Some stability and exact results in generalized Turán problems

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Abstract

Given graphs $H$ and $F$, the generalized Turán number $\text{ex}(n, H, F)$ is the largest number of copies of $H$ in $n$-vertex $F$-free graphs. Stability refers to the usual phenomenon that if an $n$-vertex $F$-free graph $G$ contains almost $\text{ex}(n, H, F)$ copies of $H$, than $G$ is in some sense similar to some extremal graph. We obtain new stability results for generalized Turán problems and derive several new exact results.

1 Introduction

A fundamental question in graph theory is the following. Given a graph $F$, what is the largest number of edges that an $n$-vertex $F$-free graph can have? This quantity is called the Turán number and is denoted by $\text{ex}(n, F)$. Turán [19] proved that $\text{ex}(n, K_{k+1}) = |E(T(n,k))|$, where the Turán graph $T(n,k)$ is the complete $k$-partite graph with each part of order $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$. Let us call a vertex or an edge of a graph color-critical if deleting that vertex or edge decreases the chromatic number. Simonovits [17] showed that if $\chi(F) = k + 1$ and $F$ has a color-critical edge, then for sufficiently large $n$, we have $\text{ex}(n, F) = |E(T(n,k))|$. Gerbner and Palmer [10] defined $F$-Turán-good graphs as graphs $H$ with $\text{ex}(n, H, F) = \mathcal{N}(H, T(n, \chi(F) - 1))$ for sufficiently large $n$. In this language Zykov's theorem implies that cliques are $K_k$-Turán-good (note that his result holds for every $n$). We say that $H$ is weakly $F$-Turán-good if $\text{ex}(n, H, F) = \mathcal{N}(H, T)$ for some complete $(\chi(F) - 1)$-partite graph $T$. In this case a straightforward optimization finds $T$ for given $H$, but we are unable to execute this optimization for general $H$. Győri, Pach and Simonovits [12] showed that complete multipartite graphs are weakly $K_k$-Turán-good. They also showed a bipartite graph which is not $K_3$-Turán-good.
Gerbner and Palmer [10] showed that $C_4$ is $F_2$-Turán-good, where $F_2$ consists of two triangles sharing a vertex. Note that $F_2$ does not have a color-critical edge, thus the Turán-graph is not edge-maximal: one more edge can be added without creating an $F_2$, but that edge is not in any $C_4$. Gerbner [7] showed that there are $F$-Turán-good graphs if and only if $F$ has a color-critical vertex.

Let us call $H$ asymptotically $F$-Turán-good if $\text{ex}(n, H, F) = (1 + o(1))N(H, T(n, \chi(F) - 1))$, and weakly asymptotically $F$-Turán-good if $\text{ex}(n, H, F) = (1 + o(1))N(H, T)$ for some complete $(\chi(F) - 1)$-partite graph $T$. A theorem of Gerbner and Palmer [10] states that if $\chi(F) = k$, then $\text{ex}(n, H, F) \leq \text{ex}(n, H, K_k) + o(n^{\lvert V(H) \rvert})$. It implies that if $\chi(H) \leq k$ and $H$ is (weakly) asymptotically $K_{k+1}$-Turán-good, then $H$ is also (weakly) asymptotically $F$-Turán-good. Another theorem of Gerbner and Palmer [11] states that paths are asymptotically $F$-Turán-good for non-bipartite graphs $F$.

In extremal graph theory problems often the graphs where a parameter takes its maximum all have a similar structure. A very common phenomenon is that graphs where the parameter is close to its maximum are in some sense close to the extremal graphs. The prime example is the following theorem.

**Theorem 1.1** (Erdős-Simonovits stability theorem [3, 4, 18]). Let $\chi(F) = k + 1$. If $G$ is an $n$-vertex $F$-free graph with $|E(G)| \geq \text{ex}(n, F) - o(n^2)$, then $G$ can be obtained from $T(n, k)$ by adding and removing $o(n^2)$ edges.

Let us turn to known stability results concerning generalized Turán problems. As most of these results are similar to Theorem 1.1 we introduce a notation.

