Explicit Form of Solution of Two Atoms
Tavis–Cummings Model

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Abstract

In this paper we consider the two atoms Tavis–Cummings model and give an explicit form to the solution of this model which will play a central role in quantum computation based on atoms of laser–cooled and trapped linearly in a cavity.

We also present a problem of three atoms Tavis–Cummings model which is related to the construction of controlled–controlled NOT operation (gate) in quantum computation.
The purpose of this paper is to give an explicit form to the solution of Tavis–Cummings model (\(\Pi\)) with one and two atoms. This model is a very important one in Quantum Optics and has been studied widely, see [2], [3] or [4] as general textbooks in quantum optics. See also recent papers [5], [6] and their references.

We are studying a quantum computation and therefore want to study the model from this point of view, namely the quantum computation based on atoms of laser–cooled and trapped linearly in a cavity. We must in this model construct a controlled NOT gate or other controlled unitary gates to perform a quantum computation, see [7] as a general introduction to this subject.

For that purpose we need the explicit form of solution of the models with one, two and three atoms. As for the model of one atom it is more or less well-known, and as for the case of two or three atoms it has not been given as far as we know.

In this paper we give it for the case of two atoms, while we could not give it for the three atoms case, so we present it as a challenging problem. Anyway, let us start.

The Tavis–Cummings model (with \(n\)–atoms) that we will treat in this paper can be written as follows (we set \(\hbar = 1\) for simplicity).

\[
H = \omega L \otimes a^\dagger a + \frac{\Delta}{2} \sum_{i=1}^{n} \sigma_i^{(3)} \otimes 1 + g \sum_{i=1}^{n} \left( \sigma_i^{(+)} \otimes a + \sigma_i^{(-)} \otimes a^\dagger \right),
\]

(1)

where \(\omega\) is the frequency of radiation field, \(\Delta\) the energy difference of two level atoms, \(a\) and \(a^\dagger\) are annihilation and creation operators of the field, and \(g\) a coupling constant, and \(L = 2^n\). Here \(\sigma_i^{(+)}\), \(\sigma_i^{(-)}\) and \(\sigma_i^{(3)}\) are given as

\[
\sigma_i^{(s)} = 1_2 \otimes \cdots \otimes 1_2 \otimes \sigma_s \otimes 1_2 \otimes \cdots \otimes 1_2 \ (i \text{ – position}) \in M(L, C)
\]

(2)

where \(s\) is +, – and 3 respectively and

\[
\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(3)

Here let us rewrite the hamiltonian \(\Pi\). If we set

\[
S_+ = \sum_{i=1}^{n} \sigma_i^{(+)} , \quad S_- = \sum_{i=1}^{n} \sigma_i^{(-)} , \quad S_3 = \frac{1}{2} \sum_{i=1}^{n} \sigma_i^{(3)},
\]

(4)
then (1) can be written as
\[ H = \omega 1_L \otimes a^\dagger a + \Delta S_3 \otimes 1 + g \left( S_+ \otimes a + S_- \otimes a^\dagger \right) \equiv H_0 + V, \] (5)
which is very clear. We note that \{S_+, S_-, S_3\} satisfy the \textit{su}(2)–relation
\[ [S_3, S_+] = S_+, \quad [S_3, S_-] = -S_-, \quad [S_+, S_-] = 2S_3. \] (6)
However, the representation \( \rho \) defined by
\[ \rho(\sigma_+) = S_+, \quad \rho(\sigma_-) = S_-, \quad \rho(\sigma_3/2) = S_3 \]
is a reducible representation of \textit{su}(2).

We would like to solve the Schrödinger equation
\[ i \frac{d}{dt} U = HU = (H_0 + V) U, \] (7)
where \( U \) is a unitary operator. We can solve this equation by using the \textbf{method of constant variation}. Let us make a brief review. The equation \( i \frac{d}{dt} U = H_0 U \) is easily solved to be
\[ U(t) = \left( e^{-i\Delta S_3} \otimes e^{-i\omega N} \right) U_0 \]
where \( N = a^\dagger a \) is a number operator and \( U_0 \) a constant unitary. By changing \( U_0 \mapsto -U_0(t) \) and substituting into (7) we have the equation
\[ i \frac{d}{dt} U_0 = g \left\{ e^{it(\Delta - \omega)} S_+ \otimes a + e^{-it(\Delta - \omega)} S_- \otimes a^\dagger \right\} U_0 \] (8)
after some algebras. Here let us assume the resonance condition
\[ \Delta = \omega, \] (9)
which makes the situation simpler. Under this condition the solution of (8) becomes
\[ U_0(t) = e^{-itg(S_+ \otimes a + S_- \otimes a^\dagger)}, \] (10)
so that the full solution of (7) is given by
\[ U(t) = \left( e^{-i\omega S_3} \otimes e^{-i\omega N} \right) e^{-itg(S_+ \otimes a + S_- \otimes a^\dagger)}, \] (11)
where we have dropped the constant unitary operator for simplicity. Therefore we have only to calculate the term (10) explicitly, which is however a very hard task \(^1\). In the

