Research Article

Algebraic Connectivity and Disjoint Vertex Subsets of Graphs

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It is well known that the algebraic connectivity of a graph is the second smallest eigenvalue of its Laplacian matrix. In this paper, we mainly research the relationships between the algebraic connectivity and the disjoint vertex subsets of graphs, which are presented through some upper bounds.

1. Introduction

A graph \(G\) is often used to model a complex network. The vertex set and the edge set of graph \(G\) are denoted by \(\mathcal{V}\) and \(\mathcal{E}\), respectively. A network is represented as an undirected graph \(G = (\mathcal{V}, \mathcal{E})\) consisting of \(N = |\mathcal{V}|\) nodes and \(E = |\mathcal{E}|\) links, respectively.

Graph theory has provided chemists with a variety of useful tools, such as in the topological structure [1–3]. The Laplacian matrix of a graph \(G\) is denoted by \(L\), and \(L = D - A\), where \(D\) is a diagonal matrix whose diagonal entries are its degrees and \(A\) is the adjacency matrix of \(G\). The Laplacian eigenvalues of a graph \(G\) are the eigenvalues of \(L\), denoted by \(0 = \mu_N \leq \mu_{N-1} \leq \cdots \leq \mu_1\), which are all real and nonnegative. The second smallest Laplacian eigenvalue \(\mu_{N-1}\) of a graph is well known as the algebraic connectivity, which was first studied by Fiedler [4]. The algebraic connectivity [5] of a graph is important for the connectivity of a graph [6], which can be used to measure the robustness of a graph. It has been emerged as an important parameter in many system problems [7–18]. Especially, the algebraic connectivity also plays an important role in the partitions of a complex network. For the literature on the algebraic connectivity of a graph [19], the reader is referred to [20, 21]. In this work, the relationships are researched between the algebraic connectivity and disjoint vertex subsets of graphs, which are presented through some upper bounds.

2. Preliminaries

Let \(x \in \mathbb{R}^n\) be a vector. Let \(B\) be an incidence matrix of \(G\). Then, \(x^T L x = \|B^T x\|_2^2 = \sum_{i,j \in E} (x_i - x_j)^2\). For any vector \(x, y \in \mathbb{R}^n\), the inner product of \(x\) and \(y\) is defined as \((x, y)\).

Lemma 1 (see [20]). For any vector \(f \in \mathbb{R}^n\), the Rayleigh inequality is as follows:

\[
\mu_{N-1} \leq \frac{(Lf, f)}{(f, f)},
\]

where \((f, c) = 0\), \(c\) is a constant, and \((Lf, f) = \sum_{e \in E} (f(v_i) - f(v_j))^2\), \(f(v_i)\) is the vector \(f\) for the node \(v_i\).

Lemma 2 (see [20]). For any vector \(f \in \mathbb{R}^n\), we have

\[
\mu_{N-1} \leq \frac{\sum_{e \in E} (f(v_i) - f(v_j))^2}{\sum_{v_i \in V} f^2(v_i)},
\]

\[
\mu_{N-1} \leq \frac{\sum_{v_i, v_j \in \mathcal{V}} (f(v_i) - f(v_j))^2}{\sum_{v_i \in V} \sum_{v_j \in \mathcal{V}} (f(v_i) - f(v_j))^2},
\]

where \(f(v_i)\) is the vector \(f\) for the node \(v_i\).
Let $A$ and $B$ be two disjoint subsets of $\mathcal{N}$, respectively. The distance between two disjoint subsets $A$ and $B$ of $\mathcal{N}$ is denoted by $h(A, B)$. For contineity, $h$ takes the place of $h(A, B)$. Let $h(u, A)$ be the distance between the node $u$ and $A$, which is the shortest distance of the node $u \in \mathcal{N}$ to a node of the set $A$. Suppose $a = |A|/N$ and $b = |B|/N$. A result on the algebraic connectivity and two partitions of graphs is presented by Alon [22] and Milman [20] below.

Lemma 3 (see [23]). For any two disjoint subsets $A$ and $B$ of $\mathcal{N}$, it holds

$$\mu_{N-1} \leq \frac{1}{Nh^2}(\frac{1}{a} + \frac{1}{b})(E - E_A - E_B),$$

where $E_A$ and $E_B$ are the number of links in the sets $A$ and $B$, respectively.

Moreover, the next step consider the case of three disjoint vertex subsets of graphs [24].

