LS-Galleries, the path model and MV-cycles

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Abstract

We give an interpretation of the path model of a representation \cite{18} of a complex semisimple algebraic group $G$ in terms of the geometry of its affine Grassmannian. In this setting, the paths are replaced by LS–galleries in the affine Coxeter complex associated to the Weyl group of $G$. To explain the connection with geometry, consider a Demazure–Hansen–Bott–Samelson desingularization $\hat{\Sigma}(\lambda)$ of the closure of an orbit $G(\mathbb{C}[[t]]).\lambda$ in the affine Grassmannian. The homology of $\hat{\Sigma}(\lambda)$ has a basis given by Bialynicki–Birula cell’s, which are indexed by the $T$–fixed points in $\hat{\Sigma}(\lambda)$. Now the points of $\hat{\Sigma}(\lambda)$ can be identified with galleries of a fixed type in the affine Tits building associated to $G$, and the $T$–fixed points correspond in this language to combinatorial galleries of a fixed type in the affine Coxeter complex. We determine those galleries such that the associated cell has a non-empty intersection with $G(\mathbb{C}[[t]]).\lambda$ (identified with an open subset of $\hat{\Sigma}(\lambda)$), and we show that the closures of the strata associated to LS-galleries are exactly the MV–cycles \cite{24}, which form a basis of the representation $V(\lambda)$ for the Langland’s dual group $G^\vee$.

Introduction

The aim of the present article is to provide a connection between the path model \cite{18} for finite dimensional representations of a connected complex semisimple algebraic group $G$ and the geometry of the affine Grassmann variety $\mathcal{G}$. As a byproduct of this interpretation, we associate in a canonical way to each path (or rather gallery) in a path model of a representation one of the algebraic cycles occurring in the work of Mirkovic and Vilonen \cite{24}. Recall that these cycles form, by \cite{24} and Vasserot \cite{26}, a canonical basis of the finite dimensional complex representations of the Langland’s dual group $G^\vee$.

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The affine Grassmannian, as a set, is defined as $G = G(K)/G(O)$, where $K = \mathbb{C}(t)$ and $O = \mathbb{C}[[t]]$. Fix a maximal torus $T \subset G$ and let $X = X(T)$ be its character group. The group $X^\vee = \text{Mor}(\mathbb{C}^*, T)$ of co-characters is the character group of the dual maximal torus $T^\vee$ of $G^\vee$. The dominant co-characters $X^\vee_+$ form an indexing set for two classes of objects. On the one hand, the finite dimensional irreducible $G^\vee$-modules $V(\lambda)$ are classified by their highest weight $\lambda \in X^\vee_+$. On the other hand, the co-characters can be viewed as points in $G$, and $G$ is the disjoint union of the $G(O)$–orbits $G_\lambda = G(O)\lambda$, $\lambda \in X^\vee_+$. The (Zariski) closure $X_\lambda = \overline{G_\lambda}$ is a finite dimensional projective variety, we fix a Demazure–Hansen–Bott–Samelson desingularization $\hat{\Sigma}(\lambda) \to X_\lambda$.

Another object which comes into the picture is the affine Tits–building $J^a$ associated to $G$. Corresponding to the choice of $T \subset G$, we have a fixed apartment $A \subset J^a$ in the Tits building, which, as a simplicial complex, is isomorphic to the affine Coxeter complex associated to the Weyl group of $G$. The link between the combinatorial part and the geometrical part of our construction is described by the diagram below:

$$
\begin{align*}
G(O)\text{–orbit } G_\lambda & \quad \hookrightarrow \quad \hat{\Sigma}(\lambda) \\
\text{minimal galleries of fixed type } \gamma_\lambda & \quad \hookrightarrow \quad \text{galleries of fixed type } \gamma_\lambda \\
\text{in the affine building } J^a & \quad \downarrow \quad \text{retraction onto the} \\
\text{affine building} & \quad \downarrow \quad \text{in } BB\text{-cells} \\
\text{positively folded combinatorial} & \quad \hookrightarrow \quad \text{combinatorial galleries of} \\
galleries of fixed type } \gamma_\lambda \text{ in } A & \quad \quad \text{fixed type } \gamma_\lambda \text{ in } A
\end{align*}
$$

which we explain now, starting with the combinatorial galleries. Assume for simplicity that $\lambda$ is regular. A combinatorial gallery joining the origin and $\lambda$ is a sequence of alcoves $(\Delta_i)_{i=0,\ldots,r}$ in $A$ such that $0 \in \Delta_0$, $\lambda \in \Delta_r$, and two consecutive alcoves have at least a codimension one face in common. Fix a minimal such gallery $\gamma_\lambda$ and let $\Gamma(\gamma_\lambda)$ denote the set of all combinatorial galleries in $A$ of the same type as $\gamma_\lambda$ (section 4) starting at the origin. We define a dimension function on the set $\Gamma(\gamma_\lambda)$ counting the number of certain hyperplanes crossed by a gallery, see section 5. The dimension of the galleries is bounded above, and we call a gallery in $\Gamma(\gamma_\lambda)$ an LS–gallery if its dimension is equal to the upper bound. We construct “folding operators” $e_\alpha, f_\alpha$ on the set of galleries $\Gamma(\gamma_\lambda)$ for all simple roots and show by analogy with the path model [18]: (see Proposition 4, Theorem 2)

**Theorem A.** The set of LS–galleries $\Gamma_{LS}(\gamma_\lambda)$ in $\Gamma(\gamma_\lambda)$ is stable under the folding operators. Let $B(\gamma_\lambda)$ be the directed colored graph having as vertices the set of LS-galleries $\Gamma_{LS}(\gamma_\lambda)$, and put an arrow $\delta \xrightarrow{\alpha} \delta'$ with color $\alpha$ between two galleries if $f_\alpha(\delta) = \delta'$. Then this graph is connected, and it is isomorphic to the crystal graph of the irreducible representation $V(\lambda)$ of $G^\vee$ of highest weight $\lambda$. Let $e(\delta)$ be the endpoint of a gallery, then

$$
\text{Char } V(\lambda) = \sum_{\delta \in \Gamma_{LS}(\gamma_\lambda)} \exp(e(\delta)).
$$
To connect the combinatorics with the geometry, recall that in terms of the affine Kac-Moody group associated to the extended Dynkin diagram of $G$, the affine Grassmannian is a generalized flag variety and $X_\lambda$ is a Schubert variety. Fixing a minimal gallery in $\mathcal{A}$ joining the origin and $\lambda$ is equivalent to fix a reduced decomposition of $\lambda$ in $W^a/W$, where $W^a$ is the affine Weyl group. In geometric terms, such a choice is equivalent to fix a Demazure–Hansen–Bott–Samelson desingularization $\pi : \hat{\Sigma}(\gamma_\lambda) \to X_\lambda$. Contou–Carrère [4] has shown that the points of an affine Bott–Samelson variety can be viewed in terms of the affine building $\mathcal{J}^a$ as the set of all galleries of the same type as $\gamma_\lambda$ starting at the origin ($= G(O)$). Using this interpretation in terms of galleries, we show that the birational desingularization map identifies the subset of minimal galleries $\hat{\mathcal{G}}_\lambda \subset \hat{\Sigma}(\gamma_\lambda)$ with the orbit $\mathcal{G}_\lambda \subset X_\lambda$.

Using the action of $T$, we get a cellular decomposition of $\hat{\Sigma}(\gamma_\lambda)$ indexed by the $T$-fixed points. In terms of the interpretation of $\hat{\Sigma}(\gamma_\lambda)$ as a set of galleries, the $T$-fixed points correspond to the galleries $\Gamma(\gamma_\lambda)$. This defines a set theoretic map $\hat{\Sigma}(\gamma_\lambda) \to \Gamma(\gamma_\lambda)$ sending a point to the center of its cell. This map has in fact also an interpretation in terms of buildings. The affine Tits–building $\mathcal{J}^a$ is the union of its apartments, and all apartments are isomorphic to $\mathcal{A}$. The theory of Tits–buildings provides a retraction of the affine building $\mathcal{J}^a$ onto the affine Coxeter complex $\mathcal{A}$. The retraction maps the set $\hat{\mathcal{G}}_\lambda$ of minimal galleries of type $\gamma_\lambda$ in $\mathcal{J}^a$ onto combinatorial galleries in $\mathcal{A}$ of type $\gamma_\lambda$. More precisely we get (see Theorem 3 and 4, Corollary 4).

**Theorem B.** The retraction induces a map $r_{\gamma_\lambda} : \hat{\mathcal{G}}_\lambda \to \Gamma^+(\gamma_\lambda)$ onto the set of all positively folded galleries of the same type as $\gamma_\lambda$. For such a gallery $\delta$, the fibre $r_{\gamma_\lambda}^{-1}(\delta)$ is naturally equipped with the structure of an irreducible quasi-affine variety, it is the intersection of a Białynicki–Birula cell of $\hat{\Sigma}(\gamma_\lambda)$ with $\hat{\mathcal{G}}_\lambda$. The dimension of the fibre is equal to the combinatorially defined dimension $\dim \delta$, and $r_{\gamma_\lambda}^{-1}(\delta)$ admits a finite decomposition into a union of subvarieties, each being a product of $\mathbb{C}^*$'s and $\mathbb{C}^*$'s. In particular, the fibre admits a canonical open and dense subvariety isomorphic to $\mathbb{C}^a \times (\mathbb{C}^*)^b$, where $b = \sharp J^-\infty(\delta)$ (section 14) and $a + b = \dim \delta$.

Let $p$ be the bijection $p : \hat{\mathcal{G}}_\lambda \to \mathcal{G}_\lambda$, and for $\delta \in \Gamma^+(\gamma_\lambda)$ write $Z(\delta)$ for the closure of the fibre in $X_\lambda$. Denote by $X^{\mu}_\lambda$ the union of the $Z(\delta)$ for all $\delta \in \Gamma^+(\gamma_\lambda)$ having $\mu$ as target:

$$Z(\delta) = p(r_{\gamma_\lambda}^{-1}(\delta)) \subset X_\lambda, \quad X^{\mu}_\lambda = \bigcup_{\delta \in \Gamma^+(\gamma_\lambda), \ e(\delta) = \mu} Z(\delta).$$

In group theoretic terms, we have $X^{\mu}_\lambda = \overline{U^-(K)\mu} \cap \mathcal{G}_\lambda$, where $U^- \subset B^-$ is the unipotent radical. The special rôle played by the LS-galleries is that the corresponding $Z(\delta)$ are precisely the irreducible components of $X^{\mu}_\lambda$ (see Theorem 4).

**Theorem C.** The irreducible components of $X^{\mu}_\lambda$ are given by the $Z(\delta)$ for $\delta$ a LS-gallery, i.e., $X_\lambda(\mu) = \bigcup_\delta Z(\delta)$, where $\delta$ runs only over all LS–galleries in $\Gamma^+(\gamma_\lambda)$ having as target $e(\delta) = \mu$. These irreducible components are precisely the MV–cycles.
As a consequence, the algebraic cycles in
\[ MV(\lambda) = \{ Z(\delta) = r_{\gamma}^{-1}(\delta) \mid \delta \text{ a LS-gallery in } \Gamma^+(\gamma) \} \]
form by [24] and [26] a canonical basis of \( V(\lambda) \), realized as the intersection homology of \( X_\lambda \).
So, Theorem C provides the representation theoretic interpretation of the combinatorial character formula in Theorem A, and the construction relates the path model to yet another construction of bases for representations (see also [20], [14], [15]).

It should be very interesting to connect directly the methods presented in this paper with the methods (using the moment map) developed by Anderson and Kogan in [1], [2].

To make the paper as self contained as possible, we recall in section 1 the definition of the affine Grassmannian and we explain how we restrict ourselves to the case of a simply connected semisimple group. The next section deals with the affine Kac-Moody group associated to \( G \) and various incarnations of the affine Grassmannian. The third section concerns the building theory, we recall the definitions of the three buildings one can associate to \( G \): the spherical building, the affine building and the building at infinity, and we introduce the retraction centered at \(-\infty\).

In section 4 we define our main objects, the combinatorial galleries. In section 5 we associate a dimension to a positively folded gallery and we introduce the LS-galleries.

To prove the majoration of the dimension of a positively folded gallery, we introduce in section 6 the folding operators, which are appropriate analogues of the root operators for the path model of a representation. We also give a combinatorial characterization of the LS-galleries, and we give the character formula for \( V(\lambda) \) in terms of LS-galleries.

In section 7 we discuss the connection between a Demazure–Hansen–Bott–Samelson resolution of the Schubert variety \( X_\lambda \) and galleries of a fixed type in the Tits–building. The last sections are devoted to the construction of the stratification of the orbit \( G_\lambda \) indexed by the positively folded galleries and the proof of the connection between MV-cycles and strata corresponding to LS-galleries.

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1 The Affine Grassmannian

Let \( G \) be a connected semisimple complex algebraic group. For a commutative \( \mathbb{C} \)-algebra \( \mathcal{R} \) let \( G(\mathcal{R}) \) be the set of \( \mathcal{R} \)-rational points of \( G \), i.e., the set of algebra homomorphisms
from the coordinate ring \( \mathbb{C}[G] \to \mathcal{R} \). Then \( G(\mathcal{R}) \) comes naturally equipped again with a group structure, for example for \( G = SL_n(\mathbb{C}) \), we can identify \( SL_n(\mathcal{R}) \) with the set of \( n \times n \)–matrices with entries in \( \mathcal{R} \) and determinant 1. Similarly, by embedding \( G \hookrightarrow SL_n(\mathbb{C}) \), we can identify \( G(\mathcal{R}) \) naturally with a subgroup of \( SL_n(\mathcal{R}) \).

Denote \( \mathcal{O} = \mathbb{C}[[t]] \) the ring of formal power series in one variable and let \( \mathcal{K} = \mathbb{C}((t)) \) be its fraction field, the field of formal Laurent series. Denote \( v : \mathcal{K}^* \to \mathbb{Z} \) the standard valuation on \( \mathcal{K} \) such that \( \mathcal{O} = \{ f \in \mathcal{K} \mid v(f) \geq 0 \} \). The loop group \( G(\mathcal{K}) \) is the set of \( \mathcal{K} \)–valued points of \( G \), we denote by \( G(\mathcal{O}) \) its subgroup of \( \mathcal{O} \)–valued points. The latter has a decomposition as a semi-direct product \( G \ltimes \mathcal{O}(1) \), where we view \( G \subset G(\mathcal{O}) \) as the subgroup of constant loops and \( \mathcal{O}(1) \) is the subgroup of elements congruent to the identity modulo \( t \). Note that we can describe \( G(\mathcal{O}) \) also as the image of \( (\text{Lie} G) \otimes_{\mathbb{C}} t \mathbb{C}[[t]] \) via the exponential map. As a set, the affine grassmannian \( \mathcal{G} \) is the quotient

\[
\mathcal{G} = G(\mathcal{K})/G(\mathcal{O}).
\]

Note that \( G(\mathcal{K}) \) and \( \mathcal{G} \) are ind–schemes and \( G(\mathcal{O}) \) is a group scheme (see [4, 16, 17, 23]). There is a model of \( \mathcal{G} \) due to Lusztig which describes \( \mathcal{G} \) as an increasing union of finite dimensional complex projective varieties \( \mathcal{G}^{(n)} \), where each \( \mathcal{G}^{(n)} \) is a subvariety of some finite dimensional Grassmann variety.

Fix a maximal torus \( T \subset G \) and Borel subgroups \( B, B^- \subset G \) such that \( T \cap B^- = T \). We denote \( \langle \cdot, \cdot \rangle \) the non–degenerate pairing between the character group \( X := \text{Mor} (T, \mathbb{C}^*) \) of \( T \) and the group \( X^\vee := \text{Mor} (\mathbb{C}^*, T) \) of co-characters. We identify the lattice \( X^\vee \) with the quotient \( T(\mathcal{K})/T(\mathcal{O}) \), so we use the same symbol \( \lambda \) for the co-character and the point in \( \mathcal{G} \). Let now \( p : G' \to G \) be an isogeny with \( G' \) being simply connected. Then \( G'(\mathcal{O}) \simeq G' \ltimes (G')^{(1)}(\mathcal{O}) \), and since \( (G')^{(1)}(\mathcal{O}) \) and \( G^{(1)}(\mathcal{O}) \) are naturally isomorphic, we see that the natural map \( p_\mathcal{O} : G'(\mathcal{O}) \to G(\mathcal{O}) \) is surjective and has the same kernel as \( p \). Let \( X' \) and \( X'^\vee \) be the character group respectively group of co-characters of \( G' \) for a maximal torus \( T' \subset G' \) such that \( p(T') = T \), then \( p : T' \to T \) induces an inclusion \( X'^\vee \hookrightarrow X^\vee \).

The quotient \( X^\vee/X'^\vee \) measures the difference between \( \mathcal{G} \) and the affine grassmannian \( \mathcal{G}' = G'(\mathcal{K})/G'(\mathcal{O}) \). In fact, \( \mathcal{G}' \) is connected, and the connected components of \( \mathcal{G} \) are indexed by \( X^\vee/X'^\vee \). The natural maps \( p_\mathcal{K} : G'(\mathcal{K}) \to G(\mathcal{K}) \) and \( p_\mathcal{O} : G'(\mathcal{O}) \to G(\mathcal{O}) \) induce a \( G'(\mathcal{K}) \)–equivariant inclusion \( \mathcal{G}' \hookrightarrow \mathcal{G} \), which is an isomorphism onto the component of \( \mathcal{G} \) containing the class of 1. Now \( G'(\mathcal{K}) \) acts via \( p_\mathcal{K} \) on all of \( \mathcal{G} \), and each connected component is a homogeneous space for \( G'(\mathcal{K}) \), isomorphic to \( G'(\mathcal{K})/\mathcal{Q} \) for some parahoric subgroup \( \mathcal{Q} \) of \( G'(\mathcal{K}) \) which is conjugate to \( G(\mathcal{O}) \) by an outer automorphism (see [4, 16] below, for a definition of parahoric subgroups).

Remark 1. So to study \( G(\mathcal{O}) \)–orbits on \( G(\mathcal{K})/G(\mathcal{O}) \) for \( G \) semisimple, without loss of generality we may assume that \( G \) is simply connected, but we have to investigate more generally \( G(\mathcal{O}) \)–orbits on \( G(\mathcal{K})/\mathcal{Q} \) for all parahoric subgroups \( \mathcal{Q} \subset G(\mathcal{K}) \) conjugate to \( G(\mathcal{O}) \) by an outer automorphism.
2 The affine Kac-Moody group

In the following let $G$ be a simply connected semisimple complex algebraic group. Let again $\mathcal{O} = \mathbb{C}[\lbrack \lbrack t \rbrack \rbrack]$ denote the ring of formal power series in one variable and let $K = \mathbb{C}(\lbrack \lbrack t \rbrack \rbrack)$ be its fraction field.

The rotation operation $\gamma : \mathbb{C}^* \to \text{Aut}(\mathcal{K})$, $\gamma(z)(f(t)) = f(zt)$ gives rise to group automorphisms $\gamma_G : \mathbb{C}^* \to \text{Aut}(G(\mathcal{K}))$, we denote $\mathcal{L}(G(\mathcal{K}))$ the semidirect product $\mathbb{C}^* \ltimes G(\mathcal{K})$.

The rotation operation on $\mathcal{K}$ restricts to an operation of $\mathcal{O}$ and hence we have a natural subgroup $\mathcal{L}(G(\mathcal{O})) := \mathbb{C}^* \ltimes G(\mathcal{O})$ (for this and the following see [16], Chapter 13).

