The Completeness of Reasoning Algorithms for Clause Sets in Description Logic $\mathcal{ALC}$

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Abstract: On the Semantic Web, metadata and ontologies are used to enable computers to read data. The Web Ontology Language (OWL) has been proposed as a standard ontological language, and various inference systems for this language have been studied. Description logics are regarded as the theoretical foundations of OWL; they provide the syntax and semantics of a formal language for describing ontologies and knowledge bases. In addition, tableau algorithms for description logics have been developed as the standard reasoning algorithms for decidable problems. However, tableau algorithms generate inefficient reasoning steps owing to their nondeterministic branching for disjunction as well as the increase in the size of models occasioned by existential quantification. In this study, we propose conjunctive normal form (CNF) concepts, which utilize a flat concept form for description logic $\mathcal{ALC}$ in order to develop algorithms for reasoning about sets of clauses. We present an efficient reasoning algorithm for clause sets where any $\mathcal{ALC}$ concept is transformed into an equivalent CNF concept. Theoretically, we prove the soundness, completeness, and termination of the reasoning algorithms for the satisfiability of CNF concepts.

1. Introduction

The Semantic Web [1] is a framework that enables computers to read information on the Web by adding metadata [2] and ontologies. It is not possible for computers to distinguish between different metadata, whether they represent the same content or not. This is solved by defining the relationships between metadata, or by defining metadata in greater detail using a standard vocabulary. The technology and representation used to realize this is called an ontology [3]. The Web Ontology Language (OWL) [4] has been proposed by the Web Ontology Working Group of the World Wide Web Consortium (W3C) as a standard language for describing ontologies on the Web. It is important to formalize the syntax and semantics of an ontological language, based on which the termination, soundness, and completeness of reasoning algorithms can be proven for ontology-based reasoning tasks.

Description logics are logical systems that establish methodologies for knowledge representation and reasoning and that specialize in the characteristics (called concepts) of nominal and adjectival vocabularies. The languages used in description logics are categorized into families based on expressivity and computational complexity, which depends on the combination of constructs and syntax. Many studies have been conducted to investigate the properties of description logics with respect to ontological languages and knowledge representation [1]. Description logics are formalized by syntax, semantics, and reasoning algorithms; the decidability and computational complexity of reasoning tasks have been shown to be the theoretical foundation for OWL. Description logics are expressive but decidable for application to reasoning, although first-order logic is not decidable (it is semi-decidable). Since the tableau method is the standard reasoning algorithm for description logics, the termination, soundness, and completeness of the tableau method are guaranteed. However, depending on the expressiveness and scale of concepts, branches that increase the number of reasoning operations and the reasoning time may occur, which may lead to inefficient reasoning.

In general, disjunctions and universal or existential quantifiers in logical expressions increase the computational complexity of the reasoning related to the expressions. For efficient reasoning, logical formulas can be transformed to a conjunctive normal form (CNF) [5] or a clausal form to simplify the derivation. Clausal forms have been used to simplify logical formulas, and many reasoning algorithms have been proposed for clausal forms. The Davis-Putnam-Logemann-Loveland (DPLL) algorithm can be applied to the clausal forms of propositional logic to efficiently decide their satisfiability. The resolution principle is applied to the clausal forms of first-order predicate logic for refutation reasoning.

The standard reasoning algorithms of description logics are based on the tableau calculus [1]. The hyper tableau algorithm [6, 7] proposed by Boris et al. is an improvement on the existing hypertableau algorithm [8] and hyper-resolution method [9]. Although the existing absorption optimizations cannot completely reduce the computational complexity caused by disjunctions, the algorithm by Boris et al. solves this problem by improving absorption. To prove the termination of a reasoning algorithm for knowledge bases containing assertions with
relations based on cyclic concepts, such as “humans who have some human children,” techniques for blocking a redundant reasoning of individuals with a repetitive set of concepts are applied. In conventional methods, ancestor pairwise blocking works to prevent model expansion only when one individual is an ancestor of the other. Anywhere pairwise blocking of the hypertableau algorithm extends this scheme, and avoids mutual blocking by defining an ordering for all individuals while generating blocking between individuals that are not ancestrally related, thereby reducing the computational complexity more efficiently.

The description logic programs [10] have been proposed as an approach that attempts to combine description logics and logic programs. In this approach, ontology descriptions in description logics and rule expressions in logic programs complement each other by translating through intermediate representations. As a result, it is possible to reason about description logics using efficient algorithms in logic programs.

In this study, we formalize a reasoning algorithm for the simplified forms of concepts in the description logic \( \text{ALC} \) (Attributive Language with Complements) [11] via the following three steps.

- A flat concept form (CNF concepts) for description logic \( \text{ALC} \) based on the conjunctive normal form is defined.
- A decidable reasoning algorithm is designed using inference rules for the clause sets of CNF concepts.
- The soundness and completeness of the algorithm for clause sets are proved by introducing restricted tableau for CNF concepts.

Essentially, we define a novel concept form (CNF concepts) in description logics based on CNF, and clause sets of this new concept form are used by reasoning algorithms. Any \( \text{ALC} \) concept can be transformed to an equivalent CNF concept by applying De Morgan’s laws, distributive properties, and so on. CNF concepts can be represented as flat concepts with clause sets, which allows reasoning algorithms to be constructed with efficient inference rules. The flow of the algorithm is as follows. First, we simplify the clauses by selecting one literal from each clause in the clause set. Next, the subconcepts \( C \) of the universal role concepts \( \forall R.C \) are added as \( \exists R.(C \cup D) \) to the existential role concepts \( \exists R.D \) to reduce all universal role concepts while preserving the equivalence of the concepts. Finally, the satisfiability of the target concept is determined by determining the satisfiability of the remaining existential role concepts. We prove the soundness and completeness of the algorithm using a restricted tableau that corresponds to reasoning for clause sets in description logic \( \text{ALC} \).

The rest of this paper is structured as follows. As a preparation, Section 2 presents basic definitions of the syntax and semantics of description logic \( \text{ALC} \). We define the conjunctive normal form in description logics and describe a reasoning algorithm for clause sets in Section 3. In Section 4, we prove the soundness, completeness, and termination of the algorithm introduced in Section 3. Some derivation examples using the algorithm are presented in Section 5. Finally, we conclude the paper and propose future work in Section 6.

2. Preliminaries

2.1. Description Logics

The concept languages of description logics constitute a family of languages with different expressive power depending on the combination of components and syntactic rules. A concept language in description logic \( \text{ALC} \) consists of a set \( \text{CN} \) of concept names \( A \), a set \( \text{RN} \) of role names \( R \), a set \( \text{IN} \) of individual names \( a \), and the logical connectives \( \sqcap \) (conjunction), \( \sqcup \) (disjunction), \( \lnot \) (negation), and the quantifiers \( \exists \) (existential) and \( \forall \) (universal). The top concept \( \top \), which contains all individuals, and the bottom concept \( \bot \), which contains nothing, are also included in \( \text{CN} \).

**Definition 1 (Syntax).** Let \( A \) be a concept name, \( R \) be a role name, and \( C, D \) be \( \text{ALC} \)-concepts. The set of \( \text{ALC} \)-concepts is defined inductively by the following rules:

\[
A \mid \top \mid \bot \mid \lnot C \mid C \sqcap D \mid C \sqcup D \mid \forall R.C \mid \exists R.C
\]

Complex concepts can be expressed by combining any concept, role names, and logical connections. As an example, “animals who have legs” can be represented as follows:

\[
\text{Animal} \sqcap \exists \text{hasPart.Leg}
\]

An interpretation \( I \) of \( \text{ALC} \) consists of a pair \((\Delta^I, \tau^I)\) of a non-empty set \( \Delta^I \) and a function \( \tau^I \) such that

- for each \( A \in \text{CN} \), \( A^I \subseteq \Delta^I \) (in particular, \( \top^I = \Delta^I \) and \( \bot^I = \emptyset \)),
- for each \( R \in \text{RN} \), \( R^I \subseteq \Delta^I \times \Delta^I \), and
- for each \( o \in \text{IN} \), \( o^I \in \Delta^I \).
Definition 2 (Semantics). Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be an interpretation. The interpretation of $\mathcal{ALC}$-concepts are defined inductively as follows:

$$(\neg C)^\mathcal{I} = \Delta^\mathcal{I} \setminus C^\mathcal{I}$$
$$(C \cap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$$
$$(C \cup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$$
$$(\forall R.C)^\mathcal{I} = \{ x \in \Delta^\mathcal{I} | \forall y [(x, y) \in R^\mathcal{I} \rightarrow y \in C^\mathcal{I}] \}$$
$$(\exists R.C)^\mathcal{I} = \{ x \in \Delta^\mathcal{I} | \exists y [(x, y) \in R^\mathcal{I} \land y \in C^\mathcal{I}] \}$$

Let $C$ be an $\mathcal{ALC}$-concept. $C$ is satisfiable if there exists an interpretation $\mathcal{I}$ such that $C^\mathcal{I} \neq \emptyset$, and otherwise, $C$ is unsatisfiable.

