A NOTE ON (SEMI)VARIOGRAM
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Abstract. (Semi)Variograms are usually discussed in the framework of stationary or intrinsically stationary processes. We retell here this piece of theory in the setting of generic Gaussian vectors and of Gaussian vectors with constant variance. We show how to re-parametrize the distribution as a function of the variogram and how to characterise all the Gaussian distribution with a given variogram.

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1. Introduction

If \((Y_t)_{t \in T}\) is a Gaussian random field, its variogram is the mapping from 2-sets \(\{s, t\}\), \(s, t \in T\), to \(\text{Var}(Y_s - Y_t) / 2\). In some applied fields, such as Geostatistics or Metrology, such a multivariate parameter is considered more telling than the process correlation \(\{s, t\} \mapsto \text{Cor}(Y_s, Y_t)\). For example, if a meaningful distance \(\{s, t\} \mapsto d(s, t)\) is available, one would like the variogram values to increase with distance to model the larger randomness at far away locations.

In this paper we discuss some general topics on variogram that was originally discussed by Matheron [8] under assumptions of stationarity and homogeneity. Modern expositions are to be found in [4, Ch. 2], [2, Ch. 2], [5, Ch.1], [6]. The present piece of research work was prompted by the need of a clear and sound methodology in the occasion of previous applied research, see [9, 10].

Our goal now is to rework the basic mathematics in order to prepare for a future better treatment of a number of items of interest e.g.,
- simulation of a Gaussian random field with given variogram;
- geometry of the Gaussian model based on the use of variograms as parameters, in the sense of [12, 11];
- parsimonious models e.g., graphical models [7], parametrized by variograms;
- Bayes approach to Kriging, especially nonparametric Bayes.

In Sec. 2 we formally discuss the case of a generic Gaussian vector and of the special case of a constant variance. In Sec. 3 we briefly discuss the connection with the case of Gaussian stationary random fields. A few conclusions are discussed in the final section.
2. Variogram of a normal vector

We first consider a generic Gaussian vector and we plan to specialise our assumptions later on.

**Definition 1.** Assume $Y \sim N_n(\mu, \Sigma)$, $\mu = (\mu_i : i = 1, \ldots, n)$, $\Sigma = [\sigma_{ij}]_{i,j=1}^n$. The (semi)variogram of $Y$ is the $n \times n$ matrix $\Gamma = [\gamma_{ij}]_{i,j=1}^n$ with

$$2\gamma_{ij} = \text{Var}(Y_i - Y_j) = (e_i - e_j)'\Sigma(e_i - e_j) = \sigma_{ii} + \sigma_{jj} - 2\sigma_{ij}.$$  

The matrix $\Gamma$ can be written

$$\Gamma = \frac{1}{2} \left( \text{vdiag}(\Sigma) \cdot 1' + 1 \cdot \text{vdiag}(\Sigma)' \right) - \Sigma$$

where $1$ is the unit column vector and $\text{vdiag}(\Sigma) = \text{diag}(\Sigma)$ is the orthogonal projector on $\text{Span}(1)$.

Let us recall the basic properties of the variogram matrix.

**Proposition 2.** The variogram $\Gamma$ is symmetric, with zero diagonal, and it is conditionally negative definite.

**Proof.** The quadratic form of

$$\Sigma = \frac{1}{2} \left( \text{vdiag}(\Sigma) \cdot 1' + 1 \cdot \text{vdiag}(\Sigma)' \right) - \Gamma$$

at $\alpha \in \mathbb{R}^n$ is

$$\alpha'\Sigma\alpha = (\alpha \cdot 1)(\alpha \cdot \text{vdiag}(\Sigma)) - \alpha'\Gamma\alpha,$$

hence $\alpha \cdot 1 = 0$ implies $\alpha'\Sigma\alpha = -\alpha'\Gamma\alpha$, in particular $\Gamma$ is negative definite conditionally to $\sum_j \alpha_j = 0$. □

**Definition 3.** A nonzero symmetric matrix which has zero diagonal and is conditionally negative definite will be called a variogram matrix.

