On deconvolution problems: numerical aspects.

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Key words: linear ill-posed problems, Volterra equations, deconvolution  
AMS subject classification: 45D05, 45L05, 45P05, 65R20, 65R30

Abstract. An optimal algorithm is described for solving the deconvolution problem of the form

\[ ku := \int_0^t k(t-s)u(s)\,ds = f(t), \quad 0 \leq t \leq T, \]

where \(k(t), \ t \geq 0\), is a kernel of linear integral equation (1.1), \(k \ast u\) is the convolution. It is important in many engineering applications, in physics, and other areas. There is a vast literature on deconvolution methods, see, for example, [6].

If the operator \(k\) in (1.1) is considered as an operator on \(X := L^\infty(0, T)\), and \(\int_0^T |k(t)|\,dt < \infty\), then \(k\) is not boundedly invertible, so problem (1.1) is ill-posed. Assume that the data \(f\) are noisy: \(f_\delta\) is given, such that \(|f - f_\delta| \leq \delta\). In this case it is natural to seek an approximate solution of equation (1.1) in the class \(Q_\delta := \{u \in X : ||ku - f_\delta|| \leq \delta\}\). However, for ill-posed equation (1.1) an arbitrary element \(u_\delta \in Q_\delta\) cannot be taken as an approximate solution to (1.1), since \(u_\delta\) is not continuous with respect to \(\delta\) in general. In order to select possible solutions one needs to use a priori information (usually available) about the solution, which may be of a quantitative or qualitative nature.

The usage of qualitative a priori information makes it possible to narrow the class of solutions, for example, to a compact set, so that the problem becomes stable under small changes in the data. This leads to a concept of a quasisolution [8]. Various algorithms for approximate determination of quasisolutions were studied in [8].
A priori information of a qualitative nature (for example, smoothness of the solution) generates different approaches. The one which is used often is variational regularization \[20, 10\], which allows one to construct stable approximate solutions to ill-posed problems by means of a stabilizing functional. The variational method has been extensively developed in \[4, 9\], and certain a priori and a posteriori choices of a regularization parameter \( \varepsilon = \varepsilon(\delta) \) have been designed and implemented \[9, 2\].

One can also find approximate solutions to (1.1) by iterations (see \[21, 11\]), taking \( x_n = R(f_\delta, x_{n-1}, \ldots, x_{n-k}) \), where \( k \leq n \). For these solutions to be stable under small changes of the data, the iteration number \( n = n(\delta) \) yielding \( x_n \) must depend on the \( \delta \) suitably.

Other important techniques in theory of ill-posed problems give regularizing operators by using Fourier, Laplace, Mellin, and other integral transforms, statistical regularization, and the dynamical systems method (DSM) \[12, 13\].

In \[14\] some general new approaches are proposed for solving an ill-posed deconvolution problem. One of these approaches is based on the following idea. Assume that the operator \( k \) in (1.1) can be decomposed into a sum \( k := A + B \), where the operator \( A^{-1}B := S \) is compact in the Banach space \( X \), in which \( k \) acts, and \( I + S \) is boundedly invertible. By the Fredholm alternative, it is equivalent to assuming that \( \mathcal{N}(I + S) = \{0\} \), where \( \mathcal{N}(A) \) is the null space of \( A \). In this case \( I + S \) is an isomorphism of \( X \) onto \( X \), \( \mathcal{R}(A) = \mathcal{R}(k) \), where \( \mathcal{R}(A) \) is the range of the operator \( A \), and

\[
ku = A(I + S)u = f_\delta.
\]

If a regularizer for \( A \) is known, then (1.2) can be solved stably by the scheme

\[
(1.3) \quad u_\delta = (I + S)^{-1}R(\delta)f_\delta,
\]

and

\[
(1.4) \quad ||u - u_\delta|| \to 0 \quad \text{as} \quad \delta \to 0.
\]

Since \( I + S \) is an isomorphism, the error \( ||v - v_\delta|| \) of the approximation of the solution of the equation \( Av = f_\delta \) by the formula \( v_\delta = R(\delta)f_\delta \) is of the same order as \( ||u_\delta - u|| \). In this paper (see sections 2 and 3) we show that the proposed method is practically efficient and works better than the variational regularization.

Theoretically the proposed method is optimal on the class of the data defined as a triple \( \{\delta, f_\delta, M_2\} \), where \( f \in C^2(0, T), ||f''|| \leq M_2 \), and \( f \) is otherwise arbitrary, \( f_\delta \in L^\infty(0, T) \) and \( ||f - f_\delta|| \leq \delta \) and \( f_\delta \) is otherwise arbitrary.

