Monomial ideals of weighted oriented graphs

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Abstract

Let \( I = I(D) \) be the edge ideal of a weighted oriented graph \( D \). We determine the irredundant irreducible decomposition of \( I \). Also, we characterize the associated primes and the unmixed property of \( I \). Furthermore, we give a combinatorial characterization for the unmixed property of \( I \), when \( D \) is bipartite, \( D \) is a whisker or \( D \) is a cycle. Finally, we study the Cohen-Macaulay property of \( I \).

Keywords: Weighted oriented graphs, unmixed property, irreducible decomposition, Cohen-Macaulay property.

1 Introduction

A weighted oriented graph is a triplet \( D = (V(D), E(D), w) \), where \( V(D) \) is a finite set, \( E(D) \subseteq V(D) \times V(D) \) and \( w \) is a function \( w : V(D) \rightarrow \mathbb{N} \). The vertex set of \( D \) and the edge set of \( D \) are \( V(D) \) and \( E(D) \), respectively. Some times for short we denote these sets by \( V \) and \( E \) respectively. The weight of \( x \in V \) is \( w(x) \). If \( e = (x, y) \in E \), then \( x \) is the tail of \( e \) and \( y \) is the head of \( e \). The underlying graph of \( D \) is the simple graph \( G \) whose vertex set is \( V \) and whose edge set is \( \{ \{x, y\} | (x, y) \in E \} \). If \( V(D) = \{x_1, \ldots, x_n\} \), then we consider the polynomial ring \( R = K[x_1, \ldots, x_n] \) in \( n \) variables over a field \( K \). In this paper we introduce and study the edge ideal of \( D \) given by \( I(D) = (x_i x_j^{w(x_i)} : (x_i, x_j) \in E(D)) \) in \( R_i \) (see Definition 3.1).

In Section 2 we study the vertex covers of \( D \). In particular we introduce the notion of strong vertex cover (Definition 2.6) and we prove that a minimal vertex cover is strong. In Section 3 we characterize the irredundant irreducible decomposition of \( I(D) \). In particular we show that the minimal monomial irreducible ideals of \( I(D) \) are associated with the strong vertex covers of \( D \). In Section 4 we give the following characterization of the unmixed property of \( I(D) \).

\[
I(D) \text{ is unmixed} \iff G \text{ is unmixed and } D \text{ has the minimal-strong property}
\]

\[
\begin{array}{c}
\text{All strong vertex covers have the same cardinality} \\
\text{All minimal vertex covers have the same cardinality} \\
\text{All strong vertex covers are minimals}
\end{array}
\]

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Furthermore, if $D$ is bipartite, $D$ is a whisker or $D$ is a cycle, we give an effective (combinatorial) characterization of the unmixed property. Finally in Section 5 we study the Cohen-Macaulayness of $I(D)$. In particular we characterize the Cohen-Macaulayness when $D$ is a path or $D$ is complete. Also, we give an example where this property depend of the characteristic of the field $K$.

2 Weighted oriented graphs and their vertex covers

In this section we define the weighted oriented graphs and we study their vertex covers. Furthermore, we define the strong vertex covers and we characterize when $V(D)$ is a strong vertex cover of $D$. In this paper we denote the set $\{x \in V \mid w(x) \neq 1\}$ by $V^+$.

Definition 2.1. A vertex cover $C$ of $D$ is a subset of $V$, such that if $(x, y) \in E$, then $x \in C$ or $y \in C$. A vertex cover $C$ of $D$ is minimal if each proper subset of $C$ is not a vertex cover of $D$.

Definition 2.2. Let $x$ be a vertex of a weighted oriented graph $D$, the sets $N^+_D(x) = \{y : (x, y) \in E(D)\}$ and $N^-_D(x) = \{y : (y, x) \in E(D)\}$ are called the out-neighborhood and the in-neighborhood of $x$, respectively. Furthermore, the neighbourhood of $x$ is the set $N_D(x) = N^+_D(x) \cup N^-_D(x)$.

Definition 2.3. Let $C$ be a vertex cover of a weighted oriented graph $D$, we define

$$L_1(C) = \{x \in C \mid N^+_D(x) \cap C^c \neq \emptyset\},$$

$$L_2(C) = \{x \in C \mid x \notin L_1(C) \text{ and } N^-_D(x) \cap C^c \neq \emptyset\} \text{ and }$$

$$L_3(C) = C \setminus (L_1(C) \cup L_2(C)),$$

where $C^c$ is the complement of $C$, i.e. $C^c = V \setminus C$.

Proposition 2.4. If $C$ is a vertex cover of $D$, then

$$L_3(C) = \{x \in C \mid N_D(x) \subset C\}.$$  

Proof. If $x \in L_3(C)$, then $N^+_D(x) \subseteq C$, since $x \notin L_1(C)$. Furthermore $N^-_D(x) \subseteq C$, since $x \notin L_2(C)$. Hence $N_D(x) \subset C$, since $x \notin N_D(x)$. Now, if $x \in C$ and $N_D(x) \subset C$, then $x \notin L_1(C) \cup L_2(C)$. Therefore $x \in L_3(C)$.

Proposition 2.5. If $C$ is a vertex cover of $D$, then $L_3(C) = \emptyset$ if and only if $C$ is a minimal vertex cover of $D$.

Proof. $\Rightarrow$) If $x \in C$, then by Proposition 2.3 we have $N_D(x) \not\subset C$, since $L_3(C) = \emptyset$. Thus, there is $y \in N_D(x) \setminus C$ implying $C \setminus \{x\}$ is not a vertex cover. Therefore, $C$ is a minimal vertex cover.

$\Leftarrow$) If $x \in L_3(C)$, then by Proposition 2.3 $N_D(x) \subseteq C \setminus \{x\}$. Hence, $C \setminus \{x\}$ is a vertex cover. A contradiction, since $C$ is minimal. Therefore $L_3(C) = \emptyset$.

Definition 2.6. A vertex cover $C$ of $D$ is strong if for each $x \in L_3(C)$ there is $(y, x) \in E(D)$ such that $y \in L_2(C) \cup L_3(C)$ with $y \in V^+$ (i.e. $w(y) \neq 1$).
Remark 2.7. Let $C$ be a vertex cover of $D$. Hence, by Proposition 2.4 and since $C = L_1(C) \cup L_2(C) \cup L_3(C)$, we have that $C$ is strong if and only if for each $x \in C$ such that $N(x) \subseteq C$, there exist $y \in N^-(x) \cap (C \setminus L_1(C))$ with $y \in V^+$. 

Corollary 2.8. If $C$ is a minimal vertex cover of $D$, then $C$ is strong.

Proof. By Proposition 2.5, we have $L_3(C) = \emptyset$, since $C$. Hence, $C$ is strong. \hfill \Box

Remark 2.9. The vertex set $V$ of $D$ is a vertex cover. Also, if $z \in V$, then $N_D(z) \subseteq V \setminus z$. Hence, by Proposition 2.14, $L_3(V) = V$. Consequently, $L_3(V) = L_2(V) = \emptyset$. By Proposition 2.14, $V$ is not a minimal vertex cover of $D$. Furthermore since $L_3(V) = V$, $V$ is a strong vertex cover if and only if for each $x \in V$.

Definition 2.10. If $G$ is a cycle with $E(D) = \{(x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_1)\}$ and $V(D) = \{x_1, \ldots, x_n\}$, then $D$ is called oriented cycle.

Definition 2.11. $D$ is called unicycle oriented graph if it satisfies the following conditions:

1) The underlying graph of $D$ is connected and it has exactly one cycle $C$.

2) $C$ is an oriented cycle in $D$. Furthermore for each $y \in V(D) \setminus V(C)$, there is an oriented path from $C$ to $y$ in $D$.

3) $w(x) \neq 1$ if $deg_G(x) \geq 1$.

Lemma 2.12. If $V(D)$ is a strong vertex cover of $D$ and $D_1$ is a maximal unicycle oriented subgraph of $D$, then $V(D')$ is a strong vertex cover of $D' = D \setminus V(D_1)$.

