ON FIRST EXTENSIONS IN $\mathcal{S}$-SUBCATEGORIES OF $\mathcal{O}$

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Abstract. We compute the first extension group from a simple object to a proper standard object and, in some cases, the first extension group from a simple object to a standard object in the principal block of an $\mathcal{S}$-subcategory of the BGG category $\mathcal{O}$ associated to a triangular decomposition of a semi-simple finite dimensional complex Lie algebra.

1. Introduction and description of the results

Bernstein-Gelfand-Gelfand category $\mathcal{O}$ associated to a triangular decomposition of a semi-simple finite dimensional complex Lie algebra $\mathfrak{g}$ is about half a century old, it originates from the classical papers [BGG1, BGG2]. However, it remains an important and intensively studied object in modern representation theory, see [Hu, CM1, CM2, KMM1, KMM2, KMM3, KMM4] for details. Category $\mathcal{O}$ has numerous analogues and generalizations, which include:

- parabolic category $\mathcal{O}$, see [RC],
- $\mathcal{S}$-subcategories in $\mathcal{O}$, see [FKM3, MS1].

Homological invariants of the above categories carry essential information about both the structure and the properties of these categories. For category $\mathcal{O}$, many homological invariants are explicitly known, see [Ma1, Ma2, CM1, CM2, KMM1, KMM2, KMM3, KMM4] and references therein. Using these results, many homological invariants for $\mathcal{S}$-subcategories in $\mathcal{O}$, for example, various projective dimensions, can be computed using the approach of [MPW, Section 4], especially using [MPW, Theorem 15]. In the present paper, inspired by the recent results from [KMM3, KMM4], we take a closer look at the first extension space between certain classes of structural object in $\mathcal{S}$-subcategories of $\mathcal{O}$.

We completely determine, in type $A$, the first extension space from a simple object to a proper standard object in the regular block of an $\mathcal{S}$-subcategory of $\mathcal{O}$ in Theorems 10 and 15. In many special cases (notably both for the dominant and the antidominant standard objects), we completely determine the first extension space from a simple object to a standard object in the regular block of an $\mathcal{S}$-subcategory of $\mathcal{O}$ in Proposition 19. We also obtain some general results which reduce the problem of computation of the first extension space from a simple object to a standard object in an $\mathcal{S}$-subcategory of $\mathcal{O}$ to a similar problem for certain objects in category $\mathcal{O}$, see Proposition 17.

The paper is organized as follows: Section 2 contains preliminaries on category $\mathcal{O}$ and its combinatorics. In Section 3, we survey some of the recent results of [KMM3, KMM4] which describe extensions from a simple highest weight module to a Verma module in category $\mathcal{O}$. In Section 4, we recall the definition and basic properties of $\mathcal{S}$-subcategories in $\mathcal{O}$. Section 5 is devoted to explicit description of the first extensions space from a simple to a proper standard object in $\mathcal{S}$-subcategories in $\mathcal{O}$ in type $A$. We also formulate a number of general results which hold in all types. In Section 6, we similarly look at the first extensions space from a simple to a standard object. We complete the paper.
with some examples in Section 7. This includes a detailed \( \mathfrak{sl}_2 \)-example (for a rank one parabolic) as well as various examples of non-trivial extension from a simple to a proper standard object for the algebra \( \mathfrak{sl}_4 \).

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## 2. Preliminaries on category \( \mathcal{O} \)

### 2.1. Category \( \mathcal{O} \)

Let \( g \) be a semi-simple finite dimensional complex Lie algebra with a fixed triangular decomposition \( g = n_- \oplus \mathfrak{h} \oplus n_+ \), see [Hu, MP] for details. Associated to this datum, we have the Bernstein-Gelfand-Gelfand category \( \mathcal{O} \) defined as the full subcategory of the category of all finitely generated \( g \)-modules, consisting of all \( \mathfrak{h} \)-diagonalizable and locally \( U(n^-) \)-finite modules, cf. [BGG1, BGG2, MP, Hu].

Simple modules in \( \mathcal{O} \) are exactly the simple highest weight modules \( L(\lambda) \), where \( \lambda \in \mathfrak{h}^* \), see [Di, Chapter 7] for details. For each such \( \lambda \), we also have in \( \mathcal{O} \) the corresponding

- Verma module \( \Delta(\lambda) \),
- dual Verma module \( \nabla(\lambda) \),
- indecomposable projective module \( P(\lambda) \),
- indecomposable injective module \( I(\lambda) \),
- indecomposable tilting module \( T(\lambda) \).

Consider the principal block \( \mathcal{O}_0 \) of \( \mathcal{O} \), which is defined as the indecomposable direct summand containing the trivial \( g \)-module \( L(0) \). Simple modules in \( \mathcal{O}_0 \) are indexed by the elements of the Weyl group \( W \) of \( g \). For \( w \in W \), we have the corresponding simple module \( L_w := L(w \cdot 0) \), where \( w \cdot - \) denotes the usual dot-action of the Weyl group on \( \mathfrak{h}^* \). We will similarly denote by \( \Delta_w, \nabla_w, P_w, I_w \) and \( T_w \) the other structural modules corresponding to \( L_w \).

We will use \( \text{Ext} \) and \( \text{Hom} \) to denote extensions and homomorphisms in \( \mathcal{O} \), respectively. The simple preserving duality on \( \mathcal{O} \) is denoted by \( \star \).

### 2.2. Graded category \( \mathcal{O} \)

The category \( \mathcal{O}_0 \) admits a \( \mathbb{Z} \)-graded lift \( \mathcal{O}^\mathbb{Z}_0 \), see [So, St]. All structural modules in \( \mathcal{O}_0 \) admit graded lifts (unique up to isomorphism and shift of grading). We will use the same notation as for ungraded modules to denote the following graded lifts of the structural modules in \( \mathcal{O}^\mathbb{Z}_0 \):

- \( L_w \) denotes the graded simple object concentrated in degree 0,
- \( \Delta_w \) denotes the graded Verma module with top in degree 0,
- \( \nabla_w \) denotes the graded dual Verma module with socle in degree 0,
- \( P_w \) is the graded indecomposable projective module with top in degree 0,
- \( I_w \) is the graded indecomposable injective modules with socle in degree 0,
- \( T_w \) is the graded indecomposable tilting module having the unique \( L_w \) subquotient in degree 0.

