Central limit theorem for linear spectral statistics of block-Wigner-type matrices

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Abstract: Motivated by the stochastic block model, we investigate a class of Wigner-type matrices with certain block structures, and establish a CLT for the corresponding linear spectral statistics via the large-deviation bounds from local law and the cumulant expansion formula. We apply the results to the stochastic block model. Specifically, a class of renormalized adjacency matrices will be block-Wigner-type matrices. Further, we show that for certain estimator of such renormalized adjacency matrices, which will be no longer Wigner-type but share long-range non-decaying weak correlations among the entries, the linear spectral statistics of such estimators will still share the same limiting behavior as those of the block-Wigner-type matrices, thus enabling hypothesis testing about stochastic block model.

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1. Introduction

The investigation into the limiting properties of large random matrices has been popular for over two decades. Many techniques [8][6][13][18] are developed to solve problems in this area. There are plenty of objects of interest, namely the empirical spectral distribution (ESD), the limiting spectral distribution (LSD), the largest eigenvalue, the linear spectral statistics (LSS), the eigenvector statistics, etc. Particularly, the linear spectral statistics have attracted lots of attention ever since the 90s [28]. Various methods are explored to study the behavior of the LSS, such as moment method [5], martingale difference method [7][9], cumulant expansion method [22]. Also there is progress from the stochastic calculus [17][20] and free probability [27]. Further, [14][15] generalize the Stein method and use second order Poincaré inequalities to prove a CLT for the LSS. Specifically, in recent years, a more in-depth understanding of the behavior of the LSS of Wigner and Wigner-type matrices has been achieved by researchers from various perspectives. [26] introduces an interpolation method for more general Wigner matrices than the ones that share the same cumulants with GOE/GUE. [12] extends the CLT to certain heavy-tailed random matrices. More recently, [21] studies the mesoscopic eigenvalue statistics of the Wigner matrices via the Green function and the local law, [16] yields a thorough analysis of fluctuations of regular functions of Wigner matrices and [11] establishes a near-optimal convergence rate for the CLT of LSS of Wigner matrices.

In the meantime, motivations are drawn from social networks and other associated random graph models, which brings the researchers’ attention to more involved matrix models. One of the most classic models in this field is the stochastic block model (SBM). In contrast to the Erdős-Rényi model in which all nodes are exchangeable, the SBM introduces inhomogeneity by dividing the nodes into different communities. In the SBM with nodes $V$ and edges $E$, all edges are undirected, and different edges are independent, in the meantime, the probability that two nodes $v_i, v_j \in V$ connect with each other is only determined by which communities $v_i$ and $v_j$ belong to. In other words, the adjacency matrix of the SBM
can be viewed as a random 0-1 matrix whose entries have block-wise constant expectations. Thus, the centered adjacency matrices of the SBMs are Wigner-type matrices with inhomogeneous variance profiles.

One of the most important questions in the SBM is community detection, which is to recover the community structure underneath via one single observation of the adjacency matrix. Further, an induced problem is to determine the number of communities. For most community detection algorithms, the number of communities needs to be given a priori as a hyperparameter. This motivates hypothesis testing for this parameter via the distributional information of certain test statistics of the model. [25] proposes a sequential test for the renormalized adjacency matrix \( \left( \frac{A_{ij} - p_{ij}}{\sqrt{np_{ij}(1-p_{ij})}} \right) \) and \( \left( \frac{A_{ij} - \hat{p}_{ij}}{\sqrt{n\hat{p}_{ij}(1-\hat{p}_{ij})}} \right) \) based on the Tracy-Widom fluctuation of the largest eigenvalue. In the same spirit, more recently in [10], Banerjee and Ma propose a hypothesis testing for the community structure via the LSS of the renormalized adjacency matrix \( \left( \frac{A_{ij} - \hat{p}_{ij}}{\sqrt{n\hat{p}_{ij}(1-\hat{p}_{ij})}} \right) \) with the method of moments approach [5] in the cases where the SBM has only one community or two asymptotically equal-sized communities. Towards another end, [1] proves a CLT for the LSS of general Wigner-type matrices via the second order Poincaré inequality without providing the explicit formulas for the asymptotic mean and covariance function.

In this paper, we establish CLTs for the class of block-Wigner-type matrices which is motivated by the renormalization \( \left( \frac{A_{ij} - p_{ij}}{\sqrt{n}} \right) \) as well as a correlated matrix model induced from the renormalization \( \left( \frac{A_{ij} - \hat{p}_{ij}}{\sqrt{n}} \right) \). We derive the explicit formulas for the asymptotic mean functions and covariance functions with the help of precise large deviation estimates of the Green function by [4] and the application of cumulant expansion formula [22][19].

**Our contributions.** We strengthen the existing results in the following ways:

1. Our block-Wigner-type matrices may have not only inhomogeneous fourth moments but also inhomogeneous second moments. This greatly extends the potential range of application of the theorem. We show that the approximately low-rank structure of the entries would reproduce itself in terms of repetitive patterns in the system of equations for key moments of \( Tr(G(z)) \) and other related higher-order structures.

2. Further, we establish a CLT for the LSS of the data-driven variation of the above matrices, which is no longer Wigner-type and shares long-range weak correlations among the entries. This yields a direct application in the SBM.

**Organization**

We first introduce a few prerequisites about our main tools and ingredients in Section 2. Then we introduce the block-Wigner-type matrices and establish the CLT for LSS of such matrices in Section 3.1. In Section 3.2, we consider the application to the SBM and establish a new CLT for LSS of a data-driven variation of the block-Wigner-type matrices. In Section 4, the outlines of the proofs of the main results are shown. In Section 5, we
apply the above 2 CLTs to the synthetic data of the SBM to show the efficiency of the theorems. Details of proofs are shown in Section A and B.

2. Preliminary

2.1. Notation

For simplicity of presentation, we will use $M$ for normalized trace $\frac{1}{N}Tr(M)$ of a $n \times n$ square matrix $M$, $\langle R \rangle$ to denote centered version $R - ER$ of a random variable $R$, and $[K] = \{1, 2, \cdots, K\}$ to represent the set of positive integers from 1 to $K$. Further, we introduce two operations related to diagonal terms: for a column vector $v = [v_1, \cdots, v_n]^\top$, $\text{Diag}[v]$ denotes the diagonal matrix whose diagonal elements are the entries of $v$, and for a $n \times n$ matrix $M$, $\text{diag}(M)$ denotes the column vector whose entries are the diagonal element of $M$. In particular, $\text{diag} \circ \text{Diag} = \text{Id}$.

2.2. Large deviation bounds from local law for Wigner-type matrices

This section gives a quick review of the large deviation bounds from the local law for general Wigner-type matrices by Ajanki et al. [3][4], which will serve as one of the main ingredients for proving our central limit theorem.

The main object of interest is the resolvent $G(z) = (H - z)^{-1}$, where $H$ is the so-called Wigner-type matrix such that $H$ is real symmetric and the entries $H_{ij}$ are independent for $i \leq j$ and centered $\mathbb{E}H_{ij} = 0$, $\forall i,j \in [n]$.

Let $S = (s_{ij})_{i,j=1}^n = (\mathbb{E}|H_{ij}|^2)_{i,j=1}^n$, then the system of the quadratic vector equations (QVE) is

$$-\frac{1}{m_i(z)} = z + \sum_{j=1}^n s_{ij}m_j(z), \quad \text{for all } i \in [n], \quad z \in \mathbb{C}_+.$$  \hspace{1cm} (1)

There exists a unique solution $m = (m_1, \ldots, m_n) : \mathbb{C}_+ \to \mathbb{C}_+^n$ of the above system on the complex upper half-plane. We refer the readers to [3] for properties of the QVE system. It is proved by Ajanki et al. [4] that under certain regularity conditions, the solution $m = (m_1, \ldots, m_n)$ of the above system of equations may serve as a good approximation for the diagonal terms $(G_{11}, \ldots, G_{nn})$ of the resolvent.

\textbf{Definition 2.1 (Stochastic domination).} Suppose $N_0 : (0, \infty)^2 \to \mathbb{N}$ is a given function, depending only on certain model parameters. For two sequences, $\varphi = (\varphi^{(N)})_N$ and $\psi = (\psi^{(N)})_N$, of non-negative random variables we say that $\varphi$ is stochastically dominated by $\psi$ if for all $\epsilon > 0$ and $D > 0$,

$$\mathbb{P}(\varphi^{(N)} > N^\epsilon \psi^{(N)}) \leq N^{-D}, \quad N \geq N_0(\epsilon, D).$$

In this case we write $\varphi \prec \psi$ or $\varphi = O_<(\psi)$.

\textbf{Lemma 2.2 (Theorem 1.7 of [4], reformulated to a macroscopic version).} Let $H$ be a Wigner-type matrix and $m$ be defined in (1). Suppose that the following assumptions hold:
A For all $n$ the matrix $S$ is flat, i.e.,

$$s_{ij} \leq \frac{C}{n}, \quad C > 0, \quad i, j \in [n];$$

B For all $n$ the matrix $S$ is uniformly primitive, i.e.,

$$\left((S^L)^{ij}\right) \geq \frac{p}{n}, \quad p > 0, \quad i, j \in [n],$$

C For all $n$ the entries $H_{ij}$ of the random matrix $H$ have bounded moments

$$E |H_{ij}|^k \leq \mu_k s_{ij}^{k/2}, \quad k \in \mathbb{N}, i, j \in [n].$$

are satisfied. Then uniformly for all $z = \tau + i\eta \in \mathbb{C}_+$ with constant order imaginary part $\eta$ or real part $\tau$ that is bounded away from the edge of the spectrum of $H$, the resolvents of the random matrices $H = H^{(n)}$ satisfy

$$\max_{i,j} |G_{ij}(z) - m_i(z)\delta_{ij}| = O_{\prec}(\frac{1}{\sqrt{n}}). \quad (2)$$

Furthermore, for any sequence of deterministic vectors $w = w^{(n)} = [w_1, \cdots, w_n] \in \mathbb{C}^n$ with $\max_i |w_i| \leq 1$, the averaged resolvent diagonal has an improved convergence rate.

$$\left|\frac{1}{n} \sum_{i=1}^{n} \bar{w}_i(G_{ii}(z) - m_i(z))\right| = O_{\prec}(\frac{1}{n}). \quad (3)$$

A direct application of Lemma 2.2 together with the trivial bound $|G_{ij}(z)| \leq \frac{1}{|z|^2}$ leads to the following corollary, whose proof is omitted.

**Corollary 2.3.** \(\forall \varepsilon > 0, K_0 \in \mathbb{N}, \) there exists a $N_{\varepsilon,K_0}$ s.t. when $n \geq N_{\varepsilon,K_0}$,

$$E|G_{ij}(z) - \delta_{ij}m_i(z)|^k \leq \frac{n^\varepsilon}{n^{k/2}},$$

$$E\left|\frac{1}{n} \sum_{i=1}^{n} \bar{w}_i(G_{ii}(z) - m_i(z))\right| \leq \frac{n^\varepsilon}{n^k}, \quad (4)$$

for $k \in [K_0]$, for any fixed $z \in \mathbb{C} \setminus \mathbb{R}$, where $w = w^{(n)} = [w_1, \cdots, w_n] \in \mathbb{C}^n$ is deterministic with $\max_i |w_i| \leq 1$.

### 2.3. Cumulant expansion

The cumulant expansion formula was first introduced to the random matrices literature by Pastur et al. [22].
Lemma 2.4. 
\[ \mathbb{E}[\xi f(\xi)] = \sum_{a=0}^{p} \frac{\kappa_{a+1}}{a!} \mathbb{E}[f^{(a)}(\xi)] + \varepsilon, \quad (5) \]

where \(|\varepsilon| \leq C \sup_t |f^{(p+1)}(t)|E[|\xi|^{p+2}]\) and \(C\) depends on \(p\) only.

The cumulant expansion formula will serve as another important tool in our analysis. In some literature, it is also known as the generalized Stein’s method.

3. Main results

3.1. CLT for LSS of block-Wigner-type matrices

We first define the random matrix model of concern. Note that the initial motivation comes from the stochastic block model. Intuitively, the block-Wigner-type matrix to be defined should be close to a symmetric block-wise i.i.d. matrix. Further, for simplicity and consistency with the SBM, we require that all the diagonal terms \(H_{ii} = 0, \forall i \in [n]\).

First, we introduce the community and the membership operator.

Definition 3.1 (Community and membership operator). Let \(\{C_k\}_{k \in [K]}\) be any partition of \([n]\) with \(K\) components, i.e.
\[
C_{k_1} \cap C_{k_2} = \emptyset, \text{ when } k_1 \neq k_2, \quad k_1, k_2 \in [K].
\]
\[
\cup_{k=1}^{K} C_k = [n].
\]

We call \(C_k\) the \(k\)-th community and define the community membership operator \(\sigma\) s.t.
\[ \sigma(i) = k \text{ iff } i \in C_k, \quad i \in [n], \quad k \in [K]. \]

For simplicity, we will use \(1_{C_k}\) to denote the (column) indicator vector of \(C_k, \forall k \in [K]\).

Further, we assume the community number \(K\) is fixed and the sizes of the communities are comparable.

Assumption 3.2. There exists \(\alpha = [\alpha_1, \cdots, \alpha_K]\), s.t.
\[ n_k := |C_k| = \alpha_k n, \forall k \in [K]. \]
\[ \sum_{k=1}^{K} \alpha_k = 1, \quad 0 < \alpha_k < 1, \quad \forall k \in [K]. \]

Definition 3.3. [block-Wigner-type Matrix] Let \(\kappa_{ij}^{(a)}\) be the \(a\)-th cumulant of \((\sqrt{n}H_{ij})\). If there exists a sequence of partitions \(\{C_k\}_{k \in [K]} = \{C_k^{(n)}\}_{k \in [K]}\), s.t.

a) \(H\) is a real symmetric matrix with mean zero and zero diagonal terms,
\[ \mathbb{E}H = 0, \quad H_{ii} = 0, \forall i \in [n]. \]
b) Assumption 3.2 is satisfied.
c The first 4 cumulants of $(\sqrt{n}H_{ij})$ will be fully determined by the partition $\{C_k\}_{k \in [K]}$ and $K \times K$ constant matrices $Q^{(2)}, Q^{(3)}, Q^{(4)}$, namely, let $\sigma$ be the membership operator induced by $\{C_k\}_{k \in [K]}$, then

$$\kappa^{(k)}_{ij} := k^{(k)}(\sqrt{n}H_{ij}) = \begin{cases} Q^{(k)}_{\sigma(i)\sigma(j)}, & \forall i \neq j \\ 0, & \forall i = j \end{cases}, \quad \forall k = 2, 3, 4,$$

and $Q^{(2)}_{kl} > 0, \forall k, l \in [K]$.

d There exists a deterministic sequence $\{v_a\}_{a \geq 5}$, s.t.

$$\mathbb{E}|\sqrt{n}H_{ij}|^a \leq v_a(\kappa^{(2)}_{ij})^{a/2}, \quad a \geq 5.$$

Then we say that $\{H_n\}$ are block-Wigner-type matrices with model parameters $(K, n, \alpha, Q^{(2)}, Q^{(3)}, Q^{(4)}, \{v_a\}_{a \geq 5})$. For simplicity, we will use $H$ for short in this paper when there is no confusion.

With our $K$-block model, one can easily check that the quadratic vector equations (1) will degenerate into the following $K$-equations.

**Proposition 3.4** (Quadratic vector equation for the block-Wigner-type matrices). Given $H(K, n, \alpha, Q^{(2)}, Q^{(3)}, Q^{(4)}, \{v_a\}_{a \geq 5})$, then for any fixed $z$, the diagonal terms of the resolvent $G = (H - z)^{-1}$ have the following approximation

$$|G_{ii}(z) - M_l(z)| = O_{\prec}(\frac{1}{\sqrt{n}}), \forall i \in C_l, \forall l \in [K], \quad (6)$$

where $M = (M_1, \ldots, M_K) : \mathbb{C}_+ \rightarrow \mathbb{C}_+^K$ is defined to be the unique solution on the complex upper half-plane of the system

$$-\frac{1}{M_l(z)} = z + \sum_{m=1}^K Q^{(2)}_{lm}\alpha_m M_m(z), \quad \text{for all} \quad l = 1, \ldots, K, \quad z \in \mathbb{C}_+. \quad (7)$$

Thus, the Stieltjes transform of the ESD converges to

$$\sum_{l=1}^K \alpha_l M_l(z),$$

and the corresponding measure $\mu_\infty$ is determined by

$$\int_{\mathbb{R}} \frac{1}{x - z} d\mu_\infty = \sum_{l=1}^K \alpha_l M_l(z).$$

**Remark.** One may find the assumption $\alpha_k = \frac{n_k}{n}$ pretty strong. In general, due to the nature of the rational number, one may only expect that $\alpha_k^{(n)} := \frac{n_k}{n} \to \alpha_k$. It then directly follows from the fact $|M_m(z)| \leq |\frac{1}{M(z)}|$ that

$$\sum_{m=1}^K Q^{(2)}_{lm}(\alpha_m^{(n)} - \alpha_m) M_m^{(n)}(z) \to 0.$$
Thus, when we consider only the leading order terms of the equations, we have

\[-\frac{1}{M_l^{(n)}(z)} = z + \sum_{m=1}^{K} Q_{lm}^{(2)} \alpha_m M_m^{(n)}(z) = z + \sum_{m=1}^{K} Q_{lm}^{(2)} \alpha_m M_m(z), \forall l \in [K], z \in \mathbb{C}_+.\]

In other words, the leading term of $M_l(z)$ and $M_l^{(n)}(z)$ will follow the same QVE on the complex upper half-plane by the uniqueness of the solution of the QVE. Then w.l.o.g. we may simply treat the case as $\alpha_k = \frac{n_k}{n}$.

One may argue that the above argument only implies that the limiting spectral distribution will be the same. We claim that it will not affect our CLT as well. Precisely, one may check the system of equations in the preceding sections and note that all the coefficients will count only up to order 1, and all the limiting functions will be fixed once the $|\alpha_k^{(n)} - \alpha_k| = o(1), \forall k$.

The minor order terms in $|\alpha_k^{(n)} - \alpha_k|$ do matter, not in our CLT, but in the normalization term $-n \int f d\mu_\infty$.

