Hiding Anomalies

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Abstract

Anomalies can be anticipated at the classical level without changing the classical cohomology, by introducing extra degrees of freedom. In the process, the anomaly does not quite disappear. We show that, in fact, it is shifted to new symmetries that come with the extra fields.

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1 Introduction

The most interesting feature of the quantisation of anomalous gauge theories lies in the fact that degrees of freedom that can be gauged away classically become propagating at the quantum level. When trying to quantise such theories there are at least two strategies one can follow. The first one consists of first integrating out the matter degrees of freedom. The result (see for examples [1, 2]) is the induced action for the gauge field. It is generically non-local, and is used thereafter as the starting point for further quantisation, i.e. integrated over to get the effective action. The second possibility is to introduce some extra fields in the anomalous theory. This line of developments started with Faddeev [3], who showed that the presence of anomalies changes first-class constraints (related to the presence of gauge-symmetries) into second-class constraints. Later it was suggested in [4] that by introducing extra degrees of freedom, one could keep the symmetries first-class, and hence anomaly-free. It was recognised that in a pathintegral approach the required new variables arise very naturally when applying the Faddeev-Popov procedure. In the case of a Yang-Mills model with chiral fermions for instance, it has been shown [5] that the integral over the gauge group does not factor out: the fermion measure has an (anomalous) dependence on the gauge variables as well, effectively producing the well-known Wess-Zumino action.

In a recent article [6] these ideas have been implemented in the Batalin-Vilkovisky quantisation scheme. In this scheme anomalies arise as the impossibility to find a local solution to the so called master equation $\frac{1}{2}(W,W) = \hbar \Delta W$ where $W$ is the quantum action i.e. the classical action $S_{ext}$, extended with sources for the BRST transformation, some extra terms in case of open gauge algebra’s, plus eventual (both renormalisation- and finite) counter-terms $\hbar M_1 + \hbar^2 M_2 + \cdots$. In a second step they add as many extra fields to the theory as there are anomalous symmetries. By choosing their transformations under the classical symmetries in a specific way, a local solution to the master equation can actually be found in this extended field space. This involves constructing a suitable $M_1$-term, which in turn makes these new fields dynamical on the quantum level. In this approach, the anomaly apparently has disappeared, where the price to pay is a (minor) change in the classical theory.

In this paper we show how these extra fields can be introduced classically
via trivial systems, which do not change the classical cohomology at all. This approach makes clear that, at the same time as introducing new fields, one also introduces extra symmetries. The extra gauge symmetries of the extra fields are properly taken into account. As a result, we will show that the anomaly which seemed to have disappeared, is in fact still present: it has merely been shifted, or hidden, in these extra symmetries which are often ignored.

2 Discussion

In this section we will describe the method. We will formulate it in the scheme of Batalin and Vilkovisky (BV) for quantising gauge-theories \([7][1]\) but will also try to convey the central idea in less technical language.

In BV one starts from an extended action \(S\), depending on the fields \(\phi^i\), on the ghost-fields \(c^a\) for the gauge-symmetries of that action, and on the anti-fields of all these fields, indicated with an asterisk, which can be viewed as sources for the BRST transformations. Assume that \(S\) is a solution of the classical master-equation \((S, S) = 0\), i.e. is BRST-invariant. A further requirement is that it has to satisfy the properness-condition, which means that ghosts have been introduced for all the gauge-symmetries. In order to keep the symmetries intact through the first quantum correction, one has to solve the master-equation at order \(\hbar\). This involves first regulating the theory, and then calculating the operator \(\Delta S\). We assume that a Pauli-Villars (PV) regularisation is used to do this, as described in \([\text{[8]}]\). Anomalies are possible when the mass term chosen for the PV-fields does not maintain (all) the original gauge symmetries. The result is always of the form

\[
(\Delta S)_{\text{reg}} = c^a A_a.
\]

If this quantity is the BRST-variation of something, than the anomaly can be countered; if not, one speaks of a genuine anomaly. We will always have in mind the latter case.

