String/FluxTube Duality on the Lightcone

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Abstract
The equivalence of quantum field theory and string theory as exemplified by the AdS/CFT correspondence is explored from the point of view of lightcone quantization. On the string side we discuss the lightcone version of the static string connecting a heavy external quark source to a heavy external antiquark source, together with small oscillations about the static string configuration. On the field theory side we analyze the weak/strong coupling transition in a ladder diagram model of the quark antiquark system, also from the point of view of the lightcone. Our results are completely consistent with those obtained by more standard covariant methods in the limit of infinitely massive quarks.

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1 Introduction

The postulate of quark confinement in QCD is usually associated with the idea that, in the theory without dynamical quarks, there is a mass gap $m_G$ (the lightest glueball mass) and the gauge field responds to a fixed quark source separated from a fixed antiquark source by a distance $L \gg m_G^{-1}$ by forming some kind of gluonic flux tube between these sources. The energy of the $q\bar{q}$ system is then expected to grow with $L$ as $U(L) \simeq T_0 L$. This picture establishes a string-like object (the flux tube) as a real physical entity in the theory of strong interactions. Of course, as soon as dynamical quarks enter the picture, the string can break, making long strings unstable. But the presence in the hadron spectrum of high spin narrow resonances on nearly linear Regge trajectories $J \simeq \alpha' M^2$, with $\alpha' = 1/2\pi T_0$, gives strong experimental evidence that the states of a long string are metastable and play an important role in the dynamics of the strong interactions.

In the theory with dynamical quarks, ’t Hooft’s large $N_c$ limit [1] provides another handle, albeit an indirect one, on quark confinement. It is based on considering a family of $SU(N_c)$ gauge groups, QCD being the case $N_c = 3$, and considering the extrapolation $N_c \to \infty$ (with $\lambda = N_c \alpha_s / 2\pi$ fixed). In this mathematical limit all S-matrix elements for processes involving only color singlet particles in the initial and final states are suppressed by powers of $1/N_c$. In particular the decay rate for any color singlet particle into any number of color singlet particles vanishes in the limit. In contrast the decay of a color singlet into color nonsinglet particles would not be suppressed. If we define quark confinement as the statement that all finite mass particles are color singlets, then it follows that in the limit $N_c \to \infty$ all finite mass hadrons would be stable noninteracting particles. In particular long strings would be unbreakable in this limit. Given the postulate of quark confinement we may arguably define the QCD string as the large $N_c$ limit of QCD. There is even hope that the limit is a quantitatively reasonable approximation to QCD, $N_c = 3$.

One of the big surprises of the duality between superstring theory on $\text{AdS}_5 \times S_5$ and $\mathcal{N} = 4$ supersymmetric Yang Mills [2] is that in the ’t Hooft limit a string description can even be applied to a gauge theory that neither confines nor possesses a mass gap. The absence of a mass gap in this case has motivated those exploring the precise nature of the AdS/CFT correspondence to supply an IR cutoff by defining the field theory on $S_3$ rather than $R^3$. Remarkably a precise correspondence holds in detail in this case as well and this has enabled many successful quantitative tests of Field/String duality [3].

However there are some physical situations which provide their own infrared cutoff (when $N_c = \infty$), and these can be safely analyzed on $R^3$. Among these is the case of fixed quark and antiquark sources separated by a distance $L$. On the string side of the duality the absence of infrared difficulties in this system has been implicitly understood ever since the computation of the large ’t Hooft coupling limit of the $q\bar{q}$ potential [4] by solving for the static string configuration that connects the two sources. The point was further underlined by Callan and Gúijosa whose study of small oscillations about this static string configuration [5] (see also [6]) implies, in the $\lambda \to \infty$ limit, the existence of many discrete energy levels above the minimum energy $-c\sqrt{\lambda}/L$ given by the static solution. From the field theory side, the existence of stable excited states seems to clash with the absence of a gap in the field theory; but this puzzle was resolved in [7] where it was shown that at $N_c = \infty$ the system decouples from all $q\bar{q}+$Glue final states whose energy is below the ionization threshold $E_{th} = 0$.

Thus the large $N_c$ limit of the $q\bar{q}$ system can be explored in a well-defined way whether or not confinement occurs. In particular, perturbation theory applied to this system is infrared safe. In this article we study various aspects of this system, especially from the point of view of light-
cone quantization. The advantage of the lightcone description is that it provides an unambiguous canonical quantization of the system on both sides of Field/String duality. On the string side, the parametrization of the worldsheet is completely fixed on the lightcone without the need for parametrization ghosts. Similarly on the field side the gauge is completely fixed without the need for Fadeev-Popov ghosts. Since the quantum evolution on both sides of the duality is with respect to the same lightcone time, the prospects for a clean and detailed comparison of the dynamics on both sides of the duality are especially bright in this formulation.

Indeed, following precisely this line of thought, the lightcone worldsheet formalism [8–10] has been developed. This formalism gives an explicit and concrete lightcone string description of the sum of the planar diagrams of field perturbation theory in lightcone gauge, with the goal of understanding field/string duality from the field theory side. It is therefore worthwhile to see how the lightcone description deals with the many aspects of the duality already understood from other starting points. On the string side, we test the application of lightcone methods by showing how the lightcone quantization of string on AdS set up in [11] leads in its semi-classical approximation to the classical small oscillation problem about a static string stretched between two sources on the AdS boundary solved by Callan and Güijosa. Much of this analysis goes through with a more general metric than AdS so we only specialize to the latter at the end. Of course, at the classical level the lightcone equations are simply coordinate transformations of the more conventional covariant equations so it is not surprising to reproduce the solutions in [5, 6]. It is nonetheless instructive to see how the equations and their solution flow from the lightcone starting point which has no constraints to solve. In fact, lightcone quantization is so tightly formulated that it is hard to describe two completely localized sources. On the string side, this is because the spatial coordinate $x^-$ is determined in terms of the other target space coordinates in such a way that fixing the string ends in those coordinates implies Neumann conditions on $x^-$, i.e. the string ends move freely in the $z$ direction. We finesse this difficulty by fixing the string ends to two separated one-branes parallel to the $z$-axis as illustrated in Fig. 1.

On the field theory side, we study the weak coupling limit of this system by employing the ladder approximation to the Bethe-Salpeter equation for fixed separated quark and antiquark sources initiated in [12]. We discuss this Feynman gauge model in lightcone coordinates, and we also construct the corresponding model in lightcone gauge. We shall see that the lightcone gauge ladder model, though different in detail from the Feynman gauge one, gives qualitatively similar results. Finally we turn to a study of the interpolation between weak and strong coupling within the Feynman gauge ladder model. This model is only meant as a rough qualitative guide to the weak/strong coupling transition, and gives at best a reasonable indication of the conformal $\mathcal{N} = 4$ case [7]. We show that for $\lambda < \lambda_c = 1/4$ there are no discrete levels between the ground state energy and threshold. For $\lambda > \lambda_c$ an infinite number of levels appear accumulating at threshold. These results have been previously reported in [7], along with a less detailed derivation of them. Of course, the ladder approximation is strictly valid only at weak coupling, so the numerical value of this critical coupling should be taken with a grain of salt. But we think the conformal case ($\mathcal{N} = 4$) should be qualitatively similar. In real QCD, of course, the 't Hooft coupling depends on the $q\bar{q}$ separation $\lambda(L)$, so we should then speak of a critical separation $L_c$. All this suggests that the confining gluonic flux tube can be usefully thought of as a more or less conventional multi-gluon bound state, a point of view advocated in [13].
2 Classical string solutions on the lightcone

