Semi-ordinary Iwasawa theory for Rankin–Selberg products

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## Contents

Chapter 1. Introduction  
1.1. The setting 4  
1.2. Conjectures and Results 5  
1.3. A summary of our results 7  
1.4. Layout 12  
1.5. Notation 13  

Chapter 2. Construction of the optimized Coleman maps 15  
2.1. Dieudonné modules and \( \varphi \)-eigenvectors 15  
2.2. Factorization of one-variable Perrin-Riou maps 16  
2.3. Two-variable Perrin-Riou maps and Coleman maps 23  
2.4. The case \( a_p(g) = 0 \) 27  

Chapter 3. Beilinson–Flach elements and \( p \)-adic \( L \)-functions 31  
3.1. Beilinson–Flach elements 31  
3.2. Euler systems of rank 2 and uniform integrality 34  
3.3. \( p \)-adic \( L \)-functions 36  

Chapter 4. Selmer groups and main conjectures 41  
4.1. Definitions of Selmer groups 41  
4.2. Classical and signed Iwasawa main conjectures 43  

Chapter 5. Applications towards main conjectures 45  
5.1. Cyclotomic main conjectures for \( f \otimes g \) 45  
5.2. Cyclotomic main conjectures for \( f \otimes g \) 46  
5.3. Main conjectures over an imaginary quadratic field where \( p \) is inert 48  

Appendix A. A divisibility criterion in regular rings 63  

Appendix B. \( p \)-adic Rankin–Selberg \( L \)-functions and universal deformations 65  
B.1. The set up 66  
B.2. The minimally ramified universal deformation representation 67  
B.3. Hida families 68  
B.4. \( p \)-adic Rankin–Selberg \( L \)-function 69  

Appendix C. Images of Galois representations attached to Rankin–Selberg convolutions 73  
C.1. Group theory 73  
C.2. Applications to families on \( GL_2 \times Res_{K/Q}GL_1 \) 74  

Bibliography 77
Abstract

Our primary goal in this article is to study the Iwasawa theory for semi-ordinary families of automorphic forms on $\text{GL}_2 \times \text{Res}_{K/Q} \text{GL}_1$, where $K$ is an imaginary quadratic field where the prime $p$ is inert. We prove divisibility results towards Iwasawa main conjectures in this context, utilizing the optimized signed factorization procedure for Perrin-Riou functionals and Beilinson–Flach elements for a family of Rankin–Selberg products of $p$-ordinary forms with a fixed $p$-non-ordinary modular form. The optimality enables an effective control on the $\mu$-invariants of Selmer groups and $p$-adic $L$-functions as the modular forms vary in families, which is crucial for our patching argument to establish one divisibility in an Iwasawa main conjecture in three variables.
CHAPTER 1

Introduction

We fix forever a prime \( p \geq 5 \). Let \( f \in S_{k_f+2}(\Gamma_1(N_f), \varepsilon_f) \) be a cuspidal eigenform which is not of CM type, where the level \( N_f \) is coprime to \( p \). We assume that \( f \) admits a \( p \)-ordinary stabilization \( f_\alpha \) with \( U_p \)-eigenvalue \( \alpha_f \). We fix an imaginary quadratic field \( K \) with discriminant \( D_K \) which is coprime to \( N_f \) and where \( p \) remains inert. We let \( c \) denote any lift of a generator of \( \text{Gal}(K/\mathbb{Q}) \) to \( G_\mathbb{Q} \). We also fix a ray class character \( \chi \) of \( K \) of order coprime to \( p \) (which we call the branch character, following Hida) and of conductor coprime to \( D_KN_f \) (but not necessarily to \( p \)). We assume that \( \chi \neq \chi^c \); i.e. we do not treat the “Eisenstein” case.

Our eventual goal is to prove divisibility results on the main conjectures for \( p \)-semi-ordinary families of automorphic motives on \( \GL_2 \times \Res_{K/\mathbb{Q}} \GL_1 \). In more explicit (but still very rough) terms, we will prove divisibilities in the Iwasawa Main Conjectures for families of Rankin–Selberg convolutions \( f_\alpha \times \theta(\psi) \), where \( \psi \) varies among algebraic Hecke characters of \( K \) with \( p \)-power conductor and \( \theta(\psi) \) is the theta-series of \( \psi \). Our results in this article can be considered as a piece of evidence towards the variational versions of Bloch–Kato conjectures for the relevant class of motives.

There are many earlier results in this direction and we record several of these which our present work extends.

1) Bertolini–Darmon in [BD05] studied the “definite” anticyclotomic Iwasawa main conjectures for the base change \( f_{/K} \) of a modular form \( f \in S_2(\Gamma_0(N_f)) \) to \( K \), where the prime \( p \) may split or remain inert in \( K/\mathbb{Q} \). The family in question consists of the self-dual twists of the Rankin–Selberg motives associated to \( f \times \theta(\psi) \) as \( \psi \) varies among Hecke characters of \( K \) with \( p \)-power conductor.

2) The work of Chida and Hsieh [CH15] generalized the work of Bertolini–Darmon eluded to above and studied the “definite” anticyclotomic Iwasawa main conjectures for the base change to \( K \) of a modular form \( f \in S_{k_f+2}(\Gamma_0(N_f)) \) of arbitrary even weight. In this work, the prime \( p \) may split or remain inert in \( K/\mathbb{Q} \).

3) Howard [How04] obtained results towards Perrin-Riou’s Heegner point (“indefinite” anticyclotomic) Main Conjecture for \( f \in S_2(\Gamma_0(N_f)) \), refining earlier results of Bertolini [Ber95]. Later in [How07], he initiated the study of “indefinite” anticyclotomic Main Conjectures in ordinary families. The nearly-ordinary family studied in op. cit. interpolates the self-dual twists of the Rankin–Selberg motives associated to \( f_\alpha \times \theta(\psi) \) as \( \psi \) varies among Hecke characters of \( K \) with \( p \)-power conductor and \( f_\alpha \) in a Hida

\[ ^1 \text{In fact, we treat somewhat a more general class of semi-ordinary Rankin–Selberg convolutions } f_\alpha \otimes g_\mu \text{ on } \GL_2 \times \GL_2, \text{ where } f_\alpha \text{ is above and } g_\mu \text{ is a non-ordinary cuspidal eigenform.} \]
family. Works of Fouquet [Fou13] and the first named author [Büy14a] take the big Heegner points introduced in [How07] as an input to obtain results towards “indefinite” anticyclotomic Main Conjectures in the multivariate setting of [How07]. In all these works, there is a priori no restriction on the local behaviour of \( p \) in the extension \( K/Q \). However, one should note that these articles have consequences towards main conjectures without \( p \)-adic \( L \)-functions and one needs to assume further that \( p \) is split in \( K/Q \) for statements that do involve \( p \)-adic \( L \)-functions.

4) Skinner–Urban [SU14] proved the Iwasawa Main Conjectures for the base change \( f/K \) of a modular form \( f \) of any weight, along the \( \mathbb{Z}_p^2 \)-tower \( K_{\infty} \) of \( K \), assuming that the prime \( p \) splits in \( K/Q \). Their work also allows a treatment with variation of modular forms in families. The families the op. cit. concerns with interpolate the Rankin–Selberg motives associated to \( f_{\alpha} \times \theta(\psi) \) as \( \psi \) varies among Hecke characters of \( K \) with \( p \)-power conductor and \( f_{\alpha} \) in a Hida family. We note that the global root number for \( f_{j/K} \) is assumed to be \(+1\) in [SU14]; this property is constant in Hida families.

5) Castella–Wan [CW20] proved similar results assuming that the global root number for \( f_{j/K} \) is \( -1 \), complementing [SU14] (again assuming that the prime \( p \) splits in \( K/Q \)).

All the results 1), 5) above concern the case when \( \chi = \chi^c \) is abelian over \( Q \). In the “non-Eisenstein” cases, there also has been some progress, albeit somewhat weaker than the case when \( \chi = \chi^c \). We record these below, noting that all of them require that the prime \( p \) is split in \( K/Q \).

6) Under the hypothesis that \( \chi \) is \( p \)-distinguished, Castella [Cas17] (when \( k_f = 0 \) and the present authors [BL18] (general \( k_f \)) proved results for the family that consists of Rankin–Selberg motives associated to \( f \times \theta(\chi \psi) \) where \( \psi \) varies among Hecke characters of \( K \) with \( p \)-power conductor. In these works, the authors descend to the anticyclotomic tower and treat definite and indefinite cases simultaneously. In [BL20a], the present authors relaxed the \( p \)-distinguished condition.

7) In some cases, the work of Wan [Wan20] complements the results of [Cas17, BL18] to obtain a proof of Iwasawa Main Conjectures (up to \( \mu \)-invariants).

8) Castella and Hsieh in [CH18] have also obtained results towards Bloch–Kato Conjectures for certain Rankin–Selberg motives \( f \times \theta(\chi \psi) \) at the central critical point, relying on Generalized Heegner cycles of Bertolini–Darmon–Prasanna (without variations in families).

In this article, we prove results towards a variety of Iwasawa main conjectures (with \( p \)-adic \( L \)-functions) in the setting when the prime \( p \) remains inert in \( K/Q \), where there has been very limited progress as compared to when \( p \) splits in \( K/Q \): To the best of our knowledge, the works of Bertolini–Darmon and Chida–Hsieh that we have recalled in 1) and 2) are the only prior results in this direction (and they concern the “definite” anticyclotomic Iwasawa theory in the scenario when \( \chi = 1 \)).

Before presenting our results in more precise form and reviewing our strategy, we recall the role the hypotheses that \( p \) splits in \( K/Q \) has played in the works we have recorded under the items 6) and 7) indicating how the approach therein breaks down in its entirety in the absence of this hypothesis.
When the prime $p$ splits in $K/\mathbb{Q}$ and the Hecke character $\chi$ is $p$-distinguished, one may attach to the Rankin–Selberg product $f_K \otimes \chi$ a full-fledged Euler system over all ray class extensions of $K$. This construction dwells on a “patching” argument that was first introduced and employed in [LLZ15] when $f$ has weight 2 and later extended in [BL18] to treat the case when $f$ is of higher weight. The hypotheses that $p$ splits in $K/\mathbb{Q}$ and the Hecke character $\chi$ is $p$-distinguished are required to identify the irreducible component of the eigencurve that contains the $p$-ordinary stabilization of $\theta(\chi)$. Moreover, this irreducible component necessarily has CM by $K$ thanks to the $p$-distinguished hypothesis, by a result of Bellaïche and Dimitrov, which interpolates the $p$-ordinary $p$-stabilizations of the theta-series of Hecke characters of $K$ with $p$-power conductor.

When $p$ is inert in $K$, the eigencurve cannot have components that have CM by $K$; c.f. [CIT16, Corollary 3.6]. Morally, this is due to the fact that slope of any non-ordinary CM form of weight $k$ is at least $(k - 1)/2$, so the refinements of the corresponding theta-series cannot be contained in a finite slope family. This is the fundamental difficulty that one has to overcome in the setting of the present article. We now explain (in very rough terms) how we circumvent this difficulty:

**Step 1** Extending [BL20b, BLLV19], we introduce Perrin-Riou functionals, the signed Coleman maps (in single cyclotomic variable) and signed Beilinson–Flach elements associated to Rankin–Selberg convolutions of the form $f_\alpha \otimes g$, where $f_\alpha$ has slope zero, whereas $g \in S_{k_g + 2}(\Gamma_1(N_g), \varepsilon_g)$ is non-ordinary at $p$ with $p \nmid N_g$. We show that these constructions in fact interpolate as $f_\alpha$ varies in a Hida family $f$. Our results in this direction are summarized in §1.3.1. These constructions allow us to prove one divisibility in the Iwasawa–Greenberg main conjectures for $f_\alpha \otimes g$ as well as $f \otimes g$ over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ (see Theorems 5.1.2 and 5.2.1 in the main body of the article).

**Step 2** We apply our constructions in **Step 1** with the particular choice when $g$ is the theta-series of a Hecke character $\psi$ of our fixed imaginary quadratic field $K$. Employing the locally restricted Euler system machinery, we first prove results towards cyclotomic Iwasawa main conjectures. Moreover, our constructions of the Perrin-Riou functionals and signed Beilinson–Flach elements are integrally optimal and as such, they permit us to vary $f_\alpha$ in the Hida family $f$ and to prove a result (Theorem 1.3.5 below) towards two variable main conjectures (where one of the variables parameterize $f$ and the other accounts for the cyclotomic variation) using a general patching criterion we establish in Appendix A.

**Step 3** We then vary $\psi$ to prove Theorem 1.3.6, which is a result towards main conjectures in three variables (where one of the variables parameterize $f$ and the other two the $\mathbb{Z}_p^2$-extension of $K$). We once again rely on the patching criterion we have noted in **Step 2** and the integral optimality of the signed objects from **Step 1** also plays a crucial role in this portion. We remark that our results towards Iwasawa main conjectures for $f_\alpha \otimes g$ in **Step 1** are restricted to the case where $k_f > k_g$. Consequently, for a fixed $f_\alpha$, our results in **Step 2** allow only the treatment of crystalline Hecke characters of fixed tame conductor and whose infinity types belong to a fixed finite set. Since there are only finitely many such characters, this is not enough to obtain results on Iwasawa main conjectures over the...
\( \mathbb{Z}_p^2 \)-extension of \( K \). This problem is overcome by varying \( f_\alpha \) in a Hida family, which in turn permits us to vary \( \psi \) in an infinite family. This is the main reason for our emphasis on the signed-splitting procedure for Beilinson–Flach elements for Hida families.

**Step 4** We use Nekovář’s (very general) descent formalism for his Selmer complexes to descend our results on the \( \mathbb{Z}_p^2 \)-extension of \( K \) to the anticyclotomic \( \mathbb{Z}_p \)-tower. This portion of our results is recorded as Theorem 1.3.7 (in the definite case) and Theorem 1.3.8 (in the indefinite case).

1.1. The setting

Throughout this article, we fix an odd prime \( p \) and embeddings \( \iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \) and \( \iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p \). Let \( f \) and \( g \) be two normalized, cuspidal eigen-forms, of weights \( k_f + 2, k_g + 2 \), levels \( N_f, N_g \), and nebentypes \( \varepsilon_f, \varepsilon_g \) respectively. We assume \( p \nmid N_f N_g \). In [BL18, BL20b], we have studied the Iwasawa theory for the Rankin–Selberg product of \( f \) and a \( p \)-ordinary \( g \) (with respect to the embeddings we fixed) as \( g \) varies in a CM Hida family.

In this article, we assume that \( f \) is ordinary at \( p \) and vary \( f \) in a Hida family \( \mathfrak{f} \), but \( g \) shall be non-ordinary at \( p \). We give finer results than those given in [BL18, BL20b]. We note that in [BL18], both \( f \) and \( g \) are assumed to be \( p \)-ordinary. So, our results in the present article do not overlap with those in op. cit. The results in [BL20b] resemble those here, but the difference is that in this paper, the non-ordinary eigenform \( g \) is dominated by the members of the Hida family \( \mathfrak{f} \).

Let \( L/\mathbb{Q}_p \) be a finite extension containing the coefficients of \( f \) and \( g \), the \( N_f N_g \)-th roots of unity as well as the roots of the Hecke polynomials of \( f \) and \( g \) at \( p \). We shall write \( \alpha_f, \beta_f, \alpha_g \) and \( \beta_g \) for these roots; we assume throughout that \( \alpha_f \) is a \( p \)-adic unit and \( \alpha_g \neq \beta_g \). For \( h \in \{ f, g \} \) and \( \lambda \in \{ \alpha, \beta \} \), we write \( h_\lambda \) for the \( p \)-stabilization of \( h \) satisfying \( U_p h_\lambda = \lambda h_\lambda \). Let \( \mathfrak{f} \) denote the Hida family passing through \( f_\alpha \). The corresponding branch of the Hecke algebra is denoted by \( \Lambda_\mathfrak{f} \).

\(^2\)When we say that members of the family \( \mathfrak{f} \) dominates \( g \), we mean that the \( p \)-adic \( L \)-function (resp., the Selmer group) which we study here interpolates the critical values of the Rankin–Selberg \( L \)-functions (resp., the Bloch–Kato Selmer groups) associated to \( f(\kappa) \otimes g \) where the classical specialization \( f(\kappa) \) has higher weight than \( g \).
1.2. Conjectures and Results

1.2.1. Main conjectures: The set up and statements. Let us fix once and for all an imaginary quadratic field $K$ where the prime $p$ is inert. Let $D_K$ denote its discriminant. We also fix a ray class character $\chi$ of $K$ with conductor dividing $fp^{\infty}$ (where we assume that $f$ is coprime to $pD_K$) and order coprime to $p$. We will also work with an algebraic Hecke character $\psi$ with infinity type $(0,k_\psi+1)$ and conductor $f$, where $k_\psi$ is a positive integer. The character $\chi$ will be fixed throughout (and will take on the role of a branch character, in the sense of Hida) whereas $\psi$ will be allowed to vary. For any ray class character $\eta$ of $K$, we shall set $\eta^c := \eta \circ c$, where $c$ is the generator of $\text{Gal}(K/Q)$.

Let $\Gamma_{\text{cyc}} = \text{Gal}(\mathbb{Q}(\mu_{p^{\infty}})/\mathbb{Q}) \cong \Delta \times \Gamma_1$, where $\Delta \cong \mathbb{Z}/(p-1)\mathbb{Z}$ and $\Gamma_1 \cong \mathbb{Z}_{p}$. Let $K_{\infty}, K_{\text{cyc}}$ and $K_{\text{ac}}$ be the $\mathbb{Z}_p$-extension, the cyclotomic $\mathbb{Z}_p$-extension and the anticyclotomic $\mathbb{Z}_p$-extension of $K$ respectively. We write $\Gamma_K = \text{Gal}(K_{\infty}/K), \Gamma_{\text{ac}} = \text{Gal}(K_{\text{ac}}/K)$. We also identify $\Gamma_1$ and $\Gamma_{\text{cyc}}$ with $\text{Gal}(K_{\text{cyc}}/K)$ and $\text{Gal}(K(\mu_{p^{\infty}})/K)$ respectively.

For any profinite abelian group group $G$, we let $\Lambda(G) := \mathbb{Z}_p[[G]]$ denote the completed group ring of $G$ with coefficients in $\mathbb{Z}_p$. We put $\Lambda_{\mathcal{O}}(G) := \Lambda(G) \otimes_{\mathbb{Z}_p} \mathcal{O}$.

We will mainly concern ourselves with the cases when $G = \Gamma_K, \Gamma_{\text{cyc}}, \Gamma_1$ or $\Gamma_{\text{ac}}$. Let us denote by

$$\Psi : G_K \rightarrow \Gamma_K \xrightarrow{\gamma \mapsto \gamma^{-1}} \Gamma_K \hookrightarrow \Lambda_{\mathcal{O}}(\Gamma_K)^\times$$

$$\Psi\gamma : G_K \rightarrow \Gamma_\gamma \xrightarrow{\gamma \mapsto \gamma^{-1}} \Gamma_\gamma \hookrightarrow \Lambda_{\mathcal{O}}(\Gamma_1)^\times$$

the tautological characters, where $? = 1, \text{ac}$. We set $\chi := \chi\Psi$ and similarly define the $\Lambda_{\mathcal{O}}(\Gamma_1)$-valued character $\chi_{\text{cyc}} := \chi\Psi_1$ and the $\Lambda_{\mathcal{O}}(\Gamma_{\text{ac}})$-valued character $\chi_{\text{ac}} := \chi\Psi_{\text{ac}}$. We shall denote by $\Lambda_{\mathcal{O}}(\Gamma_K)^G$ the free $\Lambda_{\mathcal{O}}(\Gamma_K)$-module of rank one on which $G_K$ acts via $\Psi_K$ (and similarly, we define $\Lambda_{\mathcal{O}}(\Gamma_1)^G$ for $? = 1, \text{ac}$). More generally, we put $M^G := M \otimes_{\Lambda_{\mathcal{O}}(\Gamma_1)} \Lambda_{\mathcal{O}}(\Gamma_1)^G$ for any $\Lambda_{\mathcal{O}}(\Gamma_1)$-module $M$ on which $G_K$ acts.

Throughout, we also fix a branch $f \in \mathfrak{A}_{f}[\mathfrak{q}]$ of a primitive non-CM Hida family with tame conductor $N_f$ and residually non-Eisenstein (in these sense that $\mathfrak{f}_f$ is absolutely irreducible and $p$-distinguished (in the sense that $\mathfrak{f}_f$ is not scalar), where $\Lambda_f$ is an irreducible component of Hida’s universal ordinary Hecke algebra (see Definition 1.3.1 for further details). We write $\kappa_\mathfrak{f}$ for the prime ideal of $\Lambda_f$ which specializes to $f$. We assume that $N_f$ is coprime to $D_K N_f$. We denote by $R^p_f$ Hida’s $G_{\mathbb{Q}}$-representation attached to $f$, which is free of rank 2 over $\Lambda_f$. We will assume without loss of generality that $\Lambda_f$ contains $\mathcal{O}$ and put $\Lambda_f(\Gamma_\gamma) := \Lambda(\Gamma_\gamma) \otimes_{\mathbb{Z}_p} \Lambda_f$. For each $h \in \{f, g\}$, let $V_h$ denote Deligne’s $L$-linear continuous cohomological $G_{\mathbb{Q}}$-representation attached to $h$, so that its restriction to $G_{\mathbb{Q}_p}$ has Hodge–Tate weights 0 and $-1 - k_h$ (with the convention that the Hodge–Tate weight of the cyclotomic character is 1). Let $\mathcal{O}$ denote the ring of integers of $L$. We fix a Galois-stable $\mathcal{O}$-lattice $R_h$ inside $V_h$ so that its linear dual $R^*_h = \text{Hom}(R_h, \mathcal{O})$ coincides with the Galois representation realized in the cohomology of the the modular curve $Y_1(N_h)$ with coefficients in the integral sheaf $T_{\text{Sym}}^{k_h} \mathcal{H}_{Z_p}(1)$ (see [KLZ17] §2.3 and Definition 2.1.1 below). We study the Iwasawa theory of $T_{f,g} := R^*_f \otimes R^*_g = \text{Hom}(R_f \otimes R_g, \mathcal{O})$ over the cyclotomic tower $\mathbb{Q}(\mu_{p^{\infty}})$, as well as when $f$ and $g$ vary in a Hida family and CM forms over a fixed imaginary quadratic field where $p$ is inert respectively.
For any \( \chi \) and \( \psi \) as above, we define the \( G_K \)-representations

\[
\mathfrak{T}_{f,\psi}^{\text{cy}} := R_f^* \otimes \psi \otimes \Lambda_{\text{Gr}}(\Gamma_1)^i \quad \text{and} \quad \mathfrak{T}_{f,\chi}^{\text{cy}} := R_f^* \otimes \chi \otimes \Lambda_{\text{Gr}}(\Gamma_1)^i
\]

where \( ? = K, \text{ac} \). We let \( \widetilde{H}_f^2(G_K, \mathfrak{T}_{f,\psi}^{\text{cy}}, \Delta_{\text{Gr}}) \) denote the extended Greenberg Selmer group attached to the representation \( \mathfrak{T}_{f,\psi}^{\text{cy}} \) (c.f. Definition 5.3.2) and similarly define \( \widetilde{H}_f^2(G_K, \mathfrak{T}_{f,\chi}^{\text{cy}}, \Delta_{\text{Gr}}) \). Finally, we let \( L_{p}^{\text{RS}}(f/K \otimes \psi) \) denote the Rankin–Selberg \( p \)-adic \( L \)-function given as in Definition 5.3.3.

We are finally ready to state the first of a series of three conjectures, which concerns the cyclotomic Iwasawa theory of the family \( f \otimes \psi \).

**Conjecture 1.2.1.** Suppose that \( \overline{\rho}_f \) is absolutely irreducible. Then,

\[
L_p^{\text{RS}}(f/K \otimes \chi) \cdot \Lambda_\text{Gr}(\Gamma_1) \otimes_{\Z_p} \Q_p = \text{char}_{\Lambda_\text{Gr}(\Gamma_1)} \left( \widetilde{H}_f^2(G_K, \mathfrak{T}_{f,\psi}^{\text{cy}}, \Delta_{\text{Gr}})^i \right) \otimes_{\Z_p} \Q_p.
\]

In Appendix B, we introduce the “semi-universal” Rankin–Selberg \( p \)-adic \( L \)-function

\[
L_p^{\text{RS}}(f/K \otimes \chi) \in \Lambda_\text{Gr}(\Gamma_K)
\]

in three variables, extending the work of Loeffler [Loe20] ever so slightly. With that, we can state the second conjecture, which concerns the 3-parameter family \( f \otimes \chi \).

**Conjecture 1.2.2** (Main conjectures over \( \Lambda_\text{Gr}(\Gamma_K) \)). Suppose \( \chi \) is a ray class character as above and assume that \( \overline{\rho}_f \) is absolutely irreducible. Then

\[
L_p^{\text{RS}}(f/K \otimes \chi) \cdot \Lambda_\text{Gr}(\Gamma_K) \otimes_{\Z_p} \Q_p = \text{char}_{\Lambda_\text{Gr}(\Gamma_K)} \left( \widetilde{H}_f^2(G_K, \mathfrak{T}_{f,\chi}^{\text{cy}}, \Delta_{\text{Gr}})^i \right) \otimes_{\Z_p} \Q_p.
\]

To state our third conjecture, which concerns the anticyclotomic Iwasawa theory of (families of) Rankin–Selberg products, we consider the restriction

\[
L_p^{\text{RS}}(f/K \otimes \chi) \in \Lambda_\text{Gr}(\Gamma_{\text{ac}})
\]

of the twist \( L_p^{\text{RS}}(f/K \otimes \chi) \in \Lambda_\text{Gr}(\Gamma_K) \) of Loeffler’s \( p \)-adic \( L \)-function to the anticyclotomic tower (c.f. Definition 5.3.13(ii)). We also denote by \( \mathfrak{T}_{f,\chi}^{\text{ac}} \) the central critical twist of \( \mathfrak{T}_{f,\chi}^{\text{cy}} \), which we introduce in Definition 5.3.13(i).

We assume in this portion that \( \chi \) is a ring class character (so that \( \chi^c = \chi^{-1} \)) and that the nebentype character \( \varepsilon_f \) of the Hida family \( f \) is trivial. As it is customary in the study of anticyclotomic Iwasawa theory, we shall write \( N_f = N^- \), where \( N^- \) (resp. \( N^+ \)) is a product of primes which are split (resp. inert) in \( K/\Q \).

We denote by \( \partial_{\text{cy}} L_p^{\text{ac}}(f/K \otimes \chi) \in \Lambda_\text{Gr}(\Gamma_{\text{ac}}) \) the restriction (to the anticyclotomic tower) of the derivative of \( L_p^{\text{RS}}(f/K \otimes \chi) \) in the cyclotomic direction (c.f. Definition 5.3.10). We set

\[
r(f, \chi) := \text{rank}_{\Lambda_\text{Gr}(\Gamma_{\text{ac}})} \widetilde{H}_f^2(G_K, \mathfrak{T}_{f,\chi}^{\text{ac}}, \Delta_{\text{Gr}})
\]

and call it the generic algebraic rank. Finally, we let \( \text{Reg}_{f,\chi} \subset \Lambda_\text{Gr}(\Gamma_{\text{ac}}) \) denote the \( \Lambda_\text{Gr}(\Gamma_{\text{ac}}) \)-adic regulator given as in Definition 5.3.14.

**Conjecture 1.2.3** (Anticyclotomic Main conjectures). Suppose \( \chi \) is a ring class character and \( \overline{\rho}_f \) is absolutely irreducible. Then,

\[
L_p^{\text{RS}}(f/K \otimes \chi) \cdot \Lambda_\text{Gr}(\Gamma_{\text{ac}}) \otimes_{\Z_p} \Q_p = \text{char}_{\Lambda_\text{Gr}(\Gamma_{\text{ac}})} \left( \widetilde{H}_f^2(G_K, \mathfrak{T}_{f,\chi}^{\text{ac}}, \Delta_{\text{Gr}})^i \right) \otimes_{\Z_p} \Q_p.
\]
1.3. A SUMMARY OF OUR RESULTS

i) (Definite case) Suppose $N$ is a square-free product of odd number of primes. Then $L_p^\dagger(f/K \otimes \chi_{ac}) \neq 0$; in particular, the $\Lambda_f(\Gamma_{ac})$-module $\bar{H}_I^\dagger(G_K, \Sigma, T_{f,ac}^\dagger; \Delta_{Gr})$ is torsion.

ii) (Indefinite case) Suppose $N$ is a square-free product of even number of primes. Then $L_p^\dagger(f/K \otimes \chi_{ac}) = 0$ and $r(f, \chi_{ac}) = 1 = \text{ord}_{\gamma+1} \cdot L_p^\dagger(f/K \otimes \chi)$.

Moreover,

$$\partial_{\text{cyc}} L_p^\dagger(f/K \otimes \chi) \cdot \Lambda_f(\Gamma_{ac}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \text{Reg}_{f,ac} \cdot \text{char}_{\Lambda_f(\Gamma_{ac})} \left( \bar{H}_I^\dagger(G_K, \Sigma, T_{f,ac}^\dagger; \Delta_{Gr})_{\text{tor}} \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$ 

Despite the wealth of results in the setting when $p$ is split in $K/\mathbb{Q}$, the evidence for these conjectures when $p$ is inert have been extremely scant and are limited to the “definite” anticyclotomic Iwasawa theory of Rankin–Selberg products of the form $f/K \otimes \chi$, where $\chi$ has finite order (c.f., [BD05, CH15]). As far as we are aware, there are no previous results neither in the indefinite case when $p$ remains inert, nor results which allow variation in $f$, nor in the case when $\psi$ has infinite order.

Our treatment in this article, which builds on fundamentally different techniques from those utilized in op. cit., will hopefully shed some light to these cases of main conjectures, which previously seemed inaccessible. We illustrate our results towards these conjectures in Theorems 1.3.5 (cyclotomic case), 1.3.6 (3-variable case), 1.3.7 (definite anticyclotomic case) and 1.3.8 (indefinite anticyclotomic case) below.

1.3. A SUMMARY OF OUR RESULTS

We will discuss our results in two parts: In §1.3.1, we summarize our results concerning the (optimized) signed splitting of Beilinson–Flach elements and Perrin-Riou maps. In the second part (§1.3.2), we illustrate applications of these results towards Conjectures 1.2.1, 1.2.2 and 1.2.3.

1.3.1. Optimized Perrin-Riou maps and Beilinson–Flach elements for semi-ordinary families. The first result we record in the introduction concerns the construction of certain Perrin-Riou functionals, which are eigen-projections of two-variate Perrin-Riou maps. For each choice of $\lambda, \mu \in \{\alpha, \beta\}$, we choose a $\varphi$-eigenvector of $D_{cris}(T_{f,g}) \otimes_{\mathbb{Q}} L$ associated to the eigenvalue $(\lambda_f \mu_g)^{-1}$. Given a finite unramified extension $F/\mathbb{Q}_p$, we write

$$L_{F,f,g}^{(\lambda,\mu)} : H_{Iw}^\dagger(F(\mu_{p\infty}), T_{f,g}) \to H_{\text{ord}_p}(\lambda_f \mu_g)(\Gamma_{\text{cyc}}) \otimes F$$

for the Perrin-Riou map on $H_{Iw}^\dagger(F(\mu_{p\infty}), T_{f,g})$ projecting to the chosen $\varphi$-eigenvector.

We take additional care with the choices of our $\varphi$-eigenvectors to normalize these functionals, which in turn allow us to keep track of the crucial integrality properties of the functionals (with respect to the natural lattices inside the Galois representations we work with). We also note that we are not directly working with the affinoid-valued Perrin-Riou maps of [LZ16], but rather introduce $\Lambda_f$-adic maps (as part of our proof of Theorem 2.3.10 below).

**THEOREM 1.3.1** (Theorem 2.3.10) Optimized Perrin-Riou functionals for semi-ordinary families. For $\mu \in \{\alpha, \beta\}$ and a finite unramified extension $F/\mathbb{Q}_p$, there exists
a $\Lambda_f(\Gamma_{\text{cyc}})$-morphism

$$L_{F,f,g,\mu} : H^1_{\text{Iw}}(F(\mu_{p^{\infty}}), T_{f,g}) \to \Lambda_f \otimes \mathcal{H}_{\text{ord},p}(\mu_{\text{cyc}}) \otimes F$$

whose specialization at $\kappa_0$ equals, up to a $p$-adic unit,

$$\frac{1}{\lambda_{N_f}(f) \left( 1 - \frac{\beta_f}{p\alpha_f} \right) \left( 1 - \frac{\beta_f}{\alpha_f} \right)} L^{(\alpha,\mu)}_{F,f,g}$$

where $L^{(\alpha,\mu)}_{F,f,g}$ is the $p$-stabilization of $L^{(\alpha,\mu)}_{F,f,g}$ (see Definition 2.3.4 below) and $\lambda_{N_f}(f)$ is the pseudo-eigenvalue of the Atkin–Lehner operator of level $N_f$ (which is given by the identity $W_{N_f} f = \lambda_{N_f}(f) f^*$).

We may then factorize these multivariate Perrin-Riou functionals into doubly-signed $\Lambda_f(\Gamma_{\text{cyc}})$-adic Coleman maps.

**Theorem 1.3.2 (Theorem 2.4.7, optimized factorization of Perrin-Riou maps for semi-ordinary families).** Suppose that $g$ satisfies either the Fontaine-Laffaille condition $p > k_g + 1$ or $a_p(g) = 0$. There exist a logarithmic matrix $Q^{-1}_g M_g \in M_{2 \times 2}(H(\Gamma_1))$ and a pair of $\Lambda_f(\Gamma_{\text{cyc}})$-morphisms

$$\text{Col}_{F,f,g,#}, \text{Col}_{F,f,g,\flat} : H^1_{\text{Iw}}(F(\mu_{p^{\infty}}), T_{f,g}) \to \Lambda_f(\Gamma_{\text{cyc}}) \otimes \mathcal{O}_F$$

that verify the factorization

$$\begin{pmatrix} L_{F,f,g,#} \\ L_{F,f,g,\flat} \end{pmatrix} = Q^{-1}_g M_g \begin{pmatrix} \text{Col}_{F,f,g,#} \\ \text{Col}_{F,f,g,\flat} \end{pmatrix}.$$
the modules where they take values in (c.f. Definition 3.1.3 below). This slight alteration is necessary in order to apply our signed factorization procedure.

**Theorem 1.3.3** (Theorem 3.1.3, optimized factorization of Beilinson–Flach elements for semi-ordinary families). Suppose that either \( p > k_g + 1 \) or \( a_p(g) = 0 \). Let \( N \) be the set of positive square-free integers that are coprime to \( 6pN_fN_g \). For \( m \in N \), there exists a pair of cohomology classes

\[
\text{BF}_{f,g,\ast,m}, \text{BF}_{f,g,\ast,m} \in \varpi^{-s(g)}H^1_1(Q(\mu_m), R_f^g \otimes R_g^\ast \Lambda_\mathcal{O}(\Gamma_{\text{cyc}})^{1})
\]

such that

\[
\begin{pmatrix}
\text{BF}_{f,g,\ast,m} \\
\text{BF}_{f,g,\ast,m}
\end{pmatrix} = Q_g^{-1}M_g
\begin{pmatrix}
\text{BF}_{f,g,\ast,m} \\
\text{BF}_{f,g,\ast,m}
\end{pmatrix}.
\]

Here, \( s(g) \) is a natural number that depends on \( k_g \) but is independent of \( m \).

We can bound the exponent \( s(g) \) in the denominator of the Beilinson–Flach elements uniformly for the family \( f \) and integers \( m \in N \). This is crucial for our purposes. We conjecture that the exponents \( s(g) \) in the statement of Theorem 1.3.3 are bounded independently of \( g \). In Corollary 3.2.4 below, we verify this conjecture granted the (conjectural) existence of a rank-2 Euler system (c.f. Conjecture 3.2.1), whose existence is predicted by the Perrin-Riou philosophy; see [PR88, PR98].

We next study the \( p \)-local properties of the signed Beilinson–Flach elements and prove that they form a locally restricted Euler system. We also analyze the images of the Beilinson–Flach elements under the signed Coleman maps (which give rise to what we shall call doubly-signed Rankin–Selberg \( p \)-adic \( L \)-functions), to compare them to the Loeffler–Zerbes geometric \( p \)-adic \( L \)-functions. Thanks to the careful choices of the normalizations concerning the eigen-projections of the multivariate Perrin-Riou maps, this comparison we prove is sufficiently precise and allows us to keep track of delicate integrality questions.

**Theorem 1.3.4** (Proposition 3.3.5). Suppose that either \( p > k_g + 1 \) or \( a_p(g) = 0 \). We have

\[
\text{Col}_{\varpi} \circ \text{loc}_p(\text{BF}_{f,g,\ast,1}) = 0.
\]

Furthermore,

\[
\text{Col}_{\varpi} \circ \text{loc}_p(\text{BF}_{f,g,\ast,1}) = -\text{Col}_{\varpi} \circ \text{loc}_p(\text{BF}_{f,g,\ast,1}) = D_g\delta_{k_g+1}L_p^{\text{geo}}(f,g),
\]

where \( L_p^{\text{geo}}(f,g) \) is the Loeffler–Zerbes geometric \( p \)-adic \( L \)-function introduced in Definition 3.3.4, \( \text{loc}_p \) is the localization map at \( p \), \( D_g \) is a unit in \( \Lambda_\mathcal{O}(\Gamma_1) \) and \( \delta_{k_g+1} \) is some explicit elements in \( \Lambda_\mathcal{O}(\Gamma_1) \) (see 1.3.3).

See also Remark 3.3.6 for a variant that concerns individual Rankin–Selberg convolutions, but also covers more ground in that case.

1.3.2. Main conjectures: Results. We will now summarize the applications of our constructions and results in §1.3.1 towards the Conjectures 1.2.1, 1.2.2 and 1.2.3 above.

