Algebraic \( q \)-Integration and Fourier Theory on Quantum and Braided Spaces

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Abstract

We introduce an algebraic theory of integration on quantum planes and other braided spaces. In the one dimensional case we obtain a novel picture of the Jackson \( q \)-integral as indefinite integration on the braided group of functions in one variable \( x \). Here \( x \) is treated with braid statistics \( q \) rather than the usual bosonic or Grassmann ones. We show that the definite integral \( \int_{-\infty}^{\infty} \) can also be evaluated algebraically as multiples of the integral of a \( q \)-Gaussian, with \( x \) remaining as a bosonic scaling variable associated with the \( q \)-deformation. Further composing our algebraic integration with a representation then leads to ordinary numbers for the integral. We also use our integration to develop a full theory of \( q \)-Fourier transformation \( \mathcal{F} \). We use the braided addition \( \Delta x = x \otimes 1 + 1 \otimes x \) and braided-antipode \( S \) to define a convolution product, and prove a convolution theorem. We prove also that \( \mathcal{F}^2 = S \). We prove the analogous results on any braided group, including integration and Fourier transformation on quantum planes associated to general R-matrices, including \( q \)-Euclidean and \( q \)-Minkowski spaces.

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Short title: \( q \)-Integration and Fourier Theory on Quantum Spaces

1 Introduction

This work continues a series of papers \[\int\] in which is developed a systematic approach to \( q \)-deforming physics based on the idea that the geometrical co-ordinates

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should have braid statistics. Such co-ordinates are a generalisation of usual Bose or Fermi ones with ±1 replaced by a parameter $q$, or more generally by an R-matrix. This gives us a generalisation of super-geometry as some kind of ‘braided-geometry’. Here $[1][2][3]$ introduced the basic notions of braided-matrices, braided vectors, their addition law and their covariance properties under a background quantum group, i.e. a complete covariant and braided linear algebra. $[4]$ introduced the notion of braided differentiation on braided vectors spaces as an infinitesimal translation, and also the general braided exponential map as its eigenfunctions. We refer to $[5]$ for a review of the whole braided approach. Our goal now is to take a step towards completing this programme by providing also the beginnings of a general theory of braided or $q$-deformed integration and Fourier transformation. We treat the one-dimensional case in complete detail and some aspects of the general theory. The standard quantum plane in $n$-dimensions is covered completely as well. The algebras for $q$-Euclidean and $q$-Minkowski space are known and our approach applies to these also.

The general programme of $q$-deforming physics is both a classical subject in the context of $q$-special functions, see e.g.$[9][10]$, and a currently popular one in the context of quantum groups and non-commutative geometry. While compatible with some of this previous work, the braided approach above differs in the following fundamental way: the $q$-deformation which we introduce is not directly into the co-ordinate algebra of the system but rather into non-commutativity of the tensor product algebra $\otimes_q$ of two independent copies of the system.

All of this is visible even in one dimension, where we just have one variable $x$ say to which we apply $q$-derivatives etc. Usually, one would consider $x$ at least in this case as a number since it always commutes with itself. In the braided approach however, this generator $x$ must remain an operator or abstract generator because it will not commute with other copies of itself. It was shown in $[3]$ how such braid statistics lead at once to the usual $q$-derivative $\partial_q$ and $q$-exponential. The key idea is that functions in one variable form a braided-Hopf algebra called the ‘braided-line’$[11]$. This consists of the functions $\mathbb{C}[x]$ equipped with a braided coaddition $\Delta : \mathbb{C}[x] \to \mathbb{C}[x] \otimes \mathbb{C}[y]$ sending $f(x)$ to $\Delta f = f(x+y)$, but where the two copies $q$-commute according to $qyx = xy$. We can then
define
\[(\partial_q f)(y) = \left( x^{-1} (f(x+y) - f(y)) \right) |_{x=0} = \frac{f(y) - f(0)}{1 - q y} \] (1)
which is the standard \(q\)-derivative. There is also a ‘counit’ and braided-antipode
\[
\epsilon f = f(0), \quad Sx^n = (-1)^n q^{\frac{n(n-1)}{2}} x^n
\] (2)
for evaluation at zero and \(q\)-subtraction. The braided antipode \(S\) is one of the important new ingredients in \(q\)-analysis provided by the theory of braided groups. The \(q\)-exponential is likewise understood in braided group theory as a braided-multiplicative element:
\[
\Delta e^{x\lambda}_q = e^{x\lambda}_q \otimes e^{x\lambda}_q, \quad Se^{x\lambda}_q = e^{-x\lambda}_{q^{-1}}
\] (3)
Moreover, it is this new braided point of view on \(q\)-differentials and \(q\)-exponentials which generalises at once to arbitrary dimensions and arbitrary R-matrices[4].

We want to apply these same techniques now to obtain an understanding of the celebrated Jackson \(q\)-integral \[\int_0^a d_q x\], as well as to generalise it to higher dimensions. The point of view that we are led to is a fairly radical one due to the fact that the variable \(x\) is for us a braided co-ordinate and hence not a \(\mathbb{C}\)-number as it would be in the usual point of view. We consider the limits of the Jackson integral as braided variables and hence the integral itself as operators of indefinite integration \(\int_0^x\) and \(\int_0^y f = \int_0^x f - \int_0^y f\). Then one finds
\[
\int_0^{x+y} f = \int_0^y f + \int_0^x f(( ) + y), \quad \text{if} \quad y[0, x] = q[0, x]y
\] (4)
\[
= \int_0^x f + \int_0^y f(x + ( )), \quad \text{if} \quad [0, y]x = qx[0, y]
\] (5)
where the braid statistics \(y[0, x] = q[0, x]y\) means to assume \(yx = qxy\) when computing \(\int_0^{x+y}\) and also \(yz = qzy\) in \(f(z + y)\) when computing \(\int_0^x\) with variable of integration \(z\). Similarly for the second identity. These two identities are easily verified on monomials using the well-known \(q\)-binomial theorem. We see that we can exactly think of \(\int_0^y f\) as built up by a Riemann sum provided the small interval \([0, x]\) being added is treated with braid statistics relative to the points in the existing sum. We arrive at exactly an inverse to \(q\)-differentiation from the point of view of (1). Indeed, supposing \(x\) ‘small’ we have
\[
\int_0^{x+y} f - \int_0^y f = \int_0^x f(( ) + y) \approx xf(y)
\]
as in the usual theory of integration. From these identities, one also has

\[
\begin{align*}
\int_{z+x}^{z+y} f &= \int_{y}^{y} f(z + (\ ) ) & \text{if} & \quad [x, y] z = q z [x, y] \\
\int_{x+z}^{y+z} f &= \int_{y}^{y} f((\ ) + z) & \text{if} & \quad z [x, y] = q [x, y] z
\end{align*}
\]  

(6)

which is global translation-covariance in our approach. One can think of the functions on the braided line as an infinite braided tensor product of \( \mathbb{C} \) with the tensor factors ordered lexicographically on a line cf. [8, Sec. 6]. Then the braid statistics are \( y x = q x y \) whenever \( y > x \).

This braided point of view requires us to think of integration as an operator rather than to evaluate it as a number. In Section 2 we take this further and write the integration operator as a powerseries in the \( q \)-Heisenberg algebra generated by the operator \( x \) of multiplication by the co-ordinate function, and \( \partial = \partial_{x q} \). The idea is to work abstractly in this algebra, with the question of convergence or not (i.e. of integrability of a function) becoming a property of any representation in which we might view it. We take this line also for integration \( \int_{0}^{\infty} f \), which we define by infinitely scaling up \( \int_{0}^{x} f \). So our braided \( x \) remains as a scale parameter with its braid-statistics 'diluted' to the extent that in the limit, it behaves bosonically. As before, limits are to be evaluated only in representations.

Section 3 is devoted to our first such family of representations, labelled by a parameter \( c \) and motivated by a quantum-mechanical picture of the \( q \)-Heisenberg algebra. We show that in these representations, we recover the formulae and values given by Jackson integration in its usual form. Also within these representations there is a reasonable notion of 'points' as eigenvectors of the position operator, and of 'integration regions' as sub spaces of the representation space. Our representations also provide a trace formula for Jackson integration. So far, we have developed these representations only in the 1-dimensional case.

Section 4 develops a second evaluation procedure which is like a representation in that it yields numbers, as multiples of a single undetermined integral which remains as a bosonic element in our algebra. This is done by integrating with reference to a suitable \( q \)-Gaußian function \( g \), which we also introduce. Our approach here is similar in spirit to the example of [13] [14] for integration over the entire complex plane, but somewhat different. We give a condition for integrability of a function according to our scheme. There are plenty of integrable functions in this sense, for example polynomials times our reference
Gaussian. In general terms, our algebraic approach is analogous to the treatment of path integration in physics where all integrals are computed relative to a reference Gaussian.

In Section 5 we construct integration on general \( n \)-dimensional quantum planes, in the same spirit as our treatment above. In the simple case of the standard \( SL_q(n) \) quantum planes we show that this can effectively be computed by iterating the 1-dimensional integration with integration operators \( \int_0^x \), cf. the iterated integrals in [13]. In general however, these operators from the quantum co-ordinate ring to itself are somewhat more complicated. They have \( q \)-commutation relations like the co-ordinates \( x_i \). We then give the Gaussian approach in the general R-matrix setting of quantum and braided spaces as introduced in [3]. The main theorem is Theorem 5.1 and expresses the Gaussian-weighted integral of monomials in terms of an interesting factorisation property. For example,

\[
\int g x_{i_1} x_{i_2} x_{i_3} x_{i_4} g = \int g x_{i_1} x_{i_2} g \int g x_{i_3} x_{i_4} g + \lambda^2 \int g x_{i_1} x_{a} g \int g x_{i_2} x_{b} g R_{i_3 i_4}^{a b}, \quad \lambda^4 \int g x_{i_1} x_{a} g \int g x_{b} x_{c} g R_{i_3}^{a i_4} R_{i_3}^b R_{i_3}^c R_{i_3}^d
\]

where \( g \) is the Gaussian and \( R \) the general R-matrix. \( \lambda \) is a normalisation constant. We compute the integrals on \( q \)-Euclidean and \( q \)-Minkowski spaces as examples.

Finally, in Section 6 we develop an easy application of our integration theory, making use of the translation invariance with respect to \( \Delta \) for integration on the braided-line and other braided spaces. Namely, we give a fairly complete theory of \( q \)-Fourier transforms on such spaces. While \( q \)-exponentials, and also \( q \)-sine and \( q \)-cosine are known[9], the theory that we describe would not have been possible before even in the one dimensional case, as it depends for its meaningfulness on the notion of braided-groups with \( \Delta \) and antipode (2) as introduced only in the last few years by the second author.

We do the general case first using the diagrammatic techniques associated with braided groups[8], i.e. we develop a kind of braided-calculus for Fourier transforms which works for any braided group. Again, it is algebraic and takes a concrete form with questions of convergence arising only in a realisation. In this setting, we have a convolution theorem, \( \delta \)-functions and \( \mathcal{F}^2 = S \). We then pass to the 1-dimensional case and the general multidimensional R-matrix case.

While this work was being written-up, we received a preprint[16] where the braided-translation ideas introduced in [3][4] were also used to propose the possibility of a \( q \)-Fourier transform in the Hecke case. On the other hand, these authors use a quite different proposal for translation-invariant integration, which, in one dimension, is just
ordinary integration. Hence their corresponding Fourier transform is quite different from our proposal based on Jackson integration and its generalisation to \( n \)-dimensions.

We would also like to note two recent papers \cite{17}\cite{18}, which were pointed out to us, where translation invariant integration on \( q \)-Euclidean space was considered in the context of \( SO_q(N) \)-invariant quantum mechanics. The second of these used an explicit Gaußian approach, though not in our general \( R \)-matrix setting.

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2 Algebraic integration

In this section we develop an algebraic point of view on ordinary \( q \)-integration in one variable \( x \) as, quite literally, the inverse operator to \( q \)-differentiation. We study such an operator \( \int_0^x \) and develop its elementary properties. After that, we obtain integration to \( \infty \) by scaling the indefinite one. We also explain the issues relevant to the \( n \)-dimensional case to come later.

2.1 Indefinite integration

Let us consider the algebra \( \mathcal{Q} \) generated by a multiplication operator \( x \) and a differentiation operator \( \partial \) obeying the commutation relations:

\[
\partial x - q^2 x \partial = 1 \quad \text{with} \quad q^2 < 1
\]  

(7)

The algebra is naturally represented on the algebra \( \mathcal{A} \) of polynomials or other suitable functions in \( x \), with \( \partial \) represented as in (1) except that we use conventions with \( q^2 \) rather than \( q \). It will later be instructive to consider a different representation also.

