A new proof for a generalization of a Proctor’s formula on plane partitions

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Mathematics Subject Classifications: 05A15, 05C70, 05E99

Abstract

Proctor proved a simple formula for the number of a class of plane partitions contained in a “maximal staircase”. The result is equivalent to the enumeration of lozenge tilings of a hexagon with a maximal staircase removed from some of its vertices. In this paper we give a new proof of a generalization of the Proctor’s result by using Kuo’s graphical condensation.

Keywords: tilings, perfect matchings, quartered hexagon, plane partitions, graphical condensation

1 Introduction

A plane partition is a rectangular array of non-negative integers with weakly decreasing rows and columns. The number of plane partitions contained in a $a \times b$ rectangle with entries at most $c$ is given by MacMahon’s formula $\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$ (see [8]). As a variation of this, Proctor proved a simple product formula for the number of plane partitions with entries at most $c$ which are contained in a shape with row lengths $b + a - 1, b + a - 2, \ldots, b + 1, b$ (see Corollary 4.1 in [10]).

A lozenge tiling of a region on the triangular lattice is a covering of the region by unit rhombi (or lozenges) so that there are no gaps or overlaps. We use notation $L(R)$ for the number of lozenge tilings of a region $R$ ($L(\emptyset) := 1$). The plane partitions contained in an $a \times b$ rectangle with entries at most $c$ are in bijection with lozenge tilings of the hexagon $H_{a,b,c}$ of sides $a, b, c, a, b, c$ (in cyclic order, starting from the north side). In the view of this we have an equivalent form of Proctor’s result as follows.
Theorem 1.1 (Proctor [10]). Assume that $a, b, c$ are non-negative integers so that $b \geq c$. Let $P_{a,b,c}$ be the region obtained from the hexagon $H_{a,b,c}$ by removing the “maximal staircase” from its east corner (see Figure 1.1 for $P_{3,6,4}$). Then

$$L(P_{a,b,c}) = \prod_{i=1}^{c} \left[ \prod_{j=1}^{b-c+1} \frac{a+i+j-1}{i+j-1} \prod_{j=1}^{j=b-c+i} \frac{2a+i+j-1}{i+j-1} \right],$$

(1.1)

where empty products are equal to 1 by convention.

One can find more variations and generalizations of the Proctor’s result in [1]. We consider next a different generalization of Theorem 1.1.

Let $R_{a,b,c}$ be the region described as in Figure 1.2. To precise, $R_{a,b,c}$ consists of all unit triangles on the right of the vertical symmetry axis of the hexagon of sides $2a+1, b, c, 2a+b-c+1, c, b$ (in cyclic order, starting from the north side). Figure 1.2(a) illustrates the region $R_{2,6,3}$ and Figure 1.2(b) shows the region $R_{2,5,3}$ (see the ones restricted by the bold contours). We are interested in the region $R_{a,b,c}$ with $k$ up-pointing unit triangles removed from the base ($k = \lfloor \frac{b-c+1}{2} \rfloor$). If the positions of the triangles removed are $s_1, s_2, \ldots, s_k$, then we denote by $R_{a,b,c}(s_1, s_2, \ldots, s_k)$ the resulting region (see Figures 1.2(b) and (c) for $R_{2,3,6}(2, 3)$ and $R_{2,5,3}$, respectively). The number of lozenge tilings of the region $R_{a,b,c}(s_1, \ldots, s_k)$ is given by the theorem stated below.

Theorem 1.2. Assume $a, b, c$ are non-negative integers. If $b - c = 2k - 1$ for some non-negative integer $k$, then

$$L(R_{a,b,c}(s_1, s_2, \ldots, s_k)) = \prod_{1 \leq i < j \leq k+c} \frac{s_j - s_i}{j-i} \frac{s_i + s_j - 1}{i+j-1},$$

(1.2)

where $s_{k+i} := a + \frac{b-c+1}{2} + i$, for $i = 1, 2, \ldots, c$. 

