Verification of Process Rewrite Systems in normal form

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Abstract

We consider the problem of model–checking for Process Rewrite Systems (PRSs) in normal form. In a PRS in normal form every rewrite rule either only deals with procedure calls and procedure termination, possibly with value return, (this kind of rules allows to capture Pushdown Processes), or only deals with dynamic activation of processes and synchronization (this kind of rules allows to capture Petri Nets). The model-checking problem for PRSs and action-based linear temporal logic (ALTL) is undecidable. However, decidability of model–checking for PRSs and some interesting fragment of ALTL remains an open question. In this paper we state decidability results concerning generalized acceptance properties about infinite derivations (infinite term rewritings) in PRSs in normal form. As a consequence, we obtain decidability of the model-checking (restricted to infinite runs) for PRSs in normal form and a meaningful fragment of ALTL.

1 Introduction

Automatic verification of systems is nowadays one of the most investigated topics. A major difficulty to face when considering this problem comes to the fact that, reasoning about systems in general may require dealing with infinite state models. For instance, software systems may introduce infinite states both manipulating data ranging over infinite domains, and having unbounded control structures such as recursive procedure calls and/or dynamic creation of concurrent processes (e.g. multi–treading). Many different formalisms have been proposed for the description of infinite state systems. Among the most popular are the well known formalisms of Context Free Processes, Pushdown Processes, Petri Nets, and Process Algebras. The first two are models of sequential computation, whereas Petri Nets and Process Algebra explicitly take into account concurrency. The model checking problem for these infinite state formalisms have been studied in the literature. As far as Context Free Processes and Pushdown Automata are concerned (see [1, 4, 6, 8, 13, 12, 16]), decidability of the modal $\mu$–calculus, the most powerful of the modal and temporal logics
used for verification, has been established (e.g. see [6, 13]). In [11, 10], model checking for Petri nets has been studied. The branching temporal logic as well as the state-based linear temporal logic are undecidable even for restricted logics. Fortunately, the model checking for action-based linear temporal logic (ALTL) is decidable.

Verification of formalisms which accommodate both parallelism and recursion is a challenging problem. To formally study this kind of systems, recently the formal framework of Process Rewrite Systems (PRSs) has been introduced [14]. This framework, which is based on term rewriting, subsumes many common infinite states models such as Pushdown Systems, Petri Nets, Process Algebra, etc. The decidability results already known in the literature for the general framework of PRSs concerns reachability analysis. However, the model checking of action-based temporal logic becomes undecidable. It remains undecidable even for restricted models such as those presented in [3].

In this paper we extend the known decidability results, for a relevant syntactic fragment of PRSs, to properties of infinite derivations, thus allowing for automatic verification of some interesting classes of action-based linear time properties. The fragment we consider is that of PRSs in normal form, where every rewrite rule either only deals with procedure calls and procedure termination, possibly with value return, (this kind of rules allows to capture Pushdown Processes), or only deals with dynamic activation of processes and synchronization (this kind of rules allows to capture Petri Nets).

Our result extends our previous result established in [5], and regards the decidability of two problems: the first (resp., the second) concerns generalized acceptance properties of finite derivations (resp., infinite derivations) in PRSs in normal form. As a consequence we obtain decidability of the model-checking (restricted to infinite executions) for PRSs in normal form and a meaningful ALTL fragment.

The rest of the paper is structured as follows. In Section 2, we recall the framework of Process Rewrite Systems, we summarize some decidability results for reachability problems, and our previous result for PRSs in normal form. In Section 3, it is shown how our decidability results about generalized acceptance properties of infinite derivations in PRSs in normal form can be used in model-checking for a meaningful ALTL fragment. In Section 4, we prove decidability of the two problems about finite and infinite derivations in PRSs in normal form, mentioned above. Appendix contains detailed proof of our results.

2 Process Rewrite Systems

In this section we recall the framework of Process Rewrite Systems (PRSs). We also recall the notion of Büchi Rewrite System (BRS) introduced in [5] to prove decidability of the model–checking problem for some classes of linear time properties and PRSs in normal form. We conclude this section by summarizing some decidability results on PRSs, known in the literature, that will be exploited in further sections of the paper.
2.1 Process Rewrite Systems and Büchi Rewrite Systems

In this subsection we recall the notion of Process Rewrite System, as introduced in [14]. The idea is that a process (and its current state) is described by a term. The behavior of a process is given by rewriting the corresponding term by means of a finite set of rewrite rules.

**Definition 2.1 (Process Term).** Let $\text{Var}$ be a finite set of process variables. The set $T$ of process terms over $\text{Var}$ is inductively defined as follows:

- $\text{Var} \subseteq T$
- $\varepsilon \in T$
- $t_1 \parallel t_2 \in T$, for all $t_1, t_2 \in T$
- $X.(t) \in T$, for all $X \in \text{Var}$ and $t \in T$

where $\varepsilon$ denotes the empty term, “$\parallel$” denotes parallel composition, and “$(.)$” denotes sequential composition\(^1\).

We denote by $T_{\text{SEQ}}$ the subset of terms in $T$ devoid of any occurrence of parallel composition operator, and by $T_{\text{PAR}}$ the subset of terms in $T$ devoid of any occurrence of the sequential composition operator. Notice that we have $T_{\text{PAR}} \cap T_{\text{SEQ}} = \text{Var} \cup \{\varepsilon\}$.

In the rest of the paper we only consider process terms modulo commutativity and associativity of “$\parallel$”, moreover $\varepsilon$ will act as the identity for both parallel and sequential composition. Therefore, we introduce the relation $\approx_T$, which is the smallest equivalence relation on $T$ such that for all $t_1, t_2, t_3 \in T$ and $X \in \text{Var}$:

- $t_1 \parallel t_2 \approx_T t_2 \parallel t_1$, $t_1 \parallel (t_2 \parallel t_3) \approx_T (t_1 \parallel t_2) \parallel t_3$, and $t_1 \parallel \varepsilon \approx_T t_1$.
- $X.(\varepsilon) \approx_T X$, and if $t_1 \approx_T t_2$, then $X.(t_1) \approx_T X.(t_2)$.

In the paper, we always confuse terms and their equivalence classes (w.r.t. $\approx_T$). In particular, $t_1 = t_2$ (resp., $t_1 \neq t_2$) will be used to mean that $t_1$ is equivalent (resp., not equivalent) to $t_2$.

**Definition 2.2 (Process Rewrite System).** A Process Rewrite System (or PRS, or Rewrite System) over the alphabet $\Sigma$ and the set of process variables $\text{Var}$ is a finite set of rewrite rules $\mathcal{R} \subseteq T \times \Sigma \times T$ of the form $t \xrightarrow{a} t'$, where $t \neq \varepsilon$ and $t'$ are terms in $T$, and $a \in \Sigma$.

The semantics of a PRS $\mathcal{R}$ is given by a Labelled Transition System $\langle T, \Sigma, \rightarrow \rangle$, where the set of states is the set of terms $T$ of $\mathcal{R}$, the set of actions is the alphabet $\Sigma$ of $\mathcal{R}$, and

\(^1\) [14] also allows terms of the form $t_1.(t_2)$, where $t_1$ is a parallel composition of variables. In the current context this generalization is not relevant.
the transition relation \( \rightarrow \subseteq T \times \Sigma \times T \) is the smallest relation satisfying the following inference rules:

\[
\frac{t \xrightarrow{a} t'}{(t \xrightarrow{a} t') \in \mathcal{R}} \quad \frac{t_1 \xrightarrow{a} t_1'}{t_1 \parallel t \xrightarrow{a} t_1' \parallel t} \quad \forall t \in T \quad \frac{t_1 \xrightarrow{a} t_1'}{X.(t_1) \xrightarrow{a} X.(t_1')} \quad \forall X \in \text{Var}
\]

For a PRS \( \mathcal{R} \) with set of terms \( T \) and LTS \( \langle T, \Sigma, \rightarrow \rangle \), a path in \( \mathcal{R} \) from \( t \in T \) is a path in \( \langle T, \Sigma, \rightarrow \rangle \) from \( t \), i.e. a (finite or infinite) sequence of LTS edges \( t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \xrightarrow{a_2} \ldots \) such that \( t_0 = t \) and \( t_j \xrightarrow{a_j} t_{j+1} \in \mathcal{R} \) for any \( j \). A run in \( \mathcal{R} \) from \( t \) is a maximal path from \( t \), i.e. a path from \( t \) which is either infinite or has the form \( t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} \ldots \xrightarrow{a_{n-1}} t_n \) and there is no edge \( t_n \xrightarrow{a_n} t' \in \mathcal{R} \), for any \( a_n \in \Sigma \) and \( t' \in T \). We write \( \text{run}_{\mathcal{R}}(t) \) (resp., \( \text{run}_{\mathcal{R}, \infty}(t) \)) to refer to the set of runs (resp., infinite runs) in \( \mathcal{R} \) from \( t \), and \( \text{run}_\mathcal{R}(\mathcal{R}) \) to refer to the set of all the runs in \( \mathcal{R} \).

The LTS semantics induces, for a rule \( r \in \mathcal{R} \), the following notion of one-step derivation by \( r \). The one-step derivation by \( r \) relation, \( \xrightarrow{r} \), is the least relation such that:

- \( t \xrightarrow{r} t' \), for \( r = t \xrightarrow{a} t' \)
- \( t_1 \parallel t \xrightarrow{r} t_2 \parallel t \), if \( t_1 \xrightarrow{r} t_2 \) and \( t \in T \)
- \( X.(t_1) \xrightarrow{r} X.(t_2) \), if \( t_1 \xrightarrow{r} t_2 \) and \( X \in \text{Var} \)

A finite derivation in \( \mathcal{R} \) from a term \( t \) to a term \( t' \) (through a finite sequence \( \sigma = r_1 r_2 \ldots r_n \) of rules in \( \mathcal{R} \)), is a sequence \( d \) of one-step derivations \( t_0 \xrightarrow{r_0} t_1 \xrightarrow{r_1} t_2 \xrightarrow{r_2} \ldots \xrightarrow{r_{n-1}} t_n \), with \( t_0 = t \), \( t_n = t' \) and \( t_i \xrightarrow{r_{i+1}} t_{i+1} \) for all \( i = 0, \ldots, n-1 \). The derivation \( d \) is a \( n \)-step derivation (or a derivation of length \( n \)), and for succinctness is denoted by \( t \xrightarrow{\sigma}^* t' \). Moreover, we say that \( t' \) is reachable in \( \mathcal{R} \) from term \( t \) (through derivation \( d \)). If \( \sigma \) is empty, we say that \( d \) is a null derivation.

A infinite derivation in \( \mathcal{R} \) from a term \( t \) (through an infinite sequence \( \sigma = r_1 r_2 \ldots \) of rules in \( \mathcal{R} \)), is an infinite sequence of one step derivations \( t_0 \xrightarrow{r_1} t_1 \xrightarrow{r_2} t_2 \xrightarrow{r_2} \ldots \) such that \( t_0 = t \) and \( t_i \xrightarrow{r_{i+1}} t_{i+1} \) for all \( i \geq 0 \). For succinctness such derivation is denoted by \( t \xrightarrow{\sigma}^\infty \).

Notice that there is a strict correspondence between the notion of derivation from a term \( t \) and that of path from the term \( t \). In fact, there exists a path \( t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \ldots \) from \( t_0 \) in \( \mathcal{R} \) if and only if there exists a derivation \( t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \ldots \) from \( t_0 \) in \( \mathcal{R} \), with \( a_i = \text{label}(r_i) \), for any \( i \) (where for a rule \( r \in \mathcal{R} \) with \( r = t \xrightarrow{a} t' \), \( \text{label}(r) \) denotes the label \( a \) of \( r \)).

In the following, we shall consider PRSs in a syntactical restricted form called normal form.

**Definition 2.3 (Normal Form).** A PRS \( \mathcal{R} \) is said to be in normal form if every rule \( r \in \mathcal{R} \) has one of the following forms:

**PAR rules:** Any rule devoid of sequential composition;
SEQ rules: $X \xrightarrow{a} Y, (Z), \ X.(Y) \xrightarrow{a} Z$ or $X \xrightarrow{a} Y$, or $X \xrightarrow{a} \varepsilon$.

with $X, Y, Z \in \text{Var}$. A PRS where all the rules are SEQ rules is called sequential PRS. Similarly, a PRS where all the rules are PAR rules is called parallel PRS.

The sequential and parallel fragments of PRS are significant: in [14] it is shown that sequential PRSs are semantically equivalent (via bisimulation equivalence) to Pushdown Processes, while parallel PRSs are semantically equivalent to Petri Nets. Moreover, from the fact that Pushdown systems and Petri Nets are not comparable (see [13, 7]) it follows that PRSs in normal form are strictly more expressive than both their sequential and parallel fragment. So, the following result holds:

**Proposition 2.1.** PRSs in normal form are strictly more expressive than Petri nets and Pushdown Processes.

Now, let us extend the notion of PRS to that of Büchi Process Rewrite System (BRS). Intuitively, a BRS is a PRS where we can distinguish between non–accepting rewrite rules and accepting rewrite rules.

**Definition 2.4 (Büchi Rewrite System).** A Büchi Rewrite System (BRS) over a finite set of process variables Var and an alphabet $\Sigma$ is a pair $\langle \mathcal{R}, \mathcal{R}_F \rangle$, where $\mathcal{R}$ is a PRS over Var and $\Sigma$, and $\mathcal{R}_F \subseteq \mathcal{R}$ is the set of accepting rules.

A Büchi Rewrite System $\langle \mathcal{R}, \mathcal{R}_F \rangle$ is called a BRS in normal form (resp., sequential BRS, parallel BRS), if the underlying process rewrite system $\mathcal{R}$ is a PRS in normal form (resp., parallel PRS, sequential PRS).

**Definition 2.5 (Acceptance in Büchi Rewrite Systems).** Let us consider a BRS $M = \langle \mathcal{R}, \mathcal{R}_F \rangle$. An infinite derivation $t \xrightarrow{\sigma} \mathcal{R}^*$ in $\mathcal{R}$ from $t$ is said to be accepting (in $M$) if $\sigma$ contains infinite occurrences of accepting rules.

A finite derivation $t \xrightarrow{\sigma} t'$ in $\mathcal{R}$ from $t$ is said to be accepting (in $M$) if $\sigma$ contains some occurrence of accepting rule.

**2.2 Decidability results for PRSs**

In this section we will summarize decidability results on PRSs which are known in the literature, and which will be exploited in further sections of the paper.

**Verification of ALTL (Action–based LTL)**

Given a finite set $\Sigma$ of atomic propositions LTL, the set of formulae $\varphi$ of ALTL over $\Sigma$ is defined as follows:

$$
\varphi ::= \text{true} | \neg \varphi | \varphi_1 \land \varphi_2 | \langle a \rangle \varphi | \varphi_1 U \varphi_2 | G \varphi | F \varphi
$$

where $a \in \Sigma$. 

5
In order to give semantics to ALTL formulae on a PRS $\mathcal{R}$, we need some additional notation. Given a path $\pi = t_0^{a_0} t_1^{a_1} t_2^{a_2} \ldots$ in $\mathcal{R}$, $\pi^i$ denotes the suffix of $\pi$ starting from the $i$-th term in the sequence, i.e. the path $t_i^{a_i} t_{i+1}^{a_{i+1}} \ldots$. The set of all the suffixes of $\pi$ is denoted by $\text{suffix}(\pi)$ (notice that if $\pi$ is a run in $\mathcal{R}$, then $\pi^i$ is also a run in $\mathcal{R}$, for each $i$.) If the path $\pi = t_0^{a_0} t_1^{a_1} \ldots$ is non-trivial (i.e., the sequence contains at least two terms) $\text{firstact}(\pi)$ denotes $a_0$, otherwise we set $\text{firstact}(\pi)$ to an element non in $\Sigma$, say it 0.

ALTL formulae over a PRS $\mathcal{R}$ are interpreted in terms of the set of the runs in $\mathcal{R}$ satisfying the given ALTL formula. The denotation of a formula $\varphi$ relative to $\mathcal{R}$, in symbols $[[\varphi]]_\mathcal{R}$, is defined inductively as follows:

- $[[\text{true}]]_\mathcal{R} = \text{runs}(\mathcal{R})$
- $[[\neg \varphi]]_\mathcal{R} = \text{runs}(\mathcal{R}) \setminus [[\varphi]]_\mathcal{R}$
- $[[\varphi_1 \land \varphi_2]]_\mathcal{R} = [[\varphi_1]]_\mathcal{R} \cap [[\varphi_2]]_\mathcal{R}$
- $[[\langle a \rangle \varphi]]_\mathcal{R} = \{ \pi \in \text{runs}(\mathcal{R}) \mid \text{firstact}(\pi) = a \text{ and } \pi^1 \in [[\varphi]]_\mathcal{R} \}$
- $[[\varphi_1 \varphi_2]]_\mathcal{R} = \{ \pi \in \text{runs}(\mathcal{R}) \mid \text{for some } i \geq 0, \pi^i \text{ is defined and } \pi^i \in [[\varphi_2]]_\mathcal{R}, \text{ and } \text{for all } j < i, \pi^j \in [[\varphi_1]]_\mathcal{R} \}$
- $[[G \varphi]]_\mathcal{R} = \{ \pi \in \text{runs}(\mathcal{R}) \mid \text{suffix}(\pi) \subseteq [[\varphi]]_\mathcal{R} \}$
- $[[F \varphi]]_\mathcal{R} = \{ \pi \in \text{runs}(\mathcal{R}) \mid \text{suffix}(\pi) \cap [[\varphi]]_\mathcal{R} \neq \emptyset \}$

For any term $t \in T$ and ALTL formula $\varphi$, we say that $t$ satisfies $\varphi$ (resp., satisfies $\varphi$ restricted to infinite runs) (w.r.t $\mathcal{R}$), in symbols $t \models_\mathcal{R} \varphi$ (resp., $t \models_{\mathcal{R},\infty} \varphi$), if $\text{runs}_\mathcal{R}(t) \subseteq [[\varphi]]_\mathcal{R}$ (resp., $\text{runs}_{\mathcal{R},\infty}(t) \subseteq [[\varphi]]_\mathcal{R}$).

The model-checking problem (resp., model-checking problem restricted to infinite runs) for ALTL and PRSs is the problem of deciding if, given a PRS $\mathcal{R}$, an ALTL formula $\varphi$ and a term $t$ of $\mathcal{R}$, $t \models_\mathcal{R} \varphi$ (resp., $t \models_{\mathcal{R},\infty} \varphi$). The following are well-known results:

**Proposition 2.2** (see [14, 10, 11]). The model-checking problem for ALTL and parallel PRSs, possibly restricted to infinite runs, is decidable.

**Proposition 2.3** (see [2, 4, 14]). The model-checking problem for ALTL and sequential PRSs, possibly restricted to infinite runs, is decidable.

The model-checking problem for ALTL and unrestricted PRSs is known undecidable (see [14]).

In [5] we showed that the model-checking problem for PRSs in normal form (that are more expressive than parallel and sequential PRSs) and a small fragment of ALTL is decidable. In particular, we established the following result.

**Theorem 2.1** (see [5]). Given a BRS $\langle \mathcal{R}, \mathcal{R}_F \rangle$ in normal form and a process variable $X$ it is decidable if
1. there exists an infinite accepting derivation in $\mathcal{R}$ from $X$.

2. there exists an infinite derivation in $\mathcal{R}$ from $X$, not containing occurrences of accepting rules.

3. there exists an infinite derivation from $X$, containing a finite non–null number of occurrences of accepting rules.

This result implies the decidability of the model–checking problem (restricted to infinite runs) for PRSs in normal form and the following fragment of ALTL

$$\varphi ::= F \psi \mid GF \psi \mid \neg \varphi$$ (1)

where $\psi$ denotes a ALTL propositional formula$^2$.

Thus, the following result holds

**Theorem 2.2 (see [5]).** The model–checking problem for PRSs in normal form and the fragment ALTL $\{1\}$, restricted to infinite runs from process variables, is decidable.

### 3 Multi Büchi Rewrite Systems

In this section we generalize the notion of acceptance in PRSs, as defined in 2.5, introducing the notion of Multi Büchi Rewrite System (MBRS). Intuitively, a MBRS is a PRS with a finite number of accepting components, where each component is a subset of the PRS. The goal is to extend the decidability result of theorem 2.1. As a consequence, we obtain decidability of model–checking for PRSs in normal form and a meaningful fragment of ALTL, that includes strictly the fragment $\{1\}$ defined in subsection 2.2.

**Definition 3.1 (Multi Büchi Rewrite System).** A Multi Büchi Rewrite System (MBRS) (with $n$ accepting components) over a finite set of process variables $\text{Var}$ and an alphabet $\Sigma$ is a tuple $M = \langle \mathcal{R}, \langle \mathcal{R}^A_1, \ldots, \mathcal{R}^A_n \rangle \rangle$, where $\mathcal{R}$ is a PRS over $\text{Var}$ and $\Sigma$, and $\forall i = 1, \ldots, n \mathcal{R}^A_i \subseteq \mathcal{R}$. $\mathcal{R}$ is called the support of $M$.

A MBRS $M = \langle \mathcal{R}, \langle \mathcal{R}^A_1, \ldots, \mathcal{R}^A_n \rangle \rangle$ is a MBRS in normal form (resp., sequential MBRS, parallel MBRS), if the underlying PRS $\mathcal{R}$ is in normal form (resp., is sequential, is parallel).

**Definition 3.2.** $\forall n \in \mathbb{N} \setminus \{0\}$ let us denote by $P_n$ the set $2^{\{1,\ldots,n\}}$ (i.e., the set of the subsets of $\{1,\ldots,n\}$).

**Definition 3.3 (Finite Maximal).** Let $M = \langle \mathcal{R}, \langle \mathcal{R}^A_1, \ldots, \mathcal{R}^A_n \rangle \rangle$ be a MBRS, and let $\sigma$ be a rule sequence in $\mathcal{R}$. The finite maximal of $\sigma$ as to $M$, denoted by $\Upsilon^f_M(\sigma)$, is the set $\{i \in \{1,\ldots,n\} \mid \sigma \text{ contains some occurrence of rule in } \mathcal{R}^A_i\}$. 

$^2$The set of ALTL propositional formulae $\psi$ over the set $\Sigma$ of atomic propositions is so defined: $\psi ::= \langle a \rangle \text{true} \mid \psi \land \psi \mid \neg \psi$ (where $a \in \Sigma$)
Definition 3.4 (Infinite Maximal). Let $M = (\mathcal{R}, \langle \mathcal{R}^A_1, \ldots, \mathcal{R}^A_n \rangle)$ be a MBRS, and let $\sigma$ be a rule sequence in $\mathcal{R}$. The infinite maximal of $\sigma$ as to $M$, denoted by $\Upsilon^\infty_M(\sigma)$, is the set $\{ i \in \{1, \ldots, n\} | \sigma \text{ contains infinite occurrences of some rule in } \mathcal{R}^A_i \}$.

Now, we give a generalized notion of accepting derivation in a PRS.

Definition 3.5 (Acceptance). Let $M = (\mathcal{R}, \langle \mathcal{R}^A_1, \ldots, \mathcal{R}^A_n \rangle)$ be a MBRS, and let $t \Rightarrow^*_n \sigma$ be a derivation in $\mathcal{R}$. Given $K, K^\omega \in P_n$, we say that $t \Rightarrow^*_n \sigma$ is a $(K, K^\omega)$–accepting derivation in $M$ if $\Upsilon^f_M(\sigma) = K$ and $\Upsilon^\infty_M(\sigma) = K^\omega$.

Definition 3.6. Let $\sigma_1$ and $\sigma_2$ be sequences of rules in a PRS $\mathcal{R}$.
We denote by Interleaving($\sigma_1, \sigma_2$) the set of rule sequences in $\mathcal{R}$ defined inductively in the following way (we denote by $\varepsilon$ the empty sequence):

- $\text{Interleaving}(\varepsilon, \sigma) = \{ \sigma \}$
- $\text{Interleaving}(\sigma, \varepsilon) = \{ \sigma \}$
- $\text{Interleaving}(r_1\sigma_1, r_2\sigma_2) = \{ r_1\sigma | \sigma \in \text{Interleaving}(\sigma_1, r_2\sigma_2) \} \cup \{ r_2\sigma | \sigma \in \text{Interleaving}(r_1\sigma_1, \sigma_2) \}$ where $r_1$ and $r_2$ are rules in $\mathcal{R}$.

The above definition can be extended in obvious way to an arbitrary number of rule sequences.

Now, we establish (through propositions 3.1) simple properties of rule sequences in MBRSs, important in the following. We need the following definition.

Definition 3.7. Let $\{K_h\}_{h \in N}$ be a succession of sets in $P_n$. Let us denote by $\bigoplus_{h \in N} K_h$ the subset of $P_n$ given by $\{i \mid \forall j \in N \text{ there exists a } h > j \text{ such that } i \in K_h \}$.

Proposition 3.1. Given a MBRS $M = (\mathcal{R}, \langle \mathcal{R}^A_1, \ldots, \mathcal{R}^A_n \rangle)$ and two rule sequences $\sigma$ and $\sigma'$ in $\mathcal{R}$, the following properties hold:

1. If $\sigma$ is finite, then $\Upsilon^\infty_M(\sigma) = \emptyset$.
2. If $\sigma'$ is a subsequence of $\sigma$, then $\Upsilon^f_M(\sigma') \subseteq \Upsilon^f_M(\sigma)$ and $\Upsilon^\infty_M(\sigma') \subseteq \Upsilon^\infty_M(\sigma)$.
3. If $\lambda \in \text{Interleaving}(\sigma, \sigma')$, then $\Upsilon^f_M(\lambda) = \Upsilon^f_M(\sigma) \cup \Upsilon^f_M(\sigma')$ and $\Upsilon^\infty_M(\lambda) = \Upsilon^\infty_M(\sigma) \cup \Upsilon^\infty_M(\sigma')$.
4. If $\sigma = \sigma_0\sigma_1\sigma_2 \ldots$, then $\Upsilon^f_M(\sigma) = \bigcup_{h \in N} \Upsilon^f_M(\sigma_h)$ and $\Upsilon^\infty_M(\sigma) = \bigoplus_{h \in N} \Upsilon^f_M(\sigma_h)$.
5. If $\sigma'$ is a reordering of $\sigma$, then $\Upsilon^f_M(\sigma) = \Upsilon^f_M(\sigma')$ and $\Upsilon^\infty_M(\sigma) = \Upsilon^\infty_M(\sigma')$.

In the following subsection we enunciate the two main results of the paper (proved in section 4): the one regarding acceptance properties of finite derivations in MBRSs in normal form, the second regarding acceptance properties of infinite derivations in MBRSs in normal form. Moreover, we show that the second result can be exploited for automatic verification of some meaningful (action-based) linear time properties of infinite runs in PRSs in normal form.
3.1 Model-checking of PRSs in normal form

The main result of the paper is the following:
Given a MBRS in normal form \( M = (\mathcal{R}, (\mathcal{R}_1^1, \ldots, \mathcal{R}_n^1)) \) over \( \text{Var} \) and the alphabet \( \Sigma \), given a variable \( X \in \text{Var} \) and two sets \( K, K^\omega \in P_n \) it is decidable if:

Problem 1: There exists a \((K, \emptyset)\)-accepting finite derivation in \( M \) from \( X \).

Problem 2: There exists a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( X \).

The decidability of Problem 1 is used mainly, as we’ll see, to prove decidability of Problem 2.

