Noncommuting Gauge Fields as a Lagrange Fluid

R. Jackiw  
Center for Theoretical Physics  
Massachusetts Institute of Technology  
Cambridge, MA 02139-4307, USA

S.-Y. Pi  
Physics Department, Boston University  
Boston, MA 02215, USA

A.P. Polychronakos  
Physics Dept., Rockefeller University  
New York, NY 10021-6399, USA

and  
Physics Dept., University of Ioannina  
45110 Ioannina, Greece

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Abstract

The Lagrange description of an ideal fluid gives rise in a natural way to a gauge potential and a Poisson structure that are classical precursors of analogous noncommuting entities. With this observation we are led to construct gauge-covariant coordinate transformations on a noncommuting space. Also we recognize the Seiberg-Witten map from noncommuting to commuting variables as the quantum correspondent of the Lagrange to Euler map in fluid mechanics.
I. INTRODUCTION

Noncommuting coordinates are characterized by a constant, antisymmetric tensor $\theta^{ij}$:

$$[x^i, x^j] = i\theta^{ij}. \quad (I.1)$$

Subjecting the coordinates to an infinitesimal coordinate transformation

$$\delta x = -f(x) \quad (I.2)$$

and requiring that (I.1) remain unchanged results in the condition

$$-[f^i(x), x^j] - [x^i, f^j(x)] = 0 \quad (I.3)$$

which in turn implies by (I.1) that

$$-\partial_k f^i(x)\theta^{kj} - \partial_k f^j(x)\theta^{ik} = 0. \quad (I.4)$$

The left side is recognized as the Lie derivative of a contravariant tensor

$$L_f \theta^{ij} = f^k \partial_k \theta^{ij} - \partial_k f^i \theta^{kj} - \partial_k f^j \theta^{ik} \quad (I.5)$$

with the first term on the right vanishing since $\theta$ is constant. So the noncommutative algebra (I.1) is preserved by those coordinate transformations that leave $\theta$ invariant: $L_f \theta = 0$.

To unravel the condition (I.4), we must specify whether $\theta$ possesses an inverse $\omega$:

$$\theta^{ij} \omega_{jk} = \delta^i_k. \quad (I.6)$$

An inverse can exist in even dimensions, provided $\theta$ is nonsingular, but $\omega$ will not exist in odd dimensions, where the antisymmetric $\theta$ always possesses a zero mode. We shall assume the generic situation: nondegenerate $\theta$ with no zero modes in even dimensions, one zero mode in odd dimensions.

To solve for $f$ in even dimensions, we define

$$f^i = \theta^{ij} g_j \quad (i, j = 1, \ldots, 2n). \quad (I.7)$$

This entails no loss of generality, because $\theta$ is nonsingular (by hypothesis). Then (I.4) becomes

$$\theta^{ij} \partial_k g_\ell \theta^{kj} + \theta^{j\ell} \partial_k g_\ell \theta^{ik} = 0. \quad (I.8a)$$

Because $\theta$ is nonsingular and antisymmetric, this implies

$$\partial_k g_\ell - \partial_\ell g_k = 0 \quad (I.8b)$$
or
\[ g_t = \partial_t \phi \, . \]  

(I.9)

Thus we have
\[ f^i = \theta^{ij} \partial_j \phi \]  

(I.10)

for the coordinate transformations (in even dimensions) that leave \( \theta \) invariant. Since
\[ \nabla \cdot f = 0 \, . \]  

(I.11)

the transformations are volume preserving; the Jacobian of the finite diffeomorphism is unity. However, except in two dimensions, these are not the most general volume-preserving transformations. Nevertheless, they form a group: the Lie bracket of two transformations like (I.10), \( f_1^i = \theta^{ij} \partial_j \phi_1 \) and \( f_2^i = \theta^{ij} \partial_j \phi_2 \), takes the same form, \( \theta^{ij} \partial_j (\theta^{kl} \partial_k \partial_t \phi_1 \partial_t \phi_2) \). The group is the symplectic subgroup of volume-preserving diffeomorphisms that also preserve \( \theta^{ij} \).

In two dimensions, where we can set \( \theta^{ij} = \theta \epsilon^{ij} \), the above transformations exhaust all the area-preserving transformations.

In odd dimensions, where (by assumption) \( \theta \) possesses a single zero mode, for definiteness we orient the coordinates so that the zero mode lies in the first direction (labeled \( 0 \rightarrow \) time) and \( \theta \), confined to the remaining (spatial) dimensions, is nonsingular:
\[ \theta^{\mu \nu} = \begin{pmatrix} 0 & 0 \\ 0 & \theta^{ij} \end{pmatrix} \quad (i, j = 1, \ldots, 2n) \]  

(I.12)

\[ \theta^{ij} \omega_{jk} = \delta^i_k \, . \]

The diffeomorphisms that preserve \( \theta \) now take the infinitesimal form
\[ f^\mu = \begin{cases} f(t) \\ \theta^{ij} \frac{\partial}{\partial x^j} \phi(t, x) \end{cases} \]  

(I.13)

These still form a group. Two transformations, \((f_1, \phi_1)\) and \((f_2, \phi_2)\), possess a Lie bracket of the same form (I.13), with \((f_2 \partial_t f_1 - f_1 \partial_t f_2, f_2 \partial_t \phi_1 - f_1 \partial_t \phi_2 + \theta^{kl} \partial_k \partial_t \phi_1 \partial_t \phi_2) \). But the space-time volume is not preserved: \( \partial_\mu f^\mu \neq 0 \). (Of course, at fixed time, the spatial volume is preserved.)

Unit-Jacobian diffeomorphisms also leave invariant the equations for an ideal fluid, in the Lagrange formulation of fluid mechanics, and in particular a planar (two dimensional) fluid supports area-preserving diffeomorphisms. This coincidence of invariance suggests that other aspects of noncommutativity possess analogs in the theory of fluids, whose familiar features can therefore clarify some obscurities of noncommutativity. (A similar point of view was taken by Susskind [I] in the description of the quantum Hall effect.)
In this paper we explore connections between fluid mechanics and noncommuting field theory, first in Section I for low-dimensional systems, and then in Section II for higher-dimensional theories. The natural Poisson (commutator) structure, present in the Lagrange description of a fluid, and the possibility of introducing a vector potential to describe the evolution of comoving coordinates, will be recognized as classical precursors of analogous noncommuting entities. Within this framework, we shall show how noncommuting gauge fields respond to coordinate transformations, generalizing previously established results [2]. Also we shall demonstrate that the Seiberg-Witten map between noncommuting and commuting gauge fields [3] corresponds to the mapping between the Lagrange and Euler formulations of fluid mechanics. In this context it is possible to rederive simply the explicit “solution” to the Seiberg-Witten map in even dimensions [4] and to extend it to odd dimensions.

