Sharp Semiclassical Estimates for the Number of Eigenvalues Below a Degenerate Critical Level

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Abstract: We consider the semiclassical asymptotic behaviour of the number of eigenvalues smaller than \( E \) for elliptic operators in \( L^2(\mathbb{R}^d) \). We describe a method of obtaining remainder estimates related to the volume of the region of the phase space in which the principal symbol takes values belonging to the intervals \([E'; E' + h]\), where \( E' \) is close to \( E \). This method allows us to derive sharp remainder estimates \( O(h^{1-d}) \) for a class of symbols with critical points and non-smooth coefficients.

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1 Introduction

We assume that for \( h \in [0; h_0] \) the differential operators \( A_h = a(x, hD, h) \) are self-adjoint in \( L^2(\mathbb{R}^d) \) and the symbol \( a(x, \xi, h) = \sum_{0 \leq n \leq N} h^na_n(x, \xi) \) is sufficiently regular. If \( E \in \mathbb{R} \) satisfies

\[
E < \lim \inf_{|x| + |\xi| \to \infty} a_0(x, \xi)
\]  

(1.1)

and \( h_0 \) is small enough, then the spectrum of \( A_h \) is discrete in \( ]-\infty; E] \) and it is natural to ask whether the counting function \( \mathcal{N}(A_h, E) \) (i.e. the number of eigenvalues smaller than \( E \) counted with their multiplicities) satisfies the
semiclassical asymptotic formula

\[ N(A_h, E) = (2\pi h)^{-d} c_E + O(h^{\mu-d}) \text{ as } h \to 0, \quad (1.2) \]

where \( \mu > 0 \) and

\[ c_E = \int_{a_0(x,\xi) < E} dx d\xi = \text{vol} \{ (x,\xi) \in \mathbb{R}^{2d} : a_0(x,\xi) < E \}. \quad (1.2') \]

The most powerful approach of studying semiclassical asymptotics has its origin in the microlocal analysis of L. Hörmander [12]. Since the first papers of J. Chazarain [7] and B. Helffer, D. Robert [10], this approach has been used in numerous works, cf. the monographs [9], [15], [21]. A basic result says that (1.2) holds with \( \mu = 1 \) if \( E \) is not a critical value of \( a_0 \) [i.e. \( a_0(x,\xi) = E \Rightarrow \nabla a_0(x,\xi) \neq 0 \)] and we refer to the papers T. Paul, A. Uribe [20] and M. Combescure, J. Ralston, D. Robert [8], giving more precise estimates in relation with the periodic orbits of the Hamiltonian flow of \( a_0 \).

In this paper we investigate the case when the critical set

\[ C_{E}^{a_0} = \{ (x,\xi) \in \mathbb{R}^{2d} : a_0(x,\xi) = E \text{ and } \nabla a_0(x,\xi) = 0 \} \quad (1.3) \]

is not empty and we consider elliptic operators with non-smooth coefficients. Below we enumerate different methods and works treating this problem.

– The analysis of oscillatory integrals. If \( C_{E}^{a_0} \) is a smooth manifold and the Hessian matrix of \( a_0 \) is transversely non-degenerate, then the semiclassical spectral asymptotics can be obtained from the analysis described in the paper of R. Brummelhuis, T. Paul, A. Uribe [2]. This approach was developed to study the contribution of periodic orbits under some geometrical assumptions on the flow (cf. D. Khuat-Duy [18], B. Camus [3, 4]) and recent results of B. Camus [5, 6] concern the case of a totally degenerate critical point of \( a_0 \). The oscillatory integrals being degenerate in the case of a degenerate Hessian matrix, the principal difficulty of this approach appears in suitable generalizations of the stationary phase method.

– The multiscale analysis developed by V. Ivrii [15]. This method was extended to treat elliptic operators with non-smooth coefficients in the paper V. Ivrii [17] (cf. also V. Ivrii [16] and M. Bronstein, V. Ivrii [1]).

– The approximative spectral projector method of M. A. Shubin, V. A. Turovskii [22]. The application of this method to our problem was described
in the monograph of S. Z. Levendorskii [19] and it gives remainder estimates involving the volume of suitable regions determined by \( a_0 \) in the phase space valid without any additional assumptions on the Hessian matrix of \( a_0 \). After the improvement of L. Hörmander [13], for every \( \mu < \frac{2}{3} \) one can find a constant \( C_\mu > 0 \) such that for \( h \in ]0; h_0[ \) one has

\[
\left| \mathcal{N}(A_h, E) - (2\pi h)^{-d} c_E \right| \leq C_\mu h^{-d} \mathcal{R}_{E}^{a_0}(h^\mu)
\]

where

\[
\mathcal{R}_{E}^{a_0}(h^\mu) = \text{vol} \left\{ (x, \xi) \in \mathbb{R}^{2d} : |a_0(x, \xi) - E| \leq h^\mu \right\}.
\]

The method of integrations by parts used in [23, 26]. This method allowed us to show that the estimates (1.4) are still valid for \( \mu < 1 \).

In this paper we show how to generalize the method of [26] to recover estimates with \( \mu = 1 \). Our aim is to show that for every \( \varepsilon > 0 \) it is possible to find a constant \( C_\varepsilon > 0 \) such that for \( h \in ]0; h_0[ \) one has

\[
\left| \mathcal{N}(A_h, E) - (2\pi h)^{-d} c_E \right| \leq C_\varepsilon h^{-d} \mathcal{R}_{E}^{c,a_0}(h),
\]

where

\[
\mathcal{R}_{E}^{c,a_0}(h) = h + \sup_{E' \in [E-h^{1-\varepsilon}; E+h^{1-\varepsilon}]} \text{vol} \left\{ (x, \xi) \in \mathbb{R}^{2d} : |a_0(x, \xi) - E'| \leq h \right\}.
\]

Using a regularization procedure similarly as in [24, 26] we can show that these estimates are valid for elliptic operators with coefficients which have second order derivatives Hölder continuous.

It is easy to see that one can always find constants \( C, c > 0 \) such that

\[
\text{vol} \left\{ (x, \xi) \in \mathbb{R}^{2d} : |a_0(x, \xi) - E'| \leq h \right\} \leq C h^c
\]

and the asymptotic formula (1.2) holds with \( \mu = 1 \) if additional properties of \( a_0 \) ensure the estimate (1.6) with \( c = 1 \) for \( E' \in [E-h^{1-\varepsilon}; E+h^{1-\varepsilon}] \).

The main part of this paper is devoted to the proof of a microlocal trace formula in the region

\[
\{(x, \xi) \in \mathbb{R}^{2d} : |\nabla a_0(x, \xi)| \geq h^{\delta_0}\}
\]

where \( \delta_0 \in ]0; 1/2[ \). This result allows us to derive the asymptotic formula (1.5) under the assumption that the Hessian matrix of \( a_0 \) is of rank \( \geq 2 \).
Indeed, it is easy to see (cf. Section 6) that the last assumption ensures the fact that the volume of
\[ \{(x, \xi) \in \mathbb{R}^{2d} : |\nabla a_0(x, \xi)| \leq h^{\delta_0}\} \]

is \(o(h)\) if \(\frac{1}{2} - \delta_0\) is small enough and the corresponding contribution can be included in the right hand side of (1.5) due to (1.6). The next paper [27] will present a trace formula in the region (1.7′), completing the proof of (1.5) without any hypotheses on the Hessian of \(a_0\).

Assume \(0 < r_0 < 1\). We write \(a \in C^{2+r_0}_b(\mathbb{R}^d)\) if and only if the function \(a : \mathbb{R}^d \to \Phi\) satisfies the following conditions
\[
\partial^\alpha a \in L^\infty(\mathbb{R}^d) \text{ if } |\alpha| \leq 2, \quad (1.8)
\]
\[
|\partial^\alpha a(x) - \partial^\alpha a(y)| \leq C|x - y|^{r_0} \text{ if } |\alpha| = 2, \quad x, y \in \mathbb{R}^d. \quad (1.8')
\]

Let \(m_0 \in \mathbb{N}\) and for \(\nu, \bar{\nu} \in \mathbb{N}^d, |\nu|, |\bar{\nu}| \leq m_0\) let \(a_{\nu, \bar{\nu}} = a_{\nu, \bar{\nu}} \in C^{2+r_0}_b(\mathbb{R}^d)\) be real-valued and such that
\[
\sum_{|\nu| = |\bar{\nu}| = m_0} a_{\nu, \bar{\nu}}(x)\xi^{\nu+\bar{\nu}} \geq c_0|\xi|^{2m_0} \quad (x, \xi \in \mathbb{R}^d) \quad (1.9)
\]
holds for a certain constant \(c_0 > 0\). Let \(A_h\) be the quadratic form
\[
A_h[\varphi, \psi] = \sum_{|\nu|, |\bar{\nu}| \leq m_0} (a_{\nu, \bar{\nu}}(hD)^\nu \varphi, (hD)^\bar{\nu} \psi), \quad (1.10)
\]
where \(\varphi, \psi \in C^{m_0}_0(\mathbb{R}^d), (hD)^\nu = (-ih)^{|\nu|} \frac{\partial^\nu}{\partial x^\nu}\) and \((\cdot, \cdot)\) is the scalar product of \(L^2(\mathbb{R}^d)\). Due to (1.9), \(A_h\) is bounded from below and its closure defines a self-adjoint operator \(A_h\). Usually \(A_h\) is expressed formally as
\[
A_h = \sum_{|\nu|, |\bar{\nu}| \leq m_0} (hD)^\nu (a_{\nu, \bar{\nu}}(hD)^\nu). \quad (1.10')
\]
Moreover we denote
\[
a_0(x, \xi) = \sum_{|\nu|, |\bar{\nu}| \leq m_0} a_{\nu, \bar{\nu}}(x)\xi^{\nu+\bar{\nu}}. \quad (1.11)
\]

Then we have
Theorem 1.1 Let \( a_{\nu, \bar{\nu}} \in \mathcal{C}^{2+\tau_0}_{r, 0}(\mathbb{R}^d) \) be such that (1.9) holds and let \( A_h \) be self-adjoint operators in \( L^2(\mathbb{R}^d) \) defined by (1.10). Let \( E \in \mathbb{R} \) be such that (1.1) holds (with \( a_0 \) given by (1.11)) and let \( h_0 > 0 \) be small enough.

a) If \( h \in ]0; h_0[ \) then the spectrum of \( A_h \) is discrete in \( ]-\infty; E[ \).

b) If the dimension \( d \geq 3 \), then for every \( \varepsilon > 0 \) one can find a constant \( C_\varepsilon > 0 \) such that (1.5) holds for \( h \in ]0; h_0[ \).

In this paper we show

Theorem 1.2 Let \( A_h, a_0 \) and \( E \) satisfy the assumptions of Theorem 1.1. Assume moreover that the rank of the Hessian matrix of \( a_0 \) is greater or equal 2 at every point of the critical set \( \mathcal{C}_{a_0} \). If the dimension \( d \geq 2 \), then for every \( \varepsilon > 0 \) one can find a constant \( C_\varepsilon > 0 \) such that (1.5) holds for \( h \in ]0; h_0[ \).

The proof of Theorem 1.2 presented in this paper will be used in [27] to derive Theorem 1.1.

Remark. More general behaviour of coefficients can be considered for \( x \) such that \( a_0(x, \xi) \geq E_0 > E \) holds for all \( \xi \in \mathbb{R}^d \). In particular we have assumed that the coefficients \( a_{\nu, \bar{\nu}} \) are bounded for sake of simplicity, but the same results hold for tempered variations models (cf. B. Helffer, D. Robert [11]).

Plan of the proof. We begin Section 2 by a description of the regularization of non-smooth coefficients. It allows us to define the operators \( P_h \) with smooth coefficients and Theorems 1.1, 1.2, can be deduced from a suitable microlocal trace formula for \( P_h \). The proof of the trace formula is based on the analysis of the evolution group \( \exp(itP_h/h) \) and its approximation is described in Section 3.

