On reflection positive formulation of chiral gauge theories on a lattice

Sergei V. Zenkin

aInstitute for Nuclear Research of Russian Academy of Sciences, 60th October Anniversary Prospect 7a, 117312 Moscow, Russia

A formulation of chiral gauge theories on a lattice which is both reflection positive and gauge invariant is discussed.

1. INTRODUCTION

In constructing continuum quantum gauge theories from the lattice ones the properties of reflection positivity and gauge invariance of the latter play a fundamental role (see, e.g., [1] and references therein). Reflection positivity allows one to construct the Hilbert space of states with properties therein). Reflection positivity allows one to play a fundamental role (see, e.g., [1] and references therein).

Let the gauge group \( G \) be unitary. Dynamical variables of the theory are the fermion \( 2^D/2 \)-component (Grassmannian) fields \( \psi_n, \overline{\psi}_n \), defined on lattice sites, and gauge variables \( U_{n, n + \hat{\mu}} \in G \), \( U_{n, n - \hat{\mu}} = U_{n + \hat{\mu}, n}^\dagger \) defined on lattice links. Conventionally \( U_{n, n + \hat{\mu}} = \exp[i g a A_n(n + \hat{\mu}/2)] \), where \( A \) is a gauge field which belongs to the gauge group algebra and \( g \) is gauge coupling. We shall use the representation for \( U \) introduced in ref. [3] in terms of “half gauge” variables \( W_{(n, \pm \hat{\mu})} \) associated with each pair \((n, \pm \hat{\mu})\):

\[
U_{n, n + \hat{\mu}} = W_{(n, \hat{\mu})} W_{(n + \hat{\mu}, -\hat{\mu})}^\dagger. \tag{1}
\]

For \( W \) there is no representation in terms of a local field which transforms as an irreducible representation of the group of rotation of the Euclidean space. However to allow for a perturbative consideration we introduce variables \( z_{\pm \hat{\mu}}(n) \), so that \( W_{(n, \pm \hat{\mu})} = \exp[i g a z_{\pm \hat{\mu}}(n)] \).

For a simplicity we consider right-handed fermions being singlets under the gauge group, so the gauge transformations are defined as follows:

\[
\psi_n \to (h_n P_L + P_R)\psi_n, \quad \overline{\psi}_n \to \overline{\psi}_n (h_n^\dagger P_R + P_L), \tag{2}
\]

\[
W_{(n, \pm \hat{\mu})} \to h_n W_{(n, \pm \hat{\mu})}, \tag{3}
\]

where \( h_n \in G, \ P_{L,R} = (1 \pm \gamma_D)/2 \) are chiral projecting operators. For general case of non-singlet \( P_R \psi \) see [3].

A theory with an action \( A[\psi, \overline{\psi}, W] \) is defined by the functional integrals

\[
Z^{-1} \int d\psi_n d\overline{\psi}_n \prod_{n \in \Lambda} dW_{(n, \hat{\mu})} dW_{(n, -\hat{\mu})} \cdot O[\psi, \overline{\psi}, W] e^{-A}, \tag{4}
\]
where $Z$ is the partition function of the theory, $dW_{(n, \pm \hat{\mu})}$ is the Haar measure, and $O$ is some functionals of the dynamical variables.

Let $\Lambda_{\pm}$ denote the equal parts of the lattice with $n_0 > 0$ and $n_0 < 0$, respectively, and let $r$ be such a reflection, that $r \Lambda_{\pm} = \Lambda_{\mp}$. So, the reflection does not change $n_\mu$ and $\hat{\mu}$ for $\mu \neq 0$, while $r n_0 = -n_0 + 1$, $r 0 = -0$.

Given reflection $r$, an antilinear operator $\theta$ is defined as

$$\theta[\overline{\psi}_m] = \overline{\psi}_{rm} \Gamma W_{(m, \pm \hat{\mu})} ... \psi_n,$$

where $\Gamma$ is a matrix.