Proof. We take $x \in V(D')$. Thus, by Remark 2.9, there is $y \in N_D^-(x) \cap V^+(D)$. If $y \in D_1$, then we take $D_2 = D_1 \cup \{(y, x)\}$. Hence, if $C$ is the oriented cycle of $D_1$, then $C$ is the unique cycle of $D_2$, since $deg_{D_2}(x) = 1$. If $y \notin C$, then $(y, x)$ is an oriented path from $C$ to $x$. Now, if $y \notin C$, then there is an oriented path $L$ form $C$ to $y$ in $D_1$. Consequently, $L \cup \{(y, x)\}$ is an oriented path form $C$ to $x$. Furthermore, $deg_{D_2}(x) = 1$ and $w(y) \neq 1$, then $D_2$ is an unicycle oriented graph. A contradiction, since $D_1$ is maximal. This implies $y \in V(D')$, so $y \in N_D^-(x) \cap V^+(D')$. Therefore, by Remark 2.9, $V(D')$ is a strong vertex cover of $D'$. \hfill \Box

Lemma 2.13. If $V(D)$ is a strong vertex cover of $D$, then there is an unicycle oriented subgraph of $D$.

Proof. Let $y_1$ be a vertex of $D$. Since $V = V(D)$ is a strong vertex cover, there is $y_2 \in V$ such that $y_2 \in N^-(y_1) \cap V^+$. Similarly, there is $y_3 \in N^-(y_2) \cap V^+$. Consequently, $(y_3, y_2, y_1)$ is an oriented path. Continuing this process, we can assume there exist $y_2, y_3, \ldots, y_k \in V^+$ where $(y_k, y_{k-1}, \ldots, y_2, y_1)$ is an oriented path and there is $1 \leq j \leq k - 2$ such that $(y_j, y_k) \in E(D)$, since $V$ is finite. Hence, $C = (y_k, y_{k-1}, \ldots, y_j, y_k)$ is an oriented cycle and $L = (y_j, \ldots, y_k)$ is an oriented path form $C$ to $y_1$. Furthermore, if $j = 1$, then $w(y_1) \neq 1$. Therefore, $D_1 = C \cup L$ is an unicycle oriented subgraph of $D$. \hfill \Box

Proposition 2.14. Let $D = (V, E, w)$ be a weighted oriented graph, hence $V$ is a strong vertex cover of $D$ if and only if there are $D_1, \ldots, D_s$ unicycle oriented subgraphs of $D$ such that $V(D_1), \ldots, V(D_s)$ is a partition of $V = V(D)$. 

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Proof. \( \Rightarrow \) By Lemma 2.13 there is a maximal unicycle oriented subgraph \( D_1 \) of \( D \). Hence, by Lemma 2.12 \( V(D') \) is a strong vertex cover of \( D' = D \setminus V(D_1) \). So, by Lemma 2.13 there is \( D_2 \) a maximal unicycle oriented subgraph of \( D' \). Continuing this process we obtain unicycle oriented subgraphs \( D_1, \ldots, D_s \) such that \( V(D_1), \ldots, V(D_s) \) is a partition of \( V(D) \).

\(<=\) We take \( x \in V(D) \). By hypothesis there is \( 1 \leq j \leq s \) such that \( x \in V(D_j) \). We assume \( C \) is the oriented cycle of \( D_j \). If \( x \in V(C) \), then there is \( y \in V(C) \) such that \( (y, x) \in E(D_j) \) and \( w(y) \neq 1 \), since \( \deg_{D_j}(y) \geq 2 \) and \( D_j \) is a unicycle oriented subgraph. Now, we assume \( x \notin V(C) \), then there is an oriented path \( L = (z_1, \ldots, z_r) \) such that \( z_1 \in V(C) \) and \( z_r = x \). Thus, \( (z_{r-1}, x) \in E(D) \). Furthermore, \( w(z_{r-1}) \neq 1 \), since \( \deg_{D_j}(z_{r-1}) \geq 2 \). Therefore \( V \) is a strong vertex cover. \( \square \)

3 Edge ideals and their primary decomposition

As is usual if \( I \) is a monomial ideal of a polynomial ring \( R \), we denote by \( \mathcal{G}(I) \) the minimal monomial set of generators of \( I \). Furthermore, there exists a unique decomposition, \( I = q_1 \cap \cdots \cap q_r \), where \( q_1, \ldots, q_r \) are irreducible monomial ideals such that \( I \neq \bigcap_{i \neq j} q_i \) for each \( j = 1, \ldots, r \). This is called the irredundant irreducible decomposition of \( I \). Furthermore, \( q_i \) is an irreducible monomial ideal if and only if \( q_i = (x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}) \) for some variables \( x_{i_j} \). Irreducible ideals are primary, then a irreducible decomposition is a primary decomposition. For more details of primary decomposition of monomial ideals see [6] Chapter 6. In this section, we define the edge ideal \( I(D) \) of a weighted oriented graph \( D \) and we characterize its irredundant irreducible decomposition. In particular we prove that this decomposition is an irreducible primary decomposition, i.e, the radicals of the elements of the irredundant irreducible decomposition of \( I(D) \) are different.

Definition 3.1. Let \( D = (V, E, w) \) be a weighted oriented graph with \( V = \{x_1, \ldots, x_n\} \). The edge ideal of \( D \), denote by \( I(D) \), is the ideal of \( R = K[x_1, \ldots, x_n] \) generated by \( \{x_ix_j^{w(x_j)} \mid (x_i, x_j) \in E\} \).

Definition 3.2. A source of \( D \) is a vertex \( x \), such that \( N_D(x) = N_D^+(x) \). A sink of \( D \) is a vertex \( y \) such that \( N_D(y) = N_D^-(y) \).

Remark 3.3. Let \( D = (V, E, w) \) be a weighted oriented graph. We take \( D' = (V, E, w') \) a weighted oriented graph such that \( w'(x) = w(x) \) if \( x \) is not a source and \( w'(x) = 1 \) if \( x \) is a source. Hence, \( I(D) = I(D') \). For this reason in this paper, we will always assume that if \( x \) is a source, then \( w(x_i) = 1 \).

Definition 3.4. Let \( C \) be a vertex cover of \( D \), the irreducible ideal associated to \( C \) is the ideal

\[ I_C = (L_1(C) \cup \{x_i^{w(x_j)} \mid x_j \in L_2(C) \cup L_3(C)\}) \].

Lemma 3.5. \( I(D) \subseteq I_C \) for each vertex cover \( C \) of \( D \).

Proof. We take \( I = I(D) \) and \( m \in \mathcal{G}(I) \), then \( m = xy^{w(y)} \), where \((x, y) \in D \). Since \( C \) is a vertex cover, \( x \in C \) or \( y \in C \). If \( y \in C \), then \( y \in I_C \) or \( y^{w(y)} \in I_C \). Thus, \( m = xy^{w(y)} \in I_C \). Now, we assume \( y \notin C \), then \( x \in C \). Hence, \( y \in N_D^-(x) \cap C^c \), so \( x \in L_1(C) \). Consequently, \( x \in I_C \) implying \( m = xy^{w(y)} \in I_C \). Therefore \( I \subseteq I_C \). \( \square \)
Definition 3.6. Let $I$ be a monomial ideal. An irreducible monomial ideal $q$ that contains $I$ is called a minimal irreducible monomial ideal of $I$ if for any irreducible monomial ideal $p$ such that $I \subseteq p \subseteq q$ one has that $p = q$.

Lemma 3.7. Let $D$ be a weighted oriented graph. If $I(D) \subseteq \langle x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s} \rangle$, then $\{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of $D$.

Proof. We take $J = \langle x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s} \rangle$. If $(a, b) \in E(D)$, then $ab^{w(b)} \in I(D) \subseteq J$. Thus, $x_{i_j}^{a_j}ab^{w(b)}$ for some $1 \leq j \leq s$. Hence, $x_{i_j} \in \{a, b\}$ and $\{a, b\} \cap \{x_{i_1}, \ldots, x_{i_s}\} \neq \emptyset$. Therefore $\{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of $D$.

Lemma 3.8. Let $J$ be a minimal irreducible monomial ideal of $I(D)$ where $G(J) = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. If $a_j \neq 1$ for some $1 \leq j \leq s$, then there is $(x, x_{i_j}) \in E(D)$ where $x \notin G(J)$.

