The statistics of domain walls for ground states of the 2D Ising spin glass with +1 and -1 bonds are studied for $L \times L$ square lattices with $L \leq 48$, and $p = 0.5$, where $p$ is the fraction of negative bonds, using periodic and/or antiperiodic boundary conditions. When $L$ is even, almost all domain walls have energy $E_{dw} = 0$ or 4. When $L$ is odd, most domain walls have $E_{dw} = 2$. The probability distribution of the entropy, $S_{dw}$, is found to depend strongly on $E_{dw}$. When $E_{dw} = 0$, the probability distribution of $|S_{dw}|$ is approximately exponential. The variance of this distribution is proportional to $L$, in agreement with the results of Saul and Kardar. For $E_{dw} = k > 0$ the distribution of $S_{dw}$ is not symmetric about zero. In these cases the variance still appears to be linear in $L$, but the average of $S_{dw}$ grows faster than $\sqrt{L}$. This suggests a one-parameter scaling form for the $L$-dependence of the distributions of $S_{dw}$ for $k > 0$.

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I. INTRODUCTION

There continues to be a controversy about the nature of the Ising spin glass. The Sherrington-Kirkpatrick model\textsuperscript{1} with its infinite-range interactions between the spins, is described by the Parisi replica-symmetry breaking mean-field theory\textsuperscript{2,3}. To understand models with short-range interactions on finite-dimensional lattices, however, it is necessary to include the effects of interfaces, which do not exist in a well-defined way in an infinite-range model. The droplet model of Fisher and Huse\textsuperscript{4,5,6} which starts from the domain-wall renormalization group ideas of McMillan\textsuperscript{7,8,9} and Bray and Moore\textsuperscript{10,11,12} and studies the properties of interfaces, provides a very different viewpoint on the spin-glass phase.

In two dimensions (2D), the spin-glass phase is not stable at finite temperature. Because of this, it is necessary to treat cases with continuous distributions of energies (CDE) and cases with quantized distributions of energies (QDE) separately\textsuperscript{11,13}.

In three or more space dimensions, where a spin-glass phase is believed to occur at finite temperature $T$, the general framework of thermodynamics requires that the CDE and the QDE should be treated on the same footing. The way this comes about is that in these cases the typical domain wall energy increases as a positive power of the size of the lattice. Thus the quantization energy becomes a negligible fraction of the domain wall energy for large lattices. All bond distributions behave in a qualitatively similar way, except that the QDE have finite ground state entropies\textsuperscript{6,11}.

Amoruso, Hartmann, Hastings and Moore\textsuperscript{14} have recently proposed that in 2D there is a relation

$$d_S = 1 + \frac{3}{4(3 + \theta_E)},$$

where $d_S$ is the fractal dimension of domain walls, and $\theta_E$ is the exponent which characterizes the scaling of the domain wall energy with size. For the CDE case, the existing numerical estimates of $d_S$ and $\theta_E$ satisfy Eqn. (1). However, it is unclear if Eqn. (1) should continue to be correct when the scaling exponent for spin correlations, $\eta$, is not zero. For the QDE, the current estimates\textsuperscript{15,16} find $\eta \approx 0.14$.

In three dimensions it is known from the droplet theory\textsuperscript{5,6,12} that for the QDE, which have a positive entropy at $T = 0$, in the spin-glass phase

$$d_S = 2\theta_S.$$
θ_s is the exponent for the scaling of domain wall entropy with size. Thus, for the QDE, Eqn. (1) provides a relation between the scaling of the energy and the entropy of domain walls. It is not known how to calculate \( d_s \) directly for the QDE case, so we need to use Eqn. (2) to check Eqn. (1) in that case. One might hope that this relation would also hold in 2D, even though the spin-glass order only occurs at \( T = 0 \).

For the QDE, it is known that \( \theta_E = 0.13,17 \) Then using Eqn. (1) gives \( d_s = 5/4 \), or using Eqn. (2), \( \theta_S = 5/8 \). The calculation of \( \theta_S \) by Saul and Kardar,\(^{18,19}\) found \( \theta_S = 0.49 \pm 0.02 \). Since \( d_s \) cannot be less than 1, this result was interpreted as a strong indication that \( \theta_S = 1/2 \).

