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Progress on the calculation of the spectrum from lattice calculations is reviewed. Particular emphasis is placed on discussing our ability to control possible systematic errors coming from finite volume, and extrapolations in quark mass and lattice spacing. Recent approaches based on improved actions are compared.

1. INTRODUCTION

A main goal of lattice QCD is to calculate the spectrum of light hadrons. Such a calculation would not only be a major achievement in its own right and a confirmation that QCD is the correct theory of the strong interaction, it would give us added confidence in our ability to calculate many other nonperturbative quantities that are of phenomenological interest.

It has been 15 years since the first pioneering calculations of the spectrum were done with computers capable of about 1 Megaflop, yet some very simple questions are still relevant. Can we control the systematic errors in lattice calculations? Does the quenched or valence approximation describe the real world? Does QCD with dynamical quarks describe the real world? Can we improve the lattice action and ease the computational burden? Are there hadrons in QCD that are not in the quark model, e.g., glueballs or exotics? What are the quark masses?

Fortunately, the last question is the purview of a talk given earlier by Paul Mackenzie. Subsequent sections of this paper discuss introductory material regarding systematic errors, an overview of recent major simulations, control of systematic errors, results for improvement schemes, and calculations of glueballs and exotics.

2. INTRODUCTION TO SYSTEMATIC ERRORS

There are three physical sources of systematic errors in any lattice calculation, the finite volume $V$ or box size $N_s$, the quark mass $am_q$ which is always heavier than in Nature, and the nonzero lattice spacing $a$. In addition, there may be algorithmic sources of systematic error. These would vary with the particular computational techniques used in the calculation. Chief among these are the use of the quenched or valence approximation and the issue of whether Wilson or Kogut-Susskind (aka, staggered or KS) quarks are used. A priori, we don’t know how much difference it should make to neglect the dynamics of the quarks. In fact, this is one of our basic questions. On the other hand, the two quark representations have different finite lattice spacing errors but are expected to have the same continuum limit. We would like to see this demonstrated by the calculations. Additional possible sources of systematic error include convergence criterion for iterative matrix inversions, gauge fixing accuracy when that is done, and integration step size for molecular dynamics algorithms. We will assume these additional errors are all under control.

One important lesson of recent years is that very high statistics are needed for careful study of systematic errors. For the light quark spectrum, the days of interest in qualitative (10% error) calculations are long past. Researchers should be striving for mass values with errors of about a fraction of a percent and errors in extrapolated quantities of about 1–3%. This may seem like a high standard, especially for more exploratory calculations with improved actions, but future progress requires it.

*Review talk at LATTICE96, St. Louis, USA, June 1996
3. OVERVIEW OF MAJOR SIMULATIONS

There have been a number of new calculations in the last year. To save space, we omit the traditional table of calculations, but direct the reader to the database at the WWW site mentioned at the end of this paragraph. We summarize the major simulations that have been done (over approximately the last five years) for the Wilson gauge action and either Wilson or Kogut-Susskind quarks in a series of graphs. In these graphs, we show the gauge coupling and the spatial size of the simulation. The spatial size can be shown either in lattice units $N_s$ or in terms of the physical size $aN_s$. To save space here, we only show the physical size; however, graphs of $N_s$ vs. $6/g^2$ as well as many additional graphs are available on the WWW at http://physics.indiana.edu/~sg/lat96_spectrum.html.

To determine the physical size, we set the lattice spacing by assuming that the $\rho$ mass at zero quark mass is 770 MeV. This involves an extrapolation in quark mass, and the potential for introducing an error is discussed in the next section. In Fig. 1, we show the lattice spacing as a function of $6/g^2$.

![Figure 1. Lattice spacing as a function of $6/g^2$.](image1)

![Figure 2. Box size vs. $6/g^2$ for quenched Kogut-Susskind calculations.](image2)

gauge coupling. We note that the large discrepancy between Wilson and Kogut-Susskind scales decreases as the lattice spacing does. Further, the range of lattice spacing explored with dynamical quarks and in the quenched approximation is by this measure not very different. (However, this type of summary graph does not clearly indicate the quality of the calculations. For that, we need to know more about the volume, quark masses studied and statistical quality of the results.)

