ON EXISTENCE OF $L^2$-SOLUTIONS OF COUPLED BOLTZMANN CONTINUOUS SLOWING DOWN TRANSPORT EQUATION SYSTEM

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Abstract. The paper considers a coupled system of linear Boltzmann transport equations (BTE), and its Continuous Slowing Down Approximation (CSDA). This system can be used to model the relevant transport of particles used e.g. in dose calculation in radiation therapy. The evolution of charged particles (e.g. electrons and positrons) are in practice often modelled using the CSDA version of BTE because of the so-called forward peakedness of scattering events contributing to the particle fluencies (or particle densities), which causes severe problems in numerical methods. We shall verify, after the preliminary discussion, that CSDA-type modelling is actually necessary due to hyper-singularities in the differential cross-sections of certain interactions, that is, first-order partial derivatives with respect to energy must be included into the transport part of charged particles. The existence and uniqueness of (weak) solutions is shown, under sufficient criteria and in appropriate $L^2$-based spaces, for a single (particle) CSDA-equation by using three techniques, the Lions-Lax-Milgram Theorem (variational approach), the theory of $m$-dissipative operators and the theory evolution operators (semigroup approach). The necessary a priori estimates are derived. In addition, we prove the corresponding results and estimates for the system of coupled transport equations. The related results are given for the adjoint problem as well. We also give some computational points (e.g. certain explicit formulas), and we outline a related inverse problem at the end of the paper.

1. Introduction

The Boltzmann transport equation (BTE) models changes of the number density of particles in phase space (position, velocity direction, energy). In this paper the species of particles include photons, electrons and positrons and the explored analysis of transport equations is mainly intended for dose calculation in radiation treatment planning. However, various other kinds of transport phenomena can be modelled by equations of similar type including in e.g. transport of particles in optical tomography ([4], [6]), in cosmic radiation ([50]) and in solid state physics ([77]). For general theory of linear BTE with relevant boundary conditions we refer to [20] and [2]. See also [13], [14], [22], [57] where the subject is considered from a physical point of view. For some recent issues (including certain inverse problems) related to linear BTE can be found in [50], and general non-linear aspects e.g. in [74], [9]. A thorough mathematical survey (mathematical and physical foundations, results, problems) of non-linear collision theory of particle transport is given in [77]. This survey is mainly intended to collision processes in dilute gases and plasmas but

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analogous results and problems arise in other fields of particle physics. Finally, for topics related to Monte–Carlo methods in the context of BTE, both from theoretical and practical point of view, we refer to [10], [65] and [66].

Dose calculation is of crucial importance in radiation therapy. Relevant dose calculation models require (approximate) solution of a coupled system of (linear) transport equations for fluencies (number densities in the phase space) for all considered particles. This is a difficult problem, at least from computational point of view, due to the different particle species and their dependence on a high–dimensional phase space. For that reason traditional dose calculation algorithms have applied some closed-form formulas which have their origins in analytical solutions, or Monte–Carlo derived solutions, of simplified problems. The latter however contain often empirically derived corrections to take more accurately into account the underlying particle physics ([49], [65]). Certain "factors" which account for e.g. the spatial inhomogeneities must be included to improve the accuracy of the final solution. These approaches lead to methods that are fast enough but have typically a limited accuracy. Commonly used models are based on the so-called pencil beams, or point kernels, see [7], [11], [41], [49], [72], [75] for more details. A notable exception to these approximate (deterministic) methods is the Acuros code [76], which is based on a discretization of the BTE.

In radiation therapy BTE describes the evolution of radiative particles due to scattering and absorption in tissue. The dose delivery methods can be roughly divided into two categories. In external therapy the sources (below denoted by $g$) of high energy particles (usually photons, electrons or protons) are on the patches of patient’s surface. In internal therapy, on the other hand, the sources (below denoted by $f$) are inside the patient close to the cancerous tissue. In the energy range, say up to 25 MeV, relevant for photon and electron therapy, the three species of particles whose simultaneous evolution should be taken into account in a realistic transport model, are photons, electrons and positrons. In this setting, the potential creation of (or contamination by) other heavy particles will not be taken into account since their contribution to the dose is negligible (see [65]).

The transport of relevant particles in tissue (in an appropriate energy range) can be modelled by the following linear coupled system of three BTEs

$$\omega \cdot \nabla_x \psi_j(x, \omega, E) + \Sigma_j(x, \omega, E)\psi_j(x, \omega, E) - (K_j\psi_j)(x, \omega, E) = f_j(x, \omega, E),$$

for $j = 1, 2, 3$, combined with an inflow boundary condition (for the definition of $\Gamma_-$, see section [2.1])

$$\psi_j|_{\Gamma_-} = g_j, \quad j = 1, 2, 3,$$

where for $j = 1, 2, 3$,

$$(K_j\psi_j)(x, \omega, E) = \sum_{k=1}^{3} \int_{S \times I} \sigma_{kj}(x, \omega', \omega, E', E)\psi_k(x, \omega', E')d\omega'dE'.$$

For a derivation of linear BTE, see e.g. [2], [3], [22], [67]. The first term on the left in (1) is called a convection (or advection) operator, the second term is a (total) scattering operator and the third one is a collision operator. Notice that the (total) scattering operator

$$\Sigma = \Sigma_a + \Sigma_s$$

(we drop the index $j$ here to simplify the notation) contains contribution from both the absorption (term $\Sigma_a$) and the scattering (term $\Sigma_s$), see [67], Sec. 9.1. On the
right in (1), the functions $f_j$ represent (internal) sources and $g_j$ in (2) are (inflow) boundary sources. The system is coupled through the integral operators $K_j$ (unless, of course, $\sigma_{kj} = 0$ for $j \neq k$). The solution $\psi = (\psi_1, \psi_2, \psi_3)$ of the problem (1)-(2) is a vector-valued function whose components describe the radiation fluxes of photons, electrons and positrons, respectively. Roughly speaking, the flux $\psi(x, \omega, E)$ is the flux of energy through a surface located at $x$ and normal to the direction $\omega$. The particle number density $N$, which is another usual unknown in kinetic theory, is related to $\psi$ by $\psi = \|v\| N$, where $\|v\|$ is the particle speed ([67]), which is often relativistic.

The equation (1) is a steady state counterpart of the dynamical equation

$$\frac{1}{\|v_j\|} \frac{\partial \psi_j}{\partial t} + \omega \cdot \nabla_x \psi_j + \Sigma_j \psi_j - K_j \psi = f_j, \quad j = 1, 2, 3,$$

(4)

where $v_j$ is the velocity of the $j$-th particle type. In radiation therapy related applications, it is sufficient to consider the steady state equations because the flux $\psi$ reaches the steady state nearly instantly ([12]). The existence of solutions for the problem (1), (2), as well as for the time-dependent problem (4), (2) (with an appropriate initial condition) in $L^1$-based spaces has been studied in [71] (the results of which remaining valid, after slight modifications, for any $1 \leq p < \infty$). In [71] it is assumed that the collision operator $K$ satisfies a so-called Schur criterion (for boundedness) which is not valid for all species of particle interactions.

The differential cross-sections may have singularities, or even hyper-singularities, and in these cases the integral $\int_{S \times I}$ appearing in the collision term must be interpreted as the Hadamard finite part integral. In section 2.3 we shall present some details of real, physical collision operators. In particular, we find that certain differential cross sections may contain hyper-singularities like $\frac{1}{(E' - E)^2} dE'$. These kind of singularities lead to extra partial differential (or pseudo-differential) terms in the transport equation ([38, Sec. 7.1, pp. 353–394]). The analysis reveals the exact form of transport operators, and serves as a basis for better approximations and error analysis. These considerations, however, would require further knowledge about the regularity properties of solutions, which, to our understanding, remains an open question. Likewise the existence analysis of (non-CSDA) exact (coupled) system of transport equations is open. In this work, we demonstrate how the, here considered, CSDA-approximation follows from "hyper-singular analysis" and how the CSDA equations can be derived. When the collision terms containing hyper-singularities are removed, the remaining operators are, however, in Schur form (at least approximately).

Furthermore, we notice that in the expression of $K_j$ the integration is performed only with respect to $\omega'$ and $E'$, while $x$-variable is kept fixed. Therefore, $K_j$ is only a so-called partial (hyper-singular) integral operator (cf. [5]). These facts imply that the familiar properties of (singular) integral operators (compactness, for example) are not valid even for restricted collision operators.

Due to the above mentioned hyper-singularities, the transport of new primary electrons (and positrons) is forward-peaked. This implies that the grid in numerical computations needs to be very tight in order to achieve reliable results. Traditionally, to overcome the computational complexity, one has applied to the evolution of electrons and positrons the so-called continuous slowing down approximation (CSDA). The CSDA equations are vastly applied e.g. in various cosmic radiation problems ([61], [80]). The analysis given in section 2.3 show that the exact transport model
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inherently contains first order partial differential (pseudo-differential) terms, and so the traditional use of CSDA-approximation can be justified.

The CSDA-method replaces Eqs. (1) for $\psi_j$, $j = 2, 3$ (electrons and positrons) by the following equations \([80], [41], [28]\)

$$\frac{\partial (S_{j,r} \psi_j)}{\partial E} + \omega \cdot \nabla_x \psi_j + \sum_{j,r} \psi_j - K_{j,r} \psi = f_j, \quad j = 2, 3,$$

where

$$(K_{2,r} \psi)(x, \omega, E) = \int_{S \times I} \sigma_{12}(x, \omega', \omega, E') \psi_1(x, \omega', E') d\omega' dE' + \int_{S \times I} \sigma_{22,r}(x, \omega', \omega, E') \psi_2(x, \omega', E') d\omega' dE' + \int_{S \times I} \sigma_{32}(x, \omega', \omega, E') \psi_3(x, \omega', E') d\omega' dE',$n

and

$$(K_{3,r} \psi)(x, \omega, E) = \int_{S \times I} \sigma_{13}(x, \omega', \omega, E') \psi_1(x, \omega', E') d\omega' dE' + \int_{S \times I} \sigma_{23}(x, \omega', \omega, E') \psi_2(x, \omega', E') d\omega' dE' + \int_{S \times I} \sigma_{33,r}(x, \omega', \omega, E') \psi_3(x, \omega', E') d\omega' dE'.$n

Here, for $j = 2, 3$, the functions $\Sigma_{j,r} = \Sigma_{j,r}(x, \omega, E)$ are the restricted total cross-sections, $S_{j,r} = S_{j,r}(x, E)$ are the restricted stopping powers, and $\sigma_{j,j,r}(x, \omega', \omega, E', E)$ are the restricted differential cross-sections, which do not include soft inelastic interactions \([10]\). Besides the inflow boundary condition (2), one needs to impose on the solutions $\psi_j$, $j = 2, 3$, of (5) an energy-boundary condition. We make the reasonable assumption that at some (high enough, but finite) cut-off energy $E_m > 0$ the fluxes $\psi_2, \psi_3$ vanish, i.e. we take the energy-boundary condition to be

$$\psi_2(x, \omega, E_m) = \psi_3(x, \omega, E_m) = 0,$n

(below, we often call this an initial condition as well). One could also demand the cut-off energy $E_m$ to be infinite, in which case the corresponding energy boundary condition would become

$$\lim_{E \to \infty} \psi_2(x, \omega, E) = \lim_{E \to \infty} \psi_3(x, \omega, E) = 0.$n

However, we shall restrict our discussion to the case where $E_m$ is finite. The requirement that such energy initial condition be satisfied by $\psi_2, \psi_3$, makes the overall problem mathematically well-posed, that is, under relevant (physical) assumptions the problem has a unique (weak) solution.

We give here a short heuristic derivation of CSDA based on the presentation given in \([80, pp. 14-17]\), while a more transparent justification of it will be given in section 2.3. Firstly, we decompose for $k = 2, 3$ the differential cross-section into two parts

$$\sigma_{kk}(x, \omega', \omega, E', E) = \sigma_{kk}^{at}(x, \omega', \omega, E', E) + \sigma_{kk}^{nu}(x, \omega', \omega, E', E), \quad k = 2, 3,$n

where "at" refers to interactions with atomic electrons, and "nu" refers to nuclear interactions. The term $\sigma_{kk}^{at}(x, \omega', \omega, E', E)$ contains the "problematic" features that
are to be approximated by the continuous slowing down model. Secondly, we write formally

\[ \sigma_{kk}^{at}(x, \omega', \omega, E', E) = \sum_{n=0}^{N} \sigma_{k,n}^{at}(x, E + \epsilon_n) \delta_{\omega}(\omega') \delta_{E+E_n}(E'), \]

where \( \delta_{\omega} \) is the Dirac measure on the 2-dimensional unit sphere \( S = S_2 \) concentrated at \( \omega \), and \( \delta_{E+E_n} \) is the Dirac measure on \( \mathbb{R} \) concentrated at \( E + \epsilon_n \). This approximation assumes small changes in energy but not changes in angular direction. Applying these concepts and Taylor’s formula up to first–order, we find that

\[ \int_{S \times I} \sigma_{kk}^{at}(x, \omega', \omega, E', E) \psi_k(x, \omega', E') d\omega' dE' = \sum_{n=0}^{N} \sigma_{k,n}^{at}(x, E + \epsilon_n) \psi_k(x, \omega, E + \epsilon_n) \]

\[ \approx \sum_{n=0}^{N} \sigma_{k,n}^{at}(x, E) \psi_k(x, \omega, E) + \sum_{n=0}^{N} \frac{\partial (\sigma_{k,n}^{at} \psi_k)}{\partial E}(x, \omega, E) \epsilon_n. \]  

(6)

The restricted total atomic cross-sections and the restricted stopping powers are

\[ \Sigma_{k,r}^{at}(x, E) := \sum_{n=0}^{N} \sigma_{k,n}^{at}(x, E), \quad S_{k,r}(x, E) := \sum_{n=0}^{N} \sigma_{k,n}^{at}(x, E) \epsilon_n. \]

Let \( \sigma_{kr} := \sigma_{kk}^{at} \). Then by (6) we obtain for \( k = 2, 3 \),

\[ \int_{S \times I} \sigma_{kk}(x, \omega', \omega, E', E) \psi_k(x, \omega', E') d\omega' dE' \]

\[ = \int_{S \times I} \sigma_{kk}^{at}(x, \omega', \omega, E', E) \psi_k(x, \omega', E') d\omega' dE' + \int_{S \times I} \sigma_{kk,r}(x, \omega', \omega, E', E) \psi_k(x, \omega', E') d\omega' dE' \]

\[ \approx \Sigma_{k,r}(x, E) \psi_k(x, \omega, E) + \frac{\partial (S_{k,r} \psi_k)}{\partial E}(x, \omega, E) + \int_{S \times I} \sigma_{kk,r}(x, \omega', \omega, E', E) \psi_k(x, \omega', E') d\omega' dE'. \]  

(7)

Finally, writing \( \Sigma_{j,r} := \Sigma_{j} - \Sigma_{j,r}^{at} \) and substituting (7) into (1), we obtain approximately the equations (3) for \( j = 2, 3 \).

The so-called CSDA-Fokker-Planck model includes (cf. (14)) additionally (roughly speaking) the second order partial derivatives with respect angle in the above Taylor expansion (for \( \psi_2, \psi_3 \)). It is a further approximation that in addition to small energy changes assumes small angle changes for the scattering. The hierarchy of the Boltzmann equation, its CSDA approximation and the Fokker-Planck approximation is detailed in Appendix A of [12]. Typically, the Fokker-Planck approximation is considered not valid for electrons in tissues [42]. In addition, in [50] a formal asymptotic analysis is performed that sheds light onto the validity of these approximations. For example, it was shown that the Fokker-Planck approximation is not valid for the often used Heney-Greenstein kernel.

The dose \( D(x) = (D \psi)(x) \) is calculated from the solution of the problem

\[ \omega \cdot \nabla_x \psi_1 + \Sigma_1 \psi_1 - K_1 \psi = f_1, \]  

\[ - \frac{\partial (S_{j,r} \psi_j)}{\partial E} + \omega \cdot \nabla_x \psi_j + \Sigma_{j,r} \psi_j - K_{j,r} \psi = f_j, \quad j = 2, 3, \]  

\[ \psi_j |_{\Gamma_-} = g_j, \quad j = 1, 2, 3, \]  

\[ \psi_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3, \]  

(11)
by

\[ D(x) = \sum_{j=1}^{3} \int_{S \times I} \varsigma_j(x, E) \psi_j(x, \omega, E) d\omega dE, \tag{12} \]

where \( \varsigma_j(x, E) \) are stopping powers, which in general can be different from the restricted stopping powers \( S_{j,r} \). The dose calculation is a forward problem. The determination of the external particle flux \( g = (g_1, g_2, g_3) \) and/or the distribution of internal source \( f = (f_1, f_2, f_3) \) is called inverse radiation treatment planning problem (IRTP) which is an inverse problem. It always requires a dose calculation model. We refer to [63], [71], [78] and references therein for some details concerning the IRTP-problem. In [28] the IRTP-problem has been studied in the context of CSDA-equation for a single particle (when the stopping power is independent of \( x \)). See also [10] where related spatially 3-dimensional numerical simulations (real case simulations applying finite element methods, FEM) have been explored.

This paper contains several novel contributions to the study of particle transport in tissues. In the beginning we discuss the preliminaries including details (many of which are reproductions of known results) of the so-called escape-time mapping and inflow trace theory. These tools are essential in the treated analysis. After that we consider the existence and uniqueness of solutions for a single (particle) CS DA equation

\[ -\frac{\partial (S_0 \psi)}{\partial E} + \omega \cdot \nabla_x \psi + \Sigma \psi - K \psi = f, \tag{13} \]

\[ \psi_{|_{\Gamma_-}} = g, \tag{14} \]

\[ \psi(\cdot, \cdot, E_m) = 0, \tag{15} \]

in \( L^2(G \times S \times I) \)-based spaces. Here \( G \subset \mathbb{R}^3 \) is the spatial domain, \( S = S_2 \subset \mathbb{R}^3 \) is the unit sphere and \( I = [0, E_m] \) is the energy interval. The set \( \Gamma_- \) is "the inflow boundary part of \( \partial G \)." We extend the results of [28], where a spatially homogeneous stopping power was assumed. In addition, we notice that the single CSDA equation is (symmetric) hyperbolic in nature, and so the hyperbolic theory (e.g. [58]) is applicable at least in the case where \( K = 0 \) and \( G = \mathbb{R}^3 \). For \( G \neq \mathbb{R}^3 \) the existence and regularity of solutions is more subtle because of the inflow boundary condition.

We also remark that the coupled system \((8), (9)\) is not hyperbolic.

In section 3.2 we give a variational formulation of the problem \((13), (14), (15)\) in appropriate spaces. We give sufficient conditions under which the corresponding bilinear and linear forms obey the assumptions of the so-called Lions-Lax-Milgram Theorem, which then provides an existence result for solution of the equation \((13)\). Under a certain additional assumption on traces, this solution is unique and, additionally, it satisfies the boundary and initial conditions \((14), (15)\). In Section 3.3 one applies the classical theory of (formally) dissipative operators (e.g. by [31], [43], [59]), and it is via this approach that we achieve the most satisfactory existence and uniqueness result for the problem considered. In section 3.5 we show the existence and uniqueness results for the solution of the problem \((13), (14), (15)\) by using a yet another technique, based on the theory of evolution equations and \((m-)\)-dissipative operators, but for a restricted class of collision operators \(K\).

In section 4 we extend the corresponding existence and uniqueness results for the coupled transport system \((8)-(11)\), which has not been studied in the literature, in \( L^2(G \times S \times I)^3 \)-based spaces. We also show certain a priori estimates (needed e.g. in section 7) which in particular show (under specific assumptions) that the solution
depends continuously on the data. Analogous results for the adjoint problem are formulated in section \[5\]. Section \[6\] considers certain computational aspects. The emphasis is on how to calculate numerical solution (for the forward problem), in principle, without inversion of (huge) matrices. Finally, in the last section \[7\] we outline a related IRTP-problem but its thorough study remains open for future work.

2. Preliminaries

2.1. Notations, Assumptions and Introduction of Relevant Function Spaces.

We assume that \( G \) is an open bounded connected set in \( \mathbb{R}^3 \) such that \( G \) is a \( C^1 \)-manifold with boundary (as a submanifold of \( \mathbb{R}^3 \); cf. \[44\]). In particular, it follows from this definition that \( G \) lies on one side of its boundary.

The unit outward (with respect to \( G \)) pointing normal on \( \partial G \) is denoted by \( \nu \), and the surface measure (induced by the Lebesgue measure) on \( \partial G \) is written as \( \sigma \). We let \( S = S_2 \) be the unit sphere in \( \mathbb{R}^3 \) equipped with the usual rotationally invariant surface measure \( \mu_S \).

Definition 2.1 For \((x, \omega) \in G \times S\) the escape time (in the direction \( \omega \)) \( t(x, \omega) = t_-(x, \omega) \) is defined by

\[
t(x, \omega) = \inf \{ s > 0 \mid x - s\omega \notin G \} \quad \text{for } (x, \omega) \in G \times S.
\]

Furthermore, let \( I = [0, E_m] \) where \( 0 < E_m < \infty \). We could replace \( I \) by \( I = [E_0, E_m] \) or \( I = [E_0, \infty] \) where \( E_0 \geq 0 \) but we omit this generalization here. We shall denote by \( I^o \) the interior \( ]0, E_m[ \) of \( I \). The interval \( I \) in equipped with the 1-dimensional Lebesgue measure \( \mathcal{L}^1 \), which we typically write as \( dE \) in the sense that\n
\[
\mathcal{L}^1(A) = \int_A dx, \quad \sigma(A) = \int_A d\sigma, \quad \mu_S(A) = \int_A d\omega.
\]

The escape time function \( t(\cdot, \cdot) \) is known to be lower semicontinuous in general, and continuous if \( G \) is convex, see e.g. \[71\].

We define

\[
\Gamma := (\partial G) \times S \times I,
\]

and

\[
\Gamma_0 := \{(y, \omega, E) \in \Gamma \mid \omega \cdot \nu(y) = 0\}
\]

\[
\Gamma_- := \{(y, \omega, E) \in \Gamma \mid \omega \cdot \nu(y) < 0\}
\]

\[
= \{(y, \omega) \in \partial G \times S \mid \omega \cdot \nu(y) < 0\} \times I
\]

\[
\Gamma_+ := \{(y, \omega, E) \in \Gamma \mid \omega \cdot \nu(y) > 0\}
\]

\[
= \{(y, \omega) \in \partial G \times S \mid \omega \cdot \nu(y) > 0\} \times I.
\]
Let \( \mu_G = \sigma \otimes \mu_S \otimes \mathcal{L}^1 \), written typically as \( d\sigma d\omega dE \) in the same sense as discussed above. It follows that \( \mu_G(\Gamma_0) = 0 \) and
\[
\Gamma = \Gamma_0 \cup \Gamma_- \cup \Gamma_+.
\]
Let
\[
N_0 := \{(x, \omega, E) \in G \times S \times I \mid \omega \cdot \nu(x - t(x, \omega)\omega) = 0\}, \tag{17}
\]
and
\[
D := (G \times S \times I) \setminus N_0. \tag{18}
\]
Recall that \( N_0 \) has a measure zero in \( G \times S \times I \) ([61], Theorem 3.8])

**Definition 2.2** Define escape-time mappings \( \tau_{\pm}(y, \omega) \) from boundary to boundary in the direction \( \omega \) as follows
\[
\begin{align*}
\tau_-(y, \omega) &:= \inf\{s > 0 \mid y + sw \notin G\}, \quad (y, \omega) \in \partial G \times S, \quad (19) \\
\tau_+(y, \omega) &:= \inf\{s > 0 \mid y - sw \notin G\}, \quad (y, \omega) \in \partial G \times S. \quad (20)
\end{align*}
\]
Note that for \( (y, \omega, E) \in \Gamma_- \) the vector \( (y_+, \omega, E) \) \( \tau_+(y_+, \omega) \) where \( y_+ := y + \tau_-(y, \omega)\omega \in \Gamma_+ \) and
\[
\tau_-(y, \omega) = \tau_+(y_+, \omega).
\]

**Remark 2.3** For most purposes of this paper, we could have chosen, instead of the particular measure \( \mu_S \), any positive Radon measure \( \rho \) on (Borel sets of) the unit sphere \( S \), as long as relevant additional assumptions are supposed, for example that \( \Gamma_0 \) has measure zero with respect to \( \sigma \otimes \rho \otimes \mathcal{L}^1 \), i.e. \( \int_G \chi_{\Gamma_0}(y, \omega, E)d\sigma(y)d\rho(\omega)dE = 0 \). Note that it follows from this assumption that \( N_0 \) has measure zero as well.

Especially, the main existence and uniqueness results for the solutions of the CSDA Boltzmann transport problem, as given in Sections 3 and 4 (Theorems 3.8, 3.22, 3.36, 4.8, 4.10, 4.12, along with their corollaries), as well as the non-negativity result of Subsection 3.4, remain essentially true even with this more general choice of measure on \( S \).

Below the maps \( t : G \times S \to [0, \infty] \) and \( \tau_\pm : \partial G \times S \to [0, \infty] \), will often be interpreted as maps \( t : G \times S \times I \to [0, \infty] \) and \( \tau_\pm : \Gamma \to [0, \infty] \), by dropping off the energy variable, e.g. \( t(x, \omega, E) = t(x, \omega) \).

**Lemma 2.4** Let \( (y_0, \omega_0) \in \partial G \times S \) be such that there exists a bounded open cone \( C := \{x \in \mathbb{R}^3 \mid |x|^2 - \frac{a}{\|x\|^2} < r, 0 < \|x\| < r\} \subset \mathbb{R}^3, a \neq 0 \) containing \( -\omega_0 \) such that for some \( \lambda_0 > 0 \) one has \( y_0 + \lambda_0 C \subset \mathbb{R}^3 \setminus G \). Then we have the following:

(i) The map \( t(\cdot, \cdot) \) is continuous at any point \( (x_0, \omega_0) \in G \times S \) such that \( y_0 = x_0 - t(x_0, \omega_0)\omega_0 \).

(ii) Letting \( y_+ = y_0 + \tau_-(y_0, \omega_0)\omega_0 \), one has the following limits
\[
\begin{align*}
\lim_{(x, \omega) \to (y_0, \omega_0)} t(x, \omega) &= 0, \\
\lim_{(x, \omega) \to (y_+, \omega_0)} t(x, \omega) &= \tau_-(y_0, \omega_0),
\end{align*}
\]
where \( (x, \omega) \in G \times S \) when taking the limits.
Proof. By using charts of a $C^1$-manifold with boundary and the assumption, there exists a bounded open cone $C$ containing $-\omega_0$, a $\lambda'_0 > 0$ and a neighbourhood $U$ of $y_0$ in $\mathbb{R}^3$ such that

$$y + \lambda'_0 C' \subset \mathbb{R}^3 \setminus G, \quad \forall y \in U \cap \partial G.$$ 

(i) First, let us show that $t(\cdot, \cdot)$ is lower-semicontinuous on $G \times S$. Indeed, if $(x, \omega) \in G \times S$ and $0 < s < t(x, \omega)$, we have $x - [0, s] \omega := \{x - t \omega : t \in [0, s]\} \subset G$. Thus if $(x_n, \omega_n)$ is a sequence in $G \times S$ converging to $(x, \omega)$, one has $x_n - [0, s] \omega_n \subset G$ for all large enough $n$, which implies $t(x_n, \omega_n) \geq s$, and thus $\lim \inf_{n \to \infty} t(x_n, \omega_n) \geq s$. Letting $s \to t(x, \omega)$ from the left, we conclude that $\lim \inf_{n \to \infty} t(x_n, \omega_n) \geq t(x, \omega)$, which gives the lower semi-continuity we were to show.

It remains to show that $t(\cdot, \cdot)$ is upper semi-continuous at $(x_0, \omega_0)$ which then implies that $t(\cdot, \cdot)$ is continuous at $(x_0, \omega_0)$ since we already known that $t(\cdot, \cdot)$ is lower-semicontinuous at $(x_0, \omega_0)$.

Indeed, let $(x_n, \omega_n) \to (x_0, \omega_0)$ when $n \to \infty$. Fix $\lambda \in [0, \lambda_0]$. Then

$$y_0 - \lambda \omega_0 \in y_0 + \lambda C' \subset y_0 + \lambda_0 C' \subset \mathbb{R}^3 \setminus G,$$

from which it follows that for all large enough $n$ (which we assume from now on),

$$x_n - (t(x_0, \omega_0) + \lambda) \omega_n \in y_0 + \lambda_0 C' \subset \mathbb{R}^3 \setminus G.$$ 

Therefore, $t(x_n, \omega_n) \leq t(x_0, \omega_0) + \lambda$, and hence

$$\lim \sup_{n \to \infty} t(x_n, \omega_n) \leq t(x_0, \omega_0) + \lambda,$$

for all $\lambda > 0$, which gives us the desired upper semi-continuity at $(x_0, \omega_0)$,

$$\lim \sup_{n \to \infty} t(x_n, \omega_n) \leq t(x_0, \omega_0).$$

(ii) Arguments are analogous to the ones employed in case (i). Consider first the case $(x_n, \omega_n) \to (y_0, \omega_0)$ where $(x_n, \omega_n) \in G \times S$. Then $y_0 - \lambda_0' \omega_0 \in y_0 + \lambda'_0 C' \subset \mathbb{R}^3 \setminus G$ implies that for all $n$ large enough, one has $x_n - \lambda \omega_n \in y_0 + \lambda'_0 C' \subset \mathbb{R}^3 \setminus G$ for all $0 < \lambda \leq \lambda'_0$, hence $t(x_n, \omega_n) \leq \lambda$, from which $\lim \sup_{n \to \infty} t(x_n, \omega_n) \leq \lambda$, and finally $\lim \sup_{n \to \infty} t(x_n, \omega_n) = 0$.

We then suppose that $(x_n, \omega_n) \to (y_0, \omega_0)$. Since $y_0 = y_+ - \tau_-(y_0, \omega_0) \omega_0$ and $y_0 - \lambda'_0 \omega_0 \in y_0 + \lambda'_0 C' \subset \mathbb{R}^3 \setminus G$, one has for all $n$ large enough and all $0 < \lambda \leq \lambda'_0$, that

$$x_n - (\tau_-(y_0, \omega_0) + \lambda) \omega_n \in y_0 + \lambda'_0 C' \subset \mathbb{R}^3 \setminus G,$$

whence $t(x_n, \omega_n) \leq \tau_-(y_0, \omega_0) + \lambda$, which gives the upper limit

$$\lim \sup_{n \to \infty} t(x_n, \omega_n) \leq \tau_-(y_0, \omega_0).$$

In order to obtain the corresponding lower limit, which then shows the existence and the correct value of the limit we were set out to show, notice that if $0 < \sigma < \tau_-(y_0, \omega_0)$, then $y_0 + \sigma \omega_0 \in G$, hence for all $n$ large enough $x_n - (\tau_-(y_0, \omega_0) - \sigma) \omega_n \in G$ (since $x_n - (\tau_-(y_0, \omega_0) - \sigma) \omega_n \to x_0 - (\tau_-(y_0, \omega_0) - \sigma) \omega_0 = y_0 + \sigma \omega_0$), which implies that $t(x_n, \omega_n) \geq \tau_+(y_+, \omega_0) - \sigma$, and thus $\lim \sup_{n \to \infty} t(x_n, \omega_n) \geq \tau_-(y_0, \omega_0) - \sigma$. Letting $\sigma \to 0+$ allows us to conclude that

$$\lim \inf_{n \to \infty} t(x_n, \omega_n) \geq \tau_-(y_0, \omega_0).$$

\[\Box\]
Lemma 2.5 Let \((x_0, \omega_0, E_0) \in (G \times S \times I) \cup \Gamma_+ \cup \Gamma_-.\) Then

\[
\lim_{(x, \omega, E) \to (x_0, \omega_0, E_0)} t(x, \omega) = \begin{cases} 
  t(x_0, \omega_0), & \text{if } (x_0, \omega_0, E_0) \in G \times S \times I \\
  \tau_+(x_0, \omega_0), & \text{if } (x_0, \omega_0, E_0) \in \Gamma_+ \text{ and } (x_0 - \tau_+(x_0, \omega_0)\omega_0, \omega_0, E_0) \in \Gamma_-, \\
  0, & \text{if } (x_0, \omega_0, E_0) \in \Gamma_-.
\end{cases}
\]

where \((x, \omega, E) \in G \times S \times I\) when taking the limits.

\[\text{Proof.}\] Define

\[
z_0 = \begin{cases} 
  x_0 - t(x_0, \omega_0)\omega_0, & \text{if } (x_0, \omega_0, E_0) \in G \times S \times I \\
  x_0 - \tau_+(x_0, \omega_0)\omega_0, & \text{if } (x_0, \omega_0, E_0) \in \Gamma_+ \\
  x_0, & \text{if } (x_0, \omega_0, E_0) \in \Gamma_-.
\end{cases}
\]

In each of the limits concerned, we have thus assumed that \((z_0, \omega_0, E_0) \in \Gamma_-\), which implies, using the fact that \(\overline{G}\) is \(C^1\)-manifold with boundary and \(\omega_0 \cdot \nu(z_0) < 0\), that there is an open finite cone \(C \subset \mathbb{R}^3\) and \(\lambda_0 > 0\) such that \(z_0 + \lambda_0 C \subset \mathbb{R}^3 \setminus G\) and \(-\omega_0 \in C\). The claim then follows from Lemma 2.4. Note in particular that when considering the limit \(t(x, \omega) = \tau_+(x_0, \omega_0) = \tau_-(y_0, \omega_0)\) one takes in the lemma \(y_0 = x_0 - \tau_+(x_0, \omega_0)\omega_0\) and (hence) \(y_+ = x_0\). □

Proposition 2.6 Define

\[\Gamma_+ :\) := \{(y, \omega, E) \in \Gamma_+ \mid (y - \tau_+(y, \omega)\omega, \omega, E) \in \Gamma_-\}.\]

Then \(\Gamma_+ \setminus \Gamma_+ :\) has zero-measure in \(\Gamma\) and there is a continuous extension \(\overline{t}\) of \(t : D \to \mathbb{R}\) onto \(D_- := D \cup \Gamma_- \cup (\Gamma_+ :\)) given by

\[
\overline{t}(x, \omega) = \begin{cases} 
  t(x, \omega), & \text{if } (x, \omega, E) \in D, \\
  \tau_+(x, \omega), & \text{if } (x, \omega, E) \in (\Gamma_+ :), \\
  0, & \text{if } (x, \omega, E) \in \Gamma_-.
\end{cases}
\]

\[\text{Proof.}\] The result is a straightforward consequence of the above Lemmas. □

Example 2.7 Note that in the case where \(G\) is convex and its boundary is \(C^1\)-regular the above extension \(\overline{t}\) is continuous on \(\overline{G} \times S\). For example, for the ball \(G = B(0, r)\) we have \(\tau_+(x, \omega) = 2|\langle x, \omega \rangle|\) and \(t(x, \omega) = \langle x, \omega \rangle + \sqrt{|\langle x, \omega \rangle|^2 + r^2} - |x|^2\) ([71 Example 3.1]). We find that for \((y, \omega) \in \Gamma' = \partial G \times S\)

\[
\lim_{(x, \omega) \to (y, \omega), (x, \omega) \in G \times S} t(x, \omega) = \langle y, \omega \rangle + |\langle y, \omega \rangle| \quad \text{(21)}
\]

For \((y, \omega) \in \Gamma'_+\) the projection \(\langle y, \omega \rangle\) is non-negative (and then the limit \(\langle y, \omega \rangle\) is \(\tau_+(y, \omega)\)) and for \((y, \omega) \in \Gamma'_-\) it is non-positive and then the limit \(\langle y, \omega \rangle\) is 0. Hence \(\overline{t} : \overline{G} \times S \to \mathbb{R}\) is continuous.

Due to the Proposition 2.6 we can (almost everywhere) uniquely set \(t(y, \omega) = 0\) for \((y, \omega, E) \in \Gamma_-\). For \((y, \omega, E) \in \Gamma_+\) we set \(t(y, \omega) := \tau_+(y, \omega)\). For further (unexplained) notations we refer to [71].
In the sequel we denote for \( k \in \mathbb{N}_0 \)
\( C^k(\overline{G} \times S \times I) := \{ \psi \in C^k(G \times S \times I^o) \mid \psi \text{ has continuous partial derivatives on } \overline{G} \times S \times I \} \)
and
\( D^k(\overline{G} \times S \times I) := \{ \psi \in C^k(G \times S \times I^o) \mid \psi = f|_{G \times S \times I^o}, \ f \in C^k_0(\mathbb{R}^n \times S \times \mathbb{R}) \}. \)

It is well known that these spaces are equal, i.e. \( C^k(\overline{G} \times S \times I) = D^k(\overline{G} \times S \times I) \),
for a given \( k \), if the boundary \( \partial G \) of \( G \) is of class \( C^k \) (see [29, Part 1, Lemma 5.2]).

Thus, in particular, since our standing assumption in this paper is that \( \overline{G} \) is (at least) of class \( C^1 \), we have
\[
C^1(\overline{G} \times S \times I) = D^1(\overline{G} \times S \times I).
\]

Define the (Sobolev) space \( W^2(G \times S \times I) \) by
\[
W^2(G \times S \times I) = \{ \psi \in L^2(G \times S \times I) \mid \omega \cdot \nabla_x \psi \in L^2(G \times S \times I) \}
\]
and its subspace \( W^2_1(G \times S \times I) \) by
\[
W^2_1(G \times S \times I) = \{ \psi \in W^2(G \times S \times I) \mid \frac{\partial \psi}{\partial E} \in L^2(G \times S \times I) \}.
\]

Here \( \omega \cdot \nabla_x \psi \) and \( \frac{\partial \psi}{\partial E} \) are understood in the distributional sense. In what follows,
\( \omega \cdot \nabla_x \psi \) will stand for (the distribution) \( \Omega \cdot \nabla_x \psi \), where \( \Omega : G \times S \times I \to \mathbb{R}^3; \Omega(x, \omega, E) = \omega \).
The spaces \( W^2(G \times S \times I), W^2_1(G \times S \times I) \) are equipped with the inner products, respectively
\[
\langle \psi, v \rangle_{W^2(G \times S \times I)} = \langle \psi, v \rangle_{L^2(G \times S \times I)} + \langle \omega \cdot \nabla_x \psi, \omega \cdot \nabla_x v \rangle_{L^2(G \times S \times I)}
\]
and
\[
\langle \psi, v \rangle_{W^2_1(G \times S \times I)} = \langle \psi, v \rangle_{L^2(G \times S \times I)} + \langle \omega \cdot \nabla_x \psi, \omega \cdot \nabla_x v \rangle_{L^2(G \times S \times I)} + \left\langle \frac{\partial \psi}{\partial E}, \frac{\partial v}{\partial E} \right\rangle_{L^2(G \times S \times I)}.
\]

These spaces are Hilbert spaces. Let
\[
C^1(\overline{G} \times S \times I) := \{ \phi|_{G \times S \times I} \mid \phi \in C^1_0(\mathbb{R}^3 \times S \times \mathbb{R}) \}.
\]

We have (cf. [22], [8]. The proof can also be shown by the similar considerations as in [29] pp. 11-19)

**Theorem 2.8** The space \( C^1(\overline{G} \times S \times I) \) is a dense subspace of \( W^2(G \times S \times I) \) and of \( W^2_1(G \times S \times I) \).

For \( \Gamma_- = \{ (y, \omega, E) \in (\partial G) \times S \times I \mid \omega \cdot \nu(y) < 0 \} \) we define the space of \( L^2 \)-functions with respect to the measure \( |\omega \cdot \nu| \, d\sigma d\omega \, dE \) which is denoted by \( T^2(\Gamma_-) \) that is, \( T^2(\Gamma_-) = L^2(\Gamma_-, |\omega \cdot \nu| \, d\sigma d\omega \, dE) \). \( T^2(\Gamma_-) \) is a Hilbert space and its inner product is (in this paper all functions are real-valued)
\[
\langle h_1, h_2 \rangle_{T^2(\Gamma_-)} = \int_{\Gamma_-} h_1(y, \omega, E) h_2(y, \omega, E) |\omega \cdot \nu| \, d\sigma d\omega dE.
\]

The space \( T^2(\Gamma_+) \) and its inner product of \( L^2 \)-functions on \( \Gamma_+ = \{ (y, \omega, E) \in (\partial G) \times S \times I \mid \omega \cdot \nu(y) > 0 \} \) with respect to the measure \( |\omega \cdot \nu| \, d\sigma d\omega \, dE \) is similarly defined. We denote by \( T^2(\Gamma) \) the space of \( L^2 \)-functions with respect to the measure
In addition the trace mapping $\gamma$ (\cite{73}, p. 220) or see (52) in the proof of Theorem 2.15 below) $\gamma$ that $\gamma |v$

Hence any element $\psi \in W^2(G \times S \times I)$ has well defined trace $\psi|_{\Gamma_-}$ in $L^2_{\text{loc}}(\Gamma_-, |\omega \cdot \nu| \, d\sigma d\omega dE)$ defined by

$$\psi|_{\Gamma_-} := \lim_{j \to \infty} \psi|_{\Gamma_j}$$

where $\{\psi_j\} \subset C^1(G \times S \times I)$ is a sequence such that $\lim_{j \to \infty} \|\psi_j - \psi\|_{W^2(G \times S \times I)} = 0$. In addition the trace mapping $\gamma_- : W^2(G \times S \times I) \to L^2_{\text{loc}}(\Gamma_-, |\omega \cdot \nu| \, d\sigma d\omega dE)$ such that $\gamma_-|\psi) = \psi|_{\Gamma_-}$ is continuous. Similarly one has a continuous trace mapping $\gamma_+ : W^2(G \times S \times I) \to L^2_{\text{loc}}(\Gamma_+, |\omega \cdot \nu| \, d\sigma d\omega dE)$ and so we can define (a.e. unique) the trace $\gamma(\psi)$ on $\Gamma$ for $\psi \in W^2(G \times S \times I)$.

By the Sobolev Embedding Theorem for $\psi \in C^1(G \times S \times I)$ (cf. \cite{29} p. 22), or \cite[p. 220]{73}; or see (52) in the proof of Theorem 2.15 below)

$$\|\psi(\cdot, \cdot, E)\|_{L^2(G \times S)} \leq C\left( \|\psi\|_{L^2(G \times S \times I)} + \left\|\frac{\partial \psi}{\partial E}\right\|_{L^2(G \times S \times I)} \right), \quad \forall E \in I,$$

and then the traces $\psi(\cdot, \cdot, 0)$, $\psi(\cdot, \cdot, E_m) \in L^2(G \times S)$ are well-defined for any $\psi \in W^2(G \times S \times I)$.

The trace $\gamma(\psi)$, $\psi \in W^2(G \times S \times I)$ is not necessarily in the space $T^2(\Gamma)$. Hence we define the spaces

$$\tilde{W}^2(G \times S \times I) = \{\psi \in W^2(G \times S \times I) \mid \gamma(\psi) \in T^2(\Gamma)\},$$

which is equipped with the inner product

$$\langle \psi, v \rangle_{\tilde{W}^2(G \times S \times I)} = \langle \psi, v \rangle_{W^2(G \times S \times I)} + \langle \gamma(\psi), \gamma(v) \rangle_{T^2(\Gamma)}.$$  

The space $\tilde{W}^2(G \times S \times I)$ is a Hilbert space (cf. \cite{73}). Moreover, we define subspaces $\tilde{W}^2_{\pm,0}(G \times S \times I)$ of it by

$$\tilde{W}^2_{\pm,0}(G \times S \times I) = \{\psi \in \tilde{W}^2(G \times S \times I) \mid \gamma_\pm(\psi) = 0\}.$$  

For $v \in \tilde{W}^2(G \times S \times I)$ and $\psi \in \tilde{W}^2(G \times S \times I)$ it holds the Green’s formula

$$\int_{G \times S \times I} (\omega \cdot \nabla_x \psi) v \, dx d\omega dE + \int_{G \times S \times I} (\omega \cdot \nabla_x v) \psi \, dx d\omega dE = \int_{\partial G \times S \times I} (\omega \cdot \nu) v \psi \, d\sigma d\omega dE,$$

which is obtained by Stokes Theorem for $v, \psi \in C^1(G \times S \times I)$ and then by the limiting considerations for general $v \in \tilde{W}^2(G \times S \times I)$ and $\psi \in \tilde{W}^2(G \times S \times I)$.

**Remark 2.9** By Green’s formula

$$\|\gamma(\psi)\|_{T^2(\Gamma_\pm)} \leq \|\psi\|_{W^2_{\pm}(G \times S \times I)}$$

and thus $\gamma : \tilde{W}^2_{\pm,0}(G \times S \times I) \to T^2(\Gamma_\pm)$ is bounded.
Occasionally we work in the (energy independent) spaces $L^2(G \times S)$. The corresponding Hilbert spaces $W^2(G \times S)$, $T^2(\Gamma_\pm)$, $T^2(\Gamma')$ and $\tilde{W}^2(G \times S)$ are similarly defined, where

\[
\begin{align*}
\Gamma' &:= (\partial G) \times S, \\
\Gamma'_0 &:= \{(y, \omega) \in (\partial G) \times S \mid \omega \cdot \nu(y) = 0\}, \\
\Gamma'_- &:= \{(y, \omega) \in (\partial G) \times S \mid \omega \cdot \nu(y) < 0\}, \\
\Gamma'_+ &:= \{(y, \omega) \in (\partial G) \times S \mid \omega \cdot \nu(y) > 0\}.
\end{align*}
\]

In addition, the trace $\gamma'(\psi) := \psi|_{\Gamma'}$ for $\psi \in \tilde{W}^2(G \times S)$ is defined as $\gamma(\psi)$ above.

In the context of CSDA-equations we need the following additional Hilbert spaces. Let $H$ be the completion of $C^1(\overline{G} \times S \times I)$ with respect to the inner product

\[
\langle \psi, v \rangle_H := \langle \psi, v \rangle_{L^2(G \times S \times I)} + \langle \gamma(\psi), \gamma(v) \rangle_{T^2(\Gamma)}. \tag{36}
\]

The elements of $H$ are of the form $\tilde{\psi} = (\psi, q) \in L^2(G \times S \times I) \times T^2(\Gamma)$. Actually, they are exactly elements of the closure of the graph of trace operator $\gamma : C^1(\overline{G} \times S \times I) \to C^1(\partial G \times S \times I)$ in $L^2(G \times S \times I) \times T^2(\Gamma)$. The inner product in $H$ is

\[
\langle \tilde{\psi}, \tilde{\psi}' \rangle_H = \langle \psi, \psi' \rangle_{L^2(G \times S \times I)} + \langle q, q' \rangle_{T^2(\Gamma)}, \tag{37}
\]

for $\tilde{\psi} = (\psi, q), \tilde{\psi}' = (\psi', q') \in H$.

Furthermore, let $H_1$ be the completion of $C^1(\overline{G} \times S \times I)$ with respect to the inner product

\[
\langle \psi, v \rangle_{H_1} := \langle \psi, v \rangle_{L^2(G \times S \times I)} + \langle \gamma(\psi), \gamma(v) \rangle_{T^2(\Gamma)} + \langle \gamma(\cdot, \cdot, 0), v(\cdot, \cdot, 0) \rangle_{L^2(G \times S)} + \langle \psi(\cdot, \cdot, E_m), v(\cdot, \cdot, E_m) \rangle_{L^2(G \times S)}. \tag{38}
\]

The elements of $H_1$ are of the form $\tilde{\psi} = (\psi, q, p_0, p_m) \in L^2(G \times S \times I) \times T^2(\Gamma) \times L^2(G \times S)^2$. More precisely, the elements of $H_1$ are the elements of the closure (in $L^2(G \times S \times I) \times T^2(\Gamma) \times L^2(G \times S)^2$) of the graph of the (trace) operator $\gamma \in C^1(\overline{G} \times S \times I) \to C^1(\partial G \times S \times I) \times C^1(\overline{G} \times S)^2$ defined by $\psi \mapsto (\gamma(\psi), \psi(\cdot, 0), \psi(\cdot, E_m))$. The inner product in $H_1$ is

\[
\begin{align*}
\langle \tilde{\psi}, \tilde{\psi}' \rangle_{H_1} &= \langle \psi, \psi' \rangle_{L^2(G \times S \times I)} + \langle q, q' \rangle_{T^2(\Gamma)} \\
&\quad + \langle p_0, p_0' \rangle_{L^2(G \times S)} + \langle p_m, p_m' \rangle_{L^2(G \times S)},
\end{align*}
\]

for $\tilde{\psi} = (\psi, q, p_0, p_m), \tilde{\psi}' = (\psi', q', p_0', p_m') \in H_1$.

Finally, let $H_2$ be the completion of $C^1(\overline{G} \times S \times I)$ with respect to the inner product

\[
\langle \psi, v \rangle_{H_2} := \langle \psi, v \rangle_{W^2(G \times S \times I)} + \left( \frac{\partial \psi}{\partial E}, \frac{\partial v}{\partial E} \right)_{L^2(G \times S \times I)} \\
\quad = \langle \psi, v \rangle_{L^2(G \times S \times I)} + \langle \gamma(\psi), \gamma(v) \rangle_{T^2(\Gamma)} + \left( \frac{\partial \psi}{\partial E}, \frac{\partial v}{\partial E} \right)_{L^2(G \times S \times I)}. \tag{39}
\]

Obviously $H_2 \subset \tilde{W}^2(G \times S \times I) \cap W^2_1(G \times S \times I)$ and the inner product in $H_2$ is given by \[39\].
2.2. Some Details on (Inflow) Trace Theory. We still bring up a refinement of the above explained trace theory. Let us define
\[ T^2_{\tau\pm}(\Gamma_\pm) := L^2(\Gamma_\pm, \tau\pm(\cdot, \cdot)|\omega \cdot \nu|d\sigma d\omega dE), \]
equipped with the inner product
\[ \langle h_1, h_2 \rangle_{T^2_{\tau\pm}(\Gamma_\pm)} = \int_{\Gamma_\pm} h_1(y, \omega, E)h_2(y, \omega, E)\tau\pm(\cdot, \cdot)|\omega \cdot \nu| d\sigma d\omega dE. \quad (40) \]

Suppose that \( g \in C(\Gamma_-) \) such that \( \frac{\partial g}{\partial y_i} \in C(\Gamma_-), i = 1, 2, \) where \( \frac{\partial g}{\partial y_i} \) denotes any local basis of the tangent space of \( \partial G \) (and \( \frac{\partial g}{\partial y_i} \in C(\Gamma_-) \) is to be understood in a local sense), and \( \Sigma \in C(\overline{G} \times S \times I) \) such that \( \frac{\partial \Sigma}{\partial y_j} \in C(\overline{G} \times S \times I), \) \( j = 1, 2, 3. \) Then the unique (classical) solution of the homogeneous \textit{convection-scattering equation} (recall the definition of \( D \) in (13))
\[ \omega \cdot \nabla_x \psi + \Sigma \psi = 0 \quad \text{on} \ D, \quad (41) \]
satisfying the inhomogeneous inflow boundary condition
\[ \psi(y, \omega, E) = g(y, \omega, E) \quad \forall (y, \omega, E) \in \Gamma_- \]
is given by (\textit{[11]} Theorem 3.13)
\[ \psi(x, \omega, E) = e^{-\int_0^t \xi(x, \omega, s, \tau_s, \omega, E)ds}g(x - t(x, \omega)\omega, \omega, E). \quad (43) \]
We denote by \( L_-g = \psi \) the solution of the problem (41)-(42) that is, \( L_-g \) is given by the right hand side of (43). Note that \( L_-g \) is not generally even continuous (for non-convex \( G \)) even if \( g \) happens to be smooth. We show, however, that the formula (43) gives a \textit{weak solution} of the problem (41)-(42) that is,
\[ \langle \psi, -\omega \cdot \nabla_x v + \Sigma v \rangle_{L^2(G \times S \times I)} = 0 \quad \text{for all} \ v \in C_0^\infty(G \times S \times I^0), \]
\[ \psi(y, \omega, E) = g(y, \omega, E) \quad \text{for a.e.} \ (y, \omega, E) \in \Gamma_- \]
for any \( g \in T^2_{\tau\pm}(\Gamma_-), \) and that \( L_-g \in W^2(G \times S \times I). \)

We record the following Lemma for later use.

\textbf{Lemma 2.10} Assume that \( \Sigma \in L^\infty(G \times S \times I) \) and that \( \Sigma \geq 0. \) Then for any \( g \in T^2_{\tau\pm}(\Gamma_-) \) the function \( L_-g \) defined by (13) (is measurable and) belongs to \( L^2(G \times S \times I), \) and
\[ \|L_-g\|_{L^2(G \times S \times I)} \leq \|g\|_{T^2_{\tau\pm}(\Gamma_-)}. \quad (45) \]
Moreover, equality holds here if \( \Sigma = 0. \)

\textit{Proof.} Write \( L_{\Sigma,-} \) for the lift-operator defined in (13) for a given \( \Sigma \geq 0, \) i.e. \( \psi = L_{\Sigma,-}g. \) Then
\[ \|L_{\Sigma,-}g\|_{L^2(G \times S \times I)}^2 = \int_{G \times S \times I} g(x - t(x, \omega)\omega, \omega, E)^2dxd\omega dE \]
\[ = \int_{\Gamma_-} g(y, \omega, E)^2\tau_-\omega(y)\omega \cdot \nu(y)|d\sigma(y)|d\omega dE \]
\[ = \|g\|_{T^2_{\tau\pm}(\Gamma_-)}^2. \]
where in the second step we applied the change of variables in integration explained in the proof of Theorem 2.11 below (see Remark 2.16), and noticed that \( t(y + s\omega, \omega) = s \) whenever \((y, \omega, E) \in \Gamma_\ast\). Therefore,

\[
\|L_{\gamma} - g\|^2_{L^2(G \times S \times I)} = \int_{G \times S \times I} \left( e^{-\int_0^t g(x, \omega, E) ds} g(x, \omega, E) \right)^2 dxd\omega dE
\]

\[
\leq \int_{G \times S \times I} g(x - t(x, \omega)\omega, \omega, E)^2 dxd\omega dE
\]

\[
= \|L_0 - g\|^2_{L^2(G \times S \times I)} = \|g\|^2_{L^2(\Gamma_\ast)}.
\]

Since \( t(y, \omega) = 0 \) a.e. in \( \Gamma_\ast \) we see that \( \gamma(L_\gamma - g) = g \). When we verify (Lemma 2.11) that \( \omega \cdot \nabla_x(L_\gamma - g) + \Sigma(L_\gamma - g) = 0 \) (weakly) in \( G \times S \times I \) we can conclude that \( L_\gamma - g \in W^2(G \times S \times I) \).

