The zeros of the QCD partition function

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Abstract

We establish a relationship between the zeros of the partition function in the complex mass plane and the spectral properties of the Dirac operator in QCD. This relation is derived within the context of chiral Random Matrix Theory and applies to QCD when chiral symmetry is spontaneously broken. Further, we introduce and examine the concept of normal modes in chiral spectra. Using this formalism we study the consequences of a finite Thouless energy for the zeros of the partition function. This leads to the demonstration that certain features of the QCD partition function are universal.
1 Introduction

In recent years the spectral correlators of the Dirac operator in QCD have been the object of intense study using both numerical and analytic means. These correlators contain valuable information regarding both the chiral properties of the QCD vacuum and the topological structure of the gauge fields. The relation to the chiral properties of the QCD vacuum was established by Banks and Casher [1]: The eigenvalue density of the QCD Dirac operator at eigenvalue zero is proportional to the chiral condensate and is therefore an appropriate order parameter for chiral symmetry. As a complement to the Banks-Casher relation, one has the Yang-Lee picture [2] of a phase transition. In the attempt to analyze phase transitions in statistical spin models Lee and Yang [2] introduced the concept of the zeros of the finite volume partition function in the thermodynamic limit. The volume dependence of these zeros allows finite size scaling studies and subsequent identification of universality classes. In the case of chiral symmetry, one focuses on the zeros of the partition function in the complex quark mass plane. If these zeros pinch the real axis and exhibit a constant density, a discontinuity in the partition function arises at the pinch, and chiral symmetry is spontaneously broken. The mass zeros of the partition function and the low lying eigenvalues of the Dirac operator thus contain similar information about the chiral phase transition. The relation between the two is, however, involved. The partition function and its zeros are obtained by averaging over all gauge field configurations. By contrast, the eigenvalues of the Dirac operator are given for each gauge field configuration, and only a density of eigenvalues is well-defined after averaging.

The challenge of deriving relations between the zeros of the partition function and the eigenvalues of the Dirac operator was first taken up by Leutwyler and Smilga [3]. They studied QCD in a Euclidean 4-dimensional box with side length \(L\) subject to the constraint that

\[
\frac{1}{\Lambda} \ll L \ll \frac{1}{m_\pi} .
\]

Here \(m_\pi\) is the pion mass and \(\Lambda\) is the typical QCD scale. They computed the QCD partition function for equal quark masses using the effective chiral Lagrangian and found that quark masses enter only in the rescaled combination \(m L^4 \Sigma\), where \(\Sigma\) is the chiral condensate in the chiral limit. They further observed that the partition function zeros could be thought of as average positions of the eigenvalues. While highly suggestive, these results were not completely quantitative. The situation has changed dramatically since then. The main breakthrough came with the introduction [4] of random matrix concepts in QCD which permits the study of the correlations of the eigenvalues of matrices drawn on a general weight constructed to ensure the chiral structure of each eigenvalue spectrum. The relation of random matrix theory to QCD in the limit (1) has been established through a number of universality studies [5], lattice QCD simulations [6,7], and direct calculations using the effective chiral Lagrangian [8]. (For a review of random matrix theory in QCD see [9].) In terms of the spectral correlation functions, the universal limit in which QCD and chiral random matrix theory (\(\chi\)RMT) coincide is the limit

\[
N \to \infty , \quad \lambda \to 0 , \quad m \to 0 ,
\]

in which the microscopic variables

\[
\zeta \equiv 2 N \lambda , \quad \mu \equiv 2 N m
\]

are kept fixed and \(N\) is identified as the dimensionless volume. (Here, \(\lambda\) denotes an eigenvalue of the Dirac operator and \(m\) is the dimensionless quark mass parameter.) The determination of the individual eigenvalue distributions and their most important correlators now permits direct comparison of partition function zeros and eigenvalue positions. The suggestion of Leutwyler and Smilga is remarkably accurate. The zeros and the average positions of the eigenvalues are intimately connected.
Here we shall demonstrate that this relationship can be understood as a fundamental property of the chiral ensembles. We show that the zeros are uniquely trapped by the maxima of the joint eigenvalue distribution function. This trapping appears on all scales and is thus relevant for any finite $N$ as well as in the large $N$ limit. To obtain a better understanding of the relation between the maxima of the joint eigenvalue distribution function and the zeros of the partition function, we introduce and determine the spectral “normal modes” of the chiral unitary ensemble. This provides us with a simple tool to describe the fluctuations of the eigenvalues about the maximum of the joint eigenvalue distribution.

