A classification of Thurston geometries without compact quotients

Panagiotis Konstantis∗ Frank Loose†

Abstract
We classify pairs \((M, G)\) where \(M\) is a 3–dimensional simply connected smooth manifold and \(G\) a Lie group acting on \(M\) transitively, effectively with compact isotropy group.

1 Introduction

Background. Over the last few decades the study of closed 3-manifolds was a very active topic in geometry and topology, which culminated in the remarkable proof of Thurston’s geometrization conjecture by G. Perelman. It was known since [Kne29] and [Mil62] that every closed 3-manifold \(M\) can be decomposed uniquely into prime 3-manifolds. On the other hand W. Thurston conjectured in [Thu82] that if a prime manifold is cut along some tori, which is also known as the Jaco-Shalen-Johannson Torus Decomposition, the interior admits a unique geometric structure of finite volume. This means that the universal cover of the interior is a model geometry also known as Thurston geometry. These are defined as pairs \((M, G)\) where \(M\) is a simply connected, smooth, 3-dimensional manifold and \(G\) a Lie group of diffeomorphisms of \(M\), such that \(G\) acts transitively and with compact stabilizers. Moreover, \((M, G)\) should admit compact quotients (i.e. there is a discrete subgroup \(\Gamma \subset G\) such that \(M/\Gamma\) is a smooth compact manifold) and \(G\) should be maximal in the sense that \(G\) is not contained in a bigger Lie group of diffeomorphisms with same properties.

The list of Thurston geometries may be found, e.g. in [Thu97] or in [Sco83], however it is difficult to find a stringent proof of this classification among the literature. One aim of this article is to present a complete proof of this classification following the arguments in [Thu97].

Another aim is to study model geometries without assuming the existence of compact quotients nor requiring the maximality of \(G\). We call such a pair \((M, G)\) simply a geometry. These objects may be interesting for applications in physics, since they may be used to model the spatial parts of homogeneous space-times. For example in [Kon13] they were used to study non-isotropic, homogeneous cosmological models with positive cosmological constant. Finally it should be mentioned that in [Sco83, p.474] there is a remark wherein R. S. Kulkarni carried out such classification but remained unpublished.
Overview of the classification. As we already pointed out the classification will follow the outline of the proof in [Thu97] but without using the assumption that \((M, G)\) has compact quotients. For the sake of convenience we assume that \(G\) is connected. An easy argument shows that the stabilizer \(K\) of \((M, G)\) in a point has to be connected and since \(K\) is compact, it has to be a subgroup of \(\text{SO}(3)\). It follows that the possible dimensions of \(K\) are 0, 1 and 3 since the Lie algebra of \(\text{SO}(3)\) clearly has no two-dimensional Lie subalgebra. So we divide the geometries according to the dimension of their isotropy groups:

- \(\dim K = 3\) : We call this type an isotropic geometry. Clearly we have \(K = \text{SO}(3)\) and \(M\) admits a metric of constant sectional curvature. Hence \(M\) is a space form and \(G\) is the identity component of the full isometry group of the standard metric of this space form, compare Theorem 3.8.4 (a) in [Thu97].

- \(\dim K = 1\) : We call this type an axially symmetric geometry. The isotropy group is isomorphic to \(\text{SO}(2)\) and it acts on the tangent space of a fixed point as a rotation around a unique line. This implies the existence of a \(G\)-invariant vector field \(X\) which in turn gives a foliation into geodesics of \(M\) through its flow lines. Here we give a precise argument why the space of leaves \(N\) inherits a differentiable structure from \(M\) (compare [Thu97, p. 183]). The crucial fact is that the flow acts properly on \(M\) which is proven in Corollary 2.9. Moreover the quotient map \(\pi: M \to N\) is a fiber bundle with \(\mathbb{R}\) or \(\text{SO}(2)\) as fiber group. We explain in section 2.2 that the plane field orthogonal to \(X\) with respect to any \(G\)-invariant metric is unique and defines a connection for this bundle, compare [Thu97, p. 183] and Corollary 2.14. Since this connection is \(G\)-invariant its curvature is either globally zero or not. Furthermore, from now on the classification in [Thu97] makes heavily use of the existence of compact quotients. A key point is here that if \((M, G)\) admits compact quotients, then \(\text{div} \, X\), which is the divergence of \(X\) with respect to a \(G\)-invariant metric, is zero, cf. [Thu97, p. 182]. But then it follows from Proposition 2.5 that \(X\) has to be a Killing field for a \(G\)-invariant metric which means that \(N\) admits a metric (of constant curvature) such that \(\pi: M \to N\) is a Riemannian submersion.

In section 2.3 we start to classify the geometries which allow such a flat connection. In [Thu97] there is no argument to prove this step. In our approach the key fact for this case is Proposition 2.16 where \(G\) fits into an exact sequence

\[
1 \to G' \to G \to \mathbb{R} \to 1
\]

where \(G' \in \{E_6(2), \text{SO}(3), \text{SO}^+(2, 1)\}\). Now if \((M, G)\) admits compact quotients this sequence splits through the flow of \(X\), therefore \(G = G' \times \mathbb{R}\) and we obtain the geometries of \((b_1)\) in [Thu97, p. 183] (compare Remark 2.19). Otherwise we obtain an additional geometry, cf. Example 2.20. A complete list and proof may be found in Theorem 2.21.

The classification of axially symmetric geometries with non-flat \(G\)-invariant connection is stated within section 2.4. We prove with Corollary 2.23, even though \((M, G)\) does not admit compact quotients, that \(X\) is a Killing field. This means basically that we do not obtain a new geometry here. Hence this case does not depend on the existence of compact quotients. Nonetheless there is no precise argument for this step in [Thu97] or in [Sco83]. Therefore we use our Lie group theoretical approach here. Basically we try to obtain the Lie
algebra of $G$ by means of extensions of Lie groups and Lie algebras. Afterwards we deduce the Lie group $G$ as well as the action of $G$ on $M$. This is all done in the Propositions 2.25, 2.27 and 2.28. The final list is stated in Theorem 2.29.

- $\dim K = 0$: Here $M$ is the Lie group $G$. Since we assumed $M$ to be simply connected, the Lie group $G$ is uniquely determined by its Lie algebra $\mathfrak{g}$. This was done by many authors, compare [Bia98], [Bia01], [GKKL13], [Kon13]. The 3-dimensional Lie groups which are Thurston geometries are the unimodular Lie groups, compare [Mil76, p.99]. Also in [Mil76] one finds a complete classification of unimodular Lie groups. For a more detailed discussion see section 3.

Notations and basic definitions. A geometry is a pair $(M, G)$ where $M$ is a simply connected, smooth 3-manifold and $G$ is a Lie group of diffeomorphisms acting on $M$ transitively with compact stabilizers. We assumed $G$ to be connected, but with a little more effort the classification can be done without this assumption. Using the long exact homotopy sequence, this implies that the stabilizer $K$ has to be connected, since $M$ is simply connected.

Actions of $\theta: G \times X \to X$ on smooth manifolds $X$ will be abbreviated by $\theta(g, x) := g.x$. Note that $G$ is a subgroup of $\text{Diff}(M)$, hence every element $g \in G$ is viewed as a diffeomorphism on $M$.

For this article we fix a point $m_0 \in M$ and let $K$ be its isotropy group in $G$. We denote with $\rho: K \to \text{SO}(3)$ its faithful isotropy representation. Hence

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Some groups will be of great importance in this article. We denote be $E_0(n)$ the connected subgroup of the full group of motions in euclidean space, i.e. $E_0(n) = \mathbb{R}^n \rtimes \text{SO}(n)$, where $\text{SO}(n)$ acts on $\mathbb{R}^n$ by its standard representation. Furthermore let $\text{SO}^+(n, 1)$ be the connected component of the isometry group of hyperbolic space $D^n$.

At the beginning it was mentioned that the classification of isotropic geometries is very easy. The resulting geometries are given by the pairs

\[ (\mathbb{R}^3, E_0(3)), \quad (S^3, \text{SO}(4)), \quad (D^3, \text{SO}^+(3, 1)) \]

with the standard actions as isometry groups.

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2 Axially Symmetric Geometries

The classification of axially symmetric geometries is much harder than the classification of isotropic geometries. Here the crucial fact is that every non-trivial $\text{SO}(2)$-representation on a 3-dimensional vector space has a unique decomposition into two irreducible subspaces. Even more, $\text{SO}(2)$ is acting trivially on the 1-dimensional subspace. Since the isotropy groups of an axially symmetric
geometry are isomorphic to $SO(2)$, this induces a $G$–invariant vector field $X$ on $M$ whose flow determines a foliation of $M$ into geodesics.

