Research Article

Sharp Bounds for Fractional Conjugate Hardy Operator on Higher-Dimensional Product Spaces

Zequn Wang 1, Mingquan Wei 2, Qianjun He 3, and Dunyan Yan 1

1 School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
2 School of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, Henan, China
3 School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, China

Correspondence should be addressed to Zequn Wang; wangzequn17@mails.ucas.ac.cn

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In this paper, we obtain the sharp bound for fractional conjugate Hardy operator on higher-dimensional product spaces from \( L^1 (\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) \) to the space \( wL^q (\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) \) and \( L^p (\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) \) to the space \( L^q (\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) \). More generally, the operator norm of the fractional Hardy operator on higher-dimensional product spaces from \( L^p (\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) \) to \( L^q (\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) \) is obtained.

1. Introduction

Let \( f \) be a nonnegative integrable function on \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m} \). Define the fractional conjugate Hardy operator on higher-dimensional product spaces by

\[
(H^*_{\beta_1, \beta_2, \ldots, \beta_m} f)(x) = \left[ \int_{|y_1| < s_1} \cdots \int_{|y_m| < s_m} \frac{f(y_1, \ldots, y_m)}{|B(0, |y_1|)^{\alpha_1}(\beta_1) \cdots |B(0, |y_m|)^{\alpha_m}(\beta_m)|} \, dy_1 \cdots dy_m \right]^{1/\beta_m},
\]

(1)

where \( x = (x_1, \ldots, x_m) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, \prod_{i=1}^m |x_i| \neq 0 \), and \( 0 \leq \beta_i < n_i \) with \( i = 1, 2, \ldots, m \).

Operator (1) is a generalization of classical Hardy-type operators. In 1925, Hardy first gave the following definition:

\[
Hf(x) = \frac{1}{x} \int_0^x f(y) \, dy,
\]

(2)

\[
H^* f(x) = \int_x^\infty \frac{f(y)}{y} \, dy,
\]

for \( x > 0 \).

Hardy [1] presented the classical Hardy inequalities on \( G = (0, \infty) \):

\[
\|Hf\|_{L^p(G)} \leq \frac{p}{p-1} \|f\|_{L^q(G)},
\]

(3)

\[
\|H^* f\|_{L^p(G)} \leq \|f\|_{L^q(G)},
\]

with \( 1 < p \leq \infty \), where the constants \( (p/(p-1)) \) and \( p \) are the best. Later, Christ and Grafakos [2] extended this result to higher-dimensional setting.

In 1930, Bliss [3] proved Hardy inequality with power weight as

\[
\|Hf\|_{L^p(x^\alpha)} \leq C_{pq} \|f\|_{L^q(G)} \quad \text{with } \alpha = \frac{q}{p} - 1.
\]

(4)

Inequality (4) can be stated as Theorem 1.

Theorem 1. Let \( f \) be a nonnegative integrable function on \( G \) and \( 1 < p < q < \infty \). The inequality

\[
\left( \int_0^\infty \left( \int_0^x f(t) \, dt \right)^{q/p-1} \, dx \right)^{1/q} \leq C_{pq} \left( \int_0^\infty f^p(x) \, dx \right)^{1/p},
\]

(5)

holds with the sharp constant...
where $B(\cdot, \cdot)$ is a beta function.

Inequality (5) turns into equality with the above constant if and only if
\[
f(x) = \frac{c}{(ax^p/p - 1) + 1/(q - p)}.
\]

The fractional Hardy operator on $L^q(G)$ can be shown as
\[
H_{\beta,p}f(x) = \frac{1}{x^{\beta/p}} \int_0^x f(y)dy,
\]
with $0 < \beta < 1$.

Theorem 1 shows that
\[
\|H_{\beta,p}f\|_{L^q(G)} \leq C_{pq} \|f\|_{L^p(G)},
\]
where $\beta = 1/p - 1/q$.

Lu-Zhao [4] and Persson-Samko [5] extended those one-dimension results to higher-dimensional results. For general situations, results could be found in [6-8].

Then, it is natural to consider the Hardy-type operator on product spaces, which can be defined as
\[
H_{\beta,p}f(x) = \left( \beta/p \right) \left( \frac{1}{x_{\beta/p}} \right) \int_0^x f(y_1, \ldots, y_m)dy_1 \cdots dy_m,
\]
where $x = (x_1, \ldots, x_m) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$.

