PARTIALLY HYPERBOLIC SETS WITH POSITIVE MEASURE AND ACIP FOR PARTIALLY HYPERBOLIC SYSTEMS

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Abstract. In [20] Xia introduced a simple dynamical density basis for partially hyperbolic sets of volume preserving diffeomorphisms. We apply the density basis to the study of the topological structure of partially hyperbolic sets. We show that if \( \Lambda \) is a strongly partially hyperbolic set with positive volume, then \( \Lambda \) contains the global stable manifolds over \( \alpha(\Lambda^d) \) and the global unstable manifolds over \( \omega(\Lambda^d) \).

We give several applications of the dynamical density to partially hyperbolic maps that preserve some \( \text{acip} \). We show that if \( f \) is essentially accessible and \( \mu \) is an \( \text{acip} \) of \( f \), then \( \text{supp}(\mu) = M \), the map \( f \) is transitive, and \( \mu \)-a.e. \( x \in M \) has a dense orbit in \( M \). Moreover if \( f \) is accessible and center bunched, then either \( f \) preserves a smooth measure or there is no \( \text{acip} \) of \( f \).

1. Introduction

Let \( M \) be a \( n \)-dimensional connected, closed manifold, \( r > 1 \) and \( f \in \text{Diff}^r(M) \) be a \( C^r \) diffeomorphism on \( M \). A compact \( f \)-invariant subset \( \Lambda \subset M \) is said to be partially hyperbolic if there are a nontrivial \( Tf \)-invariant splitting of \( T_x M = E^s_x \oplus E^c_x \oplus E^u_x \) for every \( x \in \Lambda \), a smooth Riemannian metric \( g \) on \( M \) for which we can choose continuous positive functions \( \nu, \tilde{\nu}, \gamma \) and \( \tilde{\gamma} \) on \( \Lambda \) with \( \nu, \tilde{\nu} < 1 \) and \( \nu < \gamma \leq \tilde{\gamma}^{-1} < \tilde{\nu}^{-1} \) such that, for all \( x \in \Lambda \) and for all unit vectors \( v \in E^s_x, w \in E^c_c \) and \( v' \in E^u_x \),

\[
\|Tf(v)\| \leq \nu(x) < \gamma(x) \leq \|Tf(w)\| \leq \tilde{\gamma}(x)^{-1} < \tilde{\nu}(x) \leq \|Tf(v')\|.
\]

The notation here is taken from [8]. Such a metric is called adapted (see [13]). If both \( E^s \) and \( E^u \) are nontrivial, then we say \( \Lambda \) is strongly partially hyperbolic. In particular the map \( f \) is called a (strongly) partially hyperbolic diffeomorphism if \( M \) itself is a (strongly) partially hyperbolic set. It is well known that \( E^s \) and \( E^u \) are uniquely integrable and tangent to the stable lamination \( \mathcal{W}^s \) and the unstable lamination \( \mathcal{W}^u \) respectively.

In [20] Xia introduced a simple dynamical density basis for general partially hyperbolic sets. Namely let \( \delta > 0 \) small, \( W^s(x, \delta) \) be the local stable manifold over \( x \in \Lambda \). Let \( B^s_n(p) = f^n W^s(f^{-n} p, \delta) \) for each \( p \in \Lambda \) and \( n \geq 0 \). The collection of sets \( \mathcal{S} = \{ B^s_n(p) : n \geq 0, p \in \Lambda \} \) is called the stable basis (see [20]). Let \( A \subset \Lambda \) be a measurable subset. A point \( p \in A \) is said to be a \( \mathcal{S} \)-density point of \( A \) if

\[
\lim_{n \to \infty} \frac{m_{W^s(p)}(B^s_n(p) \cap A)}{m_{W^s(p)}(B^s_n(p))} = 1,
\]

where \( m_{W^s(p)} \) is the leaf-volume induced by the Riemannian metric. Let \( \Lambda^d \) be the set of \( \mathcal{S} \)-density points of \( A \). Following [20] we have:

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**Proposition.** Let \( r > 1, f \in \text{Diff}^r(M) \) and \( \Lambda \) be a partially hyperbolic set with positive volume. For each measurable subset \( A \subset \Lambda \), \( m \)-a.e. point in \( A \) is \( S \)-density point of \( A \), that is, \( m(A \setminus A^d) = 0 \). In words, \( S \) forms a density basis.

This simply defined density basis turns out to be useful in the study of the topological structure of (partially) hyperbolic sets. There is an extensive literature discussing the topology of (partially) hyperbolic sets. We just name a few that are close related to the results here. Bowen showed in [1], there exists \( C^1 \) horseshoe of positive volume (It is also showed in [5] that this fat horseshoe can not exist among the \( C^2 \) diffeomorphisms). In [1] Alves and Pinheiro showed that for a diffeomorphism \( f \in \text{Diff}^r(M) \), if \( \Lambda \) is a partially hyperbolic set that attracts a positive volume set, then \( \Lambda \) contains some local unstable disk (hence \( \Lambda \) cannot be a horseshoe-like set). Under a much stronger setting, we can get a useful characterization that serves well for later applications. More precisely let \( \alpha(x) \) be the set of accumulating points of \( \{ f^n x : n \leq 0 \} \) of \( x \in M \). For \( E \subset M \), let \( \alpha(E) \) be the closure of \( \bigcup_{x \in E} \alpha(x) \). Similarly we can define \( \omega(x) \) and \( \omega(E) \). Then we have

**Theorem A.** Let \( f \in \text{Diff}^r(M) \) for some \( r > 1 \) and \( \Lambda \) be a partially hyperbolic set with positive volume. Then \( \Lambda \) contains the global stable manifolds over \( \alpha(\Lambda^d) \), that is, \( W^s(x) \subset \Lambda \) for each \( x \in \alpha(\Lambda^d) \).

In particular if \( \Lambda \) is a strongly partially hyperbolic set with positive volume, then \( \Lambda \) contains the global stable manifolds over \( \alpha(\Lambda^d) \) and the global unstable manifolds over \( \omega(\Lambda^d) \).

The argument here relies on the bounded distortion estimates and the absolute continuity of stable and unstable laminations, which fail for \( C^1 \) maps. See [3, 17]. Although \( \alpha(\Lambda^d) \) is nonempty, the volume of \( \alpha(\Lambda^d) \) could be zero (even in the hyperbolic case). In fact Fisher [11] constructed several hyperbolic sets \( \Lambda \) with nonempty interior such that \( \alpha(\Lambda^d) \) are repellers and \( \omega(\Lambda^d) \) are attractors (hence their volume must be zero).