In particular we will see that the space of leaves is a smooth manifold such that $M$ is the total space of a principal bundle over a surface. Now the 2-dimensional irreducible subspace creates a distribution of planes on $M$ which turns out to be a $G$–invariant connection on $M$. The classification will be then divided into two cases, namely if the connection is flat or not.

### 2.1 Foliation of $M$ into geodesics

We start to study the isotropy representation and the properties of a $G$–invariant vector field which comes from that representation. The flow of the vector field will give us a foliation of $M$.

Since $\dim K = 1$ we obtain $K \cong SO(2)$ and since the isotropy representation $\rho$ is faithful we formulate without proof the crucial

**Proposition 2.1.** Let $\rho: SO(2) \to V$ be a faithful representation on a 3–dimensional vector space $V$. Then there is a unique $SO(2)$–invariant decomposition $V = L \oplus W$ into irreducible subspaces where $\dim L = 1$ and $\dim W = 2$ such that $\rho$ fixes every point in $L$ and $W$ is the orthogonal complement to $L$ for every $SO(2)$–invariant metric.

It will be of great importance to decompose the $G$–invariant bilinear forms on $M$ with respect to the representation $\rho$. Therefore it is sufficient to know how the $K$–invariant endomorphisms on $(V, \rho)$ decompose. We denote by $\text{End}_\rho(V)$ the algebra of equivariant endomorphisms on $(V, \rho)$.

**Proposition 2.2.** Suppose $\rho: SO(2) \to GL(V)$ is a faithful representation on a 3–dimensional vector space $V$. If $f \in \text{End}_\rho(V)$ then it is given by

$$
\begin{pmatrix}
\alpha & 0 \\
0 & \lambda \cdot k|W
\end{pmatrix}
$$

with respect to the decomposition $V = L \oplus W$ of Proposition 2.1 where $\alpha \in \text{End}(\mathbb{R}) \cong \mathbb{R}$, $\lambda \geq 0$ and $k \in SO(2) \subseteq GL(V)$.

**Proof.** We abbreviate $K = SO(2)$. Choose a $K$–invariant scalar product $\sigma$ on $V$. For $f \in \text{End}(V)$ we denote by $f^*$ the adjoint endomorphism of $f$ with respect to $\sigma$. Note thereby that $k^* = k^{-1}$ for all $k \in K$. It is easy to see that $\text{End}_\rho(V)$ is invariant under the map $f \mapsto f^*$. Now let $f$ be a $K$–equivariant endomorphism. If $l \in L$ then by the uniqueness of $L$ as the fixed point space of $K$ we obtain $f(l) \in L$. For $w \in W$ we compute

$$
\sigma(f(w), l) = \sigma(w, f^*(l)) = 0
$$

for all $l \in L$ since $W$ is the orthogonal complement to $L$, see Proposition 2.1. Hence $f|W: W \to W$ is a $K$–equivariant map. Fix a vector $w_0 \in W$ then for every $w \in W$ there are $\lambda \geq 0$ and $k \in K$ such that $w = \lambda(k \cdot w_0)$. This implies $f|W = \lambda \cdot k|W$. 

We apply Proposition 2.1 to the case where $\rho$ is the isotropy representation of $(M, G)$ in some point of $M$. This leads to a 1–dimensional Distribution $\mathcal{D}$ on $M$ such that $\mathcal{D}_m$ is the 1–dimensional fixed point subspace of $TM_m$ as in Proposition 2.1. Moreover this line bundle is trivial:
Corollary 2.3. There is a $G$–invariant vector field $X$ on $M$ which has no zeros and for every point $m \in M$ the vector $X_m$ spans the space $\mathcal{D}_m$.

Proof. Fix a point $m_0 \in M$ and a $0 \neq \xi \in \mathcal{D}_{m_0}$. For $m \in M$ there is a $g \in G$ such that $m = g.m_0$. Define now $X_m := g.\xi$ where here we use the induced action of $G$ on $TM$. This is well–defined since any other $g' \in G$ such that $m = g'.m_0$ differs from $g$ by an element of the isotropy $K$ in $m_0$. But $\xi$ is fixed by elements of $K$, hence $X_m$ is well–defined. And since the isotropy groups of the points $m$ and $m_0$ are conjugated by $g$ we obtain that $X_m$ lies in $\mathcal{D}_m$. ■

Remark 2.4.

(a) The vector field $X$ is complete because of his $G$–invariance and moreover its flow commutes with all elements of $G$, compare Remark 2.11. So let $\Phi: \mathbb{R} \times M \rightarrow M$ be its flow. This can be viewed as an action of $\mathbb{R}$ on $M$. We will show in Proposition 2.5 that the integral curves of $X$ are geodesics for every $G$–invariant metric.

(b) If $\mu$ is a $G$–invariant metric, the length of $X$, $\sqrt{\mu(X,X)}$, as well as the divergence of $X$ with respect to $\mu$ are constant functions on $M$. Surely $\mu(X,X)$ is non–zero and $\text{div}X$ is zero if $(M,G)$ admits compact quotients (cf. [Thu97]). Moreover as the next proposition will show, $\text{div}X$ does not depend on the $G$–invariant metric and indicates when $X$ is a Killing field.

Proposition 2.5. The divergence of $X$ with respect to a $G$–invariant metric does not depend on the metric and $\text{div}X = 0$ if and only if $X$ is a Killing field. Furthermore the integral curves of $X$ are geodesics for $G$–invariant metrics.

Proof. Let $\mu$ be a $G$–invariant metric. The metric $\mu$ induces a global $G$–invariant Riemannian volume form $v_\mu$ since $M$ is orientable, which follows from the fact that $(M,G)$ has connected isotropy groups. Thus if $\mu'$ is another $G$–invariant metric and $v_{\mu'}$, its corresponding $G$–invariant volume form, then there exists a non–zero scalar $\lambda \in \mathbb{R}$ such that $v_{\mu'} = \lambda v_\mu$. Hence

$$ (\text{div}_{\mu'}X)v_{\mu'} = \mathcal{L}_X v_{\mu'} = \lambda \mathcal{L}_X v_\mu = (\text{div}_\mu X)v_{\mu'} $$

which shows that $\text{div}_\mu X$ does not depend on $\mu$.

Fix a point $m_0 \in M$ and let $K$ be the isotropy group in $m_0$. The bilinear form $\mathcal{L}_X \mu$ on $M$ is $G$–invariant and hence it is determined by the symmetric bilinear form $\beta := (\mathcal{L}_X \mu)_{m_0}$. Let $V := TM_{m_0} = L \oplus W$ be the $K$–invariant decomposition (see Proposition 2.1) and denote by $f: V \rightarrow V$ the $K$–invariant endomorphism defined as $\mu_{m_0}(f(v),w) = \beta(v,w)$ for all $v,w \in V$, which is self–adjoint with respect to $\mu_{m_0}$. Note that since $\text{div}X$ is constant we have $2\text{div}X = \text{tr} f$. With Proposition 2.2 it follows that $f$ is represented as

$$
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \text{id}_W
\end{pmatrix}
$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. For all $l \in L$ we have $\beta(l,l) = 0$ thus $\lambda_1 = 0$. Hence the vector field $X$ is a Killing field if and only if $\lambda_2 = 0$. But this is equivalent to $\text{div}X = 0$ since $\text{div}X = \frac{1}{2} \text{tr} f = \lambda_2$.

Suppose $\mu$ is a $G$–invariant metric on $(M,G)$ and $\nabla$ its Levi–Civita connection. We have $k.(\nabla_{X_{m_0}}X) = \nabla_{X_{m_0}}X$ for all $k \in K$, so there is a $\lambda \in \mathbb{R}$
such that $\nabla_{X_m} X = \lambda X_{m_0}$. But since $X$ has constant length with respect to $\mu$, the vector $\nabla_{X_m} X$ lies orthogonal to $X_{m_0}$, hence $\lambda = 0$. This shows that the integral curves of $X$ are indeed geodesics since $\nabla_X X$ is $G$–invariant.

Everything is set up to prove that the quotient of $M$ by the flow of $X$ is indeed a smooth manifold.

**Proposition 2.6.** The action of $\mathbb{R}$ through the flow of $X$ is either free or descends to a free circle action of $S^1$ on $M$.

**Proof.** Suppose the action of $\mathbb{R}$ is not free. Let $Z_m$ be the isotropy group of $\mathbb{R}$ in the point $m \in M$ and since $Z_m$ has to be a closed 0–dimensional subgroup of $\mathbb{R}$ we obtain that $Z_m$ is isomorphic to $\mathbb{Z}$. The group $Z_m$ fixes any other point on $M$ since the flow $\Phi$ and the action of $G$ commute. Hence we may pass to an induced free action $\Psi$: $\mathbb{R}/ \mathbb{Z} \times M \to M$.

