SHORT EFFECTIVE INTERVALS CONTAINING PRIMES.

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ABSTRACT. We prove that if $x$ is large enough, namely $x \geq x_0$, then there exists a prime between $x(1 - \Delta^{-1})$ and $x$, where $\Delta$ is an effective constant computed in terms of $x_0$.

1. INTRODUCTION.

In this article, we address the problem of finding short intervals containing primes. In 1845 Bertrand conjectured that for any integer $n > 3$, there always exists at least one prime number $p$ with $n < p < 2n - 2$. This was proven by Chebyshev in 1850, using elementary methods. Since then other intervals of the form $(kn, (k+1)n)$ have been investigated. We refer the reader to [1] for $k = 2$, and to [12] for $k = 3$. Assuming that $x$ is arbitrarily large, the length of intervals containing primes can be drastically reduced. To date, the record is held by Baker, Harman, and Pintz [2] as they prove that there is at least one prime between $x$ and $x + x^{0.525 + \varepsilon}$. This is an impressive result since under the Riemann Hypothesis the exponent $0.525$ can only be reduced to $0.5$. On the other hand, maximal gaps for the first primes have been checked numerically up to $4 \cdot 10^{18}$ by Oliveira e Silva et al. [14]. In particular, they find that the largest prime gap before this limit is $1476$ and occurs at $142572824437699411 = e^{41.8008...}$. The purpose of this article is to obtain an effective result of the form: for all $x \geq x_0$, there exists an $\Delta > 0$ such that the interval $(x(1 - \Delta^{-1}), x)$ contains at least one prime. In 1976 Schoenfeld’s [18, Theorem 12] gave this for $x_0 = 2010881.1$ and $\Delta = 16598$. In 2003 Ramaré and Saouter improved on Schoenfeld’s method by using a smoothing argument. They also extended the computations to many other values for $x_0$ ([16, Theorem 2 and Table 1]).

We present an example of numerical improvement this theorem allows, for instance when $x_0 = e^{59}$. Ramaré and Saouter [16] found that the interval gap was given by $\Delta = 209257759$. In [5] page 74, Helfgott mentioned an improvement of Ramaré using Platt’s latest verification of the Riemann Hypothesis [15]: $\Delta = 307779681$. Our Theorem 1.1 leads to $\Delta = 1946282821$.

We now mention an application to the verification of the Ternary Goldbach conjecture. This conjecture was known to be true for sufficiently large integers (by Vinogradov), and Liu and Wang [11] prove it for all integers $n \geq e^{3100}$. On the other hand, the conjecture was verified for the first values of $n$. In [16, Corollary 1], Ramaré and Saouter verified it for $n \leq 1.132 \cdot 10^{22}$. Very recently, Oliveira e Silva et. al. [14, Theorem 2.1] extended this limit to $n \leq 8.370 \cdot 10^{26}$. In [5, Proposition A.1.], Helfgott applied the above result on
short intervals containing primes ($\Delta = 307779681$) and found $n \leq 1.231 \cdot 10^{27}$. This allowed him to complete his proof [5] [6] of the Ternary Goldbach conjecture for the remaining integers. Here our main theorem gives:

**Corollary 1.2.** Every odd number larger than 5 and smaller than

$$1966196911 \times 4 \cdot 10^{18} = 7.864 \ldots \cdot 10^{27}$$

is the sum of at most three primes.

As of today, Helfgott and Platt [7] have announced a verification up to $8.875 \cdot 10^{30}$.

2. **Proof of Theorem**

We recall the definition of the classical Chebyshev functions:

$$\theta(x) = \sum_{p \leq x} \log p, \quad \psi(x) = \sum_{n \leq x} \Lambda(n), \quad \text{with } \Lambda(n) = \begin{cases} 1 & \text{if } n = p^k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

For each $x_0$, we want to find the largest $\Delta > 0$ such that, for all $x > x_0$, there exists a prime between $x(1 - \Delta^{-1})$ and $x$. This happens as soon as

$$\theta(x) - \theta(x(1 - \Delta^{-1})) > 0.$$

2.1. **Introduction of parameters.** We list here the parameters we will be using throughout the proof.

- * $m$ integer with $m \geq 2$,
- * $0 \leq u \leq 0.0001$, $\delta = mu$ and $0 \leq \delta \leq 0.0001$,
- * $0 \leq a \leq 1/2$,
- * $\Delta = (1 - (1 + \delta a)(1 + \delta(1 - a))^{-1} e^{-u})^{-1}$, \hspace{1cm} (2.1)
- * $X \geq X_0 \geq e^{38}$,
- * $x = e^u X(1 + \delta (1 - a)) \geq x_0 = e^u X_0 (1 + \delta (1 - a))$,
- * $y = X (1 + \delta a) = x (1 - \Delta^{-1})$.

2.2. **Smoothing the difference** $\theta(x) - \theta(y)$. We follow here the smoothing argument of [16]. Let $f$ be a positive function integrable on $(0, 1)$. We denote

$$\|f\|_1 = \int_0^1 f(t)dt,$$ \hspace{1cm} (2.2)

$$\nu(f,a) = \int_0^a f(t)dt + \int_{1-a}^1 f(t)dt,$$ \hspace{1cm} (2.3)

and

$$I_{\delta,u,X} = \frac{1}{\|f\|_1} \int_0^1 (\theta(e^u X(1 + \delta t)) - \theta(X(1 + \delta t))) f(t)dt. \hspace{1cm} (2.4)$$

Note that for all $a \leq t \leq 1 - a$, $\theta(e^u X(1 + \delta t)) - \theta(X(1 + \delta t)) \leq \theta(x) - \theta(y)$. We integrate with the positive weight $f$ and obtain:

$$\int_a^{1-a} (\theta(e^u X(1 + \delta t)) - \theta(X(1 + \delta t))) f(t)dt \leq (\theta(x) - \theta(y)) \int_a^{1-a} f(t)dt. \hspace{1cm} (2.5)$$
We extend the left integral to the interval $(0, 1)$ and use a Brun-Titchmarsh inequality to control the primes on the extremities $(0, a)$ and $(1 - a, 1)$ of the interval (see [16 page 16, line -5] or [13 Theorem 2]):

\[
\int_{t \in (0, a) \cup (1 - a, 1)} (\theta(e^u X (1 + \delta t)) - \theta(X (1 + \delta t))) f(t) dt \leq 2(1 + \delta) (e^u - 1) \log \left( \frac{e^u X}{\log (e^u - 1)} \right) \nu(f, a) X. \tag{2.6}
\]

