Genuine infinitesimal bendings of submanifolds

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Abstract

A basic question in submanifold theory is whether a given isometric immersion \( f: M^n \to \mathbb{R}^{n+p} \) of a Riemannian manifold of dimension \( n \geq 3 \) into Euclidean space with low codimension \( p \) admits, locally or globally, a genuine infinitesimal bending. That is, if there exists a genuine smooth variation of \( f \) by immersions that are isometric up to the first order. Until now only the hypersurface case \( p = 1 \) was well understood. We show that a strong necessary local condition to admit such a bending is the submanifold to be ruled and give a lower bound for the dimension of the rulings. In the global case, we describe the situation of compact submanifolds of dimension \( n \geq 5 \) in codimension \( p = 2 \).

An isometric immersion \( f: M^n \to \mathbb{R}^{n+p} \) of an \( n \)-dimensional Riemannian manifold \( M^n \) into Euclidean space with codimension \( p \) is called isometrically bendable if there is a non-trivial smooth variation \( \mathcal{F}: I \times M^n \to \mathbb{R}^{n+p} \) of \( f \) for an interval \( 0 \in I \subset \mathbb{R} \) such that \( f_t = \mathcal{F}(t, \cdot): M^n \to \mathbb{R}^{n+p} \) with \( f_0 = f \) is an isometric immersion for any \( t \in I \), that is, the metrics \( g_t \) induced by \( f_t \) satisfy \( g_t = g_0 \). The bending being trivial means that the variation is the restriction to the submanifold of a smooth one-parameter family of isometries of \( \mathbb{R}^{n+p} \).

The study of bendings of surfaces \( M^2 \) in \( \mathbb{R}^3 \) was a hot topic between geometers in the 19th century. Initially, there was no distinction between isometric variations and the ones that are only infinitesimally isometric, but that changed due to the work of Darboux by the end of that century. For a modern account of some aspects of the subject we refer to Spivak [22].

The study of isometric bendings of hypersurfaces \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3 \), goes back to the first part of the last century. In fact, the local classification of
isometrically bendable hypersurfaces is due to Sbrana \[20\] in 1909 and Cartan \[2\] in 1916. For a modern presentation of their parametric classifications, as well as for further results, see \[6\] or \[10\]. In the global case, the classification is due to Sacksteder \[18\] for compact hypersurfaces and to Dajczer and Gromoll \[7\] in the case of complete hypersurfaces.

The classical concept of an infinitesimal bending of an isometric immersion \(f : M^n \to \mathbb{R}^{n+p}\) is the infinitesimal analogue of an isometric bending and refers to smooth variations \(\mathcal{F} : I \times M^n \to \mathbb{R}^{n+p}\) that preserve lengths “up to the first order”, that is, the metrics \(g_t\) induced by \(f_t = \mathcal{F}(t, \cdot) : M^n \to \mathbb{R}^{n+p}\) satisfy \(g'_t(0) = 0\). The variational vector field \(\tau = \mathcal{F}_s \partial/\partial t|_{t=0}\) verifies

\[
\langle f_\ast X, \tau_\ast X \rangle = 0
\]

for any tangent vector fields \(X \in \mathfrak{X}(M)\). Clearly (1) is the condition for a smooth variation to preserve the metric up to the first order. If \(\tau\) is an immersion, it was said classically that the pair of submanifolds \(f\) and \(\tau\) correspond with orthogonality of corresponding linear elements; see Bianchi \[1\] or Eisenhart \[12\].

We say that a section \(\tau\) of \(f^\ast T\mathbb{R}^{n+p}\) is an infinitesimal bending of an isometric immersion \(f : M^n \to \mathbb{R}^{n+p}\) if (1) holds. Given a smooth variation whose variational vector field \(\tau\) is an infinitesimal bending, by keeping only the terms of first order of the variation we obtain the smooth variation \(\mathcal{F} : \mathbb{R} \times M^n \to \mathbb{R}^{n+p}\) with variational vector field \(\tau\) defined by \(f_t = f + t\tau\). Then (1) gives

\[
\|f_\ast sX\|^2 = \|f_\ast X\|^2 + t^2\|\tau_\ast X\|^2
\]

for any \(X \in TM\).

Of course, we always have the trivial infinitesimal bendings obtained as the variational vector field of a smooth variation by isometries of the ambient space. In other words, they are locally the restriction to the submanifold of a Killing vector field of the ambient space.

Dajczer and Rodríguez \[9\] showed that submanifolds in low codimension are generically infinitesimally rigid, that is, only trivial infinitesimal bendings are possible. In fact, they proved that well-known algebraic conditions on the second fundamental form of an immersion that give isometric rigidity also yield infinitesimal rigidity. For instance, for a hypersurface \(f : M^n \to \mathbb{R}^{n+1}\) to be infinitesimally bendable it is a necessary condition (but far from sufficient) to have at most two nonzero principal curvatures at any point. This result is already contained in the book of Cesàro \[3\] published in 1886. For higher
codimension the rather strong algebraic conditions are given in terms of the type number or the s-nullities of the immersion.

After the pioneering work of Sbrana [19] in 1908, a complete parametric local classification of the infinitesimally bendable hypersurfaces was given by Dajczer and Vlachos [11]. In particular, they showed that this class is much larger than the class of isometrically bendable ones, a fact that may be seen as a surprise. The classification in the case of complete hypersurfaces was obtained by Jimenez [15]. Infinitesimal bendings of submanifolds have also been considered by Schouten [21] in 1928.

When trying to understand the geometry of the infinitesimally bendable submanifolds in codimension larger than one the following fact has to be taken into consideration. If $\tilde{\tau}$ is an infinitesimal bending of an isometric immersion $F: \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$, $0 < \ell < p$, and $j: M^n \to \tilde{M}^{n+\ell}$ is an embedding, then $\tau = \tilde{\tau}|_{j(M)}$ is an infinitesimal bending of $f = F \circ j: M^n \to \mathbb{R}^{n+p}$. This basic observation motivates the following definitions where a more general situation is considered since certain singularities are allowed.

A smooth map $F: \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$, $0 < \ell < p$, from a differentiable manifold into Euclidean space is said to be a singular extension of a given isometric immersion $f: M^n \to \mathbb{R}^{n+p}$ if there is an embedding $j: M^n \to \tilde{M}^{n+\ell}$, $0 < \ell < p$, such that $F$ is an immersion along $\tilde{M}^{n+\ell} \setminus j(M)$ and $f = F \circ j$. Notice that the map $F$ may fail (but not necessarily) to be an immersion along points of $j(M)$. We say that an infinitesimal bending $\tau$ of an isometric immersion $f: M^n \to \mathbb{R}^{n+p}$ extends in the singular sense if there is a singular extension $F: \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$ of $f$ and a smooth map $\tilde{\tau}: \tilde{M}^{n+\ell} \to \mathbb{R}^{n+p}$ such that $\tilde{\tau}$ is an infinitesimal bending of $F|_{\tilde{M} \setminus j(M)}$ and $\tau = \tilde{\tau}|_{j(M)}$.

We point out that the necessity to admit the existence of singularities of $F$ along $j(M)$ in the above definitions was already well established in [8] and [14] for isometric bendings in both the local and global situation.

An infinitesimal bending $\tau$ of an isometric immersion $f: M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, is called a genuine infinitesimal bending if $\tau$ does not extend in the singular sense when restricted to any open subset of $M^n$. If $f$ admits such a bending we say that it is genuinely infinitesimally bendable. As one expects, trivial infinitesimal bending are never genuine. If $f(M) \subset \mathbb{R}^{n+\ell} \subset \mathbb{R}^{n+p}$, $\ell < p$, and $e \in \mathbb{R}^{n+p}$ is orthogonal to $\mathbb{R}^{n+\ell}$ then $\tau = \phi e$ for $\phi \in C^\infty(M)$ is another example of an infinitesimal bending that is not genuine.

Recall that an isometric immersion $f: M^n \to \mathbb{R}^{n+p}$ is said to be $r$-ruled if there exists an $r$-dimensional smooth totally geodesic tangent distribution
whose leaves (rulings) are mapped diffeomorphically by \( f \) to open subsets of affine subspaces of \( \mathbb{R}^{n+p} \).

**Theorem 1.** Let \( f: M^n \to \mathbb{R}^{n+p}, n > 2p \geq 4, \) be an isometric immersion and let \( \tau \) be an infinitesimal bending of \( f \). Then along each connected component of an open and dense subset either \( \tau \) extends in the singular sense or \( f \) is \( r \)-ruled with \( r \geq n - 2p \).

The following is an immediate consequence of the above result.

**Corollary 2.** Let \( f: M^n \to \mathbb{R}^{n+p}, n > 2p \geq 4, \) be a genuinely infinitesimally bendable isometric immersion. Then \( f \) is \( r \)-ruled with \( r \geq n - 2p \) along connected components of an open dense subset of \( M^n \).

We say that \( f: M^n \to \mathbb{R}^{n+p} \) is genuinely infinitesimally rigid if given any infinitesimal bending \( \tau \) of \( f \) there is an open dense subset of \( M^n \) such that \( \tau \) restricted to any connected component extends in the singular sense.

Theorem 1 also has the following two consequences.