**Proposition 4.** Let $\Gamma$ be a variogram matrix. There exist nonnegative $\mu_1, \ldots, \mu_{n-1}$ and orthonormal vectors $w_1, \ldots, w_{n-1}$ in $\text{Span}(1)^\perp$ such that

$$\Gamma = \sum_{j=1}^{n-1} \frac{\mu_j}{n} \cdot 1 \otimes 1 - \sum_{j=1}^{n-1} \mu_j w_j \otimes w_j$$

(1)

**Proof.** For each matrix $U = [u_1 \cdots u_{n-1}] \in \mathbb{R}^{n \times (n-1)}$ such that $U'^T U = I_{n-1}$ and $1' U = 0$, the matrix $\Sigma_0 = -U'^T \Gamma U \in \mathbb{R}^{n \times (n-1)}$ is nonnegative definite. It follows that $V'^T \Sigma_0 V = \text{diag}(\mu_j : j = 1, \ldots, n-1)$ for some $V \in O_{n-1}$, $\mu_j \geq 0$, $i = 1, \ldots, n-1$, hence $(U V')'^T \Gamma (U V') = \text{diag}(\mu_j : j = 1, \ldots, n-1)$. If $W = U V \in \mathbb{R}^{n \times (n-1)}$, then $W'^T W = V'^T U'^T U V = V'^T V = I_{n-1}$ and $1' W = 0$. If $W = [w_1 \cdots w_{n-1}]$, then $(w_j, -\mu_j)$, $j = 1, \ldots, n-1$ are couples of eigen-vectors and eigenvalues of $-\Gamma$. As $\Gamma$ has zero trace, then the $n$-eigenvalue of $\Gamma$ is $\sum_{j=1}^{n-1} \mu_j > 0$. Its eigen-space must be orthogonal to all $w_j$, hence it contains $\text{Span}(1)$. □

Computation of the parameters suggests that the variogram matrix carries $n(n-1)/2$ degrees of freedom, while the diagonal of $\Sigma$ carries $n$ df. Together, $\Lambda$ and $\Gamma$ carry as many degrees of freedom as $\Sigma$, i.e. $n(n-1)/2 + n = (n + 1)n/2$. More precisely, we have the following re-parametrization of $\Sigma$. 

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Proposition 5.

1. The mapping from a positive definite \( \Sigma \) to a positive diagonal \( \Lambda \) and a variogram matrix \( \Gamma \) defined by

\[
\Sigma \mapsto \left( \text{diag}(\Sigma), \frac{1}{2}(\text{vdiag}(\Sigma)\mathbf{1} + \mathbf{1}\text{vdiag}(\Sigma)') - \Sigma \right) = (\Lambda, \Gamma)
\]

is the restriction to the cone of positive definite matrices of a linear map on \( n \times n \) real matrices. It is injective with inverse

\[
(\Lambda, \Gamma) \mapsto \frac{1}{2}(\Lambda\mathbf{1}\mathbf{1}' + \mathbf{1}\Lambda\mathbf{1}') - \Gamma = \frac{1}{2}(\text{vec}(\Lambda)\mathbf{1}' + \mathbf{1}\text{vec}(\Lambda)') - \Gamma.
\]

2. The range of the mapping (1) consists of all \( \Lambda, \Gamma \) positive diagonal and symmetric, respectively, and satisfying

\[
(\beta \cdot \mathbf{1})(\beta \cdot \text{vec}(\Lambda)) \geq \beta'\Gamma\beta, \quad \beta \in \mathbb{R}^n.
\]

3. In particular, \( \Gamma \) is conditionally negative definite and

\[
n\text{Tr}(\Lambda) \geq \mathbf{1}'\mathbf{1} = \sum_{i,j=1}^{n} \gamma_{ij}.
\]

4. If the spectral decomposition (1) holds, then the condition (5) on \( \Lambda \) becomes

\[
\text{Tr}(\Lambda) \geq \sum_{j} \mu_j.
\]

Proof. (1) If \( \Sigma_i \mapsto (\Lambda_i, \Gamma_i), \ i = 1, 2, \) and \( (\Lambda_1, \Gamma_1) = (\Lambda_2, \Gamma_2) \), then \( \text{diag}(\Sigma_1) = \text{diag}(\Sigma_2) \) and \( \Sigma_1 = \Sigma_2 \) follows from \( \Gamma_1 = \Gamma_2 \).