We conclude this Introduction by recalling the two formulations of fluid dynamics [5]. The Lagrange description uses the coordinates of the particles comprising the fluid: \( X(t, \mathbf{x}) \). These are labeled by a set of parameters \( \mathbf{x} \), which are the coordinates of some initial reference configuration, e.g., \( X(0, \mathbf{x}) = \mathbf{x} \), and are called comoving coordinates. We may parameterize the evolution of \( X \) by defining

\[
X^i(t, \mathbf{x}) = x^i + \theta^{ij} \hat{A}_j(t, \mathbf{x})
\]

which loses no generality provided \( \theta \) is nonsingular. As will be seen below, \( \hat{A} \) behaves as a noncommuting, Abelian vector potential.

In the Euler description \( X \) is promoted to an independent variable and renamed \( r \). Dynamics is described by the space-time–dependent density \( \rho(t, r) \) and velocity \( \mathbf{v}(t, r) \). The two formulations are related by postulating sufficient regularity so that (single-valued) inverse functions exist:

\[
X(t, \mathbf{x}) \bigg|_{\mathbf{x} = \chi(t, r)} = r
\]

\( X(t, \mathbf{x}) \) provides a mapping of the original position \( \mathbf{x} \) to position at time \( t \): \( X = r \), while \( \chi(t, r) \) is the inverse mapping. The Eulerian density then is defined by

\[
\rho(t, r) = \int d \mathbf{x} \, \rho_0(\mathbf{x}) \delta(X(t, \mathbf{x}) - r) .
\]

where \( \rho_0(\mathbf{x}) \) is a reference density, usually taken to be homogeneous:

\[
\rho(t, r) = \rho_0 \int d \mathbf{x} \delta(X(t, \mathbf{x}) - r) .
\]

(The integral and the \( \delta \)-function carry the dimensionality of the relevant space.) Evidently this evaluates as

\[
\frac{1}{\rho(t, r)} = \frac{1}{\rho_0} \left. \det \frac{\partial X^i(t, \mathbf{x})}{\partial x^j} \right|_{\mathbf{x} = \chi(t, r)} .
\]
The Eulerian velocity is

\[ v(t, r) = \dot{X}(t, x) \bigg|_{x = \chi(t, r)} \]  \tag{I.18}

where the overdot denotes differentiation with respect to the explicit time dependence. [Evaluating an expression at \( x = \chi(t, r) \) is equivalent to eliminating \( x \) in favor of \( X \), which is then renamed \( r \).] It is also true that the current \( j = \rho v \), given in terms of Lagrange variables by

\[ j(t, r) = \rho_0 \int dx \dot{X}(t, x) \delta(X(t, x) - r) \]  \tag{I.19}

obeys a continuity equation as a consequence of the above definitions:

\[ \dot{\rho} + \nabla \cdot j = 0 . \]  \tag{I.20}

The kinetic part of the Lagrangian for the Lagrange variables is simply

\[ L_0 = \int dx \frac{1}{2} \dot{X}(t, x) \cdot \dot{X}(t, x) . \]  \tag{I.21}

This is invariant against the infinitesimal diffeomorphism \((I.2)\), provided \( X \) transforms as a scalar

\[ \delta X = f \cdot \nabla X \]  \tag{I.22}

and \( f \) is transverse, \((I.11)\). When the interaction Lagrangian is taken as

\[ L_I = -\int dx V(\det \frac{\partial X^i(t, x)}{\partial x^j}) \]  \tag{I.23}

its variation under \((I.22)\), with transverse \( f \), also vanishes so that volume-preserving diffeomorphisms remain symmetries of the interacting theory. These are not symmetries of dynamics; rather they describe redundancy in the description: the transformations \((I.2), (I.22)\) relabel the parameters \( x \); in a sense, which is made precise below, they are gauge transformations.

Although we shall not need this, we note for completeness that the equation of motion for the Lagrange variables

\[ \ddot{X}^i(t, x) = \frac{\partial}{\partial x^j} \left[ \left( \frac{\partial X^i(t, x)}{\partial x^j} \right)^{-1} V' \left( \det \frac{\partial X^k(t, x)}{\partial x^\ell} \right) \det \frac{\partial X^m(t, x)}{\partial x^n} \right] \]  \tag{I.24}

implies that the Euler velocity \( v \) satisfies an evolution equation, which follows by differentiating \((I.19)\) with respect to time, and using \((I.18), (I.20)\) and \((I.24)\),

\[ \dot{v} + v \cdot \nabla v = -\frac{1}{\rho} \nabla P(\rho) . \]  \tag{I.25}
Here the pressure $P$ is given by

$$P(\rho) = - \int dx' V' \left( \det \frac{\partial X^k(t, x)}{\partial x'} \right) \det \frac{\partial X^i(t, x)}{\partial x^j} \delta(X(t, x) - r)$$

$$= - V' \left( \frac{1}{\rho} \right).$$  \hspace{1cm} (I.26)

(The prime denotes derivation with respect to argument.)

By multiplying $L_0 + L_I$ by unity in the form $\int dr \delta(X(t, x) - r)$ and performing the $x$ integral with the help of (I.16) and (I.19) we obtain a Lagrangian in terms of Euler variables:

$$L = \frac{1}{\rho_0} \int dr \left( \frac{1}{2} \rho v^2 - \rho V \left( \frac{1}{\rho} \right) \right).$$ \hspace{1cm} (I.27)

The Euler variables $\rho, v$ do not change under the relabeling symmetry (I.22) of the Lagrangian parameters. Since these parameters are absent in the Euler formulation, these diffeomorphisms are invisible.

