Singularity Structures in Coulomb-Type Potentials in Two Body Dirac Equations of Constraint Dynamics

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Dirac’s relativistic constraint dynamics has been successfully applied to obtain a covariant non-perturbative description of QED and QCD bound states. In these applications, the interaction includes Coulomb-type potentials which lead naively in other approaches to singular relativistic corrections at short distances that require the introduction of either perturbative treatments or smoothing parameters. We describe here the unusual singularity structures of the effective potentials and wave functions that show up in the Pauli form of the Two Body Dirac Equations (TBDE) for singlet and triplet states, in both QED and QCD. We show how the TBDE formalism lead in fact to effective potentials and wave functions that are nevertheless non-singular in nature. The most noteworthy case is for the coupled triplet system \( ^3L_J \), with \( S = 1 \) and \( L = (J - 1) \) and \( (J + 1) \), when the tensor coupling is properly taken into account. Earlier work found that a nonperturbative (i.e. numerical) treatment of the full coupled system of equations was essential to obtain agreement with the perturbative spectral results for QED. In the present detailed study we find here that including the tensor coupling is essential in order that the effective potentials and wave functions are well behaved at short distances where the \( S \)-state and \( D \)-state wave functions become simply proportional to each other. Furthermore, this behavior is similar in both QED and QCD, independent of the asymptotic freedom behavior of the assumed QCD vector potential. The \( S \)-state wave functions in the triplet coupled case dip sharply to zero at the origin, unlike the usual \( S \)-state wave functions. The tensor coupling is essential to obtain the good singlet-triplet splitting even for the \( \pi-\rho \), mirroring the precise results which occurs in the nonperturbative treatment of QED. Without the inclusion of the tensor coupling, the effective \( S \)-state potential would become singular (more attractive than \(-1/4r^2\)). Low-lying as well as heavy meson states can be described well by using a simplified linear-plus-Coulomb-type QCD potential apportioned appropriately between world scalar and vector potentials. The TBDE formalism developed here may be used to study quarkonium in other environments such as those in quark-gluon plasma.

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I. INTRODUCTION

The two-body Dirac equations we discuss in this paper are based on Dirac’s constraint formalism and a minimal interaction structure for two particles in relative motion, first used by Todorov [1] and confirmed by both classical [2] and quantum field theory [3]. The constraint approach gives more than a sophisticated method for guessing relativistic wave equations for systems of bound quarks or general fermion-anti-fermion systems, since it can be readily combined with the field-theoretic machinery of the Bethe-Salpeter equation. When used with the kernel of the Bethe-Salpeter equation for QED, it combines weak-potential agreement in QED [4] with the nonperturbative structure of the field-theoretic eikonal approximation [1, 3]. The minimal interaction structure is then automatically inherited from relativistic classical [5, 6] and quantum field theory [3].

As has been demonstrated earlier, the constraint equations correspond to a “quantum-mechanical transform” [7, 8] of the Bethe-Salpeter equation. This is provided by the two coupled Dirac equations whose fully covariant interactions are determined by QED in the Feynman gauge [3, 4, 9]. Unlike most other truncations of the Bethe-Salpeter equation, the constraint approach does not require the use the awkward Coulomb gauge (whose noncovariant nature does not
allow its incorporation in covariant equations). Instead, its expansion about the BSE naturally occurs in the covariant Feynman gauge and is free from spurious infrared singularities that occur in the other approaches when that gauge is used.

In QCD with flavor independent interactions this formalism lead to spectral results in very good agreement with most of the experimental meson spectra (isoscalars such as the \( \eta, \eta', \) and \( \omega \) are not included). At the same time, as stressed in a recent publication [10], the formalism naturally accounts for the perturbative results of QED bound states, when treated in a nonperturbative manner. So far this has not been fully replicated in any other approach. In a natural way it leads not only to good singlet-triplet ground state splittings for the light as well as heavy mesons, but also a Goldstone behavior for the pion. By this we mean that the numerically computed pion mass tends to zero when the quark mass tends to zero. This is tied to the same relativistic structures that account for the nonperturbative positronium and muonium results [11].

The relativistic Two-Body Dirac equations may be written as an effective one-body wave equation [12–14]. The proper formulation of this relativistic scheme requires the successful treatment (that is a covariant elimination) of the quantum ghost states (due to the presence of the “relative time”) that first appeared in Nakanishi’s work on the Bethe-Salpeter equation [15].

The coupled constraint equations are known as the Two Body Dirac equations (TBDE) for at least two reasons. The first is that in the limit in which one of the particle masses becomes infinite, the equations reduce to the (one-body) Dirac equation. The second concerns the Pauli-form of the TBDE. It displays various spin-spin, spin-orbit, tensor and Darwin terms with energy-dependent denominators. They reduce to the Pauli-form of the Dirac equation in the static limit with the remaining spin-orbit and Darwin terms displaying similar energy-dependent denominators. We explore several related questions in this paper. How is it that the TBDE lead to a Goldstone-like behavior of the pion? How does this approach bypass many of the singularities that appear in the effective potentials in other approaches without the necessity for introducing cutoff parameters? A more complete understanding of how the TBDE are able to accomplish this will aid not only in a better understanding of its success in meson spectroscopy but will assist us in its application to two-body bound states in other environments such as those in a strongly-coupled quark gluon plasma (QGP).

In the application of TBDE to QED and QCD bound state problems, the interaction includes Coulomb-type potentials which lead naively in other approaches to singular relativistic corrections at short distances (delta functions and potentials more attractive than \(-1/4r^2\)) that require the introduction of either perturbative treatments or smoothing parameters. We would like to describe here these unusual singularity structures and show how the TBDE formalism lead in fact to effective potentials and wave functions that are nevertheless non-singular in nature.

In addition to our investigation of the singularity structures in TBDE, we also wish to test a simplified QCD based potential for future applications. Previous studies of the TBDE [9, 10, 16] made use of the QCD based Alder-Piran potential [17] giving a quite successful description for various \( qq \) states. The Adler-Piran potential contains functions with many terms and parameters divided into different sections of the spatial region. On the other hand, simple potentials such as the Cornell potential [18] appears to be adequate for many applications, although lacking in asymptotic freedom. It is therefore desirable to seek a simplified potential similar to the Cornell potential, but one containing asymptotic freedom that can be easily modified for future application of the TBDE in other quarkonium problems.

One such application is on the stability of \( qq \) states in the quark-gluon plasma which consists of quarks and antiquarks of different flavors, and gluons. The degree to which the constituents of a QGP can combine to form composite entities is an important property of the plasma. It has significant implications on the nature of the phase transition, the quark-gluon plasma equation of state, the probability of recombination of plasma constituents prior to the phase transition, and the chemical yields of the observed bound hadrons. The successes of the recombination model [19]-[28] suggests that quark partons may form bound or quasi-bound states in the quark-gluon plasma, at least at, or close to, the phase transition temperature. It is an important theoretical question as to the range of temperatures in which these quarkonia may be bound or quasi-bound. The successes of the thermal model [19]-[22],[29]-[38] for hadron yields also raises the important question whether hadrons may become bound or quasi-bound in the quark-gluon plasma. If they are indeed bound in the quark-gluon plasma, the approach to chemical equilibrium may commence in the quark-gluon plasma phase before the phase transition and the boundaries of the quark-gluon plasma phase and the hadron phase may overlap. It is therefore of interest to examine quarkonia in a medium as a function of the medium temperature. Our studies on meson spectroscopy and their singularity structures provide a zero-temperature limit as well as a relativistic framework for extension to finite temperatures.

Accordingly, we begin in Sec II with a discussion of the most often cited QCD potentials (including a simplified model for the QCD potentials to be used for the first time in this paper in conjunction with the TBDE for meson spectroscopy). It is well known how a naive use of Coulomb-type potential leads to singularity problems when one introduces relativistic spin-spin corrections. How we avoid these problems in the TBDE is the next question considered. In Sec. III, we seek out first how these problems are avoided in the Dirac equation for an external
Coulomb potential. This allows us to explore in Sec. IV the parallels between the singularity structures that occur in the Dirac and TBDE and how they each connect the different structures of the weak-potential (or perturbative) and strong-potential (or nonperturbative) forms of the two sets of equations. Those different structures, although requiring different treatments, give the same spectral results. By this we mean that a perturbative treatment of the weak-potential forms gives the same spectral results as a nonperturbative (analytic or numerical) treatment of the strong-potential or nonperturbative forms of the two sets of the equations. The example we use in Sec. V for this parallel discussion is the TBDE for electromagnetic interactions, QED. In Sec. VI we discuss the spectral results, focusing on the pion and the singularity structure of the TBDE for a simplified QCD potential model in the case of $^3S_1^3D_1$ or more generally $[^3(J-1),[^3(J+1),] mixing. In Sec. VII we summarize the results and discuss questions that may arise when attempts are made to apply the TBDE to two-body bound states in a QGP.

