Stochastic gauge fixing in $\mathcal{N} = 1$ supersymmetric Yang-Mills theory

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Abstract: The gauge fixing procedure for $N = 1$ supersymmetric Yang-Mills theory is proposed in the context of stochastic quantization method. The stochastic gauge fixing, which was first discovered by Zwanziger for Yang-Mills theory, is extended to $\text{SYM}_4$ in the superfield formalism by introducing the chiral and anti-chiral superfields as the gauge fixing functions. It is shown that the stochastic gauge fixing reproduces the probability distribution of $\text{SYM}_4$, defined by the Faddeev-Popov ansatz in the path-integral method, in the equilibrium under an appropriate choice of the stochastic gauge fixing functions. We also show the BRST invariant structure of the corresponding stochastic action in the superfield formalism which ensures the renormalizability of $\text{SYM}_4$ in this context.
1. Introduction

The analysis of $N = 1$ supersymmetric Yang-Mills theories (SYM) and their large $N$ dimensionally reduced models in terms of the Schwinger-Dyson equations or loop equations\cite{1} shows up the various dynamical aspects in these theories. In 4 dimensions, the recent matrix model description successfully reproduces the non-perturbative effects in SYM$_4$\cite{2}. In higher dimensions, large $N$ supersymmetric matrix models, which can be regarded as reduced models of SYM$_{10}$, provide constructive definitions of fundamental strings\cite{3} as well as a light-cone description of $N$ D-objects in M-theory\cite{4}.

Motivated to construct a collective field theory of Wilson loops in large $N$ reduced models of SYM, we have applied stochastic quantization method (SQM)\cite{5} to a prototype of these reduced models, i.e. a zero volume limit of Yang-Mills theory (YM)\cite{6}. The advantage of SQM approach is that it systematically incorporate a certain class of Schwinger-Dyson equations of observables into their time evolution equation with respect to the stochastic time, a generalized Langevin equation. In this respect, roughly speaking, the collective Fokker-Planck hamiltonian provides a manifestly Lorentz covariant hamiltonian constraint in the collective field theory of gauge invariant observables in the equilibrium limit\cite{6}. This is due to the stochastic time independence in the equilibrium. The integrability of the time development implies the algebraic structure of the constraints included in the collective Fokker-Planck hamiltonian. This machinery is observed in the construction of non-critical string field theories from matrix models in SQM\cite{7}.
In order to extend the collective field theory approach in Ref. [6] to supersymmetric case, we have derived a basic Langevin equation and the corresponding Fokker-Planck equation for the $N = 1$ supersymmetric Yang-Mills theory in four dimensions (SYM$_4$) in the superfield formalism. By paying a special attention to the correspondence of SYM$_4$ to lattice gauge theories, we have incorporated the non-abelian nature of SYM$_4$ in the superfield formulation of SQM which was applied to the supersymmetric U(1) gauge theory. In the superfield formalism, the underlying stochastic process, defined in terms of Itô formulation, manifestly preserves the global supersymmetry as well as the local gauge symmetry of SYM$_4$.

In SQM, as far as the expectation values of gauge invariant observables (such as Wilson loops) are concerned, it is not necessary to introduce an explicit gauge fixing procedure. For perturbative analysis with gauge variant quantities, however, we need a gauge fixing procedure to introduce a drift force for the longitudinal degrees of freedom. Such a procedure, the so-called stochastic gauge fixing procedure was first introduced to YM and its equivalence to the Faddeev-Popov ansatz in the path-integral approach was shown. It was also clarified how the extra gauge fixing term simulates the Faddeev-Popov ghost loops in the perturbative sense. The renormalization procedures in various models were investigated in two different types of regularization schemes. One is based on an action principle. The other respects the covariantly regularized Schwinger-Dyson equations which are derived from the regularized Langevin equations. For gauge theories, in particular, the renormalizability in SQM was shown in terms of the Ward-Takahashi identities which are derived from the BRST invariant structure of the stochastic action. The BRST invariance in this context was also discussed in relation to the 5 dimensional gauge invariance of YM$_4$ and in general gauge invariant structures including gravity.

For the purpose to construct a collective field theory of Wilson loops, it is not necessary to introduce a gauge fixing procedure and we postponed these issues in the previous work. In this note, we extend the arguments, given for the stochastic gauge fixing and the renormalizability of YM$_4$, to SYM$_4$ in the superfield formalism. The stochastic gauge fixing procedure is defined by introducing the chiral and the anti-chiral superfields as the gauge fixing functions that correspond to the gauge degrees of freedom. It is shown that the stochastic gauge fixing procedure in SYM$_4$ reproduces the Faddeev-Popov ghost determinant under the appropriate choice of the gauge fixing functions. We also illustrate that the non-renormalization theorem and the renormalizability of SYM$_4$, which are already well-known in the path-integral formulation, are shown by introducing the BRST invariant stochastic action in SQM approach. In order to preserve the global supersymmetry and the BRST invariance, we adopt the dimensional reduction as the regularization procedure.
2. Superfield Langevin equation for SYM₄

The superfield Langevin equation and the corresponding Fokker-Planck equation are formulated in terms of Itô stochastic calculus by noticing the similarity of SYM₄ in the superfield formalism to lattice gauge theories. In the superfield formalism, without choosing the Wess-Zumino gauge, the action for SYM₄ is non-polynomial.