**Theorem 1.3.5** (Theorem 5.3.6, cyclotomic main conjectures for \( f/K \otimes \psi \)). Suppose that \( p \geq 7 \) and \( \text{SL}_2(\mathbb{F}_p) \subset \mathcal{T}_f(\mathcal{O}_{\mathbb{Q}(\mu_{p^\infty})}) \) as well as that \( k_g \neq p - 1 \) and \( p + 1 | k_g + 1 \). Then,

\[
\varpi^{s(\psi)}H^\text{cyc}R^\Psi_p(f/K \otimes \psi) \in \text{char}_{\Lambda_\mathcal{O}(\Gamma_1)}\left(H^2_1(G_{\mathbb{K},\Sigma}, \mathcal{T}_{f,\psi}/\Delta_{\mathcal{O},\Gamma})^{1}\right)
\]
where \( s(\psi) = s(\theta(\psi)) \) is given as in Theorem 1.3.5.

Theorem 1.3.5 is in fact the special case of a more general result we prove (Theorem 5.2.1 below), where we pick \( g = \theta(\psi) \) the theta-series of the Hecke character \( \psi \) of our fixed imaginary quadratic field \( K \). We also remark that Theorem 5.3.6 relies heavily on Appendix C where we study the images of Galois representations associated automorphic forms on \( GL_2/\mathbb{Q} \times \text{Res}_{K/\mathbb{Q}} GL_1 \).

We deduce Theorem 5.2.1 from Theorem 5.1.2, which concerns individual (as opposed to families of) Rankin–Selberg products \( f \otimes g \), via a patching criterion we establish in Appendix A. This criterion rests on \([\text{Che}43, \text{Lemma 7}]\) (which characterizes the \( m \)-adic topology of complete local regular rings); we thank T. Ochiai for bringing this lemma of Chevalley to our attention. Also crucial for our patching argument is to have a uniform control over the \( \mu \)-invariants. This is achieved through our optimized constructions of the signed Beilinson–Flach elements and the optimized factorizations of Perrin-Riou functionals into signed Coleman maps (which we have summarized in 1.3.1).

In order to prove Theorem 5.1.2, we utilize a locally restricted Euler system argument. The input is the signed Beilinson–Flach classes we produce via Theorem 1.3.3. Here, the adjective “locally restricted” refers to the \( p \)-local properties of the signed Beilinson–Flach elements, c.f. Theorem 1.3.4. This argument a priori yields bounds on the signed Selmer groups (c.f. Definition 1.1.1) in terms of signed \( p \)-adic \( L \)-functions (c.f. Definition 3.3.7). It turns out that these signed Selmer groups can be identified with the classical Greenberg Selmer groups (see Corollary 1.1.3 below). Likewise, we compare the signed \( p \)-adic \( L \)-functions with the Rankin–Selberg \( p \)-adic \( L \)-functions in Remark 3.3.8(ii). This explicit comparison is also crucial for our patching argument.

The following is our main result towards the Iwasawa main conjectures over \( \Lambda_f(\Gamma_K) \) (Conjecture 1.2.2) for an imaginary quadratic field \( K \) where \( p \) remains inert.

**Theorem 1.3.6 (Theorem 5.3.11).** Suppose \( \chi \) is a ray class character as above and assume that \( p \geq 7 \) as well as that \( SL_2(\mathbb{F}_p) \subset \pi_f(G_{\mathbb{Q}(\mu_p^\infty)}) \). Assume also that the uniform boundedness condition \([\text{Bdd}(s(\psi))]\) on the variation of \( s(\psi) \) holds true. We then have the following containment in the Iwasawa main conjecture for the family \( f_K \otimes g \) of Rankin–Selberg products:

\[
L_p^{RS}(f_K \otimes \chi) \in \text{char}_{\Lambda(\Gamma_K)} \left( \widetilde{H}^2_f(G_K, \Sigma, \mathbb{F}_{\chi}^K; \Delta_{Gr})^t \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
\]

As we have remarked in the paragraph following the statement of Theorem 3.1.4, the condition on the growth of \( s(\psi) \) can be verified granted the existence of a rank-2 Euler system predicted by Conjecture 3.2.1.

We deduce Theorem 1.3.6 from Theorem 1.3.5 by varying \( \psi \) and using the patching criterion (in Appendix A) eluded to above. We note that it is crucial that we work with the Hida family \( f \) (as opposed to an individual eigenform \( f \)); see Remark 5.3.8 where we discuss this technical point. Let us assume in Theorems 1.3.7 and 1.3.8 below that the nebentype \( \varepsilon_f \) is trivial and we put \( N_f = N^+N^- \) as before, where \( N^+ \) (resp. \( N^- \)) is a product of primes which are split (resp. inert) in \( K/\mathbb{Q} \). Utilizing the descent formalism of \([\text{BL20a}, \text{§5.3.1}]\) (which relies crucially on the work of Nekovář) applied together with Theorem 1.3.6, we shall obtain divisibilities in both “definite” and “indefinite”
anticyclotomic Iwasawa main conjectures (see Conjecture 1.2.3). We underline the importance to work with Nekovář’s Selmer complexes (rather than classical Selmer groups). Among other things, it allows us to by-pass any issues that may stem from the potential existence of pseudo-null submodules of Selmer groups.

**Theorem 1.3.7** (Theorem 5.3.15) anticyclotomic main conjectures in the “definite” case. Suppose \( \chi \) is a ring class character such that \( \chi |_{\mathbb{Q}_p} \neq \chi |_{\mathbb{Q}_p} \) and \( N^- \) is a square-free product of odd number of primes. Assume that the following conditions hold true:

- \( p \geq 7 \) and \( \text{SL}_2(\mathbb{F}_p) \subset \mathcal{P}(\mathbb{Q}_{\mu_{p^\infty}}) \).
- \( \text{Bdd}_{4(\mathfrak{p})} \) is valid.

Then the \( \Lambda_{\mathcal{L}}(\Gamma_{ac}) \)-module \( \tilde{H}^2(K,\Sigma; \mathcal{T}_{\mathfrak{f},ac}^\dagger; \Delta_{Gr})^\dagger \) is torsion and the following containment in the anticyclotomic Iwasawa main conjecture for the family \( \mathfrak{f}/K \otimes \chi_{ac} \) holds:

\[
L_p^\dagger(\mathfrak{f}/K \otimes \chi_{ac}) \in \text{char}_{\Lambda_{\mathcal{L}}(\Gamma_{ac})} \left( \tilde{H}^2(K,\Sigma; \mathcal{T}_{\mathfrak{f},ac}^\dagger; \Delta_{Gr})^\dagger \right) \otimes \mathbb{Z}_p \mathbb{Q}_p .
\]

**Theorem 1.3.8** (Theorem 5.3.20) anticyclotomic main conjectures in the “indefinite” case. Suppose \( \chi \) is a ring class character such that \( \chi |_{\mathbb{Q}_p} \neq \chi |_{\mathbb{Q}_p} \) and \( N^- \) is a square-free product of even number of primes. Assume that the following conditions hold true.

- \( p \geq 7 \) and \( \text{SL}_2(\mathbb{F}_p) \subset \mathcal{P}(\mathbb{Q}_{\mu_{p^\infty}}) \).
- \( \text{Bdd}_{4(\mathfrak{p})} \) is valid.

Then:

1. \( \text{ord}_{(\gamma_{\mathfrak{f}}-1)} L_p^\dagger(\mathfrak{f}/K \otimes \chi) \geq 1 \).
2. The following containment (partial \( \Lambda_{\mathcal{L}}(\Gamma_{ac}) \)-adic BSD formula for the family \( \mathfrak{f}/K \otimes \chi_{ac} \)) is valid:

\[
\partial_{\mathcal{L}}(\mathfrak{f}/K) L_p^\dagger(\mathfrak{f}/K \otimes \chi) \in \text{Reg}_{\mathfrak{f},ac} \cdot \text{char}_{\Lambda_{\mathcal{L}}(\Gamma_{ac})} \left( \tilde{H}^2(K,\Sigma; \mathcal{T}_{\mathfrak{f},ac}^\dagger; \Delta_{Gr})^\dagger \right) \otimes \mathbb{Z}_p \mathbb{Q}_p .
\]

**Remark 1.3.9.** We show in Corollary 5.2.3 that the uniform boundedness condition \( \text{Bdd}_{4(\mathfrak{p})} \) on the variation of the exponents \( s(\psi) \) holds true granted the existence of a rank-2 Euler system that the Perrin-Riou philosophy predicts.

In the situation of Theorem 1.3.8 note that Conjecture 1.2.3(ii) predicts in addition that

\[
\text{ord}_{(\gamma_{\mathfrak{f}}-1)} L_p^\dagger(\mathfrak{f}/K \otimes \chi) \geq 1 \Rightarrow r(\mathfrak{f},\chi_{ac}) = 0 .
\]

In the setting where the prime \( p \) splits in \( K/\mathbb{Q} \), the second expected equality follows from [BL18, Theorem 3.15]. Still when the prime \( p \) splits in \( K/\mathbb{Q} \), one could also utilize [BL18, Theorem 3.30] to show that the first expected equality holds if and only if \( \text{Reg}_{\mathfrak{f},ac} \neq 0 \).

Based on the recent work of Andreatta and Iovita [AI20], it seems very likely that the inequality \( r(\mathfrak{f},\chi_{ac}) \geq 1 \) will be within reach very soon, see Remark 5.3.23 for a detailed discussion concerning this point. Note that granted this lower bound on the generic algebraic rank, the statement of Theorem 1.3.8(ii) can be recast in the following form (c.f. Corollary 5.3.22), which is further in line with the \( \Lambda_{\mathcal{L}}(\Gamma_{ac}) \)-adic Birch and Swinnerton-Dyer formula predicted by Conjecture 1.2.3(ii):

\[
\partial_{\text{cyc}} L_p^\dagger(\mathfrak{f}/K \otimes \chi) \in \text{Reg}_{\mathfrak{f},ac} \cdot \text{char}_{\Lambda_{\mathcal{L}}(\Gamma_{ac})} \left( \tilde{H}^2(K,\Sigma; \mathcal{T}_{\mathfrak{f},ac}^\dagger; \Delta_{Gr})^\dagger \right) \otimes \mathbb{Z}_p \mathbb{Q}_p .
\]
Still in the situation of Theorem 1.3.8 we note that a natural extension of Hsieh’s generic non-vanishing result [Hsi14, Theorem C] combined with our results in the present article yields the upper bound $r(f, \gamma) \leq 1$ on the generic algebraic rank. We explain this in details in Remark 5.3.24 below.

1.4. Layout

We begin our discussion in §2 with an analysis of the integrality properties of the relevant Dieudonné modules (which, among other things, require to keep track of Hida’s cohomological congruence numbers). This involves the determination of their bases which are integral with respect to the natural lattices that arise when we realize the Galois representations in question inside the étale cohomology of modular curves. Such an analysis is inevitable since the integrality properties of the Beilinson–Flach elements are relative to the same lattices, and since our methods require a good control on the $\mu$-invariants as our Galois representations vary in $p$-adic families.

With the choice of an integral basis, we explain in §2.2 how to factor cyclotomic one-variable Perrin-Riou functionals (which are the eigen-projections of the Perrin-Riou map, normalized according to a good choice of an integral eigen-basis) to signed Coleman maps. We use this input in §2.3 to do the same for the two-variable Perrin-Riou functionals (With the second variable coming from a Hida family), which interpolate those constructed in §2.2 in a precise manner. In §2.4 we consider the special case when $a_p(g) = 0$, which allows us to relax the Fontaine–Laffaille condition on $g$. This alteration is crucial for scenarios where we would like to allow variation in $g$. For our eventual goals towards Conjectures 1.2.2 and 1.2.3 we shall take $g$ to be the theta-series of a Hecke character of the imaginary quadratic field $K$ where $p$ remains inert (and vary this Hecke character); note that $a_p(g) = 0$ for such $g$.

In §3.1 we establish the analogous factorization results for the distribution-valued (non-integral) Beilinson–Flach elements into signed Beilinson–Flach elements with better integral properties. In §3.3 we study the $p$-local properties of the Beilinson–Flach elements (signed or otherwise) and recall their relation with Rankin–Selberg $p$-adic $L$-functions.

We introduce the doubly-signed Selmer groups in §4.1 which naturally arise when we employ the locally restricted Euler system machinery with signed Beilinson–Flach elements, and we compare them to their classical counterparts (i.e., the Greenberg Selmer groups). Note that we eventually (in Section 5.3.1) introduce and work with yet another group of Selmer groups (i.e., Nekovář’s extended Selmer groups), which have better base-change properties. In §4.2 we formulate signed main conjectures (which are accessible via the locally restricted Euler system signed Beilinson–Flach classes) and compare them to their classical variants (which are amenable to $p$-adic variation). An important aspect in our comparison is the integrally-optimal determination of the images of the Perrin-Riou functionals.

In §5.1 (resp., in §5.2), we discuss the consequences of our constructions in §§2–3 towards the (cyclotomic) main conjectures for Rankin–Selberg convolutions $f \otimes g$ (resp., for the family of Rankin–Selberg convolutions $f \otimes g$). In §5.3 (more particularly, in §5.3.1 and §5.3.2), we recast our results in §5.1 and §5.2 in the special case when $g = \theta(\psi)$ is the theta-series of an algebraic Hecke character $\psi$ of the imaginary quadratic field $K$ (where $p$ remains inert), and in terms of
Nekovář’s Selmer complexes. Relying on the main results of §5.3.2 (which concern a fixed choice of \( \psi \)) and the patching criterion we establish in Appendix A, we prove in §5.3.3 our main results towards the \( \Lambda_f(\Gamma_K) \)-adic main conjectures in three variables. In §§5.3.4–5.3.6, we descent to the anticyclotomic tower dwelling on the general descent formalism Nekovář has established (see also [BL20a, §5.3.1] where his results have been simplified to cover a limited scope of Galois representations that we concern ourselves with). This concludes the main body of our article.

In the first of the three appendices (Appendix A), we prove a divisibility criterion in regular rings, which plays a crucial role to patch our results for individual Rankin–Selberg convolutions \( f \otimes g \) to obtain results towards main conjectures in three-variables. The criterion we prove is based on a lemma of Chevalley; we thank Tadashi Ochiai for bringing this observation of Chevalley to our attention.

In Appendix B, we revisit Loeffler’s recent work [Loe20, §4] where he constructs \( p \)-adic Rankin–Selberg \( L \)-functions. We extend it very slightly to cover the case of minimally ramified universal deformations. This input is important even in the formulation of Conjectures 1.2.2 and 1.2.3. In Appendix C, we study the images of the Galois representations we are interested in. We record a number of sufficient conditions to ensure the validity of the image-related hypotheses in our divisibility results towards main conjectures.

1.5. Notation

We conclude our introduction with some further notation we shall rely on in the main body of our article.

**Definition 1.5.1 (The weight space and Hida’s universal Hecke algebra).**

1) Let us put \([\cdot] : \mathbb{Z}_p^\times \hookrightarrow \Lambda(\mathbb{Z}_p^\times)^\times\) for the natural injection. The universal weight character \( \kappa \) is defined as the composite map

\[
\kappa : G_Q \rightarrow \mathbb{Z}_p^\times \hookrightarrow \Lambda(\mathbb{Z}_p^\times)^\times,
\]

where \( \chi_{\text{cyc}} \) is the \( p \)-adic cyclotomic character. A ring homomorphism

\[
\kappa : \Lambda_{\text{wt}} := \Lambda(\mathbb{Z}_p^\times)^\times \rightarrow \mathcal{O}
\]

is called an arithmetic specialization of weight \( k + 2 \in \mathbb{Z} \) if the compositum

\[
G_Q \xrightarrow{\kappa} \Lambda_{\text{wt}} \xrightarrow{\kappa} \mathcal{O}
\]

agrees with \( \chi_{\text{cyc}}^k \) on an open subgroup of \( G_Q \).

2) Given an eigenform \( f \) as above, let us define

\[
\Lambda(\mathbb{Z}_p^\times)^{\text{(f)}} \cong \Lambda(1 + p\mathbb{Z}_p)
\]

as the component that is determined by the weight \( k_f + 2 \), in the sense that the map

\[
\Lambda(\mathbb{Z}_p^\times)^{\text{(f)}} \xrightarrow{(k_f)} \mathbb{Z}_p \text{ factors through } \Lambda(\mathbb{Z}_p^\times)^{\text{(f)}}.
\]

Here, for a given integer \( k \), we have written \( \langle k \rangle : \Lambda(\mathbb{Z}_p^\times) \rightarrow \mathbb{Z}_p \) to denote the group homomorphism induced from the map \( [x] \rightarrow x^k \).

3) We let \( f = \sum_{n=1}^\infty \varphi_n(f)q^n \in \Lambda[[q]] \) denote the (non-CM) branch of the primitive Hida family of tame conductor \( N_f \), which admits \( f_\alpha \) as a weight \( k_f + 2 \) specialization. Here, \( \Lambda_f \) is the branch (i.e., the irreducible component) of the Hida’s universal ordinary Hecke algebra determined by \( f_\alpha \). It is finite flat over \( \Lambda(\mathbb{Z}_p^\times)^{\text{(f)}} \) and the universal weight character \( \kappa \) restricts to a character (also denoted by \( \kappa \))

\[
\kappa : G_Q \rightarrow \Lambda_{\text{wt}} \rightarrow \Lambda(\mathbb{Z}_p^\times)^{\text{(f)}} \rightarrow \Lambda_f^\times.
\]
1. INTRODUCTION

iv) We set \( \Lambda_f(\Gamma) := \Lambda_f \hat{\otimes} \Lambda_o(\Gamma) \) for any profinite group \( \Gamma \).

1.5.1. Iwasawa algebras and distribution algebras. We fix a topological generator \( \gamma \) of \( \Gamma_1 \) and identify \( \Lambda_o(\Gamma_{\text{cyc}}) \) and \( \Lambda_o(\Gamma_1) \) with the power series rings \( \mathcal{O}[\Delta[|X|]] \) and \( \mathcal{O}[[X]] \), respectively, where \( X = \gamma^{-1} \).

We also consider \( \Lambda_o(\Gamma_{\text{cyc}}) \) and \( \Lambda_o(\Gamma_1) \) as subrings of \( \mathcal{H}(\Gamma_{\text{cyc}}) \) and \( \mathcal{H}(\Gamma_1) \) respectively, which are the rings of power series \( F \in L[\Delta[|X|]] \) (respectively \( F \in L[[X]] \)) which converge on the open unit disc \( |X| < 1 \) in \( \mathbb{C}_p \), where \( | \| \) denotes the \( p \)-adic norm on \( \mathbb{C}_p \) normalized by \( |p|_p = p^{-1} \).

For any real number \( r \geq 0 \), we write \( \mathcal{H}_r(\Gamma_{\text{cyc}}) \) and \( \mathcal{H}_r(\Gamma_1) \) for the set of power series \( F \) in \( \mathcal{H}(\Gamma_{\text{cyc}}) \) and \( \mathcal{H}(\Gamma_1) \) respectively satisfying \( \sup_p F_{\rho_i} < \infty \), where \( \rho_i = p^{-1} / p_i^r - 1 \) and \( F_{\rho_i} = \sup_{|z| \leq \rho_i} |F(z)| \). It is common to write \( F = O(\log_p^r) \) when \( F \) satisfies this condition.

For each integer \( n \geq 0 \), we put \( \omega_n(X) := (1 + X)^{p^n} - 1 \). We set \( \Phi_0(X) = X \), and \( \Phi_n(X) = \omega_n(X) / \omega_{n-1}(X) \) for \( n \geq 1 \). Let \( \chi_{\text{cyc}} \) denote the \( p \)-adic cyclotomic character.

We write \( T_w \) for the ring automorphism of \( \mathcal{H}(\Gamma_{\text{cyc}}) \) defined by \( \sigma \mapsto \chi_{\text{cyc}}(\sigma) \sigma \) for \( \sigma \in \Gamma \). If we set \( u := \chi_{\text{cyc}}(\gamma) \), then \( T_w \) maps \( X \) to \( u(1 + X)^{p^n} - 1 \). If \( m \geq 1 \) is an integer, we define

\[
\omega_{n,m}(X) = \prod_{i=0}^{m-1} T_w^{-i} (\omega_n(X)) ; \quad \Phi_{n,m}(X) = \prod_{i=0}^{m-1} T_w^{-i} (\Phi_n(X)) .
\]

We let \( \log_p \in \mathcal{H}(\Gamma_1) \) denote the \( p \)-adic logarithm. We also define

\[
\log_{p,m} = \prod_{i=0}^{m-1} T_w^{-i} (\log_p) .
\]

We also define Pollack's half logarithms

\[
\log_{p,m}^+ = \prod_{i=0}^{m-1} T_w^{-i} \left( \prod_{m=1}^{\infty} \frac{\Phi_{2m}(X)}{p} \right) ,
\]

\[
\log_{p,m}^- = \prod_{i=0}^{m-1} T_w^{-i} \left( \prod_{m=1}^{\infty} \frac{\Phi_{2m-1}(X)}{p} \right) .
\]

Finally, we set

\[
\delta_{m} = \prod_{i=0}^{m-1} T_w^{-i} X .
\]

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CHAPTER 2

Construction of the optimized Coleman maps

Throughout this chapter, we fix a pair of eigenforms $f$ and $g$ as in the introduction. Our goal in this chapter is to define the appropriate Coleman maps for $T_{f,g}$. These maps will be used to formulate Iwasawa main conjectures later on. We will slightly alter the construction of Coleman maps in [BLLV19] where the Rankin–Selberg products of two non-ordinary forms were studied. We will show that two of these Coleman families behave well as $f$ varies in a Hida family, giving rise to $\Lambda_f(\Gamma_{cyc})$-adic Coleman maps, which in turn allow us to formulate (two-variable) signed Iwasawa main conjectures which also incorporates variation in the Hida families.

We begin our discussion by fixing some notation.

2.1. Dieudonné modules and $\varphi$-eigenvectors

Definition 2.1.1.

i) Let us identify $M_{dR}(h) \otimes \mathbb{Z}_p$ with $D_{cris}(V_h)$ via the comparison isomorphism for each $h \in \{f, g\}$, where $M_{dR}(h)$ is the de Rham realization of the Scholl motive associated to $h$.

We recall (following the notation of [KLZ17, §2.8]) that $V_h$ denotes the $h$-isotypic subspace of $H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbb{Q}}}, \text{Sym}^k(\mathcal{H}_{Z_p}))(1) \otimes_{\mathbb{Z}_p} L$, whereas $V^*_h$ is the $h$-isotypic quotient of $H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbb{Q}}}, \text{Tsym}^k(\mathcal{H}_{Z_p}))(1)) \otimes_{\mathbb{Z}_p} \mathcal{O}$. We let $R^*_h \subset V^*_h$ denote the lattice that maps isomorphically to $R^*_h(-1-k_h)$ under the isomorphism $V^*_h \sim \rightarrow V^*_h(-1-k_h)$.

We write $\omega_h \in \text{Fil}^1 D_{cris}(V_h)$ to be the basis vector as defined in [KLZ20, §6.1] and write $\omega^*_h \in \text{Fil}^1 D_{cris}(V^*_h)$ for the corresponding element for the conjugate form $h^*$, (denoted by $h^*$ in Definition 6.1.2 in op. cit.), which we regard as an element of $\text{Fil}^0 D_{cris}(V^*_h)$ via the natural isomorphism $V_h \sim \rightarrow V^*_h(-1-k_h)$.

ii) For $h$ as in (i) and $\lambda \in \{\alpha, \beta\}$, we choose $v_{h,\lambda} \in D_{cris}(V^*_h)$ to be the $\varphi$-eigenvector of $D_{cris}(V^*_h)$ with eigenvalue $\lambda_h$ given by

$$v_{h,\lambda} := \frac{1}{\langle \varphi(\omega_h), \omega^*_h \rangle} (\varphi(\omega_h) - \lambda_h \omega_h),$$

where $\lambda'$ is the unique element of $\{\alpha, \beta\} \setminus \{\lambda\}$ and

$$\langle -, - \rangle : D_{cris}(V_h) \otimes D_{cris}(V^*_h) \rightarrow L$$

denotes the natural pairing.

iii) We define $\{v^*_{h,\alpha}, v^*_{h,\beta}\}$ to be the dual basis of $\{v_{h,\alpha}, v_{h,\beta}\}$.

iv) Given a pair $\lambda, \mu \in \{\alpha, \beta\}$, we set $v_{\lambda,\mu} := v^*_{f,\lambda} \otimes v^*_{g,\mu}$.
Remark 2.1.2. A direct computation using the fact that
\[ \langle \omega_h, \omega_{h*} \rangle = \langle \varphi(\omega_h), \varphi(\omega_{h*}) \rangle = 0 \]
and the duality \[ \langle \varphi(\omega_h), \omega_{h*} \rangle = \langle \omega_h, \varphi^{-1}(\omega_{h*}) \rangle \] shows that
\[ v_{h, \lambda} = \frac{\lambda_h}{\lambda_{h*}} (\omega_{h*} - \lambda_h \varphi(\omega_h)) \in D_{\text{cris}}(V_h^*) \]
where we identified \( D_{\text{cris}}(V_h) \) with \( D_{\text{cris}}(V_h^*) \) via the isomorphism in Definition 2.1.1(i)
and trough the choice of a basis \( \epsilon = \{ \zeta_{\alpha} \}_n \) of \( \mathbb{Z}_p(1) \).

For a finite unramified extension \( F \) of \( \mathbb{Q}_p \), we have a Perrin-Riou map
\[ L_{F,f,g} : H^1_{\text{Iw}}(F(\mu_{p^n}), T) \to \mathcal{H}_{k_j+k_g+2}(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} D_{\text{cris}}(T) \otimes_{\mathbb{Z}_p} F \]
as defined in [LLZ11] §3.1 and [LZ14] Appendix B. For any \( \lambda, \mu \in \{ \alpha, \beta \} \), our choice of \( \varphi \)-eigenvectors in Definition 2.1.1 gives rise to a map
\begin{equation}
L_{F,f,g}^{(\lambda,\mu)} : H^1_{\text{Iw}}(F(\mu_{p^n}), T) \to \mathcal{H}_{\text{ord}_p(\lambda_j \mu_g)}(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} F
\end{equation}
on projecting \( L_{F,f,g} \) to the \( (\lambda_j \mu_g)^{-1} \)-eigenvector \( v_{\lambda,\mu} \in D_{\text{cris}}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). When \( F = \mathbb{Q}_p \), we omit \( F \) from the notation and simply write \( L_{f,g} \) and \( L_{f,g}^{(\lambda,\mu)} \) in place of \( L_{\mathbb{Q}_p,f,g} \) and \( L_{\mathbb{Q}_p,f,g}^{(\lambda,\mu)} \), respectively.

2.2. Factorization of one-variable Perrin-Riou maps

In [BLLV19] §4, under the hypothesis that both \( f \) and \( g \) are non-ordinary at \( p \), we have constructed a logarithmic matrix attached an \( \mathcal{O} \)-basis \( B = \{ v_i : i = 1, \ldots, 4 \} \) of \( D_{\text{cris}}(T_{f,g}) \), where we recall that \( T_{f,g} = R_f^* \otimes R_g^* \), using the theory of Wach modules. This matrix is in turn used to define the bounded Coleman maps in the context of op. cit. We explain here that one may still carry out a similar construction even when \( f \) is ordinary at \( p \). The main ingredient in the construction of the logarithmic matrix in loc. cit. is Berger’s result [Ber04] proof of Proposition V.2.3], which allows one to lift a basis of \( D_{\text{cris}}(T_{f,g}) \) to an \( \mathbb{A}_L^+ \)-basis of the Wach module \( \mathcal{N}(T_{f,g}) \) (where we recall briefly that \( \mathbb{A}_L^+ = \mathcal{O}[[\pi]] \), which is equipped with an \( \mathcal{O} \)-linear actions by \( \varphi \) and \( \Gamma \) given by \( \varphi(\pi) = (1 + \pi)^p - 1 \) and \( \sigma \cdot \pi = (1 + \pi)^{\chi(\sigma)} - 1 \) for \( \sigma \in \Gamma \). Berger’s result applies whenever the following theee conditions hold:

- (FL\(_{f,g}\)) \( T_{f,g} \) verifies the Fontaine–Laffaille condition \( p > k_f + k_g + 2 \);
- (Slo) The slope of \( \varphi \) on \( D_{\text{cris}}(T_{f,g}) \) does not attain \(-k_f - k_g + 1 \) and \( 0 \) simultaneously;
- (Fil) The basis \( B \) of \( D_{\text{cris}}(T_{f,g}) \) is a filtered basis, in the sense that it gives rise to bases for the graded pieces of \( D_{\text{cris}}(T_{f,g}) \).

Remark 2.2.1. We have chosen in [BLLV19] §4.1] the basis
\[ \{ \omega_{f+}, \omega_{g+}, \varphi(\omega_{f*}) \otimes \omega_{g+}, \omega_{f*} \otimes \varphi(\omega_{g*}), \varphi(\omega_{f*}) \otimes \varphi(\omega_{g*}) \} \]
which verifies the hypothesis (Fil) under the assumption that \( k_f \leq k_g \). We note that there is a typo in the labelling of \( v_2 \) and \( v_3 \) in [BLLV19]. There is another typo in the formula of the matrix of \( \varphi \) on \( D_{\text{cris}}(T_{f,g}^*) \) (labelled \( A_h \) in loc. cit.), where \( \epsilon_h(p) \) should have been \( \epsilon_h(p)^{-1} \). The formula for \( A_0 \) and \( Q \) should also be altered.
2.2. FACTORIZATION OF ONE-VARIABLE PERRIN-RIOU MAPS

Accordingly. These typos do not affect subsequent calculations. For completeness, the correct formula for $A_0$ reads:

$$A_0 = \begin{pmatrix}
0 & 0 & 0 & (\varepsilon_f(p)\varepsilon_g(p))^{-1} \\
0 & 0 & \varepsilon_g(p)^{-1} & -\varepsilon_g(p)^{-1}a_p(f) \\
0 & -\varepsilon_f(p)^{-1} & 0 & -\varepsilon_f(p)^{-1}a_p(g) \\
1 & a_p(f) & a_g(f) & a_p(f)a_g(g)
\end{pmatrix}. $$

When $k_f \geq k_g$, we may swap the ordering of $\varphi(\omega_f^*) \otimes \omega_{g^*}$ and $\omega_{f^*} \otimes \varphi(\omega_g^*)$. This then gives a basis satisfying the condition $(\text{Fil})$.

In our current setting, the fact that $g$ is non-ordinary at $p$ ensures that $[\text{Slo}]$ holds even when $f$ is $p$-ordinary. Therefore, granted the Fontaine–Laffaille condition $(\text{Fil}_{f,g})$, we are left to determine a basis of $D_{\text{cris}}(T_f, g)$ that verifies the condition $(\text{Fil})$ in order to apply Berger’s result. We recall from [LLZ17] Lemma 3.1 that the non-ordinarity of $g$ at $p$ implies that $\omega_{g^*}$ and $\varphi(\omega_{g^*})$ form an $O$-basis of $D_{\text{cris}}(R_g^\dagger)$.

We now study an $O$-basis of $D_{\text{cris}}(R_f^\dagger)$ under the following hypothesis:

$(\Theta)$ \hspace{1cm} $f$ is not $\theta$-critical.

**Remark 2.2.2.** In more precise terms, the condition $(\Theta)$ requires that the non-ordinary $p$-stabilization $f_\beta$ of $f$ does not fall in the image of Coleman’s operator $\theta_{k_f} : S_{-k_f}(1\Gamma_1(Np)) \to S_{k_f+2}(1\Gamma_1(Np))$ on the space of overconvergent cusp forms, which is given by $(q^{\frac{p-1}{2}})^{k_f+1}$ on $q$-expansions.

A deep result of Breuil and Emerton [BE10] Theorem 1.1.3 shows that the hypothesis $(\Theta)$ is equivalent to the requirement that the $G_{\overline{Q}_p}$-restriction of $R_f^\dagger$ is not decomposable. It is expected that $(\Theta)$ holds whenever $f$ does not have complex multiplication; this is known only when $k_f = 0$ and the Hecke field of $f$ equals $Q$; see [Eme04] Theorem 1.3.

We conclude our remark recalling a result due to Ghaite and Vatsal [GV04] which concerns the validity of $(\Theta)$ in a non-CM Hida family $f$. Under mild hypotheses, [GV04] Theorem 2 shows that $(\Theta)$ is valid for all but finitely many specializations of $f$.

**Proposition 2.2.3.** Set $\eta_f^* := \alpha_f^{-1}\omega_f^* - \varphi(\omega_f^*)$. Then $\{\omega_f^*, \eta_f^*\}$ is an $O$-basis of $D_{\text{cris}}(R_f^\dagger)$.

**Proof.** By [LZ13] Corollary 5.8, there exists a $G_{\overline{Q}_p}$-stable lattice $R^\circ$ in $R_f^\dagger \otimes \overline{Q}_p$ such that the Wach module $\mathbb{N}(R^\circ)$ admits an $\mathbb{A}_f^\dagger$-basis $\{n_1, n_2\}$, with respect to which the matrix of $\varphi$ is given by

$$\begin{pmatrix}
\alpha_f^{-1} & 0 \\
\pi^{k_f+1}x_f & (\frac{\pi}{\varphi(\pi)})^{k_f+1} \varepsilon_f(p)^{-1}\alpha_f
\end{pmatrix},$$

where $x_f \in \mathbb{A}_f^\dagger$ satisfies $t^{k_f+1}x_f = k_f! + O(t^{k_f+2})$ (note that we have swapped $n_1$ and $n_2$ as compared to the notation in loc. cit.). Recall from [Ber04] Théorème III.4.4] that the Wach module is equipped with a filtration. Furthermore, $\mathbb{N}(R^\circ)/\pi$ is naturally isomorphic to the filtered $\varphi$-module $D_{\text{cris}}(R^\circ)$. Let us write $\nu_t \in D_{\text{cris}}(R^\circ)$.
for the image of \( n_i \) under this isomorphism. Then, \( \nu_1 \) generates \( \text{Fil}^0 \mathcal{D}_{\text{cris}}(R^\circ) \) and the matrix of \( \varphi \) with respect to the basis \( \{ \nu_1, \nu_2 \} \) is

\[
\begin{pmatrix}
\alpha_f^{-1} & 0 \\
0 & k_f! \beta_f^{-1}
\end{pmatrix}.
\]

Note in particular that \( \nu_2 \) is a \( \varphi \)-eigenvector with eigenvalue \( \beta_f^{-1} \) and \( \varphi(\nu_1) = \alpha_f^{-1}\nu_1 + k_f!\nu_2 \). We deduce that

\[
\nu_2 = \frac{1}{k_f!}(\alpha_f^{-1}\nu_1 - \varphi(\nu_1)).
\]

Under the running hypotheses \( \text{(FL)} \) \( \text{Corollary 2.2.3} \) \( \text{Definition 2.1.1(ii)} \) \( \text{Proposition 2.2.3} \) \( \text{Corollary 2.2.4} \), we see that there exists an \( \mathcal{O} \)-basis of \( \mathcal{D}_{\text{cris}}(R_f) \) that \( \nu_1 \) and \( \alpha_f^{-1}\nu_1 - \varphi(\nu_1) \) form an \( \mathcal{O} \)-basis of \( \mathcal{D}_{\text{cris}}(R^\circ) \).

The hypothesis \( \text{(Θ)} \) implies that \( G_{\mathbb{Q}_p} \)-stable lattices contained in \( R_f^\circ \otimes \mathbb{Q}_p \) are unique up to homothety. In particular, \( R_f^\circ = c\mathbb{R}^\circ \) for some \( c \in \mathbb{L} \). Thus, \( \mathcal{D}_{\text{cris}}(R_f^\circ) = c\mathcal{D}_{\text{cris}}(R^\circ) \). Moreover, since \( \{ \omega_f \} \) and \( \{ \nu_1 \} \) are an \( \mathcal{O} \)-basis of \( \text{Fil}^0 \mathcal{D}_{\text{cris}}(R_f^\circ) \) and \( \text{Fil}^0 \mathcal{D}_{\text{cris}}(R^\circ) \) respectively, we may identify \( \omega_f \) with \( \nu_1 \) (after multiplying either \( \omega_f \) or \( \nu_1 \) by a \( p \)-adic unit, if necessary). The result follows.

**Corollary 2.2.4.** The \( \mathcal{O} \)-module \( \mathcal{D}_{\text{cris}}(R_f) \) admits \( \{ v_{f,\alpha}, \omega_f \} \) as an \( \mathcal{O} \)-basis, where \( v_{f,\alpha} \) is given as in Definition \( \text{Definition 2.1.1(ii)} \).

**Proof.** If we replace \( f \) by \( f^* \) in Proposition \( \text{Proposition 2.2.3} \), and identify \( R_f \) with \( R_f^* \) \( (-k_f-1) \), we see that there exists an \( \varphi \)-eigenvector \( v_0 \in \mathcal{D}_{\text{cris}}(R_f) \) with \( \varphi(v_0) = \alpha_f v_0 \) such that \( \{ \omega_f, v_0 \} \) is an \( \mathcal{O} \)-basis of \( \mathcal{D}_{\text{cris}}(R_f) \). Since \( \omega_f \) is an \( \mathcal{O} \)-basis of the direct summand \( \text{Fil}^0 \mathcal{D}_{\text{cris}}(R^\circ) \) of \( \mathcal{D}_{\text{cris}}(R^\circ) \), we have \( \langle v_0, \omega_f \rangle \in \mathcal{O}^\times \). It follows from the definition of \( v_{f,\alpha} \) that \( \langle v_{f,\alpha}, \omega_f \rangle = 1 \). As both \( v_0 \) and \( v_{f,\alpha} \) lie in the same \( \varphi \)-eigenspace, which is one-dimensional, we necessarily have \( v_0 = \langle v_0, \omega_f \rangle v_{f,\alpha} \). Since \( \langle v_0, \omega_f \rangle \in \mathcal{O}^\times \) and \( \{ \omega_f, v_0 \} = \{ \omega_f, \langle v_0, \omega_f \rangle v_{f,\alpha} \} \) is an \( \mathcal{O} \)-basis of \( \mathcal{D}_{\text{cris}}(R_f) \), we have proved our claim.