Let $\chi(H) < \chi(F) = k + 1$. We say that $H$ is $F$-Turán-stable if the following holds. If $G$ is an $n$-vertex $F$-free graph with $N(H, G) \geq \text{ex}(n, H, F) - o(n^{\lvert V(H) \rvert})$, then $G$ can be obtained from $T(n, k)$ by adding and removing $o(n^2)$ edges. Theorem 1.1 states that $K_2$ is $F$-Turán-stable for every non-bipartite $F$. We say that $H$ is weakly $F$-Turán-stable if the following holds. If $G$ is an $n$-vertex $F$-free graph with $N(H, G) \geq \text{ex}(n, H, F) - o(n^{\lvert V(H) \rvert})$, then $G$ can be obtained from a complete $k$-partite graph by adding and removing $o(n^2)$ edges. We remark that if $H$ is (weakly) $F$-Turán-stable, then $H$ is (weakly) asymptotically $F$-Turán-good.

Ma and Qiu [16] obtained the following generalization of Theorem 1.1.

**Theorem 1.2** (Ma, Qiu [16]). Let $r < \chi(F)$. Then $K_r$ is $F$-Turán-stable.

A different kind of stability result is due to the author [6]. Assume that $F$ has a color-critical edge and $\chi(F) > r$. If $G$ is an $n$-vertex $F$-free graph with chromatic number more than $\chi(F) - 1$, then $\text{ex}(n, K_r, F) - N(K_r, G) = \Omega(n^{r-1})$.

Most other stability results in this area were obtained either as a lucky coincidence when the proof of a bound on $\text{ex}(n, H, F)$ gives a stronger result (e.g. $\text{ex}(n, P_4, C_5)$ in [11]), or as a lemma towards a bound on $\text{ex}(n, H, F)$ (e.g. $\text{ex}(n, C_5, K_k)$ in [14]).

A more systematic approach to this latter type of stability results is due to Hei, Hou and Liu [13]. They showed that if $\chi(H) < \chi(F)$, $F$ has a color-critical edge, the Turán graph contains the most copies of $H$ among complete $k$-partite graphs and $H$ is $F$-Turán-stable,
then $H$ is $F$-Turán-good. In fact, instead of $F$-Turán-stability, they used the following weaker property: If $G$ is an $n$-vertex $F$-free graph with $N(H,G) = \text{ex}(n,H,F)$, then $G$ can be obtained from $T(n,k)$ by adding and removing $o(n^2)$ edges. They also showed that if $F$ has a color-critical edge, then paths are $F$-Turán-stable.

Liu, Pikhurko, Sharifzadeh and Staden [15] introduced a general framework for studying graphs parameters that do not decrease by Zykov symmetrization, and proved that under some conditions, a stability result is also implied. Symmetrization does not decrease $\text{ex}(n, H, K_{k+1})$ if $H$ is complete multipartite. It is not hard to see that the additional conditions are also satisfied, thus their results imply that complete multipartite graphs are weakly $K_k$-Turán-stable.

Now we are ready to list our contributions.

**Proposition 1.3.** Let $\chi(F) = k + 1$ and $H$ be weakly asymptotically $F$-Turán-good and weakly $K_{k+1}$-Turán-stable. Then $H$ is weakly $F$-Turán-stable. Furthermore, if $H$ is asymptotically $F$-Turán-good and $K_{k+1}$-Turán-stable, then $H$ is $F$-Turán-stable.

Combining the above proposition with the known results mentioned earlier, we obtain that complete multipartite graphs and paths are $F$-Turán-stable for every $F$ with larger chromatic number.

We can extend the result of Hei, Hou and Liu [13] to the weak case.

**Theorem 1.4.** Let $\chi(F) > \chi(H)$ and assume that $F$ has a color-critical edge. If $H$ is weakly $F$-Turán-stable, then $H$ is weakly $F$-Turán-good. Moreover, the same property is also implies by the weaker assumption that there is an $n$-vertex $F$-free graph $G$ with $N(H,G) = \text{ex}(n,H,F)$ can be obtained from a complete $(\chi(F) - 1)$-partite graph by adding and removing $o(n^2)$ edges.

