A large spin limit of strings on $AdS_5 \times S^5$

in a non-compact sector

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Abstract

We study the scaling law of the energy spectrum of classical strings on $AdS_5 \times S^5$, in particular, in the $SL(2)$ sector for large $S$ ($AdS$ spin) and fixed $J$ ($S^1 \subset S^5$ spin). For any finite gap solution, we identify the limit in which the energy exhibits the logarithmic scaling in $S$, characteristic to the anomalous dimension of low-twist gauge theory operators. Our result therefore shows that the log $S$ scaling, first observed by Gubser, Klebanov and Polyakov for the folded string, is universal also on the string side, suggesting another interesting window to explore the AdS/CFT correspondence as in the BMN/Frolov-Tseytlin limit.
1 Introduction

The integrable structures underlying in the $\mathcal{N}=4$ super Yang-Mills (SYM) theory [1,2] and the string theory on $AdS_5 \times S^5$ [3] have enabled us to probe the gauge/gravity correspondence quantitatively beyond the supersymmetric sectors. At one-loop in the planar limit on the SYM side, the scaling dimension of the long operators is found to precisely match the energy of the rotating strings with large spins [4–10]. In particular, in that (Frolov-Tseytlin) limit [4,9], the scaling dimension/energy universally exhibits the BMN scaling [11]. A systematic way to show the matching between the two sides is to compare the algebraic curves and the differentials thereon which are associated with the integrability [8,10,12–17].

In the non-compact sectors, there is another interesting large charge limit: from the point of view on the string side, it is $S (AdS_5 \text{ spin}) \gg 1$ with $J (S^5 \text{ spin})$ fixed, whereas, on the gauge theory side, it corresponds to considering low-twist ($J$) operators with large Lorentz spin ($S$) [18,19]. In this case, the anomalous dimension on the gauge theory side universally behaves as $\Delta - S \sim c\lambda \log S$ at one loop, where $\lambda$ is the ’t Hooft coupling and $c$ is a constant [20–23]. On the string theory side, some classical solutions with this log $S$ scaling are found [18,19,24,25] (see also [26]), but with different dependence on $\lambda$.

The non-compact sectors are of interest, since any gauge theories, including QCD, possess such sectors and integrable structures emerge rather ubiquitously (see, e.g., [21, 27, 28]). The anomalous dimension of low-twist operators, accounting for the violation of the Bjorken scaling, is also related at large $S$ to the cusp anomalous dimension and, hence, to certain physical processes. Moreover, the analysis in the large $S$ limit with fixed $J$ may provide us important data for the higher-loop Bethe ansätze [29–35], which have been studied mainly in the large $J$ limit.

In this note, we consider the finite gap solution on the string side in the $SL(2)$ sector for large $S$ and fixed $J$. Related discussions on both string and gauge theory sides are found in [21–23,27,28,36]. In particular, detailed analyses of the spectrum of the gauge theory operators and of the higher-loop Bethe ansätze are given in [23] and [36], respectively. In terms of the algebraic curve, we identify the limit in which the energy of any finite gap solution of the string sigma model takes the form $\Delta - S \sim c\sqrt{\lambda} \log S$, where $\sqrt{\lambda}$ is the effective string tension and $c$ is a constant. Therefore, the log $S$ scaling, which is characteristic to the anomalous dimension of the low-twist gauge theory operators, is universal also on the string side, as the BMN scaling in the BMN/Frolov-Tseytlin limit. This suggests another interesting window to explore the AdS/CFT correspondence.

In the next section, following [12], we summarize the finite gap solution in the $SL(2)$ sector. In section 3, we analyze the general two-cut solution and identify the
conditions under which the log $S$ scaling emerges. In the course, we explicitly write down the period conditions for the differential. We briefly summarize the results of the corresponding integrals in Appendix. In section 4, based on the results in section 3, we discuss the general finite gap solution. We show that, in a certain limit with $S/\sqrt{\lambda} \gg 1$ and fixed $J/\sqrt{\lambda}$, the energy of any finite gap solution exhibits the log $S$ scaling, but with different dependence on $\lambda$ from the perturbative gauge theory case.

