Critical behaviour of the $O(n)$-$\phi^4$ model with an antisymmetric tensor order parameter

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Abstract. Critical behaviour of the $O(n)$-symmetric $\phi^4$-model with an antisymmetric tensor order parameter is studied by means of the field-theoretic renormalization group (RG) in the leading order of the $\varepsilon = 4 - d$-expansion (one-loop approximation). For $n = 2$ and 3 the model is equivalent to the scalar and the $O(3)$-symmetric vector models, for $n \geq 4$ it involves two independent interaction terms and two coupling constants. It is shown that for $n > 4$ the RG equations have no infrared (IR) attractive fixed points and their solutions (RG flows) leave the stability region of the model. This means that fluctuations of the order parameter change the nature of the phase transition from the second-order type (suggested by the mean-field theory) to the first-order one. For $n = 4$, the IR attractive fixed point exists and the IR behaviour is non-universal: if the coupling constants belong to the basin of attraction for the IR point, the phase transition is of the second order and the IR critical scaling regime realizes. The corresponding critical exponents $\nu$ and $\eta$ are presented in the order $\varepsilon$ and $\varepsilon^2$, respectively. Otherwise the RG flows pass outside the stability region and the first-order transition takes place.

Key words: critical behaviour, tensor order parameter, renormalization group.

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1. Introduction

Numerous physical systems reveal interesting singular behaviour in the vicinity of their critical points. Their thermodynamical and correlation functions exhibit scaling behaviour with universal critical dimensions: according to the prevailing belief, they depend only on few global characteristics of the system, like symmetry or dimension; see, e.g., the monographs [1]. Preliminary analysis of the critical behaviour is usually performed within the framework of the phenomenological Landau theory, where the free energy of the system in question is written in the simplest form dictated by the symmetry [2]. That approach predicts the type of the phase transition (first- or second-order one) but gives only approximate “mean-field” values for the critical exponents. More refined fluctuation theory applies Landau’s idea to the effective Hamiltonian of the
Critical behaviour of the $O(n)$-$\phi^4$ model with an antisymmetric order parameter

system, which is written in a form similar to a certain Euclidean field theoretic model [2]. Thus the further analysis of the problem calls for the field theoretic techniques.

The powerful and quantitative theory of the critical behaviour is provided by the field theoretic renormalization group (RG); see the monographs [3, 4, 5] and the literature cited therein. In the RG approach, possible types of critical behaviour (universality classes) are associated with infrared (IR) attractive fixed points of renormalizable field theoretic models. Most typical phase transitions (liquid-vapour systems, binary alloys, ferro- and antiferromagnets) belong to the universality class of the $O(n)$-symmetric model with quartic interaction (Euclidean $\phi^4$-model) with an $n$-component order parameter. Another important example is provided by the $U(n)$-symmetric $\phi^4$-model with a complex order parameter. That model, describing transitions in quantum gases and liquids, is in fact equivalent to the $O(2n)$-symmetric real case.

In agreement with the Landau theory that predicts a second-order transition for such systems, the RG analysis establishes therein the existence of nontrivial IR attractive fixed point in the physical range of parameters, and hence the existence of IR scaling behaviour. Its universal characteristics depend only on $n$ and $d$, the dimension of the system, and can be systematically calculated as expansions in $\varepsilon = 4 - d$, the deviation of the spatial dimension from its upper critical value $d = 4$ [3, 4, 5].

In many cases, however, description by the aforementioned, relatively simple, models appears inadequate, and one has to consider more sophisticated symmetries or more complex types of the order parameter with tensor or matrix nature. Not an exhaustive list of such phenomena includes phase transitions in systems with nontrivial crystallographic symmetry or randomly distributed impurities (see the monograph [6] for a general review and references), various transitions in liquid crystals [7, 8, 9, 10, 11], transitions between different superfluid phases in He$^3$ [12, 13, 14] and in the neutron liquid in neutron stars [15, 16], transition to superconductive state in systems with higher spins [17], models of Laplacian growth with multifractal properties [18], and so on.

As a rule, the corresponding field-theoretic models involve several types of interaction terms and hence several coupling constants (charges). The corresponding RG equations can have several fixed points with different attractive properties [9, 10, 13, 14, 15, 16, 17, 19]. This can lead to a very complicated pattern of the corresponding RG flows (solutions of the RG equations for the invariant charges) in the space of model parameters.