A theory is called reflection positive if for each functional $O$ of the form $F \theta[F]$, where $F = F[\psi, \overline{\psi}, W]$ is defined on $\Lambda_+$, the integral (4) is non-negative. The sufficient condition for a theory with an action $A$ to be reflection positive is existence of such functionals $B[\psi, \overline{\psi}, W]$ and $C[\psi, \overline{\psi}, W]$ defined on $\Lambda_+$, that $A$ can be represented in the form

$$-A = B + \theta[B] + \sum C_i \theta[C_i].$$

### 3. CONSTRUCTING THE THEORY

We proceed from the Wilson action for free massless fermions:

$$A = a^D \sum_{n_0} \overline{\psi}_n \left[ \frac{1}{2a} (\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}}) \right],$$

$$X_{n, n+\hat{\mu}} = W_{(n+\hat{\mu}, -\hat{\mu})},$$

$$X_{n, n-\hat{\mu}} = W_{(n, -\hat{\mu})},$$

while from (ii), (iii) one has

$$Y_{(n, \hat{\mu})} = \frac{1}{2} (W_{(n, \hat{\mu})} + W_{(n, -\hat{\mu})}),$$

$$Y_{(n, -\hat{\mu})} = \frac{1}{2} (W_{(n, \hat{\mu})} + W_{(n, -\hat{\mu})}).$$

So these requirements determine the action uniquely.

One can rewrite the action and the measure in the functional integrals in terms of $U_{n, n+\hat{\mu}}$ and, say, $W_{(n, \hat{\mu})}$. Then our theory is determined by functional integrals of the form

$$Z^{-1} \int \prod_{n_0} d\psi_n d\overline{\psi}_n \prod_{n \in \Lambda, \mu} dU_{n, n+\hat{\mu}} dW_{(n, \hat{\mu})}$$

$$\cdot O[\psi, \overline{\psi}, U, W] e^{-A_{gauge}} + A[\psi, \overline{\psi}, U, W],$$

where $A_{gauge}$ is a reflection positive action for gauge variables and $O$ is a gauge invariant functional of the dynamical variables.

### 4. DISCUSSION

Owing to reflection positivity this theory is unitary, but, in general, this holds for full Hilbert space including all gauge variables: either $W_{n, \hat{\mu}}$ and $W_{(n, -\hat{\mu})}$, or $U_{n, n+\hat{\mu}}$ and $W_{(n, \hat{\mu})}$. An argument for that the theory may not be unitary in the subspace of the conventional variables $\psi, \overline{\psi}, U$ is that...
explicit integrating over $W_{(\psi, 0)}$ in (11) for operators $O$ independent of such variables leads to a theory whose action does not satisfy condition (6) of reflection positivity. Therefore we must require that unpaired variables $W$ decouple. The price to be paid for this is the main question to this approach.

If $A_{\text{gauge}}$ in (11) is the Wilson plaquette action, formal limit of the full action at $a \to 0$ coincides with the action of the continuum chiral gauge theory with dynamical variables $\psi, \overline{\psi}$, and $A$ (target theory $\mathbb{R}^3$):

$$A = \int d^Dx \left[ -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \overline{\psi} \gamma_\mu (D_\mu P_L + \partial_\mu P_R) \psi \right],$$

where $D_\mu$ is covariant derivative. However decoupling $W$ at the classical level does not guarantee their decoupling in the quantum theory. This is true, however, of right-handed fermions, because action (8) has the shift symmetry $\psi_n \rightarrow \psi_n + \tau P_L$, $\overline{\psi}_n \rightarrow \overline{\psi}_n + \tau P_L$, that guarantees their decoupling in the continuum limit.

To get some idea of what happens to unpaired $W$ we consider in this formulation the chiral Schwinger model, whose perturbative solution (at least in the topologically trivial sector) is known to be exact.

