Thompson Sampling for Combinatorial Network Optimization in Unknown Environments

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Abstract—Influence maximization, item recommendation, adaptive routing and dynamic spectrum allocation all require choosing the right action from a large set of alternatives. Thanks to the advances in combinatorial optimization, these and many similar problems can be efficiently solved given that the stochasticity of the environment is perfectly known. In this paper, we take this one step further and focus on combinatorial optimization in unknown environments. All of these settings fit into the general combinatorial learning framework called combinatorial multi-armed bandit with probabilistically triggered arms. We consider a very powerful Bayesian algorithm, Combinatorial Thompson Sampling (CTS), and analyze its regret under the semi-bandit feedback model. Assuming that the learner does not know the expected base arm outcomes beforehand but has access to an exact oracle, we show that when the expected reward is Lipschitz continuous in the expected base arm outcomes CTS achieves \( O(\sum_{i=1}^{m} \log T / (p_i \Delta_i)) \) regret, where \( m \) denotes the number of base arms, \( p_i \) denotes the minimum non-zero triggering probability of base arm \( i \), \( \Delta_i \) denotes the minimum suboptimality gap of base arm \( i \) and \( T \) denotes the time horizon. In addition, we prove that when triggering probabilities are at least \( p^\ast > 0 \) for all base arms, CTS achieves \( O(1 / p^\ast \log(1/p^\ast)) \) regret independent of the time horizon. We also numerically compare CTS with algorithms that use the principle of optimism in the face of uncertainty in several combinatorial networking problems, and show that CTS outperforms these algorithms by at least an order of magnitude in the majority of the cases.

Index Terms—Combinatorial network optimization, multi-armed bandits, Thompson sampling, regret bounds, online learning.

I. INTRODUCTION

How should an advertiser promote its products in a social network to reach to a large set of users with a limited budget [2], [3]? How should a search engine suggest a ranked list of items to its users to maximize the click-through rate [4]? How should a base station allocate its users to channels to maximize the system throughput [5]? How should a mobile crowdsourcing platform dynamically assign available tasks to its workers to maximize the performance [6]? How can we identify the most reliable paths from source to destination under probabilistic link failures [7]? All of these problems require optimizing decisions among a vast set of alternatives. When the probabilistic description of the environment is fully specified, these problems—and many others—are solved using computationally efficient exact or approximation algorithms. In this paper, we focus on a much difficult and more realistic problem: How should we learn the optimal decisions in these complex problems via repeated interaction with the environment when the probabilistic description of the environment is unknown or only partially known?

It is natural to assume that the environment is unknown in many real-world applications. For instance, the advertiser may not know what probability user \( i \) will influence its neighbor \( j \) in the social network or the search engine may not know with what probability user \( i \) will click the item shown on position \( j \) beforehand. Moreover, decisions need to be made sequentially over time. For instance, the recommender system should show a new list of items to each arriving user and the base station should reallocate network resources when the channel conditions change or the users leave/enter the system. Obviously, future decisions of the learner must be guided based on what it has observed thus far, i.e., the trajectory of actions, observations and rewards generated by the learner’s past decisions. Importantly, both the cumulative reward of the learner and what it has learned so far also depend on this trajectory. Therefore, the learner needs to balance how much it earns (by exploiting the actions it believes to be the best) and how much it learns (by exploring actions it does not know much about) in order to maximize its long-term performance. In this paper, we solve the formidable task of combinatorial optimization in unknown environments by modeling it as a combinatorial multi-armed bandit (MAB).

MAB problems have a long history as they exhibit the prime example of the tradeoff between exploration and exploitation [5], [8]. In the classical MAB, at each round the learner selects an arm (action) which yields a random reward that comes from an unknown distribution. The goal of the learner is to maximize its expected cumulative reward over all rounds by learning to select arms that yield high rewards. The learner’s performance is measured by its regret with respect to an oracle that always selects the arm with the highest expected reward. It is shown that when the arms’ rewards are independent, any uniformly good policy will incur at least logarithmic in time regret [9].

Several classes of policies are proposed for the learner to minimize its regret. One example is Thompson sampling [10]–[12], which is a Bayesian method. In this method, the learner keeps a posterior distribution over the expected arm rewards, and at each round takes a sample from each arm’s posterior, and then, plays the arm with the largest sample. Reward observed from the played arm is then used to update
Thompson sampling (CTS) and analyze its regret assuming that the learner does not know the expected base arm outcomes beforehand but has access to an exact optimization oracle. Essentially, this oracle outputs an estimated optimal super arm given estimates of expected base arm outcomes as inputs. When the expected reward is Lipschitz continuous in the expected base arm outcomes, we show that CTS achieves \( O(\sum_{i=1}^m \log T/(p_i \Delta_i)) \) regret, where \( m \) denotes the number of base arms, \( p_i \) denotes the minimum non-zero triggering probability of base arm \( i \), \( \Delta_i \) denotes the minimum suboptimality gap of base arm \( i \) and \( T \) denotes the time horizon. Our bound is optimal in the sense that the dependence of the regret on the terms that appear in the bound is shown to be unavoidable in general [22]. In addition, we prove that when triggering probabilities are at least \( p^\ast > 0 \) for all base arms, CTS achieves \( O(1/p^\ast \log(1/p^\ast)) \) regret independent of the time horizon. This setting is of particular interest since it can model random behavior of users in a recommender system. For instance, a user may rate an item even when it is not in the list of recommended items as a result of an exogenous event (by rating the item on a partner website or by explicitly navigating to the item to rate it). Moreover, it is also closely linked to related work on online learning with probabilistic graph feedback [23], [24] and MAB with side observations [25]. Specifically, the models in [24] and [25] become special cases of our work when the graph is fully-connected for the one-step case and connected for the cascade case in [24] and when the probability of having an observation from any arm is non-zero in [25].

We complement our theoretical findings via extensive simulations in the following combinatorial network optimization problems: cascading bandits [4], probabilistic maximum coverage bandits [20] and influence maximization bandits [20]. For cascading bandits, we show that CTS, which uses Beta posterior on base arms significantly outperforms all competitor algorithms that use either UCB indices [4] or Thompson sampling with Gaussian posterior [26]. The latter finding emphasizes the importance of working with the correct type of posterior. For probabilistic maximum coverage bandits, we show that CTS achieves an order of magnitude improvement over combinatorial UCB (CUCB) in [20] when both algorithms use an exact oracle. For influence maximization bandits, we show a similar result even when both algorithms use an approximation oracle instead of an exact oracle.

To sum up, the main contribution of this paper is to analyze Thompson sampling for a very general combinatorial online learning framework that is comprehensive enough to model many different sequential decision-making applications defined over networks and show its optimality both theoretically and experimentally.

The rest of the paper is organized as follows. Related work is given in Section II followed by problem formulation in Section III. Applications of CMAB-PTA are detailed in Section IV. Description of CTS and regret bounds are given in Section V. Proofs of the main results are explained in Sections VI and VII. Numerical results are presented in Section VIII and concluding remarks are given in Section IX.
II. RELATED WORK

CMAB has been studied under various assumptions on the relation between super arms, base arms and rewards [17]. Here, we mainly discuss the related works that assume semi-bandit feedback as we do in our work. A version of CMAB in which the expected reward of a super arm is a linear combination of the expected outcomes of the base arms in that super arm is studied in [5]. For this problem, it is shown in [18] that a combinatorial version of UCB1 in [14] achieves $O(Km \log T / \Delta)$ gap-dependent and $O(\sqrt{KmT \log T})$ gap-free (worst-case) regrets, where $m$ is the number of base arms, $K$ is the maximum number of base arms in a super arm, and $\Delta$ is the gap between the expected reward of the optimal super arm and the second best super arm.

Later on, this setting is generalized to allow the expected reward of each super arm to be a more general function of the expected outcomes of the base arms that obeys certain monotonicity and bounded smoothness conditions [19]. The main challenge in the general case is that the optimization problem itself is NP-hard, but an approximately optimal solution can usually be computed efficiently for many special cases [27]. Therefore, it is assumed that the learner has access to an approximation oracle, which can output a super arm that has expected reward that is at least $\alpha$ fraction of the optimal reward with probability at least $\beta$ when given the expected outcomes of the base arms. Thus, the regret is measured with respect to the $\alpha \beta$ fraction of the optimal reward, and it is proven that a combinatorial variant of UCB1, called CUCB, achieves $O(\sum_{i=1}^m \log T / \Delta_i)$ regret, when the bounded smoothness function is $f(x) = \gamma x$ for some $\gamma > 0$, where $\Delta_i$ is the minimum gap between the expected reward of the optimal super arm and the expected reward of any suboptimal super arm that contains base arm $i$.

