Action-type axiomatizable classes of group representations

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Abstract

The paper adjoins the book [PV] and turns to be, in a sense, its continuation. In the book the varieties of representations had been considered. In the matter of fact, the varieties under consideration are action-type varieties. This paper studies other classes of representations, axiomatizable in the special action-type logic.

1 Background (preliminaries)

1.1 Variety $Rep - K$

Given a commutative with the unit ring $K$, consider a variety which is also a category, denoted by $Rep - K$. The algebras of this variety are two-sorted algebras. These algebras are pairs, or representations, $(V, G)$, where $V$ is $K$-module and $G$ a group, acting in $V$. The action is treated as an operation in the two-sorted algebra $(V, G)$.

The axioms are as follows:

1. the mapping $a \rightarrow a \circ g$ is a $K$-linear mapping in $V$,
2. $(a \circ g_1) \circ g_2 = a \circ g_1 g_2$,
3. $a \circ 1 = a$. 
Here 1 is the unit in $G$, $a \in V$, $g \in G$. The operation $\circ$ recovers the representation $\rho : G \to AutV$, which is sometimes identified with the algebra $(V, G)$, i.e., $\rho = (V, G)$.

The pointed identities together with the identities of groups and $K$-modules determine the variety $\text{Rep} - K$. As usual, this variety is also a category. Its morphisms are two-sorted homomorphisms

$$\mu = (\alpha, \beta) : (V, G) \to (V', G'),$$

where $\alpha : V \to V'$ a $K$-homomorphism of modules, $\beta : G \to G'$ is a homomorphism of groups, and $(a \circ g)\alpha = a^\alpha \circ g^\beta$. We have: $\text{Ker} \mu = (\text{Ker} \alpha, \text{Ker} \beta)$. Take $\text{Ker} \alpha = V_0$, $\text{Ker} \beta = H$. Then the pair $(V_0, H)$ is a congruence of the representation $(V, G)$ in the following sense: $V_0$ is a submodule in $V$, invariant under the action of the group $G$, $H$ is a normal subgroup in $G$, acting trivially in $V/V_0$. We have $(V, G)/\text{Ker} \mu = (V/V_0, G/H)$. This gives the theorem on homomorphisms.

Let us consider a free representation $W = W(X, Y)$ Here the pair $(X, Y)$ is a two-sorted set. We have: $W(X, Y) = (XKF(Y), F(Y))$, where $F(Y) = F$ is a free group over the set $Y$, $XKF = \Phi$ is a free $KF$-module over the set $X$ and $KF$ is a group algebra.

The elements of $\Phi$ have the form $w = x_1u_1 + ... + x_nu_n$, $u_i \in KF$; the elements of $F$ are written as $f = f(y_1, ..., y_m)$. The action $\circ$ is defined by the rule:

$$w \circ f = wf = x_1(u_1f) + ... + x_n(u_nf).$$

It is easy to understand that the mappings $\alpha : X \to V$ and $\beta : Y \to G$ determine the homomorphism $\mu = (\alpha, \beta) : (\Phi, F) \to (V, G)$.

Consider further the sets $\text{Hom}(W, \rho) = \text{Hom}(W, (V, G))$. In the situation of the finite $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_m\}$ the sets $\text{Hom}(W, \rho)$ are treated as affine spaces. There is a natural bijection $\text{Hom}(W, \rho) \to V^{(n)} \times G^{(m)}$. A point

$$((\alpha(x_1), ..., (\alpha(x_n)), (\beta(y_1), ..., \beta(y_m))) = ((a_1, ..., a_n), (g_1, ..., g_m)),$$

$a_i \in V$, $g_k \in G$, corresponds to the homomorphism $\mu = (\alpha, \beta) : (\Phi, F) \to (V, G)$.

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1.2 Logic in $Rep - K$.

Given $X$ and $Y$, consider a signature 

$$L = L_{X,Y} = \{\lor, \land, \neg, \exists x, \exists y, x \in X, y \in Y\}.$$ 

For the algebra $W = W(X,Y)$ we consider equalities $w \equiv 0$ and $f \equiv 1$. We treat these equalities as logical formulas. The equalities of the first kind we call action-type equalities, while the equalities of the second type we call group equalities. Denote by $LW = L_{X,Y}W(X,Y)$ the absolutely free $L$-algebra over the set of all $W$-equalities. Its elements we call formulas (elementary, first order formulas) over the free representation $W = (\Phi, F)$.

Let us consider an example of $L$-algebra. Take the set $\text{Hom}(W, (V,G))$. Let $\text{Set}(W, (V,G))$ be a system of all subsets of $\text{Hom}(W, (V,G))$. It is clear, that boolean operations are defined in $\text{Set}(W, (V,G))$ and it is a boolean algebra. Let us define also quantifiers. Let $A$ be a subset of $\text{Hom}(W, (V,G))$.

