INDEX-PROPER NONNEGATIVE SPLITTINGS OF MATRICES

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ABSTRACT. The theory of splitting is a useful tool for finding solution of a
system of linear equations. Many works are going on for singular system of linear
equations. In this article, we have introduced a new splitting called index-proper
nonnegative splitting for singular square matrices. Several convergence
and comparison results are also established. We then apply the same theory
to double splitting.

1. Introduction. Iterative methods for solving the system of linear equations

\[ \mathbf{A}\mathbf{x} = \mathbf{b}, \]

where \( \mathbf{A} \) is a real square \( n \times n \) matrix and \( \mathbf{b} \) is a real \( n \)-vector, are related to splittings
of \( \mathbf{A} \) (a splitting is an expression of the form \( \mathbf{A} = \mathbf{U} - \mathbf{V} \), where \( \mathbf{U} \) and \( \mathbf{V} \) are matrices
of same order as in \( \mathbf{A} \)). A splitting \( \mathbf{A} = \mathbf{U} - \mathbf{V} \) is called an index-proper splitting
([6]) of \( \mathbf{A} \in \mathbb{R}^{n \times n} \) if \( \mathbb{R}(\mathbf{A}^k) = \mathbb{R}(\mathbf{U}^k) \) and \( \mathbb{N}(\mathbf{A}^k) = \mathbb{N}(\mathbf{U}^k) \), where \( k = \text{ind}(\mathbf{A}) \) (see
next section for its definition), and \( \mathbb{R}(\mathbf{A}) \) and \( \mathbb{N}(\mathbf{A}) \) stand for the range of \( \mathbf{A} \)
and the kernel of \( \mathbf{A} \). It reduces to index splitting ([14]) if \( \text{ind}(\mathbf{U}) = 1 \). When \( k = 1 \), then
an index-proper splitting becomes a proper splitting ([4]). The asymptotic behavior
of the iterative sequences:

\[ \mathbf{x}^{i+1} = \mathbf{U}^D \mathbf{V} \mathbf{x}^i + \mathbf{U}^D \mathbf{b}, \quad i = 0, 1, 2, \ldots \]

and

\[ \mathbf{Y}^{j+1} = \mathbf{U}^D \mathbf{V} \mathbf{Y}^j + \mathbf{U}^D, \quad j = 0, 1, 2, \ldots, \]

where \( \mathbf{U}^D \) is the Drazin inverse of \( \mathbf{U} \), is governed by the spectral radius of the
iteration matrix \( \mathbf{U}^D \mathbf{V} \) (see next section for the definition of Drazin inverse). For an
index-proper splitting, the spectral radius of \( \mathbf{U}^D \mathbf{V} \) is strictly less than 1 if and only
if the above schemes converge to \( \mathbf{A}^D \mathbf{b} \) and \( \mathbf{A}^D \), respectively to the system \( \mathbf{A}\mathbf{x} = \mathbf{b} \).
More on index-proper splitting can be found in the recent articles [6, 7]. The aim
of this paper is to study the theory of nonnegative splitting \(^1\) for square singular
matrices using Drazin inverse. When two splittings of \( \mathbf{A} \) are given, it is of interest to compare the spectral radii
of the corresponding iteration matrices. The comparison of asymptotic rates of convergence
of the iteration matrices induced by two index-proper splittings of a given

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\(^1\) A splitting \( \mathbf{U} - \mathbf{V} \) is called nonnegative ([12]) if \( \mathbf{U}^{-1} \) exists and \( \mathbf{U}^{-1} \mathbf{V} \geq 0 \).
matrix, has been studied in the recent articles by Jena and Mishra, [6] and Jena and Pani, [7]. Jena and Mishra, [6] also obtained many results for nonnegativity of the Drazin inverse using different matrix splittings. Applications of Drazin inverse lie in many areas such as singular differential and difference equations, Markov chain, cryptography, iterative methods, multi-body dynamics and optimal control. Therefore the computation of the Drazin inverse and its properties have been an area of active research. Here, only a few articles on the Drazin inverse are mentioned, but there is a vast amount of literature on it. (See the references [2, 3, 6, 7, 14] and the references cited therein.)

In this article, we first introduce a new splitting called index-proper nonnegative splitting (see Section 3 for the definition) for square singular matrices by extending the notion of nonnegative splitting for square nonsingular matrices using the notion of Drazin inverse. We then study the convergence of this splitting and prove comparison results for different iterative schemes arising out of this splitting. At last, we apply to theory of double index-proper splitting.

