Enstrophy dissipation and vortex thinning for the incompressible 2D Navier–Stokes equations

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Received 5 June 2020, revised 11 December 2020
Accepted for publication 18 December 2020
Published 18 February 2021

Abstract
Vortex thinning is one of the main mechanisms of two-dimensional turbulence. By direct numerical simulation to the two-dimensional Navier–Stokes equations with small-scale forcing and large-scale damping, Xiao et al (2009 \textit{J. Fluid Mech.} \textbf{619} 1–44) found an evidence that inverse energy cascade may proceed with the vortex thinning mechanism. On the other hand, Alexakis and Doering (2006 \textit{Phys. Lett. A} \textbf{359} 652–657) calculated the upper bound of the bulk averaged enstrophy dissipation rate of the steady-state two dimensional turbulence. In this paper, we show that vortex thinning induces enhanced dissipation with strictly slower vanishing order of the enstrophy dissipation than $\text{Re}^{-1}$.

Keywords: Euler equations, Navier–Stokes equations, enstrophy dissipation, vortex thinning

Mathematics Subject Classification numbers: Primary 35Q35; Secondary 35B30; Tertiary 76F99.

1. Introduction

The mechanism of forward and inverse energy cascade of two-dimensional turbulence has been attracting many physicists but nevertheless is not well clarified so far. Theoretical studies of 2D turbulence usually employ statistics and sometimes impose assumptions on homogeneity isotropy. Alexakis and Doering [1] derived some rigorous upper bounds for the long time averaged bulk energy and enstrophy dissipation rates for 2D statistically stationary flows sustained...
by a variety of driving forces. In particular, they showed the enstrophy dissipation vanishes in the order $Re^{-1}$ when the external force is only on the single scale. They are emphasising that, in this $Re^{-1}$ scaling, the flow exhibits laminar behavior, since energy is concentrated at relatively long length scales (independent of Reynolds number). Thus, their result tells us that, at least, 2D turbulence behavior must provide vanishing order which is strictly slower than the order $Re^{-1}$. Let us explain more precisely. The two dimensional Navier–Stokes equations are described as follows:

$$\begin{align*}
\partial_t u - \nu \Delta u + u \cdot \nabla u &= -\nabla p + f, \quad t \geq 0, \quad x \in (\mathbb{R}/2\mathbb{Z})^2 \\
\text{div} \ u &= 0 \\
u(t = 0) &= u_0
\end{align*}$$

(1)

where $u = u(t, x) = (u_1(t, x), u_2(t, x))$, $p = p(t, x)$ and $f = (f_1(t, x), f_2(t, x))$ denote the velocity field, the pressure function of the fluid and the external force respectively. The case $\nu = 0$ is called the Euler equations. Introducing the vorticity $\omega = \nabla \times u$, we can rewrite (1) as the following 2D vorticity equations:

$$\begin{align*}
\partial_t \omega - \nu \Delta \omega + (u \cdot \nabla) \omega &= \nabla \times f, \quad x \in \mathbb{T} := (\mathbb{R}/2\mathbb{Z})^2,
\end{align*}$$

(2)

where the velocity $u$ is determined by the (periodic) 2D Biot–Savart law:

$$u(t, x) = \int_{\mathbb{T}^2} K_2(x - y) \omega(t, y) \, dy,$$

with

$$K_2(x) = \frac{1}{2\pi} \left( \frac{-x_2, x_1}{|x|^2} \right) \quad \text{(with reflections)}.$$

Assume that the external force $f$ involves only a single length scale, namely, $-\Delta f \sim k_f^2 f$. Assume further that the flow is a statistical steady state of body forced two-dimensional turbulence. Then the solution $u$ (depending on $\nu$) to the Navier–Stokes equation (1) satisfy

$$\nu \langle |\nabla \omega|^2 \rangle = \chi \leq \nu k_f^4 U^2,$$

where $\chi$ is the enstrophy dissipation, $\langle \cdot \rangle$ is some averaging (in this paper we regard it as both space and time averages) and $U$ is the representative velocity. Note that, mathematically, if the enstrophy is finite, then this $\chi$ is always vanishing for $\nu \to 0$ (see [7, proposition 2] for example).

The aim of this paper is to find a specific vorticity configuration which provides strictly slower vanishing order of the enstrophy dissipation than $Re^{-1}$, and in this case ‘vortex thinning’ should be a strong candidate for it. To be more precise, Xiao–Wan–Chen–Eyink ([15]) investigated inverse energy cascade in steady-state two-dimensional turbulence by direct numerical simulation of the two-dimensional Navier–Stokes equations with small scale forcing and large scale damping. In their numerical work, they used an alternative equation (with a damping term), and found strong evidence that inverse energy cascade may proceed with vortex thinning mechanism. According to their evidence, there is a tensile turbulent stress in directions parallel to the isolines of small-scale vorticity. Thus the small-scale circular vortex will be stretched into elliptical shape, which is nothing more than ‘vortex thinning’ phenomena.

In this paper we focus on a sequence of prescribed initial vorticity which provides vortex thinning behavior, and show that the vortex thinning behavior provides strictly slower vanishing
order of the enstrophy dissipation than \( \text{Re}^{-1} \). Now we formulate mathematically this enstrophy dissipation problem more precisely. Let us find a sequence of viscosity \( \{ \nu_n \}_n (\nu_n \rightarrow 0) \), a sequence of initial vorticity \( \{ \omega_{0n}^{(v)} \}_n \subset L^2(\mathbb{T}^2) \cap C^1(\mathbb{T}^2) \) with \( \| \omega_{0n}^{(v)} \|_{H^1} \lesssim 1 \) and a function \( \varphi \) with \( \varphi(\nu) \rightarrow 0 \) \( (\nu \rightarrow 0) \), such that the corresponding solutions to the Navier–Stokes equations \( \{ \omega_n^v \}_n \) satisfy the following: for any \( T > 0 \) fixed,

\[
\liminf_{n \to \infty} \varphi(\nu_n) \frac{1}{T} \int_0^T \int_{\mathbb{T}^2} |\nabla \omega_n^v(t, x)|^2 \, dx \, dt > 0.
\]

(3)

Since we are interested in a finite-time result, we shall neglect the external force and take \( \text{Re} = \nu^{-1} \).