The strong ’t Hooft coupling limit of the large $N_c N = 4$ supersymmetric gauge theory is described by the classical IIB superstring on $AdS_5 \times S_5$. The fixed quark antiquark system corresponds at strong coupling to a superstring stretched between two separated points on the AdS boundary. Therefore we begin by considering the open string in lightcone gauge on $AdS_5 \times S_5$ or a similar background. We set the worldsheet fermion fields to zero and focus attention on the bosonic worldsheet co-ordinates $x^\mu(\sigma, \tau), \phi(\sigma, \tau)$ of the $AdS_5$ factor. Fixing to lightcone parametrization of the worldsheet, with $x^+ = (x^0 + x^3)/\sqrt{2} = \tau$ and $P^+(\sigma, \tau) = 1$, the lightcone Hamiltonian for the remaining radial, $\phi$, and transverse, $x = (x^1, x^2)$, coordinates is 

$$H = p^- = \frac{1}{2} \int_0^{p^+} d\sigma \left[ \mathcal{P}^2 + G^2 x'^2 + G(\Pi^2 + \phi'^2) \right]. \quad (1)$$

In fact this is the Hamiltonian for a general warped metric of the form,

$$ds^2 = G(\phi) dx^\mu dx_\mu + d\phi^2 = G(\phi)(-dt^2 + dx_\perp^2 + dx_3^2) + d\phi^2. \quad (2)$$

For the AdS case $G(\phi) = e^{\phi/\gamma} = r^2/R^2$ with $4\gamma^2 = R^2 T_0 = R^2/2\pi\alpha'$. So the conformal scaling is $\phi \to \phi + c, x^\mu \to \exp[-c/2\gamma] x^\mu$. However we find it useful to keep $G(\phi)$ arbitrary until the end. As discussed in reference [14], the more general analysis is useful for a qualitative understanding of conformally broken QCD-like backgrounds, which are asymptotically near to $AdS_5$ in the UV ($r \to 0$) and terminate beyond some scale in the IR ($r \to r_0$) to give confinement.

For the purpose of finding classical solutions, it is more convenient to employ the action implied by this Hamiltonian,

$$S = \int d\tau \int_0^{p^+} d\sigma \frac{1}{2} \left[ \dot{x}^2 - G^2 x'^2 + \frac{1}{G} \dot{\phi}^2 - \frac{1}{G} \phi'^2 \right]. \quad (3)$$

We will be considering solutions in which the ends of the string are constrained to be on the boundary of AdS. In lightcone quantization, it is best to preserve $p^+$ conservation. This happens automatically in the lightcone quantization of the string, which eliminates $x^-$ in favor of the other worldsheet fields by

$$x^- = \dot{x} \cdot x' + \dot{\phi} \phi'/G + \text{fermion terms}, \quad (4)$$

so the Neumann boundary condition $x^- = 0$ follows from either Dirichlet or Neumann boundary conditions on $x, \phi$. However, one may easily impose Dirichlet conditions on $x, \phi$ at the string ends so that the string may be stretched a fixed direction $\mathbf{L}$ in the $x$ plane at the UV boundary, $\phi \to \infty$, of $AdS_5$. This geometry represents infinitely massive $D1, \bar{D1}$ brane sources as drawn in Fig. 1. Of course, the equations of motion for transverse oscillations are not affected by the boundary condition although the mode discretization does depend on boundary conditions. So the equations of motion apply equally well to $D0$ (static quarks), $D1$ and $D2$ sources.

2.1 Static solutions

The lightcone equations for static string solutions are

$$\left( G^2 x' \right)' = 0, \quad (G^2 x^2)' = 0, \quad (5)$$

$$G \frac{\partial G}{\partial \phi} x'^2 + \frac{1}{2} \frac{\partial G}{\partial \phi} \phi'^2 - (G \phi')' = 0. \quad (6)$$
Figure 1: String in AdS$_5$ fixed at the boundary as $\phi \to \infty$ on D1, $\bar{D}1$ branes aligned on the $x^3$ axis and separated by $L$ in the $x = (x^1, x^2)$ plane.

These equations imply two integrals of the motion
\[ x' = \frac{C}{G^2}, \quad C^2 + G\phi'^2 = 2\frac{p^-}{p^+} = \frac{M^2}{p^+}, \]
where $C$ is an integration constant and $M$ is the total system mass. Note that for static solutions, the lightcone quantization constraint $x^- = x' \cdot \mathbf{P} + \phi' \Pi$ implies that there is no extension in $x^-$. For an isolated quark, we would also want no extension in $x$, which would mean $C = 0$. For $q\bar{q}$ sources separated by a distance $L$ we have
\[ L = \int_0^{p^+} d\sigma x' = C \int_0^{p^+} \frac{d\sigma}{G^2}, \]
which together with the integrals of the motion determine the system mass, $M = \sqrt{2p^+p^-}$. For the AdS case $G = e^{\phi}/\gamma$ the static string's mass was found to be [11]
\[ M = 4\gamma \sqrt{G_{\text{max}}} - \frac{4\gamma^2 (2\pi)^3}{L \Gamma(1/4)^4}. \]
Which agrees with the known result [4], after we recall that $4\gamma^2 = R^2T_0 = R^2/2\pi\alpha' = \sqrt{\alpha_s N_c/\pi}$. The first divergent term is just twice the isolated quark mass, so the second finite term is the predicted interaction energy between infinitely massive quark and antiquark.

### 2.2 Small transverse oscillations

On the lightcone the ends of a static string stretched between two fixed points in transverse space automatically lie at the same longitudinal coordinate because $x^- = 0$. However, to calculate small oscillations about this static configuration on the lightcone, it is simplest to retain fixed Dirichlet conditions only on the transverse coordinates of the string ends, but allow the ends to move freely.
in the longitudinal direction, in the manner suggested in [15]. Fixing the longitudinal coordinates of the ends would impose a nonlinear constraint on the transverse oscillations which is awkward to implement, and would defeat the main purpose of using lightcone in the first place. Thus for lightcone quantization we shall instead consider a string stretched between two fixed 1-branes parallel to each other and to the z axis, and separated from each other by the distance $L$. As just explained the static solution for this physical situation is identical to that quoted in the previous subsection.

To study small oscillations, we replace $x, \phi$ by $x(\sigma) + x$, $\phi_c(\sigma) + \phi$ and expand the action to quadratic order in $x, \phi$. The classical equations of motion for $x, \phi_c$ guarantee that the linear terms in fluctuations vanish and we get

\[
S = S_c + \int d\sigma \frac{1}{2} \left[ \dot{x}^2 - \frac{C_c^2}{c^2} \dot{x}^2 + \phi_c^2 G^{-1} - \frac{1}{2} \frac{\phi_{c}\phi_{c}'}{c^2} G_{c}^{\prime} \right],
\]

where $G_c(\sigma) = G(\phi_c(\sigma))$ and $G_{c}'(\sigma) = \partial G/\partial \phi$ evaluated at $\phi = \phi_c(\sigma)$, etc. In the special AdS limit,

\[
S \to S_c + \int d\tau \frac{1}{2} \left[ \dot{x}^2 - \frac{C_{c}^2}{c^2} \dot{x}^2 + \phi_c^2 e^{-\phi_c/\gamma} - \frac{1}{2} \frac{\phi_{c}\phi_{c}'}{c^2} e^{\phi_c/\gamma} - 2 \left( e^{2\phi_c/\gamma} x_c^2 + \frac{1}{4} \frac{\phi_{c}\phi_{c}'}{c^2} e^{\phi_c/\gamma} \right) \right] .
\]

We see that the oscillations in $x$ perpendicular to $x_c$ decouple from the $\phi$ oscillations, whereas those parallel to $x_c$ have a coupling to the $\phi$ oscillations. In this subsection, we restrict attention to transverse oscillations where $x \cdot x_c = 0$. Two other independent modes will be considered in the following subsections. Letting $\omega$ be the oscillation frequency, we see that the transverse oscillations are governed by the ordinary differential equation

\[
\omega^2 f + \partial_\sigma (G_{c}^2 \partial_\sigma f) = 0, \quad f|_{\sigma=0, p^+} = 0 . \quad (12)
\]

