On the Shannon Cipher System With a Wiretapper Guessing Subject to Distortion and Reliability Requirements*

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Abstract - In this paper we discuss the processes in the Shannon cipher system with discrete memoryless source and a guessing wiretapper. The wiretapper observes a cryptogram of \( N \)-vector of ciphered messages in the public channel and tries to guess successively the vector of messages within given distortion level \( \Delta \) and small probability of error less than \( \exp\{-NE\} \) with positive reliability index \( E \). The security of the system is measured by the expected number of guesses which wiretapper needs for the approximate reconstruction of the vector of source messages. The distortion, the reliability criteria and the possibility of upper limiting the number of guesses extend the approach studied by Merhav and Arikan. A single-letter characterization is given for the region of pairs \((R_L, R)\) (of the rate \( R_L \) of the maximum number of guesses \( L(N) \) and the rate \( R \) of the average number of guesses) in dependence on key rate \( R_K \), distortion level \( \Delta \) and reliability \( E \).

Index Terms — Cryptanalysis, guessing, wiretapper, source coding with fidelity criterion, rate-distortion theory, rate-reliability-distortion dependence, Shannon cipher system.

I. INTRODUCTION

We investigate the procedure of wiretapper’s guessing with respect to fidelity and reliability criteria in the Shannon cipher system (see Fig. 1) [29].

![Fig. 1. The Shannon cipher system with a guessing wiretapper.](chart.png)

Encrypted vector of messages of a discrete memoryless stationary source must be transmitted via a public channel to a legitimate receiver. The key-vector is communicated to encrypter

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and to decrypter by special secure channel protected against wiretappers. After ciphering the vector of source messages by a key-vector, the cryptogram is sent over a public channel to a legitimate receiver, which can recover the original message on the base of the cryptogram and the same key-vector. A wiretapper that eavesdrops a public channel aims to decrypt the source messages on the base of cryptogram, within the framework of given distortion and reliability, knowing the source statistics and the encryption function but not the key. The wiretapper makes sequential guesses (suppositions), each time applying a testing mechanism by which he can learn whether the estimate is successful (is within a given distortion level). He stops if the answer is affirmative, or the number of guesses attains the prescribed limit. The restriction of the number of guesses is justified because it often happens that when some time passes the task of guessing loses its actuality or even the sense.

The guessing problem was first considered by Massey [25], then by Arikan [3] and recently by Malone and Sullivan [22]. The guessing subject to fidelity criterion was studied by Arikan and Merhav in [4], [5], for reliability criterion by Haroutunian and Ghazaryan in [10], for the Shannon cipher system with exact reconstruction of messages by wiretapper by Merhav and Arikan in [26] and by Hayashi and Yamamoto in [29]. The Shannon cipher system with wiretapper reconstructing source messages subject to fidelity criterion was examined by Yamamoto in [31]. We study a combination of these problems with additional reliability criterion and restriction of the number of guesses by a limit $L(N)$ (less or equal to the number of all messages in $\mathcal{X}^N$). The Shannon’s rate-distortion concept generalization, introduced by Haroutunian and Mekoush [15], consists in studying the rate-reliability-distortion dependence. We use the term reliability instead of the longer term error probability exponent. Applications of the reliability criterion were investigated for various multiterminal systems (see [10], [13] – [18], [23], [30]).

The security of the cipher system we measure by the expected number of guesses needed for reconstruction of the source messages. That approach was used also by Merhav and Arikan in [26] and earlier by Hellman in [21] and by Sgarro in [27], [28]. But we characterize the activity of the system also by the rate of the maximum number of wiretapper guesses, the distortion level of the approximate reconstruction of messages and the value of the reliability (exponent) $E$ in the upper estimate $\exp\{-NE\}$ of the probability of error of the wiretapper.

The objective of this paper is investigation of the optimal correlations of noted characteristics of the described model. Abstracts of results of the paper were published in [11], [12].

II. Definitions

We pass to detailed definitions. The discrete memoryless source is defined as a sequence $\{X_i\}_{i=1}^\infty$ of discrete, independent, identically distributed (i.i.d.) random variables (RVs) $X$ taking values in the finite set $\mathcal{X}$ of messages $x$ of the source. Let

$$P^* = \{P^*(x), x \in \mathcal{X}\}$$

be the source messages generating probability distribution (PD) which is supposed to be known also to the wiretapper. Let $\mathbf{X} = (X_1, X_2, \ldots, X_N)$ be a random $N$-vector. Since we study the memoryless source the probability of the vector $\mathbf{x} = (x_1, \ldots, x_N)$, a realization of the random $N$-vector $\mathbf{X}$, is

$$P^N(\mathbf{x}) = \prod_{n=1}^N P^N(x_n).$$

The key-source $\{U\}$ is given by a sequence $\{U_i\}_{i=1}^\infty$ of binary i.i.d. RVs, which take values from the set $\mathcal{U} = \{0, 1\}$. The distribution $P^* = \{1/2, 1/2\}$ is the PD of the key bits. The key-vector $\mathbf{u} = (u_1, u_2, \ldots, u_K)$ is a vector of $K$ bits and $P^K(\mathbf{u}) = 2^{-K}$. Let $\mathbf{U} = (U_1, U_2, \ldots, U_K)$ be a key-vector of $K$ binary RVs independent of the vector $\mathbf{X}$.
Denote by \( \hat{x} \) values of RV \( \hat{X} \) representing the wiretapper reconstruction of the source message with values in the finite wiretapper’s reproduction alphabet \( \hat{X} \), in general different from \( X \).

