The Plane-Width of Graphs

Marcin Kamiński,¹ Paul Medvedev,² and Martin Milanič³

¹DÉPARTEMENT D’INFORMATIQUE
UNIVERSITÉ LIBRE DE BRUXELLES
BRUXELLES, BELGIUM
E-mail: Marcin.Kaminski@ulb.ac.be

²DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF TORONTO, CANADA
E-mail: pashadag@cs.toronto.edu

³FAMNIT AND PINT
UNIVERSITY OF PRIMORSKA, KOPER, SLOVENIA
E-mail: martin.milanic@upr.si

Received December 23, 2008; Revised August 11, 2010

Published online 1 February 2011 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.20554

Abstract: Map the vertices of a graph to (not necessarily distinct) points of the plane so that two adjacent vertices are mapped at least unit distance apart. The plane-width of a graph is the minimum diameter of the image of its vertex set over all such mappings. We establish a relation between the plane-width of a graph and its chromatic number. We also connect it to other well-known areas, including the circular chromatic number and the problem of packing unit discs in the plane. © 2011 Wiley Periodicals, Inc. J Graph Theory 68: 229–245, 2011

Marcin Kamiński is Chargé de Recherches du F.R.S.—FNRS.
Contract grant sponsors: Communauté française de Belgique (to M. K.); Fonds National de la Recherche Scientifique (F.R.S.—FNRS) (to M. K.); Agencija za raziskovalno dejavnost Republike Slovenije research Program; Contract grant number: P1-0285 (to M. M.).
Journal of Graph Theory
© 2011 Wiley Periodicals, Inc.
1. INTRODUCTION

Given a simple, undirected, finite graph \( G = (V, E) \), a realization of \( G \) is a function \( r \) assigning to each vertex a point in the plane such that for each \( \{u, v\} \in E \), \( d(r(u), r(v)) \geq 1 \), where \( d \) is the Euclidean distance. The width of a realization is the maximum distance between the images of any two vertices. In this article, we introduce a new graph invariant, called the plane-width and denoted by \( \text{pwd}(G) \), which is the minimum width of all realizations of \( G \). The notion is well defined, as the plane-width of a graph can be expressed as the minimum value of a continuous function over a compact subset of \( \mathbb{R}^2 \). Also, notice that the definition of a realization allows different vertices to be mapped to the same point.

The plane-width of an edgeless graph is 0. To avoid trivialities, we only consider graphs with at least one edge. A realization of \( G \) whose width equals \( \text{pwd}(G) \) is called optimal. The image of an (optimal) realization is called an (optimal) arrangement. Given a graph \( G \), we denote by \( V(G) \) its vertex set and with \( E(G) \) its edge set. An edge \( \{u, v\} \) in a graph will be denoted by \( uv \). For terminology not defined here, we refer the reader to [9].

Related work. A notion related to the plane-width is the dilation coefficient, defined by Pisanski and Žitnik in [21] and recently considered by Horvat et al. [16] for complete graphs. Given a realization of a graph, its dilation coefficient is the ratio of the longest to the shortest edge length. For complete graphs, the minimum possible dilation coefficient coincides with the plane-width of the graph. Other notions similar to the plane-width were considered by Belk and Connelly [3], Carmi et al. [8], and Mohar [18].

2. PLANE-WIDTH OF COMPLETE GRAPHS AND ODD WHEELS

The problem of determining the plane-width of complete graphs \( K_n \) has previously appeared in the literature in different contexts: finding the minimum diameter of a set of \( n \) points in the plane such that each pair of points is at distance at least one [4], or packing non-overlapping unit discs in the plane so as to minimize the maximum distance between any two disc centers [22]. A similar well-studied problem is that of computing the smallest diameter of a circle enclosing \( n \) circles of unit diameter [13]. In this section, we review what is known about the plane-width of complete graphs and add our own results.

A. Asymptotic Behavior

The asymptotic behavior of \( \text{pwd}(K_n) \) is determined. A lower bound is provided by the following result by Bezdek and Fodor.

**Lemma 2.1** (Bezdek and Fodor [4]). For every \( n \geq 2 \), \( \text{pwd}(K_n) \geq \sqrt{2(\sqrt{3}/\pi)n} - 1 \).
They further mention that this lower bound can be asymptotically matched by mapping the vertices of $K_n$ to points of the equilateral triangular lattice. Indeed, it can be shown using straightforward geometric arguments that mapping the vertices of $K_n$ to points of the equilateral triangular lattice so that they are contained in the smallest possible circle results in an upper bound that differs from the lower bound by a small additive constant. We formalize this in the following lemma, in the proof of which we use the dual notion of the equilateral triangular lattice, i.e., the hexagonal tiling of the plane.

Lemma 2.2. For every $n \geq 2$, $\text{pwd}(K_n) \leq \sqrt{\frac{2\sqrt{3}/\pi}{n} + 4/\sqrt{3} - 1}$.

Proof. Let $\mathcal{H}$ be a hexagonal tiling of the plane with hexagons of side length $\frac{1}{\sqrt{3}}$.

Observe that the diameter of each hexagon is $d = \frac{2}{\sqrt{3}}$, the area is $\frac{\sqrt{3}}{2}$, and the centers of every two adjacent hexagons are unit distance apart. For $R > 0$, let $D(R)$ denote the disk of radius $R$ centered at the origin. To prove the lemma, we will map the vertices of $K_n$ to the centers of the hexagons in $\mathcal{H}$ that are (completely) contained within $D(R)$, for $R = \sqrt{\frac{\sqrt{3}/(2\pi)}{n} + d}$. As the centers of these hexagons are actually contained in the disk $D(R - \frac{1}{2})$, the width of the realization is at most $2R - 1$ and the lemma follows.

We only need to show that the number of hexagons contained in $D(R)$, denoted by $h(R)$, is at least $n$. For every point $p \in D(R - d)$, the disk of radius $d$ centered at $p$ is contained in $D(R)$; therefore, every hexagon from $\mathcal{H}$ containing $p$ is contained in $D(R)$ as well. Consequently, the disk $D(R - d)$ is (completely) covered by the union of those hexagons in $\mathcal{H}$ that are contained in $D(R)$. We thus have, comparing the areas:

$$\pi(R - d)^2 \leq h(R) \cdot \frac{\sqrt{3}}{2},$$

or, equivalently,

$$h(R) \geq \frac{2\pi}{\sqrt{3}} (R - d)^2 = n.$$

Together, these two bounds lead to the exact expression for the asymptotic behavior of $\text{pwd}(K_n)$.