A point \( x \) is said to be backward recurrent if \( x \in \alpha(x) \), and to be recurrent if \( x \in \alpha(x) \cap \omega(x) \). An interesting case is when most points are recurrent. This will hold in particular if \( \mu(\Lambda) > 0 \) for some absolutely continuous invariant probability measure (acip for short) \( \mu \ll m \). For simplicity we assume that \( \Lambda = \text{supp} \mu \).

**Corollary B.** Let \( f \in \text{Diff}^r(M) \) for some \( r > 1 \) and \( \Lambda \) a strongly partially hyperbolic set supporting some acip \( \mu \). Then \( \Lambda \) is bi-saturated, that is, for each point \( p \in \Lambda \), the global stable manifolds and the global unstable manifolds over \( p \) lie in \( \Lambda \).

In particular we give a dichotomy for maps \( f \in \text{Diff}^r(M) \): either \( f \) is a transitive Anosov diffeomorphism, or each \( f \)-invariant hyperbolic set \( \Lambda \) is \( \text{acip} \)-null, that is, \( \mu(\Lambda) = 0 \) for every \( \text{acip} \) \( \mu \).

**Theorem C.** Let \( f \in \text{Diff}^r(M) \) for some \( r > 1 \), \( \mu \) be an acip and \( \Lambda \) be a hyperbolic set with positive \( \mu \)-measure. Then \( \Lambda = M \) and \( f \) is a transitive Anosov diffeomorphism on \( M \).

The similar result has been proved if the \( \text{acip} \) \( \mu \) assumed to be equivalent to \( m \) (see [3, 20]). Moreover it is proved in [5] that for \( C^r \) transitive Anosov diffeomorphism, the \( \text{acip} \) must have Hölder continuous density with respect to the volume and be an ergodic (indeed Bernoulli) measure, see Remark [11]. Also note that the condition that \( \Lambda \) has positive \( \mu \)-measure for some \( \text{acip} \) is nontrivial and see [11] for counter-examples.

Theorem C motivates the analogous generalizations from hyperbolic systems to accessible partially hyperbolic systems. Recall that an \( f \)-invariant measure \( \mu \) is said to be weakly ergodic
if for $\mu$-a.e. $x$, $O(x)$ is dense in $\text{supp}(\mu)$. Following generalizes the well known result of Brin \[6\] to the presence of acip.

**Theorem D.** Let $f \in \text{SPH}^r(M)$ for some $r > 1$ be essentially accessible. If there exists some acip $\mu$ of $f$, then $\text{supp}(\mu) = M$, the map $f$ is transitive, and the acip $\mu$ is weakly ergodic. In particular $O(x)$ is dense in $M$ for $\mu$-a.e. $x \in M$.

In the following we assume $r = 2$ for simplicity. Burns and Wilkinson proved in \[8\] that if a map $f \in \text{SPH}^2(M)$ is center bunched, then every measurable bi-essential saturated set is essential bi-saturated. Applying to acip we have

**Proposition.** Let $f \in \text{SPH}^2(M)$ be essentially accessible and center bunched. If there exists some acip $\mu$, then $\mu$ must be equivalent to the volume. In particular $\mu$ is ergodic.

Note that the arguments in \[8\] still work if $f \in \text{SPH}^r(M)$ for $r > 1$, as long as we assume strong center bunching (see \[8\], Theorem 0.3). So our results also extend to this setting. Then applying the cohomologous theory developed in \[19\], we show that the acip is a smooth measure, that is, the density $\frac{d\mu}{dm}$ of $\mu$ with respect to $m$ is Hölder continuous on $M$, bounded and bounded away from zero.

**Theorem E.** Let $f \in \text{SPH}^2(M)$ be accessible and center bunched. If there exists some acip, then the acip must have Hölder continuous density with respect to the volume of $M$. In words, either $f$ preserves some smooth measure or there is no acip for $f$.

Combining the results in \[10\] we have the following direct corollary:

**Corollary F.** The set of maps that admit no acip contains a $C^1$ open and dense subset of $C^2$ strongly partially hyperbolic and center bunched diffeomorphisms. In particular the set of maps that admit no acip contains a $C^1$ open and dense subset of $C^2$ strongly partially hyperbolic diffeomorphisms with $\dim(E^c) = 1$.

Finally we remark that although the volume measure need not be $f$-invariant, there always exists some $f$-invariant measures. The density argument combines the dynamics of acip and the dynamics of volume on $M$. This is why most results of volume-preserving partially hyperbolic systems have parallel generalizations to the systems with acip.

2. **Dynamical density basis for partially hyperbolic set**

In this section we will consider a $C^r$ diffeomorphism for some $r > 1$ and a partially hyperbolic invariant set with positive volume. More precisely let $M$ be a closed and connected smooth manifold. Each Riemannian metric $g$ on $M$ induces a (geodesic) distance $d$ on $M$ and a normalized volume measure $m$ on $M$. Let $\mathcal{B}$ be the Borel $\sigma$-algebra of $M$. A Borel probability measure $\mu$ on $M$ is said to be absolutely continuous with respect to $m$, denoting $\mu \ll m$, if $\mu(A) = 0$ for each set $A \in \mathcal{B}$ with $m(A) = 0$, and to be equivalent to $m$ if $\mu \ll m$ and $m \ll \mu$. It is evident for any other Riemannian metric $g'$ compatible with $g$, the induced volume of $g'$ is equivalent to $m$.

