Power Network System Identification and Recovery Based on the Matrix Completion

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Abstract. Health detection of power networks is a complex and important issue. In order to reduce the detection complexity of the power network, a matrix recovery method based on the matrix completion has been proposed in this paper. Matrix completion has attracted considerable attentions in computer vision, system identification, and machine learning in recent years. This paper presents a survey on algorithms about matrix completion including Singular Value Threshold (SVT), Alternating Direction Method (ADM) and so on. In order to reproduce algorithms about matrix completion and evaluate the performance of these algorithms, this paper is present. The numerical experiments are conducted to evaluate the performance of algorithms. An application about image denoising is also carried out.

1. Introduction

Matrix completion in smart grid aims to recover the information such as power and current of a whole grid from a small number of measuring. Low-rank matrix in smart grid can be reconstructed if the plants or devices are numbered in series with the varying discrete time. Correspondingly, the random values of power or current of one plant in particular time will be filled in the matrix. Through matrix completion, the information of devices at other time will be reconstructed. This way can be used in forecasting load of users, fault location. Using matrix analysis is common in power system operation and control such as random matrix theory system in data processing [1]. In addition, matrix completion is very popular in identification [2], sensor networks [3] and computer vision [4]. In particular, low-rank matrix completion has attracted growing attentions in recommendation systems [5], [6]. Moreover, the structure-from-motion problem can also be modeled as matrix completion problem [7]. In the area of wireless communication, matrix completion is utilized to deal with the joint channel state information acquisition problem [8]. Matrix completion also brings benefits for adaptive beamforming in an impulsive noise environment [9].

In 2009, Candès proved that if the number m of sampled entries obeys

\[ m \geq Cn^5 r \log n \]  \hspace{1cm} (1)

where \( C \) is a positive constant, the matrix \( X \in \mathbb{R}^{n \times n} \) can be recovered, with a very high probability \( 1 - cn^{-3} log n \), by solving a simple convex optimization program [10]. Subsequently, there are many algorithms to deal with the matrix completion problem. In order to recover large scale matrix, the Singular Value Thresholding (SVT) Algorithm was proposed by Cai [11]. In addition, H. Keshavan et. al. proposed the OPTSPACE method to recover the low rank matrix [12]. Furthermore, Fixed Point
Continuation with Approximate (FPCA) SVD was studied by Ma [13]. In 2009, Toh proposed an Accelerated Proximal Gradient Algorithm (APG) for nuclear norm regularized least squares problems [15]. In the same year, Liu studied a Proximal point algorithm for nuclear norm minimization [16]. Afterwards, Alternating Direction Method (ADM) and Robust Alternative Minimization (RAM) for matrix completion were studied by Chen and Lu respectively [7], [14]. This paper is presented for evaluating the performance of these algorithms aforementioned.

2. Problem formulation
The design formulation for matrix completion problem is given by
\[
\min \text{rank}(X) \\
\text{subject to } X_{ij} = M_{ij} \ (i,j) \in \Omega
\]
(2)
where \( M \in \mathbb{R}^{n \times n} \) is square with low-rank \( r \). \( m \) entries \( \{M_{ij} : (i,j) \in \Omega\} \) are sample from a random subset \( \Omega \). In some sense, the problem formulation indicated by Eq. (2) is an NP-hard rank minimization problem.

Similar to Compressed Sensing (CS), the tight convex relation can be acquired by replacing rank \( \text{rank}(X) \) by \( \|X\|_* \). The reason is that the nuclear norm \( \{X: \|X\|_* \leq 1\} \) is the convex hull of the set of rank-one matrices. Therefore, the problem formulation of matrix completion can be rewritten as
\[
\min \|X\|_* \\
\text{subject to } X_{ij} = M_{ij} \ (i,j) \in \Omega
\]
(3)
Where \( \|X\|_* \) denotes the nuclear norm of \( X \).

\[
\|X\|_* = \sum_{i=1}^{n} \sigma_i(X)
\]
(4)
Where \( \sigma_i(X) \) is the \( i \)th largest singular value of \( X \). \( P_\Omega \) is defined as
\[
P_\Omega(X) = X_{ij} \ (i,j) \in \Omega
P_\Omega(X) = 0 \text{ elsewhere}
\]
(5)
Consequently, the problem can be expressed as
\[
\min \|X\|_* \\
\text{subject to } P_\Omega(X) = P_\Omega(M)
\]
(6)
The problem indicated by Eq. (6) is a convex problem, which can be easily solved by CVX or SeDuMi in MATLAB [11].