In the case of a circle action we get a smooth surface $N$ as the quotient $M/S^1$ (i.e. the leaf space of the foliation) and moreover the quotient map $\pi: M \to N$ is a circle principal bundle over $N$ since $S^1$ is compact. We would like to obtain a similar result in the case of a free $\mathbb{R}$ action on $M$. The key property is thereby that $\mathbb{R}$ has to act properly on $M$, i.e. the map $f: \mathbb{R} \times M \to M \times M, (t, m) \mapsto (t.m, m)$ has to be a proper map between topological spaces.

To prove this it is sufficient to show that if $(m_i)_{i \in \mathbb{N}}$ is a converging sequence in $M$ and $(t_i, m_i)_{i \in \mathbb{N}}$ converges as well then $(t_i)$ has a converging subsequence. From now on we assume a free $\mathbb{R}$ action on $M$. We define $\text{Fix}(K) = \{ m \in M : k.m = m \text{ for all } k \in K \}$ the fixed–point set of $K$. For every $G$–invariant metric $\text{Fix}(K)$ is a totally geodesic, embedded and closed submanifold of $M$, see [Kob95, p.59].

**Proposition 2.7.** The orbit $\mathbb{R}.m_0$ is the connected component of $\text{Fix}(K)$ containing $m_0$.

**Proof.** $F$ shall denote the connected component of $\text{Fix}(K)$ containing $m_0$. Then since the actions of $G$ and $\mathbb{R}$ commute (see Remark 2.11) we have $\mathbb{R}.m_0 \subset F$. Choose now a $G$–invariant metric and let $m \in F$. So there is a geodesic $\gamma$ lying in $F$, starting at $m_0$ and ending in $m$. Moreover there is a $\lambda \in \mathbb{R}$ such that $\gamma(0) = \lambda X_{m_0}$ and therefore $\gamma(t) = \exp_{m_0}(t\lambda X_{m_0}) = (t\lambda).m_0$ since $\beta :: t \mapsto t.m_0$ is the geodesic with $\beta(0) = X_{m_0}$. This shows $F \subset \mathbb{R}.m_0$.

**Corollary 2.8.** The orbits $\mathbb{R}.m$ for $m \in M$ are embedded, closed submanifolds of $M$.

**Corollary 2.9.** The action of $\mathbb{R}$ through the vector field $X$ is proper.

**Proof.** Let $(m_i)$ be a converging sequence in $M$ and let $(t_i)$ be a sequence in $\mathbb{R}$ such that $(t_i, m_i)$ converges as well. We will show that $(t_i)$ has a converging subsequence which implies that the action is proper. Choose a sequence $(g_i)$ in $G$ such that $m_i = g_i.m_0$ for all $i \in \mathbb{N}_0$. Moreover the principal bundle $p: G \to M, p(g) = g.m_0$ has compact fibers and since $(m_i)$ converges, we may pass to a subsequence such that $(g_i)$ converges in $G$ which implies that $g_i^{-1}(t_i, m_i)$ converges. If we use that the actions of $G$ and $\mathbb{R}$ commute we see that $g_i^{-1}(t_i, m_i) = t_i.m_0$ converges and with Corollary 2.8 we obtain that $(t_i)$ converges to some point in $\mathbb{R}$.
Now by the Quotient Manifold Theorem (see [Lee03, p. 218]) the space $M/R$ has a unique manifold structure such that the canonical projection $\pi: M \to M/R$ is an $R$-principal bundle. This motivates the following.

**Definition 2.10.** Let $R$ be considered above. Moreover let $\pi: N \to M$ be the smooth surface $N$ we conclude that $N$ has a unique manifold structure such that the canonical projection $\pi: M \to N$ is simply connected.

2.2 The $G$–invariant connection on $M \to N$

Now we are ready to study the canonical $G$-invariant connection on the principal bundle $H \to M \to N$.

Let $\mu$ be a $G$-invariant metric on $M$ and let $\omega$ be the dual form of $X$. Proposition 2.1 tells us that the kernel of $\omega$ does not depend on the choice of the $G$-invariant metric $\mu$. Moreover, since $\mu$ and $X$ are $G$-invariant we obtain that $\omega$ is $G$-invariant as well and we may choose $\mu$ such that $\omega(X) = 1$.

We start with some technical propositions which will be needed to prove that $\omega$ is indeed a connection.

**Remark 2.11.** Let $\Phi: \mathbb{R} \times M \to M$ be the flow of $X$. For every $g \in G$ the curves $t \mapsto \Phi_t(g.m)$ and $t \mapsto g.\Phi_t(m)$ are both integral curves of $X$ and they coincide in $t = 0$. Hence $\Phi_t(g.m) = g.\Phi_t(m)$ for all $t \in \mathbb{R}$ which shows that the action of $G$ and the flow of $X$ commute which in turn implies that the actions of $H$ and $G$ commute. Now if $m_0 \in M$, $F$ its orbit under $H$ and $K$ the isotropy group in $m_0$ then $K$ is the isotropy group for all points $m \in F$ since $H$ and $K$ commute.

**Proposition 2.12.** Let $m_0 \in M$, $F$ its orbit under $H$ and $K$ the isotropy group in $m_0$. The set

$$K_F = \{ g \in G : g.m_0 \in F \}$$

is a closed, connected Lie subgroup of $G$ and there is a natural epimorphism $\Psi: K_F \to H$ with $\ker \Psi = K$.

**Proof.** Using that $G$ and $H$ commute we obtain that $K_F$ is indeed a subgroup of $G$. Moreover $K_F$ is closed since the orbit $F$ is closed (see Corollary 2.8). We define $\Psi$ as follows: for $g \in K_F$ there is a unique $h \in H$ such that $g.m_0 = h.m_0$, so define $\Psi(g) := h$. Clearly $\Psi$ is continuous and it is a homomorphism of groups. Namely, if $g_1, g_2 \in G$ and $h_i = \Psi(g_i)$ for $i = 1, 2$, then $(g_1 g_2).m_0 = (h_1 h_2).m_0 = (\Psi(g_1) \Psi(g_2)).m_0$. Hence $\Psi$ is a Lie group homomorphism. $\Psi$ is onto since $G$ acts transitively and the kernel of $\Psi$ is the isotropy group of the $G$–action in $m_0$, namely $K$.

It remains to show that $K_F$ is connected. Therefore note that $K_F/K = H$ and since $K$ as well as $H$ are connected groups it follows also that $K_F$ has to be connected.

**Corollary 2.13.** All elements of $K_F$ commute with $K$, with other words the extension

$$1 \to \text{SO}(2) \to K_F \to H \to 1$$

is a central extension.
Proof. Consider the (smooth) conjugation map $C: K_F \to \text{Aut}(SO(2))$, $C_g(k) = gkg^{-1}$. $C$ well-defined since $K$ fixes all points on $F$. But $\text{Aut}(SO(2)) \cong \mathbb{Z}_2$ and since $K_F$ is connected we obtain $C_g = \text{id}_{SO(2)}$ for all $g \in K_F$. ■

Corollary 2.14. The 1–form $\omega$ is a $G$–invariant connection form for $\pi: M \to N$.

Proof. The Lie algebra $\mathfrak{h}$ of $H$ is spanned by the vector field $X$ and therefore we may identify $\mathfrak{h}$ with $\mathbb{R}$ through $X$. In that way $\omega$ takes values in $\mathfrak{h}$ and the fundamental vector field of $r \in \mathbb{R}$ is given by $rX$, hence $\omega(rX) = r$. We check now that $\omega$ is invariant under the action of $H$. For $h \in H$ and $\xi \in T_{m_0}M$ we obtain

$$h^*(\omega)_{m_0}(\xi) = \omega_{h,m_0}(h,\xi) = \mu_{h,m_0}(X_{h,m_0},h,\xi).$$

Now choose $g \in G$ such that $g.m_0 = m_0$ thus $g \in K_F$ where $F$ is the orbit of $m$ under $H$. Let $f: TM_{m_0} \to TM_{m_0}$ be the derivative of the diffeomorphism $m \mapsto g.m$ in $m_0$. We obtain

$$\mu_{h,m_0}(X_{h,m_0},h,\xi) = \mu_{m_0}(X_{m_0},f(\xi)).$$

Using Corollary 2.13 we see that $f$ commutes with elements of $K$, i.e. $f$ is a $K$–equivariant map. Using Proposition 2.2 we obtain $\mu_{m_0}(X_{m_0},f(\eta)) = 0$ for $\eta \in \ker \omega_{m_0}$ and we conclude $\omega$ is indeed a connection form on $\pi: M \to N$. ■

The fiber group $H$ of $\pi: M \to N$ is abelian and therefore the exterior derivative $d\omega$ represents the curvature of $\omega$. Note since $\omega$ is $G$–invariant the curvature $d\omega$ is $G$–invariant as well. Hence if $d\omega$ is zero in a single point it is zero everywhere, i.e. $\omega$ is flat. We would like to study the flat case first.

