ON SHAFAREVICH-TATE GROUPS AND THE ARITHMETIC OF FERMAT CURVES

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To Sir Peter Swinnerton-Dyer on his 75th birthday.

1. Introduction

Let \( \mathbb{Q} \) denote the field of rational numbers and \( \overline{\mathbb{Q}} \) a fixed algebraic closure of \( \mathbb{Q} \). For a fixed prime \( p \) such that \( p \geq 5 \), choose a primitive \( p \)-th root of unity \( \zeta \) in \( \overline{\mathbb{Q}} \) and let \( K = \mathbb{Q}(\zeta) \). If \( a, b \) and \( c \) are integers such that \( 0 < a, b, a + b < p \) and \( a + b + c = 0 \) let \( F_{a,b,c} \) denote a smooth projective model of the affine curve

\[
y^p = x^a(1-x)^b
\]

and let \( J_{a,b,c} \) be the Jacobian of \( F_{a,b,c} \). Then \( J_{a,b,c} \) has complex multiplication induced by the birational automorphism \( (x,y) \mapsto (x,\zeta y) \) of \( F_{a,b,c} \). Let \( \lambda \) denote the endomorphism \( \zeta - 1 \) of \( J_{a,b,c} \). Note that \( \lambda^{p-1} \) is, up to a unit in \( \mathbb{Z}[\zeta] \), multiplication by \( p \) on \( J_{a,b,c} \).

We are interested in the Shafarevich-Tate group of \( J_{a,b,c} \) over \( K \), which we denote simply by \( \Sha \). In [McC88], the first author studied the restriction of the Cassels-Tate pairing

\[
\Sha[\lambda] \times \Sha[\lambda] \rightarrow \mathbb{Q}/\mathbb{Z}
\]

and showed that \( \Sha[\lambda] \) is non-trivial in certain cases depending on the reduction type of the minimal regular model of \( F_{a,b,c} \) over \( \mathbb{Z}_p[\zeta] \). The purpose of this paper is to extend those results by carrying out higher descents, and to derive some consequences for the arithmetic of Fermat curves using the techniques of the second author.

First we recall the main result of [McC88]. The possible reduction types for \( F_{a,b,c} \) are as shown in Figure 1 [McC82], with the proper transform of the special fiber of the model (1.1) indicated. The wild type is further divided into split and non-split, according to whether the two tangent components are defined over the finite field \( \mathbb{F}_p \) or conjugate over a quadratic extension. The reduction type can be computed as follows. For a rational number \( x \) of \( p \)-adic valuation 0, let \( q(x) = (x^{p-1} - 1)/p \), viewed as an element of \( \mathbb{F}_p[\zeta] \). Then \( F_{a,b,c} \) is

\[
\begin{align*}
tame & \quad \text{if } -2abcq(a^ib^jc^k) = 0 \\
\text{wild split} & \quad \text{if } -2abcq(a^ib^jc^k) \in \mathbb{F}_p^\times \\
\text{wild non-split} & \quad \text{if } -2abcq(a^ib^jc^k) \notin \mathbb{F}_p^\times
\end{align*}
\]

Let \( M_K \) be the set of finite places of \( K \) and let \( w \) denote the unique place of \( K \) above \( p \). Define

\[
U = \{ x \in K^\times/K^\times_p : v(x) \equiv 0 \pmod{p} \text{ for all } v \in M_K \}, \quad V = K_w^\times/K_w^\times_p.
\]
Figure 1. Reduction types of $F_{a,b,c}$

| Tame          | Wild           |
|---------------|----------------|
| proper transform | proper transform |
| multiplicity 2  |                |
| $p + 1$        |                |

Let $\pi$ be the uniformizer of $K_w$ defined by

$$\pi^{p-1} = -p \quad \text{and} \quad \frac{\pi}{1-\zeta} \equiv 1 \pmod{w}.$$  

If $\kappa : \text{Gal}(K/Q) \to \mathbb{Z}_p^\times$ is the Teichmüller character, let $V(i)$ denote the intersection of the $\kappa^i$-th eigenspace of $V$ with the subgroup of $V$ generated by units congruent to 1 modulo $\pi^i$ (thus $V(i)$ is one-dimensional if $2 \leq i \leq p$).

**Theorem 1.1** ([McC88]). Suppose that $F_{a,b,c}$ is wild split, $p \equiv 1 \pmod{4}$, and the image of $U$ is non-trivial in both $V((p-1)/2)$ and $V((p+3)/2)$. Then $\text{III}(\lambda)/\lambda \text{III}(\lambda^2) \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

The condition on $U$ is satisfied if $p \nmid B_{(p-1)/2}B_{(p+3)/2}$, where $B_k$ is the $k$-th Bernoulli number. As noted in [McC88], the technique used to prove Theorem 1.1 applies to the pairing

$$\text{III}(\lambda^2) \times \text{III}(\lambda) \to \mathbb{Q}/\mathbb{Z}$$

and yields information about $\text{III}(\lambda^2)$.

**Theorem 1.2.** Suppose that either of the following conditions is satisfied:

(a) $F_{a,b,c}$ is wild-split and $p \equiv 3 \pmod{4}$

(b) $F_{a,b,c}$ is wild non-split or tame and the image of $U$ in either $V((p+1)/2)$ or $V((p+3)/2)$ is trivial.

Then $\text{III}(\lambda^2)/\lambda \text{III}(\lambda^3) = 0$, that is, $\text{III}(\lambda^3)$ is a free module over $\mathbb{Z}[\zeta]/(\lambda^3)$.

As discussed in [McC88], the hypothesis on $U$ in condition (b) of the theorem is quite mild, since for $U$ to be nontrivial in $V(k)$ with $k > 1$ and odd requires that $p$ divides $B_{p-k}$.

**Corollary 1.3.** If one of conditions (a) or (b) of Theorem 1.2 is satisfied, and if $|\text{III}(p^\infty)| < p^3$, then $\text{III}(p^\infty) = 0$.

It is natural to ask which occurs more often under the conditions of Theorem 1.2, $|\text{III}(p)| = 0$ or $|\text{III}(p)| \geq p^3$. To explore this question, we compute

$$\text{III}(\lambda^3) \times \text{III}(\lambda) \to \mathbb{Q}/\mathbb{Z}.$$
Theorem 1.4. Suppose that $p \geq 19$ is regular, $p \equiv 3 \pmod{4}$, $F_{a,b,c}$ is tame or wild non-split and

$$q(a^3 b c^3 + abc B p - 3) \not\equiv 0 \pmod{p}.$$ 

Then $\text{III}[\lambda^3] \neq 0$ (and hence, by Corollary 1.3, $|\text{III}| \geq p^3$).

The next proposition shows that, in certain cases, one can combine Theorems 1.2 and 1.4 to describe the exact structure of $X[p^\infty]$:

Theorem 1.5. Suppose that $p$, $a$, $b$, and $c$ are chosen so that the hypotheses of both Theorems 1.2 and 1.4 are satisfied. If, in addition, the free $\mathbb{Z}[\zeta]/(\lambda^3)$-module $X[\lambda^3]$ has rank 2, then

$$\text{III}[p^\infty] = \text{III}[\lambda^3] \simeq (\mathbb{Z}[\zeta]/(\lambda^3))^2.$$ 

In Section 6 we establish the following application of the above results:

Theorem 1.6. Let $p = 19$, $a = 7$, $b = 1$. Then

1. $\text{III}[p^\infty] \simeq (\mathbb{Z}[\zeta]/(\lambda^3))^2$.
2. The Mordell-Weil rank of $J_{7,1,-8}$ over $\mathbb{Q}$ equals 1.
3. The only quadratic points (i.e. algebraic points whose field of definition is a quadratic extension of $\mathbb{Q}$) on the Hurwitz-Klein curve $F_{7,1,-8}$ and also on the Fermat curve $X^{19} + Y^{19} + Z^{19} = 0$ are the ones described by Gross and Rohrlich in [GR78].

We also note that, by combining Theorem 1.4 with Faddeev’s bounds in [Fad61], one gets that the Mordell-Weil rank (over $\mathbb{Q}$) of any tame or wild non-split quotient of the Fermat curve $F_{19}$ or $F_{23}$ is at most 2.

