A Note on Clustering Aggregation

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Abstract

We consider the clustering aggregation problem in which we are given a set of clusterings and want to find an aggregated clustering which minimizes the sum of mismatches to the input clusterings. In the binary case (each clustering is a bipartition) this problem was known to be NP-hard under Turing reduction. We strengthen this result by providing a polynomial-time many-one reduction. Our result also implies that no $2^{o(n)} \cdot |I'|^{O(1)}$-time algorithm exists for any clustering instance $I'$ with $n$ elements, unless Exponential Time Hypothesis fails. On the positive side, we show that the problem is fixed-parameter tractable with respect to the number of input clusterings.

1 Clustering Aggregation for Binary Strings

The problem can be formalized as follows. We aim to find a length-$n$ binary string that minimizes the Mirkin distance to some input binary strings of length $n$. The Mirkin distance $\text{mirk}(s, s')$ [11] between two strings $s$ and $s'$ counts the number of mismatches for each pair of bits. Formally, $\text{mirk}(s, s') = |\{i, j\} : (s[i] = s[j] \land s'[i] \neq s'[j]) \lor (s[i]) \neq s[j] \land s'[i] \neq s'[j]|$. The Mirkin distance of a string $s^*$ to a sequence $S$ of strings is the sum of the Mirkin distances between $s^*$ and each string in the sequence: $\text{mirk}(s, S) = \sum_{s' \in S} \text{mirk}(s, s')$.

The Mirkin distance has an alternative definition that uses Hamming distances.

$$\text{mirk}(s, s') = d(s, s') \cdot d(\overline{s}, \overline{s'}) = d(s, s') \cdot (n - d(s, s'))$$

Note that by the above formulation, the Mirkin distance objective function is not convex. The formal statement of the problem is as follows:

Mirkin Distance Minimization

Input: A set $S$ of strings $s_1, \ldots, s_m \in \{0, 1\}^n$ and an integer $k$.

Question: Is there a string $s^* \in \{0, 1\}^n$ such that $\sum_{i=1}^{m} \text{mirk}(s^*, s_i) \leq k$?

Notations. For two binary strings $s$ and $s'$, let $s \circ s'$ denote the concatenation of $s$ and $s'$ and let $\overline{s}$ denote the complement of string $s$. By $s[i]$ we mean the value of the $i^{th}$ bit of string $s$ and we write $s[i, j]$ as shorthand of $s[i]s[j]$. Given two integers $i, j \in \{1, 2, \cdots, |s|\}$ with $i \leq j$, we use the notation $s[i]_j$ to denote the substring $s[i]s[i+1] \cdots s[j]$. 

1
Our contributions. Our main result in this paper is a tight running time bound on the Mirkin Distance Minimization problem. Specifically, we show that the problem cannot be solved in $2^{o(n+m)}$ time unless the Exponential Time Hypothesis (ETH) fails. While the upper bound in this result is not very difficult, the lower bound uses an intricate construction, which shows that the trivial brute-force $O^*(2^n)$ algorithm for the problem cannot be substantially improved. In the second part of the paper, we show that the problem is fixed-parameter tractable for the parameter $m$ of input strings, via an integer linear programming (ILP) approach.

Related work. Mirkin Distance Minimization is a special variant of the NP-hard Clustering Aggregation [6] problem (aka. Consensus Clustering [4] or Clusters Ensembles [12]) from machine learning and bioinformatics. The problem has as input a set $P$ of $m$ partitions on a set $U$ of $n$ elements and we search for a target partition $P^*$ that minimizes the Mirkin distances to all $m$ partitions. Herein, a partition on the set $U$ is an equivalent binary relation $\sim$ (i.e. reflexive, symmetric, and transitive). Thus, each partition can be represented by the equivalence classes of the corresponding equivalence relation. The Mirkin distance between two partitions is defined as the number of pairs of elements which are equivalent in one partition but non-equivalent in the other, or the other way round.

A partition with at most two equivalence classes can also be expressed as a binary string. Thus, it is straightforward to see that our problem is equivalent to Clustering Aggregation for Binary Strings, i.e. both the input and the output partitions are binary strings. Mirkin Distance Minimization has further applications in voting theory and is also studied under the name of Binary Relation Aggregation [1, 13, 14], which is related to a concept in voting known as the median relation [1]. Dörnfelder et al. [3] showed that Mirkin Distance Minimization is NP-hard under Turing reduction. We will show in this note that the problem is NP-hard by providing a many-one reduction, which also implies that the trivial brute-force $O^*(2^n)$ algorithm for the problem cannot be substantially improved.