We now specialize to the AdS case for which $G_c = r_c^2/R^2 = e^{\phi_c/\gamma}$. In order to simplify notation, we will also set $R = 1$ in what follows. Since the static solution $r_c(\sigma)$ is piecewise monotonic, it is possible to change variables from $\sigma$ to $r_c$ over each monotonic interval. Recall that

\[
r_c^2 = \frac{C_{c}^2}{4\gamma^2} \left( \frac{1}{r_{\min}^4} - \frac{1}{r_c^4} \right) . \quad (13)
\]

The fact that $\sigma$ measures $p^+$ implies that

\[
\frac{C_{c}}{\gamma} \int \frac{r_{\max}^2}{r_{\min}^2} \frac{d\chi}{\sqrt{\chi - 1/\chi}} \approx \frac{4r_{\min}^2 r_{\max}^2}{p^+} . \quad (14)
\]

Then we find, dispensing with the $c$ subscript,

\[
\frac{dr}{d\sigma} = \pm \frac{2r_{\min}^2 r_{\max}^2}{p^+} \sqrt{\frac{1}{r_{\min}^4} - \frac{1}{r_c^4}} . \quad (15)
\]

\[
\omega^2 f + \left( \frac{2r_{\min}^2 r_{\max}^2}{p^+} \right)^2 \sqrt{\frac{1}{r_{\min}^4} - \frac{1}{r_c^4}} \partial_\tau \left( \frac{1}{r_{\min}^4} - \frac{1}{r_c^4} \partial_\tau f \right) = 0 . \quad (16)
\]
Another useful coordinate is \( z \equiv R^2/r \) so with \( R = 1 \), we also write the small oscillation equation as,

\[
\omega^2 f + \left( \frac{2r_{\text{max}}}{p^+} \right)^2 \left[ (1 - z^4 r_{\text{min}}^4) \partial_z^2 f - \frac{2}{z} \partial_z f \right] = 0.
\] (17)

Now put \( \omega = 2\xi r_{\text{min}} r_{\text{max}} / p^+ \) and \( \hat{z} = r_{\text{min}} z \) to reduce the equation to that solved by Callan and Güijosa\[5\],

\[
\xi^2 f + (1 - \hat{z}^4) \partial_{\hat{z}}^2 f - \frac{2}{\hat{z}} \partial_{\hat{z}} f = 0.
\] (18)

This equation can be brought into a more familiar form,

\[
\frac{d^2 F(q)}{dq^2} + \xi^2 (\xi^4 - 1) F(q) = 0,
\] (19)

where \( F = f / (1 + \xi^2 \hat{z}^2) \), and \( q = \pm 2 \int_0^1 dt \sqrt{t^2 / (1 + \xi^2 t^2)^2} \sqrt{1 - t^4} \). With Dirichlet boundary conditions and \( r_{\text{max}} = \infty \) the exact eigenfrequency equation becomes

\[
\xi_n \sqrt{\xi_n^4 - 1} \int_0^1 \frac{t^2 dt}{[1 + \xi_n^4 t^2] \sqrt{1 - t^4}} = \frac{n\pi}{2}, \quad n = 1, 2, \ldots
\] (20)

The integral here is an elliptic integral and the equation must be solved numerically. However for large frequency it becomes elementary,

\[
\xi_n \approx \frac{(n + 1)\pi}{2} \left[ \int_0^1 \frac{dt}{\sqrt{1 - t^4}} \right]^{-1} = (n + 1) \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}.
\] (21)

The next correction is \( O(1/n) \). In fact it turns out that the exact \( n \) versus frequency curve is practically linear for \( n \geq 1 \) (see Fig. 2). This asymptotic form is the asymptote for the exact curve, the exact frequency being roughly 10% below the asymptote for \( n = 1 \).

We must now relate the \( \xi \)'s to actual physical energy levels. Since on the lightcone the Hamiltonian is \( p^- \), the \( \omega \)'s represent the excitations in this variable. To match with the semi-classical
solution in temporal quantization given below we must compare the finite excitation energies in the limit to infinitely massive source ($r_{\text{max}} \to \infty$). In the previous section in Eq. 9, we found that the $p^- = M^2/2p^+$ value of the static string solution was

$$p^-_c = \frac{1}{2p^+} \left[ 4\gamma r_{\text{max}} - \frac{4\gamma^2 (2\pi)^3}{L \Gamma(1/4)^4} \right],$$

as $r_{\text{max}} \to \infty$. To this we add the excitation energy

$$\Delta\{\xi_n\} = \sum_n N_n \omega_n \simeq \frac{4\gamma r_{\text{max}} \tau_{\text{min}}}{2p^+} \sum_n N_n \xi_n = \frac{8\gamma r_{\text{max}} (2\pi)^{3/2}}{2p^+ L \Gamma(1/4)^2} \sum_n N_n \xi_n.$$  

Thus we find (restoring the $R$ dependence by $r_{\text{max}} \to r_{\text{max}}/R$)

$$p^-_{\{\xi_n\}} \simeq \frac{1}{2p^+} \left[ (4\gamma \frac{r_{\text{max}}}{R})^2 + 8\gamma \frac{r_{\text{max}}}{R} \left( -\frac{4\gamma^2 (2\pi)^3}{L \Gamma(1/4)^4} + \frac{1}{L \Gamma(1/4)^2} \sum_n N_n \xi_n \right) + O(1) \right],$$

$$E_{\text{CM}}^{\{\xi_n\}} = \sqrt{2p^+ p^-_{\{\xi_n\}}} \simeq 4\gamma \frac{r_{\text{max}}}{R} - \frac{4\gamma^2 (2\pi)^3}{L \Gamma(1/4)^4} + \frac{1}{L \Gamma(1/4)^2} \sum_n N_n \xi_n.$$  

As we will now show this excitation spectrum for $M \to \infty$ is in exact agreement with the semi-classical approximation using temporal coordinates $x^0 = \tau$ for all $L$. Moreover, it is interesting that for flat space, where the lightcone action is quadratic and therefore the semi-classical solution to the stretched string is exact [16], the agreement with temporal quantization also requires taking the $L \to \infty$ limit. In this limit the leading excitation energy is the $1/L$ universal conformal “Lüscher term”. Interestingly in the present case of a truly conformal AdS$_5$ theory only $M \to \infty$ is needed for agreement between the leading lightcone and temporal semi-classical approximations.

### 2.3 Comparison to temporal quantization

We have so far identified only one transverse mode in our lightcone analysis. We consider here the small oscillation problem using the temporal parameters where the evolution parameter is target space time, $t = x^0 = \tau$. In this parametrization, it will be obvious that there are two degenerate modes transverse to a stretched string, and another mode for “longitudinal” oscillation. We shall return in the next sub-section to see how these two additional modes manifest themselves in a lightcone approach.

To do this most systematically, we use the phase space version of Hamilton’s principle:

$$\mathcal{L} \equiv \dot{x} \cdot \vec{P} + \dot{\phi} \Pi - P^0 - \frac{\lambda}{2} \left( \vec{P}^2 + G^2(\phi)\dot{x}^2 - P^{02} + G(\phi)(\Pi^2 + \phi'^2) \right) - \mu(x \cdot \vec{P} + \phi' \Pi),$$

where $\lambda, \mu$ are Lagrange multipliers implementing the constraints associated with parameterization invariance. The lost equation from setting $x^0 = \tau$ is $\dot{P}^0 - (\mu P^0)' = 0$. Integrating out $P^0$ gives $P^0 = 1/\lambda$. With a time dependent reparametrization of $\sigma$ we can set $\mu = 0$, and with a further time independent one we can set $\lambda = 1$ ($\mu = 0$ and $P^0 = 1/\lambda$ imply $\lambda = 0$). Doing this fixes the scale of
\[ \dot{\sigma} = \frac{1}{2} \dot{\phi}^2 + G^2(\phi) \ddot{x}^2 + G(\phi)(\dot{\Pi}^2 + \phi^2) \]  
\[ 1 = \ddot{\phi}^2 + G^2(\phi) \ddot{x}^2 + G(\phi)(\Pi^2 + \phi^2) \]  
\[ 0 = \ddot{\sigma} - \phi' \dot{\Pi} \]  
(26)