Let $\hat{\mathcal{L}}(G)$ be the affine Kac-Moody group associated to the affine Kac–Moody algebra

$$\hat{\mathcal{L}}(g) = g \otimes \mathcal{K} \oplus \mathbb{C}c \oplus Cd,$$

where $0 \to \mathbb{C}c \to g \otimes \mathcal{K} \oplus \mathbb{C}c \to g \otimes \mathcal{K} \to 0$ is the universal central extension of the loop algebra $g \otimes \mathcal{K}$ and $d$ denotes the scaling element. We have corresponding exact sequences also on the level of groups, i.e., $\hat{\mathcal{L}}(G)$ is a central extension of $\mathcal{L}(G(\mathcal{K}))$

$$1 \to \mathbb{C}^* \to \hat{\mathcal{L}}(G) \xrightarrow{\pi} \mathcal{L}(G(\mathcal{K})) \to 1$$

(see [16], Chapter 13). Denote $\mathcal{P}_\mathcal{O} \subset \hat{\mathcal{L}}(G)$ the “parabolic” subgroup $\pi^{-1}(\mathcal{L}(G(\mathcal{O})))$. We have four incarnations of the affine grassmannian:

$$\mathcal{G} = G(\mathcal{K})/G(\mathcal{O}) = \mathcal{L}(G(\mathcal{K}))/\mathcal{L}(G(\mathcal{O})) = \hat{\mathcal{L}}(G)/\mathcal{P}_\mathcal{O} = G(\mathbb{C}[t,t^{-1}])/G(\mathbb{C}[t]). \tag{1}$$

We consider now the various maximal tori and Weyl groups. Let $N = N_G(T)$ be the normalizer in $G$ of the fixed maximal torus $T \subset G$, we denote by $W$ the Weyl group $N/T$ of $G$. Let $N_K$ be the subgroup of $G(\mathcal{K})$ generated by $N$ and $T(\mathcal{K})$, let $\overline{T}$ be the standard maximal torus $\mathbb{C}^* \times T$ in $\mathcal{L}(G(\mathcal{K}))$ and denote by $\overline{N} \subset \mathcal{L}(G(\mathcal{K}))$ the extension of $N_K$.

Finally, let $T \subset \hat{\mathcal{L}}(G)$ be the standard maximal torus (such that $\pi(T) \subset \overline{T}$), and let $\mathcal{N}$ be its normalizer in $\hat{\mathcal{L}}(G)$, then we get three incarnations of the affine Weyl group:

$$W^a = N_K/T \simeq \overline{N}/\overline{T} \simeq \mathcal{N}/T$$

Let $ev : G(\mathcal{O}) \to G$ and $ev_L : \mathcal{L}(G(\mathcal{O})) \to \mathbb{C}^* \times G$ be the evaluation maps at $t = 0$, and let $B = ev^{-1}(B)$ respectively $B_L = ev_L^{-1}(\mathbb{C}^* \times B)$ be the corresponding Iwahori subgroups, and let $\hat{B} = \pi^{-1}(B_L) \subset \hat{\mathcal{L}}(G)$ be the Borel subgroup. We have the corresponding Bruhat decompositions, and note that the pair $(B, N_K)$ is a BN–pair in the loop group $G(\mathcal{K})$, and $(\hat{B}, \mathcal{N})$ is a BN–pair in the affine Kac–Moody group $\hat{\mathcal{L}}(G)$.

$$G(\mathcal{K}) = \bigcup_{w \in W^a} BwB, \quad \mathcal{L}(G(\mathcal{K})) = \bigcup_{w \in W^a} B_LwB_L, \quad \hat{\mathcal{L}}(G) = \bigcup_{w \in W^a} \hat{B}w\hat{B}.$$
by an outer automorphism of $G(K)$, by the study of $P\O$--orbits on $\hat{L}(G)/\hat{Q}$, where $\hat{Q} \subset \hat{L}(G)$ is a parabolic subgroup of $\hat{L}(G)$, conjugate to $P\O$ by a diagram automorphism of $\hat{L}(G)$. These orbits correspond exactly to each other because the kernel of $\pi$ acts trivially on $\hat{L}(G)/\hat{Q}$, $\hat{L}(G)/\hat{Q}$, and the $L(G(\O))$--orbits and the $G(\O)$--orbits in $\hat{L}(G)/\hat{Q}$ obviously coincide since by the Bruhat decomposition the orbits are parameterized by the same $T$-- respectively $\hat{T}$-- respectively $T$--fixed points in $G(K)/Q = \hat{L}(G)/\hat{Q}$.

3 Buildings, roots and characters

From an abstract point of view, a building $\mathcal{J}$ is a simplicial complex covered by some subcomplexes, called apartments, such that, each apartment is a Coxeter complex, any two simplices of $\mathcal{J}$ are always contained in an apartment, and given two apartments $A$ and $A'$ with a common maximal simplex, then there is an isomorphism $A \to A'$ fixing every simplex in $A \cap A'$ (see for example [7] or [29]).

These properties can be used to define retractions $r : \mathcal{J} \to A$. We will use the building theory later because one can identify a $G(\O)$--orbit in $G(K)/G(\O)$ with a subset of the affine building (see [33]) associated to $G$ (not needed to be simply connected). In particular, the retraction of the building onto an apartment will be the main tool to identify certain strata of a $G(\O)$--orbit with combinatorial objects in the apartment. Note that the apartment is in our case nothing else then the Coxeter complex of the affine Weyl group.

In this section we fix some notation and recall for the convenience of the reader some general facts on building theory. As references we suggest [7], [8], [25] and/or [29].

3.1 Roots and characters

We denote $\langle \cdot , \cdot \rangle$ the non–degenerate pairing between the character group $X := \text{Mor} (T, \mathbb{C}^*)$ of $T$ and its group $X^\vee := \text{Mor} (\mathbb{C}^*, T)$ of co-characters. Let $\Phi \subset X$ be the root system of the pair $(G, T)$, and, corresponding to the choice of $B$, denote $\Phi^+$ the set of positive roots, let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots, and let $\rho$ be half the sum of the positive roots. Let $\Phi^\vee \subset X^\vee$ be the dual root system, together with a bijection $\Delta \to \Delta^\vee, \alpha \mapsto \alpha^\vee$. We denote $R^\vee_+$ the submonoid of the coroot lattice $R^\vee$ generated by the positive coroots $\Phi^\vee_+$. We define on $X^\vee$ a partial order by setting $\lambda \geq \nu \iff \lambda - \nu \in R^\vee_+$, and let $X^\vee_+$ be the cone of dominant co-characters.

$$X^\vee_+ := \{\lambda \in X^\vee | \langle \lambda, \alpha \rangle \geq 0 \forall \alpha \in \Phi^+\}$$

In the following, we will deal with three buildings associated to the simply connected complex semisimple group $G$. The first building we consider is the
3.2 Spherical Building

Let us denote by $J^s$ the set of all the parabolic subgroups of $G$. The opposite relation of the inclusion between parabolic subgroups endows this set with a structure of simplicial complex. The maximal simplices given by the Borel subgroups will be called “spherical” chambers, the others, simplices or “spherical” faces. We will often drop the term “spherical” when no confusion may arise.

For every maximal torus $T$ in $G$, we define an apartment in $J^s$, by letting $A^s(T)$ be the set of all the parabolic subgroups that contain $T$. Together with those apartments, the simplicial complex $J^s$ is a building (see for example [29] or [7]). The apartments are isomorphic to the Coxeter complex $C(W, S)$ defined by the Coxeter presentation of the Weyl group $W$. The existence of the retractions in the spherical building is equivalent to the Bruhat decompositions of the group $G$.

Example 1. Denote by $C_f$ the chamber corresponding to the Borel subgroup $B$. Recall that, as a set, we can identify the set of chambers in the building with the flag variety $G/B$. The retraction $r_{C_f}: J^s \rightarrow A^s$ onto the apartment $A^s$ of center $C_f$ is defined as follows: Given a chamber $F_{B'}$ in the building associated to a Borel subgroup $B'$, by the Bruhat decomposition we can find $b \in B$ and $w \in W$ such that $B' = bwB/B$ in $G/B$, and the image by the retraction is the $T$-fixed point $wB/B$ in the Schubert cell $BwB/B$. In the language of buildings we set $r_{C_f}(F_{B'}) := F_{wBw^{-1}}$. This map is actually defined on any face since there exists only one parabolic subgroup of a given type containing a given Borel subgroup.

3.3 Affine Building

We can identify in a similar way the affine building $J^a$, as a set, with the set of all the parahoric subgroups of $G(\mathcal{K})$, that is the set of all the subgroups of $G(\mathcal{K})$ that contain a conjugate of $B$. The construction below differs from the one above, but, with the appropriate changes, it would apply in the same way also to the spherical building.

Set $\mathcal{A} := X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, the affine Weyl group $W^a$ acts on $\mathcal{A}$ as an affine reflection group. Let $H^a \subset \mathcal{A}$ be the set of affine reflection hyperplanes for the action of $W^a$ (i.e., the affine reflections with respect to these hyperplanes generate $W^a$ and the set $H^a$ is stable under the action of $W^a$). The connected components of $\mathcal{A} - H^a$ are called open alcoves, the closure of such a component is called a closed alcove or just alcove.

The hyperplanes in $H^a$ are all of the form $H_{\beta,m} = \{ a \in \mathcal{A} \mid \langle a, \beta \rangle = m \}$ for some positive root $\beta \in \Phi^+$ and $m \in \mathbb{Z}$, we denote $s_{\beta,m}$ the corresponding affine reflection. The associated closed affine halfspaces are denoted $H^+_{\beta,m} = \{ a \in \mathcal{A} \mid \langle a, \beta \rangle \geq m \}$, resp. $H^-_{\beta,m} = \{ a \in \mathcal{A} \mid \langle a, \beta \rangle \leq m \}$. We use the notation $H^+_{\beta,m}^o$, respectively $H^-_{\beta,m}^o$, for the corresponding open affine halfspaces.

Definition 1. By a face $F$ we mean a subset of $\mathcal{A}$ obtained as the intersection of closed
affine halfspaces and affine hyperplanes, the intersection running over all pairs \((\beta, m)\), \(\beta \in \Phi^+, m \in \mathbb{Z}\). By the corresponding open face \(F^o\) we mean the subset of \(F\) obtained when replacing the closed affine halfspaces in the definition of \(F\) by the corresponding open affine halfspaces.

The open faces define a partition of \(A\): we call two elements \(x, y \in A\) equivalent if for all pairs \((\beta, m)\), \(x\) and \(y\) are both in either \(H_{\beta,m}\), or \(H^+_{\beta,m}\), or \(H^-_{\beta,m}\). The equivalence classes of this relation are the open faces \(F^o\), and the closed face \(F\) is the closure of \(F^o\) in the affine subspace \(\langle F^o \rangle \text{aff}\) spanned by \(F^o\).

We call \(\langle F^o \rangle \text{aff} = \langle F \rangle \text{aff}\) the support of the (open) face, the dimension of the face is the dimension of its support. A wall of an alcove is the support of a codimension one face. In general, instead of the term hyperplane we use often the term wall, which is more common in the language of buildings. The following is well-known:

**Theorem 1.** The affine Weyl group \(W^a\) acts simply transitive on the set of all alcoves. The fundamental alcove \(\Delta_f = \{ \nu \in A \mid 0 \leq \langle x, \beta \rangle \leq 1, \forall \beta \in \Phi^+ \}\) is a fundamental domain for the action, and \(W^a\) is generated by the affine reflections

\[
S^a = \{ s_{\beta,m} \mid H_{\beta,m} \text{ is a wall for } \Delta_f \}
\]

Denote by \(S \subset S^a\) the set of reflections \(S = \{ s_{\beta} := s_{\beta,0} \mid H_{\beta,0} \text{ is a wall for } \Delta_f \}\); we can describe \(S\) also as \(S^a(0) = \{ s_{\beta,m} \in S^a \mid 0 \subset H_{\beta,m} \}\). More generally, for a face \(F\) of \(\Delta_f\) let

\[
S^a(F) = \{ s_{\beta,m} \in S^a \mid F \subset H_{\beta,m} \}.
\]

We call \(S^a(F)\) the type of \(F\), so \(S^a(0) = S\) and \(S^a(\Delta_f) = \emptyset\). For an arbitrary face \(F\) its type is defined as the type of \(F'\), where \(F'\) is the unique face of \(\Delta_f\) such that \(F = w(F')\) for some \(w \in W^a\).

**Remark 2.** The alcoves are actually the chambers of the Coxeter complex \(A^a\) associated to the Coxeter group \((W^a, S^a)\), but since we look at the same time also at the spherical complex, to not confuse the “affine” and the spherical chambers, we prefer the term alcove for the faces of maximal dimension.

We view the (spherical) Weyl group \(W \subset W^a\) as the subgroup generated by the reflections in \(S\). By an open chamber we mean always an open spherical chamber, i.e., a connected component of \(A - \bigcup_{\beta \in \Phi^+} H_{\beta,0}\), and a chamber is the closure of such a connected component. We have the dominant chamber \(\mathcal{C}_f\) corresponding to the choice of the Borel subgroup \(B\) and the anti-dominant chamber \(-\mathcal{C}_f\).

**Definition 2.** A sector \(s\) in \(A\) is a \(W^a\)-translate of a chamber. Two sectors \(s, s'\) are called equivalent if there exists a third sector \(s''\) in the intersection: \(s'' \subset s \cap s'\).
Lemma 1. The equivalence classes of sectors are in one-to-one correspondence with the spherical chambers, i.e., in every class there exists a unique spherical chamber.

For a root $\beta \in \Phi$ let $g_\beta \subset g$ be the root subspace of the complex Lie algebra $g = \text{Lie} \, G$ and fix a generator $X_\beta$. Via the exponential map $\exp : \mathbb{C} \to G$, $x \mapsto \exp(xX_\beta)$ we get a one-dimensional unipotent subgroup $U_\beta$ normalized by $T$.

Definition 3. For a real number $r$ let $U_{\beta,r} \subset U_\beta(K)$ be the unipotent subgroup

$$U_{\beta,r} = \{1\} \cup \left\{ \exp(X_\beta \otimes f) \mid f \in K^*, v(f) \geq r \right\}.$$

For a non-empty subset $\Omega \subset A$ let $\ell_\beta(\Omega) = -\inf_{x \in \Omega} \beta(x)$. We attach to $\Omega$ a subgroup of $G(K)$ by setting

$$U_\Omega := \langle U_{\beta,\ell_\beta(\Omega)} \mid \beta \in \Phi \rangle. \quad (2)$$

Example 2. If $\Omega = 0$, then we have $U_0 = G(O)$, and if $w \in W^a$, then $U_{w(\Omega)} = n_w U_\Omega n_w^{-1}$ for any representative $n_w \in N(K)$ of $w$.

To define the affine building $J^a$, let $\sim$ be the relation on $G(K) \times A$ defined by:

$$(g, x) \sim (h, y) \quad \text{if} \quad \exists n \in N(K) \text{ such that } nx = y \text{ and } g^{-1}hn \in U_x.$$ 

Definition 4. The affine building $J^a := G(K) \times A/\sim$ associated to $G$ is the quotient of $G(K) \times A$ by “$\sim$”. The building $J^a$ comes naturally equipped with a $G(K)$-action $g \cdot (h, y) := (gh, y)$ for $g \in G(K)$ and $(h, y) \in J^a$.

The map $A \to J^a$, $x \mapsto (1, x)$ is injective and $N(K)$ equivariant, we will identify in the following $A$ with its image in $J^a$.

The stabilizer $P_{(g,y)}$ of any point is a parahoric subgroup. In fact, for $x \in A$ it is the parahoric subgroup generated by the stabilizer $N_x = \{n \in N(K) \mid nx = x\}$ and $U_x$:

$$P_x = \langle U_x, N_x \rangle.$$

Example 3. If $x \in \Delta^o_f$ is in the open face of the fundamental alcove $\Delta_f$, then $P_x = B$. More generally, let $F$ be a face of the fundamental alcove and denote by $W^a(F) \subset W^a$ the subgroup generated by its type $S^a(F)$. If $x \in F^a$ is an element in the corresponding open face, then $P_x = P_F$ is the corresponding standard parahoric subgroup of type $F$:

$$P_F := \bigcup_{w \in W^a(F)} BwB.$$

Definition 5. The subsets of $J^a$ of the form $gA$ are called apartments.

Lemma 2. For any $g \in G(K)$ and $x \in A \cap g^{-1}A$, there exists a $n \in N(K)$ such that $gx = nx$, or, in other words, $G(K)x \cap A = N(K)x$ for $x \in A$. 

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So we can extend the partition of $\mathcal{A}$ into open faces into a partition of $\mathcal{J}$ by calling a subset $F^o$ of $\mathcal{J}$ an open face if it is of the form $gF^o$ for some face $F' \in \mathcal{A}$. Similarly, we call a subset of $\mathcal{J}$ a face, an alcove, a sector etc. if it is of the form $gF$, $g\Delta$, $gs$ etc. for some face, alcove, sector etc. in $\mathcal{A}$. Note that for any non-empty set $\Omega \subset \mathcal{A}$ the group $U_\Omega$ acts transitively on the set of all apartments which contain $\Omega$.

We have a Bruhat decomposition $G(K) = U_x N(K) U_y$ for any pair of elements $x, y \in \mathcal{A}$. Another important fact from the theory of buildings is that any two faces or any two alcoves are contained in a common apartment, and if $s, s'$ are two sectors, then there exist subsectors $s_1 \subset s$ and $s'_1 \subset s'$ such that $s_1$ and $s'_1$ are contained in a common apartment, and if $s$ is a sector and $\Delta$ is an alcove, then there exists a subsector $s_1 \subset s$ such that $\Delta$ and $s_1$ are in a common apartment.

We see that on the level of sets the affine building $\mathcal{J}$ is the disjoint union of $G(K)/P_F \times F^o$’s, where the $F$’s are running through the set of faces of the fundamental alcove.

**Remark 3.** The action of $G(K)$ extends to an action of the Kac-Moody group $\hat{L}(G)$ on $\mathcal{J}$ such that the stabilizer of $\Delta_f$ is the Borel subgroup $\hat{B}$ and the center acts trivially.

### 3.4 The Building at Infinity and the Retraction $r_{-\infty}$

The last building we introduce is called the spherical building at infinity, we will denote it by $\mathcal{J}_\infty$. We refer to [7] or [25] for a precise definition, we recall quickly its construction and the properties which we will need.

The apartments for $\mathcal{J}_\infty$ are the same as for $\mathcal{J}$, only the structure of the complex is now different. The chambers of the complex in an apartment are the equivalence classes of sectors. So the structure is similar to that of the spherical complex $\mathcal{J}$, only that one does not have anymore a preferred vertex for the “Weyl chambers”. Still, the apartments of $\mathcal{J}_\infty$ are isomorphic to the Coxeter complex $C(W, S)$ of the Weyl group $W$. Since the equivalence classes of sectors are in one-to-one correspondence with the spherical chambers, it makes sense to use the following notation:

**Definition 6.** The equivalence class of sectors of the anti-dominant Weyl chamber $-\mathcal{E}_f$ is called the anti-dominant chamber at $-\infty$ and is denoted $\mathcal{E}_{-\infty}$.

For any alcove $\Delta$ in the apartment $\mathcal{A} \hookrightarrow \mathcal{J}$, one can define a chamber complex map $r_{\Delta, \mathcal{A}}: \mathcal{J} \to \mathcal{A}$, called the retraction onto $\mathcal{A}$ of center $\Delta$. The retraction has the following two properties, which in addition characterize the map uniquely: the first property is that for any face $F$ of $\Delta$, $r_{\Delta, \mathcal{A}}^{-1}(F) = \{F\}$.