At the beginning of Section 4 we apply the integration by parts to check the correct trace asymptotics of the approximation constructed in Section 3. It remains to control the difference between \( \exp(itP_h/h) \) and its approximation. Our reasoning is divided in two steps. In Section 4 we observe that for every \( \varepsilon > 0 \) one can obtain suitable estimates for \( |t| \leq h^{\varepsilon} \) similarly as in [26]. In Section 5 we use a property of the wave front propagation to show that the contribution of the region (1.7) is negligible if \( |t| \geq h^{\varepsilon} \) and \( \varepsilon + \delta_0 < \frac{1}{2} \). In Section 6 we complete the proof estimating the volume of the region (1.7').
2 Regularized problem

2.1 Description of smooth operators

We assume $\frac{1}{2+r_0} < \delta_0 < \frac{1}{2}$. Let $\gamma \in C_0^\infty(\mathbb{R}^d)$ be such that $\int \gamma(x) \, dx = 1$ and $\int x^\alpha \gamma(x) \, dx = 0$ for $\alpha \in \mathbb{N}^d$ satisfying $1 \leq |\alpha| \leq 2$. We introduce $h$-dependent regularization of coefficients

$$ a_{\nu,\bar{\nu},h}(x) = \int_{\mathbb{R}^d} a_{\nu,\bar{\nu}}(y) \gamma(h^{-\delta_0}(x-y)) \, dy $$

and define formally self-adjoint differential operators

$$ P^\pm_h = \sum_{|\nu|,|\bar{\nu}| \leq m_0} (hD)\nu (a_{\nu,\bar{\nu},h}(x)(hD)\bar{\nu}) \pm h(I - h^2 \Delta)^{m_0}. $$

We write $P^\pm_h$ in the standard form

$$ P^\pm_h = \sum_{|\nu| \leq 2m_0} p^\pm_{\nu,h}(x)(hD)\nu $$

and we use the standard notation $P^\pm_h = p^\pm_h(x, hD)$ with

$$ p^\pm_h(x, \xi) = \sum_{|\nu| \leq 2m_0} p^\pm_{\nu,h}(x)\xi^\nu. $$

In Section 7 we check the following properties:

**Lemma 2.1** Let $a_0$, $A_h$ be as in Theorem 1.1 and $P^\pm_h$, $p^\pm_h$ as above.

(a) The estimates

$$ |\partial_x^\alpha \partial_\xi^\beta p^\pm_h(x, \xi)| \leq C_{\alpha,\beta}(1 + h(2+r_0-|\alpha|)\delta_0)(1 + |\xi|)^{2m_0-|\beta|} $$

hold for every $\alpha, \beta \in \mathbb{N}^d$ and

$$ |\partial_x^\alpha \partial_\xi^\beta (a_0 - p^\pm_h)(x, \xi)| \leq C_{\alpha,\beta}(h + h(2+r_0-|\alpha|)\delta_0)(1 + |\xi|)^{2m_0-|\beta|} $$

hold if $|\alpha| \leq 2$.

(b) Let $h_0 > 0$ be small enough and consider $h \in ]0; h_0]$. Then $A_h^\pm$ and the self-adjoint realizations of $P^\pm_h$ have discrete spectrum in $]-\infty; E[$. Moreover the inequalities

$$ P^+_h \leq A_h \leq P^-_h $$

hold in the sense of quadratic forms (for $h \in ]0; h_0]$).
We deduce (1.5) from suitable asymptotics for \(P^\pm_h\), observing that (2.7) and the min-max principle ensure \(\mathcal{N}(P^+_h, E) \leq \mathcal{N}(A_h, E) \leq \mathcal{N}(P^-_h, E)\). Further on we write \(P_h\) and \(p_h\) instead of \(P^+_h\) and \(P^-_h\).

### 2.2 Microlocalisation

Assume that \(\Gamma_h \subset \mathbb{R}^{2d}\) for \(h \in [0; h_0]\) and denote \(\Gamma = (\Gamma_h)_{h \in [0; h_0]}\).

For \(m \in \mathbb{R}\) and \(\delta, \delta_1 \in [0; 1]\) satisfying \(\delta + \delta_1 < 1\) we define \(S^m_{\delta, \delta_1}(\Gamma)\) writing \(b \in S^m_{\delta, \delta_1}(\Gamma)\) if and only if \(b = (b_h)_{h \in [0; h_0]}\) is a family of smooth functions \(b_h : \mathbb{R}^d \to \mathbb{C}\) such that the estimates

\[
\sup_{(x, \xi) \in \Gamma_h} |\partial_\xi^\alpha \partial_x^\beta b_h(x, \xi)| \leq C_{\alpha, \beta} h^{-m-|\alpha|\delta-|\beta|\delta_1}
\]

hold for all \(\alpha, \beta \in \mathbb{N}^d\). In the case \(\delta = \delta_1\) we abbreviate \(S^m_{\delta, \delta}(\Gamma) = S^m_\delta(\Gamma)\).

If \(b = (b_h)_{h \in [0; h_0]} \in S^m_{\delta, \delta_1}(\mathbb{R}^{2d})\) (i.e. \(\Gamma_h = \mathbb{R}^{2d}\) for all \(h\)), then writing

\[
(B_h \varphi)(x) = (2\pi h)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ix\xi/h} b_h(x, \xi) e^{-iy\xi/h} \varphi(y) d\xi d\eta
\]

for \(\varphi \in C^\infty_0(\mathbb{R}^d)\), we define the operators \(B_h = b_h(x, hD) \in B(L^2(\mathbb{R}^d))\) satisfying \(||b_h(x, hD)|| \leq Ch^{-m}||\cdot||\) where \(||\cdot||\) is the norm of the algebra of bounded operators \(B(L^2(\mathbb{R}^d))\).

Let \(1_I : \mathbb{R} \to \{0, 1\}\) be the characteristic function of the interval \(I \subset \mathbb{R}\). Then \(1_I(P_h)\) denotes the spectral projector of \(P_h\) on \(I\) and

\[
\mathcal{N}(P_h, E) = \text{tr} 1_{[0; E]}(P_h).
\]

For a given \(t_0 > 0\) we consider a standard mollifying of \(1_I\) using a real valued pair function \(\gamma_0 \in C^\infty_0([-t_0/2; t_0/2])\) and \(\gamma_1 = \gamma_0 \ast \gamma_0\) such that \(\gamma_1(0) = 1\).

The inverse \(h\)-Fourier transform of \(\gamma_1\), given by the formula

\[
\tilde{\gamma}_h(\zeta) = (2\pi h)^{-1} \int_{\mathbb{R}} \gamma_1(t) e^{it\zeta/h} dt\quad \text{for } \zeta \in \mathbb{C},
\]

defines a family of holomorphic functions satisfying \(\int_{\mathbb{R}} \tilde{\gamma}_h(\lambda)d\lambda = \gamma_1(0) = 1\) and \(\tilde{\gamma}_h(\lambda) > 0\) for \(\lambda \in \mathbb{R}\). We denote

\[
\tilde{f}_h^Z(\zeta) = \int_{\mathbb{R}} d\lambda \tilde{\gamma}_h(\zeta - \lambda)
\]

and in Section 7 we show the following result:
Lemma 2.2 In order to show the asymptotic formula (1.5) it is sufficient to fix \( c > 0 \) and to prove that for every \( l \in S^{0}_{0,0}(\mathbb{R}^{2d}) \) satisfying

\[
\text{supp } l_{h} \subset \{ v \in \mathbb{R}^{2d} : |a_{0}(v) - E| \leq c \}
\]  \hspace{1cm} (2.10)

one has the estimates

\[
\left| \text{tr } \tilde{f}_{h}^{Z}(P_{h})L_{h} - \int_{\mathbb{R}^{2d}} \frac{dv}{(2\pi h)^{d}} \tilde{f}_{h}^{Z}(\hat{p}_{h}(v))l_{h}(v) \right| \leq C_{\varepsilon} h^{-d} \sum_{1 \leq j \leq 2} R_{E_{j}}^{E_{0}}(h), \hspace{1cm} (2.11)
\]

where \( L_{h} = l_{h}(x,hD) \), \( \hat{p}_{h} = \text{Re } p_{h} \), \( Z = [E_{1}; E_{2}] \subset [E - c; E + c] \) and the constant \( C_{\varepsilon} \) is independent of \( E_{1}, E_{2} \).

2.3 Partition of the phase space

Further on \( \bar{C}, \bar{c} > 0 \) are constants and we denote

\[
\hat{\Gamma}(\bar{C}h^{\delta_{0}}) = \{ v \in \mathbb{R}^{2d} : |\nabla a_{0}(v)| \leq \bar{C} h^{\delta_{0}} \}, \hspace{1cm} (2.12)
\]

\[
\bar{\Gamma}(\bar{c}h^{\delta_{0}}) = \{ v \in \mathbb{R}^{2d} : |\nabla a_{0}(v)| > \bar{c} h^{\delta_{0}} \}. \hspace{1cm} (2.12')
\]

Using (1.1) we can find \( \Gamma_{0} \) being a compact subset of \( \mathbb{R}^{2d} \) such that

\[
\{ v \in \mathbb{R}^{2d} : |a_{0}(v) - E| \leq 2c \} \subset \bar{\Gamma}_{0}, \hspace{1cm} (2.13)
\]

where the constant \( c > 0 \) is fixed small enough. We will consider (2.11) with \( l_{h} = \hat{l}_{h} + \tilde{l}_{h} \), where

\[
\text{supp } \hat{l}_{h} \subset \hat{\Gamma}(\bar{C}h^{\delta_{0}}) \cap \bar{\Gamma}_{0}, \hspace{0.5cm} \text{supp } \tilde{l}_{h} \subset \tilde{\Gamma}(\bar{c}h^{\delta_{0}}) \cap \bar{\Gamma}_{0} \hspace{1cm} (2.14)
\]

and moreover we introduce an auxiliary cut-off function \( \tilde{l} \in C_{0}^{\infty}(\mathbb{R}^{d}) \) such that \( \tilde{l} = 1 \) on \( \bar{\Gamma}_{0} \). Further on we denote

\[
\hat{L}_{h} = \hat{l}_{h}(x,hD), \hspace{1cm} \tilde{L}_{h} = \tilde{l}_{h}(x,hD), \hspace{1cm} (2.15)
\]

where \( \hat{l}, \tilde{l} \) are as above. Then we have \( \| \hat{L}_{h}(I - \tilde{L}_{h}^{*}) \|_{\text{tr}} = O(h^{\infty}) \), where \( \| \cdot \|_{\text{tr}} \) denotes the trace class norm, \( O(h^{\infty}) \) means that \( O(h^{m}) \) holds for every \( m \in \mathbb{R} \) and \( \tilde{L}_{h}^{*} \) denotes the adjoint of \( \tilde{L}_{h} \) in \( L^{2}(\mathbb{R}^{d}) \). Using moreover the trace cyclicity we obtain

\[
\text{tr } \hat{L}_{h} \tilde{f}_{h}^{Z}(P_{h}) = \text{tr } \hat{L}_{h} \tilde{f}_{h}^{Z}(P_{h}) \tilde{L}_{h}^{*} + O(h^{\infty})
\]
and further on we keep the auxiliary cut-off $\tilde{L}_h$ to be sure that our analysis always concern operators of the trace class. We introduce
\[
\tilde{f}_h(t) = \int_{\mathbb{R}} d\lambda \ e^{-it\lambda/h} \tilde{f}_h^Z(\lambda) = \gamma_1(t) \int_{Z} d\lambda \ e^{-it\lambda/h}.
\] (2.16)

Then $\tilde{f}_h(\lambda) = \int_{\mathbb{R}} \frac{dt}{2\pi h} \ f_h^Z(t) e^{it\lambda/h}$ and consequently
\[
\text{tr} \hat{L}_h \tilde{f}_h(P_h) \hat{L}_h^* = \int_{\mathbb{R}} \frac{dt}{2\pi h} f_h(t) \text{tr} \hat{L}_h e^{itP_h/h} \hat{L}_h^*. \quad (2.17)
\]

We will prove

**Proposition 2.3** Assume that $t_0, \kappa > 0$ are small enough, $\varepsilon > 0$ and $\tilde{L}_h$, $\hat{L}_h$ are given by (2.15). Then for every $\tilde{N} \in \mathbb{N}$ one can find the operators $Q^h_N(t) \in B(L^2(\mathbb{R}^d))$ such that
\[
\sup_{-t_0 \leq t \leq t_0} |\text{tr} \ (Q^h_N(t) - \hat{L}_h e^{itP_h/h}) \hat{L}_h^*| \leq C_N h^{\tilde{N}\kappa - 5d - 1} \quad (2.18)
\]
and one has the estimates
\[
\left| \int_{\mathbb{R}} \frac{dt}{2\pi h} f_h^Z(t) \text{tr} Q^h_N(t) \hat{L}_h^* - \int_{\mathbb{R}^{2d}} \frac{dv}{(2\pi h)^d} \hat{L}_h(v) \tilde{f}_h^Z(\hat{p}_h(v)) \right| \leq C_\varepsilon h^{-\delta} \sum_{1 \leq l \leq 2} \mathcal{R}_{E_j}^{\varepsilon, \delta_0}(h), \quad (2.19)
\]
where $\hat{p}_h = \text{Re} \ p_h$, $Z = [E_1; E_2] \subset [E - c; E + c]$ and $C_\varepsilon$ is independent of $E_1, E_2$.

Using Proposition 2.3 we find that (2.11) holds if $l_h$, $L_h$ are replaced by $\tilde{L}_h$, $\hat{L}_h$. Indeed, we have $|f^Z_h(t)| \leq C |\gamma_1(t)|$ and supp $\gamma_1 \subset [-t_0; t_0]$, hence we can replace $\hat{L}_h e^{itP_h/h}$ by $Q^h_N(h)(t)$ in (2.17) with an error $O(h^{\tilde{N}\kappa - 5d - 1})$.

The construction of $Q^h_N(t)$ is presented in Section 3 and Proposition 2.3 is proved in Sections 4-5. At the end of this section we introduce classes of symbols describing properties of $\tilde{L}_h$ and $\hat{L}_h$.

