ON THE EXISTENCE OF HADAMARD DIFFERENCE SETS IN GROUPS OF ORDER 400

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ABSTRACT. This paper concerns the problem of the existence of Hadamard difference sets in nonabelian groups of order 400. By introducing a new construction method, we construct new difference sets in 20 groups. We survey non-existence results, verifying non-existence in 45 groups.

1. The aim and results

We consider the existence problem of Hadamard difference sets (HDS) in groups of order 400. These groups will be labelled by $[400, id]$, where $id$ is the group index in the “SmallGroups” library of the software packages GAP [8] and/or MAGMA [3]. The parameters explored are $(400, 190, 90)$. Several authors have addressed this problem in the abelian case which comprises 10 groups. The survey of their results can be found in [5]. It states that the decision is negative in case of 7 abelian groups, while 3 groups are reported as “not decided yet”. These groups are $C_2^2 \times C_4 \times C_5 \cong [400, 201]$, $C_2 \times C_8 \times C_5^2 \cong [400, 111]$ and $C_4^2 \times C_5^2 \cong [400, 108]$; the problem is still open.

Here we focus on the HDS existence problem in nonabelian groups of order 400. We introduce (Section 4) a new method for difference set construction. As a starter, our method needs at least one known difference set with given parameters. The bigger is the set of initial, already known difference sets, the more efficient the method becomes. Making use of the well-known “product method” [6] at the beginning of the construction procedure and then applying our method, we finally construct more than 10000 nonisomorphic $(400, 190, 90)$ difference sets. They prove to exist in 20 groups, those being $[400, id], id \in \{124, 125, 126, 129, 130, 134, 151, 152, 155, 158, 162, 205, 207, 208, 210, 211, 212, 216, 217\}$. The MAGMA records of the constructed structures can be found at the authors’ web site [13] in the folder “MAGMA_REC400”. That folder also provides a file called “Info file 400” on how to handle the included files.

Further, we make use of several necessary conditions for the existence of HDS (cf. [5]) to rule out 38 groups of order 400. Their “SmallGroups” library indices are given in Section 6. This implies, together with the results in the abelian case, that 45 out of 221 groups of order 400 have been decided not to admit HDS existence so far.

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Two \((100, 45, 20)\) difference sets, not found in the literature to our knowledge, are also obtained by the introduced construction method. The third new \((100, 45, 20)\) difference set that occurred in the course of our research is not directly connected with the method.

2. Preliminaries

Let \(\Omega\) be a nonempty finite set and \(B \subseteq 2^\Omega\) a family of nonempty subsets of \(\Omega\). Then the ordered pair \((\Omega, B)\) is an incidence structure. Elements of \(\Omega\) are called points of that incidence structure and elements of \(B\) blocks. Here we consider only simple incidence structures which do not contain repeated blocks.

**Definition 1.** Incidence structure \(D = (\Omega, B)\) with \(|\Omega| = v\) and \(|B| = b\) we call \(t\)-(\(v, k, \lambda\)) design if each block consists of exactly \(k\) points and any \(t\) distinct points are contained in exactly \(\lambda\) blocks, \(t \leq k\) and \(\lambda > 0\). A 2-(\(v, k, \lambda\)) design is called \((v, k, \lambda)\) block design and it is symmetric if \(b = v\).

Incidence structures \((\Omega_i, B_i), i = 1, 2\) are isomorphic if there exists a bijective map \(\varphi : \Omega_1 \rightarrow \Omega_2\) such that \(B_1^\varphi = B_2\). Then \(\varphi\) is called an isomorphism of incidence structures. The set of all isomorphisms of an incidence structure \((\Omega, B)\) onto itself forms its full automorphism group \(\text{Aut}(\Omega, B)\). An automorphism group of \((\Omega, B)\) is any subgroup of the full automorphism group. If there exists an automorphism group acting transitively on the set of points and blocks, then we speak of a transitive incidence structure.