Before proving these results in Sections 4.2 and 4.3, we show how a solution to these problems can be effectively employed to perform model checking of some linear time properties of infinite runs (from process variables) in PRSs in normal form. In particular we consider the following ALTL fragment, that includes strictly the fragment (1) defined in subsection 2.2:

\[
\varphi ::= F \psi \mid GF \psi \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi
\]

(1)

where \( \psi \) denotes a ALTL propositional formula. For succinctness, we denote a ALTL propositional formula of the form \( \langle a \rangle \text{ true} \) (with \( a \in \Sigma \)) simply by \( a \).

The difference with fragment (1) defined in subsection 2.2 is that, now, we allow boolean combinations of formulae of the form \( T\psi \), where \( T \) denotes a temporal operator in \{\( F, G, FG, GF \}\) and \( \psi \) is a ALTL propositional formula.

Within the fragment above, property patterns frequent in system verification can be expressed. In particular, we can express safety properties (e.g., \( Gp \)), guarantee properties (e.g., \( Fp \)), obligation properties (e.g., \( Fp \rightarrow Fq \) or \( Gp \rightarrow Gq \)), response properties (e.g., \( GFp \)), persistence properties (e.g., \( FGp \)), and finally reactivity properties (e.g., \( GFp \rightarrow GFq \)). Notice that important classes of properties like invariants, as well as strong and weak fairness constraints, can be expressed.

To prove decidability of the model–checking problem restricted to infinite runs for this fragment of ALTL we need some definitions.

Given a propositional formula \( \psi \) over \( \Sigma \), we denote by \( [[\psi]]_\Sigma \) the subset of \( \Sigma \) inductively defined as follows

- \( \forall a \in \Sigma \ [ [a] ]_\Sigma = \{a\} \)
- \( [ [\neg \psi] ]_\Sigma = \Sigma \setminus [ [\psi] ]_\Sigma \)
- \( [ [\psi_1 \land \psi_2] ]_\Sigma = [ [\psi_1] ]_\Sigma \cap [ [\psi_2] ]_\Sigma \)

Evidently, given a PRS \( \mathcal{R} \) over \( \Sigma \), a ALTL propositional formula \( \psi \) and an infinite run \( \pi \) of \( \mathcal{R} \) we have that \( \pi \in [ [\psi] ]_\mathcal{R} \) iff \( \text{firstact}(\pi) \in [ [\psi] ]_\Sigma \).

Given a rule \( r = a \rightarrow t' \in \mathcal{R} \), we say that \( r \) satisfies the propositional formula \( \psi \) if \( a \in [ [\psi] ]_\Sigma \).

We denote by \( AC_{\mathcal{R}}(\psi) \) the set of rules in \( \mathcal{R} \) that satisfy \( \psi \).

Now, we introduce a new temporal operator, denoted by \( F^+ \), whose semantic is so defined:
\[ [F^+ \varphi]_R = \{ \pi \in \text{runs}(\mathcal{R}) \mid \text{suffix}(\pi) \cap ([\varphi]_R) \neq \emptyset \}, \text{ and either } \pi \text{ is finite or there exists a } j \geq 0 \text{ such that } \forall h \geq j \pi^h \notin [[\varphi]]_R \} \]

Now, we can prove the following result

**Theorem 3.1.** The model–checking problem for PRSs in normal form and the fragment ALTL, restricted to infinite runs from process variables, is decidable.

**Proof.** Given a PRS \( \mathcal{R} \) in normal form, a variable \( X \) and a formula \( \varphi \) belonging to ALTL fragment \( [1] \), we have to prove that it’s decidable if

\[ X \models_{\mathcal{R}, \infty} \varphi \quad (2) \]

This problem is reducible to the problem of deciding if the following property is satisfied

**A.** There exists an infinite run \( \pi \), with \( \pi \in \text{runs}_{\mathcal{R}, \infty}(X) \), satisfying the formula \( \neg \varphi \), i.e. with \( \pi \in [[\neg \varphi]]_R \).

Pushing negation inward, and using the following logic equivalences

- \( G\varphi_1 \land G\varphi_2 \equiv G(\varphi_1 \land \varphi_2) \)
- \( \neg F\varphi_1 \equiv G\neg \varphi_1 \)
- \( \neg G\varphi_1 \equiv F\neg \varphi_1 \)
- \( F\varphi_1 \equiv F^+\varphi_1 \lor GF\varphi_1 \)
- \( FG\varphi_1 \equiv F^+\neg \varphi_1 \lor G\varphi_1 \) (this equivalence holds for infinite runs)

the formula \( \neg \varphi \) can be written in the following disjunctive normal form

\[ \neg \varphi \equiv \bigvee_i \left( \bigwedge_j F^+\psi_j \land \bigwedge_k GF\eta_k \land G\zeta \right) \quad (3) \]

where \( \psi_j, \eta_k \) and \( \zeta \) are ALTL propositional formulae. Evidently, we can restrict ourselves to consider a single disjunct in \( (3) \). In other words, problem in equation \( (2) \) is reducible to the problem of deciding, given a formula having the following form

\[ F^+\psi_1 \land \ldots \land F^+\psi_m \land GF\eta_1 \land \ldots \land GF\eta_{m_2} \land G\zeta^3 \quad (4) \]

if the following property is satisfied

**B.** There exists an infinite run \( \pi \), with \( \pi \in \text{runs}_{\mathcal{R}, \infty}(X) \), satisfying formula \( (4) \).

\[^3\] \( \psi_j, \eta_k \) and \( \zeta \) are ALTL propositional formulae.
Let us consider the MBRS in normal form $M = \langle \mathcal{R}, \langle \mathcal{R}_1^A, \ldots, \mathcal{R}_n^A \rangle \rangle$ where $n = m_1 + m_2 + 1$ and

$$
\begin{align*}
&\text{for all } i = 1, \ldots, m_1 \quad \mathcal{R}_i^A = AC(\psi_i) \\
&\text{for all } j = 1, \ldots, m_2 \quad \mathcal{R}_{j+m_1}^A = AC(\eta_j) \\
&\mathcal{R}_{m_1+m_2+1}^A = AC(\neg \zeta)
\end{align*}
$$

Let $K = \{1, \ldots, m_1 + m_2\}$ and $K^\omega = \{m_1 + 1, \ldots, m_1 + m_2\}$. It is easy to show that property B is satisfied if, and only if, there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ from variable $X$. From decidability of Problem 2, we obtain the assertion. 

## 4 Decidability results on MBRSs in normal form

In this section we prove the main results of the paper, namely the decidability of problems about derivations in MBRSs stated in subsection 3.1. Therefore, in subsection 4.1 we report some preliminary results on the decidability of some properties about derivations of parallel and sequential MBRSs which are necessary to carry out the proof of the main results, which are given in subsection 4.2 and 4.3.

### 4.1 Decidability results on parallel and sequential MBRSs

In this section we establish simple decidability results on properties of derivations in parallel and sequential MBRSs. These results are the basis for the decidability proof of the problems 1-2 stated in subsection 3.1.

**Proposition 4.1.** Given a parallel MBRS $M_P = \langle \mathcal{R}_P, \langle \mathcal{R}_P^A, \ldots, \mathcal{R}_P^{A,n} \rangle \rangle$ over $\text{Var}$ and the alphabet $\Sigma$, given two variables $X, Y \in \text{Var}$ and $K \in P_n$, it is decidable if

1. there exists a finite derivation in $\mathcal{R}_P^A$ of the form $X \xrightarrow{\sigma} t \parallel Y$, for some term $t$, with $|\sigma| > 0$ and $\Upsilon^f_{M_P}(\sigma) = K$.

2. there exists a finite derivation in $\mathcal{R}_P^A$ of the form $X \xrightarrow{\sigma} \epsilon$ such that $\Upsilon^f_{M_P}(\sigma) = K$.

3. there exists a finite derivation in $\mathcal{R}_P^A$ of the form $X \xrightarrow{\sigma} Y$ such that $\Upsilon^f_{M_P}(\sigma) = K$.

4. there exists a $(K, \emptyset)$-accepting finite derivation in $M_P$ from $X$.

**Proof.** We exploit the decidability of the model-checking problem for ALTL and parallel PRSs (see prop. 2.2).

For all $r \in \mathcal{R}_P$, let us denote by $\text{label}(r)$ the set $\{h \in \{1, \ldots, n\} | r \in \mathcal{R}_P^{A,h}\}$. Let us denote by $\zeta$ the set $\{\text{label}(r) | r \in \mathcal{R}_P\}$.

Let us consider the first problem. Starting from $\mathcal{R}_P$, we build a new parallel PRS $\mathcal{R}_P'$ over the alphabet $\Sigma = \{Y\} \cup \zeta$, as follows. At first, we substitute every rule $r$ in $\mathcal{R}_P$ of the form
with the rule $t \xrightarrow{\text{label}(r)} t'$. Finally, we add the rule $Y \xrightarrow{Y} Y$. The reason to add this rule is to allow to express reachability of variable $Y$ as an ALTL formula.

Now, let us assume that $K \neq \emptyset$. A similar reasoning applies if $K = \emptyset$. Let us indicate by $\varphi_1$ the following ALTL formula,

$$F(<Y > true) \land G(\neg(<Y > true) \lor <Y > (G(<Y > true)))$$

This formula is satisfied by infinite runs $\pi$ in $\mathcal{R}_p$ having the form $\pi_1 \pi_2$, where $\pi_2$ contains only occurrences of label $Y$, and $\pi_1$ doesn’t contain occurrences of label $Y$.

It’s easy to deduce that property 1 is satisfied if, and only if, there exists a run $\pi$ in $\mathcal{R}_p$ with $\pi \in \text{runs}_{\mathcal{R}_p}(X)$ satisfying the following ALTL formula:

$$\varphi := \varphi_1 \land \left( \bigwedge_{i \in K} \bigvee_{r \in \mathcal{R}_{p,i}} F(<\text{label}(r) > true) \right) \land \left( \bigwedge_{i \notin K} \bigwedge_{r \in \mathcal{R}_{p,i}} G(\neg <\text{label}(r) > true) \right)$$

Therefore, property 1 isn’t satisfied if, and only if, $\forall \pi \in \text{runs}_{\mathcal{R}_p}(X) \pi \notin [[\varphi]]_{\mathcal{R}_p}$, that is if, and only if, $X \models_{\mathcal{R}_p} \neg \varphi$.

Now, let us consider the second problem. Similarly to the problem above, starting from $\mathcal{R}_p$, we build a new parallel PRS $\mathcal{R}_p'$, this time on the alphabet $\Sigma = \text{Var} \cup \zeta$, as follows. At first, we substitute every rule $r$ in $\mathcal{R}_p$ of the form $t \xrightarrow{a} t'$ with the rule $t \xrightarrow{\text{label}(r)} t'$. Finally, $\forall Y \in \text{Var}$ we add the rule $Y \xrightarrow{Y} Y$. Notice that, by construction, a term $t$ has no successor in $\mathcal{R}_p'$ if, and only if, $t = \varepsilon$. Let us indicate by $\varphi_1$ the following ALTL propositional formula,

$$\bigvee_{Y \in \text{Var}} ( <Y > true) \lor \bigvee_{l \in \zeta} ( <l > true)$$

The negation of $\varphi_1$ means that no rule can be applied, in other words the system has terminated.

It’s easy to deduce that property 2 is satisfied if, and only if, there exists a run $\pi$ in $\mathcal{R}_p$ with $\pi \in \text{runs}_{\mathcal{R}_p}(X)$ satisfying the following ALTL formula:

$$\varphi := F(\neg \varphi_1) \land \left( \bigwedge_{i \in K} \bigvee_{r \in \mathcal{R}_{p,i}} F(<\text{label}(r) > true) \right) \land \left( \bigwedge_{i \notin K} \bigwedge_{r \in \mathcal{R}_{p,i}} G(\neg <\text{label}(r) > true) \right)$$

Therefore, property 2 isn’t satisfied if, and only if, $\forall \pi \in \text{runs}_{\mathcal{R}_p}(X) \pi \notin [[\varphi]]_{\mathcal{R}_p}$, that is if, and only if, $X \models_{\mathcal{R}_p} \neg \varphi$.

\[\text{for all } i \in K \text{ if } \mathcal{R}_{p,i} = \emptyset, \text{ then } \bigvee_{r \in \mathcal{R}_{p,i}} F(<\text{label}(r) > true) \text{ denotes false}\]
Now, let us consider the third problem. Starting from $\mathcal{R}_P$, we build a new parallel PRS $\mathcal{R}'_P$ over the alphabet $\Sigma = \{\varepsilon\} \cup \text{Var} \cup \zeta$, as follows. At first, we substitute every rule $r$ in $\mathcal{R}_P$ of the form $t \xrightarrow{\text{label}(r)} t'$ with the rule $t' \rightarrow t$. For each $Z \in \text{Var}$ we add the rule $Z \rightarrow Z$. Finally, we add the rule $Y \xrightarrow{\varepsilon} \varepsilon$. Notice that, by construction, a term $t$ has no successor in $\mathcal{R}'_P$, if, and only if, $t = \varepsilon$. Let us denote by $S$ the set of runs $\mathcal{R}'_P$, with the rule $\delta$. It's easy to deduce that property 3 is satisfied if, and only if, there exists a run $\pi$ in $\mathcal{R}'_P$, with $\pi \in \text{runs}_{\mathcal{R}'_P}(X)$ satisfying the following ALTL formula:

$$\bigvee_{Y \in \text{Var}} \left( <Y \text{ true}> \right) \lor <\varepsilon \text{ true}> \lor \bigvee_{l \in \zeta} \left( <l \text{ true}> \right)$$

Moreover, let us denote by $\varphi_2$ the following ALTL formula,

$$\bigvee_{l \in \zeta} \left( <l \text{ true}> \right) U \left( <\varepsilon \text{ true}> \neg \varphi_1 \right)$$

This formula is satisfied by runs in $\mathcal{R}'_P$, that end in $\varepsilon$ such that the last label is $\varepsilon$, with each other label in $\zeta$, and the last but one term is $Y$.

It’s easy to deduce that property 3 is satisfied if, and only if, there exists a run $\pi$ in $\mathcal{R}'_P$ with $\pi \in \text{runs}_{\mathcal{R}'_P}(X)$ satisfying the following ALTL formula:

$$\varphi := \varphi_2 \land \left( \bigwedge_{i \in K} \bigvee_{r \in \mathcal{R}'_P} F(\text{<label}(r) \text{ true})\right) \land \left( \bigwedge_{i \notin K} \bigwedge_{r \in \mathcal{R}'_P} G(\neg \text{<label}(r) \text{ true})\right)$$

Therefore, property 3 isn’t satisfied if, and only if, $\forall \pi \in \text{runs}_{\mathcal{R}'_P}(X) \pi \notin [[\varphi]]_{\mathcal{R}'_P}$, that is if, and only if, $X \models \mathcal{R}'_P \neg \varphi$.

Finally, it’s easy to prove the decidability of the fourth problem applying a reasoning similar to previous ones.

**Proposition 4.2.** Let us consider two parallel MBRSs $M_{P_1} = \langle \mathcal{R}_P, \langle \mathcal{R}_{P_{1,1}}, \ldots, \mathcal{R}_{P_{1,n}} \rangle \rangle$ and $M_{P_2} = \langle \mathcal{R}_P, \langle \mathcal{R}_{P_{2,1}}, \ldots, \mathcal{R}_{P_{2,n}} \rangle \rangle$ over Var and the alphabet $\Sigma$, and with the same support $\mathcal{R}_P$. Given a variable $X \in \text{Var}$, two sets $K, K' \subseteq P_n$, and a subset $\mathcal{R}_P$ of $\mathcal{R}_P$ it’s decidable if the following condition is satisfied:

1. There exists a derivation in $\mathcal{R}_P$ of the form $X \xrightarrow{\sigma}_{\mathcal{R}_P}$ such that $\gamma^I_{M_{P_1}}(\sigma) = K$ and $\gamma^I_{M_{P_2}}(\sigma) = K'$. Moreover, either $\sigma$ is infinite or $\sigma$ contains some occurrence of rule in $\mathcal{R}_P \setminus \mathcal{R}'_P$.

**Proof.** The proof relies on the decidability of the model-checking problem for ALTL and parallel PRSs (see prop. 2.2).

Let us consider the tuple $\langle \mathcal{R}_{P_{1,1}}, \ldots, \mathcal{R}_{P_{2,n+1}} \rangle$ where $\forall i = 1, \ldots, n \mathcal{R}_{P_{i,i}} = \mathcal{R}_{P_{i,i}}$, and $\mathcal{R}_{P_{i,i+n}} = \mathcal{R}_{P_{2,i}}$, and $\mathcal{R}_{P_{2,i+n+1}} = \mathcal{R}_P \setminus \mathcal{R}'_P$.

Let us denote by $S$ the set $\{ (K_1, K_2) \subseteq P_n \times P_n \mid K_1 \cup K_2 = K' \}$. Evidently, property 1 is equivalent to the following property:
2. There exists a derivation in $\mathcal{R}_P$ of the form $X \xrightarrow{\sigma^*} \mathcal{R}_P$ such that

2.1 $\forall i \in K \sigma$ contains some occurrence of rule in $\mathcal{R}_{P,i}$, and $\forall j \in \{1, \ldots, n\} \setminus K \sigma$ doesn’t contain occurrences of rules in $\mathcal{R}_{P,j}$.

2.2 There exists a $(K_1, K_2) \in S$ such that $\forall i \in K_1$ (resp., $\forall i \in K_2$) $\sigma$ contains infinite occurrences of rules in $\mathcal{R}_{P,i}$ (resp., contains some occurrence of rule in $\mathcal{R}_{P,i+n}$), and $\forall j \in \{1, \ldots, n\} \setminus K_1$ (resp., $\forall j \in \{1, \ldots, n\} \setminus K_2$) $\sigma$ doesn’t contain infinite occurrences of rules in $\mathcal{R}_{P,j}$ (resp., doesn’t contain occurrences of rules in $\mathcal{R}_{P,j+n}$).

2.3 Either $\sigma$ is infinite or $\sigma$ contains some occurrence of rule in $\mathcal{R}_{P,2n+1}$.

For all $r \in \mathcal{R}_P$ let us denote by $label(r)$ the set $\{h \in \{1, \ldots, 2n+1\} | r \in \mathcal{R}_{P,h}\}$. Moreover, let us denote by $\zeta$ the set $\{label(r) | r \in \mathcal{R}_P\}$. Now, we construct a new parallel $\text{PRS} \mathcal{R}_P$ over the alphabet $\zeta \cup Var$, as follows. At first, we replace every rule $r$ in $\mathcal{R}_P$ of the form $t \xrightarrow{a} t'$ with the rule $t \xrightarrow{\text{label}(r)} t'$. Finally, $\forall Y \in Var$ we add the rule $Y \rightarrow Y$. Let us consider the following $\text{ALTL}$ propositional formula,

$$\psi = \bigvee_{l \in \zeta} \left( <l > \text{true} \right)$$

Now, let us consider the following $\text{ALTL}$ formula.

$$\varphi_3 := GF \left( \bigvee_{l \in \zeta} <l > \text{true} \right) \lor \bigvee_{r \in \mathcal{R}_{P,2n+1}} F \left( <l > (FG \neg \psi) \right)$$

This formula is satisfied either from infinite runs in $\mathcal{R}_P$ containing infinite occurrences of labels associated to rules in $\mathcal{R}_P$, or runs containing some occurrence of a label associated to a rule belonging to $\mathcal{R}_P \setminus \mathcal{R}_P$, and containing a finite number of occurrences of labels related to rules in $\mathcal{R}_P$. So, formula $\varphi_3$ expresses property 2.3.

Now, let us consider the following two $\text{ALTL}$ formulae

$$\varphi_1 := \left( \bigwedge_{i \in K} \bigvee_{r \in \mathcal{R}_{P,i}} F(<l > \text{true}) \right) \land \left( \bigwedge_{i \in \{1, \ldots, n\} \setminus K} \bigwedge_{r \in \mathcal{R}_{P,i}} G(\neg <l > \text{true}) \right)^5$$

$$\varphi_2 := \bigvee_{(K_1, K_2) \in S} \left( \bigwedge_{i \in K_1} \bigvee_{r \in \mathcal{R}_{P,i}} GF(<l > \text{true}) \right) \land^6$$

5for all $i \in K$ if $\mathcal{R}_{P,i} = \emptyset$, then $\bigvee_{r \in \mathcal{R}_{P,i}} F(<l > \text{true})$ denotes false

6for all $i \in K_1$ if $\mathcal{R}_{P,i} = \emptyset$, then $\bigvee_{r \in \mathcal{R}_{P,i}} GF(<l > \text{true})$ denotes false
Let us indicate by \( \pi \) is satisfied if, and only if, there exists a run \( \pi \in \text{runs}_{\mathcal{P}}(X) \) satisfying the following ALTL formula:

\[
\varphi := \varphi_1 \land \varphi_2 \land \varphi_3
\]

Therefore, property 2 isn’t satisfied if, and only if, \( \forall \pi \in \text{runs}_{\mathcal{P}}(X) \pi \notin [\varphi]_{\mathcal{P}}, \) that is if, and only if, \( X \models_{\mathcal{P}} \neg \varphi. \)

Now, let us give an additional notion of reachability in sequential PRSs.

**Definition 4.1.** Given a sequential PRS \( \mathcal{R}_S \) over \( \text{Var} \), and given \( X, Y \in \text{Var} \), we say that \( Y \) is reachable from \( X \) in \( \mathcal{R}_S \) whether there exists a term \( t \in T \setminus \{\varepsilon\} \) of the form \( X_1.(X_2.(\ldots X_n.(Y)\ldots)) \) (with \( n \) possibly equals to zero) such that \( X \rightarrow_{\mathcal{R}_S}^* t. \)

**Proposition 4.3.** Let us consider a sequential MBRS \( M_S = (\mathcal{R}_S, (\mathcal{R}^1_S, \ldots, \mathcal{R}^n_S)) \) over \( \text{Var} \) and the alphabet \( \Sigma \). Given two variables \( X, Y \in \text{Var} \) and two sets \( K, K^\omega \in P_n \), it is decidable if

1. \( Y \) is reachable from \( X \) in \( \mathcal{R}_S \) through a derivation having finite maximal \( K \) as to \( M_S \).
2. there exists a \((K, K^\omega)\)-accepting infinite derivation in \( M_S \) from \( X \).

**Proof.** The proof relies on the decidability of the model-checking problem for ALTL and sequential PRSs (possibly restricted to infinite runs) (see prop. 2.3).

\( \forall r \in \mathcal{R}_S \) let us denote by \( \text{label}(r) \) the set \( \{h \in \{1, \ldots, n\}|r \in \mathcal{R}^A_{S,h}\} \). Moreover, let us denote by \( \zeta \) the set \( \{\text{label}(r)|r \in \mathcal{R}_S\}. \)

Let us consider the first problem. Starting from \( \mathcal{R}_S \) we build a new sequential PRS \( \mathcal{R}'_S \) over the alphabet \( \Sigma = \{Y\} \cup \zeta \), as follows. At first, we substitute every rule \( r \) in \( \mathcal{R}_S \) of the form \( t \rightarrow t' \) with the rule \( t \xrightarrow{\text{label}(r)} t' \). Finally, we add the rule \( Y \rightarrow Y \).

Let us indicate by \( \varphi_1 \) the following ALTL formula,

\[
F(< Y > true) \land G\left((-< Y > true) \lor < Y > (G(< Y > true))\right)
\]

This formula is satisfied by infinite runs \( \pi \) in \( \mathcal{R}'_S \) having the form \( \pi_1\pi_2 \) where \( \pi_2 \) contains only occurrences of label \( Y \), and \( \pi_1 \) doesn’t contain occurrences of label \( Y \).

It’s easy to deduce that property 1 is satisfied if, and only if, there exists a run \( \pi \in \mathcal{R}'_S \) with \( \pi \in \text{runs}_{\mathcal{R}'_S}(X) \) satisfying the following ALTL formula:

\[
\varphi := \varphi_1 \land \left( \bigvee_{i \in K} F(< \text{label}(r) > true)\right) \land \left( \bigvee_{i \in K} G(-< \text{label}(r) > true)\right)
\]

\(^7\)for all \( i \in K_2 \) if \( \mathcal{R}^4_{P,i+n} = \emptyset \), then \( \bigvee_{r \in \mathcal{R}^4_{P,i+n}} F(< \text{label}(r) > true) \) denotes false.

\(^8\)for all \( i \in K \) if \( \mathcal{R}^A_{S,i} = \emptyset \), then \( \bigvee_{r \in \mathcal{R}^A_{S,i}} F(< \text{label}(r) > true) \) denotes false.
Therefore, property 1 isn't satisfied if, and only if, $\forall \pi \in \text{runs}_{\mathcal{R}_S}(X) \pi \not\in \varphi$, that is if, and only if, $X \models_{\mathcal{R}_S} \neg \varphi$.

Now, let us consider the second problem. We construct a new sequential $\text{PRS } \mathcal{R}'_S$ over the alphabet $\zeta$ in the following way. We substitute every rule $r$ in $\mathcal{R}_S$ of the form $t \xrightarrow{\text{label}(r)} t'$ with the rule $t' \xrightarrow{\text{label}(r)} t$.

It's easy to deduce that property 2 is satisfied if, and only if, there exists an infinite run $\pi$ in $\mathcal{R}'_S$ with $\pi \in \text{runs}_{\mathcal{R}'_S,\infty}(X)$ satisfying the following $\text{ALTL}$ formula:

$$
\varphi := (\bigwedge_{i \in K} \bigvee_{r \in \mathcal{R}_{S,i}^A} \text{GF}(\langle \text{label}(r) > \text{true} \rangle)) \land (\bigwedge_{i \notin K} \bigvee_{r \in \mathcal{R}_{S,i}^A} \text{FG}(\neg \langle \text{label}(r) > \text{true} \rangle))^{9} 
$$

$$
(\bigwedge_{i \in K} \bigvee_{r \in \mathcal{R}_{S,i}^A} \text{F}(\langle \text{label}(r) > \text{true} \rangle)) \land (\bigwedge_{i \notin K} \bigvee_{r \in \mathcal{R}_{S,i}^A} \text{G}(\neg \langle \text{label}(r) > \text{true} \rangle))^{10}
$$

Therefore, property 1 isn't satisfied if, and only if, $\forall \pi \in \text{runs}_{\mathcal{R}'_S,\infty}(X) \pi \not\in \varphi$, that is if, and only if, $X \models_{\mathcal{R}'_S,\infty} \neg \varphi$.

**Theorem 4.1.** Let us consider two parallel MBRSs $M_{P_1} = (\mathcal{R}_P, \langle \mathcal{R}_{P,1}^A, \ldots, \mathcal{R}_{P,n}^A \rangle)$ and $M_{P_2} = (\mathcal{R}_P, \langle \mathcal{R}_{P,1}^A, \ldots, \mathcal{R}_{P,n}^A \rangle)$ with the same support $\mathcal{R}_P$, and a sequential MBRS $M_S = (\mathcal{R}_S, \langle \mathcal{R}_{S,1}^A, \ldots, \mathcal{R}_{S,n}^A \rangle)$. Given a variable $X \in \text{Var}$, two sets $K, K' \subseteq P_n$, and a subset $\mathcal{R}_P^*$ of $\mathcal{R}_P$ it's decidable if the following condition is satisfied:

1. There exists a variable $Y$ reachable from $X$ in $\mathcal{R}_S$ through a $(K', \emptyset)$-accepting derivation in $M_S$ with $K' \subseteq K$, and there exists a derivation $Y \xrightarrow{\varphi^*_{\mathcal{R}_P}} \rho$ such that $\Upsilon_{M_{P_1}}^f(\rho) = K$ and $\Upsilon_{M_{P_2}}^f(\rho) \cup \Upsilon_{M_{P_1}}^\infty(\rho) = K'$. Moreover, either $\rho$ is infinite or $\rho$ contains some occurrence of rule in $\mathcal{R}_P \setminus \mathcal{R}_P^*$.

