Admissibility and minimaxity of generalized Bayes estimators for spherically symmetric family

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Abstract: We give a sufficient condition for admissibility of generalized Bayes estimators of the location vector of spherically symmetric distribution under squared error loss. Compared to the known results for the multivariate normal case, our sufficient condition is very tight and is close to being a necessary condition. In particular we establish the admissibility of generalized Bayes estimators with respect to the harmonic prior and priors with slightly heavier tail than the harmonic prior. We use the theory of regularly varying functions to construct a sequence of smooth proper priors approaching an improper prior fast enough for establishing the admissibility. We also discuss conditions of minimaxity of the generalized Bayes estimator with respect to the harmonic prior.

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1. Introduction

We consider estimation of the $p$-dimensional location parameter of a spherically symmetric distribution. Specifically, let $X = (X_1, \ldots, X_p)'$ have a density function $f(\|x - \theta\|)$ and consider estimation of $\theta$ with a general quadratic loss function $L_Q(\theta, \delta) = (\delta - \theta)'Q(\delta - \theta) = \|\delta - \theta\|_Q^2$ for a positive definite matrix $Q$. The usual minimax estimator $X$, which is generalized Bayes, is inadmissible for $p \geq 3$ as shown in Stein (1956) in the normal case, and in Brown (1966) under more general situation, respectively. In the decision-theoretic point of view, we are interested in proposing admissible estimators dominating $X$, that is, minimax admissible estimators. Note that the dominance over $X$ means minimaxity in our setting because $X$ is minimax with a constant risk. Note also that our results hold

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In the normal case, there already exists a broad class of admissible minimax estimators. Baranchik (1970) gave a sufficient condition for minimaxity of a shrinkage estimator of the form
\[ \delta_{\phi}(X) = (1 - \phi(\|X\|)/\|X\| - 2)X. \] (1.1)
Strawderman (1971) found a subclass of proper Bayes estimators of the form (1.1) satisfying the sufficient conditions for minimaxity. Brown (1971) gave a very powerful sufficient condition for admissibility of generalized Bayes estimators. Using Brown's (1971) condition, Berger (1976), Fourdrinier et al. (1998) and Maruyama (1998, 2004) enlarged a class of admissible minimax estimators which are of the Strawderman type and generalized Bayes. For a subclass of scale mixtures of multivariate normal distributions which includes multivariate-t distribution, some proper Bayes minimax estimators were proposed by Maruyama (2003) by using Strawderman's (1971) techniques.

However, for general spherically symmetric distributions, no minimax admissible estimators of the location vectors have been derived, although for the minimaxity, various Baranchik-type sufficient conditions of the estimator (1.1) were given by Berger (1975), Brandwein and Strawderman (1978, 1991) and Bock (1985). The main reason is the lack of a standard class of generalized or proper Bayes estimators of the form (1.1) like the Strawderman type in the normal case, which allows an easy check of the minimaxity condition. Furthermore no sufficient condition for admissibility of generalized Bayes estimators has been derived. In Maruyama and Takemura (2003), we provided satisfactory solutions to these problems. In this paper we will weaken some regularity conditions assumed in Maruyama and Takemura (2003).

In Section 2, we give preliminary results including the properties of regularly varying functions and asymptotic behaviors of expected values when \( \|\theta\| \) is sufficiently large. The former is useful for constructing a very convenient sequence of proper densities approaching an improper density \( g(\theta) \), which is required in applying the method of Blyth (1951).

In Section 3, we will present a powerful sufficient condition for admissibility of generalized Bayes estimator and in particular show that the generalized Bayes estimators with respect to the harmonic prior \( g(\theta) = \|\theta\|^{2-p} \) and with respect to a prior with a slightly heavier tail
\[ g(\theta) = \|\theta\|^{2-p} \log(\|\theta\| + c), \quad c > 1, \] (1.2)
are admissible under mild regularity conditions on \( f \).

In Section 4, we show that the generalized Bayes estimator with respect to the harmonic prior is written as
\[ \left( 1 - \frac{\int_0^1 t^{p-1} F(t\|X\|) dt}{\int_0^1 t^{p-3} F(t\|X\|) dt} \right) X, \] (1.3)
where \( F(u) = \int_u^\infty s f(s) ds \). This form is simple enough to check various sufficient conditions for minimaxity and we demonstrate that (1.3) is minimax for some \( f \). We believe that (1.3)
is minimax for a broad subclass of spherically symmetric distributions. Notice that the generalized Bayes estimators with respect to priors except \( \| \theta \|^{2-p} \) do not have such simple forms as far as we know.

Our proof of admissibility is mainly based on the techniques of \cite{Brown and Hwang 1982}. \cite{Brown and Hwang 1982} considered the problem of estimating the natural mean vector of an exponential family under a quadratic loss function. Note that the intersection of their setting and our setting is the multivariate normal case. Their sufficient condition for admissibility in the normal case does not however permit \( g(\theta) \) to diverge to infinity around the origin like \( \| \theta \|^{2-p} \), while it permits \( g(\theta) \leq \| \theta \|^a \) with \( a \leq 2 - p \) for sufficiently large \( \| \theta \| \). Prior to \cite{Brown and Hwang 1982}, \cite{Brown 1971} considered the estimation in the multivariate normal case and gave a powerful sufficient condition for minimaxity which are satisfied by the harmonic prior and (1.2), but his proof was based on many advanced mathematics. Our mathematical tool is much more familiar to the readers. \cite{Brown 1971} also gave sufficient condition for inadmissibility. By using it we see that the generalized Bayes estimator with respect to \( \| \theta \|^{2-p} \log^2(\| \theta \| + 2) \) is inadmissible. Hence our sufficient condition for admissibility should be very tight and close to being a necessary condition.

Brown (1979) considered a more general problem than ours: estimation of \( \theta \) for a general density \( p(x - \theta) \) and a general loss function \( W(\delta - \theta) \). He conjectured that the prior \( g(\theta) \sim \| \theta \|^a \) with \( a \leq 2 - p \) for sufficiently large \( \| \theta \| \) leads to admissibility, regardless of the density \( p \) and the loss \( W \). Hence our results support Brown (1979)'s conjecture for the case of elliptically-contoured family and a general quadratic loss function.

Finally we notice that the most important key for our proof for admissibility is the construction of a very convenient sequence \( h_i(\theta) \) for approximating \( g(\theta) \) by \( g(\theta) h_i^\gamma(\theta) \) for \( 0 < \gamma \leq 2 \). \cite{Brown and Hwang 1982} used \( \gamma = 2 \) and

\[
h_i(\theta) = \begin{cases} 
1 & \| \theta \| \leq 1 \\
1 - \log \| \theta \| / \log i & 1 \leq \| \theta \| \leq i \\
0 & \| \theta \| > i.
\end{cases}
\]

This \( h_i(\theta) \) is not differentiable at \( \| \theta \| = 1 \) and truncated at \( \| \theta \| = i \), which makes handling and extension difficult for our purposes. Our \( h_i(\theta) \) given in Section 2 is smoother and not truncated. Furthermore our \( \gamma \) is flexible whereas \( \gamma = 2 \) in \cite{Brown and Hwang 1982}. By such a flexible \( \gamma \), we can adjust the rate of convergence of \( g(\theta) h_i^\gamma(\theta) \) so that it is just enough to be proper. We will see that choosing as small \( \gamma \) as possible is important in the main theorem, Theorem 3.1. We believe that our smooth function \( h_i(\theta) \) and an idea of flexible \( \gamma \) are very useful for showing admissibility of generalized Bayes estimators in various problems.

2. Preliminaries

In this section we prepare a sequence of proper densities using the theory of regularly varying functions and give some results on asymptotic behaviors of expected values when
the location parameter diverges to infinity. For the theory of regularly varying and slowly varying functions the readers are referred to Geluk and de Haan (1987) and Bingham et al. (1987).

2.1. Regularly varying functions

A Lebesgue measurable function \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \) which is eventually positive is called regularly varying if for some \( \alpha \in \mathbb{R} \)

\[
\lim_{{x \rightarrow \infty}} \frac{f(tx)}{f(x)} = t^\alpha, \quad \forall t > 0.
\] (2.1)

We sometimes use the notation \( f \in \text{RV}_\alpha \). The number \( \alpha \) in the above is called the index of regular variation. A function satisfying (2.1) with \( \alpha = 0 \) is called slowly varying.

Let \( \beta : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be positive, continuously differentiable, monotone decreasing, integrable (i.e. \( \int_0^\infty \beta(r)dr < \infty \)) and regular varying with index \(-1\). A typical \( \beta(\eta) \) is

\[
\frac{1}{\eta + c} \left\{ \log (\eta + c) \right\}^2 \prod_{i=0}^{n-1} \frac{1}{\log_i (\eta + c)},
\] (2.2)

where \( n \) is a positive integer, \( \log_0 (\eta + c) \equiv 1 \),

\[ \log_i (\eta + c) = \underbrace{\log \log \cdots \log}_{i} (\eta + c), \quad i \geq 1, \]

and \( c \) is chosen such that \( \log_n(c) > 0 \). Note that for (2.2)

\[
\int_\eta^\infty \beta(r)dr = \frac{1}{\log_n(\eta + c)}.
\]

The following results for \( \beta(\eta) \) satisfying the above assumptions are known from the theory of regularly varying functions.

Lemma 2.1.  1. \( \int_\eta^\infty \beta(r)dr \in \text{RV}_0 \) and \( \beta'(r) \in \text{RV}_{-2} \).
2. \( \lim_{\eta \rightarrow \infty} \eta \beta(\eta)/\int_\eta^\infty \beta(r)dr = 0 \), \( \lim_{\eta \rightarrow \infty} \eta \beta'(\eta)/\beta(\eta) = -1 \).

