ON THE DEFINITION OF NEUTROSOPHIC LOGIC

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Abstract. Smarandache (2003) introduced a new set-valued fuzzy logic called (nonstandard) neutrosophic logic by using Robinson’s non-standard analysis. However, its definition involved many errors including the illegal use of non-standard analysis. In this paper, we provide a rigorous definition of neutrosophic logic. All the errors in the original definition are addressed. We then point out some paradoxes of neutrosophic logic. Finally we formulate neutrosophic logic with no use of nonstandard analysis.

1. Introduction

Smarandache [6] introduced a new set-valued fuzzy logic, called (nonstandard) neutrosophic logic. This logic can be characterised as follows: each proposition takes a value of the form \((T, I, F)\), where \(T\), \(I\) and \(F\) are subsets of the non-standard unit interval \([-0, 1^+]\) (defined later) and represent all possible values of truthiness, indeterminacy and falsity of the proposition, respectively. Unfortunately, the original definition contains many errors including the illegal use of nonstandard analysis.

In Section 2, we first point out the errors that appear in the original definition. We then provide a rigorous definition of neutrosophic logic. All the errors in the original definition are corrected. We also point out some problems concerning neutrosophic logic, e.g., the paradox where complex propositions may have strange truth values. In Section 3, we give alternative definition of nonstandard neutrosophic logic without nonstandard analysis.

Note that, in this paper, we do not discuss the theoretical significance or the applications of neutrosophic logic. We only focus on mathematical correctness of the target theory.

We assume the reader to have basic familiarity with elementary nonstandard analysis (over the reals \(\mathbb{R}\)). We refer to Robinson [5] for the foundations of nonstandard analysis.

2. Correction of the definition

We first review the original definition of neutrosophic logic given in Smarandache [6, 7] and keep track of the errors (and ambiguity). We then provide a rigorous and corrected definition of neutrosophic logic. Finally we point out some paradoxical (counter-intuitive) behaviours of the neutrosophic logical connectives.

Before reviewing the definition, we recall the outline of the definition. The usual fuzzy logic and its minor variations use the closed unit interval \([0, 1]_\mathbb{R}\) as...
ON THE DEFINITION OF NEUTROSOPHIC LOGIC

the set of truth values. Smarandache’s (nonstandard) neutrosophic logic uses the nonstandard unit interval \([-0, 1^+]\) of the hyperreals \(\mathbb{R}^*\) instead of the closed unit interval \([0, 1]\) of the reals \(\mathbb{R}\). The nonstandard unit interval \([-0, 1^+]\) consists of three types of hyperreals: hyperreals between 0 and 1, hyperreals infinitely close to but less than 0, and hyperreals infinitely close to but greater than 1. Neutrosophic logic is then characterised as the \(\mathcal{P}(\mathbb{R}^*)^3\)-valued logic. Intuitively, for \((T, I, F) \in \mathcal{P}(\mathbb{R}^*)^3\), its components \(T\), \(I\) and \(F\) represent the set of truthiness, indeterminacy and falsity, respectively. The logical operations on \(\mathcal{P}(\mathbb{R}^*)^3\) are defined according to this intuitive idea.

2.1. Confused notations. The notations \(-a\) and \(b^+\) are used as particular hyperreal numbers.

Let \(\varepsilon > 0\) be a such infinitesimal number. [...] Let’s consider the nonstandard finite numbers \(1^+ = 1 + \varepsilon\), where “1” is its standard part and “\(\varepsilon\)” its non-standard part, and \(-0 = 0 - \varepsilon\), where “0” is its standard part and “\(\varepsilon\)” its non-standard part. ([6, p. 141]; [7, p. 9])

At the same time, the notations \(-a\) and \(b^+\) are also used as particular sets of hyperreal numbers.

Actually, by “\(-a\)” one signifies a monad, i.e., a set of hyper-real numbers in non-standard analysis:

\[ (-a) = \{ a - x \in \mathbb{R}^* \mid x \text{ is infinitesimal} \}, \]

and similarly “\(b^+\)” is a hyper monad:

\[ (b^+) = \{ b + x \in \mathbb{R}^* \mid x \text{ is infinitesimal} \}. \]

([6, p. 141]; [7, p. 9])

This confusion of notation can be found thereafter: in [7, pp.10–11], the notations \(-a\) and \(b^+\) are used as hypermonads; on the other hand, in [7, p.13], \(1^+\) is used as a particular hyperreal number.