The organization of this paper is as follows. In Section 2, we list all relevant definitions, notation and some earlier results which we use throughout the paper. The main results are given in Section 3 and Section 4. Section 3 introduces the generalization of nonnegative splitting to square singular matrices, and then discusses convergence and comparison theorems for this splitting. In Section 4, we propose the notion of double index-proper nonnegative splitting for real $n \times n$ matrices. Then convergence and comparison results for double index-proper nonnegative splitting are established.

2. Preliminaries. Throughout this article, we will deal with $\mathbb{R}^n$ equipped with its standard cone $\mathbb{R}_+^n$, and all our matrices are real square matrices of order $n$ unless stated otherwise. We denote the transpose, the null space and the range space of $A$ by $A^T$, $N(A)$ and $R(A)$, respectively. $A$ is said to be nonnegative (i.e., $A \geq 0$) if all the entries of $A$ are nonnegative, and $B \geq C$ for matrices $B$ and $C$, if $B-C \geq 0$. We also use these notation and nomenclature for vectors. Let $L, M$ be complementary subspaces of $\mathbb{R}^n$. Then $P_{L,M}$ stands for the projection of $\mathbb{R}^n$ onto $L$ along $M$. So, $P_{L,M}B = B$ if and only if $R(B) \subseteq L$, and $BP_{L,M} = B$ if and only if $N(B) \supseteq M$. The spectral radius of a matrix $A$ is denoted by $\rho(A)$, and is equal to the maximum of the moduli of the eigenvalues of $A$. For any two matrices $A$ and $B$, we have $\rho(AB) = \rho(BA)$.

The Moore-Penrose inverse of a matrix $A \in \mathbb{R}^{m \times n}$, denoted by $A^\dagger$ is the unique solution $X$ of the equations

$$AXA = A, \quad XAX = X, \quad (AX)^T = AX, \quad \text{and} \quad (XA)^T = XA.$$ 

The index of $A$ is the least nonnegative integer $k$ such that $\text{rank}(A^{k+1}) = \text{rank}(A^k)$, and we denote it by $\text{ind}(A)$. Then $\text{ind}(A) = k$ if and only if $R(A^k) \oplus N(A^k) = \mathbb{R}^n$. Also, for $l \geq k$, $R(A^l) = R(A^k)$ and $N(A^l) = N(A^k)$. The Drazin inverse of a matrix $A \in \mathbb{R}^{n \times n}$ is the unique solution $X \in \mathbb{R}^{n \times n}$ satisfying the equations: $A^k = AXA$, $X = XAX$ and $AX = XA$, where $k$ is the index of $A$. It is denoted by $A^D$. When $k = 1$, then the Drazin inverse is said to be the group inverse and is denoted by $A^\#$. While Drazin inverse exists for all matrices, the group inverse does not. It exists if and only if $\text{ind}(A) = 1$. If $A$ is nonsingular, then of course, we have $A^{-1} = A^D = A^\# = A^\dagger$. $A$ is said to be Drazin monotone if $A^D \geq 0$. Similarly, $A$ is group monotone if $A^\#$ exists and $A^\# \geq 0$. Some well-known properties of $A^D([1])$
are follows: \( R(A^k) = R(A^D); N(A^k) = N(A^D); AA^D = P_{R(A^k),N(A^k)} = A^D A. \) In particular, if \( x \in R(A^k) \) then \( x = A^D Ax. \)

We list certain results to be used in the sequel. The next two theorems deal with nonnegativity and spectral radius, and the first one is known as Perron–Frobenius theorem which states that:

**Theorem 2.1.** (Theorem 2.20, [13]) Let \( A \geq 0 \). Then \( A \) has a nonnegative real eigenvalue equal to its spectral radius.

**Lemma 2.2.** (Theorem 2.21, [13]) Let \( A \geq B \geq 0 \). Then \( \rho(A) \geq \rho(B) \).

**Lemma 2.3.** (Lemma 2.2, [11]) Let \( X = \begin{pmatrix} B & C \\ I & O \end{pmatrix} \geq 0 \) and \( \rho(B + C) < 1 \). Then \( \rho(X) < 1 \).

**Lemma 2.4.** (Corollary 3.2, [9]) Let \( B \geq 0 \) and \( x \geq 0 \) be such that \( Bx - \alpha x \geq 0 \). Then \( \alpha \leq \rho(B) \).

The next result is finite dimensional version of the corresponding result which also holds in Banach spaces.

**Theorem 2.5.** (Theorem 3.16, [13]) Let \( X \geq 0 \). Then \( \rho(X) < 1 \) if and only if \((I - X)^{-1}\) exists and \((I - X)^{-1} = \sum_{k=0}^{\infty} X^k \geq 0\).