As suggested in [11], chiral random matrix theory is not expected to describe all aspects of QCD. Only correlations below a certain energy length are expected to be in agreement with chiral random matrix theory [10]. In solid state physics, this energy is denoted as the Thouless energy. Recently [11] it was realized that the effects of a finite Thouless energy can be studied naturally using the language of spectral normal modes [12]. We thus perform a normal mode analyses of the chiral ensemble to formulate and establish certain universal features of the partition function zeros. This argument is independent of standard proofs of universality, and its general nature can shed some light on the way universality is realized.

In section 2 we show that for $N_f = 1$ the zeros of the partition function of $\chi$RMT are trapped between the maximum positions of the joint distribution function. This result holds for all scales. In section 3 we derive the normal modes of the chiral unitary ensembles and find that they are Chebyshev polynomials in the large $N$ limit. We discuss the effects of the Thouless energy in section 4. In section 5 we make the connection to the familiar microscopic spectral density. Our conclusions are contained in section 6.

### 2 Zeros of the partition function in $\chi$RMT

#### 2.1 Chiral random matrix theory

We start with the partition function of chiral random matrix theory ($\chi$RMT) for $N_f$ flavors, which is given by [4, 13]

$$ Z_{N,\beta}^{N_f,\nu}(\{m_f\}) = \int DW \prod_{f=1}^{N_f} \det(D + m_f) \exp \left[ -\frac{N \beta \Sigma^2}{2} \text{Tr}(W^\dagger W) \right] , \quad (4) $$

where $\beta$ denotes the Dyson index and $DW$ is the Haar measure over the Gaussian distributed random matrices $W$. $D$ is the analogue of the Dirac operator which has the chiral structure

$$ D = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} . \quad (5) $$

Here $W$ is a $N \times M$ matrix with $\nu = |N - M|$ playing the role of the topological charge. Without loss of generality we assume $\nu$ to be positive. The chiral condensate in the chiral limit, $\Sigma$, is related to the eigenvalue density of $D$, $\rho(\lambda)$, via the Banks–Casher relation [1]

$$ \Sigma = \lim_{\lambda \to 0} \lim_{m_f \to 0} \lim_{N \to \infty} \frac{\pi \rho(\lambda)}{N} . \quad (6) $$

The partition function is invariant under transformations $W \to U^\dagger W V$, where $U$ is a $N \times N$ matrix and $V$ a $M \times M$ matrix. Following the diagonalization $W = U^\dagger \Lambda V$, the partition function can be
expressed in terms of the eigenvalues of \( W \) (with \( \Sigma \equiv 1 \)),

\[
Z_{N_f,\beta}^{N_f,\nu}(\{m_f\}) = \left( \prod_{f=1}^{N_f} m_f^{\nu_f} \right) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{N} d\lambda_k \prod_{f=1}^{N_f} \left( \lambda_k^2 + m_f^2 \right) \lambda_k^{\beta\nu + \beta - 1} e^{-\frac{N_f}{2} \lambda_k^2} \Delta(\lambda^2)^\beta. \tag{7}
\]

The Vandermonde determinant, \( \Delta(\lambda^2) \), which is the non-trivial Jacobian of the transformation from the matrices to the eigenvalues, has the form

\[
\Delta(\lambda^2) = \prod_{k < l} (\lambda_k^2 - \lambda_l^2). \tag{8}
\]

The partition function (7) can now be written as an integral over the joint probability density \( P_{N_f,\beta}^{N_f,\nu}(\lambda_1, \ldots, \lambda_N; \{m_f\}) \) as

\[
Z_{N_f,\beta}^{N_f,\nu}(\{m_f\}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{N} d\lambda_k P_{N_f,\beta}^{N_f,\nu}(\lambda_1, \ldots, \lambda_N; \{m_f\}) \tag{9}
\]

with

\[
P_{N_f,\beta}^{N_f,\nu}(\lambda_1, \ldots, \lambda_N; \{m_f\}) = \left( \prod_{f=1}^{N_f} m_f^{\nu_f} \right) \prod_{k=1}^{N} \prod_{f=1}^{N_f} \left( \lambda_k^2 + m_f^2 \right) \lambda_k^{\beta\nu + \beta - 1} e^{-\frac{N_f}{2} \lambda_k^2} \Delta(\lambda^2)^\beta. \tag{10}
\]