**2.4 Classes of symbols $S^m_{(0)}$**

Further on $\delta_0, \bar{T}_0$ are fixed as before, we fix $\bar{C} > 1$, we denote
\[
\hat{\Gamma}_h = \hat{\Gamma}(\bar{C}h^{\delta_0}), \quad \tilde{\Gamma}_h = \mathbb{R}^{2d} \setminus \hat{\Gamma}_h = \hat{\Gamma}(\bar{C}h^{\delta_0}) \quad (2.20)
\]
and we write \( \hat{\Gamma} = (\hat{\Gamma}_h)_{h \in [0, h_0]}, \) \( \breve{\Gamma} = (\breve{\Gamma}_h)_{h \in [0, h_0]} \).

For \( m \in \mathbb{R} \) we define \( S_m^{(0)} \) writing \( b \in S_m^{(0)} \) if and only if \( b = (b_h)_{h \in [0, h_0]} \in S_m^{(0)} \cap S_m^{(0)}(\hat{\Gamma}) \) and \( \text{supp } b \subseteq \Gamma_0 \) (i.e. \( \text{supp } b_h \subseteq \Gamma_0 \) for every \( h \in [0, h_0] \)).

**Lemma 2.4** Let \( m_0 \in S_{0, \delta_0} \) satisfy \( \text{supp } m_0 \subseteq \Gamma_0 \). Then there exist \( \hat{l}, \breve{l} \in S_{0, \delta_0}^{(0)} \) such that \( l = \hat{l} + \breve{l} \), \( \text{supp } \breve{l} \subseteq \text{supp } l \) and (2.14) holds if \( \delta > 0 \) is small enough.

**Proof.** Let \( \chi_o \in C_{0}^\infty [1 - 1; 1] \) satisfy \( 0 \leq \chi_o \leq 1 \) and \( \chi_0 = 1 \) on \( [-\frac{1}{2}; \frac{1}{2}] \).

We define \( \hat{\chi} \in S_{0, \delta_0}^{(0)}(\Gamma_0) \) setting \( \hat{\chi}(v) = \prod_{\beta = 1}^{2d} \chi_0(\partial^\beta \hat{p}_h(v) \cdot 2d/h^{\delta_0}) \). Since

\[
\nabla p_h(x, \xi) = \nabla a_0(x, \xi) + o(h^{\delta_0})(1 + |\xi|^{2m_0})
\]

due to (2.6) and \( a_0 \) is real valued, (2.21) still holds if we replace \( p_h \) by \( \hat{p}_h \).

However \( \hat{\chi}(v) \neq 0 \) implies \( |\partial^\beta \hat{p}_h(v)| \cdot 2d/h^{\delta_0} \leq 1 \) if \( |\beta| = 1 \), hence we have \( |\nabla a_0(v)| \leq h^{\delta_0}(1 + o(1)) \) if moreover \( v \in \Gamma_0 \). Therefore setting \( \hat{l} = l \hat{\chi} \) we obtain \( \text{supp } \hat{l} \subset \hat{\Gamma}(\delta h^{\delta_0}) \) if \( \delta > 1 \) and consequently \( \hat{l} \in S_{0, \delta_0}^{(0)} \).

Next we define \( \breve{l} = l - \hat{l} \in S_{0, \delta_0}^{(0)} \). If \( v \in \text{supp } \breve{l} \), then \( \hat{\chi}(v) \neq 1 \) and \( |\partial^\beta p_h(v)| \cdot 2d/h^{\delta_0} \geq 1/2 \) holds for a certain \( \beta \in \mathbb{R}^{2d} \) with \( |\beta| = 1 \), hence \( |\nabla p_h(v)| \geq h^{\delta_0}/(4d^2) \). Therefore \( \text{supp } \hat{\chi} \subset \hat{\Gamma}(\delta h^{\delta_0}) \) holds if \( \delta < 1/(4d^2) \). \( \Box \)

## 3 Approximation of the evolution

### 3.1 Preliminaries

We write \( \hat{p}_h = \text{Re} p_h \) and we consider an approximation of \( \hat{L}_h e^{itP_h/h} \) in the form

\[
Q_h^N(t) = \left( e^{it\hat{p}_h/h} \sum_{0 \leq n \leq N} t^n q_{N,n,h}^c(x, hD) \right),
\]

where \( q_{N,n,h}^c \) will be described in Proposition 3.3. We introduce formally

\[
\hat{Q}_N^h(t) = \frac{d}{dt} Q_N^h(t) - iQ_N^h(t)P_h/h
\]

and require \( Q_N^h(0) = \hat{L}_h \), which allows us to express

\[
Q_N^h(t) - \hat{L}_h e^{itP_h/h} = \int_0^t d\tau \hat{Q}_N^h(t - \tau)e^{i\tau P_h/h}.
\]
To investigate (3.2) in terms of symbols we introduce the notation
\[
(\tilde{\mathcal{P}}_N)b(t) = e^{-it\hat{p}_h/h} \left( \partial_t (b_h(t)e^{it\hat{p}_h/h}) - \sum_{|\alpha| \leq N} \frac{h^{|\alpha|-1}}{\alpha!} \partial^{\alpha}_{\xi} (b_h(t)e^{it\hat{p}_h/h}) \right) \tag{3.4}
\]
if \((b_h(t))_{h \in [0,h_0]} \in S^m_{(0)}\) for \(t \in \mathbb{R}\).

### 3.2 Classes of symbols \(S^m_{(N)}\) for \(N \in \mathbb{N}\)

We use the induction with respect to \(N \in \mathbb{N}\) in the following definition:

i) if \(N = 0\) then \(S^m_{(N)} = S^m_{(0)}\) was defined at the end of Section 2;

ii) we write \((b_h)_{h \in [0,h_0]} \in S^m_{(N+1)}\) if and only if

\[
b_h(x,\xi) = b_{0,h}(x,\xi) + \sum_{|\beta| = 1} h^{-1/2} b_{\beta,h}(x,\xi) \partial^{\beta}_{\xi} \hat{p}_h(x,\xi)
\]

holds with some \((b_{\beta,h})_{h \in [0,h_0]} \in S^m_{(0)}\) for \(\beta \in \mathbb{N}^{2d}\) satisfying \(|\beta| \leq 1\).

It is clear that \(S^m_{(N)} \subset S^{m+N/2}_{(0)}\). Moreover \(b \in S^m_{(N)}\) if and only if one can write

\[
b(x,\xi) = \sum_{\{\beta \in \mathbb{N}^{2d}: |\beta| \leq N\}} b_{\beta}(x,\xi) h^{-|\beta|/2} (\nabla \hat{p}(x,\xi))^{\beta} \tag{3.6}
\]

with some symbols \(b_{\beta} \in S^m_{(0)}\), where \(\nabla \hat{p}(x,\xi) \in \mathbb{R}^{2d}\) and

\[(t_1,\ldots,t_n)^{(a_1,\ldots,a_n)} = \prod_{1 \leq j \leq n} t_j^{a_j} \text{ for } (t_1,\ldots,t_n) \in \mathbb{R}^n, \ (a_1,\ldots,a_n) \in \mathbb{N}^n.\]

Using this characterization we find

\[
b \in S^m_{(N)}, \ \tilde{b} \in S^m_{0,00}(\mathbb{T}_0) \Rightarrow b\tilde{b} \in S^m_{(N)}, \tag{3.7}
\]

\[
b \in S^m_{(N)}, \ \tilde{b} \in S^m_{(N)} \Rightarrow b\tilde{b} \in S^m_{(N+N/2)}. \tag{3.8}
\]

**Lemma 3.1** (a) If \(b \in S^m_{(N)}\) then \(\partial^\alpha b \in S^{m+|\alpha|/2}_{(N)}\) for every \(\alpha \in \mathbb{N}^{2d}\).

(b) If \(b \in S^m_{(N+1)}\) then \(\partial_k b \in S^{m+1/2}_{(N)}\) for \(k \in \{1,\ldots,d\}\).

**Proof** (a) The general statement follows by induction with respect to \(|\alpha|\)
and we consider only the case \(|\alpha| = 1\). It is clear that the assertion holds
for \(N = 0\) and reasoning by induction we assume that the assertion holds

for a given $N \in \mathbb{N}$. Let $b \in S_{(N+1)}^m$. Then (3.5) holds with $b_\beta \in S_{(N)}^m$, hence $\partial^\alpha b_\beta \in S_{(N)}^{m+1/2}$ if $|\alpha| = 1$. Due to (2.5) we have $\partial^{\alpha+\beta} \hat{\rho} \in S_{0,\delta_0}^0(\bar{\Gamma}_0)$ and writing
\[
\partial^\alpha (b_\beta h^{-1/2} \partial^\beta \hat{\rho}) = b_\beta h^{-1/2} \partial^{\alpha+\beta} \hat{\rho} + \partial^\alpha b_\beta h^{-1/2} \partial^\beta \hat{\rho} \in S_{(N+1)}^{m+1/2},
\]
we obtain the assertion of Lemma 3.1(a) for $N + 1$.

(b) We will show that for $k \in \{1, \ldots, d\}$, $|\alpha| = 1$ and $N \in \mathbb{N}$ we have
\[
b \in S_{(N)}^m \Rightarrow \partial_{\xi_k} b \partial^\alpha \hat{\rho} \in S_{(N)}^m.
\]
To begin we consider $N = 0$. Since $\partial^\alpha \hat{\rho} \in S_{0,\delta_0}^0(\bar{\Gamma}_0)$, we have
\[
b \in S_{(0)}^m \Rightarrow \partial_{\xi_k} b \in S_{0,\delta_0}^m(\hat{\Gamma}) \Rightarrow \partial_{\xi_k} b \partial^\alpha \hat{\rho} \in S_{0,\delta_0}^m(\hat{\Gamma}).
\]
It is easy to check that $\partial^\alpha \hat{\rho} \in S_{\delta_0}^{-\delta_0}(\hat{\Gamma} \cap \bar{\Gamma}_0)$ and consequently
\[
b \in S_{(0)}^m \Rightarrow \partial_{\xi_k} b \in S_{\delta_0}^{m+\delta_0}(\hat{\Gamma}) \Rightarrow \partial_{\xi_k} b \partial^\alpha \hat{\rho} \in S_{\delta_0}^m(\hat{\Gamma}).
\]
However (3.11) and (3.12) imply $\partial_{\xi_k} b \partial^\alpha \hat{\rho} \in S_{(0)}^m$, i.e. (3.10(0)) holds.

Reasoning by induction we assume that (3.10(N)) holds for a given $N \in \mathbb{N}$. If $b \in S_{(N+1)}^m$ is given by (3.5), then
\[
b_\beta \in S_{(N)}^m \Rightarrow \partial_{\xi_k} b_\beta \partial^\beta \hat{\rho} b_\beta \in S_{(N)}^m \Rightarrow \partial_{\xi_k} b \partial^\alpha \hat{\rho} b_\beta h^{-1/2} \partial^\beta \hat{\rho} \in S_{(N+1)}^m.
\]
Moreover the induction hypothesis ensures
\[
b_\beta \in S_{(N)}^m \Rightarrow \partial_{\xi_k} b_\beta \partial^\beta \hat{\rho} b_\beta \in S_{(N)}^m \Rightarrow \partial_{\xi_k} b_\beta \partial^\alpha \hat{\rho} b_\beta h^{-1/2} \partial^\beta \hat{\rho} \in S_{(N+1)}^m.
\]
Summing up (3.13) and (3.14) we obtain $\partial_{\xi_k} (b_\beta h^{-1/2} \partial^\beta \hat{\rho}) \partial^\alpha \hat{\rho} \in S_{(N+1)}^m$, which completes the proof of (3.10(N + 1)).

If $b \in S_{(N+1)}^m$ is given by (3.5) with $b_\beta \in S_{(N)}^m$, then $\partial_{\xi_k} b_\beta \partial^\beta \hat{\rho} \in S_{(N)}^m$ is ensured by (3.10(N)) and we obtain the assertion of Lemma 3.1(b) writing (3.9) with $\partial_{\xi_k}$ instead of $\partial^\alpha$. $\triangle$

3.3 Construction of the approximation

We define auxiliary classes of symbols $\mathring{S}_{(N)}^m \subset S_{(N)}^m$ for $N \in \mathbb{N} \setminus \{0\}$ as follows: we write $b \in \mathring{S}_{(N)}^m$ if and only if it is possible to find $b_j \in S_{(N-1)}^m$, $j \in \{0, 1, \ldots, d\}$, such that $b = b_0 + \sum_{1 \leq j \leq d} h^{-1/2} b_j \partial_{\xi_j} \hat{\rho}$. 12
Lemma 3.2 Let \( b \) be independent of \( t \). If \( b \in S^m_{(N)} \) then

\[
(\bar{P}_N b)(t) = \sum_{0 \leq n \leq N} t^n b_n
\]

(3.15)

holds with \( b_0 \in S^m_{(N)} \subset \tilde{S}^m_{(N+1)} \) and \( b_n \in S^m_{(N+n+1)} \) for \( n \in \{1, \ldots, \tilde{N}\} \).