3. Reasoning about Clause Sets in Description Logics

3.1. Conjunctive Normal Form of Description Logics

In this section, we define a new form of $\mathcal{ALC}$-concepts based on the conjunctive normal form. Every concept name $A$ and the negation $\neg A$ are concept literals, and if $F$ is a conjunctive normal form, then $\exists R.F$ and $\forall R.F$ are concept literals. Let us denote a concept literal $L$. A disjunction of concept literals $L_1 \sqcup \cdots \sqcup L_m$ is called a clause, denoted by $CL$. A conjunction of clauses $CL_1 \sqcap \cdots \sqcap CL_n$ is called a conjunctive normal form and is denoted as $F$.

Definition 3 (Concept Literals, Clauses, and Conjunctive Normal Forms). Let $A$ be a concept name, $R$ be a role name, $L_1, \ldots, L_m$ be concept literals, $CL_1, \ldots, CL_n$ be clauses, and $F$ be a conjunctive normal form. The set of concept literals, the set of clauses, and the set of conjunctive normal forms are inductively defined as follows:

$$L = A \mid \neg A \mid \exists R.F \mid \forall R.F$$
$$CL = L_1 \sqcup \cdots \sqcup L_m$$
$$F = CL_1 \sqcap \cdots \sqcap CL_n$$

Any $\mathcal{ALC}$-concept $C$ can be converted to a conjunctive normal form (denoted by $\text{CNF}(C)$), i.e., $C \equiv \text{CNF}(C)$, by the following laws:

1. De Morgan’s and double negation laws: the left side is transformed into the right side such that negations appear only in concept names.

$$\neg (C \cap D) \equiv \neg C \sqcup \neg D$$
$$\neg (C \sqcup D) \equiv \neg C \sqcap \neg D$$
$$\neg (\exists R.C) \equiv \forall R.\neg C$$
$$\neg (\forall R.C) \equiv \exists R.\neg C$$
$$\neg \neg C \equiv C$$

2. Distributive laws: the left side is transformed into the right side such that no conjunction is included in disjunctions.

$$(C \cap D) \sqcup E \equiv (C \sqcup E) \cap (D \sqcup E)$$
$$C \sqcup (D \sqcap E) \equiv (C \sqcup D) \sqcap (C \sqcup E)$$

3. Associative laws: the parentheses on the left or right side are deleted by transforming into $C \sqcup D \sqcup E$ and $C \cap D \cap E$, respectively.

$$(C \sqcup D) \sqcup E \equiv C \sqcup (D \sqcup E)$$
$$(C \sqcap D) \sqcap E \equiv C \sqcap (D \sqcap E)$$

A concept name $A$ and its negation $\neg A$ are complementary literals of each other. The complementary literals of concept literals $\exists R.F$ and $\forall R.F$ are $\exists R.\text{CNF}(\neg F)$ and $\forall R.\text{CNF}(\neg F)$, respectively. We denote a complementary literal of a concept literal $L$ as $\overline{L}$. 

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3.2. Reasoning Algorithms for Clause Sets

We design a reasoning algorithm for clause sets in ALC. Given an ALC-concept $C$, the conjunctive normal form\(CNF(C) = CL_1 \sqcap \ldots \sqcap CL_n\) is represented as a clause set as follows:

$$\{CL_1, \ldots, CL_n\}$$

where each clause $CL_i$ is a literal set \(\{L_1, \ldots, L_m\}\) of $L_1 \sqcup \ldots \sqcup L_m$. In particular, a clause $CL$ is called a unit clause if \(|CL|=1\), and an empty clause if \(|CL|=0\). Thus, the disjunction $CL_1 \sqcup CL_2$ is represented by $CL_1 \sqcup CL_2$, and the conjunction $F_1 \sqcap F_2$ of two clause sets $F_1, F_2$ is represented by $F_1 \sqcap F_2$. For example, the two clauses $\neg A_1 \sqcup A_2$ and $A_3 \sqcup \exists R.A_1$ are represented by the literal sets \(\{\neg A_1, A_2\}\) and \(\{A_3, \exists R.A_1\}\), respectively. Furthermore, the conjunctive normal form \((\neg A_1 \sqcup A_2) \sqcap (A_3 \sqcup \exists R.A_1)\) of those two clauses is represented as the clause set \(\{\neg A_1, A_2, A_3, \exists R.A_1\}\).

An algorithm for deciding the satisfiability of the conjunctive normal form of a concept $F = CNF(C)$ is provided by inference rules.

**Definition 4 (Inference Rules A1, A2, and A3).** Let $S_i$ be a family of clause sets, $S_{i+1}$ is derived from $S_i$ by applying one of the following rules to each clause set $F \in S_i$:

(A1) if $L \in CL$ in $F$, then

$$CL \rightarrow \{L\},$$

(A2) if $\forall R.F_1 \in CL$ in $F$, then

$$F \rightarrow F \setminus \{CL' \mid \forall R.F_1 \in CL'\} \text{ and for all } \exists R.F_2 \in CL' \in F, \exists R.F_2 \rightarrow \exists R.(F_1 \sqcup F_2),$$

(A3) if all clauses in $F$ are unit clauses of \(\{A\}, \{\neg A\}\) or \(\{\exists R.F'\}\), then

$$F \rightarrow F \setminus \{\exists R.F_1\} \text{ and add } F_1 \text{ to } S_{i+1} \text{ for some } \{\exists R.F_1\} \in F \text{ where } F \text{ is a parent of } F_1 \text{ with respect to } R.$$
3.3. Efficient Reasoning Algorithm for Clause Sets

In this section, we improve the reasoning algorithm described in the previous section by exploiting the flat clause representation. If a concept literal \( L \) is selected by inference rule \( A1 \), the clause containing \( L \) or \( \overline{L} \) can be further simplified in the derivation steps. Moreover, we attempt to avoid the branches caused by applying inference rule \( A2 \) to universal role concepts.

**Definition 5 (Inference Rules \( A1^+ \) and \( A2^+ \)).** The rules \( A1 \) and \( A2 \) introduced in Definition 4 are replaced by the following inference rules \( A1^+ \) and \( A2^+ \):

\[
(A1^+) \text{ if } L \in CL \text{ in } F, \text{ then } \\
\text{for all } CL' \in F \text{ with } L \in CL', \text{ } CL' \rightarrow \{L\}, \text{ and for all } CL' \in F \text{ with } \overline{L} \in CL', \text{ } CL' \rightarrow CL' \setminus \{\overline{L}\}, \text{ and}
\]

\[
(A2^+) \text{ if all clauses in } F \text{ are unit clauses and } \forall \, R. \, F_1 \in CL \text{ in } F, \text{ then } \\
F \rightarrow F \setminus \{CL' \in F \mid \forall \, R. \, F_1 \in CL'\} \text{ and for all } \exists \, R. \, F_2 \in CL' \text{ in } F, \text{ } \exists \, R. \, F_2 \rightarrow \exists \, R. \, (F_1 \cup F_2).
\]

If a concept literal \( L \) is selected from \( CL \) by applying inference rule \( A1^+ \), then for all clauses \( CL' \) containing the literal \( L \), the other concept literals are removed from \( CL' \), and for all clauses \( CL' \) containing the complementary literal \( \overline{L} \), the complementary literal is removed from \( CL' \). That is, if \( L \) is selected from a clause, all \( CL' \) are converted as follows:

\[
L \sqcup L_1 \sqcup \ldots \sqcup L_m \rightarrow L \\
\overline{L} \sqcup L_1 \sqcup \ldots \sqcup L_m \rightarrow L_1 \sqcup \ldots \sqcup L_m
\]

Inference rule \( A2^+ \) is restricted by the condition that all clauses are unit clauses in order to avoid applying \( A2^+ \) before \( A1^+ \).

The unsatisfiability is inherited by applying inference rules \( A1^+ \) and \( A2^+ \). Inference rule \( A1^+ \) deletes disjunctions \( L \sqcup L_1 \sqcup \ldots \sqcup L_m \) and \( \overline{L} \sqcup L_1 \sqcup \ldots \sqcup L_m \) in all clauses. Since \( L \sqcup (L \sqcup L_1 \sqcup \ldots \sqcup L_m) \) and \( (L_1 \cdot \ldots \cdot L_m) \sqcup (\overline{L} \sqcup L_1 \sqcup \ldots \sqcup L_m) \) hold, if \( L \sqcup L_1 \sqcup \ldots \sqcup L_m \) and \( \overline{L} \sqcup L_1 \sqcup \ldots \sqcup L_m \) are unsatisfiable, then \( L \) and \( L_1 \cdot \ldots \cdot L_m \) are also unsatisfiable, respectively. Furthermore, inference rule \( A2^+ \) derives the same clauses as inference rule \( A2 \).