Now we propose to add trivial systems to \(S\), one for every anomalously broken gauge-symmetry:

\[
S \rightarrow S + \alpha^a d^a.
\]

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1For a detailed account of BV, which includes the study of anomalies in that scheme, we refer to \([8][8]\).
This may be viewed as follows. We introduced extra scalar degrees of freedom \( \alpha_a \). Since they do not occur in the original action, they can always be shifted by an arbitrary amount \( \Lambda_a \). The \( d^a \) are nothing but the ghosts for these extra gauge-symmetries. Although these extra fields, and the entailing extra symmetries, are completely trivial at this point (hence 'trivial systems'), regularisation in the quantum theory will interfere with this. The new degrees of freedom clearly do not change the classical theory, as they are cohomologically trivial for the (new) cohomology-operator \((S, \cdot)\).

At this point, we are still free to specify how these newly introduced fields should transform under the original symmetries related to the \( c \)-ghosts. This choice can be encoded in the extended action by doing a canonical transformation \(^2\). Our approach then is to choose this transformation such that a PV-mass term can be constructed that is invariant under the \( c \)-symmetries.

Suppose one would like to have fields with transformations

\[
\delta_c \alpha^a = f^a(\phi, c, \alpha).
\]

This can be achieved simply by taking as generating fermion for the canonical transformation \(^3\)

\[
F = 1 - d^a \ast f^a(\phi, c, \alpha).
\]

The fact that we used a canonical transformation guarantees that the transformed extended action is still a solution of the classical master-equation, i.e. the new action is still BRST-invariant with the modified transformation rule. It is also important to remark that the extended action after the transformation still contains terms with the \( d \)-ghosts. These are necessary to ensure the properness of the action. It is clear that, if we use the extra \( \alpha \)-fields to construct a PV mass-term that is invariant under the \( c \)-symmetries, it cannot also be invariant under the \( d \)-symmetries, which shift \( \alpha \). As a result the anomalies will have been shifted to the new symmetries and

\[
(\Delta S)_{reg} = d^\beta B_\beta.
\]

\(^2\) Here we use 'canonical' in the sense of anti-brackets. It is well-known that such a canonical transformation does not change the classical cohomology.

\(^3\) Remark that the field redefinitions of [10] can be implemented in BV in an analogous way. If one wants to perform a field-redefinition given by \( \phi_i = g_i(\alpha_j, \phi'_i) \), with the inverse relation \( \phi'_i = h_i(\alpha_j, \phi_i) \) then one should take as generating fermion \( F = \phi'_i h^i + \alpha^{\alpha j} \alpha_j + d^i \ast d^i \).
It is only when one neglects the $d$-symmetries that one would conclude that there are no anomalies left.

To carry out the calculation one may still use the old ($c$ non-invariant) mass-term if one wishes to do so, for technical or other reasons. The anomaly will still be left in the $d$-symmetries, if a counterterm is added that matches the interpolation between the two different regularisations (see [9][11]). This counterterm is completely fixed, and involves an integration over a parameter that interpolates between the two mass-terms. In practise, it provides non-trivial dynamics for the variables that were introduced as a trivial system.

3 An example: $W_2$-gravity

The action under consideration is

$$S_{\text{ext}} = \int d^2 x \left[ \frac{1}{2\pi} (\partial \varphi^i \bar{\partial} \varphi^i - h \partial \varphi^i \partial \varphi^i) + \varphi_i^* \partial \varphi^i c + h^* (\bar{\partial} c - h \partial c + c \partial h) + c^* (\partial c) c \right].$$

(1)

We will only be concerned with the integral over the matterfields, and define

$$\exp -\frac{1}{h} \Gamma[h] = \int D\varphi \exp -\frac{1}{h} S_{\text{ext},\varphi^*=0}$$

It is clear that we only have one-loop diagrams, so using PV-fields the integral is completely regulated. We need to specify a massterm. We choose the obvious $S_{\text{Mass}}^{(0)} = -\frac{1}{2\pi} M^2 (\varphi^i)^2$ : for the covariant form of the action (1), it corresponds to keeping the coordinate transformations anomaly-free. In that case we find the well known Ward-identity

$$\int d^2 x (\bar{\partial} c - h \partial c + c \partial h) \frac{\partial \Gamma}{\partial h} = \frac{nh}{12\pi} \int d^2 x c \partial^3 h.$$ 