He et al. [9] had already proved the boundedness of the fractional Hardy operator on higher-dimensional product spaces.

Theorem 2. Let $1 < p < q < \infty$. Set $0 < \beta_i < n_i$ and $(1/q) = (1/p) - (\beta_i/n_i)$ with $1 \leq i \leq m$. Note $n = (n_1, n_2, \ldots, n_m)$. If $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})$, then we have
\[
\|H_{\beta_1,\ldots,\beta_m}f\|_{L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})} \leq C_{pq} \|f\|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})},
\]
where
\[
C_{pq} = \prod_{i=1}^m \left( \frac{1}{n_i} \right) \left( \frac{1}{n_i - \beta_i} \right)^{(1/q) - (1/p)}.
\]

is sharp and $C_{pq}$ is the sharp constant in (5).

In 2016, Gao et al. [10] proved the weak-type estimate for $H_{\beta_1,\ldots,\beta_m}$ with $n_1 = \cdots = n_m = n$, $\beta_1 = \cdots = \beta_m = \beta$, and $m = 1$. Their result is as follows.

Theorem 3. Let $f$ be a nonnegative integrable function on $\mathbb{R}^n$. Set $0 < \beta < n$ and $(1/q) = (1/p) - (\beta/n)$. Then, we have
\[
\|H_{\beta}^s\|_{L^p(\mathbb{R}^n)} = 1,
\]
\[
\|H_{\beta}^s\|_{L^q(\mathbb{R}^n)} = \left( \frac{q}{p} \right)^{(1/p')}.
\]

We will extend (13) to higher-dimensional product spaces.

Theorem 4. Set $0 < \beta_i < n_i$ for $i = 1, 2, \ldots, m$. Let $Q = (n_1/(n_1 - \beta_1)), (n_2/(n_2 - \beta_2)), \ldots, (n_m/(n_m - \beta_m))$. If $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})$, then we have
\[
\|H_{\beta_1,\ldots,\beta_m}^s f\|_{L^q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})} \leq \|f\|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m})},
\]
where $1$ is the sharp bound.

Next, we will generalize the results into mixed norm, which will be mentioned in Section 4.

Theorem 5. Suppose $P = (p_1, \ldots, p_m) \geq 1$ (i.e., for every $i = 1, 2, \ldots, p_i \geq 1$) and $Q = (q_1, \ldots, q_m) \geq 1$. Set $(1/p_i) - (1/q_i) = (\beta_i/n_i)$ with $0 < \beta_i < n_i$ for $i = 1, \ldots, m$. We further suppose $1 < p_i < \infty$ when $i \in I \subset \{1, \ldots, m\}$. Then, operator $H_{\beta_1,\ldots,\beta_m}^s$ is bounded from $L^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})$ to $L^q(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})$. Moreover, there holds
\[
\|H_{\beta_1,\ldots,\beta_m}^s f\|_{L^q(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})} \leq \|f\|_{L^p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})},
\]
where
\[
C_{\beta_1,\ldots,\beta_m} = \prod_{i=1}^m \left( \frac{1}{n_i} - \frac{1}{p_i} \right)^{(1/q_i) - (1/p_i)} C_{pq}.
\]

is sharp and $C_{pq}$ is defined as in (6).

2. Preliminaries

First, we can use the following lemma to reduce the dimension. This lemma is from [9].

Lemma 1. Suppose $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, |x|^l)$, with $|x|^l = |x_1|^l |x_2|^l \cdots |x_m|^l$. Define
\[
g_f(x) = \frac{1}{\omega_{n_1}} \frac{1}{\omega_{n_2}} \frac{1}{\omega_{n_m}} \int_{|x_1| = 1} \int_{|x_2| = 1} \cdots \int_{|x_m| = 1} f(|x_1|\xi_1, |x_2|\xi_2, \ldots, |x_m|\xi_m) d\xi_1 d\xi_2 \cdots d\xi_m,
\]
where $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}$ and
\[
\omega_{n_i} = \frac{2\pi^{n_i/2}}{\Gamma(n_i/2)} 1 \leq i \leq m.
\]

