Bilinear approach to Kuperschmidt super-KdV type equations

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Abstract
Hirota bilinear form and soliton solutions for the super-KdV (Korteweg–de Vries) equation of Kuperschmidt (Kuper–KdV) are given. It is shown that even though the collision of supersolitons is more complicated than in the case of the supersymmetric KdV equation of Manin–Radul, the asymptotic effect of the interaction is simpler. As a physical application it is shown that the well-known FPU problem, having a phonon-mediated interaction of some internal degrees of freedom expressed through Grassmann fields, transforms to the Kuper–KdV equation in a multiple-scale approach.

Keywords: Grassmann function, Hirota formalism, solitons

1. Introduction

Nonlinearly coupled equations containing bosonic and fermionic fields can display a very interesting phenomenology. The first systematic construction of an integrable supersymmetric hierarchy was given a long time ago by Manin and Radul [2]. Complete integrability of such equations is a very interesting and deep problem which has no definitive answer so far, both classically and quantum mechanically. However, there are a lot of interesting results on the integrability of supersymmetric nonlinear equations (supersymmetry being an extra symmetry imposed by construction which means, roughly speaking an invariance with respect to a kind of interchanging between the bosonic and fermionic fields) [1, 2]. Results on Darboux transformations [19, 20], bi-Hamiltonian structure [9], prolongation structures [10] and Painlevé property [11] and many others revealed many nontrivial and completely new aspects about such systems. We also mention here appearance of non-local integrals of motion [3], lack of unique bilinear formulation of integrable hierarchies [4–8], inelastic interactions of supersolitons [13], peculiar properties of super-Painlevé equations [14] etc.
From the point of view of solutions, the inelastic character of super-solitons interaction was observed in [12, 13] together with a dressing of fermionic phases. The study of super-soliton interaction was extended to lattice equations. Here the supersymmetry is broken due to discretisation but still the equations can be analysed using the Lax pair or super-bilinear formalism [15, 16, 22, 23].

The case of non-supersymmetric coupled nonlinear equations is more difficult. Quite interestingly, the first completely super-extension of the KdV (Korteweg–de Vries) equation was given independently by Kuperschmidt and Kulish [26, 29] before the Manin–Radul [2] paper. It is not supersymmetric and served as the first example of integrability in Grassmann algebra. Although the Lax pair and bi-Hamiltonian structure were established quickly by Kuperschmidt himself, nothing was done about the solutions so far. The only partial answer was proposed recently in the paper of Kulish and Zeitlin [30] where the IST scheme was adapted to the Kuper–KdV equation but only in the case of a Grassmann algebra with only one generator (case in which the Kuper–KdV equation is reduced to a simple coupling between an ordinary KdV equation and an ordinary linear equation). Another non-supersymmetric system was the SSH-polyacetylene model which, in the non-resonant multiple scales, goes to the purely fermionic complex-mKdV equation and, at the resonance between phononic and electronic dispersion branches, to the coupled bosonic–fermionic Yajima–Oikawa–Redekopp equation having super-dark solitons [17, 18].

In this paper we are trying to go one step further and to compute a super-soliton solution for the Kuper–KdV equation using super-Hirota bilinear formalism. We will find a bilinear system containing some auxiliary fermionic tau function. We will examine the super-solitons interaction showing some similarities/differences with the Manin–Radul supersymmetric KdV case.

2. Kuper–KdV type equations

Throughout the paper we will deal with functions depending on $x, t$ having values in the commuting (bosonic) and anticommuting (fermionic) sector of an infinite dimensional Grassmann algebra.

The Kuper–KdV equation is:

\begin{align}
    u_t - 6uu_x + u_{xxx} + 12\xi_{xx} &= 0 \\
    \xi_t + 4\xi_{xxx} - 6u\xi_x - 3u\xi &= 0
\end{align}

with $u(x, t)$ bosonic function and $\xi(x, t)$ fermionic one.

The complete integrability was established by the existence of the Lax pair [26]. Indeed if

$$L = \frac{d^2}{d\xi} + u(x, t) + \xi(x, t)\frac{d}{d\xi}^{-1}\xi(x, t)$$

then

$$L_t = [\left(\frac{L^3}{2}\right)_+, L]$$

is equivalent with the Kuper–KdV equation. $L$ is a pseudo-differential super-operator and $L^3/2$ is the purely differential part of the formal power 3/2.