Note that [16] uses the slightly larger bound

\[
2.0004 u \log \frac{X}{\log(uX)} \nu(f, a) X.
\]

Combining (2.5) and (2.6) gives for $I_{\delta,u,X}$:

\[
I_{\delta,u,X} \leq (\theta(x) - \theta(y)) \frac{\int_0^{1-a} f(t) dt}{\|f\|_1} + 2(1 + \delta) (e^u - 1) \frac{\log(e^u X(1 + \delta)) \nu(f, a)}{\log(X(e^u - 1))} \|f\|_1 X. \tag{2.7}
\]

Thus $\theta(x) - \theta(y) > 0$ when

\[
I_{\delta,u,X} - 2(1 + \delta) (e^u - 1) \frac{\log(e^u X(1 + \delta)) \nu(f, a)}{\log(X(e^u - 1))} \|f\|_1 X > 0. \tag{2.8}
\]

It remains to establish a lower bound for $I_{\delta,u,X}$. To do so, we first approximate $\theta(x)$ with $\psi(x)$. This will allow us to translate our problem in terms of the zeros of the zeta function. We use approximations proven by Costa in [3, Theorem 5]:

**Lemma 2.1.** Let $x \geq e^{38}$. Then

\[
0.999 \sqrt{x} + \sqrt{x} < \psi(x) - \theta(x) < 1.001 \sqrt{x} + \sqrt{x}. \tag{2.9}
\]

Then we have that for all $0 < t < 1,$

\[
(\psi(e^u X (1 + \delta t)) - \theta(e^u X (1 + \delta t))) - (\psi(X (1 + \delta t)) - \theta(X (1 + \delta t))) < \sqrt{X} \sqrt{1 + \delta} \left( 1.001 \frac{e^{u/2}}{X} - 0.999 + X^{-1/6}(1 + \delta)^{-1/6} \right) < \omega \sqrt{X}, \tag{2.10}
\]

where we can take, under our assumptions (2.1),

\[
\omega = 2.05022 \cdot 10^{-3}. \tag{2.11}
\]

We denote

\[
J_{\delta,u,X} = \frac{1}{\|f\|_1} \int_0^1 (\psi(e^u X (1 + \delta t)) - \psi(X (1 + \delta t))) f(t) dt. \tag{2.12}
\]

It follows from (2.10) that

\[
I_{\delta,u,X} \geq J_{\delta,u,X} - \omega \sqrt{X}. \tag{2.13}
\]

Note that [16] used older approximations from [18], which lead to $\omega = 0.0325$. To summarize, we want to find conditions on $m, \delta, u, a$ so that

\[
J_{\delta,u,X} - \omega \sqrt{X} - 2(1 + \delta) (e^u - 1) \frac{\log(e^u X(1 + \delta)) \nu(f, a)}{\log(X(e^u - 1))} \|f\|_1 X > 0. \tag{2.14}
\]

We are now left with evaluating $J_{\delta,u,X}$, which we shall do by relating it to the zeros of zeta through an explicit formula.
2.3. An explicit inequality for $J_{b,u,X}$.

Lemma 2.2. \textbf{[16] Lemma 4} Let $2 \leq b \leq c$, and let $g$ be a continuously differentiable function on $[b,c]$. We have

$$\int_b^c \psi(u)g(u)\,du = \int_b^c ug(u) - \sum \int_b^c \frac{u^\varrho}{\varrho} g(u)\,du + \int_b^c \left(2\pi - \frac{1}{2} \log(1 - u^{-2})\right) g(u)\,du.$$  \hfill (2.15)

We apply this identity to respectively $g(t) = f \left( \delta^{-1} (e^{-u}X^{-1}t - 1) \right)$, $b = e^uX$, $c = e^uX(1 + \delta)$ and $g(t) = f \left( \delta^{-1} (X^{-1}t - 1) \right)$, $b = X$, $c = X(1 + \delta)$. It follows that

$$J_{b,u,X} = \frac{(e^u - 1)X}{\|f\|_1} \int_0^1 (1 + \delta t) f(t)\,dt - \frac{1}{\|f\|_1} \sum \int_0^1 \frac{(e^u - 1)}{\varrho} X^\varrho (1 + \delta t)^\varrho f(t)\,dt$$

$$- \frac{1}{2\|f\|_1} \int_0^1 \left( \log \left( 1 - (e^uX(1 + \delta t))^{-2} \right) - \log \left( 1 - (X(1 + \delta t))^{-2} \right) \right) f(t)\,dt.$$  \hfill (2.16)

Observe that the last term is $\geq -\frac{u}{X}$. We obtain

$$\frac{J_{b,u,X}}{(e^u - 1)X} \geq \frac{\int_0^1 (1 + \delta t) f(t)\,dt}{\|f\|_1} - \sum \left( \frac{(e^u - 1)}{\varrho} \int_0^1 (1 + \delta t)^\varrho f(t)\,dt \right) X^{\varrho q \varrho - 1} - \frac{u}{2(e^u - 1)X^2}.$$  \hfill (2.17)

We obtain some small savings by directly computing the first term whereas \textbf{[16] equation (13)} use the following bound in (2.16) instead:

$$\frac{\int_0^1 (1 + \delta t) f(t)\,dt}{\|f\|_1} \geq \frac{u}{e^u - 1}.$$  \hfill (2.18)

Let $s$ be a complex number. We denote $G_{m,\delta,u}(s)$ the summand

$$G_{m,\delta,u}(s) = \frac{(e^{us} - 1)}{(e^u - 1)s} \int_0^1 (1 + \delta t)^s f(t)\,dt,$$  \hfill (2.19)

and we rewrite inequality (2.16) as

$$\frac{J_{b,u,X}}{(e^u - 1)X} \geq G_{m,\delta,u}(1) - \sum \left| G_{m,\delta,u}(\varrho) \right| X^{\varrho q \varrho - 1} - \frac{u}{2(e^u - 1)X^2}.$$  \hfill (2.20)