Unlike in ordinary calculus, which is the case \( q = 1 \), it is now possible to identify an operator \( \int_0 \) or \( \partial^{-1} \) as an element of the algebra \( \mathcal{Q} \). We start by formally writing:

\[
\int_0 \partial^{-1} = \partial^{-1} \int_0 = x (\partial x)^{-1} = x \frac{1 - q^2}{1 - (1 - q^2) \partial x}
\]  

(8)

From the action of the term \( (1 - (1 - q^2) \partial x) \):

\[
(1 - (1 - q^2) \partial x)x^r = q^{2(r+1)}x^r
\]  

(9)

we see that it is diagonal in the basis of monomials and has the eigenvalues \( q^{r+1} \) with \( r = 0, 1, 2, \ldots \). Thus we can treat this operator like a number of absolute value smaller than 1 and the expansion yields:

\[
\int_0 = (1 - q^2)x \sum_{n=0}^{\infty} (1 - (1 - q^2) \partial x)^n
\]  

(10)

We will see in Section 5 that these arguments also go through for the differential calculus on the \( SL_q(n) \) and other quantum planes. In this case, however, the function algebra \( \mathcal{A} \) generated by the co-ordinate functions on the quantum planes will be noncommutative.
To complete our operator picture, note that ordinary integration

\[ f(t) \rightarrow \int_{t_1}^{t_2} dt' f(t') \]

is really a mapping from a space of functions of one variable to a space of functions of two variables, namely the upper and the lower limits of the integration interval. In our case, we define analogously

\[ \int : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}, \quad \int := \int_0 1 - 1 \otimes \int_0. \]  

(11)

One readily checks the desired properties

\[ (\partial \otimes \text{id}) \int = \text{id} \otimes 1 \]  

(12)

\[ (\text{id} \otimes \partial) \int = -1 \otimes \text{id} \]  

(13)

\[ \int \partial = \text{id} \otimes 1 - 1 \otimes \text{id} \]  

(14)

and the global translation properties

\[ \Delta \circ \int_0 = 1 \otimes \int_0 + (\int_0 \otimes \text{id}) \circ \Delta = \int_0 \otimes 1 + (\text{id} \otimes \int_0) \circ \Delta \]  

(15)

\[ \beta_L \circ \int = (\text{id} \otimes \int) \circ \Delta, \quad \beta_R \circ \int = (\int \otimes \text{id}) \circ \Delta. \]  

(16)

Here \( \Delta \) is the coaddition for the braided line, which we view as a left or right coaction of \( \mathcal{A} \) on itself. Equation (14) says that \( \int \) is an intertwiner between these coactions and the corresponding induced tensor product coactions \( \beta_L, \beta_R \) on \( \mathcal{A} \otimes \mathcal{A} \). This is the operator description of our observations (4)–(6) from the Introduction.

### 2.2 Scaling and infinity

Let us now develop a simple method for integration ‘over the whole space’. We will not aim to extract the information of what the value of the integral is from the ‘values’ the function has in the integration interval. Instead we will obtain the integral in terms of the coefficients of the power series expansion of the function, which is the form in which our function algebra \( \mathcal{A} \) is naturally given in the quantum group framework. We later use this formalism in the \( n \)-dimensional case also.
Consider the ‘scaling’ operator:

\[ L := 1 - (1 - q^2)x\partial \quad \in \mathcal{Q} \quad (17) \]

Its action on the monomials in \( \mathcal{A} \) is

\[ L.x^r = q^{2r}.x^r \quad (18) \]

Thus, it scales the functions in \( \mathcal{A} \):

\[ L.f(x) = f(q^2x) \quad (19) \]

The inverse can also be found in \( \mathcal{Q} \):

\[ L^{-1} = \frac{1}{1 - (1 - q^2)x\partial} = \sum_{s=0}^{\infty}((1 - q^2)x\partial)^s \quad (20) \]

As we saw in Section 2.1, the integral of a function should be considered a function in its integration limits. We now have an algebraic tool at hand to scale these limits. In particular one readily checks that

\[ L^{-r}\int_0^x = x(1 - q^2) \sum_{n=-r}^{\infty} (1 - (1 - q^2)\partial x)^n \quad (21) \]

leading us to define

\[ \int_0^{x^\infty} := \lim_{r \to \infty} L^{-r}\int_0^x = x(1 - q^2) \sum_{n=-\infty}^{\infty} (1 - (1 - q^2)\partial x)^n. \quad (22) \]

For each finite \( r \), this operator remains a map from the function space \( \mathcal{A} \) into \( \mathcal{A} \). In the generic case then the integral is the limit of a power series in the generator. So, although we are integrating over the whole space, the integral

\[ I(f) := \int_0^{x^\infty} f \quad (23) \]

of a function \( f \) is still an expression in the variable \( x \). In ordinary calculus this would be a constant function, while in our case we can say at least that it is invariant under the scaling operation

\[ L^{\pm 1}I(f) = I(f) \quad (24) \]
This is obvious from its definition. In the braided picture it means that $I(f)$ is bosonic in the sense that $yI(f) = I(f)y$ even if $y$ q-commutes with $x$.

This point of view on global integration extends to the $n$-dimensional case also. We will see that one has similar scaling operators $L_i$ in the case of $SL_q(n)$ quantum planes and can use them to define global integration by infinite scaling in the same manner as above. Now the various co-ordinates $x_i$ are noncommuting even among themselves and both the indefinite integral and the global integral of a function in our noncommutative algebra remain as functions of these noncommuting variables. In this case it is even more clear that it would be naive to try to ‘evaluate’ such functions at some finite ‘point’ by simply putting in numbers for the noncommutative generators. The same function, rearranged using only the commutation relations would have another value at the same ‘point’. In general one does not have many nontrivial algebra homomorphisms from the noncommutative function algebra to the commutative ground field.

However, for physical applications it is of course very desirable to obtain actual numbers from the integration. We see at least two strategies to achieve this:

1. One could find other concrete representations and try to define ‘quasipositions’ as eigenvectors of the position operators. Due to their noncommutativity they will not be simultaneously diagonalisable, even if the $*$-structure is such that they are symmetric. Integration regions then appear as sub-Hilbert spaces of the representation space. We will demonstrate the idea in the simple one dimensional case in the next section.

2. For integration over the whole space one can use the following technique: If $g$ is a suitable function that is integrable over the whole space, it is possible to generate a space of integrable functions from it. The integral of $g$ may in general be a noncommutative expression. However, the integrals of all other integrable functions are simply multiples of the integral of $g$. The key point is that for appropriately chosen $g$ all boundary terms of partial integrations vanish, so that no new noncommutative expressions other than the integral of $g$ can appear. We will develop this technique in Section 4 for the one dimensional case, and we will then extend it to the true noncommutative case in Section 5.
3 Quasipositions representation

In the preceding section we represented the algebra of $x$ and $\partial$ as operators on the function space $\mathcal{A}$. Following the first strategy explained above, we consider another representation of the algebra $Q$ spanned now by the eigenvectors of the position operator $x$. Note that (4) allows us to choose $x$ to be symmetric. However, unlike in ordinary calculus, $\partial$ will then no longer be antisymmetric.

3.1 Construction of the representation

We construct the representation by starting with a normalised eigenvector of the position operator to the eigenvalue $c$:

$$x|v_c\rangle = c|v_c\rangle \quad \text{with} \quad \langle v_c|v_c\rangle = 1$$

(25)

In $Q$ we have

$$x(1 - (1 - q^2)\partial x) = q^{-2}(1 - (1 - q^2)\partial x)x$$

(26)

Thus the normalised vectors

$$|v_{cq^{-2r}}\rangle := N(r)(1 - (1 - q^2)\partial x)^r|v_c\rangle$$

(27)

are eigenvectors of $x$ with the eigenvalues $cq^{-2r}$. There is also the inverse operator

$$\frac{1}{1 - (1 - q^2)\partial x} = \sum_{m=0}^{\infty} ((1 - q^2)\partial x)^m$$

(28)

since the eigenvalues of $(1 - q^2)\partial x$ are $1 - q^{2r}$ i.e. smaller than 1. We can thus let $r$ in (27) run through all integers. Evidently, the scale $c$ labels the representation.

We consider the eigenvectors $|v_{cq^{2r}}\rangle$ of $x$ as the ‘quasipoints’. We can then think of general vectors as ‘functions’ on the set of quasipoints, i.e. with values as given by evaluation against quasipoint vectors. Thus

$$|f\rangle = \sum_r f_r|v_{cq^{2r}}\rangle \quad \text{has values} \quad \langle v_{cq^{2r}}|f\rangle = f_r$$

(29)

The position operator $x$ then appears as a matrix

$$x = \sum_{r,s} |v_{cq^{2r}}\rangle c q^{2r} \delta_{r,s} \langle v_{cq^{2s}}|$$

(30)
In the representation $\mathcal{A}$ that we had considered so far, the functions are polynomials or power series in the variable $x$. Each such function $f(x)$ can now uniquely be identified with a vector in the quasipoint representation. Let us first consider the constant polynomial $1 \in \mathcal{A}$. It is identified with that vector $|1\rangle$ in the quasipoint representation which maps every quasipoint onto the number 1, i.e.

$$|1\rangle = \sum_{t=-\infty}^{+\infty} |v_{cq^{2^t}}\rangle$$

(31)

Obviously at each quasipoint $|v_{cq^{2^t}}\rangle$ the value of $|1\rangle$ is $\langle v_{cq^{2^t}} |1\rangle = 1$.

Consider now an arbitrary polynomial or power series $f(x) \in \mathcal{A}$. It is identified with a vector in the quasipoint representation which is obtained as follows. We simply let $f(x)$ act, as a polynomial or power series in the position operator $x$, on the constant function $|1\rangle$

$$|f\rangle := f(x)|1\rangle$$

(32)

On the other hand, just like in ordinary calculus, of course not every function from (quasi-)points to numbers can also be expressed as a polynomial or power series.

For the matrix representation of the differentiation operator

$$\partial = \sum_{r,s} |v_{cq^{2^r}}\rangle d_{r,s} \langle v_{cq^{2^s}}|$$

(33)

equation (33) yields

$$d_{r,s} = \delta_{r,s}(cq^{2^r} - cq^{2(r+1)})^{-1} + a(r)\delta_{r+1,s}$$

(34)

Thus the commutation relation does not determine the differentiation operator completely. It is however fixed from the requirement that the differential of a constant function still vanishes: Putting equation (34) into

$$\partial|1\rangle = 0$$

(35)

fixes $a(r)$ so that the matrix elements of the differentiation operator are

$$d_{r,s} = (\delta_{r,s} - \delta_{r+1,s})(cq^{2^r}(1 - q^2))^{-1}.$$ 

(36)

As we said, even for $q < 1$ we can keep the position operator symmetric $x^\dagger = x$. However, from equation (7) it is clear that then $\partial$ will no longer be antisymmetric, as it would be
for $q = 1$. Explicitely we have
\[
x^\dagger = x = \sum_{r=-\infty}^{+\infty} |v_{cq^2r}\rangle c q^{2r} \langle v_{cq^2r}|
\]  
(37)
\[
\partial = \sum_{r,s} |v_{cq^2r}\rangle (\delta_{r,s} - \delta_{r+1,s})(c q^{2r}(1 - q^2))^{-1} \langle v_{cq^2s}|.
\]  
(38)

The hermitean conjugate of $\partial$ is thus
\[
\partial^\dagger = \sum_{r,s} |v_{cq^2s}\rangle (\delta_{r,s} - \delta_{r+1,s})(c q^{2r}(1 - q^2))^{-1} \langle v_{cq^2r}|.
\]  
(39)

In order to get the normalisation constants we now calculate the step operators and their adjoints explicitely as
\[
(1 - (1 - q^2)x)^r = \sum_s q^{2s} |v_{cq^{-2s}}\rangle \langle v_{cq^{-2(s+r)}}|
\]  
(40)
\[
((1 - (1 - q^2)x)^\dagger)^r = \sum_s q^{2s} |v_{cq^{-2(s+r)}}\rangle \langle v_{cq^{-2s}}|.
\]  
(41)

The normalisation condition and Eq (27) yield:
\[
1 = \langle v_{cq^2r}|v_{cq^2r}\rangle = N^2(r) \langle v_e|(1 - (1 - q^2)x)^r + (1 - (1 - q^2)x)^\dagger|^v_e\rangle
\]  
\[
= N^2(r) q^{4r} \sum_{s,t} |v_{cq^{-2(r+s)}}\rangle \langle v_{cq^{-2s}}|v_{cq^{-2t}}\rangle \langle v_{cq^{-2(r+t)}}|v_e\rangle
\]  
\[
= N^2(r) q^{4r}
\]  
(42)

Thus the normalisation constants in (27) are
\[
N(r) = q^{-2r}
\]  
(43)

The Hilbert space is of course isomorphic to $l^2$.

3.2 The integral operator and the Jackson integral formula

In order to calculate the matrix representation of the integral operator
\[
\int_0^1 = \sum_{r,s} |v_{cq^2r}\rangle (\int_0^1)_{r,s} \langle v_{cq^2s}|
\]  
(44)
we could represent the expression in (44) or simply invert the matrix of $\partial$ to obtain:
\[
(\int_0^1)_{r,s} = c(1 - q^2) \sum_{t=0}^{\infty} q^{2(r+t)} \delta_{r+t,s}.
\]  
(45)
Thus
\[ \int_0^\infty = c(1 - q^2) \sum_{r=-\infty}^{\infty} q^{2(r+t)} |v_{cq^r}\rangle \langle v_{cq^{r+t}}| \] (46)

Using (11), the integral from the ‘quasiposition’ \(cq^{2a}\) to the quasiposition \(cq^{2b}\) is:
\[ \int_{cq^{2a}}^{cq^{2b}} f = \langle v_{cq^{2b}} | \int_{cq^{2a}} f | v_{cq^{2a}} \rangle - \langle v_{cq^{2b}} | \int_{0}^{\infty} f | v_{cq^{2a}} \rangle \] (47)
\[ = c(1 - q^2) \sum_{t=0}^{\infty} q^{2t} \left( q^{2b} \langle v_{cq^{2(t+\alpha)}} | - q^{2a} \langle v_{cq^{2(a+t)}} \right) |f\rangle \] (48)

which, using (29) can be brought into the form of the Jackson integral:
\[ \int_{cq^{2a}}^{cq^{2b}} f = cq^{2b}(1 - q^2) \sum_{t=0}^{\infty} q^{2t} f(cq^{2b}) - cq^{2a}(1 - q^2) \sum_{t=0}^{\infty} q^{2t} f(cq^{2a}) \] (49)

### 3.3 A new trace formula for the integral

On the other hand we will now also obtain a new trace formula: From (47) we get for \(a \rightarrow \infty\) and \(b \rightarrow -\infty\):
\[ \int_0^{\infty} f = c(1 - q^2) \sum_{t=0}^{\infty} q^{2t} \langle v_{cq^{2t}}|f\rangle = (1 - q^2) \sum_{t=0}^{\infty} \langle v_{cq^{2t}}|x|f\rangle. \] (50)

Now using (32) we obtain
\[ \int_0^{\infty} f = (1 - q^2) \sum_{s,t=-\infty}^{\infty} \langle v_{cq^{2t}}|xf(x)|v_{cq^{2s}}\rangle \] (51)

Since \(xf(x)\) is diagonal we get for all functions \(f \in A\):
\[ \int_0^{\infty} f = (1 - q^2) \text{Trace}(xf(x)) \] (52)

This is now a basis independent formulation and one may expect it to be extendable to the general \(n\) dimensional case. Finite integration regions then simply mean to take the trace only over a finite dimensional sub Hilbert space of the representation space. We have in mind a similar representation theoretic approach in the \(n\)-dimensional case also, though we will not develop this explicitly here.

