Squares of congruence subgroups of the extended modular group

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SQUARES OF CONGRUENCE SUBGROUPS OF THE EXTENDED MODULAR GROUP

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Abstract. In this paper, we generalize some results related to the congruence subgroups of modular group $\Gamma$, given in [7] and [6] by Kiming, Schütt, and Verrill, to the extended modular group $\tilde{\Pi}$.

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1. INTRODUCTION

The modular group $\Gamma = PSL(2, \mathbb{Z})$ is the discrete subgroup of $PSL(2, \mathbb{R})$ generated by two linear fractional transformations

$$T(z) = \frac{1}{z}, \quad \text{and} \quad S(z) = -\frac{1}{z + 1}.$$

Then modular group $\Gamma$ has a presentation

$$\Gamma = \langle T, S \mid T^2 = S^3 = I \rangle \cong \mathbb{Z}_2 \ast \mathbb{Z}_3.$$

The extended modular group $\tilde{\Pi} = PGL(2, \mathbb{Z})$ has been defined by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the modular group $\Gamma$. The extended modular group $\tilde{\Pi}$ has a presentation, see [5].

$$\tilde{\Pi} = \langle T, S, R \mid T^2 = S^3 = R^2 = (RT)^2 = (RS)^2 = I \rangle \cong D_2 \ast \mathbb{Z}_2 \ast D_3.$$

Here $T$, $S$ and $R$ have matrix representations

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively (in this work, we identify each matrix $A$ in $GL(2, \mathbb{Z})$ with $-A$, so that they represent the same element of $PGL(2, \mathbb{Z})$). Thus the modular group $\Gamma = PSL(2, \mathbb{Z})$ is a subgroup of index 2 in the extended modular group $\tilde{\Pi}$.

Let us define $\tilde{\Pi}^m$ as the subgroup generated by the $m^{th}$ powers of all elements of $\tilde{\Pi}$, for some positive integer $m$. The subgroup $\tilde{\Pi}^m$ is called the $m^{th}$ power subgroup of $\tilde{\Pi}$. As fully invariant subgroups, they are normal in $\tilde{\Pi}$.
Then, power subgroups of the extended modular group \( \Pi \) were examined by Sahin, Ikikardes and Koruoglu in [10]. The authors showed that

\[
|\Pi : \Pi^2| = 4, \quad \Pi^2 = \Gamma^2, \\
\Pi^2 = \langle S, TST \mid (S)^3 = (TST)^3 = I \rangle = \mathbb{Z}_3 \rtimes \mathbb{Z}_3.
\]

Also, from [5], we have the following. Let

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

represent a general element of \( \Pi \). For each integer \( N \geq 1 \), we define

\[
\Pi(N) = \{ A \in \Pi \mid a \equiv d \equiv \pm 1 \text{ and } b \equiv c \equiv 0 \pmod{N} \},
\]

\[
\Gamma(N) = \Pi(N) \cap \Gamma.
\]

These are normal subgroups of finite index in \( \Pi \), and they are called as the principal congruence subgroups. If \( N > 2 \) then \( \Pi(N) = \Gamma(N) \) and if \( N = 2 \) then \( \Pi(2) \supseteq \Gamma(2) \geq \Pi(4) = \Gamma(4) \). A subgroup \( K \) of \( \Pi \) contains some \( \Pi(N) \) if and only if it contains some \( \Gamma(N) \). Such a subgroup \( K \) is called a congruence subgroup, and the level of \( K \) is the least \( n \) such that \( \Pi(N) \leq K \). Any other subgroup of finite index in \( \Pi \) is called a non-congruence subgroup.

The most important of the congruence subgroups of \( \Pi \) are

\[
\Pi_0(N) = \{ A \in \Pi \mid c \equiv 0 \pmod{N} \}
\]

and

\[
\Pi_1(N) = \{ A \in \Pi \mid a \equiv d \equiv \pm 1 \text{ and } c \equiv 0 \pmod{N} \}.
\]

From [9], it is known that

\[
\Pi_0(N) = \Gamma_0(N) \cup TR.\Gamma_0(N) \quad \text{and} \quad \Pi_1(N) = \Gamma_1(N) \cup TR.\Gamma_1(N).
\]

Also, it is clear that \( \Pi_1(N) < \Pi_0(N) \) and for \( N > 2 \), \( |\Pi_0(N) : \Pi_1(N)| = \varphi(N)/2 \) where \( \varphi \) is the Euler Phi function (for the index \( \Gamma_1(N) \) in \( \Gamma_0(N) \), see [4]).

