ULTRAMETRICS AND SURFACE SINGULARITIES

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Abstract. The present lecture notes give an introduction to works of García Barroso, González Pérez, Ruggiero and the author. The starting point of those works is a theorem of Płoski, stating that one defines an ultrametric on the set of branches drawn on a smooth surface singularity by associating to any pair of distinct branches the quotient of the product of their multiplicities by their intersection number. We show how to construct ultrametrics on certain sets of branches drawn on any normal surface singularity from their mutual intersection numbers and how to interpret the associated rooted trees in terms of the dual graphs of adapted embedded resolutions. The text begins by recalling basic properties of intersection numbers and multiplicities on smooth surface singularities and the relation between ultrametrics on finite sets and rooted trees. On arbitrary normal surface singularities one has to use Mumford’s definition of intersection numbers of curve singularities drawn on them, which is also recalled.

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1. Introduction

This paper is an expansion of my notes prepared for the course with the same title given at the International school on singularities and Lipschitz geometry, which took place in Cuernavaca (Mexico) from June 11th to 22nd 2018.

If \( S \) denotes a normal surface singularity, that is, a germ of normal complex analytic surface, a branch on it is an irreducible germ of analytic curve contained in \( S \). In his 1985 paper [21], Arkadiusz Płoski proved that if one associates to every pair of distinct branches on the
singularity $S = (\mathbb{C}^2, 0)$ the quotient $A \cdot B / m(A) \cdot m(B)$ of their intersection number by the product of their multiplicities, then for every triple of pairwise distinct branches, two of those quotients are equal and the third one is not smaller than them. An equivalent formulation is that the inverses $m(A) \cdot m(B) / A \cdot B$ of the previous quotients define an ultrametric on the set of branches on $(\mathbb{C}^2, 0)$.

Using the facts that the multiplicity of a branch is equal to its intersection number with a smooth branch $L$ transversal to it, and that a given function is an ultrametric on a set if and only if it is so in restriction to all its finite subsets, one deduces that Ploski’s theorem is a consequence of:

**Theorem A.** Let $L$ be a smooth branch on the smooth surface singularity $S$ and let $\mathcal{F}$ be a finite set of branches on $S$, transversal to $L$. Then the function $u_L : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ defined by $u_L(A, B) := (L \cdot A) \cdot (L \cdot B) / A \cdot B$ if $A \neq B$ and $u_L(A, A) := 0$ is an ultrametric on $\mathcal{F}$.

This may be seen as a property of the pair $(S, L)$ and one may ask whether it extends to other pairs consisting of a normal surface singularity and a branch on it. It turns out that this property characterizes the so-called arborescent singularities, that is, the normal surface singularities such that the dual graph of every good resolution is a tree. Namely, one has the following theorem, which combines [9, Thm. 85] and [12, Thm. 1.46]:

**Theorem B.** Let $L$ be a branch on the normal surface singularity $S$. Then the function $u_L$ defined as before is an ultrametric on any finite set $\mathcal{F}$ of branches on $S$ distinct from $L$ if and only if $S$ is an arborescent singularity.

It is possible to think topologically about ultrametrics on finite sets in terms of certain types of decorated rooted trees. In particular, any such ultrametric determines a rooted tree. One may try to describe this tree directly from the pair $(S, \mathcal{F} \cup \{L\})$, when $S$ is arborescent and the ultrametric is the function $u_L$ associated to a branch $L$ on it. In order to formulate such a description, we need the notion of convex hull of a finite set of vertices of a tree: it is the union of the paths joining those vertices pairwise.

The following result was obtained in [9, Thm. 87]:

**Theorem C.** Let $L$ be a branch on the arborescent singularity $S$ and let $\mathcal{F}$ be a finite set of branches on $S$ distinct from $L$. Then the rooted tree determined by the ultrametric $u_L$ on $\mathcal{F}$ is isomorphic to the convex hull of the strict transform of $\mathcal{F} \cup \{L\}$ in the dual graph of its preimage by an embedded resolution of it, rooted at the vertex representing the strict transform of $L$.

Even when the singularity $S$ is not arborescent, the function $u_L$ becomes an ultrametric in restriction to suitable sets $\mathcal{F}$ of branches on $S$. Those sets are defined only in terms of convex hulls taken in the so-called brick-vertex tree of the dual graph of an embedded resolution of $\mathcal{F} \cup \{L\}$, and do not depend on any numerical parameter of the exceptional divisor of the resolution, be it a genus or a self-intersection number. The brick-vertex tree of a connected graph is obtained canonically by replacing each brick – a maximal inseparable subgraph which is not an edge – by a star, whose central vertex is called a brick-vertex. One has the following generalization of Theorem C (see [12, Thm. 1.42]):
Theorem D. Let $L$ be a branch on the normal surface singularity $S$ and let $\mathcal{F}$ be a finite set of branches on $S$ distinct from $L$. Consider an embedded resolution of $\mathcal{F} \cup \{L\}$. Assume that the convex hull of its strict transform in the brick-vertex tree of the dual graph of its preimage does not contain brick-vertices of valency at least 4 in the convex hull. Then the function $u_L$ is an ultrametric in restriction to $\mathcal{F}$ and the associated rooted tree is isomorphic to the previous convex hull, rooted at the vertex representing the strict transform of $L$.

If $S$ is not arborescent, there may exist other sets of branches on which $u_L$ restricts to an ultrametric. Unlike the sets described in the previous theorem, in general they do not depend only on the topology of the dual graph of their preimage on some embedded resolution, but also on the self-intersection numbers of the components of the exceptional divisor (see [12, Ex. 1.44]).

The aim of the present notes is to introduce the reader to the previous results. Note that in the article [12, Part 2], these results were extended to the space of real-valued semivaluations of the local ring of $S$.

Let us describe briefly the structure of the paper. In Section 2 are recalled basic facts about multiplicities and intersection numbers of plane curve singularities. In Section 3 are stated two equivalent formulations of Płoski’s theorem. In Section 4 is explained the relation between ultrametrics and rooted trees mentioned above, an intermediate concept being that of hierarchy on a finite set. Using this relation, Section 5 presents a proof of Theorem A. This proof uses the so-called Eggers-Wall tree of a plane curve singularity relative to a smooth reference branch $L$, constructed using associated Newton-Puiseux series. Section 6 explains the notions used in the formulation of Theorem B, that is, those of good resolution, embedded resolution, associated dual graph and arborescent singularity. In Section 7 are described the related notions of cut-vertex and brick-vertex tree of a finite connected graph. Section 8 explains and illustrates the statement of Theorem D. In Section 9 is explained Mumford’s intersection theory of divisors on normal surface singularities, after a proof of a fundamental property of such singularities, stating that the intersection form of any of their resolutions is negative definite. In Section 10 the ultrametric inequality concerning the restriction of $u_L$ to a triple of branches is reexpressed in terms of the notion of angular distance on the dual graph of an adapted resolution. A crucial property of this distance is stated, which relates it to the cut-vertices of the dual graph. In Section 11 is sketched the proof of a theorem of pure graph theory, relating distances satisfying the previous crucial property and the brick-vertex tree of the graph. This theorem implies Theorem D.

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2. Multiplicity and intersection numbers for plane curve singularities

In this section we recall the notions of multiplicity of a plane curve singularity and intersection number of two such singularities. One may find more details in [16, Sect. 5.1] or [8, Chap. 8].

Let \( (S, s) \) be a smooth surface singularity, that is, a germ of smooth complex analytic surface. Denote by \( \mathcal{O}_{S,s} \) its local \( \mathbb{C} \)-algebra and by \( \mathfrak{m}_{S,s} \) its maximal ideal, containing the germs at \( s \) of holomorphic functions vanishing at \( s \).

A local coordinate system on \( S \) at \( s \) is a pair \((x, y) \in \mathfrak{m}_{S,s} \times \mathfrak{m}_{S,s}\) establishing an isomorphism between a neighborhood of \( s \) in \( S \) and a neighborhood of the origin in \( \mathbb{C}^2 \). Algebraically speaking, this is equivalent to the fact that the pair \((x, y)\) generates the maximal ideal \( \mathfrak{m}_{S,s} \), or that it realizes an isomorphism \( \mathcal{O}_{S,s} \simeq \mathbb{C}_{x, y} \). This isomorphism allows to see each germ \( f \in \mathcal{O}_{S,s} \) as a convergent power series in the variables \( x \) and \( y \).

A curve singularity on \( (S, s) \) is a germ \( (C, s) \hookrightarrow (S, s) \) of not necessarily reduced curve on \( S \), passing through \( s \). As the germ \( (S, s) \) is isomorphic to the germ of the affine plane \( \mathbb{C}^2 \) at any of its points, one says also that \( (C, s) \) is a plane curve singularity. A defining function of \( (C, s) \) is a function \( f \in \mathfrak{m}_{S,s} \) such that \( \mathcal{O}_{C,s} = \mathcal{O}_{S,s}/(f) \), where \( (f) \) denotes the principal ideal of \( \mathcal{O}_{S,s} \) generated by \( f \). Write then \( C = \{Z(f)\} \).

The curve singularity \( (C, s) \) may also be seen as an effective principal divisor on \( (S, s) \). This allows to write \( C = \sum_{i \in I} p_i C_i \), where \( p_i \in \mathbb{N}^* \) for all \( i \in I \) and the curve singularities \( C_i \) are pairwise distinct and irreducible. We say in this case that the \( C_i \)'s are the branches of \( C \). A branch on \( (S, s) \) is an irreducible curve singularity on \( (S, s) \).

Next definition introduces the simplest invariant of a plane curve singularity:

**Definition 2.1.** Assume that \( f \in \mathcal{O}_{S,s} \). Its multiplicity is the vanishing order of \( f \) at \( s \):

\[
\text{ms}(f) := \sup\{n \in \mathbb{N}, f \in \mathfrak{m}_{S,s}^n \} \in \mathbb{N} \cup \{\infty\}.
\]

If \( (C, s) \) is the curve singularity defined by \( f \), we say also that \( \text{ms}(C) := \text{ms}(f) \) is its multiplicity at \( s \). It is a simple exercise to check that the multiplicity of a curve singularity is independent of the function defining it. If one chooses local coordinates \((x, y)\) on \( (S, s) \), then \( \text{ms}(f) \) is the smallest degree of the monomials appearing in the expression of \( f \) as a convergent power series in the variables \( x \) and \( y \). One has \( \text{ms}(f) = \infty \) if and only if \( f = 0 \) and \( \text{ms}(f) = 1 \) if and only if \( f \) defines a smooth branch on \( (S, s) \).

The following definition describes a measure of the way in which two curve singularities intersect:

**Definition 2.2.** Let \( C, D \hookrightarrow (S, s) \) be two plane curve singularities defined by \( f, g \in \mathfrak{m}_{S,s} \). Then their intersection number is defined by:

\[
[C \cdot D] := \dim_{\mathbb{C}} \mathcal{O}_{S,s}/(f, g) \in \mathbb{N} \cup \{\infty\},
\]

where \( (f, g) \) denotes the ideal of \( \mathcal{O}_{S,s} \) generated by \( f \) and \( g \).