**Proof.** Since the sets $\{K' \subseteq P_n \mid K' \subseteq K\}$ and $\text{Var}$ are finite, the result follows directly from propositions 4.2 and 4.3.

**Theorem 4.2.** Let us consider two parallel MBRSs $M_{P_1} = (\mathcal{R}_P, \langle \mathcal{R}_{P,1}^A, \ldots, \mathcal{R}_{P,n}^A \rangle)$ and $M_{P_2} = (\mathcal{R}_P, \langle \mathcal{R}_{P,1}^A, \ldots, \mathcal{R}_{P,n}^A \rangle)$ with the same support $\mathcal{R}_P$, and a sequential MBRS $M_S = (\mathcal{R}_S, \langle \mathcal{R}_{S,1}^A, \ldots, \mathcal{R}_{S,n}^A \rangle)$. Given a variable $X \in \text{Var}$, two sets $K, K' \subseteq P_n$, and a subset $\mathcal{R}_P^*$ of $\mathcal{R}_P$ it's decidable if one of the following conditions is satisfied:

1. There exists a variable $Y$ reachable from $X$ in $\mathcal{R}_S$ through a $(K', \emptyset)$-accepting derivation in $M_S$ with $K' \subseteq K$, and there exists a derivation $Y \xrightarrow{\varphi^*_{\mathcal{R}_P}} \rho$ such that $\Upsilon_{M_{P_1}}^f(\rho) = K$ and $\Upsilon_{M_{P_2}}^f(\rho) \cup \Upsilon_{M_{P_1}}^\infty(\rho) = K'$. Moreover, either $\rho$ is infinite or $\rho$ contains some occurrence of rule in $\mathcal{R}_P \setminus \mathcal{R}_P^*$.

---

9for all $i \in K'$ if $\mathcal{R}_{S,i}^A = \emptyset$, then $\bigvee_{r \in \mathcal{R}_{S,i}^A} \text{GF}(\langle \text{label}(r) > \text{true} \rangle)$ denotes false

10for all $i \in K'$ if $\mathcal{R}_{S,i}^A = \emptyset$, then $\bigvee_{r \in \mathcal{R}_{S,i}^A} \text{F}(\langle \text{label}(r) > \text{true} \rangle)$ denotes false
2. There exists a \((K, K^\omega)\)-accepting infinite derivation in \(M_S\) from \(X\).

**Proof.** Since the sets \(\{K' \in P_n | K' \subseteq K\}\) and \(Var\) are finite, the result follows directly from propositions 4.2 and 4.3. \(\Box\)

### 4.2 Decidability results on finite derivations of \(MBRSs\) in normal form

In this section we prove the decidability of Problem 1 stated in subsection 3.1, that for clarity we recall.

**Problem 1**  
Given a \(MBRS\) in normal form \(M = \langle \Re, \langle \Re_1^A, \ldots, \Re_n^A \rangle \rangle\) over \(Var\) and the alphabet \(\Sigma\), given a variable \(X \in Var\) and a set \(K \in P_n\), to decide if there exists a \((K, \emptyset)\)-accepting finite derivation in \(M\) from \(X\).

We show that problem 1, with input a set \(K \in P_n\), can be reduced to a similar, but simpler, problem, that is a decidability problem on finite derivations restricted to parallel \(MBRSs\). In particular, we show that it is possible to construct effectively a parallel \(MBRS\) \(M^K_{\text{PAR}} = \langle \Re^K_{\text{PAR}}, \langle \Re^K_{\text{PAR},1}, \ldots, \Re^K_{\text{PAR},n} \rangle \rangle\) in such a way that Problem 1, with input the set \(K\) and a variable \(X\), is reducible to the problem of deciding if the following condition is satisfied:

- There exists a \((K, \emptyset)\)-accepting finite derivation in \(M^K_{\text{PAR}}\) from \(X\).

Since this last problem is decidable (see proposition 4.1), decidability of Problem 1 is entailed.

Before illustrating the main idea underlying our approach, we need few additional definitions and notations, which allows us to look more in detail at the structure of derivations in \(MBRSs\) in normal form. The following definition introduces the notion of level of application of a rule in a derivation:

**Definition 4.2.** Let \(t \overset{r}{\Rightarrow}_\Re t'\) be a single-step derivation in \(\Re\). We say that \(r\) is applicable at level 0 in \(t \overset{r}{\Rightarrow}_\Re t'\), if \(t = \overline{t}\|s,\ t' = \overline{t}\|s'\) (for some \(\overline{t}, s, s' \in T\)), and \(r = s \overset{a}{\Rightarrow} s'\), for some \(a \in \Sigma\).

We say that \(r\) is applicable at level \(k > 0\) in \(t \overset{r}{\Rightarrow}_\Re t'\), if \(t = \overline{t}\|X.(s),\ t' = \overline{t}\|X.(s')\) (for some \(\overline{t}, s, s' \in T\), \(s \overset{r}{\Rightarrow}_\Re s'\), and \(r\) is applicable at level \(k - 1\) in \(s \overset{r}{\Rightarrow}_\Re s'\).

The level of application of \(r\) in \(t \overset{r}{\Rightarrow}_\Re t'\) is the greatest level of applicability of \(r\) in \(t \overset{r}{\Rightarrow}_\Re t'\).

The definition above extends in the obvious way to \(n\)-step derivations and to infinite derivations. The next definition introduces the notion of subderivation starting from a term.

**Definition 4.3 (Subderivation).** Let \(\overline{t} \overset{r}{\Rightarrow}_\Re t\|X.(s) \overset{r}{\Rightarrow}_\Re t'\) be a finite or infinite derivation in \(\Re\) starting from \(\overline{t}\). The set of the subderivations \(d' = s \overset{d}{\Rightarrow}_\Re s'\) of \(d = t\|X.(s) \overset{r}{\Rightarrow}_\Re t'\) from \(s\) is inductively defined as follows:
1. if \( d \) is a null derivation or \( s = \varepsilon \), then \( d' \) is the null derivation from \( s \);
2. if \( \sigma = r\sigma' \), and \( d \) is of the form
   \[ t \| X.(Z) \xrightarrow{r} t \| Y \xrightarrow{\sigma'_r}^* \] (with \( r = X.(Z) \xrightarrow{\alpha} Y \) and \( s = Z \in \text{Var} \))
   then \( d' \) is the null derivation from \( s \);
3. if \( \sigma = r\sigma' \), and \( d \) is of the form
   \[ t \| X.(s) \xrightarrow{r} t \| X.(s') \xrightarrow{\sigma'_r}^* \] (with \( s \xrightarrow{r} s' \))
   then \( d' = s \xrightarrow{r} s' \xrightarrow{\mu'_s}^* \), where \( s' \xrightarrow{\mu'_s}^* \) is a subderivation of \( t \| X.(s') \xrightarrow{\sigma'_r}^* \) from \( s' \);
4. if \( \sigma = r\sigma' \), and \( d \) is of the form
   \[ t \| X.(s) \xrightarrow{r} t' \| X.(s) \xrightarrow{\sigma'_r}^* \] (with \( t \xrightarrow{r} t' \))
   then \( d' \) is a subderivation of \( t' \| X.(s) \xrightarrow{\sigma'_s}^* \) from \( s \).

Moreover, we say that \( d' \) is a subderivation of \( T \xrightarrow{T} t \| X.(s) \xrightarrow{\sigma'_s}^* \).

Clearly, in the definition above \( \mu \) is a subsequence of \( \sigma \). Moreover, if \( k \) is the level of application of a rule occurrence of \( \mu \) in the derivation \( d \) then, \( k > 0 \), and the level of application of this occurrence in the subderivation \( d'' = s \xrightarrow{r} s' \xrightarrow{\mu'_s}^* \) is \( k' \) with \( k' < k \).

Given a sequence \( \sigma = r_1r_2 \ldots r_n \ldots \) of rules in \( \mathcal{R} \), and a subsequence \( \sigma' = r_{k_1}r_{k_2} \ldots r_{k_n} \ldots \) of \( \sigma \), \( \sigma \setminus \sigma' \) denotes the sequence obtained by removing from \( \sigma \) all and only the occurrences of rules in \( \sigma' \) (namely, those \( r_i \) for which there exists a \( j = 1, \ldots, |\sigma'| \) such that \( k_j = i \)).

In the following, \( M_P = \langle \mathcal{R}_P, \langle \mathcal{R}_{P,1}, \ldots, \mathcal{R}_{P,n} \rangle \rangle \) denotes the restriction of \( M \) to \( \text{PAR} \) rules, that is \( \mathcal{R}_P \) (resp., \( \mathcal{R}_{P,i}^A \) for \( i = 1, \ldots, n \)) is the set \( \mathcal{R} \) (resp., \( \mathcal{R}_{i}^A \)) restricted to the \( \text{PAR} \) rules.

Let us sketch the main ideas at the basis of our technique. We show how it is possible to mimic a \((K, \emptyset)\)-accepting finite derivation in \( M \) from a variable by using only \( \text{PAR} \) rules belonging to an extension of the parallel \( \text{MBRS} \) \( M_P \), denoted by \( M_P^{K} = \langle \mathcal{R}_{P,K}, \langle \mathcal{R}_{P,K,1}, \ldots, \mathcal{R}_{P,K,n} \rangle \rangle \). More precisely, we show that

1. if \( p \xrightarrow{\sigma} \) (with \( p \in T_{PAR} \)) is a \((K, \emptyset)\)-accepting finite derivation in \( M \) with \( K \subseteq K \) then, there exists a \((K, \emptyset)\)-accepting finite derivation in \( M_P^{K} \) from \( p \), and vice versa.

So, let \( p \xrightarrow{\sigma} \) be a \((K, \emptyset)\)-accepting finite derivation in \( M \) with \( p \in T_{PAR} \) and \( K \subseteq K \). Then, all its possible subderivations contain all, and only, the rule occurrences in \( \sigma \) applied at a
level $k$ greater than 0 in $p \overset{\sigma}{\rightarrow}_k^*$. If $\sigma$ contains only PAR rule occurrences the statement $i$ is evident, since $M^K_{\text{PAR}}$ is an extension of $M_P$. Otherwise, $p \overset{\sigma}{\rightarrow}_k^*$ can be written in the form:

$$p \overset{\lambda}{\rightarrow}_k^* t\|X \overset{\rho}{\rightarrow}_k^* t\|Y.(Z) \overset{\omega}{\rightarrow}_k^*$$

(1)

where $r = X \overset{\alpha}{\rightarrow} Y.(Z)$, $\lambda$ contains only occurrences of rules in $\mathcal{R}_P$, $t \in \text{T}_{\text{PAR}}$ and $X,Y,Z \in \text{Var}$. Let $Z \overset{\rho}{\rightarrow}_k^*$ be a subderivation of $t\|Y.(Z) \overset{\omega}{\rightarrow}_k^*$ from $Z$. Evidently, $\Upsilon^f_M(\rho) \subseteq K$. Moreover, only one of the following three cases may occur:

**A** $Z \overset{\rho}{\rightarrow}_k^*$ leads to the term $\varepsilon$, and $p \overset{\sigma}{\rightarrow}_k^*$ is of the form

$$p \overset{\lambda}{\rightarrow}_k^* t\|X \overset{\rho}{\rightarrow}_k^* t\|Y.(Z) \overset{\omega}{\rightarrow}_k^* t\|Y \overset{\omega}{\rightarrow}_k^*$$

(2)

where $\rho$ is a subsequence of $\omega_1$ and $t \overset{\omega_2}{\rightarrow}_k^* \bar{t}$. The finite derivation above is $(\mathcal{K},\emptyset)$-accepting if, and only if, the following finite derivation, obtained by anticipating (by interleaving) the application of the rules in $\rho$ before the application of the rules in $\xi = \omega_1 \setminus \rho$, is $(\mathcal{K},\emptyset)$-accepting

$$p \overset{\lambda}{\rightarrow}_k^* t\|X \overset{\rho}{\rightarrow}_k^* t\|Y.(Z) \overset{\omega}{\rightarrow}_k^* \bar{t}\|Y \overset{\omega}{\rightarrow}_k^*$$

(3)

Let $\Upsilon^f_M(\rho \rho) = K' \subseteq K$. The idea is to collapse the derivation $X \overset{\rho}{\rightarrow}_k^* Y.(Z) \overset{\rho}{\rightarrow}_k^* Y$ into a single PAR rule of the form $r' = X^{K'} \rightarrow Y$, where $\Upsilon^f_M(r') = K' = \Upsilon^f_M(\rho \rho)$. Notice that in the step from (2) to (3), we exploit the fact that the property on finite derivations we are interested in is insensitive to permutations of rule applications within a derivation. Now, we can apply recursively the same reasoning to the finite derivation in $\mathcal{R}$ from $t\|Y \in \text{T}_{\text{PAR}}$

$$t\|Y \overset{\xi}{\rightarrow}_k^* \bar{t}\|Y \overset{\omega}{\rightarrow}_k^*$$

(4)

whose finite maximal as to $M$ is contained in $K$.

**B** $Z \overset{\rho}{\rightarrow}_k^*$ leads to a variable $W$, and $p \overset{\sigma}{\rightarrow}_k^*$ can be written as

$$p \overset{\lambda}{\rightarrow}_k^* t\|X \overset{\rho}{\rightarrow}_k^* t\|Y.(Z) \overset{\omega}{\rightarrow}_k^* t\|Y.(W) \overset{r'}{\rightarrow}_k^* W' \overset{\omega}{\rightarrow}_k^*$$

(5)

where $r' = Y.(W) \overset{b}{\rightarrow} W'$ (with $W' \in \text{Var}$), $\rho$ is a subsequence of $\omega_1$ and $t \overset{\omega_1}{\rightarrow}_k^* \bar{t}$. The finite derivation above is $(\mathcal{K},\emptyset)$-accepting if, and only if, the following finite derivation is $(\mathcal{K},\emptyset)$-accepting

$$p \overset{\lambda}{\rightarrow}_k^* t\|X \overset{\rho}{\rightarrow}_k^* t\|Y.(Z) \overset{\omega}{\rightarrow}_k^* t\|Y.(W) \overset{r'}{\rightarrow}_k^* W' \overset{\xi}{\rightarrow}_k^* \bar{t}\|W' \overset{\omega}{\rightarrow}_k^*$$

(6)

where $\xi = \omega_1 \setminus \rho$. Let $\Upsilon^f_M(\rho r' \rho) = K' \subseteq K$. The idea is to collapse the derivation $X \overset{\rho}{\rightarrow}_k^* Y.(Z) \overset{\rho}{\rightarrow}_k^* Y.(W) \overset{r'}{\rightarrow}_k^* W'$ into a single PAR rule of the form $r'' = X^{K'} \rightarrow W'$, where
\[ \Upsilon^f_{M_{\text{PAR}}} (r'') = K' = \Upsilon^f_M (r'' \rho). \]

Now, we can apply recursively the same reasoning to the finite derivation in \( \mathbb{R} \) from \( t \parallel W' \in T_{\text{PAR}} \)

\[ t \parallel W' \xrightarrow{\xi^*_r} 7 \parallel W' \xrightarrow{\xi^*_r} \]

whose finite maximal as to \( M \) is contained in \( K \).

C In this case \( Z \xrightarrow{\omega}_r^* \) does not influence the applicability of rules in \( \omega \setminus \rho \) in the derivation \( t \parallel Y. (Z) \xrightarrow{\omega}_r^* \) (i.e. the rule occurrences in \( \rho \) can be arbitrarily interleaved with any rule occurrence in \( \omega \setminus \rho \)). In other words, we have \( t \xrightarrow{\omega \setminus \rho}_r^* \). Moreover, \( \Upsilon^f_M (r \rho) = K' \) with \( K' \subseteq K \). Then, we keep track of the sequence \( r \rho \) by adding a new variable \( \hat{Z}_F \) (where \( \hat{Z}_F \notin \text{Var} \)) and a \( \text{PAR} \) rule of the form \( r' = X^{K'} \hat{Z}_F \), where \( \Upsilon^f_M (r') = K' = \Upsilon^f_M (r \rho) \). Now, we can apply recursively the same reasoning to the finite derivation \( t \parallel \hat{Z}_F \xrightarrow{\omega}_r^* \) in \( \mathbb{R} \) from the parallel term \( t \parallel \hat{Z}_F \), whose finite maximal as to \( M \) is contained in \( K \).

The parallel MBRS \( M^K_{\text{PAR}} = (\mathbb{R}^K_{\text{PAR}}, (\mathbb{R}^{K,A}_{\text{PAR},1}, \ldots, \mathbb{R}^{K,A}_{\text{PAR},n})) \) is defined as follows.

**Definition 4.4.** The MBRS \( M^K_{\text{PAR}} = (\mathbb{R}^K_{\text{PAR}}, (\mathbb{R}^{K,A}_{\text{PAR},1}, \ldots, \mathbb{R}^{K,A}_{\text{PAR},n})) \) is the least parallel MBRS with \( n \) accepting components, over \( \text{Var} \cup \{ \hat{Z}_F \} \) and the alphabet \( \Sigma = \Sigma \cup P_n^{11} \), satisfying the following properties:

1. \( \mathbb{R}^K_{\text{PAR}} \supseteq \{ r \in \mathbb{R} | r \text{ is a PAR rule} \} \) and \( \forall i = 1, \ldots, n, \mathbb{R}^{K,A}_{\text{PAR},i} \supseteq \{ r \in \mathbb{R}^A | r \text{ is a PAR rule} \} \)

2. Let \( r = X^a Y. (Z) \in \mathbb{R} \) and \( Z \xrightarrow{\sigma^*_K}_{\text{PAR}} \rho \) for some term \( \rho \), with \( \Upsilon^f_M (r) = K_1 \subseteq K \) and \( \Upsilon^f_{M^{K}_{\text{PAR}}} (\sigma) = K_2 \subseteq K \). Denoted by \( K' \) the set \( K_1 \cup K_2 \), then we have \( r' = X^{K'} \hat{Z}_F \in \mathbb{R}^K_{\text{PAR}} \) and \( \Upsilon^f_{M^{K}_{\text{PAR}}} (r') = K' \).

3. Let \( r = X^a Y. (Z) \in \mathbb{R} \) and \( Z \xrightarrow{\sigma^*_K}_{\text{PAR}} \varepsilon \) with \( \Upsilon^f_M (r) = K_1 \subseteq K \) and \( \Upsilon^f_{M^{K}_{\text{PAR}}} (\sigma) = K_2 \subseteq K \). Denoted by \( K' \) the set \( K_1 \cup K_2 \), then we have \( r' = X^{K'} Y \in \mathbb{R}^K_{\text{PAR}} \) and \( \Upsilon^f_{M^{K}_{\text{PAR}}} (r') = K' \).

4. Let \( r = X^a Y. (Z) \in \mathbb{R} \) then \( r' = Y. (W) \xrightarrow{b} W' \in \mathbb{R} \) and \( Z \xrightarrow{\sigma^*_K}_{\text{PAR}} W \) with \( \Upsilon^f_M (r) = K_1 \subseteq K \), \( \Upsilon^f_M (r') = K_2 \subseteq K \) and \( \Upsilon^f_{M^{K}_{\text{PAR}}} (\sigma) = K_3 \subseteq K \). Denoted by \( K' \) the set \( K_1 \cup K_2 \cup K_3 \), then we have \( r'' = X^{K'} W' \in \mathbb{R}^K_{\text{PAR}} \) and \( \Upsilon^f_{M^{K}_{\text{PAR}}} (r'') = K' \).

5. If \( r = X^{K'} Y \in \mathbb{R}^K_{\text{PAR}} \setminus \mathbb{R} \) then, \( X \xrightarrow{\sigma^*_K} t \) for some term \( t \), with \( |\sigma| > 0 \) and \( \Upsilon^f_M (\sigma) = K' \). Moreover, if \( Y \in \text{Var} \) then \( t = Y \).

\(^{11}\)let us assume that \( \Sigma \cap P_n = \emptyset \)
Lemma 4.1. The parallel MBRS \( M_{\text{PAR}}^K = \langle R_{\text{PAR}}^K, \langle R_{\text{PAR,1}}^K, \ldots, R_{\text{PAR,n}}^K \rangle \rangle \) can be effectively constructed.

Proof. Figure 1 reports the procedure BUILD-PARALLEL-MBRS\((M,K)\), which, starting from MBRS \( M \) (in normal form) and a set \( K \in P_n \), builds a parallel MBRS \( M_P = \langle R_P, \langle R_{P,1}^A, \ldots, R_{P,n}^A \rangle \rangle \). From proposition 4.1 the conditions in each of the if statements in lines 9, 16 and 27 are decidable, therefore, the procedure is effective.

Now, let us show that the algorithm terminates. It suffices to prove that the number of iterations of the repeat loop is finite. The termination condition of this loop is \( \text{flag} = \text{false} \). On the other hand, at the beginning of every iteration the \( \text{flag} \) is set to \( \text{true} \), and they will not be updated anymore. Now, the rule \( r = X \rightarrow Y, (Z) \in R \) and \( Z \xrightarrow{p} X \gamma_{\text{PAR}}^{K} \) for some term \( p \), with \( \gamma_M (r) = K_1 \subseteq K \) and \( \gamma (\sigma) = K_2 \subseteq K \). Denoted by \( K' \) the set \( K_1 \cup K_2 \), then we have to prove that \( r' = X \gamma_{\text{PAR}}^{K'} Z_F \in R_{\text{PAR}}^K \), and \( \gamma_M^{K'} (r') = K' \). From property a (notice that \( r' \notin R \)) it suffices to prove that \( r' \) is added to \( R_P \) during the computation. Let us consider the last iteration of the repeat loop. Since any update of the sets \( R_P, R_{P,1}^A, \ldots, R_{P,n}^A \) (the \( \text{flag} \) is set to \( \text{true} \)) involves a new iteration of this loop, we deduce that in this computation phase

\[
R_P = R_{\text{PAR}}^K \quad \forall i = 1, \ldots, n \quad R_{P,i}^A = R_{\text{PAR,1}, i}^K
\]

and they will not be updated anymore. Now, the rule \( r = X \rightarrow Y, (Z) \) will be examined during an iteration of the for loop in lines 6–36. From (1) it follows that during the inner for loop (lines 8–23) iteration associated to \( K_2 \), the condition of the if statement in line 9 is satisfied. Since \( R_P \) cannot be updated anymore, we deduce that the condition of the if statement in line 11 cannot be satisfied. Therefore, \( X \gamma_{\text{PAR}}^{K'} Z_F \in R_P \), and the assertion is proved.

Following a similar reasoning we can easily prove that also properties 3 and 4 in definition 4.1 are satisfied.
Algorithm BUILD-PARALLEL-MBRS($M,K$)

1 $R_P := \{ r \in \mathbb{R} \mid r \text{ is a PAR rule} \}$;
2 for $i = 1, \ldots, n$ do
3 $R_{P,i} := \{ r \in R_P \mid r \text{ is a PAR rule} \}$;
4 repeat
5 flag := false;
6 for each $r = X \xrightarrow{\sigma} Y.(Z) \in R$ such that $\Upsilon_M(r) \subseteq K$ do
7 $\ Upsilon^f_M(r)$
8 Set $K_1 = \Upsilon^f_M(r)$
9 for each $K_2 \subseteq K$ do
10 if $Z \xrightarrow{\sigma^*} P$ for some $P$ such that $\Upsilon^f_M(\sigma) = K_2$ then
11 if $X \xrightarrow{K_1} \hat{Z}_F \notin R_P$ then
12 $R_P := R_P \cup \{ X \xrightarrow{K_1} \hat{Z}_F \}$;
13 for $j = 1, \ldots, |K'|$ do
14 $\ Upsilon^A_{P,i}$
15 flag := true;
16 if $Z \xrightarrow{\sigma^*} P$ for some $P$ such that $\Upsilon^f_M(\sigma) = K_2$ then
17 Set $K' = K_1 \cup K_2 = \{ i_1, \ldots, i_{|K'|} \}$;
18 if $X \xrightarrow{K'} Y \notin R_P$ then
19 $R_P := R_P \cup \{ X \xrightarrow{K'} Y \}$;
20 for $j = 1, \ldots, |K'|$ do
21 $\ Upsilon^A_{P,i}$
22 flag := true;
23 done $\triangleright$ for
24 for each $r' = Y.(W) \xrightarrow{\sigma} W' \in R$ such that $\Upsilon_M(r') \subseteq K$ do
25 Set $K_2 = \Upsilon_M(r')$
26 for each $K_3 \subseteq K$ do
27 if $Z \xrightarrow{\sigma^*} P$ for some $P$ such that $\Upsilon_M(\sigma) = K_3$ then
28 Set $K' = K_1 \cup K_2 \cup K_3 = \{ i_1, \ldots, i_{|K'|} \}$;
29 if $X \xrightarrow{K'} W' \notin R_P$ then
30 $R_P := R_P \cup \{ X \xrightarrow{K'} W' \}$;
31 for $j = 1, \ldots, |K'|$ do
32 $\ Upsilon^A_{P,i}$
33 flag := true;
34 done $\triangleright$ for
35 done $\triangleright$ for
36 done $\triangleright$ for
37 until flag = false

Figure 1: Algorithm to turn a MBRS into a parallel MBRS.
Now, let us prove property 5 of definition 4.3.1. Let \( \bar{\tau} = X^{K'} \bar{p}' \in \mathcal{R}^{K}_{PAR} \setminus \mathcal{R} \). Therefore, \( X \in \text{Var}, \bar{p}' \in \text{Var} \cup \{ \bar{Z}_F \} \) and \( K' \in P_n \). We have to prove that \( X \xrightarrow{\bar{p}' \bar{t}} \bar{X} \) for some term \( t \), with \( |\sigma| > 0 \), \( \mathcal{Y}^f_M (\sigma) = K' \) and \( t = p' \) if \( p' \in \text{Var} \). Let us assume that \( \bar{\tau} \) is the \( n \)-th rule added to \( \mathcal{R}_P \) during the computation. Then, \( \bar{\tau} \) is added to \( \mathcal{R}_P \) during an iteration of the for loop in lines 6-36, in which a rule \( r \) of the form \( X^{a} Y.(Z) \in \mathcal{R} \) is examined. Let \( K_1 = \mathcal{Y}^f_M (r) \). The proof is by induction on \( n \).

**Base Step:** \( n = 1 \). In this phase

1. \( \mathcal{R}_P = \{ r \in \mathcal{R} | r \text{ is a PAR rule} \} \) and \( \forall i = 1, \ldots, n \mathcal{R}^A_{P,i} = \{ r \in \mathcal{R}^A_i | r \text{ is a PAR rule} \} \).

2. \( \forall \bar{r}' \in \mathcal{R}_P \) we have \( \mathcal{Y}^f_M (\bar{r}') = \mathcal{Y}^f_{M_p} (\bar{r}') \).

There are three cases:

- \( \bar{\tau} \) is added to \( \mathcal{R}_P \) in line 12. Then, \( \bar{p}' = \bar{Z}_F \notin \text{Var} \), and the condition of the if statement in line 9 is satisfied: \( Z^{s_p} \sigma \notin \mathcal{R}_P \) for some \( p \), with \( \mathcal{Y}^f_M (\sigma) = K_2 \) and \( K' = K_1 \cup K_2 \). From 1 we deduce that \( \sigma \) is a sequence of PAR rules in \( \mathcal{R} \). From 2 it follows that \( \mathcal{Y}^f_M (\sigma) = \mathcal{Y}^f_{M_p} (\sigma) \). Therefore, \( X \xrightarrow{\bar{r}} Y.(Z) \xrightarrow{\bar{p}} Y.(\bar{t}) \) with \( \mathcal{Y}^f_M (r \sigma) = K_1 \cup K_2 = K' \), and the assertion is proved.