Very recently, we [2] considered a related problem, $p$-Norm Hamming Centroid, which searches for a centroid string which minimizes the $p$-norm of its Hamming distances to the input strings, for each fixed $p > 1$. When the objective is to maximize instead of minimize the distances and when $p = 2$, the Mirkin Distance Minimization problem can be reduced to this maximization variant.

2 NP-hardness for Sum of Mirkin Distances

We show that Mirkin Distance Minimization is indeed NP-hard by utilizing a gadget that Dörnfelder et al. [3] used to enforce that for each two bits, when restricted to only these two bits, exactly half of the input strings have the same value (00 or 11) and the other half have different values (01 or 10). Algorithm 1 computes such kind of gadget. Note that each output string has length $2^\ell$. Note that, however, this type of gadget alone is not enough to devise a many-one hardness reduction. This gadget can be used to encode truth-values of variables in a reduction from 3SAT but an essential difficulty that remains is to find gadgets that encode clause satisfaction.

We show that the strings constructed by Algorithm 1 fulfills our requirement.

Proposition 1. Let $S$ be the sequence of strings constructed by Algorithm 1. Then, for each two distinct bits $i, j \in \{1, 2, \cdots, 2^\ell\}$ the following two statements hold.
Algorithm 1: Algorithm for constructing a sequence of $2^\ell$ length-$2^\ell$ binary strings such that for each two bits, half of the strings have the same value and the other half have not.

1. Build($2^\ell$):
   - if $\ell = 1$ then
     - return (00, 01)
   - else
     - $(s_1, s_2, \cdots, s_{2^\ell-1}) \leftarrow$ Build($2^{\ell-1}$) return $(s_i \circ s_i \circ s_i)_{1 \leq i \leq 2^{\ell-1}}$

2. There are $|S|/2$ strings from $S$: $a_1, a_2, \cdots, a_{|S|/2}$, such that $a_r[i] = a_r[j]$, $r \in \{1, 2, \cdots, |S|/2\}$.

3. There are $|S|/2$ strings from $S$: $b_1, b_2, \cdots, b_{|S|/2}$, such that $b_r[i] \neq b_r[j]$, $r \in \{1, 2, \cdots, |S|/2\}$.

Proof. We show the statement via induction on $\ell$. For $\ell = 1$, Algorithm 1 returns (00, 01). Our two statements follow immediately. Assume that sequence $S' = $ Build($2^{\ell-1}$) satisfies the proposition. We show that $S = $ Build($2^\ell$) also satisfies the proposition. By Algorithm 1, we have $S = (s_r \circ s_r \circ s_r)_{s_r \in S'}$.

Consider two arbitrary bits $i, j \in \{1, 2, \cdots, 2^\ell\}$. Obviously, by our induction assumption, the two statements hold if $1 \leq i, j \leq 2^{\ell-1}$ or $2^{\ell-1} + 1 \leq i, j \leq 2^\ell$. Thus, we assume that $1 \leq i \leq 2^{\ell-1}$ and $2^{\ell-1} + 1 \leq j \leq 2^\ell$ (the other case when $1 \leq j \leq 2^{\ell-1}$ and $2^{\ell-1} + 1 \leq i \leq 2^\ell$ is symmetric). By construction, $S$ consists of the strings $s_r \circ s_r$ and $s_r \circ \overline{s_r}$, $1 \leq r \leq 2^{\ell-1}$. To show the two statements, it suffices if we can show that $"(s_r \circ s_r)[i] = (s_r \circ s_r)[j]"$ if and only if $(s_r \circ \overline{s_r})[i] \neq (s_r \circ \overline{s_r})[j].$ This is equivalent to $"s_r[i] = s_r[j]"$ if and only if $s_r[i] \neq \overline{s_r}[j]$, which is obvious.

We reduce from an NP-hard variant of the 3-SAT problem, called Not-All-Equal 3-SAT (NAE-3SAT) [5], which given a set of size-three clauses asks whether there is a satisfying truth assignment such that each clause has at least one true literal and at least one false literal.

Theorem 1. Mirkin Distance Minimization is NP-hard.

Proof. As mentioned, we reduce from the NP-hard NAE-3SAT problem [5]. Let $I = (X, C)$ be an instance of NAE-3SAT, where $X = \{x_1, x_2, \cdots, x_n\}$ denotes the set of $n$ variables and $C = \{c_1, c_2, \cdots, c_m\}$ denotes a set of $m$ clauses of size three each. Without loss of generality, assume that $n = 2^\ell + 1$ for some $\ell$. We construct two groups of binary strings where each string is of length $2n$. Variables will be encoded by pairs of two consecutive bits in the string, one on odd position, one on even position. We use the gadget constructed via Algorithm 1 to enforce that these two bits will always have the same value so that 11 will correspond to setting the variable to true while 00 will correspond to setting the variable to false.