Since the right term increases with $X$, we can replace $X$ with $X_0$ for $X \geq X_0$. Note that this is also the case for the other left term for

$$\frac{\omega}{(e^u - 1)\sqrt{X}} - 2(1 + \delta) \frac{\log(e^uX(1 + \delta))}{\log(X(e^u - 1))} \frac{\nu(f,a)}{\|f\|_1}.$$  \hfill (2.21)

For simplicity we denote

$$\Sigma = \Sigma_{m,\delta,u,X} = \sum \left| G_{m,\delta,u}(\varrho) \right| X^{\varrho q \varrho - 1}.$$  \hfill (2.22)

The following Proposition gives a first inequality in terms of the zeros of zeta and conditions on $m, u, \delta, a$ (and thus $\Delta$) so that $\theta(x) - \theta(x(1 - \Delta^{-1})) > 0$:  \hfill (2.23)
Proposition 2.3. Let \( m, u, \delta, a, \Delta, X_0 \) satisfy (2.1). If \( X \geq X_0 \) and
\[
G_{m,\delta,u}(1) - \sum_{m,\delta,u,X_0} \frac{u}{2(e^u - 1)} X_0^{-2} - \frac{\omega}{(e^u - 1)} X_0^{-1/2} - \frac{2\nu(f,a)(1+\delta)\log(e^uX_0(1+\delta))}{\|f\|_1} \log(X_0(e^u-1)) > 0, \tag{2.20}
\]
then there exists a prime number between \( x(1-\Delta^{-1}) \) and \( x \).

We are now going to make this Lemma more explicit by providing computable bounds for the sum over the zeros \( \Sigma_{m,\delta,u,X_0} \).

2.4. Evaluating \( G_{m,\delta,u} \). Let \( f \) be an \( m \)-admissible function over \([0,1]\). We recall the properties it entitles according to the definition of [16]:
- \( f \) is an \( m \)-times differentiable function,
- \( f^{(k)}(0) = f^{(k)}(1) = 0 \) for \( 0 \leq k \leq m-1 \),
- \( f \geq 0 \),
- \( f \) is not identically 0.

Let \( k = 0, \ldots, m, s = \sigma + i\tau \) be a complex number with \( \tau > 0, 0 \leq \sigma \leq 1 \). We denote
\[
F_{k,m,\delta} = \frac{\int_0^1 (1+\delta t)^{1+k}|f^{(k)}(t)|dt}{\|f\|_1}. \tag{2.21}
\]

We provide here finer estimates than [16] for \( G_{m,\delta,u} \). Observe that
\[
\left| \frac{e^{us} - 1}{s} \right| = \left| \int_1^u e^{xs} dx \right| \leq \int_1^u e^{x\sigma} dx = \frac{e^{u\sigma} - 1}{\sigma}, \tag{2.22}
\]
\[
\left| \frac{e^{us} - 1}{s} \right| \leq \frac{e^{u\sigma} + 1}{\tau}, \tag{2.23}
\]
and
\[
\left| \int_0^1 (1+\delta t)^{s} f(t)dt \right| \leq \frac{1}{\delta^{k+1}} F_{k,m,\delta}. \tag{2.24}
\]

We deduce easily bounds for \( G_{m,\delta,u}(s) \) by combining (2.22) and (2.24) with respectively \( k = 0, k = 1, k = m \), and lastly by combining (2.23) and (2.24) with \( k = m \):
\[
|G_{m,\delta,u}(s)| \leq F_{0,m,\delta} \frac{e^{u\sigma} - 1}{(e^u-1)\sigma}, \tag{2.25}
\]
\[
|G_{m,\delta,u}(s)| \leq F_{1,m,\delta} \frac{e^{u\sigma} - 1}{(e^u-1)\sigma \delta \tau}, \tag{2.26}
\]
\[
|G_{m,\delta,u}(s)| \leq F_{m,m,\delta} \frac{e^{u\sigma} - 1}{(e^u-1)\sigma \delta^m \tau m}, \tag{2.27}
\]
\[
|G_{m,\delta,u}(s)| \leq F_{m,m,\delta} \frac{e^{u\sigma} + 1}{(e^u-1)\delta^m \tau m + 1}. \tag{2.28}
\]

2.5. Zeros of the Riemann-zeta function. We denote each zero of zeta \( g = \beta + i\gamma \), \( N(T) \) the number of zeros in the rectangle \( 0 < \beta < 1, 0 < \gamma < T \), and \( N(\sigma_0,T) \) the number of those in the rectangle \( \sigma_0 < \beta < 1, 0 < \gamma < T \). We assume that we have the following information.

Theorem 2.4.
- (a) A numerical verification of the Riemann Hypothesis:
  There exists \( H > 2 \) such that if \( \zeta(\beta + i\gamma) = 0 \) at \( 0 \leq \beta \leq 1 \) and \( 0 \leq \gamma \leq H \), then \( \beta = 1/2 \).
(b) A direct computation of some finite sums over the first zeros:
Let \(0 < T_0 < H\) and \(S_0 > 0\) satisfy
\[
\sum_{0 < \gamma \leq T_0 \atop \beta = 1/2} 1 \leq N_0 = N(T_0),
\]
and
\[
\sum_{0 < \gamma \leq T_0 \atop \beta = 1/2} \frac{1}{\gamma} \leq S_0.
\]

(c) A zero-free region:
There exists \(R_0 > 0\) constant, such that \(\zeta(\sigma + it)\) does not vanish in the region
\[\sigma \geq 1 - \frac{1}{R_0 \log |t|} \text{ and } |t| \geq 2.\]

(d) An estimate for \(N(T)\):
There exist \(a_1, a_2, a_3\) positive constants such that, for all \(T \geq 2\),
\[
|N(T) - P(T)| \leq R(T),
\]
where \(P(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}, R(T) = a_1 \log T + a_2 \log \log T + a_3.\)

(e) An upper bound for \(N(\sigma_0, T)\):
Let \(3/5 < \sigma_0 < 1\). Then there exist \(c_1, c_2, c_3\) constants such that, for all \(T \geq H\),
\[
N(\sigma_0, T) \leq c_1 T + c_2 \log T + c_3.\]

Note that [16] did not use any information of the type (2.30), (2.31), or (2.33). Instead they used (2.29), the fact that all nontrivial zeros satisfied \(\beta < 1/2\), and the classical bound (2.32) for \(N(T)\) as given in [17][Theorem 19]. Our improvement will mainly come from using a new zero-density of the form of (2.33).