II. QCD MODEL POTENTIALS

Previously, the authors of [10] used a sophisticated form of the static quark potential developed by Adler and Piran [17], one that has ties at all length scales to field theoretic data. Very good agreement with experimental quarkonium spectrum was obtained. On the other hand, in nonrelativistic treatments the most commonly used static quark potential for potential model studies is the Cornell potential,

$$V(r) = -\frac{\alpha_c}{r} + br.$$  \hfill (1)

Although not displaying asymptotic freedom, it does give the dominant Coulomb-like behavior as well as linear quark confinement. Early on a model was proposed by Richardson for the static potential a) depends only a single scale size ($\Lambda$), and b) interpolates in a simple way between asymptotic freedom and linear confinement [39]. Richardson’s model for the static interquark potential in momentum space is

$$\tilde{V}(q) = -\frac{16\pi}{27} \frac{1}{q^2 \ln(1 + q^2/\Lambda^2)}.$$  \hfill (2)

arising from the assumption that

$$\tilde{V}(q) = -\frac{4\alpha_s(q^2)}{3q^2},$$  \hfill (3)

(including the color factor $-4/3$). Asymptotic freedom requires that for $q^2/\Lambda^2 >> 1$,

$$\alpha_s(q^2) \to \frac{12\pi}{27} \frac{1}{\ln(q^2/\Lambda^2)}.$$  \hfill (4)

while linear confinement requires that for $\Lambda r >> 1$, $V(r) \sim \text{const} \times r$ or equivalently for $q^2/\Lambda^2 << 1$ that one must impose $\alpha_s(q^2) \sim q^{-2}$. The interpolation of Eq. (2) is not tied at all in the intermediate region and only roughly tied in the large $r$ region to any field theoretic data. Nevertheless it provides a convenient one-parameter form for the static quark potential. In coordinate space it has the form

$$V(r) = \frac{8\pi\Lambda^2 r}{27} - \frac{8\pi f(\Lambda r)}{27 r},$$  \hfill (5)

where $f(\Lambda r)$ is given by a complicated integral transform\footnote{In addition to the spin independent nonrelativistic model presented in [39] see also a relativistic extension of it given in [40].} that displays the asymptotic freedom behavior for $r \to 0$ of

$$f(\Lambda r) \to -\frac{1}{\ln \Lambda r},$$  \hfill (6)

while for $r \to \infty$,

$$f(\Lambda r) \to 1.$$  \hfill (7)
A simpler model, which we will apply in this paper and one which displays the same large and small \( r \) behavior is

\[
V(r) = \frac{8\pi A^2 r}{27} - \frac{16\pi}{27r \ln(e^2 + 1/(Ar)^2)}.
\]

(8)

It amounts to replacing Richardson’s \( f(Ar) \) by \( 2/\ln(e^2 + 1/(Ar)^2) \), having the same limits. Although not giving as good a fit to the spectra as the more closely tied QCD based potential of [17], the modified form of Eq. (8) which we use in this paper (see Eq. (75) below) does provide reasonable results for the spectrum. Furthermore its linear-plus-Coulomb-type parametrization is more convenient for extension of the quark model to high temperature environments.

Problems arise in the quark model with the above potentials if their relativistic corrections are naively grafted from semirelativistic expressions. For example, the spin-spin interaction proportional to

\[
-\nabla^2 V_{\sigma_1 \sigma_2} \frac{m_1 m_2}{m_1 m_2},
\]

(9)

would lead to a singular delta function potential that can only be treated perturbatively. Some approaches simply include cutoff parameters so that the Laplacian is not singular. How do the TBDE treat this problem? Potential energy terms such as the above arise from the second order reductions of those equations (the Pauli forms). Let us first examine how such problems are treated in a very natural way in the Schrödinger-like Pauli-form of Dirac’s original wave equation.

III. SINGULARITY STRUCTURE OF PAULI-FORM OF THE DIRAC EQUATION

The \( q-\bar{q} \) interaction in Eq. (8) contains the color-Coulomb term that is proportional to \( 1/r \) and contains a logarithmic function of \( r \). It leads naïvely to singular relativistic corrections that may render the solution singular at short distances. It is worth while to investigate Coulomb-type potential in relativistic equations. Let us be more precise in our definitions of singular potentials. Case [41], describes how potentials that are more attractive at the origin than \(-1/r^2\) must be adjusted to maintain their self-adjoint status. Let us call such potentials, attractive singular potentials. They include attractive delta functions and attractive \( 1/r^3 \) potentials that appear in spin-orbit terms. Such terms must either be treated only in perturbation theory or in cases where the coupling is strong, require adjustments, e.g. by smoothing parameters. Frank et al. [42] and Calogero [43] also discuss another category of potentials called repulsive singular potentials. These are repulsive potentials that exceed an inverse quadratic power law behavior. Strictly speaking they need not be treated using perturbation theory, although in the case of weak potentials they are most easily treated like that. We will discuss these more in context below.

Let us show how the Pauli-form of the Dirac equation with a Coulomb-type interaction contains effective potentials that are repulsively singular, when viewed in an incomplete or perturbative context. However, when viewed in a complete or nonperturbative context the effective potentials are nonetheless nonsingular. We examine for simplicity the case of the Dirac equation in a Coulomb potential \( (A = -\alpha/r) \) (instead of the more complicated forms with asymptotic freedom) for stationary states,

\[
(\alpha \cdot p + \beta m + A)\Psi = E\Psi.
\]

(10)

Then with

\[
\Psi = \left( \begin{array}{c} \phi \\ \chi \end{array} \right),
\]

(11)

we have

\[
\begin{pmatrix}
m - E + A & \sigma \cdot p \\
\sigma \cdot p & -E - m + A
\end{pmatrix}
\begin{pmatrix}
\phi \\
\chi
\end{pmatrix} = 0.
\]

(12)

Eliminating \( \chi \) we obtain

\[
\chi = \frac{1}{E + m - A} \sigma \cdot p \phi,
\]

(13)

leading to

\[
(p^2 - \frac{iA'}{(E + m - A)} \hat{\mathbf{p}} + \frac{A'}{r(E + m - A)} \sigma \cdot \mathbf{L})\phi = [(E - A)^2 - m^2]\phi.
\]

(14)
We can eliminate the first order derivative $\hat{\mathbf{r}} \cdot \mathbf{p}$ term by the substitution

$$\phi = F(r)\psi$$

if one takes

$$\frac{F'}{F} = -\frac{A'}{2(E + m - A)}.$$  

Then our equation becomes

$$\left(\mathbf{p}^2 + \frac{1}{2}\frac{\nabla^2 A}{(E + m - A)} + \frac{3}{4} \frac{(\nabla A)^2}{(E + m - A)^2} + \frac{A'}{r(E + m - A)} \sigma \cdot \mathbf{L}\right)\psi = [(E - A)^2 - m^2]\psi,$$

which for a Coulomb potential becomes

$$\left(\mathbf{p}^2 - \frac{2E\alpha}{r} - \frac{\alpha^2}{r^2} + \frac{2\pi\alpha\delta^3(\mathbf{r})}{(E + m + \alpha/r)} + \frac{2\alpha^2}{4} \frac{(E + m + \alpha/r)^2}{(E + m + \alpha/r)^2} + \frac{\alpha}{r^3(E + m + \alpha/r)} \sigma \cdot \mathbf{L}\right)\psi = (E^2 - m^2)\psi.$$

For $\alpha > 1/2$ the inverse quadratic term would lead to an overall attractive singular potential for $S$-states. If one takes the weak potential limit which the denominators are replaced by $2m$ then the spin-orbit potential is an attractive singular potential for both coupling states and must be handled by perturbative techniques. Under weak potential circumstances the delta function would be treated by perturbative techniques. Since it is repulsive, one could, in principle treat it in a nonperturbative way. It has been shown, however [44], that a nonperturbative treatment of repulsive delta functions potentials produce no effect on bound state energies. The repulsive $1/r^4$ term would require special numerical treatments (its perturbative effects on $S$-states is ill defined). Let us now compare perturbative and nonperturbative treatments of this Pauli form.

By using the atomic units $\mathbf{r} = \mathbf{x}/(E\alpha)$ the above equation takes the dimensionless coordinate space form of

$$(-\alpha^2\nabla_x^2 - \frac{2\alpha^2}{x} - \frac{\alpha^4}{x^2} + \frac{2\pi\alpha^4\delta^3(x)}{(1 + m/E + \alpha^2/x)} + \frac{3}{4} x^4(1 + m/E + \alpha^2/x)^2 + \frac{\alpha}{x^3(1 + m/E + \alpha^2/x)} \sigma \cdot \mathbf{L})\psi = \left[1 - \left(\frac{m}{E}\right)^2\right]\psi.$$  

The standard perturbative treatment retains terms through order $\alpha^4$ to arrive at the equation,

$$(-\alpha^2\nabla_x^2 - \frac{2\alpha^2}{x} - \frac{\alpha^4}{x^2} + \pi\alpha^4\delta^3(x) + \frac{\alpha^4}{2x^3} \sigma \cdot \mathbf{L})\psi = \left[1 - \left(\frac{m}{E}\right)^2\right]\psi.$$  

The standard semirelativistic spectral results through order $\alpha^4$ can be obtained by treating this as an ordinary eigenvalue problem with the last three terms on the left hand side treated as a perturbation. For the ground state this leads to

$$E = m - \frac{m\alpha^2}{2} - \frac{m\alpha^4}{8} + O(\alpha^6).$$

Note that the $\alpha^6$ term in Eq. (18) does not contribute perturbatively to this order.