This is not the only one super-extension (non-supersymmetric). There is also the Holod–Pakuliak system [28]

$$u_t = -u_{xxx} + 6uu_x + 6(\alpha\beta_{xx} - \alpha_{xx}\beta)$$

(3)
\[ \alpha_t = -4\alpha_{xxx} + 6u\alpha_x + 3u_x\alpha \]
\[ \beta_t = -4\beta_{xxx} + 6u\beta_x + 3u_x\beta \]

and the extended super-KdV equation of Geng–Wu [27]

\[ u_t = -u_{xxx} + 6uu_x + 12u\xi_x\xi + 6u_x\xi_x\xi - 3\xi_{xxx}\xi_x - 6\xi_{xx}\xi_x \]
\[ \xi_t = -4\xi_{xxx} + 3u_x\xi + 6u_\xi. \]

2.1. Bilinear form of Kuper–KdV

Consider the following nonlinear substitutions

\[ u = -2(\log F)_{xx} \]
\[ \xi(x,t) = G(x,t)/F(x,t) \]

where \( G(x,t) \) is a Grassmann odd (anticommuting) function and \( F(x,t) \) is a Grassmann even (commuting) function. The bilinear form will be:

\[ (D_x D_t + D^2_x) F \cdot F + 6D_x G \cdot G = 0 \] (4)
\[ (D_t + D^3_x) G \cdot F + 3D_x K \cdot F = 0 \]
\[ D^2_x G \cdot F - KF = 0 \]

where \( K(x,t) \) is an auxiliary odd function. It has no role in the solution but is crucial for bilinearisation.

**Proof.** Introducing \( u = -2(\log F)_{xx} \) in (1) we get after one integration with respect to \( x \)

\[ 2(\log F)_{xx} + 2(\log F)_{xxxx} + 3(2(\log F)_{xxx})^2 = 12\xi_x. \]

Using the definition of the Hirota bilinear operator (A.1) and (A.2), this relation is transformed in:

\[ \frac{(D_x D_t + D^2_x) F \cdot F}{F^2} = 12 \frac{GD_x G \cdot F}{F^3}. \]

But, because \( G \) is Grassmann odd we have \( (12GD_x G \cdot F)/F^3 = 12GG_x/F^2 = -6D_x G \cdot G/F^2 \).

Accordingly, we obtain the first bilinear equation:

\[ (D_x D_t + D^2_x) F \cdot F + 6D_x G \cdot G = 0. \]

Now we can write (2) in the following way:

\[ \xi_t + 4\xi_{xxx} - 6u\xi_x - 3u_x\xi = \xi_t + \xi_{xxx} - 3u_x\xi + 3(\xi_{xx} - u_\xi) = 0. \]

Using (A.4) we get:

\[ \frac{(D_t + D^3_x) G \cdot F}{F^2} + 3(\xi_{xx} - u_\xi) = 0. \]

Let \( K(x,t) \) be an auxiliary fermionic function such that: \( \xi_{xx} - u_\xi = K/F \) which means (using (A.3)) \( D^2_x G \cdot F - KF = 0 \). Also introducing the above we get:

\[ (D_t + D^3_x) G \cdot F + 3D_x K \cdot F = 0. \]
Now we can compute directly the soliton solutions of the bilinear system (4) as combinations of exponentials \( \exp(k_i x - \omega_i t) \) where \( k_i \) are commuting (even) invertible Grassmann numbers and \( \omega_i = \omega_i(k_i) \) is the dispersion relation (some odd parameters can enter in the definition of the dispersion relation, but this is not the case here). For the fermionic (odd) tau functions \( G(x, t) \) and \( K(x, t) \) odd parameters \( \zeta \) have to be considered. Accordingly, every super-soliton is characterised by the following triplet \( (k, \zeta, \omega(k, \zeta)) \). Of course, nobody imposes the number of odd parameters for a soliton but we consider the simplest case here where any soliton is characterised by only one \( k \) and only one \( \zeta \).

