A REVIEW ON MULTIFRACTAL ANALYSIS OF HEWITT-STROMBERG MEASURES

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ABSTRACT. We estimate the upper and lower bounds of the Hewitt-Stromberg dimensions. In particular, these results give new proofs of theorems on the multifractal formalism which is based on the Hewitt-Stromberg measures and yield results even at points $q$ for which the upper and lower multifractal Hewitt-Stromberg dimension functions differ. Finally, concrete examples of a measure satisfying the above property are developed.

1. INTRODUCTION

The multifractal analysis is a natural framework to finely describe geometrically the heterogeneity in the distribution at small scales of the measures on a metric space. The multifractal formalism aims at expressing the dimension of the level sets in terms of the Legendre transform of some free energy function in analogy with the usual thermodynamic theory. One says that $\mu$ satisfies the multifractal formalism if the Legendre transform of this free energy yields the Hausdorff dimension of the level set of the local Hölder exponent of $\mu$. For the measures we consider in this article, the thermodynamic limit does not exist: the free energy splits into two functions given by the upper and lower limits, and the Legendre transforms of both of these functions have an interpretation in terms of dimensions of the sets of iso-singularities. However in the standard formalism discontinuities of the free energy or one of its derivatives correspond to phase transitions, we are facing a new phenomenon. It would be of interest to know whether physical systems exhibiting such behavior exist.

The motivations of this paper come from several sources. The authors in [5] constructed "bad" measures whose Olsen’s multifractal functions $b_\mu$ and $B_\mu$ coincide at one or two points only. These measures can fulfill the classical multifractal formalism at one or two points only, i.e., the classical multifractal formalism does not hold. Ben Nasr et al. in [5] give two constructions: The first one provides $b_\mu$ and $B_\mu$ functions with Lipschitz regularity. The second one provides real analytic functions, but in this case, the support of the measure is a Cantor set of Hausdorff dimensions less than 1 (they use inhomogeneous Bernoulli measures). Later, Ben Nasr et al. [4] and Shen [31], revisited the first example in [5, Section 2.3], the meaning no interpretation of Olsen’s multifractal function was given in terms of dimensions. By the way, it was proven in [4] that, for some range of $\alpha$, the Hausdorff dimension of the local Hölder exponent is given by the value of the Legendre transform of $b_\mu$ at $\alpha$ and their packing dimension is the value of the Legendre transform of $B_\mu$ at $\alpha$ (This is the idea that we refine to get our results for the multifractal formalism for the Hewitt-Stromberg measure). In [31], Shen studies the main results of [4] for which the function $b_\mu$ and $B_\mu$ can be real analytic functions. Let us mention also that the authors in [15, 34, 35, 36] extended these results to some Moran measures associated with homogeneous Moran fractals. The above results were later generalized in [18, 30] to the relative multifractal formalism, and in [27] to inhomogeneous multinomial measures constructed on the product symbolic space.

Motivated by the above papers, the authors in [1, 2, 6, 28, 29] introduced and studied a new multifractal formalism based on the Hewitt-Stromberg measures. We point out that this formalism is completely parallel to Olsen’s multifractal formalism introduced in [21] which is based on the Hausdorff and packing measures. In fact, the two most important (and well-known) measures in fractal geometry are the Hausdorff measure and the packing measure. However, in 1965, Hewitt and Stromberg introduced a further fractal measure in their classical textbook [14, Exercise (10.51)]. Since then, these measures have been investigated by several authors, highlighting their importance in the study of local properties of fractals and products of fractals. One can cite, for example [3, 6, 9, 10, 11, 12, 13, 16, 22, 29, 37]. In particular, Edgar’s textbook [7, pp. 32-36] provides an excellent and systematic introduction to these measures. Such measures appears also explicitly, for example, in Pesin’s monograph [23, 5.3] and implicitly in Mattila’s text [17]. One of the purposes of this paper is to define and study a class of natural multifractal analogue of the Hewitt-Stromberg measures. While Hausdorff and packing measures are

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defined using coverings and packings by families of sets with diameters less than a given positive number $\delta$, say, the Hewitt-Stromberg measures are defined using packings of balls with a fixed diameter $\delta$.

In the present paper we estimate the upper and lower bounds of the Hewitt-Stromberg dimensions of a subset $E$ of $\mathbb{R}^n$. We apply the main results to give new (and in several cases simpler) proofs of theorems on the multifractal formalism which is based on the Hewitt-Stromberg measures, that they yield results even at points $q$ for which the the upper and lower multifractal Hewitt-Stromberg dimension functions differ. We also give some examples of a measure for which the multifractal functions are different and for which the lower and upper multifractal Hewitt-Stromberg functions are different and the lower and upper Hewitt-Stromberg dimensions of the level sets of the local Hölder exponent are given by the Legendre transform respectively of lower and upper multifractal Hewitt-Stromberg dimension functions.

The paper is structured as follows. In Section 2 we recall the definitions of the various fractal and multifractal dimensions and measures investigated in the paper. The definitions of the Hausdorff and packing measures and the Hausdorff and packing dimensions are recalled in Section 2.1, and the definitions of the Hewitt-Stromberg measures are recalled in Section 2.2, while the definitions of the Hausdorff and packing measures are well-known, we have, nevertheless, decided to include these-there are two main reasons for this: firstly, to make it easier for the reader to compare and contrast the Hausdorff and packing measures with the less well-known Hewitt-Stromberg measures, and secondly, to provide a motivation for the Hewitt-Stromberg measures. In Section 2.3 we recall the definitions of the multifractal Hewitt-Stromberg measures and separator functions, and study their properties. In particular, this section recalls earlier results on the values of the multifractal formalism based on Hewitt-Stromberg measures developed in [1, 2, 6]; this discussion is included in order to motivate our main results presented in Section 3. Section 3.1 gives some estimates of the upper and lower bounds of the Hewitt-Stromberg dimensions. In Sections 3.2-3.3 we apply the results from Section 3.1 to give simple proofs of theorems on the multifractal formalism developed in [2] and yield a result even at points $q$ for which the multifractal Hewitt-Stromberg dimension functions differ. Finally, Section 4 contains concrete examples related to these concepts.

2. Preliminaries

2.1. Hausdorff measure, packing measure and dimensions. While the definitions of the Hausdorff and packing measures and the Hausdorff and packing dimensions are well-known, we have, nevertheless, decided to briefly recall the definitions below. There are several reasons for this: firstly, since we are working in general metric spaces, the different definitions that appear in the literature may not all agree and for this reason it is useful to state precisely the definitions that we are using; secondly, and perhaps more importantly, the less well-known Hewitt-Stromberg measures (see Section 2.2) play an important part in this paper and to make it easier for the reader to compare and contrast the Hausdorff and packing measures with the less well-known Hewitt-Stromberg measures, and thirdly, in order to provide a motivation for the Hewitt-Stromberg measures. Let $(X, d)$ be a separable metric space, $E \subseteq X$ and $t > 0$. Throughout this paper, $B(x, r)$ stands for the open ball

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$ 

The Hausdorff measure is defined, for $\delta > 0$, as follows

$$\mathcal{H}^t_\delta(E) = \inf \left\{ \sum_i \left( \text{diam}(E_i) \right)^t \mid E \subseteq \bigcup_i E_i, \ \text{diam}(E_i) < \delta \right\}.$$ 

This allows to define first the $t$-dimensional Hausdorff measure $\mathcal{H}^t(E)$ of $E$ by

$$\mathcal{H}^t(E) = \sup_{\delta > 0} \mathcal{H}^t_\delta(E).$$ 

Finally, the Hausdorff dimension $\dim_H(E)$ is defined by

$$\dim_H(E) = \sup \left\{ t \geq 0 \mid \mathcal{H}^t(E) = +\infty \right\}.$$ 

The packing measure is defined, for $\delta > 0$, as follows

$$\mathcal{P}^t_\delta(E) = \sup \left\{ \sum_i (2r_i)^t \right\},$$

where $r_i$ are the radii of the balls in the packing. 

The packing dimension $\dim_P(E)$ is defined by

$$\dim_P(E) = \sup \left\{ t \geq 0 \mid \mathcal{P}^t_\delta(E) = +\infty \right\}.$$ 

These measures are used to estimate the upper and lower bounds of the Hewitt-Stromberg dimensions.
where the supremum is taken over all open balls \( B_i = B(x_i, r_i) \) such that \( r_i \leq \delta \) and with \( x_i \in E \) and \( B_i \cap B_j = \emptyset \) for all \( i \neq j \). The \( t \)-dimensional packing pre-measure \( \mathcal{P}^t(E) \) of \( E \) is now defined by
\[
\mathcal{P}^t(E) = \sup_{\delta > 0} \mathcal{P}^t_\delta(E).
\]
This makes us able to define the \( t \)-dimensional packing measure \( \mathcal{P}^t(E) \) of \( E \) as
\[
\mathcal{P}^t(E) = \inf \left\{ \sum_i \mathcal{P}^t(E_i) \mid E \subseteq \bigcup_i E_i \right\},
\]
and the packing dimension \( \dim_P(E) \) is defined by
\[
\dim_P(E) = \sup \left\{ t \geq 0 \mid \mathcal{P}^t(E) = +\infty \right\}.
\]
The reader is referred to Falconer’s book [8] for an excellent discussion of the Hausdorff measure and the packing measure.

### 2.2. Hewitt-Stromberg measures and dimensions

In this section, we recall the definitions of the Hewitt-Stromberg measures, while the definitions of the Hausdorff and packing measures are well-known, we have, nevertheless, decided to include these-there are two main reasons for this: firstly, to make it easier for the reader to compare and contrast the Hausdorff and packing measures with the less well-known Hewitt-Stromberg measures, and secondly, to provide a motivation for the Hewitt-Stromberg measures. Let \( X \) be a separable metric space and \( E \subseteq X \). For \( t > 0 \), the Hewitt-Stromberg pre-measures are defined as follows,
\[
L^t(E) = \liminf_{r \to 0} N_r(E) (2r)^t \quad \text{and} \quad H^t(E) = \sup_{F \subseteq E} L^t(F),
\]
and
\[
\mathcal{P}^t(E) = \limsup_{r \to 0} M_r(E) (2r)^t,
\]
where the covering number \( N_r(E) \) of \( E \) and the packing number \( M_r(E) \) of \( E \) are given by
\[
N_r(E) = \inf \left\{ \sharp\{I\} \mid \left\{ B(x_i, r) \right\}_{i \in I} \text{ is a family of open balls with } x_i \in E \text{ and } E \subseteq \bigcup_i B(x_i, r) \right\}
\]
and
\[
M_r(E) = \sup \left\{ \sharp\{I\} \mid \left\{ B_i = B(x_i, r) \right\}_{i \in I} \text{ is a family of open balls with } x_i \in E \text{ and } B_i \cap B_j = \emptyset \text{ for } i \neq j \right\}.
\]

Now, we define the lower and upper \( t \)-dimensional Hewitt-Stromberg measures, which we denote respectively by \( H^t(E) \) and \( P^t(E) \), as follows
\[
H^t(E) = \inf \left\{ \sum_i \mathcal{H}^t(E_i) \mid E \subseteq \bigcup_i E_i \right\}
\]
and
\[
P^t(E) = \inf \left\{ \sum_i \mathcal{P}^t(E_i) \mid E \subseteq \bigcup_i E_i \right\}.
\]
We recall some basic inequalities satisfied by the Hewitt-Stromberg, the Hausdorff and the packing measure
\[
H^t(E) \leq \xi P^t(E) \leq \xi \mathcal{P}^t(E)
\]
and
\[
H^t(E) \leq \mathcal{H}^t(E) \leq \xi \mathcal{P}^t(E) \leq \xi \mathcal{P}^t(E),
\]
where \( \xi \) is the constant that appears in Besicovitch’s covering theorem.

The reader is referred to Edgar’s book [7, pp. 32] (see also [16, 22, Proposition 2.1]) for a systematic introduction to the Hewitt-Stromberg measures.