Although some general statement (the so-called $\eta$ conjecture) can be formulated for the IR attractive fixed points in models with an $n$-component vector order parameter [19], the very existence of such points is not a necessary feature of the $\phi^4$-models. Furthermore, even in the presence of IR attractive points, the RG flow can pass outside the natural region of parameters, determined by the stability of the system, the situation usually interpreted as a first-order phase transition. It may also go to infinity, the situation that lies beyond the scope of the perturbation theory. As a result, the
predictions of the plain Landau theory can be essentially corrected.

In this connection, the Ginzburg-Landau model of superconductivity [2] is worth recalling: the one-loop analysis of the corresponding field theoretic model (actually, the electrodynamics of a charged scalar field) shows that is has admissible fixed point only for very large \( n \) [20]. The situation, however, is not completely clear: two-loop calculations with an appropriate resummation procedure suggest that the attractive point “has a chance to exist” [21]. The non-perturbative analysis of Ref. [22] also favors the second-order transition.

In a sense, opposite examples are provided by the model with a symmetric tensor order parameter and by the Potts model: according to the Landau theory, existence of a cubic term excludes the possibility of the second-order transition. On the contrary, exact two-dimensional results, numerical simulations and RG analysis suggest that for small \( n \), the phase transition is of the second order [9, 23].

In this paper we apply the field theoretic RG to the \( O(n) \)-symmetric \( \phi^4 \)-model of the real \( n \)-th rank tensor order parameter. This model can be relevant in the analysis of transitions between the nematic, cholesteric and blue phases in liquid crystals [24, 25], transitions to ferroelastic state in solids [26, 27] and transitions to superconductive state in systems with higher spins [17]. Our main motivation, however, is more theoretical, and in order to simplify and to sharpen the problem, we consider the case of a purely antisymmetric tensor. In comparison to the general \( n \)-th rank tensor case, this reduces the model to the two-charge problem, which makes the results more visible. The model is probably the simplest one with a non-vector order parameter, but remains a multicharge one and, as we will show, demonstrates the features typical of more realistic and complex situations listed above. In this connection it is also important that the cubic invariant for the purely antisymmetric tensor vanishes identically, so that the Landau theory, as conventionally applied, predicts a second-order transition, and we do not face the contradiction, mentioned above for the symmetric tensor case.

The plan of the paper is as follows. In the next three sections we formulate the model, give the corresponding Feynman rules, perform the ultraviolet (UV) renormalization and give the explicit leading-order expressions for the renormalization constants. In section 5 we present the RG equations for the renormalized Green functions and give the leading-order expressions for their coefficients (\( \beta \)-functions and anomalous dimensions). In section 6 we analyze the fixed points of the RG equations for the invariant coupling constants.

It turns out that existence of an IR attractive fixed point in the physical range of parameters is an exception rather than a rule. Such points exist for the special cases \( n = 2 \) and \( n = 3 \), when our model becomes equivalent to the scalar and the \( O(3) \)-symmetric vector models, respectively, and thus is in fact a single-charge one. For general \( n \), admissible fixed points exist only in a certain single-charge special case of the model, which appears multiplicatively renormalizable in itself (that is, closed with respect to the renormalization procedure) and thus can be studied as a separate internally consistent model (in the full-scale two-charge model, such fixed points are
saddle points).

The only admissible fixed point in the full-scale problem exists for \( n = 4 \), that is, for the minimal possible value of the rank where the model is a genuine two-charge one (for larger \( n \) that point becomes complex). Its existence means that all the Green functions of the model in the IR range can demonstrate self-similar (scaling) behaviour. The corresponding critical exponents \( \eta \) and \( \nu \) are given in section 7 in the leading order of the \( \varepsilon \)-expansion. However, for a multicharge model, even when an IR attractive fixed point is present, not every “RG flow” (solution of the RG equations for the invariant charges) approaches it in the IR asymptotic range: it can first pass outside the region of stability (an indication of the first-order transition) or go to infinity (then no definitive conclusions can be drawn within the perturbation theory).

The main conclusion is that the account of fluctuations can change the character of the phase transition for the antisymmetric order parameter from the second-order to the first-order type; this is a possible behaviour for \( n = 4 \) and the only possible one for all \( n > 4 \).