2.3 Classification of geometries with flat connections

In this part we classify the axially symmetric geometries where the canonical $G$–invariant connection is flat. The crucial fact is to deduce an extension of $G$ from the flatness of $\omega$.

Now since $\omega$ is flat and $M$ as well as $N$ are simply connected we obtain the

Corollary 2.15. If $\omega$ is flat the $H$–principal bundle $\pi: M \to N$ is isomorphic to the trivial $\mathbb{R}$–principal bundle $pr_1: N \times \mathbb{R} \to N$ endowed with the trivial connection $dt$ where $t := pr_2: N \times \mathbb{R} \to \mathbb{R}$ is the projection onto the second factor. Moreover we have

(a) The horizontal distribution is given by $\mathcal{H} = \ker dt$ and a $G$–invariant vector field is obtained by taking the vector field to the flow $\Phi: \mathbb{R} \times (N \times \mathbb{R}) \to N \times \mathbb{R}$, $\Phi(t)(n,s) = (n, s + t)$ which shall be denoted by $\partial_t$.

(b) For every $G$–invariant metric $\ker dt$ is orthogonal to $\partial_t$ and $dt$ has to be $G$–invariant.

Proposition 2.16. There is a Lie group epimorphism $\Pi: G \to \mathbb{R}$ such that $G' := \ker \Pi$ acts on $N \times 0$ effectively, transitively and with isotropy group $K$.
Proposition 2.18. Let \((N \times \mathbb{R}, G)\) be a flat geometry like in Corollary 2.15 and let \(T(N \times \mathbb{R}) = \mathcal{H} \oplus V\) as described above. If \(\mu\) is a \(G\)-invariant metric on \(N \times \mathbb{R}\) then

\[ \mu = \varepsilon \alpha + \lambda dt^2 \]

where \(\alpha\) is the pullback by \(\text{pr}_1\) of a metric of constant Gaussian curvature on \(N\) and \(\lambda = \mu(\partial_t, \partial_t)\) and \(\kappa = \text{div} \partial_t\).
Proof. By Corollary 2.15 $\mathcal{H}$ and $\mathcal{V}$ are orthogonal for any $G$–invariant metric, hence $(\Phi^t)^* (\mu)$ must have the form $\nu^* + \lambda dt^2$ where $\nu^*$ is a symmetric bilinear form such that $\nu^* (\partial_t, \xi) = 0$ for all $\xi \in T(N \times \mathbb{R})$ and $\lambda = \mu(\partial_t, \partial_t)$ (note that $dt(\partial_t) = 1$). Fix a point $(n, s) \in N \times \mathbb{R}$ and let $t \mapsto g_t$ be a smooth curve in $G$ such that $g_t \circ \Phi^t(n, s) = (n, s)$ for all $t$ and $g_0 = \text{id}$. Denote by $f_t$ the automorphism of $T(N \times \mathbb{R})_{(n, s)}$ which is given as the derivative of $g_t \circ \Phi^t$ in $(n, s)$. We obtain

$$(\Phi^t)^* (\mu)_{(n, s)} = (f_t)^* (\mu_{(n, s)})$$

for all $t$. Using Corollary 2.13 and Remark 2.11 we see that $f_t$ commutes with all elements of $K$. And moreover, using Proposition 2.2 we have $f_t|\mathcal{H}_{(n, s)} = r_t k_t|\mathcal{H}_{(n, s)}$ with $r_t > 0$ and $k_t \in K$. Taking the derivative of equation (1) we obtain for $\zeta_1, \zeta_2 \in \mathcal{H}_{(n, s)}$

$$(\Phi^t)^* (\mathcal{L}_X \mu)(\zeta_1, \zeta_2) = 2r_t \dot{r}_t \mu(\zeta_1, \zeta_2)$$

where the dot denotes the $t$–derivative of $t \mapsto r_t$. From the proof of Proposition 2.5 we know that $\mathcal{L}_X \mu(\zeta_1, \zeta_2) = \kappa \mu(\zeta_1, \zeta_2)$ and so we end up to solve the equation

$$\kappa r_t = \frac{1}{2} \dot{r}_t$$

such that $r_0 = 1$, hence $r_t = e^{\frac{1}{2} \kappa t}$. Moreover we have

$$(D\Phi^t)_{(n, s)} : TN_n \oplus \mathbb{R}(\partial_t)_{(n, s)} \to TN_n \oplus \mathbb{R}(\partial_t)_{(n, s+t)}$$

and therefore for $\zeta_1, \zeta_2 \in \mathcal{H}_{(n, s)}$

$$\mu_{(n, s)}(\zeta_1, \zeta_2) = (\Phi^t)^* (\mu)_{(n, 0)}(\zeta_1, \zeta_2) = e^{\kappa s} \mu_{(n, 0)}(\zeta_1, \zeta_2).$$

But $\mu$ restricted to $N \times 0$ is invariant under $G'$, cf. Proposition 2.16. Hence it is a metric of constant curvature on $N$. Let $\nu$ be its pullback under $p_3$. We obtain

$$\mu_{(n, s)}(\zeta_1, \zeta_2) = e^{\kappa s} \mu_{(n, 0)}(\zeta_1, \zeta_2) = e^{\kappa s} \nu_{(n, s)}(\zeta_1, \zeta_2).$$

Before we state the classification for geometries with flat canonical connections we would like to introduce a geometry, which is less known and which is an axially symmetric one with flat connection.

Remark 2.19. If we assume that $G$ admits cocompact discrete subgroups then with Proposition 2.5 $t \mapsto \Phi^t$ is a one-parameter group of isometries for any $G$-invariant metric on $M$. Moreover, $\Phi^t \in G$ for all $t \in \mathbb{R}$ since $\Phi^t$ maps $\partial_t$ to $\partial_t$. By Corollary 2.15 the flow is just a translation in the fiber, hence $t \mapsto \Phi^t$ is a section for $\Pi: G \to \mathbb{R}$. It follows that $G$ is isomorphic to $G' \times \mathbb{R}$ since $\Phi^t$ lies in the center of $G$. The resulting geometries are the one listed in [Thu82, p. 183].

Example 2.20. Let $\kappa \in \mathbb{R}$ and set $G_\kappa := E(2) \times_{\rho_\kappa} \mathbb{R}$, where $\rho_\kappa: \mathbb{R} \to \text{Aut}_0(E(2))$ is given as $(\rho_\kappa)_s(a, A) := (e^{-\frac{1}{2} \kappa s} a, A)$. For the sake of convenience we omit the subscript $\kappa$ for $\rho_\kappa$ and $G_\kappa$ but nevertheless we keep in mind that those objects depend on that number. Consider the manifold $M = \mathbb{R}^2 \times \mathbb{R}$.
where we denote by \((x,t)\) a point in \(M\) such that \(x \in \mathbb{R}^2\) and \(t \in \mathbb{R}\). The group \(G\) acts on \(M\) by

\[(a,A,s)(x,t) := \left(e^{-\kappa s}Ax + a, t + s\right).
\]

It is easy to see that \((M,G)\) is an axially symmetric geometry and the \(G\)-invariant vector field \(\partial_t\) is obtained as the vector field to the flow \(\Phi^t(x,s) = (x,s + t)\). The metric

\[\mu(x,t) = \begin{pmatrix} e^{s\kappa} & E_2 \\ 0 & 1 \end{pmatrix}
\]

is invariant under the action of \(G\). The Riemannian volume form \(v_\mu\) with respect to the standard orientation \(dx \wedge dt\) is \(e^{s\kappa}dx \wedge dt\) and it follows that

\[
\mathcal{L}_{\partial_t}v_\mu = \kappa v_\mu,
\]

hence \(\text{div}\partial_t = \kappa\). Moreover the connection form is \(dt\) which is a flat connection on the trivial \(\mathbb{R}\)-principal bundle \(\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2\).

If \(\kappa \neq 0\) then \((M,G_\kappa)\) is equivariant diffeomorphic to \((M,G_1)\). The map \(f: M \to M, f(x,t) = (x,\kappa t)\) is an \(F\)-equivariant diffeomorphism, where \(F: G_\kappa \to G_1, F(a,A,s) = (a,A,\kappa s)\). The geometries \((M,G_1)\) and \((M,G_0)\) cannot be isomorphic since there is no Lie group isomorphism between \(G_1\) and \(G_0\) (note that \(G_1\) has trivial center and \(G_0\) not).

Clearly \(G_0\) is a subgroup of \(E(3)\). The group \(G_1\) is a subgroup of the isometry group of hyperbolic 3–space, \(SO^+(3,1)\). To see this consider the upper half–space \(H = \mathbb{R}^2 \times \mathbb{R}_{>0}\) and the diffeomorphism \(F: H \to M, (x,s) \mapsto (x,-\ln s)\).

