Hirota’s virtual multi-soliton solutions of 
$N = 2$ supersymmetric KdV equations

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Abstract

We prove that Mathieu’s $N = 2$ supersymmetric Korteweg–de Vries equations with $a = 1$ or 4 admit Hirota’s $n$-supersoliton solutions, whose nonlinear interaction does not produce any phase shifts. For initial profiles that can not be distinguished from a one-soliton solution at times $t \ll 0$, we reveal the possibility of a spontaneous decay and, within a finite time, transformation into a solitonic solution with a different wave number. This paradoxal effect is realized by the completely integrable $N = 2$ super-KdV systems, whenever the initial soliton is loaded with other solitons that are virtual and become manifest through the $\tau$-function as the time grows.

Key words and phrases: Hirota’s solitons, $N = 2$ supersymmetric KdV, Krasil’shchik–Kersten system, phase shift, spontaneous decay

Introduction. Let $u(x, t; \theta_1, \theta_2)$ be a scalar complex bosonic $N = 2$ superfield. We consider Mathieu’s $N = 2$ supersymmetric Korteweg–de Vries equations [1, 2]

$$u_t = -u_{xxx} + 3(uD_1D_2u)_x + \frac{a-1}{2}(D_1D_2u^2)_x + 3au^2u_x, \quad D_i = \frac{\partial}{\partial \theta_i} + \theta_i \cdot \frac{\partial}{\partial x}$$

(1)

with $a = 4$ or $a = 1$. In this paper, we construct $n$-supersoliton solutions of (1),

$$u = A \cdot D_1D_2 \log \tau_{k_1,\ldots,k_n}(x, t; \theta_1, \theta_2), \quad A(a) = \text{const} \in \mathbb{R},$$

(2)

that possess the following properties. First, the linear truncations of Hirota’s exponential series,

$$\tau = 1 + \sum_{i=1}^n \alpha_i \exp \eta_i, \quad \eta_i = k_ix + \omega(k_i) \cdot t + \lambda(k_i) \cdot \theta_1 \theta_2, \quad k_i \in \mathbb{R}, \alpha_i \in \mathbb{R},$$

(3)

yield exact solutions of (1). Functions (3) contain no pairwise interaction ($A_{ij}(k_i, k_j) \equiv 0$, see (9) below) and no multiple-order interaction terms (respectively, $A_{\ldots ij}(k_1, \ldots, k_i) = \prod_{\alpha, i < j} A_{\alpha i j}(k_{\alpha}, k_i) \equiv 0$, see [3]). Second,
as $n$ different solitons constitute an $n$-soliton solution and obey the nonlinear superposition formulas while overlapping, they acquire no phase shifts having become distant from each other. Thus the spatially extensive solitary waves demonstrate a dimensionless behaviour of particles with elastic nonlinear collisions. We recall that a similar example of the field representation for point particles was known, e.g., for soliton solutions of the integro-differential Benjamin–Ono equation [4].

Third, we show that a fast soliton that precedes a set of other waves always decays spontaneously and, sooner or later, the observed solution converts into the slowest soliton from that set. We say that the initial soliton, which was visible at $t \ll 0$, was loaded with $n-1$ virtual solitons ahead of it. We demonstrate how the fastest soliton gains the virtual ones and becomes virtual itself, while the slowest virtual soliton manifests itself behind the others and becomes visible. Of course, these properties of the new solutions for $N = 2$ super-KdV equations (1) are radically different from superposition laws and phase shifts that are exhibited by the $n$-solitons for the standard KdV equation, see (6) below, and by the super-soliton solutions (7) derived from them. This effect, which is new to the best of our knowledge, may be useful in verifying the relevance of nonlinear $N = 2$ supersymmetric KdV equations (1) in field theory and, more generally, establishing the physical sense of the Grassmann variables $\theta_1$, $\theta_2$.

The paper is organized as follows. First, we summarize well-known properties of $n$-soliton solutions for the KdV equation. Here we also recall how these solutions are inherited by Hamiltonian $N = 2$ super-KdV systems (1) from the purely bosonic KdV. Then we formulate our main result, describing the $\tau$-functions and dispersion relations in the new $n$-supersolitons for (1). If $a = 1$, then the new super-fields yield exact $n$-soliton solutions of the Krasil’shchik–Kersten equation [5]. We establish the paradoxal properties of the $n$-solitons for (1), including their spontaneous decay followed by transition into virtual states.