2.6. Evaluating the sum over the zeros \(\Sigma_{m,\delta,u,X_0}\): We assume Theorem 2.4. We split the sum \(\Sigma_{m,\delta,u,X_0}\) vertically at heights \(\gamma = 0\) (so as to use the symmetry with respect to the \(x\)-axis) and consider
\[
\tilde{G}_{m,\delta,u}(\beta + i\gamma) = |G_{m,\delta,u}(\beta + i\gamma)| + |G_{m,\delta,u}(\beta - i\gamma)|.
\]

We then split at \(\gamma = H\) (so as to take advantage of the fact that all zeros below this horizontal line satisfy \(\beta = 1/2\)), and again at \(\gamma = T_0\) and \(\gamma = T_1\) (where \(T_1\) will be chosen between \(T_0\) and \(H\)), and consider:
\[
\Sigma_0 = \sum_{0 < \gamma \leq T_0} \tilde{G}_{m,\delta,u}(1/2 + i\gamma) X_0^{-1/2},
\]
\[
\Sigma_1 = \sum_{T_0 < \gamma \leq T_1} \tilde{G}_{m,\delta,u}(1/2 + i\gamma) X_0^{-1/2},
\]
and
\[
\Sigma_2 = \sum_{T_1 < \gamma \leq H} \tilde{G}_{m,\delta,u}(1/2 + i\gamma) X_0^{-1/2}.
\]

For the remaining zeros (those with \(\gamma > H\)), we make use of the symmetry with respect to the critical line, and we split at \(\beta = \sigma_0\) for some fixed \(\sigma_0 > 1/2\) (we will consider \(9/10 \leq \sigma_0 \leq 99/100\) for our
computations). We denote
\[
\Sigma_3 = \sum_{\gamma > H \atop \beta = 1/2} \tilde{G}_{m,\delta,u}(1/2 + i\gamma)X_0^{-1/2} + \sum_{\gamma > H \atop 1/2 < \beta \leq \sigma_0} \left( \tilde{G}_{m,\delta,u}(\beta + i\gamma)X_0^{\beta - 1} + \tilde{G}_{m,\delta,u}(1 - \beta + i\gamma)X_0^{-\beta} \right),
\]
(2.37)
\[
\Sigma_4 = \sum_{\gamma > H \atop \sigma_0 < \beta < 1} \left( \tilde{G}_{m,\delta,u}(\beta + i\gamma)X_0^{\beta - 1} + \tilde{G}_{m,\delta,u}(1 - \beta + i\gamma)X_0^{-\beta} \right).
\]
(2.38)

As a conclusion, we have
\[
\Sigma_{m,\delta,u,X_0} = \Sigma_0 + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4.
\]
(2.39)

We state here some preliminary results (see [4, equations (2.18), (2.19), (2.20), (2.21), (2.26)]).

Lemma 2.5. Let \( T_0, H, R_0, \sigma_0 \) be as in Theorem 2.4. Let \( m \geq 2 \), \( X_0 > 10 \), and \( T_1 \) between \( T_0 \) and \( H \). We define
\[
S_1(T_1) = \left( \frac{1}{2\pi} + q(T_0) \right) \left( \log \frac{T_1}{T_0} \log \frac{\sqrt{T_1T_0}}{2\pi} \right) \frac{2R(T_0)}{T_0},
\]
(2.40)
\[
S_2(m,T_1) = \left( \frac{1}{2\pi} + q(T_1) \right) \left( \frac{1 + m \log T_1^2}{m^2T_1^m} - \frac{1 + m \log H^2}{m^2H^m} \right) + \frac{2R(T_1)}{T_1^{m+1}},
\]
(2.41)
\[
S_3(m) = \left( \frac{1}{2\pi} + q(H) \right) \left( \frac{1 + m \log H^2}{m^2H^m} \right) + \frac{2R(H)}{H^{m+1}},
\]
(2.42)
\[
S_4(m,\sigma_0) = \left( c_1 \left( \frac{1}{m} + \frac{1}{c_2 \log \frac{H}{H}} \right) + \left( c_3 + \frac{c_2}{m+1} \right) \frac{1}{H} \right) \frac{1}{H^m},
\]
(2.43)
\[
S_5(X_0, m, \sigma_0) = \left( c_1 + \frac{c_2 \log \frac{H}{H}}{H} + \frac{c_3}{H} + \left( c_1 + \frac{c_2}{H} \right) \frac{R_0}{2 \log X_0} \left( \frac{\sigma_0}{\log X_0} \right)^2 - \frac{1}{H^m} \right) \frac{1}{H^m}.
\]
(2.44)

We assume Theorem 2.4. Then
\[
\sum_{T_0 < \gamma \leq T_1} \frac{1}{\gamma} \leq S_1(T_1),
\]
(2.45)
\[
\sum_{T_1 < \gamma \leq H} \frac{1}{\gamma^{m+1}} \leq S_2(m, T_1),
\]
(2.46)
\[
\sum_{\gamma > H} \frac{1}{\gamma^{m+1}} \leq S_3(m),
\]
(2.47)
\[
\sum_{\gamma > H} \frac{1}{\gamma^{m+1}} \leq S_4(m, \sigma_0). \tag{2.48}
\]

Moreover, if \( \log X_0 < R_0m(\log H)^2 \), then
\[
\sum_{\gamma > H} \frac{X_0^{\gamma^{-1}}} {\gamma^{m+1}} \leq S_5(X_0, m, \sigma_0)X_0^{\gamma^{-1}(\log X_0) \frac{R_0 \log H}{\sigma_0 \log X_0}}.
\]
(2.49)
Lemma 2.6. Let \( m, \delta, X_0 \) satisfy (2.1). We assume Theorem 2.4 If \( \log X_0 < R_0 m (\log H)^2 \), then

\[
\Sigma_{m, \delta, u, X_0} \leq B_0(m, \delta)X_0^{-1/2} + B_1(m, \delta, T_1)X_0^{-1/2} + B_2(m, \delta, T_1)X_0^{-1/2} + B_3(m, \delta) \left( X_0^{-\sigma_0 - 1} + X_0^{-\sigma_0} \right) + B_{41}(X_0, m, \delta, \sigma_0)X_0^{-\frac{1}{R_0 \log H}} \]
\[
+ B_{42}(m, \delta, \sigma_0)X_0^{-1 + \frac{1}{R_0 \log H}}, \tag{2.50}
\]

where the \( B_i \)'s are defined in (2.51), (2.54), (2.58), (2.60), (2.62), and (2.63).

