FINDING A WIDELY DIGITALLY DELICATE PRIME

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Abstract
This paper describes the first construction of an explicitly-known widely digitally delicate prime.

1. Introduction

In a recent paper, Filaseta and Southwick [2] prove the remarkable theorem that a positive proportion of primes become composite when any digit is changed, including leading zeros. Such primes are known as widely digitally delicate primes. This fact is all the more surprising when one considers that no such prime is explicitly known. Using an existing table of digitally delicate primes at the Online Encyclopedia of Integer Sequences [3], they show that there is no widely digitally delicate prime below $10^9$.

It is the aim of the current paper to remove that element of surprise. We do so by giving the first explicit example of a widely digitally delicate prime.

For a comprehensive history of related problems, see [2]. We will define a few terms for clarity.

1.1. Terminology

Definition 1. A covering system is a collection of residue classes such that every integer is in at least one congruence class in the collection.

Definition 2. A digitally delicate prime is a prime such that changing any one of its digits gives a composite. The digits under consideration do not include the leading zeros.

Definition 3. A widely digitally delicate prime is a digitally delicate prime that also becomes composite when any one of its leading zeros is changed.

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2. Finding the Covering Systems

Filaseta and Southwick use covering systems to construct a congruence class such that any digitally delicate prime in that class is a widely digitally delicate prime. They then show that a positive proportion of primes in this congruence class are digitally delicate.

Unfortunately, they do not explicitly construct the congruence class. They rely on the fact that ten specific unfactored composite numbers have at least two prime factors. In particular, any prime factor of the cyclotomic polynomial \( \Phi_n(10) \) is a prime where 10 has order \( n \). Any composite factor of \( \Phi_n(10) \) (that is not a power) is divisible by at least two distinct primes \( p_1, p_2 \) where 10 has order \( n \). The unknown primes can be used for two pieces of the covering systems.

They note that it is possible to construct an explicit congruence class, but that the resulting numbers to be tested for primality would be around 20,000 digits. By modifying their original covering systems, we only need to test numbers of slightly more than 4,000 digits. We do not reproduce the congruence in [2], but recommend a careful reading of that paper to those who want to see the exact nature of the modifications described below.

The first step in their technique constructs a congruence class \( a \mod M \) such that for each digit \( d = 1, \ldots, 9 \), the quantity \( a + d \times 10^k \) is guaranteed to be composite, for all \( k \). The recipe given for the digits 1, 2, 4, 5, 6, 7, 8, and 9 is very easy to follow — all primes are given explicitly. The modifications required are to make sure that adding \( 3 \times 10^k \) produces a composite.

The first observation is that the numbers of the form \( \Phi_n(10) \) are factors of repunits. The practice of factoring numbers is very popular on the Internet, as are repunits. Kamada [4] runs a website that collects many of these factors.

We use these factorizations to replace \( p_{242,1} \), \( p_{242,2} \), \( p_{275,1} \), \( p_{275,2} \), \( p_{363,1} \), \( p_{363,2} \), \( p_{363,3} \), \( p_{364,1} \), \( p_{364,2} \), \( p_{605,1} \), \( p_{605,2} \), \( p_{726,1} \), \( p_{726,2} \), \( p_{2420,1} \), \( p_{4356,1} \), \( p_{5808,1} \), and \( p_{5808,2} \) with explicit prime values.

Kamada’s website does not give us values for \( p_{1210,1} \), \( p_{1210,2} \), \( p_{2904,1} \), and \( p_{2904,2} \), so we have to improvise. Instead of \( p_{1210,1} \), we use a third new factor of \( \Phi_{605}(10) \), which we call \( p_{605,3} \). Because 605 is a divisor of 1210, a congruence for \( k \) modulo 605 also holds modulo 1210. Instead of \( p_{1210,2} \), we use the remaining composite factor of \( \Phi_{605}(10) \) after dividing the composite \( C_{605} \) from [2] by the three new primes. Although knowing a prime factor would give a shorter overall congruence, a composite factor is equally valid.

Instead of \( p_{2904,1} \), we use \( p_{363,3} \), and instead of \( p_{2904,2} \), we use \( p_{726,3} \).

Filaseta and Juillerat [1] have produced alternate covering systems, but the congruence class would be too large for the techniques in this paper.
3. Implementation

All of the code used in this computation is at github.com/31and8191/delicate.

In order to construct the congruence class for the prime, we put all of the congruence classes from [2] into a PARI/GP [6] script and then use the Chinese Remainder Theorem. This part of the computation takes a fraction of a second. We add the restriction \( p \equiv 1 \text{ mod } 2 \) so that we only search over odd numbers.