Note that, in physics, \( H^1 \)-norm of vorticity is called ‘palmistrophy’ (see [10, section 8] for example). The above formulation is the most strict one, since we are imposing uniformly boundedness \( \| \omega_{0n}^{(v)} \|_{H^1} \lesssim 1 \) and taking arbitrary time \( T > 0 \). These are necessary to avoid ‘trivial enstrophy dissipation’: if we choose \( \{ \omega_{0n} \}_n \) satisfying \( \| \omega_{0n} \|_{H^1} \rightarrow \infty \), and choose \( T_n (T_n \rightarrow 0, n \rightarrow \infty) \) to be \( \sup_{0 < t < T_n} \| \omega_{0n} - \omega_{0n}^{\nu}(t) \|_{H^1} \lesssim \epsilon \) (for sufficiently small \( \epsilon > 0 \)) with \( \nu_n \sim \| \omega_{0n} \|_{H^1}^{-\alpha} \), then we trivially have (cf [7, section 6])

\[
\lim_{n \to \infty} \nu_n \frac{1}{T_n} \int_0^{T_n} \int_{\mathbb{T}^2} |\nabla \omega_{n}^{\nu}(t, x)|^2 \, dx \, dt \sim 1.
\]

(4)

This is nothing more than trivial enstrophy dissipation we want to avoid.

We now construct our initial data. Given any smooth radial bump function \( 0 \leq \phi \leq 1 \) with support in the ball \( B(0, 1/4) \) let

\[
\phi_0(x_1, x_2) = \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \phi(x_1 - \varepsilon_1, x_2 - \varepsilon_2).
\]

We assume further that \( \phi = 1 \) in the ball \( B(0, 1/8) \). Clearly, the function \( \phi_0 \) is odd with respect to both \( x_1 \) and \( x_2 \). Moreover, given \( \ell_0 \geq 10 \) (to be determined later independently of \( n \); see the proof of proposition 2 below), we take \( \tilde{\omega}_0 \) to be a smooth function which is odd–odd and satisfy \( \tilde{\omega}_0 = 0 \) on \( [0, 1]^2 \backslash \{2^{-\ell_0} + 2 \in [1 - 2^{-\ell_0 + 2}] \} \) and \( \tilde{\omega}_0 = 1 \) on \( [2^{-\ell_0 + 3}, 1 - 2^{-\ell_0 + 3}]^2 \). Given some non-negative and decreasing sequence of reals \( \{ \alpha_\ell \}_{\ell \geq 1} \), we define

\[
\omega_{0n}(x) = \phi_{2\ell_0}(x) + \tilde{\omega}_0(x) + \sum_{\ell = \ell_0}^n \alpha_\ell \phi_\ell(x),
\]

(5)

where \( \phi_\ell(x) = \phi_0(2^\ell x) \). Note that the supports of \( \phi_\ell \) are disjoint. We remark that \( \tilde{\omega}_0 \) is necessary to guarantee (14). The term \( \sum_{\ell = \ell_0}^n \alpha_\ell \phi_\ell \) (‘Bourgain–Li’ bubbles after the work [4]) creates a large rate-of-strain tensor at the origin, which thins the smallest scale vortex blob \( \phi_{2\ell_0} \).

**Remark 1.** In the proof of our main result, we take

\[
a_\ell = \ell^{-1/2-\epsilon} \quad \text{for some small} \quad 0 < \epsilon < 1/4.
\]

With this choice, we see that \( \| \omega_{0n} \|_{H^1} \lesssim 1 \) for all \( n \).

We mention that the relation between the vortex thinning process and palmistrophy
\( (H^1 \text{-norm of the vorticity}) \) has already been studied. In [2, section 6.3] (see also [11]), Ayala–Protas found an initial vorticity of 2D Navier–Stokes equations (with very high Reynolds number) which attains maximum growth of palmistrophy, by using their own optimising method. They figured out that the initial vorticity has odd (in both \( x_1 \) and \( x_2 \)) type of
symmetry with two different scales formation (which seems very similar to our initial vorticity setting). On the other hand, some of mathematicians have showed that there is an initial vorticity in \( H^1 \) such that the value of \( \| \nabla \omega(t) \|_{L^2} \) (palinstrophy) to the 2D-Euler flow instantaneously blows up. More precisely, Bourgain–Li [4] and Elgindi–Jeong [5] constructed solutions to the 2D-Euler equations which exhibit norm inflation in \( H^1 \) (see also [12]). Now we state our main result.

**Theorem 1 (Lower bound on the enstrophy dissipation).** Let \( \{ \omega_{\nu}(t) \}_{\nu \in (0, \nu_0)} \) be the unique solution of the 2D Navier–Stokes equations with viscosity \( \nu > 0 \) and initial data \( \omega_{0,\nu} \) given in (5). Then, there exist a continuous function \( \varphi(\nu) \) with \( \varphi(\nu) \to 0 \) as \( \nu \to 0 \) and a sequence of viscosity constants \( \nu_n > 0 \) converging to zero, such that for some absolute constant \( \delta > 0 \),

\[
\liminf_n \varphi(\nu_n) \frac{1}{\delta} \int_0^\delta \int_{\Omega^2} \| \nabla \omega_{\nu_n}(t, x) \|^2 \, dx \, dt \gtrsim 1.
\]  

(6)

**Remark 2.** In the proof, one sees that the growth of \( \| \nabla \omega_{\nu_n} \|_{L^2} \) comes from vortex thinning of the smallest scale bubble \( \phi_{2n} \) in (5). The function \( \varphi \) can be chosen as \( \varphi(\nu) = (\log \frac{1}{\nu})^{-\alpha} \) for some absolute constant \( c_0 > 0 \). Finally, note that the sequence of initial data in (5) is convergent weakly in \( H^1 \), and moreover, we can obtain the same conclusion for the following initial data:

\[
\omega_{0,\nu}(x) = \frac{\phi_{2n}(x)}{(S_0 \delta)^{\frac{1}{2}}} + \tilde{\omega}_0(x) + \sum_{\ell = \ell_0}^n a_{\ell} \phi_{\ell}(x),
\]

where \( S_n, \delta \) and \( c_0 \) are determined in proposition 2. In this case the smallest-scale vorticity \( \frac{\omega_{0,\nu}(x)}{(S_0 \delta)^{\frac{1}{2}}} \) vanishes for \( n \to \infty \).

**Remark 3.** Using somewhat similar construction, the authors considered enhanced energy dissipation under the \( 2 + \frac{1}{2} \)-dimensional Navier–Stokes flow: small-scale horizontal vortex blob being stretched by large-scale, anti-parallel pairs of vertical vortex tubes [8]. The zeroth law states that, in the limit of vanishing viscosity, the rate of kinetic energy dissipation for solutions to the 3D incompressible Navier–Stokes equations becomes nonzero. The authors proved a version of the zeroth law satisfying the above, which implies in particular enhanced dissipation.