Note that \( C \cdot D < +\infty \) if and only if \( C \) and \( D \) do not share common branches, which is also equivalent to the existence of \( n \in \mathbb{N}^* \) such that one has the following inclusion of ideals: \( (f, g) \supseteq \mathfrak{m}_{S,s}^n \). Nevertheless, unlike the multiplicity, the intersection number \( C \cdot D \) is not always equal to the smallest exponent \( n \) having this property. For instance, if one takes \( f := x^3 \) and \( g := y^2 \), then \( C \cdot D = 6 \) but \( (f, g) \supseteq (x, y)^5 \). We leave the verification of the previous facts as an exercise.
The following proposition, which may be proved using Proposition 2.5 below, relates multiplicities and intersection numbers:

**Proposition 2.3.** If \((C, s) \leadsto (S, s)\) is a plane curve singularity, then \(C \cdot L \geq m_s(C)\) for any smooth branch \(L\) through \(s\), with equality if and only if \(L\) is transversal to \(C\). More generally, if \(D\) is a second curve singularity on \((S, s)\), then \(C \cdot D \geq m_s(C) \cdot m_s(D)\), with equality if and only if \(C\) and \(D\) are transversal.

Let us explain the notion of transversality used in the previous proposition, as it is more general than the standard notion of transversality, which applies only to smooth submanifolds of a given manifold. If \(C\) is a branch on \((S, s)\) and one chooses a local coordinate system \((x, y)\) on \((S, s)\), as well as a defining function \(f\) of \(C\), it may be shown that the lowest degree part of \(f\) is a power of a complex linear form in \(x\) and \(y\). This linear form defines a line in the tangent plane \(T_sS\) of \(S\) at \(s\), which is by definition the tangent line of \(C\) at \(s\). One may show that it is independent of the choices of local coordinates and defining function of \(C\). If \(C\) is now an arbitrary curve singularity, then its tangent cone is the union of the tangent lines of its branches. Given two plane curve singularities on the same smooth surface singularity \(S\), one says that they are transversal if each line of the tangent cone of one of them is transversal (in the classical sense) to each line of the tangent cone of the other one.

Let us pass now to the question of computation of intersection numbers. A basic method consists in breaking the symmetry between the two curve singularities, by working with a defining function of one of them and by parametrizing the other one. One has to be cautious and choose a normal parametrization, in the following sense:

**Definition 2.4.** A normal parametrization of the branch \((C, s)\) is a germ of holomorphic morphism \(\nu : (\mathbb{C}, 0) \to (C, s)\) which is a normalization morphism, that is, it has topological degree one.

For instance, if the branch \((C, 0)\) on \((\mathbb{C}^2, 0)\) is defined by the function \(y^2 - x^3\), then \(t \mapsto (t^2, t^3)\) is a normal parametrization of \(C\), but \(u \mapsto (u^4, u^6)\) is not. A normal parametrization of a branch \((C, s)\) may be also characterized by asking it to establish a homeomorphism between suitable representatives of the germs \((\mathbb{C}, 0)\) and \((C, s)\).

Normalization morphisms may be defined more generally for reduced germs \((X, x)\) of arbitrary dimension (see [16, Sect. 4.4]), by considering the multi-germ whose multi-local ring is the integral closure of the local ring \(\mathcal{O}_{X,x}\) in its total ring of fractions. Except for curve singularities, the source of a normalization morphism is not smooth in general.

The following proposition is classical and states the announced expression of intersection numbers in terms of a parametrization of one germ and a defining function of the second one (see [2, Prop. II.9.1] or [16, Lemma 5.1.5]):

**Proposition 2.5.** Let \(C\) be a branch on the smooth surface singularity \((S, s)\) and \(D\) be a second curve singularity, not necessarily reduced. Let \(\nu : (\mathbb{C}, 0) \to (C, s)\) be a normal parametrization of \(C\) and let \(g \in m_{S,s}\) be a defining function of \(D\). Then:

\[
C \cdot D = \text{ord}_t (g \circ \nu(t)) ,
\]

where \([\text{ord}_t]\) denotes the order of a power series in the variable \(t\).

**Proof.** This proof is adapted from that of [16, Lemma 5.1.5].

The order of the zero power series is equal to \(\infty\) by definition, therefore the statement is true when \(C\) is a branch of \(D\).

Let us assume from now on that \(C\) is not a branch of \(D\).
Consider a defining function \( f \in \mathfrak{m}_{S,s} \) of \( C \). By Definition 2.2:

\[
(2.6) \quad C \cdot D = \dim_{\mathbb{C}} \frac{\mathcal{O}_{S,s}}{(f, g)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{S,s}/(f)}{(g_C)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{C,s}}{(g_C)}
\]

where we have denoted by \( g_C \in \mathcal{O}_{C,s} \) the restriction of \( g \) to the branch \( C \).

Algebraically, the normal parametrization \( \nu : (\mathbb{C}, 0) \rightarrow (C, s) \) corresponds to a morphism of local \( \mathbb{C} \)-algebras \( \mathcal{O}_{C,s} \rightarrow \mathcal{C}[t] \), isomorphic to the inclusion morphism of \( \mathcal{O}_{C,s} \) into its integral closure taken inside its quotient field. In order to distinguish them, denote from now on by \( g_C \mathcal{O}_{C,s} \) the principal ideal generated by \( g_C \) inside \( \mathcal{O}_{C,s} \) and by \( g_C \mathcal{C}[t] \) its analog inside \( \mathcal{C}[t] \). One has the following equality inside the local \( \mathbb{C} \)-algebra \( \mathcal{C} \):

\[
g_C \circ \nu(t) = g_C.
\]

As a consequence:

\[
g_C \mathcal{C}[t] = t^{\text{ord}_t (g_C \circ \nu(t))} \mathcal{C}[t].
\]

Therefore:

\[
(2.7) \quad \text{ord}_t (g_C \circ \nu(t)) = -\dim_{\mathbb{C}} \frac{\mathcal{C}[t]}{g_C \mathcal{C}[t]}.
\]

By comparing equations (2.6) and (2.7), we see that the desired equality is equivalent to:

\[
(2.8) \quad \dim_{\mathbb{C}} \frac{\mathcal{O}_{C,s}}{g_C \mathcal{O}_{C,s}} = \dim_{\mathbb{C}} \frac{\mathcal{C}[t]}{g_C \mathcal{C}[t]}.
\]

The two quotients appearing in (2.8) are the cokernels of the two injective multiplication maps \( \mathcal{O}_{C,s} \rightarrow g_C \mathcal{O}_{C,s} \) and \( \mathcal{C}[t] \rightarrow g_C \mathcal{C}[t] \). The associated short exact sequences may be completed into a commutative diagram in which the first two vertical maps are the inclusion map \( \mathcal{O}_{C,s} \rightarrow \mathcal{C}[t] \):

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{O}_{C,s} & \rightarrow & \mathcal{O}_{C,s} & \rightarrow & \mathcal{O}_{C,s} / g_C \mathcal{O}_{C,s} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{C}[t] & \rightarrow & \mathcal{C}[t] & \rightarrow & \mathcal{C}[t] / g_C \mathcal{C}[t] & \rightarrow & 0
\end{array}
\]

The last vertical map is not necessarily an isomorphism. We want to show that its source and its target have the same dimension. Let us complete it into an exact sequence by considering its kernel \( K_1 \) and cokernel \( K_2 \):

\[
0 \rightarrow K_1 \rightarrow \mathcal{O}_{C,s} / g_C \mathcal{O}_{C,s} \rightarrow \mathcal{C}[t] / g_C \mathcal{C}[t] \rightarrow K_2 \rightarrow 0.
\]

For every finite exact sequence of finite-dimensional vector spaces, the alternating sum of dimensions vanishes. Therefore:

\[
\dim_{\mathbb{C}} K_1 - \dim_{\mathbb{C}} \frac{\mathcal{O}_{C,s}}{g_C \mathcal{O}_{C,s}} + \dim_{\mathbb{C}} \frac{\mathcal{C}[t]}{g_C \mathcal{C}[t]} - \dim_{\mathbb{C}} K_2 = 0.
\]

This shows that the desired equality (2.8) would result from the equality \( \dim_{\mathbb{C}} K_1 = \dim_{\mathbb{C}} K_2 \). This last equality is a consequence of the so-called “snake lemma” (see for instance [1, Prop. 2.10]), applied to the previous commutative diagram. Indeed, by this lemma, one has an exact sequence:

\[
0 \rightarrow K_1 \rightarrow \mathcal{C}[t] / \mathcal{O}_{C,s} \rightarrow \mathcal{C}[t] / \mathcal{O}_{C,s} \rightarrow K_2 \rightarrow 0.
\]
Reapplying the previous argument about alternating sums of dimensions, one gets the needed equality $\dim_{\mathbb{C}} K_1 = \dim_{\mathbb{C}} K_2$. □

Note that the previous proof shows in fact that for any abstract branch $(C, s)$, not necessarily planar, one has the equality:

\[(2.9) \dim_{\mathbb{C}} \mathcal{O}_{C, s}(g) = \text{ord}_t (g \circ \nu(t)),\]

for any $g \in \mathcal{O}_{C, s}$ and for any normal parametrization $\nu : (C, 0) \to (C, s)$ of $(C, s)$. If the branch $(C, s)$ is contained in an ambient germ $(X, s)$ and $H$ is an effective principal divisor on $(X, s)$ which does not contain the branch, then equality (2.9) shows that the intersection number of $C$ and $H$ at $s$ may be computed as the order of the series obtained by composing a defining function of $(H, s)$ and a normal parametrization of $(C, s)$.

**Example 2.10.** Consider the branches:

\[
\begin{align*}
A &:= Z(y^2 - x^3), \\
B &:= Z(y^3 - x^5), \\
C &:= Z(y^6 - x^5)
\end{align*}
\]

on the smooth surface singularity $(\mathbb{C}^2, 0)$. Denoting by $m_0$ the multiplicity function at the origin of $\mathbb{C}^2$, we have:

\[m_0(A) = 2, \quad m_0(B) = 3, \quad m_0(C) = 5,\]

as results from Definition 2.1. Using Proposition 2.5 and the fact that whenever $m$ and $n$ are coprime positive integers, $t \to (t^n, t^m)$ is a normal parametrization of $Z(y^n - x^m)$, one gets the following values for the intersection numbers of the branches $A, B, C$:

\[B \cdot C = 15, \quad C \cdot A = 10, \quad A \cdot B = 9.\]

Therefore:

\[
\begin{align*}
\frac{B \cdot C}{m_0(B) \cdot m_0(C)} &= 1, \\
\frac{C \cdot A}{m_0(C) \cdot m_0(A)} &= 1, \\
\frac{A \cdot B}{m_0(A) \cdot m_0(B)} &= \frac{3}{2}.
\end{align*}
\]

One notices that two of the previous quotients are equal and the third one is greater than them. Płoski discovered that this is a general phenomenon for plane branches, as explained in the next section.

### 3. The statement of Płoski’s theorem

In this section we state a theorem of Płoski of 1985 and a reformulation of it in terms of the notion of ultrametric.

Denote simply by $[S]$ the germ of smooth surface $(S, s)$ and by $m(A)$ the multiplicity of a branch $(A, s) \hookrightarrow (S, s)$.