(2) Let \( \Lambda \) and \( \Gamma \) be generic positive diagonal and conditionally negative definite, respectively. Then for a generic \( \alpha = \alpha_0 + \alpha_1 \), with \( \alpha_0 \cdot \mathbf{1} = 0 \) and \( \bar{\alpha} = \frac{1}{n}\alpha \cdot \mathbf{1} \), we have

\[
\alpha' \left[ \frac{1}{2}(\Lambda\mathbf{1}\mathbf{1}' + \mathbf{1}\Lambda\mathbf{1}') - \Gamma \right] \alpha = n\bar{\alpha} \alpha \cdot \text{vec}(\Lambda) - \alpha'\Gamma\alpha
\]

\[
= \begin{cases} 
-\alpha_0\Gamma\alpha_0 \geq 0 & \text{if } \bar{\alpha} = 0, \\
\alpha \cdot \text{vec}(\Lambda) - \alpha'\Gamma\alpha & \text{if } n\bar{\alpha} = 1.
\end{cases}
\]

Finally, we take \( \alpha = (\beta \cdot \mathbf{1})^{-1}\beta \) to obtain (1).

(3) Eq. (1) implies a conditionally negative definite \( \Gamma \) if \( \beta \cdot \mathbf{1} = 0 \). Otherwise, if \( \beta = \mathbf{1} \) the inequality becomes (5).

(4) If the spectral decomposition holds, then \( \mathbf{1}'\Gamma\mathbf{1} = n\sum_{j=1}^{n-1} \mu_j \).

\[\square\]

Remark. If \( \det(\Sigma) \neq 0 \), similar formulæ are obtained by considering the correlation matrix

\[
R = (\text{diag} \Sigma)^{-1/2} \Sigma (\text{diag} \Sigma)^{-1/2};
\]

viz

\[
\Gamma = \frac{1}{2}(\text{vdiag}(\Sigma)\mathbf{1}' + \mathbf{1}\text{vdiag}(\Sigma)') - (\text{diag} \Sigma)^{1/2} R (\text{diag} \Sigma)^{1/2}
\]

\[
= \Lambda^{1/2} \left( \frac{1}{2} \left( \Lambda^{1/2}\mathbf{1}\mathbf{1}'\Lambda^{-1/2} + \Lambda^{-1/2}\mathbf{1}\mathbf{1}'\Lambda^{1/2} \right) - R \right) \Lambda^{1/2}.
\]

where \( \text{diag} \Sigma = \Lambda \).

This formula is sometimes preferred in the applied literature because both \( \Gamma \) and \( R \) carry the same number of degrees of freedom and they are thought as being equivalent.
assignments. However, it is important to consider that the imputation of the coherent diagonal $\Lambda$ depends on $\Gamma$.

Given a variogram matrix $\Gamma$, Eq. (5) is a linear bound on $\Lambda$. Here, we do not discuss it in full generality, but we move to consider the case where the variance is constant. Such an assumption is of interest in applications where a minimum of stationarity must be assumed.

**Proposition 6.** Assume that the variance is constant, $\text{diag} (\Sigma) = \lambda I_n$.

1. Eq.s (2) and (3) become

$$\Sigma \mapsto (\lambda, \lambda \mathbf{1}^T - \Sigma) = (\lambda, \Gamma) \tag{7}$$

and

$$(\lambda, \Gamma) \mapsto \lambda \mathbf{1}^T - \Gamma = \Sigma, \tag{8}$$

respectively.

