Counting tree-like graphs in locally dense graphs

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Abstract

We prove that a class of graphs obtained by gluing complete multipartite graphs in a tree-like way satisfies a conjecture of Kohayakawa, Nagle, Rödl, and Schacht on random-like counts for fixed graphs in locally dense graphs. We also prove an analogous result with odd cycles replacing complete multipartite graphs. The proof uses a general information-theoretic method to prove graph homomorphism inequalities for tree-like structured graphs, which may be of independent interest.

1 Introduction

A graph homomorphism is a vertex map from a graph $H$ to another graph $G$ that preserves adjacency, and the graph homomorphism density $t_H(G)$ is the probability that a random vertex map from $H$ to $G$ is a graph homomorphism, i.e.,

$$t_H(G) := \frac{\vert \text{Hom}(H, G) \vert}{\vert V(G) \vert^{\vert V(H) \vert}}.$$ 

Many statements in extremal graph theory can be rephrased as inequalities between certain graph homomorphism densities, especially when $H$ is a fixed graph and the target graph $G$ is large, i.e., $\vert V(G) \vert \to \infty$. For example, we say that a graph sequence $G_n$ with $\vert V(G_n) \vert \to \infty$ is quasirandom if and only if

$$t_H(G_n) = (1 \pm o(1))t_{K_2}(G_n)^{|E(H)|}, \tag{1}$$

for every fixed graph $H$, that is, the $H$-count is random-like in $G$. A fundamental observation in the theory of quasirandom graphs, due to Thomason [23] and Chung, Graham, and Wilson [2], states that $G_n$ is quasirandom if and only if every subset $X \subseteq V(G)$ spans

$$\frac{1}{2}t_{K_2}(G)|X|^2 \pm o(|V(G_n)|^2) \tag{2}$$

edges. That is, we have a uniform edge density everywhere up to an error dominated by $|V(G_n)|^2$.

We can also ask if some modifications of (1) or (2) imply variations of the other. For example, we say that a graph $G$ is $(\rho, d)$-dense if every vertex subset $X \subseteq V(G)$ of size at least $\rho|V(G)|$ contains $\frac{d}{2}|X|^2$ edges. This is a weaker condition than (2), as we do not have an upper bound for the number of edges spanned by a vertex subset $X$. This means that we cannot recover (1), but it is still plausible that we can recover the lower bound

$$t_H(G_n) \geq (1 - o(1))t_{K_2}(G_n)^{|E(H)|}$$

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when $\rho$ is sufficiently small.

This question was formalised by Kohayakawa, Nagle, Rödl, and Schacht [11]. They conjectured that every $(o(1), d)$-dense graph $G$ gives a random-like count for the minimum number of any fixed graph $H$. More precisely,

**Conjecture 1.1** ([11]). Let $H$ be a graph and let $\eta$ be positive. Then there exists $\rho = \rho(\eta, H) > 0$ such that

$$t_H(G) \geq d^{|E(H)|} - \eta$$

for every sufficiently large $(\rho, d)$-dense graph $G$.

This conjecture is not an arbitrary variant of graph quasirandomness, but has natural applications to Ramsey theory. Indeed, the notion of $(\rho, d)$-dense graphs already appears in a paper of Graham, Rödl, and Ruciński [7], where they use it to bound the Ramsey number of sparse graphs. Roughly speaking, given a 2-edge-colouring of a complete graph, one colour is very dense on some vertex subset or the other colour is somewhat dense on all vertex subsets. In the first case, it is usually simple to embed a graph $H$, while in the second case, the problem reduces to an embedding problem in locally dense graphs. It is therefore of significant interest to understand when we can embed a graph $H$ in a locally dense graph and how many copies we obtain.

Conjecture 1.1 is also closely related to another beautiful conjecture of Sidorenko [19] and Erdős–Simonovits [6].

**Conjecture 1.2** (Sidorenko’s conjecture [6, 19]). Let $H$ be a bipartite graph and let $G$ be a graph. Then

$$t_H(G) \geq t_{K_2}(G)^{|E(H)|}.$$ (4)

We say that a bipartite graph has *Sidorenko’s property* if and only if (4) holds for all graphs $G$. There are a number of graphs known to have Sidorenko’s property [3, 4, 5, 9, 10, 13, 18, 19, 21], but the conjecture is still wide open.

In this paper, we focus on Conjecture 1.1. It is straightforward to see that Conjecture 1.1 is true for $H$ having Sidorenko’s property with $\rho = 1$ and $d = t_{K_2}(G)$. As a partial converse, it is shown in [4] that every $H$ satisfying Conjecture 1.1 can be used to construct a bipartite graph with Sidorenko’s property. For non-bipartite graphs, it is folklore that the complete $\ell$-partite graph $K(r_1, r_2, \ldots, r_\ell)$ on $r = r_1 + \cdots + r_\ell$ vertices satisfies the conjecture,\(^1\) and Reiher [16] settled the case where $H$ is an odd cycle. We give a new class of graphs for which Conjecture 1.1 holds. For instance, we prove that the Goldner–Harary graph shown in Figure 1 satisfies Conjecture 1.1.

To describe our general result, it is convenient to use the notion of tree decompositions, introduced by Halin [8] and developed by Robertson and Seymour [17]. A *tree decomposition* of a graph $H$ is a pair $(\mathcal{F}, \mathcal{T})$ consisting of a family $\mathcal{F}$ of vertex subsets of $H$ and a tree $\mathcal{T}$ on $\mathcal{F}$ such that

1. $\bigcup_{X \in \mathcal{F}} X = V(H)$,
2. for each $e \in E(H)$, there exists a set $X \in \mathcal{F}$ such that $e \subseteq X$, and
3. for $X, Y, Z \in \mathcal{F}$, $X \cap Y \subseteq Z$ whenever $Z$ lies on the path from $X$ to $Y$ in $\mathcal{T}$.

