, defined by $f_0(g) = f(gv_0)$, is smooth.

*Proof.* Let $\kappa : O(n) \to V_{n,m}$, $\kappa(g) = gv_0$. We make use of Proposition A.3 with $M = O(n)$, $N = V_{n,m}$, $P = \mathbb{C}$, and $\sigma = \kappa$. The result will be proved if we show that $\kappa$ is a submersion. We have $\kappa = F \circ \pi$, where $\pi$ is the quotient map (A.2) and $F$ is the diffeomorphism (A.3). Because both $\pi$ and $F$ are submersions, their composition is a submersion (see, e.g., [29, Exercise 7.2]), and the proof is complete. □
We fix the Haar measure $dv$ on $V_{n,m}$, which is left $O(n)$-invariant, right $O(m)$-invariant, and normalized by
\[ \sigma_{n,m} \equiv \int_{V_{n,m}} dv = \frac{2^m \pi^{mn/2}}{\Gamma_m(n/2)} \tag{A.4} \]
[32, p. 70]. The notation $d_\ast v = \sigma_{n,m}^{-1} dv$ is used for the corresponding probability measure. For any $v \in V_{n,m}$,
\[ \int_{V_{n,m}} f(v) d_\ast v = \int_{O(n)} f(\gamma v) d_\ast \gamma, \tag{A.5} \]
where $d_\ast \gamma$ stands for the Haar probability measure on $O(n)$.

Proposition A.5. If $f$ belongs to $L^1(V_{n,m})$ or $C^\infty(V_{n,m})$, then the average $f_{\text{ave}}(v) = \int_{O(m)} f(v\alpha) d_\ast \alpha$ belongs to $L^1(V_{n,m})$ or $C^\infty(V_{n,m})$, respectively.

Proof. The $L^1$ statement holds by Fubini’s theorem. To prove the $C^\infty$ statement, we observe that by Proposition A.1, the map $\rho : (v, \alpha) \to v\alpha$ is smooth. Hence $f(v\alpha) = (f \circ \rho)(v, \alpha)$ is smooth on $V_{n,m} \times O(m)$, and therefore $f_{\text{ave}}$ is a smooth right $O(m)$-invariant function on $V_{n,m}$. \qed

Below we give another characterization of smooth functions on $V_{n,m}$, which is probably new. Recall that in the case $m = 1$, it is customarily to define $C^\infty(S^{n-1})$ as the space of restrictions onto $S^{n-1}$ of $C^\infty$ functions on $\mathbb{R}^n \setminus \{0\}$, so that $f \in C^\infty(S^{n-1})$ if and only if the extended function $\tilde{f} : x \to f(x/|x|)$ belongs to $C^\infty(\mathbb{R}^n \setminus \{0\})$. The following theorem allows us to proceed in a similar way if $m > 1$, when the radial component is an element of the set $\Omega_m$ of positive definite symmetric $m \times m$ matrices.

Theorem A.6. [7, 22, 32] (Matrix Polar Decomposition) If $n \geq m$, then every matrix $x \in \bar{M}_{n,m}$ can be uniquely represented as
\[ x = vr^{1/2}, \quad v \in V_{n,m}, \quad r = x'x \in \Omega_m, \tag{A.6} \]
and $dx = 2^{-m}|r|^{(n-m-1)/2} dr dv$.

By this theorem, there is a one-to-one correspondence between functions $f$ on $V_{n,m}$ and their homogeneous extensions
\[ \tilde{f}(x) = f(x(x'x)^{-1/2}), \quad x \in \bar{M}_{n,m}. \]

Lemma A.7. The relations $f \in C^\infty(V_{n,m})$ and $\tilde{f} \in C^\infty(\bar{M}_{n,m})$ are equivalent.
Proof. We make use of Proposition A.3 with \( M = \mathfrak{M}_{n,m}, \quad N = V_{n,m}, \quad P = \mathbb{C}, \) and \( \sigma(x) = x(x'x)^{-1/2}. \) It suffices to show that \( \sigma : \mathfrak{M}_{n,m} \to V_{n,m} \) is a surjective submersion. Clearly, \( \sigma \) is smooth and its differential, as a linear map between tangent spaces at \( x \in \mathfrak{M}_{n,m} \) and \( v = \sigma(x) \in V_{n,m}, \) has full rank, which is equal to \( \dim V_{n,m}. \) Thus \( \sigma \) is a smooth surjective map of constant rank, and therefore (use, e.g. [29, Theorem 7.15]) it is a submersion. This completes the proof. \( \square \)

Remark A.8. By Lemma A.7, we can realize \( C^\infty(V_{n,m}) \) as the space of all functions \( f \) on \( V_{n,m}, \) for which \( \tilde{f} \in C^\infty(\mathfrak{M}_{n,m}) \) in the usual sense, as on \( \mathbb{R}^{nm}. \) This remark plays a key role in the paper because it allows us to define differential operators on \( V_{n,m} \) via homogeneous continuation.

A.2. Grassmann manifolds. We denote by \( G_{n,m} \) the Grassmann manifold of \( m \)-dimensional linear subspaces of \( \mathbb{R}^n. \) There exist several diffeomorphic realizations of \( G_{n,m}, \) see, e.g., [29, pp. 22, 234, 238 (Problem 9-14)]. The Lie group \( G = O(n) \) acts on \( G_{n,m} \) smoothly and transitively, and therefore \( G_{n,m} \) is a homogeneous \( G \)-space. If \( \xi_0 = \text{span}(e_1, \ldots, e_m) \in G_{n,m} \) is the coordinate subspace of \( \mathbb{R}^n, \) then \( G_0 = O(n-m) \times O(m) \) is the isotropy group of \( \xi_0, \) which is a closed embedded Lie subgroup of \( G \) (see, e.g., [29, Lemma 9.23]). Hence Theorem 9.24 from [29] yields the following

Proposition A.9. The maps
\[ E_1 : G/G_0 \to G_{n,m}, \quad E_2 : G/G_0 \to G_{n,n-m} \quad \text{(A.7)} \]
are \( G \)-equivariant diffeomorphisms.