2. The model

We study a model of a real antisymmetric \( n \)-th rank tensor field \( \phi = \phi_{ik}(x) \) (so that \( \phi_{ik} = -\phi_{ki} \) and \( i, k = 1, \ldots, n \)) in the Euclidean \( d \)-dimensional \( x \) space. The action functional has the form

\[
S(\phi) = S_0(\phi) + V(\phi)
\]

with the free part

\[
S_0(\phi) = \frac{1}{2} \text{tr}\{\phi(-\partial^2 + \tau_0)\phi\}
\]

and the interaction term with the two independent quartic structures

\[
V(\phi) = V_1(\phi) + V_2(\phi) = -\frac{g_{10}}{4!} \{\text{tr}(\phi^2)\}^2 - \frac{g_{20}}{4!} \text{tr}(\phi^4).
\]

Here (and in analogous formulas below) integration over the \( d \)-dimensional \( x \) space is implied; \( \partial^2 \) is the Laplace operator, \( \tau_0 \) is the deviation of the temperature (or its analog) from the critical value and \( g_{10}, g_{20} \) are the coupling constants. In the detailed notation

\[
S_0(\phi) = -\frac{1}{2} \int d\mathbf{x} \sum_{i,k=1}^{n} \phi_{ik}(\mathbf{x})(-\partial^2 + \tau_0)\phi_{ik}(\mathbf{x})
\]

and similarly for \( V(\phi) \). The cubic term \( \text{tr}(\phi^3) \) vanishes due to the antisymmetry of \( \phi \) and does not appear in the interaction.

Correlation functions (Green functions) of the model are given by the functional averages with weight \( \exp S(\phi) \). The action \( (2.1)-(2.3) \) is invariant with respect to the transformation \( \phi \rightarrow O\phi O^\dagger \), where \( O \in O(n) \) is an \( n \)-th rank orthogonal matrix (note that the antisymmetry property is preserved by this transformation).
The stability of the model requires that the interaction term (2.3) be negative for all values of $\phi$. One can check that the condition $V(\phi) < 0$ imposes the following restrictions on the coupling constants:

$$2g_{10} + g_{20} > 0, \quad ng_{10} + g_{20} > 0$$

(2.4)

for even values of $n$ and

$$2g_{10} + g_{20} > 0, \quad (n - 1)g_{10} + g_{20} > 0$$

(2.5)

for $n$ odd.

For $n = 2$ and $n = 3$ the model (2.1)-(2.3) reduces to the well-known cases: the single-component $\phi^4$-model and the $O(3)$-invariant vector model, respectively. The correspondence can be established by means of the transformations $\phi_{ik} = \varepsilon_{ik}\phi$ for $n = 2$ and $\phi_{ik} = \varepsilon_{ikl}\phi_l$ for $n = 3$, where the both $\varepsilon$’s are fully antisymmetric tensors with normalization $\varepsilon_{12} = \varepsilon_{123} = +1$. Then the both structures in $V(\phi)$ become identical to $\phi^4$ for $n = 2$ and $(\phi\phi_l)^2$ for $n = 3$, and the only remaining coupling constant is the combination $g_0 = 2g_{10} + g_{20}$. In the both cases, the stability conditions (2.4), (2.5) reduce to the single inequality $g_0 > 0$.

3. Diagrammatic techniques

The Feynman diagrammatic techniques for the model (2.1)-(2.3) is derived in a standard fashion; see e.g. [3]–[5]. In the momentum (Fourier) representation the bare propagator, determined by the free action (2.2), has the form

$$\langle \phi_{ik}\phi_{lm} \rangle_0 = \frac{J_{ik;lm}}{(p^2 + \tau_0)},$$

(3.1)

where $p = |p|$ is the wave number. The tensor

$$J_{ik;lm} = \frac{1}{2} (\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}),$$

(3.2)

built of the Kronecker $\delta$ symbols, is antisymmetric with respect to the transpositions of its indices $i \leftrightarrow k$ and $l \leftrightarrow m$, and symmetric with respect to the transposition of the pairs $ik \leftrightarrow lm$. It plays the part of the unit operation on the space of antisymmetric tensors in the sense that $J_{ik;lm}\phi_{lm} = \phi_{ik}$ and $J_{ik;lm}J_{lm;js} = J_{ik;js}$. Its “trace” with respect to the pairs of indices $J_{ik;ik} = n(n - 1)/2$ gives the number of independent components of an $n$-th rank antisymmetric tensor.