**Definition 2.** A \((v, k, \lambda)\) difference set is a subset \(\Delta \subseteq G\) of size \(k\) in a group \(G\) of order \(v\) with the property that the multiset \(\{xy^{-1} \mid x, y \in \Delta, x \neq y\}\) contains each nonidentity element of \(G\) exactly \(\lambda\) times.

Equivalently, a \((v, k, \lambda)\) difference set \(\Delta\) in group \(G\) can be identified with the element \(\sum_{d \in \Delta} d\) in the integral group ring \(\mathbb{Z}G\) which satisfies the equation \(\Delta\Delta^{-1} = k1_G + \lambda(G \setminus \{1_G\})\), where \(\Delta^{-1} = \sum_{d \in \Delta} d^{-1}\).

The development of a difference set \(\Delta \subseteq G\) is the incidence structure \(\text{dev}\Delta = (G, \{g\Delta \mid g \in G\})\). A mutual correspondence between difference sets and symmetric designs can be established in the well-known way given in the following theorem.

**Theorem 1.** Let \(\Delta \subseteq G\) be a \((v, k, \lambda)\) difference set. Then \(\text{dev}\Delta\) is a regular symmetric \((v, k, \lambda)\) design with respect to \(G \leq \text{Aut}(\Delta)\). Vice versa, let \(D = (\Omega, B)\) be a symmetric \((v, k, \lambda)\)-design with automorphism group \(G\) acting regularly on \(\Omega\). Then, for any point \(p \in \Omega\) and any block \(B \in B\), the set \(\Delta = \{g \in G \mid p^g \in B\}\) is a \((v, k, \lambda)\) difference set in \(G\).

If the automorphism group of a symmetric design acts regularly on points, then it also acts regularly on blocks.

If a difference set \(\Delta \subseteq G\) in a group \(G\) is known, the latter part of Theorem 1 provides also a procedure for obtaining difference sets in regular subgroups of \(\text{Aut}(\text{dev}\Delta)\) distinct from \(G\), if any. Difference sets obtained in this way are obviously isomorphic and inequivalent to \(\Delta\).

Parameter triples of the form
\[(1) \quad (4u^2, 2u^2 - u, u^2 - u), \quad u \in \mathbb{N},\]
determine the Hadamard family of difference sets and/or the Menon family of symmetric designs. Difference sets belonging to the family \((1)\) can be obtained by the well-known “product method” according to the following theorem.
Theorem 2 ([6, p.13]). Let $G$ be a group and $G_1, G_2$ its subgroups with the properties $G = G_1G_2$ and $G_1 \cap G_2 = \{1_G\}$. If difference sets with parameters of the type (1) exist in $G_1$ and $G_2$ for $u = u_1$ and $u = u_2$ respectively, then $G$ contains a difference set with parameters (1) for $u = 2u_1u_2$.

Denoting initial difference sets by $\Delta_1 \subseteq G_1$ and $\Delta_2 \subseteq G_2$, the product difference set $\Delta$ in group $G = G_1G_2$ is described by the formula

$\Delta := (\Delta_1 \overline{\Delta}_2) \cup (\overline{\Delta}_1 \Delta_2)$,

where $\overline{\Delta}_i = G_i \setminus \Delta_i$, $i = 1, 2$. Formula (2) generalizes the one given in [2, p. 368]. Our considered (400, 190, 90) Hadamard difference sets with $u = 10$ can be obtained by the product method from (100, 45, 20) HDSs and trivial HDSs in groups of order 4 which consist of a single element.