**Corollary 3.** Let \( f: M^n \to \mathbb{R}^{n+p}, n > 2p \geq 4, \) be an isometric immersion. If \( M^n \) has positive Ricci curvature then \( f \) is genuinely infinitesimally rigid.

**Corollary 4.** Let \( g: M^n \to S^{n+p-1}, n > 2p \geq 4, \) be an isometric immersion and let \( f = i \circ g \) where \( i: S^{n+p-1} \to \mathbb{R}^{n+p} \) denotes the umbilical inclusion. Then \( f \) is genuinely infinitesimally rigid.

A special class of ruled submanifolds are the ones with a relative nullity foliation. The relative nullity subspace \( \Delta(x) \) of \( f: M^n \to \mathbb{R}^{n+p} \) at \( x \in M^n \) is the kernel of the second fundamental form \( \alpha: TM \times TM \to N_fM \) with values in the normal bundle, that is,

\[
\Delta(x) = \{ X \in T_xM : \alpha(X,Y) = 0 \text{ for all } Y \in T_xM \}.
\]

The dimension \( \nu(x) \) of \( \Delta(x) \) is called the index of relative nullity of \( f \) at \( x \in M^n \). It is a standard fact that the submanifold is ruled by the leaves of the relative nullity distribution on any open subset of \( M^n \) where the index of relative nullity \( \nu > 0 \) is constant.

In the case of low codimension, with a substantial additional effort we obtain a better lower bound for the dimension of the rulings.
Theorem 5. Let $f: M^n \to \mathbb{R}^{n+p}$, $n > 2p$, be a genuinely infinitesimally bendable isometric immersion. If $2 \leq p \leq 5$, then one of the following holds along any connected component, say $U$, of an open dense subset of $M^n$:

(i) $f|_U$ is $\nu$-ruled by leaves of relative nullity with $\nu \geq n - 2p$.

(ii) $f|_U$ has $\nu < n - 2p$ at any point and is $r$-ruled with $r \geq n - 2p + 3$.

For $p = 2$ notice that we are always in case (i) since a $(n - 1)$-ruled submanifold in that codimension has index of relative nullity $\nu \geq n - 3$ at any point.

Dajczer and Gromoll [8] proved that along connected components of an open dense subset an isometrically deformable compact Euclidean submanifold of dimension at least five and codimension two is either isometrically rigid or is contained in a deformable hypersurface (with possible singularities) and any isometric deformation of the former is given by an isometric deformation of the latter. This result was extended by Florit and Guimarães [14] to other low codimensions. The next result of similar nature concerns infinitesimal bendings of submanifolds in codimension two.

Theorem 6. Let $f: M^n \to \mathbb{R}^{n+2}$, $n \geq 5$, be an isometric immersion of a compact Riemannian manifold with no open flat subset. For any infinitesimal bending $\tau$ of $f$ one of the following holds along any connected component, say $U$, of an open dense subset of $M^n$:

(i) The infinitesimal bending $\tau|_U$ extends in the singular sense.

(ii) There is an orthogonal splitting $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \oplus \text{span}\{e\}$ so that $f(U) \subset \mathbb{R}^{n+1}$ and $\tau|_U = \tau_1 + \tau_2$ is a sum of infinitesimal bendings that extend in the singular sense where $\tau_1 \in \mathbb{R}^{n+1}$ and $\tau_2 = \phi e$ for $\phi \in C^\infty(U)$.

It follows from the proof that the assumption on the open flat subset can be replaced by the weaker hypothesis that there is no open subset of $M^n$ where the index of relative nullity satisfies $\nu \geq n - 1$. Moreover, we will see that cases (i) and (ii) are not disjoint.

In the last section of the paper, we discuss why the local results given above also hold if the ambient space is a nonflat space form.
1 The associated tensor

In this section, we discuss several properties of a tensor associated to an infinitesimal bending called in the classical theory of surfaces the associated rotation field; for instance see [22]. For basic facts on infinitesimal bendings we refer to [9], [10], [11] and [17].

In the sequel, let $\tau$ denote an infinitesimal bending of a isometric immersion $f: M^n \to \mathbb{R}^{n+p}$. Then the section $L \in \Gamma(\text{Hom}(TM, f^*T\mathbb{R}^{n+p}))$ is the tensor defined as

$$LX = \tilde{\nabla}_X \tau$$

where $\tilde{\nabla}$ is the Levi-Civita connection in $\mathbb{R}^{n+p}$. Hence (1) can be written as

$$\langle LX, f_*Y \rangle + \langle LY, f_*X \rangle = 0 \quad (2)$$

for any $X, Y \in \mathfrak{X}(M)$.

Let $B: TM \times TM \to f^*T\mathbb{R}^{n+p}$ the symmetric tensor defined as

$$B(X, Y) = (\tilde{\nabla}_X L)Y$$

for any $X, Y \in \mathfrak{X}(M)$. If $\tau$ is an immersion notice that $B$ is nothing else than its second fundamental form.

**Proposition 7.** The tensor $B$ satisfies

$$(\tilde{\nabla}_X B)(Y, Z) - (\tilde{\nabla}_Y B)(X, Z) = -LR(X, Y)Z \quad (3)$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

**Proof.** Use that

$$(\tilde{\nabla}_X B)(Y, Z) = \tilde{\nabla}_X (\tilde{\nabla}_Y L)Z - (\tilde{\nabla}_{\nabla_X Y} L)Z - (\tilde{\nabla}_Y L)\nabla_X Z \quad (4)$$

and the definition of the curvature tensor. \qed

The metrics $g_t$ induced by $f_t = f + t\tau$ satisfy

$$\partial/\partial t|_{t=0} g_t(X, Y) = 0 \quad (5)$$

for any $X, Y \in \mathfrak{X}(M)$. Hence, the Levi-Civita connections and curvature tensors of $g_t$ verify

$$\partial/\partial t|_{t=0} \nabla^t_X Y = 0 \quad (6)$$
and
\[ \frac{\partial}{\partial t}|_{t=0} g_t(R^t(X,Y)Z,W) = 0 \]  
for any \( X, Y, Z, W \in \mathfrak{X}(M) \). Taking the derivative with respect to \( t \) at \( t = 0 \) of the Gauss formula for \( f_t \), namely, of
\[ \tilde{\nabla}_X f_t Y = f_t \nabla'_X Y + \alpha'(X,Y), \]
we obtain
\[ B(X,Y) = \frac{\partial}{\partial t}|_{t=0} \alpha'(X,Y). \]  
(8)

Taking tangent and normal components with respect to \( f \) we have
\[ B(X,Y) = f_\ast y(X,Y) + \beta(X,Y) \]
where the tensors \( y : TM \times TM \to TM \) and \( \beta : TM \times TM \to N_f M \) are also symmetric.

**Proposition 8.** The tensor \( y : TM \times TM \to TM \) satisfies
\[ \langle \alpha(X,Y), LZ \rangle + \langle y(X,Y), Z \rangle = 0 \]  
for any \( X, Y, Z \in \mathfrak{X}(M) \).

**Proof.** Given \( \eta(t) \in \Gamma(N_f M) \), let \( y_\eta \) be the tangent vector field given by
\[ f_\ast y_\eta = (\frac{\partial}{\partial t}|_{t=0} \eta(t)) f_\ast TM. \]
The derivative of \( \langle f_\ast Z, \eta(t) \rangle = 0 \) with respect to \( t \) at \( t = 0 \) yields
\[ \langle \eta, LZ \rangle + \langle y_\eta, Z \rangle = 0 \]
where \( Z \in \mathfrak{X}(M) \) and \( \eta = \eta(0) \). In particular,
\[ \langle \alpha(X,Y), LZ \rangle + \langle y_{\alpha(X,Y)}, Z \rangle = 0 \]
for any \( X, Y, Z \in \mathfrak{X}(M) \). On the other hand, we obtain from (8) that
\[ y_{\alpha(X,Y)} = y(X,Y) \]
for any \( X, Y \in \mathfrak{X}(M) \). \( \square \)
Proposition 9. The tensor \( \beta : TM \times TM \rightarrow N_fM \) satisfies
\[
\langle \beta(X, W), \alpha(Y, Z) \rangle + \langle \alpha(X, W), \beta(Y, Z) \rangle = \langle \beta(X, Z), \alpha(Y, W) \rangle + \langle \alpha(X, Z), \beta(Y, W) \rangle \tag{10}
\]
and
\[
(\nabla_X^\perp \beta)(Y, Z) - (\nabla_Y^\perp \beta)(X, Z) = \alpha(Y, \mathcal{Y}(X, Z)) - \alpha(X, \mathcal{Y}(Y, Z)) - (LR(X, Y)Z)_{N_fM} \tag{11}
\]
for any \( X, Y, Z, W \in \mathfrak{X}(M) \).

Proof. To prove (10) take the derivative with respect to \( t \) at \( t = 0 \) of the Gauss equations for \( f_t \), that is, of
\[
g_t(R^t(X, Y)Z, W) = g_t(\alpha^t(X, W), \alpha^t(Y, Z)) - g_t(\alpha^t(X, Z), \alpha^t(Y, W))
\]
and use (5), (7) and (8).