**Theorem 3.5.** Let the matrix $H := H_n$ be a sequence of block-Wigner-type matrices with model parameter $(K, n, \alpha, Q^{(2)}, Q^{(3)}, Q^{(4)}, \{v_a\}_{a \geq 5})$. Let $C_{1}(z)$ and $C_{2}(z)$ be $K \times K$ matrices defined by

\((C_{1}(z))_{kl} := Q_{kl}^{(2)} \alpha_k M_k(z) - \delta_{kl} \frac{1}{z} \delta_{kl} M_k(z),\)

and

\((C_{2}(z_1, z_2))_{kl} := Q_{kl}^{(2)} \alpha_k M_k(z_2) - \delta_{kl} \frac{1}{z_1} \delta_{kl} M_k(z_1),\)

respectively. Then the spectral empirical process $G_n = (G_n(f)) := \sum_{i=1}^{n} f(\lambda_i) - n \int f d\mu_\infty$ indexed by the set of analytic functions $\mathcal{A}$ converges weakly in finite dimension to a Gaussian process $G := \{G(f) : f \in \mathcal{A}\}$ with mean function $M(f)$ and the covariance function $V(f, g)$ to be defined below. The mean function is

\[M(f) = -\frac{1}{2\pi i} \int_{\Gamma} \text{Mean}(z) f(z) dz,\]

where $\Gamma$ is a contour that encloses the support of spectrum of $H_n$ and

\[\text{Mean}(z) = \sum_{k=1}^{K} Y_k(z),\]

where $Y(z) = [Y_1(z), \ldots, Y_K(z)]^{\top}$ is the solution of

\[C_{1}(z)Y(z) = -\frac{1}{z} \text{diag}(Q^{(2)} X(z)) + \frac{2}{z} [\alpha_1 Q^{(2)}_{11} M_1^2, \ldots, \alpha_K Q^{(2)}_{KK} M_K^2]^{\top} - \frac{1}{z} \text{diag} \left[ Q^{(4)} (\alpha_1 \alpha_m M_l^2 M_m^2)_{l,m=1}^{K} \right],\]
and $X(z) = (X_{lm}(z))_{l,m=1}^K$ is defined by

$$C_0(z)X(z) = -\frac{1}{z} \text{Diag}([\alpha_1 M_1(z), \ldots, \alpha_K M_K(z)]^\top).$$

The covariance function is

$$V(f, g) = -\frac{1}{4\pi^2} \int \int f(z_1)g(z_2)\text{Cov}(z_1, z_2)dz_1dz_2,$$

where $\text{Cov}(z_1, z_2) = \sum_{l,m=1}^K Z_{lm}$, and $Z := (Z_{lm}(z_1, z_2))_{l,m=1}^K$ satisfies the equation

$$z_1(C_0 Z)_{lm} = -\sum_{k=1}^K 2Q_{lk}^{(2)} W_{l,m,k}(z_1, z_2) + 2Q_{ll}^{(2)} M_l(z_1) X_{lm}(z_2)$$

$$- \sum_{k=1}^K Q_{lk}^{(4)} \alpha_k M_l(z_1) M_k(z_2) X_{lm}(z_2) - \sum_{k=1}^K Q_{lk}^{(4)} \alpha_l M_l(z_1) M_k(z_2) X_{km}(z_2),$$

where $W = (W_{l,m,r})$ is a $K \times K \times K$ tensor and the vector $\tilde{W}^{(l,m)} = [W_{l,m,1}, \ldots, W_{l,m,K}]^\top$ satisfies the equation

$$(z_1 C_0^2(z_1, z_2) \tilde{W}^{(l,m)}(z_1, z_2))_r = -\sum_{k=1}^K Q_{rk}^{(2)} \tilde{X}_{lk}(z_1, z_2) X_{mr}(z_2) - \delta_{rl} X_{lm}(z_2),$$

where $\tilde{X} = (\tilde{X}_{lm})_{l,m=1}^K$ satisfies

$$C_0^2(z_1, z_2) \tilde{X}(z_1, z_2) = -\frac{1}{z_1} \text{Diag}([\alpha_1 M_1(z_2), \ldots, \alpha_K M_K(z_2)]^\top).$$

### 3.2. Application to the stochastic block model: a step forward with the data-driven renormalized adjacency matrices of SBM

As mentioned in the introduction, the stochastic block model serves as one of the primary motivations for the block-Wigner-type matrices. Recall that a stochastic block model is a random graph with $n$ nodes which are divided into $K$ disjoint communities $\{C_k\}_{k=1}^K$, the size of $k$-th community $n_k = |C_k|$ satisfies assumption 3.2. The upper-triangular entries of the symmetric adjacency matrix are independent Bernoulli random variables whose parameters are determined by the community membership of the nodes. In other words, we have a $K \times K$ deterministic symmetric matrix $(\tilde{P}_{ij})_{K \times K}$, such that the symmetric adjacency matrix of the network follows the rule:

$$A_{ij} \in \{0, 1\}, \forall i, j \in [n],$$

$$A_{ii} = 0, \forall i \in [n],$$

$$p_{ij} := P(A_{ij} = 1) = \tilde{P}_{\sigma(i)\sigma(j)},$$

$$A_{ij} \perp A_{kl}, \text{ for } (i, j) \neq (k, l), \ i < j, \ k < l.$$
where \( \sigma(i) \in \{1, 2, \ldots, K\} \) is the membership operator defined by this model and indicates which community node \( i \) belongs to. We can see that it fits the description of our block-Wigner-type matrices after the renormalization:

\[
H_{ij}^{(A)} = \begin{cases} 
\frac{A_{ij} - p_{ij}}{\sqrt{n}}, & i \neq j \\
0, & i = j.
\end{cases}
\] (13)

Further, in statistical application such as the hypothesis testing on a stochastic block model, the connection probabilities \( p_{ij} \)'s are not known a priori. Instead, they need to be directly estimated from the observed graph \( (A_{ij})_{n \times n} \) as defined in (12). Assume the membership operator \( \sigma \) is known, we can define the empirical estimator

\[
\hat{p}_{ij} = \sum_{\alpha \in C_{\sigma(i)}, \beta \in C_{\sigma(j)}} \frac{A_{\alpha\beta}}{N_{\sigma(i)\sigma(j)}},
\]

for \( p_{ij} \), where \( N_{kl} \) is the total number of non-diagonal entries whose first index falls in the \( k \)-th community, and the second index lies in the \( l \)-th community, \( k, l \in [K] \). Namely

\[
N_{kl} = \begin{cases} 
n_k n_l, & \text{if } k \neq l, \\
n_k (n_k - 1), & \text{if } k = l.
\end{cases}
\] (14)

We then consider the data-driven renormalized adjacency matrix

\[
\hat{H}_{ij}^{(A)} = \begin{cases} 
\frac{A_{ij} - \hat{p}_{ij}}{\sqrt{n}}, & i \neq j \\
0, & i = j.
\end{cases}
\] (15)

It turns out that the LSS of \( H \) and \( \hat{H} \) will share similar asymptotic behavior. We have the following theorem.

**Theorem 3.6.** Let the matrix \( H^{(A)} \) be defined by (13), which is a block-Wigner-type matrix with model parameter \( (K, n, \alpha, Q^{(2)}, Q^{(3)}, Q^{(4)}; \{v_a\}_{a \geq 5}) \), and \( \hat{H}^{(A)} \) be defined via (15), then the spectral empirical process \( \hat{G}_n(f) := \sum_{i=1}^{n} f(\lambda_i(\hat{H})) - n \int f d\mu_\infty \) will share the same limiting distributions with \( G_n(f) := \sum_{i=1}^{n} f(\lambda_i(H)) - n \int f d\mu_\infty \) established in Theorem 3.5.

4. Outline of the proof

**4.1. Outline of the proof of Theorem 3.5**

Recall the classic Cauchy integral trick \( f(x) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-x} dz \), which allows us to rewrite the sum

\[
\sum_{j=1}^{n} f(\lambda_j) = -\frac{1}{2\pi i} \oint_C f(z) Tr(G(z)) dz,
\] (16)
where $\mathcal{C}$ is a contour that encloses the support of $H$ with high probability. Naturally one may expect that the behavior of the linear spectral statistics $\sum_{j=1}^n f(\lambda_j)$ will be governed by that of the quantity $\text{Tr} G(z)$.

Inspired by the previous works such as [22][23][9][7], our proof first combines the characteristic function method with the cumulant expansion to prove the finite-dimensional convergence of the process $\langle \text{Tr}(G(z)) \rangle$, then with the tightness of the process we proceed to the linear spectral statistics. To be more specific, our tasks are divided into 4 steps mainly:

- Expectation;
- Covariance;
- Normality;
- Tightness.

We use the resolvent identities $G = \frac{1}{z}(HG - I)$ so that the cumulant expansion formula could be applied. Then we use the block structure to simplify the calculations. Let $T_k := \text{Id}|_{C_k}$ be the restriction of the identity matrix on the $k$-th community $C_k$, we have the following decomposition for $\mathbb{E}\text{Tr}(G(z))$:

$$z\mathbb{E}\text{Tr}(G(z)) = \mathbb{E}\text{Tr}(HG - I)$$

$$= \mathbb{E}\left\{ -n - \frac{1}{n} \sum_{i,j=1,i\neq j}^n \kappa_{ij}^{(2)} (G_{ij}^2 + G_{ii}G_{jj}) \right\}$$

$$+ \frac{1}{n^{3/2}} \sum_{i,j=1,i\neq j}^n \frac{\kappa_{ij}^{(3)}}{2!} (2G_{ij}^3 + 6G_{ij}G_{ii}G_{jj})$$

$$- \frac{1}{n^2} \sum_{i,j=1,i\neq j}^n \frac{\kappa_{ij}^{(4)}}{3!} (6G_{ij}^4 + 36G_{ij}^2G_{ii}G_{jj} + 6G_{ii}^2G_{jj}^2)$$

$$= -n - I_{1,1} - I_{1,2} - I_{1,3} + \varepsilon I_1,$$

where

$$I_{1,1} = \frac{1}{n} \mathbb{E} \sum_{l,m=1}^K \sum_{i\in C_l, j\in C_m, i\neq j} \kappa_{ij}^{(2)} G_{ij}^2$$

$$= \frac{1}{n} \mathbb{E} \sum_{l,m=1}^K Q_{lm}^{(2)} \sum_{i\in C_l, j\in C_m} G_{ij}^2 - \frac{1}{n} \mathbb{E} \sum_{l=1}^K Q_{ll}^{(2)} \sum_{i\in C_l} G_{ii}^2$$

$$= \frac{1}{n} \mathbb{E} \sum_{l,m=1}^K Q_{lm}^{(2)} \text{Tr}(T_lGT_mG) - \frac{1}{n} \mathbb{E} \sum_{l=1}^K Q_{ll}^{(2)} \sum_{i\in C_l} G_{ii}^2.$$
\[ I_{1,2} = \frac{1}{n} \mathbb{E} \sum_{l,m=1}^{K} \sum_{i \in C_l, j \in C_m, i \neq j} \kappa_{ij}^{(2)} G_{ii} G_{jj} \]
\[ = \frac{1}{n} \mathbb{E} \sum_{l,m=1}^{K} Q_{lm}^{(2)} \sum_{i \in C_l, j \in C_m} G_{ii} G_{jj} - \frac{1}{n} \mathbb{E} \sum_{l=1}^{n} Q_{ll}^{(2)} \sum_{i \in C_l} G_{ii}^2 \]  
\[ = \frac{1}{n} \mathbb{E} \sum_{l,m=1}^{K} Q_{lm}^{(2)} \text{Tr}(T_l G(T_m G) - \frac{1}{n} \mathbb{E} \sum_{l=1}^{n} Q_{ll}^{(2)} \sum_{i \in C_l} G_{ii}^2, \] 

(19)

and

\[ I_{1,3} = \frac{1}{n} \mathbb{E} \sum_{i,j=1,i \neq j}^{n} \kappa_{ij}^{(4)} G_{ii}^2 G_{jj}^2. \]

(20)

The remainder \( \varepsilon_{I_1} \) will have a vanishing order \( O\left(\frac{1}{n}\right) \).

Though \( I_{1,3} \) can be directly estimated from the first-order approximation from the local law, approximations for \( I_{1,1} \) and \( I_{1,2} \) are not so straightforward. We need to derive new systems of equations for the quantities. To be more precise, we introduce the following lemmas.

**Lemma 4.1.** The vector

\[ X_{GTG}^{(l)}(z) = \left[ \frac{1}{n} \mathbb{E} \text{Tr}(G(z)T_l G(z)T_1), \cdots, \frac{1}{n} \mathbb{E} \text{Tr}(G(z)T_l G(z)T_K) \right]^\top \]

satisfies the following system of equations

\[ C_1(z) X_{GTG}^{(l)}(z) = B^{(l)}(z) \]

(21)

up to order 1, where

\[ B^{(l)}(z) = [0, \ldots, 0, -\frac{\alpha_l M_1(z)}{z}, 0, \ldots, 0]^\top. \]

Further, the matrix \( M_{GTG}(z) = (\frac{1}{n} \text{Tr}(G(z)T_l G(z)T_m))_{l,m=1}^{K} \) satisfies

\[ M_{GTG}(z) = -\left( Q^{(2)} - \text{Diag}\left( \frac{1}{\alpha_1 M_1^2(z)}, \cdots, \frac{1}{\alpha_K M_K^2(z)} \right)^\top \right)^{-1}. \]

(22)

**Lemma 4.2.** The vector

\[ Y(z) = [\mathbb{E} \text{Tr}(T_l G(z)) - \alpha_1 n M_1(z), \cdots, \mathbb{E} \text{Tr}(T_K G(z)) - \alpha_K n M_K(z)]^\top \]

satisfies the following equation

\[ C_0(z) Y(z) = - \frac{1}{z} \text{diag}(Q^{(2)} X(z) + \frac{2}{z} \alpha_l Q^{(2)}_{11} M_1^2, \cdots, \alpha_K Q^{(2)}_{KK} M_K^2)^\top \]

\[ - \frac{1}{z} \text{diag} \left[ Q^{(4)} (\alpha_l \alpha M_l^3 M_m^2)_{l,m=1}^{K} \right], \]

(23)
We refer the proofs to Sections A.1 and A.2.
Similarly, we may use the same techniques to calculate the covariance function $\text{Cov}(z_1, z_2) := \text{Cov}(\text{Tr}G(z_1), \text{Tr}G(z_2))$.
First we decompose the covariance function of $\text{Tr}(G)$ into the following block-wise forms:

$$
\text{Cov}(z_1, z_2) = \text{Cov}(\text{Tr}G(z_1), \text{Tr}G(z_2)) = \text{Cov}\left(\sum_{l=1}^{K} \text{Tr}(T_l G(z_1)), \sum_{m=1}^{K} \text{Tr}(T_m G(z_2))\right)
$$

where

$$
\text{Cov}_{lm}(z_1, z_2) := \text{Cov}(T_l G(z_1), T_m G(z_2)).
$$

Our primary problem is to calculate $\text{Cov}_{lm}(z_1, z_2)$ to order 1, $\forall l, m \in [K]$. Note that $\text{Cov}_{lm}(z_1, z_2) = n^2 \mathbb{E}[T_l G(z_1) T_m G(z_2) - \mathbb{E}[T_l G(z_1)] \mathbb{E}[T_m G(z_2)]]$, then we need to calculate the following expansion to the order $\frac{1}{n^2}$,

$$
\frac{1}{n^2} z_1 \text{Cov}_{lm}(z_1, z_2)
$$

$$
= z_1 \mathbb{E}[T_l G(z_1) T_m G(z_2) - \mathbb{E}[T_l G(z_1)] \mathbb{E}[T_m G(z_2)]] = z_1 \mathbb{E}[G(z_1) T_l T_m G(z_2)]
$$

$$
= \mathbb{E} H G(z_1) T_l T_m G(z_2) = \frac{1}{n} \mathbb{E} \sum_{i \in C_{l,j}} H_{ij} G_{ij}(z_1) T_m G(z_2)
$$

$$
= \frac{1}{n} \mathbb{E} \sum_{i \in C_{l,j}} \sum_{a+b=0}^{5} \frac{\kappa^{(a+b+1)} k_{ij}^{(a+b+1)}}{n^{(a+b+1)/2}a!b!} \frac{\partial^a G_{ij}(z_1) \partial^b (T_m G(z_2))}{\partial H_{ij}^{a+b}} + \varepsilon_{I_2}
$$

$$
= \sum_{a+b=0}^{5} I_{2,(a,b)} + \varepsilon_{I_2}.
$$

It turns out, only $I_{2,(0,1)}, I_{2,(1,0)}, I_{2,(1,2)}$ have $O(\frac{1}{n^2})$ contributions, which will lead to a set of systems of equations for $\{\text{Cov}_{lm}(z_1, z_2)\}_{l,m=1}^{K}$ as well. The key observation is that similar to the quantities calculated in the mean function, we will explore similar $K$-dimensional systems of equations via cumulant expansion due to the block structure in calculating the covariance function. We refer the details to Section A.3.

Section A.7 will show the proof of the normality of the linear spectral statistics. The proof for normality is relatively routine. We will adopt the following technique originated from Tikhomirov [29]. The core idea can be simplified as follows. To prove that a sequence of real random variable $R_n$ converges to a Gaussian random variable with mean zero and variance $\sigma^2$, it suffices to prove that

$$
\mathbb{E} e^{it R_n} \rightarrow e^{-\frac{1}{2} \sigma^2 t^2}.
$$
We prove alternatively that its derivative will behave similarly to that of the derivative of a characteristic function of a Gaussian distribution
\[ i\mathbb{E}R_ne^{itR_n} \rightarrow -\sigma^2i\mathbb{E}e^{itR_n}, \]
in which \( R_n \) is a real function constructed from \( Tr(G(z)) \), from which by \( HG = I + zG \) we can find the form \( HG \), thus extract the form \( Ehf(h) \). Then the cumulant expansion formula can be applied. In Section A.7, we will apply the multivariate version of the above trick to establish the normality.

In Section A.8, we establish the tightness of the process \( \langle Tr(G(z)) \rangle \) via a similar approach, then it follows from [9] that we can proceed from finite dimensional convergence of \( \{(\langle Tr(G(z_s)) \rangle)^t\}_{s=1} \) to the weak convergence of the linear spectral statistics.

4.2. Outline of the proof of Theorem 3.6

Recall that in the SBM setting, given the adjacency matrix \((A_{ij})_{i,j\in[n]}\) of a SBM, we may consider two renormalized versions \( H^{(A)} \) (13) and \( \hat{H}^{(A)} \) (15). For simplicity, we will use \( H \) and \( \hat{H} \) for short when there is no confusion.

Note that when \( i \neq j \),
\[
(\hat{H} - H)_{ij} = \frac{p_{ij} - \hat{p}_{ij}}{\sqrt{n}} = -\frac{1}{\sqrt{n}} \sum_{\alpha\in C_{\sigma(i)}, \beta\in C_{\sigma(j)}} \frac{A_{\alpha\beta} - p_{ij}}{N_{\sigma(i)\sigma(j)}}
\]
\[= -\frac{1}{N_{\sigma(i)\sigma(j)}} \sum_{\alpha\in C_{\sigma(i)}, \beta\in C_{\sigma(j)}} \frac{A_{\alpha\beta} - p_{\alpha\beta}}{\sqrt{n}} = -\frac{1}{N_{\sigma(i)\sigma(j)}} \sum_{\alpha\in C_{\sigma(i)}, \beta\in C_{\sigma(j)}} H_{\alpha\beta}. \tag{25} \]

Then by concentration inequality we know instantly that \( ||\hat{H} - H|| = o_p(\frac{\log(n)}{\sqrt{n}}) \), which implies that the limiting spectral distribution of \( \hat{H} \) will be the same as that of \( H \). However, this stand-alone bound is not sufficient for identical CLTs. To study the LSS of \( \hat{H} \), we need to follow a similar process to the one we use to prove Theorem 3.5.

Further, we investigate on the resolvents \( G(z) = (H - z)^{-1} \) and \( \hat{G}(z) = (\hat{H} - z)^{-1} \). Note also that by the resolvent identity, we have
\[
\hat{G}(z) = \sum_{k=0}^{m} G(z)[-(\hat{H} - H)G(z)]^k + \hat{G}(z)[-(\hat{H} - H)G(z)]^{m+1} \]
\[= G(z) - G(z)(\hat{H} - H)G(z) + G(z)(\hat{H} - H)G(z)(\hat{H} - H)G(z)
\]
\[- G(z)(\hat{H} - H)G(z)(\hat{H} - H)G(z)(\hat{H} - H)\hat{G}(z). \]

Further, note that by \( ||\hat{H} - H|| = o_p(\frac{\log(n)}{\sqrt{n}}) \) and \( ||G(z)|| \leq 1/|\sigma(z)| \), we expect that the higher-order expansion terms would vanish.

The proof of the Theorem 3.6 will adopt the same approach as Theorem 3.5 per se. However, we will mainly focus on the difference of the resolvents. The details of the proof can be found in Section B.
5. Numerical results

5.1. Experiments on verifying Theorem 3.5

We test our theorems under the setting of SBM (12) since the renormalized adjacency matrix (13) is naturally a block-Wigner-type matrix. Numerical experiments are conducted for the cases where $\tilde{P}$ in (12) is a matrix with identical diagonal terms $p$ and identical off-diagonal terms $q$. Under this framework, we may let both $p$ and $q$ run through the grid $\{0.1, 0.2, \cdots, 0.9\}$ to obtain a total of $9 \times 9 = 81$ stochastic block models. Given a test function, we can calculate the theoretical values of asymptotic means and asymptotic variances via Theorem 3.5. In the meantime, we are also able to generate real empirical data via Monte Carlo method with $N_r$ repetitions and get empirical means and variances for each model. Then we may compare the theoretical values and the empirical values via the 2D-mesh plots.