II. NONCOMMUTING GAUGE THEORY (PRIMARILY IN LOW DIMENSIONS)

A. Commuting theory with Poisson structure

We introduce into the Lagrange fluid description the (nonsingular) antisymmetric tensor $\theta$. This allows for a natural definition of a Poisson bracket, which may be viewed as a classical precursor of the noncommutativity of coordinates. We define the bracket by

$$\{O_1, O_2\} = \theta^{ij} \frac{\partial O_1}{\partial x^i} \frac{\partial O_2}{\partial x^j}$$ \hspace{1cm} (II.1)

so that

$$\{x^i, x^j\} = \theta^{ij}. \hspace{1cm} (II.2)$$

It follows from the definition (II.14) that

$$\{X^i, X^j\} = \theta^{ij} + \theta^{ik} \theta^{j\ell} \hat{F}_{k\ell}$$ \hspace{1cm} (II.3)

with

$$\hat{F}_{ij} = \frac{\partial}{\partial x^i} \hat{A}_j - \frac{\partial}{\partial x^j} \hat{A}_i + \{\hat{A}_i, \hat{A}_j\}. \hspace{1cm} (II.4)$$

It is seen that the structure of the gauge field $\hat{F}$ is as in a noncommuting theory, with the Poisson bracket replacing the commutator of two potentials $\hat{A}$. Also, in the limit that the deviation of $X$ from the reference configuration $x$ is small, that is, for small $\hat{A}$, we recover a conventional Abelian gauge field.

The above formulas are understood to hold either in even dimensions for a purely spatial Euclidean formulation (there is no time variable) or in odd-dimensional space-time for spatial components ($X$ and $x$ are spatial vectors, without time components).
B. Commuting transformations (even dimensions)

In even dimensions, the $\theta$-preserving transverse diffeomorphism, which also implements the reparameterization symmetry of the Lagrange fluid, acts on $X$ through the bracket [see (I.10), (I.22), and (II.1)]:

$$\delta_\phi X = \theta^{ij} \frac{\partial X}{\partial x^i} \frac{\partial \phi(x)}{\partial x^j} = \{X, \phi(x)\}.$$  (II.5)

Because $\delta X$ compares the transformed and untransformed $X$ at the same argument, $\delta \hat{A}_i = \omega_{ij} \delta X^j$ and the volume-preserving diffeomorphism (II.5) induces a gauge transformation on $\hat{A}$:

$$\delta_\phi \hat{A}(x) = \nabla \phi(x) + \{\hat{A}(x), \phi(x)\} \equiv D\phi$$  (II.6a)

$$\delta_\phi \hat{F}_{ij}(x) = \{\hat{F}_{ij}(x), \phi(x)\}.$$  (II.6b)

We see that the dynamically sterile relabeling diffeomorphism of the parameters in the Lagrange fluid leads to an equally sterile gauge transformation, under which $X$ and $\hat{F}$ transform covariantly, as in (II.5) and (II.6b).

Next we consider a diffeomorphism of the target space:

$$\delta_f X = -\hat{f}(X).$$  (II.7)

In contrast to the previous relabelings, this transformation is dynamical, deforming the fluid configuration. Quantities

$$C_n(X) = \frac{1}{2^n n! \varepsilon_{i_1 j_1 \ldots i_n j_n}} \{X^{i_1}, X^{j_1}\} \cdots \{X^{i_n}, X^{j_n}\}$$  (II.8)

which are defined in $d = 2n$ dimensions, respond to the transformation (II.7) in a noteworthy fashion. One verifies that

$$\delta_f C_n(X) = -\nabla \cdot \hat{f}(X) C_n(X)$$  (II.9)

so that transverse (volume preserving) target-space diffeomorphisms leave $C_n$ invariant. Eq. (II.3) is most easily established by recognizing that

$$C_n(X) = \text{Pfaff}\{X^i, X^j\} = \text{det}^{1/2}\{X^i, X^j\} = \text{det}^{1/2} \theta \text{det} \frac{\partial X^i}{\partial x^j}.$$  (II.10)

The significance of these transformations is evident from (I.17), which shows that $1/\rho(r) = C_n(X)|_{x=\chi(r)}$ when $\text{det}^{1/2} \theta$ is identified with $1/\rho_0$. The transformation law for $\rho$ under transverse target space diffeomorphisms becomes

$$\delta_f \rho(r) = \hat{f} \cdot \nabla \rho(r).$$  (II.11)
It follows that this transformation leaves invariant all terms in the Lagrangian that depend only on \( \rho \) [like \( L_I \) in (I.23)].

When we restrict the transverse, target-space diffeomorphisms to those that also leave \( \theta \) invariant, i.e., (I.10) (of course in two dimensions this is not a restriction), further quantities are left invariant. These are constructed as in (II.8), but with any number of brackets \( \{X^i, X^j\} \) replaced by \( \theta^{ij} \).

It is interesting to combine the diffeomorphism of the parameter space with that of the target space, for a simultaneous transformation on both spaces. To this end we chose the form of the target space transformation to coincide with that of the reparameterization/relabeling transformation.

\[
\hat{f}(X) = \theta^{ij} \frac{\partial \phi(X)}{\partial X^j}.
\]

As we shall show below, this results in a gauge-covariant coordinate transformation on the vector potential \( \hat{A} \), once a further gauge transformation is carried out. Thus we consider \( \Delta \equiv \delta\phi + \delta f \),

\[
\Delta X^i = \{X^i, \phi(x)\} - \theta^{ij} \frac{\partial \phi(X)}{\partial X^j}.
\]

[Note that any deviation of \( \hat{f}(X) \) from \( \theta^{ij} \frac{\partial \phi(X)}{\partial X^j} \) may be attributed to \( \phi \), and can be removed by a further gauge transformation.] However, covariance is not preserved in (II.13): \( X \) on the left is covariant, but on the right in the Poisson bracket there occurs \( \phi(x) \), which is not covariant. The defect may be remedied by combining \( \Delta X \) with a further gauge transformation

\[
\delta_{\text{gauge}} X = \{X, \phi(X) - \phi(x)\}
\]

so that in \( \Delta + \delta_{\text{gauge}} \equiv \hat{\delta} \) we have a covariant transformation rule:

\[
\hat{\delta} X^i = \{X^i, \phi(X)\} - \theta^{ij} \frac{\partial \phi(X)}{\partial X^j}
\]

which in turn implies that \( \hat{\delta} A \) transforms as

\[
\hat{\delta} A_i = \omega_{ij} \{X^j, \phi(X)\} - \frac{\partial \phi(X)}{\partial X^i}.
\]