The first result given below shows that \( UD_V \) and \( AD_V \) have the same eigenvectors while the next one provides a relation between their eigenvalues.

**Remark 1.** (Remark 1.7, [7]) Let \( A = U - V \) be an index-proper splitting. Then the matrices \( UD_V \), \( AD_V \), (or \( VU_D \) and \( VA_D \)) have the same eigenvectors.

**Lemma 2.6.** (Lemma 1.8, [7]) Let \( A = U - V \) be an index-proper splitting. Let \( \mu_i \), \( 1 \leq i \leq s \) and \( \lambda_j \), \( 1 \leq j \leq s \) be the eigenvalues of the matrices \( UD_V \) (\( VU_D \)) and \( AD_V \) (\( VA_D \)) respectively. Then for every \( i \), there exists \( j \) such that \( \mu_i = \frac{\lambda_j}{1 + \lambda_j} \) and for every \( j \), there exists \( i \) such that \( \lambda_j = \frac{\mu_i}{1 - \mu_i} \).

We conclude this section with the following two theorems.

**Theorem 2.7.** (Theorem 3.2, [6]) Let \( A = U - V \) be an index-proper splitting. Then
\( a) \ AA^D = UU^D = AD^A; \)
\( b) \ I - UD_V \) is invertible;
\( c) \ AD^D = (I - UD_V)^{-1} UD_V. \)

Since \( A = U - V \) is an index-proper splitting, so is \( U = A + V \). So, the following results are:

**Theorem 2.8.** (Theorem 1.6, [7])
Let \( A = U - V \) be an index-proper splitting. Then
\( a) \ I + AD_V \) and \( I + VA_D \) are invertible;
\( b) \ AD^D = (I + AD_V)U^D = U^D(I + VA_D); \)
\( c) \ UD^D = (I + AD_V)^{-1} AD^D = AD^D(I + VA_D)^{-1}; \)
\( d) \ UD_VAD_D = AD_VUD_V; \)
\( e) \ UD_VAD_V = AD_VUD_V; \)
\( f) \ VU_DAD_V = VA_DVU_D. \)
3. Index-proper nonnegative splitting. In this section, we first introduce the definition of an index-proper nonnegative splitting. After that some convergence and comparison theorems are proved under a few different sufficient conditions. Very recently, Ballarsingh and Jena [2] have introduced index-proper regular and index-proper weak regular splittings for singular square matrices. Jena [5] again studied the same theory.

**Definition 3.1.** (Definition 3.1, [2] and Definition 2.1, [5])
A splitting \( A = U - V \) of \( A \in \mathbb{R}^{n \times n} \) is called an **index-proper regular splitting** if it is an index-proper splitting such that \( U^D \geq 0 \) and \( V \geq 0 \).

**Definition 3.2.** (Definition 3.5, [2] and Definition 2.2, [5])
A splitting \( A = U - V \) of \( A \in \mathbb{R}^{n \times n} \) is called an **index-proper weak regular splitting** if it is an index-proper splitting such that \( U^D \geq 0 \) and \( U^DV \geq 0 \).

We now introduce another splitting which covers a larger class of matrices in comparison to the above two splittings.

**Definition 3.3.** A splitting \( A = U - V \) of \( A \in \mathbb{R}^{n \times n} \) is called an **index-proper nonnegative splitting** if it is an index-proper splitting such that \( U^D \geq 0 \).

When \( k = 1 \), the above definition coincides with the definition of proper G-nonnegative splitting (Definition 5.1, [10]). So index-proper nonnegative splitting is an extension of proper G-nonnegative splitting. While the theory of proper regular, proper weak regular and proper nonnegative splitting for rectangular matrices are recently studied in the articles [8] and [10].

The example given below does not have an index-proper regular or an index proper weak regular splitting, but has an index-proper nonnegative splitting. Not only that it also does not have Proper G-nonnegative splitting.

**Example 1.** Let \( A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{pmatrix} \) and \( U = 2A \). The index of \( A \) and \( U \) are

2. Also \( R(A^2) = R(U^2) \) and \( N(A^2) = N(U^2) \). Hence \( A = U - V \) is an index-proper splitting. Now \( U^D = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1/2 & 0 \\ 0 & -1 & 0 \end{pmatrix} \geq 0 \) and \( U^DV = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \geq 0 \). So \( A = U - V \) is an index-proper nonnegative splitting but not an index-proper regular and weak regular splitting as \( V \not\geq 0 \) and \( U^D \not\geq 0 \), respectively.