§1. Known $n$-solitons for the $N = 2$ super-KdV. Some classes of exact travelling wave solutions $u(x + ct + i\theta_1 \theta_2, \theta_1 + i \theta_2)$ for equations (1) with $a = -2$, $\pm 1, \pm 4$ were obtained in [6] using the symmetry invariance technique. First, a Lie subgroup of point symmetries for (1) with $a \in \mathbb{R}$ was chosen such that the respective group invariants specified the arguments of the solutions $u$ as above. It follows easily that the Taylor expansion $u = \rho_0 + (\theta_1 + i \theta_2) \cdot \rho_1$ in the second argument, with both $\rho_0$ and $\rho_1$ depending on $x + ct + i \theta_1 \theta_2$, converts (1) to the triangular system of ODE upon $\rho_0$ and $\rho_1$. The second assumption of [6] is that solutions of the first equation in this system, which is $\rho_0'' - a \rho_0^2 + i(a + 2) \rho_0 \rho_1 + c \rho_0 + \text{const} = 0$, have no movable critical points. This yields the admissible values $a = -2, \pm 1, \pm 4$. Particular solutions $\rho_0(x + ct + i \theta_1 \theta_2)$, $\rho_1 \equiv 0$, which are found in [3] Eq. (3.15) for (1) with $a \in \{1, 4\}$, are close to the one-solitons (2) with $c = \omega/k$ in (3). However, the pairwise or collective interaction of the travelling waves was not discussed in [6].

At the same time, it is clear that the $N = 2$ super-KdV equations (1), and not only with the values $a \in \{-2, 1, 4\}$ that ensure the complete integrability, possess multi-soliton solutions. Indeed, expand the bosonic $N = 2$ super-field $u$ in $\theta_1$ and $\theta_2$,

\[ u = u_0 + \theta_1 u_1 + \theta_2 u_2 + \theta_1 \theta_2 u_{12}, \tag{4} \]

and substitute (1) in (1). Then, in particular, the even components $u_0$ and $u_{12}$
satisfy the system \[ \begin{align*}
u_{0,t} + u_{0,xxx} - 3uu_{0,x} + (a + 2)(u_0u_{12})_x - (a - 1)(u_1u_2)_x &= 0, \quad (5a) \\
u_{12,t} + u_{12,xxx} + 6u_{12}u_{12;x} - (a + 2)u_0u_{0,xxx} - 3a(u_0u_{0,x} + u_0^2u_{12;x} + 2uu_{0,x}u_{12}) \\
- 3u_{0,2u_{2,x}} - (a + 2)u_{1,1,x} + 6a(u_0u_{12,x} - u_0u_{1;x}u_2 + u_0u_xu_{12}) &= 0. \quad (5b)
\end{align*} \]

Set the odd components \( u_1 = u_2 \equiv 0 \). First, if \( a = -2 \), then system \( 5 \) splits to the modified KdV equation \( 5a \) upon \( u_0 \) together with the KdV-type equation \( 5b \) with a feedback. In what follows, we do not consider the well-studied case \( a = -2 \). If, under the same assumption \( u_1 = u_2 = 0 \), we let \( a = 1 \), then system \( 5 \) is the Krasil’shchik–Kersten coupled KdV-mKdV equation \( 5a \), see also \( 7, 8, 9 \). In the next section we describe a new class of its multi-soliton solutions in Hirota’s form.

Conversely, set the components \( u_0, u_1, \) and \( u_2 \) to zero. Then for any \( a \in \mathbb{R} \) the entire \( N = 2 \) supersymmetric equation \( 1 \) reduces to the purely bosonic, scalar KdV equation

\[ u_{12,t} + u_{12,xxx} + 6u_{12}u_{12;x} = 0. \quad (6) \]

Therefore the super-function

\[ u = \theta_1 \theta_2 \cdot u_{12}(x,t), \quad (7) \]

where \( u_{12} \) satisfies \( 3 \), is a solution of \( 1 \). Reciprocally, each function \( 7 \) yields the solution \( u_{12} = D_2 D_1 (u) \) of \( 3 \). Note that for the class \( 7 \) one transfers the formulas of Bäcklund autotransformation for KdV to the \( N = 2 \) super-KdV equations \( 1 \).