The interactions $V_{1,2}(\phi)$ in (2.3) correspond to the quartic vertices with the vertex factors $(-g_{10})V^{(1)}_{abcd;ef;mn}$ and $(-g_{20})V^{(2)}_{abcd;ef;mn}$, where the tensors

$$V^{(1)}_{abcd;ef;mn} = \frac{1}{3} (J_{abcd}J_{ef;mn} + J_{abef}J_{cd;mn} + J_{abmn}J_{cd;ef})$$

(3.3)

and

$$V^{(2)}_{abcd;ef;mn} = \frac{1}{6} \left( J_{abij}J_{cd;jk}J_{ef;kp}J_{mn;pi} + J_{abij}J_{cd;jk}J_{mn;kp}J_{ef;pi} + J_{abij}J_{ef;jk}J_{mn;kp}J_{cd;pi} + J_{abij}J_{ef;jk}J_{cd;kp}J_{mn;pi} + J_{abij}J_{mn;jk}J_{ef;kp}J_{cd;pi} + J_{abij}J_{mn;jk}J_{cd;kp}J_{ef;pi} \right)$$

(3.4)
are defined such that
\[ V^{(1)}_{abcd;ef, mn} \phi_{ab} \phi_{cd} \phi_{ef} \phi_{mn} = \{ \text{tr}(\phi^2) \}^2 \]
and
\[ V^{(2)}_{abcd;ef, mn} \phi_{ab} \phi_{cd} \phi_{ef} \phi_{mn} = \text{tr}(\phi^4), \]
and such that they are antisymmetric with respect to the transpositions of the indices $a \leftrightarrow b$, $c \leftrightarrow d$ and so on, and symmetric with respect to the transpositions of the pairs $ab \leftrightarrow cd$, $ab \leftrightarrow mn$ and so on.

Thus any diagram of our model is represented as a product of two factors: the corresponding diagram for the single-component $\phi^4$-model with the corresponding symmetry coefficient and the additional $n$-dependent factor stemming from the contractions of the tensors in the propagators (3.1) and vertices (3.3), (3.4).

4. UV renormalization

The analysis of renormalizability of the model (2.1)–(2.3) is very similar to the case of the single-component $\phi^4$-model; see e.g. [3]–[5]. The model is logarithmic (the coupling constants $g_{10}$, $g_{20}$ are dimensionless) for $d = 4$. In the dimensional regularization, the UV divergences have the form of the poles in $\varepsilon = 4 - d$, deviation of the dimension of space from its upper critical value $d = 4$. Standard analysis, based on the dimensionality and symmetry considerations, shows that superficial UV divergences, whose elimination requires counterterms, are present only in the 1-irreducible Green functions $\langle \phi \phi \rangle$ and $\langle \phi \phi \phi \phi \rangle$. The needed counterterms have the same forms as the terms already present in the action and can therefore be reproduced by the multiplicative renormalization of the field and the model parameters.

The corresponding renormalized action has the form
\[ S_R(\phi) = \frac{1}{2} \text{tr}\{\phi(-Z_1 \partial^2 + Z_2 \tau)\phi\} - \frac{g_1 \mu^\varepsilon}{4!} Z_3 \{\text{tr}(\phi^2)\}^2 - \frac{g_2 \mu^\varepsilon}{4!} Z_4 \text{tr}(\phi^4). \tag{4.1} \]

Here $\tau$, $g_1$ and $g_2$ are renormalized analogs of the bare parameters (with the subscripts “o”) and $\mu$ is the reference mass scale (additional arbitrary parameter of the renormalized theory). Expression (4.1) can be reproduced by the multiplicative renormalization of the field $\phi \rightarrow \phi Z_\phi$ and the parameters:

\[ \tau_0 = \tau Z_\tau, \quad g_{01} = g_1 \mu^\varepsilon Z_{g_1}, \quad g_{02} = g_2 \mu^\varepsilon Z_{g_2}, \tag{4.2} \]

so that
\[ Z_1 = Z_\phi^2, \quad Z_2 = Z_\phi Z_\phi^2, \quad Z_3 = Z_{g_1} Z_\phi^4, \quad Z_4 = Z_{g_2} Z_\phi^4. \tag{4.3} \]

