A FULL-TWIST INEQUALITY FOR THE $\nu^+$-INVARIANT

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Abstract. Hom and Wu introduced a knot concordance invariant called $\nu^+$, which dominates many concordance invariants derived from Heegaard Floer homology. In this paper, we give a full-twist inequality for $\nu^+$. By using the inequality, we extend Wu’s cabling formula for $\nu^+$ (which is proved only for particular positive cables) to all cables in the form of an inequality. In addition, we also discuss $\nu^+$-equivalence, which is an equivalence relation on the knot concordance group. We introduce a partial order on $\nu^+$-equivalence classes, and study its relationship to full-twists.

1. Introduction

1.1. Full-twist inequality for $\nu^+$-invariant. The $\nu^+$-invariant is a non-negative integer valued knot concordance invariant defined by Hom and Wu [4]. The $\nu^+$-invariant dominates many concordance invariants derived from Heegaard Floer homology, in terms of obstructions to sliceness. In fact, Hom proves in [3] that for a given knot $K$ in $S^3$, if both $\nu^+(K)$ and $\nu^+(-K)$ are zero, then all invariants $\tau, \nu, V_k, \gamma, \varepsilon, d(S^3_{p/q}(-)), i$ and $\Upsilon(t)$ agree with their values on the unknot. (Here $-K$ denotes the orientation reversed mirror of $K$.) Hence the $\nu^+$-invariant plays a special role among knot concordance invariants derived from Heegaard Floer homology.

In this paper, we give a full-twist inequality for the $\nu^+$-invariant. To state the inequality, we first describe full-twist operations. Let $K$ be a knot in $S^3$ and $D$ a disk in $S^3$ which intersects $K$ in its interior. By performing $(−1)$-surgery along $\partial D$, we obtain a new knot $J$ in $S^3$ from $K$. Let $n = \text{lk}(K, \partial D)$. Since reversing the orientation of $D$ does not affect the result, we may assume that $n \geq 0$. Then we say that $K$ is deformed into $J$ by a positive full-twist with $n$-linking, and call such an operation a full-twist operation. The main theorem of this paper is stated as follows. (Here $\#$ denotes connected sum.)

**Theorem 1.1.** Suppose that a knot $K$ is deformed into a knot $J$ by a positive full-twist with $n$-linking. If $n = 0$, then $\nu^+(J#(-K)) = 0$. Otherwise, we have

$$\frac{(n-1)(n-2)}{2} \leq \nu^+(J#(-K)) \leq \frac{n(n-1)}{2}.$$ 

Remark. For any coprime $p, q > 0$, let $T_{p,q}$ denote the $(p, q)$-torus knot. Then we note that $\nu^+(T_{p,q}) = (p-1)(q-1)/2$ [4], and hence the inequality in Theorem 1.1 implies

$$\nu^+(T_{n,n-1}#K#(-K)) \leq \nu^+(J#(-K)) \leq \nu^+(T_{n,n+1}#K#(-K)).$$

Since both $T_{n,n-1}#K$ and $T_{n,n+1}#K$ are obtained from $K$ by a positive full-twist with $n$-linking, the inequalities are best possible.
Here we note that Theorem 1.1 gives an inequality for $J \# (-K)$ rather than $J$ and $K$. However, by subadditivity of $\nu^+$ [1], we also have the following result for $J$ and $K$.

**Theorem 1.2.** Suppose that $K$ is deformed into $J$ by a positive full-twist with $n$-linking. If $n = 0$, then $\nu^+(J) \leq \nu^+(K)$. Otherwise, we have

$$\frac{(n-1)(n-2)}{2} - \nu^+(-K) \leq \nu^+(J) \leq \frac{n(n-1)}{2} + \nu^+(K).$$

Furthermore, we can use Theorem 1.2 to obtain the following lower bound for the $\nu^+$-invariant of all cable knots (including negative cables).

**Theorem 1.3.** For any knot $K$ and coprime integers $p, q$ with $p > 0$, we have

$$\nu^+(K_{p,q}) \geq p\nu^+(K) + \frac{(p-1)(q-1)}{2},$$

where $K_{p,q}$ denotes the the $(p,q)$-cable of $K$.

Note that Wu proves in [15] that the equality holds in the case where $p, q > 0$ and $q \geq (2\nu^+(K) - 1)p - 1$. Hence Theorem 1.3 partially extends his result to arbitrary cables. Furthermore, Theorem 1.3 also enables us to extend Wu’s 4-ball genus bound for particular positive cable knots to all positive cable knots.

**Corollary 1.4.** If $\nu^+(K) = g_4(K)$, then for any coprime $p, q > 0$, we have

$$\nu^+(K_{p,q}) = g_4(K_{p,q}) = pg_4(K) + \frac{(p-1)(q-1)}{2}.$$  

As an application, for instance, we can determine the 4-ball genus for all positive cables of the knot $T_{2,3}\#T_{2,3}\#T_{2,3}\#(-T_{2,3})$. This example is used in [1] to show that $\nu^+ \neq \tau$. Remark that the $\tau$-invariant cannot determine the 4-ball genus for any positive cable of the knot. Also note that this generalizes [4, Proposition 3.5] and Wu’s result in the introduction of [13].

### 1.2. A partial order on $\nu^+$-equivalence classes.

Let $C$ denote the knot concordance group. For two elements $x, y \in C$, we say that $x$ is $\nu^+$-equivalent to $y$ if the equalities $\nu^+(x - y) = 0$ and $\nu^+(y - x) = 0$ hold. In [3], Hom proves that $\nu^+$-equivalence is an equivalence relation and the quotient has a group structure as a quotient group of $C$ (we denote it by $C_{\nu^+}$). Furthermore, it follows from [3, Theorem 1] that the invariants $\tau, \Upsilon, V_k, \nu^+$ and $d(S^3_{p/q}(:i))$ are invariant under not only knot concordance but also $\nu^+$-equivalence. In particular, we can regard these invariants as maps

$$\tau : \ C_{\nu^+} \rightarrow \mathbb{Z},$$

$$\Upsilon : \ C_{\nu^+} \rightarrow \text{Cont}([0, 2]),$$

$$V_k : \ C_{\nu^+} \rightarrow \mathbb{Z}_{\geq 0},$$

$$\nu^+ : \ C_{\nu^+} \rightarrow \mathbb{Z}_{\geq 0},$$

and

$$d(S^3_{p/q}(:i)) : \ C_{\nu^+} \rightarrow \mathbb{Q},$$

respectively. (Here, $\text{Cont}([0, 2])$ denotes the set of continuous functions from the closed interval $[0, 2]$ to $\mathbb{R}$.)