Substitution \( 7 \) further implies that equations \( 1 \) inherit the classical Hirota \( n \)-solitons \( 2 \) from the solutions

\[ u_{12} = A(a) \cdot \frac{d^2}{dx^2} \log \tau \quad (8) \]

of the KdV equation \( 3 \), see \( 3 \), where the \( \tau \)-function is

\[ \tau = \sum_{\mu_t = 0,1} \sum_{1 \leq \ell \leq n} \exp \left( \sum_{\ell = 1}^n \mu_t \cdot (\eta_\ell + \log \alpha_\ell) + \sum_{1 \leq \ell < m \leq n} \mu_\ell \mu_m \cdot \log A_{\ell m} \right), \quad (9) \]

and the dispersion and interaction (now \( \lambda \equiv 0 \)) are given through

\[ A(a) \equiv 2, \quad \omega = -k^3, \quad A_{\ell m} = (k_\ell - k_m)^2 / (k_\ell + k_m)^2, \quad \alpha_\ell \in \mathbb{R}. \quad (10) \]

**Lemma.** After each pairwise interaction, which is encoded by \( A_{\ell m}(k_\ell, k_m) \) in \( 10 \), the solitons for KdV equation \( 3 \) acquire the phase shifts, whose absolute values depend on the wave numbers \( k_\ell, k_m \) and equal

\[ \Delta \varphi_{\ell m} = 2 \log \left( |k_\ell - k_m| / |k_\ell + k_m| \right); \quad (11) \]

the signs of the shifts themselves are opposite for the \( \ell \)-th and \( m \)-th solitons. For instance, if \( k_\ell > k_m > 0 \), then the fast soliton with the wave number \( k_\ell \) is pushed forward, and the slow soliton given by \( k_m \) retards back. As \( t \to -\infty \), the two solitons move freely, still staying (respectively, as \( t \to +\infty \), having already become) distant from each other.

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The knowledge of this effect dates back to the beginning of the KdV-boom in late 60’s, see [3] and references therein. We are obliged to re-derive formula (11) in order to compare it with the application of the same reasoning to the new solutions (17), see below, of \( N = 2 \) super-equations (1). The new super-solitons demonstrate a principally different behaviour both from the analytic and physical points of view.

**Proof.** Formula (11) is particularly transparent on the level of the \( \tau \)-functions (9); without loss of generality, we assume \( n = 2 \) and thus let \( \ell = 1, m = 2 \) for a two-soliton solution of (6) given by \( k_1 > k_2 > 0 \). Let \( x_1(t) \) and \( x_2(t) \) be the coordinates of the peaks of the first and second solitons, respectively, at the times \( t \) when they are sufficiently distant from each other. For convenience, we set all \( \alpha_i := 1 \), and hence in view of (9) the \( \tau \)-function has the form

\[
\tau_{k_1,k_2}(x,t) = 1 + \exp \eta_1 + \exp \eta_2 + A_{12}(k_1,k_2) \cdot \exp(\eta_1 + \eta_2).
\]

First, let \( t \ll 0 \) be fixed. Consider the fast first soliton, which has not yet overpassed the slow second one. In the vicinity of its peak located at \( x_1(t) \), we have \( \eta_1 \approx 0 \) and, since the peak of the second soliton is still far ahead at \( x_2(t) \gg x_1(t) \) for \( t \ll 0 \), we deduce \( \eta_2(x) \ll 0 \) for \( x \approx x_1(t) \). Hence

\[
\tau_{k_1,k_2}(x,t) = 1 + \exp \eta_1 + \exp \eta_2 \cdot (1 + A_{12} \cdot \exp \eta_1) \approx 1 + \exp \eta_1 = \tau_{k_1}(\eta_1),
\]

(12)
because \( \exp \eta_2(x) \ll 0 \) uniformly as \( t \rightarrow -\infty \) on any finite interval that contains \( x_1(t) \). The asymptote is just the fast soliton in its pure form (9) without any phase shift yet.

