A Two dimensional Model of Superconductivity

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Abstract
We present a two dimensional model of superconductivity where bosonization of fermions is described by topological fermion-boson duality. The model solves the discrepancy between theoretical and empirical values of penetration depth and explains the appearence of quantized vortex lines in superconductors in accord with cyclotron motion of superconducting electrons.
First let us recall that superconductivity is defined by two basic properties of superconductors: The absence of usual resistivity relation in view of a new relation between electromagnetic field and current which is represented by London equations; and the appearance of quantized magnetic flux in superconductor, that expels the applied magnetic field. We show that both properties can be explained by a *two dimensional* microscopic (quantum) model of superconductivity, where the phenomenological London equations [1] are derived as equations of motion; and the flux quantization results from the *canonical quantization* of theory. Thus flux quantization is an invariant *two dimensional* concept, in view of its definition with respect to *surface* and in view of appearance of flux quantization on the *two dimensional* quantum Hall samples relating superconductivity with quantum Hall effect [2], which we presented recently a purely *two dimensional* model for [3]. In this sense, also the superconductivity can be considered as a two dimensional effect described by a two dimensional model. Thus also other theoretical models of superconductivity, i.e. Ginzburg-Landau model, which is conceived four dimensionally, is based on two dimensional fundations (see below). Thus following investigations of theoretical and empirical aspects of superconductivity show that basic phenomenological concepts of superconductivity, are in fact two dimensional concepts which can be given by a two dimensional theory:

With respect to theoretical aspects, note that the solutions of combined Maxwell-London equations:

\[ \Delta B_i = \frac{1}{\lambda^2} B_i, \]

are given by:

\[ B_i = B_i(\text{surface}) e^{-\frac{X}{\lambda}}, \]

where \( \lambda \) is London penetration depth and \( B_i(\text{surface}) \) is the magnetic field strength on the *two dimensional* surface of superconductor. Hence the solution \( B_i \), is dominated by the *surface* solution. This shows the *two dimensional* structure of superconductivity, even if it is applied three and four dimensional London- and Maxwell equations. Further note that the Ginzburg-Landau equations are accompanied by a boundary condition: \((-i\hbar \partial_3 - eA_3)\psi = 0\) which allows electronic currents *on the surface only*. Therefore also in Ginzburg-Landau model, the electronic dynamics which causes the superconductivity, is restricted to the electronic dynamics on the surface of superconductors only.

Note that also in the standard approach to superconductivity, the so called Fermi *surface* where the electrons motion takes place, is in any case a *surface* which can be described by two independent variables only. Thus the motion of electrons on such a momentum surface is a *two dimensional* motion [4]. We show that such a two dimensional momentum space is, just in view of its two dimensionality, equivalent
to a two dimensional configuration space. Further note that the exact solution of corresponding two dimensional Schroedinger equation results in energy levels with energy gap [4]. In other words, also the phenomenological concept of energy gap, is, from point of view of Schroedinger theory, equivalent to a two dimensional concept (see also below). These facts show the two dimensional basis of standard approach to superconductivity, although it is conceived as a three- or four dimensional approach.

We show also that our two dimensional theory is equivalent by Stokes theorem to a one dimensional theory on the contour region of two dimensional superconducting sample, hence the bosonization of fermions can be considered in this model, as a result of Hodge duality between fermions and bosons on the one dimensional contour manifold [5]. We show further that flux quantization with respect to electrons, is equivalent to the quantization of cyclotron motion of electrons; hence in this model the quantized vortex lines are due to quantized cyclotronic currents of electrons (see below).

Furthermore for dimensional structure of superconductivity in view of flux quantization, note that the flux quantization integral: \( e \int \int_{\text{surface}} F_{lm} dX^l \wedge dX^m = Nh \), \( N \in \mathbb{Z} \), is a two dimensional topologically invariant quantity: Since its invariant value (\( Nh \)), is defined with respect to a two dimensional surface with surface element \( (dX^l \wedge dX^m) \). Therefore the invariant definition of flux is entirely a two dimensional concept and the flux quantization is a two dimensional effect. Then the superconductivity which is defined by flux quantization, can be understood also as a two dimensional effect.