The second aim of this paper is to introduce a partial order on $C_{\nu^+}$ and discuss its relationship to full-twists. The precise definition of the partial order is as follows.

**Definition.** For two elements $x, y \in C_{\nu^+}$, we write $x \leq y$ if $\nu^+(x - y) = 0$. 

Note that the equality in the above definition is one of the equalities in the definition of \( \nu^+ \)-equivalence, and so this partial order seems to be very natural. In fact, we can prove the following proposition.

**Proposition 1.5.** The relation \( \leq \) is a partial order on \( C_{\nu^+} \) with the following properties:

1. For elements \( x, y, z \in C_{\nu^+} \), if \( x \leq y \), then \( x + z \leq y + z \).
2. For elements \( x, y \in C_{\nu^+} \), if \( x \leq y \), then \( -y \leq -x \).
3. For coprime integers \( p, q > 0 \), \( k \in \mathbb{Z} \geq 0 \) and \( 0 \leq i \leq p - 1 \), all of \( \tau, \nu, -Y, V_k, \nu^+ \) and \( -d(S^3_{p/q}(\cdot), i) \) preserve the partial order.

Here the third assertion in Proposition 1.5 implies that there are many algebraic obstructions to one element of \( C_{\nu^+} \) being less than another. On the other hand, the following theorem establishes similar obstructions in terms of geometric deformations.

**Theorem 1.6.** Suppose that \( K \) is deformed into \( J \) by a positive full-twist with \( n \)-linking.

1. If \( n = 0 \) or \( 1 \), then \( [J]_{\nu^+} \leq [K]_{\nu^+} \).
2. If \( n \geq 3 \), then \( [J]_{\nu^+} \not\leq [K]_{\nu^+} \). In particular, if the geometric intersection number between \( K \) and \( D \) is equal to \( n \), then \( [J]_{\nu^+} > [K]_{\nu^+} \).

Here \( [K]_{\nu^+} \) denotes the \( \nu^+ \)-equivalence class of a knot \( K \), and the symbol \( > \) means \( x \geq y \) and \( x \neq y \) for elements \( x, y \in C_{\nu^+} \).

In the above theorem, we can see that only the case of \( n = 2 \) tells us nothing about the partial order. This follows from the fact that Theorem 1.1 gives \( 0 \leq \nu^+(x - y) \leq 1 \) for \( n = 2 \) and hence we can show neither \( \nu^+(x - y) = 0 \) nor \( \nu^+(x - y) \neq 0 \).

We also mention the relationship between our partial order and satellite knots. Let \( P \) be a knot in a standard solid torus \( V \subset S^3 \) with the longitude \( l \), and \( K \) a knot in \( S^3 \). For \( n \in \mathbb{Z} \), let \( e_n : V \to S^3 \) be an embedding so that \( e(V) \) is a tubular neighborhood of \( K \) and \( \text{lk}(K, e_n(l)) = n \). Then we call \( e_n(P) \) the \( n \)-twisted satellite knot of \( K \) with pattern \( P \), and denote it by \( P(K, n) \). Furthermore, if \( P \) represents \( m \) times generators of \( H_1(V; \mathbb{Z}) \) for \( m \geq 0 \), then we denote \( w(P) := m \).

It is proved in [5, Theorem B] that the map \( [K]_{\nu^+} \rightarrow [P(K, n)]_{\nu^+} \) for any pattern \( P \) with \( w(P) \neq 0 \). We extend their theorem to all satellite knots, and show that those maps preserve our partial order.

**Proposition 1.7.** For any pattern \( P \) and \( n \in \mathbb{Z} \), the map \( P_n : C_{\nu^+} \to C_{\nu^+} \) defined by \( P_n([K]_{\nu^+}) := [P(K, n)]_{\nu^+} \) is well-defined and preserve the partial order \( \leq \).

By Proposition 1.7, we obtain infinitely many order-preserving maps on \( C_{\nu^+} \) which have geometric meaning. Now it is an interesting problem to compare these satellite maps. Theorem 1.6 tells us the relationship among the maps \( \{P_n\}_{n \in \mathbb{Z}} \) for some particular patterns.

**Corollary 1.8.** Let \( P \) be a pattern.

1. If \( w(P) = 0 \) or \( 1 \), then the inequality \( P_m(x) \geq P_n(x) \) holds for any integers \( m < n \) and \( x \in C_{\nu^+} \).
2. If the geometric intersection number between \( P \) and the meridian disk of \( V \) is equal to \( w(P) \) and \( w(P) \geq 3 \), then \( P_m(x) < P_n(x) \) for any \( m < n \) and \( x \in C_{\nu^+} \).
1.3. The idea of proofs: study of slice knots in \( \mathbb{C}P^2 \). In this subsection, we explain the idea of our proof of Theorem 1.1. We start from an interpretation of full-twist operations in 4-dimensional topology.

When a knot \( K \) is deformed into a knot \( J \) by a positive full-twist with \( n \)-linking, we can see that the knot \( J \# (-K) \) bounds a disk \( D \) in \( \text{punc}\mathbb{C}P^2 \). (Here, for a closed 4-manifold \( X \), \( \text{punc}X \) denotes \( X \) with an open 4-ball deleted.) In particular, the disk \( D \) represents \( n \) times a generator of \( H_2(\text{punc}\mathbb{C}P^2, \partial(\text{punc}\mathbb{C}P^2); \mathbb{Z}) \cong \mathbb{Z} \). In this situation, we consider Ni-Wu’s \( V_k \)-sequence [7].

**Proposition 1.9.** Suppose that a knot \( K \) in \( S^3 \) bounds a disk \( D \) in \( \text{punc}\mathbb{C}P^2 \) such that \([D, \partial D] = n\gamma \in H_2(\text{punc}\mathbb{C}P^2, \partial(\text{punc}\mathbb{C}P^2); \mathbb{Z}) \) for a generator \( \gamma \) and some integer \( n \geq 0 \).

