CHARACTERIZATIONS OF GENERALIZED POLES BY POLE CANCELLATION FUNCTIONS OF HIGHER ORDER

MUHAMED BOROGOVAC AND ANNEMARIE LUGER

Abstract. In this paper the analytic characterization of generalized poles of operator valued generalized Nevanlinna functions (including the length of Jordan chains of the representing relation) is completed. In particular, given a Jordan chain of length \( \ell \), we show that there exists a pole cancellation function of order at least \( \ell \), and, moreover, this function is of surprisingly simple form.

Key words: Pontryagin spaces, generalized Nevanlinna functions, generalized poles, pole cancellation functions, pole functions

MSC (2010) 47B50 46C20 30E99

1. Introduction

In this work certain spectral properties of operators are characterized by analytic properties of corresponding matrix (or operator) valued functions. More precisely, the focus lies on Jordan chains of self-adjoint operators in Pontryagin spaces.

As a motivation we give a short description of two approaches that lead to this question. Start with an easy example: Let \( A \) be a Hermitian matrix, then the resolvent \( (A - z)^{-1} \) is a matrix-valued function with poles precisely at the eigenvalues of \( A \). Obviously these poles are of order one, since the Hermitian matrix \( A \) cannot have any Jordan chains.

In this paper we are interested in the more general situation, where Jordan chains can appear as well as it is possible that an eigenvalue is not an isolated point of the spectrum. More precisely, let \((K, \langle \cdot, \cdot \rangle)\) be a Pontryagin space and \( A \) a self-adjoint operator (or even relation) in this space\(^1\). We are then interested in the eigenvalues of \( A \), in particular, in the structure of the algebraic eigenspaces; i.e. Jordan chains. To this end let \( H \) be a Hilbert space and \( \Gamma_0 : H \to K \) a bounded linear operator, denote its adjoint by \( \Gamma_0^\dagger \), and define the (matrix or operator valued) function \( Q \) by

\[ Q(z) := \Gamma_0^\dagger (A - z)^{-1} \Gamma_0, \quad z \in \varrho(z). \tag{1.1} \]

If in applications \( A \) is a differential operator then \( \Gamma_0 \) might act on the boundary of the domain.

Instead of investigating \( A \) itself, or its resolvent, we aim to describe the properties of Jordan chains (such as lengths and inner products between elements) by the function \( Q \) instead. Note that the singularites of \( Q \) belong to the spectrum of \( A \), however, in general the converse is not true. In order to assure equality an additional minimality assumption is needed, see Proposition 2.2.

\(^1\)Note that this property can also be described by the following condition. Let \( G \) be a self-adjoint operator in a Hilbert space such that \( \sigma(G) \cap \mathbb{R}^- \) consists of finitely many negative eigenvalues of finite multiplicity only. Then we consider operators or relations \( A \) satisfying that \( GA \) is self-adjoint in the Hilbert space.
In this formulation of the problem the focus lies on the operator $A$, whereas the function $Q$ appears as an auxiliary object. Conversely, one can also focus on functions that essentially arise from the above situation, namely \textit{generalized Nevanlinna functions}, $Q \in \mathcal{N}_\kappa(\mathcal{H})$. These are $\mathcal{L}(\mathcal{H})$-valued functions generalizing the class $\mathcal{N}_0(\mathbb{C})$, which consists of scalar functions mapping the upper half plane holomorphically into itself, see Definition 2.1.

Recall that such functions admit a realization in a Pontrygain space, see Proposition 2.2. Essentially this means that for $\alpha \in \mathbb{R}$ the function $Q \in \mathcal{N}_\kappa(\mathcal{H})$ can be written in the form

$$Q(z) = \Gamma_0(A - z)^{-1}\Gamma_0 + H(z),$$

(1.2)

with some $\Gamma_0 \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $H$ is analytic at $\alpha$. Hence the question arises how the internal spectral structure of $A$ is reflected in analytic properties of $Q$.

When dealing with such questions the following difficulties appear: An eigenvalue need not be an isolated spectral point and hence it is possible that the corresponding algebraic eigenspace is degenerate. This amounts to the fact that no “Laurent expansion” is possible, with this we mean that given $\alpha$ it might be impossible to write $Q$ as

$$Q(z) = R(z) + Q_0(z),$$

where $R$ is rational and $Q_0 \in \mathcal{N}_0(\mathcal{H})$ is a Nevanlinna function without negative squares, where Jordan chains cannot appear. As an example serves $Q(z) = \frac{\sqrt{z}}{z}$. Another major issue is the fact that both $Q$ and $\tilde{Q}(z) := -Q(z)^{-1}$ might have a generalized pole at the same point. The current paper reveals that the first difficulty, in fact, is only technical, whereas the second is intrinsic.

The problem of characterizing the eigenspace structure of $A$ in (1.2) in terms of analytic properties of $Q$ has been around from the time when generalized Nevanlinna functions have been introduced, around 40 years ago, cf. [K La]. For scalar generalized Nevanlinna functions the non-positive part of the Jordan chain was characterized in [La] whereas as an analytic characterization of the whole chain (in terms of the asymptotic behaviour of $Q$) can be found in [HLu]. The methods there are quite different from our approach.

For matrix (and operator) generalized Nevanlinna functions the problem is much harder, since the singularities lie only in “certain directions”. The appropriate tool here are so-called \textit{pole cancellation functions}. Essentially this is a vector valued function $\vec{\eta}$ vanishing at $\alpha$, the point under investigation, such that $Q(z)\vec{\eta}(z)$ is in some sense regular and does not vanish at $\alpha$, see Definitions 3.1 and 3.2.

It has partially been shown earlier, see Remark 3.4, that the existence of such a pole cancellation function (vanishing of order $\ell$) implies that $A$ has a Jordan chain (of length at least $\ell$).

Conversely, the construction of a pole cancellation function given a Jordan chain has appeared to be much more demanding. The results available either make rather restrictive assumptions or cannot cover the whole Jordan chain, and, moreover, all these constructions are very technical and do not help so much in finding a pole cancellation function in a concrete situation, see Remark 3.14 for more history of the problem.
In the present paper we not only show that for a Jordan chain of length $\ell$ there exists a pole cancellation function of order at least $\ell$, but also - not less important - it is given in a surprisingly simple form, see Theorems 3.5 and 3.12. In a concrete situation this means that there is only a finite number of parameters that have to be found, when constructing the pole cancellation function.

The article is organized as follows. After this introduction and preliminaries the main results are in Section 3. In 3.1 a short collection of results on how the existence of a pole cancellation function implies the existence of a Jordan chain is given. The core of this text is Section 3.2, where the construction of the pole cancellation function is done. We also illustrate the sharpness of this result (regarding order and the polynomial form) by several examples, collected in 3.3. The presentation is completed by a short conclusion.