Proof. By contradiction suppose there is $a_j \neq 1$ such that $(x, x_{i_j}) \in E(D)$, then $x \in M = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. We take the ideal $J' = (M \setminus \{x_{i_j}^{a_j}\})$. If $(a, b) \in E(D)$, then $ab^{w(b)} \in I(D) \subseteq J$. Consequently, $x_{i_j}^{a_j}ab^{w(b)}$ for some $1 \leq k \leq s$. If $k \neq j$, then $ab^{w(b)} \in J'$. Now, if $k = j$, then by hypothesis $a_j \neq 1$. Hence, $x_{i_j}^{a_j}b^{w(b)}$ implying $x_{i_j} = b$. Thus, $(a, x_{i_j}) \in E(D)$, so by hypothesis $a \in M \setminus \{x_{i_j}^{a_j}\}$. This implies $ab^{w(b)} \in J'$. Therefore $I(D) \subseteq J' \subseteq J$. A contradiction, since $J$ is minimal.

Lemma 3.9. Let $J$ be a minimal irreducible monomial ideal of $I(D)$ where $G(J) = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. If $a_j \neq 1$ for some $1 \leq j \leq s$, then $a_j = w(x_{i_j})$.

Proof. By Lemma 3.8 there is $(x, x_{i_j}) \in E(D)$ with $x \notin M = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. Also, $xx_{i_j}^{w(x_{i_j})} \in I(D) \subseteq J$, so $x_{i_k}^{w(x_{i_j})} \in I(D) \subseteq J$ for some $1 \leq k \leq s$. Hence, $x_{i_k}^{a_k}x_{i_j}^{w(x_{i_j})}$, since $x \notin M$. This implies, $k = j$ and $a_j \leq w(x_{i_j})$. If $a_j < w(x_{i_j})$, then we take $J' = (M' \setminus \{x_{i_j}^{a_j}\}) \cup \{x_{i_j}^{w(x_{i_j})}\}$. So, $J' \subseteq J$. Furthermore, if $(a, b) \in E(D)$, then $m = ab^{w(b)} \in I(D) \subseteq J$. Thus, $x_{i_k}^{a_k}ab^{w(b)}$ for some $1 \leq k \leq s$. If $k \neq j$, then $x_{i_k}^{a_k} \in M'$ implying $ab^{w(b)} \in J'$. Now, if $k = j$ then $x_{i_j}^{a_j}b^{w(b)}$, since $a_j > 1$. Consequently, $x_{i_j} = b$ and $x_{i_j}^{w(x_{i_j})}|m$. Then $m \in J'$. Hence $I(D) \subseteq J' \subseteq J$, a contradiction since $J$ is minimal. Therefore $a_j = w(x_{i_j})$.

Theorem 3.10. The following conditions are equivalent:

1) $J$ is a minimal irreducible monomial ideal of $I(D)$.

2) There is a strong vertex cover $C$ of $D$ such that $J = I_C$.

Proof. 2) $\Rightarrow$ 1) By definition $J = I_C$ is a monomial irreducible ideal. By Lemma 3.5 $I(D) \subseteq J' \subseteq J$, where $J'$ is a monomial irreducible ideal. We can assume $G(J') = \{x_{j_1}^{b_1}, \ldots, x_{j_s}^{b_s}\}$. If $x \notin L_1(C)$, then there is $(x, y) \in E(D)$ with $y \notin C$. Hence, $xy^{w(y)} \in I(D)$ and $y' \notin J$ for each $r \in N$. Consequently $y'^r \notin J'$ for each $r$, implying $y \notin \{x_{j_1}, \ldots, x_{j_s}\}$. Furthermore $x_{j_i}^{b_i}xy^{w(y)}$ for some $1 \leq i \leq s$, since $x_{j_i}^{w(y)} \in I(D) \subseteq J'$. This implies, $x = x_{j_i}^{b_i} \in J'$. Now, if $x \in L_2(C)$, then there is $(y, x) \in E(D)$ with $y \notin C$. Thus $y \notin J$, so $y \notin \{x_{j_1}^{b_1}, \ldots, x_{j_s}^{b_s}\}$. Also, $x^{w(x)}y \in I(D) \subseteq J'$,
then $x_{j_i}^{b_i} | x^{u(x)}$ for some $1 \leq i \leq s$. Consequently, $x_{j_i}^{b_i} | x^{u(x)}$ implies $x^{u(x)} \in J'$. Finally, if $x \in L_2(C)$, then there is $(y, x) \in E(D)$ where $y \in L_2(C) \cup L_3(C)$ and $w(y) \neq 1$, since $C$ is a strong vertex cover. So, $x^{u(y)} | y \in I(D) \subseteq J'$ implies $x_{j_i}^{b_i} | x^{u(y)}$ for some $i$. Furthermore $y \notin J = I_C$, since $y \in L_2(C) \cup L_3(C)$ and $w(y) \neq 1$. This implies $y \notin J'$ so, $x_{j_i}^{b_i} | x^{u(x)}$ then $x^{u(x)} \in J'$. Hence, $J = I_C \subseteq J'$. Therefore, $J$ is a minimal monomial irreducible of $I(D)$.

1) $\Rightarrow$ 2) Since $J$ is irreducible, we can suppose $G(J) = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. By Lemma 3.7 we have $a_j = 1$ or $a_j = w(x_{i_j})$ for each $1 \leq j \leq s$. Also, by Lemma 3.7 $C = \{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of $D$. We can assume $G(I_C) = \{x_{i_1}^{b_1}, \ldots, x_{i_s}^{b_s}\}$, then $b_j \in \{1, w(x_{i_j})\}$ for each $1 \leq j \leq s$. Now, suppose $b_k = 1$ and $w(x_{i_k}) \neq 1$ for some $1 \leq k \leq s$. Consequently $x_{i_k} \notin L_1(C)$. Thus, there is $(x_{i_k}, y) \in E(D)$ where $y \notin C$. So, $x_{i_k} y^{w(y)} \in I(D) \subseteq J$ and $x_{i_k}^{b_k} | x_{i_k} y^{w(y)}$ for some $1 \leq r \leq s$. Furthermore $y \notin C$, then $r = k$ and $a_k = a_{i_k} = 1$. Hence, $I_C \cap V(D) \subseteq J \cap V(D)$. This implies, $I_C \subseteq J$, since $a_j, b_j \in \{1, w(x_{i_j})\}$ for each $1 \leq j \leq s$. Therefore $J = I_C$, since $J$ is minimal. In particular $a_i = b_i$ for each $1 \leq i \leq s$. Now, assume $C$ is not strong, then there is $x \in L_3(C)$ such that if $(y, x) \in E(D)$, then $w(y) = 1$ or $y \in L_1(C)$. We can assume $x = x_{i_1}$, and we take $J'$ the monomial ideal with $G(J') = \{x_{i_2}^{a_2}, \ldots, x_{i_s}^{a_s}\}$. We take $(z_1, z_2) \in E(D)$. If $x_{i_j}^{a_j} | z_1 z_2^{w(2z)}$ for some $2 \leq j \leq s$, then $z_1 z_2^{w(2z)} \in J'$. Now, assume $x_{i_j}^{a_j} \in z_1 z_2^{w(2z)}$ for each $2 \leq j \leq s$. Consequently $z_2 \notin \{x_{i_2} \ldots, x_{i_s}\}$, since $a_j \notin \{1, w(x_{i_j})\}$. Also $z_1 z_2^{w(2z)} \in I(D) \subseteq J$, then $x_{i_j}^{a_j} | z_1 z_2^{w(2z)}$. But $x_{i_1} \in L_3(C)$, so $z_1, z_2 \in N_G[x_{i_1}] \subseteq C$. If $x_{i_1} = z_1$, then there is $2 \leq r \leq s$ such that $z_2 = x_{i_r}$. Thus $x_{i_r}^{a_r} | z_1 z_2^{w(2z)}$. A contradiction, then $x_{i_1} = z_2$, $z_1 \in C$ and $(z_1, x_{i_1}) \in E(D)$. Then, $w(z_1) = 1$ or $z_1 \in L_1(C)$. In both cases $z_1 \notin G(I_C)$. Furthermore $z_1 \neq z_2$ since $(z_1, z_2) \in E(D)$. This implies $z_1 \notin G(J')$. So, $z_1 z_2^{w(2z)} \notin J'$. Hence, $I(D) \subseteq J'$. This is a contradiction, since $J$ is minimal. Therefore $C$ is strong.