1. If \( n = 0 \), then \( V_0(K) = 0 \).
2. If \( n \) is odd, then for any \( 0 \leq j \leq \frac{n-1}{2} \), we have

\[
V_{nj}(K) = \frac{1}{2} \left( \frac{n-1}{2} - j \right) \left( \frac{n-1}{2} - j + 1 \right).
\]

3. If \( n \) is even and \( n > 0 \), then for any \( 0 \leq j \leq \frac{n}{2} - 1 \), we have

\[
V_{\frac{n}{2} + nj}(K) = \frac{1}{2} \left( \frac{n}{2} - j \right) \left( \frac{n}{2} - j - 1 \right).
\]

In fact, Theorem 1.1 is an immediate consequence of Proposition 1.9 and this proposition is much stronger than Theorem 1.1 when one wants obstructions to the existence of a positive full-twist between two knots. We will study the strength of Proposition 1.9 in future work.

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2. Preliminaries

In this section, we recall some invariants derived from Heegaard Floer homology.

2.1. Correction terms. Ozsváth and Szabó [10] introduced a \( \mathbb{Q} \)-valued invariant \( d \) (called the correction term) for rational homology 3-spheres endowed with a Spin\(^c\) structure. It is proved that the correction term is invariant under Spin\(^c\) rational homology cobordism. In particular, the following proposition holds.

**Proposition 2.1.** If a rational homology 3-sphere \( Y \) bounds a rational homology 4-ball \( W \), then for any Spin\(^c\) structure \( s \) over \( W \), we have

\[
d(Y, s|_Y) = 0,
\]

where \( s|_Y \) denotes the restriction of \( s \) to \( Y \).
2.2. $V_k$-sequence and $\nu^+$-invariant. The $V_k$-sequence $\nu^+ : \{V_k(K)\}_{k \geq 0}$ is a family of $\mathbb{Z}_{\geq 0}$-valued knot concordance invariants defined by Ni and Wu. In particular, $\nu^+(K) := \min \{ k \geq 0 \mid V_k(K) = 0 \}$ is known as the $\nu^+$-invariant.

In [7], Ni and Wu prove that the set $\{V_k(K)\}_{k \geq 0}$ determines all correction terms of the $p/q$-surgery along $K$ for any coprime $p, q > 0$. Let $S^3_{p/q}(K)$ denote the $p/q$-surgery along $K$. Note that there is a canonical identification between the set of Spin$^c$ structures over $S^3_{p/q}(K)$ and $\{ i \mid 0 \leq i \leq p-1 \}$. (This identification can be made explicit by the procedure in [11, Section 4, Section 7].)

Proposition 2.2 ([7 Proposition 1.6]). Suppose $p, q > 0$, and fix $0 \leq i < p$. Then

$$d(S^3_{p/q}(K), i) = d(S^3_{p/q}(O), i) - 2V_{\min}(\lfloor \frac{i}{q} \rfloor, \lfloor \frac{p+i-1}{q} \rfloor)(K),$$

where $O$ denotes the unknot and $\lfloor \cdot \rfloor$ is the floor function.

Here we modify the right hand side of the equality by using the fact that $\{V_k(K)\}_{k \geq 0}$ satisfy the inequality $V_{k+1}(K) \leq V_k(K)$ for each $k \geq 0$. In particular, for integer surgeries, we have the following formula.

Corollary 2.3. For any $p > 0$ and $0 \leq i < p$, we have

$$d(S^3_p(K), i) = d(S^3_p(O), i) - 2V_{\min}(i, p-i)(K).$$

3. Proof of Proposition 1.9

In this section, we prove Proposition 1.9. We start from the following lemma.

Lemma 3.1. Suppose that a knot $K$ satisfies the assumption of Proposition 1.9 and $n > 0$. Then $S^3_{n^2}(K)$ bounds a rational homology 4-ball.

Proof. Consider a standard handle decomposition $\overline{CP^2} = h^0 \cup h^2 \cup h^4$, where $h^i$ denotes its unique $i$-handle for $i \in \{0, 2, 4\}$. Think of punctured $\mathbb{CP}^2$ as $\overline{CP^2} \setminus \text{Int } h^0$, and suppose that $K$ lies in $\partial (\overline{CP^2} \setminus \text{Int } h^0) = -\partial h^0$. Then we can think of a tubular neighborhood of $D$ (with orientation reversed) as an $(-n^2)$-framed 2-handle $\tilde{h}^2$ attached to $h^0$ along $-K$ (see Figure 1). Let $X := h^0 \cup h^2$. Then $X$ is a codimension 0 sub-manifold of $\overline{CP^2}$ which satisfies:

(1) $H_*(X; \mathbb{Z}) \cong H_*(S^2; \mathbb{Z})$,

(2) the induced map $i_* : H_2(X; \mathbb{Z}) \to H_2(\overline{CP^2}; \mathbb{Z})$ from the inclusion is non-trivial, and

(3) $\partial X = S^3_{-n^2}(-K)$.

These imply that for the exterior $W := \overline{CP^2} \setminus \text{Int } X$, we have $\partial W = -S^3_{-n^2}(-K) \cong S^3_{n^2}(K)$ and $H_*(W; \mathbb{Q}) \cong H_*(B^4; \mathbb{Q})$. \hfill \square

In light of Proposition 2.2, Lemma 3.1 shows that the correction term of some Spin$^c$ structures over $S^3_{n^2}(K)$ are zero. In fact, it follows from [8 Proposition 4.1] that the number of such Spin$^c$ structures is at least $n$.

Lemma 3.2. If $S^3_{n^2}(K)$ bounds a rational homology 4-ball, then there exists a subset $S \subset \{ i \mid 0 \leq i < n^2 \}$ such that $|S| = n$ and for any element $i \in S$, we have $d(S^3_{n^2}(K), i) = 0$. 
We will determine the subset $S$. Let $f_p : \{i \mid 0 \leq i < p\} \rightarrow \mathbb{Q}$ be a map defined by
\[ f_p(i) = \frac{-p + (p - 2i)^2}{4p}. \]
Actually, $f_p(i)$ is the value of $d(S^3_p(O), i)$. By Corollary 2.3, we can see that if an element $i \in \{0 \leq i < n^2\}$ belongs to $S$, then $f_{n^2}(i)$ must be an even integer. So we next observe when $f_{n^2}(i)$ is an even integer.