Combined with results mentioned earlier, the above theorem implies that complete multipartite graphs $H$ are weakly $F$-Turán-good if $\chi(H) < \chi(F)$ and $F$ has a color-critical edge. This was proved by the author [8] in the case $\chi(F) = 3$. Note that if $H$ is $F$-Turán-stable, then $H$ is weakly $F$-Turán-good by the above theorem, but not necessarily $F$-Turán-good: it is possible that the extremal graph is only slightly unbalanced. This is the case for $H = K_{1,3}$ and $F = K_3$: it was shown by Brown and Sidorenko [2] that the bipartite graph with the most copies of $K_{1,3}$ is either $K_{k,n-k}$ or $K_{k+1,n-k-1}$, where $k = \lfloor \frac{n}{2} - \sqrt{(3n-4)/2} \rfloor$.

Theorem 1.4 is the special case $r = 1$ of the next theorem.

**Theorem 1.5.** Let $\chi(F) = k + 1$, $\chi(H) = k$ and assume that $F$ has a color-critical vertex. Let $r$ be the smallest number such that there is a color-critical vertex $v$ in $F$ that is adjacent to exactly $r$ vertices of one of the color classes in the $k$-coloring of the graph we obtain by

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1 Their statement is weaker. They use is later for $H$ satisfying additional properties, and they assume those properties in this theorem, but those properties are not actually used in the proof.

2 Their statement is again weaker. They use it later in the above described way, thus they only need to consider $n$-vertex $F$-free graphs $G$ with $N(P_k,G) = \text{ex}(n,P_k,F)$, but their proof also works for $G$ with $N(P_k,G) \geq \text{ex}(n,P_k,F) - o(n^k)$.
deleting $v$ from $F$. Assume that if we embed graphs of maximum degree less than $r$ into each part of a complete $k$-partite graph $T_0$, we do not obtain any copies of $H$ besides those contained in $T_0$.

Assume that there is an $n$-vertex $F$-free graph $G$ with $N(H,G) = \text{ex}(n,H,F)$, such that $G$ can be obtained from a complete $k$-partite graph $T$ by adding and removing $o(n^2)$ edges. Then $H$ is weakly $F$-Turán-good.

We remark that the assumptions of the theorem ensure that we can embed a graph with maximum degree $r-1$ into one part of $T$, but the resulting graph contains $N(H,T)$ copies of $H$.

Using stability, we can obtain a result on the structure of the extremal graphs for $\text{ex}(n,H,F)$ in some cases. Let $d_G(H,v)$ denote the number of copies of $H$ containing $v$ in $G$. Observe that for vertices $u,v$ in $T(n,k)$, we have $d_{T(n,k)}(H,v) = (1 + o(1))d_{T(n,k)}(H,u)$.

**Theorem 1.6.** Let $\chi(H) < \chi(F) = k + 1$, and assume that $H$ is $F$-Turán-stable. Let $G$ be an $n$-vertex $F$-free graph which contains $\text{ex}(n,H,F)$ copies of $F$. Then for every vertex $u \in G$ we have $d_G(H,u) \geq (1 + o(1))d_{T(n,k)}(H,u)$.

Note that for the ordinary Turán case, this is known [17]. Let us now state some of the exact results of generalized Turán problems that follow from our theorems (together with known results).

**Theorem 1.7.** (i) If $\chi(F) = k + 1$, $\chi(F)$ has a color-critical edge, $H$ is $K_{k+1}$-Turán-stable, then $H$ is weakly $F$-Turán-good. In particular, complete $r$-partite graphs and paths are weakly $F$-Turán-good. If $H$ is also $K_{k+1}$-Turán-good, then $H$ is also $F$-Turán-good.

(ii) If $\chi(F) = k + 1$, $\chi(F)$ has a color-critical vertex and $H$ is a complete $k$-partite graph with each part sufficiently large, then $H$ is weakly $F$-Turán-good.