**Proof.** We investigate two ways to evaluate \( \Sigma_0 \) and \( \Sigma_1 \). For \( \Sigma_0 \), we can either combine (2.26) with (2.30) which computes \( \sum_{0 < \gamma \leq T_0} \gamma^{-1} \), or (2.25) with (2.29) which computes \( \sum_{0 < \gamma \leq T_0} 1 \). We denote

\[
B_0(m, \delta) = \min(\Sigma_{01}(m, \delta), \Sigma_{02}(m, \delta)), \tag{2.51}
\]

with

\[
\Sigma_{01}(m, \delta) = \frac{4F_{1,m,\delta}}{(e^{u/2} + 1)\delta} S_0 \quad \text{and} \quad \Sigma_{02}(m, \delta) = \frac{4F_{0,m,\delta}}{(e^{u/2} + 1)} N_0. \tag{2.52}
\]

We obtain

\[
\Sigma_0 \leq B_0(m, \delta)X_0^{-1/2}. \tag{2.53}
\]

For \( \Sigma_1 \), we can either combine (2.26) with the bound (2.45) for \( \sum_{T_0 < \gamma \leq T_1} \gamma^{-1} \), or (2.25) with the bound (2.32) for \( N(T) \) from Theorem 2.4. We denote

\[
B_1(m, \delta, T_1) = \min(\Sigma_{11}(m, \delta, T_1), \Sigma_{12}(m, \delta, T_1)), \tag{2.54}
\]

with

\[
\Sigma_{11}(m, \delta) = \frac{4F_{1,m,\delta}}{(e^{u/2} + 1)\delta} S_1(T_1), \quad \text{and} \quad \Sigma_{12}(m, \delta) = \frac{4F_{0,m,\delta}}{e^{u/2} + 1}(N(T_1) - N_0). \tag{2.55}
\]

We obtain

\[
\Sigma_1 \leq B_1(m, \delta, T_1)X_0^{-1/2}. \tag{2.56}
\]

It follows from (2.28) and (2.46) that

\[
\Sigma_2 \leq B_2(m, \delta, T_1)X_0^{-1/2}, \tag{2.57}
\]

with

\[
B_2(m, \delta, T_1) = \frac{2F_{m,m,\delta}}{(e^{u/2} - 1)\delta^m} S_2(m, T_1). \tag{2.58}
\]

We use (2.28) to bound \( \tilde{G} \) in \( \Sigma_3 \):

\[
\Sigma_3 \leq \frac{2F_{m,m,\delta}}{(e^{u} - 1)\delta^m} \sum_{1/2 \leq \gamma \leq \gamma_0 \leq \sigma_0} \frac{(e^{u\beta} + 1)X_0^{\beta - 1} + (e^{u(1-\beta)} + 1)X_0^{-\beta}}{\gamma^{m+1}}. \tag{2.59}
\]

Note that since \( \log X_0 > u \), then \( (e^{u\beta} + 1)X_0^{\beta - 1} + (e^{u(1-\beta)} + 1)X_0^{-\beta} \) increases with \( \beta \geq 1/2 \). Moreover, we use (2.47) to bound the sum \( \sum_{\gamma \geq H \gamma^{-\beta} \geq \frac{1}{2}} \), and obtain

\[
\Sigma_3 \leq B_3(m, \delta, \sigma_0)X_0^{\sigma_0 - 1} + B_3(m, \delta, 1 - \sigma_0)X_0^{-\sigma_0}, \tag{2.59}
\]

where

\[
B_3(m, \delta, \sigma) = \frac{2F_{m,m,\delta} e^{u\sigma} + 1}{\delta^m e^{u} - 1} S_3(m). \tag{2.60}
\]
For $\Sigma_4$ we use again (2.28) to bound $\tilde{G}$ and the fact that $X_0^{\beta-1} + X_0^\beta$ increases with $\beta$. Since $\beta \leq 1 - \frac{1}{\gamma_0\log \gamma}$ and $\gamma > H$ we obtain

$$\Sigma_4 \leq \frac{2(e^n + 1)F_{m,m,\delta}}{(e^n - 1)\delta m} \left( \sum_{\gamma > H} \frac{X_0^{-\frac{1}{\gamma_0\log \gamma}}} {\gamma^{m+1}} + X_0^{-1+\frac{1}{\gamma_0\log \gamma}} \sum_{\gamma > H} \frac{1}{\gamma^{m+1}} \right).$$

We apply (2.48) and (2.49) to bound the above sums over the zeros and obtain

$$\Sigma_4 \leq B_{41}(X_0, m, \delta, \sigma_0)X_0^{-\frac{1}{\gamma_0\log(H)}} + B_{42}(m, \delta, \sigma_0)X_0^{-1+\frac{1}{\gamma_0\log \gamma}}, \quad (2.61)$$

with

$$B_{41}(X_0, m, \delta, \sigma_0) = \frac{2(e^n + 1)F_{m,m,\delta}}{(e^n - 1)\delta m}S_5(X_0, m, \sigma_0), \quad (2.62)$$

$$B_{42}(X_0, m, \delta, \sigma_0) = \frac{2(e^n + 1)F_{m,m,\delta}}{(e^n - 1)\delta m}S_4(m, \sigma_0). \quad (2.63)$$

Note that $G_{m,\delta,u}(1) = F_{0,m,\delta}$. Finally we apply Proposition 2.3 and Lemma 2.6.