Theorem 2.3 (Bateman and Erdős [2], Bezdek and Fodor [4], and Erdős [11]).

$$\lim_{n \to \infty} \frac{\text{pwd}(K_n)}{\sqrt{n}} = \sqrt{\frac{2\sqrt{3}}{\pi}} \approx 1.05.$$

Interestingly, it was conjectured by Erdős and proved by Schürmann [22] that for all sufficiently large $n$, the optimal value of $\text{pwd}(K_n)$ is not attained by any arrangement contained in a lattice.

---

1We remark that the word “lattice”, as used here, does not have its usual order theoretic meaning but refers to a discrete subgroup of the 2-dimensional Euclidean space.
TABLE I. Known values of pwd($K_n$) (in the last row rounded to three decimal places).

| n  | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|----|-----|-----|-----|-----|-----|-----|-----|
| pwd($K_n$) | 1   | 1   | $\sqrt{2}$ | $1+\sqrt{2}$ | 2 sin 72° | 2   | $(2\sin(\pi/14))^{-1}$ |
| $\approx$ | 1   | 1   | 1.414 | 1.618 | 1.902 | 2   | 2.246 |

B. Small Complete Graphs

The exact values of pwd($K_n$) are known only for $n \leq 8$, and are summarized in Table I. Clearly, pwd($K_2$) = pwd($K_3$) = 1. For $n = 3, 4, 5$, Bateman and Erdős [2] state without proof that an optimal arrangement is attained by the vertices of a regular $n$-gon with unit sides. For completeness, we give below a simple proof for $K_4$ and $K_5$ and show that the optimal arrangements are unique in both cases. Bateman and Erdős [2] also mention without proof that pwd($K_6$) = $2\sin 72°$, an optimal arrangement consisting of the center and the vertices of a regular pentagon of circumradius 1, and prove that pwd($K_7$) = 2 and that the unique optimal arrangement consists of the center and the vertices of a regular hexagon with unit sides. For $n = 8$, Bezdek and Fodor [4] prove that pwd($K_8$) = $(2\sin(\pi/14))^{-1} \approx 2.246$ and that the convex hull of every optimal arrangement of $K_8$ is the regular heptagon with unit sides. The current best upper bound on the plane-width of $K_9$ is 2.584306 by Audet et al. [1].

Proposition 2.4 (Bateman and Erdős [2]).

(a) pwd($K_4$) = $\sqrt{2}$ and the unique optimal arrangement is given by the corners of the unit square.

(b) pwd($K_5$) = $(1 + \sqrt{5})/2$ and the unique optimal arrangement is given by the corners of a regular pentagon with side length 1.

Proof. It is easy to verify that the above arrangements have the desired diameters, so what remains to show is a matching lower bound on the plane-width and a proof of uniqueness.

Suppose that a complete graph has an arrangement of width $d < (1 + \sqrt{5})/2$ and let $A$ and $B$ be two points at distance $d$. The remaining points of the arrangement must lie within the set $S$, the set of all the points at distance at least 1 and at most $d$ from both $A$ and $B$. The set $S$ is composed of two connected parts $S^-$ and $S^+$, each of diameter less than 1 (see Fig. 1).

Let us now justify that the diameter of each part is less than 1. Assume the notation as in Figure 1, where the coordinates of $A$ and $B$ are $(0, d/2)$ and $(0, -d/2)$, respectively. Let us denote by $\hat{CD}$, $\hat{DE}$, $\hat{EF}$, $\hat{FC}$ the four circular arcs (together with their endpoints) enclosing the set $S^+$. The diameter of $S^+$ can only be achieved as the distance between two points on the boundary of $S^+$. More specifically:

1. $\text{diam}(\hat{CD} \cup \hat{DE}) = \text{diam}(\hat{EF} \cup \hat{FC}) = \max\{|CD|, |CE|, |DF|\}$,
2. $\text{diam}(\hat{CD} \cup \hat{EF}) = \text{diam}(\hat{DE} \cup \hat{FC}) = \max\{|CE|, |DF|\}$,
3. $\text{diam}(\hat{CD} \cup \hat{FC}) = \max\{|CD|, |DF|\}$,
4. $\text{diam}(\hat{DE} \cup \hat{EF}) = |DF|$.

Since the proofs of all four statements are similar, we only prove (1) here. Let $U, V \in \hat{CD} \cup \hat{DE}$. Suppose first that $U \in \hat{CD}$ and $V \in \hat{DE}$. By considering the triangle $AVU$ and...
one of the triangles $ACV$ and $ADV$ (the former one if $U$ lies on or below the line $AV$, the latter one otherwise), we see that $|UV| \leq \max\{|CV|, |DV|\}$, as the angle $UAV$ is at most the angle $CAV$ (or $DAV$). Similarly, we can argue that $|CV| \leq \max\{|CD|, |CE|\}$ and $|DV| \leq |DE|$, and consequently $|UV| \leq \max\{|CD|, |CE|, |DE|\}$. If $U, V \in \widehat{CD}$, then $|UV| \leq |CD|$, and if $U, V \in \widehat{DE}$, then $|UV| \leq |DE|$.

By (1)–(4), the diameter of $S^+$ is either $|CD|, |CE|, |DE|$, or $|DF|$. By considering the triangles $DAE$ and $DAF$, we see that $|DE| < |DF|$. Furthermore, $|CD|^2 = (x_D - x_C)^2 + y_D^2$, $|CE| = x_E - x_C$, and $|DF| = 2y_D$, where $C = (x_C, 0)$, $D = (x_D, y_D)$, and $E = (x_E, 0)$. The coordinates $x_C, x_E, x_D,$ and $y_D$ are easily computed to be:

$$x_C = \sqrt{1 - d^2/4}, \quad x_E = \sqrt{3d}/2, \quad x_D = \sqrt{1 - 1/(4d^2)}, \quad y_D = \frac{d}{2} - \frac{1}{2d}.$$

After some algebraic manipulation, the system of inequalities $|CD|^2 < 1$, $|CE| < 1$, $|DF| < 1$ reduces to

$$d^4 - 4d^2 + 1 < 0, \quad d < \sqrt{3}, \quad d^2 - d + 1 < 0,$$

which holds true for every $d \in [1, (1 + \sqrt{5})/2)$. Hence, the diameter of $S^+$ (and hence of $S^-$) is less than 1.