We set: $\mu(\alpha, \beta) \in \exists xA$, if there exists $\nu = (\alpha', \beta)$ such that $\nu \in A$ and $\alpha(x_1) = \alpha'(x_1)$ for $x_1 \neq x$. Analogously we define $\exists yA$. Here $\exists x$ and $\exists y$ are quantifiers of the boolean algebra $\text{Set}(W, (V,G))$ in the sense of the following definition.

The quantifier $\exists$ of a boolean algebra $B$ is a mapping $\exists : B \to B$ with the properties:

1. $\exists 0 = 0$,
2. $a < \exists a$,
3. $\exists(a \land \exists b) = \exists a \land \exists b$.

Here 0 is a zero in $B$, and $a, b \in B$. Define now a canonical homomorphism of $L$-algebras

$$\text{Val} = \text{Val}^{W}_{(V,G)} : LW \to \text{Set}(W, (V,G)).$$

We set: $\mu = (\alpha, \beta) \in \text{Val}(w \equiv 0)$, if $w^\alpha = 0$ in $V$, $\mu = (\alpha, \beta) \in \text{Val}(f \equiv 1)$, if $f^\beta = 1$ in $G$. Since $LW$ is free, this determines $\text{Val}(u)$ for every formula $u \in LW$. Here the set of homomorphisms $\text{Val}(u)$ is called the value of the formula $u$ in the representation $(V,G)$. If $\text{Val}(u) = \text{Hom}(W, (V,G))$, then it means that the formula $u$ holds in the representation $(V,G)$.

Define further separately the logic of action (action-type logic). It is generated by equalities of action $w \equiv 0$ in the signature $L_X = \{\lor, \land, \neg, \exists x, x \in X\}$. There are no quantifiers $\exists y$ and equalities $f \equiv 1$. Denote this logic by $L_XW$. 

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1.3 Classes of representations

Consider classes $\mathcal{X}$ of representations $(V, G)$ in $Rep - K$ and, simultaneously, the sets of formulas $T$ in the logic $LW$ with countable $X$ and $Y$. We establish Galois correspondence

$$T = \mathcal{X}^* = \{ u \in LW | u \text{ holds in every } (V, G) \in \mathcal{X} \}$$

$$\mathcal{X} = T^* = \{ (V, G) | \text{ every formula from } T \text{ holds in } (V, G) \}.$$ 

A class $\mathcal{X}$ of the kind $\mathcal{X} = T^*$ is called *axiomatizable one*. Here $T$ is an arbitrary set. If $T$ consists of action-type formulas, then $\mathcal{X} = T^*$ is an *action-type axiomatizable class*.

Let us distinguish special cases.

1. $T$ consists of equalities. Then $\mathcal{X} = T^*$ is a variety of representations. If $T$ consists of action-type equalities, then $\mathcal{X} = T^*$ is an *action-type variety*. Such $\mathcal{X}$ is given also by the formulas of the form $x \circ u \equiv 0$, $u \in KF$.

2. Formulas of the kind $u_1 \lor u_2 \lor \ldots \lor u_n$, where all $u_i$ are equalities, are called *pseudo-equalities* (or *pseudo-identities*). The corresponding $\mathcal{X} = T^*$ are called pseudo-varieties. Action-type pseudo-varieties are given by the formulas of the kind $w_1 \equiv 0 \lor \ldots \lor w_n \equiv 0$.

3. Formulas of the kind $u_1 \land u_2 \land \ldots \land u_n \implies u$, where all $u_i$ are equalities, are called *quasi-equalities* (or *quasi-identities*). Formulas $w_1 \equiv 0 \land \ldots \land w_n \equiv 0 \implies w \equiv 0$ is an action-type quasi-identity. We have quasi-varieties and action-type quasi-varieties.

4. Universal formulas have the form $u_1 \lor \ldots \lor u_n \lor \neg v_1 \lor \ldots \lor \neg v_m$ where $u_i$ and $v_j$ are equalities. Universal action-type formulas have the form $w_1 \equiv 0 \lor \ldots \lor w_n \equiv 0 \lor w'_1 \neq 0 \lor \ldots \lor w'_m \neq 0$. The corresponding classes $\mathcal{X}$ are universal classes and action-type universal classes. We consider also classes $\mathcal{X}$, axiomatizable by arbitrary action-type formulas.

A class $\mathcal{X}$ we call a *saturated one* if the inclusion $(V, G) \in \mathcal{X}$ holds if and only if for the corresponding faithful representation $(V, \overline{G})$ holds $(V, \overline{G}) \in \mathcal{X}$.

A class $\mathcal{X}$ we call a *right-hereditary one*, if $(V, G) \in \mathcal{X} \implies (V, H) \in \mathcal{X}$ holds for every subgroup $H$ in $G$.

Finally, class of representations $\mathcal{X}$ is called a *right-local one*, if $(V, G) \in \mathcal{X}$ holds if all $(V, H)$ belong to $\mathcal{X}$, where $H$ are finitary-generated subgroups in $G$. 


2 Action-type axiomatizable classes

2.1 Theorem

Theorem 1. If class $\mathcal{X}$ is given by action-type formulas, then such class is saturated, right-hereditary and right-local.