2. Preliminaries

We start this section with the analytic definition of generalized Nevanlinna functions even though in the following mainly an alternative way of describing these functions, namely their realizations, will be used.

Let $(\mathcal{H}, (\cdot, \cdot))$ be a Hilbert space and denote by $L(\mathcal{H})$ the set of bounded linear operators in $\mathcal{H}$, and let $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im} z > 0\}$ denote the open upper half plane.

Definition 2.1. An operator valued function $Q : D(Q) \subset \mathbb{C} \to L(\mathcal{H})$ belongs to the generalized Nevanlinna class $\mathcal{N}_\kappa(\mathcal{H})$, if it satisfies the following properties:

- $Q$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$,
- $Q(z)^* = Q(z)$ for all $z \in D(Q)$,
- the Nevanlinna kernel
  $$N_Q(z, w) := \frac{Q(z) - Q(w)^*}{z - w}, \quad z, w \in D(Q) \cap \mathbb{C}^+$$
  has $\kappa$ negative squares, i.e. for arbitrary $n \in \mathbb{N}$, $z_1, \ldots, z_n \in D(Q) \cap \mathbb{C}^+$ and $\vec{h}_1, \ldots, \vec{h}_n \in \mathcal{H}$ the Hermitian matrix $\left(N_Q(z_i, z_j) \vec{h}_i, \vec{h}_j\right)_{i,j=1}^n$ has at most $\kappa$ negative eigenvalues, and $\kappa$ is minimal with this property.

It is well known, see eg. [KLa] and [HSW], that these functions can also be described by their realizations.

Proposition 2.2. A function $Q$ with values in $L(\mathcal{H})$ is a generalized Nevanlinna function if and only if there exist a Pontryagin space $(\mathcal{K}, [\cdot, \cdot])$, a self-adjoint relation $A$ in $\mathcal{K}$, a point $z_0 \in \partial(A) \cap \mathbb{C}^+$, and a bounded linear map $\Gamma : \mathcal{H} \to \mathcal{K}$ such that $Q$ can be written as

$$Q(z) = Q(z_0)^* + (z - z_0)\Gamma^* \left(I + (z - z_0)(A - z)^{-1}\right)\Gamma \quad \text{for all } z \in D(Q). \quad (2.1)$$

Moreover, this realization can be chosen minimal, that is,

$$\mathcal{K} = \overline{\text{span}}\left\{(I + (z - z_0)(A - z)^{-1})\Gamma \vec{h}, \; z \in \partial(A), \; \vec{h} \in \mathcal{H}\right\}.$$  

If the realization is minimal, then $Q \in \mathcal{N}_\kappa(\mathcal{H})$ if and only if the negative index of the Pontryagin space equals $\kappa$.

Recall that a linear relation can be seen as a multi-valued operator, see eg. [DS].
With the abbreviation
\[ \Gamma_z := (I + (z - z_0)(A - z)^{-1}) \Gamma \]
the following useful identities hold for \( z, w \in \mathcal{g}(A) \):

\[ \Gamma_w = (I + (w - z)(A - w)^{-1}) \Gamma_z \tag{2.2} \]
\[ \Gamma_w^+ \Gamma_z = \Gamma_z^+ \Gamma_w = \frac{Q(z) - Q(w)}{z - w} \tag{2.3} \]
\[ (A - z_0)^{-1} \Gamma_z = \frac{1}{z - z_0} \left( \Gamma_z - \Gamma_{z_0} \right) \tag{2.4} \]

**Remark 2.3.** Note that the minimal realization of a given function \( Q \) is unique up to unitary equivalence and in this case \( g(A) = \text{hol}(Q) \).

Hence in what follows, we refer to \( A \) in a minimal realization of \( Q \) as the representing relation of \( Q \) and we will be interested in its algebraic eigenspace structure.

**Definition 2.4.** A point \( \alpha \in \mathbb{C} \cup \{ \infty \} \) is called generalized pole of \( Q \in \mathcal{N}_k(\mathcal{H}) \) if \( \alpha \) is an eigenvalue of the representing relation \( A \) in a minimal realization of the form (2.1).

### 3. Algebraic eigenspace and pole cancellation functions

The main tool in the analytic characterization of generalized poles are so-called pole cancellation functions, which also have been used for “ordinary” poles of general matrix-valued meromorphic functions, cf. [GSi].

Definitions and also notions vary in the literature. We give two versions, their relation will become clear by Theorem 3.3. Here and in the following \( z \to \alpha \) denotes the non-tangential limit as \( z \) tends to \( \alpha \in \mathbb{R} \), and \( w \)-lim the weak limit, here in the definitions, in the Hilbert space \( \mathcal{H} \).

**Definition 3.1.** Let \( Q \in \mathcal{N}_k(\mathcal{H}) \) and \( \alpha \in \mathbb{R} \) be given. Denote by \( U_\alpha \) a neigbourhood of \( \alpha \). A function \( \vec{\eta} : U_\alpha \cap \mathbb{C}^+ \to \mathcal{H} \) is called pole cancellation function of \( Q \) at \( \alpha \) if the following properties are satisfied:

(A) \( w\)-lim \( z \to \alpha \) \( \vec{\eta}(z) = \vec{0} \),

(B) \( w\)-lim \( z \to \alpha \) \( Q(z) \vec{\eta}(z) =: \vec{\eta}_0 \) exists and \( \vec{\eta}_0 \neq \vec{0} \),

(C) \( \left( \frac{Q(z) - Q(w)^*}{z - w}, \vec{\eta}(z), \vec{\eta}(w) \right) \) is bounded as \( z \to \alpha \).

We say that \( \vec{\eta} \) is a strong pole cancellation function if, moreover,

(D) \( \lim_{z, w \to \alpha} \left( \frac{Q(z) - Q(w)^*}{z - w}, \vec{\eta}(z), \vec{\eta}(w) \right) \) exists.