**Remark 4.** Although the above result stated in possibly small time interval \([0, \delta]\), we can take \( \delta > 0 \) arbitrarily large simply by rescaling the initial data in the \( L^\infty \)-norm. Moreover, the same result holds in the whole space \( \mathbb{R}^2 \) with essentially no modifications of the proof; a version of the key lemma (see below) in \( \mathbb{R}^2 \) was established in [6].

**Notation.** We write \( A \lesssim B \) if there is an absolute constant \( c > 0 \) such that \( A \leq c B \). Similarly, \( A \approx B \) if \( A \lesssim B \) and \( B \lesssim A \). We define some function norms: given \( f : \mathbb{T}^2 \to \mathbb{R} \), we define the Hölder norm for \( 0 < \alpha \leq 1 \) by

\[
\| f \|_{C^\alpha} = \sup_{x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|^\alpha}, \quad \| f \|_{C^0} = \| f \|_{C^0} + \| f \|_{L^\infty}.
\]

Moreover, the \( L^2 \) based Sobolev norms are defined by

\[
\| f \|_{H^s} = \| (-\Delta)^{\frac{s}{2}} f \|_{L^2}, \quad \| f \|_{H^0}^2 = \| f \|_{L^2}^2 + \| f \|_{H^s}^2.
\]

where \((-\Delta)^{\frac{s}{2}}\) is the Fourier multiplier with symbol \( |\xi|^s \). Finally, for a vector-valued function \( F = (f_j) \), \( \| F \|_X = \sup_j \| f_j \|_X \).
2. Proof of the main result

Our result is a consequence of the ‘large Lagrangian deformation’ for the 2D Euler equations, which was first established in a work of Bourgain and Li [4]. However, their idea works only for a short time and relies on a contradiction argument. Even worse, they were not able to obtain any quantitative growth rate. To overcome these difficulties, we employ an induction argument accompanied by a careful scaling control. For additional technical improvements achieved in this paper, see the remarks following the proof of proposition 2.

Let $\eta_n = (\eta_{n,1}, \eta_{n,2})$ be the associated Lagrangian flow from $\omega_{0,n}$, which is given by

$$\partial_t \eta_n(t, x) = u_n(t, \eta_n(t, x)) \quad \text{with} \quad \eta_n(0, x) = x \in \mathbb{T}^2,$$

where $u_n$ is the corresponding velocity associated with $\omega_{0,n}$. The proof of our main result is based on the following two propositions. The first proposition makes precise what we mean by large Lagrangian deformation.

**Proposition 2 (Creation of large Lagrangian deformation).** Let $\{a_k\}_{k=1}^{\infty}$ be a bounded sequence of non-negative reals, and $\omega_n(t)$ be the solution to the 2D Euler equation with initial data as in (5) with associated Lagrangian flow $\eta_n$. Set $S_k := S_{k-1} + a_k$ for $k \geq 1$ with $S_0 := 1$. Assume that $S_k$ is divergent with $k$. Then, for some absolute constant $c_0 > 0$, we have

$$-\frac{\partial u_{n,1}}{\partial x_1}(t, 0) = \frac{\partial u_{n,2}}{\partial x_2}(t, 0) \geq S_n \mathbf{1}_{\{0 \leq t \leq S_k^{-1}\}} + \frac{1}{t} \mathbf{1}_{\{t \geq S_k^{-1}\}}$$

(7)

and

$$\left| \frac{\partial \eta_{n,1}}{\partial x_1}(t, 0) \right| = \left| \frac{\partial \eta_{n,2}}{\partial x_2}(t, 0) \right| \geq (S_n t)^{\alpha_0}$$

(8)

for all $t \in [0, \delta)$ with some absolute constant $\delta > 0$ and any $n \geq \ell_0$ where $\ell_0$ is a sufficiently large absolute constant.

**Proposition 3.** Fix $a_\ell = \ell^{-\frac{1}{4} + \delta}$ and $S_n := 1 + \sum_{k=1}^n a_k$. Let $\omega_n(t)$ be the solution of the 2D Euler equation with initial data $\omega_{0,n}$ defined in (5). Then we have

$$\|\nabla \omega_n(t)\|_{L^2} \gtrsim (S_n t)^{\frac{\alpha_0}{2}}$$

(9)

for $t \in [0, \delta]$ with the same $\delta > 0$ and $n$ as in proposition 2.

Given the above propositions, we complete the proof of the main theorem.

**Proof of theorem 1.** Since initial data is always smooth, we note that for each $\nu > 0$, the unique solution $\{\omega_{0,n}^\nu\}$ exists globally-in-time. From now on the multiplicative constants in the estimates will depend implicitly on $T > 0$. From the maximum principle $\|\omega_n(t)\|_{L^\infty} = 1$, we have

$$\|\nabla u_n(t)\|_{L^\infty} \lesssim \log \|\omega_{0,n}\|_{C^1} e^{C_1} \lesssim n$$

on the time interval $[0, T]$. Note that $\|\omega_{0,k}\|_{C^1}$ can be always controlled by taking sufficiently large $k$ (depending on $n$). Then we have, from the classical estimate (see [3] for example)

$$\|\omega_n(t)\|_{H^2} \lesssim \|\omega_{0,n}\|_{H^2} e^{C_1 t} \|\nabla u_n(t)\|_{L^\infty} dt$$

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that
\[ \| \omega_n(t) \|_{H^s} \lesssim 2^{(s)n} \]
for some constant \( c(s) > 0 \) depending only on \( s > 1 \) for any \( s \) and \( t \in [0, T] \). We note that the Navier–Stokes solutions satisfy the same bounds:
\[ \| \omega_n'(t) \|_{H^s} \lesssim 2^{(s)n} \]
with constant independent of \( \nu > 0 \). This is because we still have the maximum principle \( \| \omega_n'(t) \|_{L^\infty} \leq 1 \) for all \( t \geq 0 \) and the \( H^s \) estimate holds a fortiori for the Navier–Stokes. Taking \( s > 3 \) and by the Sobolev embedding, we obtain
\[ \| \nabla^2 \omega_n'(t) \|_{L^\infty} + \| \nabla^2 \omega_n''(t) \|_{L^\infty} \lesssim 2^{sn} \quad \text{for} \quad t \in [0, T]. \] (10)
This is elementary but is the key in the estimates below.