4. Completeness

We denote by \( \text{rol}(F) \) a set of all role names contained in a clause set \( F \). For example, \( \text{rol}((\forall R_1. \exists R_2. C_1, \neg C_2)) = \{R_1, R_2\} \). The set of subexpressions of a non-empty clause set \( F \) is the smallest set such that

1. \( F \in \text{sub}(F), \)
2. if \( F' \in \text{sub}(F) \), then \( F' \subseteq \text{sub}(F), \)
3. if \( CL \in \text{sub}(F) \), then \( CL' \in \text{sub}(F) \) for all non-empty \( CL' \subseteq CL, \)
4. if \( \{\neg A\} \in \text{sub}(F) \), then \( \{A\} \in \text{sub}(F), \) and
5. if \( \forall R. F' \in \text{sub}(F) \) or \( \exists R. F' \in \text{sub}(F) \), then \( F' \in \text{sub}(F). \)

For example, \( \text{sub}(\{\forall R_1. \{\neg A_1\}, \{A_2\}\}) = \{\{\forall R_1. \{\neg A_1\}, \{A_2\}\}\}, \{\forall R_1. \{\neg A_1\}, \{A_2\}\}, \{\neg A_1\}, \{A_2\}, \{A_1\}\}. \)

**Definition 6 (CNF Tableau).** Let \( F \) be a clause set. A CNF tableau for \( F \) is a tuple \( T = (S, L, E) \) such that \( S \) is a set of individuals, \( L : S \rightarrow 2^{\text{sub}(F)} \) is a function from individuals to subexpressions in \( \text{sub}(F) \), and \( E : \text{rol}(F) \rightarrow 2^{S \times S} \) is a function from role names to pairs of individuals, and there exists some \( s_0 \in S \) such that \( F \in L(s_0) \). For all \( s, t \in S \), it satisfies the following conditions:

1. if \( \{L\} \in L(s) \), then \( \overline{L} \notin L(s), \)
2. if \( F' \in L(s) \), then \( F' \subseteq L(s), \)
3. if \( CL \in L(s) \), there exists some \( L' \in CL \) such that \( \{L'\} \in L(s), \)
4. if \( \forall R. F' \in L(s) \) and \( (s, t) \in E(R) \), then \( F' \in L(t), \)
5. if \( \exists R.F' \in L(s) \), then \((s, t) \in E(R)\), and there exists \( t \in S \) such that \( F' \in L(t) \), and
6. if \( \forall R.F_1, \exists R.F_2 \in L(s) \), then \( \exists R.F_1 \cup F_2 \in L(s) \).

**Lemma 1.** There exists a CNF tableau \( T = (S, L, E) \) for a clause set \( F \) if and only if \( F \) is satisfiable.

**Proof.** (⇒) Let \( T = (S, L, E) \) be a CNF tableau for a clause set \( F \). Then, an interpretation \( \mathcal{I} = (\Delta^T, \mathcal{T}) \) of \( F \) can be constructed as follows:

\[ \Delta^T = S, \]

for each unit clause of \( \{A\} \in \text{sub}(F) \), \( A^T = \{s \in S \mid \{A\} \in L(s)\} \), and

for each role name \( R \in \text{rol}(F) \), \( E(R) = R^T \).

For \( \mathcal{I} \), we prove by induction on the structure of a clause set that for all \( C \) in \( \text{sub}(F) \), if \( C \in L(s) \), then \( s \in C^T \) holds. If \( \{A\} \in L(s) \), then by the definition of \( \mathcal{I} \), \( s \in A^T \) holds. If \( \{\neg A\} \in L(s) \), by Condition (1) of Definition \( \mathcal{I} \) there exists some \( L' \in CL \) such that \( \{L'\} \in L(s) \). By the induction hypothesis, \( s \in (L')^T \) holds. Thus, \( s \in CL^T \). If \( \{\forall R.F'\} \in L(s) \), then by Condition (1) of Definition \( \mathcal{I} \) for any \( t \in S \), if \((s, t) \in E(R)\), then \( F' \in L(t) \) holds. The induction hypothesis, \( t \in (F')^T \) holds. Thus, for all \((s, t) \in E(R), (s, t) \in R^T \), so by Definition \( \mathcal{I} \) \( s \in (\forall R.F')^T \). If \( \{\exists R.F'\} \in L(s) \), then by Condition (1) of Definition \( \mathcal{I} \) there exists some \( t \in S \) such that \((s, t) \in E(R)\) and \( F' \in L(t) \). By the induction hypothesis, \( t \in (F')^T \) holds. Therefore, since \((s, t) \in R^T \), by Definition \( \mathcal{I} \) \( s \in (\exists R.F')^T \). For \( F' \in L(s) \), then by Condition (2) of Definition \( \mathcal{I} \) \( CL \in L(s) \) for all \( CL \in F' \). By the induction hypothesis, \( s \in CL^T \), so \( s \in (F')^T \).

Therefore, \( F \in \text{sub}(F) \), and there exists \( s_0 \) such that \( F \in L(s_0) \) by Definition \( \mathcal{I} \) so \( s_0 \in F^T \). Hence, the interpretation \( \mathcal{I} \) satisfies \( F \).

(⇐) Assume that an interpretation \( \mathcal{I} = (\Delta^T, \mathcal{T}) \) satisfies a CNF concept \( F \). Let us construct the tableau \( T = (S, L, E) \) for \( F \) such that

\[ S = \Delta^T, \]

for each \( s \in \Delta^T \), \( L(s) = \{C \in \text{sub}(F) \cup \{\exists R.F_1 \cup F_2\} \mid \{\forall R.F_1\}, \{\exists R.F_2\} \in \text{sub}(F)\} \mid s \in C^T \)}, and

for each role name \( R \in \text{rol}(F) \), \( E(R) = R^T \).

Since \( F \) is satisfiable, \( F^T \neq \emptyset \). That is, there exists some \( s \in F^T \) such that \( s \in \Delta^T \). By definition of \( T = (S, L, E) \) above, there exists some \( s \in S \) such that \( s \in F^T \) and \( F \in L(s) \). We show that \( T \) satisfies the conditions of Definition \( \mathcal{I} \) for all \( s \). If \( \{L\} \in L(s) \) for a concept literal \( L \), then \( s \in L^T \) from the construction of \( T \). Therefore, \( s \notin \{\mathcal{T}^T, \{L\} \notin L(s) \) from the Definition \( \mathcal{I} \). For a clause set \( F' = \{CL_1, \ldots, CL_n\} \), if \( F' \in L(s) \), then \( s \in (F')^T \). Thus, for any \( CL_i \in F' \), \( s \in CL_i^T \), so \( CL_i \in L(s) \). Therefore, \( F' \subseteq L(s) \).

If \( CL \in L(s) \) for a clause (literal set) \( CL = \{L_1, \ldots, L_n\} \), then \( s \in CL^T \). Thus, there exists \( L' \in CL \) and \( s \in \{L'\} \), \( L' \in L(s) \) from the Definition \( \mathcal{I} \). If \( \{\forall R.F'\} \in L(s) \), then \( s \in (\forall R.F')^T \) for any \( s \in R^T \). Therefore, \( F' \in L(t) \). If \( \{\exists R.F'\} \in L(s) \), then \( s \in (\exists R.F')^T \), so there exists some \( (s, t) \in R^T \) and \( t \in (F')^T \). Therefore, \( F' \in L(t) \). If \( \{\forall R.F_1\}, \{\exists R.F_2\} \in L(s) \), from \( s \in (\forall R.F_1)^T, (\exists R.F_2)^T \), there exists some \( (s, t) \in R^T \) such that \( t \in F_1^T, F_2^T \), so \( s \in (\exists R.F_1 \cup F_2)^T \). Therefore, \( \{\exists R.F_1 \cup F_2\} \in L(s) \). So, \( T \) is a CNF tableau for \( F \) since \( T \) satisfies the conditions of the Definition \( \mathcal{I} \).

Lemma \( \mathcal{I} \) shows that an interpretation \( \mathcal{I} \) satisfying a clause set \( F \) can be constructed from a CNF tableau \( T \) for \( F \). Moreover, if a clause set \( F \) is satisfiable, a tableau \( T \) can be constructed from an interpretation \( \mathcal{I} \) of \( F \).

**Theorem 1 (Termination).** The reasoning algorithm with \( A_1 \), \( A_2 \), and \( A_3 \) terminates. Also, the reasoning algorithm with \( A_1^+ \), \( A_2^+ \), and \( A_3 \) terminates.

**Proof.** Let \( F \) be a clause set. Since the number of concept literals in \( F \) is finite, inference rule \( A_1 \) (or \( A_1^+ \)) will eventually become inapplicable. Inference rules \( A_2 \) (or \( A_2^+ \)) and \( A_3 \) delete at least one role name in a clause set, so it terminates in finite steps, depending on the number of role names in \( F \). Thus, the satisfiability of \( F \) can be decided in finite steps.

**Theorem 2 (Soundness of Reasoning with \( A_1 \), \( A_2 \), and \( A_3 \)).** If the reasoning algorithm with \( A_1 \), \( A_2 \), and \( A_3 \) yields a complete and clash-free family of clause sets \( S_n \) for a clause set \( F \), then \( F \) is satisfiable.
Proof. Let $S_0$ be a complete and clash-free family of clause sets derived from the initial family of clause sets $S_0 = \{ F \}$. Then, we prove that a CNF tableau $T = (S, L, E)$ for $F$ can be constructed from the nodes $S_0, S_1, \ldots, S_n$ in a derivation tree where each $S_{i+1}$ is derived from its parent node $S_i$.