For \( k \in \mathbb{Z} \), we denote by \( \langle k \rangle \) the functor which shifts the grading by \( k \), with the convention that \( \langle 1 \rangle \) maps degree 0 to degree -1. We will use \( \text{ext} \) and \( \text{hom} \) to denote
extensions and homomorphisms in \(O^0_S\), respectively. Note that, for any \(k \geq 0\) and any two structural modules \(M\) and \(N\) with fixed graded lifts \(M\) and \(N\), we have

\[
\text{Ext}^k(M, N) \cong \bigoplus_{i \in \mathbb{Z}} \text{ext}^k(M, N(i)).
\]

2.3. Combinatorics of category \(O^0_S\). Let \(H\) denote the Hecke algebra of \(W\) over \(\mathbb{Z}[v, v^{-1}]\) in the normalization of \(\mathbb{S}_W\). It has the standard basis \(\{H_w : w \in W\}\) and the Kazhdan-Lusztig basis \(\{H_w : w \in W\}\). The Kazhdan-Lusztig polynomials \(\{p_{x,y} : x, y \in W\}\) are the entries of the transformation matrix between these two bases, that is

\[
H_y = \sum_{x \in W} p_{x,y} H_x, \text{ for all } y \in W.
\]

Taking the Grothendieck group gives rise to an isomorphism of \(\mathbb{Z}[v, v^{-1}]\)-modules as follows:

\[
\text{Gr}(O^0_S) \cong H, \quad [\Delta_w] \mapsto H_w, \text{ for } w \in W.
\]

Here the \(\mathbb{Z}[v, v^{-1}]\)-module structure on \(\text{Gr}(O^0_S)\) is given by letting the element \(v\) act as \((-1)^w\). This isomorphism maps \(P_w\) to \(H_w\), for \(w \in W\).

2.4. Kazhdan-Lusztig orders and cells. Following [KL], for \(x, y \in W\), we write \(x \geq_L y\) provided that there is \(w \in W\) such that \(H_x\) appears with a non-zero coefficient in \(H_w H_x\). This defines the left pre-order on \(W\). The equivalence classes with respect to this pre-order are called left cells and the corresponding equivalence relation is denoted \(\sim_L\).

Similarly, for \(x, y \in W\), we write \(x \geq_R y\) provided that there is \(w \in W\) such that \(H_x\) appears with a non-zero coefficient in \(H_y H_w\). This defines the right pre-order on \(W\). The equivalence classes with respect to this pre-order are called right cells and the corresponding equivalence relation is denoted \(\sim_R\).

Finally, for \(x, y \in W\), we write \(x \geq_J y\) provided that there are \(w, w' \in W\) such that \(H_x\) appears with a non-zero coefficient in \(H_y H_w H_{w'}\). This defines the two-sided pre-order on \(W\). The equivalence classes with respect to this pre-order are called two-sided cells and the corresponding equivalence relation is denoted \(\sim_J\).

The two-sided pre-order induces a partial order on the set of the two-sided cells. The maps \(w \mapsto w_0 w\) and \(w \mapsto w w_0\) induce anti-involution on the poset of two-sided cells, see [BB] Chapter 6. In particular, the poset of two-sided cells has the minimum element \(\{e\}\) and the maximum element \(\{w_0\}\). In type \(A_1\), there is nothing else. Outside type \(A_1\), removing these two extreme cells, we again get a poset with the minimum and the maximum element. The new minimum element is the cell containing all simple reflections, called the small cell (see [KMMZ]), while the new maximum element is the image of the small cell under the \(w \mapsto w_0 w\) anti-involution (note that the two new extreme cells coincide in rank 2). We call this new maximum cell the penultimate cell and denote it by \(J\).

3. First extension from a simple to a Verma module in category \(O\)

In this section, we briefly summarize the results from [Ma1, KMM3, KMM4] which describe the first extension from a simple module to a Verma module.
3.1. First extension to a Verma from the anti-dominant simple. We have the usual length function \( \ell \) on the Weyl group \( W \) considered as a Coxeter group with respect to the simple reflections determined by our fixed triangular decomposition of \( g \). For \( w \in W \), the value \( \ell(w) \) is the length of a reduced expression of \( w \). We also have the content function \( c : W \to \mathbb{Z}_{\geq 0} \). For \( w \in W \), the value \( c(w) \) is the number of different simple reflections which appear in a reduced expression of \( w \) (from the Coxeter relations it follows that this number does not depend on the choice of a reduced expression).

As usual, we denote by \( w_0 \) the longest element of \( W \). The following result is [Ma1, Theorem 32].

**Theorem 1.** For \( w \in W \) and \( i \in \mathbb{Z} \), we have
\[
\dim \text{ext}^1(\Delta_{w_0}, \Delta_w(i)) = \begin{cases} 
   c(w_0 w), & \ell(w_0) - \ell(w) - 2; \\
   0, & \text{otherwise.}
\end{cases}
\]

3.2. First extension to a Verma module from other simple modules and inclusion of Verma modules. Recall the following properties of (graded) Verma modules:
- every non-zero map between two Verma modules is injective;
- for \( x, y \in W \), we have \( \text{hom}(\Delta_x, \Delta_y(d_i)) \neq 0 \) if and only if \( x \geq y \) in the Bruhat order and \( d = \ell(x) - \ell(y) \);
- \( \dim \text{Hom}(\Delta_x, \Delta_y) \leq 1 \), for all \( x, y \in W \).

The ungraded versions of these properties can be found in [Di, Chapter 7]. The graded version of the second property follows by matching the degrees using standard arguments, see e.g. [St].

In particular, each \( \Delta_x(-\ell(x)) \) injects into \( \Delta_x \), and the cokernel \( \Delta_x/\Delta_x(-\ell(x)) \) belongs to \( \mathcal{O}^W_0 \). To ease the notation, we denote the latter by \( \Delta_x/\Delta_x \). These cokernels control the first extension from non-anti-dominant simple modules to Verma modules in the following way, as observed in [KMM3].

**Proposition 2.** For each \( x, w \in W \), with \( x \neq w_0 \), we have
\[
\dim \text{ext}^1(L_x(d), \Delta_{\ell(w)}(-\ell(w))) = [\text{soc} \Delta_x/\Delta_w : L_x(d)].
\]

The proof is similar to the second part of [KMM3, Proof of Corollary 2]. A similar argument will also be given in Proposition 11.

The rest of this section describes the cokernels \( \Delta_x/\Delta_w \), for \( w \in W \). To do this, we need to dive into poset-theoretic properties of the Bruhat order. An element \( w \in W \) is called join-irreducible provided that it is not a join (supremum) of other elements, that is, there is no \( U \subseteq W \) with \( w \not\in U \) such that \( w = \bigvee U \). The set of all join-irreducible elements, denoted by \( B \), is called the base of the poset \( W \).