Theorem 3.11. If $C$ is the set of strong vertex covers of $D$, then the irredundant irreducible decomposition of $I(D)$ is given by $I(D) = \bigcap_{C \in C_s} I_C$.

Proof. By Theorem 1.3.1, there is a unique irredundant irreducible decomposition $I(D) = \bigcap_{i=1}^m I_i$. If there is an irreducible ideal $I_j'$ such that $I(D) \subseteq I_j' \subseteq I_j$ for some $j \in \{1, \ldots, m\}$, then $I(D) = (\bigcap_{x \notin J} I_i) \cap I_j'$ is an irreducible decomposition. Furthermore this decomposition is irredundant. Thus, $I_j' = I_j$. Hence, $I_1, \ldots, I_m$ are minimal irreducible ideals of $I(D)$. Now, if there is $C \in C_s$ such that $I_C \notin \{I_1, \ldots, I_m\}$, then there is $x_{i_j}^{a_j} \in I_C \setminus I_i$ for each $i \in \{1, \ldots, m\}$. Consequently, $m = \text{lcm}(x_{i_1}^{a_1}, \ldots, x_{i_m}^{a_m}) \in I_1 = I(D) \subseteq I_C$. Furthermore, if $C = \{x_{i_1}, \ldots, x_{i_k}\}$, then $I_C = (x_{i_1}^{a_1}, \ldots, x_{i_k}^{a_k})$ where $\beta_j \in \{1, w(x_{i_j})\}$. Hence, there is $j \in \{1, \ldots, k\}$ such that $x_{i_j}^{\beta_j} | m$. So, there is $1 \leq u \leq m$ such that $x_{i_j}^{\beta_j} | x_{i_j}^{a_j}$. A contradiction, since $x_{i_j}^{a_j} \notin I_C$. Therefore $I(D) = \bigcap_{C \in C_s} I_C$ is the irredundant irreducible decomposition of $I(D)$.

Remark 3.12. If $C_1, \ldots, C_s$ are the strong vertex covers of $D$, then by Theorem 3.11 $I_{C_1} \cap \cdots \cap I_{C_s}$ is the irredundant irreducible decomposition of $I(D)$. Furthermore, if $P_i = \text{rad}(I_{C_i})$, then $P_i = (C_i)$. So, $P_i \neq P_j$ for $1 \leq i < j \leq s$. Thus, $I_{C_1} \cap \cdots \cap I_{C_s}$ is an irredundant primary decomposition of $I(D)$. In particular we have $\text{Ass}(I(D)) = \{P_1, \ldots, P_s\}$.
Example 3.13. Let $D$ be the following oriented weighted graph

![Graph Image]

whose edge ideal is $I(D) = (x_1^3 x_2, x_2^4 x_3, x_3 x_4 x_5^2, x_4^2 x_5)$. From Theorem 3.10 and Theorem 3.11, the irreducible decomposition of $I(D)$ is:

$$I(D) = (x_1^3, x_3, x_5) \cap (x_2, x_3, x_5) \cap (x_2, x_4, x_5^2) \cap (x_1, x_2, x_5) \cap (x_1, x_4, x_5^2) \cap (x_1, x_2, x_3, x_4, x_5^2).$$

Example 3.14. Let $D$ be the following oriented weighted graph

![Graph Image]

Hence, $I(D) = (x_1 x_2^2, x_2 x_3^5, x_3 x_4^7)$. By Theorem 3.10 and Theorem 3.11, the irreducible decomposition of $I(D)$ is:

$$I(D) = (x_1, x_3) \cap (x_2, x_3) \cap (x_2, x_4^7) \cap (x_1, x_3^5, x_4^7) \cap (x_2^2, x_3^5, x_4^7).$$

In Example 3.13 and Example 3.14, $I(D)$ has embedding primes. Furthermore the monomial ideal $(V(D))$ is an associated prime of $I(D)$ in Example 3.13. Proposition 2.14 and Remark 3.12 give a combinatorial criterion for to decide when $(V(D)) \in \text{Ass}(I(D))$.

4 Unmixed weighted oriented graphs

Let $D = (V, E, w)$ be a weighted oriented graph whose underlying graph is $G = (V, E)$. In this section we characterize the unmixed property of $I(D)$ and we prove that this property is closed under c-minors. In particular if $G$ is a bipartite graph or $G$ is a whisker or $G$ is a cycle, we give an effective (combinatorial) characterization of this property.

Definition 4.1. An ideal $I$ is unmixed if each one of its associated primes has the same height.

Theorem 4.2. The following conditions are equivalent:

1) $I(D)$ is unmixed.

2) Each strong vertex cover of $D$ has the same cardinality.

3) $I(G)$ is unmixed and $L_3(C) = \emptyset$ for each strong vertex cover $C$ of $D$. 

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Proof. Let $C_1, \ldots, C_\ell$ be the strong vertex covers of $D$. By Remark 3.1.2, the associated primes of $I(D)$ are $P_1, \ldots, P_\ell$, where $P_i = \text{rad}(I(C_i)) = (C_i)$ for $1 \leq i \leq \ell$.

1) $\Rightarrow$ 2) Since $I(D)$ is unmixed, $|C_i| = \text{ht}(P_i) = \text{ht}(P_j) = |C_j|$ for $1 \leq i < j \leq \ell$.

2) $\Rightarrow$ 3) If $C$ is a minimal vertex cover, then by Corollary 2.3, $C \in \{C_1, \ldots, C_\ell\}$. By hypothesis, $|C_i| = |C_j|$ for each $1 \leq i < j \leq \ell$, then $C_i$ is a minimal vertex cover of $D$. Thus, by Lemma 2.5, $L_3(C_i) = \emptyset$. Furthermore $I(G)$ is unmixed, since $C_1, \ldots, C_\ell$ are the minimal vertex covers of $G$.

3) $\Rightarrow$ 1) By Proposition 2.5, $C_i$ is a minimal vertex cover, since $L_3(C_i) = \emptyset$ for each $1 \leq i \leq \ell$. This implies, $C_1, \ldots, C_\ell$ are the minimal vertex covers of $G$. Since $G$ is unmixed, we have $|C_i| = |C_j|$ for $1 \leq i < j \leq \ell$. Therefore $I(D)$ is unmixed.

Definition 4.3. A weighted oriented graph $D$ has the minimal-strong property if each strong vertex cover is a minimal vertex cover.

Remark 4.4. Using Proposition 2.5, we have that $D$ has the minimal-strong property if and only if $L_3(C) = \emptyset$ for each strong vertex cover $C$ of $D$.

Definition 4.5. $D'$ is a c-minor of $D$ if there is a stable set $S$ of $D$, such that $D' = D \setminus N_G[S]$.

Lemma 4.6. If $D$ has the minimal-strong property, then $D' = D \setminus N_G[x]$ has the minimal-strong property, for each $x \in V$.

Proof. We take a strong vertex cover $C'$ of $D' = D \setminus N_G[x]$ where $x \in V$. Thus, $C = C' \cup N_D(x)$ is a vertex cover of $D$. If $y' \in L_3(C')$, then by Proposition 2.4, $N_D(y') \subseteq C'$. Consequently, $N_D(y') \subseteq C' \cup N_D(x) = C$ implying $y' \in L_3(C)$. Hence, $L_3(C') \subseteq L_3(C)$. Now, we take $y \in L_3(C)$, then $N_D(y) \subseteq C$. This implies $y \notin N_D(x)$, since $x \notin C$. Then, $y \in C'$ and $N_D(y) \cap N_D(x) = N_D(y) \subseteq C = C' \cup N_D(x)$. So, $N_D(y) \subseteq C'$ implies $y \in L_3(C')$. Therefore $L_3(C) = L_3(C')$.

