NOTES ON BERGMAN PROJECTION TYPE OPERATOR RELATED WITH BESOV SPACE

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Abstract. Let $Qf$ be the maximal derivative of $f$ with respect to the Bergman metric $b_B$. In this paper, we will find conditions such that $(1 - \|z\|)^s(Qf)^p(z)$ is bounded on $B$. We will also find conditions such that Bergman projection type operator $P_r$ is bounded operator from $L^p(B, d\mu_r)$ to the holomorphic Besov $p$-space $B^s_p(B)$ with weight $s$.

1. Introduction

Throughout this paper, $\mathbb{C}^n$ will be the Cartesian product of $n$ copies of $\mathbb{C}$. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, the inner product is defined by $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $\|z\|^2 = \langle z, z \rangle$.

Let $\Omega$ be any bounded domain in $\mathbb{C}^n$. Let $f \in C^1(\Omega)$ and $\xi \in \mathbb{C}^n$. The maximal derivative of $f$ with respect to the Bergman metric $b_\Omega$ is defined by

$$\hat{Q}_f(z) = \sup_{\|\xi\|=1} \frac{|\langle df(z), \xi \rangle|}{b_\Omega(z, \xi)}, \quad z \in \Omega$$

where

$$\langle df(z), \xi \rangle = \sum_{i=1}^n \left[ \frac{\partial f(z)}{\partial z_i} \xi_i + \frac{\partial f(z)}{\partial \overline{z_i}} \overline{\xi_i} \right].$$

If $f \in H(\Omega)$ where $H(\Omega)$ is the set of holomorphic functions on $\Omega$, then the quantity $Qf$ is reduced to

$$Qf(z) = \sup_{\|\xi\|=1} \frac{|\langle \nabla f(z), \xi \rangle|}{b_\Omega(z, \xi)}, \quad z \in \Omega, \quad \xi \in \mathbb{C}^n$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n})$ is the holomorphic gradient of $f$.

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For Lebesgue measure $\nu$ in $\mathbb{C}^n$, let $d\lambda(z) = K(z, z) d\nu(z)$ where $K(z, w)$ is Bergman kernel. Let $\delta_{\Omega}(z)$ be the Euclidean distance from $z$ to the boundary $\partial \Omega$. For $0 < p < \infty$ and $s \in \mathbb{R}$, the holomorphic Besov $p$-space $\mathcal{B}_p^s(\Omega)$ with weight $s$ is defined by the space of all holomorphic functions $f$ on $\Omega$ such that

$$\| f \|_{p,s} = \left\{ \int_{\Omega} (Qf)^p(z) \delta_{\Omega}(z)^s d\lambda(z) \right\}^{\frac{1}{p}} < \infty.$$ 

In this paper, we will consider the case where $\Omega$ is open unit ball in $\mathbb{C}^n$. Let $B$ be the open unit ball in the complex space $\mathbb{C}^n$ and $S$ the boundary of $B$. For $z \in B, \xi \in \mathbb{C}^n$, the Bergman metric on $B$ is given by

$$b_B^2(z, \xi) = \frac{n+1}{(1-\|z\|^2)^2} \left[ (1-\|z\|^2)\|\xi\|^2 + |\langle z, \xi \rangle|^2 \right].$$

The quantity $Qf$ for the unit ball $B$ is invariant under the group $\text{Aut}(B)$ of holomorphic automorphisms of $B$. Namely, $Q(f \circ \varphi) = (Qf) \circ \varphi$ for all $\varphi \in \text{Aut}(B)$.

Let $\nu$ be the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. The Bergman space $L^2_a(B, d\nu)$ is defined to be the subspace of $L^2(B, d\nu)$ consisting of analytic functions.

