Fermionic time-reversal symmetry in a photonic topological insulator

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Supplementary Information for

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I Experimental techniques

I.1 Implementation of on-site potentials

While the femtosecond laser inscription technique is capable of directly and precisely modulating the effective index of the fabricated waveguides via the exposure parameters (pulse energy, writing velocity) [1], we followed a different approach in this work to selectively implement diagonal terms in the discrete Hamiltonian. Instead of writing detuned couplers, i.e., evanescently interacting waveguides with different effective refractive indices, we designed the trajectories of the transition sections between subsequent steps such that precisely defined differences in their overall optical path lengths allow propagating light to accumulate the same additional phases that physically detuned couplers would produce (see Fig. S1). In this vein, our method separates the couplings terms from the detunings/on-site terms, since one is realized during the steps, while the other implementation occurs in between (see Fig. S2). The technique is of particular importance for the verification of time reversal symmetry (TRS), since in order to obtain a non-zero contrast of the sine/cosine shaped intensity-phase-dependences, coupling steps 1, 3, 4 and 6 necessarily require detuning via non-zero diagonal entries of the Hamiltonian. In line with the approach described above, these were implemented via geometric path differences of 9.6\,\mu m (transitions from step 1 \(
\rightarrow 2\) and 2 \(
\rightarrow 3\)) and 9.9\,\mu m (4 \(
\rightarrow 5\) and 5 \(
\rightarrow 6\)), which would in the conventional realisation correspond to a detuning of 4/T within the couplers of steps 1, 2, 3 and 4 (see Fig. S2).

Figure S1: Three dimensional representation of a unit cell (red/blue) of the implemented waveguide structure. The unit cell is embedded in the surrounding waveguides (grey). The coupling regions are highlighted by semi-transparent ribbons. The different path lengths of the waveguides between the coupling regions are visible.
Figure S2: **Implementation of the on-site potential.** The discrete driving protocol of Fig. 2a of the main text is combined with trajectories of the waveguides between the hopping steps. The trajectories are marked by the red and blue arrows. The length of these arrows corresponds to the optical path length of the light guided by the waveguides. The asymmetric path lengths are clearly visible in the transitions from step 1 → 2 and 2 → 3.
I.2 Estimation of the experimental data correlation

The experimental data, which we presented in Fig. 3d,e,g,h of the main text, can be compared to theoretical simulations of the proposed driving protocol. The experimental images are mapped into a $8 \times 6$ matrix, by integrating over the area of the lattice sites. We determine the image normalized cross-correlation of the simulations and these reduced experimental data [2] (using $\sigma = 2$, since our edge typically has the size of $1 - 2$ lattice sites). The extracted values, given in the main text, demonstrate the excellent agreement of experiment and theory.

I.3 Additional edge state measurements

As further evidence for the predicted edge state behaviour in our system, Fig. S4 and Fig. S5 show the output intensity profiles for additional single-site excitations beyond the ones shown in Fig. 3. Note that the spin flip case (Fig. S4) is characterised both by chiral edge transport (panels a/e, b/f), as well as a flat bulk band (Fig. 3c). The latter is responsible for the localised bulk excitations (panels c/g and d/h). The more general spin-rotation case (Fig. S5) continues to support the edge states. However, owing to the non-zero curvature of their trajectories through the band diagram (Fig. 3f), these edge states exhibit non-uniform transverse (along the edge) velocities. As a result, edge state excitations remain decoupled from the bulk, but are subject to a certain degree of dispersive broadening (along the edge) as they propagate along the edges. Due to the rhomboidal unit cell, some lattice sites belong to the edge, yet are not in direct contact with the environment. Figure S3 explicitly displays the edge location for clockwise and counterclockwise edge states.

Figure S3: Edge locations. The black arrows pointing out the path of light during one driving period in the spin flip case. The colored background indicates the lattice sites which belong to the edge for a clockwise and b counterclockwise moving edge states, respectively.
Figure S4: **Additional data for the spin-flip case.** Shown are the output intensity distributions resulting from excitations of the orange-outlined lattice sites after three full driving periods. The effective wave packet trajectories are indicated by blue and red arrows for clockwise and counter-clockwise propagation, respectively. a,b, Edge excitations exhibit chiral transport, c,d, bulk excitations remain effectively localised after each driving period. e–h Corresponding numerical simulations. In comparison with the simulations a,b,c,d the experimental data e,f,g,h coincide to 0.9093, 0.9966, 0.9903, 0.9919, respectively (based on [2]).
Figure S5: **Additional data for the spin-rotation case.** Shown are the output intensity distributions resulting from edge excitations of the orange-outlined lattice sites after three full driving periods. The effective wave packet trajectories are indicated by blue and red arrows for clockwise and counter-clockwise propagation, respectively. The edge states associated with both sublattices R (panels a,b) and B (panels c,d) now exhibit non-uniform transverse velocities, as indicated by a certain amount of wave packet broadening. e–h Corresponding numerical simulations. In comparison with the simulations a,b,c,d the experimental data e,f,g,h coincide to 0.9345, 0.8111, 0.9321, 0.8917, respectively (based on [2]).
II Theory

II.1 Construction of the driving protocol

Our construction of a driving protocol with fermionic time-reversal symmetry (TRS) follows the conceptual idea depicted in Fig. S6. The driving protocol is based on the square lattice model proposed in Ref. [3], which combines the four elementary coupling patterns between adjacent lattice sites defined in Fig. S7. To denote these patterns in the real-space Hamiltonian $H(t)$ of the driving protocol, we use the shorthand graphical notation

\[
\begin{array}{ccccc}
    & & & & \\
    & & & & \\
    & & & & \\
    & & & & \\
    & & & & \\
    & & & & \\
    & & & & \\
    & & & & \\
\end{array}
\]  

(SI.1)

introduced in Fig. S7. Similarly, we write

\[
\begin{array}{c}
\sum_{k,l} (-1)^{k+l}|k,l\rangle\langle k,l|
\end{array}
\]  

(SI.2)

for a term with alternating on-site potentials. In this notation, the ket vector $|k,l\rangle$, for $k,l \in \mathbb{Z}$, denotes the state at the $k$th and $l$th lattice site in horizontal and vertical direction, respectively. Lattice sites with even $k+l$ are identified with filled circles, sites with odd $k+l$ with hollow circles.

