A 5-Dimensional Spherical Symmetric Solution in Einstein-Yang-Mills Theory With Gauss-Bonnet Term

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Abstract. We present a numerical solution on a 5-dimensional spherically symmetric space time, in Einstein-Yang-Mills-Gauss-Bonnet theory using a two point boundary value routine. It turns out that the Gauss-Bonnet contribution has a profound influence on the behaviour of the particle-like solution: it increases the number of nodes of the YM field. When a negative cosmological constant in incorporated in the model, it turns out that there is no horizon and no singular behaviour of the model. For positive cosmological constant the model has singular behaviour.

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1. Introduction

In recent years higher dimensional gravity is attracting much interest. One reason is the possibility that these higher dimensions could be detectable at CERN. The possibility that space time may have more than four dimensions is initiated by high energy physics and inspired by D-brane ideology in string theory. Our 4-dimensional space time (brane) is embedded in the 5-dimensional bulk. It is assumed that all the standard model degrees of freedom reside on the brane, where as gravity can propagate into the bulk [1]. The effect of string theory on classical gravitational physics is investigated by the low-energy effective action. If our 5-dimensional space time is obtained as an effective theory, the matter fields, for example the U(1) field, can exists in the bulk. In General Relativity(GR), gravitating non-Abelian gauge field, i.e., the Yang-Mills(YM) field, can be regarded as the most natural generalization of Einstein-Maxwell(EM) theory. In particular, particle-like, soliton-like and black hole solutions in the combined Einstein-Yang-Mills(EYM) models, shed new light on the complex features of compact object in these models. See [2] for an overview.

The reason for adding a cosmological constant to these models, was inspired by the study of the so-called AdS/CFT correspondence [3, 4], since the 5-dimensional Einstein gravity with cosmological constant gives a description of 4-dimensional conformal field theory in large N limit. Brane world scenarios predict a negative cosmological constant. There is a relationship between the FRW equations controlling the cosmological expansion and the formulas that relate the energy and entropy of the CFT [5], indicating that both sets of equations may have a common origin. In the AdS-brane cosmological models, the AdS/CFT model describes a CFT dominated universe as a co-dimension one brane, with fixed tension, in the background of an AdS black hole [6]. The brane starts out inside the black hole, passes through the horizon and keeps expanding until it reaches a maximal radius, after which it contracts and falls back into the black hole. At these moments of horizon crossing, it turns out that the FRW equation turns into an equation that expresses the entropy density in terms of the energy density and coincides with the entropy of the CFT. However, in these models, one adds on an artificial way tension into the equations. More general, one could solve the equations of Einstein together with the matter field equations, for example, the YM field and try to obtain the same correspondence.

String theory also predicts quantum corrections to classical gravity theory and the Gauss-Bonnet(GB) term is the only one leading to second order differential equations in the metric. In the 4-dimensional EYM-GB model with a dilaton field (EYMD-GB) [7, 8], it was found that the GB contribution can lead to possible new types of dilatonic black holes. Further, for a critical GB coupling \( \kappa > \kappa_{cr} \) the solutions cease to exist. The AdS/CFT correspondence can also be investigated in the Einstein-GB gravity. For a recent overview, see [9]. From the viewpoint of AdS/CFT correspondence, it is argued that the GB term in the bulk corresponds to the next leading order corrections in the \( \frac{1}{N} \) expansion of a CFT. Further, it is argued that the entropy of an Einstein-GB black hole...
A 5-Dimensional Spherical Symmetric Solution in Einstein-Yang-Mills Theory With Gauss-Bonnet Term

and the CFT entropy induced on the brane are equal in the high temperature limit.

In this paper we investigate the possibility of regular and singular solutions in the 5-dimensional EYM-GB model and the effect of a cosmological constant on the behaviour of the solutions.

2. The model

The action of the model under consideration is

\[ S = \frac{1}{16\pi} \int d^5x \sqrt{-g_5} \left[ \frac{1}{G_5} (R - \Lambda) + \kappa (R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2) - \frac{1}{g_2^2} \mathcal{T}_F \right]. \]

(1)

with \( G_5 \) the gravitational constant, \( \Lambda \) the cosmological constant, \( \kappa \) the Gauss-Bonnet coupling and \( g \) the gauge coupling. The coupled set of equations of the EYM-GB system will then become

\[ \Lambda g_{\mu\nu} + G_{\mu\nu} - \kappa GB_{\mu\nu} = 8\pi G_5 T_{\mu\nu}, \]