### A. Small $r$ Wave Function Behavior

Note that the $\delta^3(\mathbf{r})$ potential in Eq. (19) must only be treated perturbatively to obtain a nonzero result. We know, however, that the Dirac equation in this case can be solved analytically. How does that reconcile here with the appearance of these singular potentials, particularly the $\delta^3(\mathbf{r})$ potential? Let us restrict ourselves here to $S$-states to make our main point. That would mean with $u = \sqrt{4\pi x}\psi$ that we must include all terms in

$$(-\alpha^2\frac{d^2}{dx^2} - \frac{2\alpha^2}{x} - \frac{\alpha^4}{x^2} + \frac{3}{4} x^2(1 + m/E + \alpha^2)^2)u = \left[1 - \left(\frac{m}{E}\right)^2\right]u.$$
in any nonperturbative solution. Note that we have left out here the $\delta^3(x)$ term in Eq. (19) since it, together with the Coulomb potential in the denominator would yield a vanishing result for its contribution (its expectation value in any well behaved basis would give zero). That means that we must have the rather unusual circumstance here of the term which does not contribute to the weak potential form Eq. (19) (the $\alpha^6$ term) having a nonperturbative effect on the spectrum that reproduces that of the perturbative $\delta^3(x)$ term (in a perturbative expansion). A set of straight-forward but tedious manipulations show how this comes about. We first point out that the $\alpha^6$ term has a short distance behavior of that of a repulsive $r^{-2}$ behavior, so it is not in the category of a repulsive singular potential. This allows a standard type of solution. One finds that the ground state wave function and eigenvalue are given by:

$$
\begin{align*}
&u = k x^\beta (1 + m/E + \alpha^2) \exp(-\lambda x), \\
&\beta = \sqrt{1 - \alpha^2} + \frac{1}{2}, \\
&\gamma = -\frac{1}{2}, \\
&E = m \sqrt{1 - \alpha^2}, \\
&\lambda = \frac{\sqrt{(m/E)^2 - 1}}{\alpha} = \frac{1}{\sqrt{1 - \alpha^2}}.
\end{align*}
$$

We also verify that our exact solution

$$
E = m \sqrt{1 - \alpha^2} = m \left( 1 - \frac{1}{2} \alpha^2 - \frac{1}{8} \alpha^4 + O(\alpha^6) \right),
$$

agrees with the perturbative spectral results Eq. (20). Note that the small $r$ behavior of the radial part of the wave function is

$$
\psi \sim x^{\sqrt{1-\alpha^2} - 1/2}
$$

which dips toward the origin unlike the flat behavior of the nonrelativistic limit of the Pauli form or mildly singular behavior of the Dirac wave function.

In summary, the Pauli-form (18) of the Dirac equation in the weak potential approximation or perturbative form of (19) includes terms missing in the strong potential or nonperturbative form (21). Also, Eq. (21) includes terms missing in the weak potential or perturbative form (19). However, they both give rise to the same spectral results through order $\alpha^4$, with one treated in a weak potential approximation and the other treated with no approximations (whose spectra expansion yields the same result). As we shall see below, such an unusual feature (with different parts of the equation contributing to the perturbative and nonperturbative spectral evaluations) is also displayed in the Pauli-form of the TBDE of constraint dynamics.

The above exercise shows that what appears as singular in a perturbative context turns out in fact to be non-singular in a full non-perturbative treatment. For our case of the ground state, the two approaches give the same result up to order $\alpha^4$, using different parts of the effective interaction. Does this extend to the radially and orbitally excited states? Do different parts (and approximations) of the interaction used in reaching the spectral results lead to the same results for all high excited states. Although we have not shown this here, it is expected to be true since the exact (nonperturbative) Pauli form should faithfully reproduce the exact spectral results of the first order form of the Dirac equation. This would imply that one would expect the two approaches to give the same results through order $\alpha^4$.

### IV. THE TWO BODY DIRAC EQUATIONS OF CONSTRAINT DYNAMICS

Dirac constructed a quantum wave equation from a first-order wave operator that is the matrix square-root of the corresponding Klein-Gordon operator [45] in order to treat a single relativistic spin-one-half particle, free or in an external field. The TBDE of constraint dynamics extend his construction to the system of two interacting relativistic

\[\text{IV. THE TWO BODY DIRAC EQUATIONS OF CONSTRAINT DYNAMICS}\]

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2 At short distance we have $-\frac{d^2 u}{dx^2} + (\frac{3/4 - \alpha^2}{x^2})u = 0 = -\beta(\beta - 1) + 3/4 - \alpha^2$ with allowed solution $\beta = \sqrt{1 - \alpha^2} + \frac{1}{2}$. At long distance $(-\frac{d^2 u}{dx^2} - \frac{2}{x} - \frac{\alpha^2}{x^2})u = -\lambda^2 u$ the allowed solution has behavior $u \sim x^{\sqrt{1-\alpha^2}} \exp(-\lambda x)$ which forces $\gamma = -1/2$. 

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spin-one-half particles with quantum dynamics governed by a pair of compatible Dirac operators acting on a single 16-component wave function. For an extensive review of this approach, see Refs. [4, 10, 14, 16] and works cited therein. We present below a brief review.

Over thirty years ago, the relativistic constraint approach first successfully yielded a covariant yet canonical formulation of the relativistic two-body problem for two interacting spinless classical particles by applying a Hamiltonian approach introduced by Dirac [46] for handling systems with constraints. It accomplished this by introducing two constraints thereby reducing the number of degrees of freedom of the relativistic two-body problem to that of the corresponding nonrelativistic problem [47]-[50]. By this one covariantly eliminates the troublesome relative time and relative energy. The constraints used for this reduction are a pair of generalized mass shell constraints for each of the two interacting spinless particles: \( p_i^2 + m_i^2 + \Phi_i \approx 0 \). Mathematical consistency then requires that the two constraints be “compatible” in the sense that they be conserved by a covariant system-Hamiltonian. Upon quantization, the quantum version of this “compatibility condition” becomes the requirement that the quantum versions of the constraints (two separate Klein-Gordon equations on the same wave function for spinless particles) possess a commutator that vanishes when applied to the wave-function. In 1982, Crater and Van Alstine used a supersymmetric classical formulation of the single-particle Dirac equation due to Galvao and Teitelboim to successfully extend this construction to the “pseudoclassical” mechanics of two spin-one-half particles [51, 52]. Upon quantization, this scheme produces a consistent relativistic quantum mechanics for a pair of fermions interacting by a world scalar interaction and governed by two coupled Dirac equations.

For the case of two relativistic spin-one-half particles interacting through four-vector and scalar potentials, the two compatible 16-component Dirac equations ([4, 10, 14, 16]) take the form

\[
S_1 \psi = \gamma_{51} (\gamma_1 \cdot (p_1 - \hat{A}_1) + m_1 + \hat{S}_1) \psi = 0
\]  
\[
S_2 \psi = \gamma_{52} (\gamma_2 \cdot (p_2 - \hat{A}_2) + m_2 + \hat{S}_2) \psi = 0,
\]

in terms of \( S_i \) operators that in the free-particle limit become operator square roots of the Klein-Gordon operator.

The relativistic four-vector potentials \( \hat{A}_i^\mu \) and scalar potentials \( \hat{S}_i \) are effective constituent potentials that in either limit \( m_i \to \infty \) go over to the ordinary external vector and scalar potentials of the light-particle’s one-body Dirac equation. The covariant spin-dependent terms in \( \hat{A}_i^\mu \) and \( \hat{S}_i \) are recoil terms whose general forms are nonperturbative consequences of the compatibility condition

\[
[S_1, S_2] \psi = 0.
\]

This condition also requires that the potentials depend on the space-like interparticle separation only through the combination

\[
x_\perp^\mu = (\eta^{\mu\nu} + \hat{P}^\mu \hat{P}_\nu)(x_1 - x_2)_\nu
\]

with no dependence on the relative time in the c.m. frame. This separation variable is orthogonal to the total four-momentum

\[
P^\mu = p_1^\mu + p_2^\mu, \quad -P^2 = w^2.
\]

\( \hat{P} \) is the time-like unit vector

\[
\hat{P}^\mu \equiv P^\mu / w.
\]

The accompanying relative four-momentum canonically conjugate to \( x_\perp \) is

\[
p^\mu = (\epsilon_2 p_2^\mu - \epsilon_1 p_1^\mu) / w; \quad \text{where} \quad \epsilon_1 + \epsilon_2 = w, \quad \epsilon_1 - \epsilon_2 = (m_1^2 - m_2^2) / w,
\]

in which \( w \) is the total c.m. energy. The \( \epsilon_i \)'s are the invariant c.m. energies of each of the (interacting) particles. Another consequence of the compatibility condition is that the relative momentum is constrained to be orthogonal to the total four-momentum

\[
P \cdot p \psi = 0,
\]

thus providing the conjugate covariant control on the relative energy. One finds also that the vector and scalar potentials are defined in terms of two invariant functions \( S(r), A(r) \) in which \( r \) is the invariant

\[
r \equiv \sqrt{x_\perp^2}.
\]
Those potentials have the general forms

\[ \tilde{A}^\mu_i = \tilde{A}^\mu_i(A(r), p_\perp, \tilde{P}, w, \gamma_1, \gamma_2), \quad \tilde{S}_i = \tilde{S}_i(S(r), A(r), p_\perp, \tilde{P}, w, \gamma_1, \gamma_2). \quad (33) \]

The wave operators in Eqs. (25a) and (25b) operate on a single 16-component spinor

\[ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad (34) \]

in which the \( \psi_i \) are four-component spinors.

With compatibility ensured, this two-body formalism has many advantages over the traditional Bethe-Salpeter equation and its numerous three dimensional truncations. One is its simplicity. A (two step) Pauli reduction and scale transformation ([14], [4], [16], and [10]) to be described below brings these equations to this covariant Schrödinger-like form involving a four component spinor \( \psi_+ \),

\[ \left( p^2 + \Phi_w(\sigma_1, \sigma_2, p_\perp, A(r), S(r)) \right) \psi_+ = b^2(w)\psi_+, \quad (35) \]

resembling an ordinary Schrödinger equations with the interaction term \( \Phi_w \) including central-potential, Darwin, spin-orbit, spin-spin, and tensor terms. The interactions are completely local but depend explicitly on the invariant c.m. total energy \( w \). The usual invariant

\[ b^2(w) \equiv (w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2)/4w^2 \quad (36) \]

plays the role of energy eigenvalue. This invariant is the c.m. value of the square of the relative momentum expressed as a function of the invariant total c.m. energy \( w \).

Note that in the limit in which one of the particles becomes very heavy, this Schrödinger-like equation turns into the one obtained by eliminating the lower component of the ordinary one-body Dirac equation in terms of the other component (see Eq. (16)).