We now define functions \( H_i(\eta) \), \( i = 1, 2, \ldots \), based on \( \beta(r) \) by

\[
H_i(\eta) = \frac{\int_\eta^\infty e^{(\eta-r)/i} \beta(r)dr}{\int_\eta^\infty \beta(r)dr}.
\] (2.3)

These functions are very useful for constructing a sequence of proper prior densities approaching the target improper density in the next section. The properties of \( H_i \) are given in the following theorem.

Theorem 2.1.  1. \( 0 \leq H_1(\eta) \leq H_2(\eta) \leq \cdots \leq 1 \). For any fixed \( \eta \), \( \lim_{i \rightarrow \infty} H_i(\eta) = 1 \).
2. For any fixed \( i \), \( \lim_{\eta \to \infty} \int_{\eta}^{\infty} \beta(r)dr \beta(\eta)^{-1} H_i(\eta) = i \) and hence \( H_i(\eta) \in RV_{-1} \).

3. For any fixed \( \eta \), \( \lim_{i \to \infty} H_i'(\eta) = 0 \).

4. \( |H_i'(\eta)| < 2\beta(\eta)/\int_{\eta}^{\infty} \beta(r)dr \) for all \( \eta > 0 \).

5. For any \( \epsilon > 0 \), there exists \( \eta_0 \) such that \( -1 - \epsilon < \eta H_i'(\eta)/H_i(\eta) \leq 0 \) for all \( \eta \geq \eta_0 \) and for all \( i \).

**Proof.** It is obvious that \( 0 \leq H_1(\eta) \leq 1 \) and \( H_i(\eta) \) is increasing in \( i \). For fixed \( \eta \), \( H_i(\eta) \uparrow 1 \) by the monotone convergence theorem.

By integration by parts, the numerator of \( H_i(\eta) \) is written as

\[
\int_{\eta}^{\infty} e^{(\eta-r)/i} \beta(r)dr = i\beta(\eta) + i \int_{\eta}^{\infty} e^{(\eta-r)/i} \beta'(r)dr. \tag{2.4}
\]

Therefore

\[
H_i(\eta) = i\frac{\beta(\eta)}{\int_{\eta}^{\infty} \beta(r)dr} + i \frac{\int_{\eta}^{\infty} e^{(\eta-r)/i} \beta'(r)dr}{\int_{\eta}^{\infty} \beta(r)dr}. \tag{2.5}
\]

\( \frac{2.5}{2.4} \) divided by \( 2.4 \) is

\[
1 = i\frac{\beta(\eta)}{H_i(\eta) \int_{\eta}^{\infty} \beta(r)dr} + i \frac{\int_{\eta}^{\infty} e^{-(r)/i} \beta'(r)dr}{\int_{\eta}^{\infty} \beta(r)dr}.
\]

For fixed \( i \), the second term of the above equation converges to 0 as \( \eta \to \infty \) by the L’Hospital theorem. \( H_i(\eta) \in RV_{-1} \) because \( \int_{\eta}^{\infty} \beta(r)dr \in RV_0 \) and \( \beta(\eta) \in RV_{-1} \).

Using \( 2.4 \) again, differentiation of the numerator of \( H_i(\eta) \) gives

\[
\left( \int_{\eta}^{\infty} e^{(\eta-r)/i} \beta(r)dr \right)' = \frac{1}{i} \int_{\eta}^{\infty} e^{(\eta-r)/i} \beta(r)dr - \beta(\eta) = \int_{\eta}^{\infty} e^{(\eta-r)/i} \beta'(r)dr.
\]

Therefore

\[
H_i'(\eta) = \frac{\beta(\eta) \int_{\eta}^{\infty} e^{(\eta-r)/i} \beta(r)dr}{(\int_{\eta}^{\infty} \beta(r)dr)^2} - \frac{\int_{\eta}^{\infty} e^{(\eta-r)/i} \{ -\beta'(r) \} dr}{\int_{\eta}^{\infty} \beta(r)dr}. \tag{2.6}
\]

Note that \( -\beta'(r) \geq 0 \) by our assumption. Each term of the right hand side of \( 2.6 \) is nondecreasing in \( i \) and hence by the monotone convergence theorem

\[
\lim_{i \to \infty} H_i'(\eta) = \frac{\beta(\eta) \int_{\eta}^{\infty} \beta(r)dr}{(\int_{\eta}^{\infty} \beta(r)dr)^2} - \frac{\int_{\eta}^{\infty} \{ -\beta'(r) \} dr}{\int_{\eta}^{\infty} \beta(r)dr} = 0.
\]

Furthermore we have

\[
|H_i'(\eta)| < \left| \frac{\beta(\eta) \int_{\eta}^{\infty} e^{(\eta-r)/i} \beta(r)dr}{(\int_{\eta}^{\infty} \beta(r)dr)^2} \right| + \left| \frac{\int_{\eta}^{\infty} e^{(\eta-r)/i} \{ -\beta'(r) \} dr}{\int_{\eta}^{\infty} \beta(r)dr} \right|.
\]
Dividing (2.6) by (2.3), we have
\[
\eta H'_i(\eta) / H_i(\eta) = \eta \left( \frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + \frac{\int_\eta^\infty e^{-r/i} \beta'(r)dr}{\int_\eta^\infty e^{-r/i} \beta(r)dr} \right)
\]
(2.7)
\[
> \frac{\eta \beta(\eta)}{\int_\eta^\infty \beta(r)dr} - \frac{\int_\eta^\infty e^{-r/i} \left\{ -r \beta'(r) / \beta(r) \right\} \beta(r)dr}{\int_\eta^\infty e^{-r/i} \beta(r)dr}
\]
\[
> \frac{\eta \beta(\eta)}{\int_\eta^\infty \beta(r)dr} - \sup_{r > \eta} \frac{-r \beta'(r)}{\beta(r)}.
\]
By 2 of Lemma 2.1 the right hand side converges to \(-1\). This implies that for any \(\epsilon > 0\) there exists \(\eta_0\) such that \(\eta H'_i(\eta) / H_i(\eta) > -1 - \epsilon\) for all \(\eta \geq \eta_0\) and for all \(i\). Finally we will prove that \(H'_i(\eta) \leq 0\) for sufficiently large \(\eta\) independent of \(i\). By 2 of Lemma 2.1
\[
\frac{\eta \int_\eta^\infty \beta(r)dr}{\beta(\eta)} \left( \frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} \right)' = \eta \frac{\beta'(\eta)}{\beta(\eta)} + \eta \frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} \rightarrow -1
\]
and hence \(\beta(\eta) / \int_\eta^\infty \beta(r)dr\) is eventually nonincreasing. Hence by redefining \(\eta_0\) if necessary, we can assume that \(\beta(\eta) / \int_\eta^\infty \beta(r)dr\) is monotone nonincreasing for \(\eta \geq \eta_0\). By integration by parts on the numerators of each term in (2.6), we have
\[
\int_\eta^\infty e^{-r/i} \beta(r)dr = e^{-\eta/i} \int_\eta^\infty \beta(r)dr - i^{-1} \int_\eta^\infty e^{-r/i} \left\{ \int_r^\infty \beta(s)ds \right\} dr
\]
\[
\int_\eta^\infty e^{-r/i} \beta'(r)dr = -e^{-\eta/i} \beta(\eta) + i^{-1} \int_\eta^\infty e^{-r/i} \beta(r)dr
\]
and hence
\[
\left\{ \frac{i \int_\eta^\infty \beta(r)dr}{\int_\eta^\infty e^{-r/i} \left\{ \int_r^\infty \beta(s)ds \right\} dr} \right\} H'_i(\eta) = -\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + \frac{\int_\eta^\infty e^{-r/i} \beta(r)dr}{\int_\eta^\infty e^{-r/i} \left\{ \int_r^\infty \beta(s)ds \right\} dr},
\]
which is nonpositive for \(\eta \geq \eta_0\).

2.2. **Asymptotic behavior of expectations**

In the next section, we need evaluation of an asymptotic behavior of expectation
\[
E_x[\rho(\theta)] = \int_{R^p} \rho(\theta) f(\|\theta - x\|) d\theta
\]
for sufficiently large $\|x\|$, where a random vector $\theta$ has the density function $f(\|\theta - x\|)$. This is the expected value with respect to the posterior distribution. Interchanging the roles of $x$ and $\theta$, in this subsection, we consider the asymptotic behavior of expectation

$$E_\theta[\rho(X)] = \int_{R^p} \rho(x)f(\|x - \theta\|)dx$$

for sufficiently large $\|\theta\|$, where a random vector $X$ has the density function $f(\|x - \theta\|)$. We believe that this does not confuse the readers.

We discuss some notations used in the following. In addition to the Euclidean norm $\|x\|^2 = x_1^2 + \cdots + x_p^2$, we consider the norm $\|x\|^2_d = d_1^2x_1^2 + \cdots + d_p^2x_p^2$. For convenience we assume, without loss of generality, that $d_1 \geq \cdots \geq d_p \geq 1$. Under this assumption

$$\|x\| \leq \|x\|_d \leq d_1\|x\|. \quad (2.8)$$

By introducing this norm our results hold for elliptically-contoured distributions. The gradient of $\rho(x)$ is denoted by

$$\nabla \rho(x) = \left(\frac{\partial}{\partial x_1}\rho(x), \ldots, \frac{\partial}{\partial x_p}\rho(x)\right)' .$$

We also write $\nabla_j \rho(x) = (\partial/\partial x_j)\rho(x)$. Finally we write $c_p = 2\pi^{p/2}/\Gamma(p/2)$.

Now we make the following regularity conditions on the density $f$ and the function $\rho$.