By this proposition, \( G_{n,m} \) and \( G_{n,n-m} \) are diffeomorphic and every function \( f_1 \) on \( G_{n,m} \) can be identified with a function \( f_2 \) on \( G_{n,n-m}, \) so that
\[ f_1(\xi) = f_2(\xi^\perp), \quad f_2(\eta) = f_1(\eta^\perp); \quad \xi \in G_{n,m}, \quad \eta \in G_{n,n-m}. \]
In other words,
\[ f_1(\xi) = f_2(E_2(E_1^{-1}\xi)), \quad f_2(\eta) = f_1(E_1(E_2^{-1}\eta)). \]
This reasoning gives the following

Proposition A.10. \( f_1 \in C^\infty(G_{n,m}) \) if and only if \( f_2 \in C^\infty(G_{n,n-m}). \)

Proposition A.11. Given a function \( g \) on \( G_{n,m}, \) let
\[ g_0(\gamma) = g(\gamma \xi_0), \quad \gamma \in O(n), \quad \xi_0 = \text{span}(e_1, \ldots, e_m). \]
Then \( g_0 \in C^\infty(O(n)) \) if and only if \( g \in C^\infty(G_{n,m}). \)
Proof. Let $\kappa : O(n) \to G_{n,m}$, $\kappa(g) = g\xi_0$. We make use of Proposition A.3 with $M = O(n)$, $N = G_{n,m}$, $P = \mathbb{C}$, and $\sigma = \kappa$. The result will be proved if we show that $\kappa$ is a submersion. Denote $G = O(n)$, $G_0 = O(n - m) \times O(m)$. We have $\kappa = E_1 \circ \pi$, where $\pi : G \to G/G_0$ is the quotient map and $E_1$ is the diffeomorphism from (A.7). Because both $\pi$ and $E_1$ are submersions, their composition is a submersion, and the proof is complete. 

Every subspace $\xi \in G_{n,m}$ is uniquely determined by its orthonormal basis $v \in V_{n,m}$. Because all bases of the form $v_0\gamma$, $\gamma \in O(m)$, define the same subspace, we can realize $G_{n,m}$ as a quotient space

$$G_{n,m} \simeq V_{n,m}/O(m).$$

Proposition A.12. The map

$$E_3 : G_{n,m} \to V_{n,m}/O(m)$$

is an $O(n)$-equivariant diffeomorphism.

Proof. Let $G = O(n)$. By Proposition A.9, it suffices to prove this statement with $G_{n,m}$ replaced by $G/G_0$, $G_0 = O(n - m) \times O(m)$. The manifold $V_{n,m}/O(m)$ is a homogeneous $G$-space with the isotropy subgroup $G_0$ of the element $v_0O(m) \in V_{n,m}/O(m)$, $v_0 = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in V_{n,m}$. Hence, by Theorem 9.24 from [29], there exists a $G$-equivariant diffeomorphism between $G/G_0$ and $V_{n,m}/O(m)$. This completes the proof. 

Remark A.13. By Proposition A.12, we can identify functions $g \in C^\infty(G_{n,m})$ with functions $f : v \to g(\{v\})$ belonging to $C^\infty(V_{n,m})^{O(m)}$. Thus, in view of Remark A.8, there is a one-to-one correspondence between smooth functions on the Grassmannian $G_{n,m}$ and smooth functions on the matrix space $\mathfrak{g}^{n,m}$.

Owing to diffeomorphisms

$$V_{n,m}/O(m) \iff G_{n,m} \iff G_{n,n-m} \iff V_{n,n-m}/O(n-m),$$

we obtain the following statement.

Proposition A.14. There is a one-to-one correspondence

$$f \simeq f_*$$

between right $O(m)$-invariant functions $f$ on $V_{n,m}$ and right $O(n-m)$-invariant functions $f_*$ on $V_{n,n-m}$. Moreover, $f \in C^\infty(V_{n,m})^{O(m)}$ if and only if $f_* \in C^\infty(V_{n,n-m})^{O(n-m)}$. 

The next Proposition, which is a consequence of normalization, characterizes integrability properties of functions $f, f^*$, and the relevant functions on Grassmannians.

**Proposition A.15.** Given $v \in V_{n,m}$, let $\tilde{v} \in V_{n,n-m}$ be an arbitrary frame, which is orthogonal to the subspace $\xi = \{v\}$, and let $\tilde{\xi} = \{\tilde{v}\}$. If $f$ is a right $O(m)$-invariant function on $V_{n,m}$, $f^*$ is defined by (A.8), and the functions $g$ and $g^*$ are defined by

$$f(v) = g(\{v\}), \quad f^*(\tilde{v}) = g^*(\{\tilde{v}\}),$$

then

$$\int_{V_{n,m}} f(v) \, d_\ast v = \int_{V_{n,n-m}} f^*(\tilde{v}) \, d_\ast \tilde{v} = \int_{G_{n,m}} g(\xi) \, d_\ast \xi = \int_{G_{n,n-m}} g^*(\tilde{\xi}) \, d_\ast \tilde{\xi}, \quad \text{(A.9)}$$

provided that at least one of these integrals exists in the Lebesgue sense.

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Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803, USA
E-mail address: borisr@lsu.edu