Let $f \in \text{Diff}^r(M)$ for $r > 1$ and $\Lambda$ be a compact partially hyperbolic invariant set with positive volume. In the following we always assume that the stable subbundle $E^s$ is nontrivial on $\Lambda$ and $m$ is the normalized volume measure on $M$ induced by some Riemannian metric adapted to the invariant splitting (see \[13\]).
Since \( r > 1 \), it is well known that the stable bundle \( E^s \) is Hölder continuous over \( \Lambda \) (the Hölder exponent may be much smaller than \( r - 1 \), see [7] and is tangent to the stable lamination \( \mathcal{W}^s \) over \( \Lambda \). (A lamination over \( \Lambda \) is a partial foliation which may not foliate an open neighborhood of \( \Lambda \), see [14]. In case that \( \Lambda = M \), \( \mathcal{W}^s \) turns out to be a foliation.) For \( \delta > 0 \) small we use \( W^s(x, \delta) \) to denote the local stable manifold over \( x \in \Lambda \). Note that \( W^s(x, \delta) \) varies uniformly \( \alpha \)-Hölder continuously with respect to the base point \( x \in \Lambda \). For simplicity we also write \( E^s_y = T_y W^s(x) \) for all \( y \in W^s(x, \delta) \) and \( x \in \Lambda \). The invariance of \( \mathcal{W}^s \) implies that the extended distribution \( E^s \) is also invariant. The Hölder continuity of \( E^s \) ensures that the family \( \mathcal{W}^s \) is absolutely continuous. By slightly increasing \( \nu \) and decreasing \( \delta \) if necessary, we can assume that for each \( x \in \Lambda \) the following holds:

\[
\text{if } p, p' \in W^s(x, \delta), \text{ then } d(f(p), f(p')) \leq \nu(p)d(p, p').
\]

In particular we have \( fW^s(x, \delta) \subset W^s(fx, \delta, \nu(x)) \) for all \( x \in \Lambda \).

Before moving on, let’s fix some notations as in [8]. Let \( S \subset M \) be a submanifold of \( M, m_S \) be the volume measure on \( S \) induced by the restricted Riemannian metric \( g|_S \) on \( S \). In particular if \( S = W^s(x) \), we abbreviate the induced measure as \( m_{s,x} \). Denote \( m_{s,x}(A) \) the restricted submanifold measure for a measurable subset \( A \subseteq W^s(x) \). This should not be confused with conditional measures. Let \( \eta = \min\{\|Tf(v)\| : v \in TM \text{ with } \|v\| = 1\} \) and \( \overline{\eta} = \sup_{p \in \Lambda} \nu(p) \). Clearly \( 0 < \eta \leq \nu(p) \leq \overline{\eta} < 1 \) by compactness. For each \( p \in \Lambda \) let \( p_i = f^i p \) for \( i \in \mathbb{Z} \), \( \nu_0(p) = 1 \) and \( \nu_n(p) = \nu(p_{n-1}) \cdots \nu(p_0) \) for \( n \geq 1 \). Let \( B^s_n(p) = f^n W^s(p_{-n}, \delta) \). Since \( \Lambda \) is \( f \)-invariant, we have \( B^s_n(p) \subset W^s(p, \delta, \nu_n(p_{-n})) \).

Since each stable manifold is a smooth submanifold of the Riemannian manifold \( M \) and \( f \) is \( C^r \) for \( r > 1 \), the stable Jacobian \( J^s(f, x) \) of the restricted map \( T^f : E^s_x \to E^s_y \) is well defined and Hölder continuous with uniform Hölder exponent and Hölder constant. That is, there exist \( \alpha > 0 \) and \( C_0 > 0 \) such that for any \( p \in \Lambda \) and \( x, y \in W^s(p, \delta) \) we have \( |J^s(f, x) - J^s(f, y)| \leq C_0d(x, y)^\alpha \).

Also there exists \( J^* \geq 1 \) such that \( 1/J^* \leq J^s(f, x) \leq J^* \) for all \( x \in W^s(p, \delta) \) and \( p \in \Lambda \). Decreasing \( \delta \) again if necessary we assume \( C_1 = \prod_{k=0}^{\infty} (1 + J^* C_0^k \overline{\eta}^{\alpha k}) < \infty \).

Let \( B^s_n(p) = f^n W^s(p_{-n}, \delta) \) for each \( p \in \Lambda, n \geq 0 \) and \( S = \{B^s_n(p) : n \geq 0, p \in \Lambda\} \) be the stable basis. It is easy to see that \( \{B^s_n(p) : n \geq 0\} \) forms a nesting sequence of neighborhoods of \( p \in \Lambda \) relative to \( W^s(p, \delta) \) and \( B^s_n(p) \) shrinks to \( p \) as \( n \to \infty \). Note that the basis here is in leafwise sense and may has infinite eccentricity. The proposition below states that the stable basis \( S \) behaves well in the sense of [16]:

**Proposition 1.** The following properties hold for stable basis \( S \):

1. For any \( p \in \Lambda \), \( m_{s,p}(B^s_n(p)) \to 0 \) if and only if \( n \to \infty \).
2. For any \( k \geq 0 \), there exists \( c_k \geq 1 \) such that \( m_{s,p}(B^s_{n+k}(p)) \leq c_k m_{s,p}(B^s_n(p)) \) for all \( p \in \Lambda, n \geq 0 \).
3. There exists \( L \in \mathbb{N} \) such that for any \( p, q \in \Lambda, n \geq 0 \), if \( B^s_{n+L}(p) \cap B^s_{n+L}(q) \neq \emptyset \), then \( B^s_{n+L}(p) \cup B^s_{n+L}(q) \subseteq B^s_n(p) \cap B^s_n(q) \).

The properties listed above appeared in [16] (in a general setting) and is named to be **volumentrically engulfing** (also see [20] for example). The proof mainly uses distortion estimates.

Let \( A \in \mathcal{B}_\Lambda \) be a measurable subset of \( \Lambda \). Recall that a point \( x \in A \) is said to be a \( S \)-density point of \( A \) if

\[
\lim_{n \to \infty} \frac{m_{s,p}(B^s_n(p) \cap A)}{m_{s,p}(B^s_n(p))} = 1.
\]

For different \( \delta \)'s, the induced stable bases are internested (see [8] Lemma 2.1 for details). So the definition of \( S \)-density point is independent of the choice of the size of stable manifolds and the choice of adapted Riemannian metric on \( M \). Let \( A^d \) be the set of \( S \)-density points of \( A \).
For each $A \in \mathcal{B}_\Lambda$ and each $p \in \Lambda$, $A \cap W^s(p, \delta)$, the intersection of two Borel measurable subsets, is a Borel measurable subset of the submanifold $W^s(p, \delta)$. (Note that if $A$ is Lebesgue measurable, above relation will hold for $m$-a.e. $p \in \Lambda$ by Fubini Theorem.) Let us denote $A_p^d$ the set of $S$-density points of $A \cap W^s(p, \delta)$. Clearly we have $A^d = \bigcup_{p \in \Lambda} A_p^d$.