Now, if $y \in L_3(C) = L_3(C')$, then there is $z \in C' \setminus L_1(C')$ with $w(z) \neq 1$, such that $(z, y) \in E(D')$. If $z \in L_1(C)$, then there exist $z' \notin C$ such that $(z, z') \in E(D)$. Since $z' \notin C$, we have $z' \notin C'$, then $z \in L_1(C')$. A contradiction, consequently $z \notin L_1(C)$. Hence, $C$ is strong. This implies $L_3(C) = \emptyset$, since $D$ has the minimal-strong property. Thus, $L_3(C') = L_3(C) = \emptyset$. Therefore $D'$ has the minimal-strong property.

Proposition 4.7. If $D$ is unmixed and $x \in V$, then $D' = D \setminus N_G[x]$ is unmixed.

Proof. By Theorem 1.2, $G$ is unmixed and $D$ has the minimal-strong property. Hence, by 6, $G' = G \setminus N_G[x]$ is unmixed. Also, by Lemma 4.6, we have that $D'$ has the minimal-strong property. Therefore, by Theorem 1.2, $D'$ is unmixed.

Theorem 4.8. If $D$ is unmixed, then a c-minor of $D$ is unmixed.

Proof. If $D'$ is a c-minor of $D$, then there is a stable $S = \{a_1, \ldots, a_s\}$ such that $D' = D \setminus N_G[S]$. Since $S$ is stable, $D' = (\cdots ((D \setminus N_G[a_1]) \setminus N_G[a_2]) \setminus \cdots) \setminus N_G[a_s]$. Hence, by induction and Proposition 4.7, $D'$ is unmixed.

Proposition 4.9. If $V(D)$ is a strong vertex cover of $D$, then $I(D)$ is mixed.

Proof. By Proposition 2.4, $V(D)$ is not minimal, since $L_3(V(D)) = V(D)$. Therefore, by Theorem 4.2, $I(D)$ is mixed.

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Remark 4.10. If \( V = V^+ \), then \( I(D) \) is mixed.

Proof. If \( x_i \in V \), then by Remark 5.3 \( N_D^-(x_i) \neq \emptyset \), since \( V = V^+ \). Thus, there is \( x_j \in V \) such that \( (x_j, x_i) \in E(D) \). Also, \( w(x_j) \neq 1 \) and \( x_j \in V = L_3(V) \). So, \( V \) is a strong vertex cover. Hence, by Proposition 4.9, \( I(D) \) is mixed. \( \Box \)

In the following three results we assume that \( D_1, \ldots, D_r \) are the connected components of \( D \). Furthermore, \( G_i \) is the underlying graph of \( D_i \).

Lemma 4.11. Let \( C \) be a vertex cover of \( D \), then \( L_1(C) = \bigcup_{i=1}^r L_1(C_i) \) and \( L_3(C) = \bigcup_{i=1}^r L_3(C_i) \), where \( C_i = C \cap V(D_i) \).

Proof. We take \( x \in C \), then \( x \in C_j \) for some \( 1 \leq j \leq r \). Thus, \( N_D(x) = N_{D_j}(x) \). In particular \( N_D^+(x) = N_{D_j}^+(x) \), so \( C \cap N_D^+(x) = C_j \cap N_{D_j}^+(x) \). Hence, \( L_1(C) = \bigcup_{i=1}^r L_1(C_i) \).

On the other hand,

\[ x \in L_3(C) \iff N_D(x) \subseteq C \iff N_{D_j}(x) \subseteq C_j \iff x \in L_3(C_j). \]

Therefore, \( L_3(C) = \bigcup_{i=1}^r L_3(C_i) \). \( \Box \)

Lemma 4.12. Let \( C \) be a vertex cover of \( D \), then \( C \) is strong if and only if each \( C_i = C \cap V(D_i) \) is strong with \( i \in \{1, \ldots, r\} \).

Proof. \( \Rightarrow \) We take \( x \in L_3(C_j) \). By Lemma 4.11, \( x \in L_3(C) \) and there is \( z \in N_D^-(x) \cap V^+ \) with \( z \in C \setminus L_1(C) \), since \( C \) is strong. So, \( z \in N_{D_j}(x) \) and \( z \in V(D_j) \), since \( x \in D_j \). Consequently, by Lemma 4.11, \( z \in C_j \setminus L_1(C_j) \). Therefore \( C_j \) is strong.

\( \Leftarrow \) We take \( x \in L_3(C) \), then \( x \in C_i \) for some \( 1 \leq i \leq r \). Then, by Lemma 4.11 \( x \in L_3(C_i) \). Thus, there is \( a \in N_{D_i}^-(x) \) such that \( w(a) \neq 1 \) and \( a \in C_i \setminus L_1(C_i) \), since \( C_i \) is strong. Hence, by Lemma 4.11, \( a \in C \setminus L_1(C) \). Therefore \( C \) is strong. \( \Box \)

Corollary 4.13. \( I(D) \) is unmixed if and only if \( I(D_i) \) is unmixed for each \( 1 \leq i \leq r \).

Proof. \( \Rightarrow \) By Theorem 4.8 since \( D_i \) is a c-minor of \( D \).

\( \Leftarrow \) By Theorem 4.2, \( G_i \) is unmixed thus \( G \) is unmixed. Now, if \( C \) is a strong vertex cover, then by Lemma 4.11, \( C_i = C \cap V(D_i) \) is a strong vertex cover. Consequently, \( L_3(C_i) = \emptyset \), since \( I(D_i) \) is unmixed. Hence, by Lemma 4.11, \( L_3(C) = \bigcup_{i=1}^r L_3(C_i) = \emptyset \). Therefore, by Theorem 4.8, \( I(D) \) is unmixed. \( \Box \)

Definition 4.14. Let \( G \) be a simple graph whose vertex set is \( V(G) = \{x_1, \ldots, x_n\} \) and edge set \( E(G) \). A whisker of \( G \) is a graph \( H \) whose vertex set is \( V(H) = V(G) \cup \{y_1, \ldots, y_n\} \) and whose edge set is \( E(H) = E(G) \cup \{(x_1, y_1), \ldots, (x_n, y_n)\} \).

Definition 4.15. Let \( D \) and \( H \) be weighted oriented graphs. \( H \) is a weighted oriented whisker of \( D \) if \( D \subseteq H \) and the underlying graph of \( H \) is a whisker of the underlying graph of \( D \).

Theorem 4.16. Let \( H \) a weighted oriented whisker of \( D \), where \( V(D) = \{x_1, \ldots, x_n\} \) and \( V(H) = V(D) \cup \{y_1, \ldots, y_n\} \), then the following conditions are equivalents:

1) \( I(H) \) is unmixed.

2) If \( (x_i, y_i) \in E(H) \) for some \( 1 \leq i \leq n \), then \( w(x_i) = 1 \).
Proof. 2) \( \Rightarrow \) 1) We take a strong vertex cover \( C \) of \( H \). Suppose \( x_j, y_j \in C \), then \( y_j \in L_3(C) \), since \( N_D(y_j) = \{x_j\} \subseteq C \). Consequently, \( (x_j, y_j) \in E(G) \) and \( w(x_j) \neq 1 \), since \( C \) is strong. This is a contradiction by condition 2). This implies, \( |C \cap \{x_i, y_i\}| = 1 \) for each \( 1 \leq i \leq n \). So, \( |C| = n \). Therefore, by Theorem 4.2, \( I(H) \) is unmixed.

1) \( \Rightarrow \) 2) By contradiction suppose \((x_i, y_i) \in E(H) \) and \( w(x_i) \neq 1 \) for some \( i \). Since \( w(x_i) \neq 1 \) and by Remark 5.3 we have that \( x_i \) is not a source. Thus, there is \( x_j \in V(D) \), such that \((x_j, x_i) \in E(H) \). We take the vertex cover \( C = \{V(D) \setminus x_j\} \cup \{y_j, y_i\} \), then by Proposition 7.2, \( L_3(C) = \{y_i\} \). Furthermore \( N_D(x_j) \setminus C = \{x_j\} \) and \((x_j, x_i) \in E(H) \), then \( x_i \in L_2(C) \). Hence \( C \) is strong, since \( L_3(C) = \{y_i\} \). \( (x_i, y_i) \in E(G) \) and \( w(x_i) \neq 1 \). A contradiction by Theorem 4.2, since \( I(H) \) is unmixed.