Note that with a different \( U \), \( A \) may have an an index-proper regular or an index proper weak regular splitting. Thus the class of index-proper nonnegative splitting is larger than the class of index-proper regular and weak regular splittings. We next present a convergence result for index-proper nonnegative splitting under a sufficient condition \( A^DU \geq 0 \).

**Lemma 3.4.** Let \( A = U - V \) be an index-proper nonnegative splitting of \( A \in \mathbb{R}^{n \times n} \) and \( A^DU \geq 0 \). Then \( \rho(U^DV) = \frac{\rho(A^DU)^{-1} - 1}{\rho(A^DU)} < 1 \).

**Proof.** Since \( U^DV \geq 0 \), so there exists a nonnegative vector \( x \) \((x \neq 0)\) such that \( U^DVx = \rho(U^DV)x \) by Theorem 2.1. Then \( x \in R(U^D) = R(U^k) = R(A^k) \).

Therefore \( U^DUx = x \). But \( A^D = (I - U^DV)^{-1}U^D \), by Theorem 2.7 (c). So \( A^DU = (I - U^DV)^{-1}U^DU \). Then \( A^DUx = (I - U^DV)^{-1}U^DUx = (I - U^DV)^{-1}x = \frac{1}{1 - \rho(U^DV)}x \) which implies \( \frac{1}{1 - \rho(U^DV)} \geq 0 \) and is an eigenvalue of \( A^DU \). So \( 0 \leq \frac{1}{1 - \rho(U^DV)} \geq 0 \) and is an
\[
\frac{1}{1 - \rho(A^D U)} \leq \rho(A^D U) \text{ i.e., } \rho(U^D V) \leq \frac{\rho(A^D U)^{-1}}{\rho(A^D U)}. \]
Similarly, \(A^D U \geq 0\) guaranties the existence of a nonnegative vector \(y (y \neq 0)\) such that \(A^D U y = \rho(A^D U)y\). Then \(y \in R(A^k) = R(U^k)\) implies \(y = U^D U y\). Hence \((I - U^D V)^{-1}y = (I - U^D V)^{-1}U^D U y = A^D U y = \rho(A^D U)y\). Thus \(\frac{1}{\rho(A^D U)} y = y - U^D V y\) i.e., \(U^D V y = \frac{\rho(A^D U)^{-1}}{\rho(A^D U)} y\) which yields \(\rho(U^D V) \geq \frac{\rho(A^D U)^{-1}}{\rho(A^D U)}\). Therefore \(\rho(U^D V) = \frac{\rho(A^D U)^{-1}}{\rho(A^D U)} < 1\).

Now we discuss some more properties of index-proper nonnegative splitting.

**Theorem 3.5.** Let \(A = U - V\) be an index-proper nonnegative splitting. Then the following conditions are equivalent.
(a) \(A^D V \geq 0\);
(b) \(\rho(U^D V) = \frac{\rho(A^D V)}{1 + \rho(A^D V)} < 1\);
(c) \((I - U^D V)^{-1} = \sum_{k=0}^{\infty} (U^D V)^k \geq 0\);
(d) \(A^D V \geq U^D V\).

**Proof.** (a) \(\Rightarrow\) (b): We have \(A^D V \geq 0\) and \(U^D V \geq 0\). Let \(\lambda\) and \(\mu\) be any nonnegative eigenvalues of \(A^D V\) and \(U^D V\), respectively. Let \(f(\lambda) = \frac{\lambda}{1 + \lambda}\), \(\lambda \geq 0\). Then \(f\) is a strictly increasing function. Then, by Lemma 2.6, \(\mu = \frac{\lambda}{1 + \lambda}\). So, \(\mu\) attains its maximum when \(\lambda\) is maximum. But \(\lambda\) is maximum when \(\lambda = \rho(A^D V)\). As a result, the maximum value of \(\mu\) is \(\rho(U^D V)\). Hence, \(\rho(U^D V) = \frac{\rho(A^D V)}{1 + \rho(A^D V)} < 1\).

(b) \(\Rightarrow\) (a): \(A = U - V\) is an index-proper splitting yields \(A^D = (I - U^D V)^{-1}U^D\). By Theorem 2.5, the condition \(\rho(U^D V) < 1\) implies that \((I - U^D V)^{-1} = \sum_{k=0}^{\infty} (U^D V)^k\).

Therefore \(A^D V = \sum_{k=1}^{\infty} (U^D V)^k \geq 0\).

(b) \(\Rightarrow\) (c): Since \(\rho(U^D V) < 1\) and \(U^D V \geq 0\), so by Theorem 2.5 \((I - U^D V)^{-1} \geq 0\).

