ON EFFECTIVE DIMENSIONAL REDUCTION IN HYPERBOLIC SPACES

E.V. Gorbar

Bogolyubov Institute for Theoretical Physics, 03680
Kiev, Ukraine

Abstract

It is shown that the classical motion of massive particles in hyperbolic spaces $H^D$ has a bounded character in $D-1$ coordinates. Studying the Dirac equation, it is found that a bounded character of the classical motion corresponds to the effective dimensional reduction $D+1 \rightarrow 1+1$ for fermions in the infrared region in the quantum problem. This effective dimensional reduction leads to the zero critical value of coupling constant for dynamical symmetry breaking in hyperbolic spaces.

1 Introduction

It is well known that dynamical symmetry breaking (DSB) and a mass (gap) generation for fermions usually require the presence of a strong attractive interaction [1] in order to break symmetry that makes the quantitative study of DSB a difficult problem. Therefore, it is very interesting to consider the cases where DSB takes place in the regime of weak coupling. Three such examples are known.

The first is breaking of the $U(1)$ gauge symmetry in the presence of the Fermi surface when some fermion states are filled. According to the Bardeen–Cooper–Schrieffer theory of superconductivity [2], the Fermi surface is crucial for the formation of a bound state and a symmetry breaking condensate in the case of arbitrary small attraction between fermions. Indeed, according to [3, 4], the renormalization group scaling in this case is connected only with the direction perpendicular to the Fermi surface. Therefore, the effective dimension of spacetime is $1+1$ from the viewpoint of renormalization group scaling. Since a bound state can be generated for any attraction in $1+1$ dimension, this implies that the critical coupling constant is zero in this case.

The other example of DSB in the regime of weak coupling is DSB in a constant magnetic field [5, 6], where chiral symmetry is also dynamically broken for an arbitrary weak attraction. The physical reason for this is the effective dimensional reduction of spacetime for fermions in the infrared region by 2 units in a constant magnetic field (DSB in a magnetic field in spacetimes of dimension higher than four was considered in [7]). Indeed, electrons being charged particles cannot propagate in directions perpendicular to the magnetic field when their energy is much less than the Landau gap $\sqrt{|eB|}$. This leads to the effective dimensional reduction in the infrared region by 2 units.

The dynamics of fermions in hyperbolic spaces $H^D$ gives the third known example of DSB with zero critical coupling constant [8–12]. Analysing the heat kernel in spacetimes $R \times H^D$, it was shown in [13] that the zero value of critical coupling constant for DSB in this case is connected also with the effective dimensional reduction $D+1 \rightarrow 1+1$ for fermions in the infrared region. In a recent paper [14], chiral and diquark condensates were studied in the extended NJL model in the hyperbolic space $H^3$ and it was shown that negative curvature increases the values of both condensates.

The dynamics of quantum fields in hyperbolic spaces has received a great deal of attention in the literature (see, e.g., [15, 16] and the references therein). Callan and Wilczek advocated hyperbolic spaces as a geometric means to regularize the infrared divergences of quantum field theories [17]. Further, spatial sections of open Friedmann-Robertson-Walker models are hyperbolic spaces. The near-horizon optical metric of non-extreme black holes is asymptotically hyperbolic also [18–20].

Although the analysis of the heat kernel in [13] clearly showed the presence of the effective dimensional reduction in spacetimes $R \times H^D$, physical reasons for such a reduction remained unknown because the heat kernel is only an integral characteristic of the dynamics of the system. Therefore, it would be certainly desirable to clarify physical reasons of such a reduction. Fortunately, there exists a physically transparent way to demonstrate the occurrence of the effective dimensional reduction in $H^D$. According to [21], the effective dimensional reduction in the infrared region takes place in a quantum problem only when classical motion in the corresponding
problem has a bounded character with respect to the coordinates over which the dimensional reduction occurs. Let us consider, for example, the effective dimensional reduction in a constant magnetic field. In this case, a charged classical particle moves on circular orbits in the plane perpendicular to the constant magnetic field. Since the radius of orbit is proportional to energy, the particle can go to infinity only if it has infinite energy. Therefore, a charged particle of finite energy always moves only in a finite region of the plane perpendicular to the constant magnetic field. This bounded character of motion means that the system is effectively of a finite size and leads to the effective dimensional reduction by 2 units in the infrared region in the quantum problem [5]. Therefore, it is interesting to consider the classical motion in hyperbolic spaces and see whether this motion has indeed a bounded character with respect to the coordinates over which the effective dimensional reduction takes place. Briefly this motion and an analysis of the Dirac equation in hyperbolic spaces were considered in a recent paper [22]. In the present paper, we consider these problems in more detail.