Note that the definitions of (one-sided) monads have minor errors. The correct definitions are the following:

\[ (-a) := \{ a - x \in \mathbb{R}^* \mid x \text{ is positive infinitesimal} \}, \]
\[ (b^+) := \{ b + x \in \mathbb{R}^* \mid x \text{ is positive infinitesimal} \}. \]

2.2. Ambigious definition of the nonstandard unit interval. Smarandache ([6] gives no explicit definition of the nonstandard unit interval \([-0, 1^+]\) (or \([-0, 1^+]\) in [7])). He only says:

Then, we call \([-0, 1^+]\) a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. ([6, p. 141]; [7, p. 9])

Here \(-0\) and \(1^+\) are particular real numbers defined in the previous paragraph: \(-0 = 0 - \varepsilon\) and \(1^+ = 1 + \varepsilon\), where \(\varepsilon\) is a fixed non-negative infinitesimal. (Note that the phrase “infinitely small but less than 0” should be read as “infinitely close to but less than 0”. Similarly, the phrase “infinitely small but greater than 1” is intended to mean “infinitely close to but greater than 1”.)
There are two possible definitions of the nonstandard unit interval:

(1) \( \left] -0, 1^+ \right[ = \{ x \in \mathbb{R}^* \mid -0 < x < 1^+ \} \) following the unusual notation of open interval;

(2) \( \left] -0, 1^+ \right] = \{ x \in \mathbb{R}^* \mid 0 \leq x \leq 1 \} \). (This is equal to the monad \( \mu ([0, 1]) \) of the closed unit interval.)

In the first definition, it is false that hyperreals infinitely close to but less than 0 or infinitely close to but greater than 1, belong to the nonstandard unit interval. The hyperreal \( 0 - 2\varepsilon \) is infinitely close to 0 but not in \( \left] -0, 1^+ \right[ \). Similarly for \( 1 + 2\varepsilon \). Thus the first definition is not compatible with the above-quoted sentence. The second definition is better than the first one: when adopting the first one, the resulting logic depends on the choice of the positive infinitesimal \( \varepsilon \). In our definition, we adopt the second one.

The cause of the confusion and the ambiguity can be found in the following quote:

"We can consider \((-a)\) equals to the open interval \((a - \varepsilon, a)\), where \(\varepsilon\) is a positive infinitesimal number. Thus:

\((-a) = (a - \varepsilon, a)\)

\((b^+) = (b, b + \varepsilon)\)

\((-a^+) = (a - \varepsilon_1, a) \cup (a, a + \varepsilon_2), \) where \(\varepsilon, \varepsilon_1, \varepsilon_2\) are positive infinitesimal numbers. ([5, p. 10])"

Obviously it is wrong. Suppose, on the contrary, that \((-a)\) can be expressed in the form \((a - \varepsilon, a)\). Then \(a - \varepsilon\) does not belong to \((-a)\). On the other hand, \(a - \varepsilon\) is less than but infinitely close to \(a\), so \(a - \varepsilon \in (-a)\), a contradiction. This false belief well explains why Smarandache fell into the confusion of notation and why he gave only an ambiguous description of the nonstandard unit interval: if the monad could be described like above, the two definitions of the unit interval would be equivalent.

2.3. Misuse of nonstandard analysis. Let us continue to read the definition.

Let \(T, I, F\) be standard or non-standard real subsets of \(\left] -0, 1^+ \right[\),

with \(\sup T = t_{\sup}, \inf T = t_{\inf}\),

\(\sup I = i_{\sup}, \inf I = i_{\inf}\),

\(\sup F = f_{\sup}, \inf F = f_{\inf}\),

and \(n_{\sup} = t_{\sup} + i_{\sup} + f_{\sup}\),

\(n_{\inf} = t_{\inf} + i_{\inf} + f_{\inf}\).

The sets \(T, I, F\) are not necessarily intervals, but may be any real sub-unitary subsets: discrete or continuous; single-element, finite, or (countably or uncountably) infinite; union or intersection of various subsets; etc. ([8 pp. 142–143]; [9 p. 12])

Subsets of \(\left] -0, 1^+ \right[\) may have neither infima nor suprema, because the transfer principle ensures the existences of infima and suprema only for internal sets. External sets may lack suprema and/or infima. For instance, the monad \(\mu (1/2) = \{ x \in \mathbb{R}^* \mid x \approx 1/2 \}\) has neither the infimum nor the supremum. To see this, suppose, on the contrary, that \(\mu (1/2)\) has the infimum \(L = \inf \mu (1/2)\). Let \(\varepsilon\) be any positive infinitesimal. Then there is an \(x \in \mu (1/2)\) such that \(x \leq L + \varepsilon\). Since every hyperreal infinitely close to \(\mu (1/2)\) belongs to \(\mu (1/2)\), we have that \(x - 2\varepsilon \in \mu (1/2)\). Hence \(L \leq x - 2\varepsilon \leq L - \varepsilon\), a contradiction. The same applies to the supremum.