In our case of $4u^2 = 400$, the above “pure” product is just one special case of the generalized products introduced in [7, Theorem 5.1]. Namely, let $G$ be a group of order 400, let $H < G$ be its subgroup of index 4, and let $G = H \cup aH \cup bH \cup cH$ be a left coset partition of $G$. Let $\Delta$ be a difference set in $H$. Then, if the conditions of [7, Theorem 5.1] are fulfilled, $\hat{\Delta} = (H \setminus \Delta) \cup a\Delta \cup b\Delta \cup c\Delta$ is a difference set in $G$. We shall call the pair $(G, \hat{\Delta})$ a HDS of the generalized product type. Note that for group $G$ of order 400 and a HDS $(G, \hat{\Delta})$ of the generalized product type there exists at least one subgroup $H < G$ of order 100 such that the sets $g^{-1}(\Delta \cap gH)$ are difference sets in $H$ for all $g \in G$. We use this necessary condition for a HDS to be of the generalized product type in Section 5 to analyze our results.

In Section 4 we develop another method of construction, applicable to transitive incidence structures.

3. TRANSITIVE INCIDENCE STRUCTURES

We henceforth denote by $(\Omega, G, B)$ an incidence structure $(\Omega, B)$ with automorphism group $G \leq \text{Sym}(\Omega)$ acting transitively on sets $\Omega$ and $B$ and such that $B = \{B^g \mid g \in G\}, B \subseteq \Omega$. In this notation a simple fact about transitive incidence structures can be expressed in the following form.

Lemma 1. Incidence structures $I(\Omega, G, B^g)$ and $I(\Omega, G^{g^{-1}}, B)$ are isomorphic for every $\pi \in \text{Sym}(\Omega)$.

In order to prepare the introduction of a new difference set construction method in Section 4, let us consider how to obtain transitive substructures of a given transitive incidence structure $D = I(\Omega, G, B)$ related to some subgroup of $G$. Let a subgroup $H \leq G$ act transitively on $\Omega$ and in $l$ orbits on $B$, $l \in \mathbb{N}$. If we denote by $B_1, \ldots, B_l$ the representatives of $H$-orbits on $B$, then the set

$\{I(\Omega, H, B_i), i = 1, \ldots, l\}$

comprises all transitive substructures of $D$ with the automorphism group $H$. Obviously, there exist $g_i \in G, i = 1, \ldots, l$ with the property $B_i = B^{g_i}$, so the set (3) can be rewritten as $\{I(\Omega, H, B^{g_i}), i = 1, \ldots, l\}$. Lemma 1 implies that $I(\Omega, H, B^{g_i})$ is isomorphic to $I(\Omega, H^{g_i^{-1}}, B)$. Thus, we conclude that searching for transitive substructures of $D$ related to a given automorphism subgroup $H \leq G$ can be performed in a technically convenient manner by obtaining incidence structures $I(\Omega, H^g, B^g)$, where $g$ is taken from the (right) transversal of $H$ in $G$. 

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As we show in the next section, the substructures of our interest are block designs. Therefore we focus on computer search for all possible transitive substructures of a block design $D = I(\Omega, G, B)$ that are block designs themselves. A convenient way to proceed to accomplish the task is the following: first one finds, up to conjugation, all maximal subgroups $M \leq G$ that are transitive on $\Omega$. The next step is to check whether $I(\Omega, M^g, B)$ is a block design, for each $M$ found in the first step and all elements $g$ from the (right) transversal of $M$ in $G$. As groups $M^g$, for which the answer is positive, have to be further analyzed, they are stored and the procedure continues by exploring their maximal subgroups. This repeating the first step on the lower level can be continued as long as one obtains transitive block designs in the succeeding check.

4. CONSTRUCTION METHOD

The following result of Cameron and Praeger is basic for our construction method that we introduce in this section.

**Theorem 3** ([4, Proposition 1.1]). If $I(\Omega, H, B)$ is a $t-(v, k, \lambda)$ design and $H \leq G \leq \text{Sym}(\Omega)$ holds, then $I(\Omega, G, B)$ is a $t-(v, k, \lambda^*)$ design with $\lambda^* \geq \lambda$.

