A simpler condition for consistency of a kernel independence test

Arthur Gretton
January 27, 2015

Abstract
A statistical test of independence may be constructed using the Hilbert-Schmidt Independence Criterion (HSIC) as a test statistic. The HSIC is defined as the distance between the embedding of the joint distribution, and the embedding of the product of the marginals, in a Reproducing Kernel Hilbert Space (RKHS). It has previously been shown that when the kernel used in defining the joint embedding is characteristic (that is, the embedding of the joint distribution to the feature space is injective), then the HSIC-based test is consistent. In particular, it is sufficient for the product of kernels on the individual domains to be characteristic on the joint domain. In this note, it is established via a result of Lyons (2013) that HSIC-based independence tests are consistent when kernels on the marginals are characteristic on their respective domains, even when the product of kernels is not characteristic on the joint domain.

1 Introduction

The Hilbert-Schmidt Independence Criterion [4] provides a measure of dependence between random variables $X$ on domain $\mathcal{X}$, and $Y$ on domain $\mathcal{Y}$, with joint probability measure $P_{XY}$ on $\mathcal{X} \times \mathcal{Y}$. This dependence measure may be used in statistical tests of dependence [5, 6]. The simplest way to understand HSIC is as the distance between an embedding of the joint distribution and the product of the marginals, to an appropriate feature space [9, 3], which is in our case a reproducing kernel Hilbert space. The distance covariance of [12] is a special case, for a particular choice of kernel [8]. We say the feature space is characteristic when the embedding is injective, and uniquely identifies probability measures [11, 10]. A test based on HSIC is consistent when product of kernels on the domains being compared is characteristic to the joint domain [1] Theorem 3]. This is shown to be the case e.g. when Gaussian kernels are used on each of the domains.

We propose a simpler condition: namely, that the kernels on each of the individual domains $\mathcal{X}$ and $\mathcal{Y}$ should be characteristic to those domains. The result is a direct consequence of [7, Lemma 3.8]. The result is of particular interest since it may be easier to define characteristic kernels on individual domains...
than on the joint domain. For example, characteristic kernels may be defined on
the group of orthogonal matrices [2, Section 4], and on the semigroup of vectors
of non-negative reals [2, Section 5], however a kernel jointly characteristic to
both domains (i.e., to orthogonal matrix/non-negative vector pairs) is harder
to define.

2 Results

We begin with a result from [10] that characteristic, translation invariant kernels
provide injective embeddings of finite signed measures.

**Proposition 1** (Injective embeddings of finite signed measures). Let $X$ be
a Polish, locally compact Hausdorff space. Let $k(x, y)$ be a $c_0$-kernel, i.e. a
bounded kernel for which $k(x, \cdot) \in C_0(X) \quad \forall x$, where $C_0(X)$ is the class of con-
tinuous functions on $X$ that vanish at infinity. Assume $k(x, y) = k(x - y)$, i.e.
the kernel is translation invariant. Define as $F$ the RKHS induced by $k$. The
following statements are equivalent:

1. $k$ is characteristic
2. The embedding of a finite signed Borel measure $\mu \in M_b(X)$, defined as
   $$\mu \mapsto \int_X k(\cdot, x) d\mu(x),$$
   is injective.

This result may be obtained by combining [10, Proposition 2], which states
that an RKHS is $c_0$-universal iff the embedding in (1) is injective, with the result
in [10, Section 3.2] that translation invariant kernels are $c_0$-universal iff they are
characteristic.

This being the case, a minor adaptation of the proof of [7, Lemma 3.8] leads
to the following result.

**Theorem 2** (Characteristic kernels and independence measures). Let $k$ and $l$
be kernels for the respective RKHSs $F$ on $X$ and $G$ on $Y$, with respective feature
maps $\phi$ and $\psi$. Assume both $k$ and $l$ are characteristic, translation invariant
$c_0$-kernels, satisfying the conditions of Proposition 1. Define the finite signed
measure
$$\theta := P_{XY} - P_X P_Y.$$ Define the covariance operator as the embedding of this signed measure into the
tensor space
$$C_{YX} = \int_{X \times Y} \psi(y) \otimes \phi(x) d\theta(x, y).$$
Then $C_{YX} = 0$ iff $\theta = 0$.