**Proposition 2.** Let $f \in \text{Diff}^r(M)$ for some $r > 1$ and $\Lambda$ be a partially hyperbolic set with positive measure. For each subset $A \in \mathcal{B}_\Lambda$, we have

1. for each $p \in \Lambda$, $m_{s,p}$-a.e. point in $W^s(p, \delta) \cap A$ is a $S$-density point of $A$: $m_{s,p}(W^s(p, \delta) \cap A \setminus A_p^d) = 0$.
2. $m$-a.e. point of $A$ is a $S$-density point of $A$: $m(A \setminus A^d) = 0$.

Moreover if $A \in \mathcal{B}_\Lambda$ is $f$-invariant, so is $A^d$.

**Proof.** The first item follows by applying Theorem 3.1 in [16] to stable basis $S$ to each intersection $A \cap W^s(p, \delta)$. Proposition [1] ensures that $S$ forms a density basis in this leafwise sense.

Using the absolute continuity of the stable foliation $W^s$ and the relation $A^d = \bigcup_{p \in \Lambda} A_p^d$, we have $m(A \setminus A^d) = 0$. Hence $S$ also forms a density basis in the ambient sense.

For the last item, we note that each local leaf $W^s(s, \delta)$ is a $C^r$ submanifold of $M$ and the restriction of $f$ between local stable manifolds is diffeomorphic onto its image. So $p \in \Lambda$ is a $S$-density point of $A \cap W^s(p, \delta)$ (or equally, of $A$) if and only if $fp$ is a $S$-density point of $A \cap W^s(fp, \delta)$. $\square$

### 3. Topological structure of partially hyperbolic sets

In this section we give some descriptions of the topological structure of partially hyperbolic sets with positive volume. As in Section 2 we let $M$ be a connected closed manifold, $f \in \text{Diff}^r(M)$ for some $r > 1$ and $\Lambda$ a partially hyperbolic set with positive volume.

Given a Borel subset $A \subset \Lambda$, we consider a family of functions $\eta_n$ on $\Lambda$ as

$$\eta_n(x) = m_{s,x}(B_n^0(x) \setminus A)/m_{s,x}(B_n^0(x)).$$

The following result shows the increasing occupation of an invariant set $A$ in the local stable manifolds along the backward iterates of an $S$-density point of $A$.

**Lemma 3.1.** There exists a constant $C \geq 1$ such that given an $f$-invariant subset $A \in \mathcal{B}_\Lambda$, $m_{s,x,n}(W^s(x_n, \delta) \setminus A) \leq C \cdot \eta_n(x)$ for each $x \in \Lambda$ and $n \geq 0$.

**Proof.** We only need to adapt the notations in [20, Lemma 3.2], since the proof is essentially the same. Let $A$ be an invariant set and $x \in \Lambda$ be fixed. Let $B_k^0 = f^kW^s(x_n, \delta)$ and $D_k^0 = B_k^0 \setminus A$ for each $0 \leq k \leq n$. Note that $B_0^0 = W^s(x_n, \delta)$ is a local stable leaf and $B_n^0 = B_n^0(x)$ is an element in the stable basis $S$. Then using the constant $C_0$ given by [20 Page 816], we have $m_{s,x,n}(D_k^0) \leq C_0 \cdot \eta_n(x) \cdot m_{s,x,n}(B_0^0)$. Applying $D_0^0 = W^s(x_n, \delta)$ and $D_k^0 = W^s(x_n, \delta) \setminus A$, we finish the proof with a uniform constant $C_2 = C_0 \cdot \max_{p \in \Lambda} m_{s,p}(W^s(p, \delta))$. $\square$

Recall that $\alpha(x)$, the $\alpha$-set of $x$, is the set of accumulating points along the backward orbit $\{x, f^{-1}x, \cdots\}$. Let $\alpha(E)$ be the closure of $\bigcup_{x \in E} \alpha(x)$. Note that for each point $x \in \Lambda$ and each subset $E \subset \Lambda$, the sets $\alpha(x)$ and $\alpha(E)$ are compact $f$-invariant subsets of $\Lambda$.

**Theorem 3.2.** Let $f \in \text{Diff}^r(M)$ for some $r > 1$ and $\Lambda$ a partially hyperbolic set with positive volume. Then $\Lambda$ contains the global stable manifolds over $\alpha(\Lambda^d)$.

**Proof.** First let us consider $y \in \alpha(x)$ for some $x \in \Lambda^d$. Pick a sequence of times $n_i \to +\infty$ such that $x_{-n_i} \to y$ (clearly all these points are in $\Lambda$). By Lemma [1] we have $m_{s,x_{-n_i}}(W^s(x_{-n_i}, \delta) \setminus \Lambda) \leq C_2 \eta_n(x)$. (Note that $\eta_n(x) \to 0$ as $n \to \infty$.) Passing to a subsequence if necessary, we
can assume that $W^s(x_{-n_i}, \delta) \cap \Lambda$ contains a $\frac{1}{n_i}$-dense subset $E_{x_{-n_i},i}$ of $W^s(x_{-n_i}, \delta)$. Let $E = \limsup_{i \to \infty} E_{x_{-n_i},i} = \bigcap_{k \geq 1} \bigcup_{i \geq k} E_{x_{-n_i},i}$. It is clear that $E \subset \Lambda$ since $\Lambda$ is compact. By continuity of the stable manifolds, $E$ contains a dense subset of $W^s(y, \delta)$, and hence $W^s(y, \delta) \subset E$. So $W^s(y, \delta) \subset \Lambda$ for each $y \in \alpha(x)$ and each $x \in \Lambda^d$.

Still by the compactness of $\Lambda$, $W^s(y, \delta) \subset \Lambda$ for each $y \in \alpha(\Lambda^d)$. By the invariance of $\Lambda$ and $\alpha(\Lambda^d)$, the global stable manifolds $W^s(y) \subset \Lambda$ for each $y \in \alpha(\Lambda^d)$. \hfill $\square$

Similarly we consider the $\omega$-set and define $\omega(\Lambda^d)$. For a strongly partially hyperbolic set we have

**Theorem 3.3.** Let $f \in \text{Diff}^r(M)$ for some $r > 1$ and $\Lambda$ a strongly partially hyperbolic set with positive volume. Then $\Lambda$ contains the global stable manifolds over $\alpha(\Lambda^d)$ and the global unstable manifolds over $\omega(\Lambda^d)$.