In his 1985 paper [21], Płoski proved the following theorem:
Theorem 3.1. If $A, B, C$ are three pairwise distinct branches on a smooth surface singularity $S$, then one has the following relations, up to a permutation of the three fractions:

$$\frac{A \cdot B}{m(A) \cdot m(B)} \geq \frac{B \cdot C}{m(B) \cdot m(C)} = \frac{C \cdot A}{m(C) \cdot m(A)}.$$ 

Denote by $\mathcal{B}(S)$ the infinite set of branches on $S$. By inverting the fractions appearing in the statement of Theorem 3.1, it may be reformulated in the following equivalent way:

Theorem 3.2. Let $S$ be a smooth surface singularity. Then the map $\mathcal{B}(S) \times \mathcal{B}(S) \to [0, \infty)$ defined by

$$(A, B) \to \begin{cases} \frac{m(A) \cdot m(B)}{A \cdot B} & \text{if } A \neq B, \\ 0 & \text{otherwise} \end{cases}$$

is an ultrametric.

What does it mean that a function is an ultrametric? We explain this in the next section and we show how to think topologically about ultrametrics on finite sets in terms of certain kinds of decorated rooted trees. This way of thinking is used then in Section 5 in order to prove the reformulation 3.2 of Płoski’s theorem.

4. Ultrametrics and rooted trees

In this section we define the notion of ultrametric and we explain how to think about an ultrametric on a finite set in topological terms, as a special kind of rooted and decorated tree. This passes through understanding that the closed balls of an ultrametric form a hierarchy and that finite hierarchies are equivalent to special types of decorated rooted trees. For more details, one may consult [9, Sect. 3.1].

Definition 4.1. Let $(M, d)$ be a metric space. It is called ultrametric if one has the following strong form of the triangle inequality:

$$d(A, B) \leq \max\{d(A, C), d(B, C)\} \text{, for all } A, B, C \in M.$$ 

In this case, one says also that $d$ is an ultrametric on the set $M$.

In any metric space $(M, d)$, a closed ball is a subset of $M$ of the form:

$$\mathcal{B}(A, r) := \{P \in M, d(P, A) \leq r\}$$

where the center $A \in M$ and the radius $r \in [0, \infty)$ are given. As we will see shortly, given a closed ball, neither its center nor its radius are in general well-defined, contrary to an intuition educated only by Euclidean geometry.

One has the following characterizations of ultrametrics:

Proposition 4.2. Let $(M, d)$ be a metric space. Then the following properties are equivalent:

(1) $(M, d)$ is ultrametric.

(2) The triangles are all isosceles with two equal sides not less than the third side.

(3) All the points of a closed ball are centers of it.

(4) Two closed balls are either disjoint, or one is included in the other.

Proof. All the equivalences are elementary but instructive to check. We leave their proofs as exercises. \qed
Example 4.3. Consider a set $M = \{A, B, C, D\}$ and a distance function $d$ on it such that:

\[
\begin{array}{c}
d(B, C) = 1, \ d(A, B) = d(A, C) = 2, \ d(A, D) = d(B, D) = d(C, D) = 5.
\end{array}
\]

Note that one may embed $(M, d)$ isometrically into a 3-dimensional Euclidean space by choosing an isosceles triangle $ABC$ with the given edge lengths, and by choosing then the point $D$ on the perpendicular to the plane of the triangle passing through its circumcenter. Let us look for the closed balls of this finite metric space. For radii less than 1, they are singletons. For radii in the interval $[1, 2)$, we get the sets $\{B, C\}, \{A\}, \{D\}$. Note that both $B$ and $C$ are centers of the ball $\{B, C\}$, that is, $B(B, r) = B(C, r) = \{B, C\}$ for every $r \in [1, 2)$. Once the radius belongs to the interval $[2, 5)$, the balls are $\{A, B, C\}$ and $\{D\}$. Finally, for every radius $r \in [5, \infty)$, there is only one closed ball, the whole set. Figure 1 depicts the set $\{A, B, C, D\}$ as well as the mutual distances and the associated set of closed balls.

![Figure 1. The balls of an ultrametric space with four points](image)

Example 4.3 illustrates the fact that neither the center nor the radius of a closed ball of a finite ultrametric space is well-defined, once the ball has more than one element. Instead, every closed ball has a well-defined diameter:

**Definition 4.4.** The **diameter** of a closed ball in a finite metric space is the maximal distance between pairs of not necessarily distinct points of it.

The last characterization of ultrametrics in Proposition 4.2 shows that the set $\text{Balls}(M, d)$ of closed balls of an ultrametric space $(M, d)$ is a hierarchy on $M$, in the following sense:

**Definition 4.5.** A **hierarchy** on a set $M$ is a subset $\mathcal{H}$ of its power set $\mathcal{P}(M)$, satisfying the following properties:

- $\emptyset \notin \mathcal{H}$.
- The singletons belong to $\mathcal{H}$.
- $M$ belongs to $\mathcal{H}$.
- Two elements of $\mathcal{H}$ are either disjoint, or one is included into the other.

If $\mathcal{H}$ is a hierarchy on a set $M$, it may be endowed with the inclusion partial order. We will consider instead its reverse partial order $\leq_{\mathcal{H}}$ defined by:

\[
A \leq_{\mathcal{H}} B \iff A \supseteq B, \ \text{for all} \ A, B \in \mathcal{H}.
\]
Reversing the inclusion partial order has the advantage of identifying the leaves of the corresponding rooted tree with the maximal elements of the poset \((\mathcal{H}, \leq_{\mathcal{H}})\) (see Proposition 4.8 below).

When \(M\) is finite, one may represent the poset \((\mathcal{H}, \leq_{\mathcal{H}})\) using its associated Hasse diagram:

**Definition 4.6.** Let \((X, \preceq)\) be a finite poset. Its Hasse diagram is the directed graph whose set of vertices is \(X\), two vertices \(a, b \in X\) being joined by an edge oriented from \(a\) to \(b\) whenever \(a \prec b\) and the two points are directly comparable, that is, there is no other element of \(X\) lying strictly between them.

Hasse diagrams of finite posets are abstract oriented acyclic graphs. This means that they have no directed cycles, which is a consequence of the fact that a partial order is antisymmetric and transitive. Hasse diagrams are not necessarily planar, but, as all finite graphs, they may be always immersed in the plane in such a way that any pair of edges intersect transversely. When drawing a Hasse diagram in the plane as an immersion, we will use the convention to place the vertex \(a\) of the Hasse diagram below the vertex \(b\) whenever \(a \prec b\). This is always possible because of the absence of directed cycles. This convention makes unnecessary adding arrowheads along the edges in order to indicate their orientations.

**Example 4.7.** Consider the finite set \(\{1, 2, 3, 4, 6, 12\}\) of positive divisors of 12, partially ordered by divisibility: \(a \preceq b\) if and only if \(a\) divides \(b\). Its Hasse diagram is drawn in Figure 2.

![Figure 2. The Hasse diagram of the set of positive divisors of 12.](image-url)

The Hasse diagrams of finite hierarchies are special kinds of graphs:

**Proposition 4.8.** The Hasse diagram of a hierarchy \((\mathcal{H}, \leq_{\mathcal{H}})\) on a finite set \(M\) is a tree in which the maximal directed paths start from \(M\) and terminate at the singletons. Moreover, for each vertex which is not a singleton, there are at least two edges starting from it.

**Proof.** We sketch a proof, leaving the details to the reader.

The first statement results from the fact that the singletons of \(M\) are exactly the maximal elements of the poset \((\mathcal{H}, \leq_{\mathcal{H}})\), that \(M\) itself is the unique minimal element and that all the elements of a hierarchy which contain a given element are totally ordered by inclusion.

Let us prove the second statement. Consider \(B_1 \in \mathcal{H}\) and assume that it is not a singleton. This means that it is not minimal for inclusion, therefore there exists \(B_2 \in \mathcal{H}\) such that \(B_2 \subsetneq B_1\) and \(B_2\) is directly comparable to \(B_1\). Let \(A\) be a point of \(B_1 \setminus B_2\). Consider \(B_3 \in \mathcal{H}\) which contains the point \(A\), is included into \(B_1\) and is directly comparable to it. As \(A \in B_3 \setminus B_2\), this shows that \(B_3\) is not included in \(B_2\). We want to show that the two sets \(B_2\) and \(B_3\) are disjoint.
Otherwise, by the definition of a hierarchy, we would have $B_2 \subseteq B_3 \subseteq B_1$, which contradicts the assumption that $B_1$ and $B_2$ are directly comparable. □

**Example 4.9.** Consider the ultrametric space of Example 4.7, represented in Figure 1. We repeat it on the left of Figure 3. The Hasse diagram of the hierarchy of its closed balls is drawn on the right of Figure 3. Near each vertex is represented the diameter of the corresponding ball. We have added a root vertex, connected to the vertex representing the whole set. It may be thought as a larger ball, obtained by adding formally to $M = \{A, B, C, D\}$ a point $\omega$, infinitely distant from each point of $M$. This larger ball is the set $\overline{M} := M \cup \{\omega\}$.

![Diagram](image)

**Figure 3.** The tree of the hierarchy of closed balls of Example 4.9

One may formalize in the following way the construction performed in Example 4.9:

**Definition 4.10.** The tree of a hierarchy $(H, \leq_H)$ on a finite poset $M$ is its Hasse diagram, completed with a root representing the set $\overline{M} := M \cup \{\omega\}$, joined with the vertex representing $M$ and rooted at $\overline{M}$. Here $\omega$ is a point distinct from the points of $M$.

The tree of a hierarchy is a rooted tree in the following sense:

**Definition 4.11.** A rooted tree is a tree with a distinguished vertex, called its root. If $\Theta$ is a rooted tree with root $r$, then the vertex set of $\Theta$ gets partially ordered by declaring that $a \leq_r b$ if and only if the unique segment $[r, a]$ joining $r$ to $a$ in the tree is contained in $[r, b]$.

When $\Theta$ is the rooted tree of a hierarchy $H$ on a finite set $M$, then the partial order $\leq_{\overline{M}}$ defined by choosing $\overline{M}$ as root restricts to the partial order $\leq_H$ if one identifies the set $H$ with the set of vertices of $\Theta$ which are distinct from the root.

Proposition 4.8 may be reformulated in the following way as a list of properties of the tree of the hierarchy:

**Proposition 4.12.** Let $\Theta$ be the tree of a hierarchy on a finite set, and let $r$ be its root. Then $r$ is a vertex of valency 1 and there are no vertices of valency 2.

This proposition motivates the following definition:
Definition 4.13. A rooted tree whose root is of valency 1 and which does not possess vertices of valency 2 is a hierarchical tree. The hierarchy of a hierarchical tree \((\Theta, r)\) is constructed in the following way:

- Define \(M\) to be the set of leaves of the rooted tree \((\Theta, r)\), that is, the set of vertices of valency 1 which are distinct from the root \(r\).
- For each vertex \(p\) of \(\Theta\) different from the root, consider the subset of \(M\) consisting of the leaves \(a\) such that \(p \preceq_r a\).

We leave as an exercise to prove:

Proposition 4.14. The constructions of Definitions 4.10 and 4.13, which associate a hierarchical tree to a hierarchy on a finite set and a hierarchy to a hierarchical tree are inverse of each other.

As a preliminary to the proof, one may test the truth of the proposition on the example of Figure 3.