Now we add the trivial system $\{\theta^*,d\}$. We want to use it to shift the anomaly away from the $c$-symmetry into the new $d$-symmetry, which amounts to constructing a $c$-invariant massterm. Since $\delta_c S_{\text{Mass}}^{(0)} \sim c \partial (\varphi^i)^2$, we need a factor transforming as a density. Writing

$$S_{\text{Mass}}^{(1)} = -\frac{1}{2\pi} M^2 (\varphi^i)^2 e^\theta$$

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we take
$$
\delta c \theta = \partial c + c \partial \theta.
$$
The canonical transformation leading to this transformation is generated by
$$
F = 1 - d^* (\partial c + c \partial \theta).
$$
The resulting extended action, in the new coordinates, is
$$
S_{\text{ext}} = \int d^2 x \left\{ \frac{1}{2\pi} \left( \partial \varphi^i \bar{\partial} \varphi^i - h \partial \varphi^i \bar{\partial} \varphi^i \right) + \varphi^* i \bar{\partial} \varphi^i c + h^* (\bar{\partial} c - h \partial c + c \partial h) 
+ c^* (\partial c)c + \theta^* (d + \partial c + c \partial \theta) + d^* (\partial d)c \right\}.
$$
This action is in fact the same (in simpler appearance) as the action used in [6] except for the terms containing the $d$-ghost. The origin of these terms was the extra trivial system, and they are needed to assure that we have a proper extended action.

Next we calculate the anomaly due to the mattersector. As a massterm we take
$$
S_{\text{Mass}}^{(\alpha)} = -\frac{1}{2\pi} M^2 (\varphi^i)^2 e^{\theta \alpha}
$$
where $\alpha$ is a parameter interpolating between the conformal ($c$-) invariant case ($\alpha = 1$), and the coordinate transformation invariant case ($\alpha = 0$). We will not repeat here how to deduce the corresponding anomaly (follow e.g. [12, 11] for this application and [13] for the general method). The result for arbitrary $\alpha$ is
$$
\Delta \varphi S(\alpha) = n \lim_{M \to \infty} \int d^2 x \sqrt{g} \frac{1}{4\pi} \left[ -\frac{1}{2} \partial c + \frac{1}{2} \alpha (\partial c + d) \right] \left[ M^2 - \frac{1}{6} R \right]
$$
where $R$ is the Riemann scalar corresponding to the metric
$$
g^{\mu\nu} = e^{-\theta \alpha} \begin{pmatrix}
2h & -1 \\
-1 & 0
\end{pmatrix},
$$
and $n$ is the number of scalars.

The diverging term can be countered by adding a local term to the action, since $(S, \int d^2 x e^{\theta \alpha}) = \int d^2 x \partial (e^{\theta \alpha} c) - \partial c + \alpha (d + \partial c)$, or alternatively by considering several PV-fields with $\sum M^2 = 0$. The result is then
$$
(\Delta \varphi S)_{\text{reg}}(\alpha) = \frac{n}{12\pi} \int d^2 x \left\{ \partial c - \alpha (\partial c + d) \right\} \left\{ \partial^2 h - \alpha (\bar{\partial} \theta - \partial (h \theta)) \right\}.
$$
Clearly, at $\alpha = 1$, the $c$-ghost disappears from this expression, and only the $d$-ghost remains: this is one way to see that the $c$-transformation anomaly is
no longer present, but is replaced by a $d$-transformation anomaly. Another way to see this is by looking at the Ward identity

$$\int d^2 x \left( (\bar{c} - h \partial c + c \partial h) \frac{\partial \Gamma}{\partial h} + (d + \partial c + c \partial \theta) \frac{\partial \Gamma}{\partial \theta} \right)$$

$$= \frac{n \hbar}{12 \pi} \int d^2 x \left\{ \partial^2 h - \bar{\theta} \partial \theta + \partial(h \partial \theta) \right\}$$

where $\Gamma$ is the induced action computed with regulating mass term $S^{(\alpha)}_{\text{Mass}}$ at $\alpha = 1$:

$$\exp -\frac{1}{\hbar} \Gamma[h, \theta] = \int \mathcal{D}\varphi_{(\alpha=1)} \exp -\frac{1}{\hbar} S_{\text{ext,} \varphi^* = 0}. \quad (2)$$