At this point we can integrate out momenta to return to the coordinate space Lagrangian:

\[ \mathcal{L} = \frac{1}{2} \left( \ddot{x}^2 + \frac{\dot{\phi}^2}{G(\phi)} - 1 - G^2(\phi) \ddot{x}^2 - G(\phi) \phi^2 \right) \]  
\[ 1 = \ddot{x}^2 + G^2(\phi) \ddot{x}^2 + \frac{\dot{\phi}^2}{G(\phi)} + G(\phi) \phi^2 \]  
\[ 0 = \dddot{x} + \frac{\phi' \dot{\phi}}{G(\phi)} \]  
(29)

The need for these side constraints, which are absent on the lightcone, is the main obstacle to quantization in this parametrization. Classically there is of course no problem with them. If the second constraint holds, the right side of the first constraint is a constant of the motion following from the equations of motion for \( \ddot{x}, \dot{\phi} \). It can then be used in place of one of the equations of motion. Thus to find all the classical solutions we only need solve all but one of the equations.

\[ \ddot{\phi} - G^2 \phi'' + G^2 \frac{\partial G}{\partial \phi} \ddot{x}^2 - \frac{1}{2} \frac{\partial G}{\partial \phi} \left( G \phi^2 + \frac{\dot{\phi}^2}{G} \right) = 0 \]  
\[ \dddot{x} - (G^2 \dddot{x})' = 0 \]  
(32)

For small oscillations about a static solution it will be useful to use the constraint to eliminate \( \dddot{x}^2 \) from the \( \phi \) equation:

\[ \ddot{\phi} - G^2 \phi'' - \frac{3}{2} G \frac{\partial G}{\partial \phi} \phi^2 + \frac{\partial G}{\partial \phi} \left( \ddot{x}^2 + \frac{3 \phi^2}{2 G} \right) = 0 \]  
\[ \dddot{x} + \frac{1}{2 G \phi} \left[ G^2 (1 - G \phi^2) \right]' - \frac{\partial G}{\partial \phi} \left( \ddot{x}^2 + \frac{3 \phi^2}{2 G} \right) = 0 \]  
(34)

For small oscillations about a static solution the last term is second order in fluctuations and can be dropped. Without the constraints, the equations for \( \phi, \dddot{x} \) are identical in form to the lightcone equations for \( \phi, x \). But we must bear in mind that here \( \tau \) is ordinary time rather than lightcone time and \( \sigma \) is a measure of ordinary energy rather than of \( p^+ \).

For static solutions, the second constraint is automatically satisfied and the equations for \( \dddot{x} \) are immediately integrable, so it is convenient to drop the \( \phi \) equation of motion and obtain

\[ \dddot{x} = \frac{\dddot{\sigma}}{G^2}, \quad \phi^2 = \frac{1}{G} - \frac{C^2}{G^3} \]  
(36)
where $\vec{C}$ is again a constant vector. For fixed ends separated by the vector $\vec{L}$ we have

$$E = \int_0^E d\sigma = 2 \int_{\phi_{\text{min}}}^{\phi_{\text{max}}} d\phi \frac{\sqrt{G}}{\sqrt{1 - C^2/G^2}},$$

$$|\vec{L}| = 2 \int_{\phi_{\text{min}}}^{\phi_{\text{max}}} d\phi \frac{\sqrt{G}}{G^2 \sqrt{1 - C^2/G^2}},$$

where the constant $C$ can be fixed by the condition $\dot{\phi}' = 0$, leading to $C = G(\phi_{\text{min}})$. For the AdS case $G(\phi) = e^\phi/\gamma \equiv r^2$, so $d\phi = 2\gamma dr/r$, $r_{\text{min}}^2 = C$ and we obtain

$$E = 4\gamma \int_{r_{\text{min}}}^{r_{\text{max}}} dr \frac{r^2}{\sqrt{r^4 - r_{\text{min}}^4}},$$

$$|\vec{L}| = 4\gamma r_{\text{min}}^2 \int_{r_{\text{min}}}^{r_{\text{max}}} dr \frac{1}{r^2 \sqrt{r^4 - r_{\text{min}}^4}}.$$

We see that $L, E$ coincide with the lightcone evaluations of $L, \sqrt{2p^+p^-}$ as they must.

Now we come to small oscillations about the static solution $\vec{x}_c, \phi_c$. From now on call the fluctuations about these solutions $\vec{x}, \phi$. First take the case of oscillations, $\vec{x}_\perp$, transverse to $\vec{L}$. In this case the $\vec{x}_\perp$ fluctuations enter the constraints and the equation for $\phi$ quadratically, so we can consistently set $\phi = 0$, and seek solutions of

$$\ddot{\vec{x}}_\perp - (G^2(\phi_c)\vec{x}'_\perp)' = 0.$$  

As we have already mentioned this differential equation is identical in form to the lightcone equation for $x$. However, there are differences in the passage to the Callan-Güijosa form. Namely in the relation between $\omega$ and $\xi$. For ordinary time we find $\omega_{\text{temp}} = \xi r_{\text{min}}/2\gamma$ compared to the lightcone relation $\omega_{\text{LC}} = 2\xi r_{\text{min}}r_{\text{max}}/p^+$. The different prefactors here precisely account for the fact that $\hbar\omega_{\text{LC}}$ gives the semi-classical excitation energy of lightcone energy $p^-$, whereas $\hbar\omega_{\text{temp}}$ gives the excitation energy of ordinary energy $E$. This is easily seen by comparing the two lines of (24). Here we see that there are two modes of transverse oscillation with identical frequencies. On the lightcone, in contrast, we only see one manifest transverse mode. We return to the other transverse lightcone mode, related to fluctuations in $x_\parallel$, at the end of the next subsection.

Next we consider oscillations, $x_\parallel$, parallel to $\vec{L}$. For definiteness, take $\vec{L} = \hat{x} \vec{x}$, where $\hat{x}$ is also one of the lightcone transverse directions. In temporal quantization the equations to solve couple the $x_\parallel$ coordinate with $\phi$. The linearized constraints become

$$G^2(\phi_c)(x_c'^2 + 2x_c'x_\parallel') + G(\phi_c)(\phi_c' + \phi')^2 = 1,$$

$$x_c'\ddot{x}_\parallel + \frac{\phi_c'\dot{\phi}}{G(\phi_c)} = 0,$$

which then serve to express $\dot{x}_\parallel, x_\parallel'$ as explicit functions of $\phi$. Thus the longitudinal small oscillations are completely controlled by the linearized approximation of (35),

$$\ddot{\phi} + \frac{1}{G_c\phi_c'} \left[ -G_c^3\phi_c'\phi'' + G_c \frac{\partial G_c}{\partial \phi_c} \left( 1 - \frac{3}{2} G_c^2 \phi_c^2 \right) \right]' = 0,$$

9
where $G_c(\sigma) = G(\phi_c(\sigma))$.

It is useful to change variable from $\sigma$ to $x_c$, which measures the length of the stretched string, i.e., using $x_c = C/G_c^2$ and $-L/2 \leq x_c \leq L/2$. One finds that the above linearized equation can be written in a generic form\cite{14}

$$
(\partial_t^2 - v^2(x_c)\partial_{x_c}^2 + m^2(x_c))\Psi(t, x_c) = 0 ,
$$

where

$$
m^2(x_c) = (G_0^2/G_c^2)[G''_c - (3/2)G''_c/G_c] ,
$$

and $v^2 = (G_0^2/G_c^2)$. Note that the wave function has been rescaled, $\Psi = G_c^{1/2} \phi$, and that the derivatives are taken respect to $\phi$, $G'_c = \partial G/\partial \phi$, $G''_c = \partial^2 G/\partial \phi^2$ evaluated at $\phi = \phi_c$. We have also introduced $G_0 \equiv C = G(\phi_{min})$ to rationalize the notation. In deriving this result, we have made use of the fact that

$$
\phi'_c = G_c^{-3/2}(G_c^2 - G_0^2)^{1/2} .
$$

This longitudinal mode has been identified previously in Ref.\cite{14} as the “radion” mode. In particular, for a metric deformation with confinement, this mode is massive, with the mass scale set by the glueball mass\cite{4}. Here, for pure $AdS$, the frequency scales as $1/L$.