**Lemma 3.3.** If $n$ is odd, then
\[ f_{n^2}(i) \in 2\mathbb{Z} \iff i \in n\mathbb{Z}. \]
If $n$ is even, then
\[ f_{n^2}(i) \in 2\mathbb{Z} \iff i - \frac{n}{2} \in n\mathbb{Z}. \]

**Proof.** We first assume that $n$ is odd. It is easy to check that the following equalities hold:
\[ f_{n^2}(i) = \left( \frac{n - 1}{2} - \frac{i}{n} \right) \left( \frac{n - 1}{2} - \frac{i}{n} + 1 \right) \]
\[ = \frac{(n + 1)(n - 1)}{4} - i + \left( \frac{i}{n} \right)^2. \]

Then the equality (1) implies that $f_{n^2}(i) \in 2\mathbb{Z}$ if $i \in n\mathbb{Z}$, while the equality (2) implies that $f_{n^2}(i) \in \mathbb{Z}$ only if $(i/n)^2 \in \mathbb{Z}$. Since $(i/n)^2 \in \mathbb{Z}$ if and only if $i \in n\mathbb{Z}$, this proves Lemma 3.3 for odd $n$.

Next we assume that $n$ is even. We can see that the following equalities also hold:
\[ f_{n^2}(i) = \left( \frac{n}{2} - \frac{i - n}{n} \right) \left( \frac{n}{2} - \frac{i - n}{n} - 1 \right) \]
\[ = \left( \frac{n}{2} \right)^2 - i + \left( \frac{i - n}{n} \right) + \left( \frac{i - n}{n} \right)^2. \]
Let \( x = (i - n/2)/n \). Then the equality (3) implies that \( f_n^2(i) \in 2\mathbb{Z} \) if \( x \in \mathbb{Z} \), while the equality (4) implies that \( f_n^2(i) \in \mathbb{Z} \) only if \( x + x^2 \in \mathbb{Z} \). Furthermore, it is not hard to verify that \( x + x^2 \in \mathbb{Z} \) if and only if \( x \in \mathbb{Z} \). This completes the proof. \( \square \)

Now we have

\[
S \subset \{0 \leq i < n^2 \mid f_n^2(i) \in 2\mathbb{Z}\} = \begin{cases} 
\{0 \leq i < n^2 \mid i \in n\mathbb{Z}\} & (n: \text{odd}) \\
\{0 \leq i < n^2 \mid i - n/2 \in n\mathbb{Z}\} & (n: \text{even})
\end{cases}.
\]

However, for any \( n \), the order of the rightmost set is \( n \). This implies

\[
S = \begin{cases} 
\{0 \leq i < n^2 \mid i \in n\mathbb{Z}\} & (n: \text{odd}) \\
\{0 \leq i < n^2 \mid i - n/2 \in n\mathbb{Z}\} & (n: \text{even})
\end{cases}.
\]

Hence, combining the above three lemmas, we obtain the following lemma.

**Lemma 3.4.** Suppose that a knot \( K \) satisfies the assumption of Proposition 1.9 and \( n > 0 \). If \( n \) is odd, then we have

\[
d(S_n^3(K), nj) = 0 \quad (0 \leq j \leq n - 1).
\]

If \( n \) is even, then we have

\[
d(S_n^3(K), n/2 + nj) = 0 \quad (0 \leq j \leq n - 1).
\]

Now let us prove Proposition 1.9.

**Proof of Proposition 1.9.** We first suppose that \( n = 0 \). Then, the knot \( K \) bounds a null-homologous disk in punc \( \mathbb{C}P^2 \), whose intersection form is negative definite. Such a knot is studied in author’s paper [14], and it is proved that \( V_0(K) = 0 \). Hence the assertion (1) of Proposition 1.9 holds.

To prove the other two assertions, we use the equality

\[
d(S_n^3(K), i) = f_n^2(i) - 2V_i(K) \quad (0 \leq i \leq n^2/2),
\]

which is obtained by Corollary 2.3. We first consider the case where \( n > 0 \) is odd and \( 0 \leq j \leq (n - 1)/2 \). Then the inequalities \( 0 \leq nj \leq n(n - 1)/2 < n^2/2 \) hold, and hence the equality (5) and Lemma 3.4 give

\[
0 = f_n^2(nj) - 2V_{nj}(K).
\]

This equality and the equality (1) prove the assertion (2) of Proposition 1.9.

We next consider the case where \( n > 0 \) is even and \( 0 \leq j \leq n/2 - 1 \). Then the inequalities \( 0 \leq n/2 + nj \leq n^2/2 - n/2 < n^2/2 \) hold, and hence the equality (5) and Lemma 3.4 give

\[
0 = f_n^2(n/2 + nj) - 2V_{n/2 + nj}(K).
\]

This equality and the equality (3) prove the assertion (3) of Proposition 1.9. \( \square \)

**Remark.** We can prove Lemma 3.4 by taking suitable Spin \( c \) structures over \( \mathbb{C}P^2 \) and restricting them to \( W \). This alternate proof seems to be natural rather than the original proof, while the original proof also shows that any other correction term of \( S_n^3(K) \) is not lying even in \( \mathbb{Z} \).
4. PROOF OF FULL-TWIST INEQUALITIES

In this section, we prove Theorem 1.1 and Theorem 1.2. We first prove Theorem 1.1.

**Proof of Theorem 1.1.** By the assumption of Theorem 1.1 and the definition of a full-twist operation, there exists a disk \( D \) in \( S^3 \) which intersects \( K \) in its interior, and after \((-1)\)-surgery along \( \partial D \), we obtain \( J \) from \( K \). Consider a standard handle decomposition \( \mathbb{CP}^2 = h^0 \cup h^2 \cup h^4 \). Then we have an annulus \( A \) properly embedded in \( \mathbb{CP}^2 \setminus (\text{Int } h^0 \cup \text{Int } h^4) \) such that \( (\mathbb{CP}^2 \setminus (\text{Int } h^0 \cup \text{Int } h^4), A) \) is a pair of cobordism from \((\partial h^0, K)\) to \((\partial(\mathbb{CP}^2 \setminus \text{Int } h^4), J)\) (see Figure 2). Furthermore, the annulus \( A \) induces an annulus \( A' \) in \( \mathbb{CP}^2 \setminus (\text{Int } h^0 \cup \text{Int } h^4) \), which connects \( (\partial h^0, K \# (-K)) \) to \((\partial(\mathbb{CP}^2 \setminus \text{Int } h^4), J \# (-K))\) (Figure 3). Since \( K \# (-K) \) bounds a disk \( D' \) in \( h^0 \cong B^4 \), by gluing \( D' \) with \( A' \) along \( K \# (-K) \), we have a disk \( D'' \) in \( \partial(\mathbb{CP}^2 \setminus \text{Int } h^4) \) with boundary \( J \# (-K) \). Here we note that the disk \( D'' \) represents \( n\gamma \in H_2(\mathbb{CP}^2 \setminus h^4, \partial(\mathbb{CP}^2 \setminus h^4); \mathbb{Z}) \) for a generator \( \gamma \) and \( n = \text{lk}(K, \partial D) \geq 0 \).