Let us return to finite ultrametric spaces \((M, d)\). We saw that the set \(\text{Balls}(M, d)\) of its closed balls is a hierarchy on \(M\). Proposition 4.14 shows that one may think about this hierarchy as a special kind of rooted tree, namely, a hierarchical tree. This hierarchical tree alone does not allow to get back the distance function \(d\). How to encode it on the tree?

The idea is to look at the function defined on \(\text{Balls}(M, d)\), which associates to each ball its diameter (see Definition 4.4):

Proposition 4.15. Let \((M, d)\) be a finite ultrametric space. Then the map which sends each closed ball to its diameter is a strictly decreasing \([0, \infty)\)-valued function defined on the poset \((\text{Balls}(M, d), \preceq)\), taking the value 0 exactly on the singletons of \(M\). Equivalently, it is a strictly decreasing \([0, \infty)\]-valued function on the set of vertices of the tree of the hierarchy, vanishing on the set \(M\) of leaves and taking the value \(\infty\) on the root.

As an example, one may look again at Figure 3. The value taken by the previous diameter function is written near each vertex of the hierarchical tree.

If \((\Theta, r)\) is a hierarchical tree, denote by \(V(\Theta)\) its set of vertices and by \([a \land_r b]\) the infimum of \(a\) and \(b\) relative to \(\preceq_r\), whenever \(a, b \in V(\Theta)\). This infimum may be characterized by the property that \([r, a] \cap [r, b] = [r, a \land_r b]\). The following is a converse of Proposition 4.15:

Proposition 4.16. Let \((\Theta, r)\) be a hierarchical tree and \(\lambda : V(\Theta) \to [0, \infty]\) be a strictly decreasing function relative to the partial order \(\preceq_r\) on \(\Theta\) induced by the root. Assume that \(\lambda\) vanishes on the set \(M\) of leaves of \(\Theta\) and takes the value \(\infty\) at \(r\). Then the map

\[
d : M \times M \to [0, \infty) \\
(a, b) \mapsto \lambda(a \land_r b)
\]

is an ultrametric on \(M\).

Let us introduce a special name for the functions appearing in Proposition 4.16:

Definition 4.17. Let \((\Theta, r)\) be a hierarchical tree. A depth function on it is a function \(\lambda : V(\Theta) \to [0, \infty]\) which satisfies the following properties:

- it is strictly decreasing relative to the partial order \(\preceq_r\) on \(\Theta\) induced by the root \(r\);
- it vanishes on the set of leaves of \(\Theta\);
- it takes the value \(\infty\) at the root \(r\).

Note that the first two conditions of Definition 4.17 imply that a depth function vanishes exactly on the set of leaves of the underlying hierarchical tree.

One has the following analog of Proposition 4.14:
Proposition 4.18. The constructions of Propositions 4.15 and 4.16 are inverse of each other. That is, giving an ultrametric on a finite set $M$ is equivalent to giving a depth function on a hierarchical tree whose set of leaves is $M$.

It is this proposition which allows to think about an ultrametric as a special kind of rooted and decorated tree. We leave its proof as an exercise (see [3]).

5. A proof of Płoski’s theorem using Eggers-Wall trees

In this section we sketch a proof of Płoski’s theorem 3.1 using the equivalence between ultrametrics on finite sets and certain kinds of rooted trees formulated in Proposition 4.18. The rooted trees used in this proof are the Eggers-Wall trees of a plane curve singularity relative to smooth reference branches. The precise definition of Eggers-Wall trees is not given, because the proofs of the subsequent generalizations of Płoski’s theorem will be of a completely different spirit.

Instead of working both with multiplicities and intersection numbers as in Płoski’s original statement, we will work only with the latest ones.

Let $S$ be a smooth germ of surface and $L \hookrightarrow S$ be a smooth branch. Define the following function on the set of branches on $S$ which are different from $L$:

$$u_L : (B(S)\setminus\{L\})^2 \to \mathbb{R}_+$$

(5.1)

$$(A, B) \mapsto \begin{cases} (L \cdot A) \cdot (L \cdot B) & \text{if } A \neq B, \\ A \cdot B & \text{otherwise.} \end{cases}$$

In the remaining part of this section we will sketch a proof of:

Theorem 5.2. The function $u_L$ is an ultrametric.

We leave as an exercise to show using Proposition 2.3 that Theorem 5.2 implies the reformulation given in Theorem 3.2 of Płoski’s Theorem 3.1.

Our proof of Theorem 5.2 will pass through the notion of Eggers-Wall tree associated to a plane curve singularity relative to a smooth branch of reference $L$ (see the proof of Theorem 5.6 below). Let us illustrate it by an example.

Example 5.3. Consider again the branches $A = Z(y^2 - x^3), B = Z(y^3 - x^5), C = Z(y^6 - x^5)$ on $S = (\mathbb{C}^2, 0)$ of Example 2.10. Assume that the branch $L$ is the germ at 0 of the $y$-axis $Z(x)$. The defining equations of the three branches $A, B, C$ may be considered as polynomial equations in the variable $y$. As such, they admit the following roots which are fractional powers of $x$:

$$A : x^{3/2},$$

$$B : x^{5/3},$$

$$C : x^{5/6}.$$

Associate to the root $x^{3/2}$ a compact segment $\Theta_L(A)$ identified with the interval $[0, \infty]$ using an exponent function $e_L : \Theta_L(A) \to [0, \infty]$ and mark on it the point $e^{-1}_L(3/2)$ with exponent $3/2$. Define also an index function $i_L : \Theta_L(A) \to \mathbb{N}^*$, constantly equal to 1 on the interval $[e^{-1}_L(0), e^{-1}_L(3/2)]$ and to 2 on the interval $(e^{-1}_L(3/2), e^{-1}_L(\infty))]$ (see the left-most segment of Figure 4). Here the number 2 is to be thought as the minimal positive denominator of the exponent $3/2$ of the monomial $x^{3/2}$. The segment $\Theta_L(A)$ endowed with the two functions $e_L$ and $i_L$ is the Eggers-Wall tree of the branch $A$ relative to the branch $L$. It is considered as a rooted tree with root $e^{-1}_L(0)$, labeled with the branch $L$. Its leaf $e^{-1}_L(\infty)$ is labeled with the branch $A$. 

Consider analogously the Eggers-Wall trees $\Theta_L(B)$ and $\Theta_L(C)$, endowed with pairs of exponent and index functions and labeled roots and leaves (see the left part of Figure 4).

Look now at the plane curve singularity $A + B + C$. Its Eggers-Wall tree $\Theta_L(A + B + C)$ relative to the branch $L$ is obtained from the individual trees $\Theta_L(A), \Theta_L(B), \Theta_L(C)$ by a gluing process, which identifies two by two initial segments of those trees.

Consider for instance the segments $\Theta_L(A), \Theta_L(B)$. Look at the order of the difference $x^{3/2} - x^{5/3}$ of the roots which generated them, seen as a series with fractional exponents. This order is the fraction $3/2$, because $3/2 < 5/3$. Identify then the points with the same exponent $\leq 3/2$ of the segments $\Theta_L(A), \Theta_L(B)$. One gets a rooted tree $\Theta_L(A + B)$ with root labeled by $L$ and with two leaves, labeled by the branches $A, B$. The exponent and index functions of the trees $\Theta_L(A), \Theta_L(B)$ descend to functions with the same name $e_L, i_L$ defined on $\Theta_L(A + B)$. Endowed with those functions, $\Theta_L(A + B)$ is the Eggers-Wall tree of the curve singularity $A + B$.

If one considers now the curve singularity $A + B + C$, then one glues analogously the three pairs of trees obtained from $\Theta_L(A), \Theta_L(B), \Theta_L(C)$. The resulting Eggers-Wall tree $\Theta_L(A + B + C)$ is drawn on the right side of Figure 4. It is also endowed with two functions $e_L, i_L$, obtained by gluing the exponent and index functions of the trees $\Theta_L(A), \Theta_L(B), \Theta_L(C)$. Its marked points are its ends, its bifurcation points and the images of the discontinuity points of the index function of the Eggers-Wall tree of each branch. Near each marked point is written the corresponding value of the exponent function. The index function is constant on each segment $(a, b]$ joining two marked points $a$ and $b$, where $a <_L b$. Here $\leq_L$ denotes the partial order on the tree $\Theta_L(A + B + C)$ determined by the root $L$ (see Definition 4.11).

One may associate analogously an Eggers-Wall tree $\Theta_L(D)$ to any plane curve singularity $D$, relative to a smooth reference branch $L$. It is a rooted tree endowed with an exponent function $e_L : \Theta_L(D) \to [0, \infty]$ and an index function $i_L : \Theta_L(D) \to \mathbb{N}^*$. The tree and both functions are constructed using Newton-Puiseux series expansions of the roots of a Weierstrass polynomial $f \in \mathbb{C}[[x]][y]$ defining $D$ in a coordinate system $(x, y)$ such that $L = Z(x)$. The triple $(\Theta_L(D), e_L, i_L)$ is independent of the choices involved in the previous definition (see [9, Proposition 103]). One may find the precise definition and examples of Eggers-Wall trees in Section 4.3 of the previous reference and in [10, Sect. 3]. Historical remarks about this notion...
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may be found in [10, Rem. 3.18] and [11, Sect. 6.2]. The name, introduced in author’s thesis [22], makes reference to Eggers’ 1983 paper [6] and to Wall’s 2003 paper [27].

What allows us to prove Theorem 5.2 using Eggers-Wall trees is that the values \( u_{L}(A, B) \) of the function \( u_{L} \) defined by relation (5.1) are determined in the following way from the Eggers-Wall tree \( \Theta_{L}D \), for each pair of distinct branches \((A, B)\) of \( D \) (recall from the paragraph preceding Proposition 4.16 that \( A \wedge_{L} B \) denotes the infimum of \( A \) and \( B \) relative to the partial order \( \leq_{L} \) induced by the root \( L \) of \( \Theta_{L}D \)):

**Theorem 5.4.** For each pair \((A, B)\) of distinct branches of \( D \) and every smooth reference branch \( L \) different from the branches of \( D \), one has:

\[
\frac{1}{u_{L}(A, B)} = \int_{L}^{A \wedge_{L} B} \frac{de_{L}}{i_{L}}.
\]

**Example 5.5.** Let us verify the equality stated in Theorem 5.4 on the branches of Example 5.3. Looking at the Eggers-Wall tree \( \Theta_{L}(A + B + C) \) on the right side of Figure 4, we see that:

\[
\int_{L}^{A \wedge_{L} B} \frac{de_{L}}{i_{L}} = \int_{0}^{3/2} \frac{de}{1} = \frac{3}{2}.
\]

But \( 1/u_{L}(A, B) = (A \cdot B)/( (L \cdot A)(L \cdot B)) = (A \cdot B)/(m(A) \cdot m(B)) = 3/2 \), as was computed in Example 2.10. The equality is verified. We have used the fact that both \( A \) and \( B \) are transversal to \( L \), which implies that \( L \cdot A = m(A) \) and \( L \cdot B = m(B) \).

In equivalent formulations which use so-called characteristic exponents, Theorem 5.4 goes back to Smith [23, Section 8], Stolz [24, Section 9] and Max Noether [20]. A modern proof, based on Proposition 2.5, may be found in [28, Thm. 4.1.6].