Proof. Check first that the class is saturated.

Take a representation $(V, G)$ and the corresponding faithful representation $(V, G_{\overline{G}})$. Let $\beta_0 : G \to G_{\overline{G}}$ be the natural homomorphism. Then for any $a \in V$ and $g \in G$ we have $a \circ g = a \circ g^{\beta_0}$. Consider, further, homomorphisms

$$\mu_0 = (\alpha, \beta) : W \to (V, G)$$

and

$$\mu = (\alpha, \beta \beta_0) : W \to (V, G_{\overline{G}}).$$

We have the diagram

$$\begin{array}{c}
W \\
\xrightarrow{\mu_0}
\xrightarrow{\mu}
(V, G) \\
\xrightarrow{\nu = (1, \beta_0)}
(V, G_{\overline{G}})
\end{array}$$

Here $\mu = \mu_0 \nu$ and every $\mu$ can be represented in such a way.

Note that the homomorphism $\beta : F \to G$ induces the homomorphism $\beta : KF \to KG$ of group algebras. We have also $\beta_0 : KG \to K\overline{G}$, and $a \circ h = a \circ h^{\beta_0}$ for every $a \in V$ and $h \in KG$.

Take now an element $w = x_1 \circ u_1 + \ldots + x_n \circ u_n$. Then

$$w^\alpha = x_1^\alpha \circ u_1^{\beta_0} + \ldots + x_n^\alpha \circ u_n^{\beta_0} = x_1^\alpha \circ u_1^{\beta_0} + \ldots + x_n^\alpha \circ u_n^{\beta_0}. $$

From this follows that $w^\alpha \equiv 0$ holds in $(V, G)$ if and only if the same holds in $(V, G_{\overline{G}})$. In other words,

$$\mu = (\alpha, \beta \beta_0) \in Val_{(V, G)}(w \equiv 0) \iff \mu_0 = (\alpha, \beta) \in Val_{(V, G)}(w \equiv 0)$$

holds for every formula of the type $w \equiv 0$. Prove by induction, that this is true for every action-type formula. Let it be true for action-type formulas $u$ and $v$. Pass to $u \lor v$, $u \land v$, $\neg u$, $\exists x \ u$. Let $\mu \in Val_{(V, G)}(u \lor v) = Val_{(V, G)}(u) \lor Val_{(V, G)}(v)$ and let, say, $\mu \in Val_{(V, G)}(u)$. Then $\mu_0 \in Val_{(V, G)}(u)$,
\( \mu_0 \in Val_{(V,G)}(u \lor v) \). Similarly we check that \( \mu_0 \in Val_{(V,G)}(u \lor v) \implies \mu \in Val_{(V,G)}(u \lor v) \). Just the same for the case \( u \land v \).

Let now \( \mu \in Val_{(V,G)}(\neg u) = Val_{(V,G)}(u) \), i.e., \( \mu \notin Val_{(V,G)}(u) \). Hence, \( \mu \notin Val_{(V,G)}(u) \), \( \mu_0 \in Val_{(V,G)}(\neg u) \). The same for the opposite direction.

Consider now \( \exists xu. \) Let \( \mu \in Val_{(V,G)}(\exists xu) = \exists xu Val_{(V,G)}(u) \). Find \( \nu = (\alpha_1, \beta_0) \in Val_{(V,G)}(u) \), \( \alpha_1(x_1) = \alpha(x_1), x_1 \neq x \). Let us note here, that the induction proceeds by any pairs \( \mu \) and \( \mu_0 \) fitting the diagram. Take \( \nu_0 = (\alpha_1, \beta), \nu = \nu_0(1, \beta_0) \). Inclusion \( \nu \in Val_{(V,G)}(u) \) is equivalent to the inclusion \( \nu_0 \in Val_{(V,G)}(u) \). Since \( \alpha_1(x_1) = \alpha(x_1) \) for \( x_1 \neq x \), we conclude that \( \mu_0 \in Val_{(V,G)}(\exists xu) \). Similarly, \( \mu_0 \in Val_{(V,G)} \) implies \( \mu \in Val_{(V,G)} \).

Assume now that the class \( \mathcal{X} \) is given by some set of action-type formulas. Each of these formulas holds in the representation \((V,G)\) if and only if it holds in the representation \((V,G)\). Hence, the class \( \mathcal{X} \) is saturated.