---

1 Continuous functions vanishing at infinity are members of $f \in C(X)$ such that for all
   $\varepsilon > 0$ the set $\{x : |f(x)| \geq \varepsilon\}$ is compact.
2 The tensor product is defined such that $(a \otimes b) c = (b, c) \otimes a, \forall a \in F, b, c \in G.$
Proof. The result \( \theta = 0 \implies C_{YX} = 0 \) is straightforward. We now prove the other direction. For every \( f \in \mathcal{F} \) and \( B \in \sigma(Y) \), we define the finite signed Borel measure

\[
\nu_f(B) = \int_{X \times Y} \langle \phi(x), f \rangle \mathbb{I}_B(y) d\theta(x, y),
\]

where \( \mathbb{I}_B(\cdot) \) is the indicator of the set \( B \). The embedding of this measure to \( \mathcal{G} \) is injective, and is written

\[
\mu_{\nu_f} = \int \psi(y) \langle \phi(x), f \rangle d\theta(x, y)
= \int (\psi(y) \otimes \phi(x)) f d\theta(x, y)
= \left[ \int (\psi(y) \otimes \phi(x)) d\theta(x, y) \right] f
= C_{YX} f = 0,
\]

where we have used the linearity of the tensor product

\[
(a \otimes b) c = T_c(a \otimes b) = \langle h, c \rangle a.
\]

Since the embedding \( \mu_{\nu_f(B)} \) is injective, we have that \( \nu_f = 0 \). Since this is true for all \( f \in \mathcal{F} \), we have that

\[
\int_{X \times Y} \phi(x) \mathbb{I}_B(y) d\theta(x, y) = 0.
\]

Define the finite signed measure on \( A \), \( \nu_B(A) = \theta(A \times B) \). The above equation can be interpreted as the embedding of this measure to \( \mathcal{F} \),

\[
\mu_{\nu_B} = \int_{X \times Y} \phi(x) \mathbb{I}_B(y) d\theta(x, y) = 0,
\]

hence \( \nu_B = 0 \), given that the embedding \( \mu_{\nu_B} \) is injective. We conclude that \( \theta(A \times B) = 0 \) for all Borel sets \( A, B \), and hence \( \theta = 0 \).

An important point to note is that the embedding of \( \theta \) need not be characteristic to all probability measures: only the embeddings of each of the individual dimensions \( X \) and \( Y \) need be characteristic. A second point is that a consistent test still requires characteristic kernels on both domains; it is not sufficient for one domain alone to have a characteristic kernel. A simple example can be used to illustrate the resulting failure mode: \( X := \mathbb{R} \) with a characteristic kernel, \( Y := \mathbb{R} \) with the linear kernel \( l(y_1, y_2) = y_1 y_2 \), and points are distributed uniformly on a circular ring centered at the origin. The data are dependent, but HSIC with these kernels will not detect this dependence.

Acknowledgements: Thanks to Joris Mooij, Jonas Peters, Dino Sejdinovic, and Bharath Sriperumbudur for helpful discussions.
References

[1] K. Fukumizu, A. Gretton, X. Sun, and B. Schölkopf. Kernel measures of conditional dependence. In Advances in Neural Information Processing Systems 20, pages 489–496, Cambridge, MA, 2008. MIT Press.

[2] K. Fukumizu, B. Sriperumbudur, A. Gretton, and B. Schoelkopf. Characteristic kernels on groups and semigroups. In Advances in Neural Information Processing Systems 21, pages 473–480, Red Hook, NY, 2009. Curran Associates Inc.

[3] A. Gretton, K. Borgwardt, M. Rasch, B. Schoelkopf, and A. Smola. A kernel two-sample test. JMLR, 13:723–773, 2012.

[4] A. Gretton, O. Bousquet, A. J. Smola, and B. Schölkopf. Measuring statistical dependence with Hilbert-Schmidt norms. In S. Jain, H. U. Simon, and E. Tomita, editors, Proceedings of the International Conference on Algorithmic Learning Theory, pages 63–77. Springer-Verlag, 2005.

[5] A. Gretton, K. Fukumizu, C.-H. Teo, L. Song, B. Schölkopf, and A. J. Smola. A kernel statistical test of independence. In Advances in Neural Information Processing Systems 20, pages 585–592, Cambridge, MA, 2008. MIT Press.

[6] A. Gretton and L. Gyorfi. Consistent nonparametric tests of independence. Journal of Machine Learning Research, 11:1391–1423, 2010.

[7] R. Lyons. Distance covariance in metric spaces. The Annals of Probability, 41(5):3051–3696, 2013.

[8] D. Sejdinovic, B. Sriperumbudur, A. Gretton, and K. Fukumizu. Equivalence of distance-based and rkhs-based statistics in hypothesis testing. Annals of Statistics, 41(5):2263–2702, 2013.

[9] A. J. Smola, A. Gretton, L. Song, and B. Schölkopf. A Hilbert space embedding for distributions. In Proceedings of the International Conference on Algorithmic Learning Theory, volume 4754, pages 13–31. Springer, 2007.

[10] B. Sriperumbudur, K. Fukumizu, and G. Lanckriet. Universality, characteristic kernels and RKHS embedding of measures. Journal of Machine Learning Research, 12:2389–2410, 2011.

[11] B. Sriperumbudur, A. Gretton, K. Fukumizu, G. Lanckriet, and B. Schölkopf. Hilbert space embeddings and metrics on probability measures. Journal of Machine Learning Research, 11:1517–1561, 2010.

[12] G. Székely, M. Rizzo, and N. Bakirov. Measuring and testing dependence by correlation of distances. Ann. Stat., 35(6):2769–2794, 2007.