### 4 Induced integration

In this section we develop our algebraic approach to global integration based on Gaussians, in the one-dimensional case. The presence of a Gaussian weight factor in the integrand allows integration to be ‘turned back’ into differentiation and thereby computed purely algebraically.
4.1 Rapidly decreasing functions

We begin with a simple classification of a function’s behaviour at infinity, which will also prove to be useful in the \( n \)-dimensional noncommutative case. Let us say that a function \( f \in A \) is vanishing at \( \pm \infty \) if it obeys:

\[
\lim_{n \to \infty} L^{-n} f(\pm x) = 0. \tag{53}
\]

Similarly we say that a function \( f \in A \) is rapidly decreasing at infinity if we have the even stronger condition:

\[
\lim_{n \to \infty} L^{-n} x^r f(\pm x) = 0 \quad \text{for all } r = 0, 1, 2, \ldots \tag{54}
\]

An important example of a function that is rapidly decreasing at infinity is the function \( g_\alpha \) defined by the differential equation:

\[
\partial g_\alpha = -\alpha x g_\alpha \quad \text{with } \alpha > 0 \tag{55}
\]

It is of course the analogue of a Gaußian function \( e^{-\alpha x^2 / 2} \). However, although we can calculate it as a vector in representation space, it is not trivial to write it as a power series in the operator \( x \) acting on the constant function \( |1 \rangle \). Let us check that it is rapidly decreasing. Using (17) and (55) we get for any \( u(x) := x^r g_\alpha \):

\[
L.u(x) = u(q^2 x) = (1 + \alpha(1 - q^2)x^2)q^{2r} u(x) \tag{56}
\]

Thus:

\[
u(x) = \prod_{n=0}^r (1 + \alpha(1 - q^2)q^{-2n}x^2) q^{2r} u(q^{-2r}x) \tag{57}
\]

Since the product is strictly increasing and diverging as \( r \to \infty \), the function \( u(x) \) vanishes:

\[
\lim_{r \to \infty} L^{-r} x^r g_\alpha = 0 \quad \text{for all } r = 0, 1, 2, \ldots \tag{58}
\]

In suitable representations then, we will have the rapid decrease property of \( g_\alpha \) explicitly (see the appendix). Since \( g_\alpha \) is an even function it has the same behaviour at \( -\infty \). For the same reason, its integral from \( -\infty \) to \( +\infty \) is simply twice the integral from 0 to \( +\infty \).
In the following we will not actually need any details about its global integral

\[ I(\alpha) := \int_{-\infty}^{\infty} g_\alpha. \]  

(59)

In fact, we show in the Appendix that its actual value

\[ (1 - q^2)\text{Trace}(xg_\alpha) \]  

(60)

is a finite number. This number depends nontrivially on the chosen representation or, physically speaking, on the choice of the scale \( c \) that appears with the \( q \)-deformation. Note that the trace is basis independent, but representation dependent.

### 4.2 Induced integrals

Keeping in mind that \( I(\alpha) \) is not a number, but representation dependent, we will now aim at expressing the global integrals of other functions as multiples of \( I(\alpha) \).

To this end we consider

\[ \int_{-\infty}^{\infty} \partial x^r g_\alpha = 2 \lim_{n \to \infty} L^{-n} x^r g_\alpha = 0 \]  

(61)

which vanishes because \( g_\alpha \) is rapidly decreasing.

Thus

\[ 0 = \int_{-\infty}^{\infty} \partial x^r g_\alpha = \int_{-\infty}^{\infty} \left( [r] x^{r-1} g_\alpha + q^{2r} x^r \partial g_\alpha \right) \]  

(62)

where \([r] = \frac{1 - q^r}{1 - q} \) is the usual \( q \)-integer. Using (55) yields the recursion relation

\[ \int_{-\infty}^{\infty} \alpha q^{2r} x^{r+1} \partial g_\alpha = \int_{-\infty}^{\infty} [r] x^{r-1} g_\alpha \]  

(63)

and eventually

\[ \int_{-\infty}^{\infty} x^r g_\alpha = I(\alpha) \frac{[r - 1]!!}{\alpha^{r/2} q^{r/2}} \quad \text{for all } r \text{ even} \]  

(64)

and 0 for \( r \) odd. Now it is not difficult to prove that the global integral of the function \( fg_\alpha \), for any \( f \in A \) can be obtained simply from the action of the operator \( P_\alpha \in Q \):

\[ \int_{-\infty}^{\infty} fg_\alpha = I(\alpha) P_\alpha \cdot f \big|_{x=0} \]  

(65)

where

\[ P_\alpha := \sum_{r=0}^{\infty} \alpha^{-r} q^{-2r} \frac{\partial_x^{2r}}{[2r]!!}. \]  

(66)

In particular, every polynomial \( f \in A \) times the Gaußian is integrable i.e. \( P_\alpha \cdot f \big|_{x=0} \) is finite.
4.3 Global integration formula

We eventually aim at expressing the global integration over \( f \) alone. To this end let us ‘undo’ the multiplication with \( g_\alpha \) by multiplication with \( g_\alpha^{-1} \). Actually \( g_\alpha^{-1} \) can be found in \( \mathcal{A} \) as follows.

With the ansatz

\[
g_\alpha^{-1} := \sum_{r=0}^{\infty} a_r x^r \quad (67)
\]

follows from

\[
\partial g_\alpha^{-1} g_\alpha = 0 \quad (68)
\]

and (63) that

\[
g_\alpha^{-1}(x) = \sum_{r=0}^{\infty} \alpha^r q^{2(r^2-r)} \frac{[2r]!!}{[2r]!!} x^{2r} \quad (69)
\]

A simple ratio test proves that this is convergent everywhere, i.e. for all eigenvalues of \( x \).

We thus express the global integral of an arbitrary function i.e. power series \( h \in \mathcal{A} \) as a multiple of the global integral \( I(g_\alpha) \) of the Gaußian:

\[
\int_{-\infty}^{\infty} h(x) = \lim_{n \to \infty} \int_{-\infty}^{\infty} h(x) \sum_{r=0}^{n} \alpha^r q^{2(r^2-r)} \frac{[2r]!!}{[2r]!!} x^{2r} g_\alpha \quad (70)
\]

Using (64) we thus get the integral in terms of the coefficients of the power series of \( h(x) \) as

\[
I(g_\alpha) \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{[2r + 2s - 1]!!}{[2r]!!} \alpha^{-s} q^{-2(s^2+2rs+r)} h_{2s} \quad (71)
\]

where \( h(x) = \sum_{s=0}^{\infty} h_s x^s \). Note that the two summations do not in general commute. The global integral of a function \( h(x) \in \mathcal{A} \) can thus be written

\[
\int_{-\infty}^{\infty} h(x) = I(g_\alpha) S_\alpha h \big|_{x=0} \quad (72)
\]

where \( S_\alpha \in \mathcal{Q} \) is the operator

\[
S_\alpha = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{[2r + 2s - 1]!!}{[2r]!![2s]!!} \alpha^{-s} q^{-2(s^2+2rs+r)} \partial^{2s} \quad (73)
\]

For sufficiently well behaved power series in \( \mathcal{A} \), which we shall call integrable functions, the above global integral is a finite multiple of \( I(g_\alpha) \) and, using (68), we have also

\[
\int_{-\infty}^{\infty} \partial h(x) = 0 \quad (74)
\]
Recall that instead of the Gaußian one could also use another rapidly decreasing function $k$. We would then arrive at a global integration formula that expresses integrals as multiples of the global integral $I(k)$.

5 The $n$ dimensional case

We will now generalise some of our algebraic integration techniques for the $n$ dimensional case.

To begin with we show for the $SL_q(n)$ case how operators which act as indefinite integration can again be found in the algebra $Q$ generated by the differentiation and multiplication operators $\partial^i$ and $x_j$, $(i,j = 1, 2, ..., n)$. We again consider functions of rapid decrease and integration limits that are scaled to infinity.

Then we develop the general case of Gaußian induced integration for quantum planes associated to R-matrices. Unlike the one-dimensional case, we do not discuss the integrability and rapid decay properties of the Gaußians in detail, taking instead a more algebraic line. One still has precise integration formulae without knowing details of the Gaußian itself or its integral. We give the examples of $q$-Euclidean space, $q$-Minkowski space and $SL_q(n)$ quantum planes explicitly. A representation theoretic approach will be followed elsewhere, including also the problem of expressing the $n$-dimensional integral as a trace along the lines of (52).

5.1 Integration operators on $SL_q(n)$ quantum planes

The commutation relations of the $SL_q(n)$ - comodule algebra $Q$ of multiplication and differentiation operators read [19]:

\[
\begin{align*}
\partial^i \partial^j - q \partial^j \partial^i &= 0 \quad \text{for} \quad i < j \\
\partial^i x_j - q x_j \partial^i &= 0 \quad \text{for} \quad i > j \\
x_i x_j - q x_j x_i &= 0 \quad \text{for} \quad i > j \\
\partial^i x_i - q^2 x_i \partial^i &= 1 + (q^2 - 1) \sum_{j < i} x_j \partial^j
\end{align*}
\]

These relations and their complex conjugates appeared for operators on $q$-deformed Bargmann Fock space in [14][20].
We see that the first dimension can be identified with the one dimensional case that we have considered so far.

The algebra is again naturally represented on the function algebra $\mathcal{A}$ of power series in the generators $x_i$, obeying the same commutation relations as the multiplication operators $x_i$ i.e. (74). We want to identify $\int_0^{x_i}$ or $\partial^{-1}$ as an element of the algebra $\mathcal{Q}$. Writing formally
\[
\int_0^{x_i} = \partial^{-1} = x_i \frac{1 - q^2}{1 - (1 - q^2)\partial x_i}
\] (79)
we note that the last term is actually well defined as a power series. This is because the action of the operator $(1 - (1 - q^2)\partial x_i)$ is
\[
(1 - (1 - q^2)\partial x_i).x_1^{r_1} \cdots x_i^{r_i} \cdots x_n^{r_n} = (1 - q^{2(r_1+\cdots+r_{i-1})})(1 - q^{2(r_i+1)})x_1^{r_1} \cdots x_i^{r_i} \cdots x_n^{r_n}.
\] (80)
The operator is thus diagonal in this basis of $\mathcal{A}$, and we read off that its eigenvalues are all of absolute value smaller than 1. This allows its expansion as a geometrical series:
\[
\int_0^{x_i} = (1 - q^2)x_i \sum_{n=0}^{\infty} (1 - (1 - q^2)\partial x_i)^n
\] (81)
We thus have the desired properties:
\[
\partial^i \int_0^{x_i} f = f \quad \forall f \in \mathcal{A}
\] (82)
\[
\int_0^{x_i} \partial^i x_i f = x_i f \quad \forall f \in \mathcal{A}.
\] (83)
We used the factor $x_i$ in order to prevent $\int_0^{x_i}$ from acting on 0 which could otherwise occur through the annihilation of $f$ by the action of $\partial^i$. It is not very difficult to prove these properties also directly by using the above basis of ordered polynomials. Actually we see that due to this choice of basis, those derivatives that arise from the term on the rhs of (78) do not find co-ordinates to act on. This means that for ordered polynomials in the evaluation of
\[
\int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} x_1^{r_1} \cdots x_n^{r_n}
\] (84)
one can neglect this term on the rhs of (78) so that we have in every dimension the effective commutation relation
\[
\partial^i x_i - q^2 x_i \partial^i = 1
\] (85)
which is of the same form as in the one dimensional relation. It is thus a special feature of these ordered polynomials that the operators $f^{x_i}_0$ which act on them are built of operators which effectively obey the one dimensional commutation relation. Even then, the $f^{x_i}_0$ are noncommuting with each other and with the co-ordinate functions. We will find a similar phenomenon of effective decoupling of the integrations on ordered polynomials in the $SL_q(n)$ case of the $n$-dimensional Gaussian induced integration method.