(iii) Let $F$ have chromatic number at least 4 and a color-critical edge. Then $C_5$ is $F$-Turán-good.

(iv) If $F$ has a color-critical edge, then any union of cliques of order less than $\chi(F)$ is $F$-Turán-good.

(v) Let us assume that $F$ has a color-critical edge and $\chi(F) = k + 1$. The $K_{k+1}$-Turán-good graphs listed in Theorem 2.3 are also $F$-Turán-good.

We remark that (iv) generalizes a theorem of the author [9], who showed that matchings are $F$-Turán-good if $F$ has a color-critical edge.

In the next section, we state the lemmas we need. We also state and prove a proposition that give stability for several generalized Turán problem. We present the proofs of Proposition 1.3 and of Theorems 1.5, 1.6 and 1.7 in Section 3.

## 2 Preliminaries

We will use the removal lemma [5].
Lemma 2.1 (Removal lemma). If an $n$-vertex graph $G$ contains $o(n^{|V(H)|})$ copies of $H$, then we can delete $o(n^2)$ edges from $G$ to obtain an $H$-free graph.

We will combine this with a result of Alon and Shikhelman [1].

Proposition 2.2. $\text{ex}(n, K_k, F) = \Theta(n^k)$ if and only if $\chi(F) > k$.

We will also use the following lemma.

Lemma 2.3. Let us assume that $\chi(H) < \chi(F)$ and $H$ is weakly $F$-Turán-stable, i.e. $\text{ex}(n, H, F) = \mathcal{N}(H, T) - o(n^{|V(H)|})$ for some complete multipartite $n$-vertex graph $T$. Then every part of $T$ has order $\Omega(n)$.

Proof. Assume the statement does not hold and let $A_1, \ldots, A_{\chi(F)-1}$ be the parts of $T$. Let $A_1$ be the smallest part and $A_2$ be the largest part. Then $|A_2| \geq n/(\chi(F) - 1)$. Let $U$ be a subset of $A_2$ of order $n/(2(\chi(F) - 1))$ and move $U$ from $A_2$ to $A_1$ to obtain another $F$-free graph $T'$. This way we remove only those copies of $H$ that contain an edge between $U$ and $A_2$, thus $o(n^{|V(H)|})$ copies. Let us take a proper $\chi(H)$-coloring of $H$, and embed a color class into $U$, another color class into $A_2 \setminus U$, and the other color classes to $A_3, \ldots, A_{\chi(H)}$. This way we obtain $\Theta(n^{|V(H)|})$ copies of $H$ that are in $T'$ and not in $T$, thus $\mathcal{N}(H, T') > \mathcal{N}(H, T) + \Theta(n^{|V(H)|}) > \text{ex}(n, H, F)$, a contradiction completing the proof. ■

There are several results [12, 10, 7] that take $F$-Turán-good graphs as building blocks and form new $F$-Turán-good graphs.

Let $\chi(H), \chi(H') \leq k$. Let $H_1$ be the vertex-disjoint union of $H$ and $H'$. Assume first that $H$ contains a copy $X$ of $K_k$, then let $H_2$ be the graph obtained by taking $H_1$ and a clique $Y$ in $H'$, and connecting vertices of $Y$ and $X$ arbitrarily. Assume now that $H$ has a unique $k$-coloring. Let $H_3$ obtained by taking $H$ and a $K_k$ with with vertices $v_1, \ldots, v_k$, and adding additional edges such that for every $i \leq k$, there is a copy of $K_k$ in $H_3$ containing $v_i$, but not containing any $v_j$ for $j > i$ and assume that $\chi(H_3) = k$.

Theorem 2.4. (i) (Gerbner, Palmer [11]) Let $H'$ be a $K_{k+1}$-Turán-good graph and $H = K_k$. Then $H_2$ is $K_{k+1}$-Turán-good.

(ii) (Gerbner [7]) Let $H$ be a $K_{k+1}$-Turán-good graph. Then $H_3$ is $K_{k+1}$-Turán-good.

