Linear and nonlinear edge dynamics of trapped fractional quantum Hall droplets

Alberto Nardin\textsuperscript{1,}\textsuperscript{∗} and Iacopo Carusotto\textsuperscript{1}
\textsuperscript{1}INO-CNR BEC Center and Dipartimento di Fisica, Universit`a di Trento, via Sommarive 14, I-38123 Trento, Italy.

We report numerical studies of the linear and nonlinear edge dynamics of a non-harmonically confined macroscopic fractional quantum Hall fluid. In the long-wavelength and weak excitation limit, observable consequences of the fractional transverse conductivity are recovered. The first non-universal corrections to the chiral Luttinger liquid theory are then characterized: for a weak excitation in the linear response regime, cubic corrections to the linear wave dispersion and a broadening of the dynamical structure factor of the edge excitations are identified; for stronger excitations, sizable nonlinear effects are found in the dynamics. The numerically observed features are quantitatively captured by a nonlinear chiral Luttinger liquid quantum Hamiltonian that reduces to a driven Korteweg-de Vries equation in the semiclassical limit. Experimental observability of our predictions is finally discussed.

\textbf{I. INTRODUCTION}

The fractional quantum Hall (FQH) effect is one of the most fascinating concepts of modern quantum condensed matter physics \cite{1,2}. Whereas FQH states of matter were originally observed in the solid-state context of two-dimensional electron gases under strong magnetic fields, a strong experimental attention is presently devoted to synthetic quantum matter systems \cite{3} such as gases of ultracold atoms under synthetic magnetic fields \cite{4,5,6} or fluids of strongly interacting photons in nonlinear topological photonics devices \cite{7,8,9}. As it was pointed out in recent theoretical proposals \cite{10,11}, such systems typically offer a wider variety of experimental tools compared to the transport and optical probes of electronic systems. Important experimental steps towards observing FQH physics have been recently reported in both atomic \cite{27,28} and photonic systems \cite{30,31}.

One of the most exciting features of FQH liquids is the possibility of observing fractional statistics effects both in the bulk and on the edge \cite{20,21}. On this latter, in particular, gapless modes supporting fractionally charged excitations have been observed in shot-noise experiments \cite{33}: more recently, edge modes have been used as a probe of the topological state of the bulk \cite{34}, hints of generalized exclusion statistics have been highlighted \cite{35}, and a number of further intriguing properties have been anticipated \cite{36,37}. Many of these features are theoretically captured by the chiral Luttinger liquid (\(\chi\)-LL) theory \cite{38,39,40} which is expected to be an accurate description of the edge in the long-wavelength and weak excitation limits.

In this work, we investigate the physics beyond the regime of validity of the \(\chi\)-LL description and perform numerical studies of the linear and nonlinear edge dynamics of a fractional QH liquid trapped by a generic, non-harmonic external potential. As compared to our previous study of integer QH liquids \cite{11}, the strongly correlated nature of FQH liquids poses enormous technical challenges to the theoretical description and requires the development of a novel numerical approach to follow the dynamics of macroscopic FQH clouds. In particular, we focus on the neutral edge excitations (EE) that are generated by applying an external time-dependent potential to an incompressible FQH cloud.

In electronic systems generation and diagnostics of edge excitations requires ultrafast tools that are presently being developed with state-of-the-art electronic and optical technologies \cite{42,43}. On the other hand, arbitrary time-dependent potentials can be readily applied to synthetic systems and high-resolution detection tools at the single-particle level are also available \cite{3}. This suggests that our results will offer a useful guidance to the next generation of FQH experiments in a wide range of experimental platforms.

In addition to this, we expect that our results may be of interest also from a theoretical perspective: leveraging on the physical insight provided by numerical calculations, we are able to formulate a nonlinear extension of \(\chi\)-LL theory that is able to quantitatively describe the system dynamics at a much lower numerical cost. This theory offers an effective theoretical framework for future investigations of the rich nonlinear quantum dynamics of the FQH edge and is amenable to sophisticated theoretical tools for non-linear Luttinger liquids \cite{62}.