Next, consider a vicinity \( \{ x \approx x_2(t) \} \) of the peak of the slow soliton, which still goes far ahead the fast one. Now \( \eta_2 \approx 0 \) and, since \( x_2(t) \gg x_1(t) \) for \( t \ll 0 \), we have \( \eta_1(x) \gg 0 \) at any \( x \approx x_2(t) \). Consequently,

\[
\tau_{k_1,k_2}(x,t) = \exp \eta_1 \cdot \left( 1 + \exp(\eta_2 + \log A_{12}) + \exp(-\eta_1) \cdot (1 + \exp \eta_2) \right).
\]

Acting by the operator \( 2d^2/dx^2 \circ \log \) and noting that the phase \( \eta_1 \) is only linear in \( x \) (it does not contain higher powers of \( x \)), we conclude that

\[
u_{k_1,k_2}(x,t) = 2 \frac{d^2}{dx^2} \left[ (k_1 x + \omega_1 t) + \log \left( 1 + \exp(\eta_2 + 2 \log \frac{k_1-k_2}{k_1+k_2}) + \exp(-\eta_1) \cdot (1 + \exp \eta_2) \right) \right]
\]

\[
\approx \nu_{k_2}(\eta_2 + 2 \log \frac{k_1-k_2}{k_1+k_2}),
\]

since \( \exp(-\eta_1) \ll 0 \) on a finite interval around \( x_2(t) \) as \( t \rightarrow -\infty \). The negative phase shift is precisely (11); as \( t \rightarrow +\infty \), it will vanish, and hence the overall phase shift of the slow soliton will be positive.

Notice that, initially, the slow soliton is displaced forward by \( |\Delta \varphi_{k_1,k_2}| \) with respect to an identical soliton (8) with the same wave number \( k_2 \) that would arrive to the origin \( x_2(t) = 0 \) at \( t = 0 \). Thus, we recall, the system of coordinates \( (x,t) \) used in Hirota’s method is such that the center of mass for the two soliton solution achieves \( x = 0 \) at some \( t < 0 \).

\[1\] Here and everywhere below the standard symbol \( \Rightarrow \) denotes the uniform convergence of a functional sequence on a finite closed interval. The symbol \( \ll \) denotes the uniform convergence from above (in our case, to the zero function).
Second, suppose \( t \to +\infty \); by now, the fast soliton has left the slow one far behind. Therefore, in a vicinity \( \{x \approx x_2(t)\} \) of the slow peak with \( \eta_2(x) \approx 0 \), we have \( \eta_1(x) \ll 0 \) at a fixed time \( t \), whence

\[
\tau_{k_1 k_2}(x, t) = 1 + \exp \eta_2 + \exp \eta_1 \cdot (1 + A_{12} \exp \eta_2) \equiv 1 + \exp \eta_2 = \tau_{k_2}(\eta_2),
\]

because \( \exp \eta_1(x) \ll 0 \) near \( x_2(t) \) as \( t \to +\infty \).

Concentrated around \( x_1(t) \) such that \( \eta_1 = 0 \), the fast soliton shifts forward, borrowing the energy and momentum from the slowly moving obstruction. Indeed, by \( \eta_2(x_1(t)) \gg 0 \) we have

\[
\tau_{k_1 k_2}(x, t) = \exp \eta_2 \cdot \left( 1 + \exp \left( \eta_1 + \log A_{12} \right) + \exp(-\eta_2) \cdot \left( 1 + \exp \eta_1 \right) \right),
\]

which implies

\[
u_{k_1 k_2}(x, t) = 2 \frac{d^2}{dx^2} \left[ (k_2 x + \omega_2 t + \log(1 + \exp \eta_1 + 2 \log \frac{k_1 + k_2}{k_1 k_2}) + \exp(-\eta_2) \cdot (1 + \exp \eta_1)) \right] = u_{k_1}(\eta_1 = 2 \log \frac{k_1}{k_1 + k_2}) \tag{13}
\]

on \( \{x \approx x_1(t)\} \) as \( t \to +\infty \), because \( \exp(-\eta_2) \ll 0 \) there. Comparing the in- and out-going states of the two solitons, we recognize the accumulated phase shifts \( \pm \Delta \varphi_{k_1 k_2} \) for the first and second solitons, respectively.

\begin{remark}
The dispersion law in \( \ref{eq:dispersion} \) is determined by the Hirota bilinear equation for the KdV equation \( \ref{eq:KdV} \), see \[3\],

\[
D_x(D_t + D_x^3) \tau \cdot \tau = 0,
\]

where \( D_x \) and \( D_t \) are Hirota’s derivatives. We recall that a bilinear representation for the Manin–Radul \( N = 2 \) supersymmetric KdV equation \( \ref{eq:sKdV} \) with \( a = -2 \), see \[10\], was obtained in \[11\] for its expression in Inami–Kanno’s form \[12\]. The bilinear representation for the \( a = -2 \) super-KdV \( \ref{eq:sKdV} \) is such that the \( \tau \)-functions for the \( n \)-soliton solutions, assigned to the wave numbers \( k_1, \ldots, k_n \), contain the Grassmann constants \( \zeta_1, \ldots, \zeta_n \).