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\(^1\) Though Theorem 1.4 implies this folklore result, we give a simple proof for the case $H = K_n$ in Theorem 3.1.
We say that $\max_{F \in \mathcal{F}} |F| - 1$ is the width of the tree decomposition $(\mathcal{F}, \mathcal{T})$, and the minimum width over all possible tree decompositions of the given graph $H$ is called the tree-width. Let $H$ be a graph with tree width $r$ and let $(\mathcal{F}, \mathcal{T})$ be the tree decomposition that attains the minimum width. Then by adding edges and vertices to each $X \in \mathcal{F}$ if necessary, we may obtain a graph $H'$ with the same tree decomposition $(\mathcal{F}, \mathcal{T})$ such that each $H'[X]$ is isomorphic to the complete graph $K_{r+1}$ on $(r + 1)$ vertices. Such a graph $H'$ is called an $r$-tree, and the Goldner–Harary graph in Figure 1 is a typical example of a 3-tree. Our first main result is the following:

**Theorem 1.3.** Conjecture 1.1 is true if $H$ is an $r$-tree.

Theorem 1.3 is a consequence of a more general result. Given a graph $H$ and an induced subgraph $J$, a $J$-decomposition of a graph $H$ is a tree decomposition $(\mathcal{F}, \mathcal{T})$ of $H$ satisfying the following two extra conditions:

1. each induced subgraphs $H[X], X \in \mathcal{F}$, is isomorphic to $J$, and

2. for every pair $X, Y \in \mathcal{F}$ which are adjacent in $\mathcal{T}$, there is an isomorphism between the two copies $H[X]$ and $H[Y]$ of $J$ that fixes $X \cap Y$.

We call a graph $J$-decomposable if it allows a $J$-decomposition, i.e., it can be obtained by symmetrically gluing copies of $J$ in a tree-like way. If $J$ is a complete graph then the second condition on the symmetry between $H[X]$ and $H[Y], XY \in E(\mathcal{T})$, is automatically satisfied. Hence, a graph is an $r$-tree if and only if it is $K_{r+1}$-decomposable.

In [5], Conlon and the author proved that $J$-decomposable graphs have Sidorenko’s property whenever $J$ satisfies the so-called weakly norming property. We follow an analogous strategy to obtain the following result, which immediately implies Theorem 1.3.

**Theorem 1.4.** Let $r_1, r_2, \ldots, r_\ell$ be non-negative integers. Then Conjecture 1.1 is true if $H$ is a $K(r_1, r_2, \ldots, r_\ell)$-decomposable graph.

We also prove the following result, building upon the theorem of Reiher [16] on odd cycles.

**Theorem 1.5.** Conjecture 1.1 is true if $H$ is a $C_{2k+1}$-decomposable graph.

It is already shown in [5] that Conjecture 1.1 is true for $C_{2k}$-decomposable graphs. Thus, Theorem 1.5 in fact implies that every $C_k$-decomposable graph satisfies Conjecture 1.1.
2 Entropy calculus

The proofs of Theorems 1.4 and 1.5 rely on the entropy analysis applied in [4, 5]. We will use the following facts about entropy, though we refer the reader to [1] and [4] for more detailed information on entropy and conditional entropy. In what follows, logarithms will be understood to be base 2.

**Lemma 2.1.** Let $X$, $Y$, and $Z$ be random variables and suppose that $X$ takes values in a set $S$, $\mathbb{H}(X)$ is the entropy of $X$, and $\mathbb{H}(X | Y)$ is the conditional entropy of $X$ given $Y$. Then

1. $\mathbb{H}(X) \leq \log |S|$,
2. $\mathbb{H}(X | Y, Z) = \mathbb{H}(X | Z)$ if $X$ and $Y$ are conditionally independent given $Z$.

A folklore lemma, essentially implied by the Kolmogorov extension theorem, is necessary to construct the desired random variables. We refer to [22] for a modern introduction to product measure spaces and the Kolmogorov extension theorem.

**Lemma 2.2.** Let $(X_1, X_2)$ and $(X'_2, X_3)$ be random vectors. If $X_2$ and $X'_2$ are identically distributed, then there exists $(Y_1, Y_2, Y_3)$ such that $Y_1$ and $Y_2$ are conditionally independent given $Y_3$, and, for $i = 1, 2, 3$, $X_i$ and $Y_i$ are identically distributed.

Let $\mathcal{F}$ be a family of subsets of $[k] := \{1, 2, \ldots, k\}$. Partly motivated by the notion of tree decompositions, a Markov tree on $[k]$ is a pair $(\mathcal{F}, \mathcal{T})$ with $\mathcal{T}$ a tree on vertex set $\mathcal{F}$ that satisfies

1. $\bigcup_{F \in \mathcal{F}} F = [k]$ and
2. for $A, B, C \in \mathcal{F}$, $A \cap B \subseteq C$ whenever $C$ lies on the path from $A$ to $B$ in $\mathcal{T}$.

Let $V$ be a finite set and let $(X_{i:F})_{i \in \mathcal{F}}$ be a random vector indexed by pairs $(i, F)$ with $i \in F$, taking values on $V^F$. We are interested in such random vectors where ‘local’ information is ‘globally’ extendible. That is, there exist random variables $Y_1, Y_2, \ldots, Y_k$ such that, for each $F \in \mathcal{F}$, the two random vectors $(Y_{i:F})_{i \in \mathcal{F}}$ and $(X_{i:F})_{i \in \mathcal{F}}$ are identically distributed over $V^F$. If such $Y_1, \ldots, Y_k$ exist, then $(X_{i:A})_{i \in A \cap B}$ and $(X_{j:B})_{j \in A \cap B}$ must be identically distributed. Our main theorem states that the converse is also true and, moreover, the maximum entropy under such constraints can always be attained.