We use the minimal subtraction (MS) scheme, where the all renormalization constants $Z_i$ have the forms “1+ only poles in $\varepsilon$,”
\[ Z_i = 1 + \sum_{n=1}^\infty A_{in}(g_{1,2}) \varepsilon^{-n}, \tag{4.4} \]
Critical behaviour of the \( O(n) - \phi^4 \) model with an antisymmetric order parameter

with the coefficients depending only on the completely dimensionless renormalized couplings \( g_{1,2} \). The constants \( Z_{1,2} \) and \( Z_{3,4} \) are calculated directly from the two-point and four-point 1-irreducible Green functions, respectively, then the constants in (4.2) are found from the relations (4.3).

The explicit one-loop calculation gives

\[
Z_2 = 1 + \frac{1}{12\varepsilon} \left\{ (n^2 - n + 4)g_1 + (2n - 1)g_2 \right\},
\]

(4.5)

\[
Z_3 = 1 + \frac{1}{12\varepsilon} \left\{ (n^2 - n + 16)g_1 + 2(2n - 1)g_2 + 3g_1^2/g_2 \right\},
\]

(4.6)

\[
Z_4 = 1 + \frac{1}{12\varepsilon} \left\{ 24g_1 + (2n - 1)g_2 \right\},
\]

(4.7)

with the corrections of the order \( g_{1,2}^2 \). In order to simplify the coefficients, here and below we pass to the new couplings: \( g_{1,2} \rightarrow g_{1,2}/(8\pi^2) \).

One can show that the model (2.1)–(2.3) with \( g_{20} = 0 \) is multiplicatively renormalizable in itself: the interaction \( V_1 \) alone does not generate the structure \( V_2 \) in the counterterms. On the contrary, the model with \( g_{10} = 0 \) is not closed with respect to renormalization: the interaction \( V_2 \) gives rise to the both structures \( V_{1,2} \). This leads to the appearance of the coupling \( g_2 \) in the denominator of the one-loop expression (4.6).

Like in the ordinary \( \varphi^4 \)-model, the nontrivial contributions to the constant \( Z_1 \) appear only in the two-loop order:

\[
Z_1 = 1 - \frac{1}{2 \cdot 24^2\varepsilon} \left\{ (n^2 - n + 4)(4g_1^2 + g_2^2) + 8(2n - 1)g_1g_2 \right\},
\]

(4.8)

with the corrections of the order \( g_{1,2}^3 \).

5. RG equations and RG functions

The RG equations for the renormalized Green functions in a multiplicatively renormalizable model are derived in a standard fashion; see e.g. [4]. In the model (4.1) the RG equation for the renormalized \( n \)-point function \( W_n^R \) has the form:

\[
\{ \mathcal{D}_\mu + \beta_1 \partial g_1 + \beta_2 \partial g_2 - \gamma_\tau \mathcal{D}_\tau - n\gamma_\phi \} W_n^R = 0.
\]

(5.1)

where \( \mathcal{D}_x \equiv x \partial_x \) for any variable \( x \).

The RG functions (\( \beta \)-functions for the coupling constants and anomalous dimensions \( \gamma \)) are defined by the relations

\[
\gamma_i \equiv \tilde{\mathcal{D}}_\mu \ln Z_i \quad \text{for any } Z_i,
\]

(5.2)

where \( \tilde{\mathcal{D}}_\mu \) is the operation \( \mathcal{D}_\mu \) at fixed bare parameters and

\[
\beta_i \equiv \tilde{\mathcal{D}}_\mu g_i = g_i \left[ -\varepsilon - \gamma_i(g_i) \right], \quad i = 1, 2,
\]

(5.3)

where the second equalities come from the definitions and the relations (4.2). In the MS scheme the anomalous dimensions depend only on the couplings \( g_{1,2} \) and are given by simple expressions

\[
\gamma_i = -(\mathcal{D}_{g_1} + \mathcal{D}_{g_2}) A_{i1}(g_{1,2}),
\]