\section{New (virtual) \( n \)-solitons for the \( N = 2 \) super-KdV.}

We discard the idea of using any ‘Grassmann constants,’ see Remark \[1\] above. Consequently, the most general admissible form of the Hirota exponentials is

\[
\eta = k x + \omega t + \lambda \theta_1 \theta_2, \tag{14}
\]

where \( k \in \mathbb{R} \) are the wave numbers, \( \omega \) are the frequencies, and \( \lambda \in \mathbb{C} \) describes the dependence on the Grassmann variables \( \theta_1, \theta_2 \) for \( n \)-supersolitons \( u_{k_1 \ldots k_n}(x, t; \theta_1, \theta_2) \).

\begin{proposition}
  i. The travelling wave solutions with Hirota’s exponentials \( \ref{eq:exponential} \) for the \( N = 2 \) supersymmetric KdV equation \( \ref{eq:sKdV} \) with \( a = 1 \) or \( a = 4 \) are

\[
u = A \cdot D_1 D_2 \log(1 + \alpha \exp \eta), \tag{15}
\]

where the dispersion and the relation between the wave number \( k \) and the nonzero factor \( \lambda \) are

\[
\omega = -k^3, \quad \lambda = \pm ik, \quad k \in \mathbb{R}, \quad \alpha \in \mathbb{R}; \quad \begin{cases} A = 1 \text{ if } a = 1, \\ A = \frac{1}{2} \text{ if } a = 4. \end{cases} \tag{16}
\]

\end{proposition}
ii. The Hirota two-soliton solutions of (1) with \( a \in \{1, 4\} \) and \( \lambda \neq 0 \) in (14),
\[
u = A \cdot D_1 D_2 \log \left(1 + \alpha_1 \exp \eta_1 + \alpha_2 \exp \eta_2 + A_{12} \alpha_1 \alpha_2 \exp (\eta_1 + \eta_2)\right),
\]
exist if and only if relations (16) for Hirota’s exponentials (14) hold and there is no coupling between the exponentials for one-soliton solutions (15):
\[
A_{12}(k_1, k_2) \equiv 0.
\]

iii. The exact \( n \)-supersoliton solution of the \( N=2 \) super-KdV (1) with \( a \in \{1, 4\} \) and \( \lambda_i \neq 0 \) is
\[
u = -A \cdot \sum_{i=1}^{n} \lambda_i \alpha_i \exp \tilde{\eta}_i + A \cdot \theta_1 \theta_2 \frac{d}{dx} \left[ \sum_{i=1}^{n} k_i \alpha_i \exp \tilde{\eta}_i \right]. \tag{17}
\]
Here \( n \in \mathbb{N} \) is arbitrary and we put the reduced phase be
\[
\tilde{\eta} = kx + \omega t; \tag{18}
\]
then, substituting (16) for \( A(a) \), \( \omega(k) \), and \( \lambda(k) \), we expand the supersoliton (2) with the \( \tau \)-function
\[
\tau_{k_1 \ldots k_n}(x, t; \theta_1, \theta_2) = 1 + \sum_{i=1}^{n} \alpha_i \exp \tilde{\eta}_i \cdot (1 + \lambda_i \theta_1 \theta_2)
\]
in \( \theta_1 \theta_2 \), whose square is zero.

iv. The \( a = -2 \) super-equation (1) does not possess travelling wave solutions (16) other than (10) with \( \lambda \equiv 0 \), and hence it does not admit \( n \)-supersolitons (17) for any \( n \geq 1 \). This also holds for any other \( a \in \mathbb{R} \setminus \{1, 4\} \), which still do admit the \( n \)-solitons given by (7–10) but not by (15–16).

**Proof.** Once (15–17) are known, statements (i–iii) are verified by direct calculation. The exhaustive conclusion of statement (iv) is obtained using analytic environment [13].

**Remark 2.** In [14], a particular class of exact solutions was constructed for the Manin–Radul \( N=2 \) super-KP equation and for its reduction to the \( N=2 \) super-KdV equation (1) with \( a = -2 \). The components of these solutions (those are called solitinos in loc. cit.) are at most quadratic in Hirota’s exponentials \( \exp \eta_i \) for any number \( n \geq 2 \) of interacting solitinos, see [14, Eqs (39–40)].