2 Classical strings in the $SL(2)$ sector

Let us consider a classical string in the $SL(2)$ sector, namely, a string moving in $AdS_3 \times S^1$ in $AdS_5 \times S^5$ [12]. One can choose a gauge in which the coordinate field of $S^1$ takes the form

$$\phi = l\tau + m\sigma ,$$

where $\tau, \sigma$ are the world-sheet coordinates. Here both $\phi$ and $\sigma$ are periodically identified $\phi \cong \phi + 2\pi$, $\sigma \cong \sigma + 2\pi$, so that $m$ is an integer. In terms of the angular momentum ($J$) associated with $S^1$ and the effective string tension ($\sqrt{\lambda}$), $l$ is given by $l = J/\sqrt{\lambda}$. In this gauge, only the $AdS_3$ part is thus non-trivial. For a class of solutions called the finite gap solution, the motion of the string is specified by a differential

$$dp = -\pi \frac{dx}{y} \left[ l_+ f_+(x) + l_- f_-(x) + \sum_{k=1}^{K-1} b_k x^{k-1} \right]$$

on a hyperelliptic curve given by

$$y^2 = \prod_{a=1}^{2K} (x - x_a) .$$

Here, $x_a \in \mathbb{R}$, $b_k$ are constant, $l_\pm = l \pm m$, and

$$f_\pm(x) = \frac{y_\pm}{(x \mp 1)^2} + \frac{y'_\pm}{x \mp 1}$$

with $y_\pm = y|_{x=\pm 1}$ and $y'_\pm = \partial_x y|_{x=\pm 1}$. The differential has to satisfy the following period conditions:

$$\oint_{A_a} dp = 0 , \quad \int_{B_a} dp = 2\pi n_a .$$

The contours $A_a$ surround the $K - 1$ (among $K$) cuts, whereas $B_a$ traverse each cut and terminate at the infinity on each of the two sheets (see Fig.1). The full $AdS_3$ sigma model actually possesses two kinds of $SL(2)$ excitation modes associated with the left
and right multiplication of the $SL(2)$ group coordinates. For the sake of comparison to the minimal non-compact sector on the gauge side, it is enough to take $|x_a| > 1$ so that excitations in a single $SL(2)$ are present \[15\]. The asymptotic behaviors of $dp$ for $x \to \infty$ and $x \to 0$ give relations to the energy $\Delta$ and the AdS spin $S$:

$$\frac{2}{\sqrt{\lambda}}(\Delta + S) = b_{K-1},$$

$$\frac{2}{\sqrt{\lambda}}(\Delta - S) = \frac{1}{y(0)} \left[ l_+(y_+ - y'_+) + l_-(y_- + y'_-) + b_1 \right]. \quad \text{(6)}$$

3 General two-cut solutions

As a preliminary to the discussion on the general finite gap solution, we first consider the general two-cut solution. A discussion on the symmetric case is found in \[23\].

In this case, there are three independent period conditions. It is possible to carry out these integrals explicitly. The results are summarized in Appendix. One can then combine those period conditions, to obtain a simpler set:

$$2b_1 = -2 \left( \frac{l_+}{y_+} - \frac{l_-}{y_-} \right) x_0 + \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right) \left[ (x_1 x_2 + x_3 x_4 + 2) - x_{13} x_{24} \frac{E(1-k)}{K(1-k)} \right],$$

$$n_1 - n_2 = \frac{\pi}{2K(1-k)} \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right),$$

$$n_1 = \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right) \sqrt{x_{13} x_{24}} G \left( \sqrt{x_{13}/x_{23}}, k \right) - \frac{l_+}{y_+}(x_1 - 1) - \frac{l_-}{y_-}(x_1 + 1),$$

where $x_{ab} = x_a - x_b$, $2x_0 = \sum_{a=1}^4 x_a$, $k = x_{14} x_{23}/x_{13} x_{24}$.