To recall the second property, we define a distance on the set of alcoves. Two alcoves are called adjacent if they have a common codimension one face. A gallery of alcoves of length $r$ is a sequence $(\Delta_0, \ldots, \Delta_r)$ of alcoves such that $\Delta_i$ and $\Delta_{i-1}$ are adjacent for $1 \leq i \leq r$. The distance $d(\Delta, \Delta')$ of two alcoves in $\mathcal{J}$ is the minimal length of a gallery $(\Delta_0, \ldots, \Delta_r)$ joining $\Delta$ and $\Delta'$ (i.e., $\Delta_0 = \Delta$ and $\Delta_r = \Delta'$).
A gallery of alcoves \((\Delta_0, \ldots, \Delta_r)\) is called \textit{minimal} if the length is equal to the distance \(d(\Delta_0, \Delta_r)\) between the first and the last alcove. Recall ([8], section 2.3.6), two alcoves lie always in an apartment containing both, and any apartment that contains both also contains all minimal galleries joining the two.

The second important property of the retraction is that for any alcove \(\Delta'\), \(d(\Delta, \Delta') = d(\Delta, r_{\Delta, A}(\Delta'))\), so the map preserves the distance from its center. Further, \(r_{\Delta, A}\) restricts to an isomorphism of chamber complexes \(gA \simeq A\), for any apartment \(gA\), \(g \in G(K)\).

The image \(r_{\Delta, A}(\Delta')\) of an alcove \(\Delta'\) depends of course on the alcove \(\Delta\), but the image of \(\Delta'\) becomes in fact stable if \(\Delta\) is only “far away” enough from \(\Delta'\). To be more precise, consider an alcove \(\Delta' \in J^a\) and let \(-C_f\) be the anti-dominant chamber. Then there exists a sector \(s\) equivalent to \(-C_f\) such that \(s\) and \(\Delta'\) lie in a common apartment \(gA\). Let \(\Delta\) be an alcove in the sector and recall that \(r_{\Delta, A}\) restricts to an isomorphism of chamber complexes \(gA \simeq A\), which fixes of course the common sector \(s\). It follows (see for example, [25] §9.4) that \(r_{\Delta, A}(\Delta')\) is independent of the choice of \(\Delta \subset s\).

**, Definition 7.** The map \(r_{-\infty} : J^a \to A\), defined by \(r_{-\infty}(\Delta') = r_{\Delta, A}(\Delta')\) for some alcove \(\Delta \subset s\), where \(s\) is a sector, equivalent to \(-C_f\), contained in a common apartment with \(\Delta'\), is called the \textit{retraction of center} \(-\infty\).

The retraction \(r_{\Delta, A}\) can also be expressed in terms of the action of the group \(U_{\Delta}\) (see (2)). Recall that the latter operates transitively on the set of apartments containing \(\Delta\), and the retraction is in fact the projection \(r_{\Delta} : uA \to A\) that maps a face \(uF\), \(u \in U_{\Delta}, F \subset A\), onto \(F\). So the fibres of the retraction \(r_{\Delta}\) are exactly the \(U_{\Delta}\)–orbits.

Consider the retraction \(r_{-\infty}\) of center \(-\infty\). For a face \(F \in A\) and a face \(F'\) in the fibre, we can find a subsector \(s \subset -C_f\) such that for all \(\Delta \in s\) there exists an \(u \in U_{\Delta}\) so that \(F' = uF\). This means for all \(N \gg 0\) we can choose an alcove in the anti-dominant chamber such that \(\Delta \subset s\) and \(\ell_\beta(\Delta) > N\) for all positive roots. This means (compare [8], §6 and §7) that we can actually choose \(u \in U^-(K)\), where \(U^- \subset B^-\) is the unipotent radical of the opposite Borel subgroup \(B^-\). Summarizing we have the following description of the fibres of \(r_{-\infty}\) in terms of the group operation on \(J^a\):

**Proposition 1.** The fibres of \(r_{-\infty} : J^a \to A\) are the \(U^-(K)\)–orbits on \(J^a\).

### 4 Generalized Galleries

In the next three sections, we take \(G\) to be a semisimple group of any type. We first need a more general version of a gallery as the one in section 3, we will essentially follow [9].

**, Definition 8.** A \textit{generalized gallery in the affine building} is a sequence of faces \(\gamma\) in \(J^a\)

\[
\gamma = (\Gamma'_0 \subset \Gamma_0 \supset \Gamma'_1 \subset \cdots \subset \Gamma_{j-1} \supset \Gamma'_j \subset \Gamma_j \supset \cdots \supset \Gamma'_p \subset \Gamma_p \supset \Gamma'_{p+1}),
\]

such that
• the first and the last faces, $\Gamma'_{0}$ and $\Gamma'_{p+1}$, in other words, the source and the target of $\gamma$, are vertices of $\mathcal{J}^a$,
• the $\Gamma'_j$'s are faces, all of the same dimension,
• the $\Gamma'_j$'s, for $j = 1, \ldots, p$, are faces of two consecutive faces, of relative codimension one.

If such a gallery is contained in the apartment $\mathcal{A}$, it will be called a combinatorial gallery.

**Remark 4.** The last two conditions are not necessary to define the gallery model, but we will keep them throughout in the following.

For any subset $\Omega$ and any face $F$ contained in an apartment $A$ of the affine building $\mathcal{J}^a$, we say that a wall $\mathcal{H}$ separates $\Omega$ and $F$ if $\Omega$ is contained in the corresponding closed half space and $F$ is a subset of the opposite open half space. Let $E$ and $F$ be two faces in the building. In any apartment $A$ containing both of them, there exists a finite number of walls that separate $E$ and $F$. We denote this set by $\mathcal{M}(E, F)$. Note, if $E$ is not an alcove, all the walls containing $E$ but not $F$ belong to $\mathcal{M}(E, F)$.

Let $\Delta \supset E$ be an alcove in $A$, then, by [29], Proposition 2.29, there exists a unique alcove in $A$, denoted $proj_F(\Delta)$, such that any face of the convex hull of $\Delta$ and $F$ containing $F$ is contained in $proj_F(\Delta)$.

**Definition 9.** The alcove $\Delta$ is said to be at maximal distance to $F$ if the length of the minimal alcoves galleries between $\Delta$ and $proj_F(\Delta)$ is $\#\mathcal{M}(E, F)$.

Note, any of these minimal galleries will cross only once each wall of $\mathcal{M}(E, F)$. Such an alcove $\Delta \supset E$ at maximal distance to $F$ is not uniquely determined.

**Lemma 3.** The alcoves $\Delta \supset E$ in $A$ at maximal distance to $F$ are conjugate under the stabilizer of $E \cup F$.

**Proof.** Let $\Delta'$ be another alcove in $A$, the fixed apartment which contains the two faces $E$ and $F$. Suppose $\Delta' \supset E$ is at maximal distance to $F$ and $\Delta$ and $\Delta'$ are adjacent, then the support $\mathcal{H}$ of the common face contains $E$. Now for any wall $\mathcal{H}' \neq \mathcal{H}$, the relative position of $\Delta$ and $\Delta'$ is the same, i.e., they lie in the same closed halfspace. Since they lie on different sides of $\mathcal{H}$, the two can be at maximal distance to $F$ at the same time only if $F \subset \mathcal{H}$. In the general case, choose a minimal gallery $(\Delta_0 = \Delta, \ldots, \Delta_r = \Delta')$. Since $E \subset \Delta, \Delta'$, one concludes by the minimality of the gallery that $E \subset \Delta_j$ for all $j$. Further, for all walls in $\mathcal{M}(E, F)$, $\Delta$ and $\Delta'$ lie in the same closed halfspace, and hence, by the minimality of the gallery, so do all $\Delta_j$. But this implies that all the $\Delta_j$ are at maximal distance to $F$, which finishes the proof since the $\Delta_j$ are pairwise adjacent.

We have also to generalize the notion of a minimal gallery. Roughly speaking, a generalized gallery is called minimal if it can be embedded in a minimal gallery of alcoves. More precisely:
Definition 10. A generalized gallery

$$\gamma = (F_f = \Gamma_0' \subset \Gamma_0 \supset \Gamma_1' \subset \cdots \supset \Gamma_{j-1}' \supset \Gamma_j' \subset \cdots \supset \Gamma_p' \subset \Gamma_p \supset F)$$

in the affine building $J^a$ is called minimal if the following holds: let $\Delta \supset F_f$ be an alcove at maximal distance to $F$, then

(a) there exists a minimal gallery of alcoves between $\Delta$ and $\text{proj}_F(\Delta)$

$$\mu = (\Delta = \Delta_1^0 \supset \cdots \supset \Delta_{q_0}^0 \supset \Delta_{q_1}^1 \supset \cdots \supset \Delta_{q_p}^p \supset \text{proj}_F(\Delta)),$$

and for all $j$ the alcoves $\Delta_{q_0}^0 \supset \cdots \supset \Delta_{q_j}^j$ contain $\Gamma_j'$.

(b) let $A$ be an apartment containing $\gamma$, and denote $H^A$ the set of affine hyperplanes in $A$. The set $\mathcal{M}(F_f, F)$ is the disjoint union of the sets

$$\mathcal{H}_j := \{H \in H^A \mid \Gamma_j' \subset H, \ \Gamma_j \nsubseteq H\}, \ j = 0, \ldots, p.$$ 

Remark 5. A minimal gallery $\gamma$ is contained in an apartment by condition (a), and an apartment containing $\Delta$ and $\text{proj}_F(\Delta)$ contains also all minimal galleries between the two alcoves. One checks easily that if (b) holds for one apartment, then it holds for all apartments $A \supset \gamma$.

Remark 6. We will also need to consider minimal galleries of the form $(E \supset F' \subset F)$ in an apartment $A$. Such a gallery is called minimal if the set of walls $\mathcal{M}(E, F)$ is exactly the set of walls that contain $F'$ but not $F$.

If $\gamma$ is a minimal gallery (in the sense of Definition 10), then all $(\Gamma_{j-1}' \supset \Gamma_j' \subset \Gamma_j)$, $j = 1, \ldots, p$ are minimal.

Remark 7. Using the alcoves at maximal distance, we can reformulate the condition of $(E \supset F' \subset F)$ to be minimal as follows: Let $\mathcal{H}(F', F)$ be the set of walls that contain $F'$ but not $F$. Then $(E \supset F' \subset F)$ is a minimal gallery in $A$ if for any alcove $\Delta \supset E$ in $A$ at maximal distance to $F$, any minimal alcoves gallery between $\Delta$ and $\text{proj}_F(\Delta)$ has length $\#\mathcal{H}(F', F)$.

Our main objects will be minimal galleries having the vertex corresponding to $G(\mathcal{O})$ as source. Since all apartments containing this vertex are conjugate under the action of $G(\mathcal{O})$, the following definitions make sense for arbitrary galleries contained in an apartment (recall, if $\gamma$ is minimal then it is contained in at least one apartment). Our aim is to express the condition on the minimality in terms of the walls crossed by the gallery.

Let $\lambda \in X^\vee$ be a co-character and denote $H_\lambda := \bigcap_{(\lambda, \alpha) = 0} H_{\alpha, 0}$ the intersection of all the hyperplanes corresponding to the positive roots orthogonal to $\lambda$. Let $F_f$ be the face (of type $S$) of $A$ corresponding to the origin of $A$ and let $F_\lambda$ be the face corresponding to $\lambda.$
Definition 11. A combinatorial gallery joining 0 with \( \lambda \) is a generalized gallery \( \gamma \) in \( \mathcal{A} \) that starts at \( F_f \) and ends in \( F_\lambda \):

\[
\gamma = (F_f = \Gamma'_0 \subset \Gamma_0 \supset \cdots \supset \Gamma'_{j-1} \supset \Gamma'_j \subset \Gamma_j \supset \cdots \supset \Gamma'_p \subset \Gamma_p \supset F_\lambda).
\]

such that the dimension of the faces \( \Gamma'_j \)'s is always equal to \( \text{dim} \mathcal{H}_\lambda \). The latter will be called the large faces of the gallery \( \gamma \) and the \( \Gamma'_j \)'s the small faces of \( \gamma \).

Lemma 4. A combinatorial gallery \( \gamma = (F_f = \Gamma'_0 \subset \Gamma_0 \supset \cdots \supset \Gamma'_p \subset \Gamma_p \supset F_\lambda) \) joining 0 with \( \lambda \) is minimal if and only if \( \gamma \subset \mathcal{H}_\lambda \) and \( p \) is minimal in the following sense:

- for \( j = 0, 1, \ldots, p \), let \( \mathcal{H}_j \) be the set of all the affine hyperplanes \( \mathcal{H} \subset \mathbb{H}^a \) such that \( \Gamma'_j \subset \mathcal{H} \) and \( \Gamma_j \not\subset \mathcal{H} \), then the sets \( \mathcal{H}_j \) are pairwise distinct and \( \bigcup_{j \in \{0, \ldots, p-1\}} \mathcal{H}_j = \mathcal{M}(F_f, F_\lambda) \), the set of all the affine hyperplanes that separate any alcove \( \Delta \supset F_f \) at maximal distance to \( F_\lambda \) and the latter.

Proof. Suppose \( \gamma \) is minimal, then \( 0, \lambda \subset \mathcal{H}_\lambda \) implies \( \gamma \subset \mathcal{H}_\lambda \) by minimality, and the second part of the condition follows from the definition of a minimal gallery.

For the reverse implication, we construct the minimal gallery of alcoves inductively. We start with an alcove \( \Delta = \Delta^0_0 \subset F_f \) at maximal distance to \( \Gamma_0 \), set \( \Delta^0_0 = \text{proj}_{\Gamma_0} (\Delta) \). If \( p = 0 \), then the corresponding minimal gallery joining \( \Delta^0_1 \) and \( \Delta^0_0 \) has length 0 and is already the desired minimal gallery of alcoves. If \( p \geq 1 \), note that \( \Delta^0_0 \) is at maximal distance to \( \Gamma_1 \): a wall separating \( \Delta^0_1 \) and \( \Gamma_1 \) is either an element of \( \mathcal{H}_0 \) or \( \mathcal{H}_1 \), so by construction the walls in \( \mathcal{H}_1 \) separate \( \Delta^0_0 \) and \( \Gamma_1 \). But this implies that the length of a minimal gallery joining \( \Delta^0_0 \) and \( \Delta^1_0 = \text{proj}_{\Gamma_1} (\Delta^0_0) \) is \( \mathcal{H}_1 \). By repeating the procedure, we obtain the desired minimal gallery containing \( \gamma \).

Let \( \lambda \in \mathcal{X}_+^\vee \) be a dominant co-character and let \( \gamma_\lambda \) denote a minimal combinatorial gallery joining \( F_f \) with \( F_\lambda \):

\[
\gamma_\lambda = (F_f \subset \Upsilon_0 \supset \Upsilon'_1 \subset \cdots \supset \Upsilon'_{j-1} \supset \Upsilon'_j \subset \Upsilon_j \supset \cdots \supset \Upsilon'_p \subset \Upsilon_p \supset F_\lambda).
\]

Because of the minimality assumption, all the faces of \( \gamma_\lambda \) are contained in the (spherical) dominant chamber \( \mathcal{C}_f \), and \( \Upsilon_0 \) is a face of the fundamental alcove \( \Delta_f \).

Definition 12. The gallery of types associated to \( \gamma_\lambda \) is the list of the types of the faces in the gallery above:

\[
t_{\gamma_\lambda} = \text{type}(\gamma_\lambda) = (S = t'_0 \supset t_0 \supset t'_1 \supset \cdots \supset t_{j-1} \supset t'_j \supset t_j \subset \cdots \subset t'_p \supset t_p \supset t_\lambda),
\]

where \( t_\lambda \) is the type of the face \( F_\lambda \) and \( t_j \) is the type of the face \( \Upsilon_j \), \( t'_j \) the type of \( \Upsilon'_j \).

Remark 8. Fix a minimal combinatorial gallery \( \gamma_\lambda \) joining \( F_f \) with \( F_\lambda \), and consider the set of all generalized galleries \( \delta \) in the affine building \( \mathcal{J}^a \) of type \( t_{\gamma_\lambda} \). Let \( \gamma = (F_f = \Gamma'_0 \subset \cdots \supset \Gamma'_p \subset \Gamma_p \supset F_\lambda) \) be such a gallery, and assume that all \( (\Gamma_{j-1} \supset \Gamma'_j \subset \Gamma_j) \), \( j = 1, \ldots, p \) are minimal. Then one sees easily that \( \gamma \) is a minimal gallery, the fixed type comes from a minimal gallery and hence forces the gallery to be minimal.
Let $\Gamma(\gamma_\lambda)$ be the set of all the combinatorial galleries of type $t_\gamma$ and of source $F_f$, i.e., an element $\delta \in \Gamma(\gamma_\lambda)$ is a gallery starting at $F_f$:

$$\delta = (F_f \subset \Sigma_0 \supset \Sigma_1 \supset \cdots \supset \Sigma_p \supset F_\nu),$$

such that $\Sigma_j$ is a face of type $t_j$ of $\mathcal{A}$ and $\Sigma_j'$ is of type $t_j'$ and $F_\nu$ is of the same type as $F_\lambda$. In addition, such a sequence of types gives rise to a sequence of subgroups of $W^a$

$$W_0' \supset W_0 \subset W_1' \supset \cdots \supset W_{j-1}' \subset W_j', W_j \subset \cdots \subset W_p' \supset W_p,$$

where for $j = 0, \ldots, p$, $W_j'$ respectively $W_j$ is the Coxeter subgroup of $W^a$ generated by the reflections in $t_j' \subset S^a$ respectively in $t_j \subset S^a$.

These groups are also the stabilizers in $W^a$ of the faces of the fundamental alcove of the corresponding type. This provides us with another way to "encode" a gallery of type $\lambda$: The stabilizer of the $F_f$ in $W^a$ is $W = W_0'$, and the stabilizer of $\Upsilon_0$ is $W_0$, so $\Sigma_0$ is completely determined by an element in $W/W_0$. Proceeding inductively, we see $[\gamma]$:

**Proposition 2.** The set of all the combinatorial galleries of type $t_{\gamma_\lambda}$ and of source $F_f$, $\Gamma(\gamma_\lambda)$, is in bijection with the quotient

$$W \times_{W_0} W_1' \times_{W_1} \cdots \times_{W_{p-1}} W_p'/W_p,$$

of the group $W \times W_1' \times \cdots \times W_p$ by the subgroup $W_0 \times W_1 \times \cdots \times W_p$ under the action defined by $(w_0 w_0, w_0^{-1} w_1, \ldots, w_{p-1}^{-1} w_p w_p)$.

The image of a gallery $\gamma$ will be denoted by $\gamma = [\delta_0, \delta_1, \ldots, \delta_p]$. Because of the definition of the quotient we can (and will) assume that $\delta_j \in W_j'$ is the unique representative of minimal length of its class in $W_j'/W_j$. We will use freely both of the notation for a gallery.

**Example 4.** The gallery $\gamma_\lambda$ can be written as $\gamma_\lambda = [1, \tau_1^m, \ldots, \tau_p^m]$, where each $\tau_j^m \in W_j'$ is the unique minimal representative of the largest class (in the induced Bruhat ordering) in $W_j'/W_j$. The product $\tau_1^m \cdots \tau_p^m \in W^a$ is a reduced decomposition of the element that sends $\Delta_f$ to $\text{proj}_{F_f}(\Delta_f)$. All galleries of shape $[\gamma_0, \tau_1^m, \ldots, \tau_j^m]$ for $\gamma_0 \in W$ (of minimal length modulo $W_\lambda$) are minimal.

