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SPECTRUM OF THE LAPLACIAN WITH WEIGHTS

BRUNO COLBOIS AND AHMAD EL SOUFI

Abstract. Given a compact Riemannian manifold \((M, g)\) and two positive functions \(\rho\) and \(\sigma\), we are interested in the eigenvalues of the Dirichlet energy functional weighted by \(\sigma\), with respect to the \(L^2\) inner product weighted by \(\rho\). Under some regularity conditions on \(\rho\) and \(\sigma\), these eigenvalues are those of the operator \(-\rho^{-1}\text{div}(\sigma \nabla u)\) with Neumann conditions on the boundary if \(\partial M \neq \emptyset\). We investigate the effect of the weights on eigenvalues and discuss the existence of lower and upper bounds under the condition that the total mass is preserved.

1. Introduction

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\), possibly with nonempty boundary. We designate by \(\{\lambda_k(M, g)\}_{k \geq 0}\) the nondecreasing sequence of eigenvalues of the Laplacian on \((M, g)\) under Neumann conditions on the boundary if \(\partial M \neq \emptyset\). The min-max principle tells us that these eigenvalues are variationally defined by

\[
\lambda_k(M, g) = \inf_{E \in S_{k+1}} \sup_{u \in E \setminus \{0\}} \frac{\int_M |\nabla u|^2 \sigma v_g}{\int_M u^2 \rho v_g},
\]

where \(S_k\) is the set of all \(k\)-dimensional vector subspaces of \(H^1(M)\) and \(v_g\) is the Riemannian volume element associated with \(g\).

The relationships between the eigenvalues \(\lambda_k(M, g)\) and the other geometric data of \((M, g)\) constitute a classical topic of research that has been widely investigated in recent decades (the monographs [3, 4, 7, 24, 35] are among basic references on this subject). In the present work we are interested in eigenvalues of “weighted” energy functionals with respect to “weighted” \(L^2\) inner products. Our aim is to investigate the interplay between the geometry of \((M, g)\) and the effect of the weights.

Therefore, let \(\rho\) and \(\sigma\) be two positive continuous functions on \(M\) and consider the Rayleigh quotient

\[
R_{(g, \rho, \sigma)}(u) = \frac{\int_M |\nabla u|^2 \sigma v_g}{\int_M u^2 \rho v_g}.
\]

The corresponding eigenvalues are given by

\[
\mu_k^{\rho}(\rho, \sigma) = \inf_{E \in S_{k+1}} \sup_{u \in E \setminus \{0\}} R_{(g, \rho, \sigma)}(u). \tag{1}
\]

Under some regularity conditions on \(\rho\) and \(\sigma\), \(\mu_k^{\rho}(\rho, \sigma)\) is the \(k\)-th eigenvalue of the problem

\[
-\text{div}(\sigma \nabla u) = \mu \rho u \quad \text{in } M \tag{2}
\]

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with Neumann conditions on the boundary if \( \partial M \neq \emptyset \). Here \( \nabla \) and \( \text{div} \) are the gradient and the divergence associated with the Riemannian metric \( g \). When there is no risk of confusion, we will simply write \( \mu_k(\rho, \sigma) \) for \( \mu^g_k(\rho, \sigma) \).

Notice that the numbering of eigenvalues starts from zero. It is clear that the infimum of \( R_{(g, \rho, \sigma)}(u) \) is achieved by constant functions, hence \( \mu^g_0(\rho, \sigma) = 0 \) and

\[
\mu^g_1(\rho, \sigma) = \inf_{\int_M u \rho v_g = 0} R_{(g, \rho, \sigma)}(u). \tag{3}
\]

One obviously has \( \mu^g_k(1, 1) = \lambda_k(M, g) \). When \( \sigma = 1 \), the eigenvalues \( \mu_k(\rho, 1) \) correspond to the situation where \( M \) has a non necessarily constant mass density \( \rho \) and describe, in dimension 2, the vibrations of a non-homogeneous membrane (see [31, 24] and the references therein). The eigenvalues \( \mu_k(1, \sigma) \) are those of the operator \( \text{div}(\sigma \nabla u) \) associated with a conductivity \( \sigma \) on \( M \) (see [24, Chapter 10] and [2]). In the case where \( \rho = \sigma \), the eigenvalues \( \mu_k(\rho, \rho) \) are those of the Witten Laplacian \( L_{\rho} \) (see [12] and the references therein). Finally, when \( \sigma \) and \( \rho \) are related by \( \sigma = \rho^{\frac{4}{n-2}} \), the corresponding eigenvalues \( \mu^g_k(\rho, \rho^{\frac{4}{n-2}}) \) are exactly those of the Laplacian associated with the conformal metric \( \rho^{\frac{4}{n-2}} g \), that is \( \mu^g_k(\rho, \rho^{\frac{4}{n-2}}) = \lambda_k(M, \rho^{\frac{4}{n-2}} g) \).

Our goal in this paper is to investigate the behavior of \( \mu^g_k(\rho, \sigma) \), especially in the most significant cases mentioned above, under normalizations that we will specify in the sequel, but which essentially consist in the preservation of the total mass. The last case, corresponding to conformal changes of metrics, has been widely investigated in recent decades (see for instance [9, 22, 23, 26, 28, 29, 33, 34]) and most of the questions we will address in this paper are motivated by results established in the conformal setting. These questions can be listed as follows:

1. Can one redistribute the mass density \( \rho \) (resp. the conductivity \( \sigma \)) so that the corresponding eigenvalues become as small as desired?
2. Can one redistribute \( \rho \) and/or \( \sigma \) so that the eigenvalues become as large as desired?
3. If Question (1) (resp. (2)) is answered positively, what kind of constraint can one impose in order to get upper or lower bounds for the eigenvalues?
4. If Question (1) (resp. (2)) is answered negatively, what are the geometric quantities that bound the eigenvalues?
5. If the eigenvalues are bounded, what can one say about their extremal values?
6. Is it possible, in some specific situations, to compute or to have sharp estimates for the first positive eigenvalues?

In a preliminary section we deal with some technical issues concerning the possibility of relaxing the conditions of regularity and positivity of the densities. In the process, we prove a 2-dimensional convergence result (Theorem 2.1) which completes a theorem that Colin de Verdière had established in dimension \( n \geq 3 \). Question (1) is discussed at the beginning of Section 3 where we show that it is possible to fix one of the densities \( \rho \) and \( \sigma \) and vary the other one, among densities preserving the total mass, in order to produce arbitrarily small eigenvalues (Theorem 3.1). This leads us to get into Question (3) that we tackle by establishing the following Cheeger-type inequality (Theorem 3.2):

\[
\mu_1(\rho, \sigma) \geq \frac{1}{4} h_{\sigma, \sigma}(M) h_{\rho, \sigma}(M)
\]
where \( h_{\sigma,\sigma}(M) \) and \( h_{\rho,\sigma}(M) \) are suitably defined isoperimetric constants, in the spirit of what is done in [27].

Whenever a Cheeger-type inequality is proved, a natural question is to investigate a possible reverse inequality under some geometric restrictions (see [6] and the introduction of [32] for a general presentation of this issue). It turns out that in the present situation, such a reverse inequality cannot be obtained without additional assumptions on the densities. Indeed, we prove that on any given Riemannian manifold, there exists families of densities such that the associated Cheeger constants are as small as desired while the corresponding eigenvalues are uniformly bounded from below (Theorem 3.3).

Questions (2) and (4) are addressed in Section 4. A. Savo and the authors have proved in [12] that the first positive eigenvalue \( \mu_1(\rho,\rho) \) of the Witten Laplacian is not bounded above as \( \rho \) runs over densities of fixed total mass. In Proposition 4.1 we prove that, given a Riemannian metric \( g_0 \), we can find a metric \( g \), within the set of metrics conformal to \( g_0 \) and of the same volume as \( g_0 \), and a density \( \rho \), among densities of fixed total mass with respect to \( g_0 \), so that \( \mu_k^g(\rho,1) \) is as large as desired. The same also holds for \( \mu_k^g(1,\sigma) \).

However, if instead of requiring that the total mass of the densities is fixed with respect to \( g_0 \), we assume that it is fixed with respect to \( g \), then the situation changes completely. Indeed, Theorem 4.1 below gives the following estimate when \( M \) is a domain of a complete Riemannian manifold \((\tilde{M},\tilde{g}_0)\) whose Ricci curvature satisfies \( \text{Ric}_{\tilde{g}_0} \geq -(n-1)\text{ric}_0 \) (including the case \( M = \tilde{M} \) if \( \tilde{M} \) is compact): For every metric \( g \) conformal to \( g_0 \) and every density \( \rho \) on \( M \) with \( \int_M \rho v_g = |M|_g \), one has

\[
\mu_k^g(\rho,1) \leq \frac{1}{|M|_g^2} \left( A_n k^2 + B_n |M|_g^2 \right),
\]

where \( |.|_g \) and \( |.|_{g_0} \) denote the Riemannian volumes with respect to \( g \) and \( g_0 \), respectively, and \( A_n \) and \( B_n \) are two constants which depend only on the dimension \( n \).

A direct consequence of this theorem is the following inequality satisfied by any density \( \rho \) on \((M,g)\) with \( \int_M \rho v_g = |M|_g \):

\[
\mu_k^g(\rho,1) \leq A_n \left( k \left| \frac{1}{|M|_g} \right|^2 + B_n \text{ric}_0 \right),
\]

where \( \text{ric}_0 \) is a positive number such that \( \text{Ric}_g \geq -(n-1)\text{ric}_0 \) \( g \) (see Corollary 4.1).

Regarding the eigenvalues \( \mu_k^g(1,\sigma) \), we are able to prove an estimate of the same type as (5): For every positive density \( \sigma \) on \((M,g)\) with \( \int_M \sigma v_g = |M|_g \) one has (Theorem 4.2)

\[
\mu_k^g(1,\sigma) \leq A_n \left( k \left| \frac{1}{|M|_g} \right|^2 + B_n \text{ric}_0 \right),
\]

where \( A_n \) and \( B_n \) are two constants which depend only on the dimension \( n \). It is worth noting that although the estimates (5) and (6) are similar, their proofs are of different nature. That is why we were not able to decide whether a stronger estimate such as (4) holds for \( \mu_k^g(1,\sigma) \).

When \( M \) is a bounded domain of a manifold \((\tilde{M},\tilde{g})\) of nonnegative Ricci curvature (e.g. \( \mathbb{R}^n \)), the inequalities (5) and (6) give the following estimates that can be seen as extensions
of Kröger’s inequality [30]:

\[ \mu_k^g(\rho, 1) \leq A_n \left( \frac{k}{|M|_g} \right)^{\frac{2}{n}} \]

and

\[ \mu_k^g(1, \sigma) \leq A_n \left( \frac{k}{|M|_g} \right)^{\frac{2}{n}} \]

provided that \( \int_M \rho v_g = |M|_g \) and \( \int_M \sigma v_g = |M|_g \). Notice that if we follow Kröger’s approach, then we get an upper bound of \( \mu_k^g(\rho, 1) \) which involves the gradient of \( \rho \) and the integral of \( \frac{1}{\rho} \) (see [16]).

According to (5) and (6), it is natural to introduce the following extremal eigenvalues on a given Riemannian manifold \((M, g)\):

\[ \mu_k^*(M, g) = \sup_{\int_M \rho v=1} \mu_k^g(\rho, 1) \quad \text{and} \quad \mu_k^{**}(M, g) = \sup_{\int_M \sigma v=1} \mu_k^g(1, \sigma) \]

In section 5 we investigate the qualitative properties of these quantities in the spirit of what we did in [9] for the conformal spectrum, thereby providing some answers to Question (5). For example, when \( M \) is of dimension 2, we have the following lower estimate (see [9, Corollary 1]):

\[ \mu_k^*(M, g) \geq 8 \pi \frac{k}{|M|_g}. \]

This means that, given any Riemannian surface \((M, g)\), endowed with the constant mass distribution \( \rho = 1 \) (whose eigenvalues can be very close to zero), it is always possible to redistribute the mass density \( \rho \) so that the resulting eigenvalue \( \mu_k^g(\rho, 1) \) is greater or equal to \( 8 \pi \frac{k}{|M|_g} \).

It turns out that this phenomenon is specific to the dimension 2. Indeed, we prove (Theorem 5.1) that on any compact manifold \( M \) of dimension \( n \geq 3 \), there exists a 1-parameter family of Riemannian metrics \( g_\varepsilon \) of volume 1 such that

\[ \mu_k^*(M, g_\varepsilon) \leq C k \varepsilon^{\frac{n-2}{n}}, \]

where \( C \) is a constant which does not depend on \( \varepsilon \). This means that in dimension \( n \geq 3 \), there exist geometric situations that generate very small eigenvalues, regardless of how the mass density is distributed.

Regarding the extremal eigenvalues \( \mu_k^{**}(M, g) \), a similar result is proved (Theorem 5.2) which is, moreover, also valid in dimension 2.

Note however that it is possible to construct examples of Riemannian manifolds \((M, g)\) with very small eigenvalues (for the constant densities), for which \( \mu_k^*(M, g) \) and \( \mu_k^{**}(M, g) \) are sufficiently large (see Proposition 5.2).

The last part of the paper (Section 6) is devoted to the study of the first extremal eigenvalues \( \mu_1^* \) and \( \mu_1^{**} \). We give sharp estimates of these quantities for some standard examples or under strong symmetry assumptions.

2. Preliminary results

This section is dedicated to some preliminary technical results. The reason is that in order to construct examples and counter-examples, it is often more convenient to use densities that are non smooth or which vanish somewhere in the manifold. The key arguments used in the proof of these results rely on the method developed by Colin de Verdière in [14].

Let \((M, g)\) be a compact Riemannian manifold, possibly with boundary.
Proposition 2.1. Let $\rho \in L^\infty(M)$ and $\sigma \in C^0(M)$ be two positive densities on $M$. For every $N \in \mathbb{N}^*$, there exist two sequences of smooth positive densities $\rho_p$ and $\sigma_p$ such that, \( \forall k \leq N \),
\[
\mu_k(\rho_p, \sigma_p) \to \mu_k(\rho, \sigma)
\]
as $p \to \infty$.

Proof. Using standard density results, let $\rho_p$ and $\sigma_p$ be two sequences of smooth positive densities such that, $\rho_p$ converges to $\rho$ in $L^2(M)$ and $\sigma_p$ converges uniformly towards $\sigma$. Assume furthermore that $\frac{1}{2} \inf \rho \leq \rho_p \leq 2 \sup \rho$ almost everywhere and that (replacing $\sigma_p$ by $\sigma + \|\sigma_p - \sigma\|_\infty$ if necessary) $\sigma \leq \sigma_p$ on $M$. Then the sequence of quadratic forms $q_p(u) = \int_M |\nabla u|^2 \sigma_p v_g$ together with the sequence of norms $\|u\|_p^2 = \int_M u^2 \rho_p v_g$ satisfy the assumptions of Theorem 1.8 of [14] which enables us to conclude. \( \square \)

Let $M_0$ be a domain in $M$ with $C^1$-boundary and let $\rho$ be a positive bounded function on $M_0$. In order to state the next result, let us introduce the following quadratic form defined on $H^1(M_0)$:
\[
Q_0(u) = \int_{M_0} |\nabla u|^2 v_g + \int_{\partial M \setminus M_0} |\nabla H(u)|^2 v_g
\]
where $H(u)$ is the harmonic extension of $u$ to $M \setminus M_0$, with Neumann condition on $\partial M \setminus \partial M_0$ if $\partial M \setminus \partial M_0 \neq \emptyset$ (i.e. $H(u)$ is harmonic on $M \setminus M_0$, coincides with $u$ on $\partial M_0 \setminus \partial M$, and $\frac{\partial H(u)}{\partial n} = 0$ on $\partial M \setminus \partial M_0$). The function $H(u)$ minimizes $\int_{M \setminus M_0} |\nabla v|^2 v_g$ among all functions $v$ on $M \setminus M_0$ which coincide with $u$ on $\partial M_0 \setminus \partial M$. We denote by $\gamma_k(M_0, \rho)$ the eigenvalues of this quadratic form with respect to the inner product of $L^2(M_0, \rho v_g)$ associated with $\rho$, that is,
\[
\gamma_k(M_0, \rho) = \inf_{E \in S_k^0} \sup_{\substack{u \in E \setminus \{0\}} \epsilon \in S_k^0} \frac{\int_{M_0} |\nabla u|^2 v_g + \int_{\partial M \setminus M_0} |\nabla H(u)|^2 v_g}{\int_{M_0} u^2 \rho v_g}
\]
where $S_k^0$ is the set of all $k$-dimensional vector subspaces of $H^1(M_0)$.