$$G(z, k) \equiv E(z, k) + F(z, k) \frac{E(1-k)}{K(1-k)} - 1, \quad \text{(8)}$$

and $E(k), K(k), E(z, k), F(z, k)$ denote standard elliptic integrals, our conventions of which are found in Appendix. $n_2$ is expressed similarly to $n_1$, but that is not independent of the above. In the case of the symmetric cuts, i.e., $x_1 = -x_4$, $x_2 = -x_3$, these conditions reduce to those in \[23, 37\].
Now, we would like to consider the case when
\[ \gamma \equiv \Delta - J - S \] (9)
scales as \( \log S \) for large \( S/\sqrt{\lambda} \) and fixed \( J/\sqrt{\lambda}, m \). In the following, we focus on the case in which one cut is on the left side of \( x = 0 \), and the other is on the right side, i.e., \( x_3 < x_4 < -1, 1 < x_1 < x_2 \). The analysis of the case in which both cuts are on the same side of the origin is similar. For our purpose, we first note that (6) with \( K = 2 \) gives an expression of \( \gamma \) in terms of \( x_a \):
\[
\frac{2}{\sqrt{\lambda}}(\gamma + J)\left(1 - \frac{1}{\sqrt{\prod x_a}}\right) = \frac{4S/\sqrt{\lambda}}{\sqrt{\prod x_a}}
\]
(10)
\[ + l_+ \sqrt{\prod \frac{x_a - 1}{x_a}} \left(1 + \frac{1}{2} \sum x_a^{-1}ight) + l_- \sqrt{\prod \frac{x_a + 1}{x_a}} \left(1 - \frac{1}{2} \sum \frac{1}{x_a + 1}\right). \]

Second, from the first equation in (6), it follows that
\[ \sqrt{\lambda} b_1 = \mathcal{O}\left(\max(\gamma, S)\right). \] (11)

Third, the first condition in (7) then implies that, for \( S \) to be large, at least one of the following conditions has to be satisfied:
\[ x_2 \gg 1, \quad |x_3| \gg 1, \quad x_1 - 1 \ll 1, \quad |x_4 + 1| \ll 1. \] (12)

With these in mind, one can analyze the spectrum of \( \gamma \) for large \( S \). We omit details but, after some analysis, we find that \( \gamma \sim c\sqrt{\lambda} \log(S/\sqrt{\lambda}) \) for large \( S/\sqrt{\lambda} \) and fixed \( l, m \) with \( c \) a constant independent of \( J \), when the following two conditions are satisfied:

1. the inner branch points \( x_1, x_4 \) are of \( \mathcal{O}(1) \), and at least one of them approaches the singularities of \( dp \) at \( x = \pm 1 \),

2. the mode numbers \( n_1, n_2 \) are of \( \mathcal{O}(1) \).

Throughout in this paper relation by tilde (\( \sim \)) signifies equality in the leading order approximation. One can check that these conditions are equivalent to condition 1 and

\[ 2'. \quad b_1, x_2, x_3 = \mathcal{O}(S/\sqrt{\lambda}). \]

(See also the next section.) These are consistent with the result for the symmetric two-cut case [23, 37]. In addition, the independence from \( J \) is in accord with the fact on the gauge theory side that the lowest anomalous dimension, which is described by a hyperelliptic curve with degenerate two cuts, is independent of the length of the operators [23].

We however remark that, in general, large \( S \) does not necessarily mean the logarithmic scaling, but various asymptotics of \( \gamma \) for large \( S \) are allowed classically by adjusting
the parameters. To understand the difference from the gauge theory side, which always gives the log $S$ scaling, it would be useful to note the allowed values of the mode numbers $n_a$: on the string side they can be large, whereas those on the gauge theory side are bound by $J$. Condition 2 eliminates such large mode numbers. In addition, our analysis here is different from that for the symmetric two-cut case on the string side in [23], because the asymmetry of cuts, in addition to large mode numbers, is allowed. In fact, one can check that $\gamma = \mathcal{O}(S)$ with $n_1 = \mathcal{O}(S/\sqrt{\lambda})$, $n_2 = \mathcal{O}((S/\sqrt{\lambda})^2)$ when $x_{1,2} = \mathcal{O}(S/\sqrt{\lambda})$ and $x_3 = -1 - \mathcal{O}((S/\sqrt{\lambda})^{-1})$, $x_4 = -1 - \mathcal{O}((S/\sqrt{\lambda})^{-3})$.