Unlike real QCD, \( \chi \text{RMT} \) has the special feature that the partition function can be expressed in terms of the eigenvalues of the Dirac operator. This enables us to derive a number of statements regarding the zeros of the partition function. We now focus on the case \( \beta = 2 \) — the universality class of QCD with 3 colours and quarks in the fundamental representation of the gauge group. (The choice \( \beta = 1 \) corresponds to QCD with two colours in the fundamental representation; \( \beta = 4 \) describes QCD with any number of flavours and quarks in the adjoint representation of the gauge group.)

Eq. (10) expresses an evident duality between flavor and topology: The joint probability density for \( N_v \) massless flavours and \( N_f \) massive flavours depends only on \( \nu + N_v \). This relation was proven for the QCD partition function independent of \( \chi \text{RMT} \) in [14].

We now wish to determine the maximum of the joint probability distribution. This will allow us to put a tight bound on the zeros of the partition function. The chiral normal modes, to be discussed in section 3, describe fluctuations about the maximum of the joint probability distribution.

### 2.2 Extremum of the joint probability distribution

In order to determine the maximum of the joint eigenvalue probability distribution, we consider variations of \( \log P_{N_f,\beta}^{N_f,\nu} \) with respect to the eigenvalues. (We assume the eigenvalues to be ordered with \( \lambda_i < \lambda_{i+1} \).) We introduce the coordinates \( y_i = \lambda_i^2 \) and evaluate the equations

\[
\frac{\partial \log P_{N_f,\beta}^{N_f,\nu}}{\partial y_i} = 0. \tag{11}
\]

For \( N_f = 0 \) and topological sector \( \nu \) this yields

\[
\left( \nu + \frac{1}{2} \right) \frac{1}{N} y_i - 1 + \frac{1}{N} \sum_{j \neq i} \frac{2}{y_i - y_j} = 0. \tag{12}
\]
We now choose to focus on the quenched, i.e. $N_f = 0$, joint eigenvalue probability distribution. The solution to this equation reveals that the maximum of $\log P_{N_f=0,\nu}^{N_f}$ is obtained for

$$L_N^{\nu-1/2} \left( N \lambda_i^2 \right) = 0 ,$$

where $L_N^{\alpha}$ denotes the generalized Laguerre polynomials. This result follows from the observation that Laguerre’s differential equation,

$$z L_N^{\alpha}(z)'' + (\alpha + 1 - z)L_N^{\alpha}(z)' + NL_N^{\alpha}(z) = 0 ,$$

reduces to $L_N^{\nu-1/2}$ at the zeros of $L_N^{\nu-1/2}$. (The proof follows from considerations similar to those made in appendix A6 in [15]). For $\nu = 0$ we can use the fact that

$$L_N^{-1/2}(z^2) = (-1)^n n! 2^n H_{2n}(z)$$

to see that the density of eigenvalues in the $N \to \infty$ limit is precisely that of the usual Gaussian ensembles, i.e. a semicircle with support $-2 \leq \lambda \leq +2$. The partition function for $\beta = 2$ is the average of a product of fermionic determinants over $P_{N_f=0,\nu}^{N_f}$

$$Z_{N_f,\nu=0}^{N_f}(\{m_f\}) = \left\langle \prod_{f=1}^{N_f} m_f^{\nu} \prod_{i=1}^{N} (\lambda_i^2 + m_f^2) \right\rangle .$$