To this end, given two binary strings $s$ and $s'$, and an integer $i$ with $1 \leq i \leq |s| + 1$, by $ins(s, s', i)$ we mean inserting the string $s'$ into $s$ at the position $i$. For instance, $ins(0110, 00, 3) = 01001$. In particular, $ins(s, s', 1) = s' \circ s$ and $ins(s, s', |s| + 1) = s \circ s'$.

Group 1. Let $S = $ Build($2^{\ell+1}$). Then, for each integer $r \in \{1, 2, \cdots, n\}$ (representing the index of a specific variable) we introduce $2^{\ell+1}$ strings as follows. For each string $s_i \in S$, construct two strings with the forms $ins(s_i, 11, 2r - 1)$ and $ins(\overline{s_i}, 11, 2r - 1)$. Note that each of these newly constructed strings has length $2^{\ell+1} + 2 = 2n$. Let $S_r$ denote the sequence that contains all these newly introduced strings.
Group 2. For each clause $c_j \in \mathcal{C}$ let $\ell_1, \ell_2, \ell_3$ be the three literals contained in $c_j$. We define three strings $t_j^{(1)}, t_j^{(2)}, t_j^{(3)} \in \{0,1\}^{2n}$ as follows.

$$\forall i \in \{1, 2, \cdots, n\}, \forall z \in \{1, 2, 3\}:
\begin{align*}
t^{(z)}[2i-1, 2i] &= \begin{cases} 11, & \ell_z = x_i, \\
00, & \ell_z = \overline{x}_i, \\
00, & \ell_y = x_i \text{ for some } y \in \{1, 2, 3\} \setminus \{z\}, \\
11, & \ell_y = \overline{x}_i \text{ for some } y \in \{1, 2, 3\} \setminus \{z\}, \\
01, & \text{otherwise.}
\end{cases}
\end{align*}$$

Let $T_j = (t_j^{(1)}, t_j^{(2)}, t_j^{(3)})$. For instance, for clause $c_j = (\overline{x}_1, x_2, \overline{x}_3)$, the three corresponding strings are

$$t_j^{(1)} = 00 00 11 01 01 \ldots 01,$$
$$t_j^{(2)} = 11 11 11 01 01 \ldots 01,$$
$$t_j^{(3)} = 11 00 00 01 01 \ldots 01.$$

Let $L = 3m \cdot n^2$. The instance $I'$ consists of the following strings: For each $r \in \{1, 2, \cdots, n\}$, add $L$ copies of $S_r$ to $I'$. For each $j \in \{1, 2, \cdots, n\}$, add $T_j$ to $I'$. This completes the construction, which can clearly be done in polynomial time. (Note that Build$(2^\ell+1)$ takes $O(2^\ell+1) = O(n)$ time.)

We claim that the instance $I$ has a satisfying truth assignment such that each clause has a true literal and a false literal if and only if there is binary string $s$ that has a Mirkin distance of at most $L \cdot n \cdot ((2n^2-2) + 2(n-1)) \cdot (n-1) + m \cdot (3n^2 - 11)$ to the strings from $I'$.

Before we show the correctness of the construction, we present two observations which will help us to determine the solution string for $I'$.

Claim 1. Let $s^*$ be an arbitrary binary string of length $2n$. For each integer $r \in \{1, 2, \cdots, n\}$, the following holds. If $s^*[2r-1, 2r] \in \{01, 10\}$, then $\text{mirk}(s^*, S_r) = ((2n^2-2) + 2(n-1) + 2) \cdot (n-1)$. If $s^*[2r-1, 2r] \in \{00, 11\}$, then $\text{mirk}(s^*, S_r) = ((2n^2-2) + 2(n-1)) \cdot (n-1)$.

Proof. By the construction of $S_r$ (Proposition 1), we have the following.

- For each pair $\{i, j\} \subseteq \{1, 2, \cdots, 2n\} \setminus \{2r-1, 2r\}$ we have
  1. $|S_r|/2$ strings $s$ from $S_r$ such that $s[i] = s[j]$, and
  2. $|S_r|/2$ strings $s$ from $S_r$ such that $s[i] \neq s[j]$.

This means that the Mirkin distance from $s^*$ to $S_r$ regarding the pair $\{i, j\}$ is always $|S_r|/2$.

- For each bit $i \in \{1, 2, \cdots, 2n\} \setminus \{2r-1, 2r\}$, $|S_r|/2$ strings from $S_r$ have a 0 in column $i$ and $|S_r|/2$ strings from $S_r$ have a 1 in column $i$. Thus, the Mirkin distance from $s^*$ to $S_R$ regarding the pair $\{i, 2r-1\}$ (resp. $\{i, 2r\}$) is also $|S_r|/2$.