On the other hand, in [7] and [6], Kiming Schüt, and Verrill studied lifts of projective congruence subgroups. Now, we recall the following information from [7]. For a subgroup \( A \) of \( SL(2, \mathbb{Z}) \) denote by \( \overline{A} \) the image of \( A \) in \( PSL(2, \mathbb{Z}) \). A lift of \( \overline{A} \) is a subgroup of \( SL(2, \mathbb{Z}) \) that projects to \( \overline{A} \) in \( PSL(2, \mathbb{Z}) \). A lift is called a congruence lift if it is a congruence subgroup.

In [7] and [6], the authors gave some consequences of their main results for the groups generated by squares of elements in congruence subgroups. These results are

a) \( \Gamma(N)^2 \) is a congruence if and only if \( N \leq 2 \).

b) All lifts of \( \Gamma_0(N) \leq PSL(2, \mathbb{Z}) \) are congruence subgroups of \( SL(2, \mathbb{Z}) \) if and only if either \( N \in \{3, 4, 8\} \) or if \( 4 \nmid N \) and all odd prime divisors of \( N \) are congruent to 1 modulo 4.
c) All lifts of $\Gamma^1(N) \leq PSL(2, \mathbb{Z})$ are congruence subgroups of $SL(2, \mathbb{Z})$ if and only if $N \leq 4$.

The congruence and principal congruence subgroups (especially, $\Pi(2)$, $\Gamma(2)$, $\Gamma_0(N)$ and $\Gamma^1(N)$) of $\Gamma$ and $\Pi$ have been studied from various aspects in the literature, for example, number theory, modular forms, modular curves, Belyi’s theory, graph theory, (please see [1], [2], [3] and [8]).

In this paper, we generalize the above results related with congruence subgroups of $\Gamma$, given in [7] and [6], to the extended modular group $\Pi$.

2. SQUARES OF CONGRUENCE SUBGROUPS OF $\Pi$

From [5], if $N > 2$ then $\Gamma(N) = \Gamma(N)$ and so $\Pi(N)^2 = \Gamma(N)^2$. Thus, if $N > 2$ then $\Pi^2(N)$ is not a congruence. Also, from [10] and [5], $\Pi^2(1) = \Pi'$ and $\Pi(6) \leq \Pi^2(1)$ and so $\Pi^2(1)$ is a congruence subgroup. Therefore we need the following theorem.

**Theorem 1.** $\Pi(2)^2 = \Pi(4)$.

**Proof.** We know that the group structure of $\Pi(2)$ is

$$\Pi(2) = \langle TR, RSTS, RS^2TS^2 \mid (TR)^2 = (RSTS)^2 = (RS^2TS^2)^2 = I \rangle \cong \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2.$$ 

Let $a = TR, b = RSTS, c = RS^2TS^2$. Then the quotient group $\Pi(2)/\Pi(2)^2$ is the group obtained by adding the relation $X^2 = I$ for all $X \in \Pi(2)$ to the relations of $\Pi(2)$. Thus we have

$$\Pi(2)/\Pi(2)^2 \cong \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^2 = \ldots = I \rangle.$$ 

As $a^2 = b^2 = c^2 = I$, we obtain

$$\Pi(2)/\Pi(2)^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$ 

Therefore, we obtain $|\Pi(2) : \Pi(2)^2| = 8$.

