STRONGLY NEAR PROXIMITY & HYPERSPACE TOPOLOGY

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Dedicated to the Memory of Som Naimpally

Abstract. This article introduces strongly near proximity $\delta$, which represents a new kind of proximity called almost proximity. A main result in this paper is the introduction of a hit-and-miss topology on $\text{CL}(X)$, the hyperspace of nonempty closed subsets of $X$, based on the strongly near proximity.

1. Introduction

Recently, we introduced strongly far proximities [31]. Strongly far proximities make it possible to distinguish between a weaker form of farness and a stronger one associated with the Efremovič property. In this article, we introduce a framework that enables us to single out when two sets are near and when they are strongly near. This article also introduces a kind of hit and far-miss topology on the hyperspace of all non-empty closed subsets $\text{CL}(X)$ of a topological space $X$.

2. Preliminaries

Recall how a Lodato proximity is defined [20, 21, 22] (see, also, [27, 25]).

Definition 2.1. Let $X$ be a nonempty set. A Lodato proximity $\delta$ is a relation on $\mathcal{P}(X)$ which satisfies the following properties for all subsets $A, B, C$ of $X$:

- $P0) A \delta B \Rightarrow B \delta A$
- $P1) A \delta B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$
- $P2) A \cap B \neq \emptyset \Rightarrow A \delta B$
- $P3) A \delta (B \cup C) \Leftrightarrow A \delta B$ or $A \delta C$
- $P4) A \delta B$ and $\{b\} \delta C$ for each $b \in B \Rightarrow A \delta C$

Further $\delta$ is separated, if

- $P5) \{x\} \delta \{y\} \Rightarrow x = y$.

When we write $A \delta B$, we read "$A$ is near to $B$", while when we write $A \not\delta B$ we read "$A$ is far from $B$". A basic proximity is one that satisfies the Čech axioms $P0) - P3) \text{[2]} \text{§2.5, p. 439}$. Lodato proximity or LO-proximity is one of the simplest proximities. We can associate a topology with the space $(X, \delta)$ by considering as

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closed sets the ones that coincide with their own closure, where for a subset \( A \) we have

\[
\text{cl}A = \{ x \in X : x \not\in A \}.
\]

This is possible because of the correspondence of Lodato axioms with the well-known Kuratowski closure axioms [19, Vol. 1, §4.1, 20-21).

By considering the gap between two sets in a metric space (\( d(A, B) = \inf\{d(a, b) : a \in A, b \in B\} \) or \( \infty \) if \( A \) or \( B \) is empty), Efremović introduced a stronger proximity called Efremović proximity or EF-proximity [16, 15].

**Definition 2.2.** An EF-proximity is a relation on \( \mathcal{P}(X) \) which satisfies

\[
P_0 \quad \text{through} \quad P_3
\]

and in addition

\[
A /\not\in B \Rightarrow \exists E \subset X \text{ such that } A /\not\in E \text{ and } X \setminus E /\not\in B
\]

A topological space has a compatible EF-proximity if and only if it is a Tychonoff space [28, §5.3] (see, also, S. Willard on Tychonoff spaces [33, §8, 52-53]).

Any proximity \( \delta \) on \( X \) induces a binary relation over the powerset \( \exp X \), usually denoted as \( \ll \) and named the natural strong inclusion associated with \( \delta \), by declaring that \( A \) is strongly included in \( B \), \( A \ll B \), when \( A \) is far from the complement of \( B \), i.e., \( A /\not\in X \setminus B \).

By strong inclusion, the Efremović property for \( \delta \) can be written also as a betweenness property

\[
(\text{EF}) \quad \text{If } A \ll B, \text{ then there exists some } C \text{ such that } A \ll C \ll B.
\]

A pivotal example of EF-proximity is the metric proximity in a metric space \((X, d)\) defined by

\[
A \delta B \iff d(A, B) = 0.
\]

That is, \( A \) and \( B \) either intersect or are asymptotic: for each natural number \( n \) there is a point \( a_n \) in \( A \) and a point \( b_n \) in \( B \) such that \( d(a_n, b_n) < \frac{1}{n} \).

### 2.1. Hit and far-miss topologies.

Let \( \text{CL}(X) \) be the hyperspace of all non-empty closed subsets of a space \( X \). Hit and miss and hit and far-miss topologies on \( \text{CL}(X) \) are obtained by the join of two halves. Well-known examples are Vietoris topology [34, 35, 36, 37] (see, also, [1, 2, 5, 7, 8, 4, 11, 12, 13, 9, 10, 24, 23, 14, 18]) and Fell topology [17]. In this article, we concentrate on an extension of Vietoris based on the strongly far proximity.