A concrete example to give the log $S$ scaling is the case in which $|x_{2,3}| \gg 1$ and $x_1 = 1 + \delta_1^2$, $x_4 = -1 - \delta_4^2$ with $\delta_1, \delta_4 \ll 1$. In this case, $k \sim 2x_{32}/x_{23} \ll 1$. Thus, using the asymptotic forms of the elliptic integrals in (33), the period conditions are reduced to

$$\frac{\sqrt{2}}{\log(1/k)} \left( \frac{l_+}{\delta_1} + \frac{l_-}{\delta_4} \right) \sim \frac{n_1}{\arcsin \sqrt{\frac{-x_4}{x_{23}}}} \sim \frac{-n_2}{\arcsin \sqrt{\frac{x_2}{x_{23}}}} \sim \frac{2b_1}{\sqrt{-x_2x_3}}.$$

(13)

Since $n_{1,2} = \mathcal{O}(1)$, the above relation between $b_1$ and $n_{1,2}$ implies that $x_{2,3} = \mathcal{O}(b_1)$. It turns out that $b_1, x_{2,3} = \mathcal{O}(S/\sqrt{\lambda})$ for (11) to be satisfied. Together with (10), the relation between $\delta_{1,4}$ and $n_1$ then gives

$$\gamma \sim \frac{\sqrt{\lambda}n_1}{4\arcsin \sqrt{\frac{-x_4}{x_{23}}}} \log \frac{S}{\sqrt{\lambda}}.$$

(14)

The relation between $n_1$ and $n_2$ constrains the coefficient of log $S$. Taking into account this, one finds that, when $n_1 = -n_2 = 1$ and $x_2 = -x_3$, $\gamma$ takes the minimum $\gamma_{\text{min}} = (\sqrt{\lambda}/\pi) \log(S/\sqrt{\lambda})$, which agrees with the energy of the folded strings corresponding to the symmetric two-cut solutions [18, 19].

4 General finite gap solutions

In this section, we generalize the analysis in the previous section to the case of the general finite gap solution. We show that there exists a limit in which the energy of any finite gap solution for large $S/\sqrt{\lambda}$ and fixed $J/\sqrt{\lambda}, m$ behaves as $\gamma \sim c\sqrt{\lambda} \log(S/\sqrt{\lambda})$ with $c$ a constant of order 1.

Here, we would like to emphasize that our point is to show a sector-wise correspondence to the gauge theory for large $S$, as in the Frolov-Tseytlin limit [7, 8, 12–16]: The log $S$ scaling holds not only for multi-soliton solutions, whose variation from the ground state is relatively small, but also for general quasi-periodic solutions, which could be far

1 This bound may emerge also on the string side if one imposes some ‘quantization condition’ respecting the integrality of $J$. 
apart from the ground state. Such a sector-wise correspondence would also be useful to study the correspondence of the operators/solutions between the gauge/string sides, which is generally quite non-trivial (e.g., [38]). Moreover, our results may give useful insights into the asymptotic Bethe ansätze [29–35], which are expected to interpolate all the states between the gauge/string sides.

For this purpose, it is useful to introduce rescaled variables

$$\tilde{x} = \frac{x}{M}, \quad \tilde{x}_a = \frac{x_a}{M}, \quad \tilde{l}_\pm = \frac{l_\pm}{M}, \quad \tilde{b}_k = \frac{b_k}{M^{K-k}}.$$  \hspace{1cm} (15)

In terms of these, the differential $dp$ becomes

$$dp = -\pi d\tilde{x} \tilde{y} \left[ \tilde{l}_+ \tilde{f}_+(\tilde{x}) + \tilde{l}_- \tilde{f}_-(\tilde{x}) + \sum_{k=1}^{K-1} \tilde{b}_k \tilde{x}^{k-1} \right],$$  \hspace{1cm} (16)

where $\tilde{y}^2 = \prod_{a=1}^{2K}(\tilde{x} - \tilde{x}_a)$,

$$\tilde{f}_\pm(x) = \frac{\tilde{y}_\pm}{(\tilde{x} + \epsilon)^2} + \frac{\tilde{y}_\pm'}{\tilde{x} + \epsilon},$$ \hspace{1cm} (17)