Since the operator $f^{x_i}_0$ is the inverse of the operator $\partial^i$ in the representation of the algebra $Q$ as operators on the function algebra $A$, we immediately get the commutation relations:

$$\partial^j \int_0^{x_i} - q \int_0^{x_i} \partial^j = 0 \quad \text{for} \quad i < j$$  \hspace{1cm} (86)$$

$$q \partial^j \int_0^{x_i} - \int_0^{x_i} \partial^j = 0 \quad \text{for} \quad i > j$$  \hspace{1cm} (87)$$

$$\int_0^{x_i} \int_0^{x_j} - q \int_0^{x_j} \int_0^{x_i} = 0 \quad \text{for} \quad i > j$$  \hspace{1cm} (88)$$

$$x_j \int_0^{x_i} - q \int_0^{x_i} x_j = 0 \quad \text{for} \quad i \neq j$$  \hspace{1cm} (89)$$

$$x_i \int_0^{x_i} - q^2 \int_0^{x_i} x_i = \int_0^{x_i} \int_0^{x_i} +(q^2 - 1) \sum_{j<i} x_j \partial^j \int_0^{x_i} \int_0^{x_i}$$  \hspace{1cm} (90)$$

which could of course also be written in R-matrix notation. Note that (90) describes integration by parts. $f^{x_i}_0$ behaves like $x_i$.

In complete analogy with the one dimensional case one can also use scaling operators $L_i$ to define functions that vanish at infinity, rapidly decreasing functions and the scaling of integration limits to infinity. We arrive then at operators $f^{x_i}_{-\infty}$ and could define for example

$$\int = \int_{-x_1\infty}^{x_1\infty} \ldots \int_{-x_n\infty}^{x_n\infty}$$  \hspace{1cm} (91)$$

for a global integration operator over the whole space. Using the relations (88) we can always bring any of the $f^{x_i}_{-\infty}$ to the right, so translation invariance in the form $\int \partial^i = 0$ is assured. We note that integration on $SL_q(n)$ quantum planes as an iteration of 1-dimensional Jakson integrals has been proposed previously in [15] from a different point view.

This is one very concrete approach to constructing an integral given here in the $SL_q(n)$ case. Next we develop next a more powerful Gaussian approach, but will see in Sec-
tion 5.3.3 that it is consistent with the above concrete one.

5.2 Preliminaries on braided differentiation and integration

Here we collect some formulae of general braided differential calculus [4] which we will need in the next section, where we develop integration for this setting. The data we need are matrices $R, R' \in M_n \otimes M_n$ that fulfill:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (92)$$

$$R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}, \quad R'_{12}R_{13}R_{23} = R_{23}R_{13}R'_{12} \quad (93)$$

$$R_{21}R'_{12} = R'_{21}R_{12}, \quad (PR + 1)(PR' - 1) = 0 \quad (94)$$

where $P$ is the permutation matrix. Examples of matrices $R'$ can be constructed from the minimal polynomial of $PR$. We further assume that $R$ be invertible in two ways:

$$\exists R^{-1}, \tilde{R} : \quad R^{-1}_a^i b R^a_k b = \delta^i_k \delta^j_l = \tilde{R}^a_k b R^i_a b. \quad (95)$$

The braided differential calculus then implies the following commutation relations in the algebra $Q$ of multiplication and differentiation operators:

$$x_i x_j - x_j x_a R^a_i b = 0 \quad (96)$$

$$\partial^i \partial^j - R^i_a b \partial^b \partial^a = 0 \quad (97)$$

and

$$\partial^i x_j - x_a R^a_i b \partial^b = \delta^i_j \quad (98)$$

or equivalently:

$$x_i \partial^j - \tilde{R}^b_i j_a \partial^a x_b = -\tilde{R}^b_i j_a. \quad (99)$$

Our convention here and below is that repeated indices $a, b, c$ etc., are to be summed, but indices $i, j, k$ etc, remain free. Again, the algebra (96)–(99) acts naturally as operators on the function algebra generated by generators $x_1, ..., x_n$, obeying the position co-ordinate relations (96). We no longer use bold-face to denote $x_i$ as operators by multiplication.

The goal is to give a method for translation-invariant integration in this general R-matrix setting. Let us note that the question of existence of such a map $f$ is not really
an issue, at least in the global case. The position co-ordinates \( x_i \) form a braided-Hopf algebra\(^3\) and just as for usual groups and quantum groups, there are quite general ways to argue existence of \( f \). The problem is rather one of explicit evaluation, which is what we provide in the next section.

Before proceeding to this Gaussian integration, we note that one could also try to follow the concrete approach of Section 5.1. The key idea is to find partial inverses \( \int_0^{x_i} \) for the differentiation operators \( \partial^i \) in the spirit of (82)–(83). They may no longer be expressible in the associated Heisenberg algebra generated by \( \partial^i, x_j \) but (97) still yields their commutation relations

\[
\int_0^{x_j} \int_0^{x_i} -R^i_{a b} \int_0^{x_a} \int_0^{x_b} = 0. \tag{100}
\]

Next one has to scale these to infinity. Since the relations (96) are quadratic, one possibility is \( L^{-\infty} \) with \( L(x_i) = q^2 x_i \) for a parameter \( q < 1 \). Other scaling methods are also possible. Finally, one can write down products such as (91). If the scaled form of the relations (100) are sufficiently nice that one can move any of the integrations to the right, one would still have \( \int \partial^i = 0 \) as required. In general however, the required form may be more complicated and has to be analysed case by case. By contrast, the following Gaussian method yields quite general formulae without knowing details of \( R \) and \( R' \).

### 5.3 Gaussian-induced integration for general R-matrices

We have already analysed the Gaussian method in some detail in the one-dimensional case and seen that a Gaussian exists in this case, with the right properties of rapid decay etc in a suitable representation. Our strategy in this section is to assume a similar Gaussian now in \( n \)-dimensions. We will not try to give it explicitly or prove its decay properties. These are topics for further work. Remarkably, the Gaussian approach yields numbers that do not depend on these details as long as we are content to integrate only functions against this Gaussian weight.

We let \( \eta_{ij} \in M_n \) be a matrix and assume that a solution \( g_\eta \) of the equation

\[
- \eta_{ia} \partial^i g_\eta = x_i g_\eta \tag{101}
\]
exists and is (at least formally) rapidly decreasing and integrable with respect to an 
integration $f$, to be determined. We will not try to determine

$$I(g_\eta) = \int g_\eta$$

itself but only integrals of $f g_\eta$ where $f$ is a polynomial in our non-commutative position 
co-ordinates. We assume furthermore that

$$\int \partial^a f g_\eta = 0 \quad \text{for all polynomials } f \in \mathcal{A}, \quad \forall i = 1, \ldots, n \quad (102)$$

which says that we neglect boundary terms of the form (polynomial · $g_\eta$).

We show now that these assumptions uniquely determine $\int f g_\eta$ in terms of $\eta$ and our 
R-matrix data. Indeed, we show that this takes the form

$$\int f g_\eta = Z[f]I(g_\eta) \quad (103)$$

where $Z[f] \in \mathbb{C}$ for any polynomial $f \in \mathcal{A}$.

We prove the existence of the linear functional $Z$ by giving an explicit way to calculate it. Consider the global integral with an arbitrary multinomial $f = x_{i_1} \cdots x_{i_m}$ say. The 
indices here need not be distinct. Then

$$\int x_{i_1} \cdots x_{i_m} \eta = - \int x_{i_1} \cdots x_{i_{m-1}} \eta_{i_m a} \partial^a g_\eta$$

$$= - \int x_{i_1} \cdots x_{i_{m-2}} \eta_{i_m a} \left( - \tilde{R}_{i_m-1}^{b a} \cdot \eta_{i_{m-1} b} + \tilde{R}_{i_m-1}^{c a} \cdot \partial d x_c \right) g_\eta$$

using (101) and (99). We think of $x_i$ as operators acting to the right by multiplication. 
The first term here is the integral of a monomial of degree $m - 2$ times $g_\eta$. We can use 
(99) in the same way again for $x_{i_{m-2}} \partial^d$ which generates another integral of a monomial 
of degree $m - 2$ times $g_\eta$, and moves $\partial$ further to the left. We repeat this until the term 
containing $\partial$ is of the form $\int \partial \cdots g_\eta$, which then vanishes by (102). Hence $\int x_{i_1} \cdots x_{i_m} g_\eta$ is 
expressed explicitly in terms of the integration of monomials of degree two less, times $g_\eta$. 
Since $\int g_\eta$ is of the required form with $Z[1] = 1$, we see by induction that $\int x_{i_1} \cdots x_{i_m} g_\eta = Z[x_{i_1} \cdots x_{i_m}] I(g_\eta)$ where $Z[x_{i_1} \cdots x_{i_m}]$ is a complex number built from the matrices $\eta$ and 
$\tilde{R}$. This is our Gaußian integration method. Note that we did not require anywhere that 
the algebra of co-ordinates was commutative.

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Clearly, this method works whenever the differentials and co-ordinates enjoy a relation of the general form (99). Moreover, our derivation is independent of the detailed form of both $g_\eta$ and $f$. Any choice of these such that (101) and (102) hold will yield the same values $Z[f]$. This $Z[f]$ is the *Gaussian-weighted* integral of a polynomial in our co-ordinate algebra. Of course, details of $g_\eta$ would be needed if we wanted to know exactly what functions $fg_\eta$ we have integrated with respect to $\int$. Classically, this class would be dense in the set of Riemann-integrable functions if one takes the usual Euclidean metric. One would also like to know exactly how the class of integrable functions with respect to $g_\eta$ is related to the class with respect to $g_\eta'$ for a different choice of metric. One can expect that these various questions have algebraic answers the along lines developed in Section 4 for the one-dimensional case. These finer points of ‘quantum analysis’ will be investigated elsewhere.

We specialise now to an important class of metrics $\eta$, namely ones that are quantum group covariant. The co-ordinate algebra generated by the $x_i$, the algebra of differentials $\partial^i$ etc, were introduced in [3] in a manifestly covariant way under the usual matrix quantum group $t = \{t_{ij}\}$ associated to $R$. We suppose here that $R$ is of the regular type so that there is an antipode, i.e. an inverse quantum matrix $t^{-1}$. The co-ordinates transform as covectors by $x_j \rightarrow x_i t_{ij}$ and the differentials transform as vectors using $t^{-1}$. In Section 6.3 we will include a dilaton $\varsigma$ also in order to induce the correct braiding there. In order to remain in this covariant setting, we assume now that $\eta$ is invariant in the sense

$$\eta_{ab} t_{ia} t_{bj} = \eta_{ij}. \quad (104)$$

Using the properties of the dual-quasitriangular structure induced by $\lambda R$, [3], one finds easily the useful identities

$$\eta_{ja} R^l_{i \, a} k = \lambda^{-2} R^{-1} l_{i \, a} j k \eta_{lk}, \quad \eta_{ja} \tilde{R}^l_{i \, a} k = \lambda^2 R^l_{i \, a} j k \eta_{lk}. \quad (105)$$

$$\eta_{ia} \eta_{jb} R^a_{m \, a} b = R^a_{i \, j} a \eta_{am} \eta_{bn}, \quad \eta_{ia} R^a_{j \, a} k = \lambda^{-2} R^{-1} a_{i \, a} l k \eta_{ja}. \quad (106)$$

Here $\lambda$ is a constant that converts $R$ in (104) into the quantum group normalisation as explained in [3]. Using (105), we write (98) as

$$\partial_i x_j = \eta_{ij} + x_a \partial_c \lambda^{-2} R^{-1} a_{j \, c i}, \quad \text{i.e.} \quad \partial_1 x_2 = \eta_{12} + x_1 \partial_2 \lambda^{-2} (PR)^{-1}_{12} \quad (107)$$
where the right hand form is in a standard compact notation in which the numerical suffices stand for the positions of repeated tensor indices. We use bold-face here to denote an entire vector or matrix of generators. The inverse relation (99) is then clearly

\[ x_1 \partial_2 = (\eta_{12} + \partial_1 x_2)(PR)_{12} \lambda^2. \]  

(108)

Finally, we check a kind of ‘quantum integrability’ condition for our heat equation \( \partial_i g = -x_i g \). Recall that the usual integrability condition for a partial differential equation comes from requiring commutativity of partial derivatives. In our case we require (107) and find the condition for this by computing

\[
\partial_1 \partial_2 g = -\partial_1 x_2 g = -\eta_{12} - x_1 \partial_2 \lambda^{-2} (PR)_{12}^{-1} g \\
= -\eta_{12} + x_1 x_2 \lambda^{-2} (PR)_{12}^{-1} g = -\eta_{12} + x_1 x_2 \lambda^{-2} (PR')_{12}^{-1} g \\
= -\eta_{12} + x_1 x_2 \lambda^{-2} (PR')_{12}^{-1} (PR)_{12} = \eta_{12} ((PR')_{12} - 1) g + \partial_1 \partial_2 (PR')_{12} g
\]

using (107), the first of (94), and (96). For the quadratic term in derivatives to vanish, we see that we require \( \partial_i \) to obey the algebra (96) like the \( x_i \). Since \( \partial^i \) obey (17), the conditions for ‘quantum integrability’ of our quantum heat equation become

\[
\eta_{ia} \eta_{jb} R^a_{\ m} \ b = R'_{ia \ j} \eta_{am} \eta_{bm}, \quad \eta_{ab} R'_{\ i \ j} = \eta_{ij}.
\]  

(109)

These are natural conditions on the metric and hold in important examples such as \( q \)-Minkowskian space\[6\], where (109) were introduced in connection with the isomorphism between spacetime vectors and covectors.

We therefore require our invariant metric \( \eta \) to obey (109) also. In this case we can reasonably expect that a Gaussian \( g_\eta \) can be found at least for sufficiently well-behaved \( R \).

