The Quantum Hall Effect of Interacting Electrons in a Periodic Potential

Daniela Pfannkuche\textsuperscript{a,b} and A. H. MacDonald\textsuperscript{a}

\textsuperscript{a}Department of Physics, Indiana University, Bloomington, Indiana 47405, U.S.A.
\textsuperscript{b}Max-Planck-Institut für Festkörperforschung, Heisenbergstr. 1, D-50689 Stuttgart, Germany

(Received July 14, 2021)

We consider the influence of an external periodic potential on the fractional quantum Hall effect of two-dimensional interacting electron systems. For many electrons on a torus, we find that the splitting of incompressible ground state degeneracies by a weak external potential diminishes as $\exp(-L/\xi)$ at large system size $L$. We present numerical results consistent with a scenario in which $\xi$ diverges at continuous phase transitions from fractional to integer quantum Hall states which occur with increasing external potential strength.

Theoretical studies of the quantum Hall effect (QHE) have identified three different sources which can act separately or in concert to create the charged excitation energy gap responsible for dissipationless transport in a two-dimensional electron system (2DES). Kinetic energy quantization is usually primarily responsible for the integer QHE (IQHE), while interaction energy quantization is responsible for the fractional QHE (FQHE). Integer quantum Hall gaps at fractional Landau level filling factors ($\nu$) can also arise from periodic external potentials. Although this instance has not yet been realized experimentally, it has played an important role in theoretical developments, especially in giving rise to the topological picture of the QHE. In addition, considerations of the periodic external potential case have cautionary implications for the comprehensiveness of some theoretical pictures of the bulk QHE.

In this paper we address the competition between electron-electron interactions and a periodic external potential in determining the value of the quantized Hall conductance of a 2DES in the strong magnetic field ($B$) limit where all electrons lie in the lowest Landau level and Landau level mixing can be neglected. Interactions alone give rise to charge gaps at a set of rational values of $\nu$, all of which have odd denominators, and lead to fractionally quantized Hall conductivity values $\sigma_H = (e^2/h)\nu$. A periodic external potential alone splits the Landau level into subbands in an intricate way giving rise to the captivating ‘Hofstadter butterfly’ illustrations of the magnetic field dependence of the one-body spectral support. In the non-interacting electron limit, the charge gaps are the gaps in the one-body spectrum. For a given unit cell area, $A_0$, gaps occur at $\nu = \sigma + sA_0/A_0$ where $\sigma$ and $s$ are integers which depend in detail on the periodic potential, and $A_0 = \Phi_0/B = 2\pi\ell^2$ is the area penetrated by the magnetic flux quantum $\Phi_0 = hc/e$. In this case the Hall conductivity has integer quantization, $\sigma_H = (e^2/h)\sigma$. Progress in nanolithography is beginning to allow the fabrication of systems in which physical consequences of this complicated electronic structure can be studied experimentally, although the quantum Hall (QH) regime has not quite been reached. Addressing the role played by inescapable electron-electron interactions in the QH regime is a principle motivation for the present work. The charge gaps of the non-interacting periodic potential limit clearly survive to finite interaction strengths since, when all occupied subbands are filled, the non-interacting many-particle ground state is non-degenerate and separated from many-particle excited states by a gap. The weak periodic potential limit is addressed below.

The QHE of interacting electrons in a periodic external potential is most succinctly discussed by combining the toroidal geometry, in which quasiperiodic boundary conditions are applied to a finite area, with the topological picture of the quantized Hall conductance. In this picture the Hall conductance at zero temperature in $e^2/h$ units is given by the integer valued Chern index which expresses the adiabatic evolution of the phase of the ground state wavefunction under cyclic evolution of the boundary condition phases. The occurrence of a FQHE requires the many-particle ground state to be degenerate, since in that case the quantized Hall conductance depends on the average Chern number of the degenerate states. Indeed, the necessary degeneracies are a consequence of continuous translational invariance for many-electron systems on a torus and occur for every state in the Hilbert space whether or not there is a charge gap.