It follows that each of the two parts can contain at most one vertex, and so the graph contains at most four vertices. This immediately proves that $\text{pw}(K_4) \geq (1 + \sqrt{5})/2$. Moreover, for $d < \sqrt{2}$, the shortest distance between any two points lying in different parts of $S$ is $|CC'| = 2x_C = \sqrt{4 - d^2} > \sqrt{2}$. Hence, the two parts of $S$ cannot both contain a vertex, and the graph cannot contain more than 3 vertices. This proves that $\text{pwd}(K_4) \geq \sqrt{2}$.

To show uniqueness for $K_4$, note that for $d = \sqrt{2}$ there exists only one pair of points lying in different parts of $S$ such that the distance between them is not more than $\sqrt{2}$. For $K_5$, observe that for $d = (1 + \sqrt{5})/2$ there exists only one pair of points within a connected part of $S$ that are at least unit distance apart (e.g., points $D$ and $F$ in $S^+$). By mapping two of the vertices of $K_5$ to such a pair, we restrict the remaining vertex to exactly one location—the closest point in the other part of $S$.

![Figure 1](https://example.com/figure1.png)

**Figure 1.** The situation in the proof of Proposition 2.4. The shaded area is the set $S$. 

---

*Journal of Graph Theory* DOI 10.1002/jgt
C. Odd Wheels

A wheel is the graph obtained from a cycle by adding a new vertex adjacent to all the vertices of the cycle. A wheel is either odd or even according to the parity of the number of vertices in the cycle. As will become clear in the next section (by Theorem 3.1(a)), the plane-width of every even wheel is equal to 1. For odd wheels, we have already seen that the plane-width is $\sqrt{2}$ for the smallest odd wheel ($K_4$). In this section, we generalize this result by showing that the plane-width of every odd wheel is equal to $\sqrt{2}$. Odd wheels and 3-colorable graphs (see Section 3) are the only natural families of graphs whose plane-width is known.

**Proposition 2.5.** The plane-width of every odd wheel is equal to $\sqrt{2}$.

**Proof.** Let $G$ be an odd wheel. To show that $\text{pwd}(G) \leq \sqrt{2}$, consider a proper 4-coloring of $G$, and map (the vertices of) each color class to a different vertex of the unit square.

Suppose now that $\text{pwd}(G) = d < \sqrt{2}$ and consider an arrangement $A$ of $G$ of width $d$ given by a realization $r$. Let $v^*$ denote a vertex adjacent to the other vertices of $G$. Assume without loss of generality that $v^*$ is mapped to the origin. Then, all the other vertices must be mapped to points at distance at least 1 and at most $d$ from the origin. Since $d < \sqrt{2}$, the maximum angle between any two lines connecting the origin with the image of vertices $v \neq v^*$ is less than $\pi/2$. Therefore, we can assume that one of the points other than the origin lies on the positive part of the $x$-axis, and all the other points in $A$ lie in the first quadrant (otherwise we can rotate the arrangement around the origin). Now, let $P$ denote the point $(1,0)$ and let $Q$ denote the point in the first quadrant that is at distance $d$ from $P$ and at distance 1 from the origin. Furthermore, let $\ell$ denote the line through $P$ perpendicular to the line segment $PQ$, and let $\ell'$ denote the line parallel to $\ell$ passing through the point $Q$ (see Fig. 2).

We now rotate the arrangement counter-clockwise so that it lies entirely on or above the line $\ell$, and so that at least one of the points lies on $\ell$. Then, all the points of the rotated arrangement (except $v^*$) belong to the part of the first quadrant between

---

**FIGURE 2.** The situation in the proof of Proposition 2.5.
the lines $\ell$ and $\ell'$ and between the two concentric circles of respective radii 1 and $d$ centered at the origin. In particular, the arrangement (except $v^*$) is contained in the rectangle defined by $\ell$, $\ell'$, $PQ$ and the line parallel to $PQ$ and tangent to the outer circle.

Halving this rectangle produces two congruent rectangles $R$ and $R'$ with sides of length $d/2$ and $d - \sqrt{1 - d^2}/4 < d/2$ (where the last inequality follows from the assumption on $d$). Hence, each of the two rectangles is of diameter less than 1, which in turn implies that the vertices of $G$ can be partitioned into three independent sets: the vertex $v^*$, the vertices mapped to $R$, and the vertices mapped to $R' \setminus R$. This is a contradiction to the fact that odd wheels are 4-chromatic. ■

3. PLANE-WIDTH AND THE CHROMATIC NUMBER

In this section, we establish a connection between the plane-width of a graph and its chromatic number $\chi(G)$.

A. Graphs With Small Chromatic Number

For small values of the chromatic number, there is a strong relation between the plane-width of a graph and its chromatic number. The goal of this subsection is to prove the following theorem (cf. Fig. 3).

Theorem 3.1. For all graphs $G$,

(a) $\text{pwd}(G) = 1$ if and only if $\chi(G) \leq 3$,
(b) $\text{pwd}(G) \notin (1, \frac{2}{\sqrt{3}}]$,
(c) $\text{pwd}(G) \in \left(\frac{2}{\sqrt{3}}, \sqrt{2}\right]$ if and only if $\chi(G) = 4$,
(d) $\text{pwd}(G) \in (\sqrt{2}, 2]$ if and only if $\chi(G) \in \{5, 6, 7\}$.

In particular, every bipartite graph has plane-width exactly 1. Also, every graph with maximum degree 3, different from the complete graph on 4 vertices, has plane-width exactly 1. (By Brooks’ Theorem such graphs are 3-colorable.) The plane-width of every planar graph is at most $\sqrt{2}$ (as such graphs are 4-colorable), and the plane-width of graphs embeddable on a torus is at most 2 (as such graphs are 7-colorable). These bounds are tight since $\text{pwd}(K_4) = \sqrt{2}$ and $K_4$ is planar; also, $\text{pwd}(K_7) = 2$ and $K_7$ is embeddable on a torus.