With respect to empirical results note that, it is known that there is a remarkable discrepancy between the measured values and the theoretical value of London penetration depth given by: \( \lambda^2 := \frac{M_s}{\mu_0 e_s n_s^2} \) where \( M_s, e_s \) and \( n_s \) are mass, charge and volume density of Cooper pairs. This fact shows that something is wrong with the structure of London model, although it is a good phenomenological model [1]. Thus it is also a strong hint to choose a new structure for London equations, where the penetration depth can be defined in a manner which fits to its empirical values. We show, that the mentioned discrepancy can be corrected, if one uses the conjectured two dimensional model where the penetration depth is defined with respect to a two dimensional surface density.

Note also that, the ”two dimensionality” of superconducting rings, is an empirical hint for the main role played by two dimensional structures in superconductivity. Thus the two dimensional tin films play very important role in application of superconductivity, specially in superconducting electronics [4].
Taking all these theoretical and empirical facts about the main role played by the two dimensional system of electrons in superconductivity into account; one is halted to consider the superconductivity as caused by the two dimensional motion of electrons in a magnetic field, under certain conditions. In other words we argue that, in view of mentioned empirical and theoretical difficulties of three dimensional model of superconductivity, and in view of several accordencies of empirical and phenomenological aspects of superconductivity with two dimensional concepts: The superconductivity can be considered as an effect which results from the quantum behaviour of electrons on the surface of superconductors, interacting with electromagnetic field. Thus, we show that superconducting effects which are described by London equations and flux quantization, follow from the quantum electrodynamical behaviour of electrons with two degrees of freedom. Thus the theory of superconductivity can be formulated, as the two dimensional quantum electrodynamics of electrons with two degrees of freedom only.

The action of superconductivity is given by the two dimensional topologically invariant action of an electron interacting with electromagnetic potential $A_m$ or with magnetic field strength $F_{ml}$, in the single electron picture [6]:

$$S = \frac{1}{2} \left( \int_{\text{surface}} dP_m \wedge dX^m + e \int_{\text{surface}} F_{lm} dX^l \wedge dX^m \right) = \frac{1}{2} \left( \oint_{\text{contour}} P_m dX^m + e \oint_{\text{contour}} A_m dX^m \right), (1)$$

where $P_m$ and $X_m; \ l, m = 1, 2$ are the momentum and the position coordinates of an electron and the surface integral is considered over the surface of superconductor, whereas the equivalent contour integral is considered on the contour region of superconductor surface. Thus the equality represents the Stokes theorem for both kinetic and potential term of electron.

Here note that the actual motion of a physical system takes place always on a polarized phase space which contains the half of phase space variables [7], thus the action function of a system and its wave function are always functions of half of phase space variable, beside the time parameter. The best example of such a polarized phase space with half of phase space variables is the configuration space which is known, in the case that the wave function is a function on this space, as the position representation of wave function. The configuration space of superconducting system is the two dimensional surface where the position of electrons are defined. Therefore the actual action function of the two dimensional superconductivity can
be defined on the polarized phase space of the superconducting system, i.e. on the two dimensional configuration space of the system, or on the two dimensional surface of superconductor. It is in this manner that we can write the action function (1) on the surface of superconductor, since the surface represents the polarized phase space of our superconducting system.

Nevertheless the contour action (1) can be rewritten by the current density $J_m := n_e e P_m / M_e$ as:

$$S = \frac{1}{2} \left( \frac{M_e}{n_e e} \oint_{\text{contour}} J_m dX^m + e \oint_{\text{contour}} A_m dX^m \right),$$