Using (4) we have
\[
((\tilde{\nabla}_X f)(Y, Z))_{N_fM} = \alpha(X, \mathcal{Y}(Y, Z)) + (\nabla_X^\perp \beta)(Y, Z)
\]
and (11) follows from (3). \( \square \)

We discuss next the simplest examples of infinitesimal bendings.

Examples 10. (1) If \( \tau \) is a trivial infinitesimal bending of \( f : M^n \rightarrow \mathbb{R}^{n+p} \), \( p \geq 2 \), then we have from the references that
\[
\tau = Df(x) + w
\]
where \( D \) is a skew-symmetric linear transformation of \( \mathbb{R}^{n+p} \) and \( w \in \mathbb{R}^{n+p} \). Take \( \lambda \in \Gamma(f^*\mathbb{R}^{n+p}) \) such that \( F : \tilde{M}^{n+1} = M^n \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+p} \), given by \( F(x, t) = f(x) + t\lambda(x) \), is an immersion for \( t \neq 0 \). Then \( \tau \) extends in the singular sense since
\[
\tilde{\tau}(x, t) = \tau + tD\lambda
\]
is a (trivial) infinitesimal bending of \( F \) on the open subset where \( F \) is an immersion.

(2) The first normal space of \( f : M^n \rightarrow \mathbb{R}^{n+p} \) at \( x \in M^n \) is
\[
N_1(x) = \text{span}\{\alpha(X, Y) : X, Y \in T_xM\}.
\]
Then \( \tau = f_*Z + \delta \) is an infinitesimal bending if \( Z \in \mathfrak{X}(M) \) is a Killing field and \( \delta \in \Gamma(N_{1+}) \) is a smooth normal vector field.
2 Flat bilinear forms

Flat bilinear forms were introduced by J. D. Moore [16] after the pioneering work of E. Cartan to deal with rigidity questions on isometric immersions in space forms. In this paper, it is shown that they are also very helpful in the study of similar questions for infinitesimal bendings of submanifolds.

Let $V$ and $U$ be finite dimensional real vector spaces and let $W^{p,q}$ be a real vector space of dimension $p+q$ endowed with an indefinite inner product of type $(p,q)$. A bilinear form $B : V \times U \to W^{p,q}$ is said to be flat if

$$\langle B(X, Z), B(Y, W) \rangle - \langle B(X, W), B(Y, Z) \rangle = 0$$

for all $X, Y \in V$ and $W, Z \in U$. Then $X \in V$ is called a (left) regular element of $B$ if

$$\dim B_X(U) = \max \{ \dim B_Y(U) : Y \in V \}$$

where $B_X(Y) = B(X, Y)$ for any $Y \in U$. The set $RE(B)$ of regular elements of $B$ is open dense in $V$.

The following basic fact was given in [16].

**Lemma 11.** Let $B : V \times U \to W$ be a flat bilinear form. If $Y \in RE(B)$ then

$$B(X, \ker B_Y) \subset B_Y(U) \cap B_Y(U)$$

for any $X \in V$.

The next is a fundamental result in the theory of symmetric flat bilinear forms. It turns out to be false for $p \geq 6$ as shown in [5].

**Lemma 12.** Let $B : V^n \times V^n \to W^{p,q}$, $p \leq 5$ and $p + q < n$, be a symmetric flat bilinear form and set

$$N(B) = \{ X \in V : B(X, Y) = 0 \text{ for all } Y \in V \}.$$ 

If $\dim N(B) \leq n - p - q - 1$ then there is an orthogonal decomposition

$$W^{p,q} = W^{p-\ell,q-\ell}_1 \oplus W^{p-\ell,q-\ell}_2, \quad 1 \leq \ell \leq p,$$

such that the $W_j$-components $B_j$ of $B$ satisfy:
(i) \( \mathcal{B}_1 \) is nonzero and
\[
\langle \mathcal{B}_1(X, Y), \mathcal{B}_1(Z, W) \rangle = 0
\]
for all \( X, Y, Z, W \in V \).

(ii) \( \mathcal{B}_2 \) is flat and \( \dim \mathcal{N}(\mathcal{B}_2) \geq n - p - q + 2\ell \).

Proof. See [4] or [10].

3 The local results

In this section we give the proofs the local theorems in the introduction. A key ingredient is the following result due to Florit and Guimarães [14].

Proposition 13. Let \( f : M^n \to \mathbb{R}^{n+p} \) be an isometric immersion and let \( D \) be a smooth tangent distribution of dimension \( d > 0 \). Assume that there does not exist an open subset \( U \subset M^n \) and \( Z \in \Gamma(D|_U) \) such that the map \( F : U \times \mathbb{R} \to \mathbb{R}^{n+p} \) given by
\[
F(x, t) = f(x) + tf_*Z(x)
\]
is a singular extension of \( f \) on some open neighborhood of \( U \times \{0\} \). Then for any \( x \in M^n \) there is an open neighborhood \( V \) of the origin in \( D(x) \) such that \( f_*V \subset f(M) \). Hence \( f \) is \( d \)-ruled along each connected component of an open dense subset of \( M^n \).

Proof. See [10] or [14].

3.1 The first local result

We first associate to an infinitesimal bending a flat bilinear form.

Lemma 14. Let \( \tau \) be an infinitesimal bending of an isometric immersion \( f : M^n \to \mathbb{R}^{n+p} \). Then the bilinear form \( \theta : TM \times TM \to N_fM \oplus N_fM \) defined at any point of \( M^n \) by
\[
\theta(X, Y) = (\alpha(X, Y) + \beta(X, Y), \alpha(X, Y) - \beta(X, Y))
\]
is flat with respect to the inner product in \( N_fM \oplus N_fM \) given by
\[
\langle (\xi_1, \eta_1), (\xi_2, \eta_2) \rangle_{N_fM \oplus N_fM} = \langle \xi_1, \xi_2 \rangle_{N_fM} - \langle \eta_1, \eta_2 \rangle_{N_fM}.
\]
Proof. A straightforward computation shows that
\[
\frac{1}{2} ( \langle \theta(X, Z), \theta(Y, W) \rangle - \langle \theta(X, W), \theta(Y, Z) \rangle ) = \langle \beta(X, Z), \alpha(Y, W) \rangle \\
+ \langle \alpha(X, Z), \beta(Y, W) \rangle - \langle \beta(X, W), \alpha(Y, Z) \rangle - \langle \alpha(X, W), \beta(Y, Z) \rangle,
\]
and the proof follows from (10).

An isometric immersion \( f : M^n \to \mathbb{R}^{n+p} \) is called 1-regular if the first normal spaces \( N_1(x) \) have constant dimension \( k \leq p \) on \( M^n \) and thus form a subbundle \( N_1 \) of rank \( k \) of the normal bundle. Under the 1-regularity assumption we have the following equivalent statement.

Lemma 15. Assume that \( f \) is 1-regular and let \( \beta_1 : TM \times TM \to N_1 \) be the \( N_1 \)-component of \( \beta \). Then the bilinear form \( \hat{\theta} : TM \times TM \to N_1 \oplus N_1 \) defined at any point by
\[
\hat{\theta}(X, Y) = (\alpha(X, Y) + \beta_1(X, Y), \alpha(X, Y) - \beta_1(X, Y))
\]
is flat with respect to the inner product induced on \( N_1 \oplus N_1 \).

Proof of Theorem 1: Let \( \tau \) be an infinitesimal bending of \( f \). With the use of (2) and (9) we easily obtain
\[
\langle f_*X + \tilde{\nabla}_X Y, LX + \tilde{\nabla}_X LY \rangle = \langle \alpha(X, Y), \beta(X, Y) \rangle
\]
for any \( X, Y \in \mathfrak{X}(M) \).

By Lemma 14 we have at any point of \( M^n \) that the symmetric tensor \( \theta \) is flat. Given \( Y \in RE(\theta) \) at a point denote \( D = \ker \theta_Y \) where \( \theta_Y(X) = \theta(Y, X) \). Notice that \( Z \in D \) means that \( \alpha(Y, Z) = 0 = \beta(Y, Z) \).

Let \( U \subset M^n \) be an open subset where \( Y \in \mathfrak{X}(U) \) satisfies \( Y \in RE(\theta) \) and \( D \) has dimension \( d \) at any point. Lemma 11 gives
\[
\langle \theta(X, Z), \theta(X, Z) \rangle = 0
\]
for any \( X \in \mathfrak{X}(U) \) and \( Z \in \Gamma(D) \). Equivalently, the right hand side of (14) vanishes and thus
\[
\langle f_*X + \tilde{\nabla}_X Z, LX + \tilde{\nabla}_X LZ \rangle = 0
\]
for any \( X \in \mathfrak{X}(U) \) and \( Z \in \Gamma(D) \).
Assume that there exists a nowhere vanishing \( Z \in \Gamma(D) \) defined on an open subset \( V \) of \( U \) such that \( F: V \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+p} \) given by
\[
F(x, t) = f(x) + tf_\ast Z(x)
\]
is a singular extension of \( f|_V \). The map \( \tilde{\tau}: V \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+p} \) given by
\[
\tilde{\tau}(x, t) = \tau(x) + tLZ(x)
\]
is an infinitesimal bending as well as an extension of \( \tau|_V \) in the singular sense. In fact,
\[
\langle F_\ast \partial_t, \tilde{\nabla}_\partial \tilde{\tau} \rangle = \langle f_\ast Z, LZ \rangle = 0,
\]
\[
\langle \tilde{\nabla}_\partial \tilde{\tau}, F_\ast X \rangle + \langle \tilde{\nabla}_X \tilde{\tau}, F_\ast \partial_t \rangle = \langle LZ, f_\ast X + t\tilde{\nabla}_X Z \rangle + \langle LX + t\tilde{\nabla}_X LZ, f_\ast Z \rangle = 0
\]
and
\[
\langle F_\ast X, \tilde{\nabla}_X \tilde{\tau} \rangle = \langle f_\ast X + t\tilde{\nabla}_X Z, LX + t\tilde{\nabla}_X LZ \rangle = 0
\]
where the last equality follows from (15).