The structure of the article is the following. In section \(\text{II}\) we discuss the physical system under consideration (\(\text{II}\,\text{A}\)), we introduce our numerical approach for its description (\(\text{II}\,\text{B}\)), and we show some benchmark calculations (\(\text{II}\,\text{C}\)). In section \(\text{III}\) signatures of the quantized transverse conductivity of the bulk in the edge physics are highlighted and discussed within the \(\chi\)-LL picture. In section \(\text{IV}\) we start investigating effects beyond the \(\chi\)-LL description by looking at the dynamical structure factor of a anharmonically confined droplet (\(\text{IV}\,\text{A}\)), at the group velocity dispersion of the edge excitations (\(\text{IV}\,\text{B}\)) and at their nonlinear features at stronger excitation levels (\(\text{IV}\,\text{C}\)). In section \(\text{V}\) we capitalize on the numerical observations of the previous section to write a min-
nal non-linear $\chi$LL Hamiltonian whose classical limit gives a Kortweg-de Vries equation for the edge-density dynamics. In particular, we show how this generalized $\chi$LL Hamiltonian is able to reproduce all the microscopic calculations in a quantitative way. In section [VI] we discuss the experimental observability of the described physics. Finally, we give some conclusive remarks in section [VII]. The Appendices summarize additional information in support of our claims: Appendix A shows statistical information on the collected Monte Carlo data; in appendix B we comment on the protocol we used to excite the edge dynamics; Appendix C provides further details on the linear response calculations within the Hamiltonian whose classical limit gives a Kortweg-de Vries equation for the edge-density dynamics. In particular, we show how this generalized $\chi$LL Hamiltonian is able to reproduce all the microscopic calculations in a quantitative way. In section VI we discuss the experimental observability of the described physics. Finally, we give some conclusive remarks in section VII. The Appendices summarize additional information in support of our claims: Appendix A shows statistical information on the collected Monte Carlo data; in appendix B we comment on the protocol we used to excite the edge dynamics; Appendix C provides further details on the linear response calculations within the Hamiltonian whose classical limit gives a Kortweg-de Vries equation for the edge-density dynamics. In particular, we show how this generalized $\chi$LL Hamiltonian is able to reproduce all the microscopic calculations in a quantitative way. In section VI we discuss the experimental observability of the described physics. Finally, we give some conclusive remarks in section VII. The Appendices summarize additional information in support of our claims: Appendix A shows statistical information on the collected Monte Carlo data; in appendix B we comment on the protocol we used to excite the edge dynamics; Appendix C provides further details on the linear response calculations within the Hamiltonian whose classical limit gives a Kortweg-de Vries equation for the edge-density dynamics. In particular, we show how this generalized $\chi$LL Hamiltonian is able to reproduce all the microscopic calculations in a quantitative way. In section VI we discuss the experimental observability of the described physics. Finally, we give some conclusive remarks in section VII.
where we have introduced the short-hands \( z = \{ z_1, \ldots, z_N \} \) and \( D_z = dz_1 \ldots dz_N \) and we have defined the norm as \( \| \psi_{l,z} \|^2 = \int D_z |\psi_{l,z}(z)|^2 \). The integrals in both the numerator and the denominator are then performed with the Metropolis-Hastings algorithm using \( W(z) = |\psi_{l,z}(z)|^2 / |\psi_{l,0}(z)|^2 \) as the target probability distribution function [58, 59]. Since the \( \psi_{l,z}(z) \) wavefunctions have the form \( \text{consisting of a Laughlin state multiplied by a suitable polynomial of moderate degree, they share most of their zeros and their weights are concentrated in similar regions of configuration space. This feature is strongly beneficial in view of the convergence of the Monte-Carlo sampling. In principle, the matrices \( \mathbb{M} \) and \( \mathbb{H} \) obtained in this way are not exactly Hermitian, so we perform a preliminary Hermitization step before proceeding with the calculations.}

Using this method we have been able to study the dynamics of systems of up to \( N \sim 80 \) particles. In the following we will focus on results for up to 40 particles for which the statistical error of the Monte Carlo sampling is smaller (see Appendix A). As we are going to see, for this particle number, the system is in fact large enough to be in the macroscopic limit where the edge properties are independent of the system size.

\[ R_q = \sqrt{2N/\nu} \]  \cite{2}. The excited state energies successfully compare to exact diagonalization (ED) results for all particle numbers for which ED is feasible (b).

**FIG. 2:** (a) Amplitude of the edge density response after the weak \( l = 2 \) external potential has been switched off, for different filling factors \( \nu \), normalized to the one of a large IQH system. (b) DSF weights plotted against the excitation energy of each eigenstate. Within each \( l \) sector, the dashed lines are guides to the eye. MC data (black dots) are compared to the nonlinear \( \chi L L \) theory (red crosses). (c) SSF \( S_\ell \) as a function of \( l \) for the same values of \( \nu \) as in (a). Dashed lines indicate the \( \chi L L \) prediction \( S_\ell = \nu \ell \). (d) Normalized edge-mode dispersion for different \( N \). Same trap potential as in Fig. 1. In panels (b,d) the filling factor is fixed to \( \nu = 1/2 \).

### III. Quantized Transverse Conductivity

We then investigate the dynamical evolution of the system in response to a temporally short excitation. With no loss of generality \cite{3}, we assume for simplicity a radially flat potential,

\[ U(\theta, t) = U_\ell(t) e^{i \ell \theta} + \text{c.c.} \]  \cite{5}.

The force along the azimuthal direction induced by the angular gradient of \( U(\theta, t) \) generates a transverse Hall current along the radial direction, which locally changes the cloud density on the edge.

Numerical results for the linear response to a weak excitation are displayed in Fig. 2(a): in agreement with transverse conductivity quantization arguments, a clear proportionality of the response on the FQH filling factor \( \nu \) is found in the large-\( N \) limit. Quite remarkably, this limiting behaviour is accurately approached in the FQH case already for way lower particle numbers \( N \gtrsim 15 \) in the FQH than in the \( \nu = 1 \) IQH case. This conclusion is of great experimental interest as it suggests that evidence of the quantized conductivity can be observed just by probing the response of the edge of relatively small clouds.

**C. Benchmark**

A first application of the numerical MC method is illustrated in Fig. 1, where we show a radial cut of the GS density (a) and the energies of the lowest-\( l \) excited states sitting below the many-body energy gap (b,c). The density profile shows the density plateau corresponding to the incompressible bulk \( \rho_0 = \nu / (2\pi) \) and the usual oscillating structure on the edge near the classical radius.
to trap deformations, a technique of widespread use for ultracold atomic clouds [60].