The operator \( R(\delta) \), defined in (2.3) and originally proposed in \[10\] for stable numerical differentiation, yields an optimal estimate of \( f' \) in \( L^\infty(0, T) \)–norm in the following sense:

\[
\inf_T \sup_{f: ||f - f_\delta|| \leq \delta, ||f|| \leq M_2} ||Tf_\delta - f'|| \geq (2M_2\delta)^{1/2},
\]

where the infimum is taken over all, linear and non-linear, operators \( T : X \to X \), \( X = L^\infty(0, T) \), the supremum is taken over all \( f \) and \( f_\delta \) subject to the conditions \( f \in C^2(0, T), ||f''|| \leq M_2 \), \( ||f - f_\delta|| \leq \delta \), and

\[
||R(\delta)f_\delta - f'|| \leq (2M_2\delta)^{1/2},
\]

(see e.g., \[16\], [17], [14]).
This argument shows that our "deconvolution" method for stable solution of (1.1) is optimal on the above data set: the operator $R(\delta)$ gives an optimal (on the above data set) approximation of $f'$. Inversion of an isomorphism $I + S$, where $S$ is a compact operator, can be done very accurately by a projection method, for example, so that the total error of the solution is of the same order as the error obtained by applying $R(\delta)$.

2. The case $k(t) \in C^1(0,T)$

Let $k(t) \in C^1(0,T)$ and $k(0) \neq 0$. Then without loss of generality one can take $k(0) = 1$. As in [14], write (1.1) as

$$ku = \int_0^t u(s) \, ds + \int_0^t [k(t-s) - 1]u(s) := Au + Bu = f.$$  

Assume that $f(x)$ is given by its $\delta$-approximation, i.e. one knows $f_\delta(x)$ such that $||f - f_\delta||_X \leq \delta$. In the experiments of this section $\delta = 0.1$. Let $A^{-1}B := S$. Then

$$ku = A(I + S)u = f.$$  

Stable inversion of $A$ is equivalent to stable numerical differentiation of noisy data, and therefore as a regularizer $R(\delta)f_\delta$ for $A$ one can use (see [11], [17], [18], [19], and .
also \[13,10\]

\begin{equation}
(2.3) \quad R(\delta)f_\delta := \frac{f_\delta(t + h(\delta)) - f_\delta(t - h(\delta))}{2h(\delta)},
\end{equation}

with \( h(\delta) = \left(\frac{2\delta}{M_2}\right)^{1/2} \), \( ||f''||_{L_\infty(0,T)} \leq M_2 \). Hence

\begin{equation}
(2.4) \quad (I + S)u_\delta = R(\delta)f_\delta,
\end{equation}

where \( S \) is a Volterra operator: \( Su_\delta = \int_0^t k'(t - s)u_\delta(s) \, ds \). To test numerical efficiency of the above deconvolution algorithm, we take

\begin{equation}
(2.5) \quad k(y) = \exp(ay), \quad f(t) = \frac{(b + a)(\exp(at) - \cos(bt)) + (b - a) \sin(bt)}{a^2 + b^2}.
\end{equation}

Then equation (2.4) has the exact solution:

\begin{equation}
(2.6) \quad u_{\text{orig}}(t) = \sin(bt) + \cos(bt).
\end{equation}

The graphs of \( f \) and its \( \delta \)-approximation, \( f_\delta \), for \( T = 1, \ a = 1, \ b = 2\pi \), are presented in Figure 1. The perturbation was generated as a sum of five sinusoids with various periods and amplitudes in such a way that \( ||f - f_\delta||_X \leq 0.1 \). For \( \delta = 0.1 \) and for the above choice of \( f, T, a, \) and \( b \), one has \( h(\delta) = \left(\frac{2\delta}{M_2}\right)^{1/2} = 0.1253 \). Since in practice often only an estimate for \( M_2 \) may be available, our first experiment was done with the approximate value of \( h(\delta) \), namely \( h = 0.105 \). The goal of the first experiment was to compare the results obtained by the deconvolution method suggested in \[14\] and by the variational regularization with a choice of the parameter by the Morozov discrepancy principle. The integral in (1.1) was calculated by the corrected trapezoid formula (see \[7\]) with the number of node points \( n = 200 \) on the interval \([0,1]\). The graphs of \( u_{\text{disc}}(t) \) and \( u_{\text{deconv}}(t) \) as well as the graph of the original solution, \( u_{\text{orig}}(t) \), for \( h(\delta) = 0.105 \) and \( n = 200 \) are given in Figure 2. One can see from the picture that method [14] provides higher quality of reconstruction.