$\tilde{y}_\pm = \tilde{y}|_{\tilde{x}=\pm\epsilon}, \tilde{y}_\pm' = \partial_\tilde{x} \tilde{y}|_{\tilde{x}=\pm\epsilon}$, and $\epsilon = 1/M$. From (16), one finds that $\frac{2}{\sqrt{\lambda}}(\Delta + S) = M\tilde{b}_{K-1}$ and

$$\frac{2}{\sqrt{\lambda}}(\Delta - S) = \frac{1}{\tilde{y}(0)} \left[ \tilde{l}_+ \tilde{y}_+(\frac{1}{\epsilon} + \frac{1}{2} \sum_a \frac{1}{\tilde{x}_a - \epsilon}) + \tilde{l}_- \tilde{y}_-(\frac{1}{\epsilon} - \frac{1}{2} \sum_a \frac{1}{\tilde{x}_a + \epsilon}) + \epsilon \tilde{b}_1 \right].$$ \hspace{1cm} (18)

When

$$l = \frac{J}{\sqrt{\lambda}}, \quad \frac{S}{\sqrt{\lambda}} = O(M) \gg 1, \quad m, \tilde{b}_k, \tilde{x}_a = O(1),$$ \hspace{1cm} (19)

the period conditions (5) give relations among quantities of order 1; nothing is very large or small (generically). $\tilde{b}_k, \tilde{x}_a$ of order 1 solve these conditions for the winding number $m$, mode numbers and fillings (i.e., the contributions to $S$ from each cut) of order 1, giving a classical string solution. The energy is then obtained by expanding (18) up to and including terms of $O(\epsilon)$ and setting $\epsilon = 1/4\pi l$:

$$\gamma \sim \frac{\lambda}{128\pi^2 J} \left( \sum_a \frac{1}{\tilde{x}_a^2} - \sum_{a>b} \frac{2}{\tilde{x}_a \tilde{x}_b} + \frac{16\pi \tilde{b}_1}{\tilde{y}(0)} \right).$$ \hspace{1cm} (20)

Since $2\pi \tilde{b}_1 \sim 1 + 2S/J$, this $\gamma$ precisely agrees with the gauge theory result in the Frolov-Tseytlin limit (cf. Eq. (3.13) in [8]), to confirm the matching between the string and the one-loop gauge theory results shown in [12].

In the following, based on the result in the previous section, we consider the case in which $K \geq 2$,

$$M = \frac{S}{\sqrt{\lambda}} \gg 1, \quad l, m, \tilde{b}_k = O(1), \quad \tilde{x}_a = O(1) \ (a \neq 1, 2K),$$ \hspace{1cm} (21)
the innermost branch points \( \tilde{x}_1, \tilde{x}_{2K} \) are of \( \mathcal{O}(\epsilon) \) and at least one of them approaches \( \pm \epsilon \). Thus, we denote the innermost branch points by

\[
\tilde{x}_1 = \epsilon + \tilde{\delta}_1^2, \quad \tilde{x}_{2K} = -\epsilon - \tilde{\delta}_{2K}^2,
\]

where \( \tilde{\delta}_1^2, 2K \lesssim \epsilon \) and at least one of \( \tilde{\delta}_{1,2K} \ll \epsilon \). We also consider the generic case so that \( \tilde{x}_a (a \neq 1, 2K) \) do not collide each other, or the algebraic curve does not degenerate there.