Let \( \mathcal{C}_f \) denote the cohomological congruence number associated to the eigenform \( f \) (c.f. \( \text{Hid} \text{81} \) \( (0.3) \) for its definition; see also \( \text{DDT} \text{94} \) Corollary 4.19 for an elaboration in the particular case \( k_f = 0 \)).

**Corollary 2.2.5.** Let \( \eta_f' \in \mathcal{D}_{\text{cris}}(V_f)/\text{Fil}^1 \) be the unique vector that verifies \( (\eta_f', \omega_f^*)_{Y_1(N_f)} = G(\varepsilon_f^{-1}) \).

where \( (-,-)_{Y_1(N_f)} \) is a pairing induced by Poincaré duality (see the discussion in \( \text{K} \text{L} \text{Z} \text{20} \) just before the statement of Proposition 6.1.3) and \( G(\varepsilon_f^{-1}) \) denotes the Gauss sum for \( \varepsilon_f^{-1} \). Then,

\[
v_{f,\alpha} \equiv \mathcal{C}_f \eta_f' \mod \text{Fil}^1 \mathcal{D}_{\text{cris}}(V_h)
\]

up to \( p \)-adic units.

**Proof.** In terms of the \( \mathcal{O} \)-basis \( \{ v_{f,\alpha}, \omega_f \} \) of \( \mathcal{D}_{\text{cris}}(R_f) \) (c.f. Corollary \( \text{Corollary 2.2.4} \)), \( \mathcal{C}_f \) is given (up to a \( p \)-adic unit) via the chain of identities

\[
(v_{f,\alpha}, \omega_f)_{Y_1(N_f)} \cdot \mathcal{O} = (v_{f,\alpha}, W_{N_f} \omega_f)_{Y_1(N_f)} \cdot \mathcal{O} = \mathcal{C}_f \cdot \mathcal{O},
\]
where $W_{N_f}$ is the Atkin–Lehner operator and the first equality holds since $W_{N_f}f = \lambda_{N_f}f^*$ and the Atkin–Lehner pseudo-eigenvalue $\lambda_{N_f}$ is a $p$-adic unit, whereas the second equality is one of the definitions of the cohomological congruence number $\mathcal{C}_f$. Note that $G(\varepsilon_f^{-1})$ is a $p$-adic unit since $p$ does not divide the conductor of $\varepsilon_f^{-1}$. Combining (2.2) and (2.3), the result follows.

**Remark 2.2.6.** Note that the basis $\{\omega_f, \eta_f\}$ is a basis of $\mathbb{D}_{\text{cris}}(R_f^*)$ that gives rise to bases of the graded pieces of $\mathbb{D}_{\text{cris}}(R_f^*)$. Furthermore, $\varphi(\eta_f) = \beta_f^{-1} \eta_f$. In particular, the matrix of $\varphi$ with respect to this basis is given by

$$A_f = \begin{pmatrix} a_f & 0 \\ -1 & b_f^{-1} \end{pmatrix} = A_{f,0} \begin{pmatrix} 1 \\ 1/p^k_f+1 \end{pmatrix},$$

where $A_{f,0} = \begin{pmatrix} a_f & 0 \\ -1 & p^k_f+1 b_f^{-1} \end{pmatrix} \in \text{GL}_2(O)$.

**Definition 2.2.7.** We let $\{v_1, v_2, v_3, v_4\} \subset \mathbb{D}_{\text{cris}}(T_{f,g})$ denote the $O$-basis given by

$$v_1 = \omega_f \otimes \omega_{g^*}, \quad v_2 = \omega_f \otimes \varphi(\omega_{g^*}), \quad v_3 = \eta_f \otimes \omega_{g^*}, \quad v_4 = \eta_f \otimes \varphi(\omega_{g^*}).$$

Note that if $k_g \leq k_f$, this basis satisfies (Fil) otherwise, the basis $\{v_1, v_3, v_2, v_4\}$ does.

**Definition 2.2.8.**

i) Suppose $k_g \leq k_f$. We let $\{n_1, n_2, n_3, n_4\}$ denote the basis of the Wach-module $\mathbb{N}(T_{f,g})$ given by [Ber04] Proof of Proposition V.2.3, that lifts the ordered basis $\{v_1, v_2, v_3, v_4\}$ of $\mathbb{D}_{\text{cris}}(T_{f,g})$.

ii) Suppose $k_g > k_f$. We similarly define $\{n_1, n_2, n_3, n_4\}$ as the ordered basis of $\mathbb{N}(T_{f,g})$ lifting the ordered basis $\{v_1, v_2, v_3, v_4\}$ of $\mathbb{D}_{\text{cris}}(T_{f,g})$.

iii) Set $q := \varphi(\pi)/\pi \in \mathbb{A}_L^+$ and $\xi = p/(q - \pi^{p-1}) \in (\mathbb{A}_L)^\times$.

**Lemma 2.2.9.** Let $A_{f,g,0} \in \text{GL}_4(O)$ so that the matrix $A_{f,g}$ of $\varphi_{\mathbb{D}_{\text{cris}}(T_{f,g})}$ with respect to the ordered basis $\{v_1, v_2, v_3, v_4\}$ verifies

$$A_{f,g} = A_{f,g,0} \begin{pmatrix} 1 \\ 1/p^k_f+1 \\ 1/p^k_f+k_g+2 \end{pmatrix}.$$

Then the matrix of $\varphi_{\mathbb{N}(T_{f,g})}$ with respect to the ordered basis $\{n_1, n_2, n_3, n_4\}$ is given by

$$P_{f,g} = A_{f,g,0} \begin{pmatrix} \xi^{k_f+k_g+2} \\ \xi^{k_f+1}/q^{k_g+1} \\ \xi^{k_g+1}/q^{k_f+1} \\ 1/q^{k_f+k_g+2} \end{pmatrix}.$$

**Proof.** We note that [Ber04] Proposition V.2.3 applies to the representation $T_{f,g}(-k_f - k_g - 2)$. We have the identifications

$$\mathbb{N}(T_{f,g}) \times \mathbb{A}_L^{k_f+k_g+2} \rightarrow \mathbb{N}(T_{f,g}(-k_f - k_g - 2)),$$

$$\mathbb{D}_{\text{cris}}(T_{f,g}) \times \mathbb{A}_L^{k_f+k_g+2} \rightarrow \mathbb{D}_{\text{cris}}(T_{f,g}(-k_f - k_g - 2)),$$
where $t = \log(1 + \pi)$. Let us put $n^i = \pi^{k_j} + k_p + 2 \cdot n_i$ and $v^i = t^{k_j} + k_p + 2 \cdot v_i$ for $i = 1, \ldots, 4$. The matrix of $\varphi|_{\mathcal{B}_L}(T_{f,g}(-k_j - k_p - 2))$ with respect to $\{v^i\}_{i=1}^4$ equals

$$A_{f,g,0} = \begin{pmatrix} p^{k_j} + k_p & p^{k_j+1} & p^{k_p+1} & 1 \\ p^{k_p} & p^{k_p+1} & 1 & 1 \end{pmatrix},$$

then matrix of $\varphi|_{\mathcal{B}_L}(T_{f,g}(-k_j - k_p - 2))$ with respect to $\{n^i\}_{i=1}^4$ is given by

$$A_{f,g,0} = \begin{pmatrix} (\xi q)^{k_j} + k_p + 2 & (\xi q)^{k_j+1} & (\xi q)^{k_p+1} & 1 \\ (\xi q)^{k_p} & (\xi q)^{k_p+1} & 1 & 1 \end{pmatrix}.$$

We then obtain the matrix of $\varphi|_{\mathcal{B}_L}(T_{f,g})$ with respect to the basis $\{n_i\}_{i=1}^4$ using the identity $\varphi(\pi) = \eta \pi$.

**Definition 2.2.10.** We write $M_{f,g}$ for the resulting logarithmic matrix with respect to the basis $\{n_1, n_2, n_3, n_4\}$, defined as in [BLV19] §4.2. In more explicit terms,

$$M_{f,g} := m^{-1} \left( ((1 + \pi)A_{n_1, n_2, n_3, n_4}(M) \right) \in \text{Mat}_{4 \times 4}(\mathcal{H}_{k_p + k_p + 1}(\Gamma_1)),$$

where $M$ is the change of basis matrix as given in (4.2.2) of op. cit. and $m$ is the Mellin transform.

Concretely, $M_{f,g}$ is the change of basis matrix satisfying

$$((1 + \pi)\varphi(n_1) \ (1 + \pi)\varphi(n_2) \ (1 + \pi)\varphi(n_3) \ (1 + \pi)\varphi(n_4)) = (v_1 \ v_2 \ v_3 \ v_4) M_{f,g}.$$

**Definition 2.2.11.** Let $A_g$ be the matrix of $\varphi$ with respect to the basis $\{\omega_1, \varphi(\omega_2), \ldots, \varphi(\omega_4)\}$, which we factor as

$$A_g = A_{g,0} \begin{pmatrix} 1 & 0 \\ 1/p^{k_p+1} & 1 \end{pmatrix},$$

where $A_{g,0} \in \text{GL}_2(\mathcal{O})$. We write $M_g \in \text{Mat}_{2 \times 2}(\mathcal{H}_{k_p+1,1})$ for the resulting $2 \times 2$ logarithmic matrix, given as in [BLV20b] §2 (note that we may define such a matrix as long as $p > k_p + 1$).

We now describe the relation between the logarithmic matrices $M_{f,g}$ and $M_g$.

**Proposition 2.2.12.** There exist elements $u_f \in \Lambda_{\mathcal{O}}(\Gamma_1)^\times$ and $\ell_f \in \mathcal{H}_{k_p+1}(\Gamma_1)$ such that

$$M_{f,g} = \begin{pmatrix} u_f M_g & 0 \\ \ell_f T_{w_{k_p+1}} M_g \end{pmatrix}.$$  

Moreover, $\ell_f = \frac{\log u_{f,k_p+1}}{2 \pi i}$ up to multiplication by a unit of $\Lambda_{\mathcal{O}}(\Gamma_1)$.

**Proof.** Thanks to the choice of the basis $\{v_1, v_2, v_3, v_4\}$, the matrix $A_{f,g}$ is of the form

$$A_{f,g} = \begin{pmatrix} \alpha_f^{-1} A_g & 0 \\ -A_g & \beta_f^{-1} A_g \end{pmatrix}.$$  

Consequently,

$$A_{f,g,0} = \begin{pmatrix} \alpha_f^{-1} A_{g,0} & 0 \\ -A_{g,0} & \beta_f^{-1} A_{g,0} \end{pmatrix},$$
which in turn shows

\[ P_{f,g} = A_{f,g,0} \begin{pmatrix} \xi_{k_f+k_g+2} & \xi_{k_f+1}/q^{k_g+1} & \xi_{k_g+1}/q^{k_f+1} & 1/q^{k_f+k_g+2} \\ -\xi_{k_f+1}P_g & 0 & -\xi_{k_g+1}P_g & p^{k_f+1}(\beta_f q^{k_f+1})^{-1}P_g \end{pmatrix}, \]

where \( P_g \) is the matrix of \( \varphi_{[u(R_g)]} \) with respect to the Wach-module basis of \( \mathcal{B}(R_g) \) obtained by lifting the basis \( \{\omega_g, \varphi(\omega_g)\} \) via [Ber04] Proof of Proposition V.2.3. We therefore have

\[ P_{f,g}^{-1} = \begin{pmatrix} \alpha_f \xi^{-k_f-1}P_g^{-1} & 0 \\ -\varepsilon_f(p)q^{k_f+1}P_g^{-1} & p^{-k_f-1}\beta_f q^{k_f+1}P_g^{-1} \end{pmatrix}, \]

which yields

\[ A_{f,g}^{n+1} \varphi^n(P_{f,g}^{-1}) \cdots \varphi(P_{f,g}^{-1}) = \begin{pmatrix} u_{f,n} \varepsilon_n & 0 \\ v_{f,n} \varepsilon_n & \beta_f^{-1}(q^{k_f+1}/p^{k_f+1}) \cdots \varepsilon_n(q^{k_f+1}/p^{k_f+1}) \varepsilon_n \end{pmatrix}, \]

where \( u_{f,n} \) is a unit, \( v_{f,n} \) is some linear combination of \( \varphi(q^{k_f+1}/p^{k_f+1}), \cdots, \varphi^n(q^{k_f+1}/p^{k_f+1}) \) and \( \varepsilon_n \) is given by the product \( A_{f,g}^{n+1} \varphi^n(P_g^{-1}) \cdots \varphi(P_g^{-1}) \).

In what follows, we write \( F \sim G \) if \( F \) and \( G \) are two elements of \( \mathcal{H}(\Gamma_1) \) such that \( F = uG \) for some \( u \in \Lambda(\Gamma_1)^\times \). By [LLZ10] Theorem 5.4 and [LLZ17] Theorem 2.1,

\[ m^{-1}\left(1 + \pi\left(\prod_{i=1}^{n} \varphi^i(q^{k_i+1})\right)\right) \sim A_{g}^{n+1} \prod_{i=1}^{n} m^{-1}\left((1 + \pi)\varphi^i(q^{k_i+1})\varphi^i(P_g^{-1})\right) \]

\[ \sim A_{g}^{n+1} \prod_{i=1}^{n} \left(\prod_{j=0}^{k_i} T_{w_{j\Phi_1}}\right)\prod_{j=0}^{k_f} T_{w_{k_f+1}}m^{-1}\left((1 + \pi)\varphi^i(P_g^{-1})\right) \]

since the entries of \( \varphi^i(P_g^{-1}) \) are, up to units in \( \mathcal{B}(\mathcal{L})^+ \), either \( \varphi^i(q^{k_g+1}) \) or constants. The result follows from [BLLV19] (4.2.5)] on passing to limit in \( n \).

In order to define Coleman maps using the logarithmic matrix \( M_{f,g} \) to decompose the Perrin-Riou maps \( \mathcal{L}^{(\lambda,\mu)}_{F,f,g} \) we introduce the following change of basis matrices, which will allow us to pass between our chosen eigenvector basis \( \{v_{\lambda,\mu}\} \) and the integral basis of \( \mathcal{D}_{\text{cris}}(T_{f,g}) \) studied above.
Definition 2.2.13. 

i) Let \( Q_f = \begin{pmatrix} \alpha_f - \beta_f & 0 \\ \alpha_f \beta_f & -\alpha_f \beta_f \end{pmatrix} \) be the change of basis matrix from \( \{v_f^*, v_{f,\alpha}\} \) to \( \{\omega_{f,\alpha}, \eta_{f,\alpha}\} \), so that we have \( Q_f^{-1} A_f Q_f = \begin{pmatrix} \alpha_f^{-1} & 0 \\ 0 & \beta_f^{-1} \end{pmatrix} \).

ii) We set \( Q_g = \begin{pmatrix} \alpha_g & -\beta_g \\ -\alpha_g \beta_g & \alpha_g \beta_g \end{pmatrix} \), which is the change of basis matrix from \( \{v_{g,\alpha}, v_{g,\beta}\} \) to \( \{\omega_{g,\alpha}, \varphi(\omega_{g,\beta})\} \), so that we have \( Q_g^{-1} A_g Q_g = \begin{pmatrix} \alpha_g^{-1} & 0 \\ 0 & \beta_g^{-1} \end{pmatrix} \).

iii) We denote by \( Q_{f,g} \) the change of basis matrix from \( \{v_{\lambda,\mu} : \lambda, \mu \in \{\alpha, \beta\}\} \) to \( \{v_1, v_2, v_3, v_4\} \), so that we have

\[
Q_{f,g}^{-1} A_{f,g} Q_{f,g} = \begin{pmatrix} (\alpha_f \alpha_g)^{-1} & 0 \\ 0 & \beta_f \beta_g \end{pmatrix}.
\]

Remark 2.2.14. Observe that we have

\[
Q_{f,g} = \begin{pmatrix} Q_g & 0 \\ \alpha_f \beta_f Q_g & -\alpha_f \beta_f Q_g \end{pmatrix}.
\]

Combining this observation with Proposition 2.2.12, we deduce that

\[
Q_{f,g}^{-1} M_{f,g} = \begin{pmatrix} u f^{-1} Q_g^{-1} M_g & 0 \\ 0 & 0 \end{pmatrix}.
\]

Proposition 2.2.15. If \( F \) is a finite unramified extension of \( \mathbb{Q}_p \), there exists a quadruple of bounded Coleman maps

\[
\text{Col}_{F,f,g}^{(\omega,\#)}, \quad \text{Col}_{F,f,g}^{(\omega,\cdot)}, \quad \text{Col}_{F,f,g}^{(\eta,\cdot)}, \quad \text{Col}_{F,f,g}^{(\eta,\#)} : \quad H^1_{\text{Iw}}(F(\mu_{p^\infty}), T) \to \Lambda_0(\Gamma_{\text{Cyc}}) \otimes \mathcal{O}_F,
\]

which satisfies

\[
\begin{pmatrix}
L_{F,f,g}^{(\alpha,\alpha)} \\
L_{F,f,g}^{(\alpha,\beta)} \\
L_{F,f,g}^{(\beta,\alpha)} \\
L_{F,f,g}^{(\beta,\beta)}
\end{pmatrix} = Q_{f,g}^{-1} M_{f,g}
\begin{pmatrix}
\text{Col}_{F,f,g}^{(\omega,\#)} \\
\text{Col}_{F,f,g}^{(\omega,\cdot)} \\
\text{Col}_{F,f,g}^{(\eta,\cdot)} \\
\text{Col}_{F,f,g}^{(\eta,\#)}
\end{pmatrix}.
\]

In particular,

\[
\begin{pmatrix}
L_{F,f,g}^{(\alpha,\alpha)} \\
L_{F,f,g}^{(\alpha,\beta)}
\end{pmatrix} = u f^{-1} Q_g^{-1} M_g \begin{pmatrix}
\text{Col}_{F,f,g}^{(\omega,\#)} \\
\text{Col}_{F,f,g}^{(\omega,\cdot)}
\end{pmatrix}.
\]

Proof. The first assertion follows from [LLZ10, Theorem 3.5]; see also the discussion in [BLLV19, §5.1]. The second follows from the first, combined with Remark 2.2.14. \( \square \)

We conclude this subsection with the following definition of the semi-local Coleman maps, which we will later use to define local conditions.
2.3. Two-variable Perrin-Riou maps and Coleman maps

Recall from the introduction that $f$ denotes a Hida family passing through $f_\alpha$ (we remark that our Hida families are what [KLZ17] calls branches, as explained in Section 7.5 of op. cit.). To the best of the authors’ knowledge, a theory of Wach modules for families, which would allow one to decompose a three-variable Perrin-Riou map (where both $f$ and $g$ vary in families), is currently unavailable. We shall content ourselves with studying two-variable $\Lambda_f$-adic Perrin-Riou maps interpolating $L_{f,f,g}^{(\alpha,\alpha)}$ and $L_{f,f,g}^{(\alpha,\beta)}$ as $\alpha_0$ varies. We will show that these maps can be decomposed into bounded Coleman maps, deforming $\text{Col}_{f,f,g}^{(\omega,\#)}$ and $\text{Col}_{f,f,g}^{(\omega,\flat)}$ under the following weaker version of the hypothesis $(\text{FL}_{f,g})$:

$R_g^\ast$ verifies the Fontaine–Laffaille condition $p > k_g + 1$.

**Remark 2.3.1.** One advantage of working with only two of the Perrin-Riou maps $L_{f,f,g}^{(\alpha,\alpha)}$ and $L_{f,f,g}^{(\alpha,\beta)}$ and two of the Coleman maps $\text{Col}_{f,f,g}^{(\omega,\#)}$ and $\text{Col}_{f,f,g}^{(\omega,\flat)}$ is that we only need an integral basis for the Dieudonné module of a rank-two quotient of $T_{f,g}$, rather than the whole of $T_{f,g}$. In particular, not only can we weaken hypothesis $(\text{FL}_{f,g})$ to $(\text{FL}_g)$ but we may also discard the hypothesis $(\Theta)$.

We now introduce a number of objects attached to the family $f$. Throughout this subsection, $g$ is a fixed non-$p$-ordinary cuspidal eigenform as in the introduction.

**Definition 2.3.2.**

i) The prime ideal of $\Lambda_f$ that corresponds to the specialization of $f$ to $f_\alpha$ (as well as the specialization itself) will be denoted by $\kappa_0$.

ii) We write $\alpha_f$ for the $U_p$-eigenvalue of $f$. Given a specialization $\kappa$ of $\Lambda_f$, we write $f_\kappa$ and $\alpha_\kappa$ for the respective specializations of $f$ and $\alpha_f$ at $\kappa$.

iii) Let $R^\ast_f$ be the $\Lambda_f$-adic $G_{Q_p}$-representation attached to $f$ as given in [KLZ17, Definition 7.2.5] and let $F^\ast R^\ast_f$ denote the free rank-one $\Lambda_f$-module given as in Theorem 7.2.8 of op. cit.

iv) We put $T_{f,g} := R^\ast_f \otimes R^\ast_g$ and $T_{f,g}^\oplus := F^\ast R^\ast_f \otimes R^\ast_g$.

**Remark 2.3.3.**

i) As explained in [KLZ17, Theorem 7.2.3(iii)], the $G_{Q_p}$-representation $F^\ast R^\ast_f|_{G_{Q_p}}$ is unramified.

ii) The specialization of $R^\ast_f$ at $\kappa_0$ coincides with the natural $O$-lattice $R^\ast_{f,0}$ contained in Deligne’s representation attached to the $p$-stabilized eigenform $f_{\alpha_0}$, which is realized as the $f_{\alpha_0}$-isotypic Hecke eigenspace inside $H^1_{et}(Y_1(pN_f)|\overline{\mathbb{Q}}_p, \text{Sym}^{k_f}(\mathcal{H}^{\vee}_{\alpha_0})_f(1))$. We let $R^\ast_{f,0}$ denote the lattice we have introduced in Definition 2.1.1(i). This is a Galois-stable lattice contained in the $f_{\alpha_0}$-isotypic Hecke eigenspace

$H^1_{et}(Y_1(pN_f)|\overline{\mathbb{Q}}_p, \text{Sym}^{k_f}(\mathcal{H}^{\vee}_{\alpha_0})|_{f_{\alpha_0}}) \subset H^1_{et}(Y_1(pN_f)|\overline{\mathbb{Q}}_p, \text{Sym}^{k_f}(\mathcal{H}^{\vee}_{\alpha_0}))$.

iii) We recall that the lattice $R_f$ is defined in a similar manner, which is realized in the cohomology of $Y_1(N_f)$. The lattices $R_f$ and $R_{f,0}$ are related via a morphism $(\text{Pr}^{\alpha_0})^\ast: R_f \rightarrow R_{f,0}$, where $(\text{Pr}^{\alpha_0})^\ast = (\text{Pr}_1)^\ast - \frac{\beta_f}{p_1 - 1}(\text{Pr}_2)^\ast$ is the map dual to
(Pr^α)_* : R^*_f \rightarrow R^*_f\), which was studied in [KLZ17 §7.3]. Note that the map (Pr^α)^* is necessarily injective, since it induces an isomorphism R_{f[1/p]} \sim R_{f[1/p]}.

We now introduce the following p-stabilized version of the Perrin-Riou maps for the representation R^*_f \otimes R^*_g, which can be defined under the hypothesis [FLg].

**Definition 2.3.4.** Let F/\mathbb{Q}_p a finite unramified extension. For \mu \in \{\alpha, \beta\}, we define the \Lambda_{\alpha, \beta}(\Gamma_{\text{cyc}})-morphism

$$\mathcal{L}^{(\alpha, \beta)}_{F, f, g} : H^1_{\text{Iw}}(F(\mu_p^\infty), R^*_f \otimes R^*_g) \rightarrow \mathcal{H}_{\text{ord}_{p}(\mu_p)}(\Gamma_{\text{cyc}}) \otimes \mathbb{Z}_p F$$

as the compositum

$$H^1_{\text{Iw}}(F(\mu_p^\infty), R^*_f \otimes R^*_g) \xrightarrow{(\Pr_{\alpha})^*} H^1_{\text{Iw}}(F(\mu_p^\infty), R^*_f \otimes R^*_g) \xrightarrow{\mathcal{L}^{(\alpha, \beta)}_{F, f, g}} \mathcal{H}_{\text{ord}_{p}(\mu_p)}(\Gamma_{\text{cyc}}) \otimes \mathbb{Z}_p F$$

where the morphism \mathcal{L}^{(\alpha, \beta)}_{F, f, g} is given in [2.1].

**Lemma 2.3.5.** Let (Pr^α)^* : R_f \rightarrow R_{f[1/p]} be the morphism given as in Remark 2.3.3. Then (Pr^α)^* is an isomorphism if at least one of the following conditions holds true: i) \kappa_f > 0, ii) \nu_{p}(\alpha_f - \beta_f/p) = 0, iii) \overline{f} is absolutely irreducible.

**Proof.** In the situation of (i)-(ii), this is essentially [KLZ17 Prop. 7.3.1] transposed, whereas the case (iii) is [LLZ15 Prop. 4.3.6] transposed. We go over the details in the setting of (i)-(ii). We shall apply the ("dual") argument in the proof of [KLZ17 Prop. 7.3.1] to show that the cokernel of (Pr^α)^* is annihilated by \alpha_f - \beta_f/p. This will suffice to conclude the proof of our lemma when (i) or (ii) holds true.

In fact, we shall prove that the map (pr_2)_* \circ (Pr^α)^* : R_f \rightarrow R_f is given by multiplication by \alpha_f - \beta_f/p. To see that, we directly compute

$$(pr_2)_* \circ (Pr^α)^* = (pr_2)_* \circ pr^*_1 - \frac{\beta_f}{p+1}(pr_2)_* \circ pr^*_2$$

$$= a_p(f) - \frac{\beta_f(p+1)}{p} = \alpha_f - \beta_f/p$$

where the second equality is because (pr_2)_* \circ pr^*_1 = T_p and (pr_2)_* \circ pr^*_2 = p^{\beta_f}(p+1); whereas the final equality is because a_p(f) = \alpha_f + \beta_f. The proof of our lemma is now complete in the situation of (i)-(ii). The case (iii) can be treated following the proof of [LLZ15 Prop. 4.3.6] but “dualizing” the argument in op. cit. as above. \qed

The \Lambda_p-adic objects we introduce in Definition 2.3.6 below is akin to the constructions of [KLZ17 §8.2].

**Definition 2.3.6.** Let F be a finite unramified extension of \mathbb{Q}_p.

i) We define \mathbb{D}(F, F^{-} R^*_f) = (F^{-} R^*_f \otimes \mathbb{Z}_p)[\mathbb{G}_F], which is a \Lambda_p-module equipped with an operator \varphi induced by the arithmetic Frobenius action on \mathbb{Z}_p^{ur} and define the \Lambda_p^L-module \mathbb{N}(F, F^{-} R^*_f) = \mathbb{D}(F, F^{-} R^*_f)[[\pi]], with \varphi sending \pi to \{ (1 + \pi)^p - 1. There is a left inverse \psi of \varphi induced by the trace operator on \mathbb{Z}_p[[\pi]] \rightarrow \mathbb{Z}_p[[\pi]]].

ii) We define the corresponding objects for \mathbb{D}(F, T^*_g) on setting \mathbb{D}(F, T^*_g) = \mathbb{D}(F, F^{-} R^*_f) \otimes \mathbb{D}_{\text{crys}}(R^*_g) and \mathbb{N}(F, T^*_g) = \mathbb{N}(F, F^{-} R^*_f) \otimes \Lambda_p^L \cdot \mathbb{N}(R^*_g), respectively.
ii) Let $I_i$ be the congruence ideal of $f$ and let $H_i \in I_i$ denote Hida’s congruence divisor.

Lemma 2.3.8. There is an isomorphism of $\Lambda_f(\Gamma_{cyc})$-modules

$$H^1_{I_m}(F(\mu_{p^\infty}), T_{\tilde{f},g}) \cong \mathcal{N}(F, T_{\tilde{f},g})^{\psi=1}.$$  

Proof. Our proof follows closely part of the proof of [KLZ17 Theorem 8.2.3]. Since all the modules above commute with inverse limits, we may replace $F - R^*_f$ by a finitely generated $\mathbb{Z}_p$-module $M$ equipped with an unramified action by $G_{\mathbb{Q}_p}$.

Since $\mathcal{N}(F, M \otimes R^*_f)$ is now the usual Wach module of $M \otimes R^*_f$ over $F$, [Ber03 Theorem A.3] tells us that

$$H^1_{I_m}(F(\mu_{p^\infty}), M \otimes R^*_f) \cong (\pi^{-1}\mathcal{N}(F, M \otimes R^*_f))^{\psi=1}.$$  

As the Hodge–Tate weights of $M \otimes R^*_f$ are non-negative and $R^*_f$ is non-ordinary, it follows that $M \otimes R^*_f$ does not admit a subquotient isomorphic to the trivial representation. Thus, we may eliminate $\pi^{-1}$ from $\pi^{-1}\mathcal{N}(M \otimes R^*_f)$ in the isomorphism above, proving the lemma.

Equipped with the isomorphism given by Lemma 2.3.8 we may mimic the strategy of [LZ14] and [KLZ17] to define a two-variable Perrin-Riou map on $H^1_{I_m}(F(\mu_{p^\infty}), T_{\tilde{f},g})$. This is done in the proof of Theorem 2.3.10 below. Pairing with appropriate $\varphi$-eigenvectors, which we introduce below, we may deform the maps $L_{\tilde{f},f\mu}$ as $f_\alpha$ varies in a Hida family.

Definition 2.3.9.

i) Let $I_f$ be the congruence ideal of $f$ and let $H_f \in I_f$ denote Hida’s congruence divisor.

ii) Let $\eta_f : \mathcal{D}(\mathbb{Q}_p, F - R^*_f) \to \Lambda_f(\Gamma_{cyc})$ be the $\Lambda_f$-morphism given by [KLZ17 Proposition 10.1.1(2)]. Its specialization under $\kappa$ will be denoted by $\eta_\kappa$, which we identify as an element of $\mathcal{D}_{crys}(V_{\tilde{f}_k})$ via duality.

Theorem 2.3.10. For $\mu \in \{\alpha, \beta\}$ and a finite unramified extension $F/\mathbb{Q}_p$, there exists a $\Lambda_f(\Gamma_{cyc})$-morphism

$$L_{\tilde{f},f\mu} : H^1_{I_m}(F(\mu_{p^\infty}), T_{\tilde{f},g}) \to \Lambda_f \hat{\otimes} \mathcal{H}_{ord_p(\mu_\mu)}(\Gamma_{cyc}) \otimes F$$

whose specialization at $\kappa_0$ equals, up to a $p$-adic unit,

$$\frac{1}{\lambda_{N_f}(f)} \left(1 - \frac{\beta_f}{\alpha_f}ight) \left(1 - \frac{\beta_f}{\alpha_f}ight) L_{\tilde{f},f\mu}^{(\alpha,\mu)}$$

where $L_{\tilde{f},f\mu}$ is given as in Definition 2.3.5 and $\lambda_{N_f}(f)$ is the pseudo-eigenvalue of the Atkin–Lehner operator of level $N_f$ (which is given by the identity $W_{N_f,f} = \lambda_{N_f}(f) f^*)$. In particular, the specialization of $L_{\tilde{f},f\mu}$ at $\kappa_0$ agrees with $L_{\tilde{f},f\mu}$ up to multiplication by a $p$-adic unit, if at least one of the following conditions holds true: i) $k_f > 0$, ii) $v_p(\alpha_f - \beta_f/p) = 0$, iii) $\tilde{f}$ is absolutely irreducible.
Proof. Let us write $\varphi^* N(F, T_{f,g}^0)$ for the $A_f^+$-module generated by the image of $n(F, T_{f,g}^0)$ under $\varphi$ and define $\varphi^* N(M)$ similarly if $M$ is a $G_{\mathbb{Q}}$-representation whose Hodge–Tate weights are non-negative. The Perrin-Riou map $L_{F,M}$ of $M$ over $F$ is given as the composite map

$$H^1_{\text{lw}}(F(\mu_{p^\infty}), M) \xrightarrow{\sim} n(F, M)^{\psi=1} \xrightarrow{1-\varepsilon} \varphi^* n(F, M)^{\psi=0}$$

$$\xrightarrow{\left.\right|_{\text{crys}}(M) \otimes (B_{\text{rig},L}^+)^{\psi=0} \otimes F}$$

$$\xrightarrow{1\otimes m^{-1} \otimes 1} \left(\text{crys}(M) \otimes \mathcal{H}(\Gamma_{\text{cyc}}) \otimes F, \right)$$

where $B_{\text{rig},L}^+$ denotes the set of power series in $L[[\pi]]$ that converge on the open unit disk.

Since $\varphi$ acts invertibly on $A_f(\alpha_f^{-1})$, the description of $n(F, T_{f,g}^0)$ in Remark 2.3.7 tells us that

$$\varphi^* n(F, T_{f,g}^0) = A_f(\alpha_f^{-1}) \otimes \varphi^* n(F, R_g^\sigma).$$

Thanks to Lemma 2.3.8 we may define the Perrin-Riou map $L_{F,T_{f,g}^{-\theta}}$ for $T_{f,g}^\theta$ over $F$ as the composite map

$$(2.4)$$

$$H^1_{\text{lw}}(F(\mu_{p^\infty}), T_{f,g}^{-\theta}) \equiv n(F, T_{f,g}^{-\theta})^{\psi=1} \xrightarrow{1-\varepsilon} \varphi^* n(F, T_{f,g}^{\theta})^{\psi=0}$$

$$= \Lambda_f(\alpha_f^{-1}) \otimes_{\mathbb{Z}} \varphi^* n(F, R_g^\sigma)$$

$$\xrightarrow{\left.\right|_{\text{crys}}(M) \otimes (B_{\text{rig},L}^+)^{\psi=0} \otimes F}$$

$$= \left(\text{crys}(T_{f,g}^{-\theta}) \otimes (B_{\text{rig},L}^+)^{\psi=0} \otimes F\right)$$

$$\xrightarrow{1\otimes m^{-1} \otimes 1} \left(\mathcal{D}(\mathbb{Q}, T_{f,g}^\theta) \otimes \mathcal{H}(\Gamma_{\text{cyc}}) \otimes F. \right)$$

We now define $L_{F,T_{f,g}}$ by the compositum of the following arrows

$$H^1_{\text{lw}}(F(\mu_{p^\infty}), T_{f,g}) \xrightarrow{\sim} H^1_{\text{lw}}(F(\mu_{p^\infty}), T_{f,g}^{-\theta}) \xrightarrow{\sim} \mathcal{H}(\Gamma_{\text{cyc}}) \otimes \mathcal{D}(F, T_{f,g}) \otimes F$$

$$\rightarrow \Lambda_f \otimes \mathcal{H}_{\text{ord}}(\mu_{p^\infty}) \otimes \mathcal{C}(\Gamma_{\text{cyc}}) \otimes F,$$

where the first arrow is induced by the natural projection, the second arrow is given by $L_{F,T_{f,g}^{-\theta}}$ and the final arrow is the pairing with $H_f : \eta_f \otimes v_{g,\mu}$.

Given an arbitrary element $z \in H^1_{\text{lw}}(F(\mu_{p^\infty}), R_{f_a}^\sigma \otimes R_g^\sigma)$, set $z_\alpha := (Pr^\alpha)_*(z) \in H^1_{\text{lw}}(F(\mu_{p^\infty}), R_f^\sigma \otimes R_g^\sigma)$. Observe then that

$$L_{F,f_a,g}^{(\alpha, \mu)}(z) := \langle (L_{F,f,g}(z_\alpha), v_{f,\alpha} \otimes v_{g,\mu} \rangle_{(N_f, N_g)}$$

(2.5)

$$= \langle (Pr^\alpha)_* \circ L_{F,f_a,g}(z), v_{f,\alpha} \otimes v_{g,\mu} \rangle_{(N_f, N_g)}$$

$$= \langle L_{F,f_a,g}(z), (Pr^\alpha)^*(v_{f,\alpha}) \otimes v_{g,\mu} \rangle_{(N_f, N_g)}$$

where the first equality follows from the functoriality of the construction of the Perrin-Riou map and the second using the fact that $(Pr^\alpha)_*$ is the transpose of $(Pr^\alpha)^*$. We further note that the subscript $(N_f, N_g)$ (resp., $(N_f, N_g)$) signifies the levels of the pair of modular curves where one realizes $f$ and $g$ (resp., $f_a$ and $g$), on which the Poincaré duality induces the indicated pairing. Let $\eta_f \in D_{\text{crys}}(V_f)/\text{Fil}^1$ be the vector defined as in Corollary 2.2.5. It follows from [KLZ17].
Proposition 10.1.1(2)(b)] that the specialization of $H_f \cdot \eta_f$ at $\kappa$ is given by

$$
\mathcal{L}_f \cdot (\Pr^\alpha)^*(\eta'_f).
$$

The first assertion concerning the specialization of $\mathcal{L}_{E,f,g,\mu}$ at $\kappa_0$ follows on combining this fact with Corollary 2.2.3 and Equation (2.5).

The last assertion of the theorem follows from Lemma 2.3.5 since the constant $\lambda_N(f) \left(1 - \frac{\beta_f}{p\alpha_f}\right) \left(1 - \frac{\beta_f}{\alpha_f}\right)$ is a $p$-adic unit under the stated list of hypotheses. □

The following theorem can be considered as a deformed version of the second half of Proposition 2.2.1.