| \( t \) | \( u_{\text{exact}}(t) \) | \( u_{\text{disc}}(t) \) | \( u_{\text{deconv}}(t) \) |
|-------|-----------------|-----------------|-----------------|
| 0.05  | 1.26007351067010 | 0.88613219081253 | 1.611047434242 |
| 0.15  | 1.39680224666742 | 0.77345683250358 | 1.1677102714854 |
| 0.25  | 1.00000000000000 | 0.78531546607804 | 0.9756729235993 |
| 0.35  | 0.22123174208247 | 0.32264819143761 | 0.4690189004636 |
| 0.45  | -0.64203952192021 | -0.01522580641369 | -0.9401907284100 |
| 0.55  | -1.26007351067010 | -0.72058597578420 | -1.39931254313538 |
| 0.65  | -1.39680224666742 | -0.65363525334725 | -1.13246945274454 |
| 0.75  | -1.00000000000000 | -0.84181827797783 | -1.26012127085008 |
| 0.85  | -0.22123174208247 | -0.48659287989254 | -0.24854842261471 |
| 0.95  | 0.64203952192021 | -0.25478764331776 | 0.99713489435843 |
Table 1 allows one to analyze the computed values of $u_{\text{disc}}(t)$ and $u_{\text{deconv}}(t)$ for $h(\delta) = 0.1$ and $n = 10$. The regularization parameter for the variational regularization calculated by the Morozov discrepancy principle, $\varepsilon_{\text{disc}}$, is equal to 0.0275 for our particular $f_\delta$. The functions $u_{\text{disc}}(t)$ and $u_{\text{deconv}}(t)$ approximate the exact solution $u_{\text{orig}}(t)$ with the relative errors $\delta_{\text{disc}} = 0.5216$ and $\delta_{\text{deconv}} = 0.2470$, respectively, for $n = 10$.

**Table 2.**

| $n$ | $\delta_{\text{disc}}$ | $\delta_{\text{deconv}}$ |
|-----|------------------------|-------------------------|
| 10  | 0.52160739359373       | 0.24703402714545        |
| 50  | 0.47882066139400       | 0.29886579484582        |
| 100 | 0.49421933901812       | 0.29887532874922        |

The deconvolution procedure [14] is applicable when the constant $M_a$, $a > 1$ is known. Here $M_a$ is the bound on the $f^{(a)}$, $a > 0$ is a real number, and $f^{(a)}$ is
the (fractional order) derivative of $f$ (see [16] for details). Figures 3-6 show the dependence of the quality of calculations provided by the deconvolution technique for different values of $h(\delta)$ with the same $f_\delta$ that is given in Figure 1. The level of reconstruction is acceptable for all values of $h(\delta) \in (0.09, 0.3)$, but the best quality is attained for the near-optimal values: $h(\delta) = 0.1$ and $h(\delta) = 0.2$. Outside the interval $(0.09, 0.3)$ the reconstruction by the variational regularization works better because $h(\delta)$ is away from its optimal value.

Table 2 contains relative errors, $\delta_{\text{disc}}$ and $\delta_{\text{deconv}}$, for values of $n = 10, 50, 100$. In both cases the relative errors are not decaying further as $n$ increases, because the major component in these errors come from the noise level, and not from the error of the computational methods.

3. **Kernel of the type** $k(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)} + m(t), \quad 0 < \gamma < 1, \quad m(t) \in C^1$

In this section we solve (1.1) with the kernel $k(t)$ of the form

(3.1) \[ k(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)} + m(t), \quad 0 < \gamma < 1, \quad m(t) \in C^1 \]

As in [14], write (1.1) as

(3.2) \[ ku := Au + Bu = f, \]
where

\[(3.3) \quad A u := \frac{t^{\gamma-1}}{\Gamma(\gamma)} * u, \quad B u := m * u.\]

One has (15, pp.117-118) \( A^{-1} f = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{f(s)}{(t-s)^\gamma} ds. \) Since the right-hand side \( f \) is given by its \( \delta \)-approximation \( f_\delta \), \( ||f-f_\delta||_X \leq \delta \), we replace \( A^{-1} \) by the regularizer \( R_1(\delta) \) (see 14): \n
\[(3.4) \quad R_1(\delta)f_\delta := \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{R(\delta)f_\delta(s)}{(t-s)^\gamma} ds. \]