Figures 2–5 show the box size as a function of $6/g^2$ for the major simulations. The WWW site contains color graphs where the color indicates the number of lattices analyzed in each calculation. From these graphs one can immediately see that there is a general tendency for the volumes to decrease as weaker couplings are used. This leads us into our discussion of systematic errors.

4. CONTROL OF SYSTEMATIC ERRORS

In this, the longest section of this paper we consider the three physical sources of systematic
error introduced in Sec. 2. Where possible we try to include results for quenched and dynamical, and Wilson and KS calculations. In some cases, the results available from the literature, including what was presented at Lattice ’96, are insufficient to warrant extrapolation to the physical limit.

4.1. Finite size effects

We expect that finite size (FS) effects will be quark mass dependent, with the effect increasing as the quark mass decreases. If we had infinite computer resources, we would study hadron masses over a wide range of volume, quark mass and coupling. We could then see whether the effect shows proper physical scaling. That is, when we vary the coupling do we find that the volume dependence is independent of $a$ for a fixed physical quark mass (or $m_\pi/m_\rho$)? With such detailed understanding, we might actually be able to do accurate extrapolations in volume from small volume calculations. Unfortunately, we are far from this situation and can only currently hope to identify a physical volume above which the finite size effects are sufficiently small.

For quenched Wilson quarks, there have not been very extensive studies of finite volume effects. Only at $6/g^2 = 5.7$ and 6.0 are there any calculations on more than two volumes. Also, there tends to be more variation in the hopping parameter $\kappa$ chosen by different groups than there is in the staggered masses. At $6/g^2 = 5.7$, the IBM group [4] has used three sizes $N_s = 8$, 16 and 24. On those lattices, there are only two $\kappa$ values that have been used for all three volumes. They correspond to $m_\pi/m_\rho = 0.69$ and 0.86. In Fig. 6, we show their results for the nucleon mass along with a result from the APE group for $N_s = 12$ [5]. There does seem to be a FS effect between $N_s = 8$ and 16, for the heavier masses. For the lightest mass, $\kappa = 0.1675$, the difference in the masses at 16 and 24 is 0.042(18). This is a $2.6\sigma$ or 4.8% effect. (The next heavier masses for $N_s = 16$ and 24, are actually not at the same $\kappa$, which is why we have no plotting symbol for those points. However, they are used in the chiral extrapolation, which was a linear extrapolation based on the three lightest quark masses [3].) This is all
the data we have for finite size effects for the nucleon at this coupling. For the $\rho$, the data are shown in Fig. 4. There is no evidence here for an effect at $N_s = 8$. For the lightest mass, there is a difference of 0.028(8), which is a 2.5$\sigma$ or 3.3% effect. Using the $\rho$ to set the scale, the two largest lattices are 2.3 and 3.4 fm on a side. Turning our attention to $6/g^2 = 6.0$, no single group has done calculations on three volumes, but there are six calculations for $N_s = 16$ [6], 18 [7], 24 [8,9,10] and 32 [11]. The two largest sizes here correspond to 2.1 and 2.9 fm. For the nucleon, Fig. 5 shows the data. The mass difference for the lightest $\kappa$ is 0.003(16). Clearly, this is consistent with no effect, but the error in the difference is 3%, so we have not ruled out the 3.3% effect seen at 5.7. At the next heaviest $\kappa$, the difference is 0.0026(96) or a 1.6% error in the difference. At the heavier quark masses, there are some observable differences between $N_s = 24$ and 32. For instance, for $\kappa = 0.153$, the heaviest mass, the difference is 0.0150(49) which is a 1.9% or 3.1$\sigma$ effect. At $\kappa = 0.155$, the difference is 0.0113(40) which is a 1.8% or 2.8$\sigma$ effect. To save space, we do not include here the graph for the $\rho$. We merely note that for the more precise calculation at $N_s = 24$ the difference between 24 and 32 is 0.010(6) which is a 2.5% difference or 1.7$\sigma$. If we only considered the $N_s = 24$ result with the larger error bar, there would be no observable effect.