The soliton solution has the following form:

- **1-soliton solution**
  \[
  G = \zeta_1 e^{\eta_1}, \quad F = 1 + e^{\eta_1}, \quad K = \zeta_1 \frac{k^2}{4} e^{\eta_1}
  \]
  where \( \eta_1 = k_1 x - k_1^2 t \) and \( \zeta_1 \) is a free Grassmann parameter,

- **2-soliton solution**
  \[
  G = \zeta_1 e^{\eta_1} + \zeta_2 e^{\eta_2} + \zeta_1 \alpha_{12} A_{12} e^{\eta_1 + \eta_2} + \zeta_2 \alpha_{21} A_{21} e^{\eta_2 + \eta_1}
  \]
  \[
  F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2} + \zeta_1 \zeta_2 A_{12} \frac{16}{(k_1 - k_2)^3} e^{\frac{\eta_1 + \eta_2}{2}}
  \]
  \[
  K = \zeta_1 \frac{k^2}{4} e^{\eta_1} + \zeta_2 \frac{k^2}{4} e^{\eta_2} + \zeta_1 \frac{k^2}{4} A_{12} e^{\eta_1 + \eta_2} + \zeta_2 \frac{k^2}{4} A_{21} e^{\eta_2 + \eta_1},
  \]

- **3-soliton solution**
  \[
  G = \sum_{i=1}^{3} \zeta_i e^{\eta_i} + \sum_{i \neq j \neq l} \zeta_i \alpha_{ij} A_{ij} e^{\eta_i + \eta_j} (1 + \alpha_{ij} A_{ij} e^{\eta_j}) + \zeta_1 \zeta_2 \zeta_3 M_{123} e^{\eta_1 + \eta_2 + \eta_3}
  \]
  \[
  F = 1 + \sum_{i=1}^{3} e^{\eta_i} + \sum_{i < j} A_{ij} e^{\eta_i + \eta_j} + A_{12} A_{13} A_{23} e^{\eta_1 + \eta_2 + \eta_3} + \sum_{i < j \neq l} \frac{16 \zeta_i \zeta_j A_{ij}}{(k_i - k_j)^3} e^{\eta_i + \eta_j} (1 + \alpha_{ij} A_{ij} e^{\eta_j})
  \]
  \[
  K = \sum_{i=1}^{3} \frac{k^2}{4} e^{\eta_i} + \sum_{i \neq j \neq l} \frac{k^2}{4} \alpha_{ij} A_{ij} e^{\eta_i + \eta_j} (1 + \alpha_{ij} A_{ij} e^{\eta_j}) + \zeta_1 \zeta_2 \zeta_3 M_{123} Q_{123} e^{\eta_1 + \eta_2 + \eta_3},
  \]

where:

\[
A_{ij} = \left( \frac{k_i - k_j}{k_i + k_j} \right)^2, \quad \alpha_{ij} = \frac{k_i + k_j}{k_i - k_j}
\]

\[
M_{123} = 8 \prod_{i < j \neq l} (k_i - k_j) \left( \frac{\alpha_{ij} A_{ij}}{(k_i - k_j)^2} + \frac{\alpha_{il} A_{il}}{(k_i - k_l)^2} \right)
\]
\[ Q_{123} = \frac{1}{3} \sum_{i < j \neq l}^3 \left( \frac{k_j^2}{4} + \frac{A_i A_j \alpha_{ij} \alpha_{il} (k_i - k_j)^4}{4(k_j - k_i)^2 A_i A_j \alpha_{il} + 4(k_i - k_j)^2 A_i A_j \alpha_{il}} \right). \]

It is instructive to see what the difference is between the super-soliton dynamics of the Kuper–KdV equation and the supersymmetric KdV equation of Manin–Radul. Indeed the nonlinear form of the supersymmetric KdV (susy-KdV) is [21]:

\[ u_t + 6uu_x - uu_{xxx} - 3\xi \xi_x = 0 \]
\[ \xi_t + 3(\xi u)_x + \xi_{xxx} = 0. \] (5)

If \( u = 2\partial_t \ln F, \xi = G/F \) then the supersymmetric bilinear form is [12, 16]:

\[ (D_t + D_x^1)G \cdot F = 0 \]
\[ (D_t D_x + D_x^1)F \cdot F = (D_x + D_x^1)G \cdot G \]

and the two super-soliton solution has the following form [13]:

\[ F = 1 + e^{\eta t} + e^{\eta t} + A_{12}(1 + \beta_{12} \zeta_1) e^{\eta t}; \]
\[ G = \zeta_1 e^{\eta t} + \zeta_2 e^{\eta t} + \zeta_1 A_{12} e^{\eta t} + \zeta_2 A_{12} e^{\eta t}; \]