**Theorem 2.3.** Let $(\mathcal{F}, \mathcal{T})$ be a Markov tree on $[k]$ and let $(X_{i:F})_{i \in \mathcal{F}}$ be random vectors taking values on a finite set $V^F$ for each $F \in \mathcal{F}$. If $(X_{i:A})_{i \in A \cap B}$ and $(X_{j:B})_{j \in A \cap B}$ are identically distributed whenever $AB \in E(\mathcal{T})$, then there exist $Y_1, \ldots, Y_k$ with entropy

$$\mathbb{H}(Y_1, \ldots, Y_k) = \sum_{F \in \mathcal{F}} \mathbb{H}((X_{i:F})_{i \in F}) - \sum_{AB \in E(\mathcal{T})} \mathbb{H}((X_{i:A})_{i \in A \cap B})$$

(5)

such that $(Y_{i:F})_{i \in \mathcal{F}}$ and $(X_{i:F})_{i \in \mathcal{F}}$ are identically distributed over $V^F$ for all $F \in \mathcal{F}$.

Before getting into the proof, let us discuss how Theorem 2.3 relates to the classical Cauchy–Schwarz inequality. For a simple example, let $V = V(G)$ be the vertex set of a graph $G$, and let $(X, Y)$ and $(Y', Z)$ be two uniform random labelled edges. Since the distribution of $Y$ and $Y'$ are identical, Theorem 2.3 implies that there exists $(X_1, X_2, X_3)$ with entropy

$$\mathbb{H}(X_1, X_2, X_3) = \mathbb{H}(X, Y) + \mathbb{H}(Y', Z) - \mathbb{H}(Y).$$

Using basic facts on entropy (see Lemma 2.1), we have $\mathbb{H}(X, Y) = \mathbb{H}(Y', Z) = \log 2|E(G)|$, $\mathbb{H}(Y) \leq \log |V(G)|$, and $\mathbb{H}(X_1, X_2, X_3) \leq \log |\text{Hom}(K_{1,2}, G)|$, which implies $t_{K_{1,2}}(G) \geq t_{K_2}(G)^2$. This is
also an easy consequence of the Cauchy–Schwarz inequality, and we may recover many graph homomorphism inequalities obtained using the Cauchy–Schwarz inequality by letting \(|\mathcal{F}| = 2\) and \(\mathcal{T}\) the single edge tree. Hence, (5) may be seen as a tree-like extension of the Cauchy–Schwarz inequality. In fact, analogous lemmas to Theorem 2.3 have already been used in [4, 5, 12, 21] to obtain such results.

**Proof of Theorem 2.3.** Fix a leaf \(L\) of \(\mathcal{T}\) and let \(\mathcal{T}'\) be the tree \(\mathcal{T} \setminus L\) on \(\mathcal{F}' := \mathcal{F} \setminus \{L\}\). By rearranging indices, we may assume that \(L = \{t, t+1, \ldots, k\}\) for some \(t \leq k\) and that \(\mathcal{F}'\) consists of subsets of \([\ell]\) for some \(\ell \leq k\). By the inductive hypothesis, there is \(Y_1, Y_2, \ldots, Y_\ell\) such that \((Y_i)_i \in \mathcal{F}\) and \((X_i, F)_{i \in \mathcal{F}}\) are identically distributed for each \(F \in \mathcal{F}'\) and, moreover

\[
\mathbb{H}(Y_1, \ldots, Y_\ell) = \sum_{F \in \mathcal{F}'} \mathbb{H}((X_i, F)_{i \in F}) - \sum_{AB \in E(\mathcal{T}')} \mathbb{H}((X_i, A)_{i \in A \cap B}) (6)
\]

holds. Using Lemma 2.2 with \((X_1, X_2) = (Y_1, Y_2, \ldots, Y_\ell)\) and \((X'_1, X'_3) = (X_{t, L}, X_{t+1, L}, \ldots, X_{k, L})\), there exists \((Z_1, Z_2, \ldots, Z_k)\) such that \((Z_1, Z_2, \ldots, Z_{i-1})\) and \((Z_{i+1}, \ldots, Z_k)\) are conditionally independent given \((Z_i, Z_{i+1}, \ldots, Z_\ell)\). Then \(Z_i\) and \(Y_i\) are identically distributed for \(i = 1, 2, \ldots, \ell\), and \(Z_j\) and \(X_{j, L}\) are identically distributed for all \(j \in L\). By conditional independence, we obtain

\[
\mathbb{H}(Z_1, Z_2, \ldots, Z_k) = \mathbb{H}(Y_1, Y_2, \ldots, Y_\ell) + \mathbb{H}((X_j, L)_{j \in L}) - \mathbb{H}(Y_1, Y_{t+1}, \ldots, Y_\ell).
\]

Using (6) and the fact that \(\{t, t+1, \ldots, \ell\} = L \cap P\), where \(P\) is the neighbour of \(L\) in \(\mathcal{T}\), (5) follows. \(\square\)

To obtain graph homomorphism inequalities, the following corollary of Theorem 2.3, which appeared implicitly in [5], is useful.