Now we can apply Proposition 1.9 to the pair \((D'', J \# (-K))\). In particular, if \( n = 0 \), then \( V_0(J \# (-K)) = 0 \), and we have

\[
\nu^+ (J \# (-K)) = \min \{ k \geq 0 \mid V_k(J \# (-K)) = 0 \} = 0.
\]

We consider the case where \( n \) is odd. Then for any \( 0 \leq j \leq \frac{n-1}{2} \), we have

\[
V_{nj}(J \# (-K)) = \frac{1}{2} \left( \frac{n-1}{2} - j \right) \left( \frac{n-1}{2} - j + 1 \right).
\]

In particular, the equality for \( j = (n-1)/2 \) gives \( V_{\frac{n(n-1)}{2}}(J \# (-K)) = 0 \). Moreover, if \( n > 1 \), then the equality for \( j = (n-3)/2 \) gives \( V_{\frac{n(n-3)}{2}}(J \# (-K)) = 1 \). These imply that for \( n > 1 \), we have

\[
\frac{(n-1)(n-2)}{2} = \frac{n(n-3)}{2} + 1 \leq \nu^+(J \# (-K)) \leq \frac{n(n-1)}{2}.
\]

On the other hand, if \( n = 1 \), then the equality \( V_{\frac{n(n-1)}{2}}(J \# (-K)) = 0 \) directly shows

\[
\frac{(n-1)(n-2)}{2} = 0 \leq \nu^+(J \# (-K)) \leq \frac{n(n-1)}{2}.
\]
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We next consider the case where $n$ is even and $n > 0$. In this case, Proposition 1.9 gives

$$
V_{\frac{n}{2} + n j}(K) = \frac{1}{2} \left( \frac{n}{2} - j \right) \left( \frac{n}{2} - j - 1 \right)
$$

for any $0 \leq j \leq \frac{n}{2} - 1$. In particular, the equality for $j = n/2 - 1$ gives

$$
V_{\frac{n}{2}}(J#(-K)) = 0.
$$

Moreover, if $n > 2$, then the equality for $j = n/2 - 2$ gives

$$
V_{\frac{n}{2} - \frac{3}{2}}(J#(-K)) = 1.
$$

These imply that for $n > 2$, we have

$$
\frac{(n - 1)(n - 2)}{2} = \frac{n^2}{2} - \frac{3n}{2} + 1 \leq \nu^+(J#(-K)) \leq \frac{n^2}{2} - \frac{n}{2} = \frac{n(n - 1)}{2}.
$$

On the other hand, if $n = 2$, then the equality $V_{\frac{n}{2} - \frac{3}{2}}(J#(-K)) = 0$ directly shows

$$
\frac{(n - 1)(n - 2)}{2} = 0 \leq \nu^+(J#(-K)) \leq \frac{n^2}{2} - \frac{n}{2} = \frac{n(n - 1)}{2}.
$$

This completes the proof. \( \square \)

We next prove Theorem 1.2. To prove it, we use the following theorem.

**Theorem 4.1** ([1, Theorem 1.4]). For any two elements $x, y \in C_{\nu^+}$, we have

$$
\nu^+(x + y) \leq \nu^+(x) + \nu^+(y).
$$

**Proof of Theorem 1.2.** Under the assumption of Theorem 1.2 for $n > 0$, it follows from Theorem 1.1 and Theorem 4.1 that

$$
\frac{(n - 1)(n - 2)}{2} \leq \nu^+(J#(-K)) \leq \nu^+(J) + \nu^+(-K)
$$

and

$$
\nu^+(J) = \nu^+(J#(-K)#K) \leq \nu^+(J#(-K)) + \nu^+(K) \leq \frac{n(n - 1)}{2} + \nu^+(K).
$$

On the other hand, if $n = 0$, we have

$$
\nu^+(J) = \nu^+(J#(-K)#K) \leq \nu^+(J#(-K)) + \nu^+(K) = \nu^+(K).
$$

These complete the proof. \( \square \)

5. 4-Ball Genus Bound for Positive Cable Knots

In this section, we prove Theorem 1.3 and Corollary 1.4. Before proving Theorem 1.3, we note that it has been proved in [15] that the equality in Theorem 1.3 holds for sufficiently large $q$ relative to $p$.

**Theorem 5.1** ([15, Theorem 1.1]). Let $K$ be a knot and $p, q > 0$ coprime integers with $q \geq (2\nu^+(K) - 1)p - 1$. Then we have

$$
\nu^+(K_{p,q}) = p\nu^+(K) + \frac{(p - 1)(q - 1)}{2}.
$$

In our proof, we deform a given $(p, q)$-cable (with $p > 0$) into $(p, np + q)$-cable by $n$ times positive full twists for sufficiently large $n$, and apply Wu’s theorem to the $(p, np + q)$-cable.
Proof of Theorem 1.3. Without loss of generality, we can assume that \( p > 0 \). We can easily see that any cable knot \( K_{p,q} (p > 0) \) is deformed into \( K_{p,p+q} \) by a positive full-twist with \( p \)-linking. Hence, for any \( n > 0 \), by taking \( n \) times of such positive full-twists and applying Theorem 1.2, we have

\[
\nu^+(K_{p,q}) \geq \nu^+(K_{p,np+q}) - \frac{np(p-1)}{2}.
\]

Let \( n \) be a positive integer satisfying \( n \geq 2\nu^+(K) - 1 + \left\lfloor \frac{q}{p} \right\rfloor \). Then \( np + q \geq (2\nu^+(K) - 1)p - 1 \), and Theorem 5.1 gives

\[
\nu^+(K_{p,np+q}) = p\nu^+(K) + \frac{(p-1)(q-1)}{2} + \frac{np(p-1)}{2}.
\]

By combining the inequality (6) and the equality (7), we obtain the desired inequality. □

Next we prove Corollary 1.4.

Proof of Corollary 1.4. Since one can construct a slice surface for \( K_{p,q} \) from \( p \) parallel copies of a slice surface for \( K \) together with \( (p-1)q \) half-twisted bands, we have the inequality

\[
g_4(K_{p,q}) \leq pg_4(K) + \frac{(p-1)(q-1)}{2}.
\]

On the other hand, the assumption \( \nu^+(K) = g_4(K) \) and Theorem 1.3 imply

\[
g_4(K_{p,q}) \geq \nu^+(K_{p,q}) \geq pg_4(K) + \frac{(p-1)(q-1)}{2}.
\]

These complete the proof. □

6. A partial order on \( C_{\nu^+} \)

In this section, we prove Proposition 1.5 and Theorem 1.6. We first prove Proposition 1.5.