3.3. Cokernel of inclusion between Verma modules in type \( A \). In a few coming subsections we restrict to the case of type \( A \). The join-irreducible elements in \( W \) of type \( A \) are explicitly identified in [LS] as the bigrassmannian elements. An element \( w \in W \) is called bigrassmannian provided that there is a unique simple reflection \( s \) such that \( \ell(sw) < \ell(w) \) and there is a unique simple reflection \( t \) such that \( \ell(w) < \ell(wt) \). In type \( A \), the base \( B \) agrees with the set of bigrassmannian elements in \( W \).
The Kazhdan-Lusztig two-sided order is also easier in type $A$. The classical Robinson-Schensted correspondence

$$RS : S_n \rightarrow \bigsqcup_{\lambda \vdash n} SYT_{\lambda} \times SYT_{\lambda}$$

assigns to $w \in W$ a pair $RS(w) = (p_w, q_w)$ of standard Young tableaux of shape $\lambda := \text{sh}(w)$, where $\lambda$ is a partition of $n$, see [Sa, Section 3.1]. By [KL, Theorem 1.4], we have

- $x \sim_L y$ if and only if $q_x = q_y$;
- $x \sim_R y$ if and only if $p_x = p_y$;
- $x \sim_J y$ if and only if $\text{sh}(x) = \text{sh}(y)$.

The poset of all two-sided cells with respect to the two-sided order is isomorphic to the poset of all partitions of $n$ with respect to the dominance order, see [Ge].

Recall that $J$ denotes the penultimate cell with respect to the two-sided order. In type $A$, the elements in $J$ are naturally indexed by pairs of simple reflections in $W$: for any pair $(s, t)$ of simple reflections in $W$, there is a unique element $w_{s,t} \in J$ such that $w_0 = sw_{s,t} = w_{s,t}t$.

We now formulate the main result of [KMM3], that is [KMM3, Theorem 1].

**Theorem 3.**

(i) For $w \in S_n$, the module $\Delta_e / \Delta_w$ has simple socle if and only if $w \in B$.

(ii) The map $B \ni w \mapsto \text{soc}(\Delta_e / \Delta_w)$ induces a bijection between $B$ and simple subquotients of $\Delta_e$ of the form $L_x$, where $x \in J$.

(iii) For $w \in S_n$, the simple subquotients of $\Delta_e / \Delta_w$ of the form $L_x$, where $x \in J$, correspond, under the bijection from (ii), to $y \in B$ such that $y \leq w$.

(iv) For $w \in S_n$, the socle of $\Delta_e / \Delta_w$ consists of all $L_x$, where $x \in J$, which correspond, under the bijection from (ii), to the Bruhat maximal elements in the set $\{y \in B : y \leq w\}$.

Motivated by the last claim, we denote $BM(w) := \max\{y \in B : y \leq w\}$.

The socle of the cokernel of an inclusion between two arbitrary Verma modules can be described using Theorem 3. The following corollary is [KMM3, Corollary 23].

**Corollary 4.** Let $v, w \in S_n$ be such that $v < w$.

(i) The bijection from Theorem 3 induces a bijection between simple subquotients of $\Delta_v / \Delta_w$ of the form $L_x$, where $x \in J$, and $y \in B$ such that $y \leq w$ and $y \not\leq v$.

(ii) The socle of $\Delta_v / \Delta_w$ consists of all $L_x$, where $x$ corresponds to an element in $BM(w) \setminus BM(v)$.

3.4. **First extension to a Verma from other simples in type $A$.** Let $w \in B$ be such that $\ell(sw) < \ell(w)$ and $\ell(wt) < \ell(w)$, for two simple reflections $s$ and $t$. Denote by $\Phi : B \rightarrow J$ the map which sends such $w$ to $w_{s,t}$. Theorem 3 and Proposition 2 has the following consequence:
Corollary 5. Let \( x, y \in S_n \) with \( x \neq w_0 \). Then we have
\[
\dim \text{Ext}^1(L_x, \Delta_y) = \dim \text{Ext}^1(\nabla_y, L_x) = \begin{cases} 
1, & x \in \Phi(BM(y)) \\
0, & \text{otherwise}.
\end{cases}
\]

3.5. Extensions in singular blocks in type \( A \). Let \( \lambda \) be a dominant integral weight and \( O_\lambda \) the indecomposable summand of \( O \) containing \( \lambda \). If \( \lambda \) is regular, then \( O_\lambda \) is equivalent to \( O_0 \). In the general case, denote by \( W^\lambda \) the stabilizer of \( \lambda \) with respect to the dot action of \( W \). Simple objects in \( O_\lambda \) are then in a natural bijection with the cosets in \( W/W_\lambda \).

For \( w \in W \) denote by \( \mathfrak{w} \) the unique longest element in \( wW^\lambda \). Also, denote by \( w \) the unique shortest element in \( wW^\lambda \). The following claim is [KMM3, Theorem 16]:

**Theorem 6.** Let \( x, y \in S_n \) and let \( \mu \) be an integral, dominant weight. Then we have
\[
\dim \text{Ext}^1(L(x \cdot \lambda), \Delta(y \cdot \lambda)) = \begin{cases} 
c(\mathfrak{w}y) - \text{rank}(W^\lambda), & \mathfrak{w} = w_0; \\
1, & \mathfrak{w} \in \Phi(BM(y)); \\
0, & \text{otherwise}.
\end{cases}
\]

3.6. The graded picture in type \( A \). Corollary 5 admits a graded lift. Let \( s_1, \ldots, s_{n-1} \) be the simple reflections in \( S_n \) such that the corresponding Dynkin diagram is
\[
\begin{array}{cccccccc}
s_1 & s_2 & \cdots & \cdots & \cdots & \cdots & \cdots & s_{n-1}
\end{array}
\]
For \( i, j \in \{1, 2, \ldots, n-1\} \), let
\[
\mathcal{B}_j := \{ w \in B : \ell(s_i w) < \ell(w) \text{ and } \ell(ws_j w) < \ell(w) \}.
\]

The set \( \mathcal{B}_j \) consists of \( \min \{i, j, n-i, n-j\} \) elements which can be described very explicitly, see [KMM3, Subsection 4.2]. For example, here are the three elements of \( \mathcal{B}_3 \) in \( S_7 \) and their graphs:
\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 5 & 3 & 4 & 6 & 7
\end{array}, \quad \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 5 & 6 & 2 & 3 & 4 & 7
\end{array}, \quad \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
5 & 6 & 7 & 1 & 2 & 3 & 4
\end{array}, \quad \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}, \quad \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}, \quad \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\]

The elements of \( \mathcal{B}_j \) form naturally a chain with respect to the Bruhat order on \( S_n \).

This allows us to index the elements of the set \( \mathcal{B}_j \) via the tuples \((i, j, k)\), where \( 0 \leq k \leq \min \{i, j, n-i, n-j\} - 1 \), increasingly along the Bruhat order. Since Theorem 3 is gradable, see [KMM3, Proposition 22], we can lift Corollary 5 to the graded setup by Proposition 2.

**Proposition 7.** Let \( y \in S_n \) and \( x = \Phi((i, j, k)) \), where \((i, j, k) \in BM(y)\). Then the unique degree \( m \in \mathbb{Z} \) for which \( \dim \text{ext}^1(L_x(-m), \Delta_y(-\ell(y))) = 1 \) is
\[
m = \frac{(n-1)(n-2)}{2} + |i-j| + 2k.
\]

Similarly, Theorem 6 can also be graded.