Proposition 2.2. Let $M_0 \subset M$ be a domain with $C^1$-boundary and let $\rho \in L^\infty(M_0)$ be a positive density with $\inf_{M_0} \rho > 0$. Define, for every $\epsilon > 0$, the density $\rho_\epsilon \in L^\infty(M)$ by
\[
\rho_\epsilon(x) = \begin{cases} \rho(x) & \text{if } x \in M_0 \\ \epsilon & \text{otherwise.} \end{cases}
\]
Then, for every positive $k$, $\mu_k(\rho_\epsilon, 1)$ converges to $\gamma_k(M_0, \rho)$ as $\epsilon \to 0$.

Proof. The eigenvalues $\mu_k(\rho_\epsilon, 1)$ are those of the quadratic form $q(\rho) = \int_M |\nabla u|^2 v_g$, $u \in H^1(M)$, with respect to the inner product $\|u\|_\rho^2 = \int_M u^2 \rho v_g$. Set $M_\infty = M \setminus M_0$ and $\Gamma = \partial M_0 \cap \partial M_\infty = \partial M_0 \setminus \partial M$. We identify $H^1(M)$ with the space $\mathcal{H}_\epsilon = \{v = (v_0, v_\infty) \in H^1(M_0) \times H^1(M_\infty) : v_\infty|_{\Gamma} = \sqrt{\epsilon} v_0|_{\Gamma} \}$ through the map $\Psi_\epsilon(u) = (u|_{M_0}, \sqrt{\epsilon} u|_{M_\infty})$. We endow $\mathcal{H}_\epsilon$ with the inner product given by $\|\Psi_\epsilon(v_0, v_\infty)\|_\rho^2 = \int_{M_0} v_0^2 \rho v_g + \int_{M_\infty} v_\infty^2 v_g$ and consider the quadratic form $q_\epsilon(v_0, v_\infty) = \int_{M_0} |\nabla v_0|^2 v_g + \frac{1}{\epsilon} \int_{M_\infty} |\nabla v_\infty|^2 v_g$, so that, for every $u \in H^1(M)$
\[
\|\Psi_\epsilon(u)\|_\rho = \|u\|_\epsilon \quad \text{and} \quad q_\epsilon(\Psi_\epsilon(u)) = q(u).
\]
Therefore, the eigenvalues of the quadratic form $q : H^1(M) \to \mathbb{R}$ with respect to $\| \cdot \|_\varepsilon$ (i.e. \( \mu^\varepsilon_k(\varepsilon, 1) \)) coincide with those of $q_\varepsilon : \mathcal{H}_\varepsilon \to \mathbb{R}$ with respect to $\| \cdot \|_\rho$.

The space $\mathcal{H}_\varepsilon$ decomposes into the direct sum $\mathcal{H}_\varepsilon = \mathcal{K}_0^\varepsilon + \mathcal{K}_\infty^\varepsilon$ with $\mathcal{K}_0^\varepsilon = \{(v_0, v_\infty) \in \mathcal{H}_\varepsilon : v_\infty \text{ is harmonic, and } \frac{\partial v_\infty}{\partial \nu} = 0 \text{ on } \partial M \setminus \partial M_0 \text{ if } \partial M \setminus \partial M_0 \neq \emptyset \}$, and $\mathcal{K}_\infty^\varepsilon = \{(v_0, v_\infty) \in \mathcal{H}_\varepsilon : v_0 = 0 \}$ (Indeed, $v = (v_0, v_\infty) = (v_0, \sqrt{\varepsilon} H(v_0)) + (0, v_\infty - \sqrt{\varepsilon} H(v_0))$). These two subspaces are $q_\varepsilon$-orthogonal and, denoting by $\lambda_1(M_\infty)$ the first eigenvalue of $M_\infty$ under Dirichlet boundary conditions on $\Gamma$ and Neumann boundary conditions on $\partial M_\infty \setminus \Gamma$, we have, for every $v = (0, v_\infty) \in \mathcal{K}_\infty^\varepsilon$,

$$
q_\varepsilon(v) = \frac{1}{\varepsilon} \int_{M_\infty} |\nabla v_\infty|^2 v_\infty \geq \frac{1}{\varepsilon} \lambda_1(M_\infty) \int_{M_\infty} v_\infty^2 v_\infty = \frac{1}{\varepsilon} \lambda_1(M_\infty) \| v \|^2.
$$

Theorem I.7 of [14] then implies that, given any integer $N > 0$, the $N$ first eigenvalues $\mu_k(\rho_\varepsilon, 1)$ of $q_\varepsilon$ on $\mathcal{H}_\varepsilon$ are, for sufficiently small $\varepsilon$, as close as desired to the eigenvalues of the restriction of $q_\varepsilon$ on $\mathcal{K}_0^\varepsilon$.

We still have to compare the eigenvalues of $q_\varepsilon$ on $\mathcal{K}_0^\varepsilon$, that we denote $\gamma_k(\varepsilon)$, with the eigenvalues $\gamma_k(M_0, \rho)$ of $Q_0$ on $L^2(M_0, \rho v_g)$. For this, we make use of Theorem I.8 of [14]. Indeed, $\mathcal{K}_0^\varepsilon$ can be identified to $H^1(M_0)$ through $\Psi_\varepsilon^0 : u \in H^1(M_0) \mapsto (u, \sqrt{\varepsilon} H(u)) \in \mathcal{K}_0^\varepsilon$, which satisfies $\| \Psi_\varepsilon^0(u) \|_\varepsilon^2 = \int_{M_0} u^2 \rho v_g + \varepsilon \int_{M_\infty} H(u^2) v_g$ and $q_\varepsilon(\Psi_\varepsilon^0(u)) = Q_0(u) = \int_{M_0} |\nabla u|^2 v_g + \int_{M_\infty} |\nabla H(u)|^2 v_g$. Hence, we are led to compare, on $L^2(M_0)$, the eigenvalues of the quadratic form $Q_0$ with respect to the following two scalar products: $\| u \|_\rho^2 = \int_{M_0} u^2 \rho v_g$ and $\| u \|_\varepsilon^2 = \int_{M_0} u^2 \rho v_g + \varepsilon \int_{M_\infty} H(u)^2 v_g$.

Now, since $H(u)$ is a harmonic extension of $u_\Gamma$ to $M_\infty$, there exists a constant $C$, which does not depend on $\varepsilon$, such that $\int_{M_\infty} H(u)^2 v_g \leq C \int_{\Gamma} u^2 v_\beta$, where $\bar{g}$ is the metric induced on $\Gamma$ by $g$. Indeed, let $\eta$ be the solution in $M_\infty$ of $\Delta \eta = -1$ with $\eta_\Gamma = 0$ and $\frac{\partial \eta}{\partial \nu} = 0$ on $\partial M_\infty \setminus \Gamma$. Observe that we have $\eta \geq 0$ (maximum principle and Hopf Lemma) and, since $\int_{M_\infty} g(\nabla (\eta H(u)), \nabla H(u)) v_\beta = 0$, $\int_{M_\infty} g(\nabla \eta, \nabla H(u)^2) v_\beta = -2 \int_{M_\infty} \eta |\nabla H(u)|^2 v_\beta \leq 0$. Thus

$$
\int_{M_\infty} H(u)^2 v_\beta = - \int_{M_\infty} H(u) \Delta \eta v_\beta = \int_{M_\infty} g(\nabla \eta, \nabla H(u^2)) v_\beta + \int_{\Gamma} u^2 \frac{\partial \eta}{\partial \nu} v_\beta \leq c \int_{\Gamma} u^2 v_\beta
$$

where $c$ is an upper bound of $\frac{\partial \eta}{\partial \nu}$ on $\Gamma$. On the other hand, $\int_{\Gamma} u^2 v_\beta$ is controlled by $\| u \|_{H^2(\Gamma)}^2$ which in turn is controlled (using boundary trace inequalities in $M_0$) by $\| u \|_{H^1(M_0)}^2$. Finally, there exists a constant $C$ (which depends on $\text{ess inf}_{M_0} \rho$ but not on $\varepsilon$) such that $\int_{M_\infty} H(u)^2 v_g \leq C(\int_{M_0} u^2 \rho v_g + \int_{M_\infty} |\nabla u|^2 v_g)$ and, then

$$
\| u \|_{\varepsilon}^2 \leq C(\| u \|_{\rho}^2 + Q_0(u)).
$$

Since $\| u \|_{\rho}^2$ converges to $\| u \|_{\rho}^2$ as $\varepsilon \to 0$, this implies, according to [14, Theorem I.8] (see also [25, Remark 2.14]), that, for sufficiently small $\varepsilon$, the $N$ first eigenvalues $\gamma_k(\varepsilon)$ of $Q_0$ with respect to $\| \cdot \|_\varepsilon$ are as close as desired to those, $\gamma_k(M_0, \rho)$, of $Q_0$, with respect to $\| \cdot \|_\rho$. 

Recall that in dimension 2, one has

$$
\mu^\varepsilon_k(\varepsilon, 1) = \lambda_k(M, \varepsilon g).
$$

(7)
An immediate consequence of Proposition 2.2 is the following result which completes Theorem III.1 of Colin de Verdière [14].

**Theorem 2.1.** Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\) and let \(M_0 \subset M\) be a domain with boundary of class \(C^1\). Let \(g_\varepsilon\) be the a family of Riemannian metrics on \(M\), with \(g_\varepsilon = g\) on \(M_0\) and \(g_\varepsilon = \varepsilon g\) outside \(M_0\). Let \(k \geq 1\).

1. (Theorem III.1 of [14]) If \(n \geq 3\), then \(\lambda_k(M, g_\varepsilon)\) converges to \(\lambda_k(M_0, g)\) as \(\varepsilon \to 0\).
2. If \(n = 2\), then \(\lambda_k(M, g_\varepsilon)\) converges to \(\gamma_k(M_0, 0)\) as \(\varepsilon \to 0\).

From Proposition 2.1 and Proposition 2.2 we can deduce the following two corollaries:

**Corollary 2.1.** Let \(\rho \in L^\infty(M_0)\) be a positive density on a domain \(M_0 \subset M\) with boundary of class \(C^1\). There exists a family of smooth positive densities \(\rho_\varepsilon\) on \(M\) such that \(\int_M \rho_\varepsilon v_g\) tends to \(\int_{M_0} \rho v_g\) and, for every \(k \in \mathbb{N}^*\), \(\mu_k(\rho_\varepsilon, 1)\) converges to \(\gamma_k(M_0, 0)\) as \(\varepsilon \to 0\).

**Corollary 2.2.** Let \((M, g)\) be a compact manifold possibly with boundary and let \(M_0 \subset M\) be a domain with boundary of class \(C^1\). For every integer \(k > 0\) and every \(\varepsilon > 0\), there exists a positive smooth density \(\rho_\varepsilon\) on \(M\) such that \(\int_M \rho_\varepsilon v_g = |M|_g\) and

\[
\mu_k(\rho_\varepsilon, 1) \geq \frac{|M_0|_g}{|M|_g} \gamma_k(M_0, 0) - \varepsilon.
\]

**Proof.** Let \(\rho\) be the density on \(M_0\) defined by \(\rho = \frac{|M|_g}{|M_0|_g}\). We apply Corollary 2.1 taking into account that \(\gamma_k(M_0, \rho) = \frac{|M_0|_g}{|M|_g} \gamma_k(M_0, 0)\). \(\square\)

**Remark 2.1.** In dimension 2, it is clear from (7) that the problem of minimizing or maximizing \(\mu_k^g(\rho, 1)\) w.r.t. \(\rho\) is equivalent to the problem of minimizing or maximizing \(\lambda_k(M, g)\) w.r.t. conformal deformations of the metric \(g\). In dimension \(n \geq 3\), the two problems are completely different. To emphasize this difference, observe that, given a positive constant \(c\), one has

\[
\inf_{\rho \leq c} \mu_k^g(\rho, 1) \geq \frac{1}{c} \mu_k^g(1, 1) = \frac{1}{c} \lambda_k(M, g) > 0
\]

while

\[
\inf_{\rho \leq c} \lambda_k(M, \rho g) = 0.
\]

Indeed, let \(B_j, j \leq k + 1\) be a family of mutually disjoint balls in \(M\) and consider the density \(\rho_\varepsilon\) which is equal to \(c\) on each \(B_j\) and equal to \(\varepsilon\) elsewhere. According to [14, Theorem III.1], \(\lambda_k(M, \rho_\varepsilon g)\) converges as \(\varepsilon \to 0\) to the \((k + 1)\)-th Neumann eigenvalue of the union of balls which is zero.

3. Bounding the eigenvalues from below

3.1. Non existence of “density-free” lower bounds. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\), possibly with boundary, and denote by \([g]\) the set of all Riemannian metrics \(g'\) on \(M\) which are conformal to \(g\) with \(|M|_{g'} = |M|_g\). It is well known that \(\lambda_k(M, g')\) can be as small as desired when \(g'\) varies within \([g]\), i.e.
Proof of Theorem 3.1. (i): In dimension 2 one has $\mu_k(\rho, 1) = \lambda_k(M, \rho g)$ and the problem is equivalent to that of deforming conformally the metric $g$ into a metric $\rho g$ whose $k$-th eigenvalue is as small as desired. The existence of such a deformation is well known.

Assume now that the dimension of $M$ is at least 3. Let us choose a point $x_0$ in $M$. The Riemannian volume of a geodesic ball $B(x, r)$ of radius $r$ in $M$ is asymptotically equivalent, as $r \to 0$, to $\omega_n r^n$, where $\omega_n$ is the volume of the unit ball in the $n$-dimensional Euclidean space. Therefore, there exist $\varepsilon_0 \in (0, 1)$ sufficiently small and $N \in \mathbb{N}$ so that, for every $r < \frac{\varepsilon_0}{N}$ and every $x \in B(x_0, \varepsilon_0)$,

$$\frac{1}{2} \omega_n r^n \leq |B(x, r)| \leq 2 \omega_n r^n. \quad (9)$$

Fix a positive integer $k$ and let $\delta = \frac{n-2}{2} - 1$. One can choose $N \in \mathbb{N}$ sufficiently large so that, for every $\varepsilon < \frac{\varepsilon_0}{N}$, the ball $B(x_0, \varepsilon)$ contains $k$ mutually disjoint balls of radius $2 \varepsilon^{\frac{2}{n} - \delta}$ (indeed, since $\frac{n}{2} - \delta > 1$, $2 \varepsilon^{\frac{2}{n} - \delta}$ is very small compared to $\varepsilon$ as the latter tends to zero). We consider a smooth positive density $\rho_\varepsilon$ such that $\rho_\varepsilon = \frac{1}{\varepsilon^n}$ inside $B(x_0, \varepsilon)$, $\rho_\varepsilon = \varepsilon$ in $M \setminus B(x_0, 2\varepsilon)$, and $\rho_\varepsilon \leq \frac{1}{\varepsilon^n}$ elsewhere. Thanks to (9), one has

$$\int_M \rho_\varepsilon v_g \leq \frac{1}{\varepsilon^n} |B(x_0, 2\varepsilon)|_g + \varepsilon |M|_g \leq 2^{n+1} \omega_n + \varepsilon |M|_g.$$

For simplicity, we set $\alpha = \frac{n}{2} - \delta = \frac{n+2}{4}$ and denote by $x_1, \ldots, x_k$ the centers of $k$ mutually disjoint balls of radius $2\varepsilon^n$ contained in $B(x_0, \varepsilon)$.