- The Mirkin distance from $s^*$ to $S_r$ regarding the pair $\{2r-1, 2r\}$ is $|S_r|$ if $s^*[2r-1, 2r] \in \{01, 10\}$; otherwise it is zero.
In total, we have
\[
\text{mirk}(S_r, s^*) = \binom{2n - 2}{2} \cdot |S_r|/2 + 2(n - 1) \cdot |S_r|/2 + \begin{cases} |S_r|, & s^*[2r - 1, 2r] \in \{01, 10\} \\ 0, & \text{otherwise.} \end{cases}
\]
\[
= \begin{cases} \left(\binom{2n - 2}{2} + 2(n - 1) + 2\right) \cdot (n - 1), & s^*[2r - 1, 2r] \in \{01, 10\} \\ \left(\binom{2n - 2}{2} + 2(n - 1)\right) \cdot (n - 1), & \text{otherwise.} \end{cases}
\]

(of Claim 1) ⊙

Define γ: \(\{0, 1\}^n \to \{0, 1\}^{2n}\) by \(γ(e_1e_2 \cdots e_n) = (e_1e_1e_2e_2 \cdots e_ne_n)\).

Claim 2. Let \(c_j \in C\) be an arbitrary clause. Then for each \(s \in \{0, 1\}^n\), we have that \(\text{mirk}(γ(s), T_j) \geq 3n^2 - 11\), and the equality is attained if and only if the string \(s\), interpreted as a truth assignment to the variables \(x_i, i \in \{1, \ldots, n\}\), satisfies \(c_j\) with at least one true literal and at least one false literal.

Proof. Assume, without loss of generality, that the literals in \(c_j\) correspond the first, the second, and the third variable (each in either a positive or a negative form). For each string \(t_j^{(z)} \in T_j\) with \(z \in \{1, 2, 3\}\), by the definition of the Hamming distance, \(d(γ(s), t_j^{(z)}) = d(γ(s)|_1^n, t_j^{(a)}|_1^n) + d(γ(s)|_1^n, t_j^{(b)}|_1^n)\). By the definition of \(t_j^{(z)}\) regarding the positions from 7 to 2\(n\), we have that \(d(γ(s)|_1^n, t_j^{(a)}|_1^n) = n - 3\).

Assume that \(s\) satisfies \(c_j\) with the \(a^{th}\) literal being true and the \(b^{th}\) literal being false, \(a, b \in \{1, 2, 3\}\) and \(a \neq b\). Let \(c \in \{1, 2, 3\} \setminus \{a, b\}\). We distinguish two cases. If \(\ell_c\) is true under \(s\), then \(d(γ(s)|_1^n, t_j^{(a)}|_1^n) = 2 = d(γ(s)|_1^n, t_j^{(c)}|_1^n)\) while \(d(γ(s)|_1^n, t_j^{(b)}|_1^n) = 6\). If \(\ell_c\) is false under \(s\), then \(d(γ(s)|_1^n, t_j^{(a)}|_1^n) = 4 = d(γ(s)|_1^n, t_j^{(c)}|_1^n)\) while \(d(γ(s)|_1^n, t_j^{(b)}|_1^n) = 0\).

Using the alternative definition of the Mirkin distance, we have that
\[
\text{mirk}(γ(s), t_j^{(a)}) = \text{mirk}(γ(s), t_j^{(c)}) = d(γ(s), t_j^{(a)}) \cdot (2n - d(γ(s), t_j^{(a)})) = n^2 - 1,
\]
and that \(\text{mirk}(γ(s), t_j^{(b)}) = d(γ(s), t_j^{(b)}) \cdot (2n - d(γ(s), t_j^{(b)})) = n^2 - 9\). Altogether, we have that \(\text{mirk}(γ(s), T_j) = 2(n^2 - 1) + n^2 - 9 = 3n^2 - 11\).