**F1** There exist $r_0 > 0$, $L > 0$, and $s > 1$, such that $r^{p+s}f(r) \leq L$ for all $r \geq r_0$.

**B1** $\rho(x)$ is written as $\rho(x) = g(\|x\|_d)$, where $g(r)$ is continuously differentiable in $r > 0$.

**B2** There exists $r_1 \geq 1$ and $t_1 \leq t_2$ such that $g(r) > 0$ and $t_1 \leq rg'(r)/g(r) \leq t_2$ for all $r \geq r_1$.

Assumption **B2** is, for instance, satisfied by

$$g(r) = \exp \left(\int_0^r \frac{-p + \alpha \cos t}{t+1} dt \right), \text{ for } \alpha > 0$$

where we easily see $rg'(r)/g(r) = \{r/(r + 1)\}(-p + \alpha \cos r)$ and hence $t_1 = -p - \alpha$ and $t_2 = -p + \alpha$ in **B2**. Since $\lim_{r \to \infty} rg'(r)/g(r)$ exists for regularly varying $g$, we deal with a broader class of functions than the class of regularly varying functions. We will discuss more in Section 3. Note that

$$\nabla \rho(x) = \frac{g'(\|x\|_d)}{\|x\|_d} (d_1^2x_1^2, \ldots, d_p^2x_p^2)'$$

and

$$\|\nabla \rho(x)\| = \frac{|g'(\|x\|_d)|}{\|x\|_d} (d_1^4x_1^2 + \cdots + d_p^4x_p^2)^{1/2} \leq d_1 |g'(\|x\|_d)| .$$

The following lemma is useful. The proof based on the integration of $(\log g(r))' = g'(r)/g(r)$ is easy and omitted.
Theorem 2.2. Assume for large \( \| \frac{z}{y} \| \) for any \( z > y \geq r_1 \). Moreover

\[
\limsup_{y \to \infty} \sup_{0 < y \leq \beta y} \frac{\theta(z)}{\theta(y)} \leq \max(\alpha^{\ell_1}, \beta^{\ell_2})
\]

for any \( 0 < \alpha < 1 < \beta \).

We now state the following theorem concerning the asymptotic behavior of \( E_\theta[\rho(X)] \) for large \( \| \theta \|_d \).

Theorem 2.2. Assume \( \textbf{F1}, \textbf{B1} \) and \( \textbf{B2} \). For \( a = 0 \) or \( 1 \), and \( j = 1, \ldots, p \), if \( s > \max(1, -t_1 - a - p, t_2 + a) \) and \( \int_0^1 r^{p+a-1}|\theta(r)|dr < \infty \), then there exists \( \epsilon > 0 \) (say \( \epsilon = \min(1, s + t_1 + a + p)/4 \)) such that

\[
\| \theta \|_d^{\ell-a} E_\theta[\rho(X)] - \theta_j \rho(\theta) \leq C \rho(\theta)
\]

(2.9)

for \( \| \theta \|_d \geq 2d_1 \max(r_0, r_1) \). Moreover \( C \) depends on \( \rho \) (or \( \theta \)) only through \( r_1, t_1, t_2 \) and \( \{\theta(r_1)\} - \int_0^1 r^{p+a-1}|\theta(r)|dr \).

For simplicity, in the rest of the paper, we will write \( E_\theta[\rho(X)] \approx \theta_j \rho(\theta) \) if, as in Theorem 2.2 there exists \( \epsilon > 0 \) such that (2.9) is satisfied for sufficiently large \( \| \theta \|_d \).

Proof. Note that

\[
E_\theta[\rho(X)] = E_\theta[\rho(X) - \rho(\theta)]
\]

Fix \( 0 < \nu < 1 \) (set \( \nu = 1/2 \) finally). Define

\[
V_\nu = \{ x : \| x - \theta \| \leq \nu \| \theta \| \}
\]

\[
V'_\nu = \{ x : (1 - \nu)\| \theta \| \leq \| x \| \leq (1 + \nu)\| \theta \| \}
\]

\[
V''_\nu = \{ x : d_1^{-1}(1 - \nu)\| \theta \|_d \leq \| x \|_d \leq d_1(1 + \nu)\| \theta \|_d \}
\]

By (2.8) \( V_\nu \subset V'_\nu \subset V''_\nu \). Then

\[
\| \theta \|_d^{\ell-a} E_\theta[\rho(X) - \rho(\theta)]
\]

\[
\leq \| \theta \|_d^{\ell-a} \left( \int_{V_\nu} + \int_{V'_\nu} \right) \| x \|^{\ell-a} |\rho(x) - \rho(\theta)| f(\| x - \theta \|)dx
\]

\[
\leq \| \theta \|_d^{\ell-a} \sup_{x \in V'_\nu} \left( \frac{\| x \|_d}{\| \theta \|_d} \right)^{\ell-a} \int_{V_\nu} |\rho(x) - \rho(\theta)| f(\| x - \theta \|)dx + \| \theta \|_d^{\ell-a} \int_{V'_\nu} \| x \|^{\ell-a} |\rho(x)| f(\| x - \theta \|)dx
\]

\[
= I_1 + I_2 + I_3. \quad \text{(say)} \quad (2.10)
\]
Therefore we have, for $0 \leq \nu < 1$. Then for $\nu < \epsilon < 1$. Therefore we have $I_1 \leq C_1 \rho(\theta)$, where $C_1 = d_1 \epsilon^{1 - \nu} \{d_1/(1 - \nu)\} \max\{(1 - \nu)/d_1\} \{d_1(1 + \nu)\} \max(|t_1|, |t_2|)$.

Now we consider the integral outside of $V_\nu$. We only consider $\|\theta\|_d \geq \max(d_1\nu^{-1}r_0, r_1)$. Then for $x \in V_\nu^\circ$

$$\|x - \theta\| \geq \nu \|\theta\| \geq \nu d_1^{-1} \|\theta\|_d \geq r_0.$$ 

Therefore we have, for $0 \leq \alpha < s$

$$\int_{V_\nu^\circ} \|x\|^\alpha f(\|x - \theta\|)dx \leq \int_{V_\nu^\circ} \{\|x - \theta\| + \|\theta\|\}^\alpha f(\|x - \theta\|)dx$$

$$\leq (1 + 1/\nu)^\alpha \int_{V_\nu^\circ} \|x - \theta\|^\alpha f(\|x - \theta\|)dx$$

$$\leq (1 + 1/\nu)^\alpha c_p L \int_{\nu \|\theta\|}^{\infty} r^{-s + \alpha - 1} dr$$

$$= (1 + 1/\nu)^\alpha c_p L \frac{(\nu \|\theta\|)^{-s + \alpha}}{s - \alpha}$$

$$\leq C_2(\alpha) \|\theta\|_d^{\alpha - s},$$

(2.11)

where $C_2(\alpha) = (1 + 1/\nu)^\alpha c_p L(d_1/\nu)^{s - \alpha}(s - \alpha)^{-1}$. Hence for the second term $I_2$, if $s > 1$ and $0 < \epsilon < 1$, then $I_2 \leq C_2(\alpha) \rho(\theta)$ for $a = 0, 1$.

We have seen that $I_1$ and $I_2$ are bounded from above assuming only $s > 1$.

The third term $I_3$ of (2.10) is more problematic. Write

$$I_3 = \|\theta\|_d^{\alpha - s} \int_{V_\nu^\circ} \|x\|^\alpha |\rho(x)| f(\|x - \theta\|)dx$$

$$\leq \|\theta\|_d^{\alpha - s} \left( \int_{V_\nu^\circ \cap \{\|x\|_d < r_1\}} + \int_{V_\nu^\circ \cap \{r_1 \leq \|x\|_d \leq \|\theta\|_d\}} \right)$$

Consider the first integral $I_1$.

$$\sup_{x \in V_\nu^\circ} (\|x\|_d/\|\theta\|_d)^\alpha \leq d_1^\alpha (1 + \nu)^{\alpha}.$$ If $s > 1$, then $m_1 = \int_{V_\nu} \|x - \theta\| f(\|x - \theta\|) dx$ is finite. Therefore for $\|\theta\|_d \geq d_1 (1 - \nu)^{-1} r_1$ we have

$$\|\theta\|_d^\alpha \int_{V_\nu} |\rho(x) - \rho(\theta)| f(\|x - \theta\|) dx$$

$$\leq m_1 \|\theta\|_d^\alpha \int_{V_\nu} \|\nabla \rho(x)\| f(\|x - \theta\|) dx$$

$$\leq m_1 d_1 \|\theta\|_d^{\alpha - 1} \sup_{x \in V_\nu} \|\nabla \rho(x)\|$$

$$\leq m_1 d_1 \|\theta\|_d^{\alpha - 1} \sup_{x \in V_\nu} \|f(\|x\|_d)\|$$

$$\leq m_1 d_1 \|\theta\|_d^{\alpha - 1} \sup_{x \in V_\nu} \|f(\|x\|_d)\| \sup_{x \in V_\nu} \|\nabla f(\|x\|_d)\| \times \rho(\theta)$$

for $0 < \epsilon < 1$. Therefore we have $I_1 \leq C_1 \rho(\theta)$, where $C_1 = d_1 \epsilon^{1 - \nu} \{d_1/(1 - \nu)\} \max\{(1 - \nu)/d_1\} \{d_1(1 + \nu)\} \max(|t_1|, |t_2|)$.