In this work we will find that Eqn. (1) may not work for the QDE case in 2D. It appears, however, that Eqn. (2) is still correct in 2D, except when the domain wall energy, \( E_{dw} \), is zero. The actual behavior of the QDE probability distributions under finite-size scaling turns out to be more subtle than what has been assumed until recently.\(^{20,21}\) As pointed out by Wang, Harrington and Preskill,\(^{22}\) domain walls of zero energy which cross the entire sample play a special role when the energy is quantized.

We will analyze data for the \( E_{dw} \) and for the domain wall entropy, \( S_{dw} \), for the ground states (GS) of 2D Ising spin glasses obtained using a slightly modified version of the computer program of Gallucio, Loebl and Vondrác\(^{23}\) which is based on the Pfaffian method. The Pfaffians are calculated using a fast exact integer arithmetic procedure, coded in C++. Thus, there is no roundoff error in the calculation until the double precision logarithm is taken to obtain \( S_{dw} \). This extended precision is essential, in order to obtain meaningful results for entropy differences at large \( L \). An earlier version of the domain wall entropy calculation\(^{21}\) using data provided by S. N. Coppersmith\(^{24}\) was limited to small \( L \times L \) lattices with even \( L \) and came to somewhat different conclusions.

We will demonstrate that for \( L \times L \) square lattices the Edwards-Anderson\(^{25}\) (EA) model with a \( \pm J \) bond distribution has a strong correlation between \( E_{dw} \) and \( S_{dw} \) for the GS domain walls. Because of this correlation, we will need to treat domain walls of different energies as distinct classes. We will find that the scaling parameter identified by Saul and Kardar\(^{18,19}\) is the one associated with domain walls having \( E_{dw} = 0 \). It is not, however, the one which controls the dominant behavior for large \( L \).
The Hamiltonian of the EA model for Ising spins is

\[ H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j, \]  

(3)

where each spin \( \sigma_i \) is a dynamical variable which has two allowed states, +1 and -1. The \( \langle ij \rangle \) indicates a sum over nearest neighbors on a simple square lattice of size \( L \times L \). We choose each bond \( J_{ij} \) to be an independent identically distributed quenched random variable, with the probability distribution

\[ P(J_{ij}) = p\delta(J_{ij} + 1) + (1 - p)\delta(J_{ij} - 1), \]  

(4)

so that we actually set \( J = 1 \), as usual. Thus \( p \) is the concentration of antiferromagnetic bonds, and \( (1 - p) \) is the concentration of ferromagnetic bonds.

II. GROUND STATE DOMAIN WALLS

We define the GS entropy to be the natural logarithm of the number of ground states. For each sample the GS energy and GS entropy were calculated for the four combinations of periodic (P) and antiperiodic (A) toroidal boundary conditions along each of the two axes of the square lattice. We will refer to these as PP, PA, AP and AA. In the spin-glass region of the phase diagram, the variation of the sample properties for changes of the boundary conditions is small compared to the variation between different samples of the same size, except when \( p \) is close to the ferromagnetic phase boundary and the ferromagnetic correlation length becomes comparable to \( L \).

We define domain walls for the spin glass as it was done in the seminal work of McMillan. We look at differences between two samples with the same set of bonds, and the same boundary conditions in one direction, but different boundary conditions in the other direction. Thus, for each set of bonds we obtain domain wall data from the four pairs (PP,PA), (PP,AP), (AA,PA) and (AA,AP). The reader should remember that the term “domain wall”, as used in this work, refers only to this procedure. Saul and Kardar follow the same procedure used in this work, but use the term “defect” instead of “domain wall”.

The domain-wall renormalization group of McMillan is based on the idea that we are studying an effective coupling constant which is changing with \( L \). For the CDE case we can use the energy as the coupling constant. For the quantized energy case, what we need to
do is a slight generalization of this idea. We should think of the coupling constant as the free energy at some infinitesimal temperature. When we do this, the entropy contributes to the coupling constant. As we will see, the distribution of $E_{dw}$ rapidly becomes essentially independent of $L$ as $L$ becomes large, except that there are separate distributions for even $L$ and odd $L$. Under these conditions, it becomes possible to treat each value of $E_{dw}$ as a separate class, representing a different coupling constant.

The domain wall entropy, $S_{dw}$, is defined, by analogy to $E_{dw}$, to be the difference in the GS entropy when the boundary condition is changed along one direction from P to A (or vice versa), with the boundary condition in the other direction remaining fixed. $[S_{dw}]$, where the brackets $[ ]$ indicate an average over random samples of the $J_{ij}$, is expected to increase as a positive power of $L$ for any $E_{dw}$. Therefore, these coupling constants must eventually, at large enough $L$, be controlled by $[S_{dw}]$ for any $T > 0$. Of course, the value of $L$ which is needed for this to happen depends in $T$. The droplet model assumes that all these coupling constants, except for the $E_{dw} = 0$ case which has a special symmetry, are equal.