Correspondingly, by \( X^N \) and \( \hat{X}^N \) we denote the \( N \)-th order Cartesian powers of the sets \( X \) and \( \hat{X} \), by \( U^K \) – the \( K \)-th order Cartesian power of the set \( U \).

We consider a single-letter distortion measure between source and wiretapper reproduction messages:

\[
d : X \times \hat{X} \to [0; \infty).
\]

It is supposed that for every \( x \in X \) there exists at least one \( \hat{x} \in \hat{X} \) such that \( d(x, \hat{x}) = 0 \). The distortion measure between a source vector \( x \in X^N \) and a wiretapper reproduction vector \( \hat{x} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_N) \in \hat{X}^N \) is defined as an average of the corresponding component distortions:

\[
d(x, \hat{x}) = N^{-1} \sum_{n=1}^{N} d(x_n, \hat{x}_n).
\]  

(1)

Let

\[
f_N : X^N \times U^K \to W(N, K)
\]

be an encryption function with the set \( W(N, K) \) of all possible for this \( N \) and \( K \) cryptograms \( w \). This function is assumed to be invertible providing the key is given, i.e. there exists the decryption function

\[
f_N^{-1} : W(N, K) \times U^K \to X^N.
\]

We denote by \( W(N, K) \) the RV with values \( w \). For each cryptogram \( w = f_N(x, u) \) the ordered list of sequential guesses of the wiretapper

\[
G_N(w) \overset{\Delta}{=} \{ \hat{x}_1(w), \hat{x}_2(w), ..., \hat{x}_{L(N)}(w) \}, \hat{x}_l(w) \in \hat{X}^N, l = 1, 2, ..., L(N),
\]

with the limit of the number of guesses \( L(N) \leq |X|^N \), is called the guessing strategy of the wiretapper. For a given guessing strategy \( G_N(w), w \in W(N, K) \), we name guessing function and denote by \( G_N(x, w) \) the function

\[
G_N : X^N \times W(N, K) \to \{1, 2, 3, ..., L(N), L(N) + 1\},
\]

which shows index \( l \) of the first successful guessing vector \( \hat{x}_l(w) \in G_N(w) \), i.e. such minimal \( l \) that \( d(x, \hat{x}_l(w)) \leq \Delta \). In other words \( l \) is the quantity of sequential guesses of the wiretapper until the successful estimate \( \hat{x}_l(w) \) of the source vector \( x \in X^N \) is found. \( G_N(x, w) \) equals \( L(N) + 1 \) if the guessing is stopped after \( L(N) \) unsuccessful attempts.

For each distortion level \( \Delta \geq 0 \), a positive number \( L(N) \) and a guessing strategy \( G_N(w) \) let us consider two sets of vectors \( x \) of messages:

- the first is the set of those \( x \) which can be successfully deciphered by the wiretapper within \( L(N) \) guessing attempts for every key \( u \)

\[
A(w) \overset{\Delta}{=} A(L(N), G_N(w), \Delta) \overset{\Delta}{=} \{ x : \forall u, \exists l \leq L(N), f_N(x, u) = w, d(x, \hat{x}_l(w)) \leq \Delta \}
\]

\[
= \{ x : G_N(x, w) \leq L(N) \},
\]

and the other with those \( x \), which can not be deciphered by the wiretapper with necessary precision after \( L(N) \) guesses

\[
\overline{A(w)} \overset{\Delta}{=} \{ x : \exists u, \forall l \leq L(N), f_N(x, u) = w, d(x, \hat{x}_l(w)) > \Delta \}
\]
Let us denote by $R > 0$ and for all $w$ of the error probability with given reliability (exponent) $E$. With $E \to 0$ we can obtain also results corresponding to the case of error probability upper limited by given small $\varepsilon > 0$ not decreasing exponentially by $N$.

In this paper log-s and exp-s are taken to the base 2.

Let $R_K$ be the key rate:

$$R_K = N^{-1} \log 2^K = K/N.$$  

It is supposed that $L(N)$ also increases exponentially by $N$. The guessing rates pair $R_L, R$ will be called (from the point of view of cryptanalysis, i.e. the wiretapper) $(R_K, E, \Delta)$-achievable for given $E > 0$, $\Delta \geq 0$ and $R_K$, if for every encryption function $f_N$ there exists a sequence of guessing strategies $G_N(w)$ such that

$$\lim \inf_{N \to \infty} N^{-1} \log L(N) = R_L;$$

$$\lim \inf_{N \to \infty} N^{-1} \log E_{P^*, P_1^*} \{G_N(X, W)\} = R,$$

and for all $w \in W(N, K)$

$$e(L(N), G_N(w), \Delta) \leq \exp\{-NE\}.$$  

Let us denote by $R_G(P^*, R_K, E, \Delta)$ the set of all $(R_K, E, \Delta)$-achievable (for wiretapper) pairs of guessing rates $R_L, R$ and call it the guessing rates-keyrate-reliability-distortion region. The boundary of the region $R_G(P^*, R_K, E, \Delta)$ we will designate by $R_G(P^*, R_K, E, \Delta)$. It contains information on interdependence of extremal values of rates $R$ and $R_L$, so it will be convenient to conditionally name it guessing rate-keyrate-reliability-distortion function.