Theorem 4.17. Let \( D \) be a bipartite weighted oriented graph, then \( I(D) \) is unmixed if and only if

1) \( G \) has a perfect matching \( \{x_1^1, x_1^2\}, \ldots, \{x_s^1, x_s^2\} \) where \( \{x_1^1, \ldots, x_s^1\} \) and \( \{x_1^2, \ldots, x_s^2\} \) are stable sets. Furthermore if \( \{x_1^1, x_2^1\}, \{x_1^2, x_2^2\} \in E(G) \) then \( x_1^1, x_2^1 \in E(G) \).

2) If \( w(x_i^k) \neq 1 \) and \( N_D(x_i^k) = \{x_i^k, \ldots, x_i^{k+1}\} \) where \( k, k' = \{1, 2\} \), then \( N_D(x_i^k) \subseteq N_D^+(x_i^{k'}) \) and \( N_D(x_i^k) \cap V^+ = \emptyset \) for each \( 1 \leq k \leq r \).

Proof. \( \Leftarrow \) By 1) and [1] Theorem 2.5.7, \( G \) is unmixed. We take a strong vertex cover \( C \) of \( D \). Suppose \( L_3(C) \neq \emptyset \), thus there exist \( x_i^k \in L_3(C) \). Since \( C \) is strong, there is \( x_j^k \in V^+ \) such that \((x_j^k, x_i^k) \in E(D) \), \( x_j^k \in C \setminus L_1(C) \) and \( k, k' = \{1, 2\} \). Furthermore \( N_D^-(x_j^k) \subseteq C \), since \( x_j^k \notin L_1(C) \). Consequently, by 3), \( N_D(x_i^k) \subseteq N_D^+(x_i^k) \subseteq C \) and \( N_D^-(x_i^k) \cap V^+ = \emptyset \). A contradiction, since \( x_i^k \in L_3(C) \) and \( C \) is strong. Hence \( L_3(C) = \emptyset \) and \( D \) has the strong-minimal property. Therefore \( I(D) \) is unmixed, by Theorem 4.2.

\( \Rightarrow \) By Theorem 4.2, \( G \) is unmixed. Hence, by [1] Theorem 2.5.7, \( G \) satisfies 1).

If \( w(x_i^k) \neq 1 \), then we take \( C = N_D^-(x_i^k) \cup \{x_i^k | N_D(x_i^k) \subseteq N_D^+(x_i^k)\} \) and \( k' \) such that \( \{k, k'\} = \{1, 2\} \). If \( \{x_i^k, x_i^{k'}\} \in E(G) \) and \( x_i^k \notin C \), then \( x_i^k \in N_D(x_i^k) \subseteq N_D^+(x_i^k) \subseteq C \). This implies, \( C \) is a vertex cover of \( D \). Now, if \( x_i^k \in L_3(C) \), then \( N_D(x_i^k) \subseteq C \). Consequently \( N_D(x_i^k) \subseteq N_D^+(x_i^k) \) implies \( x_i^k \notin C \). A contradiction, then \( L_3(C) \subseteq N_D^+(x_i^k) \). Also, \( N_D(G)(x_i^k) \neq \emptyset \), since \( w(x_i^k) \neq 1 \). Thus \( x_i^k \in L_2(C) \), since \( N_D^-(x_i^k) \cap C = \emptyset \). Hence \( C \) is strong, since \( L_3(C) \subseteq N_D^+(x_i^k) \) and \( x_i^k \in V^+ \). Furthermore \( \{x_1^1, \ldots, x_s^1\} \) is a minimal vertex cover, then by Theorem 4.2 \( |C| = s \), since \( D \) is unmixed. We assume \( N_D^+(x_i^k) = \{x_i^k, \ldots, x_i^{k+1}\} \). Since \( C \) is minimal, \( x_i^k \notin C \) for each \( 1 \leq k \leq r \). So, \( N_D(x_i^k) \subseteq N_D^+(x_i^k) \). Now, suppose \( z \in N_D^-(x_i^k) \cap V^+ \), then \( z = x_i^{k'} \), for some \( 1 \leq k' \leq r \), since \( N_D(x_i^k) \subseteq N_D^+(x_i^k) \). We take \( C' = N_D^+(x_i^k) \cup \{x_i^k | i \notin \{i_1, \ldots, i_r\}\} \cup N_D^+(x_i^{k'}) \). Since \( N_D(x_i^k) \subseteq N_D^+(x_i^k) \) for each \( 1 \leq u \leq r \), we have that \( C' \) is a vertex cover. If \( \{x_q^k, x_q^{k'}\} \cap L_3(C) = \emptyset \), then \( x_q^k, x_q^{k'} \subseteq C' \), so \( x_q^{k'} \in N_D^+(x_i^k) \) implies \( q \notin \{i_1, \ldots, i_r\} \). Consequently, \( x_q^{k'} \in N_D^+(x_i^k) \), since \( x_q^k \in C' \). This implies, \( (x_j^k, x_j^{k'}) \in E(D) \). Moreover, \( N_D^+(x_i^{k'}) \subseteq N_D^+(x_i^k) \subseteq C' \), then \( x_i^{k'} \notin L_1(C') \) and \( x_i^k \notin L_1(C') \). Thus \( C' \) is strong, since \( x_i^{k'} \in V^+ \). Furthermore, by Theorem 4.2 \( |C'| = s \). But \( x_i^{k'} \in N_D^-(x_i^k) \) and \( x_i^k \in N_D^+(x_i^{k'}) \), hence \( x_i^{k'}, x_i^k \in C' \). A contradiction, so \( N_D(x_i^{k'}) \cap V^+ = \emptyset \). Therefore \( D \) satisfies 2).
Lemma 4.18. If the vertices of $V^+$ are sinks, then $D$ has the minimal-strong property.

Proof. We take a strong vertex cover $C$ of $D$. Hence, if $y \in L_3(C)$, then there is $(z,y) \in E(D)$ with $z \in V^+$. Consequently, by hypothesis, $z$ is a sink. A contradiction, since $(z,y) \in E(D)$. Therefore, $L_3(C) = \emptyset$ and $C$ is a minimal vertex cover.

Lemma 4.19. Let $D$ be a weighted oriented graph, where $G \simeq C_n$ with $n \geq 6$. Hence, $D$ has the minimal-strong property if and only if the vertices of $V^+$ are sinks.

Proof. $\Leftarrow$) By Lemma 4.18

$\Rightarrow$) By contradiction, suppose there is $(z,y) \in E(D)$, with $z \in V^+$. We can assume $G = (x_1,x_2,\ldots,x_n,x_1) \simeq C_n$, with $x_2 = y$ and $x_3 = z$. We take a strong vertex cover $C$ in the following form: $C = \{x_1,x_3,\ldots,x_{n-1}\} \cup \{x_2\}$ if $n$ is even or $C = \{x_1,x_3,\ldots,x_{n-2}\} \cup \{x_2,x_{n-1}\}$ if $n$ is odd. Consequently, if $x \in C$ and $N_D(x) \subseteq C$, then $x = x_2$. Hence, $L_3(C) = \{x_2\}$. Furthermore $(x_3,x_2) \in E(D)$ with $x_3 \in V^+$. Thus, $x_3$ is not a source, so, $(x_4,x_3) \in E(D)$. Then, $x_3 \in L_2(C)$. This implies $C$ is a strong vertex cover. But $L_3(C) \neq \emptyset$. A contradiction, since $D$ has the minimal-strong property.

![Graph](image)

Theorem 4.20. If $G \simeq C_n$, then $I(D)$ is unmixed if and only if one of the following conditions hold:

1) $n = 3$ and there is $x \in V(D)$ such that $w(x) = 1$.

2) $n \in \{4,5,7\}$ and the vertices of $V^+$ are sinks.

3) $n = 5$, there is $(x,y) \in E(D)$ with $w(x) = w(y) = 1$ and $D \not\simeq D_1, D \not\simeq D_2, D \not\simeq D_3$.

4) $D \simeq D_4$.