Check further that the class is right-hereditary. Take \((V,G)\) and \((V,H)\), \( H \subset G \). For every action-type formula \( u \) verify that

\[
Val_{(V,H)}(u) = Val_{(V,G)}(u) \cap Hom(W, (V,H)).
\]

Again we apply induction. The case when \( u \) is an equality \( w \equiv 0 \) is easily checked. It is also easily verified that if \( u \) and \( v \) meet the condition, then the same is true for \( u \lor v, u \land v \) and \( \neg u \). It is left to check \( \exists xu \) if there is an equality for \( u \). Take \( \mu = (\alpha, \beta) \in Val_{(V,H)}(\exists xu) = \exists xu Val_{(V,H)}(u) \). Then \( \mu \in Hom(W, (V,H)) \) and there is an element \( \nu = (\alpha_1, \beta) \) with \( \alpha_1(x_1) = \alpha(x_1) \) if \( x_1 \neq x \), \( \nu \in Val_{(V,H)}(u) \). By the condition, \( \nu \in Val_{(V,G)}(u) \) and \( \mu \in \exists xu Val_{(V,G)} = Val_{(V,G)}(\exists xu) \), \( \mu \in Val_{(V,G)}(\exists xu) \cap Hom(W, (V,H)) \). Conversely, let \( \mu \in Val_{(V,G)}(\exists xu) \cap Hom(W, (V,H)) \), \( \mu = (\alpha, \beta) \) and \( \beta \) is a homomorphism \( F \to H \). Find \( \nu = (\alpha, \beta) \in Val_{(V,G)}(u) \cap Hom(W, (V,H)) = Val_{(V,H)}(u), \alpha_1(x_1) = \alpha(x_1), x_1 \neq x \). Now \( \mu \in Val_{(V,H)}(\exists xu) \). The equality is checked. Assume now, that the action-type formula \( u \) holds in the representation \((V,G), Val_{(V,G)}(u) = Hom(W, (V,G)) \). We have \( Val_{(V,H)}(u) = Hom(W, (V,G)) \cap Hom(W, (V,H)) = Hom(W, (V,H)) \), and \( u \) holds in \((V,H)\). Thus, the class \( \mathcal{X} \) is right-hereditary.

Now it is only left to prove that the class \( \mathcal{X} \) is right-local. As earlier, assume that \( \mathcal{X} \) is given by action-type formulas. We need to show, that if \( u \) is action-type formula, \((V,G)\) is a representation, \( u \) holds on every \((V,H)\) where \( H \) is finite-generated subgroup in \( G \), then \( u \) holds in \((V,G)\) as well.

We need some auxiliary material. Let a homomorphism \( \mu = (\alpha, \beta) : W \to (V,G), W = W(X,Y) = (XKF, F) \) be given. Take a subset \( Y_0 \) in \( Y \)
and change $\beta$ by $\beta' : F \to G$, coinciding with $\beta$ on $Y_0$ and sending all the rest $y$'s to the unit. Take further $\mu' = (\alpha', \beta')$, where $\alpha'$ coincides with $\alpha$ on $X$. Take now an arbitrary $w = x_1 \circ u_1 + \ldots x_n \circ u_n$ such that supports of all $u_i$ belong to $Y_0$ (in other words all elements $u_i$ are expressed via variables from $Y_0$). Then

$$w^{\alpha'} = x_1^{\alpha'} \circ u_1^{\beta'} + \ldots x_n^{\alpha'} \circ u_n^{\beta'} = x_1^{\alpha} \circ u_1^{\beta} + \ldots x_n^{\alpha} \circ u_n^{\beta} = w^\alpha.$$  

For an arbitrary action-type formula $u$ consider its $Y$-support $\Delta_Y(u)$. We have:

$$\Delta_Y(u) = \Delta_Y(\neg u) = \Delta_Y(\exists xu),$$

$$\Delta_Y(u \lor v) = \Delta_Y(u \land v) = \Delta_Y(u) \cup \Delta_Y(v).$$

Let now $\mu = (\alpha, \beta)$ be a homomorphism and $u$ an action-type formula, $Y_0 \supset \Delta_Y(u)$, $Y_0$ be a finite set. Pass to $\mu' = (\alpha', \beta')$ by $Y_0$. The following property always takes place in these conditions:

$$\mu = (\alpha, \beta) \in Val_{(V,G)}(u) \iff \mu' = (\alpha', \beta') \in Val_{(V,G)}(u).$$

Let us check it. The property is evident for identities. Now let it be true for some action-type formula $u$. Pass to $\neg u$ and $\exists xu$. We have $\Delta_Y(\neg u) = \Delta_Y(\exists xu) = \Delta_Y(u)$. Take an arbitrary finite $Y_0$, including these supports. Let

$$\mu \in Val_{(V,G)}(\neg u), \text{ i.e., } \mu \not\in Val_{(V,G)}(u).$$

Then $\mu' \not\in Val_{(V,G)}(\neg u)$ and $\mu' \in Val_{(V,G)}(\neg u)$. The opposite is checked similarly.