Let $\delta = [\delta_0, \delta_1, \ldots, \delta_p] = (F_f \subset \Sigma_0 \supset \Sigma_1 \supset \cdots \supset \Sigma_p \supset F_\nu) \in \Gamma(\gamma_\lambda)$. The gallery $\delta$ is called folded around the small face $\Sigma_j'$ if $\delta_j \neq \tau_j^m$. Such a gallery will be obtained from $[\delta_0, \tau_1^m, \ldots, \tau_p^m]$ by applying at the places $j$, where $\delta_j \neq \tau_j^m$, some affine reflections with respect to the affine hyperplanes containing the small faces. I.e., associated to $\delta$ we have the following sequence of galleries in $\Gamma(\gamma_\lambda)$:

$$\gamma_0 = [\delta_0, \tau_1^m, \tau_2^m, \ldots, \tau_p^m], \quad \gamma_1 = [\delta_0, \delta_1, \tau_2^m, \ldots, \tau_p^m], \quad \gamma_2 = [\delta_0, \delta_1, \delta_2, \tau_3^m, \ldots, \tau_p^m],
\ldots
\gamma_{p-1} = [\delta_0, \delta_1, \ldots, \delta_{p-1}, \tau_p^m], \quad \gamma_p = \delta.$$  \hspace{1cm} (3)

Here $\gamma_1$ is obtained from the minimal gallery $\gamma_0 = \delta$ by a folding around $\Sigma_1'$, $\gamma_2$ is obtained by a folding of $\gamma_1$ around $\Sigma_2'$, etc.
Definition 13. The gallery $\delta \in \Gamma(\gamma)$ is called positively folded at $\Sigma_j'$ if for all the affine hyperplanes $H$ involved in the sequence of reflections for the folding of $\gamma_j$ around $\Sigma_j'$ to get $\gamma_j$, the image of the reflection is “separated” by $H$ from the anti-dominant chamber $C_{-\infty}$ at $-\infty$. We say that the gallery is positively folded if all foldings are positive.

Here “separated” means that there exists a representative $s$ of the class $C_{-\infty}$ such that the image of the reflection and $s$ are separated by the reflection hyperplane.

Example 5. The minimal galleries are automatically positively folded.

Let $\Gamma^+(\gamma)$ denote the subset of $\Gamma(\gamma)$ of all positively folded combinatorial galleries of type $t_\gamma$, and for a co-character $\nu$, we denote $\Gamma^+(\gamma, \nu)$ the subset of galleries $\delta$ in $\Gamma^+(\gamma)$ ending in $\nu$, i.e., $\delta = (F_{j} \subset \Sigma_0 \supset \ldots \subset \Sigma_p \supset F_\nu)$.

Example 6. Consider the group $G = SL_3(\mathbb{C})$, let $S^a = \{s_0, s_1, s_2\}$ be indexed such that the Weyl group $W$ of $G$ is generated by $\{s_1, s_2\}$. A minimal gallery joining the origin with the highest root $\beta$ is $\gamma_\beta = [1, s_0]$. The elements in the set $\Gamma^+(\gamma_\beta)$ of all positively folded galleries of the same type and source 0 are:

$$I = [1, s_0], \ II = [s_1, s_0], \ III = [s_2, s_0], \ IV = [s_2 s_1, s_0], \ V = [s_1 s_2, s_0], \ VI = [s_2 s_1 s_2, s_0]$$

which are the minimal galleries in this set, and

$$VII = [s_1 s_2, 1], \ VIII = [s_2 s_1, 1], \ IX = [s_2 s_1 s_2, 1].$$

Note: for each root we have exactly one such gallery ending in this weight (the minimal galleries), and for the zero weight we have three (the non-minimal galleries), which is one too much if one has expected to get the character of the adjoint representation of $SL_3(\mathbb{C})$. We will see in the next section how to refine the choice.
5 Dimension of galleries

We will see that the notion of the dimension of a gallery is related to the dimension of the fibre of the retraction $r_{-\infty}$ with center at $-\infty$ discussed later. Fix a positively folded gallery $\gamma \in \Gamma^+(\gamma_\lambda)$ of type $t_{\gamma_\lambda}$:

$$\gamma = (F_f = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_j \supset \Gamma_{j+1} \supset \cdots \supset \Gamma_p \supset F_\nu).$$

For $j = 0, \ldots, p$, let $\mathcal{H}_j$ be the set of all the affine hyperplanes $H$ in $\mathbb{H}^a$ such that $\Gamma_j \subset H$ and $\Gamma_j \notin H$. We say that an affine hyperplane $H$ is a load-bearing wall for $\gamma$ at $\Gamma_j$ if $H \in \mathcal{H}_j$ and $H$ separates $\Gamma_j$ from $\mathcal{C}_{-\infty}$. Note: for a positively folded gallery all folding hyperplanes are load-bearing walls by definition.

**Definition 14.** The dimension of the gallery $\gamma \in \Gamma^+(\gamma_\lambda)$ is the number of pairs $(H, \Gamma_j)$ such that $H$ is a load-bearing wall for $\gamma$ at $\Gamma_j$:

$$\text{dim}\,\gamma = \sharp\{(H, \Gamma_j) \mid H \text{ is a load-bearing wall for } \gamma \text{ at } \Gamma_j\}.$$  

**Example 7.** Let $\lambda$ be a dominant co-character and let $\gamma_\lambda$ be a minimal gallery joining $0$ with $\lambda$. If $\beta$ is a positive root such that $\langle \lambda, \beta \rangle > 0$, then the affine hyperplanes $H_{\beta,n}$ for all $0 \leq n < \langle \lambda, \beta \rangle$ are load-bearing walls. It follows that

$$\text{dim}\,\gamma_\lambda = \sum_{\beta \in \Phi^+} \langle \lambda, \beta \rangle = \langle \lambda, 2\rho \rangle = \langle 2\lambda, \rho \rangle.$$  

Next, let $\nu = \tau(\lambda)$ be a Weyl group conjugate of $\lambda$ (as usual, we identify $\tau \in W/W_\lambda$ with its representative of minimal length), and let $\gamma_\tau$ be the gallery obtained from $\gamma_\lambda$ by applying simultaneously $\tau$ to all faces of $\gamma_\lambda$. Since $0 = F_f$ is a fixed point for the action of $W$ and $\nu = \tau(\lambda)$, this is a gallery joining $0$ with $\nu$. By the minimality of $\gamma_\lambda$, this gallery is also minimal and has no foldings. So, in particular, $\gamma_\tau \in \Gamma^+(\gamma_\lambda)$. One shows easily by decreasing induction on $\langle \nu, \rho \rangle$ that

$$\text{dim}\,\gamma_\tau = \langle \lambda + \nu, \rho \rangle.$$  

Our next aim is to provide such a formula for all $\gamma \in \Gamma^+(\gamma_\lambda)$. Recall that $\Gamma^+(\gamma_\lambda, \nu)$ is the set of all galleries $\gamma$ in $\Gamma^+(\gamma_\lambda)$ ending in $\nu$, i.e., $\gamma = (F_f \subset \Gamma_0 \supset \cdots \supset \Gamma_p \supset \cdots \supset \Gamma_{p+1} \supset F_\nu)$.

**Proposition 3.** If $\gamma \in \Gamma^+(\gamma_\lambda, \nu)$, then $\text{dim}\,\gamma \leq \langle \lambda + \nu, \rho \rangle$.

**Definition 15.** A positively folded gallery $\gamma \in \Gamma^+(\gamma_\lambda, \nu)$ is called an $LS$-gallery of type $\lambda$ if $\text{dim}\,\gamma = \langle \lambda + \nu, \rho \rangle$.

The proofs of the propositions 3 and 4 (see below) will be given in the next section, where we also provide a combinatorial characterization of the LS-galleries. Denote
$\Gamma^+_LS(\gamma_\lambda, \nu)$ the set of all galleries in $\Gamma^+(\gamma_\lambda, \nu)$ that are LS-galleries. As a consequence of the description we will get the following combinatorial character formula, which is the equivalent in terms of galleries of the path character formula in \cite{13, 12}. Let $V(\lambda)$ be the irreducible complex representation of highest weight $\lambda$ for the semisimple algebraic group $G^\vee$ (the Langland’s dual group of $G$), and denote $\text{Char} V(\lambda)$ its character:

**Proposition 4.** $\text{Char} V(\lambda) = \sum_{\nu \in X^+} \sharp \Gamma^+_LS(\gamma_\lambda, \nu) \exp(\nu)$.

### 6 Root Operators

The aim of this section is to prove the dimension formula and to give a combinatorial characterization of the LS-galleries. We define now “folding” operators $f_\alpha, e_\alpha, \bar{e}_\alpha$ (for all simple roots) on the set of all combinatorial galleries $\Gamma(\gamma_\lambda)$ of a fixed type. Let $\lambda$ be a dominant co-character and fix a minimal gallery $\gamma_\lambda$ joining the origin with $\lambda$.

Let $\alpha$ be a simple root and let $\nu \preceq \lambda$ be a co-character. Fix a combinatorial gallery $\gamma \in \Gamma(\gamma_\lambda, \nu)$ of type $t_{\gamma_\lambda}$, say

$$\gamma = [\gamma_0, \gamma_1, \ldots, \gamma_p] = (F_f = \Gamma_0 \supset \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_p \supset \Gamma_{p+1} = F_\nu).$$

Let $m \in \mathbb{Z}$ be minimal such that one of the small faces $\Gamma'_k$ is contained in the hyperplane $H_{\alpha,m}$, note that $m \leq 0$. The operators $e_\alpha$ and $\bar{e}_\alpha$ are different, but the conditions for the operators to be defined are not exclusive, so it might well be that both are defined for a given gallery.

1) Suppose that $m \leq -1$. Let $k$ be minimal such that $\Gamma'_k \subset H_{\alpha,m}$, and fix $0 \leq j \leq k$ maximal such that the small face $\Gamma'_j \subset H_{\alpha,m+1}$ is contained in the hyperplane $H_{\alpha,m+1}$.

![Diagram of galleries](image)

**Definition 16.** If $\gamma = [\gamma_0, \gamma_1, \ldots, \gamma_p]$ satisfies 1), then let $e_\alpha \gamma$ be the gallery defined by:

$$e_\alpha \gamma = [\delta_0, \delta_1, \ldots, \delta_p] = (F_f \subset \Delta_0 \supset \Delta'_1 \supset \cdots \supset \Delta'_p \subset \Delta_p \supset F_\nu)$$

where $\Delta_i = \begin{cases} \Gamma_i & \text{for } i \leq j - 1, \\ s_{\alpha,m+1}(\Gamma_i) & \text{for } j \leq i \leq k - 1, \\ t_{\alpha^\vee}(\Gamma_i) & \text{for } i \geq k. \end{cases}$ and $t_{\alpha^\vee} = \text{translation by } \alpha^\vee$. 

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We define a partial inverse operator \( f_\alpha \) to the operator \( e_\alpha \).

II) Suppose that \( \langle \nu, \alpha \rangle - m \geq 1 \). Let \( j \) be maximal such that \( \Gamma'_j \subset H_{\alpha,m} \) and fix \( j \leq k \leq p + 1 \) minimal such that the small face \( \Gamma'_k \) is contained in \( H_{\alpha,m+1} \).

![Diagram](image)

**Definition 17.** If \( \gamma \) satisfies II), then let \( f_\alpha \gamma \) be the gallery defined by:

\[
f_\alpha \gamma = [\gamma_0, \gamma_1, \ldots, \gamma_p] = (F_f \subset \Delta_0 \supset \Delta'_1 \supset \cdots \supset \Delta'_p \subset \Delta_p \supset F_b)
\]

where \( \Delta_i = \begin{cases} 
\Gamma_i & \text{for } i < j, \\
s_{\alpha,m}(\Gamma_i) & \text{for } j \leq i < k, \\
t_{-\alpha}^{-\nu}(\Gamma_i) & \text{for } i \geq k.
\end{cases} \)

III) Assume that \( \gamma \) crosses \( H_{\alpha,m} \). Fix \( j \) minimal with this property, i.e., \( \Gamma'_j \subset H_{\alpha,m} \) and \( H_{\alpha,m} \) separates the faces \( \Gamma_i \) from \( C_{-\infty} \) for \( i < j \) (in the large sense, i.e., \( \Gamma_i \subset H^+_{\alpha,m} \) but not the face \( \Gamma_j \) (so \( H_{\alpha,m} \in H_j \), the set of all the affine hyperplanes \( H \) such that \( \Gamma'_j \subset H \) and \( \Gamma_j \not\subset H \)). Fix \( k > j \) minimal such that \( \Gamma'_k \subset H_{\alpha,m} \).

![Diagram](image)

**Definition 18.** If \( \gamma = [\gamma_0, \gamma_1, \ldots, \gamma_p] \) satisfies III), then let \( \overline{e}_\alpha \gamma \) be the gallery defined by:

\[
\overline{e}_\alpha \gamma = [\delta_0, \delta_1, \ldots, \delta_p] = (F_f \subset \Delta_0 \supset \Delta'_1 \supset \cdots \supset \Delta'_p \subset \Delta_p \supset F_b)
\]

where \( \Delta_i = \begin{cases} 
\Gamma_i & \text{for } i \leq j - 1 \text{ and } i \geq k, \\
s_{\alpha,m}(\Gamma_i) & \text{for } j \leq i < k.
\end{cases} \)

It follows immediately from the definition of the operators that if one of the galleries \( e_\alpha \gamma, f_\alpha \gamma \) or \( \overline{e}_\alpha \gamma \) is defined, then it is again a gallery of type \( \gamma_\lambda \).

The proof of the following simple lemma is left to the reader. Let \( \gamma \in \Gamma(\gamma_\lambda, \nu) \) be a combinatorial gallery of type \( t_\gamma \) ending in \( \nu \), let \( \alpha \) be a simple root and suppose that \( m \) is minimal such that one of the small faces \( \Gamma'_k \subset H_{\alpha,m} \) is contained in the hyperplane \( H_{\alpha,m} \).
Lemma 5.  

(i) The gallery $e_{\alpha \gamma}$ is not defined if and only if $m = 0$, and if $e_{\alpha \gamma}$ is defined, then $e_{\alpha \gamma} \in \Gamma(\gamma \lambda, \nu + \alpha^\vee)$.

(ii) The gallery $f_{\alpha \gamma}$ is not defined if and only if $m = \langle \nu, \alpha \rangle$, and if $f_{\alpha \gamma}$ is defined, then $f_{\alpha \gamma} \in \Gamma(\gamma \lambda, \nu - \alpha^\vee)$.

(iii) If $e_{\alpha \gamma}$ is defined, then $f_{\alpha}(e_{\alpha \gamma})$ is defined and equal to $\gamma$. Further, $m + 1$ is minimal such that a small face of the gallery $e_{\alpha \gamma}$ is contained in $H_{\alpha, m + 1}^+$.

(iv) If $f_{\alpha \gamma}$ is defined, then $e_{\alpha}(f_{\alpha \gamma})$ is defined and equal to $\gamma$. Further, $m - 1$ is maximal such that a small face of the gallery $f_{\alpha \gamma}$ is contained in $H_{\alpha, m - 1}^+$.

(v) Let $p$ be maximal such that $f_{\alpha \gamma}^p$ is defined and let $q$ be maximal such that $e_{\alpha \gamma}^q$ is defined, then $p - q = \langle \nu, \alpha \rangle$.

Let $\Gamma(\gamma \lambda, \text{dom})$ be the set of all combinatorial galleries $\delta$ in $\Gamma(\gamma \lambda)$ such that none of the small faces is contained in one of the hyperplanes $H_{\alpha, -1}$, or, in other words, $e_{\alpha \delta}$ is not defined for all simple roots. Set $\text{Char} \Gamma(\gamma \lambda) = \sum \exp(e(\gamma))$, the sum over all $\gamma \in \Gamma(\gamma \lambda)$.

Corollary 1. $\text{Char} \Gamma(\gamma \lambda) = \sum \text{Char} V(e(\gamma))$, the sum running over all $\gamma \in \Gamma(\gamma \lambda, \text{dom})$.

Proof. The character formula can be proved using the same arguments as in [22], so we will only give a sketch of the proof. It follows easily from Lemma 5 that $\text{Char} \Gamma(\gamma \lambda)$ is stable under the Weyl group. Using Weyl’s character formula, the above is equivalent to

$$\left( \sum_{w \in W} sgn(w) e^{w(\rho)} \right) \text{Char} \Gamma(\gamma \lambda) = \sum_{\gamma \in \Gamma(\gamma \lambda, \text{dom})} \left( \sum_{w \in W} sgn(w) e^{w(\rho + e(\gamma))} \right)$$

So both sides are stable (up to sign) under the Weyl group and hence it is sufficient to compare the coefficients of the terms corresponding to dominant weights, i.e. we have to prove for $\Omega := \{(w, \delta) \mid w \in W, \delta \in \Gamma(\gamma \lambda), w(\rho) + e(\delta) \in X_+^\vee\}$:

$$\sum_{(w, \delta) \in \Omega} sgn(w) e^{w(\rho) + e(\delta)} = \sum_{\delta \in \Gamma(\gamma \lambda, \text{dom})} e^{\rho + e(\delta)}.$$

Let $\Omega_0$ be the set of pairs $(w, \pi) \in \Omega$ such that $w$ is the identity and $\delta \in \Gamma(\gamma \lambda, \text{dom})$. Set $\Omega' := \Omega - \Omega_0$, to prove the proposition we have to show:

$$\sum_{(w, \delta) \in \Omega'} sgn(w) e^{w(\rho) + e(\delta)} = 0. \quad (\ast)$$

We will define an involution $\varphi : \Omega' \to \Omega'$ such that $\varphi(w, \delta) = (w', \delta')$ has the property:

$\text{sgn}(w) = -\text{sgn}(w')$ and $w(\rho) + e(\delta) = w'(\rho) + e(\delta')$. This implies obviously $(\ast)$ and hence the proposition.
The construction of the involution: suppose first \((w, \delta) \in \Omega'\) is such that \(w\) is not the identity. Denote by \(w(\rho) + \delta\) the gallery of source \(w(\rho)\) and target \(w(\rho) + e(\delta)\) obtained by shifting the gallery \(\delta\) by the co-character \(w(\rho)\). Since \(w(\rho) + e(\delta) \in X_\lambda^\vee\), the shifted gallery has to meet at least once a proper face of the dominant Weyl chamber \(\mathcal{C}_f\), i.e., at least one of the small faces of \(w(\rho) + \delta\) is contained in a proper face of \(\mathcal{C}_f\). If \(w\) is the identity, then the shifted gallery also has to meet a proper face \(F\) of \(\mathcal{C}_f\), the pair would otherwise be an element of \(\Omega_0\).

For a proper face \(F\) of \(\mathcal{C}_f\) denote by \(\Omega'(F)\) the set of pairs \((w, \delta) \in \Omega'\) having the following property: there exists a small face \(w(\rho) + \Gamma'_j\) of the shifted gallery \(w(\rho) + \delta\) such that the open faces \(w(\rho) + \Gamma'_j o \subset F^o\) are contained in each other, and for all \(k > j\) the open part of the small faces \(w(\rho) + \Gamma'_k o\) are contained in the interior of \(\mathcal{C}_f\).

The set \(\Omega'\) is obviously the disjoint union of the \(\Omega'(F)\), so it is sufficient to define the involution for such an \(\Omega'(F)\). Let \(\alpha\) be a simple root orthogonal to \(F\). For \((w, \delta) \in \Omega'(F)\) set \(n := \langle w(\rho), \alpha \rangle\), note that \(n \neq 0\).

If \(n < 0\), then Lemma 5 (v) implies that \(f_\alpha^{|n|}(\delta)\) is defined. Of course we have \(w(\rho) + e(\delta) = s_\alpha w(\rho) + f_\alpha^{|n|}e(\delta)\), and, by construction,

\[
\varphi(w, \delta) := (s_\alpha w, f_\alpha^{|n|}(\delta)) \in \Omega'(F).
\]

Similarly, if \(n > 0\), one sees that \(e_\alpha^n(\delta)\) is defined, we have \(w(\rho) + e(\delta) = s_\alpha w(\rho) + e_\alpha^n e(\delta)\), and, by construction,

\[
\varphi(w, \delta) := (s_\alpha w, e_\alpha^n(\delta)) \in \Omega'(F).
\]

Lemma 5 implies that \(\varphi\) is an involution, which finishes the proof. \(\bull\)

Obviously, the character of \(V(\lambda)\) occurs in the decomposition above with multiplicity one. We want to show that this character comes from the subset of \(\Lambda S\)-galleries.