The knowledge of such functional dependence is practically useful because it gives possibility to ameliorate the security of the cipher system by increasing of the key rate $R_K$, or by decreasing of the number of allowed guesses $L(N)$.

In case $E \to \infty$, $X \equiv \hat{X}$, $\Delta = 0$, and $R_L = \log |X|$ guessing rate-keyrate-reliability-distortion function becomes the guessing rate-keyrate function $R_G(P^*, R_K)$ studied by Merhav and Arikan in [26]. A problem studied by Yamamoto in the framework of the rate-distortion theory for Shannon cipher system [31] corresponds to the case $L(N) = 1$ with measuring of the security of the system by the attainable minimum distortion.

Let $P = \{P(x), x \in X\}$ be a PD on $X$ and $Q = \{Q(\hat{x} \mid x), x \in X, \hat{x} \in \hat{X}\}$ be a conditional PD on $\hat{X}$ for given $x$, also we denote by $PQ$ the marginal PD on $\hat{X}$:

$$PQ \triangleq \{PQ(\hat{x}) = \sum_x P(x)Q(\hat{x} \mid x), \hat{x} \in \hat{X}\}.$$  

For given $x \in X$ denote by $Q_P(\hat{x} \mid x)$ the conditional PD on $\hat{X}$ such that for each $\Delta$ the following condition is fulfilled: $E_{P, Q_P} d(X, \hat{X}) \triangleq \sum_x P(x)Q_P(\hat{x} \mid x)d(x, \hat{x}) \leq \Delta$.

Let $M(P, \Delta)$ be the set of all PDs $Q_P$ for given $\Delta$ and $P$.

We use the following notations for entropy, information and divergence:

$$H_P(X) \triangleq -\sum_x P(x) \log P(x),$$
\[ I_{P,Q}(X \land \hat{X}) \triangleq \sum_{x,\hat{x}} P(x)Q(\hat{x} \mid x) \log \frac{Q(\hat{x} \mid x)}{\sum_{\hat{x}} P(x)Q(\hat{x} \mid x)}, \]

\[ D(P \mid\mid P^*) \triangleq \sum_x P(x) \log \frac{P(x)}{P^*(x)}. \]

For given \( E > 0 \) consider the following set of PDs \( P \) “surrounding” the generating PD \( P^* \):

\[ \alpha (P^*, E) \triangleq \{ P : D(P \mid\mid P^*) \leq E \}. \] (5)

We denote by \( R(P, \Delta) \) the rate-distortion function for PD \( P \) (see [6], [8]):

\[ R(P, \Delta) \triangleq \min_{Q_P \in M(P, \Delta)} I_{P,Q_P}(X \land \hat{X}), \] (6)

and by \( R(P^*, E, \Delta) \) the rate-reliability-distortion function (introduced in [15]): for source with generating PD of messages \( P^* \)

\[ R(P^*, E, \Delta) \triangleq \max_{P \in \alpha (P^*, E)} R(P, \Delta). \] (7)

The first emergence of \( R(P^*, E, \Delta) \) may be explained by Theorem 2 below. But we apply it to solving of the problem under consideration.

In the next Section we formulate a theorem specifying the guessing rates-keyrate-reliability-distortion region \( R_G(P^*, R_K, E, \Delta) \). The proofs are exposed in Section IV.

III. Formulation of the Result

The main result of the paper is the complete characterization of the guessing rates-keyrate-reliability-distortion region \( R_G(P^*, R_K, E, \Delta) \). We introduce the following region:

\[ \bar{R}_G(P^*, R_K, E, \Delta) \triangleq \{(R_L, R) : \log |\mathcal{X}| \geq R_L \geq \min(R_K, R(P^*, E, \Delta)), \]

\[ R_L \geq R \geq \max_{P \in \alpha (P^*, E)} \{ \min(R_K, R(P, \Delta)) - D(P \mid\mid P^*) \} \}. \] (8)

![Figure 2](image_url)

**Fig. 2.** Schematic diagram of region \( \bar{R}_G(P^*, R_K, E, \Delta) \).

**Theorem 1:** For given PD \( P^* \) on \( \mathcal{X} \), every key rate \( R_K \geq 0 \), reliability \( E > 0 \), and permissible distortion level \( \Delta \geq 0 \),

\[ R_G(P^*, R_K, E, \Delta) = \bar{R}_G(P^*, R_K, E, \Delta). \] (10)
Theorem 1 comprises the following important particular cases. Denote by \( \bar{R}_G(P^*, R_K, E, \Delta) \) the boundary of the region \( \bar{R}_G(P^*, R_K, E, \Delta) \).