(5.4)
Critical behaviour of the $O(n)$-$\phi^4$ model with an antisymmetric order parameter

where $A_{11}$ is the coefficient in the first-order pole in $\varepsilon$ in (4.4). In our approximation $Z_r = Z_2$, $Z_{g_1} = Z_3$ and $Z_{g_2} = Z_4$, and from equation (5.4) and the explicit expressions (4.5)–(4.8) we obtain:

$$
\beta_1 = -\varepsilon g_1 + \frac{1}{12}(n^2 - n + 16)g_1^2 + \frac{1}{6}(2n - 1)g_1 g_2 + \frac{1}{4}g_2^2,
$$

$$
\beta_2 = -\varepsilon g_2 + 2g_1 g_2 + \frac{1}{12}(2n - 1)g_2^2
$$

(5.5)

with the corrections of the order $g_1^2$ and

$$
\gamma_\tau = -\frac{1}{12}\{(n^2 - n + 4)g_1 + (2n - 1)g_2\},
$$

$$
\gamma_\phi = \frac{1}{2\cdot 24^2}\{(n^2 - n + 4)(4g_1^2 + g_2^2) + 8(2n - 1)g_1 g_2\}.
$$

(5.6)

with the corrections of the order $g_1^2$.

6. Fixed points and critical regimes

Possible asymptotic regimes of a renormalizable field theoretic model are determined by the asymptotic behaviour of the system of ordinary differential equations for the so-called invariant coupling constants

$$
D_s \tilde{g}_i(s, g) = \beta_i(\tilde{g}), \quad \tilde{g}_i(1, g) = g_i.
$$

(6.1)

Here $s = k/\mu$ is a nondimensionalized momentum, $g = \{g_i\}$ is the full set of couplings and $\tilde{g}_i(s, g)$ are the corresponding invariant variables. As a rule, the IR ($s \to 0$) and UV ($s \to \infty$) behaviour of the Green functions is determined by fixed points $g_*$ of the system (6.1). The coordinates of possible fixed points are found from the requirement that all the $\beta$ functions vanish:

$$
\beta_i(g_*) = 0.
$$

(6.2)

The type of a fixed point is determined by the matrix

$$
\omega_{ik} = \partial \beta_i/\partial g_k|_{g=g_*},
$$

(6.3)

which appears in the linearized version of the system (6.1) near the given point. For IR attractive fixed points (which we are interested in here) the matrix $\omega$ is positive, i.e. the real parts of all its eigenvalues $\omega_i$ are positive.

However, as already mentioned, for $n = 2$ and $n = 3$ our model reduces to the single-charge scalar and $O(3)$-vector models, respectively. The only coupling constant appearing in the Green functions is the combination $g = 2g_1 + g_2$. From expressions (5.5), (5.6) it is easily checked that the corresponding $\beta$-function $\beta = 2\beta_1 + \beta_2$ and the anomalous dimensions $\gamma_\phi, \gamma_\tau$ depend on the only parameter $g$ and coincide, up to the notation, with the known expressions for the scalar and vector cases. An IR attractive fixed point with $\beta(g_*) = 0$, $\beta'(g_*) > 0$ in the physical range $g_* > 0$ exists for $\varepsilon > 0$.

For $n \geq 4$ we have a genuine two-charge model. In renormalized perturbation theory, the physical region of their values is given by the inequalities (2.4), (2.5) with
the replacement $g_0 \rightarrow g_i$:

$$2g_1 + g_2 > 0, \quad ng_1 + g_2 > 0 \quad \text{for even} \ n$$

$$2g_1 + g_2 > 0, \quad (n-1)g_1 + g_2 > 0 \quad \text{for odd} \ n.$$  \hfill (6.4)

Analysis of the one-loop expressions (5.5) reveals the following fixed points:

1) Gaussian (free) fixed point $g_1^* = g_2^* = 0$, UV attractive (IR repulsive) for all $n$ with the eigenvalues $\omega_{1,2} = -\varepsilon$.

2) The point

$$g_1^* = 12\varepsilon/(n^2 - n + 16), \quad g_2^* = 0.$$  \hfill (6.5)

For all $n \geq 4$ it lies in the physical region, but is a saddle point: the eigenvalues $\omega_1 = \varepsilon$ and $\omega_2 = -\varepsilon(n^2 - n - 8)/(n^2 - n + 16)$ are real and opposite in sign.