Assume that under \(s\) either all literals from \(c_j\) are true or all literals from \(c_j\) are false. For the first case, for each \(z \in \{1, 2, 3\}\), we have \(d(γ(s)|_1^n, t_j^{(z)}|_1^n) = 4\), implying \(\text{mirk}(γ(s), t_j^{(z)}) = d(γ(s), t_j^{(z)}) \cdot (2n - d(γ(s), t_j^{(z)})) = n^2 - 1\). For the other case, for each \(z \in \{1, 2, 3\}\), we have \(d(γ(s)|_1^n, t_j^{(z)}|_1^n) = 2\), implying \(\text{mirk}(γ(s), t_j^{(z)}) = d(γ(s), t_j^{(z)}) \cdot (2n - d(γ(s), t_j^{(z)})) = n^2 - 1\). Altogether, we have that \(\text{mirk}(γ(s), T_j) = 2(n^2 - 1) + n^2 - 9 = 3n^2 - 11\) (of Claim 2) ⊙

Now we are ready to show the equivalence between \(I\) and \(I'\), i.e. \(I = (X, C)\) admits a truth assignment such that each clause in \(C\) has a true literal and a false literal if and only if there is a string \(s^*\) whose Mirkin distance to the strings in \(I'\) is at most \(L \cdot n \cdot (\binom{2n - 2}{2} + 2(n - 1)) \cdot (n - 1) + m \cdot (3n^2 - 11)\).

For the “only if” direction, assume that \(s \in \{0, 1\}^n\) is a satisfying assignment for \(C\) such that each clause \(c_j \in C\) has at least one true literal and at least one false literal. Claim 2 indicates that \(γ(s)\) has Mirkin distance \(3 \cdot n^2 - 11\) to each triple in \(T_j\) that corresponds to the clause \(c_j\).
second statement in Claim 1 indicates that \( \gamma(s) \) has Mirkin distance \((2n^2-2) + 2(n-1)\) to all strings in \( S_r \) that corresponds to the variable \( x_r \). Altogether, the Mirkin distance between \( \gamma(s) \) and all strings in \( I' \) is \( m \cdot (3n^2-11) + L \cdot n \cdot (2n-1)) \cdot (n-1) \cdot (3n^2-11) \). We claim that \( s^* \) has the form \( s^* = e_1e_2\ldots e_ne_n \) with \( e_i \in \{0,1\} \) for all \( 1 \leq i \leq n \). Suppose, towards a contradiction, that \( s^* \) is not of the desired form, and let \( i \in \{1,2,\ldots,n\} \) be an integer such that \( s^*[2i-1, 2i] \in \{0,10\} \). Then, by the first statement in Claim 1, the Mirkin distance of \( s^* \) to the first group of strings in \( I' \) will be at least \( L \cdot (2n-2) + 2(n-1) + 2 \cdot (n-1) + L \cdot (n-1) \cdot ((2n-1) + 2(n-1)) \cdot (n-1) = L \cdot n \cdot (2n-2) + 2(n-1) \cdot (n-1) + 2L \cdot (n-1) \) which exceeds our distance bound \( L \cdot n \cdot ((2n-2) + 2(n-1)) \cdot (n-1) + m \cdot (3n^2-11) \) since \( L > m \cdot (3n^2-11) \) — a contradiction.

Thus, \( s^* \) has the form \( s^* = e_1e_2\ldots e_ne_n \) with \( e_i \in \{0,1\} \) for all \( 1 \leq i \leq n \). We show that \( s = e_1e_2\ldots e_n \) is a satisfying assignment for \( C \) such that each clause has at least one true literal and at least one false literal. By the above reasoning, the Mirkin distance of \( s^* \) to the second group of strings can be at most \( m \cdot (3n^2-11) \). Since there are \( m \) triples in the second group, one for each clause, the average Mirkin distance of \( s^* = \gamma(s) \) to each triple is \( 3n^2-11 \). By Claim 2 the Mirkin distance of \( s^* \) to each triple in the second group is indeed \( 3n^2-11 \), meaning that under \( s \) each clause has at least one true literal and one false literal.

As a corollary, we obtain a running time lower bound for our problem.

**Corollary 1.** Unless the Exponential Time Hypothesis fails, no \( 2^{o(n)} \cdot |I'|^{O(1)} \)-time algorithm exists for any instance \( I' \) of Mirkin Distance Minimization where \( n \) is the length of the input strings.

**Proof.** To show the statement, note that the length of the the strings that we constructed in the proof of Theorem 1 is exactly \( 2n \), where \( n \) is the number of variables in the NAE-3SAT instance. Thus, if we can show that, assuming the Exponential Time Hypothesis, NAE-3SAT does not admit a \( 2^{o(n)} \cdot |I'|^{O(1)} \)-time algorithm, where \( I \) is an NAE-3SAT instance with \( n \) variables, then our result follows.