2. The existence condition (5) on $\lambda$ becomes

$$n^2 \lambda \geq \sum_{i,j=1}^n \gamma_{ij}. \tag{9}$$

3. The correlation Eq. (6) becomes

$$\Gamma = \lambda (\mathbf{1}^T - R). \tag{10}$$

4. If $n\lambda > \mathbf{1}^T \mathbf{1}$, then $\det \Gamma \neq 0$ then $\Sigma$ is invertible and, in such a case,

$$\Gamma^{-1} = -\Sigma^{-1} - \lambda(1 - \lambda \mathbf{1} \Sigma^{-1} \mathbf{1})^{-1} \Sigma^{-1} \mathbf{11}^T \Sigma^{-1}, \tag{10}$$

$$\Sigma^{-1} = -\Gamma^{-1} - \lambda(1 - \lambda \Gamma^{-1} \mathbf{1})^{-1} \Gamma^{-1} \mathbf{11}^T \Gamma^{-1}. \tag{11}$$

**Proof.** Everything but the last item is a special case of Prop. 5. If the matrices $\Sigma$ and $\Gamma$ are both invertible, from the Sherman-Morrison formula we obtain Eq.s (10) and (11). Assume det $\Gamma \neq 0$ and $n^2 \lambda > \mathbf{1}^T \mathbf{1}$. From the spectral representation of $\Gamma$ in Eq. (1), we derive $\mathbf{1}^T \gamma^{-1} \mathbf{1} = n \left( \sum_{j=1}^{n-1} \gamma_j \right)^{-1}$. It follows from the assumption that $1 - \lambda \mathbf{1}^T \Gamma^{-1} \mathbf{1} \neq 0$, so that the Sherman-Morrison formulæ hold. □

**Remark.** The knowledge of the support of the parametrization with $\lambda$ and $\Gamma$ is crucial in the choice of a coherent apriori distribution.

Let us discuss first the case $n = 2$. We have

$$\begin{bmatrix} \sigma & \sigma_{1,2} \\ \sigma_{1,2} & \sigma \end{bmatrix} = \begin{bmatrix} \lambda & \lambda - \gamma \\ \lambda - \gamma & \lambda \end{bmatrix}, \quad \gamma = \sigma > 0,$$

and we need the sign of

$$\det \begin{bmatrix} \lambda & \lambda - \gamma \\ \lambda - \gamma & \lambda \end{bmatrix} = \lambda^2 - (\lambda - \gamma)^2 = \gamma(2\lambda - \gamma),$$

which is positive if $\lambda \geq \gamma/2$. This shows existence and shows that there is a restriction on $\lambda$ which is worthwhile to investigate further. The condition in (9) involves the lower bound $\max_\alpha 2\alpha(1 - \alpha)\gamma = \gamma/2$. If the lower bound is reached with $\lambda = \gamma/2$, hence $\det \Gamma = 0$.

Assume now $n = 3$, that is

$$\begin{bmatrix} \sigma & \sigma_{1,2} & \sigma_{1,3} \\ \sigma_{1,2} & \sigma & \sigma_{2,3} \\ \sigma_{1,3} & \sigma_{2,3} & \sigma \end{bmatrix} = \begin{bmatrix} \lambda & \lambda - \gamma_{12} & \lambda - \gamma_{13} \\ \lambda - \gamma_{12} & \lambda & \lambda - \gamma_{23} \\ \lambda - \gamma_{13} & \lambda - \gamma_{23} & \lambda \end{bmatrix},$$
with \( \Gamma \) conditionally negative definite. We have to assume \( \lambda > \gamma_{12}/2 \) and moreover we need the sign of

\[
\begin{vmatrix}
\lambda & \lambda - \gamma_{12} & \lambda - \gamma_{13} \\
\lambda - \gamma_{12} & \lambda & \lambda - \gamma_{23} \\
\lambda - \gamma_{13} & \lambda - \gamma_{23} & \lambda,
\end{vmatrix}
\]

\[-2\gamma_{12}\gamma_{13}\gamma_{23} + \lambda \left( -\gamma_{12}^2 + 2\gamma_{12}\gamma_{13} - \gamma_{13}^2 + 2\gamma_{12}\gamma_{23} + 2\gamma_{13}\gamma_{23} - \gamma_{23}^2 \right) \geq 0.
\]

The solution of such algebraic inequalities is difficult in general, but we see that the admissible values of \( \lambda \) form a semi-infinite interval. In this and other similar cases, we can use a symbolic software such as Sage \([13]\) to help with the algebra.