\[
S = \int d^4 x d^2 \theta d^2 \bar{\theta} \frac{1}{4k g^2} \text{Tr} \left( W^\alpha W_\alpha \delta^2 (\bar{\theta}) + \bar{W}_\alpha W^\alpha \delta^2 (\theta) \right). \tag{2.1}
\]

Here

\[
W_\alpha = - \frac{1}{8} D^2 e^{-2gV} D_\alpha e^{2gV}, \quad \bar{W}_\alpha = \frac{1}{8} D^2 e^{2gV} D_\alpha e^{-2gV} . \tag{2.2}
\]

We use a closely related notation in Ref. [30]. The superfield \( V \) is \( SU(N) \) algebra valued, \( V \in su(N) \); \( V = V^a t_a, [t_a, t_b] = i f_{ab}^c t_c \) and \( \text{Tr}(t_a t_b) = k \delta_{ab} \). In this non-polynomial form, in particular, we note the following correspondence of SYM₄ to lattice gauge theories.

dynamical field : \( U_\mu \in U(N) \), link variable
\( \leftrightarrow U \equiv e^{2gV} \), vector superfield \( V \in su(N) \).

local gauge transf. : \( U_\mu \rightarrow e^{i\Lambda(x)} U_\mu e^{-i\Lambda(x+\mu)} \)
\( \leftrightarrow U \rightarrow e^{-i\alpha^a(t_a)} U e^{i\alpha^a}, \) chiral superfields \( D \bar{\Lambda} = \bar{D} \bar{\Lambda} = 0 \).

time development : \( (\delta U_\mu) U_\mu^\dagger \), Maurer–Cartan form
\( \leftrightarrow (\delta U) U^{-1}, \) \( U \) is not a group element.

differential operator : \( E_a \), (left) Lie derivative, \( E_a U_\mu = t_a U_\mu \)
\( \leftrightarrow \hat{\mathcal{E}}_a \), analogue of the (left) Lie derivative.

integration measure : \( dU_\mu \), Haar measure on \( U(N) \)
\( \leftrightarrow \sqrt{G} \mathcal{D} V \), analogue of the Haar measure.

In lattice gauge theories, the stochastic process which describes the time evolution of link variables is well-established. [28] [29] To realize the correspondence and derive a basic Langevin equation for SYM₄, we introduce a derivative \( \hat{\mathcal{E}}_a^{(L)}(z) \) \( (\hat{\mathcal{E}}_a^{(R)}(z) \) ) and a measure \( \sqrt{G} \mathcal{D} V \) which are analogues of the left (right) Lie derivative and the Haar measure defined on a group manifold, respectively. \( \hat{\mathcal{E}}_a^{(L, R)}(z) \) is defined by

\[
\hat{\mathcal{E}}_a^{(L)}(z) e^{2gV(z')} = t_a e^{2gV(z')} \delta^8(z - z'), \quad \hat{\mathcal{E}}_a^{(R)}(z) e^{2gV(z')} = e^{2gV(z')} t_a \delta^8(z - z'). \tag{2.3}
\]

\[
\hat{\mathcal{E}}_a^{(L)}(z) \equiv \frac{1}{2g} L^a_b(z) \frac{\delta}{\delta V^b(z)},
\]

\[
\hat{\mathcal{E}}_a^{(R)}(z) \equiv \frac{1}{2g} L^a_b(z) \frac{\delta}{\delta V^b(z)},
\]

\[
L^a_b(z) \equiv \frac{1}{k} \text{Tr} \left( t_a \cdot \frac{2g L_V(z) - b}{1 - e^{-2g L_V(z)}} \right). \tag{2.4}
\]
where \( z \) is the superspace coordinates \( z \equiv (x, \theta, \bar{\theta}) \), \( \delta^8(z - z') \equiv \delta^4(x - x')\delta^2(\theta - \theta')\delta^2(\bar{\theta} - \bar{\theta}') \) and \( L_V X \equiv [V, X] \). These differential operators satisfy the algebra

\[
\begin{align*}
[\hat{E}^{(L)}_a(z), \hat{E}^{(L)}_b(z')] &= -i f_{abc} \delta^8(z - z') \hat{E}^{(L)}_c(z), \\
[\hat{E}^{(R)}_a(z), \hat{E}^{(R)}_b(z')] &= i f_{abc} \delta^8(z - z') \hat{E}^{(R)}_c(z),
\end{align*}
\]  

(2.5)

which imply

\[
\begin{align*}
L_a^c \partial_c L_b^d - L_b^c \partial_c L_a^d &= -2ig f_{abc} L_c^d, \\
L^c \partial_c L^d_b - L^d \partial_b L^c_a &= 2ig f_{abc} L^d_c.
\end{align*}
\]  

(2.6)

In addition to the differential operators, we also define a measure by introducing a metric \( G^{ab}(z) \equiv L_c^a(z) L^b_c(z) \) and \( G_{ab}(z) \equiv K_a^c(z) K_b^c(z) \), where \( K_a^c(z) L^b_c(z) = L_a^c(z) K^b_c(z) = \delta^a_b \). By the metric, the integration measure is defined by \( \sqrt{G} D \nu \).