Since we restrict our attention here to magnetic field strengths for which charge gaps occur in both non-interacting modulated and interacting isotropic (i.e., zero periodic potential) limits, we consider only rational filling factors $\nu = N_e/N_\phi = q/p$ where $p$ is odd. (Here $N_e$ is the number of electrons in a finite area ($A$) system and $N_\phi = AB/\Phi_0$ is the number of states in the Landau level.) A gap can be created by a periodic potential alone at $\nu = q/p$ only if $A_\phi/A_0 = t/p$ for some integer $t$. A weak periodic potential will have no effect on the FQHE provided that the ground-state degeneracy splitting which it produces vanishes in the limit of an infinite system size and the fractional charge gap remains finite. In the following paragraphs we use this criterion to demonstrate the stability of the FQHE against weak lateral potentials.

We consider a weak periodic potential of the form which has been most extensively investigated for non-interacting electron systems: $U(\vec{r}) = U_0[\cos(2\pi x/a_x) + \cos(2\pi y/a_y)]$. To be compatible with quasi-periodic
boundary conditions we choose a rectangular finite area system with sides $L_x = N_x a_x$ and $L_y = N_y a_y$, where both $N_x$ and $N_y$ are integers, $N_x N_y = t N_x / q$, and use a Landau gauge in which the vector potential is independent of the $y$ coordinate. Our analysis is based on three elementary properties which follow from translational symmetry considerations in the isotropic system limit: i) Isotropic system states can be labeled by a two-dimensional wavevector $\vec{K}$ in a rectangular magnetic Brillouin-zone with $K_x \in [0, L_y / p \ell^2]$ and $K_y \in [0, L_x / p \ell^2]$; ii) Pseudomomentum is conserved so that $(\vec{K} + \vec{i} / \rho_{\vec{G}} | \vec{K} \vec{i})$ is non zero only if $\vec{K} = \vec{K} + \vec{p}$; iii) Translation by $(L_x q / p \ell^2) \vec{y}$ in momentum space corresponds to rigid spatial translations by $(2 \pi \ell^2 / L_y) \vec{y}$ so that $E(\vec{K}, i) = E(\vec{K} + L_x \vec{y} / p \ell^2, i)$ and $(\vec{K} + L_x \vec{y} / p \ell^2, i | \rho_{\vec{G}} \vec{K} + L_x \vec{y} / p \ell^2, i) = (\vec{K} + L_x \vec{y} / p \ell^2, i | \rho_{\vec{G}} \vec{K} + L_x \vec{y} / p \ell^2, i)$ up to a phase factor independent of $i$ and $\vec{i}$. Here all wavevectors are understood to be reduced to the magnetic Brillouin-zone where necessary and $\rho_{\vec{G}} \equiv \sum_j \exp(i \vec{p} \cdot \vec{r}_j)$ is the density operator. (The external potential contribution to the Hamiltonian is of the form $H’ = (U_0 / 2) \sum_{\vec{G}} \rho_{\vec{G}}$, where $\vec{G}$ is a nearest neighbor reciprocal lattice vector.) When the FQHE occurs the $p$ degenerate ground states (at $\vec{K} = \vec{K}_0 + m \vec{y} L_x / p \ell^2$ with $m = 0, \cdots, p - 1$) are separated from all other eigenstates by an energy gap $\Delta$ which remains finite in the thermodynamic limit.

When $H’$ is included, shifts in all eigenvalues occur starting at second order in perturbation theory. It follows from property iii) above that the shifts which appear at low order are identical for each member of the degenerate ground-state manifold. Splitting of these states can result only from terms in perturbation theory in which one member of the manifold is coupled to another via a series of intermediate states at energies above the gap. It follows from property ii) above that these terms first occur at order $M_y = (L_x / p \ell^2) / (2 \pi / a_y) = [N_x / t]$ for $y$-dependent terms in the external potential and at order $M_x = (L_y / p \ell^2) / (2 \pi / a_x) = [N_y / t]$ for $x$-dependent terms in the external potential. (Here $|i/j|$ denotes the numerator of $i/j$ after elimination of common divisors.) For $U_0 << \Delta$ and fixed aspect ratio the splitting of the $p$-fold degenerate ground state manifold will therefore be $\sim \Delta (U_0 / \Delta)_{\text{min}} (M_y, M_x) \sim \Delta \exp(-L / \xi)$ where $L$ is the system size and $\xi \sim a / \ln (\Delta / U_0)$. Hence, the FQHE will survive in an infinite system as long as $U_0$ is small compared to the fractional gap $\Delta$. The exponential dependence of splitting on system size we find is much weaker than the Gaussian dependence $\sim \exp(-\nu^2 L^2 / 4 \ell^2)$ which occurs at the lowest order of perturbation theory for a random external potential and is in agreement with expectations from more heuristic arguments which apply equally well to random external potentials and are based on effective theories of the FQHE.