### 2.4 Radion and second transverse modes on the lightcone

In lightcone quantization, we have the equation of motion for $x$, and the equation of motion for $\phi$. But there are no further constraints: instead the right side of (30) is just $2\dot{x}^-$ and the right side of (31) is $x^-$:

$$
\dot{x}^- = \frac{1}{2} \left( \dot{x}^2 + G^2(\phi)x'^2 + \frac{\dot{\phi}^2}{G(\phi)} + G(\phi)\phi'^2 \right) ,
$$

$$
x^- = x' \cdot \dot{x} + \frac{\phi'\dot{\phi}}{G(\phi)} .
$$

The integrability condition for this pair of equations, that the $\sigma$ derivative of the first equals the $\tau$ derivative of the second, follows from the equations of motion for $x, \phi$. But for strictly longitudinal classical motion we do want $x^- = 0$, in which case (31) will also be imposed on the lightcone. Then the right side of (48) is a constant of the motion fixed to be $2p^-/\rho^+$ by the fact that $\dot{x}^-$ is the density of $p^-$ momentum. The equations for longitudinal oscillations on the lightcone can be made identical to those in temporal quantization by scaling lightcone $\sigma, \tau$ by a common factor $\sqrt{p^+/2p^-}$. This implies that the oscillation frequencies in the two approaches will be related by

\footnote{We thank Igor Klebanov for drawing our attention to a typo in Eq. (23) in Ref.\cite{14} in the expression for the mass. There $G_0(z)$ has a different definition by expressing the metric as $ds^2 = \frac{R^2}{z^4}[dz^2 + G^2(z)dz^2]$, which should have led to the expression

$$
m^2(x_c) = -\frac{1}{z_{max}^4} \frac{d}{dz} \left( \frac{1}{G^2} \frac{d}{dz} \right) + \frac{2z^2}{G^2} \big|_{z = z_c} .
$$

}
\[ \omega_{LC} = \omega_{ord} \sqrt{2p^- / p^+}, \]

in the limit of infinitely massive sources. In turn this implies that the total energy calculated in temporal quantization will agree with \( \sqrt{2p^+ p^-} \) calculated on the lightcone:

\[
\sqrt{2p^+(p_0^- + \hbar \omega_{LC})} \simeq \sqrt{2p^+ p_0^-} \left( 1 + \hbar \frac{\omega_{LC}}{2p_0^-} \right) = \sqrt{2p^+ p_0^-} + \hbar \omega_{ord}. \tag{50}
\]

Here \( p_0^- \) is the lightcone energy of the static solution. Thus as far as longitudinal oscillations between infinitely massive sources are concerned the two approaches agree.

Finally, we consider the second transverse oscillation from the point of view of lightcone. Then we look for solutions in which fluctuations in both transverse lightcone coordinates vanish. Then the fluctuations are completely described by fluctuations in \( \phi \). The oscillation frequencies could be determined by analyzing the \( \phi \) equation of motion. However, it is quicker to find the equation that \( x^- \) satisfies because it is a function of \( x, \phi \). Indeed for any \( x, \phi \) satisfying their respective equations it is straightforward to compute

\[
\ddot{x}^- - (G^2 x^-)' = 0. \tag{51}
\]

This is the same differential equation satisfied by \( x \). For the static solution \( x_c' = 0 \), so the small oscillation approximation implies

\[
\ddot{x}^- - (G^2_c x^-)' = 0, \tag{52}
\]

the same equation as oscillations in \( x \) perpendicular to \( L \). The only difference is that the way we have set up boundary conditions on the lightcone, Dirichlet conditions will not apply to \( x^- \). Indeed, we have Dirichlet conditions on \( x, \phi \). This clearly implies Neumann conditions on \( x^- \). Thus with the lightcone setup the eigenfrequencies for the second transverse mode are determined by the boundary value problem

\[
\omega^2 f + (G^2_c f')' = 0
\]

\[
f'|_{\sigma=0, p^+} = 0. \tag{53}
\]

By comparison with Eq. 12, the spectrum of oscillation frequencies for this second transverse mode will differ from those for the “manifest” transverse mode due only to the change in the boundary conditions.

### 3 An interpolation between weak and strong coupling

In this section we turn our attention to the field description of the dual to a free string stretched between two 1-branes on the AdS boundary: the response of gauge fields to a quark and antiquark constrained to move on two lines parallel to the \( z \)-axis in the \( \ell_H \) limit \( N_c \to \infty \). We should therefore try to sum all of the planar Feynman diagrams contributing to this process. This is a daunting prospect, for which the lightcone worldsheet formalism [8–10] might be useful. At weak coupling it suffices to sum only the ladder diagrams, a tractable subset of all planar diagrams. Here we discuss the ladder sum in both Feynman and lightcone gauge. The corresponding Bethe-Salpeter equation that sums ladder diagrams is tractable at all coupling strengths. As already noted in [7, 12] its solution at strong coupling captures many qualitative features of the known strong coupling limit in the \( \mathcal{N} = 4 \) case. This makes it an instructive model of the interpolation between weak and strong coupling.
### 3.1 Bethe-Salpeter equation for heavy sources on the lightcone

In momentum space the ladder approximation to the Bethe-Salpeter equation [17] for a massive scalar quark-antiquark system, assuming the 't Hooft limit and with \( P, p \) the total and relative momenta of the quarks, reads

\[
(m^2 + (P/2 + p)^2)(m^2 + (P/2 - p)^2)\Psi_P(p) = \frac{g^2 N_c}{2} \int \frac{-id^4k}{(2\pi)^4 k^2} (P + 2p - k)^\mu (-P + 2p - k)^\nu N_{\mu\nu}(k) \Psi_P(p - k) ,
\]

where \( N_{\mu\nu} = \eta_{\mu\nu} \) in Feynman gauge and \( N_{\mu\nu} = \eta_{\mu\nu} - k_\mu \eta_{\nu-} - k_\nu \eta_{\mu-} / k_- \) in lightcone gauge. To adapt this equation to heavy external sources, we take \( m \to \infty \) and look for mass eigenstates of total mass \( M = 2m - B \) with \( B \), the binding energy, finite. Since \( P^2 = -M^2 \), we also neglect \( p, k \) in comparison to \( P \). Then the equation reduces to

\[
\left( \frac{B^2}{4} - \frac{(p \cdot P)^2}{M^2} \right) \Psi_P(p) = -\frac{g^2 N_c}{2} \int \frac{-id^4k}{(2\pi)^4 k^2} \frac{P^\mu P^\nu N_{\mu\nu}(k)}{M^2} \Psi_P(p - k) .
\]

To simplify even further we can work in the center of mass frame \( P = (M, \vec{0}) \):

\[
\left( \frac{B^2}{4} - (p^0)^2 \right) \Psi_P(p) = -\frac{g^2 N_c}{2} \int \frac{-id^4k}{(2\pi)^4 k^2} N_{00}(k) \Psi_P(p - k) .
\]

To describe static sources with separation \( \vec{L} \) we just put \( \Psi_P = \phi(p^0) e^{i\vec{L} \cdot \vec{p}} \) and Fourier transform \( \phi(p^0) \) with respect to \( t \) obtaining

\[
\left( \frac{B^2}{4} + \partial_t^2 \right) \psi(t) = -\frac{g^2 N_c}{2} \int \frac{-id^4k}{(2\pi)^4 k^2} e^{i\vec{L} \cdot \vec{p} - i\phi t} N_{00}(k) \psi(t) .
\]

In Feynman gauge, \( N_{00} = \eta_{00} = -1 \) and the Fourier integral is elementary, yielding, after Wick rotation \( it \to t \), the Schrödinger equation analyzed in [12]:

\[
\left[ -\partial_t^2 - \frac{\lambda}{L^2 + t^2} \right] g(t) = -\frac{B^2}{4} g(t) ,
\]

where \( \lambda = \alpha_s N_c / 2\pi \). In the language of the Wilson loop, this system corresponds to a rectangular loop of width \( L \) and length \( T \to \infty \). In the context of the \( N = 4 \) AdS/CFT correspondence the authors of [12] studied this approximation as a guide to the physics of the interpolation between weak and strong coupling. Later it was shown in [18, 19] that for a circular Wilson loop the corresponding “rainbow” graph approximation is actually exact in the limit \( N_c \to \infty \), provided scalar ladder rungs are also included with a strength such that \( \lambda \) doubles: \( \lambda \to N_c \alpha_s / \pi \). This is not the case for rectangular Wilson loops, but it is natural to include the scalar rungs when applying the interpolation to the \( N = 4 \) case.