As a consequence of Theorem 5.4, we get the following strengthening of Theorem 5.2:

**Theorem 5.6.** Let \( D \) be a plane curve singularity. Denote by \( F(D) \) the set of branches of \( D \). Let \( L \) be a reference smooth branch which does not belong to \( F(D) \). Then the function \( u_{L} \) is an ultrametric in restriction to \( F(D) \) and its associated rooted tree is isomorphic as a rooted tree with labeled leaves to the Eggers-Wall tree \( \Theta_{L}D \).

**Proof.** Consider \( \Theta_{L}D \) as a topological tree with vertex set equal to its set of ends and of ramification points. Root it at \( L \). Then it becomes a hierarchical tree in the sense of Definition 4.13. The function

\[
P \rightarrow \left( \int_{L}^{P} \frac{de_{L}}{i_{L}} \right)^{-1}
\]

is a depth function on it, in the sense of Definition 4.17. Using Theorem 5.4 and Proposition 4.18, we get Theorem 5.6. \( \square \)

For more details about the proof of Ploski’s theorem presented in this section, see [9, Sect. 4.3].

### 6. An Ultrametric Characterization of Arborescent Singularities

In this section we state a generalization of Theorem 5.2 for all arborescent singularities and the fact that it characterizes this class of normal surface singularities. We start by recalling the needed notions of embedded resolution and associated dual graph of a finite set of branches contained in a normal surface singularity.

From now on, \( S \) denotes an arbitrary normal surface singularity, that is, a germ of normal complex analytic surface. Let us recall the notion of resolution of such a singularity:
Definition 6.1. Let $(S, s)$ be a normal surface singularity. A resolution of it is a proper bimeromorphic morphism $\pi : S^\pi \to S$ such that $S^\pi$ is smooth. Its exceptional divisor $E^\pi$ is the reduced preimage $\pi^{-1}(s)$. The resolution is good if its exceptional divisor has normal crossings and all its irreducible components are smooth. The dual graph $\Gamma(\pi)$ of the resolution $\pi$ is the finite graph whose set of vertices $\mathcal{P}(\pi)$ is the set of irreducible components of $E^\pi$, two vertices being joined by an edge if and only if the corresponding components intersect.

Every normal surface singularity admits resolutions and even good ones. This result, for which partial proofs appeared already at the end of the XIXth century, was proved first in the analytical context by Hirzebruch in his 1953 paper [15]. His proof was inspired by previous works of Jung [18] and Walker [26], done in an algebraic context.

Assume now that $F$ is a finite set of branches on $S$. It may be also seen as a reduced divisor on $S$, by thinking about their sum. The notion of embedded resolution of $F$ is an analog of that of good resolution of $S$:

Definition 6.2. Let $(S, s)$ be a normal surface singularity and let $\pi : S^\pi \to S$ be a resolution of $S$. If $A$ is a branch on $S$, then its strict transform by $\pi$ is the closure inside $S^\pi$ of the preimage $\pi^{-1}(A\cdot s)$. Let $F$ be a finite set of branches on $S$. Its strict transform by $\pi$ is the set or, depending on the context, the divisor formed by the strict transforms of the branches of $F$. The preimage $\pi^{-1}F$ of $F$ by $\pi$ is the sum of its strict transform and of the exceptional divisor of $\pi$. The morphism $\pi$ is an embedded resolution of $F$ if it is a good resolution of $S$ and the preimage of $F$ by $\pi$ is a normal crossings divisor. The dual graph $\Gamma(\pi^{-1}F)$ of the preimage $\pi^{-1}F$ is defined similarly to the dual graph $\Gamma(\pi)$ of $\pi$, taking into account all the irreducible components of $\pi^{-1}F$.

In the previous definition, the preimage $\pi^{-1}F$ of $F$ by $\pi$ is seen as a reduced divisor. We will see in Definition 9.16 below that there is also a canonical way, due to Mumford, to define canonically a not necessarily reduced rational divisor supported by $\pi^{-1}F$, called the total transform of $F$ by $\pi$, and denoted by $\pi^*F$.

The notion of dual graph of a resolution allows to define the following class of arborescent singularities, whose name was introduced in the paper [9], even if the class had appear before, for instance in Camacho’s work [5]:

Definition 6.3. Let $S$ be a normal surface singularity. It is called arborescent if the dual graphs of its good resolutions are trees.

Remark that in the previous definition we ask nothing about the genera of the irreducible components of the exceptional divisors.

By using the fact that any two resolutions are related by a sequence of blow ups and blow downs of their total spaces (see [14, Thm. V.5.5]), one sees that the dual graphs of all good resolutions are trees if and only if this is true for one of them.

Consider now an arbitrary branch $L$ on the normal surface singularity $S$. We may define the function $u_L$ by the same formula (5.1) as in the case when both $S$ and $L$ were assumed smooth. Intersection numbers of branches still have a meaning, as was shown by Mumford. We will explain this in Section 9 below (see Definition 9.17).

The following generalization of Theorem 5.2 both gives a characterization of arborescent singularities and extends Theorem 5.6 to all arborescent singularities $S$ and all – not necessarily smooth – reference branches $L$ on them (recall that $\mathcal{B}(S)$ denotes the set of branches on $S$):
Theorem 6.4. Let $S$ be a normal surface singularity and $L \in B(S)$. Then:

1. $u_L$ is ultrametric on $B(S) \setminus \{L\}$ if and only if $S$ is arborescent.
2. In this case, for any finite set $F$ of branches on $S$ not containing $L$, the rooted tree of the restriction of $u_L$ to $F$ is isomorphic to the convex hull of $F \cup \{L\}$ in the dual graph of the preimage of $F \cup \{L\}$ by any embedded resolution of $F \cup \{L\}$, rooted at $L$.

We do not prove in the present notes that if $u_L$ is an ultrametric on $B(S) \setminus \{L\}$, then $S$ is arborescent. The interested reader may find a proof of this fact in [12, Sect. 1.6]. The remaining implication of point (1) and point (2) of Theorem 6.4 are, taken together, a consequence of Theorem 8.1 below. For this reason, we do not give a separate proof of them, the rest of this paper being dedicated to the statement and a sketch of proof of Theorem 8.1. The notion of brick-vertex tree of a finite connected graph being crucial in this theorem, we dedicate next section to it.

By combining Theorems 5.6 and 6.4 one gets (see [9, Thm. 112]):

Proposition 6.5. Whenever $S$ and $L$ are both smooth, the Eggers-Wall tree $\Theta_L(D)$ of a plane curve singularity $D \hookrightarrow S$ not containing $L$ is isomorphic to the convex hull of the strict transform of $F(D) \cup \{L\}$ in the dual graph of its preimage by any of its embedded resolutions.

A prototype of this fact was proved differently in the author’s thesis [22, Thm. 4.4.1], then generalized in two different ways by Wall in [28, Thm. 9.4.4] (see also Wall’s comments in [28, Sect. 9.10]) and by Favre and Jonsson in [7, Prop. D.1].

7. The brick-vertex tree of a connected graph

In this section we introduce the notion of brick-vertex tree of a connected graph, which is crucial in order to state Theorem 8.1 below, the strongest known generalization of Ploski’s theorem.

Definition 7.1. A graph is a compact cell complex of dimension $\leq 1$. If $\Gamma$ is a graph, its set of vertices is denoted $V(\Gamma)$ and its set of edges is denoted $E(\Gamma)$.

In the sequel it will be crucial to look at the vertices which disconnect a given graph:

Definition 7.2. Let $\Gamma$ be a connected graph. A cut-vertex of $\Gamma$ is a vertex whose removal disconnects $\Gamma$. A bridge of $\Gamma$ is an edge such that the removal of its interior disconnects $\Gamma$. If $a, b, c$ are three not necessarily distinct vertices of $\Gamma$, one says that $b$ separates $a$ from $c$ if either $b \in \{a, c\}$ or if $a$ and $c$ belong to different connected components of the topological space $\Gamma \setminus \{b\}$.

Note that an end of a bridge is a cut-vertex if and only if it has valency at least 2 in $\Gamma$, that is, if and only if it is not a leaf of $\Gamma$. It will be important to distinguish the class of graphs which cannot be disconnected by the removal of one vertex, as well as the maximal graphs of this class contained in a given connected graph:

Definition 7.3. A connected graph is called inseparable if it does not contain cut-vertices. A block of a connected graph $\Gamma$ is a maximal inseparable subgraph of it. A brick of $\Gamma$ is a block which is not a bridge.

Note that all the bridges of a connected graph are blocks of it.

Example 7.4. In Figure 5 is represented a connected graph. Its cut-vertices are surrounded in red. Its bridges are represented as black segments. It has three bricks, the edges of each brick being colored in the same way.
By replacing each brick of a connected graph by a star-shaped graph, one gets canonically a tree associated to the given graph:

**Definition 7.5.** The brick-vertex tree $BV(\Gamma)$ of a connected graph $\Gamma$ is the tree whose set of vertices is the union of the set of vertices of $\Gamma$ and of a set of new brick-vertices corresponding bijectively to the bricks of $\Gamma$, its edges being either the bridges of $\Gamma$ or new edges connecting each brick-vertex to the vertices of the corresponding brick. Formally, this may be written as follows:

- $V(BV(\Gamma)) = V(\Gamma) \sqcup \{\text{bricks of } \Gamma\}$.
- $E(BV(\Gamma)) = \{\text{bridges of } \Gamma\} \sqcup \{[v, b], v \in V(\Gamma), b \text{ is a brick of } \Gamma, v \in V(b)\}$.

We denoted by $[\overline{v}]$ the vertex $v$ of $\Gamma$ when it is seen as a vertex of $BV(\Gamma)$ and $[\overline{\Gamma}] \in V(BV(\Gamma))$ the brick-vertex representing the brick $b$ of $\Gamma$.

The notion of brick-vertex tree was introduced in [12, Def. 1.34]. It is strongly related to other notions introduced before either in general topology or in graph theory, as explained in [12, Rems. 1.35, 2.50].

Note that whenever $\Gamma$ is a tree, $BV(\Gamma)$ is canonically isomorphic to it, as $\Gamma$ has no bricks.

**Example 7.6.** On the left side of Figure 6 is repeated the graph $\Gamma$ of Figure 5, with its cut-vertices and bricks emphasized. On its right side is represented its associated brick-vertex tree $BV(\Gamma)$. Each representative vertex of a brick is drawn with the same color as its corresponding brick. The edges of $BV(\Gamma)$ which are not bridges of $\Gamma$ are represented in magenta and thicker than the other edges.

![Figure 6](image-url)
Proposition 7.7. Let $\Gamma$ be a finite graph and $a, b, c \in V(\Gamma)$. Then $b$ separates $a$ from $c$ in $\Gamma$ if and only if $\overline{b}$ separates $\overline{a}$ from $\overline{c}$ in $BV(\Gamma)$.

We are ready now to state the strongest known generalization of Płoski’s theorem (see Theorem 8.1 below).

8. Our strongest generalization of Płoski’s theorem

In this section we formulate Theorem 8.1, which generalizes Theorem 5.6 to all normal surface singularities and all branches on them, using the notion of brick-vertex tree introduced in the previous section.