Since we are not aware of any reference that explicitly states such a running time lower bound for NAE-3SAT, we prove this by providing a simple reduction from 3SAT. 3SAT is known not to admit any sub-exponential time algorithm unless the Exponential Time Hypothesis fails [8]. Let \( J = (X,C) \) be an instance of 3SAT, where \( X = \{x_1,x_2,\ldots,x_n\} \) denotes the set of \( n \) variables and \( C = \{c_1,c_2,\ldots,c_m\} \) denotes a set of \( m \) clauses of size three each. We construct an instance \( J' = (X',C') \) of NAE-3SAT as follows. The variable set \( X' \) of \( J' \) consists of all variables from \( X \), and \( m+1 \) new variables \( y_j, 1 \leq j \leq m \), and \( z \), i.e., \( X' = X \cup \{y_j \mid 1 \leq j \leq m\} \cup \{z\} \). For each clause \( c_j \) of \( C \) let \( c_j = (\ell^1_j \lor \ell^2_j \lor \ell^3_j) \) to unify the notation. For each clause \( c_j \), we introduce to \( C' \) the following two clauses \( d_j \) and \( e_j \) with \( d_j = (\ell^1_j \lor \ell^2_j \lor y_j) \) and \( e_j = (\ell^3_j \lor z \lor \overline{y_j}) \).

This completes the construction which can be carried out in linear time. We claim that \( J = (X,C) \) admits a satisfying truth assignment \( \sigma : X \to \{T,F\} \) if and only if \( J' = (X',C') \) admits a satisfying truth assignment \( \sigma' : X' \to \{T,F\} \) such that each clause in \( C' \) has at least one true literal and at least one false literal.
For the “only if” direction, assume that $\sigma$ is a satisfying truth assignment for $J$. It is straightforward to verify that the following truth assignment $\sigma' : X' \rightarrow \{T, F\}$ is a satisfying truth assignment for $J'$ such that each clause has at least one true literal and one false literal.

For all $x_i \in X$: $\sigma'(x_i) = \sigma(x_i)$,
for all $j \in \{1, 2, \ldots, m\}$: $\sigma'(y_j) = T$ if and only if $\sigma(\ell_j^1) = \sigma(\ell_j^2) = F$,
$\sigma'(z) = F$.

For the “if” direction, assume that $\sigma' : X' \rightarrow \{T, F\}$ is a satisfying truth assignment for $J'$ such that each clause in $J'$ has at least one true literal and one false literal. We claim that the following truth assignment $\sigma : X \rightarrow \{T, F\}$ is a satisfying assignment for $J$.

For all $x_i \in X$:

\[
\sigma(x_i) = \begin{cases} 
T, & \text{if } \sigma'(x_i) \neq \sigma'(z), \\
F, & \text{otherwise.}
\end{cases}
\]

Suppose, for the sake of contradiction, that there is a clause $c_j \in C$ which is not satisfied by $\sigma$. Let $\ell_j^1, \ell_j^2$, and $\ell_j^3$ be the three literals in $c_j$. Since $c_j$ is not satisfied by $\sigma$, it follows that $\sigma(\ell_j^1) = \sigma(\ell_j^2) = \sigma(\ell_j^3) = F$. We distinguish two cases and show in each case a contradiction.

Case 1: $\ell_j^3$ is a positive literal, implying that $\sigma'(z) = \sigma'(\ell_j^3) = F$. Since $c_j$ is satisfied (contains either a true or a false literal), it follows that $\sigma'(y_j) = F$. However, since $d_j$ is satisfied, it follows that $\sigma'(y_j) = T$—a contradiction.

Case 2: $\ell_j^3$ is a negative literal, say $\ell_j^3 = \neg x_i$, implying that $\sigma'(z) = F$ and $\sigma'(x_i) = T$. Again, since $c_j$ is satisfied, it follows that $\sigma'(y_j) = F$. However, since $d_j$ is satisfied, it follows that $\sigma'(y_j) = T$—a contradiction.

We have shown the correctness of our construction. Now, observe that our constructed instance $J'$ has in total $|X'| = n + m + 1$ variables. Hence, a $2^{O(|X'|)} \cdot |J'|^{O(1)}$-time algorithm for NAE-3SAT would imply a $2^{O(n+m)} \cdot |J|^{O(1)}$-time algorithm for 3SAT, which is unlikely unless the Exponential Time Hypothesis fails [8]. In summary, this proves our running time lower bound statement for the Mirkin Distance Minimization problem.

3 An Integer Linear Program (ILP) Formulation

In this section, we show that minimizing the Mirkin distance is fixed-parameter tractable with respect to the number $m$ of input strings. To achieve this, we formulate our problem as an integer linear program with the number $\rho$ of variables upper-bounded by $2^{m^2}$, each corresponding to a pair of column types (to be defined shortly), and with polynomial number of constraints. By Lenstra [10], Kannan [9], we immediately have that our problem is solvable in time $O(\rho^{O(\rho)} \cdot L) = 2^{O(m^2 \cdot 2^{m^2})} \cdot L)$, where $L$ denotes the length of binary encoding of the input strings. We note that this integer programming approach similar to ours is applicable in many string problems whenever the columns of the input can be grouped together in order to be represented by a constant number of variables [7, 2]. The resulting mathematical programming formulation is not linear at first. We need additional tricks where reformulate such that we can safely omit the square of binary variables, and such that we can introduce some extra variables to avoid multiplications of binary variables.