For each $i \leq k$, we denote $f_i$ the function which vanishes outside $B(x_i, 2\varepsilon^n)$, equals 1 in $B(x_i, \varepsilon^n)$, and $f_i(x) = 2 - \frac{1}{\varepsilon^n} d_g(x, x_i)$ for every $x$ in the annulus $B(x_i, 2\varepsilon^n) \setminus B(x_i, \varepsilon^n)$. The norm of the gradient of $f_i$ vanishes everywhere unless inside the annulus where we have $|\nabla f_i| = \frac{1}{\varepsilon^n}$. Thus, using (9),

$$\int_M f_i^2 \rho_\varepsilon v_g \geq \frac{1}{\varepsilon^n} \int_{B(x_i, \varepsilon^n)} f_i^2 v_g = \frac{|B(x_i, \varepsilon^n)|}{\varepsilon^n} \geq \frac{1}{2} \omega_n \varepsilon^n (\alpha - 1).$$
and
\[ \int_M |\nabla f_i|^2 v_g \leq \frac{|B(x_i, 2\varepsilon^2)|}{\varepsilon^{2n}} = 2^{n+1} \omega_n \varepsilon^{n(n-2)}. \]

Thus
\[ R_{(g, \rho, 1)}(f_i) \leq 2^{n+2} \varepsilon^{n-2} = 2^{n+2} \varepsilon^{n+2}. \]

In conclusion, we have
\[ \mu_k(\rho, 1) \leq 2^{n+2} \varepsilon^{n+2} \]
and
\[ \mu_k \left( \frac{\rho}{\int_M \rho_v v_g}, 1 \right) = \mu_k(\rho, 1) \int_M \rho v_g \leq 2^{n+2} \left( \frac{2^{n+1} \omega_n \varepsilon^{n-2}}{|M|g} + \varepsilon^2 \right). \]

Letting \( \varepsilon \) tends to zero we get the result.

\( (ii) \): The proof is similar to the previous one. For \( \varepsilon \) sufficiently small, we may assume that there exist \( k + 1 \) mutually disjoint balls \( B(x_i, \varepsilon^2) \) inside a ball \( B(x_0, \varepsilon) \) and consider any function \( \sigma_\varepsilon \in \mathcal{R}_0 \) such that \( \sigma_\varepsilon = \varepsilon^2 \) inside \( B(x_0, \varepsilon) \). For each \( i \leq k + 1 \), let \( f_i \) be the function which vanishes outside \( B(x_i, 2\varepsilon^2) \), equals \( 1 \) in \( B(x_i, \varepsilon^2) \), and \( f_i(x) = 2 - \frac{1}{\varepsilon^2} d_g(x, x_i) \) in \( B(x_i, 2\varepsilon^2) \setminus B(x_i, \varepsilon^2) \). As before,
\[ \int_M f_i^2 v_g \geq \int_{B(x_i, \varepsilon^2)} f_i^2 dx \geq |B(x_i, \varepsilon^2)| \geq \frac{1}{2} \omega_n \varepsilon^{2n} \]
and
\[ \int_M |\nabla f_i|^2 \sigma_\varepsilon v_g \leq \frac{1}{\varepsilon^2} \int_{B(x_i, 2\varepsilon^2)} \sigma_\varepsilon v_g \leq \varepsilon |B(x_i, 2\varepsilon^2)| \leq 2^{n+1} \omega_n \varepsilon^{2n+1}. \]

Thus
\[ \mu_k(1, \sigma_\varepsilon) \leq \max_{i\leq k+1} \frac{\int_M |\nabla f_i|^2 \sigma_\varepsilon v_g}{\int_M f_i^2 v_g} \leq 2^{n+2} \varepsilon. \]

\( (iii) \): For sufficiently small \( \varepsilon \), let \( B(x_i, 4\varepsilon) \), \( i \leq k + 1 \), be \( k + 1 \) mutually disjoint balls of radius \( 4\varepsilon \) in \( M \). As before, we can assume that, \( \forall r \leq 4\varepsilon, \frac{1}{2} \omega_n r^n \leq |B(x_i, r)| \leq 2\omega_n r^n \). We define \( \rho_\varepsilon \) to be equal to \( \frac{1}{\varepsilon^2} \) on each of the balls \( B(x_i, \varepsilon) \) and equal to \( \varepsilon^n \) in the complement of \( \bigcup_{i=1}^{k} B(x_i, 2\varepsilon) \). For every \( i \leq k + 1 \), the function \( f_i \) defined to be equal to \( 1 \) on \( B(x_i, 2\varepsilon) \) and \( f_i(x) = 2 - \frac{1}{\varepsilon^2} d_g(x, x_i) \) in the annulus \( B(x_i, 4\varepsilon) \setminus B(x_i, 2\varepsilon) \) and zero in the complement of \( B(x_i, 4\varepsilon) \) satisfies
\[ \int_M f_i^2 \rho_\varepsilon v_g \geq \int_{B(x_i, \varepsilon)} f_i^2 \rho_\varepsilon dx = \frac{1}{\varepsilon^n} |B(x_i, \varepsilon)| \geq \frac{1}{2} \omega_n. \]

On the other hand, \( \forall p > 0 \),
\[ \int_M |\nabla f_i|^2 \rho_\varepsilon^p v_g = \varepsilon^p \int_{B(x_i, 4\varepsilon) \setminus B(x_i, 2\varepsilon)} |\nabla f_i|^2 v_g = \varepsilon^p \frac{1}{4\varepsilon^2} |B(x_i, 4\varepsilon)| \leq 2^{2n-1} \omega_n \varepsilon^{(p+1)n-2}. \]

Thus
\[ \mu_k(\rho_\varepsilon, \rho_\varepsilon^p) \leq \max_{i\leq k+1} \frac{\int_M |\nabla f_i|^2 \sigma_\varepsilon v_g}{\int_M f_i^2 v_g} \leq 2^{2n} \varepsilon^{(p+1)n-2}. \]

Regarding \( \int_M \rho_\varepsilon v_g \), it is clear that it is bounded both from above and from below by positive constants that are independent of \( \varepsilon \), which enables us to conclude. \( \square \)
3.2. Cheeger-type inequality. Theorem 3.1 tells us that it is necessary to involve other quantities than the total mass in order to get lower bounds for the eigenvalues. Our next theorem gives a lower estimate which is modeled on Cheeger’s inequality, with suitably defined isoperimetric constants, as was done by Jammes for Steklov eigenvalues [27].

Let \((M, g)\) be a compact Riemannian manifold, possibly with boundary. The classical Cheeger constant is defined by

\[
h(M) = \inf_{|D| \leq \frac{1}{4}|M|_g} \frac{|\partial D \setminus \partial M|_g}{|D|_g} = \inf_{D \subset M} \frac{|\partial D \setminus \partial M|_g}{\min\{|D|_g, |M|_g - |D|_g\}}.
\]

Given two positive densities \(\rho\) and \(\sigma\) on \(M\), we introduce the following Cheeger-type constant:

\[
h_{\rho, \sigma}(M) = \inf_{|D|_\sigma \leq \frac{1}{4}|M|_\sigma} \frac{|\partial D \setminus \partial M|_\sigma}{|D|_\rho},
\]

with \(|D|_\sigma\) (resp. \(|\partial D \setminus \partial M|_\sigma\)) is the \(n\)-volume of \(D\) (resp. the \((n-1)\)-volume of \(\partial D \setminus \partial M\)) with respect to the measure induced by \(\sigma v_g\).

**Theorem 3.2.** One has

\[
\mu_1(\rho, \sigma) \geq \frac{1}{4} h_{\sigma, \sigma}(M) h_{\rho, \sigma}(M).
\]

**Proof.** The proof follows the same general outline as the original proof by Cheeger (see [8] and [5]). We give here a complete proof in the case where \(M\) is a closed manifold. The proof in the case \(\partial M \neq \emptyset\) can be done analogously. Let \(f\) be a Morse function such that the \(\sigma\)-volume of its positive nodal domain \(\Omega^+(f) = \{ f > 0 \}\) is less or equal to half the \(\sigma\)-volume of \(M\). For every \(t \in (0, \sup f)\) excepting a finite number of values, the set \(f^{-1}(t)\) is a regular hypersurface of \(M\). We denote by \(v_g^t\) the measure induced on \(f^{-1}(t)\) by \(v_g\) and set \(P_\sigma(t) = \int_{f^{-1}(t)} \sigma v_g^t\). The level sets of \(f\) are denoted \(\Omega(t) = \{ f > t \}\) and we set \(V_\sigma(t) = \int_{\Omega(t)} \sigma v_g\) and \(V_\rho(t) = \int_{\Omega(t)} \rho v_g\). Using the co-area formula one gets

\[
\int_{\Omega^+(f)} |\nabla f| \sigma v_g = \int_0^{+\infty} P_\sigma(t) dt.
\]

On the other hand, the same co-area formula gives

\[
V_\rho(t) = \int_t^{+\infty} ds \int_{f^{-1}(s)} \frac{\rho}{|\nabla f|} v_g^s.
\]

Thus

\[
V_\rho'(t) = -\int_{f^{-1}(t)} \frac{\rho}{|\nabla f|} v_g^t.
\]

Now

\[
\int_{\Omega^+(f)} f \rho v_g = \int_0^{+\infty} dt \int_{f^{-1}(t)} f \rho v_g^t = \int_0^{+\infty} t dt \int_{f^{-1}(t)} \rho v_g^t = -\int_0^{+\infty} t V_\rho'(t) dt
\]

which gives after integration by parts

\[
\int_{\Omega^+(f)} f \rho v_g = \int_0^{+\infty} V_\rho(t) dt.
\]
Similarly, one has
\[ \int_{\Omega^+ (f)} f \sigma v_g = \int_0^{+\infty} V_\sigma (t) dt. \]
Since \( P_\sigma (t) \geq h_{\sigma, \sigma} (M) V_\sigma (t) \) and \( P_\sigma (t) \geq h_{\rho, \rho} (M) V_\rho (t) \) we deduce
\[ \int_{\Omega^+ (f)} |\nabla f| \sigma v_g \geq \max \left\{ h_{\sigma, \sigma} (M) \int_{\Omega^+ (f)} f \sigma v_g, \, h_{\rho, \rho} (M) \int_{\Omega^+ (f)} f \rho v_g \right\}. \]
Using Cauchy-Schwarz inequality we get
\[ \int_{\Omega^+ (f)} |\nabla f|^2 \sigma v_g \geq \frac{1}{4} \left( \int_{\Omega^+ (f)} |\nabla f|^2 \sigma v_g \right)^2 \geq \frac{1}{4} \frac{h_{\sigma, \sigma} (M) h_{\rho, \rho} (M) \int_{\Omega^+ (f)} f^2 \sigma v_g \int_{\Omega^+ (f)} f^2 \rho v_g}{\int_{\Omega^+ (f)} f^2 \sigma v_g} = \frac{1}{4} h_{\sigma, \sigma} (M) h_{\rho, \rho} (M) \int_{\Omega^+ (f)} f^2 \rho v_g. \] (10)

Now, let \( m \in \mathbb{R} \) be such that \( |\{ f > m \}|_\sigma = |\{ f < m \}|_\sigma = \frac{1}{2} |M|_\sigma \) (such an \( m \) is called a median of \( f \) for \( \sigma \)). Applying (10) to \( f - m \) and \( m - f \) we get
\[ \int_{\{ f > m \}} |\nabla f|^2 \sigma v_g \geq \frac{1}{4} h_{\sigma, \sigma} (M) h_{\rho, \rho} (M) \int_{\{ f > m \}} (f - m)^2 \rho v_g \]
and
\[ \int_{\{ f < m \}} |\nabla f|^2 \sigma v_g \geq \frac{1}{4} h_{\sigma, \sigma} (M) h_{\rho, \rho} (M) \int_{\{ f < m \}} (f - m)^2 \rho v_g. \]
Summing up we obtain
\[ \int_M |\nabla f|^2 \sigma v_g \geq \frac{1}{4} h_{\sigma, \sigma} (M) h_{\rho, \rho} (M) \int_M (f - m)^2 \rho v_g. \]
Since \( \int_M (f - m)^2 \rho v_g = \int_M f^2 \rho v_g + m^2 |M|_\rho - 2m \int_M f \rho v_g \), we deduce that, for every \( f \) such that \( \int_M f \rho v_g = 0 \),
\[ \int_M |\nabla f|^2 \sigma v_g \geq \frac{1}{4} h_{\sigma, \sigma} (M) h_{\rho, \rho} (M) \int_M f^2 \rho v_g \]
which, thanks to (3), implies the desired inequality. \( \square \)

**Remark 3.1.** In dimension 2, Theorem 3.2 can be restated as follows: If \((M, g)\) is a compact Riemannian surface, then
\[ \lambda_1 (M, g) \geq \frac{1}{4} \sup_{g' \in [g]} h_{g', g'} (M) h_{g, g'} (M) \] (11)
where \( h_{g, g'} (M) = \inf_{\|D\|_{g'} \leq \frac{1}{2} |M|_{g'}} \frac{|\partial D|_{g'}}{|\partial D|_g} \). Indeed, for any \( g' \in [g] \) there exists a positive \( \rho \in C^\infty (M) \) such that \( g = \rho g' \). Thus, \( \lambda_1 (M, g) = \mu^2 (\rho, 1) \) and (11) follows from Theorem 3.2. This inequality can be seen as an improvement of Cheeger’s inequality since the right-hand side is obviously bounded below by \( h_{g, g} (M)^2 \). Notice that in [6], Buser gives an example of a family of metrics on the 2-torus such that the Cheeger constant goes to zero while the first eigenvalue is bounded below. The advantage of (11) is that its right hand side does not go to zero for Buser’s example.
A natural question is to investigate a possible reverse inequality of Buser’s type (see [6, 32]). The following theorem provides a negative answer to this question.

**Theorem 3.3.** Let \((M, g)\) be a compact Riemannian manifold, possibly with boundary.

(i) There exists a family of positive densities \(\sigma_\varepsilon, \varepsilon > 0\), on \(M\) with \(\int_M \sigma_\varepsilon v_g = 1\) and such that \(h_{1, \sigma_\varepsilon}(M)h_{\sigma_\varepsilon, \sigma_\varepsilon}(M)\) goes to zero with \(\varepsilon\) while \(\mu_1(1, \sigma_\varepsilon)\) stays bounded below by a constant \(C\) which does not depend on \(\varepsilon\).

(ii) There exists a family of positive densities \(\rho_\varepsilon, \varepsilon > 0\), on \(M\) with \(\int_M \rho_\varepsilon v_g = 1\) and such that \(h_{\rho_\varepsilon, 1}(M)\) goes to zero with \(\varepsilon\) while \(\mu_1(\rho_\varepsilon, 1)\) stays bounded below by a constant \(C\) which does not depend on \(\varepsilon\).

**Proof.** We start by proving the result for the unit ball \(B^n \subset \mathbb{R}^n\) and then explain how to deduce it for any compact Riemannian manifold. For every \(r \in (0, 1)\) we denote by \(B(r)\) the ball of radius \(r\) centered at the origin and by \(A_r\) the annulus \(B^n \setminus B(r)\). In the sequel, whenever we integrate over a Euclidean set, the integration is implicitly made with respect to the standard Lebesgue’s measure.