The lower and upper Hewitt-Stromberg dimension \( \dim_{MB}(E) \) and \( \overline{\dim}_{MB}(E) \) are defined by
\[
\dim_{MB}(E) = \inf \left\{ t \geq 0 \mid H^t(E) = 0 \right\} = \sup \left\{ t \geq 0 \mid H^t(E) = +\infty \right\}
\]
and
\[ \text{dim}_{MB}(E) = \inf \left\{ t \geq 0 \mid P^t(E) = 0 \right\} = \sup \left\{ t \geq 0 \mid P^t(E) = +\infty \right\}. \]

The lower and upper box dimensions, denoted by \( \text{dim}_B(E) \) and \( \overline{\text{dim}}_B(E) \), respectively, are now defined by
\[ \text{dim}_B(E) = \liminf_{r \to 0} \frac{\log N_r(E)}{-\log r} = \liminf_{r \to 0} \frac{\log M_r(E)}{-\log r} \]
and
\[ \overline{\text{dim}}_B(E) = \limsup_{r \to 0} \frac{\log N_r(E)}{-\log r} = \limsup_{r \to 0} \frac{\log M_r(E)}{-\log r}. \]

These dimensions satisfy the following inequalities,
\[ \text{dim}_H(E) \leq \text{dim}_B(E) \leq \overline{\text{dim}}_B(E) \leq \text{dim}_P(E), \]
\[ \text{dim}_H(E) \leq \text{dim}_P(E) \leq \overline{\text{dim}}_B(E) \leq \text{dim}_H(E) \]
and
\[ \text{dim}_H(E) \leq \text{dim}_B(E) \leq \overline{\text{dim}}_B(E). \]

In particular, we have (see [8, 20])
\[ \text{dim}_{MB}(E) = \inf \left\{ \sup_i \text{dim}_B(E_i) \mid E \subseteq \bigcup_i E_i, E_i \text{ are bounded in } X \right\} \]
and
\[ \overline{\text{dim}}_{MB}(E) = \inf \left\{ \sup_i \overline{\text{dim}}_B(E_i) \mid E \subseteq \bigcup_i E_i, E_i \text{ are bounded in } X \right\}. \]

The reader is referred to [7, 8] for an excellent discussion of the Hausdorff dimension, the packing dimension, lower and upper Hewitt-Stromberg dimension and the box dimensions.

2.3. Multifractal Hewitt-Stromberg measures and dimension functions. This section gives a brief summary of the main results in [1, 2]. We recall the definitions of the lower and upper multifractal Hewitt-Stromberg measures and dimension functions. Let \( q, t \in \mathbb{R} \) and \( \mu \) be a compactly supported Borel probability measure on \( \mathbb{R}^n \). For \( E \subseteq K =: \text{supp} \mu \), the pre-measure of \( E \) is defined by
\[ P_{q,t}^\mu(E) = \limsup_{r \to 0} M_{q,t}^\mu(E)(2r)^t, \]
where
\[ M_{q,t}^\mu(E) = \sup \left\{ \sum_i \mu(B(x_i, r))^q \mid \left( B(x_i, r) \right)_i \text{ is a centered packing of } E \right\}. \]

It is clear that \( P_{q,t}^\mu \) is increasing and \( P_{q,t}^\mu(\emptyset) = 0 \). However it is not \( \sigma \)-additive. For this, by using the standard Method I construction [25, Theorem 4], we introduce the \( P_{q,t}^\mu \)-measure defined by
\[ P_{q,t}^\mu(E) = \inf \left\{ \sum_i P_{q,t}^\mu(E_i) \mid E \subseteq \bigcup_i E_i \right\}. \]

In a similar way we define
\[ L_{q,t}^\mu(E) = \liminf_{r \to 0} N_{q,t}^\mu(E)(2r)^t, \]
where
\[ N_{q,t}^\mu(E) = \inf \left\{ \sum_i \mu(B(x_i, r))^q \mid \left( B(x_i, r) \right)_i \text{ is a centered covering of } E \right\}. \]

Since \( L_{q,t}^\mu \) is not increasing and not countably sub-additive, one needs a standard modification to get an outer measure. Hence, we modify the definition as follows
\[ \overline{P}_{q,t}^\mu(E) = \sup_{F \subseteq E} L_{q,t}^\mu(F) \]
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and by applying now the standard Method I construction [25, Theorem 4], we obtain
\[
H_\mu^{q,t}(E) = \inf \left\{ \sum_i H_\mu^{q,t}(E_i) \mid E \subseteq \bigcup_i E_i \right\}.
\]

The measure \(H_\mu^{q,t}\) is of course a multifractal generalization of the lower \(t\)-dimensional Hewitt-Stromberg measure \(H^t\), whereas \(P_\mu^{q,t}\) is a multifractal generalization of the upper \(t\)-dimensional Hewitt-Stromberg measures \(P^t\).

In fact, it is easily seen that, for \(t > 0\), one has
\[
H_\mu^{0,t} = H^t \quad \text{and} \quad P_\mu^{0,t} = P^t.
\]

The following result describes some of the basic properties of the multifractal Hewitt-Stromberg measures including the fact that \(H_\mu^{q,t}\) and \(P_\mu^{q,t}\) are outer measures and summarizes the basic inequalities satisfied by these measures.

**Theorem 1.** [1] Let \(q, t \in \mathbb{R}\). Then for every set \(E \subseteq K\) we have

1. the set functions \(H_\mu^{q,t}\) and \(P_\mu^{q,t}\) are outer measures and thus they are measures on the algebra of the measurable sets.
2. There exists an integer \(\xi \in \mathbb{N}\), such that \(H_\mu^{0,t}(E) \leq \xi P_\mu^{q,t}(E)\).

The measures \(H_\mu^{q,t}\) and \(P_\mu^{q,t}\) and the pre-measure \(P_\mu^{q,t}\) assign in the usual way a multifractal dimension to each subset \(E\) of \(\mathbb{R}^n\). They are respectively denoted by \(b_\mu^q(E), B_\mu^q(E)\), and \(\Delta_\mu^q(E)\).

**Proposition 1.** [2] Let \(q \in \mathbb{R}\) and \(E \subseteq K\). Then

1. there exists a unique number \(b_\mu^q(E) \in [-\infty, +\infty]\) such that
\[
H_\mu^{q,t}(E) = \begin{cases} 
\infty & \text{if } t < b_\mu^q(E), \\
0 & \text{if } b_\mu^q(E) < t,
\end{cases}
\]
2. there exists a unique number \(B_\mu^q(E) \in [-\infty, +\infty]\) such that
\[
P_\mu^{q,t}(E) = \begin{cases} 
\infty & \text{if } t < B_\mu^q(E), \\
0 & \text{if } B_\mu^q(E) < t,
\end{cases}
\]
3. there exists a unique number \(\Delta_\mu^q(E) \in [-\infty, +\infty]\) such that
\[
\overline{P}_\mu^{q,t}(E) = \begin{cases} 
\infty & \text{if } t < \Delta_\mu^q(E), \\
0 & \text{if } \Delta_\mu^q(E) < t.
\end{cases}
\]

In addition, we have
\[
b_\mu^q(E) \leq B_\mu^q(E) \leq \Delta_\mu^q(E).
\]

The number \(b_\mu^q(E)\) is an obvious multifractal analogue of the lower Hewitt-Stromberg dimension \(\underline{\dim}_{MB}(E)\) of \(E\) whereas \(B_\mu^q(E)\) is an obvious multifractal analogues of the upper Hewitt-Stromberg dimension \(\overline{\dim}_{MB}(E)\) of \(E\). In fact, it follows immediately from the definitions that
\[
b_\mu^0(E) = \underline{\dim}_{MB}(E) \quad \text{and} \quad B_\mu^0(E) = \overline{\dim}_{MB}(E).
\]

Next, we define the separator functions \(\Delta_\mu, B_\mu, b_\mu : \mathbb{R} \to [-\infty, +\infty]\) by,
\[
\Delta_\mu(q) = \Delta_\mu^q(K), \quad B_\mu(q) = B_\mu^q(K) \quad \text{and} \quad b_\mu(q) = b_\mu^q(K).
\]

The definition of these dimension functions makes it clear that they are counterparts of the \(\tau\)-function which appears in the multifractal formalism. This being the case, it is important that they have the properties described by the physicists. The next proposition shows that these functions do indeed have some of these properties.

**Proposition 2.** [2] Let \(q \in \mathbb{R}\) and \(E \subseteq K\).

1. The functions \(q \mapsto H_\mu^{q,t}(E), P_\mu^{q,t}(E), C_\mu^{q,t}(E)\) are decreasing.
(2) The functions \( t \mapsto H^q_\mu(E), P^q_\mu(E), C^q_\mu(E) \) are decreasing.

(3) The functions \( q \mapsto b^q_\mu(E), B^q_\mu(E), \Delta^q_\mu(E) \) are decreasing.

(4) The functions \( q \mapsto B^q_\mu(E), \Delta^q_\mu(E) \) are convex.

(5) For \( q < 1 \), we have \( 0 \leq b_\mu(q) \leq B_\mu(q) \leq \Delta_\mu(q) \).

(6) For \( q = 1 \), we have \( b_\mu(q) = B_\mu(q) = \Delta_\mu(q) = 0 \).

(7) For \( q > 1 \), we have \( b_\mu(q) \leq B_\mu(q) \leq \Delta_\mu(q) \leq 0 \).

**Remark 1.** It is easy to check that we get the same values for fractal and multifractal measures and dimensions if we use just open balls or just closed balls and would not change the results (for more detail see [4, 8, 26, 32]). Note that the use of just closed balls is necessary where the use of open balls was erroneous when applying Vitali’s theorem and in this case, we need to apply the Besicovitch covering theorem. In particular, if we use closed balls, then we obtain

\[
H^t(E) \leq P^t(E) \leq \mathcal{F}^t(E)
\]

and

\[
\mathcal{H}^d(E) \leq H^t(E) \leq P^t(E) \leq \mathcal{F}^t(E).
\]

In addition, when \( q \leq 0 \) or \( q > 0 \) and \( \mu \) satisfies the doubling condition, we have

\[
H^q_\mu(E) \leq P^q_\mu(E).
\]

The upper and lower local dimensions of a measure \( \mu \) on \( \mathbb{R}^n \) at a point \( x \) are respectively given by

\[
\overline{\alpha}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}
\]

and

\[
\underline{\alpha}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},
\]

where \( B(x, r) \) denote the open ball of center \( x \) and radius \( r \). We refer to the common value as the local dimension of \( \mu \) at \( x \), and denote it by \( \alpha_\mu(x) \). For \( \alpha, \beta \geq 0 \), let us introduce the fractal sets

\[
E_\mu(\beta) = \overline{\mathcal{E}}_\mu(\beta) = \left\{ x \in K \mid \overline{\alpha}_\mu(x) \leq \beta \right\},
\]

\[
E_\mu(\alpha) = \underline{\mathcal{E}}_\mu(\alpha) = \left\{ x \in K \mid \underline{\alpha}_\mu(x) \geq \alpha \right\},
\]

\[
E_\mu(\alpha, \beta) = E_\mu(\alpha) \cap E_\mu(\beta)
\]

and

\[
E_\mu(\alpha) = E_\mu(\alpha) \cap \overline{E}_\mu(\alpha).
\]

Before stating this formally, we remind the reader that if \( \chi : \mathbb{R} \to \mathbb{R} \) is a real valued function, then the Legendre transform \( \chi^* : \mathbb{R} \to [-\infty, +\infty] \) of \( \chi \) is defined by

\[
\chi^*(y) = \inf_{x} (xy + \chi(y)).
\]

The multifractal formalism based on the measures \( H^q_\mu \) and \( P^q_\mu \) and the dimension functions \( b_\mu, B_\mu, \text{ and } \Delta_\mu \) provides a natural, unifying and very general multifractal theory which includes all the hitherto introduced multifractal parameters, i.e., the multifractal spectra functions

\[
\alpha \mapsto f_\mu(\alpha) =: \underline{\dim}_\mu B_\mu(\alpha)
\]

and

\[
\alpha \mapsto F_\mu(\alpha) =: \overline{\dim}_\mu B_\mu(\alpha).
\]

The dimension functions \( b_\mu \) and \( B_\mu \) are intimately related to the spectra functions \( f_\mu \) and \( F_\mu \), whereas the dimension function \( \Delta_\mu \) is closely related to the upper box spectrum (more precisely, to the upper multifractal
box dimension function). Let us briefly recall the notations and the main results proved in [2]. We say that the multifractal formalism which is based on the Hewitt-Stromberg measures holds if,

\[ \dim_{MB}(E_{\mu}(\alpha)) = \overline{\dim}_{MB}(E_{\mu}(\alpha)) = b_\mu^*(\alpha) = B_\mu^*(\alpha). \]

One important thing which should be noted is that there are many measures for which the multifractal formalism does not hold. In fact, one question which several measure theorists are interested in is, can we find a necessary and sufficient condition for the multifractal formalism to hold? The authors in [2] proved the following statement.

**Theorem 2.** Let \( \mu \) be a compactly supported Borel probability measure on \( \mathbb{R}^n \). Define

\[ \alpha = \sup_{0 < q} -\frac{b_\mu(q)}{q} \quad \text{and} \quad \overline{\alpha} = \inf_{0 > q} -\frac{b_\mu(q)}{q}. \]

Then,

\[ \dim_{MB}\left(E_{\mu}(\alpha)\right) \leq b_\mu^*(\alpha) \quad \text{and} \quad \overline{\dim}_{MB}\left(E_{\mu}(\alpha)\right) \leq B_\mu^*(\alpha) \quad \text{for all} \quad \alpha \in (\alpha, \overline{\alpha}). \]

It is more difficult to obtain a minoration for the dimensions of the sets described in Theorem 2. Selmi et al. [2] gave a sufficient condition for a valid multifractal formalism as follows.