This behaviour can be understood on the basis of the $\chi$LL theory [28–30, 31], with the external potential $U(\theta,t)$ minimally coupled to the edge density $\hat{\rho}(\theta)$. The system response after $U$ has been turned off can be written [C] to linear order as

$$\langle \delta \hat{\rho}(\theta,t) \rangle = \frac{1}{\pi} \Im \left[ \sum_l \int U_l(\omega) S_l(\omega) e^{i(\theta - \omega t)} d\omega \right] , \quad (6)$$

where $U_l(\omega)$ is the space-time Fourier transform of $U(\theta,t)$,

$$S_l(\omega) = \int dt \frac{e^{i\omega t}}{2\pi} \langle e^{i\hat{H}t} \delta \hat{\rho} e^{-i\hat{H}t} \delta \hat{\rho}_{-l} \rangle , \quad (7)$$

is the dynamical structure factor (DSF) restricted here to the edge mode manifold of states-- and $\delta \hat{\rho}$ is the angular Fourier transform of the edge-density variation $\delta \hat{\rho}(\theta)$. When the trap is quadratic, the edge is a prototypical $\chi$LL and the DSF is a $\delta$-peak centered at $\omega_l = \Omega l$, with $\Omega = 2\lambda$. For anharmonic traps [Fig.3(a)], $\Omega$ is still determined by the potential gradient at the cloud edge,

$$\Omega = r^{-1} \partial_r V_{\text{conf}}(r) \bigg|_{R_{cl}} \propto N(\delta-2)/2 \quad (8)$$

but at the same time the DSF broadens. Up to not-too-late times, the density response can nevertheless be accurately approximated as

$$\langle \delta \hat{\rho}(\theta,t) \rangle \simeq \frac{1}{\pi} \Im \left[ \sum_l \tilde{U}_l(\omega_l) e^{i(\theta - \omega_l t)} S_l \right] , \quad (9)$$

where $S_l = \int S_l(\omega) d\omega$ is the edge-mode static structure factor (SSF). As long as the confinement potential is not strong enough to mix with states above the many-body gap, the SSF keeps its $\chi$LL value $S_l = l\nu$ for $l \geq 0$ and zero otherwise up to $l$ values where finite-$N$ effects get important [Fig.2(c)].

IV. BEYOND CHIRAL LUTTINGER LIQUID EFFECTS

Our numerical framework is not restricted to study the response of the system to weak and long-wavelength excitations as captured by the standard chiral Luttinger liquid theory. The goal of this Section is to explore the physics beyond the $\chi$LL, namely the response of the edge to stronger and shorter wavelength perturbations.

A. Dynamical structure factor

As we have seen in the previous Section, anharmonic confinements cause the DSF to broaden [Fig.2(b)] within a finite frequency window, whose extension turns out [D] to be proportional to $l^2$ and to the curvature of the trap potential at the classical radius

$$c_0 = R_{cl}^{-1} \partial_r (r^{-1} \partial_r V_{\text{conf}}(r)) \bigg|_{R_{cl}} = \lambda \delta(\delta-2)R_{cl}^{\delta-4} , \quad (10)$$

a quantity related to the second $l$-derivative of the LLL projection of $V_{\text{conf}}(r)$, which physically corresponds to the radial gradient of the angular velocity. Like in the IQH case [41], the broadening is responsible for the decay of the oscillations at late time that is visible in Fig.4(b). However, in contrast to the IQH case, the DSF weights at fixed $l$ are non-flat: the weight is suppressed close to the high-energy threshold and peaked at the low-energy one. This behavior is in close analogy to what was found for a fermionic LL beyond the linear dispersion approximation [61–64] and will be the subject of further investigation [65].

FIG. 3: (a,b) Normalized angular velocity $\Omega$ and group velocity dispersion parameter $\alpha$ as a function of $N$ for different trap exponents $\delta$ at a constant $\nu = 1/2$. (c,d) Normalized $\alpha$ as a function of inverse filling $1/\nu$ for (c) $\delta = 4$ and different $N$, and (d) as a function of trap curvature $\propto \delta(\delta-2)$ for different fillings $\nu$ at given $N$. All points are extracted from low-$l$ fits to the numerical MC predictions for $\omega_l$ as a function of $l$.

B. Group velocity dispersion

This asymmetrical distribution of the DSF makes its center-of-mass frequency shift from the low-energy result $\omega_l \simeq \Omega l$. EE experience a wavevector-dependent frequency-shift and, thus, a finite group velocity dispersion. As shown in Fig. 2(d), the negative shift gets stronger according to a cubic law at small $l$,

$$\omega_l = \Omega l - \alpha l^3 . \quad (11)$$

Note that this cubic form is different from the quadratic Benjamin-Ono one introduced in [66, 67] and critically scrutinized on the basis of conformal field theory in [68].
Whereas the results in Fig 2(d) may suggest that the shift is a finite-size effect, a careful account of the N dependence, of the geometry and confinement parameters indicates that the effect persists in the macroscopic limit. To this purpose, we note that as N increases at fixed trapping parameters λ, δ, the cloud gets correspondingly larger as \( R_{cl} = \sqrt{2mN} \), so the effective spatial wavevector of an excitation at \( t \) decreases as \( q = l/R_{cl} \). At fixed \( q \), we expect the frequency shift to be proportional to the curvature of the confining potential in a straight-edge geometry, which in our case suggests \( \alpha \Gamma^3 = \beta_r \bar{c}_0 q^4 \), with \( \bar{c}_0 = R_{cl}^2 c_0 = \lambda \delta (\delta - 2) R_{cl}^{-2} \) and a size-independent \( \beta_r \). This functional form is validated against the numerical results in Fig 3(b-d). Panel (b) shows that \( \alpha \) is indeed proportional to \( \sqrt{N}^{5-5} \) at fixed \( \lambda \). Panels (c,d) illustrate the linear dependence on the filling factor and on the trap curvature parameter, respectively. From these data, we extract a macroscopic coefficient \( \beta_r \simeq \pi (1 - \nu) / 8 \nu \). Work is in progress to understand this result in connection with the Hall viscosity and the bulk structure factor of the FQH fluid [69].