The superfield Langevin equation for SYM\(_4\) is derived by regarding \( e^{2\vartheta V} \), instead of \( V \), as a fundamental variable

\[
\frac{1}{2g}(\Delta e^{2\vartheta V}) e^{-2\vartheta V}(\tau, z) = -\Delta \tau 2g \hat{E}^{(L)}(\tau, z) S + \Delta w(\tau, z), \\
= \frac{\Delta \tau}{2g} (e^{2\vartheta L_V} D \nu + \bar{D} \nu) + \Delta w(\tau, z), \\
(\Delta w_{ij}(\tau, z) \Delta w_{kl}(\tau, z'))_{\Delta w_{\nu}} = 2k \Delta \tau \left( \delta_{ik} \delta_{jl} - \frac{1}{N} \delta_{ij} \delta_{kl} \right) \delta^8(z - z').
\]

(2.7)

Here \( \hat{E}^{(L)}(\tau, z) = t_a \hat{E}^{(L)}_a(\tau, z) \). \( \langle \ldots \rangle_{\Delta w_{\nu}} \) denotes that the expectation value is evaluated by means of the noise correlation at the stochastic time \( \tau \). The equations of motion are defined by \( D \nu \equiv \{ D^a, W^a \} \), \( \bar{D} \nu \equiv \{ \bar{D}_a, \bar{W}^a \} \), where \( D_a \equiv e^{-2\vartheta V} D_a e^{2\vartheta V} \) and \( \bar{D}_a \equiv e^{2\vartheta V} \bar{D}_a e^{-2\vartheta V} \). We note that the reality condition\([28]\) implies \( e^{2\vartheta L_V} D \nu = \bar{D} \nu \). Therefore, we discard one of them in the perturbative calculation.

The probability distribution \( P(\tau, e^{2\vartheta V}) \) is defined by the expectation value of an arbitrary observable \( F[e^{2\vartheta V}] \)

\[
\langle F[e^{2\vartheta V}](\tau) \rangle \equiv \int F[e^{2\vartheta V}] P(\tau, e^{2\vartheta V}) \sqrt{G} D \nu.
\]

(2.8)

Here the l.h.s. is evaluated by means of the Langevin equation (2.7) and all the noise correlations at \( \tau' \leq \tau - \Delta \tau \). \( P(\tau, e^{2\vartheta V}) \) is a scalar probability density. The Fokker-Planck equation is deduced from the Langevin equation (2.7)

\[
\frac{\partial}{\partial \tau} P(\tau, e^{2\vartheta V}) = \frac{4g^2}{k} \int d^8 z \text{Tr} \hat{E}^{(L)}(z) \left( \hat{E}^{(L)}(z) + \langle \hat{E}^{(L)}(z) S \rangle \right) P(\tau, e^{2\vartheta V}).
\]

(2.9)

This shows \( P(\tau, e^{2\vartheta V}) \to e^{-S} \) in the equilibrium limit.
The Langevin equation with respect to the vector superfield appears in a geometrical expression

\[ \Delta V^a(\tau, z) = -\Delta \tau G^{ab}(\tau, z) \frac{\delta S}{\delta V^b(z)} + \Delta w^a(\tau, z), \]

\[ \langle \Delta w^a(\tau, z) \Delta w^b(\tau, z') \rangle_{\Delta w^r} = 2\Delta \tau G^{ab}(\tau, z) \delta^8(z - z'). \] (2.10)

Here we have introduced the collective noise superfield: \( \Delta w^a(\tau, z) = \Delta w^b(\tau, z) L^a_b(\tau, z). \)

In Itô calculus, the noise variable is decorrelated to the equal stochastic time dynamical variables. This means that the r.h.s. of the collective noise correlation (2.10) is not the expectation value. The corresponding Fokker-Planck equation is also expressed as

\[ \frac{\partial}{\partial \tau} P(\tau, V) = \int d^8z \frac{\delta}{\delta V^a(z)} \left\{ G^{ab}(\tau, z) \left( \frac{\delta}{\delta V^b(z)} + \frac{\delta S}{\delta V^b(z)} \right) P(\tau, V) \right\}. \] (2.11)

The Langevin equation and the noise correlation are covariant under the stochastic time independent local gauge transformation \( e^{2gV} \rightarrow e^{-igA^t} e^{2gV} e^{igA}, \) for which \( \Delta A = \Delta A^t = 0, \) provided that the noise superfield is transformed as

\[ \Delta w(\tau, z) \rightarrow e^{-igL^t A(z)} \Delta w(\tau, z). \] (2.12)

We also note that the time evolution for the vector superfield preserves its reality condition which implies \( V(\tau + \Delta \tau)^t = V(\tau + \Delta \tau). \) The constraint is satisfied if the hermitian conjugation of the noise superfield is defined by

\[ \Delta w(\tau)^t = e^{-2gL V} \Delta w(\tau) \] (2.13)

The hermiticity assignment is consistent to the transformation property of \( \Delta w. \) The hermitian conjugation of the noise components is uniquely determined from this hermiticity assignment which means that the noise superfield is not a vector superfield. Their correlation are deduced from (2.7) and (2.13). From (2.13), we establish that the time development described by (2.10) preserves the reality of the vector superfield \( V(\tau + \Delta \tau)^t = V(\tau + \Delta \tau) \) which is manifest from the reality of the collective noise.

\[ \Delta w^a = \Delta w^b L^a_b, \]

\[ = 2g \frac{\text{Tr}}{k} \left( \Delta w^t \cdot \frac{2gL V(z)}{e^{2gL V(z)} - 1} \right), \]

\[ = \Delta w^a. \] (2.14)

In the weak coupling limit, our formulation agrees with that in Ref.\[ for supersymmetric U(1) vector multiplet.
3. Stochastic gauge fixing for SYM

In SQM approach, the stochastic gauge fixing was first introduced to solve the Gribov ambiguity of a class of covariant gauge in YM in the path-integral method. The significant fact is that the extra self-interaction of the Yang-Mills field introduced by the stochastic gauge fixing term in the Langevin equation simulates the Faddeev-Popov ghost contribution while it does not affect the time evolution of the local gauge invariant observables. The extension of the stochastic gauge fixing procedure to SYM is straightforward. It is defined by the following four steps.