Figure S6: Construction of the driving protocol with TRS: Two copies ("red" and "blue") of a driving protocol with opposite chirality are combined into a centred square lattice. The red/blue sublattice structure can be associated with a pseudo-spin $1/2$, where two neighbouring lattice sites are paired ("green" oval). After rotation by $45^\circ$, this construction gives the protocol depicted in Fig. 2 in the main text.
\[ \sum_{k,l} |2k+1,2l\rangle\langle 2k,2l| + H.c. \]

\[ \sum_{k,l} |2k,2l+1\rangle\langle 2k,2l| + H.c. \]

\[ \sum_{k,l} |2k-1,2l\rangle\langle 2k,2l| + H.c. \]

\[ \sum_{k,l} |2k,2l-1\rangle\langle 2k,2l| + H.c. \]

Figure S7: Shorthand graphical notation for the four elementary coupling patterns on the square lattice.
If the four coupling patterns are arranged in a periodic sequence, as in the model from Ref. [3], the resulting driving protocol implements a Floquet topological insulator with chiral edges states, but non-trivial symmetries cannot be enforced without modification of the protocol [4].

Therefore, to construct a driving protocol with TRS, we duplicate the previous non-symmetric model and combine the two copies, as shown in Fig. S6. One copy is the mirror image of the other, such that they implement opposite chirality for states on equivalent lattice sites. For the theoretical analysis, it is convenient to associate the two copies with a pseudo-spin $1/2$, where we identify the “red” and “blue” sublattice of the centred square lattice in Fig. S6 with the “up” spin state $|\uparrow\rangle$ and “down” spin state $|\downarrow\rangle$, respectively. In this way, the coupling patterns become associated with the two spin directions. We have, for example,

$$
\begin{align*}
\uparrow\uparrow & = \sum_{k,l} |2k + 1, 2l\rangle\langle 2k, 2l| \otimes |\uparrow\rangle\langle \uparrow| + H.c. , \\
\downarrow\downarrow & = \sum_{k,l} |2k + 1, 2l\rangle\langle 2k, 2l| \otimes |\downarrow\rangle\langle \downarrow| + H.c. , \\
\end{align*}
$$

and similarly

$$
\begin{align*}
\uparrow\downarrow & = \sum_{k,l} (-1)^{k+l} |k, l\rangle\langle k, l| \otimes |\downarrow\rangle\langle \downarrow| + H.c. 
\end{align*}
$$

for the potential terms. These terms preserve the pseudo-spin direction, as expressed by the projections $|\uparrow\rangle\langle \uparrow| = \frac{1}{2}(1 + \sigma_z)$ and $|\downarrow\rangle\langle \downarrow| = \frac{1}{2}(1 - \sigma_z)$.

To connect the two pseudo-spin directions, or sublattices, steps with a pseudo-spin transformation

$$
\sigma_x = |\uparrow\rangle\langle \downarrow| + |\downarrow\rangle\langle \uparrow| ,
$$

need to be included in the driving protocol. In order to preserve TRS, these steps have to appear pairwise in symmetric position, in our case as steps 2 and 5 of the protocol.

The entire construction results in the driving protocol specified by the time-dependent Hamiltonian

$$
H(t) = H_j , \quad \text{for } (n + \frac{j-1}{6}) T \leq t < (n + \frac{j}{6}) T \quad \text{with} \quad n \in \mathbb{N} ,
$$

where the Hamiltonians $H_j$ of each step $j \in \{1, \ldots, 6\}$ are listed in Tab. S1. By construction, the Hamiltonian is Hermitian and periodic, $H(t + T) = H(t)$. Each period consists of six steps of equal duration $T/6$. Steps 1, 3, 4 and 6 leave the pseudo-spin unchanged, while steps 2, 5 involve a pseudo-spin rotation. To allow for breaking of particle-hole and chiral symmetry, steps 1, 3, 4 and 6 contain additional on-site potentials. In summary, the driving protocol has ten parameters: six couplings $c^{(j)}$, for $j \in \{1, \ldots, 6\}$, and four on-site potentials $\epsilon^{(j)}$, for $j \in \{1,3,4,6\}$. All parameters, hence also the entire Hamiltonian, are real-valued. The Hamiltonian in Eq. (SI.7) has been used in all numerical calculations presented in this work, and is the basis of the experimental implementation.

From this Hamiltonian, the Floquet propagator

$$
U(T) = U_6 U_5 U_4 U_3 U_2 U_1 
$$

(SI.8)
Table S1: Hamiltonian $H(t)$ of the driving protocol in pseudo-spin representation, using the graphical notation from Fig. S7.

| Driving protocol | $H(t)$ |
|------------------|--------|
| Step 1: $0 \leq t < \frac{1}{6} T$ | $H_1 = e^{(1)} \left( \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \right) + e^{(1)} \left( \begin{array}{c}
\uparrow \\
- \downarrow + \uparrow + \downarrow
\end{array} \right)$ |
| Step 2: $\frac{1}{6} T \leq t < \frac{2}{6} T$ | $H_2 = e^{(2)} \sigma_x$ |
| Step 3: $\frac{2}{6} T \leq t < \frac{3}{6} T$ | $H_3 = e^{(3)} \left( \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \right) + e^{(3)} \left( \begin{array}{c}
\uparrow \\
- \downarrow + \uparrow + \downarrow
\end{array} \right)$ |
| Step 4: $\frac{3}{6} T \leq t < \frac{4}{6} T$ | $H_4 = e^{(4)} \left( \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \right) + e^{(4)} \left( \begin{array}{c}
\uparrow \\
- \downarrow + \uparrow + \downarrow
\end{array} \right)$ |
| Step 5: $\frac{4}{6} T \leq t < \frac{5}{6} T$ | $H_5 = e^{(5)} \sigma_x$ |
| Step 6: $\frac{5}{6} T \leq t < T$ | $H_6 = e^{(6)} \left( \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \right) + e^{(6)} \left( \begin{array}{c}
\uparrow \\
- \downarrow + \uparrow + \downarrow
\end{array} \right)$ |

is obtained, where the six propagators for each step are defined by $U_j = \exp \left( -iH_j T/6 \right)$. For full coupling ($e^{(j)} = \pm 3\pi/T$, $e^{(j)} = 0$) the Floquet propagator in the bulk is trivial ($U(T) = \pm 1$). Especially, steps 2 and 5 correspond to a spin flip $U_{2,5} = \pm i\sigma_x$ and thus transplant states from one to the other pseudo-spin direction (see Fig. S6). The introduction of edges gives rise to pairs of edge states with opposite chirality, which move along the trajectories depicted in Fig. 2b in the main text. Note that an edge must result from a cut that preserves TRS, and does not separate lattice sites that are paired in the pseudo-spin (or red and blue sublattice) representation (see last panel in Fig. S6).