2 Classical motion in hyperbolic spaces

In this section, we study the classical motion in hyperbolic spaces. Let us first recall what hyperbolic spaces are and introduce the necessary notation. Hyperbolic spaces \( H^D \) are symmetric Riemannian spaces of constant negative curvature whose interval in the Poincaré coordinates is given by

\[
dl^2 = \frac{a^2}{x_1^2}(dx_1^2 + dx_2^2 + \ldots + dx_D^2),
\]

where \( x_1 > 0 \) and \( a \) is the curvature radius. Further, the spacetime that we consider is \( R \times H^D \) and the corresponding spacetime interval is

\[
ds^2 = c^2 dt^2 - dl^2.
\]

Relativistic motion of free particles proceeds along geodesics (see, e.g., [23]) which are extrema of the action

\[
S = -mc \int ds = -mc^2 \int \sqrt{1 - \sum_{k=1}^{D} \frac{a^2 x_k^2}{c^2 x_1^2}} dt.
\]

Since the Lagrangian in (3) does not depend explicitly on \( t \), energy

\[
E = \sum_k p_k \dot{x}_k - L = \frac{mc^2}{\sqrt{1 - \sum_k \frac{a^2 x_k^2}{c^2 x_1^2}}} \tag{4}
\]

is an integral of motion. Using this fact, it is not difficult to find the equations of motion. They are

\[
\frac{d^2 x_1}{dt^2} = \frac{\dot{x}_1^2 - \dot{x}_2^2 - \dot{x}_3^2 - \ldots - \dot{x}_D^2}{x_1}, \tag{5}
\]

where \( C_2, \ldots, C_D \) are integration constants. Substituting (11) in (6), the equation for \( x_1 \) takes the form

\[
\frac{d^2 x_1}{dt^2} = \frac{\dot{x}_1^2}{x_1} = -C^2 x_1^3, \tag{8}
\]

where \( C^2 = C_2^2 + \ldots + C_D^2 \). Making the change of variable \( z = \ln x_1 \) (note that this change of variable is unambiguous because \( x_1 > 0 \)), we find the following equation for \( z \):

\[
\frac{d^2 z}{dt^2} = -C^2 e^{2z}. \tag{9}
\]

Since this equation does not depend explicitly on \( t \), we introduce \( y(z) = \dot{z} \) and obtain the equation

\[
\frac{d y^2}{dz} = -2C^2 e^{2z}, \tag{10}
\]

which is easily integrated

\[
y^2 = -C^2 e^{2z} + A, \tag{11}
\]

where \( A \) is an integration constant. Since \( y^2 \) is positive, the right-hand side of Eq. (11) should be positive also, therefore, \( A > 0 \). Since \( y = \dot{z} \), for \( A = u^2 \), it follows from Eq. (11) that

\[
\dot{z} = \pm \sqrt{u^2 - C^2 e^{2z}}. \tag{12}
\]

