Euler-Mahonian polynomials for $C_a \wr S_n$

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1 Introduction

Let $S_n$ denote the symmetric group on $\{1, \ldots, n\}$. The classical descent number and major index statistics on $S_n$ are defined by

$$\text{des}(\sigma) := \sum_{i=1}^{n-1} \chi(\sigma(i) > \sigma(i+1))$$

and

$$\text{maj}(\sigma) := \sum_{i=1}^{n-1} i \chi(\sigma(i) > \sigma(i+1))$$

respectively, where $\sigma \in S_n$ and $\chi(p)$ equals 1 if the statement $p$ is true and 0 if $p$ is false.

The classical Eulerian polynomials $A_n(t)$ may be defined by

$$A_n(t) := \sum_{\sigma \in S_n} t^{\text{des}(\sigma)}$$

and their “maj” $q$-analogues, also known as the $q$-maj Euler-Mahonian polynomials, may be defined by

$$\text{maj} A_n(t, q) := \sum_{\sigma \in S_n} t^{\text{des}(\sigma)}q^{\text{maj}(\sigma)}. \quad (1)$$
Carlitz \cite{Car75} has given a recurrence formula for the coefficients of \( \text{maj} A_n(t, q) \): with \( \text{maj} A_n(t, q) = \sum_{s \geq 0} t^s \text{maj} A_{n,s}(q), \)

\[
\text{maj} A_{n,s}(q) = (1 + q + \cdots + q^s) \text{maj} A_{n-1,s}(q) + (q^s + q^{s+1} + \cdots + q^{n-1}) \text{maj} A_{n-1,s-1}(q). \tag{2}
\]

Gessel \cite{Ges77} has obtained the exponential generating function for the quotients \( \text{maj} A_n(t, q) \):

\[
\sum_{n \geq 0} u^n n! (1-t)(1-qt) \cdots (1-q^n t) = \sum_{s \geq 0} t^s e^{u(1+q+\cdots+q^s)}. \tag{3}
\]

Identity \( 3 \) is known as Carlitz’s identity.

Let \( C_a \) be the cyclic group of order \( a \), and let \( C_a \wr S_n \) be its wreath product with the symmetric group \( S_n \), which comprises colored permutations.

The problem of extending the distribution of \( (\text{des}, \text{maj}) \) to the hyperoctahedral group \( B_n = C_2 \wr S_n \) was first suggested by Foata. Adin, Brenti and Roichman \cite{ABR01} have given a solution to Foata’s problem in the form of two pairs of statistics, \( (\text{ndes}, \text{nmaj}) \) and \( (\text{fdes}, \text{fmaj}) \). Later, Biagioli \cite{Bia03} has given a generalization of Carlitz’s identity to the even-signed permutation group \( D_n \). More recently, Bagno \cite{Bag04} has extended the des and maj statistics to the wreath products \( C_a \wr S_n \) in two different ways, \( (\text{ndes}, \text{nmaj}) \) and \( (\text{ldes}, \text{lmaj}) \), giving two further generalizations of Carlitz’s formula.

In a recent paper \cite{RR04}, Regev and Roichman introduced the order \( <_L \) and the \( C_a \wr S_n \) statistics \( \text{des}_L \) (the \( L \)-descent number) and \( \text{min}_L \) (number of \( L \)-colored right-to-left minima), and studied the distribution of \( \text{min}_L \) on \( C_a \wr S_n \) and on the subset \( \{ \sigma \in C_a \wr S_n : \text{min}_L(\sigma) = \text{des}_L(\sigma) \} \).

Here we define the \( \text{rmaj}_{L,n} \) (\( L \)-reverse major index) statistic on \( C_a \wr S_n \) and study the distribution of \( \text{des}_L \) and the bi-statistic \( (\text{des}_L, \text{rmaj}_{L,n}) \). We obtain new wreath-product analogues of the Eulerian and \( q \)-Euler-Mahonian polynomials, and a generalization of Carlitz’s identity (see Corollary 4.5).