The next step is a C program using the PARI library. It proceeds through the congruence class, testing each number with PARI’s \texttt{ispseudoprime()} function. Once a number passes, the program tests each variation with a changed digit by using the same function. It is possible to inadvertently reject a digitally delicate number if a variation is a pseudoprime, but that is a small risk. Moreover, we are not concerned with false negatives.

That code ran for about 8 hours on 50 Xeon E5-2699 processors, each running at 2.3 GHz, before producing a positive answer. Note that this answer is a widely digitally delicate probable prime.

That brings us to the last step. I used the PARI/GP \texttt{isprime()} function to prove the primality of the number. That proof took 10 hours and 45 minutes on one of the Xeon processors described above. The prime contains 4030 digits; it is given its own section below.

4. Verification

In order to verify that the prime is widely digitally delicate, I used PARI/GP to verify that each of the prime factors of the congruence modulus is a factor of \( 10^{439084800} - 1 \). Then I verified that \( p + d \times 10^j \) for \( 1 \leq j \leq 439084800 \) and \( 1 \leq d \leq 9 \) has a non-trivial GCD with the congruence modulus.

An ECPP primality certificate is at the GitHub repository.

5. Other Computation

I used the exact congruences provided in [5] and similar code to provide examples of widely digitally delicate primes for bases 4, 5, 6, 9 and 11. Southwick had given examples for bases 2 and 3. Bases 7 and 8 should be equally easy.

I computed a list of digitally delicate primes up to \( 10^{11} \); none of them are widely digitally delicate.
6. The Prime

28589457049198700117815372437458793851550112535218876552086436334395325
908162183231616043938595598854989817448619772705429763745991194664461
10016372712342907968305357135952950110604335656194333249551496146898786
776550562056535289099234146062784906443020313053712612067781273992789520
54661565727359114807958673019425654563382405130810590043829832715380
016952742364731312668598733740946023055663765267200830877802707668563471
933771807679500366888773110101833834850681901417331345780704986628791794
326250751817496814285942909205589193464254902430887558565879364674560897
542896910325950737624441567814981362009791661429525863388175583418337750
63685420137494121103506749630474584378450982057067390182173675337406552
43294929089428282153403327225579837994125441071272039794085468380534381
1312938371746361008659659526789843416781456368648816995696037677489663210
301585644557371116185244779145237550959668086972088867036523522000563482
6611230111917376253346938362615371782983097701381566032284620855925636748
898126820650564827690207892499783431619069410841541638928928146059
6511855363945946956756089728038333262955417851244474541192580656810039197263
449463081193143879385884674684249129022230733836060868965975175033096064
034822282704608841906934855928709497283155895543194572503761740641744551
9711849978562922984965915349241212136168099241705439755139254869750171630
87309352903881552039740735951586740778501204184978455970866066251558
95009735202806090093648339377865876895075608208343448249289882987473163
15366206843020913389386172497510663145775351914284509567047337964533926
5508550853332423043989112688217307299672161009951526577264986933125572
3071908147353752196290275292634491061063247803494512628483131027582454
324305015354715576132370241860473732343086015177330810507975371414875
702723456262422212299575728920837111815181062217947234635389480536163739
4954313263452196082617263234142265869540157459361580543252076259730639827
33504435760156544640200860614624187438842722624283878554638769578358538
0586675457401100552617595512692321050024549971122364904484569497924
32431211206886086972294630725707719694695000656364962331145257799080486
343967092211316107065852719634190720095113178807135940134914662437802272
052545844144930115005039595882651286374824889559182652979334678090959581
4003523683211287320319489841731268453554502588069324055161268443331216
6295213998030002232335354419767585266713647261688469891402856910813765
5245435665843951623972774982343669566887491792446589934390152360145
4022370205987097601813242258731483730651273101910480339049326664560463
527738695382754181282843602972126278596244455275259258715441921530826
11355260292172870034744046794515625234263274775061989113520577721694133490
78439990630524613447165994122899943020211377881407919533204583820656458
References

[1] M. Filaseta and J. Juillerat, Consecutive primes which are widely digitally delicate, *Integers* **21A** (2021) #A12, 37pp.

[2] M. Filaseta and J. Southwick, Primes that become composite after changing an arbitrary digit, *Math. Comp.* **90** (2021)(328), 979–993, URL https://doi.org/10.1090/mcom/3593.

[3] O. Foundation, Inc., The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A050249, 2021.

[4] M. Kamada, Factorization of 11...11(repunit), https://stdkmd.net/nrr/repunit/, accessed 2021-04-24.

[5] J. T. Southwick, Two Inquiries Related to the Digits of Prime Numbers, ProQuest LLC, Ann Arbor, MI, 2020, URL http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:27828596, Thesis (Ph.D.)--University of South Carolina.

[6] The PARI Group, Univ. Bordeaux, *PARI/GP Version 2.11.0*, 2018, available at http://pari.math.u-bordeaux.fr/.