Now we consider the integral outside of $V_\nu$. We only consider $\|\theta\|_d \geq \max(d_1\nu^{-1}r_0, r_1)$. Then for $x \in V_\nu^\circ$

$$\|x - \theta\| \geq \nu \|\theta\| \geq \nu d_1^{-1} \|\theta\|_d \geq r_0.$$
Note that by simple change of variables we have

\[ I = I_{31} + I_{32} + I_{33}. \]

We take care of \( I_{33} \) first. Since \( \varrho(r)r^{-t_2} \) is monotone nonincreasing for \( r \geq r_1, \) \( \rho(x)||x||^{-t_2} \leq \rho(\theta)||\theta||^{-t_2} \) for \( ||x|| \geq ||\theta|| \geq r_1. \) Therefore we have,

\[ I_{33} \leq \|\theta\|^{\epsilon-a-t_2} \rho(\theta)\max(t_2,0) \int_{V^C} \|x\|^\alpha f(||x-\theta||)dx. \]

If \( 0 \leq a + t_2 < s, \) as in (2.11)

\[ \int_{V^C} \|x\|^\alpha f(||x-\theta||)dx \leq \int_{V^C} \|x\|^\alpha f(||x-\theta||)dx \leq C_2(a + t_2)\|\theta\|^\alpha. \]

and if \( a + t_2 < 0, \)

\[ \int_{V^C} \|x\|^\alpha f(||x-\theta||)dx \leq d_{1}^{-t_2-a}\|\theta||^\alpha \int_{V^C} f(||x-\theta||)dx \leq d_{1}^{-t_2-a}C_2(0)\|\theta||^\alpha. \]

Hence \( I_{33} \leq C_{33} \rho(\theta) \) where \( C_{33} = d_{1}^{\max(t_2,0)} \max(C_2(a + t_2), C_2(0)d_{1}^{-a-t_2}). \)

Next we consider \( I_{31}. \) For \( ||\theta|| \geq \max(d_1^{-1}r_0, r_1) \) and \( x \in V^C \)

\[ f(||x-\theta||) \leq L ||x-\theta||^{-p-s} \leq L(\nu||\theta||)^{-p-s} \leq L(d_1/\nu||\theta||)^{p+s}. \] (2.12)

Therefore

\[ I_{31} \leq \|\theta\|^{\epsilon-a}Ld_{1}^{p+s} \nu^{-p-s}||\theta||^{-p-s} \int_{\|x\| \leq r_1} \|x\|^{2} \rho(x)dx. \]

Note that by simple change of variables we have

\[ \frac{\partial}{\partial r} \int_{\|x\| \leq r} dx = c_{p,d} r^{p-1}, \quad c_{p,d} = c_{p} \prod_{i=1}^{p} d_{i}^{-1}. \]

Then

\[ \int_{\|x\| \leq r_1} \|x\|^{2} \rho(x)dx = c_{p,d} \int_{0}^{r_1} r^{p+a-1} |\varrho(r)| dr. \]

Therefore

\[ I_{31} \leq C_{*} \|\theta\|^{\epsilon-a-p-s} \int_{0}^{r_1} r^{p+a-1} |\varrho(r)| dr, \]

where \( C_{*} = Ld_{1}^{p+s} \nu^{-p-s}c_{p,d}. \) On the other hand for \( ||\theta|| \geq r_1, \) \( \rho(\theta) = \varrho(||\theta||) \) is bounded from below as

\[ \varrho(r_1)r_1^{-t_2}||\theta||^{t_2} \leq \varrho(||\theta||). \]

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Therefore
\[ I_{31} \leq \|\theta\|_d^{\epsilon-a-p-s-t_1} \times C_r \frac{r_{t_1}}{g(r_1)} \int_{r_1}^{r^*} r^{p+a-1} |\varrho(r)| dr \times \rho(\theta). \]

Hence if \( s \geq -t_1 - a - p \), then we can choose \( \epsilon > 0 \) (say \( \epsilon = (a + p + s + t_1)/4 \)) such that \( \epsilon - a - p - s - t_1 < 0 \) and hence \( I_{31} \leq C_{31} \rho(\theta) \) where
\[ C_{31} = C_r \frac{r_{t_1}}{g(r_1)} \int_{r_1}^{r^*} r^{p+a-1} |\varrho(r)| dr. \]

Finally we consider \( I_{32} \). Note \( \varrho(r) \leq \varrho(||\theta||_d)||\theta||_d^{\epsilon-t_1} r^{t_1} \) for \( r_1 \leq r \leq ||\theta||_d \) and \( \nu \geq 3 \).

Then
\[ I_{32} \leq ||\theta||_d^{\epsilon-a-t_1} L d_1^{p+s} \varrho^{p-s} ||\theta||_d^{-p-s} \varrho(||\theta||_d) \int_{r_1}^{||\theta||_d} \langle x \rangle r^{t_1+t_1} dr \times \rho(\theta). \]

Consider the integral \( Q = \int_{r_1}^{||\theta||_d} r^{p+a+t_1-1} dr \). If \( p + a + t_1 < 0 \), then
\[ Q \leq \frac{r_{t_1}^{t_1+a+p}}{-t_1 - a - p}. \]

Therefore as in the case of \( I_{31} \), if \( s > -t_1 - a - p \), then we can choose \( \epsilon > 0 \) (say \( \epsilon = (a + p + s + t_1)/4 \)) such that \( \epsilon - a - p - s - t_1 < 0 \) and hence
\[ I_{32} \leq r_{t_1}^{\epsilon-s} C_{32} \rho(\theta) \leq C_{32} \rho(\theta) \]

where \( C_{32} = \{1/(p + a + t_1)\} C_\ast \). If \( p + a + t_1 \geq 0 \),
\[ Q = \int_{r_1}^{||\theta||_d} r^{p+a+t_1+\epsilon_3-1-\epsilon_3} dr \leq \frac{||\theta||_d^{p+a+t_1+\epsilon_3} r_{t_1}^{\epsilon_3}}{p + a + t_1 + \epsilon_3} \]

for any \( \epsilon_3 > 0 \). Hence
\[ I_{32} \leq \frac{C_\ast}{p + a + t_1 + \epsilon_3} ||\theta||_d^{\epsilon+\epsilon_3-s} \rho(\theta). \]

If \( s > 1 \), we can choose \( \epsilon \) and \( \epsilon_3 \) (say \( \epsilon = \epsilon_3 = 1/4 \)) such that \( \epsilon + \epsilon_3 - s < 0 \) and hence
\[ I_{32} \leq C_{32} \rho(\theta), \]

where \( C_{32} = (p + a + t_1 + \epsilon_3)^{-1} C_\ast \).

We have now confirmed that if \( s > \max(1, -t_1 - a - p, a + t_2) \), there exist \( \epsilon > 0 \) and \( C = C_1 + C_2 + C_{31} + C_{32} + C_{33} \), such that \( \nu \geq (2,3) \) folds for \( ||\theta||_d \geq \max(d_1 \nu^{-1} r_0, r_1, d_1 (1 - \nu)^{-1} r_1) \), which equals to \( 2d_1 \max(r_0, r_1) \) for \( \nu = 1/2 \). □
In the next section, we need asymptotic behavior of the expectation of \( \rho(X) \times h_i^2(X) \) where \( h_i(\theta) = H_i(\|\theta\|) \) given by (2.3) and \( \gamma > 0 \).

**Corollary 2.1.** Assume \( F_1, B_1 \) and \( B_2 \). For \( a = 0 \) or \( 1 \), \( i \geq 1 \), \( \gamma > 0 \) and \( j = 1, \ldots, p \), if \( s > \max(1, \gamma - t_1 - a - p, t_2 + a) \) and \( \int_0^1 r^{p+a-1} |\varrho(r)|dr < \infty \), there exists \( \epsilon > 0 \) (say \( \epsilon = \min(1, s + t_1 + a + p - \gamma)/4 \)) such that

\[
\|\theta\|_{d}^{\epsilon-a} \left| E_{\theta}[X_{\lambda}^a \rho(X)h_i^2(X)] - \theta_i^a \rho(\theta)h_i^2(\theta) \right| < C \rho(\theta)h_i^2(\theta) \tag{2.13}
\]

for \( \|\theta\| \geq 2d_1 \max(r_0, r_1, \eta_0) \). Moreover \( C \) does not depend on \( i \).

**Proof.** Since we have

\[
\eta \{ \varrho(\eta)H_i^\gamma(\eta)/\|\varrho(\eta)H_i^\gamma(\eta)\| \} = \eta \varrho'(\eta)/\varrho(\eta) + \gamma \eta H_i^\gamma(\eta)/H_i(\eta),
\]

under Assumption \( B_2 \) and by 5 of Theorem 2.1 for \( \epsilon_1 = (s + t_1 + a + p - \gamma)/16(> 0) \) there exists \( \eta_0 \) such that

\[
t_1 - \gamma - \gamma \epsilon_1 \leq \eta \{ \varrho(\eta)H_i^\gamma(\eta)/\|\varrho(\eta)H_i^\gamma(\eta)\| \} \leq t_2
\]

for all \( \eta \geq \max(\eta_0, r_1) \). Then for \( \epsilon = \min(1, s + t_1 + a + p - \gamma)/4(> 0) \), (2.13) follows from Theorem 2.2.

For \( \lambda_1 = \max(\eta_0, r_1) \),

\[
\frac{1}{H_i(\lambda_1)\varrho(\lambda_1)} \int_0^{\lambda_1} r^{p+a-1}|H_i^\gamma(r)\varrho(r)|dr \leq \frac{1}{H_i(\lambda_1)\varrho(\lambda_1)} \int_0^{\lambda_1} r^{p+a-1}|\varrho(r)|dr
\]

which implies that \( C \) does not depend on \( i \). \qed

### 3. Admissibility

In this section, we give a sufficient condition for admissibility of the generalized Bayes estimator with respect to an elliptically symmetric prior density \( \varrho(\theta) = G(\|\theta\|) \). The assumptions on \( G \) are the following.

- **G1** \( G(\eta) \) is continuously differentiable in \( \eta > 0 \). There exist \( r_1 > 0 \), \( t_1 \leq t_2 \) such that \( t_1 \leq \eta G'(\eta)/G(\eta) \leq t_2 \) for all \( \eta \geq r_1 \).