We decompose Proposition 1.5 into the following three lemmas.

Lemma 6.1. The relation \( \leq \) is a partial order on \( C_{\nu^+} \).

Proof. To prove Lemma 6.1, it is sufficient to show that the followings hold; for all \( x, y, z \in C_{\nu^+} \), we have

1. \( x \leq x \),
2. if \( x \leq y \) and \( y \leq z \), then \( x \leq z \), and
3. if \( x \leq y \) and \( y \leq x \), then \( x = y \).

Since \( \nu^+(x-x) = \nu^+(0) = 0 \), the condition (1) holds. We next prove the condition (3). Since the assumptions \( x \leq y \) and \( y \leq x \) imply that \( \nu^+(x-y) = 0 \) and \( \nu^+(y-x) = 0 \), representatives of \( x \) and \( y \) are \( \nu^+ \)-equivalent, and we have \( x = y \).

To prove the condition (2), we use Theorem 1.1 again. The assumptions \( x \leq y \), \( y \leq z \) and Theorem 1.1 imply that

\[
\nu^+(x-z) = \nu^+((x-y) + (y-z)) \leq \nu^+(x-y) + \nu^+(y-z) = 0.
\]

This completes the proof. □

Lemma 6.2. For elements \( x, y, z \in C_{\nu^+} \), if \( x \leq y \), then \( x + z \leq y + z \) and \( -y \leq -x \).
Proof. Since $\nu^+(x - y) = 0$, we have

$$\nu^+((x + z) - (y + z)) = \nu^+(x - y) = 0$$

and

$$\nu^+((-y) - (-x)) = \nu^+(x - y) = 0.$$ 

\[\square\]

**Lemma 6.3.** For coprime integers $p, q > 0$, $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq p - 1$, all of $\tau, -\Upsilon, V_k, \nu^+$ and $-d(S_{p,q}^3(\cdot), i)$ preserve the partial order.

**Proof.** It is proved in [4] and [9, Proposition 4.7] that

$$\tau(x) \leq \nu^+(x)$$

and

$$-\Upsilon_x(t) \leq (1 - |1 - t|)\nu^+(x)$$

for any $x \in \mathcal{C}_{\nu^+}$ and $t \in [0, 2]$. Since $\tau$ and $\Upsilon$ are group homomorphisms, if $x \leq y$, then we have

$$\tau(x) - \tau(y) = \tau(x - y) \leq \nu^+(x - y) = 0$$

and

$$( - \Upsilon_x(t)) - ( - \Upsilon_y(t)) = - \Upsilon_{x-y}(t) \leq (1 - |1 - t|)\nu^+(x - y) = 0.$$ 

These imply that $\tau(x) \leq \tau(y)$ and $-\Upsilon_x \leq -\Upsilon_y$. Furthermore, Theorem 4.1 implies that if $x \leq y$, then

$$\nu^+(x) = \nu^+(y + (x - y)) \leq \nu^+(y) + \nu^+(x - y) = \nu^+(y).$$

To consider $V_k$, we use the following proposition.

**Proposition 6.4 ([4, Proposition 6.1])**. For any two elements $x, y \in \mathcal{C}_{\nu^+}$ and any $m, n \in \mathbb{Z}_{\geq 0}$, we have

$$V_{m+n}(x + y) \leq V_m(x) + V_n(y).$$

This proposition implies that if $x \leq y$, then for any $k \in \mathbb{Z}_{\geq 0}$, we have

$$V_k(x) = V_{k+0}(y + (x - y)) \leq V_k(y) + V_0(x - y) = V_k(y).$$

Finally we consider $-d(S_{p,q}^3(\cdot), i)$. By Proposition 2.2 we have

$$-d(S_{p/q}^3(x), i) = -d(S_{p/q}^3(0), i) + 2V_{\text{min}}(\|\frac{i}{q}, \|\frac{p+q-1-i}{q}\|)(x).$$

Hence if $x \leq y$, then we have

$$-d(S_{p/q}^3(x), i) \leq -d(S_{p/q}^3(0), i) + 2V_{\text{min}}(\|\frac{i}{q}, \|\frac{p+q-1-i}{q}\|)(y)$$

$$= -d(S_{p/q}^3(y), i).$$

This completes the proof. \[\square\]

**Proof of Proposition 1.5.** Proposition 1.5 immediately follows from the above three lemmas. \[\square\]

We next prove Theorem 1.6.
Proof of Theorem 1.6. First, suppose that \( n = 0 \) or 1. Then Theorem 1.1 shows \( n^+([J]_{\nu^+} - [K]_{\nu^+}) = 0 \), and hence \([J]_{\nu^+} \leq [K]_{\nu^+}\).

Next, suppose that \( n \geq 3 \). Then Theorem 1.1 shows

\[
n^+([J]_{\nu^+} - [K]_{\nu^+}) \geq \frac{(n-1)(n-2)}{2} > 0,
\]

and hence \([J]_{\nu^+} \not\leq [K]_{\nu^+}\).

Finally, suppose that \( n \geq 3 \) and the geometric intersection number between \( K \) and \( D \) is equal to \( n \). Take a small tubular neighborhood of \( D \) (denoted by \( \nu(D) \)), and think of the intersection \( K \cap \nu(D) \) as a trivial braid with index \( n \). Then \( J \) is obtained by replacing \( K \cap \nu(D) \) with the pure braid \( \Delta_n \), where

\[
\Delta_n = (\sigma_1)(\sigma_2\sigma_1) \cdots (\sigma_n \cdots \sigma_2\sigma_1)
\]

(see [2, Section 10.5]). In particular, the pure braid \( \Delta_n^2 \) has only positive crossings (Figure 4), and hence it can be deformed into the trivial braid only by changing positive crossing to negative crossing (Figure 5). Such a crossing change is realized by a positive full-twist with 0-linking, as shown in Figure 6. This implies that \( J \) is deformed into \( K \) only by positive full-twists with 0-linking. Hence, applying the above argument in the case of \( n = 0 \) repeatedly (and transitivity of the partial order), we have \([K]_{\nu^+} \leq [J]_{\nu^+}\). On the other hand, since \( n \geq 3 \), we have \([J]_{\nu^+} \not\leq [K]_{\nu^+}\). Therefore \([J]_{\nu^+} \neq [K]_{\nu^+}\). \(\square\)

7. Results on satellite knots

In this section, we prove Proposition 1.7 and Corollary 1.8.