Let \( W \subset U \) be an open subset such that a \( Z \in \Gamma(D) \) as above does not exist along any open subset of \( W \). By Proposition 13 the immersion is \( d \)-ruled along any connected component of an open dense subset of \( W \). Moreover, we have \( d = \dim D = n - \dim \text{Im(}\theta_Y\text{)} \geq n - 2p. \)

**Remark 16.** In Theorem 1 if \( f \) is 1-regular with \( \dim N_1 = q < p \) we obtain the better lower bound \( r \geq n - 2q \) since the proof still works making use of Lemma 15 instead of Lemma 14.

### 3.2 The second local result

Let \( F: \tilde{M}^{n+1} \to \mathbb{R}^{n+p} \) be an isometric immersion and let \( \tilde{\tau} \) be an infinitesimal bending of \( F \). Given an isometric embedding \( j: M^n \to \tilde{M}^{n+1} \) consider the composition of isometric immersions \( f = F \circ j: M^n \to \mathbb{R}^{n+p} \). Then \( \tau = \tilde{\tau}|_{j(M)} \) is an infinitesimal bending of \( f \). It is easy to see that
\[
B(X, Y) = \tilde{B}(X, Y) + \langle \tilde{\nabla}_X Y, F_\ast \eta \rangle \tilde{L}\eta
\]
for \( \eta \in \Gamma(N_{\tilde{M}}M) \) of unit length and \( X, Y \in \mathfrak{X}(M) \). Then (9) gives
\[
\langle \beta(X, Y), F_\ast \eta \rangle + \langle \alpha^f(X, Y), \tilde{L}\eta \rangle = 0
\]
for any $X, Y \in \mathfrak{X}(M)$. We will see that satisfying a condition of this type may guarantee that an infinitesimal bending is not genuine. In fact, this was already proved by Florit [13] in a special case.

We say that an infinitesimal bending of a given isometric immersion $f : M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, satisfies the condition (*) if there is $\eta \in \Gamma(N_f \mathbb{R}^n)$ nowhere vanishing and $\xi \in \Gamma(R)$, where $R$ is determined by the orthogonal splitting $N_f M = P \oplus R$ and $P = \text{span}\{\eta\}$, such that

$$B_\eta + A_\xi = 0$$  \hspace{1cm} (16)

where $B_\eta = \langle \beta, \eta \rangle$. We choose $\eta$ of unit length for simplicity. Thus, that (16) holds means

$$\langle \beta(X, Y), \eta \rangle + \langle \alpha(X, Y), \xi \rangle = 0$$  \hspace{1cm} (17)

for any $X, Y \in \mathfrak{X}(M)$.

The following result is of independent interest since it does not require the codimension to satisfy $p \leq 5$ as is the case in Theorem 5.

**Theorem 17.** Let $f : M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, be an isometric immersion and let $\tau$ be an infinitesimal bending of $f$ that satisfies the condition (*). Then along each connected component of an open and dense subset of $M^n$ either $\tau$ extends in the singular sense or $f$ is $r$-ruled with $r \geq n - 2p + 3$.

As before there is the following immediate consequence.

**Corollary 18.** Let $f : M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, be an isometric immersion and let $\tau$ be a genuine infinitesimal bending of $f$ that satisfies the condition (*). Then $f$ is $r$-ruled with $r \geq n - 2p + 3$ on connected components of an open dense subset of $M^n$.

When $\tau$ satisfies the condition (*) we may extend the tensor $L$ to the tensor $\bar{L} \in \Gamma(\text{Hom}(TM \oplus P, f^*\mathbb{R}^{n+p}))$ by defining

$$\bar{L}_\eta = f_*Y + \xi$$

where $Y \in \mathfrak{X}(M)$ is given by

$$\langle Y, X \rangle + \langle LX, \eta \rangle = 0$$

for any $X \in \mathfrak{X}(M)$. Then $\bar{L}$ satisfies

$$\langle \bar{L}X, \eta \rangle + \langle f_*X, \bar{L}_\eta \rangle = 0$$
for any $X \in \mathfrak{X}(M)$.

Given $\lambda \in \Gamma(f_*TU \oplus P)$ nowhere vanishing where $U$ is an open subset of $M^n$, we define the map $F: U \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+p}$ by

$$F(x,t) = f(x) + t\lambda(x).$$

(18)

Notice that $F$ is not an immersion at least for $t = 0$ at points where $\lambda$ is tangent to $U$. Then let $\tilde{\tau}: U \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+p}$ be the map given by

$$\tilde{\tau}(x,t) = \tau(x) + t\bar{L}\lambda(x).$$

(19)

We have

$$\langle F_*\partial_t, \tilde{\nabla}_{\partial_t}\tilde{\tau} \rangle = 0.$$  

Moreover, since $\langle \bar{L}\lambda, \lambda \rangle = 0$ we obtain

$$\langle \tilde{\nabla}_{\partial_t}F_*X, F_*\partial_t \rangle + \langle \tilde{\nabla}_X\tilde{\tau}, F_*\partial_t \rangle = \langle \bar{L}\lambda, f_*X \rangle + \langle LX, \lambda \rangle + tX\langle \bar{L}\lambda, \lambda \rangle = 0$$

for any $X \in \mathfrak{X}(M)$ and $t \in (-\epsilon, \epsilon)$. Thus $\tilde{\tau}$ is an infinitesimal bending of $F$ on the open subset $\tilde{U}$ of $U \times (-\epsilon, \epsilon)$ where $F$ is an immersion if and only if

$$\langle F_*X, \tilde{\nabla}_X\tilde{\tau} \rangle = 0,$$

or equivalently, if

$$\langle f_*X + t\tilde{\nabla}_X\lambda, LX + t\bar{L}\lambda \rangle = 0$$

for any $X \in \mathfrak{X}(M)$.

In the sequel we take $F$ restricted to $\tilde{U}$. By the above, in order to have that $\tilde{\tau}$ is an infinitesimal bending of $F$ the strategy is to make use of the condition (*) to construct a subbundle $D \subset f_*TM \oplus P$ such that

$$\langle f_*X + \tilde{\nabla}_X\lambda, LX + \tilde{\nabla}_X\bar{L}\lambda \rangle = 0$$

for any $X \in \mathfrak{X}(M)$ and any $\lambda \in \Gamma(D)$.

**Lemma 19.** Assume that $\tau$ satisfies the condition (*). Then

$$\langle f_*X + \tilde{\nabla}_X\lambda, LX + \tilde{\nabla}_X\bar{L}\lambda \rangle = \langle (\tilde{\nabla}_X\lambda)_R, (\tilde{\nabla}_X\bar{L})\lambda \rangle$$

(20)

where $X \in \mathfrak{X}(M)$, $\lambda \in \Gamma(f_*TM \oplus P)$ and

$$(\tilde{\nabla}_X\bar{L})\lambda = \tilde{\nabla}_X\bar{L}\lambda - \bar{L}\nabla'_X\lambda,$$

being $\nabla'$ the connection induced on $f_*TM \oplus P$. 

14
Proof. Set \( \lambda = f_s Z + \phi \eta \) where \( Z \in \mathfrak{X}(M) \) and \( \phi \in C^\infty(M) \). Then

\[
\langle f_s X + \tilde{\nabla} X \lambda, LX + \tilde{\nabla} X \tilde{\lambda} \rangle = \langle f_s (\tilde{\nabla} X \lambda)_{TM} + (\tilde{\nabla} X \lambda)_P + (\tilde{\nabla} X \lambda)_R, \tilde{\nabla} X \tilde{\lambda} \rangle + \langle \tilde{\nabla} X \lambda, LX \rangle + \langle f_s X, \tilde{\nabla} X \tilde{\lambda} \rangle + \langle f_s (\tilde{\nabla} X \lambda)_{TM}, (\tilde{\nabla} X L) Z + L \nabla_X Z + X(\phi) \tilde{L} \eta + \phi \tilde{\nabla} X \tilde{L} \eta \rangle \\
+ \langle (A_\eta X, Z) + X(\phi) \eta, (\tilde{\nabla} X L) Z + L \nabla_X Z + X(\phi) \tilde{L} \eta + \phi \tilde{\nabla} X \tilde{L} \eta \rangle \\
+ \langle (\tilde{\nabla} X \lambda)_R, \tilde{\nabla} X \tilde{\lambda} \rangle + \langle \tilde{\nabla} X \lambda, LX \rangle + \langle f_s X, \tilde{\nabla} X \tilde{\lambda} \rangle
\]  

