ON THE HAUSDORFF DIMENSION OF SOME SETS OF NUMBERS DEFINED THROUGH THE DIGITS OF THEIR $Q$-CANTOR SERIES EXPANSIONS

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ABSTRACT. Following in the footsteps of P. Erdős, A. Rényi, and T. Šalát we compute the Hausdorff dimension of sets of numbers whose digits with respect to their $Q$-Cantor series expansions satisfy various statistical properties. In particular, we consider difference sets associated with various notions of normality and sets of numbers with a prescribed range of digits.

1. Introduction

The study of normal numbers and other statistical properties of real numbers with respect to large classes of Cantor series expansions was first done by P. Erdős and A. Rényi in [4] and [5] and by A. Rényi in [13], [15], and [16] and by P. Turán in [19].

The $Q$-Cantor series expansions, first studied by G. Cantor in [3], are a natural generalization of the $b$-ary expansions. Let $N_k := \mathbb{Z} \cap [k, \infty)$. If $Q \in N_2^\mathbb{N}$, then we say that $Q$ is a basic sequence. Given a basic sequence $Q = (q_n)_{n=1}^\infty$, the $Q$-Cantor series expansion of a real number $x$ is the (unique) expansion of the form

$$x = E_0 + \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n}$$

where $E_0 = \lfloor x \rfloor$ and $E_n$ is in $\{0, 1, \ldots, q_n - 1\}$ for $n \geq 1$ with $E_n \neq q_n - 1$ infinitely often. We abbreviate (1.1) with the notation $x = E_0.E_1E_2E_3 \ldots$ w.r.t. $Q$.

A block is an ordered tuple of non-negative integers, a block of length $k$ is an ordered $k$-tuple of integers, and block of length $k$ in base $b$ is an ordered $k$-tuple of integers in $\{0, 1, \ldots, b - 1\}$.

Let

$$Q_n^{(k)} := \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}}$$

and $T_{Q,n}(x) := \left( \prod_{j=1}^{n} q_j \right) x \pmod{1}$.

A. Rényi [15] defined a real number $x$ to be normal with respect to $Q$ if for all blocks $B$ of length 1,

$$\lim_{n \to \infty} \frac{N_Q^B(B, x)}{Q_n^{(1)}} = 1.$$
If \( q_n = b \) for all \( n \) and we restrict \( B \) to consist of only digits less than \( b \), then \( (1.2) \) is equivalent to simple normality in base \( b \), but not equivalent to normality in base \( b \). A basic sequence \( Q \) is \( k \)-divergent if \( \lim_{n \to \infty} Q_n^{(k)} = \infty \), fully divergent if \( Q \) is \( k \)-divergent for all \( k \), and \( k \)-convergent if it is not \( k \)-divergent. A basic sequence \( Q \) is infinite in limit if \( q_n \to \infty \).

**Definition 1.1.** A real number \( x \) is \( Q \)-normal of order \( k \) if for all blocks \( B \) of length \( k \),

\[
\lim_{n \to \infty} \frac{N^Q_n(B, x)}{Q_n^{(k)}} = 1.
\]

We let \( N_k(Q) \) be the set of numbers that are \( Q \)-normal of order \( k \). The real number \( x \) is \( Q \)-normal if \( x \in N(Q) := \bigcap_{k=1}^{\infty} N_k(Q) \). \( x \) is \( Q \)-ratio normal of order \( k \) (here we write \( x \in \mathcal{RN}_k(Q) \)) if for all blocks \( B_1 \) and \( B_2 \) of length \( k \)

\[
\lim_{n \to \infty} \frac{N^Q_n(B_1, x)}{N^Q_n(B_2, x)} = 1.
\]

\( x \) is \( Q \)-ratio normal if \( x \in \mathcal{RN}(Q) := \bigcap_{k=1}^{\infty} \mathcal{RN}_k(Q) \). A real number \( x \) is \( Q \)-distribution normal if the sequence \( (T_{Q,n}(x))_{n=0}^{\infty} \) is uniformly distributed mod 1. Let \( \mathcal{DN}(Q) \) be the set of \( Q \)-distribution normal numbers.

It was proven in [11] that the directed graph in Figure 1 gives the complete containment relationships between these notions when \( Q \) is infinite in limit and fully divergent. The vertices are labeled with all possible intersections of one, two, or three choices of the sets \( N(Q), \mathcal{RN}(Q), \text{ and } \mathcal{DN}(Q) \). The set labeled on vertex \( A \) is a subset of the set labeled on vertex \( B \) if and only if there is a directed path from \( A \) to \( B \). For example, \( N(Q) \cap \mathcal{DN}(Q) \subseteq \mathcal{RN}(Q) \), so all numbers that are \( Q \)-normal and \( Q \)-distribution normal are also \( Q \)-ratio normal.

Note that in base \( b \), where \( q_n = b \) for all \( n \), the corresponding notions of \( Q \)-normality, \( Q \)-ratio normality, and \( Q \)-distribution normality are equivalent. This equivalence is fundamental in the study of normality in base \( b \).