Pass to $\exists xu$ with the same $Y_0$. Given $\mu = (\alpha, \beta) \in Val_{(V,G)}(\exists xu) = \exists x Val_{(V,G)}(u)$, we have $\mu' = (\alpha', \beta')$ by $Y_0$. Besides that, we can select $\mu_1 = (\alpha_1, \beta)$ such that $\mu_1 = \in Val_{(V,G)}(u)$, $\alpha_1(x_1) = \alpha(x_1)$ for $x_1 \not\in x$. Take once more $\mu'_1 = (\alpha'_1, \beta')$ for $\mu_1 = (\alpha_1, \beta)$ by $Y_0$. Now $\mu_1 \in Val_{(V,G)}(u)$ implies $\mu'_1 \in Val_{(V,G)}(u)$. Recall that $\alpha$ and $\alpha'$ coincide on the set $X$, as well as $\alpha_1$ and $\alpha'_1$. Then for $x_1 \not\in x$ we have $\alpha'_1(x_1) = \alpha_1(x_1) = \alpha(x_1)\alpha'(x_1)$. Comparing $\mu'$ with $\mu'_1$ we conclude: $\mu' \in Val_{(V,G)}(\exists xu)$. Similarly we derive $\mu' \in Val_{(V,G)}(\exists xu) \Rightarrow \mu \in Val_{(V,G)}(\exists xu).$
Let now the property holds for $u$ and $v$. Check $u \lor v$, $u \land v$. Take $Y_0 \supset \Delta_Y(u \lor v) = \Delta_Y(u) \lor \Delta_Y(v)$. Take $\mu = (\alpha, \beta)$ and pass to $\mu' = (\alpha', \beta')$ by $Y_0$.

Given $\mu \in \text{Val}_{(V,G)}(u \lor v) = \text{Val}_{(V,G)}(u) \lor \text{Val}_{(V,G)}(v)$. Let $\mu \in \text{Val}_{(V,G)}(u)$. Since $Y_0 \supset \Delta_Y(u)$, then $\mu' \in \text{Val}_{(V,G)}(u)$ and $\mu' \in \text{Val}_{(V,G)}(u \lor v)$. The rest is done in the same way. The property is checked for every $u$.

We continue the proof of the theorem. Let the formula $u$ be hold in every $(V,H)$. We have: $\text{Val}_{(V,H)}(u) = \text{Val}_{(V,G)}(u) \cap \text{Hom}(W,(V,H))$. Take an arbitrary $\mu = (\alpha, \beta)$ and pass to $\mu' = (\alpha', \beta')$ by $u$. Take $\text{Im} \beta'$ as $H$. Then $\mu' \in \text{Hom}(W,(V,H)) = \text{Val}_{(V,H)}(u)$. But then $\mu' \in \text{Val}_{(V,G)}(u)$ and this gives $\mu \in \text{Val}_{(V,G)}(u)$. This is true for every $\mu$ and $\text{Val}_{(V,G)}(u) = \text{Hom}(W,(V,G))$. The formula $u$ holds in the representation $(V,G)$.

Let now the class $\mathfrak{X}$ is given by the set $T$ of action-type formulas and for the representation $(V,G)$ every representation $(V,H)$ belongs to $T^*$, i.e., satisfies all $u \in T$. Then all these $u$ are fulfilled in $(V,G)$ and $(V,G)$ belongs to $\mathfrak{X}$. Hence, the goal local property takes place and the theorem is proved.

There arises the following main

**Problem 1.** Is it true that an axiomatized class $\mathfrak{X}$ is action-type axiomatized if and only if it is saturated, right-hereditary and right-local?

Theorem 2.1 solves the question in one direction, but the opposite claim requires additional considerations.

### 2.2 Theorem

**Theorem 2.** For varieties, pseudovarieties, quasivarieties and universal classes of representations Problem 1 has a positive solution.

Proof.

The case of varieties is studied in [PV]. Apply the general approach, also outlined in [PV].

Let $\mathfrak{X}$ be a saturated class and $G$ a group. There are representations with the acting group $G$ in $\mathfrak{X}$. Denote by $\mathfrak{X}_G$ a $G$-layer in $\mathfrak{X}$, all $(V,G)$ in $\mathfrak{X}$ are with the given group $G$.

Let now $\mathfrak{X}$ be a pseudovariety satisfying the closure conditions from Problem 1. We need to show that $\mathfrak{X}$ is given by action-type pseudoidentities. Show first that the class $\mathfrak{X}$ with these closure conditions is a pseudovariety if and only if every layer $\mathfrak{X}_G$ is a pseudovariety of $G$-modules.
Recall that the class of algebras $\Theta$ is a pseudovariety if and only if it is closed in respect to ultra-products, subalgebras and homomorphic images (for one-sorted algebras see [Ma], the generalization for multi-sorted ones see in [Gva]).

Let now $\mathcal{X}$ be a pseudovariety of representations and let this class be saturated, right-hereditary and right-local. Pass to the layer $\mathcal{X}_G$. It is evident that it is closed under subalgebras and homomorphisms and we need to check that it is closed under ultra-products.