Note that for simplicity of presentation, we will compare $L_n(f) = \sum_{i=1}^{n} f(\lambda_i)$ instead of the truncated version $G_n(f) = \sum_{i=1}^{n} f(\lambda_i) - n \int f \mu_\infty$.

Example. The following parameters are used: $K = 3$, $\alpha = [0.25, 0.25, 0.5]$, $N = 800$, $N_r = 800$. $\tilde{P} = (p_i - q_j)I + q_j11^T$, where $p_i = \frac{i}{10}$, $q_j = \frac{j}{10}$, $i,j \in [9]$. $f = x^2$.

|                  | Asymptotic | Empirical | maximal absolute difference |
|------------------|------------|-----------|----------------------------|
| Mean of $L_n(x^2)$ | ![Graph](image1.png) | ![Graph](image2.png) | 0.0195                     |
| Variance of $L_n(x^2)$ | ![Graph](image3.png) | ![Graph](image4.png) | 0.0112                     |

Table 1: Comparison of the asymptotic mean, variance and their empirical values obtained by Monte Carlo for the LSS $L_n(x^2)$. Empirical values use 800 repetitions.

One can see from Table 1 that we obtain a quite good match between theoretical and empirical means and variances.
Next, we consider 9 SBMs out of the 81 in Example 5.1 and display in Figure 1 the normal qqplots of the empirical LSS $L_n(f)$ after normalization $\frac{L_n(f) - \text{Mean}(L_n(f))}{\text{Std}(L_n(f))}$. These qqplots empirically confirm the asymptotic normality of the LSS.

**Example.** Same setting except the test function $f$ is $x^4$. Simulation results are shown in Table 2 and Figure 2. The conclusion is similar to that of Example 5.1.

### 5.2. Experiments on the data-driven matrix $\hat{H}$

We have also conducted numerical experiments for the data-driven matrix $\hat{H}$. The simulation set-up much follows the one used in Section 3.2. The main purpose is to verify whether the limiting distributions of linear spectral statistics of $\hat{H}$ would be the same as those of $H$.

Towards this end, we display relative qqplots of linear spectral statistics from $H$ and $\hat{H}$, respectively. Under distributional identity, qqplots would coincide with the identity line $y = x$.

**Example.** The SBM parameters are as follows:

- $K = 6$, $\alpha = [0.1, 0.15, 0.2, 0.25, 0.1, 0.2]$, $N = 1000$, $N_r = 1600$, $\hat{P} = (p_i - q_j)I + q_j 11^T$, where $p_i = \frac{2i-1}{10}$, $q_j = \frac{2j-1}{10}$. Test functions are $f_1 = x^4$, $f_2 = x^5$, $f_3 = \exp(x)$.

The empirical qqplots are given in Figure 3 4 5. It can be seen that these qqplots are basically on the line $y = x$, which gives a good empirical confirmation of Theorem 3.6.
|                       | Asymptotic | Empirical | Maximal absolute difference |
|-----------------------|------------|-----------|-----------------------------|
| Mean of \( L_n(x^4) \) | ![Image](image1.png) | ![Image](image2.png) | 0.3100                      |
| Variance of \( L_n(x^4) \) | ![Image](image3.png) | ![Image](image4.png) | 0.0065                      |

**Table 2**

Comparison of the asymptotic mean, variance and their empirical values obtained by Monte Carlo for the LSS \( L_n(x^4) \). Empirical values use 800 repetitions.

Fig 2: q-qplots for normalized LSS of 9 different SBMs, test function \( x^4 \)
Fig 3: qqplots of $L_n(x^4)[H] - L_n(x^4)[\hat{H}]$

Fig 4: qqplots of $L_n(x^5)[H] - L_n(x^5)[\hat{H}]$
6. Conclusion

In this paper, we consider two applicable renormalizations \( \left( A_{ij} - p_{ij} \right) / \sqrt{n} \) and \( \left( A_{ij} - \hat{p}_{ij} \right) / \sqrt{n} \) of adjacency matrices of the stochastic block models. The CLTs of linear spectral statistics for both renormalizations are derived. The situations are fundamentally different from the existing literature in the sense that \( \left( A_{ij} - p_{ij} \right) / \sqrt{n} \) induces a block-Wigner-type matrix whose LSD is no longer guaranteed to be the semicircle law but governed by the so-called quadratic vector equations introduced in [3]. And the CLT for LSS also requires finer tools from the local law estimations. Meanwhile, \( \left( A_{ij} - \hat{p}_{ij} \right) / \sqrt{n} \) is further perturbed by a low rank yet correlated structure, whose non-decaying correlations among the entries increase the difficulty of analysis.

We discuss several directions for future research. First, the CLTs introduced here are still in the dense regime of the stochastic block model. While [24] provides a more subtle analysis of the local law for the Erdős-Rényi model in the sparser regime, it makes a local law for the sparse stochastic block model possible. Thus, the CLT for LSS of SBM in the sparse regime could be doable.

Second, a natural question is that for more general Wigner-type matrices, for instance, when the patterns explore more complex structures, can we get some CLTs or non-CLTs? For instance, if the number of communities for the SBM is growing along with \( n \) or the random graph model is defined via a graphon approach [2][30], then how will the linear spectral statistics behave?
Appendix A: Detailed calculations for the proof of Theorem 3.5

In this section, we will show the details of the calculation of the mean function in Section A.1-A.2, covariance function in Section A.3-A.6, proof of normality in Section A.7, and tightness of the process in Section A.8 for the block-Wigner-type matrices $H$.

Remark. Corollary 2.3 will be extensively used in our proof. Since $\epsilon > 0$ is arbitrarily small and essentially $n^\epsilon$ can be substituted by $\log(n)^k$ for some large enough $k$ in these large-deviation bounds. Sometimes we will use $n^\epsilon$ for simplicity when it is actually $n^{k_0\epsilon}$ for some positive integer $k_0$ which is independent of $n$.

Recall that in Section 4, we decompose the mean function $\text{Tr}G(z)$ into several components. Starting from $I_{1,1}$ in (17), we need to calculate $E T_l G(z) T_m G(z)$ to order 1.

A.1. System of equations for $E G(z) T_l G(z) T_m$

Proof of Lemma 4.1. By the identity $G(z) = \frac{1}{z}(HG(z) - I)$, we have

$$E G(z) T_l G(z) T_m = \frac{1}{z} E H G(z) G(z) T_l T_m G(z) = \frac{1}{z} E H G(z) G(z) T_l T_m G(z) - \frac{1}{z} E T_l G(z).$$

Then by the cumulant expansion formula,

$$E H G(z) G(z) T_l T_m = \frac{1}{n} \sum_{ij} H_{ij} (G(z) G(z) T_l T_m)_{ij}$$

$$= \sum_{ij} \kappa_{ij}^{(2)} \frac{\partial (G(z) G(z) T_l T_m)_{ji}}{\partial H_{ij}} + \sum_{ij} \kappa_{ij}^{(3)} \frac{\partial^2 (G(z) G(z) T_l T_m)_{ji}}{\partial H_{ij}^2}$$

$$+ \sum_{ij} \kappa_{ij}^{(4)} \frac{\partial^3 (G(z) G(z) T_l T_m)_{ji}}{\partial H_{ij}^3} + \epsilon_{GTGT}.$$

where by the cumulant expansion and the trivial bound, the error term satisfies $|\epsilon_{GTGT}| \leq C \sum_{ij} \sup_t |f^{(3+1)}(t)| E \|H_{ij}\|^{3+2} = O(\frac{1}{\sqrt{n}}) thus minor.

In the meantime, note that when we take derivatives $\frac{\partial^k (G(z) G(z) T_l T_m)_{ji}}{\partial H_{ij}^k}$, the terms with the largest order of magnitude should be the ones with the form $(\cdot)_{ii}(\cdot)_{jj} \times (\cdot)$, which will be of order 1 since $\|G\| \leq \frac{1}{5z}$ and $\|T_l\| = 1, \forall l \in [K], so$

$$\sum_{ij} \frac{\kappa_{ij}^{(3)}}{2n^{5/2}} \frac{\partial^2 (G(z) G(z) T_l T_m)_{ji}}{\partial H_{ij}^2} = O(\frac{1}{\sqrt{n}}).$$

$$\sum_{ij} \frac{\kappa_{ij}^{(4)}}{3n^3} \frac{\partial^3 (G(z) G(z) T_l T_m)_{ji}}{\partial H_{ij}^3} = O(\frac{1}{n}).$$ (26)
It follows that
\[
z \mathbb{E}GT_lGT_m + \delta_{lm} \mathbb{E}T_lT = \mathbb{E}HGT_lGT_m = \frac{1}{n} \mathbb{E} \sum_{ij} H_{ij} (GT_lGT_m)_{ij}
\]
\[
= \frac{1}{n} \sum_{ij} \kappa_{ij}^{(2)} \mathbb{E} \frac{\partial (GT_lGT_m)_{ji}}{\partial H_{ij}} + O\left( \frac{1}{\sqrt{n}} \right)
\]
\[
= - \frac{1}{n^2} \sum_{ij} \kappa_{ij}^{(2)} \mathbb{E} [G_{ji} (GT_lGT_m)_{ji} + G_{jj} (GT_lGT_m)_{ii}]
\]
\[
+ (GT_lG)_{ji} (GT_m)_{ji} + (GT_lG)_{jj} (GT_m)_{ii} + O\left( \frac{1}{\sqrt{n}} \right)
\]
\[
= - \frac{1}{n^2} \sum_{k=1}^{K} \sum_{k_1, k_2 = 1}^{K} Q_{k_1 k_2}^{(2)} \mathbb{E} [G_{jj} (GT_lGT_m)_{ii} + (GT_lG)_{jj} (GT_m)_{ii}]
\]
\[
+ O\left( \frac{n^2}{\sqrt{n}} \right)
\]
\[
= - \sum_{k=1}^{K} Q_{mk}^{(2)} (\alpha_k M_k(z) \mathbb{E}GT_lGT_m + \alpha_m M_m(z) \mathbb{E}GT_lGT_k) + O\left( \frac{n^2}{\sqrt{n}} \right).
\]

If we adopt the notation
\[
X_m^{(l)} := \mathbb{E}GT_lGT_m,
\]
then we may rewrite the above system of equations as
\[
(1 + \frac{1}{z} \sum_{k=1}^{K} Q_{mk}^{(2)} \alpha_k M_k + \frac{1}{z} \alpha_m M_m Q_{mm}^{(2)} \mathbb{E}GT_lGT_m + \frac{1}{z} \alpha_m M_m \sum_{k=1, k \neq m}^{K} Q_{mk}^{(2)} X_k^{(l)}) X_m^{(l)} = -\delta_{ml} \frac{1}{z} \alpha_l M_l.
\]

Now we have the system of equations (21) for vector \([\mathbb{E}GT_lGT_1, \ldots, \mathbb{E}GT_lGT_K]\) and the system of equations (22) for matrix \((\mathbb{E}GT_lGT_k)_{l,k=1}^{K}\).

Remark. Further from the QVE (7), for \(z \in \mathbb{C}_+\) and sufficiently bounded away from the spectrum of \(H\), we have
\[
- \frac{1}{M_l(z)} = z + \sum_{m=1}^{K} Q_{lm}^{(2)} \alpha_m M_m(\mathbb{E}GT_lGT_m), \quad \text{for all} \quad l = 1, \ldots, K,
\]

one can see that for different \(l\)'s. The coefficient matrices are the same, after simplification, we have that the matrix \(M_{GTGT}(z) = (G(z)T_lG(z)T_m)_{l,m=1}^{K}\) adopts this simple explicit form
\[
M_{GTGT}(z) = -Q^{(2)} - \text{Diag}([\frac{1}{\alpha_1 M_1(z)}] \cdots [\frac{1}{\alpha_K M_K(z)}])^{-1},
\]
which is symmetric and in accordance with the tracial property which leads to
\[
\text{Tr}(GT_lGT_m) = \text{Tr}(GT_mGT_l).
\]
The next task is to identify the leading term of $1$. By Cauchy-Schwarz inequality it's easy to see that $I$, Note that we cannot calculate these terms up to the order $1$. The problem arises that the trivial upper bound $|\Im z|$, $j \in [K]$. Then when $|3z|^2 \geq \max_{k \in [K]} \alpha_k \sum_{j=1}^{K} Q^{(2)}_{kj}$, the matrix

$$
(Q^{(2)} - \text{Diag}([\frac{1}{\alpha_1 \beta_1^2(z)}, \ldots, \frac{1}{\alpha_K \beta_K^2(z)}])^T)
$$

becomes diagonal dominant, thus non-singular. Similar things happen when we are near the real axis but also sufficiently bounded away from the edge. Then we can ensure the existence and uniqueness of the solutions of our systems of equations. All we have to pay is to select a larger contour when we apply the Cauchy integral trick to proceed from the trace of the resolvent to the linear spectral statistics. Due to the homogeneity of the coefficient, similar arguments hold for other systems of equations of interest.

Specifically, we introduce the parameter $\varepsilon_0$, s.t. for $z \in \mathbb{C} \setminus B_\varepsilon(\sigma(H))$, the existence and uniqueness of the solution are guaranteed by the above mechanism.

### A.2. Leading term for $\frac{1}{n} \mathbb{E} \sum_{i,j=1}^{n} \kappa_{ij}^{(2)} G_{ii} G_{jj}$ and system of equations for $\mathbb{E}\text{Tr}(T(G))$

The next task is to identify the leading term of $\frac{1}{n} \mathbb{E} \sum_{i,j=1}^{n} \kappa_{ij}^{(2)} G_{ii} G_{jj}$. Note that we need to calculate these terms up to the order $1$. The problem arises that the trivial upper bound $|G_{jj}(z) - m_j(z)| < \frac{1}{\sqrt{n}}$ is far from enough since it only yields $\frac{1}{n} \sum_{i,j=1}^{n} \kappa_{ij}^{(2)} |G_{ii} G_{jj} - m_i m_j| < n^{1/2}$.

Recall the decomposition in (19), we further write

$$
I_{1,2} = \frac{1}{n} \mathbb{E} \sum_{k_1, k_2=1}^{K} Q_{k_1 k_2}^{(2)} \sum_{i \in C_{k_1}, j \in C_{k_2}} G_{ii} G_{jj} - \frac{1}{n} \mathbb{E} \sum_{k=1}^{K} Q_{kk}^{(2)} \sum_{i \in C_k} G_{ii}^2
$$

$$
= I_{1,2,1} + I_{1,2,2}. \quad (29)
$$

where

$$
I_{1,2,1} = \frac{1}{n} \mathbb{E} \sum_{k_1, k_2=1}^{K} Q_{k_1 k_2}^{(2)} \sum_{i \in C_{k_1}, j \in C_{k_2}} G_{ii} G_{jj} = n \mathbb{E} \sum_{k_1, k_2=1}^{K} Q_{k_1 k_2}^{(2)} T_{k_1 G} T_{k_2 G}. \quad (30)
$$

Note that we cannot calculate $I_{1,2,1}$ to the desired order directly.

Simply notice that by local law, we have

$$
\mathbb{E}[(\text{Tr}(T_k G) - \mathbb{E}\text{Tr}(T_k G))^2] \leq n^{2}. \quad (31)
$$

By Cauchy-Schwarz inequality it’s easy to see that

$$
\mathbb{E}[(\text{Tr}(T_{k_1} G) - \mathbb{E}\text{Tr}(T_{k_1} G))(\text{Tr}(T_{k_2} G) - \mathbb{E}\text{Tr}(T_{k_2} G))] \leq n^{2}. \quad (32)
$$

Then we have

$$
I_{1,2,1} = n \mathbb{E} \sum_{k_1, k_2=1}^{K} Q_{k_1 k_2}^{(2)} T_{k_1 G} T_{k_2 G} = n \sum_{k_1, k_2=1}^{K} Q_{k_1 k_2}^{(2)} \mathbb{E}T_{k_1 G} \mathbb{E}T_{k_2 G} + O(\frac{n^{2}}{n}). \quad (33)
$$
And for $I_{1,2,2}$ we can get the simple formulation

$$
I_{1,2,2} = -\frac{1}{n} \mathbb{E} \sum_{k=1}^{K} Q_{kk}^{(2)} \sum_{i \in C_k} G_{ii}^2 = -\sum_{k=1}^{K} Q_{kk}^{(2)} \alpha_k M_k^2 + O(\frac{n^\varepsilon}{\sqrt{n}}). \tag{34}
$$

In other words, again we have obtained a function of $(\mathbb{E}(Tr(T_k G)), \mathbb{E}(Tr(T_k^2 G)))$ on the RHS, note that the leading order terms of $(\mathbb{E}(Tr(T_k G)), \mathbb{E}(Tr(T_k^2 G)))$, which are of order $n$, are known. So this motivates us to derive a system of equations for the subleading order terms of $\{\mathbb{E}Tr(T_k G)\}_{k=1}^{K}$, which are of order 1.

**Proof of Lemma 4.2.** By the cumulant expansion formula, we have the following equality for $\mathbb{E}Tr(T_k G)$:

$$
\mathbb{E}Tr(T_k G) = -\frac{\alpha_k n}{z} + \mathbb{E} \frac{1}{z} Tr(HG T_k) = -\frac{\alpha_k n}{z} + \mathbb{E} \frac{1}{z} \sum_{ij} (H_{ij} e_j GT_k e_i)
$$

$$
= -\frac{\alpha_k n}{z} - \mathbb{E} \frac{1}{z} \sum_{ij} \frac{\kappa_{ij}^{(2)}}{n} [G_{jj}(GT_k)_{ii} + G_{ji}(GT_k)_{ji}] + \mathbb{E} \frac{1}{z} \sum_{ij} \frac{\kappa_{ij}^{(3)}}{2! n^{3/2}} \frac{\partial^2 e_j GT_k e_i}{\partial H_{ij}^2}
$$

$$
+ \mathbb{E} \frac{1}{z} \sum_{ij} \frac{\kappa_{ij}^{(4)}}{3! n^2} \frac{\partial^3 e_j GT_k e_i}{\partial H_{ij}^3} + \varepsilon_{I_{1,5}}
$$

$$
= -\frac{\alpha_k n}{z} - \sum_{k_1,k_2=1}^{K} \sum_{i \in C_{k_1}, j \in C_{k_2}} \frac{\kappa_{ij}^{(2)}}{n} [G_{jj}(GT_k)_{ii} + G_{ji}(GT_k)_{ji}]
$$

$$
+ \mathbb{E} \frac{1}{z} \sum_{ij} \frac{\kappa_{ij}^{(4)}}{3! n^2} \frac{\partial^3 e_j GT_k e_i}{\partial H_{ij}^3} + \varepsilon_{I_{1}}
$$

$$
= -\frac{\alpha_k n}{z} - \tilde{I}_{1,1} - \tilde{I}_{1,2} - \tilde{I}_{1,4} + \varepsilon_{I_{1}},
$$

where $\varepsilon_{I_{1}}$ consists of higher-order expansions of the formula

$$
\varepsilon_{I_{1,3}} = \frac{1}{z} \mathbb{E} \sum_{i,j} \frac{\kappa_{ij}^{(3)}}{2! n^{3/2}} 2[G_{jj} G_{ij}(GT_k)_{ii} + G_{jj} G_{ii}(GT_k)_{ji} + G_{jj} G_{jj}(GT_k)_{ii} + G_{ji} G_{jj}(GT_k)_{ji}],
$$

$$
|\varepsilon_{I_{1,5}}| \leq \sum_{i,j} C \sup_t |f^{(p+1)}(t)| |\mathbb{E}[t]|^{p+2} = O(\frac{1}{\sqrt{n}}).
$$

Then it suffices to show that

$$
\varepsilon_{I_{1,3}} = \frac{1}{z} \mathbb{E} \sum_{i \in C_{1}, j \in C_{m}} \frac{\kappa_{ij}^{(3)}}{2! n^{3/2}} 2[G_{jj} G_{ij}(GT_k)_{ii} + G_{jj} G_{ii}(GT_k)_{ji} + G_{jj} G_{jj}(GT_k)_{ii} + G_{ji} G_{jj}(GT_k)_{ji}],
$$

$$
\varepsilon_{I_{1,3}} = \frac{1}{z} \mathbb{E} \sum_{i \in C_{1}, j \in C_{m}} \frac{\kappa_{ij}^{(3)}}{2! n^{3/2}} 2[G_{jj} G_{ij}(GT_k)_{ii} + G_{jj} G_{ii}(GT_k)_{ji} + G_{jj} G_{jj}(GT_k)_{ii} + G_{ji} G_{jj}(GT_k)_{ji}].
$$
are minor. Let \( a = [G_{11}, \ldots, G_{nn}] \), \( b_k = [(T_k G)_{11}, \ldots, (T_k G)_{nn}] \).