where \( \eta = k_1 = k_1^1 + A_{ij} = (k_i - k_j)^2/(k_i + k_j)^2, \alpha_{ij} = (k_i + k_j)/(k_i - k_j), \beta_{ij} = 2/(k_i - k_j) \). One can see immediately the similarities between the forms of the two solutions (of Kuper–KdV and susy-KdV). However the asymptology of interaction is completely different. Indeed, in the reference system of soliton 1, (\( \eta_1 \) fixed and \( k_1 < k_2 \)) we obtain:

\[ \lim_{t \to \pm \infty} F = 1 + e^{\eta t}, \lim_{t \to \pm \infty} G = \zeta_1 e^{\eta t}. \]
\[ \lim_{t \to \infty} F = 1 + A_{12}(1 + 2\beta_{12} \zeta_1) e^{\eta t}, \lim_{t \to -\infty} G = \zeta_2 A_{12} e^{\eta t}. \]

Accordingly the interaction is elastic for the bosonic component with a fermionic correction of the phase shift (given by \( \beta_{12} \zeta_1 \zeta_2 \)), but for the fermionic component the interaction is not elastic. Not only is the amplitude dressed by the \( \alpha_{ij} \) but a creation of a fermionic background in the fermionic tau function appears given by \( \zeta_2 \) (however the conservation of energy is still preserved since the fermionic background can be gauged-away due to supersymmetry).

In the case of Kuper–KdV, the 2-super-soliton solution has the following form:

\[ G = \zeta_1 e^{\eta t} + \zeta_2 e^{\eta t} + \zeta_1 A_{12} e^{\eta t} + \zeta_2 A_{12} e^{\eta t} + 16 \frac{(1 - k_2)^3}{(k_1 - k_2)^3} e^{\eta t}. \]
\[ F = 1 + e^{\eta t} + e^{\eta t} + A_{12} e^{\eta t} + \zeta_1 A_{12} e^{\eta t} + \zeta_2 A_{12} e^{\eta t} + 16 \frac{(1 - k_2)^3}{(k_1 - k_2)^3} e^{\eta t}. \]

Again, considering the reference frame of the first soliton (i.e. \( \eta_1 \) fixed) and \( k_1 < k_2 \) we get:

\[ \lim_{t \to \pm \infty} F = 1 + e^{\eta t}, \lim_{t \to \pm \infty} G = \zeta_1 e^{\eta t}. \]
\[ \lim_{t \to \infty} F = 1 + A_{12} e^{\eta t}, \lim_{t \to -\infty} G = \zeta_2 A_{12} e^{\eta t}. \]
This is a simpler interaction. Asymptotically, the bosonic soliton does not feel at all the presence of the fermionic one. But the fermionic soliton has not only a phase shift but also a changing of amplitude from $\zeta_1$ to $\zeta_2$. What is really interesting is the presence of the fermionic dressing $\alpha_{ij} = (k_i + k_j)/(k_i - k_j)$ which appears both in supersymmetric KdV and Kuper–KdV. Moreover, this fermionic dressing seems to be universal in the sense that it appears in all bilinear super-equations analysed so far in literature. In the discrete setting the fermionic dressing also appears in the form $\alpha_{ij} = (e^{k_i} + e^{k_j} - 1)/(e^{k_i} - e^{k_j})$ [16]. □

**Remark.** Using the same procedure we can find the bilinear for the Holod–Pakuliak system (3). Namely if $\alpha = G_1/F, \beta = G_2/F, u = -2(\log F)_x$, then we get:

\[
(D_t + D_x^2) G_1 \cdot F + 6D_x G_1 \cdot G_2 = 0
\]

\[
(D_t + D_x^2) G_1 \cdot F + 3D_x K_1 \cdot F = 0
\]

\[
(D_t + D_x^2) G_2 \cdot F + 3D_x K_2 \cdot F = 0
\]

\[
D_x^2 G_1 \cdot F - K_1 F = 0
\]

\[
D_x^2 G_2 \cdot F - K_2 F = 0
\]

where $K_1, K_2$ are two auxiliary fermionic functions. The computation of soliton solutions goes on the same way as in the case of the one component system, except of some phases in the exponents. However the solutions turn out to be rather trivial, i.e. $G_1 = \pm G_2$ reducing the system to the one component case. So maybe a more complicated ansatz involving more Grassmann parameters is needed.