Here we show how the same arguments as used in [11] and [7] give similar statements for (weakly) asymptotically Turán-good and (weakly) Turán-stable graphs. The first statement of the next proposition is complicated because we need to emphasize that the nearly extremal complete multipartite graph is the same graph for both $H$ and $H'$.

Proposition 2.5. Let $\chi(H), \chi(H') \leq k$ and $\chi(F) > k$. Let $\text{ex}(n, H, F) = (1+o(1))\mathcal{N}(H, T)$ and $\text{ex}(n, H', F) = (1+o(1))\mathcal{N}(H', T)$ for some complete $k$-partite $n$-vertex graph $T$. Then $\text{ex}(n, H_1, F) = (1+o(1))\mathcal{N}(H_1, T)$. Let $\text{ex}(n, H, K_{k+1}) = (1+o(1))\mathcal{N}(H, T)$ and $\text{ex}(n, H', K_{k+1}) = (1+o(1))\mathcal{N}(H', T)$ for some complete $k$-partite $n$-vertex graph $T$. Then $\text{ex}(n, H_2, K_{k+1}) = (1+o(1))\mathcal{N}(H_2, T)$. If $H$ is asymptotically $K_{k+1}$-Turán-good, then $H_3$ is also asymptotically $K_{k+1}$-Turán-good. Moreover, $H_3$ is $K_{k+1}$-Turán-stable.
Furthermore, if $H$ or $H'$ are weakly $F$-Turán-stable, then $H_1$ is also weakly $F$-Turán-stable. Similarly, if $H$ or $H'$ are $K_{k+1}$-Turán-stable, then $H_2$ is also weakly $K_{k+1}$-Turán-stable.

We remark that the stability of $H_i$ is implied if we have stability for any of $H$ and $H'$. In the case of $H_3$, $H'$ is replaced by $K_k$, for which we have stability by Theorem 1.2, thus we have stability automatically for $H_3$.

**Proof.** Let $G$ be an $n$-vertex $K_{k+1}$-free graph. We count the copies of $H_i$ by picking $H$, a vertex-disjoint $H'$, and then the additional edges. Clearly picking $H$ and then picking $H'$ can be done $(1+o(1))N(H,T(n,k))$ and $(1+o(1))N(H',T(n,k))$ ways, thus it is asymptotically maximized by the Turán graph. This completes the proof for $H_1$.

For $H_2$, we consider the bipartite graph $G'$ with $X$ and $Y$ embedded into $G'$ as the two parts, with the edges of $G$ between them. It was shown in [10] that a matching covering $Y$ is missing from $G'$. On the other hand, in the Turán graph only such a matching is missing, thus the Turán graph also maximizes $G'$, meaning that the bipartite graph obtained this way in $G$ is a subgraph of the bipartite graph obtained this way in $T(n,k)$. This implies that the number of ways to pick the additional edges is maximized in $T(n,k)$ also.

For $H_3$, we pick a copy $K$ of $K_k$ and $H$. Then we need to finish the embedding of $H_3$ into $G$ by adding the additional edges. We claim that there is at most one way to do that. Indeed, we go through the vertices $v_i$ in $H$ in increasing order. When we pick $v_i$, there is a copy of $K_k$ in $H_3$ containing $v_i$ such that the other vertices are already embedded into $G$. Since $G$ is $K_{k+1}$-free, we have that those vertices have at most one common neighbor in $G$, thus there is at most one way to choose $v_i$. On the other hand, in the Turán graph there is always a way to finish the embedding (see the proof of Proposition 1.3 in [7]), thus the number of ways to pick the additional edges is maximized in $T(n,k)$ also.

Finally, if $G$ cannot be obtained from $T$ (or $T(n,k)$) by adding and removing $o(n^2)$ edges, then $N(H,G) < \alpha N(H,T)$ (or $N(H,G) < \alpha N(H,T(n,k))$) for some $\alpha < 1$ (or the same holds with $H'$ in place of $H$). Thus the above calculations give the bounds $N(H_1,G) < (1+o(1))\alpha N(H_1,T)$ and $N(H_2,G) < (1+o(1))\alpha N(H_2, T(n,k))$, a contradiction. ■

We remark that in this case we obtain that two (and by induction any number of) copies of $H$ have the same asymptotically extremal complete multipartite graph $T$, and if we have stability for $H$, then we have it for the multiple copies.