This can be readily evaluated using orthogonal polynomials, and the result for $\nu = 0$ agrees with the one presented in [16]

$$Z_{N_f,\nu=0}^{N_f}(\{m_f\}) = \prod_{f=1}^{N_f} \frac{(N + f - 1)!}{N^{N+f-1}} \frac{C_{N_f}^{N_f}(\{m_f\})}{\Delta_{N_f}(\{-m_f^2\})} ,$$

where

$$C_{N_f}^{N_f} = \text{det} \left[ L_{N+f-1}^{(0)} \begin{pmatrix} -N m_f^2 \end{pmatrix} \right]_{f,f'=1,...,N_f} ,$$

and $\Delta_{N_f}(\{-m_f^2\})$ is the Vandermonde determinant with the negative square of the $N_f$ masses as arguments. For the special case of $N_f = 1$ this yields the result

$$Z_{N=1,\nu=0}^{N}(m) = \frac{N!}{N^{N}} L_{N}^{(0)} \left( -N m^2 \right) ,$$

which is now ready for investigation. The expression [16] coincides (up to a constant) with the series expansion for the partition function derived in [17]. From the expression found there we find that

$$Z_{N=1,\nu}^{N}(m) \sim m^{|\nu|} L_{N}^{(\nu)} \left( -N m^2 \right) .$$

### 2.3 Trapping of the zeros

The closed forms given above allow us obtain information about the zeros of the partition from the spectral correlators. Specifically, we now show that the locations of the partition function zeros in $\chi$RMT are trapped by the maxima of the joint distribution function.

The Laguerre polynomials $L_N^{\alpha}(z)$ are polynomials orthogonal on the interval $[0,\infty]$ with weight function $w(z) = z^{\alpha} \exp(-z)$. They have three properties which are useful for our purpose:
\[ \nu = 0 \text{ is proportional to } L^0 \text{ eigenvalues are located at the zeros of } \ln \nu \text{.} \]

In the last section we saw that the massless joint distribution function has its maxima when the eigenvalues are located at the zeros of the Laguerre polynomials. In order to study the properties of fluctuations about this maximum, it is useful to make a Gaussian approximation to \( P_{N_f=0,\nu} \) which leads to the form

\[ \log P_{N_f=0,\nu} \approx \log P_{N_f=0,\nu} + \frac{1}{2} \delta \lambda_i C_{ij} \delta \lambda_j , \]

where \( i, j = 1, \ldots, N_f \) and \( C_{ij} \) is the covariance matrix.

3 Normal modes in \( \chi \text{RMT} \)

We have seen that the maximum of the massless joint distribution function is obtained when the eigenvalues are located at the zeros of the Laguerre polynomials. In order to study the properties of fluctuations about this maximum, it is useful to make a Gaussian approximation to \( P_{N_f=0,\nu} \) which leads to the form

\[ \log P_{N_f=0,\nu} \approx \log P_{N_f=0,\nu} + \frac{1}{2} \delta \lambda_i C_{ij} \delta \lambda_j , \]

where \( i, j = 1, \ldots, N_f \) and \( C_{ij} \) is the covariance matrix.
where $\delta \lambda_i$ is the position of the $i$th eigenvalue relative to $\bar{\lambda}_i$, its value at the collective maximum of $\log P_{N_f=0,\nu}^N$. The matrix $C$ is defined as

$$C_{ij} = \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} \log P_{N_f=0,\nu}^N,$$  \hspace{1cm} (24)

evaluated at the maximum. Concentrating again on the case $N_f = 0$, we find that the diagonal elements of $C$ are

$$C_{ii} = -2N - \frac{2\nu + 1}{\lambda_i^2} - 4 \sum_{j \neq i} \frac{(\lambda_i^2 + \lambda_j^2)}{(\lambda_i^2 - \lambda_j^2)^2},$$ \hspace{1cm} (25)

and that the off-diagonal elements are

$$C_{ij} = \frac{8\lambda_i \lambda_j}{(\lambda_i^2 - \lambda_j^2)^2}.$$ \hspace{1cm} (26)

We now consider the eigenvalue equation for the real symmetric matrix $C$:

$$\sum_{i=1}^{N} C_{ij} \phi_{j}^{(k)} = \omega_k \phi_{i}^{(k)},$$ \hspace{1cm} (27)

The eigenvectors, $\phi^{(k)}$, are the (normalized) normal modes of the $\chi$RMT spectrum. They describe the statistically independent correlated fluctuations of the eigenvalues of the random matrix about their most probable values. The normal modes provide an alternate description of the eigenvalues of any given random matrix since

$$\delta \lambda_i = \sum_{k=1}^{N} c_k \phi_i^{(k)} \text{ with } \sum_{i=1}^{N} \phi_i^{(k)} \phi_i^{(k')} = \delta_{kk'}.$$ \hspace{1cm} (28)

We can locate the eigenvalues by specifying either the $\delta \lambda_i$ or the amplitudes $c_k$ as convenient. The eigenvalues, $\omega_k$, provide a measure of the magnitude of these fluctuations.