**Lemma 2.11** Assume that \( \Sigma \in L^\infty(G \times S \times I) \) and that \( \Sigma \geq 0 \). Let \( L_\gamma : T^2(\Gamma_\ast) \to L^2(G \times S \times I) \) be defined by

\[
(L_\gamma)(x, \omega, E) = e^{-\int_0^t g(x, \omega, E) ds} g(x, \omega, E).
\]

Then in the weak sense on \( G \times S \times I \),

\[
\omega \cdot \nabla_x(L_\gamma) + \Sigma(L_\gamma) = 0.
\]

**Proof.** Given \( g \in C^1_0(\Gamma_\ast) \), choosing a sequence \( g_n \) in \( C^1_0(\Gamma_\ast) \) that converges to \( g \) in \( T^2(\Gamma_\ast) \) (the proof of the existence of this kind sequence is quite standard and is omitted), we have by the continuity of \( L_\gamma \) (see 45), that \( L_\gamma g_n \to L_\gamma g \) in \( L^2(G \times S \times I) \), hence in \( D'(G \times S \times I^c) \), where \( I^c := [0, E_m] \), from which we deduce that \( \omega \cdot \nabla_x(L_\gamma g_n) + \Sigma(L_\gamma g_n) \to \omega \cdot \nabla_x(L_\gamma g) + \Sigma(L_\gamma g) \) in \( D'(G \times S \times I^c) \). This shows that we may assume \( g \in C^1_0(\Gamma_\ast) \).

Let \( \varphi \in C^\infty(G \times S \times I^c) \). Then by Fubini’s Theorem

\[
-(\omega \cdot \nabla_x(L_\gamma)(\varphi)) = (L_\gamma)(\omega \cdot \nabla_x \varphi)
\]

\[
= \int_{G \times S \times I} e^{-\int_0^t g(x, \omega, E) ds} g(x - t(x, \omega)\omega, \omega, E)(\omega \cdot \nabla_x \varphi)(x, \omega, E)dxd\omega dE
\]

\[
= \int_{S \times I} \int_{G_\omega} \int_{J_{y,\omega}} e^{-\int_t^{y+\tau\omega} \Sigma(y + (t-s)\omega, \omega, E) ds} g(y + \tau\omega - t(y + \tau\omega, \omega)\omega, \omega, E)
\]

\[
\cdot \frac{d}{d\tau} \varphi(y + \tau\omega, \omega, E) d\tau dyd\omega dE
\]

\[
= \int_{S \times I} \int_{G_\omega} \int_{J_{y,\omega}} e^{-\int_t^{y+(\tau-s)\omega} \Sigma(y + s\omega, \omega, E) ds} g(y + \tau\omega - t(y + \tau\omega, \omega)\omega, \omega, E)
\]

\[
\cdot \frac{d}{d\tau} \varphi(y + \tau\omega, \omega, E) d\tau dyd\omega dE
\]

Here \( G_\omega \) is the orthogonal projection of \( G \) along \( \omega \),

\[
G_\omega = \{ x - (x \cdot \omega)\omega \mid x \in G \},
\]

which is an \((n-1)\)-dimensional open submanifold of \( \mathbb{R}^n \). Moreover, \( J_{y,\omega} \) is the intersection of \( G \) and the straight line \( y + \tau \omega \),

\[
J_{y,\omega} = \{ \tau \in \mathbb{R} \mid y + \tau \omega \in G \}.
\]
Notice that \( J_{y,\omega} \) is open subset of \( \mathbb{R} \), hence disjoint union of countably many open intervals \( J_{y,\omega}^i = \{ a_{y,\omega}^i, b_{y,\omega}^i \} \), \( i \in \mathbb{N} \) (we take \( J_{y,\omega}^i = \emptyset \) for all big enough \( i \) if there is only finitely many non-empty ones)

\[
J_{y,\omega} = \bigsqcup_{i \in \mathbb{N}} J_{y,\omega}^i.
\]

Noticing also that for all \( \tau \in J_{y,\omega}^i \),

\[
t(y + \tau \omega, \omega) = \tau - a_{y,\omega}^i
\]

we have

\[
- (\omega \cdot \nabla_x (L_- g))(\varphi)
= \int_{S \times I} \int_{G^0} \sum_i \int_{J_{y,\omega}^i} e^{-\int_{a_{y,\omega}^i} \Sigma(y, \omega, E) d\tau} g(y, a_{y,\omega}^i, \omega, E) \frac{d}{d\tau} \varphi(y + \tau \omega, \omega, E) d\tau d\omega dE
= \int_{S \times I} \int_{G^0} \sum_i \int_{J_{y,\omega}^i} g(y, a_{y,\omega}^i, \omega, E) \left( e^{-\int_{a_{y,\omega}^i} \Sigma(y, \omega, E) d\tau} \varphi(y + \tau \omega, \omega, E) \right)_{\tau = a_{y,\omega}^i} dy d\omega dE
+ \int_{S \times I} \int_{G^0} \sum_i \int_{J_{y,\omega}^i} \Sigma(y + \tau \omega, \omega, E) e^{-\int_{a_{y,\omega}^i} \Sigma(y, \omega, E) d\tau} g(y + \tau \omega, \omega, E) \varphi(y + \tau \omega, \omega, E) d\tau d\omega dE
= 0 + \int_{G^0} \Sigma(x, \omega, E) e^{-\int_{0}^{t(x, \omega)} \Sigma(x, \omega, E) d\tau} g(x, t(x, \omega), \omega, E) dx d\omega dE
= (\Sigma(L_- g))(\varphi),
\]

where at the 3rd equality we used the fact that \( \varphi \) vanishes on the boundary of \( G \) and \( y + a_{y,\omega}^i \in \partial G \) and \( y + b_{y,\omega}^i \in \partial G \). This completes the proof.

\[\square\]

**Remark 2.12** Note that if we assume that there exists \( c > 0 \) such that \( \Sigma \geq c \) on \( G \times S \times I \), then in the previous lemma one can take \( G \) to be unbounded as well.

Analogously to Lemma 2.11 for any \( g \in T_{\omega}^2(\Gamma_+) \) the weak solution of the problem

\[
\omega \cdot \nabla_x \psi + \Sigma \psi = 0,
\psi(y, \omega, E) = g(y, \omega, E) \quad \text{for} \quad (y, \omega, E) \in \Gamma_+
\]

is given by (note that \( (y, \omega) \in \Gamma_- \) if and only if \( (y, -\omega) \in \Gamma_+ \))

\[
(L_+ g)(x, \omega, E) := \psi(x, \omega, E) = e^{-\int_0^{t(x, \omega)} \Sigma(x, \omega, E) d\tau} g(x + t(x, \omega), \omega, E).
\]

For later use (section 3) we also treat the inhomogeneous convection-scattering equation with the homogeneous boundary data. Suppose that \( f \in C(\overline{G} \times S \times I) \) such that \( \frac{\partial f}{\partial x_j} \in C(\overline{G} \times S \times I) \), and let \( \Sigma \in C(\overline{G} \times S \times I) \) such that \( \frac{\partial \Sigma}{\partial x_j} \in C(\overline{G} \times S \times I) \), \( j = 1, 2, 3 \). Then the unique (classical) solution of the equation

\[
\omega \cdot \nabla_x \psi + \Sigma \psi = f \quad \text{on} \quad D,
\]

satisfying the homogeneous inflow boundary condition

\[
\psi(y, \omega, E) = 0 \quad \text{for} \quad (y, \omega, E) \in \Gamma_-,
\]

(46)

(47)
Lemma 2.13 Assume that $G$ is bounded, $d := \text{diag}(G)$, $\Sigma \in L^\infty(G \times S \times I)$ and that $\Sigma \geq 0$. Then for any $f \in L^2(G \times S \times I)$ the formula (48) defines $L^2(G \times S \times I)$-function $\psi = S_\Sigma f$, and

$$\|S_\Sigma f\|_{L^2(G \times S \times I)} \leq d \|f\|_{L^2(G \times S \times I)}.$$

Proof. We have

$$\|S_\Sigma f\|_{L^2(G \times S \times I)}^2 = \int_{G \times S \times I} \left( \int_0^{t(x,\omega)} e^{-\int_0^t \Sigma(x-s,\omega) ds} f(x-t,\omega, E) dt \right)^2 dxd\omega dE$$

$$\leq \int_{\Gamma_-} \left( \int_0^{t(y,\omega)} e^{-\int_0^t \Sigma(y-s,\omega) ds} f(y-(r-t),\omega, E) dr \right)^2 d\omega dE$$

where again in the second step applied the change of variables in integration explained in the proof of Theorem 2.15 below and in the fourth step applied the Cauchy-Schwartz inequality. Since $\tau_- (y, \omega) \leq d$, this gives

$$\|S_\Sigma f\|_{L^2(G \times S \times I)}^2 \leq d^2 \int_{\Gamma_-} \int_0^{\tau_- (y,\omega)} |f(y+t,\omega, E)|^2 dt d\omega dE$$

More generally, we have the following.

Lemma 2.14 Assume that $\Sigma \in L^\infty(G \times S \times I)$ and that $\Sigma \geq 0$. Then $\psi$ defined by (48) satisfies weakly in $G \times S \times I$,

$$\omega \cdot \nabla_x \psi + \Sigma \psi = f,$$

and the inflow boundary condition (17) is valid.

Proof. Due to (49) it suffices to show (50) only for $f \in C^1(G \times S \times I)$. Using the notations from the proof of Lemma 2.11 for $\varphi \in C^\infty_0(G \times S \times I^o)$ we get by the
Fubin’s Theorem

\[-(\omega \cdot \nabla_x \psi)(\varphi) = \psi(\omega \cdot \nabla_x \varphi)\]

\[= \int_{G \times S \times I} \psi(x, \omega, E)(\omega \cdot \nabla_x \varphi)(x, \omega, E)dxd\omega dE\]

\[= \int_{S \times I} \int_{G_\omega} \sum_i \int_{J^i_y, \omega} \psi(y + \tau \omega, \omega, E) \frac{d}{d\tau} \varphi(y + \tau \omega, \omega, E)d\tau dyd\omega dE\]

\[= \int_{S \times I} \int_{G_\omega} \sum_i \int_{J^i_y, \omega} \int_0^{\tau-a^i_y, \omega} e^{-\int_0^\tau \Sigma(y+(\tau-s)\omega, \omega, E)ds} f(y + (\tau - t)\omega, \omega, E)\]

\[\cdot \frac{d}{d\tau} \varphi(y + \tau \omega, \omega, E)d\tau dyd\omega dE\]

\[= \int_{S \times I} \int_{G_\omega} \sum_i \int_{J^i_y, \omega} \int_0^\tau e^{-\int_0^\tau \Sigma(y+(\tau-s)\omega, \omega, E)ds} f(y + t \omega, \omega, E)\]

\[\cdot \frac{d}{d\tau} \varphi(y + \tau \omega, \omega, E)d\tau dyd\omega dE\]

\[= \int_{S \times I} \int_{G_\omega} \sum_i \int_{J^i_y, \omega} f(y + t \omega, \omega, E) \int_t^{b^i_y, \omega} e^{-\int_s^\tau \Sigma(y+s\omega, \omega, E)ds}\]

\[\cdot \frac{d}{d\tau} \varphi(y + \tau \omega, \omega, E)d\tau dt dyd\omega dE.\]

In the last step, we changed the order of integration \(dtd\tau \rightarrow d\tau dt\), in which the domain of integration

\[\{(\tau, t) \mid \tau \in J^i_y, \omega = ]a^i_y, \omega, b^i_y, \omega[, \ t \in ]a^i_y, \omega, \tau[\}\]

changes into

\[\{(t, \tau) \mid t \in J^i_y, \omega = ]a^i_y, \omega, b^i_y, \omega[, \ \tau \in ]t, b^i_y, \omega[\}\]

as usual.

Observing that

\[\int_t^{b^i_y, \omega} e^{-\int_s^\tau \Sigma(y+s\omega, \omega, E)ds} \frac{d}{d\tau} \varphi(y + \tau \omega, \omega, E)d\tau\]

\[= \left( e^{-\int_t^{b^i_y, \omega} \Sigma(y+s\omega, \omega, E)ds} \varphi(y + \tau \omega, \omega, E) \right)_{\tau=b^i_y, \omega}\]

\[+ \int_t^{b^i_y, \omega} \Sigma(y + \tau \omega, \omega, E)e^{-\int_s^\tau \Sigma(y+s\omega, \omega, E)ds} \varphi(y + \tau \omega, \omega, E)d\tau\]

\[= - \varphi(y + t \omega, \omega, E) + \int_t^{b^i_y, \omega} \Sigma(y + \tau \omega, \omega, E)e^{-\int_s^\tau \Sigma(y+s\omega, \omega, E)ds} \varphi(y + \tau \omega, \omega, E)d\tau,\]
we obtain
\[ - (\omega \cdot \nabla_x \psi)(\varphi) \]
\[ = \int_{S \times I} \int_{G \omega} \sum_{i} \int_{J_{y,\omega}} f(y + t\omega, \omega, E)(-\varphi(y + t\omega, \omega, E) \]
\[ + \int_{S \times I} \sum_{\tau, \omega} e^{-\int_{\tau}^t \Sigma(\gamma + s\omega, \omega, E)ds} \varphi(y + \tau\omega, \omega, E)d\tau) dtdy\omega dE \]
\[ = - \int_{G \times S \times I} f(x, \omega, E)\varphi(x, \omega, E)dxd\omega dE \]
\[ + \int_{G \times S \times I} \sum_{\omega} e^{-\int_{0}^{t(x,\omega)} \Sigma(x - s\omega, \omega, E)ds} f(x - t\omega, \omega, E)dt\varphi(x, \omega, E)dxd\omega dE \]
\[ = \int_{G \times S \times I} (- f(x, \omega, E) + \sum_{\omega} \psi(x, \omega, E))\varphi(x, \omega, E)dxd\omega dE \]
which is what we set out to prove. \( \square \)

Choosing especially \( \Sigma = 0, f = 1 \) in Lemma 2.14 we find that in the weak sense \( \omega \cdot \nabla_x t = 1 \) in \( G \times S \times I \).

We are now ready to prove the inflow trace theorem. Since \( \|\omega\| = 1 \) and since the domain \( G \) is bounded we have \( \tau_{\pm}(y, \omega) \leq d \). Hence from [20, p. 252], [16] or [18] (where the result is considered for a more general \( G \)) we obtain the following theorem. For completeness we give its detailed proof.

**Theorem 2.15** The trace mappings
\[ \gamma_{\pm} : W^{2}(G \times S \times I) \rightarrow T_{\tau_{\pm}}^{2}(\Gamma_{\pm}) \]
are (well-defined) bounded surjective operators with bounded right inverses \( L_{\pm} : T_{\tau_{\pm}}^{2}(\Gamma_{\pm}) \rightarrow W^{2}(G \times S \times I) \) that is, \( \gamma_{\pm} \circ L_{\pm} = I \) (the identity). The operators \( L_{\pm} \) are called lifts.

Below we use abbreviation \( L := L_{-} \) if no confusion is possible.

**Proof.** A. For the first instance we recall an elementary estimate for (smooth) functions defined on an interval \([0, T]\) \( \subset \mathbb{R} \). Let \( f \in C^{1}([0, T]) \). Then for any \( s \in [0, T] \)
\[ f(0) = - \int_{0}^{s} f'(t)dt + f(s) \]
which implies (by the Cauchy-Schwartz’s inequality) that
\[ |f(0)|^{2} \leq 2 \left(s \int_{0}^{s} |f'(t)|^{2}dt + |f(s)|^{2}\right) \leq 2 \left(T \|f\|_{L^{2}([0, T])}^{2} + |f(s)|^{2}\right). \]  \( 51 \)
Hence integrating over \([0, T]\) we obtain the estimate
\[ T|f(0)|^{2} \leq 2 \left(T^{2} \|f\|_{L^{2}([0, T])}^{2} + \|f\|_{L^{2}([0, T])}^{2}\right). \]  \( 52 \)

B. We shall only deal with the trace \( \gamma_{-} \) because the treatment of the trace \( \gamma_{+} \) is analogous. At first we show the boundedness of \( \gamma_{-} \). Since \( C^{1}(\overline{G} \times S \times I) \) is dense in \( W^{2}(G \times S \times I) \) (by Theorem 2.8) it suffices to prove that there exists a constant \( C > 0 \) such that
\[ \|\gamma_{-}(\psi)\|_{T_{\tau_{-}}^{2}(\Gamma_{-})} \leq C \|\psi\|_{W^{2}(G \times S \times I)} \quad \forall \psi \in C^{1}(\overline{G} \times S \times I). \]  \( 53 \)
We apply the change of variables given in [71] proof of Lemma 5.8 [see also 18 Prop. 2.1]]. Assume for simplicity that \( \partial G \) has a \( C^1 \)-parametrization (which is almost global), say \( h : V \to \partial G \setminus \Gamma_1 \) where \( \Gamma_1 \) has zero surface measure. Generally we have a finite number of parametrized patches that cover \( \partial G \). Applying for each fixed \( \omega \) the change of variables (in \( x \)-variable) \( x = h(v) + t\omega = : H(v, t) \), we find that the Jacobian \( J_H \) of \( h \) is

\[
J_H(v, t) = \omega \cdot (\partial_1 h \times \partial_2 h)(v) = \omega \cdot \nu(h(v)) \left\| \left( \partial_1 h \times \partial_2 h \right)(v) \right\| ,
\]

since \( \nu(h(v)) = \frac{\partial \left( \partial_1 h \times \partial_2 h \right)(v)}{\left\| \partial \left( \partial_1 h \times \partial_2 h \right)(v) \right\|} \). We notice that \( J_H(v, t) \) depends only on \( v \) (and \( \omega \)), but not on \( t \), and hence we write it as \( J_H(v) \). Moreover, almost everywhere \( H(W) = G \), where \( W := \{ (v, t) \mid v \in V, \ 0 < t < \tau_-(h(v), \omega) \} \) and \( V := \{ v \in V \mid \omega \cdot \nu(h(v)) < 0 \} \) (which depend on \( \omega \)). Hence for any \( \psi \in C^1(\mathcal{G} \times S \times I) \)

\[
\int_{G \times S \times I} |\psi(x, \omega, E)|^2 \, dx \, d\omega \, dE = \int_{S \times I} \left( \int_G |\psi(x, \omega, E)|^2 \, dx \right) \, d\omega \, dE
\]

\[
= \int_{S \times I} \int_W |\psi(H(v, t), \omega, E)|^2 |J_H(v, t)| \, dt \, d\omega \, dE
\]

\[
= \int_{S \times I} \int_{V_\tau} \int_0^{\tau_-(h(v), \omega)} |\psi(h(v) + t\omega, \omega, E)|^2 |J_H(v, t)| \, dt \, d\omega \, dE. \quad (54)
\]

For a fixed \( (v, \omega) \in V_\tau \) and \( E \in I \) we apply [52] to the \( C^1 \)-mapping \( f : [0, \tau_-(h(v), \omega)] \to \mathbb{R} \) defined by

\[
f(t) := \psi(h(v) + t\omega, \omega, E).
\]

Noting that \( f'(t) = (\omega \cdot \nabla_x \psi)(h(v) + t\omega, \omega, E) \) we obtain the estimate

\[
\tau_-(h(v), \omega)|\psi(h(v), \omega, E)|^2 \leq 2 \left( \tau_-(h(v), \omega)^2 \int_0^{\tau_-(h(v), \omega)} |(\omega \cdot \nabla_x \psi)(h(v) + t\omega, \omega, E)|^2 \, dt \right.
\]

\[
+ \int_0^{\tau_-(h(v), \omega)} |\psi(h(v) + t\omega, \omega, E)|^2 \, dt \bigg). \quad (55)
\]

Utilizing these preliminaries and the fact that \( \tau_-(h(v), \omega) \leq d \) we get

\[
\|\gamma_-(\psi)\|_{L_2^\tau(G \times S \times I)}^2 = \int_{\Gamma_\tau} |\psi(y, \omega, E)|^2 \tau_{\gamma_-(y, \omega)} |\omega \cdot \nu(y)| \, d\sigma \, d\omega \, dE
\]

\[
= \int_{S \times I} \int_{V_\tau} |\psi(h(v), \omega, E)|^2 \tau_-(h(v), \omega) |\omega \cdot \nu(h(v))| \left\| \left( \partial_1 h \times \partial_2 h \right)(v) \right\| \, d\omega \, dE
\]

\[
= \int_{S \times I} \int_{V_\tau} |\psi(h(v), \omega, E)|^2 \tau_-(h(v), \omega) |J_H(v)| \, d\omega \, dE
\]

\[
\leq \int_{S \times I} \int_{V_\tau} 2 \left( d^2 \int_0^{\tau_-(h(v), \omega)} |(\omega \cdot \nabla_x \psi)(h(v) + t\omega, \omega, E)|^2 |J_H(v)| \, dt \right.
\]

\[
+ \int_0^{\tau_-(h(v), \omega)} |\psi(h(v) + t\omega, \omega, E)|^2 |J_H(v)| \, dt \bigg) \quad (56)
\]

where we in the last step applied [71] to \( \omega \cdot \nabla_x \psi \) and to \( \psi \). This completes the boundedness claim of \( \gamma_- \).

C. Next we prove the existence of the right inverses. Again we consider only the case of \( \gamma_- \). We choose \( \Sigma = 0 \) in Lemma 2.11 and define the right inverse
(57) \[ L_- := g(x - t(x, \omega)\omega, \omega, E). \]

Then by \( \text{(45)} \) \( L_- : T^2_{\tau_-}(\Gamma_-) \to W^2(G \times S \times I) \) is a well-defined bounded linear operator, which is even isometric embedding (which follows since \( \Sigma = 0 \)). This completes the proof.

**Remark 2.16** From the proof of the previous lemma we get the following useful formulas (when the integrals exist)

\[
\int_{\Gamma_-} g(y, \omega, E) d\sigma d\omega dE = \int_{S \times I} \int_{V_-} g(h(v), \omega, E) \| (\partial_1 h \times \partial_2 h)(v) \| d\sigma d\omega dE,
\]

\[
\text{where } V_- = \{ v \in V \mid \omega \cdot \nu(h(v)) < 0 \} \text{ (which depends on } \omega), \text{ and (see (34))}
\]

\[
\int_{G \times S \times I} |\psi(x, \omega, E)|^2 dx d\omega dE
\]

\[
= \int_{S \times I} \int_{\tau_-} \int_{\tau_-} |\psi(h(v) + t\omega, \omega, E)|^2 |\omega \cdot \nu(h(v))| \| (\partial_1 h \times \partial_2 h)(v) \| dt d\sigma d\omega dE
\]

\[
= \int_{\tau_-} \int_{\tau_-} |\psi(y + t\omega, \omega, E)|^2 |\omega \cdot \nu(y)| dt d\sigma d\omega dE.
\]

**Remark 2.17** A. For any compact set \( K \subset \Gamma_- \) we have \( \tau_-(y, \omega) \geq c_K > 0 \) for all \( (y, \omega, E) \in K \) and so the estimate \( (23) \) follows from Theorem \( (2.15) \).

B. The formula \( (57) \) gives the lift \( L_- g \) explicitly which is useful e.g. in numerical computations. Note that the lift is not unique (for example, we are able to define \( L_- = L_{\Sigma, -} \) for any \( \Sigma \) given above).

Analogously to \( L_- \) the (isometric) lift \( L_+ : T^2_{\tau_+}(\Gamma_+) \to W^2(G \times S \times I) \) can be chosen to be

\[
(L_+ g)(x, \omega, E) := g(x + t(x, -\omega)\omega, \omega, E).
\]

C. We have for any \( w \in L^2(G \times S \times I) \) and \( g \in T^2(\Gamma_-) \) (as in \( (54) \))

\[
\langle L_- g, w \rangle_{L^2(G \times S \times I)} = \int_{G \times S \times I} g(x - t(x, \omega)\omega, \omega, E)w(x, \omega, E) dx d\omega dE
\]

\[
= \int_{S \times I} \int_{\tau_-} \int_{\tau_-} g(h(v) + t(h(v) + t\omega, \omega, E)w(h(v) + t\omega, \omega, E)J_{\Sigma}(v, t)\| d\sigma d\omega dE dt
\]

\[
= \int_{S \times I} \int_{\tau_-} g(h(v), \omega, E)(L^*_w)(h(v), \omega, E)\tau_-(h(v), \omega)\| \omega \cdot \nu(h(v))\| \| \partial_1 h \times \partial_2 h \| d\omega dE dt
\]

\[
= \langle g, L^*_w \rangle_{T^2_{\tau_-}(\Gamma_-)}
\]

(60)

where we used the fact \( t(h(v) + t\omega) = t \) and defined

\[
(L^*_w)(y, \omega, E) := \frac{1}{\tau_-(y, \omega)} \int_{\tau_-} w(y + t\omega, \omega, E) dt.
\]

Hence \( L^*_w \) is the adjoint of the operator \( L_- : T^2_{\tau_-}(\Gamma_-) \to L^2(G \times S \times I) \). Applying the preceding computations we find that

\[
\| L^*_w \|_{T^2_{\tau_-}(\Gamma_-)} \leq \| w \|_{L^2(G \times S \times I)}.
\]
we recall that the adjoint
\[ T^*_a \] (1), (2). Let the solution does not exist. To demonstrate that, consider (for simplicity) the problem

\[ \gamma \psi = 0, \] (62)

where \( A \) is interpreted as a densely defined closed operator \( L^2(G \times S \times I)^3 \rightarrow L^2(G \times S \times I)^3 \times T^2_\gamma (\Gamma^-)^3 \) with \( D(A) := W^2(G \times S \times I)^3 \). We find that the adjoint \( A^* \) of \( A \) is a densely defined operator \( L^2(G \times S \times I)^3 \times T^2_\gamma (\Gamma^-)^3 \rightarrow L^2(G \times S \times I)^3 \) given by

\[ A^* \begin{pmatrix} u \\ v \end{pmatrix} = (T^* \gamma^*) \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in D(A^*) \] (63)

where \( T^* = (T^*_1, T^*_2, T^*_3) \) with \( T^*_j u := -\omega \cdot \nabla x u_j + \Sigma_j \psi_j - K_j \psi_j \) (the operator \( K_j \) is given in section 5). Supposing that \( A \) is a Fredholm operator we have \( R(A) = N(A^*)^\perp \). Hence a necessary and sufficient condition for the existence of solutions of

\[ A\psi = \begin{pmatrix} f \\ g \end{pmatrix} \] (64)

is the orthogonality criterion

\[ \langle (f, g), (u, v) \rangle_{L^2(G \times S \times I)^3 \times T^2_\gamma (\Gamma^-)^3} = 0 \] (64)

for all \((u, v) \in N(A^*)\) that is, for all \((u, v) \in D(A^*) \subset L^2(G \times S \times I)^3 \times T^2_\gamma (\Gamma^-)^3 \) which obey

\[ T^* u + \gamma^* v = 0. \] (64)

Under the due assumptions the transport operator \( T \) is (after an appropriate change of variables) a pseudo-differential operator (actually only the term \( K \) needs careful treatment). Hence the application of the pseudo-differential boundary value operator calculus is possible. We omit here the details of such approaches.

E. For \((y, \omega, E) \in \Gamma_+\) it is reasonable to set (cf. Proposition 2.6)

\[ (L_-g)(y, \omega, E):= g(y - \tau_{\gamma_+}(y, \omega)\omega, \omega, E), \quad \gamma \in T^2_\gamma (\Gamma^-). \] (62)

From [20, p. 253] (or [16]) it follows that for \( g \in T^2_\gamma (\Gamma) \), where \( \tau|_{\Gamma_+} = \tau, \tau|_{\Gamma_-} = \tau_+ \) and \( \tau|_{\Gamma_0} = 0 \), there exists an element \( \tilde{\psi} \in \tilde{W}^2(G \times S \times I) \) such that

\[ \gamma_-(\tilde{\psi}) = g|_{\Gamma_-} =: \gamma_- \quad \text{and} \quad \gamma_+(\tilde{\psi}) = g|_{\Gamma_+} =: \gamma_+ \] (62)

if and only if

\[ (g - L_-g)|_{\Gamma_+} \in L^2(\Gamma_+ \tau_{\gamma_+}^{-1}(y, \omega)\omega \cdot \nu(y)|d\sigma d\omega dE). \] (62)

G. It can also be shown that \( \gamma_- : \tilde{W}^2_{\gamma_0}(G \times S \times I) \rightarrow L^2(\Gamma_-, \tau_{\gamma_-}^{-1}(y, \omega)\omega \cdot \nu|d\sigma dE) \) is bounded (note again that for bounded \( G \) we have \( \tau_- (x, \omega) \leq d \)) and that it has a bounded right inverse \( L_- : L^2(\Gamma_-, \tau_{\gamma_-}^{-1}(y, \omega)\omega \cdot \nu|d\sigma dE) \rightarrow \tilde{W}^2_{\gamma_0}(G \times S \times I) \) ([15] or [20, p. 252]). Similar result hold for \( \gamma_+ \).
We still consider the following special case of the trace theory for an exterior of a convex bounded domain $G$. Let $G_e$ be the complement (the exterior of $G$) $G_e := \mathbb{R}^3 \setminus \overline{G}$. Then $\partial G_e = \partial G$. Denote (as above for $G$)

$$
\Gamma_{e,+} := \{(y,\omega, E) \in \partial G_e \times S \times I) = \partial G \times S \times I)| \omega \cdot \nu_e(y) > 0\},
$$

$$
\Gamma_{e,-} := \{(y,\omega, E) \in \partial G_e \times S \times I) = \partial G \times S \times I)| \omega \cdot \nu_e(y) < 0\},
$$

$$
\Gamma_e := \Gamma_{e,+} \cup \Gamma_{e,-}
$$

where $\nu_e$ is the unit outward pointing normal vector on $\partial G_e$.

We find that $\nu_e = -\nu$ and then $\Gamma_{e,+} = \Gamma_+$ and $\gamma_{e,\pm}(\psi) := \psi|_{\Gamma_{e,\pm}} = \gamma_{\pm}(\psi)$. Furthermore, let $t_e(x, \omega)$ be the escape time mapping for the domain $G_e$ and let for an element $(y,\omega, E) \in \Gamma_{e,-}$ (as above) $\tau_{e,-}(y,\omega) = \inf\{s > 0 | y + s\omega \not\in G_e\}$. We observe that for the convex set $G$ actually $\tau_{e,-}(y,\omega) = \infty$ for all $(y,\omega, E) \in \Gamma_{e,-}$.

**Theorem 2.18** The trace mappings

$$
\gamma_{e,\pm} : W^2(G_e \times S \times I) \to T^2(\Gamma_{e,\pm})
$$

are (well-defined) bounded surjective operators with bounded right inverses (lifts) $L_{e,\pm} : T^2(\Gamma_{e,\pm}) \to W^2(G_e \times S \times I)$.

**Proof.** Again it needs only to consider the trace operator $\gamma_-$. The proof runs similarly to the proof of Theorem 2.15 with following changes. Instead of estimate (52) we utilize the inequality

$$
|f(0)|^2 = \left| \int_0^\infty \frac{df(t)}{dt} \right|^2 = \left| \int_0^\infty 2f'(t)f(t) \right|^2 \leq 2\left( \int_0^\infty |f'(t)|^2dt \right)^{1/2} \left( \int_0^\infty |f(t)|^2dt \right)^{1/2} \leq \int_0^\infty |f'(t)|^2dt + \int_0^\infty |f(t)|^2dt, \quad (65)
$$

which is valid for all $f \in C^1_0([0,\infty))$ (note that $f(t) = 0$ for sufficiently large $t$).

In addition, one applies the change of variables $H$ we have $H(W) = G_e$ where $W := \{(v, t) | v \in V, \quad 0 < t < \infty\}$ and $V := \{v \in V \mid \omega \cdot \nu_e(h(v)) < 0\}$.

Let $\lambda > 0$. For any $g_e \in C(\Gamma_{e,-})$ such that $\frac{\partial g}{\partial y_i} \in C(\Gamma_{e,-})$ the (classical) solution $\Psi$ of the problem

$$
\omega \cdot \nabla_x \Psi + \lambda \Psi = 0 \quad \text{on } D_e,
$$

$$
\Psi|_{\Gamma_{e,-}} = g_e,
$$

where $D_e$ is the set corresponding to $G_e \times S \times I$, is given explicitly by (cf. (13))

$$
\Psi(x,\omega, E) = \begin{cases} e^{-\lambda t_e(x,\omega)}g_e(x - t_e(x,\omega)\omega,\omega, E), & \text{when } t_e(x,\omega) \text{ is finite} \\ 0, & \text{otherwise} \end{cases}. \quad (67)
$$

Similarly to the proof of Lemma 2.11 we find that (65) holds weakly in $G_e \times S \times I$. For $g_e \in T^2(\Gamma_{e,-})$ we define the lift explicitly by

$$
L_{e,-}g_e := \Psi. \quad (68)
$$
Then as in the proof of Lemma 5.8 given in [77] we get that \( L_{e,-}g_e \in \tilde{W}^2(G_e \times S \times I) \) and (note that \( \| \omega \cdot \nabla_x \Psi \|_{L^2(G_e \times S \times I)} = \lambda \| \Psi \|_{L^2(G_e \times S \times I)} \))

\[
\| L_{e,-}g_e \|_{W^2(G_e \times S \times I)} = \| \Psi \|_{W^2(G_e \times S \times I)} = \sqrt{1 + \lambda^2} \| \Psi \|_{L^2(G_e \times S \times I)} = \sqrt{1 + \frac{\lambda^2}{\tau_\xi}} \| g_e \|_{T^2(\Gamma_{e,-})},
\]

(69)

since (cf. Remark 2.15)

\[
\| \Psi \|_{L^2(G_e \times S \times I)}^2 = \int_{\Gamma_{e,-}} \int_0^{\tau_{e,-}(x,\omega)} (e^{-\lambda \xi} g_e(y, \omega, E))^2 |\omega \cdot \nu(y)| ds \sigma(y) d\omega dE = \frac{1}{2\lambda} \int_{\Gamma_{e,-}} g_e(y, \omega, E)^2 |\omega \cdot \nu(y)| ds \sigma(y) d\omega dE = \frac{1}{2\lambda} \| g_e \|_{T^2(\Gamma_{e,-})}^2,
\]

where \( \tau_{e,-} \) is \( \tau_\xi \) for the domain \( G_e \times S \times I \). We omit further details. \( \square \)

Remark 2.19

A. By (67) one can show that

\[ L_{e,-}(g_{e,-})|_{\Gamma_{e,+}} = 0. \]

B. The lift \( L_{e,+} : T^2(\Gamma_{e,+}) \to W^2(G_e \times S \times I) \) is given by

\[ (L_{e,+})(x, \omega, E) = \begin{cases} 
  e^{-\lambda \xi(x,\omega)} g_e(x + t_e(x, -\omega)\omega, \omega, E), & \text{when } t_e(x, -\omega) \text{ is finite}, \\
  0, & \text{otherwise}.
\end{cases} \]

We find that

\[ L_{e,+}(g_{e,+})|_{\Gamma_{e,-}} = 0. \]

C. Let \( g_e \in T^2(\Gamma_e) \) and let \( g_{e,\pm} := g_e|_{\Gamma_{e,\pm}} \). Then for

\[ \Psi := L_{e,-}(g_{e,-}) + L_{e,+}(g_{e,+}) \in W^2(G_e \times S \times I), \]

we find that \( \tilde{\Psi} \in \tilde{W}^2(G_e \times S \times I) \) and

\[ \tilde{\Psi}|_{\Gamma_{e,\pm}} = g_{e,\pm}. \]

(See Remark 2.17 Part E.)

As a corollary we show the following extension result. In the proof we explicitly construct the extension of \( \psi \) (cf. [20], p. 415, proof of Lemma 2).

Corollary 2.20 Suppose that \( G \subset \mathbb{R}^3 \) is as above and that it is convex. Then for any \( \psi \in \tilde{W}^2(G \times S \times I) \) there exists an extension \( \mathcal{E}\psi \in W^2(\mathbb{R}^3 \times S \times I) \) of \( \psi \) that is, \( \mathcal{E}\psi|_{G \times S \times I} = \psi \). In addition, the linear operator \( \mathcal{E} : \tilde{W}^2(G \times S \times I) \to W^2(\mathbb{R}^3 \times S \times I) \) is bounded.

Proof. Suppose that \( \psi \in \tilde{W}^2(G \times S \times I) \). Denote \( g := \psi|_{\Gamma} \), which belongs to \( T^2(\Gamma) = T^2(\Gamma_e) \). Let \( \tilde{\Psi} \in \tilde{W}^2(G_e \times S \times I) \) given in Remark 2.19 Part C. Define \( \mathcal{E}\psi \) by

\[
\mathcal{E}\psi := \begin{cases} 
  \psi & \text{on } G \times S \times I \\
  \gamma_\pm(\tilde{\Psi}) = \gamma_{e,\mp}(\tilde{\Psi}) & \text{on } \Gamma_\pm \\
  \tilde{\Psi} & \text{on } G_e \times S \times I
\end{cases}
\]

(70)
Then $\mathcal{E}\psi$ is in $W^2(\mathbb{R}^3 \times S \times I)$. This follows from the Green’s formula since for all $v \in C^0_0(\mathbb{R}^3 \times S \times I^0)$

\begin{align}
\int_{\mathbb{R}^3 \times S \times I} (\mathcal{E}\psi) (\omega \cdot \nabla_x v) dx d\omega dE & = \int_{G \times S \times I} (\mathcal{E}\psi) (\omega \cdot \nabla_x v) dx d\omega dE + \int_{G_e \times S \times I} (\mathcal{E}\psi) (\omega \cdot \nabla_x v) dx d\omega dE \\
& = -\int_{G \times S \times I} (\omega \cdot \nabla_x \psi) v dx d\omega dE + \int_{\partial G \times S \times I} \gamma(\psi)(v)(\omega \cdot \nu)d\sigma d\omega dE \\
& \quad - \int_{G_e \times S \times I} (\omega \cdot \nabla_x \tilde{\Psi}) v dx d\omega dE + \int_{\partial G_e \times S \times I} \gamma_e(\tilde{\Psi}) \gamma_e(v)(\omega \cdot \nu_e)d\sigma d\omega dE \\
& = -\int_{G \times S \times I} (\omega \cdot \nabla_x \psi) v dx d\omega dE - \int_{G_e \times S \times I} (\omega \cdot \nabla_x \tilde{\Psi}) v dx d\omega dE
\end{align}

(71)

where we used the facts that $\partial G = \partial G_e$ and $\nu_e = -\nu$ and so $\gamma_{\pm}(\psi) = g_{\pm} = g_{e,\mp} := \gamma_{e,\mp}(\tilde{\Psi})$. Hence $\omega \cdot \nabla_x (\mathcal{E}\psi) \in L^2(\mathbb{R}^3 \times S \times I)$, as desired.

Finally, we find that by (69) (recall Remark 2.19, Part C.)

\begin{align}
\|\mathcal{E}\psi\|_{W^2(\mathbb{R}^3 \times S \times I)} & = \|\psi\|_{W^2(G \times S \times I)} + \|\tilde{\Psi}\|_{W^2(G_e \times S \times I)} \\
& \leq \|\psi\|_{W^2(G \times S \times I)} + \|L_{e,-}(g_{e,-})\|_{W^2(G_e \times S \times I)} + \|L_{e,+}(g_{e,\mp})\|_{W^2(G_e \times S \times I)} \\
& = \|\psi\|_{W^2(G \times S \times I)} + \sqrt{\frac{1 + \lambda^2}{2\lambda}} \left( \|\gamma_+(\psi)\|_{T^2(G_e)} + \|\gamma_-(\psi)\|_{T^2(\Gamma_{e,\mp})} \right)
\end{align}

(72)

which implies the boundedness of $\mathcal{E}$. This completes the proof. \(\square\)

Let $\tilde{W}^2(G \times S \times I)$ be the completion of $C^1(\overline{G} \times S \times I)$ with respect to $\|\cdot\|_{\tilde{W}^2(G \times S \times I)}$-norm. For a convex set $G \subset \mathbb{R}^3$ have the following density result.

**Corollary 2.21** Suppose that $G \subset \mathbb{R}^3$ is as above and that it is convex. Then

\begin{align}
\tilde{W}^2(G \times S \times I) = \bar{W}^2(G \times S \times I),
\end{align}

(74)

and

\begin{align}
H_2 = \tilde{W}^2(G \times S \times I) \cap W^2_1(G \times S \times I).
\end{align}

(75)

(Definitions of the spaces $W^2_1$, $\tilde{W}^2$ and $H_2$ were given in (23), (32) and (39), respectively.)

**Proof.** At first, we deal with the claim (74). The inclusion “$\subset$” is clear and then it suffices to prove only the opposite inclusion.

Because $G$ is convex we have by Theorem 2.18

\begin{align}
\|\gamma_e(\Psi)\|_{T^2(\Gamma_e)} \leq C \|\Psi\|_{W^2(G_e \times S \times I)} \quad \forall \Psi \in W^2(G_e \times S \times I).
\end{align}

(76)

Let $\psi \in \tilde{W}^2(G \times S \times I)$ and let $\mathcal{E}\psi \in W^2(\mathbb{R}^3 \times S \times I)$ be its extension provided by Lemma 2.20. Since $C^0_0(\mathbb{R}^3 \times S \times \mathbb{R})$ is dense in $W^2(\mathbb{R}^3 \times S \times I)$ there exists a sequence $\{\Psi_n\} \subset C^0_0(\mathbb{R}^3 \times S \times \mathbb{R})$ such that $\|\Psi_n - \mathcal{E}\psi\|_{W^2(\mathbb{R}^3 \times S \times I)} \rightarrow 0$ for $n \rightarrow \infty$.

Let $\psi_n := \Psi_n|_{G \times S \times I}$. We have

\begin{align}
\|\gamma(\psi_n - \psi)\|_{T^2(\Gamma)} = \|\gamma_e(\Psi_n - \mathcal{E}\psi)\|_{T^2(\Gamma_e)}.
\end{align}
Hence we get by (76)
\[
\|\psi_n - \psi\|^2_{W^2(G \times S \times I)} = \|\psi_n - \psi\|^2_{W^2(G \times S \times I)} + \|\gamma(\psi_n - \psi)\|^2_{T^2(I)} \\
\leq \|\Psi_n - \mathcal{E}\psi\|^2_{W^2(G \times S \times I)} + C^2 \|\Psi_n - \mathcal{E}\psi\|^2_{W^2(G \times S \times I)} \\
\leq \|\Psi_n - \mathcal{E}\psi\|^2_{W^2(\mathbb{R}^3 \times S \times I)} + C^2 \|\Psi_n - \mathcal{E}\psi\|^2_{W^2(\mathbb{R}^3 \times S \times I)},
\]
which implies that \(\psi \in \tilde{W}^2(G \times S \times I)\). This completes the proof of (74).

Since \(C_0^\infty(\mathbb{R}^3 \times S \times I)\) is dense in \(W^2(\mathbb{R}^3 \times S \times I)\) the proof of (75) is quite similar and so the proof is complete.

Finally, we notice that the following continuous inclusions are valid
\[
\tilde{W}^2(G \times S \times I) \subset H; \\
\psi \mapsto (\psi, \gamma(\psi)),
\]
and
\[
\tilde{W}^2(G \times S \times I) \cap W^2(G \times S \times I) \subset H_1; \\
\psi \mapsto (\psi, \gamma(\psi), \psi(\cdot, 0), \psi(\cdot, E_m)).
\]

**Remark 2.22** Let \(\rho_1, \rho_2 : I \to \mathbb{R}\) be positive (weight) functions in \(L^\infty(I)\). We can define a linear space (more generally instead of \(W^2(G \times S \times I)\)) by
\[
W^2_{\rho_1, \rho_2}(G \times S \times I) = \{\psi \in L^2(G \times S \times I) \mid \rho_1(\omega) \cdot \nabla_x \psi \in L^2(G \times S \times I), \rho_2 \frac{\partial \psi}{\partial E} \in L^2(G \times S \times I)\},
\]
which can be equipped with the inner product
\[
\langle \psi, v \rangle_{W^2_{\rho_1, \rho_2}(G \times S \times I)} = \langle \psi, v \rangle_{L^2(G \times S \times I)} + \langle \rho_1(\omega) \cdot \nabla_x \psi, \rho_1(\omega) \cdot \nabla_x v \rangle_{L^2(G \times S \times I)} \\
+ \left(\rho_2 \frac{\partial \psi}{\partial E}, \rho_2 \frac{\partial v}{\partial E}\right)_{L^2(G \times S \times I)},
\]
rendering \(W^2_{\rho_1, \rho_2}(G \times S \times I)\) to a Hilbert space. Similar weighted spaces can be defined generalizing other spaces above. These spaces are needed e.g. in the context of time-dependent transport equations (where \(\rho_1 = \rho_2 = \sqrt{E}\)).

2.3. **On Collision Operators.** The differential cross-sections may have singularities, or even hyper-singularities, which would lead to extra (partial differential and) pseudo-differential terms in the transport equation (see Sec. 7.1, pp. 353-394). Instead of explaining systematically the underlying theory, the following slightly informal description suffices for the purposes of this work.

First of all, in the case where \(\sigma(x, \omega', \omega, E', E)\) has hyper-singularities (like Møller differential cross section given in the below example) the integral \(\int_S \int_I\) occurring in the collision operator must be understood in the sense of Cauchy principal value p.v. \(\int_S \int_I\) or more generally in the sense of Hadamard finite part integral p.f. \(\int_S \int_I\) ([38 Sec. 3.2], [17], [48], [64 pp. 104-105]). We remark that one encounters this kind of hyper-singularities frequently in physical models. In addition, we must assume that \(E_0 > 0\) in the energy interval \(I = [E_0, E_m]\), because otherwise \(K \psi\), for \(\psi \in C^\infty_0(G \times S \times I^0)\), might turn out to be (strictly) a distribution, which would increase the complexity of what is presented here. In [46 p. 7], it is reported that the differential cross sections are not necessarily valid for very small energies which supports this assumption.
Consider the following partial hyper-singular integral operator; for clarity we denote by \( S' \) and \( I' \) the set \( S \) and \( I \) when its variable is \( \omega' \) and \( E' \), respectively,

\[
(K_0\psi)(x, \omega, E) = \text{p.f.} \int_{I'} \int_{S'} \sigma(x, \omega', \omega, E') \psi(x, \omega', E')d\omega'dE'.
\] (81)

The simplest case is where \( \sigma = \sigma_0(x, \omega', \omega, E', E) \) is a measurable non-negative function \( G \times S' \times S \times (I' \times I \setminus D) \to \mathbb{R} \), where \( D = \{(E, E) \mid E \in I = I'\} \) is the diagonal of \( I' \times I \), obeying for \( E \neq E' \) the estimates

\[
\text{ess sup}_{(x, \omega)} \int_{S'} \sigma_0(x, \omega', \omega, E', E)d\omega' \leq \frac{C}{|E - E'|^\kappa},
\] (82)

\[
\text{ess sup}_{(x, \omega')} \int_S \sigma_0(x, \omega', \omega, E', E)d\omega \leq \frac{C}{|E - E'|^\kappa},
\] (83)

where \( \kappa < 1 \), meaning that \( \sigma_0(x, \omega', \omega, E', E) \) may have a so-called weak singularity with respect to energy. We see that

\[
\text{ess sup}_{(x, \omega, E) \in G \times S \times I} \int_{I'} \int_{S'} \sigma_0(x, \omega', \omega, E', E)d\omega'dE' \leq \sup_{E} C \int_{I'} \frac{1}{|E - E'|^\kappa}dE'
\]

\[
= \sup_{E} C \frac{1}{1 - \kappa}[(E_m - E)^{1-\kappa} + (E - E_0)^{1-\kappa}] \leq \frac{2CE^1}{1 - \kappa},
\] (84)

and similarly for \( \int_S \int_S \sigma_0(x, \omega', \omega, E', E)d\omega dE \). Hence we see that \( \sigma_0(x, \omega', \omega, E', E) \) satisfies conditions (151) below, and the corresponding collision operator

\[
(K_0\psi)(x, \omega, E) = \int_{I'} \int_{S'} \sigma_0(x, \omega', \omega, E', E)\psi(x, \omega', E')d\omega'dE',
\] (85)

is the usual partial Schur (singular) integral operator. It is bounded \( L^2(G \times S \times I) \to L^2(G \times S \times I) \).

Nevertheless, the collision operator \( K \) is not generally of the above form \( K_0 \). \((E', E)\)-dependence in differential cross section \( \sigma(x, \omega', \omega, E', E) \) may contain hyper-singularities of higher order, \( \frac{1}{(E' - E)^m} \), for \( m = 1, 2 \) for example; see Example 2.27 below.

Moreover, the \((\omega', \omega)\)-dependence in differential cross-sections typically contain Dirac’s \( \delta \)-distributions (on \( \mathbb{R} \)). More precisely, in \( \sigma(x, \omega', \omega, E', E) \) there may occur terms like \( \delta(\omega \cdot \omega' - \mu(E', E)) \) which require special treatment. We remark, however that \( \delta \)-distribution can be approximated by smooth functions \( \eta_\epsilon \in C_0^\infty(\mathbb{R}) \) in the sense that

\[
|\delta(\phi) - \langle \eta_\epsilon, \phi \rangle_{L^2(\mathbb{R})}| \leq \|\delta - \eta_\epsilon\|_{H^{-1}(\mathbb{R})} \|\phi\|_{H^1(\mathbb{R})}, \quad \phi \in H^1(\mathbb{R}),
\] (86)

where \( \|\delta - \eta_\epsilon\|_{H^{-1}(\mathbb{R})} \to 0 \) when \( \epsilon \to 0^+ \). Typically \( \eta_\epsilon \) is chosen to be the convolution \( \eta_\epsilon := \delta \ast \theta_\epsilon \), where \( \theta \in C_0^\infty(\mathbb{R}) \) such that \( \int_{\mathbb{R}} \theta(x)dx = 1 \), and \( \theta_\epsilon(x) := \epsilon \theta(\epsilon x) \). Hence we are able to replace \( \delta(\omega \cdot \omega' - \mu(E', E)) \), with \( \eta_\epsilon(\omega \cdot \omega' - \mu(E', E)) \) which is a well-behaved function. The standard method is first to solve the regularized problem (where \( \delta \) is replaced with \( \eta_\epsilon \)), and then retrieve existence results for the original problem using a limiting process.
We shall see that the cross section $\sigma$ may be the form (e.g. in Møller electron-electron cross-section)

$$\sigma(x, \omega', \omega, E', E) = \chi(E', E)\left(\frac{1}{(E' - E)^2}\sigma_2(x, \omega, \omega, E', E) + \frac{1}{E' - E}\sigma_1(x, \omega, \omega, E', E) + \sigma_0(x, \omega, \omega, E', E)\right)$$

(87)

where $\chi(E', E)$ is a certain product of characteristic functions. Here each of $\sigma_j(x, \omega, \omega, E', E)$, $j = 0, 1, 2$ may contain the above explained $\delta$-distributions, and hence they are not necessarily measurable functions. Denote for $j = 0, 1, 2$,

$$\langle \mathcal{K}_j \psi \rangle(x, \omega, E', E) := \int_{S'} \sigma_j(x, \omega, \omega, E', E)\psi(x, \omega', E')d\omega',$n

$$\langle \mathcal{K}_j \psi \rangle(x, \omega, E', E) := \chi(E', E)\langle \mathcal{K}_j \psi \rangle(x, \omega, E', E).$$

In the sequel the integral $\int_{S'}$ is interpreted as a distribution when needed. We shall see, according to the examples below, that at worst $K$ can be of the form (this is corresponding the Møller scattering for electrons)

$$(K\psi)(x, \omega, E) = \mathcal{H}_2(\langle \mathcal{K}_2 \psi \rangle(x, \omega, \cdot, E))(E)
+ \mathcal{H}_1(\langle \mathcal{K}_1 \psi \rangle(x, \omega, \cdot, E))(E) + \int_{I'} (\mathcal{K}_0 \psi)(x, \omega, E', E)dE',$n

(88)

where $\mathcal{H}_m$, $m = 1, 2$, are the Hadamard finite part operators with respect to $E'$-variable defined by

$$(\mathcal{H}_m u)(E) := \text{p.f.} \int_E^{E_m} \frac{1}{(E' - E)^m} u(E')dE'.$n

The expression (88) is the hyper-singular integral form of $K$.

We shall verify that (88) can be equivalently given in the "pseudo-differential form" by

$$(K\psi)(x, \omega, E) = \frac{\partial}{\partial E} \left[ \mathcal{H}_1(\langle \mathcal{K}_2 \psi \rangle(x, \omega, \cdot, E))(E) \right] - \mathcal{H}_1(\langle \frac{\partial \langle \mathcal{K}_2 \psi \rangle(x, \omega, \cdot, E)}{\partial E} \rangle(x, \omega, \cdot, E))(E)
+ \frac{\partial}{\partial E'} \left( \langle \mathcal{K}_2 \psi \rangle(x, \omega, E', E) \right)_{|E' = E}
+ \mathcal{H}_1(\langle \mathcal{K}_1 \psi \rangle(x, \omega, \cdot, E))(E) + \int_{I'} (\mathcal{K}_0 \psi)(x, \omega, E', E)dE'$$

(89)

where only $\mathcal{H}_1$ appears. This formulation reveals the nature of charged particles’ collisions and it enables among others to retrieve existence results of solutions. Note that $\mathcal{H}_1$ is well-defined (at least) for all $u \in C^\infty(I)$, $\alpha > 0$ and (cf. [17])

$$(\mathcal{H}_1 u)(E) = \int_E^{E_m} \frac{u(E') - u(E)}{E' - E}dE' + u(E)\ln(E_m - E).$$

(90)

Moreover, it can perhaps be shown that $\mathcal{H}_1$ is a zeroth-order pseudo-differential operator (cf. [38] Chapter 7).