Proof of Proposition 1.7. Let \( P \) be a pattern in the standard solid torus \( V \subset S^3 \). Note that for any integer \( n \), if \( P^n \) is a pattern obtained from \( P \) by performing \((-1/n)\)-surgery along the boundary of the meridian disk of \( V \), then \( P^n(K, 0) \) is isotopic to \( P(K, n) \) for any knot \( K \). Hence we only need to prove that \( P_0 : [K]_{\nu^+} \mapsto [P(K, 0)]_{\nu^+} \)}
is a well-defined order-preserving map for any pattern $P$. In this proof, we denote $P(K,0)$ simply by $P(K)$.

Let $-P \subset V$ denote the orientation reversed mirror of $P$. Note that $-(P(K)) = (-P)(-K)$. In order to prove Proposition [1.7] it suffices to prove that for any two knots $K$ and $J$ satisfying $V_0(K\#(-J)) = 0$, the equality

$$V_0(P(K)\#(-P(J))) = 0$$

holds.

By handle calculus, the knot $P(K)\#(-P(J)) = P(K)\#((-P)(-J))$ is described as shown in Figure 7. Let $C$ denote the (unoriented) center line of $V$ and $\natural$ boundary connected sum. We start from deforming $P\#(-P)$ into parallel copies of $C\#C$ and in $V\natural V$.

![Figure 7](image)

**Claim 1.** For any pattern $P$, there exists an oriented compact surface $S$ properly embedded in $(V\natural V) \times [0,1]$ such that

1. $S$ has genus zero,
2. $S \cap ((V\natural V) \times \{0\}) = P\#(-P)$, and
3. $S \cap ((V\natural V) \times \{1\})$ is isotopic to finitely many parallel copies of $C\#C$ with a certain orientation.

**Proof.** Identifying $S^1$ with $\mathbb{R}/\mathbb{Z}$ and $V$ with $S^1 \times [-1,1] \times [1/2,1]$ respectively. After an isotopy of $P$, we can regard $P$ as an proper embedding $P : S^1 \rightarrow V$ such that for small $\varepsilon > 0$,

1. the projection of $P(S^1)$ on $S^1 \times [-1,1] \times \{1/2\}$ is a regular projection,
2. $P(S^1) \cap \{[-\varepsilon,\varepsilon] \times [-1,1] \times [1/2,1]\} = [-\varepsilon,\varepsilon] \times \{(s_i,t_i)\}_{i=1}^n$ for some $n \in \mathbb{Z}_{>0}$,
3. $P(t) = (t,0,3/4)$ for any $t \in [1/4,3/4]$, and
4. $P^{-1}([1/4,3/4] \times \{0\} \times [1/2,3/4]) = \emptyset$.

Roughly speaking, the above conditions mean

1. $P(S^1) \subset S^1 \times [-1,1] \times \{1/2\}$ can be seen as a knot diagram,
2. $P(S^1) \cap \{[-\varepsilon,\varepsilon] \times [-1,1] \times [1/2,1]\}$ is a trivial braid,
3. $P$ contains a half of the center line $[1/4,3/4] \times \{0\} \times \{3/4\}$, and
4. $[1/4,3/4] \times \{0\} \times \{3/4\}$ does not contain the over pass for any crossing on the diagram derived from the condition (1)

respectively. Next, for two copies $V_i$ of $V$ ($i = 1,2$), we consider a diffeomorphism $f : V_i \natural V_j \rightarrow (S^1 \times [-1,1] \times [-1,1]) \setminus ((-\varepsilon/2,\varepsilon/2) \times [-1,1] \times (-1/2,1/2))$
so that

$$f|_{V_1}(r, s, t) = (r, s, t)$$

and

$$f|_{V_2}(r, s, t) = (r, s, -t)$$

respectively. In particular, $V_1$ and $V_2$ are identified with

$$S^1 \times [-1, 1] \times [1/2, 1]$$

and

$$S^1 \times [-1, 1] \times [-1, -1/2]$$

respectively. Here, boundary connected sum $V_1 \natural V_2$ is thought of as a disjoint union $V_1 \coprod V_2$ with the 1-handle

$$h^1 := (S^1 \setminus (-\varepsilon/2, \varepsilon/2)) \times [-1, 1] \times [-1/2, 1/2].$$

(See Figure 8. Note that $V_1 \natural V_2$ is the complement of the yellow region.)

Figure 8.

Then, the embedded circle $P\#(-P) : S^1 \to V_1 \natural V_2$ is described as

$$(P\#(-P))(t) = \begin{cases} 
(f|_{V_1}) \circ P \left( -\frac{1}{4} + 2(t + \frac{1}{4}) \right) & (-\frac{1}{4} \leq t \leq 0) \\
(\frac{1}{4}, 0, \frac{3}{4} - 6t) & (0 \leq t \leq \frac{1}{4}) \\
(f|_{V_2}) \circ P \left( \frac{1}{4} - 2(t - \frac{1}{4}) \right) & (\frac{1}{4} \leq t \leq \frac{1}{2}) \\
(-\frac{1}{4}, 0, -\frac{3}{4} + 6(t - \frac{1}{2})) & (\frac{1}{2} \leq t \leq \frac{3}{4})
\end{cases}.$$  

(Note that $(f|_{V_1}) \circ P(S^1)$ and $(f|_{V_2}) \circ P(-S^1)$ are connected by the band $[1/4, 3/4] \times [-3/4, 3/4]$. Now, we can see that $P\#(-P)$ bounds a ribbon disk in $S^1 \times [-1, 1] \times [-1, 1]$, which is defined so that

$$p_i \circ R(s, t) = p_i \circ (P\#(-P)) \left( -\frac{1}{4}(1 - t) \right) \left( (s, t) \in [0, 1] \times [0, 1] \right)$$

for $i = 1, 2$, and

$$p_3 \circ R(s, t) = p_3 \circ (P\#(-P)) \left( -\frac{1}{4}(1 - t) \right) \cdot (2s - 1) \left( (s, t) \in [0, 1] \times [0, 1] \right),$$

where $p_i$ denotes the $i$-th projection of $S^1 \times [-1, 1] \times [-1, 1]$. Indeed, we can verify concretely that the boundary of $R$ is equal to $P\#(-P)$. Moreover, any singularity of $R$ is contained in $\{r\} \times \{s\} \times (-1, 1)$, where $(r, s)$ is the coordinate of a double
point on the regular projection \((p_1 \times p_2) \circ P(S^3)\). Let \(1/2 < t_1 < t_2 < 1\) be the 3rd coordinate of points in \((p_1 \times p_2)^{-1}(r, s) \cap P(S^3)\). Then the singularity of \(R\) in \(\{r\} \times \{s\} \times (-1, 1)\) is equal to \(\{r\} \times \{s\} \times [-t_1, t_1]\), which is contained in \(\{r\} \times \{s\} \times [-t_2, t_2]\subset R\). Therefore, any singularity of \(R\) is ribbon.