2.7. Main Theorem.

**Theorem 2.7.** Let $m, u, \delta, a, \Delta, X_0$, and $x$ satisfy (2.1). Let $T_0, H, R_0, \sigma_0$ be as in Theorem 2.4. We assume Theorem 2.4. If $X \geq X_0$ and

$$F_{0,m,\delta} - B_0(m, \delta)X_0^{-1/2} - B_1(m, \delta, T_1)X_0^{-1/2} - B_2(m, \delta, T_1)X_0^{-1/2} - B_3(m, \delta, \sigma_0)X_0^{-\sigma_0-1} - B_3(m, \delta, 1 - \sigma_0)X_0^{-\sigma_0} - B_{41}(X_0, m, \delta, \sigma_0)X_0^{-1+\frac{1}{\gamma_0\log \gamma}} - B_{42}(m, \delta, \sigma_0)X_0^{-1+\frac{1}{\gamma_0\log \gamma}} - \frac{u}{2(e^n - 1)}X_0^{-2} - \frac{\omega}{(e^n - 1)}X_0^{-1/2} - \frac{2\nu(f,a)(1+\delta)}{\|f\|_1} \log (e^nX_0(1+\delta)) + \frac{\log \beta}{\log \gamma} > 0, \quad (2.64)$$

then there exists a prime number between $x(1-\Delta^{-1})$ and $x$.

3. Computations.

3.1. Introducing the Smooth Weight $f$. We choose the same weight as [16], that is

$$f_m(t) = (4t(1 - t))^m \text{ if } 0 \leq t \leq 1, \text{ and } 0 \text{ otherwise.}$$

We proved in [3] that a primitive of $f_m$ was providing a close to optimum weight to estimate $\psi(x)$. Thus we believe that the above weight should also be close to optimal to evaluate $\psi(y) - \psi(x)$ when $y$ is close to $x$. We recall [16] Lemma 6:

$$\|f_m\|_1 = \frac{2^{2m}(ml)!!}{(2m + 1)!}, \quad (3.1)$$

$$\|f_m^{(m)}\|_2 = \frac{2^{2m}m!}{\sqrt{2m + 1}}. \quad (3.2)$$

We now provide estimates for $F_{k,m,\delta}$ as defined in (2.21).
Lemma 3.1. Let $m \geq 2$, $\delta > 0$, and $0 < \sigma < 1$. We define
\[
\lambda_0(m, \delta) = \frac{(2m + 1)!}{2^{2m-1}(m!)^2},
\]
\[
\lambda_1(m, \delta) = \frac{(1 + \delta)(2m + 1)!}{2^{2m-1}(m!)^2},
\]
\[
\lambda(m, \delta) = \sqrt{\frac{(1 + \delta)^{2m+3} - 1}{\delta(2m + 3)}} \frac{(2m + 1)!}{m!\sqrt{2m + 1}}.
\]

Then
\[
1 \leq F_{0,m,\delta} \leq 1 + \delta,
\]
\[
\lambda_0(m, \delta) \leq F_{1,m,\delta}(\sigma) \leq \lambda_1(m, \delta),
\]
\[
F_{m,m,\delta}(\sigma) \leq \lambda(m, \delta).
\]

Proof. Inequalities (3.3) follow trivially from the fact $1 \leq (1 + \delta t) \leq 1 + \delta$.

To bound $F_{1,m,\delta}$, we note that
\[
\frac{\|f_m'\|_1}{\|f_m\|_1} \leq F_{1,m,\delta} \leq \frac{(1 + \delta)^2\|f_m'\|_1}{\|f_m\|_1}.
\]

Since $f_m'(t)$ has same sign as $1 - 2t$, we have
\[
\|f_m'\|_1 = \int_1^{1/2} f_m'(t)dt - \int_{1/2}^1 f_m'(t)dt = 2f_m(1/2) - f_m(0) - f_m(1) = 2.
\]

This together with (3.1) achieves to prove (3.4).

Lastly, for $F_{m,m,\delta}$, we apply (3.2) together with the Cauchy-Schwarz inequality:
\[
F_{m,m,\delta}(\sigma) \leq \frac{\sqrt{\int_0^1 (1 + \delta t)^{2(m+1)}dt \int_0^1 |f_m^{(m)}(t)|^2dt}}{\|f_m\|_1} = \sqrt{\frac{(1 + \delta)^{2m+3} - 1}{\delta(2m + 3)} \frac{\|f_m^{(m)}\|_2}{\|f_m\|_1}}.
\]

Note that while $F_{0,m,\delta}$ and $F_{1,m,\delta}$ can be easily computed as integrals, it is not the case for $F_{m,m,\delta}$.

The following observation helps us to compute $F_{m,m,\delta}$ directly. We recognize in the definition of $f_m^{(m)}$ the analogue of Rodrigues’ formula for the shifted Legendre polynomials:
\[
f_m^{(m)}(t) = 4^m m! P_m(1 - 2t),
\]
where $P_m(x)$ is the $m^{th}$ Legendre polynomial, and
\[
P_m(1 - 2t) = (-1)^m \sum_{k=0}^{m} \binom{m}{k} \binom{m+k}{k} (-t)^k.
\]

For each each $P_m(1 - 2t)$, we denote $r_{j,m}$, with $j = 0, \ldots, m$, its $m + 1$ roots. Since $P_m(1 - 2t)$ alternates sign between each of them, we have
\[
F_{m,m,\delta} = \frac{\int_0^1 (1 + \delta t)^{m+1} |P_m(1 - 2t)|dt}{\|f\|_1} = \frac{1}{\|f\|_1} \sum_{j=0}^{m-1} (-1)^j \int_{r_j}^{r_{j+1}} (1 + \delta t)^{m+1} P_m(1 - 2t)dt,
\]
and GP-Pari is able to compute quickly this sum of polynomial integrals.
3.2. **Explicit results about the zeros of the Riemann zeta function.** We provide here the latest values for the constants appearing in Theorem 2.4.

**Theorem 3.2.**

(a) *A numerical verification of the Riemann Hypothesis (Platt [15]):*

\[ H = 3.061 \cdot 10^{10} \]

(b) *A direct computation of some finite sums over the first zeros (using A. Odlyzko’s list of zeros):*

For \( T_0 = 1132491 \), \( N_0 = N(T_0) = 2001052 \), and \( S_0 = 11.637732363 \).