2 Preliminaries

2.1 The Group \( C_a \wr S_n \)

Let \( C_a \) be the (multiplicative) cyclic group of order \( a \): \( \alpha := e^{\frac{2\pi i}{a}} \) and \( C_a := \{ \alpha^t : 0 \leq t \leq a-1 \} \), and let \( C_a \wr S_n \) be its wreath product with \( S_n \).
Elements of $C_a \wr S_n$ can be regarded as indexed permutations or colored permutations—those permutations $\sigma$ of the set \{ $\alpha^i : 0 \leq t \leq a - 1, \ 1 \leq i \leq n$ \} satisfying
\[
\sigma(\beta^j) = \beta \sigma(j) \quad \forall \beta \in C_a, 1 \leq j \leq n.
\]

We shall write colored permutations using the window notation
\[
\sigma = [\sigma(1), \ldots, \sigma(n)]
\]
and denote
\[
|\sigma| := [|\sigma(1)|, \ldots, |\sigma(n)|] \in S_n.
\]

Note that $S_n$ is a subgroup of $C_a \wr S_n$. $C_a \wr S_n$ is a Coxeter group if and only if $a = 1$ or 2. In particular, for $a = 2$, $C_2 \wr S_n = B_n$ is the hyperoctahedral group, whose elements are the signed permutations.

### 2.2 $q$-analogues

**Definition 2.1.** For an integer $n \geq 1$, the $q$-analogue of $n$ is
\[
[n]_q := 1 + q + \cdots + q^{n-1}.
\]

**Definition 2.2.** For an integer $n \geq 0$, define
\[
(\alpha; q)_n := \begin{cases} 1, & \text{if } n = 0; \\ (1 - \alpha)(1 - \alpha q) \cdots (1 - \alpha q^{n-1}), & \text{if } n > 0. \end{cases}
\]

### 3 Statistics on $C_a \wr S_n$

In this section we present Regev and Roichman’s $<_L$ order and various statistics based on it.

**Definition 3.1 (RR04, Definition 4.4).** A subset $L \subseteq \{0, 1, \ldots, a - 1\}$ determines a linear order $<_L$ on \{ $\alpha^v j : 0 \leq v \leq a - 1, \ 0 \leq j \leq n$ \} as follows:

Let $U = \{0, \ldots, a - 1\} \setminus L$ be the complement of $L$ in $\{0, \ldots, a - 1\}$.

If $v \in L$ then $\alpha^v j <_L 0$ for every $1 \leq j \leq n$. If $v \in U$ then $\alpha^v j >_L 0$ for every $1 \leq j \leq n$. 

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For $u, v \in L$ (resp. $u \in U$) (not necessarily distinct) and $i \neq j \in [n]$, \( \alpha^u i <_L \alpha^u j \) if and only if $i > j$ (resp. $i < j$).

Then, for each $1 \leq j \leq n$, order each subset \( \{ \alpha^u j : v \in L \} \) (and each subset \( \{ \alpha^u j : v \in U \} \)) in an arbitrary linear order.

**Example 3.2.** Let $a = 4$ and $L = \{2, 3\}$, then $U = \{0, 1\}$. We can choose the following order

\[
\begin{align*}
\alpha^2 2 &<_L \alpha^2 3 <_L \alpha^3 (n-1) <_L \alpha^3 n <_L \alpha^2 (n-1) <_L \alpha^2 n <_L 0 \\
0 &<_L \alpha <_L 1 <_L \alpha 2 <_L \alpha (n-1) <_L (n-1) <_L \alpha n <_L n.
\end{align*}
\]

**Definition 3.3 ([RR04 Definition 4.6]).** Let $L \subseteq \{0, 1, \ldots, a - 1\}$.

1. The $L$-descent set of $\sigma \in C_a \wr S_n$ is

\[
\text{Des}_L(\sigma) := \{ 0 \leq i \leq n-1 : \sigma(i) >_L \sigma(i+1) \}
\]

where $\sigma(0) := 0$.