The theorem ensures that a block design can appear as a transitive substructure only within an overstructure which is a block design itself. In that sense, starting from a known difference set, say $\Delta$, we accomplish the construction of new difference sets with the same parameters proceeding in the following two steps:

1) developing a transitive overstructure (of the regular symmetric design corresponding to $\Delta$) which is a block design;

2) exploring the developed block design for sought-after (regular) subdesigns.

Let $\Delta \subseteq H$ be a given difference set and let $G$ be an overgroup of $H$, $H \leq G \leq \text{Sym}(\Omega)$. For any point $\omega \in \Omega$ let $B = \{\omega^g | g \in \Delta\}$. Then $I(\Omega, H, B)$ is the symmetric design corresponding to the starting difference set $\Delta$. Theorem 3 implies that the overstructure $D = I(\Omega, G, B)$ of $I(\Omega, H, B)$ is also a block design. The developed design $D$ remains to be explored for its regular subdesigns. The feasibility of the underlying task of obtaining regular subgroups of $G$ or, if possible, of $\text{Aut}D$ depends on the size of these groups. If the input group size is convenient, one simple command in the software MAGMA [3] returns, up to conjugation, all its regular subgroups.

In this research the choice of overgroup $G$ is made so as to ensure that at least regular subgroups of $G$, together with their transversals in $G$, stay within the reach of MAGMA. Besides, our desirable overgroup $G$ is to contain a considerable number of regular subgroups. Finally, it turns out that holomorph of $H$, denoted by $\text{Hol}(H)$, is an appropriate choice for $G$. Group $\text{Hol}(H)$ is a semidirect product of $\text{Aut}H$ acting on $H$, with the multiplication formula $(\alpha, x)(\beta, y) = (\alpha\beta, x^{\alpha}y)$ for all $x, y \in H$ and $\alpha, \beta \in \text{Aut}H$. The equation $x\alpha y = x\alpha y$ defines an action of group $\text{Hol}(H)$ on set $H$. In this action $H$ acts regularly. If the action of $H$ on $\Omega$ is regular, then $\text{Hol}(H)$ can be observed as embedded in $\text{Sym}(\Omega)$. By this fact we justify our keeping to the so far used notation in the next passage.

Upon developing the design $D = I(\Omega, \text{Hol}(H), B)$, for each regular subgroup $R \leq H$ and for every $\bar{R}$ from the conjugacy class of $R$ in $\text{Hol}(H)$, it is necessary to check whether the structure $I(\Omega, \bar{R}, B)$ is a block design. The designs detected in this search are necessarily symmetric and in the corresponding regular
groups \( \overline{R} \) difference sets are easily read off. Groups \( \overline{R} \) are handled in the convenient form \( R^g \), with \( g \) taken from the (right) transversal of \( R \) in \( Hol(H) \).

5. **New method implemented on parameters (100,45,20) and (400,190,90)**

In the research we had at disposition 15 nonisomorphic difference sets with parameters (100,45,20) known from the literature. They have been proved to exist in only two groups of order 100, seven of them [11, 12] in group [100,11] and eight [9] in group [100,12]. We applied our method on these difference sets, which resulted in obtaining two new, nonisomorphic (100,45,20) difference sets in group [100,12]. In the manner of [9] and using the same presentation

\[ [100,12] = \langle a, b, c \mid a^5 = b^5 = 1, ab = ba, c^4 = 1, c^{-1}ac = a^2, c^{-1}bc = b^3 \rangle, \]

the obtained difference sets, say \( \Delta^{(1)} \) and \( \Delta^{(2)} \), can be written in the form

\[ \Delta^{(1)} = (1 + a + a^3 + a^4) + (a^2 + a^3 + a^4) b + a^2 b^2 + (a^2 + a^4) b^3 + a^2 b^4 + \{ (1 + a + a^2 + a^4) + (a + a^2 + a^3) b + ab^2 + (a + a^2 + a^4) b^3 + (1 + a^2 + a^3 + a^4) b^4 \} c + \{ (1 + a + a^4) + (1 + a^2 + a^4) b + a^2 b^2 + a^2 b^3 + (a + a^2 + a^4) b^4 \} c^2 + \{ (1 + a^2 + a^3) + (1 + a^2 + a^4) b^2 + (a + a^2) b^3 + (1 + a) b^4 \} c^3; \]