We start with some definitions. The term plane set will mean a set of points in the plane. Given a realization $r$ of a graph $G$ and a plane set $S$, we denote by $r^{-1}(S)$ the set

\[ \chi \leq 3 \hspace{1cm} \chi = 4 \hspace{1cm} 5 \leq \chi \leq 7 \]

FIGURE 3. Relation between pwd and $\chi$ for small values of these invariants.
\[ \{ v \in V(G) : r(v) \in S \} \]. The diameter of \( S \) is defined as \( \text{diam}(S) = \sup_{x,y \in S} d(x,y) \). We say that \( S \) is \( \delta \)-small if \( d(x,y) < \delta \) for all \( x, y \in S \). Note that every set of diameter less than \( \delta \) is \( \delta \)-small, but the converse does not hold; the diameter of a \( \delta \)-small set can be \( \delta \).

The importance of 1-small sets for the purposes of relating plane-width to coloring is based on the following statement, which follows directly from the definitions.

**Observation 3.2.** Let \( G \) be a graph and \( r \) be a realization of \( G \). For every 1-small plane set, the set \( r^{-1}(S) \) is an independent set in \( G \).

The following lemma establishes a connection between the plane-width of a graph and its chromatic number.

**Lemma 3.3.** Let \( G \) be a graph and let \( \delta = 1/\text{pwd}(G) \). If every plane set of unit diameter can be partitioned into \( k \) \( \delta \)-small sets, then \( \chi(G) \leq k \).

**Proof.** Let \( G \) be a graph and let \( \delta = 1/\text{pwd}(G) \). Consider an arrangement \( A \) of \( G \) given by an optimal realization \( r \). Then, \( A \) is a plane set of diameter \( \text{diam}(A) = \text{pwd}(G) \). Suppose that every plane set of unit diameter can be partitioned into \( k \) \( \delta \)-small sets. Then, by scaling, for every \( d > 0 \), every plane set of diameter \( d \) can be partitioned into \( k \) \( (\delta d) \)-small sets. In particular, taking \( d = \text{pwd}(G) \), we can partition \( A \) into \( k \) 1-small sets \( A_1, \ldots, A_k \). By Observation 3.2, each of the sets \( r^{-1}(A_i) \) is an independent set in \( G \).

Therefore, \( \chi(G) \leq k \).

This lemma provides a method for translating upper bounds on \( \text{pwd}(G) \) into upper bounds on \( \chi(G) \), which involves showing how to partition a plane set of unit diameter into sets of smaller diameter. We now apply this technique to graphs of small plane-width.

**Lemma 3.4.** Every plane set of unit diameter can be partitioned into either of the following ways:

(a) Three \( (\frac{\sqrt{3}}{2}) \)-small sets,
(b) Four \( (\frac{\sqrt{2}}{2}) \)-small sets,
(c) Seven \( (\frac{1}{2}) \)-small sets.

**Proof.** Our proof relies on a result of Pál [19] that every plane set \( S \) of unit diameter can be enclosed by a regular hexagon whose opposite sides are at unit distance. We thus enclose \( S \) in such a hexagon \( H \), which, without loss of generality, is centered at the origin. What remains to be done is to partition the hexagon into small sets.

(a) Our proof is similar to the proof of cutting a hexagon whose opposite sides are at unit distance into three sets of diameter less than one, as given, e.g., in [6]. We cut \( H \) into three sets \( H_1, H_2, \) and \( H_3 \) of diameter \( \frac{\sqrt{3}}{2} \), as follows: denoting the sides of the hexagon by \( s_1, s_2, \ldots, s_6 \) in a cyclic order, we cut the hexagon along the lines \( l_1, l_2, l_3 \) connecting the center of the hexagon with the midpoints of the sides \( s_1, s_3, \) and \( s_5 \), respectively. For each \( i \in \{1, 2, 3\} \), we let \( H_i \) be the subset of \( H \) defined by the boundaries of \( l_i \) and \( l_{i+1} \), inclusive of \( l_i \) and exclusive of \( l_{i+1} \) (indices take modulo 3). Moreover, we assume that the center of the hexagon belongs to \( H_1 \) but not to \( H_2 \) and \( H_3 \) (see Fig. 4A). By construction, the sets \( H_1, H_2, \) and \( H_3 \) form a partition of \( H \) and
are each \((\sqrt{3}/2)\)-small. Finally, the three \((\sqrt{3}/2)\)-small sets that partition \(S\) are given by \(S_i = H_i \cap S\) for all \(i \in \{1, 2, 3\}\).

(b) We partition \(H\) into four sets \(H_1, H_2, H_3,\) and \(H_4,\) with \(H_i\) corresponding to the subset of \(H\) lying in the \(i\)th quadrant (see Fig. 4B). In order to make the sets \((\sqrt{3}/2)\)-small, we remove the point \((0,0)\) from \(H_1, H_2,\) and \((0,\frac{1}{2})\) from \(H_1, (-\frac{1}{2},0)\) from \(H_2, (0,-\frac{1}{2})\) from \(H_3,\) and \((\frac{1}{2},0)\) from \(H_4.\) These sets can be made pairwise disjoint by assigning each point that belongs to at least two sets in an arbitrary way to only one of the sets. As in the previous case, the sets \(H_i\) induce a partition of \(S\) into four \((\sqrt{3}/2)\)-small sets.

(c) Let us name the vertices of \(H\) consecutively \(p_0, \ldots, p_5\) and for \(i = 0, 1, \ldots, 5;\) let \(m_i\) be the midpoint of edge \(p_{i-1}p_i\) (indices taken modulo 6). Also, let \(q_i\) be the point at distance \((\sqrt{3} - 1)/2\) from \(m_i\) on the line segment connecting \(m_i\) and \(m_{i+3}.\) The convex hull of \(q_0, \ldots, q_5\) is a hexagon, and let \(R\) be the convex hull of \(q_0, \ldots, q_5,\) without the \(q_i\)'s. Notice that \(R\) is \((\frac{1}{2})\)-small.

Let \(R_i\) be the convex hull of \(q_i, m_i, p_i, m_{i+1}, q_{i+1},\) without points \(q_{i+1}\) and \(m_{i+1}.\) It is easy to verify that each of \(R_i\)'s is a \((\frac{1}{2})\)-small set and, as before, induce a partition of \(S\) into seven \((\frac{1}{2})\)-small sets (see Fig. 4C).

The following corollary is a direct consequence of Lemmas 3.3 and 3.4.