Recall that the notion of brick-vertex tree of a connected graph was introduced in Definition 7.5. A fundamental property of normal surface singularities is that the dual graphs of their resolutions are connected (which is a particular case of the so-called Zariski’s main theorem, whose statement may be found in [14, Thm. V.5.2]). This implies that the dual graph of the preimage (see Definition 6.2) of any finite set of branches on such a singularity is also connected. Therefore, one may speak about its corresponding brick-vertex tree. The convex hull of a finite set of vertices of it is the union of the segments which join them pairwise.

Here is the announced generalization of Theorem 5.6, which is a slight reformulation of [12, Thm. 1.42]:

**Theorem 8.1.** Let $S$ be a normal surface singularity. Consider a finite set $\mathcal{F}$ of branches on it and an embedded resolution $\pi: S^\pi \to S$ of $\mathcal{F}$. Let $\Gamma$ be the dual graph of the preimage $\pi^{-1}\mathcal{F}$ of $\mathcal{F}$ by $\pi$. Assume that the convex hull $Conv_{BV(\Gamma)}(\mathcal{F})$ does not contain brick-vertices of valency at least 4 in $Conv_{BV(\Gamma)}(\mathcal{F})$. Then for all $L \in \mathcal{F}$, the restriction of $u_L$ to $\mathcal{F} \setminus \{L\}$ is an ultrametric and the corresponding rooted tree is isomorphic to $Conv_{BV(\Gamma)}(\mathcal{F})$, rooted at $L$.

**Example 8.2.** Assume that the dual graph $\Gamma$ of $\pi^{-1}\mathcal{F}$ is as shown on the left side of Figure 7. The vertices representing the strict transforms of the branches of the set $\mathcal{F}$ are drawn arrowheaded. Note that the subgraph which is the dual graph of the exceptional divisor is the same as the graph of Figure 5. On the right side of Figure 7 is represented using thick red segments the convex hull $Conv_{BV(\Gamma)}(\mathcal{F})$. We see that the hypothesis of Theorem 8.1 is satisfied. Indeed, the convex hull contains only two brick-vertices, which are of valency 2 and 3 in $Conv_{BV(\Gamma)}(\mathcal{F})$. Note that the blue one is of valency 4 in the dual graph $\Gamma$, which shows the importance of looking at the valency in the convex hull $Conv_{BV(\Gamma)}(\mathcal{F})$, not in $\Gamma$.

![Figure 7](image-url)
As shown in [12, Ex. 1.44], the condition about valency is not necessary in general for \( u_L \) to be an ultrametric on \( \mathcal{F} \setminus \{ L \} \).

Note that we have expressed Theorem 8.1 in a slightly different form than the equivalent Theorem D of the introduction. Namely, we included \( L \) in the branches of \( \mathcal{F} \). This formulation emphasizes the symmetry of the situation: all the choices of reference branch inside \( \mathcal{F} \) lead to the same tree, only the root being changed. In fact, we will obtain Theorem 8.1 as a consequence of Theorem 10.10, in which no branch plays any more a special role.

Before that, we will explain in the next section Mumford’s definition of intersection number of two curve singularities drawn on an arbitrary normal surface singularity, which allows to define in turn the functions \( u_L \) appearing in the statement of Theorem 8.1.

9. Mumford’s intersection theory

In this section we explain Mumford’s definition of intersection number of Weil divisors on a normal surface singularity, introduced in his 1961 paper [19]. It is based on Theorem 9.1, stating that the intersection form of any resolution of a normal surface singularity is negative definite. This theorem being fundamental for the study of surface singularities, we present a detailed proof of it.

Let \( \pi : S^n \to S \) be a resolution of the normal surface singularity \( S \). Denote by \( (E_u)_{u \in \mathcal{P}(\pi)} \) the collection of irreducible components of the exceptional divisor \( E^n \) of \( \pi \) (see Definition 6.1).

Denote by:

\[
\mathcal{E}(\pi)_\mathbb{R} := \bigoplus_{u \in \mathcal{P}(\pi)} \mathbb{R}E_u
\]

the real vector space freely generated by those prime divisors, that is, the space of real divisors supported by \( E^n \). It is endowed with a symmetric bilinear form \( (D_1, D_2) \to D_1 \cdot D_2 \) given by intersecting the corresponding compact cycles on \( S^n \). We call it the intersection form. Its following fundamental property was proved by Du Val [25] and Mumford [19]:

**Theorem 9.1.** The intersection form on \( \mathcal{E}(\pi)_\mathbb{R} \) is negative definite.

**Proof.** The following proof is an expansion of that given by Mumford in [19].

The singularity \( S \) being normal, the exceptional divisor \( E^n \) is connected (this is a particular case of Zariski’s main theorem, see [14, Thm. V.5.2]). Therefore:

(9.2) The dual graph \( \Gamma(\pi) \) is connected.

Consider any germ of holomorphic function \( f \) on \( (S, s) \), vanishing at \( s \), and look at the divisor of its lift to the surface \( S^n \):

(9.3) \( (\pi^* f) = \sum_{u \in \mathcal{P}(\pi)} a_u E_u + (\pi^* f)_{str} \).

Here \((\pi^* f)_{str}\) denotes the strict transform of the divisor defined by \( f \) on \( S \). Denote also:

\[
\begin{cases}
  e_u := a_u E_u \in \mathcal{E}(\pi)_\mathbb{R}, & \text{for all } u \in \mathcal{P}(\pi), \\
  \sigma := \sum_{u \in \mathcal{P}(\pi)} e_u \in \mathcal{E}(\pi)_\mathbb{R}.
\end{cases}
\]

As \( f \) vanishes at the point \( s \), its lift \( \pi^* f \) vanishes along each component \( E_u \) of \( E^n \), therefore \( a_u > 0 \) for every \( u \in \mathcal{P}(u) \). We deduce that \((e_u)_{u \in \mathcal{P}(\pi)}\) is a basis of \( \mathcal{E}(\pi)_\mathbb{R} \) and that:

(9.5) \( e_u \cdot e_v \geq 0 \), for all \( u, v \in \mathcal{P}(\pi) \) such that \( u \neq v \).
The divisor \((\pi^* f)\) being principal, its associated line bundle is trivial. Therefore:

\[(\pi^* f) \cdot E_u = 0\]  \((\text{9.6})\)

because this intersection number is equal by definition to the degree of the restriction of this line bundle to the curve \(E_u\). By combining the relations \((\text{9.3}), (\text{9.4})\) and \((\text{9.6})\), we deduce that:

\[\sigma \cdot e_u = -a_u(\pi^* f)_{\text{str}} \cdot E_u, \text{ for every } u \in \mathcal{P}(\pi).\]  \((\text{9.7})\)

As the germ of effective divisor \((\pi^* f)_{\text{str}}\) along \(E^\pi\) has no components of \(E^\pi\) in its support, the intersection numbers \((\pi^* f)_{\text{str}} \cdot E_u\) are all non-negative. Moreover, at least one of them is positive, because the divisor \((\pi^* f)_{\text{str}}\) is non-zero. By combining this fact with relations \((\text{9.7})\) and with the inequalities \(a_u > 0\), we get:

\[
\begin{align*}
\sigma \cdot e_u &\leq 0, \text{ for every } u \in \mathcal{P}(\pi), \\
\text{there exists } u_0 \in \mathcal{P}(\pi) \text{ such that } \sigma \cdot e_{u_0} < 0.
\end{align*}
\]  \((\text{9.8})\)

Consider now an arbitrary element \(\tau \in \mathcal{E}(\pi)_{\mathbb{R}} \setminus \{0\}\). One may develop it in the basis \((e_u)_{u \in \mathcal{P}(\pi)}:\)

\[\tau = \sum_{u \in \mathcal{P}(\pi)} x_u e_u.\]  \((\text{9.9})\)

We will show that \(\tau^2 < 0\). As \(\tau\) was chosen as an arbitrary non-zero vector, this will imply that the intersection form on \(\mathcal{E}(\pi)_{\mathbb{R}}\) is indeed negative definite. The trick is to express the self-intersection \(\tau^2\) using the expansion \((\text{9.9})\), then to develop it by bilinearity and to replace the vectors \(e_u\) by \(\sigma - \sum_{v \neq u} e_v\) in a precise place:

\[
\tau^2 = \left( \sum_{u} x_u e_u \right)^2 = \\
= \sum_{u} x_u^2 \cdot e_u^2 + 2 \sum_{u < v} x_u x_v e_u \cdot e_v = \\
= \sum_{u} x_u^2 (\sigma - \sum_{v \neq u} e_v) \cdot e_u + 2 \sum_{u < v} x_u x_v e_u \cdot e_v = \\
= \sum_{u} x_u^2 (\sigma \cdot e_u) - \sum_{u < v} (x_u - x_v)^2 e_u \cdot e_v.
\]

We get the equality:

\[
\tau^2 = \sum_{u} x_u^2 (\sigma \cdot e_u) - \sum_{u < v} (x_u - x_v)^2 e_u \cdot e_v.
\]  \((\text{9.10})\)

Using the inequalities \((\text{9.5})\) and \((\text{9.8})\), we deduce that its right-hand side is non-positive, therefore the intersection form is negative semi-definite.

It remains to show that \(\tau^2 < 0\). Assume by contradiction that \(\tau^2 = 0\). Equality \((\text{9.10})\) shows that the following equalities are simultaneously satisfied:

\[
\sum_{u} x_u^2 (\sigma \cdot e_u) = 0,\]  \((\text{9.11})\)

\[
(x_u - x_v)^2 e_u \cdot e_v = 0, \text{ for all } u < v.
\]  \((\text{9.12})\)

The relations \((\text{9.12})\) imply that \(x_u = x_v\) whenever \(e_u \cdot e_v > 0\). As \(e_u = a_u E_u\) with \(a_u > 0\), the inequality \(e_u \cdot e_v > 0\) is equivalent with \(E_u \cdot E_v > 0\), that is, with the fact that \([u, v]\) is an edge of the dual graph \(\Gamma(\pi)\). This dual graph being connected (see \((\text{9.2})\)), we see that \(x_u = x_v\).
for all \( u, v \in \mathcal{P}(\pi) \). Consider now an index \( u_0 \) satisfying the second condition of relations (9.8). Equation (9.11) implies that \( x_{u_0} = 0 \). Therefore all the coefficients \( x_u \) vanish, which contradicts the hypothesis that \( \tau \neq 0 \). □

As a consequence of Theorem 9.1, one may define the dual basis \( \left( E_u^* \right)_{u \in \mathcal{P}(\pi)} \) of the basis \( (E_u)_{u \in \mathcal{P}(\pi)} \) by the following relations, in which \( \delta_{uv} \) denotes Kronecker’s delta-symbol:

\[
E_u^* \cdot E_v = \delta_{uv}, \quad \text{for all } (u, v) \in \mathcal{P}(\pi)^2.
\]

(9.13) By associating to each prime divisor \( E_u \) the corresponding valuation of the local ring \( \mathcal{O}_{S,s} \), computing the orders of vanishing along \( E_u \) of the pull-backs \( \pi^* f \) of the functions \( f \in \mathcal{O}_{S,s} \), one injects the set \( \mathcal{P}(\pi) \) in the set of real-valued valuations of \( \mathcal{O}_{S,s} \). This allows to see the index \( u \) of \( E_u \) as a valuation. Such valuations are called divisorial. If \( u \) denotes a divisorial valuation, it has a center on any resolution, which is either a point or an irreducible component of the exceptional divisor. In the second case, one says that the valuation appears in the resolution. The following notion, inspired by approaches of Favre & Jonsson [7, App. A] and [17, Sect. 7.3.6], was introduced in [12, Def. 1.6]:

**Definition 9.14.** Let \( u, v \) be two divisorial valuations of \( S \). Consider a resolution of \( S \) in which both \( u \) and \( v \) appear. Then their bracket is defined by:

\[
\langle u, v \rangle := -E_u^* \cdot E_v^*.
\]

The bracket \( \langle u, v \rangle \) may be interpreted as the intersection number of two Weil divisors on \( S \) associated to the divisors \( E_u \) and \( E_v \) (see Proposition 9.19 below). As a consequence, it is well-defined. That is, if the divisorial valuations \( u, v \) are fixed, then their bracket does not depend on the resolution in which they appear. This fact may be also proved using the property that any two resolutions of \( S \) are related by a sequence of blow ups and blow downs (see [12, Prop. 1.5]).