From the point of view of lightcone quantization, we interpret the equation a little differently. We first consider sources free to move on 1-branes parallel to the \( z \) axis separated by a distance \( L \). This means putting \( \Psi_P(p) = \phi(p^+ + p^-) e^{i\vec{L} \cdot \vec{p}} \), and Fourier transforming \( \phi(p^+, p^-) \) in the variable \( p^- \) with respect to lightcone time \( x^+ \). Recalling that \( p^0 = (p^+ + p^-) / \sqrt{2} \), this leads to

\[
\left( \frac{B^2}{4} - \frac{1}{2}(p^+ + i\partial_+)^2 \right) \psi(p^+, x^+) = -\frac{g^2 N_c}{2} \int \frac{-id^4k}{(2\pi)^4 k^2} e^{i\vec{L} \cdot \vec{p} - ik^- x^+} N_{00}(k) \psi(p^+ - k^+, x^+) .
\]
The integrations over $k^-$ and $k$ can be easily done for both the Feynman and lightcone gauge choices.

\[
D(k^+, x^+) \equiv -\int \frac{-idkd(k^-)}{(2\pi)^4k^2}N_{00}e^{iL\cdot k}e^{-x^+k^+}
\]

\[
= \frac{\theta(x^+)}{8\pi^2} \int_0^{\infty} dk^+ e^{-k^+L^2/2\tau - i(x^+ + a^+)k^+}
\]

\[
= \frac{\theta(x^+) + \theta(-x^+)}{16\pi^2} \int_{-\infty}^{0} dk^+ e^{-k^+L^2/2\tau - i(x^+ + a^+)k^+}
\]

\[
\equiv \frac{1}{4\pi^2 (L^2 + a^2)^2} \int_{-\infty}^{\infty} dk^+ e^{-k^+L^2/2\tau - i(x^+ + a^+)k^+}
\]

where $\tau \equiv ix^+$ will eventually be taken to be real. The equation now takes the form

\[
\left( B^2 - \frac{1}{2}(p^+ + i\partial_+)^2 \right)\psi(p^+, x^+) = \frac{g^2 N_c}{2} \int dk^+ D(k^+, x^+)\psi(p^+ - k^+, x^+).
\]

To see that this equation in Feynman gauge has identical content to (59), put $\psi = e^{i(x^+ + a^+)p^+}f(x^+)$, which corresponds to putting $z = a/\sqrt{2}$, to obtain

\[
\left( B^2 + \frac{1}{2}g^2 \right) f(x^+) = \frac{g^2 N_c}{2} \int dk^+ D(k^+, x^+)e^{i(x^+ + a^+)k^+}f(x^+).
\]

Then for Feynman gauge we find

\[
\int dk^+ D(k^+, x^+)e^{-i(x^+ + a^+)k^+} = \frac{\theta(x^+)}{8\pi^2} \int_0^{\infty} dk^+ e^{-k^+L^2/2\tau - i(x^+ + a^+)k^+}
\]

\[
- \frac{\theta(-x^+)}{8\pi^2} \int_{-\infty}^{0} dk^+ e^{-k^+L^2/2\tau - i(x^+ + a^+)k^+}
\]

\[
\equiv \frac{1}{4\pi^2 (L^2 + a^2)^2} \int_{-\infty}^{\infty} dk^+ e^{-k^+L^2/2\tau - i(x^+ + a^+)k^+}
\]

But since this solution fixes $z = a/\sqrt{2}$, $\sqrt{L^2 + a^2}/2$ is precisely the separation between the sources, so the change of variables $t = i\sqrt{2}(x^+ + a^2/2)$ reduces the lightcone Feynman gauge equation to (59).

For completeness we also work out $D(k^+, x^+)$ for lightcone gauge, putting $a = 0$ for simplicity.

\[
\int dk^+ D(k^+, x^+)e^{-ix^+k^+} = \frac{1}{4\pi^2} \frac{L^2}{L^2 + 2\tau^2} \int_0^{\infty} dk^+ e^{-k^+L^2/2\tau - ix^+k^+}
\]

\[
- \frac{\theta(x^+)}{8\pi^2} \int_{-\infty}^{0} dk^+ e^{-k^+L^2/2\tau - ix^+k^+}
\]

\[
\equiv \frac{1}{4\pi^2} \frac{L^2}{L^2 + 2\tau^2} \int_{-\infty}^{\infty} dk^+ e^{-k^+L^2/2\tau + ix^+k^+}.
\]

This expression is problematic because the integrals diverge logarithmically at $k^+ = 0$. Let’s regulate the integrals by inserting a factor $(k^+ / \mu)^\alpha$, and use

\[
\int_0^{\infty} \frac{dk^+}{k^+} (k^+ / \mu)^\alpha e^{-k^+A} = \Gamma(\alpha)(\mu A)^{-\alpha} \rightarrow \frac{1}{\alpha} + \Gamma'(1) - \ln(\mu A).
\]

Then one finds after some rearrangement and putting $t = \tau \sqrt{2}$,

\[
\int dk^+ D(k^+, x^+)e^{-ix^+k^+} = \frac{1}{4\pi^2} \left[ \frac{1}{t^2} \ln \frac{L^2 + t^2}{L^2} - \frac{1}{L^2 + t^2} \right] - \frac{1}{4\pi^2 t^2} \left[ \frac{1}{\alpha} + \Gamma'(1) - \ln \frac{\mu L^2}{t^2} \right].
\]
The second term on the right is divergent as $\alpha \to 0$ and violates scale invariance and is singular as $t \to 0$. Its presence indicates that the naive ladder approximation in lightcone gauge is deficient due to $k^+ = 0$ singularities. On the other hand, we know from detailed one-loop studies that in a complete gauge invariant computation these singularities are harmless, see for example [20, 21]. The ladder diagrams are not gauge invariant so it is not inconsistent for these singularities to cause problems in them. In fact, different regulators of the $k^+ = 0$ singularities give different intermediate results. It is straightforward to show, for example, that the Mandelstam-Leibbrandt (ML) principal value prescription [22, 23] applied to the Fourier transform of the lightcone gauge gluon propagator gives precisely the first term on the right of (68). That is, it interprets the divergent second term as zero. If we use the ML prescription, then the lightcone gauge version of the potential term appearing in (59) is

$$V_{\text{LCML}}(t) = -\lambda \left[ \frac{1}{t^2} \ln \frac{L^2 + t^2}{L^2} - \frac{1}{L^2 + t^2} \right].$$

(69)

In the weak coupling limit this potential can be replaced by a delta function with coefficient

$$\int dt V_{\text{LCML}}(t) = -\lambda \int dt \left[ \frac{1}{t^2} \ln \frac{L^2 + t^2}{L^2} - \frac{1}{L^2 + t^2} \right] = -\lambda \int dt \frac{1}{L^2 + t^2} = -\pi \lambda,$$

(70)

which follows immediately from integrating the first term in square brackets by parts. We see that this agrees with the integral of the analogous potential from Feynman gauge. Thus the ML treatment of lightcone gauge ladder diagrams agrees in weak coupling with Feynman gauge. In the remainder of this section we further analyze the ladder model using Feynman gauge, but the lightcone gauge a la ML should give similar results.