From the real-space Hamiltonian $H(t)$, one obtains the Bloch-Hamiltonian $H(k, t)$ in momentum space given in Table S2. With this Hamiltonian, computation of the bulk band structures in Fig. 2d and Fig. S8 (below) is straightforward.

**Pseudo-spin to lattice mapping** As mentioned before, we map the up spin state $|\uparrow\rangle$ onto the “red” and the down spin state $|\downarrow\rangle$ onto the “blue” sublattice to obtain a pure lattice model without pseudo-spin degrees of freedom, which is suitable for a photonic waveguide implementation. Now, the ket vector $|k, l, R/B\rangle$ carries the sublattice information $R/B$ in addition to the lattice site position $k, l,$
Table S2: Same as Tab. S1, now for the Bloch Hamiltonian $H(k, t)$ of the driving protocol. We use the abbreviations $k_1 = (a/2)(k_x + k_y)$ and $k_2 = (a/2)(-k_x + k_y)$, with the quasi-momentum $k = (k_x, k_y)^t$ given in the $x, y$ coordinates of Fig. 2b in the main text.

| Driving protocol | $H(k, t)$ |
|------------------|------------|
| Step 1: $0 \leq t < \frac{1}{6}T$ | $H_1(k) = \left(\begin{array}{cc} \epsilon^{(1)} & c^{(1)} e^{-ik_1} \\ c^{(1)} e^{ik_1} & -\epsilon^{(1)} \end{array}\right) \otimes |\uparrow\rangle\langle\uparrow| + \left(\begin{array}{cc} \epsilon^{(1)} & c^{(1)} e^{-ik_2} \\ c^{(1)} e^{ik_2} & -\epsilon^{(1)} \end{array}\right) \otimes |\downarrow\rangle\langle\downarrow|$ |
| Step 2: $\frac{1}{6}T \leq t < \frac{1}{5}T$ | $H_2(k) = \epsilon^{(2)} |1\rangle\langle 1| \otimes \sigma_x$ |
| Step 3: $\frac{2}{6}T \leq t < \frac{1}{4}T$ | $H_3(k) = \left(\begin{array}{cc} \epsilon^{(3)} & c^{(3)} e^{ik_1} \\ c^{(3)} e^{-ik_1} & -\epsilon^{(3)} \end{array}\right) \otimes |\uparrow\rangle\langle\uparrow| + \left(\begin{array}{cc} \epsilon^{(3)} & c^{(3)} e^{ik_2} \\ c^{(3)} e^{-ik_2} & -\epsilon^{(3)} \end{array}\right) \otimes |\downarrow\rangle\langle\downarrow|$ |
| Step 4: $\frac{3}{6}T \leq t < \frac{1}{3}T$ | $H_4(k) = \left(\begin{array}{cc} \epsilon^{(4)} & c^{(4)} e^{ik_2} \\ c^{(4)} e^{-ik_2} & -\epsilon^{(4)} \end{array}\right) \otimes |\uparrow\rangle\langle\uparrow| + \left(\begin{array}{cc} \epsilon^{(4)} & c^{(4)} e^{ik_1} \\ c^{(4)} e^{-ik_1} & -\epsilon^{(4)} \end{array}\right) \otimes |\downarrow\rangle\langle\downarrow|$ |
| Step 5: $\frac{4}{6}T \leq t < \frac{1}{2}T$ | $H_5(k) = \epsilon^{(5)} |1\rangle\langle 1| \otimes \sigma_x$ |
| Step 6: $\frac{5}{6}T \leq t < T$ | $H_6(k) = \left(\begin{array}{cc} \epsilon^{(6)} & c^{(6)} e^{ik_2} \\ c^{(6)} e^{-ik_2} & -\epsilon^{(6)} \end{array}\right) \otimes |\uparrow\rangle\langle\uparrow| + \left(\begin{array}{cc} \epsilon^{(6)} & c^{(6)} e^{ik_1} \\ c^{(6)} e^{-ik_1} & -\epsilon^{(6)} \end{array}\right) \otimes |\downarrow\rangle\langle\downarrow|$ |

and the coupling and potential terms read,

$$\sum_{k,l,|R\rangle} |2k + 1, 2l, R\rangle \langle 2k, 2l, R| + \text{H.c.} \ , \quad (\text{SI.9})$$

$$\sum_{k,l,|B\rangle} |2k + 1, 2l, B\rangle \langle 2k, 2l, B| + \text{H.c.} \ , \quad (\text{SI.10})$$

or

$$\sum_{k,l} (-1)^{k+l} |k, l, B\rangle \langle k, l, B| + \text{H.c.} \ , \quad (\text{SI.11})$$

and similarly for the remaining terms. The pseudo-spin transformation $\sigma_x$ in steps 2 and 5 is replaced by the operator

$$\Sigma_x = \sum_{k,l} |k, l, R\rangle \langle k, l, B| + \text{H.c.} \ , \quad (\text{SI.12})$$

which swaps the red and blue sublattice (see Fig. S6). In this way, we obtain the Hamiltonian of the pure lattice model specified explicitly in Table S3.

II.2 Time-reversal symmetry

TRS is defined by the relation

$$\Theta H(t) \Theta^{-1} = H(T - t) \quad (\text{SI.13})$$
Table S3: Hamiltonian $H(t)$ of the driving protocol in “red” and “blue” sublattice representation of the pseudo-spin.

| Driving protocol | $H(t)$ |
|------------------|--------|
| Step 1: $0 \leq t < \frac{1}{6}T$ | $H_1 = c^{(1)} \left( \begin{array}{c} \text{R} \text{R} \\ \text{B} \text{B} \end{array} \right) + \epsilon^{(1)} \left( \begin{array}{c} \text{R} \text{R} \\ \text{B} \text{B} \end{array} \right)$ |
| Step 2: $\frac{1}{6}T \leq t < \frac{2}{6}T$ | $H_2 = c^{(2)}$ |
| Step 3: $\frac{2}{6}T \leq t < \frac{3}{6}T$ | $H_3 = c^{(3)} \left( \begin{array}{c} \text{R} \text{R} \\ \text{B} \text{B} \end{array} \right) + \epsilon^{(3)} \left( \begin{array}{c} \text{R} \text{R} \\ \text{B} \text{B} \end{array} \right)$ |
| Step 4: $\frac{3}{6}T \leq t < \frac{4}{6}T$ | $H_4 = c^{(4)} \left( \begin{array}{c} \text{R} \text{R} \\ \text{B} \text{B} \end{array} \right) + \epsilon^{(4)} \left( \begin{array}{c} \text{R} \text{R} \\ \text{B} \text{B} \end{array} \right)$ |
| Step 5: $\frac{4}{6}T \leq t < \frac{5}{6}T$ | $H_5 = c^{(5)}$ |
| Step 6: $\frac{5}{6}T \leq t < T$ | $H_6 = c^{(6)} \left( \begin{array}{c} \text{R} \text{R} \\ \text{B} \text{B} \end{array} \right) + \epsilon^{(6)} \left( \begin{array}{c} \text{R} \text{R} \\ \text{B} \text{B} \end{array} \right)$ |