As long as $E_{dw} > 0$, the two boundary conditions which we are comparing are not on an equal footing. As Wang, Harrington and Preskill\textsuperscript{22} express the situation, the $E_{dw} > 0$ domain wall does not destroy the topological long-range order. However, in the $E_{dw} = 0$ case the two boundary conditions are on an equal footing, and the topological order is destroyed. Therefore the $E_{dw} = 0$ class of domain walls can be expected to behave in a special way, which differs from the prediction of the droplet model.

It is natural to wonder if topological long-range order can be related to replica-symmetry breaking, and if the $E_{dw} = 0$ domain walls can be described by the replica-symmetry breaking theory. We will not attempt to do this here.

It is important to realize that the meaning of a domain wall is very different when the GS entropy is positive, as in the model we study here, as compared to the standard case of a doubly degenerate ground state. In the standard case one can identify a line of bonds which forms a boundary between regions of spins belonging to the two different ground states. It is not possible, in general, to do that when there are many ground states. Despite this, we continue to use the term “domain wall”.

When $L$ is even, the energy difference, $E_{dw}$, for any pair must be a multiple of 4. When $L$ is odd, $E_{dw}$ is $4n + 2$, where $n$ is an integer. The sign of $E_{dw}$ for a pair is essentially arbitrary for $p = 1/2$. Thus we can, without loss of generality, choose all of the domain-wall
TABLE I: Domain wall energy statistics for $p = 0.5$ with even $L$. The number of random bond configurations studied for each $L$ was 500, and there are four McMillan pairs for each of these. $n_i$ is the number of domain walls of each type having $E_{dw} = k$. $f_k$ is the fraction of domain walls having $E_{dw} = k$.

| $L$ | $n_0$ | $n_4$ | $n_8$ | $f_0$  | $f_4$  |
|-----|-------|-------|-------|--------|--------|
| 8   | 1467  | 530   | 3     | 0.7335 | 0.265  |
| 12  | 1542  | 458   | 0     | 0.771  | 0.229  |
| 16  | 1515  | 484   | 1     | 0.7575 | 0.242  |
| 24  | 1578  | 422   | 0     | 0.789  | 0.211  |
| 32  | 1530  | 470   | 0     | 0.765  | 0.235  |
| 48  | 1546  | 450   | 4     | 0.773  | 0.225  |

energies to be non-negative.

III. NUMERICAL RESULTS

Our calculated statistics for $E_{dw}$ at $p = 0.5$, as a function of $L$, for even $L$ and odd $L$ are given in Table I and Table II, respectively. For each $L$, 500 distinct random configurations of bonds were studied. We obtain four McMillan pairs for each random sample, so we have 2000 sets of $E_{dw}$ and $S_{dw}$ at each $L$. For even $L > 10$ it turns out, crudely speaking, that about 77% of the time we find $E_{dw} = 0$, and 23% of the time $E_{dw} = 4$. For odd $L > 20$, $E_{dw} = 2$ about 98.5% of the time. No domain walls with energies greater than 8 were observed at any $L$ for these values of $p$. This, however, does not have much fundamental significance. The probability distribution for $E_{dw}$ is also a weak function of $p$, and a strong function of the aspect ratio of the lattice. Our results are consistent with the results of Amoruso et al.

It is interesting to note that Wang, Harrington and Preskill use an analytical argument to predict that $f_0$, the fraction of $E_{dw} = 0$ walls, is approximately 0.75, independent of $p$, in the spin-glass regime. However, the value of $f_0$ depends strongly on the aspect ratio of the lattice and it is not clear why this analytical argument should apply only when the aspect ratio is equal to one. It is also completely unclear to this author where the argument
TABLE II: Domain wall energy statistics for $p = 0.5$ with odd $L$. Column labels as in Table I.

| $L$ | $n_2$  | $n_6$  | $f_2$  |
|-----|--------|--------|--------|
| 7   | 1944   | 56     | 0.972  |
| 11  | 1960   | 40     | 0.980  |
| 15  | 1957   | 43     | 0.9785 |
| 21  | 1973   | 27     | 0.9865 |
| 29  | 1967   | 33     | 0.9835 |
| 41  | 1973   | 27     | 0.9865 |

FIG. 1: (color online) (a) Average $|S_{dw}|$ vs. $L$ for the $E_{dw} = 0$ domain walls, log-log plot. The error bars indicate one standard deviation. (b) Histogram of $|S_{dw}|$ for $E_{dw} = 0$ with $L = 48$. The vertical scale is logarithmic.

uses the fact that $E_{dw} = 0$ domain walls can only occur when $L$ is even.