**Theorem 2.3.11.** Suppose that (FL$_g$) holds. There exist $\Lambda_f(\Gamma_{\text{cyc}})$-morphisms

$$
\text{Col}_{E,f,g,\#}, \text{Col}_{E,f,g,\flat} : H^1_{\text{IW}}(F(T_g), T_g) \to \Lambda_f(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} \mathcal{O}_F
$$

such that

$$
\begin{pmatrix}
\mathcal{L}_{E,f,g,\alpha} \\
\mathcal{L}_{E,f,g,\flat}
\end{pmatrix} = Q_g^{-1} M_g \begin{pmatrix}
\text{Col}_{E,f,g,\#} \\
\text{Col}_{E,f,g,\flat}
\end{pmatrix}.
$$

**Proof.** Let $\{n_{g,1}, n_{g,2}\}$ be the Wach module basis lifting $\{\omega_{g,1}, \varphi(\omega_{g,1})\}$, given by Berger in the proof of [Ber04 Proposition V.2.3]. Then $\{(1 + \pi)\varphi(n_{g,1}),(1 + \pi)\varphi(n_{g,2})\}$ is a $\Lambda_{\mathcal{O}}(\Gamma_{\text{cyc}})$-basis of $\varphi^* \mathbb{N}(R_{g})^{\infty=0}$ by [LLZ10 Theorem 3.5]. Let us define $pr_\#$ and $pr_\flat$ to be the projection maps $\varphi^* \mathbb{N}(R_{g})^{\infty=0} \to \Lambda_{\mathcal{O}}(\Gamma_{\text{cyc}})$ given by this basis. We define for $\bullet \in \{\#, \flat\}$ the map $\text{Col}_{E,f,g,\bullet} : \Lambda_f(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} \mathcal{O}_F$

$$
H^1_{\text{IW}}(F(T_g)) \to \Lambda_f(\alpha_f^{-1}) \otimes_{\mathbb{Z}_p} \varphi^* \mathbb{N}(R_{g})^{\infty=0} \otimes_{\mathbb{Z}_p} \mathcal{O}_F
$$

$$
H_{E,R \otimes \mathcal{O}} \otimes_{\Lambda_{\mathcal{O}}(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} \mathcal{O}_F} \Lambda_f(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} \mathcal{O}_F = \Lambda_f(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} \mathcal{O}_F,
$$

where the first arrow is given by (2.4). The theorem follows from the fact that

$$
\begin{pmatrix}
(1 + \pi)\varphi(n_{g,1}) \\
(1 + \pi)\varphi(n_{g,2})
\end{pmatrix} = (\omega_{g,1}, \varphi(\omega_{g,1})) M_g = (v_{g,\alpha}, v_{g,\flat}) Q_g^{-1} M_g.
$$

□

**Convention 2.3.12.** Instead of $\mathcal{L}_{E,f,g,\mu}$ and $\text{Col}_{E,f,g,\bullet}$, we shall write $\mathcal{L}_{E,g,\mu}$ for $\mu \in \{\alpha, \beta\}$ and $\text{Col}_{E,g,\bullet}$ for $\bullet \in \{\#, \flat\}$, respectively.

**Definition 2.3.13** (Semi-local $\Lambda_f(\Gamma_{\text{cyc}})$-adic Coleman maps). For each $m \in \mathbb{N}$ and $\bullet \in \{\#, \flat\}$, we define the semi-local Coleman maps on setting

$$
\text{Col}_{E,g,\bullet,m} := \bigoplus_{\nu \mid p} \text{Col}_{E,g,\bullet,\nu}.
$$

When $m = 1$, we will simply write $\text{Col}_{E,g,\bullet}$ in place of $\text{Col}_{E,g,\bullet,1}$.

**2.4. The case $a_p(g) = 0$**

In the final portion of our paper, we will study in detail the case where $g$ is the $\theta$-series of a Hecke character of an imaginary quadratic field where $p$ remains inert. In particular, $a_p(g)$ will vanish. The goal of this section is to investigate the construction of Coleman maps under this condition. For the rest of the current section, we replace the hypothesis (FL$_g$) by the following.

(FL$_g$-$S_g$) $p > k_f + 1$ and $a_p(g) = 0$. 

Under [(FL$_{f}$-S$_g$)], we have a very explicit description of $M_g$ in terms of Pollack’s plus and minus logarithms. Furthermore, we may allow $g$ to be outside the Fontaine–Laffaille range. The corresponding objects in Definition 2.2.11 are available under [(FL$_{f}$-S$_g$)], even if $p \leq k_g + 1$, thanks to the work of Berger–Li–Zhu [BLZ04].

**Definition 2.4.1.** We let $\{n_+, n_-\}$ denote the basis of $\mathbb{N}(R_g)$ constructed in [BLZ04] §3.1 (which we have to twist by $\epsilon_q^{-1/2}$ as in §4.2 in op. cit., since the authors constructed a $p$-adic representation with trivial central character in §3.1 of op. cit.). Let $\{v_+, v_-\}$ be the basis of $D_{\text{cris}}(T_g)$ given by $n_\pm \mod \pi$.

As in the proof of Lemma 2.2.9, we note that the representation considered in op. cit. is in fact $R_g^*(-k_g - 1)$. We have the identifications

$$\mathbb{N}(R_g^*) \times_{\pi^{k_g+1}} \mathbb{N}(R_g^*(-k_g - 1)),$$

$$D_{\text{cris}}(R_g^*) \times_{\epsilon_q^{k_g+1}} D_{\text{cris}}(R_g^*(-k_g - 1)).$$

The matrix of $\varphi|_{D_{\text{cris}}(R_g^*)}$ with respect to the basis $\{v_+, v_-\}$ is given by

$$\begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix},$$

where $\eta = \epsilon_q(p)^{-1/2}$, whereas the matrix of $\varphi|_{\mathbb{N}(R_g^*)}$ with respect to $\{n_+, n_-\}$ is given by

$$\begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}.$$ 

Note that $v_+$ generates $\text{Fil}^0 D_{\text{cris}}(R_g^*)$. Therefore, upon multiplying $n_+$ by a suitable unit in $\mathcal{O}$, we may arrange that $\omega_{g^*} = v_+$.

**Definition 2.4.2.** We set $n_{g,1} := u \cdot n_+$, where $u \in \mathcal{O}_k$ is chosen so that $n_{g,1} \equiv \omega_{g^*} \mod \pi$. Furthermore, we put $n_{g,2} := u n_-$ and also define $v_{g,i} := n_{g,i}$ mod $\pi$ for $i = 1, 2$.

i) We let $A'_g$ and $P'_g$ denote the matrices of $\varphi|_{D_{\text{cris}}(R_g^*)}$ and $\varphi|_{\mathbb{N}(R_g^*)}$ with respect to the bases $\{v_{g,1}, v_{g,2}\}$ and $\{n_{g,1}, n_{g,2}\}$, respectively. More explicitly, we have

$$A'_g = \begin{pmatrix} 0 & 1 \\ -\frac{\epsilon_q(p)}{p} \end{pmatrix}$$

and

$$P'_g = \begin{pmatrix} 0 & 1 \\ -\frac{\epsilon_q(p)}{p} \end{pmatrix}.$$

ii) We finally define the logarithmic matrix associated to the basis $\{n_{g,1}, n_{g,1}\}$:

$$M'_g = m^{-1} \left( \lim_{n \to \infty} (1 + \pi)(A'_g)^{n+1} \varphi^n(P'_g)^{-1} \cdots \varphi(P'_g)^{-1} \right).$$

**Remark 2.4.3.**

i) It follows from the calculations in [LLZ10] §5.2.1 that

$$M'_g = \begin{pmatrix} a_+^{-1} & \frac{a_+^{-1}}{p} \log - \log_{p, k_g + 1} \\ a_+ \log^{+}_{p, k_g + 1} & 0 \end{pmatrix},$$

where $a_+ \in \Lambda_\mathcal{O}(\Gamma_1)^\infty$. 


ii) Note that $A'_f$ coincides with $A_f$ given in Definition 2.2.11. However, the matrix $P'_f$ differs from $P_f$ given in the proof of Proposition 2.2.13 slightly (one of the entries would differ by the scalar $\xi_{k_f+1}$). The resulting matrix $M'_g$ is therefore slightly different from $M_g$. But it can be shown under the hypothesis (FLf-Sg) that the matrix $M_g$ also has the form

$$M_g = \begin{pmatrix} 0 & b_g^+ 
 b_g^+ \log_{p,k_g+1}^- & 0 \end{pmatrix},$$

where $b_g^+ \in \Lambda_G(\Gamma_1)^\times$.

iii) Under the hypothesis (FLf-Sg), the matrix $Q_g$ simplifies to

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\
\alpha_g & -\alpha_g \end{pmatrix}.$$
Let $Q_{f,g}$ be the change of basis given as in Definition 2.2.13(iii). As in Proposition 2.2.15, the basis $\{n'_i : i = 1, \cdots, 4\}$ determines the following Coleman maps.

**Proposition 2.4.6.** If $F$ is a finite unramified extension of $\mathbb{Q}_p$, there exist bounded Coleman maps $\text{Col}^{(\omega,+)}_{F,f,g}$, $\text{Col}^{(\omega,-)}_{F,f,g}$, $\text{Col}^{(\eta,+)}_{F,f,g}$ and $\text{Col}^{(\eta,-)}_{F,f,g}$ with source $H^1_{\text{Iw}}(F(\mu_p^\infty), T)$ and target $\Lambda_{\mathcal{O}}(\Gamma_{\text{cyc}}) \otimes \mathcal{O}_F$, which satisfy

$$
\begin{pmatrix}
L^{(\alpha,\alpha)}_{F,f,g} \\
L^{(\alpha,\beta)}_{F,f,g} \\
L^{(\beta,\alpha)}_{F,f,g} \\
L^{(\beta,\beta)}_{F,f,g}
\end{pmatrix} = Q_{f,g}^{-1} M'_{f,g} 
\begin{pmatrix}
\text{Col}^{(\omega,+)}_{F,f,g} \\
\text{Col}^{(\omega,-)}_{F,f,g} \\
\text{Col}^{(\eta,+)}_{F,f,g} \\
\text{Col}^{(\eta,-)}_{F,f,g}
\end{pmatrix}.
$$

In particular,

$$
\begin{pmatrix}
L^{(\alpha,\alpha)}_{F,f,g} \\
L^{(\alpha,\beta)}_{F,f,g}
\end{pmatrix} = u'_f Q_{g}^{-1} M'_{g} 
\begin{pmatrix}
\text{Col}^{(\omega,+)}_{F,f,g} \\
\text{Col}^{(\omega,-)}_{F,f,g}
\end{pmatrix}.
$$

We note that the construction of the $\Lambda_f(\Gamma_{\text{cyc}})$-adic Perrin-Riou maps in Theorem 2.3.10 does not rely on the hypothesis $\mathbf{(FL_g)}$ and can be carried over under the following weaker version of $\mathbf{(FL_f-S_g)}$:

$\mathbf{(S_g)}$ $g$ satisfies $a_g(g) = 0$,

yielding the maps $L_{F,f,g,\alpha}$ and $L_{F,f,g,\beta}$. The proof of Theorem 2.3.11 also generalizes easily to yield the following:

**Theorem 2.4.7.** Suppose $\mathbf{(S_g)}$ holds. There exist a pair of $\Lambda_f(\Gamma_{\text{cyc}})$-morphisms

$$
\text{Col}^{F,f,g,\pm}_*: H^1_{\text{Iw}}(F(\mu_p^\infty), T_{f,g}) \rightarrow \Lambda_f(\Gamma_{\text{cyc}}) \otimes \mathcal{O}_F
$$

that verify the factorization

$$
\begin{pmatrix}
L^{(\alpha,\alpha)}_{F,f,g} \\
L^{(\alpha,\beta)}_{F,f,g}
\end{pmatrix} = Q_{g}^{-1} M'_{g} 
\begin{pmatrix}
\text{Col}^{F,f,g,\alpha}_{F,f,g} \\
\text{Col}^{F,f,g,\beta}_{F,f,g}
\end{pmatrix}.
$$
Beilinson–Flach elements and \( p \)-adic \( L \)-functions

In this chapter, we recall the unbounded Beilinson–Flach elements for the representations \( T_{f,g} = R_f^* \otimes R_g^* \) and explain how they can be decomposed into bounded elements via the logarithmic matrices defined in Chapter 2. Furthermore, we study how these elements are related to various \( p \)-adic \( L \)-functions via the Perrin–Riou maps and Coleman maps given in the previous chapter.

### 3.1. Beilinson–Flach elements

We begin this section by explaining how to obtain \( \Lambda \)-adic Beilinson–Flach elements for \( T_{f,g} \) utilizing results in [KLZ17, LZ16]. Our arguments follow very closely those presented in [BL20b, §3] (with the role of \( f \) and \( g \) in op. cit. played by \( g \) and \( f \) respectively).

**Definition 3.1.1.** Suppose \( \mu \in \{ \alpha, \beta \} \) and set \( N = \text{lcm}(N_f, N_g) \). Let \( M \geq 1 \) be an integer such that \( \mathfrak{c} \) is coprime to \( 6pMN \).

i) We write \( z_{f,\mu}^{M,j} \in H^1(\mathbb{Q}(\mu M), R_f^* \otimes R_g^* \mu(-j)) \) for the image of the Rankin–Iwasawa class \( RI_{M,pMN,1}^{\mu} \) of [LZ16, Definition 3.2.1] under the compositum of the arrows

\[
H^3_{\text{et}} \left( \mathcal{Y}(M,pMN)^2, \Lambda(\mathbb{H}_{\mathbb{Z}_p}(t))^{j_0}(2 - j) \right) \\
\longrightarrow H^1 \left( \mathbb{Q}(\mu M), H^1_{\text{et}} \left( \mathbb{Y}(pMN), \Lambda(\mathbb{H}_{\mathbb{Z}_p}(t))^{j_0}(2 - j) \right) \right) \\
\longrightarrow H^1(\mathbb{Q}(\mu M), H^1_{\text{et}}(\mathbb{Y}(pMN), \Lambda(\mathbb{H}_{\mathbb{Z}_p}(t))(1)) \boxtimes H^1_{\text{et}}(\mathbb{Y}(pMN), \text{TSym}^k \mathbb{H}_{\mathbb{Z}_p}(1)) (-j)) \\
\longrightarrow H^1(\mathbb{Q}(\mu M), H^1_{\text{ord}}(Np^\infty)^{[ka]} \otimes R_{g_{\mu}}^*(-j)) \\
\longrightarrow H^1(\mathbb{Q}(\mu M), R_f^{*,[ka]} \otimes R_{g_{\mu}}^*(-j)),
\]

which are given as in [BL20b (17)], composed with the projection

\[
H^1_{\text{ord}}(Np^\infty)^{[ka]} \longrightarrow R_f^{*,[ka]},
\]

where \( H^1_{\text{ord}}(Np^\infty)^{[ka]} \) is defined as in §3.1 of op. cit. Here, the superscript \( [ka] \) means that we have twisted the representation \( R_f \) by the \( k \)-th power of the weight character. We also note that we have dropped the left-subscript \( \mathfrak{c} \) from the notation (see Remark 3.1 of op. cit.).

ii) When \( M = mp^n \) where \( n \geq 1 \), we define \( x_{M,j}^{f \mu} \in H^1(\mathbb{Q}(\mu M), R_f^{*,[ka]} \otimes W^*_g(-j)) \) to be the image of

\[
\frac{(U_p')^{-n} \times (U_p')^{-n}}{(-1)^j j! (\lambda')^j} RI_{M,pMN,1}
\]

under the same series of maps in (i).
Theorem 3.1.2. Let $m \in \mathcal{N}$ and $\mu \in \{\alpha, \beta\}$. There exists a unique element
\[
\mathcal{B}F_{f,m}^\mu \in H^1 \left( \mathbb{Q}(\mu_m), R_\mathbb{T}^{[k_g]} \otimes R_g^\ast \otimes \mathcal{H}_{\text{ord}(\mu)}(\Gamma_{\text{cyc}})^\dagger \right)
\]
such that its image in $H^1 \left( \mathbb{Q}(\mu_{mp^n}), R_\mathbb{T}^{[k_g]} \otimes W_{g_\ast}^\ast (-j) \right)$ equals $x_{mp^n,j}^\mu$ for all $0 \leq j \leq k_g$ and $n \geq 1$.

Proof. This follows from the same proof as [BL20]. Note that the requirement in the equation (18) of op. cit. translates to
\[
H^0(\mathbb{Q}(\mu_{mp^n}), R_\mathbb{T}^{[k_g]} \otimes R_g^\ast (-j)) = 0,
\]
which follows as a consequence of the fact that $g$ is non-ordinary at $p$, whereas $f$ is a $p$-ordinary family.

Definition 3.1.3. We define
\[
\mathcal{B}F_{f,g,\mu,m} \in H^1 \left( \mathbb{Q}(\mu_m), R_\mathbb{T}^{[k_g]} \otimes R_g^\ast \otimes \mathcal{H}_{\text{ord}(\mu)}(\Gamma_{\text{cyc}})^\dagger \right)
\]
as the image of $\mathcal{B}F_{m}^\mu$ given in Theorem 3.1.2 under the weight-twisting morphism
\[
H^1 \left( \mathbb{Q}(\mu_m), R_\mathbb{T}^{[k_g]} \otimes R_g^\ast \otimes \mathcal{H}_{\text{ord}(\mu)}(\Gamma_{\text{cyc}})^\dagger \right) \rightarrow H^1 \left( \mathbb{Q}(\mu_m), R_\mathbb{T} \otimes R_g^\ast \otimes \mathcal{H}_{\text{ord}(\mu)}(\Gamma_{\text{cyc}})^\dagger \right).
\]

We recall from [BLLV19] Theorem 5.4.1 that the logarithmic matrix constructed using Berger’s Wach module basis when both $f$ and $g$ are $p$-non-ordinary leads to a partial splitting of unbounded Beilinson–Flach elements associated to the Rankin–Selberg products. The reason why we fell short of establishing the full splitting Conjecture 5.3.1 in op. cit. is the lack of information on the Beilinson–Flach elements associated to the twists $T_{f,g}(-j)$ for $j > \max(k_f, k_g)$. When $f$ is $p$-ordinary, we may allow $f_\alpha$ to vary in the Hida family $f$ to by-pass this restriction imposed by the weights and obtain a full decomposition of the unbounded Beilinson–Flach elements $\{\mathcal{B}F_{f,g,\mu,m} \mid m \in \mathcal{N}, \mu \in \{\alpha, \beta\}\}$ into bounded ones:

Theorem 3.1.4.

i) Suppose that ([FL1]) holds. For $m \in \mathcal{N}$, there exists a pair of cohomology classes
\[
\mathcal{B}F_{f,g,\#} \in H^1 \left( \mathbb{Q}(\mu_m), R_\mathbb{T} \otimes R_g^\ast \otimes \Lambda_{\mathcal{O}}(\Gamma_{\text{cyc}})^\dagger \right)
\]
such that
\[
\left( \begin{array}{c}
\mathcal{B}F_{f,g,\alpha,m} \\
\mathcal{B}F_{f,g,\beta,m}
\end{array} \right) = Q_g^{-1}M_g \left( \begin{array}{c}
\mathcal{B}F_{f,g,\#} \\\n\mathcal{B}F_{f,g,\#,m}
\end{array} \right).
\]

Here, $s(g)$ is a natural number that depends on $k_g$ but is independent of $m$.

ii) Suppose that ([S1]) holds. Then the statement holds if we replace $M_g$ by $M'_g$ and $\#$ by $\#'$, $+$ and $-$ respectively.

Proof. Let us first consider part (i). It is enough to show that $\mathcal{B}F_{m}^{\alpha}$ and $\mathcal{B}F_{m}^{\beta}$ verify the analogous factorization. In more precise wording, it suffices to show that there exists a pair
\[
\mathcal{B}F_{m}^{\#}, \mathcal{B}F_{m}^{\dagger} \in H^1 \left( \mathbb{Q}(\mu_m), R_\mathbb{T}^{[k_g]} \otimes R_g^\ast \otimes \Lambda_{\mathcal{O}}(\Gamma_{\text{cyc}})^\dagger \right)
\]
such that

\[
\left( B_{F_m}^{\alpha}, B_{F_m}^{\beta} \right) = Q_g^{-1} M_g \left( B_{F_m}^{\alpha}, B_{F_m}^{\beta} \right) .
\]

This follows from the same proof as [BL20b] Theorem 3.7, where the Hida family there is assumed to be CM, but the CM condition in fact plays no role in the argument. We overview the the main steps of the proof of [BL20b] Theorem 3.7.

For \( i = 0, \ldots, k_g \) and \( \mu \in \{ \alpha, \beta \} \), consider the cocycle

\[
c_n^{f, \mu} \in Z^1 \left( G_{Q(\mu_m^\infty)}, R_f^{[k_g]} \otimes R_g^{*} \otimes \mathcal{O}[\Gamma / \mathbb{P}^{n-1}] \right)
\]

lifting res \( \left( \mu_n^n x_{mp^n, \beta}^{f, \mu} \right) \) as in op. cit., which satisfies

\[
\left( f, 1 - j \right) \sum_{i=0}^{j} (-1)^i \cdot \left( \begin{array}{c} j \end{array} \right) 1 \otimes Tw^{-i} \cdot c_n^{f, \mu} \leq 1.
\]

Using Lemma 2.3 and Remark 2.4 in op. cit., we obtain the bounded cocycle

\[
c_n^{f, \mu} \in Z^1 \left( G_{Q(\mu_m^\infty)}, R_f^{[k_g]} \otimes R_g^{*} \otimes \mathcal{O}[\Gamma / \mathbb{P}^{n-1}] \right)
\]

whose image modulo \( Tw^{-i} \cdot \omega_{n-1} \) is \( c_n^{f, \mu} \). This in turn gives rise to a pair of cohomology classes

\[
\mathcal{O}^{\omega_0} \cdot \mu_n^n x_{mp^n, \beta}^{f, \mu} \in H^1(\mathbb{Q}(\mu_m^\infty), R_f^{[k_g]} \otimes R_g^{*} \otimes \mathcal{O}[\Gamma / \mathbb{P}^{n-1}], \omega_{n-1}^{\mu} \cdot c_n^{f, \mu} \).
\]

\( \mu \in \{ \alpha, \beta \} \).

Let \( C_{n-1} \) denote the matrix

\[
m^{-1} \left( (1 + \pi)^{n-1}(p_g^{-1}) \cdots (p_g^{-1}) \right) .
\]

The interpolative properties of the Rankin–Iwasawa classes given by Corollary 3.4 in op. cit. allow us to apply [BL20b] Proposition 2.10 to obtain a pair of bounded classes

\[
x_{#, mp^n}, x_{\beta, mp^n} \in H^1 \left( \mathbb{Q}(\mu_m^\infty), R_f^{[k_g]} \otimes R_g^{*} \otimes \mathcal{O}[\Gamma / \mathbb{P}^{n-1}] \right)
\]

(\( h_{n-1} \) denotes the map \( \mathcal{O}[\Gamma / \mathbb{P}^{n-1}] \otimes \mathcal{O}[\Gamma / \mathbb{P}^{n-1}] \), given by the matrix \( C_{n-1} \), satisfying

\[
\left( \begin{array}{c} \alpha_n^{x_{mp^n}^{f, \alpha}} \\
\beta_n^{x_{mp^n}^{f, \beta}} \end{array} \right) = Q_g^{-1} C_{n-1} \left( \begin{array}{c} \alpha_n^{x_{mp^n}^{f, \alpha}} \\
\beta_n^{x_{mp^n}^{f, \beta}} \end{array} \right) .
\]

The result then follows on passing to limit in \( n \) and setting \( B_{F_{m}}^{f, \bullet} := \lim_n x_{mp^n}^{f, \bullet} \) for \( \bullet \in \{ #, \beta \} \) with \( s(g) = s_0(g) + s_1(g) \).

We now consider part (ii). As in the first portion, for each \( \mu \in \{ \alpha, \beta \} \), we have bounded cocycles \( c_n^{f, \mu} \) interpolating res \( \left( \mu_n^n x_{mp^n, \beta}^{f, \mu} \right) \) since the inequality \( \mathcal{O}^{\omega_0} \cdot c_n^{f, \mu} \) is still valid. Observe in addition that [BL20b] Corollary 3.4 also holds without the hypothesis [FL20b]. Notice that we now have \( \alpha_g = -\beta_g \). This tells us that for \( 1 \leq m \leq n-1 \), the cocycle

\[
c_n^{f, \alpha} \pm c_n^{f, \beta} \in Z^1 \left( G_{Q(\mu_m^\infty)}, R_f^{[k_g]} \otimes R_g^{*} \otimes \mathcal{O}[\Gamma / \mathbb{P}^{n-1}] \right)
\]

\( \mu \in \{ \alpha, \beta \} \).
is divisible by the polynomial $\Phi_{m,k_g+1}$, where the parity of $m$ determines the sign above. This gives rise to a pair of bounded cocycles $c_{n}^{f,\pm}$ satisfying

$$
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
c_{n}^{f,\alpha} \\
c_{n}^{f,\beta}
\end{pmatrix} =
\begin{pmatrix}
\omega_{n,k_g+1}^{f,+}c_{n}^{f,+} \\
\omega_{n,k_g+1}^{f,-}c_{n}^{f,-}
\end{pmatrix},
$$

where $\omega_{n,k_g+1}^{f,\pm} = \prod \Phi_{n,k_g+1}$ with the products running over all even (respectively odd) integers between 1 and $n - 1$ for the sign $+$ (respectively $-$).

It follows from Remark 2.4.3 that

$$
Q_{g}^{-1}M'_{g} = \frac{1}{\alpha_{g}^{n}}\begin{pmatrix}
\sigma_{n}^{+}\omega_{n,k_g+1}^{+} & \sigma_{n}^{+}\omega_{n,k_g+1}^{+} \\
\sigma_{n}^{-}\delta_{n}\omega_{n,k_g+1}^{+} & \sigma_{n}^{+}\omega_{n,k_g+1}^{+}
\end{pmatrix} \mod \omega_{n-1,k_g+1},
$$

where $\sigma_{n}^{\pm}$ are bounded polynomials. Combining this with (3.2) gives

$$
\begin{pmatrix}
c_{n}^{f,\alpha} \\
c_{n}^{f,\beta}
\end{pmatrix} = Q_{g}^{-1}M'_{g} \begin{pmatrix}
d_{n}^{f,\pm} \\
d_{n}^{f,\pm}
\end{pmatrix},
$$

where $d_{n}^{f,\pm}$ are bounded cocycles. This in turn produces a pair of bounded cohomology classes $x_{mp,n}^{f,\pm}$ represented by $d_{mp,n}^{f,\pm}$ which verify

$$
\begin{pmatrix}
x_{mp,n}^{f,\alpha} \\
x_{mp,n}^{f,\beta}
\end{pmatrix} = Q_{g}^{-1}M'_{g} \begin{pmatrix}
x_{mp,n}^{f,+} \\
x_{mp,n}^{f,-}
\end{pmatrix}.
$$

The result now follows on setting $BF_{f,g,\pm,m}$ as the limit of $x_{mp,n}^{f,\pm}$ as $n$ tends to infinity. 

**Definition 3.1.5.** Suppose $\bullet \in \{\# , \flat\}$ under the hypothesis (FL$_{g}$) or else $\bullet \in \{+ , -\}$ under (S$_{g}$). We shall write $BF_{f,g,m}^{(\alpha,\bullet)}$ for the specialization of the element $BF_{f,g,\bullet,m}$ under $\kappa_{f,m} : \Lambda_{f} \to \mathcal{O}$ associated to the member $f_{\alpha}$ of the Hida family $f$.

We remark that we then automatically deduce from Theorem 3.1.4 the factorizations

$$
\begin{pmatrix}
BF_{f,g,m}^{(\alpha,\alpha)} \\
BF_{f,g,m}^{(\alpha,\beta)}
\end{pmatrix} = Q_{g}^{-1}M_{g} \begin{pmatrix}
BF_{f,g,m}^{(\alpha,\#)} \\
BF_{f,g,m}^{(\alpha,\flat)}
\end{pmatrix},
\begin{pmatrix}
BF_{f,g,m}^{(\alpha,\alpha)} \\
BF_{f,g,m}^{(\alpha,\beta)}
\end{pmatrix} = Q_{g}^{-1}M'_{g} \begin{pmatrix}
BF_{f,g,m}^{(\alpha,+) \\
BF_{f,g,m}^{(\alpha,-)}
\end{pmatrix},
$$

for every specialization $f_{\alpha}$ of the Hida family $f$ and integer $m \in N$.

### 3.2. Euler systems of rank 2 and uniform integrality

The goal of the current section is to analyze the variation of exponent $s(g)$ with the eigenform $g$. Our main result in this vein is Corollary 3.2.4 below.

This analysis is relevant to our attempt towards the Iwasawa main conjectures for $f_{/K} \otimes \psi$, where $K$ is an imaginary quadratic field where $p$ is inert, $f_{/K}$ is the “base-change” of $f$ to $K$ and $\psi$ is a Hecke character of $K$. See §3.3 for a detailed discussion, more particularly Remark 5.3.7 and §§5.3.8, 5.3.9 for the relevance of our discussion in §3.2.
3.2. Euler systems of rank 2 and uniform integrality

The following conjecture, which is in line with Perrin-Riou’s philosophy \([\text{PR98}]\), is the \(\Lambda\)-adic version of [BLV19 Conjecture 3.5.1]. We retain the notation of \([\S2.3]\) and the present \([\S3.1]\).

**Conjecture 3.2.1.** Suppose \(m \in \mathcal{N}\). There exists a unique element

\[
\mathbb{B}^{F,g}_m \in \bigwedge^2 H^1 \left( \mathbb{Q}(\mu_m), T\mathbb{F}_g \bar{\otimes}_{\Lambda} (\Gamma_{\text{cyc}})^{\dagger} \right)
\]

such that for \(\mu = \alpha, \beta\), we have

\[
j_\mu \circ \mathcal{L}_{\mathbb{Q}(\mu_m), f,g,\mu} \left( \mathbb{B}^{F,g}_m \right) = \mathbb{B}F_{f,g,\mu,m}.
\]

Here, we have regarded the functional \(\mathcal{L}_{\mathbb{Q}(\mu_m), f,g,\mu}\) as a map

\[
\bigwedge^2 H^1 \left( \mathbb{Q}(\mu_m), T\mathbb{F}_g \bar{\otimes}_{\Lambda} (\Gamma_{\text{cyc}})^{\dagger} \right) \xrightarrow{\mathcal{L}_{\mathbb{Q}(\mu_m), f,g,\mu}} H^1 \left( \mathbb{Q}(\mu_m), T\mathbb{F}_g \bar{\otimes}_{\Lambda} (\Gamma_{\text{cyc}})^{\dagger} \right) \otimes \mathcal{H}_{\text{ord}(\mu)} (\Gamma_{\text{cyc}})
\]

\[
\mathcal{L}_{\mathbb{Q}(\mu_m), f,g,\mu}(x \otimes y) := \mathcal{L}_{\mathbb{Q}(\mu_m), f,g,\mu} (\text{res}_p(x)) \cdot y - \mathcal{L}_{\mathbb{Q}(\mu_m), f,g,\mu} (\text{res}_p(y)) \cdot x
\]

and \(j_\mu\) is stands for the compositum of the arrows

\[
\begin{align*}
H^1 \left( \mathbb{Q}(\mu_m), T\mathbb{F}_g \bar{\otimes}_{\Lambda} (\Gamma_{\text{cyc}})^{\dagger} \right) \otimes \mathcal{H}_{\text{ord}(\mu)} (\Gamma_{\text{cyc}}) &\xrightarrow{id \times (\text{pr}^{\dagger})^*} H^1 \left( \mathbb{Q}(\mu_m), T\mathbb{F}_g \bar{\otimes}_{\Lambda} (\Gamma_{\text{cyc}})^{\dagger} \right) \xrightarrow{\mathcal{L}_{\mathbb{Q}(\mu_m), f,g,\mu}} \\
H^1 \left( \mathbb{Q}(\mu_m), T\mathbb{F}_g \bar{\otimes}_{\Lambda} (\Gamma_{\text{cyc}})^{\dagger} \right) \otimes \mathcal{H}_{\text{ord}(\mu)} (\Gamma_{\text{cyc}}) &\rightarrow
\end{align*}
\]

See [BL19, BO20] for results in support of this conjecture.

Assuming the validity of Conjecture [3.2.1], one may define the following pair of signed Euler systems without going through the technical difficulties one needs to overcome in the proof of Theorem [3.1.4].

**Definition 3.2.2.** We assume the truth of Conjecture [3.2.1].

i) Suppose that \([\text{FL}_g]\) holds and \(m \in \mathcal{N}\). For the pair of signed Coleman maps \(\text{Col}_{\mathbb{Q}(\mu_m), f,g,?}\) (with \(? = \#, \flat\)) given as in Theorem [2.3.11], we define the element

\[
\mathbb{B}^{F,g}_m \mathbb{F}_{g,?} := \text{Col}_{\mathbb{Q}(\mu_m), f,g,?} \left( \mathbb{B}^{F,g}_m \right) \in H^1 \left( \mathbb{Q}(\mu_m), T\mathbb{F}_g \bar{\otimes}_{\Lambda} (\Gamma_{\text{cyc}})^{\dagger} \right)
\]

ii) Suppose that \([\text{S}_g]\) holds and \(m \in \mathcal{N}\). For the pair of signed Coleman maps \(\text{Col}_{\mathbb{Q}(\mu_m), f,g,?}\) (with \(? = +, -\)) given as in Theorem [2.4.7], we define the element

\[
\mathbb{B}^{F,g,?}_m := \text{Col}_{\mathbb{Q}(\mu_m), f,g,?} \left( \mathbb{B}^{F,g}_m \right) \in H^1 \left( \mathbb{Q}(\mu_m), T\mathbb{F}_g \bar{\otimes}_{\Lambda} (\Gamma_{\text{cyc}})^{\dagger} \right)
\]

One may use Theorem [2.3.11] and Theorem [2.4.7] to easily prove that the elements \(\{\mathbb{B}^{F,g,?}_m\}\) verify the following factorization statement (which should be compared to the conclusions of Theorem [3.1.4]).

**Proposition 3.2.3.** We assume the truth of Conjecture [3.2.1].

i) Suppose that \([\text{FL}_g]\) holds. For every \(m \in \mathcal{N}\),

\[
\begin{pmatrix}
\mathbb{B}F_{f,g,\alpha,m} \\
\mathbb{B}F_{f,g,\beta,m}
\end{pmatrix}
= Q_{g}^{-1} M_g \begin{pmatrix}
\mathbb{B}^{F,g,\#,m} \\
\mathbb{B}^{F,g,\flat,m}
\end{pmatrix}.
\]


ii) Suppose that \((S_p)\) holds. For every \(m \in N\), we have

\[
\begin{pmatrix}
\text{BF}_{f,g,a,m} \\
\text{BF}_{f,g,b,m}
\end{pmatrix}
\begin{pmatrix}
\text{BF}_{f,g,+}^{f,g,\alpha,m} \\
\text{BF}_{f,g,-}^{f,g,\beta,m}
\end{pmatrix}
= Q_g^{-1}M_p^{f,g,\alpha,m}.
\]

**Corollary 3.2.4.** Assume the truth of Conjecture \(3.3.1\) Then,

\[
\text{BF}_{f,g,\alpha,m} = \text{BF}_{f,g,\beta,m}^{f,g,?}. \quad ? = \#^+, b, +, - \quad \text{and} \quad m \in N.
\]

In particular, the exponents \(s(g)\) in the statement of Theorem \(3.1.4\) are bounded independently of \(g\).

**Proof.** This immediate on comparing the identities we have verified in Theorem \(3.1.4\) (concerning the elements \(\text{BF}_{f,g,\alpha,m}\)) with those in Proposition \(3.2.3\) (concerning the elements \(\text{BF}_{f,g,\beta,m}\)). \(\square\)

### 3.3. \(p\)-adic \(L\)-functions

We next study the link between Beilinson–Flach classes to \(p\)-adic \(L\)-functions. By an abuse of notation, we shall identify the Beilinson–Flach classes with their images under the \(p\)-localization map. We first prove the preliminary Lemmas \(3.3.1\) and \(3.3.3\).

**Lemma 3.3.1.** Suppose \(m \in N\).

i) For \(\mu \in \{\alpha, \beta\}\), we have

\[
\mathcal{L}^{(\alpha,\mu)}_{f,g,m} \left( \text{BF}^{(\alpha,\mu)}_{f,g,m} \right) = 0.
\]

ii) Suppose that \(k_f \geq k_g\), then

\[
\mathcal{L}^{(\alpha,\alpha)}_{f,g,m} \left( \text{BF}^{(\alpha,\beta)}_{f,g,m} \right) = -\mathcal{L}^{(\alpha,\beta)}_{f,g,m} \left( \text{BF}^{(\alpha,\alpha)}_{f,g,m} \right) \in \log_{p,k_g+1} \Lambda_O(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Q}} \mathbb{Q}(m)_p,
\]

where \(\mathbb{Q}(m)_p = \mathbb{Q}(m) \otimes_{\mathbb{Q}} \mathbb{Q}_p\).

**Proof.** Part (i) follows from [LZ16, Theorem 7.1.2].

For part (ii), note that the eigenvectors \(v_{f,\lambda}\) and \(v_{g,\mu}\) given in Definition \(2.1.1\) differ from the ones given in [BLV19, §3.5] by fixed constants that are independent of \(\lambda\) and \(\mu\). Therefore, the stated equality follows from the same proof as the last assertion of Theorem \(3.6.5\) in op. cit.