(cf. Eq. (1) in the main text), where $\Theta$ is an anti-unitary operator with $\Theta^2 = 1$ for bosonic TRS and $\Theta^2 = -1$ for fermionic TRS.

For fermionic TRS we choose $\Theta = \sigma_y \mathcal{K}$, with the second Pauli matrix $\sigma_y$ and the operator of complex conjugation $\mathcal{K}$. Then, the symmetry relation (SI.13) reads

$$\sigma_y H(t) \sigma_y^{-1} = H(T - t)^*$$  \hspace{1cm} (SI.14)

(and we have $\sigma_y^{-1} = \sigma_y$). Note that the operator $\sigma_y$ only acts on the pseudo-spin degrees of freedom of $H(t)$.

The transformation of terms in the Hamiltonian $H(t)$ is straightforward, for example, $\sigma_y \uparrow \downarrow \sigma_y^{-1} = \downarrow \uparrow$, or generally

$$\sigma_y |\uparrow\rangle \langle \uparrow| \sigma_y^{-1} = |\downarrow\rangle \langle \downarrow|.$$  \hspace{1cm} (SI.15)

On the other hand, we have $\sigma_y \sigma_x \sigma_y^{-1} = -\sigma_x$ for the spin flip $\sigma_x$. Therefore, the driving protocol obeys the relation (SI.14) if and only if the conditions

$$c^{(1)} = c^{(6)}, \quad c^{(3)} = c^{(4)}, \quad \epsilon^{(1)} = \epsilon^{(6)}, \quad \epsilon^{(3)} = \epsilon^{(4)}, \quad c^{(2)} = -c^{(5)}$$  \hspace{1cm} (SI.16)

are fulfilled. Then, we have

$$\Theta H_j \Theta^{-1} = H_{7-j} \quad (j = 1, \ldots, 6)$$  \hspace{1cm} (SI.17)
for each of the steps, or equivalently

$$\sigma_y H_j \sigma_y^{-1} = H_{j-1} \quad (j = 1, \ldots, 6) \quad (SI.18)$$

since all $H_j$ are real-valued. If all parameters are non-zero, the protocol does not possess additional particle-hole or chiral symmetry, see Sec. II.4.

For the present work, we choose the parameters (spin flip case)

$$c^{(1,2,3,4,6)} = 3\pi/T, \quad c^{(5)} = -3\pi/T, \quad c^{(1,3,4,6)} = 0, \quad (SI.19)$$

and (spin rotation case)

$$c^{(1,3,4,6)} = 5\pi/(2T), \quad c^{(1,3,4,6)} = 3/(2T), \quad (SI.20)$$

II.3 Negative coupling

The condition (SI.16) implies that either the coupling $c^{(2)}$ in step 2 or $c^{(5)}$ in step 5 has to be negative, unless trivially $c^{(2)} = c^{(5)} = 0$. Negative couplings can indeed be implemented experimentally [5, 6], but we decided to circumvent the additional complexity involved in their implementation. To avoid negative couplings, we make the following observation: In steps 2, 5 of the driving protocol, of duration $\delta t$ (here $\delta t = T/6$) and with the spin matrix $c^{(2,5)}\sigma_x$, we have

$$\exp\left(-i\delta t c^{(2,5)}\sigma_x\right) = \exp\left[i n\pi \sigma_x - i\delta t \left(\frac{n\pi}{\delta t} + c^{(2,5)}\right)\sigma_x\right] \quad (SI.21)$$

$$= (-1)^n \exp\left[-i\delta t \left(\frac{n\pi}{\delta t} + c^{(2,5)}\right)\sigma_x\right] \quad (SI.22)$$

for every $n \in \mathbb{Z}$. Therefore, negative couplings $c^{(2,5)} < 0$ in these steps can be replaced by positive couplings $\frac{n\pi}{\delta t} + c^{(2,5)} > 0$ for sufficiently large $n$, without changing the driving protocol implemented in the experiment. For odd $n$, the modified protocol contains an irrelevant global phase.

In the experiment (cf. Methods section), we realise the parameters (spin flip case)

$$c^{(1,2,3,4,5,6)} = 3\pi/T, \quad c^{(1,3,4,6)} = 0, \quad (SI.23)$$

and (spin rotation case)

$$c^{(1,3,4,6)} = 5\pi/(2T), \quad c^{(1,3,4,6)} = 3/(2T), \quad (SI.24)$$

having replaced the negative coupling $c^{(5)}$ by the positive value $c^{(5)} + 6\pi/T$ in step 5 of the driving protocol. Due the global phase introduced by this replacement the Floquet quasi-energies are shifted by $\varepsilon \mapsto \varepsilon + \pi/T$, but the real space propagation remains unchanged.
II.4 Particle-hole and chiral symmetry

Particle-hole symmetry is defined by the relation

$$\Pi H(t)\Pi^{-1} = -H(t), \quad (SI.25)$$

where $\Pi$ is an anti-unitary operator for which $\Pi^2 = \pm 1$. Note that the same time argument $t$ appears on both sides of the relation, so the symmetry relation maps each step of the driving protocol

$$\Pi H_j\Pi^{-1} = -H_j \quad (j = 1, \ldots, 6) \quad (SI.26)$$
on to itself.