We have extended this result onto the entire triple \( a \in \{-2, 1, 4\} \) of integrable Hamiltonian super-equations (1). Namely, for \( N=2 \) super-KdV equations with \( a = 1 \) or 4 we revealed the \( n \)-soliton solutions (17), whose \( \tau \)-functions remain linear in \( \exp \eta_i \) for any \( n \geq 1 \). We stress that the unexpected \( n \)-solitons (17) of the \( N=2 \) super-KdV must also solve the \( N=2 \) supersymmetric KP equations that reduce to (1) with \( a \in \{1, 4\} \). Definitely, the admissible super-systems are not the Manin–Radul \( N=2 \) super-KP [10] that implies \( a = -2 \).

Finally, we note that the constraint \( u_1 = u_2 \equiv 0 \) is fulfilled for the components of super-fields (17). Therefore Proposition (11) yields exact solutions of the Krasil’shchik–Kersten equation, see (5), whenever \( a = 1 \). Likewise, it provides the \( n \)-solitons for the bosonic limit of (11) if \( a = 4 \).
Under the nonlinear superposition of $n$ solitons, exact solutions (17) of the equation (1), which are given by linear truncations (3) of Hirota’s $\tau$-functions, demonstrate the properties that are essentially different from the standard picture (11) for KdV (6), regarding both the asymptotes at the time infinity and the phase shifts for each of the $n$ solitons. We claim that

the asymptotic behaviour ($t \to \pm\infty$) of an $n$-soliton solution (17) for (1) is completely determined by the soliton that moves behind the others, and that no phase shifts at all are accumulated as the time $t$ runs from $-\infty$ to $+\infty$.

The proof goes in parallel with (12–13), see p. 4, and the reasonings are alike for all $n \geq 2$, hence we set $n = 2$ and suppose $k_{\text{max}} = k_1 > k_2 = k_{\text{min}} > 0$. Without loss of generality, we assume $\alpha_i = 1$ for all $i$. Recall that $\tau_{k_1k_2}(x, t; \theta_1, \theta_2) = 1 + \exp \eta_1 + \exp \eta_2$, where the full phases $\eta_i$ are (14) and the dispersion is specified by (16). This yields the solutions (17) of (1), where $A(a)$ is given in (10) for either $a = 1$ or $a = 4$. Notice that the essential non-constant part of a soliton (15) is, as usual, concentrated near the zero of the reduced phase (18). However, the first, purely imaginary summand in the expansion (17) does not have a peak located at $\tilde{\eta} \approx 0$. At the same time, the real coefficient of $\theta_1\theta_2$ in the one-soliton solution does have the peak at $\tilde{\eta} \approx 0$. (Let us remark that this coefficient is nothing else but one half (for $a = 1$) or a quarter (respectively, $a = 4$) of the one-soliton solution (8) for KdV, see (19).) We shall use this in what follows to refer on points $x \in \mathbb{R}$.

So, let $t \to -\infty$. In a vicinity of the peak $x_1(t)$ of the fast soliton with the maximal wave number $k_1 = k_{\text{max}}$, the reduced phase (18) is $\tilde{\eta}_1 \approx 0$, and $\tilde{\eta}_2 \ll 0$. For still distant solitons, $\exp \tilde{\eta}_2 \ll 0$ as $t \to -\infty$ on any finite interval $\{x \approx x_1(t)\}$. Therefore,

$$
\tau_{k_1k_2}(x, t; \theta_1, \theta_2) = 1 + \exp \eta_1 + \exp \eta_2 \implies 1 + \exp \eta_1 = \tau_{k_1}(\eta_1). \quad (19)
$$

This is the fast soliton with its unshifted full phase $\eta_1(x, t)$.

On the other hand, near the peak $x_2(t)$ of the slow soliton we have $\tilde{\eta}_2 \approx 0$ and $\tilde{\eta}_1 \gg 0$, whence

$$
\tau_{k_1k_2}(x, t; \theta_1, \theta_2) = \exp \eta_1 \cdot (1 + \exp(-\eta_1) \cdot (1 + \exp \eta_2)).
$$