**Proposition 8.** Let \( y = y \in S_n \) and \( x = \mathfrak{w} = \Phi((i, j, k)) \), where \((i, j, k) \in BM(y)\). Then the unique degree \( m \in \mathbb{Z} \) for which \( \dim \text{ext}^1(L(x \cdot \lambda)(-m), \Delta(y \cdot \lambda)(-\ell(y))) = 1 \) is
\[
m = \frac{(n-1)(n-2)}{2} + |i-j| + 2k.
\]
3.7. First extension to a Verma from other simples in other types. By Proposition 2, the problem again reduces to determining the (socles of) $\Delta_x/\Delta_w$, for $w \in W$. However, the latter does not seem to follow a uniformly describable pattern, in general. In particular, it is shown in [KMM4] that none of the statements in Theorem 2 is true, in general, in other types.

What remains to be true is that, for $x, w \in W$ with $x \neq w$, we have $\text{Ext}^1(L_x, \Delta_w) = 0$, unless $x \in J$. Another partial result is an upper bound. Let

$$\text{BM}_s(w) = \{ z \in \text{BM}(w) \mid sz < z \text{ and } zt < z \},$$

for $w \in W$ and $s, t$ simple reflections. The following is Theorem F(c) in [KMM4].

**Theorem 9.** Let $w \in W$ and $x \in J$. If $s, t$ are simple reflections in $W$ such that $sx > x$ and $xt > x$, then

$$\dim \text{Ext}^1(L_x, \Delta_w(d)) \leq \dim \text{Ext}^1(L_x, \Delta_b(d)),$$

for all $d \in \mathbb{Z}$, where $b$ is the join of $s \text{BM}_s(w)$. The right hand side of (1) is again bounded by

$$\dim \text{Ext}^1(L_x, \Delta_b) \leq |s \text{BM}_s(w)|.$$

In particular, $\text{Ext}^1(L_x, \Delta_w) = 0$, if $s \text{BM}_s(w) = \emptyset$.

Using further computation, it is determined in [KMM4] that

$$\dim \text{Ext}^1(L_x, \Delta_b(d))$$

is bounded by 1 in type $B$, by 2 in types $DF$, and by 3, 4, and 6 in types $E_6, E_7$ and $E_8$, respectively.

The paper [KMM4] develops several techniques to compute specific $\Delta_x/\Delta_w$. Thus for a given $w \in W$, it is often possible to determine $\text{ext}^1(L_x, \Delta_w(d))$, for all $x \in W$ and $d \in \mathbb{Z}$. See [KMM4, Section 5] for details.

4. $S$-subcategories in $O$

In this subsection we recall the definition and basic properties of $S$-subcategories in $O$ from [FKM3, MS1].

4.1. Definition. Let $p$ be a parabolic subalgebra of $\mathfrak{g}$ containing $h \oplus n_+$. We denote by $W_p$ the corresponding parabolic subgroup of $W$ and by $w_p^\text{long}$ the longest element of $W_p$. Denote by $\chi_p^\text{long}$ and $\chi_p^\text{short}$ the sets of the longest and the shortest representatives in the $W_p$-cosets from $W$, respectively. The map $w_p^\text{long} \cdot : \chi_p^\text{long} \to \chi_p^\text{short}$ is a bijection with inverse $w_p^\text{long} : \chi_p^\text{short} \to \chi_p^\text{long}$.

Recall that the parabolic category $O_0^p$ is defined in [RC] as the Serre subcategory of $O_0$ generated by all $L_w$, where $w \in \chi_p^\text{short}$.

We define the $S$-subcategory $S_0^p$ of $O_0$ as the quotient of $O_0$ modulo the Serre subcategory $Q_p$ generated by all $L_w$, where $w \not\in \chi_p^\text{long}$. We denote by $\pi_p : O_0 \to S_0$ the Serre quotient functor.

The category $S_0^p$ admits various realizations as a full subcategory of $O_0$. For example, $S_0^p$ is equivalent to the full subcategory of $O_0$ consisting of all $M$ which have a projective presentation of the form

$$X \to Y \to M \to 0,$$
such that, for each \( P_w \) appearing as a summand of \( X \) or \( Y \), we have \( w \in X_p^{\text{long}} \).
Alternatively, \( S_p^\perp \) is equivalent to the full subcategory of \( O_0 \) consisting of all \( N \) which have an injective copresentation of the form
\[
0 \to N \to X \to Y
\]
such that, for each \( I_w \) appearing as a summand of \( X \) or \( Y \), we have \( w \in X_p^{\text{long}} \).

By abstract nonsense, see [Au], \( S_p^\perp \) is equivalent to the module category over the endomorphism algebra \( A^p \) of the direct sum of all \( P_w \), where \( w \in X_p^{\text{long}} \).

We also note that, in the case \( W^p \) is of type \( A_1 \), the category \( S_0^\perp \) is the Serre quotient of \( O_0 \) by \( O_0^{\perp} \). In this case \( W = X_p^{\text{long}} \cup X_p^{\text{short}} \).

The graded version \( (S_0^\perp)^g \) of \( S_p^\perp \) is similarly defined as the Serre quotient of \( O_0^g \) by the Serre subcategory of the latter category generated by all \( L_w(i) \), where \( w \notin X_p^{\text{long}} \) and \( i \in \mathbb{Z} \). We use the same notation \( \pi_p \) for the graded Serre quotient functor.

The above alternative descriptions have the obvious graded analogues. For example, \( (S_0^\perp)^g \) is equivalent to the full subcategory of \( O_0^g \) consisting of all objects which have a projective presentation as above with indecomposable summands of the form \( P_w(i) \), where \( w \in X_p^{\text{long}} \) and \( i \in \mathbb{Z} \). Similarly for the injective copresentation.

### 4.2. Origins and motivation.

\( S \)-subcategories in \( O \) were formally defined in [FKM3]. They provide a uniform description for a number of generalizations of category \( O \) in [FKM1, FKM2, FKM4, Ma3, MiSo]. Notably, these include various categories of Gelfand-Zeitlin module, see [Ma3], and Whittaker modules, see [MiSo].

The realization of the \( S \)-subcategories in \( O \) as projectively presentable modules in \( O \) was studied in [MS1]. In particular, in [MS1] it was shown that the action of projective functors on \( S_0 \) categorifies the permutation \( W \)-module for \( W^p \), i.e., the \( W \)-module obtained by inducing the trivial \( W^p \)-module up to \( W \) (see also [MS2] for further details).

### 4.3. Stratified structure.

Here we recall some structural properties of \( S_p^\perp \) established in [FKM3, MS1].