\[
\tilde{I}_{1,1} = \frac{1}{z} \sum_{k_1, k_2 = 1}^{K} \sum_{i \in C_{k_1}, j \in C_{k_2}, i \neq j} \frac{Q_{ij}^{(2)}}{n} G_{ij} (GT_k)_{ii} \\
= \frac{1}{z} \sum_{k_1, k_2 = 1}^{K} Q_{k_1 k_2}^{(2)} \mathbb{E} \text{Tr}(T_{k_2} G) \text{Tr}(GT_k) - \frac{1}{z} \sum_{k_1} \frac{Q_{kk}^{(2)}}{n} G_{ii}^2 \\
+ \frac{1}{z} \sum_{k_2 = 1}^{K} Q_{k_1 k_2}^{(2)} [\mathbb{E} \text{Tr}(T_{k_2} G) \text{Tr}(GT_k) - \mathbb{E} \text{Tr}(T_{k_2} G) \mathbb{E} \text{Tr}(GT_k)] \\
- \frac{1}{z} \sum_{i \in C_{k}} Q_{kk}^{(2)} [G_{ii}^2 - M_k^2(z)].
\]

One may notice that similar to the cases above, we have

\[
|\mathbb{E} \text{Tr}(T_{k_2} G) \text{Tr}(GT_k) - \mathbb{E} \text{Tr}(T_{k_2} G) \mathbb{E} \text{Tr}(GT_k)| \\
\leq |\mathbb{E} [\text{Tr}(T_{k_2} G)] - \mathbb{E} \text{Tr}(T_{k_2} G)] [\text{Tr}(GT_k) - \mathbb{E} \text{Tr}(GT_k)]| \leq n^{2\varepsilon},
\]

and

\[
\mathbb{E} |G_{ii}^2(z) - M_k^2(z)| = \mathbb{E} |G_{ii}(z) - M_k(z)||G_{ii}(z) + M_k(z)| \leq \frac{n^{\varepsilon}}{\sqrt{n \Im(z)} } \frac{2}{Im(z)} \forall i \in C_{k},
\]

thus,

\[
\tilde{I}_{1,1} = \frac{1}{z} \sum_{k_2 = 1}^{K} Q_{k_1 k_2}^{(2)} \mathbb{E} \text{Tr}(T_{k_2} G) \mathbb{E} \text{Tr}(GT_k) - \frac{1}{z} Q_{kk}^{(2)} \alpha_k M_k^2 + o(1).
\]
Similarly, we can proceed to \( \tilde{I}_{1,2} \),

\[
\tilde{I}_{1,2} = \mathbb{E} \frac{1}{z} \sum_{k_1, k_2 = 1}^{K} \sum_{i \in C_{k_1}, j \in C_{k_2}, i \neq j} \frac{\kappa_{ij}^{(2)}}{n} G_{ji}(GT_k)_{ji} \\
= \mathbb{E} \frac{1}{z} \sum_{k_1, k_2 = 1}^{K} \sum_{i \in C_{k_1}, j \in C_{k_2}} \frac{Q_{k_1k_2}^{(2)}}{n} G_{ji}(GT_k)_{ji} - \mathbb{E} \frac{1}{z} \frac{Q_{kk}^{(2)}}{n} \sum_{i \in C_{k}} G_{ii}(GT_k)_{ii}
\]

(37)

\[
= \mathbb{E} \frac{1}{z} \sum_{k_2 = 1}^{K} Q_{kk}^{(2)} \mathbb{E} G_{k_2} G_{k_2} - \mathbb{E} \frac{1}{z} Q_{kk}^{(2)} \alpha_k M_k^2 + o(1).
\]

\[
\tilde{I}_{4,1} = \mathbb{E} \frac{1}{z} \sum_{l} \frac{\kappa_{ij}^{(4)}}{3! n^2} \frac{\partial^3 e_j^i}{\partial H_{ij}^3} = \mathbb{E} \frac{1}{z} \sum_{i \in C_{k}, j \in C_{l}, i \neq j} \frac{\kappa_{ij}^{(4)}}{n^2} G_{ii}^2 G_{jj}^2 + O \left( \frac{n^\epsilon}{\sqrt{n}} \right)
\]

(38)

\[
= \mathbb{E} \frac{1}{z} \sum_{l = 1}^{K} \sum_{i \in C_{k}, j \in C_{l}, i \neq j} \frac{Q_{kl}^{(4)}}{n^2} M_k^2 M_l^2 + O \left( \frac{n^\epsilon}{\sqrt{n}} \right) = \frac{1}{z} \sum_{l = 1}^{K} Q_{kl}^{(4)} \alpha_k \alpha_l M_k^2 M_l^2 + O \left( \frac{n^\epsilon}{\sqrt{n}} \right).
\]

Then we may derive the system of equation for \( Y_k = \mathbb{E} Tr(T_k G) - \alpha_k n M_k \),

\[
\alpha_k n M_k + Y_k = - \frac{\alpha_k n}{z} - \frac{1}{z} \sum_{k_2 = 1}^{K} Q_{kk}^{(2)} \left( \alpha_k n M_{k_2} + Y_{k_2} \right) \left( \alpha_k n M_k + Y_k \right) \\
- \frac{1}{z} \sum_{k_2 = 1}^{K} Q_{kk}^{(2)} \mathbb{E} G_{k_2} G_{k_2} + \frac{2Q_{kk}^{(2)} \alpha_k M_k^2}{z} - \frac{1}{z} \sum_{l = 1}^{K} Q_{kl}^{(4)} \alpha_k \alpha_l M_k^2 M_l^2.
\]

(39)

The above equation could be decomposed into two parts, one is of order \( n \), while the other is of order 1. One may easily verify that the order \( n \) part yields

\[
M_k = - \frac{1}{z} - \frac{\sum_{k_2 = 1}^{K} Q_{kk}^{(2)} \alpha_k M_{k_2} M_k}{z},
\]

which directly follows from the quadratic vector equation (7), thus canceled.

The order 1 part yields

\[
Y_k = - \frac{1}{z} \sum_{l = 1}^{K} Q_{kl}^{(2)} \left[ \alpha_l M_l Y_k + \alpha_k M_k Y_l \right] - \frac{1}{z} \sum_{l = 1}^{K} Q_{kl}^{(2)} \mathbb{E} G_{l} G_{l} + \frac{2Q_{kk}^{(2)} \alpha_k M_k^2}{z} \\
- \frac{1}{z} \sum_{l = 1}^{K} Q_{kl}^{(4)} \alpha_k \alpha_l M_k^2 M_l^2,
\]

which reformulates into our (23).
A.3. System of equations for $\text{Cov}_{lm}(z_1, z_2)$

As stated in Section 4, the estimation for $\text{Cov}(z_1, z_2)$ can be decomposed into the summation of the block-wise covariance functions $\{\text{Cov}_{lm}(z_1, z_2)\}_{l,m=1}^K$. In this subsection, we will derive the system of equations for $\{\text{Cov}_{lm}(z_1, z_2)\}_{l,m=1}^K$.

In this section and thereafter, we will use $\langle \cdot \rangle$ for centered random variables. First, note that for any two random variables $X$ and $Y$, we have

$$\mathbb{E}\langle X \rangle \langle Y \rangle = \mathbb{E}[X - \mathbb{E}X][Y - \mathbb{E}Y] = \mathbb{E}X[Y - \mathbb{E}Y] = \mathbb{E}X \langle Y \rangle,$$

then by cumulant expansion formula,

$$\frac{1}{n^2} z_1 \mathbb{E} \text{Cov}_{lm}(z_1, z_2) = z_1 \mathbb{E} \langle G(z_1) T_l \rangle \langle T_m G(z_2) \rangle = z_1 \mathbb{E} G(z_1) T_l \langle T_m G(z_2) \rangle$$

$$= \mathbb{E} H G(z_1) T_l \langle T_m G(z_2) \rangle = \frac{1}{n} \mathbb{E} \sum_{i \in C_{l,j}} H_{ij} G_{ij}(z_1) \langle T_m G(z_2) \rangle$$

$$= \frac{1}{n} \sum_{a+b=0}^\infty \mathbb{E} \sum_{i \in C_{l,j}} \kappa_{ij}^{(a+b+1)} \frac{\partial^a G_{ij}(z_1) \partial^b \langle T_m G(z_2) \rangle}{n^{a+b+1} a! b!} \partial H_{ij}^{(a+b)}$$

$$= \sum_{a+b=0}^5 I_{2,(a,b)} + \varepsilon_{I_2}. \tag{40}$$

Now we proceed to the detailed treatments for the terms $\{I_{2,(a,b)}\}_{a+b=1}^3$. It can be shown that $\sum_{a+b=4}^5 I_{2,(a,b)}$ are minor via similar calculations, the details for calculating $\sum_{a+b=4}^5 I_{2,(a,b)}$ are tedious and of minor importance thus omitted here and in the proof of (10).
Proof of (10).

\[ I_{2,(1,0)} = - \frac{1}{n} \mathbb{E} \sum_{k_1=1}^{K} \frac{Q_{k_1}^{(2)}}{n} \left[ \text{Tr}(T_{k_1} G(z_1) T_{k_1} G(z_1)) + \text{Tr}(T_{k_1} G(z_1)) \text{Tr}(T_{k_1} G(z_1)) \right] \langle T_m G(z_2) \rangle \]

\[ + \frac{2Q_{il}^{(2)}}{n^2} \mathbb{E} \sum_{i \in C_l} [G_{ii}^2(z_1) - M_i^2(z_1)] \langle T_m G(z_2) \rangle \]

\[ - \mathbb{E} \sum_{k_1=1}^{K} \frac{Q_{k_1}^{(2)}}{n} \langle T_{k_1} G(z_1) T_{k_1} G(z_1) \rangle \langle T_m G(z_2) \rangle - \mathbb{E} \sum_{k_1=1}^{K} Q_{ik_1}^{(2)} \langle T_{k_1} G(z_1) \rangle \langle T_{k_1} G(z_1) \rangle \langle T_m G(z_2) \rangle \]

\[ - \mathbb{E} \sum_{k_1=1}^{K} Q_{ik_1}^{(2)} \langle T_{k_1} G(z_1) \rangle \langle T_{k_1} G(z_1) \rangle \langle T_m G(z_2) \rangle - \mathbb{E} \sum_{k_1=1}^{K} Q_{ik_1}^{(2)} \langle T_{k_1} G(z_1) \rangle \langle T_{k_1} G(z_1) \rangle \langle T_m G(z_2) \rangle \]

\[ + O\left( \frac{n^\varepsilon}{n^{5/2}} \right) \]

\[ = - \sum_{k_1=1}^{K} Q_{ik_1}^{(2)} \alpha_{k_1} M_{k_1}(z_1) \text{Cov}_{l,m}(z_1, z_2) - \sum_{k_1=1}^{K} Q_{ik_1}^{(2)} \alpha_{k_1} M_{l}(z_1) \text{Cov}_{k_1,m}(z_1, z_2) + O\left( \frac{n^\varepsilon}{n^{5/2}} \right). \]

\[ \]

\[ I_{2,(0,1)} = \]

\[ - \frac{1}{n} \mathbb{E} \sum_{k_1=1}^{K} \sum_{i \in C_l, j \in C_{k_1}} \frac{Q_{ik_1}^{(2)}}{n} G_{ij}(z_1) \frac{1}{n} \sum_{k \in C_m} (G_{kj}(z_2) G_{ik}(z_2) + G_{ki}(z_2) G_{jk}(z_2)) \]

\[ + \frac{1}{n} \mathbb{E} \sum_{i \in C_l} \frac{Q_{il}^{(2)}}{n} G_{ii}(z_1) \frac{1}{n} \sum_{k \in C_m} (G_{ki}(z_2) G_{ik}(z_2) + G_{ki}(z_2) G_{ik}(z_2)) \]

\[ - \mathbb{E} \sum_{k_1=1}^{K} \sum_{i \in C_l, j \in C_{k_1}} \frac{2Q_{ik_1}^{(2)}}{n^3} G_{ij}(z_1) (GT_m G)_{ij}(z_2) + \mathbb{E} \sum_{i \in C_l} \frac{2Q_{il}^{(2)}}{n^3} G_{ii}(z_1) (GT_m G)_{ii}(z_2) \]

\[ - \sum_{k_1=1}^{K} \frac{2Q_{ik_1}^{(2)}}{n^3} \mathbb{E} G(z_1) T_{k_1} G(z_2) T_{k_1} M_{k_1}(z_1) (T_{k_1} G(z_2)) + \frac{2Q_{il}^{(2)}}{n^3} M_{l}(z_1) \mathbb{E} (T_l G(z_2)) + O\left( \frac{n^\varepsilon}{n^{5/2}} \right). \]

We claim that in \( I_{2,(1,0)} \) both \( \frac{1}{n} \mathbb{E} \langle T_l G(z_1) T_{k_1} G(z_1) \rangle \langle T_m G(z_2) \rangle \) and the triple-product term \( \mathbb{E} \langle T_{k_1} G(z_1) \rangle \langle T_{k_1} G(z_1) \rangle \langle T_m G(z_2) \rangle \) will be the minor terms. The second one is of order \( \frac{nc}{n^\sigma} \) by Cauchy inequality, thus minor. The first one, however, requires a little bit effort.

To get an sufficient upper bound for \( \frac{1}{n} \mathbb{E} \langle T_l G(z_1) T_{k_1} G(z_1) \rangle \langle T_m G(z_2) \rangle \), we only need to show that \( \mathbb{E} \langle T_{k_1} G(z) T_{k_2} G(z) \rangle \) is of order \( O(n^{-1}) \) for some \( t > 0 \). By intuition from the classic Wigner matrix, the essential order for \( \mathbb{E} \langle T_{k_1} G(z) T_{k_2} G(z) \rangle \) should be \( O(n^{-2}) \). We refer to the Section A.4 for the details.
Also, $I_{2,0}$ gives rise to the quantities $\mathbb{E}G(z_1)T_iG(z_2)T_mG(z_2)T_k$, which will be treated in Section A.5 and $\mathbb{E}G(z_2)T_iG(z_2)T_m$ which has already been studied in Section A.1.

$$I_{2,0} = \frac{1}{n^2} \sum_{i \in C, j} \mathbb{E} \frac{\kappa_{ij}^{(3)}}{n^{3/2}} \langle (G_{ij}(z_1))^3 + 3G_{ii}(z_1)G_{jj}(z_1)G_{ij}(z_1) \rangle \langle T_m G(z_2) \rangle$$

$$= \frac{1}{n^2} \sum_{i \in C, j} \mathbb{E} \frac{\kappa_{ij}^{(3)}}{n^{3/2}} [3G_{ii}(z_1)G_{jj}(z_1)G_{ij}(z_1)] \langle T_m G(z_2) \rangle + O\left(\frac{n^5}{n^3}\right).$$

(41)

Note that one argument for higher-order expansion terms in the cumulant expansion that we will use over and over again is that we can ignore the diagonal terms in many situations since there are only $n$ diagonal terms which are at most $O(1)$ each. To be more specific,

$$\frac{n^5}{n^{d+1}} \sum_{m=1}^{n} \frac{\partial^d G_{mm}}{\partial H_{mm}} \leq \frac{n^5}{n^{d+1}} \leq \frac{C^d n^5}{n^{d+3/2}} = o(n^{-2}), d = 2, 3,$$

then w.l.o.g. we can ignore the diagonal terms here. Further, because $\langle T_m G \rangle$ is $O\left(\frac{n^5}{n^3}\right)$, we only need to show that $\frac{1}{n^2} \sum_{i \in C, j} \frac{\kappa_{ij}^{(3)}}{n^{3/2}} G_{ij} G_{ii} G_{jj} = o\left(\frac{1}{n^{1/2}}\right)$ for any $t > 0$ to ensure that $I_{2,0}$ is vanishing.

Note that the trivial bounds for $G_{ij} G_{ii} G_{jj}$ will not be sufficient. The trick here is to apply the cumulant expansion formula one more time to get certain equation of $\frac{1}{n^2} \sum_{i \in C, j} \frac{\kappa_{ij}^{(3)}}{n^{3/2}} G_{ij} G_{ii} G_{jj} = o\left(\frac{1}{n^{1/2}}\right)$, hence the sufficient bounds.

By cumulant expansion, we have

$$\frac{1}{n^2} \sum_{i \in C, j} \frac{\kappa_{ij}^{(3)}}{n^{3/2}} G_{ij} G_{ii} G_{jj} = \mathbb{E} \frac{Q_{lm}^{(3)}}{n^{3/2}} \sum_{i \in C, j} G_{ij} G_{ii} G_{jj} = \mathbb{E} \frac{Q_{lm}^{(3)}}{n^{3/2}} \sum_{i \in C} \delta_{lm} G_{ii}^3$$

$$= \mathbb{E} \frac{Q_{lm}^{(3)}}{n^{3/2}} \sum_{i \in C, j} \sum_{k} H_{ik} G_{kj} G_{ii} G_{jj} + O(n^{-3/2})$$

$$= \mathbb{E} \frac{Q_{lm}^{(3)}}{n^{3/2}} \sum_{i \in C, j} \sum_{k} \frac{\kappa_{ik}^{(2)}}{n} \frac{\partial (G_{kj} G_{ii} G_{jj})}{\partial H_{ik}} + \mathbb{E} \frac{Q_{lm}^{(3)}}{n^{3/2}} \sum_{i \in C, j} \sum_{k} \frac{\kappa_{ik}^{(3)}}{2!n^{3/2}} \frac{\partial^2 (G_{kj} G_{ii} G_{jj})}{\partial H_{ik}^2}$$

$$+ \mathbb{E} \frac{Q_{lm}^{(3)}}{n^{3/2}} \sum_{i \in C, j} \sum_{k} \frac{\kappa_{ik}^{(4)}}{3!n^{2}} \frac{\partial^3 (G_{kj} G_{ii} G_{jj})}{\partial H_{ik}^3} + O(n^{-3/2}).$$

(42)
Then the problem becomes to derive bounds for the terms

\[
\frac{Q^{(3)}_{lm}}{n^{5/2}} \frac{1}{z} \sum_{i \in C_l, j \in C_m} \sum_{k} \frac{\kappa^{(2)}_{ik}}{n} \frac{\partial(G_{kj} G_{ii} G_{jj})}{\partial H_{ik}}
\]

\[
= \frac{Q^{(3)}_{lm}}{n^{7/2}} \frac{1}{z} \sum_{i \in C_l, j \in C_m} \sum_{k} \sum_{k_1=1}^{K} \kappa^{(2)}_{ik} \left[-(G_{ik} G_{jk} + G_{ij} G_{kk}) G_{ii} G_{jj} - 2G_{kj} G_{ii} G_{ij} G_{kj} \right]
\]

\[
= - \frac{Q^{(3)}_{lm}}{n^{7/2}} \frac{1}{z} \sum_{i \in C_l, j \in C_m} \sum_{k_1=1}^{K} Q^{(2)}_{ik_1} Tr(T_{k_1} G) G_{ij} G_{ii} G_{jj} + O(\frac{K}{n^{3/2}})
\]

\[
= - \frac{Q^{(3)}_{lm}}{n^{7/2}} \sum_{k_1=1}^{K} Q^{(2)}_{ik_1} Tr(T_{k_1} G) \frac{1}{z} \sum_{i \in C_l, j \in C_m} G_{ij} G_{ii} G_{jj} + O(\frac{Kn^\varepsilon}{n^{3/2}}).
\]