As we will see, the existence of a pole cancellation function at a point \( \alpha \) is sufficient for \( \alpha \) to be a generalized pole of \( Q \). However, in order to describe the whole algebraic eigenspace of the representing relation (not only the eigenvectors) higher order derivatives will be needed.
Definition 3.2. A pole cancellation function $\vec{\eta}$ of $Q$ at $\alpha$ is of order $\ell \in \mathbb{N}$ if $\ell$ is the maximal number such that for all $0 \leq j < \ell$ it holds

\begin{align*}
(D) \quad & \wlim_{z \to \alpha} (\vec{\eta}(z))^{(j)} = 0, \\
(E) \quad & \wlim_{z \to \alpha} (Q(z)\vec{\eta}(z))^{(j)} =: \vec{\eta}_j \text{ exist and } \vec{\eta}_0 \neq 0, \\
(F) \quad & \frac{d^{j+1}}{dz^{j+1}} \left( \frac{Q(z) - Q(w)^*}{z - w} \vec{\eta}(z), \vec{\eta}(w) \right) \text{ is bounded as } z \to \alpha.
\end{align*}

For a strong pole cancellation function $\vec{\eta}$ to be said to be of order $\ell \in \mathbb{N}$ additionally the following property has to be satisfied

\begin{align*}
(F_s) \quad & \lim_{z, w \to \alpha\alpha} \frac{d^{j+1}}{dz^{j+1}} \left( \frac{Q(z) - Q(w)^*}{z - w} \vec{\eta}(z), \vec{\eta}(w) \right) \text{ exist for all } 0 \leq j < \ell.
\end{align*}

Note that for $j = 0$ conditions (D-F) become (A-C). Also for historical reasons we have separated these two cases.

Let $Q \in \mathcal{N}_\kappa(\mathcal{H})$ and $\alpha \in \mathbb{R}$ be given. The analytic characterization of the algebraic eigenspace of the representing relation with the help of pole cancellation functions is divided into two sections, corresponding to the two implications that have to be shown.

3.1. Jordan chains by means of pole cancellation functions. In this section we assume that there exists a pole cancellation function of $Q$ at $\alpha$ of order $\ell$ as introduced in Definitions 3.1 and 3.2. Under these (or stronger) assumptions partial results for the existence of a Jordan chain of the representing relation have been obtained in [BLa, B, DLaS], and [Lu2]. The following theorem collects and completes these results, in particular, we allow weaker conditions on the pole cancellation function.

Theorem 3.3. Let $Q \in \mathcal{N}_\kappa(\mathcal{H})$ be given with a minimal realization (2.1)

\[ Q(z) = Q(z_0)^* + (z - z_0)\Gamma^+ (I + (z - z_0)(A - z)^{-1})\Gamma \]

and $\alpha \in \mathbb{R}$. Let $\vec{\eta} : U_\alpha \cap \mathbb{C}^+ \to \mathcal{H}$ satisfy (A) and (B) in Definition 3.1. Then the following holds:

1. If $\vec{\eta}$ satisfies even (C), i.e. it is a pole cancellation function, then $\alpha$ is a generalized pole of $Q$, more precisely,

\[ \Gamma_z \vec{\eta}(z) \xrightarrow{\text{weakly}} x_0, \quad \text{as } z \to \alpha, \]

where $x_0$ is an eigenvector of the representing relation $A$.

2. If the pole cancellation function $\vec{\eta}$ satisfies even (C$_a$) then

\[ \Gamma_z \vec{\eta}(z) \xrightarrow{\text{strongly}} x_0, \quad \text{as } z \to \alpha. \]

3. If the pole cancellation function $\vec{\eta}$ satisfies also (D), (E), and (F) for some $\ell \in \mathbb{N}$ then for $0 \leq j < \ell$

\begin{equation}
\frac{1}{j!} \left( \Gamma_z \vec{\eta}(z) \right)^{(j)} \xrightarrow{\text{weakly}} x_j, \quad \text{as } z \to \alpha, \tag{3.1}
\end{equation}

and $x_0, x_1, \ldots, x_{\ell-1}$ form a Jordan chain of $A$ at $\alpha$.

4. If $\vec{\eta}$ additionally satisfies (F$_s$) then the convergence in (3.1) is strong and for $0 \leq i, j < \ell$ it holds

\begin{equation}
\lim_{z, w \to \alpha\alpha} \frac{1}{j!} \frac{d^{j+1}}{dz^{j+1}} \left( \frac{Q(z) - Q(w)^*}{z - w} \vec{\eta}(z), \vec{\eta}(w) \right) = [x_i, x_j]. \tag{3.2}
\end{equation}
Proof. Recall that weak convergence can be characterized as follows, see [BLa]:

The sequence \((f_k)_{k \in \mathbb{N}}\) in the Pontryagin space \((\mathcal{K}, \langle \cdot, \cdot \rangle)\) converges weakly if and only if the sequence \(\left(\langle f_k, f_k \rangle\right)_{k \in \mathbb{N}}\) is bounded and \(\left(\langle f_k, u \rangle\right)_{k \in \mathbb{N}}\) is a Cauchy sequence for all elements \(u\) of some total subset \(U\) of \(\mathcal{K}\).

We apply this to \(\frac{1}{j!} \Gamma_z \tilde{\eta}(z)\) as \(z \to \alpha\). By conditions (2.3) and (F) we have that

\[
\left[ \frac{1}{j!} \Gamma_z \tilde{\eta}(z) \right]^{(j)} = \frac{d^j}{dz^j} \left( \frac{Q(z) - Q(z)^*}{z - \bar{w}} \tilde{\eta}(z), \tilde{\eta}(z) \right) \tag{3.3}
\]

is bounded as \(z \to \alpha\). By assumption the set \(\{ \Gamma_w \tilde{h}, w \in \mathbb{P}(A), \tilde{h} \in \mathcal{H} \}\) (and hence also the possibly smaller set \(\{ \Gamma_w \tilde{h}, w \in \mathbb{P}(A) \setminus \{ \alpha \}, \tilde{h} \in \mathcal{H} \}\)) is a total set in \(\mathcal{K}\). By (D) and (E) it follows that for all \(w \in \mathbb{P}(A) \setminus \{ \alpha \}\) and \(\tilde{h} \in \mathcal{H}\)

\[
\left[ \frac{1}{j!} \Gamma_z \tilde{\eta}(z) \right]^{(j)} , \Gamma_w \tilde{h} = \frac{d^j}{dz^j} \left( \frac{Q(z)\tilde{\eta}(z), \tilde{h}}{z - \bar{w}} \right) , \tilde{\eta}(z) = Q(w)\tilde{h} \tag{3.3.1}
\]

converges as \(z \to \alpha\). Thus for \(j < \ell\) the above statement from [BLa] implies that \(\frac{1}{j!} \left( \Gamma_z \tilde{\eta}(z) \right)^{(j)}\) converges weakly as \(z \to \alpha\), and we denote the limit element by \(x_j\).