For \( w \in X_p^{\text{long}} \), denote by
- \( L_w^p \) the object \( \pi_p(L_w) \) in \( S_p^\perp \);
- \( P_w^p \) the object \( \pi_p(P_w) \) in \( S_0^\perp \);
- \( I_w^p \) the object \( \pi_p(I_w) \) in \( S_p^\perp \);
- \( T_w^p \) the object \( \pi_p(T_w(-\ell(w^p))) \) in \( S_p^\perp \).

By construction, \( L_w^p \) is simple and \( \{ L_w^p : w \in X_p^{\text{long}} \} \) is a complete and irredundant list of representatives of simple objects in \( S_p^\perp \). The objects \( P_w^p \) and \( I_w^p \) are the corresponding indecomposable projectives and injectives in \( S_p^\perp \), respectively. For structural modules, we will use the same notation for the ungraded versions of the modules and for their graded versions. The latter are obtained by applying the graded version of \( \pi_p \) to the standard graded lifts of structural modules.

For \( w \in X_p^{\text{long}} \), denote by \( \Delta_w \) the object \( \pi_p(\Delta_w) \) in \( S_0^\perp \). Then \( \Delta_w \cong \pi_p(\Delta_w(\ell(x))) \), for all \( x \in W^p \). The object \( \Delta_w \) is called the proper standard object corresponding to the element \( w \).

Further, for \( w \in X_p^{\text{long}} \), let \( Q_w \in O_0 \) denote the quotient of \( P_w \) modulo the trace in \( P_w \) of all \( P_y \), where \( y \in W \) is such that \( y < w \) with respect to the Bruhat order and
y ≠ xw, for any x ∈ W^F. Denote by Δ_y^p the object π_y(Q_w) in S_0^p. The object Δ_y^p is called the standard object corresponding to w.

The object Δ_y^p has a filtration with subquotients Δ_y^p (up to graded shift). The length of this filtration is |W^F| + 1. More details for the graded version: for i ∈ Z, the multiplicity of Δ_y^p(−2i) as a subquotient of a (graded) proper standard filtration of Δ_y^p equals the cardinality of the set \{w ∈ W^F : ℓ(w) = i\}. Furthermore, each projective object in S_0^p has a filtration with standard subquotients.

The simple preserving duality * on O_0 induces a simple preserving duality on S_0^p which we will denote by the same symbol, see [MST] Lemma 2.12.

The above means that the underlying algebra A^p of the category S_0^p is properly stratified in the sense of [Dl]. The objects T_y^p are tilting with respect to this structure, in the sense of [AHLU]. An additional property of the algebra A^p is that each T_y^p is also cotilting. This follows from the description of tilting modules for A^p in [FKM3 Section 6] and the fact that these modules are self-dual.

5. First extension from a simple to a proper standard module in S^p

5.1. First extension from the antidominant simple. Similarly as in O, it is easy to separately treat the following special case.

Theorem 10. For y ∈ X_p^{long} and i ∈ Z, we have
\[ \text{Ext}_S^1(L_{w_0}^p, Δ_y^p) = \dim \text{Ext}_S^1(L_{w_0}^p(-\ell(w_0) + 2), Δ_y^p(-\ell(y))) = c(w_0 w_0^p y). \]

Proof. Note that L_{w_0}^p ∼= Δ_y^p is a proper standard object. The object T_y^p is both tilting and cotilting. In particular, it is the cotilting envelope of Δ_y^p. Let Q be such that the following sequence is short exact in (S_0^p)^{\otimes}:
\[ 0 \to Δ_y^p(-\ell(y)) \to T_y^p(2\ell(w_0^p) - \ell(y)) \to Q \to 0. \]

Set a := 2\ell(w_0^p) - \ell(y). As proper standard and costandard (and hence also cotilting) objects are homologically orthogonal, it follows that
\[ \dim \text{Ext}_S^1(L_{w_0}^p(i), Δ_y^p(-\ell(y))) = \dim \text{Ext}_S(L_{w_0}^p(i), Q) - \dim \text{Hom}_S(L_{w_0}^p(i), T_y^p(a)) + 1. \]

At the same time, we have w_0^p y ∈ X_p^{short}. Therefore, Δ_y^p ∼= π_p(Δ_{w_0^p y}(\ell(w_0^p))) and T_y^p ∼= π_p(T_{w_0^p y}(−\ell(w_0^p))). It follows that the sequence given by Formula 3 is obtained by applying π_p to the following short exact sequence in O:
\[ 0 \to Δ_{w_0^p y}(−\ell(w_0^p)) \to T_{w_0^p y}(−\ell(w_0^p)) \to Q' \to 0. \]

Since Q' has a Verma flag, the socle of Q' is a direct sum of copies of shifts of Δ_{w_0}, and w_0 ∈ X_p^{long}. Consequently, π_p induces isomorphisms
\[ \text{Hom}_S(L_{w_0}^p(i), Q) = \text{Hom}_O(L_{w_0}^p(i), Q') \]
and
\[ \text{Hom}_S(L_{w_0}^p(i), T_y^p(−\ell(w_0^p))) = \text{Hom}_O(L_{w_0}^p(i), T_{w_0^p y}(a)). \]

This implies that
\[ \text{Ext}_S^1(L_{w_0}^p(i), Δ_y^p(-\ell(y))) = \text{Ext}_O^1(L_{w_0}^p(i), Δ_{w_0^p y}(−\ell(w_0^p))) \]
and the claim of the theorem now follows from Theorem 11.

□
5.2. Inclusions between proper standard modules. Recall from Subsection 5.2 the properties of homomorphisms between Verma modules in \( \mathcal{O} \). Applying the functor \( \pi_p \) gives:

- every non-zero map between two proper standard objects in \( S^p_0 \) is injective;
- for \( x, y \in X^\text{long}_p \), we have \( \text{hom}_{S_0}(\Delta^p_x, \Delta^p_y(d)) \neq 0 \) if and only if \( x \geq y \) and \( d = \ell(y) - \ell(x) \);
- \( \dim \text{Hom}_{S_0}(\Delta^p_x, \Delta^p_y) \leq 1 \), for all \( x, y \in X^\text{long}_p \).

We thus obtain the canonical quotients \( \overline{\Delta}^p_{y}/\overline{\Delta}^p_{x} := \overline{\Delta}^p_{y}/(\overline{\Delta}^p_{x}(\ell(y) - \ell(x))) \). The following analogue of Proposition 6 relates these quotients to extensions from simple to proper standard objects in \( S^p_0 \).