**Proof.** To begin we show that \( b_0 \in S^m_{(N)} \). We observe that

\[
b_0 = ih^{-1}(\dot{p} - \overline{p})b + \sum_{1 \leq |\alpha| \leq \tilde{N}} \frac{h^{|\alpha|-1}}{\alpha!} \partial^\alpha(\overline{\partial_x p}(b \overline{\partial_x p}))
\]

(3.16)

and since \( p_h(x, hD) \) is self-adjoint, using (2.5) we obtain

\[
|\alpha| \leq 1 \Rightarrow \partial^\alpha(\dot{p} - \overline{p}) = 2i \operatorname{Im} \partial^\alpha p \in S^{-1}_{0, \delta_0}(\Gamma_0).
\]

(3.17)

We observe that the first term of the expression (3.16) belongs to \( S^m_{(N)} \) due to (3.17) with \( \alpha = 0 \). Then \( b \in S^m_{(N)} \Rightarrow b\partial_{x_j} \dot{p} \in S^m_{(N+1)} \) and using (3.17) with \( |\alpha| = 1 \) we obtain \( b\overline{\partial_{x_j} \dot{p}} \in S^m_{(N+1)} \). Therefore Lemma 3.1(b) ensures \( \partial_{x_j}(b\overline{\partial_{x_j} \dot{p}}) \in S^m_{(N)}, \) i.e. all terms of (3.16) with \( |\alpha| = 1 \) belong \( S^m_{(N)} \).

In the next step we consider the terms of (3.16) with \( |\alpha| \geq 2 \).

Since (2.5) ensures \( |\alpha| \geq 2 \Rightarrow \partial^\alpha p \in S^{|\alpha|-2}_{0, \delta_0}(\Gamma_0) \), we obtain

\[
b \overline{\partial_x p} \in \tilde{S}^m_{(N)} \Rightarrow h^{|\alpha|-1} \overline{\partial_x p} b \overline{\partial_x p} \in S^{-|\alpha|+1+|\alpha|/2+m+|\alpha|-2\delta_0}_{(N)}
\]

due to Lemma 3.1(a) and \( m + (|\alpha| - 2)(\delta_0 - 1/2) \leq m \) gives \( b_0 \in S^m_{(N)} \).

In order to show \( b_n \in \tilde{S}^m_{(N+n+1)} \) for \( n \in \{2, \ldots, \tilde{N}\} \) we write

\[
b_n = \sum_{1 \leq |\beta| \leq |\alpha|, \beta + \overline{\beta} \leq \alpha} h^{|\alpha|-n} b_{\beta, \overline{\beta}} (\nabla_\overline{\xi} \dot{p})^\beta \partial^\overline{\beta} (b \overline{\partial_x p}),
\]

where \( b_{\beta, \overline{\beta}} \in S^0_{0, \delta_0}(\Gamma_0) \) for \( \beta, \overline{\beta} \in \mathbb{N}^d \). More precisely: in the case \( |\beta| < n \), \( b_{\beta, \overline{\beta}} \) is a linear combination of terms \( \Pi_{1 \leq k \leq n-|\beta|} \partial_{\xi}^{\tilde{\alpha}(k)} \dot{p} \) where \( \tilde{\alpha}(k) \in \mathbb{N}^d \) are such that \( |\tilde{\alpha}(k)| \geq 2 \) for \( k \in \{1, \ldots, n-|\beta|\} \) and \( \beta + \overline{\beta} + \sum_{1 \leq k \leq n-|\beta|} \tilde{\alpha}(k) = \alpha \), implying

\[
|\alpha| \geq |\beta| + |\overline{\beta}| + 2(n - |\beta|) = 2n + |\overline{\beta} - |\beta||.
\]

(3.18)
In the case $|\beta| = n$ the symbols $b_{\beta,\bar{\beta}}$ are constant and (3.18) still holds.

Consider first the case $|\alpha| \geq 2$. Then using Lemma 3.1(a) we find
\[ h^{|\alpha|-n} \partial_{\xi}(b \partial_x \hat{p})(\nabla_{\xi} \hat{p})^\beta \in S_{(N+|\beta|)}^{m-|\alpha|+1+n+\bar{\beta}/2+m+(|\alpha|-2)\delta_0-|\beta|/2} \] (3.19)
and (3.18) ensures
\[ 1 - 2\delta_0 + (2n + |\bar{\beta}| - |\beta|)/2 + |\alpha|\delta_0 - (|\alpha| - 2)(\delta_0 - 1/2) \leq 0, \] (3.20)
i.e. all terms corresponding to $|\alpha| \geq 2$ belong to $S_{(N+n)}^{m} \subset \tilde{S}_{(N+1)}^{m}$.

To complete the proof we observe that in the case $n = |\alpha| = 1$ we have
\[ b \partial_{x_j} p \in S_{(N+1)}^{m-1/2} \Rightarrow h^{-1} b \partial_{x_j} p \partial_{\xi_j} \hat{p} \in \tilde{S}_{(N+2)}^{m}. \]

**Proposition 3.3** Let $\tilde{l} \in S_{(0)}^0$ and $\bar{N} \in \mathbb{N}$. Assume that $N \in \{0, 1, \ldots, \bar{N}\}$.

Then we can find
\[ q_{\bar{N},N}(t) = \sum_{0 \leq n \leq N} t^n q_{\bar{N},n}^o \]
where $q_{\bar{N},0}^o = \tilde{l}, \quad q_{\bar{N},1}^o \in S_{(0)}^0$, \[ q_{\bar{N},n}^o \in \tilde{S}_{(n)}^0 \quad \text{for} \quad n \in \{2, \ldots, N\} \] (3.22(N))
and
\[ \hat{P}_N q_{\bar{N},N}(t) = \sum_{N \leq n \leq N + \bar{N}} t^n \tilde{q}_{\bar{N},N,n}^o \]
(3.23(N))
holds with
\[ \tilde{q}_{\bar{N},N,n}^o \in \tilde{S}_{(n+1)}^0 \quad \text{for} \quad n \in \{N, \ldots, N + \bar{N}\}. \] (3.24(N))

**Proof.** If $N = 0$ then we take $q_{0,0}^o = \tilde{l} \in S_{(0)}^0$ and Lemma 3.2 with $b = \tilde{l}$ ensures (3.23(0)) and (3.24(0)). Next we assume that the statement of Proposition 3.3 holds for a given $N \leq \bar{N} - 1$ and using the induction hypothesis (3.23(N)) to express $\hat{P}_N q_{\bar{N},N}(t)$ we find
\[ \hat{P}_N q_{\bar{N},N}(t) = \hat{P}_N(t^{N+1} q_{\bar{N},N+1}^o) + \hat{P}_N q_{\bar{N},N}(t) = 
\]
\[ t^N \left((N + 1) q_{\bar{N},N+1}^o + \tilde{q}_{\bar{N},N,N}^o\right) + t^{N+1} \hat{P}_N q_{\bar{N},N+1}^o(t) + \sum_{N+1 \leq n \leq N + \bar{N}} t^n \tilde{q}_{\bar{N},N,n}^o. \]
To obtain (3.23(N + 1)) we cancel the term with \( t^N \) taking
\[
q^0_{N,N+1} = -\tilde{q}^0_{N,N,N} / (N + 1),
\]
which is an element of \( \tilde{S}^0_{(N+1)} \) by the induction hypothesis (3.24(N)) and (3.24(N + 1)) follows if we develop \( t^{N+1}\tilde{P}_N q^0_{N,N+1}(t) \) as in Lemma 3.2 with \( b = q^0_{N,N+1} \in S^0_{(N+1)} \). Moreover for \( N = 0 \) we have \( q^0_{N,1} = -\tilde{q}^0_{N,0,0} \in S^0_{(0)} \) (due to Lemma 3.2 with \( b = \tilde{l} \)). △

4 Quality of the approximation

This section is devoted to the proof of Proposition 2.3. To begin we introduce more notations. We write \( q \in \tilde{S}^m_{(0)} \) if and only if \( q = (q_h)_{h \in [0; h_0]} \) with \( q_h \in C^\infty(\mathbb{R}^{3d}) \) satisfying the estimates
\[
|\partial^\alpha q_h(x, \xi, y)| \leq C_\alpha h^{-m-|\alpha|\delta_0}
\]
for every \( \alpha \in \mathbb{N}^{3d} \) and \( \text{supp} \, q_h \subseteq \Omega_0 \times \mathbb{R}^d \). As before \( \hat{p}_h = \text{Re} \, p_h \) and writing
\[
(\text{Op}_{t}^h[q] \varphi)(x) = \int_{\mathbb{R}^{2d}} \frac{dyd\xi}{(2\pi h)^d} e^{i(x-y)\xi/h} e^{it\hat{p}_h(x,\xi)/h} q_h(x ,\xi , y) \varphi(y)
\]
for \( \varphi \in C^\infty_0(\mathbb{R}^d) \) we define operators on \( L^2(\mathbb{R}^d) \) such that
\[
q \in \tilde{S}^m_{(0)} \Rightarrow \sup_{-t_0 \leq t \leq t_0} ||\text{Op}_{t}^h[q]||_{\text{tr}} \leq C h^{-m-5d}.
\]
Indeed, (4.3) follows from standard estimates of pseudo-differential operators (e.g. [14, Sec. 18]), the details are given in the proof of (4.4) in [24].

We observe that
\[
\text{tr} \, \text{Op}_{t}^h[q] = \int_{\mathbb{R}^{2d}} \frac{dxd\xi}{(2\pi h)^d} e^{it\hat{p}_h(x,\xi)/h} q_h(x ,\xi , x)
\]
and for \( b_h \in C^\infty_0(\mathbb{R}^{2d}) \) we introduce the notation
\[
J_{t}^h(b) = \int_{\mathbb{R}^{2d}} \frac{dxd\xi}{(2\pi h)^d} e^{it\hat{p}_h(x,\xi)/h} b_h(x ,\xi).
\]
Using this notation and $Q^h_N(t)\tilde{L}_h^* = \sum_{0 \leq k \leq N} t^k \text{Op}_t^h[q_{\tilde{N},k}^0(x,\xi)l(y,\xi)]$ with $\tilde{l} = 1$ on supp $q_{\tilde{N},k}^0$, we find the expression

$$\text{tr} Q^h_N(t)\tilde{L}_h^* = \sum_{0 \leq k \leq N} t^k J^h_t(q_{\tilde{N},k}^0). \quad (4.6)$$

**Lemma 4.1** Let $n \in \mathbb{N}$. (a) If $b \in S^m_{(N)}$ then

$$t^n J^h_t(b) = \sum_{0 \leq k \leq n} t^k J^h_t(b_{k,n}) \quad (4.7(n))$$

holds with some $b_{k,n} \in S^m_{(\text{max}(0,N-n))}$ for $k \in \{0,\ldots,n\}$.

(b) If $b \in \tilde{S}^m_{(N+1)}$, $N \geq 1$, then (4.7(n)) holds with $b_{k,n} \in S^m_{(\text{max}(0,N-n))}$.

**Proof.** (a) Reasoning by induction we assume that the statement holds for a given $N \in \mathbb{N}$. In order to show that the statement still holds for $N+1$ instead of $N$ we consider $b \in S^m_{(N+1)}$. Then (3.5) holds with $b_\beta \in S^m_{(N)}$ and $t^n J^h_t(b_\beta)$ can be expressed in a suitable way due to the induction hypothesis. Then the integration by parts gives

$$t J^h_t(h^{-1/2}\partial^\beta \hat{p} b_\beta) = J^h_t(\hat{b}_\beta) \quad \text{with} \quad \hat{b}_\beta = h^{1/2}i\partial^\beta b_\beta \quad (4.8)$$

and Lemma 3.1(a) ensures $\hat{b}_\beta \in S^m_{(N)}$, i.e. (4.8) implies (4.7(1)). Reasoning by induction with respect to $n \in \mathbb{N}$ we obtain (4.7(n)).

(b) If $b \in \tilde{S}^m_{(N+1)}$, $\partial^\beta = \partial_{\beta_k}$, then $b_\beta \in S^m_{(N)} \Rightarrow \hat{b}_\beta = h^{1/2}i\partial^\beta b_\beta \in S^m_{(N-1)}$ due to Lemma 3.1(b). Thus (4.8) gives $t^n J^h_t(b) = t^n J^h_t(b_\beta) + t^{n-1} J^h_t(\hat{b})$ with $b_\beta \in S^m_{(N)}$, $\hat{b} \in S^m_{(N-1)}$ and we complete the proof using the assertion a) with $b_0, \hat{b}$ instead of $b$. \(\Box\)

**Proof of (2.19).** Writing the terms $t^{k-1} J^h_t(q_{\tilde{N},k}^0)$ with $k \in \{2,\ldots,\tilde{N}\}$ as described in Lemma 4.1(b) (for $N = n = k-1$ and $b = q_{\tilde{N},k}^0 \in S^0_{(k)}$), we can express (4.6) in the form

$$\text{tr} Q^h_N(t)\tilde{L}_h^* = J^h_t(\tilde{l}_h) + \sum_{1 \leq k \leq \tilde{N}} t^k J^h_t(b_{\tilde{N},k}) \quad (4.9)$$

with some $b_{\tilde{N},k} \in S^0_{(0)}$. Changing the order of integrals we find

$$\int_{\mathbb{R}} \frac{dt}{2\pi h} J^Z_h(t)J^h_t(\tilde{l}_h) = \int_{\mathbb{R}^d} \frac{dv}{(2\pi h)^d} \tilde{l}_h(v) J^Z_h(\hat{p}_h(v)).$$
It remains to consider the terms of (4.9) with \( k \in \{1, \ldots, N\} \) and to estimate
\[
\int_{\mathbb{R}} \frac{dt}{2\pi i_h} f_h^Z(t) i^k J_t^h(b_{N,k}) = \int_{\mathbb{R}^d} \frac{dv}{(2\pi i_h)^d} b_{N,k,h}(v) i^{-k} h^k (\tilde{f}_h^Z(k)(\hat{p}_h(v))).
\] (4.10)

However \( k \geq 1 \Rightarrow h^k (\tilde{f}_h^Z(k)(\lambda)) = \tilde{\gamma}_0^{(k-1)}(\frac{\lambda-E_1}{h}) - \tilde{\gamma}_1^{(k-1)}(\frac{\lambda-E_2}{h}) \) and we have \( |\tilde{\gamma}_1^{(k-1)}(\lambda)| \leq C_{k,N}(1 + |\lambda|)^{-N} \) for every \( \lambda \in \mathbb{R}, \ N \in \mathbb{N} \), hence (4.10) can be estimated by
\[
\sum_{1 \leq j \leq 2} h^{-d} \int_0^{\Gamma_{E_j}} dv \left( 1 + \frac{|\hat{p}_h(v) - E_j|}{C h} \right)^{-N},
\]
where we can choose \( C \geq 1 \) such that \( \sup_{v \in \Gamma_0} |a_0(v) - \hat{p}_h(v)| \leq \frac{1}{2} Ch \).