**Remark 9.** Let \(\delta \in \Gamma(\gamma_\lambda)\) be a combinatorial gallery having as target \(\nu = e(\delta)\) and denote \(t_\nu\) the translation by \(-\nu\). The combinatorial gallery \(t_\nu(\delta)\) starts in \(-\nu\) and ends in \(0\). Let \(\delta^*\) be the gallery having the same faces as \(t_\nu(\delta)\) but in reverse order, so the source of \(\delta^*\) is the origin, and the target is \(-\nu\). The type of \(\delta^*\) is the same as the type of \(\gamma_\lambda^*\), which is the same as the type of \(w_0(\gamma_\lambda^*)\). The latter is again a minimal gallery, but now joining the origin with the coweight \(\lambda^* = -w_0(\lambda)\), the highest weight of the representation dual to \(V(\lambda)\). On sees easily that the map \(\delta \mapsto \delta^*\) induces a bijection \(\Gamma(\gamma_\lambda) \rightarrow \Gamma(\lambda_0(\gamma_\lambda^*))\), and, if \(e_\alpha(\delta)\) respectively \(f_\alpha(\delta)\) are defined, then

\[
e_\alpha(\delta) = (f_\alpha(\delta^*))^* \quad \text{and} \quad f_\alpha(\delta) = (e_\alpha(\delta^*))^*\]

**Lemma 6.**

i) If \(\gamma \in \Gamma(\gamma_\lambda)\) and \(e_\alpha\gamma\) (respectively \(\overline{e}_\alpha\gamma\)) is defined, then \(\dim e_\alpha\gamma = \dim \gamma + 1\) (respectively \(\dim \overline{e}_\alpha\gamma = \dim \gamma + 1\), and, similarly, if \(f_\alpha\gamma\) is defined, then \(\dim f_\alpha\gamma = \dim \gamma - 1\).

ii) If \(\delta \in \Gamma^+(\gamma_\lambda)\) is such that \(\overline{e}_\alpha\gamma\) is defined, then \(\overline{e}_\alpha\gamma \in \Gamma^+(\gamma_\lambda)\).
iii) If \( \delta \in \Gamma^+(\gamma, \lambda) \) is such that \( \tilde{e}_\alpha \delta \) is not defined but \( e_\alpha \delta \) (respectively \( f_\alpha \delta \)), then \( e_\alpha \delta \) (respectively \( f_\alpha \delta \)) is again positively folded.

**Proof.** Consider first the operator \( e_\alpha \). Translations do not affect the relative position of a pair \( \Gamma'_k \subset \Gamma_k \) and \( C_\infty \) with respect to a wall \( H \supset \Gamma'_k \). Moreover, an affine reflection associated to a simple root \( \alpha \) affects the relative position only if \( \Gamma'_k \subset \mathbb{H}_{\alpha,p} \) for some \( p \in \mathbb{Z} \). So to check the dimension formula, it suffices to compare the pair \( \Gamma'_j \subset \Gamma_j \) (where \( j \) is chosen as in Definition 10) with the corresponding pair of large and small faces in the new gallery. Now, by the definition of the operator, \( \mathbb{H}_{\alpha,m+1} \) is not load-bearing for \( \gamma \) at \( \Gamma_j \), but \( \mathbb{H}_{\alpha,m+1} \) is load-bearing for \( e_\alpha \gamma \). The same arguments apply to the operator \( f_\alpha \), only in this case \( \mathbb{H}_{\alpha,m} \) is load-bearing for \( \gamma \) at \( \Gamma_j \) but not for \( f_\alpha \gamma \).

Using the same arguments as above, one sees that for the operator \( \tilde{e}_\alpha \) it suffices to consider the pairs \( \Gamma'_j \subset \Gamma_j \) and \( \Gamma'_k \subset \Gamma_k \) (\( j, k \) chosen as in Definition 18). Now \( \mathbb{H}_{\alpha,m} \) is load-bearing for \( \gamma \) and \( \tilde{e}_\alpha \gamma \) at \( \Gamma_k \), and \( \mathbb{H}_{\alpha,m} \) is load-bearing for \( \tilde{e}_\alpha \gamma \) at \( \Gamma_j \) but not for \( \gamma \). This implies the dimension formula.

To prove that \( \tilde{e}_\alpha \delta \in \Gamma^+(\gamma, \lambda) \) if \( \delta \in \Gamma^+(\gamma, \lambda) \), arguments similar to the ones above reduce the proof to check what the operator changes for the triples \( \Gamma'_{j-1} \supset \Gamma'_j \subset \Gamma_j \) and \( \Gamma_{k-1} \supset \Gamma'_k \subset \Gamma_k \). But since the additional foldings induced by \( \tilde{e}_\alpha \) at \( \Gamma'_j \) and \( \Gamma'_k \) are positive by construction, it follows that \( \tilde{e}_\alpha \gamma \in \Gamma^+(\gamma, \lambda) \).

It remains to prove iii). Note that \( \tilde{e}_\alpha \delta \) is not defined if and only if \( \tilde{e}_\alpha \delta^* \) is not defined. Since \( \delta \) is positively folded if and only if \( \delta^* \) is, by Remark 9 it suffices to give a proof for the operator \( e_\alpha \). Once again, arguments as above show that it suffices to check what the operator changes for the triples \( \Gamma'_{j-1} \supset \Gamma'_j \subset \Gamma_j \) and \( \Gamma_{k-1} \supset \Gamma'_k \subset \Gamma_k \) (\( j, k \) chosen as in Definition 18). The operator \( e_\alpha \) adds an additional positive folding at \( \Gamma'_j \), which finishes the proof for the index \( j \).

The assumption that \( \tilde{e}_\alpha \delta \) is not defined implies that \( \Gamma'_{k-1} \) and \( \Gamma_k \) are on the same side of \( \mathbb{H}_{\alpha,m} \). Let us denote by \( \Delta_{k-1} \supset \Delta'_k \subset \Delta_k \) the triple in \( e_\alpha \delta \) corresponding to \( \Gamma'_{k-1} \supset \Gamma'_k \subset \Gamma_k \). If \( \lambda \) is regular then the two alcoves \( \Delta_{k-1} \) and \( \Delta_k \) are separated by the wall spanned by \( \Delta'_k \), namely \( \mathbb{H}_{\alpha,m+2} \). If \( \lambda \) is not regular, the two large faces \( \Delta_{k-1} \) and \( \Delta_k \) are still separated by \( \mathbb{H}_{\alpha,m+2} \). And, because \( \alpha \) is simple, the possible other positive foldings at the place \( k \) in \( \delta \) are sent by \( e_\alpha \) to positive foldings.

It follows that \( e_\alpha \delta \in \Gamma^+(\gamma, \lambda) \).

We recall now Proposition 3 from section 5 and give the proof:

**Proposition 3.** If \( \delta \in \Gamma^+(\gamma, \nu, \lambda) \), then \( \dim \delta \leq \langle \lambda + \nu, \rho \rangle \).

**Proof.** We proceed by induction on \( \text{ht}(\lambda - \nu) \), the height of \( \lambda - \nu \). Note that \( \langle \lambda + \nu, \rho \rangle = 2 \langle \lambda, \rho \rangle - \text{ht}(\lambda - \nu) \). There is only one gallery \( \delta \in \Gamma^+(\gamma, \lambda) \) such that \( e(\delta) = \lambda \), it is \( \delta = \gamma \).

The dimension formula holds in this case by Example 4.

If \( \nu = e(\delta) \neq \lambda \), then \( \delta = [\delta_0, \ldots, \delta_p] \) for some \( \delta_0 \neq \delta_0 \). Let \( \lambda \) be a simple root such that \( s_\alpha \delta_0 < \delta_0 \), then either the gallery \( \delta' = \tilde{e}_\alpha \delta \) is defined and is positively folded, or \( \tilde{e}_\alpha \delta \) is not defined, but then the gallery \( e_\alpha \delta \) is defined and positively folded (Lemma 6). In
the latter case the claim follows immediately by induction and Lemma \( \Box i \). In the first case we repeat the procedure (if necessary, with another simple root) with \( \delta' \) until we get a positively folded gallery \( \delta'' \) such that still \( \nu = e(\delta'') \), \( \overline{e}_\alpha \delta' \) is not defined but \( e_\alpha \delta'' \) is. Again, the claim follows now by induction and Lemma \( \Box i \).

As a consequence of the proof above we get, in particular, a proof of Proposition \( \Box i \).

**Corollary 2.** (i) \( \dim \delta = \langle \lambda + \mu, \rho \rangle \) if and only if there exists a sequence of simple roots such that \( e_{\alpha_1} \cdots e_{\alpha_i} \delta = \gamma_\lambda \), or, equivalently, \( \delta = f_{\alpha_i} \cdots f_{\alpha_1} \gamma_\lambda \). In particular, the set of LS–galleries in \( \Gamma^+(\gamma_\lambda) \) is the subset generated from \( \gamma_\lambda \) using the operators \( f_\alpha \).

(ii) The gallery \( \delta \) is an LS–gallery if and only if \( \delta^* \) is an LS–gallery.

(iii) \( \operatorname{Char} V(\lambda) = \sum \exp(e(\gamma)) \), where the sum runs over all LS–galleries in \( \Gamma^+_{LS}(\gamma_\lambda) \).

**Proof.** If \( \delta \) is an LS–gallery, then the proof above shows (for dimension reasons) the existence of a sequence such that \( e_{\alpha_1} \cdots e_{\alpha_i} \delta = \gamma_\lambda \). Conversely, if such a sequence exists, then we get a sequence such that \( \delta = f_{\alpha_i} \cdots f_{\alpha_1} \gamma_\lambda \). Suppose we have already shown that \( \delta' = f_{\alpha_{ij}} \cdots f_{\alpha_{i1}} \gamma_\lambda \), \( 0 \leq j < t \), is an LS–gallery. Then Lemma \( \Box i \) and Proposition \( \Box i \) implies that \( \overline{e}_\alpha \delta' \) is not defined (for dimension reasons), so \( f_{\alpha_{ij+1}} \delta' \) is positively folded and an LS–gallery by Lemma \( \Box i \) and \( \Box iii \). It follows by induction that \( \delta \) is an LS–gallery. Since the set of LS–galleries is closed under the folding operators, the proof of Corollary \( \Box i \) applies also to the set of LS–galleries in \( \Gamma^+(\gamma_\lambda) \). Since \( \gamma_\lambda \) is the only dominant LS–gallery, this proves part \( \Box iii \). Part \( \Box ii \) follows from Remark \( \Box i \) part \( \Box i \) and the fact that \( \gamma_\lambda^* \) is an LS–gallery.

We will now describe the connection between the path model and the LS–galleries. Recall that the minimal galleries are of the form \([\gamma_0, \tau_1^m, \ldots, \tau_p^m]\), where \( \gamma_0 \in W \) is a representative of minimal length of a class in \( W/W_\lambda \), and each \( \tau_j^m \in W'_j \) is the unique minimal representative of the class of the longest element in \( W'_j \).

**Definition 19.** The companion \((\sigma_0, \ldots, \sigma_p)\) of a gallery \( \gamma \in \Gamma(\gamma_\lambda) \) is a sequence of Weyl group cosets in \( W/W_\lambda \) satisfying the following conditions:

- \( \sigma_0 = \gamma_0 \mod W/W_\lambda \).
- if there is no folding at \( \Gamma'_{j} \), then \( \sigma_j = \sigma_{j-1} \).
- assume that \( \gamma \) is folded at \( \Gamma'_{j} \).

(a) If \( \lambda \) is regular, then let \( s_\beta \) be the reflection used for the folding, i.e., \( \Gamma_j = s_\beta, m_\gamma \Gamma_{j-1} \) for an appropriate integer \( m_\gamma \). We set \( \sigma_j = s_\beta \sigma_{j-1} \) in \( W \).

(b) If \( \lambda \) is not regular, then more than one reflection can occur in a folding. Let \( s_{\beta_1}, \ldots, s_{\beta_r} \) be the reflections used for the folding, i.e., \( \Gamma_j = s_{\beta_1, m_1} \cdots s_{\beta_r, m_r} \Gamma_{j-1} \) for appropriate integers \( m_1, \ldots, m_r \), then set \( \sigma_j = s_{\beta_1} \cdots s_{\beta_r} \sigma_{j-1} \) in \( W/W_\lambda \).

**Remark 10.** To compare this notion with the language of the path model, note that one can find a piecewise linear path \( \pi \) starting in \( 0 \) and ending in \( \lambda \), such that all directions
are dominant rational weights and have the same stabilizer in $W$ as $\lambda$, and the image is contained in $\gamma_\lambda = (F_f \subset \Gamma_0 \supset \Gamma'_1 \subset \Gamma_1 \supset \ldots \subset \Gamma_p \supset F_\lambda)$.

**Example 8.** Assume for simplicity that $G$ is simple. Take the piecewise linear path that joins the origin with the barycenter of $\Gamma'_1$, then the barycenter of $\Gamma'_1$ with the barycenter of $\Gamma'_2$, etc, and finally joins the barycenter of $\Gamma'_p$ with $\lambda$. Note that the segments in $\Gamma_i$, $1 \leq i \leq p - 1$, are parallel to all codimension one faces of the $\Gamma_i$ except for the faces $\Gamma'_i$ and $\Gamma'_{i+1}$. So the minimality of the gallery implies that each segment is a positive rational multiple of a dominant weight. Now these may have a larger stabilizer than $\lambda$, but at least the first segment has obviously the desired property. Now by moving around the turning point on $\Gamma'_1$ (and letting the others fixed), we may still assume that the first segment is dominant rational and has the same stabilizer as $\lambda$, and we may assume in addition that this is now also true for the second segment. Continuing the finite procedure, we obtain the desired piecewise linear path.

Now the image $\pi_\gamma$ of this path in a folded gallery $\gamma$ of the same type will be a path such that its directions will be Weyl group conjugates of the directions we started with. The corresponding Weyl group elements (or rather the classes) give the companion.

**Definition 20.** A gallery is called a combinatorial LS–gallery if the following holds: for each pair $(\sigma_{i-1}, \sigma_i)$ in the companion, the sequence of folding reflections $s_{\beta_1,m_1}, \ldots, s_{\beta_r,m_r}$ can be chosen such that applied to $\sigma_{i-1}$, this gives a sequence $\sigma_{i-1} = \tau'_0 > \tau'_1 = s_{\beta_1} \tau'_0 > \ldots > \tau'_r = s_{\beta_r} \tau'_{r-1} = \sigma_i$ of Weyl group cosets (modulo $W_\lambda$) such that the length is always decreasing by one for each reflection.

**Remark 11.** The condition that the classes are decreasing in the Bruhat order implies that this is a positive folding: if $\kappa = s_\beta \tau > \tau$, then $\kappa(\lambda) < \tau(\lambda)$ in the weight ordering (i.e., $\tau(\lambda) - \kappa(\lambda)$ is a non-negative sum of positive roots), and hence $\langle \kappa(\lambda), \beta \rangle > 0$.

The special rôle played by the combinatorial LS–galleries $\gamma$ is the following: let $\alpha$ be a simple root and let $m \in \mathbb{Z}$ be minimal such that a small face of $\gamma$ is contained in $H_{\alpha,m}$.

**Lemma 7.** If $\gamma$ is a combinatorial LS–gallery and $e_\alpha \gamma$ (respectively $f_\alpha \gamma$) is well defined, then $e_\alpha \gamma$ (respectively $f_\alpha \gamma$) is again a combinatorial LS–gallery of type $t_\gamma$.

**Proof.** Suppose first that $\tilde{e}_\alpha \gamma$ is not defined. We know already (Lemma [i]iii) that if, in this case, $e_\alpha \gamma$ is defined, then it is a positively folded gallery. To see that it is in fact a combinatorial LS–gallery, one can now apply the same arguments as in [LS] on chains of Weyl group cosets to prove that the companion of $e_\alpha \gamma$ has again the desired properties. The proof for $f_\alpha$ follows in the same way using the “$*$”–operation.

It remains to show that $\tilde{e}_\alpha \gamma$ is not defined for a combinatorial LS–gallery, i.e., we have to show that the gallery does not cross the hyperplane $H_{\alpha,m}$. Let $\pi$ be a piecewise linear path corresponding to $\gamma_\lambda$ as in Remark [10] and Example [8] and denote $\pi_\gamma$ the corresponding folded path for $\gamma$. We may assume that the path has rational turning
points and is linear inside each large face of the gallery. Suppose that $\tilde{e}_a \gamma$ is defined, i.e., the gallery crosses $H_{a,m}$, let $\Gamma'_j \subset \Gamma_j$ and $\Gamma_{k-1} \subset \Gamma'_k$ be the small and large faces as in Definition 18. Let $\mu, \mu_{k-1}$ be the rational coweights that correspond to the segments of $\pi_r$ in the large faces $\Gamma_j, \Gamma_{k-1}$. Since $\gamma$ crosses $H_{a,m}$ we have $\langle \mu_j, \alpha \rangle < 0$, and since $k > j$ is minimal such that $\Gamma'_k \subset H_{a,m}$, we have $\langle \mu_{k-1}, \alpha \rangle > 0$. Recall that none of the folding hyperplanes at the small faces $\Gamma'_j, \ldots, \Gamma'_{k-1}$ is of the form $H_{a,p}$. But then such a change of sign is impossible for a combinatorial LS–gallery by the following Lemma 8 and Remark 12 which finishes the proof.

Let $\mu, \nu$ be dominant rational coweights having the same stabilizer $W'$ in $W$. For $\sigma \in W/W'$ denote $N(\sigma)$ the set of positive roots $\beta'$ such that $\langle \sigma(\mu), \beta' \rangle \leq 0$. Let $\beta \in N(\sigma)$ be such that $\ell(s_\beta \sigma) = \ell(\sigma) - 1$ in $W/W'$.

**Lemma 8.** If $\alpha$ is a simple root such that $\langle \sigma(\mu), \alpha \rangle < 0$ but $\langle s_\beta \sigma(\nu), \alpha \rangle > 0$, then $\alpha = \beta$.

**Remark 12.** Note that the positive roots occurring in the folding of a combinatorial LS–gallery $\gamma$ have exactly this property: let $\Delta_{l-1} \supset \Delta'_l \subset \Delta_l$ be two large and one small face such that $\Delta'_l \subset H_{\beta,n}$, and the affine hyperplane separates the large faces. Suppose that all foldings at the small faces $\Delta_r$, $r \leq l$ satisfy the condition for a combinatorial LS–gallery. Let $\sigma_l$ be the element in the companion for $\Delta'_l$, and let $\mu_l$ be the segment in $\Delta_{l-1}$ of an associated path $\pi_\gamma$ as in the proof above. Because any combinatorial LS–gallery is positively folded, we need $\langle \mu_l, \beta \rangle < 0$, and to satisfy the condition for a combinatorial LS–gallery, we also need $\ell(s_\beta \sigma_l) = \ell(\sigma_l) - 1$.