2.1. \( L^2 \) inviscid limit estimate on the velocity

We compare the 2D Euler and Navier–Stokes equations of the velocity:
\[
\begin{align*}
\partial_t u_n' + u_n' \cdot \nabla u_n' + \nabla p_n' &= \nu \Delta u_n', \\
\partial_t u_n + u_n \cdot \nabla u_n + \nabla p_n &= 0.
\end{align*}
\]

By
\[
\int (u_n' \cdot \nabla) (u_n' - u_n) \cdot (u_n' - u_n) = 0,
\]
then we see that
\[
\frac{1}{2} \frac{d}{dt} \| u_n' - u_n \|_{L^2}^2 + \int (u_n' - u_n) \cdot \nabla u_n \cdot (u_n' - u_n) = \nu \int \Delta u_n' \cdot (u_n' - u_n).
\]
We handle the right-hand side as follows:
\[
-\nu \int |\nabla u_n'|^2 + \nu \int \nabla u_n' \cdot \nabla u_n.
\]
Then, using the Cauchy–Schwarz inequality and Young’s inequality, we have
\[
\frac{d}{dt} \| u_n' - u_n \|_{L^2}^2 \lesssim \| \nabla u_n \|_{L^\infty} \| u_n' - u_n \|_{L^2}^2 + \nu \| \nabla u_n \|_{L^2}^2 \lesssim \nu \| u_n' - u_n \|_{L^2}^2 + \nu,
\]
which gives
\[
\| u_n' - u_n \|_{L^2}^2 \lesssim \nu 2^{sn}
\]
for \( t \in [0, T] \).
2.2. $H^1$ inviscid limit estimate

This time, we consider the 2D Navier–Stokes solutions in the vorticity form and compare it with the corresponding Euler solutions:

$$
\partial_t \omega_n^\nu + u_n^\nu \cdot \nabla \omega_n^\nu = \nu \Delta \omega_n^\nu, \\
\partial_t \omega_n + u_n \cdot \nabla \omega_n = 0.
$$

Then using previous bounds,

$$
\frac{d}{dt} \| \omega_n - \omega_n^\nu \|^2_{L^2} \lesssim \| \nabla \omega_n \|_{L^\infty} \| u_n - u_n^\nu \|_{L^2} \| \omega_n - \omega_n^\nu \|_{L^2} + \nu \| \nabla \omega_n \|^2_{L^2}
$$

Thus we have

$$
\| \nabla u_n - \nabla u_n^\nu \|_{L^2} \lesssim \| \omega_n - \omega_n^\nu \|_{L^2} \lesssim \nu^{1/2} 2^m.
$$

2.3. $H^2$ inviscid limit estimate

Let $\partial$ be a spatial derivative. Here, we consider the 2D Navier–Stokes solutions in the vorticity-gradient form and compare it with the corresponding Euler solutions:

$$
\partial_t \partial\omega_n^\nu + (\partial u_n^\nu \cdot \nabla) \omega_n^\nu + (u_n^\nu \cdot \nabla) \partial \omega_n^\nu = \nu \Delta \partial \omega_n^\nu, \\
\partial_t \partial \omega_n + (\partial u_n \cdot \nabla) \omega_n + (u_n \cdot \nabla) \partial \omega_n = 0.
$$

Again using previous bounds,

$$
\frac{d}{dt} \| \partial(\omega_n - \omega_n^\nu) \|^2_{L^2} \lesssim \| \partial u_n \|_{L^\infty} \| \nabla(\omega_n - \omega_n^\nu) \|_{L^2} \| \partial(\omega_n - \omega_n^\nu) \|_{L^2}
$$

$$
+ (\| \nabla \partial \omega_n \|_{L^\infty} + \| \nabla \partial \omega_n^\nu \|_{L^\infty}) \| u_n - u_n^\nu \|_{L^2} \| \partial(\omega_n - \omega_n^\nu) \|_{L^2}
$$

$$
+ (\| \nabla \omega_n \|_{L^\infty} + \| \nabla \omega_n^\nu \|_{L^\infty}) \| \partial(u_n - u_n^\nu) \|_{L^2} \| \partial(\omega_n^\nu - \omega_n) \|_{L^2} + \nu \| \omega_n \|^2_{H^{3/2}}
$$

$$
\lesssim \| \nabla(\omega_n - \omega_n^\nu) \|^2_{L^2} + \nu^{1/2} 2^m \| \nabla(\omega_n - \omega_n^\nu) \|_{L^2} + \nu 2^m .
$$

Here we choose $\nu \sim 2^{-c} \cdot n$ with some $c^* > 0$ large absolute constant, which guarantees that

$$
\| \nabla(\omega_n - \omega_n^\nu) \|_{L^2} \lesssim 1.
$$

2.4. Completion of the proof

Recall (9) from proposition 3:

$$
\| \nabla \omega_n(t) \|_{L^2} \gtrsim (S_n t)^{\alpha_0}
$$

for $t \in [0, \delta]$. Then

$$
\frac{1}{\delta} \int_0^\delta \| \nabla \omega_n(t) \|^2_{L^2} \, dt \gtrsim S_n^{2\alpha_0}.
$$
Then we finally have
\[
\varphi(v_n) \frac{1}{\delta} \int_0^\delta \|\nabla \omega^u_n(t)\|_2^2 \, dt \geq \varphi(v_n) \frac{1}{\delta} \int_0^\delta \|\nabla \omega_n(t)\|_2^2 \, dt - \varphi(v_n)
\]
\[
\sup_{0 < c < \delta} \|\nabla (\omega^u_n(t) - \omega_n(t))\|_2^2 \geq \varphi(v_n) S_n^{2u} - \varphi(v_n).
\]
Here we set \( \varphi(v_n) \sim (-\log(v_n))^\frac{\alpha}{\beta} \); then, observe that \( \varphi(v_n) \geq n^{-\frac{\alpha}{\beta}} \) and \( S_n \geq n^{\frac{2c}{1-\epsilon}} \) so that \( \varphi(v_n) S_n^{2u} \geq n^{\frac{2c}{1-\epsilon} (1+\epsilon)} \to \infty \) for \( \epsilon < \frac{1}{4} \). This finishes the proof.

3. Creation of Lagrangian deformation

In this section, we give the proofs of two propositions.

Proof of proposition 2. Recall that the 2D flow map associated with the initial vorticity \( \omega_{0,n} \) is denoted by \( \eta_n = (\eta_{n,1}, \eta_{n,2}) \) and it satisfies
\[
\partial_t \eta_n(t, x) = u_n(t, \eta_n(t, x)) \quad \text{with} \quad \eta_n(0, x) = x \in \mathbb{T}^2,
\]
where \( u_n \) is the corresponding velocity associated with \( \omega_{0,n} \). Then we have the following representation
\[
D \eta_n := \begin{pmatrix} \partial_1 \eta_{n,1} & \partial_2 \eta_{n,1} \\ \partial_1 \eta_{n,2} & \partial_2 \eta_{n,2} \end{pmatrix}, \quad D \eta_n^{-1} = \begin{pmatrix} \partial_2 \eta_{n,2} & -\partial_2 \eta_{n,1} \\ -\partial_1 \eta_{n,2} & \partial_1 \eta_{n,1} \end{pmatrix}.
\]
(11)

Note that we always have \( \|D \eta_n(t)\|_{L^\infty} = \|D \eta_n^{-1}(t)\|_{L^\infty} \). To begin with, we shall fix some \( n \) large and write \( \eta = \eta_n \) for simplicity. Moreover, we note that \( S_n \) is divergent in \( n \). We systematically use the following 'key lemma' due to Kiselev and Sverak [9] (a version on \( \mathbb{T}^2 \) has been established by [16]). The version presented below written in the polar coordinates is given in [6, lemma 5.1].