This analysis suggests a scenario in which a continuous phase transition between weak and strong periodic potential QH states occurs at a critical value of $U_0 = U_c$ and that $\xi$ diverges as $U$ approaches $U_c$ from below. We have attempted to test this simplest picture for the transition between fractional and IQHEs, and to obtain a quantitative estimate of the modulation strength at which the putative transition occurs, by performing numerical exact diagonalization calculations at a series of $U_0$ values including Coulombic electron-electron interactions. Here we report results for $\nu = 1/3$ and $\phi_0 / A_0 = 1/3$; this case appears to offer the greatest promise for experimental study since large charge gaps occur in both weak and strong modulation limits. For these commensurability ratios, the zero temperature Hall conductivity will change from the fractionally quantized value $\sigma_H = 1/3 e^2 / h$ to the integer quantized value $\sigma_H = 0$ at the critical modulation strength. Typical finite size results for the evolution of the low energy portion of the spectrum with modulation potential strength are shown in Fig. 1. We expect that in the thermodynamic limit the splitting of the three lowest energy states should vanish and the correlation gap to the fourth lowest energy state should remain finite for $U_0 < U_c$. For $U_0 > U_c$ we expect the gap between the first and second lowest energy states, which is proportional to $U_0$ for large $U_0$, to be finite.

![Fig. 1. Low energy part of the 4-particle spectrum](image)

All the results reported here were obtained with $a_x = a_y = a$ and up to six electrons in a system with area $A = N_x N_y a^2$. For systems with up to five electrons the finite Hamiltonian matrix could be directly diagonalized, while for the largest systems the low lying eigenenergies were obtained using a block Lanczos procedure.

In interpreting these numerical results we concentrate...
on the energy difference $W_1$ ($W_i := E_i - E_0$, where $E_0$ is the ground state energy) between the ground state and the first excited state. Explicit perturbative calculations for the finite size system show that at small modulation strengths $W_1$ is given at leading order in perturbation theory by

$$W_1 \approx nN(V/\Delta(n, N))^N\Delta(n, N)/2$$  

(1)

where $n = \min(N_x, N_y)$, $N = \max(N_x, N_y)$, $V = U_0 \exp(-\pi t/2\rho)$ is the external potential strength corrected by the form factor for lowest Landau level wavefunctions, and $\Delta(n, N) = W_3(V = 0)$ is the finite size correlation gap. In the limit $V \gg \Delta$ we find that $W_1 = \Delta_H + \delta(n, N)$ where $\delta(n, N)$ is positive, and $\Delta_H := V(3 - \sqrt{3})/2$ is the gap between the first and second subband of the Hofstadter spectrum.

$$W_1 = (L^2)^{-1}Q(L^\frac{1}{\nu} \frac{V - V_c}{V_c})$$  

(2)

where $Q$ is the scaling function, $\nu$ is the correlation length critical exponent, and $z$ is the dynamical critical exponent. The small sizes of the systems for which we are able to numerically solve the many-electron problem limit the thoroughness with which we can test this ansatz. In an analysis which neglects aspect ratio dependence of $W_1$ we use $L = \sqrt{N_x^2 + N_y^2}$ as a measure of the system size. Since the fractional gap $\Delta(n, N)$ exhibits strong size and aspect ratio dependence, we perform the scaling analysis with the modulation strength measured in units of the finite-size gap defining $v = V_0/\Delta(n, N)$.