(I): Introduce a chiral and an anti-chiral superfield, Λ and Λ†, which are stochastic time dependent, into the Langevin equation by performing the (inverse) local gauge transformation,

\[ e^{2gV} \rightarrow e^{ig\Lambda} e^{2gV} e^{-ig\Lambda}. \]

(II): Redefine the auxiliary chiral and anti-chiral superfields as

\[ \Phi = \frac{i}{g} e^{-igL\Lambda} - \frac{1}{L\Lambda} \dot{\Lambda}, \quad \bar{\Phi} = \frac{i}{g} e^{-igL\Lambda^\dagger} - \frac{1}{L\Lambda^\dagger} \dot{\Lambda}^\dagger. \] (3.1)

Here we denotes \( \dot{\Lambda} = \frac{\Delta \Lambda}{\Delta \tau} \). \( \Phi \) and \( \bar{\Phi} \) are also the chiral and the anti-chiral superfield, \( \bar{D}\Phi = D\bar{\Phi} = 0 \). Then we obtain

\[ \frac{1}{2g} (\Delta e^{2gV}) e^{-2gV} + \frac{i}{2g} \Delta \tau (\Phi - e^{2gLV} \Phi) = \frac{\Delta \tau}{2g} (e^{2gL\Lambda V} D\bar{W} + \bar{D}W) + \Delta w . \] (3.2)

(III): Define the stochastic time dependent local gauge transformation

\[ e^{2gV(\tau,z)} \rightarrow e^{-ig\Sigma^\tau(\tau,z)} e^{2gV(\tau,z)} e^{ig\Sigma^\tau(\tau,z)}, \]
\[ e^{ig\Lambda(\tau,z)} \rightarrow e^{ig\Lambda(\tau,z)} e^{ig\Sigma(\tau,z)}. \] (3.3)

Under this extended ("extended" means stochastic time dependent) local gauge transformation, the Langevin equation is covariantly transformed. In particular, the auxiliary fields are transformed as

\[ \Phi \rightarrow \frac{i}{g} e^{-igL\Sigma} - \frac{1}{L\Sigma} \dot{\Sigma} + e^{-igL\Sigma} \Phi , \]
\[ \bar{\Phi} \rightarrow \frac{i}{g} e^{-igL\Sigma^\dagger} - \frac{1}{L\Sigma^\dagger} \dot{\Sigma}^\dagger + e^{-igL\Sigma^\dagger} \bar{\Phi} . \] (3.4)

(IV): Specify the gauge fixing functions, Φ and \( \bar{\Phi} \). For example,

\[ \Phi = \frac{\xi}{8} \bar{D}^2 D^2 V, \quad \bar{\Phi} = -i \frac{\xi}{8} D^2 \bar{D}^2 V . \] (3.5)

The Langevin equation (3.2) under the gauge fixing (3.5) is a supersymmetric extension of the gauge fixed version of the YM Langevin equation. In component expansion, it includes the expression for the stochastically gauge fixed YM Langevin
equation. The stochastic gauge fixing term in (3.2) with (3.5) simulates the Faddeev-Popov ghost loops in the perturbative calculation. In fact, this gauge fixing procedure does not affect to the time development of the local gauge invariant observables. This can be seen as follows. Let us consider the time development of a functional of $e^{2gV}$, $F[e^{2gV}]$. The time evolution equation is given by

$$\Delta F[e^{2gV}] = \Delta \tau 4g^2 \int d^8z \left(-\hat{E}^{(L)}(z)S - \frac{i}{2g}(\Phi - e^{2gL\Phi} \Phi) + \hat{E}^{(L)}(z)\right) a \hat{E}^{(L)}(z) F[e^{2gV}] . \quad (3.6)$$

In this expression, the contribution of the auxiliary fields defines the infinitesimal local gauge transformation. The generator of the infinitesimal local gauge transformation is defined by

$$G(\Phi, \bar{\Phi}) \equiv i \int d^8z \left(\bar{\Phi} - e^{2gL\Phi} \Phi\right) a \hat{E}^{(L)}(z) ,$$

$$= i \int d^8z \left((\bar{\Phi}) a \hat{E}^{(L)}(z) - (\Phi) a \hat{E}^{(R)}(z)\right) . \quad (3.7)$$

This satisfies the algebra

$$\left[G(\Phi_1, \bar{\Phi}_1), G(\Phi_2, \bar{\Phi}_2)\right] = -iG([\Phi_1, \Phi_2], [\bar{\Phi}_1, \bar{\Phi}_2]) . \quad (3.8)$$

Therefore the stochastic gauge fixing term does not contributes to the time development of local gauge invariant observables while it breaks the extended local gauge invariance (3.3). This is the reason why the term includes the gauge fixing and the self-interaction of the vector superfield which simulates the Faddeev-Popov ghost contribution in the path-integral method.