Without the loss of generality, we assume the sign minus in Eq. (12) because in view of the change \( t \rightarrow -t \) both sings are equivalent. Then integrating Eq. (12), we obtain the sought solution

\[
z(t) = \ln \frac{u}{C \cosh(ut + b)}. \tag{13}
\]
where $b$ is an integration constant. Using (13), it is not difficult to integrate Eqs. (7) and get the law of motion in other $D - 1$ space coordinates $x_2, \ldots, x_D$. Thus, we find the following classical trajectories of motion in $H^D$ (geodesics):

$$
z(t) = \ln \left( x_1(t) \right) = \ln \left( \frac{u}{C \cosh (ut + b)} \right),
$$

$$
x_2(t) = \frac{uC_2}{C^2} \tanh (ut + b) + \tilde{C}_2,
$$

$$
\ldots
$$

$$
x_D(t) = \frac{uC_D}{C^2} \tanh (ut + b) + \tilde{C}_D,
$$

where $C = \sqrt{C_2^2 + C_3^2 + \ldots + C_D^2}$ and $u, b, C_2, \tilde{C}_2, \ldots, C_D, \tilde{C}_D$ are arbitrary constants (there are 2D of them and, for a particular trajectory, they are fixed by initial conditions). Finally, it is not difficult to check that energy (14) that corresponds to this motion equals $E = mu^2 / (1 - v^2 / c^2)^{1/2}$, where $v = au$. This means that $u$ cannot exceed $c/a$.

At this point, we would like to comment why we prefer to work with the $z = \ln x_1$ coordinate rather than $x_1$. In a certain sense, the $z$ coordinate is more natural from the viewpoint of metric (1) because then the corresponding contribution to the interval has the flat space form $dz^2$ and $z$ takes values on the whole real axis unlike the $x_1$ coordinate which takes values only on the positive semiaxis. It is easy to see from Eqs. (14) that the motion in the coordinate $z$ has the same character as the usual flat space motion except a time interval of order $1/u$. Indeed, according to Eqs. (14), the classical particle moves like $z(t) = ut + z_1$ for $t \ll -\frac{1}{u}$. For $|t| \leq \frac{1}{u}$, its motion differs from the familiar inertial motion in flat space. For $t \gg \frac{1}{u}$, the particle moves like $z(t) = -ut + z_2$, i.e., it changes the direction of motion and goes back to $-\infty$, where it started its motion. Thus, the motion in the coordinate $z$ has a familiar inertial character except a time interval of order $1/u$ with the resulting change of the direction of motion.

On the other hand, according to Eqs. (14), motion in $x_2, \ldots, x_D$ coordinates has a completely different character. The particle is practically motionless for almost all period of time except the time interval of order $1/u$ when it moves some finite distance. Therefore, the classical motion in these coordinates has a bounded character. On the other hand, coordinates can be arbitrarily chosen in curved spacetime, therefore, we should search for an invariant way to demonstrate a bounded character of the classical motion in hyperbolic spaces. For this, we consider Killing vector fields connected with translations in $x_2, x_3, \ldots, x_D$ coordinates.

The hyperbolic space has $D - 1$ “translational” Killing vector fields $\xi_i$ ($i = 2, D$), which, in the coordinates employed in this paper, are given by

$$
\xi_i = \frac{\partial}{\partial x_i}, \quad i = 2, \ldots, D.
$$

Note that these Killing vector fields realize translations in the corresponding coordinates and form a commuting subalgebra of Killing vector fields in the hyperbolic space. Using the normalized vector fields

$$
n_i = \frac{\xi_i}{(-\xi_1 \cdot \xi_i)^{1/2}}, \quad i = 1, \ldots, D,
$$

where the central dot denotes the scalar product in metric (1), we calculate the following reparametrization-invariant integral along a geodesic:

$$
I_i^{\text{geo}} = \int |(u_i \cdot n_i)| dt = a \int \sqrt{\frac{x_i^2}{x_1^2}} dt = \frac{\pi a |C_i|}{C},
$$