4.1. Two-dimensional example

Let us consider the continuum limit of the effective action

$$W[U, W] = -\ln \prod_{n \in \Lambda} d\psi_n d\overline{\psi}_n e^{-A[\psi, \overline{\psi}, U, W]}.$$

Then, for sufficiently smooth $A$ and $z$, we find

$$W[A, z] = \frac{1}{2} \frac{d^Dq}{(2\pi)^2} \sum_{\mu, \nu} \left[ A_\mu(-q) \left[ \delta_{\mu\nu} q^2 - q_\mu q_\nu \right] + A_\nu(-q) \left[ \delta_{\mu\nu} q^2 - q_\mu q_\nu \right] \right]$$

$$+ \Pi^{AA}_{\mu\nu}(-q) \left[ A_\mu(q) + A_\nu(q) \right] \right].$$

Here $A_\mu(q)$ is $a \to 0$ limit of the gauge invariant combination $A_\mu(q) + 2i \sin(\frac{1}{2} q_\mu a) z_\mu(q)$,

$$\Pi^{AA}_{\mu\nu}(q) = \frac{g^2}{2\pi} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right),$$

$$R^{AA}_{\mu\nu}(q) = \frac{g^2}{4\pi} \left( \epsilon_{\mu\nu q_\alpha q_\sigma} + q_\mu \epsilon_{\nu q_\alpha q_\sigma} \right),$$

and $\Pi^{AA}, \Pi^{A\bar{A}}, \Pi^A$ are some symmetrical matrices independent of $q$. From the Ward identities and the lattice rotation symmetry we have

$$\sum_\mu \Pi^{\bar{A}A}_{\mu\nu} = \sum_\nu \Pi^{\bar{A}A}_{\mu\nu} = 0,$$

and

$$\Pi_{00} = \Pi_{11}, \quad \Pi^{A\bar{A}}_{\mu\nu} = \left[ \Pi^{\bar{A}A}_{\mu\nu} \right] = -2 \Pi_{01}. \quad (17)$$

respectively. Numerically the elements of these matrices depend on infrared regularization used and do not make much sense; we started with the finite lattice which ensures such a regularization, then $\Pi_{00} = 1.959(7) g^2/(2\pi), \quad \Pi_{01} = -0.361(6) g^2/(2\pi)$.

The lessons from this example are following: The formulation ensures decoupling the doubler fermion modes in the external gauge fields. Gauge invariance of the effective action in the case of anomaly fermion contents is provided by producing a Wess-Zumino term. It involves variable $z$ instead of scalar one and reads as

$$(i/2) \int_q \sum_{\mu, \nu} \left( \tilde{A}_\mu K^{AA}_{\mu\nu} \tilde{A}_\nu - A_\mu K^{AA}_{\mu\nu} A_\nu \right).$$

Variable $z$ does not decouple in quantum theory rendering it non-invariant under continuum rotations. Therefore additional efforts are necessary for decoupling this variable.

4.2. Outlooks

Let us note that in terms of $W_{(\nu, 0)}$ and $W_{(\nu, -\bar{\mu})}$ the principal difference between vector and chiral gauge theories is that in the vector theories variables $W$ are always paired forming link variables $U$, while in chiral ones they are splitted by the Wilson term. This means that vector gauge theories have additional symmetry as compared to chiral ones. This symmetry can be defined as invariance of the theory under transformations:

$$W_{(\nu, \pm\bar{\mu})} \rightarrow W_{(\nu, \pm\bar{\mu})} g_{n \pm \bar{\mu}/2}, \quad (18)$$
where \( g_{n+\hat{\mu}/2} \in G \), other variables being non-transformed. Therefore to ensure decoupling of unpaired \( W \) we can require symmetry (18) to hold in the continuum limit.