It follows from a well known result of H. Weyl [24, 25] that \( \mathcal{DN}(Q) \) is a set of full Lebesgue measure for every basic sequence \( Q \). We will need the following result of the second author [13] later in this paper.

**Theorem 1.2.** Suppose that \( Q \) that is infinite in limit. Then \( N_k(Q) \) and \( \mathcal{RN}_k(Q) \) are of full measure if and only if \( Q \) is \( k \)-divergent. The sets \( N(Q) \) and \( \mathcal{RN}(Q) \) are of full measure if and only if \( Q \) is fully divergent.

Based on Figure 1 and Theorem 1.2, it is natural to ask for the Hausdorff dimension of the difference sets. It was proven in [12] that for every basic sequence \( Q \) that is infinite in limit

\[
\dim_H(\mathcal{DN}(Q) \setminus N(Q)) = \dim_H(\mathcal{DN}(Q) \setminus \mathcal{RN}(Q)) = 1.
\]

Using different methods we will prove the following theorem.

**Theorem 1.3.** Every non-empty difference set expressed in terms of \( N(Q), \mathcal{RN}(Q), \) and \( \mathcal{DN}(Q) \), possibly involving intersections and unions, has full Hausdorff dimension for every \( Q \) that is infinite in limit, except for the set \( N(Q) \setminus \mathcal{DN}(Q) \).

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3Early work in this direction has been done by A. Rényi [15], T. Šalát [22], and F. Schweiger [18].
It will be shown that the set \( N(Q) \setminus \mathcal{D}(Q) \) has full Hausdorff dimension for a more restricted class of basic sequences in Theorem 3.4. We should note that we can not hope to establish \( \dim_H (N(Q) \setminus \mathcal{D}(Q)) = 1 \) for all \( Q \) that are infinite in limit. This follows from the result in [13] that \( N(Q) = \emptyset \) when \( Q \) is infinite in limit and not fully divergent.

A surprising property of \( Q \)-normality of order \( k \) is that we may not conclude that \( N_k(Q) \subseteq N_j(Q) \) for all \( j < k \) like we may for the \( b \)-ary expansions. In fact, it was shown in [10] that for every \( k \) there exists a basic sequence \( Q \) and a real number \( x \) such that \( N_k(Q) \setminus \bigcup_{j=1}^{k-1} N_j(Q) \) is non-empty. Thus, we will have to be more careful in stating exactly what our theorems prove since lack of \( Q \)-normality of order 2 does not imply lack of \( Q \)-normality of order 338, for example. Furthermore, we will greatly expand on this result in Theorem 3.5 where for each natural number \( \ell \) we exhibit a class of basic sequences such that

\[
\dim_H \left( \bigcap_{j=\ell}^{\infty} N_j(Q) \setminus \bigcup_{j=1}^{\ell-1} N_j(Q) \right) = 1.
\]

For \( x = E_0.E_1 E_2 \cdots \) w.r.t. \( Q \), define the set

\[
S_Q(x) = \{ E_1, E_2, E_3, \ldots \}.
\]

P. Erdős and A. Rényi [4] proved the following theorems.

**Theorem 1.4** (P. Erdős and A. Rényi). If \( Q \) is 1-convergent, then \( S_Q(x) \) has density 0 for almost every real number \( x \).

**Theorem 1.5** (P. Erdős and A. Rényi). For \( x = E_0.E_1 E_2 \cdots \) w.r.t. \( Q \), let \( d_n(x) \) denote the number of different numbers in the sequence \( E_1, \ldots, E_n \). If \( Q \) is 1-convergent, then for almost every \( x \) we have \( \lim_{n \to \infty} \frac{d_n(x)}{n} = 1 \).
It should be noted that T. Šalát considered sets related to those mentioned in Theorem 1.4 and Theorem 1.5. We will need the following definition from [2].

**Definition 1.6.** For \( S \subseteq \mathbb{Z} \), define the mass dimension of \( S \) to be the limit
\[
\dim_{M}(S) = \lim_{n \to \infty} \frac{\log \#(S \cap (-n/2, n/2))}{\log n},
\]
if it exists.

We note that an upper mass dimension and a lower mass dimension may be defined similarly by changing the limit in Definition 1.6 to a lim sup or a lim inf.

For non-empty \( S \subseteq \mathbb{N} \), define
\[
\mathcal{W}_Q(S) = \{ x \in \mathbb{R} : S \cap \mathcal{Q}(x) = S \}.
\]

We will build on Theorem 1.4 and Theorem 1.5 by proving the following theorem.

**Theorem 1.7.** If \( Q \) is infinite in limit, \( \lim_{n \to \infty} \frac{\log q_n}{\sum_{i=1}^{n} \log q_i} = 0 \), and \( S \subseteq \mathbb{N} \) such that \( \min S < \min Q \) and \( \dim_{M}(S) \) exists, then
\[
\dim_{H}(\mathcal{W}_Q(S)) = \dim_{M}(S).
\]

T. Šalát proved in [21] that under some conditions on the basic sequence \( Q \) the set of real numbers whose digits in their \( Q \)-Cantor expansion is bounded has zero Hausdorff dimension. We remark that his result may be sharpened with his conditions weakened by use of our Lemma 2.4 instead of Satz 1 from [20]. The proof of this otherwise follows identically to his original proof, so we do not record it in this paper.