(c) \(\Rightarrow\) (d): By Theorem 2.7 (c), \(A^D = (I - U^D V)^{-1}U^D\). Post-multiplying \(V\) both sides, we get \(A^D V = (I - U^D V)^{-1}U^D V = \sum_{k=0}^{\infty} (U^D V)^k U^D V = \sum_{k=1}^{\infty} (U^D V)^k = U^D V + \sum_{k=2}^{\infty} (U^D V)^k \geq U^D V\). Since \(U^D V \geq 0\), so \(\sum_{k=2}^{\infty} (U^D V)^k \geq 0\). Hence \(A^D V \geq U^D V\).

The theorem given below generalizes Lemma 2.6, [12] to square singular matrix case. Mishra [10] also extended the same lemma to rectangular case using the Moore-Penrose inverse. But we have used the Drazin inverse here. The Drazin inverse case does not hold in general. It is true under the assumption of two extra conditions and is presented next.

**Theorem 3.6.** Let \(A = U - V\) be an index-proper nonnegative splitting with \(R(V) \subseteq R(A^k) \text{ and } N(A^k) \subseteq N(V)\). Then the following are equivalent:
(a) \(\rho(U^D V) < 1\);
(b) \(A^D U \geq U^D V\);
(c) \(A^D U \geq A^D A\);
Theorem 3.8. Drazin monotone matrix.

Proof. (a) ⇒ (b): $A = U - V$ is an index-proper splitting implies $A^D = (I - U^DV)^{-1}U^D$. Again $\rho(U^DV) < 1$ yields that $(I - U^DV)^{-1} = \sum_{k=0}^{\infty} (U^DV)^k$. Since $N(A^k) \subseteq N(V)$ i.e. $R(V^T) \subseteq R((A^k)^T)$. Hence $U^DUV^T = V^T$ which gives $VU^DU = V$. Therefore $A^D U = \sum_{i=1}^{\infty} (U^DV)^i U^DU = U^DU + \sum_{i=1}^{\infty} (U^DV)^i \geq 0$ implies $A^D U - U^DU = \sum_{i=1}^{\infty} (U^DV)^i$. Since $U^DV \geq 0$ then, $A^D U - U^DU = \sum_{i=1}^{\infty} (U^DV)^i \geq 0$.

Hence $A^D U - U^DU \geq 0$. That is $A^D U \geq U^DU$.

(b) ⇒ (c): The fact $A^D U \geq U^DU$ implies that $A^D U \geq A^D A$ since $A^D A = U^DU$.

(c) ⇒ (d): Since $U^DV \geq 0$, post-multiplying $U^DV$ to $A^D U \geq A^D A$ we get $A^D U U^DV \geq A^D A U^DU = U^DU U^DV$.

Now, using the condition $R(V) \subseteq R(A^k) = R(U^k)$, we have $UU^DV = V$. Thus $A^D V \geq U^DV$.

(d) ⇒ (e): Since $U^DV \geq 0$, so $A^D V \geq U^DV \geq 0$.

(e) ⇒ (f): From Theorem 3.5 ((a) ⇒ (b)).

(f) ⇒ (a): Obviously. $\square$

It is well known that the comparison theorems between the spectral radii of matrices are useful in analysis of rate of convergence of iterative methods induced by the splittings and iteration matrices. An accepted rule for preferring one iteration scheme to another is to choose the scheme having the smaller spectral radius of $U^DV$. Many authors such as Mishra [10], Jena et al. [8], Jena and Mishra [6], etc. have introduced various comparison results for different splittings of semimonotone (i.e. $A^1 \geq 0$) and Drazin monotone matrices. Here we discuss the case of index-proper nonnegative splitting.

Theorem 3.7. Let $A = U_1 - V_1 = U_2 - V_2$ be two convergent index-proper nonnegative splittings. If $A^D V_1 \leq A^D V_2$, then $\rho(U_1^D V_1) \leq \rho(U_2^D V_2)$.

Proof. Since $A = U_1 - V_1 = U_2 - V_2$ are two convergent index-proper nonnegative splittings, so $A^D V_i \geq 0$, $i = 1, 2$, by Theorem 3.5, (b) ⇒ (a). Then $0 \leq A^D V_1 \leq A^D V_2$. Let $\lambda_i$ be the eigenvalues of $A^D V_i$ for $i = 1, 2$. Since $\frac{\lambda_i}{1 + \rho(A^D V_i)}$ is a strictly increasing function for $\lambda_i \geq 0$, Lemma 2.2 yields $\rho(A^D V_1) \leq \rho(A^D V_2)$. Hence $\frac{\rho(A^D V_1)}{1 + \rho(A^D V_1)} \leq \frac{\rho(A^D V_2)}{1 + \rho(A^D V_2)}$ and then $\rho(U_1^D V_1) \leq \rho(U_2^D V_2)$.