**Corollary 1:** When \( E \to \infty \), and the strategy permits the total exhaustion of the wiretapper reproduction vectors set \( (R_L = \log |\mathcal{X}|) \) we get a solution of the problem suggested by Merhav and Arikan [26], concerning the reconstruction of the \( N \)-vector of messages by wiretapper within an allowed level \( \Delta \) of distortion from the true vector

\[
\lim_{E \to \infty, R_L = \log |\mathcal{X}|} R_G(P^*, R_K, E, \Delta) = \lim_{E \to \infty, R_L = \log |\mathcal{X}|} \bar{R}_G(P^*, R_K, E, \Delta) = \max_P [\min(R_K, R(P, \Delta)) - D(P||P^*)].
\]

**Corollary 2:** When \( E \to \infty \), \( \mathcal{X} \equiv \hat{\mathcal{X}} \), \( \Delta = 0 \), i.e. the wiretapper requires only the exact reconstruction of sequences of source messages, and \( R_L = \log |\mathcal{X}| \), we arrive at the result of Merhav and Arikan from [26]:

\[
\lim_{E \to \infty, \Delta = 0, R_L = \log |\mathcal{X}|} R_G(P^*, R_K, E, \Delta) = \max_P [\min(R_K, H_P(X)) - D(P||P^*)].
\]

**Corollary 3:** When \( E \to 0 \) we find that

\[
\lim_{E \to 0} \bar{R}_G(P^*, R_K, E, \Delta) = \{ (R_L, R) : R_L \geq \min(R_K, R(P^*, \Delta)), R \geq \min(R_K, R(P^*, \Delta)) \}.
\]

This means that when the error probability decays by \( N \) not exponentially the maximal number of guesses may be greater than the average number of guesses only by a factor which does not grow exponentially by \( N \).

Explicit expressions of the guessing-rate-keyrate-reliability-distortion function for particular case of binary source and Hamming distortion measure are presented together with some diagrams in [17].

**IV. Proof of Theorem 1**

The first part of this Section will be appropriated to preliminary necessary known results and tools. We apply the method of types (see [7–9]) in the proof of the theorem so let us begin with the formulation of some basic concepts, notations and relations of this method.

The type \( P \) of a vector \( x \in \mathcal{X}^N \) is a PD \( P = \{ P(x) = N(x|x)/N, x \in \mathcal{X} \} \), where \( N(x|x) \) is the number of repetitions of symbol \( x \) among \( x_1, \ldots, x_N \). The set of all PD-s \( P \) on \( \mathcal{X} \), which are types of vectors from \( \mathcal{X}^N \) for given \( N \), we denote by \( \mathcal{P}(\mathcal{X}, N) \). The set of vectors \( x \) of type \( P \) will be denoted by \( T^N_P(X) \) and also called the type.

Let \( N(x, \hat{x} | x, \hat{x}) \) be the number of repetitions of the pair \((x, \hat{x})\) in the pair of vectors \((x, \hat{x})\). The conditional type of \( \hat{x} \) for given \( x \) from \( T^N_P(X) \) is conditional PD \( Q = \{ Q(\hat{x}|x), x \in \mathcal{X}, \hat{x} \in \hat{\mathcal{X}} \} \) such that \( N(x, \hat{x}|x, \hat{x}) = N(x|x)Q(\hat{x}|x) = NP(x)Q(\hat{x}|x) \) for \( x \in \mathcal{X}, \hat{x} \in \hat{\mathcal{X}} \). The set of all vectors \( \hat{x} \in \hat{\mathcal{X}}^N \) of the conditional type \( Q \) for given \( x \in T^N_P(X) \) is denoted by \( T^N_{P,Q}(\hat{x}|x) \). The set of possible conditional types \( Q \) for all \( x \) of the type \( P \) is denoted by \( Q(\mathcal{X}, P, N) \).

We use the following well known properties of types ([7–9]):

\[
|\mathcal{P}(\mathcal{X}, N)| < (N + 1)^{|\mathcal{X}|}, \quad (11)
\]
and for each PD $P'$ on $X$

$$(N + 1)^{-|X|} \exp \{-ND(P||P')\} < P'^N(T_P^N(X)) \leq \exp \{-ND(P||P')\}. \quad (12)$$

It turns out (as coming discussion shows) that the described guessing problem is substantially interconnected with the problem of source lossy coding subject to distortion and reliability criteria. The latter, according to [15], as well as further works [14], [19], treats the Shannon rate-distortion coding in view of the error probability exponential decay with exponent $E$. This implies a more general optimal relation, rate-reliability-distortion one $R(P^*, E, \Delta)$ between the coding parameters instead of the rate-distortion function $R(P^*, \Delta)$.

![Source lossy coding system](image)

**Fig. 3.** The source lossy coding system.

For more details, let

$$f_c : \mathcal{X}^N \rightarrow \{1, 2, \ldots, C(N)\}$$

be an encoding mapping for source $N$-vectors with $C(N)$ standing for the volume of the code. A backward mapping as a decoder of source messages

$$g_c : \{1, 2, \ldots, C(N)\} \rightarrow \hat{\mathcal{X}}^N$$

is functioning with the encoder in a way to enable the probability of error for $N$ large enough be restricted as follows:

$$e(f_c, g_c, \Delta) \triangleq \sum_{x \in \mathcal{X}^N} P_x^N\{x : d(x, g_c(f_c(x))) > \Delta\} \leq \exp\{-NE\}, \quad (13)$$

where $d(x, g_c(f_c(x)))$ is distortion between transmitted source vector $x$ and its reconstruction $g_c(f_c(x))$. This distortion $d$ we supposed to be identical to defined in (1).