Then, we have
Define \( a_0 = -n - 1 \) and \( \omega_0 = p(1 + \beta) - (1 + n) \). We have
\[
\omega_0 < p - 1 \quad \text{and} \quad (a_0 + 1/q) = (\omega_0 + 1/p) - 1.
\]
Let
\[
\nu = \nu(u) = \int_{0}^{u} z^{-(\omega_0/(p-1))} \, dz = \frac{p - 1}{p - 1 - \omega_0} u^{(p-1-\omega_0)/(p-1)},
\]
\[
h(v) = h(v(u)) = g(u)u^{(\omega_0/(p-1))}.
\]
So we have \( u(\omega_0/p-1) \, du = dv \) and \( g(u)u^{\omega_0/(p-1)} = h(v) \). It follows that
\[
\int_{0}^{\infty} g^{p}(u)u^{\omega_0} \, du = \int_{0}^{\infty} g^{p}(u)u^{(\omega_0/p-1)} h^{p}(v) \, dv.
\]
Notice that
\[
\omega_0 \frac{q}{p} - \frac{q}{p} - 1 + \frac{\omega_0}{p-1} = \frac{\omega_0}{p-1} - 1 + q \left( \frac{p}{p-1} \right) - 1
\]
\[
= \frac{\omega_0}{p-1} - 1 + q \left( \frac{p}{p-1} \right),
\]
\[
= \frac{\omega_0}{p-1} - 1 + q \left( \frac{p}{p-1} \right).
\]
Using Bliss’s result, we conclude that
\[
I \leq n^{(1/p)+1/p'} \left[ \frac{1}{n} - \frac{1}{p'} \right]^{1/p'} C_{p'} \int_0^\infty t^{p} u e^{a t} \, dt.
\]
(30)

The sharp bound can be reached if and only if
\[
f(x) = \frac{c x^{-\omega/(p-1)}}{a x ((p-1-\omega)/(p-1)) + 1} \left( q^{(p'-(p-1))} \right)
\]
(31)

3. Main Result

In this section, we will give the proof of Theorems 2 and 4.

**Proof of Theorem 2.** Without loss of generality, we only need to discuss the case for \( m = 2 \). The case for \( m > 2 \) is also true by the same method.

Assume that \( f \) is a nonnegative integrable function. Fixing variable \( x_1 \), we define
\[
F(y_2, x_1) = \int_{|y_1|\leq |x_1|} \frac{f(y_1, y_2)}{|B(0, |y_1|)|^{-1-m_1}} \, dy_1.
\]
(32)

So
\[
\left| H^*_p f(x_1, x_2) \right|^q = \int_{|y_1|\leq |x_1|} \frac{F(y_2, x_1)}{|B(0, |y_2|)|^{-1-m_1}} \, dy_2 \]
(33)

By simple integral estimate, we have
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \left( H^*_p f(x_1, x_2) \right)^q \right| \, dx_1 \, dx_2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \frac{F(y_2, x_1)}{|B(0, |y_2|)|^{-1-m_1}} \right|^q \, dx_1 \, dx_2
\]
\[
\leq \int_{\mathbb{R}^n} \left| \left( C_{p,n}^* \left( \int_{\mathbb{R}^n} F(y_2, x_1)^p \, dy_2 \right) \right)^{1/q} \right| \, dx_1
\]
\[
= \left( C_{p,n}^* \right)^q \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} F(y_2, x_1)^p \, dy_2 \right)^{1/q} \, dx_1 = I.
\]

Applying generalized Minkowski’s inequality, we have
\[
I = \left( C_{p,n}^* \right)^q \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} F(y_2, x_1)^p \, dy_2 \right)^{1/q} \, dx_1 \right)^{q/p-1/q}
\]
\[
\leq \left( C_{p,n}^* \right)^q \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} F(y_2, x_1)^p \, dy_2 \right)^{1/q} \, dx_1 \right)^{q/p-1/q}
\]
\[
= \left( C_{p,n}^* \right)^q \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} F(y_2, x_1)^p \, dy_2 \right)^{1/q} \, dx_1 \right)^{q/p-1/q}
\]
\[
\leq \left( C_{p,n}^* \right)^q \left( \int_{\mathbb{R}^n} \left( C_{p,n}^* \int_{\mathbb{R}^n} f^p (y_1, y_2) \, dy_1 \right)^{1/q} \, dy_2 \right)^{q/p}
\]
\[
= \left( C_{p,n}^* C_{p,n}^* \right)^q \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f^p (y_1, y_2) \, dy_1 \, dy_2 \right)^{1/p}.
\]
(35)