So every partially hyperbolic set with positive volume is far from being a topological horseshoe-like set. Although the sets $\alpha(\Lambda^d)$ and $\omega(\Lambda^d)$ are always nonempty, we do not know how large they could be and when they could intersect with each other. This can be improved if we require that $\Lambda$ admits some recurrence.

**Definition 3.4.** A point $x$ is said to be **backward recurrent** if $x \in \alpha(x)$. The definition of **forward recurrent** is analogous. A point is said to be **recurrent** if it is both backward and forward recurrent.

**Definition 3.5.** Let $E$ be a measurable subset of $\Lambda$. Then $E$ is said to be **$s$-saturated** if for each $x \in E, W^s(x) \subset E$. Similarly we can define **$u$-saturated** sets. Then the set $E$ is **bi-saturated** if it is $s$-saturated and $u$-saturated.

**Corollary 1.** Let $f \in \text{Diff}^r(M)$ for some $r > 1$ and $\Lambda$ be a strongly partially hyperbolic set supporting some $acip$ $\mu$. Then $\Lambda$ is bi-saturated.

**Proof.** By Poincaré recurrence theorem, we have that $\mu$-a.e. $x \in \Lambda$ is recurrent, that is, $\mu(\text{Rec}_\Lambda) = 1$ where $\text{Rec}_\Lambda$ is the set of recurrent points in $\Lambda$. Also we have $\mu(\Lambda \setminus \Lambda^d) = 0$ since $\mu \ll m$ and $m(\Lambda \setminus \Lambda^d) = 0$. So $\mu(\Lambda^d \cap \text{Rec}_\Lambda) = 1$ and the closed set $\alpha(\Lambda^d)$ contains $\Lambda^d \cap \text{Rec}_\Lambda$, which is a subset of full $\mu$-measure in $\Lambda$ and hence dense in $\text{supp}\mu = \Lambda$. So $\alpha(\Lambda^d) = \Lambda$ and the set $\Lambda$ is $s$-saturated by Theorem 3.3. Similarly we can show $\Lambda$ is $u$-saturated. This completes the proof. \hfill $\square$

**4. Regularity of $acip$: hyperbolic case.**

In this section we consider the hyperbolic sets. We show that if a hyperbolic set has positive $acip$-measure, then the map is a transitive Anosov diffeomorphism. Then it is well known that the $acip$ is not only equivalent to the volume, but also has smooth density with respect to the volume. This motivates the generalization to partial hyperbolic systems in next section.

**Theorem 4.1.** Let $f \in \text{Diff}^r(M)$ for some $r > 1$, $\mu$ be an $acip$ and $\Lambda$ be a hyperbolic set with positive $\mu$-measure. Then $\Lambda = M$ and $f$ is an transitive Anosov diffeomorphism on $M$.

**Proof.** By considering $\Lambda_\mu = \Lambda \cap \text{supp}(\mu)$ and $\mu|_{\Lambda_\mu}$, if necessary, we can assume that $\Lambda = \text{supp}(\mu)$. By Corollary 1 we have that $\Lambda$ is bi-saturated. By the uniform hyperbolicity of $\Lambda$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset \bigcup_{y \in W^s(x, \delta)} W^s(y, \delta) \subset \Lambda$ for each $x \in \Lambda$. So the set $\Lambda$ is both close and open, hence coincides with the whole manifold $M$. Since the recurrent set is a dense subset of $\text{supp}(\mu) = \Lambda = M$ and is contained in the nonwandering set $\Omega(f)$, we have that $\Omega(f) = M$ and $f$ is a transitive Anosov diffeomorphism on $M$ (by spectrum decomposition theorem, see [5]). \hfill $\square$
**Remark 1.** Spectrum decomposition theorem actually implies that $f$ is mixing. Moreover by Corollary 4.13 and Theorem 4.14 in [5], $\mu$ coincides with the equilibrium state $\mu_{\phi^u}$ of the potential $\phi^u(x) = -\log(J^u(f, x))$, and has H"{o}lder continuous density with respect to $m$. Furthermore the smooth measure $\mu$ is ergodic and Bernoulli.

**Remark 2.** The regularity of $f \in \text{Diff}^r(M)$ for some $r > 1$ is an essential assumption in a two-fold sense. In [14] Robinson and Young constructed a $C^1$ Anosov diffeomorphism with non-absolutely continuous stable and unstable foliations, which does have some closed invariant set with positive volume. In [4] Bowen constructed a $C^1$ horseshoe with positive volume and absolutely continuous local stable and unstable laminations, where the bounded distortion property in Lemma 3.1 fails.

5. Regularity of acip: partially hyperbolic case

In this section we show analogous results in Section 4 hold for accessible strongly partially hyperbolic systems. Namely, let $f \in \text{SPH}^r(M)$ for some $r > 1$ be a $C^r$ strongly partially hyperbolic diffeomorphism and $m$ be the volume measure associated to some Riemannian metric adapted to the partially hyperbolic splitting. Let $W^s$ be the stable foliation tangent to the stable bundle and $W^u$ the unstable foliation tangent to the unstable bundle.

**Definition 5.1.** Let $E$ be a measurable subset of $M$. Then $E$ is said to be essentially $s$-saturated if there exists an $s$-saturated set $\hat{E}^s$ with $m(E \triangle \hat{E}^s) = 0$. Similarly we can define essentially $u$-saturated sets. The set $E$ is essentially bi-saturated if there exists a bi-saturated set $\hat{E}^{su}$ with $m(E \triangle \hat{E}^{su}) = 0$, and bi-essentially saturated if $E$ is essentially $s$-saturated and essentially $u$-saturated.

**Definition 5.2.** A strongly partially hyperbolic diffeomorphism $f : M \rightarrow M$ is said to be accessible if each nonempty bi-saturated set is the whole manifold $M$. The map $f$ is essentially accessible if every measurable bi-saturated set has either full or zero volume.

**Theorem 5.3.** Let $f \in \text{SPH}^r(M)$ be essentially accessible. If there exists some acip for $f$, then the support of the acip is the whole manifold and the map $f$ is transitive.

Before the proof, we mention that there exists a $C^1$ open set of accessible but non-transitive diffeomorphisms (see [15]).