As a conclusion we find that some interactions produce the first-order partial derivatives with respect to energy $E$ combined with the "zeroth-order" Hadamard part operator. A closer analysis of the operators $\mathcal{K}_j$ reveals that in addition, partial derivatives with respect to $\omega$ may appear. We shall demonstrate that below for $n = 2$. The problematic interactions are the primary electron-electron (considered below), primary positron-positron collisions and bremsstrahlung.
The operator (89) (or equivalently (88)) contains two features that require further study: i) The analysis of operators $K_j$, $j = 0, 1, 2$, and ii) the analysis of the Hadamard finite part operator $H_1$. The analysis of the existence of the solutions for the transport problem, in the case where these operators are included in the transport operator remains to our understanding open.

For the analysis of the examples below we need some preliminaries. Recall the definition of Hadamard finite part integral for discontinuous functions $f : [a, b] \to \mathbb{R}$ by [13], pp. 5 and 32, formulas (14) and (32) therein or [64, p. 104]. Applying these definitions (for a fixed $x$) to the function $F_x(t) := \chi_{[x,b]}(t)f(t)$, where $\chi_{[x,b]}(t)$ is the characteristic of the interval $[x, b]$, we have

$$p.f. \int_a^b \frac{F_x(t)}{t-x} dt = p.f. \int_x^b \frac{f(t)}{t-x} dt = \lim_{\epsilon \to 0} \left( \int_{x+\epsilon}^b \frac{f(t)}{t-x} dt + f(x^+) \ln(\epsilon) \right) \quad (91)$$

and

$$p.f. \int_a^b \frac{F_x(t)}{(t-x)^2} dt = \lim_{\epsilon \to 0} \left( \int_{x+\epsilon}^b \frac{f(t)}{(t-x)^2} dt + f'(x^+) \ln(\epsilon) - \frac{1}{\epsilon} f(x^+) \right). \quad (92)$$

These formulas give

$$p.f. \int_x^b \frac{1}{t-x} dt = \ln(b-x), \quad (93)$$

and

$$p.f. \int_x^b \frac{1}{(t-x)^2} dt = -\frac{1}{b-x}. \quad (94)$$

Recall the Taylor’s formula (of order $r \in \mathbb{N}_0$) for sufficiently smooth functions $f : U \to \mathbb{R}$ for open $U \subset \mathbb{R}^N$,

$$f(x) = \sum_{|\alpha| \leq r} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(x_0)(x-x_0)^\alpha + \sum_{|\alpha|=r+1} R_\alpha(x)(x-x_0)^\alpha \quad (95)$$

where the residual (in one of its variant forms) is

$$R_\alpha(x) := \frac{|\alpha|}{\alpha!} \int_0^1 (1-t)^{|\alpha|-1} \frac{\partial^\alpha f}{\partial x^\alpha}(x_0 + t(x-x_0)) dt.$$

**Lemma 2.23** Suppose that $u \in C^2([a, b])$. Then for $x \in [a, b]$

$$\frac{d}{dx} \left( p.f. \int_x^b \frac{f(t)}{t-x} dt \right) = p.f. \int_x^b \frac{f(t)}{(t-x)^2} dt - f'(x). \quad (96)$$

**Proof.** In virtue of the Taylor’s formula and (93)

$$I(x) := p.f. \int_x^b \frac{f(t)}{t-x} dt$$

$$= p.f. \int_x^b f(x) + \left( \int_0^1 f'(x + s(t-x)) ds \right) (t-x) dt$$

$$= f(x) \ln(b-x) + \int_x^b \int_0^1 f'(x + s(t-x)) ds dt. \quad (97)$$
Hence
\[ I'(x) = f'(x) \ln(b-x) - f(x) \frac{1}{b-x} - f'(x) + \int_x^b \int_0^1 f''(x+s(t-x))(1-s)dsdt \]

(98)

On the other hand by the Taylor’s formula
\[
p.f. \int_x^b \frac{f(t)}{(t-x)^2} dt = p.f. \int_x^b \frac{f(x) + f'(x)(t-x) + \left(\int_0^1 (1-s)f''(x+s(t-x))ds\right)(t-x)^2}{(t-x)^2} dt
\]
\[= - f(x) \frac{1}{b-x} + f'(x) \ln(b-x) + \int_x^b \int_0^1 (1-s)f''(x+s(t-x))dsdt \]
\[= I'(x) + f'(x) \]

(99)
which implies the claim.

Remark 2.24 We remark that the operators of the form
\[(Pu)(x,E) := p.f. \int_{E_0}^{E_m} \sigma_0(x,E,E') \frac{u(x,E,E')}{E'-E} dE', \quad u \in C_0^\infty(G \times I^o \times I^o), \quad (100)\]
can be treated as in [38, Chapter 7]. Note that in (100) the integration is over the whole interval \([E_0,E_m]\). Under relevant assumptions on \(\sigma_0\), the operators (100) can (probably) be shown to be pseudo-differential operators. In particular we recall that the partial Hilbert transform
\[(Hu)(x,E) := p.f. \int_{E_0}^{E_m} \frac{u(x,E,E')}{E'-E} dE', \quad (101)\]
is a pseudo-differential operator with symbol \(-i \text{sign}(\xi)\). Recall that the operator \(H_1\) introduced above can be presented in the form
\[(H_1u)(x,E) := p.f. \int_{E_0}^{E_m} \frac{\sigma_0(x,E,E')}{E'-E} u(x,E,E') dE', \quad u \in C_0^\infty(G \times I^o \times I^o). \]

The problematic feature in the expression of \((H_1u)(x,E)\) is that the integration is over \([E,E_m]\). Similar observations concern the operator \(H_2\). In consistence with [38], formal computations suggest that the expected symbols of \(H_j\), \(j = 1,2\), are
\[p_1(x,E,\xi) = p.f. \int_E^{\infty} \frac{\sigma_0(x,E,E')}{E'-E} e^{i(E'-E)\xi} dE' = p.f. \int_0^{\infty} \frac{\sigma_0(x,E,E+z)}{z} e^{iz\xi} dz, \quad (102)\]
and
\[p_2(x,E,\xi) = p.f. \int_0^{\infty} \frac{\sigma_0(x,E,E+z)}{z^2} e^{iz\xi} dz, \quad (103)\]
respectively. Careful analysis of these operators (whether they are correspondingly zeroth-order and first-order pseudo-differential operators, for example) remains to our knowledge open.

The next examples of the two real collision operators, relevant in e.g. radiation therapy, illustrate the above observations.
Example 2.25 \textit{Photon-electron scattering - Compton-Klein-Nishina.} This scattering process describes the collision of a photon with (free) electron, and the corresponding photon→ photon (i.e. $1 \rightarrow 1$) scattering cross-section is given by (for its derivation from QED, see [79] Section 8.7; see also [46] Section VII)\footnote{\cite{FootnoteReference} \cite{FootnoteReference}}

$$
\sigma_{11}(x, \omega', \omega, E', E) = \hat{\sigma}_{11}(x, E', E)\chi_{11}(E, E')\delta(\omega' \cdot \omega - \mu_{11}(E', E)),
$$

(104)

where

$$
\hat{\sigma}_{11}(x, E', E) := \sigma_0(x)\left( \frac{1}{E'} \right)^2 \frac{E'}{E} + \frac{1}{E'} - 1 + \mu_{11}(E', E)^2
$$

$$
\chi_{11}(E', E) := \chi_{R_+}(E - E_0)\chi_{R_+}(E - \frac{E'}{1 + 2E})\chi_{R_+}(E' - E)
$$

$$
\mu_{11}(E', E) := 1 + \frac{1}{E'} - \frac{1}{E}.
$$

Here $(\omega', E')$ and $(\omega, E)$ are, respectively, the (direction, energy) of the incident and the scattered (outgoing) photons. If the scattering angle is written as $\theta_{11}$, then $\omega \cdot \omega' = \cos(\theta_{11}) = \mu_{11}(E', E)$, and this condition is enforced by the delta-distribution term in $\sigma_{11}$.

We point out that if one defines (for a presentation more or less in this way, see [37] Appendix A.1.)

$$
\hat{\sigma}'_{11}(E', E) := \sigma_0(x)\hat{P}(E', E')\delta(E - E'\hat{\Psi}(\omega', \omega', E')),
$$

where

$$
P(E', E) := \frac{1}{1 + E'(1 - \mu_{11}(E', E))} = \frac{E}{E'},
$$

$$
\hat{\Psi}(\omega', E') := \frac{1}{1 + E'(1 - \omega' \cdot \omega)}
$$

$$
\hat{\sigma}'_{11}(E', E) := P(E', E)^2\left( P(E', E) + \frac{1}{P(E', E)} - 1 + \mu_{11}(E', E)^2 \right),
$$

Then $\sigma_0(x)\hat{\sigma}'_{11}(E', E) = E^2\hat{\sigma}_{11}(x, E', E)$, using which one can further show that the collision operator, say $\hat{K}_{11}$, corresponding to $\hat{\sigma}_{11}$ is equal to the collision operator $K_{11}$ defined by $\sigma_{11}$, i.e. $\hat{K}_{11} = K_{11}$.

We find that the operator $\hat{K}_{11}$ is given by

$$
(\hat{K}_{11}\psi)(x, \omega, E', E) = \hat{\sigma}_{11}(x, E', E)\chi_{11}(E', E) \int_{S'} \delta(\omega' \cdot \omega - \mu_{11}(E', E))\psi(x, \omega', E')d\omega'
$$

$$
= \hat{\sigma}_{11}(x, E', E)\chi_{11}(E', E) \int_0^{2\pi} \psi(x, \gamma(s), E')ds,
$$

(105)

where $\gamma = \gamma_{11}(E', E, \omega) : [0, 2\pi] \rightarrow S$ is a parametrization with (constant) speed

$$
\|\gamma'(s)\| = \sqrt{1 - \mu_{11}(E', E)^2}, \quad s \in [0, 2\pi],
$$

of the curve $\Gamma(E', E, \omega)$ which is the intersection of $S$ and the plane which is orthogonal to $\omega$ and whose distance from origin is $\mu_{11}(E', E)$, that is

$$
\Gamma(E', E, \omega) = \{\omega' \in S \mid \omega' \cdot \omega - \mu_{11}(E', E) = 0\}.
$$

For example, we can choose

$$
\gamma(s) = R(\omega)(\sqrt{1 - \mu_{11}^2} \cos(s), \sqrt{1 - \mu_{11}^2} \sin(s), \mu_{11}), \quad s \in [0, 2\pi]
$$

(106)
where \( \mu_{11} = \mu_{11}(E', E) \), and \( R(\omega) \) is any rotation matrix which maps the vector \((0, 0, 1)\) into \( \omega \).

Indeed, let \( \eta_\epsilon \subset C_0^\infty(\mathbb{R}) \) be such that \( \lim_{\epsilon \to 0} \eta_\epsilon \to \delta \) in \( H^{-1}(\mathbb{R}) \). Then for \( \psi \in C_0^\infty(G \times S \times I^0) \), we have (by definition)

\[
\int_{\gamma'} \delta(\omega' \cdot \omega - \mu_{11}(E', E)) \psi(x, \omega', E')d\omega' = \lim_{\epsilon \to 0} \int_{\gamma'} \eta_\epsilon(\omega' \cdot \omega - \mu_{11}(E', E)) \psi(x, \omega', E')d\omega'.
\]

For each \( t \in [-1, 1] \), let

\[
\gamma_t(s) := R(\omega)(\sqrt{1-t^2}\cos(s), \sqrt{1-t^2}\sin(s), t), \quad s \in [0, 2\pi],
\]

and let \( \Gamma_t \) be the curve (Jordan loop) corresponding to \( \gamma_t \), i.e. \( \Gamma_t = \gamma_t([0, 2\pi]) \). Then the (differential) surface measure \( d \mu_S(\omega) = d\omega \) on \( S \) can be disintegrated into the family \( \left( \frac{1}{\sqrt{1-t^2}} d\ell_\epsilon \right) \otimes dt \), with \( t \in [-1, 1] \), where \( d\ell_\epsilon(s) = \gamma'_t(s)ds \) is the (differential) path length measure (on \( \Gamma_t \)) along \( \gamma_t \). Note that the differential path length of the circle \( S_1(0, t) \) is \( \frac{1}{\sqrt{1-t^2}}dt \).

In virtue of the Fubini’s Theorem (in disintegration sense),

\[
\int_{\gamma'} \eta_\epsilon(\omega' \cdot \omega - \mu_{11}(E', E)) \psi(x, \omega', E')d\omega' = \int_{-1}^{1} \int_{\gamma_t} \eta_\epsilon(\gamma \cdot \omega - \mu_{11}(E', E)) \psi(x, \gamma, E') \frac{1}{\sqrt{1-t^2}}d\ell_\epsilon(\gamma)dt
\]

\[
= \int_{-1}^{1} \int_{-\pi}^{\pi} \eta_\epsilon(t - \mu_{11}(E', E)) \psi(x, \gamma_t(s), E') \frac{1}{\sqrt{1-t^2}} ||\gamma'_t(s)|| dsdt \]

\[
= \int_{-1}^{1} \int_{-\pi}^{\pi} \eta_\epsilon(t - \mu_{11}(E', E)) \psi(x, \gamma_t(s), E') dsdt \]

\[
\underset{\epsilon \to 0}{\longrightarrow} \int_{-1}^{1} \int_{0}^{2\pi} \delta(t - \mu_{11}(E', E)) \psi(x, \gamma_t(s), E') dsdt
\]

\[
= \int_{0}^{2\pi} \psi(x, \gamma_{11}(E', E)(s), E') ds.
\]

Hence combining (107) and (108) we get

\[
\int_{\gamma'} \delta(\omega' \cdot \omega - \mu_{11}(E', E)) \psi(x, \omega', E')d\omega' = \int_{0}^{2\pi} \psi(x, \gamma_{11}(E', E)(s), E') ds,
\]

as desired, since \( \gamma_{11}(E', E)(s) = \gamma(s) (= \gamma_{11}(E', E, \omega)(s)) \).

It thus follows that

\[
(K_{11}\psi)(x, \omega, E) = \int_{\gamma'} (\hat{K}_{11}\psi)(x, \omega, E', E)dE
\]

\[
= \int_{\gamma'} \chi_{11}(E', E) \sigma_{11}(x, E', E) \int_{0}^{2\pi} \psi(x, \gamma(s), E') ds.
\]

Let us approximate \( \delta \)-distribution with a smooth function, say \( \eta_\epsilon \) as above. Then

\[
(\hat{K}_{11}\psi)(x, \omega, E', E) \approx \sigma_{11}(x, E', E) \chi_{11}(E', E) \int_{\gamma'} \eta_\epsilon(\omega' \cdot \omega - \mu_{11}(E', E)) \psi(x, \omega', E')d\omega'.
\]
and so
\[(K_{11}\psi)(x,\omega, E', E) = \int_I \langle \tilde{K}_{11}\psi \rangle(x, \omega, E', E) dE \quad (110)\]
\[\approx \int_{I'} \int_{S'} \tilde{\sigma}_{11}(x, E', E) \chi_{11}(E', E) \eta_\sigma(\omega' - \mu_{11}(E', E)) \psi(x, \omega', E') d\omega' dE' \]
\[= \int_{I'} \int_{S'} \tilde{\sigma}_{11}(x, \omega', \omega, E', E) \psi(x, \omega', E') d\omega' dE' \]
\[= : (K_{11, \epsilon}\psi)(x, \omega, E), \quad (111)\]
where $K_{11, \epsilon}$ is a partial Schur integral operator satisfying the below assumptions [151].

In the case where $\psi$ has the first order derivatives (in the weak sense) with respect to $\omega$ that is, $\psi$ belongs to the mixed-norm Sobolev-Slobodevskij space $H^{2,(0,1,0)}(G \times S \times I')$. We conjecture that
\[\|K_{11, \epsilon}\psi - K_{11}\psi\|_{L^2(G \times S \times I)} \leq C \|\delta \ast \theta_\epsilon - \delta\|_{H^{-1}(\mathbb{R})} \|\psi\|_{H^{2,(0,1,0)}(G \times S \times I')} \quad (112)\]
and so the approximation of $K_{11}$ with $K_{11, \epsilon}$ would be under control.

**Remark 2.26** We make the following remark regarding the fact, as explained in Remark 2.23 that the unit sphere $S$ could have been equipped with more general Radon measure $\rho$, instead of its typical (Lebesgue-measure induced) measure $\mu_S$. Letting $\rho = H^1$ be the 1-dimensional Hausdorff measure on $S$ (see [26, pp. 7–10]), and writing
\[\sigma_{11}(x, \omega', \omega, E', E) = \chi_{11}(E', E) \frac{\sigma_{11}(x, E', E)}{\sqrt{1 - \mu_{11}(E', E)^2}} \chi_{\mathcal{M}}(\omega', \omega, E', E),\]
where $\chi_{\mathcal{M}}$ is the characteristic function of the set
\[\mathcal{M} := \{(\omega', \omega, E', E) \in S^2 \times I^2 \mid \omega' \cdot \omega - \mu_{11}(E', E) = 0\},\]
then we would have (assuming that pertinent functions are Borel integrable)
\[(K_{11}\psi)(x, \omega, E) = \int_{S^2 \times I'} \sigma_{11}(x, \omega', \omega, E', E) \psi(x, \omega', E') d\rho(\omega') dE'. \quad (113)\]
This expression has the pleasant feature that the differential cross section $\sigma_{11}(x, \omega', \omega, E', E)$ becomes a measurable function on space $G \times S^2 \times I^2$.

Indeed, the key observation here is that in (108), the last line can be written as an integral,
\[\int_0^{2\pi} \psi(x, \gamma_{11}(E', E, \omega)(s), E') ds = \frac{1}{\sqrt{1 - \mu_{11}(E', E)^2}} \int_{\Gamma(E', E, \omega)} \psi(x, \cdot, E') d\ell, \]
\[= \frac{1}{\sqrt{1 - \mu_{11}(E', E)^2}} \int_{S'} \chi_{\mathcal{M}}(\omega', \omega, E', E) \psi(x, \omega', E') d\rho(\omega') \]
where $\int_{\Gamma(E', E, \omega)} (\cdot \cdot \cdot) d\ell$ is the path integral along the curve $\Gamma(E', E, \omega)$. Hence we get the expression (113).

Finally, for the sake of completeness, let us mention that the Compton-Klein-Nishina photon→ electron (i.e. 1 → 2) cross section, with corresponding operator
$K_{12}$, is given by the following formulas

$$
\sigma_{12}(x, \omega', \omega, E', E) = \delta_{12}(x, E', E) \chi_{12}(E, E') \delta(\omega' \cdot \omega - \mu_{12}(E', E)),
$$

$$
\hat{\sigma}_{12}(x, E', E) := \hat{\sigma}_{11}(x, E', E' - E) \frac{(1 + E')^2(1 - \mu_{12}(E', E' - E))^2}{\mu_{12}(E', E)^3},
$$

$$
\mu_{12}(E', E) := \left(1 + \frac{1}{E'}\right) \sqrt{\frac{E}{E + 2}},
$$

$$
\chi_{12}(E', E) := \chi_{\mathbb{R}^+}(E - E_0) \chi_{\mathbb{R}^+}(\frac{2E'^2}{1 + 2E'} - E) \chi_{\mathbb{R}^+}(E' - E),
$$

where $(\omega', E')$ is the (direction, energy) of the incident photon, while $(\omega, E)$ is the (direction, kinetic-energy) of the (outgoing) recoil electron. If the scattering angle is written as $\theta_{12}$, then $\omega \cdot \omega' = \cos(\theta_{12}) = \mu_{12}(E', E)$.

**Example 2.27**  **Electron-electron scattering - Møller.** We denote the corresponding differential cross section by $\sigma_{22}(x, \omega', \omega, E', E)$. It has a decomposition ([21], [46], [10], [37])

$$
\sigma_{22}(x, \omega', \omega, E', E) = \sigma_{\mu}^{\mu}(x, \omega', \omega, E', E) + \sigma_{\mu}^{s}(x, \omega', \omega, E', E),
$$

where $\sigma_{\mu}^{\mu}(x, \omega', \omega, E', E)$ is corresponding to the (new) primary electrons and $\sigma_{\mu}^{s}(x, \omega', \omega, E', E)$ is corresponding to the secondary electrons. In this scattering process the spins have been averaged out, and the two electrons completely lose their identity. Therefore, categorizing the electrons as "primary" and "secondary" is simply done by assigning the electron exiting the scattering event with the highest energy to be the primary one. The scattering cross section for primary electron $\sigma_{\mu}^{\mu}(x, \omega', \omega, E', E)$ has an expression

$$
\sigma_{\mu}^{\mu}(x, \omega', \omega, E', E) = \sigma_0(x) \left(\frac{(E' + 1)^2}{E'(E' + 2)} + \frac{1}{(E' - E)^2} + \frac{1}{(E' + 1)^2}
\right.
$$

$$
- \frac{2E' + 1}{(E' + 1)^2 E(E' - E)} \chi_{\mu,\mu}(E', E) \delta(\omega' \cdot \omega - \mu_{\mu,\mu}(E', E)),
$$

where $\sigma_0(x)$ depends on the background material, and

$$
\mu_{\mu,\mu}(E', E) := \sqrt{\frac{E(E' + 2)}{E'(E' + 2)}},
$$

$$
\chi_{\mu,\mu}(E', E) := \chi_{\mathbb{R}^+}(E - E_0) \chi_{\mathbb{R}^+}(E - \frac{E'}{2}) \chi_{\mathbb{R}^+}(E' - E),
$$

while the cross section for the secondary electron $\sigma_{\mu}^{s}(x, \omega', \omega, E', E)$ is

$$
\sigma_{\mu}^{s}(x, \omega', \omega, E', E) = \sigma_0(x) \left(\frac{(E' + 1)^2}{E'(E' + 2)} + \frac{1}{(E' - E)^2} + \frac{1}{(E' + 1)^2}
\right.
$$

$$
- \frac{2E' + 1}{(E' + 1)^2 E(E' - E)} \chi_{\mu,\mu}(E', E) \delta(\omega' \cdot \omega - \mu_{\mu,\mu}(E', E)),
$$

where

$$
\mu_{\mu,\mu}(E', E) := \mu_{\mu,\mu}(E', E - E),
$$

$$
\chi_{\mu,\mu}(E', E) := \chi_{\mathbb{R}^+}(\frac{E'}{2} - E) \chi_{\mathbb{R}^+}(E - E_0).
$$
Since $\sigma_{22}^2(x, \omega', \omega, E', E) = 0$ for $E' \leq 2E$ the singularities at $E' = E$ do not cause any problems for the secondary electrons.

Write
\[
\chi_{22}(E', E) := \chi_{22,p}(E', E) + \chi_{22,s}(E', E),
\]
\[
\mu_{22}(E', E) := \begin{cases} 
\mu_{22,p}(E', E), & E' \leq 2E \\
\mu_{22,s}(E', E), & E' \geq 2E,
\end{cases}
\]
\[
\hat{\sigma}_{22,0}(x, E', E) := \sigma_0(x) \frac{(E' + 1)^2}{E'(E' + 2)} \left( \frac{1}{E^2} + \frac{1}{(E' + 1)^2} \right),
\]
\[
\hat{\sigma}_{22,1}(x, E', E) := -\sigma_0(x) \frac{2E' + 1}{E'(E' + 2)},
\]
\[
\hat{\sigma}_{22,2}(x, E', E) := \sigma_0(x) \frac{(E' + 1)^2}{E'(E' + 2)}.\]

Then we find that
\[
\sigma_{22}(x, \omega', \omega, E', E) = \chi_{22}(E', E) \left( \frac{1}{(E' - E)^2} \hat{\sigma}_{22,2}(x, E', E) \delta(\omega' \cdot \omega - \mu_{22}(E', E)) + \frac{1}{E' - E} \hat{\sigma}_{22,1}(x, E', E) \delta(\omega' \cdot \omega - \mu_{22}(E', E)) + \hat{\sigma}_{22,0}(x, E', E) \delta(\omega' \cdot \omega - \mu_{22}(E', E)) \right). (115)
\]

The operators $\mathcal{K}_{22,j}$ are for any $j = 0, 1, 2,$
\[
(\mathcal{K}_{22,j}\psi)(x, \omega, E', E) = \hat{\sigma}_{22,j}(x, E', E) \int_{S'} \delta(\omega' \cdot \omega - \mu_{22}(E, E')) \psi(x, \omega', E') d\omega'
\]
\[
= \hat{\sigma}_{22,j}(x, E', E) \int_0^{2\pi} \psi(x, \gamma(s), E') ds, (116)
\]
where $\gamma = \gamma_{22}(E', E, \omega) : [0, 2\pi] \to S$ is a parametrization with (constant) speed
\[
\|\gamma'(s)\| = \sqrt{1 - \mu_{22}(E', E)^2}, \quad s \in [0, 2\pi],
\]
of the curve
\[
\Gamma(E', E, \omega) = \{ \omega' \in S \mid \omega' \cdot \omega - \mu_{22}(E', E) = 0 \}.
\]

Writing for $j = 0, 1, 2,$
\[
(\mathcal{K}_{22,j}\psi)(x, \omega, E', E) = \chi_{22}(E', E)(\mathcal{K}_{22,j}\psi)(x, \omega, E', E),
\]
the collision operator $K_{22}$ decomposes into
\[
K_{22} = K_{22,2} + K_{22,1} + K_{22,0}
\]
where
\[
(K_{22,0}\psi)(x, \omega, E) = \int_{E'} (\hat{K}_{22,0}\psi)(x, \omega, E', E), dE' (117)
\]
Note that \( \hat{K}_{22}(\psi)(x, \omega, E) \) is the hyper-singular integral form of \( K_{\partial \omega} \).

Hence by (117), (118), (119), and (120), we see that

\[
(K_{22}(\psi)(x, \omega, E) = p.f. \int_{E'} \frac{(\hat{K}_{22}(\psi)(x, \omega, E, E') \mid E' = E) dE'}{E' - E} = p.f. \int_{E}^{E_m} \frac{(\hat{K}_{22}(\psi)(x, \omega, E, E) \mid E' = E) dE'}{E' - E} = \mathcal{H}_1((\hat{K}_{22}(\psi)(x, \omega, E))(E),
\]

which is the hyper-singular integral form of \( K_{22} \).

Applying Lemma 2.23 we see that

\[
(\hat{K}_{22}(\psi)(x, \omega, E) = \mathcal{H}_2((\hat{K}_{22}(\psi)(x, \omega, E))(E) + \mathcal{H}_1((\hat{K}_{22}(\psi)(x, \omega, E))(E)
\]

Finally, we remark that the approximative \( \hat{K}_{22,j} \) for \( j = 0, 1, 2 \) are

\[
(\hat{K}_{22,j}(\psi)(x, \omega, E) \approx \int_{S'} \tilde{\sigma}_j(x, E', E) \chi_{22}(E', E) \eta_k(\omega' \cdot E - \mu_{22}(E, E')) \psi(x, \omega, E') d\omega'
\]

Note that \( \hat{K}_{22,0} \) is the usual partial Schur integral operator. The approximations \( \hat{K}_{22,j} \) may be useful from theoretical and practical point of view.

**Example 2.28** A. When the spatial dimension \( n = 2 \),

\[
(\hat{K}_{11}(\psi)(x, \omega, E', E) = \hat{d}_{11}(x, E', E) \chi_{11}(E', E) \psi(x, \mu_{11}(E', E) \omega + \sqrt{1 - \mu_{11}(E', E)^2} \omega^\perp, E') + \psi(x, \mu_{11}(E', E) \omega - \sqrt{1 - \mu_{11}(E', E)^2} \omega^\perp, E'),
\]

where \( \omega^\perp := (-\omega_2, \omega_1) \) (the tangent vector of the unit circle \( S = S_1 \) at \( \omega \)).
B. We compute further some of the above terms. For simplicity we restrict ourselves to the case $n = 2$. In this case

\[
(\mathcal{K}_{22,2}\psi)(x, \omega, E', E) = \hat{\sigma}_2(x, E', E)\left(\psi(x, \mu_{22}(E', E)\omega + \sqrt{1 - \mu_{22}(E', E)^2}\omega^+, E') + \psi(x, \mu_{22}(E', E)\omega - \sqrt{1 - \mu_{22}(E', E)^2}\omega^-, E')\right).
\]

(122)

Denote

\[
\xi_{\pm}(E', E, \omega) := \mu_{22}(E', E)\omega \pm \sqrt{1 - \mu_{22}(E', E)^2}\omega^\perp.
\]

Then

\[
\frac{\partial}{\partial E'}\left((\mathcal{K}_{22,2}\psi)(x, \omega, E', E)\right) = \frac{\partial}{\partial E'}\left(\hat{\sigma}_2(x, E', E)(\psi(x, \xi_+(E', E, \omega), E')) + \frac{\partial}{\partial E'}\left(\hat{\sigma}_2(x, E', E)(\psi(x, \xi_-(E', E, \omega), E'))\right)\right).
\]

(123)

Furthermore,

\[
\frac{\partial}{\partial E'}\left(\hat{\sigma}_2(x, E', E)(\psi(x, \xi_+(E', E, \omega), E'))\right) = \frac{\partial}{\partial E'}(x, E')\psi(x, \xi_+(E', E, \omega), E')
\]

\[
+ \hat{\sigma}_2(x, E', E)(\nabla_{\omega}\psi)(x, \xi_+(E', E, \omega), E') \cdot \frac{\partial \xi_+}{\partial E'}(E', E, \omega)
\]

\[
+ \hat{\sigma}_2(x, E', E)\frac{\partial \psi}{\partial E'}(x, \xi_+(E', E, \omega), E')
\]

(124)

and similarly for the last term in (123). Since $\mu_{22}(E, E) = \mu_{22,p}(E, E) = 1$ we have

\[
\frac{\partial}{\partial E'}\left((\mathcal{K}_{22,2}\psi)(x, \omega, E', E)\right) \mid_{E' = E} = \frac{\partial}{\partial E'}(x, E, E)\psi(x, \omega, E)
\]

\[
+ \hat{\sigma}_2(x, E, E)(\nabla_{\omega}\psi)(x, \omega, E) \cdot \frac{\partial \xi_+}{\partial E'}(E, E, \omega) + \hat{\sigma}_2(x, E, E)\frac{\partial \psi}{\partial E'}(x, \omega, E)
\]

\[
+ \frac{\partial}{\partial E'}(x, E, E)\psi(x, \omega, E)
\]

\[
+ \hat{\sigma}_2(x, E, E)(\nabla_{\omega}\psi)(x, \omega, E) \cdot \frac{\partial \xi_-}{\partial E'}(E, E, \omega) + \hat{\sigma}_2(x, E, E)\frac{\partial \psi}{\partial E'}(x, \omega, E)
\]

\[
= 2\hat{\sigma}_2(x, E, E)\frac{\partial \psi}{\partial E'}(x, \omega, E) + \hat{\sigma}_2(x, E, E) \cdot \left(\frac{\partial \xi_+}{\partial E'}(E, E, \omega) + \frac{\partial \xi_-}{\partial E'}(E, E, \omega)\right) \cot(\nabla_{\omega}\psi)(x, \omega, E)
\]

\[
+ 2\frac{\partial}{\partial E'}(x, E, E)\psi(x, \omega, E).
\]

(125)
Finally,
\[
\frac{\partial (\mathcal{K}_{22,22} \psi)}{\partial E}(x, \omega, E) = \frac{\partial}{\partial E} \left( \hat{\sigma}_2(x, E') \psi(x, \xi_+(E', E, \omega), E') \right) + \hat{\sigma}_2(x, E') \psi(x, \xi_-(E', E, \omega), E') \cdot \frac{\partial \xi_+}{\partial E}(E', E) \\
+ \hat{\sigma}_2(x, E') \psi(x, \xi_-(E', E, \omega), E') \cdot \frac{\partial \xi_-}{\partial E}(E', E).
\]
(128)

As a conclusion we see that for \( n = 2 \)
\[
(K_{22,22} \psi)(x, \omega, E) = \frac{\partial}{\partial E} \left( \mathcal{H}_1((\mathcal{K}_{22,22} \psi)(x, \omega, \cdot, E))(E) \right) \\
- \mathcal{H}_1(\frac{\partial (\mathcal{K}_{22,22} \psi)}{\partial E}(x, \omega, \cdot, E))(E) \\
+ 2\hat{\sigma}_2(x, E) \frac{\partial \psi}{\partial E}(x, \omega, E) + 2 \frac{\partial \hat{\sigma}_2}{\partial E}(x, E) \psi(x, \omega, E) \\
+ \hat{\sigma}_2(x, E) \left( \frac{\partial \xi_+}{\partial E'}(E, E, \omega) + \frac{\partial \xi_-}{\partial E'}(E, E, \omega) \right) \cdot (\nabla_\omega \psi)(x, \omega, E),
\]
(129)

where \( \frac{\partial (\mathcal{K}_{22,22} \psi)}{\partial E} \) is computed by (128).

The corresponding collision operator can be analogously computed in the general dimension \( n \) (which we omit here). The Møller collision term produces first order partial differential terms, along with a Hadamard finite part operator. The exact form of Møller collision operator allows for accessing relevant approximation schemes for which the error analysis can be carried out. We find that the CSDA-approximation does not take into account the change of angle for the (new) primary electron during transport. Hence the angular derivative \( (\nabla_\omega) \) is missing from it. On the other hand, CSDA-Focker-Plank approximation contains also second order partial derivatives (with respect to angle) which do not show up in (129).

As an application of the above analysis we derive a CSDA-type approximation for the Møller scattering. We apply the hyper-singular integral form of the collision operator \( K \) that is (recall [20])
\[
(K_{22} \psi)(x, \omega, E) = \mathcal{H}_2((\mathcal{K}_{22,22} \psi)(x, \omega, \cdot, E))(E) \\
+ \mathcal{H}_1((\mathcal{K}_{22,21} \psi)(x, \omega, \cdot, E))(E) + \int_{E'} (\mathcal{K}_{22,0} \psi)(x, \omega, E', E) dE'.
\]
(130)

Recall also that the operators \( \mathcal{K}_j, j = 0, 1, 2 \) are
\[
(\mathcal{K}_{22,j} \psi)(x, \omega, E', E) = \hat{\sigma}_j(x, E', E) \int_{S'} \delta(\omega' \cdot \omega - \mu_{22}(E, E')) \psi(x, \omega', E') d\omega'
\]
and
\[
(\tilde{\mathcal{K}}_{22,j} \psi)(x, \omega, E', E) = \chi_{22}(E', E) (\mathcal{K}_{22,j} \psi)(x, \omega, E, E').
\]
In the case where \( E \approx E' \) (as in the case of forward peaked primary electrons) we have

\[
\mu_{22,p}(E', E) \approx 1 \tag{131}
\]

and then for \( E \leq E' \leq 2E \)

\[
(\bar{K}_{22,j}\psi)(x, \omega, E') \approx \hat{\sigma}_j(x, E', E) \int_{S_r} \delta(\omega' - \omega) \psi(x, \omega', E') d\omega' = \hat{\sigma}_j(x, E', E) \psi(x, \omega, E'). \tag{132}
\]

Assuming (131), we have an approximation

\[
(\bar{K}_{22,0}\psi)(x, \omega, E) \approx (\bar{K}_{22,0}\psi)(x, \omega, 0) := \int_0^{2E} \hat{\sigma}_0(x, E', E) \psi(x, \omega, E') dE' + \int_{2E}^{Em} (\bar{K}_{22,0}\psi)(x, \omega, E') \tag{133}
\]

Eq. (131) is the first CSDA-type approximation. When we apply the approximation

\[
\delta(\omega' - \omega - \sigma_{22}(E, E')) \approx \eta_{\mu}(\omega' - \omega - \sigma_{22}(E, E')) \tag{134}
\]

we immediately see that \( \bar{K}_{22,0} \) is the usual partial Schur integral operator.

Consider now the term \( H_1((\bar{K}_{22,1}\psi)(x, \omega, , E))(E) \). We have by (132),

\[
H_1((\bar{K}_{22,1}\psi)(x, \omega, , E))(E) = p.f. \int_E^{Em} \frac{1}{E' - E} (\bar{K}_{22,1}\psi)(x, \omega, E', E) dE' = p.f. \int_E^{2E} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \psi(x, \omega, E') dE' + \int_{2E}^{Em} \frac{1}{E' - E} (\bar{K}_{22,1}\psi)(x, \omega, E', E) dE' \approx p.f. \int_E^{2E} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \psi(x, \omega, E') dE' + \int_{2E}^{Em} \frac{1}{E' - E} (\bar{K}_{22,1}\psi)(x, \omega, E', E) dE'. \tag{135}
\]

Due to the Taylor’s formula,

\[
p.f. \int_E^{2E} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \psi(x, \omega, E') dE' \approx p.f. \int_E^{2E} \frac{1}{E' - E} (\hat{\sigma}_1(x, E, E) \psi(x, \omega, E) + R_{1,1}(E'')(E' - E)) dE' \tag{136}
\]

where

\[
R_{1,1}(E'') = \int_0^1 \frac{\partial}{\partial E'} (\hat{\sigma}_1(x, \cdot, E) \psi(x, \omega, \cdot))(E + t(E' - E)) dt. \tag{137}
\]

The second CSDA-type approximation (which is valid if \( E \approx E' \)) is that

\[
R_{1,1}(E'')(E' - E) = 0, \tag{138}
\]

which gives by (133) that approximately

\[
H_1((\bar{K}_{22,1}\psi)(x, \omega, , E))(E) \approx p.f. \int_E^{2E} \frac{1}{E' - E} \hat{\sigma}_1(x, E, E) \psi(x, \omega, E) dE' + \int_{2E}^{Em} \frac{1}{E' - E} (\bar{K}_{22,1}\psi)(x, \omega, E', E) dE' = \hat{\sigma}_1(x, E, E) \psi(x, \omega, E) \ln(E) + (K_{22,1,2}\psi)(x, \omega, E), \tag{139}
\]
where
\[
(K_{22,1,2}\psi)(x,\omega,E) := \int_{2E}^{E_m} \frac{1}{E'-E}(\mathcal{K}_{22,1}\psi)(x,\omega,E',E)dE'.
\]
Applying the approximation (134), $K_{22,1,2}$ is approximately a partial Schur integral operator, say $\tilde{K}_{22,1,2}$.

Next, consider the term $\mathcal{H}_2((\mathcal{K}_{22,2}\psi)(x,\omega,\cdot,E))(E)$. In virtue of the Taylor’s formula,
\[
\hat{\sigma}_2(x,E',E)\psi(x,\omega,E') = \hat{\sigma}_2(x,E,E)\psi(x,\omega,E) + \frac{\partial}{\partial E'}(\hat{\sigma}_2(x,\cdot,E)\psi(x,\omega,\cdot))(E)(E' - E) + R_{2,2}(E')(E' - E)^2,
\]
where
\[
R_{2,2}(E') = \frac{2}{2!} \int_0^1 (1-t)\frac{\partial^2}{\partial E'^2}(\hat{\sigma}_2(x,\cdot,E)\psi(x,\omega,\cdot))(E + t(E' - E))dt
\]
In the case where $E \approx E'$ we can omit the residual term that is,
\[
R_{2,2}(E')(E' - E)^2 = 0
\]
which is the third CSDA-type approximation. Then we get by (133), (134) approximately
\[
\mathcal{H}_2((\mathcal{K}_{22,2}\psi)(x,\omega,\cdot,E))(E)
\approx \text{p.f.} \int_E^{3E} \frac{1}{(E'-E)^2} \left( \hat{\sigma}_2(x,E,E)\psi(x,\omega,E) + \frac{\partial}{\partial E'}(\hat{\sigma}_2(x,\cdot,E)\psi(x,\omega,\cdot))(E) \right) dE'
+ \int_{2E}^{E_m} \frac{1}{(E'-E)^2} (\mathcal{K}_{22,2}\psi)(x,\omega,E',E)dE'
= -\hat{\sigma}_2(x,E,E)\psi(x,\omega,E)\frac{1}{E} + \frac{\partial}{\partial E'}(\hat{\sigma}_2(x,\cdot,E)\psi(x,\omega,\cdot))(E)\ln(E)
+ \int_{2E}^{E_m} \frac{1}{(E'-E)^2} (\mathcal{K}_{22,2}\psi)(x,\omega,E',E)dE'
= \hat{\sigma}_2(x,\cdot,E)\ln(E)\frac{\partial\psi}{\partial E}(x,\omega,E) + (K_{22,2,2}\psi)(x,\omega,E)
+ \left( -\hat{\sigma}_2(x,E,E)\frac{1}{E} + \frac{\partial\hat{\sigma}_2}{\partial E}(x,E,E)\ln(E) \right)\psi(x,\omega,E),
\]
where
\[
(K_{22,2,2}\psi)(x,\omega,E) := \int_{2E}^{E_m} \frac{1}{(E'-E)^2} (\mathcal{K}_{22,2}\psi)(x,\omega,E',E)dE'.
\]
Applying the approximation (134), $K_{22,2,2}$ is approximately a partial Schur integral operator, say $\tilde{K}_{22,2,2}$.

Combining (142), (139), (133) we obtain an approximation of the exact Møller collision operator
\[
(\tilde{K}_{22}\psi)(x,\omega,E) = \hat{\sigma}_2(x,\cdot,E)\ln(E)\frac{\partial\psi}{\partial E}(x,\omega,E)
+ \left( -\hat{\sigma}_2(x,E,E)\frac{1}{E} + \frac{\partial\hat{\sigma}_2}{\partial E}(x,E,E)\ln(E) + \hat{\sigma}_1(x,E,E)\ln(E) \right)\psi(x,\omega,E)
+ (\tilde{K}_{22,2,2}\psi)(x,\omega,E) + (\tilde{K}_{22,2,1}\psi)(x,\omega,E) + (\tilde{K}_{22,0}\psi)(x,\omega,E).
\]
Here $\tilde{K}_{22,2,2} + \tilde{K}_{22,2,1} + \tilde{K}_{22,0}$ is a partial Schur integral operator.
The above approximations are reasonable only if $\psi \in H^{(0.0,2)}(G \times S \times I^o) \cap H^{(0.1,0)}(G \times S \times I^o)$.

We end this section by bringing up some open issues. The existence and uniqueness analysis for the exact transport equation $T \psi = f$ (with the due inflow boundary and initial conditions). Here $T$ is of the form

$$T \psi = -\frac{\partial}{\partial E}\left(\mathcal{H}_1((\mathcal{K}_2 \psi)(x, \omega, \cdot, E))(E)\right) + \mathcal{H}_1\left(\frac{\partial (\mathcal{K}_2 \psi)(x, \omega, \cdot, E)}{\partial E}(E)\right) + \omega \cdot \nabla_x \psi + F \cdot \nabla_\omega \psi + \Sigma \psi$$

$$- \mathcal{H}_1((\mathcal{K}_1 \psi)(x, \omega, \cdot, E))(E) - K_0 \psi,$$

where

$$K_0 \psi = \int_{\mu} (\mathcal{K}_0 \psi)(x, \omega, E', E) dE' = \int_{\mu} \hat{\sigma}_0(x, E', E) \int_0^{2\pi} \psi(x, \gamma(s), E') ds dE',$$

and where $\gamma$ is defined similarly as $\gamma_{11}(E', E, \omega)(s)$ and $\gamma_{22}(E', E, \omega)(s)$ above.

In addition, the real, physical model is a coupled system $T = (T_1, T_2, T_3)$ of the operators like (144) and some terms may be missing in $T_j$ (cf. the system considered in this paper). The existence and uniqueness properties for the exact transport equation $T \psi = f$ with the given inflow boundary and initial conditions remains to be analysed. Potential methods are (as in this paper) Lions-Lax-Milgram Theorem, theory of maximally dissipative operators and the theory of evolution operators.

It might be useful to first approximate operator $\mathcal{K}$ by $\mathcal{K}_\epsilon$ as we have outlined above. Then the corresponding transport operator, say $T_\epsilon$, might be a pseudo-differential operator, and hence a rich calculus of pseudo-differential operators could be applied. In this context, one would seek solutions for the approximative problem $T_\epsilon \psi_\epsilon = f$, and then one could try to show that $\psi_\epsilon$ converges to a solution of $T \psi = f$.

It is also important to understand regularity of solutions of BTE in the mixed-norm (anisotropic) Sobolev-Slobodevskij spaces. This is needed e.g. in approximation analysis and, in particular, in numerical analysis (e.g. FEM). Uniqueness and regularity analysis the above derived pseudo-differential form-like expressions of collision operators might also be useful.

**Remark 2.29** At least in existence and uniqueness analysis of solutions, it is more fruitful to use the partial differential (or pseudo-differential) form of the exact transport equation. Nevertheless, the numerical methods might apply directly the hyper-singular partial integral equation (88). For instance, the Galerkin (discontinuous) finite element methods (FEM) are able to consider hyper-singular partial integral terms. These techniques are well-known e.g. in field of boundary element methods (BEM) where the hyper-singular integral kernels are emerging from single and double layer potentials. We remark that carefully chosen (special) numerical integration schemes, and the choice of basis functions are needed in computing element matrices for hyper-singular integral operators. The applicability of these methods for the problem considered remains an open question.
3. Single Continuous Slowing Down Equation

3.1. Preliminaries. At first we consider a single CSDA transport equation given by

$$-\frac{\partial(S_0\psi)}{\partial E} + \omega \cdot \nabla_x \psi + \Sigma \psi - K\psi = f \quad \text{on } G \times S \times I,$$

where the solution satisfies inflow boundary and initial value conditions

$$\psi|_{\Gamma_-} = g \quad \text{on } \Gamma_-,$$

$$\psi(\cdot, \cdot, E_m) = 0 \quad \text{on } G \times S.$$  

We assume that

$$\Sigma \in L^\infty(G \times S \times I), \quad \Sigma \geq 0$$

a.e. on $G \times S \times I$. Furthermore, the collision operator is given for $\psi \in L^2(G \times S \times I)$ by

$$(K\psi)(x, \omega, E) = \int_{S \times I} \sigma(x, \omega', \omega, E)\psi(x, \omega', E')d\omega'dE',$$

where $\sigma : G \times S^2 \times I^2 \rightarrow \mathbb{R}$ is a non-negative measurable function such that

$$\int_{S \times I} \sigma(x, \omega', \omega, E) d\omega'dE' \leq M_1,$$

$$\int_{S \times I} \sigma(x, \omega', \omega, E) d\omega'dE' \leq M_2,$$

for a.e. $(x, \omega, E) \in G \times S \times I$, and we assume that there exists $c \geq 0$ such that

$$\Sigma(x, \omega, E) - \int_{S \times I} \sigma(x, \omega', \omega, E')e^{C(E'-E)}d\omega'dE' \geq c,$$

$$\Sigma(x, \omega, E) - \int_{S \times I} \sigma(x, \omega', \omega, E')e^{C(E'-E)}d\omega'dE' \geq c,$$

for a.e. $(x, \omega, E) \in G \times S \times I$, and where the constant $c \geq 0$ is specified below (see (151)). The criterion (151) is called the Schur criterion (emerged from measure theory) and the corresponding collision operators are Schur operators. Note that in some cases we will assume $c$ to be strictly positive, $c > 0$. This assumption has been relaxed in [24] (see also [20, Remark 15, pp. 241-242]). We have by Cauchy-Schwarz inequality and (151),

$$\|K\| \leq \left\| \int_{S \times I} \sigma(\cdot, \omega', \cdot, E')d\omega'dE' \right\|_{L^\infty}^{1/2} \left\| \int_{S \times I} \sigma(\cdot, \cdot, \omega', \cdot, E')d\omega'dE' \right\|_{L^\infty}^{1/2}$$

$$\leq M_1^{1/2}M_2^{1/2},$$

where $L^\infty = L^\infty(G \times S \times I)$, $\|K\|$ is the norm of $K$ as an operator in $L^2(G \times S \times I)$, and

$$\left\| \int_{S \times I} \sigma(\cdot, \omega', \cdot, E')d\omega'dE' \right\|_{L^\infty(G \times S \times I)} = \sup_{(x, \omega, E) \in G \times S \times I} \int_{S \times I} \sigma(x, \omega', \omega, E')d\omega'dE',$$

$$\left\| \int_{S \times I} \sigma(\cdot, \cdot, \omega', E')d\omega'dE' \right\|_{L^\infty(G \times S \times I)} = \sup_{(x, \omega, E) \in G \times S \times I} \int_{S \times I} \sigma(x, \omega, \omega', E')d\omega'dE'.$$

(153)
In what follows, we assume that the stopping power \( S_0 : G \times I \to \mathbb{R} \) satisfies the following assumptions:

\[
S_0 \in L^\infty(G \times I),
\]
\[
\frac{\partial S_0}{\partial E} \in L^\infty(G \times I),
\]
\[
\kappa := \inf_{(x,E) \in G \times I} S_0(x,E) > 0,
\]
\[
\nabla_x S_0 \in L^\infty(G \times I).
\]

We remark that the assumption \((158)\) will be needed only in the context of the theory of evolution operators in section 3.5.

**Remark 3.1** Assume that \( \hat{\sigma} : G \times \mathbb{R}^3 \times I \to \mathbb{R} \) is a measurable non-negative function, and that the cross-section \( \sigma \) is (formally) of the form

\[
\sigma(x,\omega',\omega,E,E') = \hat{\sigma}(x,\omega',\omega,E)\delta(E' - E),
\]

in the sense that the collision operator \( K \) is given by

\[
(K\psi)(x,\omega,E) = \int_S \hat{\sigma}(x,\omega',\omega,E)\psi(x,\omega',E)\,d\omega', \quad \psi \in L^2(G \times S \times I). \tag{159}
\]

In this case the assumptions (151) and (152) for any \( C \) mean that

\[
\int_S \hat{\sigma}(x,\omega',\omega,E)\,d\omega' \leq M_1,
\]
\[
\int_S \hat{\sigma}(x,\omega',\omega,E)\,d\omega' \leq M_2,
\]

for a.e. \( (x,\omega) \in G \times S \times I \), and

\[
\Sigma(x,\omega,E) - \int_S \hat{\sigma}(x,\omega',\omega,E)\,d\omega' \geq c,
\]
\[
\Sigma(x,\omega,E) - \int_S \hat{\sigma}(x,\omega',\omega,E)\,d\omega' \geq c,
\]

for a.e. \( (x,\omega,E) \in G \times S \times I \). All results proved in this paper are valid for these (simplified) collision operators. The estimate \((153)\) in this case is

\[
\|K\| \leq \left\|\int_S \hat{\sigma}(\cdot,\omega',\cdot,\cdot)\,d\omega'\right\|_{L^\infty(G \times S \times I)}^{1/2} \left\|\int_S \hat{\sigma}(\cdot,\cdot,\cdot,\cdot)\,d\omega'\right\|_{L^\infty(G \times S \times I)}^{1/2} \leq M_1^{1/2}M_2^{1/2}, \tag{162}
\]

where \( \|K\| \) is the norm of \( K \) as an operator in \( L^2(G \times S \times I) \), and

\[
\left\|\int_S \hat{\sigma}(\cdot,\omega',\cdot,\cdot)\,d\omega'\right\|_{L^\infty(G \times S \times I)} = \operatorname{ess sup}_{(x,\omega,E) \in G \times S \times I} \int_S \hat{\sigma}(x,\omega',\omega,E)\,d\omega',
\]
\[
\left\|\int_S \hat{\sigma}(\cdot,\cdot,\cdot,\cdot)\,d\omega'\right\|_{L^\infty(G \times S \times I)} = \operatorname{ess sup}_{(x,\omega,E) \in G \times S \times I} \int_S \hat{\sigma}(x,\omega',\omega,E)\,d\omega'.
\]

We begin with a lemma.
Lemma 3.2  For all $\psi \in C^1(G \times S \times I)$,
\[
\left< \frac{\partial (S_0 \psi)}{\partial E}, \psi \right>_{L^2(G \times S \times I)} \leq q \|\psi\|^2_{L^2(G \times S \times I)} + \frac{1}{2} \int_{G \times S} (S_0(x, E_m)\psi^2(x, \omega, E_m) - S_0(x, 0)\psi^2(x, \omega, 0)) dxd\omega,  \tag{163}
\]
where
\[
q := \frac{1}{2} \operatorname{ess sup}_{(x, E) \in G \times I} \frac{\partial S_0}{\partial E}(x, E).  \tag{164}
\]

Proof. Integrating by parts, we have
\[
\left< \frac{\partial (S_0 \psi)}{\partial E}, \psi \right>_{L^2(G \times S \times I)} = \left< \frac{\partial S_0}{\partial E} \psi, \psi \right>_{L^2(G \times S \times I)} + \left< \frac{\partial \psi}{\partial E}, S_0 \psi \right>_{L^2(G \times S \times I)}
\]
\[
+ \int_{G \times S} (S_0(x, E_m)\psi(x, \omega, E_m)^2 - S_0(x, 0)\psi(x, \omega, 0)^2) dxd\omega,  \tag{165}
\]
and therefore
\[
2 \left< \frac{\partial (S_0 \psi)}{\partial E}, \psi \right>_{L^2(G \times S \times I)} = \left< \frac{\partial S_0}{\partial E} \psi, \psi \right>_{L^2(G \times S \times I)} + \int_{G \times S} (S_0(x, E_m)\psi(x, \omega, E_m)^2 - S_0(x, 0)\psi(x, \omega, 0)^2) dxd\omega
\]
\[
\leq 2q \|\psi\|^2_{L^2(G \times S \times I)} + \int_{G \times S} (S_0(x, E_m)\psi(x, \omega, E_m)^2 - S_0(x, 0)\psi(x, \omega, 0)^2) dxd\omega.  \tag{166}
\]
This finishes the proof. \(\square\)

Note that if $E \mapsto S_0(x, E)$ is decreasing for every $x \in G$ then $q \leq 0$ (and therefore $C$ below vanishes).

Let
\[
C := \max \{q, 0\}.  \tag{167}
\]
We make the following change of the unknown function. We replace $\psi$ by
\[
\phi(x, \omega, E) := e^{CE} \psi(x, \omega, E).  \tag{168}
\]
This substitution changes the equation (146) to (here and below by writing $e^{CE}$ we mean a function $(x, \omega, E) \mapsto e^{CE}$)
\[
\frac{\partial (S_0 \phi)}{\partial E} + \omega \cdot \nabla_x \phi + CS_0 \phi + \Sigma \phi - K_C \phi = e^{CE} f,  \tag{169}
\]
where $K_C$ is given by
\[
(K_C \phi)(x, \omega, E) = \int_{S \times I} \sigma(x, \omega', \omega, E', E) e^{C(E-E')} \phi(x, \omega', E') d\omega' dE'.  \tag{170}
\]
The inflow boundary and the initial conditions are
\[ \phi_{|\Gamma_-} = e^{CE}g, \quad (171) \]
\[ \phi(x, \omega, E_m) = 0, \quad (172) \]
the latter (initial) condition holding for a.e. \((x, \omega) \in G \times S\).