The derivation of these eigenvalues and eigenvectors can be performed as in [12]. The resulting eigenvalues are

$$\omega_k = -4kN.$$ \hspace{1cm} (29)

As in [12], we find a linear dispersion relation valid for all $k$ and $N$. The linearity of (29) is a reflection of the well-known rigidity of random matrix spectra. Furthermore, this result is independent of $\nu$. Just as in the Gaussian case, the eigenvectors are found to be Chebyshev polynomials in the large $N$ limit (i.e., with corrections of order $1/N$):

$$\phi_i^{(k)} = \sqrt{\frac{2}{N}} U_{2k-1} \left( \frac{\lambda_i}{2} \right).$$ \hspace{1cm} (30)

The normalization of the eigenvectors is

$$\int dx \rho(x) \phi^{(k)}(x) \phi^{(l)}(x) = \delta_{kl},$$ \hspace{1cm} (31)

where $\rho(x)$ is again the semicircle

$$\rho(x) = \frac{N}{\pi} \sqrt{4-x^2}.$$ \hspace{1cm} (32)
Note that only odd normal Chebyshev polynomials appear. This is a consequence of chiral symmetry, which ensures that all non-zero eigenvalues come in pairs \((-\lambda, \lambda)\). Equation (23) can now be written as

\[
\log \left( \frac{P_{Nf=0,\nu}^N}{P_{Nf=0,0}^N} \right) = \frac{1}{2} \sum_{k=1}^{N} |c_k|^2 \omega_k .
\]  

(33)

Following (28), the coefficients \(c_k\) are constructed as

\[
c_k = \sum_{i=1}^{N} \delta \lambda_i \phi_i^{(k)},
\]  

(34)

and are statistically independent. It is obvious from (33) that the mean square amplitude for the \(k\)th normal mode is

\[
\langle c_k^2 \rangle = \frac{1}{\omega_k} .
\]  

(35)

We can use the normal modes to construct a Gaussian approximation to the partition function as

\[
Z_{Nf=1,\nu}^N(m) = \int_0^\infty \prod_{k=1}^{N} dc_k \exp \left[ -\frac{1}{2} \sum_{k=1}^{N} |c_k|^2 \omega_k \right] (m^2 + (\hat{\lambda}_k + \delta \lambda_k)^2) ,
\]  

(36)

with the \(\hat{\lambda}_k\) given by (13) and \(\delta \lambda_k\) given by (28). The Gaussian approximation also permits a simple approximate calculation of the number variance. This calculation reveals that the familiar logarithmic behaviour of the number variance (i.e., the “spectral rigidity” of the random matrix ensembles) is a direct consequence of the linearity of the dispersion relation for all \(k\).

We have chosen to consider the normal modes for the case \(N_f = 0\) and \(\nu = 0\). This choice is somewhat arbitrary; it would be equally sensible to start with the \(N_f = 0\) and \(\nu = 1\) normal modes. We will employ this Gaussian approximation below to consider the sensitivity of the partition function zeros to the effects of a Thouless energy. Since the resulting shifts are small, this arbitrariness will be of no consequence.

4 Effects of a Thouless energy

Normal modes describe the correlated fluctuations of eigenvalues about their most probable values. As we have seen, the normal modes for \(\chi\)RMT obey a linear dispersion relation. By contrast, uncorrelated eigenvalues obey a quadratic dispersion relation, and the mean square amplitude of the lowest mode with \(k = 1\) is larger by a factor of \(N/k\) [12]. In QCD, it is expected that spectral correlations in a sufficiently small energy domain will follow the predictions of \(\chi\)RMT. On larger energy scales, spectral correlations die out. The characteristic energy which divides these regions is the Thouless energy, usually denoted by \(E_c\). In 4-dimensional QCD the Thouless energy is estimated to be [10]

\[
E_c/D \sim \sqrt{N} ,
\]  