The group \(G_1\) acts on \(H\) via \(F\) and the metric \(F^*(\mu)\) is invariant under this action. But \(F^*(\mu)\) is the standard hyperbolic metric on \(H\) of constant curvature equal to \(-1\), hence \(G_1 \subset SO^+(3,1)\).

**Theorem 2.21.** Let \((M,G)\) be an axially symmetric geometry and \(\pi: M \to N\) the induced principal bundle with connection form \(\omega\). Suppose \(\omega\) is flat. Then \((M,G)\) is equivariant diffeomorphic to exactly one of the following geometries

(a) \((S^2 \times \mathbb{R}, SO(3) \times \mathbb{R})\)

(b) \((D^2 \times \mathbb{R}, SO^+(2,1) \times \mathbb{R})\)

(c) \((\mathbb{R}^2 \times \mathbb{R}, E_0(2) \times \mathbb{R})\)

(d) \((\mathbb{R}^2 \times \mathbb{R}, E_0(2) \rtimes_{\rho^1} \mathbb{R})\) \[\text{[cf. Example 2.20]}\]

**Proof.** From Corollary 2.15 we know that \((M,G)\) is isomorphic to \((N \times \mathbb{R}, G)\) and the principal bundle \(\pi: M \to N\) is the trivial \(\mathbb{R}\)-bundle \(N \times \mathbb{R} \to N\). From Proposition 2.16 we have an exact sequence

\[1 \to G' \to G \to \mathbb{R} \to 1.\]

such that \(G'/K = N\). This sequence always splits as \(\mathbb{R}\) is simply connected. This implies that \(G\) is isomorphic to the semidirect product \(G' \rtimes_{\rho} \mathbb{R}\), for a Lie group homomorphism \(\rho: \mathbb{R} \to \text{Aut}_0(G')\). The isotropy group \(K\) of \(G\) acting on \(N \times \mathbb{R}\) lies in \(G' \rtimes \mathbb{R}\) as \(K \times 0\). For \(G'\) we have the possibilities \(SO(3), SO^+(2,1)\) and \(E_0(2)\).
Suppose first $G' = \text{SO}(3)$ thus $N = S^2$. The group $\text{SO}(3)$ is complete (i.e. $\text{SO}(3)$ is centerless and the automorphism group is equal to the inner automorphism group) and therefore $\rho$ is the trivial action on $G'$ which means that $G$ is $\text{SO}(3) \times \mathbb{R}$. Since the isotropy group is $K \times 0$ we have that $(M, G)$ is isomorphic to $(S^2 \times \mathbb{R}, \text{SO}(3) \times \mathbb{R})$ with the standard action. If $G' = \text{SO}^+(2, 1)$ then the connected component of its automorphism group is $\text{SO}^+(2, 1)$ itself.

Same arguments as for the case $G' = \text{SO}(3)$ apply since $\rho$ is continuous and thus the geometry is isomorphic to $(D^2 \times \mathbb{R}, \text{SO}^+(2, 1) \times \mathbb{R})$ again endowed with the standard action.

The remaining case is $G' = E_0(2)$ and $N = \mathbb{R}^2$. If $\sigma: \mathbb{R} \to G$ is the splitting map then $\rho_t$ is the conjugation with $\sigma(t)$ in $G'$, $\rho_t = \epsilon_{\sigma(t)}$ for all $t \in \mathbb{R}$. Hence the group structure of $G = E_0(2) \rtimes \mathbb{R}$ depends on the splitting map. We choose a $G$-invariant metric $\mu$ such that $\partial_t$ has length equal to 1. With Proposition 2.18 $\mu$ has the form

$$\mu = \epsilon_\alpha \delta + dt^2$$

where $\delta$ is the pullback of a flat metric on $\mathbb{R}^2$ to $\mathbb{R}^2 \times \mathbb{R}$ and $\kappa = \text{div} \partial_t$. For $t \in \mathbb{R}$ consider the map $\sigma_t: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R},$

$$\sigma_t(x, s) := \left(e^{-\frac{\kappa}{2} t} x, s + t\right)$$

which is an isometry for $\mu$ and furthermore $\sigma_t$ is contained in $G$ for all $t \in \mathbb{R}$ (note that all isometries which maps $\partial_t$ to $\partial_t$ are elements of $G$). The map $\sigma: \mathbb{R} \to G, \ t \mapsto \sigma_t$ is a Lie group homomorphism which is obviously a splitting map for the exact sequence above, since the last entry is a translation in the fiber. It follows that $\rho: \mathbb{R} \to \text{Aut}(E_0(2))$ is given by the conjugation in the diffeomorphism group with $\sigma$

$$\rho_t(a, A) = \left(e^{-\frac{\kappa}{2} t} a, A\right)$$

and therefore the action is given by

$$(a, A, t).(x, s) = \left(e^{-\frac{\kappa}{2} t} A x + a, t + s\right).$$

This geometry was described in Example 2.20.

2.4 Classification of geometries with non-flat connections

Henceforth we suppose $d\omega \neq 0$. Using the $G$-invariance of $d\omega$ we see that it is fully determined by $\delta := d\omega_{m_0}$ for $m_0 \in M$ and by assumption we have $\delta \neq 0$. From Proposition 2.2 we deduce the

**Corollary 2.22.** With respect to the decomposition $TM_{m_0} = L \oplus W$ of Proposition 2.1 $\delta$ has the form

$$\left(\begin{array}{cc}0 & 0 \\0 & \nu_2\end{array}\right)$$

where $\nu_2$ is a volume form on $W$.

**Corollary 2.23.** If $d\omega \neq 0$ then $X$ is a Killing field for any $G$-invariant metric.
Proof. With Proposition 2.5 it is sufficient to show that \( \kappa = \div X \) vanishes. Like in the proof of Proposition 2.18 we obtain

\[(\Phi^t)^*(d\omega) = e^{\kappa t}d\omega\]

using the \( G \)-invariance of \( d\omega \). Hence \( L_X d\omega = \kappa d\omega \) but on the other hand we have \( L_X d\omega = d(i_X(d\omega)) \). Since \( i_X(d\omega) \) is \( G \)-invariant it is sufficient to evaluate this 1–form in \( m_0 \). However with Corollary 2.22 we see \( i_{X_{m_0}}(\delta) = 0 \) which implies \( L_X d\omega = 0 \). Using again Corollary 2.22 \( d\omega \) restricted to the horizontal distribution is non–degenerated which forces \( \kappa = 0 \). \( \blacksquare \)

**Proposition 2.24.** The fiber group \( H \in \{ \mathbb{R}, \mathbf{SO}(2) \} \) of the principal bundle \( \pi: M \to N \) is a subgroup of the center of \( G \) and \( G' := G/H \) acts on \( N \) effectively, transitively with isotropy group isomorphic to \( \mathbf{SO}(2) \) such that \( \pi \) is an equivariant map.

**Proof.** Let \( \mu \) be a \( G \)-invariant metric. The flow of \( t \mapsto \Phi^t \) is a one–parameter subgroup of the isometry group of \((M, \mu)\) which we deduce from Corollary 2.23. But since \( \Phi^t \) maps \( X \) into \( X \) it is easy to see that \( \Phi^t \) is indeed an element of \( G \). We saw in Remark 2.11 that the actions of \( G \) and \( H \) commute which means that \( H \) is a subgroup of the center of \( G \).

The group \( G' = G/H \) acts on \( N = M/H \) by the induced action of \( G \) on \( M \): if \( [g] \in G/H \) and \([m] \in M/H\) then \([g][m] := [gm] \) which is well-defined. Clearly \( G' \) acts transitively on \( N \) since \( G \) does on \( M \). Suppose for \( g' \in G' \) we have \( g' n = n \) for all \( n \in N \). If \( g \in G \) represents \( g' \) then there is a \( h \in H \) such that \( g m_0 = h m_0 \), so there is a \( k \in K_{m_0} \) with \( g = hk \). But this would imply that \( [k] \) acts as the identity on \( N \). We claim that this means that \( k \) has to be the identity element, which proves that \( G' \) acts effectively. Let \( TM_{m_0} = L \oplus W \) be the decomposition of Proposition 2.1. By construction the differential of \( \pi: M \to N \) in \( m_0 \) namely \( D\pi_{m_0} : W \to TN_{m_0} \) \((n_0 = \pi(m_0))\) is an isomorphism. The assumption \( \pi \circ k = \pi \) implies \( k, \xi = \xi \) for all \( \xi \in W \). Thus \( k \) has to be the identity on \( M \) and therefore \( g = h \).

