ON GENERA OF QUADRATIC FORMS

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Abstract. We compute the genus of a rational quadratic form in terms of an invariant of the $C^*$-algebra $\mathcal{A}$ attached to the adelic orthogonal group of the form. Using the $K$-theory of $\mathcal{A}$, one gets a higher composition law for the quadratic forms. In particular, the Gauss composition is a special case corresponding to the binary quadratic forms.

1. Introduction

The binary quadratic forms $q(u,v) = \{au^2 + buv + cv^2 \mid a, b, c \in \mathbb{Z}\}$ were studied by C.-F. Gauss. Two forms $q$ and $q'$ are said to be equivalent, if a substitution $\{u = \alpha u' + \beta v', v = \gamma u' + \delta v' \mid \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \alpha\delta - \beta\gamma = 1\}$ transforms $q$ into $q'$. It is easy to see that the discriminant $\Delta = b^2 - 4ac$ of the form $q(u,v)$ is an invariant of the substitution and, therefore, the equivalent forms have the same discriminant. But a converse statement is false in general. Gauss showed that there exists a finite number of the pairwise non-equivalent binary quadratic forms having the same discriminant. Moreover, the equivalence classes make an abelian group $\mathcal{G}$ under a composition defined on these forms [Cassels 1978] [8, Chapter 14]. The cardinality $g = |\mathcal{G}|$ of such a group is called the genus of $q(u,v)$. An extension of the Gauss composition to the general quadratic forms $q(x) := q(x_1, \ldots, x_n) = \{\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \mid a_{ij} \in \mathbb{Z}, n \geq 1\}$ is a difficult and important problem [Bhargava 2004] [1], [2], [3] and [Bhargava 2008] [4]. We refer the reader to the excellent survey [Bhargava 2006] [5].

Let $G(K)$ be a non-compact reductive algebraic group defined over a number field $K$. Denote by $\mathbb{A}$ the ring of adeles of $K$ and by $\mathbb{A}_\infty$ a subring of the integer adeles. It is well known that $G(K)$ is a discrete subgroup of $G(\mathbb{A})$ and the double cosets $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$ is a finite set. The $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$ is an important arithmetic invariant of the group $G(K)$. For instance, if $G \cong O(q)$ is the orthogonal group of a quadratic form $q(x)$, then $|G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)| = g$ [Platonov & Rapinchuk 1994] [14, Chapter 8].

Consider a Banach algebra $L^1(G(\mathbb{A}) / G(K))$ of the integrable complex-valued functions on the homogeneous space $G(\mathbb{A}) / G(K)$ endowed with the operator norm. Recall that the addition of functions $f_1, f_2 \in L^1(G(\mathbb{A}) / G(K))$ is defined pointwise and multiplication is given by the convolution product:

$$ (f_1 * f_2)(g) = \int_{G(\mathbb{A}) / G(K)} f_1(gh^{-1}) f_2(h) dh. \quad (1.1) $$

By $\mathcal{A}$ we understand an enveloping $C^*$-algebra of the algebra $L^1(G(\mathbb{A}) / G(K))$; we refer the reader to [Dixmier 1977] [9, Section 13.9] for details of this construction.

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The aim of our note is a map from the set $G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K)$ to the $K$-theory of algebra $\mathcal{A}$. Since the $K$-theory deals with the abelian groups, such a map defines a group structure on the set $G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K)$. As an application, we consider the case $G \cong O(q)$, where $O(q)$ is the orthogonal group of the rational quadratic form $q(x)$. In this case, one gets a higher composition law for the quadratic forms and a formula for their genera. To formulate our results, we shall need the following definitions.

It is known that $\mathcal{A}$ is a stationary AF-algebra of rank $n = \text{rank } G(K)$ [13, Lemma 3.1]. Such an algebra is defined by an $n \times n$ integer matrix $A$ with $\det A = 1$ [Effros 1981] [10, Chapter 6]. Denote by $\mathcal{A} \rtimes \mathbb{Z}$ a crossed product $C^*$-algebra of algebra $\mathcal{A}$ taken by the shift automorphism of $\mathcal{A}$ [Blackadar 1986] [6, Exercise 10.11.9]. By $K_0(A)$ we understand the $K_0$-group of a $C^*$-algebra $A$. In particular, $K_0(\mathcal{A} \rtimes \mathbb{Z}) \cong \mathbb{Z}^n/(I - A)\mathbb{Z}^n$ and $|K_0(\mathcal{A} \rtimes \mathbb{Z})| = |\det (I - A)|$, where $I$ is the identity matrix [Blackadar 1986] [6, Theorem 10.2.1]. Our main results can be formulated as follows.