**Proof of (i):** For every \(\varepsilon \in (0, \frac{1}{2})\) we define a smooth nonincreasing radial density \(\sigma_\varepsilon\) on \(B^n\) such that \(\sigma_\varepsilon = \frac{1}{a}\), with \(a \in (0, 1)\) (e.g. \(a = \frac{1}{2}\)) inside \(B^n(\varepsilon)\) and \(\sigma_\varepsilon = b_\varepsilon\) in \(B^n \setminus B(2\varepsilon)\), where \(b_\varepsilon\) is chosen so that \(\int_{B^n} \sigma_\varepsilon = \omega_n\), the volume of \(B^n\). We then have

\[
\int_{B(\varepsilon)} \sigma_\varepsilon = \omega_n \varepsilon^{n-1-a} \quad \text{and} \quad \int_{A_2\varepsilon} \sigma_\varepsilon = \omega(1 - 2^n \varepsilon^n) b_\varepsilon.
\]

Since \(\int_{B^n} \sigma_\varepsilon = \omega_n\) and \(b_\varepsilon \leq \sigma_\varepsilon \leq \varepsilon^{-1-a}\) on \(B(2\varepsilon) \setminus B(\varepsilon)\), we have

\[
\omega_n \varepsilon^{n-1-a} + b_\varepsilon \omega_n (1 - \varepsilon^n) \leq \omega_n \leq \omega_n^2 \varepsilon^{n-1-a} + b_\varepsilon \omega_n (1 - 2^n \varepsilon^n),
\]

that is

\[
\frac{1 - 2^n \varepsilon^{n-1-a}}{1 - 2^n \varepsilon^n} \leq b_\varepsilon \leq \frac{1 - \varepsilon^{n-1-a}}{1 - \varepsilon^n}.
\]

(12)

Now, the Cheeger constant \(h_{\sigma_\varepsilon, \sigma_\varepsilon}(B^n)\) satisfies

\[
h_{\sigma_\varepsilon, \sigma_\varepsilon}(B^n) \leq \frac{|\partial B(2\varepsilon)|_{\sigma_\varepsilon}}{|\partial B(2\varepsilon)|_{\sigma_\varepsilon}} \leq \frac{|\partial B(2\varepsilon)|_{\sigma_\varepsilon}}{|\partial B(\varepsilon)|_{\sigma_\varepsilon}} = \frac{n b_\varepsilon \omega_n (2\varepsilon)^{n-1}}{\omega_n (1 - \varepsilon^n)} \leq n 2^{n-1} \varepsilon^n.
\]

On the other hand, for \(r_0 = (\frac{1}{2})^{\frac{1}{2}}\) we have \(|B(r_0)|_{\sigma_\varepsilon} < \omega_n (\varepsilon^{n-1-a} + \frac{1}{4} b_\varepsilon) < \frac{1}{2} \omega_n\) when \(\varepsilon\) is sufficiently small, so that

\[
h_{1, \sigma_\varepsilon}(B^n) \leq \frac{|\partial B(r_0)|_{\sigma_\varepsilon}}{|\partial B(r_0)|_{\sigma_\varepsilon}} = \frac{n \omega_n r_0^{n-1} b_\varepsilon}{\omega_n r_0^n} \leq 4 \frac{1}{2} \omega_n.
\]

Hence, the product \(h_{1, \sigma_\varepsilon}(B^n)h_{\sigma_\varepsilon, \sigma_\varepsilon}(B^n)\) tends to zero as \(\varepsilon \to 0\). Regarding the first positive eigenvalue \(\mu_1(1, \sigma_\varepsilon)\), if \(f\) is a corresponding eigenfunction, then \(\int_{B^n} f = 0\) and

\[
\mu_1(1, \sigma_\varepsilon) = \frac{\int_{B^n} \nabla f^2(\sigma_\varepsilon)}{\int_{B^n} f^2} \geq b_\varepsilon \frac{\int_{B^n} \nabla f^2}{\int_{B^n} f^2} \geq b_\varepsilon \lambda_1(B^n, g_E)
\]

with \(b_\varepsilon \geq \frac{1}{2}\) for sufficiently small \(\varepsilon\) according to (12).

Now, given a Riemannian manifold \((M, g)\), we fix a point \(x_0\) and choose \(\delta > 0\) so that the geodesic ball \(B(x_0, \delta)\) is 2-quasi-isometric to the Euclidean ball of radius \(\delta\). In the Riemannian manifold \((M, \frac{1}{2}g)\), the ball \(B(x_0, 1)\) is 2-quasi-isometric to the Euclidean
ball $B^n$. We define $\sigma_\varepsilon$ in $B(x_0, 1)$ as the pull back of the function $\sigma_\varepsilon$ constructed above, and extend it by $b_\varepsilon$ in $M \setminus B(x_0, 1)$. Because of (12), we easily see that $\int_M \sigma_\varepsilon v_\varepsilon$ stays bounded independently from $\varepsilon$. We can also check that $h_{\sigma_\varepsilon}(M)$ and $h_{\sigma_\varepsilon, \sigma_\varepsilon}(M)$ have the same behavior as before and that (since $\sigma_\varepsilon \geq b_\varepsilon \geq \frac{1}{2}$) the eigenvalue $\mu_1^{\delta_\varepsilon \varepsilon}(1, \sigma_\varepsilon)$ is bounded from below by $\frac{1}{2} \lambda_1(M, \delta^{-\varepsilon} g)$ which is a positive constant $C$ independent of $\varepsilon$. Thus, $\mu_1^\varepsilon(1, \sigma_\varepsilon) = \delta \mu_1^\varepsilon(1, \sigma_\varepsilon) \geq C \delta^2$.

Proof of (ii): As before we define the density $\rho_\varepsilon \in L^\infty(B^n)$, $\varepsilon \in (0, \frac{1}{2})$, by

$$\rho_\varepsilon = \left\{ \begin{array}{ll} \frac{1}{b_\varepsilon} & \text{if } x \in B(\varepsilon) \\ \frac{1}{1-\varepsilon^n} & \text{if } x \not\in B(\varepsilon) \end{array} \right. \quad (13)$$

so that $\int_{B^n} \rho_\varepsilon dx = \omega_n$ and $b_\varepsilon < 1$. The corresponding Cheeger constant satisfies

$$h_{\rho_\varepsilon, 1} \leq \frac{|\partial B(\varepsilon)|}{|B(\varepsilon)|} = \frac{n \omega_n \varepsilon^{n-1}}{\omega_n \varepsilon^{n-1-a}} = n \varepsilon^a,$$

which goes to zero as $\varepsilon \to 0$.

To prove that the first positive Neumann eigenvalue $\mu_1(\rho_\varepsilon, 1)$ is uniformly bounded below we will first prove that the first Dirichlet eigenvalue $\lambda_1(\rho_\varepsilon)$ satisfies

$$\lambda_1(\rho_\varepsilon) \geq \frac{1}{4} \lambda^* \quad (14)$$

where $\lambda^*$ is the first Dirichlet eigenvalue of the Laplacian on $B^n$. Indeed, let $f$ be a positive eigenfunction associated to $\lambda_1(\rho_\varepsilon)$. Such a function is necessarily a nonincreasing radial function and it satisfies (with $b_\varepsilon \leq 1$)

$$\lambda_1(\rho_\varepsilon) = \frac{\int_{B(\varepsilon)} |\nabla f|^2 + \int_{A_\varepsilon} |\nabla f|^2}{\int_{B(\varepsilon)} f^2 \rho_\varepsilon + \int_{A_\varepsilon} f^2 \rho_\varepsilon} \geq \frac{\int_{B(\varepsilon)} |\nabla f|^2 + \int_{A_\varepsilon} |\nabla f|^2}{\varepsilon^{-1-a} \int_{B(\varepsilon)} f^2 + \int_{A_\varepsilon} f^2} \quad (15)$$

For convenience we assume that $f(\varepsilon) = 1$.

If we denote by $\nu(A_\varepsilon)$ the first eigenvalue of the mixed eigenvalue problem on the annulus $A_\varepsilon$, with Dirichlet conditions on the outer boundary and Neumann conditions on the inner boundary, then it is well known that $\nu(A_\varepsilon)$ converges to $\lambda^*$ as $\varepsilon \to 0$ (see[1]). Thus, using the min-max, we will have for sufficiently small $\varepsilon$,

$$\int_{A_\varepsilon} |\nabla f|^2 \geq \nu(A_\varepsilon) \int_{A_\varepsilon} f^2 \geq \frac{1}{2} \lambda^* \int_{A_\varepsilon} f^2. \quad (16)$$

On the other hand, since $f - 1$ vanishes along $\partial B(\varepsilon)$, its Rayleigh quotient is bounded below by $\frac{1}{\varepsilon^3} \lambda^*$, the first Dirichlet eigenvalue of $B(\varepsilon)$. Thus

$$\int_{B(\varepsilon)} |\nabla f|^2 \geq \frac{1}{\varepsilon^3} \lambda^* \int_{B(\varepsilon)} (f - 1)^2 \geq \frac{1}{\varepsilon^2} \lambda^* \left( \frac{1}{\varepsilon^2} \int_{B(\varepsilon)} f^2 - 2 \int_{B(\varepsilon)} f \right) \quad (17)$$

with

$$\int_{B(\varepsilon)} f \leq \left( \omega_n \varepsilon^n \int_{B(\varepsilon)} f^2 \right)^{\frac{1}{2}}.$$
Thus, if $\omega n \varepsilon ^n \leq \frac{1}{16} \int _{B(\varepsilon)} f ^2$, then (17) yields
\[
\int _{B(\varepsilon)} |\nabla f |^2 \geq \frac{1}{2} \varepsilon ^2 \lambda ^* \int _{B(\varepsilon)} f ^2 > \frac{1}{2} \lambda ^* \varepsilon ^{-1-a} \int _{B(\varepsilon)} f ^2
\]
which, combined with (16) and (15), implies (14).

Assume now that $\omega n \varepsilon ^n \leq \frac{1}{16} \int _{B(\varepsilon)} f ^2$ and let us prove the following:
\[
\int _{A_\varepsilon } |\nabla f |^2 \geq \frac{n(n-2)}{16 \varepsilon ^{a-1}} \varepsilon ^{-a} \int _{B(\varepsilon)} f ^2 \quad \text{if } n \geq 3
\]
\[
\int _{A_\varepsilon } |\nabla f |^2 \geq \frac{1}{2} \lambda ^* \varepsilon ^{-1-a} \int _{B(\varepsilon)} f ^2 \quad \text{if } n = 2
\]
(18)
which would imply for sufficiently small $\varepsilon$,
\[
\int _{A_\varepsilon } |\nabla f |^2 \geq \frac{1}{2} \lambda ^* \varepsilon ^{-1-a} \int _{B(\varepsilon)} f ^2
\]
(19)

 enabling us to deduce (14) from (15) and (16). Indeed, since $f(\varepsilon) = 1$ and $f(1) = 0$, one has $\int _{\varepsilon} f' = -1$. Therefore, applying the Cauchy-Schwarz inequality to the product $f' = \left(f' r^{(n-1)/2} r^{-n-1/2}\right), \text{ we get}
\[
\frac{1}{n \omega n} \int _{A_\varepsilon } |\nabla f |^2 = \int _{\varepsilon} f'^2 r^{n-1} \geq \left(\int _{\varepsilon} f' \right)^2 \left(\int _{\varepsilon} 1 / r^{n-1} \right)^{-1} \geq \frac{1}{\int _{\varepsilon} 1 / r^{n-1}}
\]
with
\[
\int _{\varepsilon} 1 / r^{n-1} = \left\{ \begin{array}{ll}
\frac{1}{n-2} \left(\frac{1}{\varepsilon^{n-2}} - 1\right) & \text{if } n \geq 3\\
\frac{1}{n-2} \varepsilon ^{-n-2} & \text{if } n = 2
\end{array} \right.
\]
(20)

Therefore,
\[
\int _{A_\varepsilon } |\nabla f |^2 \geq \left\{ \begin{array}{ll}
\frac{n(n-2)}{2 \varepsilon ^{n-2}} \omega n \varepsilon ^{n-2} & \text{if } n \geq 3\\
\frac{2n}{\ln(1/\varepsilon)} & \text{if } n = 2
\end{array} \right.
\]
(21)
which gives (18) since $\omega n \varepsilon ^n \geq \frac{1}{16} \int _{B(\varepsilon)} f ^2$.

Let us check now that the first positive Neumann eigenvalue is also uniformly bounded from below. Indeed, let $f$ be a Neumann eigenfunction with $\Delta f = -\mu _1(\rho _\varepsilon, 1) \rho _\varepsilon f$. If $f$ is radial, then $\mu _1(\rho _\varepsilon, 1) \geq \lambda _1(\rho _\varepsilon) \geq \frac{1}{4} \lambda ^*$ (there exists $r_0 < 1$ with $f(r_0) = 0$ so that $f$ is a Dirichlet eigenfunction on the ball $B(r_0)$). If $f$ is not radial, then, up to averaging (or assuming that $f$ is orthogonal to radial functions), one can assume w.l.o.g. that $\int _{S^{n-1}(r)} f d\theta = 0$ for every $r < 1$. Thus, $\int _{S^{n-1}(r)} |\nabla ^0 f|^2 d\theta \geq \frac{n(n-1)}{2} \int _{S^{n-1}(r)} f ^2 d\theta$, where $\nabla ^0 f$ is the tangential part of $\nabla f$. Hence,
\[
\int _{B^n} |\nabla f |^2 = \int _{0} ^{1} r^{n-1} dr \int _{S^{n-1}(r)} |\nabla f |^2 d\theta \geq (n-1) \int _{0} ^{1} r^{n-1} dr \int _{S^{n-1}(r)} \left(\frac{f}{r}\right)^2 d\theta
\]
\[
= (n-1) \int _{B^n} \left(\frac{f}{r}\right)^2 \geq (n-1) \int _{B^n} f ^2 \rho _\varepsilon
\]
since $\rho _\varepsilon(r) \leq \frac{1}{r^2}$ everywhere. Thus, in this case, $\mu _1(\rho _\varepsilon, 1) \geq n - 1$. Finally
\[
\mu _1(\rho _\varepsilon, 1) \geq \min(n-1, \frac{1}{4} \lambda ^*).
\]
As before, this construction can be implemented in any Riemannian manifold $(M, g)$, using a quasi-isometry argument, Proposition 2.2 and Corollary 2.1. □
A relevant problem is to know if a Buser’s type inequality can be obtained in this context under assumptions on the volume of balls with respect to $\sigma$ and $\rho$.

4. Bounding the eigenvalues from above

4.1. Unboundedness of eigenvalues if only one parameter among $g, \rho, \sigma$ is fixed.

Let $(M, g_0)$ be a compact Riemannian manifold, possibly with boundary. Our first observation in this section is that the eigenvalues $\mu_k^g(\rho, \sigma)$ are not bounded from above when one quantity among $g \in [g_0], \rho \in \mathcal{R}_0, \sigma \in \mathcal{R}_0$ is fixed and the two others are varying (here $\mathcal{R}_0 = \{ \phi \in C^\infty(M) : \phi > 0 \text{ and } \int_M \phi v_{g_0} = 1 \}$).

Let us first recall that the authors and Savo have proved in [12] that on any compact Riemannian manifold $(M, g_0)$ there exists a sequence of densities $\rho_j \in \mathcal{R}_0$ such that $\mu_1^{g_0}(\rho_j, \rho_j)$ tends to $+\infty$ with $j$. In particular,

$$\sup_{f_M \rho v_{g_0} = 1, f_M \sigma v_{g_0} = 1} \mu_1^{g_0}(\rho, \sigma) \geq \sup_{f_M \rho v_{g_0} = 1} \mu_1^{g_0}(\rho, \rho) = +\infty \quad (22)$$

A natural subsequent question is: Can one construct examples of $g \in [g_0]$ and $\rho \in \mathcal{R}_0$ (resp. $\sigma \in \mathcal{R}_0$) so that $\mu_1^g(\rho, 1)$ (resp. $\mu_1^g(1, \sigma)$) is as large as desired?

**Proposition 4.1.** Let $(M, g_0)$ be a compact Riemannian manifold, possibly with boundary. Then

$$\sup_{g \in [g_0], \rho \in \mathcal{R}_0} \mu_1^g(\rho, 1) = +\infty \quad (23)$$

and

$$\sup_{g \in [g_0], \sigma \in \mathcal{R}_0} \mu_1^g(1, \sigma) = +\infty. \quad (24)$$

**Proof.** To prove (23), the idea is to deform both the metric and the density so that $\rho_\varepsilon v_{g_\varepsilon}$ becomes everywhere small. Indeed, let $V$ be an open set of $M$ with $|V|_{g_0} \geq \frac{1}{10} |M|_{g_0}$. For every $\varepsilon \in (0, 1)$, we consider a continuous density $\rho_\varepsilon$ such that $\rho_\varepsilon = \varepsilon$ on $V, \varepsilon \leq \rho_\varepsilon \leq 2$ everywhere on $M$, and $\int_M \rho_\varepsilon v_{g_\varepsilon} = 1$. Define $g_\varepsilon = \phi_\varepsilon^2 g_0$ with

$$\phi_\varepsilon^n = \frac{|M|_{g_0}}{\int_M \rho_\varepsilon^{-1} v_{g_\varepsilon}} \frac{1}{\rho_\varepsilon}$$

so that $|M|_{g_\varepsilon} = \int_M \phi_\varepsilon^n v_{g_\varepsilon} = |M|_{g_0}$ (here $n$ denotes the dimension of $M$). Now, we observe that

$$\frac{1}{\varepsilon} |M|_{g_\varepsilon} \geq \int_M \rho_\varepsilon^{-1} v_{g_\varepsilon} \geq \int_V \rho_\varepsilon^{-1} v_{g_\varepsilon} = \frac{1}{\varepsilon} |V|_{g_\varepsilon} \geq \frac{1}{10 \varepsilon} |M|_{g_\varepsilon}. $$

Thus,

$$\phi_\varepsilon^n \leq \frac{10 \varepsilon}{\rho_\varepsilon}$$

and, since $\rho_\varepsilon \leq 2$,

$$\phi_\varepsilon^n \geq \frac{\varepsilon}{\rho_\varepsilon} \geq \frac{\varepsilon}{2}.$$

Now, for any smooth function $u$ on $M$ one has (with $\frac{1}{2} \leq \phi_\varepsilon^n \leq \frac{10 \varepsilon}{\rho_\varepsilon}$)

$$\frac{\int_M |\nabla u|^2 v_{g_\varepsilon}}{\int_M u^2 \rho_\varepsilon v_{g_\varepsilon}} = \frac{\int_M |\nabla u|^2 \phi_\varepsilon^{n-2} v_{g_\varepsilon}}{\int_M u^2 \rho_\varepsilon \phi_\varepsilon^n v_{g_\varepsilon}} \geq \frac{1}{\frac{\varepsilon^2}{2^n} - 10 \varepsilon^2 \frac{1}{2^n}} \frac{\int_M |\nabla u|^2 v_{g_\varepsilon}}{\int_M u^2 v_{g_\varepsilon}}.$$
Therefore
\[ \mu_1^\sigma(\rho_\varepsilon, 1) \geq \frac{1}{2^{n-1}10^n \pi^\varepsilon} \mu_1^\sigma(1, 1) \]
which tends to infinity as \( \varepsilon \) goes to zero.