Next two comparison results are for index-proper nonnegative splittings of a Drazin monotone matrix.

Theorem 3.8. Let $A = U_1 - V_1 = U_2 - V_2$ be two index-proper nonnegative splittings of $A$. If $0 \leq A^D U_1 \leq A^D U_2$, then $\rho(U_1^D V_1) \leq \rho(U_2^D V_2) < 1$. 

4. Double index-proper nonnegative splittings. Motivated by the idea of Je-naj et al. [8] we now introduce the double index-proper splitting \( A = P - R - S \) of \( A \) to \( Ax = b \) which leads to the following iterative scheme spanned by three iterates:

\[
x^{i+1} = P^D R x^i + P^D S x^{i-1} + P^D b, \quad i = 1, 2, \ldots
\]

(2)

Then

\[
\begin{pmatrix}
x^{i+1} \\
x^i
\end{pmatrix} =
\begin{pmatrix}
P^D R & P^D S \\
I & 0
\end{pmatrix}
\begin{pmatrix}
x^i \\
x^{i-1}
\end{pmatrix} +
\begin{pmatrix}
P^D b \\
0
\end{pmatrix},
\]

where

\[
y^{i+1} = \begin{pmatrix}
x^{i+1} \\
x^i
\end{pmatrix}, \quad y^i = \begin{pmatrix}
x^i \\
x^{i-1}
\end{pmatrix}, \quad W = \begin{pmatrix}
P^D R & P^D S \\
I & 0
\end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix}
P^D b \\
0
\end{pmatrix}
\]

i.e.

\[
y^{i+1} = Wy^i + d.
\]

(3)

The iteration scheme (3) is convergent if \( \rho(W) < 1 \) and then \( A = P - R - S \) is called a convergent double splitting.

**Definition 4.1.** A double splitting \( A = P - R - S \) of \( A \) is called double index-proper nonnegative splitting if \( R(A^k) = R(P^k) \), \( N(A^k) = N(P^k) \), \( P^D R \geq 0 \) and \( P^D S \geq 0 \), where \( k \) is the index of \( A \).

Setting \( P = U \) and \( R - S = V \) in Definition 4.1 and \( k = 1 \), we obtain proper nonnegative splitting (Definition 3.1, [10]) for square matrices. We first prove a convergence result which relates convergence of single and double splitting. Note that Mishra, [10] also studied a similar result for rectangular matrices while our result discusses about only square singular matrices.

**Theorem 4.2.** Let \( A = P - R - S \) be a double index-proper nonnegative splitting with \( R(V) \subseteq R(A^k) \) of \( A \in \mathbb{R}^{n \times n} \). Then \( \rho(W) < 1 \) if and only if \( \rho(U^D V) < 1 \), where \( U = P \) and \( V = R + S \).

**Proof.** The fact \( A = P - R - S \) is a double index-proper nonnegative splitting yields \( W \geq 0 \). Then \( \rho(W) < 1 \) if and only if \( (I - W)^{-1} \) exists and \( (I - W)^{-1} \geq 0 \) by Theorem 2.5. Now \( A = P - (R + S) = U - V \) is an index-proper splitting implies...
that \((I - UDV)^{-1} = [I - PD(R + S)]^{-1}\) exists by Theorem 2.7 (b). Block matrix computation yields
\[
(I - W)^{-1} = \begin{pmatrix} [I - PD(R + S)]^{-1} & 0 \\ [I - PD(R + S)]^{-1} & I - PD R \end{pmatrix}
\]