(2)

\[ \mathcal{D}_\mu F^{\mu\nu} = 0, \]

(3)

with the Einstein tensor

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \]

(4)

and Gauss-Bonnet tensor

\[ GB_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \left( R_{\gamma\delta\lambda\sigma} R^{\gamma\delta\lambda\sigma} - 4 R_{\gamma\delta} R^{\gamma\delta} + R^2 \right) - 2 R R_{\mu\nu} + 4 R_{\mu\gamma} R^{\gamma\nu} + 4 R_{\nu\gamma} R^{\gamma\mu} \]

\[ + 4 R_{\gamma\delta} R_{\mu\nu}^{\gamma\delta} - 2 R_{\mu\nu} R_{\gamma\delta} - 2 R_{\mu\nu} R^{\gamma\delta}, \]

(5)

Further, with \( R_{\mu\nu} \) the Ricci tensor and \( T_{\mu\nu} \) the energy-momentum tensor

\[ T_{\mu\nu} = \text{Tr} F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{2} g_{\mu\nu} \text{Tr} F_{\alpha\beta} F^{\alpha\beta}, \]

(6)

and with \( F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g e^{abc} A_\mu^b A_\nu^c \), and \( \mathcal{D}_\alpha F_{\mu\nu}^a = \nabla_\alpha F_{\mu\nu}^a + g e^{abc} A_\alpha^b F_{\mu\nu}^c \) where \( A_\mu^a \) represents the YM potential.

Consider now the spherically symmetric 5-dimensional space time

\[ ds^2 = -\frac{F}{E^2} dt^2 + \frac{1}{F} dr^2 + r^2 \left( d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta^2 d\varphi^2) \right), \]

(7)

with the YM parameterization

\[ A_{t}^{(a)} = A_{t}^{(a)} = 0, A_{\psi}^{(a)} = \left( 0, 0, W \right), \]

\[ A_{\theta}^{(a)} = \left( W \sin \psi, - \cos \psi, 0 \right), \]

\[ A_{\varphi}^{(a)} = \sin \theta \left( \cos \psi, W \sin \psi, \frac{-1}{\tan \theta} \right), \]

(8)

where \( F \) and \( W \) are functions of and \( t \) and \( r \). It turns out that no time evolution of the metric component \( E \) can be found from the equations, so \( E \) depends only on \( r \). The equations become (we take \( g = 1 \))

\[ F_r = \frac{2r(1 - F) - \frac{2}{3} Ar^3 - \frac{G_5}{E}(1 - W^2)^2 - G_5 r (FW_r + E W_r^2)}{r^2 + 4\kappa(1 - F)}, \]

(9)
A 5-Dimensional Spherical Symmetric Solution in Einstein-Yang-Mills Theory With Gauss-Bonnet Term

\[ E_r = \frac{-G_5 r E (F^2 W_r^2 + E^2 W_t^2)}{F^2 (r^2 + 4 \kappa (1 - F))}, \]  
\[ F_t = \frac{-2 G_5 r F W_t W_r}{r^2 + 4 \kappa (1 - F)}, \] and

\[ W_{tt} = \frac{F^2}{E^2} W_{rr} + \frac{W_t F_t}{F} + \frac{F^2 W_r}{E^2} \left( \frac{-E_r}{E} + \frac{F_r}{F} + \frac{1}{r} \right) - \frac{2 W F (W^2 - 1)}{E^2 r^2}. \]

3. Numerical solutions

The independent field equations then read

\[ W'' = W' \left( \frac{2 r (F - 1) + \frac{2}{3} \Lambda r^3 + \frac{G_5}{r} (1 - W^2)^2}{F (r^2 + 4 \kappa (1 - F))} - \frac{1}{r} \right) + \frac{2 W (W^2 - 1)}{F r^2}, \]

\[ F' = \frac{2 r (1 - F) - \frac{2}{3} \Lambda r^3 - \frac{G_5}{r} (1 - W^2)^2 - G_5 r F W'^2}{r^2 + 4 \kappa (1 - F)}, \]

while the equation for \( E \) decouples and can be integrated:

\[ E = e^{-G_5 \int_{r=0}^{r} \frac{W'}{r^2 + 4 \kappa (1 - F)} dr}. \]

The equations are easily solved with an ODE solver and checked with MAPLE. We will take for the initial value of \( W \) the usual form \( W(0) = 1 - br^2 \). Then the other variables can be expanded around \( r = 0 \):

\[ E(0) = a + \frac{G_5 ab^2}{4 \kappa (c - 1)} r^4 + ... \quad F(0) = c + \frac{1}{4 \kappa} r^2 + .... \]

So we have 3 initial parameters and 4 fundamental constants \( \Lambda, G_5, g \) and \( \kappa \).