**Proposition 11.** For each \( x, y \in X^\text{long}_p \) with \( x \neq w_0 \), we have

\[
\dim \text{ext}^1(L^p_0(d), \overline{\Delta}^p_{y}/(-\ell(y))) = [\text{soc} \overline{\Delta}^p_{y}/\overline{\Delta}^p_{x} : L^p_0(d)].
\]

**Proof.** Let \( L := (L^p_0(d))^\text{sem} \) and suppose we have a short exact sequence

\[
0 \to \overline{\Delta}^p_{y}/(-\ell(y)) \to M \to L \to 0
\]

such that \( M \) is indecomposable. Since \( L \) is semisimple, we have

\[
\text{soc} M = \text{soc} \overline{\Delta}^p_{y}/(-\ell(y)) = L^p_{w_0}(-\ell(w_0)).
\]

Thus, the injective covers of \( \overline{\Delta}^p_{y}/\overline{\Delta}^p_{x} \) and of \( M \) coincide and are isomorphic to \( I^p_{w_0}(-\ell(w_0)) \). The latter is also isomorphic to a shift of \( P^p_{w_0} \).

Being both a tilting and a cotilting object, \( P^p_{w_0} \) has a proper standard filtration which starts with a submodule isomorphic to \( \overline{\Delta}^p_{w_0} \), up to shift. In particular, the cokernel of the inclusion

\[
0 \to \overline{\Delta}^p_{w_0}(-\ell(w_0)) \to I^p_{w_0}(-\ell(w_0))
\]

has a proper standard filtration.

As the socle of each proper standard module is a shift of \( L^p_{w_0} \), we have

\[
\text{Ext}^1(L^p_0, \overline{\Delta}^p_{w_0}) = 0,
\]

for any \( w \in X^\text{long}_p \) such that \( w \neq w_0 \). This implies that \( M \) must be a submodule of \( \overline{\Delta}^p_{w_0}(-\ell(w_0)) \). In other words, \( L \) should be a summand of the socle of the cokernel of the canonical inclusion \( \overline{\Delta}^p_{y}/(-\ell(y)) \subset \overline{\Delta}^p_{w_0}/(-\ell(w_0)) \).

On the other hand, any summand of this socle gives rise to a non-split short exact sequence as in Formula (4) (since in that case \( M \) obviously has simple socle). The claim follows. \( \square \)

5.3. Cokernel of inclusion of proper standard modules.

**Lemma 12.** Let \( x, y \in X^\text{short}_p \) be such that \( x \geq y \). Let \( z \in W \) be such that \( L_z \) appears in the socle of \( \Delta_y/\Delta_x \). Then \( z \in X^\text{long}_p \).

**Proof.** Note that \( x \in X^\text{short}_p \) is equivalent to \( sx > x \), for each \( s \in S \cap W^p \). Thus, if \( L_z \) appears in the socle of \( \Delta_y/\Delta_x \), (and thus in the socle of \( \Delta_x/\Delta_z \)) then, by [KMM3, Proposition 6], we have \( sz < z \), for each \( s \in S \cap W^p \). The latter is equivalent to \( z \in X^\text{long}_p \), as desired. \( \square \)
Proposition 13. For \( x, y \in X_p^{long} \) such that \( x \geq y \), we have
\[
\text{soc}(\Sigma^p_y / \Sigma^p_x) \cong \pi_p(\text{soc} \Delta_{w_0^p x / \Delta_{w_0^p y}}).
\]
This isomorphism holds as well for graded non-zero extensions in Theorem 15.

Proof. As mentioned in Subsection 4.3, we have the isomorphisms
\[
\Sigma^p_x \cong \pi_p(\Delta_{w_0^p x})(\ell(w_0^p)) \quad \text{and} \quad \Sigma^p_y \cong \pi_p(\Delta_{w_0^p y})(\ell(w_0^p)).
\]
Note that \( w_0^p x, w_0^p y \in X_p^{short} \). Therefore we may apply Lemma 12 to conclude that the socle of the cokernel of the inclusion \( \Delta_{w_0^p x} \subset \Delta_{w_0^p y} \) contains only \( L_z \) such that \( z \in X_p^{long} \). Now the claim of the proposition follows by applying \( \pi_p \).

5.4. Ungraded statements in type \( A \). In type \( A \), the above results can be summarized and made more precise as follows.

Proposition 14. In type \( A \), for \( x, y \in X_p^{long} \) such that \( x \geq y \), the cokernel of \( \Sigma^p_x \subset \Sigma^p_y \) is isomorphic to the (multiplicity-free) direct sum of all simples \( L_x^p \), where \( z \in X_p^{long} \) and \( z \in \Phi(BM(x) \setminus BM(y)) \).

Proof. This follows from Corollary 4 and Proposition 13.

Theorem 15. In type \( A \), let \( x, y \in X_p^{long} \). Then we have
\[
\dim \text{Ext}^1_Y(L_x^p, \Sigma^p_y) = \begin{cases} 
\ell(w_0^p y), & x = w_0; \\
1, & x \in \Phi(BM(y)); \\
0, & \text{otherwise}.
\end{cases}
\]

Proof. The case \( x = w_0 \) is covered by Theorem 10.

5.5. Graded statement in type \( A \). We can also explicitly determine the degree shifts for the graded non-zero extensions in Theorem 15.

Proposition 16. Assume we are in type \( A \). Let \( y \in X_p^{long} \) and \( x = \Phi((i, j, k)) \), for some \( (i, j, k) \in BM(y) \cap X_p^{long} \). Then the unique degree \( m \in \mathbb{Z} \) for which \( \dim \text{ext}^1(L_x^p(-m), \Sigma^p_y(-\ell(y))) = 1 \) is
\[
m = \frac{(n - 1)(n - 2)}{2} + |i - j| + 2k.
\]

Proof. This follows from Proposition 13, Proposition 11 and Proposition 7.

5.6. First extension from other simples to proper standard modules in other types. Proposition 11 and Proposition 13 translate all graded and ungraded results from \([KMM2]\) to the corresponding statements on the first extension spaces. In particular, we have
- for \( x, y \in X_p^{long} \), we have \( \text{Ext}^1(L_x^p, \Sigma^p_y) = 0 \) unless \( x \in \mathcal{J} \);
- if \( x \in \mathcal{J} \), we have
\[
\dim \text{Ext}^1(L_x^p, \Sigma^p_y) \leq |BM_t(w)|,
\]
where \( s, t \) are simple reflections in \( W \) such that \( sx > x \) and \( xt > x \).
We emphasize that the main point of giving the above bound is to have a general statement, and that the bound $|\text{BM}_T(w)|$ is a gross exaggeration in most of the cases. For computing/bounding first extension spaces between simple and proper standard modules, it is strongly recommended to ignore the bound $|\text{BM}_T(w)|$ and instead look at [KMM2, Section 5] (see also the discussion after [KMM2, Theorem F]).

6. First extension from a simple to a standard module in $S^p$

6.1. Elementary general observations. Since $w_0$ corresponds to the minimum element for the partial order with respect to which $A^p$ is stratified, the standard object $\Delta^p_{w_0}$ is a tilting object. Due to the special properties of $A^p$ mentioned at the end of Subsection 4.3, it is also a cotilting module. The simple object $L^p_{w_0}$ is a proper standard module. Therefore, due to the homological orthogonality of proper standard and cotilting modules, we have

$$\text{Ext}^i_c(L^p_{w_0}, \Delta^p_{w_0}) = 0,$$

for all $i > 0$.