We now change our point of view to consider the same problem from a different angle. We can associate the variogram with the state space description of the Gaussian vector. This is of use, for example, when a simulation is required. The following proposition is similar to Prop. \([4]\).

**Proposition 7.**

1. The matrix \( \Gamma \) is a variogram matrix if, and only if, the matrix

\[
\Sigma_0 = -\left( I - \frac{1}{n} 11' \right)' \Gamma \left( I - \frac{1}{n} 11' \right)
\]

is symmetric and positive definite. In such a case, the variogram of \( \Sigma_0 \) is \( \Gamma \).

2. If \( Y_0 \sim N_n(0, \Sigma_0) \), then its variogram is \( \Gamma \) and it is supported by \( \text{Span} \left( 1 \right)^\perp \).

**Proof.**

1. If \( \Gamma \) is a variogram matrix, then the matrix \( \Sigma_0 \) of Eq. \((12)\) is symmetric and positive definite. In fact, for a generic vector \( \alpha \) the vector \( (I - \frac{1}{n} 11')\alpha \) is orthogonal to \( 1 \), hence

\[
\alpha' \Sigma_0 \alpha = -\left( \left( I - \frac{1}{n} 11' \right) \alpha \right)' \Gamma \left( \left( I - \frac{1}{n} 11' \right) \alpha \right) \geq 0.
\]

Viceversa, assume \( \Sigma_0 \) is a covariance matrix. As \( e_i - e_j \in \text{Span} \left( 1 \right)^\perp \), the variogram of \( \Sigma_0 \) has elements

\[
(e_i - e_j)' \Sigma_0 (e_i - e_j) = -\left( I - \frac{1}{n} 11' \right)' ( -\Gamma ) \left( I - \frac{1}{n} 11' \right) (e_i - e_j) = -\gamma_{ii} + 2\gamma_{ij} = 2\gamma_{ij}.
\]

2. As \( 1' (e_i - e_j) = 0 \), then \( 1' \left( I - \frac{1}{n} 11' \right)' ( -\Gamma ) \left( I - \frac{1}{n} 11' \right) 1 = 0 \), hence the distribution of \( Y_0 \) is supported by the space \( \text{Span} \left( 1 \right)^\perp \).

\[\square\]

**Remark.** Let us derive some other equivalent expression for \( \Sigma_0 \). The \( h \)-th element of \( \text{diag} \left( \Sigma_0 \right) \) is

\[
e_h' \Sigma_0 e_h = -e_h' \left( I - \frac{1}{n} 11' \right)' \Gamma \left( I - \frac{1}{n} 11' \right) e_h = -(e_h - \frac{1}{n} 1)' \Gamma (e_h - \frac{1}{n} 1)
\]

hence

\[
\text{diag} \left( \Sigma_0 \right) = -\sum_h e_h e_h' \left( I - \frac{1}{n} 11' \right)' \Gamma \left( I - \frac{1}{n} 11' \right) e_h e_h'
\]
and also
\[ \text{diag}(\Sigma_0)\, 11' = -\sum_h e_h e_h' \left( I - \frac{1}{n} 11' \right)' \Gamma \left( I - \frac{1}{n} 11' \right) e_h' \]
\[ 11' \text{ diag}(\Sigma_0) = -\sum_h 1e_h' \left( I - \frac{1}{n} 11' \right)' \Gamma \left( I - \frac{1}{n} 11' \right) e_h e_h' . \]

Let us compute the \((i, j)\) element.
\[ e_i' \text{ diag}(\Sigma_0)\, 11' e_j = -e_i' \left( I - \frac{1}{n} 11' \right)' \Gamma \left( I - \frac{1}{n} 11' \right) e_j \]
\[ e_i' 11' \text{ diag}(\Sigma_0)\, e_j = -e_j' \left( I - \frac{1}{n} 11' \right)' \Gamma \left( I - \frac{1}{n} 11' \right) e_j . \]

The previous computation are of use in the analysis of the variogram of \(\Sigma_0\), because in this case
\[ \gamma = \frac{1}{2} \left( \text{diag}(\Sigma_0)\, 11' + 11' \text{ diag}(\Sigma_0) \right) + \left( I - \frac{1}{n} 11' \right)' \Gamma \left( I - \frac{1}{n} 11' \right) . \]

In the following Proposition we derive an additive decomposition of a generic Gaussian vector into a term whose variance is that obtained in Prop. 7 and a Gaussian vector proportional to the unit vector \(\mathbf{1}\).