We solved the equations with a two point boundary value solver.

Fig.1 Solution for \( \Lambda = 0 \) for \( \kappa = 0 \) and \( \kappa = 1 \) respectively.
A 5-Dimensional Spherical Symmetric Solution in Einstein-Yang-Mills Theory With Gauss-Bonnet Term

Fig. 2 Solution for $\Lambda = -0.006$ for $\kappa = 0$ and $\kappa = 1$ respectively.

Fig. 3 Solution for $\Lambda = 0.0025$ for $\kappa = 0$ and $\kappa = 1$ respectively.

From our numerical solutions, we see that the GB term increases the number of nodes of the Yang-Mills field. Further, we see that for positive $\Lambda$ the solution develops a singularity, while for negative $\Lambda$ it remains singular free.

A matter field term in the action will lead to an extra term inside the square root of Eq.(16), for example in the case of the 5-dimensional Einstein-Maxwell-GB model [14] and the 5-dimensional Einstein-Yang-Mills-GB model with the Wu-Yang ansatz [15]. In these models, however, there are no additional equations for the Maxwell field and YM field respectively. So an analytic solution for $F(r)$ is obtained. When one simultaneously tries to solve the Einstein equation and matter field equations, then it is not easy to obtain an analytic expression for $F(r)$, as is the case of our EYM-GB model.

However we can analyse the equation for $F(r)$ when $W$ becomes a constant:

$$F_{r} = \frac{2r(1 - F) - \frac{2\Lambda \kappa^{3}}{3}}{r^{2} + 4\kappa(1 - F)}. \tag{17}$$

The solution is

$$F(r) = 1 + \frac{r^{2}}{4\kappa} \pm \frac{1}{12\kappa} \sqrt{(9 + 12\kappa \Lambda)r^{4} + 144\kappa^{2} + 24\kappa M}, \tag{18}$$

with $M$ an integration constant. Since horizons occur where $F(r) = 0$, we can expect cosmological- and event horizons. One can easily check that the zero’s of $F(r)$ are

$$r_{h} = \pm \left(\frac{3 \pm \sqrt{9 - 2\Lambda M}}{\Lambda}\right)^{\frac{1}{2}}. \tag{19}$$
So the horizon radius depends only on suitable combinations of $\Lambda$ and $M$. The expression inside the square root becomes negative (and hence $F(r)$ is singular) for
\[
 r_s \leq \left( \frac{-8\kappa (6\kappa + M)}{4\kappa \Lambda + 3} \right)^{\frac{1}{2}}.
\] (20)
Depending on the parameters, this singular surface can be shielded by the event horizon (otherwise, it will be naked). This is well known behaviour in the models where the equation for $F(r)$ decouples from the matter field equation. One should like to prove that for negative $\Lambda$ that $F(r)$ has no zero’s and is regular everywhere in our model. This is currently under study.

4. Conclusion and outlook

A 5-dimensional spherically symmetric particle-like solution is found in the Einstein-Yang-Mills Gauss-Bonnet model. As in other studies in higher dimensional cosmological models, a negative cosmological constant seems to favor for stability and results in most cases in asymptotically anti de Sitter space time.

In our 5-dimensional EYM-GB model, we also find a profound influence of a negative cosmological constant on the behaviour of horizons. The appearance of horizons in E-GB models is not surprising. These GB black holes are found by many authors. However, the lacking of horizons in the EYM-GB model for suitable negative cosmological constant is quite new. The explanation for this behaviour must come from the YM term on the right hand side of Eq.(9). The zero’s of $F(r)$ will depend on the behaviour $W(r)$.

There could be a connection of the solution presented here with the AdS/CFT correspondence. As mentioned before, no analytic expression for $F(r)$ available. Moreover, to obtain the (n-1)-dimensional entropy on the brane, one needs the junction conditions at the brane ([5, 9]), which becomes very complicated in the EYM-GB model. The junction condition also introduces a brane tension. This tension must cancel the cosmological constant, in order to obtain the desired CFT correspondence. The contribution of YM field on the junction could have profound impact on the tension of the brane and the role of a cosmological constant could be different. So the strong influence of a small cosmological constant on the eventually formed GB black hole in our model is quite clear from the consideration mentioned above.

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A 5-Dimensional Spherical Symmetric Solution in Einstein-Yang-Mills Theory With Gauss-Bonnet Term

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