### 3.2 Ladder diagram model of the spectrum

Ref. [12] showed that in the weak coupling limit the Schrödinger equation (59) has a single bound state with $B = \pi \lambda / L$ which is just the Coulomb interaction energy $-N_c g^2 / 4\pi L = -N_c \alpha_s / 2L$ in pure QCD (or twice this for $N' = 4$) between heavy quark and antiquark in the fundamental representation. They also noted that in the strong coupling limit the Schrödinger equation is well described by its harmonic oscillator approximation

$$\left[ -\partial_t^2 - \frac{\lambda}{L^2} + \frac{\lambda}{L^4} t^2 \right] g(t) = -\frac{B^2}{4} g(t),$$

(71)

whose natural frequency is $\omega_0 = \sqrt{2\lambda} / L$, so

$$B_N = 2\sqrt{\frac{\lambda}{L^2} - \frac{\sqrt{2\lambda}}{2L^2} - N\sqrt{2\lambda}} \gtrsim 2\sqrt{\frac{\lambda}{L}} - \frac{\sqrt{2}}{2L} - N\frac{\sqrt{2}}{L}.$$  

(72)

In the case of lightcone gauge the potential $V_{\text{LCML}}(t)$ has minima away from $t = 0$ at $t = \pm x_0 L$ where $x_0$ satisfies

$$\ln(1 + x_0^2) = \frac{x_0^2 (1 + 2x_0^2)}{1 + x_0^2}$$

(73)

for which $x_0 \simeq 1.47$. In this case the harmonic approximation reads

$$V_{\text{LCML}}(t) = -\frac{\lambda}{L^2} \frac{x_0^2}{(1 + x_0^2)^2} + \frac{2\lambda}{L^4} \frac{x_0^2 - 1}{(x_0^2 + 1)^3} (t \pm x_0 L)^2 \simeq -0.216 \frac{\lambda}{L^2} + 0.0735 \frac{\lambda}{L^4} (t \pm x_0 L)^2.$$

(74)
So besides a difference in the numerical coefficients, the lightcone gauge case shows two families of nearly degenerate oscillator levels at strong coupling.

We close this subsection with an analysis of the Feynman gauge ladder model to find the coupling at which, in this model, the first discrete levels peel off the continuum. In terms of the Schrödinger equation (59) the first discrete level is the first excited state which is present only for sufficiently large coupling. To do this let us first simplify the equation by defining \( x = t/L \) and \( b = BL/2 \):

\[
\left[-\partial_x^2 - \frac{\lambda}{1 + x^2}\right] g = -b^2 g. \tag{75}
\]

When \( x \gg 1 \) the solution to this equation is just the Kelvin function \( g_\infty(x) = \sqrt{x}K_\nu(bx) \), with \( \nu = \sqrt{1/4 - \lambda} \). We want to find the smallest \( \lambda \) for which a discrete level forms just below threshold, i.e. whose eigenvalue \( b^2 \ll 1 \). In this case there is a region of \( x, 1 \ll x \ll 1/b \) for which the small argument approximation of \( K_\nu \) is valid:

\[
g_\infty(x) = \sqrt{x}K_\nu(bx) \simeq \sqrt{x} \frac{\pi}{2 \sin \pi \nu} \left[ \frac{1}{\Gamma(1 - \nu)} \left( \frac{bx}{2} \right)^{-\nu} - \frac{1}{\Gamma(1 + \nu)} \left( \frac{bx}{2} \right)^{\nu} \right]. \tag{76}
\]

Note that in the range \( 0 < \lambda < 1/4 \) for which \( \nu \) is real and positive, the second term in this approximation can be neglected.

On the other hand, in the region \( 0 < x \ll \sqrt{\lambda}/b \), the solution is well-approximated by the solution \( g_0 \) of the Schrödinger equation with \( b = 0 \)

\[
\left[-\partial_x^2 - \frac{\lambda}{1 + x^2}\right] g_0 = 0. \tag{77}
\]

Since we are searching for the first excited state, \( g \) and \( g_0 \) must vanish at \( x = 0 \). The solution of this last equation is

\[
g_0(x) = \sqrt{1 + x^2}(A_1 P_{\nu-1/2}(ix) + A_2 Q_{\nu-1/2}(ix)), \tag{78}
\]

where \( P_{\nu-1/2}, Q_{\nu-1/2} \) are the Legendre functions. It follows from the differential equation that in the region \( x \gg 1 \) it has the behavior

\[
g_0 \simeq A\sqrt{x}(x^\nu + C(\lambda)x^{-\nu}). \tag{79}
\]

If \( C(\lambda) \) is finite and \( \lambda < 1/4 \) the second term of this asymptotic behavior is negligible. And in the region of overlapping validity matching \( g_0 \) and \( g_\infty \) is impossible.

By investigating the properties of Legendre functions one finds that the condition \( g_0(0) = 0 \) implies:

\[
C(\lambda) = \frac{2^{-2\nu}\Gamma(-\nu)\Gamma(\nu - 1/2)\cos \pi(\nu/2 + 1/4)}{\Gamma(\nu)\Gamma(-\nu - 1/2)\cos \pi(-\nu/2 + 1/4)} = -\frac{2^{-2\nu}\Gamma(1 - \nu)\Gamma(3/2 + \nu)\sin \pi(1/4 - \nu/2)}{\Gamma(1 + \nu)\Gamma(3/2 - \nu)\sin \pi(1/4 + \nu/2)}. \tag{80}
\]

This result applies not only to the first excited level but to all odd parity levels. Comparing to (76) we read off the matching condition valid whenever \( 1 \ll x \ll 1/b, \sqrt{\lambda}/b \):

\[
\left( \frac{b}{2} \right)^{-2\nu} = \frac{2^{-2\nu}\Gamma(1 - \nu)^2\Gamma(3/2 + \nu)\sin \pi(1/4 - \nu/2)}{\Gamma(1 + \nu)^2\Gamma(3/2 - \nu)\sin \pi(1/4 + \nu/2)}. \tag{81}
\]
If $\sqrt{\lambda/b} \gg 1$, there is an overlapping region where $g$ is simultaneously well approximated by both $g_0$ and $g_\infty$. We see explicitly that the right side is finite in the range $0 < \lambda < 1/4$ so we conclude that there are no odd parity discrete levels with $b \ll 1$ for $\lambda < 1/4$. Since the first excited discrete level must be odd parity and since the only discrete level for weak coupling is the ground state, it follows from continuity that there are no excited discrete levels at all for these $\lambda$. Thus the critical $\lambda$ we are seeking must satisfy $\lambda_c \geq 1/4$.

Indeed it is exactly equal to $1/4$. For $\lambda > 1/4$, $\nu = i \sqrt{\lambda - 1/4}$ and both terms in the asymptotic approximations of $g_\infty$ and $g_0$ must be kept and there is sufficient flexibility to match the two asymptotic behaviors in the region of common validity. Indeed, it is easy to see that when $\nu$ is imaginary and fixed there is actually an infinite accumulation of levels approaching threshold. Let $b_1 \ll 1$ be an eigenvalue solving the matching condition. Then $b_1 e^{-\pi n/|\nu|}$ for $n$ an integer also obeys the matching condition. For all $n > 0$ these new levels are closer to threshold than $b_1$ and hence all are valid eigenvalues.