Fix a point $z \in B$. Since the functional $e_z$ given by $e_z(f) = f(z), f \in L^2_a(B, d\nu)$, is continuous, there exists a function $K(\cdot, z) \in L^2_a(B, d\nu)$ such that

$$f(z) = \int_B f(w) K(w, z) d\nu(w)$$

by the Riesz representation theorem. The function $K(z, w)$ is called the Bergman reproducing kernel in $L^2_a(B, d\nu)$. It is well known that $K(z, w) = \frac{1}{(1-\langle z, w \rangle)^{n+1}}$ (See [9]).

Let $0 < p < \infty$ and $s \in \mathbb{R}$. The holomorphic Besov $p$-spaces $\mathcal{B}_p^s(B)$ with weight $s$ is defined by the space of all holomorphic functions $f$ on the unit ball $B$ such that

$$\| f \|_{p,s} = \left\{ \int_B (Qf)^p(z)(1-\|z\|^2)^s d\lambda(z) \right\}^{\frac{1}{p}} < \infty.$$ 

Here $d\lambda(z) = K(z, z) d\nu(z) = (1-\|z\|^2)^{-n-1} d\nu(z)$ is an invariant volume measure with respect to the Bergman metric on $B$.

For a fixed $p \in (0, \infty)$, $\mathcal{B}_p^s(B)$ is an increasing family of function spaces in $s$; that is, if $-\infty < s \leq t < +\infty$, then $\mathcal{B}_p^s(B) \subset \mathcal{B}_p^t(B)$. The holomorphic Besov $p$-space $\mathcal{B}_p^s(B)$ with weight $s$ include many well
known spaces as special case. $B^\infty_p(B)$ is the usual Hardy space $H^p(B)$ for $s = n$, the Bergman space $L^p_n(B)$ for $s = n + 1$. In particular, the diagonal Besov space $B^\infty_p(B)$ are shown to be the Möbius invariant subsets of the Bloch space (See [3]).

In recent years, there have been many papers focused on studying the Besov space and its applications (See [4], [6], [7] and [10]).

In section 2, we will find conditions such that $(1 - \|z\|)^s(Qf)^p(z)$ is bounded on $B$.

The orthogonal projection operator $P$ from $L^2(B, d\nu)$ to $L^2_\alpha(B, d\nu)$ is denoted by

$$
P f(z) = \int_B f(w)K(z, w)d\nu(w).
$$

$P$ is called the Bergman projection. The Bergman projection is used in many areas related with Hankel operators and Toeplitz operators (See [1], [8], [11] and [12]).

The measure $\mu_r$ is the weighted Lebesgue measure:

$$
d\mu_r(z) = c_r(1 - \|z\|^2)^rd\nu(z)
$$

where $r > -1$ is fixed, and $c_r$ is a normalization constant such that $\mu_r(B) = 1$. Define the Bergman projection type operator $P_r$ by

$$
P rf(z) = \int_B \frac{f(w)}{(1 - \langle z, w \rangle)^{n+r+1}}d\mu_r(w).
$$

In section 3, we will find conditions such that $P_r$ is bounded operator from $L^p(B, d\mu_r)$ to the holomorphic Besov $p$-spaces $B^s_p(B)$ with weight $s$.

2. Holomorphic Besov $p$-space $B^s_p(B)$ with weight $s$

The traditional holomorphic Besov space $B^p_p(\Omega)$ is a subspace of $L^2_\alpha(\Omega)$ with semi-norm

$$
\|f\|_{B^p_p} = \left\{ \int_\Omega (\nabla f)^p(z)\delta_\Omega(z)^pd\lambda(z) \right\}^{\frac{1}{p}} < \infty
$$

where $\delta_\Omega(z)/2$ is the distance from $z$ to $\partial\Omega$. It is known that the fact $\int_\Omega \delta_\Omega(z)^{-q}d\nu(z) = \infty$ when $q \geq 1$ implies that $B^p_p(\Omega) = \mathbb{C}$ when $p \leq n$ and $\Omega$ is a smoothly bounded strictly pseudo convex domain in $\mathbb{C}^n$.