It is a consequence of Theorem 9.1 that the brackets are all non-negative (see [12, Prop. 1.4]). Moreover, by the Cauchy-Schwarz inequality applied to the opposite of the intersection form:

**Lemma 9.15.** For every \( a, b \in \mathcal{P}(\pi) \):

\[
\langle a, b \rangle^2 \leq \langle a, a \rangle \langle b, b \rangle,
\]

with equality if and only if \( a = b \).

Let now \( D \) be a Weil divisor on \( S \), that is, a formal sum of branches on \( S \). If \( D \) is principal, that is, the divisor \( (f) \) of a meromorphic germ on \( S \), then one may lift it to a resolution \( S^\pi \) as the principal divisor \( (\pi^* f) \). This divisor decomposes as the sum of an exceptional part \( (\pi^* D)_{\text{ex}} \) supported by \( E^\pi \) and the strict transform of \( D \). The crucial property of the lift \( (\pi^* f) \), already used in the proof of Theorem 9.1 (see relation (9.6)), is that its intersection numbers with all the components \( E_u \) of \( E^\pi \) vanish. In [19, Sect. II (b)], Mumford imposed this property in order to define a lift \( \pi^* D \) for any Weil divisor \( D \) on \( S \):

**Definition 9.16.** Let \( D \) be a Weil divisor on \( S \). Its total transform \( \pi^* D \) is the unique sum \( (\pi^* D)_{\text{ex}} + (\pi^* D)_{\text{str}} \) such that:

1. \( (\pi^* D)_{\text{ex}} \in \mathcal{E}(\pi)_{\mathbb{Q}} \).
2. \( (\pi^* D)_{\text{str}} \) is the strict transform of \( D \) by \( \pi \).
3. \( (\pi^* D) \cdot E_u = 0 \) for all \( u \in \mathcal{P}(\pi) \).

The divisor \( (\pi^* D)_{\text{ex}} \) supported by the exceptional divisor of \( \pi \) is the exceptional transform of \( D \) by \( \pi \).
The divisor $\pi^* D$ is well-defined, as results from Theorem 9.1. The point is to show that $(\pi^* D)_e x$ exists and is unique with the property (3). Write it as a sum $\sum_{v \in \mathcal{P}(\pi)} x_v E_v$. The last condition of Definition 9.16 may be written as the system:

$$\sum_{v \in \mathcal{P}(\pi)} (E_v \cdot E_u)x_v = -(\pi^* D)_{str} \cdot E_u, \text{ for all } u \in \mathcal{P}(\pi).$$

This is a square linear system in the unknowns $x_v$, whose matrix is the matrix of the intersection form in the basis $(E_u)_{u \in \mathcal{P}(\pi)}$. As the intersection form is negative definite, it is non-degenerate, therefore this system has a unique solution. Moreover, all its coefficients being integers, its solution has rational coordinates, which shows that $(\pi^* D)_e x \in \mathcal{E}(\pi)\mathbb{Q}$.

Using Definition 9.16 and the standard definition of intersection numbers on smooth surfaces recalled in Section 2, Mumford defined in the following way in [19, Sect. II (b)] the intersection number of two Weil divisors on $S$:

**Definition 9.17.** Let $A, B$ be two Weil divisors on $S$ without common components, and $\pi$ be a resolution of $S$. Then the **intersection number** of $A$ and $B$ is defined by:

$$A \cdot B := \pi^* A \cdot \pi^* B.$$

Using the fact that any two resolutions of $S$ are related by a sequence of blow ups and blow downs (see [14, Thm. V.5.5]), it may be shown that the previous notion is independent of the choice of resolution, similarly to that of bracket of two divisorial valuations introduced in Definition 9.14. In particular, if $S$ is smooth, one may choose $\pi$ to be the identity. This shows that in this case Mumford’s definition gives the same intersection number as the standard Definition 2.2.

**Example 9.18.** Let $S$ be the germ at the origin 0 of the quadratic cone $Z(x^2 + y^2 + z^2) \hookrightarrow \mathbb{C}^3$ (it is the so-called $A_1$ surface singularity). Let $A$ and $B$ be the germs at 0 of two distinct generating lines of the cone. One may resolve $S$ by blowing up 0. This morphism $\pi: S^\pi \to S$ separates all the generators, therefore it is an embedded resolution of $\{A, B\}$. The exceptional divisor of $\pi$ is the projectivisation of the cone, that is, it is a smooth rational curve $E$. Its self-intersection number is the opposite of the degree of the curve seen embedded in the projectivisation of the ambient space $\mathbb{C}^3$. Therefore, $E^2 = -2$. Let us compute the total transform $\pi^* A = (\pi^* A)_{str} + x E$. The imposed constraint $\pi^* A \cdot E = 0$ becomes $1 - 2x = 0$, therefore $x = 1/2$. We have used the fact that the strict transform $(\pi^* A)_{str}$ of $A$ by $\pi$ is smooth and transversal to $E$, which implies that $(\pi^* A)_{str} \cdot E = 1$.

We obtained $\pi^* A = (\pi^* A)_{str} + (1/2) E$ and similarly, $\pi^* B = (\pi^* B)_{str} + (1/2) E$. Using Definition 9.17, we get:

$$A \cdot B = \pi^* A \cdot \pi^* B =$$

$$= ((\pi^* A)_{str} + (1/2) E) \cdot ((\pi^* B)_{str} + (1/2) E) =$$

$$= (\pi^* A)_{str} \cdot (\pi^* B)_{str} + (1/2)((\pi^* A)_{str} + (\pi^* B)_{str}) \cdot E + (1/2)^2 E^2 =$$

$$= 0 + (1/2) \cdot 2 + (1/2)^2 \cdot (-2) =$$

$$= 1/2.$$

This example shows in particular that the intersection number of two curve singularities depends on the normal surface singularity on which it is computed. Indeed, the branches $A$ and $B$ are also contained in a smooth surface (any two generators of the quadratic cone are obtained as the intersection of the cone with a plane passing through its vertex). In such a surface, their intersection number is 1 instead of 1/2.

Definition 9.17 allows to give the following interpretation of the notion of bracket introduced in Definition 9.14 (see [12, Prop. 1.11]):
Proposition 9.19. Let $A, B$ be two distinct branches on $S$. Consider an embedded resolution $\pi$ of their sum. Denote by $E_a, E_b$ the components of the exceptional divisor $E^\pi$ which are intersected by the strict transforms $(\pi^* A)_{str}$ and $(\pi^* B)_{str}$ respectively. Then:
\[ A \cdot B = \langle a, b \rangle. \]

Proof. This proof uses directly Definition 9.16.

As $\pi$ is an embedded resolution of $A + B$, the strict transforms $(\pi^* A)_{str}$ and $(\pi^* B)_{str}$ are disjoint. Therefore $(\pi^* A)_{str} \cdot (\pi^* B)_{str} = 0$. Using the last condition in the Definition 9.16 of the total transform of a divisor, we know that $(\pi^* A) \cdot (\pi^* B)_{ex} = (\pi^* A)_{ex} \cdot (\pi^* B) = 0$. Combining both equalities, we deduce that:
\[
A \cdot B = (\pi^* A) \cdot (\pi^* B) = (\pi^* A) \cdot ((\pi^* B)_{ex} + (\pi^* B)_{str}) = (\pi^* A) \cdot (\pi^* B)_{str} = ((\pi^* A)_{ex} + (\pi^* A)_{str}) \cdot (\pi^* B)_{str} = (\pi^* A)_{ex} \cdot (\pi^* B)_{str} = (\pi^* A)_{ex} \cdot (\pi^* B - (\pi^* B)_{ex}) = -(\pi^* A)_{ex} \cdot (\pi^* B)_{ex} = -(-E^\pi_a) \cdot (-E^\pi_b) = \langle a, b \rangle.
\]

At the end of the computation we have used the equality $(\pi^* A)_{ex} = -E^\pi_b$, which results from the fact that $\pi$ is an embedded resolution of $A$. Indeed, this implies that $((\pi^* A)_{str} + E^\pi_a) \cdot E_u = 0$ for every $u \in \mathcal{P}(\pi)$, which shows that one has indeed the stated formula for $(\pi^* A)_{ex}$.

10. A REFORMULATION OF THE ULTRAMETRIC INEQUALITY

In this section we explain the notion of angular distance on the set of vertices of the dual graph of a good resolution of $S$. Theorem 10.2 states a crucial property of this distance, relating it to the cut-vertices of the dual graph. Then the ultrametric inequality is reexpressed in terms of the angular distance. This allows to show that Theorem 8.1 is a consequence of Theorem 10.10, which is formulated only in terms of the angular distance.

Let $\pi : S^\pi \rightarrow S$ be a good resolution of the normal surface singularity $S$. Recall that $\mathcal{P}(\pi)$ denotes the set of irreducible components of its exceptional divisor $E^\pi$. Using the notion of bracket from Definition 9.14, one may define (see [13, Sect. 2.7] and [12, Sect. 1.2]):

Definition 10.1. The angular distance is the function $\rho : \mathcal{P}(\pi) \times \mathcal{P}(\pi) \rightarrow [0, \infty)$ given by:
\[
\rho(a, b) := \begin{cases} 
-\log \frac{\langle a, b \rangle^2}{\langle a, a \rangle \langle b, b \rangle} & \text{if } a \neq b, \\
0 & \text{if } a = b.
\end{cases}
\]

The fact that the function $\rho$ takes values in the interval $[0, \infty)$ is a consequence of Lemma 9.15. The attribute “angular” was chosen by Gignac and Ruggiero because their definition in [13, Sect. 2.7] was more general, applying to any pair of real-valued semivaluations of the local ring $\mathcal{O}_{S,s}$, and that it depended only on those valuations up to homothety, similarly to the angle of two vectors. It is a distance by the following theorem of Gignac and Ruggiero [13, Prop. 1.10] (recall that the notion of vertex separating two other vertices was introduced in Definition 7.2):

Theorem 10.2. The function $\rho$ is a distance on the set $\mathcal{P}(\pi)$. Moreover, for every $a, b, c \in \mathcal{P}(\pi)$, the following properties are equivalent:
- one has the equality $\rho(a, b) + \rho(b, c) = \rho(a, c)$;
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b separates a and c in the dual graph \( \Gamma(\pi) \).