For staggered quarks, there have been more extensive studies of finite size effects in both the quenched [12,13] and dynamical cases [12,14]. For $6/g^2 = 5.7$, six lattice sizes have been studied from $N_s = 8$ to 24. For $6/g^2 = 6.0$, six lattice sizes have been studied from 6 to 32. In Fig. 6, we show the nucleon mass in lattice units vs. $N_s$. The three largest sizes correspond to 1.8, 2.6 and 3.5 fm. The finite size effect is clearly quite large at the smallest volumes, for which results are only available for the heaviest quark mass $am_{\pi} = 0.01$. The heaviest quarks here ($m_{\pi}/m_\rho = 0.51$) are comparable to the lightest for the Wilson quark calculations. Looking at the octagons, we see that $am_N(N_s = 16) - am_N(N_s = 32) = 0.0158(140)$ which is slightly above 1$\sigma$, but represents a 2.2% error in the difference. Between 24 and 32, the difference is about 0.6$\sigma$ with the same size error. However, for the lightest quark mass, $am_N(N_s = 16) - am_N(N_s = 32) = 0.080(11)$ which is a 7$\sigma$
effect and one standard deviation is also 2.2% error. Comparing only the two largest sizes, the difference is 1.7σ or 3.7±2.2%. It looks like we have some good evidence for finite size effects with a 1.8 fm box, especially at the lighter quark masses. Between 2.6 and 3.5 fm, we don’t have really strong evidence for an effect, but with an error in the difference of about 2%, there could be a few percent effect even on such large volumes.

For dynamical staggered quarks, there has been extensive study of the FS effects at two couplings[12,14,15]. This work is a few years old so we merely recall that a box size of at least 2.5 fm was needed to eliminate the FS effect for the quark masses studied there. One might need even bigger volumes for lighter quarks. For dynamical Wilson quarks, there are not enough results at different volumes to study this issue.

To summarize, for the quenched approximation, we see FS effects with a box size of under 2 fm. For the lighter quark masses, there might even be some effect on the nucleon mass for a box size as big as 2.5 fm. A box size that large or greater is strongly recommended if the nucleon mass is to be calculated. Those who ignore this advice do so at their own risk! More high statistics work is needed if we wish to determine what box size is needed to reduce effects to the 1% level for various quark masses. We are far from understanding the effects well enough to extrapolate from small volumes (say, 1.5 fm) to the infinite volume limit with 1% accuracy.

4.2. Extrapolation in Mass

The chiral extrapolation may well now be the least well understood source of systematic error. Chiral perturbation theory[16] provides an expansion for the hadron masses in terms of the quark mass; however, in the quenched approximation[17], from the work of Bernard, and Golterman[18], and Labrenz and Sharpe[19], we know that there are additional terms, e.g., a $\sqrt{m_q}$ term for the nucleon, that are important at small quark mass. The talk at this conference by Steve Sharpe[20] deals in more detail with the difficulties of chiral extrapolation.

The chiral extrapolation process is complicated because chiral perturbation theory is a small mass expansion, but our most accurate numerical data are for large mass. Thus, instead of extending our fits from small mass to large mass and adding new chiral terms as needed, we are often forced to use simple chiral forms (constant plus linear) for relatively large mass and add additional terms when the simple forms don’t work.

As an example of this problem, in Fig. 10, we
show several curves related to the chiral behavior of the IBM group's nucleon mass at $6/g^2 = 5.93$. The curves come from a cubic fit by Gupta [17] and include the possibility of a term proportional to $m_\pi$ that only appears in the quenched approximation. The three curves that converge at the $y$-axis are from the Gupta fit to the data. From top to bottom, they are the quadratic, cubic and linear truncations of the fit. The fourth curve, which diverges from the others at the $y$-axis is of the form $a + bm_\pi^2$, which is the lowest order contribution in chiral perturbation theory. The coefficients $a$ and $b$ have been adjusted to fit the previous cubic fit over the range 0.45–0.85 GeV. The two curves are nearly indistinguishable over a wider range, but at the physical pion mass or below, there is clearly a significant difference. We thus see that the extrapolation can be quite sensitive to which terms are kept in the chiral approximation.