(2)

where $n_e$ is the homogenous or spatially constant volume density of electrons. Note that despite of assertion that Ginzburg-Landau theory considers a variable density $n(r) := |\psi(r)|^2$, actually $n(r) := |\psi(r)|^2$ does not play the role of a true variable in this theory, since the true variables here are $\psi(r), A_i$ only, with respect to which the action of theory is varied in order to obtain the equations of motion of theory: Whereas the $n(r) := |\psi(r)|^2$ is not varied in this theory at all. Thus in contrary, the actual value of density in this theory is given by: $|\psi(r)|^2 = -\frac{\alpha}{\beta}$ in the thermodynamical equilibrium, where $\alpha$ and $\beta$ are dimensional constants, since also they are not varied in the theory. Otherwise the theory would possess further equations of motion with respect to variation of $\alpha, \beta$ or $|\psi(r)|^2$. Whereas the whole dynamics of Ginzburg-Landau model is described by the two equations of motion which result from the variation of action with respect to $\psi(r), A_i$ variables. In other words, although $\psi(r)$ is a variable, nevertheless $|\psi(r)|^2$ is constant in this theory, in accord with the constancy of $\alpha$ and $\beta$ in $|\psi(r)|^2 = -\frac{\alpha}{\beta}$.

Recall that also the coherence length is obtained in this theory in accord with $|\psi(r)|^2 = -\frac{\alpha}{\beta}$. Therefore even in the Ginzburg-Landau model of superconductivity the density of electrons or Cooper pairs is actually constant.

Note further that in view of the equivalence: $\frac{M_e}{n_e e^2} = \frac{M_s}{n_s e_s^2}$ where the index $s$ denotes Cooper pairs, the action (2) is also valid for Cooper pair of electrons. Thus the two dimensional model applies also as a Cooper pair model, if one replaces $J_m, X_m$, and $\frac{M}{n_e e}$ by $J^s_m, X^s_m$ and $\frac{M_s}{n_s e_s}$, respectively.

On the surface of superconductor, the two position variables of electron, i.e. $X^l$, are the actual variables of superconducting system of electrons. Therefore the Euler-Lagrange equations of system (2):

$$\frac{\partial L}{\partial X^l} = \frac{\partial L}{\partial \partial_m X^l} = 0$$

are given for a variation of $S$ with respect to $X^l$ by:
\[ \epsilon_{lm} \partial_l J_m = -\frac{1}{\mu_0 \lambda^2} B \quad \epsilon_{lm} := -\epsilon_{ml} = 1 \quad (3) \]

in accord with \( dX^l := \frac{\partial X^l}{\partial X^m} \, dX^m = \partial_m X^l \, dX^m \) and \( \frac{\partial L}{\partial X^l} = 0 \), where \( B := \epsilon_{lm} \partial_l A_m \) and \( \lambda \) is the London penetration depth: \( \lambda^2 := \frac{M_e}{\mu_0 n_e e^2} \) which is defined here for electrons. Note that \( \frac{\partial X^l}{\partial X^m} \) is in general not constant, i.e. for a cyclotronic motion with \( X^1 := r \cos \alpha \) and \( X^2 := r \sin \alpha \), or also on a curved manifold, it is not constant.

Nevertheless, in accord with \( (dX^l = \dot{X}^l \, dt) \), one may consider the time parameter \( t \), instead of \( X^l \), as the variable of system. Then the equations of motion of system are given by:

\[ \partial_t J_m = \frac{1}{\mu_0 \lambda^2} E_m \quad (4) \]

where \( E_m := -\partial_t A_m \).

These are London equations which are derived here as equations of motion from a two-dimensional action function \( (1) = (2) \). It shows that the theory of superconductivity in accord with London equations, can be described by a two-dimensional model for electrons with only two degrees of freedom.

We show now the flux quantization, as a result of canonical quantization of action \( (1) \) in the sense of Bohr-Sommerfeld quantization [8]:

The action \( (1) \) can be rewritten also by:

\[ S = \int \int_{\text{surface}} d\pi_m \wedge dX^m = \oint_{\text{contour}} \pi_m dX^m , \quad (5) \]

where \( \pi_m := \frac{1}{2} (P_m + eA_m) \).