Lim [Lim95] has also stated a result attempting to improve on [McC88] in certain cases. However, in Section 6, we show that the hypotheses of Propositions A and B of [Lim95] are never simultaneously satisfied.

2. Formulas for the pairings

We recall the situation and notation of [McC88]. For $\phi \in \mathcal{O}_K$ and $F$ a field containing $K$, we denote by $\delta = \delta_{\phi,F}$ the coboundary map $J(F) \to H^1(F, J[\phi])$. We have the $\phi$-Sehmer group $S_{\phi} \subset H^1(K, J[\phi])$, defined to be the subgroup whose specialization to each completion $K_v$ of $K$ lies in the image of $\delta_{\phi,K_v}$, and which sits in an exact sequence

$$0 \to J(K)/\phi J(K) \to S_{\phi} \to \text{III}[\phi] \to 0.$$ 

For $\phi, \psi \in \text{End}(J)$, we have a pairing

$$S_{\phi} \times S_{\psi} \to \mathbb{Q}/\mathbb{Z},$$

described in [McC88], which is a lift of the restriction of the Cassels pairing to $\text{III}[\phi] \times \text{III}[\psi]$. An expression for the pairing (2.1) is given in [McC88], under a certain splitting hypothesis.

We use this formula to derive formulas for the pairings (1.5) and (1.6). The formula for (1.5) is a straightforward consequence of Theorem 2.6 in [McC88]; the formula for (1.6) takes more work. The point is that $J[\lambda^3] \subset J(K)$ [Gre81], so that it is possible to express the pairings (1.2) and (1.3) as purely local pairings at $w$, as explained in [McC88]. However, by [Gre81] and [Kur92], the $\lambda^4$-torsion on $J_{a,b,c}$ is not in general defined over $K$, introducing an essentially global aspect to the calculation of (1.6).
In particular, we have \( \hat{\lambda} \) by what follows without mention, in cases where we are dealing with a module killed by \( \lambda^5 \). Furthermore, it suffices to prove Theorems 1.2 and 1.4 with this new choice of \( \lambda \). Since \( \lambda/\hat{\lambda} \) is a unit, \( \III[\lambda] = \III[\hat{\lambda}] \), and we can proceed by computing the pairing \([\mathbb{R}]_{\hat{\lambda}}\) with \( \phi = \lambda^k \) and \( \psi = \hat{\lambda} \).

The local formula for the Cassels-Tate pairing is expressed in terms of certain local descent maps as follows. Given a pair \((\psi,\phi)\) with \( \psi \phi = 1 \), \( \lambda = \phi \lambda \phi^{-1} \), and \( \psi = \phi^k \), then we have exact sequences

\[
\prod_{v \mid \infty} H^2(K_v/F_v, \mu_p) \rightarrow H^2(K/F, \mu_p) \rightarrow C[p] \rightarrow 0,
\]

which follows from Theorem 2.6 in \([\text{McC}88]\), with \( \phi = \lambda^2 \) and \( \psi = \lambda \).

**Proposition 2.1.** Let \( a \in S_\lambda \), \( b \in S_{\hat{\lambda}}, \ a_w = \delta(x_w), \ x_w \in J(K_w) \). Then

\[
\zeta_p^{(a,b,2)} = (\iota, p_3(x_w), b_w).
\]

**Proof.** This follows from Theorem 2.6 in \([\text{McC}88]\), with \( \phi = \lambda^2 \) and \( \psi = \lambda \).

For a number field \( F \) we denote by \( \mathcal{O}_F^\times \) the ring of \( p \)-integers in \( F \). If \( F \subset K(p) \), then we have exact sequences

\[
0 \rightarrow \mathcal{O}_F^\times /\mathcal{O}_F^{\times, p} \rightarrow H^1(K(p)/F, \mu_p) \rightarrow C[p] \rightarrow 0
\]

and

\[
0 \rightarrow C/pC \rightarrow H^2(K(p)/F, \mu_p) \rightarrow Br(F)[p] \rightarrow 0,
\]

which follows from Theorem 2.6 in \([\text{McC}88]\), with \( \phi = \lambda^2 \) and \( \psi = \lambda \).
Lemma 2.2. Each element of \( H^1(K(p)/K, J[\lambda^k]) \) lifts to \( H^1(K, J[\lambda^{k+1}]) \). Furthermore, if \( p \) is regular, it lifts to \( H^1(K(p)/K, J[\lambda^{k+1}]) \).

Proof. Let \( a \in H^1(K(p)/K, J[\lambda^k]) \), and let \( \delta a \in H^2(K(p)/K, J[\lambda]) \) be the coboundary of \( a \) for the sequence
\[
0 \to J[\lambda] \to J[\lambda^{k+1}] \to J[\lambda^k] \to 0.
\]
Then the inflation of \( \delta a \) in \( H^2(K, J[\lambda]) \simeq H^2(K, \mu_p) = \text{Br}(K)[p] \) has zero invariant at every place not dividing \( p \). Thus it is zero, by the Brauer-Hasse-Noether theorem (since there is only one place of \( K \) dividing \( p \)). For the second statement, we argue in the same way, using (2.3). \( \blacksquare \)

We recall the definition of \( \langle \cdot, \cdot \rangle_3 \). Let \( a \in S_\lambda^3 \) and \( b \in S_\lambda \). Lift \( a \) to an element \( a_1 \) of \( \text{Br}^1(K, J[\lambda^3]) \) (possible by Lemma 2.2). For each place \( v \) of \( K \), lift \( a_v \) to an element \( a_{v,1} \) that is in the image of \( \delta \). Then \( a_{1,v} - a_{v,1} \) is the image of an element \( c_v \in H^1(K_v, J[\lambda]) \), and
\[
\langle a, b \rangle = \sum_v c_v \cup b_v
\]
where the cup product is with respect to the local pairing
\[
H^1(K_v, J[\lambda]) \times H^1(K_v, J[\lambda]) \to \mathbb{Q}/\mathbb{Z}.
\]
If \( p \) is regular, \( L/K \) is totally ramified at \( w \), and there is a unique extension of \( w \) to \( L \), which we also denote by \( w \). Our calculation of \( \langle \cdot, \cdot \rangle_3 \) uses the following lemma.

Lemma 2.3. Suppose \( p > 5 \) is regular, and let \( L \) be as in (2.2). Then
1. the map \( H^1(K(p)/L, \mu_p) \to H^1(L_w, \mu_p) \) is injective
2. the norm map \( N_{L/K} : \mathcal{O}_L^{\times} \to \mathcal{O}_K^{\times} \) is surjective
3. \( H^1(K(p)/L, \mu_p)^G(i) = 0 \) if \( i \) is odd and \( i \neq 1 \), or if \( i = p - 1 \).

Proof. Let \( H_K \) (resp. \( H_L \)) be the Hilbert class field of \( K \) (resp. \( L \)). Since \( L/K \) is unramified outside \( p \), and there is only prime of \( L \) above \( p \), it follows that \( \text{Gal}(H_L/L)/(\sigma - 1) \simeq \text{Gal}(H_K/K) \). Therefore \( p \) does not divide the order of the class group \( C_L \simeq \text{Gal}(H_L/L) \). The injectivity statement follows, since anything in the kernel would generate an unramified Kummer extension of \( L \) of degree \( p \). Furthermore, every unit of \( L \) is a local norm everywhere except possibly at the prime above \( p \), and therefore is a local norm there also by the product formula. Thus it is a global norm. The surjectivity of the norm map follows by a standard argument using \( \text{Gal}(L/K) \) cohomology of the sequences
\[
1 \to \mathcal{O}_L^{\times} \to L^{\times} \to P_L \to 1
\]
and
\[
1 \to P_L \to I_L \to C_L \to 1,
\]
where \( I_L \) and \( P_L \) are the groups of ideals and principal ideals respectively. Finally, by (2.4), \( H^1(K(p)/K, \mu_p) = \mathcal{O}_K^{\times}/\mathcal{O}_K^{\times,p} \) and \( H^1(K(p)/L, \mu_p) = \mathcal{O}_L^{\times}/\mathcal{O}_L^{\times,p} \). Furthermore, the cokernel of \( \mathcal{O}_K^{\times}/\mathcal{O}_K^{\times,p} \) in \( (\mathcal{O}_L^{\times}/\mathcal{O}_L^{\times,p})^G \) is \( H^2(L/K, \mu_p) \simeq \mathbb{Z}/p\mathbb{Z} \), with \( \text{Gal}(K/\mathbb{Q}) \) acting via \( \kappa^{p-3} \), since it acts on \( G \) via \( \kappa^3 \). Since \( p - 3 \) is even and \( (\mathcal{O}_K^{\times}/\mathcal{O}_K^{\times,p})(i) = 0 \) if \( i \) is odd and \( i \neq 1 \), or if \( i = p - 1 \), the third statement of the lemma follows. \( \blacksquare \)

