Efficiently stabbing convex polygons and variants of the Hadwiger-Debrunner $(p,q)$-theorem

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Stabbing convex polygons
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For every choice of $p$ sets, some $q$ have a common intersection.

Family of 6 convex sets with the $(5, 4)$-property
**Hadwiger-Debrunner \((p, q)\)-Theorem**

**Theorem (Hadwiger and Debrunner)**

If \(\mathcal{F}\) is a finite family of convex sets such that:

- \(\mathcal{F}\) has the \((p, q)\) property;
- \(p\) and \(q\) are "close enough";

Then \(\mathcal{F}\) can be stabbed with \(p - q + 1\) points.

Family of 6 convex sets with the \((5, 4)\)-property
Theorem (Hadwiger and Debrunner)
If $\mathcal{F}$ is a finite family of convex sets such that:

- $\mathcal{F}$ has the $(p, q)$ property;
- $p$ and $q$ are "close enough";

Then $\mathcal{F}$ can be stabbed with $p - q + 1$ points.

Family of 6 convex sets with the $(5, 4)$-property
Problem

How fast can we compute such $p - q + 1$ points for a family of $n$ polygons in the plane?
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**Restriction:**
Polygons of constant size.
A first algorithm

A family of $n$ convex polygons with the $(p, q)$-property.
A first algorithm

Candidate points: leftmost point in pairwise intersections.
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First stabbing point: rightmost candidate point.
A first algorithm

Delete stabbed polygons and restart for next stabbing point.
If $p$ and $q$ are constants:

Runtime for this algorithm

$\approx$

Time to compute the first stabbing point
A first algorithm

If $p$ and $q$ are constants:

Runtime for this algorithm

\[\approx\]

Time to compute the first stabbing point

Naively: $\mathcal{O}(n^2)$ time.
Call $s_1 =$ first stabbing point.
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**Using a technique by Chan (1999):**

Decide if $s_1$ lies to the right of a vertical line in $T(n)$ time

$\implies$

Compute the first stabbing point in $O(T(n))$ expected time
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$\implies$

Compute the first stabbing point in $O(T(n))$ expected time

Can we decide this in subquadratic time?
The decision problem

**Rephrasing the question**
Are there any two intersecting polygons whose intersection lies entirely to the right of some vertical line $\ell$?

**W.l.o.g.**
All polygons intersect $\ell$. 
The decision problem

Agarwal et al. (2002):
Counting the number of pairwise intersections between $n$ convex polygons of constant size can be done in $O(n^{4/3} \log^{2+\epsilon}(n))$ time.

We can use this to test if two polygons intersect exclusively to the right of $\ell$. 
The decision problem

6 pairwise intersections

5 pairwise intersections
Theorem
We can compute \( p - q + 1 \) points stabbing \( F \) in \( O(n^{4/3} \log^{2+\epsilon}(n)) \) expected time.

For polyhedra in \( \mathbb{R}^3 \), a similar method yields an algorithm running in \( O(n^{13/5+\epsilon}) \) expected time.
Ordered-Helly System
A set system with sufficient conditions to carry out a proof similar to the one for the Hadwiger-Debrunner Theorem.
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With this structure, an analogue to the Hadwiger-Debrunner $(p, q)$-theorem can be proven:
Ordered-Helly System
A set system with sufficient conditions to carry out a proof similar to the one for the Hadwiger-Debrunner Theorem.

With this structure, an analogue to the Hadwiger-Debrunner \((p, q)\)-theorem can be proven:

**Theorem**
If \(\mathcal{F}\) is a family of sets of an Ordered-Helly System such that:
- \(\mathcal{F}\) has the \((p, q)\) property;
- \(p\) and \(q\) are "close enough";
Then \(\mathcal{F}\) can be stabbed with \(p - q + 1\) points.
Crucial condition: Existence of a Helly number

$S$ has Helly number $h$:
If $\mathcal{F} \subseteq S$ such that not all sets in $\mathcal{F}$ share a common point, then some $h$ sets in $\mathcal{F}$ do not share a common point.
Crucial condition: Existence of a Helly number

$S$ has Helly number $h$:
If $F \subset S$ such that not all sets in $F$ share a common point, then some $h$ sets in $F$ do not share a common point.

Helly’s theorem: for convex sets in the plane, $h = 3$. 
**Crucial condition: Existence of a Helly number**

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Helly’s theorem: for convex sets in the plane, $h = 3$. 
Examples of Ordered-Helly Systems

- Convex sets in $\mathbb{R}^d$;
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- Convex sets in $\mathbb{R}^d$;
- Convex sets in $\mathbb{R}^d \times \mathbb{Z}^k$;
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- Subtrees of a given tree;
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- Ideals of a given Poset;
Examples of Ordered-Helly Systems

- Convex sets in $\mathbb{R}^d$;
- Convex sets in $\mathbb{R}^d \times \mathbb{Z}^k$;
- Subtrees of a given tree;
- Ideals of a given Poset;
- Abstract convex geometries.
Open questions

• How to these methods translate when dropping the condition that \( p \) and \( q \) are "close enough"?
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• Can we get similar runtimes for polygons of non-constant size?

Thank you for your attention.
**Definition of \((p, q)\)-property**

\(\mathcal{F}\) has the \((p, q)\)-property if \(|\mathcal{F}| \geq p\) and for every choice of \(p\) sets in \(\mathcal{F}\) there exist \(q\) among them which have a common intersection.

**Theorem (Hadwiger and Debrunner)**

Let \(p \geq q \geq d + 1\) and \((d - 1)p < d(q - 1)\), and let \(\mathcal{F}\) be a finite family of convex sets in \(\mathbb{R}^d\).

If \(\mathcal{F}\) has the \((p, q)\)-property, then there exist \(p - q + 1\) points in \(\mathbb{R}^d\) stabbing \(\mathcal{F}\).
A base set $B$ with a total order $\preceq$, a family $C \subset \mathcal{P}(B)$ of "convex sets" and a family $D \subset C$ of "compact sets", such that:

1. $D$ is closed under intersections;
2. For all non-empty $S \in D$, there exists a minimum $x \in S$ with respect to $\preceq$;
3. For all $t \in B$, we have $\{x \in B \mid x \preceq t \text{ and } x \neq t\} \in C$;
4. There exists a Helly number $h$ on $C$.

**Theorem**
Let $p \geq q \geq h$ and $(h - 2)p < (h - 1)(q - 1)$, and let $F$ be a finite subfamily of $C$.
If $F$ has the $(p, q)$-property, then there exists a set of $p - q + 1$ elements in $B$ stabbing $F$. 