Then, Bohr-Sommerfeld quantization of this system is given by:

\[ S = \int \int_{\text{surface}} d\pi_m \wedge dX^m = \oint_{\text{contour}} \pi_m dX^m = N \hbar , \quad N \in \mathbb{Z} , \quad (6) \]

or by:

\[ S_1 = \int \int_{\text{surface}} dP_m \wedge dX^m = \oint_{\text{contour}} P_m dX^m = N \hbar , \quad (7) \]
\[ S_2 = \int_{\text{surface}} F_{lm} dX^l \wedge dX^m = \oint_{\text{contour}} A_m dX^m = N \frac{\hbar}{e} \quad (8) \]

The Bohr-Sommerfeld quantization postulate (7) is equivalent to the commutator quantization:

\[ [\hat{P}_m, \hat{X}_m] = -i \hbar [8], \]

describing the quantum behaviour of electron, whereas the Bohr-Sommerfeld quantization (8) describes the flux quantization in quantum units of \( \frac{\hbar}{e} \). Thus flux quantization can be considered as the canonical quantization of present two dimensional electrodynamics. Thus Bohr-Sommerfeld quantization (8) is equivalent to the commutator quantization \( e[\hat{A}_m, \hat{X}_m] = -i \hbar [8], \) describing the quantum behaviour of electromagnetic potential with respect to electrons (see also below). By a comparison of these commutators: \([\hat{P}_m, \hat{X}_m] = e[\hat{A}_m, \hat{X}_m] = -i \hbar,\) one obtains: \( \hat{P}_m = e\hat{A}_m,\) which recalls either the flux quantization condition of vanishing of velocity of electrons:

\[ V_m = (M_e)^{-1} (P_m - eA_m) = 0 \]

on the region of contour integration in the standard approach, or the boundary condition of Ginzburg-Landau theory.

For bosonization note that, the topological invariance of flux quantization is enough hint about the important role played by topology in superconductivity. Thus, as we show, the bosonization of electrons is a topological property of two dimensional model with boundary: Since in this case, as it is obvious from the action (1), the two dimensional model is equivalent to a one dimensional model on the one dimensional contour region. In order to investigate the standard topology of this model, one has to use the differential form representation of fermions and bosons. Then we attach odd differential forms to fermions and even differential forms to bosons, as the usual attachment [9]. Therefore, in our one dimensional model which is represented by the contour action on the one dimensional contour manifold, the fermions are considered as one-forms \( \Omega^1 := \Omega_m dX^m \) which obey the Fermi statistics in accord with exterior algebra of forms:

\[ [\Omega^1_1, \Omega^1_2]+ = 0, \quad \text{in view of exterior algebra: } dX^l \wedge dX^m + dX^m \wedge dX^l = 0. \]

Further we consider bosons in our one dimensional case, as even, i.e. zero-forms \( \Omega^0 \) or scalar functions which obey the Bose statistics in accord with:

\[ [\Omega^0_1, \Omega^0_2]_- = 0, \quad \text{since zero forms commute always. In other words we have the following mathematically well defined differential form representations for fermions } (f_m \in \Omega^1) \text{ and bosons } (b_m \in \Omega^0) \text{ in our one dimensional case, which obey: } [\Omega^1_1, \Omega^1_2]+ = [f_1, f_2]+ = 0 \text{ or } f_1 f_2 = -f_2 f_1 \text{ and } [\Omega^0_1, \Omega^0_2]_- = [b_1, b_2]_- = 0 \text{ or } b_1 b_2 = b_2 b_1, \text{ in accord with the standard statistics.} \]
On the other hand, as on any manifold, also on the one dimensional contour manifold where electrons are concentrated in this model, there exists the so called Hodge duality between differential forms. In other words, on a m-dimensional manifold a r-form $\Omega^r$ is dual to a $(m - r)$-form $\Omega^{m-r}$, i. e.: $\Omega^r = \ast \Omega^{m-r}$ [5]. Hence on a one-dimensional manifold one-forms are dual to zero-forms: $\Omega^0 = \ast \Omega^1$: In other words on the one dimensional contour manifold of superconductor, zero-form bosons and one-form fermions are dual to each other, i. e. $b_m = \ast f_m$. Therefore fermions obey on the superconducting contour manifold ($1D$) the Bose statistics, in view of their Hodge duality with bosons; since by $b_m = \ast f_m$ the Bose statistics relation: $[b_1, b_2] = 0$ can be rewritten by: $[f_1, f_2] (1D) = 0$, in view of commutativity of boson zero forms with fermion one forms and $\ast^2 \equiv 1$: Thus, in view of the absence of two forms on a one dimensional manifold and the fact that the commutator of one forms are two forms, i. e. in view of $[\Omega^1, \Omega^1] \in \Omega^2$, the commutator of fermion one forms should vanish on the one dimensional contour manifold: ($1D$), i. e. $[\Omega^1, \Omega^1] (1D) \equiv 0$; so that they should obey the Bose statistics on the contour manifold: $[f_1, f_2] (1D) = 0$ or $(f_1 f_2 = f_2 f_1) (1D)$. This is the prove of Bose statistics of fermions in the contour manifold. Therefore electrons as fermion one forms on the contour manifold, obey the Bose statistics and behave themselves as bosons, occupying the ground state collectively in very low temperatures. This explains the reason for the bosonization of electrons on the contour region of superconductors.