### 3 Proofs

Let us start with the proof of Proposition 1.3 that we restate here for convenience.

**Proposition.** Let $\chi(F) = k + 1$ and $H$ be weakly asymptotically $F$-Turán-good and weakly $K_{k+1}$-Turán-stable. Then $H$ is weakly $F$-Turán-stable. Furthermore, if $H$ is asymptotically $F$-Turán-good and $K_{k+1}$-Turán-stable, then $H$ is $F$-Turán-stable.
Proof. Let $G$ be an $n$-vertex $F$-free graph. We start by applying the removal lemma and Proposition 2.2 to obtain a $K_{k+1}$-free graph $G_0$ by removing $o(n^2)$ edges. As we removed $o(n^{V(H)})$ copies of $H$ this way, we have that $\mathcal{N}(H,G_0) \geq ex(n,H,F) - o(n^{V(H)}) = (1 + o(1))\mathcal{N}(H,T)$ for some complete $k$-partite graph $T$. As $H$ is weakly $K_{k+1}$-Turán-stable, this means that $G_0$ can be obtained from $T$ by adding and removing $o(n^2)$ edges, thus so does $G$. The furthermore part follows the same way, $T = T(n,k)$ in that case. \hfill \blacksquare

We continue with the proof of Theorem 1.5 that we restate here for convenience.

**Theorem.** Let $\chi(F) = k+1$, $\chi(H) = k$ and assume that $F$ has a color-critical vertex. Let $r$ be the smallest number such that there is a color-critical vertex $v$ in $F$ that is adjacent to exactly $r$ vertices of one of the color classes in the $k$-coloring of the graph we obtain by deleting $v$ from $F$. Assume that if we embed graphs of maximum degree less than $r$ into each part of a complete $k$-partite graph $T_0$, we do not obtain any copies of $H$ besides those contained in $T_0$.

Assume that there is an $n$-vertex $F$-free graph $G$ with $\mathcal{N}(H,G) = ex(n,H,F)$, such that $G$ can be obtained from a complete $k$-partite graph $T$ by adding and removing $o(n^2)$ edges. Then $H$ is weakly $F$-Turán-good.

Proof. Let $k = \chi(F) - 1$ and $G$ be an $n$-vertex $F$-free graph with $\mathcal{N}(H,G) = ex(n,H,F)$, that can be obtained from a complete $k$-partite graph $T$ with parts $V_1, \ldots, V_k$ by adding and removing $o(n^2)$ edges. We pick $T$ such that we need to add and remove the least number of edges this way. In particular, every vertex $v \in V_i$ is connected at least as many vertices in every $V_j$ a in $V_i$ (otherwise we could move $v$ to $V_j$).

Let $E$ denote the set of edges in $G$ that are not in $T$ (i.e., those inside a part $V_i$). Let $r(u)$ denote the number of edges incident to $u$ in $T$ that are not in $G$, i.e. the missing edges between $u$ and vertices in other part. Then we have $\sum_{u \in V(G)} r(u) = o(n^2)$, thus there are $o(n)$ vertices $u$ with $r(u) = \Omega(n)$. Let $A$ denote the set of vertices with $r(u) = o(n)$ and $A_i = A \cap V_i$, then $|A_i| = |V_i| - o(n)$. By Lemma 2.3, we have that $|A_i| = \Omega(n)$. For $u \in V_i \setminus A_i$, we have that $u$ is adjacent to $\Omega(n)$ vertices in every $V_j$, thus in every $A_j$.

Recall that there are edges $v_1, \ldots, v_r$ in $F$ such that by deleting these edges we obtain a $k$-partite graph with $v, v_1, \ldots, v_r$ in the same part. Let $f_1$ denote the order of that part and $f_2, \ldots, f_k$ denote the order of the other parts.