\[ \Delta^{(2)} = (1 + a^2 + a^3 + a^4) + a^2 b + (1 + a) b^2 + (1 + a) b^3 + a^2 b^4 + (a + a^2 + a^3 + a^4) b + (a + a^2 + a^3) b^2 + (1 + a + a^2 + a^4) b^3 + a^2 b^4 + (a + a^2 + a^3 + a^4) b + b^2 + (a + a^4) b^3 + a^2 b^4 + (a + a^2 + a^4) + (a^2 + a^3 + a^4) b + b^2 + (a + a^4) b^3 + a^2 b^4 \} c^3. \]

One finds \( Aut \, dev \Delta^{(i)} \cong [100,12], i = 1,2 \). Consequently, \( \Delta^{(1)} \) and \( \Delta^{(2)} \) are not isomorphic to the difference sets given in [9] whose developments have full automorphism groups of order 200, 300 and 1200. Herewith the number of the known nonisomorphic difference sets in group [100,12] increases to 10, which improves the result of [9].

Next we applied the product method, explained in Section 2, on the input factors comprising all already known inequivalent (100,45,20) difference sets (their number exceeds that of the nonisomorphic ones) and two trivial \((4,1,0)\) difference sets in groups of order 4. Among the developments of the resulting (product) difference sets, there were 383 nonisomorphic (400,190,90)-symmetric designs. These designs are given in the file “SD400” on the site [13]. Exploring regular subgroups of their full automorphism groups yielded an additional number of inequivalent (400,190,90) difference sets. Because of the large number and size of the constructed structures, we list here only the 17 nonabelian groups of order 400 in which they exist: \([400, id], id \in \{124, 125, 126, 129, 130, 134, 151, 152, 155, 158, 159, 162, 205, 210, 211, 216, 217 \}\). On the next level we used the mentioned 383 (400,190,90)-symmetric designs as an initial set to launch our new, two-step construction method. In the course of the construction procedure, step \(3^0\) ends in the distinction of regular symmetric designs which can again be subjected to step \(1^0\), i.e. developed into overstructures to be explored for regular subdesigns, etc. In our case the combinatorial task increased in scope as many nonisomorphic regular symmetric designs emerged. Without having exhausted all possibilities of consecutive developing overstructures and checking their substructures, we stopped the procedure at the stage when the number of nonisomorphic (400,190,90)-symmetric designs involved exceeded 10000, but the
absence of new groups appearing in the process was evident and indicative to us. The obtained designs prove the existence of HDSs in 3 more nonabelian groups of order 400, namely groups

\[ (4) \quad [400, id], \quad id \in \{207, 208, 212\}. \]

They appear as automorphism groups of 113, 47 and 258 of our designs, respectively. The corresponding HDSs in groups (4) cannot be obtained by the product method, but only by the generalized products method [7].

Not many different regular automorphism groups turned up when compared to the great number of the developed designs. This might lead to a conclusion that nonabelian groups of order 400 are more likely not to admit a HDS existence than the opposite. An interesting question is whether or not all our constructed difference sets can be obtained by the previously known construction methods; in fact, it is sufficient to check the generalized products method. In regard to this question we analyzed the whole set of exactly 10045 nonisomorphic (400 symmetric) designs reached in the construction process. The set is contained in the file “DS400” [13]. It takes approximately 30 minutes for MAGMA users to load and reconstruct all the designs. We considered each design \( D \) focusing on the set \( \{ G < \text{Aut} D \mid G \text{ is regular} \} \) of all its regular automorphism groups, together with the corresponding difference sets denoted by \( \Delta \). We checked the possibility that at least one among the pairs \((G, \Delta)\) is a HDS of the generalized product type employing the necessary condition given in Section 2. A HDS \((G, \Delta)\) of the generalized product type has the property that the subsets \(g^{-1}(\Delta \cap gH) \subset H, g \in G\), are difference sets in \( H \) for all subgroups \( H < G \) of order 100. Eventually, we detected 703 symmetric designs whose full automorphism group does not contain any regular subgroup \( G \) with the corresponding HDS \((G, \Delta)\) of the generalized product type. These 703 designs, with underlying HDSs not of the product type, are given in the file “NPDS400” [13]. The process of examining subgroups of order 100 revealed a new difference set, say \( \Delta^{(3)} \), in group [100, 11]. In the manner of [12] and using the same presentation