To recognize this transformation more clearly, we present it as

\[
\hat{\delta} A_i = \omega_{ij} \{X^j, X^k\} \frac{\partial \phi(X)}{\partial X^k} - \frac{\partial \phi(X)}{\partial X^i}
\]

and use (II.3) to find

\[
\hat{\delta} A_i = \theta^{kt} \tilde{F}_{it} \frac{\partial \phi(X)}{\partial X^k} = f^k(X) \tilde{F}_{ki}.
\]
Note that in the final expression (II.17b) the response of $\hat{A}$ is entirely covariant: it involves the covariant curvature $\hat{F}$ and the diffeomorphism function $f$ evaluated on the covariant argument $X$. This expression is precisely the gauge-covariant coordinate transformation, which was previously derived [3], in a setting that differs from the present in several ways. First, [3] dealt with a noncommuting theory, but we shall presently extend the above to the noncommuting case. Second, in [3] the transformation was not required to leave $\theta$ invariant. With transformations that change $\theta$, $\hat{\delta}$ includes the contribution $-(L_f^\theta)\partial/\partial\theta^{ij}$, and the action of the $\theta$ derivative is evaluated by the Seiberg-Witten equation [3]. Finally, in order to simplify ordering problems, $f$ was restricted to be at most linear in its argument, and the noncommutative formula corresponding to (II.17b) involved a (star) anticommutator.

C. Noncommuting theory with star products (even dimensions)

The above development may be taken over directly into a noncommutative field theory by replacing Poisson brackets by $-i$ times (star) commutators, so that (II.2) goes over into (I.1). Eq. (I.14) remains and (II.3), (II.4) become

$[X^i, X^j]_\ast = i\theta^{ij} + i\theta^{ik}\theta^{j\ell}\hat{F}_{k\ell}$

$\hat{F}_{ij} = \frac{\partial}{\partial x^i}\hat{A}_j - \frac{\partial}{\partial x^j}\hat{A}_i - i[\hat{A}_i, \hat{A}_j]_\ast$. (II.18)

The covariant transformation rules (II.13) and (II.16) may be used in the noncommutative context, provided a sensible ordering prescription is set for $\phi(X)$. This we do as follows. Define

$\Phi = \int d\mathbf{x} \phi(\mathbf{X})$ (II.20a)

where $\phi(\mathbf{X})$ is a series of (star) powers of $\mathbf{X}$:

$\phi(\mathbf{X}) = c + c_{ij}X^i \ast X^j + \frac{1}{2}c_{ijk}X^i \ast X^j \ast X^k + \cdots$. (II.20b)

[We are not concerned about convergence of the integral (II.20a), since we are interested in local quantities like (II.20b) or (II.23) below.] The integration over $\mathbf{x}$ (the argument of $\mathbf{X}$) ensures that $\Phi$ is invariant (in an operator formalism the integral becomes the trace of the operators). The $c$-coefficients in (II.20b) enjoy cyclic invariance (so that $\Phi$ and $\phi$ possess the same number of free parameters). Also we require $\phi$ to be Hermitian. [This ensures, e.g., that $c_{ij}$ is real symmetric; that Re $c_{ijk}$ is entirely symmetric and that Im $c_{ijk}$ is entirely antisymmetric (which is impossible in two dimensions).] Then (II.13) and (II.16) become

$\hat{\delta}X^i = -i[X^i, \phi(\mathbf{X})]_\ast - \theta^{ij}\frac{\delta\Phi}{\delta X^j}$ (II.21)

$\hat{\delta}\hat{A}_i = -i\omega_{ij}[X^j, \phi(\mathbf{X})]_\ast - \frac{\delta\Phi}{\delta X^i}$. (II.22)
where now the last entries employ a functional derivative:

\[
\frac{\delta \Phi}{\delta X^i} = c_i + c_{ij}X^j + c_{ijk}X^j \ast X^k + \ldots
\]  

(II.23)

In two dimensions, the ordering prescription (II.20) and its consequence (II.23) preserve the invariance of the \([X^i, X^j]\) commutator against the target space diffeomorphism [last term in (II.21)]. Thereby a property of the classical Poisson bracket [c.f. (II.9) at \(n = 1\)] is maintained in the noncommuting theory.

With \(\phi(X)\) at most quadratic in \(X\) (\(f\) at most linear) as in [2], one readily verifies the result in that paper

\[
\delta \hat{A}_i = \frac{1}{2} \{ f^j(X) \ast \hat{F}_{ji} + \hat{F}_{ji} \ast f^j(X) \}.
\]  

(II.24)

But with more general \(\phi\) (\(f\) containing quadratic and higher powers) there arise further reordering terms.

D. Commuting and noncommuting transformations (odd dimensions)

In odd dimensions, with the \(\theta\)-preserving transformation function given by (I.13), the relabeling transformation on the base space is

\[
\delta_\phi X(t, x) = \theta^{ij} \frac{\partial}{\partial x^j} \phi(t, x) \frac{\partial}{\partial x^i} X(t, x) + f(t) \frac{\partial}{\partial t} X(t, x)
\]  

(II.25)

\[
= \{ X(t, x), \phi(t, x) \} + f(t) \dot{X}(t, x).
\]

The fluid coordinate \(X\) has components only in the spatial directions. Here the Poisson bracket is defined with the nonsingular \(\theta^{ij}\).