Next we show that \(x_0, \ldots, x_{\ell - 1}\) is a Jordan chain of \(A\) at \(\alpha\). With (2.4) we obtain for \(0 \leq j < \ell\)

\[
(A - z_0)^{-1} x_j = \text{w-lim}_{z \to \alpha} (A - z_0)^{-1} \frac{1}{j!} \left( \Gamma_z \tilde{\eta}(z) \right)^{(j)}
\]

\[
= \text{w-lim}_{z \to \alpha} \frac{d^j}{dz^j} \left( \frac{1}{j!} (z - z_0)^{-1} (\Gamma_z \tilde{\eta}(z) - \Gamma_{z_0} \tilde{\eta}(z)) \right).
\]

Taking into account (D) this further equals

\[
= \text{w-lim}_{z \to \alpha} \frac{d^j}{dz^j} \Gamma_z \tilde{\eta}(z) = \frac{1}{j!} (z - z_0)^{-1} \Gamma_z \tilde{\eta}(z) = \frac{1}{j!} (z - z_0)^{-1} \Gamma_z \tilde{\eta}(z) = \frac{1}{j!} (z - z_0)^{-1} \Gamma_z \tilde{\eta}(z)
\]

\[
= \sum_{k=0}^{j-1} \frac{(-1)^j}{(j-k)! (z - z_0)^{j-k+1}} x_k.
\]

For \(j = 0\) this is

\[
(A - z_0)^{-1} x_0 = \frac{1}{\alpha - z_0} x_0.
\]

Assume now \(x_0 = 0\), then

\[
0 = [x_0, \Gamma \tilde{h}] = \lim_{z \to \alpha} \Gamma_z \tilde{\eta}(z), \Gamma \tilde{h} = \lim_{z \to \alpha} \frac{Q(z) - Q(z)^*}{z - \bar{w}} \tilde{\eta}(z), \tilde{h} = \frac{(\tilde{\eta}, \tilde{h})}{\alpha - z_0}
\]

for all \(\tilde{h} \in \mathcal{H}\), which would imply \(\tilde{\eta} = \tilde{h} = 0\) in contradiction to (B). Hence \(x_0\) is an eigenvector of \(A\) at \(\alpha\), and (I) is shown.

Rewriting

\[
(A - z_0)^{-1} x_j = \sum_{k=0}^{j} \frac{(-1)^j}{(\alpha - z_0)^{j-k+1}} x_k, \quad \text{for } j = 1, \ldots, \ell - 1
\]
by using this relation again backwards for \( j - 1 \) yields

\[
(A - z_0)^{-1}x_j = \frac{1}{\alpha - z_0} (x_j - (A - z_0)^{-1}x_{j-1}),
\]

and hence \( x_0, \ldots, x_{m-1} \) forms a Jordan chain of \( A \) at \( \alpha \), which proves (3).

In order to show (2) and (4) a similar characterization for strong convergence is used, namely where the condition that \( ([f_k, f_n])_{k, n \in \mathbb{N}} \) is bounded is substituted by the requirement that \( ([f_k, f_n])_{k, n \in \mathbb{N}} \) is a Cauchy-sequence. Then instead of (3.3) we note that by (F_s) we have that

\[
\left[ \frac{1}{j!} \left( \Gamma z \eta(z) \right)^{(j)} \left( \Gamma w \eta(w) \right)^{(j)} \right] = \frac{1}{(j!)^2} \frac{\partial^2}{\partial z^j \partial w^j} \left( \frac{Q(z) - Q(w)^*}{z - w} \eta(z), \eta(w) \right)
\]

converges as \( z, w \to \alpha \) and hence

\[
\frac{1}{j!} \left( \Gamma z \eta(z) \right)^{(j)} \to \frac{1}{j!} \left( \Gamma z \eta(z) \right)^{(j)} \quad \text{strongly, as } z, w \to \alpha.
\]

From (3) we already know that \( x_0, \ldots, x_{m-1} \) is a Jordan chain of \( A \) and hence (2) and the first statement of (4) is shown.

Finally, as by (F_s) the limit in (3.2) exists it follows that for \( 0 \leq i, j < \ell \)

\[
\frac{d^{i+j}}{dz^i \partial w^j} \left( \frac{Q(z) - Q(w)^*}{z - w} \eta(z), \eta(w) \right) = \left[ \frac{1}{j!} \left( \Gamma z \eta(z) \right)^{(i)} \frac{1}{j!} \left( \Gamma w \eta(w) \right)^{(j)} \right]
\]

converges to \([x_i, x_j]\) as \( z, w \to \alpha \), which completes the proof.

We conclude this section by some comments.

**Remark 3.4.** Statement (1) was basically shown in [BLa], only property (C) was replaced by a stronger property (but weaker than (C_s)), namely that the limit

\[
\lim_{z \to \alpha} \left( \frac{Q(z) - Q(z)^*}{z - \bar{z}} \eta(z), \eta(z) \right)
\]

exists. Already there (and explicitly mentioned in [DLaS]) only the weaker condition corresponding to (C) is actually used in the proof. However, the above limit plays an important role there, as it is shown that

\[
\lim_{z \to \alpha} \left( \frac{Q(z) - Q(z)^*}{z - \bar{z}} \eta(z), \eta(z) \right) \leq [x_0, x_0]
\]

and hence the non-positivity of this limit implies that \( x_0 \) is a non-positive element, which was an important issue in these papers. Statement (2) can be found in [DLaS].

Statement (3) and (4) are new in this generality. Originally such results have been proven in [BL] and [Lu2]. In [BL] holomorphy conditions were used, as only meromorphic \( Q \) are treated there. In the general case the proof was given in [Lu2], however, using (F_s) and an (unnecessary) strong variant of (E). Note that with (F) instead of (F_s) only inequalities could have been achieved in (3.2).
3.2. Pole cancellation functions by means of Jordan chains. The main result of this paper is Theorem 3.5 together with Theorem 3.12.

Recall that a function $Q \in \mathcal{N}_h(\mathcal{H})$ is called regular if there exists a point $\gamma \in \mathbb{C}^+$ such that $Q(\gamma)$ is boundedly invertible. For a regular function $Q$ a point $\alpha \in \mathbb{C} \cup \{\infty\}$ is called generalized zero of $Q$ if it is a generalized pole of $\hat{Q}(z) := -Q(z)^{-1}$.

**Theorem 3.5.** Let the regular generalized Nevanlinna function $Q$ be given with a minimal realization (2.1)

$$Q(z) = Q(z_0)^* + (z - \overline{z}_0)\Gamma^+ (I + (z - z_0)(A - z)^{-1})\Gamma$$

and assume that $\alpha \in \mathbb{R}$ is not a generalized zero of $Q$.

If $\alpha \in \mathbb{R}$ is a generalized pole of $Q$, that is $\alpha \in \sigma_p(A)$, and $x_0, x_1, \ldots, x_{\ell-1}$ is a Jordan chain of $A$ at $\alpha$, then

$$\tilde{\eta}(z) := (z - \overline{z}_0)Q(z)^{-1}\Gamma^+ \left(x_0 + (z - \alpha) x_1 + \ldots + (z - \alpha)^{\ell-1}x_{\ell-1}\right). \quad (3.4)$$

is a strong pole cancellation function of $Q$ at $\alpha$ of order at least $\ell$.