**Proposition 8.** Let \(Y \sim N_n(\mu, \Sigma)\) with variogram \(\Gamma\). Let \(Y_0 = (I - \frac{1}{n} 11') Y\) be the projection of \(Y\) onto \(\text{Span}(\mathbf{1})^\perp\) so that we can write \(Y = Y_0 + \overline{Y}\), where each component of \(\overline{Y}\) is the empirical mean \(\frac{1}{n} Y\).

(1) The distribution of \(Y_0\) is \(N_n(\mu - \frac{1}{n} 11' \mu, \Sigma_0)\), with \(\Sigma_0 = - \left( I - \frac{1}{n} 11' \right) \Gamma \left( I - \frac{1}{n} 11' \right)\), that is it depends on the mean and the variogram only.

(2) The distribution of \(\frac{1}{n} 11' Y\), conditionally to \(Y_0\), is Gaussian with mean
\[ \frac{1}{n} 11' \mu + l' \left( I - \frac{1}{n} 11' \right) (Y - \mu) \]
and variance
\[ \frac{\sum_i \lambda_i}{n} - \frac{\sum_{i,j} \gamma_{ij}}{n^2} + l' \left( I - \frac{1}{n} 11' \right) \Gamma \left( I - \frac{1}{n} 11' \right) l , \]
where \(l\) is a vector such that
\[ \frac{1}{n} 11' \mu + l' \left( I - \frac{1}{n} 11' \right) (Y - \mu) = \mathbb{E} \left( \frac{1}{n} 11' Y \bigg| Y_0 \right) . \]

**Proof.**
(1) The variance of \(Y_0\) is
\[ \left( I - \frac{1}{n} 11' \right) \Sigma \left( I - \frac{1}{n} 11' \right) = \left( I - \frac{1}{n} 11' \right) \left( \frac{1}{2}(\Lambda 11' + 11' \Lambda) - \Gamma \right) \left( I - \frac{1}{n} 11' \right) = - \left( I - \frac{1}{n} 11' \right) \Gamma \left( I - \frac{1}{n} 11' \right) . \]
(2) We have $\mathbb{E} \left( \frac{1}{n} Y | Y_0 \right) = \frac{1}{n} l' \mu + l' (Y_0 - \mathbb{E}(Y_0))$, if the vector $l \in \mathbb{R}^n$ is such that $\text{Cov} \left( \frac{1}{n} Y, Y_0 \right) = l' \text{Var}(Y_0)$, that is

$$
\frac{1}{n} l' \Sigma \left( I - \frac{1}{n} 11' \right) = l' \left( I - \frac{1}{n} 11' \right) \Sigma \left( I - \frac{1}{n} 11' \right),
$$

or, in terms of $\Lambda = \text{diag} \Sigma$ and the variogram $\Gamma$,

$$
\frac{1}{2} l' \Lambda \left( I - \frac{1}{n} 11' \right) = \frac{1}{n} l' \Gamma \left( I - \frac{1}{n} 11' \right) - l' \left( I - \frac{1}{n} 11' \right) \Gamma \left( I - \frac{1}{n} 11' \right),
$$

The variance of $\frac{1}{n} Y$ is

$$
\frac{1}{n^2} l' \Sigma l = \frac{1}{n^2} (n1' \Lambda 1 - 1' \Gamma 1) = \frac{\sum_j \lambda_j}{n} - \frac{\sum_{i,j} \gamma_{i,j}}{n^2},
$$

and the variance of $l' Y_0$ is

$$
l' \Sigma_0 l = -l' \left( I - \frac{1}{n} 11' \right) \Gamma \left( I - \frac{1}{n} 11' \right) l.
$$

The conclusion follows from the conditioning formula for Gaussian vectors.

\[\square\]

Remark. The previous proposition suggest an algorithm for the simulation when the variogram is given, by generating first the deviations from the general mean by using the covariance $\Sigma_0$, then from the conditional distribution of the general mean, given the deviations. It should be noted that in case of a stationary variance $\Sigma = \text{diag} \Sigma$, the distance between the locations, while the covariance vanishes. In the stationary case, the variance of the difference between values measured in two locations is increasing with the distance between the locations, while the covariance vanishes. In the stationary case, these assumptions are still valid; therefore, we can use the results of the previous section, together with a further characterisation of variograms, which is based on the following theorem.

