Field Theory of the Fractional Quantum Hall Effect-I

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(To appear in COMPOSITE FERMIONS, edited by Olle Heinonen.)

(March 24, 2022)

Abstract

We present a Chern-Simons (CS) theory for fractional quantum Hall states, in which flux attachment is followed by an enlargement of Hilbert space to include $n$ magneto-plasmon oscillators and $n$ constraints, $n$ being the number of electrons. The simplest approximation yields correlated wavefunctions for fractions $\nu = p/(2ps + 1)$. For $\nu = 1/(2s + 1)$ we obtain multi-quasihole wavefunctions with correct gaussian and normalization factors, operators that create quasiholes, and a possible Read operator. Our composite fermion and boson operators attach flux as well as the correlation hole to electrons, i.e., bind them to zeros. We perform a further canonical transformation which decouples the the particles and oscillators in the long wavelength limit. The $n$ constraints now act on just the particles, restricting them to the Lower Landau level (LLL). Their kinetic energy vanishes In the noninteracting limit, they couple to the external field through a magnetic moment $e/2m$, carry charge $e^* = e/(2ps + 1)$ and a dipole moment which alone survives when $\nu = 1/2$. When a Coulombic interaction is turned on, a nonzero $1/m^* \simeq (2\nu)^{2/3}e^2l_0/6$ arises. Density-density correlation functions and Hall conductivity, additive over the particles and oscillators, are calculated within an approximation for $\nu = 1/2$. The formalism makes precise and unifies many prevalent notions regarding composite fermions and bosons, quasiparticles, cyclotronic physics, $m$ and $m^*$.

I. INTRODUCTION

The spectacular discovery of the Fractional Quantum Hall Effect (FQHE) by Tsui, Störmer and Gossard\textsuperscript{1} has sparked a remarkable amount of activity among theorists and experimentalists.\textsuperscript{2,3} The theoretical efforts fall into two broad categories: those that seek trial wavefunctions that capture the essential physics, for example references\textsuperscript{4–10}, and those that attempt to construct a theory that starts with electrons in a magnetic field subject to Coulomb interactions and try to work their way towards the experiments through a series of approximations. Chern-Simons (CS)\textsuperscript{11} field theories play a central role in the second category\textsuperscript{12–17}. The present article, also of this genre, is an elaboration of a recent letter\textsuperscript{18}. It relies on its predecessors, extends them, unifies many miscellaneous ideas that have been in the air and exposes the underlying physics in a particularly transparent way.

We follow the standard notation in which the electrons, $n$ in number, have a bare (band) mass $m$, charge $-e$, a complex coordinate $z = x + iy$, move in unit area in the $x - y$ plane,
and are in a field $\mathbf{B} = -\hat{z}B$ pointing down the $z$-axis. We set $\hbar = c = 1$. The inverse filling fraction

$$\nu^{-1} = \frac{eB}{2\pi n}$$

(1)
gives the number of flux quanta per electron. In the cases of interest, $\nu < 1$, it will be assumed all spins are polarized.

Let us first neglect interactions. All the electrons will populate the Lowest Landau Level (LLL). The ground state will be macroscopically degenerate and the first nonzero energy excitation will be at the cyclotron frequency

$$\omega_c = \frac{eB}{m}.$$  

(2)
The effective mass of the particles will be infinite:

$$\frac{1}{m^*} = 0$$  

(3)
reflecting the quenching of kinetic energy.

Let us now turn on a Coulomb interaction, weak in the sense

$$\frac{e^2}{l_0} << \omega_c$$  

(4)
where the cyclotron length

$$l_0 = (eB)^{-1/2}$$  

(5)
is a measure of the typical interparticle spacing and the size of the orbits, and where the dielectric constant $\varepsilon$ has been suppressed. One may reasonably expect that in the limit $m \to 0$ all higher Landau levels and any reference to $m$ will disappear from the picture; that the degenerate manifold of the LLL states will broaden out to a width controlled by the effective mass $1/m^* \approx e^2l_0$, of Coulombic origin. Thus consider Laughlin’s wavefunctions at $\nu = 1/3$ for the ground state, and state with a quasihole at the origin:

$$\Psi_{1/3} = \prod_{i<j}(z_i - z_j)^3 \exp - \sum_i |z_i|^2/(4l_0^2)$$  

(6)
$$\Psi_{1/3,qh} = \prod_k z_k \prod_{i<j}(z_i - z_j)^3 \exp - \sum_i |z_i|^2/(4l_0^2).$$  

(7)
Note that only $l_0$ enters the wavefunctions and $m$ does not. This is true also of the subsequent functions written down by Haldane, Halperin, Jain and Rezayi-Read.

From these wavefunctions one has gleaned many of the basic features of the FQHE such as the incompressibility, fractional charge of the quasiparticles, and their fractional statistics. These successes notwithstanding, there are many reasons to pursue the hamiltonian approach in parallel. If it can yield these wavefunctions in some approximation, the nature of that approximation will be physically revealing. One can calculate other wavefunctions and correlation functions at general $(q, \omega)$, one can couple the system to impurities to study the effects of disorder, or to a confining potential to study the physics at the edge. The underlying theory will tell us how the charge of the excitations gets renormalized from $e$.
in the case of free electrons to $e^* = e/(2ps + 1)$ for the principal fractions $p/(2ps + 1)$, and in particular how it vanishes for the case of $\nu = 1/2$, becoming dipolar as anticipated by Read. It should also tell us who carries the Hall current if the quasiparticle (the composite fermion) is neutral. It should shed light on the order parameter with algebraic order proposed by Girvin and MacDonald and studied by ZHK, and the order parameter with long-range order proposed by Read.

The theory must also address the question of the bare and renormalized masses in some detail. The notion that low energy physics can be described with no reference to $m$ or higher Landau levels is plagued with subtleties as emphasized by Girvin, MacDonald and Platzman (GMP). First, the current operator depends explicitly on $m$ while the Hall conductance $\sigma_{xy}$ does not. For $m$ to drop out, one must necessarily invoke virtual transitions to the next Landau level with matrix elements of order $m$. Next, one must understand how $m$ gives way to $m^*$ in the rest of the low energy physics and how the latter is to be calculated in terms of the interactions. Even if one begins with a phenomenological $m^*$, there are places where $m$ enters. For example, the cyclotron pole is determined by $m$ according to Kohn’s theorem. In the Fermi-liquid-like state of $\nu = 1/2$, Simon and Halperin argue that it is necessary to bring $m$ back by invoking a suitable Landau parameter $f_1$ and the relation

$$\frac{1}{m} = \frac{1}{m^*} + \frac{f_1}{2\pi},$$

(8)

a scheme that was briefly alluded to in HLR. Now one may argue that the Kohn mode is part of the high energy physics and an effective low energy theory is not obliged to explain it. But as Simon, Stern and Halperin (SSH) point out, $m$ enters the low energy physics as well. Consider the zero point energy of $eB/2m$ per electron, which one may ignore as a constant. Suppose we impose on the system a small slowly varying field $\delta B(r)$. This amounts to imposing a potential $e\delta B(r)/2m$, the response to which is surely $m$ dependent. Also, if for any reason the density becomes nonuniform, there will be a current due to incomplete cancellation of cyclotron currents:

$$j_{mag} = \frac{e}{2m} \hat{z} \times \nabla n.$$  

(9)

Both these effects imply that the charge-current correlator $K_{01} = K_{10}$ has a piece $(iqe/2m)K_{00}$ as $q \to 0$. SSH show that these effects can be reproduced by appending to each particle a magnetic moment $e/2m$. It is desirable to have this occur naturally within a computational scheme.

The variant of CS theory we propose here solves many of these problems.

To set the context, let us begin by recalling the basic idea behind CS theories, using as a concrete example $\nu = 1/3$ and the goal of getting Laughlin’s wavefunction $\Psi_{1/3}$. Clearly, the more we know of the answer the less we have to calculate. The usual bosonic CS theory of Zhang, Hansson and Kivelson (ZHK) begins by writing the wavefunction for the state as

$$\Psi_{1/3}(r_i) = [\text{phase of } \Psi_{1/3}] \Psi_{CS}(r_i)$$  

(10)

and then solving for $\Psi_{CS}$ which describes CS bosons. At mean field level the bosons condense into the zero momentum state, giving $\Psi_{CS} = 1$ and we obtain just the phase of $\Psi_{1/3}$. Each
particle sees three flux tubes in the others, but does not avoid them with the triple zeros, a correlation essential to the success of Laughlin’s wavefunction. These zeros, which imply correlation holes, arise upon going beyond mean-field and including fluctuations. This is also true for fermionic CS theories of Lopez and Fradkin, Kalmeyer and Zhang, Halperin, Lee and Read (HLR) and Kwon, Marston and Houghton, though wave functions have not been explicitly worked out this way for the case of $\nu = 1/2$. For the same reason, the composite bosons and fermions all carry the full charge of the electrons at mean-field level and the fractional charges of the quasiparticles are again to be found in the fluctuations. An exception is the theory of Rajaraman and Sondhi. These authors include all of $\Psi_{1/3}$ as a prefactor in Eqn.(10) so that at mean field level they obtain all of $\Psi_{1/3}$. On the other hand, the fluctuations about mean field are described by a complex CS field whose real and imaginary parts come from the phase and modulus in the prefactor. To go beyond mean-field theory using diagrams one must face the fact that the Hamiltonian is not hermitian. However the formalism has already been applied successfully to questions involving solitons.

The method we propose manages essentially to append as a prefactor both the modulus and phase of $\Psi_{1/3}$ (and its generalization to fractions of the form $p/(2ps + 1)$) without the use of complex vector potentials. The correlation holes give the right charge $e^*$ to the quasiparticles. The theory has two sectors, and with one (desirable) exception, $m$ appears in one and $m^*$ in the other.

Setting aside details for later, here is how this is done. Recall (or go and read, if you are too young) the work of Bohm and Pines on the electron gas in three dimensions. Besides its fermionic quasiparticles and particle-hole excitations, the gas has plasmons in the spectrum. These arise as poles in the density-density correlations. The poles have a negligible imaginary part (i.e., the plasmons are very long-lived) till the electron-hole continuum runs into the plasmon pole at large enough $q$. Rather than treat the small $q$ plasmons as composites built out of the original degrees of freedom, Bohm and Pines elevate them to elementary particles with their own set of states and operators in an enlarged Hilbert space. The plasmons couple to the electrons in a way that reproduces the pole and residue in the density-density correlations. Having introduced the plasmons as elementary objects, one must ensure that the particles do not duplicate them. To keep the problem unchanged in the enlarged Hilbert space, only states obeying a set of constraints (one for each $q$ at which a plasmon was introduced) are deemed physical. The constraints freeze all fermionic collective degrees of freedom that were assigned to plasmons. Placing the plasma oscillators in their ground states gives, upon projection to the physical states, a correlation factor to the fermionic wavefunction, which is simply the exponential of the Coulomb potential energy associated with each configuration. If the principal coupling between the fermions and plasmons is removed by a canonical transformation, it is found that the fermion mass gets renormalized upwards while the plasmon pole is fixed at the classical value as $q \to 0$. The number of plasmon modes introduced is treated as a variational parameter and is a small fraction of

1A CS theory with complex electric and magnetic fields was briefly discussed up by Girvin in the closing article in Reference (2).
the number of fermions.

We adapt this scheme to our problem with the magnetoplasmons playing the role of plasmons. For a variety of reasons we pick \( n_0 \), the number of oscillators, to equal \( n \), the number of particles. Putting the magnetoplasmons in their ground states gives the correlation zeros of the trial wavefunctions. Our field operators for composite bosons and fermions create the electron as well as its correlation hole, the latter being described by collective coordinates. When we decouple the oscillators from the particles, \( 1/m^* \) gets renormalized to zero in the noninteracting limit and picks up a finite value when interactions are turned on. The zeroth order approximation to \( 1/m^* \) is readily evaluated. The decoupling, performed in the long-distance limit and the RPA, also changes the charge of the quasiparticles from \( e \) to \( e^* \). Upon decoupling, the \( n \) constraints act on the particles alone, the reduction in degrees of freedom being exactly what it takes to get LLL physics. This is confirmed by the fact that the density (limited to the sector in which the oscillators are in their ground states) now obeys the magnetic translation algebra of GMP\(^{23}\). All dependence on \( m \) is contained in the oscillator sector with one exception: the particles couple to the external magnetic field with a moment \( e/2m \), the need for which was discovered by SSH\(^{27}\). We find that in the clean limit, all the Hall current is carried by the oscillators, which takes care of the nagging question of who carries the Hall current when the composite fermions become neutral at \( \nu = 1/2 \). Our formalism gives a precise meaning to the very useful concept of the composite fermion. All this will be explained as we go along.

The plan of the paper is as follows: In Section II we recall the key ideas behind flux attachment in both the wavefunctions and CS approaches. Section III describes our CS representation which results from enlarging the Hilbert space and doing a canonical transformation to reach what we call the middle representation (MR). In Section IV we derive correlated ground state wavefunctions for \( \nu = 1/(2m+1), 1/2 \) and the Jain series \( p/(2p+1) \). (The case \( p/(2ps+1) \) calls for straightforward generalization and we do not discuss it in any detail.) For the first case, described by a bosonic CS theory, we display operators that create quasielectrons and quasiholes and a candidate for the Read operator\(^{22}\). We show that our composite boson and fermion operators create particles in which an electron is bound to zeroes, and not just flux quanta. In Section V we perform the canonical transformation to the final representation (FR) in which the particles and oscillators are decoupled in a long-distance approximation. We exhibit the substantial mass and charge renormalization. We pay special attention the gapless case \( \nu = 1/2 \) which arises in the limit \( p \to \infty \) and compare it to the work of HLR. Discussions and conclusions follow in Section VI.

**II. FLUX ATTACHMENT AND CHERN-SIMONS THEORIES**

In two space dimensions one has the remarkable freedom to alter particle statistics by attaching point flux tubes\(^{31,32}\). In particular, a fermion carrying an odd/even number of flux quanta will behave like a boson/fermion. This fact is exploited as follows. Let us consider \( \nu = 1/3 \). In the noninteracting limit, the ground state is macroscopically degenerate. Trading the electrons for composite bosons (CBs), each carrying three quanta of statistical flux opposed to the external field, one ends up on the average with composite bosons in zero field. Alternatively, trading them for composite fermions (CFs) carrying two quanta (opposing the
external field) each, one obtains CFs with one quantum each on the average, i.e., enough to fill exactly one Landau level. In either case we have a nondegenerate starting point for including the effect of interactions. The notion of dealing with the flux on the average to get a simple starting point was first exploited by Laughlin in studying anyon superconductivity. Of course one must deal with the (possibly singular) fluctuations to complete the analysis of the problem.

Let us see how flux attachment is implemented in building wavefunctions and CS field theories.