If \( Q \) is infinite in limit and not fully divergent, then \( \lambda(\mathcal{R}_N(Q)) = 0 \). We will show as a consequence of the following theorem that \( \dim_{H}(\mathcal{R}_N(Q)) = 1 \) whenever \( Q \) is infinite in limit.

**Theorem 1.8.** If \( Q \) is infinite in limit, then \( \dim_{H}(\mathcal{R}_N(Q) \cap \mathcal{D}_N(Q) \setminus \mathcal{N}(Q)) = 1 \).

Lastly, we remark that some of the techniques developed in this paper and Lemma 2.4 are used to study fractals associated with normality preserving operations in [1].

### 2. Lemmas

Let \((n_k)\) be a sequence of positive integers and \((c_k)\) be a sequence of positive numbers such that \( n_k \geq 2, 0 < c_k < 1, n_1 c_1 \leq \delta \), and \( n_k c_k \leq 1 \), where \( \delta \) is a positive real number. For any \( k \), let \( D_k = \{(i_1, \ldots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\} \), and \( D = \bigcup D_k \), where \( D_0 = \emptyset \). If \( \sigma = (\sigma_1, \ldots, \sigma_k) \in D_k \), \( \tau = (\tau_1, \ldots, \tau_m) \in D_m \), put \( \sigma * \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_m) \).

**Definition 2.1.** Suppose \( J \) is a closed interval of length \( \delta \). The collection of closed subintervals \( \mathcal{F} = \{ J_\sigma : \sigma \in D \} \) of \( J \) has homogeneous Moran structure if:

1. \( J_\emptyset = J \);
2. \( \forall k \geq 0, \sigma \in D_k, J_{\sigma+1}, \ldots, J_{\sigma+n_k+1} \) are subintervals of \( J_\sigma \) and \( \hat{J}_{\sigma+i} \cap \hat{J}_{\sigma+j} = \emptyset \) for \( i \neq j \);
3. \( \forall k \geq 1, \forall \sigma \in D_{k-1}, 1 \leq j \leq n_k, c_k = \frac{\lambda(J_{\sigma+i})}{\lambda(J_{\sigma+j})} \).
Suppose that $\mathcal{F}$ is a collection of closed subintervals of $J$ having homogeneous Moran structure. Let $E(\mathcal{F}) = \bigcap_{k \geq 1} \bigcup_{F \in \mathcal{F}_k} F_k$. We say $E(\mathcal{F})$ is a homogeneous Moran set determined by $\mathcal{F}$, or it is a homogeneous Moran set determined by $J$, $(n_k)$, $(c_k)$. We will need the following theorem of D. Feng, Z. Wen, and J. Wu from [6].

**Theorem 2.2** (D. Feng, Z. Wen, and J. Wu). If $S$ is a homogeneous Moran set determined by $J$, $(n_k)$, $(c_k)$, then

\[
\liminf_{k \to \infty} \frac{-\log n_1 n_2 \cdots n_k}{k} \leq \dim H(S) \leq \liminf_{k \to \infty} \frac{-\log c_1 c_2 \cdots c_k}{k}.
\]

Given basic sequences $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$, sequences of non-negative integers $s = (s_i), t = (t_i), u = (u_i)$, and $F = (F_i)$, and a sequence of sets $I = \{I_k\}$ such that $I_k \subseteq \{0, 1, \cdots, \beta_i - 1\}$, define the set $\Theta(\alpha, \beta, s, t, u, F, I)$ as follows. Let $Q = Q(\alpha, \beta, s, t, u) = (q_n)$ be the following basic sequence:

\[
(2.1) \quad \left[ [\alpha_1]^{s_1} [\beta_1]^{t_1} \right]^{u_1} \left[ [\alpha_2]^{s_2} [\beta_2]^{t_2} \right]^{u_2} \left[ [\alpha_3]^{s_3} [\beta_3]^{t_3} \right]^{u_3} \cdots.
\]

Define the function

\[
i(n) = \min \left\{ t : \sum_{i=1}^{i-1} u_i (s_i + t_i) < n \right\}.
\]

Set

\[
\Phi_{\alpha}(i, c, d) = \sum_{j=1}^{i-1} v_j s_j + c s_i + d
\]

where $0 \leq c < v_i$ and $0 \leq d < s_i$ and let the functions $i_\alpha(n), c_\alpha(n)$, and $d_\alpha(n)$ be such that $\Phi_{\alpha}^{-1}(n) = (i_\alpha(n), c_\alpha(n), d_\alpha(n))$. Note this is possible since $\Phi_{\alpha}$ is a bijection from $U = \{(i, c, d) \in \mathbb{N}^3 : 0 \leq c < v_i, 0 \leq d < s_i\}$ to $\mathbb{N}$. Define the functions

\[
G(n) = \sum_{j=1}^{i(n)-1} v_j s_j + t_j + c_\alpha(n) \left( s_{i(n)} + t_{i(n)} \right) + d_\alpha(n)
\]

and $g(n) = \min \{ t : G(t) \geq n \}$. Note that $i_\alpha(g(n)) = i(n)$ and $c_\alpha(g(n)) = c_\alpha(n)$.