We claim that every vertex in $V_i$ is adjacent to less than $r$ vertices of $A_i$. Assume otherwise, without loss of generality let $uu_1, \ldots, uu_r$ be edges with $u \in V_i, u_1, \ldots, u_r \in A_i$. Let $B_i$ denote the neighborhood of $u$ in $A_i$, then $|B_i| = \Omega(n)$ and every vertex of $A \setminus A_i$ is connected to $|B_i| - o(n)$ vertices of $B_i$. We pick $f_1 - r - 1$ other vertices in $A_i$. These $f_1$ vertices have $|B_i| - o(n)$ common neighbors in $B_2$, we pick $f_2$ of them, and so on. For every $i$, we pick $f_i$ vertices from $B_i$ that are joined to every vertex picked earlier. This is doable, since all but $o(n)$ vertices of $B_i$ are connected to each of the vertices picked earlier. This way we find a copy of $F$ in $G$, a contradiction.

Let $X$ be a smallest set of vertices inside $V(G) \setminus A$ such that every $K_{1,r}$ inside $E$ contains at least one vertex of $X$. By the above, $\sum_{u \in X} r(u) = \Omega(n|X|)$. On the other hand, there are at most $\binom{|X|}{2}$ edges of $E$ inside $X$, and at most $|X|(r-1)$ edges of $E$ go out from $X$.\hfill \Box
Consider a set $S$ of $H$-free graphs $G$. Let $f$ be one of the partite sets of the resulting Turán graph. Let $v$ be a vertex in $G$ such that $|v| > 0$. In particular in the setting of Theorem 1.4, if an $F$-free $n$-vertex graph $G$ has chromatic number more than $k$, then $G$ contains $\Omega(n^{\chi(H)-1})$ copies of $H$.

Let us continue with Theorem 1.6 that we restate here for convenience.

**Theorem.** Let $\chi(H) < \chi(F) = k + 1$, and assume that $H$ is $F$-Turán-stable. Let $G$ be an $n$-vertex $F$-free graph which contains $\Omega(n^{\chi(H)-1})$ copies of $F$. Then for every vertex $u \in G$ we have $d_G(H, u) \geq (1 + o(1))d_{T(n,k)}(H, v)$.

**Proof.** $G$ can be transformed to $T(n, k)$ by adding and removing $o(n^2)$ edges. Let $V_1$ be one of the partite sets of the resulting Turán graph. Let $f(v)$ denote the number of copies of $H$ that are removed this way and contain $v$. Then we have $\sum_{v \in V(G)} f(v) = o(n^{\chi(H)})$. Consider a set $S$ of $|V(F)|$ vertices in $V_1$ such that $\sum_{v \in S} f(v)$ is minimal. Then by averaging $\sum_{v \in S} f(v) \leq \frac{|S|}{|V_1|} \sum_{v \in V_1} f(v) = o(n^{\chi(H)-1})$.

Now we apply a variant of Zykov’s symmetrization [20]. Let us consider copies of $H$ that contain exactly one vertex $s$ of $S$, and if $sv$ is an edge of the copy of $H$, then $v$ is in the common neighborhood of $S$ in $G$ (i.e., we do not use the edges from $s$ to vertices not in the common neighborhood of $S$). Let $d_G(H, S)$ denote the number of such copies. Observe that each vertex of $S$ is in $\frac{d_G(H, S)}{|S|}$ such copies of $H$.

Let $x$ denote the number of copies of $H$ that contain $u$ and a vertex from $S$, then $x = O(n^{\chi(F)-2})$. If $d_G(H, u) < \frac{d_G(H, S)}{|S|} - x$, then we remove the edges incident to $u$ from $G$ and connect $u$ to the common neighborhood of $S$. This way we do not create any copy of $F$, as the copy should contain $u$, but $u$ could be replaced by any vertex of $S$ that is not already in the copy to create a copy of $F$ in $G$. We removed $d_G(H, u)$ copies of $H$, but added at least $\frac{d_G(H, S)}{|S|} - x$ copies, a contradiction.