**Corollary 3.5.** For every graph \(G,\)

(a) If \(\text{pwd}(G) \leq \frac{2}{\sqrt{3}},\) then \(\chi(G) \leq 3,\)

(b) If \(\text{pwd}(G) \leq \sqrt{2},\) then \(\chi(G) \leq 4,\)

(c) If \(\text{pwd}(G) \leq 2,\) then \(\chi(G) \leq 7.\)

Now we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1 (a) and (b).** Any 3-colorable graph admits a realization of width 1 by assigning (the vertices of) each color class to a different vertex of the equilateral triangle with side length 1. On the other hand, if \(\text{pwd}(G) \leq \frac{2}{\sqrt{3}},\) then Corollary 3.5(a) gives that \(\chi(G) \leq 3.\) In turn, this implies that \(\text{pwd}(G) \leq 1.\)

(c) and (d). Observe that 4-colorable graphs admit a realization of width \(\sqrt{2},\) by mapping (the vertices of) each color class to a different vertex of the unit square. Similarly, 7-colorable graphs admit a realization of width 2, by mapping each color class \(C_1, \ldots, C_6\) to a different vertex of the regular hexagon \(H\) of side length 1, and (the
vertices of) the remaining color class to the center of \( H \). Together with Corollary 3.5, these observations imply the theorem.

**B. Graphs With Large Chromatic Number**

In this section, we study the asymptotic behavior of \( \text{pwd}(G) \) as \( \chi(G) \to \infty \). We have seen in Theorem 2.3 that \( \text{pwd}(K_n) = O(\sqrt{n}) \). Now we prove, more generally, that the relation \( \text{pwd}(G) = \Theta(\sqrt{\chi(G)}) \) holds for arbitrary graphs as \( \chi(G) \to \infty \).

**Lemma 3.6.** For every graph \( G \),

\[
\chi(G) \leq (\frac{2}{\sqrt{3}} \cdot \text{pwd}(G) + \sqrt{7})^2.
\]

**Proof.** Let \( G \) be a graph, and consider an arrangement \( A \) of \( G \) given by an optimal realization of width \( d = \text{pwd}(G) \). We can use the result of Pál [19] to enclose the arrangement in a regular hexagon \( H \) whose opposite sides are at distance \( d \). Let \( t = \lceil 2d/3 \rceil \), and let \( T \) be a hexagonal tiling of the plane with hexagons of side length \( d/(3t) \). The hexagon \( H \) (and with it \( A \)) can be translated and rotated so that \( H \) is contained in the union of \( 6(\frac{t+1}{2}) + 1 \) hexagons from \( T \). (First, rotate \( H \) until it becomes parallel to the hexagons from \( T \), then translate it until each corner of \( H \) coincides with the center of a hexagon in \( T \). For an example, see Fig. 5.) Moreover, by the choice of \( t \), these \( 6(\frac{t+1}{2}) + 1 = 3t^2 + 3t + 1 \) hexagons can be turned into a collection of 1-small sets whose union contains \( H \). This results in a proper coloring of \( G \) with \( 3t^2 + 3t + 1 \) colors. Since \( t < 2/3 \cdot \text{pwd}(G) + 1 \), this expression can be bounded from above by \( ((\frac{2}{\sqrt{3}}) \cdot \text{pwd}(G) + \sqrt{7})^2 \).

The upper bound in Lemma 3.6 can be improved by using, instead of the hexagon \( H \), a set similar to a more sophisticated universal cover (a set having a subset congruent to any given plane set of unit diameter); see e.g. [14].
Theorem 3.7.  There exists a constant C > 0 such that for every graph G,
\[
\frac{\sqrt{3}}{2} - C \leq \text{pwd}(G) \leq \sqrt{\frac{2}{\pi}} \sqrt{\chi(G)} + C.
\]

Proof.  Theorem 2.3 together with Corollary 5.2 gives the upper bound. The lower bound follows from Lemma 3.6. ■

Some questions regarding the plane-width of a graph can be answered via chromatic number by applying Theorem 3.7. For instance, the plane-width of almost every random graph (in the $G_{n,p}$ model with a fixed $p \in (0,1)$) is $\Theta(\sqrt{n/\log(n)})$ (since the chromatic number of almost every random graph is $\Theta(n/\log(n))$ [5]). Another example is the existence of graphs of arbitrarily large plane-width and girth (as there are graphs of arbitrarily large chromatic number and girth [12]).

4. PLANE-WIDTH AND CIRCULAR CHROMATIC NUMBER

A c-circular realization of a graph $G$ is a mapping which assigns each vertex of $G$ to a point on a circle of circumference $c$ so that two adjacent vertices are mapped at distance at least 1 as measured along the circumference of the circle. The circular chromatic number of a graph $G$, denoted by $\chi_c(G)$, is defined as
\[
\chi_c(G) = \inf\{c : G \text{ admits a } c\text{-circular realization}\}.
\]

Due to the fact that for all graphs $G$, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, the circular chromatic number can be seen as a refinement of the chromatic number. The circular chromatic number is a well-studied graph parameter (see [24] for a survey). In this section, we will establish a connection between the circular chromatic number and plane-width. This will allow us to apply some known results to obtain the following theorem, which should be viewed as complementary to Theorem 3.1.

Theorem 4.1.  For every $\varepsilon > 0$, there exists
(a) A 4-chromatic graph $G$ such that $\text{pwd}(G) < \frac{2}{\sqrt{3}} + \varepsilon$,
(b) A 5-chromatic graph $G$ such that $\text{pwd}(G) < \sqrt{2} + \varepsilon$,
(c) An 8-chromatic graph $G$ such that $\text{pwd}(G) < 2 + \varepsilon$.

We first use the circular chromatic number of a graph to upper bound its plane-width.

Lemma 4.2.  For all graphs $G$, $\text{pwd}(G) \leq \left[ \sin \left( \frac{\pi}{\chi_c(G)} \right) \right]^{-1}$.

Proof.  Let $c = \chi_c(G)$, and consider a c-circular realization of $G$. (This is possible, since the infimum in the definition of the circular chromatic number is attained [24].) Observe that scaling this circular realization by a factor of
\[
\gamma = \frac{\pi/c}{\sin(\pi/c)}
\]
results in a mapping $r$ of the vertices of $G$ to points on a circle (and thus in the plane) such that the Euclidean distance of the images of two adjacent vertices is at least 1. Therefore, $r$ is a realization of $G$, and its width is at most the diameter of the scaled circle $\gamma \cdot c / \pi = [\sin (\pi / c)]^{-1}$.