where $u_i$ is the tangent vector to the geodesic and integration in $t$ goes from $-\infty$ to $+\infty$. Clearly, Eq. (17) describes the geodesic motion in $x_i$ ($i = 2, D$) coordinates and, what is important, it does not depend on the choice of coordinates because $n_i$ are expressed through Killing vector fields. Since $|C_i| < C$, the quantities $I_i^{\text{geo}}$ are always less than $L = \pi a$. This is an important result. First of all, note that if we fix by hand $x_1$ and integrate in Eq. (17) over $dx_1$ (of course, the corresponding curve is not a geodesics), then the integral in (17) will be infinite. Therefore, a priori, one may expect the space $I_2^{\text{geo}} \times \ldots \times I_D^{\text{geo}}$ to be a subspace of $(D - 1)$-dimensional Euclidean space $R^{D-1}$ unbounded in all $D - 1$ directions. However, according to Eq. (17), this is not correct and the space $I_2^{\text{geo}} \times \ldots \times I_D^{\text{geo}}$ connected with geodesic motions is a $(D - 1)$-dimensional cube $L^{D-1}$. Since $I_i^{\text{geo}}$ are invariant with respect to the change of coordinates, we conclude that geodesic motion in hyperbolic spaces has a bounded character in $D - 1$ directions. Finally, let us emphasize that this result does not mean that geodesic motion takes place in a certain finite region in $x_2, \ldots, x_D$ coordinates. Eqs. (14) imply, for example, that one can connect any two points with diffe-
rent \(x_i\) by a geodesics because \(|C_i|/C^2\) can be arbitrary large unlike \(|C_i|/C\) that enters Eq. (12), which is always bounded by unity.

At this point, it is instructive to calculate the spatial interval of a geodesics. Obviously, it is invariant with respect to coordinate transformations that do not involve the time coordinate. Using Eqs. (11) and (14), we find

\[
\ell^{(\text{geo})} = \int_{t_a}^{t_b} \sqrt{\sum_{k=1}^{D} \frac{a^2 x_k^2}{x_1^2}} \, dt = u a \int_{t_a}^{t_b} dt. \tag{18}
\]

Obviously, it diverges as \(t_a\) or \(t_b\) tends to infinity. Finally, note that we can consider \(\ell_1^{(\text{geo})} = a \int \sqrt{x_1^2} \, dt\), which is an analog of Eq. (11) for motion in the \(x_1\) coordinate and which, like \(\ell^{(\text{geo})}\), diverges. Using \(\zeta_1 = \partial_{x_1}\), this integral can be represented also as

\[
\ell_1^{(\text{geo})} = \int |\langle u_\parallel \cdot n_1\rangle| \, dt, \tag{19}
\]

where \(n_1 = \zeta_1 / (-\zeta_1 \cdot \zeta_1)^{1/2}\) is the normalized vector field which is orthogonal to the vector fields \(n_i\) \((i = 2, D)\). Although \(\zeta_1\) is similar to \(\xi_*\) given by Eq. (15), it is not a Killing vector field in hyperbolic spaces. Nevertheless, being orthogonal to all \(\xi_*\) vector fields, it is, in fact, uniquely defined. Therefore, \(\ell_1^{(\text{geo})}\) has a clear geometric meaning. Note that the vector fields \(\zeta_1\) and \(\zeta_*\) form a commuting algebra of vector fields equivalent to the algebra of “translational” Killing vector fields of D-dimensional Euclidean space. Therefore, perhaps, the most clear signature of a bounded character of motion in hyperbolic spaces is provided by the ranges of values of \(\ell_1^{(\text{geo})}\) and \(\ell_i^{(\text{geo})}\).

### 3 Solutions of the Dirac equation and the effective dimensional reduction

In the previous section, we showed that the classical motion in hyperbolic spaces has a bounded character in \(D - 1\) coordinates. According to [21], this should lead to the effective dimensional reduction \(D + 1 \rightarrow 1 + 1\) in the quantum problem. In order to show this, we consider in this section solutions of the Dirac equation in hyperbolic spaces (see also [15, 19] where different approaches to the derivation of solutions are used).