Furthermore, define $C_\alpha(n) = \left( \sum_{j=1}^{i(n)-1} u_j \right) + c_\alpha(n)$.

We consider the condition on $n$

\[
(2.2) \quad \left( n - \sum_{j=1}^{i(n)-1} v_j s_j + t_j \right) \mod (s_{i(n)} + t_{i(n)}) \geq s_{i(n)}.
\]

Define the intervals

\[
V(n) = \begin{cases} 
I_{i(n)} & \text{if condition (2.2) holds} \\
[F_{G(n)}, F_{G(n)} + 1] & \text{else} 
\end{cases}
\]

That is, we choose digits from $I_{i(n)}$ in positions corresponding to the bases obtained from the sequence $\beta$ and choose a specific digit from $F$ for the bases obtained from the sequence $\alpha$. Set

\[
\Theta(\alpha, \beta, s, t, u, F, I) = \{ x = 0.E_1 E_2 \cdots \text{ w.r.t. } Q : E_n \in V(n) \}.
\]

We will need the following basic lemma to prove Lemma [2.4] and elsewhere in this paper.
Lemma 2.3. Let $L$ be a real number and $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be two sequences of positive real numbers such that
\[
\sum_{n=1}^\infty b_n = \infty \text{ and } \lim_{n \to \infty} \frac{a_n}{b_n} = L.
\]
Then
\[
\lim_{n \to \infty} \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} = L.
\]

Lemma 2.4. Given basic sequences $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$, sequences of non-negative integers $s = (s_i), t = (t_i), v = (v_i)$, and $F = (F_i)$, and a sequence of sets $I = (I_i)$ such that $I_i \subseteq \{0, 1, \ldots, \beta_i - 1\}$ such that the following conditions hold:

\[
\begin{align*}
(2.3) & \quad \lim_{n \to \infty} \frac{s_n \log \alpha_n}{\sum_{j=1}^{s_n} t_j \log \beta_j} = 0; \\
(2.4) & \quad \lim_{n \to \infty} \frac{s_n \log \alpha_n}{\sum_{j=1}^{s_n} t_n \log \beta_j} = 0.
\end{align*}
\]

Then
\[
\dim_H(\Theta(\alpha, \beta, s, t, v, F, I)) = \gamma := \lim_{n \to \infty} \frac{\log |I_n|}{\log \beta_n}.
\]

Proof. Note that $\Theta(\alpha, \beta, s, t, v, F, I)$ is a homogeneous Moran set with
\[n_k = \begin{cases} |I_k| & \text{if } q_k = \beta_{i(k)} \\ 1 & \text{if } q_k = \alpha_{i(k)} \end{cases}\]
and $c_k = \frac{1}{q_k}$. Thus
\[
dim_H(\Theta(\alpha, \beta, s, t, v, F, I)) \geq \lim_{n \to \infty} \inf_{k \to \infty} -\log c_1 c_2 \cdots c_{k+1} n_{k+1} + \frac{\sum_{j=1}^{i(n)-1} u_j t_j \log \beta_j + \sum_{j=1}^{b(n)} t_j \log |I_j|}{\sum_{j=1}^{i(n)} u_j t_j \log \beta_j + \sum_{j=1}^{b(n)} t_j \log |I_j|} + \sum_{j=1}^{i(n)} \frac{t_j \log \beta_j + \log \alpha_j}{t_j \log \beta_j + \log \alpha_j}
\]
which follows from Lemma 2.3.

For a sequence of real numbers $X = (x_n)$ with $x_n \in [0, 1)$ and an interval $I \subseteq [0, 1]$, define $A_n(I, X) = \#\{i \leq n : x_i \in I\}$. We will need the following standard definition and lemma that we quote from [8].
Definition 2.5. Let $X = (x_1, \cdots, x_N)$ be a finite sequence of real numbers. The number
\[
D_N = D_N(X) = \sup_{0 \leq \alpha \leq \beta \leq 1} \left| \frac{A_N([\alpha, \beta), X)}{N} - (\beta - \alpha) \right|
\]
is called the discrepancy of the sequence $\omega$.

It is well known that a sequence $X$ is uniformly distributed mod 1 if and only if $D_N(X) \to 0$.

Lemma 2.6. Let $x_1, x_2, \cdots, x_N$ and $y_1, y_2, \cdots, y_N$ be two finite sequences in $[0,1)$. Suppose $\epsilon_1, \epsilon_2, \cdots, \epsilon_N$ are non-negative numbers such that $|x_n - y_n| \leq \epsilon_n$ for $1 \leq n \leq N$. Then, for any $\epsilon \geq 0$, we have
\[
|D_N(x_1, \cdots, x_N) - D_N(y_1, \cdots, y_N)| \leq 2\epsilon + \frac{\mathcal{N}(\epsilon)}{N},
\]
where $\mathcal{N}(\epsilon)$ denotes the number of $n$, $1 \leq n \leq N$, such that $\epsilon_n > \epsilon$.