It implies that
\[
\left\| H^*_p f \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)} \leq \left( C_{p,n}^* C_{p,n}^* \right) \left\| f \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)}.
\]
(36)

On the other hand, by Lemma 1, we only need to prove the situation when \( f \) is a nonnegative radial smooth function with compact support so that we can separate the variable.
Using the similar method of Theorem 6, it is easy to find that the best constant can be reached when

\[
 f(x) = \frac{c x^{-\frac{\epsilon n_i}{(p-1)q}}}{\left(\frac{\alpha}{Q} \left(\frac{p-1-w_n}{p-1}\right) \left(\frac{q/p-1}{q/p-1} + 1\right) \right)^{\frac{q}{q/p-1}}}
\]

\[
, \quad \frac{c x^{-\frac{\epsilon n_i}{(p-1)q}}}{\left(\frac{\alpha}{Q} \left(\frac{p-1-w_n}{p-1}\right) \left(\frac{q/p-1}{q/p-1} + 1\right) \right)^{\frac{q}{q/p-1}}}
\]

(37)

\[
\left(H^*_{\beta_1, \beta_2} f\right)(x_1, x_2) = \int_{|x_1|=|x_2|} \int_{|y_1|=|y_2|} f(y_1, y_2) \left|B(0, |y_1|)^{1-(\beta_1/n_1)} B(0, |y_2|)^{1-(\beta_2/n_2)} \right| dy_1 dy_2.
\]

(38)

Using Theorem 4 and Fubini’s theorem, it implies that

\[
\left\|H^*_{\beta_1, \beta_2} f\right\|_{L^\infty((n/n_1), n_1) \rightarrow (R^n)} = \sup_{\lambda > 0} \lambda \left|\int_{|x_1|=|x_2|} f(y_1, y_2) \left|B(0, |y_1|)^{1-(\beta_1/n_1)} B(0, |y_2|)^{1-(\beta_2/n_2)} \right| dy_1 dy_2 \left(\frac{\epsilon n_i}{n_1-\beta_2}\right)\right|
\]

\[
\leq 1 \cdot \frac{1}{|y_1|=|y_2|} \int_{R^n} f(y_1, y_2) \left|B(0, |y_1|)^{1-(\beta_1/n_1)} B(0, |y_2|)^{1-(\beta_2/n_2)} \right| dy_1 dy_2.
\]

(39)

\[
= \sup_{\lambda > 0} \lambda \left|\int_{|x_1|=|x_2|} f(y_1, y_2) \left|B(0, |y_1|)^{1-(\beta_1/n_1)} B(0, |y_2|)^{1-(\beta_2/n_2)} \right| dy_1 dy_2 < \lambda_2 \left(\frac{\epsilon n_i}{n_1-\beta_2}\right)\right|
\]

\[
\leq 1 \cdot \int_{R^n} f(y_1, y_2) dy_1 dy_2.
\]

(40)

Therefore, we have

\[
\left\|H^*_{\beta_1, \beta_2} f\right\|_{L^\infty((n/n_1), n_1) \rightarrow (R^n \times R^n)} \leq 1 \cdot \left\|f\right\|_{L^1(R^n \times R^n)}.
\]

(41)

On the other hand, define

\[
f\epsilon(x) = \begin{cases} |x|^{-(\beta_1 + \epsilon)} & , x \geq 1, \\ 0 & , |x| < 1. \end{cases}
\]