We have already given general arguments for the integral \( \int \). Putting these assumptions together we have:

**Theorem 5.1** Let \( R, R' \) and an invariant matrix \( \eta \) obey the conditions above. Then

\[ Z[\{x_1 \cdots x_m\}] = \left( \int x_1 \cdots x_m g_\eta \right) \left( \int g_\eta \right)^{-1} \]

is a well-defined linear functional from the algebra (94) generated by the co-ordinates \( \{x_i\} \) to \( \mathbb{C} \) and can be computed inductively by

\[ Z[1] = 1, \quad Z[\mathbf{x}] = 0, \quad Z[\mathbf{x} \cdot \mathbf{x}_2] = \eta_{12} (PR)_{12} \lambda^2 \]
\[ Z[x_1 \cdots x_m] = \sum_{i=0}^{m-2} Z[x_1 \cdots \hat{x}_{i+1} x_{i+2} \cdots x_m] Z[x_{i+1} x_{i+2}] (PR)_{i+2} \cdots (PR)_{m-1} m \lambda^{2(m-2-i)} \]

where \( \hat{\cdot} \) denotes omission. We call \( Z \) the Gaussian-weighted integral functional on the braided space. It is invariant under the background quantum group.

**Proof** The formulae here come from the Gaussian integration method as explained above. We have

\[
\int x_1 \cdots x_m g_\eta = - \int x_1 \cdots x_{m-1} \partial_m g_\eta = - \int x_1 \cdots x_{m-2} (-\eta_{m-1} m + \partial_{m-1} x_m) (PR)_{m-1} m \lambda^2 g_\eta
\]

much as before, but this time using (108) and the compact notation. We repeat this again for the \( x_{m-2} \partial_{m-1} \) etc., until we reach \( \int \partial = 0 \). For \( m = 2 \) we have \( Z[x_1 x_2] \) in terms of \( \eta \), which we then use to give the general form stated. The invariance corresponds to the assumption that \( \int, g_\eta \) should be invariant under the quantum group. □

This gives a way to compute the integral knowing only the R-matrix data and \( \eta \). In principle, one can in fact take Theorem 5.1 as an inductive definition of \( Z \), verifying directly from the initial data (92)–(94) and (104), (109) that it is well-defined and invariant. Thus, to see invariance, we use the FRT relations for the quantum group [21]cf[22] in the form \( PR t_1 t_2 = t_1 t_2 PR \). Then

\[
Z[x_1 x_2 t_1 t_2] = \lambda^2 \eta_{12} (PR)_{12} t_1 t_2 = \lambda^2 \eta_{12} t_1 t_2 (PR)_{12} = Z[x_1 x_2]
\]

using (104) for the last equality. We then prove the general case in the same way by induction using the stated formula for \( Z[x_1 \cdots x_m] \) in terms of lower ones and \( PR \). We can likewise verify directly that \( Z[x_1 x_2] \) is well-defined as

\[
Z[x_1 x_2 PR'] = \lambda^2 \eta_{12} PPR' = \lambda^2 \eta_{12} PR' PR = Z[x_1 x_2]
\]

using (14) and (103). For the higher orders the direct proof is rather involved and will not be attempted here. In lieu of a formal proof, we have given general arguments based on integrability of (101) to expect this for any reasonable \( R, R', \eta \). Since the construction of \( Z \) is purely algebraic, any further subtleties about convergence of powerseries and ‘quantum
analysis’ which would be needed for \( f \) and \( g \eta \) themselves, do not apply. Moreover, it is easy enough to compute \( Z \) by our formula and check in each example that it is well-defined.

We remark also that the factorisation in Theorem 5.1 (and a similar one involving \( \tilde{R} \) in the general non-invariant case) is reminiscent of the well-known factorisation property of the \( S \)-matrix in certain exactly solvable quantum statistical systems. It could be viewed as some kind of quantum or braided correlation function and \( Z[x_1x_2] \) as the 2-point function. On the other hand, the \( x_i \) are for us are non-commutative position co-ordinates, such as those of \( q \)-Euclidean and \( q \)-Minkowski space.

5.3.1 Example of \( q \)-Euclidean space

For \( q \)-Euclidean space, we use the definition in \([7]\) as twisting \( \bar{M}_q(2) \) of the usual \( 2 \times 2 \) quantum matrices. We have generators \( a, b, c, d \) and relations

\[
ba = qab, \quad ca = q^{-1}ac, \quad da = ad, \quad db = q^{-1}bd \quad dc = qcd \\
bc = cb + (q - q^{-1})ad.
\]

This is actually isomorphic to the usual \( M_q(2) \) by a permutation of the generators, so one can regard the following as integration on this with its additive structure as introduced in \([23]\).

It is easy to find the quantum metric from the data in \([7]\), to which we refer the reader. The \( R, R' \) are built from two copies of the usual \( SL_q(2) \) R-matrix. The metric is built from the invariant metric associated to each of these and comes out as

\[
\eta_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -q^{-1} & 0 \\ 0 & -q & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]

(111)

It is easy to see that it obeys the conditions (109) needed for integrability. We need \( \lambda = q^{-1} \) to connect with the quantum group normalisation.

Using Theorem 5.1, we find that the ‘two-point function’ comes out as proportional to the metric,

\[
Z[x_i x_j] \equiv \frac{\int x_i x_j g}{\int g} = q^{-4} \eta_{ij}
\]

(112)

while the next lowest order is therefore

\[
Z[x_i x_j x_k x_l] \equiv \frac{\int x_i x_j x_k x_l g}{\int g} = q^{-8} \eta_{ij} \eta_{kl} + q^{-10} \eta_{ia} \eta_{jb} R^d_{k \ l} + q^{-12} \eta_{ia} \eta_{bc} R^b_{j \ d} R^c_{k \ l}
\]

(113)
where $R = R_+$ in [7]. Similarly for higher orders, each involving more powers of $\eta$. It is a non-trivial fact that these are indeed well-defined linear maps $M_q(2) \rightarrow \mathbb{C}$. This can easily be checked for lower orders by computer calculations. We have done so using the computer package REDUCE. The non-zero integrals of quartic expressions come out as

$$Z[abcd] = -q^{-1}, \quad Z[acbd] = -q^3, \quad Z[a^2d^2] = q^2 + 1$$

$$q^2 Z[b^2c^2] = q^{-2} Z[c^2b^2] = Z[bc^2b] = Z[c^2b] = q^2 + 1, \quad q^2 Z[bc^2] = Z[c^2b] = q^4 + 1,$$

times an overall factor $q^{-10}$ and plus the others needed for consistency under the first five relations in (110). One can see that the remaining relation in (110) is respected.

We can also consider the spacetime and radius co-ordinates

$$t = \frac{a - d}{t}, \quad x = c - qb, \quad y = \frac{c + qb}{t}, \quad z = a + d, \quad r^2 = ad - q^{-1}bc$$
in terms of the matrix generators $a, b, c, d$. Then

$$Z[t^2] = Z[z^2] = \frac{2}{q^4}, \quad Z[x^2] = Z[y^2] = q^2 Z[r^2] = \frac{[2]}{q^4}$$

$$\frac{1}{2} Z[t^4] = \frac{1}{2} Z[z^4] = \frac{3[2]}{q^{10}}, \quad \frac{1}{2} Z[x^4] = \frac{1}{2} Z[y^4] = q^4 Z[r^4] = \frac{[3]!}{q^{10}}$$

where $[n] \equiv \frac{q^n - 1}{q - 1}$ is the usual $q$-integer. These results agree with the classical values after setting $q = 1$ where $r^2 = \frac{1}{4}(t^2 + x^2 + y^2 + z^2)$. We see that the Gaußian-weighted integral $Z$ has a degree of spherical symmetry even in the non-commutative case.

We remark that in this example (and other cases such as the $q$-Minkowski space in the next section) one has the additional $R$-matrix relations

$$R_{\alpha}^{\beta} R_{\beta}^{\gamma} = q^{-2} \eta^{\alpha \gamma}$$

$$\eta_{\alpha \beta} R_{\beta}^{\gamma} R_{\gamma}^{\alpha} = q^{2} \eta_{\alpha \beta}$$

with $\eta^{\alpha \beta}$ being the inverse to $\eta_{\alpha \beta}$. In this case one can compute the Gaußian explicitly as

$$g = \sum_{r=0}^{\infty} \frac{1}{[r]!} \left( \frac{x \cdot x}{1 + q^{-2}} \right)^r = e^{\frac{x \cdot x}{q^2}}$$

where $[r] = (1 - q^{2r})/(1 - q^2)$ and $x \cdot x = x_a x_b \eta^{ab}$. We use (114) to compute the commutation relations

$$\partial_j (x \cdot x) = (1 + q^{-2}) x_j + q^2 x \cdot x \partial_j$$

(117)
and (115) to show that $x \cdot x$ commutes with $x_j$, after which (116) follows. This makes contact with the specific Gaußian for $SO_q(n)$-quantum planes in [18]. Our $R$-matrix formulation is however, more general.

5.3.2 Example of $q$-Minkowski space

For $q$-Minkowski space, we use the definition as the algebra $BM_q(2)$ of $2 \times 2$ braided hermitian matrices [1] with generators $a, b, c, d$ and relations

$$ba = q^2 ab, \quad ca = q^{-2} ac, \quad da = ad, \quad bc = cb + (1 - q^{-2}) a(d - a)$$

$$db = bd + (1 - q^{-2}) ab, \quad cd = dc + (1 - q^{-2}) ca$$

(118)

The $R, R'$ description and the necessary metric are also known. We use the recent formulation of [3], but see also [24] [25]. The quantum metric we need is

$$\eta_{ij} = \begin{pmatrix} 0 & 0 & -q^{-2} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 - q^{-2} \end{pmatrix}$$

(119)

and is shown in [3] to obey the key equations (109) for integrability. Here $\lambda = q^{-1}$.

From Theorem 5.1 the ‘two-point function’ again comes out as proportional to the metric and with the same factors as in the Euclidean case, i.e. we have (112)--(113) again. The metric is now (119) and $R = R_L$ in [3]. Similarly for higher orders in $\eta$. Again, it is a non-trivial fact that these are indeed well-defined linear maps $BM_q(2) \to \mathbb{C}$. This has been verified to low orders using REDUCE. For the the non-zero integrals of quartic expressions, we have

$$Z[abcd] = Z[abdc] = Z[adbc] = Z[bcad] = -q^{-2}, \quad Z[acbd] = Z[acdb] = Z[adcb] = -q^2$$

$$q^2 Z[bdcd] = q^2 Z[dbdc] = Z[cdbd] = Z[dcbd] = 1 - q^2$$

$$q^4 Z[bcd^2] = q^4 Z[bdcd] = q^4 Z[dbdc] = q^2 Z[cd^2b] = 1 - q^4$$

$$q^2 Z[bd^2c] = Z[cd^2b] = Z[dcbd] = Z[d^2cb] = -\frac{1}{2} Z[d^4] = -(q - q^{-1})^2$$

$$Z[ad^3] = (1 + 2q^2)(1 - q^{-2}), \quad q^4 Z[bcbc] = q^2 Z[cbcb] = q^4 + 1$$

$$Z[a^2d^2] = q^4 Z[b^2c^2] = Z[c^2b^2] = q^2 Z[bc^2b] = q^2 Z[cb^2c] = q^2 + 1$$

times an overall factor $q^{-10}$ and plus the others cases needed for consistency under the first three relations in (118). One can see that the remaining three relations in (118) are respected.
We can also consider the spacetime and radius co-ordinates

\[ t = q^{-1}a + qd, \quad x = b + c, \quad y = \frac{b - c}{t}, \quad z = d - a, \quad r^2 = ad - q^2 cb \]

in terms of the matrix generators \( a, b, c, d \). Then

\[ Z[t^2] = Z[r^2] = \frac{[2]}{q^4}, \quad Z[x^2] = Z[y^2] = Z[z^2] = -\frac{[2]}{q^6} \]

\[ \frac{1}{2}Z[z^4] = \frac{3[2]}{q^{12}}, \quad \frac{1}{2}Z[t^4] = \frac{1}{2}q^4Z[x^4] = \frac{1}{2}q^4Z[y^4] = Z[r^4] = \frac{[3]!}{q^{10}} \]

which is consistent at \( q = 1 \) with \( r^2 = \frac{1}{4}(t^2 - x^2 - y^2 - z^2) \). We see that the Gaussian-weighted integral \( Z \) is quite similar to the Euclidean one in its values, except for the sign in the spacelike directions.

This \( q \)-Minkowski space is a striking example of our algebraic machinery because classically of course, the Gaussian has far from rapid decay in the space-like directions! The integrals \( \int x_i x_j x_k x_l g \) etc, and \( \int g \) are both infinite but their ratio \( Z[f] = \frac{\int fg}{\int g} \) is perfectly well-defined by the algebraic formulae in Theorem 5.1. This is familiar in the context of path integrals in quantum theory, but applies just as well for Gaussian-weighted integrals in spacetime. Moreover, it applies just as well in the \( q \)-deformed non-commutative setting.

From an algebraic point of view, the structure of \( q \)-Minkowski space is in fact strictly related to the \( q \)-Euclidean structure by a ‘quantum Wick rotation’ [4].

### 5.3.3 Example of \( SL_q(n) \) quantum planes

Here we demonstrate the Gaussian method for the \( SL_q(n) \) quantum plane, using the usual Euclidean metric \( \eta_{ij} = \alpha^{-1}\delta_{ij} \). One can check that even though this is not invariant, the special form of the relations (75)–(76) is such that (101) is consistent with the relations of the algebras, i.e. that the equation is ‘quantum integrable’. As before, we do not actually compute the Gaussian \( g \).