The relation $g_2^* = 0$ remains valid to all orders in $\varepsilon$. This is a consequence of the fact that the model (2.1) with $g_2 = 0$ is “closed with respect to renormalization,” see the end of section 4. For the single-charge model with the only interaction $V_1$ in (2.1) this point is IR attractive with the only relevant eigenvalue $\omega_1 = \varepsilon$.

3) Two fully nontrivial points with the both nonvanishing coordinates:

$$g_{1*} = -6\varepsilon\frac{(4n^2 - 4n - 143) \pm (2n - 1)\sqrt{(-8n^2 + 8n + 97)}}{(4n^4 - 8n^3 - 123n^2 + 127n + 1696)};$$

$$g_{2*} = 12\varepsilon\frac{(2n - 1)(n^2 - n - 20) \pm 12\sqrt{(-8n^2 + 8n + 97)}}{(4n^4 - 8n^3 - 123n^2 + 127n + 1696)}. \quad \hfill (6.6)$$

For all $n \geq 5$, however, these points are complex and thus cannot be reached by the RG flow (6.1) with real initial data. The only exception is the case $n = 4$, when the expressions (6.6) become real and take on the following simple forms:

$$g_{1*} = 12\varepsilon/17, \quad g_{2*} = -12\varepsilon/17 \quad \hfill (6.7)$$

for the plus sign in front of the square root in (6.6) and

$$g_{1*} = 9\varepsilon/11, \quad g_{2*} = -12\varepsilon/11 \quad \hfill (6.8)$$

for the minus sign. The both points lie in the stability region (6.4). The first point is IR attractive with the eigenvalues $\omega_1 = \varepsilon$, $\omega_2 = \varepsilon/17$, while the second one is a saddle point with $\omega_1 = \varepsilon$, $\omega_2 = -\varepsilon/11$.

7. Discussion and Conclusion

We conclude that for the genuine two-charge cases $n \geq 4$ with the both interactions $V_{1,2}$ an IR attractive fixed point in the stability region exists only for $n = 4$; in the one-loop approximation it is given by the expression (6.7). For the model with the only interaction $V_1$ there is an IR attractive point (6.5) for all $n$.

Existence of an IR attractive fixed point implies existence of scaling behaviour for all the Green functions, described by the two main independent critical exponents [3, 4, 5]

$$\eta = 2\gamma_{\phi}^*, \quad 1/\nu = 2 + \gamma_{\tau}^*,$$  \hfill (7.1)
where \( \gamma_i^* = \gamma_i(g_1^*, g_2^*) \) are the values of anomalous dimensions \( \langle 5.2 \rangle \) at the fixed point in question.

For the fixed point \( \langle 6.5 \rangle \) from the explicit leading-order expressions \( \langle 5.6 \rangle \) one obtains

\[
\eta = \frac{(n^2 - n + 4)}{(n^2 - n + 16)^2} \varepsilon^2, \quad 1/\nu = 2 - \frac{(n^2 - n + 4)}{(n^2 - n + 16)} \varepsilon, \tag{7.2}
\]

which for \( n = 4 \) gives

\[
\eta = \frac{\varepsilon^2}{49}, \quad 1/\nu = 2 - \frac{4\varepsilon}{7}. \tag{7.3}
\]

For the fixed point \( \langle 6.7 \rangle \) one has

\[
\eta = \frac{6\varepsilon^2}{289}, \quad 1/\nu = 2 - \frac{9\varepsilon}{17}. \tag{7.4}
\]

while for \( \langle 6.8 \rangle \) one obtains

\[
\eta = \frac{5\varepsilon^2}{242}, \quad 1/\nu = 2 - \frac{5\varepsilon}{11}. \tag{7.5}
\]

All these expressions have the corrections of order \( O(\varepsilon^3) \) for \( \eta \) and \( O(\varepsilon^2) \) for \( 1/\nu \).

For a single-charge model, the invariant coupling constant \( \tilde{g} \) always lies in the interval \((0, g^*)\) and necessarily tends to the IR attractive fixed point \( g^* \) as \( s = k/\mu \) tends to zero. For a multicharge model, even when an IR attractive fixed point is present, not every RG trajectory (solution of the system \( \langle 6.1 \rangle \)) will approach it for \( s = k/\mu \to 0 \). A trajectory may first pass outside the region of stability (given by the inequalities \( \langle 6.4 \rangle \) in our model), the situation usually interpreted as a first-order phase transition. It may also go to infinity within the stability region, the situation in which the perturbation theory becomes unapplicable. Thus for every IR fixed point \( g^* \) one can introduce the notion of its basin of attraction: the set of all initial data \( g \) for which the solution of the system \( \langle 6.1 \rangle \) approaches \( g^* \) in the limit \( s = k/\mu \to 0 \).