(37)

where \(D\) is the mean level spacing. This behaviour has been verified in lattice studies [18]. The connection between the Thouless energy and the normal modes of the eigenvalue spectrum has been investigated in [11] for the case of sparse matrices. There it was found that “almost all” normal modes obey the linear dispersion relation discussed above with remarkable accuracy. Reflecting the presence of a Thouless energy, a small fraction (i.e., approximately \(1/\sqrt{N}\)) of long wave length modes are strongly enhanced. The mean square amplitude of the longest wave length mode with \(k = 1\) approaches the value appropriate for uncorrelated eigenvalues. Since such enhancement can be readily
incorporated in our Gaussian approximation to the partition function (36), normal modes provide us with a natural and convenient tool to study the effects of a Thouless energy on the zeros of the partition function. So far we have seen that the most probable eigenvalues are located at the zeros of the Laguerre polynomial \( L_{-1/2} \) and that the zeros of the partition function are given by the zeros of \( L_N^0 \). The Gaussian approximation (36) allows us to see how this result is modified by the presence of a Thouless energy. To mimic the effects of the Thouless energy, we shall enhance the long wavelength modes in the partition function.

Our aim is to demonstrate that the zeros of the partition function in the microscopic region remain virtually unaffected even if the enhancement of the soft modes is substantial. Since we are concerned only with the microscopic zeros, every long wavelength mode contributes to a “breathing” of the spectrum. In order to investigate the influence of the soft modes, it is sufficient to evaluate the strength (i.e., mean square amplitude) of this effective breathing.

For concreteness we start with \( \sqrt{N} \) longest wavelengths modes with fluctuations as given by the Gaussian approximation. Let \( \bar{\lambda}_i \) denote the values at the maximum of \( P_N^{f=0, \nu=0} \). The fluctuations introduced in the Gaussian approximation by the \( \sqrt{N} \) longest wavelengths modes are

\[
\bar{\lambda}_i \rightarrow \bar{\lambda}_i \left( 1 + \sum_{k=1}^{\sqrt{N}} \eta^{(k)} c_k \right) \equiv \bar{\lambda}_i s . \tag{38}
\]

From (39) we know that the normal modes are \( \sqrt{\frac{2}{N}} U_{2k-1} \left( \lambda_i / 2 \right) \). This gives for the linear coefficient

\[
\eta^{(k)} = \sqrt{\frac{2}{N}} k (-1)^{k+1} . \tag{39}
\]

Additionally we know from (26) that the normal modes are linearly independent. With this information the variance of \( s \) becomes in leading order of \( 1/N \)

\[
\langle s^2 \rangle - \langle s \rangle^2 = \langle s^2 \rangle - 1 \sim \sum_{k=1}^{\sqrt{N}} k^2 \frac{1}{N^2} \sum_{k=1}^{\sqrt{N}} k = \frac{1}{N^2} \sqrt{N} (\sqrt{N} + 1) \sim \frac{1}{N} . \tag{40}
\]

Here we used the facts that the linear terms vanish and that \( \langle c_k^2 \rangle = 1/|\omega_k| \) with (29) for the quadratic terms. The result is simply a \( O(1/N) \) correction.

In order to study the effects of a Thouless energy, we now enhance the mean square amplitudes of these \( \sqrt{N} \) soft modes by a factor of \( N/k^2 \). This factor provides a smooth interpolation from the behaviour of uncorrelated soft modes (for \( k = 1 \)) to that of \( \chi \) RMT (for \( k = \sqrt{N} \)). This interpolation is completely consistent with the results of [11]. We now find that

\[
\langle s^2 \rangle - \langle s \rangle^2 \sim \sum_{k=1}^{\sqrt{N}} k^2 \frac{1}{N^2} \frac{N}{k^2} \sim \frac{\log(N)}{2N} . \tag{41}
\]

The result still shows strong suppression in \( N \). The decision to single out \( \sqrt{N} \) soft modes for enhancement is not essential. We could declare any fraction of the long wavelength modes which vanishes in the \( N \to \infty \) limit as “soft”. A similar interpolation between the limits of uncorrelated and \( \chi \) RMT modes will always lead to a value of \( \langle s^2 \rangle - 1 \) which vanishes as \( N \to \infty \). In short, the effects of a Thouless energy are expected to have a negligible effect on \( \langle \lambda_i^2 \rangle \) over the entire microscopic spectrum.