Recently, it is shown in [28] that Thompson sampling can achieve $O(\sum_{i=1}^m \log T / \Delta_i)$ regret for the general CMAB under a Lipschitz continuity assumption on the expected reward, given that the learner has access to an exact computation oracle, which outputs an optimal super arm when given the set of expected base arm outcomes. Moreover, it is also shown that in general the learner cannot guarantee sublinear regret when it only has access to an approximation oracle. Since the setting studied in this paper is a special case of ours, for our theoretical analysis we also assume that the learner uses an exact computation oracle. Nevertheless, we show in Section VIII that in practice CTS works well even when used with an approximation oracle. Another related work on CMAB [29] considers a new smoothness condition termed the Gini-weighted smoothness on the expected reward. For some problem types, this leads to regret bounds with better dependency on the sizes of super arms as compared to the common linear dependency of existing algorithms.

Different from CMAB, papers on CMAB-PTA assume that the expected reward is a function of expected outcomes of the triggered base arms, which is a random superset of base arms in the selected super arm. For this problem, it is shown in [20] that logarithmic regret is achievable when the expected reward function has the $L_\infty$ bounded smoothness property. However, this bound depends on $1/p^*$, where $p^*$ is the minimum non-zero triggering probability. Later, it is shown in [22] that under a more strict smoothness assumption on the expected reward function, called triggering probability modulated bounded smoothness, it is possible to achieve regret that does not depend on $1/p^*$. It is also shown in this work that the dependence on $1/p^*$ is unavoidable for the general case. In another work [30], CMAB-PTA is considered for the case when the arm triggering probabilities are all positive, and it is shown that both CUCB and CTS achieve bounded regret. However, their $O((1/p^*)^2)$ bound has a much worse dependence on $p^*$ than our $O(1/p^* \log(1/p^*))$ bound.

Apart from the works mentioned above, numerous other works also tackle related online learning problems. For instance, [31] considers matroid bandits, which is a special case of CMAB where the super arms are given as independent sets of a matroid with base arms being the elements of the ground set, and the expected reward of a super arm is the sum of the expected outcomes of the base arms in the super arm. Another example is cascading bandits [4], which is a special case of CMAB-PTA, where each super arm corresponds to a ranked list of items and base arms are triggered according to a user click model. A plethora of papers exist on UCB based policies for variants of these two models (see e.g., [32] for a variant of matroid bandits and [33] and [34] for variants of cascading bandits.) Apart from these, [26] considers Thompson sampling with Gaussian posterior for cascading bandits and proves that the worst-case regret is $O(\sqrt{KmT})$. We show in Section VIII that CTS significantly outperforms their algorithm for cascading bandits. We think that this is the case in practice because Beta posterior is more suitable in modeling click probabilities compared to Gaussian posterior.

Several other works focus on contextual CMAB [34–36], CMAB with adversarial rewards [37,38] and CMAB with knapsacks [39]. Most recently there has been a surge of interest in analyzing CMAB under the full-bandit feedback setting, where the learner only observes the reward of the selected super arm but not the outcomes of the base arms [40,41]. For instance, [41] uses a sampling method based on Hadamard matrices to estimate base arm rewards from full-bandit feedback. On the other hand, [42] considers a more general feedback model where the learner observes a linear combination of base arm’s rewards.

Table I compares our work with the most closely related work with a comparison to our work.

| Publ. | Algorithm | Oracle | PTAs | Regret Bound |
|------|-----------|--------|------|--------------|
| [19] | CUCB      | Approx.| No   | $O(\sum_{i=1}^m \log T / \Delta_i)$ |
| [20] | CUCB      | Approx.| Yes  | $O(\sum_{i=1}^m \log T / (p_i \Delta_i))$ |
| [22] | CUCB      | Approx.| Yes  | $O(\sum_{i=1}^m \log T / \Delta_i)$ |
| [28] | CTS       | Exact  | No   | $O(\sum_{i=1}^m \log T / \Delta_i)$ |
| [20] | CUCB & CTS | Approx.| Yes† | $O(1/p^*)$ |
| Ours | CTS       | Exact  | Yes† | $O(\sum_{i=1}^m \log T / (p_i \Delta_i))$ |

*Under the triggering probability modulated bounded smoothness assumption.
†The case when the arm triggering probabilities are all positive.
papers in terms of the assumptions and the regret bounds. Finally, we would like to note that Thompson sampling is also analyzed for a parametric CMAB model given a prior with finite support in [43], and a contextual CMAB model with a Bayesian regret metric in [44]. Unlike these works, we adopt the models in [20] and [28], work in a setting where there is an unknown but fixed parameter (expected outcome) vector, and analyze the expected regret.

III. PROBLEM FORMULATION

CMAB-PTA is a decision making problem where the learner interacts with its environment through \( m \) base arms, indexed by the set \( [m] := \{1, 2, ..., m\} \) sequentially over rounds indexed by \( t \in [T] \). In this paper, we consider the model introduced in [20] and borrow the notation from [28]. In this model, the following events take place in order in each round \( t \):

- The learner selects a subset of base arms, denoted by \( S(t) \), which is called a super arm.
- \( S(t) \) causes some other base arms to probabilistically trigger based on a stochastic triggering process, which results in a set of triggered base arms \( S'(t) \) that contains \( S(t) \).
- The learner obtains a reward that depends on \( S'(t) \) and observes the outcomes of the base arms in \( S'(t) \).

Next, we describe in detail the base arm outcomes, the super arms, the triggering process, the reward, the observation (feedback) model and the regret.

A. Base Arm Outcomes

In each round \( t \), the environment draws a random outcome vector \( \mathbf{X}(t) := (X_1(t), X_2(t), \ldots, X_m(t)) \) from a fixed probability distribution \( D \) on \([0, 1]^m\) independent of the previous rounds, where \( X_i(t) \) represents the outcome of base arm \( i \). \( D \) is unknown by the learner, but it belongs to a class of distributions \( \mathcal{D} \) which is known by the learner. We define the mean outcome (parameter) vector as \( \mu := (\mu_1, \mu_2, \ldots, \mu_m) \), where \( \mu_i := \mathbb{E}_D[X_i(t)] \), and use \( \mu_S \) to denote the projection of \( \mu \) on \( S \) for \( S \subseteq [m] \).

Since CTS computes a posterior over \( \mu \), the following assumption is made to have an efficient and simple update of the posterior distribution.

**Assumption 1.** The outcomes of all base arms are mutually independent, i.e., \( D = D_1 \times D_2 \times \cdots \times D_m \).

Note that this independence assumption holds in many applications, including the influence maximization problem with independent cascade influence propagation model [21].

B. Super Arms and the Triggering Process

The learner is allowed to select \( S(t) \) from a subset of \( 2^m \) denoted by \( \mathcal{I} \), which corresponds to the set of feasible super arms. Once \( S(t) \) is selected, all base arms \( i \in S(t) \) are immediately triggered. These arms can trigger other base arms that are not in \( S(t) \), and those arms can further trigger other base arms, and so on. At the end, a random superset \( S'(t) \) of \( S(t) \) is formed that consists of all triggered base arms as a result of selecting \( S(t) \). We have \( S'(t) \sim D^{\text{trig}}(S(t), \mathbf{X}(t)) \), where \( D^{\text{trig}} \) is the probabilistic triggering function that describes the triggering process. For instance, in the influence maximization problem, \( D^{\text{trig}} \) may correspond to the independent cascade influence propagation model defined over a given influence graph [21]. The triggering process can also be described by a set of triggering probabilities. For each \( i \in [m] \) and \( S \in \mathcal{I} \), \( p_i^S \) denotes the probability that base arm \( i \) is triggered when super arm \( S \) is selected given that the arm outcome distribution is \( D' \in D \). For simplicity, we let \( p_i^S = p_i^{D,S} \), where \( D \) is the true arm outcome distribution. Let \( \mathcal{S} := \{i \in [m] : p_i^S > 0\} \) be the set of all base arms that could potentially be triggered by super arm \( S \), which is called the triggering set of \( S \). We have that \( S(t) \subseteq S'(t) \subseteq \mathcal{S} \subseteq [m] \). We define \( p_i := \min_{S \in \mathcal{I}, i \in S} p_i^S \) as the minimum nonzero triggering probability of base arm \( i \), and \( p^* := \min_{i \in [m]} p_i \) as the minimum nonzero triggering probability.

C. Reward

At the end of round \( t \), the learner receives a reward that depends on the set of triggered arms \( S'(t) \) and the outcome vector \( \mathbf{X}(t) \), which is denoted by \( R(S'(t), \mathbf{X}(t)) \). For simplicity of notation, we also use \( R(t) = R(S'(t), \mathbf{X}(t)) \) to denote the reward in round \( t \). Note that whether a base arm is in the selected super arm or is triggered afterwards is not relevant in terms of the reward. We assume that the expected reward depends on the mean outcome vector in a specific way by making the following mild assumptions about the expected reward function. We note that these assumptions are standard in the CMAB literature [20], [28] and hold for the networking applications given in Section IV. The first assumption states that the expected reward is only a function of \( S(t) \) and \( \mu \).

**Assumption 2.** The expected reward of super arm \( S \in \mathcal{I} \) only depends on \( S \) and the mean outcome vector \( \mu \), i.e., there exists a function \( r \) such that

\[
\mathbb{E}[R(t)] = \mathbb{E}_{S'(t) \sim D^{\text{trig}}(S(t), \mathbf{X}(t))} \mathbb{E}_{\mathbf{X}(t) \sim D}[R(S'(t), \mathbf{X}(t))] = r(S(t), \mu).
\]

In order to learn the best action, we require the estimate of the expected reward vector to converge to the true expected reward vector as the number of observations increases. This can be done when the expected reward varies smoothly with the mean outcome vector. Below, we state a form of continuity for the expected reward.