While particle-hole symmetry with $\Pi^2 = -1$ would require a modified version of our driving protocol [7], particle-hole symmetry with $\Pi^2 = 1$ is satisfied if the on-site potentials are set to zero. To see this, we choose

$$\Pi = \begin{pmatrix} \begin{array}{c} \uparrow \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \downarrow \downarrow \uparrow \end{array} & - \begin{array}{c} \downarrow \uparrow \downarrow \uparrow \\ \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \end{array} \end{pmatrix} K. \quad (SI.27)$$

The coupling terms in the Hamiltonian transform as

$$\Pi \begin{pmatrix} \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \\ \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \end{array} \end{pmatrix} \Pi^{-1} = \begin{pmatrix} \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \\ \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \end{array} \end{pmatrix}, \quad \Pi \begin{pmatrix} \begin{array}{c} \uparrow \downarrow \downarrow \downarrow \\ \downarrow \uparrow \downarrow \uparrow \downarrow \end{array} \end{pmatrix} \Pi^{-1} = \begin{pmatrix} \begin{array}{c} \uparrow \downarrow \downarrow \downarrow \\ \downarrow \uparrow \downarrow \uparrow \downarrow \end{array} \end{pmatrix}, \quad \text{or} \quad \Pi \sigma_x \Pi^{-1} = -\sigma_x \quad (SI.28)$$

Comparison with Eq. (SI.26) indicates that all of them trivially satisfy particle-hole symmetry without any restriction on the values of $\epsilon^{(j)}$. On the other hand, we have

$$\Pi \begin{pmatrix} \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \\ \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \end{array} \end{pmatrix} \Pi^{-1} = \begin{pmatrix} \begin{array}{c} \downarrow \downarrow \downarrow \downarrow \\ \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \end{array} \end{pmatrix}, \quad \text{and} \quad \Pi \begin{pmatrix} \begin{array}{c} \uparrow \downarrow \downarrow \downarrow \\ \downarrow \uparrow \downarrow \uparrow \downarrow \end{array} \end{pmatrix} \Pi^{-1} = \begin{pmatrix} \begin{array}{c} \uparrow \downarrow \downarrow \downarrow \\ \downarrow \uparrow \downarrow \uparrow \downarrow \end{array} \end{pmatrix} \quad (SI.29)$$

for the on-site terms. Therefore, the driving protocol fulfills the symmetry relation (SI.26) if and only if the on-site potentials $\epsilon^{(j)} = 0$ vanish. In the present work, we want to exclusively probe the effects that fermionic time-reversal symmetry has on the driving protocols. For this reason, we include on-site terms to break particle-hole symmetry.

Since the driving protocol has time-reversal symmetry, but no particle-hole symmetry, the product of the two, which is chiral symmetry, is also broken [8].

II.5 Bulk invariants and symmetry-protected topological phases

In order to clearly separate the four topological invariants discussed in the main text, Chern number $C$, Kane-Mele invariant $\nu_{KM}$, Floquet winding number $W$ and Floquet TRS invariant $\nu_{TR}$, we give an overview of their definition and relevance for (symmetry-protected) topological edge states. For a brief summary, see Tab. S4.

II.5.1 Chern number $C$

The topological classification of time-independent systems without additional symmetries employs the integer-valued Chern number [15]

$$C = \frac{1}{2\pi i} \int_{BZ} d\mathbf{k}^2 \nabla_k \times \langle \psi | \nabla_k | \psi \rangle. \quad (SI.30)$$
Table S4: Overview of the discussed topological invariants.

| Invariant | System type | Values | Occurrence | This work |
|-----------|-------------|--------|------------|-----------|
| $\mathcal{C}$ | Static | No symmetry | $\mathbb{Z}$ | [9, 10] | $\mathcal{C} = 0$ |
| $\nu_{\text{KM}}$ | Static | Fermionic TRS | $\mathbb{Z}_2$ | [11, 12] | $\nu_{\text{KM}} = 0$ |
| $\mathcal{W}$ | Floquet | No symmetry | $\mathbb{Z}$ | [13, 14] | $\mathcal{W} = 0$ |
| $\nu_{\text{TR}}$ | Floquet | Fermionic TRS | $\mathbb{Z}_2$ | This work | $\nu_{\text{TR}} = 1$ |

The abbreviation $\text{BZ}$ denotes integration over the entire Brillouin zone. The value of the Chern number corresponds to the net-chirality of edge states. When evaluated for the individual bands of a Floquet system, the Chern number is calculated from the eigenvectors $|\psi(k)\rangle$ of the Floquet-Bloch propagator $U(k, T)$. In Floquet systems, it usually fails to correctly predict the number of edge states [3, 16] due to the periodicity of the quasi-energy. For the numerical computation of the Chern number, we use the algorithm from Ref. [17].

II.5.2 Kane-Mele invariant $\nu_{\text{KM}}$

The topological classification of time-independent systems with fermionic TRS employs the $\mathbb{Z}_2$-valued Kane-Mele invariant [18]

$$
\nu_{\text{KM}} = \frac{1}{2\pi i} \left[ \int_{\text{BZ}_{1/2}} \text{d}k \, (\nabla_k \times \langle \psi | \nabla_k | \psi \rangle) \right] \mod 2 \, . \tag{SI.31}
$$

The abbreviations $\text{BZ}_{1/2}$ or $\partial \text{BZ}_{1/2}$ now denote integration over half of the Brillouin zone or over its boundary, respectively. A non-zero value of this invariant implies the existence of a pair of symmetry-protected edge states with opposite chirality. Again, when evaluated for the individual bands of a Floquet system, the Kane-Mele invariant is calculated from the eigenvectors of the Floquet-Bloch propagator. Now, symmetry-protected edge states can appear even when the Kane-Mele invariant is zero [4, 19, 20], which is indeed the case for our driving protocol. For the numerical computation of the Kane-Mele invariant, we use the algorithm from Ref. [21].

II.5.3 Winding Number $\mathcal{W}$

The topological classification of Floquet systems without additional symmetries employs the integer-valued winding number [3]

$$
\mathcal{W}(\varepsilon) = \frac{1}{8\pi^2} \int_0^T \text{d}t \int_{\text{BZ}} \text{d}k \, 2 \text{Tr} \left( U_\varepsilon^\dagger \partial_t U_\varepsilon \left[ U_\varepsilon^\dagger \partial_{k_x} U_\varepsilon, U_\varepsilon^\dagger \partial_{k_y} U_\varepsilon \right] \right) \, . \tag{SI.32}
$$

This invariant counts the net-chirality of edge states in the band gap at quasi-energy $\varepsilon$. Conceptually, it replaces the Chern number of time-independent systems.
as the relevant invariant for Floquet systems. The modified propagator $U_\varepsilon(k, t)$ is constructed from the Floquet-Bloch propagator $U(k, t)$ as follows:

$$U_\varepsilon(k, t) = \begin{cases} U(k, 2t) & \text{if } 0 \leq t \leq \frac{T}{2} \\ V_\varepsilon(k, 2T - 2t) & \text{if } \frac{T}{2} < t \leq T \end{cases},$$

where $V_\varepsilon(k, t) = \exp(t \log_\varepsilon U(k, T))$. The branch cut of the complex logarithm is chosen along the line from zero to $\exp(-i\varepsilon T)$, i.e., the eigenvalues of $i \log_\varepsilon U(k, T)$ are elements of the interval $(T\varepsilon - 2\pi, T\varepsilon]$.