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Figure 1.2: Obtaining the region $R_{a,b,c}$ from a hexagon ((a) and (b)). Two examples of the region $R_{a,b,c}(s_1, s_2, \ldots, s_k)$ ((c) and (d)).
If $b - c = 2k$ for some non-negative integer $k$, then

$$L(R_{a,b,c}(s_1, s_2, \ldots, s_k)) = \prod_{i=1}^{k+c} \frac{s_i}{2i-1} \prod_{1 \leq i < j \leq k+c} \frac{s_j - s_i}{j - i} \frac{s_i + s_j}{i + j},$$

(1.3)

where $s_{k+i} := a + \frac{b-c}{2} + i$, for $i = 1, 2, \ldots, c$.

By specializing $k = b - c$ and $s_i = i$, for $i = 1, 2, \ldots, k$, the region $P_{a,b,c}$ is obtained from $R_{a,b,c}(1, 2, \ldots, k)$ by removing forced lozenges on the lower-left corner (see Figure 1.3). Thus

$$L(R_{a,b,c}(1, 2, \ldots, k)) = L(P_{a,b,c}),$$

and The Proctor’s Theorem 1.1 follows from Theorem 1.2.

Remark 1.3. We enumerated the lozenge tilings of the region $R_{a,b,c}(a_1, a_2, \ldots, a_k)$ in [6] under the name quartered hexagon (see Theorem 3.1, equations (3.1) and (3.2)). In particular, we identified the lozenge tilings of $R_{a,b,c}(a_1, a_2, \ldots, a_k)$ with certain families of non-intersecting paths on $\mathbb{Z}^2$, then used Linström-Gessel-Viennot Theorem (see [7], Lemma 1; or [12], Theorem 1.2) to turn the number of path families to the determinant of a matrix whose entries are binomial coefficients, and evaluated the determinant.

A perfect matching of a graph $G$ is collection of edges so that each vertex of $G$ is incident to exactly one selected edge. The dual graph $G$ of a region $R$ on the triangular lattice is the graph whose vertices are unit triangle in $R$ and whose edges connect precisely two unit triangles sharing an edge. We have a bijection between the tilings of a region $R$ and the perfect matchings of its dual graph $G$. We use notation $M(G)$ for the number of perfect matching of a graph $G$.

In this paper, we give a new proof of Theorem 1.2 by using a technique called graphical condensation first introduced by Eric Kuo [5]. In particular, we will employ the following theorem in our proof.
Theorem 1.4 (Kuo [5]). Let \( G = (V_1 \cup V_2, E) \) be a planar bipartite graph, and \( V_1 \) and \( V_2 \) its vertex classes. Assume that \( x, y, z, t \) are four vertices appearing on a face of \( G \) in a cyclic order. Assume in addition that \( a, b, c \in V_1, d \in V_2, \) and \(|V_1| = |V_2| + 1\). Then
\[
M(G - \{y\}) M(G - \{x, z, t\}) = M(G - \{x\}) M(G - \{y, z, t\}) + M(G - \{t\}) M(G - \{x, y, z\}).
\]  
(1.4)

2 Proof of Theorem 1.2

We only prove (1.2), as (1.3) can be obtained by a perfectly analogous manner.

It is easy to see that if \( a = 0 \) then the region \( R_{a,b,c}(s_1, s_2, \ldots, s_k) \) has only one tiling (see Figure 2.1(a)). On the other hand, since now \( \{s_1, \ldots, s_k\} = [k] \), the right hand side of the equality (1.2) is also equal to 1, then (1.2) holds for \( a = 0 \). Moreover, if \( b = 0 \), then \( c = 1, k = 0 \), and the region has the form as in Figure 2.1(b). In this case, the region has also a unique tiling; and it is easy to verify that the right hand side of (1.2) equals 1. Thus, we can assume in the remaining of the proof that \( a, b \geq 1 \).

We will prove (1.2) by induction on \( a + b \).

If \( a + b \leq 2 \), then we have \( a = b = 1 \). It is easy to see that there are only 2 possible shapes for the region \( R_{a,b,c} \) (i.e. the region before removed triangles from the base) as in Figures 2.1(c) and (d). Then it is routine to verify (1.2) for \( a = b = 1 \).

For the induction step, we assume that (1.2) is true for any region with \( a + b < l \), for some \( l \geq 3 \), then we need to show that it is also true for any region \( R_{a,b,c}(s_1, \ldots, s_k) \) with \( a + b = l \).