**Theorem 3.** Let \( q \in \mathbb{R} \) and suppose that \( \mathcal{H}^{q,B_\mu(q)}_{\underline{\nu}}(K) > 0 \). Then,

\[ \dim_{MB}\left(E_\mu(-B_\mu^+(q), -B_\mu^-(q))\right) \geq \begin{cases} -qB_\mu^+(q) + B_\mu(q), & \text{if} \ q \geq 0, \\ -qB_\mu^-(q) + B_\mu(q), & \text{if} \ q \leq 0. \end{cases} \]

In particular, if \( B_\mu \) is differentiable at \( q \), one has

\[ \dim_{MB}\left(E_\mu(-B_\mu^+(q))\right) = \overline{\dim}_{MB}\left(E_\mu(-B_\mu^+(q))\right) = B_\mu^+(\alpha) = B_\mu^+(\alpha). \]

Conversely, if \( \dim_{MB}\left(E_\mu(-B_\mu^+(q))\right) \geq B_\mu^+(\alpha) = B_\mu^+(\alpha) \), then \( b(q) = B(q) \).

From the last part, when \( B_\mu^+(q) \) exists, \( b(q) = B(q) \), known as an analogue of Taylor regularity condition, is the necessary condition for a valid multifractal formalism. Nevertheless, we don’t know if the weaker condition \( b(q) = B(q) \) is sufficient to obtain the conclusion of the first part of Theorem 3.

# 3. Results and New Proofs

In this section we present our main results: we estimate the upper and lower bounds of the Hewitt-Stromberg dimensions of a subset \( E \) of \( \mathbb{R}^n \). We apply these results to give a new (and in several cases simpler) proof of Theorems 2 and 3 and provide some results even at points \( q \) for which the multifractal dimension functions \( b_\mu(q) \) and \( B_\mu(q) \) differ.

## 3.1. Estimates for the Hewitt-Stromberg dimensions

Let \( \nu \) be a compactly supported Borel probability measure on \( \mathbb{R}^n \) with \( \mathcal{K} = \text{supp} \nu \). Throughout this paper, we denote for any \( E \subseteq \mathcal{K} \),

\[ \pi(E) = \mathcal{H}^{1,0}_{\nu}(E). \]

**Theorem 4.** Let \( E \) be a bounded subset of \( \mathbb{R}^n \) and suppose that \( B^1_{\nu}(E) \leq 0 \), then

\[ \dim_{MB}(E) \leq \sup_{x \in E} \alpha_\nu(x) \quad \text{and} \quad \overline{\dim}_{MB}(E) \leq \sup_{x \in E} \overline{\alpha}_\nu(x). \]

**Proof.** Take \( \alpha > \sup_{x \in E} \alpha_\nu(x) \) and \( \varepsilon > 0 \). It follows from \( B^1_{\nu}(E) \leq 0 \) that \( P^1_{\nu,\varepsilon}(E) = 0 \), then we can choose a sequence \( (E_k)_k \) such that \( E = \bigcup_k E_k \),

\[ \sum_k P^1_{\nu,\varepsilon}(E_k) < 1 \quad \text{and} \quad \sum_k P^1_{\nu,\varepsilon}(E_k) = 0. \]

Fix \( k \in \mathbb{N} \). Let \( F \subseteq E_k \) and \( \delta > 0 \). For all \( x \in F \) we can find \( \lambda_x \geq 2 \) and \( \frac{\delta}{\lambda_x} < r_x < \delta \), such that

\[ \nu(B(x, r_x)) > r_x^{\alpha}. \]
The family \( \left( B(x, r_i) \right)_{x \in F} \) is a centered \( \delta \)-covering of \( F \). Then, we can choose a finite subset \( J \) of \( \mathbb{N} \) such that the family \( \left( B(x_i, r_i) \right)_{i \in J} \) is a centered \( \delta \)-covering of \( F \). Take \( \lambda = \max \{ \lambda_i; \ i \in J \} \), then for all \( i \in J \), we have

\[
\nu(B(x_i, \delta)) \geq \nu(B(x_i, r_{x_i})) > r_{x_i}^\alpha \geq \left( \frac{\delta}{\lambda} \right)^\alpha.
\]

Since \( \left( B(x_i, \delta) \right)_{i \in J} \) is a centered covering of \( F \), then by using Besicovitch’s covering theorem, we can construct \( \xi_n \) finite sub-families \( \left( B(x_{ij}, \delta) \right)_j, \ldots, \left( B(x_{\xi_n j}, \delta) \right)_j \), such that each

\[
F \subseteq \bigcup_{i=1}^{\xi_n} B(x_{ij}, \delta)
\]

and \( B_{ij} = B(x_{ij}, \delta) \) is a packing of \( F \) and such that \( \nu(B_{ij}) > \left( \frac{\delta}{\lambda} \right)^\alpha \). Observing that

\[
N_\delta(F)(2\delta)^{\alpha+\varepsilon} = \left( \sum_{i,j} (2\delta)^{\alpha+\varepsilon} \leq (2\lambda)^\alpha \sum_{i,j} \nu(B_{ij}) (2\delta)^\varepsilon \leq (2\lambda)^\alpha \xi_n M_\nu^1 \alpha (F)(2\delta)^\varepsilon
\]

It follows from this that

\[
L^{\alpha+\varepsilon}(F) \leq (2\lambda)^\alpha \xi_n \overline{P}_\nu^{1,\varepsilon}(F) \leq (2\lambda)^\alpha \xi_n \overline{P}_\nu^{1,\varepsilon}(E_k)
\]

which implies that

\[
\overline{H}^{\alpha+\varepsilon}(E_k) \leq (2\lambda)^\alpha \xi_n \overline{P}_\nu^{1,\varepsilon}(E_k)
\]

and

\[
\overline{H}^{\alpha+\varepsilon}(E) \leq \sum_k \overline{H}^{\alpha+\varepsilon}(E_k) \leq (2\lambda)^\alpha \xi_n \sum_k \overline{P}_\nu^{1,\varepsilon}(E_k) = 0.
\]

We therefore conclude that

\[
\dim_{MB}(E) \leq \alpha + \varepsilon \quad \text{for all} \quad \varepsilon > 0.
\]

Finally, we get

\[
\dim_{MB}(E) \leq \sup_{x \in E} \omega_\nu(x).
\]

Now, take \( \alpha > \sup_{x \in E} \omega_\nu(x) \) and \( \varepsilon > 0 \). Since \( \overline{B}_\nu^1 \alpha (E) \leq 0 \), we have \( \overline{P}_\nu^{1,\varepsilon}(E) = 0 \), then there exists \( (E_j)_j \) such that \( E = \bigcup_j E_j \),

\[
\sum_j \overline{P}_\nu^{1,\varepsilon}(E_j) < 1 \quad \text{and} \quad \sum_j \overline{P}_\nu^{1,\varepsilon}(E_j) = 0.
\]

For all \( x \in E \), we can therefore choose \( \delta > 0 \) such that, for all \( 0 < r < \delta \), we have

\[
\nu(B(x, r)) > r^\alpha.
\]

Put the set

\[
E_m = \left\{ x \in E \mid \text{for all} \ r < \frac{1}{m}, \ \nu(B(x, r)) > r^\alpha \right\}.
\]

Fix \( m \in \mathbb{N} \) and \( 0 < r < \min(\delta, \frac{1}{m}) \). Let \( \left( B_i = B(x_i, r) \right)_i \) be a packing of \( E_j \cap E_m \). Then

\[
M_\nu(E_j \cap E_m)(2r)^{\alpha+\varepsilon} \leq 2^\alpha \sum_i \nu(B_i)(2r)^\varepsilon \leq 2^\alpha M_\nu^1 \alpha (E_j)(2r)^\varepsilon
\]

and

\[
\overline{P}^{\alpha+\varepsilon}(E_j \cap E_m) \leq 2^\alpha \overline{P}_\nu^{1,\varepsilon}(E_j) = 0.
\]

This clearly implies that

\[
\overline{P}^{\alpha+\varepsilon}(E_m) \leq \sum_j \overline{P}^{\alpha+\varepsilon}(E_j \cap E_m) \leq 2^\alpha \sum_j \overline{P}_\nu^{1,\varepsilon}(E_j) = 0.
\]

We deduce the result from the fact that \( E_m \not\nearrow E \). This completes the proof of Theorem 4. \( \square \)

It always needs an extra condition to obtain a lower bound for the dimensions of sets.
Theorem 5. Let $E \subseteq \mathcal{H}$ and assume that $\pi(E) > 0$, then

$$\dim_{MB}(E) \geq \esssup_{x \in E} \alpha_\nu(x) \quad \text{and} \quad \overline{\dim}_{MB}(E) \geq \esssup_{x \in E} \overline{\alpha}_\nu(x),$$

where the essential bounds being related to the measure $\pi$.

Proof. Let $\alpha < \esssup_{x \in E} \alpha_\nu(x)$. Consider the set

$$F = \left\{ x \in E \left| \liminf_{r \to 0} \log \frac{\nu(B(x,r))}{\log r} > \alpha \right. \right\}.$$  

It is clear that $\pi(F) > 0$. For all $x \in E$ we can find $\delta > 0$ such that for all $0 < r < \delta$, we have

$$\nu(B(x,r)) \leq r^\alpha.$$  

Now, let $(F_j)_j$ be a countable partition of $F$. We put forward the set

$$F_{jp} = \left\{ x \in F_j \left| r < \frac{1}{p}, \nu(B(x,r)) \leq r^\alpha \right. \right\}.$$  

Fix $p \in \mathbb{N}$ and $G$ be a subset of $F_{jp}$. Let $0 < r < \min(\delta, \frac{1}{p})$ and $(B_i = B(x_i,r))_{i \in \{1, \ldots, N\nu(G)\}}$ be a centered covering of $G$, then

$$N_{\nu,G}(2r)^0 \leq \sum_{i} \nu(B_i) \leq 2^{-\alpha} N_{\nu,G}(2r)^\alpha.$$  

This clearly implies that

$$L_{\nu,G}^1(2r)^0 \leq 2^{-\alpha} L_{\nu,G}^1(2r)^0 \leq 2^{-\alpha} \overline{H}_{\nu,G}^1(F_{jp})$$  

and

$$H_{\nu,G}^1(F_{jp}) \leq 2^{-\alpha} \overline{H}_{\nu,G}^1(F_{jp}).$$  

Since $F_j = \bigcup_{p} F_{jp}$ for all $j$ and $\pi(F) > 0$, by making $\delta \to 0$, we obtain

$$0 < H_{\nu,G}^0(F) \leq \sum_{j} \sum_{p} H_{\nu,G}^0(F_{jp}) \leq 2^{-\alpha} \sum_{j} \sum_{p} \overline{H}_{\nu,G}^0(F_{jp})$$  

which implies that

$$0 < H^\alpha(F) \leq H^\alpha(E).$$  

We therefore conclude that

$$\dim_{MB}(E) \geq \alpha \quad \text{for all} \quad \alpha < \esssup_{x \in E} \alpha_\nu(x).$$  

Let $\alpha < \esssup_{x \in E} \overline{\alpha}_\nu(x)$. Consider the set

$$F = \left\{ x \in E \left| \limsup_{r \to 0} \log \frac{\nu(B(x,r))}{\log r} > \alpha \right. \right\},$$

then $\pi(F) > 0$. Fix $G \subseteq F$, then for all $x \in G$ and all $\delta > 0$, we can find a positive real number $0 < r_x < \delta$ such that

$$\nu(B(x,r_x)) \leq r_x^\alpha.$$  

The family $(B(x,r_x))_{x \in G}$ is a centered $\delta$-covering of $G$. Then, we can choose a finite subset $J$ of $\mathbb{N}$ such that the family $(B(x_i,r_x))_{i \in J}$ is a centered $\delta$-covering of $G$. Take $r = \max\{r_x, \quad i \in J\}$, then $(B(x_i,r))_{i \in J}$ is a centered covering of $G$. By using Besicovitch’s covering theorem, we can construct $\xi_n$ finite sub-families $(B(x_{ij},r))_j$, $i \leq \xi_n$ such that each

$$G \subseteq \bigcup_{i=1}^{\xi_n} \bigcup_j B(x_{ij},r),$$
\(B_{ij} = B(x_{ij}, r)\) is a packing of \(G\) and \(\nu(B_{ij}) \leq r^\alpha\). Then we conclude from this that
\[
N_{\nu, r}^1(G)(2r)^0 \leq \sum_{i,j} \nu(B_{i,j}) \leq 2^{-\alpha} \xi_n M_{r}(G)(2r)^\alpha.
\]
Which implies that
\[
L_{\nu}^{1,0}(G) \leq 2^{-\alpha} \xi_n \bar{P}^\alpha(G) \leq 2^{-\alpha} \xi_n \bar{P}^\alpha(F)
\]
and
\[
\bar{H}_{\nu}^{1,0}(F) \leq 2^{-\alpha} \xi_n \bar{P}^\alpha(F).
\]
Assume that \(F = \bigcup_i F_i\), then
\[
0 < \bar{H}_{\nu}^{1,0}(F) \leq \sum_i \bar{H}_{\nu}^{1,0}(F_i) \leq 2^{-\alpha} \xi_n \sum_i \bar{P}^\alpha(F_i).
\]
Finally, we conclude that
\[
P^\alpha(E) \geq P^\alpha(F) > 0 \quad \text{and} \quad \dim_{MB}(E) \geq \alpha.
\]
This completes the proof of Theorem 5. \(\square\)