If \((I - W)^{-1} \geq 0\), then \([I - PD(R + S)]^{-1} \geq 0\). By Theorem 2.5, we obtain \(\rho(UDV) = \rho(PD(R + S)) < 1\).
Conversely, if \(\rho(UDV) = \rho(PD(R + S)) < 1\), then \([I - PD(R + S)]^{-1} \geq 0\). We will now show that \((I - W)^{-1}\) exists. Let \((I - W)x = 0\) yields \(\begin{pmatrix} I - PD R & -PD S \\ -I & I \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0\), where \(x = (y, z)^T\). Then, we have \((I - PD R)y = PD Sz\) and \(y = z\). So \(y = (PD R + PD S)y = [PD(R + S)]y = UDV y\). Hence \(y \in R(UD) = R(A^k)\). Also, \(y = UDV y\) gives \(Uy = UUD V y = V y\) as \(R(V) \subseteq R(A^k)\). Thus \(Uy = V y\) which yields \((U - V)y = A y = 0\). Therefore \(y \in N(A) \subseteq N(A^k)\) which in turn implies \(x = 0\) and then we get \(x = 0\). Hence \(I - W\) is invertible. Also \([I - PD(R + S)]^{-1}(I - PD R) = \sum_{j=0}^{\infty}(PD(R + S))^j - \sum_{j=0}^{\infty}(PD(R + S))^jPD R = I + PD(R + S) - PD R + [PD(R + S)]^2 - PD(R + S)PD R + \cdots = I + PD S + PD(R + S)(PD S) + [PD(R + S)]^2(PD S) + \cdots \geq I \geq 0\) since \(PD R\) and \(PD S\) are positive. The condition \([I - PD(R + S)]^{-1}(I - PD R) \geq 0\) yields \((I - W)^{-1} \geq 0\). Thus \(\rho(W) < 1\).

A convergence result of a double index-proper nonnegative splitting under the condition \(A^DP \geq 0\) is shown next.

**Theorem 4.3.** Let \(A = P - R - S\) be a double index-proper nonnegative splitting of \(A\) such that \(A^DP \geq 0\). Then \(\rho(W) < 1\).

**Proof.** \(A = P - R - S\) is a double index-proper nonnegative splitting implies \(W = \begin{pmatrix} PD R & PD S \\ I & O \end{pmatrix} \geq 0\). Setting \(U = P\) and \(V = R + S\), we get \(A = U - V\) is an index-proper nonnegative splitting. Then by Lemma 3.4, we have \(\rho(PD(R + S)) = \rho(UDV) < 1\). By Lemma 2.3, it now follows that \(\rho(W) < 1\).

Now another convergence theorem for double index-proper nonnegative splitting.

**Theorem 4.4.** Let \(A^D(R + S) \geq 0\). If \(A = P - R - S\) is a double index-proper nonnegative splitting of \(A\), then \(\rho(W) < 1\).

**Proof.** The proof of this theorem is same as above. By taking \(U = P\) and \(V = R + S\), we get \(A = U - V\) is an index-proper nonnegative splitting. Then, by (a) \(\Rightarrow\) (b) of Theorem 3.5, we get \(\rhoPD(R + S)) = \rho(UDV) < 1\). Again, by Lemma 2.3, it now follows that \(\rho(W) < 1\).

A comparison result is shown next which discusses comparison of two different system of linear equations having two different double index-proper nonnegative splitting under a few sufficient conditions. The proof is analogous to Theorem 4.9, [10].
Theorem 4.5. Let $A_1$ and $A_2$ be two singular matrices with $N(A_1^k) = N(A_2^k)$, where $k$ is the index of $A$. Suppose that $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be their double index-proper nonnegative splitting such that $A_1^D P_1 \geq 0$ and $A_2^D P_2 \geq 0$. If $P_1^D A_1 \geq P_2^D A_2$ and $P_1^D R_1 \geq P_2^D R_2$, then $\rho(W_1) \leq \rho(W_2) < 1$.

Proof. By Theorem 4.3, we have $\rho(W_i) < 1$ for $i = 1, 2$. If $\rho(W_1) = 0$, then our claim holds trivially. Suppose that $\rho(W_1) \neq 0$. Since $A_1$ and $A_2$ possesses double index-proper nonnegative splitting, so $W_1 \geq 0$ and $W_2 \geq 0$. Now, applying Lemma 2.1 to $W_1$, we get $W_1 x = \rho(W_1) x$ i.e.,

$$P_1^D R_1 x_1 + P_1^D S_1 x_2 = \rho(W_1) x_1$$

$$x_1 = \rho(W_1) x_2.$$ 

Now $N(A_1^k) = N(A_2^k)$ implies $R(A_1^k)^T = R(A_2^k)^T$ which yields $R(P_1^k)^T = R(P_2^k)^T$. Then $P_1^D P_1 = P_2^D P_2$. The conditions $P_1^D R_1 \geq P_2^D R_2$ and $0 < \rho(W_1) < 1$ imply $(P_2^D R_2 - P_1^D R_1)x_1 \geq \frac{1}{\rho(W_1)}(P_2^D R_2 - P_1^D R_1)x_1$. Therefore

$$W_2 x - \rho(W_1) x = \begin{pmatrix} P_2^D R_2 x_1 + P_2^D S_2 x_2 - \rho(W_1) x_1 \\ x_1 - \rho(W_1) x_2 \end{pmatrix}$$