**Assumption 3.** (Lipschitz continuity) There exists a constant \( B > 0 \), such that for every super arm \( S \) and every pair of mean outcome vectors \( \mu \) and \( \mu' \), we have

\[
|r(S, \mu) - r(S, \mu')| \leq B||\mu_S - \mu'_S||_1
\]

where \( || \cdot ||_1 \) denotes the \( l_1 \) norm.

D. Observation Model

We consider the semi-bandit feedback model, where at the end of round \( t \), the learner observes the individual outcomes of
the triggered arms, denoted by \( Q(S'(t), X(t)) := \{ (i, X_i(t)) : i \in S'(t) \} \). Again, for simplicity of notation, we also use \( Q(t) = Q(S'(t), X(t)) \) to denote the observation at the end of round \( t \). Based on this, the only information available to the learner when choosing the super arm to select in round \( t + 1 \) is its observation history, given as \( F_t := \{ (S(r), Q(r)) : r \in [t] \} \).

In short, the tuple \([m], \mathcal{I}, D, D^{\text{trig}}, R\) constitutes a CMAB-PTA problem instance. Among the elements of this tuple only \( D \) is unknown to the learner.

### E. Regret

In order to evaluate the performance of the learner, we define the set of optimal super arms given an \( m \)-dimensional parameter \( \theta \) as \( \text{OPT}(\theta) := \arg\max_{S \in \mathcal{I}} r(S, \theta) \). We use \( \text{OPT} := \text{OPT}(\mu) \) to denote the set of optimal super arms given the true mean outcome vector \( \mu \). On this, we let \( S^* \) to represent a specific super arm in \( \arg\min_{S \in \text{OPT}} |S| \), which is the set of super arms that have triggering sets with minimum cardinality among all optimal super arms. We also let \( k^* := |S^*| \) and \( \tilde{k}^* := |\tilde{S}^*| \).

Next, we define the suboptimality gap due to selecting super arm \( S \in \mathcal{I} \) as \( \Delta_S := r(S^*, \mu) - r(S, \mu) \), the maximum suboptimality gap as \( \Delta_{\text{max}} := \max_{S \in \mathcal{I}} \Delta_S \), and the minimum suboptimality gap of base arm \( i \) as \( \Delta_i := \min_{S \in \mathcal{I} - \text{OPT}} |S^*| \). The goal of the learner is to minimize the (expected) regret over the time horizon \( T \), given by

\[
\text{Reg}(T) := E \left[ \sum_{t=1}^{T} (r(S^*, \mu) - r(S(t), \mu)) \right]
= E \left[ \sum_{t=1}^{T} \Delta_{S(t)} \right].
\]

### IV. NETWORKING APPLICATIONS

Here, we introduce three networking applications of CMAB-PTA: cascading bandits, probabilistic maximum coverage bandits and influence maximization bandits. Numerical experiments given in Section VIII explore specific cases of all these problems that are generated either synthetically or from real-world data.

#### A. Cascading Bandits

1) **Disjunctive Form for Search Engine Optimization:** In the disjunctive form of the cascading bandit problem [4], a search engine outputs a list of \( K \) web pages for each of its \( R \) users among a set of \( L \) web pages. Then, the users examine their respective lists, and click on the first page that they find attractive. If all pages fail to attract them, they do not click on any page. The goal of the search engine is to maximize the number of clicks.

This problem can be modeled as an instance of CMAB-PTA as follows. The base arms are page-user pairs \((i, j)\), where \( i \in [L] \) and \( j \in [R] \). User \( j \) finds page \( i \) attractive independent of other users and other pages with probability \( p_{i,j} \). The super arms consist of \( R \)-many \( K \)-tuples, where each \( K \)-tuple represents the list of pages shown to a user. Given a super arm \( S \), let \( S(k, j) \) denote the \( k \)th page that is selected for user \( j \). Then, the triggering probabilities can be written as

\[
p_i^S := \begin{cases} 1 & \text{if } i = S(1) \\ \prod_{k'=1}^{k-1} P_S(k') & \text{if } \exists k \neq 1 : i = S(k) \\ 0 & \text{otherwise} \end{cases}
\]

and the probabilistic reliability of routing path \( S \)—in other words, the expected reward—becomes

\[
r(S, p) = \prod_{k=1}^{K} P_S(k) .
\]
B. Probabilistic Maximum Coverage Bandits

Consider a special case of the probabilistic maximum coverage problem, where an online shopping site advertises \( K \) items that are selected from a catalog of \( L \) items to its \( R \) users. Each user inspects all of the items that are advertised and likes one of the attractive items. The users do not like any item if none of the items attract them. The goal of the shopping site is to maximize the number of likes. Analogous to cascading bandits, in this problem, base arms are item-user pairs \((i, j)\), where \( i \in [L] \) and \( j \in [R] \). User \( j \) finds item \( i \) attractive independent of other users and other items with probability \( p_{i,j} \).

The super arms are the set of all pairs \((i, j)\) such that item \( i \) is the element of a size-\( K \) subset of \([L]\).

This can also model the problem of allocating orthogonal channels to secondary users in a cognitive radio network [3]. Consider a special case of the probabilistic maximum coverage problem, where \( G = (V, E) \), and \( V \) represents the set of nodes and \( E \) is the set of edges. The learner selects and triggers a set of nodes, and the expected reward of super arm \( r \) can be written as

\[
E(S, p) = \sum_{j=1}^{R} \left( 1 - \prod_{i=1}^{L} (1 - p_{i,j}^*) \right)
\]

for which Assumption 3 holds when \( B = 1 \).

C. Influence Maximization Bandits

In the influence maximization problem with the independent cascade model [21], the learner is given a directed graph denoted by \( G = (V, E) \), where \( V \) is the set of nodes and \( E \) is the set of edges. The learner selects and triggers a set of nodes \( S \subseteq V \) such that \( |S| = K \), where \( K \) is one of the problem parameters. This is the first iteration of a diffusion process. In each subsequent iteration, a node that was triggered in the previous iteration might trigger another node \( j \) that is not triggered yet if \( j \) is adjacent to one of its outgoing edges. This happens with probability \( p_{i,j} \) independently from the states of all other nodes. The diffusion process ends when no new node triggers in an iteration. The goal of the learner is to maximize—the initial decision of nodes—the number of triggered nodes at the end of the diffusion process.

The problem can be modeled as a CMAB problem with PTAs, where base arms are edges \((i, j) \in E \) and super arms are the set of all edges \((i, j)\) such that \( i \in S \) and \( j \in S \). Assumption 3 holds as proven in Lemma 6 of [20].

V. The Learning Algorithm and Main Results

A. Combinatorial Thompson Sampling

CTS is a Bayesian algorithm that selects super arms by sampling from posterior distributions of base arms. Its pseudocode is given in Algorithm 1. We assume that the learner has access to an exact computation oracle, which takes as input an \( m \)-dimensional parameter vector \( \theta \) and the problem structure \(([m], \mathcal{T}, D^{\text{neg}}, R)\), and outputs a super arm, denoted by Oracle(\( \theta \)) such that \( \text{Oracle}(\theta) \in \text{OPT}(\theta) \). CTS keeps a Beta posterior over the mean outcome of each base arm. At the beginning of round \( t \), for each base arm \( i \) it draws a sample \( \theta_i(t) \) from its posterior distribution. Then, it forms the parameter vector in round \( t \) as \( \theta(t) := (\theta_1(t), \ldots, \theta_m(t)) \), gives it to the exact computation oracle, and selects the super arm \( S(t) = \text{Oracle}(\theta(t)) \). At the end of the round, CTS updates the posterior distributions of the triggered base arms using the observation \( Q(t) \).