The isotropy group \( K' \in n_0 = \pi(m_0) \) of \( G' \) acting on \( N \) is the image of \( K \) under the quotient map \( p: G \to G' \). Note that \( K' \) has to be connected and 1–dimensional for the same reasons \( K \) is connected. Clearly we have \( p(K) \subseteq K' \), so we have to show that \( p(K) \) is injective. For \( k \in K \) suppose \( p(k) = e \). Then, as above, we have \( k \in H \) which implies that \( k \) is the neutral element, since \( k \) has a fixed point but \( H \) is acting freely on \( M \). \( \blacksquare \)

As in the case of a flat connection there are three possibilities for the pair \((N, G')\). Moreover Proposition 2.24 implies a central extension

\[1 \to H \to G \to G' \to 1.\]

We handle each case of \((N, G')\) separately, since different techniques are required to determine the geometries \((M, G)\). We start with the spherical case, i.e. \((N, G') = (S^2, \mathbf{SO}(3))\).

**Proposition 2.25.** Let \((N, G') = (S^2, \mathbf{SO}(3))\). Then \((M, G)\) is isomorphic to \((S^2, \mathbf{U}(2))\).
Proof. First we determine the fiber group. Suppose that $H = \mathbb{R}$. Then there would be a global section $s : S^2 \to M$ and $s^*(d\omega)$ would be a volume form on $S^2$, see Corollary 2.22. But this leads to a contradiction by Stokes’ theorem since $d(s^*(\omega)) = s^*(d\omega)$ is exact, hence $H = \text{SO}(2)$. The central extension

$$1 \to \text{SO}(2) \to G \to \text{SO}(3) \to 1$$

implies a central extension for their Lie algebras

$$0 \to \mathbb{R} \to \mathfrak{g} \to \mathfrak{so}(3) \to 0.$$ 

Using Whitehead’s second lemma (see [Wei97, p.246]) the central extension splits and is isomorphic to the trivial one, since it is central. Hence the universal covering group of $G$ is $\mathbb{R} \times \text{SU}(2)$ and $G \cong (\text{SU}(2) \times \mathbb{R})/\pi_1(G)$ where $\pi_1(G)$ is a discrete subgroup of center $\mathbb{R} \times \mathbb{Z}_2$. Applying the long exact homotopy sequence to the principal bundles $M \to S^2$ and $G \to \text{SO}(3)$ we obtain that $\pi_1(G)$ is isomorphic to $\mathbb{Z}$. Let $(c, d)$ be a generator of $\pi_1(G)$ for $c \in \mathbb{R}$ and $d \in \mathbb{Z}_2 = \{\pm 1\}$. Then $c \neq 0$ since $\pi_1(G)$ is isomorphic to $\mathbb{Z}$. Moreover $d = -1$ since otherwise $G$ would be isomorphic to $\text{SO}(2) \times \text{SU}(2)$ and the quotient $G/K$ could not be simply connected since $K$ has to lie in $\text{SU}(2)$ (note that $K \cap H$ has to be trivial). The map $\mathbb{R} \times \text{SU}(2) \to \text{U}(2)$, $(t, S) \mapsto e^{\pi t} S$ induces an isomorphism between $(\mathbb{R} \times \text{SU}(2))/\pi_1(G)$ and $\text{U}(2)$, hence $G = \text{U}(2)$.

Since $K$ is isomorphic to $\text{U}(1)$ we may assume (after a conjugation in $\text{U}(2)$) that $K$ consists of diagonal matrices. To see this, note that $K$ is abelian and hence all elements are simultaneously diagonalizable. Let $\Phi : \text{U}(1) \to K \subset \text{U}(2)$ be an isomorphism. Then there are homomorphisms $\Phi_i : \text{U}(1) \to \text{U}(1)$ such that

$$\Phi(z) = \begin{pmatrix} \Phi_1(z) & 0 \\ 0 & \Phi_2(z) \end{pmatrix}.$$ 

Moreover there are $n, m \in \mathbb{Z}$ such that $\Phi_1(z) = z^n$ and $\Phi_2(z) = z^m$. Since $\Phi$ is an isomorphism we obtain $(n, m) = (1, 1)$, $(n, m) = (1, 0)$ or $(n, m) = (0, 1)$. But $n = m = 1$ would mean, that $K$ lies in the center which is a contradiction to $K \cap H$ is trivial. The quotient of the remaining possibilities is $S^3$ and $\text{U}(2)$ acting on it by its standard action on $\mathbb{C}^2$.

Remark 2.26. For the next case it is useful to remind some facts about the Lie group $\text{SL}$ which we define as the universal cover group of $\text{SO}^+(2, 1)$. The group $\text{SO}^+(2, 1)$ can be identified as a manifold with the circle bundle of the standard hyperbolic plane which is topologically given as $D^2 \times S^1$, so the fundamental group of $\text{SO}^+(2, 1)$ is isomorphic to $\mathbb{Z}$. But this implies that the center of $\text{SL}$ is isomorphic to $\mathbb{Z}$ since $\text{SO}^+(2, 1)$ is centerless. Moreover this implies that $\text{SL}$ is topologically $\mathbb{R} \times D^2$ which fibers over $D^2$ such that the projection $\mathbb{R} \times D^2 \to D^2$ is equivariant with respect to $\text{SL}$ and $\text{SO}^+(2, 1)$.

The group $\mathbb{R} \times \text{SL}$ has a natural action on $\text{SL}$: elements of $\mathbb{R}$ act as translations in the fibers of $\text{SL} \to D^2$ which cover the identity and $\text{SL}$ acts by group multiplication. This action descends to an action of $\Gamma := (\mathbb{R} \times \text{SL})/\mathbb{Z}(1, 1)$ on $\text{SL}$ where $(1, 1)$ is an element of the center $\mathbb{R} \times \mathbb{Z}$ such that $1 \in \mathbb{Z}$ generates the fundamental group $\pi_1(\text{SO}^+(2, 1)) \subset \text{SL}$. A more detailed discussion of $(\text{SL}, \Gamma)$ can be found in [Sco83] which is known as $\text{SL}$-geometry.
Proposition 2.27. If \((N, G^1) = (D^2, SO^+(2, 1))\) then \((M, G)\) is isomorphic to the \(SL\)-geometry \((SL, \Gamma)\).

Proof. Since \(D^2\) is contractible the principal bundle \(M \rightarrow N\) is the trivial \(\mathbb{R}\)-principal bundle \(\mathbb{R} \times D^2 \rightarrow D^2\). This implies that \(\pi_2(M)\) is trivial and using the long homotopy sequence for the principal bundle \(SO(2) \rightarrow G \rightarrow M\) we obtain that \(\pi_1(G)\) is isomorphic to \(\mathbb{Z}\). Similarly to Proposition 2.25, we obtain the central extension

\[
0 \rightarrow \mathbb{R} \rightarrow g \rightarrow \mathfrak{sl}(2, \mathbb{R}) \rightarrow 0
\]

which has to be isomorphic to the trivial one

\[
0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \times \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{sl}(2, \mathbb{R}) \rightarrow 0
\]

since \(\mathfrak{sl}(2, \mathbb{R})\) is semisimple. One then easily checks that the following diagram commutes

\[
\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\pi_1(\mathbb{R}) & \pi_1(G) & \pi_1(SO^+(2, 1)) & 1 & 1 \\
\downarrow & \downarrow & \downarrow & & \\
\mathbb{R} \times SL & SL & 1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow & & \\
G & SO^+(2, 1) & 1 & 1 & 1 \\
\end{array}
\]

The center of \(\mathbb{R} \times SL\) is isomorphic to \(\mathbb{R} \times \mathbb{Z}\) where \(\mathbb{Z}\) is the group \(\pi_1(SO^+(2, 1)) \subset SL\). Moreover the projection map \(\text{pr}_2: \mathbb{R} \times SL \rightarrow SL\) induces an isomorphism on the fundamental groups \((\text{pr}_2)_*: \pi_1(G) \rightarrow \pi_1(SO^+(2, 1))\) which is simply the restriction of \(\text{pr}_2\) to \(\pi_1(G)\) seen as a subgroup of \(\mathbb{R} \times SL\). Hence a generator of \(\pi_1(G)\) must be of the form \((c, 1) \in \mathbb{R} \times \mathbb{Z}\) where 1 is a generator of \(\mathbb{Z} = \pi_1(SO^+(2, 1))\). However we may exclude \(c = 0\) since otherwise \(G\) would be isomorphic to \(\mathbb{R} \times SO^+(2, 1)\) which would be a geometry with flat \(G\)-invariant connection. Therefore we assume that \((1, 1) \in \mathbb{R} \times \mathbb{Z}\) is a generator of \(\pi_1(G)\), since if \(c \neq 1\) we may consider the automorphism \(\Phi: \mathbb{R} \times SL \rightarrow \mathbb{R} \times SL, \Phi(x, a) = (x/c, a)\) which induces an isomorphism between \((\mathbb{R} \times SL)/(\mathbb{Z}(c, 1))\) and \((\mathbb{R} \times SL)/(\mathbb{Z}(1, 1))\).