Therefore, we have

$$d_G(H, u) \geq d_G(H, S) - x \geq d_{T(n,k)}(H, S) - \sum_{v \in S} f(r, v) - x = d_{T(n,k)}(H, S) - o(n^{\chi(H)-1}).$$

Since $S \subseteq V_1$, the common neighborhood of $S$ in $T(n, k)$ is the same as the neighborhood of any vertex of $S$, thus $d_{T(n,k)}(H, S) = d_{T(n,k)}(H, s)$, completing the proof.

We continue with the proof of Theorem 1.7 which is too long to restate here.

**Proof of Theorem 1.7.** The first sentence of (i) follows from combining Proposition 1.3 and Theorem 1.4. In the case $H$ is also $K_{k+1}$-Turán-good, we have that the Turán graph maximizes the number of copies of $H$ among complete $k$-partite $n$-vertex graphs. If $H$ is complete multipartite, we apply the result of Liu, Pikhurko, Sharifzadeh and Staden [15] mentioned
in the introduction, stating that $H$ is $K_{k+1}$-Turán-stable. If $H$ is a path, we apply the result of Hei, Hou and Liu \[13\].

To prove (ii), we combine Proposition \[1.3\] and Theorem \[1.5\] with the result of Liu, Pikhurko, Sharifzadeh and Staden \[15\]. We need to show that the assumption on $H$ of Theorem \[1.5\] is satisfied. Let $u, v \in U_1$ and assume that $uv$ is an edge in a copy of $H$. Without loss of generality, $u \in V_1$, $v \in V_2$. Then at most $2r - 2$ other vertices of $U_1$ can be in $H$. This means that every $U_i$ contains either at most one color class of $H$ or at most $2r$ vertices of $H$. If each color class of $H$ has order more than $2r$, this is impossible.

To prove (iii), we use results of Lidický and Murphy \[14\]. They proved that for $k \geq 3$, $C_5$ is $K_{k+1}$-Turán-good and $K_{k+1}$-Turán-stable. This implies that $C_5$ is weakly $F$-Turán-stable by Proposition \[1.3\] hence $C_5$ is weakly $F$-Turán-good by Theorem \[1.4\] where $\chi(F) = k + 1$. A weakly $F$-Turán-good and $K_{k+1}$-Turán-good graph is clearly $F$-Turán-good, since the complete $k$-partite graph with the most copies of $C_5$ is the Turán graph.

To prove (iv), we apply induction on the number of components. Let us assume that the statement holds for graphs with at most $\ell$ components, let $H$ be the vertex-disjoint union of $\ell$ cliques and $H'$ be another clique. Then we can use Proposition \[2.5\] to show that the vertex-disjoint union $H_1$ of $H$ and $H'$ is $F$-Turán-stable. Then Theorem \[1.4\] implies that $H_1$ is weakly $F$-Turán-good. Let us assume that the extremal complete multipartite graph $T$ contains parts $A$ and $B$ with $|A| < |B| - 1$. Then we move a vertex from $B$ to $A$.

We claim that the number of copies of $H_1$ does not decrease this way. Indeed, every copy of $H_1$ intersects $A \cup B$ in a matching and some isolated vertices. Matchings are $K_3$-Turán-good (first shown in \[12\]), thus their number does not decrease this way. Clearly, such intersections are extended the same number of times to a copy of $H_1$ with vertices from the other parts, hence the number of copies of $H_1$ also does not decrease. Repeating this we eventually arrive to the Turán graph without decreasing the number of copies of $H_1$, which completes the proof.

To prove (v), recall that in Section 2, we described how to obtain graphs $H_2$ and $H_3$ starting from $H$ and $H'$. These are generalizations of the constructions in \[11, 9\]. Therefore, applying Proposition \[2.5\] we obtain that the graphs listed are $K_{k+1}$-Turán-stable. Theorem \[1.4\] imply that they are weakly $F$-Turán-good. A weakly $F$-Turán-good and $K_{k+1}$-Turán-good graph is clearly $F$-Turán-good, completing the proof.

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