If $\Omega$ is the unit ball $B$ in $\mathbb{C}^n$ and $\nu$ is the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(\Omega) = 1$, we can find the following result.
Theorem 2.1. Let $n \geq 2$ and $0 < p \leq 2n$. If $f \in H(B)$ and
\[
\int_B (Qf)^p(z) d\lambda(z) < \infty,
\]
then $f$ is constant.

Proof. See [3], Lemma 2.11.

These results show that the above semi-norm is not natural when $p \leq n$. In this paper, we will consider the holomorphic Besov $p$-space $B^s_p(B)$ with weight $s$.

Let $a \in B$ and let $P_a$ be the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by $a$, which is given by $P_0 = 0$, and
\[
P_a z = \frac{(z, a)}{(a, a)} a, \quad \text{if } a \neq 0.
\]

Let $Q_a = I - P_a$. Define $\varphi_a$ on $B$ by
\[
\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - \|a\|^2} Q_a z}{1 - (z, a)}.
\]

Theorem 2.2. For every $a \in B$, $\varphi_a$ has the following properties:

(i) The identity
\[
1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1 - \|a\|^2)(1 - (z, w))}{(1 - (z, a))(1 - (a, w))}
\]
holds for all $z \in \overline{B}, w \in \overline{B}$.

(ii) The identity
\[
1 - \| \varphi_a(z) \|^2 = \frac{(1 - \|a\|^2)(1 - \|z\|^2)}{|1 - (z, a)|^2}
\]
holds for every $z \in \overline{B}$.

(iii) $\varphi_a$ is a homeomorphism of $\overline{B}$ onto $\overline{B}$.

Proof. See [9], Theorem 2.2.2.

Theorem 2.3. For $z \in B$, $c$ is real, $t > -1$, define
\[
I_{c,t}(z) = \int_B \frac{(1 - \|w\|^2)^t}{|1 - (z, w)|^{n+1+c+t}} d\nu(w).
\]

Then,

(i) $I_{c,t}(z)$ is bounded in $B$ if $c < 0$;

(ii) $I_{0,t}(z) \approx -\log(1 - \|z\|^2)$ as $\|z\| \to 1^-$;

(iii) $I_{c,t}(z) \approx (1 - \|z\|^2)^{-c}$ as $\|z\| \to 1^-$ if $c > 0$. 

Proof. See [9], Proposition 1.4.10.

Lemma 2.4. If $f$ is holomorphic and $\frac{Qf(w)}{1-\|w\|^2}$ is Lebesgue integrable on $B$, then

$$Qf(0) \leq (n+1) \int_B Qf(w)(1-\|w\|^2)^n d\lambda(w).$$

Proof. By the definition of Bergman metric,

$$b_B^2(z,\xi) = (n+1) \frac{(1-\|z\|^2)\|\xi\|^2 + |(z,\xi)|^2}{(1-\|z\|^2)^2} \leq (n+1) \frac{(1-\|z\|^2)\|\xi\|^2 + \|z\|^2\|\xi\|^2}{(1-\|z\|^2)^2} \leq (n+1) \frac{\|\xi\|^2}{(1-\|z\|^2)^2}.$$ 

By the mean value theorem,

$$f(t\eta) = \int_B f \circ \varphi_{t\eta}(w) d\nu(w)$$

for $f \in H(B), \eta \in B$ and $t \in [0,1]$.