Algorithm 1 Combinatorial Thompson Sampling (CTS).

```plaintext
1: For each base arm \( i \), let \( a_i = 1, b_i = 1 
2: for \, t = 1, 2, \ldots, do 
3: For each base arm \( i \), draw a sample \( \theta_i(t) \) from Beta distribution \( \beta(a_i, b_i) \); let \( \theta(t) := (\theta_1(t), \ldots, \theta_m(t)) \)
4: Select super arm \( S(t) = \text{Oracle}(\theta(t)) \), get the observation \( Q(t) \)
5: for all \((i, X_i) \in Q(t)\) do
6: \( Y_i \leftarrow 1 \) with probability \( X_i \), 0 with probability \( 1 - X_i \)
7: \( a_i \leftarrow a_i + Y_i \)
8: \( b_i \leftarrow b_i + (1 - Y_i) \)
9: end for
10: end for
```

B. Regret of CTS for the General Case

**Theorem 1.** Under Assumptions 2, 2, and 3 for all \( D \), the regret of CTS by round \( T \) is bounded as

\[
\text{Reg}(T) \leq \sum_{i=1}^{m} \max_{S \in \mathcal{S} - \text{OPT} \, |S| = K} 16B^2|\tilde{S}| \log T 
+ \left( 3 + \frac{\tilde{K}^2}{(1 - \rho)p^*\varepsilon^2} + \frac{2[p^* < 1]}{\rho^2p^*} \right) m \Delta_{\text{max}} 
+ \alpha \frac{8\tilde{k}^*}{p^*\varepsilon^2} \left( \frac{4}{\varepsilon^2} + 1 \right) \tilde{k}^* \log \frac{\tilde{k}^*}{\varepsilon^2} \Delta_{\text{max}} 
\]

for all \( \rho \in (0, 1) \), and for all \( \varepsilon \in (0, 1/\sqrt{\varepsilon}) \) such that \( \forall S \in \mathcal{T} - \text{OPT}, \Delta_S > 2B(k^* + 2)\varepsilon \), where \( B \) is the Lipschitz constant.
constant in Assumption 3. \( \alpha > 0 \) is a problem independent constant that is also independent of \( T \), and \( \bar{K} := \max_{S \in I} |\bar{S}| \) is the maximum triggering set size among all super arms.

We compare the result in Theorem 1 with [29], which shows that the regret of CUCB is \( O(\sum_{i \in [m]} \log T/p_i \Delta_i) \) given an \( l_\infty \) bounded smoothness condition on the expected reward function, when the bounded smoothness function is \( f(x) = \gamma x \). When \( \varepsilon \) is sufficiently small, the regret bound in Theorem 1 is asymptotically equivalent to the regret bound for CUCB (in terms of the dependence on \( T, p_i \) and \( \Delta_i, i \in [m] \)).

For the case with \( p^* = 1 \) (no probabilistic triggering), the regret bound in Theorem 1 matches with the regret bound in Theorem 1 in [28] (in terms of the dependence on \( T \) and \( \Delta_i, i \in [m] \)). As a final remark, we note that Theorem 3 in [22] shows that the \( 1/p_i \) term in the regret bound that multiplies the \( \log T \) term is unavoidable in general.

C. Regret of CTS for Strictly Positive Triggering Probabilities

We improve the regret bound in Theorem 1 when all triggering probabilities are strictly positive.

**Theorem 2.** Under Assumptions 1, 2, and 3, for all \( \forall i \in [m], S \in I, p^{1, S}_i \geq p^* > 0 \), the regret of CTS by round \( T \) is bounded as

\[
\text{Reg}(T) \leq \max \left\{ 16mB \sqrt{\frac{e}{(1-p^*)^2}}, \right. \\
\left. \max_{S \in I-\text{OPT}} \left\{ \frac{128mB^2 |\bar{S}|}{(1-p^*)^2 \Delta_S - 2B(k^{*2}+2)\varepsilon} \times \log \left( \frac{4B|\bar{S}|}{(1-p^*)^2 \Delta_S - 2B(k^{*2}+2)\varepsilon} \right) \right\}, \right. \\
\left. \left( 5 + \frac{\bar{K}^2}{(1-p^*)^2 \varepsilon^2} + \frac{2l(p^* < 1)}{\rho^2 p^*} \right) m \Delta_{\text{max}} \right. \\
\left. + \alpha \frac{8\bar{K}^*}{p^* \varepsilon^2} \left( \frac{4}{\varepsilon^2} + 1 \right) \log \frac{\bar{K}^*}{\varepsilon^2} \Delta_{\text{max}} \right. \\
\text{for all } \rho \in (0, 1), \text{ and for all } \varepsilon \in (0, 1/\sqrt{e}) \text{ such that } \forall S \in I - \text{OPT}, \Delta_S > 2B(k^{*2}+2)\varepsilon, \text{ where } B \text{ is the Lipschitz constant in Assumption 3. } \alpha > 0 \text{ is a problem independent constant that is also independent of } T, \text{ and } \bar{K} := \max_{S \in I} |\bar{S}| \text{ is the maximum triggering set size among all super arms.} \\
\]

Note that having all triggering probabilities strictly positive makes the exploration aspect of the MAB problem trivial. No matter which actions the learner takes, all base arms provide occasional feedback. As a result of this, the upper bound for the expected regret becomes independent of the time horizon \( T \). We compare the result of Theorem 2 with [30], which shows a similar bound for CTS in the exact same setting. While the bound in [30] is on order \( O(1/p^*)^4 \) with respect to \( p^* \), the bound in Theorem 2 is on order \( O(1/p^* \log(1/p^*)) \).

As a final remark, we observe that our bound in Theorem 2 does not match the lower bound on order \( O(\log(1/p^*)) \) proven for a special case of our setting given in Theorem 1 of [25], where the reward only depends on the selected arm. This is due to the fact that the setting in [25] satisfies the triggering probability modulated bounded smoothness condition given in [22] in addition to our Assumption 3. Considering how this condition was necessary to get rid of the \( 1/p^* \) term in the previous upper bound given in [20], it might be the case that an upper bound on order \( O(\log(1/p^*)) \) instead of order \( O(1/p^* \log(1/p^*)) \) is possible under the triggering probability modulated bounded smoothness condition, which is more restrictive than Assumption 3 (one necessitates the other).

VI. PROOF OF THEOREM 1

Before delving into the details, we give a high-level sketch of the proof. First, we identify four events that can be contributing factors when a suboptimal decision is made:

- at least one base arm has an estimated expected outcome that deviates too much from its true expected outcome,
- at least one base arm is triggered much less than the total number of rounds it could have been triggered,
- at least one base arm has a posterior sample that deviates too much from its estimated expected outcome,
- the (posterior) sample vector and the expected outcome vector are too close to each other when projected onto the triggering set of the super arm that is played.

The fact that the last event leads to suboptimal decisions might be counterintuitive since we would expect making better decisions when the samples are close to the expected outcomes. However, in Lemma 2 we show that making a suboptimal decision despite having a sample vector that is close to the expected outcome vector when projected onto the triggering set of the super arm that is played can only mean the same vectors are distant from each other when projected onto some subset of the triggering set of the optimal super arm.

Next, we show that all four of these events can only occur a finite number of times independent of the time horizon when a suboptimal decision is made (for the first two events in Section VI-D1 for the third event in Section VI-D2 and for the last event in Section VI-D3). Then, in Section VI-D3 we shift our focus to the rounds when none of those events occur, which implies that, for a suboptimal decision to be made, there must be an under-explored base arm. More specifically, for at least one base arm, the number rounds that it could have been triggered must be smaller than a threshold that is logarithmic with respect to the time horizon. This, in turn, leads to a logarithmic upper bound on the expected regret.

We would also like to emphasize that our regret analysis is significantly different from the analysis in [28] (without probabilistic triggering) in the following aspects: First of all, our bad events \( \mathcal{E}_{Z,1}(\theta) \) and \( \mathcal{E}_{Z,2}(\theta) \) given in Section VI-A are defined in terms of subsets \( Z \) of \( S^* \) rather than \( S^* \). Secondly, we need to relate the number of times base arm \( i \) is in the triggering set of the selected super arm \( (M_i(t)) \) with the number of times it is triggered \( (N_i(t)) \), which requires us to define events \( B_{Z,i}(t) \) for \( i \in [m] \) as given in [3], and use them in the regret decomposition. This introduces new challenges in bounding \( 10 \), where we make use of a variable called trial threshold \( (L_i(S(t))) \) given in [2] to show that \( 10 \)
cannot happen when \( M_i(t) > L_i(S(t)), \forall i \in \tilde{S}(t) \). We also need to take probabilistic triggering into account when proving Lemmas [1] and [2]. For instance, in Lemma [2], we define a new way to count the number of times \( E_{Z,1}(\theta) \wedge E_{Z,2}(\theta) \) happens for all \( Z \subseteq \tilde{S}^* \) such that \( Z \neq \emptyset \).

A. Preliminaries for the Proof

The complement of set \( S \) is denoted by \( \neg S \) or \( S^c \). The indicator function is given as \( \mathbb{I}\{\cdot\} \). \( M_i(t) := \sum_{\tau=1}^{t-1} \mathbb{I}\{i \in \tilde{S}(\tau)\} \) denotes the number of times base arm \( i \) is in the triggering set of the selected super arm, i.e., it is tried to be triggered, \( N_i(t) := \sum_{\tau=1}^{t-1} \mathbb{I}\{i \in S'(\tau)\} \) denotes the number of times base arm \( i \) is triggered, and \( \mu_i(t) := \frac{1}{N_i(t)} \sum_{\tau < t, i \in S'(\tau)} Y_i(\tau) \) denotes the empirical mean outcome of base arm \( i \) until round \( t \), where \( Y_i(t) \) is the Bernoulli random variable with mean \( X_i(t) \) that is used for updating the posterior distribution that corresponds to base arm \( i \) in CTS. We define

\[
\ell(S) := \frac{2 \log T}{\frac{1}{|S|} \sum_{i \in S} (i^2 + \varepsilon)^2}
\]

as the sampling threshold of super arm \( S \),

\[
L_i(S) := \frac{\ell(S)}{1 - \rho} \rho_i
\]

as the trial threshold of base arm \( i \) with respect to super arm \( S \), and \( L_i^{\max} := \max_{S \in \mathcal{Z} - \text{OPT}, i \in S} L_i(S) \).