The projective-injective object $I^p_{w_0}$ is a tilting object and is thus the tilting envelope of the standard object $\Delta^p_{w_0}$. Therefore the cokernel of the inclusion $\Delta^p_{w_0} \to I^p_{w_0}$ has a standard filtration. As the socle of each standard object is isomorphic to $L^p_{w_0}$, it follows that the only simple object appearing in the socle of the cokernel of the above inclusion is $L^p_{w_0}$. Consequently,

$$\text{Ext}^i_c(L^p_x, \Delta^p_{w_0}) = 0,$$

for all $x \in X^\text{long}_p \setminus \{w_0\}$.

We will generalize this result below in Subsection 6.3.

6.2. Reduction to category $O$. The following statement reduces the problem of computing first extensions from simple to standard objects in $S$ to the problem of computing first extensions between certain modules in $O$.

**Proposition 17.** For $x, y \in X^\text{long}_p$ and $i \in \mathbb{Z}$, we have an isomorphism

$$\text{ext}^i_c(L^p_x, \Delta^p_y(i)) \cong \text{ext}^1(L_x, Q_y(i)).$$

**Proof.** The functor $\pi_p$ connects $O_0$ and $S^p$. Since $\pi_p(L_x) = L^p_x$ and $\pi_p(Q_y) = \Delta^p_y$, we need to show that the socle of the cokernel $C_y$ of the natural inclusion $Q_y \hookrightarrow I_{w_0}$ only contains simples of the form $L_z$, where $z \in X^\text{long}_p$.

Let $a$ be the semi-simple part of $p$. For $w \in X^\text{long}_p$, the module $Q_w$ is obtained by parabolic induction (from $p$ to $g$) of a projective-injective module in the category $O$ for $a$, see [MS1, Proposition 2.9]. In particular, $Q_w$ is (an infinite) direct sum of projective-injective modules in the category $O$ for $a$. Since $I_{w_0} = P_{w_0}$ has a filtration whose subquotients are various $Q_w$'s, the module $I_{w_0}$ is (an infinite) direct sum of projective-injective module in the category $O$ for $a$. Consequently, the module $C_y$ is (an infinite) direct sum of projective injective module in the category $O$ for $a$. In particular, for any simple root $\alpha$ of $a$, the action of a non-zero element in $\alpha$, on any simple submodule $L_z$ of $C_y$ is injective.

This means that $sz > z$, for any simple reflection $s \in W^p$, and hence $z \in X^\text{long}_p$ as asserted. Now the statement of the proposition follows by comparing the long exact sequence obtained by applying $\text{Hom}^i_c(L_x, - (i))$ to the short exact sequence

$$0 \to Q_y \to I_{w_0} \to C_y \to 0$$

with the image of this long exact sequence under $\pi_p$. \hfill $\square$

As a corollary, we have the following general observation:
Corollary 18. For \( x, y \in X_p^{long} \) and \( i \in \mathbb{Z} \). If \( \text{ext}^1(L_p^\lambda, \Delta_y^\lambda(i)) \neq 0 \), then \( x \in \mathcal{J} \cup \{w_0\} \).

Proof. By Proposition 17, we need to show that the assumption \( \text{ext}^1(L_x, Q_y(i)) \neq 0 \) implies \( x \in \mathcal{J} \cup \{w_0\} \). The module \( Q_w \) has a Verma flag, by construction. From [KMM3, Proposition 3] it follows that \( \text{ext}^1(L_x, \Delta_w(i)) \neq 0 \), for \( w \in W \), implies \( x \in \mathcal{J} \cup \{w_0\} \). As any non-zero extension from \( L_x \) to \( Q_y(i) \) must induce a non-zero extension from \( L_x \) to one of the Verma subquotients of \( Q_y(i) \), the claim of the corollary follows.

6.3. The case of standard modules which can be obtained using projective functors. An element \( w \in W \) is called \((p-)\)special provided that the subgroup \( w^{-1}W^p w \) is parabolic, that is, there exists a parabolic subgroup \( W^p \) of \( W \) such that \( W^p w = wW^p \). For example, any \( w \in W^p \), in particular \( w_0^p \), is special. Also, \( w_0 \) is special, for we can choose \( W^p = w_0W^p w_0 \).

Proposition 19. Let \( x, y \in X_p^{long} \) and assume that \( y \) is special.

(i) We have
\[
\text{Ext}_\mathcal{O}^1(L_p^\lambda, \Delta_y^\lambda) \cong \text{Ext}_\mathcal{O}^1(L(x \cdot \lambda), \Delta(y \cdot \lambda)),
\]
where \( \lambda \) is an integral dominant weight which has the dot-stabilizer \( W^\tilde{p} \).

(ii) Under the additional assumption \( x \neq w_0 \), we have
\[
\dim \text{Ext}_\mathcal{O}^1(L_p^\lambda, \Delta_y^\lambda) = \dim \text{Ext}_\mathcal{O}^1(L(x \cdot \lambda), \Delta(y \cdot \lambda)) = [\text{soc} \Delta_{x}/\Delta_{w_0^p y} : I_x].
\]

Proof. Let \( x, y \) be as above and let \( \tilde{p} \) be such that \( W^p y = yW^\tilde{p} \). Let \( \tilde{w}_0 \) be the longest element in \( W^\tilde{p} \). We have \( Q_w \cong \theta_{\tilde{w}_0} \Delta_w \), since both sides are characterized as the quotient of \( P_w \) with a filtration where the factors are exactly \( \Delta_z(\ell(w) + \ell(z)) \) for \( z \in W'^u w = wW^\tilde{p} \) (with multiplicity one). Let \( \lambda \) be a dominant integral weight for which \( W^\tilde{p} \) is the dot-stabilizer. Let \( \mathcal{O}_\lambda \) be the corresponding block of \( \mathcal{O} \). Consider the corresponding projective functors
\[
\theta_{\tilde{w}_0}^{an} : \mathcal{O}_0 \to \mathcal{O}_\lambda \quad \text{and} \quad \theta_{\tilde{w}_0}^{out} : \mathcal{O}_\lambda \to \mathcal{O}_0
\]
of translation onto and out of the \( W^\tilde{p} \)-wall, respectively. These functors are biadjoint and \( \theta_{\tilde{w}_0} \cong \theta_{\tilde{w}_0}^{an} \circ \theta_{\tilde{w}_0}^{out} \). In particular, for \( x \in X_p^{long} \), we have
\[
\text{Ext}_\mathcal{O}^1(L_x, Q_w) \cong \text{Ext}_\mathcal{O}^1(\theta_{\tilde{w}_0}^{an} L_x, \theta_{\tilde{w}_0}^{out} \Delta_w).
\]

Now we prove the second equality in the second statement, where the first equality is obtained from the first claim. If \( x \neq w_0 \) then the proof of Proposition 11 (or of Proposition 2) identifies the value \( \dim \text{Ext}_\mathcal{O}^1(L(x \cdot \lambda), \Delta(y \cdot \lambda)) \) with the value \( [\text{soc} \Delta(\lambda)/\Delta(y \cdot \lambda) : L(x \cdot \lambda)] \). The latter agrees with \( [\text{soc} \Delta_{x}/\Delta_{w_0^p y} : L_x] \) by [KMM3, Proposition 15] since \( w_0^p y = y\tilde{w}_0 \) is the shortest element in \( W^p y = yW^\tilde{p} \).