3. Results

We will compute the Hausdorff dimension of difference sets formed by taking unions or intersections of the sets $N(Q)$, $\mathbb{N}(Q)$, and $\mathcal{D}N(Q)$. Every other similar result will follow as a corollary of one of these theorems, by using similar techniques, or by Figure II.

Proof of Theorem [1.5] Let $P = (p_i)$ with $p_i = [\log i] + 2$ and $\xi \in N(P)$ with $\xi = \cdot F_1 F_2 \cdots$ w.r.t. $P$. Fix a sequence $X = (x_n)$ that is uniformly distributed modulo 1. Define the sequences
\[
\nu_n = \min \left\{ t : \frac{\sum_{i=0}^{n-1} \log q_{i(n-1)+i}}{\sum_{j=0}^{n-1} \log q_{i(n-1)+j}} < \frac{1}{n}, \forall j \geq t \right\};
\]
\[
\nu_{n,k} = \min \left\{ t : \frac{Q^{(k)}_{n(k)}}{\sum_{j=1}^{n} P^{(k)}_{i-k+1}} < \frac{1}{n}, \forall j \geq t \right\};
\]
$L_0 = 0$;
$L_n = \max \left\{ \min \{ t : \log(q_j) > n, \forall j \geq t \}, L_{n-1} + n^2, L_{n-1} + \nu_n, \max_{k \leq n} \{ \nu_{n,k} \} \right\}$
and set $i(n) = \max \{ j : L_j \leq n \}$. Note that $\nu_n$ and $\nu_{n,k}$ exist since $Q$ is infinite in limit and $P$ is fully divergent. Define the set
\[
S = \bigcup_{n=1}^{\infty} \{ L_n, L_n + 1, \cdots, L_n + n - 1 \}.
\]
Note that this set has density 0 since
\[
\frac{\sum_{i=1}^{n} i}{\sum_{i=1}^{n} i + t_i} \leq \frac{\sum_{i=1}^{n} i}{\sum_{i=1}^{n} i + t_i^2} \to 0 \text{ as } n \text{ goes to infinity.}
\]
Define the intervals
\[
V(n) = \begin{cases} 
[F_n - L_i, F_n - L_i + 1) & \text{if } n \in [L_i, L_i + 1, \cdots, L_i + i] \\
[x_n q_n - \omega_n, x_n q_n + \omega_n) \cap [\lceil \log i(n) \rceil, q_n - 1] & \text{else}
\end{cases}
\]
Note that by the definition of $\Lambda_Q$, we have

$$\lim \frac{N^Q_n(B_1, x)}{N^S_n(B_2, x)} = \lim \frac{\sum_{i=1}^{i(n)} N^P_{i-k+1}(B_1, \xi) + O(1)}{\sum_{i=1}^{i(n)} N^P_{i-k+1}(B_2, \xi) + O(1)} = \frac{N^P_{n-k+1}(B_1, \xi)}{N^P_{n-k+1}(B_2, \xi)} = 1.$$

Thus $x \in \mathcal{R}(Q)$.

Consider the sequence $Y = \left(\frac{E_n}{q_n}\right)$. For $n \in \mathbb{N}\setminus S$, we have $\left|\frac{E_n}{q_n} - x_n\right| < \frac{\epsilon}{q_n}$, which tends to 0 as $n$ goes to infinity. We therefore have for $\epsilon > 0$ that $\overline{N}(\epsilon) = O(1) + \#S \cap \{1, \cdots, N\}$. Thus by Lemma 2.6

$$|D_N(X) - D_N(Y)| < 2\epsilon + \frac{O(1)}{N} + \frac{\#S \cap \{1, \cdots, N\}}{N} \to 2\epsilon$$

as $N$ tends to infinity. Since the inequality holds for all $\epsilon > 0$, we have that $\left(\frac{E_n}{q_n}\right)$ is uniformly distributed mod 1. Thus $x \in \mathcal{D}(Q)$.

Note that

$$\lim_{n \to \infty} \frac{N^Q_n(B, x)}{\sum_{i=1}^{i(n)} P_i^{(k)}} = 1.$$

However,

$$\lim_{n \to \infty} \frac{Q_n^{(k)}}{\sum_{i=1}^{i(n)} P_i^{(k)}} = 0$$

by the definition of $L_n$, so $x \not\in \mathcal{N}(Q)$. Thus $\Lambda_Q \subseteq \mathcal{R}(Q) \cap \mathcal{D}(Q) \setminus \mathcal{N}(Q)$.