**Proof.** The following proof is a slightly adapted version of the proof of Proposition 1 in [10], so we will skip some details. For a given $\beta \in N(\sigma)$, instead of decomposing $N(\sigma)$ into two sets, one decomposes it into three sets. Let $E(\sigma) := \{ \theta \in N(\sigma) \mid \langle \sigma(\mu), \theta \rangle = 0, s_\beta \theta < 0 \}$ and $A(\sigma) := \{ \theta \in N(\sigma) \mid s_\beta \theta > 0 \}$, $B(\sigma) := \{ \theta \in N(\sigma) \mid \langle \sigma(\mu), \theta \rangle < 0, s_\beta \theta < 0 \}$, then $N(\sigma)$ is the disjoint union of $A(\sigma)$, $B(\sigma)$ and $E(\sigma)$. One checks easily that $A(s_\beta \sigma) = s_\beta A(\sigma), E(s_\beta \sigma) = -s_\beta E(\sigma)$ and $B(s_\beta \sigma) \cup \{ \beta \} \subset B(\sigma)$. One shows then: if $\ell(s_\beta \sigma) = \ell(\sigma) - 1$, then $E(\sigma) = \emptyset$ and $B(\sigma) = B(s_\beta \sigma) \cup \{ \beta \}$.

Let $\alpha, \beta$ be as above and suppose $\alpha \neq \beta$. If $s_\beta(\alpha) > 0$, then $s_\beta(\alpha) \in A(s_\beta \sigma)$. But note that $\alpha, \beta \in N(\sigma)$ implies $\alpha + k\beta \in C(\sigma)$, the real convex cone spanned by $N(\sigma)$. Since $s_\beta(\alpha) > 0$, this shows $s_\beta(\alpha) \in N(\sigma)$ and hence $s_\beta(s_\beta(\alpha)) = \alpha \in N(s_\beta \sigma)$, in contradiction to the assumption. So $s_\beta(\alpha) < 0$, and hence $\alpha \in B(\sigma) - \{ \beta \} = B(s_\beta \sigma)$, in contradiction to the assumption.

Using the operators $e_a$, we can transform a LS–gallery $\gamma$ into a LS–gallery $\gamma'$ having $\sigma_0 = 1$ for the companion, and hence $\gamma' = \gamma_\lambda$. It follows:

**Corollary 3.** The set of LS-galleries in $\Gamma^+(\gamma_{\lambda})$ coincides with the set of combinatorial LS-galleries.
For the gallery $\gamma_\lambda$ let $\pi : [0, 1] \to A$ be a piecewise linear path as in Remark [10]. By comparing the definition of the folding operators $e_\alpha, f_\alpha$ on galleries with the definition of the root operators $e_\alpha, f_\alpha$ on piecewise linear paths, one sees easily that the set of LS-galleries in $\Gamma^+(\gamma_\lambda)$ is exactly the set of galleries obtained in the following way: let $B(\pi)$ be the path model for the representation $V(\lambda)$ obtained by applying the root operators to $\pi$. To a path $\eta \in B(\pi)$ we associate the gallery $\gamma_\eta$ defined as the sequence of faces gone through by $\eta$. Then $\Gamma^+_{LS}(\gamma_\lambda) = \{ \gamma_\eta \mid \eta \in B(\pi) \}$, and this identification is equivariant with respect to the operators $e_\alpha, f_\alpha$. As a consequence we get (by [19], [14], [15]):

**Theorem 2.** Let $B(\gamma_\lambda)$ be the directed colored graph having as vertices the set of LS-galleries in $\Gamma^+(\gamma_\lambda)$, and put an arrow $\delta \xrightarrow{\alpha} \delta'$ with color $\alpha$ between two galleries if $f_\alpha(\delta) = \delta'$. Then this graph is connected, and it is isomorphic to the crystal graph of the irreducible representation $V(\lambda)$ of $G^\vee$ of highest weight $\lambda$.

### 7 Variety of Galleries

From now on, we take $G$ to be simply connected. By Remark [11] this assumption is not a restriction. In this section, we give two equivalent definitions of the Bott-Samelson variety, and we point out the first step of the connection between galleries and MV-cycles. Let $\lambda \in \mathcal{X}_+^\vee$ be a dominant co-character and let $t_\lambda = \text{type}(F_\lambda)$ be the type of the vertex of $A$ associated to $\lambda$. Let $Q_\lambda$ be the parahoric subgroup of type $t_\lambda$ containing $B$. The Bott-Samelson variety $\hat{\Sigma}(\gamma_\lambda)$ occurs as a natural resolution (see [16]) of the “Schubert variety” $X_\lambda = \overline{G_\lambda}$, where $G_\lambda = G(\mathcal{O})\lambda \hookrightarrow G(\mathcal{K})/Q_\lambda$ (or $G_\lambda = P_0\lambda \hookrightarrow \hat{L}(G)/\hat{Q}_\lambda$ in terms of the Kac-Moody group, see section 2). By a resolution we mean that $\hat{\Sigma}(\gamma_\lambda)$ is a smooth algebraic variety and the morphism $\pi : \hat{\Sigma}(\gamma_\lambda) \to X_\lambda$ is birational and proper.

Let

$$t_{\gamma_\lambda} = (S = t_0' \supset t_0 \subset t_1' \supset \cdots \supset t_{j-1}' \subset t_j' \supset \cdots \subset t_p' \supset t_p \subset t_\lambda).$$

be the gallery of types of the fixed minimal combinatorial gallery $\gamma_\lambda$ joining $F_\lambda$ with $F_\lambda$. Consider first the usual definition of $\hat{\Sigma}(\gamma_\lambda)$ in terms of $\hat{L}(G)$ (in [16] §7.1, S. Kumar gives the definition only for the case where the $t_j$’s are trivial, but it makes also sense in our case). For $0 \leq j \leq p$, let us denote by $\hat{P}_j$ (resp. $\hat{Q}_j$) the parabolic subgroup of type $t_j'$ (resp. $t_j$) containing $B$, of course $P_0 = P_\mathcal{O}$.

**Definition 21.** The Bott-Samelson variety $\hat{\Sigma}(\gamma_\lambda)$ is defined as

$$\hat{\Sigma}(\gamma_\lambda) = P_\mathcal{O} \times \hat{Q}_0 \times \hat{P}_1 \times \hat{Q}_1 \cdots \times \hat{P}_p \times \hat{Q}_p,$$

i.e. the algebraic (complex) variety defined as the quotient of the group $P_\mathcal{O} \times \hat{P}_1 \times \cdots \times \hat{P}_p$ by the subgroup $Q_0 \times Q_1 \cdots \times \hat{Q}_p$ under the (right) action given by $g \cdot q = (g_0 q_0, g_0^{-1} g_1 q_1, \ldots, g_{p-1}^{-1} g_p q_p)$ where $q = (q_0, q_1, \ldots, q_p) \in Q_0 \times Q_1 \cdots \times \hat{Q}_p$ and $g = (g_0, g_1, \ldots, g_p) \in P_\mathcal{O} \times P_1 \times \cdots \times \hat{P}_p$. 

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The Bott-Samelson variety \(\hat{\Sigma}(\gamma_\lambda)\) is a smooth, projective (complex) variety of dimension \(\dim (\hat{\Sigma}(\gamma_\lambda)) = 2\langle \lambda, \rho \rangle\). We will denote by \(g = [g_0, g_1, ..., g_p]\) the points of this variety and we call them *galleries* (the reason will become apparent soon). The Bott-Samelson variety naturally comes also equipped with a proper and birational morphism

\[ \pi : \hat{\Sigma}(\gamma_\lambda) \to X_\lambda \]

by associating to a gallery \(g = [g_0, g_1, ..., g_p]\) the class of the product \(g_0 g_1 \cdots g_p\) in the affine Grassmannian \(\hat{\mathcal{L}}(G)/\mathcal{Q}_\lambda\). The following Definition–Proposition is a slight generalization of a result of C. Contou-Carrère [9].

**Definition-Proposition 1.** For \(0 \leq j \leq p\), let us denote by \(P_j\) (resp. \(Q_j\)) the parahoric subgroup of \(G(\mathcal{K})\) of type \(t'_j\) (resp. \(t_j\)) containing \(B\). The Bott-Samelson variety \(\hat{\Sigma}(\gamma_\lambda)\) is the closed subvariety of the product

\[
G(\mathcal{K})/\mathcal{Q}_0 \times \left( \prod_{1 \leq j \leq p} G(\mathcal{K})/P_j \times G(\mathcal{K})/Q_j \right) \times G(\mathcal{K})/\mathcal{Q}_\lambda
\]

given by all the sequences of parahoric subgroups of the shape

\[
(G(\mathcal{O}) \supset \mathcal{Q}'_0 \supset P'_1 \supset Q'_1 \subset \cdots \subset P'_p \supset Q'_p \subset \mathcal{Q}'_\lambda),
\]

where \(\text{type}(P'_j) = t'_j\), \(\text{type}(Q'_j) = t_j\) and \(\mathcal{Q}'_\lambda\) is a subgroup associated to a vertex of type \(t_\lambda\). In other words, the set of points of \(\hat{\Sigma}(\gamma_\lambda)\) is given by the set of all the galleries in the affine building \(J^a\) of type \(t_{\gamma_\lambda}\) starting at \(F_f\):

\[
g = (F_f \subset F_0 \supset F'_1 \subset F'_1 \supset \cdots \supset F'_p \subset F'_p \supset F'_\lambda),
\]

where the type of each face corresponds to the type of the corresponding parahoric subgroup.

**Proof.** In [9], Contou-Carrère proves the proposition in the case of a Bott-Samelson variety built of parabolic subgroups of a reductive group scheme, but it applies also to the Kac-Moody case. Let us recall that the Kac-Moody group acts on the building. The equivalence between the two definitions is then obtained by the isomorphism which associates to a point \([g_0, g_1, ..., g_p]\) in \(\hat{\Sigma}(\gamma_\lambda)\) the gallery

\[
g = (F_f \subset F_0 \supset F'_1 \subset F'_1 \supset \cdots \supset F'_p \subset F'_p \supset F'_\lambda)
\]

where, for \(0 \leq j \leq p\), \(F_j = g_0 g_1 \cdots g_j \mathcal{Q}_j\) (\(\mathcal{Q}_j\) is the only face of the fundamental alcove \(\Delta_f\) of type \(t_j\)). The faces \(F'_j\) of the gallery \(g\) are uniquely determined by the \(F_j\)’s and the type \(t_{\gamma_\lambda}\).

\[ \bullet \]
Remark 13. The identification of galleries of type $t_{\gamma}$ and of source $F_f$ with the points in the Bott-Samelson variety allows us to view the combinatorial galleries $\Gamma(\gamma)$ of type $t_{\gamma}$ and of source $F_f$ as points in $\hat{\Sigma}(\gamma)$, they are precisely the $T$–fixed points. Here the action of $G(O)$ on the galleries is the action induced by the operation on the building. Since $G(O)$ operates on $J^a$ by simplicial maps, the action preserves the type of a gallery, and, if the gallery is minimal, then so are all galleries in the orbit.

Further, in this setting, the morphism $\pi : \hat{\Sigma}(\gamma) \rightarrow X_{\lambda}$ $\pi$ is just the restriction of the projection on the last factor. Hence, it maps the minimal gallery $\gamma$ onto the point $\lambda$, so $\pi$ induces a morphism between $G(O)\gamma_{\lambda}$ and $G_{\lambda}$. Now one sees easily that all minimal galleries of type $t_{\gamma}$ are in one $G(O)$–orbit, so by the birationality of $\pi$ we get:

Lemma 9. The desingularization map $\pi : \hat{\Sigma}(\gamma) \rightarrow X_{\lambda}$ induces a bijection between the set of all minimal galleries $\hat{G}_{\lambda}$ of type $t_{\gamma}$ in $\hat{\Sigma}(\gamma)$ and the open orbit $G_{\lambda} \subset X_{\lambda}$.

The retraction $r_{-\infty}$ extends naturally to galleries by applying $r_{-\infty}$ simultaneously to all faces in the gallery. Since $r_{-\infty}$ is a map of simplicial complexes, it preserves the type, i.e., the image of a gallery of type $t_{\gamma}$ is a combinatorial gallery of type $t_{\gamma}$. Let $\Gamma(\gamma)$ be the set of all combinatorial galleries of type $t_{\gamma}$, i.e., the set of $T$–fixed points (Remark 13) in $\hat{\Sigma}(\gamma)$. The retraction map on the galleries can be described geometrically as follows:

Proposition 5. The retraction with center at $-\infty$ induces a map $\hat{r}_{-\infty} : \hat{\Sigma}(\gamma) \rightarrow \Gamma(\gamma)$.

The fibres $\hat{\Sigma}(\gamma_{\lambda}) = \hat{r}_{-\infty}^{-1}(\delta)$, $\delta \in \Gamma(\gamma)$, are naturally endowed with the structure of a locally closed subvariety, each of them being isomorphic to an affine space. In fact, the $\hat{\Sigma}(\gamma_{\lambda})$ are the Bialynicki-Birula cells $\{x \in \hat{\Sigma}(\gamma) \mid \lim_{s \rightarrow 0} \eta(s)x = \delta\}$ of center $\delta$ in $\hat{\Sigma}(\gamma)$, associated to a generic one–parameter subgroup $\eta$ of $T$ in the anti–dominant Weyl chamber.

Proof. The map $\hat{r}_{\gamma}$ is well–defined, it remains to describe the fibres. Let $\delta$ be a combinatorial gallery in $\Gamma(\gamma)$, $\delta = (F_f = \Delta_0 \subset \Delta_0 \supset \Delta_1 \subset \cdots \subset \Delta_p \supset \Delta_{p+1} = F_\nu)$, and suppose that $g = (F_f = F_0' \subset F_0 \supset \cdots \supset F_p' \subset F_p \supset F_\nu)$ is a gallery of type $t_{\gamma}$ which retracts onto $\delta$. By Proposition 1 we know that there exist hence $u_i \in U^-(K)$, $i = 0, \ldots, p$, such that $F_i = u_i\Delta_i$, and, since $g$ is a gallery (of type $t_{\gamma}$), we have necessarily $u_0\Delta_0' = \Delta_0'$ and $u_{i-1}\Delta_i' = u_i\Delta_i'$, $i = 1, \ldots, p$. Conversely, given any sequence $(u_0, \ldots, u_p)$ of elements in $U^-(K)$ satisfying $u_{i-1}^{-1}u_i \in Stab_{G(K)}(\Delta_i')$, the stabilizer of the face in $G(K)$, for $i = 0, \ldots, p$ (where we set $u_{-1} = 1$), then

$$g = (F_f = \Delta_0' \subset u_0\Delta_0' \supset u_1\Delta_1' \subset u_1\Delta_1 \supset \cdots \supset u_p\Delta_p' \subset u_p\Delta_p \supset u_p\Delta_{p+1} = u_pF_\nu) \quad (4)$$

is a gallery of type $t_{\gamma}$, which retracts onto $\delta$. The maximal torus $T$ normalizes $U^-(K)$ and $(tu_{i-1}t^{-1})(tu_i^{-1}t^{-1}) \in Stab_{G(K)}(\Delta_i')$ for $t \in T$ if and only if this holds for $u_{i-1}u_i^{-1}$, so the fibres of the retraction map $\hat{r}_{\gamma}$ are stable under the natural $T$–action on $\hat{\Sigma}(\gamma)$.  

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Let \( \eta : \mathbb{C}^* \to T \) be a one-parameter subgroup in generic position in the open antidominant Weyl chamber, so \( \lim_{s \to 0} (\eta(s) u \eta(s)^{-1}) = 1 \) for \( u \in U^-(\mathcal{K}) \). If \( g \) is as in \( \mathfrak{H} \), then \( \lim_{s \to 0} \eta(s)g = \delta \) because the \( \Delta_j, \Delta'_j \) are \( T \)-fixed points in the building, and hence

\[
\eta(s)g = (\eta(s)F_f = \eta(s)\Delta_0 \supset \eta(s)u_0\Delta_0 \supset \eta(s)u_1\Delta_1 \supset \cdots \supset \eta(s)u_p\Delta_p \supset \eta(s)u_pF_{\nu}) = (F_f \supset (\eta(s)u_0\eta(s)^{-1})\Delta_0 \supset (\eta(s)u_1\eta(s)^{-1})\Delta_1 \supset \cdots \supset (\eta(s)u_p\eta(s)^{-1})\Delta_p \supset (\eta(s)u_p\eta(s)^{-1})F_{\nu}),
\]

which implies the claim. The \( T \)-fixed points in \( \hat{\Sigma}(\gamma) \) are the combinatorial galleries in \( \Gamma(\gamma) \). Since \( \eta \) is generic, we can assume that the \( T \)-fixed points and the \( \eta \)-fixed points coincide. So, by \( \mathfrak{M} \) and \( \mathfrak{E} \), the subsets

\[
C(\delta) = \{ x \in \hat{\Sigma}(\gamma) \mid \lim_{s \to 0} \eta(s)x = \delta \}
\]

form a decomposition of \( \hat{\Sigma}(\gamma) \) into locally closed subsets, each of which is isomorphic to an affine space, and the calculation above shows that the \( C(\delta) \) are the fibres of \( \hat{r}_{\gamma} \).

By Lemma \( \mathfrak{M} \) the object of our main interest, the orbit \( G_{\lambda} \), can be identified with the minimal galleries \( \hat{G}_{\lambda} \) in \( \hat{\Sigma}(\gamma) \). Here the special role of the positively folded galleries and its connection with the MV-cycles becomes apparent:

**Theorem 3.** The restriction of \( \hat{r}_{\gamma} \) induces a map \( r_{\gamma} : G_{\lambda} \to \Gamma^+(\gamma) \). Further, the union \( \bigcup_{\delta \in \Gamma^+(\gamma), \nu} r_{\gamma}^{-1}(\delta) \) of the fibres over the galleries in \( \Gamma^+(\gamma) \) with target \( \nu \) is the (set theoretic) intersection \( U^-(\mathcal{K})\nu \cap G_{\lambda} \).

**Proof.** By Proposition \( \mathfrak{H} \) we know that the fibres of \( r_{-\infty} : \mathcal{J}^a \to \mathcal{A} \) are the \( U^-(\mathcal{K}) \)-orbits. If \( \delta \in \hat{\Sigma}(\gamma) \) is a point (or rather a gallery), then \( r_{-\infty}(\pi(\delta)) \) is the target of the gallery \( \hat{r}_{\gamma}(\delta) \). So the union of the fibres \( r_{\gamma}^{-1}(\delta) \) with \( \delta \in \Gamma(\gamma, \nu) \) is precisely the set theoretic intersection of the orbit \( U^-(\mathcal{K})\nu \) with the orbit \( G_{\lambda} \).

To finish the proof of the theorem, it remains to show that is suffices to consider positively folded galleries, but this follows from the next lemma.

**Lemma 10.** If \( \delta = [\delta_0, \delta_1, \ldots, \delta_p] = (F_f \supset \Delta_0 \supset \Delta'_1 \supset \cdots \supset \Delta_{j-1} \supset \Delta'_j \supset \Delta_j \supset \cdots \supset F_{\nu}) \) is not a positively folded gallery, then \( \hat{r}_{\gamma}^{-1}(\delta) \cap \pi^{-1}(G_{\lambda}) = \emptyset \).

**Proof.** The assumption means that there exists an index \( j \in \{1, \ldots, p\} \) and a wall \( \mathcal{H} \in \mathcal{H}_j \) such that \( \delta_j \neq \tau_j^m \) and the image of the folding around \( \mathcal{H} \) is not separated from \( \mathcal{C}_{-\infty} \) by \( \mathcal{H} \). Particularly, the large face \( \Delta_j \) is not separated from \( \mathcal{C}_{-\infty} \) by \( \mathcal{H} \) and, because we fold around \( \mathcal{H} \), the two consecutive large faces \( \Delta_{j-1} \) and \( \Delta_j \) are on the same side of this wall. Suppose that \( j \) is the first of its kind.