- \( \bar{\tau} \) is added to \( \mathcal{R}_P \) in line 19. Then, \( \bar{p}' = Y \in \text{Var} \), and the condition of the if statement in line 16 is satisfied: \( Z^{s_p} \sigma \notin \mathcal{R}_P \) with \( \mathcal{Y}^f_M (\sigma) = K_2 \) and \( K' = K_1 \cup K_2 \). From 1 we deduce that \( \sigma \) is a sequence of PAR rules in \( \mathcal{R} \). From 2 it follows that \( \mathcal{Y}^f_M (\sigma) = \mathcal{Y}^f_{M_p} (\sigma) \). Therefore, \( X \xrightarrow{\bar{r}} Y.(Z) \xrightarrow{\bar{p}} Y.(\bar{t}) \) with \( \mathcal{Y}^f_M (r \sigma) = K_1 \cup K_2 = K' \), and the assertion is proved.

- \( \bar{\tau} \) is added to \( \mathcal{R}_P \) by the inner for loop in lines 24-35, when a rule \( \bar{r}' \) of the form \( Y.(W) \xrightarrow{b} W' \) is examined. Then, \( \bar{p}' = W' \in \text{Var} \), and \( \bar{\tau} \) is added to \( \mathcal{R}_P \) in line 30. So, the condition of the if statement in line 27 is satisfied: \( Z^{s_p} \sigma \notin \mathcal{R}_P \) with \( \mathcal{Y}^f_M (\sigma) = K_3 \) and \( K' = K_1 \cup K_2 \cup K_3 \), where \( K_2 = \mathcal{Y}^f_M (\bar{r}') \). From 1 we deduce that \( \sigma \) is a sequence of PAR rules in \( \mathcal{R} \). From 2 it follows that \( \mathcal{Y}^f_M (\sigma) = \mathcal{Y}^f_{M_p} (\sigma) \). Therefore, \( X \xrightarrow{\bar{r}} Y.(Z) \xrightarrow{\bar{p}} Y.(W) \xrightarrow{\bar{t}} W' \) with \( \mathcal{Y}^f_M (r \sigma \bar{r}') = K_1 \cup K_2 \cup K_3 = K' \), and the assertion is proved.

**Induction Step:** \( n > 1 \). Let \( \mathcal{R}_P \) (resp., \( \mathcal{R}^A_{P,i} \) for \( i = 1, \ldots, n \)) be the set of the rules in \( \mathcal{R}^{K}_{PAR} \setminus \mathcal{R} \) (resp., \( \mathcal{R}^{K,A}_{PAR,i} \setminus \mathcal{R} \) for \( i = 1, \ldots, n \)) after \( n - 1 \) rules have been added to \( \mathcal{R}_P \). Then, in this phase we have

- \( \mathcal{R}_P = \{ r \in \mathcal{R} | r \text{ is a PAR rule} \} \cup \mathcal{R}_P, \) and \( \forall i = 1, \ldots, n \mathcal{R}^A_{P,i} = \{ r \in \mathcal{R}^A_i | r \text{ is a PAR rule} \} \cup \mathcal{R}^A_{P,i} \).

- \( \forall \bar{r} = i \xrightarrow{K} \bar{t} \in \mathcal{R}_P \) we have \( \mathcal{Y}^f_{M_p} (\bar{r}) = \hat{K} \).

- \( \forall \bar{r}' \in \{ r \in \mathcal{R} | r \text{ is a PAR rule} \} \) we have \( \mathcal{Y}^f_{M_p} (\bar{r}') = \mathcal{Y}^f_{M_p} (\bar{r}') \).
From the inductive hypothesis, we deduce easily that the following property is satisfied

3. If \( \hat{\gamma} \xrightarrow{r} \hat{\nu} \), where \( \hat{\nu} \) is a parallel term, then \( \hat{\gamma} \xrightarrow{r} \hat{\nu} \) for some term \( \hat{\nu} \), with \( \Upsilon^f_M(\rho) = \Upsilon^f_{M_P}(\sigma) \). Moreover, if \( \hat{\nu} \) doesn’t contain occurrences of \( \hat{\gamma} \) then \( \hat{\nu} = \hat{\nu}' \).

As before, there are three cases:

- \( \Psi \) is added to \( \Re_P \) in line 12. Then, \( p' = \hat{\gamma} \notin \text{Var} \), and the condition of the if statement in line 9 is satisfied: \( Z \xrightarrow{\sigma} p \) for some term \( p \), with \( \Upsilon^f_{M_P}(\sigma) = K_2 \) and \( K' = K_1 \cup K_2 \). From 3 it follows that \( Z \xrightarrow{\sigma} t \) for some \( t \), with \( \Upsilon^f_M(\rho) = \Upsilon^f_{M_P}(\sigma) \). Therefore, \( X \xrightarrow{\tau} Y. (Z) \xrightarrow{\rho} Y. (t) \) with \( \Upsilon^f_M(\rho) = K_1 \cup K_2 = K' \), and the assertion is proved.

- \( \Psi \) is added to \( \Re_P \) in line 19. Then, \( p' = Y \in \text{Var} \), and the condition of the if statement in line 16 is satisfied: \( Z \xrightarrow{\sigma} \varepsilon \) with \( \Upsilon^f_{M_P}(\sigma) = K_2 \) and \( K' = K_1 \cup K_2 \). From 3 it follows that \( Z \xrightarrow{\sigma} \varepsilon \), with \( \Upsilon^f_M(\rho) = \Upsilon^f_{M_P}(\sigma) \). Therefore, \( X \xrightarrow{\tau} Y. (Z) \xrightarrow{\rho} Y \) with \( \Upsilon^f_M(\rho) = K_1 \cup K_2 = K' \), and the assertion is proved.

- \( \Psi \) is added to \( \Re_P \) by the inner for loop in lines 24-35, when a rule \( r' \) of the form \( Y. (W) \xrightarrow{b} W' \) is examined. Then, \( p' = W' \in \text{Var} \), and \( \Psi \) is added to \( \Re_P \) in line 30. So, the condition of the if statement in line 27 is satisfied: \( Z \xrightarrow{\sigma} W \) with \( \Upsilon^f_{M_P}(\sigma) = K_3 \) and \( K' = K_1 \cup K_2 \cup K_3 \), where \( K_2 = \Upsilon^f_M(r') \). From 3 it follows that \( Z \xrightarrow{\sigma} \varepsilon \), with \( \Upsilon^f_M(\rho) = \Upsilon^f_{M_P}(\sigma) \). Therefore, \( X \xrightarrow{\tau} Y. (Z) \xrightarrow{\rho} Y. (W) \xrightarrow{b} W' \) with \( \Upsilon^f_M(\rho r') = K_1 \cup K_2 \cup K_3 = K' \), and the assertion is proved.

Finally, it’s easy to show that \( M^K_{PAR} \) is the least parallel MBRS over \( \text{Var} \) and the alphabet \( \Sigma \) satisfying properties 1-5 of definition 4.1.

Remark 4.1. By construction the following properties hold:

- \( \forall r \in \Re \cap \Re^K_{PAR} \text{ we have } \Upsilon^f_{M^K_{PAR}}(r) = \Upsilon^f_M(r) \).

- \( \forall r = X \xrightarrow{K'} \rho \in \Re^K_{PAR} \setminus \Re \text{ we have } \Upsilon^f_{M^K_{PAR}}(r) = K' \).

Soundness and completeness of the procedure described above is stated by the following theorem, whose proof is reported in appendix (section B).

Theorem 4.3. For all \( X \in \text{Var} \) there exists a \( (K, \emptyset) \)-accepting finite derivation in \( M \) from \( X \) if, and only if, there exists a \( (K, \emptyset) \)-accepting finite derivation in \( M^K_{PAR} \) from \( X \).

This result, together with proposition 4.1, allow us to conclude that Problem 1, stated at the beginning of this section, is decidable.
4.3 Decidability results on infinite derivations of MBRSs in normal form

In this section we prove the decidability of Problem 2 stated in subsection 3.1 that for clarity we recall.

**Problem 2** Given a MBRS in normal form $M = \langle \mathcal{R}, \langle \mathcal{R}_1^A, \ldots, \mathcal{R}_n^A \rangle \rangle$ over $\text{Var}$ and the alphabet $\Sigma$, given a variable $X \in \text{Var}$ and two sets $K, K^\omega \in \mathcal{P}_n$, to decide if there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ from $X$.

Let us observe that a necessary condition for the existence of a $(K, K^\omega)$-accepting infinite derivation in $M$ is that $K \supseteq K^\omega$.

The proof of decidability is by induction on $|K| + |K^\omega|$.

**Base Step:** $|K| = 0$ and $|K^\omega| = 0$. Let $M_F = \langle \mathcal{R}, \mathcal{R}_F \rangle$ be the BRS with $\mathcal{R}_F = \bigcup_{i=1}^n \mathcal{R}_i^A$. Given an infinite derivation $X \xrightarrow{\mathcal{R}_F^*} \emptyset$ in $\mathcal{R}$ from a variable $X$ then, this derivation is $(\emptyset, \emptyset)$-accepting in $M$ if, and only if, it doesn’t contain occurrences of accepting rules in $M_F$. So, the decidability result follows from theorem 2.1.

**Inductive Step:** $|K| + |K^\omega| > 0$. From the inductive hypothesis, for each $K' \subseteq K$ and $K^\omega \subseteq K^\omega$ with $|K'| + |K^\omega'| < |K| + |K^\omega|$ the result holds. In other words, it is decidable if there exists a $(K', K^\omega)$-accepting infinite derivation in $M$ from a variable $X$. Starting from this assumption we show that problem 2, with input the sets $K$ and $K^\omega$, can be reduced to (a combination of) two similar, but simpler, problems: the first is a decidability problem on finite derivation in $M$ and a variable $X$; the second is a decidability problem on infinite derivations restricted to parallel MBRSs; the second is a decidability problem on infinite derivations restricted to sequential MBRSs. More precisely, we show that it is possible to construct effectively two parallel MBRSs $M_{K,K^\omega}^{\text{PAR}} = \langle \mathcal{R}_{\text{PAR}}, \langle \mathcal{R}_{\text{PAR}}^K, \mathcal{R}_{\text{PAR}}^{K^\omega} \rangle \rangle$ and $M_{K,K^\omega}^{\text{PAR,\infty}} = \langle \mathcal{R}_{\text{PAR}}, \langle \mathcal{R}_{\text{PAR}}^K, \mathcal{R}_{\text{PAR,\infty}}^{K^\omega} \rangle \rangle$ with the same support, and a sequential MBRS $M_{K,K^\omega}^{\text{SEQ}} = \langle \mathcal{R}_{\text{SEQ}}^K, \langle \mathcal{R}_{\text{SEQ}}^K, \mathcal{R}_{\text{SEQ}}^{K^\omega} \rangle \rangle$, in such a way that Problem 2, with input the sets $K$ and $K^\omega$ and a variable $X \in \text{Var}$, is reducible to one of two following problems depending if $K \supset K^\omega$ or $K = K^\omega$.

**Problem 3** ($K \supset K^\omega$). To decide if the following condition is satisfied:

- There exists a variable $Y \in \text{Var}$ reachable from $X$ in $\mathcal{R}_{\text{SEQ}}^K$ through a $(K', \emptyset)$-accepting derivation in $M_{\text{SEQ}}^K$ with $K' \subseteq K$, and there exists a derivation $Y \xrightarrow{\mathcal{R}_{\text{PAR}}^*} \emptyset$ such that $\Upsilon_f^{M_{\text{PAR}}^K,K^\omega}(\rho) = K$ and $\Upsilon_f^{M_{\text{PAR}}^K,K^\omega}(\rho) \cup \Upsilon_f^{M_{\text{PAR}}^K,K^\omega}(\rho) = K^\omega$. Moreover, either $\rho$ is infinite or $\rho$ contains some occurrence of rule in $\mathcal{R}_{\text{PAR}}^K \setminus \mathcal{R}_{\text{PAR}}^K$ (where $\mathcal{R}_{\text{PAR}}^K$ is the support of the parallel MBRS $M_{\text{PAR}}^K$ defined in the previous subsection).

**Problem 4** ($K = K^\omega$). To decide if one of the following conditions is satisfied:

- There exists a variable $Y \in \text{Var}$ reachable from $X$ in $\mathcal{R}_{\text{SEQ}}^K$ through a $(K', \emptyset)$-accepting derivation in $M_{\text{SEQ}}^K$ with $K' \subseteq K$, and there exists a derivation $Y \xrightarrow{\mathcal{R}_{\text{PAR}}^*} \emptyset$ such that $\Upsilon_f^{M_{\text{PAR}}^K,K^\omega}(\rho) = K$ and $\Upsilon_f^{M_{\text{PAR}}^K,K^\omega}(\rho) \cup \Upsilon_f^{M_{\text{PAR}}^K,K^\omega}(\rho) = K^\omega$. Moreover, either $\rho$ is infinite or $\rho$ contains some occurrence of rule in $\mathcal{R}_{\text{PAR}}^K \setminus \mathcal{R}_{\text{PAR}}^K$ (where $\mathcal{R}_{\text{PAR}}^K$ is the support of the parallel MBRS $M_{\text{PAR}}^K$ defined in the previous subsection).
such that $\Upsilon_{K,K}^f(\rho) = K$ and $\Upsilon_{M_{PAR}}^\infty(\rho) \cup \Upsilon_{M_{PAR,\infty}}^f(\rho) = K^\omega$. Moreover, either $\rho$ is infinite or $\rho$ contains some occurrence of rule in $\mathcal{R}_{PAR}^{K,K} \setminus \mathcal{R}_{PAR}^K$.

- There exists a $(K, K^\omega)$-accepting infinite derivation in $M_{SEQ}^K$ from $X$.

Since these last problems are decidable (see theorems 4.1 and 4.2), decidability of Problem 2 is entailed.

Before illustrating the main idea underlying our approach, we need the following definition.

**Definition 4.5.** Let us denote by $\Pi_{PAR,\infty}^{K,K}$ the set of derivations $t \Rightarrow_{\mathcal{R}}^\ast$ in $\mathcal{R}$ not satisfying the following property:

- There exists a subderivation of $t \Rightarrow_{\mathcal{R}}^\ast$ that is a $(K, K^\omega)$-accepting infinite derivation in $M$.

In the following we use a new variable $\hat{Z}_{\infty}$, and denote by $T$ (resp., $T_{PAR}$, $T_{SEQ}$) the set of process terms (resp., the set of terms in which no sequential composition occurs, the set of terms in which no parallel composition occurs) over $Var \cup \{\hat{Z}_F, \hat{Z}_{\infty}\}$.

Let us sketch the main ideas at the basis of our technique. At first, let us focus on the class of derivations $\Pi_{PAR,\infty}^{K,K}$, showing how it is possible to mimic a $(K, K^\omega)$-accepting infinite derivation in $M$ from a variable, belonging to this class, by using only PAR rules belonging to extensions of the parallel MBRS $M_{PAR}^K$ computed by the algorithm of lemma 4.1 (with input $M$ and $K$). More precisely, we'll show, as anticipated, that it is possible construct two parallel extensions of $M_{PAR}$ with the same support, denoted by $M_{PAR} = \langle \mathcal{R}_{PAR,1}, \ldots, \mathcal{R}_{PAR,n} \rangle$ and $M_{PAR,\infty} = \langle \mathcal{R}_{PAR,1}, \ldots, \mathcal{R}_{PAR,\infty,1}, \ldots, \mathcal{R}_{PAR,\infty,n} \rangle$ (with $\mathcal{R}_{PAR,i} \supseteq \mathcal{R}_{PAR,i}$ and $\mathcal{R}_{PAR,\infty,i} \cap \mathcal{R}_{PAR,i} = \emptyset$ for $i = 1, \ldots, n$), in such way that the following condition holds:

**i.** There exists a $(\overline{K}, \overline{K}^\omega)$-accepting derivation in $M$ belonging to $\Pi_{PAR,\infty}^{K,K}$ of the form $p \Rightarrow_{\mathcal{R}}^\ast$ with $p \in T_{PAR}$, $\overline{K} \subseteq K$ and $\overline{K}^\omega \subseteq K^\omega$ if, and only if, there exists a derivation $p \Rightarrow_{\mathcal{R}_{PAR}}^\ast$ in $\mathcal{R}_{PAR}^{K,K}$ from $p$ such that $\Upsilon_{M_{PAR}}^f(\rho) = \overline{K}$ and $\Upsilon_{M_{PAR,\infty}}^\infty(\rho) \cup \Upsilon_{M_{PAR,\infty}}^f(\rho) = \overline{K}^\omega$. Moreover, if $\sigma$ is infinite then, either $\rho$ is infinite or $\rho$ contains some occurrence of rule in $\mathcal{R}_{PAR}^{K,K} \setminus \mathcal{R}_{PAR}^K$, and vice versa.

So, let $p \Rightarrow_{\mathcal{R}}^\ast$ be a $(\overline{K}, \overline{K}^\omega)$-accepting derivation in $M$ belonging to $\Pi_{PAR,\infty}^{K,K}$ with $p \in T_{PAR}$, $\overline{K} \subseteq K$ and $\overline{K}^\omega \subseteq K^\omega$. Then, all its possible subderivations contain all, and only, the rule occurrences in $\sigma$ applied at a level $k$ greater than 0 in $p \Rightarrow_{\mathcal{R}}^\ast$. If $\sigma$ contains only PAR rule occurrences the statement **i** is evident, since by construction (remember that

\[\text{\textsuperscript{12}}\hat{Z}_F\] is the variable used in the previous subsection for the construction of the parallel MBRS $M_{PAR}^K$.}
$\mathcal{R}_{PAR}^K$ contains all PAR rules of $\mathcal{R}$, we have $p \overset{r}{\Rightarrow}_{K,K,\omega}^*$ with $\Upsilon_{M_{PAR}}^f(\sigma) = \mathcal{K}$, $\Upsilon_{M_{PAR}}^\infty(\sigma) = \emptyset$. Otherwise, $p \overset{r}{\Rightarrow}_K^*$ can be written in the form:

$$p \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* t||X \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* t||Y.(Z) \overset{r}{\Rightarrow}_{\mathcal{R}_r}^*$$

where $r = X \overset{\alpha}{\Rightarrow} Y.(Z)$, $\lambda$ contains only occurrences of PAR rules in $\mathcal{R}$, $t \in T_{PAR}$ and $X,Y,Z \in \text{Var}$. Let $Z \overset{r}{\Rightarrow}_{\mathcal{R}_r}$ be a subderivation of $t||Y.(Z) \overset{r}{\Rightarrow}_{\mathcal{R}_r}$ from $Z$. Since $p \overset{r}{\Rightarrow}_{\mathcal{R}_r}$ is in $\Pi_{PAR,\infty}$, $Z \overset{r}{\Rightarrow}_{\mathcal{R}_r}$ is not a $(K,K,\omega)$-accepting infinite derivation in $M$. More precisely, $\Upsilon_{M}(\rho) \subseteq K$, $\Upsilon_{M}^\infty(\rho) \subseteq K(\omega)$ (since $\rho$ is a subsequence of $\sigma$) and $|\Upsilon_{M}(\rho)| + |\Upsilon_{M}^\infty(\rho)| < |K| + |K|\omega|$. Thus, only one of the following four cases may occur:

A $Z \overset{r}{\Rightarrow}_{\mathcal{R}_r}$ leads to the term $\varepsilon$, and $p \overset{r}{\Rightarrow}_{\mathcal{R}_r}$ is of the form

$$p \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* t||X \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* t||Y.(Z) \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* Y \overset{r}{\Rightarrow}_{\mathcal{R}_r}^*$$

where $\rho$ is a subsequence of $\omega_1$ and $t \overset{\omega_1}{\Rightarrow}_{\mathcal{R}_r}^* T$. The derivation above is $(\mathcal{K}, \mathcal{K''})$-accepting in $M$ if, and only if, the following derivation is $(\mathcal{K}, \mathcal{K''})$-accepting in $M$

$$p \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* t||X \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* t||Y.(Z) \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* Y \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* Y \overset{r}{\Rightarrow}_{\mathcal{R}_r}^*$$

where $\xi = \omega_1 \setminus \rho$. Let us consider the derivation $X \overset{r}{\Rightarrow}_{\mathcal{R}_r} Y.(Z) \overset{r}{\Rightarrow}_{\mathcal{R}_r} Y$ where $\Upsilon_{M}(\rho) \subseteq K$. From properties of $M_{PAR}^K$ (see lemma $[1,2]$ in appendix) there exists a derivation of the form $X \overset{r}{\Rightarrow}_{\mathcal{R}_r} Y$. Such that $\Upsilon_{M_{PAR}}^f(\eta) = \Upsilon_{M}(\rho)$. Now, we can apply recursively the same reasoning to the derivation in $\mathcal{R}$ from $t||Y \in T_{PAR}$

$$t||Y \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* Y \overset{r}{\Rightarrow}_{\mathcal{R}_r}^*$$

which belongs to $\Pi_{PAR,\infty}$ and whose finite (resp., infinite) maximal as to $M$ is contained in $K$ (resp., $K''$).

B $Z \overset{r}{\Rightarrow}_{\mathcal{R}_r}$ leads to a variable $W$, and $p \overset{r}{\Rightarrow}_{\mathcal{R}_r}$ can be written as

$$p \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* t||X \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* t||Y.(Z) \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* Y.(W) \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* W' \overset{r}{\Rightarrow}_{\mathcal{R}_r}^*$$

where $r' = Y.(W) \overset{b}{\Rightarrow} W'$ (with $W' \in \text{Var}$), $\rho$ is a subsequence of $\omega_1$ and $t \overset{\omega_1}{\Rightarrow}_{\mathcal{R}_r}^* T$. The derivation above is $(\mathcal{K}, \mathcal{K''})$-accepting if, and only if, the following derivation is $(\mathcal{K}, \mathcal{K''})$-accepting

$$p \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* t||X \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* t||Y.(Z) \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* Y.(W) \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* W' \overset{r}{\Rightarrow}_{\mathcal{R}_r}^* W' \overset{r}{\Rightarrow}_{\mathcal{R}_r}^*$$

where $\xi = \omega_1 \setminus \rho$. Let us consider the derivation $X \overset{r}{\Rightarrow}_{\mathcal{R}_r} Y.(Z) \overset{r}{\Rightarrow}_{\mathcal{R}_r} Y.(W) \overset{r}{\Rightarrow}_{\mathcal{R}_r} W'$ where $\Upsilon_{M}(\rho') \subseteq K$. From properties of $M_{PAR}^K$ (see lemma $[B,2]$ in appendix) there exists
a derivation of the form $X \xrightarrow{\rho \eta}_f W'$ such that $\Upsilon^f_{M^f_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}} (\eta) = \Upsilon^f_M (\rho \eta \rho)$. Now, we can apply recursively the same reasoning to the derivation in $\mathbb{R}$ from $t \parallel W' \in T_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$

$$t \parallel W' \xrightarrow{\xi \eta}_f t \parallel W' \equiv \zeta \eta_f$$

which belongs to $\Pi^{K,K\omega}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$ and whose finite (resp., infinite) maximal as to $M$ is contained in $K$ (resp., $K\omega$).

C $Z \xrightarrow{\rho \eta}_f$ is finite and does not influence the applicability of rule occurrences in $\omega \setminus \rho$ in the derivation $t \parallel Y. (Z) \xrightarrow{\omega \eta}_f$. In other words, we have $t \xrightarrow{\omega \eta \rho}_f$. Moreover, $\Upsilon^f_M (\rho \eta \rho) \subseteq K$. Let us consider the finite derivation $X \xrightarrow{t \eta}_f Y. (Z) \xrightarrow{\rho \eta}_f$. From properties of $M^f_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$ (see lemma B.2 in appendix) there exists a finite derivation of the form $X \xrightarrow{\eta \rho \eta \rho}_f \mathcal{P}$ with $\mathcal{P} \in T_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$ and $\Upsilon^f_{M^f_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}} (\eta) = \Upsilon^f_M (\rho \eta \rho)$. Now, we can apply recursively the same reasoning to the derivation $t \parallel \mathcal{P} \xrightarrow{\rho \eta \rho}_f$ in $\mathbb{R}$ from $t \parallel \mathcal{P} \in T_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$, which belongs to $\Pi^{K,K\omega}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$ and whose finite (resp., infinite) maximal as to $M$ is contained in $K$ (resp., $K\omega$).

D $Z \xrightarrow{\rho \eta}_f$ is infinite. From the definition of subderivation we have $t \xrightarrow{\omega \eta \rho}_f$. Moreover, $\Upsilon^f_M (\rho) = K_1 \subseteq K$, $\Upsilon^f_M (\rho) = K_1 \subseteq K\omega$, $\Upsilon^f_M (\rho) = K_2 \subseteq K$, $\Upsilon^f_M (\rho) = \emptyset$ and $|K_1| + |K_1\omega| < |K| + |K\omega|$. From our assumptions (inductive hypothesis) it is decidable if there exists a $(K_1, K_1\omega)$-accepting infinite derivation in $M$ from variable $Z$. Then, we keep track of the infinite sequence $\rho \eta$ by adding the new variable $\hat{Z}_\omega$ (where $\hat{Z}_\omega \notin \text{Var}$) and a PAR rule of the form $r' = X \xrightarrow{K', K\omega} \hat{Z}_F$ with $K' = K_1 \cup K_2$, $\Upsilon^f_{M^f_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}} (r') = K'$ and $\Upsilon^f_{M^f_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}} (r') = K\omega$. Now, we can apply recursively the same reasoning to the derivation $t \parallel \hat{Z}_\omega \xrightarrow{\rho \eta \rho}_f$ in $\mathbb{R}$ from $t \parallel \hat{Z}_\omega$ (where $t \parallel \hat{Z}_\omega \in T_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$), which belongs to $\Pi^{K,K\omega}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$ and whose finite (resp., infinite) maximal as to $M$ is contained in $K$ (resp., $K\omega$).

In other words, all subderivations in $p \xrightarrow{\sigma \eta}_f$ are abstracted away by PAR rules not belonging to $\mathbb{R}$, according to the intuitions given above.

By the parallel MBRS $M^f_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$ we keep track of subderivations of the forms A, B and C. In order to simulate subderivations of the form D, we need to add additional PAR rules in $M^f_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$. The following definition provides an extension of $M^f_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$ suitable for our purposes.