Let \(\pi: \mathbb{R} \times SL \rightarrow G\) be the quotient map. Then \(\widetilde{SL}\) acts on \(\mathbb{R} \times D^2\) via \(\pi\) as a subgroup of \(\mathbb{R} \times \widetilde{SL}\). We claim that \(\widetilde{SL}\) acts transitively on \(\mathbb{R} \times D^2\). Let \(p: G \rightarrow SO^+(2, 1)\) be the quotient map of \(G \rightarrow G/H = SO^+(2, 1)\) (see Proposition 2.24) and let \(\pi+: SL \rightarrow SO^+(2, 1)\) be the universal cover map. Then we have \(p \circ \pi = \pi+ \circ \text{pr}_2\) since the diagram above commutes. For \((t, x) \in \mathbb{R} \times D^2\) the action of \(a \in SL\) is then defined by \(\pi(0, a).(t, x)\). The second entry is given by \(\pi_+(a), x\) due to the fact that \(p \circ \pi = \pi+ \circ \text{pr}_2\) and since the projection \(\mathbb{R} \times D^2 \rightarrow D^2\) is
equivariant (see Proposition 2.24). Hence there is a map \( f : \mathbb{R} \times D^2 \times \mathbf{SL} \to \mathbb{R} \) such that
\[
\pi(0, a). (t, x) = (f(t, x, a), \pi_+(a)x)
\]
with the property
\[
f(t, x, ab) = f(t, x, b), \pi_+(b)x, a)
\]
for \( a, b \in \mathbf{SL} \). The differential of \( \pi(0, a) \) as a map from \( M \) to itself maps \( \partial_t \) into itself since \( \pi(0, a) \in G \). And this implies that if \( x \) and \( a \) are fixed the map \( f \) is a translation in the fiber, i.e. there is a smooth function \( s : D^2 \times \mathbf{SL} \to \mathbb{R} \) such that \( f(t, x, a) = t + s(x, a) \). Then from the property of \( f \) above we obtain
\[
s(x, ab) = s(\pi_+(b)x, a) + s(x, b)
\]
for \( a, b \in \mathbf{SL} \). Note moreover that for \( z \in \pi_1(\mathbf{SO}^+(2, 1)) \cong \mathbb{Z} \) with \( z \neq 0 \) the element \([0, z]) \in G = (\mathbb{R} \times \mathbf{SL}/\pi_1(G)) \) is equal to \([(-z, 0])\), hence \((0, z) \) acts on \( \mathbb{R} \times D^2 \) as constant translation, i.e. \( \pi(0, z).(t, x) = (t - z, x) \) for all \((t, x)\).

Fix a point \( x_0 \in D^2 \) and let \( K \) be the isotropy group in that point of \( \mathbf{SO}^+(2, 1) \). The subgroup \( \tilde{K} := (\pi_+)^{-1}(K) \) is isomorphic to \( \mathbb{R} \). To see this note the following: Let \( Z = \pi_1(\mathbf{SO}^+(2, 1)) \subset \mathbf{SL} \) then \( \tilde{K}/Z = K \) and \( \mathbf{SL}/\tilde{K} = \mathbf{SO}^+(2, 1)/K = D^2 \), hence \( \tilde{K} \) is connected. Furthermore \( Z \) is a discrete subgroup of \( \tilde{K} \) which excludes the case \( \tilde{K} = \mathbf{SO}(2) \).

Now consider the map \( \Psi : \tilde{K} \to \mathbb{R}, a \mapsto \Psi(a) := s(x_0, a) \). Using the fact, that \( \pi_+(\tilde{K}) \) fixes the point \( x_0 \), it follows from (*) that \( \Psi \) is a homomorphism. Since \( \tilde{K} \) is isomorphic to \( \mathbb{R} \), \( \Psi \) is either trivial or an isomorphism. Suppose \( \Psi \) is trivial. Then every element in \( 0 \times Z = 0 \times \pi_1(\mathbf{SO}^+(2, 1)) \subset \mathbf{SL}/\tilde{K} \subset \mathbb{R} \times \mathbf{SL} \) would act as the identity on \( \mathbb{R} \times D^2 \) which then in turn has to lie in \( \ker \pi \) but this is a contradiction since the group \( \ker \pi = \pi_1(G) \) is generated by \((1, 1)\).

Finally it is easy to see that \( \mathbf{SL} \) acts transitively on \( \mathbb{R} \times D^2 \) and since the isotropy group has to be a point, we may identify \( \mathbb{R} \times D^2 \) with \( \mathbf{SL} \).

If we sum up the results in this proposition we obtain the same action of \( \mathbb{R} \times \mathbf{SL} \) on \( \mathbf{SL} \) as for the \( \mathbf{SL} \)-geometry which descends to an action of \( \Gamma \) on \( \mathbf{SL} \).

\textbf{Proposition 2.28.} Suppose \((N, G') = (\mathbb{R}^2, E_0(2))\). Then \((M, G)\) is isomorphic to the \( \mathbf{Nil} \)-geometry.

\textbf{Proof.} The principal bundle \( M \to \mathbb{R}^2 \) is the trivial \( \mathbb{R} \)-principal bundle \( \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \) since \( \mathbb{R}^2 \) is contractible. Repeating the arguments of the preceding two propositions we obtain \( \pi_1(G) = \mathbb{Z} \) and the central extension of Lie Algebras
\[
0 \to \mathbb{R} \to g \to \mathfrak{e}(2) \to 0.
\]

The second Lie algebra cohomology group \( H^2(\mathfrak{e}(2); \mathbb{R}) \) (where \( \mathbb{R} \) is the trivial \( \mathfrak{e}(2) \)-module) is isomorphic to \( \mathbb{R} \). Every element in \( H^2(\mathfrak{e}(2); \mathbb{R}) \) has a unique representative \( \omega_\lambda \) given by
\[
\begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
for \( \lambda \in \mathbb{R} \), which shows also the isomorphism between the cohomology group and \( \mathbb{R} \). If \( \omega_\lambda \in H^2(\varepsilon(2); \mathbb{R}) \) the isomorphism class of central extensions are of the form

\[
0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \times_{\omega_\lambda} \varepsilon(2) \longrightarrow \varepsilon(2) \longrightarrow 0.
\]

(See [Wei97] for the definition of the Lie Algebra \( \mathbb{R} \times_{\omega_\lambda} \varepsilon(2) \).)

We consider first the trivial central extension

\[
0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \times \varepsilon(2) \longrightarrow \varepsilon(2) \longrightarrow 0.
\]

The universal cover group \( \widetilde{E}_0(2) \) of \( E_0(2) \) is given as the semidirect product \( \mathbb{R}^2 \times_{\rho} \mathbb{R} \) where \( \rho: \mathbb{R} \to GL(\mathbb{R}^2) \) and \( \rho_0 \) is the rotation around the origin with rotation angle \( \theta \). Then the center is isomorphic to \( \mathbb{Z} \) and embedded as \( z \mapsto (0, 2\pi z) \). The Lie group \( \mathbb{R} \times \widetilde{E}_0(2) \) has Lie algebra \( \mathbb{R} \times \varepsilon(2) \) and its center is isomorphic to \( \mathbb{R} \times \mathbb{Z} \). Like in Proposition 2.27 a generator of \( \pi_1(\mathbb{Z}_c) \) has to have the form \( \gamma_c := (e, 0, 2\pi) \) for \( c \in \mathbb{R} \). The groups \( (\mathbb{R} \times \widetilde{E}_0(2))/(\mathbb{Z}_c) \) are isomorphic to \( (\mathbb{R} \times \widetilde{E}_0(2))/(\mathbb{Z}_c) \) and an isomorphism is induced by the linear map \( \varphi_c: \mathbb{R} \times \widetilde{E}_0(2) \to \mathbb{R} \times \widetilde{E}_0(2) \) (seen as vector spaces) defined by the matrix

\[
\begin{pmatrix}
1 & 0 & -c/2\pi \\
0 & E_2 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

A simple calculation shows that \( \varphi_c \) is a Lie group isomorphism such that \( \varphi(\gamma_c) = \gamma_0 \). Thus the group \( G \) is isomorphic to \( \mathbb{R} \times \widetilde{E}_0(2) \) which cannot be the case since the induced \( G \)-invariant connection is flat.