(21)

for any \( X \in \mathfrak{X}(M) \). Using (9) and (17) we obtain

\[
\langle (\tilde{\nabla} X \lambda)_{TM}, (\tilde{\nabla} X L) Z + L \nabla_X Z \rangle = -\langle L(\tilde{\nabla} X \lambda)_{TM}, \alpha(X, Z) \rangle - \phi \langle A_\eta X, L \nabla_X Z \rangle
\]

(22)

and

\[
\langle (\tilde{\nabla} X \lambda)_{TM}, X(\phi) \tilde{L} \eta + \phi \tilde{\nabla} X \tilde{L} \eta \rangle = \phi \langle (\tilde{\nabla} X \lambda)_{TM}, \nabla_X Y \rangle \\
- X(\phi) \langle L(\tilde{\nabla} X \lambda)_{TM}, \eta \rangle - \phi \langle \alpha(X, (\tilde{\nabla} X \lambda)_{TM}), \xi \rangle
\]

(23)

where for the first term in the right hand side of (23) we have

\[
\langle (\tilde{\nabla} X \lambda)_{TM}, \nabla_X Y \rangle = X \langle (\tilde{\nabla} X \lambda)_{TM}, Y \rangle - \langle \nabla_X (\tilde{\nabla} X \lambda)_{TM}, Y \rangle \\
= -X \langle L(\tilde{\nabla} X \lambda)_{TM}, \eta \rangle + \langle L \nabla_X (\tilde{\nabla} X \lambda)_{TM}, \eta \rangle \\
= -\langle (\tilde{\nabla} X L)(\tilde{\nabla} X \lambda)_{TM}, \eta \rangle - \langle L(\tilde{\nabla} X \lambda)_{TM}, \tilde{\nabla} X \eta \rangle \\
= \langle \alpha(X, (\tilde{\nabla} X \lambda)_{TM}), \xi \rangle - \langle L(\tilde{\nabla} X \lambda)_{TM}, \tilde{\nabla} X \eta \rangle.
\]

(24)

Moreover,

\[
\langle \eta, (\tilde{\nabla} X L) Z + L \nabla_X Z \rangle = -\langle \alpha(X, Z), \xi \rangle + \langle \eta, L \nabla_X Z \rangle,
\]

(25)

\[
\langle \eta, X(\phi) \tilde{L} \eta + \phi \tilde{\nabla} X \tilde{L} \eta \rangle = -\phi \langle \tilde{\nabla} X \eta, \tilde{L} \eta \rangle \\
= -\phi \langle L A_\eta X, \eta \rangle - \phi \langle \nabla_X \tilde{L} \eta, \xi \rangle
\]

(26)

and

\[
\langle \tilde{\nabla} X \lambda, LX \rangle + \langle f_s X, \tilde{\nabla} X \tilde{\lambda} \rangle = -\langle \tilde{\nabla} X \lambda, \tilde{L} \lambda \rangle - \langle \lambda, \tilde{\nabla} X L X \rangle \\
= -\langle \nabla_X X, \tilde{L} \lambda \rangle - \langle \alpha(X, X), \tilde{L} \lambda \rangle - \langle \lambda, L \nabla_X X \rangle - \langle \lambda, (\tilde{\nabla} X L) X \rangle = 0.
\]

(27)
Now a straightforward computation replacing (22) through (27) in (21) yields
\[ \langle f_*X + \tilde{\nabla}_X \lambda, LX + \tilde{\nabla}_X \lambda \rangle = \langle (\tilde{\nabla}_X \lambda)_R, \tilde{\nabla}_X \lambda \rangle - \langle L(\tilde{\nabla}_X \lambda)_T, \alpha(X, Z)_R \rangle \]
- \phi \langle L(\tilde{\nabla}_X \lambda)_T, \nabla_X \eta \rangle - \langle \alpha(X, Z), \tilde{L}(\tilde{\nabla}_X \lambda)_P \rangle - \phi \langle \nabla_X \tilde{\eta}, \tilde{L}(\tilde{\nabla}_X \lambda)_P \rangle
= \langle (\tilde{\nabla}_X \lambda)_R, (\tilde{\nabla}_X \lambda)_R \rangle. \quad \square

In view of (20) the next step is to construct a subbundle \( D \subset f_*TM \oplus P \) such that
\[ \langle (\tilde{\nabla}_X \lambda)_R, (\tilde{\nabla}_X \lambda)_R \rangle = 0 \quad (28) \]
for any \( X \in \mathfrak{X}(M) \) and \( \lambda \in \Gamma(D) \).

**Lemma 20.** Assume that \( \tau \) satisfies the condition \((*)\). Then the bilinear form \( \varphi: TM \times f_*TM \oplus P \rightarrow R \oplus R \) defined by
\[ \varphi(X, \lambda) = ((\tilde{\nabla}_X \lambda)_R, (\tilde{\nabla}_X \lambda)_R) \]
is flat with respect to the indefinite inner product given by
\[ \left\langle \left\langle (\xi_1, \mu_1), (\xi_2, \mu_2) \right\rangle \right\rangle_{R \oplus R} = \langle \xi_1, \xi_2 \rangle_R - \langle \mu_1, \mu_2 \rangle_R. \]

**Proof.** We need to show that
\[ \Theta = \left\langle \left\langle \varphi(X, \lambda), \varphi(Y, \delta) \right\rangle - \left\langle \varphi(X, \delta), \varphi(Y, \lambda) \right\rangle \right\rangle = 0 \]
for any \( X, Y \in \mathfrak{X}(M) \) and \( \lambda, \delta \in f_*TM \oplus P \). We have
\[ \frac{1}{2} \Theta = \langle (\tilde{\nabla}_X \lambda)_R, (\tilde{\nabla}_Y \lambda)_R \rangle + \langle (\tilde{\nabla}_X \lambda)_R, (\tilde{\nabla}_Y \lambda)_R \rangle - \langle (\tilde{\nabla}_X \lambda)_R, (\tilde{\nabla}_Y \lambda)_R \rangle - \langle (\tilde{\nabla}_X \lambda)_R, (\tilde{\nabla}_Y \lambda)_R \rangle.
\]
Clearly \( \Theta = 0 \) if \( \lambda, \delta \in \Gamma(P) \). If \( \lambda, \delta \in \mathfrak{X}(M) \), then
\[ \frac{1}{2} \Theta = \langle \alpha(X, \lambda)_R, (\tilde{\nabla}_Y \lambda)_R \rangle + \langle \alpha(Y, \delta)_R, (\tilde{\nabla}_X \lambda)_R \rangle \]
- \langle \alpha(X, \delta)_R, (\tilde{\nabla}_Y \lambda)_R \rangle - \langle \alpha(Y, \lambda)_R, (\tilde{\nabla}_X \lambda)_R \rangle
= \langle \alpha(X, \lambda)_R, (\tilde{\nabla}_Y \lambda)_R \rangle + \langle \alpha(Y, \lambda)_R, \tilde{L}\eta \rangle
+ \langle \alpha(Y, \lambda)_R, (\tilde{\nabla}_X \lambda)_R \rangle - \langle \alpha(Y, \lambda)_R, \tilde{L}\eta \rangle
- \langle \alpha(X, \lambda)_R, (\tilde{\nabla}_Y \lambda)_R \rangle + \langle \alpha(Y, \lambda)_R, \tilde{L}\eta \rangle
- \langle \alpha(Y, \delta)_R, (\tilde{\nabla}_X \lambda)_R \rangle + \langle \alpha(Y, \lambda)_R, \tilde{L}\eta \rangle.
Using first (17) and then (10) we obtain

\[
\frac{1}{2} \Theta = \langle \alpha(X, \lambda), \beta(Y, \delta) \rangle + \langle \alpha(Y, \delta), \beta(X, \lambda) \rangle \\
- \langle \alpha(X, \delta), \beta(Y, \lambda) \rangle - \langle \alpha(Y, \lambda), \beta(X, \delta) \rangle = 0.
\]

Finally, we consider the case \( \lambda = \eta \) and \( \delta = Z \in \mathfrak{H}(M) \). Then

\[
\frac{1}{2} \Theta = \langle \nabla_{\frac{1}{2}} \eta, ((\tilde{\nabla}_Y L) Z)_R \rangle - \langle A_{\eta} Y, Z \rangle \langle \nabla_{\frac{1}{2}} \eta, \tilde{L} \eta \rangle + \langle \alpha(Y, Z)_R, ((\tilde{\nabla}_X L) \eta)_R \rangle \\
- \langle \nabla_{\frac{1}{2}} \eta, ((\tilde{\nabla}_X L) Z)_R \rangle + \langle A_{\eta} X, Z \rangle \langle \nabla_{\frac{1}{2}} \eta, \tilde{L} \eta \rangle - \langle \alpha(X, Z)_R, ((\tilde{\nabla}_Y L) \eta)_R \rangle.
\]