**Theorem 1.1.** There exists a one-to-one map

$$\phi : G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K) \to K_0(\mathcal{A} \rtimes \mathbb{Z}).$$

(1.2)

In particular, the map $\phi^{-1}$ defines on the set $G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K)$ the structure of an abelian group, so that $\phi$ is an isomorphism of the groups.

**Corollary 1.2.** If $G \cong O(q)$ is the orthogonal group of a rational quadratic form $q(x)$, then formula (1.2) defines a natural composition of the equivalence classes of the form $q(x)$. In particular, the genus of $q(x)$ is given by the formula $g = |\det (I - A)|$, where $A$ is an integer matrix attached to the algebra $\mathcal{A}$.

The article is organized as follows. In Section 2 we briefly review the AF-algebras, crossed products and algebraic groups over the ring of adeles. Theorem 1.1 and corollary 1.2 are proved in Section 3. An illustration of corollary 1.2 can be found in Section 4.

2. Preliminaries

The $C^*$-algebras are covered in [Dixmier 1977] [9]. For the AF-algebras we refer the reader to [Effros 1981] [10]. The K-theory of $C^*$-algebras is the subject of [Blackadar 1986] [6]. The monograph [Platonov & Rapinchuk 1994] [14] is an excellent introduction to the algebraic groups and their arithmetic.

2.1. AF-algebras.

2.1.1. $C^*$-algebras. The $C^*$-algebra is an algebra $A$ over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $||ab|| \leq ||a|| ||b||$ and $||a^*a|| = ||a||^2$ for all $a, b \in A$. Any commutative $C^*$-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on a locally compact Hausdorff space $X$; otherwise, the algebra $A$ represents a noncommutative topological space.

2.1.2. Crossed products. Let $\sigma : A \to A$ be an automorphism of the $C^*$-algebra $A$. The idea of a crossed product $C^*$-algebra $A \rtimes \mathbb{Z}$ by $\sigma$ is to embed $A$ into a larger $C^*$-algebra in which the automorphism $\sigma$ becomes an inner automorphism. Namely, let $G$ be a locally compact group and $\sigma$ a continuous homomorphism from $G$ into $\text{Aut } (A)$. A covariant representation of the triple $(A, G, \sigma)$ is a pair of
representations \((\pi, \rho)\) of \(A\) and \(G\) on a Hilbert space, such that \(\rho(g)\pi(a)\rho(g)^* = \pi(\sigma_g(a))\) for all \(a \in A\) and \(g \in G\). The covariant representation defines an algebra whose completion in the operator norm is called a crossed product \(C^*\)-algebra \(A \rtimes_\sigma G\).

2.1.3. AF-algebras. An AF-algebra (Approximately Finite \(C^*\)-algebra) is defined to be the norm closure of an ascending sequence of finite dimensional \(C^*\)-algebras \(M_n\), where \(M_n\) is the \(C^*\)-algebra of the \(n \times n\) matrices with entries in \(C\). The index \(n = (n_1, \ldots, n_k)\) represents a semi-simple matrix algebra \(M_n = M_{n_1} \oplus \cdots \oplus M_{n_k}\). The ascending sequence can be written as

\[ M_1 \overset{\varphi_1}{\rightarrow} M_2 \overset{\varphi_2}{\rightarrow} \ldots, \quad (2.1) \]

where \(M_i\) are the finite dimensional \(C^*\)-algebras and \(\varphi_i\) the homomorphisms between such algebras. If \(\varphi_i = \text{Const}\), then the AF-algebra \(A\) is called stationary. The shift automorphism of a stationary AF-algebra corresponds to the map \(i \mapsto i+1\) in (2.1) [Effros 1981] [10, p.37].

The homomorphisms \(\varphi_i\) can be arranged into a graph as follows. Let \(M_i = M_{i_1} \oplus \cdots \oplus M_{i_k}\) and \(M_{i'} = M_{i'_1} \oplus \cdots \oplus M_{i'_k}\) be the semi-simple \(C^*\)-algebras and \(\varphi_i : M_i \rightarrow M_{i'}\) the homomorphism. One has two sets of vertices \(V_{i_1}, \ldots, V_{i_k}\) and \(V_{i'_1}, \ldots, V_{i'_k}\) joined by \(a_{rs}\) edges whenever the summand \(M_{i_r}\) contains \(a_{rs}\) copies of the summand \(M_{i'_s}\) under the embedding \(\varphi_i\). As \(i\) varies, one obtains an infinite graph called the Bratteli diagram of the AF-algebra. The \(A = (a_{rs})\) is called a matrix of the partial multiplicities; an infinite sequence of \(A_i\) defines a unique AF-algebra. If \(\text{rank} (A_i) = n = \text{Const}\), then the AF-algebra is said to be of rank \(n\). If the AF-algebra \(A\) is stationary, then \(A_i = A\), where \(A\) is an \(n \times n\) matrix with the non-negative entries.