3.2. **Proof of Theorems 2 and 3.** We present some intermediate results, which will be used in the proof of Theorems 2 and 3. Let \(\mu\) be a compactly supported Borel probability measure and \(\nu\) be a finite Borel measure on \(\mathbb{R}^n\) with \(K =: \text{supp} \mu = \text{supp} \nu\). For \(\zeta = (\mu, \nu), E \subseteq K\) and \(q, t \in \mathbb{R}\), we define
\[
\bar{P}^{q,t}_\zeta(E) = \limsup_{r \to 0} M^{q,\tau}_{\zeta,r}(E)(2r)^t,
\]
where
\[
M^{q,\tau}_{\zeta,r}(E) = \sup \left\{ \sum_i \mu(B(x_i, r))^q \nu(B(x_i, r)) \left| B(x_i, r) \right. \right\}; \text{ is a centered packing of } E\right\}.
\]
Next we define the measure \(\bar{P}^{q,t}_\zeta\) by
\[
\bar{P}^{q,t}_\zeta(E) = \inf \left\{ \sum_i \bar{P}^{q,t}_{\zeta}(E_i) \left| E \subseteq \bigcup_i E_i \right. \right\}.
\]
In a similar way we define
\[
L^{q,t}_\zeta(E) = \liminf_{r \to 0} N^{q,\tau}_{\zeta,r}(E)(2r)^t,
\]
where
\[
N^{q,\tau}_{\zeta,r}(E) = \inf \left\{ \sum_i \mu(B(x_i, r))^q \nu(B(x_i, r)) \left| B(x_i, r) \right. \right\}; \text{ is a centered covering of } E\right\}.
\]
Since \(L^{q,t}_\zeta\) is not increasing and not countably sub-additive, one needs a standard modification to get an outer measure. Hence, we modify the definition as follows
\[
\bar{H}^{q,t}_\zeta(E) = \sup_{F \subseteq E} L^{q,t}_\zeta(F)
\]
and
\[
\bar{H}^{q,t}_\zeta(E) = \inf \left\{ \sum_i \bar{H}^{q,t}_{\zeta}(E_i) \left| E \subseteq \bigcup_i E_i \right. \right\}.
\]
The functions \(\bar{H}^{q,t}_\zeta\) and \(\bar{P}^{q,t}_\zeta\) are metric outer measures. In addition, those measures assign, in the usual way, a multifractal dimension to each subset \(E\) of \(\mathbb{R}^n\). They are respectively denoted by \(b^{q}_\zeta(E), B^{q}_\zeta(E)\) and \(\Delta^{q}_\zeta(E)\). More precisely, we have
\[
b^{q}_\zeta(E) = \inf \left\{ t \in \mathbb{R} \left| \bar{H}^{q,t}_\zeta(E) = 0 \right. \right\},
\]
\[
B^{q}_\zeta(E) = \inf \left\{ t \in \mathbb{R} \left| \bar{P}^{q,t}_\zeta(E) = 0 \right. \right\},
\]
\[
\Delta^{q}_\zeta(E) = \inf \left\{ t \in \mathbb{R} \left| \bar{P}^{q,t}_\zeta(E) = 0 \right. \right\}.
\]
Finally, we immediately conclude that 
\[ b_\zeta^\alpha(E) \leq B_\zeta^\alpha(E) \leq \Delta_\zeta^\alpha(E). \]
Next, we define the multifractal dimension functions \( b_\zeta, B_\zeta \) and \( \Delta_\zeta : \mathbb{R} \to [-\infty, +\infty] \) by
\[ b_\zeta : q \mapsto b_\zeta^\alpha(K), \quad B_\zeta : q \mapsto B_\zeta^\alpha(K) \quad \text{and} \quad \Delta_\zeta : q \mapsto \Delta_\zeta^\alpha(K). \]

**Proposition 3.** Set \( f(t) = B_\zeta(t) \) and suppose that \( f(0) = 0 \) and \( \pi(K) > 0 \). Then
\[ \pi \left( \left[ E_\mu(-f'_+(0), -f'_-(0)) \right] \right) = 0, \]
where \( f'_- \) and \( f'_+ \) are the left and right hand sides derivatives of the function \( f \).

**Proof.** We must now prove that
\[ \pi \left( \left\{ x \in K \mid \alpha_\mu(x) > -f'_-(0) \right\} \right) = 0. \]
The proof of the statement
\[ \pi \left( \left\{ x \in K \mid \alpha_\mu(x) < -f'_+(0) \right\} \right) = 0 \]
is identical to the proof of the statement in the first part and is therefore omitted.

Fix \( \alpha > -f'_-(0) \), then there exist \( \beta, t > 0 \) such that \( \alpha > \beta > -f'_-(0) \) and \( f(-t) < \beta t \). It follows immediately from this that \( P_{\zeta}^{-t,\beta t}(K) = 0 \). We can choose a countable partition \( (E_j)_j \) of \( K \), i.e., \( K = \cup_j E_j \) such that
\[ \sum_j P_{\zeta}^{-t,\beta t}(E_j) \leq 1 \quad \text{and} \quad P_{\zeta}^{-t,\alpha t}(E_j) = 0, \quad \forall j. \]

Now put
\[ F = \left\{ x \in K \mid \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} > \alpha \right\}. \]
If \( x \in F \), for all \( \delta > 0 \), there exists \( 0 < r_\delta < \delta \) such that
\[ \mu(B(x, r)) \leq r_\delta^\alpha. \]
Fix \( F_j = E_j \cap F \) and \( G_j \subset F_j \). For all \( \delta > 0 \) and all \( j \), the family \( \left( B(x_j, r_{j,i}) \right)_{x_j \in G_j} \) is a centered \( \delta \)-covering of \( G_j \). Then, we can choose a finite subset \( J \) of \( \mathbb{N} \) such that the family \( \left( B(x_{j,i}, r_{j,i}) \right)_{i \in J} \) is a centered \( \delta \)-covering of \( G_j \). Take \( r_j = \max(r_{j,i}, \ i \in J) \), then \( \left( B(x_{j,i}, r_{j}) \right)_{i \in J} \) is a centered covering of \( G_j \). From Besicovitch’s covering theorem, we can construct \( \xi_n \) finite sub-families \( \left( B(x_{j_{i,k}}(r_j), r_j) \right)_{k} \) such that each
\[ \left( B_{j_{i,k}} = B(x_{j_{i,k}}, r_j) \right) \]
is a packing of \( G_j \) and \( \mu(B_{j_{i,k}}) \leq r_j^\alpha \). We have
\[ N^1_{\nu, r_j}(G_j)(2r_j)^0 \leq \sum_{i,k} \nu(B_{j_{i,k}}) = \sum_{i,k} \mu(B_{j_{i,k}})^{-\phi} \mu(B_{j_{i,k}})^{t} \nu(B_{j_{i,k}}) \leq 2^{\alpha t} \sum_{i,k} \mu(B_{j_{i,k}})^{-\phi} \mu(B_{j_{i,k}})(2r_j)^\alpha t. \]
We deduce that
\[ L^1_\nu(G_j) \leq 2^{\alpha t \xi_n} P_{\zeta}^{-t,\alpha t}(E_j). \]
It follows that
\[ \mathcal{H}^1_\nu(F_j) \leq 2^{\alpha t \xi_n} P_{\zeta}^{-t,\alpha t}(E_j) = 0. \]
Finally, we immediately conclude that
\[ \pi(F) = H^1_\nu(F) \leq \sum_j \mathcal{H}^1_\nu(F_j) = 0. \]
This completes the proof of Proposition 3.

**Proposition 4.** With the same notations and hypotheses as in the previous proposition, we have

\[
\dim_{MB} \left( E_{\mu} \left( -f'_-, f'_+ \right) \right) \geq \inf \left\{ \alpha(x) \mid x \in E_{\mu} \left( -f'_-, f'_+ \right) \right\}
\]

and

\[
\dim_{MB} \left( E_{\mu} \left( -f'_-, f'_+ \right) \right) \geq \inf \left\{ \alpha(x) \mid x \in E_{\mu} \left( -f'_-, f'_+ \right) \right\}.
\]

**Proof.** It follows immediately from Theorem 5 and Proposition 3.

**Proof of Theorems 2 and 3.** We can now prove Theorems 2 and 3. For \( q \leq 0 \), take

\[
\nu(B(x,r)) = \mu(B(x,r))^{q(2r)^{B_{\mu}(q)}}.
\]

By a simple calculation, we get

\[
B_{\zeta}(t) = B_{\mu}(q+t) - B_{\mu}(q)
\]

and if \( x \in E_{\mu}(\alpha) \), we have

\[
\alpha(x) = q\alpha(x) + B_{\mu}(q) \leq q\alpha + B_{\mu}(q).
\]

Theorem 4 clearly implies that

\[
\dim_{MB} \left( E_{\mu}(\alpha) \right) \leq \inf_{q \leq 0} q\alpha + B_{\mu}(q).
\]

The proof of the statement

\[
\dim_{MB} \left( E_{\mu}(\alpha) \right) \leq \inf_{q \geq 0} q\alpha + B_{\mu}(q)
\]

is very similar to the proof of the first statement and is therefore omitted.

We now turn to lower bound theorems. If moreover we suppose that \( H^{B_{\mu}(q)}_{\mu}(K) > 0 \), then clearly we have \( \pi(K) > 0 \). By taking Proposition 3 into consideration, we get

\[
\pi \left( E_{\mu} \left( -B'_{\mu_+}(q), -B'_{\mu_-}(q) \right) \right) > 0.
\]

It follows immediately from Proposition 4 that

\[
\dim_{MB} \left( E_{\mu} \left( -B'_{\mu_+}(q), -B'_{\mu_-}(q) \right) \right) \geq \left\{ \begin{array}{ll}
-qB'_{\mu_+}(q) + B_{\mu}(q), & \text{if } q \geq 0 \\
-qB'_{\mu_-}(q) + B_{\mu}(q), & \text{if } q \leq 0.
\end{array} \right.
\]

Finally, it follows from this and since \( B_{\mu} \) is differentiable at \( q \) that

\[
\dim_{MB} \left( E_{\mu}(-B'_{\mu}(q)) \right) = \dim_{MB} \left( E_{\mu}(-B'_{\mu}(q)) \right) = B_{\mu}^*(-B'_{\mu}(q)) = b_{\mu}^*(-B'_{\mu}(q)),
\]

which yields the desired result.

**Theorem 6.** The previous results hold if we replace the multifractal function \( B_{(.)} \) by the function \( \Delta_{(.)} \).
3.3. A result for which the Hewitt-Stromberg dimensions differ. We now focus on the case when the upper and the lower Hewitt-Stromberg dimension functions do not necessarily coincide, i.e., $B_{\mu}(q) \neq b_{\mu}(q)$ for $q \neq 1$. Consider the sets, for all $\alpha, \beta \geq 0$

$$E(\alpha, \beta) = \left\{ x \in K \mid \alpha_{\mu}(x) \leq \alpha \text{ and } \beta \leq \overline{\sigma}_{\mu}(x) \right\} \text{ and } E(\alpha) = E(\alpha, \alpha).$$

For a real function $\varphi$, we set

$$\varphi^{-}(q) = \limsup_{t \to 0} \frac{\varphi(q-t) - \varphi(q)}{-t}$$

and

$$\varphi^{+}(q) = \limsup_{t \to 0} \frac{\varphi(q+t) - \varphi(q)}{t}.$$

**Theorem 7.** For $q \in \mathbb{R}$, assume that $H_{\mu}^{b_{\mu},b_{\mu}'}(K) > 0$, then we have

$$\overline{\dim}_{MB}(E(-b_{\mu}(q), -b_{\mu}^{+}(q))) \geq \begin{cases} -q b_{\mu}^{+}(q) + b_{\mu}(q), & \text{if } q \geq 0, \\ -q b_{\mu}^{-}(q) + b_{\mu}(q), & \text{if } q \leq 0. \end{cases}$$

In addition, if $b_{\mu}$ is differentiable at $q$, one has

$$\overline{\dim}_{MB}(E(-b_{\mu}'(q))) \geq -q b_{\mu}'(q) + b_{\mu}(q).$$

Theorem 7 is a consequence from Theorem 5 and the following proposition.