We compute \( Z[f] \) where \( f \) is a polynomial in our non-commuting co-ordinates \( x_i \). Without loss of generality we can assume that its terms are conveniently ordered in decreasing co-ordinate number. Then the Gaussian method gives on each such term:

\[ Z[f]I(g) = \int x^n \cdots x^2 x^1 g \]
\[ \begin{align*}
&= -\alpha^{-1} \int x_n^{r_n} \cdots x_2^{r_2} x_1^{r_1-1} \partial^1 g \\
&= \alpha^{-1} q^{-2r_1} [r_1 - 1] \int x_n^{r_n} \cdots x_2^{r_2} x_1^{r_1-2} g \\
&= \alpha^{-r_1/2} q^{-2(r_1+\ldots+r_n)/2} [r_1 - 1]!! \int x_n^{r_n} \cdots x_2^{r_2} x_1^{r_1-2} g
\end{align*} \]
for \( r_1 \) even, and 0 for \( r_1 \) odd. Inductively then, we have
\[ Z[x_n^{r_n} \cdots x_2^{r_2} x_1^{r_1}] = \alpha^{-(r_1+\ldots+r_n)/2} q^{-(r_1^2+\ldots+r_n^2)/2} [r_1 - 1]!! \ldots [r_n - 1]!! \] (120)

The choice of ordering ensured that the derivatives produced through the rhs. of (78) did not find co-ordinates to act on. In this way the problem decomposed into \( n \) integrations where each of them is performed effectively like a 1 dimensional integration. Such an effective decomposition was of course to be expected since such a phenomenon already appeared in the calculation that had lead to the effective equation (85). Let us stress, however, that we ordered the polynomials only for convenience. We would have come to exactly the same result if they were not especially ordered. Our formalism, being adjusted to the noncommutative setting, takes care of that automatically and works also in the general case even where no ordering can be found that would decouple the \( n \)-dimensional integration as it does here.

We recall also that we gave an explicit candidate for \( f \) in (91) in this \( SL_q(n) \) case. Using it above would of course give the same answer since \( Z \) is independent of the exact form of \( f \) as long as it is translation-invariant. On the other hand, we could also compute this particular \( f \) another way. Namely, consider
\[ \int_{-\infty}^{\infty} f_n(x_n) g_n(x_n) \cdots \int_{-\infty}^{\infty} f_1(x_1) g_1(x_1) \] (121)
where the functions \( g_1,\ldots,g_n \) are Gaußians, each dependent only on one co-ordinate and fulfilling \( \partial^i g_i = -\alpha x_i g_i \). This is actually the opposite ordering from the one that facilitated the calculation in (89). With this choice, we can now integrate using the Gaußian method in each dimension separately as in Section 4.2. This gives
\[ c \int_{-\infty}^{\infty} g_n(x_n) \cdots \int_{-\infty}^{\infty} g_1(x_1) \] (122)
Comparing Section 4.2 with (120) we see that \( c = Z[f_n(x_n) \cdots f_2(x_2) f_1(x_1)] \). Thus we conclude that in the \( SL_q(n) \) case of the global integral of \( f(x_1,\ldots,x_n)g \) can also be obtained
as the result of \( n \) independent Gaussian integrations, namely by writing \( f \) as a sum of ordered terms and proceeding as in \([21]\). This is a kind of \( q \)-direct product theorem which says that we can compute our Gaussian-induced integrals as a product of lower-dimensional ones provided suitable attention is paid to ordering and \( q \)-factors. This can be expected to be a general phenomenon for \( R \)-matrices of Hecke type which are the ‘gluing’ of lower-dimensional \( R \)-matrices.

6 Fourier transform on braided spaces

The main work in this paper has been to develop a fairly general theory of translation-invariant \( q \)-integration on braided vector spaces. The underlying theory of braided-linear algebra, braided-coaddition, braided-differentiation and the exponential map on these spaces has already been introduced (by the second author\([2][3][4]\)) and includes functions in 1 variable (the braided line), the quantum plane \( yx = qxy \), \( q \)-Minkowski space, etc.

We are now in a position to combine our integration theory with this existing theory and pick off an easy application, namely the theory of Fourier transformation on such spaces.

We develop this first in an abstract diagrammatic language that works for any braided group and translation-invariant ‘integration’ functional on it. This diagrammatic technique has been developed in \([8]\) and several other papers by the second author, and is by far the easiest way to prove our results. After that, we examine how the theory looks in the 1-dimensional and \( n \)-dimensional cases.

Note that the theory of quantum Fourier transform on quantum groups and certain self-dual braided groups coming from quantum groups by transmutation was already covered in \([27][28]\). Our diagrammatic definitions are compatible with this but formulated now without the need for tensor categories with left and right duals and reconstruction theorems etc. The present more general theory is needed to cover the braided vector spaces such as quantum planes etc, which were not treated before. It should be stressed that the abstract Fourier theory is quite straightforward and the non-trivial part lies in the construction of the integration itself.
Figure 1: (a) Axioms of a braided-Hopf algebra or ‘braided group’ (b) Axioms for dual braided group $B^*$ (c) The coevaluation map as an abstract exponential

6.1 Diagrammatic Fourier transform

Let $B$ be a braided-Hopf algebra or ‘braided group’ as introduced by the second author [1]. We use the notation of [2] where the product is written as $\cdot = \triangleright$, the coproduct as $\Delta = \wedge$ and the braiding as $\Psi = \times$. Other maps are written as boxes or labelled nodes with the appropriate number of inputs and outputs. All maps are written as flowing generally downwards. The axioms of a braided-Hopf algebra are then recalled in Figure 1 (a). Here $\epsilon$ is the counit and $\eta$ the unit. These have trivial braiding with everything and hence can be written as going to/from nothing. In the diagrammatic language, the complex numbers and other bosonic objects need no strings attached. The rules of braided algebra are that sliding nodes under strings etc (without cutting any strings) does not change the result of going from the top to the bottom.

We suppose that $B$ has a left dual Hopf algebra $B^*$ in the sense that there is an evaluation pairing $\langle , \rangle = \triangledown : B^* \otimes B \to \mathbb{C}$ obeying certain properties. Namely, there should also be a coevaluation $\text{coev} = \cap : \mathbb{C} \to B \otimes B^*$ such that we have the ‘double-bend axioms’ $\cap = |$ as the identity map $B \to B$ and $\cap = |$ as the identity map $B^* \to B^*$. The product in $B$ should be related to the coproduct in $B^*$ and vice-versa in the manner
recalled in Figure 1 (b). These elementary concepts from braided group theory are all that we will need. An introduction to these concepts and methods is in [8].

Our first observation is an elementary one: applying the bend-straightening axioms to the pairing of $B$ and $B^*$ in Figure 1(b), we obtain that $\exp = \coev$ obeys

\begin{align}
(\Delta \otimes \text{id}) \exp &= \exp_{23} \exp_{13}, \\
(id \otimes \Delta) \exp &= \exp_{13} \exp_{12}
\end{align}

which we have written in diagrammatic form in Figure 1(c). If we think of the coproduct as ‘addition’ in $B$ or $B^*$ (which will be exactly its role in our examples) we see that the coevaluation always obeys the characteristic property of an exponential. If $\{e_a\}$ is any basis of $B$ and $\{f^a\}$ a dual basis then $\exp = \sum e_a \otimes f^a$ is the corresponding braided exponential. In the infinite-dimensional case it means of course that $\exp$ is a formal power-series, but one can still proceed by working order by order in a deformation parameter etc in the manner well-known for the universal $R$-matrix of a quantum group.

The role of the pairing $\langle , \rangle$ itself is to provide an action of $B^*$ on $B$ by evaluation against the coproduct (the coregular representation) as already explained in [8]. This action plays the role of differentiation in our abstract picture. Thus the notion of duality of braided-Hopf algebras has two pieces, evaluation and coevaluation. When we think of $\Delta$ as ‘coaddition’, these become differentiation and exponentiation respectively. We will of course demonstrate all this concretely in our examples in Sections 6.2 and 6.3.

Next, we assume that we have a left integral $\int : B \to \mathbb{C}$ on $B$. This is required to obey

\[(\text{id} \otimes f) \circ \Delta = \eta \otimes f,\]

which is just the usual definition of translation invariance under the coproduct. We also require that the map has trivial braiding with other objects, i.e. can be represented as a map from $B$ into a free node. In practice, all our constructions are covariant under a background quantum group which induces the braiding, and this last condition is the assertion that the integral is invariant under the background quantum group.

The left integral is the final ingredient we need for a Fourier transform. It also allows one to define a new ‘convolution’ product on $B$:

**Theorem 6.1** We introduce a convolution product on a braided-Hopf algebra by $* =$
Figure 2: (a) Proof of associativity of the convolution product (in box) for a braided-Hopf algebra equipped with a left integra (b) Lemma needed in proof

\[(f \circ \otimes \text{id}) \circ (S \otimes \text{id}) \circ \Delta. \text{ Then } \ast \text{ is an associative product } B \otimes B \to B.\]

**Proof** We write the definition of \( \ast \) diagrammatically in the box in Figure 2 (a) and prove that it is associative. For the first equality we used the coassociativity of \( \Delta \), for the third that \( S \) is a braided-anti-algebra homomorphism, and for the fourth we used a lemma proven separately in part (b). For the lemma we use the axioms in Figure 1 of a Hopf algebra to introduce an antipode loop, and then the left-invariance of \( \int \).

Given a left integral, we define the abstract *Fourier transform operator* \( \mathcal{F} : B^* \to B \) and abstract *delta-functions* on \( B^* \) in the obvious way by

\[ \mathcal{F} = (\int \circ \otimes \text{id}) \circ \exp, \quad \delta^* = \mathcal{F} \circ \eta = (\int \otimes \text{id}) \circ \exp. \tag{124} \]

**Proposition 6.2** The Fourier transform operator \( \mathcal{F} : B \to B^* \) intertwines the standard right coregular representation of \( B^* \) on \( B \) in \([8]\) with the action by right multiplication in \( B^* \).
Figure 3: Fourier transformation (lower box) maps the usual action of $B^*$ by vector fields (upper box) to multiplication in $B^*$

Proof  This is given in Figure 3. The left upper box is the right coregular action of $B^*$ on $B$ from [8][5]. The lower left box is $F$. We use the lemma in Figure 2(b). Then we use (123) from Figure 1(c). For the last equality we use $\exp = \bigland$ and the double-bend axiom for the coevaluation. □

The right-regular action here is the one introduced in [8] and already computed for matrix braided groups in [5]. It corresponds to the vector fields on $B$ generated by elements of $B^*$ acting by ‘translation’ on the underlying braided group, and always respects the product of $B$ (which becomes a right braided-module algebra). The role of the antipode is to convert the left action of these differential operators to an action from the right. Proposition 6.2 is the fundamental property of Fourier transform.

Another useful property is

$$\Delta \circ \mathcal{F} = (\mathcal{F} \otimes \text{id}) \circ (\cdot \otimes \text{id}) \circ (\text{id} \otimes \exp)$$

which follows at once from (123). This is useful for computing the differential of a Fourier transform. The Fourier transform also behaves well with respect to convolution:

Theorem 6.3  The Fourier transform maps the opposite convolution product from Theorem 6.1 for $B$ to the usual product of $B^*$.

Proof  The opposite convolution product means $\ast \circ \Psi$ and is also associative. The proof that $\mathcal{F}$ is then an algebra homomorphism from this to $B^*$ is in Figure 4. The second equality is (123). The third is the lemma in Figure 2(b). □
Figure 4: Fourier transformation maps convolution $\ast \circ \Psi$ to the product of $B^*$

To complete our picture we show that the Fourier transform is invertible. For this we need to assume that we can also make a Fourier transform on $B^*$, which we ensure by providing ourselves with a (right-handed) integral $f^*: B^* \to \mathbb{C}$ in addition to the integral above on $B$. If $B$ is our quantum position space then $B^*$ is our quantum momentum space and we require that we can integrate over this too. We define

$$
\text{vol} = \int^{*} \delta^{*} = (\int \otimes \int^{*}) \circ \exp, \quad \mathcal{F}^{*} = (\text{id} \otimes \int \circ \cdot) \circ (\Psi \circ \text{id}) \circ (\text{id} \otimes \exp)
$$

where $\mathcal{F}^{*}$ is the version of Fourier-transform on $B^*$ that we need. It is a right-handed version (in contrast to our left-handed one above) converted to a left-handed setting by $\Psi$.

**Proposition 6.4** $\mathcal{F}^{*}\mathcal{F} = S \text{vol}$, where $S$ is the braided-antipode.

**Proof** This is in Figure 5(a). The second equality is (123), the third is the lemma in Figure 2(b) and last is a useful lemma which we give separately in part (b). In the proof of the lemma we use the double-bend axiom for $\exp = \bigwedge$ and then (123), followed by the assumption that $f^*$ is a right-integral. $\Box$

One can show from this and a similar calculation that if $S$ is invertible then so is $\mathcal{F}$, with inverse $S^{-1} \circ \mathcal{F}^{*}$. The braided-antipode $S$ here plays the role of the minus sign in usual Fourier theory, while $\text{vol}$ plays the role of $2\pi$. One can also define a delta-function on $B$ by $\delta = (\text{id} \otimes f^*) \circ \exp$ and show that

$$
(\text{id} \otimes \int)(\Psi \circ \text{id}) \circ \Delta \circ \delta = S \text{vol}
$$

(127)
which is the characteristic evaluation property of delta-functions. This is given in Figure 5(a) also as the right-hand half of the proof. One can show by the same methods that if our integrals are both left and right invariant (the nicest possible case) or at least are invariant under $S$, then $\delta$ is the identity for the convolution product.