### A. Schrödinger-Like Form of the TBDE

From classical [2] or quantum field theories [3] for separate scalar and vector interactions one can show that the spin independent part of the quasipotential \( \Phi_w \) involves the difference of squares of the invariant mass and energy potentials \( M_i \) and \( E_i \) respectively

\[ M_i^2 = m_i^2 + 2m_wS + S^2; \quad E_i^2 = \varepsilon_i^2 - 2\varepsilon_wA + A^2 \quad (37) \]

\[ M_i^2 - E_i^2 = 2m_wS + S^2 + 2\varepsilon_wA - A^2 - b^2(w). \quad (38) \]

where

\[ m_w = \frac{m_1m_2}{w}, \quad \varepsilon_w = \frac{(w^2 - m_1^2 - m_2^2)}{2w}, \quad (39) \]

are respectively the relativistic reduced mass and energy of the fictitious particle of relative motion introduced by Todorov [1, 12] and satisfy the effective one-body Einstein condition

\[ \varepsilon_w^2 - m_w^2 = b^2(w). \quad (40) \]

The Pauli reductions performed in [4] did not lead to the uncoupled four component structure in Eq. (35). Using steps on the TBDE (Eqs. (25a) and (25b)) analogous to those taken for the Dirac equation in Eqs. (11)-(13) yielded a Schrödinger-like equation for the four component upper-upper \( \psi_1 \) component

\[
\begin{align*}
\{ p^2 + 2m_wS + S^2 + 2\varepsilon_wA - A^2 \\
+ \Phi_D p + \Phi_D' + \Phi_{SO1}L \cdot \sigma_1 + \Phi_{SO2}L \cdot \sigma_2 + \Phi_SS\sigma_1 \cdot \sigma_2 + \Phi_TS_T \} \psi_1 \\
+ \{ \Phi_SS'\sigma_1 \cdot \sigma_2 + \Phi_T'S_T \} \psi_4 \\
= b^2(w)\psi_1,
\end{align*}
\]
\[ (41) \]
coupled to a Schrödinger-like wave equation\(^3\) for the our component lower-lower component \(\psi_4\)

\[
\begin{align*}
(p^2 + 2m_w S + S^2 + 2\varepsilon_w A - A^2) \\
+ \tilde{\Phi}_D i \tau \cdot p + \tilde{\Phi}_D + \tilde{\Phi}_{SO1} L \cdot \sigma_1 + \tilde{\Phi}_{SO2} L \cdot \sigma_2 + \tilde{\Phi}_{SS} \sigma_1 \cdot \sigma_2 + \tilde{\Phi}_T S_T \} \psi_4 \\
+ \{ \tilde{\Phi}'_S \sigma_1 \cdot \sigma_2 + \tilde{\Phi}'_T S_T \} \psi_1 \\
= b^2(w) \psi_4.
\end{align*}
\]

These equations can be solved nonperturbatively for QED (\(S = 0\)) or quark model calculations since everyone of the quasipotential terms \(\Phi\) (including the Darwin pieces \(\Phi_D\)) is quantum mechanically well defined (less singular than \(-1/4r^2\)).\(^4\) The work in [4] showed in detail for QED and related quantum field theories how a nonperturbative (numerical) treatment of these equations gave the same spectral results as a perturbative treatment of the weak potential forms. Just as in the case of the one-body Dirac equation shown above, we found that different parts of the two body wave equation appear to be important for the non-perturbative calculation that are not important for the perturbative calculation. In our work below we shall use the uncoupled Pauli form of Eqs. (25a) and (25b) of the general form of Eq. (35) to stress these unusual differences, particularly for the coupled triplet states.

Eqs. (25a) and (25b) contain an important hidden hyperbolic structure\(^5\). To reveal and employ it we introduce two independent invariant functions \(L(x_\perp)\) and \(G(x_\perp)\), in terms of which the invariant mass and energy potentials take the forms:

\[
\begin{align*}
M_1 &= m_1 \cosh L(S, A) + m_2 \sinh L(S, A), \\
M_2 &= m_2 \cosh L(S, A) + m_1 \sinh L(S, A),
\end{align*}
\]

\[
\begin{align*}
E_1 &= \varepsilon_1 \cosh G(A) - \varepsilon_2 \sinh G(A), \\
E_2 &= \varepsilon_2 \cosh G(A) - \varepsilon_1 \sinh G(A).
\end{align*}
\]

Strictly speaking, the forms in Eq. (37) are for scalar and time-like vector interactions. Eqs. (41),(42),(53), and (57) below involve combined scalar and electromagnetic-like vector interactions (this amounts to working in the Feynman gauge with the simplest relation between space- and time-like parts, see [9],[10]). In that case the mass and energy potentials in place of Eq. (37) are respectively

\[
M_i^2 = m_i^2 + \exp(2G)(2m_w S + S^2),
\]

so that

\[
\exp(L(S, A)) = \sqrt{m_1^2 + \exp(2G)(2m_w S + S^2) + \sqrt{m_2^2 + \exp(2G)(2m_w S + S^2)}} / m_1 + m_2,
\]

and

\[
E_i^2 = \exp(2G)(\varepsilon_1 - A)^2,
\]

with

\[
\exp(2G(A)) = 1/(1 - 2A/w) \equiv G.
\]

In terms of \(G\) and the constituent momenta \(p_1\) and \(p_2\), the individual four-vector potentials take the suggestive forms\([10]\)

\[
\begin{align*}
A_1 &= [1 - \cosh(G)]p_1 + \sinh(G)p_2 - \frac{i}{2}(\partial \exp G \cdot \gamma_2)\gamma_2, \\
A_2 &= [1 - \cosh(G)]p_2 + \sinh(G)p_1 + \frac{i}{2}(\partial \exp G \cdot \gamma_1)\gamma_1,
\end{align*}
\]

\(^3\) For the explicit forms of the various quasipotentials see [4], Eqs. (4.14a) and (4.14b).

\(^4\) We note that the mid \(r\)-range singularity structures that occur in the Breit equation do not appear in the TBDE. [53]
In terms of the three sets of invariants (45)-(48) the coupled TBDE (25a) and (25b) then take the form

\[ S_1 \psi = (-G\beta_1 \Sigma_1 \cdot \mathcal{P}_2 + E_1 \beta_1 \gamma_{51} + M_1 \gamma_{51} - G\frac{i}{2} \Sigma_2 \cdot \partial (G \beta_1 + L \beta_2) \gamma_{51} \gamma_{52}) \psi = 0, \]

\[ S_2 \psi = (G\beta_2 \Sigma_2 \cdot \mathcal{P}_1 + E_2 \beta_2 \gamma_{52} + M_2 \gamma_{52} + G\frac{i}{2} \Sigma_1 \cdot \partial (G \beta_2 + L \beta_1) \gamma_{51} \gamma_{52}) \psi = 0, \]

(50)
in which

\[ \mathcal{P}_i = p - \frac{i}{2} \Sigma_i \cdot \partial G \Sigma_i, \]

(51)
depending on gamma matrices with standard block forms (see Eq. (2.28) in [10] for their explicit forms) and where

\[ \Sigma_i = \gamma_{5i} \beta_i \gamma_{\perp i}. \]

(52)
The Klein-Gordon like factors appearing in the Pauli forms (41), (42), (53), and (57) of Eq. (50) are as in Eq. (38).

Eqs. (41) and (42), the corresponding Schrödinger-like equations decouple ([10, 16, 55, 56]). We obtain [10, 57]

\[
\begin{align*}
|p^2 + 2m_w S + S^2 + 2\varepsilon_w A - A^2 + m(r) & - \frac{L \cdot (\sigma_1 + \sigma_2)}{r} F' + l'(r) L \cdot (\sigma_1 - \sigma_2) + i q'(r) L \cdot (\sigma_1 \times \sigma_2) \\
+ 2F' i \hat{r} \cdot p + i K' (\sigma_1 \cdot \hat{r} \sigma_2 \cdot p + \sigma_2 \cdot \hat{r} \sigma_1 \cdot p) \\
+ k(r) \sigma_1 \cdot \sigma_2 + n(r) \sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r} | \phi_+ & = b^2 (w) \phi_+,
\end{align*}
\]

(53)

where

\[
\begin{align*}
k(r) = & \frac{1}{2} \nabla^2 G - \frac{1}{2} G^2 - G' F' - \frac{1}{2} \frac{G'}{r} + \frac{K'}{r}, \\
n(r) = & \nabla^2 K - \frac{1}{2} \nabla^2 G + \frac{3(G' \cdot 2K')}{2r} + F'(G'' - 2K'), \\
m(r) = & -\frac{1}{2} \nabla^2 G + \frac{3}{4} G'^2 + \mathcal{G} F' - K'^2,
\end{align*}
\]

(54)

and the prime symbol stands for \(d/dr\). We have used the abbreviations

\[
\begin{align*}
F = & \frac{1}{2} \log \frac{D}{\varepsilon_2 m_1 + \varepsilon_1 m_2} - G, \\
D = & E_2 M_1 + E_1 M_2, \\
K = & (G + L), \\
l'(r) = & -\frac{1}{2} \frac{E_2 M_2 - E_1 M_1}{2r} (L - G)', \\
q'(r) = & \frac{1}{2} \frac{E_1 M_1 - E_2 M_2}{2r} (L - G)'.
\end{align*}
\]

(55)

We work in the c.m. frame in which \(\hat{P} = (1, 0)\) and \(\hat{r} = (0, \hat{r})\) and then further define four component wave functions \(\psi_\pm, \eta_\pm\) related to the above by [57]

\[
\begin{align*}
\phi_\pm = & \exp (F + K \sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r}) \psi_\pm = \exp F (\cosh K + \sinh K \sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r}) \psi_\pm, \\
\chi_\pm = & \exp (F + K \sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r}) \eta_\pm = \exp F (\cosh K + \sinh K \sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r}) \eta_\pm.
\end{align*}
\]