It is clear that the region \( \{v \in \mathbb{R}^d : |\hat{p}_h(v) - E_j| \geq \frac{1}{3} h^{1-\varepsilon}\} \) gives a contribution \( O(h^{N\varepsilon-d}) \) and it remains to consider the regions
\[
\Gamma_{E_j}^{h,n} = \{v \in \mathbb{R}^d : C(n-1/2)h \leq \hat{p}_h(v) - E_j \leq C(n+1/2)h\},
\] (4.11)
where \( n \in \mathbb{Z} \) is such that \( |n| < h^{-\varepsilon}/(2C) \). However
\[
\Gamma_{E_j}^{h,n} \subset \tilde{\Gamma}_{E_j}^{h,n} = \{v \in \mathbb{R}^d : C(n-1)h \leq a_0(v) - E_j \leq C(n+1)h\}
\]
and to complete the proof we observe that
\[
\sum_{|n| < h^{-\varepsilon}/(2C)} \int_{\Gamma_{E_j}^{h,n}} dv \left( 1 + \frac{|\hat{p}_h(v) - E_j|}{C h} \right)^{-N} \leq \left( 3 + \sum_{n=1}^{\infty} \frac{2}{n N} \right) \sup_{|n| < h^{-\varepsilon}/(2C)} \text{vol} \tilde{\Gamma}_{E_j}^{h,n}
\]
can be estimated by \( C \varepsilon \mathcal{R}_{E_j}^{e,a_0}(h) \). \( \Delta \)

Our proof of (2.18) will use

**Proposition 4.2** Let \( \delta_0 + \frac{1}{2} < \mu < 1 \) and \( 0 < \kappa \leq \min\{\mu - \delta_0 - \frac{1}{2}, \frac{1-\mu}{2}\} \).

Then
\[
\sup_{\{t \in \mathbb{R} : |t| \leq h^{1-\varepsilon}\}} \left| \text{tr} \left( Q_N^h(t) - \tilde{L}_h e^{itP_h/h} \tilde{L}_h^* \right) \right| \leq C_N h^{N\kappa-5d-1}.
\] (4.12)
To begin we describe the form of $\tilde{Q}_N^h(t)$. Since $P = P^* = p(x, hD)^*$, we have

$$Q_N^h(t)P_h = Op_T^h[q_{N,N}(t, x, \xi)\overline{p(y, \xi)}]$$

and the standard Taylor’s development of $\overline{p(\cdot, \xi)}$ in $x$, followed by integrations by parts based on $(x-y)^\alpha e^{i(x-y)\xi/h} = (-ih)^{\alpha} \partial_\xi^\alpha (e^{i(x-y)\xi/h})$, gives

$$\tilde{Q}_N^h(t) = Op_T^h[(\tilde{P}_N q_{N,N})(t, x, \xi) + r_N(t, x, \xi, y)] \quad (4.13)$$

with

$$r_{N,h}(t, x, \xi, y) = h^{-1}e^{-it\tilde{p}_h(x, \xi)/h}(\bar{N} + 1) \int_0^1 d\sigma (1 - \sigma)^{\bar{N}} \tilde{r}_{N,\sigma,h}(t, x, \xi, y),$$

$$\tilde{r}_{N,\sigma,h}(t, x, \xi, y) =$$

$$\sum_{|\alpha| = \bar{N} + 1} \frac{(-ih)^{\alpha}}{\alpha!} \partial_\xi^\alpha \left[ (q_{N,N,h}(t)e^{it\tilde{p}_h/h})(x, \xi) \partial_\xi^\beta \tilde{p}_h(x + \sigma(y - x), \xi) \right]. \quad (4.14)$$

We describe the properties of $r_N(t)$ introducing new classes of symbols.

We define $\tilde{S}^m_{\mu,(N)}$ for $N \in \mathbb{N}$ setting $\tilde{S}^m_{\mu,(0)} = \tilde{S}^m_{(0)}$ and writing $b \in \tilde{S}^m_{\mu,(N+1)}$ if and only if

$$b_h(x, \xi, y) = b_{0,h}(x, \xi, y) + \sum_{|\beta|=1} b_{\beta,h}(x, \xi, y) h^{-\mu/2} \partial_\xi^\beta \tilde{p}_h(x, \xi) \quad (4.15)$$

holds with some $b_\beta \in \tilde{S}^m_{\mu,(N)}$ for $\beta \in \mathbb{N}^{2d}$ satisfying $|\beta| \leq 1$. Then

$$\tilde{S}^m_{1,(N)} \subset \tilde{S}^m_{\mu,(N)} \subset \tilde{S}^{m+(1-\mu)N/2}_{\mu,(N)} \quad (4.16)$$

and we adopt the following convention: every $(b_h)_{h \in [0; h_0]} \in \tilde{S}^m_{\mu,(N)}$ is identified with $(x, \xi, y) \rightarrow b_h(x, \xi)$ defining an element of $\tilde{S}^m_{\mu,(N)}$.

To deduce (4.12) using (3.3) we consider $t = h^{1-\mu} \bar{t}$, $\tau = h^{1-\mu} \bar{\tau}$ and $\mathcal{V} = \{(\bar{t}, \bar{\tau}) \in ([-t_0; t_0] \setminus \{0\})^2 : 0 \leq \bar{\tau}/\bar{t} \leq 1\}$. We denote $U^h_s = e^{isP/h^\mu}$ and we observe that (4.12) follows from

$$\sup_{(\bar{t}, \bar{\tau}) \in \mathcal{V}} \left| \text{tr} \left( \tilde{Q}_N^h(t)U^h_{\bar{\tau}} \tilde{L}^*_h \right)_{t = h^{1-\mu}(\bar{t} - \bar{\tau})} \right| \leq C_N h^{\mathcal{S}N - 5d - 1}. \quad (4.17)$$
However using the form of \( q_{N,N}^{(\mu)}(t) \) in (4.14) we obtain the expressions of the form considered in the proof of Lemma 3.2 and applying (3.19), (3.20) with \(|\alpha| = \tilde{N} + 1\) we find \( r_{N}(t) = \sum_{n=0}^{2\tilde{N}} t^{n} r_{N,n}^{0} \) with

\[
r_{N,n}^{0} \in \tilde{S}_{1,(2\tilde{N}+1)}^{-(\tilde{N}+1)(1/2-\delta_{0})} \subset \tilde{S}_{\mu,(2\tilde{N}+1)}^{-(\tilde{N}+1)(\mu-1/2-\delta_{0})}
\]  

(4.18)

[the inclusion follows from (4.16)]. Due to (3.23(\(\tilde{N}\)) we have

\[
t = h^{1-\mu} \tilde{t} \implies \langle \hat{P}_{N} q_{N,N}^{(\mu)}(t) \rangle = \sum_{N \leq n \leq 2N} \tilde{t}^{n} q_{N,N,n}^{(\mu)}
\]  

(4.19)

with \( q_{N,N,n,h}^{(\mu)} = h^{\mu(1-\mu)} q_{N,N,n,h}^{0} \) belonging to \( \tilde{S}_{1,(n+1)}^{-(\tilde{N}+1)(1-\mu)} \subset \tilde{S}_{\mu,(n+1)}^{-(\tilde{N}+1)(1-\mu)/2} \).

Combining (4.18) and (4.19) we find the expression

\[
t = h^{1-\mu} \tilde{t} \implies \tilde{Q}_{N}^{h}(t) = \sum_{0 \leq n \leq 2N} \tilde{t}^{n} \text{Op}_{h}^{\frac{\tilde{t}}{h}}[ (1 - \tilde{t}/h)^{n} b_{N,n}^{(\mu)} ]
\]  

(4.20)

with \( b_{N,n}^{(\mu)} \in \tilde{S}_{\mu,(n+1)}^{\kappa} \) similarly as in the formula (4.14) of [26] and following [26] we denote

\[
J_{h,\tilde{t}}^{h}(b, Y) = \text{tr} \left( \text{Op}_{h}^{\frac{\tilde{t}}{h}}[b] U_{\tilde{t}}^{h} Y_{\tilde{t}}^{h} \right)_{t = h^{1-\mu}(\tilde{t}-\tilde{t})}
\]  

(4.21)

if \( b \in \tilde{S}_{\mu,N}^{m} \) and \( Y = (Y_{h,\tilde{t}}^{h})_{(h,\tilde{t}) \in [0, h_{0}] \times V} \subset B(L^{2}(\mathbb{R}^{d})) \). Due to (4.20) the estimate (4.17) follows from

\[
\sup_{(\tilde{t},\tilde{t}) \in V} | \tilde{t}^{n} J_{h,\tilde{t}}^{h}(b_{N,n}^{(\mu)}, \tilde{L}^{*}) | \leq C_{N} h^{\kappa - 5d - 1}.
\]  

(4.22)

Next we observe that the properties of operators \( P_{h} \) given in Lemma 2.1 allow us to follow the reasoning of Sections 5-6 in [26]. More precisely: let \( T_{j} = h^{\mu/2-1} x_{j}, T_{-j} = h^{\mu/2} \partial_{x_{j}}, P_{\pm j} = [ih^{-\mu} P, T_{\pm j}] \) for \( j \in \{1, \ldots, d\} \) and write \( B \in \Psi_{\delta_{0}}^{m} \) if and only if \( B = (b_{h}(x, hD))_{h \in [0, h_{0}]} \) holds with \( (b_{h})_{h \in [0, h_{0}]} \in S_{\delta_{0}}^{m} \). Using \( \mu > \delta_{0} + \frac{1}{2} > 2\delta_{0} \) it is easy to check that for every \( B \in \Psi_{\delta_{0}}^{0} \) we have \([B, T_{\pm j}] \in \Psi_{\delta_{0}}^{0}\) and \([B, P_{\pm j}] \in \Psi_{\delta_{0}}^{-\kappa} \) with \( \kappa > 0 \). Thus it is easy to check that using \( T_{\pm j}, P_{\pm j} \) as above and \( \Psi_{\delta_{0}}^{0} \) instead of \( \Psi^{0} \) in the definition of \( \mathcal{V} \), we can follow the reasoning of the proof of Proposition 4.2 of [26] and we obtain
Proposition 4.3 Let $K = K(N, n) \in \mathbb{N}$ be large enough. Then one can write
\[
\tilde{t}^n J_{\tilde{t}, \tilde{\tau}}^b(b_{\tilde{N}, n}, \tilde{L}^*) = \sum_{1 \leq k \leq K} J_{\tilde{t}, \tilde{\tau}}^b(b_{k, n}, Y_{k, n}) \quad \text{for } (\tilde{t}, \tilde{\tau}) \in V, \tag{4.23}
\]
where $b_{k, n} \in \tilde{S}_{(1)}^{\tilde{N}}$ (for $k \in \{1, ..., K\}$) and $Y_{k, n} = (Y_{\tilde{t}, \tilde{\tau}, n, k}^h)_{(h, \tilde{t}, \tilde{\tau}) \in [0; h_0] \times V}$ is a bounded subset of $B(L^2(\mathbb{R}^d))$ (for $k \in \{1, ..., K\}$).