We show the procedure for constructing a CNF tableau $T = (S, L, E)$ as follows. First, we define the set of ordinal numbers of clause sets in $S_n$:

- $S = \{ 0, 1, \ldots, |S_n| \}$

Let $S_k = \{ F_0, \ldots, F_m \}$ for each $k \leq n$ where each $F_{i+1}$ is added from $F_0, \ldots, F_i$ in the derivation tree (i.e., a clause set of $F_0, \ldots, F_i$ is a parent of $F_{i+1}$). Then, the $i$-th clause set in $S_k$ is defined as follows:

$$S_k(i) = \begin{cases} F_i & \text{if } i \leq m \\ \emptyset & \text{otherwise} \end{cases}$$

Second, we define the function $L : S \to 2^{\mathcal{R}(F)} \cup \{ \forall \mathcal{R}(F) \}$ from $S$ to clause sets, and families of clause sets in $S_0, S_1, \ldots, S_n$ as follows:

$$L_F(i) = \{ S_k(i) \in \{ S_0(i), \ldots, S_n(i) \} | S_k(i) \neq \emptyset \},$$

$$L^+_F(i) = \{ F' \subseteq S_k(i) | F' \neq \emptyset \text{ and } S_k(i) \in L_F(i) \},$$

$$L_{CL}(i) = \{ CL' \subseteq S_k(i) | S_k(i) \in L_F(i) \},$$

$$L^+_CL(i) = L^+_F(i) \cup L_{CL}(i) \cup L_{\forall R.F}.$$ 

where $L_{\forall R.F}$ is the set of unit clauses $\{ \forall R.F' \}$ of all universal role concepts $\forall R.F'$ to which inference rule $A2$ is applied in the $i$-th clause set $S_k(i)$ for each $k \in \{ 0, \ldots, n \}$. Finally, we define $E : rol(F) \to 2^{\mathcal{R}(F)} \cup 2^{\mathcal{R}(F)}$ from a parent-child relationship of clause sets in $S_n$.

- $E(R) = \{ (i, j) \in S \times S | F_i \text{ is a parent of } F_j \text{ with respect to } R \}$

We show that $T$ satisfies the conditions of Definition 1. Let $S_0 = \{ F \}$ be the initial family of clause sets. By the above definition of $T = (S, L, E)$, there exists $0 \in S$ and $S_0(0) = F = L^+_F(0) \subseteq L(0)$. Condition (1): If $L(i) \subseteq L(i)$, then since $S_n$ is clash-free and also $S_0, S_1, \ldots, S_{n-1}$ are clash-free, there is no $S_k(i)$ (for all $0 \leq k \leq n$) containing both $L(i), L(i)$. Therefore, $\{ T \}$ is not in $L(i)$. Condition (2): If $F' \subseteq L(i)$, then $F' \neq \emptyset$ by definition, so $F' \subseteq L^+_F(i)$, and every $CL' \subseteq F'(\subseteq S_k(i) \subseteq L_F(i))$ is in $L_{CL}(i)$. Therefore, $F' \subseteq L(i)$. Condition (3): If $CL' \subseteq L(i)$, then there is some $F' \subseteq L_F(i)$ with $CL' \subseteq F'$. Since $F' = S_k(i)$, inference rule $A1$ (or $A1^+$) can be applied to derive a clause `unit` $\{ L \} \subseteq CL' \subseteq S_k(i)$ such that $\{ L \} \subset S_n(i)$. Hence, $\{ L \} \subseteq L_F(i)$ (or $A1^+$) is not applied to $CL' \subseteq F'$, then inference rule $A2$ is applied to some $\forall R.F \subseteq CL' \subseteq \forall R.F_i$ of $E(R)$. Thus, $\{ L \} \subseteq L(i)$ for some concept literal $L = \forall R.F_i$ in $CL'$. Condition (4): If $(i, j) \in E(R)$, then $S_{k-1}(j) = F''$ is derived from $S_k(i)$ (i.e., $S_k(i)$ is the parent of $F''$) containing $\exists \mathcal{R}.F''$ by inference rule $A3$. If $\exists \mathcal{R}.F'' \subseteq L(i)$, then the condition of $A3$ implies that $S_k(i)$ does not contain $\exists \mathcal{R}.F''$. For some $0 \leq k' < k$, $L_{k'}(i)$ contains a superset of $\exists \mathcal{R}.F''$. Since $\exists \mathcal{R}.F'' \subseteq S_k(i)$ is derived from $\forall R.F ''$ by inference rule $A2$, $F'' \subseteq F''$ holds. Therefore, $F'' \subseteq F'' \subseteq S_{k-1}(j) \subseteq S_k(i)$ and $S_{k-1}(j) \subseteq L_F(i)$. So, $F'' \subseteq L^+_F(i)$. Hence, $F'' \subseteq L(i)$. Condition (5): If $\exists \mathcal{R}.F' \subseteq L(i)$, then some $S_k(i) \subseteq L_F(i)$ contains $\exists \mathcal{R}.F'$ since $\exists \mathcal{R}.F' \subseteq L_C(i)$, and inference rule $A3$ is applied to $\exists \mathcal{R}.F'$, leading to $S_{k+1}(j) = F'$. Therefore, since $S_k(i)$ is the parent of $F'$, $(i, j) \in E(R)$ and $F' \subseteq L(i)$ holds. Condition (6): If $\forall R.F_1, \exists \mathcal{R}.F_2 \subseteq L(i)$, then $\forall R.F_1, \exists \mathcal{R}.F_2 \subseteq L_{CL}(i)$. So, some $S_k(i) \subseteq L_F(i)$ contains $\forall R.F_1, \exists \mathcal{R}.F_2$, and inference rule $A2$ is applied to derive $\forall R.F_1 \cup F \subseteq S_{k+1}(i)$ from $\exists \mathcal{R}.F_2$. Accordingly, $\forall R.F_1 \cup F \subseteq L_F(i)$, and $T = (S, L, E)$ satisfies the conditions 1-6.

Therefore, since there exists a CNF tableau for $F$, it is satisfiable by Lemma 1.

Theorem 3 (Completeness of Derivation with $A1$, $A2$, and $A3$). If a clause set $F$ is satisfiable, the reasoning algorithm with $A1$, $A2$, and $A3$ yields a complete and clash-free family of clause sets.

Proof. Suppose that an interpretation $I = (\mathcal{A}, \cdot, \cdot)$ satisfies $F$. By Lemma 1 there exists a CNF tableau $T = (S, L, E)$ for $F$ such that $F \subseteq L(s_0)$ for some $s_0 \in S$.

A derivation tree is generated by applying inference rules $A1$, $A2$, and $A3$ to the initial family of clause set $S_0 = \{ F \}$. As a result, multiple complete families of clause sets are derived from the branches by inference rules $A1$ and $A2$. By selecting a complete concept set $S_n$ from them, we define the set of ordinal numbers of clause sets in $S_n$:

- $S_n = \{ 0, \ldots, |S_n| \}$
We obtain a sequence \( S_0, S_1, \ldots, S_n \) from the derivation tree where \( S_0 \) is the root node, \( S_n \) is a leaf node, and \( S_i \) is the parent node of \( S_{i+1} \). Let \( S_k = \{ F_0, \ldots, F_m \} \) for each \( k \leq n \) where each \( F_{i+1} \) is added from \( F_0, \ldots, F_i \) in the derivation tree (i.e., a clause set of \( F_0, \ldots, F_1 \) is a parent of \( F_{i+1} \)). Then, the \( i \)-th clause set in \( S_k \) is defined as follows.

\[
S_k(i) = \begin{cases} 
F_i & \text{if } i \leq m \\
\emptyset & \text{otherwise}
\end{cases}
\]

We define a function \( \pi : S'_i[S_n] \rightarrow S \) to connect between \( S'_i[S_n] \) and \( S \) as follows:

1. \( \pi(0) = s_0 \), and
2. if \( \pi(i) = s \) and \( \exists R.F' \in S_k(i) \) such that \( S_{k+1}(j) = F' \) is added from \( S_k(i) \) by inference rule A3, then for some \( t \in S \) with \( F' \in L(t) \) and \( (s, t) \in E(R) \), \( \pi(j) = t \).

To avoid the branching of A1 and A2, we introduce deterministic inference rule A1 that uniquely determines a concept literal \( L \) in some \( CL \in S_k(i) \) by following the CNF tableau \( T \) with \( L \in L(\pi(i)) \). In addition, A1 is applied in preference to inference rule A2 whenever possible. Then, A2 is applied to only to unit clauses. It leads to a derivation path as a sequence \( S_0, S_1, \ldots, S_n \). If \( S_n \) is clash-free, then the reasoning algorithm yields a complete and clash-free family of clause sets. We show that \( S_0(\cup \cdots \cup S_n) \subseteq L(\pi(i)) \). Since each \( S_k(i) \) (except for \( S_0(0) = F \)) is added by inference rules, all the clauses added to each \( S_k(i) \) have to be included in \( L(\pi(i)) \). We prove this by induction on the depth \( k \) of a derivation tree.