(56)

This (second step) will yield equations which in the limit when one mass becomes extremely large reduce to the Schrödinger like or Pauli-forms discussed in the previous section. In analogy to what occurred there the decoupled form of the Schrödinger-like equation for \(\psi_+\) has the convenient property that the coefficients of the first order relative
mixture terms $2F'\mathbf{i} \cdot \mathbf{p} + iK'(\sigma_1 \cdot \hat{r} \sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \hat{r} \sigma_1 \cdot \mathbf{p})$ in Eq. (53) vanish. We obtain [57] the detailed form of Eq.

$$\{ \frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} + 2m_w S + S^2 + 2\varepsilon_w A - A^2 + \Phi_D $$

$$- 3\Phi_{SS} - \Phi_{ST} \} v_0 + 2\sqrt{j(j+1)}(\Phi_{SOD} - \Phi_{SOX}) v_1$$

$$= b^2 v_0,$$

$$\Phi_D = \frac{2F'(\cosh 2K - 1)}{r} + F'^2 + K'^2 + \frac{2K' \sinh 2K}{r} - \nabla^2 F$$

$$- \frac{2(\cosh 2K - 1)}{r^2} + m(r),$$

$$\Phi_{SO} = - \frac{F'}{r} - \frac{F'(\cosh 2K - 1)}{r} - \frac{(\cosh 2K - 1)}{r^2} + \frac{K' \sinh 2K}{r},$$

$$\Phi_{SOD} = (l' \cosh 2K - q' \sinh 2K),$$

$$\Phi_{SOX} = (q' \cosh 2K - l' \sinh 2K),$$

$$\Phi_{SS} = k(r) - \frac{F' \sinh 2K}{r} - \frac{F'(\cosh 2K - 1)}{r} + \frac{K' \sinh 2K}{r} + \frac{K'(\cosh 2K - 1)}{r}$$

$$- \frac{\sinh 2K}{r^2} - \frac{(\cosh 2K - 1)}{r^2},$$

$$\Phi_{ST} = n(r) + \frac{3F' \sinh 2K}{r} + \frac{F'(\cosh 2K - 1)}{r} + \frac{K' \sinh 2K}{r} - \frac{3K'(\cosh 2K - 1)}{r}$$

$$- \frac{\nabla^2 K}{r^2} + \frac{3 \sinh 2K}{r^2} + \frac{K'}{r^2} + \frac{F' \sinh 2K}{r}.$$  

For equal mass singlet states, the hyperbolic terms cancel. The spin-orbit difference terms in general produce spin mixing. Note that the spin-spin term includes through the part $k(r)$ a piece $\frac{1}{2} \nabla^2 \mathcal{G}$ that for $|A| << w/2$ goes like $-\nabla^2 A / 2w$. When placed as a correction to the potential in the nonrelativistic limit (so that it must be divided by $2\mu$) its contribution would be $-\nabla^2 A \sigma_1 \cdot \sigma_2 / (m_1 m_2)$ which, as discussed earlier in Eq. (9), would produce a singular delta function for a Coulomb-like potential.

B. Radial Equations

The following are radial eigenvalue equations corresponding to Eq. (57) [57]. For a general singlet $^1J_j$ wave function $v_0$ coupled to a general triplet $^3J_j$ wave function $v_1$, the wave equation

$$\{ - \frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} + 2m_w S + S^2 + 2\varepsilon_w A - A^2 + \Phi_D $$

$$- 3\Phi_{SS} - \Phi_{ST} \} v_0 + 2\sqrt{j(j+1)}(\Phi_{SOD} - \Phi_{SOX}) v_1$$

$$= b^2 v_0,$$

is coupled to

$$\{ - \frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} + 2m_w S + S^2 + 2\varepsilon_w A - A^2 + \Phi_D $$

$$- 2\Phi_{SO} + \Phi_{SS} + \Phi_{ST} - 2\Phi_{SOT} \} v_1 + 2\sqrt{j(j+1)}(\Phi_{SOD} + \Phi_{SOX}) v_0$$

$$= b^2 v_1.$$
For a general $s = 1$, $j = l + 1$ wave function $u_+$ coupled to general $s = 1$, $j = l - 1$ wave function $u_-$ the equation

$$\begin{align*}
\{-\frac{d^2}{dr^2} + \frac{j(j-1)}{r^2} + 2m_wS + S^2 + 2\varepsilon_wA - A^2 + \Phi_D & \\
+ 2(j-1)\Phi_{SO} + \Phi_{SS} + \frac{1}{2j+1}\Phi_{ST} + \frac{2(j-1)}{2j+1}\Phi_{SOT} \}u_+ & \\
+ \frac{2\sqrt{j(j+1)}}{2j+1}(\Phi_{ST} - 2(j+2)\Phi_{SOT})u_- & \\
= b^2u_+,
\end{align*}$$

is coupled to

$$\begin{align*}
\{-\frac{d^2}{dr^2} + \frac{(j+1)(j+2)}{r^2} + 2m_wS + S^2 + 2\varepsilon_wA - A^2 + \Phi_D & \\
- 2(j+2)\Phi_{SO} + \Phi_{SS} - \frac{1}{2j+1}\Phi_{ST} + \frac{2(j+2)}{2j+1}\Phi_{SOT} \}u_- & \\
+ \frac{2\sqrt{j(j+1)}}{2j+1}(\Phi_{ST} + 2(j-1)\Phi_{SOT})u_+ & \\
= b^2u_-.
\end{align*}$$

(V. THE BEHAVIOR OF SINGLET AND TRIPLET WAVE FUNCTIONS AT SMALL $r$ IN QED)

In QED bound state problems, as has been shown in detail in [4] for $A = -\alpha/r$ the nonperturbative (numerical) treatment of the coupled Schrödinger-like forms in Eqs. (41) and (42) gave the same spectral results as a perturbative treatment of the weak potential form (which in turn gave the same results as older standard methods). We note that this agreement was for a number of radial and orbitally excited states for equal and unequal mass as well as the ground state. In this section we examine the structures of the four-component decoupled forms of Eq. (57) focusing primarily on the equal mass $^3S_1$ and $^1S_0$ states.

For the equal mass singlet states, there is no spin-mixing and with $A = -\alpha/r$, $S = 0$ we obtain

$$\begin{align*}
\{-\frac{d^2}{dr^2} - \frac{2\varepsilon_w\alpha}{r} - \frac{\alpha^2}{r^2} \}v_0 = b^2v_0.
\end{align*}$$

As was shown in [58] where this equation first appeared, it has an exact solution with the eigenvalue given by

$$w = m\sqrt{2 + 2/\sqrt{1 + \alpha^2/(n + \sqrt{(l + 1/2)^2 - \alpha^2 - l - 1/2)^2}}}
$$

$$= 2m - m\alpha^2/4n^2 - m\alpha^4/2n^3(1/(2l + 1) - 11/32/n) + O(\alpha^6),$$

(64)

or $w = 2m - m\alpha^2/4 - 21m\alpha^4/64$ for the ground state. The small $r$ wave radial function behaviors $v_0 \sim r^{1/2}\sqrt{1 - 4\alpha^2} + \frac{1}{2}$ which implies the mildly singular behavior of $\psi_0 = v_0/r \sim r^{1/2}\sqrt{1 - 4\alpha^2} - \frac{1}{2}$. Of course, its perturbative treatment gives the same results as the exact one through order $\alpha^4$.

The ground state triplet c.m. energy (excluding the annihilation contribution) perturbatively is [4] $w = 2m - m\alpha^2/4 - m\alpha^4/192$. This result, unlike the singlet case has not been obtained as an expansion of an exact analytic result. Unlike the one-body Dirac equation, the triplet states do not possess an exact spectral solution for the TBDE. However, it has been verified that a nonperturbative (numerical) solution of the TBDE does produce a result that agrees with the perturbative evaluation [4].

We are more interested here in the behavior of the potentials in Eqs. (61) and (62) and the resultant wave functions at small $r$. That, we show, can be determined analytically. By using the effective potentials in those equations we find that at small $r$ Eqs. (61) and (62) become ($A = -\alpha/r$, $S = 0$) for general $j$

$$\begin{align*}
\{-\frac{d^2}{dr^2} - \frac{\alpha^2}{r^2} + \frac{(j+1)(j-\frac{1}{4(2j+1)})}{r^2} - \frac{2j(j+1)}{2j+1}\frac{1}{r} \sqrt{\frac{1}{2}r^2\alpha} \}&u_+ & \\
+ \frac{2\sqrt{j(j+1)}}{2j+1}(\frac{1}{8r^2} + \frac{j+1}{r^2} \sqrt{\frac{1}{2}r^2\alpha} \}&u_- & = 0,
\end{align*}$$

(65)
and
\[
\left\{ -\frac{d^2}{dr^2} - \frac{\alpha^2}{r^2} + \frac{j(j+1)}{r^2} + \frac{1}{(2j+1)} \right\} u_+ + 2j(j+1) \frac{1}{(2j+1)r^2} \sqrt{\frac{1}{2r\omega}} \alpha u_+ \\
+ \frac{2\sqrt{j(j+1)}}{2j+1} \left\{ -\frac{1}{8r^2} - \frac{j}{r^2} \sqrt{\frac{1}{2r\omega}} \alpha \right\} u_+ = 0.
\]
(66)