To prove (24) we first observe that, for any positive density \( \sigma \), one has, \( \forall u \in C^2(M) \),
\[ R(\sigma g_0, 1, \sigma)(u) = R(\sigma g_0, \sigma \frac{2}{n}, \sigma \frac{2}{n})(u) \]
Thus,
\[ \mu_k^{\sigma g_0}(1, \sigma) = \mu_k^{\sigma}(\sigma \frac{2}{n}, \sigma \frac{2}{n}) \]
According to [12], there exists on \( M \) a sequence \( \sigma_j \) of positive densities such that \( \int_M \sigma_j^\frac{2}{n} v_{g_0} = |M| \) and \( \mu_k^{\sigma_j}(\sigma_j \frac{2}{n}, \sigma_j \frac{2}{n}) \) tends to infinity with \( j \). We set \( g_j = \sigma_j g_0 \in [g_0] \). Hölder inequality implies that
\[ \int_M \sigma_j v_{g_0} \leq \left( \int_M \sigma_j^\frac{2}{n} v_{g_0} \right)^\frac{n}{2} |M|^{1-\frac{2}{n}} = |M| \]
Setting \( \sigma_j' = \frac{\sigma_j}{\int_M \sigma_j v_{g_0}} \in \mathcal{R}_0 \) we get
\[ \mu_k^{\sigma_j'}(1, \sigma_j') = \frac{1}{\int_M \sigma_j v_{g_0}} \mu_k^{\sigma_j}(1, \sigma_j) \geq \mu_k^{\sigma_j}(1, \sigma_j) = \mu_k^{\sigma_j'}(\sigma_j \frac{2}{n}, \sigma_j \frac{2}{n}) \]
which proves that \( \mu_k^{\sigma_j}(1, \sigma_j') \) tends to infinity with \( j \).

\[ \square \]

4.2. Upper bounds for \( \mu_k(\rho, 1) \) and \( \mu_k(1, \sigma) \). Let \( (M, g) \) be a compact Riemannian manifold of dimension \( n \geq 2 \), possibly with boundary. According to the result by Hassanezhad [23] one has, when \( M \) is a closed manifold,
\[ \lambda_k(M, g) \leq \frac{1}{|M|^\frac{2}{n}} \left( A_n k^\frac{2}{n} + B_n V([g])^\frac{2}{n} \right) \tag{25} \]
where \( A_n \) and \( B_n \) are two constants which only depend on \( n \), and \( V([g]) \) is a conformally invariant geometric quantity defined as follows:
\[ V([g]) = \inf\{|M|_{g_0} : g_0 \text{ is conformal to } g \text{ and } Ric_{g_0} \geq -(n-1)g_0\} \]
where \( Ric_{g_0} \) is the Ricci curvature of \( g_0 \). Now for every positive \( \rho \) such that \( \int_M \rho v_g = 1 \), we have \( V([\rho^\frac{n}{2} g]) = V([g]) \), \( |M|_{\rho^\frac{n}{2} g} = |M|_g \) and \( \lambda_k(M, \rho^\frac{n}{2} g) = \mu_k^{\rho^\frac{n}{2}}(\rho, \rho^\frac{n}{2} g) \). Hence, the inequality (25) implies that for every positive \( \rho \) such that \( \int_M \rho v_g = 1 \),
\[ \mu_k^{\rho^\frac{n}{2}}(\rho, \rho^\frac{n}{2} g) \leq \frac{1}{|M|^\frac{2}{n}} \left( A_n k^\frac{2}{n} + B_n V([g])^\frac{2}{n} \right) \tag{26} \]
This estimate is in contrast to what happens for the Witten Laplacian where we have \( \sup_{\int_M \rho v_g = 1} \mu_1^\rho(\rho, \rho) = +\infty \) (see [12]).

Our aim in this section is to discuss the boundedness of \( \mu_k^{\rho}(\rho, \sigma) \) in the two remaining important cases: \( \mu_k^{\rho}(\rho, 1) \) and \( \mu_k^{\rho}(1, \sigma) \). In [12, Theorem 2.1] it has been shown that the use of the GNY (Grigor’yan-Netrusov-Yau) method [22] leads to the following estimate
\[ \mu_k^{\rho}(\rho, 1) \int_M \rho v_g \leq C([g]) \left( \frac{k}{|M|^\frac{2}{n}} \right)^\frac{2}{n} \tag{27} \]
where $C([g])$ is a constant which only depends on the conformal class of the metric $g$.

This approach fails in the dual situation where $\sigma$ is varying while $\rho$ is fixed. Indeed, the GNY method leads to an upper bound of $\mu_k^g(1, \sigma)$ in terms of the $L^{(n-2)/n}$-norm of $\sigma$ (instead of the $L^1$-norm). However, using the techniques developed by Colbois and Maerten in [13], it is possible to obtain an inequality of the form

$$\mu_k^g(1, \sigma) \leq C(M, g) \left( \frac{k}{|M|_g} \right)^{\frac{2}{n}} \int_M \sigma v_g$$

(28)

where $C(M, g)$ is a geometric constant which does not depend on $\sigma$ (unlike (27), this method of proof does not allow to obtain a conformally invariant constant instead of $C(M, g))$.

In what follows, we will establish inequalities of the type (26) for $\mu_k^g(\rho, 1)$ and $\mu_k(1, \sigma)$.

**Theorem 4.1.** Let $M$ be a bounded open domain possibly with boundary of class $C^1$ of a complete Riemannian manifold $(\tilde{M}, \tilde{g}_0)$ of dimension $n \geq 2$ (with $\tilde{M} = M$ if $\partial M = \emptyset$). Assume that $\text{Ric}_{\tilde{g}_0} \geq -(n-1)\tilde{g}_0$ and let $g_0 = \tilde{g}_0|_M$. For every metric $g$ conformal to $g_0$ and every density $\rho$ with $\int_M \rho v_g = 1$, one has

$$\mu_k^g(\rho, 1) \leq \frac{1}{|M|_g^{\frac{n}{2}}} \left( A_n k^{\frac{2}{n}} + B_n |M|_g^{\frac{2}{n}} \right)$$

(29)

where $A_n$ and $B_n$ are two constants which depend only on the dimension $n$.

In the particular case where $(M, g)$ is a compact manifold without boundary, we can apply Theorem 4.1 with $M = \tilde{M}$ and get immediately the following estimate which extends (25):

$$\mu_k^g(\rho, 1) \leq \frac{1}{|M|_g^{\frac{n}{2}}} \left( A_n k^{\frac{2}{n}} + B_n V([g])^{\frac{2}{n}} \right).$$

(30)

On the other hand, if $\tilde{g}$ is a metric on $\tilde{M}$ and if $\text{ric}_0$ is a positive number such that $\text{Ric}_{\tilde{g}} \geq -(n-1)\text{ric}_0 \tilde{g}$, then the metric $\tilde{g}_0 = \text{ric}_0 \tilde{g}$ satisfies $\text{Ric}_{\tilde{g}_0} \geq -(n-1)\tilde{g}_0$ and $|M|_{g_0} = \text{ric}_0^{n/2}|M|_g$, where $g = \tilde{g}|_M$ and $g_0 = \tilde{g}_0|_M$. Thus, we get

**Corollary 4.1.** Let $M$ be a bounded open domain possibly with boundary of class $C^1$ of a complete Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $n \geq 2$ (with $\tilde{M} = M$ if $\partial M = \emptyset$) and let $g = \tilde{g}|_M$. For every density $\rho$ with $\int_M \rho v_g = 1$, one has

$$\mu_k^g(\rho, 1) \leq A_n \left( \frac{k}{|M|_g} \right)^{\frac{2}{n}} + B_n \text{ric}_0$$

(31)

where $\text{ric}_0 > 0$ is such that $\text{Ric}_g \geq -(n-1)\text{ric}_0 \tilde{g}$. In particular, $\forall k \geq |M|_g^{\frac{2}{n}} \text{ric}_0^{\frac{n}{2}}$,

$$\mu_k^g(\rho, 1) \leq C_n \left( \frac{k}{|M|_g} \right)^{\frac{2}{n}}$$

(32)

with $C_n = A_n + B_n$. 
Inequalities (30) and (31) are conceptually much stronger than (27), especially since they lead to a Kröger type inequality (32) for every $k$ exceeding an explicit geometric threshold, independent of $\rho$ (it is well known that if the Ricci curvature is not nonnegative, then an inequality like (32) cannot hold for every $k$, see [13, Remark 1.2(iii)]).

**Theorem 4.2.** Let $M$ be a bounded open domain possibly with boundary of class $C^1$ of a complete Riemannian manifold $(\hat{M}, \hat{g})$ of dimension $n \geq 2$ (with $\hat{M} = M$ if $\partial M = \emptyset$) and let $g = \hat{g}|_M$. For every positive density $\sigma$ on $M$ with $\int_M \sigma v_\sigma = 1$ one has

$$\mu_k^g(1, \sigma) \leq A_n \left( \frac{k}{|M|_g} \right)^{\frac{2}{n}} + B_n \text{Ric}_0$$

where $\text{Ric}_0 > 0$ is such that $\text{Ric}_{\hat{g}} \geq -(n-1)\text{Ric}_0 \hat{g}$ and where $A_n$ and $B_n$ are two constants which depend only on $n$. In particular, $\forall k \geq |M|_g \text{Ric}_{\hat{g}}^\frac{2}{n}$,

$$\mu_k^g(1, \sigma) \leq C_n \left( \frac{k}{|M|_g} \right)^{\frac{2}{n}}$$

with $C_n = A_n + B_n$.

**Proof of Theorem 4.1.** We consider the metric measured space $(M, d_0, \nu)$ where $d_0$ is the restriction to $M$ of the Riemannian distance on $(\hat{M}, \hat{g}_0)$, and $\nu = \rho v_\sigma$. Since $\text{Ric}_{\hat{g}_0} \geq -(n-1)g_0$, the space $(M, d_0, \nu)$ satisfies a $(2, N; 1)$–covering property for some fixed $N$ (see [23]). Therefore, we can apply Theorem 2.1 of [23] and find a family of $3(k+1)$ pairs of sets $(F_j, G_j)$ of $M$ with $F_j \subset G_j$, such that the $G_j$’s are mutually disjoint and $\nu(F_j) \geq \frac{\nu(M)}{2(c(k+1))}$, with $c = c(n)$ is a constant which depends only on $n$. Moreover, each pair $(F_j, G_j)$ satisfies one of the following properties:

- $F_j$ is an annulus $A$ of the form $A = \{ r < d_0(x, a) < R \}$, and $G_j = 2A = \{ \frac{r}{2} < d_0(x, a) < 2R \}$, with outer radius $2R$ less than 1,

- $F_j$ is an open set $V \subset M$ and $G_j = V^\nu = \{ x \in M; \ d_0(x, V) < r_0 \}$, with $r_0 = \frac{1}{1600}$.

Let us start with the case where $F_j$ is an annulus $A = A(a, r, R) = \{ r < d_0(x, a) < R \}$ and $G_j = 2A$. To such an annulus we associate the function $u_A$ supported in $2A = \{ \frac{r}{2} < d_0(x, a) < 2R \}$ and such that

$$u_A(x) = \begin{cases} \frac{2}{r}d_0(x, a) - 1 & \text{if } \frac{r}{2} \leq d_0(x, a) \leq r \\ 1 & \text{if } x \in A \\ 2 - \frac{4r}{R}d_0(x, a) & \text{if } R \leq d_0(x, a) \leq 2R \end{cases}$$

Since $u_A$ is supported in $2A$ we get, using Hölder’s inequality and the conformal invariance of $|\nabla^g u_A|^n v_\sigma$,

$$\int_M |\nabla^g u_A|^2 v_\sigma = \int_{2A} |\nabla^g u_A|^2 v_\sigma \leq \left( \int_{2A} |\nabla^g u_A|^n v_\sigma \right)^\frac{2}{n} \left( \int_{2A} v_\sigma \right)^{1-\frac{2}{n}}$$

$$= \left( \int_{2A} |\nabla^{\hat{g}_0} u_A|^n v_{\hat{g}_0} \right)^\frac{2}{n} |2A|_g^{1-\frac{2}{n}}.$$
where
\[ |\nabla^{g_0} u|_{A} \equiv \begin{cases} \frac{2}{r} & \text{if } \frac{r}{2} \leq d_0(x, a) \leq r \\ 0 & \text{if } r \leq d_0(x, a) \leq R \\ \frac{1}{R} & \text{if } R \leq d_0(x, a) \leq 2R \end{cases} \]
we get
\[ \int_{2A} |\nabla^{g_0} u_{A}|^{\alpha} v_{g_0} \leq \left( \frac{2}{r} \right)^{n} |B(a, r)|_{g_0} + \left( \frac{1}{R} \right)^{n} |B(a, 2R)|_{g_0} \leq 2^{n+1} \Gamma(g_0) \]
where
\[ \Gamma(g_0) = \sup_{x \in M, t \in (0, 1)} \frac{|B(x, t)|_{g_0}}{t^{n}} \]
(here \(B(x, t)\) stands for the ball of radius \(t\) centered at \(x\) in \((M, d_0)\)). Notice that since \(Ric_{g_0} \geq -(n-1)\hat{g}_0\), the constant \(\Gamma(g_0)\) is bounded above by a constant that depends only on \(n\) (Bishop-Gromov inequality). Hence,
\[ \int_{M} |\nabla^{g} u_{A}|^{2} v_{g} \leq C(n)|2A|_{g}^{1-\frac{2}{n}} \]
where \(C(n) \geq 2^{n+1} \Gamma(g_0)\). On the other hand, we have
\[ \int_{M} u_{A}^{2} \rho v_{g} \geq \int_{A} \rho v_{g} = \nu(A) \geq \frac{\nu(M)}{c^{2}(k+1)}. \]
Thus
\[ R_{(g, \rho, 1)}(u_{A}) = \frac{\int_{M} |\nabla^{g} u_{A}|^{2} v_{g}}{\int_{M} u_{A}^{2} \rho v_{g}} \leq A_{n} \frac{|2A|_{g}^{1-\frac{2}{n}}}{\nu(M)}(k+1) \]
for some constant \(A_{n}\).