For a predefined pair $\Delta \geq 0$ and $E > 0$ the rate-reliability-distortion function $R(P^*, E, \Delta)$ specifies the minimum achievable code rate $R \geq 0$ as a number to satisfy the inequality

$$N^{-1} \log C(N) \leq R + \varepsilon \quad (\text{where } \varepsilon > 0 \text{ is arbitrarily chosen beforehand})$$

for every code $(f_c, g_c)$, which validates (13) kept $N$ appropriately large.

The analytics for $R(P^*, E, \Delta)$ is given by the following theorem – a result constituting the inverse to the Marton’s exponent function from [24].

**Theorem 2** [15]: For every $E > 0$, $\Delta \geq 0$ and $\varepsilon > 0$, $\delta > 0$ there exists a sequence of such $N$-length block codes $(f_c, g_c)$ for source with alphabet $\mathcal{X}$, generating PD $P^*$, and reproduction alphabet $\hat{\mathcal{X}}$ that whenever $N \geq N_0(|\mathcal{X}|, \varepsilon, \delta)$, then

$$e(f_c, g_c, \Delta) \leq \exp\{-N(E + \delta)\}$$

and

$$N^{-1} \log C(N) \leq R(P^*, E, \Delta) + \varepsilon$$

with $R(P^*, E, \Delta)$ defined in (6), (7).

Conversely, for every sequence of codes satisfying (13) the volume $C(N)$ cannot be too small:

$$\liminf_{N \to \infty} N^{-1} \log C(N) \geq R(P^*, E, \Delta).$$
Theorem 2 is exposed with detailed proof in [18]. The derivation of Theorem 2 can be also observed from a more general result in [14] on robust descriptions system by eliminating all the encoders except one. We only note here that the proof is based on a random coding lemma about covering of types of vectors, which is a modification of the covering lemmas from [1], [2], [8], [10], [14], [16].

The proof of the following Proposition, which we have intention to apply in solution of our guessing problem and which concerns with coding of the vectors \( x \) of a separate type \( P \) can constitute the essential part of the proof of Theorem 2.

**Proposition:** For each given type \( P \in \mathcal{P}(X, N) \), every \( x \in \mathcal{T}^N_P(X) \), \( \Delta \geq 0 \), arbitrary \( \varepsilon > 0 \) and \( N \geq N_0(P, \varepsilon) \) there exists a sequence of such \( N \)-block codes \((f_{c,P}, g_{c,P})\) of a volume \( C(P, N) \), that 

\[
N^{-1} \log C(P, N) \leq R(P, \Delta) + \varepsilon,
\]

where \( R(P, \Delta) \) is defined in (6) and, conversely, for every such code 

\[
\liminf_{N \to \infty} N^{-1} \log C(P, N) \geq R(P, \Delta).
\]

We are ready now to proceed to the proof of Theorem 1. We intend to prove that for every \( R_K > 0, E > 0, \Delta > 0 \) the following inclusions are valid

\[
\mathcal{R}_G(P^*, R_K, E, \Delta) \supseteq \mathcal{\tilde{R}}_G(P^*, R_K, E, \Delta) \supseteq \mathcal{R}_G(P^*, R_K, E, \Delta),
\]

from where (10) follows.

The first inclusion in (13) is the converse kind statement from the viewpoint of the security of the system and the direct statement from the point of view of cryptanalysis. We have to prove that there exists a guessing strategy the parameters \( R_L, R \) of which meet conditions (8) and (9).

Now to prove the first inclusion in (14) consider a guessing strategy that ignores the cryptogram. Represent \( X^N \) as a union of vectors of various types

\[
X^N = \bigcup_{P \in \mathcal{P}(X, N)} \mathcal{T}^N_P(X).
\]

We frequently consider without additional mentioning PDs \( P \) from \( \mathcal{P}(X, N) \), which are types for given \( N \). When \( N \to \infty \) these types converge to the corresponding arbitrary PD-s from \( \mathcal{P}(X) \).

Based on the positive assertion of the Proposition independently of a received \( w \) the wiretapper can consider the collection of all possible decoding vectors as the guessing strategy for \( x \in \mathcal{T}^N_P(X) \)

\[
\mathcal{G}_N(w) = \{\hat{x}_1(w), \hat{x}_2(w), \ldots, \hat{x}_{C(N,P)}(w)\}.
\]

Using the right inequality in (12) and definition (5) of the set \( \alpha(P^*, E) \) we can bound above the probability of appearance of the source sequences of types \( P \) beyond \( \alpha(P^*, E + \delta) \) for some \( \delta > 0 \) and \( N \) large enough as follows:

\[
P^{*N}(\bigcup_{P \notin \alpha(P^*, E + \delta)} \mathcal{T}^N_P(X)) \leq (N + 1)^{|X|} \exp\{-N \min_{P \notin \alpha(P^*, E + \delta)} D(P||P^*)\}
\]
\[
\leq \exp\{-NE - N\delta + |X| \log(N + 1)\} \leq \exp\{-NE\}.
\]
Therefore, to obtain the desired low level of $e(L(N), G_N(w), \Delta)$ it is sufficient that wiretapper constructs the guessing strategy $G_N(w)$ only for vectors of types $P$ from $\alpha(P^*, E + \delta)$.