- **G1’** By redefining \( r_1 \) if necessary in **G1**, \( t_1 \) and \( t_2 \) can be taken as same sign.

- **G1”** \( G'(\eta) \) is continuously differentiable in \( \eta > 0 \). There exist \( t_3 \leq t_4 \) such that \( t_3 \leq \eta G''(\eta)/G'(\eta) \leq t_4 \) for all \( \eta \geq r_1 \).

- **G2** \( \int_0^\eta \eta^{p-1}G(\eta)d\eta = \infty \) and there exists \( 0 < \gamma \leq 2 \) such that \( \int_1^\infty \eta^{p-1}G(\eta)H_i^\gamma(\eta)d\eta < \infty \).

- **G3** \( \lim_{\eta \to 0} \eta G'(\eta)/G(\eta) = t_0(> 1 - p) \).

- **FG1** \( \int_0^\infty r^{p-2}f(r)G(r)dr < \infty \) and \( \int_0^\infty r^{p-2}F(r)G(r)dr < \infty \).
We discuss some implications of these assumptions. By the assumption \( \mathbf{G3} \),
\[
\int_0^1 \eta^{p-1}G(\eta)d\eta < \infty \quad \text{and} \quad \int_0^1 \eta^{p-1}|G'(\eta)|d\eta < \infty.
\]
From the former integrability and \( \mathbf{G2} \), the improperness of \( g \) occurs only at infinity. By 2 of Theorem 2.1 and the assumption \( \mathbf{G2} \), \( g(\theta)H^1_\gamma(\|\theta\|_d) \) for any fixed \( i \) is integrable and hence becomes a proper probability density by standardization. Since \( H_i(\cdot) \) approaches 1 as \( i \to \infty \), \( g(\theta)H^1_\gamma(\|\theta\|_d) \) is a sequence of proper densities approaching \( g(\theta) \), which is essential for using Blyth’ method.

By \( \mathbf{G1} \) and Lemma 2.2, \( G(\eta) = O(\eta^2) \). Therefore if \( t_2 < -p \), then \( g(\theta) = G(\|\theta\|_d) \) is a proper prior. Since we are considering an improper \( g(\theta) \), we assume \( t_2 \geq -p \) from now on. Moreover \( \eta^{t_1}/G(\eta) = O(1) \) by the assumption \( \mathbf{G1} \). Since \( H_1 \in \text{RV}_1 \), \( \eta^{p-1}\eta^{t_1}H^2_\gamma(\eta) \) for \( t_1 > 2 - p \) is not integrable at infinity and \( \eta^{p-1}G(\eta)H^2_\gamma(\eta) \) is not so either. Hence we also assume \( t_1 \leq 2 - p \). We now discuss when the integrability of \( \int_0^1 \eta^{p-1}G(\eta)H^2_\gamma(\eta)d\eta \) in \( \mathbf{G2} \) holds and the relationship with \( \mathbf{G1} \). When we take \( \beta(\eta) \) as in 2.2, we easily see that there exists \( L_1 \) such that \( H_1(\eta) \leq L_1(\eta + c)^{-1}\{\log(\eta + c)\}^{-1} \). If \( t_2 \leq 2 - p \), there clearly exists \( L_2 \) such that \( G(\eta) \leq L_2\eta^{2-p} \) for \( \eta > 1 \). Hence
\[
\int_1^\infty \eta^{p-1}G(\eta)H^2_\gamma(\eta)d\eta \leq L_1^2L_2 \int_1^\infty (\eta + c)^{-1}\{\log(\eta + c)\}^{-2}d\eta
\]
\[
= L_1^2L_2(\log(1 + c))^{-1} < \infty,
\]
which shows that there exists \( \gamma (= 2) \) for the integrability in \( \mathbf{G2} \). On the other hand, if \( t_2 > 2 - p \) then the integrability in \( \mathbf{G2} \) may not be apparent. But the condition \( t_2 \leq 2 - p \) is not a necessary condition for the integrability.

If \( G(\eta) \) is regularly varying, then for any \( \epsilon > 0 \), we can choose \( r_1, t_1, t_2, t_3, t_4, \) such that \( t_2 - t_1 < \epsilon \) and \( t_4 - t_3 < \epsilon \) for all \( r_1 \). However in \( \mathbf{G1} \) we are allowing the case that \( \liminf_{\eta \to -\infty} \eta G'(\eta)/G(\eta) \) is strictly less than \( \limsup_{\eta \to -\infty} \eta G'(\eta)/G(\eta) \). Hence we are dealing with a broader class of \( G(\eta) \) than the class of regularly varying functions. It should also be noted that \( t_4 \) and \( t_3 \) are not always smaller than \( t_2 \) and \( t_1 \), respectively. See Geluk and de Haan (1987) for the detail.

The generalized Bayes estimator \( \delta_g \) with respect to the improper density \( g(\theta) \) is written as
\[
\delta_g(x) = \frac{\int_{R^p} \theta f(||x - \theta||)g(\theta)d\theta}{\int_{R^p} f(||x - \theta||)g(\theta)d\theta}
\]
\[
= x + \frac{\int_{R^p} (\theta - x)f(||x - \theta||)g(\theta)d\theta}{\int_{R^p} f(||x - \theta||)g(\theta)d\theta}
\]
\[
= x + \frac{\int_{R^p} F(||x - \theta||)\nabla g(\theta)d\theta}{\int_{R^p} f(||x - \theta||)g(\theta)d\theta}, \quad (3.1)
\]
which is well-defined if both \( \int_{R^p} F(||x - \theta||)\nabla g(\theta)d\theta \) and \( \int_{R^p} f(||x - \theta||)g(\theta)d\theta \) are integrable for all \( x \). These are guaranteed by the assumption \( \mathbf{FG1} \) and Lemma 3.1 in the below. Write
\[
m(\psi | x) = \int_{R^p} \psi(\theta)f(||\theta - x||)d\theta
\]
Lemma 3.1. \[ M(\psi|x) = \frac{1}{C_f} \int_{R^p} \psi(\theta)F(\|\theta - x\|)d\theta \]

where \( C_f = \{\pi^{p/2}/\Gamma(p/2+1)\} \int_0^\infty z^{p+1} f(z)dz \). Notice that \( F(\cdot)/C_f \) is a probability density function because

\[
\int_{R^p} \|y - \theta\|^\alpha F(\|y - \theta\|)dy = \int_{R^p} \|y\|^\alpha \{ \int_{|y|} \rho(\|y\|)sds \}dy \\
= c_p \int_0^\infty r^{p-1+\alpha} \int_r^\infty sf(s)dsdr \\
= c_p \int_0^\infty r^{p+1+\alpha} \int_1^\infty tf(rt)dt\alpha \\
= c_p \int_1^\infty t\{ \int_0^\infty r^{p+1+\alpha} f(rt)dr \}dt \\
= c_p \int_1^\infty t^{-p-\alpha} dt \cdot \int_0^\infty z^{p+1+\alpha} f(z)dz \\
= \frac{c_p}{p + \alpha} \int_0^\infty z^{p+1+\alpha} f(z)dz.
\]

Then \( \delta_g \) is written as

\[ \delta_g(x) = x + C_f \frac{M(\nabla g|x)}{m(g|x)}. \]

Note that by \( G1 \) the \( j \)-th element of \( \nabla g \) is given by

\[ \nabla_j g(\theta) = d_j^2 \theta_j \frac{C'(\|\theta\|_d)}{\|\theta\|_d}. \]

We also write

\[ h_i(x) = H_i(\|x\|_d). \]

Now we state the following lemma in preparation of our main theorem.

**Lemma 3.1.**

1. Assume \( G1, G3 \) and \( F1 \). Then

\[
m(g|x) \approx g(x) \quad \text{if } s > \max(1, -t_1 - p, t_2) \quad (3.2)
\]
\[
m(gh_i|x) \approx g(x)h_i(x) \quad \text{if } s > \max(1, \gamma - t_1 - p, t_2) \quad (3.3)
\]
\[
M(gh_i|x) \approx g(x)h_i(x) \quad \text{if } s > 2 + \max(1, \gamma - t_1 - p, t_2) \quad (3.4)
\]
\[
M(\|\theta\|_{d-1}^{-1}|x) \approx g(x)|x|_{d-1}^{-1} \quad \text{if } s > 2 + \max(1, 1 - t_1 - p, t_2 - 1) \quad (3.5)
\]
\[
M(gh_i^\gamma \|\theta\|_{d-1}^{-1}|x) \approx g(x)h_i(x)|x|_{d-1}^{-1} \quad \text{if } s > 2 + \max(1, \gamma + 1 - t_1 - p, t_2 - 1). \quad (3.6)
\]

2. Assume \( G1', G1'', G3 \) and \( F1 \). Then

\[
M(\nabla_j g|x) \approx \nabla_j g(x) \quad \text{if } s > 2 + \max(1, -t_3 - p, t_4) \quad (3.7)
\]
\[
M(\nabla_j gh_i|x) \approx \nabla_j g(x)h_i(x) \quad \text{if } s > 2 + \max(1, \gamma - t_3 - p, t_4). \quad (3.8)
\]
3. \(M(\nabla g|x)/m(g|x)\) is bounded in \(x\) if \(s > 2 + \max(1, 1 - t_1 - p, t_2 - 1)\).

Proof. When we consider the asymptotic behavior of \(M(\cdot|x)\), that is, the expectation under the probability density \(F(\|\theta - x\|)/C_f\), we have only to substitute \(s\) for \(s - 2\) in order to have corresponding results for Theorem 2.2 and Corollary 2.1 because under the assumption \(F1\) there exists \(L_1\) such that \(r^{p+s-2}F(r) < L_1\) for all \(r \geq r_0\). We easily see that (3.2), (3.5) and (3.7) follow from Theorem 2.2 and that (3.3), (3.4), (3.6) and (3.8) follow from Corollary 2.1.