3. Algorithms

3.1. Single Value Decomposition
Based on SVD, a naive algorithm was proposed. Considering the SVD of \( P_\Omega(X) \), we can obtain
\[
M^E = \sum_{i=1}^{n} \sigma_i u_i v_i^T
\]
(7)
if we only consider the singular value from 1st to \( r \)th, the matrix can be recovered by
\[
\hat{M} = \frac{n^2}{m} \sum_{i=1}^{r} \sigma_i u_i v_i^T
\]
(8)

3.2. Single Value Thresholding
In Ref. [12], Cai showed that the sequence \( \{X^k\} \) of the following problem converges to unique solution of the problem formulation indicated by Eq. (6).
\[
\min \|X\|_* + 0.5 \|X\|_F^2 \\
\text{subject to } P_\Omega(X) = P_\Omega(M)
\]
(9)
where \( \|X\|_F^2 = \text{tr}(X^TX) \). Furthermore, the Lagrangian for this problem is given by
The saddle point of the Lagrangian $L(X, Y)$ is solved by two-line iterations [12].

$$X^k = D_t(Y^{k-1})$$

$$Y^k = Y^{k-1} + \delta_k P_\Omega(M - X^k)$$  \hspace{1cm} \text{(11)}$$

where

$$D_t(\Sigma) = \text{diag}\{(\sigma_i - \tau)^+\}$$  \hspace{1cm} \text{(13)}$$

$\sigma_i$ denotes the nonzero singular value of $X$ and $t_+ = \max(0, t)$.

### 3.3. Fixed Point Continuation

Considering the sampled matrix with noise, the problem formulation can be expressed as

$$\min \|X\|_*$$

subject to $\|P_\Omega(X) - P_\Omega(M)\|_2^2$. \hspace{1cm} \text{(14)}$$

and its Lagrangian version can be written as

$$\min \mu \|X\|_* + 0.5 \|P_\Omega(X) - P_\Omega(M)\|_2^2.$$  \hspace{1cm} \text{(15)}$$

The fixed point iterative algorithm for solving Eq. (15) is the following two-line algorithm.

$$Y^k = X^k - \tau P_\Omega(X^k - M)$$

$$X^{k+1} = D_t(\mu Y^k)$$  \hspace{1cm} \text{(16)}$$

### 3.4. Accelerate Proximal Gradient

Similar to FPCA, the APG algorithms consider the nuclear norm regularized linear least squares problem. Specifically, the APG algorithm is shown in Alg. 1.

### Algorithm 1 Accelerate Proximal Gradient

**Initialize** $\mu > 0, X^0 \in \mathbb{R}^{n \times n}, t^0 = 1$

**for** $k = 0, 1, 2, ...$

**do**

$$Y^k = X^k + \frac{\tau^{k-1} - 1}{\tau^k}(X^k - X^{k-1})$$

$$G^k = Y^k - \frac{1}{\tau^k}(P_\Omega(Y^k) - P_\Omega(M))$$

$$X^{k+1} = D_{\mu/\tau^k}(G^k)$$

$$t^{k+1} = \frac{1 + \sqrt{1 + 4(t^k)^2}}{2}$$

**end for**

### 3.5. Alternating Direction Method

Similar to FPCA and APG, the sampled matrix with noise is considered in ADM algorithm. However,
the penalty factor is considered in ADM. To be specific, Eq. (14) can be reformulated into the linearly constrained problem as the following.

\[
\begin{align*}
\text{minimize} & \quad \|X\|_* \\
\text{subject to} & \quad X - Y = 0 \\
X & \in \mathbb{R}^{n \times n} \\
Y & \in \mathbb{U} := \{ U \in \mathbb{R}^{n \times n} | \| \mathcal{P}_\Omega(X) - \mathcal{P}_\Omega(M) \|_2 \leq \theta \} 
\end{align*}
\]

Thus, the Lagrangian function is

\[
\mathcal{L}(X, Y, Z, \beta) = \|X\|_* + \tau \left( Z^T (X - Y) \right) + \frac{\beta}{2} \|X - Y\|^2
\]

Then, the ADM algorithm can be described in Alg. 2.