\(^1\)the situation is very similar to that of [11]
following we set
\[ A = S_+ \otimes a + S_- \otimes a^\dagger \]  
(12)
for simplicity. We can determine \( e^{-itgA} \) for \( n = 1 \) (one atom case) and \( n = 2 \) (two atoms case) completely. Let us show.

**One Atom Case**  In this case \( A \) in (12) is written as
\[ A = \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix} . \]  
(13)
Since
\[ A^2 = \begin{pmatrix} aa^\dagger & 0 \\ 0 & a^\dagger a \end{pmatrix} = \begin{pmatrix} N + 1 & 0 \\ 0 & N \end{pmatrix} \equiv D \]  
(14)
with the number operator \( N \) and
\[ A^{2j} = D^j, \quad A^{2j+1} = D^j A, \quad \text{for} \quad j \geq 0, \]
so we have
\[
e^{-itgA} = \sum_{j=0}^{\infty} \frac{(-itg)^{2j}}{(2j)!} A^{2j} + \sum_{j=0}^{\infty} \frac{(-itg)^{2j+1}}{(2j+1)!} A^{2j+1} \\
= \sum_{j=0}^{\infty} (-1)^j \frac{(tg)^{2j}}{(2j)!} D^j - i \sum_{j=0}^{\infty} (-1)^j \frac{(tg)^{2j+1}}{(2j+1)!} D^j A \\
= \sum_{j=0}^{\infty} (-1)^j \frac{(tg\sqrt{D})^{2j}}{(2j)!} - i \frac{1}{\sqrt{D}} \sum_{j=0}^{\infty} (-1)^j \frac{(tg\sqrt{D})^{2j+1}}{(2j+1)!} A \\
= \begin{pmatrix} \cos (tg\sqrt{N+1}) & -i \frac{\sin(tg\sqrt{N+1})}{\sqrt{N+1}} a \\ -i \frac{\sin(tg\sqrt{N})}{\sqrt{N}} a^\dagger & \cos (tg\sqrt{N}) \end{pmatrix}. \]
(15)
We obtained the explicit form of solution. However, this form is more or less well–known, see for example [3], Chapter 14. Because this solution is very convenient, there are many applications, see the textbook [3].

We note that (15) can be decomposed as
\[
\begin{pmatrix} \cos (tg\sqrt{N+1}) & -i \frac{\sin(tg\sqrt{N+1})}{\sqrt{N+1}} a \\ -i \frac{\sin(tg\sqrt{N})}{\sqrt{N}} a^\dagger & \cos (tg\sqrt{N}) \end{pmatrix} = \begin{pmatrix} \cos (tg\sqrt{N+1}) & -i \frac{\sin(tg\sqrt{N+1})}{\sqrt{N+1}} a \\ -i a^\dagger \frac{\sin(tg\sqrt{N+1})}{\sqrt{N+1}} & \cos (tg\sqrt{N}) \end{pmatrix}.
\]
\[
\begin{pmatrix}
1 & -i a^\dagger \tan(\tg \sqrt{N+1}) & 0 \\
-i a^\dagger \frac{\tan(\tg \sqrt{N+1})}{\sqrt{N+1}} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos(\tg \sqrt{N+1}) & 0 & 1 - i \frac{\tan(\tg \sqrt{N+1})}{\sqrt{N+1}} a \\
0 & \frac{1}{\cos(\tg \sqrt{N})} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(16)

We leave the check to the readers.

This is a Gauss decomposition of unitary operator. This may be used to construct a theory of “quantum” representation of a non–commutative group, which is now under consideration.