Estimating the statistical uncertainties in the data precisely is not trivial, due to the fact that the values of $E_{dw}$ obtained from the same set of bonds with the four different pairs of boundary conditions are not statistically independent. An upper bound on the statistical uncertainties is obtained by counting the number of samples, rather than the number of McMillan pairs of boundary conditions.

The probability distribution of $S_{dw}$ for the cases where $E_{dw} = 0$ should be symmetric about 0, and our statistics are consistent with this. If we look at the $L$-dependence of
\[ \theta_S(0) = 0.500 \pm 0.020 \] (5)

for \( E_{dw} = 0 \). The result of Saul and Kardar\textsuperscript{18,19} obtained by looking at the distribution of \( S_{dw} \) for all values of \( E_{dw} \) combined, was \( \theta_S = 0.49 \pm 0.02 \). To obtain this exponent, Saul and Kardar fit their data at small values of \( S_{dw} \). When \( L \) is even, which was the case for all of their data, this part of the data belongs almost entirely to the \( E_{dw} = 0 \) component.\textsuperscript{21}

The calculated means and skewness of these essentially symmetric distributions for \( S_{dw} \) is, naturally, consistent with zero, but their kurtosis is not. The reason for this is shown in Fig. 1(b), which is a histogram for \(|S_{dw}|\) of \( E_{dw} = 0 \) when \( L = 48 \). We see that the distribution is approximately exponential, and therefore far from Gaussian. The computed kurtosis of this \( L = 48 \) distribution is 2.0, somewhat less than the value of 3 which would be found for an exact two-sided exponential distribution. The basic shape of these distributions is similar for the smaller values of \( L \), with the width of each distribution given by the square root of its variance.

When \( E_{dw} \) is not zero, the relative signs of \( E_{dw} \) and \( S_{dw} \) are not arbitrary. Having chosen \( E_{dw} \) to be nonnegative, we then find that, when \( E_{dw} \) is positive, it turns out that \( S_{dw} \) is usually positive. In Fig. 2(a) we show the behavior of \( |S_{dw}| \) for the cases where \( E_{dw} = k \), with \( k = 2, 4 \) and 6, as a function of \( L \). We see that for \( k > 0 \), the average value of \( S_{dw}(L) \)
TABLE III: Scaling exponents for the first and second cumulants of the $S_{dw}$ distributions. $\theta_S$ is the scaling exponent for $[S_{dw}]$, and $\phi_S$ is the scaling exponent for the variance of $(S_{dw})$.

| $E_{dw}$ | $\theta_S$       | $\phi_S$       |
|---------|------------------|----------------|
| 0       | $0.500 \pm 0.020$| $0.992 \pm 0.047$|
| 2       | $0.565 \pm 0.019$| $0.972 \pm 0.051$|
| 4       | $0.584 \pm 0.015$| $1.107 \pm 0.047$|
| 6       | $0.617 \pm 0.062$| $0.85 \pm 0.28$  |

grows approximately as $L^{0.58}$. More precisely, least-squares fits to the form

$$[S_{dw}(L)] = AL^{\theta_S}$$

(6)
gives the results for $\theta_S(k)$ shown in Table III. The result for $k = 6$ is rather uncertain, due to the small number of examples of this type. These results are consistent with the prediction of droplet theory$^8$ that $\theta_S$ should be independent of $k$ (aside from the $k = 0$ case, which is clearly exceptional). However, there also appears to be a tendency for $\theta_S(k)$ to increase as $k$ increases. Therefore, the possibility that $\theta_S \to 5/8$ as $k \to \infty$, which would be consistent with Eqn. (1), cannot be excluded by these data.

Because of the large GS degeneracy in the $\pm J$ Ising spin glass, one does not know how to compute $d_S$ directly for this model. However, if we use the droplet model prediction, that there is a single value for $\theta_S$, the result is not consistent with Eqn. (1). The author’s opinion is that Eqn. (1) must be generalized when $\eta > 0$.