This theorem explains the importance of cut-vertices of the dual graph \( \Gamma(\pi) \) for understanding the angular distance.

Theorem 10.2 is a reformulation of the following theorem, which was first proved by in [9, Prop. 79, Rem. 81] for arborescent singularities, then in [13, Prop. 1.10] for arbitrary normal surface singularities (see also [12, Prop. 1.18] for a slightly different proof):

**Theorem 10.3.** Let \( a, b, c \in \mathcal{P}(\pi) \). Then:

\[
(a, b) \cdot (b, c) \leq (b, a) \cdot (a, c),
\]

with equality if and only if \( b \) separates \( a \) and \( c \) in the dual graph \( \Gamma(\pi) \).

Theorem 10.3 may be also reformulated in terms of spherical geometry using the spherical Pythagorean theorem (see [12, Prop. 1.19.III]).

Using Proposition 9.19 and Definition 10.1 of the angular distance, one may reformulate in the following way the ultrametric inequality for the restriction of the function \( u_L \) to a set of three branches:

**Proposition 10.4.** Let \( L, A, B, C \) be pairwise distinct branches on \( S \). Consider an embedded resolution of their sum and let \( E_L, E_A, E_B, E_C \) the irreducible components of its exceptional divisor which intersect the strict transforms of \( L, A, B \) and \( C \) respectively. Then the following (in)equality are equivalent:

1. \( u_L(A, B) \leq \max\{u_L(A, C), u_L(B, C)\} \).
2. \( (A \cdot B) \cdot (L \cdot C) \geq \min\{(A \cdot C)(L \cdot B), (B \cdot C)(L \cdot A)\} \).
3. \( (a, b) \cdot (l, c) \geq \min\{(a, c) \cdot (l, b), (b, c) \cdot (l, a)\} \).
4. \( \rho(a, b) + \rho(l, c) \leq \max\{\rho(a, c) + \rho(l, b), \rho(b, c) + \rho(l, a)\} \).

We leave the easy proof of this proposition to the reader. It uses the definitions of the function \( u_L \), of the angular distance, as well as Proposition 9.19. Note that excepted the first one, all the inequalities are symmetric in the four branches \( L, A, B, C \). The fourth one is a well-known condition in combinatorics, whose name was introduced by Bunemann in his 1974 paper [4]:

**Definition 10.5.** Let \((X, \delta)\) be a finite metric space. One says that it satisfies the **four points condition** if whenever \( a, b, c, d \in X \), one has the following inequality:

\[ \delta(a, b) + \delta(c, d) \leq \max\{\delta(a, c) + \delta(b, d), \delta(a, d) + \delta(b, c)\} \].

In the same way in which a finite ultrametric may be thought as a special kind of decorated *rooted* tree (see Proposition 4.18), a finite metric space satisfying the four points condition may be thought as a special kind of decorated *unrooted* tree (see [3]):

**Proposition 10.6.** The metric space \((X, \delta)\) satisfies the four points condition if and only if \( \delta \) is induced by a length function on a tree containing the set \( X \) among its set of vertices. If, moreover, one constrains \( X \) to contain all the vertices of the tree of valency 1 or 2, then this tree is unique up to a unique isomorphism fixing \( X \).

Let us introduce supplementary vocabulary in order to deal with the special trees appearing in Proposition 10.6:

**Definition 10.7.** Let \( X \) be a finite set. An **\( X \)-tree** is a tree whose set of vertices contains the set \( X \) and such that each vertex of valency at most 2 belongs to \( X \). If \((X, \delta)\) is a finite metric space which satisfies the four points condition, then the unique \( X \)-tree characterized in Proposition 10.6 is called the **tree hull** of \((X, \delta)\).
The basic idea of the proof of Proposition 10.6 is that an $X$-tree is characterized by the shapes of the convex hulls of the quadruples of points of $X$, and that those shapes are determined by the cases of equality in the 12 triangle inequalities and the 3 four points conditions associated to each quadruple. In Figure 8 are represented the five possible shapes. For instance, the $H$-shape is the generic one, characterized by the fact that one has no equality in the previous inequalities.

![Figure 8. The possible shapes of an $X$-tree, when $X$ has four elements.](image)

Let us come back to our normal surface singularity $S$. One has the following property (see [12, Prop. 1.24]):

**Proposition 10.8.** Let $\mathcal{F}$ be a finite set of branches on $S$. If $u_L$ is an ultrametric on $\mathcal{F} \setminus \{L\}$ for one branch $L$ in $\mathcal{F}$, then the same is true for any branch of $\mathcal{F}$.

By Proposition 10.4, if $u_L$ is an ultrametric on $\mathcal{F} \setminus \{L\}$ for one branch $L$ in $\mathcal{F}$, then one has the symmetric relation (2) for every quadruple of branches of $\mathcal{F}$ containing $L$. The subtle point of the proof of Proposition 10.8 is to deduce from this fact that (2) is satisfied by all quadruples.

Given Proposition 10.8, it is natural to try to relate the rooted trees associated to the ultrametrics obtained by varying $L$ among the branches of $\mathcal{F}$. By looking at quadruples of branches from $\mathcal{F}$, one may prove using Propositions 10.4 and 10.8 that:

**Proposition 10.9.** Let $\mathcal{F}$ be a finite set of branches on $S$. Consider an embedded resolution of $\mathcal{F}$ such that the map associating to each branch $A$ of $\mathcal{F}$ the component $E_a$ of the exceptional divisor intersected by its strict transform is injective. Denote by $\mathcal{F}^\pi$ the set of divisorial valuations $a$ appearing in this way. Then:

1. The function $u_L$ is an ultrametric on $\mathcal{F} \setminus \{L\}$ for some branch $L$ in $\mathcal{F}$ if and only if the angular distance $\rho$ satisfies the four points condition in restriction to the set $\mathcal{F}^\pi$.

2. Assume that the previous condition is satisfied. Then the rooted tree associated to $u_L$ on $\mathcal{F} \setminus \{L\}$ is isomorphic to the tree hull of $(\mathcal{F}^\pi, \rho)$ by an isomorphism which sends each end marked by a branch $A$ of $\mathcal{F}$ to the vertex $a$ of the tree hull.

Proposition 10.9 implies readily that Theorem 8.1 is a consequence of the following fact (see [12, Cor. 1.40]):

**Theorem 10.10.** Let $S$ be a normal surface singularity. Consider a set $\mathcal{G}$ of vertices of the dual graph $\Gamma$ of a good resolution $\pi : S^\pi \to S$ of $S$. Assume that the convex hull $\text{Conv}_{\text{BV}(\Gamma)}(\mathcal{G})$ of $\mathcal{G}$ in the brick-vertex tree of $\Gamma$ does not contain brick-vertices of valency at least 4 in $\text{Conv}_{\text{BV}(\Gamma)}(\mathcal{G})$. Then the restriction of the angular distance $\rho$ to $\mathcal{G}$ satisfies the four points condition and the associated tree hull is isomorphic as a $\mathcal{G}$-tree to $\text{Conv}_{\text{BV}(\Gamma)}(\mathcal{G})$.

In turn, Theorem 10.10 is a consequence of a graph-theoretic result presented in the next section (see Theorem 11.1).
11. A theorem of graph theory

In this final section we state a pure graph-theoretical theorem, which implies Theorem 10.10 of the previous section. As we explained before, that theorem implies in turn our strongest generalization of Ploski’s theorem, that is, Theorem 8.1.

Theorem 10.10 is a consequence of Theorem 10.2 and of the following graph-theoretic result:

**Theorem 11.1.** Let $\Gamma$ be a finite connected graph and $\delta$ be a distance on the set $V(\Gamma)$ of vertices of $\Gamma$, such that for every $a, b, c \in V(\Gamma)$, the following properties are equivalent:

- one has the equality $\delta(a, b) + \delta(b, c) = \delta(a, c)$;
- $b$ separates $a$ and $c$ in $\Gamma$.

Let $X$ be a set of vertices of $\Gamma$ such that the convex hull $\text{Conv}_{\text{BV}(\Gamma)}(X)$ of $X$ in the brick-vertex tree of $\Gamma$ does not contain brick-vertices of valency at least 4 in $\text{Conv}_{\text{BV}(\Gamma)}(X)$. Then $\delta$ satisfies the 4 points condition in restriction to $X$ and the tree hull of $(X, \delta)$ is isomorphic to $\text{Conv}_{\text{BV}(\Gamma)}(X)$ as an $X$-tree.

The idea of the proof of Theorem 11.1 is to show that, under the given hypotheses, the equalities among the triangle inequalities and four points conditions are as described by the brick-vertex tree. It is written in a detailed way in [12, Thm. 1.38].

![Diagram](image.png)

**Figure 9.** A convex hull of four vertices

**Example 11.2.** Let us consider again the connected graph $\Gamma$ of Example 7.6. Look at its vertices $a, b, c, d$ shown on the left of Figure 9. The corresponding vertices $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ of the brick-vertex tree $\text{BV}(\Gamma)$ are shown on the right side of the figure. Denoting $X := \{a, b, c, d\}$, the convex hull $\text{Conv}_{\text{BV}(\Gamma)}(X)$ is also drawn on the right side using thick red segments. We see that the hypothesis of Theorem 11.1 about the valencies of brick-vertices is satisfied, as the only brick-vertex contained in $\text{Conv}_{\text{BV}(\Gamma)}(X)$ is of valency 3 in this convex hull.

As shown by the $F$-shape of $\text{Conv}_{\text{BV}(\Gamma)}(X)$, one should have the following equalities and inequalities in the four points conditions concerning $X$:

\[(11.3) \quad \delta(a, d) + \delta(b, c) = \delta(a, c) + \delta(b, d) > \delta(a, b) + \delta(c, d).\]

Let us prove that this is indeed the case. Consider the cut vertex $v$ of $\Gamma$ shown on the left side of Figure 9. As it separates $a$ from $d$, we have the equality $\delta(a, d) = \delta(a, v) + \delta(v, d)$. As $v$ does not separate $a$ from $b$, we have the strict inequality $\delta(a, v) + \delta(b, v) > \delta(a, b)$. Using similar
equalities and inequalities, we get:
\[ \delta(a, d) + \delta(b, c) = \\
= (\delta(a, v) + \delta(v, d)) + (\delta(b, v) + \delta(v, c)) = \\
= (\delta(a, v) + \delta(v, c)) + (\delta(b, v) + \delta(v, d)) = \\
= \delta(a, c) + \delta(b, d) = \\
= (\delta(a, v) + \delta(b, v)) + (\delta(v, d) + \delta(v, c)) = \\
> \delta(a, b) + \delta(c, d). \]

The (in)equalities (11.3) are proved.

One proves similarly the triangle equalities \( \delta(a, b) + \delta(b, c) = \delta(a, c) \), \( \delta(a, b) + \delta(b, d) = \delta(a, d) \) and the fact that one has no equality among the triangle inequalities concerning the triple \( \{a, c, d\} \), which shows that the tree hull of \( (X, \delta) \) has indeed an \( F \)-shape, with the vertices \( a, b, c, d \) placed as in \( \text{Conv}_{\mathbb{B}(\Gamma)}(X) \).

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