**Proposition 5.** Let $\varphi(t) = b_{\xi}(t)$ and suppose that $\varphi(0) = 0$ and $\pi(K) > 0$. Then

$$\pi\left( \left\{ x \in K \mid \alpha_{\mu}(x) > -\varphi^{-}(0) \right\} \right) = 0$$

and

$$\pi\left( \left\{ x \in K \mid \overline{\sigma}_{\mu}(x) < -\varphi^{+}(0) \right\} \right) = 0.$$

**Proof.** We will prove the first assertion. The proof of the second statement is identical to the proof of the statement in the first part and is therefore omitted.

Let $\alpha > -\varphi^{-}(0) = \liminf_{t \to 0} \frac{\varphi(t)}{t}$, then there exists $t > 0$ such that $\alpha t > \varphi(-t)$. It is clear that $H_{\xi}^{-,\alpha t}(K) = 0$. Consider the following set

$$E = \left\{ x \in K \mid \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} > \alpha \right\}.$$

For all $x \in E$, we can therefore choose $\delta > 0$ such that, for all $0 < r < \delta$, we have

$$\mu(B(x, r)) < r^\alpha.$$

Now, let $(E_{j})_{j}$ be a countable partition of $E$. We put forward the set

$$E_{m_j} = \left\{ x \in E_j \mid \text{ for all } r < \frac{1}{m_j}, \mu(B(x, r)) < r^\alpha \right\}.$$

Fix $F \subset E_{m_j}$ and $0 < r < \min(\delta, \frac{1}{m_j})$. Let $(B_i = B(x_i, r))_{i}$ be a centered covering of $F$. Then

$$N_{\nu, r}^{1}(F)(2r)^{0} \leq \sum_{i} \nu(B_i) = \sum_{i} \mu(B_i)^{-\alpha} \mu(B_i)^{1} \nu(B_i) \leq 2^{-\alpha t} \sum_{i} \mu(B_i)^{-\alpha} \nu(B_i)(2r)^{\alpha t}.$$

It follows immediately from this that

$$N_{\nu, r}^{1}(F)(2r)^{0} \leq 2^{-\alpha t} N_{\xi}^{-1}(F)(2r)^{\alpha t}.$$

Letting $r$ tend to 0, gives

$$L_{\nu, 0}^{1}(F) \leq 2^{-\alpha t} L_{\xi}^{-1, \alpha t}(F) \leq 2^{-\alpha t} \Pi_{\xi}^{-1, \alpha t}(E_{m_j}).$$
We therefore conclude that
\[ \mathcal{H}_\nu^{1,0}(E_{m_j}) \leq 2^{-\alpha t} \mathcal{H}_\zeta^{-t,\alpha t}(E_{m_j}) \]
and
\[ \mathcal{H}_\nu^{1,0}(E) \leq \sum_j \sum_m \mathcal{H}_\nu^{1,0}(E_{m_j}) \leq 2^{-\alpha t} \sum_j \sum_m \mathcal{H}_\zeta^{-t,\alpha t}(E_{m_j}) . \]

Finally, we get
\[ \mathcal{H}_\nu^{1,0}(E) \leq 2^{-\alpha} \mathcal{H}_\zeta^{-t,\alpha t}(E) \leq 2^{-\alpha} \mathcal{H}_\zeta^{-t,\alpha t}(K) = 0. \]
This completes the proof of Proposition 5. \[ \square \]

**Remark 2.** Let \((\mathcal{X}, \rho)\) be a separable metric space, \(\mathcal{B}\) stand for the set of balls of \(\mathcal{X}\), and \(\mathcal{F}\) for the set of maps from \(\mathcal{B}\) to \([0, +\infty)\). The set of \(\mu \in \mathcal{F}\) such that \(\mu(B) = 0\) implies \(\mu(B') = 0\) for all \(B' \subseteq B\) will be denoted by \(\mathcal{F}^*\). For such a \(\mu\), one defines its support \(\text{supp} \mu\) to be the complement of the set \(\bigcup\{B \in \mathcal{B} \mid \mu(B) = 0\}\). If moreover, we assume \((\mathcal{X}, \rho)\) having the Besicovitch property then all the above results hold for any function \(\mu\) in \(\mathcal{F}^*\).

**Remark 3.** Notice that our formalism is adapted with the vectorial multifractal formalism introduced by Peyri`ere in [24] and in particular, the mixed multifractal formalism introduced in [19]. A valuation on the metric space \((\mathcal{X}, \rho)\) is a real function \(\xi\) defined on the set of balls of \(\mathcal{X}\), subject to the condition that it goes to \(+\infty\) as the radius goes to 0. In [24], Peyri`ere defined a more general multifractal formalism by considering quantities of type
\[ \sum_i e^{-\left(\langle q,\chi(x_i,r_i)\rangle + t\xi(x_i,r_i)\right)} \]
where \(\chi : \mathcal{X} \times [0, +\infty) \to \mathbb{E}'\) is a function, \(\mathbb{E}'\) is the dual of a separable real Banach space, \(\langle , \rangle\) is the duality bracket between \(\mathbb{E}\) and \(\mathbb{E}'\) and \((\mathcal{X}, \rho)\) is a metric space where the Besicovitch covering theorem holds. Here we can introduce a multifractal formalism based on the vectorial Hewitt-Stromberg measures and that this formalism is completely parallel to Peyri`ere’s multifractal formalism which based on the vectorial Hausdorff and packing measures: While the vectorial measures are defined using coverings and packings by families of sets with diameters less than a given positive number \(r\), say, the Hewitt-Stromberg measures are defined using packings of balls with a fixed diameter \(r\). More precisely, by considering the quantities of type
\[ \sum_i e^{-\left(\langle q,\chi(x_i,r)\rangle + t\xi(x_i,r)\right)}, \]
then the results of [2] and all above main theorems hold for this vectorial (mixed) multifractal formalism for some prescribed functions \(\chi\) and \(\xi\).

4. SOME EXAMPLES

This section discusses more motivations and examples related to these concepts. We present an intermediate result, which will be used in the proof of our results.

**Lemma 1.** We have
\[ \Delta^q_\xi(E) = \limsup_{r \to 0} \frac{\log M^q_{\xi,r}(E)}{-\log r} = \limsup_{r \to 0} \frac{1}{-\log r} \log \left( \sup \left\{ \mu(B(x_i,r))^{q \nu(B(x_i,r))} \mid (B(x_i,r))_i \text{ is a packing of } E \right\} \right) . \]

**Proof.** Suppose that
\[ \limsup_{r \to 0} \frac{\log M^q_{\xi,r}(E)}{-\log r} > \Delta^q_\xi(E) + \epsilon \text{ for some } \epsilon > 0. \]
Then we can find \(\delta > 0\) such that for any \(r \leq \delta\),
\[ M^q_{\xi,r}(E) (2r)^{\Delta^q_\xi(E)+\epsilon} > 2^{\Delta^q_\xi(E)+\epsilon} \]
and then
\[ \mathfrak{P}^q_{\xi} \Delta^q_\xi(E)+\epsilon(E) \geq 2^{\Delta^q_\xi(E)+\epsilon} > 0 \]
which is a contradiction. We therefore infer
\[ \limsup_{r \to 0} \frac{\log M^q_{c,r}(E)}{-\log r} \leq \Delta^q(E) + \epsilon \quad \text{for any} \quad \epsilon > 0. \]

The proof of the following statement
\[ \limsup_{r \to 0} \frac{\log M^q_{c,r}(E)}{-\log r} \geq \Delta^q(E) - \epsilon \quad \text{for any} \quad \epsilon > 0 \]
is identical to the proof of the above statement and is therefore omitted.  

Moran sets: Let us recall the class of homogeneous Moran sets (see for example [33, 34, 35]). We denote by \( \{n_k\}_{k \geq 1} \) a sequence of positive integers and \( \{c_k\}_{k \geq 1} \) a sequence of positive numbers satisfying
\[ n_k \geq 2, \quad 0 < c_k < 1, \quad n_k c_k \leq 1 \quad \text{for} \quad k \geq 1. \]

Let \( D_0 = \emptyset \), and for any \( k \geq 1 \), set
\[ D_{m,k} = \left\{ (i_m, i_{m+1}, \ldots, i_k) ; \quad 1 \leq i_j \leq n_j, \quad m \leq j \leq k \right\} \quad \text{and} \quad D_k = D_{1,k}. \]

Define \( D = \bigcup_{k \geq 1} D_k \). If
\[ \sigma = (\sigma_1, \ldots, \sigma_k) \in D_k \]
and
\[ \tau = (\tau_1, \ldots, \tau_m) \in D_{k+1,m}, \]
then we denote
\[ \sigma \ast \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_m). \]

Definition 1. Let \( J \) be a closed interval such that \( |J| = 1 \). We say the collection \( \mathcal{F} = \{J_\sigma, \sigma \in D\} \) of closed subsets of \( J \) fulfills the Moran structure if it satisfies the following conditions:
(a) \( J_\emptyset = J \).
(b) For all \( k \geq 0 \) and \( \sigma \in D_k, J_{\sigma + 1}, J_{\sigma + 2}, \ldots, J_{\sigma + n_{k+1}} \) are subintervals of \( J_\sigma \), and satisfy that
\[ J_{\sigma + i}^o \cap J_{\sigma + j}^o = \emptyset \quad \text{for all} \quad i \neq j, \]
where \( A^o \) denotes the interior of \( A \).
(c) For any \( k \geq 1, \sigma \in D_{k-1}, c_k = \frac{|J_{\sigma + j}|}{|J_\sigma|}, \quad 1 \leq j \leq n_k \) where \( |A| \) denotes the diameter of \( A \).

Let \( \mathcal{F} \) be a collection of closed subintervals of \( J \) having homogeneous Moran structure. The set
\[ E(\mathcal{F}) = \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_\sigma \]
is called an homogeneous Moran set determined by \( \mathcal{F} \). It is convenient to denote \( M(J_{\{n_k\}}, \{c_k\}) \) for the collection of homogeneous Moran sets determined by \( J, \{n_k\} \) and \( \{c_k\} \).

Remark 4. If \( \lim_{k \to +\infty} \sup_{\sigma \in D_k} |J_\sigma| > 0 \), then \( E \) contains interior points. Thus the measure and dimension properties will be trivial. We assume therefore \( \lim_{k \to +\infty} \sup_{\sigma \in D_k} |J_\sigma| = 0 \).

Now, we consider a class of homogeneous Moran sets \( E \) which satisfy a special property called the strong separation condition \( (SSC) \), i.e., take \( J_\sigma \in \mathcal{F} \). Let \( J_{\sigma + 1}, J_{\sigma + 2}, \ldots, J_{\sigma + n_{k+1}} \) be the \( n_{k+1} \) basic intervals of order \( k + 1 \) contained in \( J_\sigma \) arranged from the left to the right, then we assume that for all \( 1 \leq i \leq n_{k+1} - 1 \),
\[ \text{dist}(J_{\sigma + i}, J_{\sigma + (i+1)}) \geq \Delta_k |J_\sigma|, \]
where \( (\Delta_k)_k \) is a sequence of positive real numbers, such that
\[ 0 < \Delta = \inf_k \Delta_k < 1. \]
4.1. **Example 1.** In this example, we discuss our multifractal structures of a class of regularity Moran fractals associated with the sequences of letters such that the frequency exists. Let $A = \{a, b\}$ be a two-letter alphabet, and $A^*$ the free monoid generated by $A$. Let $F$ be the homomorphism on $A^*$, defined by $F(a) = ab$ and $F(b) = a$. It is easy to see that $F^n(a) = F^{n-1}(a)F^{n-2}(a)$. We denote by $|F^n(a)|$ the length of the word $F^n(a)$, thus

$$F^n(a) = s_1s_2 \cdots s_{|F^n(a)|}, \quad s_i \in A.$$  

Therefore, as $n \to \infty$, we get the infinite sequence

$$\omega = \lim_{n \to +\infty} F^n(a) = s_1s_2s_3 \cdots s_n \cdots \in \{a, b\}^\infty$$  

which is called the Fibonacci sequence. For any $n \geq 1$, write $\omega_n = \omega|_n = s_1s_2 \cdots s_n$. We denote by $|\omega_n|_a$ the number of the occurrence of the letter $a$ in $\omega_n$, and $|\omega_n|_b$ the number of occurrence of $b$. Then $|\omega_n|_a + |\omega_n|_b = n$.