The question is now how we can evaluate the effect on the zeros from the enhanced long wave length modes. To this end we introduce the distribution function of the fluctuations

\[
P_N(s) \equiv \frac{1}{N} s^7 e^{-\frac{s^2}{2 \sigma^2}} . \tag{42}
\]
and fix \( \gamma \) and \( \sigma \) by the value of \( \langle s^2 \rangle \) found above. \( \mathcal{N} \) is the normalization. In the RMT case we have \( \gamma = N/2 - 1 \) and \( \sigma^2 = 2(N+1)/N^2 \), while in the case where the long wavelength modes are enhanced we have \( \gamma = N/\log(N) - 1 \) and \( \sigma^2 = \log(N)(1 + \log(N)/(2N))/N \).

The effect of the first \( \sqrt{N} \) normal modes is different for large and small eigenvalues. Whereas for the smallest \( \sqrt{N} \) eigenvalues it amounts to a breathing it means incoherent fluctuations for the larger ones. We now evaluate the effect of this breathing on the microscopic zeros. Recall that the partition function for \( N_f = 1 \) and \( \nu = 0 \) is the average of the fermionic determinant with respect to the joint probability distribution, and that \( \prod_{i=1}^{N} (m^2 - \lambda_i^2) = L_N^{-1/2}(Nm^2) \). In the microscopic limit where the quantity \( \mu \equiv 2N m \) is fixed we have

\[
\frac{L_N^{-1/2} (m^2 - \lambda_i^2)}{4N \sigma^2} = \frac{(-1)^N}{N! 2^{2N}} \frac{1}{\sqrt{N} \pi} \cos \left( \frac{\mu}{\sqrt{N}} \right) . \tag{43}
\]

To investigate the correction to the microscopic partition function zeros under the influence of the \( \sqrt{N} \) longest wavelength normal modes we thus have to consider the following integral

\[
Z(\mu) \sim \frac{1}{N K} \int_0^\infty ds \exp \left( \gamma \log s - \frac{s^2}{2\sigma^2} \right) \cos \left( \frac{\mu}{s} \right) s^{2\sqrt{N}} , \tag{44}
\]

where the normalization factor \( K \) is

\[
K = \int_0^\infty ds P_N(s) s^{2\sqrt{N}} . \tag{45}
\]

The factor \( s^{2\sqrt{N}} \) comes from the rescaling in the fermionic determinant. For the evaluation of \( 44 \) we make use of the series expansion of the cosine

\[
\cos \left( \frac{x}{s} \right) = \cos(x) + x \sin(x) \ (s - 1) + O(s - 1)^2 . \tag{46}
\]

This yields

\[
Z(\mu) \sim \cos(\mu) + \mu \sin(\mu) \frac{\sqrt{2} \sigma \Gamma \left( \left( \gamma + 2\sqrt{N} + 2 \right)/2 \right)}{\Gamma \left( \left( \gamma + 2\sqrt{N} + 1 \right)/2 \right)} + \ldots . \tag{47}
\]

Since \( \gamma \gg \sqrt{N} \), we can use that \( 19 \)

\[
\frac{\Gamma(z + a)}{\Gamma(z + b)} \sim z^{a-b} \left[ 1 + \frac{(a-b)(a+b-1)}{2z} + O \left( \frac{1}{z^2} \right) \right] , \tag{48}
\]

and finally find that in the case where we consider \( \sqrt{N} \) RMT long wavelength modes the correction term to \( \cos(\mu) \) is of order \( 1/\sqrt{N} \), and in the case of \( \sqrt{N} \) enhanced wavelength modes it is of order \( \log N/\sqrt{N} \). In both cases the zeros of the partition function are unaffected in the microscopic limit.

These results suggest a new kind of “universality” of the microscopic partition function. As a specific example this universality shows that the microscopic zeros are unaffected even if the \( \sqrt{N} \) longest wavelength normal modes are enhanced in such a way that they interpolate between Poissonian and RMT statistics.