3. ‘Spinning’ FPU-problem

In this section we are going to give a possible physical model for the Kuper–KdV equation. Consider a chain of Fermi–Pasta–Ulam nonlinear oscillators (defined through $u_n(t)$) having also some internal degrees of freedom expressed through real Grassmann fields $\psi_n(t)$. We assume that in addition to nonlinear interaction we have also a phononic mediated interaction of nearest neighbours Grassmann fields (similar to the celebrated Su–Schriffer–Heeger model of polyacetylene dynamics, although in our model there is no hopping of electrons) The Hamiltonian has the form:

\[
H = \sum_n \left( \frac{p_n^2}{2m} + \frac{1}{2} \alpha^2 (u_{n+1} - u_n)^2 + \frac{\beta}{3} (u_{n+1} - u_n)^3 + i\gamma \psi_n \psi_{n+1} (1 + \delta (u_{n+1} - u_n)) \right).
\]

The equations of motion are given by the following formulae [24]:

\[
\ddot{u}_n = -\partial H/\partial u_n \quad \dot{\psi}_n = \partial H/\partial \psi_n
\]

\[
\ddot{u}_n = \alpha^2 (u_{n+1} + u_{n-1} - 2u_n) + \beta (u_{n+1} + u_{n-1} - 2u_n) (u_{n+1} - u_{n-1}) + i\gamma \delta \psi_n (\psi_{n+1} + \psi_{n-1})
\]

\[
\ddot{\psi}_n = \gamma (\psi_{n+1} - \psi_{n-1}) + \gamma \delta (u_{n+1} \psi_{n+1} + u_{n-1} \psi_{n-1}) - \gamma \delta u_n (\psi_{n+1} + \psi_{n-1}).
\]
The imaginary number $i$ in the Hamiltonian and first equation is important from the physical point of view showing the real character of the equation (even though all the fields are real), because $(i\psi_n\psi_{n+1})^* = -i\psi_n^*\psi_{n+1}^* = i\psi_n^*\psi_{n+1}$.

We want to compute the rigorous continuum limit through the method of multiple scales. Indeed in the linear regime we have two branches of dispersion related to the bosonic and fermionic field

$$\omega_b^2 - 4\alpha^2 \sin^2(k/2) = 0 \quad \text{and} \quad \omega_f + 2\gamma \sin k = 0.$$ 

In the continuum limit the wave length become large compared to $n$ so we can consider $k = k_0\epsilon$. Accordingly,$$
\omega_b^2(\epsilon) = \omega_b^2(1) - \gamma k_0^2 \epsilon^3 t/3 + O(\epsilon^5)$$

suggesting the following stretched variables

$$x = \epsilon(n + 2\gamma t), \quad \tau = \frac{\gamma}{3} \epsilon^3 t.$$ 

Also, because of the nonlinear interaction the fields also will be modified to

$$u_n(t) \rightarrow \epsilon^p \Phi(x, \tau), \quad \psi_n(t) \rightarrow \epsilon^q \zeta(x, \tau)$$

where $p, q$ are numbers to be determined in asymptotic balance. From the first equation we get:

$$\frac{\gamma^2}{9} \epsilon^{p+6} \Phi_{\tau\tau} + \frac{4\gamma^2}{3} \epsilon^{p+4} \Phi_{xx} + 4\gamma^2 \epsilon^{p+2} \Phi_{xxx} = \alpha^2 \epsilon^{p+2} \Phi_{xxx} + \frac{\alpha^2}{12} \epsilon^{p+4} \Phi_{xxxx} +$$

$$+ 2\beta \epsilon^{p+3} \Phi_x \Phi_{xx} + i\gamma \delta \epsilon^{q+2} \zeta_{xx}$$

while the second fermionic equation gives:

$$\frac{\gamma}{3} \epsilon^{q+3} \zeta_{\tau\tau} + 2\gamma \epsilon^{q+1} \zeta_x = \gamma \epsilon^q (2\epsilon \zeta_x + \frac{\epsilon^3}{3} \zeta_{xxx} + ... )$$

$$+ \gamma \delta \epsilon^{q+q}(2\zeta \Phi + \epsilon^2 (\zeta \Phi_{xx} + 2\zeta_x \Phi_x + \zeta \Phi_x) + ... ) - \gamma \delta \epsilon^{q+q}(2\zeta \Phi + \epsilon^2 \zeta_x \Phi + \frac{\epsilon^4}{12} \zeta_{xxx} \Phi + ...)$$