Consider an \( m \)-dimensional parameter vector \( \theta \). Similar to [28], given \( Z \subseteq \tilde{S}^* \), we say that the first bad event for \( Z \), denoted by \( E_{Z,1}(\theta) \), holds when all \( \theta' = (\theta_Z^c, \theta_{Z^c}) \) such that \( \|\theta_Z - \mu_{\tilde{S}}\|_2 \leq \varepsilon \) satisfies the following properties:

- \( Z \subseteq \text{Oracle}(\theta') \).
- Either \( \text{Oracle}(\theta') \in \text{OPT} \) or \( \|\theta'_Z - \mu_{\text{Oracle}(\theta')}\|_1 > \Delta_{\text{Oracle}(\theta')}^\text{Opt} - (k^2 + 1)\varepsilon \).

Given the same parameter vector \( \theta \), the second bad event for \( Z \) is defined as \( E_{Z,2}(\theta) := \|\theta_Z - \mu_{\tilde{S}}\|_2 > \varepsilon \).

In addition, similar to the regret analysis in [28], we will make use of the following events when bounding the regret:

\[
A(t) := \{S(t) \not\in \text{OPT}\}
\]

\[
B_{1,1}(t) := \left\{ \hat{\mu}_i(t) - \mu_i > \varepsilon \right\}
\]

\[
B_{1,2}(t) := \left\{ N_i(t) \leq (1 - \rho) p_i M_i(t) \right\}
\]

\[
B(t) := \left\{ \exists i \in \tilde{S}(t) : B_{1,1}(t) \vee B_{1,2}(t) \right\}
\]

\[
C(t) := \left\{ \exists i \in \tilde{S}(t) : \theta_i(t) - \hat{\mu}_i(t) > \frac{2 \log T}{N_i(t)} \right\}
\]

\[
D(t) := \left\{ \|\theta_{\tilde{S}}(t) - \mu_{\tilde{S}}\|_1 > \frac{\Delta_{\tilde{S}}(t)}{B} - (k^2 + 1)\varepsilon \right\}
\]

B. Regret Decomposition

Using the definitions of the events given in (3)-(7), the regret can be upper bounded as follows:

\[
\text{Reg}(T) = \sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{A(t)\} \times \Delta_{S(t)}]
\]

\[
\leq \sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{B(t) \wedge A(t)\} \times \Delta_{S(t)}]
\]

\[
+ \sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{C(t) \wedge A(t)\} \times \Delta_{S(t)}]
\]

\[
+ \sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{-B(t) \wedge C(t) \wedge D(t) \wedge A(t)\} \times \Delta_{S(t)}]
\]

\[
+ \sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{-D(t) \wedge A(t)\} \times \Delta_{S(t)}]
\]

\[
\leq \sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{B(t) \wedge A(t)\} \times \Delta_{S(t)}]
\]

\[
+ \sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{C(t) \wedge A(t)\} \times \Delta_{S(t)}]
\]

\[
+ \sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{-B(t) \wedge C(t) \wedge D(t) \wedge A(t)\} \times \Delta_{S(t)}]
\]

\[
+ \sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{-D(t) \wedge A(t)\} \times \Delta_{S(t)}]
\]

The regret bound in Theorem [1] is obtained by bounding each term in the above decomposition. All events except the one specified in (10) can only happen a small (finite) number of times. Every time (10) happens, there must be base arms in the triggering set of the selected super arm which are “tried” to be triggered less than the trial threshold. These “under-explored” base arms are the main contributions of the regret, and their contribution depends on how many times they are “tried.” Moreover, every time these base arms are tried, their contribution to the future regret decreases. Thus, by summing up these contributions we obtain a logarithmic bound for (10).

In the proof, we will make use of the facts and lemmas that are introduced in the following section.

C. Facts and Lemmas

**Fact 1.** (Lemma 4 in [28]) When CTS is run, the following holds for all base arms \( i \in [m] \):

\[
\Pr \left[ \theta_i(t) - \hat{\mu}_i(t) > \frac{2 \log T}{N_i(t)} \right] \leq \frac{1}{T}
\]

\[
\Pr \left[ \hat{\mu}_i(t) - \theta_i(t) > \frac{2 \log T}{N_i(t)} \right] \leq \frac{1}{T}
\]

We will make use of the following technical lemmas. Their proofs can be found in the supplemental document of the conference version of the paper [1].

**Lemma 1.** When CTS is run, we have for all \( i \in [m] \) and \( \rho \in (0, 1) \):

\[
\mathbb{E}[\{t : 1 \leq t \leq T, i \in \tilde{S}(t), |\hat{\mu}_i(t) - \mu_i| > \varepsilon \vee B_{1,2}(t)\}] \leq 1 + \frac{1}{(1 - \rho) p_i^2 \varepsilon^2} + \frac{2\varepsilon^2 p_i^2 - 1}{p_i^2}.
\]

**Lemma 2.** Suppose that \( \neg D(t) \wedge A(t) \) happens. Then, there exists \( Z \subseteq \tilde{S}^* \) such that \( Z \neq \emptyset \) and \( E_{Z,1}(\theta(t)) \) holds.

**Lemma 3.** When CTS is run, for all \( Z \subseteq \tilde{S}^* \) such that \( Z \neq \emptyset \), we have

\[
\sum_{t=1}^{T} \mathbb{E}[\mathbb{I}\{E_{Z,1}(\theta(t)), E_{Z,2}(\theta(t))\}] \leq 13 \alpha_2 \cdot \frac{|Z|}{p_i} \cdot \left( \frac{2^2 |Z| + 3 \log |Z|}{\varepsilon^2 |Z|^2 + 2} \right)
\]

where \( \alpha_2 \) is a problem independent constant.
D. Main Part of the Proof

1) Bounding $\mathbf{[8]}$: Using Lemma $\mathbf{[1]}$, we have
\[
\sum_{t=1}^{T} \mathbb{E}[I\{B(t) \land A(t)\} \times \Delta_{S(t)}] 
\leq \Delta_{\max} \sum_{i=1}^{m} \mathbb{E}\left[ \left| t : 1 \leq t \leq T, i \in \tilde{S}(t), |\hat{\mu}_i(t) - \mu_i| > \frac{\varepsilon}{K} \lor B, 2(t) \right| \right] 
\leq \left( 1 + \frac{\tilde{K}^2}{(1 - \rho)p^*\varepsilon^2} + \frac{2\mathbb{E}(p^* < 1)}{\rho_p^*p^*} \right) m\Delta_{\max}.
\]

2) Bounding $\mathbf{[8]}$: By Fact $\mathbf{[1]}$ we have
\[
\sum_{t=1}^{T} \mathbb{E}[I\{C(t) \land A(t)\} \times \Delta_{S(t)}] 
\leq \Delta_{\max} \sum_{i=1}^{m} \sum_{t=1}^{T} \Pr\left[ |\theta_i(t) - \hat{\mu}_i(t)| > \sqrt{\frac{2\log T}{N_i(t)}} \right] 
\leq 2m\Delta_{\max}.
\]

3) Bounding $\mathbf{[10]}$: For this, we first show that event $\neg B(t) \land \neg C(t) \land D(t) \land A(t)$ cannot happen when $M_i(t) \leq L_i(S(t))$, $\forall i \in \tilde{S}(t)$. To see this, assume that both $\neg B(t) \land \neg C(t) \land A(t)$ and $M_i(t) \leq L_i(S(t))$, $\forall i \in \tilde{S}(t)$ hold. Then, we have
\[
||\theta_{S(t)}(t) - \hat{\mu}_{S(t)}(t)||_1 = \sum_{i \in \tilde{S}(t)} |\theta_i(t) - \hat{\mu}_i(t)| 
\leq \sum_{i \in \tilde{S}(t)} \sqrt{\frac{2\log T}{N_i(t)}} 
\leq \sum_{i \in \tilde{S}(t)} \sqrt{\frac{2\log T}{(1 - \rho)p_iM_i(t)}} 
\leq \sum_{i \in \tilde{S}(t)} \frac{2\log T}{(1 - \rho)p_iL_i(S(t))} 
= \sum_{i \in \tilde{S}(t)} \frac{\Delta_{S(t)}}{2B(\tilde{S}(t)) - \tilde{\kappa}^2 + 2\varepsilon} 
= \sum_{i \in \tilde{S}(t)} \frac{\Delta_{S(t)}}{2B(\tilde{S}(t)) - \tilde{\kappa}^2 + 2\varepsilon} 
\leq \Delta_{S(t)} \frac{\tilde{\kappa}^2 + 2\varepsilon}{2B} \leq \Delta_{S(t)} \frac{\tilde{\kappa}^2 + 2\varepsilon}{2B} 
\]
where $\mathbf{[12]}$ holds when $\neg C(t)$ happens, $\mathbf{[13]}$ holds when $\neg B(t)$ happens, and $\mathbf{[14]}$ holds by the definition of $L_i(S(t))$.