5 The microscopic limit

So far we have been discussing the joint distribution function, the zeros of the partition function, and the \( N \) positions that specify the collective maximum. Here we link this to the more familiar
microscopic-eigenvalue density. For a finite \( N \) the eigenvalue density is found from the joint distribution function as

\[
\rho_N^{N_f,\nu}(\lambda, \{m_f\}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\lambda_2 \cdots d\lambda_N P_N^{N_f,\nu}(\lambda, \lambda_2, \lambda_3, \ldots, \lambda_N; \{m_f\}) .
\] (49)

The double-microscopic spectral density is then defined as \[4, 20, 21\]

\[
\rho_{s}^{N_f,\nu}(\zeta; \mu_1, \ldots, \mu_{N_f}) \equiv \lim_{N \to \infty} \frac{1}{N} \rho_N^{N_f,\nu}\left(\frac{\zeta}{N}, \frac{\mu_1}{N}, \ldots, \frac{\mu_{N_f}}{N}\right),
\] (50)

and similarly for all other spectral correlations. The functional form of the microscopic eigenvalue density has been derived in \[21\] and for \( N_f = 1 \) and topological charge \( \nu \) it reads

\[
\rho_{s}^{N_f=1,\nu}(\zeta; \mu) = \frac{|\zeta|}{2} \left( J_{\nu-1}(\zeta)^2 + J_{\nu-2}(\zeta)J_{\nu}(\zeta) \right) - |\zeta| \frac{J_{\nu-2}(\zeta)[\mu I_{\nu}(\mu)J_{\nu-2}(\zeta) + \zeta I_{\nu-2}(\mu)J_{\nu}(\zeta)]}{(\zeta^2 + \mu^2)J_{\nu-2}(\mu)} .
\] (51)

The partition function in the microscopic limit for one flavor is proportional to \( I_{\nu}(\mu) \). The extrema of (51) are given by

\[
\frac{\partial \rho_{s}^{(N_f=1,\nu)}}{\partial \zeta}(\zeta; \mu) = 0 ,
\] (52)

and are obviously functions of \( \mu \). If the derivative in (52) is evaluated at \( \zeta = \mu \) then the solutions of (52) corresponding to the maxima of \( \rho_{s}^{(N_f=1,\nu)} \) coincide with the zeros of the \( N_f = 1 \) microscopic partition function, \( Z_{s}^{N_f=1,\nu}(m) \), for an imaginary argument. To conclude: The peaks of the microscopic eigenvalue density are in one to one correspondence with the zeros of the partition function and by the trapping proven in section 2.3 also to the positions which maximize the joint probability distribution.

### 6 Conclusions and outlook

In the present paper we have established an intimate relationship between zeros of the partition function and the spectral properties of the Dirac operator. The relation is derived within chiral random matrix theory and applies to QCD Dirac spectra and partition function zeros near the origin (in the microscopic regime). Through the introduction of spectral normal modes we have tested the validity of the relationship when a finite Thouless energy is introduced. The observed independence of the microscopic zeros compliments the existing universality studies. The present study treats one flavour. For more flavours with degenerate masses, it is known that the zeros of the microscopic partition function are not confined to the imaginary axis \[22\]. While this makes the relation between the eigenvalues and the zeros somewhat less direct, there is no reason to expect that the normal mode analysis should not apply for any number of flavours.

We remark that the normal mode analysis is generically applicable and not a special feature of random matrix theory. In particular, the normal mode analysis lends itself to a study of the spectral properties of the Dirac operator in lattice QCD. Such a study would be truly interesting in that it would shed new light on the role of random matrix like correlations in lattice gauge theories. Almost all of the normal modes are expected to be given by random matrix theory while only the very long wavelength modes are determined by the detailed dynamics.

In a broader perspective this study may also be seen as the first step towards the establishment of a one-to-one correspondence between the zeros and the most likely eigenvalue positions whenever the short range spectral correlations are random matrix like. If such a general relation were established then it could be used to argue that the critical exponents for the Yang-Lee edge in QCD must coincide...
with the ones for the gap in the spectral density of the Dirac operator. Such relationship, if true, would allow for substantial simplifications when trying to determine critical exponents by lattice techniques.

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