Note further that the quantum state of electrons in superconductivity is manifested by flux quantization which is accompanied, as we showed with quantization of total energy of electron, as the sum of quantized kinetic energy: $\oint_{\text{contour}} P_m dX^m = \oint_{\text{contour}} P_m \dot{X}^m dt = Nh$ and the quantized potential energy: $e \oint_{\text{contour}} A_m dX^m = Nh$. Thus we have by quantization of action (1) the quantization of energy levels of electron. This means that the ground state and the excited states of electron are quantized and separated from each other, in accord with the separation of allowed energy levels. In other words, even if the kinetic energy vanishes at zero temperature, the existence of potential and the quantization of potential energy in this model, quantizes the electron energy and separates the ground state of electron from excited states. Thus, the energy spectrum of electron in certain potentials possesses energy gaps [10]. Hence, in this model, there exists always, i. e. even at zero temperature, an energy gap between the ground state and the excited states, in accord with the quantized potential energy.
Note also that, the formula for energy gap is the same as the formula for the energy levels of bound state of a quantized two-dimensional system in Schrödinger theory [4]. Thus also in the standard approach, the electrons on the Fermi surface are executing essentially two-dimensional motion (in momentum space) [4]. Nevertheless, just in the two-dimensional case, the momentum is given by: \( P_m = eA_m = eB \cdot X^l \epsilon_{ml} \), in accord with above arguments and in view of constancy of \( B \) in this case: Thus, \( A_m = eB \cdot X^l \epsilon_{ml} \) is the general definition of two-dimensional electromagnetic potential on a two-dimensional manifold, up to a constant, in view of constancy of magnetic field strength on the two-dimensional manifold [11]. Therefore, in the two-dimensional case under consideration, the momentum space variables \( P_m \) are replaced by the configuration space variables \( X^l \), in view of constancy of \( B \) in \( P_m = eB \cdot X^l \epsilon_{ml} \); and the two-dimensional Fermi surface of such a system becomes equivalent to the two-dimensional configuration space of system. Therefore the two-dimensional motion of electrons on the Fermi surface is indeed nothing else than the motion on the two-dimensional configuration space of system, or on its surface.

These facts underline the two-dimensional nature of superconductivity and the two dimensionality of energy gap conception which is used sometimes instead of concept of separated energy levels in quantized systems.

Further the two-dimensional model of superconductivity can describe the appearance of quantized vortex lines in a superconductor, in view of canonical equivalence of flux quantization and the cyclotron motion of electrons in the two-dimensional model:

Note that on the one hand, we may rewrite the quantized electromagnetic action \( S_2 \) in (8) by:

\[
S_2 = B \int \int_{\text{surface}} \epsilon_{lm} dX^l \wedge dX^m = \int \oint_{\text{contour}} A_m dX^m = N \frac{h}{e} ,
\]

in view of constancy of magnetic field \( B := \epsilon_{lm} F_{lm} = (\text{constant}) \) on the two-dimensional manifold [11]. On the other hand, this quantization can be compared, as a canonical quantization, with the general formula for canonical quantization [8]:

\[
S = \int \int d\Pi_l \wedge dQ_l = \int \Pi_l dQ_l = Nh, \quad N \in \mathbb{Z} ,
\]

which is equivalent to the canonical quantum commutator: \( \left[ \Pi_l , \hat{Q}_m \right] = -i\hbar \delta_{lm} \) [8]. By this comparison, i.e. by \( \Pi_l = A_l = eB \cdot X^m \epsilon_{lm} \) and \( Q_l = X_l \), it is obvious that the canonical quantization postulate (9)
is equivalent to the commutator quantization postulate:

\[ B[\hat{X}_l, \hat{X}_m] = -i\epsilon_{lm} \frac{\hbar}{e}, \]  