Let \( N' = \sum_{i=1}^{p-1} i\sigma^i \). where \( C \) is the ideal class group of \( F \).
Proposition 2.4. Suppose $p$ is regular. Let $a \in S_{\lambda^3}$, $b \in S_{\lambda}$, $a_w = \delta(x_w)$, $x_w \in J(K_w)$. Write $\lambda^2 a$, regarded as an element of $\mathcal{O}_K^\times / \mathcal{O}_K^{\times p}$, as $N_{L/K} \epsilon$ for some $\epsilon \in \mathcal{O}_L^\times$. Then there exists a $\lambda^4$-torsion point $P_4$, and an element $c_w \in K_w^\times$ such that

$$\zeta^{p(a,b)} = (c_w, b_w)$$

and the image of $c_w$ in $L_w$ satisfies

$$c_w = \iota_{P_4}(x_w)^{-1} N' \epsilon.$$  

Proof. Since $p$ is regular, $S_{\lambda} \subset \mathcal{O}_K^\times / \mathcal{O}_K^{\times p}$, so $b$ has zero component in the $\kappa^i$ eigenspace if $i$ is odd and if $i = p - 1$ (the possibility $i = 1$ is eliminated by local conditions, see \cite{Mcc88}, 6.2). Thus, by equivariance properties of the Hilbert eigenspace if $\lambda$, then there exists a $\lambda$-eigenspace if $\lambda$.

We now construct a candidate for $a_1^\prime$. Under the identification (2.3) between $J[\lambda]$ and $H^1(K(p)/L, J[\lambda]) = (K^\times / K^{\times p})^3$, $a$ corresponds to an element $(x_1, x_2, x_3) \in (K^\times / K^{\times p})^3$, and $\lambda^2 a = x_1$. Furthermore, in the identification

$$H^1(L, J[\lambda^4]) \simeq (L^\times / L^{\times p})^4,$$

the action of $\sigma$ on $H^1(L, J[\lambda^4])$ is intertwined with

$$(t_1, t_2, t_3, t_4) \mapsto (t_1^\sigma, t_2^\sigma, t_3^\sigma, t_4^\sigma), \quad t_i \in L^\times / L^{\times p}.$$  

Thus $(t_i)$ is fixed by $G$ if

$$t_i^\sigma = t_i, \quad i = 1, 2, 3, \quad \text{and} \quad t_4^\sigma = t_1^{-1}.$$  

By hypothesis, $x_1 = N_{L/K} \epsilon$, $\epsilon \in \mathcal{O}_L^\times$. Then

$$(2.9) \quad a_1^\prime = (x_1, x_2, x_3, N' \epsilon)$$

is an equivariant lift of $(x_1, x_2, x_3)$.

Now let $a_{w,1}$ be the local lift of $a$ given by $a_{w,1} = \delta_4(x_w)$. Then

$$(2.10) \quad \text{res}_{L/K_w}(a_{w,1}) = (x_1, x_2, x_3, \iota_{P_4}(x_w)).$$  

Thus, from equations (2.8), (2.3), and (2.10), we get

$$\text{res}_{L/K_w}(c_w) = \text{res}_{L/K_w}(a_{1,w} - a_{w,1}) = \iota_{P_4}(x_w)^{-1} \eta N' \epsilon.$$
Since \( \eta \in H^1(K(p)/L, \mu_p)^G \), its projection onto an eigenspace \((K^\times_p/K^\times_{p-1})(i)\) with \( i > 1 \) odd is trivial. Thus, by the remarks at the beginning of the proof, we may ignore the contribution of \( \eta \), and the proposition follows. ■

3. The local approximation

Let \( P_i \) be as in the previous section, \( i = 1, 2, 3, 4 \), let \( D_i \) be a divisor on \( F_{a,b,c} \), and let \( f \) be a function whose divisor is \( pD_i \). Take \( D_i \) and \( f \) to be defined over \( K = \mathbb{Q}(\zeta) \) if \( i = 1, 2, 3 \) and over \( L = K(\zeta_3^{1/p}) \) if \( i = 4 \). The maps \( \nu_{P_i} \) in Propositions 2.1 and 2.4 are computed by evaluating \( f \) on certain divisors.

We recall the notion of congruence used in (3.1). If \( Y \) is a one-dimensional affinoid defined over an extension \( F \) of \( \mathbb{Q}_p \) with uniformizer \( \pi_F \), we let \( A(Y) \) be the ring of rigid analytic functions on \( Y \), \( M(Y) \) the quotient field of \( A(Y) \), and \( D(Y) \) the module of Kahler-differentials of \( M(Y) \). We define sub-\( \mathcal{O}_F \)-modules

\[
\begin{align*}
A^0(Y) &= \{ f \in A(Y) : f(x) \leq 1 \text{ for all } x \in Y(\mathbb{C}_p) \} \\
M^0(Y) &= \{ f/g : f \in A^0(Y), g \in A^0(Y) \setminus \pi_F A^0(Y) \} \\
D^0(Y) &= \{ f \mathrm{d}g : f, g \in M^0(Y) \}.
\end{align*}
\]

If \( f, g \in A(Y), c \in F \), we say \( f \equiv g \ (\text{mod} \ c) \) if \( (f-g) \in cA^0(Y) \), and similarly we define the notion of congruence on \( Y \) in \( M(Y) \) and \( D(Y) \). In order to deduce from (3.1) information about power series expansions of \( f \) on closed discs in \( Y \), we need the following lemmas.

**Lemma 3.1.** Suppose that \( Y \) is a one-dimensional affinoid over a finite extension \( F \) of \( \mathbb{Q}_p \), \( Y \) has good reduction, and \( Z \) is an affinoid contained in \( Y \), isomorphic to a closed disc. If \( \omega \in D^0(Y) \) is a differential with at worst simple poles on \( Y \) that is regular on \( Z \), then \( \omega \in D^0(Z) \).

**Proof.** Since \( Z \) is isomorphic to a closed disc, it is contained in a residue class \( U \) of \( Y \) (or is equal to \( Y \), in which case there is nothing to prove). It is clear from the definitions that \( D^0(Y)|_U = D^0(U) \), hence \( \omega \in D^0(U) \). Furthermore, since \( Y \) has good reduction, \( U \) is isomorphic to an open disc. Choose a parameter \( t \) for \( U \) such that \( Z \) is the disc \( |t| \leq |c| < 1 \) for some \( c \in F \), and write

\[
\omega = \frac{g}{\sqrt[p]{c}} \mathrm{d}t + \sum_{i=1}^n \frac{a_i}{t-b_i} \mathrm{d}t, \quad g \in \mathcal{O}_F[[t]], a_i, b_i \in \mathcal{O}_F, |c| < |b_i| < 1.
\]

Expanding the polar terms in powers of \( t/b_i \) and setting \( t = cs \), we get \( \omega = f \mathrm{d}s \) for some \( f \in \mathcal{O}_F[[s]] \). Since \( s \) is a parameter on \( Z \), this proves the lemma. ■

**Lemma 3.2.** Suppose that \( f \) is a function whose divisor is divisible by \( p \). Let \( Y \) be an affinoid with good reduction contained in \( F_{a,b,c} \) and let \( Z \) be a \( p \)-adic disc contained in \( Y \) such that there is a function on \( F_{a,b,c} \) restricting to a parameter
on \( Z \). If \( \omega \) satisfies the congruence (3.4) \((\text{mod} \ p)\), then it satisfies the same congruence \((\text{mod} \ p)\).