(42)
Let
\[ F(x_1, x_2) = f_{\epsilon_1}(x_1) f_{\epsilon_2}(x_2) = f_{\epsilon_1}(|x_1|) f_{\epsilon_2}(|x_2|), \]
\[
\|F\|_{L^1(\mathbb{R}^n_+ \times \mathbb{R}^n_+)} = \left\| f_{\epsilon_1} \right\|_{L^1(\mathbb{R}^n_+)} \left\| f_{\epsilon_2} \right\|_{L^1(\mathbb{R}^n_+)} = \frac{n_1 \nu_{n_1}}{(\beta_1 + n_1)/e - n_1} \frac{n_2 \nu_{n_2}}{(\beta_2 + n_2)/e - n_2}
\]
\[
(43)
\]
We have
\[
H_{\beta_1}^* f_1(x_i) = (\gamma)^{\beta_i/(n-1)} \int_{|x_i|} \left| y \right|^{-(\beta_i + n_1)} (\beta_i - \beta_i) \chi_{|x_i|} \chi_{|y_i|} d\gamma_i \]
\[
(44)
\]
If \(|x_i| \geq 1\), we have
\[
H_{\beta_1}^* f_i(x_i) = n_1 \nu_{n_1} \frac{|x_i|^{\beta_i - (\beta_i + n_1) \epsilon_i}}{(\beta_i + n_1)/e - \beta_i) (45)
\]
If \(|x_i| < 1\), we have
\[
H_{\beta_1}^* f_i(x_i) = \frac{n_1 \nu_{n_1}}{(\beta_i + n_1)/e - \beta_i)} (46)
\]
Since that when \(|x_i| < 1\) and \(\epsilon\) is small enough,
\[
H_{\beta_1}^* f_i(x_i) \leq \frac{n_1 \nu_{n_1}}{\epsilon_i} (47)
\]
We obtain that
\[
\left| \left\{ x_1 \in \mathbb{R}^n_+ : |H_{\beta_1}^* f_1(x_1, x_2) > \lambda_1 \right\} \right| = \left| B(0, 1) \cap \left\{ x_1 \in \mathbb{R}^n_+ : \frac{n_1 \nu_{n_1}}{(\beta_1 + n_1)/e - \beta_1} \left| H_{\beta_1}^* f_1(x_2) > \lambda_1 \right\} \right| \]
\[
+ \left| B(0, 1) \cap \left\{ x_1 \in \mathbb{R}^n_+ : \frac{n_1 \nu_{n_1}}{(\beta_1 + n_1)/e - \beta_1} \left| H_{\beta_1}^* f_1(x_2) > \lambda_1 \right\} \right| \]
\[
(48)
\]
Set
\[
C_{\epsilon_1} = \frac{n_1 \nu_{n_1}}{(\beta_1 + n_1)/e - \beta_1)} (49)
\]
Notice that when \(\lambda_1 > C_{\epsilon_1}\), \(\left\{ x_1 \in \mathbb{R}^n_+ : |H_{\beta_1}^* f_1(x_1) > \lambda_1 \right\} \) = \(\emptyset\). If \(\epsilon_1\) is small enough, \(C_{\epsilon_1}\) tends to zero. Hence, when \(\epsilon_1\) is small enough, we obtain that
\[
I_0 = \sup_{0 < \lambda_1 < C_{\epsilon_1}} \int_{|x_1|} \left| \frac{n_1 \nu_{n_1}}{(\beta_1 + n_1)/e - \beta_1)} \right|^n \int_{|x_2|} \left| \frac{n_1 \nu_{n_1}}{(\beta_1 + n_1)/e - \beta_1)} \right|^n \int_{|x_2|} \left| \frac{n_1 \nu_{n_1}}{(\beta_1 + n_1)/e - \beta_1)} \right|^n \int_{|x_2|} \left| \frac{n_1 \nu_{n_1}}{(\beta_1 + n_1)/e - \beta_1)} \right|^n \int_{|x_2|} \left| \frac{n_1 \nu_{n_1}}{(\beta_1 + n_1)/e - \beta_1)} \right|^n
\]
Using the same method for \(x_2\), we obtain that
Definition 1. Let \( (X_n, S_n, \mu_n) \) be \( n \) given, totally \( \sigma \)-finite measure spaces, for \( 1 \leq i \leq n \). \( P = \{p_1, p_2, \ldots, p_n\} \) is a given \( n \)-tuple with \( 1 \leq p_i \leq \infty \). The set \( I \) satisfies \( I \subseteq \{1, \ldots, n\} \). A function \( f(x_1, x_2, \ldots, x_n) \) measurable in the product spaces \( (X_1, S_1, \mu_1), \ldots, (X_n, S_n, \mu_n) \) is said to belong to the space \( L^P(X) \) if the number obtained after subsequently taking successfully the mixed norm where we take \( p_i \)-norm for \( i \in I \) and weak \( p_j \)-norm for \( j \in \{1, \ldots, n\} \setminus I \). In natural order, it is finite. The number so obtained will be denoted by \( \|f\|_{L^P(X)} \), finite or not.