Note that in Eq. (65), the potential for the \( u_+ \) wave function contains an attractive term that is proportional to \( r^{-5/2} \). Without the coupling to the \( u_- \) wave function, Eq. (65) would lead to a singular \( u_+ \) wave function. However, the potential for the \( u_- \) wave function in Eq. (66) contains a repulsive term that is proportional to \( r^{5/2} \). The wave function \( u_+ \) is thus prevented from collapsing to the center due to the coupling of \( u_+ \) to \( u_- \) in Eq. (65). In fact, because of the coupling, the short distance behavior of \( u_+ \) and \( u_- \) becomes
\[
u_+(r) = r^\lambda, \\
u_-(r) = \frac{j r^\lambda}{\sqrt{j(j+1)}} = \frac{j u_+(r)}{\sqrt{j(j+1)}}.
\]
(67)

which lead to an exact cancellation of the singular attractive \( r^{-5/2} \) terms in Eqs. (65) and (66) with their repulsive \( r^{-5/2} \) counterparts. The power index \( \lambda \) with the correct behavior at the origin is
\[
\lambda = \sqrt{j(j+1) - \alpha^2} + \frac{1}{2},
\]
so that at short distance the correct behavior of the wave function is
\[
u_+(r) = r^{(1/2+\sqrt{j(j+1)-\alpha^2})}, \\
u_-(r) = \frac{j r^{(1/2+\sqrt{j(j+1)-\alpha^2})}}{\sqrt{j(j+1)}}.
\]
(69)

The corresponding radial parts of the wave functions would be
\[
\psi_+(r) = r^{\frac{1}{2}(j+1)-\alpha^2 - \frac{1}{2}}, \\
\psi_-(r) = \frac{j r^{\frac{1}{2}(j+1)-\alpha^2 - \frac{1}{2}}}{\sqrt{j(j+1)}}.
\]
(70)

Focusing on \( j = 1 \), both of these wave functions would show an unusual feature of a dip not only in the \( D \) state but also in the \( S \) state. This is in contrast to what occurs in the singlet case where there is the mildly singular behavior or in the nonrelativistic case where the behavior is flat. Also, whereas a limit of \( \alpha = 1/2 \) is placed on the coupling in the singlet case, here the limit is \( \sqrt{2} \). The unusual dips of the \( S \) - and \( D \)-wave functions for QED also shows up in numerical solution of the QCD case, as will be shown in Figs. (3) and (4).

For the special case of \( j = 0 \) (\( ^3P_0 \)) the short distance behavior is
\[
\left\{ -\frac{d^2}{dr^2} - \frac{\alpha^2}{r^2} \right\} u_0 = 0,
\]
(71)
just as it is for the \( ^1S_0 \) case.

In the case of the coupled \( ^1J, ^3J \) states, in the general unequal mass case there will be spin mixing. However, in the limit of small \( r \), the mixing term vanishes leaving us with the uncoupled short distance behavior for the wave equations (59) and (60) for \( j > 0 \) of
\[
\left\{ -\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} - \frac{\alpha^2}{r^2} \right\} v_{s=0} = 0,
\]
\[
\left\{ -\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} - \frac{\alpha^2}{r^2} - \frac{1}{4r^2} \right\} v_{s=1} = 0.
\]
(72)

The radial wave functions behave as
\[
v_{s=0} \rightarrow r^{(1/2+\sqrt{j(j+1)-\alpha^2})}, \\
v_{s=1} \rightarrow r^{(1/2+\sqrt{j(j+1)-\alpha^2})}.
\]
(73)
The corresponding radial parts of the total wave functions would be

$$\psi_{s=0} \rightarrow r^{(-1/2 + \sqrt{(j+1/2)^2 - \alpha^2})},$$
$$\psi_{s=1} \rightarrow r^{(-1/2 + \sqrt{j(j+1) - \alpha^2})},$$

(74)

and have acceptable behavior for $j > 0$. Notice that unlike the tensor mixing case, there is no short distance connection in the spin mixing case between the wave function scales as in Eq. (67). For $j = 0$, only the first wave function is relevant.

Next we examine the singularity structure of the TBDE in the case of QCD bound states. The short distance behaviors (which we attribute to the invariant $A(r)$) of QED and QCD are well known to have a crucial distinction. Renormalization group arguments show that in QED the asymptotic behavior displays a singularity structure\(^5\) at finite (though very large) energy that invalidates perturbation theory [60]. In contrast similar renormalization group arguments show for QCD the asymptotic behavior displays a structure\(^6\) that at high energy strongly validates perturbation theory (asymptotic freedom) [60]. For this reason we have not included a running coupling constant above in the QED application.

VI. SIMPLE QCD BASED MODEL FOR QUARK-ANTI-QUARK BOUND STATES.

In [10] a fully relativistic calculation of the meson spectrum was made using the above TBDE with the invariants $A(r)$ and $S(r)$ determined from a relativistic extension of the nonrelativistic Adler-Piran [17] static quark potential. That paper investigated how well the relativistic constraint approach performs in comparison with selected alternatives including those of Godfrey and Isgur [61] when used to produce a single fit of experimental results over the whole meson spectrum. The authors of [10] found that the fit provided by the two-body Dirac model for the entire meson spectrum competes with the best fits to partial spectra provided by the others and does so with the smallest number of interaction functions. Furthermore this is done without additional cutoff parameters necessary to make some of the other approaches numerically tractable.

In this section we examine the spectral results produced by a relativistic version of a simpler model motivated by the static quark potential of Richardson [39]. It is not our purpose here to improve upon the results of [10] but rather to take advantage of the simpler structure of the model to explore certain details of the wave function behavior generated by the TBDE along the lines discussed in the above sections for QED. This will also allow us to more readily make changes in the invariants $A(r)$ and $S(r)$ determined from this simpler model to facilitate investigations on quarkonium stability in the presence of a quark-gluon plasma.

A. The Model and Spectral Results

For our potential model we consider here a slightly more generalized form of Eq. (8),

$$V(r) = \frac{8\pi\Lambda^2 r}{27} - \frac{16\pi}{27r\ln(K\epsilon^2 + B/(\Lambda^2)^2)}.$$  (75)

This has the feature of giving a variable parameter $K$ to the long distance Coulomb behavior as well as (through $B$) an effective QCD parameter $\Lambda$ distinct from the one placed in the linear potential. In our QCD spectral work we assign the confining piece of this potential to the invariant $S$ that controls the scalar potential and the Coulomb-like piece to the invariant $A$ that controls the vector potential (see Eqs. (38),(43),(44), and (49)). Thus with $r = \sqrt{x^2}$ the equations

$$S(r) = \frac{8\pi\Lambda^2 r}{27},$$
$$A(r) = \frac{16\pi}{27r\ln(K\epsilon^2 + B/(\Lambda^2)^2)}.$$  (76)

---

5 The running coupling constant in spinor QED is given by [59] $\alpha_R(q^2) \rightarrow \alpha_R(m^2)/(1 - \alpha_R(m^2)\ln(-q^2/m^2))/3\pi$.

6 The running coupling constant in QCD is given by [59] $1/\alpha_s(q^2) \rightarrow 1/\alpha_s(\mu^2) + (33 - 2n_f)\ln(-q^2/\mu^2)/12\pi$.\n
---
together with the Schrödinger-like form Eq. (57) of our TBDE (50) define the covariant formalism for our QCD spectral work. With the parameter values listed in Table I we obtain the following spectral results shown in Table II and Figure 1. The mass of the $u$ and $d$ quarks are only 55 MeV. We observe that the masses of the low-lying mesons are well reproduced in the TBDE treatment with the minimum number of parameters. The singlet-triplet splittings of $\pi$-$\rho$ and $\eta_c$-$J/\psi$ are well reproduced. The agreement of the experimental masses with theory is not as impressive as that using the Adler-Piran potential. (Note that as with the Adler-Piran potential, the ground state singlet/triplet splitting for the charmonium system appears too large, while the that between the ground and first excited states appears too small. The same problem also occurs for the bottomonium system with the recently observed $\eta_b$ [62].) The simplified potential has the advantage of simplicity and ease of adoptive modification that can be useful for other applications of the TBDE to other quarkonium problems.

![Meson Masses](image)

**FIG. 1:** (Color online) Comparison of experimental and theoretical masses obtained with the Two-Body Dirac Equations.

We next consider some interesting and unexpected behaviors of the solutions of the equations in the next few subsections.

| Parameter | Value |
|-----------|-------|
| $A$       | 0.4218 GeV |
| $B$       | 0.05081 |
| $K$       | 4.198 |
| $m_u$     | 0.0557 GeV |
| $m_d$     | 0.0553 GeV |
| $m_s$     | 0.2499 GeV |
| $m_c$     | 1.476 GeV |
| $m_b$     | 4.844 GeV |

**TABLE I:** Quark Model Parameter Values
equation for \( S \) does the bound state mass. Thus our analysis here bears on the dynamics of the chiral limit. The bound state pion mass comes about. In earlier work \[10\], it was shown numerically that as the quark masses tend to zero, so also masses that give the best fit in our model here are on the order of 50

The rest mass squared (0.00302 GeV\(^2\)) but nearly cancel, leaving a small \( b^2 \) which corresponds to a pion mass of 0.159