Now, in the second situation, where \(F_{j}\) is an open set \(V\) and \(G_{j} = V^{r_{0}}\), we introduce the function \(u_{V}\) defined to be equal to 1 inside \(V\), 0 outside \(V^{r_{0}}\) and proportional to the \(d_{0}\)-distance to the outer boundary in \(V^{r_{0}} \setminus V\). We have, since \(u_{V} = 1\) in \(V\) and \(|\nabla^{g_0} u_{V}|\) is equal to \(\frac{1}{r_{0}}\) almost everywhere in \(V^{r_{0}} \setminus V\) and vanishes in \(V\) and in \(M \setminus V^{r_{0}}\),
\[ \int_{M} u_{V}^{2} \rho v_{g} \geq \int_{V} \rho v_{g} = \nu(V) \geq \frac{\nu(M)}{c^{2}(k+1)} \]
and
\[ \int_{M} |\nabla^{g} u_{V}|^{2} v_{g} \leq \left( \int_{V^{r_{0}}} |\nabla^{g} u_{V}|^{n} v_{g_0} \right)^{\frac{2}{n}} |V^{r_{0}}|_{g}^{1-\frac{2}{n}} = \left( \int_{V^{r_{0}}} |\nabla^{g_0} u_{V}|^{n} v_{g_0} \right)^{\frac{2}{n}} |V^{r_{0}}|_{g}^{1-\frac{2}{n}} \]
\[ \leq \frac{|V^{r_{0}}|_{g}^{\frac{2}{n}} |V^{r_{0}}|_{g}^{1-\frac{2}{n}}}{r_{0}^{2}} \]
Thus
\[ R_{(g, \rho, 1)}(u_{V}) \leq B_{n} \frac{|V^{r_{0}}|_{g}^{\frac{2}{n}} |V^{r_{0}}|_{g}^{1-\frac{2}{n}}}{\nu(M)}(k+1) \]
where \(B_{n} = \frac{c^{2}}{r_{0}^{2}}\) is a constant which depends only on \(n\).
In conclusion, to each pair \((F_j, G_j)\) we associate a test function \(u_j\) supported in \(G_j\) and satisfying either \(R_{(g, \rho, 1)}(u_j) \leq A_n \frac{|G_j|_g^{1-\frac{2}{n}}}{\nu(M)} (k + 1)\) or \(R_{(g, \rho, 1)}(u_j) \leq B_n \frac{|G_j|_g^{\frac{2}{n}}|G_j|_g^{1-\frac{2}{n}}}{\nu(M)} (k + 1)\), that is

\[
R_{(g, \rho, 1)}(u_j) \leq A_n \frac{|G_j|_g^{1-\frac{2}{n}}}{\nu(M)} (k + 1) + B_n \frac{|G_j|_g^{\frac{2}{n}}|G_j|_g^{1-\frac{2}{n}}}{\nu(M)} (k + 1).
\]

Now, observe that since \(\sum_{j \leq 3(k+1)} |G_j|_{g_0} \leq |M|_{g_0}\) and \(\sum_{j \leq 3(k+1)} |G_j|_g \leq |M|_g\), there exist at least \(k + 1\) sets among \(G_1, \ldots, G_{3(k+1)}\) satisfying both \(|G_j|_{g_0} \leq \frac{|M|_{g_0}}{k+1}\) and \(|G_j|_g \leq \frac{|M|_g}{k+1}\). This leads to a subspace of \(k+1\) disjointly supported functions \(u_j\) whose Rayleigh quotients are such that

\[
R_{(g, \rho, 1)}(u_j) \leq A_n \frac{|M|_g^{1-\frac{2}{n}}}{\nu(M)} (k + 1) + B_n \frac{|M|_g^{\frac{2}{n}}|G_j|_g^{1-\frac{2}{n}}}{\nu(M)} (k + 1)
\]

with \(\nu(M) = \int_M \rho v_g = |M|_g\). The desired inequality then immediately follows thanks to (1).

**Proof of Theorem 4.2.** First, observe that it suffices to prove the theorem when \(\text{ric}_0 = 1\) (i.e. \(\text{Ric}_{\tilde{g}} \geq -(n-1)\tilde{g}\)). Indeed, the Riemannian metric \(\tilde{g}_0 = \text{ric}_0 \tilde{g}\) satisfies \(\text{Ric}_{\tilde{g}_0} \geq -(n-1)\tilde{g}_0\) and \(|M|_{g_0} = (\text{ric}_0)^{n/2}|M|_g\), with \(g_0 = g_0 \circ g\). Hence, the inequality

\[
\mu_k^{g_0}(1, \sigma) \leq A_n \left( \frac{k}{|M|_{g_0}} \right)^\frac{2}{n} + B_n
\]

implies

\[
\mu_k^g(1, \sigma) = \text{ric}_0 \mu_k^{g_0}(1, \sigma) \leq \text{ric}_0 \left( A_n \left( \frac{k}{|M|_{g_0}} \right)^\frac{2}{n} + B_n \right) = A_n \left( \frac{k}{|M|_g} \right)^\frac{2}{n} + B_n \text{ric}_0.
\]

Therefore, assume that \(\text{ric}_0 = 1\) and consider the metric measured space \((M, d, v_g)\) where \(d\) is the restriction to \(M\) of the Riemannian distance of \((M, \tilde{g})\). The proof relies on the method developed by Colbois and Maerten [13] as presented in Lemma 2.1 of [11]. Applying Bishop-Gromov Theorem, we deduce that there exist two constants, \(C_n\) and \(N_n\), depending only on \(n\), such that, \(\forall x \in M\) and \(\forall r \leq 1\),

- \(|B(x, r)|_g \leq C_n r^n\)
- \(B(x, Ar)\) can be covered by \(N_n\) balls of radius \(r\)

where \(B(x, r)\) stands for the ball in \(M\) of radius \(r\) with respect to the distance \(d\).

Let \(k_0\) be the smallest integer such that \(2(k_0 + 1) > \frac{|M|_k}{4C_nN_n^2}\). For every \(k \geq k_0\) we define \(r_k\) by

\[
r_k^n = \frac{|M|_g}{8C_nN_n^2 (k + 1)} \leq 1
\]

which means that, \(\forall x \in M\),

\[
|B(x, r_k)|_g \leq C_n r_k^n \leq \frac{|M|_g}{8N_n^2 (k + 1)}.
\]
Thus, we can apply Lemma 2.1 of [11] and deduce the existence of $2(k + 1)$ measurable subsets $A_1, \ldots, A_{2(k+1)}$ of $M$ such that, $\forall i \leq 2(k+1)$, $|A_i| \geq \frac{|M|}{4Nn(k+1)}$ and, for $i \neq j$, $d(A_i, A_j) \geq 3r_k$. To each set $A_j$ we associate the function $f_j$ supported in $A_j^k = \{x \in M : d(x, A_j) < r_k\}$ and defined to be equal to 1 inside $A_j$ and proportional to the distance to the outer boundary in $A_j^k \setminus A_j$. The length of the gradient $|\nabla g f_j|$ is then equal to $\frac{1}{r_k}$ almost everywhere in $A_j^k \setminus A_j$ and vanishes elsewhere, so that we get

$$R_{(g,1,\sigma)}(f_j) = \frac{\int_{A_j^k} |\nabla g f_j|^2 \sigma v_g}{\int_{A_j^k} f_j^2 v_g} \leq \frac{1}{r_k} \frac{\int_{A_j^k} \sigma v_g}{|A_j|} \leq \frac{4N_n}{r_k^2} \frac{\int_{A_j^k} \sigma v_g}{|M|}(k + 1)$$

which gives, after replacing $r_k$ by its explicit value,

$$R_{(g,1,\sigma)}(f_j) \leq A_n \frac{\int_{A_j^k} \sigma v_g}{|M|^{1+\frac{2}{n}}}(k + 1)^{1+\frac{2}{n}}.$$

for some constant $A_n$. Now, since $\sum_{j \leq 2(k+1)} \int_{A_j^k} \sigma v_g \leq \int_M \sigma v_g$, there exist at least $k + 1$ sets among the $A_j$’s such that $\int_{A_j^k} \sigma v_g \leq \frac{|M|}{k+1}$. This leads to a subspace of $k+1$-disjointly supported functions $f_j$ whose Rayleigh quotients are such that

$$R_{(g,1,\sigma)}(f_j) \leq A_n \frac{\int_M \sigma v_g}{|M|^{1+\frac{2}{n}}}(k + 1)^{\frac{2}{n}}.$$

Consequently, we have thanks to (1), for all $k \geq k_0$,

$$\mu_k^g(1, \sigma) \leq A_n \frac{\int_M \sigma v_g}{|M|^{1+\frac{2}{n}}}(k + 1)^{\frac{2}{n}} = A_n \left(\frac{k + 1}{|M|}\right)^{\frac{2}{n}}$$

since we have assumed that $\int_M \sigma v_g = |M|$. On the other hand, for every $k \leq k_0$, one obviously has (since $k_0 + 1 \leq \frac{|M|}{4CnN_n^2}$)

$$\mu_k^g(1, \sigma) \leq \mu_{k_0}^g(1, \sigma) \leq A_n \left(\frac{k_0 + 1}{|M|}\right)^{\frac{2}{n}} \leq A_n \left(\frac{1}{4CnN_n^2}\right)^{\frac{2}{n}}.$$

Denoting by $B_n$ the latter constant we obtain, for every $k \geq 0$,

$$\mu_k^g(1, \sigma) \leq A_n \left(\frac{k}{|M|}\right)^{\frac{2}{n}} + B_n.$$

□

5. Extremal eigenvalues

Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 2$, possibly with boundary. In [9], we introduced the following conformally invariant quantities that we named “conformal eigenvalues”: For every $k \in \mathbb{N}$, $\lambda_k^c(M, [g])$ is defined as the supremum of $\lambda_k(M, g')$ when $g'$ runs over all metrics of unit volume which are conformal to $g$ (or,
equivalently, \( \lambda_k^c(M, [g]) = \sup \lambda_k(M, g') |M|^\frac{2}{k} \) when \( g' \) runs over all metrics conformal to \( g \). Thus, we can write

\[
\lambda_k^c(M, [g]) = \sup_{f_M, \rho \nu_g = 1} \lambda_k(M, \rho \frac{1}{n-2} g) = \sup_{f_M, \rho \nu_g = 1} \mu_k^c(\rho, \rho \frac{1}{n-2}).
\]

We investigated in [9] some of the properties of the conformal eigenvalues such as the existence of a universal lower bound, and proved that

\[
\lambda_k^c(M, [g]) \geq \lambda_k^c(S^n, [g_s]) \geq n\alpha_n k^\frac{2}{n}
\]

where \( \alpha_n = (n + 1)\omega_{n+1} \) is the volume of the standard sphere. Moreover, we proved that the gap between two consecutive conformal eigenvalues satisfies the following estimate:

\[
\lambda_{k+1}^c(M, [g]) \geq \lambda_k^c(M, [g]) \geq n^2 \alpha_n.
\]

Actually, these properties were established in the context of closed manifolds. However, they remain valid in the context of bounded domains, under Neumann boundary conditions, without the need to change anything to the proofs. In this regard, we can point out the following curious phenomenon that all bounded Euclidean domains have the same conformal spectrum.

**Proposition 5.1.** For every bounded domain \( \Omega \subset \mathbb{R}^n \) with \( C^1 \)-boundary one has

\[
\lambda_k^c(\Omega, [g_E]) = \lambda_k^c(B^n, [g_E])
\]

where \( g_E \) is the Euclidean metric.

For \( k = 1 \) we have \( \lambda_1^c(\Omega, [g_E]) = n\alpha_n^2 \) (see Corollary 6.1 below).

**Proof.** Let us first observe that if \( \Omega \) is a proper subset of \( \Omega' \), then \( \lambda_k^c(\Omega, [g_E]) \leq \lambda_k^c(\Omega', [g_E]) \). Indeed, given a metric \( g = f g_E \) conformal to \( g_E \) on \( \Omega \), we extend it to \( \Omega' \) in a metric \( g' \) conformal to \( g_E \). For every \( \varepsilon > 0 \), we multiply \( g' \) by the function \( f_\varepsilon \) which is equal to 1 on \( \Omega \) and equal to \( \varepsilon \) on \( \Omega' \setminus \Omega \) and apply Theorem 2.1. In dimension \( n \geq 3 \), this theorem tells us that \( \lambda_k(\Omega', f_\varepsilon g') \) converges to \( \lambda_k(\Omega, g) \). Since the volume of \( (\Omega', f_\varepsilon g') \) converges to the volume of \( (\Omega, g) \), we deduce that \( \lambda_k(\Omega, g) \Omega^{\frac{2}{n}} g \leq \lambda_k^c(\Omega', [g_E]) \). In dimension 2, we obtain that \( \lambda_k(\Omega', f_\varepsilon g') \) converges to the \( k \)-th eigenvalue of the quadratic form \( \int_\Omega |\nabla u|^2 v_g + \int_{\Omega' \setminus \Omega} |\nabla H(u)|^2 v_g \). This quadratic form is clearly larger than the Dirichlet energy \( \int_\Omega |\nabla u|^2 v_g \) on \( \Omega \) so that its \( k \)-th eigenvalue is bounded below by \( \lambda_k(\Omega, g) \). Again, this implies that \( \lambda_k(\Omega, g) \leq \lambda_k^c(\Omega', [g_E]) \).

Now, since \( \Omega \) is open and bounded, there exist two positive radii \( r_1 \) and \( r_2 \) so that

\[
B^n(r_1) \subset \Omega \subset B^n(r_2)
\]

where \( B^n(r_1) \) and \( B^n(r_2) \) are two concentric Euclidean balls. Using the observation above we get

\[
\lambda_k^c(B^n(r_1), [g_E]) \leq \lambda_k^c(\Omega, [g_E]) \leq \lambda_k^c(B^n(r_2), [g_E]).
\]

Since the balls \( B^n(r_1) \) and \( B^n(r_2) \) are homothetic to the unit ball \( B^n \), one necessarily has \( \lambda_k^c(B^n(r_1), [g_E]) = \lambda_k^c(B^n(r_2), [g_E]) = \lambda_k^c(B^n, [g_E]) \) which enables us to conclude.
As a consequence of the upper bounds given in the previous section, it is natural to introduce the following extremal eigenvalues:

$$\mu_k^*(M, g) = \sup_{f_{M, \rho v_g=1}} \mu_k^0(\rho, 1) = \sup_{\rho} \mu_k^0(\rho, 1) \int_M \rho \, v_g$$

$$\mu_k^{**}(M, g) = \sup_{f_{M, \sigma v_g=1}} \mu_k(1, \sigma) = \sup_{\sigma} \mu_k(1, \sigma) \int_M \sigma \, v_g$$

A natural question is whether properties such as (36) and (37) may occur for $\mu_k^*(M, g)$ and $\mu_k^{**}(M, g)$. Observe that these quantities are not invariant under metric scaling since

$$\mu_k^*(M, r^2 g) = r^{-2} \mu_k^*(M, g) \text{ and } \mu_k^{**}(M, r^2 g) = r^{-2} \mu_k^{**}(M, g).$$

Hence, we will assume that the volume of the manifold is fixed.

In the particular case of manifolds $(M, g)$ of dimension 2 one has for every $\rho$, $\mu_k^0(\rho, 1) = \lambda_k(M, \rho g)$. Thus,

$$\mu_k^*(M, g) = \frac{\lambda_k^0(M, g)}{|M|_g}$$

and we deduce from (36) and (37) that any 2-dimensional Riemannian manifold $(M, g)$ satisfies

$$\mu_k^*(M, g) \geq \frac{8\pi k}{|M|_g}$$

and

$$\mu_k^*(M, g) - \mu_k^*(M, g) \geq \frac{8\pi}{|M|_g}.$$

The following theorem shows that the 2-dimensional case is in fact exceptional. Indeed, it turns out that any compact manifold of dimension $n \geq 3$ can be deformed in such a way that $\mu_k^*(M, g)$ becomes as small as desired.

**Theorem 5.1.** Let $M$ be a compact manifold of dimension $n \geq 3$. There exists on $M$ a one-parameter family of metrics $g_\varepsilon$, $\varepsilon > 0$, of volume 1 such that

$$\mu_k^*(M, g_\varepsilon) \leq C \varepsilon^{\frac{n-2}{n}},$$

where $C$ is a constant which does not depend on $\varepsilon$ or $k$.

Similarly, we have the following result for the supremum with respect to $\sigma$.

**Theorem 5.2.** Let $M$ be a compact manifold of dimension $n \geq 2$. There exists on $M$ a one-parameter family of metrics $g_\varepsilon$, $\varepsilon > 0$, of volume 1 such that

$$\mu_k^{**}(M, g_\varepsilon) \leq C \varepsilon^{2\frac{n-1}{n}}$$

where $C$ is a constant which depends only on $n$.