It was noted long ago by Halperin that the $3n$ zeros of Laughlin’s wavefunction are all tied to the location of the particles, while only $n$ are required to do so by Fermi statistics. The electron and the triple zero are attracted to each other by the Coulomb force, and behave like a neutral boson, which condenses in the Laughlin state, as emphasized by Read. Thus one can write

$$\Psi_{1/3} = \prod_{i<j} (z_i - z_j)^3 \exp \left[ -\sum_i |z_i|^2/(4l_0^2) \right] \equiv \chi_1^3 \cdot 1, \quad (11)$$

where $\chi_1^3$ attaches three flux quanta to the CB whose condensed wavefunction is simply the factor of unity.

On the other hand for fractions of the form $p/(2ps + 1)$ one does not have such explicit wavefunctions. Here is where Jain’s notion of composite fermion (CF) comes to the rescue. First he notes that we can just as well write

$$\Psi_{1/3} = \prod_{i<j} (z_i - z_j)^3 \exp -\sum_i |z_i|^2/(4l_0^2) = \chi_1^2 \cdot \chi_1. \quad (12)$$

The picture now is that the electrons have been traded for CF’s which carry two flux quanta opposed to the external field, and fill exactly one Landau level in the weakened mean-field. The first factor of $\chi_1^2$ attaches two flux quanta and the last stands for the filled Landau Level of the CF. Now if $\nu = p/(2ps + 1)$, with $p > 1$, bosons in zero field are not attainable, but Jain’s idea composite fermion (CF) idea is still applicable. By attaching $2s$ units of flux opposed to the external one, the electron problem gets mapped on average to that of composite fermions which see

$$\frac{1}{\nu^*} = \frac{1}{\nu} - 2s = 1/p \quad (13)$$

units of flux each, i.e., fill $p$ Landau levels. For example if $\nu = 2/5$, we have $p = 2$, $s = 1$ and the Jain wavefunction is

$$\Psi_{2/5} = \mathcal{P} \chi_1^2 \chi_2 \quad (14)$$

where $\chi_2$ describes two filled Landau levels and $\mathcal{P}$ projects the wavefunction to the LLL. (The magnetic lengths appearing in the gaussians in $\chi_1$ and $\chi_2$ must be chosen to reflect the magnetic fields associated with filling one and two levels respectively.) When $\nu = 1/2$, (the $s = 1$, $p \to \infty$ limit) the external and attached fluxes cancel and we end up with

$$\Psi_{1/2} = \mathcal{P} \chi_1^2 |FS\rangle \quad (15)$$
where $|FS\rangle$ stands for the Fermi Sea. This function was proposed independently and studied in detail by Rezayi and Read (RR)\textsuperscript{10}. They showed that a gap is not needed for the success of the idea.

It is important to note that there is no unique notion of flux attachment in the wavefunction approach. For example, instead of using the factor $\chi_1$ to attach a unit of flux one could use a factor that did not contain the gaussian, or even the zero when the particles approached each other, by making the replacement

$$ (z_i - z_j) \rightarrow \frac{(z_i - z_j)}{|z_i - z_j|}. \quad (16) $$

If the latter choice had been made for $\nu = 1/3$ one would have lost two of the three zeros that produced the excellent correlations as the particles approached each other. Of course, such a wavefunction would have been a much poorer approximation to the true ground state than one where the correlation zeros were present. There is no reason to make the poor choice in the quest for wave functions, but as we shall now see, the choices are rather limited in the CS approach to which we now turn.

In contrast to the, flux attachment in CS field theories is a lot more restrictive, and is usually carried out as follows. In first quantization one introduces a wavefunction for the CS particles in terms of the electronic wavefunction as follows:\textsuperscript{31}

$$ \Psi_e = \prod_{i<j} \frac{(z_i - z_j)^l}{|z_i - z_j|^l} \Psi_{CS}. \quad (17) $$

where $l$ is the number of flux quanta to be attached. The CS particles are to be quantized as fermions/bosons, for $l =$even/odd since the prefactor produces a factor $(-1)^l$ under particle exchange. The prefactor introduces a gauge field $a$ in the Schrödinger equation for $\Psi_{CS}$. The field obeys

$$ \frac{\nabla \times a}{2\pi l} = \sum_i \delta^2(r - r_i). \quad (18) $$

In second quantization the CS transformation is represented by the operator relation

$$ \psi_e(r) = \exp \left[ il \int \frac{\hat{z} \times (r - r')}{|r - r'|^2} \rho(r')d^2r' \right] \psi_{CS}(r') \quad (19) $$

where $\psi_e$ and $\psi_{CS}$ are the field operators that annihilate the electron and the CS composite particle respectively. (They are not to be confused with the wavefunctions $\Psi$). Also,

$$ \rho(r) = \psi_e^\dagger(r) \psi_e(r) = \psi_{CS}^\dagger(r) \psi_{CS}(r) \quad (20) $$

is the density. The hamiltonian(density), before turning on electron-electron interactions (which will be added later), is

$$ H_{CS} = \psi_{CS}^\dagger \frac{|(-i\nabla + eA + a)|^2}{2m} \psi_{CS} \quad (21) $$
where $A$ is the external vector potential, $m$ is the bare mass and

$$
\frac{\nabla \times a}{2\pi l} = \psi^\dagger \psi.
$$

(22)

Since $A$ describes a magnetic field pointing down the $z$-axis, and the CS field has its flux of $l$ quanta per particle pointing up, they can be played off against each other. There are two kinds of $\nu$ for which things are particularly simple. If $\nu = 1/(2s+1)$, by attaching $2s+1$ quanta per particle, we can trade the electrons for bosons that see zero net field on the average. If $\nu = 1/2s$, by adding $2s$ quanta per particle we can trade the electrons for fermions in zero average field. Hereafter we use the convention that the average values have been removed from $a$ and $\psi^\dagger \psi = \rho$ (the density) both of which henceforth refer to normal-ordered quantities. (The normal ordering will occasionally be made explicit). The bosons condense in zero field and the fermions form a Fermi sea (FS), both of which are nice nondegenerate starting points explored by Zhang et al. (bosonic) and Kalmeyer and Zhang, HLR, Kwon et al (fermionic). At the mean-field level the wavefunctions one obtains are

$$
\Psi_{1/2} = \prod_{i<j} \frac{(z_i - z_j)}{i} \Psi_{CS}
$$

(23)

where $\Psi_{CS}$ is the Fermi Sea (FS) for $\nu = 1/2$ and $\Psi_{CS} = 1$ for $\nu = 1/(2s+1)$. The multiple zeros expected in the correlated wavefunctions are absent. To obtain these one needs to analyze fluctuations about the mean field.

When $\nu = p/(2ps+1)$, by attaching $2s$ opposing flux quanta per electron, we can cancel enough of the external flux so that what remains, $eA^*$, is enough to fill exactly $p$ Landau levels:

$$
eB^* = \frac{eB}{2ps+1}.
$$

(24)

These are the fractions considered by Jain. Indeed Jain’s case reduces to the two previous ones when $p = 1$ and $p = \infty$ respectively. The corresponding field theory was analyzed by Fradkin and Lopez, who computed some response functions and also showed that fluctuations produced the modulus of the Laughlin function in the limit of particle separation large compared to $l_0$.

Let us summarize: In the quest for wavefunctions we can build in not just the phase, but all of $\prod (z_i - z_j)$, or even the ubiquitous gaussian factors into the process of flux attachment. In contrast, the standard way of attaching flux in CS theories leads only to the phase of the zeros at mean-field level, while the zeros themselves emerge from the fluctuations. It is possible to obtain the zeros in a field theory at mean-field level: however, doing so leads to a complex vector potential and a non-hermitian hamiltonian, as pointed out by Rajaraman and Sondhi who analyzed this possibility in some depth. Let us now turn to our version of the CS theory and see how correlated wave functions are obtained.
III. OUR CS THEORY

Let us recall the CS hamiltonian (density) before introducing any interactions:

\[ H_{CS} = \psi_{CS}^\dagger \frac{|(-i \nabla + eA^* + a)|^2}{2m} \psi_{CS}. \]  (25)

Here \( a \) refers to the fluctuations about the mean (which has been used to reduce \( A \) down to \( A^* \)), and obeys

\[ \frac{\nabla \times a}{2\pi l} =: \psi^\dagger \psi :. \]  (26)

To make explicit the fact that \( a \) is really a dependent field, let us rewrite this as

\[ H = \frac{1}{2m} \psi_{CS}^\dagger (-i \nabla + eA^* + (\nabla \times)^{-1}2\pi lp)^2 \psi_{CS} \]  (27)

where the inverse curl is uniquely defined by its Fourier transform if we demand that the answer be transverse. If you find this form of \( H \) forbidding, it is our intention; Eqn.(25) hides the complexity of the CS hamiltonian.

We are now ready to enlarge the Hilbert space. Consider a disk (in momentum space) of radius \( Q = k_F \), the Fermi momentum of the electrons, which contains as many points as there are electrons. For each \( q \) in this disc, we associate a canonical pair of fields \( a(q), P(q) \)

\[ [a(q), P(q')] = (2\pi)^2 \delta^2(q + q'). \]  (28)

These define a pair of longitudinal and transverse vector fields

\[ P(q) = i\hat{q} P(q) \quad a(q) = -i\hat{z} \times \hat{q} a(q). \]  (29)

The Hamiltonian density of Eqn. (27) is completely equivalent to

\[ H = \frac{1}{2m} \psi_{CS}^\dagger (-i \nabla + eA^* + a + (\nabla \times)^{-1}2\pi lp)^2 \psi_{CS} \]  (30)

provided we restrict our selves to states in the larger space obeying

\[ a(q) |_{physical} = 0 \quad q < Q. \]  (31)

In other words, \( [H, a] = 0 \) allows us to find simultaneous eigenstates of \( H \) and \( a \), and the eigenstates with \( a = 0 \) solve the original problem.

Let us note for future use how we are to extract wavefunctions for the CS particles from any solution in the bigger Hilbert space. Let \( \Psi_{CS}(x) \) denote the CS wavefunction. In the larger space we have, in obvious schematic notation, the following resolution of the identity

\[ I = \int dx da|x a \rangle \langle a x| \]  (32)

where \( x \) denotes all the particle coordinates and \( a \) stands for \( a(q), [0 < q \leq Q] \). The projection operator to the physical sector is

\[ \varphi = \int dx da|xa\rangle \langle ax| \frac{\delta(a)}{\delta(0)}. \]  (33)
This means that if $\Psi(x,a)$ is a generic wavefunction, then the projected version is

$$\Psi_p(x,a) = \frac{\delta(a)}{\delta(0)} \Psi(x,a) = \frac{\delta(a)}{\delta(0)} \Psi(x,0). \quad (34)$$

Indeed one can show that every physical vector must have such a form.

Let $\Psi_p(x,a)$ obey the eigenvalue equation (in first quantization)

$$H(x,p,a) \Psi_p(x,a) = E \Psi_p(x,a) \quad (35)$$

where $H(x,p,a)$ is the hamiltonian with $a$ added on as in Eqn.(30). Such eigenfunctions will exist since $H$ commutes with the constraint. Feeding in Eqn.(34) for the wavefunction, moving the delta function through $H$ to its left using the commutativity, and canceling it from both sides, we obtain

$$H(x,p,0) \Psi(x,0) = E \Psi(x,0) \quad (36)$$

establishing that

$$\Psi_{CS}(x) = \Psi(x,0). \quad (37)$$

In other words $\Psi_{CS}$ is embedded within $\Psi_p(x,a)$ as follows:

$$\Psi_p(x,a) = \frac{\delta(a)}{\delta(0)} \Psi_{CS}(x). \quad (38)$$

Let us now return to the Hamiltonian density of Eqn. (30) which contains the very nasty inverse curl. We deal with it through the following unitary transformation

$$U = \exp\left\{\sum_q iP(-q)\frac{2\pi l}{q} \rho(q)\right\}. \quad (39)$$

In the above and what follows $q$ sums stand for integrals

$$\sum_q = \int \frac{d^2q}{4\pi^2} \quad (40)$$

and the vector nature of $q$ is often suppressed. Under the action of $U$

$$U^\dagger a(q)U = a(q) - \frac{2\pi l \rho(q)}{q} \quad (41)$$

$$\psi_{CS}(x) = \psi_{CP}(x) \exp\left\{\sum_q iP(-q)\frac{2\pi l}{q} e^{-iqx}\right\} \quad (42)$$

$$\psi_{CS}^\dagger(-i\nabla)\psi_{CS}(x) = \psi_{CP}^\dagger(x)(-i\nabla + 2\pi l P(x)) \psi_{CP}(x) \quad (43)$$

$$H = \frac{1}{2m} \psi_{CP}^\dagger(-i\nabla + eA^\dagger + a + 2\pi l P + \delta a)^2 \psi_{CP} \quad (44)$$

$$0 = (a - \frac{2\pi l \rho}{q})|\text{physical}\rangle \quad 0 < q \leq Q \quad (45)$$
where $\delta a$ refers to the dependent short range vector potential $(\nabla \times)^{-1} 2\pi l \rho$ for $q > Q$ that did not get cancelled by the unitary transformation, and $\psi_{CP}$ refers to the composite particle (boson or fermion) field. The above description of the problem, after the canonical transformation by $U$, will be referred to as the middle representation (MR). We argue that $\Psi_{CP}$ better describes the composite bosons and fermions alluded to in the literature than does $\Psi_{CS}$. Whereas the latter are associated with particles carrying just flux tubes, the former are associated with particles that carry flux tubes and the correlation holes, i.e., describe electrons bound to zeros, as we shall see.