For the target space diffeomorphism we again take the formula (II.12), so that the combined, noncovariant transformation \(\Delta \equiv \delta_\phi + \delta_t\) reads

\[
\Delta X^i = \{ X^i, \phi(t, x) \} + f(t) \dot{X}^i(t, x) - \theta^{ij} \frac{\partial \phi(t, X)}{\partial X^j}.
\]  

(II.26)

This is modified by the gauge transformation

\[
\delta_{\text{gauge}} X = \{ X, \phi(t, X) - \phi(t, x) \} - \{ X, f(t) \hat{A}_0(t, x) \}
\]  

(II.27)

resulting in the covariant transformation \(\Delta + \delta_{\text{gauge}} \equiv \hat{\delta}:\)

\[
\hat{\delta} X^i = \{ X^i, \phi(t, X) \} - \theta^{ij} \frac{\partial \phi(t, X)}{\partial X^j} + f(t) DX^i.
\]  

(II.28)

Here \(DX^i = \dot{X}^i + \{ \hat{A}_0, X^i \}\), where \(\hat{A}_0\) is a connection introduced to render the time derivative covariant against time-dependent gauge transformations, generated by \(\phi\). This is achieved when the gauge transformation law for \(\hat{A}_0\) is

\[
\delta_\phi \hat{A}_0 = \dot{\phi} + \{ \hat{A}_0, \phi \}.
\]  

(II.29)
The spatial components of the vector potential are introduced as before in (II.14)

$$DX^i = \theta^{ij} \left( \hat{A}_j - \partial_j \hat{A}_0 + \{ \hat{A}_0, \hat{A}_j \} \right) = \theta^{ij} \hat{F}_{0j}.$$  \hspace{1cm} (II.30)

The covariant transformation law of \( \hat{A} \) follows from (II.19), (II.28), and (II.30):

$$\hat{A}_i = \omega_{ij} \{ X^j, \phi(t, X) \} - \frac{\partial \phi(t, X)}{\partial X^i} + \omega_{ij} f(t) DX^j = f^j(t, X) \hat{F}_{ji} + f(t) \hat{F}_{0i} = f^\mu(t, X) \hat{F}_{\mu i}.$$  \hspace{1cm} (II.31)

It remains to fix the transformation law of \( \hat{A}_0 \). This requires specifying \( \delta_t \hat{A}_0 \). Since

$$\delta_t \hat{A}_i = - \frac{\partial \phi(t, X)}{\partial X^i}$$  \hspace{1cm} (II.32)

it is natural to take

$$\delta_t \hat{A}_0 = - \frac{\partial \phi(t, X)}{\partial t}.$$  \hspace{1cm} (II.33)

(The derivative acts on the first argument only.) Thus we have from (II.29) and (II.33)

$$\Delta \hat{A}_0 = \frac{\partial \phi(t, x)}{\partial t} + \{ \hat{A}_0, \phi(t, x) \} - \frac{\partial \phi(t, X)}{\partial t}.$$  \hspace{1cm} (II.34)

After adding to this a gauge transformation generated by \( \phi(t, X) - \phi(t, x) \) we are left with

$$\hat{A}_0 = \frac{\partial \phi(t, X)}{\partial X^i} DX^i + \{ \hat{A}_0, \phi(t, x) \}$$  \hspace{1cm} (II.35)

Eqs. (II.31) and (II.33) coincide with the formula obtained in a conventional commuting gauge theory.

Similar results follow within the noncommuting formalism, once the now familiar ordering prescription is given for \( \phi(t, X) \) and \( \Phi = \int dx \phi(t, X) \). In the noncommutative formalism (II.31) and (II.33) are regained, up to reordering terms.

### E. Seiberg-Witten map

To construct the Seiberg-Witten map in two Euclidean dimensions, we (temporarily) introduce a time dependence in the fluid variables (but not into the diffeomorphism functions – only spatial variables are transformed) and observe from (I.21) that \( (\rho, \rho_\mu) \) form a conserved 3-vector \( j^\alpha \) [also true in the noncommuting theory when an ordered definition for \( \delta(X(t, x) - r) \) is given – this will be provided below]. Therefore, the dual of \( j^\alpha, \varepsilon_{\mu \nu \alpha} j^\alpha \), satisfies a Bianchi
identity and can be presented as the curl of a potential, apart from additive and multiplicative constants:

\[ \varepsilon_{\mu\nu\alpha} j^\alpha \propto F_{\mu\nu} + \text{constant} \]  
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \]  

II.36  

II.37

Note \( j^\alpha, F_{\mu\nu}, A_\mu \) are ordinary functions, even in the noncommuting setting, since the noncommuting variables \( X \) are integrands (in an operator formalism, their trace is involved). In particular, the spatial tensor is determined by \( \rho \):

\[ \frac{\partial}{\partial r^i} A_j (r) - \frac{\partial}{\partial r^j} A_i (r) = F_{ij} (r) = -\varepsilon_{ij} (\rho - \rho_0) = -\varepsilon_{ij} \rho_0 \left( \int dx \, \delta (X (x) - r) - 1 \right) . \]  

II.38

(The time dependence is now suppressed.) \( X \) contains \(  \hat{A} \), as in (I.14), but it is (noncommuting) gauge covariant, so the integral in (II.38) is (noncommuting) gauge invariant. Therefore, (II.38) serves to define an (inverse) Seiberg-Witten map between the noncommuting (hatted) and commuting (unhatted) variables. The additive \( (\varepsilon_{ij} \rho_0) \) and multiplicative \( (-1) \) constants are fixed by requiring agreement at small \( \hat{A} \). It still remains to give a proper ordering to the \( \delta \)-function containing \( X \). This we do by a Fourier transform prescription:

\[ \int dr \ e^{i k \cdot r} F_{ij} (r) = -\varepsilon_{ij} \rho_0 \int dx \left( e^{i k \cdot X (x)} - e^{i k \cdot x} \right) \]  

II.39

and the ordering (Weyl ordering) is defined by the expansion of the exponential in (star product) powers.

When the exponential \( e^{i k \cdot X} \equiv 1 + i k \cdot X - \frac{1}{2} k \cdot X \star k \cdot X + \cdots \) is written explicitly in terms of \( \hat{A} \): \( \exp (i (k_i x^i + \theta k_i \varepsilon_{ij} \hat{A}_j)) \), factoring the exponential into \( e^{i k \cdot x} \) times another factor involves the Baker-Hausdorff lemma, and leads to an open Wilson line integral \( \mathcal{W} \). In that form (II.39) is seen to coincide with the known solution to the Seiberg-Witten map \( \Psi \), which is now also recognized as nothing but an instance of the Lagrange→Euler map of fluid mechanics. (See also \( \mathcal{W} \).)