Evidently $\Lambda_Q$ is a homogeneous Moran set with $n_k = |V(k)|$ and $c_k = \frac{1}{q_k}$. Thus

$$\dim_{\text{H}}(\Lambda_Q) \geq \lim_{k \to \infty} \frac{\log n_1 \cdots n_k}{\sum_{i=1}^{i(n)} \chi_{\mathbb{N}\setminus S}(i) (1 - \epsilon_i) \log q_i} = \lim_{n \to \infty} \left(1 - \frac{\sum_{j=1}^{i(n)} \sum_{k=0}^{j-1} \log q_{L_j+k} - \sum_{j=1}^{i(n)} \sum_{k=0}^{j-1} \log q_{L_j+k}}{\sum_{n=0}^{n-1} \log q_{L_n+i}} \right) = 1.$$
by the definition of $L_n$. Thus
\[ \dim_H (\Lambda_Q) = 1 \text{ and } \dim_H (\mathcal{RN}(Q) \cap \mathcal{DN}(Q) \setminus \mathcal{N}(Q)) = 1. \]

\[ \square \]

Corollary 3.1. If $Q$ is infinite in limit, then $\dim_H (\mathcal{RN}(Q)) = 1$.

Theorem 3.2. If $Q$ is infinite in limit, then
\[ \dim_H \left( \mathcal{RN}(Q) \setminus \left( \bigcup_{j=1}^{\infty} \mathcal{N}_j(Q) \cup \mathcal{DN}(Q) \right) \right) = 1. \]

Proof. The proof is the same as Theorem 1.8 but with $X = (x_n)$ a sequence that is not uniformly distributed mod 1. \[ \square \]

Theorem 3.3. If $Q$ is infinite in limit, then
\[ \dim_H \left( \mathcal{DN}(Q) \setminus \bigcup_{j=1}^{\infty} \mathcal{RN}_j(Q) \right) = 1. \]

Proof. The proof is the same as Theorem 1.8 but we choose $\xi = E_1 E_2 \cdots$ w.r.t. $P$ such that the digit 0 never occurs. \[ \square \]

We will need to refer to the following four conditions.

(3.1) \[ \lim_{n \to \infty} \frac{t_n \alpha_k}{s_n \beta_n} = 0; \]

(3.2) \[ \lim_{n \to \infty} \frac{t_n \alpha_k}{s_n \beta_n} > 0; \]

(3.3) \[ \lim_{n \to \infty} \frac{\alpha_k}{s_n} = 0; \]

(3.4) \[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\nu_i s_i}{\nu_i (s_i + t_i)} = 0. \]

Theorem 3.4. Suppose that $Q = Q(\alpha, \beta, s, t, v)$ is infinite in limit, $k$-divergent (resp. fully divergent), and satisfies conditions (2.3), (2.4), (3.1) for all $k$, (3.3), and (3.4). If $\alpha_i = o(\beta_i)$, then
\[ \dim_H \left( \bigcap_{j=1}^{k} \mathcal{N}_j(Q) \setminus \mathcal{DN}(Q) \right) = 1 \text{ (resp. } \dim_H (\mathcal{N}(Q) \setminus \mathcal{DN}(Q)) = 1). \]

Proof. We will prove the statement for when $Q = (q_n)$ is fully divergent. The proof for when $Q$ is $k$-divergent follows similarly. Define the basic sequence $P$ by
\[ P = [\alpha_1]^{s_1 \nu_1} [\alpha_2]^{s_2 \nu_2} [\alpha_3]^{s_3 \nu_3} [\alpha_4]^{s_4 \nu_4} \cdots. \]
We note that $P$ is fully divergent since $Q$ is fully divergent. By Theorem 1.2, there exists a real number $\xi = E_0 E_1 E_2 \cdots$ w.r.t. $P$ that is an element of $N(P)$. Set
\[ I_i = \left\{ \alpha_i, \alpha_i + 1, \ldots, \left\lfloor \beta_i^{-1/\log \beta_i} \right\rfloor + 1 \right\} \]
and $F_i = E_i$. Note that $\lim_{n \to \infty} \log |I_n| / \log \beta_n = 1$, so $\dim_H (\Theta(\alpha, \beta, s, t, v, F, I)) = 1$ by Lemma 2.4. We now wish to show that
\[ \Theta(\alpha, \beta, s, t, v, F, I) \subseteq \mathcal{N}(Q) \setminus \mathcal{DN}(Q). \]
Let \( k \) and \( n \) be natural numbers, \( B \) be a block of length \( k \), and \( x \in \Theta(\alpha, \beta, s, t, v, F, I) \). We wish to show that

\[
N_{g(n)}^P(B, \xi) - kC_\alpha(g(n)) \leq N_{g(n)}^Q(B, x) \leq N_{g(n)}^P(B, \xi) + O(1).
\]

Let \( m \) be the maximum digit in the block \( B \). Since \( \min I_i \to \infty \), we know that there are only finitely many indices \( i \) such that \( m > \min I_i \). Thus, there are at most finitely many occurrences of \( B \) starting at position \( n \) when \( q_n = \beta_i(n) \). If every occurrence of \( B \) in \( \xi \) occurs at the corresponding place in \( x \), then we have

\[
N_{g(n)}^P(B, \xi) + O(1) = N_{g(n)}^Q(B, x).
\]

If some of the occurrences of \( B \) in \( \xi \) do not occur in the corresponding places in \( x \), then we have \( N_{g(n)}^Q(B, x) \leq N_{g(n)}^P(B, \xi) \).