3. Main Result

Let $A$, $B$, and $C$ be the subsets of $\mathcal{N}$, respectively, where their numbers of nodes are, respectively, $|A|$, $|B|$, and $|C|$. Assume $a = |A|/N$, $b = |B|/N$, and $c = |C|/N$. Let $h(u, A)$, $h(u, B)$, and $h(u, C)$ be the distances from the node $u \in \mathcal{N}$ to subsets $A$, $B$, and $C$ of $\mathcal{N}$, respectively. Suppose $h_s = \min\{h(A, B), h(A, C), h(B, C)\}$. Now, we construct a function $g(u)$ related to node $u$ as follows, where the constructed function is referred to the book [25]:

$$g(u) = \frac{1}{\frac{1}{h_s^2}} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \cdot \min\{h_s, h(u, A), h(u, B), h(u, C)\}$$

Case 1. If the node $u$ belongs to any one subset of $\{A, B, C\}$, then

$$0 = \min\{h_s, h(u, A), h(u, B), h(u, C)\},$$

$$g(u) = \frac{1}{3} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) > 0.$$

Case 2. If the node $u \in \mathcal{N} - \{A, B, C\}$, then we can see that

$$\frac{\min\{h_s, h(u, A), h(u, B), h(u, C)\}}{h_s} \leq 1,$$

$$g(u) = \frac{1}{3} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \cdot \min\{h_s, h(u, A), h(u, B), h(u, C)\} > 0.$$

By Case 1 and Case 2, $(g, c) \neq 0$ holds. In contrast, if $(g, c) = 0$, then $g(u) = 0$ for each $u \in \mathcal{N}$ and $g - \overline{g} = 0$, which is a contradiction with $f = g - \overline{g} \neq 0$. From the definition $g(u)$, for any two adjacent nodes $u$ and $v$, we have

$$|g(u) - g(v)| \leq \frac{1}{9h_s} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

Our main result is as follows.

Theorem 1. Let $A$, $B$, and $C \in \mathcal{N}$ be three disjoint subsets of $\mathcal{N}$. Let $E_A$ and $E_B$ and $E_C$ be the numbers of links in the sets $A$ and $B$ and $C$, respectively. Then,

$$\mu_{N-1} \leq \frac{1}{81Nh_s^2} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) (E - E_A - E_B - E_C)$$

$$= \frac{1}{81Nh_s^2} \left( \frac{1}{N_A} + \frac{1}{N_B} + \frac{1}{N_C} \right) (E - E_A - E_B - E_C).$$

Proof. For subsets $A, B, and C$, by Lemma 2, we have

$$\sum_{u \in A} (f(u) - f(v))^2 = \sum_{u \in A} (g(u) - g(v))^2 = \sum_{u,v \in A \cup B \cup C} (g(u) - g(v))^2 \leq \frac{1}{81Nh_s^2} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) (E - E_A - E_B - E_C).$$

where $\overline{g} = 0$, and since the coordinates of the center of gravity of the three regions are the average of the triangle region, then the vectors $g(u, A) + g(u, B) + g(u, C) = 0$. The sum of the vectors of the center of gravity of the triangle to the vertices is equal to 0 [25]. The center of gravity is analogous to the mean or average from statistics [6, 31, 32].
\[
\sum_{n \in N} f^2(n) \geq \sum_{n \in A} (g(n) - \bar{g})^2 + \sum_{n \in B} (g(n) - \bar{g})^2 + \sum_{n \in C} (g(n) - \bar{g})^2
\]

\[
= |A| \left( \frac{1}{3} a + \frac{1}{b} + \frac{1}{c} - \bar{g} \right)^2 + |B| \left( \frac{1}{3} a + \frac{1}{b} + \frac{1}{c} - \bar{g} \right)^2 + |C| \left( \frac{1}{3} a + \frac{1}{b} + \frac{1}{c} - \bar{g} \right)^2
\]

\[
= \frac{1}{9} \left( A + |B| + |C| \right) \left( \frac{1}{3} a + \frac{1}{b} + \frac{1}{c} \right)^2
\]

\[
= N^2 \frac{1}{9} (1 + 1 + 1) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq N^2 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)
\]

By the above inequalities and Lemma 2, it arrives that

\[
\mu_{N-1} \leq \frac{1}{81 N^2 h^2} \left( \frac{1}{3} a + \frac{1}{3} b + \frac{1}{3} c \right) (E - E_A - E_B - E_C)
\]

\[
= \frac{1}{81 N h^2} \left( \frac{1}{N_A} + \frac{1}{N_B} + \frac{1}{N_C} \right) (E - E_A - E_B - E_C).
\]