2. The $L$-descent number is

\[
\text{des}_L(\sigma) := |\text{Des}_L(\sigma)|.
\]

**Definition 3.4.** The $L$-reverse major index of $\sigma \in C_a \wr S_n$ is

\[
\text{rmaj}_{L,n}(\sigma) := \sum_{i \in \text{des}_L(\sigma)} n - i.
\]

**Remark 3.5.** In the $a = 2$ case, the descent set is often defined as

\[
\text{Des}_L(\sigma) := \{ 1 \leq i \leq n : \sigma(i) >_L \sigma(i+1) \}
\]

where $\sigma(n+1) := 0$ (see for example [FH97]). It is easy to see that $\text{Des}_L(\sigma) = \{ n - i : i \in \text{Des}_{\{1, \ldots, n\} \setminus L}(\sigma, [n, n-1, \ldots, 1]) \}$, so $\text{des}_L(\sigma) := |\{\text{Des}_L(\sigma)\}| = \text{des}_{\{1, \ldots, n\} \setminus L}(\sigma, [n, \ldots, 1])$ and $\text{maj}_L(\sigma) := \sum_{i \in \text{des}_L(\sigma)} i = \text{rmaj}_{L,n}(\sigma, [n, \ldots, 1])$.

Since multiplication by $[n, \ldots, 1]$ is an involution of $C_a \wr S_n$, we get that the bi-statistics $(\text{des}_L, \text{maj}_L)$ and $(\text{des}_{\{1, \ldots, n\} \setminus L}, \text{rmaj}_{\{1, \ldots, n\} \setminus L,n})$ have the same distribution on $C_a \wr S_n$. Thus the results in the following sections can be easily adapted to the “tilde” statistics.

**Remark 3.6.** For $\sigma \in S_n$, let $\text{rmaj}_n(\sigma) := \text{rmaj}_{\emptyset,n}(\sigma)$. A bijective argument shows that $\text{maj}$ and $\text{rmaj}_n$ are equidistributed on $\{ \sigma \in S_n : \text{des}(\sigma) = s \}$ for every $s$. Thus in ([P]), maj can be replaced by $\text{rmaj}_n$. 

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**Definition 3.7.** Define $\phi_n : C_a \wr S_{n-1} \times \{0, \ldots, n-1\} \times \{0, \ldots, a-1\} \to C_a \wr S_n$ by
\[
\phi_n(\sigma, r, t) := [\sigma_1, \ldots, \sigma_r, \alpha^t n, \sigma_{r+1}, \ldots, \sigma_{n-1}]
\]
where $\sigma = [\sigma_1, \ldots, \sigma_{n-1}]$.

It is easy to see that $\phi_n$ is a bijection.

**Lemma 3.8.** Let $\sigma = [\sigma_1, \ldots, \sigma_{n-1}] \in C_a \wr S_{n-1}$, $L \subseteq \{0, \ldots, a-1\}$ and $\text{Des}_L(\sigma) = \{i_1, \ldots, i_s\}$, $i_1 < \cdots < i_s$ ($s = \text{des}_L(\sigma)$). Let $\{i_{s+1}, \ldots, i_n\} = \{0, \ldots, n-1\} \setminus \text{Des}_L(\sigma)$, $n-1 = i_{s+1} > \cdots > i_n$ (i.e. the non-descents of $\sigma$, from right to left). Then for $1 \leq k \leq n$ and $t \in \{0, \ldots, a-1\}$,
\[
\text{des}_L(\phi_n(\sigma, i_k, t)) = \begin{cases} 
  s, & \text{if } k < s+1 \text{ or } k = s+1, t \notin L; \\
  s+1, & \text{if } k > s+1 \text{ or } k = s+1, t \in L.
\end{cases}
\]
and
\[
\text{rmaj}_{L,n}(\phi_n(\sigma, i_k, t)) = \begin{cases} 
  \text{rmaj}_{L,n-1}(\sigma) + k, & \text{if } t \in L; \\
  \text{rmaj}_{L,n-1}(\sigma) + k - 1, & \text{if } t \notin L.
\end{cases}
\]