FIG. 4: (a) Colorplot of the density near the edge at \( t = 12 \times 10^3 \) after a strong perturbation; white (black) lines are iso-density contours for the perturbed (unperturbed) system. (b,c) Time-evolution of the fundamental and second harmonic spatial Fourier components of the edge density variation of \( N = 30 \) (red) and \( N = 9 \) (yellow) clouds. ED data for \( N = 9 \) are shown as brown dashed lines as a benchmark. Dotted black lines and black dots indicate respectively the solution of the semi-classical equation (13) and of the quantum model \( \hat{H}_{\chi}^{\nu LL} \). Insets show a magnified view of the dynamics at early times. Same trap parameters as in Fig 1 filling factor \( \nu = 1/2 \).

### C. Non-linear dynamics

When the excitation strength increases, nonlinear effects start to play an important role in the edge mode evolution. Numerical results illustrating this physics are displayed in Fig 4 panel (a) shows the density profile of the cloud edge after a relatively long evolution time past a sinusoidal perturbation with given \( l \). In contrast to the weak excitation case discussed above where the density profile keeps at all times a plane-wave form proportional to \( \cos(\Omega t - \omega_0 t) \), here a marked forward-bending of the waveform is visible, leading to a sawtooth-like profile. Upon angular Fourier transform, this asymmetry corresponds to the appearance of higher spatial harmonics.

The physical mechanism underlying the nonlinearity can be understood in analogy with the IQH case [41]. Because of the incompressibility condition, a local variation \( \delta \rho (\theta) \) of the radially-integrated angular density must correspond to a variation of the cloud radius \( \delta R (\theta) \) proportional to \( \delta \rho (\theta) / (\rho_0 R_{cl}) \). This then leads to a variation of the local angular velocity

\[
\Omega (\theta) = r^{-1} \partial_t V_{\text{conf}} |_{R(\theta)} \simeq \Omega + (2\pi c_0 / \nu) \delta \rho . \quad (12)
\]

This nonlinear effect can be combined with the group dispersion and the perturbation potential \( U(\theta, t) \) discussed above into a single semiclassical evolution equation.

For simplicity, we formulat the equation in terms of the 1D density variation \( \sigma (\zeta, t) = \delta \rho (\theta, t) / R_{cl} \), with \( \zeta = R_{cl} \theta \) being the physical position along the edge. The resulting evolution equation

\[
\frac{\partial \sigma}{\partial t} = -\left[ v_0 + 2\pi \bar{c}_0 / \nu \right] \frac{\partial \sigma}{\partial \zeta} - \beta_r \bar{c}_0 \frac{\partial^3 \sigma}{\partial \zeta^3} - \nu \frac{\partial U}{\partial \zeta} \quad (13)
\]

has the form of a driven classical KdV equation [70, 71] whose coefficients only involve macroscopic parameters such as the linear speed \( v_0 = R_{cl} \Omega \) determined by the transverse response to the inward trapping force at the cloud edge, \( v_0 \sim - \partial_r V_{\text{conf}}(r) |_{R_{cl}} \), the confinement potential curvature \( \bar{c}_0 \sim \partial^2 V_{\text{conf}}(r) |_{R_{cl}} \), namely the radial gradient of the trapping force; the FQH filling \( \nu \).

As one can see in the time evolution of the spatial Fourier components of the density shown in Fig 4(b,c), the semiclassical equation accurately reproduces the numerical evolution up to relatively long times, where the forward-bending due to the density dependent speed of sound is well visible. At later times, the broadening of the DSF discussed above starts to play a dominant role, giving rise to the collapse and revival features visible in the plots.

### V. NON-LINEAR CHIRAL LUTTINGER LIQUID THEORY

In order to properly capture these last features, quantum effects must be included in the theoretical description. In this perspective, the semiclassical evolution (13) can be seen as the classical limit of the Heisenberg equation for the density operator of a \( \chi LL \) supplemented with a group velocity dispersion term and a forward-scattering non-linearity. This suggests the following form for the lowest non-universal corrections to the quantum
\( \chi_{LL} \) Hamiltonian for our FQH fluid,

\[
\hat{H}_{\chi_{LL}}^N = \int d\zeta \left[ \frac{\pi v_0}{\nu} \sigma^2 - \frac{\pi \beta_0 \sigma_0}{\nu} \left( \frac{\partial \sigma}{\partial \zeta} \right)^2 + \frac{2\pi^2 \sigma_0}{3\nu^2} \sigma^3 + U(\zeta, t) \sigma \right]
\]

where the density operator of the chiral edge mode obeys the usual \( \chi_{LL} \) commutation rules \[\chi_{LL}\] and (14).