(c) *A zero-free region (Kadiri [8, Theorem 1.1]):*

\[ R_0 = 5.69693 \]

(d) *An estimate for \( N(T) \) (Rosser [17, Theorem 19]):*

\[ a_1 = 0.137, \ a_2 = 0.443, \ a_3 = 1.588. \]

(e) *An upper bound for \( N(\sigma,T) \) (Kadiri [10]): For all \( T \geq H \),

\[ N(\sigma,T) \leq c_1T + c_2\log T + c_3, \]

where the \( c_i \)'s are given in Table 1.

| \( \sigma \) | \( c_1 \) | \( c_2 \) | \( c_3 \) |
|---|---|---|---|
| 0.90 | 5.8494 | 0.4659 | \(-1.7905 \cdot 10^{11}\)|
| 0.91 | 5.6991 | 0.4539 | \(-1.7444 \cdot 10^{11}\)|
| 0.92 | 5.5564 | 0.4426 | \(-1.7007 \cdot 10^{11}\)|
| 0.93 | 5.4206 | 0.4318 | \(-1.6592 \cdot 10^{11}\)|
| 0.94 | 5.2913 | 0.4215 | \(-1.6196 \cdot 10^{11}\)|
| 0.95 | 5.1680 | 0.4116 | \(-1.5819 \cdot 10^{11}\)|
| 0.96 | 5.0503 | 0.4023 | \(-1.5458 \cdot 10^{11}\)|
| 0.97 | 4.9379 | 0.3933 | \(-1.5114 \cdot 10^{11}\)|
| 0.98 | 4.8304 | 0.3848 | \(-1.4785 \cdot 10^{11}\)|
| 0.99 | 4.7274 | 0.3766 | \(-1.4470 \cdot 10^{11}\)|

Note that [17, Theorem 19] was recently improved by T. Trudgian in [19, Corollary 1] with \( a_1 = 0.111, \ a_2 = 0.275, \ a_3 = 2.450 \). Our results are valid with either Rosser’s or Trudgian’s bounds.

3.3. **Understanding the contribution of the low lying zeros.** We assume Theorem 3.2 and that

\[ m \geq m_0 = 5, \delta < \delta_0 = 2 \cdot 10^{-8}, \text{ and } T_1 = 10^9 \] \hspace{1cm} (3.6)

(this would be consistent with the values we choose in Table 2). We observe that

\[ B_0(m, \delta) = \Sigma_{02} \text{ and } B_1(m, \delta,T_1) = \Sigma_{12}. \]

where \( \Sigma_{02} \) and \( \Sigma_{12} \) are defined in (2.52) and (2.55) respectively. In other words, it turns out that we obtain a smaller bound for the sum over the small zeros \( (0 < \gamma < T) \) by using \( N(T) \) directly instead of evaluating
\[ \sum_{0 \leq \gamma \leq T} \gamma^{-1}. \] This essentially comes from the fact that our choice of parameters insures us with \( \delta \ll \frac{F_{1,m,\delta} S_0}{F_{0,m,\delta} N_0} \) and \( \delta \ll \frac{F_{1,m,\delta} S_1(T)}{F_{0,m,\delta} N(T_1) N_0} \). We first prove the inequality

\[ \frac{S_1(t)}{N(t)} \geq c_0 \log \frac{t}{t}. \quad (3.7) \]

**Proof.** We denote

\[
\begin{align*}
\text{Proof. We denote} & \quad w_1 = \frac{1}{2} \left( \frac{1}{2\pi} + q(T_0) \right) = 0.0795 \ldots, \quad w_2 = -\log(2\pi) \left( \frac{1}{2\pi} + q(T_0) \right) = -0.2925 \ldots, \\
& \quad w_3 = \left( \frac{1}{2\pi} + q(T_0) \right) \left( \frac{-\log^2(T_0)}{2} + \log(T_0) \log(2\pi) \right) + \frac{2R(T_0)}{T_0} = -11.3860 \ldots, \\
& \quad v_1 = \frac{1}{2\pi} = 0.1591 \ldots, \quad v_2 = -\frac{\log(2\pi)}{2\pi} - 1 = -1.2925 \ldots, \quad v_3 = a_1 = 0.137, \\
& \quad v_4 = a_2 = 0.443, \quad v_5 = a_3 + \frac{7}{8} = 2.463.
\end{align*}
\]

and

\[ S_1(t) = w_1 (\log t)^2 + w_2 \log t + w_3, \quad P(t) + R(t) = v_1 t \log t + v_2 t + v_3 \log t + v_4 \log \log t + v_5. \]

We have from (2.40) and Theorem 3.2 (d) that

\[ \frac{S_1(t)}{N(t)} \geq \frac{S_1(t)}{P(t) + R(t)} = \frac{w_1 (\log t)^2 + w_2 \log t + w_3}{v_1 t \log t + v_2 t + v_3 \log t + v_4 \log \log t + v_5}. \]

Since \( t > t_1 = 10^9 \), we deduce the bound

\[ \frac{S_1(t)}{N(t)} \geq c_0 \log \frac{t}{t}, \quad (3.8) \]

where

\[ c_0 = \frac{w_1 + \frac{w_2}{\log t_1} + \frac{w_3}{(\log t_1)^2} \log t_1}{v_1 + \frac{v_2}{t_1 \log t_1} + \frac{v_3}{t_1 \log t_1} + \frac{v_4}{t_1 \log t_1} + \frac{v_5}{t_1 \log t_1}} \geq 0.7508. \quad (3.9) \]

We now establish that \( \Sigma_0 + \Sigma_{11}, \Sigma_0 + \Sigma_{12}, \) and \( \Sigma_0 + \Sigma_{11} \) are all larger than \( \Sigma_{02} + \Sigma_{12} \). We make use of Lemma 3.1 to provide estimates for the \( F_{k,m,\delta} \)'s of (3.8), and of the assumptions (3.6) on \( m, \delta, T_1 \). 