At this level of abstraction, our theory here works for Hopf algebras in any braided category. This includes of course the (more standard) theory for ordinary Hopf algebras, as well as [27][28] for the self-dual braided groups $B = H$ given by transmutation of quantum enveloping algebras $H$. In the latter case $\exp = (S \otimes \text{id})(R_{21}R_{12})$ where $R$ is the universal R-matrix and $S$ the ordinary antipode of $H$.

### 6.2 1-dimensional case

In this section we consider how these abstract constructions look in the one-dimensional case where $B = \mathbb{C}[x]$ the functions in one formal variable $x$ as studied in Sections 1–4. This time we make full use of the Hopf algebra structure as it was introduced in [11], namely

$$\Delta x^m = \sum_r \left[ \frac{m}{r} q^r \right] x^r \otimes x^{m-r}, \quad Sx^m = (-1)^m q^{\frac{m(m-1)}{2}} x^m, \quad \epsilon x^m = 0. \quad (128)$$

We use conventions with $q$ rather than $q^2$ as used before.
For the dually-paired Hopf algebra $B^*$ we take the same Hopf algebra with variable $v$ say and analogous structure. Both live formally in the braided category of $\mathbb{Z}$-graded algebras with $x$ of degree $|x| = 1$ and $v$ of degree $|v| = -1$. The braiding has a factor $q^{|x||v|}$ so that

$$\Psi(x \otimes x) = qx \otimes x, \quad \Psi(v \otimes v) = qv \otimes v, \quad \Psi(x \otimes v) = q^{-1}v \otimes x, \quad \Psi(v \otimes x) = q^{-1}x \otimes v.$$  

This shows up when we consider two or more copies of the algebras. For example, the braided tensor products $\mathbb{C}[x] \otimes \mathbb{C}[y]$ with $y$ a copy of $x$, $\mathbb{C}[v] \otimes \mathbb{C}[w]$ with $w$ a copy of $v$ and $\mathbb{C}[x] \otimes \mathbb{C}[v]$ have the relations

$$yx = qxy, \quad vw = qvw, \quad vx = q^{-1}xv \quad \text{(129)}$$

respectively, since $vx \equiv (1 \otimes v)(x \otimes 1) = \Psi(v \otimes x) = q^{-1}x \otimes v \equiv q^{-1}xv$ etc. The commutation relations depend on exactly which algebra one is working with (i.e. the order). For example, $\mathbb{C}[v] \otimes \mathbb{C}[x]$ is a different algebra. This is why the braiding notation with $\Psi = \times$ is useful as a way to keep track of the $q$-factors in a completely coherent way. The coproduct above is by definition the linear one extended as an algebra homomorphism to $\mathbb{C}[x] \otimes \mathbb{C}[y]$, so $\Delta f = f(x + y)$.

The pairing we take between $B, B^*$ is

$$\langle f(v), g(x) \rangle = \epsilon \circ f(\partial)g, \quad \text{i.e.} \quad \langle v^m, x^n \rangle = \delta_{m,n}[m; q]! \quad \text{(130)}$$

and the corresponding coevaluation, which is our abstract exponential, is

$$\exp = e^{|x|v} = \sum_{m=0}^{\infty} \frac{x^m v^m}{[m; q]!} = \sum_{m=0}^{\infty} \frac{(xv)^m}{[m; q^{-1}]!} \equiv e^{|x|v} q^{-1} \quad \text{(131)}$$

as an element of $\mathbb{C}[x] \otimes \mathbb{C}[v]$. If we consider this with the braided tensor product algebra (129) then $x(vx) = q(xv)x$ giving the right hand form if one wants to work in this algebra. From the diagrammatic point of view, however, it is often more convenient to keep the ordering with the symbol $|$ as on the left. One has the properties for an exponential in Figure 1(c) as

$$e_q^{x+y|v} = e_q^{y|v} e_q^{x|v}, \quad e_q^{x|v+w} = e_q^{w|v} e_q^{x|v}, \quad e_q^{x|v} | e_q^{x|v} = e_q^{x|v} | e_q^{x|v} \quad \text{(132)}$$
where the spaces are to insert the other factor in each term of the exponential. For example, the first case lives in $\mathbb{C}[x] \otimes \mathbb{C}[y] \otimes \mathbb{C}[v]$. Since $e_q^y|v$ is bosonic, it commutes with $x$ in this algebra we can also write it as shown. Similarly for the second half. Note also that it would be more conventional to consider $v$ as an ordinary number, so that $\exp = e_q^{xv}$, but this would not work in the general $n$-dimensional case and would also not be consistent with the second half ($132$).

It is clear that $e_q^x\partial f(y) = f(x + y)$, which implies in turn that

$$
(id \otimes \int)\Delta f = \int_{-y^\infty}^{y^\infty} f(x + y) = 1 \int_{-y^\infty}^{y^\infty} f(y), \quad \int_{0}^{y^\infty} f(x + y) \equiv 1 \int_{0}^{y^\infty} f(y) \quad (133)
$$

whenever our class of functions is considered to have the appropriate boundary conditions. We will emphasize the first case but both are possible at our algebraic level (and correspond to Fourier or Laplace transforms in appropriate representations). This provides our left-invariant integral in position space $x$. Note that we also required in the abstract theory that $f$ be bosonic. This is exactly the property of being bosonic in (24). For braiding a second copy $y$ past $f_{-x^\infty}^\infty$ gives $x$ a power of $q$, but the dependence of the integral on the scaling parameter is exactly modulo such powers.

The abstract Fourier theory in Section 6.1 then looks as follows. The convolution product, Fourier transform and the right coregular representation $\langle \rangle$ are

$$(f * g)(y) = \int_{-x^\infty}^{x^\infty} f(x)S_xg(x + y) \quad (134)$$

$$
\mathcal{F}(f)(v) = \int_{-x^\infty}^{x^\infty} f(x)e_q^{x|v} \quad (135)
$$

$$
f \langle g = (Sg(\partial|L^{-1}))f = g(-\partial \circ L^{-1})f \quad (136)
$$

where $S_x$ acts as in (128) on the $x$ variable and $L$ is the scaling by $q$ operator. The formula for $\langle \rangle$ here are from the upper box in Figure 3 as

$$
f \langle v^m = (-1)^m \partial^m q^{\frac{m(m+1)}{2}} \circ L^{-m}(f) = (S\partial^m)f
$$

where the braiding of $f$ past $v^m$ is the origin of the $L^{-m}$ factor. We then apply the antipode to $v^m$ and evaluate the pairing from (130) with $\Delta f$. The second form for $\langle \rangle$ uses $L^{-1} \circ \partial = q\partial \circ L^{-1}$.
The fundamental property of Fourier transform from Proposition 6.2 comes out as

$$\mathcal{F}(f \circ g)(v) = \mathcal{F}(f)(v)g(v)$$  \hfill (137)

One could verify this directly using the $q$-Leibniz rule and integration by parts. Thus,

$$\int_{-\infty}^{\infty} f(x) e_{q}^{x|v} v^{m} = \int_{-\infty}^{\infty} \partial^{m} e_{q}^{x|v} = \int_{-\infty}^{\infty} ((-\partial \circ L^{-1})^{m} f)(x)e_{q}^{x|v} = \int_{-\infty}^{\infty} (f \circ \partial v^{m})e_{q}^{x|v}$$

As a concrete demonstration of this braided Fourier calculus, we compute two examples. For our first example, we can start in the algebra $C[x] \otimes C[y] \otimes C[v]$ and compute

$$f(y) \circ e_{q}^{x|v} = \sum_{m=0}^{\infty} \frac{x^{m}}{[m, q]!} f(q^{m}y) \circ v^{m} = \sum_{m=0}^{\infty} \frac{x^{m}}{[m, q]!} (S\partial^{m})f(y)$$

from the above. Alternatively, the same starting point can be viewed in $C[y] \otimes C[x] \otimes C[v]$ and computed from the upper left box in Figure 3 as follows. Since $e_{q}^{x|v}$ is bosonic, $f(y)$ braids past it without change. We then apply $S_{v}$ to obtain $e_{q}^{-x|v} \otimes f(y)$. The coproduct and evaluation via (130) gives $e_{q}^{-x|\partial} f(y)$ which is the same as before. Either way, Proposition 6.2 then tells us that

$$\int_{-\infty}^{\infty} (S_{x}f(x + y))e_{q}^{y|v} = \int_{-\infty}^{\infty} f(y)e_{q}^{y|v} e_{q}^{x|v}. \hfill (138)$$

The lemma in Figure 2(c) is related to this and looks like

$$\int_{-\infty}^{y} (S_{x}f(x + y))g(y) = \int_{-\infty}^{y} f(y)g(x + y) \hfill (139)$$

for general $f, g$. In these formulae, $S_{x}f(x + y)$ plays the role of $f(-x + y)$.

For a second example, we compute the Fourier transform of our Gaussian $g_{\alpha}$ from Section 4. Thus

$$\partial_{v}\mathcal{F}(g_{\alpha}) = \partial_{v}\int_{-\infty}^{\infty} g_{\alpha}e_{q}^{x|v} = \int_{-\infty}^{\infty} g_{\alpha}xe_{q}^{x|v}$$

$$= \frac{1}{\alpha} \int_{-\infty}^{\infty} (-\partial g_{\alpha})e_{q}^{x|v} = \frac{1}{\alpha} (\mathcal{F} \circ L(g_{\alpha}))v = \frac{1}{\alpha} (L^{-1} \circ \mathcal{F}(g_{\alpha}))v$$

so that $\mathcal{F}(g_{\alpha})$ is of the same general Gaussian type (for a slightly different operator) and with an inverted decay factor. One can obtain the same result using derivatives acting
from the right. In both cases the action of the differential is immediate from (125), which looks as
\[ \mathcal{F}(f)(v + w) = \mathcal{F}(f e_q^x) \).

Here \( w \) is understood to lie to the far right. This in turn is immediate from (132).

To complete the picture, we assume now a similar but right-handed integral in momentum space, i.e. such that
\[ (\int \otimes \text{id}) \Delta f \equiv \int_{-v}^{v} f(v + w) = 1 \int_{-v}^{v} f(v); \quad \int_{0}^{v} f(v + w) = 1 \int_{0}^{v} f(v). \quad (140) \]

Then
\[ \mathcal{F}^*(f) = \sum_{m=0}^{\infty} \frac{x^m}{[m; q]}! \int_{-v}^{v} L^{-m}(f)(v)v^m = \int_{-v}^{v} f(v)e_q^x \quad (141) \]
\[ \mathcal{F}^* \mathcal{F}(f) = \int_{-v}^{v} \int_{-x}^{x} f(x)e_q^x e_q^y = S f(y) \int_{-x}^{x} \int_{-v}^{v} e_q^x e_q^y = S f(y) \text{ vol} \quad (142) \]

where the first from of \( \mathcal{F}^* \) is the definition in Figure 5(a) while the second is written assuming we are in the algebra \( \mathbb{C}[[x] \otimes \mathbb{C}[v]] \). The first expression for \( \mathcal{F}^* \mathcal{F} \) is its definition and Proposition 6.4 asserts that this coincides with the other expressions in our abstract setting. Here \( \int_{-v}^{v} e_q^x = \delta(x) \) is our abstract delta-function.

We have shown in this subsection that there is a coherent \( q \)-calculus for Fourier transforms in one braided variable. One has to be careful about ordering but this can be taken care of systematically by appealing to the diagrammatic method. One can further proceed to fix representations of the system and examine convergence of limits and other issues normally associated with harmonic analysis, for example with \( x \) represented as a real number and \( v \) imaginary or vice versa.

### 6.3 \( n \)-dimensional case

We now give the general \( n \)-dimensional case of the above, i.e. in the same level of generality as in Section 5.3. We take for \( B \) the algebra of braided covectors generated by \( x = \{x_i\} \) as in (96) and for \( B^* \) the algebra of braided vectors generated by \( v = \{v^i\} \) with similar relations \( R^i v_2 v_1 = v_1 v_2 \). We use the standard compact notation (as in Theorem 5.1) where numerical suffices stand for the position of the matrix or tensor indices. Our goal is to see that the various conditions needed in Section 6.1 are satisfied in this context. We
use the differentials and braided-exponentials from \cite{[4]} and the integration theory from Section 5.3 above.

We need first the full braided-Hopf algebra structure of the braided covector and vector algebras. The required braid statistics are \cite{[2]}

\begin{equation}
\begin{align*}
y_1x_2 &= x_2y_1R, \quad w_1v_2 = Rv_2w_1, \quad v_1x_2 = x_2R^{-1}v_1, \quad x_1Rv_2 = v_2x_1
\end{align*}
\end{equation}

if \(y\) is a second copy of the covector algebra and \(w\) a second copy of the vector algebra. These relations correspond to the braiding

\begin{equation}
\begin{align*}
\Psi(x_1 \otimes x_2) &= x_2 \otimes x_1 R, \quad \Psi(v_1 \otimes v_2) = Rv_2 \otimes v_1 \\
\Psi(v_1 \otimes x_2) &= x_2 \otimes R^{-1}v_1, \quad \Psi(x_1 \otimes Rv_2) = v_2 \otimes x_1
\end{align*}
\end{equation}

Recall that in the framework of \cite{[2]}, everything is manifestly covariant under \(x \to xt\varsigma\) and \(v \to \varsigma^{-1}t^{-1}v\) when there is an associated quantum group with matrix generator \(t\) and dilaton \(\varsigma\). It plays the role of the \(Z\)-grading in the 1-dimensional case and induces the braiding \(\Psi\) between any two comodule-algebras in a coherent way. The dilaton is needed when the quantum group normalization factor \(\lambda\) is not 1.