The operator \( R(\delta) \) in (3.4) is defined by formula (2.3) with \( h = \textbf{0.12} \). One gets

\[(3.5) \quad (I + S)u_\delta = R_1(\delta)f_\delta. \]

and

\[(3.6) \quad Su_\delta := A^{-1}B u_\delta = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{m(0)u_\delta(s) + \int_0^s m'(s-p)u_\delta(p)dp}{(t-s)^\gamma} ds. \]

The goal of the experiment was to compare two numerical methods for solving (1.1)-(3.1): deconvolution method (3.5)-(3.6) and variational regularization with a choice of the parameter by the discrepancy principle.

The function

\[ f(t) = \frac{t^\gamma}{\Gamma(1+\gamma)} \left(1 - \frac{2t^2}{(1+\gamma)(2+\gamma)}\right) + \frac{t^3}{3} \left(1 - \frac{t^2}{10}\right), \quad t \in [0,1], \]
was chosen as the solution to direct problem \([1.1] - [3.1]\) with \(m(t) = t^2\) and the model function 
\[u_{\text{exact}}(t) = 1 - t^2.\]
Then for the numerical tests the noisy function 
\[f_{\delta}, \|f - f_{\delta}\|_X \leq \delta, \delta = 0.1,\]
was used. The graphs of \(f\) and \(f_{\delta}\) for \(\gamma = 0.1\) are given in Figures 7 and 8.

Table 3.

| \(t\)   | \(u_{\text{exact}}(t)\)   | \(u_{\text{disc}}(t)\)   | \(u_{\text{deconv}}(t)\)   |
|-------|----------------|----------------|----------------|
| 0.05  | 0.99000000000000 | 0.9336156127658 | 1.0028144943820 |
| 0.15  | 0.96000000000000 | 0.8598375148008 | 0.98540317766041 |
| 0.25  | 0.91000000000000 | 0.78772680932067 | 0.9344277603698 |
| 0.35  | 0.84000000000000 | 0.7136282095483 | 0.85896131974999 |
| 0.45  | 0.75000000000000 | 0.63504318796224 | 0.7611793602457 |
| 0.55  | 0.64000000000000 | 0.55028612377430 | 0.6496777696250 |
| 0.65  | 0.51000000000000 | 0.45824705747747 | 0.5065483593430 |
| 0.75  | 0.36000000000000 | 0.35823345537370 | 0.3500454315540 |
| 0.85  | 0.19000000000000 | 0.2497168725846 | 0.1749329988351 |
| 0.95  | 0 | 0.13214536398250 | -0.01840021170081 |
Figures 10-12 illustrate the numerical performance of method (3.5)-(3.6) and variational regularization for $\gamma = 0.1$. The solutions evaluated by formulas (3.5)-(3.6) and by variational regularization for $n = 10$ and $\gamma = 0.1$ are also presented in Table 3. The results obtained for our particular test problem show that for small values of $\gamma$ the deconvolution approach is superior to variational regularization both in terms of accuracy and stability. However as $\gamma$ is getting bigger, the efficiency of the deconvolution method (as well as the efficiency of variational regularization) is getting worse. This is happening because when $\gamma$ is close to 1, the ill-posedness of problem (1.1)-(3.1) grows due to the errors in calculations of the singular integral. One can compare Figures 9-12 and 13-16. Moreover, as $\gamma$ changes from 0.1 to 0.9, method (3.5)-(3.6) becomes very sensitive to slight variations of $h(\delta)$. To illustrate this phenomena, we present the dependence of relative errors and discrepancies on $h(\delta)$ for $\gamma = 0.1$ and $\gamma = 0.5$ in Figures 17 and 18. For $\gamma = 0.1$ the relative error of the deconvolution method remains less than 10% when $h(\delta) \in (0.05, 0.12)$, while for $\gamma = 0.5$ the relative error is only small for $h = 0.1$.

Finally, it is important to mention that CPU time for both methods, (3.5)-(3.6) and variational regularization, is approximately the same and it is very small: about 3-4 milliseconds for $n = 200$.

**Conclusion.** The paper presents numerical results of the implementation of the deconvolution method developed by AGR and presented together with other results
in [14]. The method is shown to be optimal in the sense explained in Section 1. The numerical results confirm the theoretical results on which the method is based. It is shown that the method is more accurate than the variational regularization method with the regularization parameter chosen by the discrepancy method.

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