In the meantime,

\[
\frac{Q^{(3)}_{lm}}{n^{5/2}} \frac{1}{z} \sum_{i \in C_l, j \in C_m} \sum_{k} \frac{\kappa^{(3)}_{ik}}{n^{3/2}} \frac{\partial^2(G_{kj} G_{ii} G_{jj})}{\partial H_{ik}^2}
\]

\[
= \sum \{\text{at least two of } i, j, k \text{ would be the same}\} + \sum \{i, j, k \text{ are mutually different}\}
\]

\[
= O(\frac{n^\varepsilon}{n^{3/2}}),
\]

while the trivial upper bound is already sufficient for the 4-th order term

\[
\frac{Q^{(3)}_{lm}}{n^{5/2}} \frac{1}{z} \sum_{i \in C_l, j \in C_m} \sum_{k} \frac{\kappa^{(4)}_{ik}}{n^2} \frac{\partial^3(G_{kj} G_{ii} G_{jj})}{\partial H_{ik}^3} = O(n^{-3/2}).
\]

Thus, from above estimations we know

\[
\frac{1}{n} \mathbb{E} \sum_{i \in C_l, j \in C_m} \frac{\kappa^{(3)}_{ij}}{n^{3/2}} G_{ij} G_{ii} G_{jj}
\]

\[
= - \frac{Q^{(3)}_{lm}}{n^{5/2}} \sum_{k_1=1}^{K} Q^{(2)}_{ik_1} \frac{1}{n} \mathbb{E} \sum_{i \in C_l, j \in C_m} G_{ij} G_{ii} G_{jj} + O(\frac{K n^\varepsilon}{n^{3/2}}),
\]

instantly we come to the conclusion that

\[
\mathbb{E} \sum_{i \in C_l, j \in C_m} G_{ij} G_{ii} G_{jj} = O(K n^{1+\varepsilon}).
\]
Thus, \( \frac{1}{n} \mathbb{E} \sum_{i \in C_l, j \in C_m} \kappa_{ij}^{(3)} \frac{G_{ij} G_{ii} G_{jj}}{n^{3/2}} \) is minor and \( I_{2,(2,0)} \) is also minor.

\[
I_{2,(1,1)} = \frac{1}{n} \mathbb{E} \sum_{i \in C_l, j \in C_m} \frac{2 \kappa_{ij}^{(3)}}{2! n^{3/2}} (G_{ij}^{2}(z_1) + G_{ii}^{2}(z_1) G_{jj}(z_1)) \frac{1}{n} \sum_{k \in C_m} [G_{kj}(z_2) G_{ik}(z_2) + G_{k(2)}(z_2) G_{jk}(z_2)]
\]

\[
= \frac{1}{n^{7/2}} \sum_{k_1=1}^{K} \sum_{i \in C_l, j \in C_{k_1}} 2Q_{ik_1}^{(3)} G_{ii}^{2}(z_1) G_{ij}(z_1) (GT_m G)_{ij}(z_2) + O\left( \frac{n^5}{n^{5/2}} \right).
\]

Simply note that \( |G_{ii}(z)| \leq \left| \frac{1}{|z|} \right|, \quad ||G(z)|| \leq \left| \frac{1}{|z|} \right| \) and \( ||T_m|| = 1 \), we have

\[
\mathbb{E} \frac{1}{n^{7/2}} \sum_{i \in C_l, j \in C_{k_1}} Q_{ik_1}^{(3)} G_{ii}^{2}(z_1) G_{ij}(z_1) (GT_m G)_{ij}(z_2)
\]

\[
= \mathbb{E} \frac{1}{n^{7/2}} \sum_{i \in C_l, j \in C_{k_1}} Q_{ik_1}^{(3)} \text{diag}(G(z_1)) \times (GT_m G)_{ij}(z_2) \times \text{diag}(G(z_1)) = O(n^{-5/2}).
\]

Further,

\[
I_{2,(0,2)} = - \mathbb{E} \sum_{k_1=1}^{K} \sum_{i \in C_l, j \in C_{k_1}} \kappa_{ik_1}^{(3)} n^{7/2} G_{ij}(z_1) \sum_{k \in C_m} (G_{kj} G_{ij} G_{ik} + G_{kj} G_{ii} G_{jk} + G_{k(2)} G_{jj} G_{ik} + G_{k(2)} G_{ji} G_{jk})
\]

\[
= - \mathbb{E} \sum_{k_1=1}^{K} \sum_{i \in C_l, j \in C_{k_1}} \kappa_{ik_1}^{(3)} n^{7/2} G_{ij}(z_1) [(GT_m G)_{ij} G_{ij} + (GT_m G)_{ij} G_{ii} + (GT_m G)_{ij} G_{jj}
\]

\[
+ (GT_m G)_{ij} G_{ji}(z_2)] = O\left( \frac{n^5}{n^{5/2}} \right),
\]

where we use the fact that

\[
\sum_{i \in C_l, j \in C_{k_1}} Q_{ik_1}^{(3)} G_{ii} (GT_m G)_{ij} (GT_m G)_{jj} | = \frac{Q_{ik_1}^{(3)}}{n^{7/2}} (\text{diag}(T_i G) \times (GT_m G) \times \text{diag}(T_{k_1} G_{m} G))
\]

\[
= O\left( \frac{1}{n^{5/2}} \right).
\]
\[
I_{2,(3,0)} = \frac{1}{n} \mathbb{E} \sum_{k_1=1}^{K} \sum_{i \in C_1, j \in C_{k_1}, i \neq j} \frac{Q_{ik_1}^{(4)}}{n^3}(G_{ij}^4 + 6G_{ij}^2G_{jj} + G_{ii}^2G_{jj}^2)(T_m G(z_2))
\]

\[
= \sum_{k_1=1}^{K} \sum_{i \in C_1, j \in C_{k_1}, i \neq j} \frac{Q_{ik_1}^{(4)}}{n^3} M_i^2 M_{k_1}^2 \mathbb{E}(T_m G(z_2))
\]

\[
+ \sum_{k_1=1}^{K} \sum_{i \in C_1, j \in C_{k_1}, i \neq j} \frac{Q_{ik_1}^{(4)}}{n^3} \mathbb{E}(G_{ii}^2G_{jj}^2 - M_i^2 M_{k_1}^2)(T_m G(z_2)) + O \left( \frac{n^\varepsilon}{n^3} \right)
\]

\[= O \left( \frac{n^{2\varepsilon}}{n^{5/2}} \right). \]

\[
I_{2,(2,1)} = -\frac{3}{n} \mathbb{E} \sum_{i \in C_1, j} \frac{2\kappa_{ij}^{(4)}}{3!n^2}\big[G_{ij}^3(z_1) + G_{ij}(z_1)G_{ii}(z_1)G_{jj}(z_1)\big]
\]

\[
\times \frac{3}{n} \mathbb{E} \sum_{k \in C_m} \big[G_{ki}(z_2)G_{jk}(z_2) + G_{kj}(z_2)G_{ik}(z_2)\big]
\]

\[= -\frac{3}{n^2} \mathbb{E} \sum_{i \in C_1, j} \frac{2\kappa_{ij}^{(4)}}{n^2}\big[G_{ij}^3(z_1) + G_{ij}(z_1)G_{ii}(z_1)G_{jj}(z_1)\big]2(GT_m G)_{ij}(z_2)
\]

\[= O \left( \frac{n^{\varepsilon}}{n^{5/2}} \right). \]

\[
I_{2,(1,2)} = -\frac{3}{n} \mathbb{E} \sum_{i \in C_1, j} \frac{2\kappa_{ij}^{(4)}}{3!n^2}\big[G_{ij}^2(z_1) + G_{ii}(z_1)G_{jj}(z_1)\big]
\]

\[
\times \frac{1}{n}\big[(GT_m G)_{ij}G_{ij} + (GT_m G)_{ii}G_{jj} + (GT_m G)_{ij}G_{ji} + (GT_m G)_{jj}G_{ii}\big](z_2)
\]

\[= -\mathbb{E} \sum_{k=1}^{K} \frac{Q_{ik}^{(4)}}{n^2}\big[\alpha_k M_t(z_1)M_k(z_1)(GT_m G T_k)(z_2)M_k(z_2)\big]
\]

\[+ \alpha_t M_t(z_1)M_k(z_1)(GT_m G T_k)(z_2)M_t(z_2)\big] + O \left( \frac{n^{\varepsilon}}{n^{5/2}} \right). \]

\[
I_{2,(0,3)} = -\frac{1}{n} \mathbb{E} \sum_{i \in C_1, j} \frac{\kappa_{ij}^{(4)}}{n^2}\big[G_{ij}(z_1)\big]^2 \frac{1}{n}\big[G_{ij}^2(G^2)_{jj} + G_{ij}(G^2)_{jj} + G_{ii}G_{jj}(G^2)_{ij} + G_{ii}G_{jj}(G^2)_{ij}
\]

\[+ G_{jj}G_{ij}(G^2)_{ii} + G_{jj}G_{ii}(G^2)_{ji} + G_{ji}G_{jj}(G^2)_{ii} + G_{ji}G_{jj}(G^2)_{ji}\big](z_2)
\]

\[= O \left( \frac{n^{\varepsilon}}{n^{5/2}} \right). \]
Similarly, we can derive that $\sum_{a+b=4} I_{2,(a,b)}$ is also minor. Further note that
\[
\varepsilon_{I_2} = \frac{1}{n} \sum_{i,j} C \sup_t |f^{(5+1)}(t)||E[|\xi|^{5+2}] = O\left( \frac{1}{n^{5/2}} \right).
\]

To conclude, the covariance function $\{\text{Cov}_{lm}(z_1, z_2)\}_{l,m \in [K]}$ satisfies the following system of equations to order 1
\[
z_1 \text{Cov}_{lm}(z_1, z_2) = - \sum_{k_1=1}^{K} Q_{lk_1}^{(2)} \alpha_{k_1} M_{k_1}(z_1) \text{Cov}_{k_1m}(z_1, z_2) - \sum_{k_1=1}^{K} Q_{lk_1}^{(2)} \alpha_l M_l(z_1) \text{Cov}_{k_1m}(z_1, z_2)
- E \sum_{k_1=1}^{K} 2Q_{lk_1}^{(2)} G(z_1) T_l G(z_2) T_m G(z_2) T_{k_1} + 2Q_{ll}^{(2)} M_l(z_1) E(T_l G T_m G)(z_2)
- \sum_{k=1}^{K} Q_{lk}^{(4)} [\alpha_k M_l(z_1) M_k(z_1) E(G T_m G)(z_2) M_k(z_2)]
+ \alpha_l M_l(z_1) M_k(z_1) E(G T_m G T_k)(z_2) M_l(z_2)\]

Now we have derived the system of equations for $\{\text{Cov}_{lm}(z_1, z_2)\}_{K \times K}$. However, several questions still need to be answered. First we will show that terms with the form $E[\langle T_l G T_m G \rangle \langle T_k G T_j G \rangle]$ would be minor in Section A.4. Second, we will establish a system of equations for $E(\langle T_l G(z_1) T_m G(z_2) T_r G(z_2) \rangle)$ in Section A.5.

**A.4. Bound for $E[\langle T_l G T_m G \rangle]^2$**

Now we show that is $E[\langle T_l G T_m G \rangle]^2$ of minor order for any $l, m \in [K]$. We start from the trivial bound that $E[\langle T_l G T_m G \rangle]^2 = O(1)$. Again, we apply the cumulant expansion formula
to $E(T_l G T_m G) \langle T_r G^* T_s G^* \rangle$.

$$E(T_l G T_m G) \langle T_r G^* T_s G^* \rangle$$

$$= \frac{1}{nz} \sum_{ij} H_{ij} (T_l G T_m)_{ji} (T_r G^* T_s G^*) - \frac{1}{z} \delta_m E(T_l G^* T_s G^*)$$

$$= \mathbb{E} \left[ \frac{1}{nz} \sum_{ij} \kappa_{ij}^2 \left( \frac{\partial^2 e_i^j (GT_l GT_m) e_i}{\partial H_{ij}} \right) (T_r G^* T_s G^*) \right] + \mathbb{E} \left[ \frac{1}{nz} \sum_{ij} \kappa_{ij}^2 \left( \frac{\partial^2 T_r G^* T_s G^*}{\partial H_{ij}} \right) \right] + o \left( \frac{1}{n^2} \right)$$

$$= \mathbb{E} \sum_{ij} \kappa_{ij}^2 \left[ (GT_l GT_m)_{ji} (GT_l GT_m)_{ji} + (GT_l GT_m)_{ij} (GT_l GT_m)_{ij} \right] \langle T_r G^* T_s G^* \rangle + O \left( \frac{1}{n} \right)$$

$$+ \frac{1}{nz} \sum_{ij} \kappa_{ij}^2 \left( \frac{(GT_l GT_m)_{ji} (GT_l GT_m)_{ji} - (GT_l GT_m)_{ij} (GT_l GT_m)_{ij}}{n} \right) \langle T_r G^* T_s G^* \rangle$$

$$+ \frac{1}{nz} \sum_{ij} \kappa_{ij}^2 \left( \frac{(GT_l GT_m)_{ji} (GT_l GT_m)_{ji} - (GT_l GT_m)_{ij} (GT_l GT_m)_{ij}}{n} \right) \langle T_r G^* T_s G^* \rangle$$

$$= \mathbb{E} \left( \sum_{k_1 k_2} \frac{Q_{k_1 k_2}^2}{z} (GT_l GT_m)_{k_1} (T_l G^* T_s G^*) \right) + O \left( \frac{1}{n} \right)$$

$$+ \mathbb{E} \left( \sum_{k_1} \frac{Q_{k_1 k_2}^2}{z} (GT_l GT_m)_{k_1} (T_l G^* T_s G^*) \right) + O \left( \frac{1}{n} \right)$$

$$= \mathbb{E} \left( \sum_{k_2} \frac{Q_{m k_2}^2}{z} (GT_l GT_m)_{k_2} (T_l G^* T_s G^*) \right) + O \left( \frac{1}{n} \right)$$

$$+ \mathbb{E} \left( \sum_{k_2} \frac{Q_{m k_2}^2}{z} (GT_l GT_m)_{k_2} (T_l G^* T_s G^*) \right) + O \left( \frac{1}{n} \right)$$
Fix \( l, r, s \), then we can see that we will have the system of equation for vector \( \mathbf{R} = [R_1, \cdots, R_K]^{\top} \)

\[
C_{o1} \star \mathbf{R} = \left[ O\left( \frac{1}{n} \right), \cdots, O\left( \frac{1}{n} \right), \frac{1}{z} \mathbb{E} T_i G (T_i G^\ast T_s G^\ast) + O\left( \frac{1}{n} \right), O\left( \frac{1}{n} \right), \cdots, O\left( \frac{1}{n} \right) \right]^{\top},
\]

where

\[
R_m = \mathbb{E} \langle GT_i G_m \rangle \langle G_s^\ast T_s G^\ast \rangle, m \in [K].
\]

For simplicity of illustration, we will not distinguish \( \langle GT_i G_m \rangle \) from \( \langle T_i G^\ast T_s G^\ast \rangle \) below. Note further that \( \|C_{o1}\| = O(1) \) and \( C_{o1} \) is non-degenerate. Our bound for \( \mathbb{E} \|\langle T_i G T_m \rangle\|^2 \) then essentially comes down to the two parts, \( \frac{1}{z} \mathbb{E} T_i G (T_i G^\ast T_s G^\ast) \) and an \( O\left( \frac{1}{n} \right) \) which comes from the terms \( \frac{1}{n^2} \sum_{ij} \frac{\kappa^{(j)}(T_i G T_m) e_i}{\partial H_j^e} \langle T_i G_s^\ast T_s G^\ast \rangle \) and \( \frac{1}{n^2} \sum_{ij} \frac{\kappa^{(i)}(T_i G T_m) e_i}{\partial H_j^e} \langle T_i G^\ast T_s G^\ast \rangle \), while the reminder part of the above expression contributes only with an order \( O\left( \frac{1}{n^2} \right) \).

Note that

\[
|\mathbb{E} T_i G (T_i G^\ast T_s G^\ast)|^2 = |\mathbb{E} \langle T_i G \rangle \langle T_i G^\ast T_s G^\ast \rangle|^2 \leq \mathbb{E} |\langle T_i G \rangle|^2 \mathbb{E} |\langle T_i G^\ast T_s G^\ast \rangle|^2,
\]

by (3) we instantly know that \( \mathbb{E} |\langle T_i G \rangle|^2 \mathbb{E} |\langle T_i G^\ast T_s G^\ast \rangle|^2 \leq \mathbb{E} |\langle T_i G \rangle|^2 = O\left( \frac{n^2}{n^2} \right) \). Thus, \( \mathbb{E} |\langle T_i G T_m \rangle|^2 = O\left( \frac{n^2}{n^2} \right) \). Note that once we obtain this bound for \( \mathbb{E} |\langle T_i G T_m \rangle|^2 \), we know that they are minor, then we may claim that the system of equations (46) is an order 1 matrix equation, the solution should be also of order 1. Now the bound for \( \mathbb{E} |\langle T_i G \rangle|^2 \) is improved to \( O\left( \frac{1}{n^2} \right) \) and \( \mathbb{E} |\langle T_i G T_m \rangle|^2 = O\left( \frac{1}{n} \right) \).

Then we know that \( \frac{1}{z} \mathbb{E} T_i G (T_i G^\ast T_s G^\ast) \) will contribute to the bound via an \( O\left( \frac{1}{n} \right) \). Now suppose that \( \mathbb{E} |\langle T_i G T_m \rangle|^2 = O\left( \frac{1}{n^2} \right) \) for some \( t \geq 0 \). Then we know that \( \frac{1}{z} \mathbb{E} T_i G (T_i G^\ast T_s G^\ast) \) will then contribute to the bound via an \( O\left( \frac{1}{n^2} \right) \). And we may repeat the above process as long as we can establish the bounds simultaneously for \( \frac{1}{n^2} \sum_{ij} \frac{\kappa^{(j)}(T_i G T_m) e_i}{\partial H_j^e} \langle T_i G_s^\ast T_s G^\ast \rangle \) and \( \frac{1}{n^2} \sum_{ij} \frac{\kappa^{(i)}(T_i G T_m) e_i}{\partial H_j^e} \langle T_i G^\ast T_s G^\ast \rangle \), which is totally applicable since they also share a similar structure of the form \( O\left( \frac{1}{n} \right) \langle T_i G^\ast T_s G^\ast \rangle \). Thus, we may improve the bound from \( O\left( \frac{1}{n^2} \right) \) to \( O\left( \frac{1}{n^2 + 1} \right) \) and then improve the next sequence of improved bounds \( O\left( \frac{1}{n^2} \right), O\left( \frac{1}{n^2} \right), O\left( \frac{1}{n^2} \right), \cdots \), until we reach the limit \( O\left( \frac{1}{n^2} \right) \).