### Table III: Selected Portions of Meson Spectrum

| Meson      | Exp.(GeV) ±MeV Theory |
|------------|-----------------------|
| \( \pi^0 \): ud \( 1^1S_0 \) | 0.140 0.0 0.159 |
| \( \rho^0 \): ud \( 1^3S_1 \) | 0.767 1.2 0.792 |
| \( K^0 \): \( \bar{s}d \) \( 1^1S_0 \) | 0.494 0.0 0.493 |
| \( K^0 \): \( \bar{u}d \) \( 1^3S_1 \) | 0.498 0.0 0.488 |
| \( K^* \): \( \bar{s}d \) \( 1^3S_1 \) | 0.892 0.2 0.903 |
| \( K^* \): \( \bar{u}d \) \( 1^3S_1 \) | 0.896 0.2 0.901 |
| \( \phi \): \( \bar{s}d \) \( 1^3S_1 \) | 1.019 0.0 1.025 |
| \( D^0 \): \( \bar{s}d \) \( 1^1S_0 \) | 1.865 0.5 1.840 |
| \( D^+ \): \( \bar{s}d \) \( 1^3S_1 \) | 1.869 0.5 1.845 |
| \( D^{*0} \): \( \bar{u}d \) \( 1^3S_1 \) | 2.010 0.5 1.981 |
| \( D^{*+} \): \( \bar{u}d \) \( 1^3S_1 \) | 2.067 0.5 1.979 |
| \( D_s \): \( \bar{s}d \) \( 1^3S_0 \) | 1.968 0.6 1.965 |
| \( D_s^* \): \( \bar{s}d \) \( 1^3S_1 \) | 2.112 0.7 2.112 |
| \( \eta_c \): \( \bar{c}c \) \( 1^1S_0 \) | 2.980 2.1 2.978 |
| \( \psi \): \( \bar{c}c \) \( 1^3S_1 \) | 3.097 0.0 3.140 |
| \( \psi \): \( \bar{c}c \) \( 2^3S_1 \) | 3.686 0.0 3.689 |
| \( \eta \): \( \bar{c}c \) \( 1^1P_1 \) | 3.526 0.2 3.522 |
| \( \chi_0 \): \( \bar{c}c \) \( 1^3P_0 \) | 3.415 1.0 3.436 |
| \( \chi_1 \): \( \bar{c}c \) \( 1^3P_1 \) | 3.510 0.1 3.515 |
| \( \chi_2 \): \( \bar{c}c \) \( 1^3P_2 \) | 3.556 0.1 3.541 |
| \( \phi \): \( \bar{c}c \) \( 1^3S_1 \) | 3.638 5.0 3.591 |
| \( \psi \): \( \bar{c}c \) \( 1^3D_1 \) | 3.771 2.5 3.804 |
| \( B^- \): \( \bar{b}u \) \( 1^1S_0 \) | 5.279 1.8 5.249 |
| \( B^0 \): \( \bar{b}d \) \( 1^1S_0 \) | 5.279 1.8 5.248 |
| \( B^{*0} \): \( \bar{b}u \) \( 1^3S_1 \) | 5.325 1.8 5.299 |
| \( B_s \): \( \bar{b}s \) \( 1^1S_0 \) | 5.366 2 5.360 |
| \( B_s^* \): \( \bar{b}u \) \( 1^3S_1 \) | 5.412 3.3 5.420 |
| \( B_s^* \): \( \bar{b}s \) \( 1^3S_0 \) | 6.286 4.0 6.276 |
| \( \eta_b \): \( \bar{b}b \) \( 1^1S_0 \) | 9.389 0.2 9.345 |
| \( \eta \): \( \bar{b}b \) \( 1^3S_1 \) | 9.460 4.0 9.484 |
| \( \eta \): \( \bar{b}b \) \( 2^3S_1 \) | 10.023 0.3 10.033 |
| \( \eta \): \( \bar{b}b \) \( 3^3S_1 \) | 10.355 0.5 10.360 |

### Table II: Selected Portions of Meson Spectrum

#### B. Behaviors of singlet and triplet solution to bound state equations

1. **Detailed Analysis of the Pion and Rho Bound States.**

One of the most unusual features of the spectral results of the TBDE is how they produce the large \( \pi-\rho \) mass splitting. First we point out that unlike most other potential models, the up and down quark masses do not take on the values typically seen of about 300 MeV. Just as with the more detailed Adler-Piran potential \[17\], the quark masses that give the best fit in our model here are on the order of 50 \(-\) 60 MeV. It is of interest to see how the small pion mass comes about. In earlier work \[10\], it was shown numerically that as the quark masses tend to zero, so also does the bound state mass. Thus our analysis here bears on the dynamics of the chiral limit. The bound state equation for \( S \) states has the general form of Eq. (59) and in this case reduces to

\[
\left\{ \frac{d^2}{dr^2} + 2m_S + S^2 + 2\varepsilon_w A - A^2 + \Phi_D - 3\Phi_{SS} - \Phi_{ST} \right\} \psi_0 = 0
\]

\[
\equiv \left\{ -\frac{d^2}{dr^2} + \Phi \right\} \psi_0 = \left\{ -\frac{d^2}{dr^2} + 2m_S + S^2 + 2\varepsilon_w A - A^2 + \Phi_{SD} \right\} \psi_0
\]

\[
\equiv \left\{ -\frac{d^2}{dr^2} + \Phi_{SI} + \Phi_{SD} \right\} = b^2 \psi_0.
\]
GeV. The pion as a quark-antiquark system has large kinetic energies and potential energies that counterbalance each other. When one looks at the potential energy $\Phi = -0.8475$ GeV$^2$ from various contributions, one notes that the spin-independent contribution $\langle \Phi_{SI} \rangle = \langle 2m_w S \rangle + \langle S^2 \rangle + \langle 2m_w A \rangle + \langle A^2 \rangle = -0.3832$ GeV$^2$ while the combined Darwin and spin-dependent contributions are $\langle \Phi_{SD} \rangle = \langle \Phi_D - 3\Phi_{SS} - \Phi_{ST} \rangle = -0.4643$ GeV$^2$. They are of the same order of magnitude. Among the three contributions to $\langle \Phi_{SD} \rangle$, the Darwin term $\langle \Phi_D \rangle$ and $\langle -3\Phi_{SS} \rangle$ nearly cancel the $\langle -\Phi_{ST} \rangle$ contributions. The small mass of the pion arises in no small measure from delicate cancellations of these large contributions from the Darwin, spin-spin, and tensor interactions, as well as the balance of kinetic energy and potential energies. Such does not occur with the other mesons. (See for example the rho meson below). As seen in Fig. 1 there is nothing unusual in the pion wave function in constraint dynamics.

![Wave functions of π and η mesons](image)

**FIG. 2:** Wave functions of $\pi$ and $\eta$ mesons

| Term                  | Value   |
|-----------------------|---------|
| $\Phi$                | -0.8475 |
| $2m_w S$              | 0.0103  |
| $\langle S^2 \rangle$ | 0.0942  |
| $2m_w A$              | -0.0598 |
| $\langle A^2 \rangle$ | -0.4279 |
| $\langle -\Phi_{SI} \rangle$ | -0.3832 |
| $\langle \Phi_D \rangle$ | -3.804  |
| $\langle -3\Phi_{SS} \rangle$ | -2.895  |
| $\langle -\Phi_{ST} \rangle$ | 6.235   |
| $\langle \Phi_{SD} \rangle$ | -0.4643 |
| $b'$                  | 0.0033  |

**TABLE III:** Expectation values of various terms in (77) (in GeV$^2$) that contribute to the pion eigenvalue equation (77).

By contrast now we present a similar table for the $\rho$ meson. The coupled equations are
There is some cancellation but not nearly to the extent that occurs in the pion. There is left a significant same sign. The off diagonal tensor terms
\[\langle \Phi_{ST} \rangle \]
and potential portions which corresponds to a computed rho mass of 0.796 GeV. Note that by itself, the potential) is negative and large compared to the rest mass. Thus the positive behavior for our coupled QCD equations, corresponding to mesons such as the J/ψ and the ρ mesons.

Table IV below lists the expectation values of the various parts (units in GeV²). The quark masses make up only a small portion of the pion mass. As with the pion, the rho is a highly relativistic, but unlike the pion, its relativistic nature is not hidden in large cancellations. The constituent kinetic portion \( \langle -\frac{d^2}{dr^2} \rangle_{++}, \langle -\frac{d^2}{dr^2} \rangle_{+-}, \) and potential portions \( \langle \Phi_{ST} \rangle_{++}, \langle \Phi_{ST} \rangle_{+-}, \langle \Phi_{ST} \rangle_{--} \) are huge compared with the rest mass squared (0.00302 GeV²). There is some cancellation but not nearly to the extent that occurs in the pion. There is left a significant \( b^2 = 0.1411 \) which corresponds to a computed rho mass of 0.796 GeV. Note that by itself, the S–wave portion (kinetic plus potential) is negative and large compared to the rest mass. Thus the positive D–wave portion is crucial to bring the rho mass in line with the observed value. The two spin-independent contributions \( \langle \Phi_{SD} \rangle_{++} = \langle \Phi_{SD} \rangle_{+-} = 0.3154 \) GeV² differ by an order of magnitude as do the diagonal Darwin and spin-dependent contributions \( \langle \Phi_{SD} \rangle_{++} = \langle \Phi_{SD} \rangle_{+-} = 0.3154 \) GeV², \( \langle \Phi_{SD} \rangle_{--} = \langle \Phi_{SD} \rangle_{++} = 0.3154 \) GeV². The two sets differ by roughly a factor of 2 and are of the same sign. The off diagonal tensor terms \( \langle \Phi\rangle_{+-} = 2\sqrt{2} \langle \Phi_{ST} \rangle_{+-}, \langle \Phi\rangle_{++} = 2\sqrt{2} \langle \Phi_{ST} \rangle_{++} \) are both quite large but of opposite sign so their overall effects almost cancel.

2. Detailed analysis of the vector meson potentials and wave functions.

As may be anticipated from the dip behavior displayed in Eq. (70) for the case of QED interactions we anticipate that a similar structure may appear for the QCD wave functions. This is borne out by Figs. 3 and 4 for the S and D state contributions to the J/ψ and the ρ mesons. In the equations below the analytic origin of the dip behavior is shown from the behaviors of the quasipotential contributions.