Suppose we have a physical wavefunction $\Psi^{MR}_{\varphi}(x, a)$ in the MR:

$$
\Psi^{MR}_{\varphi}(x, a) = \frac{\delta(a - \frac{2\pi l l}{q})}{\delta(0)} \Psi^{MR}(x, a)
$$

What is the corresponding CS wavefunction? Given that $U$ implements a translation of $a(q)$ by $\frac{2\pi l l}{q}$, it follows that

$$
\Psi^{MR}_{\varphi}(x, a) = \Psi^{MR}(x, a + \frac{2\pi l l}{q}) = \frac{\delta(a)}{\delta(0)} \Psi^{MR}(x, a + \frac{2\pi l l}{q}) = \frac{\delta(a)}{\delta(0)} \Psi^{MR}(x, a + \frac{2\pi l l}{q})
$$

from which follows, upon invoking Eqn. (38), the invaluable result

$$
\Psi^{CS}_{\varphi}(x) = \Psi^{MR}(x, \frac{2\pi l l}{q}).
$$

The hamiltonian in Eqn. (44)

$$
H = \frac{1}{2m} \psi_{CP}^{\dagger} (-i \nabla + e A^* + a + 2\pi l P + \delta a) \psi_{CP}
$$

has a local gauge invariance under time-independent transformations:

$$
\psi_{CP} \rightarrow e^{2\pi il \Lambda} \psi_{CP}
$$

$$
P \rightarrow P - \nabla \Lambda
$$

where $\Lambda$ has only Fourier modes with $q \leq Q$. The constraint merely states that physical states must be singlets of the generator of these transformations:

$$
\langle \frac{\nabla \times a}{2\pi l} - : \psi^{\dagger} \psi : | \text{physical} \rangle = 0.
$$

We may also write the above as

$$
\langle \frac{q a}{2\pi l} - : \rho(q) : | \text{physical} \rangle = 0 \quad 0 < q \leq Q.
$$

Had we chosen $Q$ to be infinite, $\delta a$ would have vanished and we would have entered what is called the Weyl or $a_0 = 0$ gauge. In the Bohm-Pines case, $Q$ is very small compared to $k_F$ and chosen as a variational parameter. In our problem the choice $Q = k_F$ recommends itself over all others repeatedly, as we shall see. The gauge we use is a nonstandard one, tailor made for this problem. Sometimes one is asked how a choice of gauge could matter, given that the physics is gauge invariant. This misses the point, which is that one can use the gauge
freedom to highlight some specific feature of the theory. Thus in Yang-Mills theories, the Coulomb and unitary gauges display the physical degrees of freedom, the Feynman-Lorentz gauges the Lorentz invariance, the $R_\xi$ gauge the renormalizability etc.

We now explore our hamiltonian Eqn.(44), expanding it as follows (and dropping the subscript CP):

$$
H = \frac{1}{2m} |(-i \nabla + eA^*)\psi|^2 + \frac{n}{2m}(a^2 + 4\pi^2 l^2 P^2) + (a + 2\pi l P) \cdot \frac{1}{2m} \psi^\dagger (i \leftrightarrow \nabla + eA^*) \psi + \frac{\delta a}{2m} \cdot \psi^\dagger (i \leftrightarrow \nabla + eA^*) \psi + \psi^\dagger \psi \frac{2m}{2m} (2(a + 2\pi l P) + \delta a) \cdot \delta a \equiv H_0 + H_I + H_{II} + H_{sr}
$$

in obvious notation, $H_{sr}$ being the terms associated with the nondynamical short-range gauge field $\delta a$. Note that we are yet to add interactions.

### IV. WAVEFUNCTIONS FOR GROUND STATES AND EXCITATIONS

In this section we will show how the above hamiltonian leads to some well known correlated wavefunctions upon making the simplest approximation. Let us first consider the fractions $\nu = 1/(2s + 1)$ and $1/2s$ for which $A^* = 0$. After this we will turn to Jain’s principal fractions $p/(2ps + 1)$. We will focus on the case $s = 1$ since larger values of $s$ do not tell us anything new.

#### A. $\nu = 1/3$

For this case we trade the fermions for bosons by attaching $l = 3$ flux quanta. *Let us focus our attention on just $H_0$ in Eqn.(54).* It reads

$$
H_0 = \frac{1}{2m} |(-i \nabla)\psi|^2 + \frac{n}{2m} (a^2 + (6\pi P)^2).
$$

There are no cross terms between $a$ and $P$ since they are transverse and longitudinal respectively. Given that $(a, P)$ are canonically conjugate, we see that the second term describes harmonic oscillators of cyclotron frequency

$$
\frac{6\pi n}{m} = \frac{eB}{m} = \omega_c.
$$

The oscillator hamiltonian may be written as

$$
H_{osc} = \sum_q A^\dagger(q)A(q) \omega_c
$$
where

\[ A(q) = \frac{(a(q) + 6\pi i P(q))}{\sqrt{12\pi}}. \] (60)

We are now ready to discuss the upper limit \( Q \) that controls the number of plasma oscillators. Given that there are only \( n \) electrons in a plane with a total of \( 2n \) degrees of freedom, the number of independent oscillators is bounded by \( 2n \). We choose the number of oscillators to be introduced to be equal to the number of electrons, i.e., \( Q = k_F \). There are many reasons for this choice, only one of which is intelligible at this point. Recall that we trying to describe LLL physics for the low energy sector along with high-energy cyclotronic physics. The oscillators clearly correspond to the latter. To pay for them, the particle sector has to give up \( n_0 = n \) degrees of freedom, which is precisely what it takes to project from the full Hilbert space to the LLL. Although mathematically any value of \( Q \) is allowed (being different choices of generalized gauges) our choice will prove the most suitable.

Our quest for the ground state of the full interacting problem can be described in two equivalent ways.

Firstly, we can try a variational approach using a simple product wavefunction in the enlarged Hilbert space, initially paying no attention to the constraint. The reason is the following. Suppose one were capable of finding the average of the Hamiltonian in arbitrarily complicated variational wave functions. Then the one with the lowest energy in the enlarged Hilbert space will be the true ground state. However, since the Hamiltonian is gauge invariant, we expect the ground state to be gauge invariant as well, since we know of no case where gauge invariance is spontaneously broken. Thus in the variational quest for the ground state, we can forget gauge invariance; the winner will have that feature. Of course, we can can only carry out the variational procedure for product states, so all the above considerations can be expected to hold approximately. We will use for this purpose the ground state of \( H_0 \) which we can write down by inspection: the composite bosons all condense into the zero momentum state with wavefunction \( \Psi_{CB}(x) = 1 \) and the oscillators occupy their ground states:

\[ \Psi_{osc} = \exp \left[ -\sum_q \frac{1}{12\pi} a^2(q) \right] \] (61)

giving us the MR wavefunction

\[ \Psi^{MR}(x, a) = \exp \left[ -\sum_q \frac{1}{12\pi} a^2(q) \right] \cdot 1 \] (62)

Not only does this minimize \( H_0 \), both \( H_I \) and \( H_{II} \) have zero average in this state: Thus this wave function minimizes \( H_0 + H_I + H_{II} \) among product wavefunctions. In addition, any density-density interaction (which will go as \( a^2q^2V(q) \) since \( \rho(q) \approx qa \) by the constraint) and \( H_{sr} \) will make some difference only at very short distances, where the wavefunction will be found to have excellent correlations that keep the particles apart.

Equivalently, we can think of the product wavefunction as a zeroth order starting point for a perturbative solution based on \( H_0 \). From this vantage, the approximation has a chance of being relevant only in the presence of electron-electron interactions: without them there
is no unique ground state in the exact solution and the product ground state of $H_0$ must necessarily get destabilized by the neglected terms. Thus interactions are needed for our approach to work, even though they are not being explicitly treated. Note also that $H_0$ does not correspond to free electrons; for that we need all of $H$ in Eqn. (58). By dropping all terms but $H_0$, we have gone from the free problem with degeneracies to one with a unique ground state.

There is just one problem: the wave function in Eqn. (62) is in the enlarged Hilbert space, whereas the physical wavefunction must be constrained to lie in the physical subspace. We implement the constraint by simply multiplying the above wavefunction by a delta function, i.e., by projection. As explained in the steps leading up to Eqn. (48), $\Psi_{CS}$ is obtained by setting $a(q) = 6\pi \rho(q)/q$ in $\Psi^{MR}(x,a)$:

$$\Psi_{CS} = \exp \left[ - \sum_q 3\pi : \rho(q) : \frac{1}{q^2} : \rho(-q) : \right]$$

(63)

$$= \exp \left[ \frac{3}{2} \int dx \int dy (\sum_i \delta(x - x_i) - n) \ln |x - y| (\sum_j \delta(x - x_j) - n) \right]$$

(64)

$$= \exp \left[ \frac{3}{2} (\sum_{i,j} \ln |x_i - x_j| - 2n \sum_i \int dx \ln |x - x_i| + \text{constants}) \right]$$

(65)

$$= \prod_{i<j} |z_i - z_j|^3 \exp \left[ - \sum_j |z_j|^2 / 4l_0^2 \right].$$

(66)

The steps connecting to the penultimate and ultimate lines are from Kane et al. The integral over $x$ in the penultimate line may be interpreted as the potential energy of a point charge at $x_i$, due to a uniform charge density $4\pi n$ and the two-dimensional Coulomb potential $V(|x - y|) = -\frac{1}{\pi} \ln |x - y|$. (To find the potential at $x$, it helps to first find the field by invoking the two-dimensional Gauss’s law $\oint E \cdot dr = \int d^2 x \rho(x)$ and then integrate.)

Putting back the phase factors from Eqn. (17), gives us Laughlin’s $\Psi_{1/3}$.

Let us recapitulate. We began by showing that given an exact gauge invariant eigenfunction of the enlarged Hamiltonian, we could get the physical one by dropping the delta function of the constraint that it must necessarily have as a prefactor (Eqns. (38) and (48)). In the end we found an approximate variational wavefunction $\Psi^{MR}(x,a)$ that one could argue had a good energy, but was not gauge invariant, i.e., did not vanish outside the constraint. We then projected it to the physical subspace by appending the delta function $\delta(\chi)$ ($\chi$ being the constraint) and evaluating the product wavefunction on the constrained subspace to get $\Psi_{CS}$. Could not multiplication by $\delta(\chi)$ ruin the good energetics? In general yes, but not here: given that $\Psi^{MR}(x,a)$ is an approximate eigenstate of the exact $H$, i.e., obeys the wave equation

$$H \Psi^{MR}(x,a) \simeq E \Psi^{MR}(x,a),$$

(67)

Due to the cut-off $Q$, the form of $\Psi$ we get is good only for $|z_i - z_j| >> 1/Q$. To continue this form all the way in to get a triple zero, we must invoke the LLL condition.
we can premultiply both sides by $\delta(\chi(q))$, *commute it through $H$* to verify that $\delta(\chi(q))\Psi_{MR}(x,a)$ obeys the same equation.

Our trick for extracting the ground state can be extended to find a few other states as follows. Consider the sector in which there is a vortex at the origin. Among all such electronic wavefunctions there must be one with lowest energy, the ground state in this sector. Let us then write electronic wavefunction as

$$\Psi_{vortex}^e = \prod_j e^{i\sum_j \theta_j} \Psi'_e$$

and subject $\Psi'_e$, which is free of this vortex, to the previous CS transformation. Instead of Eqn.(30) we will end up with

$$H = \frac{1}{2m}\psi^\dagger(-i\nabla + eA^* + a + (\nabla \times)^{-1}6\pi\rho + \frac{e\theta}{r})^2\psi$$

the last term in brackets being due to the vortex prefactor.

The unitary transformation, which must now get rid of the last two terms in the bracket, is

$$U = \exp\left[\frac{Q}{q} \sum_i P(-q)\left(\frac{6\pi}{q}\rho(q) + \frac{2\pi}{q}\right)\right].$$

The bosonic wavefunction will again be unity since the external flux is zero on average, including the point flux tube at the origin. The oscillator wavefunction will still be the gaussian,

$$\Psi_{osc} = \exp\left[-\sum_q \frac{1}{12\pi}a^2(q)\right]$$

but the constraint will now be modified to

$$a(q) = \frac{6\pi\rho(q)}{q} + \frac{2\pi}{q}.$$ 

because of the modified $U$.

It is clear that we have now placed a phantom charge of size $1/3$ at the origin. The plasma will screen it and produce a correlation hole of charge $−1/3$. This is verified by evaluating the corresponding wavefunction using the same procedure as before to obtain

$$\Psi'_{CS}(x,\eta) = \prod_j |z_j - \eta| e^{-|\eta|^2/4t_0^2} \prod_{i<j} |z_i - z_j|^3 \exp\left[-\sum_j |z_j|^2/4t_0^2\right]$$

where we have displayed the answer for the case in which the vortex is at $\eta$ rather than the origin, to show how even the gaussian factor associated with its location appears naturally. Note that the magnetic length associated with it obeys $t_0^2 = 3l_0^2$, appropriate that of a charge $−1/3$ object. Going back to the electronic wavefunction, $|z_j - \eta|$ will become $z_j - \eta$ giving us Laughlin’s quasihole state, again with the understanding that we continue down to $z - \eta = 0$ using the LLL condition.
To create two quasiholes, at $\eta_1$ and $\eta_2$, we do the obvious extension of the preceding and find
\[
\Psi'_CS(x, \eta_1, \eta_2) = |\eta_1 - \eta_2|^{1/3} e^{-(|\eta_1|^2 + |\eta_2|^2)/4l_0^2} \prod_j |z_j - \eta_1| \prod_j |z_j - \eta_2| \prod_{i<j} |z_i - z_j|^3 \exp \left[ - \sum_j |z_j|^2/4l_0^2 \right].
\] (74)

Observe that the wavefunction contains the normalization factor $|\eta_1 - \eta_2|^{1/3}$ of the two-quasihole state. If we follow Halperin and drop the mod sign (by a singular gauge transformation) we can get a pseudo wavefunction for the quasiholes and infer their fractional statistics.

It is possible to assign operators that create the quasiholes in the following sense. Consider the action of the operator
\[
\psi(0) = \exp \left[ \sum_q \frac{2\pi i}{q} P(-q) \right].
\] (75)
on the unprojected ground state which we denote by
\[
|Osc = 0, \Psi_{CB} = 1\rangle.
\] (76)
Its action is one of translation in $a$:
\[
e^{-a^2(q)/12\pi} \rightarrow e^{-[a(q) + \frac{2\pi}{q}]^2/12\pi}
\] (77)
If we now obtain the projection, $\wp\psi(\eta)|Osc = 0, \Psi_{CB} = 1\rangle$: we end up with
\[
\Psi'_CS(x, 0) = \prod_j |z_j| \prod_j |z_j - z_j|^3 \exp \left[ - \sum_j |z_j|^2/4l_0^2 \right].
\] (78)

It follows that the operator which creates a quasihole at the point $\eta$ is
\[
\psi_{qh}(\eta) = e^{i \sum_j \theta_{j, \eta}} \exp \left[ \sum_q \frac{2\pi i}{q} P(-q)e^{i\eta q} \right].
\] (79)
where $\theta_{j, \eta}$ is the angle between the x-axis and the vector connecting $\eta$ to $z_j$.

Note that the projection is done only at the very end. In other words, to create two quasiholes, we need
\[
\wp\psi_{qh}(\eta_2) \psi_{qh}(\eta_1) |Osc = 0, \Psi_{CB} = 1\rangle
\] (80)
and not
\[
\wp\psi_{qh}(\eta_2) \wp\psi_{qh}(\eta_1) |Osc = 0, \Psi_{CB} = 1\rangle
\] (81)
In fact $\wp\psi_{qh}\wp\psi$ has zero matrix elements because $\psi_{qh}$ is not gauge invariant.