Let \( A := \{s_1, s_2, \ldots, s_k\} \) be a set of positive integers, we define the operators \( \Delta \) and \( \star \) by setting
\[
\Delta(A) := \prod_{1 \leq i < j \leq k} (a_j - a_i) \quad \text{(2.1)}
\]
and
\[
\star(A) := \prod_{1 \leq i < j \leq k} (a_i + a_j - 1). \quad \text{(2.2)}
\]

\(^1\)We use the notation \([k]\) for the set \( \{1, 2, \ldots, k\} \) of all positive integers not exceed \( k \).
Then one can re-write (1.2) in terms of the above operators as:

\[ L(R_{a,b,c}(s_1, \ldots, s_k)) = \frac{\Delta(S) \star (S)}{\Delta([k + c]) \star ([k + c])}. \quad (2.3) \]

where \( S := \{s_1, s_2, \ldots, s_{k+c}\} \) and \([k + c] := \{1, 2, \ldots, k + c\}\). From this stage we will work on this new form of the equality (1.2).

We first consider three special cases as follows:

(i) If \( c = 0 \), then by considering forced lozenges as in Figure 2.1(f), we get

\[ L(R_{a,b,0}(s_1, s_2, \ldots, s_k)) = L(R_{a-q,b-1,1}(s_1, s_2, \ldots, s_k)), \quad (2.4) \]

where \( q = a + \frac{b-c+1}{2} - a_k \). Then (2.3) follows from the induction hypothesis for the region on the right hand side of (2.4).

(ii) If \( k = 0 \), then \( b = c - 1 \); and we get the region \( P_{a,b,b} \) is obtained from the region \( R_{a,b+1}(\emptyset) \) by removing forced lozenges along its base. Thus, (2.3) follows from Proctor’s Theorem 1.1.

(iii) Let \( d := a + \frac{b-c+1}{2} \) (so \( s_{k+i} = d + i \)). We consider one more special case when \( a_k = d \). By removing forced lozenges again, one can transform our region into the region \( R_{a,b-1,c+1}(s_1, \ldots, s_{k-1}) \), then we get again (2.3) by induction hypothesis for the latter region (see Figure 2.1(e)).

From now on, we assume that our region \( R_{a,b,c}(s_1, \ldots, s_k) \) has the two parameter \( k \) and \( c \) positive (so \( b = c + 2k - 1 \geq 2 \)), and that \( a_k < d \).

Now we consider the region \( R \) obtained from \( R_{a,b,c}(s_1, \ldots, s_k) \) by recovering the unit triangle at the position \( s_1 \) on the base. We now apply Kuo’s Theorem 1.4 to the dual graph \( G \) of \( R \), where the unit triangles corresponding to the four vertices \( x, y, z, t \) are chosen as in Figure 2.2 (see the shaded triangles). In particular, the triangles corresponding to \( x \) and \( y \) are at the positions \( s_1 \) and \( d \) on the base; and the ones corresponding to \( z, t \) are on the upper-right corner of the region.

One readily sees that the six regions that have dual graphs appearing in the equation (1.4) of Kuo’s Theorem have some lozenges, which are forced into any tilings. Luckily, by removing such forced lozenges, we still get new regions of \( R_{a,b,c} \)-type. In particular, after removing forced lozenges from the region corresponding to \( G - \{x\} \), we get the region \( R_{a,b-1,c+1}(s_2, s_3, \ldots, s_k) \) (see the region restricted by bold contour in Figure 2.3(a)). This implies that

\[ M(G - \{x\}) = L(R_{a,b-1,c+1}(s_2, a_3, \ldots, a_k)). \quad (2.5) \]

Similarly, we have five more equalities corresponding to other graphs in (1.4):

\[ M(G - \{y\}) = L(R_{a,b,c}(s_1, s_2, \ldots, s_k)) \quad (2.6) \]

\[ M(G - \{z\}) = L(R_{a+1,b-2,c}(s_2, s_3, \ldots, s_k)) \quad (2.7) \]
Figure 2.2: Region to which we apply Kuo’s graphical condensation

\[ M(G - \{y, z, t\}) = L(R_{a,b-2,c-1}(s_1, s_2, \ldots, s_k)) \] (see Figure 2.3(d)), \hspace{1cm} (2.8)

\[ M(G - \{x, z, t\}) = L(R_{a,b-2,c}(s_2, s_3, \ldots, s_k)) \] (see Figure 2.3(e)), \hspace{1cm} (2.9)

\[ M(G - \{x, y, t\}) = L(R_{a-1,b,c}(s_1, s_2, \ldots, s_k)) \] (see Figure 2.3(f)). \hspace{1cm} (2.10)