Proof. $\Rightarrow$) By Theorem 4.12 $D$ has the minimal-strong property and $G$ is unmixed. Then, by Exercise 2.4.22, $n \in \{3,4,5,7\}$. If $n = 3$, then by Remark 4.10 $D$ satisfies 1). If $n = 7$, then by Lemma 4.18 $D$ satisfies 2). Now suppose $n = 4$ and $D$ does not satisfies 2), then we can assume $x_1 \in V^+$ and $(x_1,x_2) \in E(D)$. Consequently, $(x_4,x_1) \in E(G)$, since $w(x_1) \neq 1$). Furthermore, $C = \{x_1,x_2,x_3\}$ is a vertex cover with $L_3(C) = \{x_2\}$. Thus, $x_1 \in L_2(C)$ and $(x_1,x_2) \in E(D)$ so $C$ is strong. A contradiction, since $C$ is not minimal. This implies $D$ satisfies 2). Finally suppose $n = 5$. If $D \simeq D_1$, then $C_1 = \{x_1,x_2,x_3,x_5\}$ is a vertex cover with $L_3(C_1) = \{x_1,x_2\}$. Also $(x_5,x_1),(x_3,x_2) \in E(D)$.
with \( x_5, x_3 \in V^+ \). Consequently, \( C_1 \) is strong, since \( x_5, x_3 \in L_2(C_1) \). A contradiction, since \( C_1 \) is not minimal. If \( D \cong D_2 \), then \( C_2 = \{x_1, x_2, x_4, x_5\} \) is a vertex cover where \( L_3(C_2) = \{x_1, x_5\} \) and \( (x_2, x_1), (x_1, x_5) \in E(D) \) with \( x_2, x_1 \in V^+ \). Hence, \( C_2 \) is strong, since \( x_2, x_1 \notin L_1(C_2) \). A contradiction, since \( C_2 \) is not minimal. If \( D \cong D_3 \), \( C_3 = \{x_2, x_3, x_4, x_5\} \) is a vertex cover where \( L_3(C_3) = \{x_3, x_4\} \) and \( (x_4, x_3), (x_5, x_4) \in E(D) \) with \( x_4, x_5 \in V^+ \). Thus, \( C_3 \) is strong, since \( x_4, x_5 \notin L_1(C_3) \). A contradiction, since \( C_3 \) is not minimal. Now, since \( n = 5 \) and by 3) we can assume \( (x_2, x_3) \in E(D) \), \( x_2, x_3 \in V^+ \) and there are not two adjacent vertices with weight 1. Since \( x_2 \in V^+ \), \( (x_1, x_2) \in E(D) \). Suppose there are not 3 vertices \( z_1, z_2, z_3 \) in \( V^+ \) such that \( (z_1, z_2, z_3) \) is a path in \( G \), then 
\[
\text{w}(x_4) = \text{w}(x_1) = 1.
\]
Furthermore, \( \text{w}(x_2) \neq 1 \), since there are not adjacent vertices with weight 1. So, \( C_4 = \{x_2, x_3, x_4, x_5\} \) is a vertex cover of \( D \), where \( L_3(C_4) = \{x_3, x_4\} \). Also \( (x_2, x_3) \in E(G) \) with \( \text{w}(x_2) \neq 1 \). Hence, if \( (x_3, x_4) \in E(D) \) or \( (x_5, x_4) \in E(D) \), then \( C_4 \) is strong, since \( x_3, x_5 \in V^+ \). But \( C_4 \) is not minimal. Consequently, \( (x_2, x_4), (x_4, x_5) \in E(D) \) and \( D \cong D_4 \). Now, we can assume there is a path \( (z_1, z_2, z_3) \) in \( D \) such that \( z_1, z_2, z_3 \in V^+ \). Since there are not adjacent vertices with weight 1, we can suppose there is \( z_4 \in V^+ \) such that \( L = (z_1, z_2, z_3, z_4) \) is a path. We take \( (z_3) = V(D) \setminus V((L)) \) and we can assume \( (z_2, z_3) \in E(D) \). This implies, \( (z_1, z_2), (z_5, z_1) \in E(D) \), since \( z_1, z_2 \in V^+ \). Thus, \( C_5 = \{z_1, z_2, z_3, z_4\} \) is a vertex cover with \( L_3(C_5) = \{z_2, z_3\} \). Then \( C_5 \) is strong, since \( (z_1, z_2), (z_2, z_3) \in E(D) \) with \( z_2 \in L_2(C_5) \) and \( z_1 \in L_2(C_5) \). A contradiction, since \( C_5 \) is not minimal.

\[ \Leftarrow \] If \( n \in \{3, 4, 5, 7\} \), then by [11 Exercise 2.4.22] \( G \) is unmixed. By Theorem 4.2, we will only prove that \( D \) has the minimal-strong property. If \( D \) satisfies 2), then by Lemma 4.18 \( D \) has the minimal-strong property. If \( D \) satisfies 1) and \( C \) is a strong vertex cover, then by Proposition 2.14 \( \|C\| \leq 2 \). This implies \( C \) is minimal. Now, suppose \( n = 5 \) and \( C' \) is a strong vertex cover of \( D \) with \( \|C'\| \geq 4 \). If \( D \cong D_4 \), then \( x_2, x_5 \notin L_3(C') \), since \( (N_D(x_2) \cup N_D(x_5)) \cap V^+ = \emptyset \). So \( N_D(x_2) \not\subseteq C' \) and \( N_D(x_5) \not\subseteq C' \). Consequently, \( x_1 \notin C' \) implies \( C' = \{x_2, x_3, x_4, x_5\} \). But \( x_4 \in L_3(C') \) and \( N_D(x_4) = \emptyset \). A contradiction, since \( C' \) is strong. Now assume \( D \) satisfies 3). Suppose there is a path \( L = (x_1, x_2, x_3) \) in \( G \) such that \( w(x_1) = w(x_2) = w(x_3) = 1 \). We can suppose \((x_4, x_3) \in E(D) \) where \( V(D) \setminus V(L) = \{x_4, x_5\} \). Since \( w(x_1) = w(x_3) = 1, x_2 \notin L_3(C') \). If \( x_2 \notin C' \), then \( C' = \{x_1, x_3, x_4, x_5\} \) and \( x_4 \notin L_3(C') \). But \( N_D(x_4) = \{x_3\} \) and \( w(x_3) = 1 \). A contradiction, hence \( x_2 \in C' \). We can assume \( x_3 \notin C' \), since \( x_2 \notin L_3(C') \). This implies \( C' = \{x_1, x_2, x_4, x_5\} \) and \( L_3(C') = \{x_1, x_2\} \). Thus \( (x_5, x_3) \in E(D) \), \( x_5, x_4 \in V^+ \). Consequently \((x_3, x_4) \in E(D) \), since \( x_4 \in V^+ \). A contradiction, since \( D \not\cong D_2 \). Hence, there are not three consecutive vertices whose weights are 1. Consequently, since \( D \) satisfies 3), we can assume \( w(x_1) = w(x_2) = 1, w(x_3) \neq 1 \) and \( w(x_5) \neq 1 \). If \( w(x_4) = 1 \), then \( x_3, x_5 \notin L_3(C') \) since \( N_D(x_3, x_5) \cap V^+ = \emptyset \). This implies \( N_D(x_3) \not\subseteq C' \) and \( N_D(x_5) \not\subseteq C' \). Then \( x_4 \notin C' \) and \( C' = \{x_1, x_2, x_3, x_5\} \). Thus, \( (x_5, x_1), (x_3, x_2) \in E(D) \), since \( L_3(C') = \{x_1, x_2\} \). Consequently, \((x_4, x_3), (x_4, x_3) \in E(D) \), since \( x_5, x_3 \in V^+ \). A contradiction, since \( D \not\cong D_1 \). So, \( w(x_4) \neq 1 \) and we can assume \((x_5, x_4) \in E(D) \), since \( x_4 \in V^+ \). Furthermore \((x_1, x_3) \in E(D) \), since \( x_5 \in V^+ \). Hence, \((x_3, x_4) \in E(D) \), since \( D \not\cong D_3 \). Then \((x_2, x_3) \in E(D) \), since \( x_3 \in V^+ \). This implies \((x_1, x_2, x_3, x_5) \notin L_3(C') \), since \( N_D(x_i) \cap V^+ = \emptyset \) for \( i \in \{1, 2, 3, 5\} \). A contradiction, since \( |C'| \geq 4 \). Therefore \( D \) has the minimal-strong property.
5 Cohen-Macaulay weighted oriented graphs

In this section we study the Cohen-Macaulayness of $I(D)$. In particular we give a combinatorial characterization of this property when $D$ is a path or $D$ is complete. Furthermore, we show the Cohen-Macaulay property depends on the characteristic of $K$.