Applying the operator $A(a) \cdot \mathcal{D}_1\mathcal{D}_2 \circ \log$ to the $\tau$-function, we obtain the asymptote of the two-soliton solution on a finite interval near $x_2(t)$:

$$
u_{k_1k_2}(x, t; \theta_1, \theta_2) = A \mathcal{D}_1\mathcal{D}_2 \left[ (k_1 x + \omega_1 t) + \log(1 + \lambda_1 \theta_1 \theta_2) + \log \left( 1 + \exp(-\eta_1) \cdot (1 + \exp \eta_2) \right) \right] \implies -A\lambda_1 = \text{const}.
$$

In other words, the solution near $x_2(t)$ does not vary under a small finite motion left or right along $x$ at $t \ll 0$. For this reason, we say that the slow soliton has not yet manifested its presence, remaining virtual. We also say that the visible fast soliton, which starts being the slower soliton(s) in the beginning of the elastic scattering process, is loaded with the virtual soliton(s). Note that the asymptotic behaviour of $u_{k_1\ldots k_n}(x, t; \theta_1, \theta_2)$ at early times $t \ll 0$ is analogous for any $n \geq 2$. 

Now consider $t \to +\infty$. The slow soliton with the wave number $k_2 = k_{\min}$ becomes visible. Near its peak at $x_2(t)$, we have $\tilde{\eta}_2 \approx 0$ and $\tilde{\eta}_1 \ll 0$ as soon as the fast soliton has gone sufficiently far ahead. Consequently,

$$\tau_{k_1k_2}(x; t; \theta_1, \theta_2) = 1 + \exp \eta_1 + \exp \eta_2 \equiv 1 + \exp \eta_2 = \tau_{k_2}(\eta_2),$$

because $\exp \tilde{\eta}_1 \ll 0$ near $x_2(t)$ as $t \to +\infty$.

And the fast soliton becomes virtual. Indeed, near its peak at $x_1(t)$ such that $\tilde{\eta}_1 \approx 0$, we have $\tilde{\eta}_2 \gg 0$ and $\exp(-\tilde{\eta}_2) \ll 0$ as $t \to +\infty$, whence

$$\tau_{k_1k_2}(x; t; \theta_1, \theta_2) = \exp \eta_2 \cdot (1 + \exp(-\eta_2) \cdot (1 + \exp \eta_1))$$

yields

$$u_{k_1k_2}(x, t; \theta_1, \theta_2) = AD_1D_2 \left[(k_2x + \omega_2 t) + \log(1 + \lambda_2 \theta_1 \theta_2) + \log(1 + \exp(-\eta_2) \cdot (1 + \exp \eta_1))\right]$$

$$\equiv -A\lambda_2 = \text{const.} \quad (20)$$

This value is the limit $\lim_{x \to +\infty} u_{k_2}(x, t)$ determined by the slow soliton that now goes behind the fast. The constant does not alter under a small finite motion left or right from the point $\tilde{\eta}_1 = 0$, where the peak of the fast soliton would be located at any $t \gg 0$.

We have described the “vanishing” of the fast soliton. Now the visible slow soliton is loaded with the virtual fast soliton.

It remains to note from (19) that none of the solitons, neither the visible fast that becomes virtual, nor the virtual slower soliton that becomes visible, acquires any phase shift. The proof is complete. \(\blacksquare\)

Let us summarize our main result.

**Theorem 2.**

i. As $t \to -\infty$, the time asymptote of the $n$-soliton (17) for the super-KdV (1) is given by the fastest soliton with the maximal wave number $k_{\max}$, which is loaded with the virtual slower solitons that correspond to $k_2, \ldots, k_{\min} < k_{\max}$. These virtual solitons are invisible at $t \ll 0$.

ii. As $t \to +\infty$, the asymptote of the same solution $u_{k_1 \ldots k_n}(x, t; \theta_1, \theta_2)$ for (1) is the slowest soliton with the minimal wave number $k_{\min}$, which is loaded with faster solitons determined by $k_{\max}, \ldots, k_{n-1} > k_{\min}$. These $n-1$ virtual solitons remain (for $k_2, \ldots, k_{n-1}$) or become (respectively, for $k_{\max}$) invisible at $t \gg 0$.

iii. None of the solitons, neither the visible, which goes behind the others, nor the virtual ones in front of it, acquires any phase shift under the evolution in time $t \in (-\infty, +\infty)$.