Plugging the above six equalities (2.5) – (2.10) in (1.4), we have the following recurrence

\[
L(R_{a,b,c}(s_1, s_2, \ldots, s_k)) L(R_{a,b-2,c}(s_2, s_3, \ldots, s_k)) \\
= L(R_{a,b-1,c+1}(s_2, s_3, \ldots, a_k)) L(R_{a,b-2,c-1}(s_1, s_2, \ldots, s_k)) \\
+ L(R_{a+1,b-2,c}(s_2, s_3, \ldots, s_k)) L(R_{a-1,b,c}(s_1, s_2, \ldots, s_k)).
\] \hspace{1cm} (2.11)

The five regions other than \( R_{a,b,c}(s_1, s_2, \ldots, s_k) \) in the above recurrence (2.11) have their \((a + b)\)-parameter less than \( l \). Therefore, by induction hypothesis, we get

\[
L(R_{a-1,b,c}(s_1, \ldots, s_k)) = \frac{\Delta(S \cup \{d\} - \{s_1\}) \star (S \cup \{d\} - \{s_1\})}{\Delta([k + c]) \star ([k + c])},
\] \hspace{1cm} (2.12)

\[
L(R_{a,b-2,c}(s_1, \ldots, s_k)) = \frac{\Delta(S - \{s_{k+c}\}) \star (S - \{s_{k+c}\})}{\Delta([k + c - 1]) \star ([k + c - 1])},
\] \hspace{1cm} (2.13)

\[
L(R_{a+1,b-2,c}(s_2, \ldots, s_k)) = \frac{\Delta(S - \{s_1\}) \star (S - \{s_1\})}{\Delta([k + c - 1]) \star ([k + c - 1])},
\] \hspace{1cm} (2.14)

\[
L(R_{a-1,b,c}(s_1, \ldots, s_k)) = \frac{\Delta(S \cup \{d\} - \{s_{k+c}\}) \star (S \cup \{d\} - \{s_{k+c}\})}{\Delta([k + c]) \star ([k + c])},
\] \hspace{1cm} (2.15)
Figure 2.3: Obtaining the recurrence for $L(R_{a,b,c}(s_1, \ldots, s_k))$. 
Similarly, we get
\[ L(R_{a,b-2,c}(s_2, \ldots, s_k)) = \frac{\Delta(S \cup \{d\} - \{s_1, s_{k+c}\}) \star (S \cup \{d\} - \{s_1, s_{k+c}\})}{\Delta([k+c-1]) \star ([k+c-1])}. \tag{2.16} \]

By the above five equalities (2.12) – (2.16) and the recurrence (2.11), we only need to show that
\[
1 = \frac{\Delta(S \cup \{d\} - \{s_1\})\Delta(S - \{s_{k+c}\}) \star (S \cup \{d\} - \{s_1\}) \star (S - \{s_{k+c}\})}{\Delta(S) \Delta(S \cup \{d\} - \{s_1, s_{k+c}\}) \star (S) \star (S \cup \{d\} - \{s_1, s_{k+c}\})} + \frac{\Delta(S \cup \{d\} - \{s_{k+c}\})\Delta(S - \{s_1\}) \star (S \cup \{d\} - \{s_{k+c}\}) \star (S - \{s_1\})}{\Delta(S) \Delta(S \cup \{d\} - \{s_1, s_{k+c}\}) \star (S) \star (S \cup \{d\} - \{s_1, s_{k+c}\})}, \tag{2.17} \]

and (2.3) follows.