**Definition 4.6.** By $M^f_{\Pi^{K,K\omega}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}} = (\mathbb{R}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}, (\mathbb{R}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}, \ldots, \mathbb{R}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}))$ and $M^f_{\Pi^{K,K\omega}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}} = (\mathbb{R}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}, (\mathbb{R}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}, \ldots, \mathbb{R}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}))$ we denote the parallel MBRSs over $\text{Var}$ and $\{\hat{Z}_F, \hat{Z}_\omega\}$ and the alphabet $\Sigma \cup P_n \cup P_n \times P_n$, defined by $M$ and $M^f_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}_{\Pi^{P:\Pi}}}}$ in the following way:
By construction the following properties hold:

- \( \mathcal{R}_{\mathcal{P}AR}^{K,K^\omega} = \mathcal{R}_{\mathcal{P}AR}^{K} \cup \{ X \xrightarrow{K,K^\omega} \hat{Z}_\infty \mid \hat{K} \subseteq K, \hat{K}^\omega \subseteq K^\omega \}, \) there exists a rule \( r = \alpha \rightarrow \beta \cdot (Z) \in \mathcal{R} \) and an infinite derivation \( Z \xrightarrow{\alpha}^* \) such that
  \[ |\gamma^f_M(\sigma)| + |\gamma^\infty_M(\sigma)| < |K| + |K^\omega| \text{ and } \gamma^f_M(\sigma) \cup \gamma^f_M(r) = \hat{K} \text{ and } \gamma^\infty_M(\sigma) = \hat{K}^\omega \}

- \( \mathcal{R}_{\mathcal{P}AR,i}^{K,K^\omega} = \mathcal{R}_{\mathcal{P}AR,i}^{K} \cup \{ X \xrightarrow{K,K^\omega} \hat{Z}_\infty \in \mathcal{R}_{\mathcal{P}AR}^{K,K^\omega} \mid i \in \hat{K} \} \) for all \( i = 1, \ldots, n \)

- \( \mathcal{R}_{\mathcal{P}AR,i}^{K,K^\omega,A} = \mathcal{R}_{\mathcal{P}AR,i}^{K,A} \cup \{ X \xrightarrow{K,K^\omega} \hat{Z}_\infty \in \mathcal{R}_{\mathcal{P}AR}^{K,K^\omega} \mid i \in \hat{K} \} \) for all \( i = 1, \ldots, n \)

From the inductive hypothesis on the decidability of problem 2 for sets \( \hat{K}, \hat{K}^\omega \in \mathcal{P}_n \) such that \( \hat{K} \subseteq K, \hat{K}^\omega \subseteq K^\omega \) and \( |\hat{K}| + |\hat{K}^\omega| < |K| + |K^\omega| \), it follows that \( M_{\mathcal{P}AR}^{K,K^\omega} \) and \( M_{\mathcal{P}AR,\infty}^{K,K^\omega} \) can be built effectively. Thus, the following result holds.

**Lemma 4.2.** \( M_{\mathcal{P}AR}^{K,K^\omega} \) and \( M_{\mathcal{P}AR,\infty}^{K,K^\omega} \) can be built effectively.

**Remark 4.2.** By construction, the following properties hold:

- for all \( r \in \mathcal{R}_{\mathcal{P}AR}^{K} \) we have \( \gamma^f_{M_{\mathcal{P}AR}^{K,K^\omega}}(r) = \gamma^f_{M_{\mathcal{P}AR}^{K}}(r) \) and \( \gamma^f_{M_{\mathcal{P}AR}^{K,K^\omega,\infty}}(r) = \emptyset \).
- for all \( r \in \mathcal{R}_{\mathcal{P}AR}^{K,K^\omega} \cap \mathcal{R} \) we have \( \gamma^f_{M_{\mathcal{P}AR}^{K,K^\omega}}(r) = \gamma^f_{M_{\mathcal{P}AR}^{K}}(r) \) and \( \gamma^f_{M_{\mathcal{P}AR}^{K,K^\omega,\infty}}(r) = \emptyset \).
- for all \( r = X \xrightarrow{K,K^\omega} \hat{Z}_\infty \in \mathcal{R}_{\mathcal{P}AR}^{K,K^\omega} \) we have \( \gamma^f_{M_{\mathcal{P}AR}^{K,K^\omega}}(r) = \hat{K} \) and \( \gamma^f_{M_{\mathcal{P}AR}^{K,K^\omega,\infty}}(r) = \hat{K}^\omega \).

Now, let us go back to Problem 2 and consider a \((K,K^\omega)\)-accepting infinite derivation in \( M \) from a variable \( X \) of the form \( X \xrightarrow{K} Y \), and non belonging to \( \Pi_{\mathcal{P}AR,\infty}^{K,K^\omega} \). In this case, the derivation \( X \xrightarrow{\alpha}^*_r \) can be written in the form \( X \xrightarrow{\alpha}^*_r t \parallel Y \cdot Z \xrightarrow{\beta}^*_r \) with \( Z \in \text{Var} \), and such that there exists a subderivation of \( t \parallel Y \cdot Z \xrightarrow{\beta}^*_r \) from \( Z \) that is a \((K,K^\omega)\)-accepting infinite derivation in \( M \). To manage this kind of derivation, we build, starting from the MBRSs \( M \) and \( M_{\mathcal{P}AR}^{K} \), a sequential MBRS \( M_{\mathcal{S}E_Q}^{K} \) according to the following definition:

**Definition 4.7.** By \( M_{\mathcal{S}E_Q}^{K} = (\mathcal{R}_{\mathcal{S}E_Q}^{K}, \{ \mathcal{R}_{\mathcal{S}E_Q,1}^{K,A}, \ldots, \mathcal{R}_{\mathcal{S}E_Q,n}^{K,A} \}) \) we denote the sequential MBRS over \( \text{Var} \) and the alphabet \( \sum = \sum \cup \mathcal{P}_n \) so defined:

- \( \mathcal{R}_{\mathcal{S}E_Q}^{K} = \{ X \xrightarrow{\alpha} Y \cdot (Z) \in \mathcal{R} \} \cup \{ X \xrightarrow{K^\prime} Y \mid X, Y \in \text{Var}, K^\prime \subseteq K \text{ and there exists a derivation } X \xrightarrow{\alpha}^*_r p \parallel Y \text{ in } \mathcal{R}_{\mathcal{P}AR}^{K} \text{ for some } p \in \mathcal{T}_{\mathcal{P}AR}, \text{ with } |\sigma| > 0 \text{ and } \gamma^f_{M_{\mathcal{P}AR}^{K}}(\sigma) = K^\prime \} \)
- \( \mathcal{R}_{\mathcal{S}E_Q,i}^{K,A} = \{ X \xrightarrow{\alpha} Y \cdot (Z) \in \mathcal{R}^A_{\mathcal{S}E_Q} \} \cup \{ X \xrightarrow{K^\prime} Y \in \mathcal{R}_{\mathcal{S}E_Q}^{K} \} \forall i \in K^\prime \} \) for all \( i = 1, \ldots, n \)

**Remark 4.3.** By construction the following properties hold:
• for all \( r \in \mathcal{R} \cap \mathcal{R}^K_{\text{SEQ}} \) we have \( \mathcal{Y}_M^f (r) = \mathcal{Y}_{M^K_{\text{SEQ}}}^f (r) \).

• for all \( r = X^K \leadsto Y \in \mathcal{R}^K_{\text{SEQ}} \setminus \mathcal{R} \) we have \( \mathcal{Y}_{M^K_{\text{SEQ}}}^f (r) = K' \).

**Lemma 4.3.** \( M^K_{\text{SEQ}} \) can be built effectively.

**Proof.** The result follows directly from the definition of \( M^K_{\text{SEQ}} \) and proposition \[ \] \[ \square \]

Soundness and completeness of the procedure described above is stated by the following two theorems, whose proof is reported in appendix (section C).

**Theorem 4.4.** Let us assume that \( K \neq K^\omega \). Given \( X \in \text{Var} \), there exists a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( X \) if, and only if, the following property is satisfied:

- There exists a variable \( Y \in \text{Var} \) reachable from \( X \) in \( \mathcal{R}^K_{\text{SEQ}} \) through a \((K', \emptyset)\)-accepting derivation in \( M^K_{\text{SEQ}} \) with \( K' \subseteq K \), and there exists a derivation \( Y \xrightarrow{\rho^*} \mathcal{Y}_{M_{\text{PAR}}^K, K^\omega} \)
  such that \( \mathcal{Y}_{M_{\text{PAR}}^K, K^\omega}^f (\rho) = K \) and \( \mathcal{Y}_{M_{\text{PAR}}^K, K^\omega}^\infty (\rho) \cup \mathcal{Y}_{M_{\text{PAR}}^K, K^\omega}^f (\rho) = K^\omega \). Moreover, either \( \rho \) is infinite or \( \rho \) contains some occurrence of rule in \( \mathcal{R}_{\text{PAR}}^K \setminus \mathcal{R}_{\text{PAR}}^K \).

**Theorem 4.5.** Let us assume that \( K = K^\omega \). Given \( X \in \text{Var} \), there exists a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( X \) if, and only if, one of the following properties is satisfied:

1. There exists a variable \( Y \in \text{Var} \) reachable from \( X \) in \( \mathcal{R}^K_{\text{SEQ}} \) through a \((K', \emptyset)\)-accepting derivation in \( M^K_{\text{SEQ}} \) with \( K' \subseteq K \), and there exists a derivation \( Y \xrightarrow{\rho^*} \mathcal{Y}_{M_{\text{PAR}}^K, K^\omega} \)
  such that \( \mathcal{Y}_{M_{\text{PAR}}^K, K^\omega}^f (\rho) = K \) and \( \mathcal{Y}_{M_{\text{PAR}}^K, K^\omega}^\infty (\rho) \cup \mathcal{Y}_{M_{\text{PAR}}^K, K^\omega}^f (\rho) = K^\omega \). Moreover, either \( \rho \) is infinite or \( \rho \) contains some occurrence of rule in \( \mathcal{R}_{\text{PAR}}^K \setminus \mathcal{R}_{\text{PAR}}^K \).

2. There exists a \((K, K^\omega)\)-accepting infinite derivation in \( M^K_{\text{SEQ}} \) from \( X \).

These two results, together with theorems \[ \] and \[ \] allow us to conclude that Problem 2, stated at the beginning of this subsection, is decidable.

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APPENDIX

A Definitions and simple properties

In this section we give some definitions and deduce simple properties that will be used in sections B and C for the proof of theorems 4.1-4.3.

In the following $\mathit{Var}$ denotes the set of variables $\mathit{Var} \cup \{\tilde{Z}_F, \tilde{Z}_\infty\}$, $T$ denotes the set of terms over $\mathit{Var}$, and $T_{PAR}$ (resp., $T_{SEQ}$) the set of terms in $T$ not containing sequential (resp., parallel) composition.

**Definition A.1.** The set of subterms of a term $t \in T$, denoted by $\mathit{SubTerms}(t)$, is defined inductively as follows:

- $\mathit{SubTerms}(\varepsilon) = \{\varepsilon\}$
- $\mathit{SubTerms}(X) = \{X\}$, for all $X \in \mathit{Var}$
- $\mathit{SubTerms}(X.(t)) = \mathit{SubTerms}(t) \cup \{X.(t)\}$, for all $X.(t) \in T$ with $t \neq \varepsilon$
- $\mathit{SubTerms}(t_1 | t_2) = \bigcup_{(t_1', t_2') \in S} (\mathit{SubTerms}(t_1') \cup \mathit{SubTerms}(t_2')) \cup \{t_1 | t_2\}$, with $S = \{(t_1', t_2') \in T \times T \mid t_1', t_2' \neq \varepsilon \text{ and } t_1 | t_2 = t_1' | t_2'\}$ and $t_1, t_2 \in T \setminus \{\varepsilon\}$.

**Definition A.2.** The set of terms obtained from a term $t \in T$ substituting an occurrence of a subterm $st$ of $t$ with a term $t' \in T$, denoted by $t[st \rightarrow t']$, is defined inductively as follows:

- $t|t \rightarrow t' = \{t'\}$
- $X.(t)[st \rightarrow t'] = \{X.(s) \mid s \in t[st \rightarrow t']\}$, for all terms $X.(t) \in T$ with $t \neq \varepsilon$ and $st \in \mathit{SubTerms}(X.(t)) \setminus \{X.(t)\}$
- $t_1 | t_2[st \rightarrow t'] = \{t'' \mid t' \mid (t_1', t_2') \in T \times T, t_1', t_2' \neq \varepsilon, t_1 | t_2 = t_1 | t_2', st \in \mathit{SubTerms}(t_1'), t'' \in t_1'[st \rightarrow t']\}$, for all $t_1, t_2 \in T \setminus \{\varepsilon\}$ and $st \in \mathit{SubTerms}(t_1 | t_2) \setminus \{t_1 | t_2\}$.

**Definition A.3.** For a term $t \in T$, the set of terms $\mathit{SEQ}(t)$ is the subset of $T_{SEQ} \setminus \{\varepsilon\}$ defined inductively as follows:

- $\mathit{SEQ}(\varepsilon) = \emptyset$
- $\mathit{SEQ}(X) = \{X\}$, for all $X \in \mathit{Var}$
- $\mathit{SEQ}(X.(t)) = \{X.(t') \mid t' \in \mathit{SEQ}(t)\}$, for all $X \in \mathit{Var}$ and $t \in T \setminus \{\varepsilon\}$
- $\mathit{SEQ}(t_1 | t_2) = \mathit{SEQ}(t_1) \cup \mathit{SEQ}(t_2)$.
For a term \( t \in T_{\text{SEQ}} \setminus \{\varepsilon\} \) having the form \( t = X_1(X_2(\ldots X_n(Y)\ldots)) \), with \( n \geq 0 \), we denote the variable \( Y \) by \( \text{last}(t) \). Given two terms \( t, t' \in T_{\text{SEQ}} \setminus \{\varepsilon\} \), with \( t = X_1(X_2(\ldots X_n(Y)\ldots)) \) and \( t' = X_1'(X_2'(\ldots X'_n(Y')\ldots)) \), we denote by \( t \circ t' \) the term \( X_1(X_2(\ldots X_n(X_1'(X_2'(\ldots X'_k(Y')\ldots)))\ldots)) \). Notice that \( t \circ t' \) is the only term in \( t[Y \rightarrow t'] \), and that the operation \( \circ \) on terms in \( T_{\text{SEQ}} \setminus \{\varepsilon\} \) is associative.

**Proposition A.1 (see [5]).** The following properties hold:

1. If \( t \xrightarrow{\rho^*}_\mathcal{R} t' \) and \( t \in \text{SubTerms}(s) \), for some \( s \in T \), then it holds \( s \xrightarrow{\rho^*}_\mathcal{R} s' \) for all \( s' \in s[t \rightarrow t'] \);

2. If \( t \xrightarrow{\rho^*}_\mathcal{R} \) is an infinite derivation in \( \mathcal{R} \) and \( t \in \text{SubTerms}(s) \), for some \( s \in T \), then it holds \( s \xrightarrow{\rho^*}_\mathcal{R} \).

**Proposition A.2 (see [5]).** If \( t, t' \in T_{\text{SEQ}} \setminus \{\varepsilon\} \) such that \( \text{last}(t) \xrightarrow{\rho^*}_\mathcal{R} t' \), then it holds that

1. \( t \xrightarrow{\rho^*}_\mathcal{R} t \circ t' \);

2. \( t'' \circ t \xrightarrow{\rho^*}_\mathcal{R} t'' \circ t \circ t' \) for all \( t'' \in T_{\text{SEQ}} \setminus \{\varepsilon\} \).

**Lemma A.1.** Let \( t \| X.(s) \xrightarrow{\sigma^*}_\mathcal{R} \) be a derivation in \( \mathcal{R} \), and let \( s \xrightarrow{\sigma^*}_\mathcal{R} \) be a subderivation of \( t \| X.(s) \xrightarrow{\sigma^*}_\mathcal{R} \) from \( s \). Then, the following properties are satisfied:

1. If \( s \xrightarrow{\sigma^*}_\mathcal{R} \) is infinite, then it holds that \( t \xrightarrow{\sigma^\lambda\sigma^*}_\mathcal{R} \). Moreover, if \( t \| X.(s) \xrightarrow{\sigma^*}_\mathcal{R} \) is in \( \Pi_{P AR,\infty}^{K,K^\omega} \), then also \( t \xrightarrow{\sigma^\lambda\sigma^*}_\mathcal{R} \) is in \( \Pi_{P AR,\infty}^{K,K^\omega} \).

2. If \( s \xrightarrow{\sigma^*}_\mathcal{R} \) leads to \( \varepsilon \), then the derivation \( t \| X.(s) \xrightarrow{\sigma^*}_\mathcal{R} \) can be written in the form

\[
 t \| X.(s) \xrightarrow{\sigma^*}_\mathcal{R} t'' \| X \xrightarrow{\sigma^2^*}_\mathcal{R}
\]

where \( t \xrightarrow{\lambda^*}_\mathcal{R} t' \) with \( \sigma_1 \in \text{Interleaving}(\lambda, \sigma') \). Moreover, if \( t \| X.(s) \xrightarrow{\sigma^*}_\mathcal{R} \) is in \( \Pi_{P AR,\infty}^{K,K^\omega} \), there is a derivation of the form \( t \| X \xrightarrow{\lambda^*}_\mathcal{R} t'' \| X \xrightarrow{\sigma^2^*}_\mathcal{R} \) belonging to \( \Pi_{P AR,\infty}^{K,K^\omega} \).

3. If \( s \xrightarrow{\sigma^*}_\mathcal{R} \) leads to a term \( s' \neq \varepsilon \) one of the following conditions is satisfied:

- \( t \xrightarrow{\sigma\sigma^*}_\mathcal{R} \). If \( t \| X.(s) \xrightarrow{\sigma^*}_\mathcal{R} \) is in \( \Pi_{P AR,\infty}^{K,K^\omega} \), then also \( t \xrightarrow{\sigma\sigma^*}_\mathcal{R} \) is in \( \Pi_{P AR,\infty}^{K,K^\omega} \). Moreover, if \( t \| X.(s) \xrightarrow{\sigma^*}_\mathcal{R} \) is finite and leads to \( \top \), then \( \top = X.(s')\|t' \) where \( t \xrightarrow{\sigma^*}_\mathcal{R} t' \).

- \( s' = W \in \text{Var} \) and the derivation \( t \| X.(s) \xrightarrow{\sigma^*}_\mathcal{R} \) can be written in the form

\[
 t \| X.(s) \xrightarrow{\sigma^2^*}_\mathcal{R} t'' \| X.(W) \xrightarrow{\sigma^*}_\mathcal{R} t'\|W' \xrightarrow{\sigma^2^*}_\mathcal{R}
\]

where \( r = X.(W)\xrightarrow{\sigma} W' \in \mathcal{R} \), and \( t \xrightarrow{\lambda^*}_\mathcal{R} t' \) with \( \sigma_1 \in \text{Interleaving}(\lambda, \sigma') \). Moreover, if \( t \| X.(s) \xrightarrow{\sigma^*}_\mathcal{R} \) is in \( \Pi_{P AR,\infty}^{K,K^\omega} \), there is a derivation of the form \( t \| W' \xrightarrow{\lambda^*}_\mathcal{R} t'' \| W' \xrightarrow{\sigma^2^*}_\mathcal{R} \) belonging to \( \Pi_{P AR,\infty}^{K,K^\omega} \).

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Proof. The assertion follows easily from the definition of subderivation.

Lemma A.2. Let \( p \xrightarrow{s}^* t \parallel Y.(s) \xrightarrow{r}^* \parallel_t \) with \( s \neq \varepsilon \) and \( p \in T_{PAR} \). Then, \( p \xrightarrow{s}^* t \parallel Y.(s) \) can be written in the form

\[
p \xrightarrow{s}^* t \parallel Z \xrightarrow{s}^* t \parallel Y.(Z') \xrightarrow{r}^* t \parallel Y.(s)
\]

with \( r = Z \rightarrow Y.(Z') \), and

\[
Z' \xrightarrow{r'}^* s \quad \text{and} \quad t' \xrightarrow{r''}^* t
\]

with \( \sigma_2 \in \text{Interleaving}(\sigma', \sigma'') \). Moreover, the following property is satisfied:

\[\text{A.1}\]

Let \( s \xrightarrow{\omega'}^* \parallel_t \) be a subderivation of \( t \parallel Y.(s) \xrightarrow{\omega}^* \parallel_t \) from \( s \). Then, the derivation

\[
Z' \xrightarrow{r'}^* s \xrightarrow{\omega'}^* \parallel_t
\]

is a subderivation of \( t' \parallel Y.(Z') \xrightarrow{r}^* t \parallel Y.(s) \xrightarrow{\omega}^* \parallel_t \) from \( Z' \).

Proof. The assertion follows easily by induction on the length of \( \sigma \).

B Proof of Theorem 4.3

In order to prove theorem 4.3 we need the following two lemmata B.1 - B.2.

Lemma B.1. Let \( p \xrightarrow{s}^* p' \parallel p'' \) with \( p, p', p'' \in T_{PAR} \), \( p' \) not containing occurrences of \( \hat{Z} \) and \( \hat{Z} \), and \( p'' \) not containing occurrences of variables in \( \text{Var} \). Then, there exists a \( t \in T \) such that \( p \xrightarrow{p'}^* p' \parallel t \) with \( \Upsilon_M(p) = \Upsilon_M(p') \), and \( |\rho| > 0 \) if \( |\sigma| > 0 \).

Proof. The proof is by induction on \( |\sigma| \).

Base Step: \( |\sigma| = 0 \). In this case the assertion is obvious.

Induction Step: \( |\sigma| > 0 \). In this case the derivation \( p \xrightarrow{s}^* p' \parallel p'' \) can be written in the form

\[
p \xrightarrow{s}^* \parallel_t \xrightarrow{r}^* p' \parallel p'' \quad \text{with} \quad |\sigma'| < |\sigma|, \ r \in R_{PAR} \text{ and } \parallel_t, p', p'' \in T_{PAR}
\]

Moreover, \( \parallel_t \) doesn’t contain occurrences of \( \hat{Z} \) and \( \hat{Z} \), and \( p' \) doesn’t contain occurrences of variables in \( \text{Var} \). From the inductive hypothesis, there exists a \( t \in T \) such that \( p \xrightarrow{r}^* \parallel_t \parallel_t \parallel_t \parallel_t \) with \( \Upsilon_M(p') = \Upsilon_M(p') \).

There are two cases:

1. \( r \) is a \( \text{PAR} \) rule of \( \mathcal{R} \). From remark 4.1 \( \Upsilon_M(p') = \Upsilon_M(p') \). Moreover, \( p' = p'' \) and \( \parallel_t \xrightarrow{r}^* p' \). Then, we deduce that \( p \xrightarrow{s}^* \parallel_t \xrightarrow{r}^* \parallel_t \parallel_t \) with \( \Upsilon_M(p') = \Upsilon_M(p') \), and the assertion is proved.

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2. \( r \in \mathcal{R}_{P\text{AR}}^K \setminus \mathcal{R} \). There are two subcases:

- \( r = X \overset{K'}{\rightarrow} Y \) with \( X, Y \in \text{Var} \) and \( K' \in P_n \). From remark 4.1, \( \Upsilon^f_{M^K_{P\text{AR}}} (r) = K' \).

  From property 5 in the definition of \( M^K_{P\text{AR}} \), we have \( X \overset{\rho''}{\leftrightarrow}_r Y \) with \( \Upsilon^f_M (\rho'') = K' \) and \( |\rho''| > 0 \). Moreover, \( \overline{\rho''} = p'' \) and \( \overline{\rho} \Rightarrow^r_{\mathcal{R}_{P\text{AR}}} \rho' \). Then, we deduce that \( p \overset{\rho'*}{\rightarrow}_r \overline{\rho} \parallel \overline{\rho''} \parallel t \) with \( \Upsilon^f_M (\rho' \rho'') = \Upsilon^f_{M^K_{P\text{AR}}} (\sigma' r) \), and the assertion is proved.

- \( r = X \overset{K'}{\rightarrow} Z_F \) with \( X \in \text{Var} \), and \( \Upsilon^f_{M^K_{P\text{AR}}} (r) = K' \). From property 5 in the definition of \( M^K_{P\text{AR}} \), we deduce that \( X \overset{\rho''}{\leftrightarrow}_r t \), for some term \( t \), with \( \Upsilon^f_M (\rho'') = \Upsilon^f_{M^K_{P\text{AR}}} (r) \) and \( |\rho''| > 0 \). Evidently, \( p'' = \overline{\rho''} \parallel Z_F \) and \( \overline{\rho'} = p' \parallel X \). Then, we deduce that \( p \overset{\rho'*}{\rightarrow}_r \overline{\rho'} \parallel t \) with \( \Upsilon^f_M (\rho' \rho'') = \Upsilon^f_{M^K_{P\text{AR}}} (\sigma' r) \), and the assertion is proved.

**Lemma B.2.** Let \( p \overset{\sigma^*}{\rightarrow}_r t \parallel p' \) with \( p, p' \in T_{P\text{AR}} \) and \( \Upsilon^f_M (\sigma) \subseteq K \). Then, there exists an \( s \in T_{P\text{AR}} \) such that \( p \overset{\rho'}{\rightarrow}_r s \parallel p' \) with \( \Upsilon^f_M (\rho) = \Upsilon^f_{M^K_{P\text{AR}}} (p) \), and \( s = \varepsilon \) if \( t = \varepsilon \).

**Proof.** The proof is by induction on the length of finite derivations \( p \overset{\sigma^*}{\rightarrow}_r \) in \( \mathcal{R} \) from terms in \( T_{P\text{AR}} \) with \( \Upsilon^f_M (\sigma) \subseteq K \).

**Base Step:** \( |\sigma| = 0 \). In this case the assertion is obvious.

**Induction Step:** \( |\sigma| > 0 \). The derivation \( p \overset{\sigma^*}{\rightarrow}_r \) can be written in the form

\[
p \overset{r}{\rightarrow}_r \overline{t} \overset{\rho'}{\rightarrow}_r t \parallel p'
\]

with \( r \in \mathcal{R}, |\sigma'| < |\sigma| \) and \( \Upsilon^f_M (\sigma') \subseteq K \). There are two cases:

1. \( r \) is a PAR rule. Then, we have \( \overline{t} \in T_{P\text{AR}} \). From the inductive hypothesis, there exists a \( s \in T_{P\text{AR}} \) such that \( \overline{t} \overset{\rho'}{\rightarrow}_r s \parallel p' \) with \( \Upsilon^f_M (\sigma') = \Upsilon^f_{M^K_{P\text{AR}}} (\rho') \), and \( s = \varepsilon \) if \( t = \varepsilon \). By construction, \( r \in \mathcal{R}_{P\text{AR}}^K \). From remark 4.1, it follows that \( \Upsilon^f_M (r) = \Upsilon^f_{M^K_{P\text{AR}}} (r) \). By proposition 3.1, we obtain \( p \overset{\sigma^*}{\rightarrow}_r \overline{t} \overset{\rho'}{\rightarrow}_r s \parallel p' \) with \( \Upsilon^f_M (r \sigma') = \Upsilon^f_{M^K_{P\text{AR}}} (r \rho') \), and \( s = \varepsilon \) if \( t = \varepsilon \). Hence, the assertion is proved.

2. \( r = Z \overset{a}{\rightarrow} Y (Z') \) with \( \Upsilon^f_M (r) \subseteq K \). Then, we have \( p = p'' \parallel Z \) and \( \overline{t} = p'' \parallel Y (Z') \), with \( p'' \in T_{P\text{AR}} \). From (1), let \( Z' \overset{\lambda^*}{\rightarrow} t_1 \) a subderivation of \( \overline{t} = p'' \parallel Y (Z') \overset{\rho'}{\rightarrow}_r \), from \( Z' \). Evidently, \( \Upsilon^f_M (\lambda) \subseteq K \). From lemma A.1, we can distinguish three subcases:
• $t_1 \neq \varepsilon$ and $p'' \Rightarrow_{r}^{\sigma} t'$. Moreover, $t||p' = t'||Y.\langle t_1 \rangle$, $t' = p'||t''$ for some term $t''$, and $t = t''||Y.\langle t_1 \rangle$. In particular, $t \neq \varepsilon$. Since $\Upsilon^f_M(\sigma' \setminus \lambda) \subseteq K$, from the inductive hypothesis there exists an $s \in T_{PAR}$ such that $p'' \Rightarrow_{s}^{\sigma} \Upsilon^f_M(\sigma' \setminus \lambda) = \Upsilon^f_{M_PAR}(\rho')$. Moreover, since $\Upsilon^f_M(\lambda) \subseteq K$, from the inductive hypothesis we have that $Z' \Rightarrow_{s}^{\sigma} s \in T_{PAR}$, with $\Upsilon^f_{K^f_{PAR}}(\rho'') = \Upsilon^f_M(\lambda)$. Since $\Upsilon^f_{M_PAR}(\rho'') = \Upsilon^f_M(\lambda) \subseteq K$ and $\Upsilon^f_M(r) \subseteq K$, from property 2 in the definition of $M^K_{PAR}$ we deduce that $r' = Z'K^fZ_F \in \mathfrak{R}^K_{PAR}$ with $K' = \Upsilon^f_M(\lambda) \cup \Upsilon^f_M(r)$. Then, by proposition 3.1 we have $p = p''||Z \Rightarrow_{s}^{\sigma} p''||Z_F' \Rightarrow_{s}^{\sigma} s||p'\Rightarrow_{s}^{\sigma} \Upsilon^f_{M_PAR}(\rho'') = \Upsilon^f_M(\lambda(\sigma' \setminus \lambda)) = \Upsilon^f_M(\sigma)$, and the assertion is proved.