The adjoint operator $\psi_{qh}(\eta)^\dagger$ will append to the ground state wavefunction the factor $\prod_j (z_j - \eta)^{-1}$ which will indeed have a charge $1/3$. However the resulting wavefunction does not belong to the LLL since negative powers of $z$ are not allowed. Thus our formalism does not solve this problem with quasiparticles, first encountered by Laughlin.
Let us summarize what has just been done. In the CS representation, the hamiltonian
did not contain any spurious degrees of freedom, but was very nonlinear due to the $(\nabla \times)^{-1}$
terms that corresponded to the nondynamical field $a$. We moved to the MR in which
the oscillator degrees of freedom were introduced. The hamiltonian, which now contained
new canonical degrees of freedom, but no long-distance $(q < Q)$ nondynamical fields, had
invariance under time-independent gauge transformations. Physical states had to be gauge
singlets to compensate for the new degrees of freedom. We found an approximate variational
product ground state $|\text{Osc}=0, \Psi_{CB}=1\rangle$, which was the ground state of $H_0$, a nongauge
invariant part of $H$ with a unique ground state that could be read off by inspection. We then
projected this ground state using $\wp$ to get Laughlin’s wavefunction $\Psi_{1/3}/3$(x). This procedure
succeeds only because of particle-particle interactions which render the true ground state
unique and nondegenerate. Continuing, we saw that $\psi_{qh}$ acting on $|\text{Osc}=0, \Psi_{CB}=1\rangle$
produced states, which, upon projection, led to normalized quasiparticle states. It appears
that there is a dictionary between the eigenstates of $H_0$ and the full interacting hamiltonian:

$$|\text{Osc}=0, \Psi_{CB}=1\rangle \xrightarrow{\wp} \Psi_{1/3}(x)$$

$\psi_{qh}(\eta_k) \cdots \psi_{qh}(\eta_1)|\text{Osc}=0, \Psi_{CB}=1\rangle \xrightarrow{\wp} \Psi_{1/3, qh}(x, \eta_1, \cdots \eta_k)$

All these states may be viewed as ground states in sectors with a given number of vortices.
We do not know if the correspondence carries over to excited states.

We can now understand how we have managed to get the phase and modulus of $\Psi_{1/3}$ in the
middle representation. Recall Eqn.(42)

$$\psi^\dagger_{CB}(x) = \psi^\dagger_{CS}(x) \exp[\sum_q iP(-q)\frac{6\pi}{q}e^{-iqx}].$$

Thus $\psi_{CB}$ creates a CS boson and introduces a triple quasihole of charge $-1$, thereby
creating the extra particle along with its correlation hole. Starting from Halperin’s original
observation, all physical pictures, particularly Read’s22,20, have emphasized that electrons
bind to zeros of the wavefunction, not just the vortices in the phase, i.e, the flux tubes. In fact
only a zero with an accompanying charge deficit will attract an electron. This is the reason
we prefer to use the term Composite Boson for the particle created by $\psi^\dagger_{CB}$ rather than by
$\psi^\dagger_{CS}$. The first factor of $\psi^\dagger_{CB}$carries the flux, i.e., the phase, and the second, the magnitude,
i.e., the correlation zeros, of $\Psi_{1/3}$. (The latter has the effect of introducing the hole only
upon projection.) Crucial to the success of all of this is the assignment of independent
degrees of freedom to collective coordinates, rather than trying to express them in terms of
particle coordinates. It is only after $a(q)$ became a canonical coordinate whose conjugate
momentum $P$ could be used to translate it, that we could so readily produce the correlation
hole.

The preceding arguments suggest that $\psi_{CB}$ is a candidate for being Read’s operator. Let recall his reasoning for the choice of condensate. If we take a Laughlin state with $N+1$
electrons and kill one at $z$, it looks like an $N$ particle state with a unit charge correlation
hole there. Schematically

$$\langle N+1|\psi^\dagger_{\text{electron}}(z)U(z)^3|N\rangle \neq 0$$
where $U(z)$ in first quantization appends a factor $\prod_i (z - z_i)$, i.e., creates a quasihole at $z$. Thus $\psi_{\text{electron}}^\dagger(z)U(z)^3$ condenses. Given the phase relationship between $\psi_{\text{electron}}$ and $\psi_{CS}$, this means $\psi_{CS}^\dagger(z)|U(z)|^3$ condenses. Comparing to Eqn. (84) suggests $\psi_{CB}$ is the Read operator. There is, however, the caveat that it acts in the enlarged Hilbert space rather than the physical subspace. While it is true that

$$\varphi \exp \left[ \sum_q iP(-q) \frac{6\pi}{q} e^{-iqz} \right] |\text{Osc} = 0, \Psi_{CB} = 1\rangle = \prod_j |z - z_j|^3 \psi_{1/3} = |U(z)|^3 \psi_{1/3}$$

it is not true that $\exp[\sum_q iP(-q) \frac{6\pi}{q} e^{-iqz}]$ acting directly on the physical state $\psi_{1/3}$ produces the correlation hole. Thus it is the preimage of $|U|^3$ (in the big space, before projection), but not an operator which acting on $\psi_{1/3}$ produces this factor.

There is another related way to say this. As far as $H_0$, which is a nongauge hamiltonian goes, it is indeed true that

$$\langle \text{Osc} = 0, \Psi_{CB} = 1 | \psi_{CB}^\dagger(\eta_2) \psi_{CB}(\eta_1) | \text{Osc} = 0, \Psi_{CB} = 1 \rangle \to 1 \text{ as } |\eta_1 - \eta_2| \to \infty. \quad (87)$$

On the other hand, within the gauge theory defined by $H$, $\psi_{CB}$ is a gauge dependent operator which cannot have a mean value in the true gauge invariant ground state. (Introducing the line integral between the points $\eta_1$ and $\eta_2$ does not help; it makes it the correlator of $\psi_{CS}$, which has the algebraic order found by Girvin and MacDonald.)

### B. $\nu = \frac{1}{2}$

For $\nu = \frac{1}{2}$ we choose $l = 2$ flux quanta, cancel the external field, and end up with fermions in zero field on the average. An analysis of $H_0$ as in the previous subsection will give us (upon putting back the phase factors of the CS transformation)

$$\psi_{1/2} = \prod_{i<j}(z_i - z_j)^2 \exp \left[ - \sum_j |z_j|^2 / 4 l_0^2 \right] \cdot |FS\rangle \quad (88)$$

which differs from the Rezayi-Read wavefunction by the absence of projection to the LLL. This issue will be addressed in the next section.

### C. The Jain series $\nu = \frac{p}{2p+1}$

In this case we we follow Jain and attach two units of flux per particle to reduce $eA$ down to $eA^*$. Since $A$ corresponded to $1/\nu$ flux quanta per particle and $A^*$ to $1/p$ quanta per particle

$$\frac{A^*}{A} = \frac{1}{2p+1}. \quad (89)$$

Now $H_0$ takes the form

$$H_0 = \frac{1}{2m} |(-i \nabla + eA^*)\psi|^2 + \frac{n}{2m} (a^2 + (4\pi P)^2). \quad (90)$$
First consider the oscillators. They are not at the right (cyclotron) frequency but at
\[ \omega_0 = \frac{4\pi n}{m} = \frac{2p}{2p+1} \omega_c = 2\nu \omega_c. \]  (91)
Nonetheless they lead to good wavefunctions. Their ground state wavefunction, upon projection, gives a factor
\[ \prod_{i<j} |z_i - z_j|^2 \exp \left[ - \sum_j \frac{2\nu |z_j|^2}{4l_0^2} \right]. \]  (92)
The derivation proceeds exactly as in the case of \( \nu = 1/2 \), except that we replace \( 4\pi n \) by \( 2\nu/l_0^2 \).

The particles fill \( p \) Landau levels. Let us write their wavefunction as
\[ \Psi_{CF} = \chi_p = \exp \left[ - \sum_j \frac{|z_j|^2}{4l_0^2} \right] f_p(z, z^*) \]  (93)
where \( f \) is a polynomial and \( l_0^2 = (2p+1)l_0^2 \). This leads to a wavefunction
\[ \Psi_{CS} = \prod_{i<j} |z_i - z_j|^2 \exp \left[ - \sum_j \frac{|z_j|^2}{4l_0^2} \right] f_p(z, z^*). \]  (94)

It is very gratifying that the two gaussian factors combine very naturally to give the right magnetic length because of the relation
\[ 2\nu + \frac{1}{2p+1} = 1. \]  (95)

Note that there is no further projection to the LLL. (Jain has pointed out that even without the projection, the wavefunction is primarily in the LLL in the sense that the kinetic energy per particle is only slightly higher than \( \hbar \omega_c \).) In any event we consider the question of projection to the LLL next.

V. TRANSFORMATION TO THE FINAL REPRESENTATION

The MR has yielded many wavefunctions with good correlations. There are, however, some weaknesses. First of all, the particles have too much kinetic energy: in the absence of Coulomb interactions, they should have no kinetic energy: \( 1/m^* = 0 \). There is as yet no evidence of this mass renormalization. The oscillators, on the other hand, have the right frequency only when \( A^* = 0 \), i.e., for \( \nu = 1/(2s+1), 1/2 \). (Despite all this, the wavefunctions are good because the particle mass and oscillator frequency do not enter them, only the magnetic length does. As a result, as long as there is some nonzero \( 1/m^* \) in the end, these wavefunctions remain good. In the absence of interactions, \( 1/m^* \) indeed vanishes and the nondegenerate ground state of \( H_0 \) becomes irrelevant. However, with interactions, it can be a good approximation to the true ground state).
Another problem has to do with the charge of the particles. The composite fermions are supposed to have a charge 
\[ e^* = e/(2p+1) \] for \( \nu = p/(2p+1) \). One way to obtain this result is to imagine injecting a composite fermion by adding an electron and adiabatically injecting the two quanta of flux. During the injection, there is tangential electric field. While the physics may be complicated near the fermion, on a circle of large radius the effect of this field is to produce a radial current determined by the bulk Hall conductance. The integrated charge outflow to infinity or to the edge, i.e., the charge of the correlation hole, is readily found to be \(-2\nu e\). Adding to the charge of the injected electron, the composite fermion charge becomes

\[ e^* = e(1 - \frac{2p}{2p+1}) = \frac{e}{2p+1}. \]

The same argument predicts that the composite boson in \( \nu = 1/3 \) is neutral since in that case we add three flux quanta and 3\( \nu \) electronic units are driven to infinity.

For \( \nu = 1/2 \) these arguments imply neutral composite fermions. Read\textsuperscript{22,23} gives more specific details based on the analysis of

\[ \psi_{RR} = \mathcal{P} \prod_{i<j} (z_i - z_j)^2 \exp \left[ -\sum_j |z_j|^2/4l_0^2 \right] \cdot |FS\rangle. \]

where \( |FS\rangle \) denotes the Fermi sea. Without the \( |FS\rangle \) factor, each electron sits in the middle of the double correlation hole and the complex is neutral. The role of the Slater determinant is to produce the antisymmetry. It contains factors of the form \( e^{i\mathbf{k} \cdot \mathbf{r}} = e^{i(kz^* + k^*z)/2} \). Since \( z^* \) acts like \( 2l_0^2 \partial/\partial z \) upon projection by \( \mathcal{P}\textsuperscript{20} \), its action on the Jastrow factor is to split the double zero and move the electron off the center of the correlation hole. The shift in \( z \) is \( ikl_0^2 \) and the dipole moment associated with a fermion of momentum \( k \) is \( el_0^2 \hat{z} \times k \). Note that projection to the LLL is essential to the above picture.

The reason all these features are not evident in the MR is that the particles are coupled to the oscillators and neither is a true quasiparticle with its ultimate (or nearly ultimate) mass, charge etc. One way to deal with this is to integrate out one in favor of the other. We take the more symmetric route of eliminating the coupling between them. This will be done within the following approximation scheme:

- We will work at long distances. Thus if any quantity has an expansion in powers of \( q \) we will keep just the leading term. This involves the neglect of derivative couplings.

- When we encounter the density operator in a product with other operators, we will use the RPA:

\[ \sum_j e^{i(q-k)x_j} \simeq n(2\pi)^2\delta^2(q-k). \]

Thus we will ignore \( H_{sr} \), the short range piece due to \( \delta a \), and the non-RPA term \( H_{II} \).

The analysis of \( \nu = \frac{p}{2p+1} \) will be carried out in first quantization. The particles are fermions.
The Hamiltonian $H_0 + H_I$ reads

$$H = \sum_j \frac{\Pi_j^2}{2m} + \sum_q A^\dagger(q)A(q)\omega_0 + \theta\omega_0 \sum_q \left[ c^\dagger(q)A(q) + A^\dagger(q)c(q) \right] \equiv T + H^{osc} + H_I$$

where

$$\Pi = p + eA^*$$

(100)

$$\nabla \times eA^* = -eB^* = \frac{eB}{2p + 1}$$

(101)

$$A(q) = \frac{1}{\sqrt{8\pi}}[a(q) + 4\pi iP(q)]$$

(102)

$$\theta = \sqrt{\frac{2\pi}{4\pi \hbar}}$$

(103)

$$\omega_0 = \frac{4\pi \hbar}{m} = \frac{2p}{2p + 1} \omega_c$$

(104)

$$c(q) = \hat{q} - \sum_j \Pi_j e^{-iqx_j}$$

(105)

$$V_\pm = V_x \pm iV_y$$

(106)

$$[A(q), A^\dagger(q')] = (2\pi)^2\delta^2(q - q')$$

(107)

It is useful to know that

$$[\Pi_-, \Pi_+] = -2eB^*,$$

(108)

and that in our approximation

$$[c(q), c^\dagger(q')] = \hat{q} \hat{q}' \sum_j \left[ \Pi_+^j, \Pi_-^j \right] e^{-i(q - q')x_j} + O(q)$$

(109)

$$\approx 2eB^* n(2\pi)^2\delta^2(q - q').$$

(110)

The other commutators vanish to this order in $q$:

$$[c(q), c(q')] = [c^\dagger(q), c^\dagger(q')] = O(q, q').$$

(111)

The coupling $H_I$ will be eliminated by a canonical transformation

$$U(\lambda_0) = e^{iS_0\lambda_0} = \exp \left[ \lambda_0 \theta \sum_q \left( c^\dagger(q)A(q) - A^\dagger(q)c(q) \right) \right]$$

(112)

where $\lambda_0$ is to be chosen appropriately. The operators of the MR, called $\Omega^{old}$, will be related to those of the FR (with no superscripts) by

$$\Omega^{old} = e^{-iS_0\lambda_0}\Omega e^{iS_0\lambda_0}.$$  

(113)

We will also define

$$\Omega(\lambda) = e^{-iS_0\lambda}\Omega e^{iS_0\lambda}$$

(114)
so that
\[ \Omega(\lambda_0) = \Omega^{\text{old}} \quad \Omega(0) = \Omega \] (115)