**Lemma 3.3** Assume that the conditions (149), (151) and (152) are valid. Then
\[ \Sigma - K_C : L^2(G \times S \times I) \to L^2(G \times S \times I) \]
is a bounded operator and it satisfies the following accretivity condition
\[ \langle (\Sigma - K_C)\phi, \phi \rangle_{L^2(G \times S \times I)} \geq \varepsilon \|\phi\|^2_{L^2(G \times S \times I)}, \quad \phi \in L^2(G \times S \times I). \quad (173) \]

*Proof.* Since \( I = [0, E_m] \), where \( E_m < \infty \), we find that the conditions (151) are valid also for \( \sigma_C(x, \omega', \omega, E', E) := \sigma(x, \omega', \omega, E', E)e^{C(E-E')} \), possibly with different \( M_1, M_2 \). This observation combined with assumptions (149), (152), implies the result by a fairly simple application of Cauchy-Schwarz inequality; see [20, Theorem 4, p. 241]. \( \square \)

**Remark 3.4** In the study below, the standing assumption for the (inflow) boundary condition is \( g \in T^2(\Gamma_-) \) (single equation), or \( g \in T^2(\Gamma_-)^3 \) (coupled equation), however we point out that in some parts, one could do with the more general assumption \( g \in T^2_{\Gamma_-}(\Gamma_-) \), or \( g \in T^2(\Gamma_-)^3 \). In this paper we omit these generalizations.

### 3.2. Existence of Solutions for a Single Continuous Slowing Down Equation by a Variational Formulation

We shall consider the existence of solutions for the problem (169), (171), (172) by applying the so-called Lions-Lax-Milgram Theorem (or generalized Lax-Milgram Theorem) which is based on the variational formulation. Hence we begin with some computations which lead one to find the related bilinear and linear forms. In the following we denote
\[ f = e^{CE}f, \quad g = e^{CE}g. \quad (174) \]

Let \( f_- (f_+) \) be the negative (positive) part of a function. Recall that
\[ f = f_+ - f_- \quad \text{and} \quad |f| = f_+ + f_- . \quad (175) \]

Applying the Green’s formula [35] and integrating by parts, we have for and \( \phi \in C^1(\overline{G} \times S \times I) \) that satisfies the equation (169) and for any \( v \in C^1(\overline{G} \times S \times I) \),
\[
\begin{align*}
&\quad - \left\langle \frac{\partial (S_0\phi)}{\partial E}, v \right\rangle_{L^2(G \times S \times I)} + \left\langle \omega \cdot \nabla_x \phi, v \right\rangle_{L^2(G \times S \times I)} + \langle CS_0\phi, v \rangle_{L^2(G \times S \times I)} \\
&\quad + \langle \Sigma \phi, v \rangle_{L^2(G \times S \times I)} - \langle K_C \phi, v \rangle_{L^2(G \times S \times I)} \\
&= \left\langle \phi, S_0 \frac{\partial v}{\partial E} \right\rangle_{L^2(G \times S \times I)} - \int_{G \times S} S_0 \phi v \bigg|_{E=E_m} \, dx \, d\omega \\
&\quad - \langle \phi, \omega \cdot \nabla_x v \rangle_{L^2(G \times S \times I)} + \int_{G \times S \times I} (\omega \cdot v) \phi v \sigma d\omega \, dE \\
&\quad + \langle \phi, CS_0 v \rangle_{L^2(G \times S \times I)} + \langle \phi, \Sigma^* v \rangle_{L^2(G \times S \times I)} - \langle \phi, K^* v \rangle_{L^2(G \times S \times I)} \\
&= \langle f, v \rangle_{L^2(G \times S \times I)} \quad (176)
\end{align*}
\]
Assuming that the inflow boundary condition $\phi|_{r_-} = g$ and the initial condition $\phi(\cdot, \cdot, E_0) = 0$ are valid, the equation (176) is equivalent to

\[
\left\langle \phi, S_0 \frac{\partial v}{\partial E} \right\rangle_{L^2(G \times S \times I)} + \langle \phi(\cdot, \cdot, 0), S_0(\cdot, \cdot, 0) v(\cdot, \cdot, 0) \rangle_{L^2(G \times S)} - \langle \phi, \omega \cdot \nabla v \rangle_{L^2(G \times S \times I)} + \int_{\partial G \times S} (\omega \cdot \nu) + \phi v d\sigma d\omega dE + \langle \phi, CS_0 v \rangle_{L^2(G \times S \times I)} + \langle \phi, K^* v \rangle_{L^2(G \times S \times I)} = \langle f, v \rangle_{L^2(G \times S \times I)} + \int_{\partial G \times S \times I} (\omega \cdot \nu) - \mathbf{g} v d\sigma d\omega dE.
\]

Clearly

\[
\int_{\partial G \times S} (\omega \cdot \nu) - \mathbf{g} v d\sigma d\omega dE = \langle \mathbf{g}, \gamma_- (v) \rangle_{T^2(\Gamma_-)}
\]

and

\[
\int_{\partial G \times S} (\omega \cdot \nu) + \phi v d\sigma d\omega dE = \langle \gamma_+ (\phi), \gamma_+ (v) \rangle_{T^2(\Gamma_+)}.
\]

One thus deduces that the relevant bilinear from $B$ and linear form $F$ are

\[
B(\phi, v) = \left\langle \phi, S_0 \frac{\partial v}{\partial E} \right\rangle_{L^2(G \times S \times I)} - \langle \phi, \omega \cdot \nabla v \rangle_{L^2(G \times S \times I)} + C \langle \phi, S_0 v \rangle_{L^2(G \times S \times I)} + \langle \phi, (\Sigma^* - K^* C) v \rangle_{L^2(G \times S \times I)} + \langle \gamma_+ (\phi), \gamma_+ (v) \rangle_{T^2(\Gamma_+)} + \langle \phi(\cdot, \cdot, 0), S_0(\cdot, \cdot, 0) v(\cdot, \cdot, 0) \rangle_{L^2(G \times S)}
\]

and

\[
F(v) = \langle f, v \rangle_{L^2(G \times S \times I)} + \langle \mathbf{g}, \gamma_- (v) \rangle_{T^2(\Gamma_-)}.
\]

The variational equation corresponding to the problem (169), (171), (172) (in the classical sense) is

\[
B(\phi, v) = F(v) \quad \forall v \in C^1(\overline{G} \times S \times I).
\]

We show that the bilinear form $B : C^1(\overline{G} \times S \times I) \times C^1(\overline{G} \times S \times I) \to \mathbb{R}$ obeys the following boundedness and coercivity conditions:

**Theorem 3.5** Suppose that the assumptions (149), (151), (152) (with $C = \max\{g, 0\}$ and $C > 0$) and (155), (156), (157) are valid. Then there exists a constant $M > 0$ such that

\[
|B(\phi, v)| \leq M \|\phi\|_{H_1} \|v\|_{H_2} \quad \forall \phi, v \in C^1(\overline{G} \times S \times I)
\]

and

\[
B(\phi, \phi) \geq c \|\phi\|^2_{H_1} \quad \forall \phi \in C^1(\overline{G} \times S \times I)
\]
where we assume that
\[ c' := \min\left\{ \frac{1}{2}, \frac{\kappa}{2}, c \right\} \quad (185) \]
is strictly positive. Note that \( \kappa \) is defined in (157) and \( c \) in (152). (Recall that the spaces \( H_1 \) and \( H_2 \) were defined in equations (38) and (39), respectively.)

**Proof.** A. At first we show the boundedness of \( B(\cdot, \cdot) \). The assumptions (155), (156) and the fact that the (energy) interval \( I \) is bounded imply by the Sobolev Embedding theorem (see (31)) that \( S_0(\cdot, 0) \in L^\infty(G) \). Hence we find that
\[
|B(\phi, v)| \leq \|\phi\|_{L^2(G \times S \times I)} \|S_0\|_{L^\infty(G \times I)} \left\| \frac{\partial \phi}{\partial E} \right\|_{L^2(G \times S \times I)} \\
+ \|\phi\|_{L^2(G \times S \times I)} \left\| \omega \cdot \nabla_x v \right\|_{L^2(G \times S \times I)} \\
+ C \|S_0\|_{L^\infty(G \times I)} \|\phi\|_{L^2(G \times S \times I)} \|v\|_{L^2(G \times S \times I)} \\
+ \|\phi\|_{L^2(G \times S \times I)} \|(\Sigma - K_C)^*\| \|v\|_{L^2(G \times S \times I)} \\
+ \|\gamma(\phi)\|_{T^2(I)} \|\gamma(v)\|_{T^2(I)} \\
+ \|\phi(\cdot, \cdot, 0)\|_{L^2(G \times S)} \|S_0(\cdot, 0)\|_{L^\infty(G)} \|v(\cdot, \cdot, 0)\|_{L^2(G \times S)}.
\]
This implies the assertion (183) by the definition of the spaces \( H_1 \), \( H_2 \), and after observing that Sobolev Embedding theorem (see (31)) implies the existence of a constant \( C' \geq 0 \) such that
\[
\|w(\cdot, \cdot, 0)\|_{L^2(G \times S)} \leq C' \|w\|_{H_2}, \quad \forall w \in H_2.
\]
B. We verify the coercitivity (184). Integrating by parts we have
\[
\left\langle \phi, S_0 \frac{\partial \phi}{\partial E} \right\rangle_{L^2(G \times S \times I)} = - \left\langle \phi, \frac{\partial (S_0 \phi)}{\partial E} \right\rangle_{L^2(G \times S \times I)} \\
+ \left\langle \phi(\cdot, \cdot, E_m), S_0(\cdot, E_m) \phi(\cdot, \cdot, E_m) \right\rangle_{L^2(G \times S)} - \left\langle \phi(\cdot, \cdot, 0), S_0(\cdot, 0) \phi(\cdot, \cdot, 0) \right\rangle_{L^2(G \times S)}.
\]
Using the Green’s formula (35) we have
\[
- \left\langle \phi, \omega \cdot \nabla_x \phi \right\rangle_{L^2(G \times S \times I)} = \left\langle \omega \cdot \nabla_x \phi, \phi \right\rangle_{L^2(G \times S \times I)} - \int_{\partial G \times S \times I} (\omega \cdot \nu) \phi^2 d\sigma d\omega dE
\]
which implies
\[
\left\langle \omega \cdot \nabla_x \phi, \phi \right\rangle_{L^2(G \times S \times I)} = \frac{1}{2} \int_{\partial G \times S \times I} (\omega \cdot \nu) \phi^2 d\sigma d\omega dE \\
= \frac{1}{2} \int_{\partial G \times S \times I} ((\omega \cdot \nu)_+ - (\omega \cdot \nu)_-) \phi^2 d\sigma d\omega dE \\
= \frac{1}{2} \left( \|\gamma_+(\phi)\|^2_{T^2(I_+)} - \|\gamma_-(\phi)\|^2_{T^2(I_-)} \right).
\]
By the definition (167) of \( C \) and assumption (157), we have
\[
C = \frac{\max\{q, 0\}}{\kappa} \geq \frac{\max\{q, 0\}}{S_0} \geq \frac{q}{S_0}
\]
a.e. and hence
\[
C \left\langle \phi, S_0 \phi \right\rangle_{L^2(G \times S \times I)} \geq q \left\| \phi \right\|^2_{L^2(G \times S \times I)}.
\]
Taking Lemmas 3.2 and 3.3 into account, one can thus estimate

\[ \| \tilde{B}(\phi, V) \|_{L^2(G \times S \times I)} \leq M \| \tilde{\phi} \|_{H^1} \| v \|_{H^2} \quad \forall \tilde{\phi} \in H_1, \ v \in H_2 \]  

and

\[ \| \tilde{B}(v, V) \|_{H^1} \geq c ' \| v \|_{H^2} \quad \forall v \in H_2. \]

We see that actually

\[ \tilde{B}(\phi, V) = \left\langle \phi, S_0 \nabla_{x} \right\rangle_{L^2(G \times S \times I)} + \left\langle \phi, \omega \cdot \nabla_{x} v \right\rangle_{L^2(G \times S \times I)} + \left\langle \phi, \Sigma - K \right\rangle_{L^2(G \times S \times I)} + \left\langle \phi, \gamma_{+}(v) \right\rangle_{L^2(\Gamma_{+})} + \left\langle p_0, S_0 v \right\rangle_{L^2(G \times S \times I)} , \]

\[ \text{where } \tilde{\phi} = (\phi, q, p_0, p_m) \in H_1 \text{ and } v \in H_2. \]  

In addition, since for \( v \in C^1(G \times S \times I) \) we have \( \| \gamma_{-}(v) \|_{L^2(\Gamma_{-})} \leq \| \gamma(v) \|_{L^2(\Gamma_{-})} \), it follows that

\[ |F(v)| \leq |\langle f, v \rangle_{L^2(G \times S \times I)}| + |\langle g, \gamma_{-}(v) \rangle_{L^2(\Gamma_{-})}| \leq ||f||_{L^2(G \times S \times I)} ||v||_{L^2(G \times S \times I)} + ||g||_{L^2(\Gamma_{-})} ||\gamma(v)||_{L^2(\Gamma_{-})} \]

and therefore, since \( C^1(G \times S \times I) \) is dense in \( H_1 \), the linear form \( F : C^1(G \times S \times I) \to \mathbb{R} \) has a unique bounded extension, which we still denote by \( F \),

\[ F : H_1 \to \mathbb{R}; \quad F(\tilde{\phi}) = \langle f, \phi \rangle_{L^2(G \times S \times I)} + \langle g, q \rangle_{L^2(\Gamma_{-})} , \]

when \( \tilde{\phi} = (\phi, q, p_0, p_m) \in H_1 \). Recall also that the embedding \( H_2 \subset H_1 \) is continuous.

We need the following result, so called Lions-Lax-Milgram Theorem (generalized Lax-Milgram Theorem).

**Theorem 3.6** Let \( X \) and \( Y \) be Hilbert spaces, with \( Y \) continuously embedded into \( X \). Assume that \( B(\cdot, \cdot) : X \times Y \to \mathbb{R} \) is a bilinear form satisfying the following properties with \( M \geq 0 \), \( c > 0 \),

\[ |B(u, v)| \leq M \| u \|_X \| v \|_Y \quad \forall u \in X, \ v \in Y \quad \text{(boundedness)} \]
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and

$$B(v, v) \geq c\|v\|_X^2 \quad \forall v \in Y \quad \text{(coercivity).} \quad (197)$$

Suppose that $F : X \to \mathbb{R}$ is a bounded linear form. Then there exists $u \in X$ (possibly non-unique) such that

$$B(u, v) = F(v) \quad \forall v \in Y. \quad (198)$$

**Proof.** See e.g. [73, p. 403] or [35, p. 234]. □

Let

$$P(x, \omega, E, D) \phi := -\frac{\partial (S_0 \phi)}{\partial E} + \omega \cdot \nabla \phi$$

The space

$$\mathcal{H}_P(G \times S \times I^\circ) := \{ \phi \in L^2(G \times S \times I) \mid P(x, \omega, E, D) \phi \in L^2(G \times S \times I) \text{ in the weak sense} \} \quad (199)$$

is a Hilbert space when equipped with the inner product (cf. section 3.3)

$$\langle \phi, v \rangle_{\mathcal{H}_P(G \times S \times I^\circ)} = \langle \phi, v \rangle_{L^2(G \times S \times I)} + \langle P(x, \omega, E, D) \phi, P(x, \omega, E, D) v \rangle_{L^2(G \times S \times I)}.$$

With this notation, the equations (146) and (169) can be written as

$$P(x, \omega, E, D) \psi + \Sigma \psi - K \psi = f,$$

and

$$P(x, \omega, E, D) \psi + CS_0 \phi + \Sigma \phi - K_C \phi = e^{CE} f,$$

respectively.

In the context if Lions-Lax-Milgram Theorem we shall make use of the following assumption which we call as TC:

**Assumption TC.** Let $\gamma_\pm(\phi) = \phi|_{\Gamma_\pm}$ and $\gamma_m(\phi) := \phi(\cdot, \cdot, E_m)$, $\gamma_0(\phi) := \phi(\cdot, \cdot, 0)$. The assumption is that the linear maps

$$\gamma_\pm : \mathcal{H}_P(G \times S \times I^\circ) \to L^2_{\text{loc}}(\Gamma_\pm, |\omega \cdot \nu| \sigma d\sigma d\omega dE),$$

$$\gamma_m : \mathcal{H}_P(G \times S \times I^\circ) \to L^2_{\text{loc}}(G \times S),$$

$$\gamma_0 : \mathcal{H}_P(G \times S \times I^\circ) \to L^2_{\text{loc}}(G \times S),$$

are well-defined and continuous.

**Remark 3.7** In the case where $S = S(E)$ is independent of $x$ and $S \in C(I)$ one can show that the assumption TC holds. The proof can be based on the techniques of the proof of Theorem 2.15 and Example 6.1. If $S = S(x, E)$ depends also on $x$, we conjecture that it suffices only to assume that $S$ is regular enough.

Let

$$P'(x, \omega, E, D)v = S_0 \frac{\partial v}{\partial E} - \omega \cdot \nabla_x v$$
be the formal transpose of $P(x, \omega, E, D)$. Making the assumption \textbf{TC} the (extended) Green formula

$$\int_{G \times S \times I} (P(x, \omega, E, D) \phi)v \, dx d\omega dE - \int_{G \times S \times I} (P'(x, \omega, E, D)v) \phi \, dx d\omega dE$$

$$= \int_{\partial G \times S \times I} (\omega \cdot v) \phi \, d\sigma d\omega dE$$

$$+ \int_{G \times S} (S_0(\cdot, 0)\phi(\cdot, \cdot, 0)v(\cdot, \cdot, 0) - S_0(\cdot, E_m)\phi(\cdot, \cdot, E_m)v(\cdot, \cdot, E_m)) \, dx \omega \quad (200)$$

is valid for all $\phi, v \in \mathcal{H}_p(G \times S \times I^o)$ for which $(\text{supp}(v)) \cap \partial(G \times S \times I)$ is a compact subset of $\Gamma_\perp \cup \Gamma_+ \cup (G \times S \times \{E_m\}) \cup (G \times S \times \{0\})$. Moreover, (200) holds for $\phi, v \in \mathcal{H}_p(G \times S \times I^o)$ when $\gamma_+(\phi) \in T^2(\Gamma_\perp)$ and $\gamma_m(\phi), \gamma_0(\psi) \in L^2(G \times S)$. We omit the proof of both these claims.

We are now in position to formulate and prove the following theorem.

\textbf{Theorem 3.8} Suppose that the assumptions (149), (151), (152) (with $C = \max\{q, 0\}$ and $c > 0$) and (155), (156), (157) are valid. Let $f \in L^2(G \times S \times I)$ and $g \in T^2(\Gamma_\perp)$. Then the following assertions hold.

(i) The variational equation (see (193), (195))

$$\tilde{B}(\tilde{\phi}, v) = F(v) \quad \forall v \in H_2, \quad (201)$$

has a solution $\tilde{\phi} = (\phi, q, p_0, p_m) \in H_1$.

Furthermore, $\phi \in \mathcal{H}_p(G \times S \times I^o)$ and it is a weak (distributional) solution of the equation (169).

(ii) Suppose that additionally the assumption \textbf{TC} holds. Then a solution $\phi$ of the equation (169) obtained in part (i) is a solution of the problem (169), (171), (172).

In addition, we have $q|_{\Gamma_\perp} = \gamma_+(\phi)$ and $p_0 = \phi(\cdot, \cdot, 0)$, when $\tilde{\phi} = (\phi, q, p_0, p_m)$ is a solution in $H_1$ obtained in part (i).

(iii) Under the assumptions imposed in part (ii), any solution $\phi \in \mathcal{H}_p(G \times S \times I^o)$ of the problem (169), (171), (172) that further satisfies

$$\phi|_{\Gamma_\perp} \in T^2(\Gamma_\perp) \quad \text{and} \quad \phi(\cdot, \cdot, 0) \in L^2(G \times S), \quad (202)$$

is unique and obeys the estimate

$$\|\phi\|_{H_1} \leq \frac{1}{c'} \left( \|f\|_{L^2(G \times S \times I^o)} + \|g\|_{T^2(\Gamma_\perp)} \right), \quad (203)$$

where $c'$ is given in (185).

\textit{Proof.} The proof is based on "variations" and it is quite standard.

(i) We apply Theorem 3.6 with $X = H_1$, $Y = H_2$, and with $B(\cdot, \cdot) = \tilde{B}(\cdot, \cdot)$ and $F$ given by (193) and (195), respectively. As mentioned above $\tilde{B}(\cdot, \cdot)$ satisfies (196) and (197), while $F$ is a bounded linear functional, hence Theorem 3.6 guarantees the existence of a solution $\hat{\phi} = (\phi, q, p_0, p_m) \in H_1$ such that (201) holds.

We verify that $\phi \in L^2(G \times S \times I)$ is a weak solution of the equation (169). Let $I^o := [0, E_m]$. From (201) it follows that

$$\tilde{B}(\tilde{\phi}, v) = F(v), \quad \forall v \in C_0^\infty(G \times S \times I^o). \quad (204)$$
Since for $v \in C_0^\infty (G \times S \times I^0)$ we have $v (\cdot, \cdot, 0) = v (\cdot, \cdot, E_m) = 0$ and $v |_{\Gamma} = 0$, we see from (193) that

$$\tilde{B} (\tilde{\phi}, v) = \left\langle \phi, S_0 \frac{\partial v}{\partial E} \right\rangle_{L^2 (G \times S \times I)} - \left\langle \phi, \omega \cdot \nabla_x v \right\rangle_{L^2 (G \times S \times I)} + \left\langle CS_0 \phi, v \right\rangle_{L^2 (G \times S \times I)} + \left\langle (\Sigma - K_C) \phi, v \right\rangle_{L^2 (G \times S \times I)}$$

$$= F (v) = \langle f, v \rangle_{L^2 (G \times S \times I)},$$

(205)

for all $v \in C_0^\infty (G \times S \times I^0)$, which means that (169) holds in the weak sense.

Since $\phi \in L^2 (G \times S \times I)$, and by the above

$$\langle \phi, P' (x, \omega, E, D) v \rangle_{L^2 (G \times S \times I)} = \langle -(CS_0 + (\Sigma - K_C)) \phi + f, v \rangle_{L^2 (G \times S \times I)},$$

we see that $\phi \in \mathcal{H}_P (G \times S \times I^0)$.

(ii) Suppose that the assumption TC holds and that $\tilde{\phi} = (\phi, q, p_0, p_m) \in H_1$ satisfies (201). Then for all $v \in H_2$

$$\left\langle \phi, S_0 \frac{\partial v}{\partial E} \right\rangle_{L^2 (G \times S \times I)} - \left\langle \phi, \omega \cdot \nabla_x v \right\rangle_{L^2 (G \times S \times I)} + C \left\langle \phi, S_0 v \right\rangle_{L^2 (G \times S \times I)} + \left\langle \phi, (\Sigma^* - K_C) v \right\rangle_{L^2 (G \times S \times I)}$$

$$+ \left\langle \phi, \Sigma^* (v + \cdot, \cdot) \right\rangle_{T^2_{(\Gamma_+)}} + \left\langle p_0, S_0 (\cdot, 0) v (\cdot, \cdot, 0) \right\rangle_{L^2 (G \times S)}$$

$$= \tilde{B} (\tilde{\phi}, v) = \langle f, v \rangle_{L^2 (G \times S \times I)} + \langle g, \gamma_+(v) \rangle_{T^2_{(\Gamma_+)}},$$

(206)

Recall that $\Gamma' = \partial G \times S$ and $\Gamma'_- = \{(y, \omega) \in \partial G \times S \mid \omega \cdot v (y, \omega) < 0\}$. Choose any $\eta \in C_0^\infty (I^0)$ and $\theta \in C^1 (\overline{G} \times S)$ such that $\supp (\theta) \cap \Gamma'$ is a compact subset of $\Gamma'$. Then $w (x, \omega, E) := \theta (x, \omega) \eta (E) \in C^1 (\overline{G} \times S \times I)$ and $w (\cdot, \cdot, 0) = w (\cdot, \cdot, E_m) = 0$, $w |_{\Gamma_+} = 0$. Hence $\langle p_0, S_0 (\cdot, 0) w (\cdot, \cdot, 0) \rangle_{L^2 (G \times S)} = 0$ and $\langle q, \gamma_+(v) \rangle_{T^2_{(\Gamma_+)} = 0}$, and so by (203) for these $w$,

$$\left\langle \phi, S_0 \frac{\partial w}{\partial E} \right\rangle_{L^2 (G \times S \times I)} - \left\langle \phi, \omega \cdot \nabla_x w \right\rangle_{L^2 (G \times S \times I)}$$

$$+ \left\langle \phi, CS_0 w \right\rangle_{L^2 (G \times S \times I)} + \left\langle \phi, (\Sigma^* - K_C) w \right\rangle_{L^2 (G \times S \times I)}$$

$$= \langle f, w \rangle_{L^2 (G \times S \times I)} + \langle g, \gamma_-(w) \rangle_{T^2_{(\Gamma_-)}},$$

(207)

Since the solution $\phi$ obtained in part (i) belongs to $\mathcal{H}_P (G \times S \times I^0)$, we have by virtue of the Green’s formula (200) and (201) that

$$\langle f, w \rangle_{L^2 (G \times S \times I)} + \langle g, \gamma_- (w) \rangle_{T^2_{(\Gamma_-)}}$$

$$= \left\langle - \frac{\partial (S_0 \phi)}{\partial E} + \omega \cdot \nabla_x \phi + CS_0 \phi + (\Sigma - K_C) \phi, w \right\rangle_{L^2 (G \times S \times I)} + \langle \gamma_- (\phi), \gamma_- (w) \rangle_{T^2_{(\Gamma_-)}}$$

$$= \langle f, w \rangle_{L^2 (G \times S \times I)} + \langle g, \gamma_-(w) \rangle_{T^2_{(\Gamma_-)}},$$

(208)

and hence

$$\langle \gamma_- (\phi), \gamma_- (w) \rangle_{T^2_{(\Gamma_-)}} = \langle g, \gamma_- (w) \rangle_{T^2_{(\Gamma_-)}},$$

(209)

for any $w$ of the form as chosen above. This clearly implies that $\gamma_- (\phi) = g \in T^2_{(\Gamma_-)}$.

Next, choose $\tilde{\eta} \in C^1 (I)$ such that $\tilde{\eta} (0) = 0$ and choose $\tilde{\theta} \in C_0^\infty (G)$. Let $\tilde{w} := \tilde{\eta} \tilde{\theta}$. Then by similar calculation as above, we see that the Green’s formula (200) and (203) imply

$$\langle \phi (\cdot, \cdot, E_m), S_0 (\cdot, E_m) \tilde{w} (\cdot, \cdot, E_m) \rangle_{L^2 (G \times S)} = 0$$
for these $\tilde{w}$. Consequently, $\phi(\cdot, \cdot, E_m) = 0$ a.e. in $G \times S$ (since $S_0 \geq \kappa > 0$, as desired.

Finally, if $v \in C^1(\overline{G} \times S \times I)$, we have

$$\langle f, v \rangle_{L^2(G\times S \times I)} + \langle g, \gamma_-(v) \rangle_{T^2(\Gamma_-)} = F(v) = \bar{B}(\tilde{\phi}, v)$$

$$= \langle \phi, P'(x, \omega, E, D)v \rangle_{L^2(G\times S \times I)} + \langle CS_0 \phi, v \rangle_{L^2(G\times S \times I)} + \langle (\Sigma - K_C)\phi, v \rangle_{L^2(G\times S \times I)}$$

$$+ \langle q_{\Gamma_+}, \gamma_+(v) \rangle_{T^2(\Gamma_+)} + \langle p_0, S_0(\cdot, 0)v(\cdot, \cdot, 0) \rangle_{L^2(G\times S)}$$

$$= \langle T_C \phi, v \rangle_{L^2(G\times S \times I)} + \langle q_{\Gamma_+} - \gamma_+(\phi), \gamma_+(v) \rangle_{T^2(\Gamma_+)} + \langle \gamma_-(\phi), \gamma_-(v) \rangle_{T^2(\Gamma_-)}$$

$$+ \langle p_0 - \phi(\cdot, \cdot, 0), S_0(\cdot, 0)v(\cdot, \cdot, 0) \rangle_{L^2(G\times S)} + \langle \phi(\cdot, \cdot, E_m), S_0(\cdot, E_m)v(\cdot, \cdot, E_m) \rangle_{L^2(G\times S)}$$

where on the second to last phase we wrote $T_C \phi = (P(x, \omega, E, D) + CS_0 + \Sigma - K_C)\phi$ and used Green’s formula (200), and on the last phase we made use of the already proven facts: $T_C \phi = f$, $\gamma_-(\phi) = g$, and $\phi(\cdot, \cdot, E_m) = 0$. Thus it holds

$$\langle q_{\Gamma_+} - \gamma_+(\phi), \gamma_+(v) \rangle_{T^2(\Gamma_+)} + \langle p_0 - \phi(\cdot, \cdot, 0), S_0(\cdot, 0)v(\cdot, \cdot, 0) \rangle_{L^2(G\times S)} = 0,$$

for all $v \in C^1(\overline{G} \times S \times I)$, which clearly implies that $q_{\Gamma_+} = \gamma_+(\phi)$ and $p_0 = \phi(\cdot, \cdot, 0)$.

(iii) By assumption, $\phi_{\Gamma_+} \in T^2(\Gamma_+)$ and $\phi(\cdot, \cdot, 0)$, $\phi(\cdot, \cdot, E_m) \in L^2(G \times S)$, and moreover $\phi \in \mathcal{H}_P(G \times S \times I^0)$. These properties allow us to apply the Green’s formula (200), which in combination with the fact that

$$P'(x, \omega, E, D)\phi = -P(x, \omega, E, D)\phi - \frac{\partial S_0}{\partial E} \phi,$$

leads us to

$$\langle P(x, \omega, E, D)\phi, \phi \rangle_{L^2(G\times S \times I)} = -\langle P(x, \omega, E, D)\phi, \phi \rangle_{L^2(G\times S \times I)} - \langle \frac{\partial S_0}{\partial E} \phi, \phi \rangle_{L^2(G\times S \times I)}$$

$$+ \|\gamma_+(\phi)\|_{T^2(\Gamma_+)}^2 - \|\gamma_-(\phi)\|_{T^2(\Gamma_-)}^2$$

$$+ \langle S_0(\cdot, 0)\gamma_0(\phi), \gamma_0(\phi) \rangle_{L^2(G\times S)} - \langle S_0(\cdot, E_m)\gamma_m(\phi), \gamma_m(\phi) \rangle_{L^2(G\times S)}.$$

Using this equation, and performing estimations as in the proof of Theorem 3.3, allows us to deduce the inequality

$$\langle f, \phi \rangle_{L^2(G\times S \times I)} + \langle g, \gamma_-(\phi) \rangle_{T^2(\Gamma_-)} \geq c' \|\phi\|_{H^1}^2,$$

from which the desired estimate (203), and therefore uniqueness of solutions, follow.

\[\square\]

**Remark 3.9** Suppose that the assumption TC is valid and that $\phi \in \mathcal{H}_P$ such that

$$\phi_{\Gamma_-} \in T^2(\Gamma_-) \quad \text{and} \quad \phi(\cdot, \cdot, E_m) \in L^2(G \times S). \quad (210)$$

Then at least in some cases one is able to show that (cf. [10])

$$\phi_{\Gamma_+} \in T^2(\Gamma_+) \quad \text{and} \quad \phi(\cdot, \cdot, 0) \in L^2(G \times S). \quad (211)$$

This would make the assumption of part (iii) of Theorem 3.8 superfluous. We omit further considerations of this issue here.
Similarly as above we see that the variational equation corresponding to the original problem (146), (147), (148) is

$$\tilde{B}_0(\tilde{\psi}, v) = F_0(v) \quad \forall v \in H_2,$$

where $\tilde{\psi} \in H_1$ and $\tilde{B}_0(\cdot, \cdot)$ is the continuous extension onto $H_1 \times H_2$ of the bilinear form $B_0(\cdot, \cdot) : C^1(G \times S \times I) \times C^1(G \times S \times I) \to \mathbb{R}$ defined by (that is, the bilinear form (182) with $C = 0$)

$$B_0(\psi, v) = \left( \psi, S_0 \frac{\partial v}{\partial E} \right)_{L^2(G \times S \times I)} - \langle \psi, \omega \cdot \nabla_x v \rangle_{L^2(G \times S \times I)} + \langle \psi, (\Sigma^* - K^*)v \rangle_{L^2(G \times S \times I)}$$

$$+ \langle \gamma_+(\tilde{\psi}), \gamma_+(v) \rangle_{T^2(\Gamma_+)} + \langle \psi(\cdot, \cdot, 0), S_0(\cdot, 0)v(\cdot, \cdot, 0) \rangle_{L^2(G \times S \times I)}.$$

(212)

The linear form $F_0 : C^1(G \times S \times I) \to \mathbb{R}$ is given by

$$F_0(v) = \langle f, v \rangle_{L^2(G \times S \times I)} + \langle g, \gamma_-(v) \rangle_{T^2(\Gamma_-)}.$$

and it admits a unique extension to a bounded linear form $F_0 : H_1 \to \mathbb{R}$. Note that the bilinear form (212) is not necessarily coercive that is (184) does not necessarily hold, which justifies the need for the change of unknown $\phi = e^{CE}\psi$ performed above.

We have the following immediate corollary for the existence of solutions of the original CSDA-problem.

**Corollary 3.10** Suppose that the assumptions (149), (151), (152) (with $c > 0$), (155), (156) and (157) are valid. Let $f \in L^2(G \times S \times I)$ and $g \in T^2(\Gamma_-)$. Then the following assertions hold.

(i) The variational equation

$$\tilde{B}_0(\tilde{\psi}, v) = F_0(v) \quad \forall v \in H_2$$

(213)

has a solution $\tilde{\psi} = (\psi, q, p_0, p_m) \in H_1$.

Furthermore, $\psi \in H_P(G \times S \times I)$ and it is a weak (distributional) solution of the equation (146).

(ii) Suppose that additionally the assumption $\text{TC}$ holds. Then a solution $\psi$ of the equation (146) obtained in part (i) is a solution of the problem (146), (147), (148).

In addition, we have $q_\Gamma \psi_\Gamma = \gamma_+(\tilde{\psi})$ and $p_0 = \psi(\cdot, \cdot, 0)$, when $\tilde{\psi} = (\psi, q, p_0, p_m)$ is a solution in $H_1$ obtained in part (i).

(iii) Under the assumptions imposed in part (ii) any solution $\psi \in H_P(G \times S \times I^c)$ of the problem (146), (147), (148) that further satisfies

$$\psi|_{\Gamma_+} \in T^2(\Gamma_+) \quad \text{and} \quad \psi(\cdot, \cdot, 0) \in L^2(G \times S),$$

(214)

is unique and obeys the estimate

$$\|\psi\|_{H_1} \leq \frac{e^{CE_m}}{c} \left( \|f\|_{L^2(G \times S \times I)} + \|g\|_{T^2(\Gamma_-)} \right).$$

(215)

(Recall that $C$ is defined in (167), $c$ in (185) and that $E_m$ is the cutoff energy.)

**Proof.** Let $f \in L^2(G \times S \times I)$ and $g \in T^2(\Gamma_-)$. Since $I$ is finite interval we see that $f := e^{CE}f \in L^2(G \times S \times I)$ and $g := e^{CE}g \in T^2(\Gamma_-)$, where $C = \max_{i=0}^{\infty} \frac{\kappa_i}{\kappa_i}$ (see (167), (171)). By Theorem 3.8 the variational problem (211) has a solution $\phi = (\phi, q', p'_0, p'_m) \in H_1$, from which one deduces without difficulty that $\tilde{\psi} = (\psi, q, p_0, p_m) := e^{CE}\phi \in H_1$ is a solution of the variational problem (213). Similarly, the fact that $\phi \in H_P(G \times S \times I^c)$ implies that $\psi \in H_P(G \times S \times I^c)$. This can
be seen by substituting $e^{-CE}v \in H_2$ instead of $v$ into (201). Besides the estimate (215), all the claims are consequences of the corresponding items in Theorem 3.8.

Recalling that $f = e^{CE}f$, $g = e^{CE}g$ the estimate (215) is obtained as follows,

$$
\| \psi \|_{H_1} = \| e^{-CE} \phi \|_{H_1} \leq \| \phi \|_{H_1} \leq \frac{1}{c'} \left( \| f \|_{L^2(G \times S \times I)} + \| g \|_{T^2(\Gamma_-)} \right) \\
\leq \frac{e^{CE}}{c'} \left( \| f \|_{L^2(G \times S \times I)} + \| g \|_{T^2(\Gamma_-)} \right).
$$

(216)

This completes the proof. □

Remark 3.11 A. Let

$$
T \psi := - \frac{\partial (S_0 \psi)}{\partial E} + \omega \cdot \nabla_x \psi + \Sigma \psi - K \psi
$$

and let

$$
T^* v = S_0 \frac{\partial v}{\partial E} - \omega \cdot \nabla_x v + \Sigma^* v - K^* v
$$

be the formal transpose of $T$. Furthermore, let the assumptions (149), (151), (152), (155), (156) and (157) of Theorem 3.8 be valid, and let $\tilde{\psi} = (\psi, q, p_0, p_m) \in H_1$ be a solution of (213) (guaranteed by Corollary 3.10). By (212), we find that for any $v \in C^1(G \times S \times I)$ for which the adjoint boundary conditions $v(\cdot, \cdot, 0) = 0$ and $\gamma_+(v) = 0$ hold, we have

$$
\langle \psi, T^* v \rangle = \tilde{B}_0(\psi, v) = \langle f, v \rangle_{L^2(G \times S \times I)} + \langle g, \gamma_-(v) \rangle_{T^2(\Gamma_-)},
$$

(217)

that is

$$
\langle \psi, T^* v \rangle - \langle T \psi, v \rangle_{L^2(G \times S \times I)} = \langle g, \gamma_-(v) \rangle_{T^2(\Gamma_-)}.
$$

(218)

This means that $\tilde{\psi} \in H_1$ is a weak solution of the boundary (initial) value problem (146), (147), (148), a terminology which goes back to [43], [62]. Also the terminology that boundary initial values are weakly valid is used. The validity of the trace theorems and the Green formula (200) with $q = \gamma(\psi)$, $p_0 = \psi(\cdot, \cdot, 0)$, $p_m = \psi(\cdot, \cdot, E_m)$ are keys for obtaining well-defined solutions (that is, solutions for which the boundary (initial) values really hold). Trace theorems always demand geometrical treatments where the smoothness of the boundary $\partial G$ is essential.

B. We also remark that the following generalized Green formula (see e.g. [59, Theorems 1 and 7])

$$
\langle T \psi, v \rangle_{L^2(G \times S \times I)} - \langle T^* v, \psi \rangle_{L^2(G \times S \times I)} = (A_\nu \psi)(v), \quad \psi \in H_P(G \times S \times I),
$$

(219)

for $v \in C^1(G \times S \times I)$, is valid where $A_\nu \psi$ is interpreted as an element of the dual $H^{1/2}(\partial(G \times S \times I))^*$. The value $A_\nu \psi$ is obtained by (uniquely) extending the bilinear form (cf. 200)

$$(A_\nu \psi)(v) := \int_{\partial G \times S \times I} (\omega \cdot \nu) v \psi \, d\sigma d\omega dE

+ \int_{G \times S} (S_0(\cdot, 0) \psi(\cdot, \cdot, 0) v(\cdot, \cdot, 0) - S_0(\cdot, E_m) \psi(\cdot, \cdot, E_m) v(\cdot, \cdot, E_m)) \, dx d\omega,
$$

where $\psi$, $v \in C^1(G \times S \times I)$. 

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Remark 3.12 Define the transport operator

$$T_C \phi := -\frac{\partial(S_0 \phi)}{\partial E} + \omega \cdot \nabla_x \phi + CS_0 \phi + (\Sigma - K_C) \phi.$$ 

Then from the estimate \[184\) it follows that for all $\phi \in C^1(\mathbb{G} \times S \times I)$,

$$c' \|\phi\|_{H_1} \leq \|T_C \phi\|_{L^2(G \times S \times I)} + \|\gamma(-\phi)\|_{T^2(G \times I)} + \|S_0\|_{L^\infty(G \times I)} \|\phi(\cdot, \cdot, E_m)\|_{L^2(G \times S)}.$$ 

This a priori estimate might be (as is standard) an appropriate starting point for developing regularity results of solutions at least in the case where $G = \mathbb{R}^3$. For these kind of transport equations more regularity must be sought in mixed-norm (anisotropic) Sobolev-Slobodevskij spaces $H^{2(s_1,s_2,s_3)}(G \times S \times I^2)$ where $s_j \geq 0$.

Regularity results are needed e.g. in considerations of various approximation errors and in convergence analysis of numerical schemes. The regularity analysis remains open.

Remark 3.13 The variational methods used above apply (after relevant modifications of the assumptions) also to more general problems of the form

$$a \frac{\partial \psi}{\partial E} + F \cdot \nabla_\omega \psi + \omega \cdot \nabla_x \psi + (\Sigma - K) \psi = f$$

$$\psi|_{\Gamma_-} = g, \quad \psi(\cdot, \cdot, E_m) = 0,$$ 

(220)

where $a = a(x, \omega, E)$, $F = (F_1(x, \omega, E), F_2(x, \omega, E), F_3(x, \omega, E))$ and $\nabla_\omega$ is the gradient operator with respect to the Riemannian metric on the sphere $S$ induced by the Euclidean metric on $\mathbb{R}^3$. Notice that since $\nabla_\omega \psi$ is tangent to $S$, we may assume that $\omega \cdot F(x, \omega, E) = 0$ for all $(x, \omega, E)$. More explicitly, if $\phi \in C^1(S)$, and if $\tilde{\phi} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is given by $\tilde{\phi}(y) = \phi(\frac{\omega}{||\omega||})$, then

$$\nabla_\omega \phi = \nabla_y \tilde{\phi} - (\omega \cdot \nabla_y \tilde{\phi}) \omega$$

where $\nabla_\omega \phi = ((\nabla_\omega \phi)_1, (\nabla_\omega \phi)_2, (\nabla_\omega \phi)_3)$ and $\omega = (\omega_1, \omega_2, \omega_3)$ as elements of $\mathbb{R}^3$.

In this context the space $H_2$ must be replaced with the space $\mathcal{H}_2$ defined as follows. The space $\mathcal{H}_2$ is the completion of $C^1(G \times S \times I)$ with respect to the inner product

$$\langle \psi, v \rangle_{\mathcal{H}_2} + \langle F \cdot \nabla_\omega \psi, F \cdot \nabla_\omega v \rangle_{L^2(G \times S \times I)}.$$ 

This observation is based on the fact that

$$2 \langle F \cdot \nabla_\omega \psi, \psi \rangle_{L^2(G \times S \times I)} = \langle d \psi, \psi \rangle_{L^2(G \times S \times I)},$$ 

where

$$d := -\text{div}_S(F),$$

and $\text{div}_S$ is the divergence operator on $S$ with respect to its Riemannian metric.

The equation (220) in velocity coordinates $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ can be written as

$$\tilde{F} \cdot \nabla_v \Psi + v \cdot \nabla_x \Psi + (\tilde{\Sigma} - \tilde{K}) \Psi = \tilde{f},$$ 

(221)

which is known as the (linear) Vlasov-Boltzmann equation.

Similar observations concern the methods used in the next sections. Nevertheless, the $m$-dissipativity of the operator $\psi \mapsto \frac{1}{\alpha} (F \cdot \nabla_\omega \psi + \omega \cdot \nabla_x \psi + C \psi)$, for $C$ large enough (in relevant modified spaces), requires extra analysis.

Finally notice that no boundary condition is needed in (220) with respect to $\omega$-variable because the unit sphere $S$ is a compact manifold without boundary.
3.3. **Existence Results Based on m-dissipativity.** In this section we apply an alternative method based on the results of dissipative first-order partial differential operators. Let

\[ P(x, \omega, E, D)\phi := -\frac{\partial (S_0 \phi)}{\partial E} + \omega \cdot \nabla_x \phi + CS_0 \phi \]

where \( S_0 \in C^1(\overline{G} \times S \times I) \) and \( C \) is constant. Recall that the formal transpose (adjoint) of \( P(x, \omega, E, D) \) is

\[ P'(x, \omega, E, D)\nu := S_0 \frac{\partial \nu}{\partial E} - \omega \cdot \nabla_x \nu + CS_0 \nu. \quad (222) \]

It should be pointed out that here the notation for \( P(x, \omega, E, D) \) and \( P'(x, \omega, E, D) \) differs from that used in section 3.2 in that here the operators \( P(x, \omega, E, D) \) and \( P'(x, \omega, E, D) \) also include the term \( CS_0 \).

Define linear operators \( P, P' : L^2(G \times S \times I) \rightarrow L^2(G \times S \times I) \) with domains of definition \( D(P) \), \( D(P') \) by setting

\[ D(P) := D^1(\overline{G} \times S \times I), \quad P\phi := P(x, \omega, E, D)\phi, \quad (223) \]

and

\[ D(P') := C_0^\infty(G \times S \times I^\circ), \quad P'\nu := P'(x, \omega, E, D)\nu. \]

Clearly both \( P \) and \( P' \) are densely defined.

Let \( P'_{\phi} : L^2(G \times S \times I) \rightarrow L^2(G \times S \times I) \) be the adjoint operator of \( P' \). Then \( \phi \in L^2(G \times S \times I) \) is said to be a weak solution of

\[ P\phi = f, \quad f \in L^2(G \times S \times I) \quad (224) \]

if and only if

\[ \phi \in D(P'_{\phi}) \quad \text{and} \quad P'_{\phi}\phi = f. \]

Since the adjoint \( P'_{\phi} \) is a closed operator, the space

\[ \mathcal{H}_P(G \times S \times I^\circ) := \{ \phi \in L^2(G \times S \times I) \mid P(x, \omega, E, D)\phi \in L^2(G \times S \times I) \text{ in the weak sense} \} \quad (225) \]

is a Hilbert space when equipped with the inner product

\[ \langle \phi, \nu \rangle_{\mathcal{H}_P(G \times S \times I^\circ)} := \langle \phi, \nu \rangle_{L^2(G \times S \times I)} + \langle P(x, \omega, E, D)\phi, P(x, \omega, E, D)\nu \rangle_{L^2(G \times S \times I)}. \]

Notice that

\[ \mathcal{H}_P(G \times S \times I^\circ) = D(P'_{\phi}), \quad (226) \]

when \( D(P'_{\phi}) \) is equipped with the graph norm of \( P'_{\phi} \).

We say that \( \phi \in L^2(G \times S \times I) \) is a strong solution of \( (224) \) (without boundary conditions) if there exists a sequence \( \{ \phi_n \} \subset D^1(\overline{G} \times S \times I) \) (= \( D(P) \)) such that

\[ \| \phi - \phi_n \|_{L^2(G \times S \times I)} + \| P\phi_n - f \|_{L^2(G \times S \times I)} \rightarrow 0 \quad \text{when} \ n \rightarrow \infty. \]

Let \( \tilde{P} : L^2(G \times S \times I) \rightarrow L^2(G \times S \times I) \) be the smallest closed extension (closure) of \( P \). Then one sees that \( \phi \) is a strong solution (without boundary conditions) if and only if \( \phi \in D(\tilde{P}) \) and \( \tilde{P}\phi = f \).
Definition of the strong solution is needed. One says that an inflow boundary (initial) value problems. To this end, the following modified definition holds:

\[\{\phi_n\} \subset W^2(G \times S \times I) \cap H^1(I, L^2(G \times S))\]

such that

\[\|\phi - \phi_n\|_{L^2(G \times S \times I)} + \|P(x, \omega, E, D)\phi_n - f\|_{L^2(G \times S \times I)} \to 0, \quad \text{when } n \to \infty,\]

and

\[\phi_n|_{\Gamma} = 0, \quad \phi_n(\cdot, \cdot, E_m) = 0.\]  

(227)

Define a linear operator \(P_0 : L^2(G \times S \times I) \to L^2(G \times S \times I)\) by

\[D(P_0) := \{\phi \in W^2(G \times S \times I) \cap H^1(I, L^2(G \times S)) \mid \phi|_{\Gamma} = 0, \quad \phi(\cdot, \cdot, E_m) = 0\}\]

\[P_0\phi := P(x, \omega, E, D)\phi.\]  

(228)

Let \(\tilde{P}_0 : L^2(G \times S \times I) \to L^2(G \times S \times I)\) be the smallest closed extension (closure) of \(P_0\) (the existence of which can be seen using the argument of Remark 3.14). Then one sees that \(\phi\) is a strong solution with the homogeneous inflow boundary conditions if and only if \(\phi \in D(\tilde{P}_0)\) and \(\tilde{P}_0\phi = f\). Moreover, one sees that every strong solution with homogeneous inflow boundary conditions is a weak solution of \(P(x, \omega, E, D)\phi = f\), that is

\[\tilde{P}_0 \subset P^*\]  

(229)

Since \(\tilde{P}_0\) is a closed operator, the space

\[\mathcal{H}_{P_0}(G \times S \times I^c) := D(\tilde{P}_0)\]  

(230)

Remark 3.14 To see that the closure \(\tilde{P}\) exists, notice that if \(\phi_n \in D(P)\), \(f \in L^2(G \times S \times I)\) and \(\phi_n \to 0\), \(P\phi_n \to f\) in \(L^2(G \times S \times I)\), then for every \(v \in C_0^\infty(G \times S \times I^c)\) we have

\[\langle f, v \rangle_{L^2(G \times S \times I)} = \lim_{n \to \infty} \langle P(x, \omega, E, D)\phi_n, v \rangle_{L^2(G \times S \times I)}\]

\[= \lim_{n \to \infty} \langle \phi_n, P'(x, \omega, E, D)v \rangle_{L^2(G \times S \times I)} = 0,
\]

which implies \(f = 0\).

One immediately sees that every strong solution of (224) is its weak solution. When the boundary \(\partial G\) is smooth enough the converse is also true, the result which goes back to Friedrich [30], or [59, Proposition 1].

Theorem 3.15 Let \(G \subset \mathbb{R}^n\) be an open bounded subset, lying on one side of its boundary, and with boundary of class \(C^1\). Then every weak solution of the equation (224) is its strong solution (without boundary conditions). In other words, \(D^1(G \times S \times I)\) is dense in \(\mathcal{H}_P(G \times S \times I^c)\).

Proof. See [30], or [59, Proposition 1].

The claim of the above theorem can also be equivalently stated as

\[\tilde{P} = P^*\]  

(229)
is a Hilbert space when equipped with the inner product
\[ \langle \phi, v \rangle_{\mathcal{H}_0(G \times S \times I)} := \langle \phi, v \rangle_{L^2(G \times S \times I)} + \left\langle \tilde{P}_0 \phi, \tilde{P}_0 v \right\rangle_{L^2(G \times S \times I)}. \]

**Remark 3.16** When \( \phi \in \mathcal{H}_0(G \times S \times I^\circ) \) we say that the (homogeneous) initial and boundary conditions
\[ \phi_{| \Gamma_-} = 0, \quad \phi(\cdot, \cdot, E_m) = 0 \]
are valid in the strong sense.

We show the \( m \)-dissipativity of \( \tilde{P}_0 \) using the theory of evolution operators presented in section 3.5 below.

**Theorem 3.17** Suppose that
\[ S_0 \in C^2(I, L^\infty(G)), \]
\[ \kappa := \inf_{(x, E) \in G \times I} S_0(x, E) > 0, \]
\[ \nabla_x S_0 \in L^\infty(G \times I), \]
\[ -\frac{\partial S_0}{\partial E} + 2CS_0 \geq 0. \]

Then
\[ R(I + \tilde{P}_0) = L^2(G \times S \times I) \]
and
\[ \left\langle \tilde{P}_0 \phi, \phi \right\rangle_{L^2(G \times S \times I)} \geq 0, \quad \forall \phi \in D(\tilde{P}_0). \]

**Proof.** We apply Theorem 3.36 (see below) with \( K = 0 \), \( \Sigma = CS_0 + 1 \), \( g = 0 \). Let \( f \in C_0^\infty(G \times S \times I^\circ) \). Then it follows that the problem
\[ -\frac{\partial (S_0 \phi)}{\partial E} + \omega \cdot \nabla_x \phi + (CS_0 + 1)\phi = (I + P(x, \omega, E, D))\phi = f, \quad \phi(\cdot, \cdot, E_m) = 0, \]
has a unique solution \( \phi \in C(I, \bar{W}^2_{-0}(G \times S)) \cap C^1(I, L^2(G \times S)) \). We find that
\[ \{ \psi \in C(I, \bar{W}^2_{-0}(G \times S)) \cap C^1(I, L^2(G \times S)) \mid \psi(\cdot, \cdot, E_m) = 0 \} \subset D(P_0), \]
and so for any \( f \in C_0^\infty(G \times S \times I) \) the equation \( (I + P_0)\phi = f \) has a solution. Since \( C_0^\infty(G \times S \times I^\circ) \) is dense in \( L^2(G \times S \times I) \) we find that the range \( R(I + P_0) \) is dense that is,
\[ R(I + P_0) = L^2(G \times S \times I). \]
As in the proof of Lemma 3.2 (note that here the assumptions are somewhat weaker), we have for all \( \phi \in D(P_0) \), (we write \( L^2 = L^2(G \times S \times I) \))
\[ \langle P_0 \phi, \phi \rangle_{L^2} = \left\langle \left(-\frac{\partial (S_0 \phi)}{\partial E}, \phi \right)_{L^2} + \langle \omega \cdot \nabla_x \phi, \phi \rangle_{L^2} + \langle CS_0 \phi, \phi \rangle_{L^2} \right. \]
\[ = \left( \left(-\frac{1}{2} \frac{\partial S_0}{\partial E} + CS_0 \right) \phi, \phi \right)_{L^2} + \frac{1}{2} \langle \phi, \phi \rangle_{T^2(\Gamma_+)} + \frac{1}{2} \int_{G \times S} S_0(\cdot, 0) \phi(\cdot, \cdot, 0)^2 \, dx \, d\omega, \]
\[ (239) \]
which in combination with the assumptions (232), (234) implies
\[ \langle P_0 \phi, \phi \rangle_{L^2(G \times S \times I)} \geq 0, \quad \forall \phi \in D(P_0). \]

If \( \phi \in D(\tilde{P}_0) \), choose a sequence \( \phi_n \in D(P_0) \) such that \( \phi_n \to \phi \) and \( P_0 \phi_n \to \tilde{P}_0 \phi \)
in \( L^2(G \times S \times I) \) when \( n \to \infty \). By the above inequality, we have
\[ \langle \tilde{P}_0 \phi, \phi \rangle_{L^2(G \times S \times I)} = \lim_n \langle P_0 \phi_n, \phi_n \rangle_{L^2(G \times S \times I)} \geq 0, \]
which gives (236).