On the other hand, the total number of places up to position \( n \) where \( B \) can occur in the \( P \)-Cantor series expansion of \( \xi \) but \( B \) does not occur in the corresponding positions in the \( Q \)-Cantor series expansion of \( x \) is at most \( kC_\alpha(n) \), the total length of the last \( k \) terms of the substrings \([\alpha_i]^n\) of \( P \). Thus

\[
N_{g(n)}^Q(B, \xi) - kC_\alpha(g(n)) \leq N_{g(n)}^Q(B, x) \leq N_{g(n)}^P(B, \xi) + O(1).
\]

Many of the following calculations use Lemma 2.3. Note that

\[
P_n^{(k)} = \sum_{j=1}^{i(n)-1} \frac{s_jv_j}{\alpha_j^k} + \frac{s_{i(n)}b_{\alpha(n)}}{\alpha_{i(n)}^k}
\]

and

\[
Q_n^{(k)} = \left( \sum_{j=1}^{i(n)-1} \frac{(s_j - k)v_j}{\alpha_j^k} + \frac{(t_j - k)v_j}{\beta_j^k} \right) + \left( \sum_{j=1}^{i(n)-1} \frac{v_j}{\alpha_j^k} \right) \frac{1}{\beta_j^k}.
\]

Note that by 2.3 and 2.4, we have that

\[
\lim_{n \to \infty} \frac{Q_n^{(k)}}{P_n^{(k)} g(n)} = \lim_{n \to \infty} \frac{\sum_{j=1}^{i(n)-1} \frac{(s_j - k)v_j}{\alpha_j^k} + \frac{(t_j - k)v_j}{\beta_j^k}}{\sum_{j=1}^{i(n)-1} \frac{v_j}{\alpha_j^k}} \frac{c(n)(s_{i(n)} - k)}{\alpha_{i(n)}^k} + \frac{c(n)(t_{i(n)} - k)}{\beta_{i(n)}^k} = 1.
\]

Thus

\[
\lim_{n \to \infty} \frac{Q_n^{(k)}}{P_n^{(k)} g(n)} = \lim_{n \to \infty} \left( \sum_{j=1}^{i(n)-1} \frac{(s_j - k)v_j}{\alpha_j^k} + \frac{(t_j - k)v_j}{\beta_j^k} \right) + \frac{c(n)(s_{i(n)} - k)}{\alpha_{i(n)}^k} + \frac{c(n)(t_{i(n)} - k)}{\beta_{i(n)}^k}.
\]

Furthermore, we have that

\[
\lim_{n \to \infty} \frac{C_\alpha(g(n))}{P_n^{(k)} g(n)} = \lim_{n \to \infty} \frac{\sum_{j=1}^{i(n)-1} v_j}{\sum_{j=1}^{i(n)-1} \frac{v_j}{\alpha_j^k}} \frac{c(n)}{\alpha_{i(n)}^k} \frac{k}{s_n} = \lim_{n \to \infty} \frac{\alpha_n^k}{s_n - k} \frac{1}{\alpha_n^k} = 0.
\]
Since $\xi \in N(P)$, we have that
\[
\lim_{n \to \infty} \frac{N_n^Q(B, x)}{Q_n^{(k)}} = \lim_{n \to \infty} \frac{N_n^Q(B, x)}{P_n^{(k)}} = 1.
\]
Therefore, $x \in N(Q)$.

For $n$ where $q_n = \beta_i(n)$, we have
\[
\frac{E_n}{q_n} \leq \frac{1 - \log^{-1/2} \beta_i(n)}{\beta_i(n)} \to 0 \text{ as } n \to \infty.
\]
Up to position $n$ there are at least $\sum_{j=1}^{i(n)} v_j t_i + c(n) t_i$ such places where (3.5) holds. By (3.4), we have
\[
\lim_{n \to \infty} \frac{\sum_{j=1}^{i(n)} v_j t_i + c(n) t_i}{n} = 1
\]
so the sequence $\left(\frac{E_n}{q_n}\right)$ is not uniformly distributed mod 1. Thus $x \not\in DN(Q)$ and $\Theta(\alpha, \beta, s, t, v, F, I) \subseteq N(Q) \backslash DN(Q)$, which implies that $\dim_H(N(Q) \backslash DN(Q)) = 1$.

**Theorem 3.5.** Suppose that $Q = Q(\alpha, \beta, s, t, v)$ is infinite in limit, fully divergent, and satisfies conditions (2.3), (2.4), (3.1) for $k \geq \ell$, (3.2) for $\ell < k$, and (3.3). Then
\[
\dim_H \left( \bigcap_{j=\ell}^{\infty} N_j(Q) \backslash \bigcup_{j=1}^{\ell-1} N_j(Q) \right) = 1.
\]

**Proof.** Define the same basic sequence $P$ and sequences $I$ and $F$ as in the proof of Theorem 3.4. The same arguments regarding the asymptotics of $N_n^Q(B, x)$ for $x \in \Theta(\alpha, \beta, s, t, F, I)$ hold, so
\[
\lim_{n \to \infty} \frac{N_n^Q(B, x)}{P_n^{(k)}} = 1.
\]
But since (3.1) holds for $k \geq \ell$, we have that
\[
\lim_{n \to \infty} \frac{Q_n^{(k)}}{P_n^{(k)}} = 1 + \lim_{n \to \infty} \frac{t_n \alpha_n^k}{s_n \beta_n^k} = 1.
\]
Thus $x$ is $Q$-normal of orders greater than or equal to $\ell$. □

**Example 3.6.** Set $\alpha_n = \lfloor \log \log(n+2) \rfloor + 2$, $\beta_n = \lfloor \log n \rfloor + 2$, $s_n = \lfloor \log n \rfloor$, $t_n = n$, and $v_n = 2^n$. Then the conditions of Theorem 3.4 are satisfied.