Note that by the assumptions \(G1\) and \(G3\) there exists \(L_2\) such that \(|\eta G'(\eta)/G(\eta)| < L_2\) for all \(\eta > 0\). Then \(M(\nabla g|x)/m(g|x) \leq d_1 M(G'\|x\|)/m(g|x) \leq L_2 d_1 M(G/\|\theta\|\|x\|)/m(g|x)\).

The value at \(x = 0\) is clearly bounded under the assumption \(FG1\) and the value at \(\|x\|_d \to \infty\) is bounded by (3.2) and (3.5) if \(s > 2 + \max(1, 1 - t_1 - p, t_2 - 1)\).

By part 3 of Lemma 3.1, \(M(\nabla g|x)/m(g|x)\) is bounded in \(x\) and hence the risk function of \(\delta_g\) is finite because

\[
R(\theta, \delta_g) = E[\|X - \theta + C_f M(\nabla g|X)/m(g|X)\|_Q^2] \\
\leq Q_{\text{max}} E[\|X - \theta + C_f M(\nabla g|X)/m(g|X)\|^2] \\
\leq 2Q_{\text{max}} \{E[\|X - \theta\|^2] + C_f^2 E[M(\nabla g|X)/m(g|X)]^2\},
\]

where \(Q_{\text{max}}\) is the largest eigenvalue of \(Q\).

Now we state the main theorem of this paper.

**Theorem 3.1.**

1. Assume \(G1, G2, G3, F1, FG1\). Then the generalized Bayes estimator with respect to \(g\) is admissible if \(s > 2 + \max(1, 1 + \gamma - t_1 - p)\) and \(t_2 < 2 - p\).

2. We also assume \(G1'\) and \(G1''\). Then the generalized Bayes estimator with respect to \(g\) is admissible if \(s > 2 + \max(1, 3 - t_1 - p, t_2, 2 - t_3 - p, t_4)\) and \(t_2 \geq 2 - p\).

Although the moment conditions for \(s\) in the theorem above looks complicated, it is just from the assumptions \(G1, G1'\) and \(G1''\) which make our class of \(G\) broader than the class of regularly varying functions. We see that the condition reduces to \(s > 3\) for regularly varying functions \(G\). Before giving a proof of the main theorem we present it as a corollary.

**Corollary 3.1.** Suppose that \(G(\eta)\) is regularly varying with index \(k\) for \(-p \leq k \leq 2 - p\). Assume \(G3, F1\) with \(s > 3\) and \(FG1\).

1. Assume \(-p \leq k < 2 - p\) and \(G(\eta)\) is continuously differentiable. Then the generalized Bayes estimator with respect to \(g\) is admissible.

2. Assume \(k = 2 - p\) and \(G(\eta)\) satisfies

\[
G(\eta) \leq \eta^{2-p}\{\int_\eta^\infty \beta(r)dr\}^2\{\eta\beta(\eta)\}^{-1} \text{ for } \eta \geq 1,
\]

and is twice continuously differentiable. Then the generalized Bayes estimator with respect to \(g\) is admissible.
Proof. When \(-p \leq k < 2 - p\), we can take \(t_1 = k - \epsilon/3\) and \(t_2 = k + \epsilon/3\) for any \(0 < \epsilon < \{3/2\}(2 - p - k)\) in \(G_1\). Let \(\gamma = k + p + 2\epsilon/3\). Clearly \(0 < \gamma < 2\) and it satisfies the integrability in \(G_2\). Since \(t_2 < 2 - p\), we have only to apply part 1 of Theorem 3.1 for this paper.

Next we consider the case \(k = 2 - p\). Note that there exists \(L_1\) such that \(H_1(\eta) \leq L_1\beta(\eta)/\int_0^\infty \beta(r)dr\) by part 2 of Lemma 2.1. So if \(G(\eta)\) satisfies (3.9), we have

\[
\int_1^\infty \eta^{p-1}G(\eta)H_1^2(\eta)d\eta \leq L_1 \int_1^\infty \beta(\eta)d\eta < \infty,
\]

which shows that the integrability in the assumption \(G_2\) is guaranteed. We can take \(t_1 = 2 - p - \epsilon\), \(t_2 = 2 - p + \epsilon\), \(t_3 = 1 - p - \epsilon\) and \(t_4 = 1 - p + \epsilon\) for any \(\epsilon > 0\) in the assumptions \(G_1\), \(G_1^*\). Since \(t_2 \geq 2 - p\), we have only to apply part 2 of Theorem 3.1. Since

\[3 - t_1 - p = 1 + \epsilon, \quad t_2 \leq 1 + \epsilon, \quad 2 - t_3 - p = 1 + \epsilon, \quad t_4 \leq \epsilon\]

the moment condition for \(k = 2 - p\) is \(s > 3\).

In particular the boundary case in (3.9) by taking \(\beta(\eta)\) in (2.2) was the motivating one for this paper.

Corollary 3.2. Assume \(F1\) with \(s > 3\) and \(FG1\). Then the generalized Bayes estimator with respect to \(||\theta||^2 - p \prod_{i=0}^n \log_i(||\theta|| + c)\), where \(n\) is a nonnegative integer and \(\log_n(c) > 0\), is admissible.

In the normal case, Brown (1971)'s sufficient conditions for admissibility and inadmissibility are known. He showed that the generalized Bayes estimator with respect to \(g\) is admissible if

\[
\int_{||x||_d > 1} ||x||^{2 - 2p} m(g|x) - 1\,dx
\]

(3.10)
diverges and inadmissible if (3.10) converges. By Lemma 3.1 we see that

\[
(1/2)G(||x||_d) < m(g|x) < 2G(||x||_d)
\]

for sufficiently large \(||x||_d\) and hence that \(G(\eta) \leq \eta^{2-p} \prod_{i=0}^n \log_i(\eta + c)\) leads to admissibility and \(G(\eta) \geq \eta^{2-p} \prod_{i=0}^n \log_i(\eta + c)\log^2_n(\eta + c)\) leads to inadmissibility. Therefore our sufficient condition in Theorem 3.1 is very close to being necessary.

We also notice that the prior density suggested in Corollary 3.2 becomes \(|\theta| \prod_{i=0}^n \log_i(|\theta| + c)\) and \(\prod_{i=0}^n \log_i(||\theta||_d + c)\) for \(p = 1, 2\) respectively, which are thicker than the Lebesgue measure. In the normal case, Brown (1971) has already pointed it out.

Furthermore we indicate that our moment condition \(s > 3\) is very tight because, as pointed out in Perng (1970), admissibility requires the existence of moment one degree higher than what is needed for finite risk in various estimation problems.

Now we give a proof of Theorem 3.1.
Proof of Theorem 3.1 Let \( \delta_{gi} \) denote the Bayes estimator with respect to the proper prior density \( g(\theta)h_i^\gamma(\theta) \). Then the Bayes risk difference of \( \delta_g \) and \( \delta_{gi} \) with respect to the density \( g(\theta)h_i^\gamma(\theta) \) is written as

\[
\Delta = \int_{\mathbb{R}^p} [R(\theta, \delta_g) - R(\theta, \delta_{gi})] g(\theta)h_i^\gamma(\theta) d\theta 
\]

\[
= \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} \left[ \|\delta_g - \theta\|^2_Q - \|\delta_{gi} - \theta\|^2_Q \right] f(||x - \theta||) g(\theta)h_i^\gamma(\theta) d\theta dx 
\]

\[
= \int_{\mathbb{R}^p} \left[ \|\delta_g\|^2_Q - \|\delta_{gi}\|^2_Q \right] \int_{\mathbb{R}^p} f(||x - \theta||) g(\theta)h_i^\gamma(\theta) d\theta 
\]

\[
- 2(\delta_g - \delta_{gi})Q' \int_{\mathbb{R}^p} \theta f(||x - \theta||) g(\theta)h_i^\gamma(\theta) d\theta \right\} dx 
\]

\[
= \int_{\mathbb{R}^p} \|\delta_g - \delta_{gi}\|^2_Q \left\{ \int_{\mathbb{R}^p} f(||x - \theta||) g(\theta)h_i^\gamma(\theta) d\theta \right\} dx 
\]

\[
= C_j^2 \int_{\mathbb{R}^p} \left\| \frac{M(\nabla g|x)}{m(g|x)} - \frac{M(\nabla gh_i^\gamma|x)}{m(gh_i^\gamma|x)} \right\|_Q^2 m(gh_i^\gamma|x) dx 
\]

\[
= C_j^2 \int_{\mathbb{R}^p} \left[ \frac{\nabla g|x}{m(g|x)} - \frac{\nabla gh_i^\gamma|x}{m(gh_i^\gamma|x)} \right] m(gh_i^\gamma|x) dx 
\]

In the same way as in [Brown and Hwang (1982)], we have

\[
\Delta \leq 2C_j^2 Q_{\text{max}} \int_{\mathbb{R}^p} \left\| \frac{M(\nabla g|x)}{m(g|x)} - \frac{M(\nabla gh_i^\gamma|x)}{m(gh_i^\gamma|x)} \right\|_Q^2 m(gh_i^\gamma|x) dx 
\]

\[
+ 2C_j^2 Q_{\text{max}} \int_{\mathbb{R}^p} \left\| \frac{M g(gh_i^\gamma|x)}{m(gh_i^\gamma|x)} \right\|_Q^2 m(gh_i^\gamma|x) dx 
\]

\[
= 2C_j^2 Q_{\text{max}} (B_i + A_i). \quad \text{(say)} 
\]