**Corollary.** The iterated spatial and temporal limits of the pure imaginary first summand in (17) are not permutable: as $x, t \to +\infty$, one has\(^2\)

$$\lim_{t \to +\infty} \lim_{x \to +\infty} \sum_{i=1}^n \lambda_i \alpha_i \exp \tilde{\eta}_i = \lambda(k_{\max}) \neq \lambda(k_{\min}) = \lim_{x \to +\infty} \lim_{t \to +\infty} \sum_{i=1}^n \lambda_i \alpha_i \exp \tilde{\eta}_i$$

\(^2\)Note that the limit in $x$ in the right-hand side is taken before the peak $x_1(t)$ of the slowest $n$-th soliton passes through a point $x$, which means that the exponentially vanishing tail of all solitons has not arrived yet.

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This also shows that the solution \( u_{k_1 \cdots k_n}(x, t; \theta_1, \theta_2) \) is not a Schwarz function in \( x \). Clearly, as \( x \to -\infty \) at any \( t \), the limit of the first summand in (17) equals zero (we assume \( k_1 > 0 \) everywhere in the text). The real coefficient of \( \theta_1 \theta_2 \) in (17) tends to \( A(a) \frac{d}{dx}(\text{const}) = 0 \) as \( x \to \pm \infty \) at any \( t \).

The initial limit as \( x \to +\infty \) at any finite time \( t \) is due to the visible fastest soliton with the wave number \( k_{\text{max}} \), which is loaded with virtual slower soliton(s). The spatial limit as \( x \to +\infty \) for the out-going state at large times \( t \gg 0 \) is specified by the slowest soliton, which is loaded with the virtual faster waves and which progressively occupies the entire \( x \)-axis.

**Remark 3.** The \( a = 4 \) super-KdV equation (1) possesses twice as many Hamiltonian functionals and hence twice as many symmetries as the other two cases \( a \in \{-2, 1\} \). Therefore there are twice as many commuting flows as in the hierarchies with \( a = -2 \) or \( a = 1 \). In particular, between the spatial translation \( \partial/\partial x \) and the translation \( \partial/\partial t \), which is determined by (11) in the \( a = 4 \) super-KdV hierarchy, there is the \( N = 2 \) supersymmetric ‘Burgers’ flow [15],

\[
\dot{u} = D_1 D_2 u_x + uu_x. \tag{21}
\]

Proposition 1 and Theorem 2 are literally reproduced for (21). The \( n \)-supersolitons are (17) with the following dispersion relations in Hirota’s \( \tau \)-function (3),

\[
A = 2, \quad \omega = -ik^2, \quad \lambda = ik, \quad \alpha \in \mathbb{R}. \tag{22}
\]

**Conclusion.** The new \( n \)-soliton solutions (17) satisfy \( N = 2 \) super-KdV equation (1) with \( a \in \{1, 4\} \) or, to be even more precise, its bosonic limit, which is the Krasil’shchik–Kersten system if \( a = 1 \). Recall that the \( t \to -\infty \) asymptote of our solutions (17) is given by the fastest soliton with the maximal wave number \( k_{\text{max}} \), which is loaded with \( n - 1 \) virtual slower solitons ahead of it. We have proved that, looking at the initial profile and recognizing it as a one-soliton solution up to any precision of measurements, one can predict neither the out-going state nor even its asymptote \(-iAk_{\text{min}}\) at \( t \gg 0 \) as \( x \to +\infty \).

For an observer, the fast soliton \( (k_1 = k_{\text{max}}) \) is subject to a spontaneous decay. The profile transforms into a collection of nonlinear–overlapping waves \( (k_1 > k_2 > \cdots > k_n) \) that, finally, constitute what one sees as another soliton with a different wave number \( k_n = k_{\text{min}} \). For a given number of solitons \( n \geq 2 \) and for \textit{a priori} given wave numbers \( k_{\text{min}} \) and \( k_{\text{max}} \), this can be realized along any trajectory, which is a point in the configuration space

\[
\mathbb{R}^{n-2}_+ \times \mathbb{R}^{n-1} \ni (k_2, k_2 - k_3, \ldots, k_{n-2} - k_{n-1}) \times (\alpha_2, \ldots, \alpha_n),
\]

since over \( \mathbb{C} \) one can always rescale \( \alpha_1 = 1 \).

The absence of the phase shifts, supplemented with the presence of hierarchies of conservation laws for Hamiltonian \( N = 2 \) supersymmetric KdV equations (11), demonstrates that the spatially extensive solitary waves behave in elastic collisions as dimensionless particles (as massive material points). In this paper, we established the possibility of their spontaneous decay, and we formulated the laws for transformation of observed particles into the virtual ones and \textit{vice versa}.
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