Finally, from (236) it follows that
\[ (I + \tilde{P}_0)\phi \in L^2(G \times S \times I), \quad \forall \phi \in D(\tilde{P}_0). \]  
and therefore \( R(I + \tilde{P}_0) \) is closed in \( L^2(G \times S \times I) \). This result, the observation that \( R(I + P_0) \subset R(I + \tilde{P}_0) \), and (238) show that \( R(I + \tilde{P}_0) = L^2(G \times S \times I) \). The proof is complete. \( \square \)

**Theorem 3.17** says that:

**Corollary 3.18** The operator \(-\tilde{P}_0 : L^2(G \times S \times I) \to L^2(G \times S \times I)\) is \( m \)-dissipative, or equivalently \( \tilde{P}_0 \) is \( m \)-accretive.

**Proof.** See e.g. [19, p. 340]. \( \square \)

**Remark 3.19** The general theory of initial boundary value problems of symmetric formally dissipative first order partial differential operators can alternatively be applied to show the \( m \)-dissipativity of \(-\tilde{P}_0\). The (classical) results for positive symmetric initial boundary value problems can be found in [43], Theorem 3.2 and discussion in section 4 therein; [31], together with discussion in section 17 therein; [62] and [59]. The spatial domain there is replaced with \( G \times S \) (here the additional smooth compact manifold \( S \) without boundary does not affect to the conclusions; for the construction of the Friedrich's mollifier on \( S \) we refer to [33]). The above references are, however, valid only for problems in which the dimension of the kernel \( \text{Ker}(A_0) \), given below, is constant on \( \Gamma \) (i.e. \( A_0 \) has constant multiplicity), and thus we are not able to directly apply them. Some results for variable multiplicity can be found e.g. in [60], [51], [52] and [68] together with their references. The latter researches require the an additional assumption which concerns the "transition with a non-zero derivative" on the smooth \((6 - 2)\)-dimensional manifold \( \Gamma_0 \). For simplicity we deal only with the case where \( S_0 = S_0(x) \) is independent of \( E \) and use formulations of [51]. More general formulations can be based on [52] and related references but we omit them here.

We make the change of variables and of the unknown function
\[ \Phi(x, \omega, E) := \phi(x, \omega, E_m - E) \]  
and denote
\[ \tilde{f}(x, \omega, E) = f(x, \omega, E_m - E). \]
Making the changes, we find that the problem \((I + \tilde{P}_0)\phi = f\) is equivalent to \( \phi \) satisfying the equation
\[ \frac{\partial \Phi}{\partial E} + \frac{1}{S_0} \omega \cdot \nabla_x \Phi + \frac{1}{S_0} (C S_0 + 1) \Phi = \frac{1}{S_0} \tilde{f} =: f, \]
on \( G \times S \times I \), along with satisfying the following inflow boundary and initial value conditions
\[
\Phi|_{\Gamma-} = 0, \quad \Phi(\cdot, \cdot, 0) = 0. \tag{243}
\]

Define
\[
Q(x, \omega, E, D)\Phi := \frac{1}{S_0} \omega \cdot \nabla_x \Phi + \frac{1}{S_0}(CS_0 + 1)\Phi
\]
and the boundary matrix
\[
A_\nu(z) = \frac{1}{S_0} \omega \cdot \nu(y), \quad z = (y, \omega, E) \in \Gamma.
\]

We assume that \( \partial G = \{x \in \mathbb{R}^3 \mid f(x) = 0\} \) is a level set of class \( C^\infty \), i.e. \( f \in C^\infty(\mathbb{R}^3) \) and \( \nabla_x f(x) \neq 0 \) whenever \( f(x) = 0 \), and we also assume that \( S_0 \in C^\infty(G) \).

The outward unit normal vector field \( \nu(y) \) on \( \partial G \) is given by
\[
\nu(y) = \frac{(\nabla_x f)(y)}{\| (\nabla_x f)(y) \|}.
\]

The additional assumption is as follows. Suppose that there exist \( C^\infty \)-functions \( h \) and \( A : V_{y_0} \times S \to \mathbb{R} \) such that
\[
A_f(x, \omega) := \frac{1}{S_0(x)} \omega \cdot \nabla_x f(x) = -h(x, \omega)A(x, \omega) \tag{244}
\]
\[
(V_{y_0} \times S) \cap \Gamma_+ = \{(y, \omega) \in \partial G \times S \mid h(y, \omega) > 0\}, \tag{245}
\]
\[
(V_{y_0} \times S) \cap \Gamma_- = \{(y, \omega) \in \partial G \times S \mid h(y, \omega) < 0\}, \tag{246}
\]
\[
(V_{y_0} \times S) \cap \Gamma_0 = \{(y, \omega) \in \partial G \times S \mid h(y, \omega) = 0\}. \tag{247}
\]

Let
\[
A_h(x, \omega) := \frac{1}{S_0(x)}(\omega \cdot \nabla_x h)(x, \omega).
\]

The additional assumption is
\[
A(y, \omega) \text{ and } A_h(y, \omega) \text{ are positive definite on } (V_{y_0} \times S) \cap \Gamma_0. \tag{248}
\]

For example, in the case of the ball \( G = B(0, 1) \subset \mathbb{R}^3 \) we can choose \( f(x) = 1 - \|x\|^2 \). Then \( A_f(y, \omega) = -2 \frac{1}{S_0(y)}(y \cdot \omega) \) and we choose \( h(y, \omega) = 2(y \cdot \omega) \), \( A(y, \omega) = \frac{1}{S_0(y)} > 0 \). Moreover, we find that \( A_h(y, \omega) = 2 \frac{1}{S_0(y)} > 0 \). Hence the stated assumptions can be met for the case of the ball \( G = B(0, 1) \).

The following result holds in this context. Suppose that \( \partial G \) is in the class \( C^1 \) and that the assumption \( [248] \) holds. Furthermore, suppose that \( S_0 \in C^1(\overline{G} \times I) \) such that
\[
S_0 > 0 \text{ on } \overline{G} \times I. \tag{249}
\]

Then
\[
C_0^\infty(G \times S \times I^0) \subset R(I + P_0) \tag{250}
\]
and
\[
\left< \tilde{P}_0 \phi, \phi \right>_{L^2(G \times S \times I)} \geq 0, \quad \forall \phi \in D(\tilde{P}_0). \tag{251}
\]
Since our equation is scalar valued it is symmetric in the sense of [51]. We choose the linear subspace of [51] by

\[ M(y, \omega) = \begin{cases} \mathbb{R}, & (y, \omega) \in \Gamma' \cup \Gamma_0^\prime \\ \{0\}, & (y, \omega) \in \Gamma'_- \end{cases} \]

Then we find that \( M(y, \omega) \) is maximal positive in the sense of [51].

Due to the Theorem 5.5 of [51], for any \( f \in C^\infty_0(G \times S \times I^c) \) the equation

\[ \partial \Phi \partial E + Q(x, \omega, E, D)\Phi = f \]

has a unique strong solution \( \Phi \) satisfying the initial and boundary values. Then \( \phi(x, \omega, E) := \Phi(x, \omega, E_m - E) \) is the solution of \((I + P_0)\phi = f\). This completes the proof of (250). The inequality (251) can be shown similarly as above and so the conclusion follows.

**Remark 3.20** The previous theorem has the following generalization which can be applied for more general transport problems. Let

\[ P(x, \omega, E, D)\phi = -S_0 \frac{\partial \phi}{\partial E} + F_1 \cdot \nabla_x \phi + F_2 \cdot \nabla_\omega \phi + a\phi \]

be the first order partial differential operator with coefficients \( S_0, F_1, F_2, a \in C^1(G \times S \times I) \). Assume that \( \partial G \) is in the class \( C^2 \) and that \( \Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_0 \) where \( \Gamma_0 \) has the zero surface measure. In addition, we assume that \( \Gamma_\pm \) are open in \( \partial G \times S \times I^\circ \). Finally, suppose that

\[ \frac{\partial S_0}{\partial E} - \text{div}_x(F_1) - \text{div}_\omega(F_2) + 2a \geq 0, \quad (252) \]

and that

\[ \inf_{(x, \omega, E) \in G \times S \times I} S_0(x, \omega, E) > 0, \quad (253) \]

Then

\[ R(I + \tilde{P}_0) = L^2(G \times S \times I) \]

and

\[ \left\langle \tilde{P}_0 \phi, \phi \right\rangle_{L^2(G \times S \times I)} \geq 0, \quad \forall \phi \in D(\tilde{P}_0). \quad (256) \]

**Remark 3.21** In certain cases, the \( m \)-dissipativity of \(-\tilde{P}_0\)-like operator can even be proved by using explicit formulas for the solution. We only sketch the idea here.

Suppose that \( S_0 = S_0(E) \) is independent of \( x \) and that \( a = a(x, \omega) \in C^1(G \times S) \) is independent of \( E \). Let

\[ P(x, \omega, E, D)\phi = -S_0 \frac{\partial S_0}{\partial E} + \omega \cdot \nabla_x \phi + a\phi \]

Assume that \( f \in C^\infty_0(G \times S \times I^c) \). Then by the formula (541) below, the solution of the problem

\[ P(x, \omega, E, D)\phi = f, \quad \phi|_{\Gamma_-} = 0, \quad \phi(\cdot, \cdot, E_m) = 0 \]

is given by

\[ \phi(x, \omega, E) = \frac{1}{S_0(E)} \left( \int_0^{r(x, \omega, E)} e^{-\int_0^r a(x-\tau \omega, \omega) d\tau} \tilde{f}(x - s \omega, \omega, R(E) + s) ds \right), \quad (258) \]
We find that there exists a constant $C_1 > 0$ such that

$$
\|\phi\|_{L^2(G \times S \times I)} \leq C_1 \|f\|_{L^2(G \times S \times I)}.
$$

(259)

Let $f \in L^2(G \times S \times I)$ and let $\{f_n\} \subset C_0^\infty(G \times S \times I^c)$ be a sequence such that $\|f_n - f\|_{L^2(G \times S \times I)} \rightarrow 0$ when $n \rightarrow \infty$. Define

$$
\phi_n(x) := \frac{1}{S_0(x)} \left( \int_0^{r(x, s, E)} e^{-\int_0^s a(x, y, \omega, R(E) + s)ds} \tilde{f}_n(x - s\omega, \omega, R(E) + s)ds \right).
$$

We find that $\phi_n \in C^0([\bar{G} \times S \times I]) \cap H^1(G \times S \times I^c)$, $\phi_n|_{\Gamma_-} = 0$, $\phi_n(\cdot, \cdot, E_m) = 0,$ and then $\phi_n \in D(P_0)$. In showing that $\phi_n \in H^1(G \times S \times I^c)$ notice that

$$
\tilde{f}_n(x - t(x, \omega)\omega, \omega, R(E) + t(x, \omega)) = 0, \text{ if } R(E_m) - R(E) > t(x, \omega),
$$

and so $\frac{\partial \phi_n}{\partial x_j}, \frac{\partial \phi_n}{\partial \omega_i}$ do not appear in $\frac{\partial \phi_n}{\partial x_j}, \frac{\partial \phi_n}{\partial \omega_i}$.

By (259) there exists $\phi \in L^2(G \times S \times I)$ such that $\|\phi_n - \phi\|_{L^2(G \times S \times I)} \rightarrow 0$. In addition, $\|P_0\phi_n - f\|_{L^2(G \times S \times I)} = \|f_n - f\|_{L^2(G \times S \times I)} \rightarrow 0$, $\|P_0\phi = f\$. Hence $R(\hat{P}_0) = L^2(G \times S \times I)$. Consequently, in this special case these methods (based only on explicit solution formulas) give an alternative proof for the surjectivity of $\hat{P}_0$ (and hence together with (251) the $m$-dissipativity of $-\hat{P}_0$; cf. the treatise of A0 in \cite{71} proof of Theorem 4.7).

We return to the existence and uniqueness of solutions for the following problem. Given $f \in L^2(G \times S \times I)$, find $\phi \in L^2(G \times S \times I)$ such that

$$
-\frac{\partial(S_0\phi)}{\partial E} + \omega \cdot \nabla_x \phi + CS_0\phi + \Sigma \phi - K_C \phi = f,
$$

$$
\phi|_{\Gamma_-} = 0, \quad \phi(\cdot, \cdot, E_m) = 0.
$$

(260)

Let (for clarity, we have included here the subscript $C$ into $P$)

$$
P_C(x, \omega, E, D)\phi := -\frac{\partial(S_0\phi)}{\partial E} + \omega \cdot \nabla_x \phi + CS_0\phi
$$

$$
= -S_0\frac{\partial \phi}{\partial E} + \omega \cdot \nabla_x \phi - \frac{\partial S_0}{\partial E} \phi + CS_0\phi.
$$

(261)

We shall seek a strong solution of (260) with the homogeneous inflow boundary conditions. Using the above notations, the problem (260) is equivalent to

$$
(\hat{P}_{C,0} + \Sigma - K_C)\phi = f,
$$

where $\phi \in D(\hat{P}_{C,0})$. The following arguments are analogous to those used in \cite{71} Section 5.3].
Theorem 3.22 Suppose that the assumptions (149), (151), (152), (231), (232) and (233) are valid with $C = \frac{\max\{q, 0\}}{\kappa}$ and $c > 0$. Then for every $f \in L^2(G \times S \times I)$ the problem (260) has a unique strong solution $\phi \in D(\tilde{P}_{C,0}) = \mathcal{H}_{P_{C,0}}(G \times S \times I^\circ)$ with homogeneous inflow boundary conditions.

Proof. Recall that $q = \frac{1}{2} \sup_{(x,E) \in G \times I} \frac{\partial S_0}{\partial E}(x, E)$ (see ([64])). For $C = \frac{\max\{q, 0\}}{\kappa}$ we have
\[
\frac{\partial S_0}{\partial E} \leq 2q = 2\kappa \frac{q}{\kappa} \leq 2S_0 \max\{\frac{q}{\kappa}, 0\} = 2S_0C,
\]
that is
\[
-\frac{\partial S_0}{\partial E} + 2CS_0 \geq 0, \quad (262)
\]
and hence by Corollary 3.18, the operator $-\tilde{P}_{C,0} : L^2(G \times S \times I) \rightarrow L^2(G \times S \times I)$ is $m$-dissipative. On the other hand, due to Lemma 3.3 the bounded operator $-(\Sigma - K_C) + cI : L^2(G \times S \times I) \rightarrow L^2(G \times S \times I)$ is dissipative. These facts allow us to conclude that $-\tilde{P}_{C,0} - (\Sigma - K_C) + cI : L^2(G \times S \times I) \rightarrow L^2(G \times S \times I)$ is $m$-dissipative ([53] Theorem 4.3 and Corollary 3.3], or [71] Theorem 4.4]). This implies, as $c > 0$, that $R(cI - (\Sigma - K_C)) = R(\tilde{P}_{C,0} + \Sigma - K_C) = L^2(G \times S \times I)$, and so the existence of solutions follows.

Because $c > 0$ and because $-\tilde{P}_{C,0} - (\Sigma - K_C) + cI$ is dissipative, we have
\[
\left\| (\tilde{P}_{C,0} + \Sigma - K_C)\phi \right\|_{L^2(G \times S \times I)} \geq c \left\| \phi \right\|_{L^2(G \times S \times I)}, \quad \forall \phi \in D(\tilde{P}_{C,0}), \quad (263)
\]
which implies the uniqueness of the solution. This completes the proof. □

Remark 3.23 We note that the inequality (263) implies that for all $f \in L^2(G \times S \times I)$
\[
\left\| (-\tilde{P}_{C,0} + \Sigma - K)^{-1}f \right\|_{L^2(G \times S \times I)} \leq \frac{1}{c} \left\| f \right\|_{L^2(G \times S \times I)}, \quad (264)
\]
or in other words, the solution of the problem (260) satisfies
\[
\left\| \phi \right\|_{L^2(G \times S \times I)} \leq \frac{1}{c} \left\| f \right\|_{L^2(G \times S \times I)}. \quad (265)
\]

The next result addresses the case with inhomogeneous inflow boundary data. We begin with a lemma (see Lemmas 5.8 and 5.11 in [71]).

Lemma 3.24 Let $d := \text{diam}(G) < \infty$ (diameter of $G$). Then for any $g \in H^1(I, T^2(\Gamma_-'))$ (recall $\Gamma_-$ from section 2) we have $Lg \in H^1(I, L^2(G \times S))$, where
\[
(Lg)(x, \omega, E) := g(x - t(x, \omega)\omega, \omega, E),
\]
is the lift of $g$ (see Lemma 2.10 and Eq. (57)), and
\[
\left\| Lg \right\|_{H^1(I, L^2(G \times S))} \leq \sqrt{d} \left\| g \right\|_{H^1(I, T^2(\Gamma_-'))}. \quad (266)
\]

Proof. Taking into account that $|\tau_-| \leq d$ on $\Gamma_-$, we have by Lemma 2.10 (with $\Sigma = 0$),
\[
\left\| Lg \right\|_{L^2(G \times S \times I)} \leq \sqrt{d} \left\| g \right\|_{T^2(\Gamma_-)} = \sqrt{d} \left\| g \right\|_{L^2(I, T^2(\Gamma_-'))}. \quad (267)
\]
On the other hand, \( \frac{\partial (Lg)}{\partial E} = L \frac{\partial g}{\partial E} \), and therefore one has, again by Lemma 2.10 (with \( \Sigma = 0 \)),

\[
\left\| \frac{\partial (Lg)}{\partial E} \right\|_{L^2(G \times S \times I)} \leq \sqrt{d} \left\| \frac{\partial g}{\partial E} \right\|_{L^2(I, T^2(\Gamma'_-))}.
\]

This completes the proof. \( \square \)

**Theorem 3.25** Suppose that the assumptions (149), (151), (152), (231), (232) and (233) are valid with \( C = \max(q,0) \) and \( c > 0 \). Furthermore, suppose that \( f \in L^2(G \times S \times I) \), and \( g \in H^1(I, T^2(\Gamma'_-)) \) is such that the compatibility condition

\[
g(\cdot, \cdot, E_m) = 0
\]

holds. Then the problem

\[
-\frac{\partial (S_0 \phi)}{\partial E} + \omega \cdot \nabla_x \phi + CS_0 \phi + \Sigma \phi - K_C \phi = f,
\]

\[
\phi|_{\Gamma_-} = g, \quad \phi(\cdot, \cdot, E_m) = 0,
\]

has a unique solution

\[
\phi \in \mathcal{H}_{P_{C,0}}(G \times S \times I^c) + L(H^1(I, T^2(\Gamma'_-))) \subset \mathcal{H}_{P_{C}}(G \times S \times I^c),
\]

(266)

**Proof.** Substitute \( \phi \) in the problem (267) by \( u := \phi - Lg \) to obtain

\[
-\frac{\partial (S_0 u)}{\partial E} + \omega \cdot \nabla_x u + CS_0 u + \Sigma u - K_C u,
\]

\[
= f + \frac{\partial (S_0 Lg)}{\partial E} - CS_0 (Lg) - \Sigma (Lg) + K_C (Lg) =: \tilde{f}(x, \omega, E),
\]

where we have used the fact that \( \omega \cdot \nabla_x (Lg) = 0 \) (see Lemma 2.11 with \( \Sigma = 0 \)). On the other hand,

\[
u|_{\Gamma_-} = \phi|_{\Gamma_-} - (Lg)|_{\Gamma_-} = g - g = 0,
\]

and, using the compatibility condition (266),

\[
u(\cdot, \cdot, E_m) = \phi(\cdot, \cdot, E_m) - g(x - t(x, \omega)\omega, \omega, E_m) = 0.
\]

We find that the assumptions guarantee that \( \tilde{f} \in L^2(G \times S \times I) \) (for more details, see the proof of Corollary 3.26 below), and hence by Theorem 3.22 the problem (for \( u \))

\[
-\frac{\partial (S_0 u)}{\partial E} + \omega \cdot \nabla_x u + CS_0 u + \Sigma u - K_C u = \tilde{f},
\]

\[
u|_{\Gamma_-} = 0, \quad u(\cdot, \cdot, E_m) = 0,
\]

has a unique solution \( u \in \mathcal{H}_{P_{C,0}}(G \times S \times I^c) \). It follows that then

\[
\phi := u + Lg \in \mathcal{H}_{P_{C,0}}(G \times S \times I^c) + L(H^1(I, T^2(\Gamma'_-))),
\]

is the wanted unique solution of (267).

Finally, the last inclusion in (268) is justified by the inclusion (see (226), (229), (230))

\[
\mathcal{H}_{P_{C,0}}(G \times S \times I^c) \subset \mathcal{H}_{P_{C}}(G \times S \times I^c),
\]

and by the fact that (see the proof of Corollary 3.26 below)

\[
\mathcal{P}_C(x, \omega, E, D) Lg = -\frac{\partial (S_0 Lg)}{\partial E} + CS_0 (Lg) \in L^2(G \times S \times I),
\]
where the equality \( \omega \cdot \nabla (L\bar{g}) = 0 \) has been used again. \( \Box \)

We additionally obtain the following a priori estimate.

**Corollary 3.26** Under the assumptions of Theorem 3.25 the solution \( \phi \) of the problem (267) satisfies, with a constant \( C_1 \geq 0 \), the estimate

\[
\|\phi\|_{L^2(G \times S \times I)} \leq C_1 \left( \|f\|_{L^2(G \times S \times I)} + \|g\|_{H^1(I,T^2(\Gamma'))} \right). \tag{269}
\]

**Proof.** By estimate (264), we have

\[
\|\phi\|_{L^2(G \times S \times I)} = \|u + Lg\|_{L^2(G \times S \times I)} \leq \frac{1}{c} \|\overline{f}\|_{L^2(G \times S \times I)} + \|Lg\|_{L^2(G \times S \times I)}. \tag{270}
\]

By Lemma 3.24,

\[
\|Lg\|_{L^2(G \times S \times I)} = \|Lg\|_{L^2(I,L^2(G \times S))} \leq \|Lg\|_{H^1(I,L^2(G \times S))} \leq \sqrt{d} \|g\|_{H^1(I,T^2(\Gamma'))},
\]

where \( d = \mathrm{diam}(G) < \infty \), and similarly,

\[
\left\| \frac{\partial (Lg)}{\partial E} \right\|_{L^2(G \times S \times I)} \leq \left\| \frac{\partial (Lg)}{\partial E} \right\|_{L^2(I,L^2(G \times S))} \leq \sqrt{d} \|g\|_{H^1(I,T^2(\Gamma'))}.
\]

Finally, due to these estimates, Lemma 3.3 and the fact that \( \omega \cdot \nabla (Lg) = 0 \), one has

\[
\left\| \hat{f} \right\|_{L^2} \leq \|f\|_{L^2} + \left\| S_0 \frac{\partial (Lg)}{\partial E} + \left( \frac{\partial S_0}{\partial E} - CS_0 \right) Lg - (\Sigma - K_C) Lg \right\|_{L^2} \leq \|f\|_{L^2} + \sqrt{d} \left( \left\| \frac{\partial S_0}{\partial E} \right\|_{L^\infty} + (C + 1) \|S_0\|_{L^\infty} + \|\Sigma - K_C\| \right) \|g\|_{H^1(I,T^2(\Gamma'))},
\]

where we wrote, unambiguously, \( L^2 = L^2(G \times S \times I) \) and \( L^\infty = L^\infty(G \times I) \) in order to compress the formulas. This proves the estimate (269) as claimed. \( \Box \)

For the original problem we get

**Corollary 3.27** Suppose that the assumptions (149), (151), (152), (231), (232) and (233) are valid with \( C = \max_{r=0}^{\text{max}} \) and \( c > 0 \). Furthermore, suppose that \( f \in L^2(G \times S \times I) \), and \( g \in H^1(I,T^2(\Gamma')) \) is such that

\[
g(\cdot,\cdot,E_m) = 0. \tag{271}
\]

Then the problem

\[
- \frac{\partial (S_0 \psi)}{\partial E} + \omega \cdot \nabla \psi + \Sigma \psi - K \psi = f, \\
\psi|_{\Gamma} = g, \quad \psi(\cdot,\cdot,E_m) = 0, \tag{272}
\]

has a unique solution \( \psi \in \mathcal{H}_p(G \times S \times I) \). In addition, there exists a constant \( C_1' > 0 \) such that an a priori estimate

\[
\|\psi\|_{L^2(G \times S \times I)} \leq C_1' \left( \|f\|_{L^2(G \times S \times I)} + \|g\|_{H^1(I,T^2(\Gamma'))} \right), \tag{273}
\]

holds.

**Proof.** Recalling (see [108], [174]) that \( \psi \) is a solution of the problem (272) with data \((f,g)\) if and only if \( \phi = e^{CE} \psi \) is a solution of the problem (267) with data \((f,g) = (e^{CE} f, e^{CE} g)\), we deduce the claims from Theorem 3.25 and Corollary 3.26. \( \Box \)
3.4. On the Non-negativity of Solutions. We will outline an argument to show that the solution $\psi$ obtained by Corollary 3.27, when the assumptions of it are valid, is non-negative if the data $f$ and $g$ are non-negative. Using a change of unknown $\phi = e^{CE} \psi$, with $C$ given in (167), we return the problem to the one considered in Theorem 3.25 that is

$$-\frac{\partial (S_0 \phi)}{\partial E} + \omega \cdot \nabla \phi + CS_0 \phi + \Sigma \phi - KC\phi = f \quad \text{on } G \times S \times I,$$

$$\phi|_{\Gamma_\cdot} = g \quad \text{on } \Gamma_-,$$

$$\phi(\cdot, \cdot, E_m) = 0 \quad \text{on } G \times S.$$

(274)\hspace{1cm} (275)\hspace{1cm} (276)

Clearly, the positivity condition on data, $f \geq 0$ and $g \geq 0$, is equivalent to having $f \geq 0$ and $g \geq 0$. Recall that

$$P_C(x, \omega, E, D) \phi := -\frac{\partial (S_0 \phi)}{\partial E} + \omega \cdot \nabla \phi + CS_0 \phi.$$

A. In the first instance, we assume that $g = 0$. Let $\tilde{P}_{C,0}$ be the smallest closed extension defined above. Using the these notations, the problem (274)-(276) is equivalent to

$$(\tilde{P}_{C,0} + \Sigma - KC) \phi = f$$

(277)

where $\phi \in D(\tilde{P}_{C,0})$. Let

$$T_{C,0} := \tilde{P}_{C,0} + \Sigma - KC.$$

Since the operator $-T_{C,0} + cI$ is $m$-dissipative (see the proof of Theorem 3.22, and recall that $c > 0$ is a part of the assumption (152)), it generates a contraction $C^0$-semigroup $G_c(t)$, (cf. [25], [19]), and hence $-T_{C,0}$ generates a (contraction) $C^0$-semigroup $G(t) = G_c(t)e^{-ct}$, $t \geq 0$, such that $\|G(t)\| \leq e^{-ct}$ for all $t \geq 0$. In addition, the solution $\phi$ of (277) is given by (see [25] Chapter II, Theorem 1.10)

$$\phi = T^{-1}_{C,0} f = (0 - (-T_{C,0}))^{-1} f = \int_0^\infty G(t)f dt.$$

(278)

We decompose the operator $-T_{C,0}$ as follows

$$-T_{C,0} = B_0 + A_0 - (\Sigma + CS_0 I) + KC,$$

(279)

where linear operators $B_0$ and $A_0$ are defined by

$$D(A_0) := \tilde{W}^2_{-0}(G \times S \times I), \quad A_0 \phi := -\omega \cdot \nabla \phi$$

$$D(B_0) := \{ \phi \in W^2_{1}(G \times S \times I) \mid \phi(\cdot, \cdot, E_m) = 0 \}, \quad B_0 \phi := \frac{\partial (S_0 \phi)}{\partial E}.$$

The semi-groups generated by the last three components are given by (below $H$ is the Heaviside function)

$$T_{KC}(t) = e^{tcI},$$

$$T_{-((\Sigma + CS_0 I))}(t) = e^{-t(\Sigma + CS_0 I)},$$

$$(T_{A_0}(t)f)(x, \omega, E) = H(t(x, \omega) - t)f(x - t\omega, \omega, E),$$

(280)

and they are clearly all of positive type.

Also, the semi-group $T_{B_0}(t)$ generated by $B_0$ is of positive type. Indeed, letting $(U(t))(x, \omega, E) := (T_{B_0}(t)f)(x, \omega, E)$ for a given $f \in D(B_0)$, then $U$ satisfies the
Cauchy problem

\[ \frac{\partial U}{\partial t} - B_0 U = 0, \quad U(0) = f, \]

or equivalently,

\[ \frac{\partial U}{\partial t} - \frac{\partial (S_0 U)}{\partial E} = 0, \quad U(0) = f. \quad (281) \]

The solution of \((281)\) can be written in the form (cf. Example 6.1 below)

\[ U(x, \omega, E, t) = (U(t))(x, \omega, E) \]

\[ = H(R_x(E_m) - R_x(E) - t) \frac{S_0(x, R_x^{-1}(R_x(E) + t))}{S_0(x, E)} f(x, \omega, R_x^{-1}(R_x(E) + t)), \]

where \(H\) is the Heaviside function and

\[ R_x(E) := \int_0^E \frac{1}{S_0(x, \tau)} d\tau, \quad E \in [0, E_m]. \]

In other words,

\[(T_{B_0}(t)f)(x, \omega, E) \]

\[ = H(R_x(E_m) - R_x(E) - t) \frac{S_0(x, R_x^{-1}(R_x(E) + t))}{S_0(x, E)} f(x, \omega, R_x^{-1}(R_x(E) + t)), \]

and therefore \(T_{B_0}(t)\) is evidently of positive type.

Due to \((231)\) and \((232)\) there is \(M > 0\) such that \(0 < \kappa \leq S_0 \leq M\) a.e. on \(G \times I\). For fixed \(E \in I\), letting \(s_E(t) := R_x^{-1}(R_x(E) + t) - E\), we have \(s_E(0) = 0\), and \(s'_E(t) = S_0(x, s_E(t) + E)\), and hence the estimates

\[ \kappa t \leq s_E(t) \leq Mt, \quad \forall t \geq 0 \]

hold, and thus in particular,

\[ E \leq \kappa t + E \leq R_x^{-1}(R_x(E) + t). \]

Assumption \((234)\), written in the form \(\frac{1}{S_0} \frac{\partial S_0}{\partial E} \leq 2C\) yields after integration from \(E\) to \(E'\), where \(E \leq E'\),

\[ \frac{S_0(x, E')}{S_0(x, E)} \leq e^{2C(E'-E)}, \]

and hence by the above,

\[ \frac{S_0(x, R_x^{-1}(R_x(E) + t))}{S_0(x, E)} \leq e^{2C s_E(t)} \leq e^{2C M t}, \]

On the other hand,

\[ J_{x,t}(E) := \frac{\partial}{\partial E} R_x^{-1}(R_x(E) + t) = \frac{R'_x(E)}{R_x'(R_x^{-1}(R_x(E) + t))} = \frac{S_0(x, R_x^{-1}(R_x(E) + t))}{S_0(x, E)}, \]

we therefore we obtain the following estimate for \(T_{B_0}(t), t \geq 0\),

\[ \|T_{B_0}(t)f\|^2_{L^2(G \times S \times I)} \leq e^{2C M t} \int_{G \times S \times I} |\overline{f}(x, \omega, R_x^{-1}(R_x(E) + t))|^2 J_{x,t}(E) dx d\omega dE \]

\[ \leq e^{2C M t} \int_{G \times S \times I} |f(x, \omega, E')|^2 dx d\omega dE' = e^{2C M t} \|f\|^2_{L^2(G \times S \times I)} \]

where \(\overline{f}\) is the extension by zero of \(f\) onto \(G \times S \times [0, \infty[\).
ON EXISTENCE OF $L^2$-SOLUTIONS FOR COUPLED CSDA

The above computations show that for any $n \in \mathbb{N}$, we have
\[
\left\| T_{B_0}(t/n)T_{A_0}(t/n)T_{-((\Sigma + CS_0)I)}(t/n)T_{K_C}(t/n) \right\|^n \leq e^{\mu C_M + \mu \|\Sigma - CS_0 I\| + \|K_C\|}.
\]

Therefore, by Trotter’s product formula ([25, Corollary 5.8, p. 227]) we have for $f \geq 0$, and for all $t \geq 0$ that
\[
G(t)f = \lim_{n \to \infty} \left[ T_{B_0}(t/n)T_{A_0}(t/n)T_{-((\Sigma + CS_0)I)}(t/n)T_{K_C}(t/n) \right]^n f \geq 0,
\]
and thus $\phi \geq 0$ by (278). This implies that $\psi = e^{-CE}\phi \geq 0$. (See [20] Section XXI-§2, Proposition 2, pp. 226-227, and [71, Theorem 5.16]).

B. Suppose that $g \geq 0$ is more general. Moreover, in accordance with Theorem 3.23 (or Corollary 3.27), we assume that $g \in H^1(I, T^2(\Gamma'_-))$ for which $g(E_m) = 0$. We decompose the solution as follows. Let $u$ be the solution of the problem
\[
-\frac{\partial (S_0 u)}{\partial E} + \omega \cdot \nabla_x u + CS_0 u + \Sigma u = 0,
\]
\[
\left. u \right|_{\Gamma_-} = g, \quad u(\cdot, \cdot, E_m) = 0,
\]
and let $w$ be the solution of the problem
\[
-\frac{\partial (S_0 w)}{\partial E} + \omega \cdot \nabla_x w + CS_0 w + \Sigma w - K_C w = f + K_C u,
\]
\[
\left. w \right|_{\Gamma_-} = 0, \quad w(\cdot, \cdot, E_m) = 0.
\]
Then $\phi := w + u$ is the solution of (274)-(276).

Since $g \geq 0$, one can show that the solution $u$ of the problem (283) is positive, i.e. $u \geq 0$ (see Remark 3.28 below). For example, in the special case of Example 6.1 given below, where $S(x, E) = S(E)$ does not depend on $x$, and $\Sigma(x, \omega, E) = \Sigma(x, E)$ does not depend on $\omega$, the solution $u$ can be expressed explicitly in the form
\[
u = \frac{1}{S_0(E)} H(R(E_m) - Q(x, \omega, E)) e^{-\int_0^t \Sigma(x, s\omega, \omega) ds} \tilde{g}(x - t(x, \omega) \omega, \omega, Q(x, \omega, E)),
\]
where
\[
R(E) := \int_0^E \frac{1}{S_0(\tau)} d\tau,
\]
\[
Q(x, \omega, E) := R(E) + t(x, \omega),
\]
\[
\tilde{g}(y, \omega, \eta) := S_0(R^{-1}(\eta))g(y, \omega, R^{-1}(\eta)),
\]
From (285) it is clear that $u \geq 0$ whenever $g \geq 0$.

Finally, because $f \geq 0$, and $K_C u \geq 0$ as $u \geq 0$, it follows from the part A above that $w \geq 0$, where $w$ is the solution of the problem (281). This allows us to conclude that $\phi = w + u \geq 0$, and so $\psi \geq 0$ as desired.

The same argument is valid for the coupled system considered below.

Remark 3.28 We sketch here a more general proof that, under the standing assumptions of this section, the solution $u$ of the problem (283) is positive, when $g \geq 0$.

We allow $S_0(x, E)$ to depend on both $x$ and $E$, but, in order to make the technicalities easier, we replace the assumption (281) by a slightly stronger one, namely
\[
S_0 \in C^2(I, C_b(G)),
\]
where $C_b(G)$ is the set of bounded continuous functions on $G$, equipped with the norm $\| \cdot \|_{L^\infty(G)}$, making it a closed subspace of $L^\infty(G)$. Clearly, assumption (286) implies (231). In fact, for the argument below to work, it suffices to assume $S_0 \in C^1(I, C_b(G))$.

Since $\frac{\partial S_0}{\partial E} \in C(I, C_b(G))$, there is a constant $M > 0$ such that $\| \frac{\partial S_0}{\partial E}(E) \|_{L^\infty(G)} \leq M$, and hence for every $E, E' \in I$, it holds

$$\|S_0(E') - S_0(E)\|_{L^\infty(G)} \leq M|E - E'|.$$ 

For a given $(x, \omega) \in G \times S$, letting

$$P_{x,\omega} : [0, t(x, \omega)] \times I \to \mathbb{R}; \quad P_{x,\omega}(t, E) := S_0(x - t\omega, E),$$

we see that $P$ is continuous, and

$$|P_{x,\omega}(t, E') - P_{x,\omega}(t, E)| \leq M|E' - E|,$$

i.e. the map $(t, E) \mapsto P_{x,\omega}(t, E)$ satisfies Lipschitz condition on $I$ uniformly with respect to $[0, t(x, \omega)]$.

Therefore, Cauchy-Lipschitz (or Picard-Lindelöf) theorem (cf. [39, Chapter IV, Proposition 1.1]) implies that for every $(x, \omega, E) \in G \times S \times I$ the problem

$$\dot{\gamma}(t) = S_0(x - t\omega, \gamma(t)), \quad t \in [0, t(x, \omega)],$$

$$\gamma(0) = E,$$

has a unique solution $\gamma \in C^1([0, \overline{\tau}])$ defined on the maximal interval $[0, \overline{\tau}]$, such that $\gamma(t) \in I$ for all $t \in [0, \overline{\tau}]$.

We write this solution as $\gamma(x, \omega, E, t)$, and write $\overline{\tau}(x, \omega, E)$ for the end-point of its maximal interval of existence. Because $S_0 \geq \kappa > 0$ on $G \times I$ (by (232)), we have

$$\gamma(x, \omega, E, t') \geq \gamma(x, \omega, E, t) + \kappa(t' - t), \quad (287)$$

whenever $t \leq t'$, and therefore one can see that (i) or (ii) (or both) below holds:

(i) $\gamma(x, \omega, E, \overline{\tau}(x, \omega, E)) = E_m$, \quad or \quad (ii) $\overline{\tau}(x, \omega, E) = t(x, \omega).$ \quad (288)

Denote for $(x, \omega, E, t) \in G \times S \times I \times [0, \overline{\tau}(x, \omega, E)]$,

$$\Gamma(x, \omega, E, t) := (x - t\omega, \omega, \gamma(x, \omega, E, t)),$$

and notice that $\Gamma(x, \omega, E, 0) = (x, \omega, E)$. It follows from (288) that

$$\beta(x, \omega, E) := \Gamma(x, \omega, E, \overline{\tau}(x, \omega, E)) \in \Gamma_- \cup (G \times S \times \{E_m\}), \quad (289)$$

for all $(x, \omega, E) \in G \times S \times I$.

Below we shall understand that $S_0(\Gamma(x, \omega, E, t))$ means $S_0(x - t\omega, \gamma(x, \omega, E, t))$, and similarly for $\frac{\partial S_0}{\partial E}$, since $S_0(x, E)$ is assumed not to depend on $\omega$.

Finally, if $\varphi : G \times S \times I \to \mathbb{R}$ is smooth enough (say $C^1$), then

$$\frac{\partial}{\partial t} \left( \varphi(\Gamma(x, \omega, E, t)) \right) = S_0(\Gamma(x, \omega, E, t)) \frac{\partial \varphi}{\partial E}(\Gamma(x, \omega, E, t)) - (\omega \cdot \nabla_x \varphi)(\Gamma(x, \omega, E, t)),$$

from which one can deduce that the solution $u$ of the problem (283) is given by

$$u(x, \omega, E) = h(x, \omega, E)e^{-\int_0^t \gamma(x, \omega, E, t)dt}W(\Gamma(x, \omega, E, t))dt,$$

where

$$W(x, \omega, E) := \frac{\partial}{\partial E}S_0(x, E) + CS_0(x, E) + \Sigma(x, \omega, E),$$

and notice that $\Gamma(x, \omega, E, 0) = (x, \omega, E)$, and therefore one can see that (i) or (ii) (or both) below holds:
and (recall \(289\))

\[
h(x, \omega, E) := \begin{cases} 
0, & \text{if } \beta(x, \omega, E) \in G \times S \times \{E_m\}, \\
g(\beta(x, \omega, E)), & \text{if } \beta(x, \omega, E) \in \Gamma_-. 
\end{cases}
\]

Since \( h \geq 0 \) by the assumption \( g \geq 0 \), the explicit formula \(290\) for the solution \( u \) clearly implies that \( u \geq 0 \), which is what we were to show.

3.5. An Existence Result Based on the Theory of Evolution Equations. Due to the existence result (part (i)) of Corollary 3.10 one has,

\[
-\frac{\partial(S_0 \psi)}{\partial E} + \omega \cdot \nabla_x \psi \in L^2(G \times S \times I),
\]

but we do not know \textit{a priori} if \( \frac{\partial \psi}{\partial E} \in L^2(G \times S \times I) \), or if \( \omega \cdot \nabla_x \psi \in L^2(G \times S \times I) \). Hence it is not known if \( \psi \in W^1_2(G \times S \times I) \), and so the regularity assumption in parts (ii)-(iii) of Corollary 3.10 is not in general guaranteed. Here we give an alternative approach for a less general problem \(146, 147, 148\), which gives more regularity for its solution \( \psi \), and in particular provides the regularity criterion \( \psi \in W^1_2(G \times S \times I) \) used above.

We assume that the collision operator is of the form

\[
(K \psi)(x, \omega, E) = \int_S \tilde{\sigma}(x, \omega', \omega, E) \psi(x, \omega', E) d\omega'.
\]

(291)

This situation corresponds to the case where the differential cross-section is of the form \( \sigma(x, \omega', \omega, E', E) = \tilde{\sigma}(x, \omega', \omega, E) \delta(E' - E) \) (cf. Remark 3.1). Then \( K \) can be written as \( (K \psi)(E) = K(E) \psi(E) \), where \( \psi(E)(x, \omega) := \psi(x, \omega, E) \) and where for any fixed \( E \in I \), the linear operator \( K(E) \) is defined by

\[
(K(E) \phi)(x, \omega) = \int_S \tilde{\sigma}(x, \omega', \omega, E) \phi(x, \omega') d\omega', \quad \phi \in L^2(G \times S).
\]

For a condition related to the property \( (K \psi)(E) = K(E) \psi(E) \), see \[25\] Chapter VI, Prop. 9.13. By an argument similar to the one leading to the estimate \(153\), we have for all \( E \in I \),

\[
\|K(E)\| \leq \left\| \int_S \tilde{\sigma}(\cdot, \omega', \cdot, E) d\omega' \right\|^{1/2}_{L^\infty(G \times S)} \left\| \int_S \tilde{\sigma}(\cdot, \cdot, \omega', E) d\omega' \right\|^{1/2}_{L^\infty(G \times S)},
\]

(292)

where \( \|K(E)\| \) is the norm of \( K(E) \) as an operator in \( L^2(G \times S) \). Hence under the assumption \(160\) the operator \( K(E) : L^2(G \times S) \to L^2(G \times S) \) is bounded and

\[
\|K(E)\| \leq M_1^{1/2} M_2^{1/2},
\]

uniformly for \( E \in I \).

Consider the problem \(146, 147, 148\), where \( K \) is of the particular form given in \(291\). We shall assume that the restricted cross-sections \( \Sigma, \tilde{\sigma} \) and the stopping power \( S_0 \) satisfy somewhat different assumptions than in the previous section. We make the following change of variables and of the unknown function

\[
\tilde{\psi}(x, \omega, E) := \psi(x, \omega, E_m - E),
\]

\[
\phi := e^{-CE} \tilde{\psi},
\]

(293)
and denote
\[ \tilde{S}(x, E) = S_0(x, E_m - E), \]
\[ \tilde{\Sigma}(x, \omega, E) = \Sigma(x, \omega, E_m - E) \]
\[ \tilde{\sigma}(x, \omega, \omega', E) = \tilde{\sigma}(x, \omega, \omega', E_m - E) \]
\[ \tilde{f}(x, \omega, E) = f(x, \omega, E_m - E) \]
\[ \tilde{g}(y, \omega, E) = g(y, \omega, E_m - E) \]
\[ (\tilde{K}\phi)(x, \omega, E) = \int_S \tilde{\sigma}(x, \omega', \omega, E)\phi(x, \omega', E)d\omega', \quad (294) \]

with \( \phi \in L^2(G \times S \times I) \) in the definition of \( \tilde{K} \). Making the changes in \( (294) \), we find that the problem \( (146), (147), (148) \) is equivalent to \( \phi \) satisfying the equation
\[ \frac{\partial \phi}{\partial E} + \frac{1}{S} \omega \cdot \nabla_x \phi + C\phi + \frac{1}{S} \frac{\partial \tilde{S}}{\partial E} \phi + \frac{1}{S} \tilde{\Sigma} \phi - \frac{1}{S} \tilde{K} \phi = \frac{1}{S} e^{-CE} \tilde{f}, \quad (295) \]
on \( G \times S \times I \), along with satisfying the following inflow boundary and initial value conditions,
\[ \phi|_{\Gamma'} = e^{-CE} \tilde{g}, \quad (296) \]
\[ \phi(\cdot, \cdot, 0) = 0. \quad (on \ G \times S). \quad (297) \]

Furthermore, define for any fixed \( E \in I \) and \( C \geq 0 \) the linear operator \( A_C(E) : L^2(G \times S) \to L^2(G \times S) \) with domain \( D(A_C(E)) \) by (here \( \tilde{S}(E) = \tilde{S}(\cdot, E) \) and \( \tilde{\Sigma}(E) = \tilde{\Sigma}(\cdot, \cdot, E) \)),
\[ D(A_C(E)) = \bar{W}_2^2\phi(G \times S) := \{ \phi \in \bar{W}_2^2(G \times S) \mid \gamma'_C(\phi) = 0 \}, \]
\[ A_C(E)\phi = -\left( \frac{1}{S(E)} \omega \cdot \nabla_x \phi + C\phi + \frac{1}{S(E)} \tilde{\Sigma}(E)\phi + \frac{1}{S(E)} \frac{\partial \tilde{S}}{\partial E}(E)\phi - \frac{1}{S(E)} \tilde{K}(E)\phi \right), \quad (298) \]

and a function \( f(E) : G \times S \to \mathbb{R} \) such that
\[ f(E)(x, \omega) = \frac{1}{S(x, E)} e^{-CE} \tilde{f}(x, \omega, E), \]
where
\[ (\tilde{K}(E)\phi)(x, \omega) = \int_S \tilde{\sigma}(x, \omega', \omega, E)\phi(x, \omega', E)d\omega', \quad \phi \in L^2(G \times S), \]
and where \( \Gamma' = \{(y, \omega) \in \partial G \times S \mid \omega \cdot \nu(y) < 0 \} \), while \( \gamma'_C : \bar{W}_2^2(G \times S) \to \Gamma' \); \( \gamma'_C(\psi) = \psi|_{\Gamma'} \) is the trace mapping (see section 2.1).

We interpret \( \phi \) as a mapping \( I \to L^2(G \times S) \) by defining \( \phi(E)(x, \omega) := \phi(x, \omega, E) \). Assuming that \( \phi(E) \in D(A_C(E)) \) for any \( E \in I \) (which takes care of the inflow boundary condition) the problem \( (295), (296), (297) \) for \( g = 0 \) can be put into the abstract form
\[ \frac{\partial \phi}{\partial E} - A_C(E)\phi = f(E), \quad \phi(0) = 0. \quad (299) \]

We recall the following result from the theory of evolution equations.

**Theorem 3.29** Suppose that \( X \) is a Banach space and that for any fixed \( t \in [0, T] \) the operator \( A(t) : X \to X \) is linear and closed, with domain \( D(A(t)) \subset X \). In addition, we assume that the following conditions hold:
(i) The domain \( D := D(A(t)) \) is independent of \( t \) and is a dense subspace of \( X \).
(ii) The operator \( A(t) \) is \( m \)-dissipative for any fixed \( t \in [0, T] \).
(iii) For every \( u \in D \), the mapping \( f_u : [0, T] \to X \) defined by \( f_u(t) := A(t)u \) is in \( C^1([0, T], X) \).
(iv) \( f \in C^1([0, T], X) \) and \( u_0 \in D \).

Then the (evolution) equation
\[
\frac{\partial u}{\partial t} - A(t)u = f, \quad u(0) = u_0,
\]
has a unique solution \( u \in C([0, T], D) \cap C^1([0, T], X) \). In addition, the solution is given by
\[
u(t) = U(t, 0)u_0 + \int_0^t U(t, s)f(s)ds
\]
where \( U(t, s) : X \to X, 0 \leq t \leq s \leq T \), is a family of bounded operators, strongly continuous in \((t, s)\), called the (two-parameter) evolution system of operators of \( A(t) \), \( t \in [0, T] \), and \( U(\cdot, s)u_0 \) solves (for a fixed \( s \)) for every \( u_0 \in D \) the Cauchy problem
\[
\frac{\partial}{\partial t}(U(t, s)u_0) - A(t)U(t, s)u_0 = 0, \quad U(s, s)u_0 = u_0.
\]

**Proof.** See [69, Theorem 4.5.3, pp. 89-106], [53, pp. 126-182], [25, pp. 477-496]. \( \square \)

**Remark 3.30** We make a brief remark concerning the meaning of the claim in Theorem 3.29 that \( u \in C([0, T], D) \), since here the topology of \( D \) needs to be specified in order to speak of continuity of maps into \( D \). In fact, in [69] Corollary to Theorem 4.4.2, pp. 102-103] one equips \( D \) with the graph norm of \( A(0) \). Recall that for \( t \in [0, T] \), the graph norm of \( A(t) \) on \( D(A(t)) =: D \) is defined as \( \|v\|_{A(t)} := \|v\|_X + \|A(t)v\|_X \), for \( v \in D \).

In Theorem 3.29 the definition of the concept of a solution of (301) comprises that \( u(t) \in D \) for any \( t \in [0, T] \). Define a closed (see [71]) densely defined linear operator \( A_0(E) : L^2(G \times S) \to L^2(G \times S) \) by
\[
D(A_0(E)) = \tilde{W}^2_0(G \times S) =: D \quad \text{(independent of } E \text{)}
\]
\[
A_0(E)\phi = -\frac{1}{S(E)}\omega \cdot \nabla_x \phi.
\]

We begin with a lemma on dissipativity properties of \( A_0(E) \).

**Lemma 3.31** Suppose that the assumptions \([155], [157] \) and \([158] \) are valid, and let
\[
C_0 := \frac{1}{2} \kappa^{-2} \|\nabla_x S\|_{L^\infty(G \times I)}.
\]

Then for any \( C \geq C_0 \) the operator \( A_0(E) - CI : L^2(G \times S) \to L^2(G \times S) \) is \( m \)-dissipative for all \( E \in I \).

**Proof.** A. Dissipativity. We have for \( \phi \in D \),
\[
\omega \cdot \nabla_x \left( \frac{\phi}{S(E)^{1/2}} \right) = \frac{1}{S(E)^{1/2}} \omega \cdot \nabla_x \phi + \omega \cdot \nabla_x \left( \frac{1}{S(E)^{1/2}} \right) \phi
\]
and $\nabla_x \left( \frac{1}{S(E)^{1/2}} \right) = -\frac{1}{2} \tilde{S}(E)^{-3/2} \nabla_x \tilde{S}(E)$. By the assumptions \[157]\ and \[158]\ we get that $\frac{\phi}{S(E)^{1/2}} \in \tilde{W}_0^2(G \times S) = D$. From the Green’s formula \[35]\ we obtain (since $\omega \cdot \nu > 0$ on $\Gamma^*_+$ and since $\phi|_{\Gamma^*_+} = 0$) that for $\phi \in D$,

$$
\left\langle \omega \cdot \nabla_x \left( \frac{\phi}{S(E)^{1/2}} \right), \frac{\phi}{S(E)^{1/2}} \right\rangle_{L^2(G \times S)} = \frac{1}{2} \int_{\partial G \times S} (\omega \cdot \nu) \frac{\phi^2}{S(E)} d\sigma d\omega
$$

$$= \frac{1}{2} \int_{\Gamma^*_+} (\omega \cdot \nu) \frac{\phi^2}{S(E)} d\sigma d\omega \geq 0. \quad (306)$$

That is why by \[305]\ we have

$$\langle -A_0(E)\phi, \phi \rangle_{L^2(G \times S)} = \int_{G \times S} \frac{1}{S(E)} (\omega \cdot \nabla_x \phi) \phi d\omega dx$$

$$= \int_{G \times S} \frac{1}{S(E)^{1/2}} (\omega \cdot \nabla_x \phi) \frac{1}{S(E)^{1/2}} \phi d\omega dx$$

$$= \int_{G \times S} \omega \cdot \nabla_x \left( \frac{\phi}{S(E)^{1/2}} \right) \left( \frac{\phi}{S(E)^{1/2}} \right) d\omega dx - \int_{G \times S} \omega \cdot \nabla_x \left( \frac{1}{S(E)^{1/2}} \right) \frac{\phi^2}{S(E)^{1/2}} d\omega dx$$

$$\geq - \int_{G \times S} \omega \cdot \nabla_x \left( \frac{1}{S(E)^{1/2}} \right) \frac{\phi^2}{S(E)^{1/2}} d\omega dx$$

$$\geq - \left\| \nabla_x \left( \frac{1}{S(\cdot)^{1/2}} \right) \frac{1}{S(\cdot)^{1/2}} \right\|_{L^\infty(G \times I)} \left\| \phi \right\|_{L^2(G \times S)}^2$$

$$\geq - \frac{1}{2} \kappa^{-2} \left\| \nabla_x \tilde{S} \right\|_{L^\infty(G \times I)} \left\| \phi \right\|_{L^2(G \times S)}^2$$

$$= - C_0 \left\| \phi \right\|_{L^2(G \times S)}^2, \quad (307)$$

and hence

$$\langle (A_0(E) - CI)\phi, \phi \rangle_{L^2(G \times S)} = \langle A_0(E)\phi, \phi \rangle_{L^2(G \times S)} - C \left\| \phi \right\|_{L^2(G \times S)}^2$$

$$\leq (C_0 - C) \left\| \phi \right\|_{L^2(G \times S)}^2.$$

Choosing $C \geq C_0$, one finds that $A_0(E) - CI$ is dissipative for any $E \in I$.