Using the above connection, we are able to translate a theorem by Vince about the existence of graphs with arbitrary rational value of the circular chromatic number into a result about the existence of $k$-chromatic graphs with a bounded plane-width.

**Theorem 4.3** (Vince [23]). For every rational number $q \geq 2$, there exists a graph $G$ with $\chi_c(G) = q$.

**Lemma 4.4.** For every $k \geq 3$ and every $\varepsilon > 0$, there exists a $k$-chromatic graph $G$ such that $\text{pwd}(G) < [\sin(\pi / (k-1))]^{-1} + \varepsilon$.

**Proof.** Fix $k \geq 3$ and $\varepsilon > 0$. Let $\delta \in (0, 1)$ be a number such that

$$[\sin(\pi / (k-1+\delta))]^{-1} < [\sin(\pi / (k-1))]^{-1} + \varepsilon.$$ 

Furthermore, let $q$ be a rational number such that $q \in (k-1, k-1+\delta)$. By Theorem 4.3, there exists a graph $G$ of circular chromatic number $q$. Since $k-1 < q < k$ and $\chi(G) = \lceil \chi_c(G) \rceil$, we conclude that $\chi(G) = k$. To upper bound the plane-width of $G$, we use Lemma 4.2:

$$\text{pwd}(G) \leq \frac{1}{\sin \left( \frac{\pi}{\chi_c(G)} \right)} = \frac{1}{\sin \left( \frac{\pi}{q} \right)} \leq \frac{1}{\sin \left( \frac{\pi}{k-1+\delta} \right)} < \frac{1}{\sin \left( \frac{\pi}{k-1} \right)} + \varepsilon.$$ 

The second inequality follows from the fact that the function $f : x \mapsto (\sin(\pi / x))$ is non-decreasing for $x \geq 2$.

Notice that for large $\chi_c(G)$, the bound of Lemma 4.2 (and hence of Lemma 4.4) becomes very weak since it grows linearly in $\chi_c(G)$, whereas Theorem 3.7 tells us that $\text{pwd}(G)$ grows as the square root of $\chi_c(G)$. However, for small $\chi_c(G)$, we can still get meaningful bounds. In particular, we can prove Theorem 4.1.

**Proof of Theorem 4.1.** Parts (a) and (b) are a direct consequence of Lemma 4.4. For part (c), though the Lemma also gives a bound, we can get a tighter one with the following argument.

Given $\varepsilon > 0$, let $C$ be a circle of diameter $2 + \varepsilon$. Let $n \geq 2$ be the smallest positive integer such that

$$(2 + \varepsilon) \sin \left( \frac{n}{6n+1} \cdot \pi \right) \geq 1. \quad (1)$$

Consider a graph $G$ whose vertex set consists of $6n+1$ points $p_1, \ldots, p_{6n+1}$ spread equidistantly on $C$. Two vertices of $G$ are joined by an edge if and only if the Euclidean distance between the corresponding points is at least 1. It follows from Equation (1) that two vertices $p_i$ and $p_j$ with $i < j$ are adjacent if and only if $\min\{j-i, 6n+1+i-j\} \geq n$. By a result of Vince [23], the circular chromatic number of $G$ is $(6n+1)/n = 6 + 1/n$, which implies that $\chi(G) = \lceil \chi_c(G) \rceil = 7$. Now, let $G^*$ be the graph whose vertex set
consists of the center of \( C \), together with the points on \( C \) corresponding to the vertices of \( G \). Moreover, let \( E(G^*) = E(G) \cup E' \), where \( E' \) denotes the set of edges connecting the center of \( C \) to all other vertices. By construction, we have \( \chi(G^*) = \chi(G) + 1 = 8 \). Moreover, the defining collection of points gives an arrangement of \( G^* \) of width less than \( 2 + \varepsilon \).

Notice that Theorem 4.1 implies the existence of graphs \( G \) such that every optimal arrangement of \( G \) is of cardinality strictly greater than \( \chi(G) \).

5. PLANE-WIDTH AND GRAPH OPERATIONS

In this section, we study the effect of certain graph operations on the plane-width.

A. Homomorphisms and Perfect Graphs

A homomorphism of a graph \( G \) to a graph \( H \) is an adjacency-preserving mapping, that is a mapping \( \phi : V(G) \rightarrow V(H) \) such that \( \phi(u)\phi(v) \in E(H) \) whenever \( uv \in E(G) \). We say that a graph \( G \) is homomorphic to a graph \( H \) if there exists a homomorphism of \( G \) to \( H \).

It follows from definitions that \( \text{pwd}(G) \) is the minimum value of \( p \) such that there exists a graph homomorphism from \( G \) to the graph whose vertex set is some subset of \( \mathbb{R}^2 \) of diameter \( p \) and two vertices are adjacent if the Euclidean distance between them is at least 1. This observation together with the transitivity of the homomorphism relation immediately imply the following lemma.

Lemma 5.1. Let \( G \) be a graph homomorphic to a graph \( H \). Then, \( \text{pwd}(G) \leq \text{pwd}(H) \).

Corollary 5.2. For all graphs \( G \), \( \text{pwd}(K_{\omega(G)}) \leq \text{pwd}(G) \leq \text{pwd}(K_{\chi(G)}) \), where \( \omega(G) \) denotes the maximum size of a clique in \( G \).

Proof. Both inequalities follow directly from Lemma 5.1, since \( K_{\omega(G)} \) is homomorphic to \( G \), and \( G \) is homomorphic to \( K_{\chi(G)} \).

Corollary 5.2 establishes the importance of the complete graphs for determining the plane-width of arbitrary graphs. For graphs \( G \) whose chromatic number coincides with their maximum clique size, the plane-width is a function of the chromatic number: \( \text{pwd}(G) = \text{pwd}(K_{\chi(G)}) \). This is the case, for instance, for perfect graphs. Notice however that the relation \( \text{pwd}(G) = \text{pwd}(K_{\chi(G)}) \) also holds for some non-perfect graphs, for example for odd wheels. It is an open question to determine whether for non-bipartite graphs, the chromatic number is a function of the plane-width (see Problem 1).