**Theorem 2.4.** Let \(G, H\), and \(J\) be graphs. Suppose that \(H\) is \(J\)-decomposable and \(\text{Hom}(J, G)\) is non-empty. Fix a \(J\)-decomposition \((\mathcal{F}, \mathcal{T})\) of \(H\). Then the following inequality holds:

\[
t_H(G) \geq \frac{t_J(G)^{|\mathcal{F}|}}{\prod_{XY \in E(\mathcal{T})} t_H(X \cap Y)(G)}.
\]

**Proof of Theorem 2.4.** By definition, a \(J\)-decomposition \((\mathcal{F}, \mathcal{T})\) is a Markov tree on \(V(H)\). Let \((X_i, F)_{i \in \mathcal{F}}\) be the uniform random homomorphism in \(\text{Hom}(J, G)\). Then both \((X_i, A)_{i \in A \cap B}\) and \((X_j, B)_{j \in A \cap B}\) are supported on the set \(\text{Hom}(H[A \cap B], G)\). Moreover, they are identically distributed, because the distributions are projected from the uniform distribution on \(\text{Hom}(J, G)\) in the same way by the symmetry condition. Thus, by Theorem 2.3, there exists \((Y_v)_{v \in V(H)}\) such that

\[
\mathbb{H}((Y_v)_{v \in V(H)}) = |\mathcal{F}| \log |\text{Hom}(J, G)| - \sum_{AB \in E(\mathcal{T})} \mathbb{H}((X_i, A)_{i \in A \cap B}). (8)
\]

Since each \((Y_u, Y_v)\), \(uv \in E(H)\), is identically distributed with \((X_{u,F}, X_{v,F})\) for \(F \in \mathcal{F}\) containing \(u, v\), \((Y_u, Y_v)\) is supported on \(E(G)\). Thus, \((Y_v)_{v \in V(H)}\) can be seen as a random homomorphism from \(H\) to \(G\). Now (8) gives

\[
\log |\text{Hom}(H, G)| \geq |\mathcal{F}| \log |\text{Hom}(J, G)| - \sum_{AB \in E(\mathcal{T})} \mathbb{H}((X_i, A)_{i \in A \cap B})
\]

\[
\geq |\mathcal{F}| \log |\text{Hom}(J, G)| - \sum_{AB \in E(\mathcal{T})} \log |\text{Hom}(H[A \cap B], G)|.
\]

Rescaling by subtracting \(|V(H)| \log |V(G)|\) on both sides, we obtain the inequality (7). \(\square\)
3 Counting $K(r_1, r_2, \ldots, r_\ell)$-decomposable graphs

As a warm-up, we begin by proving the folklore fact that Conjecture 1.1 is true for the case $H = K_r$. It will also be technically helpful in what follows.

**Theorem 3.1.** Given $\eta > 0$ and positive integer $r$, there exists $\rho = \rho(\eta, r) > 0$ such that

$$t_{K_r}(G) \geq d^{r(r-1)/2} - \eta$$

for every sufficiently large $(\rho, d)$-dense graph $G$.

**Proof.** We use induction on $r$ starting from the trivial base case $r = 2$ when $\rho = 1$ suffices. Suppose that $K_r$ is the complete graph on $[r]$. Let $\rho > 0$ be such that

$$t_{K_r}(J) \geq d^{r(r-1)/2} - \eta/2$$

whenever $J$ is a sufficiently large $(\rho, d)$-dense graph. We may assume that $\rho < \eta/2r$. Let $G$ be a $(\rho^2, d)$-dense graph on $n$ vertices. Denote by $U$ the set of vertices $v$ in $G$ such that $\deg(v) \geq \rho n$. Let $c(v)$ be the number of homomorphisms $\phi$ from $K_{r+1}$ to $G$ such that $\phi(1) = v$. Observe that for any $W \subseteq V(G)$ of size $|W| \geq \rho n$, the induced subgraph $G[W]$ is $(\rho, d)$-dense. Thus,

$$|\text{Hom}(K_{r+1}, G)| = \sum_{v \in V(G)} c(v) \geq \sum_{u \in U} |\text{Hom}(K_r, G[N(u)])| \geq \sum_{u \in U} (d^{r(r-1)/2} - \eta/2)|N(u)|^r \geq \frac{(d^{r(r-1)/2} - \eta/2)}{|U|^{r-1}} \left( \sum_{u \in U} |N(u)| \right)^r,$$

where the last inequality follows from convexity. Since

$$\sum_{u \in U} |N(u)| = 2|E(G)| - \sum_{v \notin U} |N(v)| \geq (d - \rho)n^2,$$

we obtain

$$|\text{Hom}(K_{r+1}, G)| \geq \frac{1}{|U|^{r-1}}(d^{r(r-1)/2} - \eta/2)(d^r - \rho n)2^r \geq (d^{r(r+1)/2} - \eta)n^{r+1}. \quad \Box$$

To prove $K_{r+1}$ satisfies Conjecture 1.1, we have only used the fact that $K_r$ satisfies the conjecture. Hence, our proof also implies that we may add an apex vertex to any graph satisfying Conjecture 1.1 to obtain another:

**Theorem 3.2.** Let $\tilde{H}$ be the graph obtained by adding a vertex to $H$ which is adjacent to all vertices in $H$. If Conjecture 1.1 is true for $H$, then so it is for $\tilde{H}$.

However, the classical approach above does not give a direct comparison between $t_{K_{r+1}}(G)$ and $t_{K_r}(G)$. When applying Theorem 2.4, the main difficulty often lies in bounding the terms $t_{H[X\cap Y]}(G)$ from above in terms of $t_J(G)$. The following lemma gives the control needed to prove Theorem 1.4.

**Lemma 3.3.** Given $\delta > 0$ and positive integers $\ell, r_1, r_2, \ldots, r_\ell$, let $r = \sum_{i=1}^{\ell} r_i$. Then there exists $\rho = \rho(\delta, r_1, r_2, \ldots, r_\ell)$ such that

$$t_{K(r_1, r_2, \ldots, r_\ell)}(G) \geq (d^{r-r_1} - \delta)t_{K(r_1-1, r_2, \ldots, r_\ell)}(G).$$

for every sufficiently large $(\rho, d)$-dense graph $G$.  