Also note that the above argument only yields a bound \( \mathbb{E} |\langle T_i G T_m \rangle|^2 = O\left( \frac{n^2}{n^2} \right) \), \( \delta > 0 \). However, we can further derive system of equations for all \( \mathbb{E} O\left( \frac{1}{n} \right) \langle T_i G^\ast T_s G^\ast \rangle \) terms above individually and establish bounds individually, since now \( \langle T_i G^\ast T_s G^\ast \rangle \) would either break after differentiation or remain the way they are to reproduce terms of the form \( \mathbb{E} O\left( \frac{1}{n} \right) \langle T_i G^\ast T_s G^\ast \rangle \). However, note that \( \mathbb{E} |\langle T_i G T_m \rangle|^2 = O\left( \frac{n^2}{n^2} \right) \) will cease to produce higher-order structures. We then may obtain compact bounds for those \( \mathbb{E} O\left( \frac{1}{n} \right) \langle T_i G^\ast T_s G^\ast \rangle \) again via the system of equations approach. The whole process is repetitive and tedious, thus omitted. Then we can conclude that those \( \mathbb{E} O\left( \frac{1}{n} \right) \langle T_i G^\ast T_s G^\ast \rangle \) above are of order \( O\left( \frac{1}{n^2} \right) \) and further conclude that \( \mathbb{E} |\langle T_i G T_m \rangle|^2 = O\left( \frac{1}{n^2} \right) \) can be achieved.
A.5. System of equations for $\mathbb{E}T_1 G(z_1) T_m G(z_2) T_r G(z_2)$

Now we want to derive the system of equations that $\{\mathbb{E}T_1 G(z_1) T_m G(z_2) T_r G(z_2)\}_{m,r=1}^K$ satisfies to order 1. Easy to observe that the contribution of higher-order cumulants would vanish and we have

\[
z_1 \mathbb{E}G(z_1) T_1 G(z_2) T_m G(z_2) T_r = \mathbb{E}(HG(z_1) - I) T_1 G(z_2) T_m G(z_2) T_r
\]

\[
= \mathbb{E}HG(z_1) T_1 G(z_2) T_m G(z_2) T_r - \delta_{n} \mathbb{E}G(z_2) T_1 G(z_2) T_m,
\]

where

\[
\mathbb{E}HG(z_1) T_1 G(z_2) T_m G(z_2) T_r = \frac{1}{n} \mathbb{E} \sum_{i,j} H_{ij}(G(z_1) T_1 G(z_2) T_m G(z_2) T_r)_{j i}
\]

\[
= - \frac{1}{n} \mathbb{E} \sum_{i,j}^{K} \frac{K_{ij}}{n} \left[ G_{ji}(z_1) (G(z_1) T_1 G(z_2) T_m G(z_2) T_r)_{j i} + G_{jj}(z_1) (G(z_1) T_1 G(z_2) T_m G(z_2) T_r)_{i i} \right]
\]

\[
- \frac{1}{n} \mathbb{E} \sum_{i,j}^{K} \frac{K_{ij}}{n} \left[ (G(z_1) T_1 G(z_2))_{ji} (G(z_2) T_m G(z_2) T_r)_{j i} + (G(z_1) T_1 G(z_2))_{jj} (G(z_2) T_m G(z_2) T_r)_{i i} \right]
\]

\[
- \frac{1}{n} \mathbb{E} \sum_{i,j}^{K} \frac{K_{ij}}{n} \left[ (G(z_1) T_1 G(z_2) T_m G(z_2))_{ji} (G(z_2) T_r)_{j i} + (G(z_1) T_1 G(z_2) T_m G(z_2))_{jj} (G(z_2) T_r)_{i i} \right]
\]

\[
+ O\left(\frac{n^\varepsilon}{\sqrt{n}}\right)
\]

\[
= - \frac{1}{n^2} \mathbb{E} \sum_{k_1,k_2=1}^{K} Q_{k_1 k_2}^{(2)} \text{Tr}(T_{k_2} G(z_1)) \text{Tr}(G(z_1) T_1 G(z_2) T_m G(z_2) T_r T_{k_1})
\]

\[
- \frac{1}{n^2} \mathbb{E} \sum_{k_1,k_2=1}^{K} Q_{k_1 k_2}^{(2)} \text{Tr}(G(z_1) T_1 G(z_2) T_{k_2}) \text{Tr}(G(z_2) T_m G(z_2) T_r T_{k_1})
\]

\[
- \frac{1}{n^2} \mathbb{E} \sum_{k_1,k_2=1}^{K} Q_{k_1 k_2}^{(2)} \text{Tr}(G(z_1) T_1 G(z_2) T_m G(z_2) T_{k_2}) \text{Tr}(G(z_2) T_r T_{k_1}) + O\left(\frac{n^\varepsilon}{\sqrt{n}}\right)
\]

\[
= - \sum_{k_2=1}^{K} Q_{r k_2}^{(2)} \mathbb{E}(T_{k_2} G(z_1)) \mathbb{E}(G(z_1) T_1 G(z_2) T_m G(z_2) T_r)
\]

\[
- \sum_{k_2=1}^{K} Q_{r k_2}^{(2)} \mathbb{E}(G(z_1) T_1 G(z_2) T_{k_2}) \mathbb{E}(G(z_2) T_m G(z_2) T_r)
\]

\[
- \sum_{k_2=1}^{K} Q_{r k_2}^{(2)} \mathbb{E}(G(z_1) T_1 G(z_2) T_m G(z_2) T_{k_2}) \mathbb{E}(G(z_2) T_r) + O\left(\frac{n^\varepsilon}{\sqrt{n}}\right),
\]

the last line follows from the fact that $\mathbb{E}|\text{Tr}G T_m G| = O\left(\frac{n^\varepsilon}{\sqrt{n}}\right)$. 
Remark. One may notice that if we fix \( l \) and \( m \), we will get the system of equations for \( \{ \mathbb{E}G(z_1)T_lG(z_2)T_mG(z_2)T_k \}_{k=1}^K \) with a fixed coefficient matrix
\[
\text{Diag}([-\frac{1}{M_1(z)}, \ldots, -\frac{1}{M_K(z)}]) + Q \ast \text{diag}([\alpha_1 M_1, \ldots, \alpha_K M_K]).
\]

It then becomes apparent that the coefficient matrix is actually universal regardless of \( l \) and \( m \), which suggests that we can slice the tensor \( \{ \mathbb{E}G(z_1)T_lG(z_2)T_mG(z_2)T_k \}_{l,m,r=1}^K \) into \( K \) matrices in which each satisfies a matrix equation and we can actually do the slicing in any of the 3 directions. We believe that this phenomenon discloses certain supersymmetric patterns explored by the higher-order tensors \( \{ \mathbb{E}G(z_1)T_lG(z_2)T_mG(z_2)T_k \}_{l,m,r=1}^K \).

Also, the system of equations for \( \{ \mathbb{E}G(z_1)T_lG(z_2)T_mG(z_2)T_k \}_{l,m,r=1}^K \) induces another question. We still need to calculate \( \{ \mathbb{E}G(z_1)T_lG(z_2)T_m \}_{l,m=1}^K \).

### A.6. System of equations for \( \mathbb{E}G(z_1)T_lG(z_2)T_m \)

**Lemma A.1.** The vector \( X^{(l)}_{G_{1TG2T}}(z_1, z_2) := [\mathbb{E}G(z_1)T_lG(z_2)T_1, \ldots, \mathbb{E}G(z_1)T_lG(z_2)T_K]^\top \) satisfies the following equation
\[
\text{Co}_2(z_1, z_2) X^{(l)}_{G_{1TG2T}}(z_1, z_2) = B^{(l)}_{(z_1, z_2)},
\]

where
\[
B^{(l)}_{(z_1, z_2)} = [0, \ldots, 0, -\frac{\alpha_l M_l(z_2)}{z_1}, 0, \ldots, 0]^\top.
\]

**Proof.** Analog to the case of \( \mathbb{E}G(z)T_lG(z)T_m \), we derive the following system of equation
\[
\mathbb{E}G(z_1)T_lG(z_2)T_m
\]
\[
= \frac{1}{nz_1} \sum_{i,j} H_{ij}(G(z_1)T_l G(z_2)T_m)_{ji} - \delta_{lm} \frac{1}{z_1} \alpha_l M_l(z_2) + O(\frac{n^\varepsilon}{\sqrt{n}})
\]
\[
= - \frac{1}{z_1} \sum_{k=1}^K Q_{mk_1}^{(2)} \alpha_k M_k(z_1) \mathbb{E}G(z_1)T_lG(z_2)T_m
\]
\[
- \frac{1}{z_1} \sum_{k=1}^K Q_{mk_1}^{(2)} \alpha_m M_k(z_2) \mathbb{E}G(z_1)T_lG(z_2)T_k - \delta_{lm} \frac{1}{z_1} \alpha_l M_l(z_2) + O(\frac{n^\varepsilon}{\sqrt{n}}).
\]

Reorganizing the proof with (7) yields the matrix form. \( \square \)

Similar to \( M_{GTGT} \), for \( M_{G1TG2T}(z_1, z_2) := \{ \mathbb{E}G(z_1)T_lG(z_2)T_m \}_{l,m=1}^K \), we have the following simplified form
\[
M_{G1TG2T}(z_1, z_2) = - (Q^{(2)} - \text{Diag}(\frac{1}{\alpha_1 M_1(z_1) M_1(z_2)}, \ldots, \frac{1}{\alpha_K M_K(z_1) M_K(z_2)})^\top)^{-1}, \quad (47)
\]
which, again, is in accordance with the fact that the matrix $M_{GITGZT}(z_1, z_2)$ should be symmetric

$$
Tr(G(z_1)T_iG(z_2)T_m) = Tr(G(z_1)T_iG(z_2)T_m)^\top = Tr(T_mG(z_2)T_iG(z_1))
= Tr(G(z_1)T_mG(z_2)T_i).
$$

### A.7. Proof of normality

In this subsection and Section B.3 only, we will use $i$ for the unit imaginary number $\sqrt{-1}$.

To recover the covariance structure and prove normality, it’s natural to adopt the following setting as in [22] which can be viewed as variation of the Tikhomirov-Stein method. We will prove that for any integer $q$ and arbitrary collection $z_1, \ldots, z_q$ of complex numbers from $\mathbb{C}\setminus B_{\varepsilon_0}(\sigma(H)), \varepsilon_0$ is taken as in the Proof of Lemma 4.1 to ensure the uniqueness and existence of the solution. The joint probability distribution of random variables $(\langle G(z_1) \rangle, \ldots, \langle G(z_q) \rangle)$ converges as $n \to \infty$ to the $q$-dimensional Gaussian distribution with zero mean and the covariance matrix $[g(z_s, z_t)]_{s,t=1}^q$ specified in the following section.

**Proof.** Note that to prove that the process $Tr(G(z))$ converges to a Gaussian process, we need to prove the real and imaginary parts of $Tr(G(z))$ are jointly Gaussian in the limiting sense, so the first thing to do is to construct an adequate process.

Let $\gamma(z) = \gamma^{(n)}(z) := \Re\langle TrG(z) \rangle$ and $\theta(z) = \theta^{(n)}(z) = \Im\langle TrG(z) \rangle$, further

$$
\Psi(z, c) = \begin{cases}
\gamma(z) & \text{if } c = \gamma \\
\theta(z) & \text{if } c = \theta
\end{cases}
$$

and

$$(a(c), b(c)) = \begin{cases}
(1/2, 1/2) & \text{if } c = \gamma \\
(1/2i, 1/2i) & \text{if } c = \theta,
\end{cases}$$

then $\mathbb{E}\{\Psi(z, c)\} = 0$. And now we wanna prove that $\forall$ fixed $q \in \mathbb{Z}_{+}, \{z_s\}_{s=1}^q \in \{\mathbb{C}\setminus B_{\varepsilon_0}(\sigma(H))\}_{q}$, \{c_s\}_{s=1}^q \in \{\gamma, \theta\}^q$, the joint probability distribution of random variables $\Psi(z_1, c_1), \ldots, \Psi(z_q, c_q)$ is the $q$-dimensional Gaussian distribution with zero mean and covariance matrix

$$
\mathbb{E} \{\Psi(z_s, c_s) \Psi(z_t, c_t)\} = a(c_s) a(c_t) g(z_s, z_t) + a(c_s) b(c_t) g(z_s, z_t^*) + a(c_t) b(c_s) g(z_s^*, z_t) + b(c_s) b(c_t) g(z_s^*, z_t^*),
$$

where $g(z_s, z_t)$ should be in accordance with our covariance function $Cov(z_s, z_t)$ previously defined in Section A.3. Then we need to consider the characteristic function of these random variables $\Psi(z_1, c_1), \ldots, \Psi(z_q, c_q)$, which we shall write in the form

$$
e_q = e_q^{(n)}(T_q, C_q, Z_q) := \prod_{s=1}^q \exp\left\{i\tau_s [a(c_s) Tr\langle G(z_s) \rangle + b(c_s) Tr\langle G(z_s^*) \rangle]\right\},$$
where $T_q = (\tau_1, \ldots, \tau_q)$, $C_q = (c_1, \ldots, c_q)$, $Z_q = (z_1, \ldots, z_q)$. For simplicity, we shall use $e_q^{(n)}(T_q, C_q, Z_q)$ when there is no confusion. Also, we would use $a_s$ and $b_s$ to denote $a(c_s)$ and $b(c_s)$. Instantly, we have

$$\frac{\partial}{\partial \tau_s} \mathbb{E} \{e_q\} = i \mathbb{E} \{e_q [a_s \text{Tr}(G(z_s)) + b_s \text{Tr}(G(z_s^*))]\},$$

and our main goal is to show that there exist sequences of coefficients of the covariance matrices $(\Sigma_{st}^{(n)})_{s,t=1}^{q}$ s.t. for each fixed $T_q$

$$\lim_{n \to \infty} \mathbb{E} \{|e_q^{(n)}[a_s \text{Tr}(G(z_s)) + b_s \text{Tr}(G(z_s^*))]| - i \sum_{t=1}^{q} \tau_s \Sigma_{st}^{(n)} \mathbb{E} \{e_q^{(n)}\} \rangle = 0, \ z \in \mathbb{C} \setminus \mathbb{R},$$

and further, the limits of all these coefficients exist

$$\Sigma_{st} = \lim_{n \to \infty} \Sigma_{st}^{(n)}$$

and are in accordance with our previous results on the covariance function.

First, we need to calculate $\mathbb{E} \{e_q \text{Tr}(G(z))\}$. By resolvent identity and cumulant expansion, we have

$$\mathbb{E} \{e_q \langle TrG(z) \rangle\} = \mathbb{E} \{\langle e_q \rangle \text{Tr}G(z)\} = \mathbb{E} \langle e_q \rangle \sum_{j=1}^{n} G_{ii}(z) = \frac{1}{z} \mathbb{E} \langle e_q \rangle \sum_{j=1}^{n} (HG)_{ii}(z)$$

$$= z^{-1} \sum_{j,m=1}^{3} \sum_{a+b=1}^{\kappa_{mj}^{(a+b+1)}} \frac{(a+b)!}{(a+b)!} \mathbb{E} \left[ \frac{\partial^{a+b} \langle e_q \rangle \partial G_{jm}}{\partial H_{mj}^{a+b}} \right] + \varepsilon_{I_3,mj}.\]$$

Note that higher-order expansion terms vanish, namely $\varepsilon_{I_3} \leq \sum_{m,j} |\varepsilon_{I_3,mj}| = O(\frac{1}{\sqrt{n}})$, thus minor.
We begin with
\[
z I_{3,(1,0)} = -\frac{1}{n} \mathbb{E} \sum_{j,m} \kappa_{jm}^{(2)} (G_{jm}^2 + G_{jj} G_{mm}) \langle e_q \rangle
\]

\[
= -\frac{1}{n} \mathbb{E} \sum_{k,l=1}^K Q_{kl}^{(2)} [\text{Tr}(T_k G) \text{Tr}(T_l G) - \mathbb{E} [\text{Tr}(T_k G) \text{Tr}(T_l G)]] e_q
\]

\[
= -\frac{1}{n} \mathbb{E} \sum_{k,l=1}^K Q_{kl}^{(2)} \left\{ [\text{Tr}(T_k G) - \mathbb{E} [\text{Tr}(T_k G)] [\text{Tr}(T_l G) - \mathbb{E} [\text{Tr}(T_l G)]] e_q + e_q \text{Tr}(T_k G) \mathbb{E} [\text{Tr}(T_l G)]
\right.
\]

\[
+ e_q [\text{Tr}(T_k G) - \mathbb{E} [\text{Tr}(T_k G)] \mathbb{E} [\text{Tr}(T_l G)] + e_q [\text{Tr}(T_k G) - \mathbb{E} [\text{Tr}(T_k G)] \mathbb{E} [\text{Tr}(T_l G)] + e_q \mathbb{E} [\text{Tr}(T_k G) \text{Tr}(T_l G)]
\right\}
\]

\[
= -\frac{1}{n} \mathbb{E} \sum_{k,l=1}^K Q_{kl}^{(2)} \left\{ e_q [\text{Tr}(T_k G) - \mathbb{E} [\text{Tr}(T_k G)] \mathbb{E} [\text{Tr}(T_l G)]
\right.
\]

\[
+ e_q [\text{Tr}(T_k G) - \mathbb{E} [\text{Tr}(T_k G)] \mathbb{E} [\text{Tr}(T_l G)] + e_q \mathbb{E} [\text{Tr}(T_k G) \text{Tr}(T_l G)]
\right\} + o(1)
\]

\[
= -\sum_{k,l=1}^K Q_{kl}^{(2)} \left\{ \alpha_k M_k \mathbb{E} [\langle T_k G \rangle e_q] + \alpha_l M_l \mathbb{E} [\langle T_l G \rangle e_q] \right\} + o(1).
\]

In other words, we observe a system of equations structure for \( \{ \mathbb{E} [\langle \text{Tr} T_i G \rangle e_q] \}_{i=1}^K \) here. Though we don’t need to derive the system explicitly, we still need to compare the system of equations for \( \{ \mathbb{E} [\langle \text{Tr} T_i G \rangle e_q] \}_{i=1}^K \) with (46). \( I_{3,(1,0)} \) above shows the matching between the coefficient parts. Later we will compare the constant parts of both systems.

Before we proceed further, we need to calculate \( \frac{\partial e_q}{\partial H_{jm}} \). Note that
\[
\frac{\partial \langle e_q \rangle}{\partial H_{jm}} = \frac{\partial e_q}{\partial H_{jm}} = \partial \left\{ \prod_{s=1}^q \exp \left\{ i \tau_s \left[ a_s \text{Tr}(G(z_s)) + b_s \text{Tr}(G(z_s^*)) \right] \right\} \right\}
\]

\[
= -e_q \left[ \sum_{s=1}^q i \tau_s (2 a_s (G^2)_{mj}(z_s) + 2 b_s (G^2)_{mj}(z_s^*)) \right].
\]
It easily follows that

\[ I_{3,(0,1)} = \mathbb{E} \sum_{j,m=1}^{n} \frac{\kappa_{jm}^{(2)}}{n^z} G_{jm} \frac{\partial}{\partial H_{jm}} \left[ \sum_{s=1}^{q} i \tau_s (2a_s(G^2)_{mj}(z_s) + 2b_s(G^2)_{mj}(z_s^*)) \right] \]

\[ = - \frac{1}{n^z} \mathbb{E} \sum_{j,m} \kappa_{jm}^{(2)} G_{jm} e_q \left[ \sum_{s=1}^{q} i \tau_s (2a_s(G^2)_{mj}(z_s) + 2b_s(G^2)_{mj}(z_s^*)) \right] \]

\[ = - \frac{1}{n^z} \mathbb{E} \sum_{j,m} \sum_{k,l=1}^{K} Q_{kl}^{(2)} \left[ i \tau_s 2 a_s Tr(T_k G(z) T_l G^2(z_s)) + i \tau_s 2 b_s Tr(T_k G(z) T_l G^2(z_s^*)) \right] \]

\[ + \frac{1}{n^z} \mathbb{E} e_q \sum_{s=1}^{q} (i \tau_s) \sum_{k=1}^{K} M_k(z) \left[ 2 a_s Tr(T_k G^2(z_s)) + 2 b_s Tr(T_k G^2(z_s^*)) \right] . \]

\[ I_{3,(0,2)}, I_{3,(1,1)}, I_{3,(0,3)}, I_{3,(3,0)}, I_{3,(2,1)}, I_{3,(3,3)} \] are minor and the detailed calculations are omitted here.