**Two Atoms Case** In this case \( A \) in (12) is written as

\[
A = \begin{pmatrix}
0 & a & a & 0 \\
0 & a^\dagger & 0 & a \\
0 & a^\dagger & 0 & a \\
0 & a^\dagger & a^\dagger & 0
\end{pmatrix}.
\]

(17)

We would like to look for the explicit form of solution like (15), so we must find a relation like (14). It is not difficult to see

\[
A^2 = \begin{pmatrix}
2(N+1) & 0 & 0 & 2a^2 \\
0 & 2N+1 & 2N+1 & 0 \\
0 & 2N+1 & 2N+1 & 0 \\
2(a^\dagger)^2 & 0 & 0 & 2N
\end{pmatrix},
\]

(18)

\[
A^3 = 2 \begin{pmatrix}
0 & (2N+3)a & (2N+3)a & 0 \\
(2N+1)a^\dagger & 0 & 0 & (2N+1)a \\
(2N+1)a^\dagger & 0 & 0 & (2N+1)a \\
0 & (2N-1)a^\dagger & (2N-1)a^\dagger & 0
\end{pmatrix}.
\]

(19)
From (19) we find a clear relation

\[
A^3 = \begin{pmatrix}
2(2N + 3) \\
2(2N + 1) \\
2(2N + 1) \\
2(2N - 1)
\end{pmatrix}
\begin{pmatrix}
2(2N + 1) \\
2(2N + 1) \\
2(2N - 1)
\end{pmatrix}
\begin{pmatrix}
0 & a & a \\
a^\dagger & 0 & 0 \\
a^\dagger & 0 & 0 \\
0 & a^\dagger & a^\dagger
\end{pmatrix}
\equiv DA.
\]

(20)

This is our key observation. From this it is easy to see

\[
A^{2j} = D^{j-1} A^2 \quad \text{for} \quad j \geq 1, \quad A^{2j+1} = D^j A \quad \text{for} \quad j \geq 0,
\]

so that we have

\[
e^{-itgA} = 1 + \sum_{j=1}^{\infty} \frac{(-itg)^{2j}}{(2j)!} A^{2j} + \sum_{j=0}^{\infty} \frac{(-itg)^{2j+1}}{(2j + 1)!} A^{2j+1} - i \sum_{j=0}^{\infty} \frac{(tg)^{2j+1}}{(2j + 1)!} D^j A
\]

\[
= 1 + \sum_{j=1}^{\infty} (-1)^j \frac{(tg)^{2j}}{(2j)!} D^{j-1} A^2 - i \sum_{j=0}^{\infty} (-1)^j \frac{(tg)^{2j+1}}{(2j + 1)!} D^j A
\]

\[
= 1 + \frac{1}{D} \sum_{j=1}^{\infty} (-1)^j \frac{(tg\sqrt{D})^{2j}}{(2j)!} - \frac{1}{\sqrt{D}} \sum_{j=0}^{\infty} (-1)^j \frac{(tg\sqrt{D})^{2j+1}}{(2j + 1)!}
\]

\[
= 1 + \frac{1}{D} \left\{ -1 + \cos \left( tg\sqrt{D} \right) \right\} A^2 - i \frac{1}{\sqrt{D}} \sin \left( tg\sqrt{D} \right) A
\]

(21)

or more explicitly

\[
e^{-itgA} = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix}
\]

(22)

where

\[
a_{11} = \frac{N + 2 + (N + 1)\cos \left( \frac{tg\sqrt{2(2N + 3)}}{2N + 3} \right)}{2N + 3}, \quad a_{12} = a_{13} = -i \frac{\sin \left( \frac{tg\sqrt{2(2N + 3)}}{2(2N + 3)} \right)}{\sqrt{2(2N + 3)}} a,
\]

\[
a_{14} = \frac{-1 + \cos \left( \frac{tg\sqrt{2(2N + 3)}}{2N + 3} \right)}{2N + 3} a^2, \quad a_{21} = a_{31} = -i \frac{\sin \left( \frac{tg\sqrt{2(2N + 1)}}{2(2N + 1)} \right)}{\sqrt{2(2N + 1)}} a^\dagger,
\]

\[
a_{23} = a_{32} = a_{43} = a_{44} = 0.
\]
\[ a_{22} = a_{33} = \frac{1 + \cos \left( t g \sqrt{2(2N + 1)} \right)}{2}, \quad a_{23} = a_{32} = \frac{-1 + \cos \left( t g \sqrt{2(2N + 1)} \right)}{2}, \]
\[ a_{41} = -1 + \cos \left( t g \sqrt{2(2N - 1)} \right) (a^\dagger)^2, \quad a_{42} = a_{43} = -i \frac{\sin \left( t g \sqrt{2(2N - 1)} \right)}{\sqrt{2(2N - 1)}} a^\dagger, \]
\[ a_{44} = \frac{N - 1 + N \cos \left( t g \sqrt{2(2N - 1)} \right)}{2N - 1}. \]

This is our main result in this paper. The explicit form has not been known in the literature as far as we know.

Since the Tavis–Cummings model has a kind of universal characteristic and the explicit form of solution was given, there must be many applications to Quantum Optics, Mathematical Physics and etc. In the forthcoming paper [12] we will apply this to the construction of controlled–unitary operations (gates) in quantum computation (see for example [7]) based on atoms of laser–cooled and trapped linearly in a cavity.