Here we comment on the meaning of the extended local gauge transformation. In a weak coupling region, (3.3) reads, $\Phi \rightarrow \Phi + \dot{\Sigma} + ig[\Phi, \Sigma]$. This indicates a covariant derivative of the gauge parameter in the direction of the stochastic time. In this respect, we may interpret the extended local gauge transformation as a 5-dimensional local gauge transformation for which the chiral and the anti-chiral superfield play a role of 5-th component of the gauge field. For YM case, this 5-dimensional extension of the local gauge transformation in SQM is found in Ref.\[22]\. We note however, (3.3) is not a 5-dimensional supersymmetric extension that may be reduced from a 5-dimensional super-Poincaré invariance. The stochastic time has a distinct meaning from other 4-dimensional space-time coordinates and the global supersymmetry is the 4-dimensional one.

4. Equivalence to the Faddeev-Popov ansatz

In YM case, the equivalence of the stochastic gauge fixing procedure to the Faddeev-Popov prescription in the path-integral method was shown based on the BRST invariance of the Faddeev-Popov probability distribution.\[13]\[14]\. Here we give a proof...
for SYM$_4$ in the superfield formalism that the stochastic gauge fixing procedure proposed in the previous section reproduces the Faddeev-Popov distribution given in the path-integral method. The proof is a simple extension of the YM case. To show the equivalence, we introduce the BRST operator to define the BRST invariant distribution in the path-integral method.[25]

$$
\delta_{BRS} = \int d^8z \left\{ \bar{B}^a \frac{\delta}{\delta \phi^a} + B^a \frac{\delta}{\delta \phi^a} - \frac{1}{2} [\phi \times \phi]^a \frac{\delta}{\delta \phi^a} - \frac{1}{2} \bar{\phi} \times \bar{\phi} \right\},
$$

(4.1)

where $[A \times B]^a \equiv f^{abc} A^b B^c$. The Faddeev-Popov probability distribution is given by

$$
P_{FP} \equiv \int_{gh} e^{-(S + \delta_{BRS} F)} ,
$$

(4.2)

where $\int_{gh} \equiv \int d^8 \bar{D} \bar{D} \bar{D} c \bar{D} \bar{c} \bar{D} \bar{D} \bar{c}$ . The partition function is given by $Z = \int \mathcal{D}V P_{FP}$. The function $F$ is defined by

$$
F = \int d^8z \left\{ f_a (\bar{f}^a + \frac{\alpha}{2} \bar{B}^a) + \bar{c}_a (f^a + \frac{\alpha}{2} B^a) \right\}, \quad f^a = \bar{D}^2 V, \bar{f}^a = D^2 V .
$$

(4.3)

We show that the gauge fixed version of the Fokker-Planck equation

$$
\hat{P}[\tau, e^{2gV}] = 4g^2 \int d^8z \hat{\xi}_a^{(L)}(z) \left( \hat{\xi}_a^{(L)}(z) + (\hat{\xi}_a^{(L)}(z) S) + \frac{i}{2g} (\bar{F} - e^{2gLV} \Phi) \right) P[\tau, e^{2gV}] ,
$$

(4.4)

has a solution $P_{eq} = P_{FP}$ in the equilibrium; $\lim_{\tau \to -\infty} P(\tau) = P_{eq}, \hat{P}_{eq} = 0$, provided that the gauge fixing functions are defined by

$$
\Phi^a(z) \equiv g^2 \int_{gh} dz' c^a(z) \left\{ \hat{\xi}_b^{(R)}(z') (\hat{\xi}_b^{(R)}(z') F) e^{-(S + \delta_{BRS} F)} \right\} P_{FP}^{-1} ,
$$

$$
\bar{\Phi}^a(z) \equiv g^2 \int_{gh} dz' c^a(z) \left\{ \hat{\xi}_b^{(L)}(z') (\hat{\xi}_b^{(L)}(z') F) e^{-(S + \delta_{BRS} F)} \right\} P_{FP}^{-1} .
$$

(4.5)

Substituting the distribution, $P_{FP}$, into the Fokker-Planck equation (4.4), the r.h.s. of (4.4) reads

$$
\int d^8z \hat{\xi}_a^{(L)}(z) \left\{ - \left( \hat{\xi}_a^{(L)}(z) \int_{gh} \left( \hat{\delta}_{BRS} F \right) e^{-(S + \delta_{BRS} F)} \right) + \frac{i}{2g} (\bar{F}(z) - e^{2gLV} \Phi(z))^a P_{FP} \right\} .
$$

(4.6)

The 1st term in the r.h.s. of (4.6) reads

$$
- \frac{1}{2} \int_{gh} d^8z \hat{\xi}_a^{(L)}(z) \left\{ g f^{abc} B^b \hat{\xi}_c^{(L)}(z) F + \hat{\delta}_{BRS} \hat{\xi}_a^{(L)}(z) F \right\} e^{-(S + \delta_{BRS} F)}
$$

$$
- \frac{1}{2} \int_{gh} d^8z \hat{\xi}_a^{(R)}(z) \left\{ g f^{abc} B^b \hat{\xi}_c^{(R)}(z) F + \hat{\delta}_{BRS} \hat{\xi}_a^{(R)}(z) F \right\} e^{-(S + \delta_{BRS} F)} .
$$