Next we consider the central extension of

\[
0 \to \mathbb{R} \to \mathbb{R} \times \varepsilon(2) \to \varepsilon(2) \to 0.
\]

There is a basis \( (e_0, e_1, e_2, e_3) \) of \( \mathbb{R} \times \varepsilon(2) \) such that the Lie brackets are expressed by the relations

\[
[e_0, e_i] = 0, \quad [e_1, e_2] = e_0, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1,
\]

for \( i = 1, 2, 3 \). This means that \( e_0 \) spans the center and the other the \( \varepsilon(2) \) part.

Now it is easy to see that \( \mathbb{R} \times \varepsilon(2) \) is isomorphic to \( \text{nil} \rtimes \sigma \mathbb{R} \) where \( \text{nil} \) is the 3-dimensional Heisenberg Lie algebra and \( \sigma: \mathbb{R} \to \text{Der}(\text{nil}) \) is defined as \( \sigma_{e_3} = \text{ad}_{e_3} \) (see e.g. [Kon13, p.115]). Thus we obtain the central extension

\[
0 \longrightarrow \mathbb{R} \longrightarrow \text{nil} \rtimes \mathbb{R} \longrightarrow \varepsilon(2) \longrightarrow 0.
\]

where the inclusion is defined by \( \lambda \mapsto \lambda e_0 \) and the projection is given by \( e_0 \mapsto 0 \) and \( e_i \mapsto e_i \) for \( i = 1, 2, 3 \). Integrating this semidirect product turns out to be a very hard task, therefore we guess the representation \( \rho: \mathbb{R} \to \text{Nil} \). Comparing this situation with the \( \text{Nil} \)-geometry in [Sco83] we define (cf. [Sco83, p. 467])

\[
\rho_0(\mathbf{x}, z) = \left( R_\theta(\mathbf{x}, z + \frac{1}{2}s(ey^2 - cx^2 - 2sxy)) \right)
\]

where \( \mathbf{x} = (x, y) \), \( R_\theta(\mathbf{x}) \) rotates \( \mathbf{x} \) through \( \theta \) in \( \mathbb{R}^2 \), \( s = \sin \theta \) and \( c = \cos \theta \). A straightforward calculation shows that the Lie algebra of \( \text{Nil} \rtimes \mathbb{R} \) is \( \text{nil} \rtimes \rho \mathbb{R} \) (the computations were done e.g. in [Kon13, p. 115]). The center is again isomorphic
to $\mathbb{R} \times \mathbb{Z}$ and it is embedded in $\text{Nil} \times \mathbb{R}$ as $(z, l) \mapsto (0, 0, z, l)$. The generator of $\pi_1(G)$ has to have the form $\gamma_c = (0, c, 2\pi)$. The groups $(\text{Nil} \times \mathbb{R})/\mathbb{Z}\gamma_c$ are isomorphic to $(\text{Nil} \times \mathbb{R})/\mathbb{Z}\gamma_0 = \text{Nil} \times \text{SO}(2)$ and an isomorphism is e.g. the linear map $\varphi_c$ given by the matrix

$$
\begin{pmatrix}
E_2 & 0 & 0 \\
0 & 1 & -c/2\pi \\
0 & 0 & 1
\end{pmatrix},
$$

where we regard $\text{Nil} \times \mathbb{R}$ as $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$. We conclude $G = \text{Nil} \times \text{SO}(2)$. Repeating the arguments from the previous proposition, we see that $\tilde{\mathcal{E}}_0(2)$ is acting like $a \cdot (t, x) = (t + s(x,a), \pi(a).x)$ where $\pi : \tilde{\mathcal{E}}_0(2) \to \mathcal{E}_0(2)$ is the universal cover homomorphism. But restricting $\Psi$ to $\ker \pi' = \pi_1(G)$ one obtains that $\Psi$ has to be the zero map, since it is either an isomorphism or trivial. Consequently this implies that the isotropy group of $\text{Nil} \times \text{SO}(2)$ is the $\text{SO}(2)$–part. Moreover this shows that $M$ is actually the Heisenberg group $\text{Nil}$, which clarifies how $G$ is acting on $M$.

Finally the other central extensions defined by $\omega_\lambda$ for $\lambda \neq 0, 1$ are weakly isomorphic to the one defined by $\omega_1$. Here two central extensions $0 \to h \to g \to \mathfrak{k} \to 0$ and $0 \to h \to g' \to \mathfrak{k} \to 0$ are called weakly isomorphic if the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & h & \longrightarrow & g & \longrightarrow & \mathfrak{k} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & h & \longrightarrow & g' & \longrightarrow & \mathfrak{k} & \longrightarrow & 0
\end{array}
$$

commutes and the vertical maps are isomorphisms. Clearly two weakly isomorphic extensions induce isomorphic geometries $(M, G)$.

We sum up the list of geometries with non–flat connection $\omega$ in

**Theorem 2.29.** If $(M, G)$ is an axially symmetric geometry such the the $G$–invariant connection is non–flat. Then $(M, G)$ is equivariant diffeomorphic to exactly one the following geometries:

(a) $(S^3, \text{U}(2))$,

(b) $(\text{SL}, \Gamma)$,

(c) $(\text{Nil}, \text{Nil} \times \text{SO}(2))$.

For a detailed discussion of these geometries see [Sco83].

### 3 Lie groups as geometries

If the stabilizer of $G$ is trivial we may identify $G$ with $M$ through the group action. Since we assumed that $M$ is simply connected we may linearize the problem and classify the 3-dimensional Lie algebras.

This was done first by Luigi Bianchi in the 19th century, see [Bia98] or [Bia01]. This approach was translated in [GKKL13] to a more modern, coordinate free language. Here the Lie algebras are divided into two classes: in
the unimodular ones and the non-unimodular. Geometrically speaking this means that if $G$ admits a compact quotient, then $G$ has to be unimodular, compare [Mil76, p. 99]. Thus the non–unimodular Lie groups cannot posses cocompact subgroups and therefore they do not appear in Thurston’s list.

Another way to classify the 3-dimensional Lie algebras is to consider the derived Lie algebra $g' = [g, g]$ which isomorphism class is an invariant for the isomorphism class of $g$. In particular the dimension of $g'$ does not change under Lie algebra isomorphisms. This approach may be found in full detail [Kon13, p. 67] OR in [KKL]. We would like to describe briefly this approach and the resulting 3-dimensional Lie algebras.

- $\dim g = 0$: Then $g$ is abelian and $g = \mathbb{R}^3$.
- $\dim g = 1$: We obtain the short exact sequence of Lie algebras
  $$0 \rightarrow \mathbb{R} \rightarrow g \rightarrow \mathbb{R}^2 \rightarrow 0.$$  
  The resulting Lie algebras are $\mathfrak{h}^2 \times \mathbb{R}$ where $\mathfrak{h}^2$ is the non–abelian 2-dimensional Lie algebra and the Lie algebra of the Heisenberg group.
- $\dim g = 2$: We obtain the short exact sequence of Lie algebras
  $$0 \rightarrow \mathbb{R}^2 \rightarrow g \rightarrow \mathbb{R} \rightarrow 0.$$  
  Obviously this sequence always splits. Suppose $e_3$ is a non-zero vector in $\mathbb{R}$. A splitting map determines the Lie brackets of $e_3$ with an element of $g' = \mathbb{R}^2$. Since all other Lie brackets are trivial, the linear map $\text{ad}_{e_3}: g' \rightarrow g'$ determines the Lie algebra. The remaining task is to determine what linear maps $\text{ad}_{e_3}$ produce isomorphic Lie algebras, which is an easy exercise (compare [Kon13, p. 69]). There are 3 types of Lie algebras and two of them come with a continuous family of non-isomorphic Lie algebras.

4 The complete list of geometries

In this last section we would like to write down the final list of 3-dimensional geometries.

**Theorem 4.1.** Let $(M, G)$ be a geometry. Then $(M, G)$ is equivariant diffeomorphic to exactly one of the following geometries:

(i) isotropic geometries ($\dim K = 3$).

- $(\mathbb{R}^3, E_0(3))$  
- $(S^3, \text{SO}(4))$  
- $(D^3, H_0(3))$

(ii) axially symmetric geometries ($\dim K = 1$).

| flat | non-flat |
|------|----------|
| $(S^2 \times \mathbb{R}, \text{SO}(3) \times \mathbb{R})$ | $(S^2, U(2))$ |
| $(D^2 \times \mathbb{R}, \text{SO}^+(2, 1) \times \mathbb{R})$ | $(\text{SL}, \Gamma)$ |
| $(\mathbb{R}^2 \times \mathbb{R}, E_0(2) \rtimes_{\rho_i} \mathbb{R}), i = 0, 1$ | $(\text{Nil}, \text{Nil} \rtimes \text{SO}(2))$ |

(iii) Lie groups in dimension 3 ($\dim K = 0$), compare the list in [Kon13, p. 73].
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