**Proof.** Let $\mu$ be an acip of $f$. Then the support $\text{supp}(\mu)$ of $\mu$ is a strongly partially hyperbolic set supporting $\mu$, hence is a bi-saturated set by Corollary 1. Essential accessibility of $f$ implies that $m(\text{supp}(\mu)) = 1$. Hence $\text{supp}(\mu) = M$ since $\text{supp}(\mu)$ is closed.

Suppose on the contrary that $f$ is not transitive. That is, there exists an $f$-invariant nonempty open set $U$ such that $M \setminus \overline{U} \neq \emptyset$. So the set $\Lambda = M \setminus U$ is $f$-invariant, closed with nonempty interior. Hence $\mu(\Lambda) > 0$ and $\mu|_{\Lambda}$ is again an acip. Corollary 1 implies that $\Lambda$ is bi-saturated. Since $f$ is essentially accessible, we have $m(\Lambda) = 1$ and $m(U) = 0$. This contradicts the openness of $U$.

Generally for a transitive map $f$, the set $\text{Tran}_f$ of points with dense orbit is measure-theoretic meagre (although topological residual). In [18] Section 5.7 they extracted following property which can be viewed as a stronger form of transitivity (or a weak form of ergodicity).

**Definition 5.4.** An $f$-invariant measure $\mu$ is said to be weakly ergodic if the set of points with dense orbit in $\text{supp}(\mu)$ has full $\mu$-measure.

Clearly that ergodicity implies weak ergodicity, and weak ergodicity implies the transitivity of the subsystem $(f, \text{supp}(\mu))$. In the following we show some analogous results in [6] [9] [18] hold for
acip. To this end let us introduce some necessary notations. Let \( \mu \) be an acip of \( f \in \text{SPH}^r(M) \) for some \( r > 1 \) and \( \phi = \frac{d\mu}{dm} \) be the Radon-Nikodym density of \( \mu \) relative to \( m \). Note that the Jacobian \( J_f : M \to \mathbb{R}, x \mapsto \text{Jac}(D f : T_x M \to T_{f(x)} M) \) is a Hölder continuous function, bounded and bounded away from 0 on \( M \). Now for each measurable subset \( A \subset M \) we have:

\[
\int_A \phi(x)dm(x) = \mu(A) = \mu(fA) = \int_{fA} \phi(y)dm(y) = \int_A \phi(fx)J_f(x)dm(x).
\]

So the following holds:

(5.1) \( \phi(fx)J_f(x) = \phi(x) \) for \( m \)-a.e. \( x \in M \).

Let us consider the set \( E = \{ x \in M : \phi(x) > 0 \} \). Clearly \( E \) is measurable and \( m(E) > 0 \). By (5.1) we see that \( E \) is also \( f \)-invariant. Restricted to the set \( E \), the measure \( m|_E \) is equivalent to \( \mu \). So ‘(P) for \( m \text{-a.e. } x \in E \)’ is the same as ‘(P) for \( \mu \text{-a.e. } x \in E \)’. In this case we will say ‘(P) a.e. \( x \in E \)’ for short.

**Proposition 3.** Let \( f \in \text{SPH}^r(M) \), \( \mu \) be an acip with density \( \phi \) and \( E = \{ x \in M : \phi(x) > 0 \} \). Then \( E \) is bi-essentially saturated.

Note that all essential saturations are defined with respect the volume. If \( f \) is volume preserving, then every invariant set is always bi-essentially saturated by Hopf argument. See [7, Lemma 6.3.2] and [18, Theorem 5.5].

**Proof.** It suffices to prove that \( E \) is essentially \( s \)-saturated. Let \( B_n^s(x) = f^nW^s(x, \cdot, \delta) \) and \( E_d \) be the set of \( S \)-density points of \( E \). By Proposition 2 we have \( m(E \setminus E_d) = 0 \).

Consider the functions \( \eta_n(x) = m_{W^s(x)}(B_n^s(x) \setminus E)/m_{W^s(x)}(B_n^s(x)) \) for \( n \geq 1 \). So \( \eta_n(x) \rightarrow 0 \) as \( n \rightarrow +\infty \) for a.e. \( x \in E \). For each \( \epsilon > 0 \), there exists a subset \( E_\epsilon \subset E \) with \( m(E \setminus E_\epsilon) < \epsilon \) on which \( \eta_n \) converges uniformly to zero. For a recurrent point \( x \in E_\epsilon \), let \( n_i \) be the forward recurrent times of \( x \) with respect to \( E_\epsilon \). Then each \( n_i \) is recurrent.

By Lemma 3.1 there exists a uniform constant \( C_2 \) such that for the point \( y = f^{n_i}x \) and \( n = n_i \) the following holds:

\[
m_{W^s(x)}(W^s(x, \delta) \setminus E) \leq C_2 \cdot \eta_{n_i}(f^{n_i}x).
\]

Passing \( n_i \) to \( \infty \) we have \( m_{W^s(x)}(W^s(x, \delta) \setminus E) = 0 \) for a.e. \( x \in E_\epsilon \).

Since \( \epsilon \) can be arbitrary small, we have

\[
m_{W^s(x)}(W^s(x, \delta) \setminus E) = 0
\]

for a.e. \( x \in E \). Since \( E \) is \( f \)-invariant and \( f \) is smooth between leaves of \( W^s \), \( m_{W^s(x)}(f^{-n}W^s(f^n x, \cdot, \delta) \setminus E) = 0 \) for each \( n \geq 1 \) and a.e. \( x \in E \). Hence \( m_{W^s(x)}(W^s(x) \setminus E) = 0 \) for a.e. \( x \in E \). It follows from the absolute continuity of \( W^s \) that \( E \) is essentially \( s \)-saturated. Similarly we can show \( E \) is essentially \( u \)-saturated. This completes the proof. \( \square \)

**Theorem 5.5.** Let \( f \in \text{SPH}^r(M) \) be essentially accessible. Then every acip is weakly ergodic. In particular if \( \mu \) is an acip, then the orbit \( O(x) \) is dense in \( M \) for \( \mu \text{-a.e. } x \in M \).

This result is well known if the system is volume preserving (see [6, 9, 18]). The idea of the proof is similar to Lemma 5 in [9]. Also see Proposition 5.17 in [18].

**Proof.** Let \( \phi \) be the density of \( \mu \) with respect to \( m \) and \( E = \{ x \in M : \phi(x) > 0 \} \). By Proposition 3 we have \( E \) is bi-essentially saturated. Hence \( \overline{E} = \text{supp}(\mu) = M \) by Theorem 5.3 since \( f \) is essentially accessible.