The proofs of these theorems rely on the construction below. It is worth noticing that the one-parameter family of metrics $g_\varepsilon$ we will exhibit can be chosen within a fixed conformal class. Actually, we start with a Riemannian metric $g_0$ on $M$ that we conformally deform in the neighborhood of a point.

**The construction.** We start with a metric $g_0$ on $M$ and choose a sufficiently small open set $V \subset M$ so that $g_0$ is 2-quasi-isometric to a flat metric in $V$. Since the eigenvalues
corresponding to two quasi-isometric metrics are “comparable”, we can assume w.l.o.g. that the metric $g_0$ is flat inside $V$. Therefore, there exists a positive $\delta$ so that $V$ contains a flat (Euclidean) ball of radius $\delta$. After a possible dilation, we can assume that $\delta = 1$. We deform this unit Euclidean ball into a long capped cylinder (i.e. an Euclidean cylinder of radius $\delta$ closed by a spherical cap). This construction is standard and is explained, for example, in [20, pp. 3856-57]. We can even do it through a conformal deformation of $g_0$, as explained in [10, pp. 718-719]. Therefore, we obtain a family of Riemannian manifolds $(M, g_\varepsilon)$ so that $M$ is the union of three parts

$$M = M_0 \cup C \cup S_0^n$$

with
- $M_0$ is an open subset of $M$ and $g_\varepsilon$ does not vary with $\varepsilon$ on $M_0$,
- $(C, g_\varepsilon)$ is isometric to the cylinder $[0, \frac{1}{\varepsilon}] \times S^{n-1}$ of length $\frac{1}{\varepsilon}$ (with $0 < \varepsilon \leq 1$),
- $S_0^n$ is a round hemisphere of radius 1 which closes the end of the cylinder $C$ and $g_\varepsilon|_{S_0^n}$ is the round metric (and is independent of $\varepsilon$).

The only varying parameter in this construction is the length $\frac{1}{\varepsilon}$ of the cylinder $(C, g_\varepsilon)$. Notice that the volume of $(M, g_\varepsilon)$ is not equal to 1, but we will make a suitable scaling at the end of the proof.

In order to bound the eigenvalues $\mu_k^{2n}(\rho, 1)$ from above, we will use the GNY method [22]. To this end, we need a uniform control (w.r.t. $\varepsilon$) of the packing constant (see [22, Definition 3.3 and Theorem 3.5]) and of the volume growth of balls in $(M, g_\varepsilon)$. This will be done in the following lemmas. For this purpose, we introduce the connected open subset $\tilde{M}_0 \subset M$ obtained as the union of $M_0$ and the part of the cylinder which corresponds to $(0, 3d_0) \times S^{n-1} \subset [0, \frac{1}{\varepsilon}] \times S^{n-1}$, where $d_0$ is the diameter of $M_0$.

**Lemma 5.1** (volume growth of balls). There exist two positive constants $C_1$ and $C_2$, independent of $\varepsilon$, such that, for every ball $B_\varepsilon(x, r)$ in $(M, g_\varepsilon)$ we have

$$|B_\varepsilon(x, r)|_{g_\varepsilon} \leq \begin{cases} 
C_1 r^n & \text{if } r \leq 2d_0 \\
C_2 r & \text{if } r \geq 2d_0 
\end{cases}$$

(39)

**Proof.** If $B_\varepsilon(x, r) \cap M_0 = \emptyset$, then $B_\varepsilon(x, r)$ is isometric to a geodesic ball of radius $r$ of the capped cylinder and an obvious calculation shows that (39) holds true with two constants $C_1$ and $C_2$ independent of $\varepsilon$ (in fact, we can compare the volume of $B_\varepsilon(x, r)$ with the volume of $(-r, r) \times S^{n-1}$ to get $|B_\varepsilon(x, r)|_{g_\varepsilon} \leq Ar$ for some positive $A$). If $B_\varepsilon(x, r) \cap M_0 \neq \emptyset$ and $r < 2d_0$, then $B_\varepsilon(x, r)$ is contained in $\tilde{M}_0$. Hence, there exists a constant $C$, depending only on $\tilde{M}_0$, such that $|B_\varepsilon(x, r)|_{g_\varepsilon} \leq Cr^n$. If $B_\varepsilon(x, r) \cap M_0 \neq \emptyset$ and $r \geq 2d_0$, then $B_\varepsilon(x, r)$ is contained in the union of a ball $B(x_0, 2d_0) \subset \tilde{M}_0$ centered at a point $x_0 \in M_0$ and a ball of radius $r' \leq r$ contained in the cylindrical part. Thus, $|B_\varepsilon(x, r)|_{g_\varepsilon} \leq C2^n d_0^n + Ar \leq C_2 r$ for some positive $C_2$ which does not depend on $\varepsilon$. \(\square\)

**Lemma 5.2.** There exists a constant $N$, independent of $\varepsilon$, such that any ball of radius $r > 0$ in $(M, g_\varepsilon)$ can be covered by $N$ balls of radius $\frac{r}{2}$.

**Proof.** Let $B_\varepsilon(x, r)$ be a ball of radius $r$ in $(M, g_\varepsilon)$. If $B_\varepsilon(x, r) \cap M_0 = \emptyset$, then, since $(M \setminus M_0, g_\varepsilon)$ is isometric to the capped cylinder whose Ricci curvature is everywhere
nonnegative, $B_c(x,r)$ can be covered by $N_E$ balls of radius $\frac{r}{2}$, where $N_E$ is the packing constant of the Euclidean space $\mathbb{R}^n$ (Bishop-Gromov theorem).

Assume that $B_c(x,r) \cap M_0 \neq \emptyset$. If $r < 2d_0$, then $B_c(x,r)$ is contained in $\tilde{M}_0$. Thus, $B_c(x,r)$ can be covered by $N(M_0)$ balls of radius $\frac{r}{2}$, where $N(M_0)$ is the the packing constant of $\tilde{M}_0$. If $r \geq 2d_0$, then $B_c(x,r)$ is contained in the union of a ball $B_c(x_0,2d_0) \subset \tilde{M}_0$ centered at a point $x_0 \in M_0$ and a ball of radius $r' \leq r$ contained in the capped cylinder. Again, $B_c(x,r)$ can be covered by $N_E + N(M_0)$ balls of radius $\frac{r}{2}$.

**Proof of Theorem 5.1.** Let $\rho$ be a positive density on $M$ with $\int_M \rho v_{g_\varepsilon} = 1$. Applying [22, Theorem 3.5] to the metric measured space $(M, d_\varepsilon, \rho v_{g_\varepsilon})$, where $d_\varepsilon$ is the Riemannian distance associated to $g_\varepsilon$, we deduce the existence of $k+1$ annuli $A_1, \ldots, A_{k+1}$ such that $\int_{A_j} \rho v_{g_\varepsilon} \geq \frac{|M|_{g_\varepsilon}}{C_k}$ and $2A_1, \ldots, 2A_{k+1}$ are mutually disjoint. Here, $C$ should depends on the packing constant of $(M, g_\varepsilon)$, but since the latter is dominated independently of $\varepsilon$, thanks to Lemma 5.2, we can assume that $C$ is independent of $\varepsilon$.

To each annulus of the form $A = B_c(x,R) \setminus B_c(x,r)$ we associate a function $u_A$ defined as in (35). We obtain

$$R_{(g_\varepsilon,\rho,1)}(u_A) = \frac{\int_{2A} |\nabla u_A|^2 g_{g_\varepsilon}}{\int_{2A} u_A^2 g_{g_\varepsilon}} \leq \frac{4}{r^2} |B_c(x,r)|_{g_\varepsilon} + \frac{1}{r^2} |B_c(x,2R)|_{g_\varepsilon}.$$

Using Lemma 5.1 we get for every $r > 0$,

$$\frac{1}{r^2} |B_c(x,r)|_{g_\varepsilon} \leq \begin{cases} C_1 r^{n-2} \leq C_1 d_0^{n-2} & \text{if } r \leq 2d_0 \\ \frac{C_2}{r} \leq \frac{C_2}{2d_0} & \text{if } r \geq 2d_0 \end{cases} \tag{40}$$

Therefore, there exists a constant $C'$ which depends on $C_1$, $C_2$ and $d_0$ (but independent of $\varepsilon$), such that

$$R_{(g_\varepsilon,\rho,1)}(u_A) \leq \frac{C'}{\int_A \rho v_{g_\varepsilon}}.$$

Consequently, the $k+1$ annuli $A_1, \ldots, A_{k+1}$ provide $k+1$ disjointly supported functions satisfying $R_{(g_\varepsilon,\rho,1)}(u_{A_j}) \leq \frac{C'}{\int_{A_j} \rho v_{g_\varepsilon}} \leq \frac{C'k}{|M|_{g_\varepsilon}}$. Thus,

$$\mu_k^{g_\varepsilon} (\rho, 1) \leq \frac{C''}{|M|_{g_\varepsilon}} \frac{k}{|M|_{g_\varepsilon}}.$$

In order to obtain a family of metrics of volume 1 we set $g'_\varepsilon = \frac{1}{|M|_{g_\varepsilon}} g_{g_\varepsilon}$. Hence, for any $\rho$ such that $\int_M \rho v_{g'_\varepsilon} = \int_M \rho v_{g_\varepsilon} = 1$, we have

$$\mu_k^{g'_\varepsilon} (\rho, 1) = |M|_{g'_\varepsilon}^{2/n} \mu_k^{g_\varepsilon} (\rho, 1) \leq C'' \frac{k}{|M|_{g_{g_\varepsilon}}^{1-\frac{2}{n}}}. $$

But $|M|_{g_\varepsilon} \geq |C|_{g_\varepsilon} \geq \frac{\mu_{\text{min}}}{\varepsilon}$. Thus

$$\mu_k (M, g'_\varepsilon) \leq C k \varepsilon^{1-\frac{2}{n}}.$$

□
Proof of Theorem 5.2. Let \((M, g_\varepsilon)\) be as in the construction above and let \(\sigma\) be such that \(\int_M \sigma v_{g_\varepsilon} = |M|_{g_\varepsilon}\). The cylindrical part \((C, g_\varepsilon)\) of \((M, g_\varepsilon)\) can be decomposed into \(2(k+1)\) small cylinders \(C_j \approx \left[ \frac{1}{2(k+1)\varepsilon}, \frac{3(k+1)}{2(k+1)\varepsilon} \right] \times \mathbb{S}^{n-1}\), \(j = 0, \ldots, 2k+1\), of length \(\frac{1}{2(k+1)\varepsilon}\). At least \((k+1)\) cylinders among \(C_0, \ldots, C_{2k+1}\) have a measure with respect to \(\sigma\) which is less or equal to \(\frac{|M|_{g_\varepsilon}}{k+1}\). To each such \(C_j\) we associate a function \(f\) with support in \(C_j\) and which is defined in \(C_j\), through the obvious identification between \(C_j\) and \([0, \frac{1}{2(k+1)\varepsilon}] \times \mathbb{S}^{n-1}\), as follows: \(\forall (t, z) \in [0, \frac{1}{2(k+1)\varepsilon}] \times \mathbb{S}^{n-1} \approx C_j\),

\[
f(t, z) = \begin{cases} 6(k+1)\varepsilon t & \text{if } 0 \leq t \leq \frac{1}{6(k+1)\varepsilon} \\
1 & \text{if } \frac{1}{6(k+1)\varepsilon} \leq t \leq \frac{2}{6(k+1)\varepsilon} \\
-6(k+1)\varepsilon t + 3 & \text{if } \frac{2}{6(k+1)\varepsilon} \leq t \leq \frac{1}{2(k+1)\varepsilon}\end{cases} \tag{41}
\]

We have

\[
\int_M f^2 v_{g_\varepsilon} \geq \int_{\left[ \frac{1}{6(k+1)\varepsilon}, \frac{2}{6(k+1)\varepsilon} \right] \times \mathbb{S}^{n-1}} f^2 v_E = \frac{n\omega_n}{6(k+1)\varepsilon}
\]

where \(v_E\) is the standard product measure. On the other hand, the norm of the gradient of \(f\) is supported in \(C_j\) and is dominated by \(6(k+1)\varepsilon\). Thus,

\[
\int_M |\nabla f|_{g_\varepsilon}^2 \sigma v_{g_\varepsilon} \leq (6(k+1)\varepsilon)^2 \int_{C_j} \sigma v_{g_\varepsilon} \leq (6(k+1)\varepsilon)^2 \frac{|M|_{g_\varepsilon}}{k+1} = 36(k+1)\varepsilon^2 |M|_{g_\varepsilon}
\]

and the Rayleigh quotient of \(f\) satisfies

\[
R_{(g_\varepsilon, 1, \sigma)}(f) \leq \frac{216(k+1)^2 \varepsilon^3 |M|_{g_\varepsilon}}{n\omega_n}.
\]

Consequently, the \(k+1\) chosen cylinders provide \(k+1\) disjointly supported functions satisfying the last inequality, which yields

\[
\mu_k^{g_\varepsilon}(1, \sigma) \leq C|M|_{g_\varepsilon}(k+1)^2 \varepsilon^3
\]

with \(C = \frac{216}{n\omega_n}\). Setting \(g'_\varepsilon = \frac{1}{|M|_{g_\varepsilon}}g_\varepsilon\), we get

\[
\mu_k^{g'_\varepsilon}(1, \sigma) = |M|_{g'_\varepsilon}^{\frac{1}{2}}\mu_k^{g_\varepsilon}(1, \sigma) \leq C\varepsilon^3 |M|_{g_\varepsilon}^{1+\frac{1}{2}}(k+1)^2
\]

with \(|M|_{g_\varepsilon} = |\tilde{M}_0|_g + |C|_{g_\varepsilon} + \frac{1}{2}n\omega_n \leq A\varepsilon\) for some constant \(A\). Thus

\[
\mu_k^{g'_\varepsilon}(M, g'_\varepsilon) \leq C'\varepsilon^2(\frac{3}{\pi}(k+1)^2).
\]

\[ \square \]

Remark 5.1. The same type of construction used in the proof of Theorems 5.1 and 5.2 allows us to prove the existence of a family of bounded domains \(\Omega_\varepsilon \subset \mathbb{R}^n\) of volume 1 such that \(\mu_k^\varepsilon(\Omega_\varepsilon, g_\varepsilon)\) (resp. \(\mu_k^{g'_\varepsilon}(\Omega_\varepsilon, g_\varepsilon)\)) goes to zero with \(\varepsilon\). This is to be compared with the result of Proposition 5.1.

We end this section with the following proposition in which we show how to produce examples of manifolds \((M, g_\varepsilon)\) of fixed volume for which the ratio \(\frac{\mu_k^\varepsilon(M, g_\varepsilon)}{\lambda_k(M, g_\varepsilon)}\) (resp. \(\frac{\mu_k^{g'_\varepsilon}(M, g'_\varepsilon)}{\lambda_k(M, g'_\varepsilon)}\)) tends to infinity as \(\varepsilon \to 0\).
Proposition 5.2. Let $M$ be a compact manifold and let $A$ be a positive constant.

(i) There exists a family of metrics $g_\varepsilon$ of volume 1 on $M$ and a constant $A > 0$ such that
$$\forall \varepsilon \in (0, 1), \lambda_1(M, g_\varepsilon) \leq \varepsilon \text{ while } \mu_1^\ast(M, g_\varepsilon) \geq A.$$ 

(ii) There exists a family of metrics $g_\varepsilon$ of volume 1 on $M$ and a constant $A > 0$ such that
$$\forall \varepsilon \in (0, 1), \lambda_1(M, g_\varepsilon) \to 0 \text{ while } \mu_1^\ast(M, g_\varepsilon) \geq A.$$ 

Proof. (i) Let us start with a Riemannian metric $g$ of volume one on $M$ such that an open set $V$ of $M$ is isometric to the Euclidean ball of volume $\frac{1}{2}$. By a standard argument (Cheeger Dumbbell construction), one can deform the metric $g$ outside $V$ in a metric $g_\varepsilon$ of volume 1 such that $\lambda_1(M, g_\varepsilon) \leq \varepsilon$. Applying Corollary 2.2 with $M_0 = V$, we get $\mu_1^\ast(M, g_\varepsilon) \geq |V|_g \lambda_1(V, g_\varepsilon) = \frac{1}{2} \lambda_1(V, g)$. Since $\lambda_1(V, g) = (2\omega_n)^\frac{n}{2} \lambda_1(B^n, g_E)$, where $B^n$ is the unit Euclidean ball, we get the desired inequality with $A = \frac{1}{2}(2\omega_n)^\frac{n}{2} \lambda_1(B^n, g_E)$.