Let us put \(\mathcal{L} = \mathcal{L}^{(\alpha,\alpha)}_{f,g,m} \left( \text{BF}^{(\alpha,\beta)}_{f,g,m} \right)\). An analysis of the denominators of the Perrin-Riou map and the Beilinson–Flach class shows that \(\mathcal{L} \in \mathcal{H}_{k_g+1} \mathcal{O}(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}(m)_p\). It therefore remains to show that \(\mathcal{L}\) is divisible by \(\log_{p,k_g+1}\). To do so, it suffices to show that \(\mathcal{L}\) vanishes at all characters of the form \(\chi^j\theta\), where \(0 \leq j \leq k_g\) and \(\theta\) is a finite Dirichlet character on \(\Gamma\). Let \(F = \mathbb{Q}(m)_v\), where \(v\) is a prime of \(\mathbb{Q}(m)_p\) above \(p\). Note that the natural image of \(\text{BF}^{(\alpha,\mu')}_{f,g,m}\) in \(H^1(F(\mu_{p^n}), T(-j) \otimes_{\mathcal{O}} \mathcal{L})\) falls within \(H^1_E(F(\mu_{p^n}), T(-j) \otimes_{\mathcal{O}} L)\) for all \(n \geq 0\) and \(0 \leq j \leq k_g\) (by [KLZ17, Proposition 3.3.3] and that \(H^1_E = H^1\) for the representation \(T(-j) \otimes_{\mathcal{O}} L\). The interpolative property of the Perrin-Riou map then shows that \(\mathcal{L}\) does vanish at the aforementioned set of characters. \(\square\)

We henceforth work under the following additional hypothesis on \(\Lambda_\mathcal{F}\): (Reg) \(\Lambda_\mathcal{F}\) is a regular ring.
Remark 3.3.2. Note that if one is content to work over a sufficiently small open disc in the weight space (rather than the entire weight space itself), this condition on the coefficients can be ensured by replacing \( \Lambda_f \) with the restriction of the universal Hecke algebra to this open disc.

Under \((\text{Reg})\) we have the following \( \Lambda_f \)-adic version of Lemma 3.3.1.

**Lemma 3.3.3.** Suppose that \((\text{Reg})\) holds. Let \( m \in \mathbb{N} \).

i) For \( \mu \in \{ \alpha, \beta \} \), we have

\[
L_{t,g,\mu, m}(BF_{t,g,\mu, m}) = 0.
\]

ii) We have

\[
L_{t,g,\alpha, m}(BF_{t,g,\alpha, m}) = -L_{t,g,\beta, m}(BF_{t,g,\alpha, m}) \in \log_{p,k_g+1} \Lambda_f(\Gamma_{\text{cyc}}) \otimes \mathbb{Z}_p \mathbb{Q}(m)_p.
\]

**Proof.** Let \( X_{\text{cr}}^{(k_g)} \subset \text{Spec}(\Lambda_f) \) denote the set of prime ideals of the form \( P_k = \ker(\kappa) \), where \( \kappa \) is a crystalline (necessarily classical) specialization of \( \Lambda_f \) such that the weight of the eigenform \( f_k \) is at least \( k_g + 2 \). Define \( J \) to be the set of all finite intersections of elements of \( X_{\text{cr}}^{(k_g)} \). Since \( X_{\text{cr}}^{(k_g)} \) is Zariski dense in \( \text{Spec}(\Lambda_f) \), hypothesis \((\text{Reg})\) tells us that

\[
\Lambda_f = \varinjlim_{I \in J} \Lambda_f/I.
\]

Let \( v \) be a fixed prime of \( \mathbb{Q}(m) \) above \( p \) and write \( F = \mathbb{Q}(m)_v \). For \( I \in J \), let

\[
F^{-}R_{t}^{\ast} = F^{-}R_{t} \otimes \mathbb{Q}(m)_{F,I} \quad \text{and write}
\]

\[
L_{F,T_{t}^{\ast}} : H^1_{\text{tw}}(F(\mu_{p_{\infty}}), T_{t}^{-, \emptyset}) \rightarrow \mathbb{D}_{\text{cris}}(T_{t}^{-, \emptyset}) \otimes \mathcal{H}(\Gamma_{\text{cyc}}) \otimes F
\]

for the Perrin-Riou map of \( F^{-}R_{t}^{\ast} \) over \( F \). By Remark 2.3.7 and the fact that \( \varphi \) acts invertibly on \( \Lambda_f(\alpha_f^{-1}) \), we have

\[
\varphi^*N(F, T_{t}^{-, \emptyset})^{\psi=0} = \Lambda_f(\alpha_f^{-1}) \otimes_{\mathcal{O}} \varphi^*N(R_{t}^{\ast, \emptyset})^{\psi=0} \otimes \mathbb{Z}_p \mathcal{O}_F
\]

\[
= \left( \lim_{I} \Lambda_f(\alpha_f^{-1})/I \right) \otimes_{\mathcal{O}} \varphi^*N(R_{t}^{\ast, \emptyset})^{\psi=0} \otimes \mathbb{Z}_p \mathcal{O}_F.
\]

Therefore, we may realize the Perrin-Riou map \( L_{F,T_{t}^{\ast}} \) in the proof of Theorem 23.4.1 as

\[
\lim_{I} L_{F,T_{t}^{\ast}} : H^1_{\text{tw}}(F(\mu_{p_{\infty}}), T_{t}^{-, \emptyset}) \rightarrow H^1_{\text{tw}}(F(\mu_{p_{\infty}}), T_{t}^{-, \emptyset}) \rightarrow \left( \lim_{I} \mathbb{D}_{\text{cris}}(T_{t}^{-, \emptyset}) \otimes \mathcal{H}(\Gamma_{\text{cyc}}) \otimes F = \mathbb{D}(\mathbb{Q}_p, T_{t}^{-, \emptyset}) \otimes \mathcal{H}(\Gamma_{\text{cyc}}) \otimes F \right)
\]

For \( \mu \in \{ \alpha, \beta \} \), we write \( L^{(\alpha, \mu)}_{F,T_{t}^{\ast}} \) for the map defined by \( L_{F,T_{t}^{\ast}} \) paired with the \( \varphi \)-eigenvector \( (H_{t} \cdot \eta_{t} \mod I) \otimes v_{g, \mu} \). Then, on taking inverse limits again, we have the morphism

\[
\left( \lim_{I} L^{(\alpha, \mu)}_{F,T_{t}^{\ast}} \right) : H^1_{\text{tw}}(F(\mu_{p_{\infty}}), T_{t}^{-, \emptyset}) \
\rightarrow \left( \lim_{I} \Lambda_f/I \right) \otimes_{\mathcal{O}} \mathcal{H}_{\text{ord}(\mu_{g})}(\Gamma_{\text{cyc}}) \otimes \mathbb{Z}_p F.
\]
This allows us to rewrite $L_{F,f,g,\mu}$ as
\[
\left(\lim_{\xi \to 1} \mathcal{L}_{F,T_{f,g}}^{(\alpha,\mu)}\right) \circ \text{Pr}_{f,g},
\]
where $\text{Pr}_{f,g}$ denotes the natural projection
\[
H_{lw}^1(\mathbb{Q}_p(\mu_p^{-\infty}), T_{f,g}) \longrightarrow H_{lw}^1(\mathbb{Q}_p(\mu_p^{-\infty}), T_{f,g}^{-\emptyset}).
\]
Therefore, it is enough to show that
\begin{itemize}
  \item[i)] For $\mu \in \{\alpha, \beta\}$, we have
  \[
  \mathcal{L}_{F,T_{f,g}}^{(\alpha,\mu)}(\text{BF}_{f,g,\mu,v}) = 0.
  \]
  \item[ii)] We have
  \[
  \mathcal{L}_{F,T_{f,g}}^{(\alpha,\beta)}(\text{BF}_{f,g,\mu,v}) = -\mathcal{L}_{F,T_{f,g}}^{(\alpha,\beta)}(\text{BF}_{f,g,\alpha,v}) \in (\Lambda_{f}/I) \otimes \log_{p,k_{\mu}+1} \Lambda_{\mathcal{O}}(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} F,
  \]
  where $\text{BF}_{f,g,\mu,v}$ denotes the image of $\text{BF}_{f,g,\mu,m}$ in $H^1(F, T_{f,g}^{-\emptyset} \otimes \mathcal{H}_{\text{ord}(\mu_{\nu})}(\Gamma_{\text{cyc}}))^i$.
\end{itemize}
Suppose that $I = \bigcap_i P_{n_i}$. For $\tau, \mu \in \{\alpha, \beta\}$, we have the commutative diagram
\[
\begin{array}{c}
\bigoplus_i H_{lw}^1(F(\mu_{p^{-\infty}}), T_{f,g}^{-\emptyset} \otimes \mathcal{H}_{\text{ord}(\tau_i)}(\Gamma_{\text{cyc}})^i) \xrightarrow{\mathcal{L}_{F,T_{f,g}}^{(\alpha,\mu)}} (\Lambda_{f}/I) \otimes \mathcal{O} \otimes \Lambda_{\mathcal{O}}(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} F, \\
\bigoplus_i H_{lw}^1(F(\mu_{p^{-\infty}}), F^{-R_{f_i}} \otimes R_{g_i}^e \otimes \mathcal{H}_{\text{ord}(\tau_i)}(\Gamma_{\text{cyc}})^i) \xrightarrow{\mathcal{L}_{F,T_{f,g}}^{(\alpha,\mu)}} (\bigoplus_i \Lambda_{f}/P_{n_i}) \otimes \mathcal{O} \otimes \Lambda_{\mathcal{O}}(\Gamma_{\text{cyc}}) \otimes_{\mathbb{Z}_p} F,
\end{array}
\]
where the vertical arrows are injective and $\mathcal{L}_{F,T_{f,g}}^{(\alpha,\mu)}$ are defined in a similar manner as $\mathcal{L}_{F,f,g}^{(\alpha,\mu)}$. Lemma 3.3.1 says that (i) and (ii) hold for all $i$. The result follows. \(\square\)

In particular, we see that Lemma 3.3.1(ii) holds without the assumption that $k_f \geq k_g$, since we may first directly work with $f$ and specialize to $f_{\alpha}$.

**Definition 3.3.4.**

\begin{itemize}
  \item[i)] We define the geometric Rankin–Selberg $p$-adic $L$-function $L_{p,\text{geo}}^{\text{geo}}(f_{\alpha}, g) \in \Lambda_{\mathcal{O}}(\Gamma_{\text{cyc}}) \otimes_{\mathcal{O}} L$ as the unique element satisfying
  \[
  \mathcal{L}_{f,g}^{(\alpha,\mu)}(\text{BF}_{f,g,\mu,1}) = \frac{\log_{p,k_{\mu}+1}}{\mu'_{\mu} - \mu_{\mu}} \cdot L_{p,\text{geo}}^{\text{geo}}(f_{\alpha}, g)
  \]
  for $\mu, \mu' \in \{\alpha, \beta\}$ with $\mu \neq \mu'$.
  \item[ii)] We define the $\Lambda_{f}$-adic geometric Rankin–Selberg $p$-adic $L$-function
  \[
  L_{p,\text{geo}}^{\text{geo}}(f, g) \in \Lambda_{f}(\Gamma_{\text{cyc}}) \otimes_{\mathcal{O}} L
  \]
  as the unique element satisfying
  \[
  \mathcal{L}_{f,g,\mu}(\text{BF}_{f,g,\mu,1}) = \frac{\log_{p,k_{\mu}+1}}{\mu'_{\mu} - \mu_{\mu}} \cdot L_{p,\text{geo}}^{\text{geo}}(f, g),
  \]
  where $\mu$ and $\mu'$ are as in (i).
\end{itemize}

We refer the reader to the explicit reciprocity law of Loeffler–Zerbes in [LZ16, Theorem 9.3.2] for interpolation properties of the $p$-adic $L$-function $L_{p,\text{geo}}^{\text{geo}}(f, g)$.

**Proposition 3.3.5.**
3.3. $p$-ADIC $L$-FUNCTIONS

i) Suppose that $(FL_g)$ holds. For $m \in \mathcal{N}$ and $\bullet \in \{\#, \flat\}$, we have

$$\text{Col}_{f,g,\bullet,m}(BF_{f,g,\bullet,m}) = 0.$$ 

Furthermore,

$$\text{Col}_{f,g,\#,m}(BF_{f,g,\#,m}) = -\text{Col}_{f,g,\flat,m}(BF_{f,g,\#,m})$$

$$= \frac{\alpha_g \beta_g}{(\alpha_g - \beta_g)} \det M_g \mathcal{L}_{f,g,\alpha,m}(BF_{f,g,\beta,m}).$$

When $m = 1$,

$$\text{Col}_{f,g,\#,1}(BF_{f,g,\#,1}) = -\text{Col}_{f,g,\flat,1}(BF_{f,g,\#,1}) = D_g \delta_{k_g + 1} L^\text{geo}_p(f,g),$$

where $D_g$ is a unit in $\Lambda^1_{\mathcal{O}(1)}$.

ii) If $(S_g)$ holds, the analogous results where one replaces $\#$ and $\flat$ by $+$ and $-$ hold true.

Proof. We only prove part (i) since part (ii) can be proved in a similar manner.

Consider the anti-symmetric matrix

$$\begin{pmatrix}
\mathcal{L}_{f,g,\alpha,m}(BF_{f,g,\alpha,m}) & \mathcal{L}_{f,g,\alpha,m}(BF_{f,g,\beta,m}) \\
\mathcal{L}_{f,g,\beta,m}(BF_{f,g,\alpha,m}) & \mathcal{L}_{f,g,\beta,m}(BF_{f,g,\beta,m})
\end{pmatrix}
= \begin{pmatrix}
0 & \mathcal{L}_{f,g,\alpha,m}(BF_{f,g,\beta,m}) \\
-\mathcal{L}_{f,g,\alpha,m}(BF_{f,g,\beta,m}) & 0
\end{pmatrix},$$

as given in Lemma 3.3.3. Recall from Proposition 2.2.15 that

$$Q^{-1}_g M_g \begin{pmatrix}
\text{Col}_{f,g,\#,m}(BF_{f,g,\#,m}) \\
\text{Col}_{f,g,\#,m}(BF_{f,g,\#,m})
\end{pmatrix} = \begin{pmatrix}
\text{Col}_{f,g,\flat,m}(BF_{f,g,\#,m}) \\
\text{Col}_{f,g,\flat,m}(BF_{f,g,\#,m})
\end{pmatrix}. $$

Thus, together with (3.3), we may rewrite the left-hand side of (3.6) as

$$Q^{-1}_g M_g \begin{pmatrix}
\text{Col}_{f,g,\#,m}(BF_{f,g,\#,m}) \\
\text{Col}_{f,g,\#,m}(BF_{f,g,\#,m})
\end{pmatrix} (Q^{-1}_g M_g)^t.$$

This in turn implies

$$\text{Col}_{f,g,\#,m}(BF_{f,g,\#,m}) = \text{Col}_{f,g,\#,m}(BF_{f,g,\#,m}) = 0$$

and

$$\text{Col}_{f,g,\#,m}(BF_{f,g,\#,m}) = -\text{Col}_{f,g,\#,m}(BF_{f,g,\#,m}) = \frac{1}{\det(Q^{-1}_g M_g)} \mathcal{L}_{f,g,\alpha,m}(BF_{f,g,\beta,m})$$

as required.

The assertion on the case $m = 1$ follows from (3.3) and the fact that $\det(M_g)$ is, up to a unit of $\Lambda^1_{\mathcal{O}(1)}$, given by $\log_{p,k_g + 1}/p^{k_g + 1} \delta_{k_g + 1}$; see [BL20b, Lemma 2.7] and [LLZ11, Corollary 3.2].

Remark 3.3.6. The exact same proof gives the analogous statements for the Coleman maps in Proposition 2.2.17 and the bounded Beilinson–Flach elements from Definition 3.1.5. Namely, for $\bullet \in \{\#, \flat\}$ and $m \in \mathcal{N}$,

$$\text{Col}_{f,g,m}(BF_{f,g,m}) = 0.$$
Furthermore,
\[
\text{Col}_{f,g,m}(\omega, \#) = -\text{Col}_{f,g,m}(\omega, \♭) = \frac{\alpha_g \beta_g}{(\alpha_g - \beta_g) \det M_g} \rho_g(\alpha, \alpha) \text{BF}_{f,g,m}(\alpha, \beta).
\]

We end this section by defining the following bounded \( p \)-adic \( L \)-functions.

**Definition 3.3.7.**

i) Suppose \((\text{FL}_{f,g})\) holds and set \( \mathfrak{C} = (\mathfrak{F}_1, \mathfrak{F}_2) \) be an ordered pair, where
\[
\mathfrak{F}_1 \in \{(\omega, \#), (\omega, \♭), (\eta, \#), (\eta, \♭)\},
\]
\[
\mathfrak{F}_2 \in \{(\omega, \#), (\omega, \♭)\}
\]
with \( \mathfrak{F}_1 \neq \mathfrak{F}_2 \). We define the signed (geometric) \( p \)-adic \( L \)-function
\[
L_{\text{geo}}^{\mathfrak{F}_1}(f,g) := \text{Col}_{f,g}(\mathfrak{F}_2) \in \omega^{-s(g)} \Lambda_\mathcal{O}(\Gamma_{\text{cyc}}).
\]

ii) If \((\text{FL}_{f-S})\) holds, we define similar objects on replacing \( \# \) and \( \♭ \) by the symbols \( + \) and \( - \).

**Remark 3.3.8.**

i) The fact that \( L_{\mathfrak{C}}^{\text{geo}}(f,g) \in \omega^{-s(g)} \Lambda_\mathcal{O}(\Gamma_{\text{cyc}}) \) follows from the integrality of the Coleman maps and Theorem 3.1.4.

ii) Suppose that either \((\text{FL}_{f})\) or \((\text{S}_{f})\) (instead of \((\text{FL}_{f,g})\) and \((\text{FL}_{f-S})\)) holds and that \((\text{Reg})\) holds. We may still define the objects in Definition 3.3.7 as long as \( \mathfrak{F}_1 \in \{(\omega, \bullet), (\omega, \circ)\} \) using the appropriate specializations of the \( \Lambda_\mathcal{F} \)-adic Coleman maps and Beilinson–Flach classes, where \( (\bullet, \circ) \) denotes \((\#, \♭)\) or \((+, -)\). This results in only two choices of \( \mathfrak{C} \) (with \( \mathfrak{F}_1 = (\omega, \circ) \) and \( \mathfrak{F}_2 = (\omega, \bullet) \) or vice versa). It follows from (3.4) and Remark 3.3.6 that
\[
L_{\mathfrak{C}}^{\text{geo}}(f,g) = \pm D_{\gamma} \delta_{\kappa+1} L_{p}^{\text{geo}}(f,\alpha, g).
\]

We may also define the \( \Lambda_\mathcal{F} \)-adic \( L \)-function given by
\[
L_{\mathfrak{C}}^{\text{geo}}(f,g) = \pm D_{\gamma} \delta_{\kappa+1} L_{p}^{\text{geo}}(f,g) \in \omega^{-s(g)} \Lambda_\mathcal{F}(\Gamma_{\text{cyc}}),
\]
where \( \mathfrak{F}_1 = (\omega, \circ) \) and \( \mathfrak{F}_2 = (\omega, \bullet) \) or vice versa.
CHAPTER 4

Selmer groups and main conjectures

We are now ready to define the various Selmer groups arising from the Coleman maps we defined in §3 and relate them to the $p$-adic $L$-functions we defined in §3.3.

4.1. Definitions of Selmer groups

Recall that $T_{f,g} = R_f^* \otimes R_g^*$ and for a crystalline specialization $f$ of the Hida family $f$ (corresponding to the ring homomorphism $\kappa_f : \Lambda_f \to \mathcal{O}$), we have set $T = R_f^* \otimes R_g^*$. We shall define discrete Selmer groups for $T_{f,g}^\vee(1)$ and $T_{f,g}^\vee(1)$ determined by our signed Coleman maps. Throughout this section, suppose either (FL$_f$) or (S$_g$) holds and write $\{\bullet, \circ\}$ for $\#$ or $\{+, -\}$ accordingly. We also assume that the hypothesis [Reg] holds.

**Definition 4.1.1.**

i) Let $\mathfrak{C}_\omega := (\omega, \bullet, (\omega, \circ))$. The discrete Selmer group $\text{Sel}_{\mathfrak{C}_\omega}(T_{f,g}^\vee(1)/\mathbb{Q}(\mu_{p^\infty}))$ is given by the kernel of the restriction map

$$H^1(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee(1)) \to \prod_{v \mid \mathfrak{p}} H^1(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee(1)) \times \prod_{v \not\mid \mathfrak{p}} H^1(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee(1)),$$

where $v$ runs through all primes of $\mathbb{Q}(\mu_{p^\infty})$, and for $v \mid \mathfrak{p}$ the local condition $H^1_{\mathfrak{C}_\omega}(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee(1))$ is the orthogonal complement of

$$\ker(\text{Col}_{f,g}^{(\omega, \bullet)}) \cap \ker(\text{Col}_{f,g}^{(\omega, \circ)})$$

under the local Tate pairing.

ii) Suppose that either (FL$_f$) or (S$_g$) holds. We then define the dual Selmer group $\text{Sel}_{\mathfrak{C}_\omega}(T_{f,g}^\vee(1)/\mathbb{Q}(\mu_{p^\infty}))$ for $\mathfrak{C} = (\mathfrak{F}_1, \mathfrak{F}_2)$, where $\mathfrak{F}_i \in \{\omega, \bullet, (\omega, \circ), (\eta, \bullet), (\eta, \circ)\}$ with $\mathfrak{F}_1 \neq \mathfrak{F}_2$, by the local condition at $p$ given by the orthogonal complement of

$$\ker(\text{Col}_{f,g}^{\mathfrak{F}_1}) \cap \ker(\text{Col}_{f,g}^{\mathfrak{F}_2}).$$

**Definition 4.1.2.** The discrete Selmer group $\text{Sel}_{\mathfrak{C}_\omega}(T_{f,g}^\vee(1)/\mathbb{Q}(\mu_{p^\infty}))$ is given by the kernel of the restriction map

$$H^1(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee(1)) \to \prod_{v \mid \mathfrak{p}} H^1(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee(1)) \times \prod_{v \not\mid \mathfrak{p}} H^1(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee(1)),$$

where $v$ runs through all primes of $\mathbb{Q}(\mu_{p^\infty})$, and for $v \mid \mathfrak{p}$ the local condition $H^1_{\mathfrak{C}_\omega}(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee(1))$ is the orthogonal complement of $\ker(\text{Col}_{f,g,\bullet}) \cap \ker(\text{Col}_{f,g,\circ})$ under the local Tate pairing.
Since \( L \) where \( D \) and \( Q \), the first assertion of the lemma now follows. The proof of the second assertion is

\[
\text{Likewise, for any crystalline specialization } H \text{ of the Hida family } f, \text{ the intersection }
\ker \left( \text{Col}_{f,g}^{(\omega)} \right) \cap \ker \left( \text{Col}_{f,g}^{(\mathfrak{s})} \right)
\]

equals

\[
\text{im} \left( H_{1w}(Q_p(\mu_{p^\infty}), F^+ R^*_f \otimes R^*_g) \hookrightarrow H_{1w}(Q_p(\mu_{p^\infty}), T_{f,g}) \right).$
\]

**Proof.** Let \( L_{T_{f,g}}^{-,\mathfrak{s}} \) be the Perrin-Riou map on \( H_{1w}^1(Q_p(\mu_{p^\infty}), T_{f,g}^{-,\mathfrak{s}}) \) defined in the proof of Theorem 2.3.11 for \( F = Q_p \). It is injective since

\[
\ker \left( L_{T_{f,g}}^{-,\mathfrak{s}} \right) \subset \ker \left( \text{Pr}_{f,g}(z) \right) = 0,
\]

where the last equality is a consequence of the fact that the representation \( T_{f,g}^{-,\mathfrak{s}} \) does not admit the trivial representation as a sub-representation.

By Theorem 2.3.11 \( z \in \ker \left( \text{Col}_{f,g}^{(\omega)} \right) \cap \ker \left( \text{Col}_{f,g}^{(\mathfrak{s})} \right) \) if and only if \( z \) lies inside \( \ker \left( L_{T_{f,g}}^{-,\mathfrak{s}} \right) \) if and only if \( z \in \ker \left( \text{Pr}_{f,g} \right) \). This is equivalent to the condition that \( L_{T_{f,g}}^{-,\mathfrak{s}} \circ \text{Pr}_{f,g}(z) = 0 \), where \( \text{Pr}_{f,g} \) denotes the natural projection

\[
H_{1w}^1(Q_p(\mu_{p^\infty}), T_{f,g}) \rightarrow H_{1w}^1(Q_p(\mu_{p^\infty}), T_{f,g}^{-,\mathfrak{s}}),
\]

since \( L_{f,g,\alpha} \) and \( L_{f,g,\beta} \) are defined by projecting \( L_{T_{f,g}}^{-,\mathfrak{s}} \circ \text{Pr} \) to the two chosen \( \varphi \)-eigenvectors in \( D(Q_p, T_{f,g}^{-,\mathfrak{s}}) \). Thus, the injectivity of \( L_{T_{f,g}}^{-,\mathfrak{s}} \) implies that

\[
\ker \left( \text{Col}_{f,g}^{(\omega)} \right) \cap \ker \left( \text{Col}_{f,g}^{(\mathfrak{s})} \right) = \ker \left( \text{Pr}_{f,g} \right).
\]

The first assertion of the lemma now follows. The proof of the second assertion is similar. \( \square \)

We recall the definition of the classical Greenberg Selmer groups.

**Definition 4.1.5.** The classical Greenberg Selmer group for \( T_{f,g}^\vee \) over \( Q_p(\mu_{p^\infty}) \), denoted by \( \text{Sel}_{Gr}(T_{f,g}^\vee (1)/Q(\mu_{p^\infty})) \), is defined as the kernel of

\[
H^1(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee (1)) \rightarrow \prod_{v|p} H^1(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee (1)) \times \prod_{v|p} H^1(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee (1)),
\]

where \( H^1_{Gr}(\mathbb{Q}(\mu_{p^\infty}), T_{f,g}^\vee (1)) \) is the orthogonal complement of

\[
\text{im} \left( H_{1w}^1(Q_p(\mu_{p^\infty}), F^+ R^*_f \otimes R^*_g) \hookrightarrow H_{1w}^1(Q_p(\mu_{p^\infty}), T_{f,g}) \right)
\]

under the local Tate duality. We also define the classical Greenberg Selmer group \( \text{Sel}_{Gr}(T_{f,g}^\vee (1)/Q(\mu_{p^\infty})) \) similarly.

We deduce from Lemma 4.1.4 the following corollary.
4.2. Classical and signed Iwasawa main conjectures

We now formulate Iwasawa main conjectures relating the $p$-adic $L$-functions defined in Definition 3.3.7 to the Selmer groups in Definition 4.1.1.

**Conjecture 4.2.1.** Suppose that either $(\mathbf{FL}_{f,g})$ or $(\mathbf{FL}_f-S_g)$ holds. For $C = (\mathfrak{F}_1, \mathfrak{F}_2)$, where

$$\mathfrak{F}_1 \in \{ (\omega, \cdot), (\omega, \circ), (\eta, \cdot), (\eta, \circ) \}, \quad \mathfrak{F}_2 \in \{ (\omega, \cdot), (\omega, \circ) \}$$

with $\mathfrak{F}_1 \neq \mathfrak{F}_2$ and $\theta$ a character of $\Delta$, the $\theta$-isotypic component $e_\theta \text{Sel}_C(T_{f,g}^\dagger(1)/\mathbb{Q}(\mu_{p^\infty}))$ is a cotorsion $\Lambda_C(\Gamma_1)$-module. Furthermore,

$$\text{char}_{\Lambda_C(\Gamma_1)} e_\theta \text{Sel}_C(T_{f,g}^\dagger(1)/\mathbb{Q}(\mu_{p^\infty}))^\vee \equiv (e_\theta \varpi^{s(g)} L^{\text{geo}}_p(f, g)/I_C),$$

where $I_C = \text{det}(\text{Im}(\text{Col}_{\mathfrak{F}_1} \oplus \text{Col}_{\mathfrak{F}_2}))$.

**Remark 4.2.2.**

i) When $\mathfrak{F}_1 \in \{ (\eta, \cdot), (\eta, \circ) \}$, the $p$-adic $L$-functions $L^{\text{geo}}_C(f, g)$ does not have a simple description (as far as we are aware).

ii) It follows from [BLLV19] Appendix that $I_C$ is a product of powers of $\text{Tw}^{-1}X$, $i = 0, 1, \ldots, k_f + k_g + 1$.

iii) Consider the special case when $C = C_{\omega}$. Since $\{ \omega_f \otimes \omega_g, \omega_f \otimes \varphi(\omega_g) \}$ is an $O$-basis of $\mathbb{D}_{\text{crys}}(\mathcal{F}_f R^*_f \otimes R^*_g)$, it follows from [PR94] §3.4 that

$$\det \left( \text{Im} \left( L_{f,g}^{(\alpha, \alpha)} \oplus L_{f,g}^{(\alpha, \beta)} \right) \right) = \frac{1}{\text{det} Q_g} \prod_{i=0}^{k_g} \left( \log_p \frac{\log_p \chi(\Gamma_{\text{cyc}})}{\log_p \chi(\Gamma_{\text{cyc}})} - i \right) \Lambda_C(\Gamma_{\text{cyc}})$$

$$= \frac{\log_p k_g + 1}{\text{det} Q_g} \Lambda_C(\Gamma_{\text{cyc}}).$$

Hence, we deduce from the last assertions of Propositions 2.2.12 and 2.4.6 that

$$I_C = \frac{\log_p k_g + 1}{\text{det} M_g} \Lambda_C(\Gamma_{\text{cyc}}) = \delta_{k_g + 1} \Lambda_C(\Gamma_{\text{cyc}}).$$

In view of Proposition 3.3.3 (which tells us that $D_g \in \Lambda_C(\Gamma_1)^{\times}$) and Remark 3.3.8(i), Conjecture 4.2.1 can be rephrased as the conjectural identity

$$\text{char}_{\Lambda_C(\Gamma_1)} e_\theta \text{Sel}_C(T_{f,g}^\dagger(1)/\mathbb{Q}(\mu_{p^\infty})) = (e_\theta \varpi^{s(g)} L^{\text{geo}}_p(f, g))$$

for every character $\theta$ of $\Delta$.

We finish this chapter by stating a two-variable $(\Lambda_C(\Gamma_1))$-adic Iwasawa main conjecture.
**Definition 4.2.3.** Suppose $M$ is a finitely generated torsion $\Lambda_f(\Gamma_1)$-module. We define its characteristic ideal on setting

$$\text{char}_{\Lambda_f(\Gamma_1)}(M) := \prod_P \text{length}_{\Lambda_f(\Gamma_1)}(M_P),$$

where the product runs over all height-one primes of $\Lambda_f(\Gamma_1)$.

**Conjecture 4.2.4.** Suppose that either (FL$_g$) or (S$_g$) holds and that (Reg) holds. For any character $\theta$ of $\Delta$, the $\Lambda_f(\Gamma_1)$-module $e_\theta \text{Sel}_{\mathcal{C}_\omega}(T_{f,g}(1)/\mathbb{Q}(\mu_{p^n}))^{\vee}$ is torsion. Furthermore,

$$\text{char}_{\Lambda_f(\Gamma_1)}e_\theta \text{Sel}_{\mathcal{C}_\omega}(T_{f,g}(1)/\mathbb{Q}(\mu_{p^n}))^{\vee} = (e_\theta \omega^{a(g)} L_p^{\text{geo}}(f, g)) / I_{\mathcal{C}_\omega}.$$

**Remark 4.2.5.** By Corollary 4.1.6 and Remarks 3.3.8(ii) and 4.2.2(iii), Conjecture 4.2.4 is equivalent to the assertion that

$$\text{char}_{\Lambda_f(\Gamma_1)}e_\theta \text{Sel}_{\mathcal{C}_\omega}(T_{f,g}(1)/\mathbb{Q}(\mu_{p^n}))^{\vee} = (e_\theta \omega^{a(g)} L_p^{\text{geo}}(f, g)).$$
CHAPTER 5

Applications towards main conjectures

We now employ the bounded Beilinson–Flach elements we have constructed in §3.1 to show that one inclusion in the Main Conjectures 4.2.1 and 4.2.4 holds true (under mild hypotheses concerning the images of the Galois representations under consideration). In §5.3, we discuss implications of these results towards main conjectures for Rankin–Selberg convolutions $f_K \otimes \psi$ where $K$ is an imaginary quadratic field in which $p$ remains inert and $\psi$ is an algebraic Hecke character of $K$.

5.1. Cyclotomic main conjectures for $f \otimes g$

Let $f \in S_{k_f+2}(\Gamma_1(N_f))$ and $g \in S_{k_g+2}(\Gamma_1(N_g))$ be primitive eigenforms as in §1.1, where $N_f$ and $N_g$ are coprime and $k_f > k_g \geq 0$. We recall also that we are working under the assumption that $f$ is $p$-ordinary and $g$ is non-$p$-ordinary (relative to the embedding $\iota_p$).

Recall the Galois representation $T_{f,g} = R^* f \otimes R^* g$, which is a free $O$-module of rank 4, where $O$ is the ring of integers of a finite extension $L$ of $\mathbb{Q}_p$ containing the images $\iota_p(K_f)$ and $\iota_p(K_g)$ of the Hecke fields of $f$ and $g$, as well as the roots of the Hecke polynomials of $f$ and $g$ at $p$.

Suppose throughout §5.1 that the residual representations $\overline{\rho}_f$ associated to $f$ is absolutely irreducible. Assume in addition that at least one of the following conditions holds true:

1. $p > k_g + 1$.
2. $a_p(g) = 0$, $k_g \in [p + 1, 2p - 2]$.
3. $a_p(g) = 0$, $p + 1 \nmid k_g + 1$ and $k_g \geq 2p - 1$.

REMARK 5.1.1. We note that any one of the hypotheses $[\text{Irr}_1]$, $[\text{Irr}_2]$ guarantees that $\overline{\rho}_g|_{G_{O_p}}$ is absolutely irreducible; c.f. [Edi92] in the situation of $[\text{Irr}_1]$ and [Ber10] Théorème 3.2.1 in the remaining cases. Since $\overline{\rho}_f|_{G_{O_p}}$ (and therefore, also $\overline{\rho}_f|_{G_{O_p} \otimes \omega}$) is reducible, we conclude that neither $\overline{\rho}_f|_{G_{O_p}}$ nor $\overline{\rho}_f|_{G_{O_p} \otimes \omega^{-1}}$ is isomorphic to $\overline{\rho}_g|_{G_{O_p}}$, granted the truth of either one of the hypotheses $[\text{Irr}_1]$ or $[\text{Irr}_3]$. This in turn ensures the validity of the following “non-anomaly” condition:

$$H^0(\mathbb{Q}_p, T_{f,g}) = 0 = H^2(\mathbb{Q}_p, T_{f,g}).$$

We consider the following big image condition.

1. $p$-ordinary and $g$-non-$p$-ordinary (relative to the embedding $\iota_p$).

Our main result towards the validity of cyclotomic main conjectures for the Rankin–Selberg convolution $f \otimes g$ is the following:
5. APPLICATIONS TOWARDS MAIN CONJECTURES

Theorem 5.1.2. Suppose that the residual representation $\rho_f$ is absolutely irreducible. Assume also that one of the hypotheses $[[\text{Irr}_1]]$ as well as the condition $[[\text{Tr}_{f,g}]]$ hold true. Then for any character $\theta$ of $\Delta$ we have the following containment in the Iwasawa main conjecture (11):

$$e_\theta \varpi^{s(g)} L_{p}^{\text{geo}}(f,g) \in \text{char}_{\Lambda_c(\Gamma')} e_\theta \left( \text{Sel}_{G}(T_{f,g}(1)/\mathbb{Q}(\mu_{p^{\infty}}))^\vee \right),$$

where the integer $s(g)$ is given as in the statement of Theorem 5.1.4.

Proof. Given the locally restricted Euler system $\{\varpi^{s(g)} \text{BF}_{f,g,m}\}_{m \in \mathcal{N}}$ of signed Beilinson–Flach elements (c.f. §3.1), where

- $\alpha = \alpha_f$ is the root of the Hecke polynomial of $f$ at $p$ such that $\nu_p(\alpha)$ is a $p$-adic unit,
- $* \in \{\# , \beta\}$ if $[[\text{Irr}_1]]$ holds and $* \in \{+, -\}$ if $[[\text{Irr}_2]]$ or $[[\text{Irr}_3]]$ holds,

the proof of this theorem is identical to the proof of [BLL19] Theorem 6.2.4, in view of Corollary 11.16 (which identifies the signed Selmer group with the Greenberg Selmer group) and Remark 3.3.8 (which compares the signed $p$-adic $L$-functions to the geometric Rankin–Selberg $p$-adic $L$-function).

We recall that this argument builds on the locally restricted Euler system machinery developed in [BL15] Appendix A (more specifically, the conclusion of Theorem A.14 in op. cit.; see also [Büy09, Büy10, Büy14b, Büy18] for earlier incarnations and applications of this machinery in a variety of contexts). We note that the adjective “locally restricted” refers to the $p$-local property (3.7) of the classes $\{\text{BF}_{f,g,m}\}_m$, which asserts that

$$\text{loc}_p \left( \text{BF}_{f,g,m}^{\alpha,*} \right) \in \ker \left( \text{Col}_{f,g,m}^{(\alpha,*)} \right).$$

\[ \square \]

Remark 5.1.3.

i) By the explicit reciprocity law of Loeffler–Zerbes [LZ16] Theorem 9.3.2], the $p$-adic $L$-function $L_{p}^{\text{geo}}(f,g)$ evaluated at $\chi^j$ equals (up to explicit fudge-factors) the complex $L$-value $L(f,g,1+j)$ whenever $k_j < j \leq k_f$. If $j > \frac{k_f + 2k_g + 1}{2}$, then $1 + j$ falls within the range of absolute convergence for $L(f,g,s)$. Hence, we have $L(f,g,1+k_f) \neq 0$ when $k_f - k_g \geq 3$. It is also easy to see in that case that the explicit fudge-factors are non-vanishing as well. In particular, $L_{p}^{\text{geo}}(f,g) \neq 0$.

ii) We have proved in Theorem 5.1.2 one inclusion of Conjecture 4.2.1 with the choice of $\mathcal{E} = \mathcal{C}_\omega$. The same proof can be carried over to the other choices of $\mathcal{E}$ in Conjecture 4.2.1 without any difficulties.

5.2. Cyclotomic main conjectures for $f \otimes g$

Suppose that the newform $g \in S_{k_g+2}(\Gamma_1(N_g))$ is as in §5.1 We let $f$ denote the primitive Hida family of tame level $N_f$ (where we still assume that $N_f$ is coprime to $N_g$), whose basic properties were outlined in §§4.1 and 2.3.