6.4. A type A formula. By Subsection 3.5, Proposition 19 completely computes the first extension between simple and standard in \( S \)-subcategories in type \( A \).

Proposition 20. Let \( x, y \in X_p^{long} \) with \( y \) special and assume we are in type \( A \). Then
\[
\dim \text{Ext}_\mathcal{O}^1(L_p^\lambda, \Delta_y^\lambda) = \begin{cases} 
 c(\overline{\tau}) - \text{rank}(W^p), & \overline{\tau} = w_0; \\
 1, & \overline{\tau} \in \Phi(BM(g)); \\
 0, & \text{otherwise}.
\end{cases}
\]
The graded version of this claim is obtained in the obvious way using the shifts described in Subsection 3.6.

7. Examples

7.1. $\text{sl}_3$-example. Consider the case of the Lie algebra $\text{sl}_3$. In this case we have $W = S_3 = \{e, s, t, st, ts, w_0 = sts = tst\}$. Let $p$ be such that $W^p = \{e, s\}$. With such a choice, we have

\[ x^\text{long}_p = \{s, st, w_0\} \quad \text{and} \quad x^\text{short}_p = \{e, t, ts\} \]

and the Hasse diagrams for the (opposite of the) Bruhat order on $W$, $x^\text{long}_p$ and $x^\text{short}_p$ are as follows:

\[ e \quad s \quad t \quad st \quad ts \quad w_0 \]

\[ e \quad s \quad t \quad st \quad ts \quad w_0 \]

If we denote $L^p_x$ simply by $x$, then the subquotients of the graded filtrations of the indecomposable projectives in $\mathcal{S}_0$ are as follows:

\[ \begin{array}{ccc}
P^p_{w_0} & P^p_{st} & P^p_s \\
\text{w_0} & \text{st} & \text{s} \\
\text{st} & \text{w_0} & \text{sts} \\
\text{st} & \text{w_0} & \text{sts} \\
\text{w_0} & \text{st} & \text{w_0} \\
\text{w_0} & \text{w_0} & \text{w_0} \\
\text{w_0} & \text{w_0} & \text{w_0} \\
\text{w_0} & \text{w_0} & \text{w_0} \\
\end{array} \]

The (graded and unique) Loewy filtrations of the proper standard modules are as follows:

\[ \begin{array}{ccc}
\Sigma^p_{w_0} & \Sigma^p_{st} & \Sigma^p_s \\
\text{w_0} & \text{st} & \text{s} \\
\text{w_0} & \text{w_0} & \text{sts} \\
\text{w_0} & \text{w_0} & \text{sts} \\
\text{w_0} & \text{w_0} & \text{w_0} \\
\text{w_0} & \text{w_0} & \text{w_0} \\
\text{w_0} & \text{w_0} & \text{w_0} \\
\end{array} \]

We note that all proper standard modules are multiplicity-free and hence the corresponding module diagrams are well-defined. This is not the case for the indecomposable projectives $P^p_{w_0}$ and $P^p_{st}$ which are not even graded multiplicity-free. The projective $P^p_s$ is not multiplicity-free but it is graded multiplicity-free and hence its module diagram is well-defined as well as the algebra $A^p$ is positively graded.

The following table contains information on the dimension and the degree shift for the extension spaces from a simple object to a proper standard object in the format $(d, m)$.
for the formula $\dim \text{ext}^1_{S_0} (L^p_x, \Sigma^p_y(m))$:

| $x \setminus y$ | $s$ | $st$ | $w_0$ |
|-----------------|-----|------|-------|
| $s$             | $-$ | $(1, 1)$ | $-$ |
| $st$            | $-$ | $-$ | $(1, 1)$ |
| $w_0$           | $(2, 0)$ | $(2, -1)$ | $(1, 2)$ |

Note that $s$ and $w_0$ are special while $st$ is not. The following table contains information on the dimension and the degree shift for the extension spaces from a simple object to a standard object in the format $(d, m)$ for the formula $\dim \text{ext}^1_{S_0} (L^p_x, \Delta^p_y(m))$:

| $x \setminus y$ | $s$ | $st$ | $w_0$ |
|-----------------|-----|------|-------|
| $s$             | $-$ | $(1, 1)$ | $-$ |
| $st$            | $-$ | $-$ | $(1, 1)$ |
| $w_0$           | $(1, 2)$ | $(1, 1)$ | $-$ |

7.2. $sl_3$-example. The Lie algebra $sl_3$ is the smallest Lie algebra for which there are non-trivial Kazhdan-Lusztig polynomials. These non-trivial KL-polynomials also contribute to a non-trivial extension from a simple module to a Verma module.

We have $W = S_4$ and let $s_1$, $s_2$ and $s_3$ be the simple reflections with the corresponding Dynkin diagram

$s_1 \sim \sim s_2 \sim \sim s_3$.

As pointed out in [KMM3] Subsection 1.3, we have the following fact (which we present here in the graded version):

$\text{ext}^1 (L_{s_2 w_0} (-3), \Delta_{s_2} (-1)) \cong \mathbb{C}$.

Note that $s_2 w_0$ is a longest representative in the cosets $W^p \setminus W$ for the choices of a parabolic subgroups $W^p$ in $W$ given by the following subsets of simple roots:

$\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}$.

We denote the corresponding parabolic subalgebras by $p_i$, for $i = 1, 2, 3, 4$. Consequently, we have:

$\text{ext}^1_{S_1} (L^p_{s_2 w_0} (-3), \Sigma^p_{s_2} (-1)) \cong \mathbb{C}$,

$\text{ext}^1_{S_2} (L^p_{s_2 w_0} (-3), \Sigma^p_{s_1, s_2} (-2)) \cong \mathbb{C}$,

$\text{ext}^1_{S_3} (L^p_{s_2 w_0} (-3), \Sigma^p_{s_2} (-2)) \cong \mathbb{C}$,

$\text{ext}^1_{S_4} (L^p_{s_2 w_0} (-3), \Sigma^p_{s_1, s_2, s_3} (-3)) \cong \mathbb{C}$.

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