We see that the finite size numerical calculations reproduce the expected behavior in both weak and strong modulation potential limits. We have performed a finite size scaling analysis to show that these results are consistent with a continuous quantum phase transition occurring between fractional ($\sigma_H = 1/3e^2/h$) and integer ($\sigma_H = 0$) QH states at intermediate modulation strengths. We identify the inverse gap, $W_1^{-1}$, with the correlation time $\xi_T$ beyond which the properties of the system are sensitive to the periodic potential. The correlation time is finite for $V_0$ larger than a critical value $V_c$ and diverges in the thermodynamic limit as $V_0$ approaches $V_c$ from above. $W_1$ should obey the finite size scaling ansatz

$$(L^2)^{-1}Q(L^\frac{1}{\nu} \frac{V - V_c}{V_c})$$  

In Fig. 2 we plot $LW$ as a function of $v$ for various system geometries. Assuming that $z = 1$ as in other quantum Hall phase transitions, the approximate crossing of these curves allows us to estimate the critical dimensionless modulation strength to be $v_c \approx 0.63$. The smallness and limited range of our system sizes do not allow us to convincingly identify the correlation length critical exponent $\nu$. In Fig. 3 we show the scaling function $LW_1 = Q$ plotted as a function of $L(v - v_c)/v_c$, i.e. assuming $\nu = 1$. Deviations from scaling clearly seen in these curves become noticeably worse if $\nu$ is altered by more than $\sim 0.2$. 

FIG. 2. Gap between the ground state and the first excited state $W \equiv W_1$ scaled by the dimensionless system size $L = \sqrt{N_x^2 + N_y^2}$ as a function of the reduced modulation amplitude $v = V/\Delta(n, N)$. Different curves represent different system sizes. The approximate crossing of these curves allows us to estimate the critical reduced modulation strength $v_c$.

FIG. 3. Scaling behavior of the energy gap near the critical point. Curves for the $1 \times N$-system sizes are parallel, but shifted with respect to each other due to next-to-leading order finite size corrections. The $2 \times N$-systems show a different slope. Inset: Non-critical finite-size corrections in the argument of the scaling function improve the scaling behavior.
Data points for system sizes $N_x \times N_y = 1 \times 4$ and $1 \times 5$ fall on the same curve, consistent with the scaling ansatz. However, the curves for the $2 \times 2$ and $2 \times 3$, systems although parallel to each other, have a different slope than found for the $1 \times N$ systems. We can partially compensate for this large finite size effects by assuming a non-critical size dependence in the argument of the scaling function which is suggested by the small $V$ behavior of $W_I$ (see Eq. (4)). For example, plotting $Q$ as a function of $n^{1/3} L (v - v_c)/v_c$, as shown in the inset of Fig. 2, improves the scaling behavior somewhat.

Our perturbative expression for the correlation length $\xi$ suggests that the divergence of the fractional state correlation length $\xi$ and the vanishing of the fractional Hall gap ($\Delta = W_3$) should occur simultaneously. In our finite-size simulations we find that $\Delta$ has a minimum for $V_0 \sim V_c$, but at the system sizes we are able to study, no compelling evidence that these minimum gaps vanish in the thermodynamic limit emerges.

Finally we comment on the experimental implications of our work. Broadly we emphasize that interactions will always be important at weak modulation strengths in any experimental investigation of a Hofstadter butterfly system. More particularly, we propose that continuous phase transitions between integer and FQH ground states will frequently occur as modulation strengths are weakened. Numerical exact diagonalization calculations for very small systems, while not able to offer compelling evidence, are consistent with this suggestion. For the case of $\nu = 1/3$ and one electron per unit cell of the periodic potential we estimate that the phase transition will occur when the ratio of the modulation strength to the fractional Hall gap is $\sim 0.7$. $T \neq 0$ finite-temperature scaling analysis of the phase transition $\xi$ implies that the Hall conductivity will change from $\sigma_H = 1/3e^2/h$, to $\sigma_H = 0$ over an interval of modulation strength which vanishes as $T \to T_c$, becoming increasingly sharp as the temperature is lowered. If our limited finite size data is indicative, $\nu z \sim 1$ rather than $\sim 0.4$ as found experimentally in field driven integer quantum Hall transitions. Simultaneously, the longitudinal conductivity will change from $\sigma_I = 0$ (corresponding to dissipationless transport when the fractional gap separates the ground state from the excited states) to a finite value at the transition point and back to $\sigma_I = 0$ when the Hofstadter gap opens at modulation strengths larger than $V_c$. These behaviors would signal the continuous quantum phase transition envisaged here. For a system with a carrier density $n = 4 \times 10^{19} \text{cm}^{-2}$ the commensurability ratios studied here would be realized with a potential period $a \sim 50\text{nm}$ and a field $B \sim 5\text{Tesla}$. We would then predict that the transition between integer and fractional QHEs would occur for a modulation strength $\sim 0.1\text{meV}$. These periods, modulation strengths, and fields are not far removed from what can be achieved with current lithographic technology. The physics of the Hofstadter butterfly QH system, when it is finally realized, will be greatly enriched by electron-electron interaction effects.