If \( k = 0 \), then \( S_0 = \{ F \} \). For \( \pi(0) = s_0, F \in L(\pi(0)) \), so \( F \subseteq L(\pi(0)) \) from Condition 2 of Definition 6. Hence, \( CL \in L(\pi(0)) \) for all \( CL \in F(= S_0(0)) \).

If \( k > 0 \), then inference rules A1, A2, and A3 are applied to \( S_k \).

If deterministic A1 is applied to \( CL \subset S_k(i) \in S_k \), then for some \( L \in CL \) according to tableau \( T, \{ L \} \in S_{k+1}(i) \). By the induction hypothesis, \( CL \in L(\pi(i)) \), and so by Condition 3 of Definition 6 and the deterministic A1, \( \{ L \} \in L(\pi(i)) \) holds. If A2 is applied to \( \{ \exists R.F_1 \}, \{ \exists R.F_2 \} \in S_k(i) \), then \( \exists R.F \in S_k(i) \) is replaced with \( \{ \exists R.F_1 \cup \exists R.F_2 \} \in S_{k+1}(i) \). Since \( \{ \exists R.F_1 \}, \{ \exists R.F_2 \} \in L(\pi(i)) \) by the induction hypothesis, \( \{ \exists R.F_1 \cup \exists R.F_2 \} \in L(\pi(i)) \) by Condition 3 of Definition 6. If A3 is applied to \( \{ \exists R.F \} \in S_k(i) \), then \( F \) is added to \( S_k+1 \) (i.e., \( S_{k+1}(j) = F \)). By the induction hypothesis, since \( \{ \exists R.F \} \in L(\pi(i)) \), there exists \( t \in S \) such that \( \pi(i), t \in E(R) \) and \( F \) is added from Condition 3 of Definition 6. If \( \pi(j) = t \), so \( F \in L(\pi(j)) \). By Condition 2 in Definition 6, \( F \in L(\pi(j)) \). Hence, since \( S_0(\cup \cdots \cup S_n) \subseteq L(\pi(i)) \) for all \( i \in S'_i[S_n] \), from Condition 1 in Definition 6, \( S_0(\cup \cdots \cup S_n) \) is clash-free. Therefore, the complete family of clause sets \( S_n \) is clash-free.

Next, we show the completeness of the efficient reasoning algorithm with inference rules A1+, A2+, and A3. We define a restricted set of CNF tableaux by revising Condition 3 of Definition 6.

**Definition 7 (Restricted CNF Tableau).** Let \( F \) be a clause set \( F \). A restricted CNF tableau for \( F \) is a tuple \( T = (S, L, E) \) such that \( S \) is a set of individuals, \( L : S \rightarrow 2^{\text{sub}(F)} \) is a function from individuals to subexpressions in \( \text{sub}(F) \), and \( E : \text{rol}(F) \rightarrow 2^{S \times S} \) is a function from role names to pairs of individuals, and there exists some \( s_0 \in S \) such that \( F \in L(s_0) \). For all \( s, t \in S \), it satisfies the following conditions:

1. if \( \{ L \} \in L(s) \), then \( \{ T \} \notin L(s) \),
2. if \( F' \in L(s) \), then \( F' \subseteq L(s) \),
3. if \( CL \in L(s) \), there exists some \( L' \in CL \) such that \( L' \in L(s) \) and for all \( CL' \in L(s), CL' \setminus \{ T \} \in L(s) \),
4. if \( \exists R.F' \in L(s) \) and \( (s, t) \in E(R) \), then \( F' \in L(t) \),
5. if \( \exists R.F' \in L(s) \), then \( (s, t) \in E(R) \), and there exists \( t \in S \) such that \( F' \in L(t) \),
6. if \( \exists R.F_1 \), \( \exists R.F_2 \) \( \in L(s) \), then \( \exists R.F_1 \cup \exists R.F_2 \) \( \in L(s) \).

**Lemma 2.** There exists a restricted CNF tableau \( T = (S, L, E) \) for a clause set \( F \) if \( F \) is satisfiable.

**Proof.** Assume that \( F \) is satisfiable. Let \( L(s) \) be as in the proof of Lemma 1. Then, \( L(s) \) satisfies the conditions in Definition 6. We prove that \( L(s) \) satisfies the Condition 3 of Definition 7. If \( CL \in L(s) \), then \( s \in CL \), so there exists some \( L' \in CL \) and \( s \in \{ L' \} \). Therefore, \( \{ L' \} \in L(s) \). In addition, if \( \{ T \} \notin CL' \) then \( CL' \setminus \{ T \} = CL' \in L(s) \) for any \( CL' \in L(s) \). If \( \{ T \} \in CL' \), then \( CL' \neq \{ T \} \) since \( \{ T \} \notin L(s) \) from the Condition 1. So, \( \{ T \} \notin \{ H \} \) since \( s \in \{ H \} \). Thus, there exists some \( L' \in CL' \) (except for \( T \)) and \( s \in \{ L' \} \). Hence, \( CL' \setminus \{ T \} \in L(s) \) by the definition of \( L(s) \) since \( s \in (CL' \setminus \{ T \}) \). Therefore, \( L(s) \) satisfies the conditions of Definition 7.
Theorem 4 (Soundness of Reasoning with $A1^+$, $A2^+$, and $A3$). If the reasoning algorithm with $A1^+$, $A2^+$, and $A3$ yields a complete and clash-free family of clause sets $S_n$ for a clause set $F$, then $F$ is satisfiable.

Proof. Let $S_0$ be a complete and clash-free family of clause sets derived from the initial family of clause sets $S_0 = \{F\}$. Then, we prove that a restricted CNF tableau $T = (S, L, E)$ for $F$ can be constructed from the nodes $S_0, S_1, \ldots, S_n$ in a derivation tree where each $S_{k+1}$ is derived from its parent node $S_k$.

We show the procedure for constructing a restricted CNF tableau $T = (S, L, E)$ as follows. First, we define the set of ordinal numbers of clause sets in $S_n$.

- $S = \{0, 1, \ldots, |S_n|\}$

Let $S_k = \{F_0, \ldots, F_m\}$ for each $k \leq n$ where each $F_{i+1}$ is added from $F_0, \ldots, F_i$ in the derivation tree (i.e., a clause set of $F_0, \ldots, F_i$ is a parent of $F_{i+1}$). Then, the $i$-th clause set in $S_k$ is defined as follows.

$$S_k(i) = \begin{cases} F_i & \text{if } i \leq m \\ \emptyset & \text{otherwise} \end{cases}$$

Second, we define the function $L : S \to 2^{\text{sub}(F)}$: $L(F) = \{\exists F_1 \cup F_2 \mid \forall R.F_1, \exists R.F_2 \in \text{sub}(F)\}$ from $S$ to clauses, clause sets, and families of clause sets in $S_0, \ldots, S_n$ as follows:

$$L_F(i) = \{S_k(i) \mid S_k(i) \not= \emptyset\},$$

$$L'_F(i) = \{F_i \subseteq S_k(i) \mid F_i \not= \emptyset \text{ and } S_k(i) \in L_F(i)\},$$

$$L_{CL}(i) = \{CL_i \mid S_k(i) \subseteq L_{CL}(i)\},$$

$$L(i) = L'_F(i) \cup L_{CL}(i) \cup L_{NR.F}(i).$$

where $L_{NR.F}(i)$ is the set of unit clauses $\forall R.F'$ of all universal role concepts $\forall R.F'$ to which inference rule $A2^+$ is applied in the $i$-th clause set $S_k(i)$ for each $k \in \{0, \ldots, n\}$. Finally, we define $E : \text{rol}(F) \to 2^{S \times S}$ from a parent-child relationship of clause sets in $S_n$.

- $E(R) = \{(i, j) \in S \times S \mid F_i \text{ is a parent of } F_j \text{ with respect to } R\}$

We show that $T$ satisfies the conditions of Definition [7]. Let $S_0 = \{F\}$ be the initial family of clause sets. By the above definition of $T = (S, L, E)$, there exists $0 \in S$ and $S_0(0) = F \in L_F(0) \subseteq L(0)$. Condition (1): If $L \in L(i)$, then since $S_n$ is clash-free and also $S_1, \ldots, S_{n-1}$ are clash-free, there is no $S_k(i)$ for $(0 \leq k \leq n)$ containing both $L, \{T\}$. Therefore, $\{T\}$ is not in $L(i)$. Condition (2): If $F_i \in L(i)$, then $F_i \not= \emptyset$ by definition, so $F_i \in L_F(i)$, and every $CL_i \subseteq F_i \subseteq L_{CL}(i)$. Therefore, $F_i \subseteq L(i)$. Condition (3): If $CL_i \subseteq L(i)$, then $CL_i \subseteq L_{CL}(i)$, and there is some $F_i \in L_{CL}(i)$ with $CL_i \subseteq F_i$. Since $F_i \in S_k(i)$, inference rule $A1^+$ can be applied to derive a unit clause $L \in CL_i$ such that $L \in S_k(i)$. Hence, $\{L\} \subseteq L_{CL}(i)$ ($\subseteq L(i)$). In addition, if $L \not\subseteq CL_i$ for all $CL_i \subseteq L(i) \subseteq L_{CL}(i)$, then $S_{k+1}(i)$ containing $CL_i \setminus \{T\}$ is derived from $S_k(i)$ containing $L \in CL_i$ by inference rule $A1^+$ leading to $\forall R.F'$. Condition (4): If $(i, j) \in E(R)$, then $S_{k+1}(j) = F''$ is derived from $S_j(i)$ (i.e., $S_j(i)$ is the parent of $F''$) containing $\{\exists R.F''\}$ by inference rule $A3$. If $\forall R.F' \in L(i)$, then the condition of $A3$ implies that $S_j(i)$ does not contain $\forall R.F'$. So, for some $0 \leq k' < k$, $S_{k'}(i)$ contains $\forall R.F'$. Since $\{\exists R.F''\}$ in $S_k(i)$ is derived from $\forall R.F'$ by inference rule $A2^+$, $F' \subseteq F''$ holds.