Key lemma. Assume the vorticity on \( \mathbb{T}^2 \) is bounded and odd with respect to both axis. Then \( u = \nabla^3 \Delta^{-1} \omega \) satisfies
\[
u(t, r, \theta) = \begin{pmatrix} -\cos \theta \\ \sin \theta \end{pmatrix} r I(t, r) + r B(t, r, \theta)
\]
(12)
for \( |r| \leq 1/2 \), where
\[
I(t, r) := \frac{4}{\pi} \int_0^{\pi/2} \int_{2r}^{r^{\pi/2}} \frac{\sin(2\theta')}{s} \omega(t, s, \theta') \, ds \, d\theta'
\]
and
\[
\|B(t)\|_L^\infty \leq C \|\omega(t)\|_L^\infty
\]
(13)
for some absolute constant \( C \geq 0 \).

For simplicity, we shall set \( I(t) := I(t, 0) \) which is well-defined as the vorticities we consider is always vanishing in a small neighborhood of the origin. We now give a brief outline of the
argument. The goal is to estimate the time integration of the ‘key integral’ \( I(t) \) for some time interval \([0, t^\ast]\), since then we deduce from the ODE

\[
\frac{d}{dt} \partial_2 \eta_{n,2}(t,0) = \partial_2 u_{n,2}(t,0) \partial_2 \eta_{n,2}(t,0), \quad \partial_2 \eta_{n,2}(0,0) = 1
\]

(the term \( \partial_1 u_{n,2}(t,\eta_{n,2}(t,0)) \partial_2 \eta_{n,1}(t,0) \) vanishes since \( \eta_{n,2}(t,0) = 0 \) and \( \partial_1 u_{n,2} \) is odd in its variables for any \( t \)) and

\[
\partial_2 u_{n,2}(t,0) \gtrsim I(t,0) - C
\]

that (assuming \( I(t,0) > 2C \))

\[
|\partial_1 \eta_{n,1}(t^\ast,0)| \gtrsim \exp \left( \frac{1}{2} \int_0^{t^\ast} I(t) dt \right).
\]

The assumption

\[
I(t,0) > 2C
\]  

(14)

is easily guaranteed by taking \( \ell_0 \) large, since the contribution to \( I \) from \( \tilde{\omega}_0 \) diverges as \( \ell_0 \to +\infty \) (see [9]). In turn, we may write

\[
I(t) = \sum_{k=1}^n I_k(t) + I_{2n}(t)
\]

where

\[
I_{2n}(t) := \frac{4}{\pi} \int_0^{\pi/2} \int_0^{1/2} \frac{\sin(2\theta)}{r} \phi_{2n}(\eta^{-1}(t))(r,\theta) dr d\theta
\]

and

\[
I_k(t) := \frac{4}{\pi} \int_0^{\pi/2} \int_0^{1/2} \frac{\sin(2\theta)}{r} \phi_k(\eta^{-1}(t))(r,\theta) dr d\theta
\]

are the contributions to \( I(t) \) from the bubble initially located at the scales \( 2^{-2n} \) and \( 2^{-k} \) respectively. Our strategy is to establish the following assertion inductively in \( k \): the ‘shape’ of the \( k \)th bubble essentially remains the same within the time scale \( t_k := c_1/S_k \) for some absolute constant \( c_1 > 0 \). Here what we mean by shape will be made precise later; for now we just mention that as a consequence, it follows that

\[
I_k(t) \gtrsim c I_k(0) \quad \text{for} \quad t \in [0, t_k], \tag{15}
\]

where \( c > 0 \) is a universal constant. Assuming for a moment that (15) holds, we obtain that

\[
\int_0^{t_k} I_k(t) dt \gtrsim t_k I_k(0) \gtrsim \frac{a_k}{S_k}.
\]

Then we have (since \( I_{2n}(t) \) is non-negative, we just drop it off)

\[
\int_0^{t^\ast} I(t) dt \gtrsim \sum_{k=1}^n \int_0^{\min(t, t^\ast)} I_k(t) dt \gtrsim \sum_{1 \leq k < n; c^* k} \frac{a_k}{S_k}
\]
owing to the non-negativity of each $I_k(t)$. We now observe that, by approximating the sum with a Riemann integral of the function $f(x) = 1/x$,

$$
\sum_{k=1}^{n} \frac{a_k}{S_k} \approx \log(S_n).
$$

(16)

More precisely, we have

$$
\sum_{k=1}^{n} \frac{a_k}{S_k} \leq 1 + \int_{a_1}^{S_n} \frac{dx}{x} = 1 + \ln(S_n) - \ln(a_1).
$$

On the other hand, under the assumption that $\{a_k\}$ is decreasing, we have $2f(S_{k+1}) \geq f(S_k)$ and

$$
2 \sum_{k=1}^{n} \frac{a_k}{S_k} \geq \int_{a_1}^{S_n} \frac{dx}{x} = \ln(S_n) - \ln(a_1).
$$

This gives (16). Now taking $k^*$ be the smallest number satisfying

$$
S_{k^*} t^* > c_1,
$$

(which exists by taking $n$ larger if necessary, since $t^* > 0$ and we are assuming that the sequence $S_{k^*}$ is divergent for $k^* \to \infty$)

$$
\sum_{k=1}^{k^*} \frac{a_k}{S_k} \approx \log(S_{k^*}) \approx \log \left( \frac{c_1}{t^*} \right).
$$

This gives (8) with some positive constant $c_0$. Similarly, assuming (15) for now, we obtain that when $0 \leq t \leq S_n^{-1}$,

$$
\partial_2 u_{\alpha,2}(t,0) \gtrsim S_n,
$$

and for $t \geq S_n^{-1}$,

$$
\partial_2 u_{\alpha,2}(t,0) \gtrsim \sum_{k,j \geq t} I_k(0) \gtrsim t^{-1},
$$

which is simply (7). Hence it only remains to prove that

$$
I_k(t) \gtrsim I_k(0), \quad t \in [0, t_4]
$$

uniformly in $k$. Below we shall formulate and prove a claim which implies the above lower bound.