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Proof. Suppose \( r_1 \geq 2 \). Then the complete \( \ell \)-partite graph \( K(r_1, r_2, \ldots, r_\ell) \) can be obtained by gluing two copies of \( K(r_1 - 1, r_2, \ldots, r_\ell) \) along their subgraphs induced on each vertex set minus a vertex in the first colour, which is isomorphic to \( K(r_1 - 2, r_2, \ldots, r_\ell) \). Hence, by the Cauchy–Schwarz inequality or Theorem 2.4 with \(|\mathcal{F}| = 2\), we have

\[
t_{K(r_1, r_2, \ldots, r_\ell)}(G) \geq \frac{t_{K(r_1 - 1, r_2, \ldots, r_\ell)}(G)^2}{t_{K(r_1 - 2, r_2, \ldots, r_\ell)}(G)}.
\]

Here we do not worry about the case \( t_{K(r_1 - 2, r_2, \ldots, r_\ell)}(G) = 0 \), because by Theorem 3.1 it is always positive. Repeating this inequality gives the following log-convexity:

\[
t_{K(r_1, r_2, \ldots, r_\ell)}(G) \geq \frac{t_{K(r_1 - 1, r_2, \ldots, r_\ell)}(G)}{t_{K(r_1 - 2, r_2, \ldots, r_\ell)}(G)} \geq \cdots \geq \frac{t_{K(1, r_2, \ldots, r_\ell)}(G)}{t_{K(r_1)}(G)}.
\]

Thus, the goal reduces to the case \( r_1 = 1 \). We claim that, given \( \eta > 0 \), there exists \( \rho > 0 \) such that

\[
t_{K(1, r_2, \ldots, r_\ell)}(G) \geq (d^{r_2} - \eta)t_{K(r_2+1,\ldots,r_\ell)}(G)
\]

whenever \( G \) is \((\rho,d)\)-dense. If the claim is true, then using the claim repeatedly to reduce the number of colour classes and applying (9) to reduce the number of vertices in each class yields

\[
t_{K(r_1, r_2, \ldots, r_\ell)}(G) \geq \frac{(d^{r_2} - \eta)t_{K(r_2+1,\ldots,r_\ell)}(G)}{t_{K(r_1)}(G)} \geq \cdots \geq \frac{(d^{r_2+\cdots+r_\ell} - (\ell - 1)\eta)}{t_{K(r_1)}(G)}.
\]

Since both \( K(r_\ell+1) \) and \( K(r_\ell) \) are just isolated vertices, taking \( \eta = \delta/(\ell - 1) \) is enough to conclude.

Thus, it remains to prove the claim. Let \( H = K(r_2 + 1, r_3, r_4, \ldots, r_\ell) \) and let \( \rho = \eta^{1/(1+r_2)} \).

For a homomorphism \( \phi \) from \( K(r_3, r_4, \ldots, r_\ell) \) to \( G \), define \( C_\phi \) to be the set of vertices \( v \) such that \( \phi(K(r_3, \ldots, r_\ell) \cup \{v\}) \) is again a homomorphic copy of \( K(1, r_3, r_4, \ldots, r_\ell) \) to \( G \). Denote by \( \Phi \) the set of homomorphisms \( \phi \in \text{Hom}(H,G) \) such that \( |C_\phi| \geq \rho m \). Then we have

\[
|\text{Hom}(K(1, r_2, r_3, \ldots, r_\ell), G)| \geq \sum_{\phi \in \Phi} |\text{Hom}(K_{r_2}, G[C_\phi])| \geq d^{r_2} \sum_{\phi \in \Phi} |C_\phi|^{r_2+1},
\]

where the last inequality follows from the fact \( t_{K_{r_2}}(J) \geq t_{K_{r_2}}(J)^r \) for any graph \( J \). The inequality above together with

\[
\sum_{\phi \in \Phi} |C_\phi|^{r_2+1} = |\text{Hom}(H, G)| - \sum_{\phi \in \Phi} |C_\phi|^{r_2+1} \geq |\text{Hom}(H, G)| - \rho^{1+r_2}|V(G)|^{1+r_2+r_3+\cdots+r_\ell}
\]

finishes the proof of the claim.

An immediate corollary is that we may compare \( t_{K(r_1, r_2, \ldots, r_\ell)}(G) \) and \( t_{K(s_1, s_2, \ldots, s_\ell)}(G) \) by repeatedly applying Lemma 3.3 whenever \( r_i \geq s_i \geq 0 \) for \( i = 1, 2, \ldots, \ell \).

**Corollary 3.4.** Suppose \( \delta > 0 \) and \( \ell, r_1, \ldots, r_\ell \), and \( s_1, s_2, \ldots, s_\ell \) are positive integers with \( r_i \geq s_i \), \( i = 1, 2, \ldots, \ell \). Let \( r = |E(K(r_1, r_2, \ldots, r_\ell))| \) and \( s = |E(K(s_1, s_2, \ldots, s_\ell))| \). Then there exists \( \rho \) such that

\[
t_{K(r_1, r_2, \ldots, r_\ell)}(G) \geq (d^{r-s} - \delta)t_{K(s_1, s_2, \ldots, s_\ell)}(G),
\]

whenever \( G \) is a sufficiently large \((\rho,d)\)-dense graph.
Proof of Theorem 1.4. Let $K = K(r_1, r_2, \ldots , r_\ell)$ for brevity and let $(\mathcal{F}, \mathcal{T})$ be a $K$-decomposition of a graph $H$. For each edge $e = XY \in E(\mathcal{T})$, denote by $H(e)$ the complete multipartite subgraph of $H$ induced on $X \cap Y$. By Theorem 3.1 we know that Hom$(H(e), G)$ is non-empty. Hence, Theorem 2.4 gives

$$t_H(G) \geq \frac{t_K(G)^{|\mathcal{F}|}}{\prod_{e \in E(\mathcal{T})} t_{H(e)}(G)}.$$  \hfill (10)