Therefore, $F' \subseteq F'' \subseteq S_{k+1}(j)$ and $S_k+1(j) \subseteq L(j)$. If $F'' \subseteq L_{F'}(j)$, then $\forall R.F' \in L(j)$. Condition (5): If $\exists R.F' \subseteq L(i)$, then some $S_i(i) \in L_{F'}(i)$ contains $\{\exists R.F'\}$ since $\{\exists R.F'\} \subseteq L_{CL}(i)$, and inference rule $A3$ is applied to $\{\exists R.F'\}$, leading to $S_{k+1}(j) = F'$. Therefore, since $S_j(i)$ is the parent of $F'$, $(i, j) \in E(R)$ and $F' \subseteq L(j)$ holds. Condition (6): If $\forall R.F_1, \exists R.F_2 \in L(i)$, then $\forall R.F_1, \{\exists R.F_2\} \subseteq L_{CL}(i)$, and inference rule $A3$ is applied to $\{\exists R.F_2\}$, leading to $S_{k+1}(j) = F'$. Therefore, since $S_j(i)$ is the parent of $F'$, $(i, j) \in E(R)$ and $F' \subseteq L(j)$ holds. Therefore, since there exists a restricted CNF tableau for $F$, it is satisfiable by Lemma [2].

Theorem 5 (Completeness of Derivation with $A1^+$, $A2^+$, and $A3$). If a clause set $F$ is satisfiable, the reasoning algorithm with $A1^+$, $A2^+$, and $A3$ yields a complete and clash-free family of clause sets.

Proof. Suppose that an interpretation $I = (\Delta^I, T)$ satisfies $F$. By Lemma [2] there exists a restricted CNF tableau $T = (S, L, E)$ for $F$ such that $F \in L(s_0)$ for some $s_0 \in S$.

A derivation tree is generated by applying inference rules $A1^+$, $A2^+$, and $A3$ to the initial family of clause set $S_0 = \{F\}$. As a result, multiple complete families of clausal sets are derived from the branches by inference rule $A1^+$. By selecting a complete concept set $S_n$ from them, we define the set of ordinal numbers of clausal sets in $S_n(S_0)$.
• \( S'_{[S_n]} = \{0, \ldots, |S_n|\} \)

We obtain a sequence \( S_0, S_1, \ldots, S_n \) from the derivation tree where \( S_0 \) is the root node, \( S_n \) is a leaf node, and \( S_i \) is the parent node of \( S_{i+1} \). Let \( S_k = \{F_0, \ldots, F_m\} \) for each \( k \leq n \) where each \( F_{i+1} \) is added from \( F_0, \ldots, F_i \) in the derivation tree (i.e., a clause set of \( F_0, \ldots, F_i \) is a parent of \( F_{i+1} \)). Then, the \( i \)-th clause set in \( S_k \) is defined as follows.

\[
S_k(i) = \begin{cases} 
F_i & \text{if } i \leq m \\
\emptyset & \text{otherwise}
\end{cases}
\]

We define a function \( \pi : S'_{[S_n]} \rightarrow S \) to connect between \( S'_{[S_n]} \) and \( S \) as follows:

1. \( \pi(0) = s_0 \), and
2. if \( \pi(i) = s \) and \( \{R,F'\} \in S_k(i) \) such that \( S_{k+1}(j) = F' \) is added from \( S_k(i) \) by inference rule A3, then

To avoid the branching of \( A_1^+ \), we introduce deterministic inference rule \( A_1^+ \) that uniquely determines a concept literal \( L \in S_k(i) \) by following the restricted CNF tableau \( T \) with \( L \in L(\pi(i)) \). Inference rule \( A_2^+ \) only applies to a set of unit clauses, unlike inference rule \( A_2 \). It leads to a derivation path as a sequence \( S_0, S_1, \ldots, S_n \). If \( S_n \) is clash-free, then the reasoning algorithm yields a complete and clash-free family of clause sets. We show that \( S(i) \cup \cdots \cup S_n(i) \subseteq L(\pi(i)) \). Since each \( S_k(i) \) (except for \( S_0(0) = F \)) is added by inference rules, all the clauses added to each \( S_k(i) \) have to be included in \( L(\pi(i)) \). We prove this by induction on the depth \( k \) of a derivation tree.

If \( k = 0 \), then \( S_0 = \{F\} \). For \( \pi(0) = s_0 \), \( F \in L(\pi(0)) \), so \( F \subseteq L(\pi(0)) \) from Condition 2 of Definition 7. Hence, \( C(L) \subseteq L(\pi(0)) \) for all \( C(L) \in F(= S_0(0)) \).

If \( k > 0 \), then inference rules \( A1^+, A2^+, \) and A3 are applied to \( S_k \).

If deterministic \( A1^+ \) is applied to \( C(L) \in S_k(i) \) for each \( L \) in \( C(L) \) according to restricted CNF tableau \( T \), \( \{L\} \in S_{k+1}(i) \). By the induction hypothesis, \( C(L) \in L(\pi(i)) \), and so by Condition 3 of Definition 7 and the deterministic \( A1^+ \), \( \{L\} \in L(\pi(i)) \) holds. In addition, \( C(L) \setminus \{L\} \in S_{k+1}(i) \) for all \( C(L) \in S_k(i) \) such that \( L \notin C(L) \). By the induction hypothesis, \( C(L) \in L(\pi(i)) \). By Condition 3 of Definition 7 \( C(L) \setminus \{L\} \in L(\pi(i)) \).

If \( A2^+ \) and A3 are applied, they are proved as well. Hence, since \( S(i) \cup \cdots \cup S_n(i) \subseteq L(\pi(i)) \) for all \( i \in S'_{[S_n]} \), from Condition 11 in Definition 7 \( S(i) \cup \cdots \cup S_n(i) \) is clash-free. Therefore, the complete family of clause sets \( S_n \) is clash-free.

5. Example of Reasoning

We provide some examples concerning deciding the satisfiability of a CNF concept using the reasoning algorithm with inference rules A1, A2, and A3. In addition, we present an example of efficient derivation by the reasoning algorithm with inference rules \( A1^+, A2^+ \), and A3.

Example 1 (CNF Concepts). We consider the following \( ALC \) concept \( F \):

\[
F = (\text{Animal} \sqcup (\text{Black} \sqcap \forall \text{hasPart.Small}))) \sqcap \\
(\neg \text{Animal} \sqcup \exists \text{hasPart}.\text{Leg} \sqcap \neg \text{Small})) \sqcap \neg \exists \text{hasPart}.\text{Leg} \sqcap \exists \text{hasPart}.\text{Wing})
\]

As explained in Section 3 the concept \( F \) is transformed into the conjunctive normal form as follows:

\[
\text{CNF}(F) = (\text{Animal} \sqcup \text{Black}) \sqcap (\text{Animal} \sqcup \exists \text{hasPart}.\text{Small})) \sqcap \\
(\neg \text{Animal} \sqcup \exists \text{hasPart}.\text{Leg} \sqcap \neg \text{Small})) \sqcap (\exists \text{hasPart}.\neg \text{Leg} \sqcap \forall \text{hasPart}.\neg \text{Wing})
\]

Example 2 (Derivation with A1, A2, and A3). Let \( A, B, S, L, W, \) and \( h \) be concept names \( \text{Animal}, \text{Black}, \text{Small}, \text{Leg}, \text{Wing}, \) and a role name \( \text{hasPart} \), respectively. A derivation tree from the initial family of clausal sets \( S_0 = \{\text{CNF}(F)\} \) to \( S_{10} \) is shown in Figure 4.