Alternatively, the winding number $\mathcal{W}$ may be expressed as the sum

$$\mathcal{W}(\varepsilon) = \sum_{i=1}^{p} N_i(\varepsilon) \hat{C}_i \quad \text{(SI.33)}$$

over all degeneracy points $i = 1, \ldots, p$ of the Floquet-Bloch propagator $U(k, t)$ that occur during time-evolution [19, 20]. To each degeneracy point, we assign a topological charge $\hat{C}_i$, given as a Chern number, and a weight factor $N_i(\varepsilon)$ that ensures that only the degeneracy points in the gap $\varepsilon$ contribute to the sum. Now, the Chern numbers $\hat{C}_i$ and weight factors $N_i(\varepsilon)$ are calculated from the eigenvectors and eigenvalues of the Floquet-Bloch propagator $U(k, t)$ for all $0 \leq t \leq T$. For the numerical evaluation of the $\mathcal{W}$-invariant, we use the algorithm from Ref. [22].

### II.5.4 TRS invariant $\nu_{\text{TR}}$

In Floquet systems with fermionic TRS, the degeneracy points of the Bloch propagator appear in pairs with opposite topological charge, and cancel each other in the expression for the $\mathcal{W}$-invariant (SI.33). The appropriate $\mathbb{Z}_2$-valued invariant for these systems [19, 20],

$$\nu_{\text{TR}}(\varepsilon) = \sum_{i=1}^{p/2} N_i(\varepsilon) \hat{C}_i \mod 2 \quad \text{(SI.34)}$$

counts only one partner of each symmetric pair of degeneracy points, as indicated by the upper summation limit $p/2$. A non-zero value of $\nu_{\text{TR}}(\varepsilon)$ implies the existence of symmetry-protected edge states with opposite chirality in the band gap at quasi-energy $\varepsilon$. Conceptually, this invariant serves the same role for Floquet systems as the Kane-Mele invariant for time-independent systems. For the numerical evaluation of the $\nu_{\text{TR}}$-invariant, we use the algorithm from Ref. [22].

### II.6 Topological consideration of a ribbon geometry

In a finite sample, symmetry-protected topological phases manifest themselves through chiral edge states. In our experiment, as well as in the numerical simulations, the edges of the sample run along either $-45^\circ$ (“$x$-axis”) or $+45^\circ$ (“$y$-axis”) on the centred square lattice, as indicated in Fig. 3 and Fig. S6. Note that the edges have to preserve TRS and thus may not separate lattice sites that are paired in the pseudo-spin representation.
Figure S8: Floquet bands and symmetry-protected counter-propagating topological edge states for the spin flip (left panel) and spin rotation (central and right panel) case. Included are the values of the Kane-Mele invariant of the Floquet bands and the $\nu_{\text{TR}}$-invariant in the central gap.

Figure S8 shows the edge states on a semi-infinite ribbon, together with the Floquet bands of the bulk, using the parameters of our driving protocol in Eq. (SI.23) (spin flip case) or Eq. (SI.24) (spin rotation case). In Fig. S8, the ribbon is 15 unit cells wide, and we only show the edge states on one of the two edges. Numerically, the edge states and bulk bands are computed from diagonalisation of the Floquet propagator on the ribbon after one driving period $T$, evaluated as a function of the momentum $k_{x/y}$ parallel to the edges along the $x$-axis or $y$-axis. Note that we include the shift $\varepsilon \rightarrow \varepsilon + \pi/T$ of Floquet quasi-energies that appears through the replacement $c^{(5)} \rightarrow c^{(5)} + 6\pi/T$ of the negative parameter $c^{(5)}$ by a positive value as we switch from the parameters in Eqs. (SI.20), (SI.19) to the experimental parameters in Eqs. (SI.24), (SI.23) (see Sec. II.3). Accordingly, the gap appears at quasi-energy $\varepsilon = 0$.

Through the bulk-edge correspondence the existence of chiral edge states coincides with a non-zero value of the respective bulk invariants, as collected in Sec. II.5. The present situation is characterised by the values listed in Table S4. Since $C = 0$ and $W = 0$ by TRS, edge states have to appear in counter-propagating pairs. Since $\nu_{\text{KM}} = 0$ but $\nu_{\text{TR}} \neq 0$ an odd number of counter-propagating pairs of edge states has to be present in the gap between the Floquet bands. Note that this combination of invariants corresponds to an anomalous Floquet topological phase [3, 16].

Counter-propagating edge states are indeed observed in Fig. S8 (here, a single pair). In both cases, the edge states exist independently of the direction of the edge, as required for (symmetry-protected) topological states. In the spin flip case, the Floquet bands are perfectly flat and the dispersion of the edge states is linear. Changing the parameters of the driving protocol from the spin flip to the spin rotation case, the Floquet bands acquire dispersion but the topological invariants do not change since the gap does not close. Alternatively, we could note that the number of crossings of the edge state dispersion at the invariant momenta $k_{x,y} = 0, \pi/a$, and hence the number of counter-propagating edge states, is protected by TRS through Kramers degeneracy. Indeed, these two viewpoints are equivalent due to the bulk-edge correspondence. The pair of counter-propagating edge states observed here in momentum space gives rise to the propagating modes observed in real space in the experiment (see Figs. 3, S4, S5).
II.7 Probing fermionic time-reversal symmetry

To check the TRS relation (SI.13) experimentally, we reverse the sample as described in the main text. As we derive now, this allows us to probe fermionic and bosonic TRS.

Reversing the sample does not correspond to reversing time, but to reversing the order of steps of the driving protocol. Therefore, if the forward propagator is given by Eq. (SI.8), the backward propagator is

$$\tilde{U}(T) = U_1 U_2 U_3 U_4 U_5 U_6 .$$  \hfill (SI.35)

Here, we consider only one period of the driving protocol. Generalisation to several periods is straightforward.