4. Implementation issues

In the implementation of the aforementioned algorithms, the stopping criterion for Singular Value Thresholding (SVT) is

\[
\frac{\| \mathcal{P}_\Omega(X^k - M) \|_F}{\| \mathcal{P}_\Omega(M) \|_F} \leq \text{tol}
\]

while the stopping criterion for Fixed Point Continuation (FPC) and Alternating Direction Method is

\[
\frac{\| X^{k+1} - X^k \|_F}{\| X^k \|_F} \leq \text{tol}
\]

Algorithm 2 Alternating Direction Method

\[
\begin{align*}
\text{Initialize} & \quad \beta > 0, X^0, Y^0, Z^0, \gamma \in (0, \frac{\sqrt{5} + 1}{2}) \\
& \quad \text{for} \ k = 0, 1, 2, \ldots \\
& \quad \text{do} \\
& \quad \quad Y^{k+1} = M_{ij} \text{ if } (i, j) \in \Omega, Y^k = X^k - \frac{1}{\beta} Z^k \text{ if } (i, j) \notin \Omega \\
& \quad \quad A^{k+1} = Y^{k+1} - \frac{1}{\beta} Z^k \\
& \quad \quad X^{k+1} = \mathcal{D}_{1/\beta}(A^{k+1}) \\
& \quad \quad Z^{k+1} = Z^k - \gamma \beta(X^{k+1} - Y^{k+1}) \\
& \quad \quad \text{end for}
\end{align*}
\]

where \( \text{tol} \) is the given accuracy. Moreover, the stopping condition for Accelerate Proximal Gradient (APG) is

\[
\frac{\| S^k \|_F}{\tau^k \max\{1, \| X^k \|_F\}} \leq \text{tol}
\]

where

\[
S^{k+1} = \tau^k (Y^k - X^{k+1}) + \mathcal{P}_\Omega(X^{k+1} - Y^k)
\]

Additionally, the accuracy of the matrix completion is accessed by
5. Numerical experiments

To evaluate the performance of different algorithms, we conducted a series of numerical experiments for a variety of matrix sizes $n$, ranks $r$ and numbers of entries $m$. Each algorithm is repeated 100 times to get the average. We generated $M$, an $n \times n$ matrix of rank $r$, by generating two $n \times r$ factors $M_L$ and $M_R$ with independent identical distribution Gaussian entries and setting $M = M_L M_R^T$. The performance of different algorithms with different ranks is illustrated in Table 1.

Table 1. ALGORITHMS COMPARISON.

| Algorithms | Rank | Time  | RERR    | ITER |
|------------|------|-------|---------|------|
| CVX 3      | 12.81| $2.0 \times 10^{-9}$ |       |      |
| CVX 5      | 12.78| $4.7 \times 10^{-4}$ |       |      |
| CVX 7      | 14.53| $7.2 \times 10^{-3}$ |       |      |
| CVX 9      | 14.20| $5.3 \times 10^{-2}$ |       |      |
| CVX 11     | 13.32| 0.19  |         |      |
| SVT 3      | 7.24 | $9.76 \times 10^{-4}$ | 1021   |      |
| SVT 5      | 30.32| $6.20 \times 10^{-3}$ | 4268   |      |
| SVT 7      | 66.06| $8.13 \times 10^{-2}$ | 9993   |      |
| SVT 9      | 65.46| 0.27  |         |      |
| SVT 11     | 65.68| 0.42  |         |      |
| FPC 3      | 0.12 | 0.75  |         | 16    |
| FPC 5      | 0.12 | 0.76  |         | 16    |
| FPC 7      | 0.11 | 0.77  |         | 16    |
| FPC 9      | 0.10 | 0.77  |         | 16    |
| FPC 11     | 0.11 | 0.78  |         | 16    |
| APG 3      | 0.18 | 0.63  |         | 22    |
| Method | r | s | time (s) | cost (10^{-4}) |
|-------|---|---|----------|----------------|
| APG   | 5 | 0.17 | 0.66 | 22 |
| APG   | 7 | 0.16 | 0.69 | 21.9 |
| APG   | 9 | 0.15 | 0.70 | 21.7 |
| APG   | 11 | 0.15 | 0.71 | 21.3 |
| ADM   | 3 | 0.21 | 1.48 | 30.7 |
| ADM   | 5 | 0.23 | 9.29 | 33.5 |
| ADM   | 7 | 0.38 | 1.15 | 60.2 |
| ADM   | 9 | 0.61 | 5.80 | 96.1 |
| ADM   | 11 | 0.49 | 0.19 | 77.1 |

In this simulation, the matrix size is 100 × 100, and the sample number of entries \( m \) is 3500.

The next simulation is image recovery. Figure 1 is the Original Figure with 512 × 512.

Figure 2 is the Sampling Figure with \( m = 26000 \).

Figure 3 is the Recovery Figure with ADM algorithms. The initial parameters are same to the first numerical experiments of ADM algorithms. We can recover the figure with 10% sampling entries.

6. Conclusion

In this paper, series of algorithms about matrix completion are presented. Meanwhile, the metric of performance evaluation is also given. The numerical experiments are conducted to evaluate the performance of algorithms. An application about image denoising is also carried out. A power network with enormous data which conforms to the characteristic of these algorithms can be identified and recovered in this way.
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