The factor $(z - \overline{z})$ is not necessary for the above statement, however, as detailed in the following corollary, it allows us to recover the original Jordan chain from the pole cancellation function as in Theorem 3.3.

**Corollary 3.6.** With the notation from Theorem 3.5 for $0 \leq i, j < \ell$ it holds

$$\frac{1}{j!} (\Gamma_z \tilde{\eta}(z))^{\text{strongly}} \left( x_j, \text{ as } z \to \alpha, \right)$$

and

$$\lim_{z,w \to \alpha} \frac{1}{i! j!} \frac{d^{i+j}}{dz^i dw^j} \left( \frac{Q(z) - Q(w)^*}{z - \overline{w}} \tilde{\eta}(z), \tilde{\eta}(w) \right) = [x_i, x_j].$$

**Remark 3.7.** If $Q$ can be written in the simpler form (1.1) then $\tilde{\eta}$ in Theorem 3.5 can be chosen as $\tilde{\eta}(z) := Q(z)^{-1}\Gamma^+(x_0 + (z - \alpha) x_1 + \ldots + (z - \alpha)^{\ell-1}x_{\ell-1})$.

The proof of Theorem 3.5 is based on two technical lemmas. Recall first (see eg. [La1, LaLa]) that if $Q \in \mathcal{N}_h(\mathcal{H})$ has the realization

$$Q(z) = Q(z_0)^* + (z - \overline{z}_0)\Gamma^+ (I + (z - z_0)(A - z)^{-1})\Gamma$$

then the inverse function $\hat{Q}$ has the realization

$$\hat{Q}(z) = \hat{Q}(z_0)^* + (z - \overline{z}_0)\hat{\Gamma}^+ (I + (z - z_0)(\hat{A} - z)^{-1})\hat{\Gamma},$$

where

$$\hat{\Gamma} := \Gamma \hat{Q}(z_0) = -\Gamma Q(z_0)^{-1} \quad (3.5)$$

and

$$(\hat{A} - z)^{-1} = (A - z)^{-1} + \Gamma_z \hat{Q}(z) \Gamma_z^+, \quad (3.6)$$

**Lemma 3.8.** With the above notations and the definition

$$\hat{\Gamma}_z := (I + (z - z_0)(\hat{A} - z)^{-1})\hat{\Gamma}$$

it holds

$$\hat{\Gamma}_z = \Gamma_z \hat{Q}(z) \quad (3.7)$$

and

$$(A - z)^{-1} = (\hat{A} - z)^{-1} + \hat{\Gamma}_z \hat{Q}(z) \hat{\Gamma}_z^+. \quad (3.8)$$
Proof. We start from the definition of $\hat{\Gamma}$ and use (3.6) and (2.3).

\[
\hat{\Gamma}_z = -\left(I + (z - z_0)(\hat{A} - z)^{-1}\right)\Gamma Q(z_0)^{-1}
\]
\[
= -\left(I + (z - z_0)(A - z)^{-1} - (z - z_0)\Gamma_z Q(z)^{-1}\Gamma_z^+\right)\Gamma Q(z_0)^{-1}
\]
\[
= -\Gamma_z Q^{-1}(z_0) + \Gamma_z Q(z)^{-1}(Q(z) - Q(z_0))Q(z_0)^{-1} = \Gamma_z \hat{Q}(z).
\]
This implies
\[
\hat{\Gamma}_z Q(z)\Gamma_z^+ = \Gamma_z \hat{Q}(z)Q(z)\Gamma_z^+ = -\Gamma_z \hat{Q}(z)\Gamma_z^+,
\]
and hence also (3.8) is shown. \qed

In what follows, given a Jordan chain $x_0, \ldots, x_{\ell-1}$ of $A$ at $\alpha$ we use the following abbreviation
\[
x(z) := x_0 + (z - \alpha)x_1 + \ldots + (z - \alpha)^{\ell-1}x_{\ell-1}.
\]
Note that then
\[
(A - z_0)^{-1}x(z) = \frac{1}{z - z_0}(x(z) + (z - \alpha)^{\ell}(A - z_0)^{-1}x_{\ell-1}). \tag{3.9}
\]

Lemma 3.9. With the above notations it holds
\[
\hat{\Gamma}_z \Gamma^+ x(z) = \frac{1}{z - z_0}\left(x(z) + (z - \alpha)^{\ell}(I + (z - z_0)(\hat{A} - z)^{-1})(A - z_0)^{-1}x_{\ell-1}\right). \tag{3.10}
\]

Proof. The definition of $\hat{\Gamma}_z$ gives
\[
-\hat{\Gamma}_z \Gamma^+ x(z) = (I + (z - z_0)(\hat{A} - z)^{-1})\Gamma Q(z_0)^{-1}\Gamma^+ x(z).
\]
In order to rewrite $\Gamma Q(z_0)^{-1}\Gamma^+$ we use (3.7) and an identity as (2.2) for $\hat{A}$
\[
\Gamma Q(z_0)^{-1}\Gamma^+ = \hat{\Gamma} Q(z_0)Q(z_0)^{-1}(\frac{1}{z_0} + (z_0 - z_0)(\hat{A} - z_0)^{-1})\hat{\Gamma}_z^+ = \hat{\Gamma}_z Q(z_0)\hat{\Gamma}_z^+
\]
\[
= (I + (z_0 - z_0)(\hat{A} - z_0)^{-1})\hat{\Gamma}_z Q(z_0)\hat{\Gamma}_z^+.
\]
Applying the resolvent identity and using relation (3.8) we obtain
\[
-\hat{\Gamma}_z \Gamma^+ x(z) = (I + (z - z_0)(\hat{A} - z)^{-1})((A - z_0)^{-1} - (\hat{A} - z_0)^{-1})x(z)
\]
With (3.9) and again the resolvent identity for $\hat{A}$ this further equals
\[
\frac{1}{z - z_0}\left(I + (z - z_0)(\hat{A} - z)^{-1}\right)(x(z) + (z - \alpha)^{\ell}(A - z_0)^{-1}x_{\ell-1}) - (\hat{A} - z)^{-1}x(z)
\]
\[
= \frac{1}{z - z_0}\left(x(z) + (z - \alpha)^{\ell}(I + (z - z_0)(\hat{A} - z)^{-1})(A - z_0)^{-1}x_{\ell-1}\right)
\]
and hence the lemma is proved. \qed