In the present situation, a general TRS operator can be written as $\Theta = \sigma K$, with a unitary spin-$1/2$ matrix $\sigma$ such that $\sigma\sigma^* = \pm \mathbb{1}_2$. For such a general operator, the TRS relation (SI.13) is valid if and only if

$$\sigma H_j \sigma^{-1} = H_{7-j} \quad (j = 1, \ldots, 6)$$  \hfill (SI.36)

for the Hamiltonians $H_j$ of each step (cf. Eqs. (SI.17), (SI.18)). Here, we use that the $H_j$ are real-valued in our driving protocol, which allows us to drop the complex conjugation $K$. Equivalently, we have

$$\sigma U_j \sigma^{-1} = \exp \left( -i (T/6) \sigma H_j \sigma^{-1} \right)$$

$$= \exp \left( -i (T/6) H_{7-j} \right) = U_{7-j}$$  \hfill (SI.37)

$$\sigma U(T) \sigma^{-1} = (\sigma U_6 \sigma^{-1}) \cdots (\sigma U_1 \sigma^{-1}) = U_1 \cdots U_6 = \tilde{U}(T) .$$  \hfill (SI.39)

Now suppose we use in the experiment the input state

$$|\psi_{\text{in}}(\phi)\rangle = |k_0,l_0,R\rangle + e^{i\phi}|k_0,l_0,B\rangle ,$$  \hfill (SI.40)

with finite amplitude on an adjacent red (R) and blue (B) site and a relative phase $\phi$. If the state propagates through the sample, with forward propagation illustrated in Fig. 4a, the intensities of the waveguides measured at the output facet are given by the state

$$|\psi_{\text{out}}(\phi)\rangle = U(T)|\psi_{\text{in}}(\phi)\rangle = \sum_{k,l} \left( \psi_{k,l,R}(\phi)|k,l,R\rangle + \psi_{k,l,B}(\phi)|k,l,B\rangle \right) .$$  \hfill (SI.41)

The amplitudes $\psi_{k,l,R/B}(\phi)$ occurring here could be computed with the Hamiltonian $H(t)$. Summing over the red or blue sites, respectively, we obtain the output intensities

$$I^R(\phi) = \sum_{k,l} |\psi_{k,l,R}(\phi)|^2 , \quad I^B(\phi) = \sum_{k,l} |\psi_{k,l,B}(\phi)|^2 ,$$  \hfill (SI.42)
Figure S9: Probing the output intensities from Fig. 4b for fermionic TRS with \( \sigma = \sigma_y \) (panel a) or bosonic TRS with \( \sigma = \sigma_z \) (panel b). According to Table S5, it should hold \( \tilde{I}^R(\phi) = I^R(\pi - \phi) \) in case a and \( \tilde{I}^B(\phi) = I^B(\pi + \phi) \) in case b if the respective TRS is realised. Clearly, the relation for case a is satisfied but for case b is not.

shown in Fig. 4b and Fig. S9.

If, alternatively, the input state propagates through the reversed sample, i.e. with backward propagation illustrated in Fig. 4c, the output is given by the state

\[
|\tilde{\psi}_{\text{out}}(\phi)\rangle = \tilde{U}(T)|\psi_{\text{in}}(\phi)\rangle = \Sigma U(T)\Sigma^{-1}|\psi_{\text{in}}(\phi)\rangle ,
\]

now with different output intensities \( \tilde{I}^R(\phi) \), \( \tilde{I}^B(\phi) \). The operator \( \Sigma \) that appears here is the mapping of the pseudo-spin operator \( \sigma \) onto the red/blue sublattice structure of the waveguide implementation (cf. Eq. (SI.12)). In bra-ket notation, it is

\[
\Sigma = \sum_{k,l} \left( \sigma_{\uparrow\uparrow}|k, l, R\rangle\langle k, l, R| + i \sigma_{\downarrow\uparrow}|k, l, B\rangle\langle k, l, R| - i \sigma_{\downarrow\downarrow}|k, l, R\rangle\langle k, l, B| \right)
\]

for

\[
\sigma = \begin{pmatrix}
\sigma_{\uparrow\uparrow} & \sigma_{\downarrow\uparrow} \\
\sigma_{\downarrow\uparrow} & \sigma_{\downarrow\downarrow}
\end{pmatrix}.
\]

From Eq. (SI.43) we see that the relation between the output intensities \( I^R(\phi) \), \( I^B(\phi) \) for forward propagation and \( \tilde{I}^R(\phi) \), \( \tilde{I}^B(\phi) \) for backward propagation depends entirely on the operator \( \sigma \) that determines \( \Sigma \). Conversely, if the relation between the output intensities is known from the experiment, the possible choices of \( \sigma \) can be deduced.

The relevant possibilities are listed in Table S5. Note that a global phase of the operator \( \sigma \) drops out of the TRS relation (SI.13) due to complex conjugation, and is therefore not included in the table. For example, with \( \sigma = \sigma_y \) we have

\[
\Sigma \equiv \Sigma_y \equiv \sum_{k,l} \left( i|k, l, B\rangle\langle k, l, R| - i|k, l, R\rangle\langle k, l, B| \right)
\]
Table S5: Relation between output intensities in forward and backward propagation for the four relevant choices of the operator $\sigma$ in the general TRS relation.

| $\sigma$ | $\hat{I}^R(\phi)$ | $\hat{I}^B(\phi)$ |
|---------|------------------|------------------|
| $\mathbb{1}$ | $I^R(\phi)$ | $I^B(\phi)$ |
| $\sigma_x$ | $I^B(\phi)$ | $I^R(\phi)$ |
| $\sigma_y$ | $I^B(\pi - \phi)$ | $I^R(\pi - \phi)$ |
| $\sigma_z$ | $I^R(\pi + \phi)$ | $I^B(\pi + \phi)$ |

and thus

$$\Sigma^{-1}_y |\psi_{in}(\phi)\rangle = -ie^{i\phi} |\psi_{in}(-\phi + \pi)\rangle$$  \hspace{1cm} (SI.48)

for the input state while, according to Eq. (SI.43),

$$|\tilde{\psi}_{out}(\phi)\rangle = -ie^{i\phi} \Sigma_y |\psi_{out}(-\phi + \pi)\rangle$$  \hspace{1cm} (SI.49)

$$= e^{i\phi} \sum_{k,l} \left( \psi_{k,l,R}(-\phi + \pi)|k,l,B\rangle - \psi_{k,l,B}(-\phi + \pi)|k,l,R\rangle \right)$$

for the output state. The phases $\pm e^{i\phi}$ drop out, but the output intensities on the red and blue sublattice are swapped by $\Sigma_y$. Therefore, we get the relations $\hat{I}^R(\phi) = I^B(-\phi + \pi)$, $\hat{I}^B(\phi) = I^R(-\phi + \pi)$ given in Table S5.