Before presenting the formulation, we observe a useful property of an optimal solution that allows us to introduce only binary variables, one for each column type. Herein, given a non-empty
Lemma 1. Let \( S \) be a sequence of \( m \) strings, each of length \( n \), and let \( s^* \) be a solution with minimum Mirkin distance to \( S \). If two distinct columns \( j \) and \( j' \) with \( j, j' \in [n] \) have the same type, then it holds that \( s^*[j] = s^*[j'] \).

Proof. Towards a contradiction, suppose that \( s^*[j] \neq s^*[j'] \). We will show that making these two columns have the same bit, either zero or one, will result in a better solution. Let \( s^*_{00} \) (resp. \( s^*_{11} \)) be a string that we obtain from \( s^* \) by replacing with 00 (resp. 11) the bits at positions \( j \) and \( j' \). Formally, we have \( s^*_{00}[j,j'] = 00 \) and \( s^*_{11}[j,j'] = 11 \), and for each \( \ell \in [n] \setminus \{j,j'\} \), we have \( s^*_{00}[\ell] = s^*_{11}[\ell] = s^*[\ell] \). Given two strings \( s \) and \( t \), we define a function \( f \) that computes the Mirkin distance from \( s \) to \( S \) subtracted by the Mirkin distance between \( t \) to \( S \):

\[
f(s,t,S) := \sum_{s_i \in S} \text{mirk}(s,s_i) - \sum_{s_i \in S} \text{mirk}(t,s_i).
\]

To obtain a contradiction, we show that \( s^* \) is not an optimal solution by showing that

\[
f(s^*, s^*_{00}, S) + f(s^*, s^*_{11}, S) > 0,
\]

because this implies that

\[
\sum_{s_i \in S} \text{mirk}(s^*, s_i) > \sum_{s_i \in S} \text{mirk}(s^*_{00}, s_i) \quad \text{or} \quad \sum_{s_i \in S} \text{mirk}(s^*, s_i) > \sum_{s_i \in S} \text{mirk}(s^*_{11}, s_i).
\]

For each input string \( s_i \in S \), let \( d_i \) denote the Hamming distance between \( s^* \) and \( s_i \), restricted to the columns that are neither \( j \) nor \( j' \). We show that \( f(s^*, s^*_{00}, S) + f(s^*, s^*_{11}, S) > 0 \).

\[
f(s^*, s^*_{00}, S) + f(s^*, s^*_{11}, S) = 2 \sum_{s_i \in S} \text{mirk}(s^*, s_i) - \sum_{s_i \in S} \text{mirk}(s^*_{00}, s_i) - \sum_{s_i \in S} \text{mirk}(s^*_{11}, s_i)
= 2 \sum_{s_i \in S} (d_i + 1)(n - d_i - 1) - \sum_{s_i \in S} d_i(n - d_i) - \sum_{s_i \in S} (d_i + 2)(n - d_i - 2)
- \sum_{s_i \in S} (d_i + 2)(n - d_i - 2) - \sum_{s_i \in S} d_i(n - d_i)
= \sum_{s_i \in S} (2d_i + 1)(n - d_i - 1) - d_i(n - d_i) - (d_i + 2)(n - d_i - 2) = 2m > 0.
\]

By our reasoning before, this implies that \( s^* \) is not an optimal solution, a contradiction.

By Lemma 1, for each two distinct types of columns, we only need to store whether the output string will have the same value at columns that correspond to these two types. Let \( n' \) denote the number of different (column) types in \( S \). Then, \( n' \leq \min(2^m, n) \). Enumerate the \( n' \) column types as \( t_1, \ldots, t_{n'} \). Below we identify a column type with its index for easier notation. Using this, we can encode the set \( S \) succinctly by introducing a constant \( e[j] \) for each column type \( j \in [n'] \) that denotes the number of columns with type \( j \). Analogously, given an optimal solution string \( s^* \), by
Lemma 1 we can also encode this string \( s^* \) via a binary vector \( x \in \{0,1\}^{n'} \), where for each column type \( j \in \{n'\} \) we use \( x[j] \) to indicate whether the columns that correspond to the type have zeros or ones. Note that this encodes all essential information in a solution, since the actual order of the columns is not important.