**Step 1.** We will show that for each open ball \( B \), \( O(x) \cap B \neq \emptyset \) for \( m \text{-a.e. } x \in E \). To this end we first consider \( G(B) \), the subset of points \( x \) which has a neighborhood \( U \) of \( x \) such that \( O(y) \cap B \neq \emptyset \) for \( m \text{-a.e. } y \in U \cap E \). Evidently \( G(B) \) is a nonempty open subset (and \( f \)-invariant).
Claim. \( G(B) \) is bi-saturated. So \( m(G(B)) = 1 \) since \( f \) is essentially accessible.

**Proof of Claim.** Let us prove \( G(B) \) is \( s \)-saturated. It suffices to show that \( q \in G(B) \) for each \( q \in W^s(z, \delta) \) and each \( p \in G(B) \), where the size \( \delta \) is fixed. So the justification lies in a local foliation box \( X \) of \( W^s \) around \( p \). Note that we can replace \( E \) by its saturate \( \hat{E}^s \) in the definition of \( G(B) \) since \( E \) is essentially \( s \)-saturated. For a point \( x \in X \), denote \( W^s_x(x) \) the component of \( W^s(x) \cap X \) that contains \( x \). Let \( U \) be a small neighborhood of \( p \) with \( \hat{O}(y) \cap B \neq \emptyset \) for \( m \)-a.e. \( y \in U \cap \hat{E}^s \). Let \( R \) be the set of recurrent points \( z \in U \cap \hat{E}^s \) whose orbits enter \( B \). Note that \( m(U \cap \hat{E}^s \backslash R) = 0 \) since \( m|_E \) is equivalent to the invariant measure \( \mu \) and \( m(E \Delta \hat{E}^s) = 0 \). So we can pick a smooth transverse \( T \) of \( W^s_X \) in \( U \) such that \( T \cap W^s(p) \neq \emptyset \) and \( m_T(\hat{E}^s \backslash R) = 0 \), where \( m_T \) is the induced volume on \( T \) (It is helpful to keep in mind that \( \hat{E}^s \) is not only essentially \( s \)-saturated, but \( s \)-saturated). Now we have

(I) For each \( y \in W^s_X(x) \) and \( x \in R \), we have \( \mathcal{O}(y) \cap B \neq \emptyset \). This follows from that \( d(f^n x, f^n y) \rightarrow 0 \) and the recurrence of \( x \): the orbit of \( x \) will enters \( B \) infinite many times.

(II) The set \( \bigcup_{x \in T \cap R} W^s_X(x) \) has full \( m \)-measure in the set \( \bigcup_{x \in T \cap \hat{E}^s} W^s_X(x) \). This follows from that both sets are measurable and \( W^s \)-saturated, \( W^s_X \) is an absolutely continuous lamination of \( X \) and \( m_T(\hat{E}^s \backslash R) = 0 \).

(III) The set \( \bigcup_{x \in T} W^s_X(x) \) contains an open neighborhood \( V \) of \( q \). This follows from that the holonomy maps along \( W^s_X \) are homeomorphisms.

Also note that \( \bigcup_{x \in T \cap \hat{E}^s} W^s_X(x) = \left( \bigcup_{x \in T} W^s_X(x) \right) \cap \hat{E}^s \). So \( \mathcal{O}(y) \cap B \neq \emptyset \) for \( m \)-a.e. \( y \in V \cap \hat{E}^s \). This implies \( q \in G(B) \) and hence \( G(B) \) is \( s \)-saturated. Similarly we have \( G(B) \) is also \( u \)-saturated and hence \( m(G(B)) = 1 \) by the essential accessibility of \( f \). This completes the proof of Claim.

Now let \( F(B) = \{ x \in E : \mathcal{O}(x) \cap B \neq \emptyset \} \). We need to show that \( m(E \backslash F(B)) = 0 \). To derive a contradiction we assume \( m(E \backslash F(B)) > 0 \) and \( p \in G(B) \) be a Lebesgue density point of \( E \backslash F(B) \) (here we use \( m(G(B)) = 1 \)). So there exists an open neighborhood \( U \) of \( p \) such that \( \mathcal{O}(x) \cap B \neq \emptyset \) for a.e. \( x \in U \cap E \). Then we have \( m(U \cap E \backslash F(B)) = 0 \). But this is impossible since we choose \( p \) as a Lebesgue density point of \( E \backslash F(B) \). So we have \( m(E \backslash F(B)) = 0 \) for each open ball \( B \).

**Step 2.** Since \( M \) is compact, there exists a countable collection of open balls \( \{ B_n : n \geq 1 \} \) which forms a basis of the topology on \( M \). Let \( F(B_n) \) be given by Step 1. We have \( m(E \backslash F) = 0 \) where \( F = \bigcap_{n \geq 1} F(B_n) \). Now for each \( x \in F \), \( \mathcal{O}(x) \cap B_n \neq \emptyset \) for each \( n \geq 1 \). So the orbit \( \mathcal{O}(x) \) is dense in \( M \) for each point \( x \in F \). Equivalently we see \( \mu \)-a.e. \( x \in M \) has a dense orbit. So the \( acip \) \( \mu \) is weakly ergodic. This completes the proof. \( \square \)

A natural question is, if \( f \in \text{SPH}^+(M) \) is essentially accessible and preserves some \( acip \) \( \mu \), is \( \mu \) an ergodic measure? This is related to the uniqueness of \( acip \). Clearly uniqueness of \( acip \) forces the ergodicity of \( acip \). On the other hand, let us assume that exists two \( acip \)'s: \( \mu = \phi m \) and \( \nu = \psi m \). Let \( E = \{ x : \phi(x) > 0 \} \) and \( F = \{ x : \psi(x) > 0 \} \). If \( m(E \Delta F) > 0 \) we can further assume \( E \) and \( F \) are disjoint. Proposition \( \Box \) implies that both \( E \) and \( F \) are bi-essentially saturated (and nontrivial). In particular none of them can be essentially bi-saturated.

We do not know whether such example can exist, or a bi-essentially saturated set is automatically essentially bi-saturated. A sufficient condition for this property is center bunching. From now on we assume \( r = 2 \) for simplicity.