Proof. (Theorem 3.5) We start by rewriting $\bar{\eta}(z)$ from (3.4), where, in particular, (3.10) is used in the fourth equality
\[
\bar{\eta}(z) = (z - z_0)Q(z)^{-1}\Gamma^+ x(z)
\]
\[
= (z - z_0)(Q(z_0)^{-1} - (z - z_0)\hat{\Gamma}_z^+\hat{\Gamma}_z)\Gamma^+ x(z)
\]
\[
= (z - z_0)\hat{\Gamma}_z^+(-x(z) - (z - z_0)\hat{\Gamma}_z \Gamma^+ x(z))
\]
\[
= (z - z_0)(z - \alpha)\hat{\Gamma}_z^+\left(I + (z - z_0)(\hat{A} - z)^{-1}\right)(A - z_0)^{-1}x_{\ell-1} \tag{3.11}
\]
The crucial observation is that due to the assumption that \( \alpha \) is not a generalized zero of \( Q \), it holds as \( z \to \alpha \)
\[
(z - \alpha)(\hat{A} - z)^{-1} \xrightarrow{\text{strongly}} 0
\]
and, moreover, for \( j < \ell \)
\[
\frac{d^j}{dz^j} \left((z - \alpha)^\ell (\hat{A} - z)^{-1}\right) \xrightarrow{\text{strongly}} 0.
\]

In what follows we are going to show that \( \vec{\eta} \) satisfies the conditions from Definitions 3.1 and 3.2.

(A, D) Formula (3.11) for \( \vec{\eta} \) and the above observations imply \( \vec{\eta}^{(j)} \to \vec{0} \) for \( j < \ell \).

(B, E) The definition of \( \vec{\eta} \) gives
\[
\left(Q(z)\vec{\eta}(z)\right)^{(j)} = \left((z - z_0)\Gamma^+(x_0 + (z - \alpha)x_1 + \ldots + (z - \alpha)^{\ell-1}x_{\ell-1})\right)^{(j)}
\]
and hence the limits in (E) exist even strongly, in particular,
\[
Q(z)\vec{\eta}(z) = (z - z_0)\Gamma^+ x(z) \to (\alpha - z_0)\Gamma^+ x_0, \quad \text{as } z \to \alpha.
\]

Let us assume \( \Gamma^+ x_0 = \vec{0} \). As \( x_0 \) is an eigenvector of \( A \) at \( \alpha \) this implies
\[
[\left(I + (z - z_0)(A - z)^{-1}\right)\Gamma\vec{h}, x_0] = \frac{\alpha - z_0}{\alpha - z} (\vec{h}, \Gamma^+ x_0) = 0
\]
for all \( \vec{h} \in H \) and \( z \in \rho(A) \) and minimality would hence imply \( x_0 = 0 \). This contradicts that \( x_0 \) is an eigenvector and thus \( \Gamma^+ x_0 \neq \vec{0} \).

(Cs, Fs) In order to show the existence of the limits we note that
\[
\left(Q(z)\vec{\eta}(z) - Q(z)\vec{\eta}(w)\right)_{z - w} = [\Gamma_z\vec{\eta}(z), \Gamma_w\vec{\eta}(w)]
\]
and rewrite \( \Gamma_z\vec{\eta}(z) \) with the help of (3.9) and (3.10)
\[
\Gamma_z\vec{\eta}(z) = -(z - z_0)\hat{\Gamma}_z Q(z)Q(z)^{-1}\Gamma^+ x(z) = -(z - z_0)\hat{\Gamma}_z \Gamma^+ x(z) = x(z) + (z - \alpha)^\ell I + (z - z_0)(\hat{A} - z)^{-1}(A - z_0)^{-1}x_{\ell-1}
\]
Hence \( \Gamma_z\vec{\eta}(z) \) can be written as
\[
\Gamma_z\vec{\eta}(z) = x(z) + h_\ell(z), \quad (3.12)
\]
where for \( 0 \leq j < \ell \)
\[
\frac{d^j}{dz^j} h_\ell(z) \xrightarrow{\text{strongly}} 0 \quad \text{as } z \to \alpha.
\]
This implies
\[
[\Gamma_z\vec{\eta}(z), \Gamma_w\vec{\eta}(w)] = [x(z) + h_\ell(z), x(w) + h_\ell(w)]
\]
and hence the limits in (Cs), (Fs) exist.

This finishes the proof. \( \square \)
Proof. (Corollary 3.6) Formula (3.12) implies for $0 \leq j < \ell$
\[ \frac{1}{j!} \left( \Gamma_z \eta(z) \right)^{(j)} = \frac{1}{j!} (x(z) + h_\ell(z))^{(j)} \to x_j \text{ as } z \to \alpha. \]
and
\[ \lim_{z,w \to \alpha} \frac{1}{i!j!} \frac{d^{i+j}}{dz^i dw^j} \left( Q(z) - Q(w)^* \eta(z), \eta(w) \right) = \lim_{z,w \to \alpha} \frac{1}{i!j!} \left[ \Gamma_z \eta(z), \Gamma_w \eta(w) \right] = \lim_{z,w \to \alpha} \frac{1}{i!j!} \left[ x(z), x(w) \right] = [x_i, x_j], \]
which finishes the proof. \hfill \Box

Remark 3.10. Note that in the last statement of the corollary also pole cancellation functions corresponding to different Jordan chains can be employed. This makes it possible to describe all inner products in the algebraic eigenspace.

In the main theorem it was shown that given a Jordan chain $x_0, x_1, \ldots, x_{\ell-1}$ the function $\eta(z)$ is of order at least $\ell$. So the question arises how the order of $\eta$ is related to the maximality of the Jordan chain. We summarize our observations in the following corollary.

Corollary 3.11. Let the pole cancellation function $\eta$ be given as in Theorem 3.5.
1. If $x_0, \ldots, x_{\ell-1}$ is a maximal Jordan chain of $A$ then $\eta$ has order $\ell$. Conversely, if $\eta$ has order $\ell$, then $x_0, \ldots, x_{\ell-1}$ need not be maximal.
2. If $x_0, \ldots, x_{\ell-1}$ is not maximal then the order of $\eta$ is $\ell$ or larger than $\ell$ and there are examples for both situations.

Proof. (1) follows from Corollary 3.6. Example 3.15(a) shows (2) and examples for (3) are given in Example 3.15(a) and (b), respectively. \Box

In Theorem 3.5 there are still restrictive assumptions, namely that $Q$ is regular and $\alpha$ is not a generalized zero. These restrictions are removed in the following theorem.

Theorem 3.12. Let $Q \in \mathcal{N}_\kappa(\mathcal{H})$ be given and assume that $\alpha \in \mathbb{R}$ is a generalized pole of $Q$ such that the representing relation $A$ has a Jordan chain at $\alpha$ of length $\ell$.