In order to have a nontrivial limit we have to cancel the terms of order $\epsilon^{p+2}$ in the bosonic equation which means $\alpha^2 = 4\gamma^2$. Maximal balance principle [31] gives $p = 1$ and $q = 3/2$ and when we put $\epsilon \rightarrow 0$ we get:

$$\zeta_{\tau} = \zeta_{xxx} + \frac{\delta}{3} (u_x \zeta + 2 \zeta_x u)$$

$$\Phi_{\tau\tau} = \frac{1}{4} \Phi_{xxxx} + \frac{3\beta}{\gamma^2} \Phi_x \Phi_{xx} + \frac{3i\delta}{4\gamma} \zeta_{xx}.$$ 

Taking $\Phi_x = u(x, t), \tau \rightarrow 4\tau$ we get exactly the Kuper–KdV equation (up to definition of parameters $\beta, \gamma, \delta$).

**Remark.** Defining stretched variables using the other branch of the dispersion, the final result is not changed. Also there is no way to find supersymmetric KdV (5) from the asimptology no matter how one chooses the parameters.
In addition we have to point out that the above lattice Hamiltonian is not modelling a phonon mediated interaction between nearest neighbour spins. In the pseudoclassical description developed by Berezin and Marinov [24] the spin is represented as an even noninvertible (nilpotent) Grassmann quantity (the vector product is not zero since the components anticommutes):

\[ \vec{S}_n = -\frac{i}{2} (\vec{\psi}_n \times \vec{\psi}_n) \]

where the vector \( \vec{\psi}_n \) has three Grassmann odd components \( (\phi_n, \chi_n, \sigma_n) \) and the Hamiltonian is given by:

\[ H = \sum_n \left( \frac{p_n^2}{2m} + \frac{1}{2} \alpha^2 (u_{n+1} - u_n)^2 + \frac{\beta}{3} (u_{n+1} - u_n)^3 + i\gamma (\vec{S}_n \cdot \vec{S}_{n+1}) (1 + \delta (u_{n+1} - u_n)) \right). \]

The equations of motion are:

\[ \frac{d^2}{dt^2} u_n = \alpha^2 (u_{n+1} + u_{n-1} - 2u_n) + \beta (u_{n+1} + u_{n-1} - 2u_n) (u_{n+1} - u_{n-1}) + i\gamma \delta \vec{S}_n \cdot (\vec{S}_{n-1} - \vec{S}_{n+1}) \]

\[ \frac{d}{dt} \vec{S}_n = \vec{S}_n \times (\vec{S}_{n+1} + \vec{S}_{n-1}) + \gamma \delta (u_{n+1} - u_n) (\vec{S}_n \times \vec{S}_{n+1}) + \gamma \delta (u_n - u_{n-1}) (\vec{S}_n \times \vec{S}_{n-1}) \]

and in the continuum limit these equations go to a coupled Boussinesq–Heisenberg ferromagnet system in Grassmann algebra. Its integrability and solutions are open problems.

**Appendix. Hirota operator for superfunctions**

The Hirota derivative is defined for superfunctions in the same way as for ordinary functions:

\[ D^n_{x} a \cdot b = (\partial_x - \partial_y)^n a(x) b(y)|_{x=y} \]

for any Grassmann odd or Grassmann even superfunctions \( a(x), b(x) \) However unlike the ordinary case \( D^n_{x} a \cdot a = 0, D^n_{x} a \cdot a \neq 0 \) if \( a \) is Grassmann odd.

Also [25]:

\[ 2(\log b)_{xx} = \frac{D^2_{x} b \cdot b}{b^2}, \quad 2(\log b)_{xxt} = \frac{D_{x} D_{x} b \cdot b}{b^2} \quad (A.1) \]

\[ 2(\log b)_{xxxx} = \frac{D^4_{x} b \cdot b}{b^2} - 3\left(\frac{D^2_{x} b \cdot b}{b^2}\right)^2 \quad (A.2) \]

\[ \partial^2_{x} (\frac{a}{b}) = \frac{D_{x} a \cdot b}{b^2}, \quad \partial^2_{x} (\frac{a}{b}) = \frac{D^2_{x} a \cdot b}{b^2} - \frac{a D^2_{x} b \cdot b}{b^2}, \quad (A.3) \]

\[ \partial^3_{x} (\frac{a}{b}) = \frac{D^3_{x} a \cdot b}{b^2} - \frac{3 D_{x} a \cdot b D^2_{x} b \cdot b}{b^2} \quad (A.4) \]

are valid for any superfunction \( a(x, t) \) and any invertible Grassmann even superfunction \( b(x, t) \)

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