Proof of Proposition 4.2. We observe that (4.22) follows from Proposition 4.3 similarly as in [26]. Indeed, using the expression (4.23) it suffices to write
\[
|J_{\tilde{t}, \tilde{\tau}}^b(b_{k, n}, Y_{k, n})| \leq \|\text{Op}_t[b_{k, n}]_{t=\tilde{t}, \tilde{\tau}}\|_{\text{tr}} \|Y_{\tilde{t}, \tilde{\tau}, n, k}^h\| \leq C h^{\tilde{N} - 5d - 1/2},
\]
where we used (4.3) with $q = b_{k, n} \in \tilde{S}_{(1)}^{\tilde{N}} \subset \tilde{S}_{(0)}^{1/2 - \tilde{N}}$. △

5 Proofs of Proposition 4.4

Proof of (4.25). Let $\kappa$ be as in Proposition 4.2 and
\[
b = (b_h)_{h \in [0; h_0]} \in S_{h_0}^m(\mathbb{R}^{2d}) \text{ with } \text{supp } b_h \subset \tilde{\Gamma}(\tilde{c} h^{\delta_0}) \cap \Gamma_0. \tag{5.1}
\]
We will show that
\[
\sup_{\{t \in \mathbb{R}: \tilde{t}^{\frac{1}{2} - \mu} \leq |t| \leq t_0\}} |\text{tr } \tilde{L}_h e^{itP_h / h} \tilde{L}_h^*| = O(h^{\infty}), \tag{4.24}
\]
\[
\sup_{\{t \in \mathbb{R}: \tilde{t}^{\frac{1}{2} - \mu} \leq |t| \leq t_0\}} |\text{tr } Q_{\tilde{N}}^h(t) \tilde{L}_h^*| = O(h^{\infty}). \tag{4.25}
\]
that the assumptions (5.1) ensure the existence of \( b_\beta \in S^{m}_{S_0^m}(\mathbb{R}^{2d}) \) such that \( b = \sum_{|\beta|=1} b_\beta h^{-\delta_0} \partial \beta \hat{p} \) and \( \text{supp} b_\beta \subset \text{supp} b \). The integration by parts gives

\[
J^h_t(b) = \sum_{|\beta|=1} t^{-1} H^h_t(h^{1-\delta_0} \partial \beta b_\beta).
\tag{5.3}
\]

The induction hypothesis applied to \( h^{1-\delta_0} \partial \beta b_\beta \in S^{m+2\delta_0-1}_0(\mathbb{R}^{2d}) \) ensures

\[
h^{1-\mu} \leq |t| \leq t_0 \Rightarrow |t^{-1} J^h_t(h^{1-\delta_0} \partial \beta b_\beta)| \leq C h^{-(1-\mu)+\nu-\delta_0-1}.
\tag{5.4}
\]

Since \( \mu - 2 \delta_0 > \mu - \delta_0 - \frac{1}{2} \geq \kappa \), it is clear that (5.3-4) imply (5.2\((n+1))\). \( \triangle \)

Let \( \hat{\mathbf{v}} : \mathbb{R}^{2d} \to \mathbb{R}^{2d} \) be the Hamiltonian flow of \( \hat{\mathbf{p}}_h \), i.e. \( t \to \hat{\mathbf{v}}^h_t(v) \) satisfies

\[
\frac{d}{dt} \hat{\mathbf{v}}^h_t(v) = \mathbf{J} \nabla \hat{\mathbf{p}}_h(\hat{\mathbf{v}}^h_t(v)), \quad \hat{\mathbf{v}}^h_t(v)_{|t=0} = v,
\]

where \( \mathbf{J} = \begin{pmatrix} 0_{\mathbb{R}^d} & I_{\mathbb{R}^d} \\ -I_{\mathbb{R}^d} & 0_{\mathbb{R}^d} \end{pmatrix} \).

**Lemma 5.1** Assume that \( t_0 > 0 \) is small enough. Then

\[
-t_0 \leq t \leq t_0 \Rightarrow |\hat{\mathbf{v}}^h_t(v) - v| \geq |t \nabla \hat{\mathbf{p}}_h(v)|/2.
\tag{5.5}
\]

**Proof.** Set \( M^h_t(v) = \int_0^1 ds \mathbf{J} \nabla d\hat{\mathbf{p}}_h(v + s(\hat{\mathbf{v}}^h_t(v) - v)) \). Then

\[
\mathbf{J} \nabla \hat{\mathbf{p}}_h(\hat{\mathbf{v}}^h_t(v)) - \mathbf{J} \nabla \hat{\mathbf{p}}_h(v) = M^h_t(v)(\hat{\mathbf{v}}^h_t(v) - v)
\tag{5.6}
\]

and \( \frac{d}{dt}(\hat{\mathbf{v}}^h_t(v) - v) = M^h_t(v)(\hat{\mathbf{v}}^h_t(v) - v) + \mathbf{J} \nabla \hat{\mathbf{p}}_h(v) \). Therefore introducing the solution of the linear homogeneous system

\[
\frac{d}{dt} R^h_{t,\tau}(v) = M^h_t(v) R^h_{t,\tau}(v), \quad R^h_{t,\tau}(v)|_{t=\tau} = I
\]

we obtain \( \hat{\mathbf{v}}^h_t(v) - v = \int_0^t d\tau R^h_{t,\tau}(v) \mathbf{J} \nabla \hat{\mathbf{p}}_h(v) \), which ensures

\[
-1 \leq t \leq 1 \Rightarrow |\hat{\mathbf{v}}^h_t(v) - v| \leq C_1 |t \nabla \hat{\mathbf{p}}_h(v)|.
\tag{5.7}
\]

Using \( \int_0^t d\tau \frac{d}{d\tau}(\hat{\mathbf{v}}^h_t(v) - v - \tau \mathbf{J} \nabla \hat{\mathbf{p}}_h(v)) = \int_0^t d\tau \mathbf{J} (\nabla \hat{\mathbf{p}}_h(\hat{\mathbf{v}}^h_t(v)) - \nabla \hat{\mathbf{p}}_h(v)) \) and (5.6) we obtain

\[
|\hat{\mathbf{v}}^h_t(v) - v - t \mathbf{J} \nabla \hat{\mathbf{p}}_h(v)| \leq \int_0^t d\tau |M^h_t(v)||\hat{\mathbf{v}}^h_t(v) - v|.
\tag{5.8}
\]

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Using (5.7) to estimate the right hand side of (5.8) we obtain

\[ -1 \leq t \leq 1 \Rightarrow |\dot{\theta}_t^h(v) - v - tJ\nabla \hat{p}_h(v)| \leq C_2 t^2 |J\nabla \hat{p}_h(v)|. \tag{5.9} \]

Writing \(|\dot{\theta}_t^h(v) - v| \geq |tJ\nabla \hat{p}_h(v)| - |\dot{\theta}_t^h(v) - v - tJ\nabla \hat{p}_h(v)|\) and using (5.9) we obtain (5.5) if \(|t| \leq \min\{1, 1/(2C_2)\}\). \(\triangle\)

For \(h \in ]0; h_0]\) let \(\Omega_h\) be a set of parameters. We say that the family \((b_{\omega,h})_{(\omega,h) \in \Omega_h \times ]0; h_0]}\) is bounded in \(S_0^m(\Gamma)\) if and only if the estimates

\[
\sup_{\omega \in \Omega_h} \sup_{v \in \Gamma_h} |\partial^\alpha b_{\omega,h}(v)| \leq C_{\alpha} h^{-m-|\alpha| \delta} \tag{5.10}
\]

hold for all \(\alpha \in \mathbb{N}^{2d}\).

**Lemma 5.2** Assume that \(l, l \in S_0^1(\mathbb{R}^{2d})\) satisfy

\[
\text{dist}(\text{supp } \bar{l}_h, \text{supp } (1 - l_h)) \geq h^\delta. \tag{5.11}
\]

Let \(\bar{L}_h = \bar{l}_h(x, hD)\) and \(L_h(t) = (l_h \circ \dot{\theta}_t^h)(x, hD)\). Then

\[
||(I - L_h(t))e^{itP_h} \bar{L}_h|| = O(h^\infty). \tag{5.12}
\]

**Proof.** We observe that for \(\alpha \in \mathbb{N}^{2d}\) such that \(|\alpha| \leq 1\), the matrix elements of \((\partial^\alpha \dot{\theta}_t^h)(t,h)\) are bounded families in \(S_0^0(\mathbb{T}_0)\) and it is easy to check that for every \(l \in S_0^m(\mathbb{R}^{2d})\), the family \((l_h \circ \dot{\theta}_t^h)(t,h)\in [-t_0; t_0] \times ]0; h_0]\) is bounded in \(S_0^m(\mathbb{T}_0)\). Then the properties of operators \(P_h\) given in Lemma 2.1 allow us to follow the proof of Lemma 5.1 and Proposition 5.2 of [24] with \(|t| \leq t_0\) instead of \(|t| \leq 2h^{b_0}\). \(\triangle\)

**Proof of (4.24).** We can consider a family of balls \(\{B(\bar{v}_{n,h}, h^\delta)\}_{n \in \{1, ..., N(h)\}}\) covering \(\text{supp } \bar{l} \subset \bar{\Gamma}(\bar{c} h^{b_0}) \cap \mathbb{T}_0\) with \(\bar{v}_{n,h} \in \bar{\Gamma}(\bar{c} h^{b_0}) \cap \mathbb{T}_0\) for \(n \in \{1, ..., N(h)\}\) and \(N(h) \leq C h^{-2\delta}\). Using a suitable partition of unity we decompose

\[
\bar{l}_h = \sum_{1 \leq n \leq N(h)} \bar{l}_{n,h} \text{ with supp } \bar{l}_{n,h} \subset B(\bar{v}_{n,h}, h^\delta),
\]

where \((\bar{l}_{n,h})_{(n,h) \in \{1, ..., N(h)\} \times ]0; h_0]}\) is bounded in \(S_0^0(\mathbb{R}^{2d})\).
Let \((\tilde{l}_{n,h})(n,h) \in \{1, \ldots, N(h)\} \times [0; h_0]\), \((l_{n,h})(n,h) \in \{1, \ldots, N(h)\} \times [0; h_0]\) be bounded in \(S_0^0(\mathbb{R}^{2d})\) and such that \(\text{supp} \ l_{n,h} \subset B(\bar{v}_{n,h}, 2h\delta)\), \(\text{supp} \ \tilde{l}_{n,h} \subset B(\bar{v}_{n,h}, 4h\delta)\). Using the trace cyclicity and assuming \(\tilde{l}_{n,h} = 1\) on \(B(\bar{v}_{n,h}, h\delta)\) we find

\[
\text{tr} \ \hat{L}_h \ e^{itP_h/h_0} \hat{L}_h^* = \sum_{1 \leq n \leq N(h)} \text{tr} \ e^{itP_h/h_0} \hat{L}_h^* \hat{L}_{n,h}
\]

where we have denoted \(\hat{L}_{n,h} = \tilde{l}_{n,h} (x, hD)\) and \(\tilde{l}_{n,h} = l_{n,h} (x, hD)\).

We assume \(l_{n,h} = 1\) on \(B(\bar{v}_{n,h}, 3h\delta)\) and introduce \(L_{n,h}(t) = (l_{n,h} \circ \hat{\vartheta}_t)(x, hD)\).

Since the assertion of Lemma 5.2 still holds with \(l_{n,h}\) and \(\tilde{l}_{n,h}\) instead of \(l_h\) and \(\tilde{l}_h\), (5.13) can be written as

\[
\sum_{1 \leq n \leq N(h)} \text{tr} \ L_{n,h}(t) e^{itP_h/h} \tilde{L}_{n,h} \tilde{L}_h^* \tilde{L}_{n,h} + O(h^\infty).
\]

To complete the proof it suffices to check that

\[
h^{1-\mu} \leq |t| \leq t_0 \Rightarrow \text{supp} \ \tilde{l}_{n,h} \cap \text{supp} \ (l_{n,h} \circ \hat{\vartheta}_t) = \emptyset
\]

holds for a certain \(\delta \in [0; 1/2]\). Indeed, (5.15) ensures \(||L_{n,h} L_{n,h}(t)|| = O(h^\infty)\) and (5.14) is \(O(h^\infty)\) due to the trace cyclicity.