The Dirac equation in hyperbolic spaces \(H^D\) has the form

\[
\left( i \gamma^0 \partial_t + \frac{ix_1}{a} \gamma^1 \partial_1 + ... + \frac{ix_1}{a} \gamma^D \partial_D \right.
\]

\[
- \frac{i}{2a} \gamma^1(\gamma^1 - m) \psi = 0. \tag{20}
\]

In order to solve this equation, we multiply it by \(i \partial + m\) and obtain the following second order differential equation:

\[
- \partial^2_t + \frac{(D - 1)^2}{4a^2} - \frac{D - 2}{a^2} x_1 \partial_1
\]

\[
\frac{x_1^2}{a^2} (\partial_2^2 + ... + \partial_D^2) - \frac{x_1}{a^2} \gamma^1 \gamma^2 \partial_2 - \frac{x_1}{a^2} \gamma^1 \gamma^3 \partial_3 - ... - \frac{x_1}{a^2} \gamma^1 \gamma^D \partial_D - m^2) \psi = 0. \tag{21}
\]

Obviously, we can seek solution in the form \(\psi = e^{-iEt + ip_2 x_2 + ... + ip_D x_D} f(x_1)\). Then we have

\[
(E^2 + \frac{(D - 1)^2}{4a^2} - \frac{D - 2}{a^2} x_1 \partial_1 + \frac{x_1^2 \partial_1^2}{a^2}
\]

\[
- \frac{x_1^2}{a^2} (p_2^2 + ... + p_D^2) - \frac{x_1}{a^2} \gamma^1 \gamma^2 p_2 - i \frac{x_1}{a^2} \gamma^1 \gamma^3 p_3 - ... - i \frac{x_1}{a^2} \gamma^1 \gamma^D p_D).
\]

Further, the Dirac \(\gamma\) matrices are present in this equation only in the operator

\[
A = -i \frac{x_1}{a^2} \gamma^1 \gamma^2 p_2 - i \frac{x_1}{a^2} \gamma^1 \gamma^3 p_3 - ... - i \frac{x_1}{a^2} \gamma^1 \gamma^D p_D.
\]

Since this is a Hermitian operator and its square

\[
A^2 = \frac{x_1^2}{a^2} (p_2^2 + ... + p_D^2)
\]

is a unit matrix, it can be diagonalized and obviously its eigenvalues are equal to

\[
\sigma \frac{x_1}{a^2} \sqrt{p_2^2 + ... + p_D^2}, \tag{23}
\]

where \(\sigma = \pm\). Making the change of variable \(x = x_1 \sqrt{p_2^2 + ... + p_D^2}\), we obtain

\[
\left( E^2 + \frac{x_1^2}{a^2} (-1 + \partial_x^2) + \frac{(D - 1)^2}{4a^2} \right.
\]

\[
- \frac{(D - 2)x_1}{a^2} \partial_1 + \frac{\sigma x_1}{a^2} - m^2 \right) f(x) = 0. \tag{24}
\]
The absence of any dependence on $p_2,\ldots,p_D$ in this equation is remarkable because it means that energy does not depend on them, i.e., energy is the same for any $p_2,\ldots,p_D$. Eq. (24) has the form of equation of an $(1+1)$-dimensional problem and it is not difficult to find its spectrum $E = \pm \sqrt{\nu^2/a^2 + m^2}$, where $\nu$ takes values in $(0, +\infty)$. The effective $(1+1)$-dimensional form (24) of the Dirac equation means the effective dimensional reduction of the dynamics of fermions in hyperbolic spaces. From the mathematical viewpoint, this reduction is related to the spherical and scale symmetries of the $H^D$ metric written in the Poincaré coordinates. Indeed, the spherical symmetry of the $x_2,\ldots,x_D$ part of metric (1) reduces the dependence of energy on $p_2,\ldots,p_D$ to the dependence on only one invariant $p_1^2 = p_1^2 + \ldots + p_D^2$ and then the symmetry of metric (1) with respect to scale transformations $x_k \to x_k/\lambda_k (k = 1,D)$ (recall the change of variable $x = x_1\sqrt{p_1^2 + \ldots + p_D^2}$ that we performed after Eq. (23)) eliminates any dependence on $p_2,\ldots,p_D$ in Eq. (24) for eigenfunctions.