Since

\[
\langle \nabla_{\frac{1}{2}} \eta, \tilde{L} \eta \rangle = \langle \tilde{\nabla}_X \eta, \tilde{L} \eta \rangle + \langle A_{\eta} X, \tilde{L} \eta \rangle = -\langle \eta, \tilde{\nabla}_X \tilde{L} \eta \rangle - \langle L A_{\eta} X, \eta \rangle \\
= -\langle \eta, (\tilde{\nabla}_X \tilde{L}) \eta \rangle
\]

we obtain

\[
\frac{1}{2} \Theta = \langle \nabla_{\frac{1}{2}} \eta, (\tilde{\nabla}_Y L) Z \rangle - \langle \nabla_{\frac{1}{2}} \eta, (\tilde{\nabla}_X L) Z \rangle \\
+ \langle \alpha(Y, Z), (\tilde{\nabla}_X \tilde{L}) \eta \rangle - \langle \alpha(X, Z), (\tilde{\nabla}_Y \tilde{L}) \eta \rangle.
\]

For the first term using (4), (9) and (17) we obtain

\[
\langle \nabla_{\frac{1}{2}} \eta, (\tilde{\nabla}_Y L) Z \rangle = X \langle \eta, (\tilde{\nabla}_Y L) Z \rangle - \langle \eta, \tilde{\nabla}_X (\tilde{\nabla}_Y L) Z \rangle + \langle A_{\eta} X, (\tilde{\nabla}_Y L) Z \rangle \\
= -X \langle \alpha(Y, Z), \tilde{L} \eta \rangle - \langle \alpha(Y, Z), L A_{\eta} X \rangle \\
- \langle \eta, (\tilde{\nabla}_X B)(Y, Z) + (\tilde{\nabla}_{\tilde{\nabla}_X Y} L) Z + (\tilde{\nabla}_Y L) \tilde{\nabla}_X Z \rangle \\
= -\langle (\nabla_{\frac{1}{2}} \alpha)(Y, Z), \tilde{\nabla}_X \tilde{L} \eta \rangle + \langle \alpha(Y, Z), (\tilde{\nabla}_X Y, Z) + \alpha(Y, \nabla X Z), \tilde{L} \eta \rangle \\
- \langle \eta, (\tilde{\nabla}_X B)(Y, Z) + (\tilde{\nabla}_{\tilde{\nabla}_X Y} L) Z + (\tilde{\nabla}_Y L) \tilde{\nabla}_X Z \rangle \\
- \langle \alpha(Y, Z), \tilde{\nabla}_X \tilde{L} \eta \rangle - \langle \alpha(Y, Z), (\tilde{\nabla}_X \tilde{L}) \eta \rangle + \langle A_{\eta} \eta, X, \tilde{L} \eta \rangle + \langle A_{\alpha(Y, Z)} X, \tilde{L} \eta \rangle \\
= -\langle (\nabla_{\frac{1}{2}} \alpha)(Y, Z), \tilde{L} \eta \rangle - \langle \eta, (\tilde{\nabla}_X B)(Y, Z) \rangle \\
- \langle \alpha(Y, Z), (\tilde{\nabla}_X \tilde{L}) \eta \rangle - \langle LA_{\alpha(Y, Z)} X, \eta \rangle.
\]

Likewise, we have

\[
\langle \nabla_{\frac{1}{2}} \eta, (\tilde{\nabla}_X L) Z \rangle = -\langle (\nabla_{\frac{1}{2}} \alpha)(X, Z), \tilde{L} \eta \rangle - \langle \eta, (\tilde{\nabla}_Y B)(X, Z) \rangle \\
- \langle \alpha(X, Z), (\tilde{\nabla}_Y \tilde{L}) \eta \rangle - \langle LA_{\alpha(X, Z)} Y, \eta \rangle.
\]
From (3) and the Codazzi equation

$$(\nabla^\perp_X^\perp \alpha)(Y, Z) = (\nabla^\perp_Y^\perp \alpha)(X, Z)$$

we obtain

$$\frac{1}{2} \Theta = \langle L(R(X, Y)Z - A_{\alpha(Y, Z)}X + A_{\alpha(X, Z)}Y), \eta \rangle.$$ 

Hence $\Theta = 0$ from the Gauss equation. \hfill \square

**Proof of Theorem 17:** By Lemma 20 there is a flat bilinear form $\varphi$. Let $U$ be an open subset of $M^n$ where there is $Y \in \mathfrak{X}(U)$ such that $Y \in RE(\varphi)$ and $D = \ker \varphi_Y$ has dimension $d$ at any point. Then Lemma 11 gives

$$\langle \langle \varphi(X, \lambda), \varphi(X, \lambda) \rangle \rangle = 0$$

for any $X \in \mathfrak{X}(U)$ and $\lambda \in \Gamma(D)$. Notice that this implies that (28) holds for any $\lambda \in \Gamma(D)$. Whenever there is a nonvanishing $\lambda \in \Gamma(D)$ on an open subset $V \subset U$ such that (18) defines a singular extension of $f|_V$, then $\tau|_V$ extends in the singular sense by means of (19).

Let $W \subset U$ be an open subset where $\lambda \in \Gamma(D)$ as above does not exist along any open subset of $W$. Hence $D$ must be a tangent distribution on $W$, and Proposition 13 gives that $f|_W$ is $d$-ruled on connected components of an open dense subset of $W$. Moreover, the dimension of the rulings is bounded from below by $n + 1 - \dim Im(\varphi_Y) \geq n - 2p + 3$. \hfill \square

**Proof of Theorem 5:** We work on the open dense subset of $M^n$ where $f$ is 1-regular on any connected component. Consider an open subset of a connected component where the index of relative nullity is $\nu \leq n - 2p - 1$ at any point. Lemma 12 applies and thus the flat bilinear form $\hat{\theta}$ in (13) decomposes at any point as $\hat{\theta} = \theta_1 + \theta_2$ where $\theta_1$ is as in part (i) of that result. Hence, on any open subset where the dimension of $\mathcal{S}(\theta_1) = \mathcal{S}(\hat{\theta}) \cap \mathcal{S}(\hat{\theta})^\perp$ is constant there are smooth local unit vector fields $\zeta_1, \zeta_2 \in N_1$ such that $(\zeta_1, \zeta_2) \in \mathcal{S}(\theta_1)$. Equivalently,

$$\langle \beta(X, Y), \zeta_1 + \zeta_2 \rangle + \langle \alpha(X, Y), \zeta_1 - \zeta_2 \rangle = 0$$

(29)

for any $X, Y \in \mathfrak{X}(M)$. Then $\zeta_1 + \zeta_2 \neq 0$ since otherwise $\zeta_1 - \zeta_2 \in N_1^\perp$. Hence $\tau$ satisfies the condition $(\ast)$ and the proof follows from Corollary 18. \hfill \square
4 The global result

The first two results are of independent interest.

**Proposition 21.** Let $\tau$ be an infinitesimal bending of $f : M^n \to \mathbb{R}^{n+p}$ and let $\theta$ be the flat bilinear form defined by (12). Denote $\nu^*(x) = \dim \Delta^*(x)$ at $x \in M^n$ where

$$\Delta^*(x) = N(\theta)(x) = \Delta \cap N(\beta)(x).$$

Then, on any open subset of $M^n$ where $\nu^*$ is constant the distribution $\Delta^*$ is totally geodesic and its leaves are mapped by $f$ onto open subsets of affine subspaces of $\mathbb{R}^{n+p}$.

**Proof.** From (9) we have $\Delta \subset N(\beta)$. Then (11) and the Gauss equation give

$$(\nabla^\perp_X \beta)(Z, Y) = (\nabla^\perp_Z \beta)(X, Y) = 0$$

for any $X, Y \in \Gamma(\Delta^*)$ and $Z \in \mathfrak{X}(M)$. Let $\nabla^* = (\nabla^\perp, \nabla^\perp)$ be the compatible connection in $N_f M \oplus N_f M$. Hence

$$0 = (\nabla^*_X \theta)(Z, Y) = \theta(Z, \nabla_X Y)$$

for any $X, Y \in \Gamma(\Delta^*)$ and $Z \in \mathfrak{X}(M)$. Thus $\Delta^* \subset \Delta$ is totally geodesic. $\square$

On an open subset of $M^n$ where $\nu^* > 0$ is constant consider the orthogonal splitting $TM = \Delta^* \oplus E$ and the tensor $C : \Gamma(\Delta^*) \times \Gamma(E) \to \Gamma(E)$ defined by

$$C(S, X) = C_S X = -(\nabla_X S)_E$$

where $S \in \Gamma(\Delta^*)$ and $X \in \Gamma(E)$. Since $\Delta^* \subset \Delta$ is totally geodesic, the Gauss equation gives

$$\nabla_T C_S X = C_S C_T + C_{\nabla_T S}$$

for any $S, T \in \Gamma(\Delta^*)$. In particular, we have

$$\frac{D}{dt} C_{\gamma'} = C_{\gamma'}^2$$

(30)

along a unit speed geodesic $\gamma$ contained in a leaf of $\Delta^*$.