Recall that $\Lambda_f$ stands for the branch of Hida’s universal (ordinary) Hecke algebra. Without loss of generality, we may (and will) assume that $\Lambda_f$ contains $\mathcal{O}$ as a subring. We assume that $\Lambda_f$ is a regular ring (c.f., Remark 3.3.2).

Recall the Galois representation $T_{f,g} = R_f^* \otimes R_g^*$, which is a free $\Lambda_f$-module of rank 4. We consider the following big image condition on the representation $T_{f,g}$. 

by definition.

in particular, also that

for some suitable choice of a natural number

using the conclusion of Theorem 5.1.2 for varying

It follows from \[ MR04 \]

Let us also choose an element

as the condition

absolutely irreducible. Assume also that one of the hypotheses

\((\tau \otimes g)\) holds true. For any character \(\theta\) of \(\Delta\), we have the following containment in Conjecture [4.2.2] up to \(\mu\)-invariants:

\[
e_\theta L^{geo}_p(f, g) \in \text{char}_{\Lambda_\cdot}(\Gamma_1) e_\theta \left( \text{Sel}_{\text{Gr}}(T_{f,g}(1)/Q(\mu_{p^\infty}))^\vee \right) \otimes_{\mathcal{O}} L.
\]

**** PROOF. ** Let us fix \(\theta\) as in the statement of our theorem and assume \(e_\theta L^{geo}_p(f, g) \neq 0\) without loss of generality, since there is nothing to prove otherwise. Put \(\mathcal{L} := \mathcal{L}_{\mu_1} e_\theta L^{geo}_p(f, g)\) where \(\mu_1\) is the unique natural number with \(\mathcal{L} \in \Lambda_\cdot(\Gamma_1) \setminus e_\theta \Lambda_\cdot(\Gamma_1)\).

Let us also choose an element \(S \in \Lambda_\cdot(\Gamma_1) \setminus e_\theta \Lambda_\cdot(\Gamma_1)\) such that

\[
e_\theta \mathcal{L}^{\mu_2}(\Gamma_1) S = \text{char}_{\Lambda_\cdot(\Gamma_1)} e_\theta \left( \text{Sel}_{\text{Gr}}(T_{f,g}(1)/Q(\mu_{p^\infty}))^\vee \right)
\]

for some suitable choice of a natural number \(\mu_2\). We contend to prove that \(S | \mathcal{L}\), using the conclusion of Theorem [5.1.2] for varying \(f\) and the divisibility criterion established in Appendix [A]. This will in turn prove that

\[
e_\theta \mathcal{L}^{\mu_2}(f, g) \in \text{char}_{\Lambda_\cdot(\Gamma_1)} e_\theta \left( \text{Sel}_{\text{Gr}}(T_{f,g}(1)/Q(\mu_{p^\infty}))^\vee \right).
\]

We will see along the way \(\mu_2 \leq \mu_1 + s(g)\). This combined with \([1.1]\) will verify the required containment.

Let us choose any sequence \(\{P_i = \ker(\kappa_i)\}_{i=1}^\infty \subset \text{Spec}(\Lambda_\cdot)\) of distinct height-one primes of \(\Lambda_\cdot\), where each \(\kappa_i : \Lambda_\cdot \to \mathcal{O}\) is an \(\mathcal{O}\)-valued crystalline specialization of weight \(\geq 0\) (in the sense that the overconvergent eigenform \(f_\kappa\) is a classical eigenform which is \(p\)-old and has weight \(\geq 2\)). For each positive integer \(i\), let us put \(\mathcal{L}_i := \kappa_i(\mathcal{L}) \in \Lambda_\cdot(\Gamma_1)\) and similarly define \(S_i \in \Lambda_\cdot(\Gamma_1)\).

Let us fix an index \(i\). We will explain that

\[
e_\theta \mathcal{L}^{\mu_2}(S_i) \text{ divides } \text{char}_{\Lambda_\cdot(\Gamma_1)} e_\theta \left( \text{Sel}_{\text{Gr}}(T_{f_i,g}(1)/Q(\mu_{p^\infty}))^\vee \right).
\]

It follows from [MR04] Lemma 3.5.3 that

\[
\text{Sel}_{\text{Gr}}(T_{f,g}(1)/Q(\mu_{p^\infty}))^\vee / P_i \text{Sel}_{\text{Gr}}(T_{f,g}(1)/Q(\mu_{p^\infty}))^\vee \cong \text{Sel}_{\text{Gr}}(T_{f_i,g}(1)/Q(\mu_{p^\infty}))^\vee.
\]

We therefore infer that

\[
\text{char}_{\Lambda_\cdot(\Gamma_1)} e_\theta \left( \text{Sel}_{\text{Gr}}(T_{f,g}(1)/Q(\mu_{p^\infty}))^\vee [P_i] \right) \cdot (e_\theta \mathcal{L}^{\mu_2}(S_i))
\]

\[
= \text{char}_{\Lambda_\cdot(\Gamma_1)} e_\theta \left( \text{Sel}_{\text{Gr}}(T_{f_i,g}(1)/Q(\mu_{p^\infty}))^\vee \right),
\]

in particular, also that

\[
e_\theta \mathcal{L}_i \mid \text{char}_{\Lambda_\cdot(\Gamma_1)} e_\theta \left( \text{Sel}_{\text{Gr}}(T_{f_i,g}(1)/Q(\mu_{p^\infty}))^\vee \right).
\]

Observe also that

\[
e_\theta \mathcal{L}_i = e_\theta L^{geo}_p(f_\kappa, g)
\]

by definition.
Under the assumptions of our theorem, the hypotheses of Theorem \[5.1.2\] are valid with the choice \( f = f(\kappa_1) \), for any positive integer \( i \). It follows from Theorem \[5.1.2\] (applied with \( f = f(\kappa_1) \))
\[
\text{char}_{\Lambda_{\Gamma}((g_1)^{\epsilon \theta}} \left( \text{Sel}_{G_\mathfrak{R}}(T_{G(\kappa_1), g}(1)/\mathbb{Z}(\mu_p)) \right) \right) \mid \epsilon_\theta \tau^s(g) L^\text{geo}_p(f(\kappa_1), g).
\]
This, combined with \[5.2\] and \[5.3\], allows us to conclude that \( \tau^{s(g)} L^\text{geo}_p(f(\kappa_1), g) \). Since \( \mu \)-invariants of \( \mathcal{L}_1 \) and \( S_1 \) are both zero (by definition), it follows that \( \mu_2 \leq s(g) + \mu_1 \), and also that
\[
\text{(5.4)}
\]
We may now use Proposition \[A.0.1\] (with the choices \( p \) the weight space, \( \psi \), and let \( \kappa \) denote an \( R \)-type Hecke character with infinity type \((0, k_\psi)\) and conductor \( f_\psi \). As a start, in \$5.3.2\$, we shall recast (in Theorem \[5.3.5\] and Theorem \[5.3.6\]) our Theorems \[5.1.2\] and \[5.2.1\] above as results towards the cyclotomic main conjectures for the Rankin–Selberg products \( f/K \otimes \psi \) and \( f/K \otimes \psi \) of the base change of a (family of) non-CM form(s) to \( K \) and the Hecke character \( \psi \).

In \$5.3.3\$, we explain how to interpolate the conclusions of Theorem \[5.3.6\] under a very natural assumption (which we verify granted the existence of a rank-2 Euler system in the present setting in Section \[3.2\]) to a divisibility statement in the Iwasawa main conjectures over \( \Lambda_{\Gamma}(\Gamma_K) \); see Theorem \[5.3.1\] for our main result in this direction. Utilizing the descent formalism of \[BL20a\] \$5.3.1\$ (which relies crucially on the work of Nekovář \[Nek06\]), we shall obtain divisibilities in both “definite” and “indefinite” Iwasawa main conjectures over \( \Lambda_{\Gamma}(\Gamma_{ac}) \), where \( \Gamma_{ac} \) is the Galois group of the anticyclotomic \( \mathbb{Z}_p \)-extension \( \mathbb{K}_{ac}/K \).

\[5.3.1\] The setting.\ Let \( D_K \) denote the discriminant of the imaginary quadratic number field \( K \) which we have fixed above, and let \( \mathcal{O}_K \) denote its ring of integers. We let
\[
\psi : \kappa_K^x/K^x \rightarrow \mathbb{C}^x
\]
denote an \( A_0 \)-type Hecke character with infinity type \((0, k_\psi + 1)\) (where \( k_\psi \) is a natural number) and conductor \( f \), where \( f \) is coprime to \( p \). We let \( \psi \) denote the Galois character associated (via the geometrically normalized Artin map of global class field theory) to the \( p \)-adic avatar of \( \psi \). We say that \( \psi \) is a crystalline Hecke character to mean that \( \psi \) is crystalline at \( p \). In \$5.3.2\$ we will work with a fixed choice of crystalline \( \psi \), whereas in \$5.3.3\$ we will allow \( \psi \) vary among crystalline Hecke characters (but keeping \( f \) fixed) to prove our main results.
5.3. MAIN CONJECTURES OVER AN IMAGINARY QUADRATIC FIELD WHERE p IS INERT

We put
\[ \theta(\psi) := \sum_{(a,\ell)=1} \psi(a)q^{N_a} \in S_{k_\ell+2}(\Gamma_1(N_\ell)) \]
where \( N_\ell = |D_K|N_\ell \). In what follows, we shall replace \( q \) in §5.3.1 with \( \theta(\psi) \). Note that since we have assumed that \( p \) is inert in \( K/Q \), it follows that \( a_p(\theta(\psi)) = 0 \), i.e., the eigenform \( \theta(\psi) \) is indeed \( p \)-non-ordinary.

**Definition 5.3.1.** Let us define the Dirichlet character \( \varepsilon_\psi \) on setting
\[ \varepsilon_\psi := \varepsilon_K \psi|_{\mathbb{A}^1} \mathbb{N}^{-k_\ell-1} \]
where \( \mathbb{N} \) is the norm character on \( \mathbb{A}^1 \) and \( \varepsilon_K \) is the quadratic character associated to \( K/Q \). Then \( \varepsilon_\psi \) is the nebentype character of the cuspidal eigen-form \( \theta(\psi) \).

Let \( \varepsilon_f \) denote the nebentype of the eigenform \( f \) (as well as the Hida family \( f \) we have fixed).

We let \( f \in S_{k_\ell+2}(\Gamma_1(N_\ell), \varepsilon_\ell) \) be a primitive eigenforms as in §5.3.1 where \( N_\ell \) and \( N_\ell \) are coprime and \( k_\ell > k_\ell \). We recall also that we are working under the assumption that \( f \) is \( p \)-ordinary; we denote by \( f^a \) its \( p \)-ordinary specialization.

Thanks to our assumptions that \( f, D_K \) and \( N_\ell \) are pairwise coprime, observe that \( \varepsilon_\psi f^{\varepsilon_\psi} f \) can never be the trivial character.

We now let \( \mathcal{O} \) be the ring of integers of a finite extension \( \text{Frac}(\mathcal{O}) \) of \( \mathbb{Q}_p \) containing the images \( t_p(K) \) and \( t_p(\psi(\mathbb{A}^1)) \) of the Hecke fields, as well as the roots of the Hecke polynomials of \( f \) and \( \theta(\psi) \) at \( p \).

As before, we let \( \Lambda_f \) denote the branch of Hida’s universal (ordinary) Hecke algebra which admits \( f^a \) as a specialization. We again assume (without loss of generality) that \( \Lambda_f \) contains \( \mathcal{O} \) as a subring and that \( \Lambda_f \) is a regular ring (c.f., Remark 3.3.2).

We assume throughout §5.3 that the residual representation \( \overline{\rho}_f = \overline{\rho}_f \) is irreducible. The Galois representation \( T_{f,\theta(\psi)} := R^*_f \otimes \overline{\rho}_f(\psi) \), which is a free \( \mathcal{O} \)-module of rank 4, can be (and will be) identified with \( \text{Ind}_{K/f} T_{f,\psi} \), where \( T_{f,\psi} := R^*_f \otimes \psi^{-1} \).

We set \( T_{f,\psi} := T_{f,\psi} \otimes \Lambda_K(\Gamma_\ell)^{cyc} \), where \( \Lambda_K(\Gamma_\ell)^{cyc} \) is the \( \Lambda_K(\Gamma_\ell)^{cyc} \)-module of rank 4 on which \( G_K \) acts via the character \( G_K \rightarrow \Gamma_\ell \rightarrow \lambda_\ell \rightarrow \Lambda_K(\Gamma_\ell)^{cyc} \), and where \( G_K \) acts on this tensor product diagonally. We similarly define \( \overline{T}_{f,\psi} \).

We analogously define the “big” Galois representations \( T_{f,\psi} := T_{f,\psi} \otimes \Lambda_K(\Gamma_{\text{ac}})^{cyc} \), \( \overline{T}_{f,\psi} := T_{f,\psi} \otimes \Lambda_K(\Gamma_{\text{ac}})^{cyc} \), \( \overline{T}_{f,\psi} := T_{f,\psi} \otimes \Lambda_K(\Gamma_{\text{ac}})^{cyc} \), \( \overline{T}_{f,\psi} := T_{f,\psi} \otimes \Lambda_K(\Gamma_{\text{ac}})^{cyc} \), and \( \overline{T}_{f,\psi} \).

We next define the Selmer complexes associated to these \( G_K \)-representations. We will choose to work with these Selmer complexes (rather than their classical counterparts, the Greenberg Selmer groups) due to the utility of Nekovář’s base change and descent formalism (which we shall crucially rely on in §5.3.3 5.3.0).

**Definition 5.3.2.** Let \( \Sigma \) denote the set of places of \( K \) which divide \( p|N_\ell|D_K| \). In what follows, \( ? \) stands for any one of the symbols \( \{ \text{ac}, \text{cyc}, K \} \). The complex
\[ \overline{\text{Ri}}_{f_r}(G_K, \Sigma, T_{f,\psi}; \Lambda_{G_1}) \in D_{\text{b}}(\Lambda_K(\Gamma_{\text{ac}})^{\text{Mod}}) \]
is the Greenberg Selmer complex given by the local conditions which are unramified for all primes in \( \Sigma \) that are coprime to \( p \), and which is given by the strict Greenberg conditions with the choice
\[ j^+_p : F^+_r \overline{T}_{f,\psi} := F^+_r R^*_f \otimes \psi^{-1} \otimes \Lambda_K(\Gamma_{\text{ac}})^{cyc} \rightarrow \overline{T}_{f,\psi} \]
at the unique prime of $K$ above $p$ (which we shall abusively denote by $p$ as well). We shall denote its cohomology by $\tilde{H}_f^\bullet(G_{K,\Sigma}, \mathbb{T}_f, \psi; \Delta_{Gr})$.

We similarly define the complex $\tilde{R}_f^\bullet(G_{K,\Sigma}, \mathbb{T}_f, \psi; \Delta_{Gr})$ as well as its cohomology groups $\tilde{H}_f^\bullet(G_{K,\Sigma}, \mathbb{T}_f, \psi; \Delta_{Gr})$.

Let us denote the set of places of $\mathbb{Q}$ that lie below the set $\Sigma$ also with $\Sigma$. Attached to the $G_{K,\Sigma}$-representation $T_{f,\theta(\psi)}$, we may similarly define a Selmer complex $\tilde{R}_f^\bullet(G_{K,\Sigma}, T_{f,\theta(\psi)}; \Delta_{Gr})$ given by the local conditions which are unramified for all primes in $\Sigma$ that are coprime to $p$, and which is given by the strict Greenberg condition with the choice

$$J^+_p : F^+ T_{f,\theta(\psi)} := F^+ R^+_f \otimes R^\psi(\psi) \to T_{f,\theta(\psi)}$$

at $p$. Shapiro’s lemma then induces an isomorphism

$$\mathcal{S} : \tilde{R}_f^\bullet(G_{K,\Sigma}, T_{f,\theta(\psi)}; \Delta_{Gr}) \xrightarrow{\sim} \tilde{R}_f^\bullet(G_{K,\Sigma}, T_{f,\psi}; \Delta_{Gr}).$$

**Proposition 5.3.3.** Let us denote by $\mathbf{1}$ the trivial character of $\Delta$. We have,

$$e_1 \text{char}_{\Lambda'(T_{cyc})} \left( \text{Sel}_{\text{Gr}}(T_{f,\theta(\psi)}(1)/\mathbb{Q}(\mu_p)) \right)^\vee = \text{char}_{\Lambda'(\mathbb{G}_m)} \left( \tilde{H}_f^2(G_{K,\Sigma}, T_{f,\psi}; \Delta_{Gr})^\vee \right).$$

**Proof.** By control theorems and the isomorphism $\mathcal{S}$ induced by Shapiro’s lemma, it suffices to prove that

$$|\text{Sel}_{\text{Gr}}(T_{f,\theta(\psi)}(1)/\mathbb{Q}(\mu_p))| = |\tilde{H}_f^2(G_{K,\Sigma}, T_{f,\psi}; \Delta_{Gr})^\vee|$$

for infinitely many characters $\eta$ of $\Gamma_1$. This fact was proved in [BL20a, Lemma 5.7 and Lemma 5.8. Note that the set up in op. cit. a priori requires $p$ be split in $K/\mathbb{Q}$, but this assumption plays no role in the proofs of Lemma 5.7 and Lemma 5.8. It is also worth noting we are using the fact that $\eta \circ \iota = \eta^{-1}$ (and this is the reason for the use of the twisted $\Gamma_1$-action in the definitions of $\mathbb{T}_f^\psi$). \qed

**Definition 5.3.4.** We set

$$L_p^{RS}(f_{/K} \otimes \psi) := e_1 L_p^{geo}(f, \theta(\psi)) \in \Lambda'(\mathbb{G}_m)$$

and similarly define $L_p^{RS}(f_{/K} \otimes \psi) \in \Lambda'(\mathbb{G}_m)$.

**5.3.2. Cyclotomic main conjectures in the inert case.** We consider the following big image condition:

$$(\text{Full}) \quad \text{SL}_2(\mathbb{F}_p) \subset T_f(G_{K,\mu_p})$$

In Theorem [C.2.4], we explain that the condition $[\text{Full}]$ together with the assumption that $p \geq 7$ is sufficient to ensure the validity of the hypothesis $[T_{f,\theta(\psi)}]$ when $g = \theta(\psi)$.

The following divisibility statement in the cyclotomic main conjecture for the Rankin–Selberg product $f_{/K} \otimes \psi$ is a reformulation of Theorem [5.1.2] in view of Proposition 5.3.3 and Definition 5.3.4. Note also that the hypothesis that $k_g \neq p - 1$ and $p + 1 \nmid k_g + 1$ guarantees that one of $[\text{Irr}_1], [\text{Irr}_3]$ holds true.

**Theorem 5.3.5.** Suppose that $\mathbb{G}_f$ is absolutely irreducible as well as that $k_g \neq p - 1$ and $p + 1 \nmid k_g + 1$. Assume also that $p \geq 7$ and $[\text{Full}]$ holds true. Then,

$$\varpi^{\omega(g)} L_p^{RS}(f_{/K} \otimes \psi) \in \text{char}_{\Lambda'(\mathbb{G}_m)} \left( \tilde{H}_f^2(G_{K,\Sigma}, T_{f,\psi}; \Delta_{Gr})^\vee \right).$$

Using Theorem 5.3.5 as $\alpha$ varies in the Hida family $f$, we have the following divisibility statement in the cyclotomic main conjecture for the family $f_{/K} \otimes \psi$:
5.3. MAIN CONJECTURES OVER AN IMAGINARY QUADRATIC FIELD WHERE \( p \) IS INERT

**Theorem 5.3.6.** Suppose that \( \mathfrak{p}_f \) is absolutely irreducible, as well as that \( k_g \neq p - 1 \) and \( p + 1 \mid k_g + 1 \). Assume also that \( p \geq 7 \) and the tautological characters. Put \( \chi \mid \Lambda \) above \( p \) conductor dividing \( p \) as before and we shall only provide a brief sketch. We assume that \( \tilde{\chi} \mid \Lambda \), the control theorem for Nekovář’s extended Selmer groups (c.f., [Nek06], Corollary 8.10.2) to reduce to the validity of Theorem 5.3.5 for infinitely many crystalline specializations \( f \). Thanks to our running assumptions, this holds true. □

**Remark 5.3.7.** We will use Proposition A.0.1 to patch the conclusions of Theorem 5.3.6 as \( \psi \) varies among crystalline Hecke characters (with conductor dividing \( f \)) to a statement towards a 3-variable main conjecture over \( \Lambda_{\psi}(\Gamma_K) \), assuming in addition that the integers \( s(\psi) \) that appear in the statement of Theorem 5.3.7 are uniformly bounded as \( \psi \) varies (see §5.3.8 where we employ this idea; see also §§5.3.9–5.3.10 for applications in the anticyclotomic main conjectures in the inert case, both in the definite and the indefinite setting).

We recall that by Corollary 5.3.8, the exponents \( s(\psi) \) are uniformly bounded as \( \psi \) varies, granted the existence of a rank-2 Euler system that the Perrin-Riou philosophy predicts.

**Remark 5.3.8.** In contrast to the discussion in Remark 5.3.7, the conclusions of Theorem 5.3.6 as \( \psi \) varies among crystalline Hecke characters (with fixed conductor dividing \( f \)) cannot be patched to a statement towards a 2-variable main conjecture over \( \Lambda_{\psi}(\Gamma_K) \), since there are only finitely many \( \psi \) verifying the conditions of Theorem 5.3.7 as this theorem requires \( k_g < k_f \). This is primarily the reason for our emphasis on the signed-splitting procedure for Beilinson–Flach elements for families (c.f. §7.1) and on Theorem 5.3.6.

5.3.3. Results on the 3-variable main conjectures over an imaginary quadratic field where \( p \) is inert. Let us fix a ray class character \( \chi \) of \( K \) with conductor dividing \( f_{p_{\infty}} \) (where we assume that \( f \) is as before) and order coprime to \( p \). We assume that \( \tilde{\chi} \mid \Lambda_{p_2} \neq \tilde{\chi}^\vee \mid \Lambda_{p_2} \); in particular, the conductor of \( \chi \) is necessarily divisible by \( p \). Note that \( Q_{p^2} \) stands for the completion of \( K \) at its unique prime above \( p \). Let us denote by

\[
\Psi : G_K \to \Gamma_K \xrightarrow{\gamma \mapsto \gamma^{-1}} \Gamma_K \hookrightarrow \Lambda_{\psi}(\Gamma_K)^\times
\]

\[
\Psi_1 : G_K \to \Gamma_1 \xrightarrow{\gamma \mapsto \gamma^{-1}} \Gamma_1 \hookrightarrow \Lambda_{\psi}(\Gamma_1)^\times
\]

the tautological characters. Put \( \Psi := \tilde{\chi} \Psi \) and \( r_\Psi := \text{Ind}_{K/\mathbb{Q}}(\tilde{\chi}) \).

**Definition 5.3.9.** Let us fix an \( \mathcal{O} \)-valued character \( \rho \) of \( \Gamma_K \). Denote by \( x_\rho : \Lambda_{\psi}(\Gamma_K) \to \Lambda_{\psi}(\Gamma_1) \) the unique homomorphism of \( \mathcal{O} \)-algebras which induces the isomorphism

\[
\Psi \otimes_{x_\rho} \Lambda_{\psi}(\Gamma_1) \xrightarrow{\sim} \rho^{-1} \otimes \Psi_1.
\]

In explicit terms, \( x_\rho(\gamma) = \rho(\gamma) \Psi_1(\gamma) \) for every \( \gamma \in \Gamma_K \). We let \( X_\rho := \ker(x_\rho) \subset \Lambda_{\psi}(\Gamma_K) \) denote the corresponding height-one prime ideal.
Suppose \( \psi \) is a crystalline Hecke character as before, with the additional requirement that \( \hat{\chi} \psi \) factors through \( \Gamma_K \). We will repeatedly make use of the following observation:

\[
(5.5) \quad \chi \otimes_{x \in \mathbb{Bdd}} \Lambda_O(\Gamma_1) = \hat{\chi} \Psi \otimes_{x \in \mathbb{Bdd}} \Lambda_O(\Gamma_1) \sim \hat{\chi} \otimes \hat{\psi}^{-1} \otimes \Psi_1 = \hat{\psi}^{-1} \otimes \Psi_1.
\]

Given \( \rho \) as in Definition \( 5.3.9 \), we shall also denote the induced homomorphism \( \Lambda_f(\Gamma_K) \to \Lambda_f(\Gamma_1) \) by the same symbol \( x_\rho \).

**Remark 5.3.10.** Since \( X_\rho \) is a height-one prime of the regular ring \( \Lambda_O(\Gamma_K) \), it is principal. In this remark, we will describe a certain generator of \( X_\rho \), which will be useful in what follows.

We first describe the kernel of the natural continuous surjection of \( \mathcal{O} \)-algebras

\[
(5.6) \quad x_0 : \Lambda_O(\Gamma_K) \longrightarrow \Lambda_O(\Gamma_1),
\]

which is also a height-one (therefore principal) prime ideal of \( \Lambda_O(\Gamma_K) \). Let \( \psi_{ac} : \Gamma_{ac} \to \Gamma_K \) be the verschiebung map, given by \( \gamma \mapsto \gamma^{-1} \), where \( \gamma \in \Gamma_{ac} \) and \( c \) is the generator of \( \text{Gal}(\mathbb{K}/\mathbb{Q}) \). Let us put \( \gamma_0 = \psi_{ac}(\gamma_{ac}) \), where \( \gamma_{ac} \) is a topological generator of \( \Gamma_{ac} \). Then \( x_0(\gamma_0) = 1 \) and hence \( (\gamma_0 - 1) \subset \ker(x_0) \). Since both \( \ker(x_0) \) and \( (\gamma_0 - 1) \) are height one primes, it follows that \( \ker(x_0) = (\gamma_0 - 1) \).

Abusing the language, we shall also put \( x_0 := \gamma_0 - 1 \).

Fix now an \( \mathcal{O} \)-valued character \( \rho \) of \( \Gamma_K \) as in Definition \( 5.3.1 \). We will consider the topological sub-algebra \( \mathcal{O}[[\gamma_0 - 1]] = \mathcal{O}[[x_0]] \subset \Lambda_O(\Gamma_K) \), with its prime ideal generated by \( f_\rho := \gamma_0 - \rho(\gamma_0) \subset \mathcal{O}[[x_0]] \). We will also regard \( \rho \) as a ring homomorphism \( \mathcal{O}[[x_0]] \to \mathcal{O} \), given by \( \gamma_0 \mapsto \rho(\gamma_0) \). Via the containment \( \mathcal{O}[[x_0]] \subset \Lambda_O(\Gamma_K) \), we may and will treat \( f_\rho \) as an element of \( \Lambda_O(\Gamma_K) \).

We next check that \( X_\rho \subset \Lambda_O(\Gamma_K) \) is generated by \( f_\rho \). To see that, we first observe that

\[
\rho(\Psi_1(\gamma_0 - \rho(\gamma_0)) = \rho(\Psi_1(\gamma_0) - \rho(\gamma_0) = \rho(\gamma_0)\Psi(\gamma_0) - \rho(\gamma_0) = 0,
\]

which means that \( f_\rho = \gamma_0 - \rho(\gamma_0) \in \ker(f_\rho) = X_\rho \). Since both \( (f_\rho) \) and \( X_\rho \) are height-one primes of \( \Lambda_O(\Gamma_K) \) and \( (f_\rho) \subset X_\rho \), it follows that \( (f_\rho) = X_\rho \), as required.

We consider the following uniform boundedness condition on the variation of the possible denominators \( s(\psi) \) as the Hecke character \( \psi \) varies:

\( \text{(Bdd}_{s(\psi)}) \) There exists a sequence of crystalline Hecke characters \( \psi_i \) such that the Galois characters \( \hat{\psi}_i \) factor through \( \Gamma_K \), and the infinity type of \( \psi_i \) equals \( (0, k_i + 1) \) where \( k_i \in \mathbb{N} \) and \( k_i \to \infty \) as \( i \to \infty \) for which the collection of integers \( \{ s(\psi_i) \} \) is bounded independently of \( i \).

We recall that by Corollary \( 3.2.4 \) the condition \( \text{(Bdd}_{s(\psi)}) \) holds true granted the existence of a rank-2 Euler system that the Perrin-Riou philosophy predicts.

**Theorem 5.3.11.** Suppose \( \chi \) is a ray class character as above and assume that \( \overline{\mathcal{I}_i} \) is absolutely irreducible.

1) There exists an element \( L_p^{RS}(f/K \otimes \psi) \in \Lambda_f(\Gamma_K) \) with the following interpolation property: For any crystalline Hecke character \( \psi \) as in \( \{ 5.3.1 \} \) and such that \( \hat{\chi} \psi \) factors through \( \Gamma_K \), we have

\[
x \chi \psi^{-1} \left( L_p^{RS}(f/K \otimes \chi) \right) = L_p^{RS}(f/K \otimes \psi),
\]

where the specialization map \( x_{\chi \psi}^{-1} \) is the one described in Definition \( 5.3.9 \).
ii) Assume that \( p \geq 7 \) as well as that the conditions \([\text{Full}]\) and \([\text{Bdd}_{\psi}(\psi)]\) hold true. We then have the following containment in the Iwasawa main conjecture for the family \( f_{\mathbf{K}} \otimes \chi \) of Rankin–Selberg products:

\[
L_p^\text{RS} (f_{\mathbf{K}} \otimes \chi) \in \text{char}_{\Lambda(\Gamma_K)} \left( \overline{\mathcal{H}}^2_{\chi}(G_K \Sigma, \mathbb{T}_{\text{f}_{\chi}}; \Delta_{\text{Gr}})^{\dagger} \right) \otimes \mathbb{Z}_p \mathbb{Q}_p.
\]  

In particular, the containment \([5.7]\) holds assuming only \( p \geq 7 \) and the validity of \([\text{Full}]\) if Conjecture \([5.2.1]\) (on the existence of rank-2 Euler systems) holds true.

**Proof of Theorem 5.3.11.**

i) The existence of \( L_p^\text{RS} (f_{\mathbf{K}} \otimes \chi) \) is an almost direct consequence of the recent work \([\text{Loc20}]\), extending the results of op. cit. slightly to construct a p-adic L-function

\[
L_p^\text{RS} (f_{\mathbf{K}} \otimes \rho^\text{univ}) \in \Lambda_f \otimes R^\text{univ}(r_{\chi})
\]

where \( r_{\chi} := \text{Ind}_{K/\mathbb{Q}} \chi^{-1} \) and \( \rho^\text{univ} \) is the minimally ramified universal deformation representation, and where \( R^\text{univ}(r_{\chi}) \) is the minimally ramified universal deformation ring of \( r_{\chi} \). This construction is carried out in detail (following Loeffler’s work very closely) in Appendix B. The key point is that our running assumptions on \( \chi \) imply that

- The representations \( r_{\chi} |_{G_{\chi(p)}} \) and \( r_{\chi} |_{G_{\mathbb{Q}p}} \) are both absolutely irreducible (since we assumed that \( \chi \not\equiv \chi^e \) and that the order of \( \chi \) is prime to \( p \));
- The lift \( r_\chi = \text{Ind}_{K/\mathbb{Q}} \chi \) is minimally ramified (since we assumed that the order of \( \chi \) is prime to \( p \)).

The second property induces (by the universality of \( \rho^\text{univ}_{r_{\chi}} \)) a continuous ring homomorphism \( \phi_{x} : R^\text{univ}(r_{\chi}) \to \Lambda_{\tau}(\Gamma_K) \) and the p-adic L-function \( L_p^\text{RS} (f_{\mathbf{K}} \otimes \chi) \) is defined as the image of \( L_p^\text{RS} (f_{\mathbf{K}} \otimes \rho^\text{univ}_{r_{\chi}}) \) under the map \( \text{id} \otimes \phi_{x} : \Lambda_f \otimes R^\text{univ}(r_{\chi}) \to \Lambda_f(\Gamma_K). \)

ii) The proof of this portion is very similar to the proof of Theorem 5.2.1. Let us put \( \xi := \xi_{\mu_1} L^\text{RS}_p (f_{\mathbf{K}} \otimes \chi) \) where \( \mu_1 \) is the unique natural number with \( \xi \in \Lambda_f(\Gamma_K) \setminus p \Lambda_f(\Gamma_K) \). Let us also choose an element \( S \in \Lambda_f(\Gamma_K) \setminus p \Lambda_f(\Gamma_K) \) such that

\[
\xi_{\mu_2} \Lambda_f(\Gamma_K) S = \text{char}_{\Lambda_{\mu}(\Gamma_K)} \left( \overline{\mathcal{H}}^2_{\chi}(G_K \Sigma, \mathbb{T}_{\text{f}_{\chi}}; \Delta_{\text{Gr}})^{\dagger} \right)
\]

for the suitable choice of a (uniquely determined) natural number \( \mu_2 \). Let us put \( s := \sup_{n=1}^{\infty} \{ s(\psi_i) \} \), where \( \psi_i \) are the Hecke characters given as in the statement of our theorem. We will prove that \( S | 2 \), using the conclusion of Theorem 5.3.9 applied with \( \psi = \psi_i \) and the divisibility criterion established in Appendix A. This will in turn prove the validity of the containment in the statement of our theorem.

As in Remark 5.3.10 (whose notation we shall adopt in the remainder of this proof), we will consider the topological subring \( \mathcal{O}[[x_0]] \subset \Lambda_{\mathbb{Q}}(\Gamma_K) \), together with its sequence of prime ideals generated by \( x_0 - \rho_i (x_0) \subset \mathcal{O}[[x_0]] \), where we have put \( \rho_i := \chi \psi_i \) (likewise \( f_i = f_{\rho_i} := x_0 - \rho_i (x_0) \), \( x_i = x_{\rho_i} : \Lambda_{\mathbb{Q}}(\Gamma_K) \to \Lambda_{\mathbb{Q}}(\Gamma_1) \)) and \( X_i = X_{\rho_i} \subset \Lambda_{\mathbb{Q}}(\Gamma_K) \)) to ease notation. As before, we shall also denote by \( x_i \) the ring map

\[
\Lambda_f(\Gamma_K) \to \Lambda_f(\Gamma_K) \xrightarrow{id \otimes x_i} \Lambda_f(\Gamma_1)
\]

as well as its kernel (which equals \( f_i \Lambda_f(\Gamma_K) = X_i \Lambda_f(\Gamma_K) \)) also by \( X_i \). Let us write \( \mathcal{L}_i := x_i (\xi) \in \Lambda_f(\Gamma_1) \) and similarly \( S_i := x_i (S) \). Recall that the morphisms \([5.8]\)
induce the morphisms of $G_K$-representations
\begin{equation}
x_i : T_{f,\chi}^K \rightarrow \mathbb{T}_{f,\psi_i}^{\text{cyc}},
\end{equation}

Let us fix an index $i$. It follows from the control theorem for Selmer complexes [Nek06 Corollary 8.10.2] that
\[
\tilde{H}_i^2(G_K, \mathbb{T}_{f,\chi}^{\text{cyc}}; \Delta_{\text{Gr}})^i / X_i \tilde{H}_i^2(G_K, \mathbb{T}_{f,\psi_i}^{\text{cyc}}; \Delta_{\text{Gr}})^i \rightarrow \tilde{H}_i^2(G_K, \mathbb{T}_{f,\psi_i}^{\text{cyc}}; \Delta_{\text{Gr}})^i.
\]

We therefore infer that
\[
\text{char}_{\Lambda_\omega(\Gamma_i)} \left( \tilde{H}_i^2(G_K, \mathbb{T}_{f,\chi}^{\text{cyc}}; \Delta_{\text{Gr}})^i[X_i] \right) \cdot (\varpi^{\mu_2} S_i)
\]
\[
= \text{char}_{\Lambda_\omega(\Gamma_i)} \left( \tilde{H}_i^2(G_K, \mathbb{T}_{f,\psi_i}^{\text{cyc}}; \Delta_{\text{Gr}})^i \right),
\]
in particular, also that
\begin{equation}
\varpi^{\mu_2} S_i \mid \text{char}_{\Lambda_\omega(\Gamma_i)} \left( \tilde{H}_i^2(G_K, \mathbb{T}_{f,\chi}^{\text{cyc}}; \Delta_{\text{Gr}})^i \right)
\end{equation}
as we contended to prove. Moreover, we have
\begin{equation}
\varpi^{\mu_1} \mathfrak{L}_i = L_{p_1}^{\text{RS}}(f_K \otimes \psi_i)
\end{equation}
thanks to the interpolative property of the universal (geometric) $p$-adic $L$-function $L_{p_1}^{\text{RS}}(f_K \otimes \chi)$. Under the assumptions of our theorem, the hypotheses of Theorem 5.3.6 are valid with the choice $\psi = \psi_i$, for any positive integer $i$. It follows from Theorem 5.3.6 (applied with $\psi = \psi_i$)
\[
\text{char}_{\Lambda_\omega(\Gamma_i)} \left( \tilde{H}_i^2(G_K, \mathbb{T}_{f,\chi}^{\text{cyc}}; \Delta_{\text{Gr}})^i \right) \mid \varpi^{s+\mu_1} \mathfrak{L}_i.
\]
This, combined with (5.2) and (5.3), allows us to conclude that $\varpi^{\mu_2} S_i \mid \varpi^{s+\mu_1} \mathfrak{L}_i$. Since $\mu$-invariants of $\mathfrak{L}_i$ and $S_i$ are both zero (by definition), it follows that $\mu_2 \leq s + \mu_1$, and also that
\[
S_i \mid \mathfrak{L}_i.
\]
We may now use Proposition A.0.1 (with the choices $R = \Lambda_\delta(\Gamma_K)$ and $R_0 := \mathbb{Z}_p[\gamma_0 - 1]$ given as in Remark 5.3.10 and $F = S, \sigma = \mathfrak{L}$), we conclude that $S \mid \mathfrak{L}$, as required.