We now pass to construction of such strategy. It is possible to enumerate types $P$ from $\alpha(P^*, E + \delta)$ as $P_1, P_2, \ldots, P_{[\alpha(P^*, E + \delta)]}$ according to nondecreasing values of corresponding rate-distortion functions $R(P_i, \Delta)$ (for the sake of expressions simplicity we shall write only $i$ instead of $P_i$ in $R(i, \Delta)$, $T_i^N(X)$ and so on):

$$R(1, \Delta) \leq R(2, \Delta) \leq \ldots \leq R([\alpha(P^*, E + \delta)], \Delta).$$

(15)

We designate by $Q_i^\text{in}$ such conditional PD from $\mathcal{M}(i, \Delta)$ that (see (6) and (15))

$$C(i, N) = \exp\{N(\min_{Q_i, \mathcal{M}(i, \Delta)} I_{i,Q_i}(X \wedge \hat{X}) + \varepsilon)\} = \exp\{N(R(i, \Delta) + \varepsilon)\}.$$ 

Let for fixed $i$ the set $\{\hat{x}_{i,m} \in T_{i,Q_i^\text{in}}(X), m = 1, \ldots, C(i, N)\}$ be such a collection of decoding vectors that, according to the Proposition, for $N$ large enough the set

$$\{x : x \in T_{i,Q_i^\text{in}}(X \mid \hat{x}_{i,m}), f_{c,i}(x) = m, m = 1, \ldots, C(i, N)\},$$

be a code for $T_i^N(X)$. Let us consider the following guessing strategy ignoring the cryptogram $w$:

$$G_N^*(w) \triangleq \{\hat{x}_{1,m}, m = 1, \ldots, C(1, N)\}, \ldots, \{\hat{x}_{L(N,P),m}, m = 1, \ldots, C(L(N,P), N)\}.$$

The number of required guesses $G_N^*(x, w)$ for $x \in T_i^N(X)$, $P_i \in \alpha(P^*, E + \delta)$ and for each $w$ is upper bounded for $N$ large enough (see (6) and (15))

$$G_N^*(x, w) \leq C(i, N) \leq \exp\{N(R(i, \Delta) + \varepsilon)\},$$

and due to (7) for every $x$ of type $P$ from $\alpha(P^*, E + \delta)$ independently of $w$ (independently of $u$):

$$G_N^*(x, w) \leq (N + 1)^{|X|} \exp\{N(\max_{P_i \in \alpha(P^*, E + \delta)} R(i, \Delta) + \varepsilon)\} \leq \exp\{N(R(P^*, E + \delta + 2\varepsilon))\}. $$

Sometimes, especially when $\Delta = 0$, or $R_K$ is small, it may be appropriate for the wiretapper to carry out the key-search attack :

$$G_N^{**}(w) \triangleq \{f_{1}^{-1}(w, u_1), f_{2}^{-1}(w, u_2), \ldots, f_{N}^{-1}(w, u_2^K)\},$$

where $u_1, u_2, \ldots, u_2^K$ is an arbitrary numbering of all key-vectors of length $K$. Therefore, for any given cryptogram $w$, the number of required guesses $G_N^{**}(x, w)$ is upper bounded by the number of all key-vectors

$$G_N^{**}(x, w) \leq \exp K = \exp\{NR_K\}.$$ 

This strategy gives to the wiretapper the exact $\hat{x} = x$ with the error probability equal to 0, but it remains to note that for each $x \in T_P^N(X)$ when $R_K \geq R(P, \Delta)$ there is no sense to guess key-vector $u$. That is why in that case the wiretapper may ignore $w$.

When $\exp\{-K\} > \exp\{-NE\}$ (the probability of each possible key is greater than the desirable error probability) the wiretapper has to test all $\exp K$ keys, that is in this case $R_L = R_K$, and $E = \infty$. The average rate $R$ is defined from the equality

$$R = \lim_{N \to \infty} N^{-1} \log[2^{-1}(\exp\{NR_L\} + 1)].$$
Thus, it follows that in the present instance
\[ R = R_L = R_K, \]  
(16)

hence (8), (9) and left inclusion in (14) are in force.