This work was supported in part by the National Science Foundation under grant DMR-9416906. D.P. appreciates financial support by the Deutsche Forschungsgemeinschaft. The authors are grateful for helpful interactions with Steve Girvin, Qian Niu, Erik Sørensen and Ulrich Zülicke.

1. See for example A.H. MacDonald, in Les Houches Session LXI, Physique Quantique Mésoscopique, edited by E. Akkermans et al.
2. D.J. Thouless et al., Phys. Rev. Lett. 49, 405 (1982).
3. For recent experiments see T. Schössler et al., Europhys. Lett. 33, 683 (1996) and R.R. Gerhardts, D. Weiss, and U. Wulf, Phys. Rev. B 43, 5192 (1991); D. Weiss et al., Surf. Science 263, 314 (1992).
4. Q. Niu, D.J. Thouless, and Y.-S. Wu, Phys. Rev. B 31, 3372 (1985); J.E. Avron and R. Seiler, Phys. Rev. Lett. 54, 259 (1985); D.P. Arovas et al., Phys. Rev. Lett. 60, 619 (1988).
5. For non-interacting electrons in a periodic potential the bulk Hall conductivity can be evaluated from the Kubo formula. The Landauer-Buttiker edge state picture does not always yield values for the quantized Hall conductance in agreement with bulk Hall conductivities. See for example B. L. Johnson and G. Kirzenson, Phys. Rev. Lett. 69, 672 (1992); P. Streda et al., Phys. Rev. B 50, 11955 (1994) and work cited therein. It also appears unlikely that composite fermion Chern-Simons mean-field theory would generally yield correct Hall conductivity values in the presence of a periodic potential, although this question has not been explored in detail as far as we are aware.
6. D. Langbein, Phys. Rev. 180, 633 (1969); D.R. Hofstadter, Phys. Rev. B 14, 2239 (1976); D. Pfannkuche and R.R. Gerhardts, Phys. Rev. B 46, 12606 (1992).
7. A.H. MacDonald, Phys. Rev. B 28, 6713 (1983). Note that at rational values of $\nu$ this equation does not fix the integers $\sigma$ and $s$ uniquely.
8. The FQHE in a periodic potential has been studied previously by A. Kol and N. Read, Phys. Rev. B 48, 8890 (1993). These authors concentrated on the implications of Chern-Simons mean-field approximation for possible novel QH states which are stable only when both interactions and a periodic potential are present. See also Z. Téšanović, F. Axel, and B.I. Halperin, Phys. Rev. B 39, 8525 (1989).
9. Q. Niu, Phys. Rev. B 34, 5093 (1986).
10. F.D.M. Haldane, Phys. Rev. Lett. 55, 2095 (1985), F.D.M. Haldane and E.H. Rezayi, Phys. Rev. B 31, 2529 (1985).
11. D.J. Thouless, Phys. Rev. B 40, 12304 (1989).
12. X.G. Wen and Q. Niu, Phys. Rev. B 41, 9377 (1990).
13. The transition appears similar to the QH to Mott insulator transition studied by X.-G. Wen and Y.-S. Wu, Phys. Rev. Lett. 50, 1501 (1983). In our case, however, the insulating behavior at strong modulation does not require electron-electron interactions.
14. For a pedagogical review of finite size scaling analyses of continuous quantum phase transitions with applications to quantum Hall transitions see S.L. Sondhi et al., submitted to Rev. Mod. Phys. (1996).
15. H.P. Wei et al., Phys. Rev. Lett. 61, 1294 (1988), S. Koch et al., Phys. Rev. B 43, 6828 (1991).