To construct the Seiberg-Witten map in (2+1)-dimensional space-time we consider the conserved current, defined in (I.16) and (I.19), except that now the time dependence is retained throughout and the derivative is gauged with \( \hat{A}_0 \):

\[ j (t, r) = \int dx \, (X + \{ \hat{A}_0, X \}) \delta (X - r) . \]  

II.40

The operator ordering is prescribed in momentum space with the exponential (Weyl) ordering and (II.40) in the noncommuting theory becomes

\[ j (t, k) \equiv \int dr \, e^{i k \cdot r} j (t, r) = \int dx \, e^{i k \cdot X} (\dot{X} - i [\hat{A}_0, X]) . \]  

II.41
Note that the commutator does not contribute to current conservation because it is separately transverse:

\[ \int dx e_*^{ik} X [\hat{A}_0, k \cdot X]_* = \int dx \hat{A}_0 [k \cdot X, e_*^{ik} X]_* = 0. \]  

(II.42)

Therefore the 3-current is conserved as before. Its dual, \( \varepsilon_{\mu\nu\alpha} f^\alpha \) satisfies the Bianchi identity, so the Seiberg-Witten mapping reads

\[ \int dr e^{ik \cdot r} \left(1 - \frac{1}{2} \theta^{ij} F_{ij} \right) = \int dx e_*^{ik} X \]

\[ \int dr e^{ik \cdot r} F_{0i} = \omega_{ij} \int dx e_*^{ik} X (\dot{X}^j - i[\hat{A}_0, X^j]_*) \]

\[ = \int dx e_*^{ik} X \hat{F}_{0i}. \]  

(II.43)

Formulas (II.39) and (II.43) may be verified by comparison with the explicit \( O(\theta) \) Seiberg-Witten map, which for field strengths reads

\[ F_{\mu\nu} = \hat{F}_{\mu\nu} - \theta^{\alpha\beta} (\hat{F}_{\alpha\mu} \hat{F}_{\beta\nu} - \hat{A}_\alpha \partial_\beta \hat{F}_{\mu\nu}). \]  

(II.44)

Upon setting \( \theta^{\alpha0} = 0, \theta^{ij} = \theta \varepsilon^{ij} \) and

\[ e_*^{ik}(x^i + \theta \varepsilon^{ij} \hat{A}_j) = e^{ik} \left(1 + i \theta k_i \varepsilon^{ij} \hat{A}_j - \frac{1}{2} \theta^2 k_i k_m \varepsilon^{ij} \varepsilon^{mn} \hat{A}_j \hat{A}_n \right) \]  

(II.45)

it is recognized that (II.39) and (II.43) reproduce (II.44).

III. SEIBERG-WITTEN MAP IN HIGHER DIMENSIONS

A. Even dimensions

In dimensions higher than three the correspondence between the Bianchi identity and the conservation of particle current is lost. The derivation of the Seiberg-Witten map calls for higher conserved currents, whose duals are two-forms.

The introduction of such currents can be motivated by starting again from the commutative particle density \( \rho \) as expressed in (I.16b) and its inverse \( \rho^{-1} \) as expressed in (I.17). Their product

\[ 1 = \int dx \delta(X - r) \det \frac{\partial X^i(x)}{\partial x^j} \]  

(III.1)

is independent of the fluid profile \( X(x) \) and constitutes a topological invariant. The Jacobian determinant in the above can be expressed in terms of the square-root determinant (Pfaffian) of the antisymmetric matrix \( \{X^j, X^k\} \):

\[ 1 = \frac{\rho_0}{2^n n!} \int dx \delta(X - r) \varepsilon_{i_1, j_1, \ldots, i_n, j_n} \{X^{i_1}, X^{j_1}\} \cdots \{X^{i_n}, X^{j_n}\} = \rho_0 \int dx \delta(X - r) C_n(X) \]  

(III.2)
where, in analogy with the 2-dimensional case, we identified Pfaff(\(\theta\)) with \(1/\rho_0\). Removing all \(n\) Poisson brackets from the above recovers the full density \(\rho\). The removal of a single Poisson bracket \(\{X^i, X^j\}\), then, produces a sort of residual density \(\rho_{ij}\) in the corresponding dimensions, which becomes a candidate for the Seiberg-Witten commutative field strength:

\[
\rho_{ij} = \frac{\rho_0}{2^{n-1}(n-1)!} \int dx \delta(X - r) \varepsilon_{i,j,i_2,j_2,...,i_n,j_n} \{X^{i_2}, X^{j_2}\} \cdots \{X^{i_n}, X^{j_n}\}
\] (III.3)

The current dual to \(\rho_{ij}\), in momentum space,

\[
J^{j_1...j_{2n-2}} = \frac{\rho_0}{2^{n-1}(n-1)!} \int dx e^{i k \cdot X} \{X^{[j_1}, X^{j_2]} \cdots [X^{j_{2n-3}}, X^{j_{2n-2]}\}
\] (III.4)

(the indices are fully antisymmetrized) is gauge-invariant and conserved, ensuring that \(\rho_{ij}\) satisfies the Bianchi identity.

The corresponding current in the noncommutative case can be written by turning products into star-products and Poisson brackets into \((-i\times)\) star-commutators. The ordering of the exponential and other factors above has to be fixed in a way which ensures that the obtained current is conserved. Various such orderings are possible. For definiteness, we pick the ordering corresponding to the choice made in [4]:

\[
J^{j_1...j_{2n-2}} = \frac{\rho_0}{(2i)^{n-1}} \int dx \int_0^1 ds_1 \cdots \int_0^1 ds_{n-1} \delta \left(1 - \sum_{i=1}^{n-1} s_i\right) e^{i s_1 k \cdot X} [X^{[j_1}, X^{j_2]} \cdots [X^{j_{2n-3}}, X^{j_{2n-2]}\]
\] (III.5)

This corresponds to Weyl-ordering the exponential and distributing it in all possible ways between the different commutators. Note that the volume of the \(s_i\)-integration space reproduces the factor \(1/(n - 1)!\) present in (III.4).

To express compactly the above and facilitate the upcoming derivations, we introduce antisymmetric tensor notation. We define the basis one-tensors \(v_j\) representing the derivative vector field \(\partial_j\), and corresponding one-forms \(dx^j\). We consider the fundamental one-tensor \(X\) and the one-form \(k\)

\[
X = X^j v_j, \quad k = k_j dx^j
\] (III.6)

All tensor products will be understood as antisymmetric, i.e.,

\[
v_j v_k \equiv \frac{1}{2} (v_j \wedge v_k - v_k \wedge v_j)
\] (III.7)

etc., which amounts to considering \(v_j\) and \(dx^k\) as anticommuting quantities. Scalar products are given by the standard contraction

\[
v_j \cdot dx^k = \delta^j_k
\] (III.8)
We also revert to operator notation, dispensing with star products and writing \( \text{Tr} \) for \( \rho_0 \int dx \).