**Corollary 5.3.12.** In the situation of Theorem 5.3.11(ii), the $\Lambda_\delta(\Gamma_K)$-module $\tilde{H}_i^2(G_K, \mathbb{T}_{f,\chi}^K; \Delta_{\text{Gr}})$ is torsion and $\tilde{H}_i^1(G_K, \mathbb{T}_{f,\chi}^K; \Delta_{\text{Gr}}) = 0$.

**Proof.** By the global Euler–Poincaré characteristic formula, the $\Lambda_\delta(\Gamma_K)$-module $\tilde{H}_i^2(G_K, \mathbb{T}_{f,\chi}^K; \Delta_{\text{Gr}})$ is torsion if and only if $\tilde{H}_i^1(G_K, \mathbb{T}_{f,\chi}^K; \Delta_{\text{Gr}})$ is. Under our running assumptions (which guarantee that the residual representation $\mathbb{T}_{f,\chi}^K$ is irreducible), the latter condition is equivalent to the vanishing of $\tilde{H}_i^1(G_K, \mathbb{T}_{f,\chi}^K; \Delta_{\text{Gr}})$. It therefore suffices to verify the first assertion.

In view of Theorem 5.3.11 it suffices to show that $L_{p_1}^{\text{RS}}(f_K \otimes \chi) \neq 0$. Thanks to its interpolation property, one reduces to checking that the complex $L$-function $L(f(\kappa) \otimes g, 1 + j)$ does not vanish for at least one choice of a crystalline specialization $\kappa$, a crystalline Hecke character $\psi$ of infinity type $(0, k_\psi + 1)$ ($g = \theta(\psi)$) such that $\chi \psi$ factors through $\Gamma_K$, and an integer $j$ with $k_\psi + 1 \leq j \leq \kappa$. This obviously can be arranged, e.g. making sure that $1 + j$ falls within the range of absolute convergence.
5.3.4. Anticyclotomic main conjectures in the inert case: general setup. In this section, we will descend our Theorem 5.3.11 to the anticyclotomic tower, using the formalism in [BL20a, §5.3.1] (which is, essentially, due to Nekovář). In particular, we continue to work in the setting of §5.3.3 and retain the notation therein.

Our treatment will naturally break into two threads: The first will concern the definite case, where we generically have $W(f(k)/K, \kappa/2) \neq -1$ for the global root number at the central critical points of the crystalline specializations $f(k)$. The second will be a treatment of the indefinite case, where we have $W(f(k)/K, \kappa/2) = -1$. We shall assume throughout §5.3.3 that the nebentype character $\varepsilon_f$ is trivial. At the expense of ink and space, one could also the more general scenario where one assumes only that $\varepsilon_f$ admits a square-root. We will also assume that $\chi$ is anticyclotomic, in the sense that $\chi^c = \chi^{-1}$.

The action of $\text{Gal}(K/\mathbb{Q})$ on $\Gamma_K$ gives a natural decomposition

$$\Gamma_K = \Gamma_K^+ \times \Gamma_K^- = \Gamma_{\text{ac}} \times \Gamma_1,$$

that the action of $\text{Gal}(K/\mathbb{Q})$ on $\Gamma_K$ (by conjugation) determines, where $\Gamma_K^\pm$ is the $\pm 1$-eigenspace for this action. We shall treat both $\Lambda_{\text{ac}}(\Gamma_{\text{ac}})$ and $\Lambda_{\text{cyc}}(\Gamma_{\text{cyc}})$ both as a subring and a quotient ring of $\Lambda(\Gamma_K)$ via the identification (5.12). Along these lines, we shall also identify $\Lambda_f(\Gamma_K)$ with $\Lambda_f(\Gamma_{\text{ac}})[[\Gamma_{\text{cyc}}]]$.

We put $\gamma_{\text{ac}} := \gamma \otimes_{\Lambda(\Gamma_{\text{ac}})} \Lambda(\Gamma_{\text{ac}})$ and $\gamma_{\text{cyc}} := \chi \otimes_{\Lambda(\Gamma_{\text{ac}})} \Lambda(\Gamma_{\text{ac}})$. We define the $p$-adic $L$-function $L_p^{\text{RS}}(f/K \otimes \gamma)$ under the obvious canonical projection. Let us fix a topological generator $\gamma$ (resp., $\gamma_-$) of $\Gamma_1$ (resp., of $\Gamma_{\text{ac}}$) such that $\{\gamma \times \text{id}_{\Gamma_1}, \text{id}_{\Gamma_{\text{ac}}} \times \gamma_+\}$ topologically generates $\Gamma_{\text{ac}} \times \Gamma_1 = \Gamma_K$.

**Definition 5.3.13.** We recall the universal weight character $\chi : G_Q \to \Lambda_{\text{cyc}}^\times$. We denote its square-root by $\chi^{1/2}$. In what follows, $? \in \{K, \text{ac}\}$.

i) We let $\text{Tw}_f : \Lambda_f(\Gamma_?) \to \Lambda_f(\Gamma_?)$ denote the $\Lambda_f$-linear morphism induced by $\gamma \mapsto \chi^{1/2}(\gamma)\gamma$ for each $\gamma \in \Gamma_?$. We set $L_p^f(f/K \otimes \gamma) := \text{Tw}_f(L_p^{\text{RS}}(f/K \otimes \gamma))$ and similarly define $L_p^f(f/K \otimes \gamma_{\text{ac}})$.

ii) We put $T_{f,\xi}^\dagger := T_{f,\xi}^K(-\chi^{1/2})$ and set $T_{f,\xi,\text{ac}}^\dagger := T_{f,\xi}^\dagger \otimes_{\Lambda(\Gamma_{\text{ac}})} \Lambda(\Gamma_{\text{ac}})$. Observe that $T_{f,\xi}^\dagger \otimes_{\Lambda(\Gamma_{\text{ac}})} \Lambda(\Gamma_{\text{ac}})$. We shall call $T_{f,\xi,\text{ac}}^\dagger$ the central critical twist of $T_{f,\xi}^\dagger$. We set $T_{f,\xi,\text{ac}}^\dagger := T_{f,\xi,\text{ac}}^\dagger \otimes_{\Lambda(\Gamma_{\text{ac}})} \Lambda(\Gamma_{\text{ac}})^\dagger$. Via the decomposition [5.72] we will identify $T_{f,\xi}^\dagger$ with $T_{f,\xi,\text{ac}}^\dagger$.

Thanks to our running hypothesis that $\varepsilon_f = 1$, Poincaré duality induces a perfect pairing

$$T_{f,\xi,\text{ac}}^\dagger \otimes_{\Lambda(\Gamma_{\text{ac}})} T_{f,\xi,\text{ac}}^\dagger \to \Lambda(\Gamma_{\text{ac}})(1),$$

namely, the Galois representation $T_{f,\xi,\text{ac}}^\dagger$ is conjugate self-dual.

We recall the Selmer complexes $R\Gamma_f(G_{K,\Sigma}, T_{f,\xi}^\dagger; \Delta_{\text{Gr}})$ ($? = \text{ac}, K$) that we have introduced in Definition 5.3.2. In an identical manner, we also have the Selmer complexes

$$\widehat{R\Gamma}_f(G_{K,\Sigma}, T_{f,\xi}^\dagger; \Delta_{\text{Gr}}) \in D_{\text{H}}(\Lambda(\Gamma_{\text{ac}})^\text{Mod})$$

In the more general scenario when $\varepsilon_f = \eta_f^2$ for some Dirichlet character $\eta_f$, the same conclusion is valid if one defines $\widehat{T}_{f,\xi,\text{ac}}^\dagger := T_{f,\xi,\text{ac}}^K(-\chi^{1/2} \eta_f^{-1}).$
Then the primes (this is what we refer to as the definite case).

Assume that the following conditions hold true:

\( (\text{5.15}) \quad L_f(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \in \text{Det}(\Lambda_f(\Gamma_{ac})_{\text{Mod}}) \)

whose cohomology groups we denote by \( \tilde{H}_f^1(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \), \( \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \) and \( \tilde{H}_f^3(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \), respectively. Twisting formalism (c.f., [Rub00], Lemma 6.1.2) shows that

(5.14) \quad \text{Tw}_{\mathbf{f}} \left( \text{char} \left( \tilde{H}_f^1(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \right) \right) = \text{char} \left( \tilde{H}_f^1(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \right) .

**Definition 5.3.14.** We let

\( h_{\text{Nek}}^\Lambda : \tilde{H}_f^1(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \otimes \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \rightarrow \Lambda_f(\Gamma_{ac}) \)

denote the \( \Lambda_f(\Gamma_{ac}) \)-adic (cyclotomic) height pairing given as [Nek06, §11.1.4], with \( \Gamma = \Gamma_{\text{cyc}} \). We define the \( \Lambda_f(\Gamma_{ac}) \)-adic regulator \( \text{Reg}_{\mathbf{f}, \text{ac}} \) by setting

\[
\text{Reg}_{\mathbf{f}, \text{ac}} := \text{char}_{\Lambda_f(\Gamma_{ac})} \left( \text{coker} \left( \tilde{H}_f^1(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \rightarrow \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \right) \right)
\]

where \( \text{adj} \) denotes adjunction. Note that \( \text{Reg}_{\mathbf{f}, \text{ac}} \) is non-zero if and only if \( h_{\text{Nek}}^\Lambda \) is non-degenerate.

**5.3.5. Anticyclotomic main conjectures: definite/inert case.** We retain the notation and hypotheses of §5.3.4. Let us write \( N_f = N^+ N^- \) where \( N^+ \) (resp., \( N^- \)) is divisible by only those primes which split (resp., remain inert) in \( K/\mathbb{Q} \). We assume in §5.3.5 that \( N^- \) is a square-free product of an odd number of primes (this is what we refer to as the definite case).

**Theorem 5.3.15.** Suppose \( \chi \) is a ring class character such that \( \bar{\chi}_{|G_{Gp}} \neq \bar{\chi}^\epsilon_{|G_{Gp}} \).

Assume that the following conditions hold true:

- \( p \geq 7 \) and the condition \( [\text{Full}] \) holds true.

- \( [\text{Bdd}(\psi)] \)

Then the \( \Lambda_f(\Gamma_{ac}) \)-module \( \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \) is torsion and the following containment in the anticyclotomic Iwasawa main conjecture for the family \( \mathbf{f}/K \otimes \mathcal{G}_{ac} \) holds:

(5.15) \quad L_f^1(\mathbf{f}/K \otimes \mathcal{G}_{ac}) \in \text{char}_{\Lambda_f(\Gamma_{ac})} \left( \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr})^\epsilon \right) \otimes \mathbb{Z}_p \mathcal{G}_p .

We recall that by Corollary 3.2.3, the condition \( [\text{Bdd}(\psi)] \) holds true granted the existence of a rank-2 Euler system that the Perrin-Riou philosophy predicts.

**Proof of Theorem 5.3.15.** It follows from the fundamental base change property for Selmer complexes [Nek06, Corollary 8.10.2], it follows that

\( \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr})/\left( \gamma_{\text{cyc}} - 1 \right) \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) \leftarrow \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr}) .
\]

Thence, if we let \( \pi_{ac} : \Lambda_f(\Gamma_K) \rightarrow \Lambda_f(\Gamma_{ac}) \) denote the canonical morphism, we conclude that

\[
\text{char}_{\Lambda_f(\Gamma_{ac})} \left( \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr})^\epsilon [\gamma_{\text{cyc}} - 1] \right) \cdot \pi_{ac} \text{char}_{\Lambda_f(\Gamma_K)} \left( \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr})^\epsilon \right) = \text{char}_{\Lambda_f(\Gamma_{ac})} \left( \tilde{H}_f^2(\Gamma_K, \mathbb{Q}_p^\times ; \Delta_{Gr})^\epsilon \right) .
\]
5.3. MAIN CONJECTURES OVER AN IMAGINARY QUADRATIC FIELD WHERE $p$ IS INERT

We deduce using this together with Theorem [5.3.11(ii)] combined with [5.14] and the definition of $L_p(f_K \otimes \mathcal{f}_{ac})$ that

$$L_p(f_K \otimes \mathcal{f}_{ac}) \cdot \text{char}_{A_\Gamma(\Gamma_{ac})} \left( \widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})|_{\gamma_{cyc} - 1} \right)$$

(5.16)

$$
\subset \text{char}_{A_\Gamma(\Gamma_{ac})} \left( \widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})|^{\gamma_{cyc} - 1} \right) \otimes \mathbb{Q}_p.
$$

Under our running hypothesis, [Hun17] Theorem C shows that $L_p(f_K \otimes \mathcal{f}_{ac}) \neq 0$. This also shows (again using Theorem [5.3.11(ii)] together with (5.14) and the definition of $L_p(f_K \otimes \mathcal{f}_{ac})$)

$$\gamma_{cyc} - 1 \not\in \text{char}_{A_\Gamma(\Gamma_{ac})} \left( \widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})|^{\gamma_{cyc} - 1} \right),$$

which in turn shows that the $A_\Gamma(\Gamma_{ac})$-module $\widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})|^{\gamma_{cyc} - 1}$ is torsion and therefore,

$$\text{char}_{A_\Gamma(\Gamma_{ac})} \left( \widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})|^{\gamma_{cyc} - 1} \right) \neq 0.$$

This fact combined with (5.10) shows that $\text{char}_{A_\Gamma(\Gamma_{ac})} \left( \widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})|^{\gamma_{cyc} - 1} \right) \neq 0$, thence also that the $A_\Gamma(\Gamma_{ac})$-module $\widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})$ is torsion. This concludes the proof of our first assertion.

We now prove the containment (5.19), which is an improved version of (5.16). The proof of [BL20a, Theorem 5.32(ii)] (which builds on Proposition 5.24 in op. cit., which itself is a translation of the general results in [Nek06, §11.7.11]) applies verbatim to show that

$$\pi_{ac} \text{char}_{A_\Gamma(\Gamma_{ac})} \left( \widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})\right) = \text{char}_{A_\Gamma(\Gamma_{ac})} \left( \widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})\right).$$

We note that our morphism $\pi_{ac}$ coincides with $\partial_{cyc}$ in op. cit. thanks to (5.10). We remark that in order to apply [BL20a, Proposition 5.24], it suffices to verify that both $A_\Gamma(\Gamma_{ac})$-modules $\widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})$ and $\widetilde{H}_2^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})$ are torsion (which then ensures the non-degeneracy of the height pairing $b_{\text{Nek}_{ac}} \otimes K, \Sigma$). We have checked the latter above and the fact that $\widetilde{H}_1^2(G_K, \Sigma, \mathcal{T}_{\tilde{f}_{ac}}; \Delta_{Gr})$ is torsion follows from this and the global Euler–Poincaré characteristic formulae.

Theorem [5.3.11(ii)] together with (5.18) combined with (5.14) and the fact that $\pi_{ac}(L_p^0(f_K \otimes \mathcal{f}))/L_p^0(f_K \otimes \mathcal{f}_{ac})$ (which follows from definitions) conclude the proof of (5.15).

5.3.6. Anticyclotomic main conjectures: indefinite/inert case. We retain the notation and hypotheses of [5.3.4]. As in the previous subsection, let us write $N_f = N^+ N^-$ but assume (in contrast with the previous subsection) that $N^-$ is a square-free product of even number of primes (the adjective "indefinite" is in reference to this condition). In this scenario, we have

$$\varepsilon(f(\kappa) \otimes \psi, (\kappa + k_g + 1)/2) = -1 \quad (\kappa > k_g)$$

(5.19)

for all crystalline specialization $f(\kappa)$ and crystalline Hecke characters $\psi$ (c.f. [5.3.1] to recall our conventions) with infinity type $(0, k_g + 1)$ verifying $k_g < \kappa$. Here, $\varepsilon(f(\kappa) \otimes \psi, s)$ is the global root number for the Rankin–Selberg $L$-function $L(f(\kappa) \times \theta(\psi), s) = L(f(\kappa) \otimes K, \psi, s)$; c.f. [Jac72, §15] (see also [BDP13, §4.1] for a detailed summary which we rely on in the present discussion). We further remark that
\( s = (\kappa + k_g + 1)/2 \) is the central critical point. In particular, the \( L\)-function
\( L(f(\kappa) \times \theta(\psi), s) \) vanishes to odd order at its central critical point.

In the complementary scenario where \( k_g > \kappa \), we have

\[
(5.20) \quad \varepsilon(f(\kappa)/K \otimes \psi, (\kappa + k_g + 1)/2) = +1 \quad (\kappa < k_g)
\]

for all crystalline specialization \( f(\kappa) \) and crystalline Hecke characters \( \psi \) whose
\( \infty \) type \( (0, k_g + 1) \) verifies \( k_g > \kappa \) (c.f. the discussion in [BDP13], Page 1036).

One may recast (5.19) and (5.20) in terms of anticyclotomic characters as fol-

\[
\text{Definition 5.3.17.} \quad \text{We set}
\]

\[
r(f, g_{\text{ac}}) := \text{rank}_{\Lambda_{(\Gamma_{ac})}} \left( H^1_I(G_K, \Sigma, \Sigma_{f_{\text{ac}}}) \right),
\]

\[
\text{Proposition 5.3.18.} \quad \text{We have}
\]

\[
(5.22) \quad r(f, g_{\text{ac}}) \leq r(f, g). \]

with equality if and only if the height pairing \( h_N^{\text{Nek}} \) is non-degenerate.

Proof. The first equality is a consequence of the Euler-Poincaré characteristic formula. The inequality (5.22) follows from the first equality combined with the control theorem for Nekovář’s extended Selmer groups (c.f., [Nek06], Corollary 8.10.2). It remains to prove the asserted criterion for equality.

It follows from [Nek06] Proposition 11.7.6(vii) that

\[
\text{length}_{\Lambda_{(\Gamma_{(\gamma+1)})}} \left( H^1_I(G_K, \Sigma, \Sigma_{f_{\text{ac}}}) \right)
\]

\[
(5.23) \quad \geq \text{length}_{\Lambda_{(\Gamma_{ac})}} \left( H^1_I(G_K, \Sigma, \Sigma_{f_{\text{ac}}}) \right)
\]

with equality if and only if the height pairing \( h_N^{\text{Nek}} \) is non-degenerate. Hence,

\[
r(f, g) = \text{length}_{\Lambda_{(\Gamma_{(\gamma+1)})}} \left( H^1_I(G_K, \Sigma, \Sigma_{f_{\text{ac}}}) \right)
\]

\[
\geq \text{length}_{\Lambda_{(\Gamma_{ac})}} \left( H^1_I(G_K, \Sigma, \Sigma_{f_{\text{ac}}}) \right)
\]

\[
= \text{rank}_{\Lambda_{(\Gamma_{ac})}} \left( H^1_I(G_K, \Sigma, \Sigma_{f_{\text{ac}}}) \right) = r(f, g_{\text{ac}})
\]
Remark 5.3.19. One expects that the height pairing $h^{\text{Nek}}_{f,\text{ac}}$ is always non-degenerate (equivalently, $\text{Reg}_{f,\text{ac}} \neq 0$); see Bur15, BD20 for progress in this direction (in a setting that unfortunately has no overlap with the scenario we have placed ourselves in).

Our main result in 5.3.4 is Theorem 5.3.20 below, which is a partial $\Lambda_f(\Gamma_{ac})$-adic BSD formula.

**Theorem 5.3.20.** Suppose $\chi$ is a ring class character such that $\widehat{\chi}|_{\mathbb{Q}_p} \neq \chi^r|_{\mathbb{Q}_p}$.

Assume that the following conditions hold true.

- $\mathcal{R}_f$ is absolutely irreducible.
- $p \geq 7$ and the condition (Full) holds true.
- $[\text{Bdd}_{\psi}(\psi)]$ is valid.

Then:

i) $\text{ord}_{\gamma-1}(L_p^1(f/K \otimes \chi)) \geq 1$.

ii) The following containment (partial $\Lambda_f(\Gamma_{ac})$-adic BSD formula for the family $f/K \otimes \chi_{ac}$) is valid:

\[
\partial^{(f,\text{ac})}_{\text{cyc}} L_p^1(f/K \otimes \chi) \in \text{Reg}_{f,\text{ac}} \cdot \text{char}_{\Lambda_f(\Gamma_{ac})} \left( \widehat{H}^2_f(G_K,\Sigma, T^f_{f,\text{ac}}; \Delta_{Gr})_{\text{tor}} \right) \otimes \mathbb{Z}_p \mathbb{Q}_p.
\]

We recall that by Corollary 3.2.4, the condition $[\text{Bdd}_{\psi}(\psi)]$ holds true granted the existence of a rank-2 Euler system that the Perrin-Riou philosophy predicts.

**Proof of Theorem 5.3.20.**

i) Using the interpolation properties of the $p$-adic $L$-function $L^R_p(f/K \otimes \chi)$ and the definition of $L^1_p(f/K \otimes \chi_{ac})$ (c.f. Definition 5.3.13(i)) and (5.21), it follows that

\[
L^1_p(f/K \otimes \chi_{ac})(\kappa, \chi^r) = L(f/K, \psi, \kappa/2) = 0
\]

(where $a \equiv b$ means $a = cb$ for some $c \in \mathbb{C}_p$) for all crystalline specializations $f(\kappa)$ and anticyclotomic (not necessarily crystalline) Hecke characters $\psi$ of infinity type $(m - \kappa/2, \kappa/2 - m)$ with $0 < m < \kappa$. This shows that $L^1_p(f/K \otimes \chi_{ac}) = 0$, which is equivalent to the assertion in this part of our theorem.

ii) If we combine Theorem 5.3.11(ii) with (5.14) and the definition of the twisted $p$-adic $L$-function $L^p(f \otimes \chi)$, we reduce to proving that

\[
\partial^{(f,\text{ac})}_{\text{cyc}} \text{char}_{\Lambda_f(\Gamma_{ac})} \left( \widehat{H}^2_f(G_K,\Sigma, T^f_{f,\text{ac}}; \Delta_{Gr})^{\dagger} \right)
\]

\[
\subset \text{Reg}_{f,\text{ac}} \cdot \text{char}_{\Lambda_f(\Gamma_{ac})} \left( \widehat{H}^2_f(G_K,\Sigma, T^f_{f,\text{ac}}; \Delta_{Gr})_{\text{tor}}^{\dagger} \right) \otimes \mathbb{Z}_p \mathbb{Q}_p.
\]

If $r(f, \chi) \geq r(f, \chi_{ac})$, we have

\[
\partial^{(f,\text{ac})}_{\text{cyc}} \text{char}_{\Lambda_f(\Gamma_{ac})} \left( \widehat{H}^2_f(G_K,\Sigma, T^f_{f,\text{ac}}; \Delta_{Gr})^{\dagger} \right) = 0 = \text{Reg}_{f,\text{ac}}
\]

where the second equality follows from Proposition 5.3.18. In other words, (5.25) trivially holds true when $r(f, \chi) \neq r(f, \chi_{ac})$. 


We will therefore assume in the remainder of our proof that \( r(f, g) = r(f, g_{\text{ac}}) \).

We contend that
\[
\partial_{\text{cyc}}(f, g) \cdot \text{char}_{\Lambda_f}(\Gamma_K) \left( \widetilde{H}_f^2(\mathcal{G}, \mathbb{T}^\dagger; \Delta_{\text{Gr}}'') \right) \subset \text{Reg}_{f, \text{ac}} \cdot \text{char}_{\Lambda_f}(\Gamma_\text{ac}) \left( \widetilde{H}_f^2(\mathcal{G}, \mathbb{T}^\dagger_{f, \text{ac}}; \Delta_{\text{Gr}}') \right) \otimes \mathbb{Z}_p Q_p.
\]

We will verify (5.26) using [BL20a, Proposition 5.24], which is a simplification of Nekovář’s result [Nek06, §11.7.11]. To apply Proposition 5.24 in [BL20a], we need to check that the following properties hold true.

1. The height pairing \( h_{f, \text{ac}}^{\text{Nek}} \) is non-degenerate.
2. The Tamagawa factors denoted by \( \text{Tam}_{\text{ac}}(\mathbb{T}^\dagger_{f, \text{ac}}, P) \) in [BL20a] (c.f. Definition 5.21 in op. cit.) vanish for every height-one prime \( P \) of \( \Lambda_f(\Gamma_{\text{ac}}) \).
3. For \( i = 0, 3 \), we have \( \widetilde{H}_f^i(\mathcal{G}, \mathbb{T}^\dagger; \Delta_{\text{Gr}}) = 0 = \widetilde{H}_f^i(\mathcal{G}, \mathbb{T}^\dagger_{f, \text{ac}}; \Delta_{\text{Gr}}') \).
4. Both \( \Lambda_f(\Gamma_K) \)-modules \( \widetilde{H}_f^i(\mathcal{G}, \mathbb{T}^\dagger; \Delta_{\text{Gr}}) \) and \( \widetilde{H}_f^i(\mathcal{G}, \mathbb{T}^\dagger_{f, \text{ac}}; \Delta_{\text{Gr}}') \) are torsion.

The first property holds thanks to Proposition 5.3.18 (since \( r(f, g) = r(f, g_{\text{ac}}) \)). The second property follows from [Nek06, Corollary 8.9.7.4] applied with \( T = \mathbb{T}^\dagger_f \) and \( \Gamma = \Gamma_{\text{ac}} \). The third property is immediate thanks to our running assumptions, which guarantee that the residual representation \( \mathbb{T}^\dagger_{f, \text{ac}} \) is absolutely irreducible. The final property is Theorem 5.3.12 combined with (5.14). This completes the proof.

\[ \text{REMARK 5.3.21.} \quad \text{In the situation of Theorem 5.3.20, one expects that} \]
\[ \text{ord}_{\gamma_+ - 1}(L^\dagger_{f, \text{ac}}(f, K) \otimes \mathbb{Q}) = 1 \quad \text{if and only if} \quad r(f, g_{\text{ac}}) \neq 0. \]

In the setting where the prime \( p \) splits in \( K/\mathbb{Q} \), the second expected equality follows from [BL18, Theorem 3.15]. When the prime \( p \) splits in \( K/\mathbb{Q} \), one could also utilize [BL18, Theorem 3.30] to show that the first expected equality holds if and only if \( \text{Reg}_{f, \text{ac}} \neq 0 \).

\[ \text{COROLLARY 5.3.22.} \quad \text{In the setting of Theorem 5.3.20, suppose in addition that} \]
\[ r(f, g_{\text{ac}}) \geq 1. \quad \text{Then,} \]
\[
\partial_{\text{cyc}}L^\dagger_{f, \text{ac}}(f, K \otimes \mathbb{Q}) \in \text{Reg}_{f, \text{ac}} \cdot \text{char}_{\Lambda_f}(\Gamma_\text{ac}) \left( \widetilde{H}_f^2(\mathcal{G}, \mathbb{T}^\dagger_{f, \text{ac}}; \Delta_{\text{Gr}}') \right) \otimes \mathbb{Z}_p Q_p.
\]

\[ \text{PROOF.} \quad \text{Combining Proposition 5.3.18 and Theorem 5.3.11 ii together with (5.14) and the definition of the twisted \( p \)-adic \( L \)-function \( L^\dagger_{f, \text{ac}}(f, \mathbb{Q}) \), it follows that} \]
\[ \partial_{\text{cyc}}L^\dagger_{f, \text{ac}}(f, \mathbb{Q}) = 0 \quad \text{if} \quad \text{and} \quad r(f, g_{\text{ac}}) \geq 1; \quad \text{and hence, there is nothing to prove when} \]
\[ r(f, g_{\text{ac}}) \geq 1. \quad \text{We have therefore reduced to proving (5.27) when} \]
\[ r(f, g_{\text{ac}}) = 1, \quad \text{and this is precisely (5.24) in this case.} \]

The containment (5.27) is in line with Perrin-Riou’s anticyclotomic main conjectures. Based on the recent work of Andreatta and Iovita [A120], it seems that the necessary tools to prove that \( r(f, g_{\text{ac}}) \geq 1 \) might be available. We discuss this point in the following remark.
Remark 5.3.23. Suppose that we are in the setting of Theorem 5.3.20. If \( r(f, g_{ac}) = 0 \), it follows from control theorems for Selmer complexes that there exists \( N_f \in \mathbb{Z}^+ \) such that \( \tilde{H}^1_f(G_{K, \Sigma}, T_{F, \psi}(-\kappa/2); \Delta_{Gr}) = 0 \) for all crystalline anticyclotomic Hecke characters \( \psi \) of infinity type \((-a, a)\) with \( a > N_f \) and such that \( \chi \psi \) factors through \( \Gamma_K \). This implies (again by the control theorems for Selmer complexes) that for each \( \psi \) as above, there exists another constant \( M_{f, \psi} \) such that \( \tilde{H}^1_f(G_{K, \Sigma}, T_{F, \psi}(-\kappa/2); \Delta_{Gr}) = 0 \) for all (classical) crystalline weights \( \kappa > M_{f, \psi} \).

In light of the recent work of Andreatta and Iovita \( \text{AI20} \) towards Bertolini–Darmon–Prasanna type formulae in the inert case, one expects that

\[
\text{rank} \tilde{H}^1_f(G_{K, \Sigma}, T_{F, \psi}(-\kappa/2); \Delta_{Gr}) \geq 1,
\]

for \( \kappa > 0 \), which would contradict the discussion above and prove that \( r(f, g_{ac}) \geq 1 \). In other words, granted a slight extension of the work of Andreatta–Iovita (which we expect to be established in the near future), the containment \( \text{AI20} [5.27] \) in Perrin-Riou’s anticyclotomic main conjecture is valid.

Remark 5.3.24. Suppose that we are still in the setting of Theorem 5.3.20. We will explain how an improvement of Hsieh’s generic non-vanishing result \( \text{AI20} [5.3.23] \) can be used to prove the opposite containment (to what we have discussed in Remark 5.3.23) \( r(f, g_{ac}) \leq 1 \).

Suppose that the following non-vanishing statement (in the flavor of those proved in \( \text{Gre85} \)) holds true: There exist a crystalline specialization \( f(\kappa) \) and infinitely many crystalline anticyclotomic Hecke characters \( \psi \) of infinity type \((m - \kappa/2, \kappa/2 - m)\) such that \( L(f(\kappa)/K, \psi, \kappa/2) \neq 0 \) (necessarily with \( m < 0 \)). One expects that this is valid for any \( \kappa \). We can then prove using \( \text{LZ16} \) [Theorem 8.1.4] that the Bloch–Kato Selmer group \( H^1_f(K, T_{F, \psi}(-\kappa/2)) \) vanishes. Moreover, it is easy to see that

\[
\text{rank} \tilde{H}^1_f(G_{K, \Sigma}, T_{F, \psi}(-\kappa/2); \Delta_{Gr}) - \text{rank} H^1_f(K, T_{F, \psi}(-\kappa/2)) \leq 1.
\]

This in turn shows that \( \text{rank} \tilde{H}^2_f(G_{K, \Sigma}, T_{F, \psi}(-\kappa/2); \Delta_{Gr}) \leq 1 \) for all \( \kappa \) and \( \psi \) as above. By the control theorems for Selmer complexes, we conclude that \( r(f, g_{ac}) \leq 1 \).

---

5One could even prove this once the \( p \)-adic \( L \)-functions of \( \text{AI20} \) are interpolated as the eigenform \( f \) varies in a Hida family.

6This result asserts in our set up that given a crystalline anticyclotomic Hecke character \( \phi \) of infinity type \((m - \kappa/2, \kappa/2 - m)\) and for which \( \phi \) factors through \( \Gamma_K \), we have \( L(f(\kappa)/K, \eta \phi, \kappa/2) \neq 0 \) for all but finitely many characters \( \eta \) of \( \Gamma_{ac} \), under the additional requirement that all local root numbers of \( f(\kappa)/K \otimes \phi \) are \( +1 \) at every place of \( K \).
A divisibility criterion in regular rings

Suppose $R$ is a complete local regular ring of dimension $n + 2$ ($n \geq 1$) and with mixed characteristic $(0, p)$. Let $\mathfrak{m}$ denote its maximal ideal. Assume that $x_0, x_1, \cdots, x_n$ is a regular sequence in $R$ such that $p \not\in (x_0, x_1, \cdots, x_n)$. Let $(\varpi) \subset R$ denote the unique height-one prime of $R$ that contains $p$. The injective ring homomorphism

$$\iota_0 : \mathbb{Z}_p[[x_0]] \rightarrow R$$

endows $R$ with the structure of a flat $R_0 := \mathbb{Z}_p[[x_0]]$-module.

Suppose that $F, G \in R$ are two non-zero elements with the following properties:

a) $\mathfrak{m} \nmid FG$.

b) $x_0 \nmid F$.

c) There exists a collection of irreducible elements $\{f_i\}_{i=1}^{\infty} \subset \mathbb{Z}_p[[x_0]]$ which are pairwise coprime, with the property that the image of $G$ under the natural map

$$R \rightarrow R/(f_i, F) = R_0/R_0f_i \otimes_{\iota_0: R_0 \rightarrow R} R/(F)$$

is zero. Here and also below, we write $x$ in place $\iota_0(x)$ for $x \in R_0$.

**Proposition A.0.1.** If $F$ and $G$ are as above, then $F$ divides $G$ over $R$.

**Proof.** For each positive integer $n$, let us define $g_n := f_1 \cdots f_n$, so that we have $Rg_{n+1} \subset Rg_n =: b_n$.

Observe that we have an injection

$$R_0/R_0g_n \hookrightarrow \prod_{i=1}^{n} R_0/R_0f_i$$

for each positive integer $n$. Since we assumed $x_0 \nmid F$ and $\mathfrak{m} \nmid F$, and since $R$ is a UFD, it follows that $R/(F)$ is flat as an $R_0$-module. Hence, we have an injection

$$R/(g_n, F) = R_0/R_0g_n \otimes_{\iota_0: R_0 \rightarrow R} R/(F) \hookrightarrow \prod_{i=1}^{n} \left( R_0/R_0f_i \otimes_{\iota_0: R_0 \rightarrow R} R/(F) \right) = \prod_{i=1}^{n} R/(f_i, F).$$

This injection together with property (c) above shows that $G \in (g_n, F)$ for every positive integer $n$.

We will next prove that $\cap_n (g_n, F) = (F)$ and this will conclude the proof of our proposition.

To that end, suppose $a \in \cap_n (g_n, F)$, so that for every positive integer $n$, there exists $r_n, s_n \in R$ with $a = r_ng_n + Fs_n$. We first prove that the sequence
A. A DIVISIBILITY CRITERION IN REGULAR RINGS

$s_n$ converges $m$-adically. Since the ring $R$ is complete with respect to the $m$-adic topology, it suffices to prove that $\{s_n\}$ is a Cauchy sequence. Suppose we are given any positive integer $M$. According to [Che43, Lemma 7], there exists $N \in \mathbb{Z}^+$ with $g_n \in m^M$ for all $n \geq N$. This in turn means that

$$(s_n - s_m) \cdot F = r_m g_m - r_n g_n \in m^M$$

for every $n, m > N$. This concludes the proof that the sequence $\{s_n\}$ is Cauchy, and therefore convergent. Let us put $s := \lim_{n \to \infty} s_n \in R$. The lemma of Chevalley also shows that $\lim_{n \to \infty} g_n = 0$, and in turn also that

$$a = \lim_{n \to \infty} a = \lim_{n \to \infty} r_n g_n + F s_n = \lim_{n \to \infty} r_n g_n + \lim_{n \to \infty} F s_n = F s$$

concluding the proof that $a \in (F)$. This shows that $\cap_n (g_n, F) \subset (F)$, and in turn also $\cap_n (g_n, F) = (F)$, as required. \qed
APPENDIX B

$p$-adic Rankin–Selberg $L$-functions and universal deformations

In this appendix, we will go over the work of Loeffler’s recent work \cite{Loe20}, where he constructs $p$-adic Rankin–Selberg $L$-functions and extend it slightly to cover the case of minimally ramified universal deformation representation.

We make crucial use of Loeffler’s construction to study $h_{/K} \otimes \chi$ where $h_{/K}$ is the base change of a $p$-ordinary $p$-stabilized cuspidal eigenform $h$ to the imaginary quadratic field $K$ where $p$ is inert and $\chi$ is a ray class character of $K$.

To be more precise, let us put $K(p^{\infty}) = \bigcup_n K(p^n)$, where $K(p^n)$ is the ray class extension modulo $p^n$. Let us also set $H_{p^{\infty}} := \text{Gal}(K(p^{\infty})/K)$. In this scenario, one is interested in the theta-series $\theta(\chi \psi)$, as $\psi$ ranges over Hecke characters whose $p$-adic avatars $\hat{\psi}$ give rise to (via global class field theory) a character of $H_{p^{\infty}}$. These are CM forms, which are non-ordinary at $p$ (since $p$ is inert in $K/\mathbb{Q}$). It is easy to see that the eigencurve cannot contain families of CM forms; c.f. \cite[CIT16, Corollary 3.6]{CIT16}. This is due to the fact that slope of any non-ordinary CM form of weight $k$ is at least $(k-1)/2$, so the refinements of $\theta(\chi \psi)$ cannot be contained in a finite slope family. All this tells us that the $p$-adic $L$-function which can have any bearing to our situation cannot descend from a construction over the eigencurve. In other words, one should work instead over the universal deformation space.

Indeed, in contrast to \cite[CIT16, Corollary 3.6]{CIT16}, one can interpolate the Galois representations $\{\text{Ind}_{K/\mathbb{Q}} \chi \psi\}_\psi$ to a $p$-adic family of Galois representations $\text{Ind}_{K/\mathbb{Q}} \chi \Psi$, where $\Psi$ is the tautological (universal) $p$-adic Hecke character, which is given as the compositum

$$G_K \rightarrow H_{p^{\infty}} \rightarrow \Lambda_{\mathcal{O}}(H_{p^{\infty}})^\times.$$

Loeffler’s work \cite{Loe20} gives rise to a $p$-adic $L$-function

$$L_p^{RS}(f_{/K} \otimes \Psi) \in \frac{1}{H_f} \Lambda_{\mathcal{O}}(H_{p^{\infty}})^\times$$

(where $H_f$ is the congruence number of $f$, in the sense of Hida). To be more precise, in order to obtain the $p$-adic $L$-function $L_p(f_{/K} \otimes \Psi)$ above, one in fact first constructs a $p$-adic $L$-function

$$L_p^{RS}(f_{/K} \otimes \Psi) \in \frac{1}{H_f} \Lambda_{\mathcal{O}}(H_{p^{\infty}})^\times,$$

(where $f$ is the unique Hida family through $f$ and $H_f$ is Hida’s congruence ideal) and specializes $f$ to $f$.