The general pattern of the fixed points and RG flows in the \( g_1-g_2 \) plane for the case \( n = 4 \) is shown on figure 1. In the one-loop approximation it does not depend on \( \varepsilon \) in the coordinates \( g_{1,2}/\varepsilon \). One can see that the IR attractive fixed point \( \mathbf{B} \) with the coordinates \( \langle 6.7 \rangle \) lies between the two saddle points \( \mathbf{A} \) and \( \mathbf{C} \), with the coordinates \( \langle 6.5 \rangle \) and \( \langle 6.8 \rangle \), respectively. The origin houses the IR repulsive Gaussian point \( \mathbf{O} \).

Nevertheless, the RG flow approaches the point \( \mathbf{B} \) if the initial data for the system \( \langle 6.1 \rangle \) lie in the vast basin of attraction (light grey). Otherwise, the RG flow crosses the border of the stability region (given by the inequality \( \langle 6.4 \rangle \) with \( n = 4 \)), thus coming into unphysical region (dark grey).

It is worth noting that the basin of attraction lies entirely below the \( g_2 = 0 \) axis. Indeed, for the initial data with \( g_2 > 0 \) (and thus \( g_2 > -4g_1 \) due to \( \langle 6.4 \rangle \)) the RG flow has no chance to reach the point \( \mathbf{B} \); it cannot cross the axis \( g_2 = 0 \) because the function \( \beta_2 \) vanishes there for all \( g_1 \): \( \beta_2(g_2 = 0) = 0 \), see expression \( \langle 5.5 \rangle \). Note that \( \beta_1(g_1 = 0) \neq 0 \), so that crossing the axis \( g_1 = 0 \) is allowed.

We may conclude that the IR behaviour in our model for \( n = 4 \) is non-universal in the sense that it depends on the choice of the couplings \( g_{1,2} \). If they belong to the basin
Critical behaviour of the $O(n)$-$\phi^4$ model with an antisymmetric order parameter

Figure 1. RG flows for $n = 4$.

of attraction for the IR point (in particular, this implies $g_2 < 0$), the phase transition is of the second order (and thus the scaling regime takes place). Otherwise the RG flows pass outside the stability region. For $n > 4$, there is no attractive fixed points and only the second possibility can realize. This means that the account of fluctuations changes the nature of the phase transition from the second-order type (suggested by the mean-field theory) to the first-order type.

Surprisingly enough, the general situation is quite similar to the case of complex antisymmetric order parameter [17], although the models are not equivalent (in contrast to the real and complex vector cases). For $n > 4$, our RG flow is similar to that of the model with a third-rank tensor order parameter, discussed in [11] in connection with the isotropic-to-tetrahedratic transitions in liquid crystals.

In Ref. [19], the following “$\eta$ conjecture” was formulated for a general $\phi^4$-model with the vector order parameter: if the IR attractive fixed point is present, it corresponds to the fastest decay of correlations (that is, to the largest value of the exponent $\eta$). It is easily checked that the expressions (7.3)–(7.5) agree with that conjecture for our tensor model (although the numerical values of $\eta$ are very close to each other for all the three points). It should be stressed that the proof given in [19] in the lowest order of the $\varepsilon$-expansion does not apply to our case, because the $\beta$-functions (5.5) cannot be derived
from a potential, that is, they do not have the form $\beta_i = \partial U/\partial g_i$ with a certain function $U = U(g_{1,2})$. Thus we have obtained an independent confirmation of the $\eta$ conjecture for our special case of the tensor model.

Of course, it is not impossible that all these results will change somehow when the higher-order contributions to the RG functions are taken into account; cf. [21] for the electrodynamics of a charged scalar field. In order to exclude (or to confirm) such a possibility, one has to calculate the RG functions beyond the leading-order approximation and to apply an appropriate resummation procedure. This work is already in progress.

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Critical behaviour of the $O(n)$-$\phi^4$ model with an antisymmetric order parameter

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