Actually, the conclusion above involves a choice: working with the invariant $\theta$-dependent mass term, and not adding a finite counterterm $M_1$. If one prefers to work with the conventional mass term, for example in Feynman diagrams, or because one wants to use techniques with the usual OPE’s, this can be done without changing the resulting induced action if an interpolating counterterm is added. The expression for this counterterm can be obtained by considering that

$$\frac{\partial}{\partial \alpha} \log \det(-\Box + M^2 T(\alpha)) = tr \left( \frac{M^2 \frac{\partial T(\alpha)}{\partial \alpha}}{-\Box + M^2 T(\alpha)} \right)$$

with $\Box = \partial \bar{\partial} - \partial h \partial$, and applying the technique of [14]. This leads straight-forwardly to the conjectured formula for the counterterm $M_1$ given in [3]

$$M_1(\alpha) = -\frac{1}{2} \int_0^1 d\alpha \lim_{M^2 \to \infty} \text{str} \ T^{-1}(\alpha) \frac{\partial T}{\partial \alpha} e^{\mathcal{R}(\alpha)/M^2},$$

where $\mathcal{R}$ is completely fixed. Here, $\mathcal{R}(\alpha) = e^{\alpha \theta} \Box$ and in the present case the formula is exact since there is only one loop. Therefore we also have, when using the mass term $S^{(\alpha)}_{\text{Mass}}$, for any $\alpha$, the following formula for the same induced action as in eq. (2):

$$\exp -\frac{1}{\hbar} \Gamma[h, \theta] = \int \mathcal{D}\varphi_{(\alpha)} \exp -\frac{1}{\hbar} (S_{\text{ext,} \varphi^* = 0} + \hbar M_1(\alpha)),$$

$$M_1(\alpha) = \frac{n}{12 \pi} \int d^2 x \left\{ \frac{1}{2}(1 - \alpha^2)(\partial \theta \bar{\partial} \theta - h \partial \theta \partial \theta) + (1 - \alpha)h \partial^2 \theta. \right\}$$
The \( \alpha \)-independence can be checked for example by the fact that

\[
(\Delta S)_{\text{reg}}(\alpha) = (\Delta S)_{\text{reg}}(1) + (S, M_1(\alpha)).
\]

In particular, for \( \alpha = 0 \), we have the regularisation that does not depend on the extra \( \theta \)-field. The matter field integral then does not depend on \( \theta \) either. In that case, the \( M_1 \) term completely exhibits the behaviour of the anomalous degree of freedom. The variable \( \theta \), originally introduced as part of a trivial system, acquires dynamics by quantum corrections if we require the c-symmetry – including it’s action on the extra field \( \theta \) – to be anomaly-free. The \( M_1 \) term is actually the Liouville action, which is not at all surprising if we realise that \( \delta_c \theta \) is precisely the residual reparametrisation transformation law for the Liouville mode after a chiral gaugefixing.

Remark that in the quantum action \( W = S_{\text{ext}} + \hbar M_1(0) \) the matterfields and the Liouville field \( \theta \) can be treated on an equal footing if we rescale \( \theta \to \frac{1}{\sqrt{\hbar}} \theta, \theta^* \to \sqrt{\hbar} \theta^* \). This redefinition clearly doesn’t change the antibrackketoperator and we find (only retaining the matter- and Liouville transformations under the c- symmetry) that

\[
W = \int d^2x \frac{1}{2\pi} (\partial \varphi^i \bar{\partial} \varphi^i - \hbar \partial \varphi^i \partial \varphi^i) + \varphi_i^* \partial \varphi^i c \\
+ \frac{n}{24\pi} (\partial \theta \bar{\partial} \theta - \hbar \{\partial \theta \bar{\partial} \theta - 2\sqrt{\hbar} \partial^2 \theta\}) \\
+ \theta^* \{\partial \theta c + \sqrt{\hbar} \partial c\}
\]