2.1.4. \(K_0\)-groups. For a \(C^*\)-algebra \(A\), let \(V(A)\) be the union of projections in the \(k \times k\) matrix \(C^*\)-algebra with entries in \(A\) taken over all \(k = 1, 2, \ldots, \infty\). The projections \(p, q \in V(A)\) are equivalent, if there exists a partial isometry \(u\) such that \(p = u^*u\) and \(q = uu^*\). The equivalence class of projection \(p\) is denoted by \([p]\). The equivalence classes of orthogonal projections can be made to a semigroup by putting \([p] + [q] = [p + q]\). The Grothendieck completion of this semigroup to an abelian group is called the \(K_0\)-group of the algebra \(A\). The functor \(A \rightarrow K_0(A)\) maps a category of the \(C^*\)-algebras into the category of abelian groups, so that projections in the algebra \(A\) correspond to a positive cone \(K_0^+ \subset K_0(A)\) and the unit element \(1 \in A\) corresponds to an order unit \(u \in K_0(A)\).

Let \(A\) be the stationary AF-algebra given by an \(n \times n\) matrix \(A\). Denote by \(\sigma : A \rightarrow A\) the shift automorphism of \(A\). Consider a crossed product \(C^*\)-algebra \(A \rtimes \mathbb{Z}\) defined by the automorphism \(\sigma\). The \(K_0\)-group of the crossed product \(A \rtimes \mathbb{Z}\) is given by the formula:

\[ K_0(A \rtimes \mathbb{Z}) \cong \frac{\mathbb{Z}^n}{(I-A)\mathbb{Z}^n}, \quad \text{where } I = \text{diag} (1, 1, \ldots, 1). \quad (2.2) \]

2.2. Algebraic groups over adeles.

2.2.1. Algebraic groups. An algebraic group is an algebraic variety \(G\) together with (i) an element \(e \in G\), (ii) a morphism \(\mu : G \times G \rightarrow G\) given by the formula \((x, y) \mapsto xy\) and (iii) a morphism \(i : G \rightarrow G\) given by the formula \(x \mapsto x^{-1}\) with respect to which the set \(G\) is a group. If \(G\) is a non-compact variety, the algebraic
group $G$ is said to be non-compact. The $G$ is called a $K$-group if $G$ is a variety defined over the field $K$ and if $\mu$ and $i$ are defined over $K$. In what follows, we deal with the linear algebraic groups, i.e. the subgroups of the general linear group $GL_n$.

An algebraic group $G$ is called reductive if the unipotent radical of $G$ is trivial. An informal equivalent definition says that $G$ is reductive if and only if a representation of $G$ is a direct sum of the irreducible representations.

2.2.2. Orthogonal group of a quadratic form. Let $q(x) = \{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \mid a_{ij} \in \mathbb{Z}, n \geq 1\}$ be a quadratic form. Roughly speaking, the orthogonal group is a subgroup of the $GL_n$ which preserves the form $q(x)$. Namely, let $A = (a_{ij})$ be a symmetric matrix attached to the quadratic form $q(x)$. The $O(q) = \{g \in GL_n \mid g A g = A\}$ is called an orthogonal group and $SO(q) = \{g \in O(q) \mid \det g = 1\}$ is called a special orthogonal group of the form $q(x)$.