Using the bound $t_K(G)/t_{K(e)}(G) \geq d|E(K)| - |E(H(e))| - \delta$ from Corollary 3.4 and the fact $|E(\mathcal{T})| = |\mathcal{F}| - 1$, we obtain

$$t_H(G) \geq t_K(G) \prod_{e \in E(\mathcal{T})} (d|E(K)| - |E(H(e))| - \delta) \geq (d|E(K)| - \delta)^{\prod_{e \in E(\mathcal{T})} (d|E(K)| - |E(H(e))| - \delta)}.$$  

As $|E(H)| = |\mathcal{F}||E(K)| - \sum_{e \in E(\mathcal{T})} |E(H(e))|$, we have

$$t_H(G) \geq d|E(H)| - \delta |\mathcal{F}|,$$

and hence it is enough to take $\delta = \eta/|\mathcal{F}|$.  \hfill $\Box$

4 Counting $C_{2r+1}$-decomposable graphs

In proving Theorem 1.5, we will again follow the same proof strategy. In order to apply Theorem 2.4, the key will be to prove appropriate graph homomorphism inequalities between odd cycles and the paths they contain. In what follows, denote by $P_\ell$ the path of length $\ell$, that is, with $\ell$ edges. The key lemma in this section is:

**Lemma 4.1.** Given $\delta > 0$ and positive integers $\ell$ and $r$ with $\ell \leq 2r$, there exists $\rho = \rho(\delta, r, \ell)$ such that

$$t_{C_{2r+1}}(G) \geq (d - \delta)t_{P_\ell}(G)^{2r/\ell}$$  \hfill (11)

whenever $G$ is sufficiently large and $(\rho, d)$-dense.

There are two ingredients in the proof. The first is the following lemma by Reiher [16]:

**Lemma 4.2** (Lemma 2.1 in [16]). Let $G$ be a $(\rho, d)$-dense graph on $n$ vertices, and let $f : V(G) \rightarrow [0,1]$ be a function satisfying $\sum_{v \in V(G)} f(v) \geq \rho n$. Then

$$\sum_{uv \in E(G)} f(u)f(v) \geq \frac{d}{2} \left( \sum_{v \in V} f(v) \right)^2 - n.$$

This lemma essentially means that the $(\rho, d)$-denseness condition bootstraps itself to a continuously relaxed version, which will be used to deduce (11) for the case $\ell = 2r$. The second ingredient is the following lemma that enables us to reduce all the other cases with $\ell < 2r$ into $\ell = 2r$.

**Lemma 4.3.** For any graph $G$ and positive integers $\ell < 2r$, the following homomorphism inequality holds:

$$t_{P_\ell}(G) \leq t_{P_{2r}}(G)^{2\ell/2r}.$$  \hfill (12)
Proof. We will repeatedly use the inequality

$$t_{P_{k+1}}(G) \leq t_{P_{2k}}(G)^{1/2}t_{P_{2r}}(G)^{1/2}$$

that follows from the Cauchy–Schwarz inequality. If (12) holds for all paths of even length, then by $t_{P_{2k+1}}(G) \leq t_{P_{2k}}(G)^{1/2}t_{P_{2k+2}}(G)^{1/2}$ we are done. Thus, we may assume that our path is of even length $2\ell$. We claim that the sequence $t_{P_{2k}}(G)$, $k = 1, 2, \ldots, r$, is log-convex. Observe that the Cauchy–Schwarz inequality gives log-convexity for adjacent terms, i.e.,

$$t_{P_{2k}}(G) \leq t_{P_{2(k+1)}}(G)^{1/2}t_{P_{2(k-1)}}(G)^{1/2}.$$

Repeatedly applying above for $1 < k < r$ gives

$$\prod_{k=1}^{r} t_{P_{2k}}(G)^{a_k} \leq \prod_{k=1}^{r} t_{P_{2k}}(G)^{b_k},$$

where $a_i, b_i$ are non-negative numbers such that $\sum_{k=1}^{r} a_k = \sum_{k=1}^{r} b_k = 1$, $b_k = \frac{1}{2}(a_{k-1} + a_{k+1})$ for $3 \leq k \leq r - 2$, $b_2 = \frac{1}{2}a_3$, $b_{r-1} = \frac{1}{2}a_{r-2}$, $b_1 = a_1 + \frac{1}{2}a_2$, and $b_r = a_r + \frac{1}{2}a_{r-1}$. The exponent vector $(a_1, a_2, \ldots, a_r)$ at each step can be seen as a probability distribution on $[r]$. Then the recurrence relation above gives a Markov chain, where $1$ and $r$ are absorbing states and the transition probabilities from the other states to the adjacent ones are $1/2$. It is well-known that, starting from $\ell$, the distribution converges to $a_1 = (r - \ell)/(r - 1)$, $a_\ell = (\ell - 1)/(r - 1)$, and $a_i = 0$ for $1 < i < r$. Hence, we obtain

$$t_{P_{2r}}(G) \leq t_{P_2(G)^{r-1}}t_{P_2(G)}^{\frac{r-1}{2}}.$$

Using the natural $P_2$-decomposition of $P_{2r}$, Theorem 2.4 gives $t_{P_2(G)^r} \leq t_{P_2(G)}$. Thus, it follows that

$$t_{P_{2r}}(G) \leq t_{P_{2}}(G)^{r-1}t_{P_{2}}(G)^{\frac{r-1}{2}} \leq t_{P_{2}}(G)^{\frac{r}{2}}. \quad \square$$

By Lemma 4.3, it remains to show that there exists $\rho > 0$ such that

$$t_{G_{2\ell+1}}(G) \geq (d - \delta)t_{P_{2\ell}}(G)$$

whenever $G$ is $(\rho, d)$-dense. The proof of this inequality is essentially the same as that of Reiher [16] proving Conjecture 1.1 for odd cycles, but we include it for completeness.