$$|\langle \nabla f(0), \eta \rangle| = \left| \int_B \langle \nabla f(-w), \left[ \frac{d}{dt} \varphi_{t\eta}(w) \right]_{t=0} \rangle d\nu(w) \right|$$

$$= \left| \int_B \langle \nabla f(-w), \eta - \langle w, \eta \rangle w \rangle d\nu(w) \right|$$

$$\leq \int_B \left[ \langle \nabla f(-w), \frac{\eta - \langle w, \eta \rangle w}{\|\eta - \langle w, \eta \rangle w\|} \rangle \right] b_B(-w, \eta - \langle w, \eta \rangle w) d\nu(w)$$

$$\leq (n+1) \int_B Qf(w)b_B(-w, \eta - \langle w, \eta \rangle w) d\nu(w)$$

$$\leq (n+1) \int_B \frac{Qf(w)}{1-\|w\|^2} d\nu(w)$$

$$\leq (n+1) \int_B (1-\|w\|^2)^n Qf(w) d\lambda(w).$$

Thus,

$$Qf(0) \leq (n+1) \int_B Qf(w)(1-\|w\|^2)^n d\lambda(w).$$

$\square$
THEOREM 2.5. Let $1 < p < \infty$. If $s$ is a real number such that $-np < s < n$ and $\| f \|_{p,s} < \infty$, then $(1 - \| z \|^2)^s(Qf)^p(z)$ is bounded on $B$.

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$ where $q > 1$. If $s < n$, then $t = (n - s)q + s - n - 1 > -1$. If $-np < s$, then $nq + sq - s > 0$. By Theorem 2.3,

$$
\left( \int_B \frac{(1 - \| \xi \|^2)^{(n-s)q}}{|1 - \langle z, \xi \rangle|^{2nq}} d\lambda(\xi) \right)^{1/q}
= \left( \int_B \frac{(1 - \| \xi \|^2)^t}{|1 - \langle z, \xi \rangle|^{n+1+t+(nq+sq-s)} d\nu(\xi)} \right)^{1/q}
\approx (1 - \| z \|^2)^{-n-s/s/q}
\approx (1 - \| z \|^2)^{-n-s/p}.
$$

By Lemma 2.4,

$$
Qf(0) \leq (n+1) \int_B Qf(w)(1 - \| w \|^2)^n d\lambda(w).
$$

Put $\xi = \varphi_z(w)$. By Theorem 2.2,

$$
Qf(z) = Q(f \circ \varphi_z)(0)
\leq (n+1) \int_B Q(f \circ \varphi_z)(w)(1 - \| w \|^2)^n d\lambda(w)
\leq (n+1) \int_B Qf(\xi)(1 - \| \varphi_z(\xi) \|^2)^n d\lambda(\xi)
\leq (n+1) \int_B Qf(\xi) \frac{(1 - \| z \|^2)^n(1 - \| \xi \|^2)^n}{|1 - \langle z, \xi \rangle|^{2n}} d\lambda(\xi)
\leq (n+1)(1 - \| z \|^2)^n \left( \int_B (Qf)^p(\xi)(1 - \| \xi \|^2)^s d\lambda(\xi) \right)^{1/p}
\left( \int_B \frac{(1 - \| \xi \|^2)^{(n-s)q}}{|1 - \langle z, \xi \rangle|^{2nq}}(1 - \| \xi \|^2)^s d\lambda(\xi) \right)^{1/q},
$$

where the last inequality follows from Hölder inequality for $\frac{1}{p} + \frac{1}{q} = 1$. This implies that

$$
Qf(z) \leq C(1 - \| z \|^2)^{-s/p} \| f \|_{p,s}
$$

for some constant $C$. This shows that if $s$ is a real number such that $-np < s < n$ and $\| f \|_{p,s} < \infty$, then $(1 - \| z \|^2)^s(Qf)^p(z)$ is bounded on $B$. \qed
3. **Bounded Bergman projection type operator related with Besov space**

In [10], Timoney showed that the linear space of all holomorphic function $f : B \rightarrow \mathbb{C}$ which satisfy

$$\sup_{z \in B} (1 - \|z\|^2) \| \nabla f(z) \| < \infty$$

is equivalent to the space of all holomorphic function which satisfy

$$\sup_{z \in B} Qf(z) < \infty.$$

**Theorem 3.1.** Let $p > 2n$ and $s > n$. Then for every $f \in H(B)$,

$$\int_B (Qf)^p(z)(1 - \|z\|^2)^s d\lambda(z) \approx \int_B \| \nabla f(z) \|^p (1 - \|z\|^2)^{p+s} d\lambda(z)$$