**Example 1.** Figure 1 describes the graphs \( G_1 \) and \( G_2 \), each with \( N = 7 \) nodes, \( L = 10 \) links, and a diameter \( \rho = 4 \). For \( G_1 \) subsets, \( A = \{v_1, v_2, v_5, v_6\}, B = \{v_2\}, \) and \( C = \{v_4\} \). For \( G_2 \) subsets, \( A = \{u_1, u_2, u_3\}, B = \{u_3\}, C = \{u_5, u_6, u_7\}, \) and \( h = 0.5 \). Their algebraic connectivity [33] and their upper bounds on (11) are as follows. For the \( G_1 \) and \( G_2 \) aplanar matrices,
Proposition 1. Let $A, B,$ and $C \in V$ be three disjoint subsets of $V$. Suppose $h_s = 1$ and $D = V - A - B - C$. Let $m_A$, $m_B$, $m_C$, and $m_D$ be the number of links in the sets $A, B, C,$ and $D$, respectively. Then,

$$\mu_{N-1} \leq \frac{(1/81)((1/a) + (1/b) + (1/c))^2(m - m_A - m_B - m_C - m_D)}{(|A| + |B| + |C|)((1/a) + (1/b) + (1/c) - \bar{g})^2 + |D|[2/9((1/a) + (1/b) + (1/c)) - \bar{g}]^2},$$

where

$$\bar{g} = \frac{1}{n}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(|A| + |B| + |C| + \frac{2}{9}|D|),$$

in which $\bar{g}$ is the average of the $A, B, C, D$ field.

Proof. For subsets $A, B,$ and $C$, by Lemma 2, we have

$$\sum_{u,v \in E} (f(u) - f(v))^2 = \sum_{u,v \in E} (g(u) - g(v))^2 = \sum_{u,v \notin E - (E_A \cup E_B \cup E_C \cup E_D)} (g(u) - g(v))^2 \leq \frac{1}{81}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2(m - m_A - m_B - m_C - m_D).$$

where links of the sets in the node sets $A, B, C,$ and $D$ are $E_A, E_B, E_C,$ and $E_D$, respectively. From (2), we obtain

$$\sum_{n \notin F} f^2(n) = \sum_{n \in \{A \cup B \cup C \cup D\}} f^2(n) = \sum_{n \in A} (g(n) - \bar{g})^2 + \sum_{n \in B} (g(n) - \bar{g})^2 + \sum_{n \in C} (g(n) - \bar{g})^2 + \sum_{n \in D} (g(n) - \bar{g})^2$$

$$= |A|\left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \bar{g}\right]^2 + |B|\left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \bar{g}\right]^2 + |C|\left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \bar{g}\right]^2 + |D|\left[\left(\frac{2}{9}\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \bar{g}\right]^2$$

$$= (|A| + |B| + |C|)\left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \bar{g}\right]^2 + |D|\left[\frac{2}{9}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \bar{g}\right]^2.$$
By direct computation, we have
\[
\overline{\gamma} = \frac{1}{n} \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) \left( |A| + |B| + |C| + \frac{2}{n} |D| \right). \tag{21}
\]

By the above equalities and Lemma 2, inequality (17) holds.

But, we note that the algebraic connectivity \([34, 35]\), \(\mu_{N-1}\), should not be seen as a strict disconnection or a robustness metric \([36]\).

Example 2. For example, Figure 2 describes the graphs \(G_3\) and \(G_4\), with \(n = 9\), \(m = 12\), and diameter 6. By direct calculation, for \(G_3\) subsets, \(A = \{v_1, v_3, v_4\}, B = \{v_2, v_5, v_6\}, C = \{v_3\}\), and \(D = \{v_2, v_3, v_5\}\), and for \(G_4\) subsets, \(A = \{u_1, u_4, u_5\}, B = \{u_4, u_5, u_6\}, C = \{u_4\}\), and \(D = \{u_4, u_5\}\). Their algebraic connectivity \(G_3\) is 0.4798 and \(G_4\) is 0.4817, respectively. Their Laplacian matrices \(L(G_3)\) and \(L(G_4)\), for \(G_3\) upper bounds on \(\mu_{N-1}(G_3) \geq 0.431\) and for \(G_4\) upper bounds on \(\mu_{N-1}(G_4) \geq 0.357\).

Theorem 1 and Proposition 1 are two completely different situations. The theorem hypothesis is that \(A, B, C \in \mathcal{N}\) be three disjoint subsets of \(\mathcal{N}\). The proposition supposes that \(A, B, C \in \mathcal{V}\) be three disjoint subsets of \(\mathcal{V}\) and \(h_i = 1\) and \(D = V - A - B - C\). In other words, the proposition has constraints. Moreover, it is not the same as the four disjoint subsets of \(\mathcal{N}\).

Data Availability

All data, models, and codes generated or used during the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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