Step 1: some preparations

We make some simple observations regarding the evolution of the bubbles. Recall from the definition of $\phi_0$ that restricted on to the positive quadrant, there exist ‘rectangles’

$$
\mathcal{R}_0 = \{(r, \theta) : \tau_1 < r < \tau_2, \bar{\theta}_1 < \theta < \bar{\theta}_2\}
$$

and

$$
\mathcal{R}_0 = \{(r, \theta) : \zeta_1 < r < \zeta_2, \bar{\theta}_1 < \theta < \bar{\theta}_2\}
$$
such that
\[ \phi_0 = 1 \text{ on } B_0, \quad \text{and} \quad \phi_0 = 0 \text{ outside of } \overline{B}_0. \]

We may set
\[ \frac{1}{2} < r_1 < r_2 < 2 \]
and
\[ \frac{\pi}{6} < \theta_1 < \theta_2 < \frac{\pi}{3}. \]

Now by simple scaling, with the \(2^{-k}\)-scaled rectangles \(B_k\) and \(\overline{B}_k\), we have
\[ \phi_k = 1 \text{ on } B_k, \quad \text{and} \quad \phi_k = 0 \text{ outside of } \overline{B}_k, \]
still restricted on the first quadrant (more precisely on \([0,1]^2\)). This time, take an even smaller rectangle:
\[ \overline{B}_0^* = \left\{ (r, \theta) : r_1^* < r < r_2^*, \theta_1^* < \theta < \theta_2^* \right\} \subset \overline{B}_0, \]
where we may set
\[ L_1^* = \frac{2r_1 + r_2}{3}, \quad L_2^* = \frac{r_1 + 2r_2}{3} \]
and similarly
\[ \theta_1^* = \frac{2\theta_1 + \theta_2}{3}, \quad \theta_2^* = \frac{\theta_1 + 2\theta_2}{3}. \]

Then as before define
\[ \overline{B}_k^* := \left\{ (r, \theta) : r_1^* < 2^k r < r_2^*, \theta_1^* < \theta < \theta_2^* \right\}. \]

Moreover, define
\[ \overline{A}_k := \left\{ (r, \theta) : 2^{-k-1} < r < 2^{1-k}, 0 < \theta < \frac{\pi}{2} \right\}. \]

We shall now prove the following

Claim. In the time interval \([0, t_k]\), the \(k\)th bubble remains \(a_k\) on the rectangle \(\overline{B}_k^*\) and vanishes outside \(\overline{A}_k\). Here \(t_k := c_1/S_k\) with \(c_1 > 0\) independent of \(k\).

This is what we mean by retaining the same ‘shape’. We now rewrite the evolution of the trajectories in polar coordinates, using (12). Given some \(x \in [0,1]^2\), we shall express the point \(\eta(t,x)\) using \(|\eta|\) and \(\theta(\eta)\). Then,
\[ \frac{d}{dt} |\eta| = u(t, \eta) \cdot \left( \frac{\cos(\theta(\eta))}{\sin(\theta(\eta))} \right) \]
\[ = \left| \eta \right| \left( \cos(2\theta(\eta))I(t, |\eta|) + (\cos(\theta(\eta))B_1 + \sin(\theta(\eta))B_2) \right) \]
and
\[ \frac{d}{dt} \theta(\eta) = \frac{u(t, \eta)}{|\eta|} \cdot \left( \frac{\sin(\theta(\eta))}{\cos(\theta(\eta))} \right) \]
\[ = \sin(2\theta(\eta))I(t, |\eta|) + (\sin(\theta(\eta))B_1 + \cos(\theta(\eta))B_2) \]
\]
where $B = (B_1, B_2)$ is from (12).

Step 2: Induction base case $k = 1$

To proceed, we recall a simple estimate of Yudovich (see e.g. [5] for a proof):

**Lemma 4.** Let $\omega(t) \in L^\infty([0, \infty) : L^\infty(\mathbb{T}^2))$ be a solution of the 2D Euler equations on $\mathbb{T}^2$, and $\eta$ be the associated flow map. Then for some absolute constant $c > 0$, we have

$$|x - x'|^{1 + c\|\omega\|_{L^\infty}} \leq |\eta(t, x) - \eta(t, x')| \leq |x - x'|^{1 - c\|\omega\|_{L^\infty}}, \quad (19)$$

for all $0 \leq t \leq 1$ and $|x - x'| \leq 1/2$.

We shall be concerned with the bubble $\phi_1$ and the trajectories $\eta(t, x)$ where $x \in \text{supp}(\phi_1)$.

Using the Yudovich estimate (19) with $s' = 0$, we see that such trajectories are trapped inside the region $\{2^{-2} \leq r\}$ during $[0, t_1]$ by choosing $c_1 > 0$ depending only on $\|\omega\|_{L^\infty}$ in (19). Similarly, trajectories starting from $\bigcup_{k \geq 1} \text{supp}(\phi_k)$ cannot cross the circle $\{r = 2^{-2}\}$. This results in a naive bound

$$I_1(t, |\eta|) \leq I_1(t, 2^{-2}) \lesssim a_2^{-2}\|\phi_1(\eta^{-1}(t))\|_{L^1} \lesssim a_1$$

on the same time interval, where

$$I_2(t, r) : = \frac{4}{\pi} \int_{0}^{t/2} \int_{2r}^{t^{1/2}} \frac{\sin(2\theta)}{s} a_2\phi_1(\eta^{-1}(t)(s, \theta))ds\,d\theta. \quad (20)$$

We use this to obtain slightly improved estimates on $|\eta|$ in (17):

$$\left| \frac{d}{dt} \ln \frac{1}{|\eta|} \right| \lesssim S_1$$

using $|B| \lesssim 1$. This guarantees that, given any small $\epsilon > 0$, by taking $c_1 = c_1(\epsilon) > 0$ small enough if necessary, we have

$$\left| \ln \frac{|\eta(0, x)|}{|\eta(t, x)|} \right| < \epsilon, \quad t \in [0, t_1],$$

recalling that $t_1 = c_1/S_1$. For the angle, we simply use

$$\left| \frac{d}{dt} \theta(\eta) \right| \lesssim S_1$$

to deduce

$$|\theta(\eta(t, x)) - \theta(\eta(0, x))| < \epsilon$$

again for $t \in [0, t_1]$ by taking $c_1 > 0$ smaller if necessary. Thus, a suitable choice of $\epsilon > 0$ (depending only on $t_1, t_2, \theta_1, B_2$) finishes the proof of the Claim for the case $k = 1$.