Then there exists a strong pole cancellation function $\eta(z)$ of order at least $\ell$ of the form
\[ \eta(z) = (Q(z) + S)^{-1} \tilde{p}(z), \]
where $S = S^* \in \mathcal{L}(\mathcal{H})$ and $\tilde{p}(z)$ is an $\mathcal{H}$-valued polynomial of degree $\leq \ell$.

Proof. It follows from the proof of Theorem 3.7 in [Lu2] that for every $Q \in \mathcal{N}_\kappa(\mathcal{H})$ and $\alpha \in \mathbb{R}$ there exists a self-adjoint $S \in \mathcal{L}(\mathcal{H})$ such that $\alpha$ is not a generalized pole of $Q + S$. It is easy to check that a pole cancellation function of $Q$ at $\alpha$ is a pole cancellation function of $Q + S$ and conversely (and all the limits in the definitions coincide). Then we choose $\eta$ as the pole cancellation function constructed in Theorem 3.5 for $Q + S$. \hfill \Box
Remark 3.13. In a straightforward way these results can be used for characterizing generalized zeros, as well as for $\alpha = \infty$.

We want to point out that the assumption in Theorem 3.5 is not only technical. Indeed, if $\alpha$ is a generalized zero of $Q$ then the function $\overline{\eta}$ from Theorem 3.5 need not be a pole cancellation function or if it is, its order can be less than $\ell$ from the construction, see Example 3.17. Moreover, in this case it might happen that it is not possible to choose a pole cancellation function of the form $\overline{\eta}(z) = Q(z)^{-1}p(z)$, with a suitable polynomial $p$. See Example 3.10 for an illustration of this fact.

We finish this section with a historic remark.

Remark 3.14. Several results have originally been proven for generalized zeros (not poles), but they immediately can be translated for generalized poles, as generalized zeros of $Q$ by definition are generalized poles of the inverse function $\hat{Q}$.

The first result of an analytic characterization for non-scalar functions can be found in [BL]. There a generalized zero is characterized by the existence of a function $\phi$ with properties that in our notation correspond to the fact that $Q(z)^{-1}\phi(z)$ is a pole cancellation function of $Q$, but it is even mentioned that in the general situation they cannot characterize the multiplicity of the generalized pole. In other papers assumptions are made that guarantee that the algebraic eigenspace is not degenerate and hence ortho-complemented, namely [B] deals with meromorphic functions, not necessarily generalized Nevanlinna functions, whereas in [DLS] embedded eigenvalues are considered, but the further assumptions even rule out the existence of Jordan chains of length $> 1$.

In [Lu2] a different approach (via the factorization of $Q$) enables to avoid such assumptions, but the drawback there is that a pole cancellation function can be constructed only for the non-positive part of a Jordan chain (this might be only half the chain) and, moreover, this construction is quite complicated and non-constructive.

3.3. Examples. Here we collect the examples that were referred to in the text. For simplicity we use the simpler form of $\overline{\eta}$ mentioned in Remark 3.7. Note that all realizations given here are chosen minimal.

Example 3.15. Let $\mathcal{K} = \mathbb{C}^2$ and the inner product be defined by the Gram matrix $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the operator $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is self-adjoint in $\mathcal{K}$.

(a) With $\Gamma_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ define the $\mathcal{N}_1(\mathbb{C}^2)$-function

$$Q_1(z) := \Gamma_1^+(A - z)^{-1}\Gamma_1 = \begin{pmatrix} 0 & -\frac{1}{z} \\ -\frac{1}{z} & -\frac{1}{z^2} \end{pmatrix}.$$  

Choosing the non-maximal Jordan chain consisting of the vector $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ only we obtain

$$\overline{\eta}_1(z) := Q_1(z)^{-1}\Gamma_1^+x_0 = \begin{pmatrix} -z \\ 0 \end{pmatrix} \quad \text{and} \quad Q_1(z)\overline{\eta}_1(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Hence $\overline{\eta}_1$ is of order 1.
(b) With \( \Gamma_2 := \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) define the \( \mathcal{N}_1(\mathbb{C}) \)-function
\[
Q_2(z) := \Gamma_2^+(A - z)^{-1} \Gamma_2 = \frac{2z - 1}{z^2}.
\]
Choosing again the Jordan chain \( x_0 \) we now obtain
\[
\bar{\eta}_2(z) := Q_2(z)^{-1} \Gamma_2^+ x_0 = \frac{z^2}{2z - 1} \text{ and } Q_2(z) \bar{\eta}_2(z) = 1.
\]
So we found that the order of \( \bar{\eta}_2 \) equals 2, even if this pole cancellation function was constructed by a Jordan chain of (non-maximal) length 1.

This can be explained by the fact that also for the Jordan chain \( x_0, x_1 \) with \( x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \ker \Gamma_2^+ \) it holds
\[
\bar{\eta}_2(z) = Q_2(z)^{-1} \Gamma_2^+ (x_0 + (z - \alpha)x_1) = Q_2(z)^{-1} \Gamma_2^+ x_0.
\]

**Example 3.16.** Consider the function
\[
\hat{Q}_3(z) := \begin{pmatrix} z^2 \sqrt{z} \\ 1 - \frac{1}{z^2 \sqrt{z}} \end{pmatrix} \in \mathcal{N}_2(\mathbb{C}^2),
\]
for which \( \alpha = 0 \) is both a generalized pole and zero. Let us assume that there exists a pole cancellation function of \( \hat{Q}_3 \) at \( \alpha = 0 \) of the form \( \bar{\eta}(z) = Q_3(z)^{-1} \begin{pmatrix} p_1(z) \\ p_2(z) \end{pmatrix} \)
where \( p_1 \) and \( p_2 \) are polynomials of the form
\[
\begin{align*}
p_1(z) &= a_0 + a_1 z + a_2 z^2 + O(z^3) \\
p_2(z) &= b_0 + O(z),
\end{align*}
\]
which gives
\[
\bar{\eta}(z) = \frac{1}{2} \begin{pmatrix} a_0 + a_1 z + a_2 z^2 + O(z) \\ a_0 + O(z) \end{pmatrix} \text{ as } z \to 0
\]
and property (A) implies \( a_0 = a_1 = a_2 = b_0 = 0 \). Hence \( \hat{Q}_3(z) \bar{\eta}(z) = \bar{p}(z) \to 0 \) as \( z \to 0 \) and \( \bar{\eta} \) cannot be a pole cancellation function for \( \hat{Q}_3 \). However, it is easy to check that \( \alpha = 0 \) is a generalized pole and
\[
\hat{Q}_3(z)^{-1} \begin{pmatrix} -z^2 \sqrt{z} \\ 1 \end{pmatrix}
\]
is a pole cancellation function for \( \hat{Q}_3 \) at \( \alpha = 0 \).