Over the past two years, several groups have been working particularly hard to understand the chiral extrapolations. The LANL [21], MILC [22], and SCRI [23] collaborations have looked at vector mesons and the nucleon. The JLQCD collaboration [24], Kim and Sinclair [25], Kim and Ohta [26], and Mawhinney [27] have concentrated on the question of whether it is possible to observe quenched chiral logarithms in the pion mass.

Among the first three groups mentioned above, the LANL and SCRI groups have results for hadrons with unequal mass quarks; hence for the mesons they have $n(n + 1)/2$ mesons for $n$ quark masses. Thus, they have more data and more degrees of freedom when doing their fits. This allows them more freedom in choosing the range of quark mass included in the chiral fit. MILC, on the other hand, has a very wide range of quark masses in its calculations (a factor of 16 from lightest to heaviest). However, with five quark masses and only hadrons constructed from equal mass quarks, there is limited freedom to play with the range of mass included in the fits. One of the difficulties in reviewing the chiral extrapolations is that for quenched calculations hadron masses for different quark masses are correlated. To get a proper goodness of fit for the chiral extrapolation, those correlations must be known, yet they are rarely published along with the hadron masses. (It must be admitted, however, that not every group includes these correlations in their chiral extrapolations.)

At Lattice ’95, John Sloan [28] presented evidence from the SCRI collaboration that fits to the $\rho$ mass as a function of the $\pi$ mass can be extended to higher quark mass if either either $C_3 m_\pi^2$...
or $C_4 m_q^4$ is added to $C_0 + C_2 m_q^2$. For instance, with the cubic term, the range $0.58 < m_\pi/m_\rho < 0.93$ can be fit, but without it, the range is $0.58 - 0.77$. In terms of $m_\pi$, the ranges are very roughly $0.5 - 1.3$ GeV and $0.5 - 0.8$ GeV, respectively. Similar results have been seen in Refs. [11,21], where the $\rho$ was fit as a function of quark mass and a linear function was sufficient to describe the data only over the range $0.58 < m_\pi/m_\rho < 0.7$. With a higher power of quark mass, either $m_q^{3/2}$ or $m_q^2$, the fit could be extended to $m_\pi/m_\rho = 0.84$. Neither group has been able to clearly distinguish between the two higher powers; however, the first group notes that a cubic seems to follow the data better than a quartic in the high mass region where the fit is poor.

Turning to the Kogut-Susskind calculations, MILC [22], with its wide range of quark masses and small error bars, had great difficulty in fitting its rho and nucleon masses with a single term in addition to $M_0 + b m_q$. Twelve different fitting functions were studied. They are up to four parameter fits, and include terms such as $\sqrt{m_q}$ and $m_q \ln m_q$, which are quenched chiral terms, as well as $m_q^{3/2}$, $m_q^2$ and $m_q^2 \ln m_q$, which are higher order terms that appear in both quenched and ordinary chiral perturbation theory. Figure 11 gives an example of the variation of the nucleon mass with the choice of fitting function. The size of the plotting symbols is proportional to the confidence level of the fit. Any visible symbol is a “reasonable” fit. In the presence of a $\sqrt{m_q}$ term, the nucleon mass decreases and the error in the extrapolated value increases. Looking at the other volumes and couplings, a combined confidence level (CCL) for all the cases can be calculated. Adding to $M_0 + b m_q$ (fit 5) a single power, $m_q^{3/2}$ ($m_q^2$) gives a confidence of $3 \times 10^{-10}$ ($10^{-24}$). (These are fits 6 and 7). There are six functions that have a CCL $> 0.01$. Five functions are four parameter fits, to fit the five masses. The only three parameter fit, $M_0 + b m_q + c m_q \ln m_q$ (fit 9), has a CCL of 1%. Among the four parameter fits, adding both $m_q^{3/2}$ and $m_q^2$ (fit 8) to constant plus linear has the best CCL, 0.18, but is it not that much better than adding $\sqrt{m_q}$ in place of one of the higher powers, which give 0.12–0.13 (fits 2, 3). Fit 12 contains two higher order chiral terms, $m_q^2$ and $m_q^2 \ln m_q$. Thus, it is not clear whether MILC may be seeing some evidence for a quenched chiral effect in the nucleon mass.