Proof of Lemma 4.1. Let $|V(G)| = n$ and let $q(v)$ be the normalised number of walks of length $r$ starting from $v$, i.e., we divide the number of walks by $n^{r-1}$. Denote by $U := \{u : q(u) > \rho n\}$ the set of vertices with large $q(u)$. Then

$$\frac{1}{n^{2r-2}}|\text{Hom}(P_{2r}, G)| = \sum_{u \in U} q(u)^2 + \sum_{u \notin U} q(u)^2,$$

and hence $\sum_{u \in U} q(u)^2 \geq |\text{Hom}(P_{2r}, G)|/n^{2(r-1)} - \rho^2 n^3$. On the other hand, let $f_u(v)$ be the number of walks of length $r$ from $u$ to $v$. Then by definition $q(u) = \sum_{v \in V(G)} f_u(v)$, and $\sum_{v \in V(G)} f_u(v) > \rho n$ whenever $u$ is in $U$. For each $u \in U$, Lemma 4.2 gives

$$2 \sum_{vw \in E(G)} f_u(v)f_u(w) \geq d \left( \sum_{v \in V(G)} f_u(v) \right)^2 - 2n \geq dq(u)^2 - 2n.$$
Summing this inequality over all \( u \in U \) is at most the normalised number of homomorphisms 
\[ \frac{1}{n^{2r-2}} | \text{Hom}(C_{2r+1}, G) | \geq \frac{d}{n^{2r-2}} | \text{Hom}(P_{2r}, G) | - \rho^2 n^3 - 2n^2. \]

Taking \( \rho = \sqrt{\delta}/2 \) and \( n > 4/\delta \) finishes the proof.

**Proof of Theorem 1.5.** Given \( \eta > 0 \), choose \( \delta > 0 \) such that \( d^{2r+1} - \delta \geq (d - \eta/|E(H)|)^{2r+1} \). For brevity let \( \epsilon = \eta/|E(H)| \). By Reiher’s theorem [16] on odd cycles,\(^2\) there exists \( \rho = \rho(r, \delta) > 0 \) such that
\[ t_{C_{2r+1}}(G) \geq d^{2r+1} - \delta \geq (d - \epsilon)^{2r+1}. \]  

(13)

Let \((\mathcal{F}, \mathcal{T})\) be a \( C_{2r+1} \)-decomposition of \( H \) and let \( e_{XY} \) be the number of edges in \( H[X \cap Y] \) for \( XY \in E(\mathcal{T}) \). Each \( H[X \cap Y], XY \in E(\mathcal{T}) \), is a vertex-disjoint union of paths, and thus by Lemma 4.1 we obtain the upper bound
\[ t_H[X \cap Y](G) \leq \left( \frac{t_{C_{2r+1}}(G)}{d - \epsilon} \right)^{e_{XY}/2r} \leq t_{C_{2r+1}}(G)^{e_{XY}/(2r+1)}, \]

where the last inequality follows from (13). Combining this bound with Theorem 2.4 we have
\[ t_H(G) \geq \frac{t_{C_{2r+1}}(G)^{|\mathcal{F}|}}{\prod_{XY \in E(\mathcal{T})} t_H[X \cap Y](G)} \geq t_{C_{2r+1}}(G)^{|\mathcal{F}|} e_{XY} \sum_{XY \in E(\mathcal{T})} e_{XY} = t_{C_{2r+1}}(G)^{|E(H)|}. \]

Together with (13) this gives \( t_H(G) \geq (d - \epsilon)^{|E(H)|} \geq d^{|E(H)|} - \eta. \)

\[ \square \]

5 Concluding remarks

An approximate version of Conjecture 1.1 for graphs with bounded tree-width. Given a tree decomposition \((\mathcal{F}, \mathcal{T})\) of \( H \), we may assume that each leaf \( X \) of \( \mathcal{T} \) contains an edge \( e \) that is not contained in any other \( F \in \mathcal{F} \), since otherwise the leaf is redundant. By mapping each leaf to such an edge while removing leaves successively, we see that \( |\mathcal{F}| \leq |E(H)| \). Thus, for each \( m \)-edge graph with tree-width \( t \) and a tree decomposition \((\mathcal{F}, \mathcal{T})\) with the minimum width, it is enough to add \((t+1)|\mathcal{F}| \leq (t+1)m\) edges to obtain a \( t \)-tree. This gives an approximate version of Conjecture 1.1 for graphs with bounded tree-width.

**Corollary 5.1.** Let \( H \) be a graph with \( m \) edges and tree-width \( t \). Given \( \eta > 0 \), there exists \( \rho = \rho(m,t) > 0 \) such that
\[ t_H(G) \geq d\left(\frac{(t+1)^2}{2}+1\right)^m - \eta \]

for every sufficiently large \((\rho,d)\)-dense graph \( G \).

Open cases for Conjecture 1.1. Using Theorems 1.4, 1.5, 3.1, 3.2, and the theorem of Reiher [16] on odd cycles, one may check that Conjecture 1.1 is true for \( H \) with at most 4 vertices. Many graphs on 5 vertices can also be verified to satisfy the conjecture, but we do not know how to handle a 5-cycle with a chord, which is perhaps the simplest open case.

\(^2\) Obviously, it also follows from Lemma 4.1 for the case \( \ell = 2r \), which rephrases Reiher’s argument.
Question 5.2. Is Conjecture 1.1 true for $H$ isomorphic to $C_5$ with a chord?

In [5], Conlon and the author proved an analogous result that bipartite graphs obtained naturally from faces of a regular polytope are (weakly) norming. It might also be possible to verify Conjecture 1.1 for highly symmetric graphs.

Question 5.3. Is Conjecture 1.1 true for $H$ isomorphic to the 1-skeleton of a regular polytope? In particular, is it true for an icosahedron or a dodecahedron?