**Proof.** We have

\[ (\Sigma_0 + \Sigma_{11}) - (\Sigma_0 + \Sigma_{12}) = \frac{4}{e^{u/2} + 1} \left( \frac{F_{1,m,\delta}}{\delta} (S_0 + S_1(T_1)) - F_{0,m,\delta} N(T_1) \right) \]

\[ > \frac{4(1 + \delta) N(T_1)}{e^{u/2} + 1} \left( \frac{(2m_0 + 1)!}{2^{2m_0 - 1} \delta_0 (1 + \delta_0)} \frac{1}{P(t_1) + R(t_1)} - \frac{S_0}{c_0 \log t_1} 1 - 1 \right) > 0, \]

since the right term between brackets is \( > 2.4796 - 1 > 0 \). We have

\[ (\Sigma_0 + \Sigma_{12}) - (\Sigma_0 + \Sigma_{11}) = \left( \frac{S_0}{\delta} F_{1,m,\delta} - N_0 F_{0,m,\delta} \right) \frac{4}{e^{u/2} + 1} \]

\[ > \frac{4(1 + \delta) N_0}{e^{u/2} + 1} \left( \frac{(2m_0 + 1)!}{2^{2m_0 - 1} \delta_0 (1 + \delta_0) N_0} - 1 \right) > 0 \]
since the right term between brackets is $> 1574 - 1$. Finally,

$$(\Sigma_{02} + \Sigma_{11}) - (\Sigma_{02} + \Sigma_{12}) = \frac{4}{e^{\pi/2} + 1} \left( \frac{F_{1,m,\delta}}{\delta} S_1(T_1) - F_{0,m,\delta}(N(T_1) - N_0) \right)$$

$$> \frac{4(1 + \delta)(N(T_1) - N_0)}{e^{\pi/2} + 1} \left( \frac{(2m_0 + 1)!}{2^{2m_0 - 1}(m_0)!} \delta_0(1 + \delta_0) \left( \frac{S_1(t_1)}{c_0 \log t_1} - N_0 \right) - 1 \right) > 0$$

since the right term between brackets is $> 1.3737 - 1$. \qed

The values for $T_1$ and $a$ given in the next table are rounded down to the last digit.

**Table 2.** For all $x \geq x_0$, there exists a prime between $x(1 - \Delta^{-1})$ and $x$.

| log $x_0$ | $m$ | $\delta$ | $T_1$ | $\sigma_0$ | $a$ | $\Delta$ |
|-----------|-----|----------|-------|------------|-----|---------|
| log($4 \cdot 10^{18}$) | 5   | 3.580 $\cdot 10^{-8}$ | 272 519 712 | 0.92 | 0.2129 | 36 082 898 |
| 43        | 5   | 3.349 $\cdot 10^{-8}$  | 291 316 980 | 0.92 | 0.2147 | 38 753 947 |
| 44        | 6   | 2.330 $\cdot 10^{-8}$  | 488 509 984 | 0.92 | 0.2324 | 61 162 616 |
| 45        | 7   | 1.628 $\cdot 10^{-8}$  | 797 398 875 | 0.92 | 0.2494 | 95 381 241 |
| 46        | 8   | 1.134 $\cdot 10^{-8}$  | 1 284 120 197 | 0.92 | 0.2651 | 148 306 019 |
| 47        | 9   | 8.080 $\cdot 10^{-9}$  | 1 996 029 891 | 0.92 | 0.2836 | 227 619 375 |
| 48        | 11  | 6.000 $\cdot 10^{-9}$  | 3 204 848 430 | 0.93 | 0.3050 | 346 582 570 |
| 49        | 15  | 4.682 $\cdot 10^{-9}$  | 5 415 123 831 | 0.93 | 0.3275 | 518 958 776 |
| 50        | 20  | 3.889 $\cdot 10^{-9}$  | 8 466 793 105 | 0.93 | 0.3543 | 753 575 355 |
| 51        | 28  | 3.625 $\cdot 10^{-9}$  | 12 399 463 961 | 0.93 | 0.3849 | 1 037 917 449 |
| 52        | 39  | 3.803 $\cdot 10^{-9}$  | 16 139 006 408 | 0.93 | 0.4127 | 1 313 524 036 |
| 53        | 48  | 4.088 $\cdot 10^{-9}$  | 18 290 358 817 | 0.93 | 0.4301 | 1 524 171 138 |
| 54        | 54  | 4.311 $\cdot 10^{-9}$  | 19 412 056 863 | 0.93 | 0.4398 | 1 670 398 039 |
| 55        | 56  | 4.386 $\cdot 10^{-9}$  | 19 757 119 193 | 0.93 | 0.4445 | 1 770 251 249 |
| 56        | 59  | 4.508 $\cdot 10^{-9}$  | 20 210 075 547 | 0.93 | 0.4481 | 1 838 818 070 |
| 57        | 59  | 4.506 $\cdot 10^{-9}$  | 20 219 045 843 | 0.93 | 0.4496 | 1 886 389 443 |
| 58        | 61  | 4.590 $\cdot 10^{-9}$  | 20 495 459 359 | 0.93 | 0.4514 | 1 920 768 795 |
| 59        | 61  | 4.589 $\cdot 10^{-9}$  | 20 499 925 573 | 0.93 | 0.4522 | 1 946 282 821 |
| 60        | 61  | 4.588 $\cdot 10^{-9}$  | 20 504 393 735 | 0.93 | 0.4527 | 1 966 196 911 |
| 150       | 64  | 4.685 $\cdot 10^{-9}$  | 21 029 543 983 | 0.96 | 0.4641 | 2 442 159 714 |

($\log(4 \cdot 10^{18}) = 42.8328 \ldots$)

3.4. **Verification of the Ternary Goldbach conjecture.**

**Proof of Corollary [L2]** Let $N = 4 \cdot 10^{18}$. We follow Oliveira e Silva, Herzog and Pardi [14]'s argument where the authors computed all the prime gaps up to $4 \cdot 10^{18}$. From Table 2, we have that for $x = e^{90}$ and $\Delta = 1 966 090 061$, there exists at least one prime in the interval $(x - x/\Delta, x]$. This one has length $5.8082 \cdot 10^{16}$. Then $N\Delta = 7.8647 \cdot 10^{22}$ and we may infer that the gap between consecutive primes up to $N\Delta$ can be no larger than $N$ (since $N\Delta/\Delta = N$). The corollary follows by using all the odd primes up to $N\Delta$ to extend the minimal Goldbach partitions of 4, 6, \ldots, $N$ up to $N\Delta$ (the method of computation is explained in [14 Section 1]). We also note that $N + 2 = 211 + (N - 209)$ and $N + 4 = 313 + (N - 309)$, where 211, 313, $N - 209$, and $N - 309$ are all prime. Thus, there is at least one way to write each odd number greater than 5 and smaller than $N\Delta$ as the sum of at most 3 primes. \qed
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