**Proof.** With notation as in [McC88], we have
\[
\frac{df}{f} = \omega + p\eta, \quad \eta \in D^0(Y).
\]
Let \( g \) be a function on \( F_{a,b,c} \) such that \( f/g^p \) is regular on \( Z \) (we can construct \( g \) using a parameter on \( Z \) as in the hypotheses). Since a suitable scalar multiple of \( g \) is in \( M^0(Y) \), \( d\log g \in D^0(Y) \). Thus \( \eta - d\log g \in D^0(Y) \) and is also regular on \( Z \), and hence is in \( D^0(Z) \) by Lemma 3.1. Thus
\[
\frac{df}{f} \equiv \frac{df}{f} - p\frac{dg}{g} = \omega + p(\eta - \frac{dg}{g}) \equiv \omega \quad \text{(mod} \ p)\).
\]

We now apply these considerations to the affinoid \( Y \) introduced in [McC88], which is defined as follows. Let \( s \) and \( t \) be the functions on \( F_{a,b,c} \) defined by
\[
\begin{align*}
x &= -\frac{a}{c}(1 + \pi^{(p-1)/2}s) \\
y &= (-1)^ca^bc^c(1 + \pi t).
\end{align*}
\]
Let \( Y \) be the affinoid defined over \( L \) by the inequalities
\[
|t| \leq |\pi^{-1}|, \quad |s| \leq 1.
\]
A basis of holomorphic differentials on \( F_{a,b,c} \) is
\[
\omega_k = E_k \frac{x^{[ka]} (1-x^{[ka]})}{y^k} dx, \quad k \in H_{a,b,c},
\]
for some constants \( E_k \) and where \( H_{a,b,c} \) is a certain subset of \( \{1, 2, \ldots, p-1\} \) of cardinality \((p-1)/2 \) (\( H_{a,b,c} \) can be identified with the CM-type of \( F_{a,b,c} \)). We can and do choose the constants \( E_k \) so that \( \omega_k \) has expansion
\[
\omega_k \equiv ds \quad (\text{mod} \ \pi_L),
\]
(note that this normalization is different from that of [McC88]). Now, \( P_1 \) is the \( \lambda \)-torsion point represented by the divisor \((0,0) - \infty \), and we choose \( f_1 = x \). In [McC88] it was shown that
\[
\frac{df_1}{f_1} \equiv \pi^{(p-1)/2} \sum_{k \in H_{a,b,c}} b_k \omega_k \quad (\text{mod} \ p)
\]
for some \( p \)-adic integers \( b_k \), satisfying
\[
\sum_{k \in H_{a,b,c}} b_k k^i \equiv \begin{cases} F & i = 0 \\
0 & 1 \leq i \leq (p-3)/2 \end{cases} \quad (\text{mod} \ \pi_K), \quad F \in \mathbb{Z}/p\mathbb{Z}^\times.
\]
Note that although it was assumed that \( F_{a,b,c} \) is wild split in Section 5 of [McC88], there is nothing in the definition of \( Y \) or the calculation showing (3.5) and (3.6) that uses this assumption. It is only at the end of that section that the assumption comes in.
Lemma 3.3. If
\[ \sum_{k \in H_{a,b,c}} u_k \omega_k \equiv \sum_{k \in H_{a,b,c}} v_k \omega_k \pmod{\pi^{n+(p-3)/2}} \]
then
\[ u_k \equiv v_k \pmod{\pi^n}, \quad k \in H_{a,b,c}. \]

Proof. See pages 658–659 of [McC88]. ■

Proposition 3.4. We have
\[ \frac{df_3}{f_3} \equiv \sum_{k \in H_{a,b,c}} c_k \omega_k \pmod{p}, \quad c_k \equiv 0 \pmod{\pi^{(p-5)/2}} \]
and
\[ \frac{df_4}{f_4} \equiv \sum_{k \in H_{a,b,c}} d_k \omega_k \pmod{p}, \quad d_k \equiv -\pi^{(p-7)/2} \frac{b_k}{k^3} \pmod{\pi^{(p-5)/2}}, \]
where the \( b_k \) are as in equation (3.7).

Proof. We have
\[ \lambda^2 \frac{df_3}{f_3} \equiv \frac{df_1}{f_1} \pmod{p}. \]
Since \( \zeta_+ \omega_k = \zeta^{-k} \omega_k \), we have \( \lambda_k \omega_k = \lambda^\sigma \omega_k \), for some \( \sigma \in \text{Gal}(K/\mathbb{Q}_p) \). Hence it follows from Lemma 3.3 and (3.5) that
\[ \lambda^{2\sigma} c_k \equiv \pi^{(p-1)/2} b_k \pmod{\pi^{(p+1)/2}}. \]
Thus \( c_k \equiv 0 \pmod{\pi^{(p-5)/2}} \), as claimed. Similarly, we have \( \lambda^3 \frac{df_4}{f_4} \equiv \frac{df_1}{f_1} \), so \( \lambda^3 d_k \equiv \pi^{(p-1)/2} b_k \pmod{\pi^{(p+1)/2}} \). Furthermore, since \( \zeta^\sigma = \zeta^{-k} \), it follows from our choice of \( \lambda \) that \( \lambda^\sigma / \pi \equiv -k \pmod{\pi} \) for \( 1 \leq k \leq p - 1 \), so we get equation (3.7). ■

Lemma 3.5.
\[ - \sum_{k \in H_{a,b,c}} \frac{b_k}{k^3} \equiv F(q(a^ab^c)^3 + abcB_{p-3}) \pmod{\pi}, \]
where \( F \) is as in (3.6).

Proof. Let \( n = (p-1)/2 \). Define
\[ \Gamma_k(x_1, \ldots, x_n) = \det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \\ x_1^{n-1+k} & x_2^{n-1+k} & \ldots & x_n^{n-1+k} \end{bmatrix}. \]

Then an elementary linear algebra calculation using (3.6) gives
\[ \sum_{k \in H_{a,b,c}} \frac{b_k}{k^3} \equiv F \Gamma_3(H_{a,b,c}^{-1}) / \Gamma_0(H_{a,b,c}^{-1}) \pmod{\pi}. \]
Let \( S_i(x_1, \ldots, x_n) \) be the \( i \)-th symmetric function. Then
\[ \Gamma_3 = \Gamma_0(S_1^3 - 2S_1S_2 + S_3). \]
This can be proved by the usual method: the determinant vanishes if $x_i = x_j$ for $i \neq j$, or if there is a polynomial of degree $n+2$ vanishing on the $x_i$, and with no term of degree $n-1$, $n$, or $n+1$. Thus, if the roots of the polynomial are $x_1, \ldots, x_n, \alpha, \beta$, then

$$\alpha + \beta + S_1 = 0$$
$$S_2 + (\alpha + \beta)S_1 + \alpha\beta = 0$$
$$\alpha\beta S_1 + \alpha S_2 + \beta S_2 + S_3 = 0$$

Eliminating $\alpha$ and $\beta$ gives the condition $S_1^2 - 2S_1S_2 + S_3 = 0$. Now, we have

(3.8) \[ S_1(H_{a,b,c}^{-1}) = -q(a^b c^c) \]
(3.9) \[ S_2(H_{a,b,c}^{-1}) = 0 \]
(3.10) \[ S_3(H_{a,b,c}^{-1}) = -\frac{B_{p-3}}{3} (a^3 + b^3 + c^3) \equiv -abcB_{p-3}. \]

It is explained in McC, Lemma 5.24, how equation (3.8) follows from [Van20, 17]; equation (3.9) follows from parity considerations; and equation (3.10) follows from [Van20, 16], in exactly the same way as (3.8) follows from [Van20, 17].