There are two workarounds:
are infinitely close to $f(a, b)$, respectively. Of course, $h$ is continuous everywhere. The standard unit interval $[0, 1]$ is closed under $f, g$ and $h (\cdot, \cdot, \cdot)$. Here we only prove the case of $h (1, \cdot, \cdot, \cdot)$. Since $h (1, a, b)$ is monotonically decreasing for $a$, we have that $\min_{(a, b) \in [0, 1]^2} h (1, a, b) = \min_{b \in [0, 1]} h (1, 1, b) = 0$. Similarly, since $h (1, a, b)$ is monotonically increasing for $b$, we have that $\max_{(a, b) \in [0, 1]^2} h (1, a, b) = \max_{a \in [0, 1]} h (1, a, 1) = 1$. Thus $h \left( \{1\} \times [0, 1] \times [0, 1] \right) \subseteq [0, 1]$.

Now, let $a, b \in \cdot - 0, 1^+$ and $c \in 1^+$. Choose $a', b' \in [0, 1]$ infinitely close to $a, b$, respectively. Of course, $c$ is infinitely close to 1. By the nonstandard characterisation of continuity (see Theorem 4.2.7 of [2]), $f^* (a, b), g^* (a, b), h^* (c, a, b)$ are infinitely close to $f (a', b'), g (a', b'), h (1, a', b') \in [0, 1]$, respectively. Hence $f^* (a, b), g^* (a, b), h^* (c, a, b) \in \cdot - 0, 1^+$. Let $A, B \in V$. Then $A \odot B = f^* (A \times B)$, $A \ominus B = g^* (A \times B)$ and $A \oslash B = h^* (1^+ \times A \times B)$ belong to $V$.

According to the original definition ([6, p. 143]), neutrosophic logic is the $V^3$-valued (extensional) logic. Each proposition takes a value of the form $(T, I, F) \in V^3$, where $T$ represents possible values of truthiness, $I$ indeterminacy, and $F$ falsity. The
logical connectives $\land, \lor, \rightarrow$ are defined as follows:

$$(T_1, I_1, F_1) \land (T_2, I_2, F_2) = (T_1 \odot T_2, I_1 \odot I_2, F_1 \odot F_2),$$
$$(T_1, I_1, F_1) \lor (T_2, I_2, F_2) = (T_1 \oplus T_2, I_1 \oplus I_2, F_1 \oplus F_2),$$
$$(T_1, I_1, F_1) \rightarrow (T_2, I_2, F_2) = (T_1 \ominus T_2, I_1 \ominus I_2, F_1 \ominus F_2).$$

Note that our definitions of $\ominus$ and $\odot$ are different from the original ones. Smarandache uses the following operations instead of $\ominus$ and $\odot$ (see Smarandache [6, p. 145]):

$$A \circ B = (A \oplus B) \ominus (A \odot B),$$
$$A \circ B = 1^+ \ominus A \oplus (A \odot B),$$

where $\ominus, \odot, \oplus$ are the elementwise subtraction, multiplication and addition of sets. There are at least two reasons why the original definition is not good. The first is pre-mathematical. When calculating, for example, $A \circ B = (A \oplus B) \ominus (A \odot B)$, the second and the third occurrences of $A$ can take different values, despite that they represent the same proposition. The same applies to $B$ and the calculation of $A \circ B$ obviously. The second is mathematical. $\lor$ is not closed under those operations:

$$2 = 1 + 1 - 0 \cdot 1 \in \{0, 1\} \circ \{1\},$$
$$2 + \varepsilon = 1 + \varepsilon - 0 + 1 \cdot 1 \in \{0, 1\} \circ \{1\},$$

where $\varepsilon$ is positive infinitesimal. Because of this, the following ad-hoc workaround is needed:

$$[... \text{ if, after calculations, one obtains number } < 0 \text{ or } > 1 \text{, one replaces them by } -0 \text{ or } 1^+, \text{ respectively. (2, p. 145)}]$$

2.5. Paradoxical phenomena. Consider the $\lor$–valued logic, where each proposition takes a truth value $T \in \mathcal{V}$. Each neutrosophic logical connectives was defined componentwise. In other words, neutrosophic logic is the 3–fold product of the $\lor$–valued logic. Hence neutrosophic logic cannot be differentiated from the $\lor$–valued logic by equational properties (see Burris and Sankappanavar [2, Lemma 11.3 of Chapter II]).