**Vietoris topology**

Let \( X \) be an Hausdorff space. The Vietoris topology on \( \text{CL}(X) \) has as subbase all sets of the form

- \( V^- = \{ E \in \text{CL}(X) : E \cap V \neq \emptyset \} \), where \( V \) is an open subset of \( X \),
- \( W^+ = \{ C \in \text{CL}(X) : C \subset W \} \), where \( W \) is an open subset of \( X \).

The topology \( \tau_{V^-} \) generated by the sets of the first form is called hit part because, in some sense, the closed sets in this family hit the open sets \( V \). Instead, the topology \( \tau_{V^+} \) generated by the sets of the second form is called miss part, because the closed sets here miss the closed sets of the form \( X \setminus W \).

The Vietoris topology is the join of the two part: \( \tau_V = \tau_{V^-} \vee \tau_{V^+} \). It represents the prototype of hit and miss topologies.
The Vietoris topology was modified by Fell. He left the hit part unchanged and in the miss part, $\tau_F^+$ instead of taking all open sets $W$, he took only open subsets with compact complement.

**Fell topology:**

$$\tau_F = \tau_V^- \lor \tau_F^+.$$

It is possible to consider several generalizations. For example, instead of taking open subsets with compact complement, for the miss part we can look at subsets running in a family of closed sets $\mathcal{B}$. So we define the hit and miss topology on $\text{CL}(X)$ associated with $\mathcal{B}$ as the topology generated by the join of the hit sets $A^-$, where $A$ runs over all open subsets of $X$, with the miss sets $A^*$, where $A$ is once again an open subset of $X$, but more, whose complement runs in $\mathcal{B}$.

Another kind of generalization concerns the substitution of the inclusion present in the miss part with a strong inclusion associated with a proximity. Namely, when the space $X$ carries a proximity $\delta$, then a proximity variation of the miss part can be displayed by replacing the miss sets with far-miss sets $A^{++}$:

$$E\triangleleft\delta A \iff E \in \text{CL}(X) : E \ll_\delta A.$$  

Also, in this case we can consider $A$ with the complement running in a family $\mathcal{B}$ of closed subsets of $X$. Then the hit and far-miss topology $\tau_{\delta,\mathcal{B}}$, associated with $\mathcal{B}$ is generated by the join of the hit sets $A^-$, where $A$ is open, with far-miss sets $A^{++}$, where the complement of $A$ is in $\mathcal{B}$.

Fell topology can be considered as well as an example of hit and far-miss topology. In fact, in any EF-proximity, when a compact set is contained in an open set, it is also strongly contained.

### 3. Main Results

When we consider a proximity $\delta$, it can happen that $A\delta B$ and they have empty intersection or, if $X$ is a topological space, there could be only one point in the intersection of their closures. We need something more. We want to talk about strong nearness [30] (see, also, [29, 33, 32]) for subsets that at least have some points in common. For this reason, we introduce a new kind of proximity, that we call almost proximity.

**Definition 3.1.** Let $X$ be a topological space. We say that the relation $\overset{\delta}{\triangleleft}$ on $\mathcal{P}(X)$ is a strongly near proximity called an almost proximity, provided it satisfies the following axioms. Let $A,B,C \subset X$ and $x \in X$:

$\begin{align*}
&N0) \not\emptyset \overset{\delta}{\triangleleft} A, \forall A \subset X, \text{ and } X \overset{\delta}{\triangleleft} A, \forall A \subset X \\
&N1) A \overset{\delta}{\triangleleft} B \iff B \overset{\delta}{\triangleleft} A \\
&N2) A \overset{\delta}{\triangleleft} B \Rightarrow A \cap B \not\emptyset \\
&N3) \text{If int}(B) \text{ and int}(C) \text{ are not equal to the empty set, } A \overset{\delta}{\triangleleft} B \text{ or } A \overset{\delta}{\triangleleft} C \Rightarrow A \overset{\delta}{\triangleleft} (B \cup C) \\
&N4) \text{int}A \cap \text{int}B \not\emptyset \Rightarrow A \overset{\delta}{\triangleleft} B
\end{align*}$

When we write $A \overset{\delta}{\triangleleft} B$, we read $A$ is strongly near $B$. 
Example 3.2. A simple example of almost proximity that is also a Lodato proximity for nonempty sets $A, B \subset X$ is $A \overset{\delta}{=} B \iff A \cap B \neq \emptyset$.

Example 3.3. Another example of almost proximity is given for nonempty sets $A, B \subset X$ in Fig. 3.1, where $A \overset{\delta}{=} B \iff \text{int}(A) \cap \text{int}(B) \neq \emptyset$. This is not a usual proximity.

Example 3.4. We can define another almost proximity for nonempty sets $D, E \subset X$ in Fig. 3.1 in the following way: $D \overset{\delta}{=} E \iff \text{int}(D) \cap \text{int}(E) \neq \emptyset$ or $\text{int}(E) \cap D \neq \emptyset$.