The nonlinear \( \chi_{LL} \) Hamiltonian (14) can be diagonalized by expanding the operators on the underlying bosonic modes with much less effort than the full many-body problem. The excellent agreement between this procedure and the diagonalization of the full microscopic Hamiltonian is visible in the eigenenergy spectrum shown in Fig. 4(b,c). A similarly successful agreement is shown in Fig. 2(b) for the DSF and for the complete time evolution in Fig. 1(d); analogous agreement is shown in \( E_\text{LL} \) for additional observables. All together, these results strongly support the quantitatively predictive power of the nonlinear \( \chi_{LL} \) model.

VI. EXPERIMENTAL OBSERVABILITY

We conclude the work with a brief discussion of the actual relevance of our predictions in view of experiments with synthetic quantum matter systems, in particular trapped atomic gases for which an artillery of experimental tools is already available.

As several strategies to induce synthetic magnetic fields are nowadays well established, from rotating traps to combinations of optical and magnetic fields, the open challenge is to reach sufficiently low atomic filling factors and sufficiently low temperatures to penetrate the fractional quantum Hall regime: an intense work is being devoted to this issue from both the theoretical and experimental sides, and promising preliminary observations have appeared in the literature. Once the desired many-body state is generated, arbitrary confinement potentials can be generated with optical techniques and the response to rotating potentials of the form can be measured via the same tools used, e.g., to study surface excitations of rotating superfluid clouds.

Most remarkably, we have shown in Fig. 2 that this measurement provides a precise measurement of the transverse conductivity already for moderate cloud sizes \( N \sim 10 \). This suggests that a smoking gun of the topological nature of the many-body state can be obtained in strongly correlated atomic clouds with realistic sizes trapped in fast rotating potentials.

While transverse conductivity features are independent of the shape of the confinement potential, both the group velocity dispersion and the nonlinear effects crucially depend on the trap anharmonicity that also helps stabilizing the cloud at large rotation speeds close to the centrifugal limit. A rough estimate of the maximum potential curvature \( \sigma_0 \) that the FQH liquid can stand before being significantly affected is set by the many-body gap over the squared magnetic length. Since both the group velocity dispersion and the nonlinearity terms in the scale proportionally to the curvature \( \sigma_0 \) and the chiral dynamics factors out as a rigid translation at \( v_0 \), such an upper bound on \( \sigma_0 \) does not impose any restriction on the observability of interesting effects due to their interplay. It only requires that the dynamics is followed on a temporal scale much longer than the inverse many-body gap, a condition which is anyway automatically enforced upon working with a correlated many-body state.

VII. CONCLUSIONS

In this work we have reported a numerical study of the linear and nonlinear edge dynamics of a fractional quantum Hall cloud of macroscopic size. In addition to highlighting effects of direct experimental interest to characterize fractional quantum Hall fluids both in atomic or photonic synthetic matter and in electronic systems, our numerical results suggest and quantitatively validate an easily tractable formalism based on a nonlinear \( \chi_{LL} \) Hamiltonian. Once supplemented with tunneling processes between FQH edges, this formalism holds great promise in view of using FQH fluids as a novel platform for nonlinear quantum optics of EE with exotic statistics.

Acknowledgments

We acknowledge financial support from the H2020-FETFLAG-2018-2020 project “PhoQuS” (n.820392). IC acknowledges financial support from the Provincia Autonoma di Trento and from the Q@TN initiative. Continuous discussions with Elia Macaluso, Zeno Bacciconi and Daniele De Bernardis are warmly acknowledged.

Appendix A: Statistics of the sampling

In order to estimate the statistical error of the Monte Carlo sampling, we performed some statistical analysis on the numerical data. In particular, we split the calculations of our observables into \( M = 250 \) groups for the same droplet configuration. The obtained results are treated as a population of which we studied the statistics.
FIG. 5: (a) Eigenenergy spectrum (with errorbars) for a \( N = 25, \nu = 1/2 \) FQH cloud confined by a \( \delta = 4 \) quartic potential. (b) Magnified view on the statistical errors on the eigenenergies. The panels on the right show histograms for the \( M = 250 \) Monte Carlo realizations of the energy spectrum in each \( l \)-sector. Each point is obtained by an independent run.

The average energies

\[
E_{t,n} = \frac{1}{M} \sum_{i=1}^{M} e_{t,n}[i] \quad \text{(A1)}
\]

discussed later. Since these latter are very small and almost invisible on panel (a), we have replotted them separately in panel (b). Histograms of the \( M = 250 \) samples for the eigenstate energies at a few values of \( l \) are shown in the right panels.

The same analysis has been repeated for the DSF; the results for the DSF weights are shown in Fig. 6(a). Again, the error bars are too small to be seen by eye on that scale. Histograms of the \( M = 250 \) samples for a few \( l \) components of the DSF are shown in the right panels. Error propagation then yields small but sizeable errorbars on the central frequency \( \omega_l \), in particular at \( l = 1 \), as shown in panel (b).

Appendix B: Excitations with a radial dependence

Since these latter are very small and almost invisible on panel (a), we have replotted them separately in panel (b). Histograms of the \( M = 250 \) samples for the eigenstate energies at a few values of \( l \) are shown in the right panels.

The same analysis has been repeated for the DSF; the results for the DSF weights are shown in Fig. 6(a). Again, the error bars are too small to be seen by eye on that scale. Histograms of the \( M = 250 \) samples for a few \( l \) components of the DSF are shown in the right panels. Error propagation then yields small but sizeable errorbars on the central frequency \( \omega_l \), in particular at \( l = 1 \), as shown in panel (b).