B. We still have to show that $R(\lambda I - (A_0(E) - CI)) = L^2(G \times S)$ for (any) $\lambda > 0$. The equation

$$\left( \lambda I - (A_0(E) - CI) \right) \phi = f, \quad (308)$$

means that $\phi \in \tilde{W}_0^2(G \times S)$ and that

$$\frac{1}{S(E)} \omega \cdot \nabla_x \phi + (\lambda + C)\phi = f, \quad (309)$$

which is equivalent to

$$\omega \cdot \nabla_x \phi + (\lambda + C)\bar{S}(E)\phi = \bar{S}(E)f, \quad (310)$$

since by \[155]\ and \[157]\ $f \in L^2(G \times S)$ if and only if $\bar{S}(E)f \in L^2(G \times S)$. Let $B_0 : L^2(G \times S) \rightarrow L^2(G \times S)$ be a linear operator with domain $D(B_0)$ defined by

$$D(B_0) = \tilde{W}_0^2(G \times S),$$

$$B_0\phi = -\omega \cdot \nabla_x \phi.$$
Then $B_0$ is $m$-dissipative \cite{71,20}. Let $\lambda': = \kappa(\lambda + C)$. The equation \eqref{310} is equivalent to

\[(\lambda' I - (B_0 + B_1))\phi = \tilde{S}(E)f\]  

where $B_1$ is defined by

\[B_1\phi = -((\lambda + C)\tilde{S}(E) - \lambda')\phi.\]

It is clear that the operator $B_1 : L^2(G \times S) \to L^2(G \times S)$ is bounded, and since $(\lambda + C)\tilde{S}(E) - \lambda' \geq 0$ by the assumption \eqref{157} and the definition of $\lambda'$, it follows that $B_1$ is dissipative. These observations imply that $B_0 + B_1$ is $m$-dissipative (cf. \cite{25} Chapter III, or \cite{71} Theorem 4.2). Since \eqref{308} is equivalent to \eqref{311} as explained above, this shows that $R(\lambda I - (A_0(E) - CI)) = L^2(G \times S)$, and thus completes the proof. \hfill $\square$

For $E \in I$, let $A_1(E) : L^2(G \times S) \to L^2(G \times S)$ be the linear operator

\[A_1(E)\phi := -\frac{1}{S(E)}\tilde{\Sigma}(E)\phi - \frac{1}{S(E)}\frac{\partial \tilde{S}}{\partial E}(E)\phi + \frac{1}{S(E)}\tilde{K}(E)\phi.\]

We have the following uniform bound for the family of operators \{ $A_1(E)$ | $E \in I$ \}.

\begin{lemma}
Under the assumptions \eqref{149}, \eqref{150} and \eqref{160} the operator $A_1(E)$ is bounded for any fixed $E \in I$, and collectively they obey a uniform bound,

\[\sup_{E \in I} \| A_1(E) \| \leq \kappa^{-1}\left( \| \tilde{\Sigma} \|_{L^\infty(G \times S \times I)} + \| \frac{\partial \tilde{S}}{\partial E} \|_{L^\infty(G \times I)} + M_1^{1/2}M_2^{1/2} \right) =: C_0' < \infty,\]

\end{lemma}

where $M_j \geq 0$, $j = 1, 2$ are as in \eqref{160}.

\textbf{Proof.} The (uniform) estimate follows immediately from the assumptions and the estimate \eqref{292}. \hfill $\square$

Using the above notations we have for $C := C_0 + C_0'$ the following decomposition

\[A_C(E) = A_0(E) - C_0I - C_0'I + A_1(E).\]

Recall that $C_0$ and $C_0'$ were defined in \eqref{304} and \eqref{312}. Since by Lemma 3.32

\[\langle (-C_0'I + A_1(E))\phi, \phi \rangle_{L^2(G \times S \times I)} = -C_0' \| \phi \|_{L^2(G \times S \times I)}^2 + (A_1(E))\phi, \phi \rangle_{L^2(G \times S \times I)} \]

\[\leq -C_0' \| \phi \|_{L^2(G \times S \times I)}^2 + C_0' \| \phi \|_{L^2(G \times S \times I)}^2 = 0,
\]

we see that $-C_0'I + A_1(E)$ is bounded and dissipative. On the other hand, according to Lemma 3.31, $A_0(E) - C_0I$ is $m$-dissipative. Hence $A_C(E)$ is $m$-dissipative for any $E \in I$ (cf. \cite{25} Chapter III, Theorem 2.7, or \cite{71} Theorem 4.2). We record this observation into the next lemma.

\begin{lemma}
For $C = C_0 + C_0'$ and for every fixed $E \in I$, the operator $A_C(E)$ is $m$-dissipative.
\end{lemma}

We shall assume that

\[\tilde{\sigma} \in C(I, L^\infty(G \times S, L^1(S'))) \cap C(I, L^\infty(G \times S')) \subseteq L^1(S)),\]
where \( \hat{\sigma} \) interpreted as an element of \( C(I, L^\infty(G \times S, L^1(S'))) \) is
\[
\hat{\sigma}(E)(x, \omega)(\omega') = \hat{\sigma}(x, \omega', \omega, E),
\]
and when interpreted as an element of \( C(I, L^\infty(G \times S', L^1(S))) \) it is
\[
\hat{\sigma}(E)(x, \omega')(\omega) = \hat{\sigma}(x, \omega', \omega, E).
\]
Furthermore, in order to avoid ambiguity, we have denoted by \( S' \) (resp. \( S \)) the unit sphere in \( \mathbb{R}^3 \) for the variable \( \omega' \) (resp. \( \omega \)). We find that the conditions (315) are then satisfied valid for \( \overline{\sigma} \) with
\[
M_1 := \max_{E \in I} ||\overline{\sigma}(E)||_{L^\infty(G \times S, L^1(S'))} = ||\overline{\sigma}||_{C(I, L^\infty(G \times S, L^1(S'))),}
\]
\[
M_2 := \max_{E \in I} ||\overline{\sigma}(E)||_{L^\infty(G \times S', L^1(S))} = ||\overline{\sigma}||_{C(I, L^\infty(G \times S', L^1(S))}.}
\]
Moreover, for all \( E_1, \ E_2 \in I \),
\[
\begin{align*}
\text{ess sup}_{(x, \omega') \in G \times S} \int_{S'} |\overline{\sigma}(x, \omega', \omega, E_1) - \overline{\sigma}(x, \omega', \omega, E_2)| \ d\omega' &= ||\overline{\sigma}(E_1) - \overline{\sigma}(E_2)||_{L^\infty(G \times S, L^1(S'))} \\
\text{ess sup}_{(x, \omega') \in G \times S'} \int_{S} |\overline{\sigma}(x, \omega', \omega, E_1) - \overline{\sigma}(x, \omega', \omega, E_2)| \ d\omega &= ||\overline{\sigma}(E_1) - \overline{\sigma}(E_2)||_{L^\infty(G \times S', L^1(S))}.
\end{align*}
\]
(314)
Supposing additionally that
\[
\hat{\sigma} \in C^1(I, L^\infty(G \times S, L^1(S'))) \cap C^1(I, L^\infty(G \times S', L^1(S))),
\]
then for any fixed \( E \) the operator \( \frac{\partial K}{\partial E}(E) : L^2(G \times S) \to L^2(G \times S) \) defined by
\[
\left( \frac{\partial K}{\partial E}(E)\phi \right)(x, \omega) := \int_{S} \frac{\partial \overline{\sigma}}{\partial E}(x, \omega', \omega, E) \phi(x, \omega') \ d\omega'
\]
is a bounded operator and
\[
\begin{align*}
\text{ess sup}_{(x, \omega') \in G \times S} \int_{S'} |\frac{\partial \overline{\sigma}}{\partial E}(x, \omega', \omega, E_1) - \frac{\partial \overline{\sigma}}{\partial E}(x, \omega', \omega, E_2)| \ d\omega' &= \left\| \frac{\partial \overline{\sigma}}{\partial E}(E_1) - \frac{\partial \overline{\sigma}}{\partial E}(E_2) \right\|_{L^\infty(G \times S, L^1(S'))} \\
\text{ess sup}_{(x, \omega') \in G \times S'} \int_{S} |\frac{\partial \overline{\sigma}}{\partial E}(x, \omega', \omega, E_1) - \frac{\partial \overline{\sigma}}{\partial E}(x, \omega', \omega, E_2)| \ d\omega &= \left\| \frac{\partial \overline{\sigma}}{\partial E}(E_1) - \frac{\partial \overline{\sigma}}{\partial E}(E_2) \right\|_{L^\infty(G \times S', L^1(S))}.
\end{align*}
\]
(317)
The following lemma justifies the notation given in (316).

**Lemma 3.34** Under the assumption (315), for any fixed \( \phi \in L^2(G \times S) \) the map
\[
k_\phi : I \to L^2(G \times I); \quad k_\phi(E) = K(E)\phi
\]
is in \( C^1(I, L^2(G \times I)) \) and \( \frac{\partial k_\phi}{\partial E}(E) = \frac{\partial K}{\partial E}(E)\phi \).

**Proof.** If \( \beta : G \times S' \times S \times I \to \mathbb{R} \) is measurable, and one defines for any \( \phi \in L^2(G \times S) \) such that the integral converges,
\[
L_\beta(E)\phi := \int_{G \times S} \beta(x, \omega', \omega, E)\phi(x, \omega') \ d\omega \ dx,
\]
then for every \( E \in I \) one has (compare to (292)),
\[
\|L_\beta(E)\phi\|^2_{L^2(G \times S)} \leq \|\beta(E)\|_{L^\infty(G \times S, L^1(S'))} \|\beta(E)\|_{L^\infty(G \times S', L^1(S))} \|\phi\|^2_{L^2(G \times S)},
\]
(318)
with the conventions \( \beta(E)(x, \omega)(\omega') = \beta(E)(x, \omega')(\omega) = \beta(x, \omega', \omega, E) \) as above.
For a fixed $E_0 \in I$, taking $\beta(x, \omega', \omega, E) = \overline{\sigma}(x, \omega', \omega, E) - \overline{\sigma}(x, \omega', \omega, E_0)$, we have $k_{\phi}(E) - k_{\phi}(E_0) = L_\beta(E)\phi$, and the above estimate shows that $k_{\phi}$ is continuous at $E_0$, by assumption \(315\). Similarly, one sees that $E \mapsto \frac{\partial K}{\partial E}(E)\phi$, is continuous at $E_0$, by choosing $\beta(x, \omega', \omega, E) = \overline{\sigma}(x, \omega', \omega, E) - \overline{\sigma}(x, \omega', \omega, E_0)$.

Finally, the differentiability of $k_{\phi}$ at $E_0$ is obtained by taking in the above estimate,

$$\beta(x, \omega', \omega, E) = \frac{\overline{\sigma}(x, \omega', \omega, E) - \overline{\sigma}(x, \omega', \omega, E_0)}{E - E_0} - \frac{\partial \overline{\sigma}}{\partial E}(x, \omega', \omega, E_0),$$

whence $\frac{k_{\phi}(E) - k_{\phi}(E_0)}{E - E_0} - \frac{\partial \overline{K}}{\partial E}(E_0)\phi = L_\beta(E)\phi$, and we have $k'_{\phi}(E_0) = \frac{\partial \overline{K}}{\partial E}(E_0)\phi$. This completes the proof of the lemma. \qed

**Remark 3.35** In fact, the proof of the previous lemma shows that $\overline{K}$ as a map $I \to L(L^2(G \times S))$ is $C^1$, where for a Banach space $Z$ the set $L(Z)$ is the space of bounded linear operators $Z \to Z$ equipped with the uniform operator norm topology.

With the above notations and results at hand, we are ready to state the central result of this section.

**Theorem 3.36** Suppose that the assumptions \([149], [157], [158]\) and \([160]\) are valid, that $\overline{\sigma} \geq 0$, and that

\begin{align}
\Sigma &\in C^1(I, L^\infty(G \times S)), \\
S_0 &\in C^2(I, L^\infty(G)), \\
\overline{\sigma} &\in C^1(I, L^\infty(G \times S, L^1(S')) \cap C^1(I, L^\infty(G \times S', L^1(S))).
\end{align}

Let $f \in C^1(I, L^2(G \times S))$ and let $g \in C^2(I, T^2(\Gamma_-))$ which satisfies the compatibility condition

$$g(E_m) = 0.$$ \(322\)

Then the problem \([146], [147], [148]\) has a unique solution $\psi \in C(I, \hat{W}^2(G \times S)) \cap C^1(I, L^2(G \times S))$.

If in addition the assumptions \([161]\) (with $c > 0$) are also valid, the estimate

$$\|\psi\|_{H_1} \leq \frac{c}{c'} \left( \|f\|_{L^2(G \times S \times I)} + \|g\|_{T^2(\Gamma_-)} \right).$$ \(323\)

holds. (The constants $\kappa$, $q$ and $c'$ were defined in \([157], [161]\) and \([183]\), respectively.)

**Proof.** A. Assume at first that $g = 0$. We make the change of variables and the change of unknown function as above by setting $\hat{\psi}(x, \omega, E) = \psi(x, \omega, E_m - E)$ and $\phi = e^{-CE}\hat{\psi}$. Choose $C = C_0 + C_0'$ (see \([304], [314]\)). Then, as observed above, the problem \([146], [147], [148]\) can be cast into an equivalent form (see \([299]\))

$$\frac{\partial \phi}{\partial E} - A_C(E)\phi = f(E), \quad \phi(0) = 0,$$ \(324\)

where the domain $D(A_C(E)) = \hat{W}^2\_0(G \times S)$; $D$ of definition of $A_C(E)$ is independent of $E$. We have, moreover, demonstrated (see Lemma \([3.33]\)) that the (densely defined) operator $A_C(E) : L^2(G \times S) \to L^2(G \times S)$ is $m$-dissipative for any fixed $E \in I$. 

The assumptions (319), (320), (321) imply that for any fixed $\phi \in D$ the mapping

$$h_\phi : I \to L^2(G \times S); \quad h_\phi(E) := A_C(E)\phi,$$

is differentiable and

$$h_\phi'(E) = -\frac{\partial}{\partial E}\left(\frac{1}{S(E)}\right)\omega \cdot \nabla_x \phi - \frac{\partial}{\partial E}\left(\frac{1}{S(E)}\Sigma(E)\right)\phi - \frac{\partial}{\partial E}\left(\frac{1}{S(E)}\partial \hat{S}\right)\phi + \frac{\partial}{\partial E}\left(\frac{1}{S(E)}\right)\tilde{K}(E)\phi + \frac{1}{S(E)}\partial \tilde{K}(E)\phi,$$

where $\frac{\partial \tilde{K}(E)}{\partial E}\phi$ is defined in (316), and the derivative $\frac{\partial}{\partial E}(\tilde{K}(E)\phi) = \frac{\partial \tilde{K}(E)}{\partial E}\phi$ is provided by Lemma 3.34. By assumptions (319), (320), (321) we thus see that $h_\phi$ is in $C^1(I, L^2(G \times S))$.

By Theorem 3.29 there exists a unique solution $\phi \in C(I, W^2_0(G \times S)) \cap C^1(I, L^2(G \times S))$ of (324). Then $\psi(x, \omega, E) := e^{(E_m-E)}\phi(x, \omega, E_m - E)$ is the required solution for the problem (146), (147), (148) for $g = 0$.

B. Suppose that more generally $g \in C^2(I, T^2(\Gamma_-))$ and that the compatibility condition (322) holds. By Lemma 5.10 there exists a lift $Lg \in C^2(I, \hat{W}^2(G \times S))$ for which $\gamma_-(Lg) = g$, and $\langle Lg, \cdot, E_m \rangle = 0$ (follows from (322)), and furthermore $\omega \cdot \nabla_x (Lg) = 0$. Substituting into the problem (146), (147), (148) the function $u := \psi - Lg$ for $\psi$ we obtain the following problem for $u$,

$$-\frac{\partial(S_0u)}{\partial E} + \omega \cdot \nabla_x u + \Sigma u - Ku = \tilde{f},$$

$$u|_{\Gamma_-} = 0,$$

$$u(\cdot, \cdot, E_m) = 0,$$

where

$$\tilde{f} := f - \left(-\frac{\partial(S_0(Lg))}{\partial E} + \Sigma(Lg) - K(Lg)\right),$$

and we have $\tilde{f} \in C^1(I, L^2(G \times S))$ under the assumption (319), (320), (321).

By Part A of the proof, the problem (146), (147), (148) (for $g = 0$) has a unique solution $u \in C(I, \hat{W}^2_0(G \times S)) \cap C^1(I, L^2(G \times S))$. As argued above, $\psi := u + Lg$ is then desired unique solution for the problem (146), (147), (148), for the given, arbitrary $g \in C^2(I, T^2(\Gamma_-))$.

We claim that the solution $\psi$ belongs to $W^2_0(G \times S \times I)$. Indeed, since $\psi \in C^1(I, L^2(G \times S))$, we have $\psi \in L^2(G \times S \times I)$ and $\frac{\partial \psi}{\partial E} \in L^2(G \times S \times I)$. On the other hand, since $\psi$ solves (146), these imply, together with the assumptions made, that $\omega \cdot \nabla_x \psi \in L^2(G \times S \times I)$, which confirms the claim.

Hence, under the additional assumptions (161), the estimate (323) follows from Corollary 3.10 (part (iii)). This completes the proof.

The term compatibility condition used above for the assumption (322) for $g$ comes from the observation that since the solution $\psi$ is to satisfy (148) and we have $\psi \in C(I, \hat{W}^2(G \times S))$, it follows that $\psi(E_m) = 0$, and therefore $0 = \psi(E_m)|_{\Gamma_-} = g(E_m)$.

Remark 3.37 Strictly speaking, in the proof of Theorem 3.36 above we did not fully address the claim that $u \in C(I, \hat{W}^2_0\omega(G \times S))$, since, as pointed out in Remark 3.30 the common domain $D = D(A_C(E)) = \hat{W}^2_0\omega(G \times S)$ of the operators $A_C(E)$, $E \in I$, is to be equipped with the graph norm of $A_C(0)$ when applying Theorem 3.29. However, the norm $\|\cdot\|_{\hat{W}^2_0\omega(G \times S)}$ of $\hat{W}^2_0\omega(G \times S)$ is equivalent to the graph
norm $\|\cdot\|_{AC(E)}$ in $D = \tilde{W}^2_{-\delta}(G \times S)$, for every $E \in I$, and in particular for $E = 0$. This can be readily seen from the estimates, for $\phi \in D$,

$$
\|A_C(E)\phi\|_{L^2(G \times S)} \leq \frac{1}{\kappa} \left( \|\omega \cdot \nabla_x \phi\|_{L^2(G \times S)} + C_2 \|\phi\|_{L^2(G \times S)} \right),
$$

$$
\|\omega \cdot \nabla_x \phi\|_{L^2(G \times S)} \leq C_1 \|A_C(E)\phi\|_{L^2(G \times S)} + C_2 \|\phi\|_{L^2(G \times S)},
$$

where

$$
C_1 := \left\| \hat{S}(E) \right\|_{L^\infty}, \quad C_2 := CC_1 + \left\| \hat{\Sigma}(E) \right\|_{L^\infty} + \left\| \frac{\partial \hat{S}}{\partial E}(E) \right\|_{L^\infty} + \left\| \hat{K}(E) \right\|,
$$

and we have used a short hand notation $L^\infty$ for $L^\infty(G \times S)$. Notice also that $\|\gamma_-(\phi)\|_{T^2(\Gamma_-)} = 0$ when $\phi \in \tilde{W}^2_{-\delta}(G \times S)$.

Let $H^m(I, X)$, $m \in \mathbb{N}_0$ be the Sobolev space for Hilbert space valued functions $I \rightarrow X$. We have the following corollary.

**Corollary 3.38** Suppose that the assumptions (149), (157), (158), (160) and (319), (320) and (321) of Theorem 3.36 are valid, and $\tilde{\sigma} \geq 0$. Let $f \in H^2(I, L^2(G \times S))$ and $g \in H^3(I, T^2(\Gamma_-))$ which satisfies the compatibility condition

$$
g(E_m) = 0. \tag{326}
$$

Then the problem (146), (147), (148) has a unique solution $\psi \in C(I, \tilde{W}^2(G \times S)) \cap C^1(I, L^2(G \times S))$.

If in addition the assumptions (161) (with $c > 0$) are also valid, the estimate (323) holds.

**Proof.** By the Sobolev Embedding Theorem

$$
H^m(I, X) \subset C^j(I, X) \text{ for } m > j + \frac{1}{2}
$$

and then the assertion follows from Theorem 4.12

**Remark 3.39** The evolution equation based approach given above can be generalized for $L^p$-theory when $1 \leq p < \infty$. The approach based on the Lions-Lax-Milgram Theorem (section 3.2) is limited to the Hilbert space structure, and can therefore be only applied for $p = 2$. However, some (recent) generalizations for reflexive Banach spaces of Lions-Lax-Milgram theory might allow methods of section 3.2 to be generalized also for $1 < p < \infty$.

### 3.6. On the Existence of Solutions for Volterra Type Collision Operators.

In the previous section, we assumed that the collision operator $K$ is of the form $(K\psi)(x, \omega, E) = \int_{\mathbb{S}} \sigma(x, \omega', \omega, E)\psi(x, \omega', E)d\omega'$ in order to avoid integration over $I$ with respect to $E'$. Considerations were founded on the fact that $K\psi$ had a representation $(K\psi)(E) = K(E)\psi(E)$. However, for some collision operators of special type, also integration with respect to $E'$ is possible in evolution operator based approaches. In this section, we give a short and formal description of such a technique for Volterra type collision operators.
Consider the problem (146), (147), (148) with \( g = 0 \),
\[
-\frac{\partial \psi}{\partial E} + \frac{1}{S_0(E)} \omega \cdot \nabla_x \psi + \frac{1}{S_0(E)} \Sigma(E) \psi - \frac{1}{S_0(E)} \frac{\partial S_0}{\partial E}(E) \psi - \frac{1}{S_0(E)} K \psi = \frac{1}{S_0(E)} f,
\]
\( \psi_{T_0} = 0, \quad \psi(\cdot, \cdot, E_m) = 0. \)

For simplicity we denote \( \frac{1}{S_0(E)} f \) and \( \frac{1}{S_0(E)} K \) again by \( f \) and \( K \). Assume that \( I = [0, \infty] =: \mathbb{R}_+ \) and that \( K \) is of the Volterra type operator (cf. [25, pp. 447-452])

\[
(K \psi)(x, \omega, E) = \int_0^E \int_S \sigma(x, \omega', \omega, E - E', E) \psi(x, \omega', E') d\omega' dE'.
\]

In other words, \( K \psi = \int_{S \times I} \tilde{\sigma}(\cdot, \omega', \cdot, \cdot, \cdot) \psi(\cdot, \omega', \cdot, \cdot, E') d\omega' dE' \) for a differential cross section \( \tilde{\sigma} \) of the form \( \tilde{\sigma}(x, \omega', \omega, E', E) = \chi_I(E - E') \sigma(x, \omega', \omega, E - E', E) \), where \( \chi_I \) is the characteristic function of \( I \). Assume additionally that \( \sigma \) has a decomposition

\[
\sigma(x, \omega', \omega, E - E', E) = \sigma_1(x, \omega', \omega, E) \sigma_2(E - E').
\]

Let \( K_1(E) \) (for a fixed \( E \)) be a linear operator \( L^2(G \times S) \rightarrow L^2(G \times S) \) defined by

\[
(K_1(E) \phi)(x, \omega) = \int_S \sigma_1(x, \omega, \omega, \omega, E) \phi(x, \omega') d\omega'.
\]

Then we find that

\[
(K \psi)(x, \omega, E) = \int_0^E \sigma_2(E - E') K_1(E) \psi(E') dE'.
\]

Define an extended space by \( \mathcal{X} := L^2(G \times S) \times L^1(\mathbb{R}_+, L^2(G \times S)) \) and a linear operator \( A(E) : \mathcal{X} \rightarrow \mathcal{X} \) for a fixed \( E \) by (here the argument of \( \mathbb{R}_+ \) is denoted by \( s \))

\[
D(A(E)) = D := W^2,0(G \times S) \times H^{1,1}(\mathbb{R}_+, L^2(G \times S)),
\]

\[
A(E) := \left( \begin{array}{c} A(E) \\ B(E) \frac{d}{ds} \end{array} \right),
\]

where

\[
H^{1,1}(\mathbb{R}_+, L^2(G \times S)) := \{ F \in L^1(\mathbb{R}_+, L^2(G \times S)) \mid F' \in L^1(\mathbb{R}_+, L^2(G \times S)) \},
\]

is the domain of the derivative operator \( \frac{d}{ds} : L^1(\mathbb{R}_+, L^2(G \times S)) \rightarrow L^1(\mathbb{R}_+, L^2(G \times S)) \), the linear operator \( A(E) : L^2(G \times S) \rightarrow L^2(G \times S) \) with domain \( W^2,0(G \times S) \) (independent of \( E \)) is given by,

\[
A(E) \phi = -\left( \frac{1}{S_0(E)} \omega \cdot \nabla_x \phi + \frac{1}{S_0(E)} \Sigma(E) \phi - \frac{1}{S_0(E)} \frac{\partial S_0}{\partial E}(E) \phi \right),
\]

the linear operator \( \delta_0 \) is bounded and defined by

\[
\delta_0 : H^{1,1}(\mathbb{R}_+, L^2(G \times S)) \rightarrow L^2(G \times S); \quad \delta_0 F = F(0),
\]

and finally \( B(E) \) is the bounded linear operator defined by

\[
B(E) : L^2(G \times S) \rightarrow L^1(\mathbb{R}_+, L^2(G \times S)); \quad (B(E) \phi)(s) = \sigma_2(s) K_1(E) \phi.
\]

Let \( \left( \phi \begin{array}{c} \eta \\ F \end{array} \right) \in \mathcal{X} \) and let \( \left( \psi \begin{array}{c} X \\ F \end{array} \right) \in H^1(\mathbb{R}_+, \mathcal{X}) \) (note that \( \psi : \mathbb{R}_+ \rightarrow L^2(G \times S) \), \( F : \mathbb{R}_+ \rightarrow L^1(\mathbb{R}_+, L^2(G \times S)) \); here the argument of \( \psi \) and \( F \) is denoted by \( E \)). In
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In this extended setting, for a given $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \in C(\mathbb{R}_+, \mathcal{X})$, the validity of the evolution equation

$$\frac{\partial}{\partial E} \begin{pmatrix} \psi \\ F \end{pmatrix} - A(E) \begin{pmatrix} \psi \\ F \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \left( \begin{pmatrix} \psi \\ F \end{pmatrix} \right)(0) = \begin{pmatrix} \phi \\ \eta \end{pmatrix},$$

(327)

implies (from the first row of the matrix equation) that

$$\frac{\partial \psi}{\partial E} - A(E) \psi - K \psi = f, \quad \psi_T = 0, \quad \psi(0) = \phi.$$  

(328)

In some cases the evolution system of the operators $A(E)$ has properties which enable one to deduce the existence of solutions for the transport problem

$$\frac{\partial \psi}{\partial E} - A(E) \psi - K \psi = f, \quad \psi(0) = \phi.$$ 

(329)

Note that carrying out the change of variables as above, one is able to replace the initial condition $\psi(0) = \phi$ with $\psi(E_m) = \phi$.

The basis of the idea is as follows. Assume, for example, that $S_0, \Sigma$ and $\sigma_1$ are independent of $E$. Then the operators $A(E), K_1(E), B(E)$ and $A(E)$ are independent of $E$ as well. Let $T(E)$ be the $C^0$-semigroups generated by $A$. Define a linear operator $R(E): L^1(\mathbb{R}_+, L^2(G \times S)) \to L^2(G \times S)$ by

$$R(E)\eta := \int_0^E T(E - s) \eta(s)ds.$$ 

Denote by $S(E)$ the (left) translation semigroup on $L^1(\mathbb{R}_+, L^2(G \times S))$,

$$(S(E)\eta)(s) := \eta(E + s).$$  

(330)

Finally, let

$$T(E) := \begin{pmatrix} T(E) & R(E) \\ 0 & S(E) \end{pmatrix}.$$ 

Then $T(E)$ is the $C^0$-semigroup generated by $\begin{pmatrix} A & \delta_0 \\ 0 & \sigma_0 \end{pmatrix} : \mathcal{X} \to \mathcal{X}$ ([25, pp. 437-438]). Furthermore, letting $S(E) : \mathcal{X} \to \mathcal{X}$ be the $C^0$-semigroup generated by the perturbed operator

$$A := \begin{pmatrix} A & \delta_0 \\ 0 & d \sigma_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix},$$

it follows that $\begin{pmatrix} \psi \\ F \end{pmatrix}(E) := S(E) \begin{pmatrix} \phi \\ \eta \end{pmatrix}$ is the solution of (327) for $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The above semigroups obey the following formula, for every $\begin{pmatrix} \phi \\ \eta \end{pmatrix} \in \mathcal{X}$: ([25, Corollary III.1.7])

$$S(E) \begin{pmatrix} \phi \\ \eta \end{pmatrix} = T(E) \begin{pmatrix} \phi \\ \eta \end{pmatrix} + \int_0^E T(E - s)B \begin{pmatrix} \phi \\ \eta \end{pmatrix} ds,$$

$$= T(E) \begin{pmatrix} \phi \\ \eta \end{pmatrix} + \int_0^E T(E - s)B \begin{pmatrix} \psi \\ F \end{pmatrix}(s)ds,$$  

(331)

where $B := \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$. 

Choose $\eta := f$ and $\left(\begin{array}{c} F_1 \\ F_2 \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right)$. Applying the above facts we have

$$
\left(\begin{array}{c} \psi \\ F \end{array} \right)(E) = S(E) \left(\begin{array}{c} \phi \\ f \end{array} \right)
= \left(\begin{array}{cc} T(E) & R(E) \\ 0 & S(E) \end{array} \right) \left(\begin{array}{c} \phi \\ f \end{array} \right) + \int_0^E \left(\begin{array}{cc} T(E-s) & R(E-s) \\ 0 & S(E-s) \end{array} \right) \left(\begin{array}{c} 0 \\ B_\psi(s) \end{array} \right) ds
$$

\begin{equation}
= \left(\begin{array}{cc} T(E) & R(E) \\ 0 & S(E) \end{array} \right) \left(\begin{array}{c} \phi \\ f \end{array} \right) + \int_0^E \left(\begin{array}{cc} T(E-s) & R(E-s) \\ 0 & S(E-s) \end{array} \right) \left(\begin{array}{c} 0 \\ B_\psi(s) \end{array} \right) ds, \tag{332}
\end{equation}

from which the last row gives us

$$
F(E) = S(E)f + \int_0^E S(E-s)B_\psi(s)ds, \tag{333}
$$

and then (recall (330) and that $(B_\psi(s))(t) = \sigma_2(t)K_t\psi(s)$)

$$
F(E)(0) = f(E) + \int_0^E (S(E-s)(B_\psi(s)))(0)ds = f(E) + \int_0^E \sigma_2(E-s)K_t\psi(s)ds. \tag{334}
$$

Combining (334) and (328) we finally see that $\psi$ is a solution of the transport problem (329).

Finally, we note that in the case where $\left(\begin{array}{c} \psi \\ F \end{array} \right) \in D(A)$, we have $\psi(E) \in D(A) = \tilde{W}^2_2(G \times S)$, for any $E \geq 0$, and thus in particular, $\psi|_{\Gamma_-} = 0$. We find by the first row of matrix equation (332) that $\psi(0) = T(0)\phi + R(0)f = \phi$.

It might be worth attempting to generalize this method under less restrictive assumptions, especially for the case where $S_0$, $\Sigma$, $\sigma_1$ are allowed to be $E$-dependent.

4. Existence of Solutions for the Coupled System

In this section, we consider the coupled transport problem. For simplicity denote $\Sigma_j := \Sigma_{j,r}$, $S_j := S_{j,r}$, $\sigma_{j,r} := \sigma_{j,j}$ for $j = 2, 3$. Let $f = (f_1, f_2, f_3) \in L^2(G \times S \times I)^3$ and $g = (g_1, g_2, g_3) \in T^2(\Gamma_-)^3$.

We deal with the following coupled system of integro-partial differential equations for $\psi = (\psi_1, \psi_2, \psi_3)$ on $G \times S \times I$,

$$
\omega \cdot \nabla_2 \psi_1 + \Sigma_{1} \psi_1 - K_1 \psi = f_1, \tag{335}
$$

$$
- \frac{\partial(S_j \psi_j)}{\partial E} + \omega \cdot \nabla_2 \psi_j + \Sigma_j \psi_j - K_j \psi = f_j, \quad j = 2, 3. \tag{336}
$$

In order to guarantee uniqueness of solutions, we moreover impose the inflow boundary condition on $\Gamma_-$,

$$
\psi_j|_{\Gamma_-} = g_j, \quad j = 1, 2, 3, \tag{337}
$$

and initial value (or energy boundary) condition on $G \times S$,

$$
\psi_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3, \tag{338}
$$

where $E_m$ is the cut-off energy. As mentioned in the introduction the problem (335)- (338) is an approximation of the problem (11), (2).
We assume that the total (restricted) cross sections \( \Sigma_j : G \times S \times I \rightarrow \mathbb{R} \), for 
\( j = 1, 2, 3 \), are functions such that
\[
\Sigma_j \in L^\infty(G \times S \times I), \quad \Sigma_j \geq 0, \quad j = 1, 2, 3. \quad (339)
\]
Furthermore, we assume that the differential (restricted) cross sections \( \sigma_{kj} : G \times S^2 \times I^2 \rightarrow \mathbb{R} \), \( k, j = 1, 2, 3 \), are measurable functions such that
\[
\begin{align*}
\sum_{k=1}^{3} \int_{S \times I} \sigma_{kj}(x, \omega', \omega, E', E) d\omega' dE' &\leq M_1, \quad \text{a.e. } G \times S \times I, \quad j = 1, 2, 3, \\
\sum_{k=1}^{3} \int_{S \times I} \sigma_{jk}(x, \omega, \omega', E, E') d\omega' dE' &\leq M_2, \quad \text{a.e. } G \times S \times I, \quad j = 1, 2, 3, \quad (340)
\end{align*}
\]
\( \sigma_{kj} \geq 0 \), \( \text{a.e. } G \times S^2 \times I^2 \), \( k, j = 1, 2, 3 \).

Define the (restricted) scattering operator \( \Sigma_j \) and the (restricted) collision operator \( K_j \) corresponding to the particle type \( j \), for \( j = 1, 2, 3 \) and \( \psi_j \in L^2(G \times S \times I) \), as follows
\[
(\Sigma_j \psi_j)(x, \omega, E) = \Sigma_j(x, \omega, E) \psi_j(x, \omega, E), \quad (341)
\]
and for \( \psi = (\psi_1, \psi_2, \psi_3) \in L^2(G \times S \times I)^3 \),
\[
(K_j \psi)(x, \omega, E) = \sum_{k=1}^{3} \int_{S \times I} \sigma_{kj}(x, \omega', \omega, E', E) \psi_k(x, \omega', E') d\omega' dE'. \quad (342)
\]
Furthermore, we define for \( \psi = (\psi_1, \psi_2, \psi_3) \in L^2(G \times S \times I)^3 \),
\[
\Sigma \psi = (\Sigma_1 \psi_1, \Sigma_2 \psi_2, \Sigma_3 \psi_3) \quad (343)
\]
and
\[
K \psi = (K_1 \psi, K_2 \psi, K_3 \psi). \quad (344)
\]
One immediately sees that \( \Sigma : L^2(G \times S \times I)^3 \rightarrow L^2(G \times S \times I)^3 \) is a bounded linear operator. In addition, by applying Hölder’s inequality we have the following (cf. \[20\], pp. 227-228) and \[71\] Theorem 5.2 for \( p = 1 \); see also \([153]\), \([154]\)).

**Theorem 4.1** The linear operator \( K : L^2(G \times S \times I)^3 \rightarrow L^2(G \times S \times I)^3 \) is bounded and
\[
\|K\| \leq \max_{j=1,2,3} \left\| \sum_{k=1}^{3} \int_{S \times I} \sigma_{kj}(\cdot, \omega', \cdot, E', \cdot) d\omega' dE' \right\|_{L^\infty}^{1/2} \left\| \sum_{k=1}^{3} \int_{S \times I} \sigma_{jk}(\cdot, \cdot, \omega', \cdot, E') d\omega' dE' \right\|_{L^\infty}^{1/2} \\
\leq M_1^{1/2} M_2^{1/2}, \quad (345)
\]
where \( L^\infty = L^\infty(G \times S \times I) \).
We assume that functions $S_j : G \times I \to \mathbb{R}$, $j = 2, 3$, the so-called restricted stopping powers, satisfy the following assumptions:

\begin{align}
S_j & \in L^\infty(G \times I), \\
\frac{\partial S_j}{\partial E} & \in L^\infty(G \times I), \\
\kappa_j := \inf_{(x,E)\in G \times I} S_j(x,E) & > 0, \\
\nabla_x S_j & \in L^\infty(G \times I),
\end{align}

(346) (347) (348) (349)

Note that (348) implies that in $G \times I$,

\[
\frac{1}{S_j} \leq \frac{1}{\kappa_j}.
\]

(350)

We point out that the assumption (349) will, in fact, be needed only in section 4.3 when considering (a special case of) the problem (335)-(338) with in the context of the theory of evolution operators (see Theorem 4.12, Eq. (425)).

In order to prove some accretivity properties for the scattering-collision operator $\Sigma - K : L^2(G \times S \times I)^3 \to L^2(G \times S \times I)^3$ (or, equivalently, dissipativity properties for the operator $-\Sigma + K$) we assume that the cross-sections $\Sigma_j$, $\sigma_{jk}$, satisfy the following condition: There exists $c \geq 0$ such that for a.e. $(x,\omega,E) \in G \times S \times I$, and for every $j = 1, 2, 3$,

\[
\Sigma_j(x,\omega,E) - 3 \sum_{k=1}^{3} \int_{S \times I} \sigma_{jk}(x,\omega,\omega',E,E')d\omega'dE' \geq c,
\]

(351) and

\[
\Sigma_j(x,\omega,E) - 3 \sum_{k=1}^{3} \int_{S \times I} \sigma_{kj}(x,\omega',\omega,E,E')d\omega'dE' \geq c.
\]

(352)

(See [20, pp. 241] for one particle, [70] for coupled system, and [71] within $L^1$-theory.)

The following accretivity result holds (see Lemma 3.3).

**Theorem 4.2** Let the assumptions (339), (340), (351) and (352) be valid. Then for every $\psi \in L^2(G \times S \times I)^3$,

\[
\langle (\Sigma - K)\psi, \psi \rangle_{L^2(G \times S \times I)^3} \geq c \|\psi\|_{L^2(G \times S \times I)^3}^2.
\]

(353)

Notice that the estimate (353) is equivalent to the property that for every $\lambda > 0$, and $\psi \in L^2(G \times S \times I)^3$,

\[
\|(\lambda I - (-\Sigma + K + cI))\psi\|_{L^2(G \times S \times I)^3} \geq \lambda \|\psi\|_{L^2(G \times S \times I)^3};
\]

(354)

which means that the operator $-\Sigma + K + cI : L^2(G \times S \times I)^3 \to L^2(G \times S \times I)^3$ is dissipative ([25, Section II.3.b], or [53, Section 1.4]).

**Remark 4.3** In certain situations one may assume that the restricted cross-sections are of the form

\[
\sigma_{kj}(x,\omega',\omega,E',E) = \bar{\sigma}_{kj}(x,\omega',\omega,E)\delta(E - E'),
\]

(355)
where $\delta$ is the Dirac measure at zero. Hence $\sigma_{kj} \not\in L^\infty(G \times S^2 \times I^2)$, but the collision operators are simpler,

$$(K_j \psi)(x, \omega, E) = \sum_{k=1}^{3} \int_{S} \sigma_{kj}(x, \omega', \omega, E) \psi_k(x, \omega', E) d\omega',$$  

(356)

where $\psi = (\psi_1, \psi_2, \psi_3) \in L^2(G \times S \times I)^3$. In the assumptions, (330) and (351), (352), the integrals $\int_{S \times I} \sigma_{jk}(x, \omega', \omega, E') d\omega' dE'$ (resp. $\int_{S \times I} \sigma_{jk}(x, \omega, \omega', E') d\omega' dE'$) over $S \times I$ are to be replaced with $\int_{S} \sigma_{kj}(x, \omega', \omega, E) d\omega'$ (resp. $\int_{S} \sigma_{jk}(x, \omega, \omega', E) d\omega'$) over $S$.

For the coupled BTE system [1], [2] we formulate the following result which is a slight modification of results given in [71]. Note that it is valid only for Schur collision operators and hence it does not govern completely the particle transport including charged particles, such as applications in radiation therapy.

**Theorem 4.4** Suppose that the assumptions (330), (340), (351), (352) are valid, and that $c$ is strictly positive. Then for every $f \in L^2(G \times S \times I)^3$ and $g \in T^2_{\tau_-}(\Gamma_-)^3$ the following assertions hold.

(i) The boundary value problem

$$\omega \cdot \nabla_x \psi_j + \Sigma_j \psi_j - K_j \psi = f_j,$$

$$\psi_j|_{\Gamma_-} = g_j,$$  

(357)

for $j = 1, 2, 3$, has a unique solution $\psi \in W^2(G \times S \times I)^3$.

(ii) There exists a constant $C > 0$ such that

$$\|\psi\|_{W^2(G \times S \times I)^3} \leq C \left( \|f\|_{L^2(G \times S \times I)^3} + \|g\|_{T^2_{\tau_-}(\Gamma_-)^3} \right).$$  

(358)

(iii) If $f \geq 0$ and $g \geq 0$, then $\psi \geq 0$, i.e. the solution $\psi$ is non-negative for non-negative data $f, g$.

**Proof.** The assertions follow from the considerations expressed in [71] noting that in Lemma 5.8 (see its proof) of [71] we actually have

$$\|Lg\|_{L^2(G \times S \times I)} \leq \|g\|_{T^2_{\tau_-}(\Gamma_-)}; \text{ for } g \in T^2_{\tau_-}(\Gamma_-).$$

We omit details here. $\square$

The corresponding result for time-dependent coupled system of BTEs has been proven in [71] as well.

### 4.1. Existence of Solutions Based on Variational Formulation

As before, we perform a change of unknown functions, by setting

$$\phi_j := e^{CE} \psi_j, \quad j = 1, 2, 3,$$  

(359)

where the constant $C$ will be fixed below. This transforms the problem (335)-(338) into an equivalent form, with transport equation on $G \times S \times I$,

$$\omega \cdot \nabla_x \phi_1 + \Sigma_1 \phi_1 - K_{1,C} \phi = f_1,$$  

(360)

$$-\frac{\partial (S_j \phi_j)}{\partial E} + \omega \cdot \nabla_x \phi_j + CS_j \phi_j + \Sigma_j \phi_j - K_{j,C} \phi = f_j, \quad j = 2, 3,$$  

(361)

boundary condition on $\Gamma_-,$

$$\phi|_{\Gamma_-} = g.$$  

(362)
and initial condition on $G \times S$,

$$\phi_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3,$$

(363)

where $\phi = (\phi_1, \phi_2, \phi_3)$ and

$$f_j = e^{C_e} f_j, \quad g_j = e^{C_e} g_j, \quad j = 1, 2, 3.$$

The operator $K_C = (K_{1,C}, K_{2,C}, K_{3,C})$ in (360), (361) above is given by

$$K_{j,C}\phi := \sum_{k=1}^{3} \int_{S \times I} \sigma_{kj,C}(x, \omega', \omega, E, E') \phi_k(x, \omega', E') d\omega' dE',$$

(364)

where the corresponding differential cross-sections are

$$\sigma_{kj,C}(x, \omega', \omega, E', E) := \sigma_{kj}(x, \omega', \omega, E', E) e^{C(E - E')}, \quad j, k = 1, 2, 3.$$

(365)

**Remark 4.5** We could perform a more refined change of the unknown functions by setting $\phi_1 := \psi_1$, $\phi_j := e^{C_j} \psi_j$. In this case, the modified differential cross-sections $\sigma_{kj,C}$ would be

$$\sigma_{11,C} = \sigma_{11}, \quad \sigma_{1k,C} = \sigma_{1k} e^{C_k E}, \quad k = 2, 3,$$

$$\sigma_{k1,C} = \sigma_{k1} e^{-C_k E'}, \quad \sigma_{kj,C} = \sigma_{kj} e^{C_k E - C_k E'}, \quad j, k = 2, 3.$$

We omit further considerations of such refinement in this work.

In the rest of the section, work under the assumptions (339), (340), (346), (347), (348) (as already mentioned, the assumption (349) will be needed only in section 4.3) and suppose furthermore that (for $C$ given below in (370) and) for some $c \geq 0$ the estimates

$$\Sigma_j(x, \omega, E) - \sum_{k=1}^{3} \int_{S \times I} \sigma_{kj,C}(x, \omega', \omega, E, E') d\omega' dE' \geq c,$$

(366)

$$\Sigma_j(x, \omega, E) - \sum_{k=1}^{3} \int_{S \times I} \sigma_{kj,C}(x, \omega', \omega, E', E) d\omega' dE' \geq c,$$

(367)

hold for a.e. $(x, \omega, E) \in G \times S \times I$.

**Remark 4.6** Note that if $\sigma_{kj}$ were (cf. Remark 4.3) of the form $\sigma_{kj}(x, \omega', \omega, E', E) = \tilde{\sigma}_{kj}(x, \omega', \omega, E) \delta(E - E')$ then

$$\int_{S \times I} \sigma_{kj,C}(x, \omega', \omega, E', E) d\omega' dE' = \int_{S} \tilde{\sigma}_{kj}(x, \omega', \omega, E) d\omega',$$

for any $C$, and hence the conditions (366), (367) would be independent of $C$. All the considerations below, after obvious adaptations, are valid for this simplified case.

At first, we apply the variational formulations to deduce existence of solutions. Recall that the inner product in $L^2(G \times S \times I)^3$ is given by

$$\langle \phi, v \rangle_{L^2(G \times S \times I)^3} = \sum_{j=1}^{3} \langle \phi_j, v_j \rangle_{L^2(G \times S \times I)} ,$$

and analogously in other products of inner product spaces. Integrating by parts and applying the Green’s formula (35) we find (similarly as in section 3.2) that the
bilinear form $B(\cdot, \cdot) : C^1(\overline{G} \times S \times I)^3 \times C^1(\overline{G} \times S \times I)^3 \to \mathbb{R}$ and the linear form $F : C^1(\overline{G} \times S \times I)^3 \to \mathbb{R}$ corresponding to the problem (366)-(363) are

$$B(\phi, v) = \sum_{j=2,3} \left( \phi_j, S_j \frac{\partial v_j}{\partial E} \right)_{L^2(G \times S \times I)} - \left( \phi, \omega \cdot \nabla_x v \right)_{L^2(G \times S \times I)}$$

$$+ \sum_{j=2,3} C \left( \phi_j, S_j v_j \right)_{L^2(G \times S \times I)} + \left( \phi, (\Sigma^* - K_C) v \right)_{L^2(G \times S \times I)}$$

$$+ \langle \gamma_+(\phi), \gamma_+(v) \rangle_{T^2(\Gamma_+)} + \sum_{j=2,3} \langle \phi_j(\cdot, \cdot, 0), S_j(\cdot) v_j(\cdot, \cdot, 0) \rangle_{L^2(G \times S)}, \quad (368)$$

and

$$F(v) = \langle f, v \rangle_{L^2(G \times S \times I)^3} + \langle g, \gamma_-(v) \rangle_{T^2(\Gamma_-)^3}. \quad (369)$$

For $j = 2, 3$, let

$$q_j := \frac{1}{2} \text{ess sup} \left( \frac{\partial S_j}{\partial E}(x, E) \right).$$

Moreover, define

$$C_j := \max \left\{ q_j, 0 \right\}, \quad j = 2, 3,$$

$$C := \max \{ C_1, C_2 \}. \quad (370)$$

The appropriate Hilbert spaces are defined as

$$\mathcal{H} := H \times H_1 \times H_1,$$

$$\mathcal{H} := \mathfrak{W}^2(G \times S \times I) \times H_2 \times H_2,$$

(371) where the space $\mathfrak{W}^2(G \times S \times I)$ was defined just before Corollary 2.21. Recall from section 2.1 that the elements of $\mathcal{H}$ are of the form

$$\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3) = ( (\phi_1, q_1), (\phi_2, q_2, p_{02}, p_{m2}), (\phi_3, q_3, p_{03}, p_{m3})),

with $\phi_i \in L^2(G \times S \times I), q_i \in T^2(\Gamma)$ for $i = 1, 2, 3,$ and $p_{0j}, p_{mj} \in L^2(G \times S)$ for $j = 2, 3.$ Moreover, $\mathcal{H} \subset \mathcal{H}$ through the continuous embedding

$$v = (v_1, v_2, v) \mapsto ( (v_1, \gamma(v_1)), q(v_2), q(v_3)),

q(v_j) := (q_1, q_2, q_3, q_i(\cdot, \cdot, 0), v_1(v_2, \cdot, v_3)), \quad j = 2, 3.$$

(372) In spaces $\mathcal{H}$ and $\mathcal{H}$ we use respectively the inner products

$$\langle \phi, v \rangle_{\mathcal{H}} = \langle \phi_1, v_1 \rangle_H + \sum_{j=2,3} \langle \phi_j, v_j \rangle_{H_1}$$

and

$$\langle \phi, v \rangle_{\mathcal{H}} = \langle \phi_1, v_1 \rangle_{\mathfrak{W}^2(G \times S \times I)} + \sum_{j=2,3} \langle \phi_j, v_j \rangle_{H_2}. \quad (373)$$

The bilinear form $B : C^1(\overline{G} \times S \times I)^3 \times C^1(\overline{G} \times S \times I)^3 \to \mathbb{R}$ has the following boundedness and coercivity properties.

**Theorem 4.7** Suppose that the assumptions (339)-(340), (344), (347), (348) are valid and that (366), (367) hold for $c > 0$ and for $C$ given in (370). Then there exists $M > 0$ such that

$$|B(\phi, v)| \leq M \|\phi\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \quad \forall \phi, v \in C^1(\overline{G} \times S \times I)^3,$$
and 
\[ \mathbf{B}(\phi, \phi) \geq c' \|\phi\|_{\mathcal{H}}^2 \quad \forall \phi \in C^1(\mathcal{G} \times S \times I)^3, \]  
(374)
where
\[ c' := \min\left\{ \frac{1}{2}, \frac{\kappa_2}{2}, \frac{\kappa_3}{2}, c\right\}. \]
(375)
In addition, for all \( v \in C^1(\mathcal{G} \times S \times I)^3 \),
\[ |\mathbf{F}(v)| \leq (\|f\|_{L^2(\mathcal{G} \times S \times I)^3} + \|g\|_{T^2(\Gamma_-)^3}) \|v\|_{\mathcal{H}}. \]
(376)

**Proof.** The boundedness \( (373) \) can be seen as in the proof of Theorem 3.5. The assumptions \( (366), (367) \) imply by Theorem 4.2 that
\[ ((\Sigma - K_C)\phi, \phi)_{L^2(\mathcal{G} \times S \times I)^3} \geq c\|\phi\|_{L^2(\mathcal{G} \times S \times I)^3}^2. \]
(377)
Hence we see as in the proof of Theorem 3.5 that for \( C = \max\{C_1, C_2\} \) the coercitivity \( (374) \) holds with the stated \( c' > 0 \). The estimate \( (376) \) is immediate (see \( (191) \)) and so the proof is finished. \( \square \)

Due to the above theorem, the bilinear form \( \mathbf{B}(\cdot, \cdot) \) has a unique bounded extension \( \tilde{\mathbf{B}}(\cdot, \cdot) : \mathcal{H} \times \tilde{\mathcal{H}} \to \mathbb{R} \) which satisfies
\[ |\tilde{\mathbf{B}}(\tilde{\phi}, v)| \leq M \|\tilde{\phi}\|_{\mathcal{H}} \|v\|_{\tilde{\mathcal{H}}} \quad \forall \tilde{\phi} \in \mathcal{H}, \ v \in \tilde{\mathcal{H}}, \]
(378)
and
\[ \tilde{\mathbf{B}}(v, v) \geq c' \|v\|_{\tilde{\mathcal{H}}}^2 \quad \forall v \in \tilde{\mathcal{H}}. \]
(379)
Likewise, \( (376) \) implies that the linear form \( \mathbf{F} \) has a unique bounded extension \( \mathcal{H} \to \mathbb{R} \), which we still denote by \( \mathbf{F} \). The variational equation corresponding to the problem \( (360), (363) \) is
\[ \tilde{\mathbf{B}}(\tilde{\phi}, v) = \mathbf{F}(v) \quad \forall v \in \tilde{\mathcal{H}}. \]
(380)

For the coupled BTE we have the following variational existence theorem, to be compared with Theorem 3.8 for single particle transport.

**Theorem 4.8** Suppose that the assumptions \( (339), (340), (346), (347), (348) \) are valid, and that \( (360), (367) \) hold for \( c > 0 \) and for \( C \) given in \( (370) \). Let \( f \in L^2(\mathcal{G} \times S \times I)^3 \) and \( g \in T^2(\Gamma_-)^3 \). Then the following assertions hold.

(i) The variational equation
\[ \tilde{\mathbf{B}}(\tilde{\phi}, v) = \mathbf{F}(v) \quad \forall v \in \tilde{\mathcal{H}}, \]
(381)
has a solution \( \tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) \in \mathcal{H} \). Writing \( \tilde{\phi}_1 = (\phi_1, q_1), \tilde{\phi}_j = (\phi_j, q_j, p_{0j}, p_{mj}), \) \( j = 2, 3 \), and \( \phi = (\phi_1, \phi_2, \phi_3) \in L^2(\mathcal{G} \times S \times I)^3 \), then \( \phi \in \mathcal{H}_{p}(\mathcal{G} \times S \times I^3) \) (see \( (393) \)) and it is a weak (distributional) solution of the system of equations \( (360), (361) \) and \( \phi_1 \in W^2(\mathcal{G} \times S \times I) \).

(ii) Suppose that additionally the assumption TC holds (p. 19). Then a solution \( \phi \) of the equations \( (360), (361) \) obtained in part (i) is a solution of the problem \( (360), (363) \).