For the first clause \( \text{Animal} \sqcup \text{Black} (= \{A, B\}) \) of the concept \( \text{CNF}(F) \in S_0 \), \( S_1 = \{F_1\} \) is derived by applying inference rule A1 for the concept literal \( \text{Animal} (= A) \) (ex-1).

\[
F_1 = \text{Animal} \sqcup (\text{Animal} \sqcup \exists \text{hasPart}.\text{Small})) \sqcap \\
(\neg \text{Animal} \sqcup \exists \text{hasPart}.\text{Leg} \sqcap \neg \text{Small})) \sqcap (\exists \text{hasPart}.\neg \text{Leg} \sqcap \forall \text{hasPart}.\neg \text{Wing})
\]

For the second clause \( \text{Animal} \sqcup \exists \text{hasPart}.\text{Small} (= \{A, \forall h.\{S\}\}) \) of \( F_1 \in S_1 \), \( S_2 = \{F_2\} \) is derived by applying rule A1 for \( \text{Animal} (= A) \) (ex-2).

\[
F_2 = \text{Animal} \sqcup \text{Animal} \sqcup (\neg \text{Animal} \sqcup \exists \text{hasPart}.\text{Leg} \sqcap \neg \text{Small})) \sqcap (\exists \text{hasPart}.\neg \text{Leg} \sqcap \forall \text{hasPart}.\neg \text{Wing})
\]

\[
= \text{Animal} \sqcup (\neg \text{Animal} \sqcup \exists \text{hasPart}.\text{Leg} \sqcap \neg \text{Small})) \sqcap (\exists \text{hasPart}.\neg \text{Leg} \sqcap \forall \text{hasPart}.\neg \text{Wing})
\]
For the second clause \( \neg Animal \cup \exists hasPart.(Leg \cap \neg Small) \) (= \{\neg A, \exists h.\{\{L\}, \{\neg S\}\}\}) of \( F_2 \in \mathcal{S}_2 \), \( \mathcal{S}_3 = \{F_3\} \) is derived by applying inference rule A1 for \( \neg Animal \) (= \{\neg A\}) (ex-3).

\[ F_3 = Animal \cap \neg Animal \cap (\exists hasPart. \neg Leg \cup \forall hasPart. \neg Wing) \]

In this case, \( F_3 \) is unsatisifiable because it contains a clash (i.e., \( Animal \) and its negation \( \neg Animal \)). Hence, another concept literal occurring in \( \mathcal{S}_2 \) is selected in an application of inference rule A1. That is, for the second clause \( \neg Animal \cup \exists hasPart.(Leg \cap \neg Small) \) (= \{\neg A, \exists h.\{\{L\}, \{\neg S\}\}\}) of \( F_2 \in \mathcal{S}_2 \), \( \mathcal{S}_4 = \{F_4\} \) is derived by applying inference rule A1 for the concept literal \( \exists hasPart.(Leg \cap \neg Small) \) (= \( \exists h.\{\{L\}, \{\neg S\}\}\)) (ex-4).

\[ F_4 = Animal \cap \exists hasPart.(Leg \cap \neg Small) \cap (\forall hasPart. \neg Leg \cup \forall hasPart. \neg Wing) \]

For the third clause \( \forall hasPart. \neg Leg \cup \forall hasPart. \neg Wing \) (= \( \forall h.\{\{L\}, \forall h.\{\{\neg W\}\}\}\)) of \( F_4 \in \mathcal{S}_4 \), \( \mathcal{S}_5 = \{F_5\} \) is derived by applying inference rule A2 for the concept literal \( \forall hasPart. \neg Leg \) (= \( \forall h.\{\{\neg L\}\}\)) (ex-5).

\[ F_5 = Animal \cap \exists hasPart.(Leg \cap \neg Small) \cap (\forall hasPart. \neg Leg) \]

For the third clause \( \forall hasPart. \neg Leg \cup \forall hasPart. \neg Wing \) (= \( \forall h.\{\{\neg L\}\}, \forall h.\{\{\neg W\}\}\)) of \( F_5 \in \mathcal{S}_5 \), \( \mathcal{S}_6 = \{F_6\} \) is derived by applying inference rule A3 for \( \forall hasPart.(\neg Leg \cap \neg Small) \) in \( F_5 \) (ex-6).

\[ F_6 = Animal \cap \forall hasPart.(\neg Leg \cap \neg Small) \]

For the second clause \( \exists hasPart.(\neg Leg \cap \neg Small) \) (= \( \exists h.\{\{\neg L\}, \{\neg S\}\}\)) in \( \mathcal{S}_6 \), \( \mathcal{S}_7 = \{F_7, F_8\} \) is derived by applying inference rule A3 for \( F_6 \) where \( \mathcal{S}_8 = \{F_8\} \) is the parent of \( F_8 \) (ex-7).

\[ F_7 = Animal \]
\[ F_8 = \neg Leg \cup \neg Small \]

In this case, \( F_8 \) is unsatisifiable because it contains a clash (i.e., \( Leg \) and its negation \( \neg Leg \)). So, another concept literal occurring in \( \mathcal{S}_6 \) is selected in an application of inference rule A1. That is, for the third clause \( \forall hasPart. \neg Leg \cup \forall hasPart. \neg Wing \) (= \( \forall h.\{\{\neg L\}\}, \forall h.\{\{\neg W\}\}\)) of \( F_4 \in \mathcal{S}_4 \), \( \mathcal{S}_9 = \{F_9\} \) is derived by applying rule A1 for the concept literal \( \forall hasPart. \neg Wing \) (= \( \forall h.\{\{\neg W\}\}\)) (ex-8).

\[ F_9 = Animal \cap \exists hasPart.(\neg Leg \cap \neg Small) \cap (\forall hasPart. \neg Wing) \]

For the third clause \( \forall hasPart. \neg Wing \) (= \( \forall h.\{\{\neg W\}\}\)) of \( F_9 \in \mathcal{S}_9 \), \( \mathcal{S}_10 = \{F_{10}\} \) is derived by applying rule A2 for all existential role concepts (i.e., \( \exists hasPart.(\neg Leg \cap \neg Small) \) (= \( \exists h.\{\{L\}, \{\neg S\}\}\)) (ex-9).

\[ F_{10} = Animal \cap \exists hasPart.(-Wing \cap \neg Leg \cap \neg Small) \]

For the second clause \( \exists hasPart.(-Wing \cap \neg Leg \cap \neg Small) \) (= \( \exists h.\{\{\neg W\}, \{\neg S\}\}\)) of \( F_{10} \in \mathcal{S}_9 \), \( \mathcal{S}_{10} = \{F_{11}, F_{12}\} \) is derived by applying inference rule A3 for \( F_{10} \) where \( \mathcal{S}_{11} = \{F_{11}\} \) is the parent of \( F_{11} \) (ex-10).

\[ F_{11} = Animal \]
\[ F_{12} = \neg Wing \cap \neg Leg \cap \neg Small \]

In this case, \( \mathcal{S}_{10} = \{F_{11}, F_{12}\} \) is complete and clash-free. So we can decide that the concept \( F \) is satisfiable.

**Example 3 (Derivation with \( A_1^+, A_2^+, \) and \( A_3 \)).** A derivation tree from the initial family of clause sets \( \mathcal{S}_0' = \{\text{CNF}(F)\} \) to \( \mathcal{S}_0'' \) is shown in Figure 2. For the first clause of the concept \( \text{CNF}(F) \in \mathcal{S}_0' \), \( \mathcal{S}_1' = \{F_1'\} \) (ex-1) is derived by applying inference rule \( A_1^+ \) for the concept literal \( Animal \) (= \( \{A\}\)).

\[ F_1' = Animal \cap Animal \cap \exists hasPart.(Leg \cap \neg Small) \cap (\forall hasPart. \neg Leg \cup \forall hasPart. \neg Wing) \]

For the second clause \( \forall hasPart. \neg Leg \cup \forall hasPart. \neg Wing \) (= \( \forall h.\{\{\neg L\}\}, \forall h.\{\{\neg W\}\}\)) of \( F_1' \in \mathcal{S}_1' \), \( \mathcal{S}_2' = \{F_2'\} \) is derived by applying inference rule \( A_1^+ \) for the concept literal \( \forall hasPart. \neg Leg \) (= \( \forall h.\{\{\neg L\}\}\)) (ex-2).

\[ F_2' = Animal \cap \forall hasPart.(Leg \cap \neg Small) \cap (\forall hasPart. \neg Leg) \]

For the third clause \( \forall hasPart. \neg Leg \) (= \( \forall h.\{\{\neg L\}\}\)) of \( F_2' \in \mathcal{S}_2' \), \( \mathcal{S}_3' = \{F_3'\} \) is derived by applying inference rule \( A_2^+ \) for all existential role concepts (i.e., \( \exists hasPart.(\neg Leg \cap \neg Small) \) (= \( \exists h.\{\{L\}, \{\neg S\}\}\)) (ex-3).

\[ F_3' = Animal \cap \exists hasPart.(-Leg \cap \neg Leg \cap \neg Small) \]