FIG. 6: (a) DSF weights \(|\langle 0 | \delta \hat{P}_{l,n} | 0 \rangle|^2\) (with errorbars) for a \( N = 25, \nu = 1/2 \) FQH cloud confined by a \( \delta = 4 \) quartic potential. (b) Suitably normalized first moment \( \omega_l \) of the DSF (with errorbars). The panels on the right show histograms for the \( M = 250 \) Monte Carlo realizations of the DSF weights in each \( l \)-sector.

The average energies

\[
E_{t,n} = \frac{1}{M} \sum_{i=1}^{M} e_{t,n}[i] \quad \text{(A1)}
\]

are shown in Fig. 6(a) with their standard errors

\[
\sigma(E_{t,n}) = \left( \frac{1}{M(M-1)} \sum_{i=1}^{M} (e_{t,n}[i] - E_{t,n})^2 \right)^{1/2} \quad \text{(A2)}
\]

The picture presented in the main text remains valid under reasonable approximations even when the externally applied excitation depends on the radial coordinate. The external potential couples to the density (apart for a time-dependent additive constant which is anyway irrelevant for the dynamics) via

\[
\hat{V}(t) = \int U(\mathbf{r}; t) \delta \hat{\rho}(\mathbf{r}) \, d^2\mathbf{r}. \quad \text{(B1)}
\]

For edge excitations, the support of the density variation \( \delta \hat{\rho}(\mathbf{r}) \) is exponentially localized near the edge, \( r \approx R_e \); if the excitation is constant over the width of the edge

\[
\hat{V}(t) = \int U(\mathbf{r}; t) \delta \hat{\rho}(\mathbf{r}) \, d^2\mathbf{r}. \quad \text{(B1)}
\]
mode, we can approximate
\[ \dot{U}(t) \simeq \int U(R_{cl}, \theta; t) \left( \int \delta \dot{\rho}(r) \, r \, dr \right) d\theta = \int U(R_{cl}, \theta; t) \delta \dot{\rho}(\theta) \, d\theta, \]  

(B2)

which indeed yields a minimal coupling between the edge density variation and an effectively azimuthal excitation. For this formula to remain valid for a radially-dependent potential, we can expect that the potential \( U \) to reach the bulk on one side and overlap with the whole edge on the other side. This condition is needed for the quantized transverse Hall current to flow from the bulk towards the edge during the excitation time, so that the edge density variation is proportional to the macroscopic bulk transverse conductivity set by the filling fraction.

To validate this physical picture, we compare the calculations presented in the main text for a radially constant transverse conductivity set by the filling fraction. For this formula to remain valid for a radially-dependent density variation and an effectively azimuthal excitation, which indeed yields a minimal coupling between the edge density (bottom panels). The fundamental coupling (B2) is a good one, especially at small \( l \), so the simpler form (B2) is an accurate effective description also for the more general coupling (B1).

Appendix C: Linear response within the \( \chi \)LL theory

The key observable we consider is the edge density variation defined as
\[ \delta \dot{\rho}(\theta) = \int_0^\infty \left( \hat{\psi}^\dagger(r) \hat{\psi}(r) - \langle \hat{\psi}^\dagger(r) \hat{\psi}(r) \rangle \right) r \, dr \]  

(C1)

where the bra-kets denote the expectation value on the ground state and \( \hat{\psi}^\dagger(r) \) is the particle-creation operator at position \( r \).

Within linear response theory, the edge density variation induced by the external perturbing potential of the form (B2) reads
\[ \langle \delta \dot{\rho}(\theta, t) \rangle = -i \left\langle \left[ \delta \dot{\rho}(\theta, t), \int_{-\infty}^t \dot{V}(t') dt' \right] \right\rangle \]  

(C2)

where the system is assumed to be initially in its ground state at \( t \to -\infty \), higher order terms \( O(U^2) \) have been neglected and the tilde indicate interaction picture with respect to the unperturbed \( U = 0 \) Hamiltonian.

With straightforward algebra, the above formula can be rewritten as
\[ \langle \delta \dot{\rho}(\theta, t) \rangle = 2 \Re \int_{-\infty}^t dt' \int d\theta' U(\theta', t') \langle \delta \dot{\rho}(\theta') e^{-i(H-E_0)(t-t')} \delta \dot{\rho}(\theta) \rangle. \]  

(C3)

Introducing the Fourier transforms
\[ \left\{ \delta \dot{\rho}(\theta) = \frac{1}{2\pi} \sum_{l \neq 0} e^{il\theta} \delta \dot{\rho}_l \right\} \]  

(C4)

this can be reformulated as
\[ \langle \delta \dot{\rho}(\theta, t) \rangle = \int_{-\infty}^t dt' \frac{1}{(2\pi)^2} \sum_{l \neq 0} e^{il\theta} U_l(t) \mathcal{C}_l(t-t') \]  

(C5)

where the rotational invariance of the ground state has been used to remove a summation, \( \mathcal{C}_{l\nu} = \mathcal{C}_l \delta_{l\nu} \) with
\[ \mathcal{C}_l(t) = \left\langle \delta \dot{\rho}_l e^{-i(H-E_0)t} \delta \dot{\rho}_{-l} \right\rangle. \]  

(C6)

If we are interested in the late time dynamics of the system once the perturbation pulse has gone (\( U_l(t) \to 0 \) for late times), we can replace the upper boundary of the time integral with \( t \to \infty \), use the convolution theorem and write
\[ \langle \delta \dot{\rho}(\theta, t) \rangle = \frac{1}{\pi} \Re \left[ \sum_l e^{il\theta} \int U_l(\omega) S_l(\omega) e^{-i\omega t} \, d\omega \right] \]  