Finally, we simply write
\[
\int (n-1)\equiv \frac{1}{(2i)^{n-1}} \int_0^1 ds_1 \cdots \int_0^1 ds_{n-1} \delta \left( 1 - \sum_{i=1}^{n-1} s_i \right)
\]

Overall, the current in (III.3) is written as the rank-\((2n-2)\) antisymmetric tensor \( J \)
\[
J = \text{Tr} \int_{(n-1)} e^{is_1 k \cdot X} \cdots e^{is_{n-1} k \cdot X} \tag{III.10}
\]
and its conservation is expressed by the contraction \( k \cdot J = 0 \). The contraction of \( k \) acts on each \( X \) in a graded fashion. Using cyclicity of trace and invariance under relabeling the \( s_i \), this becomes
\[
k \cdot J = (n-1) \text{Tr} \int_{(n-1)} e^{is_1 k \cdot X} [k \cdot X, X] e^{is_2 k \cdot X} \cdots e^{is_{n-1} k \cdot X} \tag{III.11}
\]
Using the identity
\[
[e^{is k \cdot X}, X] = \int_0^s ds_1 e^{is_1 k \cdot X} [ik \cdot X, X] e^{i(s-s_1)k \cdot X} \tag{III.12}
\]
we can absorb the \( s_1 \)-integration in (III.11) and bring it in the form
\[
k \cdot J = -\frac{1}{2} \text{Tr} \int_{(n-2)} [e^{is_2 k \cdot X}, X] XX \cdots e^{is_{n-1} k \cdot X} XX \tag{III.13}
\]
Finally, using once more the cyclicity of trace, we see that the above contraction vanishes. This proves that the tensor \( J \) is conserved and, as a consequence, its dual \( \rho_{jk} \) satisfies the Bianchi identity. As in the 2-dimensional case, we put
\[
F_{jk}(k) = \rho_{jk}(k) - \omega_{jk} \delta(k)
\]
and recover the commuting Abelian field strength, which can, in turn, be expressed in terms of a (commutative) abelian potential \( A_j \).

In the above manipulations we freely used cyclicity of trace. In general this is dangerous, since the commuted operators may not be trace class. Assuming, however, that \( X \) becomes asymptotically \( x \) for large distances, the presence of the exponentials in the integrand ensures that this operation is permissible.

As mentioned previously, the fully symmetric ordering is not the only one that leads to an admissible \( \rho_{jk} \). As an example, in the lowest-dimensional nontrivial case \( d = 4 \) we can alter the ordering by splitting the commutator as
\[
J^{jk} = \frac{1}{2i} \text{Tr} \left\{ e^{ik \cdot X} [X^j, X^k] \right\} \quad \rightarrow \quad J^{jk} = -i \text{Tr} \int_0^1 ds f(s) e^{is k \cdot X} e^{(1-s)k \cdot X} X^k \tag{III.15}
\]
If \( f(s) = -f(1-s) \) the above will be antisymmetric in \((j, k)\) and conserved, as can explicitly be verified. Further, if \( f(s) \) satisfies
\[
\int_0^1 ds (2s - 1) f(s) = 1 \tag{III.16}
\]
then (III.15) will also have the correct commutative limit. We obtain an infinity of solutions depending on a function of one variable \( f(s) \). This arbitrariness reflects the fact that the Seiberg-Witten equations are not integrable and therefore the solution for \( \theta = 0 \) depends on the path in the \( \theta \)-space taken for integrating the equations. For \( d = 4 \) the parameter space is a plane and the path from a given \( \theta \) to \( \theta = 0 \) on the plane can be parametrized by a function of a single variable, just like \( J_{jk}^j \). The various solutions are related through field redefinitions.

\section*{B. Odd dimensions}

The situation in odd dimensions differs in that we need to specify separately the components of the conserved current in the commutative and noncommutative directions. For \( d = 2n + 1 \) the current is of rank \( 2n - 1 \) and it can be constructed by a procedure analogous to the even-dimensional case: We start from the expression for the total particle current \( j^\mu \) and \( J \) and introduce \( 2n - 2 \) commutators, one less than the number which would fully saturate it to \((1, \nu)\). The temporal components \( J_{0j_1\ldots j_{2n-2}} \) can be expressed as a rank-(2\(n-2\)) antisymmetric spatial tensor \( J^0 \), while the spatial components \( J_{j_0j_1\ldots j_{2n-2}} \) can be expressed as a rank-(2\(n-1\)) antisymmetric tensor \( J \). Their fully ordered expressions are
\[
J^0 = \frac{1}{n-1} \text{Tr} \int_{(n-1)}^\infty e^{i s_1 k \cdot X} e \ldots e^{i s_{n-1} k \cdot X} 
\tag{III.17}
\]
\[
J = \text{Tr} \int (n) e^{i s_0 k \cdot X} e^{i s_1 k \cdot X} e \ldots e^{i s_{n-1} k \cdot X} 
\tag{III.18}
\]
The above expressions can be unified by introducing a temporal component for the field \( X^\mu \), namely \( X^0 \equiv t \) (which is obviously commutative), and extending the one-tensor \( X \) also to include \( X^0 v_0 \). Further, we can Fourier transform in time and define \( k = k_\mu dx^\mu \) to also include the frequency \( k_0 \). Then the corresponding (space-time) \((2n-1)\)-tensor \( J \) acquires the form
\[
J = \int dt \int (n) e^{i s_1 k \cdot X} D e^{i s_2 k \cdot X} e \ldots e^{i s_{n} k \cdot X} 
\tag{III.19}
\]
\( X^0 \) is absent in \( XX \) and, since \( DX^0 = 1 \), only \( s_0 + s_1 \) appears in the temporal component of \( J \); integrating over \( s_1 \) reproduces the factor \( 1/(n-1) \) appearing in (III.17).