Thus we use the Reidemeister-Schreier process to find the presentation of the subgroup $\Pi(2)^2$. Now we choose $\Sigma = \{I, a, b, c, ab, ac, bc, abc\}$ as a Schreier transversal for $\Pi(2)^2$. According to the Reidemeister-Schreier method, we can form all possible products:

- $I.a.(a)^{-1} = I,$  $I.b.(b)^{-1} = I,$  $I.c.(c)^{-1} = I,$
- $a.a.(I)^{-1} = I,$  $a.b.(ab)^{-1} = I,$  $a.c.(ac)^{-1} = I,$
- $b.a.(ab)^{-1} = babab,$  $b.b.(I)^{-1} = I,$  $b.c.(bc)^{-1} = I,$
- $c.a.(ac)^{-1} = cacac,$  $c.b.(bc)^{-1} = cbcb,$  $c.c.(I)^{-1} = I,$
- $ab.a.(b)^{-1} = abab,$  $ab.b.(a)^{-1} = I,$  $ab.c.(abc)^{-1} = I,$
- $ac.a.(c)^{-1} = acac,$  $ac.b.(abc)^{-1} = abcba,$  $ac.c.(a)^{-1} = I,$
- $bc.a.(abc)^{-1} = bacbca,$  $bc.b.(c)^{-1} = bcbc,$  $bc.c.(b)^{-1} = I,$
- $abc.a.(bc)^{-1} = abcacb,$  $abc.b.(ac)^{-1} = abcbca,$  $abc.c.(ab)^{-1} = I,$
as $a^{-1} = a$, $b^{-1} = b$, and $c^{-1} = c$. Also, since $(baba)^{-1} = abab$, $(acac)^{-1} = acac$, $(bcbc)^{-1} = bcbc$, $(bcaca)^{-1} = abacbc$ and $(acbca)^{-1} = abcbca$, the generators of $\Pi(2)^2$ are $abab = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, $acac = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, $bcbc = \begin{pmatrix} 5 & -4 \\ -4 & -3 \end{pmatrix}$, $abcacb = \begin{pmatrix} -7 & -12 \\ -4 & -7 \end{pmatrix}$ and $abcbca = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$.

From [7, Lemma 32], $\Pi(2)^2 = \Gamma(4)$. As $\Gamma(4) = \Pi(4)$, we obtain $\Pi(2)^2 = \Pi(4)$. $\square$

Using the above results, we have the following.

**Proposition 1.** $\Pi(N)^2$ is a congruence if and only if $N \leq 2$.

Now we present some results related with the congruence subgroups $\Pi_0(N)$ and $\Pi_1(N)$ of $\Pi$. To do this, we suppose that

$$A = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix} \pmod{N}$$

is an element of $\Gamma_0(N)$. Then

$$TR.A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} -x & * \\ 0 & x^{-1} \end{pmatrix} \pmod{N}$$

is an element of $\Pi_0(N)$. Therefore

$$(TRA)^2 = \begin{pmatrix} -x & * \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} -x & * \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} x^2 & * \\ 0 & x^{-2} \end{pmatrix} \pmod{N}$$

is an element of $\Gamma_0(N)^2$. Thus, we get $\Pi_0(N)^2 = \Gamma_0(N)^2$.

Similarly to the case $\Pi_0(N)$, if

$$B = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}$$

is an element of $\Gamma_1(N)$, then

$$TR.B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}$$

is an element of $\Pi_1(N)$. Therefore

$$(TRB)^2 = \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}$$

is an element of $\Gamma_1(N)^2$ and so we obtain $\Pi_1(N)^2 = \Gamma_1(N)^2$.

On the other hand, if $\Pi_0(N)$ and $\Pi_1(N)$ are not congruence, then $\Pi_0(N)^2$ and $\Pi_1(N)^2$ are not congruence, since any lift of $\Pi_0(N)$ (or $\Pi_1(N)$) necessarily contains $\Pi_0(N)^2$ (or $\Pi_1(N)^2$), from [7, Lemma 5]. Consequently, we have the following.
Corollary 1. a) $\Pi_0(N)^2$ is not congruence if and only if either $N \notin \{3, 4, 8\}$ or if $4 \mid N$ and all odd prime divisors of $N$ are congruent to $3$ modulo $4$.

b) $\Pi_1(N)^2$ is not congruence if and only if $N > 4$.

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