Quenched chiral logarithms have also been sought in the $\pi$ channel. This particle has the advantage that its mass is the most precisely determined. The lowest order chiral prediction is

$$m_\pi^2 = A m_q.$$  

Kim and Sinclair [28] did a quenched calculation at $6/g^2 = 6.0$ and went to very light quark mass. Kuramashi et al. [29] noticed the failure of the above relationship. In subsequent work, Kim and Sinclair [25] varied the lattice size to demonstrate control of the finite size effects. However, it was noted last year [22] that at stronger couplings than 6.0, the rise in $m_\pi^2/m_q$ as the quark mass decreases occurs even for quite heavy quarks, thus making one wonder if the rise is truly a chiral effect. Mawhinney [27] studied the $\pi$ mass and chiral condensate in both quenched and dynamical

\[ \text{fit number} \]

\[ \text{extrapolated } m_\pi^2 \]

\[ 6/g^2 = 5.85, N_s = 24 \]

\[ 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6 \]

\[ 0.7, 0.8, 0.9, 1.0 \]
quark configurations. Varying the valence mass used to make the measurements, he found linear behavior, but with a non-zero intercept for $m_\pi^2$ and $\bar{\chi}\chi$. This, he interpreted as a FS effect, rather than a quenched chiral logarithm.

Figure 12. $m_\pi^2/m_\eta$ vs. $(m_\pi/m_\rho)^2$ for quenched staggered calculations.

Figure 12 summarizes the current results for couplings from 5.7–6.5. The horizontal axis is $(m_\pi/m_\rho)^2$, which makes it easier to compare different couplings than in an earlier plot using the scale dependent $am_\eta$ on the x-axis. Where there is data for different volumes is it all plotted to show the FS effects. It would certainly be valuable to have a lighter mass at 6.15 and heavier masses at 6.5; however, it now looks like that at 6.0 and weaker coupling there may be a broad flat region at intermediate quark mass.

The JLQCD collaboration [24] has presented a very interesting graph in which they display the parameter $\delta$ of the chiral logarithm as a function of lattice spacing. They fit the $\pi$, including non-degenerate quark and antiquark to the four parameter form

$$\langle m_\pi \rangle^2 = \frac{(m_1 + m_2)a}{2} \cdot 2A.$$
higher order terms are necessary; on the other hand, underestimation of errors may call for additional terms to explain spurious variations that are merely statistical.

The Edinburgh plot remains a very valuable tool to summarize the results of a spectrum calculation. At the conference, eight such plots were shown including three each for quenched staggered and Wilson calculations where there is so much data that presenting it all on one graph would confuse rather than enlighten. The full set of graphs may be found at the WWW site. In Fig. 14, we present results from the MILC and IBM groups who have results at several couplings. The upper plot includes a curve showing the extrapolation to \( a = 0 \). In Fig. 15, we show results for dynamical quarks. There are new results from SCRI [33], SESAM [34] and \( T\chi L\) [35] for dynamical Wilson quarks presented at this conference.

The quenched staggered simulations are generally going further toward the chiral limit than the Wilson. Dynamical Wilson calculations are very far from the chiral limit \( (m_\pi/m_\rho \gtrsim 0.6) \); staggered calculations have gone closer to the chiral limit,
but not at the weakest coupling studied (5.7).

4.3. Extrapolation in lattice spacing

The extrapolation in lattice spacing is well understood compared to the chiral case. For Wilson quarks we expect errors of order $a$, while for Kogut-Susskind quarks the errors should be of order $a^2$. In Fig. 16, we show the extrapolation to zero lattice spacing based on data from the MILC collaboration. The chiral extrapolations for the rho and nucleon were based on the form $M + am_q + b m_q^{3/2} + cm_q^2$. The strong coupling calculation at $6/g^2 = 5.54$ done this year and the analytic strong coupling result of Kluberg-Stern et al.[33] give confidence that the corrections are of order $a^2$. The two extrapolations either include or ignore the 5.54 data. Only the the largest volume data at 5.7 and 5.85 is used for the fit. For dynamical quarks, there are not sufficient results in the literature where both FS effects and the chiral extrapolation are under control. Thus, we make no attempt to produce a similar plot.