**Definition 5.6.** A strongly partially hyperbolic diffeomorphism \( f \) is **center bunched** if the functions \( \nu, \tilde{\nu} \) and \( \gamma, \tilde{\gamma} \) given in \( \square \) can be chosen so that: \( \nu < \gamma \tilde{\gamma} \) and \( \tilde{\nu} < \gamma \tilde{\gamma} \).

**Proposition 4** (Corollary 5.2 in \( \Box \)). Let \( f \in \text{SPH}^2(M) \) be center bunched. Then every measurable bi-essentially saturated subset is essentially bi-saturated.
Corollary 2. Let $f \in \text{SPH}^2(M)$ be essentially accessible and center bunched. If there exists some acip, then the acip must be equivalent to the volume.

Proof. Let $\mu$ be an acip and $\phi$ be the density of $\mu$ with respect to $m$. We showed that $E = \{x \in M : \phi(x) > 0\}$ is bi-essentially saturated. Center bunching implies that $E$ is also essentially bi-saturated. Since $f$ is essentially accessible and $m(E) > 0$, $m(E) = 1$ and hence $\mu$ is equivalent to the volume $m$. □

Remark 3. In [8], a map $f$ is said to be volume preserving if $f$ preserves some invariant measure $\mu$ that is equivalent to the volume. They proved that if $f \in \text{SPH}^2(M)$ is essentially accessible, center bunched and preserves some $\mu$ equivalent to the volume, then the measure $\mu$ is ergodic (and Kolmogorov). It is well known that ergodic measures either coincide or absolutely singular with respect to each other. So by Corollary 2 if $f \in \text{SPH}^2(M)$ is essentially accessible and center bunched, then either $f$ is volume preserving in the board sense, or there exists no acip at all.

Followed by Corollary 2 we get that the density $\phi = \frac{d\mu}{dm}$ of an acip is positive a.e. on $M$. Now we use Cohomologous Theory developed in [19] to show the smoothness of the density of $\mu$. Namely let $\psi : M \to \mathbb{R}$ be a potential on $M$ and consider the cohomologous equation on $M$:

$$\psi = \Psi \circ f - \Psi.$$

Proposition 5 (Theorem A, part II and III, in [19]). Let $f \in \text{SPH}^2(M)$ be accessible, center bunched, and volume-preserving. Let $\psi : M \to \mathbb{R}$ be a Hölder continuous potential. If there exists a measurable solution $\Psi$ such that (5.2) holds for $x \in M$, then there is a Hölder continuous solution $\Phi$ of (5.2) with $\Phi = \Psi$ a.e. $x \in M$.

Let $\psi = -\log J_f$ and $\Psi = \log \phi$. Now $\psi$ is a $C^1$ function and $\Psi$ is a well defined measurable function. Corollary 2 implies that $\Psi$ is a measurable solution of the cohomologous equation (5.2). Then applying Proposition 5 we get a Hölder continuous solution $\Phi$ of (5.2) which coincides with $\Psi$ a.e.. It is evident that $\mu = e^\Phi m$ and the density $e^\Phi$ is bounded and bounded away from zero on $M$. Such a measure $\mu$ is called a smooth measure. So we have

Theorem 5.7. Let $f \in \text{SPH}^2(M)$ be accessible and center bunched. If there exists some acip, then the acip must have a Hölder continuous density with respect to the volume of $M$ which is also bounded and bounded away from zero. In words, either $f$ preserves a smooth measure or there is no acip of $f$.

In particular center bunching holds whenever $E^c$ is one-dimensional. As a corollary, we obtain:

Corollary 3. Let $f \in \text{SPH}^2(M)$ be accessible and $\text{dim}(E^c) = 1$. Then either $f$ preserves a smooth measure or there is no acip of $f$.

Let $\text{CB}^2(M) \subset \text{SPH}^2(M)$ be the collection of $C^2$ strongly partially hyperbolic diffeomorphisms that are center bunched. Clearly $\text{CB}^2(M)$ forms an open subset of $\text{SPH}^2(M)$. Applying Theorem 5.7 and the result in [10] we have

Theorem 5.8. The set of maps that admit no acip contains a $C^1$ open and dense subset of $\text{CB}^2(M)$. In particular the set of maps that admit no acip contains a $C^1$ open and dense subset of $C^2$ strongly partially hyperbolic diffeomorphisms with $\text{dim}(E^c) = 1$.

The main obstruction for $C^2$ density in Theorem 5.8 is that we do not know whether stable accessibility is $C^2$ dense in $\text{SPH}^2(M)$.

Proof. Dolgopyat and Wilkinson proved in [10] that there is a $C^1$ dense subset of stable accessible diffeomorphisms in $\text{SPH}^2(M)$ (also $C^1$ dense in $\text{CB}^2(M)$). Starting with arbitrary $f \in \text{CB}^2(M)$, we first perturb it to a stable accessible one, say $f_1$. By $C^1$ closing lemma, there exists $f_2 \in \text{CB}^2(M)$
CB²(M) close to f₁ that has some periodic point. We can assume that f₂ is also stable accessible since we can make it arbitrary close to f₁. By Franks’ Lemma [12] we can assume that the periodic point p is hyperbolic with period k and the Jacobian of DF² : TₓM → TₓM has absolute value different from 1. These properties hold robustly for all maps in a small neighborhood U ⊂ CB²(M) of f₂.

Let g ∈ U and pₓ be the continuation of p. By the choice of U, we know that g is accessible and center bunched. If g admits some \( \text{acip} \) µ, then by Theorem 5.7 µ = φm for some Hölder continuous function φ which is bounded and bounded away from zero. By Equation (5.1) we have φ(pₓ) = Jgᵏ(pₓ)φ(gᵏpₓ) = Jgᵏ(pₓ)φ(pₓ). This is impossible since |Jgᵏ(pₓ)| ≠ 1 and φ(pₓ) ≠ 0. So each g ∈ U admits no \( \text{acip} \). Hence there exists an open set U close to f in which each map admits no \( \text{acip} \). This finishes the proof.

**Remark 4.** It is well known that among \( C^2 \) Anosov diffeomorphisms the ones that admits no \( \text{acip} \) are open and dense, see [5, Corollary 4.15]. This is due to the fact that there are many periodic points for every Anosov diffeomorphisms. Recently Avila and Bochi [2] proved that a \( C^1 \)-generic map in \( C^1(M, M) \) has no \( \text{acip} \). In particular a \( C^1 \)-generic map in \( \text{Diff}^1(M) \) has no \( \text{acip} \).

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