(iii) Under the assumptions imposed in part (ii), any solution \( \phi \) of the problem \( (360), (363) \) that further satisfies
\[ \phi_{\Gamma_+} \in T^2(\Gamma_+)^3 \quad \text{and} \quad \phi(\cdot, \cdot, 0) \in L^2(\mathcal{G} \times S)^3, \]
(382)
is unique and obeys the estimate

\[ \| \phi \|_{L^c} \leq \frac{1}{c'} (\| f \|_{L^2(G \times S \times I)^3} + \| g \|_{T^2(\Gamma^-)^3}) , \]  

where \( c' \) is given in (375).

**Proof.** The proofs of items (i)-(iii) are analogous to the proofs of the corresponding items in Theorem 3.8. Note that the claim that \( \phi_1 \in W^2(G \times S \times I) \) in part (i) follows from noticing that by (369) we have \( \omega \cdot \nabla \phi_1 = -\Sigma_1 \phi_1 + K_1 \phi_1 + \phi_1 \), and the right hand side belongs to \( L^2(G \times S \times I) \).

The problem (335)-(338) also admits a variational formulation. Define bilinear \( \tilde{B}_0(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \) by

\[
\tilde{B}_0(\tilde{\psi}, v) = \sum_{j=2,3} \left( \langle \tilde{\psi}_j, S_j \frac{\partial \psi}{\partial E} \rangle_{L^2(G \times S \times I)} - \langle \psi, \omega \cdot \nabla v \rangle_{L^2(G \times S \times I)} + \langle \psi, (\Sigma^* - K^*) v \rangle_{L^2(G \times S \times I)} \right.
+ \langle q, \gamma_+(v) \rangle_{T^2(\Gamma^+)^3} + \sum_{j=2,3} \langle p_{0j}, S_j \psi \rangle_{L^2(G \times S)} \right) ,
\]

where \( \tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3) \), \( \tilde{\psi}_1 = (\psi_1, q_1) \), \( \tilde{\psi}_j = (\psi_j, q_j, p_{0j}, p_{mj}), \) \( j = 2, 3 \), and \( \psi = (\psi_1, \psi_2, \psi_3) \), \( q = (q_1, q_2, q_3) \). In other words, \( \tilde{B}_0 \) is the unique extension of (368) with \( C = 0 \). Then by part (ii) of Theorem 3.8, a given \( \tilde{\psi} \in W^2(G \times S \times I) \times W^2(G \times S \times I)^2 \) is a solution of the problem (335)-(338) if (and only if) the variational equation

\[ \tilde{B}_0(\tilde{\psi}, v) = F_0(v), \quad \forall v \in \mathcal{H}, \]

holds. Here \( \tilde{\psi} \) is identified with \( \tilde{\psi} \in \mathcal{H} \) through the mapping given in (372), and

\[ F_0(v) = \langle f, v \rangle_{L^2(G \times S \times I)^3} + \langle g, \gamma_- (v) \rangle_{T^2(\Gamma^-)^3}. \]

We have the following corollary for the original problem.

**Corollary 4.9** Suppose that the assumptions of Theorem 3.8 are valid. Let \( f \in L^2(G \times S \times I)^3 \) and \( g \in T^2(\Gamma_-)^3 \). Then the following assertions hold.

(i) The variational equation

\[ \tilde{B}_0(\tilde{\psi}, v) = F_0(v) \quad \forall v \in \mathcal{H} \]  

has a solution \( \tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3) \in \mathcal{H} \). Writing \( \tilde{\psi}_1 = (\psi_1, q_1), \tilde{\psi}_j = (\psi_j, q_j, p_{0j}, p_{mj}), \) \( j = 2, 3 \), and \( \psi = (\psi_1, \psi_2, \psi_3) \in L^2(G \times S \times I)^3 \), then \( \psi \in H_p(G \times S \times I) \) (see (393)) and it is a weak (distributional) solution of the system of equations (335)-(338), and \( \psi_1 \in W^2(G \times S \times I) \).

(ii) Suppose that additionally the assumption TC holds (p. 49). Then a solution \( \psi \) of the equations (335), (336) obtained in part (i) is a solution of the problem (335)-(338).

(iii) Under the assumptions imposed in part (ii), any solution \( \psi \) of the problem (335)-(338) that further satisfies

\[ \psi|_{\Gamma^+} \in T^2(\Gamma^+)^3 \quad \text{and} \quad \psi(\cdot, \cdot, 0) \in L^2(G \times S)^3, \]

is unique and obeys the estimate

\[ \| \psi \|_{L^c} \leq \frac{e^{C_E m}}{c'} \left( \| f \|_{L^2(G \times S \times I)^3} + \| g \|_{T^2(\Gamma^-)^3} \right). \]  

(Recall that \( C \) is defined in (370), \( c' \) in (375) and that \( E_m \) is the cutoff energy.)
Proof. A solution $\psi$ of the problem (335)-(338) is obtained from a solution $\phi$ of the problem (360)-(363) by taking $\psi = e^{-C_0} \phi$. Note that if $\phi_1 \in W^2(G \times S \times I)$, then $\psi_1 \in W^2(G \times S \times I)$ as well. The rest of the proof proceeds in exactly the same way as that for Corollary 3.10 (of course, one uses Theorem 4.8 instead of Theorem 3.8). \hfill \Box

4.2. Existence of Solutions Based on $m$-dissipativity. The method of section 3.3 can extended to the case if coupled system in a straightforward manner. Let us state what the problem to be solved for $3.3$ can extended to the case if coupled system in a straightforward manner. Let $\phi$ be in $L^2(G \times S \times I)^3$, and $g \in T^2(\Gamma_-) \times H^1(I, T^2(\Gamma_-))^2$, find $\phi = (\phi_1, \phi_2, \phi_3) \in L^2(G \times S \times I)^3$ which satisfies the system of equations of $G \times S \times I$,

$$
\omega \cdot \nabla \phi_1 + \Sigma_1 \phi_1 - K_{1,C} \phi = f_1,
$$

(389)

$$
- \frac{\partial (S_j \phi_j)}{\partial E} + \omega \cdot \nabla \phi_j + CS_j \phi_j + \Sigma_j \phi_j - K_{j,C} \phi = f_j, \quad j = 2, 3,
$$

(390)

the boundary condition on $\Gamma_-$,

$$
\phi|_{\Gamma_-} = g,
$$

(391)

and the initial condition on $G \times S$,

$$
\phi_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3.
$$

(392)

Let

$$
P_1(x, \omega, E, D) \phi_1 := \omega \cdot \nabla_x \phi_1,
$$

$$
P_{C,j}(x, \omega, E, D) \phi_j := - \frac{\partial (S_j \phi_j)}{\partial E} + \omega \cdot \nabla_x \phi_j + CS_j \phi_j, \quad j = 2, 3,
$$

$$
P_C(x, \omega, E, D) \phi := (P_1(x, \omega, E, D) \phi_1, P_{C,2}(x, \omega, E, D) \phi_2, P_{C,3}(x, \omega, E, D) \phi_3).
$$

When $C = 0$ we write $P(x, \omega, E, D) := P_0(x, \omega, E, D)$, and define

$$
\mathcal{H}_P(G \times S \times I^c) := \{ \psi \in L^2(G \times S \times I)^3 \mid P(x, \omega, E, D) \psi \in L^2(G \times S \times I)^3 \text{ in the weak sense} \}.
$$

(393)

The operator $P_{C,0}$ is defined in the same way as $P_{C,0}$ in section 3.3, namely it is the smallest closed extension (closure) of $P_{C,0}$, where

$$
D(P_{C,0}) := \{ \phi \in \dot{W}^2(G \times S \times I) \times (\dot{W}^2(G \times S \times I) \cap H^1(I, L^2(G \times S)))^2 \mid \phi|_{\Gamma_-} = 0, \quad \phi_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3 \}
$$

$$
P_{C,0} \phi := P_C(x, \omega, E, D) \phi.
$$

Using these notations, the problem (389)-(392) with $g = 0$, in the strong sense, is equivalent to

$$
(\tilde{P}_{C,0} + \Sigma - K_C) \phi = f,
$$

where $\phi \in D(\tilde{P}_{C,0})$. 

We assume that the stopping powers for $j = 2, 3$ satisfy
\[ S_j \in C^2(I, L^\infty(G)), \]
\[ \kappa_j := \inf_{(x,E) \in I \times 1} S_j(x, E) > 0, \tag{395} \]
\[ \nabla_x S_j \in L^\infty(G \times I). \tag{396} \]

We give here only the following concluding result, which can be proven using the methods of section 3.3 and those of [71, Section 5.3]. Recall that by the conventions adopted above, $f_j = e^{CE} f_j$ and $g_j = e^{CE} g_j$, where $C$ is as defined in (370).

**Theorem 4.10** Suppose that the assumptions (339), (340), (366), (367) (with $c > 0$) and (394), (395), (396) are valid. Furthermore, suppose that $f \in L^2(G \times S \times I)^3$, and $g \in T^2(\Gamma_-) \times H^1(I, T^2(\Gamma'_-))^2$ is such that the compatibility condition
\[ g_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3, \tag{397} \]
holds. Then the problem (389)-(392) has a unique solution $\psi \in H^1_{\text{loc}}(G \times S \times I)$. In addition, there exists a constant $C_1 > 0$ such that an \textit{a priori} estimate
\[ \|\psi\|_{L^2(G \times S \times I)^3} \leq C_1 \left( \|f\|_{L^2(G \times S \times I)^3} + \|g\|_{T^2(\Gamma_-) \times H^1(I, T^2(\Gamma'_-))^2} \right), \tag{398} \]
holds.

### 4.3. Existence of Solutions Based on the Theory of Evolution Equations.

For collision operator of special type, the existence result based on the theory of evolution operators is valid also for the coupled system. One of the important features of this approach is that it yields more regularity for the solution.

We assume that the collision operator $K = (K_1, K_2, K_3)$ is of the form (see Remark 4.3)
\[ (K_j \psi)(x, \omega, E) = \sum_{k=1}^{3} \int_{S} \bar{\sigma}_{kj}(x, \omega', \omega, E) \psi_k(x, \omega', E) d\omega', \quad j = 1, 2, 3. \tag{399} \]

For a fixed $E \in I$ we define bounded linear operators $L^2(G \times S) \to L^2(G \times S)$ by
\[ (\Sigma_1(E) v)(x, \omega) := \Sigma_1(x, \omega, E) v(x, \omega), \]
\[ (\bar{\Sigma}_1(E) v)(x, \omega) := \int_{S} \bar{\sigma}_{11}(x, \omega', \omega, E) v(x, \omega') d\omega'. \]

In order to avoid ambiguity, below we often denote by $S'$ the sphere $S$ for the variable $\omega'$, while $S$ is reserved for $\omega$. For example, saying that $h$ is a function $S \to (S' \to \mathbb{R})$ (resp. $S' \to (S \to \mathbb{R})$) means that we would write $h(\omega)(\omega')$ (resp. $h(\omega')(\omega)$).

**Lemma 4.11** Suppose that
\[ \Sigma_1 \in C^1(I, L^\infty(G \times S)), \tag{400} \]
\[ \bar{\sigma}_{11} \in C^1(I, L^\infty(G \times S, L^1(S')) \cap C^1(I, L^\infty(G \times S', L^1(S))), \tag{401} \]
\[ \Sigma_1 \geq 0, \quad \bar{\sigma}_{11} \geq 0, \tag{402} \]
and that for some \( c > 0 \) the following hold a.e. \( (x, \omega, E) \in G \times S \times I \),

\[
\Sigma_1(x, \omega, E) - \int_S \tilde{\sigma}_{11}(x, \omega, \omega', E) d\omega' \geq c, \tag{403}
\]

\[
\Sigma_1(x, \omega, E) - \int_S \tilde{\sigma}_{11}(x, \omega', \omega, E) d\omega' \geq c. \tag{404}
\]

Then for every \( q \in C^4(I, L^2(G \times S)) \) and every \( E \in I \) the problem

\[
\omega \cdot \nabla_x v + \Sigma_1(E) v - \mathcal{K}_1(E) v = q(E),
\]

\[v|_{\partial I'} = 0,\tag{405}\]

has a unique solution \( v = v(E) \in W^{2,0}_2(G \times S) \), and

\[
\|v(E)\|_{L^2(G \times S)} \leq \frac{1}{c} \|q(E)\|_{L^2(G \times S)}, \quad \forall E \in I. \tag{406}
\]

In addition, \( v(\cdot) : I \to L^2(G \times S) \) belongs to \( C^1(I, L^2(G \times S)) \) and for all \( E \in I \),

\[
\left\| \frac{\partial v}{\partial E}(E) \right\|_{L^2(G \times S)} \leq \frac{1}{c} \left( \left\| \frac{\partial q}{\partial E}(E) \right\|_{L^2(G \times S)} + \frac{1}{c} \left\| \frac{\partial \Sigma_1}{\partial E}(E) \right\|_{L^\infty(G \times S)} \right) \|q(E)\|_{L^2(G \times S)}
\]

\[
+ \frac{1}{c} \left\| \frac{\partial \mathcal{K}_1}{\partial E}(E) \right\|_{L^2(G \times S)} \|q(E)\|_{L^2(G \times S)}, \tag{407}
\]

where

\[
\left( \frac{\partial \mathcal{K}_1}{\partial E}(E)v \right)(x, \omega) := \int_S \frac{\partial \tilde{\sigma}_{11}}{\partial E}(x, \omega', \omega, E)v(x, \omega') d\omega'.
\]

**Proof.** That the problem \([405]\) has, for every fixed \( E \in I \), a unique solution \( v = v(E) \), and that the estimate \([406]\) holds can be proven similarly to \([71, \text{Corollary } 5.15]\) for the existence of solutions, see also \([20, \text{Lemma 4, p. 241}]\).

A. At first we show that \( v \in C^1(I, L^2(G \times S)) \). Let \( E_1, E_2 \in I \). Since \( v(E) \in W^{2,0}_2(G \times S) \) for all \( E \in I \), it follows that \( (v(E_1) - v(E_2))|_{\partial I'} = 0 \). Moreover, \( v(E_1) - v(E_2) \) satisfies the equation

\[
\omega \cdot \nabla_x \left( v(E_1) - v(E_2) \right) + \Sigma_1(E_1) \left( v(E_1) - v(E_2) \right) - \mathcal{K}_1(E_1) \left( v(E_1) - v(E_2) \right)
\]

\[= q(E_1) - q(E_2) - (\Sigma_1(E_1) - \Sigma_1(E_2)) v(E_2) + (\mathcal{K}_1(E_1) - \mathcal{K}_1(E_2)) v(E_2). \tag{408}\]

Therefore, one can apply the estimate \([406]\), obtaining

\[
\|v(E_1) - v(E_2)\|_{L^2(G \times S)}
\]

\[
\leq \frac{1}{c} \left( \|q(E_1) - q(E_2)\|_{L^2(G \times S)} + \|\Sigma_1(E_1) - \Sigma_1(E_2)\|_{L^2(G \times S)} \right)
\]

\[
+ \left\| \mathcal{K}_1(E_1) - \mathcal{K}_1(E_2) \right\|_{L^2(G \times S)} \|v(E_2)\|_{L^2(G \times S)} \right) \tag{409}
\]

where (see \([292]\) or \([313]\))

\[
\|\mathcal{K}_1(E_1) - \mathcal{K}_1(E_2)\| \leq \|\tilde{\sigma}_{11}(E_1) - \tilde{\sigma}_{11}(E_2)\|_{L^2(G \times S, L^1(S'))}^{1/2} \|\tilde{\sigma}_{11}(E_1) - \tilde{\sigma}_{11}(E_2)\|_{L^2(G \times S', L^1(S))}^{1/2}. \tag{410}\]

The continuity of \( v : I \to L^2(G \times S) \) follows immediately from these estimates and assumptions \([100], [101]\).
B. Next we verify that \( v \in C^1(I, L^2(G \times S)) \) and that the estimate \( [107] \) holds. For a map \( k : G \times S \times I \to \mathbb{R} \), which we identify as a map \( I \to (G \times S \to \mathbb{R}); \)
\( k(E)(x, \omega) = k(x, \omega, E) \) whenever appropriate, define for \( h \neq 0 \) small enough
\[
(\delta_h k)(E) := \frac{k(E + h) - k(E)}{h}
\]
where at \( E \)-boundaries \( E = 0 \) or \( E = E_m \) we take \( h > 0 \) and \( h < 0 \), respectively. Since \( v(E') \in \dot{W}^{2,0}(G \times S) \) for all \( E' \in I \), we have \( (\delta_h v)(E)|_{E' = 0} = 0 \), and from the equation \( [405] \) we obtain (from \( [408] \), with \( v \)
\[
\omega \cdot \nabla_x ((\delta_h v)(E)) + \Sigma_1(E) (\delta_h v)(E) - \overline{K}_1(E) (\delta_h v)(E) = (\delta_h q)(E) - (\delta_h \Sigma_1)(E) v(E + h) + (\delta_h \overline{K}_1)(E) v(E + h),
\]
where for any \( w \in L^2(G \times S), \)
\[
(\delta_h \overline{K}_1)(E) w := \int_S \frac{\tilde{\sigma}_{11}(x, \omega', \omega, E + h) - \tilde{\sigma}_{11}(x, \omega', \omega, E)}{h} w(x, \omega') d\omega'
\]
\[
= \int_S (\delta_h \tilde{\sigma}_{11})(E)(x, \omega', \omega) w(x, \omega') d\omega'.
\]
To show that the limit \( \lim_{n \to 0} (\delta_h v)(E) \) exists in \( L^2(G \times S) \) it suffices to verify that \( \lim_{n \to \infty} (\delta_h v)(E) \) exists in \( L^2(G \times S) \) for any sequence \( \{h_n\} \) tending to zero as \( n \to \infty \), or equivalently that \( \{(\delta_h v)(E)\} \) is a Cauchy sequence in \( L^2(G \times S) \) for any sequence \( \{h_n\} \).

We see by \( [411] \) that for \( n, m \in \mathbb{N}, \)
\[
\omega \cdot \nabla_x ((\delta_h v)(E) - (\delta_h v)(E')) + \Sigma_1(E) ((\delta_h v)(E) - (\delta_h v)(E')) - \overline{K}_1(E) ((\delta_h v)(E) - (\delta_h v)(E')) = (\delta_h q)(E) - (\delta_h \Sigma_1)(E) v(E + h_n) - (\delta_h \Sigma_1)(E) v(E + h_m))
\]
\[
+ ((\delta_h \overline{K}_1)(E) v(E + h_n) - (\delta_h \overline{K}_1)(E) v(E + h_m))
\]
\[
= \phi_{h_n, h_m}(E).
\]

The following estimates hold (see \( [409], [410] \))
\[
||((\delta_h \Sigma_1)(E) v(E + h_n) - (\delta_h \Sigma_1)(E) v(E + h_m))||_{L^2(G \times S)}
\]
\[
\leq ||(\delta_h \Sigma_1)(E) - (\delta_h \Sigma_1)(E)||_{L^\infty(G \times S)} ||v(E + h_n)||_{L^2(G \times S)}
\]
\[
+ ||(\delta_h \Sigma_1)(E)||_{L^\infty(G \times S)} ||v(E + h_n) - v(E + h_m)||_{L^2(G \times S)},
\]
and
\[
||((\delta_h \overline{K}_1)(E) v(E + h_n) - (\delta_h \overline{K}_1)(E) v(E + h_m))||_{L^2(G \times S)}
\]
\[
\leq ||(\delta_h \overline{K}_1)(E) - (\delta_h \overline{K}_1)(E)||_{L^2(G \times S)}
\]
\[
+ ||(\delta_h \overline{K}_1)(E)||_{L^2(G \times S)} ||v(E + h_n) - v(E + h_m)||_{L^2(G \times S)},
\]
where
\[
||((\delta_h \overline{K}_1)(E)|| \leq ||(\delta_h \tilde{\sigma}_{11})(E)||_{L^\infty(G \times S, L^1(S'))}^{1/2} ||(\delta_h \tilde{\sigma}_{11})(E)||_{L^\infty(G \times S', L^1(S))}^{1/2},
\]
and
\[
||((\delta_h \overline{K}_1)(E) - (\delta_h \overline{K}_1)(E)||
\]
\[
\leq ||(\delta_h \tilde{\sigma}_{11})(E_1) - (\delta_h \tilde{\sigma}_{11})(E_2)||_{L^\infty(G \times S, L^1(S'))}^{1/2} ||(\delta_h \tilde{\sigma}_{11})(E_1) - (\delta_h \tilde{\sigma}_{11})(E_2)||_{L^\infty(G \times S', L^1(S))}^{1/2}.
\]
Therefore, since we assume \((400), (401)\) and \(q \in C^1(I, L^2(G \times S))\), and since \(v \in C(I, L^2(G \times S))\) by part A of the proof, we find that \(f_{h, n, m}(E)\) in \((412)\) converges to zero in \(L^2(G \times S)\) when \(n, m \to \infty\). This fact combined with the estimate obtained by applying \((406)\),

\[
\|\delta_h v(E) - (\delta_{h, n} v(E))\|_{L^2(G \times S)} \leq \frac{1}{c} \|f_{h, n, m}(E)\|_{L^2(G \times S)},
\]

shows that \(\{(\delta_h v(E))\}\) is a Cauchy sequence, and so \(\frac{\partial v}{\partial E}(E)\) exists (in \(L^2(G \times S)\)) for every \(E \in I\).

Applying the estimate \((406)\) to \((\delta_h v)(E)\) which satisfies \((411)\), we get

\[
\|\delta_h v(E)\|_{L^2(G \times S)} \leq \frac{1}{c} \left( \|\delta_h q(E)\|_{L^2(G \times S)} + \|\delta_h \Sigma_1(E)\|_{L^\infty(G \times S)} \|v(E + h)\|_{L^2(G \times S)} + \|\delta_h K_1(E)\| \|v(E + h)\|_{L^2(G \times S)} \right).
\]

Under the standing assumptions (and the fact that \(v\) is continuous), the inequality \((413)\) gives in the limit \(h \to 0\),

\[
\left\|\frac{\partial v}{\partial E}(E)\right\|_{L^2(G \times S)} \leq \frac{1}{c} \left( \left\|\frac{\partial q}{\partial E}(E)\right\|_{L^2(G \times S)} + \left\|\frac{\partial \Sigma_1}{\partial E}(E)\right\|_{L^\infty(G \times S)} \left\|v(E)\right\|_{L^2(G \times S)} + \left\|\frac{\partial K_1}{\partial E}(E)\right\| \left\|v(E)\right\|_{L^2(G \times S)} \right),
\]

and thus, using \((406)\) once more, we obtain \((407)\) as claimed.

It remains to be shown that \(\frac{\partial v}{\partial E} \in C(I, L^2(G \times S))\). By letting \(h \to 0\) in equation \((411)\) we get that \(\omega \cdot \nabla_x \left(\frac{\partial v}{\partial E}(E)\right) \in L^2(G \times S)\) for all \(E \in I\), and

\[
\omega \cdot \nabla_x \left(\frac{\partial v}{\partial E}(E)\right) + \Sigma_1(E) \frac{\partial v}{\partial E}(E) - K_1(E) \left(\frac{\partial v}{\partial E}(E)\right) = \frac{\partial v}{\partial E}(E) - \frac{\partial \Sigma_1}{\partial E}(E)v(E) + \frac{\partial K_1}{\partial E}(E)v(E).
\]

In addition, \((\delta_h v)(E) \to \frac{\partial v}{\partial E}(E)\) in \(W^2(G \times S \times I)\) when \(h \to 0\), and therefore, since \((\delta_h v)(E)\big|_{E \to E'} = 0\), we conclude that \((\frac{\partial v}{\partial E}(E))\big|_{E \to E'} = 0\) (by applying inflow trace results; see e.g. Remark 2.9). Finally, an application of \((406)\) yields, as in \((408)\), \((409)\) (we write \(L^2 = L^2(G \times S)\), \(L^\infty = L^\infty(G \times S)\) in order to slightly compress the formulas),

\[
\left\|\frac{\partial v}{\partial E}(E_1) - \frac{\partial v}{\partial E}(E_2)\right\|_{L^2} \leq \frac{1}{c} \left( \left\|\frac{\partial q}{\partial E}(E_1) - \frac{\partial q}{\partial E}(E_2)\right\|_{L^2} + \left\|\frac{\partial \Sigma_1}{\partial E}(E_1) - \frac{\partial \Sigma_1}{\partial E}(E_2)\right\|_{L^\infty} \left\|v(E_2)\right\|_{L^2} + \left\|\frac{\partial \Sigma_1}{\partial E}(E_2)\right\| \left\|v(E_2) - v(E_1)\right\|_{L^2} + \left\|\frac{\partial K_1}{\partial E}(E_1) - \frac{\partial K_1}{\partial E}(E_2)\right\| \left\|v(E_2) - v(E_1)\right\|_{L^2} \right),
\]

where \(E_1, E_2 \in I\) and \(E_1 < E_2\).
and one can further estimate (see (110)),
\[ \left\| \frac{\partial K_1}{\partial E}(E_1) - \frac{\partial K_1}{\partial E}(E_2) \right\| \leq \left\| \frac{\partial \tilde{\sigma}_{11}}{\partial E}(E_1) - \frac{\partial \tilde{\sigma}_{11}}{\partial E}(E_2) \right\|^{1/2}_{L(S,S')} \left(\frac{\partial \tilde{\sigma}_{11}}{\partial E}(E_1) - \frac{\partial \tilde{\sigma}_{11}}{\partial E}(E_2) \right) \left(\frac{\partial \tilde{\sigma}_{11}}{\partial E}(E_1) - \frac{\partial \tilde{\sigma}_{11}}{\partial E}(E_2) \right)^{1/2}_{L(S,S')}, \]

where we have written \( L(S,S') := L^\infty(G \times S, L^1(S')) \) and \( L(S',S) := L^\infty(G \times S', L^1(S)) \). This implies the continuity of \( \frac{\partial K}{\partial E} \) that we were to demonstrate, and completes the proof.

We are ready to formulate, in the context of the theory of evolution operators, the key existence result for the coupled transport problem (335)-(338), assuming the collision operator \( K = (K_1, K_2, K_3) \) is of the (special) form (399).

**Theorem 4.12** Assume that the cross-sections \( \Sigma_j, \tilde{\sigma}_{jk}, \) where \( j, k = 1, 2, 3 \), satisfy
\[ \Sigma_j \in C^1(I, L^\infty(G \times S)), \quad (419) \]
\[ \tilde{\sigma}_{jk} \in C^1(I, L^\infty(G \times S, L^1(S'))) \cap C^1(I, L^\infty(G \times S', L^1(S))), \quad (420) \]
\[ \Sigma_j \geq 0, \quad \tilde{\sigma}_{jk} \geq 0, \quad (421) \]
and that for some \( c > 0 \),
\[ \Sigma_1(x, \omega, E) - \int_S \tilde{\sigma}_{11}(x, \omega, \omega', E) d\omega' \geq c, \quad (422) \]
\[ \Sigma_1(x, \omega, E) - \int_S \tilde{\sigma}_{11}(x, \omega', \omega, E) d\omega' \geq c, \quad (423) \]
for a.e. \( (x, \omega, E) \in G \times S \times I \). Furthermore, assume that the stopping powers \( S_j \), where \( j = 2, 3 \), satisfy
\[ S_j \in C^2(I, L^\infty(G)), \quad (424) \]
\[ \nabla_x S_j \in L^\infty(G \times I), \quad (425) \]
\[ \kappa_j := \inf_{(x,E) \in G \times I} S_j(x, E) > 0. \quad (426) \]

Let \( f \in C^1(I, L^2(G \times S)^3) \) and let \( g \in C^2(I, T^2(\Gamma^\prime)^3) \) which satisfies the compatibility condition
\[ g_j(E_m) = 0, \quad j = 2, 3. \]

Then the problem (335)-(338) has a unique solution \( \psi \in \tilde{W}^2(G \times S \times I) \times (C(I, \tilde{W}^2(G \times S)^2) \cap C^1(I, L^2(G \times S)^2)) \). In particular, \( \psi \in \tilde{W}^2(G \times S \times I) \times (\tilde{W}^2(G \times S \times I) \cap W^2(G \times S \times I))^2 \).

If in addition, for some \( c > 0 \), the inequalities
\[ \Sigma_j(x, \omega, E) - \sum_{k=1}^3 \int_S \tilde{\sigma}_{jk}(x, \omega, \omega', E) d\omega' \geq c, \quad (427) \]
\[ \Sigma_j(x, \omega, E) - \sum_{k=1}^3 \int_S \tilde{\sigma}_{kj}(x, \omega', \omega, E) d\omega' \geq c, \quad (428) \]
hold for \( j = 1, 2, 3 \) and for a.e. \( (x, \omega, E) \in G \times S \times I \), then the solution \( \psi \) satisfies the estimate (388).
At first we notice that by the assumption (420), for a.e. \((x, \omega, E) \in G \times S \times I,\)
\[
\sum_{k=1}^{3} \int_{S} \tilde{\sigma}_{jk}(x, \omega', \omega, E) d\omega' \leq \sum_{k=1}^{3} \sup_{E \in I} \|\tilde{\sigma}_{jk}(E)\|_{L^{\infty}(G \times S, L^{1}(S))} =: M_1 < \infty, \\
(429)
\]
\[
\sum_{k=1}^{3} \int_{S} \tilde{\sigma}_{jk}(x, \omega', \omega, E) d\omega \leq \sum_{k=1}^{3} \sup_{E \in I} \|\tilde{\sigma}_{jk}(E)\|_{L^{\infty}(G \times S', L^{1}(S))} =: M'_1 < \infty. \\
(430)
\]

We begin by treating the special case where \(g = 0.\) Recall that the system of equations of interest on \(G \times S \times I\) for \(\psi = (\psi_1, \psi_2, \psi_3)\) is
\[
\omega \cdot \nabla_{x} \psi_1 + \Sigma_1 \psi_1 - K_1 \psi = f_1 \\
\frac{\partial (S_j \psi_j)}{\partial E} + \omega \cdot \nabla_{x} \psi_j + \Sigma_j \psi_j - K_j \psi = f_j, \quad j = 2, 3. \\
(432)
\]
Equation (431) can be written as
\[
\omega \cdot \nabla_{x} \psi_1 + \Sigma_1 \psi_1 - \overline{K}_1 \psi_1 - \overline{K} \hat{\psi} = f_1, \\
(433)
\]
where
\[
\hat{\psi} := (\psi_2, \psi_3),
\]
and
\[
\overline{K}_1 \psi_1 := \int_{S} \tilde{\sigma}_{11}(x, \omega', \omega, E) \psi_1(x, \omega', E) d\omega' = \overline{K}_1(E) \psi_1(E) \\
\overline{K} \hat{\psi} := \sum_{k=2}^{3} \int_{S} \tilde{\sigma}_{k1}(x, \omega', \omega, E) \psi_k(x, \omega', E) d\omega' = \overline{K}(E) \hat{\psi}(E),
\]
when we define
\[
\overline{K}_1(E) v := \int_{S} \tilde{\sigma}_{11}(x, \omega', \omega, E) v(x, \omega') d\omega', \quad v \in L^2(G \times S),
\]
\[
\overline{K}(E) u := \sum_{k=2}^{3} \int_{S} \tilde{\sigma}_{k1}(x, \omega', \omega, E) u_k(x, \omega') d\omega', \quad u = (u_2, u_3) \in L^2(G \times S)^2.
\]
For any \(q \in L^2(G \times S \times I)\) the problem
\[
\omega \cdot \nabla_{x} \psi_1 + \Sigma_1 \psi_1 - \overline{K}_1 \psi_1 = q \\
\psi_{1\Gamma_-} = 0 \\
(434)
\]
has a unique solution and
\[
\|\psi_1\|_{L^2(G \times S \times I)} \leq \frac{1}{c} \|q\|_{L^2(G \times S \times I)}. \\
(435)
\]
This can also be proven similarly to [71, Corollary 5.15] (for the existence of solutions cf. also [20, Lemma 4, p. 241]).

Define a linear operator \(T_{1,0} : L^2(G \times S \times I) \to L^2(G \times S \times I)\) with domain \(D(T_{1,0})\) by
\[
D(T_{1,0}) := \overline{W}_{-0}(G \times S \times I) \\
T_{1,0} \psi_1 := \omega \cdot \nabla_{x} \psi_1 + \Sigma_1 \psi_1 - \overline{K}_1 \psi_1. \\
(436)
\]
Then \(\psi_1 = T_{1,0}^{-1} q\) (exists and) is the solution of (434), and it follows from (435) that
\[
\|T_{1,0}^{-1} q\|_{L^2(G \times S \times I)} \leq \frac{1}{c} \|q\|_{L^2(G \times S \times I)}, \quad \forall q \in L^2(G \times S \times I). \\
(437)
\]
Equations (432) can be written as
\[ -\frac{\partial (S_j \psi_j)}{\partial E} + \omega \cdot \nabla_x \psi_j + \Sigma_j \psi_j - \hat{K}_j \hat{\psi} - \hat{K}_{1,j} \psi_1 = f_j, \quad j = 2, 3 \] (438)
where
\[ \hat{K}_j \hat{\psi} := \sum_{k=2}^{3} \int_{S} \sigma_{kj}(x, \omega', \omega, E) \psi_k(x, \omega', E) d\omega' = \hat{K}_j(E) \hat{\psi}(E) \]
\[ \hat{K}_{1,j} \psi_1 := \int_{S} \sigma_{1j}(x, \omega', \omega, E) \psi_1(x, \omega', E) d\omega' = \hat{K}_{1,j}(E) \psi_1(E) \]
when one defines
\[ \hat{K}_j(E) u := \sum_{k=2}^{3} \int_{S} \sigma_{kj}(x, \omega', \omega, E) u_k(x, \omega') d\omega', \quad u = (u_2, u_3) \in L^2(G \times S)^2, \]
\[ \hat{K}_{1,j}(E) v := \int_{S} \sigma_{1j}(x, \omega', \omega, E) v(x, \omega') d\omega', \quad v \in L^2(G \times S). \]
Therefore, if \( \psi = (\psi_1, \psi_2, \psi_3) = (\psi_1, \hat{\psi}) \) is a solution of (431), (432), we have by equation (433),
\[ \psi_1 = T_{1,0}^{-1}(f_1 + \hat{K}_j \hat{\psi}) \] (439)
and hence by (438) the function \( \hat{\psi} \) satisfies the equation, for \( j = 2, 3 \),
\[ -\frac{\partial (S_j \psi_j)}{\partial E} + \omega \cdot \nabla_x \psi_j + \Sigma_j \psi_j - \hat{K}_j \hat{\psi} - \hat{K}_{1,j}(T_{1,0}^{-1}(\hat{K}_j \hat{\psi})) = f_j + \hat{f}_j, \] (440)
where we wrote
\[ \hat{f}_j := \hat{K}_{1,j}(T_{1,0}^{-1} f_1), \quad j = 2, 3. \]
Consider the term \( \hat{K}_{1,j}(T_{1,0}^{-1}(\hat{K}_j \hat{\psi})) \). We find that for a fixed \( E \in I \)
\[ \hat{K}_{1,j}(T_{1,0}^{-1}(\hat{K}_j \hat{\psi})) = \hat{K}_{1,j}(E) T_{1,0}(E)^{-1}(\hat{K}(E) \hat{\psi}(E)) \]
where for every \( E \in I \) the linear operator \( T_{1,0}(E) : L^2(G \times S) \rightarrow L^2(G \times S) \) with domain \( D(T_{1,0}(E)) \) is defined to be
\[ D(T_{1,0}(E)) := \hat{W}_0^2(G \times S) \]
\[ T_{1,0}(E) v := \omega \cdot \nabla_x v + \Sigma_1(E) v - K_1(E) v. \]
By (the proof of) Lemma (4.11), the operator \( T_{1,0}(E) \) is invertible and
\[ \| T_{1,0}(E)^{-1} \hat{q} \|_{L^2(G \times S)} \leq \frac{1}{c} \| \hat{q} \|_{L^2(G \times S)}, \quad \forall \hat{q} \in L^2(G \times S). \] (441)
Define linear operators \( Q_j(E) : L^2(G \times S)^2 \rightarrow L^2(G \times S), \quad j = 2, 3, E \in I, \) by setting
\[ Q_j(E) u := \hat{K}_{1,j}(E) T_{1,0}(E)^{-1}(\hat{K}(E) u), \]
and let \( Q(E) := (Q_1(E), Q_2(E)) \). Then we have by (441) and (4.29), (4.30) (see e.g. (292) that for all \( E \in I \),
\[ \| Q_j(E) \| \leq \| \hat{K}_{1,j}(E) \| \| T_{1,0}(E)^{-1} \| \| \hat{K}(E) \| \leq \frac{M_1 M_1'}{c}, \] (442)
where the operator norms used are taken, in an obvious way, with respect to the space \( L^2(G \times S) \) and its product \( L^2(G \times S)^2 \).
Using assumption (420) and Lemma (4.11) we find that for any fixed \( u \in L^2(G \times S)^2 \)
the mapping \( h_u : I \rightarrow L^2(G \times S)^2 \) given by \( h_u(E) := Q(E)u \) belongs to \( C^1(I, L^2(G \times S)^2) \). Similarly, we find that \( \hat{f} := (\hat{f}_1, \hat{f}_2) \in C^1(I, L^2(G \times S)^2) \), since we have
\[
\hat{f}_j(E) = \hat{K}_{1,j}(E)T_{1,0}(E)^{-1}(f_1(E)), \quad E \in I, \ j = 2, 3,
\]
where \( \hat{f}_j(E)(x, \omega) = \hat{f}_j(x, \omega, E) \).

Let \( C := \max\{C_2, C_3\} \) where for \( j = 2, 3, \)
\[
C_j := \frac{1}{2} \kappa_j^{-2} \| \nabla_x S_j \|_{L^\infty(G \times I)} + \kappa_j^{-1} \left( \| \Sigma_j \|_{L^\infty(G \times S \times I)} + \left\| \frac{\partial S_j}{\partial E} \right\|_{L^\infty(G \times I)} + \sqrt{M_1M'_1 + \frac{M_1M'_1}{c}} \right).
\]
Replacing \( \hat{\psi} \) with \( \hat{\phi}(x, \omega, E) := e^{-CE} \hat{\psi}(x, \omega, E_m - E) \) (as in section 3.5) we find that the system (443) is equivalent to
\[
\frac{\partial \hat{\phi}}{\partial E} - A_C(E)\hat{\phi} = F(E), \quad \hat{\phi}(0) = 0,
\]
where
\[
\phi = (\phi_2, \phi_3),
\]
\[
A_C(E)\hat{\phi} = (A_{C,2}(E)\hat{\phi}, A_{C,3}(E)\hat{\phi}),
\]
\[
F = (F_2, F_3),
\]
and for \( j = 2, 3, \)
\[
F_j(E) := \frac{1}{S_j(E)} e^{-CE} \left( \hat{f}_j(x, \omega, E) + \tilde{f}_j(x, \omega, E) \right),
\]
and
\[
A_{C,j}(E)\hat{\phi} := -\left( \frac{1}{S_j(E)} \omega \cdot \nabla_x \phi_j + C \phi_j + \frac{1}{S_j(E)} \hat{\Sigma}_j(E)\phi_j + \frac{1}{S_j(E)} \frac{\partial \hat{S}_j}{\partial E}(E)\phi_j \right.
\]
\[
- \left. \frac{1}{S_j(E)} \tilde{K}_j(E)\hat{\phi} - \frac{1}{S_j(E)} \tilde{Q}_j(E)\hat{\phi} \right).
\]
Here \( \tilde{S}_j(x, E) := S_j(x, E_m - E) \) and similarly for other expressions equipped with ”tilde”.

Considering \( A_C(E) \) as an (unbounded) operator \( L^2(G \times S)^2 \rightarrow L^2(G \times S)^2 \) with domain
\[
D(A_C(E)) = \tilde{W}_{2,0}^2(G \times S) \times \tilde{W}_{2,0}^2(G \times S),
\]
we get by applying Theorem (4.29) along with computations analogous to the ones done in section 3.5 that the system (443) has a unique solution \( \hat{\phi} \in C(I, \tilde{W}_{2,0}^2(G \times S)^2) \cap C^1(I, L^2(G \times S)^2), \) which satisfies the homogeneous boundary and initial conditions, \( \hat{\phi}|_{\Gamma_-} = 0, \hat{\phi}(\cdot, \cdot, 0) = 0. \)

Then \( \hat{\psi}(x, \omega, E) = e^{C(E_m - E)}\hat{\phi}(x, \omega, E_m - E) \), and \( \psi_1 \) is obtained from (439) that is, \( \psi_1 = T_{1,0}^{-1}(f_1 + \tilde{K}\hat{\psi}) \), giving us the solution \( \psi = (\psi_1, \psi_2, \psi_3) = (\psi_1, \hat{\psi}) \) for homogeneous (inflow) boundary, and initial condition data, that is \( \psi|_{\Gamma_-} = 0, \psi_j(\cdot, \cdot, E_m) = 0 \) for \( j = 2, 3, \) which is what we were looking for.
By applying the lifts, the existence of a unique solution of the problem (335)-(338) satisfying the inhomogeneous boundary condition

$$\psi_{j|\Gamma_} = g_j, \quad j = 1, 2, 3,$$

(445)

is obtained as in section 3.5 as well (see the proof of Theorem 3.36). The estimate (388) under the assumptions (427), (428) follows from Corollary 4.9. This completes the proof.

**Corollary 4.13** Suppose that the assumptions (419)-(423) of Theorem 4.12 are valid. Furthermore, suppose that $f \in H^2(I, L^2(G \times S)^3)$ and $g \in H^3(I, T^2(\Gamma_-)^3)$ which satisfies the compatibility condition

$$g_j(E_m) = 0, \quad j = 2, 3.$$

Then the problem (335)-(338) has a unique solution $\psi \in \tilde{W}^2(G \times S \times I) \times (C^1(I, L^2(G \times S^2)) \cap C(I, \tilde{W}(G \times S^2)))$. If, in addition, (427) and (428) are valid, the estimate (388) holds.

**Proof.** Follows from the Sobolev Embedding Theorem as in Corollary 3.38. □

**Remark 4.14** The evolution operator based approach given above can be generalized for $L^p$-theory when $1 \leq p < \infty$.

### 4.4. Note on Reflective Boundary Conditions.

Let $R := (R_1, R_2, R_3) : T^2(\Gamma_+)^3 \to T^2(\Gamma_-)^3$ be a linear, possibly unbounded operator with domain $D(R)$. In some problems the inflow boundary condition

$$\psi|\Gamma_- = g,$$

(446)

of the problem (335)-(338) is substituted with a more general, so-called **reflective boundary condition** (cf. [20, Chapter XXI, Appendix of §2, p. 249-262])

$$\psi|\Gamma_- = R(\psi|\Gamma_+) + g,$$

(447)

where $g \in T^2(\Gamma_-)^3$. We call $R$ a **reflection operator**. Note that when $R = 0$, we obtain (440).

We consider here one specific type of reflection operator $R = R_0$ which is of interest in applications where backscattering from outside the region $G$ is of importance. Let $G' \subset \mathbb{R}^3$ be an open bounded set such that $\overline{G'}$ is a $C^1$-manifold with boundary (i.e. $G'$ has the same regularity properties as $G$) and such that $\overline{G} \subset G'$. Let $G_e := G' \setminus \overline{G}$. Then

$$\partial G_e = \partial^1 G_e \cup \partial^2 G_e,$$

where $\partial^1 G_e := \partial G$ and $\partial^2 G_e := \partial G_e \setminus \partial^1 G_e$. Let

$$\Gamma_e := (\partial G_e) \times S \times I = \Gamma_{e,+} \cup \Gamma_{e,-} \cup \Gamma_{e,0},$$

where on the right hand side, the decomposition of $\Gamma_e$ into the three disjoint subsets corresponds to $\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_0$ when considering $G_e \times S \times I$ instead of $G \times S \times I$. Finally, we can decompose $\Gamma_{e,+}$ and $\Gamma_{e,-}$ respectively as

$$\Gamma_{e,\pm} = \Gamma_{e,\pm}^1 \cup \Gamma_{e,\pm}^2,$$

where for $j = 1, 2,$

$$\Gamma_{e,\pm}^j := (\partial^j G_e) \times S \times I) \cap \Gamma_{e,\pm}.$$
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Notice that $\Gamma_{1,-} = \Gamma_+$, while $\Gamma_{e,+} = \Gamma_-$. We assume (for simplicity) that $f = 0$. Consider the problem \([335]-[338]\) that is, find $\psi = (\psi_1, \psi_2, \psi_3)$ that satisfies on $G \times S \times I$ the system of transport equations,

$$\begin{align}
\omega \cdot \nabla_x \psi_1 + \Sigma_{11} \psi_1 - K_1 \psi &= 0, \\
- \frac{\partial(S_j \psi_j)}{\partial E} + \omega \cdot \nabla_x \psi_j + \Sigma_{jj} \psi_j - K_j \psi &= 0, \quad j = 2, 3
\end{align}\tag{448}$$

under the (inflow) boundary condition on $\Gamma_-$,

$$\psi_j|_{\Gamma_-} = g_j, \quad j = 1, 2, 3, \tag{450}$$

and the initial condition on $G \times S$,

$$\psi_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3. \tag{451}$$

Since $\Gamma_{e,-} = \Gamma_+$, we find that the flux $\psi|_{\Gamma_+}$ is an inflow boundary source for the domain $G_e$, on the part $\partial^1 G_e = \partial G$ of its boundary. Suppose that $G_e$ does not contain any extra internal or boundary sources. Then the transport of particles in $G_e$ is governed by the system of equations on $G_e \times S \times I$,

$$\begin{align}
\omega \cdot \nabla_x \Psi_1 + \Sigma_{e,1} \Psi_1 - K_{e,1} \Psi &= 0, \\
- \frac{\partial(S_{e,j} \Psi_j)}{\partial E} + \omega \cdot \nabla_x \Psi_j + \Sigma_{e,j} \Psi_j - K_{e,j} \Psi &= 0, \quad j = 2, 3
\end{align}\tag{452}$$

for $\Psi = (\Psi_1, \Psi_2, \Psi_3)$, along with the boundary conditions

$$\begin{align}
\Psi_j|_{\Gamma_{e,-}^1} &= \psi_j|_{\Gamma_+} \quad \text{on } \Gamma_{e,-}^1 = \Gamma_+, \\
\Psi_j|_{\Gamma_{e,-}^2} &= 0 \quad \text{on } \Gamma_{e,-}^2, \quad j = 1, 2, 3, \tag{455}
\end{align}$$

and the initial condition

$$\Psi_j(x, \omega, E_m) = 0 \quad \text{on } G_e \times S \times I, \quad j = 2, 3. \tag{456}$$

Above $\Sigma_{e,j}$, $\sigma_{e,kj}$ and $S_{e,j}$ are the (restricted) cross-sections and the (restricted) stopping powers for the medium inside $G_e$, and

$$\begin{align}
(K_{e,j} \Psi)(x, \omega, E) &:= \sum_{k=1}^3 \int_{S \times I} \sigma_{e, kj}(x, \omega', \epsilon, E') \Psi_k(x, \omega', E') d\omega' dE',
\end{align}$$

for $\Psi \in L^2(G_e \times S \times I)^3$.

In this setup, we define a reflection operator $R = R_b$ by setting (recall that $\Gamma_{e,+} = \Gamma_-$)

$$\begin{align}
R_b(\psi|_{\Gamma_+}) &:= \Psi|_{\Gamma_{e,+}^1} = \Psi|_{\Gamma_-}, \\
D(R_b) &:= \{ \psi|_{\Gamma_+} \mid \psi \text{ is a solution of } \text{(448)} \text{ for some } g \in T^2(\Gamma_-)^3 \}. \tag{457}
\end{align}$$

If the assumptions of Theorem 1.8 hold for $G'$ in place of $G$ (and for the respective cross-sections, stopping powers, $\Sigma_{j}'$, $\sigma_{k,j}'$, $S_j'$), the operator $R_b$ is a linear operator $T^2(\Gamma_+)^3 \to T^2(\Gamma_-)^3$ with domain of definition $D(R_b) \subset T^2(\Gamma_+)^3$. In general, $R_b$ is not bounded.

The meaning of this definition is that $R_b(\psi|_{\Gamma_+})$ models an extra source on $\Gamma_-$ (i.e. an inflow boundary source for $G$) due to backscattering of particles from the given
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The flux $u = (u_1, u_2, u_3)$ contributed by this inflow source is governed by the system of equations

$$\omega \cdot \nabla x u_1 + \Sigma_1 u_1 - K_1 u = 0,$$

$$- \frac{\partial (S_j u_j)}{\partial E} + \omega \cdot \nabla x u_j + \Sigma_j u_j - K_j u = 0, \quad j = 2, 3,$$

on $G \times S \times I$, such that on $\Gamma_-$,

$$u|_{\Gamma_-} = R_b(\psi|_{\Gamma_+}),$$

and for almost every $(x, \omega) \in G \times S$,

$$u_j(x, \omega, E_m) = 0, \quad j = 2, 3.$$

We point out that if $\psi \in D(R_b)$ and if $u$ solves the problem (458)-(461), and is such that

$$R_b(u|_{\Gamma_+}) = 0,$$

holds, i.e. no particles are backscattered repeatedly from $G_e$ into $G$, then for $\varphi := \psi + u$ we have

$$\varphi|_{\Gamma_-} = \psi|_{\Gamma_-} + u|_{\Gamma_-} = g + R_b(\psi|_{\Gamma_+}) = g + R_b((\psi + u)|_{\Gamma_+}) = g + R_b(\varphi|_{\Gamma_+}),$$

and hence $\varphi = (\varphi_1, \varphi_2, \varphi_3) = \psi + u$ is a solution of the following problem on $G \times S \times I$, with boundary and initial conditions holding on $\Gamma_-$ and $G \times S$, respectively,

$$\omega \cdot \nabla x \varphi_1 + \Sigma_1 \varphi_1 - K_1 \varphi = 0,$$

$$- \frac{\partial (S_j \varphi_j)}{\partial E} + \omega \cdot \nabla x \varphi_j + \Sigma_j \varphi_j - K_j \varphi = 0,$$

$$\varphi|_{\Gamma_-} = R_b(\varphi|_{\Gamma_+}) + g,$$

$$\varphi_j(\cdot, \cdot, E_m) = 0,$$

where $j = 2, 3$. This shows that solving for $\varphi$ (in place of $\psi$) the problem (433), (449), (451) under boundary condition (447), is equivalent to solving first for $\psi$ the problem (448)-(451), and then for $u$ the problem (458)-(461) under the additional condition (462).

We will not explore the question of the existence of solutions for the problem (463)-(466) (or for (458)-(462)) in this paper.

5. Adjoint Transport Problem

We will discuss briefly the adjoint version of the transport problem (335)-(338), and related operators. Write

$$T_1 \psi := \omega \cdot \nabla x \psi_1 + \Sigma_1 \psi_1 - K_1 \psi,$$

$$T_j \psi := - \frac{\partial (S_j \psi_j)}{\partial E} + \omega \cdot \nabla x \psi_j + \Sigma_j \psi_j - K_j \psi, \quad j = 2, 3,$$

and define a (densely defined) linear operator $T : L^2(G \times S \times I)^3 \rightarrow L^2(G \times S \times I)^3$ by

$$D(T) := \{ \psi \in L^2(G \times S \times I)^3 \mid T_j \psi \in L^2(G \times S \times I), \ j = 1, 2, 3 \},$$

$$T \psi := (T_1 \psi, T_2 \psi, T_3 \psi).$$

(467)
Let \( f \in L^2(G \times S \times I)^3 \) and \( g \in T^2(\Gamma_-)^3 \). The problem (335)-(338) can be expressed equivalently as the problem

\[
T\psi = f, \quad \psi|_{\Gamma_-} = g, \quad \psi_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3.
\]

As in section 3.2, an application of integration by parts and the Green’s formula (35) implies

\[
\langle T\psi, v \rangle_{L^2(G \times S \times I)^3} = \langle \psi, T^* v \rangle_{L^2(G \times S \times I)^3} \quad \forall v \in C_0^1(G \times S \times I^c),
\]

where \( T^* v = (T_1^* v, T_2^* v, T_3^* v) \), and

\[
T_1^* v := -\omega \cdot \nabla \psi_1 + \Sigma_1^\ast \psi_1 - K_1^\ast \psi_1,
\]

\[
T_j^* v := S_j \frac{\partial \psi_j}{\partial E} - \omega \cdot \nabla \psi_j + \Sigma_j^\ast \psi_j - K_j^\ast \psi_j, \quad j = 2, 3.
\]

Moreover, we have \( \Sigma_j^\ast = \Sigma_j \) and for \( v \in L^2(G \times S \times I)^3, j = 1, 2, 3, \)

\[
\langle K_j^\ast v \rangle_{x, \omega, E} = \sum_{k=1}^{3} \int_{S \times I} \sigma_{jk}(x, \omega, \omega', E, E') \left( \psi_k(x, \omega', E') d\omega' / dE' \right)
\]

Let \( f^* \in L^2(G \times S \times I)^3 \) and \( g^* \in T^2(\Gamma_+)^3 \). The adjoint problem of (468) (or equivalently (197)-(199)) is defined by\( cf. \ [2, pp. 24-28] \)

\[
T^* \psi^* = f^*, \quad \psi^*|_{\Gamma_+} = g^*, \quad \psi_j^*(\cdot, \cdot, 0) = 0, \quad j = 2, 3,
\]

or more explicitly

\[
-\omega \cdot \nabla \psi_1^* + \Sigma_1^\ast \psi_1^* - K_1^\ast \psi_1^* = f_1^*,
\]

\[
S_j \frac{\partial \psi_j^*}{\partial E} - \omega \cdot \nabla \psi_j^* + \Sigma_j^\ast \psi_j^* - K_j^\ast \psi_j^* = f_j^*, \quad j = 2, 3,
\]

holding a.e. on \( G \times S \times I \), together with the outflow boundary and initial values

\[
\psi^*|_{\Gamma_+} = g^* \quad \text{a.e. on } \Gamma_+,
\]

\[
\psi_j^*(\cdot, \cdot, 0) = 0 \quad \text{a.e. on } G \times S, \quad j = 2, 3.
\]

The adjoint problem has various kind of applications both in the existence theory of solutions and in computations. At the end of this section we give an example which is related to the dose calculation in radiation therapy. We refer also to computations concerning a related optimal control problem considered in \([71, \text{ Section 7}]\), where the adjoint problem plays a significant role. In addition, recall that the concept of Green distribution (or Green function) is usually founded on the theory of adjoint problem. We also point out that the adjoint field \( \psi^* \) obeying Eqs. (472), (473) is sometimes called an importance function (cf. \([23, \text{ Ch. 5, Sec. V}]\)).