4 Conclusion

Studying the classical motion of massive particles and solving the Dirac equation in hyperbolic spaces $H^D$, we clarified the physical reasons for the effective dimensional reduction $D + 1 \to 1 + 1$ for fermions in the infrared region in hyperbolic spaces.

We showed that the classical motion has a bounded character in $D - 1$ coordinates and, according to Eq. (17), the physical system is effectively of a finite size with respect to these coordinates. On the other hand, the classical motion in the $z = \ln x_1$ coordinate has a familiar inertial character except a finite interval of time when the particle changes its direction of motion and goes back to $-\infty$, where it started its motion. Therefore, for any classical trajectory with nonzero velocity, motion in the $x_1$ coordinate ensures that $t_1^{(geo)}$ is infinite unlike $t_i^{(geo)}$ which describe geodesic motions in other $D - 1$ coordinates.

Solving the Dirac equation in spacetimes $R \times H^D$, we showed that due to the spherical and scale symmetries the initial Dirac problem is reduced to an effective $(1+1)$-dimensional problem. This result agrees with the conclusions of [21] that the effective dimensional reduction in a quantum problem is connected with a bounded character of the classical motion with respect to the coordinates over which the reduction takes place.

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References

[1] P.I. Fomin, V.P. Gusynin, V.A. Miransky, and Yu.A. Sitenko, Rivista del Nuovo Cimento 6 (1983) 1.
[2] J. Bardeen, L.N. Cooper, and J.R. Schrieffer, Phys. Rev. 108 (1957) 1175.
[3] R. Shankar, Rev. Mod. Phys. 66 (1994) 129.
[4] J. Polchinski, Proceedings of the 1992 TASI, ed. by J. Harvey and J. Polchinski - Singapore: World Scientific, 1993 [hep-th/9210046].
[5] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys. Rev. Lett. 73 (1994) 3499; Phys. Rev. D52 (1995) 4718.
[6] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys. Lett. B349 (1995) 477; Phys. Rev. D52 (1995) 4747; Nucl. Phys. B462 (1996) 249.
[7] E.V. Gorbar, Phys. Lett. B491 (2000) 305.
[8] I. Sachs and A. Wipf, Phys. Lett. B326 (1994) 105.
[9] S. Kanemura and H.-Y. Sato, Mod. Phys. Lett. A11 (1996) 785.
[10] T. Inagaki, Int. J. Mod. Phys. A11 (1996) 4561.
[11] E. Elizalde, S. Leseduarte, S.D. Odintsov, and Yu.I. Shil’nov, Phys. Rev. D53 (1996) 1917.
[12] T. Inagaki, T. Muta, and S.D. Odintsov, Prog. Theor. Phys. Suppl. 127 (1997) 93.
[13] E.V. Gorbar, Phys. Rev. D61 (2000) 024013.

[14] D. Ebert, A.V. Tyukov, and V.Ch. Zhukovsky, arXiv:0808.2961 [hep-th].

[15] A.A. Bytsenko, G. Cognola, L. Vanzo, and S. Zerbini, Phys. Rep. 266 (1996) 1.

[16] U. Moschella and R. Schaeffer, Class. Quant. Grav. 24 (2007) 3571.

[17] C.G. Callan and F. Wilczek, Nucl. Phys. B340 (1990) 366.

[18] I. Sachs and S.N. Solodukhin, Phys. Rev. D64 (2001) 124023.

[19] G.W. Gibbons and C.N. Warnick, arXiv:0809.1571 [hep-th].

[20] Z. Haba, arXiv:0809.1592 [hep-th].

[21] D.J. O'Connor, C.R. Stephens, and B.L. Hu, Ann. Phys. (NY) 190 (1989) 310.

[22] E.V. Gorbar and V.P. Gusynin, Ann. Phys. (NY) 323 (2008) 2132.

[23] L.D. Landau and E.M. Lifshits, Course of Theoretical Physics, Vol.2: The Classical Theory of Fields (Reed Publishing, Oxford, 1980).