For even parity states, we impose vanishing derivative $g_0'(0) = 0$ and following the same steps arrive at

$$C^{\text{even}}(\lambda) = -\frac{2^{-2\nu \Gamma(1 - \nu) \Gamma(3/2 + \nu)} \sin \pi (1/4 + \nu/2)}{\Gamma(1 + \nu) \Gamma(3/2 - \nu) \sin \pi (1/4 - \nu/2)}. \quad (82)$$

Putting $\nu = i |\nu|$ we find

$$\frac{\sin \pi (1/4 + \nu/2)}{\sin \pi (1/4 - \nu/2)} = \frac{1 - ie^{-|\nu|}}{1 + ie^{-|\nu|}}, \quad \frac{\sin \pi (1/4 - \nu/2)}{\sin \pi (1/4 + \nu/2)} = -i \frac{1 + ie^{-|\nu|}}{1 - ie^{-|\nu|}}. \quad (83)$$

At strong coupling these two factors differ by a sign. Thus the even levels near threshold are uniformly interleaved between the odd levels at strong coupling: $E_n \simeq -b_1 e^{-\pi n/|\nu|}$, in agreement with a slightly different argument in [7].

We conclude that an infinite number of discrete levels appear for any $\lambda > \lambda_c = 1/4$. Referring back to the relation between $\lambda$ and $\alpha_s$, we find the new states coming in for

$$\frac{N_c \alpha_s}{\pi} > \frac{1}{2}. \quad (84)$$

For $N_c = 3$ this gives $\alpha_s^c = \pi/6 \simeq 0.523$, slightly bigger than $1/2$. Though not tiny, this value of $\alpha_s$ is small enough to hope that radiative corrections to the ladder approximation may be under enough control near the critical value to allow us to assess whether a similar transition appears in large $N_c$ QCD itself. In that case of course the coupling depends on the separation $\alpha_s(L)$, and we should instead speak of a critical separation, above which extra valence gluons may start playing a decisive role in the confining gluonic flux tube [13].

### 3.3 A one gluon truncation of the ladder model

Some insight into the physics of the discrete levels in the ladder approximation can be gleaned by considering the intermediate states included in the sum. Although ladders are an iteration of single gluon exchange, there are unitarity cuts through them that include arbitrary numbers of gluons. Because of the restriction to ladder diagrams these gluons do not interact with each other except for a veto for one crossing another. This will change with the inclusion of other planar diagrams. We suggest that the discrete levels are a reflection of these states with high numbers of gluons.
As evidence for this we study a truncation of the ladder sum that removes the contribution of intermediate states with more than one gluon. To do this we first present the Bethe-Salpeter equation in the form of an integral equation:

\[ \psi(T,U) = \lambda \int dt du \theta(T-t)\theta(U-u) \frac{1}{(t-u)^2 + L^2} \psi(t,u) . \]  

With \( t < T \), \( t \) and \( T \) mark times for two sequential gluon emission from (absorption by) one source whereas \( u \) and \( U \), with \( u < U \), mark times for the absorption by (emission from) the other. However, multi-gluon states are still included because the integral includes regions with \( t > U \) or \( u > T \). We can veto this possibility by including two more step functions in the integrand:

\[ \psi(T,U) = \lambda \int dt du \theta(T-t)\theta(T-u)\theta(U-u)\theta(U-t) \frac{1}{(t-u)^2 + L^2} \psi(t,u) . \]  

Differentiating this modified equation with respect to \( T \) and \( U \) leads to an integro-differential equation:

\[ \frac{\partial^2 \psi}{\partial T \partial U} = \lambda \delta(T-U) \int du \theta(T-u)\theta(U-u) \frac{1}{(u-U)^2 + L^2} (\psi(U,u) + \psi(u,U)) . \]  

Energy eigenstates correspond to the ansatz \( \psi(T,U) = e^{-E(T+U)/2} f(T-U) \), resulting in the equation

\[ \left( \frac{E^2}{4} - \frac{\partial^2}{\partial T^2} \right) f(T) = \lambda \delta(T) \int_{-\infty}^{0} du \frac{1}{u^2 + L^2} (f(u) + f(-u)) e^{-Eu/2} , \]  

which tacitly assumes we are searching for a bound state \( E < 0 \). Clearly the Fourier transform of \( f \) is given by

\[ \tilde{f}(\omega) = \int dT e^{i\omega T} f(T) = \frac{\lambda C}{\omega^2 + E^2/4} \]  

\[ C = \int_{-\infty}^{0} du \frac{1}{u^2 + L^2} (f(u) + f(-u)) e^{-Eu/2} . \]  

Then

\[ f(T) = \int d\omega e^{-i\omega T} \tilde{f}(\omega) = C \frac{\lambda}{|E|} e^{-|ET|/2} . \]  

Plugging this into the expression for \( C \) leads to the eigenvalue equation

\[ \frac{1}{2\lambda} = \frac{1}{|E|} \int_{0}^{\infty} du \frac{1}{u^2 + L^2} e^{-|E|u} = \frac{1}{|E|L} \int_{0}^{\infty} du \frac{1}{u^2 + 1} e^{-|E|Lu} \sim \begin{cases} \frac{\pi}{2|E|L^2} & \text{for } |E|L \ll 1 \\ \frac{1}{L^2E^2} & \text{for } |E|L \gg 1 \end{cases} . \]  

Clearly the right side is a monotonically decreasing function of \( |E| \) so there is exactly one solution for any \( \lambda > 0 \). The weak and strong coupling limits of the energy eigenvalue are \( E = -\pi\lambda/L \), \( E = -\sqrt{2\lambda}/L \) respectively. This one gluon truncation of the Bethe-Salpeter equation then loses the discrete levels but still produces the \( \lambda \rightarrow \sqrt{\lambda} \) replacement for weak to strong coupling. This exercise shows the importance of multi gluon states in generating the discrete levels that populate the gap.
4 Discussion

We have examined the spectrum of the string/flux tube in strong and weak coupling respectively to understand the qualitative features of the AdS/CFT correspondence. At first it is surprising that in the strong coupling limit of a conformal theory with a pure AdS background and no confinement, one has a discrete spectrum. However as explained in [7] this can be understood in weak coupling as a consequence of the large $N_c$ limit. This paper goes on to compare the results for this configuration using both the lightcone gauge and temporal gauge on the strong coupling side and lightcone and Feynman gauge in weak coupling. In strong coupling to quadratic order we find the equations of motion for both the transverse and longitudinal modes with frequencies inversely proportional to the separation of the sources $1/L$.

The ladder approximation we used to analyze the field side of the duality is obviously inadequate for large 't Hooft coupling. Nonetheless, it is encouraging that its strong coupling limit is qualitatively similar to the exact AdS/CFT results. The lowest energy eigenvalue has the $\sqrt{\lambda}/L$ coupling dependence in both cases although the dimensionless coefficients disagree for obvious reasons. Further in both cases the gap between the lowest energy and the continuum is populated with many discrete levels. As shown in [7] there is also an exact match at strong coupling of the density of near threshold states of the ladder model and the exact strong coupling results given by the AdS/CFT correspondence. The big qualitative difference on that score is the density of states near the ground state for strong coupling. The AdS/CFT correspondence shows that these levels are those of a string, namely those of an infinite number of oscillators with frequencies $\omega_n$, $n = 1, 2, \cdots$. In contrast, the sum of ladder diagrams in Feynman gauge implies discrete levels of a single harmonic oscillator. Presumably, more complicated planar diagrams which include interactions between the many gluons in the intermediate states will remove this discrepancy.

Finally recall that the basic analysis on the string side was formulated for a more general metric which can describe QCD like models of confining backgrounds. If the background approaches AdS$_5$ in the UV, the same spectral analysis will hold for small L. Of course here a discrete spectrum is required by confinement. It is interesting however to consider the effects of confinement and/or a running coupling $\lambda(L)$ in this limit and to try to disentangle these effects from the large $N_c$ effects that are responsible for the discrete spectrum in the conformal limit. Indeed lattice investigations for pure Yang-Mills theory (or quarkless QCD) have revealed a rather intricate spectrum [24] for the small L stretched string (or what is called gluelumps [25]), which present a challenge to the understanding of short strings in an approximately conformal background. These results are not expected to be affected dramatically by taking the large $N_c$ limit. Thus a search for states corresponding to the radion modes and/or fermionic degrees of freedom should be encouraged. This could help substantially to guide the construction of more realistic models of the QCD string dual to pure Yang-Mills theory.

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