$$= \begin{pmatrix} P_2^D R_2 x_1 + P_2^D S_2 x_2 - P_1^D R_1 x_1 - P_1^D S_1 x_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} (P_2^D R_2 - P_1^D R_1)x_1 - \frac{1}{\rho(W_1)}(P_1^D S_1 - P_2^D S_2)x_1 \\ 0 \end{pmatrix}$$

$$\geq \begin{pmatrix} \frac{1}{\rho(W_1)}(P_2^D R_2 - P_1^D R_1)x_1 - \frac{1}{\rho(W_1)}(P_1^D S_1 - P_2^D S_2)x_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\rho(W_1)}[P_2^D (R_2 + S_2) - P_1^D (R_1 + S_1)] x_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\rho(W_1)}[P_2^D (P_2 - A_2) - P_1^D (P_1 - A_1)] x_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\rho(W_1)}[P_1^D A_1 - P_2^D A_2] x_1 \\ 0 \end{pmatrix}.$$ 

The condition $P_1^D A_1 \geq P_2^D A_2$ now yields that $W_2 x - \rho(W_1) x \geq 0$ i.e., $W_2 x \geq \rho(W_1) x$. Thus, by Lemma 2.4, we have $\rho(W_1) \leq \rho(W_2) < 1 \square$

The next example shows that the converse of Theorem 4.5 is not true.

Example 2. Let $A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix}$. Here $k = 2$ and $N(A_1^2) = N(A_2^2)$, $A_1^D \geq 0$ and $A_2^D \geq 0$. Set $P_1 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 4 & 0 \end{pmatrix}$,

$R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Again, $P_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 6 & 0 \end{pmatrix}$.
We have now then double index-proper nonnegative splittings of Theorem 4.6. Therefore $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ are two double index-proper nonnegative splittings. We then have $0.5 = \rho(W_1) \leq \rho(W_2) = 0.5 < 1$, but $P_1^D A_1 \nless P_2^D A_2$ and $P_1^D R_1 \nless P_2^D R_2$.

Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double index-proper nonnegative splitting of $A$. Then, we have $W_1 = \begin{pmatrix} P_1^D R_1 & P_1^D S_1 \\ I & O \end{pmatrix} \geq 0$ and $W_2 = \begin{pmatrix} P_2^D R_2 & P_2^D S_2 \\ I & O \end{pmatrix} \geq 0$. We finally conclude with a comparison theorem for a single system of equation where the coefficient matrix $A$ has two different double index-proper nonnegative splitting.

**Theorem 4.6.** Let $A \geq 0$ and $A = P_1 - R_1 - S_1$ and $A = P_2 - R_2 - S_2$ be two double index-proper nonnegative splittings of $A$. If $P_1^D \geq P_2^D$ and $P_1^D R_1 \geq P_2^D R_2$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 \leq \rho(W_2) < 1$.

*Proof.* We have now $W_1 \geq 0$ and $W_2 \geq 0$. Now, applying Lemma 2.1 to $W_1$, we get $W_2 x = \rho(W_2) x$ i.e.,

$$W_2 x = \begin{pmatrix} P_2^D R_2 & P_2^D S_2 \\ I & O \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \rho(W_2) x = \begin{pmatrix} P_2^D x_1 + P_2^D S x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \rho(W_2) x_1 \\ \rho(W_2) x_2 \end{pmatrix}.$$

We then have

$$W_1 x - \rho(W_2) x = \begin{pmatrix} P_1^D R_1 & P_1^D S_1 \\ I & O \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \rho(W_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (P_1^D R_1 - P_2^D R_2) x_1 + \frac{1}{\rho(W_2)} (P_1^D S_1 - P_2^D S_2) x_1 \\ x_1 - \rho(W_2) x_2 \end{pmatrix}.$$

The conditions $P_1^D R_1 \geq P_2^D R_2$ and $0 < \rho(W_2) < 1$ again imply

$$W_1 x - \rho(W_2) x \leq \frac{1}{\rho(W_2)} \begin{pmatrix} (P_1^D P_1 - P_2^D P_2) x_1 + (P_1^D S_1 - P_2^D S_2) x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} (P_1^D P_1 - P_2^D P_2) x_1 + (P_1^D S_1 - P_2^D S_2) x_1 \\ 0 \end{pmatrix}.$$

We have used $P_1^D P_1 = P_2^D P_2$. Thus, by Lemma 2.4, we have $\rho(W_1) \leq \rho(W_2) < 1$ for $0 \leq \rho(W_2) < 1$. 

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