Using the Cauchy-Schwartz inequality for \( A_i \), we have

\[
A_i = \gamma^2 \int_{\mathbb{R}^p} \left\| \frac{M(gh_i^{\gamma-1}\nabla h_i|x)}{m(gh_i^\gamma|x)} \right\|^2 \left\{ m(gh_i^\gamma|x) \right\}^{-1} dx 
\]

\[
\leq \gamma^2 \int_{\mathbb{R}^p} \frac{M(g|x)}{m(gh_i^\gamma|x)} M(gh_i^{\gamma-2}|\nabla h_i|^2|x) dx 
\]

\[
\leq \gamma^2 \int_{\mathbb{R}^p} \frac{M(g|x)}{m(gh_i^\gamma|x)} M(gh_i^{\gamma-2}|\nabla h_i|^2|x) dx 
\]

for \( 0 < \gamma \leq 2 \). The ratio \( M(gh_i^\gamma|x)/m(gh_i^\gamma|x) \) is bounded from above by \( M(g|x)/m(gh_i^\gamma|x) \) and hence the value at \( x = 0 \) is clearly bounded under the assumption FG1. By (3.3) and (3.4), we have

\[
\lim_{\|x\| \to \infty} \frac{M(gh_i^\gamma|x)}{m(gh_i^\gamma|x)} = 1 
\]
uniformly in $i$. This implies that there exists $c_1$ such that $M(gh_i^2|x)/m(gh_i^2|x) < c_1$ for all $x$ and for all $i$. Then

$$A_i \leq \gamma^2 c_1 \int_{R^p} M(gh_i^{-2}\|\nabla h_i\|^2|x)dx$$

$$= \gamma^2 c_1 \int_{R^p} \{1/C_f\}F(\|x - \theta\|)dx \int_{R^p} g(\theta)h_i^{-2}(\theta)\|\nabla h_i(\theta)\|^2d\theta$$

$$= \gamma^2 c_1 \int_{R^p} g(\theta)h_i^{-2}(\theta)\|\nabla h_i(\theta)\|^2d\theta.$$  

By 4 of Theorem 2.1 we have $\|\nabla h_i(\theta)\| < 2d_1\beta(\|\theta\|_d)/\int_0^{\infty} \beta(r)dr$ and together with 2 of Theorem 2.1 $\|\nabla h_i(\theta)/h_i(\theta)\|$ for all $\theta$ is bounded from above by $L_3$ independent of $i$. Therefore

$$A_i \leq \gamma^2 c_1 L_3^2 \int_{R^p} g(\theta)h_i(\theta)d\theta,$$

which is bounded by the assumption G2. Furthermore $\|\nabla h_i(\theta)\| \to 0$ as $i \to \infty$ by 3 of Theorem 2.1. Therefore by the dominated convergence theorem $A_i$ converges to 0 as $i \to \infty$.

Next we consider $B_i$. $M(\nabla g|x)$ and $M(\nabla gh_i^2|x)$ at $x = 0$ are zero vectors because $g$ and $h_i^2$ are function of $\|\theta\|_d$. So the integrand of $B_i$ is bounded around $x = 0$. For the asymptotic property of the integrand of $B_i$, we need to distinguish two cases: $t_2 < 2 - p$ and $t_2 \geq 2 - p$. When $t_2 < 2 - p$, we can bound the norm in the integrand of $B_i$ from above somewhat roughly. Using (3.3), (3.5) and (3.6) in Lemma 3.1 and noting that $|\eta G'(|\eta|)/G(|\eta|)| < L_2$ by the assumptions G1 and G3 we have

$$\frac{1}{d_i^2} \left| \frac{M(\nabla_j g|x)}{m(g|x)} - \frac{M(\nabla_j gh_i^2|x)}{m(gh_i^2|x)} \right| < \frac{M(|G'||x)}{m(g|x)} + \frac{M(|G'|h_i^2|x)}{m(gh_i^2|x)}$$

$$< L_2 \left( \frac{M(G||\theta|-1|x)}{m(g|x)} + \frac{M(Gh_i^2||\theta|-1|x)}{m(gh_i^2|x)} \right)$$

$$< c\|x\|_d^{-1}$$

for all sufficiently large $\|x\|_d$ and for all $i$. When $t_2 \geq 2 - p$, we have to bound it from above more strictly. By (3.2), (3.3), (3.5) and (3.8) in Lemma 3.1 we have

$$\frac{1}{d_i^2} \left| \frac{M(\nabla_j g|x)}{m(g|x)} - \frac{M(\nabla_j gh_i^2|x)}{m(gh_i^2|x)} \right|$$

$$= \left| \frac{G'(|x|_d)x_j + O(|x|_d^{1-\epsilon_0})}{G(|x|_d)|x|_d} \right| - \frac{G'(|x|_d)x_j + O(|x|_d^{1-\epsilon_0})}{G(|x|_d)|x|_d}$$

$$< c\|x\|_d^{-1-\epsilon_0}$$

$$< H_1(\|x\|_d)$$
for some $\epsilon_0$, for all sufficiently large $\|x\|_d$ and for all $i$. Moreover $m(gh_i^d|x) \leq m(g|x)$ and $m(g|x) < 2G(\|x\|_d)$ for all sufficiently large $\|x\|_d$ by (3.2). Therefore there exist $C_1, C_2, C_3$ and $C_4$ such that the integrand of $B_i$ is less than

\[
\begin{cases}
\min\{C_1, C_2\|x\|_d^{2+t_2}\} & \text{if } t_2 < 2 - p \\
\min\{C_3, C_4G(\|x\|_d)H_4^2(\|x\|_d)\} & \text{if } t_2 \geq 2 - p.
\end{cases}
\]

Therefore $B_i$ converges to 0 as $i \to \infty$ by the dominated convergence theorem.

Finally we confirm that we use (3.2)–(3.6) for $t_2 < 2 - p$ and (3.2)–(3.4), (3.7) and (3.8) for $t_2 \geq 2 - p$. Note also max$(1, t_2) = 1$ for $t_2 < 2 - p$ in the moment condition.

4. The generalized Bayes estimator with respect to the harmonic prior and its minimaxity

In this section, we show that the generalized Bayes estimator with respect to the harmonic prior has a form simple enough to check some sufficient conditions for minimaxity under the quadratic loss function $L_f(\theta, \delta) = \|\delta - \theta\|^2$ given in early studies. We demonstrate that it is minimax for some $f$.

In (3.1), the generalized Bayes estimator can be also written as

$$
\delta_\theta(x) = x + C_f \nabla_x M(g|x) \frac{m(g|x)}{m(g|x)}.
$$

For $p \geq 3$ and $g(\theta) = \|\theta\|^{2-p}$, we have

\[
m(g|x) = \int_{R^p} f(\|x - \theta\|)\|\theta\|^{2-p}d\theta = \int_{R^p} f(\|\eta\|)\|x - \eta\|^{2-p}d\eta
\]

\[
= c_{p-1} \int_0^\infty \int_0^\pi f(\lambda)(\lambda^2 + 2\lambda r \cos \varphi + r^2)\lambda^{p-1} \sin^{p-2}\varphi d\lambda d\varphi
\]

\[
= c_{p-1} r^2 \int_0^\infty \int_0^\pi f(rt)(1 + 2t \cos \varphi + t^2)\lambda^{p-1} \sin^{p-2}\varphi dt d\varphi
\]

\[
= c_p \left( r^2 \int_0^1 t^{p-1} f(rt)dt + r^2 \int_1^\infty t f(rt)dt \right)
\]

\[
= c_p \left( \left[-t^{p-2}F(rt)\right]_0^1 + (p - 2) \int_0^1 t^{p-3}F(rt)dt + F(r) \right)
\]

\[
= c_p(p - 2) \int_0^1 t^{p-3}F(rt)dt,
\]

where $r = \|x\|$. The fifth equality in the above equation follows from the relation

\[
\int_0^\pi (1 + 2t \cos \varphi + t^2)\lambda^{p-1} \sin^{p-2}\varphi d\varphi = B(p/2 - 1/2, 1/2) \min(t^{2-p}, 1),
\]
which is proved in Lemma 4.1 in the end of this section. In the same way, we have
\[ C_f \nabla_x M(g(x)) = -x c_p(p - 2) \int_0^1 t^{p-3} F(rt) dt. \]

Hence the generalized Bayes estimator is written as \( \delta_*(X) = (1 - \phi_*(\|X\|)/\|X\|^2)X \), where
\[ \phi_*(r) = r^2 \int_0^1 t^{p-3} F(rt) dt / \int_0^1 t^{p-1} F(rt) dt. \]

Some properties of the behavior of \( \phi_*(r) \) are easily derived as follows.

**Theorem 4.1.**
1. \( \lim_{r \to \infty} \phi_*(r) = (p - 2)E_0(\|X\|^2)/p. \)
2. \( \phi_*(r) \) is nondecreasing in \( r \) for any \( f \).
3. \( \phi_*(r)/r^2 \) is nonincreasing in \( r \) if \( F(t)\{t^2 f(t)\}^{-1} \) is nonincreasing.

**Proof.** \( \phi_*(r) \) can be written as \( \int_0^1 t^{p-1} F(t) dt / \int_0^1 t^{p-3} F(t) dt \) and we have
\[ \lim_{r \to \infty} \phi_*(r) = \int_0^\infty t^{p-1} F(t) dt / \int_0^\infty t^{p-3} F(t) dt = p - 2 \int_0^\infty t^{p+1} f(t) dt / \int_0^\infty t^{p-1} f(t) dt = p - 2 E_0(\|X\|^2). \]

The derivative of \( \phi_*(r) \) is calculated as
\[ \phi'_*(r) = \frac{r^{p-3} F(r)}{(\int_0^r t^{p-3} F(t) dt)^2} \int_0^r (r^2 - t^2) t^{p-3} F(t) dt, \]
which is nonnegative for any \( f \). The derivative of \( \phi_*(r)/r^2 \) is calculated as
\[ \frac{d}{dr} \left( \phi_*(r)/r^2 \right) = r \left( \int_0^1 t^{p-3} F(rt) dt \right)^{-2} \left( \int_0^1 t^{p-1} F(rt) dt \int_0^1 t^{p-1} f(rt) dt \right. \]
\[ - \int_0^1 t^{p+1} f(rt) dt \int_0^1 t^{p-3} F(rt) dt. \]

If \( F(t)\{t^2 f(t)\}^{-1} \) is nonincreasing, the right-hand side of the equality above is nonpositive by the covariance inequality.