Example 1. For an illustration, let \( S = \{0000, 0001, 1110\} \). The set \( S \) has two different column types, represented by \( (0,0,1)^T \), call it type 1, and \( (0,1,0)^T \), call it type 2. There are three columns of type 1 and one column of type 2. An optimal solution 0001 with minimum Mirkin distance four for \( S \) can be encoded by two binary variables \( x[1] = 0 \) and \( x[2] = 1 \).

**Integer Linear Program Formulation.** Using the binary variables \( x \) that represent a solution \( s^* \) that has the same values in the columns of the same type, we can reformulate the Hamming distance between the two strings \( s_i \) and \( s^* \) as follows. For the sake of simplicity, we let \( s_i[j] = 1 \) if the column type of column \( j \) has one in the \( i \)th row and \( s_i[j] = 0 \) if it has zero in the \( i \)th row.

\[
d(s_i, s^*) = \sum_{j=1}^{n'} e[j] \cdot |s[j] - x[j]| = \sum_{j=1}^{n'} e[j](s_i[j] + (1 - 2s_i[j]) \cdot x[j])
\]

Then the Mirkin distance between \( x \) and \( s_i \) can be formulated as follows, where \( w_i = \sum_{j=1}^{n'} e[j] \cdot s_i[j] \) denotes the number of ones in string \( s_i \) and \( c_i[j] = 1 - 2s_i[j] \in \{1, -1\} \), i.e. \( c_i[j] = 1 \) if \( s_i[j] = 0 \) and \( c_i[j] = -1 \) if \( s_i[j] = 1 \).

\[
mirk(s_i, s^*) = d(s_i, s^*) \cdot (n - d(s_i, s^*)) = (w_i + \sum_{j=1}^{n'} e[j] \cdot c_i[j] \cdot x[j]) \cdot (n - w_i - \sum_{j=1}^{n'} e[j] \cdot c_i[j] \cdot x[j])
\]

\[
= n(w_i + \sum_{j=1}^{n'} e[j] \cdot c_i[j] \cdot x[j]) - (w_i + \sum_{j=1}^{n'} e[j] \cdot c_i[j] \cdot x[j])^2
\]

\[
= n \cdot w_i - w_i^2 + \sum_{j=1}^{n'} (n \cdot c_i[j] - 2w_i \cdot c_i[j] - e[j]) \cdot e[j] \cdot x[j]
\]

\[
- \sum_{j \neq j'} e[j] \cdot e[j'] \cdot e[j] \cdot c_i[j'] \cdot x[j] \cdot x[j'].
\] (1)

The last equation holds since \( c_i[j]^2 = 1 \) and since \( x[j] \) is binary, implying that \( x[j] = x[j]^2 \). The resulting formulation is not linear since the components in the last sum are products of two binary variables. Nevertheless, we can introduce additional binary variables to linearize it. For each two distinct column types \( j \) and \( j' \) we introduce a binary variable \( y([j,j']) \) which shall have the value \( y([j,j']) = x[j] \cdot x[j'] \). We can achieve this by introducing the following constraints:

\[
\forall j, j' \in \{n'\}, j \neq j': \quad y([j,j']), x[j] \in \{0,1\}, \quad y([j,j']) \leq x[j], \quad y([j,j']) \leq x[j'], \quad x[j] + x[j'] - y([j,j']) \leq 1.
\] (2a) (2b) (2c) (2d)
Now we can replace each product of two binary variables in (1) with a corresponding variable:

\[
mirk(s_i, s^*) = n \cdot w_i - w_i^2 + \sum_{j=1}^{n'} (n \cdot c_i[j] - 2w_i \cdot c_i[j] - e[j]) \cdot e[j] \cdot x[j]
- \sum_{\{j,j'\} \subseteq \{1, \ldots, n'\}, j \neq j'} e[j] \cdot e[j'] \cdot c[j] \cdot c[j'] \cdot y[\{j, j'\}].
\]  

Combining (2a)–(2d) with the following constraint

\[
\sum_{i=1}^{m} (n \cdot w_i - w_i^2 + \sum_{j=1}^{n'} (n \cdot c_i[j] - 2w_i \cdot c_i[j] - e[j]) \cdot e[j] \cdot x[j]
- \sum_{\{j,j'\} \subseteq \{1, \ldots, n'\}, j \neq j'} e[j] \cdot e(j') \cdot c[j] \cdot c[j'] \cdot y[\{j, j'\}]) \leq k,
\]

we obtain an ILP with at most \(2^{m^2}\) binary variables and \(5n^2\) constraints. By the result of Lenstra [10], we immediately obtain the following.

**Theorem 2.** **Mirkin Distance Minimization** can be solved in \(2^{O(m^2 \cdot 2^{m^2})} \cdot n^2\) time.

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