B. Disjoint Union and Join of Graphs

Given two graphs \( G \) and \( H \), we denote by \( G \uplus H \) the disjoint union of \( G \) and \( H \). Lemma 5.1 yields the lower bound \( \text{pwd}(G \uplus H) \geq \max\{\text{pwd}(G), \text{pwd}(H)\} \). We now give an upper bound.

Theorem 5.3. For every two graphs \( G \) and \( H \),

\[
\text{pwd}(G \uplus H) \leq \max(\text{pwd}(G), \text{pwd}(H), \frac{1}{\sqrt{3}}(\text{pwd}(G) + \text{pwd}(H))).
\]
Proof. We enclose some optimal arrangements of $G$ and $H$ in regular hexagons with opposite sides at a distance of $\text{pwd}(G)$ and $\text{pwd}(H)$, respectively (using the result of Pál [19] as before). We center both hexagons (and their corresponding arrangements) at the origin, and we rotate one of the hexagons so that its edges are parallel to the other one. In this arrangement, the maximum distance is achieved by either two points from $G$, two points from $H$, or from one point in $G$ and one point in $H$. For the last case, this distance is maximized by two points in opposite corners of their respective hexagons, with the distance being the sum of the halves of the diameters of the hexagons. 

Given two graphs $H_1$ and $H_2$, let $H_1 \oplus H_2$ denote the graph (the join of $H_1$ and $H_2$) obtained by making every vertex of $H_1$ adjacent to every vertex of $H_2$.

Theorem 5.4. For every two graphs $H_1$ and $H_2$,

$$\text{pwd}(H_1 \oplus H_2) \leq \text{pwd}(H_1) + \text{pwd}(H_2) + 1.$$  

Proof. For $i = 1, 2$, let $A_i$ be an arrangement of $H_i$, and let $a_i, b_i$ be two points of $A_i$ at distance $\text{pwd}(H_i)$. Place $S_1$ and $S_2$ in such a way that $b_1, a_1, a_2, b_2$ are collinear and placed on the line $\ell$ in this order, with $a_1, a_2$ being at distance 1.

We will show that this is an arrangement of $H_1 \oplus H_2$. Let $\ell_i$ be the line perpendicular to $\ell$ and passing through $a_i$, for $i = 1, 2$. Lines $\ell_1, \ell_2$ divide the plane into three parts. Notice that the part not containing $b_1$ or $b_2$ does not contain any point of $A_1$ or $A_2$, respectively. If it did, the distance between that point and $b_1$ (or $b_2$) would be greater than the diameter of $A_1$ (or $A_2$).

Now we will show that the diameter of $A_1 \cup A_2$ is at most $\text{pwd}(H_1) + \text{pwd}(H_2) + 1$. Consider two points $x_1 \in A_i$, for $i = 1, 2$. From triangle $x_1, a_1, a_2$, the distance between $x_1$ and $a_2$ should be at most $\text{pwd}(H_1) + 1$. Now from the triangle $x_1, x_2, a_2$, the distance between $x_1$ and $x_2$ should be at most $\text{pwd}(H_1) + \text{pwd}(H_2) + 1$. 

6. GENERALIZATIONS

A. Other Norms

One possible generalization of the plane-width is to consider distance measures different from the Euclidean norm. If the plane is equipped with the $\ell_p$ norm, for some $1 \leq p \leq \infty$, we denote the corresponding plane-width of the graph by $\text{pwd}_p(G)$.

Theorem 6.1.

(a) For every graph $G$ and every $p \in [1, \infty)$,

$$2^{-1/p} \sqrt{\chi(G)} - 2^{-1/p} \leq \text{pwd}_p(G) \leq 2^{1/p} \sqrt{\chi(G)}.$$

(b) For every graph $G$, $\sqrt{\chi(G)} - 1 \leq \text{pwd}_\infty(G) \leq \sqrt{\chi(G)}$.

Sketch of Proof. For the lower bound, let us consider an optimal realization of $G$. The image of $V(G)$ through the optimal realization is of diameter $\text{pwd}_p(G)$. For any point set of diameter $d$, there exists an axis-aligned square of side length $d$ containing this set of points. Hence, the points of the optimal arrangement can be enclosed by a...
square of side length $\text{pwd}_p(G)$. Next, we divide the square into 1-small squares arranged in a grid. The number of 1-small squares is an upper bound on the chromatic number of $G$ and the lower bounds in (a) and (b) follow.

For the upper bound, we arrange a $[\sqrt{\chi(G)}] \times [\sqrt{\chi(G)}]$ square grid of points in the plane, where the distance between any two neighboring points in the grid is 1. This grid contains an arrangement of $K_{\chi(G)}$. The length of the diagonal of the grid gives an upper bound on $\text{pwd}_p(K_{\chi(G)})$, and—by Corollary 5.2—an upper bound on $\text{pwd}_p(G)$ for (a) and (b).

\[ \blacksquare \]

B. Other Dimensions

We can also consider realizations of graphs in higher dimensions. When the realization maps vertices of the graph to $\mathbb{R}^d$, the corresponding version of the plane-width is denoted by $\text{pwd}^{(d)}(G)$. It is easy to see that $\text{pwd}^{(1)}(G) = \chi(G) - 1$; hence, we can view $\text{pwd}^{(d)}(G)$ as a multi-dimensional generalization of the chromatic number. In the case of 2 dimensions, as proved in Theorem 3.7, $\text{pwd}^{(2)}(G) = \Theta(\sqrt{\chi(G)})$. One can show—using methods similar to those in the proof of Theorem 6.1—that in general, $\text{pwd}^{(d)}(G) = \Theta(\chi(G)^{1/d})$.

In 1932, Borsuk presented the following conjecture.

**Borsuk’s Conjecture 1** (Borsuk [7]). For every $d \geq 1$, every convex body in $\mathbb{R}^d$ can be partitioned into $d+1$ sets of smaller diameter.

The conjecture was disproved for $d \geq 298$ [17, 15], but holds true for $d = 2$ [6, 7] and for $d = 3$ [10, 20]. As a result, we obtain the following theorem (analogous to Theorem 3.1(a)).

**Theorem 6.2.** For all graphs $G$, $\text{pwd}^{(3)}(G) \leq 1$ if and only if $\chi(G) \leq 4$.

The technique of partitioning sets of unit diameter into $k\delta$-small sets can be applied to three-dimensional point sets to obtain results similar to Theorem 3.1 for $\text{pwd}^{(3)}$.