We start with the flow equations
\[
\frac{dA(q, \lambda)}{d\lambda} = -\theta c(q, \lambda) \tag{116}
\]
\[
\frac{dc(q, \lambda)}{d\lambda} = 2eB^*n\theta A(q, \lambda). \tag{117}
\]

It is understood that \(0 < q \leq Q\) in the above and what follows. The flow equations have the solution
\[
A(q, \lambda) = \cos \mu \lambda A(q) - \frac{\theta}{\mu} \sin \mu \lambda c(q) \tag{118}
\]
\[
c(q, \lambda) = \cos \mu \lambda c(q) + \frac{\mu}{\theta} \sin \mu \lambda A(q) \tag{119}
\]
\[
\mu^2 = 2eB^*n\theta^2 = \frac{1}{2p}. \tag{120}
\]

Consider the kinetic energy, which may be written, using Eqn.(108) as
\[
T = \sum_j \Pi_j^j \Pi_j^J + \sum_j \frac{eB^*}{2m} \tag{121}
\]

The second term, which does not evolve to the order we are working in, will turn out to describe the magnetic moment coupling of SSH\[\text{[27]}\]. The first evolves as per
\[
\frac{dT}{d\lambda} = \frac{eB^*\theta}{m} \sum_q \left( A(q, \lambda)c(q, \lambda) + c^\dagger(q, \lambda)A(q, \lambda) \right) \tag{122}
\]

so that
\[
T(\lambda) = T + \frac{eB^*\theta}{m} \sum_q \left[ \sin 2\lambda\mu \left( c(q)A(q) + A^\dagger(q)c(q) \right) + \frac{1 - \cos 2\mu \lambda}{2\mu} \left( \frac{\mu}{\theta} A^\dagger(q)A(q) - \frac{\theta}{\mu} c^\dagger(q)c(q) \right) \right] \tag{123}
\]

The oscillator hamiltonian and \(H_I\) assume the following form in terms of FR operators:
\[
H_{\text{osc}} = \omega_0 \sum_q \left( A^\dagger(q)A(q) \cos^2 \mu \lambda - (A^\dagger(q)c(q) + c^\dagger(q)A(q)) \frac{\theta}{2\mu} \sin 2\mu \lambda + \frac{\theta^2}{\mu^2} \sin^2 \mu \lambda c^\dagger(q)c(q) \right) \tag{124}
\]

and
\[
H_I = \theta \omega_0 \sum_q \left[ \cos 2\mu \lambda (c^\dagger(q)A(q) + A^\dagger(q)c(q)) + \sin 2\mu \lambda \left( \frac{\mu}{\theta} A^\dagger(q)A(q) - \frac{\theta}{\mu} c^\dagger(q)c(q) \right) \right]. \tag{125}
\]

We will now choose \(\lambda = \lambda_0\) such that cross terms between the particles and oscillators vanish. Gathering the coefficients of \(A^\dagger(q) c(q)\) from \(T, H_{\text{osc}}\) and \(H_I\) we find that they add to zero if
\[
\tan \lambda_0 \mu = \mu = \frac{1}{\sqrt{2p}} \tag{126}
\]
This completely fixes the canonical transformation.

We now ask what the frequency of the oscillators is and find that the coefficient of the $A^\dagger A$ term is exactly $\omega_c = eB/m!$! This agreement with Kohn’s theorem confirms the soundness of our decoupling transformation whose only free parameter has already been chosen.

What about the particles? The $c^\dagger c$ term has coefficient $-\frac{1}{2mn}$. Thus the total particle kinetic energy is

$$T = \sum_j \frac{\Pi^j_+ \Pi^j_+}{2m} + \sum_j \frac{eB^*}{2m} - \frac{1}{2mn} \sum_i \sum_j \sum_q \frac{Q}{q} \Pi^i_+ e^{-iq(x_i-x_j)} \Pi^j_+.$$  \hspace{1cm} (127)

We also need to transform the constraint

$$\rho^{\text{old}}(q) = \frac{qa^{\text{old}}(q)}{4\pi} \hspace{1cm} (128)$$

on physical states by transforming both sides of the equation. The flow equation

$$\frac{d\rho(q, \lambda)}{d\lambda} = \frac{q}{\sqrt{8\pi}} (A(q, \lambda) + A^\dagger(-q, \lambda)) \hspace{1cm} (129)$$

can be integrated to give

$$\rho^{\text{old}}(q) = \rho(q) + \frac{q}{\sqrt{8\pi}} \left( \frac{\sin \mu \lambda_0}{\mu} (A(q) + A^\dagger(-q)) - \frac{\theta}{\mu^2} (1 - \cos \mu \lambda_0) (c(q) + c^\dagger(-q)) \right) \hspace{1cm} (130)$$

while previous results for $A$ tell us that

$$\frac{qa^{\text{old}}(q)}{4\pi} = \frac{q}{\sqrt{8\pi}} \left[ \cos \mu \lambda_0 (A(q) + A^\dagger(-q)) - \frac{\theta}{\mu} \sin \mu \lambda_0 (c(q) + c^\dagger(-q)) \right]. \hspace{1cm} (131)$$

Remarkably, the terms involving the oscillators match on both sides upon using $\tan \mu \lambda_0 = \mu$ and the constraint involves only the particles:

$$\rho(q) = -\frac{i l_0^2}{\cos \mu \lambda_0 \sin^2 \mu \lambda_0} \sum_j (q \times \Pi_j) e^{-iqx_j} = -\frac{i l_0^2}{\cos \mu \lambda_0 (1 + \cos \mu \lambda_0)} \sum_j (q \times \Pi_j) e^{-iqx_j} \hspace{1cm} (132)$$

This is very fortunate since it is no use decoupling the particles in the hamiltonian if the constraint still couples them.

Now we calculate the transformation of the current

$$J_+^{\text{old}}(q) = \sum_j \left[ \frac{\Pi_j}{m} e^{-iqx_j} + \frac{n}{m} \sqrt{8\pi} \hat{q}_+ A(q) \right]^{\text{old}} \hspace{1cm} (133)$$

to the FR. (Note that the non-RPA part of it has been dropped). We find

$$J_+(q) = \frac{\hat{q}_+ \omega_0}{\sqrt{2\pi} \cos \mu \lambda_0} A(q) = \frac{\hat{q}_+ \omega_c \cos \mu \lambda_0}{\sqrt{2\pi}} A(q). \hspace{1cm} (134)$$

Remarkably the entire contribution comes from the oscillators; the part proportional to $\Pi$ just cancels out.
To summarize, in the FR, for the fraction \( \nu = p/(2p + 1) \)

\[
H = \sum_j \frac{\Pi^j_\perp}{2m} + \sum_j \frac{eB^*}{2m} \sum_i \sum_q \frac{Q}{q} \Pi^i_\perp e^{-iq(x_i - x_j)} + \sum_q A(q) A(q) \omega_c (135)
\]

\[
\chi(q) = \rho(q) + \frac{i l_0^2}{\cos \mu \lambda_0 (1 + \cos \mu \lambda_0)} \sum_j (q \times \Pi_j) e^{-iqx_j} = 0 \quad \text{(constraint)} \quad (136)
\]

\[
J_{\alpha}^{old}(q) = \frac{\hat{q}_\alpha \omega_0}{\sqrt{2\pi} \cos \mu \lambda_0} A(q) = (J_{\alpha}^{old}(-q))^\dagger \quad (137)
\]

\[
\rho^{old}(q) = \frac{q \cos \mu \lambda_0}{\sqrt{8\pi}} (A(q) + A^\dagger(-q)) + \rho(q) \quad (138)
\]

\[\tan \mu \lambda_0 = \frac{1}{\sqrt{2p}} = \mu \quad (139)\]

where once again we are assuming \( 0 < q \leq Q \). Outside this region there is no difference between old and new variables. Note that the transformations do not require that \( Q \) have any particular value.

Consider Eqn. (135) for the hamiltonian, focusing on the particle sector. We see that decoupling the oscillator has lead to the third term. In it is a one-particle piece corresponding to \( i = j \), those with \( i \neq j \) being additional interactions. When we combine this \( i = j \) term with the first term, we see the mass gets renormalized to:

\[
\frac{1}{m^*} = \frac{1}{m} (1 - \frac{1}{n} \sum_q). \quad (140)
\]

As the upper limit \( Q \) grows, the particle mass increases and the kinetic energy gets quenched. In the noninteracting case we are studying, the kinetic energy must be fully extinguished. *Note that our choice \( Q = k_F \), i.e., \( n_{osc} = n \) accomplishes this exactly.*

In addition, this is the right number of constraints, all in the particle sector, to restrict it to the LLL. A smaller choice of \( Q \) gives the particles kinetic energy of order \( 1/m \) and also does not impose enough constraints to project to the LLL. One is free to use a smaller value for \( Q \), but one will have a harder time going from such a hamiltonian to the correct behavior for the noninteracting case. A larger value \( Q > k_F \) leads to a negative \( 1/m^* \), clearly a bad starting point when we go on to introduce interactions. Note that the kinetic energy of the fermions is quenched independent of their momentum. We do not know what set of diagrams, if any, corresponds to carrying out this canonical transformation. Recently Chari, Haldane and Yang worked on a gauge invariant approximation scheme for mass renormalization in a theory with flux tubes of size \( 1/\Lambda \). Unfortunately, to leading order in \( \Lambda/k_F \), the mass shift went the wrong way. Presumably higher order terms (which are surely needed since the point flux limit is \( \Lambda/k_F \rightarrow \infty \)) will fix this.

Once the kinetic energy is quenched all that is left of the \( i = j \) term is

\[
\sum_j \frac{eB^*(r_j)}{2m}. \quad (141)
\]
where we have emphasized that $B^*$ (a constant so far) is to be evaluated at $r_j$ as per our calculation. Suppose we change the external field by $\delta A$. Repeating the analysis with this field will lead back to the above formula, but with an extra piece $\frac{e\delta B}{2m}$ per particle. This result also follows if $\delta B$ varies very very slowly in space, say between one galaxy and the next. This is exactly the coupling of a magnetic moment $e/2m$, mandated by SSH.

We did not have to attach it by hand, it emerged naturally. Note that the $m$ in this term did not get renormalized. We could also write this term as follows

$$\sum_j \frac{e\delta B(r_j)}{2m} = \int d^2 x \frac{e}{2m} \rho(x) \nabla \delta A = -\int d^2 x \delta A \cdot j_{mag}$$

where

$$j_{mag} = \frac{e}{2m} \hat{z} \times \nabla n$$

is the current associated with uncancelled cyclotronic currents. It follows that if we add such a $\delta A$, it will lead to a charge density $\delta \rho = K_{00} \frac{e\delta B}{2m}$. Taking the ratio of the applied vector potential to the induced $\delta \rho$, we obtain

$$K_{01} = iq \frac{e}{2m} K_{00}.$$  

Note that we did not calculate $K_{00}$, but merely showed that $K_{01}$ has a piece proportional to it with a factor $\frac{iqe}{2m}$. Let us now turn to the $i \neq j$ pieces in Eqn.(133). We treat the $q$ sum as follows:

$$\sum_{q>Q} \exp -iq(r_i - r_j) = \delta^2(r_i - r_j) - \sum_{q>Q} \exp -iq(r_i - r_j)$$

The delta function vanishes on spinless fermion wavefunctions or on hard-core boson wavefunctions, and can safely be dropped. (Note that when $H$ acts on a wavefunction to its right, we can move this delta function through the $\Pi_j$ to get the desired zero since the exchange produces corrections of higher order in $q$.) We have thus reduced the $i \neq j$ pieces to a short-range interaction to be lumped with (the RPA part of) $H_{\delta a}$, the contribution from the short range CS field $\delta a$ that was not made dynamical. What is their collective role? We know that they cannot feed back to our zeroth order hamiltonian since a vanishing kinetic energy for LLL fermions is an exact result in the noninteracting theory. On the other hand, in the case of $\nu = 1/2$, we can show that these terms affect the large $q$ sector in the following very desirable way. In RPA, the large $q$ fields (including the term we shifted from small to large $q$) conspire to produce the plasmon pole with the right position and residue in the density-density correlation, as shown in Appendix A. The issue is the following: There are no oscillators for $q > Q$. The fermion mass has been renormalized to $m^*$, but they still have to produce the cyclotron pole at $eB/m$ at large $q$ via a collective mode. One therefore needs a Landau parameter $f_1$ to restore the correct cyclotron pole. The $i \neq j$ interactions (when shifted to large $q$) fulfill exactly that role. It is very satisfying that although our oscillators no longer exist for $q > Q$, the physics (plasmon pole and residue to order $q^2$) is continuous across this border.
A. Electronic charge density

Consider next Eqn. (138) for the charge density

$$\rho_{\text{old}}(q) = \frac{q \cos \mu \lambda_0}{\sqrt{8\pi}} (A(q) + A^\dagger(-q)) + \rho(q) - \frac{i l_0^2}{1 + \cos \mu \lambda_0} \left( \sum_j (q \times \Pi_j) e^{-iqx_j} \right)$$  \hspace{1cm} (146)

and the operator that is equivalent in the physical subspace

$$\frac{q a_{\text{old}}(q)}{4\pi} = \frac{q \cos \mu \lambda_0}{\sqrt{8\pi}} \left[ (A(q) + A^\dagger(-q)) \right] - \frac{i l_0^2}{\cos \mu \lambda_0} \left( \sum_j (q \times \Pi_j) e^{-iqx_j} \right).$$  \hspace{1cm} (147)

In the MR, any combination of the form

$$\gamma \rho_{\text{old}}(q) + (1 - \gamma) \frac{q a_{\text{old}}(q)}{4\pi}$$  \hspace{1cm} (148)

is an acceptable definition of the physical charge density. (For example, HLR use this freedom to write the Coulomb interaction entirely in terms of \(a\).) We could canonically transform any such combination to represent charge density in the restricted physical space. In an exact calculation there will be nothing to choose between them. In our approximation scheme, however, the following combination stands out:

$$\rho_{\text{preferred}} = \frac{1}{2p + 1} \rho_{\text{old}}(q) + \frac{2p}{2p + 1} \frac{q a_{\text{old}}(q)}{4\pi}$$  \hspace{1cm} (149)

$$= \frac{q}{\sqrt{8\pi}} \cos \mu \lambda_0 (A(q) + A^\dagger(-q)) + \frac{\rho(q)}{2p + 1} - \frac{i l_0^2}{4} \left( \sum_j (q \times \Pi_j) e^{-iqx_j} \right).$$  \hspace{1cm} (150)

To see what is special about it, consider \(\bar{\rho}\), its projection to the LLL, which clearly corresponds to the ground state of the oscillators, and is obtained by dropping the terms linear in \(A\) and \(A^\dagger\):

$$\bar{\rho}(q) = \frac{\rho(q)}{2p + 1} - i l_0^2 \left( \sum_j (q \times \Pi_j) e^{-iqx_j} \right).$$  \hspace{1cm} (151)

Now we know from the work of Girvin and Jack and GMP that the projected density must obey the algebra of magnetic translations:

$$[\bar{\rho}(q), \bar{\rho}(q')] = i l_0^2 \left( q \times q' \right) \bar{\rho}(q + q').$$  \hspace{1cm} (152)

in the small \(q\) truncation of the structure constant, a truncation which satisfies the Jacobi identity. Our \(\bar{\rho}\) obeys this algebra. There are two points that should be noted in this connection.