The next result provides a way to transport information along geodesics contained in leaves of the nullity of $\theta$. This technique has been widely used, for instance, see [8], [14] and [15].
Proposition 22. Let $\nu^* > 0$ be constant on an open subset $U \subset M^n$. If $\gamma: [0,b] \to M^n$ is a unit speed geodesic such that $\gamma([0,b))$ is contained in a leaf of $\Delta^*$ in $U$, then $\Delta^*(\gamma(b)) = \mathcal{P}_0^b(\Delta^*(\gamma(0)))$ where $\mathcal{P}_0^t$ is the parallel transport along $\gamma$ from $\gamma(0)$ to $\gamma(t)$. In particular, we have $\nu^*(\gamma(b)) = \nu^*(\gamma(0))$ and the tensor $C_{\gamma'}$ extends smoothly to $[0,b]$.

Proof. We mimic the proof of Lemma 27 in [14]. Let the tensor $J: E \to E$ be the solution in $[0,b)$ of

$$\frac{D}{dt} J + C_{\gamma'} \circ J = 0$$

with initial condition $J(0) = I$. We have from (30) that $D^2 J/ dt^2 = 0$, and hence $J$ extends smoothly to $\mathcal{P}_0^b(E(0))$ in $\gamma(b)$. Let $Y$ and $Z$ be parallel vector fields along $\gamma$ such that $Y(t) \in E(t)$ for each $t \in [0,b)$. Since $\gamma' \in \Delta^*$, it follows from (11) that $\nabla^*_{\gamma'} \theta(JY, Z) = (\nabla^*_Y \theta)(\gamma', Z)$. This and the definition of $J$ imply that $\theta(JY, Z)$ is parallel along $\gamma$. In particular $J$ is invertible in $[0,b]$. By continuity $\mathcal{P}_0^b(\Delta^*(\gamma(0))) \subset \Delta^*(\gamma(b))$, and since $Z(0)$ is arbitrary, then $\mathcal{P}_0^b(\Delta^*(\gamma(0))) = \Delta^*(\gamma(b))$. Finally we extend the tensor $C_{\gamma'}$ to $[0,b]$ as $-DJ/dt \circ J^{-1}$.

Lemma 23. Let $f: M^n \to \mathbb{R}^{n+p}$, $p \leq 5$ and $n > 2p$ be an isometric immersion of a compact Riemannian manifold and let $\tau$ be an infinitesimal bending of $f$. Then, at any $x \in M^n$ there is a pair of vectors $\zeta_1, \zeta_2 \in N_f M(x)$ of unit length such that $(\zeta_1, \zeta_2) \in (\mathcal{S}(\theta))^\bot(x)$ where

$$\mathcal{S}(\theta)(x) = \text{span} \{\theta(X,Y) : X,Y \in T_xM\}.$$ 

Moreover, on any connected component of an open dense subset of $M^n$ the pair $\zeta_1, \zeta_2$ at $x \in M^n$ extend to smooth vector fields $\zeta_1$ and $\zeta_2$ parallel along $\Delta^*$ that satisfy the same conditions.

Proof. We claim that the subset of points $U$ of $M^n$ where there is no such a pair, that is, where the metric induced on $(\mathcal{S}(\theta))^\bot$ is positive or negative definite, is empty. It is not difficult to see that $U$ is open. From Lemma [12] we have $\nu^* > 0$ in $U$. Let $V \subset U$ be the open subset where $\nu^* = \nu_0^*$ is minimal. Take $x_0 \in V$ and a unit speed geodesic $\gamma$ in $M^n$ contained in a
maximal leaf of $\Delta^*$ with $\gamma(0) = x_0$. Since $M^n$ is compact, there is $b > 0$ such that $\gamma([0,b)) \subset V$ and $\gamma(b) \notin V$. Proposition 22 gives $\nu^*(\gamma(b)) = \nu_0^*$ which implies $\gamma(b) \notin U$. Hence, there are unit vectors $\zeta_1, \zeta_2 \in N_\gamma M(\gamma(b))$ such that $(\zeta_1, \zeta_2) \in (S(\theta))^\perp(\gamma(b))$.

Let $\zeta_i(t)$ be the parallel transport along $\gamma$ of $\zeta_i$, $i = 1, 2$. Then

$$\langle \langle \theta(X, Y), (\zeta_1, \zeta_2) \rangle \rangle = \langle \langle A_{\zeta_1 - \zeta_2} + B_{\zeta_1 + \zeta_2} X, Y \rangle \rangle.$$

It follows from (9) and (11) that

$$(\nabla^*_T \theta)(X, Y) = (\nabla^*_X \theta)(T, Y) \tag{31}$$

where $T \in \Gamma(\Delta^*)$ extends $\gamma'$ and $X, Y \in \mathcal{X}(M)$. Along $\gamma$ this gives

$$\frac{D}{dt} e_{\zeta_1, \zeta_2} = e_{\zeta_1, \zeta_2} C_{\gamma'} = C_{\gamma'} e_{\zeta_1, \zeta_2}$$

where $e_{\zeta_1, \zeta_2} = A_{\zeta_1 - \zeta_2} + B_{\zeta_1 + \zeta_2}$ and $C_{\gamma'}$ denotes the transpose of $C_{\gamma'}$. Moreover, by Proposition 22 this ODE holds on $[0, b]$. Given that $e_{\zeta_1, \zeta_2}(\gamma(b)) = 0$, then $e_{\zeta_1, \zeta_2}$ vanishes along $\gamma$. This is a contradiction and proves the claim.

We have from (31) that

$$(\nabla^*_T \theta)(X, Y) = -\theta(\nabla_X T, Y) \in \Gamma(S(\theta))$$

for any $T \in \Gamma(\Delta^*)$ and $X, Y \in \mathcal{X}(M)$. Thus $S(\theta)$ is parallel along the leaves of $\Delta^*$. Let $U_0$ be a connected component of the open dense subset of $M^n$ where the dimension of $\Delta^*$, $S(\theta)$, $S(\theta) \cap S(\theta)^\perp$ and the index of the metric induced on $S(\theta)^\perp \times S(\theta)^\perp$ are all constant. Hence on $U_0$ the vector fields $\zeta_1, \zeta_2$ can be taken parallel along the leaves of $\Delta^*$. \hfill $\square$

For an hypersurface $f : M^n \to \mathbb{R}^{n+1}$ we have

$$(\bar{\nabla}_X L)Y = (B_N X, Y)N + f_* Y(X, Y) \tag{32}$$

where $N$ is a unit vector field normal to $f$. The next result follows from Theorem 13 in [11] and was fundamental in [13].

**Lemma 24.** An infinitesimal bending $\tau$ of $f : M^n \to \mathbb{R}^{n+1}$ is trivial if and only if $B_N = 0$.

21
Proof of Theorem 6: We assume that there is no open subset of $M^n$ where the index of relative nullity satisfies $\nu \geq n - 1$. By Lemma 23, on connected components of an open dense subset of $M^n$ there are $\zeta_1, \zeta_2 \in \Gamma(N_f M)$ with $\|\zeta_1\| = \|\zeta_2\| = 1$ parallel along the leaves of $\Delta^*$ and such that

$$\langle \theta(X, Y), (\zeta_1, \zeta_2) \rangle = 0$$

for any $X, Y \in \mathfrak{X}(M)$. It follows from (12) that (29) holds on connected components of an open dense subset of $M^n$. Let $U \subset M^n$ be an open subset where $\zeta_1, \zeta_2$ are smooth and $\zeta_1 + \zeta_2 \neq 0$. Thus $\tau|_U$ satisfies the condition ($\star$). Let $V \subset U$ be an open subset where $\tau$ is a genuine infinitesimal bending. By Corollary 18 we have that $f$ is $(n - 1)$-ruled on each connected component $V$ of an open dense subset of $V$. Since our goal is to show that $V$ is empty we assume otherwise.

Proposition 13 and the proof of Theorem 17 yield that the rulings on $V$ are determined by the tangent subbundle $D = \ker \varphi_Y$ where $\varphi$ was given in Lemma 20 and $Y \in RE(\varphi)$. Also from that proof $\dim \text{Im}(\varphi_Y) = 2$ and therefore $\text{Im}(\varphi_Y) = \mathbb{R} \oplus \mathbb{R}$ where $N_f M = P \oplus \mathbb{R}$ as in Lemma 20. Lemma 11 gives

$$\varphi_X(D) \subset \text{Im}(\varphi_Y) \cap \text{Im}(\varphi_Y) = \{0\}$$

for any $X \in \mathfrak{X}(M)$, that is, $D = N(\varphi)$. In particular, from the definition of $\varphi$ it follows that $D \subset N(\alpha_R)$. Hence, by dimension reasons either $N(\alpha_R) = TM$ or $D = N(\alpha_R)$. Next we contemplate both possibilities.

Let $V_1 \subset V$ be an open subset where $N(\alpha_R) = TM$ holds, that is, $N_1 = P$. Thus $N_1$ is parallel relative to the normal connection since, otherwise, the Codazzi equation gives $\nu = n - 1$, and that has been ruled out. Hence $f|_{V_1}$ reduces codimension, that is, $f(V_1)$ is contained in an affine hyperplane $\mathbb{R}^{n+1}$. Decompose $\tau = \tau_1 + \tau_2$ where $\tau_1$ and $\tau_2$ are tangent and normal to $\mathbb{R}^{n+1}$, respectively. It follows that $\tau_1$ is an infinitesimal bending of $f|_{V_1}$ in $\mathbb{R}^{n+1}$. Since $\tau$ satisfies the condition ($\star$) then Lemma 24 gives that $\tau_1$ is trivial, that is, the restriction of a Killing vector field of $\mathbb{R}^{n+1}$ to $f(V_1)$. Extending $\tau_2$ as a vector field normal to $\mathbb{R}^{n+1}$ it follows that $\tau|_{V_1}$ extends in the singular sense and this is a contradiction.