**Example 3.7.** Fix some integer $\ell$. Set $\alpha_n = \lfloor \log \log(n+2) \rfloor + 2$, $\beta_n = \lfloor \log n \rfloor + 2$, $s_n = \lfloor \log n \rfloor$, $t_n = \left\lfloor \left( \frac{\beta_n}{\alpha_n} \right)^{\ell+1} \right\rfloor s_n$, and $v_n = 2^n$. Then the conditions of Theorem 3.5 are satisfied.

**Proof of Theorem 1.7.** Let $\gamma = \dim_M(S)$, $\alpha_i = 2$, $\beta_i = q_i$, $s_i = 0$, $t_i = 1$, $v_i = 1$, $F_i = 0$, and $I_i = S \cap \{0, \ldots, q_i - 2\}$. 
Then (2.3) and (2.4) clearly hold. Note that 

\[ W_Q(S) \subseteq \Theta(\alpha, \beta, s, t, v, F, I), \]

so \( \dim H(W_Q(S)) \leq \gamma. \)

To get a lower bound, we construct a subset of \( W_Q(S) \) with Hausdorff dimension \( \gamma. \) To do this, let \( T \subseteq \mathbb{N} \) be an infinite set that is sparse enough such that

\[
\lim_{k \to \infty} \frac{\sum_{i=1}^{k} \chi_{T}(i) \log \#(S \cap \{0, \ldots, q_i - 2\})}{\sum_{i=1}^{k} \log \#(S \cap \{0, \ldots, q_i - 2\})} = 0.
\]

Note that such a \( T \) exists since \( \lim_{k \to \infty} \sum_{i=1}^{k} \log \#(S \cap \{0, \ldots, q_i - 2\}) = \infty. \)

Let \( f : T \to S \) be a surjective function such that for all \( t \in T, \) we have \( q_t > f(t). \)

Such an \( f \) exists since \( \min S < \min Q, T \) is infinite, and \( Q \) is infinite in limit.

Consider the homogeneous Moran set \( C \) with \( n_k = \begin{cases} 1 & \text{if } k \in T \\ \#S \cap \{0, \ldots, q_k - 2\} & \text{else} \end{cases} \)

and \( c_k = \frac{1}{q_k} \), described as follows: If \( k \in T \), then for any \( x \in C \), \( E_k(x) = f(k). \)

Otherwise, \( E_k(x) \in S \cap \{0, \ldots, q_k - 2\}. \) Since \( f \) is surjective, we have that for any \( x \in C \) that \( S_Q(x) = S \), so \( C \subseteq W_Q(S). \) But

\[
\dim H(C) \geq \liminf_{k \to \infty} \frac{\log n_1 \cdots n_k}{\log c_1 \cdots c_k + 1} = \lim_{k \to \infty} \frac{\sum_{i=1}^{k} \chi_{T}(i) \log \#(S \cap \{0, \ldots, q_i - 2\})}{\sum_{i=1}^{k} \log q_i + \log q_{k+1}} = \lim_{k \to \infty} \frac{\sum_{i=1}^{k} \log \#(S \cap \{0, \ldots, q_i - 2\})}{\sum_{i=1}^{k} \log q_i} = \lim_{k \to \infty} \frac{\log \#(S \cap \{0, \ldots, q_k - 2\})}{\log q_k} = \gamma.
\]

Thus \( \dim H(W_Q(S)) \geq \gamma, \) so we have \( \dim H(W_Q(S)) = \gamma. \)

\[ \square \]

4. Further Problems

**Problem 4.1.** For which irrational \( x \) does there exist a basic sequence \( Q \) where \( x \in \mathbb{R}N(Q) \cap \mathbb{D}N(Q) \setminus \mathbb{N}(Q). \) The same question may be asked about several of the other sets discussed in this paper. We remark that it is already known that for every irrational \( x \) there exist uncountably many basic sequences \( Q \) where \( x \in \mathbb{D}N(Q). \) See [9].

**Problem 4.2.** Prove that the conclusions of Theorem 3.4 and Theorem 3.5 hold for all \( Q \) that are infinite in limit and fully divergent.

**Problem 4.3.** In [12] sufficient conditions are given under countable intersections of sets of the form \( \mathbb{D}N(Q) \setminus \bigcup_{j=1}^{\infty} \mathbb{R}N_j(Q) \) have full Hausdorff dimension. Surely a similar result holds for many of the sets described in this paper. Necessary and sufficient conditions similar to conditions found in the paper of W. M. Schmidt [17] may be possible.
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