Let \(R' := R \cap V_1 \cap V_2\). Then \(\partial R\) consists of \(P\#(-P)\) and \(n\) parallel copies of \(\partial([-\varepsilon/2, \varepsilon/2] \times \{0\} \times [-1/2, 1/2])\) (with a certain orientation). It is not hard to see that \(\partial([-\varepsilon/2, \varepsilon/2] \times \{0\} \times [-1/2, 1/2])\) is isotopic to the connected sum of the longitude of \(V\) in \(\partial(V_3 V)\), and hence it is isotopic to \(C\#C\) in \(V_3 V\). Since \(R'\) is a ribbon surface, we can construct a cobordism \(S\) in \((V_1 V_2) \times [0, 1]\) from \(P\#(-P)\) to \(n\) times parallel copies of \(C\#C\), which is homeomorphic to \(R'\). □

Figure 9. A ribbon disk \(R\) for the Whitehead double

For a knot \(K\) in \(S^3\), let \(W_0(K)\) denote the 4-manifold obtained by attaching a 0-framed 2-handle to \(S^3 \times [0, 1]\) along \(K \subset S^3 \times \{1\}\).

Claim 2. The knot \((-P(K)\#(-P(J))) \subset -(S^3 \times \{0\})\) bounds a disk in the 4-manifold \(W_0(K\#(-J))\).

Proof. We can extend the embedding of \(P\#(-P)\) shown in Figure 7 to \(V_3 V\), as shown in Figure 10. By using the cobordism in Claim 1, we have a cobordism in \(S^3 \times [0, 1]\) with genus zero which connects \(P(K)\#(-P(J))\) to the link shown in Figure 11. Furthermore, it follows from elementary handle calculus that the link in Figure 11 is isotopic to the \((n, 0)\)-cable of \(K\#(-J)\) for some positive integer \(n\). By attaching a 0-framed 2-handle along \(K\#(-J) \subset S^3 \times \{1\}\) and capping off the cable \((K\#(-J))_{n, 0}\) with \(n\) parallel copies of the core of the 2-handle, we obtain a disk in \(W_0(K\#(-J))\) with boundary \(-P(K)\#(-P(J))) \subset -(S^3 \times \{0\})\). □

Attach a \((-1)\)-framed 2-handle to \(W_0(K)\) along \(-P(K)\#((-P(-J))) \subset -(S^3 \times \{0\})\), and cap off the disk in Claim 2 with the core. Then the self-intersection of the resulting sphere is \(-1\), and hence we can blow down the sphere. Let \(W\) denote the resulting cobordism. Then we can see that

\[ \partial W = -S^3_1(P(K)\#(-P(J))) \amalg S^3_0(K\#(-J)).\]

Taking a properly embedded arc in \(W\) from \(-S^3_1(P(K)\#(-P(J)))\) to \(S^3_0(K\#(-J))\) and removing the small open tubular neighborhood of the arc, we obtain a compact oriented 4-manifold \(X\) with boundary \(-S^3_1(P(K)\#(-P(J)))\#S^3_0(K\#(-J))\).
Moreover, it follows from elementary homology theory that
\[ H_\ast(X; \mathbb{Z}) \cong H_\ast(S^2 \times D^2; \mathbb{Z}). \]

Now we apply the following theorem to \(-X\).

**Theorem 7.1** ([10, Corollary 9.13]). Suppose that \(Y\) is a 3-manifold with \(H_1(Y; \mathbb{Z}) \cong \mathbb{Z}\). If \(Y\) bounds an integer homology \(S^2 \times D^2\), then \(d_{-1/2}(Y) \geq -1/2\).

By Theorem 7.1 and Corollary 2.3, we have
\[
-1/2 \leq d_{-1/2}(-S_0(K\#(-J))) + d(S_1(P(K)\#(-P(J)))) = -1/2 + d(S_1(J\#(-K))) - 2V_0(P(K)\#(-P(J))) = -1/2 + 2V_0(K\#(-J)) - 2V_0(P(K)\#(-P(J))).
\]

Since \(V_0(K\#(-J)) = 0\), we have
\[
0 \leq V_0(P(K)\#(-P(J))) \leq V_0(K\#(-J)) = 0
\]
and hence \(V_0(P(K)\#(-P(J))) = 0\). \(\square\)

Finally, we prove Corollary 1.8.

**Proof of Corollary 1.8**. If \(D\) denotes the image of the meridian disk of \(V\) by the embedding \(e_n\), then it is easy to see that \(D\) intersects \(P(K, n)\) in its interior, \(\text{lk}(P(K, n), D) = w(P)\) and \(P(K, n)\) is deformed into \(P(K, n + 1)\) by a positive full-twist along \(D\).

We first suppose that \(w(P) = 0\) or 1. Let \(x \in C_{\nu^+}\), the symbol \(K\) denote a representative of \(x\) and \(m, n\) integers with \(m < n\). Then, by applying Theorem 1.6 to the pair \((P(K, m), D)\) repeatedly, we have
\[
P_m(x) = [P(K, m)]_{\nu^+} \geq [P(K, m + 1)]_{\nu^+} \geq \ldots [P(K, n)]_{\nu^+} = P_n(x).
\]
Next we suppose that \( w(P) \geq 3 \) and the geometric intersection number between the pattern \( P \) and the meridian disk of \( V \) is equal to \( w(P) \). Then the embedding \( e_n \) preserves the number of intersection points, and hence the geometric intersection number between \( D \) and \( P(K,n) \) is also equal to \( n \). By applying Theorem 1.6 to the pair \( (P(K,n), D) \) repeatedly, we have
\[
P_m(x) = [P(K,m)]_{\nu^+} < [P(K,m+1)]_{\nu^+} < [P(K,n)]_{\nu^+} = P_n(x).
\]
This completes the proof. 

\[\square\]

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