We give some necessary remarks for the space \( L^P(X) \):

1. If the set \( I = \{1, \ldots, n\} \), we call \( L^P(X) \) strong mixed norm space, which is also denoted by \( L^P(X) \) or \( L^{P_1 \cdots P_n}(X) \).
2. If the set \( I \) is empty, we call \( L^P(X) \) weak mixed norm space, which is also denoted by \( wL^P(X) \) or \( wL^{P_1 \cdots P_n}(X) \).
3. The spaces \( L^P(X) \) is a quasi-normed space for \( P \geq 1 \).

For more properties, we refer readers to [7].

There is a basic lemma which plays an important role in the proof of our main theorems.

Lemma 2. Let \( (X, S, \mu) \) be defined as in the above definitions. If \( p_0 \geq \cdots \geq p_i \geq 1 \) and \( f \in L^{P_0 \cdots P_i}(X) \), then \( f \in L^{P_{i+1} \cdots P_n}(X) \). Moreover, there holds

\[
\|f\|_{L^{P_{i+1} \cdots P_n}(X)} \leq C_{P_{i+1} \cdots P_n} \|f\|_{L^{P_0 \cdots P_i}(X)}.
\]
Using Theorem 3, the following inequality holds:

$$\left\| \int_{|y_1|>|x_1|} f(\cdot, y_2) \frac{d y_2}{|B(0, |y_2|)|^{1-(\beta_1/n)}} \right\|_{L^p(\mathbb{R}^n)} \leq \left( \frac{d_1}{p_2} \right)^{1/p_2} \left\| f \right\|_{L^{p_1}(\mathbb{R}^n \times \mathbb{R}^n)}$$

Combining above estimates, we obtain the desired result.

On the other hand, we take $f_1$ the sharp function on $\mathbb{R}^n$ given in Theorem 2 and $f_2$ the sharp function on $\mathbb{R}^n$ given in [10]. Define

$$F(x_1, x_2) = f_1(x_1) f_2(x_2).$$

By the definition of norm of general operator, we have

$$\left\| H_{\beta_1, \beta_2}^* f \right\|_{L^{q_1}(\mathbb{R}^n \times \mathbb{R}^n)} = \sup_{f \in L^{p_1}(\mathbb{R}^n \times \mathbb{R}^n) \neq 0} \frac{\left\| H_{\beta_1, \beta_2}^* f \right\|_{L^{q_1}(\mathbb{R}^n \times \mathbb{R}^n)}}{\left\| f \right\|_{L^{p_1}(\mathbb{R}^n \times \mathbb{R}^n)}} \geq \frac{\left\| H_{\beta_1, \beta_2}^* f_1 f_2 \right\|_{L^{q_1}(\mathbb{R}^n \times \mathbb{R}^n)}}{\left\| f_1 f_2 \right\|_{L^{p_1}(\mathbb{R}^n \times \mathbb{R}^n)}}$$

(60)

$$= \frac{\left\| H_{\beta_1, \beta_2}^* f_1 \right\|_{L^{q_1}(\mathbb{R}^n)}}{\left\| f_1 \right\|_{L^{p_1}(\mathbb{R}^n)}} \frac{\left\| H_{\beta_1, \beta_2}^* f_2 \right\|_{L^{q_1}(\mathbb{R}^n)}}{\left\| f_2 \right\|_{L^{p_1}(\mathbb{R}^n)}}$$

$$= C_{\beta_1, \beta_2, p_1, p_2} \left( \frac{d_1}{p_2} \right)^{1/p_2}$$

Combining those, we finish the proof of Theorem 7. □

**Data Availability**

The data used to support the study are available upon request to the corresponding author.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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