![Figure 16. Extrapolation of $m_N/m_\rho$ as a function of $am_\rho$ for quenched KS quarks.](image)

5. IMPROVEMENT

Quite a number of groups are attempting various improvement schemes. Improved or perfect actions[34,35] offer the hope of allowing lattice calculations on much coarser lattices, which could save a tremendous amount of effort for spectrum calculations, and possibly allow us to calculate new quantities that are currently too costly to compute. There were 59 contributions to the conference that had the string “improve” in the abstract. Actually, one of them only claimed to have improved statistics, but the others all dealt with some attempt to improve the action beyond the usual Wilson gauge action and either Wilson or Kogut-Susskind quarks. Probably an entire hour would be insufficient to review all of this work. This is especially true in view of the wide variety of calculations done. I have tried to avoid having this talk be a “laundry list” of each group’s calculations. When possible I prepared summary graphs that show data from several groups. With the variety of improvements, it is hard to know which calculations to compare.

Among the various improvement schemes, the one with the longest history is the Sheikholeslami-Wohlert[36] or “clover” quark action with the Wilson gauge action. Four groups have been actively studying this scheme in the quenched approximation. (Any calculation mentioned below without a reference should be assumed to be a presentation at Lattice ’96.) Allton, Gimenez, Giusti and Rapuano[7] have completed a series of runs at $6/g^2 = 6.0, 6.2$ and 6.4, with ensembles ranging from 200 to 400 lattices. They also have results for ordinary Wilson quarks. Decay constants are an important focus of this work. Bhattacharya and Gupta have studied weak matrix elements using the clover action, while Stephenson presented results with a non-perturbative clover coefficient[37]. Finally, the UKQCD collaboration has studied the couplings 5.7, 6.0 and 6.2, and has compared tadpole-improvement with no improvement for 5.7.

The clover action in gauge configurations that include the effects of dynamical Kogut-Susskind quarks has been studied by the SCRI group consisting of Collins, Edwards, Heller and Sloan.
They use the old HEMCGC lattices at $6/g^2 = 5.6$, $am_g = 0.01$. Borici and de Forcrand have investigated an improvement scheme based on blocked lattices.

Recently, schemes based on improved gluonic actions have been extended to include quarks. Alford, Klassen and Lepage have investigated various “highly improved” quark actions. The SCRI group mentioned above has done an extensive series of calculations for six gauge couplings with the clover action. Fiebig and Woloshyn have studied a quark action that includes next-nearest neighbor interactions. The MILC collaboration has studied Kogut-Susskind quarks in improved glue, as well as a third nearest neighbor interaction due to Naik. Finally, Morningstar and Peardon have done glueball calculations using improved glue on an anisotropic lattice. One of the potential difficulties with coarse lattices is that it may be very difficult to fit the hadron propagators. The plateau in an effective mass plot is determined by physical considerations, so if it is 10 lattice spacing on a lattice with $a = 0.1$fm, it will only be 2.5 lattice spacings if $a = 0.4$fm. With an anisotropic lattice that has a larger spatial lattice spacing, it may be possible to reap most of the benefit in computational cost without paying the price of not being able to fit the masses.

Bock presented a nice talk in which he compared four different improvement approaches all at the coupling corresponding to the thermal crossover for $N_t = 2$. He did spectrum calculations on $6^3 \times 16$ lattices and plotted ratios of $m_N$, $m_\Delta$ and $\sqrt{\sigma}$ to $m_\rho$, where $\sigma$ is the string tension. For the “Cornell” gauge action, he investigated three quark actions, Wilson, clover and D234. For the “Iwasaki-Yoshié” gauge action, he used the D234 action. He found good agreement for the ratios for all cases except the Wilson quark action.