Caveats from marginal constraints. When using Theorem 2.3, the major caveat is how to find random variables $(X_{i,F})_{i \in F}$ that agree on the marginals $(X_{j,A})_{j \in A \cap B}$. In fact, the symmetry condition in the definition of the $J$-decomposition is tailored to satisfy the marginal constraints. However, for non-isomorphic graphs $H_1$ and $H_2$ containing the same induced subgraph $J$, it is often hard to find distributions on $\text{Hom}(H_1, G)$ and $\text{Hom}(H_2, G)$ that agree on the natural projection to $\text{Hom}(J, G)$.

For example, it is possible to generate a random copy of a tree in such a way that the projection of the distribution onto a subtree agrees with the distribution generated by the same algorithm, which leads to the definition of strongly tree-decomposable graphs used in [4]. Another example is the theorem of Li and Szegedy [13] proving that, if $H$ has Sidorenko’s property, we may assume that the projection of a uniform random homomorphic copy of $H$ onto a single edge is again uniform. Therefore, it is possible to glue graphs having Sidorenko’s property on a single edge while preserving the property. Likewise, if it is possible to generate random copies of $K_r$ and $K_s$ that have the same marginals on $K_t$ for every $t \leq \min(r, s)$ in locally dense graphs, then it may be possible to prove Conjecture 1.1 for all chordal graphs.

Homomorphism domination exponent. Though graph homomorphism inequalities have appeared in many contexts for decades, Kopparty and Rossman [12] recently initiated studying a general type of domination inequalities between the number of graph homomorphisms. A normalised version of their questions in [12] is essentially to determine an optimal exponent $c = c(H, J)$ for graphs $H$ and $J$ such that

$$t_H(G) \geq t_J(G)^c$$

holds for all graphs $G$. For example, Lemma 4.3 gives the optimal domination exponent between an even path $H$ and its subgraph $J$. On the other hand, it is no longer true if $H$ is a path of odd length and $J$ is a subgraph of $H$, as noted by London [14]. More generally, the notion of (weakly) norming graphs introduced by Hatami [9] and Lovász [15] and studied further by Conlon and the author [5] gives a wide range of such inequalities.

To guarantee the existence of the exponent $c$ in (14), it is often assumed that there exists a homomorphism from $H$ to $J$. However, as seen in Lemma 3.3 or in Lemma 4.1, this condition is not necessary when proving analogous inequalities provided $G$ is locally dense. In [12], Kopparty and Rossman obtained such optimal exponents for some graphs for example some paths and 2-trees. This type of question, or more generally, determining exponents $c_1, c_2, \cdots, c_k$ for $H$ and its subgraphs $J_1, J_2, \cdots, J_k$ such that

$$t_H(G) \geq t_{J_1}(G)^{c_1}t_{J_2}(G)^{c_2}\cdots t_{J_k}(G)^{c_k},$$

are useful in deducing inequalities from Theorem 2.4, as done in Lemma 3.3 or Lemma 4.1. Hence, it will be interesting to see more inequalities of a similar type, especially if the equality holds when $G$ is the Erdős-Rényi random graph $G(n, p)$.

\footnote{In fact, many of their results are technically different from (14) because of the normalisation issue.}
Hypergraph generalisation. Another application of Theorem 2.3 is to common graphs. A graph $H$ is common if every 2-edge-colouring of a complete graph $K_n$ contains at least $2^{1-|E(H)|/n^{|V(H)|}}$ (labelled) monochromatic copies of $H$. A graph $H$ is a triangle-edge (resp. triangle-vertex) tree if it has a $K_3$-decomposition $(F, T)$ with $|X \cap Y| = 2$ (resp. $|X \cap Y| = 1$) for each $XY \in E(T)$. In [20], Sidorenko proved the following:

**Theorem 5.4 ([20]).** Every triangle-edge tree or triangle-vertex tree is common.

This can also be obtained by a hypergraph generalisation of Theorem 2.4. For the case of triangle-vertex trees, let $G_1$ and $G_2$ be 3-uniform hypergraphs of which the edge sets are red triangles and blue triangles, respectively, and let $H'$ be the auxiliary 3-uniform hypergraph by putting a hyperedge for every triangle. Then we get $t_{H'}(G_i) \geq t_\Delta(G_i)^{|E(H')|}$ by Theorem 2.4, $i = 1, 2$, where $\Delta$ is the single 3-edge graph. By Jensen’s inequality, we obtain

$$t_\Delta(G_1)^{|E(H')|} + t_\Delta(G_2)^{|E(H')|} \geq 2 \left( \frac{t_\Delta(G_1) + t_\Delta(G_2)}{2} \right)^{|E(H')|}.$$

Now the fact that $K_3$ is common implies that $H$ is common, since $t_\Delta(G_1) + t_\Delta(G_2)$ is the number of monochromatic triangles in the given 2-edge-colouring and $|E(H')| = 3|E(H)|$.

The setting for triangle-edge trees is slightly different. We again consider the auxiliary hypergraph $H'$ constructed by triangles of $H$, but construct a 3-uniform hypergraph $G$ by putting a hyperedge for every monochromatic triangle, either red or blue. Note that every copy of the hypergraph $H'$ in $G$ corresponds to a monochromatic copy of graph $H$, as every triangle in $H$ must have the same colour. Then Theorem 2.4 gives

$$t_{H'}(G) \geq t_\Delta(G)^{|E(H')|},$$

and the commonness of a triangle implies $t_{H'}(G) \geq (1/4)^{|E(H')|}$. As $|E(H)| = 2|E(H')| + 1$, we obtain the bound

$$t_{H'}(G) \geq (1/2)^{|E(H)|-1},$$

and thus $H$ is common.

Though the natural hypergraph generalisation of Sidorenko’s conjecture is false even for tripartite linear 3-uniform hypergraph $H$ [18], it would still be interesting to find more applications of Theorem 2.4 to counting fixed hypergraphs.

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