\[ [100, 11] = \langle a, b, c \mid a^5 = b^3 = 1, ab = ba, c^4 = 1, c^{-1}ac = a^2, c^{-1}bc = b^2 \rangle, \]

we write

\[ \Delta^{(3)} = (1 + a^3) + b + (1 + a + a^2) b^2 + (1 + a^3 + a^4) b^3 + a^2 b^4 + \left\{ (1 + a^2 + a^3 + a^4) + (1 + a^2 + a^3) b + (a^2 + a^3 + a^4) b^2 + (1 + a + a^2 + a^3) b^3 + ab^4 \right\} c + \left\{ (1 + a^2 + a^4) + (1 + a^4) b + (1 + a + a^3) b^2 + ab^3 + b^4 \right\} c^2 + \left\{ (a + a^3 + a^4) + (1 + a^4) b + (1 + a) b^3 + (a + a^2 + a^4) b^4 \right\} c^3. \]

One finds \( \text{Aut} \Delta^{(3)} \cong [100, 11] \), so \( \Delta^{(3)} \) is not isomorphic to the 6 difference sets given in [12]. It also proves not to be isomorphic to the first known \((100, 45, 20)\) difference set of Smith [11].

Previously mentioned file “SD400” [13] contains 383+3 symmetric designs that evidence the existence of HDSs in all 20 groups listed in Section 1.

6. Results on the non-existence and final survey

In this section we make use of different necessary group conditions for HDS existence to rule out some nonabelian groups of order 400.

The following theorem by Davis and Jedwab presents the first non-existence criterion considered. Its application is shown in the subsequent example. Let us recall that, given positive integers \( m \) and \( s, m \) is said to be self-conjugate modulo \( s \) if
The well-known Dillon’s “dihedral trick” [6] states that if group
\[ D \]
has a difference set, then any abelian group with \( A \) a semi-direct product of \( A \) and \( B \) has a difference set. It is known that HDSs do not exist in groups
\[ C_2 \]
and Sylow 5-subgroup of \( G/H \) is cyclic. Then \( m \leq 2r^{-1}h \), where \( r \) is the number of distinct prime divisors of \( \gcd(m, w) \).

Example 1. Let \( G \) be a group of order 400 and let \( H \triangleleft G \) have the property
\[ G/H = C_2^2 \times C_{25}. \]
In the notation of Theorem 4 we have \( u = 10 \), \( h = 4 \), \( w = 100 \) and \( \exp(G/H) = 50 \). Then \( m = 5 \) (not coprime to 100) satisfies the conditions of the theorem. It is self-conjugate modulo 50 \((5^1 \equiv -1(\text{mod}2))\), \( \gcd(m, w) = 5 \) and Sylow 5-subgroup of \( G/H \) is cyclic. On the other side, because \( r = 1 \), we have \( 2r^{-1}h = 4 < 5 = m \), which, according to the theorem, contradicts the existence of a HDS in \( G \).

The example above applies to the groups \([400, id], id \in \{20, 21, 22, 23, 24, 25, 26, 27, 45, 46, 47, 48, 55\}\) and rules them out regarding HDS existence. The bold group indices refer to abelian groups.