- The magnetic algebra is expected upon projection to the LLL only in the original electronic problem, i.e., between physical states. In evaluating a product or commutator, we must therefore restrict intermediate states to be physical. The exceptions are operators that do not mix physical and unphysical states, i.e., gauge invariant ones. These can be multiplied freely in the full space. Since we computed the \(\bar{\rho}\) commutator with no restriction on intermediate states, we expect that \(\bar{\rho}\) is gauge invariant and that we can freely multiply our \(\bar{\rho}\) of Eqn. (150) in the enlarged space without worrying about constraints.
The algebra contains terms up to cubic order in \( q \) in the right hand side. These can arise from the terms we have kept and terms we have neglected. The truncation we have chosen has the virtue that the algebra is satisfied with the terms we have, with no need to appeal to higher order terms. Presumably the leading higher order terms cancel out in this preferred combination.

For these reasons, we shall use this \( \bar{\rho}(q) \) and the corresponding \( \rho^{\text{old}} \):

\[
\rho^{\text{old}} = \frac{q \cos \mu \lambda_0}{\sqrt{8\pi}} (A(q) + A^\dagger(-q)) + \frac{1}{2p+1} \sum_j e^{-iqx_j} - il_0^2 \sum_j (q \times \Pi_j) e^{-iqx_j}
\]  

(153)

when we need a formula for the electronic charge density. Since the particle kinetic energy has vanished, the entire hamiltonian will be given by the Coulomb interaction written in terms of the above density. It will evidently be gauge invariant.

Our formula for the charge tells us that the CF has charge \( 1/(2p+1) \). When \( p = \infty \) i.e., \( \nu = 1/2 \), this part of the charge vanishes and we are left with just the second dipolar piece. The latter couples to an external scalar potential \( \Phi \) as per

\[
H_{\text{ext}} = (-e) \sum_q \rho(-q) \Phi(q) = -iel_0^2 \sum_q \sum_j (q \times p_j) e^{iqx_j} \Phi(q) = \sum_j -iel_0^2 \hat{z} \times p_j \cdot E(x_j)(154)
\]

where \( E \) is the electric field. Note the anticipated value of the dipole moment.

We have argued that our charge density is gauge invariant in that it does not mix physical and unphysical states. This means it must either commute with the constraints or have a commutator proportional to them. Indeed we find

\[
[\bar{\rho}(q), \chi(q')] = il_0^2 (q \times q') \chi(q + q').
\]  

(155)

Our joy is tempered by the algebra of constraints. We find

\[
[\chi(q), \chi(q')] = il_0^2 \frac{(q' \times q)}{\cos^2 \mu \lambda_0} \left[ \rho(q + q') + \frac{i l_0^2}{(1 + \cos \mu \lambda_0)^2} \sum_j ((q + q') \times \Pi_j) e^{-i(q + q')x_j} \right].(156)
\]

Note that the operator in the right hand side is not quite the constraint; the term of order \( (q \times q')(q + q') \) has the wrong coefficient. In contrast to the charge algebra which closed and was correct even to order \( (q \times q')(q + q') \), the constraint algebra closes only to leading order if we use the truncated expressions. Thus we have a situation where the Faddeev-Popov method applies (the hamiltonian commutes with the constraints to give constraints which in turn form an algebra) only to leading order.

What if we use the projected version of Eqn. (146) for the charge? The reader may verify that GMP algebra is satisfied and the constraints commute with the hamiltonian, both to

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3 Usually we say that an operator is gauge invariant if it commutes with the constraints, which are the generators of the gauge transformations. However if we merely require that the operator not turn physical to unphysical states, it is sufficient if its commutator with the constraints is proportional to the constraints. The Faddeev-Popov method applies to this general case.
leading order \((q \times q')\). In this version the charge of the CF will not be transparent, but the conservation laws and Ward identities will be. These comments apply to a models obtained by truncating operators to the orders calculated. Non-RPA and higher order terms in \(q\) can modify the commutators\(^\text{\[40\]}\). We do not pursue this issue for the gapped states for which we will propose an approximation scheme that ignores the constraints.

Of all the results that flowed from the canonical transformation, only \(1/m^* = 0\) requires \(Q = k_F\). For example the CF charge \((1/(2p + 1))\) and dipole moment are true for any \(Q\)\(^\text{\[41\]}\).

**B. Hall Conductance**

A common question one has for \(\nu = 1/2\) is who carries the Hall current if the CF is neutral. The answer, for general \(\nu = p/(2p + 1)\), and not just for \(\nu = 1/2\), is that the oscillators carry the entire Hall current in the clean limit considered here. We show this as follows.

- We couple the system to an external potential \(\Phi(q)\) using the formula Eqn.\(\text{\[153\]}\) for \(\rho^{\text{old}}\).
- We recall that in the passage to the FR, the transport current due to particle motion got exactly canceled so that finally:

\[
J_{+}^{\text{old}}(q) = \frac{\hat{q}_+ \omega_c \cos \mu \lambda_0}{\sqrt{2\pi}} A(q).
\]

\(\text{(157)}\)

- In view of the above, we focus on just the oscillator sector, find \(\langle A \rangle\) due to the potential and from it, \(\langle J \rangle\).

So we begin with

\[
H_{\text{osc}} = \sum_q \omega_c A^\dagger(q) A(q) - e \sum_q \Phi(q) \rho^{\text{old}}(-q)
\]

\(\text{(158)}\)

\[
= \sum_q \omega_c A^\dagger(q) A(q) - e \sum_q \Phi(q) \frac{q}{\sqrt{8\pi}} (A(-q) + A^\dagger(q)) \cos \mu \lambda_0
\]

\(\text{(159)}\)

\[
= \sum_q \omega_c \left[ A^\dagger(q) - \frac{qe \cos \mu \lambda_0}{\sqrt{8\pi \omega_c}} \Phi(-q) \right] \cdot \left[ A(q) - \frac{qe \cos \mu \lambda_0}{\sqrt{8\pi \omega_c}} \Phi(q) \right] + \text{const.}
\]

\(\text{(160)}\)

The new ground state of the oscillators is found by shifting \(A\) as follows:

\[
\langle A(q) \rangle = \frac{qe}{\sqrt{8\pi \omega_c}} \cos \mu \lambda_0 \Phi(q).
\]

\(\text{(162)}\)

The ground state electromagnetic current is

\[
\langle (-e) J_{+}^{\text{old}}(q) \rangle = (-e) \frac{\hat{q}_+ \omega_c \cos \mu \lambda_0}{\sqrt{2\pi}} \langle A(q) \rangle = -\frac{e^2}{\hbar} q_+ \nu \Phi(q)
\]

\(\text{(163)}\)
where we have reinstated $\hbar = 1 = h/2\pi$ to show that we have the correct value $\sigma_{xy} = \nu e^2/h$. By translation invariance (in the clean system) the Hall conductance has to have this value. What is significant is that it all comes from the oscillators. In general we can write in our approximation scheme the following expression for longitudinal and Hall conductivities

$$\sigma_{\mu\nu} = \sigma_{\mu\nu}^{osc} + \sigma_{\mu\nu}^{particles}$$

(164)
since the charge $\rho^{old}$ and Hamiltonian separates into pieces, one for the oscillators and one for the particles, as shown in our letter (14). (The particle part happened to be negligible for the Hall conductance in the clean system. When we turn on interactions, the particle current will acquire a piece proportional to $e^2$. ) This additivity of $\sigma_{\mu\nu}$, the has also been independently obtained by D.H. Lee (42), who started with the bosonic theory. In contrast, in the usual CF fermion theories in which the CF is fully charged, one adds resistivities. The concerns of Lee et al. (43) on the value of $\sigma_{xy}^{CF}$ therefore do not therefore apply to our approach.

VI. INTRODUCING INTERACTIONS

So far we have been considering just the noninteracting theory. A unique ground state appropriate to an interacting theory emerged in the MR upon keeping just $H_0$ and dropping $H_I$ and $H_{II}$. This led to good wavefunctions, but bad dispersion relations for the oscillators (wrong frequency at $q = 0$ except when $A^* = 0$) and a particle kinetic energy of order $1/m$ instead of zero. We then dealt with $H_I$ and $H_{II}$: eliminating $H_I$ by the canonical transformation and dropping $H_{II}$ as non-RPA. This led to oscillators of the right frequency and particles with no kinetic energy and just a magnetic moment $e/2m$ which coupled to variations in the background $B$ field. These are the correct results for the noninteracting case.

We are now going to include interactions, illustrating the procedure with a Coulomb interaction cut off in the ultraviolet at $q = Q$ so that it may be described entirely by our oscillators. (This rounding off of the Coulomb potential at short distances may be a realistic description of the experimental system with finite thickness.) The full Hamiltonian is:

$$H_T = \sum_q \omega_c A(q) A^\dagger(q) + \sum_i \frac{eB^* q}{2m} + \frac{1}{2} \sum_q \rho^{old}(-q) \frac{2\pi e^2}{q} \rho^{old}(q)$$

(165)

$$\rho^{old}(q) = \frac{q \cos \mu \lambda_0}{\sqrt{8\pi}} (A(q) + A^\dagger(-q)) + \sum_i e^{-iqx_j} \frac{e^2}{2p + 1} - i\frac{d}{d_i} (q \times \Pi_j) e^{-iqx_j}.$$  

(166)

When we expand out $\rho^{old}(-q) \rho(q) \rho^{old}(q)$ we will get a piece that renormalizes the oscillator frequency to order $e^2q$ (which for short range interactions would have been order $e^2q^2v(q)$) as per Kallin and Halperin (44), a mixing of order $e^2q$ between oscillators and particles and a $\bar{\rho}(-q)\frac{\pi e^2}{q} \rho(q)$ term in the particle sector. The mixing can be eliminated by a further canonical transformation, which can be done to lowest order in $e^2q$. It will modify all previous formulas for $\rho^{old}$, $J$ and so on to terms of higher order. We ignore these changes in our analysis, which is limited to lowest nontrivial order in $q$ and $e^2$. 

29
Before going to the particle sector for a detailed analysis let us note that if in the density-density correlation $K_{\rho \rho}$ we consider just the oscillator part of $\rho_{old}$:

$$\rho_{old} \simeq \frac{q \cos \mu \lambda_0}{\sqrt{8\pi}} (A(q) + A^\dagger(-q))$$

(167)

we find the cyclotron pole

$$K_{00}(q, \omega) \simeq \frac{q^2 \omega_c \cos^2 \mu \lambda_0}{4\pi(\omega_c^2 - (\omega + i\eta)^2)}$$

(168)

whose residue obeys the sum rule

$$\int_0^\infty \text{Im} K_{00}(\omega) \omega d\omega = \frac{q^2 n\pi}{2m}.$$  (169)

Since we work in unit volume $n = N$, the particle number.

Turning to the particle sector, the hamiltonian is just

$$H_{\text{part}} = \sum_q \bar{\rho}(-q) \frac{\pi e^2}{q} \rho(q).$$

(170)

(We have dropped the $eB^*/2m$ term assuming that the external field is homogeneous).

Recall that GMP began with the above hamiltonian, the LLL assumption, and the magnetic algebra of the $\bar{\rho}$’s, and proceeded to derive the spectrum of collective excitations in the Single Mode Approximation. What is new here, as far as low energy physics is concerned? The first point is that we have not just the algebra, but a concrete realization of the algebra in terms of canonical variables in terms of which we can try to do explicit calculations. This is similar in spirit to the recent work of Haldane and Pasquier who started with an algebraic formulation of the LLL problem and then switched to a concrete realization of the algebra in terms of canonical operators. Next, our approach derives the representation from first principles, and shows how a field theory in the full Hilbert space leads to LLL physics upon identifying the plasma oscillators and then freezing them. Finally there are many low energy responses depending on $m$, which we obtain by starting in the full space and systematically separating high and low energy degrees of freedom.

The particle hamiltonian, written out fully, (dropping the $e\delta B/2m$ term) is

$$H_{\text{part}} = \sum_q \left( \frac{\rho(-q)}{2p+1} + i l_0^2 \sum_j (q \times \Pi_j) e^{iqx_j} \right) \frac{\pi e^2}{q} (q \rightarrow -q)$$

(171)

$$= l_0^4 \sum_q \sum_{i,j} \frac{\pi e^2}{q} (q \times \Pi_j) e^{iq(x_j-x_i)} (q \times \Pi_i)$$

$$+ \sum_q \sum_{i,j} \frac{\pi e^2}{q} (2p+1)^2 + i l_0^2 \sum_q \sum_{i,j} \left\{ \frac{\pi e^2}{q} (q \times \Pi_j) e^{iq(x_j-x_i)} - h.c. \right\}.$$  (172)

This hamiltonian affords a concrete, microscopic, and detailed realization of composite fermions. Not only does it describe particles that see the weaker field $B^*$ that is just right to fill $p$ Landau levels, it also describes particles with local charge $e^* = e/(2p+1)$. The
main attraction of Jain’s CF picture is that one can think of the original problem in terms of roughly noninteracting particles. Since our description shows the right charge and a mass of Coulombic origin at tree level, we may expect that only small corrections separate what we see in the hamiltonian from what finally happens.

The hamiltonian is, however, not that of free fermions in the weaker field. Indeed, this is not to be expected, because the Hilbert space of free fermions is too big to describe the LLL physics that is left after we have frozen the oscillators. That is why we have constraints to reduce the size of the Hilbert space and the hamiltonian has to be gauge invariant under the transformations they generate (to the approximation we are working in). The free particle hamiltonian does not have this property.

However $H_{\text{part}}$ contains within it a free particle piece which could serve as a starting point for an approximation. It is given by the $i = j$ part of the dipole-dipole term

$$H_{\text{part}}^\text{free} = l_0^4 \sum_q Q \sum_j \frac{\pi e^2}{2m^*} (q^2 |\Pi_j|^2 \sin^2 \theta_{q,\Pi})$$

(173)

$$= \sum_j \frac{|\Pi_j|^2}{2m^*}$$

(174)

$$\frac{1}{m^*} = l_0^4 \sum_q \frac{\pi e^2}{2} q^2 \sin^2 \theta_{q,\Pi} = \frac{e^2 l_0}{6} (2 \nu)^{3/2}$$

(175)

where we have used the fact that $Q^2 = 4 \pi n = e B \cos^2 \mu \lambda_0$ and $\cos^2 \mu \lambda_0 = 2p/(2p + 1) = 2\nu$. Notice how the interaction has generated an effective $1/m^*$. Its exact value is not universal and depends on the rounded Coulomb interaction employed.