(4.7)
While the 2nd term in the r.h.s. of (4.6) is evaluated as

$$\frac{ig}{2} \int_{gh} d^8 z d^8 z' c^a(z) \tilde{E}_a(z) \left\{ \tilde{E}_b(z') (\tilde{E}_b(z') F) e^{-(S+\delta_{BRS} F)} - \frac{ig}{2} \right\},$$

$$= \frac{1}{2} \int_{gh} d^8 z \tilde{\delta}_{BRS} \left\{ \tilde{E}_a(z) (\tilde{E}_a(z) F) e^{-(S+\delta_{BRS} F)} + \tilde{E}_a(z) (\tilde{E}_a(z) F) e^{-(S+\delta_{BRS} F)} \right\} + \frac{1}{2} \int_{gh} d^8 z \left\{ \tilde{\delta}_{BRS} \tilde{E}_a(z) (\tilde{E}_a(z) F) e^{-(S+\delta_{BRS} F)} + \tilde{\delta}_{BRS} \tilde{E}_a(z) (\tilde{E}_a(z) F) e^{-(S+\delta_{BRS} F)} \right\},$$

$$= - \text{ the 1st term on the r.h.s. of (4.4).} \quad (4.8)$$

Hence we conclude that the distribution $P_{FP}$ is a stationary solution of the Fokker-Planck equation. In the proof, we have used the commutation relations

$$[\tilde{\delta}_{BRS}, \tilde{E}_a(z)] = -gf^{abc} \tilde{E}_c(z), \quad [\tilde{\delta}_{BRS}, \tilde{E}_a(z)] = -gf^{abc} \tilde{E}_c(z) \quad (4.9)$$

5. Stochastic action and the BRST invariance

In the last of this note, we illustrate the renormalizability of SYM$_4$ in SQM approach. For the renormalization procedure in SQM, it is convenient to introduce an action principle, the so-called stochastic action, by the path-integral representation of SQM approach.\[16\] \[17\] \[18\] \[19\] To show the renormalizable structure of SYM in this context is also an extension of YM case where a BRST invariant structure is introduced. In particular, in Ref.\[21\], the renormalizability of the BRST invariant stochastic action is shown in terms of the Ward-Takahashi identities. One of the W-T identities is inherited from the hidden supersymmetry of the Parisi-Souslas type\[32\] which appears in the path-integral representation of the Langevin equation.\[16\] The other comes from the BRST invariance of the local gauge symmetry. Since the extension of this argument to SYM$_4$ is straightforward, we do not repeat the standard procedure. Instead, we only show the prescription and the results. The BRST invariant stochastic action is introduced in the following steps.\[3\]

1. Start from the integral representation of the gaussian noise correlation of the noise superfield in (2.7). Insert the unity of the $\delta$-functional of the Langevin equation (3.2). The integral representation of the $\delta$-functional requires an auxiliary vector superfield $\tilde{n}^a(z)$. Then, by integrating out the noise superfield, we obtain a stochastic
action as a functional of the vector superfield \( V^a \), the auxiliary vector superfield \( \tilde{\pi}^a \), the chiral and the anti-chiral superfields, \( \Phi \) and \( \bar{\Phi} \), respectively. After the redefinition of the auxiliary field \( \tilde{\pi}^a \equiv L^a \pi^a \), \( \pi^a \) becomes a vector superfield, \( \pi^{a\dagger} = \pi^a \).

(II): As we have shown in (3.3), the stochastic action is also invariant under the extended local gauge transformation provided that the auxiliary vector superfield is transformed as \( \pi^a \rightarrow e^{-igL^a} \pi^a \). Therefore, we can introduce a BRST invariance by a standard prescription for the extended local gauge invariance of the stochastic action. The field contents are given by \( \{ (V^a, \pi_a), (\Phi^a, \bar{B}^a, e^a, \bar{c}_a) \} \).

\( \Phi^a, \bar{B}^a, e^a, \bar{c}_a \) are chiral and \( \bar{\Phi}^a, \bar{B}^a, \bar{c}_a \) are anti-chiral superfield, respectively. We also require the gauge conditions, \( \Phi^a - \lambda^a(V) = \bar{\Phi}^a - \bar{\lambda}^a(V) = 0 \).

(III): It is possible to consistently truncate the BRST transformation by integrate out the auxiliary chiral and anti-chiral superfields, \( \Phi \) and \( \bar{\Phi} \), and the Nakanishi-Lautrup superfields, \( B \) and \( \bar{B} \). There remains a nilpotent truncated BRST transformation in the extended phase space \( \{ (V^a, \pi_a), (e^a, \bar{c}_a), (\bar{c}^a, c^a) \} \).

\[
\delta_{\text{BRST}} V^a = -\lambda \frac{i}{2} L^a c^b + \chi \frac{ic^b}{2} L^a, \\
\delta_{\text{BRST}} \pi_a = \lambda \frac{i}{2} \partial_a L^c \pi_c c^b - \chi \frac{i}{2} \partial_c L^c \pi_c, \\
\delta_{\text{BRST}} e^a = -\lambda \frac{g}{2} [c \times c]^a, \\
\delta_{\text{BRST}} \bar{c}_a = -\lambda \frac{g}{2} [\bar{c} \times \bar{c}]^a, \\
\delta_{\text{BRST}} \bar{c}^a = -\lambda \frac{\pi_b L^b}{2} - \chi [\bar{c} \times c]^a, \\
\delta_{\text{BRST}} c^a = \lambda \frac{\pi_b L^b}{2} - \chi [c \times \bar{c}]_a. \\
\]  

(5.1)