We now define $p$-adic discs in $Y$, to which we apply Lemma 3.2. Let $X$ be the sub-affinoid of $Y$ defined by $|t| \leq 1$ in the wild case and by $|s| \leq |\pi_K|$ in the tame case. Let $E_w$ be the quadratic unramified extension of $K_w$. If $F_{a,b,c}$ is wild, $X$ is isomorphic to a union of two closed discs, which are defined over $K_w$ in split case and over $E_w$ in the non-split case. Furthermore, $T = t$ is a parameter on each disc. If $F_{a,b,c}$ is tame, then $X$ is isomorphic to a union of $p$ closed discs defined over $K_w$, and $T = s/\pi_K$ is a parameter on each disk. For proofs of these facts we refer the reader to McC (where $T = s'$ in the tame case). We denote by $Z$ be any of the discs that are components of $X$, with parameter $T$. We can write

$$f_i|X = C_i u_i(T) v_i(T^p) g_i(T^p), \quad i = 1, 2, 3, 4,$$

where $u_i$ and $v_i$ are unit power series with constant term 1 and integer coefficients, $u_i$ has no terms in $T^p$, and $g_i$ is a monic polynomial with integer coefficients. Furthermore, these conditions uniquely determine the $u_i$, $v_i$ and $g_i$. Then

(3.11) \[ \frac{df_i}{f_i} = \frac{du_i}{u_i} \pmod{Z}. \]

For a $p$-adic field $H$ we denote by $U_H[[T]]$ the power series in $O_H[[T]]$ which are congruent to 1 modulo the maximal ideal in $O_H[[T]]$.

**Theorem 3.6.** Let $Z$ be any of the discs that are components of $X$ and let $T$ be a parameter on $Z$. Then

(3.12) \[ u_i = 1 + \pi^{(p+3)/2 - i} D_i T + O(\pi^{(p+5)/2 - i} T), \]

where $|D_i| \leq 1$, $i = 1, 2, 3, 4$. Furthermore, $|D_1| = 1$, and under the hypotheses of Theorem 3.2, $|D_1| = 1$ and

$$\frac{D_4}{D_1} \equiv q(a^b c^c)^3 + abcB_{p-3}. \]

Finally, $u_i$ for $i = 1, 2, 3$ are defined over $E_w$, and

$$u_4 \equiv 1 + \pi^{(p-5)/2} D_4 T U_{E_w}[[T]] + \pi^{(p+1)/2} \pi_{L}^{-3} E D_4 T U_{E_w}[[T]],$$

where $E \in \mathbb{Z}/p\mathbb{Z}^\times$ is independent of the triple $(a, b, c)$.
Proof. In both wild and tame cases we have \( \omega_k \equiv \pi D dT \pmod{2} \) for all \( k \in H_{a,b,c} \), with \( D \in \mathbb{Z}/p\mathbb{Z}^\times \) independent of \( k \). This follows from our normalization (3.4), since in the tame case we have
\[
(3.13) \quad s = \pi T,
\]
and in the wild case it follows from \( \text{McC88}, (5.6) \), where it is shown that the expansion of \( s \) in terms of \( t \) on either of the discs in \( X \) is
\[
(3.14) \quad s^2 = -\frac{a^2b^2c^2}{ac} + \frac{2b}{ac}(t^p - t) + O(t^2).
\]
The statements about \( D_1 \) follow from (3.2), (3.13) (in the tame case) and (3.14) (in the wild case), since \( f_1 = x \). The statement about \( D_2 \) was proved in \( \text{McC88}, \text{Theorem 5.13} \). Although this theorem is stated only for the wild-split case, the consequence (3.12) is easily seen to hold in the other cases as well (the part of Theorem 5.13 specific to the wild-split case translates into the statement \(|D_2| = 1 \) in the current notation, and we do not need it here). The statements about \( D_3 \) and \( D_4 \) follow from Proposition 3.4, Lemma 3.5, and (3.11). The statement about the ratio \( D_4/D_1 \) follows from (3.5), the case \( i = 0 \) of (3.6), (3.7), and Lemma 3.5, taking note of the normalization (3.4) and the fact that \( ds \equiv \text{unit} \times \pi dT \pmod{\pi^2} \). The statements about the fields of definition follow from the fact that \( f_i \) is defined over \( K \) for \( i = 1, 2, 3 \) and \( f_4 \) is defined over \( L \), and that the discs \( Z \) are always defined over \( E_w \). The final statement follows from considerations of ramification theory. Locally, we have \( \eta_{p-3} = 1 + a\pi p-3 \) modulo \( p \)-th powers, so the (upper) conductor of \( L_w/K_w \) is 3. Now, it follows from the properties of the \( P_i \) that \( u_{4i-1} \equiv u_1 \) modulo \( \mathcal{O}_{F_w}[[T]] \times \mathcal{O}_{F_w}[[T^p]] \), and, since \( u_1 \equiv 1 + \pi^{(p+1)/2} T D_1 U_{F_w}[[T]] \), this implies the final statement with \( E \) such that \( (\sigma - 1)\pi L^{-3} \equiv E^{-1} \pmod{\pi L} \).

4. Computation of the Cassels pairing

Recall the local descent maps
\[
\delta_i = \iota_{P_i} \times \cdots \iota_{P_1} : J(K_w) \to (K_w^\times/K_w^{\times p})^i
\]
described in Section 3. We start by observing a couple of properties that follow from the choice of \( P_i \) made in Section 3. First, we have
\[
(4.1) \quad \iota_{P_i} \circ \lambda = \iota_{P_{i-1}}, \quad i = 2, 3, 4.
\]
Second, for \( i = 1, 2, 3 \) we have, from eigenspace considerations,
\[
(4.2) \quad \iota_{P_i}(J(K_w)(k)) \subset V(k-i+1).
\]
Let \( A \subset J(K_w) \) be the subgroup generated by divisors supported on the discs \( |T| \leq |\pi K| \) in \( X \). Let
\[
V[i,j] = \bigcup_{i \leq k \leq j} V(i).
\]
Note that \( V(i) = 0 \) for \( i > p \).

Proposition 4.1. We have
\[
(4.3) \quad \iota_{P_i}(A) \subset V[(p + 5)/2 - i, p], \quad i = 1, 2, 3.
\]
If \( F_{a,b,c} \) is wild split, we have
\[
(4.4) \quad \iota_{P_i}(J(K_w)) \subset V[(p + 1)/2 - i, p], \quad i = 1, 2, 3.
\]
If $F_{a,b,c}$ is wild non-split or tame, we have
\begin{equation}
\iota_{P_i}(J(K_w)) \subset V[(p+3)/2-i,p] \quad i = 1, 2, 3.
\end{equation}

Furthermore, in the case $i = 1$, the containments in (4.3) and (4.5) are equalities.

**Proof.** The containments in (4.3) follow immediately from Proposition 3.6, as does the claim that the inclusion is equality in the case $i = 1$. Now Faddeev [Fad61] proved that

$$
\text{im } \iota_{P_i} = \begin{cases} 
V[(p-1)/2] \cup V[(p+3)/2,p] & F_{a,b,c} \text{ is wild split} \\
V[(p+3)/2,p] & \text{otherwise}
\end{cases}
$$

This implies the statements (4.4) and (4.3) in the case $i = 1$, and also that the image of $A$ in $J(K_w)/\lambda J(K_w)$ has codimension 1, and the eigenvalue of the quotient is $\kappa^{(p-1)/2}$ in the wild non-split case and $\kappa^{(p+3)/2}$ in the other cases. The remaining statements now follow from (4.1), (4.2), and (4.3).

**Proposition 4.2.** If $F_{a,b,c}$ is wild non-split or tame, then
\begin{equation}
\delta_2(J(K_w)) = V[(p+1)/2,p]^2.
\end{equation}

**Proof.** From (4.1) with $i = 2$ we have
\begin{equation}
\text{im } \delta_2 \cap (K^\times/K^\times p)^2 \times 1 = \text{im } \delta_1 \times 1 = V[(p+1)/2,p] \times 1
\end{equation}

Furthermore, given $u \in V[(p+3)/2,p]$, we can find $a \in A$ such that $\iota_{P_2}(a) = u$, and $\iota_{P_2}(a) \in V[(p+1)/2,p] \subset \text{im } \iota_{P_1}$. Thus, modifying $a$ by $\lambda J(K_w)$, we can choose it so that $\iota_{P_2}(a) = 1$. Hence
\begin{equation}
\text{im } \delta_2 \supset 1 \times V[(p+3)/2,p].
\end{equation}

Now, it follows from local duality that $\text{im } \delta_2$ must be maximal isotropic with respect to the cup product pairing on $(K^\times/K^\times p)^2 = H^1(K,J[\lambda^2])$ induced by the Weil pairing on $J[\lambda^2]$. Since $\lambda^2 = 1$, the Weil pairing is skew symmetric. Thus the pairing on $(K^\times/K^\times p)^2$ is a non-zero multiple of $((a_1,b_1),(a_2,b_2)) \mapsto <(a_1,a_2)_w(b_1,a_2)_w^{-1}>$, where $<,>$ denotes the Hilbert symbol at $w$. The only maximal isotropic subgroup satisfying (4.6) and (4.7) is the one given in the statement of the proposition.