7. OPEN PROBLEMS

We have introduced the plane-width of a graph and studied some of its basic properties, including its behavior under certain graph operations and its relation to the chromatic number. Though asymptotically we have shown that $\text{pwd}(G) = \Theta(\sqrt{\chi(G)})$, the connection on a finer scale remains unclear. Specifically, we believe the following to be the major open problem:

**Problem 1.** Let $\mathcal{P} = \{\text{pwd}(G) : G \text{ is a graph}\}$. Determine whether there exists a function (a monotone function) $f : \mathcal{P} \to \mathbb{N}$ such that for every non-bipartite graph $G$, $f(\text{pwd}(G)) = \chi(G)$.

We have seen that complete graphs play an important role in bounding the plane-width (Corollary 5.2). The following is therefore a natural subproblem of Problem 1:

**Problem 2.** Determine whether $\text{pwd}(K_n) < \text{pwd}(K_{n+1})$ holds for all $n \geq 3$.
It follows from Corollary 5.2 that for every natural number $k$, the maximum plane-width over all $k$-chromatic graphs is attained by the complete graph on $k$ vertices. We are also interested in obtaining more insight into the opposite question:

**Problem 3.** For $k \geq 2$ what is the value of $\inf \{ \text{pwd}(G) : \chi(G) = k \}$? Which graphs or graph families realize these values?

In Sections 3 and 4, we have shown that the above infima are equal to 1 for $k \in \{2, 3\}$, $\frac{2}{\sqrt{3}}$ for $k = 4$, $\sqrt{2}$ for $k = 5$, and 2 for $k = 8$, and described some graph families realizing these bounds.

**ACKNOWLEDGMENTS**

We are very grateful to Daria Schymura, Carsten Thomassen, and David Wood for their valuable comments, encouragement, and discussions. We also appreciate many helpful comments from the anonymous reviewers, including an idea for simplifying the proof of Proposition 2.5.

M. K. gratefully acknowledges support from the Actions de Recherche Concerťees (ARC) fund of the Communauté française de Belgique and from the Fonds National de la Recherche Scientifique (F.R.S.—FNRS). P. M. gratefully acknowledges the support of the NRW Graduate School, the AG Genominformatik group at Bielefeld, and the German Academic Exchange Service (DAAD) Research Grant. M. M. gratefully acknowledges the support by the group “Combinatorial Search Algorithms in Bioinformatics” funded by the Sofja Kovalevskaja Award 2004 of the Alexander von Humboldt Stiftung and the German Federal Ministry of Research and Education, and by “Agencija za raziskovalno dejavnost Republike Slovenije”, research program P1-0285. Most of the work was done while the second and third authors were at the Universit¨at Bielefeld.

**REFERENCES**

[1] C. Audet, P. Hansen, F. Messine, and S. Perron, The minimum diameter octagon with unit-length sides: Vincze’s wife’s octagon is suboptimal, J Comb Theory Ser A 108 (2004), 63–75.

[2] P. Bateman and P. Erdős, Geometrical extrema suggested by a lemma of Besicovitch, American Math Monthly 58 (1951), 306–314.

[3] M. Belk and R. Connelly, Realizability of graphs, Discrete and Comput Geometry 37 (2007), 125–137.

[4] A. Bezdek and F. Fodor, Minimal diameter of certain sets in the plane, J Comb Theory Ser A 85 (1999), 105–111.

[5] B. Bollobás, The chromatic number of random graphs, Combinatorica 8 (1988), 49–55.

[6] V. Boltjansky and I. Gohberg, Results and Problems in Combinatorial Geometry, Cambridge University Press, Cambridge, 1985.

[7] K. Borsuk, Über die Zerlegung einer Euklidischen $n$-dimensionalen Vollkugel in $n$ Mengen, Verh Internat Math-Kongr Zürich 2 (1932), 192.
[8] P. Carmi, V. Dujmović, P. Morin, and D. R. Wood, Distinct distances in graph drawings, Electronic J Combinatorics 15 (2008), R107.
[9] R. Diestel, Graph Theory, Electronic edn 2005, Springer, Berlin, 2005.
[10] H. G. Eggleston, Covering a three-dimensional set with sets of smaller diameter, J London Math Soc 30 (1955), 11–24.
[11] P. Erdős, Some combinatorial problems in geometry, Geometry and Differential Geometry, Lecture Notes in Mathematics 792, Springer, Berlin, 1980, pp. 46–53.
[12] P. Erdős, Graph theory and probability, Canad J Math 11 (1959), 34–38.
[13] R. L. Graham, B. D. Lubachevsky, K. J. Nurmela, and P. R. J. Östergård, Dense packing s of congruent circles in a circle, Discrete Math 181 (1998), 139–154.
[14] P. Hansen, Small universal covers for sets of unit diameter, Geometriae Dedicata 42 (1992), 205–213.
[15] A. Hinrichs and C. Richter, New sets with large Borsuk numbers, Discrete Math 270 (2003), 137–147.
[16] B. Horvat, T. Pisanski, and A. Žitnik, The dilation coefficient of a complete graph, Croat Chem Acta 82 (2009), 771–779.
[17] J. Kahn and G. Kalai, A counterexample to Borsuk’s conjecture, Bull Amer Math Soc 29 (1993), 60–62.
[18] B. Mohar, Chromatic number of a nonnegative matrix, IMFM Preprint, 39 (2001), 785, in preparation, http://www.imfm.si/preprinti/PDF/00785.pdf
[19] J. Pál, Über ein elementares Variationsproblem, Kgl Danske Vid Selskab Mat Fys Medd 3 (1920), 1–35.
[20] J. Perkal, Sur la subdivision des ensembles en parties de diamètre inférieur, Colloq Math 1 (1947), 45.
[21] T. Pisanski and Žitnik A. Representing graphs and maps, Topics in Topological Graph Theory, Series: Encyclopedia of Mathematics and Its Applications 129, Cambridge University Press, Cambridge, 2009.
[22] A. Schürmann, On extremal finite packings, Discrete Comput Geom 28 (2002), 389–403.
[23] A. Vince, Star chromatic number, J Graph Theory 12 (1988), 551–559.
[24] X. Zhu, Circular chromatic number: A survey, Discrete Math 229 (2001), 371–410.