Obviously this can be achieved by adding to the original action a set of gauge invariant counterterms, so that the theory remains gauge invariant under gauge transformations (2), (3) at any \( a \), but becomes invariant under both transformations (2), (3) and (18) only in the continuum limit. As our two-dimensional example shows the number and explicit structure of such counterterms may crucially depend on whether the theory is anomaly one or not. Indeed, in the case of anomaly free fermion contents, i.e. when counterpart of \( K_{\bar{A}\bar{A}} \) in (14) vanishes, symmetry (18) is restored in the continuum limit by local counterterms of the form

\[
\sum_{n,\mu,\nu,\eta_1,\eta_2} W^\dagger_{(n,\eta_1,\hat{\mu})} Z_{\mu\nu}^{\eta_1 \eta_2} W_{(n,\eta_2,\hat{\mu})},
\]

where \( \eta_{1,2} = \pm \). Note that these counterterms are reflection positive. Owing to relations (16), (17) tuning only two parameters is needed. Then in the continuum limit we come to the Euclidean and gauge invariant in the conventional sense theory (Jackiw’s parameter \( a \) being \( a = 1 \)). But to render the anomaly theory to be invariant under (18) a non-local counterterm is needed, as it follows from (14), (15).

Similar picture is also expected to hold in four dimensions with the gauge fields being dynamical. A likely scenario is that in anomaly free case the theory is determined by tuning a few relevant parameters corresponding to gauge invariant local counterterms, while for an anomaly theory an infinite set of counterterms, including non-local ones, is required. An argument in favour this is that the original action involves no couplings, except the gauge ones and in the terms of variables \( \psi, \bar{\psi}, z_\mu, \) and \( z_{-\mu} \) the theory is renormalizable by power counting. Therefore at small non-renormalized coupling the well-known perturbative results should be reproduced. Then for non-abelian gauge group the values of those relevant parameters could be determined perturbatively due to asymptotical freedom. Certainly, if a chiral gauge theories do exist beyond the perturbation theory and are unique we shall come in the continuum limit to the result of Rome approach \( ^1, ^2 \), but with gauge fixing in principle being unnecessary and with less number of counterterms. In other words reflection positivity and gauge invariance hopefully allows one to project in the continuum limit enlarged Hilbert space (where the theory is unitary at any \( a \)) to the physical one by gauge invariant way not violating the unitarity. This crucially differs this formulation from models with the Wilson-Yukawa couplings \( ^3 \).

Of course, there is a lot of work to do for establishing actual status of this formulation, first of all, perturbative calculations in four dimensions should be done.

5. ACKNOWLEDGMENTS

I am grateful to J. Jersák, H. Joos, M. Lüscher, I. Montvay, G. Münster, D. Petcher, E. Seiler, L. Stufler, T. E. Tomboulis, and M. Tsypin for useful discussions and suggestions. It is a pleasure to thank J. Smit and the organizers of Lattice ’92 for financial support giving me the opportunity to attend the conference.

REFERENCES

1. E. Seiler, Lecture Notes in Physics 159 (Springer-Verlag, 1982).
2. I. Montvay, Phys. Lett. B199 (1987) 89;
L. Lin, I. Montvay, H. Wittig, G. Münster, Nucl. Phys. B335 (1991) 511.
3. S. V. Zenkin, Univ. of Zaragoza preprint DFTUZ/92/7 (1992), to be published in Phys. Lett. B.
4. D. Brydges, J. Fröhlich, E. Seiler, Ann. Phys. (N.Y.) 121 (1979) 227.
5. S. V. Zenkin, Mod. Phys. Lett. A6 (1991) 151; Univ. of Pisa preprint IFUP-TH 45/90 (1990).
6. A. Borrelli, L. Maiani, G. C. Rossi, R. Sisto, M. Testa, Nucl. Phys. B333 (1990) 335.
7. M. F. L. Golterman, D. N. Petcher, Phys. Lett. B225 (1989) 159.
8. R. Jackiw, R. Rajaraman, Phys. Rev. Lett. 54 (1985) 1219.
9. L. Maiani, G. C. Rossi, M. Testa, Phys. Lett. B 261 (1991) 479.