Define a subspace $V_{\text{global}} \subset V$ by
\begin{equation}
V_{\text{global}} = \bigoplus_{\text{i even, } 2 \leq i \leq p-3} V(i).
\end{equation}

**Proposition 4.3.** Assume $p \geq 11$ and $F_{a,b,c}$ is wild non-split or tame. There exists a point $x \in A$ such that $\iota_{P_2}(x)$ generates $V((p+5)/2)$ and $\iota_{P_i}(x) \in V_{\text{global}}$ for $i = 2, 3$.

**Proof.** It follows from Proposition 4.1 that $A$, regarded as a $Z_p[\xi]$-submodule of $J(K_w)$, has codimension at most 1. Hence $\delta_3(A)$ has codimension at most 3 as a $F_p$-vector space in $\delta_3(J(K_w))$. By Proposition 4.1 we can choose $x \in A$ such that $\iota_{P_3}(x)$ generates $V((p+5)/2)$. This condition leaves freedom to modify $x$ by anything in $\lambda J(K_w)$, which would change $\delta_3(x)$ by anything in $\text{im } \delta_2$. Thus, modifying $x$ as needed, we can ensure that $\iota_{P_2}(x) \in V_{\text{global}}$, $i = 2, 3$. The number of degrees of freedom in performing this modification is equal to the dimension of $\text{im } \delta_2 \cap V_{\text{global}}$, which is at least 4 if $p \geq 11$, by Proposition 4.2. Thus we can ensure that $x$ remains in $A$ when making the modification.
Computation of Cassels pairing for Theorem 1.2. We show here that the pairing \( \langle \cdot, \cdot \rangle \) is trivial under the hypotheses of Theorem 1.2. In the next section we explain how this implies the theorem.

Denote by \( \ell_i : S_{\lambda_i} \to J(K_u) / \lambda J(K_u) \) the localization map. We claim that, under the hypotheses of Theorem 1.2, \( \iota_{p_i}(\ell_1(S_{\lambda_i})) \subset V((p+3)/2, p) \) or \( \iota_{p_i}(\ell_1(S_{\lambda_i})) \subset V((p+1)/2) \cup V((p+5)/2, p) \). Now, \( V(i) \) pairs non trivially with \( V(j) \) under Hilbert pairing if and only if \( i + j \equiv p \pmod{p-1} \). Thus, it follows from our claim and from (4.4) that \( \iota_{p_i}(\ell_3(J(K_u))) \) pairs trivially with \( \iota_{p_i}(\ell_1(J(K_u))) \).

To see the claim, note that if hypothesis (i) of Theorem 1.2 is satisfied, namely that \( F_{a,b,c} \) is wild-split and \( p \equiv 3 \pmod{4} \), then, by (4.3), \( \ell_1(S_{\lambda_i}) \subset V((p-1)/2) \cup V((p+3)/2, p) \). Furthermore, we can eliminate \( V((p-1)/2) \) as a possibility, because \( \ell_w \) factors through \( H^1(K(p) / K, \mu_p) \). Since \( (p-1)/2 \) is odd, it follows from (2.5) that \( \ell_w \) can have nontrivial image in \( V((p-1)/2) \) only if \( C((p-1)/2) \) is nontrivial, which would imply \( p \mid B_{(p+1)/2} \). This never happens if \( p \equiv 3 \pmod{4} \).

If hypothesis (ii) of Theorem 1.2 is satisfied, namely that \( F_{a,b,c} \) is wild non-split or tame and the image of \( U \) in either \( V((p+1)/2) \) or \( V((p+3)/2) \) is trivial, then the claim follows immediately from (4.3).

Proof of Theorem 1.4. We exhibit \( a \in S_{\lambda} \) and \( b \in S_{\lambda} \) which pair nontrivially under the Cassels pairing.

Recall that \( S_{\lambda} \subset H^1(K(p) / K, J(K)) \), and since \( p \) is regular, this latter group is isomorphic to \( \bigoplus O^\times_K / O^\times_K(p) \) for \( i \leq 3 \). The Selmer group is the subgroup obtained by imposing the local conditions at \( w \). Since \( (p+1)/2 \) is even, we can choose an element \( b \in O^\times_K / O^\times_K(p) \) which generates \( V((p+1)/2) \), and \( b \) satisfies the local condition by Proposition 4.1, so \( b \in S_{\lambda} \).

As for \( a \), by Proposition 4.3, (4.3) there exists \( a_w = (a_{w,1}, a_{w,2}, a_{w,3}) = \delta_3(x), x \in A \), such that \( a_{w,1} \) generates \( V((p+5)/2) \) and \( a_{w,2}, a_{w,3} \in V_{\text{global}} \). Choose \( a_i \in O^\times_K / O^\times_K(p) \) specializing to \( a_{w,i} \) for \( i = 1, 2, 3 \) and define \( a \in S_{\lambda} \) by \( a = (a_1, a_2, a_3) \).

Now, by Lemma 2.3, \( \lambda_2 a = a_1 \in V((p+5)/2) \) is the norm of a global unit \( \epsilon \) in \( O^\times_L \), and by Proposition 2.4, \( s \) the Cassels pairing of \( a \) and \( b \) is the Hilbert pairing \( \langle c_w, b_w \rangle \), where

\[
\langle c_w, b_w \rangle = \iota_{p_i}(x)^{-1} N^\epsilon \in L^\times_w / L_w^\times p.
\]

To prove that the pairing is nontrivial, it suffices to show that \( c_w \) is not a \( p \)-th power, and for that it suffices to show that its image in \( L^\times_w / L_w^\times p \) is nontrivial. We may assume without loss of generality that \( c_w, N^\epsilon, \) and \( \iota_{p_i}(x) \) are eigenvectors for a lift \( \Delta \) of \( \Delta = \text{Gal}(K_w / \mathbb{Q}_p) \) to \( \text{Gal}(L_w / \mathbb{Q}_p) \), with eigenvalue \( \kappa^{(p-1)/2} \).

Since \( x \in A \), we may choose a divisor \( D \) supported on \( |T| < |\pi| \) such that \( a_{w,i} = f_i(D), 1 \leq i \leq 3 \). Since \( D \) is supported on \( |T| < |\pi| \), we have

\[
\langle u, v \rangle = f_4(D) \equiv u_4(D) \pmod{L_w^\times p(1 + \pi p O_{L_w})}.
\]

From the galois properties of the \( P_i \), we have

\[
\langle u, v \rangle \in (L_w^\times / L_w^\times p)((p+1)/2)), \quad \langle \sigma - 1 u, v \rangle = v,
\]

where \( v \) is the image in \( L_w^\times / L_w^\times p \) of a generator of \( V((p+5)/2) \). Since \( p > 19 \), \( (p+5)/2 \) is less than \( p-3 \), and thus \( \sigma - 1 \neq 0 \). Thus the subspace of \( L_w^\times / L_w^\times p \) satisfying the conditions (1.9) is two-dimensional, with generators \( u_1 \) and \( u_2 \), where \( u_1 \) is the image of a generator of \( V((p-1)/2) \) with expansion \( u_1 = 1 + \pi^{(p-1)/2} + O(\pi^{(p-1)/2}) \), and \( u_2 \in (L_w^\times / L_w^\times p)((p-1)/2) \) has expansion \( u_2 = 1 + \pi_p^{(p+5)/2} \pi_{L_p}^{-3} + O(\pi^{(p+5)/2}) \).
The argument is standard: Since
\[ N \]
the groups
\[ N \]
there is an integer
\[ N \]
Then
\[ \lambda \]
z
only if for every non-zero integer
\[ n \]
Lemma 5.2. \[ X \]
x
infinitely divisible. Now if
denote the quotient group
\[ T \]
Tate pairing induces a perfect pairing
\[ n \]
for any positive integer
\[ n \]
Thus, from (4.10), we see that by varying \( t \) appropriately we may ensure that \( v \), and hence \( c_{uv} \), varies in \( L_x^\infty / L_x^{\infty p} \), and, in particular, takes on nonzero values. Hence there exists a choice of \( t \) such that the pairing is nontrivial for the curve \( F_{a',b',c'} \).
However, this curve is isomorphic to \( F_{a,b,c} \), and hence the pairing must be nontrivial in that case as well. \( \blacksquare \)

5. Shafarevich-Tate groups

The proofs of Theorem 1.2 and Theorem 1.5 follow from the computations of the Cassels-Tate pairing by means of the following proposition.