Let $I$ be a set and $D$ a filter of subsets in $I$. We have a representation $(V_\alpha, G)$ for every $\alpha \in I$. Take a cartesian product $((\prod_\alpha V_\alpha), G^I)$ and let $(V_0, H)$ be a congruence determined by the filter $D$. The corresponding filtered product is $((\prod_\alpha V_\alpha)/V_0, G^I/H)$. It belongs to the class $\mathcal{X}$ if $D$ is an ultrafilter. Since $\mathcal{X}$ is saturated, the representation $(\prod_\alpha V_\alpha)/V_0, G^I)$ also belongs to $\mathcal{X}$. Embed now $G \rightarrow G^I$ as a diagonal. Since $\mathcal{X}$ is hereditary, the $((\prod_\alpha V_\alpha)/V_0, G)$ belongs to $\mathcal{X}$ and, consequently, to $\mathcal{X}_G$. It is easy to understand, that this representation is isomorphic to ultra-product by the filter $D$ of all $(V_\alpha, G)$ with the fixed $G$. Hence, all $\mathcal{X}_G$ are pseudovarieties.

Let now the class $\mathcal{X}$ be saturated, right-hereditary and all $\mathcal{X}_G$ be pseudovarieties. Show that $\mathcal{X}$ are pseudovarieties. Check first, that the class $\mathcal{X}$ is hereditary and closed under homomorphisms. Take $(V, G) \in \mathcal{X}$ and let $(V_0, H)$ be a subrepresentation. From right-hereditaryty follows that $(V_0, G)$ belongs to $\mathcal{X}$. Consider further a layer $\mathcal{X}_H$. It is a pseudovariety and, consequently, $(V_0, H) \in \mathcal{X}_H$ and $(V_0, H) \in \mathcal{X}$.

Let now a surjective homomorphism $\mu = (\alpha, \beta) : (V, G) \rightarrow (V_1, G_1)$, $(V, G) \in \mathcal{X}$ be given. The homomorphism $\beta : G \rightarrow G_1$ determines the representation $(V_1, G)$. We have a homomorphism $(\alpha, 1) : (V, G) \rightarrow (V_1, G)$. Since the layer $\mathcal{X}_G$ is closed under homomorphisms, then $(V_1, G) \in \mathcal{X}_G$ and $(V_1, G) \in \mathcal{X}$. The class $\mathcal{X}$ is saturated. This means that $(V_1, G_1) \in \mathcal{X}$.

It is left to check that the class $\mathcal{X}$ is ultra-closed. Let $I$ be a set, $D$ ultrafilter on $I$. Given $(V_\alpha, G_\alpha) \in \mathcal{X}$, $\alpha \in I$, take $G = \prod_\alpha G_\alpha$. We have $\pi_\alpha : G \rightarrow G_\alpha$ determining the representation $(V_\alpha, G)$. It is saturated. Consider the layer $\mathcal{X}_G$ and take in it an ultra-product by $D$. We have: $((\prod V_\alpha)/D, G) \in \mathcal{X}_G$ and hence it belongs to $\mathcal{X}$. Ultra-product by $D$ of all $(V_\alpha, G_\alpha)$ is a homomorphic image of the representation $((\prod V_\alpha)/D, G)$. Since we have checked closure in respect to homomorphisms, then the corresponding ultra-product contains in $\mathcal{X}$. Hence, $\mathcal{X}$ is a pseudovariety with the given closure conditions.

Let further $\mathcal{X}$ be a pseudovariety. Prove that $\mathcal{X}$ is determined by action-
type pseudo-identities. Take a countable set $Y$ and a free group $F = F(Y)$. Consider a layer $\mathfrak{X}_F$. It is a pseudovariety and it is determined by pseudo-identities. Take the corresponding set of pseudo-identities and let it be indexed by the set $I$. We identify $I$ with that set.

Consider pseudo-identities

$$w_i^k \equiv 0 \lor \ldots \lor w_{n_k}^k \equiv 0, \quad k \in I.$$ 

Denote by $u^k$ the given $k$-th pseudo-identity, $u^k = u^k(x_1, \ldots, x_n; y_1, \ldots, y_m)$. Besides, we have $w_i^k = w_i^k(x_1, \ldots, x_n; y_1, \ldots, y_m)$. We consider all $x_1, \ldots, x_n$ as variables while $y_1, \ldots, y_m$ are constants. Show that the same set $I$ with the varying $y$'s determines the class $\mathfrak{X}$. Take a representation $(V, F)$ and show that it belongs to the class $\mathfrak{X}$ if and only if all pseudo-identities of the given set hold in it.

Let $(V, G) \in \mathfrak{X}$. Consider a homomorphism $\mu = (\alpha, \beta) : W = W(X, Y) = (X K F, F) \to (V, G)$. Using $\beta : F \to G$, determine a representation $(V, F)$. The case of pseudovarieties is completed. The reasoning of the same type is used for quasi-varieties and universal classes.

Consider pseudovarieties in more detail, taking into account their special role in algebraic geometry [PT].

It is known (see [Ma], [GL]) that if $\mathfrak{X}$ is a class of algebras and $qVar(\mathfrak{X})$ is a quasivariety generated by the class $\mathfrak{X}$, then

$$qVar(\mathfrak{X}) = SCC_{up}(\mathfrak{X}).$$
Here $S$ is an operator of taking of subalgebras, $C$ cartesian products and $C_{up}$ an operator of taking ultra-products. This implies also that it is enough to check that the corresponding class is closed in respect to the operator $S$ and arbitrary filtered products. Besides, we need to assume that there are one-element subalgebras in the class.