**Example 3.17.** The function
\[
Q_4(z) = \begin{pmatrix} \frac{1}{(z+1)^3} \\ \frac{1}{z} \frac{1}{(z+1)^3} \end{pmatrix}
\]
is a generalized Nevanlinna function for which \( \alpha = 0 \) is both a pole and zero as also
\[
\hat{Q}_4(z) = \begin{pmatrix} \frac{(z-1)(2z+1)}{z(z+1)^2} \\ \frac{z(z+2)}{z(z+1)^2} \frac{(z-1)(z+1)^3}{z^2(z+2)} \frac{(z-1)(z+1)^3}{z^2(z+2)} \frac{(z-1)(z+1)^3}{z^2(z+2)} \end{pmatrix}
\]
has a pole at 0.

We are going to show that \( \bar{\eta} \) defined as in (3.3) not necessarily is a pole cancellation function. To this end we need a minimal representation of \( Q_4 \).
Let $\mathcal{K} = \mathbb{C}^6$ and the inner product in $\mathcal{K}$ be defined by the Gram matrix

$$G = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
\end{pmatrix}. $$

Then

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}$$

is self-adjoint in $\mathcal{K}$. With $\Gamma_4 = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & -\frac{1}{2} \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$ it follows $Q_4(z) = \Gamma_4^+ (A-z)^{-1} \Gamma_4$. Any Jordan chain of length 3 is of the form

$$\vec{x}_0 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x}_1 := \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x}_2 := \begin{pmatrix} b \\ a \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $a, b \in \mathbb{C}$. Hence $\vec{\eta}(z) = Q_4(z)^{-1} \Gamma_4^+ (\vec{x}_0 + z \vec{x}_1 + z^2 \vec{x}_2)$, which equals

$$\vec{\eta}(z) = \begin{pmatrix}
(z-1) (2az^3 + z^2(4a - 4b + 6) + z(a - 2b + 6) + 1) \\
(z+1)^3 (z^2(a + 2b - 2) + z(a - 2b + 3) + 1) \\
2(z+2) \\
\end{pmatrix}.$$  

Hence for no Jordan chain of length 3 the function $\vec{\eta}$ is a pole cancellation function, as already condition (A) is not satisfied.

In this example the order of the zero $\alpha = 0$ is larger than the order of the pole $\alpha = 0$. In a similar way examples can be constructed where $\vec{\eta}$ still is a pole cancellation function, but its order is reduced. In those examples the order of the pole is still larger than the order of the zero.
4. Conclusions

The results in this paper - in particular - the construction of the pole cancellation function of higher order gives a complete answer to the longstanding problem of an analytic characterization of generalized poles including multiplicities. The concrete form of the pole cancellation functions appears to be much simpler than expected. In the following theorem we summarize the situation.

**Theorem 4.1.** Let $Q \in N_\kappa(H)$ and $\alpha \in \mathbb{R}$ be given. Then the following statements are equivalent:

I. The point $\alpha$ is a generalized pole of $Q$ and there exists a Jordan chain of the representing relation of length $\ell$.

II. There exists a pole cancellation function of $Q$ at $\alpha$ of order at least $\ell$.

III. There exists a strong pole cancellation function of $Q$ at $\alpha$ of order at least $\ell$.

IV. There exist $S = S^* \in \mathcal{L}(H)$ and an $H$-valued polynomial $\vec{p}(z)$ of degree $\leq \ell$ such that $(Q(z) + S)^{-1}\vec{p}(z)$ is a strong pole cancellation function of $Q$ at $\alpha$ of order at least $\ell$.

The main result of this paper concerns the implications $I \Rightarrow IV$ and $II \Rightarrow I$. But we want to mention that also $II \Rightarrow III$ is new, even for $\ell = 0$.

**REFERENCES**

[B] M. Borogovac, *Multiplicites of nonsimple zeros of meromorphic matrix functions of the class $N_{n,n}^\kappa$*, Math.Nachr. 153 (1991), 69–77.

[BLa] M. Borogovac and H. Langer, *A characterization of generalized zeros of negative type of matrix functions of the class $N_{n,n}^\kappa$*, Oper. Theory Adv. Appl. 28 (1988), 17–26.

[DaLa] K. Daho and H. Langer: *Matrix functions of the class $N_\kappa$*, Math.Nachr. 120 (1985), 275–294.

[DaLaS] A. Dijksma, H. Langer, and H.S.V. de Snoo: *Eigenvalues and pole functions of Hamiltonian systems with eigenvalue depending boundary conditions*, Math. Nachr. 161 (1993), 107–154.

[DS] A. Dijksma and H.S.V. de Snoo: *Symmetric and selfadjoint relations in Krein spaces*, Integr. Equ. Oper. Theory 24 (1997), 145–166.

[GSi] I.C. Gohberg and E.I. Sigal: *An operator generalization of the logarithmic residue theorem and the theorem of Roche*, Math. USSR-Sh. 13 (1971), 603-652.

[HLu] S. Hassi and A. Luger, *Generalized zeros and poles of $N_\kappa$-functions: On the underlying spectral structure*, Methods Funct. Anal. Topology. 12 (2006) 2, 131–150.

[HSW] S. Hassi, H.S.V. de Snoo, and H. Woracek: *Some interpolation problems of Nevanlinna-Pick type*, Oper. Theory Adv. Appl. 106 (1998), 201–216.

[KLa] M.G. Krein and H. Langer, *Uber einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume $l_\kappa$ zusammenhängen, I. Einige Funktionenklassen und ihre Darstellungen*, Math. Nachr. 77 (1977), 187–236.

[La] H. Langer, *A characterization of generalized zeros of negative type of functions of the class $N_\kappa$*, Oper. Theory Adv. Appl. 17 (1986), 201–212.

[LaLu] H. Langer and A. Luger, *A class of $2 \times 2$-matrix functions*, Glas. Mat. Ser. III, 35(55) (2000), 149–160.

[Lu1] A. Luger, *A factorization of regular generalized Nevanlinna functions*, Integr. Equ. Oper. Theory 43 (2002), 326–345.

[Lu2] A. Luger, *A characterization of generalized poles of generalized Nevanlinna*, Math. Nachr. 279 (2006), 891–910.

Muhamed Borogovac, Actuarial Department, Boston Mutual Life Insurance Company, 120 Royall Street, Canton, MA 02021, Phone 781-770-0317
E-mail address: Muhamed.Borogovac@bostonmutual.com

Annemarie Luger, Department of Mathematics, Stockholm University, SE - 106 91 Stockholm, Sweden
E-mail address: luger@math.su.se