**Proof.** We consider the three possible cases:

**Case 1.** $1 \leq k \leq s$. In this case,
\[
\tilde{\sigma} := \phi_n(\sigma, i_k, t) = [\sigma_1, \ldots, \sigma_{i_1}, \ldots, \sigma_{i_k}, \alpha^t n, \sigma_{i_k+1}, \ldots, \sigma_{i_s}, \ldots, \sigma_{n-1}],
\]
thus the descents to the right of $\sigma_{i_k+1}$ are shifted one position to the right, and the $k-1$ descents to the left of $\sigma_{i_k}$ remain in place. If $t \in L$, then $\sigma_{i_k} > L \alpha^t n < L \sigma_{i_{k+1}}$, so the descent at $i_k$ is also shifted one position to the right. If $t \notin L$, then $\sigma_{i_k} < L \alpha^t n > L \sigma_{i_{k+1}}$, leaving the descent at $i_k$. The contribution to $\text{rmaj}_{L,n}(\tilde{\sigma})$ of each descent shifted one position to the right is the same as its contribution to $\text{rmaj}_{L,n-1}(\sigma)$, whereas the contribution to $\text{rmaj}_{L,n}(\tilde{\sigma})$ of each descent left in place is 1 more than its contribution to $\text{rmaj}_{L,n-1}(\sigma)$.

**Case 2.** $k = s+1$. In this case,
\[
\tilde{\sigma} := \phi_n(\sigma, n-1, t) = [\sigma_1, \ldots, \sigma_{n-1}, \alpha^t n],
\]
thus all $s = k-1$ descents remain in place, each contributing 1 more to $\text{rmaj}_{L,n}(\tilde{\sigma})$ than to $\text{rmaj}_{L,n-1}(\sigma)$. If and only if $t \in L$, $\sigma_{n-1} > L \alpha^t n$ so there is an additional descent at $n-1$, which contributes 1 to $\text{rmaj}_{L,n}(\tilde{\sigma})$. 


Case 3. $s + 1 < k \leq n$. In this case,
\[
\bar{\sigma} := \phi_n(\sigma, i_k, t) = [\sigma_1, \ldots, \sigma_{i_r}, \ldots, \sigma_{i_k}, \alpha^t n, \sigma_{i_k+1}, \ldots, \sigma_{i_r+1}, \ldots, \sigma_{n-1}]
\]
where $r$ is the number of descents to the left of $\sigma_{i_k}$, whence
\[
i_k = (n - 1) - (k - (s + 1)) - (s - r) = n - k + r.
\]
In this case the descents to the right of $\sigma_{i_k}$ are shifted one position to the right, and the $r$ descents to the left of $\sigma_{i_k}$ remain in place. The contribution to $\text{rmaj}_{L,n}(\bar{\sigma})$ of each descent shifted one position to the right is the same as its contribution to $\text{rmaj}_{L,n-1}(\sigma)$, whereas the contribution to $\text{rmaj}_{L,n}(\bar{\sigma})$ of each descent left in place is 1 more than its contribution to $\text{rmaj}_{L,n-1}(\sigma)$. If $t \in L$, then there is an additional descent at $i_k = n - k + r$, whose contribution to $\text{nrmaj}_{L,n}(\bar{\sigma})$ is $k - r$. If $t \notin L$, then the additional descent is at $i_k + 1 = n - k + r + 1$, contributing $k - r - 1$ to $\text{nrmaj}_{L,n}(\bar{\sigma})$.

4 $C_a \wr S_n$ $q$-maj Euler-Mahonian Polynomials

In this section we define $q$-maj Euler-Mahonian polynomials for $C_a \wr S_n$ and give generalizations of the results by Carlitz and Gessel.

For $L \subseteq \{0, \ldots, a - 1\}$, let $\text{maj}_{A_{a,L,n}}(t, q)$ be the generating polynomial for $C_a \wr S_n$ by the bi-statistic $(\text{des}_L, \text{maj}_{L,n})$, i.e.
\[
\text{maj}_{A_{a,L,n}}(t, q) := \sum_{\sigma \in C_a \wr S_n} t^{\text{des}_L(\sigma)} q^{\text{maj}_{L,n}(\sigma)}.
\]

**Remark 4.1.** For $n = 1$,
\[
\text{maj}_{A_{a,L,1}}(t, q) = \sum_{t=0}^{a-1} t^{\text{des}_L([a^t])} q^{\text{maj}_{L,1}([a^t])} = \ell t q + (a - \ell)
\]
where $\ell = |L|$ depends only on the number of elements in $L$ and not on the choice of elements.