To obtain (5.15) we observe that

\[
\text{dist}(\text{supp} \ l_{n,h}, \text{supp} (l_{n,h} \circ \hat{\vartheta}_t)) \geq \left| \hat{\vartheta}_{-t}(\bar{v}_{n,h}) - \bar{v}_{n,h} \right| - (1 + C_0)h\delta.
\]

Since \(\bar{v}_{n,h} \in \mathcal{G}(\hat{c} h^{\delta_0}) \cap \mathcal{T}_0 \Rightarrow |\nabla \hat{p}_h(\bar{v}_{n,h})| \geq \hat{c} h^{\delta_0}/2\), Lemma 5.1 ensures

\[
h^{1-\mu} \leq |t| \leq t_0 \Rightarrow |\hat{\vartheta}_{-t}(\bar{v}_{n,h}) - \bar{v}_{n,h}| \geq |t \nabla \hat{p}_h(\bar{v}_{n,h})|/2 \geq \hat{c} h^{1-\mu+\delta_0}/4
\]

and (5.16) implies (5.15) if we take \(\delta \in [1 - \mu + \delta_0; 1/2]\). △
6 End of the proof of Theorem 1.2

Since $d^2a_0$ is continuous, we can assume that $c > 0$ is small enough to ensure

$$v \in C_E^{a_0}(c) \Rightarrow \text{rank}(d^2a_0(v)) \geq 2,$$  \hfill (6.1)

where

$$C_E^{a_0}(c) = \{v \in \mathbb{R}^{2d} : |a_0(v) - E| + |\nabla a_0(v)| \leq 2c \}.$$  

Until now we have proved

$$\left| \text{tr} \tilde{f}_h^Z(P_h) \tilde{l}_h - \int_{\mathbb{R}^{2d}} \frac{dv}{(2\pi h)^d} \tilde{f}_h^Z(\hat{p}_h(v)) \tilde{l}_h(v) \right| \leq C_\varepsilon h^{-d} \sum_{1 \leq j \leq 2} R_{E_j}^{a_0}(h) \leq C \varepsilon h^{-d} \sum_{1 \leq j \leq 2} R_{E_j}^{a_0}(h) \leq C \varepsilon h^{-d} R_{E_j}^{a_0}(h).$$  \hfill (6.2)

and we can deduce Theorem 1.2 from Lemma 2.2 if we show

**Proposition 6.1** Let $l$ be as in Lemma 2.2, $\hat{l}$ as in Lemma 2.4 and $\tilde{l}_h = \hat{l}_h(x,hD)$. If (6.1) holds and

$$\frac{1}{2} \left( 1 - \frac{1}{4m_0-1} \right) < \delta_0 < \frac{1}{2},$$  \hfill (6.3)

then

$$\text{tr} \tilde{l}_h f_h^Z(P_h) = o(h^{1-d}),$$  \hfill (6.4)

$$\int_{\mathbb{R}^{2d}} \tilde{l}_h(\hat{f}_h^Z \circ \hat{p}_h) = o(h).$$  \hfill (6.5)

The proof of Proposition 6.1 uses the following trace norm estimate

**Lemma 6.2** Let $\tilde{l}_h = \hat{l}_h(x,hD)$ with $\hat{l} \in S_0^0(\mathbb{R}^{2d})$ and denote

$$\Gamma_h = \text{supp} \hat{l}_h + B(0,h^{\delta_0}) = \{v \in \mathbb{R}^{2d} : \text{dist}(v, \text{supp} \hat{l}_h) < h^{\delta_0} \}.$$  \hfill (6.6)

Then

$$||\tilde{l}_h||_{tr} \leq C h^{-d} \text{vol} \Gamma_h.$$

**Proof.** Let $B_h = b_h(x,hD)$ with $b \in S_0^0(\mathbb{R}^{2d})$. It is well known (cf. e.g. [21]) that the Hilbert-Schmidt norm

$$||B_h||_{HS} = (2\pi)^{-d} \left( \int_{\mathbb{R}^{2d}} |b_h(v)|^2 h^{-d} dv \right)^{1/2} \leq C h^{-d/2} (\text{vol} [\text{supp} b_h])^{1/2}.$$  

We can assume $b_h = 1$ on $\text{supp} \hat{l}_h$ and $\text{supp} b_h \subset \Gamma_h$. Therefore we have

$$||\tilde{l}_h(I - B_h)||_{tr} = O(h^{\infty})$$

and we complete the proof writing

$$||\tilde{l}_h B_h||_{tr} \leq ||\tilde{l}_h||_{HS}||B_h||_{HS} \leq C h^{-d} \left( \text{vol} [\text{supp} \tilde{l}_h] \cdot \text{vol} [\text{supp} b_h] \right)^{1/2}. \quad \triangle$$

Instead of Proposition 6.1 it suffices to show
**Proposition 6.3** Let \( l \) be as in Lemma 2.2 and \( \hat{\ell} \) as in Lemma 2.4. If \( \Gamma_h \) is given by (6.6), then

\[
\text{vol} \Gamma_h = O(h^{\delta_0(4m_0-1)/(2m_0-1)}). \tag{6.7}
\]

Indeed, since (6.3) ensures \( \delta_0(4m_0-1)/(2m_0-1) > 1 \), it is obvious that (6.7) implies (6.5) and using Lemma 6.2 we obtain similarly (6.4).

To begin the proof of Proposition 6.3 we introduce the following notation:

\[
\begin{aligned}
\partial_j &= \partial_{x_j} \quad \text{if } j \in J^+ = \{1, \ldots, d\} \\
\partial_j &= \partial_{\xi_{-j}} \quad \text{if } j \in J^- = \{-1, \ldots, -d\}.
\end{aligned}
\]

Let \( \bar{v} \in \mathbb{R}^{2d} \) be such that \( \text{rank}(d^2a_0(\bar{v})) \geq 2 \). Then there exist \( j(1, \bar{v}), j(2, \bar{v}) \in J = J^+ \cup J^- \) such that \( \nabla \partial_{j(1, \bar{v})}a_0(\bar{v}) \neq 0 \) for \( k \in \{1, 2\} \) and the angle \( \angle(\nabla \partial_{j(1, \bar{v})}a_0(\bar{v}), \nabla \partial_{j(2, \bar{v})}a_0(\bar{v})) \neq 0 \). Therefore we can find two linearly independent vectors \( e_1, \bar{v}, e_2, \bar{v} \in \mathbb{R}^{2d} \) satisfying

\[
e_{k, \bar{v}} \cdot \nabla \partial_{j(1, \bar{v})}a_0(\bar{v}) = \theta_{k, \bar{v}} > 0, \tag{6.8}
\]

\[
e_{k, \bar{v}} \notin \{(0, \xi) \in \mathbb{R}^{2d} : \xi \in \mathbb{R}^d\} \tag{6.9}
\]

for \( k \in \{1, 2\} \). Since \( d \geq 2 \) we can find

\[
e_3, \bar{v} \in \{(0, \xi) \in \mathbb{R}^{2d} : \xi \in \mathbb{R}^d\} \tag{6.10}
\]

such that the system \( (e_{k, \bar{v}})_{k \in \{1, 2, 3\}} \) is linearly independent.

**Lemma 6.4** Let \( \bar{v} \in \mathbb{R}^{2d} \) be such that \( \text{rank}(d^2a_0(\bar{v})) \geq 2 \) and let \( \varepsilon_0 > 0 \) be small enough. If \( \hat{\Gamma}_h \) is given by (2.20) and \( v \in \mathbb{R}^{2d} \), then the Lebesgue measure of

\[
\{ s \in \mathbb{R} : se_{k, \bar{v}} + v \in B(\bar{v}, \varepsilon) \cap \hat{\Gamma}_h \} \tag{6.11(k)}
\]

can be estimated by \( C_0 h^{\rho_k} \), where

\[
\rho_1 = \rho_2 = \delta_0, \quad \rho_3 = \delta_0/(2m_0 - 1) \tag{6.12}
\]

and the constant \( C_0 \) is independent of \( (h, v) \in [0; h_0] \times \mathbb{R}^{2d} \).
Let us check that Proposition 6.3 follows from Lemma 6.4. To begin we observe that (2.10) gives
\[ v \in \Gamma_h \subset \text{supp } l_h + B(0, h^{\delta_0}) \Rightarrow |a_0(v) - E| \leq c + Ch^{\delta_0} \]
and using \( \Gamma_h \subset \text{supp } \hat{l}_h \subset \hat{\Gamma}_h \) we can choose \( h_0 > 0 \) small enough to ensure \( \Gamma_h \subset C_{\hat{E}}^{a_0}(c) \) for \( h \in ]0; h_0[ \). The family \( \{ B(\tilde{v}, \varepsilon_0) \}_{\tilde{v} \in C_{\hat{E}}^{a_0}(c)} \) contains a finite covering of \( C_{\hat{E}}^{a_0}(c) \) and it suffices to show that for every \( \tilde{v} \in C_{\hat{E}}^{a_0}(c) \) one has
\[ \text{vol } B(\tilde{v}, \varepsilon_0) \cap \hat{\Gamma}_h \leq \bar{C}_\varepsilon h^{p_1 + p_2 + p_3}. \quad (6.13) \]
The system \( (e_{k,\tilde{v}})_{k \in \{1,2,3\}} \) can be completed to a basis \( (e_{k,\tilde{v}})_{k \in \{1,\ldots,2d\}} \) and let \( (e_{k,\tilde{v}}^*)_{k \in \{1,\ldots,2d\}} \) denote the dual basis in \( (\mathbb{R}^{2d})^* \). Let \( (e_k)_{k \in \{1,\ldots,2d\}} \) denote the canonical basis of \( \mathbb{R}^{2d} \) and let \( (e_k^*)_{k \in \{1,\ldots,2d\}} \) be its dual. Then the set \( (6.11(k)) \) has the form
\[ e_{k,\tilde{v}}^*(B(\tilde{v}, \varepsilon_0) \cap \hat{\Gamma}_h - v) = e_k^*(W_\varepsilon(B(\tilde{v}, \varepsilon_0) \cap \hat{\Gamma}_h - v)), \]
where \( W_\varepsilon \in \text{Hom}(\mathbb{R}^{2d}, \mathbb{R}^{2d}) \) is the matrix of the corresponding change of variables. The assertion of Lemma 6.4 allows us to estimate the measure of \( \{ s \in \mathbb{R} : s e_k + v \in W_\varepsilon(B(\tilde{v}, \varepsilon_0) \cap \hat{\Gamma}_h) \} \)
by \( \bar{C}_\varepsilon h^{p_k} \) for \( (h, v) \in ]0; h_0[ \times \mathbb{R}^{2d} \), hence the Fubini’s theorem gives
\[ \text{vol } W_\varepsilon(B(\tilde{v}, \varepsilon_0) \cap \hat{\Gamma}_h) \leq \bar{C}_\varepsilon h^{p_1 + p_2 + p_3}. \quad (6.15) \]
However \( \text{vol } B(\tilde{v}, \varepsilon_0) \cap \hat{\Gamma}_h = | \det W_\varepsilon |^{-1} \text{vol } W_\varepsilon(B(\tilde{v}, \varepsilon_0) \cap \hat{\Gamma}_h) \) and it is clear that (6.13) follows from (6.15).

**Proof of Lemma 6.4.** To begin we observe that the set \( (6.11(k)) \) is included in the interval
\[ \Delta_{k, \varepsilon, v} = \{ s \in \mathbb{R} : s e_k + v \in B(\tilde{v}, \varepsilon_0) \}. \quad (6.16) \]
For \( k \in \{1, \ldots, 2d\} \) let \( u_{k, \varepsilon, v} : \Delta_{k, \varepsilon, v} \to \mathbb{R} \) be defined by the formula
\[ u_{k, \varepsilon, v}(s) = \partial_j(k, \varepsilon_0) a_0(s e_k + v). \quad (6.17) \]
By the definition of \( \hat{\Gamma}_h \) we find that the set \( (6.11(k)) \) is included in
\[ \{ s \in \Delta_{k, \varepsilon, v} : -\bar{C} h^{\delta_0} \leq u_{k, \varepsilon, v}(s) \leq \bar{C} h^{\delta_0} \}. \quad (6.18(k)) \]
We claim that using (6.8) we can ensure
\[ s \in \Delta_{k, \bar{v}, \bar{v}} \Rightarrow \frac{d}{ds} u_{k, \bar{v}, \bar{v}}(s) > \theta_{k, \bar{v}}/2, \]  
(6.19)
for \( k \in \{1, 2\} \) if \( \varepsilon_{\bar{v}} > 0 \) is fixed small enough. It is clear that (6.19) implies the fact that (6.18(3)) defines an interval of length smaller than \( 2\bar{C}h^{\delta_0}/\theta_{k, \bar{v}} \).

In order to prove (6.19) we observe that
\[ \text{Lemma 6.5} \]

hence
\[ \frac{d}{ds} u_{k, \bar{v}, \bar{v}}(0) = \theta_{k, \bar{v}} > 0. \]

Moreover
\[ s \in \Delta_{k, \bar{v}, \bar{v}} \Rightarrow \left| \frac{d}{ds} u_{k, \bar{v}, \bar{v}}(s) - \frac{d}{ds} u_{k, \bar{v}, \bar{v}}(0) \right| \leq C|se_{k, \bar{v}} + v - \bar{v}|^{\tau_0} \leq C\varepsilon_{\bar{v}}^{\tau_0} \]
and (6.19) follows if \( \varepsilon_{\bar{v}} \) is such that \( C\varepsilon_{\bar{v}}^{\tau_0} < \theta_{k, \bar{v}}/2. \)

To estimate the measure of (6.11(3)) we observe that (6.18(3)) holds if we take \( u_{3, \bar{v}, \bar{v}}(s) = e_{3, \bar{v}} \cdot \nabla \partial_j (k, \bar{v}) a_0(se_{k, \bar{v}} + v) \), which is polynomial of degree \( 2m_0 - 1 \) due to (6.10) and the ellipticity hypothesis (1.9) ensures \( (\frac{d}{ds})^{2m_0-1} u_{3, \bar{v}, \bar{v}}(s) = (\frac{d}{ds})^{2m_0-1} u_{3, \bar{v}, \bar{v}}(0) \geq c_0 > 0. \) Thus to complete the proof of Lemma 6.4 it suffices to show

**Lemma 6.5** Let \( (F_\omega)_{\omega \in \Omega} \) be a family of polynomials of order \( m \in \mathbb{N} \) and
\[ \Delta^h_\omega = \{ s \in \mathbb{R} : |F_\omega(s)| < (Ch)^{\delta_0} \}. \]

If the \( m \)-th derivative satisfies \( |F_\omega^{(m)}(0)| \geq 1 \), then the Lebesgue measure of \( \Delta^h_\omega \) can be estimated by \( C_m h^{\delta_0/m} \), where \( C_m \) is independent of \( \omega \in \Omega \).