Using all this, prove the theorem for quasivarieties. Let us prove that if $\mathcal{X}$ is a saturated, right-hereditary and right-local quasivariety, then this $\mathcal{X}$ is an action-type quasivariety.

As earlier, select the following two items of the proof. Check first of all that if $\mathcal{X}$ satisfies the pointed closure conditions, then $\mathcal{X}$ is a quasivariety if and only if every layer $\mathcal{X}_G$ is a quasivariety. Then using the layer $\mathcal{X}_F$ and closure conditions, prove that the class $\mathcal{X}$ is determined by action-type quasivarieties. The first item of the proof copies the one for pseudo-varieties with the only difference that we work with filtered products instead of cartesian and ultra-products. The second item follows the idea used for pseudo-varieties. The same reasoning with two items we use for universal classes. Now the theorem 2.2 is proved.

3 General theorem

3.1 Some general considerations

In the previous theorem we used the fact that if the class $\mathcal{X}$ is given by axioms of a special kind, then every layer $\mathcal{X}_G$ is determined by the axioms of the same kind as well. However we cannot say anything similar if the type of the axioms is not fixed. We cannot claim, that if the class $\mathcal{X}$ is axiomatized and meets the demanded closure conditions, then the layer $\mathcal{X}_F$ is also axiomatized. We will study this obstacle later on.

Let us note that it is convenient to use the following terminology. We call a class $\mathcal{X}$ strictly saturated if it is saturated, right-hereditary and right-local.

3.2 The theorem

Theorem 3. Let $\mathcal{X}$ be an axiomatized class of representations and a layer $\mathcal{X}_F$ also axiomatized. Such $\mathcal{X}$ is action-type axiomatized if and only if it is strictly saturated.
Proof. Let us make some additional remarks on the layer $\mathfrak{X}_F$. First of all, we denote by $\text{Rep} - KF$ the class of all $K$-representations of the free group $F$. The class $\text{Rep} - KF$ of all $(V, F)$ is a subvariety in $\text{Rep} - K$.

Every representation $(V, F)$ we consider now as a one-sorted algebra, whose morphisms have the type $\alpha = (\alpha, 1) : (V, F) \rightarrow (V_1, F)$, and they commute with the action of the group $F$. The free representation in $\text{Rep} - KF$ is the same $(XKF, F)$. Logic of the representation $\text{Rep} - KF$ consists of action-type formulas but we consider these formulas from another perspective: we view variables of the set $Y$ as constants, they are immutable. If $u$ is an action-type formula in the logic over $\text{Rep} - K$, then the same $u$ in the logic $\text{Rep} - KF$ is denoted by $u^0$. If $T$ is the set of action-type formulas, then correspondingly we consider $T^0$. Let the set $T$ determines a class $X$. Whether we can claim that $T^0$ determines the layer $\mathfrak{X}_F$? Probably not. We can claim that every $u^0 \in T^0$ holds in every $(V, F) \in \mathfrak{X}_F$, but we cannot claim that if $T^0$ holds in some $(V, F)$, then $T$ holds in $(V, F) \in \mathfrak{X}$ as well. There are no general arguments for the feature of the layer $\mathfrak{X}_F$ to be axiomatizable.

Let us pass to the proof of the theorem 3.2. Let the conditions of this theorem hold for $X$ and the layer $\mathfrak{X}_F$ is axiomatized by a set of formulas $T^0$ where $T$ is a set of action-type formulas, then correspondingly we consider $T^0$. Let the set $T$ determines a class $\mathfrak{X}$. Whether we can claim that $T^0$ determines the layer $\mathfrak{X}_F$? Probably not. We can claim that every $u^0 \in T^0$ holds in every $(V, F) \in \mathfrak{X}_F$, but we cannot claim that if $T^0$ holds in some $(V, F)$, then $T$ holds in $(V, F) \in \mathfrak{X}$ as well. There are no general arguments for the feature of the layer $\mathfrak{X}_F$ to be axiomatizable.

Let us pass to the proof of the theorem 3.2. Let the conditions of this theorem hold for $\mathfrak{X}$ and the layer $\mathfrak{X}_F$ is axiomatized by a set of formulas $T^0$ where $T$ is a set of action-type formulas. Show that this $T$ determines the class $\mathfrak{X}$. We use the same reasoning as before.

Take a representation $(V, G)$. Let it belong to $\mathfrak{X}$. Check that all formulas from $T$ hold in $(V, G)$.

Consider a homomorphism

$$\mu = (\alpha, \beta) : (XKF, F) \rightarrow (V, G).$$

Applying $\beta : F \rightarrow G$, pass to the representation $(V, F)$. Let $\text{Im} \beta = H$. We have $(V, H)$, and this representation belongs to $\mathfrak{X}$ since it is right-hereditary. The class $\mathfrak{X}$ is saturated, hence the representation $(V, F)$ belongs to $\mathfrak{X}$. But then it belongs to the layer $\mathfrak{X}_F$ and all axioms of the set $T^0$ hold in it.