The following is a generalization of (2).
Proposition 4.2. With $\text{maj} A_{a,L,n}(t, q) = \sum_{s \geq 0} t^s \text{maj} A_{a,L,n,s}(q)$, the coefficients $\text{maj} A_{a,L,n,s}(q)$ satisfy the recurrence

$$\text{maj} A_{a,L,n,s}(q) = (a[s + 1]q - \ell) \text{maj} A_{a,L,n-1,s}(q) + (aq^s[n - s]q + \ell q^n) \text{maj} A_{a,L,n-1,s-1}(q)$$

(4)

where $\ell = |L|$. 

Proof. By definition,

$$\text{maj} A_{a,L,n,s}(q) = \sum_{\tilde{\sigma} \in C_a \wr S_n \ \text{des}_L(\tilde{\sigma}) = s} q^{\text{maj}_L(\tilde{\sigma})}.$$ 

By the bijectivity of $\phi_n$ and Lemma 3.8,

$$\{ \tilde{\sigma} \in C_a \wr S_n : \text{des}_L(\tilde{\sigma}) = s \} = \phi_n(A \uplus B \uplus C \uplus D)$$

where $\uplus$ denotes disjoint union and

$$A := \{ (\sigma, i_k, t) : \sigma \in C_a \wr S_{n-1}, \ \text{Des}_L(\sigma) = \{i_1, \ldots, i_s\}, \ 1 \leq k \leq s, \ t \in L \}$$

$$B := \{ (\sigma, i_k, t) : \sigma \in C_a \wr S_{n-1}, \ \{0, \ldots, n-1\} \setminus \text{Des}_L(\sigma) = \{i_s, \ldots, i_n\}, \ s \leq k \leq n, \ t \in L \}$$

$$C := \{ (\sigma, i_k, t) : \sigma \in C_a \wr S_{n-1}, \ \text{Des}_L(\sigma) \uplus \{n - 1\} = \{i_1, \ldots, i_{s+1}\}, \ 1 \leq k \leq s + 1, \ t \notin L \}$$

$$D := \{ (\sigma, i_k, t) : \sigma \in C_a \wr S_{n-1}, \ \{0, \ldots, n - 2\} \setminus \text{Des}_L(\sigma) = \{i_{s+1}, \ldots, i_n\}, \ s + 1 \leq k \leq n, \ t \notin L \}.$$ 

Note that in the definition of $A$ and $C$, $\text{des}_L(\sigma) = s$, whereas in the definition of $B$ and $D$, $\text{des}_L(\sigma) = s - 1$. 

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By the second part of Lemma 3.8,
\[
\sum_{\sigma \in C_\mathcal{A}_L} q^\text{maj}_{L,n}(\sigma) = \sum_{(\sigma,i,k,t) \in A} q^\text{maj}_{L,n}(\phi_n(\sigma,i,k,t)) + \sum_{(\sigma,i,k,t) \in B} q^\text{maj}_{L,n}(\phi_n(\sigma,i,k,t)) + \sum_{(\sigma,i,k,t) \in C} q^\text{maj}_{L,n}(\phi_n(\sigma,i,k,t)) + \sum_{(\sigma,i,k,t) \in D} q^\text{maj}_{L,n}(\phi_n(\sigma,i,k,t))
\]

\[
= \sum_{\sigma \in C_\mathcal{A}_L} \text{des}_L(\sigma) = s \sum_{(\sigma,i,k,t) \in A} q^\text{maj}_{L,n-1}(\sigma) + \sum_{(\sigma,i,k,t) \in B} q^\text{maj}_{L,n-1}(\sigma) + \sum_{(\sigma,i,k,t) \in C} q^\text{maj}_{L,n-1}(\sigma) + \sum_{(\sigma,i,k,t) \in D} q^\text{maj}_{L,n-1}(\sigma)
\]