**Proof.** We drop the index \( \omega \) and for \( k \in \{0, ..., m\} \) we set
\[ \Delta^k_h = \{ s \in \mathbb{R} : |F^{(k)}(s)| < (Ch)^{\delta_k} \}, \]
where \( \delta_k = \delta_0(1 - k/m) \). We can write \( \Delta^h_0 \) as the union of
\[ \Delta^h_{(j_1,j_2,...,j_m)} = \Delta^h_j \cap \Delta^h_{j_1} \cap \Delta^h_{j_2} \cap \ldots \cap \Delta^h_{j_m}, \]
where \( j_k \in \{1, -1\} \), \( \Delta^h_{j} = \Delta^h_k \) and \( \Delta^h_{-j} = \mathbb{R} \setminus \Delta^h_k \).

Since \( \delta_m = 0 \), the assumption \( |F^{(m)}(0)| \geq 1 \) implies \( \Delta^h_0 = \emptyset \) and for every \( (j_1, ..., j_m) \in \{1, -1\}^m \) we can find \( k \in \{1, ..., m\} \) such that
\[ \Delta^h_{(j_1,j_2,...,j_m)} \subset \Delta^h_{k-1} \setminus \Delta^h_k \]

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However $F^{(k)}$ is a polynomial of order $m-k$, $\Delta^h_k$ is a union of at most $m-k$ intervals and

$$\Delta^h_{k-1} \setminus \Delta^h_k = \bigcup_{1 \leq j \leq (k,h)} \Delta^h_{k,j}$$

where $j(k, h) \leq 2(m - k)$ and $\Delta^h_{k,j}$ are intervals such that

$$s \in \Delta^h_{k,j} \Rightarrow |F^{(k-1)}(s)| < (Ch)^{\delta_{k-1}} \text{ and } |F^{(k)}(s)| \geq (Ch)^{\delta_k}.$$ 

It is clear that the length of $\Delta^h_{k,j}$ is less than $2(Ch)^{\delta_{k-1} - \delta_k} = 2(Ch)^{\delta_0/m}$. △

7 Appendix

Proof of Lemma 2.1. For $\alpha \in \mathbb{N}^d$ we denote $\gamma_h^\alpha(x) = (\partial^\alpha \gamma)(h^{-\delta_0} x) h^{-\delta_0}$. Further on $\alpha = \alpha' + \alpha''$ with $|\alpha''| \leq 2$ and dropping the indices $\nu, \bar{\nu}$ we write

$$\partial^\alpha a_h(x) = \partial^{\alpha'+\alpha''} a_h(x) = \int_{\mathbb{R}^d} \partial^{\alpha''} a(y) \gamma_h^{\alpha'}(x-y) h^{-\delta_0|\alpha'|} dy. \quad (7.1)$$

If $|\alpha| \leq 2$ then it is clear that $\partial^\alpha a_h = O(1)$ follows from (7.1) with $\alpha' = 0$. Further on we assume $|\alpha| \geq 3$ and $|\alpha''| = 2$. Then $|\alpha'| \geq 1 \Rightarrow \int \gamma_h^{\alpha'}(x-y) dy = 0$ and (7.1) still holds if $\partial^{\alpha''} a(y)$ is replaced by $\partial^{\alpha''} a(y) - \partial^{\alpha''} a(x)$. Therefore

$$|\partial^\alpha a_h(x)| \leq \int_{\mathbb{R}^d} |\partial^{\alpha''} a(y) - \partial^{\alpha''} a(x)||\gamma_h^{\alpha'}(x-y)| h^{-\delta_0|\alpha'|} dy \leq C \int_{\mathbb{R}^d} |y-x|^{r_0} |\gamma_h^{\alpha'}(x-y)| h^{-\delta_0|\alpha'|} dy = Ch^{(r_0 - |\alpha'|)\delta_0} \int_{\mathbb{R}^d} |z|^{r_0} |\gamma^{(\alpha')} (z)| dz,$$

i.e. we obtain $\partial^\alpha a_h = O(h^{r_0+2-|\alpha'|}\delta_0)$ and (2.5) follows.

If $|\alpha''| = 2$ then we can estimate

$$\partial^{\alpha''} a_h(x) - \partial^{\alpha''} a(x) = \int_{\mathbb{R}^d} (\partial^{\alpha''} a(y) - \partial^{\alpha''} a(x)) \gamma_h^0(x-y) dy \quad (7.3)$$

by $O(h^{r_0\delta_0})$ similarly as the right hand side of (7.2) with $\alpha' = 0$.

In the next step we estimate (7.3) when $|\alpha''| = 1$. Since $\int y \gamma(y) dy = 0$, we can replace $\partial^{\alpha''} a(y) - \partial^{\alpha''} a(x)$ by

$$\partial^{\alpha''} a(y) - \partial^{\alpha''} a(x) - (y - x) \cdot \nabla \partial^{\alpha''} a(x) \quad (7.4)$$
in the right hand side of (7.3). We can express (7.4) as
\[
\int_0^1 ds \ (y - x) \cdot (\nabla \partial^\alpha a(x + s(y - x)) - \nabla \partial^\alpha a(x)), \tag{7.4'}
\]
hence its absolute value is \(O(|y - x|^{1+r_0})\) and
\[
|\partial^\alpha a_h(x) - \partial^\alpha a(x)| \leq C \int_{\mathbb{R}^d} |y - x|^{1+r_0} |\gamma_h^0(x - y)| dy = C_{r_0} h^{(1+r_0)\delta_0}.
\]
At the beginning of Section 2 we assumed \((2 + r_0)\delta_0 > 1\), hence it is easy to see that the proof of (2.6) will be complete if we show \(a_h - a = O(h^{2+r_0})\).
However writing (7.3) with \(a'' = 0\) and using \(\int y^\alpha \gamma(y) dy = 0\) when \(1 \leq |\alpha| \leq 2\) we can replace \(a(y) - a(x)\) by
\[
a(y) - \sum_{|\alpha| \leq 2} (y - x)^\alpha \partial^\alpha a(x)/\alpha! = \sum_{|\alpha| = 2} (y - x)^\alpha \int_0^1 ds 2(1 - s) (\partial^\alpha a(x + s(y - x)) - \partial^\alpha a(x))/\alpha!.
\]
Since the last expression is \(O(|y - x|^{2+r_0})\), we obtain
\[
|a_h(x) - a(x)| \leq C \int_{\mathbb{R}^d} |y - x|^{2+r_0} |\gamma_h^0(x - y)| dy = C_{r_0} h^{(2+r_0)\delta_0}.
\]
The proof of the assertion b) is described in Appendix of [24].

**Proof of Lemma 2.2.** Due to (2.7) and the min-max principle, it suffices to show that (1.5) holds with \(\mathcal{N}(P_h \pm, E)\) instead of \(\mathcal{N}(A_h, E)\). We drop \pm and we observe that it suffices to prove
\[
\mathcal{N}(P_h, E) = (2\pi h)^{-d} c_E^h + O(h^{-d}) R_E^{\varepsilon,a_0}(h). \tag{7.5}
\]
where \(c_E^h = \text{vol} \{ v \in \mathbb{R}^{2d} : \hat{\mu}_h(v) \leq E \}\). Indeed, due to (2.6) one has
\[
|c_E^h - c_E| \leq \text{vol} \{ v \in \mathbb{R}^{2d} : |a_0(v) - E| \leq Ch \} \leq C \varepsilon R_E^{\varepsilon,a_0}(h).
\]
Let \(g \in C_0^\infty([E - c; E + c]), l_h = g^2 \circ \hat{\mu}_h\) and \(L_h = l_h(x, hD)\). Then reasoning as in Section 3 of [24] we find \(||g^2(P_h) - L_h||_{W} = O(h^{1-d})\) and combining this estimate with (2.11) we have
\[
\text{tr} (g^2 \tilde{T}_h)(P_h) = \int_{\mathbb{R}^{2d}} \frac{dv}{(2\pi h)^d} (g^2 \tilde{T}_h)(\hat{\mu}_h(v)) + O(h^{-d}) \sum_{1 \leq j \leq 2} R_E^{\varepsilon/2,a_0}(h). \tag{7.6}
\]
if $Z = [E_1; E_2] \subset [E - c; E + c]$. Next we observe that

$$|(\tilde{f}_h^Z - \mathbb{1}_Z)(\lambda)| \leq C_N \sum_{1 \leq j \leq 2} \left(1 + \frac{|\lambda - E_j|}{h}\right)^{-N}$$  \tag{7.7}$$

holds for every $N \in \mathbb{N}$, hence reasoning as in Section 4 we obtain

$$\left(2\pi h\right)^{-d} \int_{\mathbb{R}^{2d}} (g^2(\tilde{f}_h^Z - \mathbb{1}_Z)) \circ \hat{p}_h = O(h^{-d}) \sum_{1 \leq j \leq 2} \mathcal{R}^{\xi/2, a_0}_{E_j}(h).$$  \tag{7.8}$$

Taking $Z = [E'; E' + h] \subset [E - c; E + c]$ we obtain

$$\left(2\pi h\right)^{-d} \int_{\mathbb{R}^{2d}} (g^2(\tilde{f}_h^{E'; E' + h})) \circ \hat{p}_h = O(h^{-d}) \mathcal{R}^{\xi/2, a_0}_{E'}(h).$$  \tag{7.9}$$

Indeed, due to (7.8) it suffices to observe that (7.9) holds if $\tilde{f}_h^{E'; E' + h}$ is replaced by $\mathbb{1}_{[E'; E' + h]}$. As a consequence of (7.6) and (7.9) we find

$$\text{tr} (g^2(\tilde{f}_h^{E'; E' + h}))(P_h) = O(h^{-d}) \mathcal{R}^{\xi/2, a_0}_{E'}(h).$$  \tag{7.10}$$

We assume moreover $g \geq 0$ and $g = 1$ in a neighbourhood of $E$. Let $\tilde{g} \in C_0^\infty(-\infty; E]$ satisfy $\tilde{g} + g^2 = 1$ on $[\min\{\inf \hat{p}_h, \inf \sigma(P_h)\}; E]$. Then

$$\epsilon_{E}^h = \int_{\mathbb{R}^{2d}} \tilde{g} \circ \hat{p}_h + \int_{\mathbb{R}^{2d}} (g^2 \mathbb{1}_{[E - c; E]}) \circ \hat{p}_h,$$  \tag{7.11}$$

$$\mathcal{N}(P, E) = \text{tr} \tilde{g}(P_h) + \text{tr} (g^2 \mathbb{1}_{[E - c; E]})(P_h).$$  \tag{7.12}$$

However reasoning as in Section 3 of [24] we obtain

$$\text{tr} \tilde{g}(P_h) = \left(2\pi h\right)^{-d} \int_{\mathbb{R}^{2d}} \tilde{g} \circ \hat{p}_h + O(h^{1-d}),$$  \tag{7.13}$$

hence in order to obtain (7.5) it suffices to show

$$\text{tr} (g^2 \mathbb{1}_{[E - c; E]})(P_h) = \left(2\pi h\right)^{-d} \int_{\mathbb{R}^{2d}} (g^2 \mathbb{1}_{[E - c; E]}) \circ \hat{p}_h + O(h^{-d}) \mathcal{R}^{\xi, a_0}_{E}(h)$$  \tag{7.14}$$

and due to (7.6), (7.8), it is clear that (7.14) follows from

$$\text{tr} (g^2(\tilde{f}_h^{E - c; E} - \mathbb{1}_{[E - c; E]}))(P_h) = O(h^{-d}) \mathcal{R}^{\xi, a_0}_{E}(h).$$  \tag{7.15}$$
To begin the proof of (7.15) we observe that modulo $O(h^\infty)$ we can estimate the left hand side of (7.15) by

$$\text{tr} \, g^2 (P_h) \left( 1 + \frac{|P_h - E|}{h} \right)^{-N} \leq \sum_{k \in \mathbb{Z}} \frac{2 \text{tr} \left( g^2 \mathbb{I}_{[E+h; E+(k+1)h]} \right) (P_h)}{(1 + \min\{|k|, |k+1|\})^N}$$  \hspace{1cm} (7.16)

due to (7.7) and clearly the contribution of $\sum_{|k| \geq h^{-\epsilon/2}}$ is $O(h^{N/2-d})$.

Next we observe that $\tilde{\gamma}_1 > 0$ allows us to find a constant $C_0 > 0$ such that $\mathbb{I}_{[E'; E'+h]} \leq C_0 \tilde{f}_h^{[E'; E'+h]}$, hence the contribution of $\sum_{|k| < h^{-\epsilon/2}}$ in the right hand side of (7.16) can be estimated by

$$C \sup_{|k| < h^{-\epsilon/2}} \text{tr} \left( g^2 \tilde{f}_h^{[E+h; E+(k+1)h]} \right) (P_h) \leq C_\varepsilon h^{-d} \sup_{|k| < h^{-\epsilon/2}} \mathcal{R}_{E+h}^{\varepsilon,0} (h)$$  \hspace{1cm} (7.17)

due to (7.10). It is clear that (7.17) can be estimated by $O(h^{-d})\mathcal{R}_{E}^{\varepsilon,0} (h)$. $\Delta$

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