**Definition 5.1.** The weighted oriented graph $D$ is Cohen-Macaulay over the field $K$ if the ring $R/I(D)$ is Cohen-Macaulay.

**Remark 5.2.** If $G$ is the underlying graph of $D$, then $\text{rad}(I(D)) = I(G)$.

**Proposition 5.3.** If $I(D)$ is Cohen-Macaulay, then $I(G)$ is Cohen-Macaulay and $D$ has the minimal-strong property.

**Proof.** By Remark 5.2, $I(G) = \text{rad}(I(D))$, then by [4, Theorem 2.6], $I(G)$ is Cohen-Macaulay. Furthermore $I(D)$ is unmixed, since $I(D)$ is Cohen-Macaulay. Hence, by Theorem 4.2, $D$ has the minimal-strong property.

**Example 5.4.** In Example 3.13 and Example 3.14 $I(D)$ is mixed. Hence, $I(D)$ is not Cohen-Macaulay, but $I(G)$ is Cohen-Macaulay.

**Conjecture 5.5.** $I(D)$ is Cohen-Macaulay if and only if $I(G)$ is Cohen-Macaulay and $D$ has the minimal-strong property. Equivalently $I(D)$ is Cohen-Macaulay if and only if $I(D)$ is unmixed and $I(G)$ is Cohen-Macaulay.

**Proposition 5.6.** Let $D$ be a weighted oriented graph such that $V = \{x_1, \ldots, x_k\}$ and whose underlying graph is a path $G = (x_1, \ldots, x_k)$. Then the following conditions are equivalent.

1) $R/I(D)$ is Cohen-Macaulay.

2) $I(D)$ is unmixed.

3) $k = 2$ or $k = 4$. In the second case, if $(x_2, x_1) \in E(D)$ or $(x_3, x_4) \in E(D)$, then $w(x_2) = 1$ or $w(x_3) = 1$ respectively.

**Proof.** 1) $\Rightarrow$ 2) By [4, Corollary 1.5.14].

2) $\Rightarrow$ 3) By Theorem 4.17, $G$ has a perfect matching, since $D$ is bipartite. Consequently $k$ is even and $\{x_1, x_2\}, \{x_3, x_4\}, \ldots , \{x_{k-1}, x_k\}$ is a perfect matching. If $k \geq 6$, then by Theorem 4.17 we have $\{x_2, x_5\} \in E(G)$, since $\{x_2, x_3\}$ and $\{x_4, x_5\} \in E(G)$. A contradiction since $\{x_2, x_5\} \notin E(G)$. Therefore $k \in \{2, 4\}$. Furthermore by Theorem 4.17 $w(x_2) = 1$ or $w(x_3) = 1$ when $(x_2, x_1) \in E(D)$ or $(x_3, x_4) \in E(D)$, respectively.

3) $\Rightarrow$ 1) We take $I = I(D)$. If $k = 2$, then we can assume $(x_1, x_2) \in E(D)$. So, $I = (x_1^w(x_2)) = (x_1) \cap (x_2^w(x_2))$. Thus, by Remark 3.12, $\text{Ass}(I) = \{(x_1), (x_2)\}$. This implies, $\text{ht}(I) = 1$ and $\text{dim}(R/I) = k - 1 = 1$. Also, $\text{depth}(R/I) \geq 1$, since $(x_1, x_2) \notin \text{Ass}(I)$. Hence, $R/I$ is Cohen-Macaulay. Now, if $k = 4$, then $\text{ht}(I) = \text{ht}(\text{rad}(I)) = \text{ht}(I(G)) = 2$. Consequently, $\text{dim}(R/I) = k - 2 = 2$. Furthermore one of the following sets $\{x_2 - x_1^w(x_1), x_3 - x_3^w(x_3)\}, \{x_2 - x_1^w(x_1), x_4 - x_3^w(x_3)\}, \{x_1 - x_2^w(x_2), x_4 - x_3^w(x_3)\}$ is a regular sequence of $R/I$, then $\text{depth}(R/I) \geq 2$. Therefore, $I$ is Cohen-Macaulay.

**Theorem 5.7.** If $G$ is a complete graph, then the following conditions are equivalent.
1) \(I(D)\) is unmixed.

2) \(I(D)\) is Cohen-Macaulay.

3) There are not \(D_1, \ldots, D_s\) unicycles orientes subgraphs of \(D\) such that \(V(D_1), \ldots, V(D_s)\) is a partition of \(V(D)\)

Proof. We take \(I = I(D)\). Since \(I(G) = \text{rad}(I)\) and \(G\) is complete, \(\text{ht}(I) = \text{ht}(I(G)) = n - 1\).

1) \(\Rightarrow\) 3) Since \(\text{ht}(I) = n - 1\) and \(I\) is unmixed, \((x_1, \ldots, x_n) \notin \text{Ass}(I)\). Thus, by Remark 3.12, \(V(D)\) is not a strong vertex cover of \(D\). Therefore, by Proposition 2.14, \(D\) satisfies 3).

3) \(\Rightarrow\) 2) By Proposition 2.14, \(V(D)\) is not a strong vertex cover of \(D\). Consequently, by Remark 3.12, \((x_1, \ldots, x_n) \notin \text{Ass}(I)\). This implies, \(\text{depth}(R/I) \geq 1\). Furthermore, \(\dim(R/I) = 1\), since \(\text{ht}(I) = n - 1\). Therefore \(I\) is Cohen-Macaulay.

2) \(\Rightarrow\) 1) By [1, Corollary 1.5.14].

If \(D\) is complete or \(D\) is a path, then the Cohen-Macaulay property does not depend of the field \(K\). It is not true in general, see the following example.

**Example 5.8.** Let \(D\) be the following weighted oriented graph:

![Graph Image]

Hence,

\[
I(D) = (x_1^2 x_4, x_2^2 x_8, x_3^2 x_5, x_4^2 x_9, x_5^2 x_{10}, x_6^2 x_5, x_7^2 x_6, x_8^2 x_7, x_9^2 x_10, x_1^2 x_3, x_2^2 x_9, x_4 x_8, x_6 x_7, x_5 x_{11}, x_5 x_9, x_5 x_{11}, x_6 x_8, x_6 x_{10}, x_7 x_{11}, x_9 x_{11}).
\]

By [5, Example 2.3], \(I(G)\) is Cohen-Macaulay when the characteristic of the field \(K\) is zero but it is not Cohen-Macaulay in characteristic 2. Consequently, \(I(D)\) is not Cohen-Macaulay when the characteristic of \(K\) is 2. Also, \(I(G)\) is unmixed. Furthermore, by Lemma 4.18, \(I(D)\) has the minimal-strong property, then \(I(D)\) is unmixed. Using Macaulay2 [2] we show that \(I(D)\) is Cohen-Macaulay when the characteristic of \(K\) is zero.
References

[1] I. Gitler, R. H. Villarreal, *Graphs, Rings and Polyhedra*, Aportaciones Mat. Textos, 35, Soc. Mat. Mexicana, México, 2011.

[2] D. Grayson and M. Stillman, *Macaulay2*, 1996. Available via anonymous ftp from math.uiuc.edu.

[3] J. Herzog and T. Hibi, *Monomial Ideals*, Graduate Texts in Mathematics. 260, Springer, 2011.

[4] J. Herzog, Y. Takayama, N. Terai, *On the radical of a monomial ideal*, Arch. Math. 85 (2005), 397–408.

[5] A. Madadi and R. Zaare-Nahandi, *Cohen-Macaulay r-partite graphs with minimal clique cover*, Bull. Iranian Math. Soc. 40 (2014) No. 3, 609–617.

[6] R. H. Villarreal, *Monomial Algebras, Second Edition*, Monographs and Research Notes in Mathematics, Chapman and Hall/CRC, 2015.