\[
= \ell q[s]_q \sum_{\sigma \in C_\mathcal{A}_L} q^\text{maj}_{L,n-1}(\sigma)
\]

\[
+ \ell q^s[n - s + 1]_q \sum_{\sigma \in C_\mathcal{A}_L} q^\text{maj}_{L,n-1}(\sigma)
\]

\[
+ (a - \ell)[s + 1]_q \sum_{\sigma \in C_\mathcal{A}_L} q^\text{maj}_{L,n-1}(\sigma)
\]

\[
+ (a - \ell)q^s[n - s]_q \sum_{\sigma \in C_\mathcal{A}_L} q^\text{maj}_{L,n-1}(\sigma)
\]

\[
= (a[s + 1]_q - \ell) \text{maj}_{A_L} A_{n-1,s}
\]

\[
+ (aq^s[n - s]_q + \ell q^n) \text{maj}_{A_L} A_{n-1,s-1}.
\]

By Proposition 4.2 and Remark 4.1, \( \text{maj}_{A_L} A_{n-1,s} \) does not depend on the choice of elements in \( L \) but only on their number \( \ell = |L| \). Therefore the polynomials

\[
\text{maj}_{A_L} A_{\ell}(t,q) := \text{maj}_{A_L} A_{n-1,s}(t,q) \quad |L| = \ell
\]

and

\[
\text{maj}_{A_L} A_{\ell}(t,q) := \text{maj}_{A_L} A_{n,s}(t,q) \quad |L| = \ell
\]

are well-defined.

The following lemma is a generalization of \([FH04\text{ equation (10.3)}]\).
Lemma 4.3. For every three integers $n > 0$, $a > 0$, $0 \leq \ell \leq a$,

$$(1 - q)^{\text{maj}} A_{a,\ell,n}(t, q) = (a - (1 - q)\ell)(1 - tq^n)^{\text{maj}} A_{a,\ell,n-1}(t, q) - aq(1 - t)^{\text{maj}} A_{a,\ell,n-1}(tq, q).$$  

(5)

Proof. Multiply both sides of (4) by $(1 - q)t^s$ and sum over all $-\infty < s < \infty$ to get

$$(1 - q)^{\text{maj}} A_{a,\ell,n}(t, q)$$

$$= \sum_{s=-\infty}^{\infty} (a(1 - q^{s+1}) - (1 - q)\ell)t^s^{\text{maj}} A_{a,\ell,n-1,s}(q)$$

$$+ \sum_{s=-\infty}^{\infty} (aq^s(1 - q^{n-s}) + (1 - q)\ell q^n)t^s^{\text{maj}} A_{a,\ell,n-1,s-1}(q)$$

$$= \sum_{s=-\infty}^{\infty} (a - (1 - q)\ell)t^s^{\text{maj}} A_{a,\ell,n-1,s}(q)$$

$$+ \sum_{s=-\infty}^{\infty} t((aq^n + (1 - q)\ell q^n)t^s + a(\ell t)^s)^{\text{maj}} A_{a,\ell,n-1,s}(q)$$

$$= \sum_{s=-\infty}^{\infty} ((a - (1 - q)\ell)(1 - tq^n)t^s - aq(1 - t)(\ell t)^s)^{\text{maj}} A_{a,\ell,n-1,s}(q)$$

$$= (a - (1 - q)\ell)(1 - tq^n)^{\text{maj}} A_{a,\ell,n-1}(t, q) - aq(1 - t)^{\text{maj}} A_{a,\ell,n-1}(tq, q).$$  

\[ \square \]

Proposition 4.4. For every three integers $n > 0$, $a > 0$, $0 \leq \ell \leq a$,

$$\frac{\text{maj} A_{a,\ell,n}(t, q)}{(t; q)_{n+1}} = \sum_{s \geq 0} t^s(a[s + 1]q - \ell)^n.$$  

(6)

Proof. By induction on $n$. For $n = 1$,

$$\frac{\text{maj} A_{a,\ell,1}(t, q)}{(t; q)_2} = \frac{a - \ell(1 - tq)}{(1 - t)(1 - tq)} = \sum_{s \geq 0} t^s(a[s + 1]q - \ell).$$

For $n > 1$, divide both sides of (5) by $(1 - q)(t; q)_{n+1}$ and use the induction
As a corollary, we get a generalization of (3).