It follows from [33] that $\lim_{n \to +\infty} \frac{|\omega_n|_a}{n} = \eta$, where $\eta^2 + \eta = 1$.

Let $0 < r_a < \frac{1}{2}, 0 < r_b < \frac{1}{3}, r_a, r_b \in \mathbb{R}$. In the above Moran construction, let

$$|J| = 1, \quad n_k = \begin{cases} 2, & \text{if } s_k = a \\ 3, & \text{if } s_k = b \end{cases}$$

and

$$c_k = \begin{cases} r_a, & \text{if } s_k = a \\ r_b, & \text{if } s_k = b \end{cases}, \quad 1 \leq j \leq n_k.$$  

Then we construct the homogeneous Moran set relating to the Fibonacci sequence and denote it by $E := E(\omega) = (J, \{n_k\}, \{c_k\})$. By the construction of $E$, we have

$$|J_\sigma| = r_a^{|\omega_k|_a}r_b^{|\omega_k|_b}, \quad \forall \sigma \in D_k.$$  

Let $P_a = (P_{a_1}, P_{a_2}), P_b = (P_{b_1}, P_{b_2}, P_{b_3})$ be probability vectors, i.e.,

$$P_{a_i} > 0, \quad P_{b_i} > 0, \quad \text{and} \quad \sum_{i=1}^2 P_{a_i} = 1, \quad \sum_{i=1}^3 P_{b_i} = 1.$$  

For any $k \geq 1$ and any $\sigma \in D_k$, we know $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ where

$$\sigma_k \in \begin{cases} \{1, 2\}, & \text{if } s_k = a \\ \{1, 2, 3\}, & \text{if } s_k = b. \end{cases}$$

For $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$, we define $\sigma(a)$ as follows: let $\omega_k = s_1s_2 \cdots s_k$ and $e_1 < e_2 < \cdots < e_{|\omega_k|_a}$ be the occurrences of the letter $a$ in $\omega_k$, then $\sigma(a) = \sigma_{e_1}\sigma_{e_2} \cdots \sigma_{e_{|\omega_k|_a}}$. Similarly, let $\delta_1 < \delta_2 < \cdots < \delta_{|\omega_k|_b}$ be the occurrences of the letter $b$ in $\omega_k$, then $\sigma(b) = \sigma_{\delta_1}\sigma_{\delta_2} \cdots \sigma_{\delta_{|\omega_k|_b}}$.

Let

$$P_{\sigma(a)} = P_{\sigma_{e_1}}P_{\sigma_{e_2}} \cdots P_{\sigma_{e_{|\omega_k|_a}}},$$

and

$$P_{\sigma(b)} = P_{\sigma_{\delta_1}}P_{\sigma_{\delta_2}} \cdots P_{\sigma_{\delta_{|\omega_k|_b}}}.$$  

Obviously

$$\sum_{\sigma \in D_k} P_{\sigma(a)}P_{\sigma(b)} = 1.$$  

Let $\mu$ be a mass distribution on $E$, such that for any $\sigma \in D_k$,

$$\mu(J_\sigma) = P_{\sigma(a)}P_{\sigma(b)}.$$  

Now we define an auxiliary function $\beta(q)$ as follows: For each $q \in \mathbb{R}$ and $k \geq 1$, there is a unique number $\beta_k(q)$ such that

$$\sum_{\sigma \in D_k} \left( P_{\sigma(a)}P_{\sigma(b)} \right)^q |J_\sigma|_{\beta_k(q)} = 1.$$
By a simple calculation, we get
\[
\beta_k(q) = -\log \left( \sum_{i=1}^{2} P^q_{a_i} \right) - \frac{k - |\omega_k|_a}{|\omega_k|_a} \log \left( \sum_{i=1}^{3} P^q_{b_i} \right) \log r_a + \frac{k - |\omega_k|_a}{|\omega_k|_a} \log r_b.
\]

Clearly, for any \( k \geq 1 \) we have \( \beta_k(1) = 0 \). Thus \( \beta_k(q) < 0 \) for all \( q \) and \( \beta_k(q) \) is a strictly decreasing function. Our auxiliary function is
\[
\beta(q) = \lim_{k \to +\infty} \beta_k(q) = \log \left( \sum_{i=1}^{2} P^q_{a_i} \right) - \eta \log \left( \sum_{j=1}^{3} P^q_{b_j} \right),
\]
where \( \eta^2 + \eta = 1 \). The function \( \beta(q) \) is strictly decreasing, smooth, \( \lim_{q \to +\infty} \beta(q) = -\infty \) and \( \beta(1) = 0 \).

Then we have the following result,

**Theorem 8.** Suppose that \( E \) is a homogeneous Moran set satisfying \( (SSC) \) and \( \mu \) is the Moran measure on \( E \). Then,
\[
\overline{\dim}_{MB}(E_\mu(-\beta'(q))) = \overline{\dim}_{MB}(E_\mu(-\beta(q))) = \beta^*(-\beta'(q)).
\]

**Proof.** Given \( q, t \in \mathbb{R} \), and let \( \nu \) be a probability measure on \( E \) such that for any \( k \geq 1 \) and \( \sigma_0 \in D_k \),
\[
\nu(J_{\sigma_0}) = \frac{\mu(J_{\sigma_0})}{\sum_{\sigma \in D_k} \mu(J_{\sigma})}. \]

It follows from Proposition 2 that
\[
\mathcal{B}_1^\nu(E_\mu(-\beta'(q))) \leq \mathcal{B}_\nu(1) = 0.
\]

It result from Theorem 4 that
\[
\overline{\dim}_{MB}(E_\mu(-\beta'(q))) \leq -q\beta'(q) + \beta(q). \quad (4.1)
\]

Now, by using Lemma 1, we define the function \( f \) by
\[
f(t) = \lim_{r \to 0} \sup \frac{1}{-\log r} \log \left( \sup \left\{ \sum_i \mu(B(x_i, r)) \nu(B(x_i, r)) \left| B(x_i, r) \right| \text{ is a packing of supp } \mu \right\} \right).
\]

Then \( f(t) = \lim_{k \to +\infty} f_k(t) \), where \( f_k(t) \) is a unique number such that
\[
\sum_{\sigma \in D_k} (\mu(J_{\sigma}))^t \nu(J_{\sigma}) |J_{\sigma}|^{f_k(t)} = 1.
\]

Which implies that
\[
\sum_{\sigma \in D_k} (\mu(J_{\sigma}))^{q+t} |J_{\sigma}|^{f_k(t) + \beta_k(q)} = 1.
\]

A straightforward calculation shows that
\[
f_k(t) = \frac{-\log \left( \sum_{i=1}^{2} P^{q+t}_{a_i} \right) - \frac{k - |\omega_k|_a}{|\omega_k|_a} \log \left( \sum_{i=1}^{3} P^{q+t}_{b_i} \right)}{\log r_a + \frac{k - |\omega_k|_a}{|\omega_k|_a} \log r_b} - \beta_k(q)
\]
and
\[
f(t) = \lim_{k \to +\infty} f_k(t) = \beta(q + t) - \beta(q).
\]

It is clear that \( f(0) = 0 \), \( f'(0) \) exists and equal to \( \beta'(q) \). We can see from the construction of measure \( \nu \) that
\[
0 < \lim_{k \to +\infty} \sum_{\sigma \in D_k} \mu(J_{\sigma})^q |J_{\sigma}|^{\beta(q)} < +\infty
\]
and then
\[
\nu(\text{supp } \mu) > 0 \Rightarrow \pi(\text{supp } \mu) > 0.
\]
We therefore conclude from Proposition 4 that
\[
\dim_{MB}(E_\mu(-\beta'(q))) \geq -q\beta'(q) + \beta(q). \tag{4.2}
\]
Thus, the result is a consequence from (4.1) and (4.2).

4.2. Example 2. In the following, our results are for a particular type of fractal called a non-regularity Moran fractal associated with the sequences of which the frequency of the letter does not exists. We first define a Moran measure on the homogeneous Moran sets \(E\). Let \(\{p_{i,j}\}_{i,j=1}^{n_i}\) be the probability vectors (i.e., \(p_{i,j} > 0\) and \(\sum_{j=1}^{n_i} p_{i,j} = 1\) for all \(i = 1, 2, 3, \ldots\)) and suppose that \(p_0 = \inf\{p_{i,j}\} > 0\). Let \(\mu\) be a mass distribution on \(E\), such that for any \(J_\sigma (\sigma \in D_k)\)
\[
\mu(J_\sigma) = p_{1,\sigma_1}p_{2,\sigma_2}\cdots p_{k,\sigma_k}.
\]

Now, we define an auxiliary function \(\beta_k(q)\) as follows: for all \(k \geq 1\) and \(q \in \mathbb{R}\), there is a unique number \(\beta_k(q)\) satisfying
\[
\sum_{\sigma \in D_k} p_{\sigma}^k \beta_k(q) = 1.
\]

Set
\[
\underline{\beta}(q) = \liminf_{k \to +\infty} \beta_k(q) \quad \text{and} \quad \overline{\beta}(q) = \limsup_{k \to +\infty} \beta_k(q),
\]
Let \(t_k\) be a sequence of integers such that
\[
t_1 = 1, \ t_2 = 3 \quad \text{and} \quad t_{k+1} = 2t_k, \ \forall \ k \geq 3.
\]
Define the family of parameters \(n_i, c_i\) and \(p_{i,j}\) as follows:
\[
n_1 = 2, \ n_i = \begin{cases} 
3, & \text{if } t_{2k-1} \leq i < t_{2k}, \\
2, & \text{if } t_{2k} \leq i < t_{2k+1}.
\end{cases}
\]

For \(0 < r_a < \frac{1}{3}\) and \(0 < r_b < \frac{1}{3}\), let
\[
c_1 = r_a, \ c_i = \begin{cases} 
r_b, & \text{if } t_{2k-1} \leq i < t_{2k}, \\
r_a, & \text{if } t_{2k} \leq i < t_{2k+1}.
\end{cases}
\]

Let \((p_{a,j})_{j=1}^{2}\) and \((p_{b,j})_{j=1}^{3}\) be two probability vectors. Define
\[
p_{i,j} = p_{a,j}, \quad \text{for all } 1 \leq j \leq 2
\]
and
\[
p_{i,j} = \begin{cases} 
p_{b,j}, & \text{if } t_{2k-1} \leq i < t_{2k}, \ 1 \leq j \leq 3, \\
p_{a,j}, & \text{if } t_{2k} \leq i < t_{2k+1}, \ 1 \leq j \leq 2.
\end{cases}
\]
Then, we have
\[
\beta_k(q) = \frac{\log \sum_{\sigma \in D_k} \mu(J_\sigma)^q}{-\log(c_1 \cdots c_k)}.
\]
Finally, if \(N_k\) is the number of integers \(i \leq k\) such that \(p_{i,j} = p_{a,j}\), then
\[
\liminf_{k \to +\infty} \frac{N_k}{k} = \frac{1}{3},
\]
\[
\limsup_{k \to +\infty} \frac{N_k}{k} = \frac{2}{3}
\]
and
\[
\beta_k(q) = -\frac{N_k}{k} \log r_a + \left(1 - \frac{N_k}{k}\right) \log r_b.
\]
We can then conclude that
\[
\beta(q) = \min \left\{ \frac{1}{3} \log \sum_{j=1}^{2} p_{a,j}^q + \frac{2}{3} \log \sum_{j=1}^{3} p_{b,j}^q, \ \frac{2}{3} \log \sum_{j=1}^{2} p_{a,j}^q + \frac{1}{3} \log \sum_{j=1}^{3} p_{b,j}^q \right\}.
\]
and
\[ \overline{\beta}(q) = \max \left\{ \frac{1}{3} \log \sum_{j=1}^{2} p_{a,j}^q + \frac{2}{3} \log \sum_{j=1}^{3} p_{b,j}^q, \frac{1}{3} \log r_a + \frac{2}{3} \log r_b \right\} - \frac{1}{3} \log \sum_{j=1}^{2} p_{a,j}^q + \frac{2}{3} \log \sum_{j=1}^{3} p_{b,j}^q \].

It is obvious that \( \beta(q) \leq \beta_k(q) \leq \overline{\beta}(q) \) for all \( k \geq 2 \), the functions \( \beta \) and \( \overline{\beta} \) are strictly decreasing, \( \beta(0) = \overline{\beta}(0) = 0 \) and \( \overline{\beta}(q) > \beta(q) \) for all \( q \neq 1 \) (see Fig. 1).