This causes some paradoxical phenomena. Let $A$ be a (true) proposition with value $\{\{1\} , \{0\} , \{0\}\}$ and let $B$ be a (false) proposition with value $\{\{0\} , \{0\} , \{1\}\}$. Usually we expect that the falsity of the conjunction $A \land B$ is $\{1\}$. However, its actual falsity is $\{0\}$. We expect that the indeterminacy of the negation $\lnot A$ is $\{0\}$. However, its actual indeterminacy is $1^+$ (see Smarandache [6, p. 145] for the definition of the negation).

This problem has been addressed in Rivieccio [4].

3. Neutrosophic logic without nonstandard analysis

3.1. Nonarchimedean fields. Let $K$ be an ordered field. It is well-known that the ordered semiring $\mathbb{N}$ of natural numbers can be canonically embedded into $K$ by sending $n \rightarrow 1_k + \cdots + 1_k$. This embedding can be uniquely extended to the ordered ring $\mathbb{Z}$ of integers and to the ordered field $\mathbb{Q}$ of rational numbers. Thus we may assume without loss of generality that $\mathbb{Q} \subseteq K$. An element $x \in K$ is said to be infinitesimal (relative to $\mathbb{Q}$) if for any positive $q \in \mathbb{Q}$, $|x| \leq q$. For instance,
the unit of the addition $0_K$ is trivially infinitesimal. The ordered field $K$ is called nonarchimedean if it has nonzero infinitesimals.

**Example 3.1.** Every ordered subfield of the real field $\mathbb{R}$ is archimedean. Conversely, every archimedean ordered field can be (uniquely) embedded into $\mathbb{R}$ (see Blyth [1, Theorem 10.21]).

**Example 3.2.** The hyperreal field $\mathbb{R}^*$ is a nonarchimedean ordered field. Generally, every proper extension $K$ of $\mathbb{R}$ is nonarchimedean: Let $x \in K \setminus \mathbb{R}$. There are two cases. Case I: $x$ is infinite (i.e. its absolute value $|x|$ is an upper bound of $\mathbb{R}$). Then its reciprocal $1/x$ is nonzero infinitesimal. Case II: $x$ is finite. Then the set \{ $y \in \mathbb{R} \mid y < x$ \} is nonempty and bounded in $\mathbb{R}$. So it has the supremum $x^\circ$. The difference $x - x^\circ$ is nonzero infinitesimal. Hence $K$ is nonarchimedean.

**Example 3.3** (cf. Gelbaum and Olmsted [3, pp. 15–16]). Let $K$ be an ordered field. Let $K(X)$ be the field of rational functions over $K$. Define an ordering on $K(X)$ by giving a positive cone:

\[ 0 \leq \frac{f(X)}{g(X)} \iff 0 \leq \text{the leading coefficients of } f(X) \]

\[ \text{the leading coefficients of } g(X). \]

Then $K(X)$ forms an ordered field having nonzero infinitesimals (relative to not only $\mathbb{Q}$ but also $K$) such as $1/X$ and $1/X^2$. Hence $K(X)$ is nonarchimedean.

### 3.2. Alternative definition of neutrosophic logic.

Comparing with other nonarchimedean fields, one of the essential features of the hyperreal field $\mathbb{R}^*$ is the transfer principle, which states that $\mathbb{R}^*$ has the same first-order properties as $\mathbb{R}$. On the other hand, the formulation of neutrosophic logic does not depend on the transfer principle. The use of nonstandard analysis is not essential for this logic, and can be eliminated from its definition.

Fix a nonarchimedean ordered field $K$. For $x, y \in K$, $x$ and $y$ are said to be infinitely close (denoted by $a \approx b$) if $a - b$ is infinitesimal. We say that $x$ is roughly smaller than $y$ (and write $x \lessdot y$) if $x < y$ or $x \approx y$. For $a, b \in K$ the set $]^{-a}, b^+[K$ is defined as follows:

\[ ]^{-a}, b^+[K = \{ x \in K \mid a \lessdot x \lessdot b \}. \]

Let $V_K$ be the power set of $]^{-0}, 1^+[K$. The $K$–valued neutrosophic logic is defined as the $V_K$–valued logic. The rest of the definition is completely the same as the case $K = \mathbb{R}^*$.

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