(C7)

where
\[ U_l(\omega) = \int \frac{dt}{2\pi} e^{i\omega t} U_l(t) \]  

(C8)

\[ S_l(\omega) = \int \frac{dt}{2\pi} e^{i\omega t} \mathcal{C}_l(t). \]  

(C9)

Combining (C9) with (C6) allows to recover the edge dynamic structure factor. As long as the confinement and perturbing potentials are weak enough not to excite states above the many-body gap, we can introduce a projector onto these states only and rewrite
\[ S_l(\omega) = \sum_n \left[ \delta (\omega - \omega_{l,n}) \right] |\langle 0 | \delta \dot{\rho}_l | l, n \rangle|^2 \]  

(C10)

where \( |0\rangle \) is the Laughlin ground state and \( \omega_{l,n} = E_{l,n} - E_0 \) the excitation energy of state \( |l, n\rangle \) with respect to the ground state.

Integrating over the frequencies in (C10) (restriction to energies below the many-body gap is automatically
enforced by the projector onto the low-energy subspace) one obtains the edge static structure factor

$$S_l = \sum_n |\langle 0 | \delta \rho_l | l, n \rangle|^2$$  \hspace{1cm} (C11)

which is invariant under a deformation of the many-body Hamiltonian as long as the gap is not closed, so that a unitary transformation between the “new” eigenstates $| l, n \rangle$ and the “old” ones $| l, n \rangle$ is well defined. Hence, in the long wavelength/low energy limit the edge static structure factor maintains its $\chi_{LL}$ value, namely $S_l = \nu l$ when $l \geq 0$ and 0 otherwise, reflecting the chirality of the system.

Assuming a narrowly peaked DSF at $\omega \approx \omega_l$ and including the $\chi_{LL}$ form of $S_l$, we can approximate [C7] as

$$\langle \delta \rho(\theta, t) \rangle = -\frac{\nu}{\pi} \frac{\partial}{\partial \theta} \sum_{l>0} \Re \left[ e^{i(\theta - \omega t)} \tilde{U}_l(\omega_l) \right] :$$  \hspace{1cm} (C12)

this formula explicitly displays the proportionality of the edge response to the FQH filling factor and is the key of our proposed measurement scheme of the transverse conductivity. Of course, this formula is only valid up to not-too-large times, namely as long as the DSF broadening is not resolved, $\Delta E_l t \ll 1$.

Note finally that the solution of the semiclassical equation introduced in the main text [Eq.(1) there] perfectly matches this result as long as the nonlinear velocity term can be neglected.

**Appendix D: Broadening of the dynamical structure factor of edge modes**

When the cloud is non-harmonically confined with $\delta \neq 2$, we have seen in the main text that the DSF broadens within a finite frequency window, whose width can be easily estimated by looking at the difference $\Delta E_l$ between the largest and smallest energies in a given angular momentum $l$ sector. The corresponding states have in fact a non-vanishing DSF weight $| \langle 0 | \delta \rho_l | l, n \rangle |^2$ and these energies thus correspond to the thresholds of the DSF.

In close analogy to to the IQH, we expect the DSF to broaden $\propto c l^2$. Here we verify this scaling. In particular, data in Fig.11 suggest the following simple form

$$\Delta E_l = \mu_m c \frac{l(l-1)}{2}.$$  \hspace{1cm} (D1)

The proportionality $c \propto \frac{\delta^{l-4}}{\nu}$ is visible from the $N$ dependence in each $l$ sector. Since all data have been normalized by $\Delta E_{l=2}$ (at a fixed number of particles, $N = 10$), the proportionality to $l(l-1)/2$ can be instead read out by looking at the first point on $y$-axis. Notice that, apart for the $m$-dependent proportionality factor, the result in [D1] is exactly the same as in the IQH case, where the lower (upper) threshold corresponds to a particle (hole) created just above (below) the Fermi surface.

**Appendix E: Quantitative comparison between the microscopic dynamics and the non-linear $\chi_{LL}$ model Hamiltonian**

To further support the nonlinear $\chi_{LL}$ model, the numerically calculated microscopic time-evolution was compared with the results of the nonlinear $\chi_{LL}$ model for different observables. To this purpose, the free param-
eters of the model have been determined according to the scaling formulas discussed in the text, without any additional fine-tuning. In particular, for a $\delta = 4$ quartic confinement the angular velocity of the edge modes is set by $\Omega = 4\lambda R_l^2$; we have a size-independent curvature $c = 8\lambda$ which determines both the cubic phonon dispersion shift coefficient $\alpha = \pi\lambda/R_{cl}$ and the strength of the nonlinearity.

The time-evolution of the spatial Fourier transform of the edge density variation calculated by the full numerics and by the $\chi LL$ model are compared in Fig. 9. A very good agreement can be seen, which gets slightly worse at larger angular momenta $l$: this small deviation may be caused by a higher-order correction of the phonon dispersion (beyond the cubic term considered here) and by the increasing difficulty in accurately sampling the matrix elements of the perturbation Hamiltonian between higher-$l$ subspaces. Note that the cubic correction to the phonon dispersion is essential to correctly capture the late-time dynamics, in particular of the harmonic components at $2l$ and $3l$ (see yellow triangles in Fig. 9). Of course, the nonlinear terms are even more essential, as they are responsible for the very appearance of a finite amplitude in the harmonic components.