• $t_1 = \varepsilon$ and the derivation $p''||Y.(Z') \Rightarrow_{s}^{\sigma} t||p'$ can be written in the form

$$p''||Y.(Z') \Rightarrow_{s}^{\sigma} t||Y \Rightarrow_{s}^{\sigma} t||p' \quad \text{with} \quad p'' \Rightarrow_{s}^{\sigma} t', \quad \text{and} \quad \sigma_1 \in \text{Interleaving}(\lambda, \sigma'_1) \quad (2)$$

Now, $Z' \Rightarrow_{s}^{\sigma} \varepsilon$ with $|\lambda| < |\sigma|$ and $\Upsilon^f_M(\lambda) \subseteq K$. From the inductive hypothesis, we have $Z' \Rightarrow_{s}^{\sigma} \varepsilon$ such that $\Upsilon^f_M(\lambda) = \Upsilon^f_{M_PAR}(\rho)$. From property 3 in the definition of $M^K_{PAR}$ it follows that $r' = Z'K^fZ_F \in \mathfrak{R}^K_{PAR}$ where $K' = \Upsilon^f_M(r) \cup \Upsilon^f_{M_PAR}(\rho)$ and $\Upsilon^f_{M_PAR}(r') = K'$. Now, we have $p''||Y \Rightarrow_{s}^{\sigma} t'||Y \Rightarrow_{s}^{\sigma} t||p'$ with $\Upsilon^f_M(\sigma_1) \subseteq K$ and $|\sigma_1| < |\varepsilon|$. From the inductive hypothesis, there exists a $s \in T_{PAR}$ such that $p''||Y \Rightarrow_{s}^{\sigma} s||p'$, with $\Upsilon^f_{M_PAR}(\rho') = \Upsilon^f_M(\sigma_1\sigma_2)$, and $s = \varepsilon$ if $t = \varepsilon$. After all, considering proposition 3.1 we have $p = p''||Z \Rightarrow_{s}^{\sigma} p''||Y \Rightarrow_{s}^{\sigma} s||p' \Rightarrow_{s}^{\sigma} \Upsilon^f_M(\rho'') = \Upsilon^f_M(\lambda(\sigma'_1\sigma_2)) = \Upsilon^f_M(\sigma)$, and $s = \varepsilon$ if $t = \varepsilon$. So, the assertion is proved.

• $t_1 = W \in Var$ and the derivation $p''||Y.(Z') \Rightarrow_{s}^{\sigma} t||p'$ can be written in the form

$$p''||Y.(Z') \Rightarrow_{s}^{\sigma} t'||Y.\langle W \rangle \Rightarrow_{s}^{\sigma} t'||W'' \Rightarrow_{s}^{\sigma} t||p' \quad (3)$$

with $p'' \Rightarrow_{s}^{\sigma} t'$, $r' = Y.\langle W \rangle \Rightarrow_{s}^{\sigma} W' \quad \text{and} \quad \sigma_1 \in \text{Interleaving}(\lambda, \sigma'_1) \quad (4)$

Now, $Z' \Rightarrow_{s}^{\sigma} W$ with $|\lambda| < |\sigma|$ and $\Upsilon^f_M(\lambda) \subseteq K$. From the inductive hypothesis, we have $Z' \Rightarrow_{s}^{\sigma} W$ such that $\Upsilon^f_M(\lambda) = \Upsilon^f_{M_PAR}(\rho)$. From property 4 in the definition of $M^K_{PAR}$, considering that $r = Z\Rightarrow_{s}^{\sigma} Y.\langle Z' \rangle \in \mathfrak{R}$ and $r' = Y.\langle W \rangle \Rightarrow_{s}^{\sigma} W' \in \mathfrak{R}$ with $\Upsilon^f_M(r) \subseteq K$ and $\Upsilon^f_{M_PAR}(\rho)$, it follows that $r'' = Z'K^fW' \in \mathfrak{R}^K_{PAR}$ where $K' = \Upsilon^f_M(rr') \cup \Upsilon^f_{M_PAR}(\rho)$ and $\Upsilon^f_{M_PAR}(rr'') = K'$. Now, we have $p''||W'$
\( \varphi^1 \sigma_1 t' \parallel W' \varphi_2^2 t \parallel p' \) with \( \Upsilon_M^f(\sigma_1' \sigma_2) \subseteq K \) and \( |\sigma_1' \sigma_2| < |\sigma| \). From the inductive hypothesis there exists a \( s \in T_{PAR} \) such that \( p'' \parallel W'' \varphi_{PAR}^s s \parallel p' \), with \( \Upsilon_M^f(\rho') = \Upsilon_M^f(\sigma_1' \sigma_2) \), and \( s = \varepsilon \) if \( t = \varepsilon \). After all, considering proposition 3.1, we obtain \( p = p'' \parallel Z \varphi_{PAR}^r s \parallel p' \), with \( \Upsilon_{M_{PAR}}^f(\rho') = \Upsilon_M^f(\rho' \lambda \sigma_1' \sigma_2) \), and \( s = \varepsilon \) if \( t = \varepsilon \). So, the assertion is proved.

At this point, theorem 4.3 follows directly from lemmata B.1–B.2.

C Proof of Theorems 4.4 and 4.5

In order to prove theorems 4.4 and 4.5 we need the following lemmata C.1–C.7

To prove lemma C.2 we use a mapping for coding pairs of integers by single integers. In particular, we consider the following bijective mapping from \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{N} \), that is a primitive recursive function [9]

\[ < > : (x, y) \in \mathbb{N} \times \mathbb{N} \to 2^z(2y + 1) - 1 \]

Let \( \ell \) (resp. \( \varphi \)) be the first component (resp. the second component) of \( < >^{-1} \). The following properties are satisfied:

1. \( \forall x, y \in \mathbb{N} \ \ell(<x, y>) = x \) and \( \varphi(<x, y>) = y \).
2. \( \forall z \in \mathbb{N} \ <\ell(z), \varphi(z)> = z \).
3. \( \forall z \in \mathbb{N} \ \ell(z), \varphi(z) \leq z \).
4. \( \forall z, z' \in \mathbb{N} \ \text{if } z > z' \text{ and } \ell(z) = \ell(z') \text{ then } \varphi(z) > \varphi(z'). \)

Now, we introduce a new function \( next : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) defined by primitive recursion in the following way

\[
next(x, 0) = (x, 0)
\]

\[
next(x, y + 1) = \begin{cases} 
(\ell(y), \varphi(y) + 1) & \text{if } next(x, y) = (\ell(y), \varphi(y)) \\
next(x, y) & \text{otherwise}
\end{cases}
\]

For all \( x, y \in \mathbb{N} \) let us denote by \( next_x(y) \) the second component of \( next(x, y) \). The following lemma establishes some properties of \( next \).

**Lemma C.1.** The function \( next \) satisfies the following properties:

1. \( \forall x, y \in \mathbb{N} \ \text{if } y \leq x \text{ then } next(x, y) = (x, 0) \).
2. \( \forall x, y \in \mathbb{N} \ next(x, y) = (x, z_{x,y}) \) for some \( z_{x,y} \in \mathbb{N} \).
Now, let us consider property 1. We prove it by induction on $y$.

**Proof.** At first, let us consider property 1. We prove it by induction on $y$. By construction, for $y = 0$ we have $next(x, 0) = (x, 0)$. Now, let $0 < y \leq x$. From the inductive hypothesis $next(x, y - 1) = (x, 0)$. So, it suffices to prove that $next(x, y) = next(x, y - 1)$. By absurd, let us assume that $next(x, y) \neq next(x, y - 1)$. Then, by construction we have $next(x, y - 1) = (ℓ(y - 1), ϕ(y - 1))$ and $next(x, y) = (ℓ(y - 1), ϕ(y - 1) + 1)$. Therefore, $x = ℓ(y - 1)$. But, $x > y - 1 \geq ℓ(y - 1)$. So, we obtain an absurd.

Now, let us consider property 2. We prove it by induction on $y$. By construction, for $y = 0$ we have $next(x, 0) = (x, 0)$. Now, let $y > 0$. From the inductive hypothesis $next(x, y - 1) = (x, m)$ for some $m \in N$. Now, by construction either $next(x, y) = next(x, y - 1) = (x, m)$ or $next(x, y) = (x, m + 1)$. In both cases the assertion is satisfied.

Now, let us consider property 3. By construction, for $x, y \in N$ either $next_x(y + 1) = next_x(y)$ or $next_x(y + 1) = next_x(y) + 1$. So, the assertion is satisfied.

Now, let us consider property 4. From property 3 $y_1 < y_2$. The proof is by induction on $y_2 - y_1$.

**Base Step:** $y_2 - y_1 = 1$. So, $y_2 = y_1 + 1$. From hypothesis $next(x, y_1) \neq next(x, y_2)$. Therefore, by construction we deduce that $next(x, y_1) = (ℓ(y_1), ϕ(y_1))$ and $next(x, y_2) = (ℓ(y_1), ϕ(y_1) + 1)$. So, setting $k = y_1$ we have $next(x, k) = (ℓ(k), ϕ(k))$, $ϕ(k) = next_x(y_2)−1$ and $y_1 ≤ k < y_2$. The assertion is proved.

**Induction Step:** $y_2 - y_1 > 1$. From property 3 there are two cases:

- $next_x(y_2 - 1) = next_x(y_2)$. So, $next_x(y_1) < next_x(y_2 - 1)$. Since $(y_2 - 1) - y_1 < y_2 - y_1$, from the inductive hypothesis there exists a $k \in N$ such that $next(x, k) = (ℓ(k), ϕ(k))$, $ϕ(k) = next_x(y_2 - 1) − 1$ and $y_1 ≤ k < y_2 − 1$. Since $next_x(y_2 - 1) = next_x(y_2)$ we obtain the assertion.

- $next_x(y_2 - 1) < next_x(y_2)$. We reason as in the base step.

Now, let us consider property 5. The proof is by induction on $n$. By construction, for $n = 0$ we have $next(x, 0) = (x, 0)$. So, in this case the assertion is satisfied. Now, let $n > 0$. From the inductive hypothesis there exists a $y \in N$ such that $next(x, y) = (x, n - 1)$. From property 4, we deduce that it suffices to prove that there exists a $m > n - 1$ such that
next(x, z) = (x, m) for some z ∈ N. By absurd we assume that this property isn’t satisfied. From property 2 and 3, we deduce that
\[ \forall z \geq y \ next(x, z) = (x, n - 1) \] (1)

Let \( k = <x, n - 1> \). Then, \( x = \ell(k) \) and \( n - 1 = \varphi(k) \). There are two cases:

- \( k \geq y \). From (1) we have \( next(x, k) = (\ell(k), \varphi(k)) \). By construction, we obtain \( next(x, k + 1) = (\ell(k), \varphi(k) + 1) = (x, n) \) in contrast with (1).

- \( k < y \). Now, \( next(x, k) = (\ell(k), \overline{n}) \). From property 3, it follows that \( \overline{n} \leq \varphi(k) \). There are two subcases:

  - \( \overline{n} = \varphi(k) \). So, \( next(x, k) = (\ell(k), \varphi(k)) \). By construction, we obtain \( next(x, k + 1) = (\ell(k), \varphi(k) + 1) = (x, n) \). So, we have \( k + 1 \leq y \) and \( next_x(k + 1) > next_x(y) \) in contrast with property 3.

  - \( \overline{n} < \varphi(k) \). In other words, we have \( k < y \) and \( next_x(k) < next_x(y) = \varphi(k) \). From property 4, there exists a \( k' \geq k \) such that \( next(x, k') = (\ell(k'), \varphi(k')) \) and \( \varphi(k') = next_x(y) - 1 = \varphi(k) - 1 \). From property 2, \( \ell(k') = x \). So, we have \( \ell(k) = \ell(k'), k' \geq k \) and \( \varphi(k') < \varphi(k) \). This is in contradiction with properties of \( \varphi \) and \( \ell \).

Now, let us consider property 6. Let \( x \in N \). So, \( x = <\ell(x), \varphi(x)> \). From property 5 there exist two integers \( y, z \in N \) such that \( next(\ell(x), y) = (\ell(x), \varphi(x) + 1) \) and \( next(\ell(x), z) = (\ell(x), \varphi(x)) \) where \( y > z \). Since \( next_{\ell(x)} \) is crescent there exists the greatest \( \overline{z} \) such that \( next(\ell(x), \overline{z}) = (\ell(x), \varphi(x)) \). In particular, \( next(\ell(x), \overline{z} + 1) \neq next(\ell(x), \overline{z}) \). From definition of \( next \) it follows that \( next(\ell(x), \overline{z}) = (\ell(\overline{z}), \varphi(\overline{z})) \). Therefore, \( (\ell(x), \varphi(x)) = (\ell(\overline{z}), \varphi(\overline{z})) \). From this we deduce that \( \overline{z} = x \), and the assertion is proved.

Finally, let us consider property 7. Let \( x, i \in N \) with \( i \neq \ell(x) \). By absurd let us assume that \( next(i, x + 1) \neq next(i, x) \). Then, by construction \( next(i, x) = (\ell(x), \varphi(x)) \). From property 2 we obtain \( i = \ell(x) \), an absurd. \( \square \)

Now, we give the notion of Interleaving of a succession of rule sequences in a PRS \( \mathcal{R} \). To formalize this concept and facilitate the proof of some connected results, we redefine the notion of sequence rule. Precisely, a sequence rule in \( \mathcal{R} \) can be seen as a mapping \( \sigma : N' \rightarrow \mathcal{R} \) where \( N' \) can be a generic subset of \( N \). In particular, this facilitates the formalization of the notion of subsequence. A rule sequence \( \sigma' : N'' \rightarrow \mathcal{R} \) is a subsequence of \( \sigma : N' \rightarrow \mathcal{R} \) iff \( N'' \subseteq N' \) and \( \sigma' = \sigma|_{N''} \), that is \( \sigma' \) is the restriction of \( \sigma \) to set \( N'' \).

Given a rule sequence \( \sigma : N' \rightarrow \mathcal{R} \), we denote by \( pr(\sigma) \) the set \( N' \).

Given a set \( N' \), subset of \( N \), we denote by \( min(N') \) the smallest element of \( N' \).

Finally, given two rule sequences \( \sigma \) and \( \sigma' \), we say that they are disjoint if \( pr(\sigma) \cap pr(\sigma') = \emptyset \).

**Definition C.1.** Let \( \{\rho_h\}_{h \in N} \) be a succession of rule sequences in a PRS \( \mathcal{R} \). The Interleaving of \( \{\rho_h\}_{h \in N} \), denoted by \( \text{Interleaving}(\{\rho_h\}) \), is the set of rule sequences \( \sigma \) in \( \mathcal{R} \) such
that there exists an injective mapping \( M_\sigma : \bigcup_{h \in \mathbb{N}} \{ h \} \times \text{pr}(\rho_h) \rightarrow N \) (depending on \( \sigma \)) satisfying the following properties (where \( \Delta \) is the set \( \bigcup_{h \in \mathbb{N}} \{ h \} \times \text{pr}(\rho_h) \))

- \( \forall h \in N \ \forall n, n' \in \text{pr}(\rho_h) \) with \( n < n' \) then \( M_\sigma(h, n) < M_\sigma(h, n') \).
- \( \text{pr}(\sigma) = M_\sigma(\Delta) \).
- \( \forall (h, n) \in \Delta \) we have \( \sigma(M_\sigma(h, n)) = \rho_h(n) \).

**Proposition C.1.** Let \( M = (\mathbb{R}, \{ \mathbb{R}_1^A, \ldots, \mathbb{R}_n^A \}) \) be a MBRS and let \( \{ \sigma_h \}_{h \in \mathbb{N}} \) be a succession of rule sequences in \( \mathbb{R} \). Then, \( \forall \pi \in \text{Interleaving}(\{ \sigma_h \}) \) we have

1. \( \Upsilon^f_M(\pi) = \bigcup_{h \in \mathbb{N}} \Upsilon^f_M(\sigma_h) \).
2. \( \Upsilon^\infty_M(\pi) = \bigcup_{h \in \mathbb{N}} \Upsilon^\infty_M(\sigma_h) \cup \bigoplus_{h \in \mathbb{N}} \Upsilon^f_M(\sigma_h) \).

**Proof.** We prove property 2. In similar way it’s possible to prove property 1. Let \( \Delta = \bigcup_{h \in \mathbb{N}} \{ h \} \times \text{pr}(\sigma(h)) \). From hypothesis there exists an injective mapping \( M_\pi : \Delta \rightarrow N \) such that \( \text{pr}(\pi) = M_\pi(\Delta) \), and for all \( (h, k) \in \Delta \) \( \pi(M_\pi(h, k)) = \sigma_h(k) \).

Let \( i \in \{ 1, \ldots, n \} \). We have to prove that

\[
i \in \Upsilon^\infty_M(\pi) \Leftrightarrow i \in \bigcup_{h \in \mathbb{N}} \Upsilon^\infty_M(\sigma_h) \cup \bigoplus_{h \in \mathbb{N}} \Upsilon^f_M(\sigma_h).
\]

(\( \Rightarrow \)). Let \( i \in \Upsilon^\infty_M(\pi) \). So, \( \pi \) contains infinite occurrences of a rule \( r \in \mathbb{R}_i^A \). Therefore, the set \( \{ k \in M_\pi(\Delta) | \pi(k) = r \} \) is infinite. Then, the set \( \{ (h, k) \in \Delta | \sigma_h(k) = r \} \) is infinite. There are two cases:

- \( \exists h \in \mathbb{N} \) such that the set \( \{ j \in \text{pr}(\sigma_h) | \sigma_h(j) = r \} \) is infinite. Therefore, \( i \in \Upsilon^\infty_M(\sigma_h) \)

- The set \( \{ h \in \mathbb{N} | \sigma_h \text{ contains some occurrence of } r \} \) is infinite. Therefore, \( i \in \bigoplus_{h \in \mathbb{N}} \Upsilon^f_M(\sigma_h) \).

In both cases the result holds.

(\( \Leftarrow \)). Let \( i \in \bigcup_{h \in \mathbb{N}} \Upsilon^\infty_M(\sigma_h) \cup \bigoplus_{h \in \mathbb{N}} \Upsilon^f_M(\sigma_h) \). There are two cases:

- \( \exists h \in \mathbb{N} \) such that \( i \in \Upsilon^\infty_M(\sigma_h) \). Since \( M_\pi \) is injective, the set \( \{ k \in M_\pi(\Delta) | \pi(k) \in \mathbb{R}_i^A \} \) is infinite. So, \( i \in \Upsilon^\infty_M(\pi) \).

- The set \( \{ h \in \mathbb{N} | \sigma_h \text{ contains some occurrence of a rule in } \mathbb{R}_i^A \} \) is infinite. Since \( M_\pi \) is injective, it follows that \( i \in \Upsilon^\infty_M(\pi) \).

**Lemma C.2.** Let \( p \overset{\Delta}{\Rightarrow}_M^* \) with \( p \in T_{\text{PAR}} \). Then, there exists in \( \mathbb{R} \) a derivation from \( p \) of the form \( p \overset{\Delta}{\Rightarrow}_M^* \) such that \( \Upsilon^f_M(\delta) = \Upsilon^f_{M_{K,K_w}}(\sigma) \) and \( \Upsilon^\infty_M(\delta) = \Upsilon^\infty_{M_{K,K_w}}(\sigma) \cup \Upsilon^f_{M_{K,K_w}}(\sigma) \).

Moreover, if \( \sigma \) is infinite or contains some occurrence of rule in \( \mathbb{R}_{\text{PAR}}^K \setminus \mathbb{R}_{\text{PAR}}^K \) then, \( \delta \) is infinite.
Proof. For the proof we use the following property

A. Let \( p'\|p'' \xrightarrow{s} \ \mathcal{R}_{PAR}^{K,K^\omega} \) with \( p', p'' \in T_{PAR} \) and \( p'' \) not containing variables in \( Var \). Then \( p' \xrightarrow{s} \mathcal{R}_{PAR}^{K,K^\omega} \).

Property A follows easily from the observation that the left-hand side of each rule in \( \mathcal{R}_{PAR}^{K,K^\omega} \) doesn’t contain occurrences of \( \hat{Z}_F \) and \( \hat{Z}_\infty \).

Let \( \lambda \) be the subsequence of \( \sigma \) containing all, and only, the occurrences of rules in \( \mathcal{R}_{PAR}^{K,K^\omega} \setminus \mathcal{R}_{PAR}^{K} \). Let us assume that \( \lambda \) is infinite. In similar way we reason if \( \lambda \) is finite. Now, \( \lambda = r_0 r_1 r_2 \ldots \), where \( \forall h \in N r_h \in \mathcal{R}_{PAR}^{K,K^\omega} \setminus \mathcal{R}_{PAR}^{K} \). Moreover, \( \sigma \) can be written in the form \( \rho_0 \rho_1 \rho_2 \rho_2 \ldots \), where \( \sigma \setminus \lambda = \rho_0 \rho_1 \rho_2 \ldots \) and \( \forall h \in N \rho_h \) is a rule sequence (possibly empty) in \( \mathcal{R}_{PAR}^{K} \). For all \( h \in N \) we denote by \( \sigma^h \) the suffix of \( \sigma \) starting at \( \rho_h \).

Now, we prove that there exists a succession of terms in \( T_{PAR}, (p_h)_{h \in N} \), a succession of variables \( (X_h)_{h \in N} \) and a succession of terms \( (t_h)_{h \in N} \) such that:

i. \( p_0 = p \).

ii. \( \forall h \in N \ p_h \xrightarrow{\sigma^h} \mathcal{R}_{PAR}^{K,K^\omega} \).

iii. \( \forall h \in N \ p_h \xrightarrow{\sigma^h} \mathcal{R}_{PAR}^{K,K^\omega} \) with \( p_h \parallel t_h \| X_h \) with \( \Upsilon_M^f(\eta_h) = \Upsilon_M^{K,K^\omega}(\rho_h) \).

iv. \( \forall h \in N \ X_h \xrightarrow{\sigma^h} \mathcal{R}_{PAR}^{K,K^\omega} \) with \( \pi_h \) infinite, \( \Upsilon_M^f(\pi_h) = \Upsilon_M^{K,K^\omega}(\rho_h) \) and \( \Upsilon_M^\infty(\pi_h) = \Upsilon_M^{K,K^\omega}(\rho_h) \).

Setting \( p_0 = p \), property ii is satisfied for \( h = 0 \). So, let us assume that the statement is true \( \forall h = 0, \ldots, k \). Then, it suffices to prove that

B. there exists a \( p_{k+1} \in T_{PAR} \), a term \( t_k \) and a variable \( X_k \) such that \( p_k \xrightarrow{\sigma^k} \mathcal{R}_{PAR}^{K,K^\omega} \) with \( p_k \parallel t_k \| t_k \| X_k \), \( p_{k+1} \xrightarrow{\sigma^{k+1}} \mathcal{R}_{PAR}^{K,K^\omega} \), and \( X_k \xrightarrow{\sigma^k} \mathcal{R}_{PAR}^{K,K^\omega} \) with \( \pi_k \) infinite. Moreover, \( \Upsilon_M^f(\eta_k) = \Upsilon_M^{K,K^\omega}(\rho_k) \), \( \Upsilon_M^f(\pi_k) = \Upsilon_M^{K,K^\omega}(\rho_k) \), \( \Upsilon_M^\infty(\pi_k) = \Upsilon_M^{K,K^\omega}(\rho_k) \).