**Proof.** See [3], Lemma 2.8.

Let $L^2_{a,r} = L^2_a(B, d\mu_r)$ be the subspace of $L^2(B, d\mu_r)$ consisting of analytic functions. If we equip $L^2_{a,r}$ with the norm $\|f\|_2 = \sqrt{\int_B |f|^2 d\mu_r}$, then $L^2_{a,r}$ is a Banach space for each $r > -1$.

Fix a point $z \in B$. Since the functional $e_z$ given by $e_z(f) = f(z), f \in L^2_{a,r}$, is continuous, there exists a function $k_{r,z} \in L^2_{a,r}$ such that

$$f(z) = \int_B f(w) k_{r,z}(w) d\mu_r(w)$$

by the Riesz representation theorem. The function $K_{r,w}(z,w) = \overline{k_{r,z}(w)}$ is called the weighted Bergman kernel. Also it is well known that

$$K_{r,w}(z,w) = \frac{1}{(1 - \langle z, w \rangle)^{r+n+1}}$$

(See [9]). It was shown in [5] that if $f \in L^1_{a,r}, r > -1$, then

$$f(z) = \int_B \frac{f(w)}{(1 - \langle z, w \rangle)^{n+r+1}} d\mu_r(w).$$

Suppose $1 \leq p < +\infty$ and $r > 0$. Let $L^p_{a,r}$ be the subspace of $L^p(B, d\mu_r)$ consisting of analytic functions. Define Bergman projection type operator $P_r$ by

$$P_r f(z) = \int_B \frac{f(w)}{(1 - \langle z, w \rangle)^{n+r+1}} d\mu_r(w).$$

Since $P_r f = f$ for all analytic $f$ in $L^1(B, d\mu_r)$, $P_r$ is a projection from $L^1(B, d\mu_r)$ onto $L^1_{a}(B, d\mu_r)$. 


In [2], the author proved that $P_r$ is a bounded projection operator from $L^p(B, d\nu)$ onto $L^p_\nu(B, d\nu)$.

In the proof of Theorem 3.2, we will use $C_{n, r}$ to denote constant depending only on $n$ and $r$, but it is not always the same at each appearance.

**Theorem 3.2.** Let $p > 2n$ and $r > 0$. If $f \in L^p(B, d\mu_r)$, then
\[ \| P_r f \|_{p, s} \leq C_{n, r} \| f \|_{L^p(B, d\mu_r)} \]
for $s > 2n + r + 1$.

**Proof.** Differentiating under the integral sign, we obtain
\[ \frac{\partial}{\partial z_j} (P_r f)(z) = (n + r + 1) \int_B \frac{f(w)(-w_j)}{(1 - \langle z, w \rangle)^{n+r+2}} d\mu_r(w) \]
for $j = 1, 2, \cdots, n$. This shows that
\[ \| \nabla P_r f(z) \| \leq C_{n, r} \int_B \frac{|f(w)|}{(1 - \langle z, w \rangle)^{n+r+2}} d\mu_r(w) . \]

Let $\frac{1}{p} + \frac{1}{q} = 1$. By the Hölder inequality,
\[ \| \nabla P_r f(z) \|^p \leq \left( C_{n, r} \int_B \frac{|f(w)|}{(1 - \langle z, w \rangle)^{n+r+2}} d\mu_r(w) \right)^p \]
\[ = C_{n, r} \int_B |f(w)|^p d\mu_r(w) \left( \int_B \frac{1}{(1 - \langle z, w \rangle)^{q(n+r+2)}} d\mu_r(w) \right)^{p/q} . \]