Now, the type of TRS realised by the driving protocol can be determined conclusively from the experimental data in Fig. 4b in the main text. In the experimental data we observe that (i) the output intensities on the red and blue sublattice are swapped and (ii) a phase shift $\phi \mapsto \pm \phi + \pi$ occurs when reversing the probe. Observation (i) rules out all possibilities for TRS apart from the choices $\sigma = \sigma_x$ or $\sigma = \sigma_y$, which are the only operators with purely off-diagonal elements as required for the swapping of intensities. Observation (ii) rules out all possibilities for TRS apart from the choices $\sigma = \sigma_y$ or $\sigma = \sigma_z$, which are the only operators leading to a phase shift $\phi \mapsto \pm \phi + \pi$. In combination, we are left with the choice $\sigma = \sigma_y$ of fermionic TRS.

For a final check of fermionic TRS, the experimental data are reproduced in Fig. S9 in direct correspondence to the relations from Table S5. Note that we have $I^R(\phi) = 1 - I^B(\phi)$ and $\hat{I}^R(\phi) = 1 - \hat{I}^B(\phi)$ for the normalised output intensities, such that the data in Fig. 4b fully determine the four functions entering these relations. Fig. S9 clearly shows that (only) the choice $\sigma = \sigma_y$ is compatible with the experimental data: Within the limit of experimental uncertainties, we have $\hat{I}^R(\phi) = I^B(\pi - \phi)$ (hence also $\hat{I}^B(\phi) = I^R(\pi - \phi)$) for normalised output intensities). Therefore, probing fermionic TRS results in a positive result: The experimental data for the output intensities are compatible with $-$ and only with $-$ fermionic TRS.
III Stability of the $\mathbb{Z}_2$ topological insulator

III.1 Experimental study of multi wavelength excitation

The coupling coefficient of evanescently coupled waveguides depends crucially on the excitation wavelength [1]. All of the samples discussed in this work are designed at an operation wavelength of 633 nm. By using a white light laser source (NKT SUPERK EXTREME) and a narrow wavelength filter (PHOTON ETC LLTF-SR-VIS-HP8), we are able to excite the structures in the wavelength range 600 – 670 nm. Hence, perturbing the photonic structure around the designed operating wavelength. We study the influence of the wavelength on two edge states at the lower $x$-edge of our structure, one clockwise and one counterclockwise (see Fig. S10a). When operated at an off-design wavelength (as shown for 626 nm in Fig. S10b), the efficiency with which the protocol confines light to the chiral edge channels systematically drops, and a certain fraction of the initial excitation can escape to the bulk. Nevertheless, more than 70% confinement is achieved over a range of 20 nm (see Fig. S10c), corresponding to a bandwidth of more than 3% of the carrier wavelength 633 nm. For comparison, the well established C-band in infrared telecommunications (1530 – 1565 nm) spans approximately 2.3% of its carrier wavelength [23]. This is another indication that the underlying topology indeed provides robustness to the counterpropagating edge states.

![Figure S10: Probing the stability of the edge states by exciting the lattice with off-design wavelength.](image)

The blue/red data points indicate the excitation of a clockwise/counter-clockwise moving edge state, respectively. The grey bar indicates the region, where more than 70% remains confined at the edge. The error bars are estimated by using the signal-to-noise ratio of the experimental image.
III.2 Bandstructure analysis under the influence of specific perturbations

The key ingredient for the $\mathbb{Z}_2$ topological insulator is the presence of fermionic TRS. If this symmetry is broken, the topological protection is lost, as illustrated in the bottom left panel of Fig. 1 in the main text. In order to highlight how such symmetry breaking affects the topological phase of our system (compare Fig. 3 in the main text), we change the system parameters and either preserve (see Figs. S11a,b) or break (see Figs. S11c,d) fermionic TRS. Since our driving protocol mainly depends on two parameters, the coupling and the detuning, we illustrate the effects of symmetry preservation/breaking for both cases. We choose the parameters in accordance to the experimental realisation (see Eq. (SI.20)) and add changes in steps one and six.

Figure S11: Analysis of the stability of the $\mathbb{Z}_2$ topological insulator with counterpropagating edge states under changes that preserve or break fermionic TRS. The band structure shows the stability of edge states under a, a symmetric offset in the coupling and b a symmetric offset in the detuning, which are both changes that preserve the symmetry. If the c coupling or d detuning is changed only in step one of the driving protocol, which breaks any TRS symmetry, a gap (marked in gray) is visible in the edge state dispersion.
In order to preserve fermionic TRS it is crucial that parameter changes are symmetric with respect to the steps of the driving protocol (see Eq. (SI.16)). The resulting band structures are displayed in Figs. S11a,b, where we add a constant offset to the coupling
\[ c^{(1)} \rightarrow c^{(1)} + \frac{\pi}{T} \quad \text{and} \quad c^{(6)} \rightarrow c^{(6)} + \frac{\pi}{T}, \] (SI.50)
as shown in Fig. S11a, or a constant offset to the detuning (see Fig. S11b)
\[ \epsilon^{(1)} \rightarrow \epsilon^{(1)} + \frac{3}{T} \quad \text{and} \quad \epsilon^{(6)} \rightarrow \epsilon^{(6)} + \frac{3}{T}. \] (SI.51)
Here, the counter-propagating edge states are still present, due to the protection by fermionic TRS.

In order to break fermionic TRS symmetry we change the parameters only in the first step, hence destroying any TRS. The resulting effect on the edge states is displayed in Figs. S11c,d, where we add a constant offset to the coupling in the first step
\[ c^{(1)} \rightarrow c^{(1)} + \frac{\pi}{T}, \] (SI.52)
as shown in Fig. S11c, or to the detuning in the first step (see Fig. S11d)
\[ \epsilon^{(1)} \rightarrow \epsilon^{(1)} + \frac{3}{T}. \] (SI.53)
Now, the edge states do no longer traverse the band gap (emphasised by a gray box), but return into the band they originated from. Hence, they do not give rise to scatter-free transport anymore, which is a direct consequence of the loss of symmetry protection. As expected, the $\mathbb{Z}_2$ topological phase with counter-propagating edge states is only protected as long as the fermionic TRS is preserved.

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