Let \( g \in \hat{r}_{\gamma}^{-1}(\delta) \) be a gallery that retracts on \( \delta \). We can assume that \( g \) has already been retracted onto \( \delta \) up to the index \( j - 1 \), meaning that the gallery is of the form

\[
g = [\delta_0, \delta_1, \ldots, \delta_{j-1}, g_j, \ldots, g_p] = (F_f \supset \Delta_0 \supset \cdots \supset \Delta_{j-1} \supset \Delta'_j \subset F_j \supset F_{j+1} \cdots \supset F_{p+1}),
\]
with \( F_j = \delta_0 \delta_1 \cdots \delta_{j-1} g_j \mathcal{F}_{Q_j} \), where \( g_j \in \hat{P}_j/Q_j \) and \( \mathcal{F}_{Q_j} \) is the face of type \( t_j \) contained in the fundamental alcove.

Now let us assume that \( g \) is minimal, then \( g^j = (\Delta_{j-1} \supset \Delta_j' \subset F_j) \) is also minimal. That implies that \( \Delta_{j-1} \) and \( F_j \) are separated by the wall image of \( H \) in any apartment \( A \) containing \( g^j \). But, incarnating the retraction \( r_{-\infty} \) using a suitable far away alcove \( \Sigma \in A \cap \mathcal{A} \) on the same side of \( H \) than \( \Delta_{j-1} \), we know that the retraction \( r_{-\infty} = r_{\Sigma,A} \) preserves the distances from \( \Sigma \) and reduces to an isomorphism of complex chambers \( A \cong \mathcal{A} \). So, \( \Delta_{j-1} \) and \( r_{-\infty}(F_j) \) are also separated by the wall \( H \), which is not possible if we want \( r_{-\infty}(F_j) = \Delta_j \). Hence, the lemma is proved. \( \bullet \)

8 An open covering of \( \hat{\Sigma}(\gamma_\lambda) \)

The open subsets of the covering of \( \hat{\Sigma}(\gamma_\lambda) \) will be centered at the combinatorial galleries \( \Gamma(\gamma_\lambda) \). This covering is a slight generalization of the one considered by J. Tits [28]. The open sets will be built using the root subgroups of the Kac-Moody group \( \hat{L}(G) \).

**Definition 22.** For any real root \( \eta \) in the root system of the Kac-Moody group \( \hat{L}(G) \), there exists a one-dimensional root subgroup \( U_\eta \), isomorphic to \( (\mathbb{C},+) \) (see [16], 6.2.7) having as Lie algebra the corresponding root subspace in \( \hat{L}(g) \). We will denote the elements of this subgroup by \( p_\eta(x) \) for \( x \in \mathbb{C} \).

The parametrizations can be chosen to have the following property: for any \( w \in W^a \), any (real) root \( \eta \) and any complex number \( a \), \( w p_\eta(a) w^{-1} = p_{w(\eta)}(a) \). For any two subgroups \( \hat{P} \supset \hat{Q} \supset \hat{B} \) such that \( W^a_{\hat{P}} \) is finite, let us consider the affine open neighborhood of \( 1 \) in \( \hat{P}/\hat{Q} \),

\[
U^- (\hat{P}/\hat{Q}) = \prod_{\eta < 0, \eta \text{ real}} U_\eta.
\]

If \( w \in W^a_{\hat{P}}/W^a_{\hat{Q}} \) (again we identify \( w \) with a representative of minimal length in its class), let us consider the following sets of real roots

\[
R^+(w) = \left\{ \eta > 0 \mid U_{w^{-1}(\eta)} \not\subset \hat{Q} \right\} \quad \text{and} \quad R^-(w) = \left\{ \theta < 0 \mid w(\theta) < 0, \ U_\theta \in U^- (\hat{P}/\hat{Q}) \right\},
\]

then \( w U^- (\hat{P}/\hat{Q}) = U^+(w) w U^- (w) \) is an affine open neighborhood of the point \( w \in \hat{P}/\hat{Q} \), where

\[
U^+(w) = \prod_{\eta \in R^+(w)} U_\eta \quad \text{and} \quad U^-(w) = \prod_{\theta \in R^-(w)} U_\theta.
\]

Note that if \( w' \) is any element in the class of \( w \) in \( W^a_{\hat{P}}/W^a_{\hat{Q}} \), then \( R^+(w) = R^+(w') \). In addition, the set of positive real roots \( R^+(w) \Pi (-R^-(w)) \) indexes the walls of \( \mathcal{A} \) that contain \( F_P \), but not \( w F_Q \) (meaning that the set of associated “affine” roots, in \( \mathcal{A} \), does).
Definition 23. Let $\delta \in \Gamma(\gamma, \lambda)$ be a combinatorial gallery of type $t_{\gamma, \lambda}$:

\[
\delta = [\delta_0, \delta_1, ..., \delta_r] = (F_f = \Delta'_0 \subset \Delta_0 \supset \cdots \Delta'_{j-1} \subset \Delta_j \supset \cdots \subset \Delta_p \supset F'_v).
\]

We define a subset $U_{\delta}$ of $\hat{\Sigma}(\gamma, \lambda)$ as follows,

\[
U_{\delta} = \left\{ [g_0, g_1, ..., g_p] \in \hat{\Sigma}(\gamma, \lambda) \mid g_j \in U^+(\delta_j)\delta_jU^-(\delta_j) \}.
\]

Now $\hat{\Sigma}(\gamma, \lambda)$ is a sequence of locally Zariski-trivial fibrations, having partial flag varieties as fibres. So $U_{\delta}$ is actually an affine open subset of $\hat{\Sigma}(\gamma, \lambda)$ centered at $\delta$, and $U_{\delta}$ is isomorphic to $U^+(\delta_0)\delta_0U^-(\delta_0) \times \cdots \times U^+(\delta_p)\delta_pU^-(\delta_p)$.

Remark 14. For any combinatorial gallery $\delta$, the cell $C(\delta) = \hat{\gamma}_{\lambda}^{-1}(\delta)$ (Proposition 5) is a subvariety of the open set $U_{\delta}$.

9 Minimal Galleries

Recall that the definition of a minimal gallery is actually valid in any apartment. Further, given any two faces of the building, there exists at least one apartment that contains those two faces, and any minimal gallery between them is contained in this apartment. Recall also that the Kac-Moody group acts on the affine building via simplicial maps.

To characterize the minimal galleries as a subset of $\hat{\Sigma}(\gamma, \lambda)$, let us begin with the case of a minimal gallery of alcoves. For the following proposition see [5], 2.1.9:

Proposition 6. Let $\mu = (\Delta_f, \Delta_1, ..., \Delta_t)$ be a sequence of alcoves in $\mathcal{J}^a$. Then $\mu$ is a minimal gallery of alcoves if and only if there exists $b_j \in \mathcal{B}$ and $s_j \in S^a$, for $j = 1, ..., t$, such that

\[
\Delta_j = b_1s_1 \cdots b_js_j\Delta_f
\]

and $s_1s_2 \cdots s_t$ is a reduced decomposition of an element in $W^a$.

Remarks 15. 1) By an abuse of notation we denote by $s_j$ the simple reflection in $W^a$ as well as the representative $s_j = p_{\alpha_j}(1)p_{-\alpha_j}(-1)p_{\alpha_j}(1)$ in the Kac-Moody group (here $\alpha_j$ is the simple root in the root system of $\hat{L}(G)$ corresponding to $s_j$).

2) The $b_j$ are elements of the root subgroups, more precisely, $b_j = p_{\eta_j}(a_j)$ for some $a_j \in \mathbb{C}$. Note that any element in $U^+(s_1 \cdots s_j)$ can be written as $p_{n_1}(a_1)p_{n_2}(a_2) \cdots p_{n_{j-1}}(a_{j-1})p_{n_j}(a_j)$ for some complex numbers $a_i$’s.

3) Minimal galleries of alcoves not starting at $\Delta_f$ can be obtained using the action of the Kac-Moody group.

4) In a minimal gallery of alcoves (of length $t$), any two consecutive alcoves are separated by a wall, and all these walls are distinct, so the gallery crosses exactly $t$ distinct walls in any apartment containing it.
We come back to the general case. We cut the gallery of types into smaller blocks, i.e., we consider the following gallery of types \( \varepsilon = (\theta \subset \kappa' \supset \kappa) = (t_{j-1} \subset t_j' \supset t_j) \) for some \( j \in \{1, ..., p\} \), where \( t_{j+1} = (S = t_0' \supset t_0 \subset \cdots \supset t_{j-1} \subset t_j' \supset t_j \subset \cdots \supset t_p \subset t_0) \). Let \( \hat{P} \) and \( \hat{Q} \) be the parabolic subgroups containing \( \hat{B} \) of type \( \kappa' \) and \( \kappa \), respectively. Let also \( P \) and \( Q \), the corresponding parahoric subgroups. We know that \( P/Q = \bigcup_{w \in W_{\kappa'}^+/W_{\kappa}^+} U^+(w) \). We denote by \( \tau_m \) the smallest representative of the class of the longest element in \( W_{\kappa'}^+/W_{\kappa}^+ \). In addition, as a corollary of the Proposition 6 and the remarks above, we get:

**Lemma 11.** Let \( g = (E \supset F' \subset F) \) be a gallery of type \( \varepsilon \), starting at a face \( E \) of the fundamental alcove \( \Delta_f \). If \( F = xF_Q \), with \( x \in U^+() \), then \( g \) is a minimal gallery.

**Proof.** By Proposition 6 we can construct a minimal gallery of alcoves of length \( r = \ell(\tau_m) \) between \( \Delta_f \supset E \) and the alcove \( x\Delta_f = \text{proj}_E(\Delta_f) \). This shows that \( \Delta_f \) is at maximal distance from \( F \). Since any such minimal gallery is obtained from one of the form above using the operation of the stabilizer of \( E \cup F \), the implication follows (see Lemma 3 and Remark 7).

We will also need the following partial converse statement.

**Proposition 7.** If a gallery \( g = (E \supset F' \subset F) \) of type \( \varepsilon \), starting at a face \( E \) of the fundamental alcove \( \Delta_f \), is minimal then \( F = xF_Q \) where \( x \in U^+() \) and \( y' \in Q_E = \text{Stab}(E) \) (a subgroup of \( \hat{P} \)).

**Proof.** Let \( A \) be an apartment containing \( g \). By definition, the gallery is minimal in \( A \) if, for any alcove \( \Delta \supset E \) at maximal distance to \( F \) in \( A \), the walls separating \( \Delta \) from \( F \) are exactly the ones that contain \( F' \) but not \( F \). Let us denote by \( \mathcal{H}(F', F) \) this set of walls and by \( r \) its cardinality.

Denote \( \Delta \supset E \) an alcove at maximal distance to \( F \) in \( A \). Since \( g = (E \supset F' \subset F) \) is minimal, any minimal gallery of alcoves between \( \Delta \supset E \) and the alcove \( \text{proj}_E(\Delta) \) is of length \( r \). Let \( \mu = (\Delta = \Delta_0, \Delta_1, ..., \Delta_r = \text{proj}_E(\Delta)) \) be such a gallery. Since \( \Delta \supset E \subset \Delta_f \), there exists an element \( y \in Q_E \) such that \( y\mu = (\Delta_f = y\Delta_0, y\Delta_1, ..., y\Delta_r) \) is a minimal alcoves' gallery starting at \( \Delta_f \). Hence, by Proposition 6 there exists \( x \in U^+() \) such that \( y\mu = x\Delta_f \). In particular, \( F = y^{-1}xF_Q \), hence the proposition is proved.

Let \( w \in W_{\kappa'}/W_{\kappa} \), by abuse of notation we still denote by \( w \) the unique minimal representative in \( W_{\kappa'}/W_{\kappa} \) of its class. The following proposition expresses morally the “trace” of minimal galleries on the open set \( U^+(w) \) of \( \hat{P}/\hat{Q} \). We use the results of Deodhar.

**Proposition 8.** The intersection \( U^+(w) \cap U^+(\tau_m) \) decomposes into a finite disjoint union of subvarieties, each of which is isomorphic to a product of \( \mathbb{C} \) and \( \mathbb{C}^* \).
Proof. Let $w_\tau$ be the longest element in $W^a_{\hat{\rho}}$, then $U^+(\tau^m)\tau^m = U^+(w_\tau)w_\tau$ modulo $Q$. Next let $v \in W^a_{\hat{\rho}}$ be such that $w_\tau = wv$ and $\ell(w) + \ell(v) = \ell(w_\tau)$, then $U^+(w)U^+(w_\tau)w_\tau = U^+(w_\tau)w_\tau$. Hence we have:

$$U^+(w)wU^-(w) \cap U^+(\tau^m)\tau^m = U^+(w)(wU^-(w) \cap U^+(w_\tau)w_\tau).$$

Let $\hat{\mathcal{B}}^{-}$ is the subgroup of the Kac-Moody group $\hat{\mathcal{L}}(G)$ generated by the torus $\mathcal{T}$ and all the root subgroups associated to negative roots, then

$$wU^-(w) \cap U^+(w_\tau)w_\tau = wU^-(w) \cap wP^- wU^-(\hat{P}/\hat{Q}) = wP^- wU^-(w) \cap U^-(\hat{P}/\hat{Q}) = wP^- (U^+(v^{-1})v^{-1} \cap U^-(\hat{P}/\hat{Q})) = wP^- (\hat{B}v^{-1} \cap \hat{B}^- \hat{Q}/\hat{Q}).$$

By [1], the intersection $\hat{B}v^{-1} \cap \hat{B}^- \hat{Q}/\hat{Q}$ in $\hat{P}/\hat{Q}$ decomposes into a finite disjoint union of subvarieties isomorphic to products of $\mathbb{C}$ and $\mathbb{C}^*$, which implies the proposition. 

**Remark 16.** In the Deodhar decomposition of $\hat{B}v^{-1} \cap \hat{B}^- \hat{Q}/\hat{Q}$, there is exactly one subvariety of maximal dimension which consists only of copies of $\mathbb{C}^*$ (this is the one which corresponds to the subexpression $(1,1,...,1)$ of $v^{-1}$). In our situation, this is exactly the open and dense subset where the variables associated to the root subgroups in $U^-(w)$ are taken to be nonzero:

$$U^+(w)wU^-(w)(\mathbb{C}^*) \subsetneq U^+(w)wU^-(w) \cap U^+(\tau^m)\tau^m.$$

### 10 The Cell Associated to a Positively Folded Gallery

In this section, we describe the fibre $C(\delta) = \hat{r}_{\lambda}^{-1}(\delta)$ of the retraction $\hat{r}_{\lambda} : \hat{\Sigma}(\gamma_\lambda) \to \Gamma(\gamma_\lambda)$ over a positively folded gallery $\delta \in \Gamma^+(\gamma_\lambda, \nu)$, as a subvariety of the affine open subset $U_\delta$ of $\hat{\Sigma}(\gamma_\lambda)$. Let us fix

$$\delta = [\delta_0, \delta_1, \ldots, \delta_p] = (F_f = \Delta' \subset \Delta_0 \supset \Delta'_1 \subset \cdots \Delta_j \supset \Delta'_j \subset \Delta_j \supset \cdots \subset \Delta_p \supset F_\nu).$$

For $j \in \{0, \ldots, p\}$, we denote by $\tau^m_j$ the “element of maximal length” in $W^j_j/W^j_j$, let $\ell_j$ be the length (see Example [4] in Section [3]). We fix a numbering on the set of walls containing $F_{p_j}$, but not $\delta_j F_{Q_j}$ (note that this set is equal to $(\delta_0 \cdots \delta_{j-1})^{-1} H_j$). The numbering is chosen such that the indexing set can be decomposed into

$$I_j = I^+_j \sqcup I^-_j = \{j_1, \ldots, j_{\ell(\delta_j)}\} \sqcup \{j_{\ell(\delta_j)+1}, \ldots, j_{p_j}\},$$

so that $R^+(\delta_j) = \{\eta_{j_i} \mid i \in I^+_j\}$ and $R^-(\delta_j) = \{\theta_{j_i} \mid i \in I^-_j\}$ (notation as in Section [3]).
Let \( J_{-\infty}(\delta) \subset I_0 \cdots I_p \) be the subset of all the indices corresponding to a load-bearing wall of \( \delta \). We have a decomposition \( J_{-\infty}(\delta) = J_{-\infty}^+(\delta) \cap J_{-\infty}^-(\delta) \), where we define

\[
J_{-\infty}^+(\delta) = J_{-\infty}(\delta) \cap (I_0^+ \cdots I_p^+) = \bigcup_{j=0}^p (J_{-\infty}(\delta) \cap I_j^+),
\]

and similarly for \( J_{-\infty}^-(\delta) \). Observe that \( J_{-\infty}^-(\delta) = I_0^- \cdots I_p^- \) since the gallery \( \delta \) is positively folded, and \( \dim \delta = \frac{1}{2} J_{-\infty}(\delta) \) (see Section 4 for the definition of \( \dim \delta \)).

**Lemma 12.** Let \( \delta = [\delta_0, \delta_1, \ldots, \delta_p] \in \Gamma^+(\gamma_\lambda) \) be a positively folded gallery. Then \( C(\delta) = \hat{r}_{\gamma_\lambda}^{-1}(\delta) \subset \hat{\Sigma}(\gamma_\lambda) \) is an affine cell of dimension \( \dim \delta \). It is the subvariety of \( \mathcal{U}_\delta \) consisting of all the galleries \( [g_0, g_1, \ldots, g_p] \) such that

\[
g_j \in \left( \prod_{i \in J_{-\infty}(\delta) \cap I_j^+} \mathcal{U}_{\theta_{ji}}(\delta_j) \right) \prod_{i \in I_j^-} \mathcal{U}_{\theta_{ji}}.
\]

The fibre \( r_{\gamma_\lambda}^{-1}(\delta) \subset \mathcal{G}_\lambda \) is obtained as the intersection \( \hat{r}_{\gamma_\lambda}^{-1}(\delta) \cap \hat{\mathcal{G}}_\lambda \), so we have:

**Corollary 4.** The fibre over a positively folded gallery \( \delta \) of the map \( r_{\gamma_\lambda} : \mathcal{G}_\lambda \to \Gamma^+(\gamma_\lambda) \) is naturally endowed with the structure of an irreducible quasi-affine variety. More precisely, it is an open dense subset of an affine space of dimension \( \dim \delta \).

Before we proceed with the proof of the lemma, we shall discuss the case where \( \lambda \) is a regular dominant co-character. In this case, \( \delta = [\delta_0, \delta_1, \ldots, \delta_p] \) is such that \( \delta_0 \in W \) and \( \delta_j = 1 \) or \( s_{ij} \), a simple reflection associated to the simple (Kac-Moody) root \( \alpha_{ij} \). Equivalently, the \( \Delta_j \)'s are alcoves and the \( \Delta_j' \)'s are faces of codimension 1 in two consecutive alcoves. Further, the set \( I_j \) is reduced to a single element, it corresponds to the wall which is the support of \( \Delta_j' \). So, in case \( \lambda \) is regular, the lemma has the following form.