\textbf{Corollary 4.5.} For every \(a > 0\), \(0 \leq \ell \leq a\),
\[
\sum_{n \geq 0} \frac{n!}{n!} \frac{\text{maj} A_{a,\ell,n}(t, q)}{(t; q)_{n+1}} = \sum_{s \geq 0} t^s e^{u(a[s+1]_q - \ell)}.
\]

\section{\(C_a \wr S_n\) Eulerian Polynomials}

For \(L \subseteq \{0, \ldots, a-1\}\), let \(A_{a,L,n}(t)\) be the generating polynomial for \(C_a \wr S_n\) by the statistic \(\text{des}_L\), i.e.
\[
A_{a,L,n}(t) := \sum_{\sigma \in C_a \wr S_n} t^{\text{des}_L(\sigma)}.
\]

Clearly \(A_{a,L,n}(t) = \text{maj} A_{a,L,n}(t, 1)\) and the polynomials
\[
A_{a,\ell,n}(t) := A_{a,L,n}(t) \quad |L| = \ell
\]
are well-defined.

\textbf{Proposition 5.1.} 1. With \(A_{a,\ell,n}(t) = \sum_{s \geq 0} t^s A_{a,\ell,n,s}\), the coefficients \(A_{a,\ell,n,s}\) satisfy the recurrence
\[
A_{a,\ell,n,s} = (a(s+1) - \ell) A_{a,\ell,n-1,s} + (a(n-s) + \ell) A_{a,\ell,n-1,s-1}.
\]
2. \[ A_{n,a,\ell}(t) = (a - \ell + (a(n-1) + \ell)t)A_{a,\ell,n-1}(t) + at(1-t)A'_{a,\ell,n-1}(t). \] (9)

3. \[ \frac{A_{a,\ell,n}(t)}{(1-t)^{n+1}} = \sum_{s \geq 0} t^s(a(s + 1) - \ell)^n. \] (10)

4. \[ \sum_{n \geq 0} \frac{u^n}{n!} A_{a,\ell,n}(t) = \sum_{s \geq 0} t^s e^{u(a(s+1) - \ell)} = \frac{e^{(a-\ell)u}}{1-te^{au}}. \] (11)

5. \[ \sum_{n \geq 0} \frac{u^n}{n!} A_{a,\ell,n}(t) = \frac{1-t}{-te^{(1-t)u} + e^{(a-\ell)u(t-1)}}. \] (12)

Proof. (8), (10) and (11) follow from (4), (6) and (7) respectively by setting \( q = 1 \). (12) follows from (11) by substituting \((1-t)u\) for \( u \) and multiplying both sides by \( 1 = t \). Finally, to see (9), multiply both sides of (8) by \( t^s \) and take the sum over all \( s \geq 0 \) to get

\[
A_{a,\ell,n}(t) = \sum_{s \geq 0} (a(s + 1) - \ell)t^sA_{a,\ell,n-1,s} + \sum_{s \geq 0} (a(n - s) + \ell)t^sA_{a,\ell,n-1,s-1}
\]

\[
= \sum_{s \geq 0} (a(s + 1) - \ell)t^sA_{a,\ell,n-1,s} + \sum_{s \geq 0} (a(n - s - 1) + \ell)t^sA_{a,\ell,n-1,s}
\]

\[
= \sum_{s \geq 0} (a - \ell + (a(n-1) + \ell)t)t^sA_{a,\ell,n-1,s}
\]

\[
+ t \sum_{s \geq 0} a(1-t)s^{s-1}A_{a,\ell,n-1,s}
\]

\[
= (a - \ell + (a(n-1) + \ell)t)A_{a,\ell,n-1}(t) + at(1-t)A'_{a,\ell,n-1}(t).
\]

\[\square\]

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