Here \( \lambda \) is a Grassmann parameter. The BRST invariant stochastic action is given by

\[
Z = \int D\pi Dc D\bar{c} D\bar{c} Dc' D\bar{c}' e^{-K_{\text{BRST}}}, \\
K \equiv \int d^8\sigma d\tau \text{Tr} \left[ \frac{1}{2\kappa} G^{ab} \pi_a \pi_b - i \pi_a \left( \bar{V}^a + \frac{1}{4gk} \left( L^a b \left( D^a W^b \right) + \left( D_a W^a \right) b \right) \right) \right], \\
K_{\text{BRST}} \equiv K - \int d^8\sigma d\tau \left[ \bar{c}^a \delta_{\text{BRST}} \bar{c}^a + \frac{1}{k} \delta_{\text{BRST}} (c \bar{c}^a) + c^a \bar{c}^a + \frac{1}{k} \delta_{\text{BRST}} (c' \bar{c}^a) \right].
\]  

(5.2)

Here \( \delta_{\text{BRST}} \equiv \lambda \delta_{\text{BRST}} \). For the multiplicative renormalization, we have introduced a scaling parameter of the stochastic time, \( \kappa \). For the gauge fixing functions, we have chosen

\[
\chi = \frac{i\xi}{8} D^2 D^2 V, \quad \bar{\chi} = \frac{i\bar{\xi}}{8} D^2 \bar{D}^2 V.
\]  

(5.3)

For example, in the Feynman gauge \( \xi = 1 \), we obtain the simplest super propagators

\[
\langle V^a(\tau, z)V^b(\tau', z') \rangle = \int \frac{d^4k d\omega}{(2\pi)^3} e^{i k \cdot (x-x') + i \omega \cdot (\tau-\tau')} \frac{1}{(i\omega + k^2)(-i\omega + k^2)} \delta^2(\theta - \theta') \delta^2(\bar{\theta} - \bar{\theta}').
\]
\[ \langle V^a(\tau, z) \pi^b(\tau', z') \rangle = \int \frac{d^4 k d \omega}{(2\pi)^3} e^{i k (x-x') + i \omega (\tau-\tau')} \frac{i}{(i \omega + k^2)^2} \delta^2(\theta - \theta') \delta^2(\bar{\theta} - \bar{\theta}') , \]
\[ \langle \pi^a(\tau, z) \bar{\pi}^b(\tau', z') \rangle = 0 . \]  
(5.4)

In the path-integral approach, it is convenient to perform the perturbative analysis with supergraphs which ensure the non-renormalization theorem in SYM. The perturbative calculation with the supergraphs is also applicable by using the super propagators defined in (5.4) in the context of SQM. For the renormalizability, the argument we illustrate below is essentially the same as that given in Ref. \[25\] for SYM in the path-integral method. In order to respect the BRST invariance, we adopt the dimensional reduction\[27\][26] as the regularization procedure in perturbative analysis. Because of the retarded nature of the ghost superfield propagators, the ghost loops do not contribute to the renormalization of the vector multiplet sector of the 1-P-I Green’s functions. While the existence of the BRST invariance is necessary for the manifest renormalizability of SYM in this context. We note that the BRST invariant stochastic action takes a form of the Legendre transformation. In (5.2), the sum of the terms which include the derivative with respect to the stochastic time is invariant under the BRST transformation, namely, \[ \delta_{BRST} \left( \int \bar{V} d\tau [ i \pi_a \dot{V}^a + \bar{c}_a \dot{\bar{c}}^a + c_a' \bar{c}^a] \right) = 0 . \] This is the reason why we have introduced the scaling parameter \( \kappa \). The multiplicative renomalizability requires this additional coupling constant \( \kappa \) which is shown by the Ward-Takahashi identity of the BRST invariance.

The BRST symmetry and the simple power counting argument indicates the multiplicative renormalizability. Let us estimate the divergence of a Feynman diagram \( G(V, I, E) \) with \( V \) vertices, \( I \) internal lines, \( E \) external lines and \( L \equiv I - V + 1 \) loops. More precisely, we define,
\[ I \equiv I_{V\pi} + I_{VV} , \quad E \equiv E_{\pi} + E_{V} . \]
\[ \begin{align*}
I_{\pi} : & \# \text{ external lines of } \pi , \quad E_{\pi} : \# \text{ external lines of } V \\
I_{V\pi} : & \# \text{ internal lines of } \langle V \pi \rangle , \quad I_{VV} : \# \text{ internal lines of } \langle V V \rangle \\
V_{1,n} : & \# \text{ vertices with } \pi \text{ and } n \text{ } V , \quad V_{2,n} : \# \text{ vertices with two } \pi \text{ and } n \text{ } V .
\end{align*} \] (5.5)

We have also topological relations
\[ \sum_{n=2} n(V_{1,n} + V_{2,n}) = 2I_{VV} + I_{V\pi} + E_{V} , \quad \sum_{n=2} (V_{1,n} + 2V_{2,n}) = I_{V\pi} + E_{\pi} . \] (5.6)

The Feynman diagram \( G(V, I, E) \) may be evaluated as
\[ G(V, I, E) \approx \left( \int d^4 x_i d \tau_i \right)^V \left( \int d^2 \theta_j d^2 \bar{\theta}_j \right)^V \left( \langle V_i V_j \rangle \right)^{I_{VV}} \left( \langle V_i \pi_j \rangle \right)^{I_{V\pi}} \left( \langle V \pi \rangle \right)^{E_{\pi}} \left( \langle \pi \rangle \right)^{E_{V}} \]
\[ \approx \left( \int d^2 \theta d^2 \bar{\theta} \right)^{V-1} (\delta^2(\theta_{ij}) \delta^2(\bar{\theta}_{ij})) \left( \int d^4 k d \omega \right) \left( \frac{1}{k^2} \right)^L \left( \frac{1}{\pm i \omega + k^2} \right)^{I_{VV} + I_{V\pi}} \]
\[ \times \left( \int d^2 \theta d^2 \bar{\theta} \right)^V \left( \delta^2(\theta_{ij}) \delta^2(\bar{\theta}_{ij}) \right)^I \left( D_\alpha D_\beta D_\gamma D_\delta \right) \sum_{n=2} V_{1,n} . \] (5.7)
Thus we estimate the degree of the ultra-violet divergence of the Feynman diagram,

\[ 4L - 2I_{VV} - 2I + 2\left( \sum_{n=2} V_{1,n} \right) = 4 - 2E_\pi. \]  