Keeping in mind such braiding \(\Psi\) or the corresponding braid statistics relations, one can add vectors etc, along familiar lines. Thus \(x + y\) obeys the correct relations and statistics with other objects, if \(x, y\) do \cite{[3],[4]}. Abstractly, there is a braided-Hopf algebra structure

\begin{equation}
\begin{align*}
\Delta x &= x \otimes 1 + 1 \otimes x, \quad Sx = -x, \quad \epsilon x = 0
\end{align*}
\end{equation}

extended via the above braiding. The higher powers are given by braided R-binomial coefficients\cite{[4]}. Similarly for \(v + w\). These are the main ideas of braided-linear algebra as introduced by the second author. We refer to \cite{[8]} for an introduction.

Next, we recall the basic formulae for R-differentiation and R-exponentiation introduced in \cite{[4]}. Namely, following the same line of reasoning as in the braided derivation of the \(q\)-derivative, one finds

\begin{equation}
\partial^i(x_1 \cdots x_m) = e^i x_2 \cdots x_m [m; R]_{1\cdots m}
\end{equation}

where \(e^i\) is a basis covector \((e^i)_j = \delta^i_j\) and

\([m; R] = \sum_{k=1}^{m-1} (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12} \cdots (PR)_{m-1,m}\)
is a certain braided integer matrix living in the \( m \)-fold matrix tensor product. Here \( P \) is the usual permutation matrix. The one-dimensional case \( R = (q) \) recovers the usual \( q \)-derivative, while the 2-dimensional case for the \( SL_2 \) R-matrix recovers \([13]\). It was also shown in \([4]\) that the \( \partial^i \) obey the relations of a braided-vector algebra (like the \( v^i \)) and the braided Leibniz rule \([19]\).

Crucial for us now is that this differentiation defines a duality pairing of braided vectors and covectors by

\[
\langle f(v), g(x) \rangle = \epsilon \circ f(\partial)g(x).
\]

With this pairing, we can say that the braided vectors \( v \) are the braided Hopf-algebra dual of the braided covectors \([3]\). We assume that the pairing is non-degenerate, which is true for standard deformations and other generic R-matrices. In this case, we are in the abstract setting of Section 6.1. The coevaluation likewise exists in the general case, as a formal power-series.

Also introduced in \([4]\) was a formula for the R-exponential or coevaluation in this setting. This was given explicitly in the universal or free case where \( R' = P \) so that no relations at all need be assumed among the \( x_i \) or among the \( v^i \) separately. Then \([4]\)

\[
\exp_R(x|v) = \sum_{m=0}^{\infty} x_1 \cdots x_m [m; R]_{1 \cdots m}^{-1} \cdots [2; R]_{m-1, m}^{-1} v_m \cdots v_1
\]

is an eigenfunction of \( \partial^i \) and generates finite translations \([4]\)

\[
\partial^i \exp_R(x|v) = \exp_R(x|v)v^i, \quad \exp_R(x|\partial)f(y) = f(x + y).
\]

Here the eigenvalues \( v^i \) are elements of a non-commutative algebra, such as the vector algebra where the \( \partial^i \) live. Finally, one has also the characteristic properties \([4]\)

\[
\exp_R(x + y | v) = \exp_R(y|v) \exp_R(x|v), \quad \exp_R(x|v + w) = \exp_R(x|w) \exp_R(x|v)
\]

where \( ( | ) \) denotes a space for \( \exp_R(y|v) \) and \( \exp_R(x|w) \) respectively to be inserted in each term of the exponential. In fact, the R-exponential is invariant under the transformation by \( t \) and hence bosonic in the sense that its braid-statistics with anything else is trivial. So we can also write \([150]\) more simply as in \([123]\) and \([132]\) if the appropriate braided
tensor product algebra is understood. Finally, \( \exp \) fits together with the pairing (147) as

\[
\langle f(v), \exp_R(x|w) \rangle = f(w), \quad \langle \exp_R(x|v), g(y) \rangle = g(x)
\] (151)

since it is an eigenfunction of the \( \partial^i \) operators. Here the pairing in the second case is between \( v \) and \( y \).

It is obvious that in the one-dimensional case with \( R = (q) \), the universal \( R \)-exponential collapses to the more usual \( q \)-exponential in the previous subsection. Moreover, the general braided spaces with \( R' \) are quotients of the \( n \)-dimensional free one, and the appropriate \( \exp_R \) is obtained by projecting down our universal one. The only subtlety here is that \( F(m; R) = [m; R]^{-1} \cdots [2; R]^{-1} \) need not be invertible as soon as there are relations among the \( x_i \): all that is really needed in the proofs in [3] is that

\[
\prod_{i=1}^{m} x_i [m; R]^{-1} \cdots [2; R]^{-1} F(m; R)^{m} \cdots v_1 = \prod_{i=1}^{m} x_i [m; q^2]^{-1} \cdots [2; q^2]^{-1} F(m; q^2)^{m} \cdots v_1
\]

For example, an \( R \)-matrix is called \textit{Hecke} if

\[
R_{21} = q^2 R^{-1} + (q^2 - 1) P
\] (152)

and in this case one has[3] a braided vector space with \( R' = q^{-2} R \). Since \( P R' v_2 v_1 = v_2 v_1 \), we know without any calculation that \( F(m; R) = F(m; q^2) = [m; q^2]^m \) in this case. Hence the \( q \)-exponential introduced in [3] collapses in the Hecke case to

\[
\exp_R(x|v) = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{m} x_i [m; q^2]^{-1} \cdots [2; q^2]^{-1} F(m; q^2)^{m} \cdots v_1}{[m; q^2]^m} = e^{x \cdot v}_{q^{-2}}.
\]

Here the second form follows trivially from \( x_1 (x \cdot v) = q^2 (x \cdot v) x_1 \), which is valid in the Hecke case as an easy consequence of the braid-statistics relations (143):

\[
x_1 (x_2 \cdot v_2) = q^{-2} x_2 x_2 R v_2 = q^{-2} x_2 x_1 (q^2 R_{21}^{-1} + (q^2 - 1) P) v_2 = (x_2 \cdot v_2) v_1 + (1 - q^{-2}) x_1 (x_2 \cdot v_2).
\]

This second Hecke form of the R-exponential (148) was stressed recently in [16], who also repeated some of the general derivation of [3] in terms of \( x \cdot v \). We would like to stress that this collapsing of the R-exponential into an ordinary \( q \)-exponential is only valid in the simplest cases such as the Hecke one. Some physically interesting braided vector spaces, such as \( q \)-Minkowski space do have an addition law[3], but the relevant \( R \) is not Hecke.
In this important case, U. Meyer has observed that \( \exp_R(x|v) = e^{x^t v} \) nevertheless holds if \( v \) is a null vector\(^{[29]}\).

Next, we need a left-invariant integration. Here we use \( \int \) as constructed implicitly in Theorem 5.1 for these braided covector spaces. We gave the construction for general \( R, R' \) and an invariant metric \( \eta_{ij} \) subjects to some reasonable conditions. Invariance of the metric ensures that \( f \) is invariant too. This means that it is bosonic in the sense that it has trivial braiding, as we assumed in Section 6.1. To know \( f \) explicitly, we need to assume that the corresponding Gaussian and its inverse can be constructed. Assuming this, we know by construction that the integral of \( \partial f \) vanish. Hence from \(^{[149]}\) we conclude global translation invariance too. Alternatively, one can formulate all our theory directly in terms of the invariant linear functional \( Z \) and consider expressions with \( f \) only as a useful notational. We can now conclude the basic Fourier theory in this setting too. We have

\[
\begin{align*}
    f \ast g(y) &= \int f(x) S_x g(x + y) \quad (153) \\
    \mathcal{F}(f)(v) &= \int f(x) \exp_R(x|v) \quad (154)
\end{align*}
\]

with \( \mathcal{F} \) an algebra homomorphism from \( \ast \circ \Psi \) to the pointwise product for \( v \). We also have the fundamental property

\[
\mathcal{F}(f \triangleleft g)(v) = \mathcal{F}(f)(v)g(v) \quad (155)
\]

where \( \triangleleft \) is the right coregular action of \( v \). There are also similar constructions for a right-integral on the braided vector algebras which then provides for the inverse Fourier transform.

It remains only to compute the right coregular action \( \triangleleft \) in our setting. We do this from the upper left box in Figure 3. From \(^{[144]}\) we have

\[
\begin{align*}
    x_i \triangleleft v^j &= \tilde{R}^{a}_{i} \ b_{b}(\langle -v^b, x_a \rangle) = -\tilde{R}^{a}_{i} \ b_{a} \equiv -u_i^j, \quad \text{i.e.} \quad x_1 R \triangleleft v_2 = -\text{id} \quad (156)
\end{align*}
\]

where the first form is explicit and the second is in the compact notation. We note that the matrix \( u \) here implements the square of the antipode of the background quantum
group. For the more general case we compute

$$
\Psi(x_1 \cdot \cdot \cdot x_m \otimes v^i) = v^a \otimes x_1 \cdot \cdot \cdot x_m \left( \tilde{R}_{m+1} \cdot \cdot \cdot \tilde{R}_{m+1} \right)^i_a
$$

$$
\Psi(x_1 \cdot \cdot \cdot x_m R_{m+1} \cdot \cdot \cdot R_{m+1} \otimes v_{m+1}) = v_{m+1} \otimes x_1 \cdot \cdot \cdot x_m.
$$

(157)

The braiding here is obtained in the manner explained in [2]. The pairing (147) then gives

$$(x_1 \cdot \cdot \cdot x_m) \triangleright v^i$$

in terms of $\tilde{R}$ or

$$(x_1 \cdot \cdot \cdot x_m R_{m+1} \cdot \cdot \cdot R_{m+1}) \triangleright v^{m+1} = -\partial_{m+1} (x_1 \cdot \cdot \cdot x_m)$$

(158)

in terms of $R$. We know from the theory of braided groups in [8, Sec. 4] that $\triangleright$ is necessarily a right action of the braided vector algebra and extends to products of the $x$ as a braided module-algebra. In the present case this means that it is a right-handed braided-derivation

$$(f(x)g(x)) \triangleright v^i = f(x) \triangleright \Psi(g(x) \otimes v^i) + f(x)(g(x) \triangleright v^i).$$

(159)

One can also take a Heisenberg algebra point of view as we did in Section 5 for the usual derivatives. Following the construction analogous to [4] for the usual $\partial^i$, we have

$$
\frac{\partial}{\partial x_i} \frac{v^j}{\partial x^j} = -\tilde{R}^j_i \tilde{R}^a_b \frac{v^a}{\partial x^b} = -u^j_i, \quad \mathrm{i.e.} \quad v^j \frac{\partial}{\partial x^j} - x_a R^a_i \frac{v^j}{\partial x^b} = \delta^i_j
$$

(160)

as a right-handed version of (98), where $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}$ denote operators acting on the position co-ordinates from the right, by multiplication and $\triangleright$ respectively.

Some applications of this theory will be explored elsewhere. In particular, we see that we now have the main ingredients needed to do classical field theory and elements of quantum field theory (such as computing braided-Feynman diagrams) on braided spaces, as part of a general programme of $q$-deforming physics.

7 Appendix

Let us examine the Gaussian function in the quasiposition representations of Section 3. It is the solution of

$$
(\partial + \alpha x)|g_\alpha \rangle = 0
$$

(161)
where
\[ |g_\alpha\rangle = \sum_{s=-\infty}^{+\infty} g_{\alpha s} |v_{cq^{2s}}\rangle. \] (162)

From
\[ \partial + \alpha x = \sum_r \left[ |v_{cq^{2r}}\rangle \left( \frac{1}{cq^{2r}(1 - q^2)} + \alpha cq^{2r} \right) \langle v_{cq^{2r}}| - |v_{cq^{2r}}\rangle \frac{1}{cq^{2r}(1 - q^2)} \langle v_{cq^{2(r+1)}}| \right] \] (163)
we get
\[ g_{\alpha s+1} = (1 + \alpha cq^{4s}(1 - q^2))g_{\alpha s} \] (164)

The normalisation of \( g_\alpha \) be \( \langle v_c|g_\alpha\rangle = 1. \) Thus
\[ g_{\alpha r} = \prod_{n=1}^{r} (1 + \alpha cq^{4n}(1 - q^2)) \quad \text{for } r = 1, 2, ... \] (165)
which is convergent for \( r \to \infty. \) Similarly one obtains
\[ g_{\alpha r} = \prod_{n=1}^{-r} \frac{1}{1 + \alpha cq^{-4n}(1 - q^2)} \quad \text{for } r = -1, -2, ... \] (166)
which is convergent to zero for \( r \to -\infty. \) One can easily check that we even have
\[ (x^t g_\alpha)_r = cq^{2r} \prod_{n=1}^{-r} \frac{1}{1 + \alpha cq^{-4n}(1 - q^2)} \to 0 \quad \text{for } r \to -\infty \] (167)
which proves the rapid decay property of \( g_\alpha. \)

Let us now also check its integrability:
\[ \int_{-\infty}^{+\infty} g_\alpha = 2(1 - q^2) \text{Trace}(x g_\alpha) \]
\[ = 2c(1 - q^2) \left[ 1/2 + \sum_{r=1}^{\infty} q^{2r} \prod_{n=1}^{r} (1 + \alpha cq^{4n}(1 - q^2)) + \sum_{r=1}^{\infty} q^{-2r} \prod_{n=1}^{r} (1 + \alpha cq^{-4n}(1 - q^2))^{-1} \right] \]
Both the infrared and the ultraviolet part of the integral are easily checked to be convergent by using a ratio test.

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