The backgroundcharge corrections to the energy-momentum tensor, accompanied by quantumcorrected transformationrules, appear naturally in this context. We also see that their implementation in the BV scheme involves a slight subtlety: the expansion in \( \hbar \) contains actually half integral powers

\[
W = S_{\text{ext}} + \sqrt{\hbar} M_1 \frac{1}{2} + \hbar M_1 + ...
\]

We also wish to emphasize the close connection between regularisation and the (finite) counterterm \( M_1 \). One can choose any regularisation (any value of \( \alpha \)), provided one adds the appropriate \( M_1 \). Certainly, specifying only \( M_1 \) without fixing the regularisation is insufficient: any of the above \( M_1(\alpha) \) may be right. Given a specific regularisation, \( M_1 \) is determined by ”external” requirements that one imposes for the quantumtheory. In the
above, this requirement was chosen to be: absence of anomalies for the
$c$-transformation.

The same ideas can easily be applied to other cases. Let us consider
chiral fermions in two dimensions, coupled to a gauge field. One starts from the
action

$$S[\psi, A] = \frac{1}{2\pi} \int d^2 x \psi^t (\bar{\partial} - A) \psi,$$

and the classical symmetry is given by $\delta \psi = c \psi$, and $\delta A = \bar{\partial} c + [c, A]$. This chiral gauge symmetry can be maintained at the quantum level if one
supplements the theory with the group element $g$ as an extra field, not related
to $A$, with the transformation rule $\delta g = (c + d) g$. Again, since this field
is not present in the action we started from, this entails extra symmetries
(with $d$-ghost) that change $g$ arbitrarily. The fermions can only be regulated
using Pauli-Villars fields of both chiralities. One can again find an invariant
massterm

$$S^{(1)}_{\text{mass}}[\psi, \xi] = -\frac{M}{4\pi} \int d^2 x \left( \psi^t g \xi - \xi^t g^{-1} \psi \right).$$

Note that a similar construction was used also in [15], in the case of abelian
chiral electrodynamics in four dimensions. In this paper the importance of
working with a regularised theory from the outset, was also emphasized.
Technically, the calculation proceeds exactly as in [11] (but there the group
element $g$ and the gauge field $A$ were related), so we do not repeat it here.
Interpolating to the chiral-non-invariant regularisation characterised by the
massterm

$$S^{(0)}_{\text{mass}}[\psi, \xi] = -\frac{M}{4\pi} \int d^2 x \left( \psi^t \xi - \xi^t \psi \right)$$

one picks up the counterterm

$$M_1[g, A] = -S^-[g] - \frac{1}{2\pi} \int d^2 x \operatorname{tr}\{\partial g \ g^{-1} A\}$$

$$S^-[g] = \frac{1}{4\pi} \int d^2 x \operatorname{tr}\{\partial g^{-1} \bar{\partial} g\} - \frac{1}{12\pi} \int d^3 x \varepsilon^{\alpha\beta\gamma} \operatorname{tr}\{g, g^{-1} g, g^{-1} g, g^{-1} g\}$$

This last term is of course the well-known Wess-Zumino-Witten action. In the
context of adding the (anomalous) quantum degrees of freedom, it has been
proposed before in [3] or for the abelian case in [12]. As mentioned before,
it is important to realise that it’s explicit expression is valid only when a
specific regularisation is used. Our approach shows how this counterterm
originates from the fact that an invariant regularisation exists.
4 Summary

Following the idea that the existence of an anomaly means that extra degrees of freedom start propagating, we enlarge the set of fields in the theory with an extra field for every anomalously broken symmetry. Together with the extra fields, extra gauge-symmetries are introduced in the theory, which act trivially. We choose the transformation rules of the extra fields under the original symmetries in such a way, that a regularisation exists that is invariant under those symmetries. Adopting that regularisation, those symmetries become anomaly-free. We find however that the anomalies have not disappeared altogether: they have been shifted to the extra symmetries, which at their introduction had a completely trivial action.

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