In contrast to the short distance behavior displayed in Eqs. (65) and (66) in the QED case, the short distance behavior for our coupled QCD equations, corresponding to mesons such as the J/ψ and ρ mesons is
\[
\begin{align*}
&\langle -\frac{d^2}{dr^2} \rangle_{++} + \langle -\frac{d^2}{dr^2} + \frac{4}{r^2} \rangle_{--} = 0.3085 - 0.2812 \\
&\langle \Phi \rangle_{++} - \langle \Phi \rangle_{--} = -0.4835 - 0.04923 \\
&\langle 2m_w S \rangle_{++} - \langle 2m_w S \rangle_{--} = 0.00263 + 0.000571 \\
&\langle S^2 \rangle_{++} - \langle S^2 \rangle_{--} = 0.1631 + 0.04457 \\
&\langle 2x_w A \rangle_{++} - \langle 2x_w A \rangle_{--} = -0.2091 - 0.02109 \\
&\langle -A^2 \rangle_{++} - \langle -A^2 \rangle_{--} = -0.1247 - 0.00944 \\
&\langle \Phi_{SI} \rangle_{++} - \langle \Phi_{SI} \rangle_{--} = -0.1680 + 0.01461 \\
&\langle \Phi_{D} \rangle_{++} - \langle \Phi_{D} \rangle_{--} = -0.2790 - 0.04360 \\
&\langle \Phi_{SS} \rangle_{++} - \langle \Phi_{SS} \rangle_{--} = -0.0691 - 0.03033 \\
&\langle \Phi_{ST} \rangle_{++} - \langle \Phi_{ST} \rangle_{--} = 0.03273 - 0.02086 \\
&\langle -6\Phi_{SOT} \rangle_{++} - \langle -6\Phi_{SOT} \rangle_{--} = 0.6177 + 0.3088 \\
\end{align*}
\]

TABLE IV: Expectation values of various terms in Eqs. (78) and (79) (in GeV${}^2$) that contribute to the rho eigenvalue equations (78 and 79).

\[
\begin{align*}
-&\frac{d^2}{dr^2} + \left( \frac{8\pi}{27} \right)^2 + 4 \left( -1 + \frac{1}{\ln(\Lambda r)} \right) \sqrt{\frac{2\pi}{27 \epsilon r \ln(\Lambda r)}} + \frac{11}{6r^2} - \frac{5}{3r^2} \ln(\Lambda r) - \frac{1}{6r^2 \ln^2(\Lambda r)} \right] u_+ \\
+ &\frac{2\sqrt{2}}{3} 2 \left( -1 - \frac{1}{\ln(\Lambda r)} \right) \sqrt{\frac{2\pi}{27 \epsilon r \ln(\Lambda r)}} - \frac{1}{8r^2} + \frac{1}{2r^2} \ln(\Lambda r) - \frac{1}{8r^2 \ln^2(\Lambda r)} \right] u_- \\
= &\ 0,
\end{align*}
\]
FIG. 4: Wave functions for the $\rho$ meson. The insert in the upper right corner gives an expanded view of the wave functions near the origin.

\[
-\frac{d^2}{dr^2} + \frac{6}{r^2} + \left(\frac{8\pi}{27}\right)^2 + \frac{4}{3r^2}(1 - \frac{1}{\ln(Ar)}) \sqrt{-\frac{2\pi}{27\varepsilon r \ln(Ar)}} - \frac{49}{12r^2} - \frac{2}{3r^2 \ln(Ar)} - \frac{1}{12r^2 \ln^2(Ar)} |u_-| + \frac{2\sqrt{2}}{3} \left(\frac{1}{r^2} + \frac{1}{r^2 \ln(Ar)}\right) \sqrt{-\frac{2\pi}{27\varepsilon r \ln(Ar)}} - \frac{1}{8r^2} + \frac{1}{2r^2 \ln(Ar)} - \frac{1}{8r^2 \ln^2(Ar)} |u_+| = 0.
\]

Since each inverse power of the logarithm dampens the corresponding singular contribution, the most singular of the terms that remain are

\[
-\frac{d^2}{dr^2} - \frac{11}{6r^2} \sqrt{-\frac{2\pi}{27\varepsilon r \ln(Ar)}} + \frac{2\sqrt{2}}{3} \left(-\frac{1}{r^2} - \frac{2\pi}{27\varepsilon r \ln(Ar)} - \frac{1}{8r^2}\right) |u_-| = 0,
\]

and

\[
-\frac{d^2}{dr^2} + \frac{6}{r^2} + \frac{4}{3r^2} \left(-\frac{2\pi}{27\varepsilon r \ln(Ar)} - \frac{49}{12r^2} - \frac{2\pi}{3r^2 \ln(Ar)} - \frac{1}{12r^2 \ln^2(Ar)}\right) |u_+| = 0.
\]

Let us assume a short distance behavior of

\[
u_+(r) = f(r), \quad u_-(r) = \sqrt{\frac{2\pi}{27\varepsilon r \ln(Ar)}} \frac{u_+(r)}{\sqrt{2}}.
\]

Then just as in the case of QED, the more singular terms (here $\sim \frac{1}{r^2} \sqrt{-\frac{2\pi}{27\varepsilon r \ln(Ar)}}$) cancel among them selves and we are left with

\[
-\frac{d^2}{dr^2} + \frac{11}{6r^2} f(r) + \frac{2}{3} \left(-\frac{1}{8r^2}\right) f(r) = 0.
\]
and

\[-\frac{d^2}{dr^2} + \frac{23}{12r^2}f(r) + \frac{4}{3}\left(-\frac{1}{8r^2}\right)f(r) = 0.\]  

(87)

which have the identical behavior of

\[-\frac{d^2}{dr^2} + \frac{7}{4r^2}f(r) = 0,\]  

(88)

or

\[f(r) \sim r^{(1/2+\sqrt{2})}\]  

(89)

The corresponding radial parts of the wave functions would be

\[
\psi_+ (r) = r^{\sqrt{2} - \frac{1}{2}},
\]

\[
\psi_- (r) = \frac{r^{\sqrt{2} - \frac{1}{2}}}{\sqrt{2}},
\]

(90)

compared to the corresponding radial parts of the QED wave functions of

\[
\psi_+ (r) = r^{\sqrt{2} - \alpha^2 - \frac{1}{2}},
\]

\[
\psi_- (r) = \frac{r^{\sqrt{2} - \alpha^2 - \frac{1}{2}}}{\sqrt{2}}.
\]

(91)

Numerically, one finds results that approach those of Eq. (90) in terms of the S/D ratio and power behavior.

\[\text{VII. CONCLUSION AND FURTHER REMARKS}\]

We have shown how the TBDE of constraint dynamics handles the problem of effective potentials that are singular in the weak potential limit. The most noteworthy feature is that different portions of the quasipotential \(\Phi\) contribute to the perturbative and nonperturbative treatment of spectral effects. The results we found for the coupled \(^3S_1-^3D_1\) system demonstrates that this effect extends not only to different terms of a given equation, but also bridges the divide between the various coupled components of the wave function and effective potential. We find unexpectedly, that the behavior of the \(j=1\), \(S\) and \(D\) waves (and more generally for arbitrary \(j\)) are simply proportional very near the origin, with a common power-law behavior.

We have introduced a new QCD potential that has many of the features of the Adler-Piran and Richardson potentials but with a much simpler parametrization. Although not giving results as good as the former, it nevertheless yields a meson mass spectrum that agrees reasonably well with experiment. We examine in particular how various contributions to the pion eigenvalue equation can lead to a pion of a small rest mass. The detailed treatment of the pion shows a unique feature of it relativistic behavior, namely a behavior that superficially appears nonrelativistic but on further analysis displays extremely large relativistic and nearly canceling contributions to and from the potential. This gives some insight into how the potential model leads to a small pion mass for small quark mass, an important consequence of spontaneous symmetry breaking. By contrast our Table IV shows why mesons with other quantum numbers, would not display such a small quark mass behavior for the bound state energy.

Another unexpected behavior for the QCD wave functions and effective potentials is that at short distance, \(j=1\), \(S\) and \(D\) waves for the \(J/\psi\) mesons are not significantly different from that which appears in QED, in spite of the asymptotic freedom behavior that occurs in QCD. It may be worthwhile to investigate what observable features of these bound states (both in QED and QCD) may reflect these unexpected connections between the two coupled wave functions.

In addition to applications to meson spectroscopy, our work has implications for studies related to quarkonia at finite temperature and in a quark-gluon plasma. We review some detail of the important aspects since our analysis was motivated by the prospect of its future application to QGP. Recent spectral analyses of quarkonium correlators indicated that \(J/\psi\) may be stable up to \(1.6T_c\) where \(T_c\) is the phase transition temperature \([63, 64]\). Subsequently, there has been renewed interest in quarkonium states in quark-gluon plasma as Zahed and Shuryak suggested that \(QQ\) states with light quarks may be bound up to a few \(T_c\) \([65]\). Since the potential model can be used to evaluate many more quantities than the lattice gauge spectral function analysis it provides the simplest way to examine the
quarkonia problem in QGP. The potential model also lends itself to extrapolation into unknown regions of quark masses and temperatures.

Within the potential model, the quark drip line at which quarkonia begins to be unbound has been estimated using non-relativistic quark models [66, 67]. A more definitive investigation of the stability of quarkonium states, especially those with light quarks, necessitates the use of the relativistic formalism developed here. The analytic structure of the $A$ and $S$ invariants given in Eq. (76) lend themselves to practical modeling at finite temperatures, while giving an adequate account of the zero temperature limit. The relativistic one developed here provides a more realistic model than the most often used Cornell-type linear-plus color-Coulomb for the case of $T = 0$. It will be of interest to investigate the composite properties of the plasma and to determine the region of temperatures in which different quarkonia become unbound using a temperature dependent extension of the simple quark model presented in this paper. An important problem for future work in relation to the present paper is to determine how best to apportion these temperature dependent potentials between the $A(r)$ and $S(r)$ invariants.

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