Let us first analyze the \( A_1 \)-period condition for one of the innermost cuts. To evaluate the integral, we note that, for \( \tilde{y} \in (\tilde{x}_1, \tilde{x}_2) \),

\[
\tilde{y} \sim c\tilde{y}^{(2)}, \quad \sum_{k=1} \tilde{b}_k \tilde{x}^{k-1} \sim c'.
\]

Here and in the following, we denote by \( c, c' \) numbers of order 1 and, by the superscript \( (2) \), quantities for the two-cut case corresponding to the two innermost cuts. For example, \( (\tilde{y}^{(2)})^2 = (\tilde{x} - \tilde{x}_1)(\tilde{x} - \tilde{x}_2)(\tilde{x} - \tilde{x}_{2K-1})(\tilde{x} - \tilde{x}_{2K}) \). In addition, \( \tilde{y}_\pm \sim c\tilde{y}_\pm^{(2)}, \quad \tilde{y}_\pm'/\tilde{y}_\pm \sim \tilde{y}_\pm^{(2)'}/\tilde{y}_\pm^{(2)} + c \) and, hence,

\[
\tilde{f}_\pm (\tilde{x}) \sim c\tilde{f}_\pm^{(2)} (\tilde{x}) + \frac{c'\tilde{y}_\pm^{(2)}}{\tilde{x} \mp \epsilon}.
\]

It turns out that the difference of \( \tilde{f}_\pm \) and \( \tilde{f}_\pm^{(2)} \) (the second terms on the right-hand side) gives terms of order \( \epsilon \) in the integral. The integral involving \( \tilde{f}_\pm^{(2)} \) can be read off from (30). Retaining the terms relevant under the conditions (21) and (22), the \( A_1 \)-period condition becomes

\[
0 \sim K(1 - k) - \left( \frac{\tilde{I}_+}{\tilde{y}_+^{(2)}} + \frac{c'\tilde{I}_-}{\tilde{y}_-^{(2)}} \right) \sqrt{\tilde{x}_{13}\tilde{x}_{24}} E(1 - k).
\]

From the asymptotic behaviors of the elliptic integrals (33), it then follows that

\[
\frac{c\tilde{I}_+}{\tilde{y}_+^{(2)}} + \frac{c'\tilde{I}_-}{\tilde{y}_-^{(2)}} \sim \log \frac{S}{\sqrt{\lambda}},
\]

and that \( \min(\tilde{\delta}_1^2, \tilde{\delta}_{2K}^2) \sim c\epsilon / \log^2 \epsilon \).

It is straightforward to take into account other period conditions. We first note that \( \tilde{f}_\pm \) in \( dp \) are neglected in the integrals, since they give contributions of order \( \epsilon^2/\tilde{\delta}_{1,2K} \ll 1 \) and hence subleading to others. Also, for these period conditions, \( \tilde{x}_{1,2K} \) can be set to zero at the leading approximation, since \( \tilde{x} - \tilde{x}_{1,2K} \sim \tilde{x} \) in evaluating the integrals. Thus, these conditions give relations among order 1 quantities, namely, \( \tilde{b}_k, \tilde{x}_a (a \neq 1, 2K) = \mathcal{O}(1) \) and

\[
n_a = \mathcal{O}(1).
\]
We are now ready to estimate $\gamma$. In the present case, the dominant contribution to $\gamma$ comes from the terms with $1/(\bar{x}_1 - \epsilon)$ or $1/(\bar{x}_{2K} + \epsilon)$. Therefore,

$$\gamma \sim \frac{\sqrt{\lambda}}{2g(0)} \left( \frac{\bar{t}_+ \bar{y}_+}{\bar{x}_1 - \epsilon} + \frac{\bar{t}_- \bar{y}_-}{\bar{x}_{2K} + \epsilon} \right) \sim c' \sqrt{\lambda} \left( \frac{\bar{t}_+ (2)}{\bar{y}_+ (2)} + \frac{\bar{t}_- (2)}{\bar{y}_- (2)} \right)$$

$$\sim c \sqrt{\lambda} \log S \sqrt{\lambda}.$$  \hfill (28)

In sum, we have shown that, in the limit given by (21) and (22), the energy of the classical string scales as $\gamma \sim c \sqrt{\lambda} \log S$ with $c$ a constant, generalizing the results in [18, 19, 24, 25]. Therefore, the log $S$ scaling, which is characteristic to the anomalous dimension of the low-twist gauge theory operators, is universal also for the classical string in the $SL(2)$ sector, as the BMN scaling in the BMN/Frolov-Tseytlin limit [10]. This suggests another interesting window to explore the AdS/CFT correspondence. Our analysis may be extended to other non-compact sectors along the line of [13–17].