Furthermore, for each almost proximity, we assume the following relation:

\[ N5) \ x \in \text{int}(A) \Rightarrow \{x\} \overset{\delta}{=} A \]
\[ N6) \ \{x\} \overset{\delta}{=} \{y\} \iff x = y \]

So, for example, if we take the almost proximity of example 3.3 we have that

$A \overset{\delta}{=} B \iff \text{int}(A) \cap \text{int}(B) \neq \emptyset$,

provided $A$ and $B$ are not singletons and if $A = \{x\}$, then $x \in \text{int}(B)$ and if $B$ is also a singleton, then $x = y$. It turns out that if $A \subset X$ is an open set, then each point that belongs to $A$ is strongly near $A$.

Now we want to define a new kind of hit and far-miss topology on $\text{CL}(X)$ by using almost proximities. That is, let $\tau_3$ be the hit and far-miss topology having as subbase the sets of the form:

- $V^- = \{E \in \text{CL}(X) : E \cap V \neq \emptyset\}$, where $V$ is an open subset of $X$,
- $A^{++} = \{E \in \text{CL}(X) : E \not{\delta} X \setminus A\}$, where $A$ is an open subset of $X$.

where $\delta$ is a Lodato proximity compatible with the topology on $X$, and $\tau^\delta$ the strongly hit and far-miss topology having as subbase the sets of the form:

- $V^- = \{E \in \text{CL}(X) : E \overset{\delta}{=} V\}$, where $V$ is an open subset of $X$,
- $A^{++} = \{E \in \text{CL}(X) : E \not{\delta} X \setminus A\}$, where $A$ is an open subset of $X$.

It is possible to define strongly hit and miss topology in many other ways by simply mixing strongly hit sets with miss sets or also strongly miss sets in a manner similar to that given in [31].

Theorem 3.5. If the space $X$ is $T_1$, the strongly hit and far-miss topology is an admissible topology.

Proof. To achieve the result we need a $T_1$ space $X$ in order to work with closed singletons. The result about the far-miss part is already known. So look at the strongly hit part. We want that the canonical injection $i : X \rightarrow \text{CL}(X)$, defined by $i(x) = \{x\}$, is a homeomorphism. It is continuous because $i^{-1}(V^-) = \{x \in X : x \overset{\delta}{=} V\} = V$. Moreover it is open in the relative topology on $i(X)$ because, for each open set $A$, $i(A) = A^\delta \cap i(X)$.

Remark 3.6. Observe that the usual hit topology is a special strongly hit topology when taking as almost proximity either the one in example 3.2 or the one in example 3.3.

Now we consider comparisons between usual hit and far miss topologies and strongly hit and far miss ones. To this purpose we need the following lemma.
Lemma 3.7. Let $A$ and $H$ be open subsets of a $T_1$ topological space $X$. Then $A^{-} \subseteq H^\circ$ implies $A \subseteq H$.

Proof. By contradiction, suppose $A \notin H$. Then there exists $a \in A$ such that $a \notin H$. Hence $\{a\} \cap A \neq \emptyset$ while $a \notin H$ by property $(N2)$. This is absurd. \hfill $\square$

Now let $\tau_5$ be the hypertopology having as subbase the sets of the form $V^-$, where $V$ is an open subset of $X$, and let $(\tau^\circ)^{-}$ the hypertopology having as subbase the sets of the form $V^\circ$, again with $V$ an open subset of $X$.

Theorem 3.8. Let $X$ be a $T_1$ topological space. The hypertopologies $\tau_5$ and $(\tau^\circ)^{-}$ are not comparable.

Proof. First, we prove that $\tau_5 \neq (\tau^\circ)^{-}$. Take $A^{-} \in \tau_5$ and $E \in A^{-}$, with $E \in CL(X)$. We ask whether there exists an open set $H$ such that $E \in H^{-} \subset A^{-}$. To this purpose, consider the almost proximity of example 3.3 and let $E$ be such that int$(E) = \emptyset$. Then $E \cap A \neq \emptyset$, but int$(E) \cap H = \emptyset$ for each $H \subset X$.

Conversely, we want to prove that $(\tau^\circ)^{-} \neq \tau_5$. Suppose $X = \mathbb{R}^2$ and consider, as before, the almost proximity of example 3.3. Take the closed set $E = B(O,2)$, that is the closed ball with the origin as center and radius 2, and the open set $H$ as in Fig. 3.2. So $E \not\subseteq H$. By lemma 3.7 to identify an open set $A$ such that $E \in A^{-} \subseteq H^\circ$, we need to consider $A \subseteq H$. But now it is possible to choose $C \in A^{-}$ such that $C \notin H^\circ$, for each $A \subset H$. For example, we can take $C = B(O,1) \cup S^1(O,s)$, where $S^1(O,s)$ is the circumference with the origin as center and radius $s$ for a suitable value of $s$ (see, e.g., Fig. 3.2).

\hfill $\square$

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