[1] J. K. Jain, “Composite Fermions”, Cambridge University Press. doi:10.1017/CBO9780511607561
[2] D. Tong, “Lectures on the Quantum Hall Effect” (2016), available as arXiv: 1606.06687.
[3] T. Ozawa, H. M. Price, Topological quantum matter in synthetic dimensions, Nature Reviews Physics 1, 349 (2019).
[4] N. Cooper, Adv. Phys. 57, 539 (2008).
[5] N. Goldman, G. Juzeliunas, P. Öhberg and I. B. Spielman, Light-induced gauge fields for ultracold atoms, Rep. Prog. Phys. 77, 126401 (2014).
[6] I. Bloch, J. Dalibard, W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[7] N. R. Cooper, J. Dalibard, I. B. Spielman, Rev. Mod. Phys. 91, 015005 (2019).
[8] I. Carusotto and C. Ciuti, Quantum fluids of light, Rev. Mod. Phys. 85, 299 (2013).
[9] T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, Topological photonics, Rev. Mod. Phys. 91, 015006 (2019).
[10] I. Carusotto, A. A. Houck, A. J. Kollár, P. Roushan, D. I. Schuster, J. Simon, Photonic materials in circuit quantum electrodynamics, Nature Physics, 16, 268-279 (2020).
[11] B. Paredes, P. Fedichev, J. I. Cirac, and P. Zoller, 1/2-Anysons in Small Atomic Bose-Einstein Condensates, Phys. Rev. Lett. 87, 010402 (2001).
[12] N. R. Cooper, Steven H. Simon, “Signatures of Fractional Exclusion Statistics in the Spectroscopy of Quantum Hall Droplets” Phys. Rev. Lett. 114, 106802 (2015).
[13] R. O. Umucalılar, I. Carusotto, Quantum Hall Fractions in Rotating Bose-Einstein Condensates, Phys. Rev. Lett. 91, 030402 (2003).
[14] M. Roncaglia, M. Rizzi, J. Dalibard, Sci. Rep. 1, 43 (2011).
[15] N. R. Cooper, J. Dalibard, Phys. Rev. Lett. 110, 185301 (2013).
[16] N. Rougerie, S. Serfaty, and J. Yngvason, Quantum Hall states of bosons in rotating anharmonic traps, Phys. Rev. A 87, 023618 (2013).
[17] A. G. Morris, D. L. Feder, “Gaussian Potentials Facilitate Access to Quantum Hall States in Rotating Bose Gases”, Phys. Rev. Lett. 99, 240401 (2007).
[18] N. Goldman, J. Dalibard, A. Dauphin, F. Gerbier, M. Lewenstein, P. Zoller, I. B. Spielman, PNAS 110, 6736 (2013).
[19] C. Repellin, N. Goldman, Detecting fractional Chern insulators through circular dichroism, Phys. Rev. Lett. 122, 166801 (2019).
[20] R. O. Umucalılar, E. Macaluso, T. Comparin, and I. Carusotto, Time-of-Flight Measurements as a Possible Method to Observe Anyonic Statistics, Phys. Rev. Lett. 120, 230403 (2018).
[21] E. Macaluso, T. Comparin, R. O. Umucalılar, M. Gerster, S. Montangero, M. Rizzi, and I. Carusotto, Charge and statistics of lattice quasiholes from density measurements: A tree tensor network study, Phys. Rev. Research 2, 013145 (2020).
[22] R.O. Umucalılar, M. Wouters, I. Carusotto, Probing few-particle Laughlin states of photons via correlation measurements, Phys. Rev. A 89 (2014).
[23] R.O. Umucalılar, I. Carusotto, Generation and spectroscopic signatures of a fractional quantum Hall liquid of photons in an incoherently pumped optical cavity, Phys. Rev. A 95, 053808 (2017).
[24] A. Muñoz de las Heras, E. Macaluso, I. Carusotto, Anyonic molecules in atomic fractional quantum Hall liquids: a quantitative probe of fractional charge and anyonic statistics, Physical Review X 10, 041058 (2020).
[25] M. Racianu, F. N. Unal, E. Anisimovas, and A. Eckardt, Creating, probing, and manipulating fractionally charged excitations of fractional Chern insulators in optical lattices, Phys. Rev. A 98, 063621 (2018).
[26] A. Stern, Ann. Phys. 323, 1, 204-249 (2008).
[27] N. Gemelke, E. Sarajlic, S. Chu, Rotating few-body atomic systems in the fractional quantum Hall regime, arXiv:1007.2677.
[28] M. E. Tai, A. Lukin, M. Rispoli, R. Schittko, T. Menke, D. Borgnia, P. M. Preiss, F. Grusdt, A. M. Kaufman, and M. Greiner, Microscopy of the interacting Harper-Hofstadter model in the two-body limit, Nature 546, 519 (2017).
[29] J. Léonard, J. Kwan, et al., private communication (2022).
[30] P. Roushan, C. Neill, A. Megrant, Y. Chen, R. Babbush, R. Barends, B. Campbell, Z. Chen, B. Chiaro, A. Dunsworth, A. Fowler, E. Jeffrey, J. Kelly, E. Lucero, J. Mutus, P. J. J. O’Malley, M. Neeley, C. Quintana, D. Sank, A. Vainsencher, J. Wenner, T. White, E. Kapit, H. Neven, and J. Martinis, Chiral ground-state currents of interacting photons in a synthetic magnetic field, Nat.