Further, note that

\[ \frac{\partial^2 e_q}{\partial H_{jm}^2} = \frac{\partial}{\partial H_{jm}} \left\{ -e_q \left[ \sum_{s=1}^{q} i \tau_s (2a_s(G^2)_{mj}(z_s) + 2b_s(G^2)_{mj}(z_s^*)) \right] \right\} \]

\[ = e_q \left[ \sum_{s=1}^{q} (2a_s(G^2)_{mj}(z_s) + 2b_s(G^2)_{mj}(z_s^*)) \right]^2 \]

\[ + e_q \left\{ \sum_{s=1}^{q} i \tau_s [2a_s((G^2)_{mm} G_{jj} + (G^2)_{mj} G_{jm})(z_s) + 2a_s(G_{mm}(G^2)_{jj} + G_{mj}(G^2)_{jm})(z_s)] \right\} \]

Thus,

\[ I_{3,(1,2)} \]

\[ = - \frac{1}{n^z} \sum_{j,m} \frac{\kappa_{jm}^{(4)}}{2} (G^2_{jm} + G_{jj} G_{mm}) e_q \left[ \sum_{s=1}^{q} i \tau_s [2a_s(G^2)_{mm}(z) G_{jj}(z_s) \right. \]

\[ \left. + 2a_s G_{mm}(z)(G^2)_{jj}(z_s^*) + 2b_s(G^2)_{mm}(z) G_{jj}(z_s) + 2b_s G_{mm}(z)(G^2)_{jj}(z_s^*)] \right] \]

\[ = - \frac{1}{n^z} \sum_{k,l=1}^{K} Q_{kl}^{(4)} M_k(z) M_l(z) e_q \left[ \sum_{s=1}^{q} i \tau_t (a_t M_k(z_t) \alpha_k Tr(G^2(z_t) T_l) \right. \]

\[ \left. + a_t M_l(z_t) \alpha_t Tr(T_k G^2(z_t^*)) + b_t M_k(z_t) \alpha_k Tr(T_l G^2(z_t)) + b_t M_l(z_t^*) \alpha_t Tr(T_k G^2(z_t^*)) \right] . \]

Then we may conclude that

\[ \mathbb{E} \{ e_q [a (c_s) \langle G(z_s) \rangle + b (c_s) \langle G(z_s^*) \rangle] \} \]
After we establish the finite dimensional convergence, it remains to show that the process
\( \text{Tr}(G(z)) \), \( z \in \mathbb{C} \setminus B_{c_0}(\sigma(H)) \) is tight.

\[ - a_s \sum_{k,l=1}^{K} Q_{kl}^{(2)} \left\{ \alpha_k M_k(z_s) \mathbb{E}[\langle T_l G(z_s) \rangle e_q] + \alpha_l M_l(z_s) \mathbb{E}[\langle T_k G(z_s) \rangle e_q] \right\} \]
\[ - b_s \sum_{k,l=1}^{K} Q_{kl}^{(2)} \left\{ \alpha_k M_k(z_s^*) \mathbb{E}[\langle T_l G(z_s^*) \rangle e_q] + \alpha_l M_l(z_s^*) \mathbb{E}[\langle T_k G(z_s^*) \rangle e_q] \right\} \]
\[ - \frac{a_s}{n z_s} \mathbb{E} e_q \sum_{t=1}^{q} \sum_{k,l=1}^{K} Q_{kl}^{(2)} \left[ i \tau_l 2 a_t \text{Tr}(T_k G(z_s) T_l G^2(z_t)) + i \tau_t 2 b_t \text{Tr}(T_k G(z_s) T_l G^2(z_t)) \right] \]
\[ + \frac{a_s}{n z_s} \mathbb{E} e_q \sum_{t=1}^{q} \sum_{k,l=1}^{K} Q_{kl}^{(2)} \left[ i \tau_l 2 a_t \text{Tr}(T_k G(z_s^*) T_l G^2(z_t)) + i \tau_t 2 b_t \text{Tr}(T_k G(z_s^*) T_l G^2(z_t)) \right] \]
\[ - \frac{a_s}{n z_s} \mathbb{E} e_q \sum_{t=1}^{q} \sum_{k,l=1}^{K} Q_{kl}^{(4)} \left[ i \tau_l 2 a_t \text{Tr}(T_k G^2(z_s^*) T_l G^2(z_t)) + i \tau_t 2 b_t \text{Tr}(T_k G^2(z_s^*) T_l G^2(z_t)) \right] \]
\[ + \frac{a_s}{n z_s} \mathbb{E} e_q \sum_{t=1}^{q} \sum_{k,l=1}^{K} Q_{kl}^{(4)} \left[ i \tau_l 2 a_t \text{Tr}(T_k G^2(z_s^*) T_l G^2(z_t)) + i \tau_t 2 b_t \text{Tr}(T_k G^2(z_s^*) T_l G^2(z_t)) \right] \]

where

\[ \Sigma st = \mathbb{E} \{ X (z_s, c_s) X (z_t, c_t) \} = a(c_s) a(c_t) \text{Cov}(z_s, z_t) + a(c_s) b(c_t) \text{Cov}(z_s, z_t^*) + a(c_t) b(c_s) \text{Cov}(z_s^*, z_t) + b(c_s) b(c_t) \text{Cov}(z_s^*, z_t^*) . \]
In other words, we will show that
\begin{align}
\mathbb{E}[\text{Tr}(G(z_1)) - \text{Tr}(G(z_2))]^2 = O(|z_1 - z_2|^2).
\end{align}
(49)

Simply note that
\begin{align}
\mathbb{E}[\text{Tr}(G(z_1)) - \text{Tr}(G(z_2))]^2 = \mathbb{E}[\text{Tr}(G(z_1)(z_1 - z_2)G(z_2))]^2,
\end{align}
(50)
so we only need to show that \( \mathbb{E}[\text{Tr}(G(z_1)G(z_2))]^2 = O(1) \). It suffices to show that
\begin{align}
\mathbb{E}[\text{Tr}(T_lG(z_1)T_mG(z_2))]^2 = O(1), \forall l, m \in [K],
\end{align}
which has been proved in Section A.4. Therefore tightness is established.

\[ \square \]

**Appendix B: Proof of Theorem 3.6**

Similar to the proof of Theorem 3.5, we first derive the mean function \( \mathbb{E}\text{Tr}(\hat{H} - z)^{-1} \) in Section B.1 and the covariance function \( \text{Cov}(\text{Tr}(\hat{G}(z_1)), \text{Tr}(\hat{G}(z_2))) \) in Section B.2. Then we discuss the normality and the tightness for this data-driven renormalized case in Sections B.3 and B.4.

**B.1. Mean function \( \mathbb{E}\text{Tr}(\hat{H} - z)^{-1} \)**

By the fact that \( \|\hat{H} - H\| = o_p\left(\frac{\log(n)}{\sqrt{n}}\right) \) and the resolvent expansion formula. Note also that \( \|\hat{H} - H\| \) is essentially bounded, we have
\begin{align}
\mathbb{E}\text{Tr}(\hat{G}(z)) = \mathbb{E}\text{Tr}(G(z)) - \mathbb{E}\text{Tr}[G(z)(\hat{H} - H)G(z)] \\
+ \mathbb{E}\text{Tr}[G(z)(\hat{H} - H)G(z)(\hat{H} - H)G(z)] + o\left(\frac{\log(n)^3}{\sqrt{n}}\right).
\end{align}
(51)

\( \mathbb{E}\text{Tr}(G(z)) \) has been investigated in the previous sections. So we only need to estimate \( \mathbb{E}\text{Tr}[G(z)(\hat{H} - H)G(z)] \) and \( \mathbb{E}\text{Tr}[G(z)(\hat{H} - H)G(z)(\hat{H} - H)G(z)] \).

\begin{align}
\mathbb{E}\text{Tr}[G(z)(\hat{H} - H)G(z)] = \mathbb{E}\text{Tr}(\hat{H} - H)G(z)^2 = \mathbb{E}\sum_{i,j=1}^{n}(\hat{H}_{ij} - H_{ij})(G^2)_{ji}
= -\mathbb{E}\sum_{i,j=1}^{n} \sum_{\alpha \in C_{\sigma(i)}, \beta \in C_{\sigma(j)}} \frac{H_{\alpha\beta}}{N_{\sigma(i)\sigma(j)}}(G^2)_{ji}
= -\mathbb{E}\sum_{k,l=1}^{K} \sum_{i \in C_k} \sum_{\beta \in C_l} \sum_{\alpha \in C_i} \frac{H_{\alpha\beta}}{N_{kl}}(G^2)_{ji}
= -\mathbb{E}\sum_{k,l=1}^{K} \frac{1}{N_{kl}} \sum_{i \in C_k} \sum_{\beta \in C_l} \sum_{d=1}^{\infty} \frac{\kappa_{\alpha\beta}^{(d+1)}}{\partial n \partial H_{\alpha\beta}^d} = \mathbb{E}\sum_{d=1}^{\infty} J_{1,d},
\end{align}
where

\[ J_{1,d} := - \sum_{k,l=1}^{K} \frac{1}{N_{kl}} \sum_{i \in C_k} \sum_{j \in C_l} \sum_{\alpha \in C_k} \sum_{\beta \in C_l} \frac{\kappa_{\alpha \beta}^{(d+1)}}{d!n^{d+1}} \frac{\partial^d (G^2)_{ji}}{\partial H_{\alpha \beta}}. \]

\[ \mathbb{E} J_{1,1} \]

\[ = \mathbb{E} \sum_{k,l=1}^{K} \frac{Q_{kl}^{(2)}}{nN_{kl}} [1_{C_l}G^21_{C_k}G1_{C_k} + 1_{C_l}G^21_{C_k}G1_{C_k} + 1_{C_l}G1_{C_k}G^21_{C_k} + 1_{C_l}G1_{C_k}G1_{C_k} + 1_{C_l}G1_{C_k}G^21_{C_k}] \]

\[ + \mathbb{E} \sum_{k,l=1}^{K} \frac{1}{N_{kk}} \sum_{j \in C_k} \sum_{\alpha \in C_k} \frac{\kappa_{\alpha}^{(2)}}{n} [(G^2)_{j\alpha}G_{\alpha j} + (G^2)_{j\alpha}G_{\alpha j} + G_{j\alpha}(G^2)_{\alpha j} + G_{j\alpha}(G^2)_{\alpha j}] \]

\[ = \mathbb{E} \sum_{k,l=1}^{K} \frac{Q_{kl}^{(2)}}{nN_{kl}} [Tr(T_kGT_jG)1_{C_l}G1_{C_k} + Tr(T_jGT_kG)1_{C_k}G1_{C_k} + 1_{C_l}G1_{C_k}Tr(T_kGT_jG)] \]

\[ + \mathbb{E} \sum_{k,l=1}^{K} \frac{2}{N_{kk}} \sum_{k,l=1}^{K} \frac{Q_{kl}^{(2)}}{n} (1_{C_k}(GT_kG^2)1_{C_k} + 1_{C_k}(GT_kG^2)1_{C_k}) \]

\[ = O(\frac{1}{n}). \]

Now further consider \( \mathbb{E} J_{1,2} \).

\[ - \mathbb{E} J_{1,2} = \sum_{k,l=1}^{K} \mathbb{E} \frac{Q_{kl}^{(3)}}{2!N_{kl}n^{3/2}} \]

\[ \times \sum_{i \in C_k} \sum_{j \in C_l} [(GT_kGT_jG^2)_{ji} + Tr(T_jG)(GT_kG^2)_{ji} + Tr(T_kG)(GT_jG^2)_{ji} + (GT_kGT_jG^2)_{ji}] \]

\[ + (GT_kG^2T_jG)_{ji} + Tr(T_jG^2)(GT_kG)_{ji} + Tr(T_kG^2)(GT_jG)_{ji} + (GT_kG^2T_jG)_{ji} \]

\[ + (G^2T_kGT_jG)_{ji} + Tr(T_jG)(G^2T_kG)_{ji} + Tr(T_kG)(G^2T_jG)_{ji} + (G^2T_kGT_jG)_{ji}] \]

\[ - \mathbb{E} \sum_{k,l=1}^{K} \frac{1}{N_{kk}} \sum_{i \in C_k} \sum_{\alpha \in C_k} \frac{Q_{\alpha i}^{(2)}}{2!n^{3/2}} [G_{j\alpha}G_{\alpha j}(G^2)_{\alpha i} + G_{j\alpha}G_{\alpha j}(G^2)_{\alpha i} + G_{j\alpha}G_{\alpha j}(G^2)_{\alpha i} + G_{j\alpha}G_{\alpha j}(G^2)_{\alpha i}] \]

\[ + G_{j\alpha}G_{\alpha j}G_{\alpha i} + G_{j\alpha}G_{\alpha j}G_{\alpha i} + G_{j\alpha}(G^2)_{\alpha a}G_{\alpha i} + G_{j\alpha}(G^2)_{\alpha a}G_{\alpha i} + G_{j\alpha}(G^2)_{\alpha a}G_{\alpha i} + G_{j\alpha}(G^2)_{\alpha a}G_{\alpha i} \]

\[ + (G^2)_{\alpha a}G_{\alpha j}G_{\alpha i} + (G^2)_{\alpha a}G_{\alpha j}G_{\alpha i} + (G^2)_{\alpha a}G_{\alpha j}G_{\alpha i} + (G^2)_{\alpha a}G_{\alpha j}G_{\alpha i}] \]

\[ = O(\frac{1}{n^{3/2}}). \]
Similarly, decompose $EJ_{1,3}$ into $\{EJ_{1,3}^{kl}K, l = 1\}

\begin{align*}
EJ_{1,3}^{kl} &= \frac{1}{N_{kl}} \sum_{i \in C_k} \sum_{a \in C_k} \sum_{j \in C_l} \sum_{\beta \in C_l} \kappa_{\alpha\beta}^{(4)} 3!n^2 \left[ e_j^I G^2(e_\alpha e'_\beta + e_\beta e'_\alpha)G(e_\alpha e'_\beta + e_\beta e'_\alpha)G(e_\alpha e'_\beta + e_\beta e'_\alpha)Ge_i 
+ e_j^I G(e_\alpha e'_\beta + e_\beta e'_\alpha)G^2(e_\alpha e'_\beta + e_\beta e'_\alpha)G(e_\alpha e'_\beta + e_\beta e'_\alpha)Ge_i 
+ e_j^I G(e_\alpha e'_\beta + e_\beta e'_\alpha)G(e_\alpha e'_\beta + e_\beta e'_\alpha)G^2(e_\alpha e'_\beta + e_\beta e'_\alpha)Ge_i 
+ e_j^I G(e_\alpha e'_\beta + e_\beta e'_\alpha)G(e_\alpha e'_\beta + e_\beta e'_\alpha)Ge_i \right] 
= O\left(\frac{1}{n^2}\right).
\end{align*}

Note that the normalizing constant is of order $\frac{1}{n^4}$, while the summation is over 4 independent indices $i, j, \alpha, \beta$. Further note that each term in the summation will be in the form $G^k_{j1} G^k_{j2} G^k_{k1} G^k_{k2}$, where the integers $1 \leq s_1, s_2, s_3, s_4 \leq 2, s_1 + s_2 + s_3 + s_4 = 5$, for each pair of $(t_1, t_2)$ and $(t_3, t_4)$, they are either $(\alpha, \beta)$ or $(\beta, \alpha)$. Note that we have odd number of $\alpha$'s and $\beta$'s and the the first of all 8 indices is $j$, with the last one to be $i$, so at least two won’t be diagonal terms when all four indices are different, note also that only one of $G^2$ could appear, which means at least one of $G_{j\alpha}, G_{j\beta}, G_{\alpha i}$ or $G_{\beta i}$ would appear in any of the products, which yields a order of $O_{\prec}(\frac{1}{\sqrt{n}})$, thus minor.

To be more precise, we have

\begin{align*}
E \frac{1}{N_{kl}} \sum_{i \in C_k} \sum_{a \in C_k} \sum_{j \in C_l} \sum_{\beta \in C_l} \kappa_{\alpha\beta}^{(4)} 3!n^2 (G^2)_{ja} G_{\beta \beta} G_{\alpha \alpha} G_{\beta i} 
= E \frac{1}{N_{kl}} \sum_{i \in C_k} \sum_{a \in C_k} \sum_{j \in C_l} \sum_{\beta \in C_l} \kappa_{\alpha\beta}^{(4)} 3!n^2 [(G^2)_{ja} (G_{\beta \beta} G_{\alpha \alpha} - M_k M_l)] G_{\beta i} + (G^2)_{ja} M_k M_l G_{\beta i} 
= E \frac{1}{N_{kl}} \sum_{a \in C_k} \sum_{\beta \in C_l} \kappa_{\alpha\beta}^{(4)} \frac{n^\varepsilon}{3!n^2} \sqrt{n} \sum_{j \in C_l} (G^2)_{ja} G_{\beta i} + O\left(\frac{1}{n^2}\right) = O\left(\frac{n^\varepsilon}{n^{3/2}}\right).
\end{align*}

For higher-order expansion terms of $EJ_{1,d}^{kl}$, $d \geq 4$, simply notice that the normalizing constant would be of order $O(\frac{1}{n^d})$, while the summation is over 4 indices with any of the terms to be of $O(1)$ due to the trivial bound $\|G(z)\| \leq \frac{1}{3(z)}$, thus minor.
Then we proceed to $\mathbb{E} Tr[G(z)(\hat{H} - H)G(z)(\hat{H} - H)G(z)]$.

$$\mathbb{E} Tr[G(z)(\hat{H} - H)G(z)(\hat{H} - H)G(z)] = \mathbb{E} Tr[(\hat{H} - H)G(\hat{H} - H)G^2]$$

$$= \mathbb{E} \sum_{i,j} \sum_{\alpha \in C_{\sigma(i)}} \sum_{\beta \in C_{\sigma(j)}} \frac{-H_{\alpha\beta}}{N_{\sigma(i)\sigma(j)}} (G(\hat{H} - H)G^2)_{ji}$$

$$= \mathbb{E} \sum_{k,l} \frac{1}{N_{kl}} \frac{Q_{kl}^{(2)}}{n} 1_{C_1} [G(E_{B(k,l)} + E_{B(l,k)})G(\hat{H} - H)G^2] 1_{C_k}$$

$$+ \mathbb{E} \sum_{k,l} \frac{1}{N_{kl}} \frac{Q_{kl}^{(2)}}{n} 1_{C_1} [G(E_{B(k,l)} + E_{B(l,k)})G^2] 1_{C_k}$$

$$+ \mathbb{E} \sum_{k,l} \frac{1}{N_{kl}} \frac{Q_{kl}^{(2)}}{n} 1_{C_1} [G(\hat{H} - H)G^2 (E_{B(k,l)} + E_{B(l,k)})G] 1_{C_k}$$

$$+ \mathbb{E} \sum_{k,l} \frac{1}{N_{kl}} \frac{Q_{kl}^{(2)}}{n} 1_{C_1} [G(\hat{H} - H)G(E_{B(k,l)} + E_{B(l,k)})G^2] 1_{C_k}$$

$$+ \mathbb{E} \sum_{i,j} \sum_{\alpha \in C_{\sigma(i)}} \sum_{\beta \in C_{\sigma(j)}} \frac{-1}{N_{\sigma(i)\sigma(j)}} \sum_{d=2}^{\infty} \frac{\kappa_{\alpha\beta}^{(d+1)}}{d! n^{\frac{d+1}{2}}} \frac{\partial^d}{\partial H^d_{\alpha\beta}} (G(\hat{H} - H)G^2)_{ji}$$

$$= o(1),$$

where $E_{B(k,l)}$ indicates the block matrix $1_{C_k} 1_{C_l}^T$.

Among the above terms we know that only

$$\mathbb{E} \sum_{k,l} \frac{1}{N_{kl}} \frac{Q_{kl}^{(2)}}{n} 1_{C_1} [G(E_{B(k,l)} + E_{B(l,k)})G^2] 1_{C_k}$$

$$= \mathbb{E} \sum_{k,l} \frac{1}{N_{kl}} \frac{Q_{kl}^{(2)}}{n} [\text{Tr}(T_i G)\delta_{ik}\text{Tr}(T_k GT_i G) + \text{Tr}(T_i G)\text{Tr}(T_k GT_k G)]$$

is $O(\frac{1}{n})$, while the others are $O(\frac{1}{n^{\frac{3}{2}}})$. 