2.2.3. Ring of adeles. The adeles is a powerful tool describing the Artin reciprocity for the abelian extensions of a number field $K$. In intrinsic terms, the ring of adeles $\mathbb{A}$ of a field $K$ is a subset of the direct product $\prod K_v$ taken over almost all places $K_v$ of $K$ endowed with the natural topology. The embeddings $K \hookrightarrow K_v$ induce a discrete diagonal embedding $\mathbb{K} \hookrightarrow \mathbb{A}$, the image of such is a ring of the principal adeles of $\mathbb{A}$. The ring of integral adeles $\mathbb{A}_\infty := \prod O_v$, where $O_v$ is a localization of the ring $O_K$ of the integers of the field $K$. The Artin reciprocity says that there exists a continuous homomorphism $\mathbb{A}_\infty \to \text{Gal}(K^{ab}/K)$, where $K^{ab}$ is a group of the invertible adeles (the idele group) and $\text{Gal}(K^{ab}/K)$ is the absolute Galois group of the abelian extensions of $K$ endowed with a profinite topology. On the other hand, there exists a canonical isomorphism $\mathbb{A}_\infty^\times \backslash \mathbb{A}_\infty^\times /K^\times \to \text{Cl}(K)$, where $\mathbb{A}_\infty^\times$ ( $\mathbb{A}_\infty^\times$ and $K^\times$, resp.) is a group of units of the ring $\mathbb{A}_\infty$ ( $\mathbb{A}$ and $O_K$, resp.) and $\text{Cl}(K)$ is the ideal class group of the field $K$.

2.2.4. Algebraic groups over adeles. The algebraic groups over the ring of adeles can be viewed as an analog of the Artin reciprocity for the non-abelian extensions of the field $K$ and, therefore, we deal with a noncommutative arithmetic [Platonov & Rapinchuk 1994] [14, p. 243]. Such groups are a starting point of the Langlands program [Langlands 1978] [12].

Let $G$ be an algebraic group. It is known that the double cosets $G(\mathbb{A}_\infty) \backslash G(\mathbb{A})/G(K)$ is a finite set [Borel 1963] [7]. In particular, if $G \cong GL_n$ then $|G(\mathbb{A}_\infty) \backslash G(\mathbb{A})/G(K)| = h_K$ and if $G \cong O(n)$ then $|G(\mathbb{A}_\infty) \backslash G(\mathbb{A})/G(K)| = g$, where $h_K = |\text{Cl}(K)|$ is the class number of the field $K$ and $g$ is the genus of quadratic form $q(x)$, respectively. The cardinality of the set $G(\mathbb{A}_\infty) \backslash G(\mathbb{A})/G(K)$ is a difficult open problem otherwise.

If $G$ is a non-compact $K$-group, then the principal class $N(\mathbb{A}) := G(\mathbb{A}_\infty)G(K)$ is a normal subgroup of the group $G(\mathbb{A})$ [Platonov & Rapinchuk 1994] [14, Proposition 8.8]. In this case $\mathcal{G} := G(\mathbb{A})/N(\mathbb{A})$ is a finite abelian group, such that $|\mathcal{G}| = |G(\mathbb{A}_\infty) \backslash G(\mathbb{A})/G(K)|$.

3. Proofs

3.1. Proof of theorem 1.1. We shall split the proof in a series of lemmas.

Lemma 3.1. There exists a canonical endomorphism $\epsilon : G(\mathbb{A}) \to G(\mathbb{A})$, such that:

(i) $\epsilon(G(\mathbb{A})) = N(\mathbb{A})$, where $N(\mathbb{A}) = G(\mathbb{A}_\infty)G(K)$ is the principal class of the group $G(\mathbb{A})$;
FIGURE 1.

(ii) the map $\alpha := \text{Id} - \epsilon$ is an automorphism of the group $G(A)$.

Proof. (i) Since $N(A)$ is a normal subgroup of the group $G(A)$, there exists a canonical short exact sequence of the form:

$$1 \rightarrow G(A)/N(A) \rightarrow G(A) \xrightarrow{\epsilon} N(A) \rightarrow 1,$$

where $\epsilon$ is a surjective homomorphism. Since $N(A) \subseteq G(A)$, we conclude that $\epsilon$ is an endomorphism of the group $G(A)$ and $\epsilon(G(A)) = N(A)$. Item (i) of lemma 3.1 is proved.

(ii) Consider a restriction $\epsilon'$ of the map $\epsilon : G(A) \rightarrow G(A)$ to the subgroup $N(A) \subseteq G(A)$. From (3.1) one gets an exact sequence:

$$G(A) \xrightarrow{\epsilon} N(A) \xrightarrow{\pi} 1,$$

where $\text{im } \epsilon = \ker \pi \cong N(A)$. It is easy to see, that $\pi = \epsilon'$. In other words, we have $\epsilon' = 0$.

To show that $\alpha = \text{Id} - \epsilon$ is an automorphism, we shall verify that $\alpha^{-1} := \text{Id} + \epsilon$ is an inverse of $\alpha$. Indeed, $(\text{Id} - \epsilon)(\text{Id} + \epsilon) = \text{Id}^2 - \epsilon^2 = \text{Id}$. Therefore, the map $\alpha$ is invertible, i.e. an automorphism. Item (ii) of lemma 3.1 is proved.