![Figure 1. The multifractal functions \( \overline{\beta} \) and \( \beta \).](image)

At last, we compute the dimension of the level sets \( E_\mu(\alpha) \) (see Fig. 2).

**Theorem 9.** Suppose that \( E \) is a homogeneous Moran set satisfying (SSC) and \( \mu \) is the Moran measure on \( E \). Let \( q \in \mathbb{R} \) and assume that \( \overline{\beta}'(q) \) (resp. \( \beta'(q) \)) exists. Then,

\[ \dim_{MB}(E_\mu(-\beta'(q))) = \beta^*(\overline{\beta}(q)) \quad \text{provided} \quad 0 < \liminf_{k \to +\infty} \sum_{\sigma \in D_k} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)} < +\infty \]

and
\[ \overline{\dim}_{MB}(E_\mu(-\beta'(q))) = \overline{\beta}^*(\beta(q)) \quad \text{provided} \quad 0 < \limsup_{k \to +\infty} \sum_{\sigma \in D_k} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)} < +\infty. \]

**Proof.** For \( q \in \mathbb{R} \), let \( \nu \) be a probability measure on \( \text{supp} \mu \) such that for any \( k \geq 1 \) and \( \sigma_0 \in D_k 
\]

\[ \nu(J_{\sigma_0}) = \frac{\mu(J_{\sigma_0})^q |J_{\sigma_0}|^{\beta(q)}}{\sum_{\sigma \in D_k} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)}}. \]

Proposition 2 implies that
\[ B^1_{\nu}(E_\mu(-\beta'(q))) \leq B_{\nu}(1) = 0. \]

It follows from Theorem 4 that
\[ \dim_{MB}(E_\mu(-\beta'(q))) \leq -q\beta'(q) + \beta(q). \]

We can see from the construction of measure \( \nu \) that \( \nu(E_\mu(-\beta'(q))) > 0 \) (we can see also [34]) which implies that \( \pi(E_\mu(-\beta'(q))) > 0 \). We therefore conclude from Theorem 5 that
\[ \dim_{MB}(E_\mu(-\beta'(q))) \geq -q\overline{\beta}'(q) + \overline{\beta}(q) \]

which yields the desired result. The proof of the second statement is identical to the proof of the first statement and is therefore omitted. \( \square \)

**Remark 5.** Note that the model of Yuan [36] (Huang et al. [15]) can be adapted to our general theory which provides a non-trivial sample with a full description of the refined multifractal analysis based on multifractal Hewitt-Stromberg measures.
Let $J = [0, 1]$, $n_i = 2$ and $\mathcal{N} := \{N_k\}_{k \in \mathbb{N}}$ be an increasing sequence of integers with $N_0 = 0$ and $
olimits \lim_{k \to +\infty} \frac{N_{k+1}}{N_k} = +\infty$. Fix four real numbers $A, B, p, \tilde{p}$ with $A > B > 2$ and $0 < p, \tilde{p} \leq 1/2$. Now for every $i \in \mathbb{N}$, we define $c_i$ and $\{P_{i,j}\}_{1 \leq j \leq n_i}$, as follows:

$$c_i = \begin{cases} 
\frac{1}{A}, & \text{if } N_{2k} < i \leq N_{2k+1}, \\
\frac{1}{B}, & \text{if } N_{2k+1} < i \leq N_{2k+2}.
\end{cases}$$

and

$$P_{i,j} = \begin{cases} 
p, & \text{if } N_{2k} < i \leq N_{2k+1} \text{ and } j = 1, \\
1 - p, & \text{if } N_{2k} < i \leq N_{2k+1} \text{ and } j = 2, \\
\tilde{p}, & \text{if } N_{2k+1} < i \leq N_{2k+2} \text{ and } j = 1, \\
1 - \tilde{p}, & \text{if } N_{2k+1} < i \leq N_{2k+2} \text{ and } j = 2.
\end{cases}$$

**Figure 2.** The Hewitt-Stromberg dimensions of the set $E_\mu(\alpha)$. 
Define now the following functions

\[ \beta_1 : \mathbb{R} \to \mathbb{R}, \quad q \mapsto \frac{\log(p^q + (1-p)^q)}{\log A}, \]

and

\[ \beta_2 : \mathbb{R} \to \mathbb{R}, \quad q \mapsto \frac{\log(\tilde{p}^q + (1-\tilde{p})^q)}{\log B}. \]

We can conclude that

\[ b_\mu(q) = \min \left\{ \beta_1(q), \beta_2(q) \right\} \]

and

\[ B_\mu(q) = \max \left\{ \beta_1(q), \beta_2(q) \right\}. \]

If \(-\log(1-\tilde{p}) < -\log \tilde{p}\), then for all \(\alpha \in \left[ -\log(1-\tilde{p}) \log B, \min\left\{-\log \tilde{p} \log A, -\log \tilde{p} \log B \right\} \right]\)

\[ \dim_{MB}(E_\mu(\alpha)) = b_\mu^*(\alpha), \]

and for \(\alpha \in \left\{ B'_\mu(q) : q \in \mathbb{R} \text{ and } B_\mu \text{ is differentiable at } q \right\}\) we have

\[ \overline{\dim}_{MB}(E_\mu(\alpha)) = B_\mu^*(\alpha). \]

Even if \(p = \tilde{p}\),

\[ \liminf_{k \to +\infty} \frac{\sharp \left\{ 1 \leq i \leq k; \ c_i = \frac{1}{A} \right\}}{k} = 0 \]

and

\[ \limsup_{k \to +\infty} \frac{\sharp \left\{ 1 \leq i \leq k; \ c_i = \frac{1}{A} \right\}}{k} = 1, \]

then the phenomena are the same, i.e., for \(\alpha \in \left[ -\log(1-p) \log B, \frac{-\log p}{\log A} \right]\), we have

\[ \dim_{MB}(E_\mu(\alpha)) = b_\mu^*(\alpha), \]

and for \(\alpha \in \left[ -\frac{-\log(1-p)}{\log B}, \frac{-\log p}{\log A} \right] \cup \left( \frac{-p \log p - (1-p) \log(1-p)}{\log A}, \frac{-p \log p - (1-p) \log(1-p)}{\log B} \right) \), we have

\[ \overline{\dim}_{MB}(E_\mu(\alpha)) = B_\mu^*(\alpha). \]

This implies that the results of Theorem 9 hold for some \((\alpha, \beta) \neq \left( -\beta'(q), -\beta'(q) \right)\).

In the following, we give some examples of a measure for which the lower and upper multifractal Hewitt-Stromberg functions are different (see Fig. 3) and the Hewitt-Stromberg dimensions of the level sets of the local Hölder exponent \(E_\mu(\alpha)\) are given by the Legendre transform respectively of lower and upper multifractal Hewitt-Stromberg functions (see Fig. 4). In particular, we prove that our multifractal formalism \([2, \text{Theorem 8}]\) holds for these measures.
4.3. **Example 3.** Take $0 < p < \hat{p} \leq 1/2$ and a sequence of integers

$$1 = t_0 < t_1 < \cdots < t_n < \cdots,$$

such that

$$\lim_{n \to +\infty} \frac{t_{n+1}}{t_n} = +\infty.$$ 

The measure $\mu$ assigned to the diadic interval of the $n$-th generation $I_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n}$ is

$$\mu(I_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n}) = \prod_{j=1}^n \varpi_j,$$

where

$$\begin{cases}
  \text{if } t_{2k-1} \leq j < t_{2k} \text{ for some } k, & \varpi_j = p \text{ if } \varepsilon_j = 0, \ \varpi_j = 1 - p \ \text{otherwise}, \\
  \text{if } t_{2k} \leq j < t_{2k+1} \text{ for some } k, & \varpi_j = \hat{p} \text{ if } \varepsilon_j = 0, \ \varpi_j = 1 - \hat{p} \ \text{otherwise}.
\end{cases}$$

We observe that

$$\sum_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n} \mu(I_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n})^q = (p^q + (1 - p)^q)^{N_n}(\hat{p}^q + (1 - \hat{p})^q)^{n-N_n},$$

where $N_n$ is the number of integers $j \leq n$ such that $\varpi_j = p$. It is clear that $\liminf_{n \to +\infty} (N_n/n) = 0$ and $\limsup_{n \to +\infty} (N_n/n) = 1$. Now, for $q \in \mathbb{R}$, we define

$$\tau(q) = \log_2 (p^q + (1 - p)^q) \quad \text{and} \quad \tau(q) = \log_2 (\hat{p}^q + (1 - \hat{p})^q).$$

Then

$$\liminf_{n \to +\infty} \frac{1}{n} \log \sum_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n} \mu(I_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n})^q = \min \{ \tau(q), \tau(q) \}$$

and

$$\limsup_{n \to +\infty} \frac{1}{n} \log \sum_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n} \mu(I_{\varepsilon_1\varepsilon_2\cdots\varepsilon_n})^q = \max \{ \tau(q), \tau(q) \}.$$ 

It results from [2, Proposition 2] and [4, 5] that

$$b_\mu(q) = \min \{ \tau(q), \tau(q) \} \quad \text{and} \quad B_\mu(q) = \Delta_\mu(q) = \max \{ \tau(q), \tau(q) \}.$$ 

This gives

$$\begin{cases}
  b_\mu(q) = \tau(q) < \tau(q) = B_\mu(q) = \Delta_\mu(q), & \text{for } 0 < q < 1, \\
  b_\mu(q) = \tau(q) < \tau(q) = B_\mu(q) = \Delta_\mu(q), & \text{for } q < 0 \text{ or } q > 1
\end{cases}$$

and

$$b_\mu(q) = B_\mu(q) = \Delta_\mu(q), \quad \text{for } q \in \{0, 1\}.$$ 

![Figure 3](image)  

**Figure 3.** The relation between the graphs of $b_\mu$ and $B_\mu$.

Given $0 < p, \hat{p} < 1$, define the mixed entropy function

$$h(\hat{p}, p) := -\hat{p} \log_2 p - (1 - \hat{p}) \log_2 (1 - p).$$

Then

$$-\tau'(\pm \infty) = h(0, p) \leq -\tau'(1) = h(p, p) \leq -\tau'(0) = h(1/2, p) \leq h(1, p) = -\tau'(-\infty).$$
and
\[-\tau'(\infty) = h(0, \hat{p}) \leq -\tau'(1) = h(\hat{p}, \hat{p}) \leq -\tau'(0) = h(1/2, \hat{p}) \leq h(1, \hat{p}) = -\tau'(\infty).\]

Now, we have the following result,

**Theorem 10.** Assume that \(\alpha \in \left( -\tau'(\infty), -\tau'(\infty) \right).\)

1. For \(\alpha \notin \left[ -b_{\mu-}^*(0), -b_{\mu-}^*(0) \right] \cup \left[ -b_{\mu+}^*(1), -b_{\mu-}^*(1) \right],\) we have
   \[\dim_M (E_\mu(\alpha)) = \dim_{MB}(E_\mu(\alpha)) = b_\mu^*(\alpha).\]

2. For \(\alpha \notin \left[ -B_{\mu+}^*(0), -B_{\mu-}^*(0) \right] \cup \left[ -B_{\mu+}^*(1), -B_{\mu-}^*(1) \right],\) we have
   \[\dim_P (E_\mu(\alpha)) = \overline{\dim}_{MB}(E_\mu(\alpha)) = B_\mu^*(\alpha).\]

![Figure 4. The Hewitt-Stromberg dimensions of the set \(E_\mu(\alpha)\).](image)

**Proof.** We can construct a new probability measure \(\nu\) on the diadic interval of the \(n\)-th generation just as \(\mu\), but replacing \((p, \hat{p})\) with \((s, \hat{s})\) such that
\[s \log p + (1 - s) \log (1 - p) = \hat{s} \log \hat{p} + (1 - \hat{s}) \log (1 - \hat{p}),\]
\[\log \frac{1 - p}{1 - \hat{p}} < s \log \frac{1 - p}{p} < \log \frac{1 - p}{\hat{p}}.\]

and
\[\alpha = -s \log_2 p - (1 - s) \log_2 (1 - p) = s \log_2 \frac{1 - p}{p} - \log_2 (1 - p).\]

From Lemma 1, we can define the function \(f\) by
\[f(t) = \limsup_{r \to 0} \frac{1}{-\log r} \log \left( \sup \left\{ \mu(B(x, r)) \nu(B(x, r)) \mid \left( B(x, r) \right), \text{ is a packing of } K \right\} \right).\]

Then, it is easy to compute
\[f(t) = \limsup_{n \to +\infty} \frac{1}{n} \log_2 \sum_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n} \mu(I_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}) \nu(I_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n})\]
\[= \limsup_{n \to +\infty} \frac{1}{n} \log_2 \left( \left( p^t r + (1 - p)^t (1 - r) \right)^N_n \left( \hat{p}^t \hat{s} + (1 - \hat{p})^t (1 - \hat{s}) \right)^{n - N_n} \right)\]
\[= \limsup_{n \to +\infty} \left( \frac{N_n}{n} \log_2 \left( p^t r + (1 - p)^t (1 - r) \right) + \left( 1 - \frac{N_n}{n} \right) \log_2 \left( \hat{p}^t \hat{s} + (1 - \hat{p})^t (1 - \hat{s}) \right) \right)\]
\[= \log_2 \max \left\{ (p^t r + (1 - p)^t (1 - r)), (\hat{p}^t \hat{s} + (1 - \hat{p})^t (1 - \hat{s})) \right\}.\]
It is clear that \( f(0) = 0 \), and the method of choosing \((s, \hat{s})\) insures that \( f'(0) \) exists and is equal to \(-\alpha\).