Consider the variational formulation of the adjoint problem. As in sections 3 and 4 (for \( C = 0 \)) we find that the bilinear form \( B_0^* (\cdot, \cdot) : C^1(G \times S \times I)^3 \times C^1(G \times S \times I)^3 \) and the linear form \( F^* \) corresponding to the adjoint problem are

\[
B_0^*(\psi^*, v) = - \sum_{j=2,3} \left\langle \psi_j^*, \frac{\partial (S_j \psi_j)}{\partial E} \right\rangle_{L^2(G \times S \times I)} + \left\langle \psi^*, \omega \cdot \nabla \psi \right\rangle_{L^2(G \times S \times I)^3}
\]

\[
+ \left\langle \psi^*, (\Sigma - K) \psi \right\rangle_{L^2(G \times S \times I)^3} + \left\langle \gamma_- (\psi^*), \gamma_- (v) \right\rangle_{T^2(\Gamma_-)^3}
\]

\[
+ \sum_{j=2,3} \left\langle \psi_j^*(\cdot, \cdot, E_m), S_j(\cdot, E_m) v_j(\cdot, \cdot, E_m) \right\rangle_{L^2(G \times S)},
\]

(476)
where $\psi^* = (\psi_1^*, \psi_2^*, \psi_3^*)$ and $v = (v_1, v_2, v_3)$, and
\[
F_0^*(v) = \langle f^*, v \rangle_{L^2(G \times S \times I)^3} + \langle g^*, \gamma_+(v) \rangle_{T_2(\Gamma_+)^3}.
\]  
(477)

Similarly as in Theorem 4.7 we find that
\[
|B_0^*(\psi^*, v)| \leq M \|\psi^*\|_H \|v\|_{\hat{H}} \quad \forall \psi^*, v \in C^1(G \times S \times I)^3
\]
and
\[
|F_0^*(v)| \leq \left( \|f^*\|_{L^2(G \times S \times I)^3} + \|g^*\|_{T_2(\Gamma_+)^3} \right) \|v\|_H \quad \forall v \in C^1(G \times S \times I)^3.
\]  
(479)
The bilinear form $B_0^*(\cdot, \cdot)$ has a unique extension $B_0^*(\cdot, \cdot) : H \times \hat{H} \to \mathbb{R}$ (and $\tilde{B}_0^*(\cdot, \cdot)$ has an explicit expression in the same way as in (384)) which satisfies
\[
|\tilde{B}_0^*(\tilde{v}^*, v)| \leq M \|\tilde{v}^*\|_H \|v\|_{\hat{H}} \quad \forall \tilde{v}^* = (\psi^*, q^*, p_0^*, p_m^*) \in H, \ v \in \hat{H}.
\]  
(480)
Moreover, the linear form $F_0^*$ has a unique extension to a bounded linear form $H \to \mathbb{R}$ which we still denote by $F_0^*$. The variational equation corresponding to the adjoint problem (472), (473), (474), (475) is then valid, and that (366), (367) hold for $c > 0$ and that (366), (367) hold for $c$.

Proposition 5.1 For all $\psi, \tilde{\psi}^* \in \hat{H}$ one has
\[
\tilde{B}_0^*(\tilde{\psi}^*, \psi) = \tilde{B}_0(\psi, \psi^*),
\]  
(482)
where $\tilde{B}_0(\cdot, \cdot)$ is the bilinear form (384).

The relation (182) is the justification for the term adjoint problem. For the existence of solutions of the adjoint problem, we formulate the following result.

Theorem 5.2 Suppose that the assumptions (339), (340), (346), (347), (348) are valid, and that (366), (367) hold for $c > 0$ and for $C$ given in (370). Let $f^* \in L^2(G \times S \times I)^3$ and that $g^* \in T_2(\Gamma_+)^3$. Then the following assertions hold.

(i) The variational equation
\[
\tilde{B}_0^*(\tilde{v}^*, v) = F_0^*(v) \quad \forall v \in \hat{H},
\]  
(483)
has a solution $\tilde{v}^* = (\tilde{\psi}_1^*, \tilde{\psi}_2^*, \tilde{\psi}_3^*) \in \hat{H}$. Writing $\tilde{\psi}_1^* := (\psi_1^*, q_1^*, p_{0j}^*, p_{mj}^*), \ j = 2, 3,$ and $\psi^* = (\psi_1^*, \psi_2^*, \psi_3^*) \in L^2(G \times S \times I)^3$, then $\psi^* \in H_{p_j}(G \times S \times I)$ (see (480)) is a weak (distributional) solution of the system of equations (472), (473), (474), and $\psi_1^* \in W^2(G \times S \times I)$.

(ii) Suppose that additionally the assumption TC holds (p. 19). Then a solution $\psi^*$ of the equations (472), (473) obtained in part (i) is a solution of the problem (472), (474).

(iii) Under the assumptions imposed in part (ii), any solution $\psi^*$ of the problem (472), (475) that further satisfies
\[
\psi_{\Gamma_+}^* \in T_2(\Gamma_+)^3 \quad \text{and} \quad \psi(\cdot, \cdot, E_m) \in L^2(G \times S)^3.
\]  
(484)
is unique and obeys the estimate
\[
\|\psi^*\|_H \leq \frac{C_{E_m}}{c'} \left( \|f^*\|_{L^2(G \times S \times I)^3} + \|g^*\|_{T_2(\Gamma_+)^3} \right).
\]  
(485)
(Recall that $C$ is defined in (370), $c'$ in (373) and that $E_m$ is the cutoff energy.)
Note that if $\psi^*$ is a solution of the problem (472)-(475), then it is a solution of the variational problem (483) and vice versa.

The method founded on the $m$-dissipativity (see sections 3.3 and 4.2) can be applied also to the adjoint problem. Define

$$P_1^*(x, \omega, E, D)\psi_1^* := -\omega \cdot \nabla_x \psi_1^*,$$

$$P_j^*(x, \omega, E, D)\psi_j^* := S_j \frac{\partial \psi_j^*}{\partial E} - \omega \cdot \nabla_x \psi_j^*, \quad j = 2, 3,$$

$$P^*(x, \omega, E, D)\psi^* := (P_1^*(x, \omega, E, D)\psi_1^*, P_2^*(x, \omega, E, D)\psi_2^*, P_3^*(x, \omega, E, D)\psi_3^*),$$

and the space

$$\mathcal{H}_{P^*}(G \times S \times I^o) := \{\psi^* \in L^2(G \times S \times I)^3 \mid P^*(x, \omega, E, D)\psi^* \in L^2(G \times S \times I)^3 \text{ in the weak sense}\}. \quad (486)$$

The relevant operator here is the smallest closed extension (closure) $\tilde{P}_0^*$ of the operator $P_0^*$ defined by

$$D(\tilde{P}_0^*) := \{\psi^* \in \tilde{W}^2(G \times S \times I) \times (\tilde{W}^2(G \times S \times I) \cap H^1(I, L^2(G \times S))^2 \mid \psi^*_j |_{T^+} = 0, \quad \psi^*_j(\cdot, \cdot, 0) = 0, \quad j = 2, 3\}$$

$$\tilde{P}_0^*\phi := P^*(x, \omega, E, D)\phi.$$ 

When $g^* = 0$, the problem (472)-(475) is equivalent, in the strong sense, to

$$(\tilde{P}_0^* + \Sigma^* - K^*)\psi^* = f^*, \quad \Sigma^*\psi^* = (\Sigma_1^*\psi_1^*, \Sigma_2^*\psi_2^*, \Sigma_3^*\psi_3^*), \quad K^*\psi^* = (K_1^*\psi^*, K_2^*\psi^*, K_3^*\psi^*).$$

The result analogous to Theorem 4.10 is the following.

**Theorem 5.3** Suppose that the assumptions (393), (394), (395), (396) (with $c > 0$) and (394), (395), (396) are valid. Furthermore, suppose that $f^* \in L^2(G \times S \times I)^3$ and $g^* \in T^2(\Gamma_+) \times H^1(I, T^2(\Gamma_+))^2$ is such that the compatibility condition

$$g^*_j(\cdot, \cdot, E_0) = 0, \quad j = 2, 3, \quad (487)$$

holds. Then the problem (472)-(475) has a unique solution $\psi^* \in \mathcal{H}_{P^*}(G \times S \times I^o)$.

In addition, there exists a constant $C_1 > 0$ such that *a priori* estimate

$$\|\psi^*\|_{L^2(G \times S \times I)^3} \leq C_1 \left( \|f^*\|_{L^2(G \times S \times I)^3} + \|g^*\|_{T^2(\Gamma_+) \times H^1(I, T^2(\Gamma_+))^2} \right), \quad (488)$$

holds.

The existence result analogous to Theorem 4.10 (based on the theory of evolution operators) holds also for the adjoint problem, and it guarantees that $\psi^* \in \tilde{W}^2(G \times S \times I) \times (\tilde{W}^2(G \times S \times I) \cap W^1_3(G \times S \times I))^2$. In this case, one assumes that $K$ takes the form (399), and that consequently its adjoint version $K^* = (K_1^*, K_2^*, K_3^*)$ is,

$$(K^*_j\psi)(x, \omega, E) = \sum_{k=1}^3 \int_S \delta_{kj}(x, \omega, \omega', E)\psi_k(x, \omega', E) d\omega', \quad j = 1, 2, 3. \quad (489)$$

The adjoint version of Theorem 4.10 can be formulated as follows.
Theorem 5.4 Suppose that the adjoint collision operator is of the form \((459)\), and that the assumptions \((419)-(423)\) of Theorem 4.12 are valid for \(\Sigma_j\), \(\sigma_{jk}\) and \(S_j\). Furthermore, suppose that \(f^* \in C^1(I, L^2(G \times S)^3)\) and \(g^* \in C^2(I, T^2(T'_3)^3)\) which satisfies the compatibility condition

\[ g_j^*(0) = 0, \quad j = 2, 3. \]

Then the problem \((472)-(475)\) has a unique solution \(\psi^* \in \hat{W}^2(G \times S \times I) \times \left(C(I, \hat{W}^2(G \times S)^2) \cap C^1(I, L^2(G \times S)^3)\right)\) in particular, \(\psi^* \in \hat{W}^2(G \times S \times I) \times \left(\hat{W}^2(G \times S \times I) \cap W^2_3(G \times S \times I)\right)^2\).

If, in addition, the conditions \((427), (428)\) are valid, then the solution \(\psi^*\) satisfies the estimate \((485)\).

It is clear that a Sobolev space version of the above theorem analogous to Corollary 4.13 holds for the adjoint problem as well.

Example 5.5 In radiation therapy the absorbed dose from the particle field \(\psi = (\psi_1, \psi_2, \psi_3)\) is defined by the functional

\[ D(x) = (D\psi)(x) := \sum_{j=1}^{3} \int_{S \times I} \zeta_j(x, E) \psi_j(x, \omega, E) d\omega dE, \quad (490) \]

where \(\psi\) is the solution of \((497)-(499)\). We see that

\[ (D\psi)(x) = \sum_{j=1}^{3} \langle \zeta_j(x, \cdot), \psi_j(x, \cdot, \cdot) \rangle_{L^2(S \times I)} = \langle \zeta(x, \cdot), \psi(x, \cdot, \cdot) \rangle_{L^2(S \times I)^3}. \quad (491) \]

Define for any fixed \(x \in G\) distribution \(T_{\psi, x}\) on \(S \times I^{\circ}\) by

\[ T_{\psi, x} \varphi := \langle \zeta_j(x, \cdot), \varphi \rangle_{L^2(S \times I)} \quad \varphi \in C_0^\infty(S \times I^{\circ}), \quad j = 1, 2, 3. \quad (492) \]

Then for all \(\psi_j = \phi_j \otimes \varphi_j\), where \(\phi_j \in C_0^\infty(G)\), \(\varphi_j \in C_0^\infty(S \times I^{\circ})\), and where \((\phi_j \otimes \varphi_j)(x, \omega, E) := \phi_j(x) \varphi_j(\omega, E)\), we find that

\[ \sum_{j=1}^{3} (\delta_x \otimes T_{\psi, x})(\psi_j) = \sum_{j=1}^{3} \delta_x(\phi_j) T_{\psi, x}(\varphi_j) = \sum_{j=1}^{3} \phi_j(x) \int_{S \times I} \zeta_j(x, E) \varphi_j(\omega, E) d\omega dE \]

\[ = \sum_{j=1}^{3} \int_{S \times I} \zeta_j(x, E) (\phi_j \otimes \varphi_j)(x, \omega, E) d\omega dE = (D\psi)(x). \quad (493) \]

Hence also for a general element \(\psi = (\psi_1, \psi_2, \psi_3) \in C_0^\infty(G \times S \times I^{\circ})\) one has

\[ (D\psi)(x) = \sum_{j=1}^{3} (\delta_x \otimes T_{\psi, x})(\psi_j) =: T_x(\psi), \quad (494) \]

where \(T_x\) is a distribution on \(G \times S \times I^{\circ}\) with values in \(\mathbb{R}^3\).

The following discussion will be formal. Let \(x \in G\) be fixed. Assume that there exists a sufficient regular solution \(\Psi^*_x\) to the variational problem

\[ \hat{B}_0^*(\Psi^*_x, v) = T_x(v), \quad \forall v \in \mathcal{H}. \quad (495) \]

Furthermore, assume that \(\psi\) (which depends on \(f, g\)) is a sufficient regular solution of \((335)-(338)\) and that \(\Psi^*_x = \Psi^*_x(x^\prime, \omega, E)\) is a solution of \((495)\) such that \((482)\) holds.
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Then we have

$$(D\psi)(x) = T_x(\psi) = B_0^*(\Psi_x, \psi) = B_0(\psi, \Psi_x^*) = F_0(\Psi_x^*)$$

$$= \langle f, \Psi_x^* \rangle_{L^2(G \times S \times I)^3} + \langle g, \gamma(-\Psi_x^*) \rangle_{T^2(\Gamma_-)^3},$$

(496)

implying that the dose can be obtained with the help of $\Psi_x^*$. That is why, when the equation (495) is solved once, one can obtain the dose $(D\psi)(x)$ at $x \in G$ for any $f$ and $g$ from (496).

6. Some Notes on Computational Methods

6.1. A Decomposition of Solutions Corresponding to Primary and Secondary Particles. Consider the system of transport equations as above,

$$\omega \cdot \nabla_x \psi_1 + \Sigma_1 \psi_1 - K_1 \psi = f_1,$$

$$-\frac{\partial (S_j \psi_j)}{\partial E} + \omega \cdot \nabla_x \psi_j + \Sigma_j \psi_j - K_j \psi = f_j, \quad j = 2, 3,$$

(497)

holding a.e. on $G \times S \times I$, together with the inflow boundary and initial values

$$\psi_{\Gamma_-} = g \quad \text{a.e. on } \Gamma_-,$$

$$\psi_j(\cdot, \cdot, E_m) = 0 \quad \text{a.e. on } G \times S, \ j = 2, 3.$$  

(498)  

(499)

The solution $\psi = (\psi_1, \psi_2, \psi_3)$ for this problem can be decomposed as follows. Let $u = (u_1, u_2, u_3)$ be the solution of the problem without collisions

$$\omega \cdot \nabla_x u_1 + \Sigma_1 u_1 = f_1,$$

$$-\frac{\partial (S_j u_j)}{\partial E} + \omega \cdot \nabla_x u_j + \Sigma_j u_j = f_j, \quad j = 2, 3,$$

(500)

together with the inflow boundary and initial values

$$u_{\Gamma_-} = g,$$

(501)

$$u_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3.$$

(502)

Furthermore, let $w = (w_1, w_2, w_3)$ be the solution of the problem

$$\omega \cdot \nabla_x w_1 + \Sigma_1 w_1 - K_1 w = K_1 u - \frac{\partial (S_j w_j)}{\partial E} + \omega \cdot \nabla_x w_j + \Sigma_j w_j - K_j w = K_j u, \quad j = 2, 3,$$

(503)

together with homogeneous inflow boundary and initial values

$$w_{\Gamma_-} = 0,$$

(504)

$$w_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3.$$  

(505)

Then we find that $\psi = u + w$ is the solution of (497)-(499). This corresponds to decomposing the evolution of the particle field $\psi$ obeying the full CSDA Boltzmann transport problem (497)-(499) in terms of the evolution of the primary (uncollided) particles, represented by $u$, and of secondary (collided) particles, represented by $w$. The method of decomposing $\psi = u + w$ in this way is useful e.g. in constructing numerical solutions, and is known, for example in neutron transport theory, under the name collided-uncollided split (cf. the recent work [36] and references therein).

To explain a bit this terminology, notice that the primary, uncollided field $u$ obeys (500) which does not involve the collision operator $K = (K_1, K_2, K_3)$, and $u$ contains
a direct contribution from the external (boundary) sources \( g \) (through \( (501) \)). On the other hand, the field \( u \) right after collision as modelled by the term \( Ku \), acts as an internal source in the equation \( (503) \) for the secondary field \( w \), while external sources do not contribute to \( w \) directly (a fact captured by \( (504) \)).

Note especially that the system \( (500)-(502) \) is uncoupled, in that the different particle species (photon, electron, positron that we consider here) evolve independently of each other. In some cases the primary component \( u \) can be calculated exactly such as the following example shows.

**Example 6.1** Suppose that \( \Sigma_1 \in L^2(G \times S \times I), \Sigma_1 \geq c > 0 \) for some constant \( c \), and that, \( \Sigma_j(x, \omega, E) = \Sigma_j(x, \omega) \) (i.e. \( \Sigma_j \) does not depend on \( E \)) and \( \Sigma_j \in L^2(G \times S), \Sigma_j \geq c > 0, \) for \( j = 2, 3 \). Furthermore, suppose that \( S_j(x, E) = S_j(E) \), \( j = 2, 3 \) (i.e. \( S_j \) is independent of \( x \)) and that \( S_j : I \to \mathbb{R}_+ \) are continuous, strictly positive functions. Finally, let \( f_1 \in L^2(G \times S \times I), g_1 \in T^2(\Gamma_-), \) and for \( j = 2, 3 \) let \( f_j \in H^1(I, L^2(G \times S)), g_j \in H^1(I, T^2(\Gamma_-)) \), such that \( g_j(E_m) = 0 \) (compatibility condition). Define \( R_j : I \to \mathbb{R} \) by

\[
R_j(E) := \int_0^E \frac{1}{S_j(\tau)} d\tau, \quad j = 2, 3. \tag{506}
\]

Let \( r_{m,j} := R_j(E_m) \). Then \( R_j : I \to [0, r_{m,j}] \) are continuously differentiable and strictly increasing bijections. Let \( R_j^{-1} : [0, r_{m,j}] \to I \) be their inverses. We denote the argument of \( R_j^{-1} \) on \([0, r_{m,j}]\) by \( \eta \), i.e. \( E = R_j^{-1}(\eta) \) (or equivalently \( \eta = R_j(E) \)).

Consider first the (primary) uncoupled problem,

\[
\begin{align*}
-\frac{\partial(S_j u_j)}{\partial E} + \omega \cdot \nabla_x u_j + \Sigma_j u_j &= f_j, \\
u_j|_{\Gamma_-} &= g_j, \quad u_j(\cdot, \cdot, E_m) &= 0, \tag{507}
\end{align*}
\]

where \( j = 2, 3 \). We perform a well-known change of variables (see e.g. \( [28], [61] \)) in the problem \( (507) \) by defining a new unknowns \( v_j \), for \( j = 2, 3 \), by setting

\[
v_j(x, \omega, \eta) := S_j(R_j^{-1}(\eta)) u_j(x, \omega, R_j^{-1}(\eta)), \tag{508}
\]

i.e.

\[
v_j(x, \omega, R_j(E)) = S_j(E) u_j(x, \omega, E). \tag{509}
\]

Then we find that

\[
\frac{\partial(S_j u_j)}{\partial E} = \frac{\partial v_j}{\partial \eta} R_j(E) = \frac{\partial v_j}{\partial \eta} \frac{1}{S_j(E)} = \frac{\partial v_j}{\partial \eta} \frac{1}{S_j(R_j^{-1}(\eta))}. \tag{510}
\]

and so, after writing

\[
\tilde{f}_j(x, \omega, \eta) := S_j(R_j^{-1}(\eta)) f_j(x, \omega, R_j^{-1}(\eta)), \quad (x, \omega, \eta) \in G \times S \times [0, r_{m,j}],
\]

\[
\tilde{g}_j(y, \omega, \eta) := S_j(R_j^{-1}(\eta)) g_j(y, \omega, R_j^{-1}(\eta)), \quad (y, \omega, \eta) \in \tilde{\Gamma}_{-j},
\]

where

\[
\tilde{\Gamma}_{-j} := \{ (y, \omega, \eta) \in \partial G \times S \times [0, r_{m,j}] \mid \omega \cdot \nu(y) < 0 \},
\]

we see that the problem \( (507) \) is equivalent to

\[
-\frac{\partial v_j}{\partial \eta} + \omega \cdot \nabla_x v_j + \Sigma_j v_j = \tilde{f}_j \quad \text{a.e. on } G \times S \times [0, r_{m,j}], \tag{511}
\]
subject to inflow boundary and initial value conditions,

\[ v_j |_{\tilde{\Gamma}_{-j}} = \tilde{g}_j \quad \text{a.e. on } \tilde{\Gamma}_{-j}, \quad \text{for } j = 1, 2, \]

\[ v_j (\cdot, \cdot, r_{m,j}) = 0 \quad \text{a.e. on } G \times S, \quad j = 2, 3. \]  

Notice that \( \tilde{g}_j(r_{m,j}) = 0 \) since \( g_j(E_m) = 0 \). The original unknowns \( u_j, j = 2, 3 \), are given in terms of \( v_j \) by

\[ u_j(x, \omega, E) = \frac{1}{S_j(E)} v_j(x, \omega, R_j(E)). \]  

The problem \((511)-(513)\) can be solved explicitly, at least formally. The solution \( v_j \) of \((511)\) is the sum \( v_{1,j} + v_{2,j} \) of solutions \( v_{1,j} \) and \( v_{2,j} \) of the following problems

\[ - \frac{\partial v_{1,j}}{\partial \eta} + \omega \cdot \nabla_x v_{1,j} + \Sigma_j v_{1,j} = \tilde{f}_j, \quad v_{1,j} |_{\tilde{\Gamma}_{-j}} = 0, \quad v_{1,j}(\cdot, \cdot, r_{m,j}) = 0, \]  

and

\[ - \frac{\partial v_{2,j}}{\partial \eta} + \omega \cdot \nabla_x v_{2,j} + \Sigma_j v_{2,j} = 0, \quad v_{2,j} |_{\tilde{\Gamma}_{-j}} = \tilde{g}_j, \quad v_{2,j}(\cdot, \cdot, r_{m,j}) = 0, \]  

where the (partial differential) equations are to be satisfies on \( G \times S \times [0, r_{m,j}] \), the (inflow) boundary conditions on \( \tilde{\Gamma}_{-j} \) and the initial (energy) conditions on \( G \times S \).

The solution of \((515)\) is (cf. \[20\], Ch. XXI, Sec. 3.2, pp. 233-235, or \[71\], proof of Theorem 6.3); replace first \( \eta \) by \( r_{m,j} - \eta \) on \( \tilde{\Gamma}_{-j} \) and one sees that

\[ v_{2,j}(x, \omega, \eta) = H(r_{m,j} - \eta - t(x, \omega))e^{\int_0^t(x,\omega) - \Sigma_j(x-s\omega,\omega)ds} \tilde{g}_j(x - t(x, \omega)\omega, \omega, \eta + t(x, \omega)), \]

where \( H \) is the Heaviside function. By performing similar computations as in the proof of Lemma \[2.11\] one sees that \( v_{2,j} \) defined by \((517)\) is in fact a weak (distributional) solution of \((516)\). Moreover, \( v_{2,j} \) satisfies (in generalized sense) the inflow boundary condition, since \( t(y, \omega) = 0 \) on \( \tilde{\Gamma}_{-j} \) (see Lemma \[2.5\]) and \( r_{m,j} - \eta > 0 \) (therefore \( H(r_{m,j} - \eta - t(y, \omega)) = 1 \) on \( \tilde{\Gamma}_{-j} \)), as well as the initial (energy) condition, since \( t(x, \omega) > 0 \) on \( G \times S \) (hence \( H(r_{m,j} - \eta - t(x, \omega)) = 0 \) for all \( \eta \) close to \( r_{m,j} \)).

The solution of \((515)\), on the other hand, is obtained as follows. Let \( V_{1,j}(x, \omega; \eta) := v_{1,j}(x, \omega, r_{m,j} - \eta) \). Then the problem \((515)\) is equivalent to

\[ \frac{\partial V_{1,j}}{\partial \eta} + \omega \cdot \nabla_x V_{1,j} + \Sigma_j V_{1,j} = F_j, \quad V_{1,j} |_{\tilde{\Gamma}_{-j}} = v_{1,j}(\cdot, \cdot, 0) = 0, \]

where \( F_j(x, \omega, \eta) := \tilde{f}_j(x, \omega, r_{m,j} - \eta) \). Let \( B_0 : L^2(G \times S) \rightarrow L^2(G \times S) \) be a densely defined operator (as in section \[3.5\]) such that

\[ D(B_0) = W^2_{r,0}(G \times S), \quad B_0 \psi = -\omega \cdot \nabla_x \psi. \]

Then \( B_0 \) generates a contraction \( C^0 \)-semigroup \( T(\eta) \), and in fact for \( h \in L^2(G \times S) \) we have (cf. \[20\] Ch. XXI, Sec. 2.2, p. 222, or \[71\] proof of Theorem 5.15)

\[ (T(\eta)h)(x, \omega) = H(t(x, \omega) - \eta)h(x - \eta \omega, \omega), \]
where \( H \) is the Heaviside function. The problem (518) can be put into the abstract form

\[
\frac{\partial V_{i,j}}{\partial \eta} - (B_0 - \Sigma_j)V_{i,j} = F_j, \quad V_{i,j}(0) = 0,
\]

where \( (V_{i,j}(\eta))(x, \omega) = V_{i,j}(x, \omega, \eta) \) and \((F_j(\eta))(x, \omega) = F_j(x, \omega, \eta)\). The \( C^0 \)-semigroup \( G(\eta) \) generated by \( B_0 - \Sigma_j \) is (by the Trotter’s formula) given by

\[
(G(\eta)h)(x,\omega) = e^{-\int_0^\eta \Sigma_j(x-\tau\omega,\omega)d\tau}H(t(x,\omega) - \eta)h(x - \eta\omega,\omega).
\]

Hence the solution \( V_{i,j} \) is (cf. [25, p. 439], [53, pp. 105-108])

\[
V_{i,j}(\eta) = \int_0^\eta G(\eta - s)F_j(s)ds,
\]

and thus

\[
v_{i,j}(x, \omega, \eta) = V_{i,j}(x, \omega, r_{m,j} - \eta) = \int_0^{r_{m,j} - \eta} (G(r_{m,j} - s)F_j(s))(x, \omega)ds
\]

\[
= \int_0^{r_{m,j} - \eta} e^{-\int_0^{r_{m,j} - \eta - s} \Sigma_j(x-\tau\omega,\omega)d\tau} \cdot H(t(x,\omega) - (r_{m,j} - \eta - s))\tilde{f}_j(x - (r_{m,j} - \eta - s)\omega,\omega, r_{m,j} - s)ds.
\]

This can be shown to be a weak (distributional) solution of (515) by a similar argument as in the proof of Lemma 2.14.

Moreover, the weak solution of the (primary) problem

\[
\omega \cdot \nabla_x u_1 + \Sigma_1 u_1 = f_1,
\]

\[
u_1|_{\Gamma_-} = g_1,
\]

is given by (see Lemmas 2.11 and 2.14)

\[
u_1(x, \omega, E) = \int_0^{t(x,\omega)} e^{-\int_0^t \Sigma_1(x-s\omega,\omega,E)ds}f_1(x - t\omega,\omega, E)dt + e^{-\int_0^{t(x,\omega)} \Sigma_1(x-s\omega,\omega,E)ds}g_1(x - t(x,\omega)\omega,\omega, E).
\]

Hence the explicit solution of the total primary problem

\[
\omega \cdot \nabla_x u_1 + \Sigma_1(x, \omega, E)u_1 = f_1,
\]

\[-\frac{\partial(S_ju_j)}{\partial E} + \omega \cdot \nabla_x u_j + \Sigma_j u_j = f_j, \quad j = 2, 3\]

\[u_1|_{\Gamma_-} = g\]

\[u_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3,
\]

is given by \( u = (u_1, u_2, u_3) \), where \( u_1 \) is obtained from (523) and \( u_j, j = 2, 3 \) are obtained from formulas (514), (517) and (522), recalling that \( v_j = v_{1,j} + v_{2,j}, j = 2, 3 \).

6.2. A Solution Based on the Neumann Series. Consider the transport problem (497), (498), (499). Setting \( \phi = e^{CE\psi} \) as in Section 4, we recall that the problem then takes the equivalent form (360) - (363). Denote

\[
T_{1,C} \phi := \omega \cdot \nabla_x \phi_1 + \Sigma_1 \phi_1 - K_{1,C} \phi
\]

\[
T_{j,C} \phi := -\frac{\partial(S_j \phi_j)}{\partial E} + \omega \cdot \nabla_x \phi_j + CS_j \phi_j + \Sigma_j \phi_j - K_{j,C} \phi, \quad j = 2, 3,
\]
and define a (densely defined) closed linear operator $T_C : L^2(G \times S \times I)^3 \to L^2(G \times S \times I)^3$ by setting
\[
D(T_C) := \{ \phi \in L^2(G \times S \times I)^3 \mid T_j, C \phi \in L^2(G \times S \times I), j = 1, 2, 3 \}
\]
\[
T_C \phi := (T_{1, C} \phi, T_{2, C} \phi, T_{3, C} \phi).
\]
(526)
Let $f \in L^2(G \times S \times I)^3$ and $g \in T^2(\Gamma_-)^3$. In the case where $\phi$ is regular enough, say $\phi \in \tilde{W}^2(G \times S \times I) \times (\tilde{W}^2(G \times S \times I) \cap W^2_1(G \times S \times I))^2$, the problem (497), (498), (499) can be expressed equivalently as
\[
T_C \phi = f, \quad \phi_{\Gamma_-} = g, \quad \phi_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3,
\]
where $f = e^{CE} f$, $g = e^{CE} g$ as in Section 4.
Assume that $g \in H^1(I, T^2(\Gamma_-)^3)$ such that $g_j(E_m) = 0, j = 2, 3$. Then $g \in H^1(I, T^2(\Gamma_-)^3)$ and $g_j(E_m) = 0, j = 2, 3$. Applying on $g$ the lift operator $L$ given by
\[
((Lg)(x))(x, \omega) := g(E)(x - t(x, \omega) \omega, \omega) = g(x - t(x, \omega) \omega, \omega, E),
\]
we have $Lg \in H^1(I, \tilde{W}^2(G \times S)^3)$, and it satisfies (cf. [71, Lemma 5.11])
\[
\omega \cdot \nabla_x (Lg) = 0, \quad (Lg)_{\Gamma_-} = g.
\]
Furthermore, the condition $g_j(E_m) = 0$ implies that
\[
(Lg_j)(\cdot, \cdot, E_m) = 0, \quad j = 2, 3.
\]
(528)
Denoting
\[
P_1(x, \omega, E, D) \phi_1 := \omega \cdot \nabla_x \phi_1 + \Sigma_1 \phi_1
\]
\[
P_{j, C}(x, \omega, E, D) \phi_j := -\frac{\partial(S_j \phi_j)}{\partial E} + \omega \cdot \nabla_x \phi_j + CS_j \phi_j + \Sigma_j \phi_j, \quad j = 2, 3,
\]
(529)
and
\[
P_C(x, \omega, E, D) \phi := (P_1(x, \omega, E, D) \phi_1, P_{2, C}(x, \omega, E, D) \phi_2, P_{3, C}(x, \omega, E, D) \phi_3),
\]
\[
K_C \phi := (K_{1, C} \phi, K_{2, C} \phi, K_{3, C} \phi),
\]
we find that $T_C = P_C - K_C$. To simplify the notation, we shall write below $T = T_C$, $K = K_C$, and $P = P_C$.
Let $P_0$ be the densely defined linear operator acting in $L^2(G \times S \times I)^3$ such that
\[
D(P_0) := \{ \phi \in \tilde{W}^2(G \times S \times I) \times (\tilde{W}^2(G \times S \times I) \cap W^2_1(G \times S \times I))^2 \mid \phi_{\Gamma_-} = 0, \phi(\cdot, \cdot, E_m) = 0 \},
\]
\[
P_0 \phi := P \phi.
\]
(530)
Furthermore, let $\tilde{P}_0 : L^2(G \times S \times I)^3 \to L^2(G \times S \times I)^3$ be the smallest closed extension of $P_0$. Writing $u := \phi - Lg$, we see that $\phi = u + Lg$ is a solution of (527) if and only if
\[
(P - K)(u + Lg) = f, \quad u_{\Gamma_-} = 0, \quad u_j(\cdot, \cdot, E_m) = 0, \quad j = 2, 3,
\]
(531)
or equivalently,
\[
\tilde{P}_0 u = Ku + f - (P - K)(Lg).
\]
(532)
Now suppose that for some $k = 0, 1, 2, \ldots$ the following assumptions hold:
(A1) $\mathcal{T} := f - (P - K)(Lg) \in H^k(I, L^2(G \times S)^3)$,
(A2) $Ku \in H^k(I, L^2(G \times S)^3)$ for all $u \in L^2(G \times S \times I)^3$, and
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(A3) $\tilde{P}_0^{-1}$ exists as an operator $H^k(I, L^2(G \times S)^3) \to L^2(G \times S \times I)^3$.

These assumptions can be met if the data (and the geometry) are regular enough; see the example below.

Then Eq. (532) gives

$$u = \tilde{P}_0^{-1}Ku + \tilde{P}_0^{-1}\tilde{f},$$

which is equivalent to

$$(I - Q)u = \tilde{P}_0^{-1}\tilde{f},$$

where

$$Q := \tilde{P}_0^{-1}K.$$ 

If 1 belongs to the resolvent set $\rho(Q)$ of $Q$ we thus have

$$u = (I - Q)^{-1}\tilde{P}_0^{-1}\tilde{f},$$

and therefore

$$\phi = (I - Q)^{-1}\tilde{P}_0^{-1}\tilde{f} + Lg,$$

where we recall that $\tilde{f} = f - (P - K)(Lg)$.

Assuming that $Q : L^2(G \times S \times I)^3 \to L^2(G \times S \times I)^3$ is bounded and that

$$\|Q\| < 1,$$

which implies in particular that $1 \in \rho(Q)$, the solution $u$ of (532) can be computed through \textit{Neumann series}

$$u = \sum_{k=0}^{\infty} Q^k(\tilde{P}_0^{-1}\tilde{f}) = \sum_{k=0}^{\infty} (\tilde{P}_0^{-1}K)^k(\tilde{P}_0^{-1}\tilde{f}).$$

Finally, the solution $\phi$ of (527) is then

$$\phi = \sum_{k=0}^{\infty} (\tilde{P}_0^{-1}K)^k(\tilde{P}_0^{-1}\tilde{f}) + Lg,$$

from which the solution $\psi$ of the original problem (497)-(499) is obtained by

$$\psi = e^{-CE}\phi.$$

This method is known in the neutron transport community under the name \textit{source iteration} \cite{45}. It can also be used to prove existence, uniqueness and positivity of solutions for the standard transport equation (see e.g. \cite{27}).

\textbf{Example 6.2} In the case treated in Example 5.1 choose $C = 0$ and $k = 0$. Then a bounded inverse $\tilde{P}_0^{-1} : L^2(G \times S \times I)^3 \to L^2(G \times S \times I)^3$ exists and, in fact, can be explicitly computed using formulas given in Example 6.1 (choose $\tilde{g} = 0$ in Eqs.
the problem), which tend to be large due to the dimensionality of the problem, without any explicit inversion of matrices (coming from a chosen discretization of approximation of it by considering in the series (539) only finitely many terms, 

\[ \begin{aligned}
(\Phi_0^{-1}h)_1(x, \omega, E) &= \int_0^{\tau(x, \omega)} e^{-\int_0^t \Sigma_1(x-s\omega, \omega, E)ds} h_1(x - t\omega, \omega, E)dt, \\
(\Phi_0^{-1}h)_j(x, \omega, E) &= \frac{1}{S_2(E)} \left( \int_0^{r_{m,j} - R_j(E)} e^{-r_{m,j} - R_j(E) - t} \Sigma_2(x - t\omega, \omega)dt \\
& \quad \cdot H(t(x, \omega) - (r_{m,j} - R_j(E) - s)) \tilde{h}_j(x - (r_{m,j} - R_j(E) - s)\omega, \omega, r_{m,j} - s)ds \right),
\end{aligned} \]

for \( j = 2, 3 \), and where \( H \) is the Heaviside function, and 

\[ \tilde{h}_j(x, \omega, \eta) = S_i(R_i^{-1}(\eta)) \tilde{h}_i(x, \omega, R_i^{-1}(\eta)), \quad i = 1, 2, 3. \]

Hence under the stated assumptions the solution \( \psi = \phi \) is obtained from \((539)\) (note that \( f = f \), \( g = g \) for \( C = 0 \)) and \((540)\). This shows that the assumption (A1)-(A3) can indeed be met (here for \( k = 0 \)).

It is also worth noticing that the above expression for \((\Phi_0^{-1}h)_j, j = 2, 3\), can be simplified into

\[ \begin{aligned}
(\Phi_0^{-1}h)_j(x, \omega, E) &= \frac{1}{S_2(E)} \left( \int_0^{\min \{r_{m,j} - R_j(E), \tau(x, \omega)\}} e^{-\int_0^t \Sigma_2(x - t\omega, \omega)dt} \tilde{h}_j(x - t\omega, \omega, R_j(E) + t)dt \right).
\end{aligned} \]

The Neumann series based method enables one to compute the solution, or an approximation of it by considering in the series \((539)\) only finitely many terms, without any explicit inversion of matrices (coming from a chosen discretization of the problem), which tend to be large due to the dimensionality of the problem, which is 6: there are 3 spatial (\( x \)), 2 angular (\( \omega \)) and one energy (\( E \)) dimensions. Sufficient criteria for the condition \( \|Q\| < 1 \) must be retrieved.

**Remark 6.3** The applicability of the above method for general cases remains open. One possibility to apply formulae like \((540)\) for spatially inhomogeneous substance (that is, \( S_j \) and \( \Sigma_j \) are dependent on \( x \in G \)) is to apply *domain decomposition method* in such a way that \( S_j \) are assumed to be constant in subdomains.

**Example 6.4** In this example, we write out Eq. \((541)\) in special case of constant \( S_0 \geq 0 \) and \( \Sigma \geq 0 \). Moreover, we consider a single particle CSDA transport equation only. Let

\[ P(x, \omega, E, D)u := -\frac{\partial(S_0u)}{\partial E} + \omega \cdot \nabla_x u + \Sigma u, \]

In this case, \( R(E) = \int_0^E \frac{1}{S_0} d\tau = \frac{1}{S_0} E \), and Eq. \((541)\) gives, when \( S_0 > 0 \), using the notation \( \eta(E) := (E_m - E)/S_0 \) and noticing that \( r_m := R(E_m) = \frac{E_m}{S_0} \),

\[ (\Phi_0^{-1}h)(x, \omega, E) = \int_0^{\min \{\eta(E), \tau(x, \omega)\}} e^{-\Sigma s} h(x - s\omega, \omega, E + S_0 s)ds, \]
for $h \in L^2(G \times S \times I)$. It is clear that this last formula gives the correct (explicit) expression for $P_0^{-1}$ also in the case where $S_0 = 0$, if we make the convention that $\eta(E) = +\infty$ for all $E \in I$ when $S_0 = 0$.

6.3. An Approximative Solution Based on the Theory of Evolution Equations. In this section we for simplicity restrict ourselves to a single particle CSDA-equation

$$-\frac{\partial(S_0 \psi)}{\partial E} + \omega \cdot \nabla_x \psi + \Sigma \psi - K \psi = f,$$

(542)

$$\psi|_{\Gamma_-} = g, \quad \psi(\cdot, \cdot, E_m) = 0.$$  

(543)

Suppose that the assumptions of Theorem 3.25 are valid.

Another method to compute approximately the solution of the problem (542), (543) which avoids the explicit inversions of matrices, can be (formally) described as follows. Note that we will be using throughout this section the notations of Section 3.4. After the change of unknown $\phi = e^{CE} \psi$ the problem is

$$T_C \phi = f, \quad \phi|_{\Gamma_-} = g, \quad \phi(\cdot, \cdot, E_m) = 0,$$

(544)

where

$$T_C \phi := -\frac{\partial(S_0 \phi)}{\partial E} + \omega \cdot \nabla_x \phi + CS_0 \phi + \Sigma \phi - K_C \phi.$$

Assume that $g \in H^1(I, T^2(\Gamma^-))$ and that $g(\cdot, \cdot, E_m) = 0$ on $G \times S$. Let $u := \phi - L(g)$. Then $u$ satisfies

$$T_{C,0} u = \tilde{f}, \quad u|_{\Gamma_-} = 0, \quad u(\cdot, \cdot, E_m) = 0,$$

(545)

where

$$\tilde{f} := f - T_C(L(g)).$$

By the Trotter’s formula, the semigroup $G(t)$ generated by $T_{C,0}$ is given by

$$G(t) \tilde{f} = \lim_{n \to \infty} \left( T_{B_0}(t/n)T_{A_0}(t/n)T_{-(\Sigma+CI)}(t/n)T_{K_C}(t/n) \right)^n \tilde{f},$$

(546)

where the convergence is uniform on compact $t$-intervals $[0, T]$. Note that the individual semi-groups $T_{B_0}(t)$, $T_{A_0}(t)$, $T_{-(\Sigma+CI)}(t)$ and $T_{K_C}(t)$ contributing to this expression can be computed explicitly (see section 3.4). Hence we get

$$\psi = \int_0^{\infty} G(t) \tilde{f} dt \approx \int_0^{T} \tilde{G}(t) \tilde{f} dt$$

$$= \int_0^{T} \lim_{n \to \infty} \left( T_{B_0}(t/n)T_{A_0}(t/n)T_{-(\Sigma+CI)}(t/n)T_{K_C}(t/n) \right)^n \tilde{f} dt$$

$$\approx \int_0^{T} \left( T_{B_0}(t/n_0)T_{A_0}(t/n_0)T_{-(\Sigma+CI)}(t/n_0)T_{K_C}(t/n_0) \right)^{n_0} \tilde{f} dt,$$

(547)

for large enough $T$ and $n_0$. On the other hand, the semi-group $T_{K_C}$ generated by the bounded operator $K_C$ can be approximately computed from

$$T_{K_C}(t) \approx \sum_{k=0}^{N_0} \frac{1}{k!}(tK_C)^k,$$

for large enough $N_0$. 

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We point out that this semi-group theory-based approach, unlike the one given in the previous section, does not require extra assumptions on cross-sections, like the ones imposed by the condition (537).

**Remark 6.5** Assume that $K$ is of the form

$$(K\psi)(x, \omega, E) = \int \sigma(x, \omega', \omega, E)\psi(x, \omega', E) d\omega'.$$

Furthermore, as in Example 6.1 we assume that $S_0 = S_0(E)$ (independent of $x$) and we define $R(E) := \int_0^E \frac{1}{S_0(\tau)} d\tau$, $\eta := R(E)$, $r_m := R(E_m)$ and $\tilde{I} := R(I) = [0, r_m]$. Let

$$v(x, \omega, \eta) := S_0(R^{-1}(\eta))\psi(x, \omega, R^{-1}(\eta)), \\
\tilde{\Sigma}(x, \omega, \eta) := \Sigma(x, \omega, R^{-1}(\eta)), \\
\tilde{\sigma}(x, \omega', \omega, \eta) := \sigma(x, \omega', \omega, R^{-1}(\eta)), \\
(\tilde{K}v)(x, \omega, \eta) := \int \tilde{\sigma}(x, \omega', \omega, \eta)v(x, \omega', \eta) d\omega', \\
\tilde{f}(x, \omega, \eta) := S_0(R^{-1}(\eta))f(x, \omega, R^{-1}(\eta)), \\
\tilde{g}(x, \omega, \eta) := S_0(R^{-1}(\eta))g(y, \omega, R^{-1}(\eta)).$$

As a further simplification, suppose that $g = 0$. After some technical considerations, the problem (542), (543) can be cast into the abstract form

$$\frac{\partial V}{\partial \eta} - A(\eta)V = F(\eta), \quad V(0) = 0,$$

(548)

where $V(\eta) := v(r_m - \eta)$, $F(\eta) := \tilde{f}(\cdot, r_m - \eta)$. In the case where appropriate assumptions are valid (cf. Theorem 3.29), the solution of (548) is given by

$$V(\eta) = \int_0^\eta U(\eta, s)F(s)ds,$$

(549)

where $U(\eta, s) : L^2(G \times S) \to L^2(G \times S)$, $0 \leq \eta \leq s \leq r_m$, is the evolution family of operators $A(\eta)$, $\eta \in \tilde{I}$.

By making use of the explicit expressions for the semi-groups $T_{B_0}(s)$ (here $B_0$ is as in the proof of Lemma 3.31), $T_{\Sigma(\eta)}(s)$, and $T_{\tilde{K}(\eta)}(s)$, there exist some approximative methods (e.g. Cauchy-Peano approximations) for computing the family of evolution operators $U(\eta, s)$ (cf. 34). This gives an approach for calculating $V$ approximately by

$$V(\eta) \approx \tilde{V}(\eta) := \int_0^\eta \tilde{U}(\eta, s)F(s)ds,$$

where $\tilde{U}(\eta, s)$ is an approximation of $U(\eta, s)$. The solution $\psi$ would then be approximated as

$$\psi(x, \omega, E) \approx \frac{1}{S_0(E)} \tilde{V}(r_m - R(E)).$$

(550)

For general $g$ the idea remains the same. It is worth studying this approach under more general assumptions as well.
7. An Outlook on the Inverse Radiation Treatment Problem

As mentioned above, in radiation therapy the dose absorbed from particle field \( \psi = (\psi_1, \psi_2, \psi_3) \) is defined by

\[
D(x) = (D \psi)(x) := \sum_{j=1}^{3} \int_{S \times I} \varsigma_j(x, E) \psi_j(x, \omega, E) d\omega dE, \tag{551}
\]

where \( \varsigma_j \in L^\infty(G \times I) \), \( \varsigma_j \geq 0 \) are the so-called (total) stopping powers.

Also, it is worth recalling that the component fields of \( \psi \), relevant to photon and electron radiation therapy, are \( \psi_1 = \) photons, \( \psi_2 = \) electrons and \( \psi_3 = \) positrons. Clearly, nothing we have said above depends on the number, or designation (to certain particle species) of fields treated (or even the dimensionality of the spaces \( G, S \) or \( I \) in fact), with the exception that typically only charged particle fields (are assumed to) obey CSDA version of the transport equation (cf. (5)), while non-charged particles obey the standard linear BTE (cf. (1)). Thus with very minor modifications, and in particular if one is interested in radiation therapy, what will and has been said works, in principle, equally well in proton (and ion) therapy framework as well.

We find that \( D : L^2(G \times S \times I)^3 \rightarrow L^2(G) \) is a bounded linear operator and its adjoint operator \( D^* : L^2(G) \rightarrow L^2(G \times S \times I)^3 \) is simply a multiplication type operator,

\[
D^* d = (\varsigma_1, \varsigma_2, \varsigma_3) d, \quad \text{for } d \in L^2(G). \tag{552}
\]

We describe shortly an optimization problem related to inverse radiation treatment planning. We restrict ourselves to external radiation therapy in which the particles are inflowing through the patch(es) of patient surface. This means that in the transport problem \( f = 0 \) (i.e. the internal particle source vanishes) and \( g \) (the inflow particle flux) is the variable to be controlled. Conversely, for the internal radiation therapy problems one sets \( g = 0 \), and \( f \) would be the variable to be controlled. Anyhow, the results presented below would be analogous in this situation. We refer to [71, Section 7] and to the references therein for a more detailed exposition of inverse problem (optimization) in this setting. Let \( g \in T^2(\Gamma_-)^3 \) and \( \psi = \psi(g) \in \bar{W}^2(G \times S \times I) \times (\bar{W}^2(G \times S \times I) \cap W_1^2(G \times S \times I))^2 \) be the solution of the variational equation (see (382), (385) and (386))

\[
\tilde{B}_0(\psi(g), v) = F_0(v) \quad \forall v \in \tilde{H}, \tag{553}
\]

where (since \( f = 0 \))

\[
F_0(v) = (F_0 g)(v) := \sum_{j=1}^{3} \int_{\partial G \times S \times I} (\omega \cdot \nu)_- g v j d\sigma d\omega dE = \langle g, \gamma_-(v) \rangle_{T^2(\Gamma_-)^3}. \tag{554}
\]

The deposited dose is then

\[
D(x) = (D(\psi(g)))(x), \quad x \in G. \tag{555}
\]

We shall also denote \( D(g) := D(\psi(g)) \).

Denote the target region by \( T \subset G \), the critical organ region by \( C \subset G \) and the normal tissue region by \( N \subset G \). Then \( G = T \cup C \cup N \) where the union is mutually disjoint. Suppose that \( d_T \in L^2(T) \), \( d_C \in L^2(C) \), \( d_N \in L^2(N) \) are given dose distributions in the respective regions (for example, they may be constants).
We define a strictly convex object (cost) function $J : X \to \mathbb{R}$ by (see [71])

$$J(g) = c_T \|d_T - D(g)\|_{L^2(T)}^2 + c_C \|d_C - D(g)\|_{L^2(C)}^2 + c_N \|d_N - D(g)\|_{L^2(N)}^2 + c \|g\|_X^2,$$  \hspace{1cm} (556)

where $X := T^2(\Gamma_-) \times H^1(I, T^2(\Gamma_-))^2$ equipped with the inner product

$$\langle g, h \rangle_X := \langle g_1, h_1 \rangle_{T^2(\Gamma_-)} + \sum_{j=2}^{3} \int_I \left\langle \frac{\partial g_j}{\partial E}, \frac{\partial h_j}{\partial E} \right\rangle_{T^2(\Gamma_-)} dE.$$

Let $Y$ be a closed subspace of $X$ defined by (here we denote $g(y, \omega, E) := (g(E))(y, \omega)$)

$$Y := \{ g \in X | g_j(\cdot, \cdot, E_m) = 0, \ j = 2, 3 \},$$  \hspace{1cm} (557)

A relevant admissible set (of controls) is

$$U_{ad} = \{ g \in Y | g \geq 0 \text{ a.e. on } G \times S \times I \},$$  \hspace{1cm} (558)

which is a closed convex subset of $X$.

Suppose that the assumptions of Theorem [110] are valid. Furthermore, suppose that $c_j \in H^2(I, L^\infty(G)), \ j = 1, 2, 3$. Then the minimum $\min_{g \in U_{ad}} J(g)$ exists. In addition, a necessary and sufficient condition that $\overline{g} \in U_{ad}$ is the minimum point, is that the following relations hold (cf. [71] proof of Theorem 7.7)

$$- \langle \gamma(-\psi^*), w \rangle_{T^2(\Gamma_-)^3} + c \langle \overline{g}, w \rangle_X \geq 0 \ \forall \ w \in U_{ad},$$  \hspace{1cm} (559)

$$- \langle \gamma(-\psi^*), \overline{g} \rangle_{T^2(\Gamma_-)^3} + c \|\overline{g}\|_X^2 = 0,$$  \hspace{1cm} (560)

$$\mathbf{B}_0(\psi, v) = (\overline{g}, v)_{T^2(\Gamma_-)^3}, \ \forall v \in \mathcal{H},$$  \hspace{1cm} (561)

where $\mathbf{B}_0(\cdot, \cdot)$ is given in (381), and $\mathbf{B}_0^*(\cdot, \cdot)$ is the extension onto $\mathcal{H} \times \mathcal{H}$ of $B_0^*$ given in (476).

We emphasize that here the described solution $\overline{g}$ of the optimal control problem can be used only as an initial point for the actual treatment planning where global optimization (see e.g. [53]) is required. The determination of a carefully chosen initial point for a large dimensional global optimization scheme is very essential for achieving (time savings and) satisfactory results ([55]).

We also notice that if we contented ourselves with so-called mild solutions ([53, p. 146]), then the existence of an optimal control $\overline{g}$ (the admissible set being a subset of $T^2(\Gamma_-)^3$), together with the explicit formula $\overline{g} = \frac{1}{c}(\gamma(-\psi^*))_+$ for it, could be proven under quite weak assumptions. Here $(h)_+$ denotes the positive part of a function $h$. In any case, the validity of estimates such as (398) is essential for guaranteeing that $\mathcal{D}$ be a bounded linear operator in appropriate spaces. The use of mild solutions, however, has the drawback that the inflow boundary conditions are not necessarily satisfied by the solutions (and thus the solutions might be non-physical).

We remark that, for example with respect to $x$-variable the solution is generally at most in $H^2(s,0,0)(G \times S \times I^2)$ with $s < \frac{3}{2}$ where $H^2(s,0,0)(G \times S \times I^2)$ is the mixed-norm Sobolev-Slobodeckij space with fractional index $s$ and Lebesgue index 2 (with respect to $x$-variable).

Finally, we notice that the system of the above variational equations can be implemented within e.g. a FEM-type numerical scheme to get numerical solutions for the optimal control $\overline{g}$. In virtue of Cea’s estimate, the boundedness and coercivity (after
replacing the unknown $\psi$ by $\phi = e^{CE_0}\psi$) of the bilinear forms $B_0(\cdot, \cdot), B^*_0(\cdot, \cdot)$ guarantee the convergence of the scheme in principle. Nevertheless, the above mentioned limited regularity of solutions of transport problems implies that the standard local interpolation results are not necessarily applicable and more advanced analysis (e.g. in choosing relevant basis functions) is needed.

We omit in this paper further discussion of the inverse radiation treatment problem which was outlined above. Nevertheless, we refer to [28] for related treatments.

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