Lemma 3.2. A $C^*$-algebra corresponding to the space $G(A_\infty)/G(A)/G(K)$ is the crossed product $\mathcal{A} \rtimes \sigma Z$, where $\sigma$ is an automorphism of $\mathcal{A}$ induced by the automorphism $\alpha$ of $G(A)$, see item (ii) of lemma 3.1.

Proof. Let $F : G(A)/G(K) \rightarrow \mathcal{A}$ be a map from the homogeneous space $G(A)/G(K)$ to the $C^*$-algebra $\mathcal{A}$ defined by formula (1.1). Let $\hat{\alpha} : G(A)/G(K) \rightarrow G(A)/G(K)$ be an automorphism of $G(A)/G(K)$ defined as a composition of the automorphism $\alpha : G(A) \rightarrow G(A)$ with the projection $G(A) \rightarrow G(A)/G(K)$. Consider a commutative diagram in the Figure 1. The closure of arrows in the diagram gives an extension of the map $F$ to the double coset space $G(A_\infty)/G(A)/G(K)$. Lemma 3.2 follows.

Remark 3.3. The automorphism $\sigma$ is a power of the shift automorphism $\sigma_0$ of the AF-algebra $\mathcal{A}$, see Section 2.1.3. Thus, in general we have $\sigma = \sigma_0^k$, where $k \geq 1$.

Lemma 3.4. There exists a one-to-one map of the form:

$$\phi : G(A_\infty)/G(A)/G(K) \rightarrow K_0(\mathcal{A} \rtimes \sigma Z).$$

(3.3)
Figure 2.

Proof. Since the K-theory is a functor on the category of $C^*$-algebras, one gets from Figure 1 a commutative diagram shown in Figure 2, where $\sigma_*$ is the action of $\sigma$ on the $K_0$-group of algebra $\mathcal{A}$. We refer the reader to [Blackadar 1986] [6, Theorem 10.2.1].

It is easy to see, that the map $\phi$ on the diagram is a bijection. Indeed, since both sets are finite, let us assume that $|G(A_\infty)\setminus G(A)/G(K)| > |K_0(\mathcal{A} \rtimes_\sigma \mathbb{Z})|$. From the diagram in Figure 2 one gets an exact sequence of the form:

$$1 \rightarrow K_0(\mathcal{A}) \xrightarrow{1-\sigma_*} K_0(\mathcal{A}) \rightarrow G(A_\infty)\setminus G(A)/G(K) \rightarrow 1.$$  \hfill (3.4)

The (3.4) would imply that there exists more than $|K_0(\mathcal{A} \rtimes_\sigma \mathbb{Z})|$ distinct embeddings of the AF-algebra $\mathcal{A}$ into the crossed product $\mathcal{A} \rtimes_\sigma \mathbb{Z}$. This is a contradiction, which implies that the map $\phi : G(A_\infty)\setminus G(A)/G(K) \rightarrow K_0(\mathcal{A} \rtimes_\sigma \mathbb{Z})$ must be one-to-one. Lemma 3.4 is proved.

Corollary 3.5. The map $\phi^{-1}$ defines on the set $G(A_\infty)\setminus G(A)/G(K)$ the structure of an abelian group extending $\phi$ to an isomorphism of the groups.

Proof. Recall that $K_0(\mathcal{A} \rtimes_\sigma \mathbb{Z}) \cong \frac{\mathbb{Z}^n}{(I-A)\mathbb{Z}^n} \cong \frac{\mathbb{Z}}{p_1^{n_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{n_k}\mathbb{Z}}$, \hfill (3.5)

where $A$ is an $n \times n$ integer matrix defining the AF-algebra $\mathcal{A}$, the $p_i$ are prime and $n_i$ are positive integer numbers. One can take a generator $x_i$ in each cyclic group $\mathbb{Z}/p_i^{n_i}\mathbb{Z}$ and let $\phi^{-1}(x_i)$ be a generator of the abelian group structure on the set $G(A_\infty)\setminus G(A)/G(K)$. It is clear that the map $\phi$ defines an isomorphism between the two abelian groups. Corollary 3.5 is proved.

Theorem 1.1 follows from lemma 3.4 and corollary 3.5.