Step 3: Completing the induction

We now assume that for some $k_0 > 1$ the Claim has been proved for $k = 1, \ldots, k_0 - 1$. We are now concerned with the trajectories $\eta(t, x)$ where $t \leq t_{k_0}$ and $x \in \text{supp}(\phi_{k_0})$. The induction hypothesis guarantees that, as long as $2^{-(k_0 + 1)} < |\eta| < 2^{-(k_0 - 1)}$, we have that

$$\left| \frac{d}{dt} \ln \frac{1}{|\eta|} \right| \lesssim S_{k_0}$$

...
simply because $t_k$ is decreasing with $k$ and the hypothesis ensures that the contribution of $a_k \phi_k \circ \eta^{-1}(t)$ to the key integral (20) is bounded by $ca_k$ with some $c$ independent of $k$, for $k = 1, \ldots, k_0 - 1$. More precisely

$$I_k(t, |\eta|) \lesssim a_k 2^{-2k}\|\phi_k(\eta^{-1}(-))\|_{L^1} \lesssim a_k.$$  

Strictly saying, here, we use the even smaller rectangles $B^*_k$. Thus $c$ is depending on $\epsilon > 0$. This implies that

$$\left| \ln \frac{|\eta(0, x)|}{|\eta(t, x)|} \right| < \epsilon, \quad t \in [0, t_0]$$

for the same $\epsilon$ and $c_1$. Similarly, we can deduce

$$|\theta(\eta(t, x)) - \theta(\eta(0, x))| < \epsilon$$

on $[0, t_0]$. The proof of Claim is complete, which finishes the proof.

Remark 5. A few remarks are in order.

- **Large Lagrangian deformation occurs at the origin.** Proposition 2 shows that for bubbles satisfying $S_n \to \infty$ as $n \to \infty$, large Lagrangian deformation must occur, and it occurs even within a time interval that shrinks to zero for $n$ large. We emphasise that we can pinpoint the location of large Lagrangian deformation to be the origin (which was an open problem to the best of our knowledge), while using contradiction arguments it is possible (see [4, 5]), with less work, to show existence of large Lagrangian deformation (somewhere in the domain).

- **Dichotomy for bubbles.** Note that in the case when the sequence $a_k$ is summable, the initial vorticity belongs to the critical Besov space $B^0_{\infty,1}$ uniformly in $n$ (for the rigorous calculation, see [12] for example). There is uniqueness and existence in this space $B^0_{\infty,1}$ ([14]), which in particular guarantees that the corresponding velocity gradient is uniformly bounded in $n$ for a short time interval. Therefore, we have the following dichotomy for bubbles: short-time large Lagrangian deformation occurs if and only if the sequence $\{a_k\}$ is not summable.

- **Sharpness of the growth rate.** It can be shown that with the data in (5), we have

$$|D\eta(t, 0)| \leq C(c_2), \quad t \in [0, c_2/S_n]$$

for any fixed constant $c_2 > 0$. This follows from the well-posedness of the Euler equations with vorticity in $B^0_{\infty,1}$ ([13]) and the fact that $|\omega|_{B^0_{\infty,1}} \sim S_n$. Comparing this with (8), one sees that the lower bound is sharp at least during this time scale. Hence we must wait a bit longer to see large deformation at the origin.

- **Case of the continuum.** Our considerations equally apply well to the ‘continuum’ version of the bubbles; that is, we may take locally

$$\omega_{\text{cont}}(r, \theta) = \varphi_{\frac{r}{2^{-k}}}(g(r)\chi(\theta)), \quad 0 \leq r < 1/2$$

where $\chi \geq 0$ on $\theta \in [0, \pi/2]$ and $\chi(\theta) = -\chi(-\theta) = -\chi(\pi - \theta)$, and $g \geq 0$ is a bounded continuous function on $[0, 1/2] \to [0, 1]$. Here $a_k$ corresponds to $g(2^{-k})$ and $S_k$ to $\int_{1/2}^1 g(r)r^{-1}dr$. For an example, in the case $g(r) = |\ln r|^{-1/2 - \epsilon}$, $\omega \in (2\pi)\sin(2\theta)$ belongs to $H^1$ (considered explicitly in [5]), and using the method in this paper one can show that the corresponding solution escapes $H^1$ without appealing to a contradiction argument.
Proof of proposition 3. Recall that proposition 2 established creation of large Lagrangian deformation at the origin for the solution of 2D Euler with initial data \( \omega_{0\eta} \). We now prove that the deformation persists on the support of \( \phi_{2n} \) and use this observation to complete the proof of proposition 3.

For convenience, we set \( \omega_{0\eta}^\delta = \phi_{2n} \) and \( \omega_{0\eta}^\delta(t) := \omega_{0\eta}^\delta \circ \eta_n^{-1}(t) \). We observe that for any \( t \), \( \|\omega_{0\eta}(t)\|_{H^1} \geq \|\omega_{0\eta}^\delta(t)\|_{H^1} \). From the definition, \( \|\omega_{0\eta}^\delta\|_{H^1} \geq 1 \). Moreover, we note that for \( t \in [0, \delta] \), the support of \( \omega_{0\eta}^\delta(t) \) is contained in the ball of radius \( 2^{-n} \) centered at the origin. This can be easily shown using lemma 4 (we choose \( \delta = \frac{1}{2\pi|\omega_{0\eta}^\delta|_{H^1}} \)).