By Theorem 2.3,
\[ \int_B \frac{1}{(1 - \langle z, w \rangle)^{q(n+r+2)}} d\mu_r(w) \]
\[ = c_r \int_B \frac{(1 - \| w \|^2)^r}{(1 - \langle z, w \rangle)^{q(n+r+2)}} d\nu(w) \]
\[ = c_r \int_B \frac{(1 - \| w \|^2)^r}{(1 - \langle z, w \rangle)^{n+1+r+1+(q-1)(n+r+2)}} d\nu(w) \]
\[ \approx (1 - \| z \|^2)^{-1-(q-1)(n+r+2)} . \]

By Theorem 3.1,
\[
\| P_r f \|_{p,s}^p = \int_B (QP_r f)(z)(1 - \| z \|^2)^s d\lambda(z) \\
\approx \int_B (1 - \| z \|^2)^p \| \nabla P_r f(z) \|^p (1 - \| z \|^2)^s d\lambda(z) \\
\leq C_{n,r} \| f \|_{L^p(B, d\mu_r)}^p \int_B (1 - \| z \|^2)^p \\
\left( \int_B \frac{1}{|1 - \langle z, w \rangle|^{q(n+1)r+2}} d\mu_r(w) \right)^{p/q} (1 - \| z \|^2)^s d\lambda(z) \\
\leq C_{n,r} \| f \|_{L^p(B, d\mu_r)}^p \\
\int_B (1 - \| z \|^2)^{p+s-n-1-(p/q)(1+(q-1)(n+r+2))} d\nu(z).
\]

If \( s > n - p + (p/q)(1 + (q - 1)(n + r + 2)) = 2n + r + 1 \), then

\[
\| P_r f \|_{p,s} \leq C_{n,r} \| f \|_{L^p(B, d\mu_r)}. 
\]

References

[1] J. Arazy, S. D. Fisher, and J. Peetre, Hankel operators on weighted Bergman spaces, Amer. J. Math. 110 (1988), 989-1054.
[2] K. S. Choi, Notes On the Bergman Projection type operators in \( C^n \), Commun. Korean Math. Soc. 21 (2006), no. 1, 65-74.
[3] K. T. Hahn, Bloch-Besov spaces and the boundary behavior of their functions, Lecture Notes series (Seoul Nat. Univ.) 21 (1993), no. 1.
[4] K. T. Hahn and E. H. Youssfi, M-harmonic Besov p-spaces and Hankel operators in the Bergman space on the unit ball in \( C^n \), Jour. Manuscripta Math. 71 (1991), no. 1, 67-81.
[5] K. T. Hahn and K. S. Choi, Weighted Bloch spaces in \( C^n \), J. Korean Math. Soc. 35 (1998), 177-189.
[6] S. G. Krantz and S-Y. Li, On the decomposition theorems for Hardy spaces in domains in \( C^n \) and applications, J. Fourier Anal. Appl. 2 (1995), 65-107.
[7] S.-Y. Li and W. Loo, Characterization for Besov spaces and applications, Part I, J. Math. Anal. Appl. 310 (2005), 477-491.
[8] D. H. Luecking, A Technique for characterizing Carleson measures on Bergman spaces, Proc. Amer. Math. Soc. 87 (1983), 656-660.
[9] W. Rudin, Function theory in the unit ball of \( C^n \), Springer Verlag, New York, 1980.
[10] R. M. Timoney, Bloch functions of several variables, J. Bull. London Math. Soc. 12 (1980), 241-267.
[11] K. H. Zhu, Duality and Hankel operators on the Bergman spaces of bounded symmetric domains, J. Funct. Anal. 81 (1988), 260-278.
[12] K. H. Zhu, *Multipliers of BMO in the Bergman metric with applications to Toeplitz operators*, J. Funct. Anal. 87 (1989), 31-50.

[13] K. H. Zhu, *Operator theory in function spaces*, Marcel Dekker, New York, (1990).

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