Now we can estimate the bounds of the dimensions of the level sets. Given \( 0 < s < 1 \) and define the entropy function
\[
H(s) := -s \log_2 s + (s - 1) \log_2 (1 - s).
\]
The strong law of large numbers shows that
\[
\lim_{n \to +\infty} \frac{\log \nu(B(x, 2^{-n}))}{-n} = \min \left\{ H(s), H(\hat{s}) \right\}
\]
and
\[
\limsup_{n \to +\infty} \frac{\log \nu(B(x, 2^{-n}))}{-n} = \max \left\{ H(s), H(\hat{s}) \right\}
\]
for \( \nu \)-almost every \( x \). So it deduces from Theorem 5 that
\[
\dim_{MB}(E_{\mu}(\alpha)) \geq \min \left\{ H(s), H(\hat{s}) \right\}
\]
and
\[
\overline{\dim}_{MB}(E_{\mu}(\alpha)) \geq \max \left\{ H(s), H(\hat{s}) \right\}.
\]
To compute \( H(s) \) and \( H(\hat{s}) \), set
\[
q = \frac{\log \frac{1-s}{s}}{\log \frac{1-p}{p}}
\]
then
\[
\tau'(q) = \frac{p^q \log_2 p + (1-p)^q \log_2 (1-p)}{p^q + (1-p)^q} = \frac{\log_2 p + \left(\frac{1-p}{p}\right)^q \log_2 (1-p)}{1 + \left(\frac{1-p}{p}\right)^q} = \frac{\log_2 p + \left(\frac{1-s}{s}\right) \log_2 (1-p)}{1 + \left(\frac{1-s}{s}\right)} = s \log_2 p + (1-s) \log_2 (1-p) = f'(0) = -\alpha.
\]

Which implies that
\[
\tau(q) - q\tau'(q) = \log_2 \left( p^q + (1-p)^q \right) - q\tau'(q)
\]
\[
= \log_2 p^q \left( 1 + \left(\frac{1-p}{p}\right)^q \right) - q\tau'(q)
\]
\[
= q \log_2 p + \log_2 \left( 1 + \left(\frac{1-p}{p}\right)^q \right) - q\tau'(q)
\]
\[
= q \log_2 p + \log_2 \left( 1 + \frac{1-s}{s} \right) - q\tau'(q)
\]
\[
= q \log_2 p - \log_2 s - q \left( s \log_2 p + (1-s) \log_2 (1-p) \right)
\]
\[
= -\log_2 s + q \left( (1-s) \log_2 p - (1-s) \log_2 (1-p) \right)
\]
\[
= -\log_2 s - (1-s) \log_2 \left(\frac{1-s}{s}\right)
\]
\[
= -s \log_2 s - (1-s) \log_2 (1-s) = H(s).
\]
Also, set

\[ \hat{q} = \frac{\log \frac{1+\hat{r}}{\hat{r}}}{\log \frac{1+\hat{r}}{\hat{p}}} \]

with the very same arguments, we have

\[ \tau'(\hat{q}) = -\alpha \quad \text{and} \quad \tau(\hat{q}) - \hat{q}\tau'(\hat{q}) = H(\hat{s}). \]

Thus

\[ H(s) = \tau(q) - q\tau'(q) = \tau'(q)|\tau'(q)|, \]

and

\[ H(\hat{s}) = \tau(\hat{q}) - \hat{q}\tau'(\hat{q}) = \tau'(\hat{q}) \]

which give the lower bounds of the dimensions of the level sets, i.e.,

\[ \dim_{MB}(E_\mu(\alpha)) \geq \min \{ \tau'(q), \tau'(\hat{q}) \} \]

and

\[ \overline{\dim}_{MB}(E_\mu(\alpha)) \geq \max \{ \tau'(q), \tau'(\hat{q}) \} \]

But we have also the opposite inequalities:

In order to have \( \tau(q) = b_\mu(q) \), we must have \( 0 < q < 1 \), which means

\[ -\tau'(1) = h(p, p) < \alpha < h(1/2, p) = -\tau'(0). \]

In order to have \( \tau(\hat{q}) = b_\mu(\hat{q}) \), we must have \( \hat{q} < 0 \) or \( \hat{q} > 1 \), which means

\[ \alpha > h(1/2, \hat{p}) = -\tau'(0) \quad \text{or} \quad \alpha < h(\hat{p}, \hat{p}) = -\tau'(1). \]

In order to have \( \tau(\hat{q}) = b_\mu(\hat{q}) \), we must have \( 0 < \hat{q} < 1 \), which means

\[ -\tau'(1) = h(\hat{p}, \hat{p}) < \alpha < h(1/2, \hat{p}) = -\tau'(0). \]

In order to have \( \tau(q) = b_\mu(q) \), we must have \( q < 0 \) or \( q > 1 \), which means

\[ \alpha > h(1/2, p) = -\tau'(0) \quad \text{or} \quad \alpha < h(p, p) = -\tau'(1). \]

Now, put

\[ I = \left( -\tau'(\infty), -\tau'(-\infty) \right) \cap \left[ -\tau'(0), -\tau'(0) \right] \cup \left[ -\tau'(1), -\tau'(1) \right] \]

and

\[ J = \left( -\tau'(\infty), -\tau'(-\infty) \right) \cap \left[ -\tau'(0), -\tau'(0) \right] \cup \left[ -\tau'(1), -\tau'(1) \right]. \]

It is easy to verify that \( I, J \subseteq (\alpha, \overline{\alpha}) \). Finally, it follows from Theorem 2 that

\[ \dim_{MB}(E_\mu(\alpha)) \leq b_\mu(\alpha) \quad \text{and} \quad \overline{\dim}_{MB}(E_\mu(\alpha)) \leq \overline{b}_\mu(\alpha), \]

which yields the desired result. \( \square \)

4.4. Example 4. For \( \mathcal{X} = \{0, 1, 2, 3\} \), we consider \( \mathcal{X}^* = \bigcup_{n \geq 0} \mathcal{X}^n \), the set of all finite words on the 4-letter alphabet \( \mathcal{X} \). Let \( w = \varepsilon_1 \cdots \varepsilon_n \) and \( v = \varepsilon_{n+1} \cdots \varepsilon_{n+m} \), denote by \( wv \) the word \( \varepsilon_1 \cdots \varepsilon_{n+m} \). With this operation, \( \mathcal{X}^* \) is a monoid whose identity element is the empty word \( \varepsilon \). If a word \( v \) is a prefix of the word \( w \), we write \( v \preceq w \).

This defines an order on \( \mathcal{X}^* \) and endowed with this order, \( \mathcal{X}^* \) becomes a tree whose root is \( \varepsilon \). At last, the length of a word \( w \) is denoted by \( |w| \). If \( w \) and \( v \) are two words, \( w \wedge v \) stands for their largest common prefix. It is well known that the function \( d := dist(w, v) = 4^{-|w \wedge v|} \) defines an ultra-metric distance on \( \mathcal{X}^* \). The completion of \( (\mathcal{X}^*, d) \) is a compact space which is the disjoint union of \( \mathcal{X}^* \) and \( \partial \mathcal{X}^* \), whose elements can be viewed as infinite words. Each finite word \( w \in \mathcal{X}^* \) defines a cylinder \( |w| = \{x \in \partial \mathcal{X}^* \mid w \wedge x \} \), which can also be viewed as a ball. Let \( a_i, b_j \in (0, 1), i, j \in \{1, 2, 3, 4\} \) satisfying

\[ \sum_{i=1}^{4} a_i = \sum_{j=1}^{4} b_j = 1 \]

and \( (t_k) \) be a sequence of integers such that

\[ t_1 = 1, \quad t_k < t_{k+1} \quad \text{and} \quad \lim_{k \to +\infty} \frac{t_{k+1}}{t_k} = +\infty. \]
We define the measure $\mu$ on $\partial \mathcal{K}^*$ such that for every cylinder $[\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n]$, one has

$$\mu([\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n]) = \prod_{j=1}^{n} p_j,$$

where

$$\begin{cases}
\text{if } t_{2k-1} \leq j < t_{2k} \text{ for some } k, p_j = a_{\varepsilon_j+1}, \\
\text{if } t_{2k} \leq j < t_{2k+1} \text{ for some } k, p_j = b_{\varepsilon_j+1}.
\end{cases}$$

Then

$$b_\mu(q) = \inf \left\{ \log_4 \left( a_1^2 + a_2^2 + a_3^2 + a_4^2 \right), \log_4 \left( b_1^2 + b_2^2 + b_3^2 + b_4^2 \right) \right\}$$

and

$$B_\mu(q) = \sup \left\{ \log_4 \left( a_1^2 + a_2^2 + a_3^2 + a_4^2 \right), \log_4 \left( b_1^2 + b_2^2 + b_3^2 + b_4^2 \right) \right\}.$$ 

The functions $b_\mu$ and $B_\mu$ are analytic and their graphs differ except at two points where they are tangent, with $b_\mu(0) = B_\mu(0)$, $b_\mu(1) = B_\mu(1)$, and $B_\mu(q) > b_\mu(q)$ for all $q \neq 0, 1$ (see Figure 3). Moreover $b_\mu$ and $B_\mu$ are convex and $B_\mu^*(R)$ and $b_\mu^*(R)$ both are intervals of positive length (for more detail see [31]).

Now, we suppose that $a_1 < b_1$. Then by construction of the measure $\mu$, the graph of $\log_4 \left( a_1^2 + a_2^2 + a_3^2 + a_4^2 \right)$ is always on top of the graph of $\log_4 \left( b_1^2 + b_2^2 + b_3^2 + b_4^2 \right)$. So, we get

$$b_\mu(q) = \log_4 \left( b_1^2 + b_2^2 + b_3^2 + b_4^2 \right)$$

and

$$B_\mu(q) = \log_4 \left( a_1^2 + a_2^2 + a_3^2 + a_4^2 \right).$$

**Theorem 11.** We assume that $\alpha \in (-\log_4 b_4, -\log_4 b_1)$. Then

$$\dim_H(E_\mu(\alpha)) = \dim_{MB}(E_\mu(\alpha)) = b_\mu^*(\alpha)$$

and

$$\dim_P(E_\mu(\alpha)) = \dim_{MB}(E_\mu(\alpha)) = B_\mu^*(\alpha).$$

**Proof.** The proof is very similar to the one of Theorem 10. We can see also [31] for the estimates of the Hausdorff and packing dimensions of the set $E_\mu(\alpha)$. \qed

**Remark 6.** Here $\mathcal{K}(\mathbb{R}^n)$ denotes the family of non-empty compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric, and $\mathcal{P}(\mathbb{R}^n)$ denotes the family of Radon measures on $\mathbb{R}^n$ equipped with the weak topology. The study of the descriptive set-theoretic complexity of the maps

$$\mathcal{K}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \times \mathbb{R} \rightarrow [\infty, +\infty] : (K, \mu, q) \mapsto H^q_{\mu}(K),$$

$$\mathcal{K}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \times \mathbb{R} \rightarrow [\infty, +\infty] : (K, \mu, q) \mapsto P^q_{\mu}(K),$$

$$\mathcal{K}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \times \mathbb{R} \rightarrow [\infty, +\infty] : (K, \mu, q) \mapsto b_\mu^q(K),$$

$$\mathcal{K}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \times \mathbb{R} \rightarrow [\infty, +\infty] : (K, \mu, q) \mapsto B_\mu^q(K),$$

$$\mathcal{K}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \times \mathbb{R} \rightarrow [\infty, +\infty] : (K, \mu, q) \mapsto \Delta_\mu^q(K)$$

and the multifractal structure of product measures and dimensions (note that Edgar and Zindulka in [7, 37] studied the structure of the Hewitt-Stromberg measures and dimensions on cartesian products in the case $q = 0$) will be achieved in further works.
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