From the inductive hypothesis we have \( p_k \xrightarrow{\sigma^k} \mathcal{R}_{PAR}^{K,K^\omega} \). The derivation \( p_k \xrightarrow{\sigma^k} \mathcal{R}_{PAR}^{K,K^\omega} \) can be written in the form

\[
p_k \xrightarrow{\sigma^k} \mathcal{R}_{PAR}^{K,K^\omega} \ p'\|p'' \ | X \xrightarrow{\sigma^k} \mathcal{R}_{PAR}^{K,K^\omega} \ p'\|p'' \| \hat{Z}_\infty \xrightarrow{\sigma^{k+1}} \mathcal{R}_{PAR}^{K,K^\omega}
\]

where \( r_k = X^{K',K^\omega}_K \hat{Z}_\infty \) with \( X \in Var \) and \( K', K^\omega \in P_n \). Moreover, \( p' \) doesn’t contain occurrences of \( \hat{Z}_F \) and \( \hat{Z}_\infty \), and \( p'' \) doesn’t contain occurrences of variables in \( Var \). From the definition of \( \mathcal{R}_{PAR}^{K,K^\omega} \) we have \( X \xrightarrow{\sigma^k} \mathcal{R}_{PAR}^{K,K^\omega} \) with \( \pi_k \) infinite, \( \Upsilon_M^f(\pi_k) = K' \) and \( \Upsilon_M^\infty(\pi_k) = K^\omega \). From remark 1.2 we have \( \Upsilon_M^{K,K^\omega}(r_k) = K' \) and \( \Upsilon_M^{K,K^\omega}(r_k) = K^\omega \). From property A it follows that \( p' \xrightarrow{\sigma^{k+1}} \mathcal{R}_{PAR}^{K,K^\omega} \). Since \( \rho_k \) is a rule sequence in \( \mathcal{R}_{PAR}^{K} \), from lemma 1.1 it follows...
that \( p_k \xrightarrow{\eta_k}_R p' \| t \| X \) for some term \( t \) and \( \Upsilon^f_M(\eta_k) = \Upsilon^f_{M^R_{\scriptscriptstyle{PAR}}} (\rho_k) \). From remark \[4.2\] we deduce that \( \Upsilon^f_M(\eta_k) = \Upsilon^f_{M^R_{\scriptscriptstyle{PAR}}} (\rho_k) \). Setting \( p_{k+1} = p' \), \( t_k = t \) and \( X_k = t \) we obtain that property \( B \) is satisfied. Thus, properties i-iv are satisfied. Now, let us consider \( \forall h \in N \) the infinite derivation \( X_h \xrightarrow{\pi} R \). It can be written in the form

\[
S(h,0) \xrightarrow{r(h,0)} s(h,0) \xrightarrow{r(h,1)} s(h,1) \xrightarrow{r(h,2)} \ldots
\]

where \( s(h,0) = X_h \) and \( \forall k \in N \ r(h,k) \in R \). For all \( k \in N \) we denote by \( \tau_k \) the rule \( r(\ell(k),\varphi(k)) \). For all \( h,k \in N \) we denote by \( s_k(h) \) the term \( s_{next(h,k)} \). Now, we show that \( \forall k \in N \) the following result holds

\[
p_{k+1} \| t_0 \| \ldots \| t_k \| s_0(k) \| s_1(k) \| \ldots \| s_k(k) \| s_{k+1}(k) \| s_{k+1}(k+1) \xrightarrow{r_{k+1}} R
\]

(2)

Since \( \forall k \in N \ s_k(k) = s_{next(k,h)} \), from property 1 of lemma \[C.1\] it follows that \( s_k(k) = s_{(k,0)} = X_k \). From iii we deduce that

\[
p_{k+1} \| t_0 \| \ldots \| t_k \| s_0(k) \| s_1(k) \| \ldots \| s_k(k) \| s_{k+1}(k) \| s_{k+1}(k+1) \xrightarrow{r_{k+1}} R
\]

(3)

So, to obtain (2) it suffices to prove that

\[
s_0(k) \| s_1(k) \| \ldots \| s_k(k) \| s_{k+1}(k) \| s_{k+1}(k+1) \xrightarrow{\tau_k} s_0(k+1) \| s_1(k+1) \| \ldots \| s_k(k+1)
\]

(4)

From property 6 of lemma \[C.1\] \( \forall k \in N \ next(\ell(k),k) = (\ell(k),\varphi(k)) \). Moreover, \( next(\ell(k),k+1) = (\ell(k),\varphi(k)+1) \). Therefore, we have \( s_{\ell(k)}(k) = s_{(\ell(k),\varphi(k))} \xrightarrow{\tau_k} s_{(\ell(k),\varphi(k)+1)} = s_{\ell(k)}(k+1) \). From property 7 of lemma \[C.1\] \( \forall i \neq \ell(k) \ next(i,k+1) = next(i,k) \). So, \( \forall i \neq \ell(h) \ s_i(k+1) = s_i(h) \). Since \( k \leq h \), we obtain evidently (4). So, (2) is satisfied \( \forall k \in N \). Moreover, since \( s_0(0) = X_0 \), we have

\[
p = p_0 \xrightarrow{\eta_0}_R p_1 \| t_0 \| s_0(0)
\]

(5)

Setting \( \delta = \eta_0 \eta_1 \tau_0 \eta_2 \tau_1 \eta_3 \tau_2 \ldots \), from (2) and (5) we obtain that \( p \xrightarrow{\delta} R \) with \( \delta \) infinite. So, to obtain the assertion it remains to prove that \( \Upsilon^f_M(\delta) = \Upsilon^f_{M^R_{\scriptscriptstyle{PAR}}} (\sigma) \) and \( \Upsilon^f_{\infty}(\delta) = \Upsilon^f_{\infty}(\sigma) \) \( \cup \ \Upsilon^f_{M^R_{\scriptscriptstyle{PAR}}} (\sigma) \). Let \( \mu = \tau_0 \tau_1 \tau_2 \ldots \). Let us observe that \( \mu \in Interleaving(\{ \pi_h \}) \). From iii-iv, propositions \[3.1\] and \[C.1\] and remembering that \( \sigma = \rho_0 \rho_0 \rho_1 \rho_1 \ldots \), we obtain

\[
\Upsilon^f_M(\delta) = \bigcup_{h \in N} \Upsilon^f_M(\eta_h) \cup \Upsilon^f_M(\mu) = \bigcup_{h \in N} \Upsilon^f_{M^R_{\scriptscriptstyle{PAR}}} (\rho_h) \cup \bigcup_{h \in N} \Upsilon^f_M(\pi_h) =
\]

\[
\bigcup_{h \in N} \Upsilon^f_{M^R_{\scriptscriptstyle{PAR}}} (\rho_h) \cup \bigcup_{h \in N} \Upsilon^f_{M^R_{\scriptscriptstyle{PAR}}} (r_h) = \Upsilon^f_{M^R_{\scriptscriptstyle{PAR}}} (\sigma).
\]
We can now distinguish the following two cases:

1. If \( r \cdot Z \in \mathbb{R} \), from remark 3.2 \( \forall r \in \mathbb{R} \), \( \mathcal{Y}_{M_{PAR}}(r) = \emptyset \). Remembering that \( \lambda = r_0r_1r_2 \ldots \), from iii–iv and propositions 3.1, 3.2 we obtain

\[
\mathcal{T}_M^\infty(\delta) = \bigoplus_{h \in \mathbb{N}} \mathcal{Y}_M(\eta_h) \cup \mathcal{Y}_M^\infty(\mu) = \bigoplus_{h \in \mathbb{N}} \mathcal{Y}_{M_{PAR}}(\rho_h) \cup \bigcup_{h \in \mathbb{N}} \mathcal{Y}_M(\pi_h) \cup \bigoplus_{h \in \mathbb{N}} \mathcal{T}_M(\pi_h) = \\
\mathcal{Y}_{M_{PAR}}(\sigma \setminus \lambda) \cup \bigcup_{h \in \mathbb{N}} \mathcal{Y}_{M_{PAR}}(\rho_h) \cup \bigoplus_{h \in \mathbb{N}} \mathcal{T}_{M_{PAR}}(\rho_h) = \\
\mathcal{Y}_{M_{PAR}}(\sigma \setminus \lambda) \cup \mathcal{Y}_{M_{PAR}}(\sigma) \cup \mathcal{Y}_{M_{PAR}}^\infty(\lambda) = \mathcal{Y}_{M_{PAR}}^\infty(\sigma) \cup \mathcal{Y}_{M_{PAR}}^\infty(\sigma).
\]

This concludes the proof.

\[\square\]

**Lemma C.3.** Let \( t, t' \in T_{SEQ} \) and \( s \) be any term in \( T \) such that \( t \in SEQ(s) \). The following results hold.

1. If \( t \overset{\sigma}{\rightarrow} \mathcal{Y}_{SEQ}^f t' \), then there exists a \( s' \in T \) with \( t' \in SEQ(s') \) such that \( s \overset{\sigma}{\rightarrow} s' \), with \( \mathcal{Y}_M^f(\sigma) = \mathcal{Y}_{SEQ}^f(\sigma) \) and \( |\sigma| > 0 \).

2. If \( \overset{\sigma}{\rightarrow} \mathcal{Y}_{SEQ}^f \), \( t' \) with \( t \neq \varepsilon \), then there exists a \( s' \in T \) with \( t' \in SEQ(s') \) such that \( s \overset{\sigma}{\rightarrow} s' \), with \( \mathcal{Y}_M^f(\rho) = \mathcal{Y}_{SEQ}^f(\rho) \), and \( |\rho| > 0 \) if \( |\sigma| > 0 \).

3. If \( t \overset{\sigma}{\rightarrow} \mathcal{Y}_{SEQ}^f \) is a \((K, K^\omega)\)-accepting infinite derivation in \( M_{SEQ}^K \) from \( t \in T_{SEQ} \), then there exists a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( s \).

**Proof.** At first, we prove property 1. We use the following properties, whose proof is immediate. Let \( t \in SEQ(s) \), \( s \in T \) and \( t = X_1.(X_2.(\ldots X_n.(Y) \ldots)) \), with \( n \geq 0 \). Then

A. if \( st \in T_{SEQ} \setminus \{\varepsilon\} \) and \( t' = X_1.(X_2.(\ldots X_n.(st) \ldots)) \), then there exists a \( s' \in s[Y \rightarrow st] \) (notice that \( Y \) is a subterm of \( s \)) such that \( t' \in SEQ(s') \).

B. if \( Z \in \text{Var} \), \( st' \in T \) and \( st = st' || Z \), then there exists a \( s' \in s[Y \rightarrow st] \) such that \( X_1.(X_2.(\ldots X_n.(Z) \ldots)) \in SEQ(s') \).

We can now, distinguish the following two cases:

- \( r = Y \overset{a}{\rightarrow} Z_1.(Z_2) \in \mathbb{R} \). From remark 3.3 \( \mathcal{Y}_M^f(r) = \mathcal{Y}_{SEQ}^f(r) \). Moreover, \( t = X_1.(X_2.(\ldots X_n.(Y) \ldots)) \) and \( t' = X_1.(X_2.(\ldots X_n.(Z_1.(Z_2)) \ldots)) \). Let \( s \in T \) be such that \( t \in SEQ(s) \). From A above, there exists a \( s' \in s[Y \rightarrow Z_1.(Z_2)] \) such that \( t' \in SEQ(s') \). Since \( Y \overset{\varepsilon}{\rightarrow} Z_1.(Z_2) \), by proposition 3.1 it follows that \( s \overset{\varepsilon}{\rightarrow} s' \), and the thesis is proved.
• \( r = Y^{K'}Z \) with \( Y, Z \in \text{Var} \) and \( \Upsilon_{M_{\text{SEQ}}}^f (r) = K' \). Moreover, \( t = X_1.(X_2.(\ldots X_n.(Y) \ldots)) \) and \( t' = X_1.(X_2.(\ldots X_n.(Z) \ldots)) \). From the definition of \( \mathcal{R}^K_{\text{SEQ}} \) there exists a derivation in \( \mathcal{R}^K_{\text{PAR}} \) of the form \( Y \xrightarrow{\rho^*}_{\mathcal{R}^K_{\text{PAR}}} p \parallel Z \) for some \( p \in T_{\text{PAR}} \), with 
\[ \Upsilon_{M_{\text{PAR}}}^f (\sigma) = \Upsilon_{M_{\text{SEQ}}}^f (r) \text{ and } |\sigma| > 0. \]
From lemma \[ \exists \] there exists a term \( st \) such that 
\[ Y \xrightarrow{\rho^*}_{\mathcal{R}^K_{\text{SEQ}}} \parallel Z \text{ with } \Upsilon_{M}(\rho) = \Upsilon_{M_{\text{PAR}}}^f (\sigma) \text{ and } |\rho| > 0. \]
So, \( \Upsilon_{M}(\rho) = \Upsilon_{M_{\text{SEQ}}}^f (r) \). Let \( s \in T \) be such that \( t \in \text{SEQ}(s) \). From property \[ \exists \] above, there exists a \( s' \in s[Y \parallel st \parallel Z] \) such that \( t' \in \text{SEQ}(s') \). Since \( Y \xrightarrow{\rho^*}_{\mathcal{R}^K_{\text{SEQ}}} \parallel Z \), by proposition \[ \exists \] we conclude that \( s \xrightarrow{\rho^*}_{\mathcal{R}^K_{\text{SEQ}}} s' \) with \( |\rho| > 0. \) Hence, the thesis.

Property 2 can be easily proved by induction on the length of \( \sigma \), and using property 1. It remains to prove property 3. The infinite derivation \( t \xrightarrow{\rho^*}_{\mathcal{R}^K_{\text{SEQ}}} \) can be written in the form 
\[ t_0 \xrightarrow{\rho^*}_{\mathcal{R}^K_{\text{SEQ}}} t_1 \xrightarrow{\rho^*}_{\mathcal{R}^K_{\text{SEQ}}} t_2 \xrightarrow{\rho^*}_{\mathcal{R}^K_{\text{SEQ}}} \ldots \]
where \( t_0 = t \) and \( \forall i \in N r_i \in \mathcal{R}^K_{\text{SEQ}} \). Let \( s \in T \) such that \( t \in \text{SEQ}(s) \). From property 1 there exists a \( s_1 \in T \) with \( t_1 \in \text{SEQ}(s_1) \) such that \( s \xrightarrow{\lambda_0^*} s_1 \text{ with } \Upsilon_{M}(\lambda_0) = \Upsilon_{M_{\text{SEQ},K}}(r_0) \) and \( |\lambda_0| > 0. \) Iterating such reasoning we deduce that there exists a succession of terms, 
\[ (s_h)_{h \in N} \text{ such that for all } h \in N \]
\[ s_h \lambda_h^* s_{h+1} \text{ with } \Upsilon_{M}(\lambda_h) = \Upsilon_{M_{\text{SEQ}}}^f (r_h), |\lambda_h| > 0 \text{ and } s_0 = s. \]

Therefore, 
\[ s = s_0 \lambda_0^* s_1 \lambda_1^* s_2 \cdot \cdot \cdot \lambda_h^* s_{h+1} \cdot \cdot \cdot \]

Let \( \rho = \lambda_0 \lambda_1 \lambda_2 \ldots. \) So, \( s \xrightarrow{\rho^*}_{\mathcal{R}^K_{\text{SEQ}}} \) is an infinite derivation in \( \mathcal{R} \) from \( s \) with \( t \in \text{SEQ}(s) \). Moreover, by proposition \[ \exists \] we obtain 
\[ \Upsilon_{M}(\rho) = \bigcup_{h \in N} \Upsilon_{M}(\lambda_h) = \bigcup_{h \in N} \Upsilon_{M_{\text{SEQ}}}^f (r_h) = \Upsilon_{M_{\text{SEQ}}}^f (\sigma) = K. \]

\[ \Upsilon^\infty_{M}(\rho) = \bigoplus_{h \in N} \Upsilon_{M}(\lambda_h) = \bigoplus_{h \in N} \Upsilon_{M_{\text{SEQ}}}^f (r_h) = \Upsilon_{M_{\text{SEQ}}}^\infty (\sigma) = K^\omega. \]

This proves the thesis. \( \square \)

**Proposition C.2.** Let \( \sigma \) be a rule sequence in \( \mathcal{R} \) and \( \{ \rho_h \}_{h \in N} \) be a succession of subsequences of \( \sigma \) two by two disjoints and such that \( \bigcup_{h \in N} \text{pr}(\rho_h) = \text{pr}(\sigma). \) Then, \( \sigma \in \text{Interleaving}(\{ \rho_h \}). \)

**Proof.** Setting \( \Delta = \bigcup_{h \in N} \{(h) \times \text{pr}(\rho_h)\} \), let us consider the following mapping 
\[ M_\sigma : (h, n) \in \Delta \rightarrow n \]

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Since for all \( h, h' \in N \) with \( h \neq h' \) we have \( pr(\rho_h) \cap pr(\rho_{h'}) = \emptyset \), it follows that \( M_{\sigma} \) is an injective mapping. Let us observe that \( M_{\sigma}(h, n_1) < M_{\sigma}(h, n_2) \) if \( n_1 < n_2 \). Moreover, \( pr(\sigma) = M_0(\Delta) \), and \( \forall (h, n) \in \Delta \) we have \( \sigma(M_{\sigma}(h, n)) = \sigma(n) = \rho_h(n) \). From definition \( \text{C.1} \) we obtain the assertion. \( \square \)

**Lemma C.4.** Let \( p \xrightarrow{\mathcal{R}} \) be a \( (\mathcal{K}, \mathcal{K}_\omega) \)-accepting non-null derivation in \( M \) belonging to \( \Pi_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}} \), with \( p \in T_{\mathcal{K}, \mathcal{K}_\omega} \subseteq \mathcal{K} \) and \( \mathcal{K}_\omega \subseteq \mathcal{K}_\omega \). Then, there exists a derivation \( p \xrightarrow{\mathcal{R}} \) in \( \mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}} \) from \( p \) such that

a. \( \Upsilon^f_{\mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}}} (\rho) = \mathcal{K} \)

b. \( \Upsilon^\infty_{\mathcal{K}, \mathcal{K}_\omega} (\rho) \cup \Upsilon^f_{\mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}}} (\rho) = \mathcal{K}_\omega \)

c. If \( \sigma \) is infinite then, either \( \rho \) is infinite or \( \rho \) contains some occurrence of rule in \( \mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}} \setminus \mathcal{K}_{\mathcal{K}} \).

**Proof.** At first, let us prove the following property

d. There exists a \( p' \in T_{\mathcal{K}, \mathcal{K}_\omega} \), a non-empty finite rule sequence \( \lambda \) in \( \mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}} \), and a non-empty subsequence \( \eta \) (possibly infinite) of \( \sigma \) such that:

1. \( \min(pr(\eta)) = \min(pr(\sigma)) \) (i.e. the first rule occurrence in \( \eta \) is the first rule occurrence in \( \sigma \)).
2. \( p \xrightarrow{\lambda} p' \)
3. \( \Upsilon^f_{\mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}}} (\lambda) = \Upsilon^f_{\mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}}} (\eta) \)
4. \( \Upsilon^f_{\mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}}} (\lambda) = \Upsilon^\infty_{\mathcal{K}_\omega} (\eta) \)
5. \( p' \xrightarrow{\eta} \) and this derivation is in \( \Pi_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}} \).
6. If \( \sigma \) is infinite then, either \( \sigma \setminus \eta \) is infinite or \( \lambda \) is a rule in \( \mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}} \setminus \mathcal{K}_{\mathcal{K}} \).

Now, let us show that from property **d** the thesis follows. So, let us assume that property **d** is satisfied. Since \( \sigma \setminus \eta \) is a subsequence of \( \sigma \), we have \( \Upsilon^f_{\mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}}} (\sigma \setminus \eta) \subseteq \mathcal{K} \) and \( \mathcal{K}_\omega \subseteq \mathcal{K}_\omega \). Thus, if \( \sigma \neq \eta \) we can apply property **d** to the derivation \( p' \xrightarrow{\sigma \setminus \eta} \). Iterating this reasoning it follows that there exists a \( m \in N \cup \{\infty\} \), a succession \( \{p_h\}_{h=0}^{m+1} \) of terms in \( T_{\mathcal{K}, \mathcal{K}_\omega} \), a succession \( \{\lambda_h\}_{h=0}^{m} \) of non-empty finite rule sequences in \( \mathcal{R}_{\mathcal{K}, \mathcal{K}_\omega}^{\mathcal{K}} \), two successions \( \{\eta_h\}_{h=0}^{m} \) and \( \{\eta_h\}_{h=0}^{m} \) of non-empty rule sequences in \( \mathcal{R} \) such that

7. \( p = p_0 \) and \( \sigma = \sigma_0 \).
8. for all \( h = 0, \ldots, m \) \( \eta_h \) is a subsequence of \( \sigma_h \) and \( \min(pr(\eta_h)) = \min(pr(\sigma_h)) \).
9. for all \( h = 0, \ldots, m - 1 \) \( \sigma_{h+1} = \sigma_h \setminus \eta_h \).

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10. for all \( h = 0, \ldots, m \) \( p_h \xrightarrow{\lambda_h} \rho \_K,K\omega^{_\text{PAR}} p_{h+1} \).

11. for all \( h = 0, \ldots, m \) \( \Upsilon^f_{M,K,K\omega^{_\text{PAR}}}(\lambda_h) = \Upsilon^f_M(\eta_h) \) and \( \Upsilon^f_{M,K,K\omega^{_\text{PAR},\infty}}(\lambda_h) = \Upsilon^\infty_M(\eta_h) \).

12. for all \( h = 1, \ldots, m \) \( p_h \xrightarrow{\sigma_h} \rho \).

13. If \( m \) is finite then, \( \sigma_m = \eta_m \).

14. If \( \sigma \) is infinite then, either \( m \) is infinite or there exists an \( h \) such that \( \lambda_h \) is a rule in \( \Re^K,K\omega^{_\text{PAR}} \).

By setting \( \rho = \lambda_0 \lambda_1 \ldots \) we have that \( p \xrightarrow{\rho} \rho \_K,K\omega^{_\text{PAR}} \). From property 14 it follows that if \( \sigma \) is infinite then, either \( \rho \) is infinite or \( \rho \) contains some occurrence of rule in \( \Re^K,K\omega^{_\text{PAR}} \). So, property c is satisfied. It remains to prove properties a and b. Let us assume that \( m = \infty \). The proof in the case where \( m \) is finite is simpler. From properties 7-9 \( \eta_0, \eta_1, \ldots \) are non empty subsequences of \( \sigma \) two by two disjoints. Since \( \sigma \) is infinite, we can assume that \( \text{pr}(\sigma) = N \).

15. \( \sigma \in \text{Interleaving}(\{\eta_h\}) \)

From proposition \[\text{(4.2)}\] it suffices to prove that \( \forall h \in N \) there exists a \( i \in N \) such that \( h \in \text{pr}(\eta_i) \). From properties 8-9 it follows that \( \forall h \in N \) \( \text{min}(\text{pr}(\sigma_h)) < \text{min}(\text{pr}(\sigma_{h+1})) \). Let \( h \in N \), then there exists the smallest \( i \in N \) such that \( h \notin \text{pr}(\sigma_i) \). Since \( \sigma_0 = \sigma \), \( i > 0 \) and \( h \in \text{pr}(\sigma_{i-1}) \). Since \( \sigma_{i-1} = \sigma_i \setminus \eta_{i-1}, h \notin \text{pr}(\sigma_i) \) and \( h \in \text{pr}(\sigma_{i-1}) \), it follow that \( h \in \text{pr}(\eta_{i-1}) \). Thus, property 15 holds.

From properties 11, 15, and propositions \[\text{(3.1)}\] and \[\text{(C.1)}\] it follows that

\[
\Upsilon^f_{M,K,K\omega^{_\text{PAR}}}(\rho) = \bigcup_{h \in N} \Upsilon^f_{M,K,K\omega^{_\text{PAR}}}(\lambda_h) = \bigcup_{h \in N} \Upsilon^f_M(\eta_h) = \Upsilon^f_M(\sigma) = \overline{K}.
\]

Therefore, property a holds. Moreover,

\[
\Upsilon^\infty_{M,K,K\omega^{_\text{PAR}}}(\rho) \cup \Upsilon^f_{M,K,K\omega^{_\text{PAR},\infty}}(\rho) = \bigoplus_{h \in N} \Upsilon^f_{M,K,K\omega^{_\text{PAR}}}(\lambda_h) \cup \bigcup_{h \in N} \Upsilon^f_{M,K,K\omega^{_\text{PAR},\infty}}(\lambda_h) = \bigoplus_{h \in N} \Upsilon^f_M(\eta_h) \cup \bigcup_{h \in N} \Upsilon^\infty_M(\eta_h) = \Upsilon^\infty_M(\sigma) = \overline{K}^\omega.
\]

Therefore, property b is satisfied.

At this point, it remains to prove property d. The derivation \( p \xrightarrow{\sigma} \rho \) can be written in the form

\[
p \xrightarrow{r} t \xrightarrow{\sigma} \rho \]

(1)

Since each subderivation of \( t \xrightarrow{\sigma} \rho \) is also a subderivation of \( p \xrightarrow{\sigma} \rho \), it follows that \( t \xrightarrow{\sigma} \rho \) is in \( \Pi^K,K\omega^{_\text{PAR},\infty} \). There are two cases:
1. \( r \) is a \( \text{PAR} \) rule. Then, we have that \( t \in T_{\text{PAR}} \) and \( r \in \mathbb{R}_{\text{PAR}}^{K,K^\omega} \). From remark 1.2, \( \Upsilon_M^f_{M_{\text{PAR}}} (r) = \Upsilon_M^j (r) \), and \( \Upsilon_M^{f_{K\omega}} (r) = \emptyset = \Upsilon_M^\infty (r) \). Moreover, \( t \equiv^r_{\mathbb{R}} \) in \( \Pi_{\text{PAR},\infty}^{K,K^\omega} \) with \( \sigma' = \sigma \setminus r \). Thus, since \( \sigma' \) is infinite if \( \sigma \) is infinite, property \( \text{d} \) follows, setting \( p' = t, \lambda = r \) and \( \eta = r \).

2. \( r = Z^\alpha_{\text{Y}}Y(Z') \). So, \( p = p''|Z \) and \( t = p''|Y(Z') \) with \( p'' \in T_{\text{PAR}} \). From (1), let \( Z' \not\overset{\mu}{\preceq} \) a subderivation of \( t = p''|Y(Z') \not\overset{\sigma'}{\preceq} \) from \( Z' \). From lemma \( \text{A.1} \) we can distinguish four subcases:

- \( Z' \not\overset{\mu}{\preceq} \) is infinite, and \( p'' \not\overset{\sigma'}{\preceq} \). Moreover, \( p'' \not\overset{\sigma'}{\preceq} \) in \( \Pi_{\text{PAR},\infty}^{K,K^\omega} \). Clearly, for every \( \overline{p} \in T_{\text{PAR}} \) we have that \( p''|\overline{p} \not\overset{\sigma'}{\preceq} \) and this derivation belongs to \( \Pi_{\text{PAR},\infty}^{K,K^\omega} \). From hypothesis, we have that \( (\Upsilon_M^f (\nu), \Upsilon_M^\infty (\nu)) \neq (K, K^\omega), \Upsilon_M^j (\nu) \subseteq K \) and \( \Upsilon_M^\infty (\nu) \subseteq K^\omega \). Hence, \( |\Upsilon_M^f (\nu)| + |\Upsilon_M^\infty (\nu)| < |K| + |K^\omega| \). Moreover, \( r = Z^\alpha_{\text{Y}}Y(Z') \) with \( \Upsilon_M^f (r) \subseteq K \). From the definition of \( \mathbb{R}_{\text{PAR}}^{K,K^\omega} \), it follows that \( r' = Z^\alpha_{\text{Y}}Y^\infty_\infty \in \mathbb{R}_{\text{PAR}}^{K,K^\omega} \setminus \mathbb{R}_{\text{PAR}}^K \) where \( K_1 = \Upsilon_M^f (\nu) \cup \Upsilon_M^j (r) \) and \( K^\omega_1 = \Upsilon_M^\infty (r) \). From remark \( \text{1.2} \) we have that \( \Upsilon_M^{f_{K,K^\omega}} (r') = K_1 \) and \( \Upsilon_M^{f_{K,K^\omega}} (r') = K^\omega_1 \). Therefore, we deduce that \( p = p''|Z \not\overset{\sigma'}{\preceq} \) and \( p'' \not\overset{\sigma'}{\preceq} \) and this derivation is in \( \Pi_{\text{PAR},\infty}^{K,K^\omega} \). Since \( \sigma' \setminus \nu = \sigma \setminus r \nu \) and \( \Upsilon_M^{f_{K,K^\omega}} (r') = \Upsilon_M^\infty (r) = \Upsilon_M^\infty (r) \), property \( \text{d} \) follows, setting \( p' = p''|\hat{Y}^\infty_\infty, \lambda = r' \) and \( \eta = r' \).

- \( Z' \not\overset{\mu}{\preceq} \) leads to a term \( t_1 \neq \varepsilon \), and \( p'' \not\overset{\sigma'}{\preceq} \). Moreover, \( p'' \not\overset{\sigma'}{\preceq} \) is in \( \Pi_{\text{PAR},\infty}^{K,K^\omega} \). Since \( \Upsilon_M^j (\nu) \subseteq K \), from lemma \( \text{B.2} \) there exists a \( \overline{p} \in T_{\text{PAR}} \) such that \( Z' \not\overset{\mu}{\preceq} \overline{p} \), where \( \Upsilon_M^{f_{\overline{p}}} (\gamma) = \Upsilon_M^j (\nu) \). Since \( \Upsilon_M^j (r) \subseteq K \), from the definition of \( \mathbb{R}_{\text{PAR}}^K \) it follows that \( r' = Z^\alpha_{\text{Y}}Y^\infty_\infty \in \mathbb{R}_{\text{PAR}}^{K,K^\omega} \) with \( \Upsilon_M^{f_{\overline{p}}} (r') = K' \), where \( K' = \Upsilon_M^j (r) \). By construction \( r' \in \mathbb{R}_{\text{PAR}}^{K,K^\omega} \) and from remark \( \text{1.2} \) \( \Upsilon_M^{f_{K,K^\omega}} (r') = K' \) and \( \Upsilon_M^{f_{K,K^\omega}} (r') = \emptyset = \Upsilon_M^\infty (r) \), and \( \sigma' \setminus \nu \) is infinite if \( \sigma \) is infinite, property \( \text{d} \) follows, setting \( p' = p''|\hat{Y}^\infty_\infty, \lambda = r' \) and \( \eta = r' \).