If
\[ \exp\{ -NE \} \geq \exp\{ -K \} = \exp\{ -NR_K \} \]

the wiretapper can examine fewer than \( \exp K \) keys. S/he can guess successively with such rate of maximum number of guesses \( R_L \) that
\[ \exp\{ NR_L \} \exp\{ -NR_K \} \geq 1 - \exp\{ -NE \}. \]

Consequently for any small \( \varepsilon > 0 \) and sufficiently large \( N \)

\[ \exp\{ NR_L \} \geq \exp\{ NR_K \}\{ 1 - \exp\{ -NE \} \} \geq \exp\{ N(R_K - \varepsilon) \}. \]

With the inequality \( R_K \geq R_L \), evident for the key searching, we obtain that in this case again \( R_L = R_K \). But if the wiretapper tests \( \exp NR_K \) keys then the average number of guesses again is equal to \( 2^{-1}(\exp\{ NR_K \} + 1) \). It means that (16) is valid and (14) holds.

Combining these two guessing strategies as \( G_N^*(w) \), when strategy \( G_N^*(w) \), or \( G_N^{**}(w) \) with the least number of guesses is applied, we conclude that for a given cryptogram \( w \) the number of sequential wiretapper guesses for the source vector \( x \in T_i^N(X) \), \( P_i \in \alpha(P^*, E + \delta) \), for \( N \) large enough is upper bounded as follows

\[ G_N^*(x, w) \leq \min\{ \exp K, \exp\{ N(R(i, \Delta) + \varepsilon) \} \} = \exp\{ N \min (R_K, R(i, \Delta) + \varepsilon) \}. \]

Hence, for \( N \) large enough, (see (7)) the required decrease of error probability is attainable by the wiretapper if

\[ L(N) \leq \max_{P \in \alpha(P^*, E + \delta)} \exp\{ N \min (R_K, R(i, \Delta) + \varepsilon) \} \]
\[ = \exp\{ N \min (R_K, R(P^*, E + \delta, \Delta) + \varepsilon) \}. \]

Taking into account the independence of appearing of key-vectors and source message vectors and using (12) and (11), we can derive for \( N \) large enough the upper estimate for the average number of guesses:

\[
E_{P^*, P_1^*}\{ G_N^{**}(X, W) \}
\]

\[
= \sum_{u \in U^K} P_1^{*k}(u) \sum_{i : P_i \in \alpha(P^*, E + \Delta) \cap \mathcal{P}(X, N)} \sum_{x \in T_i^N(X)} P^{*N}(x) G_N^{**}(x, f_N(x, u))
\]

\[
\leq \sum_{u \in U^K} P_1^{*k}(u) \sum_{P \in \alpha(P^*, E + \Delta) \cap \mathcal{P}(X, N)} \sum_{x \in T_i^N(X)} P^{*N}(x) \exp\{ N \min (R_K, R(P, \Delta) + \varepsilon) \}
\]

\[
= \sum_{P \in \alpha(P^*, E + \Delta) \cap \mathcal{P}(X, N)} \exp\{ N \min (R_K, R(P, \Delta) + \varepsilon) \} P^{*N}(T_i^N(X))
\]

\[
\leq \sum_{P \in \alpha(P^*, E + \Delta) \cap \mathcal{P}(X, N)} \exp\{ N(-D(P||P^*) + \min (R_K, R(P, \Delta) + 2\varepsilon)) \}
\]

\[
\leq \max_{P \in \alpha(P^*, E + \Delta)} \exp\{ N(-D(P||P^*) + \min (R_K, R(P, \Delta) + 2\varepsilon)) \}
\]

\[
= \exp\{ N \max_{P \in \alpha(P^*, E + \Delta)} (-D(P||P^*) + \min (R_K, R(P, \Delta) + 2\varepsilon)) \}.
\]
Therefore there exists a guessing strategy the rates of which \( R_L, R \) meet the inequalities
\[
R_L \leq \min \left( R_K, R(P^*, E + \delta, \Delta) + \varepsilon \right), \tag{17}
\]
\[
R \leq \max_{P \in \alpha(P^*, E + \delta)} \left(-D(P || P^*) + \min (R_K, R(P, \Delta) + 2\varepsilon)\right). \tag{18}
\]
The pairs of values in right hand side correspond to the points in region \( \tilde{\mathcal{R}}_G(P^*, R_K, E + \delta, \Delta) \), it means that all points from \( \tilde{\mathcal{R}}_G(P^*, R_K, E + \delta, \Delta) \) will be \( (R_K, E + \delta, \Delta) \)-achievable for wiretapper as well. Since \( \varepsilon \) and \( \delta \) can be made arbitrarily small and all present expressions are continuous in \( E \), we can consider arbitrary PDs \( P \) in (17) and (18) and thus obtain the left inclusion in (14).

Now we will prove the right inclusion in (14)
\[
\tilde{\mathcal{R}}_G(P^*, R_K, E, \Delta) \supseteq \mathcal{R}_G(P^*, R_K, E, \Delta).
\]

To prove this it is necessary to show that rates \( R_L \) and \( R \) of every guessing strategy with keyrate \( R_k \), reliability \( E \), and distortion level \( \Delta \) for arbitrary encryption algorithm must meet the right inequalities, correspondingly, in (8) and (9). This is a converse statement from the point of view of cryptographer.

It is supposed that the wiretapper knows algorithms of ciphering and deciphering. We may assume also that the guesser knows the type \( P \) of the source message \( x \), for such an informed guesser any lower bounds on \( L(N) \) and \( E_{P^*, P_K}(G^*_N(X, W)) \) are lower bounds for uninformed guesser too.