We also know that $||\mu_{\tilde{S}(t)}(t) - \mu_{\tilde{S}(t)}(t)||_1 \leq \varepsilon$, when $\neg B(t)$ happens. Then, $||\theta_{\tilde{S}(t)}(t) - \hat{\mu}_{\tilde{S}(t)}(t)||_1 \leq ||\theta_{\tilde{S}(t)}(t) - \mu_{\tilde{S}(t)}(t)||_1 + ||\hat{\mu}_{\tilde{S}(t)}(t) - \mu_{\tilde{S}(t)}(t)||_1 \leq \frac{\Delta_{S(t)}}{2B} - (\tilde{\kappa}^2 + 1)\varepsilon < \frac{\Delta_{S(t)}}{2B} - (\tilde{\kappa}^2 + 1)\varepsilon$, which implies that $\neg D(t)$ happens. Thus, we conclude that when $\neg B(t) \land \neg C(t) \land D(t) \land A(t)$ happens, then there exists some $i \in \tilde{S}(t)$ such that $M_i(t) \leq L_i(\tilde{S}(t))$. Let $S_1(t)$ be the base arms $i$ in $\tilde{S}(t)$ such that $M_i(t) > L_i(S(t))$, and $S_2(t)$ be the other base arms in $\tilde{S}(t)$. By the result above, $S_2(t) \neq \emptyset$. Next, we show that
\[
\Delta_{S(t)} \leq 2B \sum_{i \in S_2(t)} \sqrt{\frac{2\log T}{(1 - \rho)p_iM_i(t)}}.
\]
This holds since,
\[
\frac{\Delta_{S(t)}}{B} - (\tilde{\kappa}^2 + 1)\varepsilon \leq \sum_{i \in S(t)} |\theta_i(t) - \hat{\mu}_i(t)| 
\leq \sum_{i \in S_1(t)} |\theta_i(t) - \hat{\mu}_i(t)| + \sum_{i \in S_2(t)} |\theta_i(t) - \hat{\mu}_i(t)| + \varepsilon 
\leq \sum_{i \in S_1(t)} |\theta_i(t) - \hat{\mu}_i(t)| \leq \sum_{i \in S_2(t)} \sqrt{\frac{2\log T}{(1 - \rho)p_iM_i(t)}}.
\]
Fix $i \in [m]$. For $w > 0$, let $\eta_w$ be the round for which $i \in S_2(\eta_w)$ and $|\{t \leq \eta_w : i \in S_2(t)\}| = w$, and $w^i(T) := |\{t \leq T : i \in S_2(t)\}|$. We have $i \in S_2(\eta_w)$ for all $w > 0$, which implies that $M_i(\eta_{w+1}) \geq w$. Moreover, by the definition of $S_2(t)$, we know that $M_i(t) \leq L_i(S(t)) \leq L_i^{\max}$ for $i \in S_2(t)$, $t \leq T$. These two facts together imply that $w^i(T) \leq L_i^{\max}$ with probability 1.

Consider the round $\tau_1^i$ for which $i \in \tilde{S}(t)$ for the first time, i.e., $\tau_1^i := \min\{t : i \in \tilde{S}(t)\}$. We know that $M_i(\tau_1^i) = 0 \leq L_i(S)$ for all $S$, hence $i \in S_2(\tau_1^i)$. Since $\forall t < \tau_1^i$, $i \notin \tilde{S}(t)$, and $i \notin \tilde{S}(t)$ implies $i \notin S_2(t)$, we conclude that $\tau_1^i = \eta_1^i$. We also observe that $\neg B(t)$ cannot happen for $t \leq \tau_1^i = \eta_1^i$, since $N_i(t) > (1 - \rho)p_iM_i(t) = 0$ cannot be true when $N_i(t) \leq M_i(t) = 0$. Then,
\[
\sum_{t=1}^{T} \mathbb{E}[I\{\neg B(t) \land \neg C(t) \land D(t) \land A(t)\} \times \Delta_{S(t)}] 
\leq \mathbb{E}\left[ \sum_{t=1}^{T} \mathbb{E}[I\{\neg B(t)\}2B \sqrt{\frac{2\log T}{(1 - \rho)p_iM_i(t)}}] \right] 
= \mathbb{E}\left[ \sum_{i=1}^{m} \sum_{t=1}^{T} \mathbb{I}\{i \in S_2(t), \neg B(t)\} \right] \times 2B \sqrt{\frac{2\log T}{(1 - \rho)p_iM_i(t)}}.
\]
in the regret decomposition, which are given in the sections computed by summing the bounds derived for terms (8)-(11).

4) Bounding (11): From Lemma 2, we know that

\[
\sum_{t=1}^{T} \mathbb{E}[\{\neg D(t) \land A(t)\}] \leq \Delta_{\text{max}} \sum_{S \subseteq \tilde{S}^*, Z \neq \emptyset} \left( \sum_{i=1}^{T} \mathbb{E}[\{E_{Z,1}(\theta(t)), E_{Z,2}(\theta(t))\}] \right)
\]

since \(\neg D(t) \land A(t)\) implies \(E_{Z,1}(\theta(t))\) for some \(Z \subseteq \tilde{S}^*\), and \(E_{Z,1}(\theta(t)) \land \neg E_{Z,2}(\theta(t))\) implies either \(\neg A(t)\) or \(D(t)\).

From Lemma 3 we have

\[
\sum_{Z \subseteq \tilde{S}^*, Z \neq \emptyset} \left( \sum_{i=1}^{T} \mathbb{E}[\{E_{Z,1}(\theta(t)), E_{Z,2}(\theta(t))\}] \right) \\
\leq \sum_{Z \subseteq \tilde{S}^*, Z \neq \emptyset} 13\alpha_2', |Z| \cdot \frac{2^{2|Z|+3} \log |Z|}{p^*} \\
\leq 13\alpha_2' \frac{8k^*}{p^* \epsilon^2} \log \frac{\tilde{k}^*}{\epsilon^2} \sum_{Z \subseteq \tilde{S}^*, Z \neq \emptyset} \frac{2^{2|Z|}}{\epsilon^2 |Z|^2} \\
\leq 13\alpha_2' \frac{8k^*}{p^* \epsilon^2} \left( \frac{4}{\epsilon^2} + 1 \right) \frac{\tilde{k}^*}{\epsilon^2} \log \frac{\tilde{k}^*}{\epsilon^2}.
\]

5) Summing the Bounds: The regret bound for CTS is computed by summing the bounds derived for terms (8)-(11) in the regret decomposition, which are given in the sections above:

\[
\text{Reg}(T) \leq \sum_{i=1}^{m} 4B \sqrt{\frac{2 \log T}{(1-\rho)p_i M_i(t)}} + 3 \left( \frac{K^2}{(1-\rho)p^* \epsilon^2} + \frac{2I(p^* < 1)}{\rho^2 p^*} \right) m \Delta_{\text{max}} + 13\alpha_2' \frac{8k^*}{p^* \epsilon^2} \left( \frac{4}{\epsilon^2} + 1 \right) \tilde{k}^* \log \frac{\tilde{k}^*}{\epsilon^2} \Delta_{\text{max}}
\]

where \(\alpha := 13\alpha_2'\).

VII. PROOF OF THEOREM 2

While the proof of Theorem 2 is similar to the proof of Theorem 1, there are subtle differences that necessitates using a quite different proof technique. Different from Theorem 1, we need to update the definitions of the sampling and trial thresholds so that they depend on the most recent round index \(t\) rather than the time horizon \(T\).

In Section VII-B, similar to Section VI-D3 in the proof of Theorem 1, we observe that, for at least one base arm, \(M_{i}(t)\) must be smaller than the trial threshold for a suboptimal decision to be made. When the trial threshold—which is now logarithmic with respect to \(t\)—is independent of the time horizon, \(M_{i}(t)\)—which is equal to \(t\) for all base arms when all triggering probabilities are positive—surpasses it in a round that is also independent of the time horizon. In other words, we show in Section VII-B that suboptimal decisions can only be made up to a time-horizon-independent round, which was not the case in Section VI-D3. The rest of the proof of Theorem 1 remains largely unchanged except for some small tweaks to accommodate this crucial update.

A. Preliminaries for the Proof

We update the definition of the sampling threshold as

\[
\ell(S, t) := \frac{4 \log t}{\frac{\Delta_S}{2B|S|} - \frac{k^{\tilde{S}^*} + 2 \epsilon}{|S|}}
\]

and the definition of event \(C(t)\) as

\[
C(t) := \exists i \in \tilde{S}(t) : |\theta_i(t) - \hat{\mu}_i(t)| > \frac{4 \log t}{N_i(t)}
\]

We also define

\[
L := \max \left\{ e, \max_{S \in \mathcal{I} - \text{OPT}} \frac{4}{(1- \rho)p^* \left( \frac{\Delta_S}{2B|S|} - \frac{k^{\tilde{S}^*} + 2 \epsilon}{|S|} \right)^2} \right\}
\]

so that \(L_i(S, t) \leq L \log t\) for all base arms \(i \in [m]\), all super arms \(S \in \mathcal{I} - \text{OPT}\), and all rounds \(t \in [T]\) (\(t\) in \(L_i(S, t)\) represents dependence of the trial threshold on time due to \(\ell(S, t)\)). Finally, note that having non-zero triggering probabilities implies that \(M_{i}(t) = t\) for all base arms \(i \in [m]\).