- \( Z' \not\overset{\mu}{\preceq} \) leads to \( \varepsilon \) and the derivation \( p''|Y(Z') \not\overset{\sigma'}{\preceq} \) can be written in the following form

\[
p''|Y(Z') \not\overset{\sigma'}{\preceq} t''|Y \not\overset{\sigma'}{\preceq} \text{ with } p'' \not\overset{\sigma'}{\preceq} t' \text{ and } \sigma_1 \in \text{Interleaving}(\nu, \sigma'_1)
\]  

(2) Moreover, \( p''|Y \not\overset{\sigma'}{\preceq} t''|Y \not\overset{\sigma'}{\preceq} \) and this derivation is in \( \Pi_{\text{PAR},\infty}^{K,K^\omega} \). Since \( Z' \not\overset{\mu}{\preceq} \varepsilon \) and \( \Upsilon_M^j (\nu) \subseteq K \), from lemma \( \text{B.2} \) \( Z' \not\overset{\mu}{\preceq} \varepsilon \) with \( \Upsilon_M^{f_{\overline{p}}} (\chi) = \Upsilon_M^j (\nu) \). Since \( r = Z^\alpha_{\text{Y}}Y(Z') \) with \( \Upsilon_M^j (r) \subseteq K \), from the definition of \( \mathbb{R}_{\text{PAR}}^K \) it follows that \( r' = Z^\alpha_{\text{Y}}Y \in \mathbb{R}_{\text{PAR}}^K \) where \( K' = \Upsilon_M^j (r) \) and \( \Upsilon_M^{f_{\overline{p}}} (r') = K' \). By construction
Let us assume that $K \neq K^\omega$. Then, the following result holds.

**Lemma C.5.** Let us assume that $K \neq K^\omega$. Given a variable $X \in \text{Var}$ and a $(K, K^\omega)$-accepting infinite derivation in $M$ from $X$, the following property is satisfied:

1. There exists a variable $Y \in \text{Var}$ reachable from $X$ in $R^K_{\text{SEQ}}$ through a $(K', \emptyset)$-accepting derivation in $M^K_{\text{SEQ}}$ with $K' \subseteq K$, and there exists a derivation $Y \Rightarrow^*_{R^K_{\text{PAR}}} Y'$ such that $\gamma^f_{M^K_{\text{PAR}}} K^\omega(\rho) = K$ and $\gamma^\infty_{M^K_{\text{PAR}}} K^\omega(\rho) \cup \gamma^f_{M^K_{\text{PAR}}} K^\omega(\rho) = K^\omega$. Moreover, either $\rho$ is infinite or $\rho$ contains some occurrence of rule in $R^K_{\text{PAR}} \setminus R^K_{\text{PAR}}$.

**Proof.** Since $K \neq K^\omega$ and $K \supseteq K^\omega$, it follows that $K \supseteq K^\omega$. Let $d = X \Rightarrow^*_{R} \in (K, K^\omega)$-accepting infinite derivation in $M$ from $X$. Evidently, $K \setminus K^\omega = \{i \in \{1, \ldots, n\} \mid \sigma \}$ contains a finite non–null number of occurrences of rules in $R^A_1$. Then, for all $i \in K \setminus K^\omega$ it’s defined the greatest application level, denoted by $h_i(d)$, of occurrences of rules of $R^A_i$ in the derivation $d$. The proof is by induction on $\max_{i \in K \setminus K^\omega} \{h_i(d)\}$.

**Base Step:** $\max_{i \in K \setminus K^\omega} \{h_i(d)\} = 0$. In this case it follows that each subderivation of $d = X \Rightarrow^*_{R}$ does not contain occurrences of rules in $\bigcup_{i \in K \setminus K^\omega} R^A_i$. So, $d$ is belonging to $\Pi^K_{\text{PAR}}$. Then, from lemma **C.4** we obtain the assertion setting $Y = X$. 

\[ r' \in R^K_{\text{PAR}} \] and from remark **1.2** $\gamma^f_{M^K_{\text{PAR}}} (r') = K'$ and $\gamma^f_{M^K_{\text{PAR}}} (r') = \emptyset$. Since $\sigma \setminus r\nu = \sigma_1\sigma_2$, $\gamma^f_{M^K_{\text{PAR}}} (r') = \emptyset = \gamma^\infty_{M}(r\nu)$, and $\sigma_1\sigma_2$ is infinite if $\sigma$ is infinite, property $d$ follows. Setting $p' = p''\|Y$, $\lambda = r'$ and $\eta = r\nu$.

- $Z' \Rightarrow^*_{R} \in \text{Var}$ and the derivation $p''\|Y(Z') \Rightarrow^*_{R}$ can be written in the following form

\[ p''\|Y(Z') \Rightarrow^*_{R} t'\|Y(W) \Rightarrow^*_{R} t'\|W'' \Rightarrow^*_{R} (3) \]

with $p'' \Rightarrow^*_{R} t'$, $r' = Y(W) \Rightarrow^* W'$ and $\sigma_1 \in \text{Interleaving}(\lambda, \sigma')$ (4)

Moreover, $p''\|W' \Rightarrow^*_{R} t'\|W' \Rightarrow^*_{R}$ and this derivation is in $\Pi^K_{\text{PAR}}$. Since $Z' \Rightarrow^*_{R} W$ and $\gamma^f_{M}(\nu) \subseteq K$, from lemma **B.2** we have that $Z' \Rightarrow^*_{R} W$ with $\gamma^f_{M}(\lambda) = \gamma^f_{M}(\nu)$. Since $r = Z\Rightarrow^*Y(Z') \in \mathbb{R}$ and $r' = Y(W) \Rightarrow^* W'$, where $\gamma^f_{M}(\nu) \subseteq K$ and $\gamma^f_{M}(\nu) \subseteq K$, from the definition of $R^K_{\text{PAR}}$ it follows that $r'' = Z\Rightarrow^* W' \in R^K_{\text{PAR}}$ where $K' = \gamma^f_{M}(\nu) \cup \gamma^f_{M,K^\omega}(\lambda) = \gamma^f_{M}(\nu\nu')$ and $\gamma^f_{M,K^\omega}(\nu) = K'$. By construction, $r'' \in R^K_{\text{PAR}}$, and from remark **1.2** $\gamma^f_{M,K^\omega}(\nu) = K'$ and $\gamma^f_{M,K^\omega}(\nu) = \emptyset$. Since $\sigma \setminus r\nu = \sigma_1\sigma_2$, $\gamma^f_{M,K^\omega}(\nu) = \emptyset = \gamma^\infty_{M}(r\nu r')$, and $\sigma_1\sigma_2$ is infinite if $\sigma$ is infinite, property $d$ follows for setting $p' = p''\|W'$, $\lambda = r''$ and $\eta = r\nu r'$. 

\[ \square \]
Induction Step: \( \max_{i \in K \setminus K^\omega} \{ h_i(d) \} > 0 \). If \( d = X \overset{\sigma_i^*}{\rightarrow}_r \) is in \( \Pi_{\text{PAR}, \infty}^{K, K^\omega} \), from lemma C.4 we obtain the assertion setting \( Y = X \). Otherwise, from lemma A.2 it follows that the derivation \( X \overset{\sigma_i^*}{\rightarrow}_r \) can be written in the form

\[
X \overset{\sigma_i^*}{\rightarrow}_r \ t || Z \overset{\gamma_r}{\rightarrow} t || Y.(Z') \overset{\sigma_i^*}{\rightarrow}_r
\]

where \( r = Z \overset{a}{\rightarrow} Y.(Z') \), and there exists a subderivation of \( t \parallel Y.(Z') \overset{\sigma_i^*}{\rightarrow}_r \) from \( Z' \), namely \( d' = Z' \overset{\sigma_i^*}{\rightarrow}_r \) that is a \( (K, K^\omega) \)-accepting infinite derivation in \( M \). Evidently, \( \max_{i \in K \setminus K^\omega} \{ h_i(d') \} < \max_{i \in K \setminus K^\omega} \{ h_i(d) \} \). By inductive hypothesis, the thesis holds for the derivation \( d' \). Therefore, it suffices to prove that \( Z' \) is reachable from \( X \) in \( R_{\text{SEQ}}^K \) through a \( (K', \emptyset) \)-accepting derivation in \( M_{\text{SEQ}}^K \) with \( K' \subseteq K \). From lemma B.2 applied to the derivation \( X \overset{\sigma_i^*}{\rightarrow}_r \ t || Z \) where \( \Upsilon_M(\sigma_1) \subseteq K \), there exists a \( p \in T_{\text{PAR}} \) such that \( X \overset{\sigma_i^*}{\rightarrow}_r \ p \parallel Z \) with \( \Upsilon_M(K, K^\omega)(\rho_1) = \Upsilon_M(\sigma_1) \). From the definition of \( R_{\text{SEQ}}^K \) we obtain that \( X \overset{\sigma_i^*}{\rightarrow}_r \ Z \overset{\gamma_r}{\rightarrow} Y.(Z') \), with \( \Upsilon_{M_{\text{SEQ}}}(\gamma) = \Upsilon_{M_{\text{SEQ}}}(\rho_1) \) and \( \Upsilon_{M_{\text{SEQ}}}(r) = \Upsilon_M(r) \subseteq K \). So, \( \Upsilon_{M_{\text{SEQ}}}(\gamma r) \subseteq K \). This concludes the proof.

Now, let us assume that \( K = K^\omega \). The next two lemmata manage this case.

Lemma C.6. Let \( i \in K \), \( X \in \text{Var} \) and \( X \overset{\sigma_i^*}{\rightarrow}_r \) be a \( (K, K^\omega) \)-accepting infinite derivation in \( M \) from \( X \). Then, one of the following conditions is satisfied:

1. There exists a variable \( Y \in \text{Var} \) reachable from \( X \) in \( R_{\text{SEQ}}^K \) through a \( (K', \emptyset) \)-accepting derivation in \( M_{\text{SEQ}}^K \) with \( K' \subseteq K \), and there exists a derivation \( Y \overset{\sigma_i^*}{\rightarrow}_r \) such that \( \Upsilon_{M_{\text{SEQ}}}(\rho) = K \) and \( \Upsilon_{M_{\text{SEQ}}}(\rho) \cup \Upsilon_{M_{\text{SEQ}}}(\rho) = K^\omega \). Moreover, either \( \rho \) is infinite or \( \rho \) contains some occurrence of rule in \( R_{\text{PAR}}^{K, K^\omega} \setminus R_{\text{PAR}}^K \).

2. There exists a variable \( Y \in \text{Var} \) reachable from \( X \) in \( R_{\text{SEQ}}^K \) through a \( (K_i, \emptyset) \)-accepting derivation in \( M_{\text{SEQ}}^K \) with \( \{ i \} \subseteq K_i \subseteq K \), and there exists a \( (K, K^\omega) \)-accepting infinite derivation in \( M \) from \( Y \).

Proof. The proof is by induction on the level \( k \) of application of the first occurrence of a rule \( r \) of \( R_i^A \) in a \( (K, K^\omega) \)-accepting infinite derivation in \( M \) from a variable.

Base Step: \( k = 0 \). If \( X \overset{\sigma_i^*}{\rightarrow}_r \) is in \( \Pi_{\text{PAR}, \infty}^{K, K^\omega} \), from lemma C.4 property 1 follows, setting \( Y = X \). Otherwise, from lemma A.2 it follows that the derivation \( X \overset{\sigma_i^*}{\rightarrow}_r \) can be written in the form

\[
X \overset{\sigma_i^*}{\rightarrow}_r \ t || Z \overset{\gamma_r}{\rightarrow} t || Y.(Z') \overset{\sigma_i^*}{\rightarrow}_r
\]

where \( r' = Z \overset{a}{\rightarrow} Y.(Z') \), and there exists a subderivation of \( t \parallel Y.(Z') \overset{\sigma_i^*}{\rightarrow}_r \) from \( Z' \), namely \( Z' \overset{\sigma_i^*}{\rightarrow}_r \) that is a \( (K, K^\omega) \)-accepting infinite derivation in \( M \). By noticing that every rule occurrence in \( \sigma_i^* \) is applied to a level greater than zero in \( X \overset{\sigma_i^*}{\rightarrow}_r \) and that we are considering the case where \( k = 0 \), it follows that \( r \) must occur in the rule sequence \( \sigma_1 r'(\sigma_2 \setminus \sigma_2') \). From
In lemma A.1, we have $t \xrightarrow{\sigma_2} \sigma_2'$. Therefore, there exists a derivation of the form $X \xrightarrow{\sigma_2} t \| Z \xrightarrow{\sigma_2} t \| Y. (Z')$ with $\{i\} \subseteq \Upsilon_M^f (\lambda r') \subseteq K$. From lemma B.2, applied to the derivation $X \xrightarrow{\sigma_2} t \| Z \xrightarrow{\sigma_2} t \| Y. (Z')$, there exists a $p \in T_{PAR}$ such that $X \xrightarrow{p} \sigma^r_{PAR}$, with $\Upsilon_M^f (\rho) = \Upsilon_M^f (\lambda r')$. From the definition of $\mathcal{R}_K^\omega$ we have that $\Upsilon_M^f (\rho) = \Upsilon_M^f (\lambda r')$. Therefore, $\Upsilon_M^f (\mu r') = \Upsilon_M^f (\lambda r')$. Thus, there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ from $Z'$. This is exactly what property 2 states.

**Induction Step:** $k > 0$. If $X \xrightarrow{\sigma_3}$ is in $\Pi_{PAR, \infty}^K$, from lemma C.3 property 1 follows, setting $Y = X$. Otherwise, from lemma A.2 it follows that the derivation $X \xrightarrow{\sigma_3}$ can be written in the form

$$X \xrightarrow{\sigma_3} Z \xrightarrow{\sigma_3} t \| Y. (Z') \xrightarrow{\sigma_3}$$

where $r' = Z \xrightarrow{\alpha} Y. (Z')$, and there exists a subderivation of $t \| Y. (Z') \xrightarrow{\sigma_3}$ from $Z'$, namely $Z' \xrightarrow{\sigma_3}$, that is a $(K, K^\omega)$-accepting infinite derivation in $M$. There can be two cases:

- The rule sequence $\sigma_1 r' (\sigma_2 \setminus \sigma_1')$ contains the first occurrence of $r$ in $\sigma$. In this case, the thesis follows by reasoning as in the base step.

- $\sigma_1'$ contains the first occurrence of $r$ in $\sigma$. Clearly, this occurrence is the first occurrence of a rule of $\mathcal{R}_K^\omega$ in the $(K, K^\omega)$-accepting infinite derivation $Z' \xrightarrow{\sigma_3}$, and it is applied to level $k'$ in $Z' \xrightarrow{\sigma_3}$ with $k' < k$. By inductive hypothesis, the thesis holds for the derivation $Z' \xrightarrow{\sigma_3}$. Therefore, it suffices to prove that $Z'$ is reachable from $X$ in $\mathcal{R}_K^\omega$ through a $(K', \emptyset)$-accepting derivation in $M_\omega^K$ with $\{i\} \subseteq K \subseteq K$. From lemma B.2 applied to the derivation $X \xrightarrow{\sigma_3} t \| Z$, there exists a $p \in T_{PAR}$ such that $X \xrightarrow{p} \sigma^r_{PAR}$. From the definition of $\mathcal{R}_K^\omega$ we obtain that $X \xrightarrow{p} \sigma^r_{PAR} \xrightarrow{\sigma_3} Z \xrightarrow{\sigma_3} Y. (Z')$ with $\Upsilon_M^f (\mu r') = \Upsilon_M^f (\lambda r')$. So, $\Upsilon_M^f (\mu r') \subseteq K$. This concludes the proof.

**Lemma C.7.** Let $X \in \text{Var}$ and $X \xrightarrow{\sigma_3}$ be a $(K, K^\omega)$-accepting infinite derivation in $M$ from $X$. Then, one of the following conditions is satisfied:

1. There exists a variable $Y \in \text{Var}$ reachable from $X$ in $\mathcal{R}_K^\omega$ through a $(K', \emptyset)$-accepting derivation in $M_\omega^K$ with $K' \subseteq K$, and there exists a derivation $Y \xrightarrow{p}$ such that $\Upsilon_M^f (\rho) = K$ and $\Upsilon_M^\infty (\rho) \cup \Upsilon_M^f (\rho) = K^\omega$. Moreover, either $\rho$ is infinite or $\rho$ contains some occurrence of rule in $\mathcal{R}_K^\omega \setminus \mathcal{R}_K$.
2. There exists a variable $Y \in Var$ reachable from $X$ in $\mathcal{R}_{SEQ}^K$ through a $(K, \emptyset)$-accepting derivation in $M_{SEQ}^K$, and there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ from $Y$.

Proof. It suffices to prove that, assuming that property 1 is not satisfied, property 2 must hold. If $|K| = 0$, property 2 is obviously satisfied. So, let us assume that $|K| > 0$. Let $K = \{j_1, \ldots, j_{|K|}\}$, and for all $p = 1, \ldots, |K|$ let $K_p = \{j_1, \ldots, j_p\}$. Let us prove by induction on $p$ that for all $p = 1, \ldots, |K|$ the following property is satisfied (assuming that property 1 isn’t satisfied):

a There exists a variable $Y$ reachable from $X$ in $\mathcal{R}_{SEQ}^K$ through a $(K', \emptyset)$-accepting derivation in $M_{SEQ}^K$ with $K_p \subseteq K' \subseteq K$, and there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ from $Y$.

Base Step: $p = 1$. Considering that property 1 isn’t satisfied, the result follows from lemma C.6 setting $i = j_1$.

Induction Step: $1 < p \leq |K|$. From the inductive hypothesis there exists a $t \in T_{SEQ} \setminus \{\varepsilon\}$ such that $X \xrightarrow{\Delta^*_K} t$ with $K_{p-1} \subseteq \Gamma^f_{M_{SEQ}}(\rho) \subseteq K$, and there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ of the form $last(t) \xrightarrow{\Delta^*_K}$. From lemma C.6 applied to the derivation $last(t) \xrightarrow{\eta^*_g}$ and considering that property 1 isn’t satisfied, it follows that there exists a $\overline{t} \in T_{SEQ} \setminus \{\varepsilon\}$ such that $last(t) \xrightarrow{\eta^*_g} \overline{t}$ with $\{j_p\} \subseteq \Gamma^f_{M_{SEQ}}(\overline{\rho}) \subseteq K$, and there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ from $last(\overline{t})$. So, we have $X \xrightarrow{\Delta^*_K} t \circ \overline{t}$ with $K_p \subseteq \Gamma^f_{M_{SEQ}}(\rho \overline{\rho}) \subseteq K$. Therefore, setting $Y = last(\overline{t})$, we obtain the assertion.

From property a, since $K_{|K|} = K$, the thesis follows.

C.1 Proof of Theorem 4.4

Let us assume that $K \neq K^\omega$. Given $X \in Var$, we have to prove that there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ from $X$ if, and only if, the following property is satisfied:

- There exists a variable $Y \in Var$ reachable from $X$ in $\mathcal{R}_{SEQ}^K$ through a $(K', \emptyset)$-accepting derivation in $M_{SEQ}^K$ with $K' \subseteq K$, and there exists a derivation $Y \xrightarrow{\Delta^*_K} \lambda$ such that $\Gamma^f_{M_{PAR}}(\rho) = K$ and $\Gamma^\infty_{M_{PAR}}(\rho) \cup \Gamma^f_{M_{PAR}}(\rho) = K^\omega$. Moreover, either $\rho$ is infinite or $\rho$ contains some occurrence of rule in $\mathcal{R}_{PAR}^K$.

($\Rightarrow$) The result follows directly from Lemma C.5

($\Leftarrow$) From hypothesis we have

1. $X \xrightarrow{\Delta^*_K} t$ with $t \in T_{SEQ} \setminus \{\varepsilon\}$, $last(t) = Y$ and $\Gamma^f_{M_{SEQ}}(\lambda) \subseteq K$. 

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2. \( Y \xrightarrow{\rho}_s^{K,K^\omega} \) with \( Y_f^{M,P_{AR},K,K^\omega}(\rho) = K \) and \( Y^\infty_{M,P_{AR},K,K^\omega}(\rho) \cup Y_f^{M,P_{AR},K,K^\omega}(\rho) = K^\omega \). Moreover, either \( \rho \) is infinite or \( \rho \) contains some occurrence of rule in \( \mathcal{R}_{PAR} \setminus \mathcal{R}_{PAR} \).

Since \( X \in SEQ(X) \), from condition 1 and lemma \( \text{C.3} \) it follows that there exists a \( s \in T \) such that \( t \in SEQ(s) \) and \( X \xrightarrow{s}_r \) \( s \) with \( \eta \subseteq K \). From condition 2 and lemma \( \text{C.2} \) it follows that there exists a \( (K,K^\omega)-\)accepting infinite derivation in \( M \) of the form \( Y \xrightarrow{\sigma}_s^\omega \). Since \( Y \in SubTerms(s) \), from proposition \( \text{A.1} \) we have that \( s \xrightarrow{s} \). After all, we obtain \( X \xrightarrow{\eta}_s \) \( s \xrightarrow{\sigma}_s^\omega \), that is a \( (K,K^\omega)-\)accepting infinite derivation in \( M \) from \( X \). This concludes the proof.

C.2 Proof of Theorem 4.5

Let us assume that \( K = K^\omega \). Given \( X \in Var \), we have to prove that there exists a \( (K,K^\omega)-\)accepting infinite derivation in \( M \) from \( X \) if, and only if, one of the following properties is satisfied:

1. There exists a variable \( Y \in Var \) reachable from \( X \) in \( \mathcal{R}_{SEQ}^K \) through a \( (K',\emptyset)-\)accepting derivation in \( M_{SEQ}^K \) with \( K' \subseteq K \), and there exists a derivation \( Y \xrightarrow{\rho}_s^{K,K^\omega} \) such that \( Y_f^{M,P_{AR},K,K^\omega}(\rho) = K \) and \( Y^\infty_{M,P_{AR},K,K^\omega}(\rho) \cup Y_f^{M,P_{AR},K,K^\omega}(\rho) = K^\omega \). Moreover, either \( \rho \) is infinite or \( \rho \) contains some occurrence of rule in \( \mathcal{R}_{PAR} \setminus \mathcal{R}_{PAR} \).

2. There exists a \( (K,K^\omega)-\)accepting infinite derivation in \( M_{SEQ}^K \) from \( X \).

\((\Rightarrow)\) It suffices to prove that, assuming that condition 1 does not hold, condition 2 must hold. Under this hypothesis, we show that there exists a succession of terms \( (t_h)_{h \in N} \) in \( T_{SEQ} \setminus \{\varepsilon\} \) satisfying the following properties:

i. \( t_0 = X \)

ii. for all \( h \in N \) \( \text{last}(t_h) \xrightarrow{\rho}_s^{K,K^\omega} t_{h+1} \) with \( Y_f^{M,K}_{SEQ}(\rho_h) = K \).

iii. for all \( h \in N \) there exists a \( (K,K^\omega)-\)accepting infinite derivation in \( M \) from \( \text{last}(t_h) \).

iv. for all \( h \in N \) \( \text{last}(t_h) \) is reachable from \( X \) in \( \mathcal{R}_{SEQ}^K \) through a \( (K',\emptyset)-\)accepting derivation in \( M_{SEQ}^K \) with \( K' \subseteq K \).

For \( h = 0 \) properties iii and iv are satisfied, by setting \( t_0 = X \). So, assume the existence of a finite sequence of terms \( t_0,t_1,\ldots,t_h \) in \( T_{SEQ} \setminus \{\varepsilon\} \) satisfying properties i-iv. It suffices to prove that there exists a term \( t_{h+1} \) in \( T_{SEQ} \setminus \{\varepsilon\} \) satisfying iii and iv, and that \( \text{last}(t_h) \xrightarrow{\rho}_s^{K,K^\omega} t_{h+1} \) with \( Y_f^{M,K}_{SEQ}(\rho_h) = K \). From the inductive hypothesis, \( \text{last}(t_h) \) is reachable from \( X \) in \( \mathcal{R}_{SEQ}^K \) through a \( (K',\emptyset)-\)accepting derivation in \( M_{SEQ}^K \) with \( K' \subseteq K \), and there exists a \( (K,K^\omega)-\)accepting infinite derivation in \( M \) from \( \text{last}(t_h) \). From lemma \( \text{C.1} \) applied to variable \( \text{last}(t_h) \), and the fact that condition 1 does not hold, it follows that there exists
a term \( t \in T_{SEQ} \setminus \{\varepsilon\} \) such that \( \text{last}(t_h) \xrightarrow{\rho_h} t \) with \( \Upsilon^f_{M^{SEQ}_K}(\rho_h) = K \), and there exists a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( \text{last}(t) \). Since \( \text{last}(t_h) \) is reachable from \( X \) in \( \mathcal{R}^K_{SEQ} \) through a \((K', \emptyset)\)-accepting derivation in \( M^{K}_{SEQ} \) with \( K' \subseteq K \), it follows that \( \text{last}(t) \) is reachable from \( X \) in \( \mathcal{R}^K_{SEQ} \) through a \((K, \emptyset)\)-accepting derivation in \( M^{K}_{SEQ} \). Thus, setting \( t_{h+1} = t \), we obtain the result.

Let \( (t_h)_{h \in \mathbb{N}} \) be the succession of terms in \( T_{SEQ} \setminus \{\varepsilon\} \) satisfying properties i-iv. Since in this case \( |K| > 0 \) (remember that \( |K| + |K^\omega| > 0 \)), we have \( |\rho_h| > 0 \) for all \( h \in \mathbb{N} \). Then, by property 1 of proposition A.2 we obtain that for every \( h \in \mathbb{N} \)

\[
 t_h \xrightarrow{\rho_h} t_h \circ t_{h+1} \text{ with } \Upsilon^f_{M^{K}_{SEQ}}(\rho_h) = K
\]

From property 2 of proposition A.2 we have that for all \( h \in \mathbb{N} \)

\[
 t_0 \circ t_1 \circ \ldots \circ t_h \xrightarrow{\rho_h} t_0 \circ t_1 \circ \ldots \circ t_h \circ t_{h+1}
\]

Therefore,

\[
 X = t_0 \xrightarrow{\rho_0} t_0 \circ t_1 \xrightarrow{\rho_1} t_0 \circ t_1 \circ t_2 \xrightarrow{\rho_2} \ldots \xrightarrow{\rho_{h-1}} t_0 \circ t_1 \circ \ldots \circ t_h \xrightarrow{\rho_h} t_0 \circ t_1 \circ \ldots \circ t_{h+1} \xrightarrow{\rho_{h+1}} t_0 \circ t_1 \circ \ldots \circ t_{h+1} \circ t_{h+2} \xrightarrow{\rho_0} \ldots
\]

is an infinite derivation in \( \mathcal{R}^K_{SEQ} \) from \( X \). Setting \( \delta = \rho_0 \rho_1 \ldots \), from ii and proposition 3.1, we obtain that

\[
 \Upsilon^f_{M^{K}_{SEQ}}(\delta) = \bigcup_{h \in \mathbb{N}} \Upsilon^f_{M^{K}_{SEQ}}(\rho_h) = K.
\]

\[
 \Upsilon^\infty_{M^{K}_{SEQ}}(\delta) = \bigoplus_{h \in \mathbb{N}} \Upsilon^f_{M^{K}_{SEQ}}(\rho_h) = K = K^\omega.
\]

Hence, condition 2 holds.

\(\leftarrow\) At first, let us assume the condition 2 holds. Then, since \( X \in SEQ(X) \), the result follows directly from lemma C.3. Assume that condition 1 holds instead. Then, we reason as in the proof of theorem 4.4.

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