3. Stationarity

Let $G$ be an additive topological locally compact group e.g., $\mathbb{Z}$ or $\mathbb{R}$ with the ordinary sum $x + y$. A centered Gaussian random process $(Y(x))_{x \in G}$ is stationary if $\text{Cov} \left( Y(x), Y(y) \right) = \text{Cov} \left( Y(x - y), Y(0) \right) = C(x - y)$. The autocovariance function $C$ is positive definite, that is $\sum_{i,j=0}^n \alpha_i \alpha_j C(x_i - x_j) \geq 0$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in G$, $\alpha \in \mathbb{R}^n$. The process is intrinsically stationary if $\text{Var} \left( Y(x) - Y(y) \right) = \text{Var} \left( Y(x - y) - Y(0) \right) = 2\gamma(x - y)$. The variogram function $\gamma$ is conditionally negative definite, i.e. the matrix $\Gamma = [\gamma(x_i - x_j)]_{i,j=1}^n$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in G$, is conditionally negative definite, as in Prop. 3

We plan to discuss, in a paper currently in progress, the existence of an intrinsically stationary process $Y$ given a conditionally negative definite function and we want to characterise specific classes of variogram functions e.g., those which are increasing (if an order is available) and bounded as $x \to \infty$. Increasing and bounded variograms are considered especially adapted to Geostatistics. In fact, D.G. Krige himself assumed that the variance of the difference between values measured in two locations is increasing with the distance between the locations, while the covariance vanishes. In the stationary case, these assumptions are still valid; therefore, we can use the results of the previous section, together with a further characterisation of variograms, which is based on the following theorem.
Proposition 9 ([11 Th. 6.1.8]). Let $\gamma : G$ and $f(0) \geq 0$. Then $\gamma$ is conditionally negative definite if, and only if, for all finite sequence $x_1, \ldots, x_n$, the matrix $A = [\gamma(x_i - x_j) - \gamma(x_i) - \gamma(-x_j)]_{i,j=1}^n$ is negative definite.

**Proof.** If the matrix $A$ is negative definite and $\sum_i \alpha_i = 0$, then

$$0 \geq \sum_{i,j=1}^n \alpha_i \alpha_j (\gamma(x_i - x_j) - \gamma(x_i) - \gamma(-x_j)) = \sum_{i,j=1}^n \alpha_i \alpha_j \gamma(x_i - x_j)$$

Viceversa, from generic $x_1, \ldots, x_n$, $\alpha_1, \ldots, \alpha_n$, define $x_{n+1} = 0$ and $\alpha_{n+1} = -\sum_i \alpha_i$, then write the condition for conditional negativity.

Finally, in this setting one must take advantage of the harmonic representation of positive definite functions.

4. **Conclusions and future developments**

When dealing with Kriging meta-models, it is mandatory to provide a description of how the responses are correlated, since the goodness of the Kriging predictions in untried points strongly depends on the Gaussian random field. The correlation quantifies the smoothness of the response function and there are two approaches in literature. The first one is the use of the Spatial Correlation Function, SCF, (typical of the Design and Analysis of Computer Experiments); the second one, proposed by Matheron, is based on the use of the variogram. In this paper the equivalence between variogram and SPC is proved for stationary and intrinsically stationary processes. The use of the variogram is favourite because it does not require a parametric approach as the correlation estimation does.

Further developments to be published later and to be presented in forthcoming conferences, concern the use of the variogram for detecting technological signature in manufactured parts and a benchmark of different approaches (parametric and not parametric approach of the variogram and the Artificial Neuronal Networks) in the capability evaluation of the turbine features in order to maximise the performances (minimisation of the fuel consumption).
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