B. Main Part of the Proof

We use the same regret decomposition as Theorem 1 given in (11) with the updated definition for \(C(t)\). We only bound terms (9) and (10) since those are the only ones affected from the updated definitions.
1) Bounding \( \mathbb{E}_{\mathcal{C}} \): Using Fact [1] we obtain
\[
\sum_{i=1}^{T} \mathbb{E}[\mathcal{C}(t) \wedge \mathcal{A}(t)] \times \Delta_{S(t)} \leq \Delta_{\max} \sum_{i=1}^{m} \sum_{t=1}^{T} \Pr \left( |\theta_i(t) - \mu_i(t)| \geq \frac{4 \log t}{N_i(t)} \right)
\]
\[
\leq \Delta_{\max} \sum_{i=1}^{m} \sum_{t=1}^{T} \frac{2}{t^2} \leq 4m \Delta_{\max} .
\]

2) Bounding \( \mathbb{E}_{\mathcal{D}} \): Similar to Theorem [1] we first show that event \( \neg \mathcal{B}(t) \wedge \neg \mathcal{C}(t) \wedge \mathcal{D}(t) \wedge \mathcal{A}(t) \) cannot occur when \( t > L_i(S(t), t), \forall i \in \hat{S}(t) \). To see this, assume that both \( \neg \mathcal{B}(t) \wedge \neg \mathcal{C}(t) \wedge \mathcal{A}(t) \) and \( t > L_i(S(t), t), \forall i \in \hat{S}(t) \) hold. Then, we must have
\[
\| \theta_{\hat{S}(t)}(t) - \mu_{\hat{S}(t)}(t) \|_1 = \sum_{i \in \hat{S}(t)} |\theta_i(t) - \mu_i(t)|
\]
\[
\leq \sum_{i \in \hat{S}(t)} \sqrt{\frac{4 \log t}{N_i(t)}} \tag{16}
\]
\[
\leq \sum_{i \in \hat{S}(t)} \frac{4 \log t}{(1 - \rho)p_t t} \tag{17}
\]
\[
\leq \sum_{i \in \hat{S}(t)} \frac{4 \log t}{l(S(t), t)} \tag{18}
\]
\[
= \sum_{i \in \hat{S}(t)} \left( \frac{\Delta_{S(t)}}{2B|S(t)|} - \frac{\hat{k}^2 + 2}{|S(t)|} \right)
\]
\[
= \frac{|\hat{S}(t)|}{\Delta_{S(t)}} \left( \frac{\Delta_{S(t)}}{2B|\hat{S}(t)|} - \frac{\hat{k}^2 + 2}{|\hat{S}(t)|} \right)
\]
\[
= \frac{\Delta_{\hat{S}(t)}}{2B} - \left( \hat{k}^2 + 2 \right) \varepsilon
\]

where \( \hat{S}(t) \) holds when \( \neg \mathcal{C}(t) \) occurs, \( \hat{S}(t) \) holds when \( \neg \mathcal{B}(t) \) occurs, and \( \hat{S}(t) \) holds by the definition of \( L_i(S(t), t) \). We also know that \( \| \mu_{\hat{S}(t)}(t) - \mu_{\hat{S}(t)}(t) \|_1 \leq \varepsilon \), when \( \neg \mathcal{B}(t) \) happens. Then, \( |\theta_{\hat{S}(t)}(t) - \mu_{\hat{S}(t)}(t) - \mu_{\hat{S}(t)}(t)| \leq \Delta_{\hat{S}(t)} /
(2B) - \left( \hat{k}^2 + 2 \right) \varepsilon < \frac{\Delta_{\hat{S}(t)}}{2B} - \left( \hat{k}^2 + 2 \right) \varepsilon \), which implies that \( \neg \mathcal{D}(t) \) happens.

Thus, we conclude that when \( \neg \mathcal{B}(t) \wedge \neg \mathcal{C}(t) \wedge \mathcal{D}(t) \wedge \mathcal{A}(t) \) occurs, then there exists some \( i \in \hat{S}(t) \) such that \( t \leq L_i(S(t), t) \). Let \( S_1(t) \) be the base arms \( i \in \hat{S}(t) \) such that \( t > L_i(S(t), t) \), and \( S_2(t) \) be the other base arms in \( \hat{S}(t) \). By the result above, \( S_2(t) \neq \emptyset \).

We also conclude that, for some \( i \in \hat{S}(t), t \leq L_i(S(t), t) \leq L \log t \). This inequality can only hold when \( t \leq t^* \), where \( t^* \) is the greatest root of \( t = L \log t \). The equation can be written in the alternative form \( \exp(t/L) = t \), whose roots are given by \( t = -LW(-1/L) \), where \( W(\cdot) \) is the Lambert W function. Indeed for \( -1/e \leq x < 0 \), there are two real values for \( W(x) \). As our solution, we take the lowest value (this branch is usually denoted as \( W_{-1}(x) \)) since it corresponds to the greatest root of \( t = L \log t \). Thus, we set \( t^* = -LW_{-1}(-1/L) \).

Then, by using the lower bound for \( W_{-1}(x) \) given in [45], we obtain \( t^* \leq L(\log L + \sqrt{2L \log L - 2}) \leq 2L \log L \).

Next, we show that
\[
\Delta_{S(t)} \leq 4B \sum_{i \in S_2(t)} \frac{2 \log L}{(1 - \rho)p_t t} .
\]

This holds since,
\[
\frac{\Delta_{S(t)}}{B} - \left( \hat{k}^2 + 1 \right) \varepsilon < \sum_{i \in S_2(t)} |\theta_i(t) - \mu_i(t)|
\]
\[
\leq \sum_{i \in S_2(t)} |\theta_i(t) - \mu_i(t)| + \sum_{i \in S_2(t)} |\theta_i(t) - \mu_i(t)| + \varepsilon
\]
\[
\leq |S_2(t)| \left( \frac{\Delta_{S(t)}}{2B|S_2(t)|} - \left( \hat{k}^2 + 1 \right) \varepsilon \right)
\]
\[
+ \sum_{i \in S_2(t)} |\theta_i(t) - \mu_i(t)| + \varepsilon
\]
\[
\leq \frac{\Delta_{S(t)}}{2B} - \left( \hat{k}^2 + 1 \right) \varepsilon + \sum_{i \in S_2(t)} \sqrt{\frac{4 \log t}{N_i(t)}}
\]
\[
\leq \frac{\Delta_{S(t)}}{2B} - \left( \hat{k}^2 + 1 \right) \varepsilon + \sum_{i \in S_2(t)} \sqrt{\frac{8 \log L}{(1 - \rho)p_t t}}
\]

Then,
\[
\sum_{t=1}^{T} \mathbb{E}[\neg \mathcal{B}(t) \wedge \neg \mathcal{C}(t) \wedge \mathcal{D}(t) \wedge \mathcal{A}(t)] \times \Delta_{S(t)}
\]
\[
= \sum_{t=1}^{T} \mathbb{E}_{\mathcal{C}} \qquad \mathbb{E}_{\mathcal{D}} \qquad \mathbb{E}_{\mathcal{B}}
\]
\[
\leq 4B \sum_{i \in S_2(t)} \sqrt{\frac{2 \log L}{(1 - \rho)p_t t}}
\]
\[
\leq 16mB \log L \sqrt{\frac{L}{(1 - \rho)p^*}}
\]

VIII. Numerical Results

In this section, we compare CTS with other state-of-the-art CMAB algorithms in three different applications: cascading bandits, probabilistic maximum coverage bandits, and
include influence maximization bandits introduced in Section IV. We compare the performance of CTS with CUCB in 20 in all settings. For the first two problems, we assume that all algorithms have access to an exact computation oracle that computes the estimated optimal super arm in each round. On the other hand, for the third problem, we assume that all algorithms use an approximation oracle. For cascading bandits only, we also compare CTS with algorithms specifically designed for this setting: CascadeKL-UCB in 4 and TS-Cascade in 26. The former uses the principle of optimism under the face of uncertainty to compute Kullback-Leibler divergence based UCBs while the latter uses Thompson sampling with Gaussian posterior over the base arms.

A. Cascading Bandits

We consider the disjunctive case with \( L = 100, R = 20 \) and \( K = 5 \), and generate \( p_{i,j} \)s by sampling uniformly at random from \([0, 1]\). We run both CTS and CUCB for 1600 rounds, and report their regrets averaged over 1000 runs in Fig. 1 where error bars represent the standard deviation of the regret (multiplied by 10 for visibility). In this setting CTS significantly outperforms CUCB by achieving a final regret that is no more than 5% of the final regret of CUCB. Relatively bad performance of CUCB can be explained by excessive number of explorations due to the UCBs that stay high for a large number of rounds.