Let $V_2 \subset V$ be an open subset where $D = N(\alpha_R)$. By assumption $D \neq \Delta$. Let $\hat{D}$ be the distribution tangent to the rulings in a neighborhood $V'_2$ of $x_0 \in V_2$. From Proposition 13 we have $D(x_0) = \hat{D}(x_0)$. Let $W \subset V'_2$ be an open subset where $D \neq \hat{D}$, that is, where $D$ is not totally geodesic. Then there are two transversal $(n - 1)$-dimensional rulings passing through any
point \( y \in W \). It follows easily that \( N_1 = P \) on \( W \). As above we obtain that \( \tau|_W \) extends in the singular sense, leading to a contradiction. Let \( V_3 \subset V_2 \) be the interior of the subset where \( D \) is totally geodesic. On \( V_3 \) the Codazzi equation gives

\[
\nabla^\perp_X \alpha(Z, Y) \in \Gamma(P)
\]

for all \( X, Y \in \Gamma(D) \) and \( Z \in \mathcal{X}(M) \). Thus \( R \) is parallel along \( D \) relative to the normal connection. We have from Proposition 4 in [8] that \( f \) admits a singular extension

\[
F(x, t) = f(x) + t\lambda(x)
\]

for \( \lambda \in \Gamma(f_*TM \oplus P) \) as a flat hypersurface. Moreover, \( F \) has \( R \) as normal bundle and \( \partial_t \) belongs to the relative nullity distribution. Then \( (\nabla_X \lambda)_R = 0 \) for any \( X \in \mathcal{X}(V_3) \). Hence (28) is satisfied and thus \( \tau|_{V_3} \) extends in the singular sense. This is a contradiction which shows that \( V \) is empty, and hence also is \( \tilde{V} \).

It remains to consider the existence of an open subset \( U' \subset M^n \) where \( \zeta_1, \zeta_2 \) are smooth and \( \zeta_1 + \zeta_2 = 0 \). It follows from (29) that \( \zeta_1 - \zeta_2 \perp N_1 \). Once more, we obtain that \( f(U') \subset \mathbb{R}^{n+1} \). Thus, we have an orthogonal decomposition of \( \tau|_{U'} \) as in part (ii) of the statement and \( \tau_1, \tau_2 \) extend in the singular sense as follows:

(i) \( \tilde{\tau}_1(x, t) = \tau_1(x) \) to \( F: U \times \mathbb{R} \rightarrow \mathbb{R}^{n+2} \) where \( F(x, t) = f(x) + te \).

(ii) For instance locally as \( \tilde{\tau}_2(x, t) = \tau_2(x) \) to \( F: U \times I \rightarrow \mathbb{R}^{n+2} \) where \( F(x, t) = f(x) + tN \) being \( N \) is a unit normal field to \( f|_U \) in \( \mathbb{R}^{n+1} \).

\[ \square \]

**Remarks 25.** (1) In case (ii) of Theorem 6 if \( \tau_1 \) is trivial then \( \tau_1 \) and \( \tau_2 \) extend in the same direction, and hence \( \tau \) also does. Therefore we are also in case (i).

(2) Notice that for \( p = 2 \) we have shown as part of the proof that an infinitesimal bending of a submanifold without flat points as in in part (ii) of Theorem 5 cannot be genuine.

### 5 Nonflat ambient spaces

In this section we argue for the following statement:

*Theorems 4, 5 and 17 hold if the Euclidean ambient space is replaced by a nonflat space form.*
Let \( f: M^n \to Q_c^{n+p} \) be an isometric immersion where \( Q_c^{n+p} \) denotes either the sphere \( S_c^{n+p} \) or the hyperbolic space \( \mathbb{H}_c^{n+p} \) of sectional curvature \( c \neq 0 \). Then we say that \( \tau \in \Gamma(f^*\mathfrak{Q}_c^{n+p}) \) is an infinitesimal bending of \( f \) if (1) is satisfied in terms of the connection in \( Q_c^{n+p} \). And now that \( f \) is \( r \)-ruled means that there is an \( r \)-dimensional smooth totally geodesic distribution whose leaves are mapped by \( f \) to open subsets of totally geodesic submanifolds of the ambient space \( Q_c^{n+p} \).

In the sequel, for simplicity we also denote by \( f \) the composition of the immersion with the umbilical inclusion of \( Q_c^{n+p} \) into \( O_n^{n+p+1} \), where \( O_n^{n+p+1} \) stands for either Euclidean or Lorentzian flat space depending on whether \( c > 0 \) or \( c < 0 \), respectively.

Let \( \tau \) be an infinitesimal bending of \( f \) and let \( F: I \times M^n \to Q_c^{n+p} \) be a smooth variation such that \( f_t = F(t, \cdot): M^n \to Q_c^{n+p} \) verifies \( f_0 = f \) and having \( \tau \) as variational vector field. In this case we still have that (5), (6) and (7) hold. And also as before, associated to \( \tau \) we have the tensors

\[
LX = \tilde{\nabla}X \tau \quad \text{and} \quad B(X,Y) = (\tilde{\nabla}X L)Y
\]

where \( X,Y \in \mathfrak{X}(M) \) and \( \tilde{\nabla} \) denotes the connection in \( Q_c^{n+p} \). Now

\[
B(X,Y) = f_*Y(X,Y) + \beta(X,Y) + c(f_*Y, \tau) f_*X - c(X,Y) \tau
\]

where the tensors \( Y: T M \times T M \to T M \) and \( \beta: T M \times T M \to N_f M \) are the tangent and normal components of \( \partial/\partial t \big|_{t=0} \alpha^t \), respectively, and \( \alpha^t \) is the second fundamental form of \( f_t \) as a submanifold in \( Q_c^{n+p} \). In particular, we have that (10) holds.

In this case, an infinitesimal bending of \( f \) is said to satisfy the condition (*) if there is \( \eta \in \Gamma(N_f M) \) of unit length and \( \xi \in \Gamma(R) \), where \( R \) is determined by the orthogonal splitting \( N_f M = P \oplus R \) and \( P = \text{span}\{\eta\} \), such that

\[
B_\eta + A_\xi + c(\tau, \eta) I = 0
\]

where \( B_\eta = \langle \beta, \eta \rangle \).

The cone over an isometric immersion \( f: M^n \to Q_c^{n+p} \) is defined by

\[
\hat{f}: \hat{M}^{n+1} = (0, \infty) \times M^n \to Q_c^{n+p+1} \quad \text{where} \quad (s, x) \mapsto s f(x).
\]

Notice that \( \partial_s \) lies in the relative nullity of \( \hat{f} \) and that \( N_f \hat{M} \) is the parallel transport of \( N_f M \) along the lines parametrized by \( s \). Observe that if \( c < 0 \),
then the cone over $f$ is a Lorentzian submanifold of $\mathbb{L}^{n+p+1}$ and hence $N^\hat{f} \hat{M}$ has positive definite metric.

If $\tau$ is an infinitesimal bending of $f$, it is easy to see that $\hat{\tau}(s, x) = s\tau(x)$ is an infinitesimal bending of $\hat{f}$ in $O^{n+p+1}$, that is, $\hat{\tau}$ is a vector field that satisfies (1) with respect to the connection in $O^{n+p+1}$. Moreover, if $\tau$ satisfies the condition (*) then $\hat{\tau}$ satisfies the condition (*) for the flat ambient space.

Let $\hat{f}$ be the cone over an immersion $f$ in $Q^{n+p+c}$. Notice that the parameter $s$ defines lines parallel to the position vector. Thus, if the map $\hat{f} + t\lambda$, is a singular extension of $\hat{f}$ for some vector field $\lambda$ then the intersection of its image with $Q^{n+p}_c$ determines a singular extension of $f$.

Consider the maps

$$\hat{F}(t, s, x) = \hat{f}(s, x) + t\lambda(s, x) \quad \text{and} \quad \hat{\tau}'(t, s, x) = \hat{\tau}(s, x) + tL\lambda(s, x)$$

as in the proofs of Theorems 1 and 17. Notice that

$$\langle \hat{F}(t, s, x), \hat{\tau}'(t, s, x) \rangle = \langle \hat{f}(s, x) + t\lambda(s, x), \hat{\tau}(s, x) + tL\lambda(s, x) \rangle$$

$$= st\langle f(x), L\lambda \rangle + st\langle \lambda, \tau \rangle$$

$$= 0$$

where for the last equality we used $L\partial_s = \tau(x)$. Then we have that $\hat{\tau}'$ is orthogonal to the position vector $\hat{F}$. From this we have that if $\hat{F}$ determines a singular extension of $\hat{f}$ then $\tau$ extends in the singular sense.

As in the proofs of Theorems 1 and 17, if there is no $\lambda$ as above that determines a singular extension of $\hat{f}$ we conclude that $\hat{f}$ is ruled. Finally, observe that being $\hat{f}$ the cone over $f$, then these rulings determine rulings of $f$.

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