**Proposition 5.1.** For all positive integers \( m \) and \( n \), the restriction of the Cassels-Tate pairing induces a perfect pairing
\[
(\text{III}[\lambda^m]/(\text{III}[\lambda^{n+m}])) \times (\text{III}[\lambda^n]/(\lambda^m\text{III}[\lambda^{n+m}])) \rightarrow \mathbb{Q}/\mathbb{Z}.
\]

Let \( \text{III}_{div} \) denote the maximal divisible subgroup of \( \text{III} \), i.e. \( x \in \text{III}_{div} \) if and only if for every non-zero integer \( n \) there exists \( y \in \text{III} \) such that \( x = ny \). Let \( \text{III}_{red} \) denote the quotient group \( \text{III}/\text{III}_{div} \). Note that:

**Lemma 5.2.** \( \text{III}_{div} \) is a divisible group in the usual sense, i.e. for every non-zero \( n \in \mathbb{Z} \) multiplication by \( n \) on \( \text{III}_{div} \) is surjective.

**Proof.** The argument is standard: Since \( \text{III}[m] \) is finite for all non-zero \( m \in \mathbb{Z} \), the groups \( N\text{III}[Nm], N > 0 \), stabilize for sufficiently large \( N \). Thus for every \( m \) there is an integer \( N(m) \) such that an element of \( \text{III}[m] \) is divisible by \( N(m) \) it is infinitely divisible. Now if \( x \in \text{III}_{div}[m] \) and \( n > 0 \), choose \( y \in \text{III}[Nm(nm)nm] \) such that \( N(nm)ny = x \). Then \( y' = N(nm)y \) is in \( \text{III}_{div}[nm] \) and \( ny' = x \). \( \blacksquare \)

Note that since \( \zeta \) is an automorphism of \( \text{III} \) it preserves \( \text{III}_{div} \), and hence so does \( \mathbb{Z}[\zeta] \). Furthermore, since \( \lambda^{p-1} \) is a unit times \( p \) in \( \mathbb{Z}[\zeta] \), \( \text{III}_{div} \) is divisible by \( \lambda^n \) for any positive \( n \).

**Lemma 5.3.** The exact sequence
\[
0 \rightarrow \text{III}_{div} \rightarrow \text{III} \rightarrow \text{III}_{red} \rightarrow 0
\]
induces by restriction an exact sequence
\[
0 \rightarrow \text{III}_{div}[\lambda^n] \rightarrow \text{III}[\lambda^n] \rightarrow \text{III}_{red}[\lambda^n] \rightarrow 0
\]
for any positive integer \( n \).

**Proof.** Only the surjectivity is in question. Let \( x \in \text{III}_{red}[\lambda^n] \). Lift \( x \) to \( y \in \text{III} \). Then \( \lambda^ny = z \in \text{III}_{div} \). By Lemma 5.2, we can find \( w \in \text{III}_{div} \) such that \( \lambda^nw = z = \lambda^ny \). But then \( y - w \in \text{III}[\lambda^n] \) and \( y - w \) reduces to \( x \) in \( \text{III}_{red} \). \( \blacksquare \)
It is well-known that $\Sha[p^{\infty}]$ is a finite group and that the Cassels-Tate pairing induces a perfect pairing

$$[\cdot, \cdot] : \Sha[p^{\infty}] \times \Sha[p^{\infty}] \rightarrow \mathbb{Q}/\mathbb{Z}. $$

We now have the following lemma:

**Lemma 5.4.** The annihilator of $\Sha[p^{\infty}]$ with respect to the latter pairing equals $\Sha[p^{\infty}]$, for all positive integers $m$.

**Proof.** It is clear from the definition of the pairing given in [McC88], for example, and from the functoriality properties of the Weil pairing, that $[\mathfrak{a}, \mathfrak{a}'] = [\mathfrak{a}, \zeta^{-1} \mathfrak{a}']$. Hence, if $\hat{\lambda} = \zeta^{-1} - 1$, then $\Sha[p^{\infty}]$ annihilates $\Sha[p^{\infty}]$. Since $\hat{\lambda}/\lambda$ is a unit in $\mathbb{Z}[\zeta]$, we have $\hat{\lambda} \Sha[p^{\infty}] = \Sha[p^{\infty}]$. So the kernel $H$ on the right factor of the restricted pairing

$$\Sha[p^{\infty}] \rightarrow \mathbb{Q}/\mathbb{Z},$$

contains $\Sha[p^{\infty}]$. Note that the kernel on the left factor of the latter pairing is trivial. Therefore, the cardinalities of $\Sha[p^{\infty}]$ and $\Sha[p^{\infty}]/H$ are equal. But

$$|\Sha[p^{\infty}]| = |\Sha[p^{\infty}]| \cdot |\Sha[p^{\infty}]|;$$

hence $H = \Sha[p^{\infty}]$. $\blacksquare$

**Lemma 5.5.** For all positive integers $m$ and $n$, the restriction of the Cassels-Tate pairing induces a perfect pairing

$$(\Sha[p^{\infty}])/(\Sha[p^{\infty}] \Sha[p^{\infty}]) \times (\Sha[p^{\infty}])/(\Sha[p^{\infty}] \Sha[p^{\infty}]) \rightarrow \mathbb{Q}/\mathbb{Z}. $$

**Proof.** By Lemma 5.4, the annihilator of $\Sha[p^{\infty}]$ in $\Sha[p^{\infty}]$ equals

$$\Sha[p^{\infty}] \cap \Sha[p^{\infty}] = \Sha[p^{\infty}] \Sha[p^{\infty}],$$

and the assertion follows. $\blacksquare$

**Proof of Proposition 5.1.** By Lemma 5.5, it suffices to show that for all $m$ and $n$ the groups $\Sha[p^{\infty}]/(\Sha[p^{\infty})]$ and $\Sha[p^{\infty}]/(\Sha[p^{\infty}])$ are isomorphic. By Lemma 5.2, we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & \Sha[p^{\infty}] & \rightarrow & \Sha[p^{\infty}] & \rightarrow & \Sha[p^{\infty}] & \rightarrow & 0 \\
\downarrow \alpha \quad & \downarrow \beta \quad & \downarrow \gamma \quad & \downarrow \lambda \\
0 & \rightarrow & \Sha[p^{\infty}] & \rightarrow & \Sha[p^{\infty}] & \rightarrow & \Sha[p^{\infty}] & \rightarrow & 0
\end{array}
$$

where the horizontal sequences are exact. By the snake lemma, we get an exact sequence

$$0 \rightarrow \text{Ker}(\alpha) \rightarrow \text{Ker}(\beta) \rightarrow \text{Ker}(\gamma) \rightarrow \text{Coker}(\alpha) \rightarrow \text{Coker}(\beta) \rightarrow \text{Coker}(\gamma) \rightarrow 0.$$ 

By Lemma 5.2, we have $\text{Coker}(\alpha) = 0$, hence $\text{Coker}(\gamma)$ is isomorphic to $\text{Coker}(\beta)$, and this completes the proof. $\blacksquare$

**Proof of Theorem 1.2.** By the structure theorem for torsion modules over Dedekind domains we have a $\mathbb{Z}[\zeta]$-module decomposition

$$\Sha[\zeta] \cong (\mathbb{Z}[\zeta]/(\lambda))^t_1 \oplus (\mathbb{Z}[\zeta]/(\lambda^2))^t_2 \oplus (\mathbb{Z}[\zeta]/(\lambda^3))^t_3.$$
where \( t_1, t_2 \) and \( t_3 \) are non-negative integers. The computations in the previous section show that the pairing (obtained by restricting the Cassels-Tate pairing)

\[
\text{III}[\lambda^2] \times \text{III}[\lambda] \to \mathbb{Q}/\mathbb{Z}
\]

is trivial. By Proposition 5.1 (for \( m = 2 \) and \( n = 1 \)), we get that the groups \( \text{III}[\lambda^2]/(\lambda^2 \text{III}[\lambda^3]) \) and \( \text{III}[\lambda]/(\lambda^2 \text{III}[\lambda^3]) \) are both trivial. But then

\[
(\mathbb{Z}[\xi]/(\lambda))^t_1 \oplus (\mathbb{Z}[\xi]/(\lambda))^t_2 \oplus (\mathbb{Z}[\xi]/(\lambda))^t_3 \simeq \text{III}[\lambda] = \lambda^2 \text{III}[\lambda^3] \simeq (\mathbb{Z}[\xi]/(\lambda))^t_3
\]

so \( t_1 = t_2 = 0 \), which proves the claim. \( \Box \)