In this appendix, we review (and slightly extend, along the lines of the suggestion in \cite[§5.1]{Loe20}) Loeffler’s construction in a setting just a tad more general than what is required to obtain the $p$-adic $L$-function in (B.2) (we still work under...
a simplifying “minimality” condition). All results herein are due to Loeffler (but mistakes are ours) and we claim no originality.

## B.1. The set up

We let \( \mathcal{F} \) denote the residue field of \( \mathcal{O} \) (where \( \mathcal{O} \) is the ring of integers of a finite extension \( L \) of \( \mathbb{Q}_p \), as in the main body of this article) and let \( \overline{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(\mathcal{F}) \) be a two-dimensional, odd, irreducible Galois representation. It is known (thanks to Khare–Wintenberger) that \( \overline{\rho} \) is modular. We assume in addition the following conditions.

1. The restriction \( \overline{\rho}|_{G_{\mathbb{Q}(\mu_p)}} \) of \( \overline{\rho} \) to \( G_{\mathbb{Q}(\mu_p)} \) is irreducible.
2. Either \( \overline{\rho}|_{G_{\mathbb{Q}_p}} \) is irreducible, or else if we have

\[
\overline{\rho}^{\text{s.s.}} = \chi_{1,p} \oplus \chi_{2,p}
\]

for the semisimplification of \( \overline{\rho} \), then \( \chi_{1,p}/\chi_{2,p} \) is different from \( 1 \) or \( \omega^{\pm 1} \), where \( \omega \) is the Teichmüller character giving the action of \( G_{\mathbb{Q}_p} \) on \( \mu_p \).

We let \( N \) denote the Artin conductor of \( \overline{\rho} \); this is a positive integer coprime to \( p \).

**Remark B.1.1.** In applications, we shall set \( \overline{\rho} = \overline{\rho}_g \), where \( g \) is the non-ordinary newform in the main body of the present article and the integer \( N \) will be the level \( N_g \) of \( g \). In other words, whenever we refer to Appendix B, we will assume that \( g \) is minimally ramified at all primes dividing \( N_g \), in the sense of [Dia97, §3]. It should be possible to relax this assumption, but we will content to work under this assumption since it covers our case of interest when \( g \) arises as the \( \theta \)-series of a Hecke character of an imaginary quadratic field where \( p \) is inert.

**Example B.1.2.** Let \( K \) be an imaginary quadratic field as above where \( p \) remains inert and let \( \chi \) be a ray class character of \( K \) as in §5.3.3, of conductor \( p \chi \). We shall always assume that both the order of \( f \) and \( f \chi \) are coprime to \( p \).

For applications towards Iwasawa main conjectures for \( f_K \otimes \chi \) and \( f_K \otimes \chi \) (where \( f \) and \( f \) are as in the main body of this article), we shall take \( \overline{\rho} := \text{Ind}_{K/\mathbb{Q}}^F . \) As in §5.3.3, we shall assume that

\[
(\text{non-Eis}) \quad \chi|_{G_{\mathbb{Q}_p}} \neq \chi|_{G_{\mathbb{Q}_p}} \quad \text{where } c \text{ is the generator of } \text{Gal}(K/\mathbb{Q})
\]

and \( \chi^c(\sigma) = \chi(\sigma c^{-1}) \).

Let us also put \( N = |D_K|/|N_{K/\mathbb{Q}}| \chi \), where \( D_K/\mathbb{Q} \) is the discriminant of \( K/\mathbb{Q} \).

1. We explain that \( \overline{\rho} = \text{Ind}_{K/\mathbb{Q}}^F \) verifies the hypothesis \( (\overline{\rho}1) \). First of all, since \( p \) doesn’t divide the order of \( \chi \), it follows from the condition \( (\text{non-Eis}) \) that

\[
\overline{\chi} \neq \overline{\chi}^c.
\]

Let us set \( H_1 := G_K \), \( H_2 := G_{\mathbb{Q}(\mu_p)} \) and \( G = G_{\mathbb{Q}} \). Observe that both \( H_1 \) and \( H_2 \) are normal subgroups of \( G \). Since \( K \) and \( \mathbb{Q}(\mu_p) \) are linearly disjoint, it follows that \( H_1 H_2 = G \) and \( H_1 \cap H_2 = G_{\mathbb{Q}(\mu_p)} \). Moreover, \( c \) lifts canonically to a generator of \( \text{Gal}(\mathbb{Q}(\mu_p)/K(\mu_p)) = H_2/H_1 \cap H_2 \), which we still denote by \( c \). This discussion is

\[\text{except that in this appendix, we denote the Galois character that one associates to } \chi \text{ also by } \chi, \text{ rather than } \overline{\chi}.\]
summarized in the following diagram.

Then by Mackey theory, we have an isomorphism
\[ \overline{\rho}_{|H_2} = \text{Ind}_{H_1 \cap H_2}^{H_2} \chi \]
which is irreducible thanks to (B.3) and because \( H_2 / H_1 \cap H_2 \) is generated by \( c \). This concludes the proof that (71) holds true.

ii) The condition (72) holds true thanks to our running hypothesis (non-Eis) together with our assumption that the order of \( \chi \) is prime to \( p \).

B.2. The minimally ramified universal deformation representation

Let us fix a representation \( \overline{\rho} \) as in §B.1. We let \( S \) denote the set of places of \( \mathbb{Q} \) consisting of the archimedean place and all primes dividing \( N_p \) and set \( G_{\mathbb{Q}, S} := \text{Gal}(\mathbb{Q}_S / \mathbb{Q}) \), the Galois group of the maximal extension \( \mathbb{Q}_S \) of \( \mathbb{Q} \) unramified outside \( S \). In what follows, we shall consider \( \overline{\rho} \) as a representation of \( G_{\mathbb{Q}, S} \).

Our summary in this subsection is essentially identical to the discussion in [Loc20, §3] (where \( N = 1 \)), since we will only consider minimally ramified deformations.

**Definition B.2.1.**

i) We let \( R(\overline{\rho}) \) denote the minimally ramified universal deformation ring of \( \overline{\rho} \), parametrizing deformations of \( \overline{\rho} \) to \( G_{\mathbb{Q}, S} \)-representations to complete local Noetherian \( \mathcal{O} \)-algebras with residue field \( \mathbb{F} \) which are minimally ramified at primes dividing \( N \) in the sense of [Dia97, §3]. We put
\[ \rho^{\text{univ}} : G_{\mathbb{Q}, S} \to \text{GL}(V(\overline{\rho})^{\text{univ}}) \cong \text{GL}_2(R(\overline{\rho})) \]
to denote the minimally ramified universal deformation representation. We set \( X(\overline{\rho}) := \text{Spf} R(\overline{\rho}) \) and call it the minimally ramified universal deformation space.

ii) If \( g \) is a classical newform of level \( Np^s \) for some non-negative integer \( s \) such that \( \overline{\rho}_g = \overline{\rho} \), then \( \rho_g \) determines a \( \overline{\mathbb{Q}}_p \)-valued point of \( X(\overline{\rho}) \). Such points of the minimally ramified universal deformation space will be called classical.

iii) We say that a \( \overline{\mathbb{Q}}_p \)-valued point of \( X(\overline{\rho}) \) is nearly-classical of weight \( k \) if the said point corresponds to the Galois representation \( \rho \) for which we have \( \rho \cong \rho_g \otimes \chi_{\text{cyc}}^t \) for some (uniquely determined) classical newform \( g \) of weight \( k \) as in (ii) and (a uniquely determined) integer \( t \).

**Proposition B.2.2.** For any integer \( k \geq 2 \), the nearly-classical points of weight \( k \) are Zariski-dense in the minimally ramified universal deformation space.
We let $\lambda$ write (B.5) and Emerton [Böc01] required statement as a consequence of a fundamental result due to Böckle and Emerton [Eme11, Theorem 1.2.3].

**Definition B.2.3.** Let us write $\det \overline{\rho} = \epsilon^{(N)}(p)$, where $\epsilon^{(N)}$ (resp., $\epsilon^{(p)}$) is a character of conductor $N$ (resp., of conductor $p$). We let $\epsilon := G_{Q,S} \to W(\mathbb{F}) \leftarrow O^\times$ denote the Teichmüller lift of $\epsilon^{(N)}$ to the Witt vectors $W(\mathbb{F})$ of $\mathbb{F}$.

i) We let $k : \mathbb{Z}_p^\times \to \mathcal{R}(\overline{\rho})^\times$ denote the sequence determined via the formal identity

$$\sum_{p\mid n} t_n n^{-s} = \prod_{\ell \neq p} \det (1 - \ell^{-s}\text{Frob}_\ell^{-1}|(V(\overline{\rho}))^{\text{uni}})|t_\ell|^{-1}.$$  

We set

$$G^{[p]} := \sum_{p\mid n} t_n q^n \in \mathcal{R}(\overline{\rho})[[q]].$$

**B.3. Hida families**

Let $f \in S_{k_f + 2}(\Gamma_1(N_f), \varepsilon_f)$ be a cuspidal newform which admits a $p$-ordinary $p$-stabilization $f_\alpha \in S_{k_f + 2}(\Gamma_1(N_f) \cap \Gamma_0(p), \varepsilon_f^{(p)})$, as in the main body of our article (where $\varepsilon_f^{(p)}$ is the non-primitive Dirichlet character modulo $N_f p$ associated to $\varepsilon_f$).

Also as before, we let $f \in S^{\text{ord}}(N_f, \Lambda)$ denote the unique primitive $\Lambda$-adic ordinary eigenform of tame level $N_f$ which admits $f_\alpha$ as a specialization, where $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is the weight space. We choose a generator $H_f$ of the congruence module Hida has associated to $f$ in [Hid88, §4].

Mimicking Hida’s construction in [Hid88] (7.5), we define

$$\lambda_f : S^{\text{ord}}(N_f, \Lambda) \otimes_\Lambda \Lambda_f \xrightarrow{<1, \varepsilon_f> \cdot \frac{1}{H_f}} \Lambda_f,$$

where $1_f \in h^{\text{ord}}(N_f) \otimes_\Lambda \text{Frac}(\Lambda_f)$ is the idempotent that Hida in op. cit. denotes by $1_{\text{Frac}(\Lambda_f)}$ (and $h^{\text{ord}}(N_f)$ is Hida’s universal Hecke algebra of tame level $N_f$), which corresponds to the choice of the primitive form $f$ as a branch of the space of the universal ordinary branch with tame level $N_f$, and where the pairing $<,>$ is the one given in [Hid88] (7.3). Note that the image of $\lambda_f$ a priori lands in $\text{Frac}(\Lambda_f)$, since a priori $1_f \in h^{\text{ord}}(N_f) \otimes_\Lambda \text{Frac}(\Lambda_f)$. However, Hida explains that by the very definition of the congruence ideal in [Hid88] (4.3), it follows that $H_f \cdot 1_f \in h^{\text{ord}}(N_f) \otimes_\Lambda \Lambda_f$. For any integer $M$ divisible by $N_f$ and coprime to $p$,

we will write $\lambda_f^{(M)}$ for the compositum

$$\lambda_f^{(M)} : S^{\text{ord}}(M, \Lambda) \otimes_\Lambda \Lambda_f \xrightarrow{\text{tr}_{N_f}^M} S^{\text{ord}}(N_f, \Lambda) \otimes_\Lambda \Lambda_f \xrightarrow{\lambda_f} \frac{1}{H_f} \Lambda_f$$

where $\text{tr}_{N_f}^M$ is the trace map.
B.4. \( p \)-adic Rankin–Selberg \( L \)-function

Let \( \mathcal{P} : G_{Q,S} \to \GL_2(\mathbb{F}) \) be as in §B.1 and recall that \( N \) denotes its Artin conductor and \( \rho^{\text{univ}} \) its minimal universal deformation. We also fix \( f \) and \( f \) as in §B.3, with the additional assumption that \( \gcd(N_f, N_g) = 1 \). Following [Loe20 §4], we will define a \( p \)-adic Rankin–Selberg \( L \)-function

\[
L_p(\rho_f \otimes \rho^{\text{univ}}) \in \frac{1}{H_f} \Lambda_f \otimes R(\mathcal{P}).
\]

Note that thanks to this simplification together with the minimality condition on the deformation problem we consider on \( \mathcal{P} \), the bad primes require no additional treatment.

Let us put \( M = NN_f \). We enlarge \( O \) so that it contains \( M \)th roots of unity.

We set \( \zeta_M := \iota(e^{2\pi/M}) \) where we recall that \( \iota : \mathbb{C} \to \mathbb{C}_p \) is the isomorphism we have fixed at the start of this article.

**Definition B.4.1.** Consider the universal \( p \)-depleted Eisenstein series

\[
E_k[p] := \sum_{\substack{p \nmid n \nmid n \geq 1}} \left( \sum_{d,k-1} (\zeta_{M}^d + (-1)^k \zeta_{M}^{-d}) \right) q^n \in R(\mathcal{P})[[q]].
\]

We define the \( p \)-adic Rankin–Selberg \( L \)-function \( L_p(\rho_f \otimes \rho^{\text{univ}}) \) on setting

\[
L_p(\rho_f \otimes \rho^{\text{univ}}) := \lambda_f^{(M)} \left( e^{\text{ord}} \left( \mathcal{G}_p^{[p]} \otimes E_k[p] \right) \right).
\]

The \( p \)-adic \( L \)-function \( L_p(\rho_f \otimes \rho^{\text{univ}}) \) interpolates the critical values of Rankin–Selberg \( L \)-series. We formulate this as a statement which relates \( L_p(\rho_f \otimes \rho^{\text{univ}}) \) to the “geometric” \( p \)-adic \( L \)-functions of Loeffler and Zerbes, which we have recalled in §3.1. For a more “honest” version of the interpolation property in a particular case of interest involving the special values of the Rankin–Selberg \( L \)-series, see Theorem B.4.5.

**Theorem B.4.2.** Suppose \( g \in \mathcal{X}(\mathcal{P})(L) \) be an \( L \)-valued classical point. Then

\[
L_p(\rho_f \otimes \rho^{\text{univ}})(g) = L_p^{\text{geo}}(f,g).
\]

**Proof.** This is immediate thanks to the interpolation properties of both sides, which one obtains via the Rankin–Selberg integral formulae, c.f. [Loe18 Proposition 2.10]. The theorem is proved in a manner similar to [Loe18 Theorem 6.3]; see also [Loe20 Theorem 4.5].

**B.4.1. \( p \)-adic \( L \)-functions over an imaginary quadratic field where \( p \) is inert.** In this paragraph, we specialize to the setting of Example B.1.2. To that end, we let \( K \) be an imaginary quadratic field as above where \( p \) remains inert and let \( \chi \) be a ray class character of \( K \) of conductor \( f_\chi \). Let us put \( N = |D_{K/Q}| N_{K/Q} f_\chi \), where \( D_{K/Q} \) is the discriminant of \( K/Q \).

We assume that

- the order of \( \chi \) and \( f_\chi \) are coprime to \( p \);
- The non-Eisenstein hypothesis [non-Eis] holds true.

We will consider the representation \( \mathcal{P} := \Ind_{K/Q}. \) Recall that we have checked in Example B.1.2 that \( \mathcal{P} \) verifies the hypotheses [71] and [72] so that the general
We write \( f \) let \( f \) factors through \( \Gamma \).

Let \( K_\infty \subset K(p^\infty) \) denote the maximal \( \mathbb{Z}_p \)-power extension of \( K \) and put \( \Gamma_K := \text{Gal}(K_\infty/\mathbb{Q}) \). Consider the tautological character

\[
\Psi : G_K \rightarrow \Gamma_K \hookrightarrow \Lambda_\infty(\Gamma_K)^\times.
\]

Observe that

\[
\rho_\Psi := \text{Ind}_{K/\mathbb{Q}} \chi_\Psi : G_K \rightarrow \text{GL}_2(\Lambda_\infty(\Gamma_K))
\]

is a representation of \( G_K \) with coefficients in the complete local Noetherian \( \mathcal{O} \)-algebra \( \Lambda_\infty(\Gamma_K) \). Moreover, \( \rho_\Psi \) is lifts \( \mathcal{P} \) and it is evidently minimally ramified. By the universality of \( \rho_{\text{univ}} \), there exists a unique homomorphism of local rings

\[
(B.6) \quad \pi_\Psi : \mathcal{P}(\mathcal{P}) \longrightarrow \Lambda_\infty(\Gamma_K) \quad \text{with} \quad \rho_\Psi = \rho_{\text{univ}} \otimes \pi_\Psi \Lambda_\infty(\Gamma_K).
\]

**Definition B.4.3.** We set

\[
L_p^{\text{RS}}(f/K \otimes \chi_\Psi) := \pi_\Psi \left( L_p(\rho_\Psi \otimes \rho_{\text{univ}}) \right) \in \frac{1}{H_p} \Lambda_\infty(\Lambda_\infty(\Gamma_K)).
\]

We will conclude this appendix recording an interpolative property that characterizes the \( p \)-adic \( L \)-function \( L_p^{\text{RS}}(f/K \otimes \chi_\Psi) \). Before that, we define the relevant collection of Hecke characters for the said interpolation formula.

**Definition B.4.4.** Given a Hecke character \( \psi \) of \( K \), let us write \( \infty(\psi) \in \mathbb{Z}^2 \) for the infinity-type of \( \psi \). For any integer \( k \geq -1 \), we put

\[
\Sigma^{(1)}_{\text{crit}}(k) := \{ \text{Hecke characters } \psi \text{ of } K : \infty(\psi) = (\ell_1, \ell_2) \text{ with } 0 \leq \ell_1, \ell_2 \leq k \}.
\]

**Theorem B.4.5.** Let \( k \geq 0 \) be an integer such that \( \kappa \equiv k_f \mod (p - 1) \) and let \( f(\kappa) \) denote the unique specialization of \( f \) of weight \( \kappa + 2 \) which is \( p \)-old. We put \( f'(\kappa) \in S_{\kappa+2}(\Gamma_1(N_f)) \) for the newform whose \( p \)-ordinary \( p \)-stabilization is \( f(\kappa) \).

We write \( \alpha(\kappa) \) for the \( U_p \)-eigenvalue on \( f(\kappa) \) and put \( \beta(\kappa) := p^{\kappa+1} \langle f(\kappa) / \alpha(\kappa) \rangle \).

Let \( \psi \in \Sigma^{(1)}_{\text{crit}}(\kappa) \) be a Hecke character whose associated \( p \)-adic Galois character \( \widehat{\psi} \) factors through \( \Gamma_K \) and is crystalline at \( p \). Then,

\[
L_p^{\text{RS}}(f/K \otimes \chi_\Psi)(f(\kappa), \psi) = \frac{(1 - \alpha(\kappa)^{-2} \psi(p))(1 - p^{-2} \beta(\kappa)^{-1} \psi^{-1}(p))}{(1 - \beta(\kappa)/p \alpha(\kappa))(1 - \beta(\kappa)/\alpha(\kappa))} \\
\times \frac{\ell_{\kappa+1-\ell_1+\ell_2} M_{\kappa+1-\ell_1+\ell_2}}{2^{\ell_1+\ell_2+\kappa} \ell_{\kappa+1-\ell_1+\ell_2}} \\
\times \frac{L(f(\kappa))_{N_f}}{8^{\ell_1+\ell_2+\kappa} \ell_{\kappa+1-\ell_1+\ell_2} \ell_{\kappa+1-\ell_1+\ell_2}}.
\]

**Proof.** This follows on translating the interpolation formula for \( L_p(\rho_\Psi \otimes \rho_{\text{univ}}) \) that we alluded to above in the proof of Theorem B.3.2.

**Remark B.4.6.** One may give a direct construction of \( L_p^{\text{RS}}(f/K \otimes \chi_\Psi) \), without going through the minimally ramified universal deformation \( \rho_{\text{univ}} \). We briefly outline this alternative construction in this remark.

Let us write \( K(f_{\kappa} \p) = \bigcup_n K(f_{\kappa} \p^n) \) denote the ray class extension of \( K \) modulo \( f_{\kappa} \p^n \). We let \( H_{f_{\kappa} \p} := \lim_{\leftarrow n} H_{f_{\kappa} \p^n} \) denote the ray class group modulo \( f_{\kappa} \p^n \), which
we identify with $\text{Gal}(K(f_p^\infty)/K)$ via the geometrically normalized Artin reciprocity map $\mathfrak{A}$. Let us write
\[ H_{f_p^\infty} = \Delta \times H_{f_p^\infty}^{(p)} \]
where $\Delta$ is a finite group and $H_{f_p^\infty}^{(p)} \cong \mathbb{Z}_p^\times$, so that $\mathfrak{A}(H_{f_p^\infty}^{(p)}) = \Gamma_K$. Given $b \in H_{f_p^\infty}$, we write $[b] \in \Lambda_{\mathcal{O}(\Gamma_K)}^\times$ for the image of $b$ under the compositum
\[ H_{f_p^\infty} \rightarrow H_{f_p^\infty}^{(p)} \rightarrow \mathfrak{A} \rightarrow \Lambda_{\mathcal{O}(\Gamma_K)}^\times. \]
We define the $p$-depleted universal $\theta$-series for the branch character $\chi$ on setting
\[ G^{[p]}_{\chi} := \sum_{\substack{b \in \mathcal{O}_K \\ (b,p) = 1}} \chi(b)[b]q^N b \in \Lambda_{\mathcal{O}(\Gamma_K)}[[q]]. \]
If one uses $G^{[p]}_{\chi}$ in Definition B.4.1 in place of $G^{[p]}_{\rho}$, one obtains the $p$-adic $L$-function $L^{\text{RS}}_{p}(f/K \otimes \chi \Psi)$. 


APPENDIX C

Images of Galois representations attached to Rankin–Selberg convolutions

In this appendix, we study the hypotheses \( (\tau f \otimes g) \) and \( (\tau f \otimes g) \) when \( g = \theta(\psi) \) is a crystalline CM form, and provide sufficient conditions for their validity. It will be clear to reader that our arguments here draw greatly from the earlier works \cite{Fis02, Hid15, Lan16, Loe17, CLM20} on the subject.

Throughout this appendix, we assume that \( p \geq 7 \).

C.1. Group theory

Let \( R \) be a complete local ring with finite residue field and residue characteristic \( p \). Let \( m \) denote the maximal ideal of \( R \) and \( \mathbb{F} := R/m \) its residue field.

**Lemma C.1.1.** There is a unique maximal normal subgroup \( N_0 \) of \( SL_2(R) \), and the quotient group \( SL_2(R)/N_0 \) is isomorphic to \( PSL_2(\mathbb{F}) \). In particular, the quotient \( SL_2(R)/N_0 \) is non-solvable.

**Proof.** The argument we record here was suggested to us by Jackie Lang. For an ideal \( J \trianglelefteq R \) we let \( GC(J) \) denote the set of all the matrices in \( GL_2(R) \) which reduce to a scalar matrix modulo \( J \). We also let \( SC(J) \) denote the matrices of determinant 1 that reduce to the identity matrix modulo \( J \).

It follows from \cite{Kli61, Theorem 3(ii)} that if \( N \) is a proper normal subgroup of \( SL_2(R) \), then there is an ideal \( J \trianglelefteq R \) such that

\[
SC(J(N)) \subset N \subset GC(J(N)) \cap SL_2(R).
\]

Note that since \( N \) is not equal to \( SL_2(R) \) and \( N \) contains \( SC(J(N)) \), it follows that the ideal \( J(N) \trianglelefteq R \) must be a proper ideal and hence, it is contained in the maximal ideal \( m \) of the local ring \( R \). \( N \) is therefore contained in the subgroup

\[
GC(m) \cap SL_2(R) \supset GC(J(N)) \cap SL_2(R).
\]

of \( SL_2(R) \). Note that

\[
GC(m) \cap SL_2(R) = \ker (SL_2(R) \longrightarrow PSL_2(\mathbb{F})) =: N_0.
\]

We have therefore proved that every proper normal subgroup of \( SL_2(R) \) is contained in \( N_0 \), as required. \( \square \)

\(^7\)We would like to express our gratitude to Jackie Lang for carefully reading through this appendix and pointing out several inaccuracies, which helped us improve our exposition drastically. In particular, we thank her for bringing the papers of Klingenberg \cite{Kli60, Kli61} to our attention and explaining relevant portions of the results in her joint work \cite{CLM20} with Conti and Medvedovsky (which lead to a significant simplification of some of our statements and proofs).
Lemma C.1.2. Let $G_1$ and $G_2$ be two groups and $H < G_1 \times G_2$ such that for each $i = 1, 2$, the natural projection map $pr_i : H \rightarrow G_i$ is surjective. Suppose in addition that $G_2$ is solvable and $G_1$ admits a unique proper maximal normal subgroup $N$ with $G_1/N$ is non-solvable. Then $H = G_1 \times G_2$.

Proof. This is an easy application of Goursat’s lemma, which we recall for the convenience of the reader. Let $H < G_1 \times G_2$ be a subgroup such that the natural projection map $pr_i : H \rightarrow G_i$ is surjective ($i = 1, 2$). Set $N_2 := \ker(pr_2)$ and $N_1 := \ker(pr_1)$. Let us identify $N_2$ (respectively, $N_1$) as a normal subgroup of $G_2$ (respectively, of $G_1$). Then the image of $H$ in $G_1/N_1 \times G_2/N_2$ is the graph of an isomorphism $G_1/N_1 \cong G_2/N_2$. It therefore suffices to prove that $N_1 = G_1$ (equivalently, $N_2 = G_2$) for $N_1$ as above.

Suppose on the contrary that $N_1 \not\subset G_1$ is a proper normal subgroup. Then by definition, $N_1 < N$ and $G_1/N_1$ admits $G_1/N$ as a quotient. In particular, $G_1/N_1$ is non-solvable. But since $G_1/N_1 \cong G_2/N_2$ and $G_2/N_2$ is solvable (since $G_2$ is), this contradiction concludes our proof that $N_1$ cannot be a proper subgroup, and in turn also the proof our lemma.

C.2. Applications to families on $GL_2 \times \Res_{K/\mathbb{Q}}GL_1$

Let $f$ be (a non-CM branch of) a Hida family as in the main body of this article (c.f. Definition [5.1]). We also fix a crystalline Hecke character $\psi$ of our fixed imaginary quadratic field $K$, given as in [5.3]. We will freely use our notation concerning $f$ and $\psi$ from the main text.

In particular, recall the local ring $\Lambda_f$, which is a module-finite flat $\Lambda_\mathbb{Q}(1+p\mathbb{Z}_p)$-algebra, which we assume contains the ring of integers $\mathcal{O}$ of a sufficiently large extension of $\mathbb{Q}_p$. Let us denote by $k$ the residue field of $\Lambda_f$. Recall also the theta-series $g := \theta(\psi) \in S_{k+2}(\Gamma_1(N_f), \epsilon_\psi)$, whose associated Galois representation $\rho_{\theta(\psi)}$ giving the action on $R^1_{\theta(\psi)}$ can be explicitly described as follows:

\[
\begin{array}{c|c}
\sigma & \rho_{\theta(\psi)}(\sigma) \\
\hline
\sigma \in G_K & \left( \begin{array}{cc}
\hat{\psi}^{-1}(\sigma) & 0 \\
0 & \hat{\psi}^{-1}(\sigma c^{-1})
\end{array} \right) \\
\sigma \in G_\mathbb{Q} \setminus G_K & \left( \begin{array}{cc}
0 & \hat{\psi}^{-1}(\sigma c) \\
\hat{\psi}^{-1}(\sigma) & 0
\end{array} \right)
\end{array}
\]

(C.1)

Throughout [C.2] we shall assume that $\rho_f$ is residually full, in the sense that the condition

(Full) \quad $SL_2(\mathbb{F}_p) \subset \overline{\theta_f(G_\mathbb{Q}(\mu_p^\infty))}$

holds true. The condition [Full] is often (but not always) satisfied.

Let us put $\mathcal{G}_1 := \ker(\det \circ \rho_f)$. Note that $\mathcal{G}_1$ is a subgroup of $G_\mathbb{Q}(\mu_p^\infty)$ with finite index. If we have $\epsilon_f = 1$ for the tame nebentype of the family $f$, then $\mathcal{G}_1 = G_\mathbb{Q}(\mu_p^\infty)$. In general, if we let $K_1 := \mathbb{Q}(\epsilon_f)$ denote the abelian (in fact, cyclic) extension cut out by $\epsilon_f$, then $\mathcal{G}_1 = G_{K_1}(\mu_p^\infty)$. Note also that $\mathbb{Q}(\epsilon_f) \cap K = \mathbb{Q}$, thanks to our assumption that $(N_f, D_K) = 1$.

Lemma C.2.1. [Full] holds, then in fact

$SL_2(\mathbb{F}_p) \subset \overline{\theta_f(\mathcal{G}_1)}$.

\[8\text{We do not make this precise, but it at least contains the image of } \hat{\psi}.\]
PROOF. Let us put \( \mathcal{G}_0 := G_{\mathbf{Q}(\mu_{p_\infty})} \) to ease notation. Note that \( \mathcal{G}_0/\mathcal{G}_1 \) is a finite abelian group. In particular, the groups

\[
\SL_2(\mathbb{F}_p)/\overline{\mathcal{R}}(\mathcal{G}_1) \cap \SL_2(\mathbb{F}_p) < \overline{\mathcal{R}}(\mathcal{G}_0)/\overline{\mathcal{R}}(\mathcal{G}_1)
\]

are abelian as well. Moreover, since \( \overline{\mathcal{R}}(\mathcal{G}_1) \cap \SL_2(\mathbb{F}_p) \) is a normal subgroup of \( \SL_2(\mathbb{F}_p) \) and \( p \geq 7 \), it follows that either \( \overline{\mathcal{R}}(\mathcal{G}_1) \cap \SL_2(\mathbb{F}_p) < \{ \pm 1 \} \) or else \( \overline{\mathcal{R}}(\mathcal{G}_1) \) contains \( \SL_2(\mathbb{F}_p) \). In the former scenario, it would follow (using the fact that the quotient group \( \SL_2(\mathbb{F}_p)/\overline{\mathcal{R}}(\mathcal{G}_1) \cap \SL_2(\mathbb{F}_p) \) is abelian) that \( \PSL_2(\mathbb{F}_p) \) is abelian, which is absurd. We therefore conclude that the latter scenario holds, which is the assertion of our lemma.

Let us put \( \varepsilon_\psi = \eta_\psi \varepsilon_K \) where \( \eta_\psi \) is a Dirichlet character of conductor coprime to \( p D_K \). We define \( K_2 := \mathbf{Q}(\eta_\psi) \) as the field cut out by \( \eta_\psi \) and set \( K' := K_1 K_2 \). We put \( \mathcal{G} := G_{K'(\mu_{p_\infty})} \). It is a normal subgroup of \( \mathcal{G}_1 \) with finite index and moreover, the quotient \( \mathcal{G}_1/\mathcal{G} \) is abelian (in fact cyclic, but we won’t need that).

Note that \( K' \) is still linearly disjoint from \( K(\mu_{p_\infty}) \) (since the conductors of \( \eta_\psi \) and \( \varepsilon_\psi \) are both coprime to \( p D_K \)). In particular, \( \mathcal{G} \not\subset G_{K(\mu_{p_\infty})} \). We shall use this observation crucially in the proof of Theorem C.2.4.

**Lemma C.2.2.** If \((\text{Full})\) holds, then in fact

\[
\SL_2(\mathbb{F}_p) \subset \overline{\mathcal{R}}(\mathcal{G}).
\]

**Proof.** The proof of this lemma is identical to that of Lemma C.2.1: One simply replaces \( \mathcal{G}_0 \) with \( \mathcal{G}_1 \) and \( \mathcal{G}_1 \) with \( \mathcal{G} \) everywhere in the proof.

**Proposition C.2.3.** Let \( f \) be a classical specialization of \( f \) and let \( \rho_f \) denote the \( G_{\mathbf{Q}} \)-representation giving rise to the action on \( R_f^* \). We then have,

\[
\rho_f \times \rho_{\theta(\psi)}(\mathcal{G}) = \rho_f(\mathcal{G}) \times \rho_{\theta(\psi)}(\mathcal{G}),
\]

\[
\rho_f \times \rho_{\theta(\psi)}(\mathcal{G}) = \rho_f(\mathcal{G}) \times \rho_{\theta(\psi)}(\mathcal{G}).
\]

**Proof.** We will verify the first assertion using Lemma C.1.2 with the choices \( H := \rho_f \times \rho_{\theta(\psi)}(\mathcal{G}), \ G_1 := \rho_f(\mathcal{G}), \) and \( G_2 := \rho_{\theta(\psi)}(\mathcal{G}). \) Once we check that the conditions of Lemma C.1.2 are satisfied with these choices, our proposition follows at once from this lemma.

Indeed, \( H \) is clearly a subgroup of \( G_1 \times G_2 \) and the natural projections \( \pr_i : H \to G_i \) \((i = 1, 2)\) are surjective. Moreover, since \( \rho_{\theta(\psi)}(G_K) \) is abelian, it follows that \( G_2 \) is solvable. It follows from [CLM20] Proposition 6.7] that \( G_1 = \SL_2(A) \) for some finite \( \mathbf{Z}_p \)-algebra \( A \) (in particular, \( A \) is a complete local ring). We can now invoke Lemma C.1.1 to see that \( G_1 \) too verifies the conditions required in Lemma C.1.2 and thereby concludes the proof of our first assertion.

The proof of the second part proceeds in an identical manner, using the fact that the local ring \( A_0 = A_0(\rho) \) given as in [CLM20] \( \S 1 \) with \( A = A_f \) and \( \rho = \rho_f \) is a complete local ring with finite residue field.
Recall that $\varepsilon_\psi$ is the nebentype character for $\theta(\psi)$. It is a Dirichlet character of conductor dividing $|D_K|N$; see Definition 5.3.1 for its explicit description.

**Theorem C.2.4.** Let $\mathfrak{f}$ be a non-CM Hida family with trivial nebentype and suppose $\psi$ is a crystalline Hecke character as above. Suppose that $p \geq 7$ and the condition $\text{(Full)}$ holds true.

i) Let $f_\alpha$ be a crystalline specialization of $\mathfrak{f}$, which arises as the $p$-stabilization of a cuspidal eigen-newform $f$. Then the condition $\text{(\ref{C.1})}$ is satisfied with $g = \theta(\psi)$.

ii) The condition $\text{(\ref{C.2})}$ holds true with $g = \theta(\psi)$.

**Proof.** Since the proofs of both parts are very similar, we shall only provide details for the proof of (i). Let us denote by $\mathfrak{t}$ the natural map

$$
\text{GL}_2(\mathcal{O}) \times \text{GL}_2(\mathcal{O}) \cong \text{GL}(R_f^+) \times \text{GL}(R_{\theta(\psi)}^+) \xrightarrow{\mathfrak{t}} \text{GL}(T_{f,\theta(\psi)}) \cong \text{GL}_4(\mathcal{O}).
$$

We shall prove that there exist $M_1 \in \rho_f(\mathcal{G}) \subset \text{GL}(R_f^+)$ and $M_2 \in \rho_{\theta(\psi)}(\mathcal{G}) \subset \text{GL}(R_{\theta(\psi)}^+)$ such that the minimal polynomial of $\mathfrak{t}(M_1, M_2)$ equals $(X - 1)^2(X + 1)^2$.

The proof of our theorem then follows from Proposition C.2.3.

As we have explained in the proof of Proposition C.2.3, the group $\rho_f(\mathcal{G})$ contains $M_1 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Moreover, it follows from the explicit description C.1 of $\rho_{\theta(\psi)}$ that the group $\rho_{\theta(\psi)}(\mathcal{G})$ contains an element $M_2$ of the form $\begin{pmatrix} 0 & x \\ x^{-1} & 0 \end{pmatrix}$ for some $x \in \mathcal{O}^\times$. Indeed, for any $\sigma \in \mathcal{G}$, we have

$$
\text{det } \rho_{\theta(\psi)}(\sigma) = \eta_f(\sigma)\varepsilon_K(\sigma) = \varepsilon_K(\sigma),
$$

where the second equality follows from the definition of the group $\mathcal{G}$. In particular, if $\sigma \in \mathcal{G} \setminus G_K(\mu_p\infty)$, then $\text{det } \rho_{\theta(\psi)}(\sigma) = -1$. (Recall our remark in the paragraph preceding the statement of Lemma C.2.2 that $\mathcal{G} \setminus G_K(\mu_p\infty)$ is non-empty.) Given the description of the representation $\rho_{\theta(\psi)}$ in C.1, it follows that $\rho_{\theta(\psi)}(\sigma)$ has necessarily the required form for any $\sigma \in \mathcal{G} \setminus G_K(\mu_p\infty)$.

Note that $M_2$ is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and in turn, $\mathfrak{t}(M_1, M_2)$ is conjugate to the upper triangular matrix

$$
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 \\
0 & 0 & 0 & -1
\end{pmatrix}.
$$

The proof of our theorem is now complete. \qed

---

It follows that $A_0(\rho)$ is a $W(k_0)$-algebra. To complete the proof that $A_0(\rho)$ is local, we need to verify that every element in $A_0(\rho) \setminus m_0$ is invertible in $A_0(\rho)$. Indeed, if $x \in m_0$, then $1/(1 + x)$ is a power series in $x$, hence also in $A_0(\rho)$ since $A_0(\rho)$ is closed. Hence, $1 + m_0 \subset A_0(\rho)^\times$. But every element $a \in A_0(\rho) \setminus m_0$ factors as $a = um$ where $u \in W(k_0)$ (the Teichmüller representative) and $m \in 1 + m_0$. This concludes the proof of that $A_0(\rho) \setminus m_0 \subset A_0(\rho)^\times$. 
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