First, we want to simplify the first ratio in the first term on the right hand side of (2.17). We re-write it as
\[
\frac{\Delta(S \cup \{d\} - \{s_1\})\Delta(S - \{s_{k+c}\})}{\Delta(S) \Delta(S \cup \{d\} - \{s_1, s_{k+c}\})} = \frac{\Delta(S \cup \{d\} - \{s_1\})}{\Delta(S)} \cdot \frac{\Delta(S - \{s_{k+c}\})}{\Delta(S \cup \{d\} - \{s_1, s_{k+c}\})}. \tag{2.18} \]

The ratio \( \frac{\Delta(S \cup \{d\} - \{s_1\})}{\Delta(S)} \) has its numerator and denominator almost the same, except for some terms involving \( s_1 \) or \( d \). Cancelling out all common terms of the numerator and denominator, we have
\[
\frac{\Delta(S \cup \{d\} - \{s_1\})}{\Delta(S)} = \frac{\prod_{i=2}^{k}(d - s_i) \prod_{i=1}^{c}(s_{k+i} - d)}{\prod_{i=2}^{k+c}(s_i - s_1)}.
\]

Similarly, we get
\[
\frac{\Delta(S - \{s_{k+c}\})}{\Delta(S)} = \frac{1}{\prod_{i=1}^{k+c-1}(s_{k+c} - s_i)}
\]
and
\[
\frac{\Delta(S \cup \{d\} - \{s_1, s_{k+c}\})}{\Delta(S)} = \frac{\prod_{i=2}^{k}(d - s_i) \prod_{i=1}^{c-1}(s_{k+i} - d)}{\prod_{i=2}^{k+c}(s_i - s_1) \prod_{i=2}^{k+c-1}(s_{k+c} - s_i)}.
\]

Thus, we obtain
\[
\frac{\Delta(S \cup \{d\} - \{s_1\})\Delta(S - \{s_{k+c}\})}{\Delta(S) \Delta(S \cup \{d\} - \{s_1, s_{k+c}\})} = \frac{\Delta(S \cup \{d\} - \{s_1\})}{\Delta(S)} \cdot \frac{\Delta(S - \{s_{k+c}\})}{\Delta(S \cup \{d\} - \{s_1, s_{k+c}\})} = \frac{s_{k+c} - d}{s_{k+c} - s_1}. \tag{2.19} \]

By the same trick, we can simply the first ratio in the second term on the right hand side of (2.17) as
\[
\frac{\Delta(S \cup \{d\} - \{s_{k+c}\})\Delta(S - \{s_1\})}{\Delta(S) \Delta(S \cup \{d\} - \{s_1, s_{k+c}\})} = \frac{\Delta(S \cup \{d\} - \{s_{k+c}\})}{\Delta(S)} \cdot \frac{\Delta(S - \{s_1\})}{\Delta(S \cup \{d\} - \{s_1, s_{k+c}\})} = \frac{d - s_1}{s_{k+c} - s_1}. \tag{2.20} \]
Next, we simply the second ratio in each term on the right hand side of (2.3). By replacing the operator $\Delta$ by the operator $\star$, the whole simplifying-process works in the same way with each term $(s_j - s_i)$ replaced by $(s_i + s_j - 1)$. Thus, we get

$$\frac{\star(S \cup \{d\} - \{s_1\}) \star(S - \{s_{k+c}\})}{\star(S) \star(S \cup \{d\} - \{s_1, s_{k+c}\})} = \frac{s_{k+c} + d - 1}{s_{k+c} + s_1 - 1}$$

(2.21)

and

$$\frac{\star(S \cup \{d\} - \{s_{k+c}\}) \star(S - \{s_1\})}{\star(S) \star(S \cup \{d\} - \{s_1, s_{k+c}\})} = \frac{d + s_1 - 1}{s_{k+c} + s_1 - 1}.$$  

(2.22)

Finally, by (2.19)–(2.22), we can simplify the equation (2.17) to

$$1 = \frac{s_{k+c} - d}{s_{k+c} - s_1} \frac{s_{k+c} + d - 1}{s_{k+c} + s_1 - 1} + \frac{d - s_1}{s_{k+c} - s_1} \frac{d + s_1 - 1}{s_{k+c} + s_1 - 1},$$

(2.23)

which is obviously true with $s_{k+c} = d + c$. This means that (2.17) holds, and so does (1.2).

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