(11)

which is the defining commutator for the \textit{two dimensional} cyclotron motion of electron in the magnetic field \( B \). Thus the quantization units in flux quantization and cyclotron motion are also the same: \( (\frac{\hbar}{e}) \). In this sense, the flux quantization and the cyclotron motion are equivalent quantization relations for one and the same \textit{two dimensional motion} of electrons in a constant magnetic field: Where the flux quantization is the topological or global description of such a magnetic quantization in accord with Bohr-Sommerfeld postulate for the electromagnetic field: 

\[ S_2 = B \int \int_{\text{surface}} \epsilon_{lm} dX^l \wedge dX^m = \oint_{\text{contour}} A_m dX^m = N \frac{\hbar}{e}, \]

whereas the cyclotron motion is the local description of the same magnetic quantization in accord with: 

\[ B[\hat{X}_m, \hat{X}_l] = -i\epsilon_{lm} \frac{\hbar}{e}. \]

In other words, as like as any other effect, the magnetic quantization has its local and global aspects which are manifested by cyclotron motion and flux quantization, respectively. Hence, in superconductivity, the flux quantization is accompanied by the local cyclotron motion of electrons. Thus superconducting electrons in a magnetic flux, execute a cyclotron motion which causes an opposite magnetic flux that compensates the other applied flux.

Note that such a canonical equivalence between flux quantization in superconductivity and the cyclotron motion of electrons, explains the appearance of vortex lines and their quantization in superconductors in a canonical manner: If one considers the cyclotron currents of electrons in superconductors, as vortex lines which are quantized in units of \( (\frac{\hbar}{e}) \), in view of cyclotron commutator quantization: 

\[ B[\hat{X}_m, \hat{X}_l] = -i\epsilon_{lm} \frac{\hbar}{e}, \]

or in accord with the equivalent flux quantization. In other words the quantized vortex lines are due to quantized cyclotron motion of electrons or to flux quantization. Thus such an accordence of flux quantization with the two dimensional cyclotron motion manifests again the two dimensionality of flux quantization and superconductivity.

In conclusion note that the \textit{two dimensional model} of superconductivity has the advantage to present a theoretical value for penetration depth which fits better to the empirical values of penetration depth, than the theoretical values of other models. In other words our model presents a reliable method to correct the discrepancy between empirical and theoretical values of penetration depth which appears in other
models [12]: Thus considering the definition of penetration depth by: 
\[ \lambda^2 := \frac{M_s}{\mu_0 n_s e_s^2} \] in London and related models, it is obvious that in view of the fixed value of \( \frac{M_s}{\mu_0 n_s e_s^2} \), the only magnitude which can be responsible for such a discrepancy is the value of density: \( n_s \). Hence by the change of value of the density only, one can obtain a theoretical value of penetration depth that fits to its empirical values. Thus a change in the density value, can be reached by the circumstance that, in a two dimensional model of superconductivity, only surface electrons and hence surface density of electrons is relevant. Note that the above definition of \( \lambda \) in London and related models results just, in view of use of three dimensional volume density in these models, since \( \text{dim}(n_{s(London)}) = \text{dim}(n_{(s3)}) = L^{-3}, \text{dim}(M_s) = L^{-1}, \text{dim}(\mu_0) = \text{dim}(e_s) = L^0 \) and \( \text{dim}(\lambda) = L^1 \), in geometric units; so that the quantity \( \frac{M_s}{\mu_0 n_{(s3)} e_s^2} \) is here of dimension: \( L^2 \). Whereas in two dimensional model, the dimension of density of electrons is: \( \text{dim}(n_{(s2)}) = \text{dim}(n_{(e2)}) = L^{-2} \) and the quantity \( \frac{M_s}{\mu_0 n_{(s2)} e_s^2} \) is of dimension: \( L \). Thus, in a two dimensional model of superconductivity, the penetration depth can be defined by \( \lambda := \frac{M_s}{\mu_0 n_{(s2)} e_s^2} \) only, which is equal to \( \lambda := \frac{M_s}{\mu_0 n_{(e2)} e_s^2} \). This change fits the theoretical value of \( \lambda \) to its empirical values: Recall that the empirical values for penetration depth are several times larger than the theoretical value given by \( \lambda_{(3D)} := \left( \frac{M_s}{\mu_0 n_{(s3)} e_s^2} \right)^{\frac{1}{2}} \) in London and related models. Then, if one estimates the surface density by: \( n_{(s2)} = \frac{n_{(s3)}^2}{3} \) with respect to the volume density which is known to be about \( 10^{21}/\text{cm}^3 \) in experiments [13]. One obtains, in accord with the known value of \( \frac{M_s}{\mu_0 e_s^2} \), a theoretical value of penetration depth \( \lambda := \frac{M_s}{\mu_0 n_{(s2)} e_s^2} \) in the two dimensional model, which is about ten times larger than its value in the three dimensional London model. Then the value of penetration depth in the two dimensional model fits to the empirical values of penetration depth which are about five times larger than those in the London model.