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\[ S_0 = \{\{A, B\}, \{A, \forall h.\{\neg S\}\}, \{\neg A, \exists h.\{L, \{\neg S\}\}, \forall h.\{\neg L\}, \forall h.\{\neg W\}\}\}\]  
\hspace{2em} A1 (ex-1)

\[ S_1 = \{\{A\}, \{A, \forall h.\{S\}\}, \{\neg A, \exists h.\{L, \{\neg S\}\}, \forall h.\{\neg L\}, \forall h.\{\neg W\}\}\}\]  
\hspace{2em} A1 (ex-2)

\[ S_2 = \{\{A\}, \{\neg A, \exists h.\{L, \{\neg S\}\}, \forall h.\{\neg L\}, \forall h.\{\neg W\}\}\}\]  
\hspace{2em} A1 (ex-3)

\[ S_3 = \{\{\neg A\}, \forall h.\{\neg L\}, \forall h.\{\neg W\}\}\]  
\hspace{2em} A1 (ex-4)

\[ S_4 = \{\{A\}, \forall h.\{L, \{\neg S\}\}, \forall h.\{\neg L\}, \forall h.\{\neg W\}\}\]  
\hspace{2em} A1 (ex-5)

\[ S_5 = \{\{A\}, \exists h.\{L, \{\neg S\}\}, \forall h.\{\neg L\}\}\]  
\hspace{2em} A2 (ex-6)

\[ S_6 = \{\{A\}, \exists h.\{\neg L, \{\neg S\}\}\}\]  
\hspace{2em} A3 (ex-7)

\[ S_7 = \{\{\neg A\}, \forall h.\{\neg L\}, \forall h.\{\neg W\}\}\]  
\hspace{2em} A1 (ex-8)

\[ S_8 = \{\{A\}, \exists h.\{L, \{\neg S\}\}, \forall h.\{\neg L\}\}\]  
\hspace{2em} A2 (ex-9)

\[ S_9 = \{\{A\}, \exists h.\{\neg W, \{\neg S\}\}\}\]  
\hspace{2em} A3 (ex-10)

\[ S_{10} = \{\{\neg A\}, \{\neg L, \{L, \{\neg S\}\}\}\}\]  
\hspace{2em} SAT

Figure 1: A derivation tree with A1, A2, and A3
For the second clause $\exists \text{hasPart.}(\neg \text{Leg} \land \text{Leg} \land \neg \text{Small}) (= \{\exists h, \{(\neg L), \{L\}, \{(\neg S)\})\})$ of $F_3' \in S_3'$, $S_3' = \{F_4', F_5'\}$ is derived by applying inference rule A3 for $F_3'$ where $F_3'$ is the parent of $F_1'$ (ex-4').

$$F_4' = \text{Animal}$$
$$F_5' = (\neg \text{Leg} \sqcup \text{Leg} \land \neg \text{Small})$$

In this case, $F_3'$ is unsatisfiable because it contains a clash (i.e., Leg and its negation $\neg$Leg). Therefore, another concept literal occurring in $S_3'$ is selected in an application of inference rule A1+ (ex-2'). That is, for the third clause $\forall \text{hasPart.}(\neg \text{Leg} \sqcup \forall \text{hasPart.}(\neg \text{Wing} (= \{\forall h, \{(\neg L), \forall h, \{(\neg W)\}\})$ of $F_1' \in S_1'$, $S_3' = \{F_6'\}$ is derived by applying inference rule A1+ for the concept literal $\forall \text{hasPart.}(\neg \text{Wing} (= \forall h, \{(\neg W)\})\) (ex-5').

$$F_6' = \text{Animal} \sqcap \exists \text{hasPart.}(\text{Leg} \land \neg \text{Small}) \sqcap \forall \text{hasPart.}(\neg \text{Leg})$$

For the third clause $\exists \text{hasPart.}(\neg \text{Wing} (= \{\forall h, \{(\neg W)\})$ of $F_6' \in S_3'$, $S_3' = \{F_7'\}$ is derived by applying inference rule A2+ for all existential role concepts (i.e., $\exists \text{hasPart.}(\text{Leg} \land \neg \text{Small}) (= \{\exists h, \{(L), \{(\neg S)\}\})\) (ex-6').

$$F_7' = \text{Animal} \sqcap \exists \text{hasPart.}(\neg \text{Wing} \land \text{Leg} \land \neg \text{Small})$$

For the second clause $\exists \text{hasPart.}(\neg \text{Wing} \land \text{Leg} \land \neg \text{Small}) (= \{\exists h, \{(\neg W), \{L\}, \{(\neg S)\}\})$ of $F_1' \in S_1'$, $S_1' = \{F_8', F_9'\}$ is derived by applying inference rule A3 for $F_1'$ where $F_1'$ is the parent of $F_9'$ (ex-7').

$$F_8' = \text{Animal}$$
$$F_9' = \neg \text{Wing} \land \text{Leg} \land \neg \text{Small}$$

In this case, $S_1' = \{F_8', F_9'\}$ is complete and clash-free. So we can decide that the concept $F$ is satisfiable.

Remark 1. Inference rule A1 selects a concept literal $L \in CL$ in a clause $CL$ but does not handle the concept literal $L$ or the complementary literal $\neg L$ in other clauses. This process may cause redundant derivation steps or backtracking. On the other hand, inference rule A1+ selects a concept literal $L \in CL$ in a clause $CL$ and simultaneously processes all clauses containing $L$ or $\neg L$. As a result of this rule, every clause containing $L$ is transformed into the unit clause $\{L\}$ without any contradiction. In addition, the complementary literal $\neg L$ is removed from every clause containing $L$ and is then not selected in later derivation steps. Therefore, A1+ is more efficient than A1.

Remark 2. Inference rule A2 can be applied to any clause (not limited to a unit clause), thus expanding the choice of inference rules in the early stages. If A2 selects a concept literal $L$, then the selection is equivalent to applying inference rule A1 to $L$ beforehand. By restricting the applications of A2, inference rule A2+ can be applied for a unit clause after inference rule A1+ is applied. This reduces the complexity of derivation because the selection of a concept literal in each clause is minimized by applying inference rule A1+.

6. Conclusion

In this paper, we formalized CNF concepts in a conjunctive normal form for the description logic $\mathcal{ALC}$ where any $\mathcal{ALC}$ concept can be transformed to a CNF concept. To decide the satisfiability of a CNF concept, we designed a decidable reasoning algorithm for clause sets in $\mathcal{ALC}$. In particular, inference rules A1, A2, and A3 in our proposed reasoning algorithm provide efficient derivation steps owing to the clausal form of $\mathcal{ALC}$ concepts. Furthermore, to improve the efficiency of the reasoning algorithm, inference rules A1 and A2 were improved to A1+ and A2+, thus reducing further derivation steps. By formalizing (restricted) CNF tableaux based on the semantics of CNF concepts, we proved the termination, soundness, and completeness of the two reasoning algorithms using A1, A2, and A3 as well as A1+, A2+, and A3, respectively. The theoretical results are expected to lead to some applications of the proposed techniques for clausal reasoning to description logics—such as the resolution principle and solvers for the Boolean satisfiability problem (SAT solvers).

Our future work will include the formalization of a conjunctive normal form and its reasoning algorithm for more expressive description logics corresponding to OWL. In addition, the proposed conjunctive normal form in $\mathcal{ALC}$ will be applied to fast SAT solvers to implement the reasoning algorithm for ontologies and knowledge bases.
Figure 2: An efficient derivation tree with $A_1^+$, $A_2^+$, and $A_3$

References

[1] Franz Baader, Diego Calvanese, Deborah McGuinness, Daniele Nardi, and Peter Patel-Schneider. The Description Logic Handbook: Theory, Implementation, and Applications, 2nd Edition. Cambridge University Press, 2007.

[2] Tamraparni Dasu and Theodore Johnson. Exploratory Data Mining and Data Cleaning, volume 479. John Wiley & Sons, 2003.

[3] Vishal Jain and Mayank Singh. Ontology Development and Query Retrieval using Protége Tool. International Journal of Intelligent Systems and Applications, 9(9):67–75, 2013.

[4] Pascal Hitzler, Markus Krötzsch, Bijan Parsia, Peter F Patel-Schneider, Sebastian Rudolph, et al. OWL 2 web ontology language primer. W3C recommendation, 27(1):123, 2009.

[5] Joao P Marques-Silva and Karem A Sakallah. GRASP: A Search Algorithm for Propositional Satisfiability. IEEE Transactions on Computers, 48(5):506–521, 1999.

[6] Boris Motik, Rob Shearer, and Ian Horrocks. Hypertableau Reasoning for Description Logics. Journal of Artificial Intelligence Research, 36:165–228, 2009.

[7] Boris Motik, Rob Shearer, and Ian Horrocks. A Hypertableau Calculus for SHIQ. In Diego Calvanese, Enriso Franconi, Volker Haarslev, Domenico Lembo, Boris Motik, Sergio Tessaris, and Amy-Yasmin Turhan, editors, Proc. of the 20th Int. Workshop on Description Logics (DL 2007), pages 419–426, Brixen/Bressanone, Italy, June 8–10 2007. Bozen/Bolzano University Press.

[8] Peter Baumgartner, Ulrich Furbach, and Ilkka Niemelä. Hyper Tableaux. In John Alan Robinson, editor, Proceedings of the 2nd International Conference on Logic Programming, pages 408–409, London, 1986. MIT Press.

[9] Peter Baumgartner, Ulrich Furbach, and Ilkka Niemelä. Hyper Tableaux. In European Workshop on Logics in Artificial Intelligence, pages 1–17. Springer, 1996.

[10] Benjamin N Grosf, Ian Horrocks, Raphael Volz, and Stefan Decker. Description Logic Programs: Combining Logic Programs with Description Logic. In Proceedings of the 12th international conference on World Wide Web, pages 48–57, 2003.
[11] Manfred Schmidt-Schauß and Gert Smolka. Attributive concept descriptions with complements. *Artificial intelligence*, 48(1):1–26, 1991.