Read has argued that the origin of the kinetic energy is the binding of the CF to the correlation hole, and that the effective mass is the curvature of this potential energyread1. (The curvature in real space gives the mass because the momentum of the fermion is a measure of its dipole moment.) If we consider the potential we have used, (the Coulomb interaction cut off at $Q$ in momentum space, so that it is rounded off to a finite value at $r = 0$), and calculate its curvature at $r = 0$, we find that it is given precisely by the above formula. Of course the bare potential gives just the bare mass which can be renormalized by the other terms in $H_{\text{part}}$. It is encouraging that numerical work of Morf and d’Ambrumenil gives $1/m^* = .2e^2 l_0$ for the untruncated Coulomb interaction at $\nu = 1/2$ compared to our $e^2 l_0/6$ for the truncated Coulomb. In any event, the aim of the above calculation is merely to show that we have a means of computing $1/m^*$ in a zeroth order approximation and that it has the right size and physical origin. To our knowledge our earlier work was the first instance this was done. The exact value for a given potential is nonuniversal and dependent on small and large $q$’s and beyond the reach of our theory.

Note that the kinetic energy that comes from the $i = j$ term does not constitute the electron’s self-energy, but rather the interaction energy between the electron and its correlation hole produced by the (deficit of) other electrons. One may ask why we did not banish the $i = j$ term of the Coulomb interaction in the MR from the outset. The problem is that we only know how to canonically transform the full charge density, summed over all particles. This density obeys the magnetic algebra and functions of it are gauge invariant. So our strategy has been to add the $i = j$ term in the MR, where it constitutes an innocuous
chemical potential shift, work with the product of the full densities in constructing the interaction hamiltonian and then transform the latter. The magnitude of the kinetic energy so generated fits our expectations based on Read’s arguments and points to the soundness of the procedure.

Returning to general fractions, we have to understand why the CF are weakly interacting and may be described approximately by $H_{\text{free}}^{\text{part}}$, for this is essential to the success of the CF picture of Jain. The smallness of $e^2$ does not matter since it is a prefactor to the whole hamiltonian. Looking at the neglected terms, we see two possible small dimensionless parameters: $1/p$ and $q l_0$. Our preliminary analysis of matrix elements (relying on the work of Dai et al. and Chen et al.) shows that there is huge window (which gets larger as $\nu \rightarrow 1/2$) where the dipole can dominate the monopole because $q l_0^* \sqrt{p}$ is large even though $q l_0$ is small, $l_0^*$ being the magnetic length appropriate to the reduced field $A^*$. Not having completed the detailed calculation we merely outline a possible strategy. Suppose we want $K_{00}(q)$, the retarded $\rho^{\text{old}} - \rho^{\text{old}}$ correlation function. Since $\rho^{\text{old}}$ does not link physical to unphysical states, its two point function in a gauge invariant ground state can be calculated without regard to the constraint on intermediate states. However we do not have access to this gauge invariant ground state. We must argue that the numerical value of the correlation will not be too different in an approximation to the ground state. For the latter we may use the $p$-filled Landau level ground state of $H_{\text{part}}^{\text{free}}$. We could also include the neglected interactions terms in Eqn.(172) in an RPA calculation. We are in the process of evaluating this strategy by computing some correlations that are known numerically or experimentally.

A. $\nu = 1/2$

We now focus on $\nu = 1/2$ to which a great deal of attention has been given following the work of HLR. They took the notion of a Fermi surface at $\nu = 1/2$ seriously and deduced many experimental consequences which have been verified.

HLR calculated numerous response functions in the RPA working in the CS representation we began with (before the introduction of the oscillator coordinates). Let us focus on the following results:

\[
\int_0^\infty \text{Im} K_{00} \omega \, d\omega \simeq q^4
\]

(176)

\[
\int_0^\infty \text{Im} K_{00} \, d\omega \simeq q^3 \log q
\]

(177)

\[
\int_0^\infty \text{Im} K_{00} \omega^{-1} \, d\omega \simeq 1/\nu(q) = q \quad \text{(for Coulomb)}
\]

(178)

where the plasmon contribution has been removed.

What does our formalism predict for this response function? The effective hamiltonian, the constraint $\chi(q)$ and $\bar{\rho}$ are:

\[
H_{\text{part}} = \sum_i \frac{p_i^2}{2m^*} + \frac{l_0^2}{2} \sum_{i,j \neq i} \sum_q \frac{2\pi e^2}{q} (q \times p_i)(q \times p_j) e^{iq(r_i - r_j)}
\]

(179)

\[
\chi(q) = \sum_i e^{-iqr_i} + \frac{i l_0^2}{2} \sum_i (q \times p_i) e^{-iqr_i} = 0
\]

(180)
\[ \bar{\rho} = -i\ell_0^2 \sum_i (q \times p_i) e^{-iqr_i} \]  

(181)

To calculate \( K_{00} \), we need an approximation scheme. One possibility we explored starts with the \( p^2/2m^* \) term in Eqn. (179) and includes the other terms and constraint in the RPA. While this appears reasonable and familiar, there are some dangers in this case, as will be explained shortly. The result in the small \( q \) limit is,

\[
K_{00} = \frac{\omega_0 q^2}{4\pi(\omega_0^2 - (\omega + i\eta)^2)} + \frac{q^2\ell_0^2 m^*}{2\pi} \left[ \frac{1}{2} - x^2 + \theta(x^2 - 1) \right] x \sqrt{x^2 - 1} + i\theta(1 - x^2)x \sqrt{1 - x^2} \quad x = \frac{\omega}{qv^*} 
\]

\[ \equiv K_{00,osc} + K_{00,CF} \]  

(182)

We find that the answer is not changed (in the small \( q \) limit) as we change the constraint from \( \sum_i e^{-iqr_i} + \frac{i\ell_0^2}{2} \sum_i (q \times p_i)e^{-iqr_i} = 0 \) to \( \sum_i e^{-iqr_i} = 0 \).

The above formula includes the plasmon contribution as well. Note that the CF contribution has no structure other than the particle-hole branch cut. It follows from the above that

\[
\int_0^\infty \text{Im} K_{CF,00}(\omega^\alpha d\omega) = k_F^{-1}(m^*)^{-\alpha} q^{3+\alpha} \frac{\Gamma(1 + \alpha/2)}{8\sqrt{\pi} \Gamma(5/2 + \alpha/2)} \quad \alpha = 0, \pm 1 
\]

(183)

The ratio of the \( \alpha = 1 \) and \( \alpha = 0 \) integrals, which gives the average frequency of the excitations, goes as

\[ <\omega> \simeq q. \]

(184)

The \( q \) dependence of the first two sum rules agrees with that of HLR (up to logarithms) but the last is quite different: in our case the compressibility vanishes as \( q^2 \) and is independent of the interaction, while theirs goes as \( 1/v(q) \simeq q \) for the Coulomb case. In the last sum rule very small \( \omega \)'s dominate. In HLR, this region is controlled by the overdamped mode, which goes as \( \omega \simeq iq^3v(q) \). This mode arises due to magic gauge cancellations in the infrared.

We did not see such a mode or such magic cancellations. This could be due to an improper treatment of the constraints by our RPA. Could it be that if the constraints are treated properly, negative powers of \( q \) appear in the correlation function and neutralize the \( q^2 \) coming from the dipolar nature of charge?

Let us look at our constraints and the gauge transform they generate. While we do not know them exactly, there is no doubt that as \( q \to 0 \), they reduce to

\[ \chi(q) = \sum_i e^{-iqr_i} = 0. \]  

(185)

Their action on the particle coordinate and momenta are as follows:

\[ r_j \to r_j \quad p_j \to p_j + q e^{-iqr_j} \]  

(186)

Thus \( \chi \) essentially shifts the momenta of the particles. This means that the Fermi circle cannot be a ground state– it can be moved around by a gauge transformation. This is
reminiscent of Haldane’s finding that the particles like to cluster around each other in momentum space, but with no preferred origin. In our analysis this reflects a subset of the gauge symmetries of the effective Hamiltonian.

Now, in any gauge theory one resorts to gauge fixing by choosing an operator whose commutator with $\chi$ is non zero and freezing its value, say to zero. For example, in QED, the constraint $\nabla \cdot \mathbf{E} = 0$ is accompanied by a gauge fixing condition $\nabla \cdot \mathbf{A} = 0$. Such gauge fixing kills all spurious fluctuations. In our case it may justify our fixing the Fermi sea arbitrarily at the origin. On the other hand, in a gauge theory, even after gauge fixing, there can be genuine gauge invariant fluctuations of arbitrarily low energy. We did not see them in our treatment, and the question is whether they would arise in a different approximation and if so, produce inverse powers of $q$ that lead to a finite static compressibility for short range forces and and one that vanishes as $q$ for the Coulomb case.

Recently Halperin and Stern have given an existence proof that such a thing can happen, and indeed happens in a model that arises within our formalism. Consider the noninteracting Hamiltonian Eqn. (135), drop the oscillator part and the magnetic moment part, and set $\nu = 1/2$ so that $\Pi = p$. Then

$$H = \sum_i \frac{p_i^2}{2m} - \frac{1}{2mn} \sum_i \sum_j Q \sum_q p_i e^{-iq(x_i-x_j)} p_j$$

(187)

In the $Q = 0$ limit, the Hamiltonian is invariant under a uniform shift of all momenta. For this reason choose $Q$ very small. The formula for charge is still dipolar since that does not depend on $Q$. Halperin and Stern compute the density-density correlation in this model and show that inverse powers of $q$ indeed appear due to gauge cancellations and modify render the system compressible.

The authors do not claim that this is the problem we set out to solve – it has explicit $1/m$ dependence, no Coulomb interactions, describes particles that are bound to fat flux tubes and do not obey any particular statistics except outside a distance $1/Q$. The role of this calculation, as stated above, is to demonstrate that inverse powers of $q$ can and do appear in a model with the same gauge symmetries, at least as $q \to 0$ and that dipolar particles can be compressible. The authors however state that the same conclusion applies to the model at hand and promise details soon. This result was also announced by D.H. Lee in a revised version of Ref. (42).

Note that even if we regain the HLR formulas for correlations, the physical picture is now much more satisfactory. We have dipolar fermions and a special Hamiltonian whose symmetries allow for compressibility despite the dipolar coupling. This Hamiltonian and its symmetries are not postulated but derived from the microscopic theory. We also have oscillators which contribute additively to Hall conductance to complete the picture. Thus we reconcile many seemingly contradictory properties of the CF’s.

One can seek other approximations besides our RPA and see if such cancellations occur. Another possibility is to use Eqn. (136) for charge so that

$$\bar{\rho} = \left[ \sum_j e^{-iqx_j} - \frac{i l_0^2}{2} \sum_j q \times p_j e^{-iqx_j} \right].$$

(188)
In this case the constraint
\[
\chi = \left[ \sum_j e^{-i qx_j} + \frac{il_0^2}{2} \sum_j q \times p_j e^{-i qx_j} \right]
\] (189)
commutes with the charge (and hence the hamiltonian) to leading order and gauge invariance will be easier to implement.

In the meantime we can ask if there are kinematical regions where our RPA analysis may hold independent of how well it treats the constraints. The first two moments of \( K_{00} \) and intuitive arguments suggest that higher the frequency, the better the prospects. With this in mind we compared our results to the surface acoustic waves (SAW) experiments of Willett et al (1993) in the frequency range of \( \approx 3.7 - 6.1 \text{GHz} \).

The fractional shift in the velocity of SAW, as explained to us by Simon, is given by
\[
\frac{\delta v_s}{v_s} = 3.2 \cdot 10^{-4} \left[ 1 - \Re(v(q) K_{00}(q, \omega)) \right]
\] (190)
where \( v(q) = 2\pi e^2/q \) and \( 3.2 \cdot 10^{-4} \) is a material dependent constant referred to as \( \alpha^2/2 \) in the literature and \( \Re \) denotes the real part. The formula is set up so that \( \delta v_s = 0 \) for a free fermion system in zero field, for which \( \Re K_{00}(q, \omega) = 1/v(q) \).

Using our result for \( K_{00} \) we obtain
\[
\frac{\delta v_s}{v_s} = 3.2 \cdot 10^{-4} \left( 1 - \frac{2\pi e^2 m^*}{\varepsilon q 4\pi (ql_0)^2} \right).
\] (191)
Feeding in
\[
\frac{1}{m^*} = \frac{Ce^2 l_0}{\varepsilon}.
\] (192)
and
\[
q = \frac{\omega}{v_s} = \frac{2\pi \cdot 10^9 \tilde{f}}{v_s}
\] (193)
where \( \tilde{f} \) is the frequency in GHz, we get
\[
10^4 \cdot \frac{\delta v_s}{v_s} = 3.2 - \frac{0.036}{C} \tilde{f}
\] (194)
upon using \( v_s = 3000 \text{m/s} \), and \( k_F = 1/l_0 = 93 \mu \text{m}^{-1} \). Notice that the dielectric constant cancelled out.

Extracting from Fig. 29, of Willett’s 1997 review, the points
\[
(\tilde{f}, 10^4 \cdot \frac{\delta v_s}{v_s}) = (3.7, 1.85), (4.3, 1.66), (5.4, 1.37), (6.1, 1.17)
\]
we obtained a least square fit
\[
(\delta v_s/v_s) \cdot 10^4 \approx 2.9 - .28 \tilde{f}.
\] (195)
Upon comparing to Eqn. (194), the slope translates to a value of \( C \approx 0.13 \). (For reference, the zeroth order value for the rounded Coulomb potential was \( C \approx 0.17 \).) The intercept, given by the on the material dependent constant \( 3.2 \cdot 10^{-4} \) is within \( \approx 10\% \) of the data.

We are aware that comparing theory to experiment (rather than to another theory) is lot more involved. We do not claim to understand in any depth why our RPA should work in this kinematical region. But we are intrigued by this agreement with experiment and present it as such. We propose to study other response functions in this kinematical region to pursue this question.
The aim of this paper was to start with the problem of planar electrons of mass $m$, charge $-e$ in a perpendicular magnetic field $-B$ and see how far we could go towards understanding the FQHE within some approximation scheme. We discussed the fractions $\nu = p/(2ps + 1)$ focusing on $s = 1$, the extension to other values being very direct.

We began by following the standard procedure of attaching point flux tubes to go from the electronic to the CS representation. In the hamiltonian description this led to the introduction of the nondynamical CS field $(\nabla \times)^{-1}2\pi l \rho$ when $l$ quanta were attached. The average value of this (due to the average charge) cancelled some or all of the external field $A$ leaving behind an $A^*$ which either vanished for $\nu = 1/2$ or $\nu = 1/(2m+1)$; or for $\nu = p/2p+1$ was just right to fill $p$ Landau level a la Jain.