References

[1] We mean by phenomenological relations, those relations which are not derived as equations of motion from some invariant action function. Note that London equations can not be derived as equations of motion from a three- or four dimensional invariant action function, although they are conceived phenomenologically, as three dimensional equations, in view of the involved volume density.
[2] R.E. Prange and S.M. Girvin, ed., "The quantum Hall effect", Graduate Texts in Contemporary Physics (Springer, New York, 1987): R.E. Prange, 1. Introduction.

[3] F. Ghaboussi: "Quantization of Cyclotron Motion and Quantum Hall Effect", Europhys. Let. 47(5), (1999) 621-627. For relation between superconductivity and FQHE see: R. B. Laughlin, Phys. Rev. Lett., 60, (1988), 2677-2680.

[4] V. Z. Kresin and S. A. Wolf: "Fundamentals Of Superconductivity", (Plenum Press 1990).

[5] For topological and differential geometric concepts related with physics see a. o.: M. Nakahara: "Geometry, Topology and Physics" (Adam Hilger, 1990); C. Nash: "Differential Topology and Quantum Field Theory", (Academic Press 1991).

[6] In the absence of spin and electron-electron interaction, the single electron picture is always available. For non-interacting multi-electron case one should replace the constant charge $e$ by the constant $Q := N'e$, where $N'$ is the total number of electrons on the sample.

[7] N. Woodhouse, "Geometric Quantization", (Oxford University, Clarendon Press, 1980, 1990).

[8] The canonical quantization of a system represented by the action $S = \int \int d\Pi_i \wedge dQ_i$, can be given in two equivalent ways: Either by posulating the quantization of action, in the Bohr-Sommerfeld sense, i. e. by: $S = \int \int d\Pi_i \wedge dQ_i = \oint \Pi_i dQ_i = Nh$, $N \in \mathbb{Z}$. Or by postulating the quantum commutator for operators of canonically conjugate variables: $[\Pi_i, \hat{Q}_j] = -i\hbar \delta_{ij}$.

[9] For the attachment of differential forms to physical quantities, see: E. Witten, Nucl. Phys. B202 (1982) 253-316.

[10] C. P. Poole, Jr., H. A. Farach, R. J. Creswick: "Superconductivity", (Academic Press 1995).

[11] The curvature or field strength two form and its tensor components $F_{\ell m}$ or $(B := \epsilon_{\ell m} F_{\ell m})$ are constant on a two dimensional manifold, by definition, since there exists no three form on a two dimensional manifold.

[12] Note that although the mentioned discrepancy is known for London model explicitly, nevertheless in view of the fact that London equations results from Ginzburg-Landau model and this one refers
completely to BCS theory, then the mentioned discrepancy is given for these models also. Thus such a discrepancy exists for any model which is based on a three dimensional density of electrons and Cooper pairs, that allows a \textit{quadratic} definition of penetration depth of London type.

[13] K. Holzer, et al., Phys. Rev. Lett., 67, (1991), 271-274. See also Ref. [10].