The above current is obviously gauge invariant. We shall prove that it is also conserved, that is, it satisfies \( k \cdot J = 0 \). The contraction is
\[ k \cdot J = \int dt \, \text{Tr} \left\{ e^{is_1kX} k \cdot D X e^{is_2kX} XX \cdots e^{is_nkX} XX \\ - \sum_{m=2}^{n} e^{is_1kX} D X XX \cdots e^{is_mkX} [k \cdot X, X] e^{is_{m+1}kX} \cdots XX \right\} \] (III.20)

(with \( s_{n+1} = 0 \)). By formula (III.12) and a similar one for the covariant time derivative, the above can be rewritten as

\[ k \cdot J = \int dt \, \text{Tr} \left\{ D e^{is_1kX} e^{is_2kX} XX \cdots e^{is_{n-1}kX} XX \\ - \sum_{m=2}^{n-1} e^{is_1kX} D X XX \cdots [e^{is_mkX}, X] XX \cdots XX \right\} \] (III.21)

Due to the cyclicity of trace, the sum above telescopes and only the first term of the \( m = 2 \) commutator and the second term of the \( m = n - 1 \) commutator survive. Altogether we obtain

\[ k \cdot J = \int dt \, \text{Tr} \int_{(n-1)} \left( D e^{is_1kX} + DXX + XDX \right) e^{is_2kX} XX \cdots e^{is_{n-1}kX} XX \\
= \int dt \, \text{Tr} \int_{(n-1)} D e^{is_1kX} XX \cdots e^{is_{n-1}kX} XX \\
= \int dt \, \text{Tr} \int_{(n-1)} \frac{1}{n-1} D e^{is_1kX} XX \cdots e^{is_{n-1}kX} XX \\
= \int dt \, \frac{d}{dt} \text{Tr} \int_{(n-1)} \frac{1}{n-1} e^{is_1kX} XX \cdots e^{is_{n-1}kX} XX \\
= 0 \]

which proves the conservation of \( J \). Its dual \( \rho_{\mu\nu} \) satisfies the \((2n + 1)\)-dimensional Bianchi identity and can be used to define the commutative Abelian field strength

\[ F_{ij}(k) = \rho_{ij}(k) - \omega_{ij}(k) \] (III.23)

\[ F_{0i}(k) = \rho_{0i}(k) \] (III.24)

In the above we gave separate derivations of the Seiberg-Witten map for even and odd dimensions. The two can be unified by demonstrating that each case can be obtained as a dimensional reduction of the other in one more dimension. This is treated in the next section.

C. Dimensional reduction

It is quite straightforward to see that the even dimensional Seiberg-Witten map is obtained from the \( d = 2n + 1 \) map by dimensional reduction. We assume a time-independent
configuration in which \( X^j \) \( (j = 1, \ldots, 2n) \) do not depend on \( t \) and \( A_0 \) vanishes. In this case \( DX \) vanishes and so does \( J \) in (III.18); only the component \( J^0 \) in (III.17) survives, reproducing the \( 2n \)-dimensional solution.

The reduction from a fully noncommutative \( d = 2n + 2 \) case to the \( d = 2n + 1 \) case is only slightly subtler. For concreteness, we shall take \( t \equiv x^0 \) to be canonically conjugate to the last dimension, call it \( z \equiv x^{2n+1} \), which will be reduced; that is,

\[
[t, z] = i\theta_0 \quad (\theta_0 = \theta^{0,2n+1}), \quad [t, x^i] = [z, x^i] = 0 \quad (i = 1, \ldots, 2n)
\] (III.25)

This can always be achieved with an orthogonal rotation of the \( x^\mu \). The reduced configuration consists of taking all fluid coordinates other than \( X^{2n+1} \) to be independent of \( x^{2n+1} \) and, further, the gauge potential corresponding to \( z = x^{2n+1} \) to vanish. Specifically,

\[
X^i = X^i(x, t) \\
X^0 = t \\
X^{2n+1} = z + \theta_0A_0(x, t)
\] (III.26-28)

With this choice the corresponding field strengths become

\[
[X^i, X^j] = i\theta^i j + i\theta^{ik}\theta^j \hat{F}_{k\ell} \\
[X^i, X^0] = 0 \\
[X^i, X^{2n+1}] = i\theta_0(D_0X^i - i[X^i, A_0]) = i\theta^i j \theta^{2n+1,0} \hat{F}_{j0}
\] (III.29-31)

with \( \hat{F}_{\mu\nu} \) \( (\mu, \nu = 0, \ldots, 2n) \) the field strength of a noncommutative \( d = 2n + 1 \) theory.

The corresponding \( d = 2n + 2 \) Seiberg-Witten map reduces to the \( d = 2n + 1 \) map. Indeed, the current \( J \) in (III.10), now, is a rank-\( n \) antisymmetric tensor. When all its indices are spatial \( (1, \ldots, 2n) \) it becomes a fully saturated topological invariant, that is, a constant; this reproduces a constant \( \rho_{0,2n+1} \). When one of its indices is 0 and the rest are spatial it vanishes, leading to \( \rho_{i,2n+1} = 0 \). When one of its indices is \( 2n + 1 \) and the rest are spatial it reproduces expression (III.18). Finally, when two of its indices are \( 0,2n+1 \) and the rest are spatial it reproduces (III.17), recovering the full commuting \( (2n+1) \)-dimensional Abelian field strength.

We stress that the above reductions are not the most general ones. Indeed, mere invariance of the fluid configuration with respect to translations in the extra dimension does not require the vanishing of the gauge field in the corresponding direction. This means that we could choose \( X^0 = t + H(x, t) \) (instead of \( X^0 = t \)) in both \( d = 2n + 1 \) and \( d = 2n + 2 \). The corresponding reduced theory contains an extra Higgs scalar in the adjoint representation of the (noncomutative) \( U(1) \) gauge group. Our Seiberg-Witten map in this situation reproduces, with no extra effort, the space-time derivatives of a corresponding commuting ‘Higgs’ scalar.
We conclude by remarking that the above complete reduction scheme (\(2n+2 \to 2n+1 \to 2n \to \ldots\)) is reminiscent of the topological descent equations relevant to gauge anomalies. This may prove fruitful in the analysis of noncommutative topological actions and the mapping of topologically nontrivial configurations [8].

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