At this point we took the crucial step of enlarging the Hilbert space to include the canonical pair $a(q), P(q)$ which described transverse and longitudinal vector potentials with $0 < q \leq Q = k_F$. Thus the number of additional degrees of freedom equaled the number of electrons $n$. This led to $n$ constraints on physical states $(a(q)|physical > = 0)$. We made a further canonical transformation that got rid of the nasty field $(\nabla \times)^{-1}2\pi l \rho$ and changed the constraint to $qa = 2\pi l \rho$. The hamiltonian in this middle representation (MR) did not contain any dependent fields. We found that if we dropped the terms $H_I$ and $H_{II}$ we obtained a solvable model of $n$ plasma oscillators described by $a, P$ and $n$ particles, either free or in a field just right to fill $p$ Landau Levels. The ground state (product) wavefunction for this case, duly projected to the physical sector using the constraint, gave us the well known correlated wavefunctions of Jain and Rezayi-Read, (except for the overall projection to the LLL). The oscillator part of the product was the Jastrow factor associated with flux attachment, but it came with the magnitude and phase of the correlation zeros, as well as the gaussian factors with right magnetic lengths.

We identified operators that created Laughlin’s quasiholes. The corresponding wavefunctions had the right Gaussian factors for the quasiholes, and the normalization factor from which one could infer their statistics. We constructed the operator that could be identified with the Read order parameter, but pointed out some limitations of this identification. Our composite boson and fermion operators implemented the physical picture extracted by studying Laughlin’s wavefunction– that electrons like to bind to zeroes of the wave function. Only the zeros produce a charge deficit to which the electron is attracted; it is not attracted to just flux, i.e., the phase of the zeroes. It appeared as if every nondegenerate ground state of the interacting theory had a counterpart in the free theory of oscillators and particles upon projection. All these results were obtained before turning on any explicit $e - e$ interaction although it was understood that such an interactions exists in order to stabilize the unique ground state.

There were however some problems with the MR that did not surface in the quest for wavefunctions. The plasmons had the wrong frequency (unless $A^*$ vanished) and the particles had kinetic energy of order $1/m$, when they were supposed to have none (in the noninteracting case) or one of order $1/m^* \approx e^2 l_0$ in the presence of interactions. To cure this, we performed yet another transformation to the final representation (FR) to decouple the oscillators and particles. This was done in the small $q$ limit and within an RPA applied to the final variables. The result was as follows.
First, the constraints were entirely in the particle sector. Denying \( n \) particles in two dimensions \( n \) degrees of freedom was just right to project out the LLL physics. The particles and oscillators separated in the hamiltonian. The particle energy completely vanished except for the coupling of a magnetic moment \( e/2m \) to the external field, a coupling whose existence had been predicted by SSH. Thus the desired quenching of particle KE, i.e., \( 1/m^* = 0 \) was accomplished. The oscillators ended up with the correct frequency. The formula for \( \rho^{old} \), the charge density of the electrons, contained three pieces: one linear in the oscillators, one proportional to the particle density with a charge \( 1/(2p + 1) \) and a dipole piece. Thus we regained the correct charge of the CF and the dipolar nature of the charge for \( \nu = 1/2 \) as anticipated by Read. If we dropped the oscillator part of the charge (valid in the oscillator ground state sector) we obtained a projected charge \( \tilde{\rho} \) which obeyed the magnetic translation algebra of LLL charge density first noted by GMP \( ^{23} \) (in the small \( q \) limit). This projected density was gauge invariant i.e., did not mix physical and unphysical states, and could be multiplied freely in the enlarged space. Charge-charge correlations and Hall conductance were additive over the oscillators and particles since the formulae for charge and hamiltonian were. The oscillators were found carry the full Hall current in the clean limit we are considering.

We did not try to obtain improved wavefunctions from the FR since the transformations back to the electronic wavefunction were no longer simple. This simply corresponds to the fact that projection to the LLL (accomplished by the passage to the FR in our scheme and by the \( \mathcal{P} \) operator by Jain and Rezayi-Read) does something very complicated to the wavefunctions we derived in Section IV. Since we went from the MR to FR in first quantization, our main results– quenching of kinetic energy and the reappearance of an interaction dependent \( 1/m^* \), the value of \( e^* \), the constraints, additivity of response function over particles and oscillators– all apply to the bosonic case (\( \nu = 1/(2s + 1) \)) as well. Also all the results in the FR, except for the cancellation of \( 1/m \) in the noninteracting case, were insensitive to the choice \( Q = k_F \). \( ^{41} \)

We turned on the Coulomb interaction and found that the particle sector now contained a kinetic energy term. A zeroth order value for \( 1/m^* \) was easily found. It had the right order of magnitude and physical origin. Our hamiltonian had a gauge symmetry generated by the constraints. In the infrared limit, these transformations were seen to be a translation in momentum space, reminiscent of the drifting Fermi sea seen by Haldane. We proposed a computational scheme that was quite reliable for gapped states. For \( \nu = 1/2 \) we described an RPA calculation of \( K_{00} \), the retarded \( \rho^{old} - \rho^{old} \) correlator. Two truncations of the constraints gave the same answer. Two of the \( K_{00} \) moments were the same as that of HLR (up to logarithms) while the compressibility vanished as \( q^2 \) in our case versus approaching \( 1/v(q) \). We pointed out that the compressibility formula could be altered by gauge cancellations that we did not find, cancellations that have been shown to occur in a related model by Halperin and Stern. \( ^{51} \) We emphasized that even if in the one ends up with the HLR correlation functions, our method has facilitated the reconciliation many seemingly contradictory properties of the composite fermion such as its dipolar charge and compressibility.

We went on and compared our results to the SAW experiments of Willett \( et \ al \) on the expectation that perhaps in that kinematical region the precise treatment of constraints would be unimportant. We presented the agreement with data as an intriguing fact that needs to be understood.
In summary, we have proposed a way to extract a low energy theory for the FQH states. Composite bosons and fermions with the frequently quoted properties arise at tree level in the FR, but one finds that the oscillators are needed for a complete description. The most important ingredients are explicit expressions for the hamiltonian and charge-current operators of the CF and CB. Our description relies on a small $q$ and RPA approximation (in the final variables), the precise nature of which we do not fully understand. The same applies to the gauge constraints or gauge symmetry in the FR, of which we are sure only of the extreme small $q$ limit. As we, along with others, continue to work on clarifying these issues, we have provided here many details of our machinery in the hope that they may be fruitfully employed by a wider audience.

Acknowledgements

In writing this long version, we have once again profited from the generosity of numerous colleagues, especially S.M. Girvin, J. Jain, D.H. Lee, N. Read and S. Simon. We are particularly grateful to Bert Halperin for his extensive and insightful remarks and comments. As for errors that crept in despite the above, we will blame each other. We are pleased to acknowledge the NSF grants DMR-9311949 (GM) and DMR-9415796 (RS).

APPENDIX A

We show here that the short distance terms $H_{\delta a}$ along with the $i \neq j$ terms in the third sum in Eqn. (135) that we shifted to large $q$ with a change of sign, conspire to produce the plasmon pole of correct location and residue in the case $\nu = 1/2$, when $\Pi = p$. Let us focus on just this part of the hamiltonian, again keeping only zeroth order terms in $q$. (Note that we do not claim $q$ in this region is small, only that we are working consistently to a given order in $q$.) Recalling that in this region $\rho_{\text{old}} = \rho$,

$$H_{sr} = \frac{1}{2mn} \sum_{q=Q}^{\infty} \sum_{i} \sum_{j \neq i} p_i \cdot p_j e^{-i q (r_i - r_j)} + \sum_{i} \sum_{q=Q}^{\infty} \frac{p_i}{m} e^{i q r_i} \delta a(q) + \frac{n}{2m} \sum_{q=Q}^{\infty} \delta a(q) \delta a(-q) \tag{196}$$

We now switch to the functional formalism. We perform a Hubbard-Stratonovich transformation of the first term using a field $V = i q V_L - i \hat{z} \times \hat{q} V_T$. The lagrangian density is (upon dropping the $\delta$ symbols in front of the fields, all of which have $q > Q$)

$$L = \sum_{p} \bar{\psi}(p) \left( i \partial_0 - \frac{p^2}{2m} \right) \psi(p)$$

$$+ \sum_{p} \sum_{q=Q}^{\infty} \bar{\psi}(p + q) \psi(p) \left[ \frac{(i q \cdot p) V_L(q)}{m} + \frac{(i q \times p) V_T(q)}{m} + \frac{(i q \times p) a(q)}{m} - a_0(q) \right]$$

$$+ \sum_{q=Q}^{\infty} \left[ -\frac{n}{2m} a(q) a(-q) + \frac{n}{2m} (V_L^2 + V_T^2) + a_0(-q) \frac{q}{4\pi} a(q) \right] \tag{197}$$

It is now straightforward to calculate the inverse RPA matrix propagator for the 4-component field $D = (a_0, a, V_L, V_T) \equiv (D_0, D_t, D_L, D_T)$ using

$$D^{-1} = D_0^{-1} + \Pi^0 \tag{198}$$
where $\Pi^0$ is the free-field response function. In particular

$$[\Pi^0]_{00}(q, \omega) = -\int_0^{2\pi} \frac{d\theta}{4\pi^2} \frac{k_F q \cos \theta}{\omega - v^* q \cos \theta + i\eta} \tag{199}$$

while the $TT$ (or $tt$), $LL$, and $0L/L0$ components differ by the additional factors of $(k_F/m)^2 \sin^2 \theta$, $(k_F/m)^2 \cos^2 \theta$ and $\pm i(k_F/m) \cos \theta$ respectively in the numerator of the integrand. We do not list these integrals; they may be found (for the time-ordered instead of retarded functions) in the Appendix of Reference [13]. Notice that even though $1/m^*$ (or $v^*$) vanishes here, we keep it in the denominator since the answers can have $m^*$ in the numerator. Also since we are working to leading order in $q$, we do not keep the $q^2/2m^*$ in the energy denominator. These functions are needed only in the limit

$$x = (\omega/qv^*) \rightarrow \infty. \tag{200}$$

The inverse matrix in this limit is

$$D^{-1} = \begin{bmatrix}
-\frac{m^*}{4\pi x^2} & \frac{q}{4\pi x^2} & -\frac{im^* k_F}{4\pi m} & \frac{0}{4\pi m} \\
\frac{q}{4\pi x^2} & \frac{n}{m} - \frac{m^* n}{4x^2 m^2} & 0 & \frac{m^* n}{4x^2 m^2} \\
-\frac{im^* k_F}{4\pi m} & 0 & \frac{n}{m} - \frac{3m^* n}{4x^2 m^2} & 0 \\
\frac{0}{4\pi m} & \frac{m^* n}{4x^2 m^2} & 0 & \frac{n}{m} - \frac{m^* n}{4x^2 m^2}
\end{bmatrix} \tag{201}$$

It is now straightforward to verify that as $x \rightarrow \infty$ (which is the only region we have here) this propagator has only a plasmon pole at $\omega_0$. The density-density correlation, related to $D$ by

$$K = K^0 - K^0 D K^0 \tag{202}$$

(where $K^0$ is the noninteracting limit) has the following matrix element:

$$K_{00} = \frac{\omega_0 q^2}{4\pi} \frac{1}{\omega_0^2 - (\omega + i\varepsilon)^2} \tag{203}$$

which obeys the f-sum rule

$$\int_0^\infty \text{Im}K(\omega)\omega d\omega = \frac{q^2 n\pi}{2m}. \tag{204}$$

It is important to note that if we had not shifted the $i \neq j$ pieces from small to large $q$ the plasma pole would have ended up at $4\pi n/\sqrt{mm^*}$. This term would have also been unwelcome at small $q$ since the fermions are to have zero hamiltonian. All in all, the shifting from small to large $q$ seems decidedly the right thing to do.
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35 Fradkin suggests in Ref. (11) Eqn. (10.9.13), another possible choice of operator. Adapting it to the present case of time-independent transformations, (replacing the three-dimensional integral in the formula by the two-dimensional one) we find it collapses to the previously considered line integral.
36 "A gauge invariant way to evaluate quasiparticle effective mass in fermionic systems with gauge interactions " A. Raghav Chari, F.D.M. Haldane and Kun Yang cond-mat 9707055. See also "Bare effective mass in finite $\nu = 1/2$ systems" A. Raghav Chari and F.D.M. Haldane, cond-mat9709081.
37 We thank Nick Read for pointing out that this truncation is natural in that it obeys the Jacobi identities.
38 This are some subtle issues here. Both $\rho^{old}$ and $a$ are gauge invariant, as is any linear combination of them. This can change if we drop the $A$ terms in $\rho^{old}$, to project to the LLL. We are doing this LLL projection, and also truncating in $q$ and dropping non-RPA terms. There is no guarantee that a combination that does not mix physical and unphysical states should exist within this truncation, though fortunately it does. Finally, although gauge invariant operators will not mix physical and unphysical states and can be multiplied freely in testing their algebra, the converse is not true— it can be that $\rho^{preferred}$ mixes physical and unphysical states, but miraculously has the right commutator. We have discarded this possibility. This expression for charge also seems to have very special matrix elements between the $p$ and $p + 1$ Landau levels.
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41 The neglected non-RPA terms need some clarification. A non-RPA term may be discarded as oscillating in $\bar{\rho}$ or $\chi$ but may combine with another non-RPA term in the evaluation of a product or commutator to give a nonoscillating term. An analogy might help. In one dimension, when we write the nonrelativistic lattice Fermi field in terms of a Dirac field obtained by linearizing near the Fermi points, the latter will come with oscillating factors $e^{2ikF_n}$ at site $n$, leading to terms that oscillate as $e^{2ikF_n}$ in the non-relativistic charge density (along with terms that do not oscillate). The oscillating terms may be dropped if we are adding a term proportional to the charge density in the hamiltonian. On the other hand, in the charge-charge interaction, these will produce terms that go as $e^{4ikF_n}$.
which may have to be kept when \( k_f = \pi/2 \) to take care of umklapp processes. The correct prescription (say in computing a commutator) is therefore to keep all terms till we have computed the operator of interest and then throw out a terms.

We thank B. Halperin for emphasizing this. One value of this observation is found in the work of Halperin and Stern in Ref. (51).

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**51** Note that our RPA is done on the particles in the FR rather than in the MR. The former are weakly coupled (dipolar or with charge \( e^* \)) so that one may expect RPA to work better on them. We dropped \( H_{II} \) in the MR as RPA for pedagogical reasons. The dedicated reader may verify that if we keep it and transform it ot the FR, it will have no RPA pieces.