Note added: After this work was completed, we were informed by Sergey Frolov that he, with Matthias Staudacher, reached the same conclusion by a different method.

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A Period conditions for general two-cut solutions

In this appendix, we summarize the period conditions for the general two-cut case. For the $A$-period condition, we first observe an identity

$$- y \left( \frac{y}{x \mp 1} \right)' = y \pm f_\pm (x) + (x_0 \mp 2)(x \mp 1) - (x \mp 1)^2,$$  \hfill (29)
where \(2x_0 = \sum_{a=1}^{4} x_a\). The A-period condition is then found to be

\[
0 = \frac{1}{2\pi i} \int_A dp = \frac{2}{2\pi i} \int_{x_1}^{x_2} \frac{dp}{dx} dx
= \left[ \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right) (x_1 x_2 + x_3 x_4 + 2) - 2 \left( \frac{l_+}{y_+} - \frac{l_-}{y_-} \right) x_0 - 2b_1 \right] \frac{1}{\sqrt{x_1 x_2}} K(1 - k)
\]

\[
+ \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right) \sqrt{x_1 x_2} E(1 - k),
\]

(30)

where \(x_{ab} = x_a - x_b\), and \(k = x_{14} x_{23}/x_{13} x_{24}\). \(K(k), E(k)\) are the elliptic integrals, our conventions of which are

\[
F(z, k) = \int_0^z \frac{dx}{\sqrt{(1 - x^2)(1 - kx^2)}}, \quad E(z, k) = \int_0^z dx \sqrt{\frac{1 - kx^2}{1 - x^2}},
\]

(31)

and \(K(k) = F(1, k), E(k) = E(1, k)\).

To evaluate the B-period conditions, we first make a change of variables \(u = 1/x\), and consider \(\left( \frac{\sqrt{Q(u)}}{u+1} \right)'\) where \(Q(u) = u^4 y^2 = \prod_{a=1}^{4} (1 - x_a u)\). Using identities similar to [29], we obtain

\[
n_1 = \frac{1}{2\pi} \int_{B_1} dp = \frac{2}{2\pi} \int_{x_1}^{\infty} \frac{dp}{dx} dx
= \left[ \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right) (x_1 x_4 + x_2 x_3 + 2) - 2 \left( \frac{l_+}{y_+} - \frac{l_-}{y_-} \right) x_0 - 2b_1 \right] \frac{1}{\sqrt{x_1 x_2}} F(\sqrt{\frac{x_13}{x_23}}, k)
\]

\[
+ \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right) \sqrt{x_1 x_2} E(\sqrt{\frac{x_13}{x_23}}, k) - \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right) x_1 + \left( \frac{l_+}{y_+} - \frac{l_-}{y_-} \right),
\]

\[
n_2 = -\frac{1}{2\pi} \int_{B_2} dp = -\frac{2}{2\pi} \int_{x_1}^{\infty} \frac{dp}{dx} dx
= \left[ \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right) (x_1 x_4 + x_2 x_3 + 2) - 2 \left( \frac{l_+}{y_+} - \frac{l_-}{y_-} \right) x_0 - 2b_1 \right] \frac{1}{\sqrt{x_1 x_2}} F(\sqrt{\frac{x_24}{x_23}}, k)
\]

\[
+ \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right) \sqrt{x_1 x_2} E(\sqrt{\frac{x_24}{x_23}}, k) + \left( \frac{l_+}{y_+} + \frac{l_-}{y_-} \right) x_4 - \left( \frac{l_+}{y_+} - \frac{l_-}{y_-} \right).
\]

(32)

These are related by \((x, p, m, x_a) \leftrightarrow (-x, -p, -m, -x_{4-a+1})\).

In the main text, we use the asymptotic behaviors for small \(k\) such as

\[
E(1 - k) \sim 1, \quad K(1 - k) \sim \frac{1}{2} \log(1/k),
\]

\[
E(z, k) \sim F(z, k) \sim \arcsin z.
\]

(33)
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