Given abelian group \( A \), by \( D(A) \) we denote the generalized dihedral group, a semi-direct product of \( A \) by \( C_2 \) in which \( C_2 \) acts on \( A \) by inverting its elements. The well-known Dillon’s “dihedral trick” [6] states that if group \( D(A) \) contains a difference set, then any abelian group with \( A \) as a subgroup of index 2 also contains a difference set. It is known that HDSs do not exist in groups \( C_{16} \times C_{25}, C_{16} \times C_5^2 \) and \( C_2^4 \times C_5^2 \). Accordingly, the dihedral trick rules out generalized dihedral groups \( D(C_8 \times C_{25}) \cong [400, 8], D(C_8 \times C_5^2) \cong [400, 95] \), and \( D(C_2^3 \times C_5^2) \cong [400, 220] \) regarding HDS existence. Next elimination criterion we considered is the one given in the theorem of Alexander et al. [1] which generalizes Dillon’s trick using difference list over a generalized dihedral group.

Definition 3 ([10, p.108]). Let \( H \) be a group of order \( w \). An element \( S = \sum_{h \in H} s_h h \) in the integral group ring \( \mathbb{Z}H \) is called a \( (w, k, s, \lambda) \) difference list over \( H \) if \( s \) and \( k \) are positive integers, \( \lambda \) and the \( s_h \) are non-negative, \( \sum_{h \in H} s_h = k \), and
\[ SS^{-1} = (k - \lambda)1_H + s\lambda H. \]

In the special case when \( s = 1 \) and all the coefficients \( s_h = 0 \) or 1, \( S \) is a \( (w,k,\lambda) \) difference set in \( H \). A simple example of difference list is obtained by extending any group homomorphism \( \varphi : G \rightarrow H \) to the group ring homomorphism \( \hat{\varphi} : \mathbb{Z}(G) \rightarrow \mathbb{Z}(H) \) in the natural way, i.e. defining
\[ \hat{\varphi}\left(\sum_{g \in G} \alpha_g g\right) = \sum_{g \in G} \alpha_g \varphi(g). \]

Let \( G \) be a group with \( (v,k,\lambda) \) difference set \( \Delta \) and let \( N \triangleleft G \) be a normal subgroup of \( G \) of order \( s \) and index \( w \). If \( \pi : G \rightarrow G/N \) is the natural homomorphism onto the quotient group modulo \( N \), then \( \hat{\pi}(\Delta) \) is a \( (w,k,\lambda) \) difference list over \( G/N \).

Theorem 5 ([1, Theorem 4.2]). Let \( K \) be an abelian group with subgroup \( A \) of index 2. If \( (w,k,\lambda) \) difference list exists over \( D(A) \), then a \( (w,k,\lambda) \) difference list exists also over \( K \).
Example 2. In Example 1 it is shown that \((100, 190, 4, 90)\) difference list does not exist over \(C_2^2 \times C_{25}\). The above theorem implies that \((100, 190, 4, 90)\) difference list does not exist over \(D(C_2 \times C_{25}) \cong D_{100}\).

The non-existence of \((100, 190, 4, 90)\) difference list over \(D_{100}\) implies that HDS can not exist in groups of order 400 with quotient groups isomorphic to \(D_{100}\). These groups are \([400, id], \ id \in \{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 54\}\).

The following table is a report on the so far known status of groups of order 400 regarding HDS existence. The positive decision is confirmed by direct construction in 20 groups. Besides previously mentioned 7 abelian groups, the negative decision includes at least 38 nonabelian groups. Bold indices refer to abelian groups.

\[
\begin{array}{|c|c|}
\hline
\text{(400, 190, 90) HDS exist in groups [400, id],} \\
id \in \{124, 125, 126, 129, 130, 134, 151, 152, 155, 158, 159, 162, 205, 207, 208, 210, 211, 212, 216, 217\}, \\
\hline
\text{(400, 190, 90) HDS do not exist in groups [400, id],} \\
id \in \{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 51, 54, 55, 95, 220, 221\}. \\
\hline
\end{array}
\]

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