In addition, we also consider the same class of problems \( B_{LB}(L, K, p, \Delta) \) as in 4, where \( R = 1 \) and the probability that the user finds page \( j \) attractive is given as

\[
p_{i,j} = \begin{cases} 
p & \text{if } j \leq K \\
p - \Delta & \text{otherwise}
\end{cases}
\]

Similar to 4, we set \( p = 0.2 \) and vary other parameters, namely \( L, K, \) and \( \Delta \). We run both CTS and CUCB for 100000 rounds in all problem instances, and report their regrets averaged over 20 runs in Table 1.

In addition to CUCB, we compare CTS against CascadeUCB1 and CascadeKL-UCB given in 4, and TS-Cascade given in 26 as well. Note that regrets of CUCB and CascadeUCB1 matches very closely as two algorithms are essentially the same when CUCB is applied to cascading bandits except for some minor differences in the initialization stage and how UCBs larger than 1 are handled. We observe that CTS outperforms all other algorithms in all problem instances by achieving a regret that is at most 44% of the regret of all other algorithms. For CTS, we also see that the regret increases as the number of pages \( (L) \) increases, it decreases as the number of recommended items \( (K) \) increases, and it increases as \( \Delta \) decreases, which are very similar to the major observations that are made in 4.

B. Probabilistic Maximum Coverage Bandits

Our experimental setup for this case is based on MovieLens dataset 47 as in 30. The dataset contains 20 million movie ratings that are assigned between January 1995 and March 2015. Out of this, we only use the ones that are assigned between March 2014 and March 2015. In the experiments, the recommender chooses \( K = 3 \) movies out of \( L = 30 \) movies, which include 10 of the most rated movies, 10 of the least rated movies and 10 randomly selected movies from the dataset. These 30 movies are rated by \( R = 57369 \) users.

In total, there are 20 genres in the dataset. Each movie belongs to at least one genre. We take genre information into account to define attraction probabilities. For this, we create a 20-dimensional vector \( g_i \) for each movie \( i \in [L] \), where \( g_{ik} = 1 \) if the movie belongs to genre \( k \) and 0 otherwise. Using these vectors, we calculate a genre preference vector \( u_j \) for each user \( j \in [R] \) as

\[
u_j = \frac{\sum_{i \in L_j} g_i}{|L_j|} + \epsilon_j
\]

where \( L_j \) is the set of movies that user \( j \) rated and \( \epsilon_j \) is a random vector such that \( \epsilon_{jk} = |\chi_{jk}| \) for \( \chi_{jk} \sim N(0, 0.05) \). The noise \( \epsilon_j \) is introduced to model exploratory behavior of the user. Finally, defining \( \tilde{g}_j = g_j/\|g_j\| \) and \( \tilde{u}_j = u_j/\|u_j\| \) as the normalized versions of the vectors we have defined, the attraction probabilities are calculated as

\[
p_{i,j} = 0.2 \times \frac{(\tilde{g}_i, \tilde{u}_j)r_i}{\max_{i \in [L]} r_i}
\]

where \( r_i \) is the average rating of movie \( i \).

We run both CTS and CUCB for 1000 rounds, and report their regrets averaged over 10 runs in Fig 2 where error bars represent standard deviation of the regret (multiplied by 100 for visibility). We consider two cases with \( p^* = 0.01 \) and \( p^* = 0.05 \). For both cases, CTS significantly outperforms CUCB by achieving a final regret that is no more than 9% of the final regret of CUCB.

C. Influence Maximization Bandits

We consider a directed version of the Facebook network dataset 48 that consists of 15k edges and 3120 nodes. Since, the dataset does not contain influence probabilities, we

\footnote{While the probabilistic maximum coverage problem is NP-hard, here we focus on a small-scale problem and use an exact computation oracle.}
artificially generate them by setting $p_{i,j} = 1/|\mathcal{N}_i|$ where $\mathcal{N}_i$ represents the set of outgoing neighbors of node $i$. We assume that in each round the learner selects a seed set of $K = 30$ nodes and this set forms the selected super arm. Moreover, we assume that the influence propagates—starting from the seed set—according to the independent cascade model \cite{21}, which is one of the most widely used influence propagation models. We adopt the edge-level feedback model in which the learner both observes the set of influenced nodes and the influence outcomes of the outgoing edges of these nodes.

Since the problem itself is NP-hard, an exact computation oracle is computationally infeasible for the given graph size. Nevertheless, many computationally efficient approximation algorithms exist for the influence maximization problem \cite{49}. Due to its computational efficiency and good performance in practice, we set the learner to use TIM+ as the approximation oracle. When given as input an influence graph with $n$ nodes and $m$ edges, the influence probabilities on these edges and parameters $\varepsilon$ and $\ell$, TIM+ is guaranteed to return an $\alpha = (1 - 1/e - \varepsilon)$-approximate solution with probability at least $\beta = 1 - 3n^{-\ell}$ and with time complexity $O((K + \ell)(n + m)\log n/\varepsilon^2)$. For all experiments, we set $\varepsilon = 0.1$ and $\ell = 1$. Since the learner uses an approximation oracle, instead of the regret given in \cite{1}, we consider the $(\alpha, \beta)$-approximation regret as given in \cite{20} in the remainder of this section.

We run both CTS and CUCB for 5000 rounds and report their regrets averaged over 10 runs in Fig. 3. Here, error bars represent standard deviation of the regret multiplied by 10 for visibility. Note that in these simulations, we consider the realized regret of the learner’s actions instead of the expected regret as we do in the other experiments. This is once again due to the complexity of the problem and the difficulty in calculating expected regret. Again, it is observed that CTS significantly outperforms CUCB by achieving a final regret that is no more than 16% of the final regret of CUCB. Relatively bad performance of CUCB is due to the fact that the considered time horizon is not long enough for CUCB to efficiently explore all base arms. It is observed that the UCBs of many base arms remain above 1 even at the end of 5000 rounds. As an algorithm that is based on the principle of optimism in the face of uncertainty, CUCB’s performance completely depends on the confidence sets it uses to calculate the UCB indices, and this example shows that these confidence sets are not tight enough to guarantee fast convergence.

### IX. Conclusion

We analyzed the regret of CTS for CMAB-PTA and proved order optimal gap-dependent regret bounds when the expected reward function is Lipschitz continuous without assuming monotonicity. Our bounds include the $1/p^*$ term that is unavoidable in general. Future work includes deriving regret bounds under more strict assumptions on the expected reward function such as the triggering probability modulated bounded

| $L$ | $K$ | $\Delta$ | CTS | CUCB | CascadeUCB | CascadeKL-UCB | TS-Cascade |
|-----|-----|--------|-----|-------|-----------|-------------|------------|
| 16  | 2   | 0.15   | 155.4 ± 14.1 | 1284.1 ± 52.4 | 1300.6 ± 46.8 | 360.6 ± 23.4 | 381.1 ± 16.8 |
| 16  | 4   | 0.15   | 103.2 ± 9.0  | 998.9 ± 33.2  | 993.6 ± 32.8  | 267.3 ± 20.6 | 281.0 ± 11.8 |
| 16  | 8   | 0.15   | 52.1 ± 9.8   | 549.5 ± 16.8  | 546.4 ± 11.7  | 150.3 ± 15.6 | 137.9 ± 8.8  |
| 32  | 2   | 0.15   | 321.4 ± 18.9 | 2718.8 ± 61.2 | 2676.4 ± 59.4 | 749.2 ± 34.2 | 752.9 ± 49.9 |
| 32  | 4   | 0.15   | 252.2 ± 17.0 | 2227.0 ± 55.4 | 2232.1 ± 46.6 | 617.4 ± 39.9 | 612.3 ± 15.2 |
| 32  | 8   | 0.15   | 155.4 ± 25.7 | 1531.0 ± 21.9 | 1525.4 ± 30.0 | 420.6 ± 27.5 | 385.0 ± 16.3 |
| 16  | 2   | 0.075  | 276.9 ± 50.7 | 2057.6 ± 79.6 | 2065.4 ± 87.4 | 709.0 ± 60.4 | 688.3 ± 78.5 |
| 16  | 4   | 0.075  | 205.4 ± 25.7 | 1496.5 ± 65.2 | 1512.4 ± 87.0 | 546.3 ± 53.5 | 557.9 ± 45.0 |
| 16  | 8   | 0.075  | 113.1 ± 40.4 | 719.4 ± 53.7  | 717.5 ± 44.2  | 266.1 ± 32.4 | 273.8 ± 30.7 |

**TABLE II**

**REGRETS OF CTS AND CUCB WITH THEIR STANDARD DEVIATIONS FOR VARIOUS PROBLEM INSTANCES.**

Fig. 2. Regrets of CTS and CUCB for the probabilistic maximum coverage bandit problem.

Fig. 3. Regrets of CTS and CUCB for the influence maximization bandit problem.
smoothness condition given in [23] to get rid of the $1/p^2$ term. Another important research direction is to derive a worst-case bound on the regret of CTS for CMAB-PTA.

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