For each type \( P \) the principal is the relation of two numbers: \( NR_K = K < NR(P, \Delta) \), or \( K \leq NR(P, \Delta) \). In the first occasion the key search is preferable for the wiretapper, in the second situation s/he can guess ignoring the cryptogram. In fact the wiretapper uses cryptogram \( w \) only after guessing of key-vector \( u \).

Let us start with the case
\[
R_K < R(P, \Delta). \tag{19}
\]

Denote by \( \tilde{G}_N(w, P) \) a guessing strategy of the wiretapper that for any encryption function guarantees small error probability: \( \epsilon(L(N), \tilde{G}_N(w, P), \Delta) \leq \exp\{\alpha E\} \). Regardless the source probability distribution the optimal guessing strategy under the condition (19) is the key-search attack. The wiretapper can then find the exact \( x \) applying description function \( f^{−1}_N \) on the key vector and \( w \). Of course it is supposed that guessing of the exact \( x \) is also acceptable for the wiretapper. We already know that in this case the minimum values for \( R \) and \( R_L \) meet inequalities (8), (9).

Now let us consider the best strategy when \( P \in \alpha(P^*, E + \delta) \) and
\[
\exp K \geq \exp\{NR(P, \Delta)\}. \tag{20}
\]

We also know that when \( R_k \geq R(P, \Delta) \) the wiretapper can guess each \( x \in T^N_P(X) \) with distortion \( \Delta \) and error probability less than \( \exp\{NR(P, \Delta)\} \) guesses, so key-search as demanding longer work is not preferable. The question is: does another guessing strategy with less than \( \exp\{NR(P, \Delta)\} \) guesses exist? But every guessing strategy \( \{\tilde{x}_1(w), \tilde{x}_2(w), \ldots, \tilde{x}_{L(N, P)}(w)\} \) ignoring \( w \) may be considered as a list for the source encoding satisfying distortion and reliability criteria, so according to the converse statement of the Proposition for \( N \) large enough \( L(N, P) \) cannot be taken less than \( \exp\{NR(P, \Delta)\} \).

Thus the numbers less than \( \exp\{N \min(R_K, R(P, \Delta))\} \) cannot be considered as limit \( L(N, P) \), and for the common guessing strategy inequality (8) is in force.
By averaging we obtain lower estimate for the expected number of guesses:

\[
\begin{align*}
E_{P^*, P^t} \{ G_N (X, W) \} \\
= E_{P^t} \{ E_{P^*} \{ G_N (X, W) \} \} \\
\geq \sum_{u \in U^K} P^{*K} (u) \sum_{P \in \alpha (P^*, E + \delta)} \sum_{x \in T_N^N (X)} P^{*N} (x) G_N (x, w) \\
\geq \sum_{u \in U^K} P^{*K} (u) \sum_{P \in \alpha (P^*, E + \delta)} \sum_{x \in T_N^N (X) \cap A(w)} P^{*N} (x) G_N (x, w) \\
= \sum_{u \in U^K} P^{*K} (u) \sum_{P \in \alpha (P^*, E + \delta)} P^{*N} (A(w)) P^{*N} \left( T_N^N (X) \right) \\
\times \sum_{x \in T_N^N (X) \cap A(w)} \text{Pr} \{ \hat{x}_i (w) | x \in T_N^N (X) \cap A(w) \} \\
\geq \sum_{u \in U^K} P^{*K} (u) \sum_{P \in \alpha (P^*, E + \delta)} (1 - \exp \{-NE\}) \exp \{-ND(P \parallel P^*)\} \\
\times \exp \{ N \min (R_K, \min_{Q_P \in \mathcal{M}(P, \Delta)} I_{P,Q_P} (X \wedge \hat{X}) - \varepsilon) \} \\
\geq \exp \{ N \max_{P \in \alpha (P^*, E + \delta)} \min (R_K, R(P, \Delta) - D(P \parallel P^*) - 2\varepsilon) \}.
\end{align*}
\]

In this calculation \( P \) is type, but with growing of \( N \) it approaches arbitrary PD \( P \). Hence for \( N \) large enough

\[
R_L \geq N^{-1} \log L(N) - \varepsilon \geq \min (R_K - \varepsilon, R(P^*, E + \delta, \Delta) - 2\varepsilon),
\]

\[
R \geq N^{-1} \log E_{P^*, P^t} \{ G_N (X, W) \} - \varepsilon \\
\geq \max_{P \in \alpha (P^*, E + \delta)} \left( \min (R_K, R(P, \Delta)) - D(P \parallel P^*) - 2\varepsilon \right).
\]

Granting arbitrariness of \( \varepsilon \) and \( \Delta \) we obtain (8) and (9).

It rest to remark that comparison of cases (19) and (20) shows that in condition (19) it is not possible to guess with \( \Delta \neq 0 \) and have smaller number of guesses, because approximate guessing will need more than \( \exp \{ NR(P, \Delta) \} \) guesses, i.e. more than \( \exp \{ NR_K \} \), which is enough for the exact reconstruction. Therefore the proof of the right inclusion in (14) is completed.

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