As shown by Saul and Kardar$^{18,19}$ the variance of $S_{dw}$ when $E_{dw} = 0$ increases with $L$ in approximately a linear fashion. Calculating the variance of these distributions, and using linear least squares fits on the log-log plot shown in Fig. 2(b), we find that assuming the increase of the variance with $L$ is a power law gives the results shown in Table III. These numbers are reasonably consistent with the hypothesis that the scaling exponent for the variance of the $S_{dw}$ distributions is equal to 1, independent of $E_{dw}$. It is also interesting to observe that the magnitude of the variance, and not merely the slope of the fit, seems to be independent of $E_{dw}$. Except in the special $E_{dw} = 0$ case, $2\theta_S$ is greater than 1. Therefore, the exponent $d_S$ should be controlled by $\theta_S$, as predicted by Eqn. (2).
IV. SCALING OF THE DISTRIBUTIONS

In Fig. 3 we show histograms for the $S_{dw}$ distributions for $E_{dw} = 2$ at $L = 41$ and $E_{dw} = 4$ at $L = 48$. In contrast to the $E_{dw} = 0$ case, the skewness and kurtosis of the $S_{dw}$ distributions for $E_{dw} > 0$ are both small. It is possible that these distributions become Gaussian in the large $L$ limit. However, the author is not aware of any reason why this must happen.

The basic shapes of the histograms in Fig. 3(a) and Fig. 3(b) appear to be the same. Since $2\theta_S > 1$, it seems that the histogram for the $E_{dw} = 4$ case can be mapped onto the histogram for the $E_{dw} = 2$ case at a larger $L$. A way of expressing this is that for large $L$ the $S_{dw}$ histograms for $E_{dw} = k > 0$ should obey one-parameter scaling in the dimensionless variables

$$g_k(L) = \frac{[S_{dw}]^2}{[(S_{dw})^2] - [S_{dw}]^2}.$$  \hfill (7)

If $\theta_S$ is independent of $k$, then, assuming $\phi_S = 1$,

$$g_k(L) = (L/L_k)^{2\theta_S - 1},$$  \hfill (8)

where we define $L_k$ by the condition $g_k(L_k) = 1$.

What we have learned is that in this model there appear to be two distinct classes of domain walls, the $E_{dw} = 0$ domain walls and the $E_{dw} > 0$ domain walls. As we have seen,
the \( E_{dw} > 0 \) domain walls behave in a way which appears to be essentially consistent with the predictions of the droplet model, but the \( E_{dw} = 0 \) domain walls do not. This difference in behavior is due to the symmetry of the \( E_{dw} = 0 \) case, which forces the average \( S_{dw} \) to be zero.

For an \( E_{dw} > 0 \) domain wall, a large contribution to \( S_{dw} \) comes from the shift in the average GS entropy with the shift in the GS energy. What remains to be understood is why \( |S_{dw}| \) should scale with \( L \) in the way predicted by the droplet model. The conventional derivation of the droplet model uses the assumption that the GS is unique, up to a reversal of the entire state, in an essential way. What follows immediately from this is that \( \eta = 0 \). An extension of the droplet model to the more general case was given by Fisher and Huse. However, the author hopes that by now he has convinced the reader that a better understanding of the \( \eta > 0 \) case is needed.

V. SUMMARY

We have studied the statistics of domain walls for ground states of the 2D Ising spin glass with +1 and -1 bonds for \( L \times L \) square lattices with \( L \leq 48 \), and \( p = 0.5 \), where \( p \) is the fraction of negative bonds, using periodic and/or antiperiodic boundary conditions, for both even and odd \( L \). Under these conditions, most domain walls have an energy \( E_{dw} < 8 \). The probability distribution of the entropy, \( S_{dw} \), is found to depend strongly on \( E_{dw} \), but it appears possible to parameterize this dependence in a simple way. The results for \( S_{dw} \) do not appear to agree quantitatively with the prediction of Amoruso, Hartmann, Hastings and Moore, Eqn. (1). Our results for \( |S_{dw}| \) when \( E_{dw} = 0 \) agree with those of Saul and Kardar, but in addition we find that the distributions are close to being exponential in that case, even in the limit of large \( L \). Due to the special role of the \( E_{dw} = 0 \) domain walls, we can understand the difference between the scaling exponent found by Saul and Kardar and the prediction of the droplet model.

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