(5.8)

Three types of counter terms, which are constrained from the Ward-Takahashi identities of the BRST symmetry (5.1), are necessary to cancel the divergences.

- \( E_\pi = 2 \), logarithmic divergences → two \( \pi \) and infinite number of \( V \);
- \( \pi^2 V^n, n = 0, 1, 2, ... \)
- \( E_\pi = 1 \), logarithmic divergences → one \( \pi \) and infinite number of \( V \);
- \( \pi\dot{V}, \pi D^\alpha \bar{D}^2 D_\alpha V, ... \)
- \( E_\pi = 0 \), no Feynman diagrams.

For \( E_\pi = 0 \) case, since the superfield propagator \( \langle V\pi \rangle \) is a retarded one, the loop of the internal line \( \langle V\pi \rangle \) does not appear. This means that the Feynman diagram includes at least one external line of \( \pi \). This is also shown by a Ward-Takahashi identity derived as a consequence of the Parisi-Soula\( s \) type hidden supersymmetry in the dimensional reduction regularization. We note that, for \( E_\pi = 1 \), the divergence appears to be quadratic, however the BRST invariance reduces the divergence to logarithmic one. As we have already mentioned, since the ghost superfield propagators are also retarded type, the ghost loops do not contribute to the Feynman diagrams with no ghost external lines. The power counting of the Feynman diagram with ghost external lines is similar to what we have explained for vector superfields. It indicates that the new counter term of ghost superfields which does not appear in (5.2) is not necessary. Therefore we conclude that the stochastic action (5.2) is multiplicatively renormalizable by the wave function renormalizations and the renormalizations of the couplings, \( g \) and \( \kappa \).

6. Discussion

In this paper, we have introduced the stochastic gauge fixing procedure for \( N=1 \) SYM\(_4\) in SQM approach in which the superfield formalism preserves the local gauge symmetry as well as the global supersymmetry in the underlying stochastic process in the sense of Itô calculus. We have also shown that the contribution of the Faddeev-Popov ghost in the path-integral approach is simulated by the stochastic gauge fixing term and the Faddeev-Popov determinant is reproduced in the equilibrium limit. It is also shown that the non-renormalization theorem and the perturbative renormalizability of SYM\(_4\) in the superfield formalism hold in terms of the BRST invariant stochastic action principle in the dimensional reduction regularization. The extension to introduce matter multiplets, such as scalar chiral superfield, is straightforward.

There remain some questions which are not addressed. In this paper, we have discussed the renormalizability based on the BRST invariance. Since the interaction is non-polynomial, it would be more convenient to introduce the background
field method for perturbative calculations in SQM approach\cite{13}. In the background field method of SYM$_4$ in the path-integral approach,\cite{26} the chiral superfields (for example, the Nakanishi-Lautrup fields and Faddeev-Popov ghosts in SYM$_4$ without chiral matter superfields) are defined by the chiral condition with respect to the background covariant derivative. This causes a non-trivial contribution of the Nakanishi-Lautrup chiral superfields. In order to cancel it, we need an additional ghost, Nielsen-Kallosh type ghost.\cite{33} It is unclear whether, in the present formulation, the contribution of the Nielsen-Kallosh ghost can be simulated by defining the stochastic gauge fixing function in terms of the background covariant derivative instead of $D_{\alpha}$ and $\bar{D}_{\dot{\alpha}}$. We note that this is a particular situation inherent in the background field method. If we respect the BRST invariance of the stochastic action and do not use the background field method, then the perturbative analysis is not suffered from the problem. However the background field method is useful, for example, in evaluating anomalies in SYM\cite{34} and we must clarify how the Nielsen-Kallosh type ghost contribution is reproduced in SQM approach. We will report on this near future. For the evaluation of anomalies in the background field method, it may be also necessary to specify a regularization procedure which respects the background local gauge invariance. One is the dimensional reduction.\cite{26} Another possibility is, for example, to introduce a superfield regulator on the noise superfield. Such procedures were discussed for non-supersymmetric cases\cite{20} and it might be possible to extend the analysis to supersymmetric case.

As we have mentioned in the introduction, Our main interest is to apply SQM to large $N$ reduced models of supersymmetric gauge theories and construct collective field theories of local gauge invariant observables, such as Wilson loops. It may be interpreted as a field theory of fundamental strings or the hamiltonian constraint on the string field theory in some reduced models. For this purpose, the explicit gauge fixing procedure is not necessary. However, in the manifestly Lorentz covariant approach of string field theories, the BRST charge defined in the first quantization of strings plays an essential role and the string field is spanned with the target space coordinates and the ghosts. Therefore we’d like to clarify whether the collective field theory constructed in SQM also simulates such contribution of the auxiliary and ghost string states. We hope our analysis shed light on these future problems.

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