Our final graph, Fig. 17, compares results for $m_N/m_\rho$ from a variety of quenched calculations. For comparison, the results of Fig. 16 are repeated. The point above $am_\rho = 0$ plotted with a fancy plus symbol is the extrapolated value for Wilson quarks presented by the IBM group (1.278±0.068). Their extrapolation was based on only three of the crosses. From the left, the first point is their result for $6/g^2 = 6.17$, the next is at 6.0 from Bhattacharya et al., the next is IBM’s result at 5.93. The next two points directly above each other are both 5.7 results from IBM. They used several sources and had two volumes at this coupling. For the smaller volume, the results are 1.459(34) (plotted) and 1.371(38), for combined results of sink sizes 0, 1, 2 and for results from sink size 4 alone, respectively. On the larger volume, they found 1.397(36) (plotted) and 1.373(64), respectively. The IBM continuum extrapolation was based upon the smaller volume, sink 0, 1, 2 results. (That volume is more comparable to the physical volume of their weaker coupling calculations.) Any of the other three results would clearly decrease the fitted slope for $m_N/m_\rho$ and result in a larger value of the ratio at $a = 0$. Bhattacharya et al. have performed the continuum extrapolation using their result and the sink 4, smaller volume result and find 1.38(7) in the continuum limit. Clearly, the 5.7 value plays a crucial role in the extrapolation, and having more results at other couplings would improve our understanding.
Results other than the KS points denoted by diamonds, Wilson points denoted by crosses and the strong coupling KS result all involve an improved gluon action. The SCRI results, with six couplings, comprise the most extensive results at this point. The lattice spacing dependence of the mass ratio is consistent with a quadratic correction. It is not consistent with a linear dependence. Results for the Wilson quark action for these gauge configurations are also available. Most of the other improved glue results fall well below the KS result in ordinary glue (diamond above $am_q = 1.2$). The results for the D234 and D234(2/3) actions from Alford, Klassen and Lepage\cite{12} are promising, but the errors are too large to determine the $a$ dependence. The results from Fiebig and Woloshyn on the next-nearest-neighbor action may indicate a stronger dependence on $a$, but again it would be useful to have greater precision.

The two points denoted by a fancy cross compare the ordinary Kogut-Susskind and Naik quark actions in the same improved glue configurations. Their proximity indicates that most of the improvement comes from the glue, not from the quark action.

6. GLUEBALLS AND EXOTICS

There was not sufficient time in my talk for a discussion of glueballs or exotics; however, there were some interesting works that should be advertised. Bali presented a poster on new results for glueballs in dynamical Wilson quark configurations. Lee and Weingarten presented a talk and poster session regarding scalar quarkonium and further evidence that $f(1710)$ is a glueball. Luo described recent glueball mass calculations done in a Hamiltonian formalism.

The UKQCD collaboration has been studying hybrid mesons recently\cite{13} and two papers appeared shortly before the conference, one being a review by Michael\cite{14}. Toussaint presented a poster session describing recent attempts of the MILC collaboration to calculate exotic masses.

Also of interest were two contributions related to the $\eta'$. Massetti described a bermion calculation of the $\pi-\eta'$ splitting and Venkataraman described an $\eta'$ calculation done in configurations with $N_f = 0, 2$ and 4 dynamical staggered quarks.

7. CONCLUDING REMARKS

A great deal of interesting work is being done (much more than can be adequately described here, so don’t forget to check the WWW site). We are on the verge of answering some of the questions posed at the beginning of the talk, especially as regards the quenched or valence approximation. To answer those questions will require additional work in order to demonstrate control of systematic errors. We need:

- Very high statistical accuracy (< 1%)
- Large volumes
- A wide range of quark masses, with special attention to the chiral region and chiral extrapolations
- Better understanding of relevant terms in the chiral expansion.

Various improvement schemes are being pursued, but it is too soon to say which approach is best. There is certainly considerable evidence for a faster approach to the $a \rightarrow 0$ limit with several schemes. (Although it is not yet clear that all the schemes have the same limit!) Improvement schemes are supposed to greatly ease the computational burden. If that is so, then a higher degree of statistical accuracy should be expected in such studies. Such accuracy as well as reliable values for the computational effort required for these calculations are necessary to decide which scheme is best.

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