**Proof of Theorem 1.3.** Let

\[
\text{III}[\lambda^3] \simeq (\mathbb{Z}[\xi]/(\lambda))^a \oplus (\mathbb{Z}[\xi]/(\lambda^2))^b \oplus (\mathbb{Z}[\xi]/(\lambda^3))^c \oplus (\mathbb{Z}[\xi]/(\lambda^4))^d.
\]

If we show that \( d = 0 \), then \( \lambda^3 \) annihilates III[\( \lambda^3 \)], therefore III[\( \lambda^4 \)] = III[\( \lambda^3 \)]. By induction, this implies III[\( p^\infty \)] = III[\( \lambda^\infty \)] = III[\( \lambda^3 \)]. So assume \( d \geq 1 \). Since the Cassels-Tate pairing on \( \text{III}[\lambda^3] \times \text{III}[\lambda] \) is non-trivial, Proposition 5.1 implies that \( \text{III}[\lambda^3]/(\lambda^2 \text{III}[\lambda^3]) \) has dimension \( \geq 2 \) over \( \mathbb{F}_p \). Now

\[
\lambda \text{III}[\lambda^4] \simeq (\mathbb{Z}[\xi]/(\lambda))^b \oplus (\mathbb{Z}[\xi]/(\lambda^2))^c \oplus (\mathbb{Z}[\xi]/(\lambda^3))^d.
\]

Counting \( \mathbb{F}_p \)-dimensions, we get \( 6 - (b + 2c + 3d) \geq 2 \), therefore \( b + 2c + 3d \leq 4 \). This implies \( d = 1 \) and \( c = 0 \). Therefore,

\[
\text{III}[\lambda^4] \simeq (\mathbb{Z}[\xi]/(\lambda))^a \oplus (\mathbb{Z}[\xi]/(\lambda^2))^b \oplus (\mathbb{Z}[\xi]/(\lambda^3))^c \oplus (\mathbb{Z}[\xi]/(\lambda^4))^d.
\]

This implies that

\[
(\mathbb{Z}[\xi]/(\lambda))^2 = \lambda^2(\mathbb{Z}[\xi]/(\lambda^3))^2 \simeq \lambda^2 \text{III}[\lambda^3] \subseteq \lambda^2 \text{III}[\lambda^4] \simeq \mathbb{Z}[\xi]/(\lambda^2),
\]

a contradiction. \( \Box \)

### 6. Tame Reduction

Although it is not strictly necessary for Theorem 1.4, we take the opportunity to prove a general lemma on tame reduction, since it clears up some confusion in the literature. In Lim95, an attempt was made to improve the result of McC88 on the existence of non-trivial elements in III[\( \lambda \)] in the wild split case, under the additional hypothesis that the Jacobian of the Fermat curve in question is non-simple. However, as Lemma 6.1 shows, non-simple Jacobian and wild split reduction over \( \mathbb{Z}_p[\xi] \) are incompatible properties, so the Mordell-Weil rank estimates given in the last section of Lim95 are incorrect. As far as we can tell, the problem lies in the use of the function \( q(x) \) which computes the reduction type (see the introduction). Here as well as in McC88, \( q \) is evaluated on triples \( (a, b, c) \) of integers such that \( 0 < a, b, a + b < p \) and \( a + b + c = 0 \). In Lim95, however, \( q \) is evaluated on triples \( (a, b, c) \) such that \( 0 < a, b, a + b < p \) and \( a + b + c = p \). While it does not make any difference which of the two types of triples one chooses to define the curve \( F_{a,b,c} \), it does make a difference which type of triple one uses to evaluate \( q \) and hence the reduction type. We have the following lemma:

**Lemma 6.1.** Let \( (a, b, c) \) be such that \( J_{a,b,c} \) is non-simple. Then \( F_{a,b,c} \) has tame reduction over \( \mathbb{Z}_p[\xi] \).
Proof. By \cite{KR78}, \(J_{a,b,c}\) is non-simple if and only if \(p \equiv 1 \pmod{3}\) and \(F_{a,b,c}\) is isomorphic to \(F_{r,-(r+1)}\), where \(r^2 + r + 1 = 0\) in \(\mathbb{F}_p\). By definition of \(q(x)\), it therefore suffices to show that \((r + 1)^{(r+1)(p-1)} - r^{(p-1)} \equiv 0 \pmod{p^2}\). Since 6 divides \(p - 1\), it suffices to show that
\[
(r + 1)^6(r+1) - r^{6r} \equiv 0 \pmod{p^2}.
\]
Let \(k\) be an integer such that \(r^2 + r + 1 = pk\). Then \((r + 1)^2 = pk + r\). Therefore,
\[
(r + 1)^6 = (pk + r)^3 \equiv (r^3 + 3r^2pk) \pmod{p^2}.
\]
Hence, \((r+1)^6(r+1) \equiv (3^3 + 3r^2pk)^{r+1} \equiv (3^3r + 3r^2pk + 1)^{r+1} \pmod{p^2}\). Now note that, since \(r\) is a cube root of unity modulo \(p\), \(r^{3r^2}r^2(r+1) \equiv -r \pmod{p}\), so \(3r^2pk(r+1)r^{3r} \equiv -3rpk \pmod{p^2}\). Hence, \((r+1)^6(r+1) \equiv (3^3r + 3r^2pk - 3rpk) \pmod{p^2}\). Therefore,
\[
(r + 1)^6(r+1) - r^{6r} \equiv (r^3(r^3 - 3r) - 3rpk) \pmod{p^2}.
\]
Since \(r^3 = pk(r-1) + 1\), we get \(r^{3r} \equiv (rpk(r-1) + 1) \pmod{p^2}\), so \(r^3 - 3r^3 \equiv -pk(r-1)^2 \pmod{p^2}\). Hence,
\[
(r + 1)^6(r+1) - r^{6r} \equiv -pk(r^3(r-1)^2 + 3r) \pmod{p^2}.
\]
Since \(r^3(r-1)^2 + 3r \equiv 0 \pmod{p}\), this proves the proposition. ■

Proof of Theorem 1.6. By Lemma 6.1, the reduction is tame in this case. By Theorem 1.4 and Proposition 5.1, the \(\mathbb{F}_p\)-dimension of \(\mathbb{H}[\lambda]/(\lambda^3\mathbb{H}[\lambda^4])\) is \(\geq 2\). In particular, the \(\mathbb{F}_p\)-dimension of \(\mathbb{H}[\lambda]\) is \(\geq 2\). Since \(p\) is regular, the results of Faddeev \cite{Fad61} show that the Selmer group \(S\) is 3-dimensional over \(\mathbb{F}_p\). On the other hand, Gross and Rohrlich \cite{GR78} have shown that the Mordell-Weil rank of \(J_{7,1,-8}\) over \(\mathbb{Q}\) is non-zero. Therefore, the rank equals 1 and \(\mathbb{H}[\lambda]\) is 2-dimensional over \(\mathbb{F}_p\). By Theorem 1.2, it follows that \(\mathbb{H}[\lambda^3]\) has rank 2 over \(\mathbb{Z}[\zeta]/(\lambda^3)\). Theorem 1.3 then implies that \(\mathbb{H}[p^\infty] \simeq (\mathbb{Z}[\zeta]/(\lambda^3))^2\). The statement about quadratic points on \(F_{7,1,-8}\) and on the Fermat curve \(X^3 + Y^3 + Z^9 = 0\) follows immediately from Corollary 2.2 and Theorem 1.3 of \cite{Zeo02}. ■

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