B.2. Covariance function \( \text{Cov}(\text{Tr}(\hat{G}(z_1)), \text{Tr}(\hat{G}(z_2))) \)

To calculate this covariance function, first we need to do a decomposition.

\[
\mathbb{E}\text{Cov}(\text{Tr}(\hat{G}(z_1)), \text{Tr}(\hat{G}(z_2)))
= \mathbb{E}(\text{Tr}(\hat{G}(z_1)) - \mathbb{E}\text{Tr}(\hat{G}(z_1))(\text{Tr}(\hat{G}(z_2)) - \mathbb{E}\text{Tr}(\hat{G}(z_2)))
= \mathbb{E}\{[(\text{Tr}(\hat{G}(z_1)) - \text{Tr}(G(z_1)) - (\mathbb{E}\text{Tr}(\hat{G}(z_1)) - \mathbb{E}\text{Tr}(G(z_1)) + (\text{Tr}(G(z_1)) - \mathbb{E}\text{Tr}(G(z_1)))]
\times [(\text{Tr}(\hat{G}(z_2)) - \text{Tr}(G(z_2)) - (\mathbb{E}\text{Tr}(\hat{G}(z_2)) - \mathbb{E}\text{Tr}(G(z_2)) + (\text{Tr}(G(z_2)) - \mathbb{E}\text{Tr}(G(z_2)))]\}
= \mathbb{E}[a_1a_2 + a_1b_2 + a_1c_2 + b_1a_2 + b_1b_2 + b_1c_2 + c_1a_2 + c_1b_2 + c_1c_2],
\]

where
\[
a_i = \text{Tr}(\hat{G}(z_i)) - \text{Tr}(G(z_i)), \ i = 1, 2,
b_i = -\mathbb{E}(a_i), \ i = 1, 2,
c_i = \text{Tr}(G(z_i)) - \mathbb{E}\text{Tr}(G(z_i)), \ i = 1, 2.
\]

Instantly, we know that \( \mathbb{E}b_1c_2 = \mathbb{E}b_2c_1 = 0. \)

First, we consider \( \mathbb{E}a_1a_2. \)

\[
\mathbb{E}[a_1a_2] = \mathbb{E}(\text{Tr}(\hat{G}(z_1)) - \text{Tr}(G(z_1))(\text{Tr}(\hat{G}(z_2)) - \text{Tr}(G(z_2))
= \mathbb{E}[\text{Tr}(-G(z_1)(\hat{H} - H)G(z_1) + G(z_1)(\hat{H} - H)G(z_1)(\hat{H} - H)G(z_1) + o_P(\frac{\log(n)}{n})] \times
\text{Tr}(-G(z_2)(\hat{H} - H)G(z_2) + G(z_2)(\hat{H} - H)G(z_2)(\hat{H} - H)G(z_2) + o_P(\frac{\log(n)}{n})] \times
- G(z_2)(\hat{H} - H)G(z_2)(\hat{H} - H)G(z_2) + o_P(\frac{\log(n)}{n})].
\]

The problem would be that there would be too many terms (including the \( \frac{1}{\sqrt{n}} \) terms) that need to be calculated provided with trivial bound \( \mathbb{E}|\text{Tr}G(z)(\hat{H} - H)G(z)|^2 = O(1). \)
Also, note that

\[
\mathbb{E} \text{Tr}(G(z)\hat{H} - H)(\hat{H} - H)G(z)) \text{Tr}(G(z^*)(\hat{H} - H)G(z^*))
\]

\[
= \mathbb{E} \sum_{i,j} \sum_{\alpha \in C_{\sigma(i)}, \beta \in C_{\sigma(j)}} \frac{H_{\alpha \beta}}{N_{\sigma(i)\sigma(j)}} (G(z))^2(g)_{ji} \text{Tr}(G(z^*)\hat{H} - H)G(z^*))
\]

\[
= \mathbb{E} \sum_{k,l=1}^{K} \sum_{i \in C_k, j \in C_l} \frac{1}{N_{kl}} \sum_{d=2}^{\infty} \frac{Q_{d+1}^{(d+1)}}{n^{d+1}} \frac{\partial^d}{\partial H_{\alpha \beta}} [\text{Tr}(G(z^*)\hat{H} - H)G(z^*))G^2(z))]_{ji}
\]

\[
+ \mathbb{E} \sum_{k,l=1}^{K} \sum_{i \in C_k, j \in C_l} \frac{1}{N_{kl}} \sum_{d=2}^{\infty} \frac{Q_{d+1}^{(d+1)}}{n^{d+1}} \frac{\partial^d}{\partial H_{\alpha \beta}} [\text{Tr}(G(z^*)\hat{H} - H)G(z^*))G^2(z))]_{ji}
\]

\[
= O\left(\frac{1}{\sqrt{n}}\right) + \mathbb{E} \sum_{k,l=1}^{K} \sum_{i \in C_k, j \in C_l} \frac{1}{N_{kl}} \sum_{d=2}^{\infty} \frac{Q_{d+1}^{(d+1)}}{n^{d+1}} \frac{\partial^d}{\partial H_{\alpha \beta}} [\text{Tr}(G(z^*)\hat{H} - H)G(z^*))G^2(z))]_{ji}
\]

\[
= O\left(\frac{\log(n)}{\sqrt{n}}\right).
\]

Hence, we need a more efficient bound for $\mathbb{E}|\text{Tr}(G(z)(\hat{H} - H)G(z)|^2$.

\[
\mathbb{E} \text{Tr}(G(z)\hat{H} - H)(\hat{H} - H)G(z)) \text{Tr}(G(z^*)(\hat{H} - H)G(z^*))
\]

\[
= \mathbb{E} \sum_{k,l=1}^{K} \sum_{i \in C_k, j \in C_l} \frac{Q_{d+1}^{(d+1)}}{n^{d+1}} \frac{\partial^d}{\partial H_{\alpha \beta}} [\text{Tr}(G(z^*)\hat{H} - H)G(z^*))G^2(z))]_{ji}
\]

\[
+ \mathbb{E} \sum_{k,l=1}^{K} \sum_{i \in C_k, j \in C_l} \frac{1}{N_{kl}} \sum_{d=2}^{\infty} \frac{Q_{d+1}^{(d+1)}}{n^{d+1}} \frac{\partial^d}{\partial H_{\alpha \beta}} [\text{Tr}(G(z^*)\hat{H} - H)G(z^*))G^2(z))]_{ji}
\]

\[
= O\left(\frac{1}{\sqrt{n}}\right) + \mathbb{E} \sum_{k,l=1}^{K} \sum_{i \in C_k, j \in C_l} \frac{1}{N_{kl}} \sum_{d=2}^{\infty} \frac{Q_{d+1}^{(d+1)}}{n^{d+1}} \frac{\partial^d}{\partial H_{\alpha \beta}} [\text{Tr}(G(z^*)\hat{H} - H)G(z^*))G^2(z))]_{ji}
\]

\[
= O\left(\frac{\log(n)}{\sqrt{n}}\right).
\]
Thus, by Cauchy inequality we can show that
\[
\mathbb{E}[a_1a_2] = \mathbb{E}(Tr\hat{G}(z_1) - TrG(z_1))(Tr\hat{G}(z_2) - TrG(z_2)) = \mathbb{E}[Tr(-G(z_1)(\hat{H} - H)G(z_1) + G(z_1)(\hat{H} - H)G(z_1))(\hat{H} - H)G(z_1)] + o_p(\frac{\log(n)}{\sqrt{n}})) \times \\
= O\left(\frac{\log(n)}{\sqrt{n}}\right).
\]

In the meantime, by Section B.1, we know
\[
\mathbb{E}[a_1b_1] = -\mathbb{E}(Tr\hat{G}(z_1) - TrG(z_1))\mathbb{E}(Tr\hat{G}(z_2) - TrG(z_2)) = O\left(\frac{\log(n)}{n}\right),
\]
\[
\mathbb{E}[a_2b_1] = -\mathbb{E}(Tr\hat{G}(z_2) - TrG(z_2))\mathbb{E}(Tr\hat{G}(z_1) - TrG(z_1)) = O\left(\frac{\log(n)}{n}\right),
\]
\[
\mathbb{E}[b_1b_2] = \mathbb{E}(Tr\hat{G}(z_2) - TrG(z_2))\mathbb{E}(Tr\hat{G}(z_1) - TrG(z_1)) = O\left(\frac{\log(n)^2}{n}\right).
\]

While \(\mathbb{E}c_1c_2\) is also known by Section A.3, we only need to consider \(\mathbb{E}a_1c_2\).

\[
|\mathbb{E}a_1c_2|^2 = |\mathbb{E}(Tr(\hat{G}(z_1) - G(z_1)))\mathbb{E}(Tr(G(z_2)) - ET(Tr(G(z_2))))|^2 \\
\leq \mathbb{E}|Tr(\hat{G}(z_1) - G(z_1))|^2\mathbb{E}|Tr(G(z_2)) - ET(Tr(G(z_2)))|^2 \\
\]

Recall the system of equations for \(\{Cov_{lm}\}_{l=1}^K\), \(\forall m \in [K]\) in A.3, note that the entries of the coefficient matrices are of order 1, thus
\[
\mathbb{E}|Tr(G(z_2)) - ET(Tr(G(z_2)))|^2 = O(1).
\]

So we have \(\mathbb{E}[a_1c_2] = O\left(\frac{\log(n)}{\sqrt{n}}\right)\). Similarly, \(\mathbb{E}[a_2c_1] = O\left(\frac{\log(n)}{\sqrt{n}}\right)\). Then we see that only \(\mathbb{E}c_1c_2\) will count, which means that we will have exactly the same covariance function as in Section A.3. It remains to show the normality.

**B.3. Proof of normality for the data-driven version**

Proof. The main procedure is exactly the same as that in Section A.7. For simplicity we will mainly focus more on the difference, some of the overlapping details will not be stated. Let \(\hat{\gamma}(z) = \hat{\gamma}^{(n)}(z) := \Re(\hat{G}(z))\) and \(\hat{\theta}(z) = \hat{\theta}^{(n)}(z) = \Im(\hat{G}(z))\),

\[
\hat{\Psi}(z, c) = \begin{cases} \hat{\gamma}(z), & \text{if } c = \gamma \\ \hat{\theta}(z), & \text{if } c = \theta \end{cases}
\]
and extend the definition of \( \hat{a}(c) \) and \( \hat{b}(c) \), s.t.

\[
(\hat{a}(c), \hat{b}(c)) = \begin{cases} (1/2, 1/2) & \text{if } c = \gamma \\ (1/2i, 1/2i) & \text{if } c = \theta \end{cases}
\]

Apparently \( \mathbb{E}\{\hat{\Psi}(z, c)\} = 0 \). Then our goal is to prove that \( \forall \) fixed \( q \in \mathbb{Z}_+ \), \( \left\{ z_s \right\}_{s=1}^q \in \{\mathbb{C}\setminus B_{\varepsilon_0}(\sigma(H))\}^q \), \( \left\{ c_s \right\}_{s=1}^q \in \{\gamma, \theta\}^q \), the joint probability distribution of random variables \( \hat{\Psi}(z_1, c_1) \), ..., \( \hat{\Psi}(z_q, c_q) \) is the \( q \)-dimensional Gaussian distribution with zero mean and feasible covariance matrix. Then we consider the characteristic function of \( \hat{\Psi}(z_1, c_1), \ldots, \hat{\Psi}(z_q, c_q) \),

\[
\hat{\epsilon}_q^{(n)}(T_q, C_q, Z_q) = \prod_{s=1}^q \exp \left\{ i\tau_s \left[ \hat{a}(c_s) Tr(\hat{G}(z_s)) + \hat{b}(c_s) Tr(\hat{G}(z_s^*)) \right] \right\}.
\]

where \( T_q = (\tau_1, \ldots, \tau_q) \), \( C_q = (c_1, \ldots, c_q) \), \( Z_q = (z_1, \ldots, z_q) \). And we will simply use \( \hat{\epsilon}_q \) when there is no confusion.

By the resolvent identity, we have

\[
z \sum_{j=1}^n \mathbb{E}\left\{ \langle \hat{\epsilon}_q \rangle \hat{G}_{jj} \right\} = \sum_{j=1}^n \mathbb{E}\left\{ \langle \hat{\epsilon}_q \rangle (G - G(\hat{H} - H)G + G(\hat{H} - H)G(\hat{H} - H)G \right.

\nu \hat{G}(\hat{H} - H)G(\hat{H} - H)G(\hat{H} - H)G)_{jj} \right\} \]

\[
= J_3^{(1)} + J_3^{(2)} + J_3^{(3)} + O\left(\frac{\log(n)}{\sqrt{n}}\right).
\]

where

\[
J_3^{(1)} = \sum_{j=1}^n \mathbb{E}\left\{ \langle \hat{\epsilon}_q \rangle \right\} = \sum_{j=1}^n \mathbb{E}\left\{ \langle \hat{\epsilon}_q \rangle G_{jm} H_{mj} \right\}
\]

\[
= z^{-1} \sum_{j,m=1}^n \left( \sum_{d=0}^p \kappa_{jm}^{(d+1)} \mathbb{E}\left[ \frac{\partial^d \langle \hat{\epsilon}_q \rangle G_{jm}}{\partial H_{mj}^d} \right] + \varepsilon_{jm} \right) = \sum_{a+b=1}^3 J_3^{(1)} + \varepsilon_{d(1)}.
\]

Note that by Cauchy inequality and adopting the same way we deal with \( e_q \), we can show that all the terms whose counterparts vanish in the case of \( e_q \) will still vanish here. First, we need to approximate the derivatives.

\[
\frac{\partial Tr\hat{G}}{\partial H_{jm}} = \frac{\partial Tr(G - G(\hat{H} - H)G + G(\hat{H} - H)G(\hat{H} - H)G)}{\partial H_{jm}} \frac{\log(n)}{\sqrt{n}}.
\]

\[
\frac{\partial \langle \hat{\epsilon}_q \rangle}{\partial H_{jm}} = \mathbb{E}\{\hat{\epsilon}_q\} \frac{\partial \sum_{s=1}^q \left( i\tau_s \left[ \hat{a}(c_s) Tr(\hat{G}(z_s)) + \hat{b}(c_s) Tr(\hat{G}(z_s^*)) \right] \right)}{\partial H_{jm}}.
\]
One should note that truncating the infinite expansions to get approximation of the derivatives like this is always dangerous. However, note that the form of the higher-order expansion terms are always clear in the sense that $(\hat{H} - H)$ will contribute one more $\frac{\log(n)}{\sqrt{n}}$. Also in our setting (13), $\forall i, j \in [n]$, $H_{ij}$ is the averaging of centered Bernoulli random variable, thus always bounded. So we may use a finite expansion here. We can see that

$$J_{3,(0,1)}^{(1)} = -\frac{1}{nz} \sum_{j,m} \kappa_{jm}^{(2)} G_{jm} \hat{e}_q \left[ \sum_{s=1}^{q} i \tau_s (2 \hat{a}_s \frac{\partial \text{Tr} \hat{G}(z_s)}{\partial H_{jm}} + 2 \hat{b}_s \frac{\partial \text{Tr} \hat{G}(z_s^*)}{\partial H_{jm}}) \right].$$

Comparing with $I_{3,(0,1)}$, it’s not hard to see that as long as we can prove

$$\frac{1}{n} \sum_{j,m} \kappa_{jm}^{(2)} \frac{\partial \text{Tr} G(z_s)(\hat{H} - H)G(z_s)}{\partial H_{jm}} = o(1),$$

and

$$\frac{1}{n} \sum_{j,m} \kappa_{jm}^{(2)} \frac{\partial \text{Tr} G(z_s)(\hat{H} - H)G(z_s)(\hat{H} - H)G(z_s)}{\partial H_{jm}} = o(1),$$

the non-vanishing contribution of the terms to the covariance terms would be the same as in Section A.7.

Easy to see that

$$\sum_{m,j} G_{mj} \kappa_{mj}^{(2)} \frac{3!}{n} \frac{\partial^2 \langle \hat{e}_q \rangle}{\partial H_{jm}} = O\left(\frac{\log(n)}{\sqrt{n}}\right)$$

is minor. So are the other components generated by the reminder terms of the derivatives.

Similar things happen when we consider the analog of $I_{3,(1,2)}$

$$J_{3,(1,2)}^{(1)} = -\frac{1}{n^2z} \sum_{j,m} \frac{\kappa_{jm}^{(4)}}{3!} 3(G_{jm}^2 + G_{jj}G_{mm}) \frac{\partial^2 \langle \hat{e}_q \rangle}{\partial H_{jm}^2},$$

the repetitive $O\left(\frac{\log(n)}{\sqrt{n}}\right)$ factors introduced by $(\hat{H} - H)$ make the terms generated by the difference between $\hat{G}$ and $G$ minor.

Then it remains to show that $J_{3}^{(2)}$ and $J_{3}^{(3)}$ are minor.

$$J_{3}^{(2)} := -\sum_{j=1}^{n} \mathbb{E} \{ \langle \hat{e}_q \rangle (G(\hat{H} - H)G)_{jj} \} = \sum_{j,m=1}^{n} \mathbb{E} \{ \hat{e}_q \} \sum_{\alpha \in C_{\sigma(m)} \beta \in C_{\sigma(j)}} \frac{H_{\alpha\beta}}{N_{\sigma(m)}\sigma_j} (G^2)_{mj},$$

$$= \sum_{k,l=1}^{K} \sum_{m,\alpha \in C_{k,j} \beta \in C_{l}} \mathbb{E} Q_{kl}^{(2)} \langle \hat{e}_q \rangle (G_{m\alpha}(G^2)_{\beta j} + G_{m\beta}(G^2)_{\alpha j} + (G^2)_{m\alpha}(G_{\beta j}) + (G^2)_{m\beta}G_{\beta j})$$

$$+ O\left(\frac{\log(n)}{n}\right) = O\left(\frac{\log(n)}{n}\right).$$
$$J_3^{(3)} := \sum_{j=1}^{n} \mathbb{E}[\langle \hat{e}_q \rangle (G(\hat{H} - H)G(\hat{H} - H)G)_{jj}] = \sum_{j,m=1}^{n} \mathbb{E}(\langle \hat{e}_q \rangle (\hat{H} - H)_{jm}(G(\hat{H} - H)G^2)_{mj}$$

$$= - \sum_{j,m=1}^{n} \mathbb{E}(\langle \hat{e}_q \rangle \sum_{\alpha \in C_{\sigma(j)}, \beta \in C_{\sigma(j)}} \frac{H_{\alpha\beta}}{N_{\sigma(j)}(\hat{H} - H)G^2})_{mj}$$

$$= \sum_{k,l=1}^{K} \sum_{m,\alpha \in C_{k,j}, \beta \in C_{l}} \mathbb{E} \frac{Q_{kl}^{(2)}}{N_{kln}} \langle \hat{e}_q \rangle (G_{\sigma(m)}(\hat{H} - H)G^2)_{\beta j} + G_{\sigma(m)}(\hat{H} - H)G^2)_{\alpha j}$$

$$+ (G(\hat{H} - H)G_{\sigma(m)}(G^2)_{\beta j} + (G(\hat{H} - H)G_{\sigma(m)}(G^2)_{\alpha j}$$

$$+ (G(\hat{H} - H)G^2)_{\sigma(m)}G_{\beta j} + (G(\hat{H} - H)G^2)_{\sigma(m)}G_{\alpha j} + \frac{1}{N_{k}}(G(E_{B(k,l)} + E_{B(l,k)})G^2)_{mj}$$

$$+ O(\frac{\log(n)}{n})$$

$$= O(\frac{\log(n)}{n}).$$

Thus, we may also conclude that the covariance function would be the same as that of Theorem 3.5 and the normality follows.

**B.4. Tightness of the process $\langle Tr\hat{G}(z)\rangle$**

Similarly, after we establish the finite dimensional convergence, it left to show that the process $Tr\langle \hat{G}(z)\rangle, z \in \mathbb{C}\setminus B_{\epsilon_0}(\sigma(\hat{H}))$ is tight. We will show that

$$\mathbb{E}|Tr\langle \hat{G}(z_1)\rangle - Tr\langle \hat{G}(z_2)\rangle|^2 = O(|z_1 - z_2|^2). \quad (53)$$

Again note that

$$\mathbb{E}|Tr\langle \hat{G}(z_1)\rangle - Tr\langle \hat{G}(z_2)\rangle|^2$$

$$= \mathbb{E}|Tr\langle \hat{G}(z_1)\hat{G}(z_2)\rangle|^2|z_1 - z_2|^2.$$

and we can break down the question to boundedness of $\mathbb{E}|Tr\langle T_m \hat{G}(z_1)T_m \hat{G}(z_2)\rangle|^2$ and adopt a similar approach to Section A.4. The details are omitted here.

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