**Lemma 13.** Assume \( \lambda \) regular. Let \( \delta = [\delta_0, \delta_1, \ldots, \delta_p] \in \Gamma^+(\gamma_\lambda) \) be positively folded gallery. Then \( C(\delta) \) is the subvariety of \( \mathcal{U}_\delta \) consisting of all the galleries \( [g_0, g_1, \ldots, g_p] \) such that

\[
g_0 \in \delta_0 \prod_{\beta < 0, \delta(\beta) < 0} \mathcal{U}_\beta \text{ and } g_j = \begin{cases} 
\delta_j & \text{if } j \notin \delta_{-\infty}(\delta) \\
p_{-\alpha_j}(a_j), a_j \in \mathbb{C}, & \text{if } j \in \delta_{-\infty}(\delta) \\
p_{\alpha_j}(a_j)s_{ij}, a_j \in \mathbb{C}, & \text{if } j \in \delta_{+\infty}(\delta).
\end{cases}
\]

**Proof.** Consider first the condition on \( g_0 \). We want to characterize the alcoves of the form \( g_0(\Delta_f) \) containing 0 which retract onto \( \delta_0(\Delta_f) \) by \( r_{-\infty} \). In this case we can identify the retraction centered at \( -\infty \) with the retraction onto the apartment \( A^a \) centered at \( w_0(\Delta_f) \), i.e., \( r_{-\infty}(g_0(\Delta_f)) = r_{w_0(\Delta_f), A^a}(g_0(\Delta_f)) \). So the problem amounts to find all the spherical chambers that retract on the chamber \( \delta_0(\mathcal{C}_f) \) via the retraction \( r_{-\mathcal{C}_f} = \)
\[ r_{w_0(\mathcal{C}_f),\mathcal{A}^*} \] onto the spherical apartment, centered at \( w_0(\mathcal{C}_f) = -\mathcal{C}_f \). But we know that \( r_{-\mathcal{C}_f}(\delta_0(\mathcal{C}_f)) = r_{-\mathcal{C}_f}^{-1}(\delta_0 w_0(\mathcal{C}_f)) = B^{-1} \delta_0 w_0 \) in \( G/B^- \) (recall Example 1). In addition,

\[
B^{-1} \delta_0 w_0 = \left( \prod_{\eta < 0, (\delta_0 w_0)^{-1}(\mathcal{O}) > 0} U_{\eta} \right) \delta_0 w_0 = \delta_0 \prod_{\beta < 0, \delta_0(\beta) < 0} U_{\beta} w_0.
\]

Using the isomorphism \( G/B^- \cong G/B \) given by right multiplication by \( w_0 \), we see that the alcoves retracting onto \( \delta_0(\Delta_f) \) via \( r_{-\infty} \) are of the form \( g_0(\Delta_f), g_0 \in \delta_0 \prod_{\beta < 0, \delta_0(\beta) < 0} U_{\beta} \).

The conditions on \( g_j \) for \( j \in \{1, \ldots, p\} \) follow easily from the definition of the retraction \( r_{-\infty} \). Let \( g = [g_0, \ldots, g_p] \) be a gallery, we want to know when \( g \) retracts onto \( \delta \). We can assume that \( g \) has been retracted until the index \( j - 1 \), i.e.,

\[
g = [\delta_0, \delta_1, \ldots, \delta_{j-1}, g_j, \ldots, g_p] = (F_j \subset \Delta_0 \supset \Delta_1 \supset \cdots \supset \Delta_{j-1} \supset \Delta_j \subset F_j \supset \cdots \supset F_p),
\]

with \( F_j = \delta_0 \delta_1 \cdots \delta_{j-1} g_j(\Delta_f) \). If \( F_j = \Delta_j \), then this corresponds to \( a_j = 0 \) in the first or the second condition. Otherwise the gallery \( (\Delta_{j-1} \supset \Delta_j \subset F_j) \) (resp. \( (s_\beta(\Delta_{j-1} \supset \Delta_j \subset F_j)) \)) is minimal in any apartment containing it, where \( s_\beta \) is the reflection defined by the wall \( H_j \) supporting the face \( \Delta_j \). Let \( A \) be an apartment containing \( F_j \), the wall \( H_j \) and a representative sector in the equivalence class of \( \mathcal{C}_\infty \) in \( \mathcal{A} \). Let us reincarnate the retraction center at \(-\infty \) into \( r_{-\infty} = r_{\Sigma, A} \), where \( \Sigma \in \mathcal{A} \cap A \) is a sufficiently far away alcove from \( \Delta_j \).

If \( j \notin J_{-\infty}(\delta) \), then \( H_j \) is not load-bearing at this place. In particular, \( \Sigma \) and \( \Delta_j \) are not separated by \( H_j \). But \( \Sigma \) and \( F_j \) are separated by \( H_j \) in \( A \). Hence, because of the properties of the retraction, \( r_{\Sigma, A}(F_j) \neq \Delta_j \), unless \( g_j = \delta_j \) (and \( F_j = \Delta_j \)). Now, if \( j \in J_{-\infty}(\delta) \), the wall \( H_j \) is load-bearing and \( \delta_j = 1 \). So this time the only condition on \( g_j \in \mathcal{P}_j/\mathcal{Q}_j = U^{-}(\delta_j)\hat{\mathcal{Q}}_j/\hat{\mathcal{Q}}_j \Pi s_{i_j}\hat{\mathcal{Q}}_j/\hat{\mathcal{Q}}_j \) is that \( g_j \neq s_{i_j} \). Hence, \( F_j \) retracts on \( \Delta_j \) if and only if \( g_j = p_{-\alpha_{i_j}}(a_j) \), for some complex number \( a_j \). The case where \( j \in J_{-\infty}^{\infty}(\delta) \) is treated similarly and corresponds to the decomposition \( \mathcal{P}_j/\mathcal{Q}_j = U^{+}(\delta_j)s_{i_j}\hat{\mathcal{Q}}_j/\hat{\mathcal{Q}}_j \Pi \hat{\mathcal{Q}}_j/\hat{\mathcal{Q}}_j \), this finishes the proof of the lemma.

**Proof of Lemma 12** The proof for \( g_0 \) is similar to the proof in the regular case, the computation takes now place in the quotients \( G(\mathcal{O})/\mathcal{Q}_0 \cong G/P_{\lambda} \) and \( G/P_{\lambda}^- \), where \( P_{\lambda}^- \) is the parabolic subgroup of type \( t_0 \) containing \( B^- \). So, applying the retraction \( r_{-\mathcal{C}_f} = r_{w_0(\mathcal{C}_f),\mathcal{A}^*} \) to the spherical faces of type \( t_0 \), we obtain that \( g_0 F_{\mathcal{Q}_0} \) retracts onto \( \delta_0 F_{\mathcal{Q}_0} \) if and only if

\[
g_0 \in \delta_0 \prod_{\beta \in R^{-}(\delta_0)} U_{\beta}, \text{ where } \beta \in R^{-}(\delta_0) = \{ \beta < 0, \delta_0(\beta) < 0, U_{\beta} \in U^{-}(G/P_{\lambda}) \}.
\]

Note that \( J_{-\infty}(\delta) \cap I_0^+ = \emptyset \). Next consider the case \( j > 0 \). Let \( g = [g_0, g_1, \ldots, g_p] \in \hat{\Sigma}(\gamma_{\lambda}) \) be a gallery. Again, we can retract step by step, i.e. we can assume that \( g \) has been retracted to a gallery \( g' \) up to the index \( j - 1 \), so

\[
g' = [\delta_0, \delta_1, \ldots, \delta_{j-1}, g_j, \ldots, g_p] = (F_j \subset \Delta_0 \supset \Delta_1 \supset \cdots \supset \Delta_{j-1} \supset \Delta_j \subset F_j \supset F_{j+1} \cdots \supset F_p),
\]

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with $F_j = \delta_0 \delta_1 \cdots \delta_{j-1} g_j F_{G_j}$ and $g_j \in U^+(\delta_j) \delta_j U^-(\delta_j)$. Hence, there exists some complex numbers $a_i$ for $i \in I_j^+$, and $b_k$ for $k \in I_j^-$ such that we can write $g_j$ as follows

$$g_j = p_{\zeta_1}(a_1) s_{\zeta_1} \cdots p_{\zeta_\ell(\delta_j)}(a_{\ell(\delta_j)}) s_{\zeta_{\ell(\delta_j)}} \prod_{k \in I_j^-} p_{\theta_k}(b_k),$$

(5)

where $s_{\zeta_1} \cdots s_{\zeta_{\ell(\delta_j)}}$ is a reduced decomposition of $\delta_j$ and

$$R^+(\delta_j) = \left\{ \zeta_1, s_{\zeta_1}(\zeta_2), \ldots, s_{\zeta_1} \cdots s_{\zeta_{\ell(\delta_j)-1}}(\zeta_{\ell(\delta_j)}) \right\}.$$

Assume first $b_k = 0$ for all $k$. By Proposition 6 the alcove gallery $\mu = (\Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_{\ell(\delta_j)})$, where $\Gamma_i = \delta_0 \delta_1 \cdots \delta_{j-1} p_{\zeta_1}(a_1)s_{\zeta_1} \cdots p_{\zeta_{\ell(\delta_j)}}(a_{\ell(\delta_j)}) s_{\zeta_{\ell(\delta_j)}}, \Delta_f$, is a minimal gallery between $\Gamma_0 \supset \Delta_{j-1}$ and $\Gamma_{\ell(\delta_j)} \supset F_j$. Because of the minimality, $F_j$ retracts on $\Delta_j$ if and only if $\mu$ retracts onto the minimal gallery $\xi = (\Xi_0, \Xi_1, \ldots, \Xi_{\ell(\delta_j)})$ in $A$, where $\Xi_i = \delta_0 \delta_1 \cdots \delta_{j-1} s_{\zeta_1} \cdots s_{\zeta_{\ell(\delta_j)}} \Delta_f$. Again, we can assume that $\mu$ has been retracted onto $\xi$ until the index $i - 1$, so we have to find out how the alcove $\Gamma_i' = \delta_0 \delta_1 \cdots \delta_{j-1} s_{\zeta_1} \cdots s_{\zeta_{i-1}} p_{\zeta_1}(a_1)s_{\zeta_{i}} \Delta_f$ may retract onto $\Xi_i$. But we are in the situation where we can apply the discussion concluding the previous proof. Hence, if $i \in J_{-\infty}(\delta) \cap I_j^+$, $a_i$ can be any complex number, otherwise $a_i$ must equal 0. Therefore, we get the expected conditions on the variables associated to indices in $I_j^+$.

To conclude, the fact that there is no condition on the $b_k$’s in [5] is a consequence of an analogue computation as in the case $j = 0$ and the assumption that the gallery is positively folded. Thus, the lemma is proved.

**Remark 17.** The definition of load-bearing wall with respect to $-\infty$ makes sense for any gallery and the previous description of the fibre also extends to arbitrary combinatorial galleries. Therefore, one gets a description in terms of galleries of the Białynicki-Birula cellular decomposition of the Bott-Samelson variety $\hat{\Sigma}(\gamma_\lambda)$.

### 11 LS-galleries and MV-cycles

We describe an affine open subset of $r_{\gamma_\lambda}^{-1}(\delta) = C(\delta) \cap G_\lambda$, isomorphic to a product of $\mathbb{C}$’s and $\mathbb{C}^*$’s. Thereby, we can associate to each LS-gallery $\delta$ an open dense subset of a Mirkovic-Vilonen cycle $Z(\delta)$. We first deal for simplicity with the regular case.

**Proposition 9.** Let $\lambda$ be a regular dominant co-character and let $\delta = [\delta_0, \delta_1, \ldots, \delta_p] \in \Gamma^+(\gamma_\lambda)$. Then $r_{\gamma_\lambda}^{-1}(\delta)$ is a subvariety of $U_\delta$ isomorphic to a product of $\mathbb{C}$’s and $\mathbb{C}^*$’s. More precisely, the fibre consists of all the galleries $[g_0, g_1, \ldots, g_p]$ such that

$$g_0 \in \delta_0 \prod_{\beta < 0, \beta(\delta) < 0} U_\beta \text{ and } g_j = \begin{cases} \delta_j & \text{if } j \notin J_{-\infty}(\delta), \\ p_{-\alpha_i}(a_j), a_j \neq 0 & \text{if } j \in J_{-\infty}(\delta), \\ p_{\alpha_j}(a_j)s_{\delta_j} & \text{if } j \in J_{-\infty}(\delta). \end{cases}$$

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Proof. Combining the Propositions 6 and 7 we get the following simple criterion in the regular case: a gallery of alcoves \((\Delta_f \supset F' \subset \Delta)\), where \(F'\) is a face of codimension 1 associated to the simple root \(\alpha\), is minimal if and only if \(\Delta = x \Delta_f\) with \(x \neq 1\) in \(P_\alpha/B\). This applies to each step in the gallery \([g_0, g_1, ..., g_p]\), and the only condition to add in order for the latter to be minimal is the one of the lemma.

In the general case the situation is slightly more complicated, but we still have:

**Theorem 4.** Let \(\lambda\) be a dominant co-character and let \(\nu \prec \lambda\) be an arbitrary co-character. For each positively folded gallery \(\delta \in \Gamma^+(\gamma_\lambda, \nu)\), the fibre \(r_{\gamma_\lambda}^{-1}(\delta)\) admits a decomposition into a union of subvarieties, each of which is isomorphic to a product of \(w\gamma_i\). More generally, let \(\alpha\) be two consecutive galleries in the sequence of positive foldings \([\delta_0, \delta_1, ..., \delta_p]\) and \(\delta \in \Gamma^+(\gamma_\lambda, \nu)\) of maximal dimension, i.e., \(\delta\) a LS-gallery.

**Corollary 5.** The MV-cycles corresponding to \(\nu\) (i.e., the irreducible components of \(\overline{U-(K)\nu \cap G_\lambda}\)) are given by the closures \(\overline{Z(\delta)} = O(\delta)\) for \(\delta \in \Gamma^+(\gamma_\lambda, \nu)\).

**Proof of the theorem.** The first definition of \(\hat{\Sigma}(\gamma_\lambda)\) (Definition 21) allows to iterate the use of the Propositions 6 and 7. The theorem is then obtained using Remark 10 and the same arguments as above in the regular case.

It is not yet clear how to express the inclusion relations between the closures of the fibres in terms of the combinatorics of the galleries. For simplicity assume \(\lambda\) is a regular and fix \(w \in W\), then one sees easily for \(\delta' = [w', s_{i_1}, ..., s_{i_p}]\) and \(\delta = [w, s_{i_1}, ..., s_{i_p}]\):

\[
r_{\gamma_\lambda}^{-1}(\delta') \subset \overline{r_{\gamma_\lambda}^{-1}(\delta)} \Leftrightarrow w \leq w'.
\]

More generally, let \(\delta, \gamma\) be two consecutive galleries in the sequence of positive foldings starting with \(w\gamma_\lambda = [w, s_{i_1}, ..., s_{i_p}]\), where the \(s_{i_j}\)'s are the simple reflections associated to the simple (Kac-Moody) root \(\alpha_{i_j}\) (for the sequence of foldings, see Section 4 (3)). More precisely, there exists an index \(k\) such that

\[
\delta = [w, \delta_1, ..., \delta_{k-1}, s_{i_k}, s_{i_{k+1}}, ..., s_{i_p}] \quad \text{and} \quad \gamma = [w, \delta_1, ..., \delta_{k-1}, 1, s_{i_{k+1}}, ..., s_{i_p}].
\]

**Proposition 10.** If \(\delta, \gamma\) are as in (6), then \(r_{\gamma_\lambda}^{-1}(\delta) \subset \overline{r_{\gamma_\lambda}^{-1}(\gamma)}\).
Proof. Let \( g = [g_0, g_1, \ldots, g_{k-1}, p_{-\alpha_1}(a_{k}), g_{k+1}, \ldots, g_p] \) be a point in \( r_{\gamma_{\lambda}}^{-1}(\gamma) \), where \( g_j = p_{\alpha_{ij}}(a_j)s_{ij} \), for \( j \geq i + 1 \) and some complex number \( a_j \) (which might be zero). For any positive (Kac-Moody) root \( \alpha \) and any complex number \( a \), one has \( p_{-\alpha}(a) = p_{\alpha}(a^{-1})s_{\alpha}h_{\alpha}p_{\alpha}(a^{-1}) \) (see [27]) for some torus element \( h_{\alpha} \). So, \( g \) may be written as

\[
g = [g_0, g_1, \ldots, g_{k-1}, p_{\alpha_{ik}}(a_k^{-1})s_{ik}h_{\alpha_{ik}}p_{\alpha_{ik}}(a_k^{-1}), g_{k+1}, \ldots, g_p].
\]

But the last part of this gallery is minimal, so it is stable by \( p_{\alpha_{ik}}(a_k^{-1}) \), and hence

\[
g = [g_0, g_1, \ldots, g_{k-1}, p_{\alpha_{ik}}(a_k^{-1})s_{ik}, g'_{k+1}, \ldots, g_p'],
\]

where \( g'_j = p_{\alpha_{ij}}(a'_j)s_{ij}, a'_j = bJa_j \)

for \( j \geq i + 1 \) and some \( b_j \in \mathbb{C}^* \). Therefore, \( r_{\gamma_{\lambda}}^{-1}(\delta) \subset r_{\gamma_{\lambda}}^{-1}(\gamma) \). \( \blacksquare \)

Corollary 6. If \( \delta, \gamma \) are two consecutive LS–galleries in a sequence of foldings, then the associated MV–cycles are contained in each other: \( Z(\delta) \subset Z(\gamma) \).

Remark 18. The same kind of arguments (using the open subvariety \( O(\delta) \subset r_{\gamma_{\lambda}}^{-1}(\delta) \)) apply also in the general case, i.e., Proposition \( \Box \) and Corollary \( \Box \) hold for an arbitrary dominant co-character \( \lambda \) and \( \delta \in \Gamma^+(\gamma_{\lambda}) \).

Example 9. Let us take a simple example to illustrate this section. Recall the setting of Example \( \Box \) \( G = SL_3(\mathbb{C}) \), \( S^a = \{s_0, s_1, s_2\} \) and we let \( \lambda = \beta = \alpha_1 + \alpha_2 \) be the highest root. We set \( \gamma_{\beta} = [1, s_0] \). The non-minimal positively folded galleries in \( \Gamma^+(\gamma_{\beta}) \) are

\[
\delta_{12} = [s_1s_2, 1], \quad \delta_{21} = [s_2s_1, 1], \quad \delta_{121} = [s_1s_2s_1, 1].
\]

If we apply the theorem, we get:

\[
r_{\gamma_{\beta}}^{-1}(\delta_{12}) = \left\{ [s_1s_2p_{-\alpha_1}(a), p_{-\alpha_0}(b)] \in \hat{\Sigma}(t_{\gamma_{\beta}}), b \neq 0 \right\},
\]

\[
r_{\gamma_{\beta}}^{-1}(\delta_{21}) = \left\{ [s_2s_1p_{-\alpha_2}(a), p_{-\alpha_0}(b)] \in \hat{\Sigma}(t_{\gamma_{\beta}}), b \neq 0 \right\},
\]

and

\[
r_{\gamma_{\beta}}^{-1}(\delta_{121}) = \left\{ [s_1s_2s_1p_{-\alpha_0}(b)] \in \hat{\Sigma}(t_{\gamma_{\beta}}), b \neq 0 \right\}.
\]

The two Mirkovic-Vilonen cycles associated to the pair \( (\lambda = \beta, \nu = 0) \) are obtained as the closure of the two varieties \( r_{\gamma_{\beta}}^{-1}(\delta_{12}) \) and \( r_{\gamma_{\beta}}^{-1}(\delta_{21}) \), and one can show that \( r_{\gamma_{\beta}}^{-1}(\delta_{121}) \) is in the closure of those two. Using Proposition \( \Box \) \( \Box \) and, for the inclusions of the \( Z(\delta_{ij}) \), some arguments similar to the ones above, we get the following inclusion relations (note the very suggestive similarity of the diagram on the left with the crystal graph):

\[
Z([s_1s_2, s_0]) \subset Z(\delta_{12}) \subset Z([s_1, s_0]) \quad \text{and} \quad Z([s_2s_1, s_0]) \subset Z(\delta_{21}) \subset Z([s_2, s_0])
\]

\[
Z([s_1s_2, s_0]) \cup Z([s_2s_1, s_0]) \subset Z([1, s_0]) \quad \text{and} \quad Z([s_1s_2, s_0]) \cup Z([s_1s_2, s_0]) \subset Z([1, s_0]).
\]
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