Here we deviate a bit from the main stream and consider the following property.

Given a homomorphism $\mu = (\alpha, \beta) : (XKF, F) \rightarrow (V, G)$, pass, as earlier, to $(V, F)$. Then $\alpha \in \text{Val}_{(V, F)}(u^0) \iff \mu \in \text{Val}_{(V, G)}(u)$ for every action-type formula $u$. Prove this by induction. Let first $u$ be $u \equiv 0$, $w = x_1 \circ u_1 + \ldots + x_n \circ u_n$. Then $w^\alpha = x_1^\alpha \circ u_1 + \ldots + x_n^\alpha \circ u_n = x_1^\alpha \circ u_1^\beta + \ldots + x_n^\alpha \circ u_n^\beta$. Thus, $w^\alpha = 0$ holds in $(V, F)$ if and only if the same holds in $(V, G)$. The property is checked for elementary formulas.
Let the property be checked for some \( u \). Verify it for \( \neg u \) and \( \exists xu \). Take \( \alpha \in Val_{(V,F)}(\neg u) \), i.e., \( \alpha \not\in Val_{(V,F)}(u^0) \), and \( \mu \not\in Val_{(V,G)}(u) \), i.e., \( \mu \in Val_{(V,G)}(\neg u) \). The same for the opposite direction.

Let now \( \alpha \in Val_{(V,F)}(\exists xu) = \exists xVal_{(V,F)}(u^0) \). Find \( \alpha' \in Val_{(V,F)}(u^0) \) with \( \alpha'(x_1) = \alpha(x_1) \) for \( x_1 \neq x \). We have \( \mu = (\alpha, \beta) \). Take \( \mu' = (\alpha', \beta) = (\alpha, 1)(1, \beta) \). Since \( \alpha' \in Val_{(V,F)}(u^0) \) and \( (V,F) \) is built from \( (V,G) \) by \( \beta \), then \( \mu' \in Val_{(V,G)}(u) \). Together with \( \alpha'(x_1) = \alpha(x_1) \) this gives \( \mu \in Val_{(V,G)}(\exists xu) \).

Let \( \mu \in Val_{(V,G)}(\exists xu) \), \( \mu = (\alpha, \beta) \). Take \( \mu' = (\alpha', \beta) \in Val_{(V,G)}(u) \), \( \alpha'(x_1) = \alpha(x_1) \), \( x_1 \neq x \). We have \( \alpha' \in Val_{(V,F)}(u^0) \) and \( \alpha \in Val_{(V,F)}(\exists xu) \). The cases \( u \lor v \) and \( u \land v \) are checked in the same way. The property is checked.

Let us return to the proof of the theorem. Let now \( u \in T \), \( u^0 \in T^0 \). Then the formula \( u^0 \) holds in the representation \( (V,F) \), and for every \( \alpha : (XKF,F) \to (V,F) \) we have \( \alpha \in Val_{(V,F)}(u^0) \). Apply this to the initial \( \mu = (\alpha, \beta) \). Then \( \mu \in Val_{(V,G)}(u) \). This holds for every \( \mu \) and \( Val_{(V,G)}(u) = Hom(W,(V,G)) \). The formula \( u \) holds in \( (V,G) \).

Let now the set \( T \) holds in \( (V,G) \). We need to check that \( (V,G) \in \mathfrak{X} \). Since \( \mathfrak{X} \) is right-local, it is enough to verify that \( (V,H) \in \mathfrak{X} \) for every finitely generated subgroup \( H \subset G \).

Consider such \( H \) and a surjection \( \beta F \to H \). Take \( (V,F) \) by it. For an arbitrary \( \alpha X \to V \) and pass to \( KF \)-homomorphism \( \alpha : XKF \to V \). We have also \( \mu = (\alpha, \beta) : W \to (V,G) \).

As we have seen, \( \alpha \in Val_{(V,F)}(u^0) \iff \mu = (\alpha, \beta) \in Val_{(V,G)}(u) \) holds for every formula \( u^0 \in T^0 \), \( u \in T \). Since \( \mu \in Val_{(V,G)}(u) \) for every \( \mu \), then \( \alpha \in Val_{(V,F)}(u^0) \) for every \( \alpha : XKF \to V \). This means that the set \( T^0 \) holds in \( (V,F) \). Since \( T^0 \) determines the layer \( \mathfrak{X}_F \), then \( (V,F) \in X_F \), and, hence, \( (V,F) \in \mathfrak{X} \). This implies that \( (V,H) \in \mathfrak{X} \) and, therefore, we have \( (V,G) \in \mathfrak{X} \).

The theorem is proved in one direction, and what’s more, the initial class \( \mathfrak{X} \) need not be axiomatizable. The opposite direction follows from the theorem 2.1, and here we do not demand the layer \( \mathfrak{X}_F \) to be axiomatizable.

The main Problem 1 remains open in full generality.

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