(ii) Let $g$ be a Riemannian metric on $M$ such that an open subset $V$ of $M$ is isometric to the capped cylinder $C = (-2, 2) \times S^{n-1}$ closed by a spherical cap. We will deform the metric $g$ inside $V$ so that $(M, g_\varepsilon)$ looks like a Cheeger dumbbell (thus $\lambda_1(M, g_\varepsilon) \to 0$ as $\varepsilon \to 0$) and associate to $g_\varepsilon$ a family of densities such that $\mu_1^\ast(1, \sigma_\varepsilon) \geq A > 0$. Indeed, the metric on the cylinder $C = (-2, 2) \times S^{n-1}$ is given in coordinates $(t, x) \in (-2, 2) \times S^{n-1}$ by $g_\varepsilon(t, x) = dt^2 + \gamma_\varepsilon^2(t) g_{S^{n-1}}$ with $\gamma_\varepsilon(-t) = \gamma_\varepsilon(t)$ and

$$\gamma_\varepsilon(t) = \begin{cases} 
\varepsilon & \text{if } t \in [0, \frac{1}{2}]
\in (\varepsilon, 1) & \text{if } t \in [\frac{1}{2}, 1]
1 & \text{if } t \in [1, 2)
\end{cases} \quad (42)$$

We do not change the metric $g$ outside $V$. We endow $(M, g_\varepsilon)$ with the density $\sigma_\varepsilon$ given by $\sigma_\varepsilon(t, x) = \frac{1}{\gamma_\varepsilon(t)^{n-1}}$ on the cylinder $C$ and extended by 1 outside $C$.

It is well known that $\lambda_1(M, g_\varepsilon) \to 0$ as $\varepsilon \to 0$. Let us study $\mu_1^\ast(1, \sigma_\varepsilon)$. One has for every $f \in C^\infty(M)$

$$\int_M |\nabla^\varepsilon f|_{g_\varepsilon}^2 \sigma_\varepsilon v_{g_\varepsilon} = \int_{M \cap C} |\nabla f|_{g}^2 v_g + \int_{-2}^{2} dt \int_{S^{n-1}} |\nabla^\varepsilon f|_{g_\varepsilon}^2 \sigma_\varepsilon(t) \gamma_\varepsilon(t)^{-n-1} v_{S^{n-1}}$$

$$= \int_{M \cap C} |\nabla f|_{g}^2 v_g + \int_{-2}^{2} dt \int_{S^{n-1}} |\nabla^\varepsilon f|_{g_\varepsilon}^2 v_{S^{n-1}},$$

where $v_{S^{n-1}}$ denotes the volume form on the sphere $S^{n-1}$. Now, observe that $|\nabla^\varepsilon f|_{g_\varepsilon}^2$ can be estimated as follows:

$$|\nabla^\varepsilon f|_{g_\varepsilon}^2 = \left( \frac{\partial f}{\partial t} \right)^2 + |\nabla_0 f|_{g_\varepsilon}^2 \gamma_\varepsilon(t)^{-2} \geq \left( \frac{\partial f}{\partial t} \right)^2 + |\nabla_0 f|_{g}^2 = |\nabla f|_{g}^2$$

where $\nabla_0 f$ is the tangential part of the gradient of $f$ w.r.t. $S^{n-1}$. Therefore,

$$\int_M |\nabla^\varepsilon f|_{g_\varepsilon}^2 \sigma_\varepsilon v_{g_\varepsilon} \geq \int_{M \cap C} |\nabla f|_{g}^2 v_g + \int_{-2}^{2} dt \int_{S^{n-1}} |\nabla f|_{g}^2 v_{S^{n-1}} = \int_M |\nabla f|_{g}^2 v_g.$$ 

On the other hand (since $\gamma_\varepsilon(t)^2 \leq 1$)

$$\int_M f^2 v_{g_\varepsilon} \leq \int_M f^2 v_g.$$
In conclusion, for every \( f \in C^\infty(M) \), one has
\[
R_{(g_\varepsilon,1,\sigma_\varepsilon)}(f) \geq R_{(g,1,1)}(f).
\]

It follows, thanks to the min-max principle, that
\[
\mu^*_1(1, \sigma_\varepsilon) \geq \lambda_1(M,g).
\]
The last point is to suitably rescale \( g_\varepsilon \) and \( \sigma_\varepsilon \). For this purpose, just observe that
\[
\int_M \sigma_\varepsilon v_{g_\varepsilon} = |M|_g \text{ and } \frac{1}{2}|M|_g \leq |M|_{g_\varepsilon} \leq |M|_g.
\]

6. Examples

In this section we describe situations in which we can compute or give explicit estimates for the first extremal eigenvalues. Let \( (M,g) \) be a compact Riemannian manifold of dimension \( n \geq 2 \), possibly with a nonempty boundary.

**Proposition 6.1.** Assume that there exists a conformal map \( \phi \) from \( (M,g) \) to the standard \( n \)-dimensional sphere \( S^n \). Then,
\[
\lambda_1^c(M,g) = n\alpha_n^{\frac{2}{n}}
\]
and
\[
\mu^*_1(M,g) \leq n \left( \frac{\alpha_n}{|M|_g} \right)^{\frac{2}{n}}
\]
where \( \alpha_n \) is the volume of the unit Euclidean \( n \)-sphere. Moreover, if \( n = 2 \), then the equality holds in (44).

Notice that when \( (M,g) \) is the standard sphere \( S^n \), then the equality holds in (44) (see Corollary 6.3 below).

**Proof of Proposition 6.1.** Let us first prove (44). Let \( \rho \) be a density on \( M \) with \( \int_M \rho v_g = 1 \). Given any nonconstant map \( \phi = (\phi_1, \ldots, \phi_{n+1}) : (M,g) \rightarrow S^n \), a standard argument tells us that there exists a conformal diffeomorphism \( \gamma \in \text{Conf}(S^n) \) such that \( \psi = \gamma \circ \phi \) satisfies \( \int_M \psi_j \rho v_g = 0 \), \( j = 1, \ldots, n+1 \) (see for instance [21, Proposition 4.1.5]). Thus, \( \forall j \leq n+1 \),
\[
\mu_1(\rho,1) \int_M \psi_j^2 \rho v_g \leq \int_M |\nabla \psi_j|^2 v_g
\]
(see (3)) and, summing up w.r.t. \( j \),
\[
\mu_1(\rho,1) \int_M \rho v_g \leq \int_M |d\psi|^2 v_g \leq \left( \int_M |d\psi|^n v_g \right)^{\frac{2}{n}} |M|_g^{1-\frac{2}{n}}.
\]
Since \( \psi = \gamma \circ \phi \) is a conformal map, \( \int_M |d\psi|^n v_g \) is nothing but \( n^{\frac{n}{2}} \) times the volume of \( \psi(M) \subset S^n \) with respect to the standard metric \( g_s \) of \( S^n \) (indeed, \( \psi^* g_s = \frac{1}{n} |d\psi|^2 g \)). Therefore,
\[
\mu_1(\rho,1) \int_M \rho v_g \leq n |\psi(M)|_g^{\frac{n}{2}} |M|_g^{-\frac{n}{2}} \leq n \left( \frac{\alpha_n}{|M|_g} \right)^{\frac{2}{n}}
\]
which proves (44).

Using the same arguments we can prove the inequality \( \lambda_1^c(M,g) \leq n\alpha_n^{\frac{2}{n}} \). The reverse inequality follows from [9, Theorem A].
It is well known that the Euclidean space $\mathbb{R}^n$ and the hyperbolic space $\mathbb{H}^n$ are conformally equivalent to open parts of the sphere $\mathbb{S}^n$. This leads to the following corollary.

**Corollary 6.1.** Let $\Omega$ be a bounded domain of the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$ or the sphere $\mathbb{S}^n$, endowed with the induced metric $g_s$. One has
\[
\lambda_1^s(\Omega, g_s) = n\alpha_n \frac{2}{n}.
\]
and
\[
\mu_1^s(\Omega, g_s) \leq n \left( \frac{\alpha_n}{|\Omega|} \right) \frac{2}{n}.
\]
Moreover, the following equality holds in dimension 2:
\[
\mu_1^s(\Omega, g_s) = \lambda_1^s(\Omega, g_s)|\Omega|^{-1} = \frac{8\pi}{|\Omega|}.
\]

**Remark 6.1.** Let $D$ be the unit disc in $\mathbb{R}^2$ and let $\rho_t = \frac{4t}{(t^2 + 1)} |z|^2$. Then
\[
\mu_1^s(D, g_E) = \lim_{t \to \infty} \mu_1^{g_E} \left( \frac{\rho_t}{\int \rho_t dx}, 1 \right) = 8.
\]
Indeed, the map $\phi_t(z) = \frac{1}{|z|^2 + 1}(2tz, t^2|z|^2 - 1)$ identifies $(D, \frac{4t}{(t^2 + 1)^2} g_E)$ with a spherical cap $C_t$ in $\mathbb{S}^2$ whose radius goes to $\pi$ as $t \to \infty$. Hence, $\mu_1^{g_E}(\rho_t, 1) \int_D \rho_t dx = \mu_1(C_t)|C_t|$ which converges to $8\pi$ as $t \to \infty$.

**Proposition 6.2.** Assume that there exists a map $\phi : (M, g) \to \mathbb{S}^p$ from $(M, g)$ to the standard $p$-dimensional sphere $\mathbb{S}^p$ satisfying both $\int_M \phi v_g = 0$ and $|d\phi|^2 \leq \Lambda$ for some positive constant $\Lambda$. Then
\[
\mu_1^{**}(M, g) \leq \Lambda. \tag{45}
\]

**Proof.** One has, for every $j \leq p + 1$,
\[
\mu_1(1, \sigma) \int_M \phi_j^2 v_g \leq \int_M |\nabla \phi_j|^2 \sigma v_g
\]
and, summing up w.r.t. $j$,
\[
\mu_1(1, \sigma)|M|_g \leq \int_M |d\phi|^2 \sigma v_g \leq \Lambda \int_M \sigma v_g
\]
which implies (45). \qed

If $(M, g)$ be a compact homogeneous Riemannian manifold, and if $\phi_1, \ldots, \phi_p$ is an $L^2$-orthonormal basis of the first eigenspace of the Laplacian, then both $\sum_{i \leq p} \phi_i^2$ and $|d\phi|^2 = \sum_{i \leq p} |d\phi_i|^2$ are constant on $M$. This enables us to apply Proposition 6.2 and get the following

**Corollary 6.2.** Let $(M, g)$ be a compact homogeneous Riemannian manifold. Then
\[
\mu_1^{**}(M, g) = \mu_1(M, g)
\]
In other words, on a compact homogeneous Riemannian manifold, $\mu_1(1, \sigma)$ is maximized when $\sigma$ is constant.
Example 6.1. In [19], it is proved that if \( \Gamma = \mathbb{Z}e_1 + \mathbb{Z}e_2 \subset \mathbb{R}^2 \) is a lattice such that \( |e_1| = |e_2| \), then the corresponding flat metric \( g_r \) on the torus \( \mathbb{T}^2 \) satisfies \( \mu_1^*(\mathbb{T}^2, g_r) = \lambda_1(\mathbb{T}^2, g_r) |\mathbb{T}^2|_{gr}^{-1} \). A higher dimensional version of this result was also established in [18]. Since a flat torus is a 2-dimensional homogeneous Riemannian manifold, we have the following equalities

\[
\lambda_1^*(\mathbb{T}^2, g_r) |\mathbb{T}^2|_{gr}^{-1} = \mu_1^*(\mathbb{T}^2, g_r) = \mu_1^{**}(\mathbb{T}^2, g_r) = \lambda_1(\mathbb{T}^2, g_r).
\]

Nevertheless, whereas we always have \( \mu_1^{**}(\mathbb{T}^2, g_r) = \mu_1(\mathbb{T}^2, g_r) \), it follows from [9, Theorem A] that when the length ratio \( |e_2|/|e_1| \) of the vectors \( e_1 \) and \( e_2 \) is sufficiently far from 1, then \( \mu_1^*(\mathbb{T}^2, g_r) = \lambda_1(\mathbb{T}^2, g_r) |\mathbb{T}^2|_{gr}^{-1} > \lambda_1(\mathbb{T}^2, g_r) \).

Recall that a map \( \phi = (\phi_1, \ldots, \phi_{p+1}) : (M, g) \to \mathbb{S}^p \) is harmonic if and only if its components \( \phi_1, \ldots, \phi_{p+1} \) satisfy

\[
\Delta_g \phi_j = -|d\phi|^2 \phi_j, \quad j = 1, \ldots, p+1.
\]

The stress-energy tensor of a map \( \phi \) is a symmetric covariant 2-tensor defined for every tangent vectorfield \( X \) on \( M \) by: \( S_\phi(X, X) = \frac{1}{2} |d\phi|^2 |X|^2_g - |d\phi(X)|^2 \). In [15, Theorem 3.1] it is proved that if the stress-energy tensor of a harmonic map \( \phi \) is nonnegative, then, for every conformal diffeomorphism \( \gamma \) of the sphere \( \mathbb{S}^p \) one has

\[
\int_M |d(\gamma \circ \phi)|^2 v_g \leq \int_M |d\phi|^2 v_g.
\]

Moreover, the strict inequality holds if \( \gamma \) is not an isometry and if \( S_\phi \) is positive definite at some point. Observe that if \( \phi : (M, g) \to \mathbb{S}^p \) is a conformal map or a horizontally conformal map, then \( S_\phi \) is nonnegative (see [15]).

Proposition 6.3. Assume that there exists a harmonic map \( \phi : (M, g) \to \mathbb{S}^p \) with nonnegative stress-energy tensor. Then,

\[
\mu_1^*(M, g) \leq \int_M |d\phi|^2 v_g. \tag{46}
\]

Proof. Let \( \rho \) be a positive density on \( M \). As before, we know that there exists \( \gamma \in Conf(\mathbb{S}^n) \) such that \( \psi = \gamma \circ \phi \) satisfies \( \int_M \psi_j \rho \, v_g = 0, \ j = 1, \ldots, n+1 \). Thus

\[
\mu_1(\rho, 1) \int_M \psi_j \rho \, v_g \leq \int_M |\nabla \psi_j|^2 v_g
\]

and, summing up w.r.t. \( j \),

\[
\mu_1(\rho, 1) \int_M \rho \, v_g \leq \int_M |d(\gamma \circ \phi)|^2 v_g \leq \int_M |d\phi|^2 v_g
\]

which implies (46). \( \Box \)

A particular case of Proposition 6.3 is when there exists a harmonic map \( \phi : (M, g) \to \mathbb{S}^p \) which is homothetic. In this case, \( S_\phi = \frac{\omega_{n+1}}{\omega_n} |d\phi|^2 g \) and \( |d\phi|^2 \) is constant and coincides with an eigenvalue \( \lambda_k(M, g) \) for some \( k \geq 1 \). For example, if \( (M, g) \) is a compact isotropy irreducible homogeneous space (e.g. a compact rank-one symmetric space) and if \( \phi_1, \ldots, \phi_p \) is an \( L^2 \)-orthonormal basis of the first eigenspace of the Laplacian, then \( \phi = \left(\frac{\text{Vol}}{p}\right)^{\frac{1}{2}} (\phi_1, \ldots, \phi_p) \) is a harmonic map from \( (M, g) \) to \( \mathbb{S}^p \) which is homothetic and
satisfies $|d\phi|^2 = \lambda_1(M, g)$. Proposition 6.3 then implies that $\mu^*_1(M, g) = \lambda_1(M, g)$. On the other hand, the second author and Ilias [17] proved that in this situation we also have $\lambda^c_1(M, g) = \lambda_1(M, g)$. Consequently, we have the following

**Corollary 6.3.** Let $(M, g)$ be a compact isotropy irreducible homogeneous space. Then

$$\lambda^c_1(M, g) |M|^\frac{2}{g} = \mu^*_1(M, g) = \mu^{**}_1(M, g) = \lambda_1(M, g).$$

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