Now we consider the minimaxity of \( \delta_* \). We present a brief list of known sufficient conditions for minimaxity given in previous papers, for the estimator of the form \( \phi_*(X) \) with nonnegative and nondecreasing \( \phi(r) \).

| Author                  | \( p \) | \( \phi(r)/r^2 \) | upper bound of \( \phi \) |
|-------------------------|--------|-------------------|-----------------------------|
| general                 |        |                   | \( 2(p - 2) \inf_{s \in T} F(s)/f(s) \) |
| Berger (1975)           | \( p \geq 3 \) | \( \infty \) | \( 2(p - 2)(p E_0(\|X\|^{-2}))^{-1} \) |
| Brandwein (1978)        | \( p \geq 4 \) | \( \infty \) | \( 2(p - 2)(p E_0(\|X\|^{-2}))^{-1} \) |
| Brandwein and Strawderman (1978) | \( p \geq 4 \) | \( \infty \) | \( 2 p (p + 2) E_0(\|X\|^{-2})^{-1} \) |
| Ralescu et al. (1992)   | \( p = 3 \) | \( \infty \) | \( 0.93 (E_0(\|X\|^{-2}))^{-1} \) |
| F(t)/f(t) is nondecreasing |        |                   | \( 2 E_0(\|X\|^{-2})^{-1} \) |
| Back (1985)             | \( p \geq 4 \) | \( \infty \) | \( 2 E_0(\|X\|^{-2})^{-1} \) |
| scale mixtures of multivariate normal |        |                   | \( 2 E_0(\|X\|^{-2})^{-1} \) |
| Strawderman (1974)      | \( p \geq 3 \) | \( \infty \) | \( 2 E_0(\|X\|^{-2})^{-1} \) |
In the table, $U = \{ t \geq 0 | f(t) > 0 \}$ and an arrow $\nabla$ means nonincreasing. It is noted that $f(t)$ is nonincreasing in $t$ if $F(t)/f(t)$ is nondecreasing in $t$ and that $F(t)/f(t)$ is nondecreasing in $t$ if $f$ is a scale mixtures of multivariate normal.

Combining Theorem 4.1 and the table above, we can derive a sufficient condition for minimaxity of $\delta_*(X)$ and we state it in the following theorem for $p \geq 4$.

**Theorem 4.2.**

1. Assume $t^{-2}F(t)/f(t)$ is nonincreasing.

   (a) $\delta_*$ is minimax if $E_0[\|X\|^2]E_0[\|X\|^{-2}] \leq 2$.

   (b) Assume also $f(t)$ is nonincreasing. Then $\delta_*$ is minimax if $(p^2 - 4)E_0[\|X\|^2]E_0[\|X\|^{-2}] \leq 2p^2$.

   (c) Assume also $F(t)/f(t)$ is nondecreasing. Then $\delta_*$ is minimax if $(p - 2)E_0[\|X\|^2]E_0[\|X\|^{-2}] \leq 2p$.

2. Assume $0 < \inf_{s \in U} F(s)/f(s) < \infty$. Then $\delta_*$ is minimax if $E_0[\|X\|^2] \leq 2p \inf_{s \in U} F(s)/f(s)$.

   [Berger (1975) and Bock (1985)] gave several examples of $f$, checked the monotonicity of $f(t)$, $F(t)/f(t)$, and $t^{-2}F(t)/f(t)$ and calculated an upper bound of $\phi(r)$. In this paper we give just two examples but we believe that the estimator $\delta_*(X)$ is minimax for a broad class of spherically symmetric distributions.

**Example 4.1.** We consider $f(s) = s^\alpha \exp(-\beta s^2)$ for $\alpha, \beta > 0$. We have

$$\frac{F(t)}{t^2 f(t)} = \int_1^\infty u^{\alpha + 1} \exp(\beta t^2 (1 - u)) du,$$

which is decreasing in $t$. By an integration by parts, we have

$$\frac{F(t)}{f(t)} = \frac{1}{2\beta} + \frac{\alpha}{2\beta} \int_1^\infty s^{\alpha - 1} \exp(-\beta s^2) ds$$

$$= \frac{1}{2\beta} + \frac{\alpha}{2\beta} \int_1^\infty u^{\alpha - 1} \exp(\beta t^2 (1 - u^2)) du$$

and hence $\inf_{t \geq 0} F(t)/f(t) = (2\beta)^{-1}$. We also have $E_0(\|X\|^2) = (p/2 + \alpha/2)/\beta$ and $E_0(\|X\|^{-2})^{-1} = (p/2 + \alpha/2 - 1)/\beta$. Therefore the generalized Bayes estimator is minimax if $\alpha \leq p$ for $p \geq 3$ by [Berger (1973)]’s conditions and if $\alpha \geq 4 - p$ for $p \geq 4$ by [Brandwein (1973)]’s conditions regardless of $\beta$. Hence the estimator for $p \geq 4$ is minimax regardless of $\alpha$ and $\beta$.

**Example 4.2.** We consider $f(t) = \exp(-t^2/2) - a \exp(-t^2/\{2b\})$ for $0 < a \leq 1$, $0 < b < 1$. Note that if $a \leq b$ then $f$ is unimodal and if $a > b$ then $f$ is not. We easily see that $\inf_{t \geq 0} F(t)/f(t) = 1$ and that $E_0[\|X\|^2] = p(1 - ab^{p/2 + 1})/(1 - ab^{p/2})$. Because $(1 - ab^{p/2 + 1})/(1 - ab^{p/2}) \leq 2$ for $0 < a \leq 1$, $0 < b < 1$, the generalized Bayes estimator is minimax by [Berger (1973)].
The following lemma is stated in a more general form in 3.036 of Gradshteyn and Ryzhik (2000), but it is incorrectly stated with an errata posted on the book’s web page. Maruyama pointed out this error and he is acknowledged in the errata for 3.036. Since a derivation of the formula is not easily accessible, we provide our own proof.

**Lemma 4.1.** For $\alpha > -\frac{1}{2}$ and $|a| < 1$,

$$\int_0^\pi (1 + 2a \cos \varphi + a^2)^{-\alpha} \sin^{2\alpha} \varphi d\varphi = B(\alpha + 1/2, 1/2).$$

**Proof.** Let $g(\varphi) = (1 + 2a \cos \varphi + a^2)^{-1} \sin^2 \varphi$. Then we have the derivative

$$g'(\varphi) = 2 \sin \varphi \frac{(a \cos \varphi + 1)(\cos \varphi + a)}{(1 + 2a \cos \varphi + a^2)^2}.$$

We see that $g(\varphi)$ is monotone increasing from $g(0) = 0$ to $g(\arccos(-a)) = 1$ and decreasing from $g(\arccos(-a)) = 1$ to $g(\pi) = 0$. Therefore we have

$$\int_0^\pi (1 + 2a \cos \varphi + a^2)^{-\alpha} \sin^{2\alpha} \varphi d\varphi = \left(\int_0^{\arccos(-a)} + \int_{\arccos(-a)}^\pi \right) (1 + 2a \cos \varphi + a^2)^{-\alpha} \sin^{2\alpha} \varphi d\varphi$$

$$= \int_0^{\arccos(-a)} (1 + 2a \cos \varphi + a^2)^{-\alpha} \sin^{2\alpha} \varphi d\varphi$$

$$+ \int_0^{\arccos(a)} (1 - 2a \cos \rho + a^2)^{-\alpha} \sin^{2\alpha} \rho d\rho$$

$$= \int_0^1 t^{\alpha}(d\varphi/dt)dt + \int_0^1 s^{\alpha}(dp/ds)ds,$$  \hspace{1cm} (4.2)

where $t = (1 + 2a \cos \varphi + a^2)^{-1} \sin^2 \varphi$ and $s = (1 - 2a \cos \rho + a^2)^{-1} \sin^2 \rho$. Here $(d\varphi/dt)$ and $(dp/ds)$ are calculated as

$$d\varphi/dt = \frac{1}{2tA(t)} \left(1 - \{at - A(t)\}^2 \right)^{1/2}$$

$$d\rho/ds = \frac{1}{2sA(s)} \left(1 - \{as + A(s)\}^2 \right)^{1/2},$$

where $A(t) = (1 - t)^{1/2}(1 - a^2 t)^{1/2}$. Let

$$h(t) = \frac{1}{2tA(t)} \left\{ \left(1 - \{at - A(t)\}^2 \right)^{1/2} + \left(1 - \{at + A(t)\}^2 \right)^{1/2} \right\}.$$  \hspace{1cm} 

Then we have $h^2(t) = \{2tA(t)^{-2}(2 - 2a^2 t^2 - 2A(t)^2 + 2B(t))$, where

$$B(t) = \left(1 - \{at - A(t)\}^2 - \{at + A(t)\}^2 + \{a^2 t^2 - A^2(t)\}^2 \right)^{1/2}.$$
\[ = t(1 - a^2), \]

which implies \( h(t) = t^{-1/2} (1 - t)^{-1/2} \). Therefore we get

the right hand side of \( (4.2) = \int_0^1 t^\alpha h(t) dt = B(\alpha + 1/2, 1/2). \)

\[ \square \]

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