3.2. Proof of corollary 1.2. Recall that if $G \cong O(q)$ is the orthogonal group of a quadratic form $q(x)$, then the equivalence classes of $q(x)$ correspond one-to-one to the double cosets $G(A_\infty)\setminus G(A)/G(K)$, see Section 2.2.4. But in view of the corollary 3.5, the set $G(A_\infty)\setminus G(A)/G(K)$ has the natural structure of an abelian group defined by the formula (3.3). In particular, the group operation defines a composition law for the equivalence classes of quadratic form $q(x)$. The first part of corollary 1.2 is proved.
To express the genus $g$ of the form $q(x)$ in terms of an invariant of the algebra $\mathcal{A}$, recall that $g = |G(\mathcal{A}_\infty) \backslash G(\mathcal{A})/G(K)|$, see Section 2.2.4. But $|G(\mathcal{A}_\infty) \backslash G(\mathcal{A})/G(K)| = |K_0(\mathcal{A} \rtimes \sigma \mathbb{Z})|$ and one gets from the formulas (3.5):

$$g = |K_0(\mathcal{A} \rtimes \sigma \mathbb{Z})| = \left| \frac{\mathbb{Z}^n}{(I - A)\mathbb{Z}^n} \right| = \left| \frac{\mathbb{Z}}{p_1^{n_1} \mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{n_k} \mathbb{Z}} \right| = p_1^{n_1} \cdots p_k^{n_k} = |\det (I - A)|,$$

(3.6)

The genus formula of corollary 1.2 follows from the equations (3.6).

4. Binary quadratic forms

Let $n = 2$, i.e. $q_\Delta(x)$ is an indefinite binary quadratic form of discriminant $\Delta$, where $\Delta > 0$ is a square-free integer. In view of corollary 1.2 and remark 3.3, we are looking for a matrix $A$ with $\det A = 1$, such that

$$g(q_{f^2 \Delta}(x)) = |\det (I - A^k)|,$$

(4.1)

where $f \geq 1$ is a conductor of the quadratic form and $k \geq 1$ is an integer.

Remark 4.1. The left and right sides of equation (4.1) depend only on the integers $f$ and $k$, respectively. Indeed, the left side depends only on the conductor $f$, if the discriminant $\Delta$ is a fixed square-free integer. For the right side, we have the easily verified equalities $|\det (I - A^k)| = tr (A^k) - 2 = 2T_k \left( \frac{1}{2} tr (A) \right) - 2 = 2T_k (\frac{1}{2} \sqrt{\Delta + 4}) - 2$, where $T_k(x)$ is the Chebyshev polynomial of degree $k$.

Let $f$ and $k$ be the least integers satisfying equation (4.1). Recall that the equivalence classes of the quadratic form $q_{f^2 \Delta}(x)$ correspond one-to-one to the (narrow) ideal classes of the order $R_f := \mathbb{Z} + fO_K$ in the real quadratic field $K := \mathbb{Q}(\sqrt{\Delta})$. The number $h_{R_f}$ of such classes is calculated by the formula:

$$h_{R_f} = h_K \frac{f}{e_f} \prod_{p \mid f} \left( 1 - \left( \frac{\Delta}{p} \right) \frac{1}{p} \right),$$

(4.2)

where $h_K$ is the class number of the field $K$, $e_f$ is the index of the group of units of the order $R_f$ in the group of units of the ring $O_K$, $p$ is a prime number and $\left( \frac{\Delta}{p} \right)$ is the Legendre symbol [Hasse 1950] [11, pp. 297 and 351]. Thus the genus of the binary quadratic form of discriminant $\Delta > 0$ is given by the formula:

$$g(q_\Delta(x)) = \frac{|\det (I - A^k)|}{e_f \prod_{p \mid f} \left( 1 - \left( \frac{\Delta}{p} \right) \frac{1}{p} \right)}.$$

(4.3)

Example 4.2. Let $G \cong O(q)$ be the orthogonal group of the quadratic form

$$q(u, v) = u^2 + 3uv + v^2.$$

(4.4)

The discriminant is $\Delta = 5$ and therefore the $q(u, v)$ is an indefinite quadratic form. The AF-algebra $\mathcal{A}$ corresponding to the group $G(\mathcal{A})$ is represented by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

(4.5)

The Bratteli diagram of algebra $\mathcal{A}$ is shown in Figure 3. To calculate the genus of $q(u, v)$, we shall use formula (4.3) with $f = k = 1$. Namely, one gets

$$g = |\det (I - A)| = \left| \det \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} \right| = 1.$$

(4.6)
Figure 3. Bratteli diagram of the AF-algebra $\mathcal{A}$.

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