Recalling the simple estimate
\[
\|\nabla u_n(t)\|_{C^{0,\frac{1}{2}}} \lesssim \|\omega_n(t)\|_{C^{0,\frac{1}{2}}} \lesssim 2^{\frac{n}{2}},
\]
we have
\[
|\nabla u_n(t, x) - \nabla u_n(t, 0)| \lesssim 2^{\frac{n}{2}} |x|^\frac{1}{2},
\]
so that for \( x \in \operatorname{supp}(\omega_n^\delta(t)) \),
\[
|\nabla u_n(t, x) - \nabla u_n(t, 0)| \lesssim 1. \tag{21}
\]
In particular, since \( \partial_1 u_{n,2}(t, 0) = \partial_2 u_{n,1}(t, 0) = 0 \) by the odd symmetry,
\[
|\partial_1 u_{n,2}(t, x)| = |\partial_2 u_{n,1}(t, x)| \lesssim 1. \tag{22}
\]
We now fix some \( x \in \operatorname{supp}(\omega_n^\delta) \), and write for \( \eta = \eta(t, x) \) the system of ODEs satisfied by \( \partial_2 \eta_1(t) \) and \( \partial_2 \eta_2(t) \) (dropping the subscript \( n \) for simplicity)
\[
\begin{align*}
\frac{d}{dt} \partial_2 \eta_1(t) &= \partial_1 u_2(t, \eta) \partial_2 \eta_1(t) + \partial_2 u_1(t, \eta) \partial_2 \eta_2(t), \\
\frac{d}{dt} \partial_2 \eta_2(t) &= \partial_1 u_2(t, \eta) \partial_2 \eta_1(t) + \partial_2 u_2(t, \eta) \partial_2 \eta_2(t).
\end{align*} \tag{23}
\]
Using the lower bound of \( \partial_2 u_2(t, 0) \) given in (7) together with (21) and (22), we can estimate from the above that
\[
\partial_2 \eta_2(t, x) \gtrsim (S_n t)^{\frac{1}{2n}}. \tag{24}
\]
To prove this, it suffices to propagate in time the following estimate
\[
\partial_2 \eta_2(t) \gtrsim 10 |\partial_2 \eta_1(t)| \tag{25}
\]
using (23), since then we obtain from the equation for \( \partial_2 \eta_2 \) that
\[
\frac{d}{dt} \partial_2 \eta_2(t) \geq (\partial_2 u_2(t, 0) - 10 C |\partial_2 \eta_2(t)|) \partial_2 \eta_2(t) \geq \frac{1}{2} |\partial_2 u_2(t, 0) | \partial_2 \eta_2(t), \tag{26}
\]
(with \( n \) sufficiently large and \( \delta > 0 \) small so that \( \partial_2 u_2(t, 0) \geq 20 C \) on \( t \in [0, \delta] \) and integrating in time,
\[
\partial_2 \eta_2(t) \geq \exp \left( \frac{1}{2} \int_0^t \partial_2 u_2(\tau, 0) d\tau \right),
\]
which gives (24) from (7). We give details of the proof of (25) (cf [8, lemma 2.4]). For simplicity we fix some \( x \) and set \( B = \partial_2 \eta(t, x) \) and \( A = \partial_1 \eta(t, x) \). We need to propagate in time that (i) \( B - 10A \geq 0 \) and (ii) \( B + 10A \geq 0 \). Note that when both (i) and (ii) holds, we have \( B \geq 0 \) in particular. Moreover, \( \partial_2 u_2(t, \eta) > 0 > \partial_1 u_1(t, \eta) = -\partial_2 u_2(t, \eta) \). At \( t = 0 \), \( B = 1 \) and \( A = 0 \) so that (i) and (ii) are trivially satisfied with strict inequalities. By continuity in time, both inequalities hold for some time interval. Assume towards contradiction that there exists some \( 0 < t' \leq \delta \) such that either (i) or (ii) fails. Then there exists some maximal time interval \( [0, T] \) with \( 0 < T < t' \) such that (i) and (ii) holds. At \( t = T \), either \( B = 10A \) or \( B = -10A \). Let us consider the case \( B = 10A \) at \( t = T \). Since \( B \geq 10|A| \) until \( t = T \), we clearly have \( B, A \geq 0 \) at \( t = T \). Furthermore, using (26), we actually deduce that \( B > 0 \) at \( t = T \). From (23), evaluating at \( t = T \)

\[
\frac{d}{dt}(B - 10A) = (\partial_2 u_2 - 10\partial_2 u_1)(B + 10A) + (100\partial_2 u_1 + \partial_1 u_2)A \\
\geq (\partial_2 u_2 - 10\partial_2 u_1)(B + 10A) - |100\partial_2 u_1 + \partial_1 u_2|A \\
\geq \frac{1}{2}(\partial_2 u_2 - 10\partial_2 u_1)(B + 10A) > 0
\]

using that \( B(t = T) > 0 \) and

\[
2|100\partial_2 u_1 + \partial_1 u_2|A \leq (\partial_2 u_2 - 10\partial_2 u_1)10A \leq (\partial_2 u_2 - 10\partial_2 u_1)(B + 10A).
\]

This follows from (21), (22), and taking \( n \) sufficiently large and \( \delta > 0 \) small if necessary so that \( \partial_2 u_2 > 400|\partial_2 u_1| \). Moreover, the right-hand side is continuous in time, so that \( \frac{d}{dt}(B - 10A) > 0 \) for some time interval containing \( t = T \). This is a contradiction to maximality of \( T \). In the case \( B = -10A \) at \( t = T \), we simply repeat the same argument, observing that

\[
\frac{d}{dt}(B + 10A) = (\partial_2 u_2 + 10\partial_2 u_1)(B - 10A) + (100\partial_2 u_1 + \partial_1 u_2)A > 0.
\]

Finally,

\[
\nabla \omega_0^\delta(t) = \nabla \omega_0^\delta \circ \eta_n^{-1} \nabla \eta_n^{-1}
\]

and taking the second component and recalling the representation formula for \( \nabla \eta_n^{-1} \) in terms of \( \nabla \eta_n \) (see (11)) and \( \| \nabla \omega_0^\delta \|_{L^2} \gtrsim 1 \), we obtain

\[
\partial_2 \omega_0^\delta(t) = \partial_2 \omega_0^\delta \circ \eta_n^{-1} \partial_1 \eta_n^{-1} - \partial_1 \omega_0^\delta \circ \eta_n^{-1} \partial_2 \eta_n^{-1},
\]

and integrating over the region where \( |\partial_2 \omega_0^\delta| \gtrsim 2^{2n} \) and \( |\partial_1 \omega_0^\delta| < \frac{1}{2}|\partial_2 \omega_0^\delta| \) [such a region has area of size \( O(2^{-4n}) \)], we obtain the lower bound

\[
\| \omega(t) \|_{H^1} \geq \| \partial_2 \omega_0^\delta(t) \|_{L^2} \gtrsim \inf_{x \in \text{supp}(\omega_0^\delta)} |\partial_1 \eta_n^{-1}| \gtrsim (S_n(t))^{\frac{3}{2}},
\]

which finishes the proof.

4. Conclusion

We prepared small-scale vortex blob and multi-scale odd-symmetric vortices for the initial data, and showed that the corresponding 2D Euler flow creates vortex thinning. In turn, using
this thinning, we showed that the corresponding 2D Navier–Stokes flow induces the enstrophy dissipation with strictly slower decaying rate than $\text{Re}^{-1}$.

Acknowledgments

We thank Professor A Mazzucato for inspiring communications and telling us about the article [7]. Research of TY was partially supported by Grant-in-Aid for Young Scientists A (17H04825), Grant-in-Aid for Scientific Research B (15H03621, 17H02860, 18H01136, 18H01135 and 20H01819), Japan Society for the Promotion of Science (JSPS). IJ has been supported by a KIAS Individual Grant MG066202 at Korea Institute for Advanced Study, the Science Fellowship of POSCO TJ Park Foundation, and the National Research Foundation of Korea grant (No. 2019R1F1A1058486). We sincerely thank the anonymous referees for a careful reading of the earlier manuscript and providing valuable comments as well as pointing out several errors.

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