**Three Atoms Case** In this case \( A \) in (12) is written as

\[
A = \begin{pmatrix}
0 & a & a & 0 & a & 0 & 0 & 0 \\
a^\dagger & 0 & 0 & a & 0 & a & 0 & 0 \\
a^\dagger & 0 & 0 & a & 0 & 0 & a & 0 \\
0 & a^\dagger & a^\dagger & 0 & 0 & 0 & 0 & a \\
a^\dagger & 0 & 0 & 0 & 0 & a & a & 0 \\
0 & a^\dagger & 0 & 0 & a^\dagger & 0 & 0 & a \\
0 & 0 & a^\dagger & 0 & a^\dagger & 0 & 0 & a \\
0 & 0 & 0 & a^\dagger & 0 & a^\dagger & a^\dagger & 0
\end{pmatrix} = \begin{pmatrix}
\tilde{A} & a_{14} \\
a^\dagger_{14} & \tilde{A}
\end{pmatrix},
\] (23)

where \( \tilde{A} \) is \( A \) in (17).

We would like to look for the explicit form of solution like (15) or (22). However, we could not find a relation like (14) or (20) for (23) in spite of much effort (we have calculated \( A^2, \cdots, A^5 \)). We encourage the readers to tackle this problem.

We note that the solution is deeply related to the construction of controlled–controlled NOT operation (gate) in quantum computation, so the explicit form of it is needed.
We conclude this paper by making a comment on our target. The Tavis–Cummings model is based on two energy levels of atoms. However, an atom has in general infinitely many energy levels, so it is natural to use this possibility. We are studying a quantum computation based on multi-level systems of atoms (a qudit theory), [8]–[13]. Therefore we would like to extend the Tavis–Cummings model based on two–levels to a model based on multi–levels. This is a very challenging task!

**Appendix**

In this appendix we show an another approach to obtain the result in the two atoms case which may be useful in the three atoms one. Our method is to reduce the $4 \times 4$–matrix $A$ in (17) to a $3 \times 3$–matrix $B$ in (24) to make our calculation easier. For that aim we prepare the following two matrices

\[
T = \begin{pmatrix}
1 & 1 \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
1
\end{pmatrix}, \quad S = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}.
\]

Then it is easy to see

\[
S (TAT^+)^\dagger S^\dagger = (ST) A (ST)^\dagger = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2}a & 0 \\
0 & \sqrt{2}a^\dagger & 0 & \sqrt{2}a \\
0 & 0 & \sqrt{2}a^\dagger & 0
\end{pmatrix} \equiv \begin{pmatrix}
0 \\
0 \\
0 \\
B
\end{pmatrix}
\]

where

\[
B \equiv \begin{pmatrix}
0 & \sqrt{2}a & 0 \\
\sqrt{2}a^\dagger & 0 & \sqrt{2}a \\
0 & \sqrt{2}a^\dagger & 0
\end{pmatrix} = J_+ \otimes a + J_- \otimes a^\dagger. \quad (24)
\]
\{J_+, J_-\} are just generators of (spin one) irreducible representation of (3). Therefore to calculate $e^{-itgA}$ we have only to do $e^{-itgB}$. The method is almost similar. Namely, 

$$B^2 = 2 \begin{pmatrix} N+1 & 0 & a^2 \\ 0 & 2N+1 & 0 \\ a^{\dagger 2} & 0 & N \end{pmatrix}, \quad B^3 = \begin{pmatrix} 2(2N+3) \\ 0 \\ 2(2N-1) \end{pmatrix}$$

so we obtain

$$e^{-itgB} = 1 + \frac{1}{D} \left\{ -1 + \cos \left( tg\sqrt{D} \right) \right\} B^2 - i \frac{1}{\sqrt{D}} \sin \left( tg\sqrt{D} \right) B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \quad (25)$$

where

$$b_{11} = \frac{N+2 + (N+1) \cos \left( tg\sqrt{2(2N+3)} \right)}{2N+3}, \quad b_{12} = -i \frac{\sin \left( tg\sqrt{2(2N+3)} \right)}{\sqrt{2N+3}} a,$$

$$b_{13} = -1 + \cos \left( tg\sqrt{2(2N+3)} \right) a^2, \quad b_{21} = -i \frac{\sin \left( tg\sqrt{2(2N+1)} \right)}{\sqrt{2N+1}} a^\dagger,$$

$$b_{22} = \cos \left( tg\sqrt{2(2N+1)} \right), \quad b_{23} = -i \frac{\sin \left( tg\sqrt{2(2N+1)} \right)}{\sqrt{2N+1}} a,$$

$$b_{31} = -1 + \cos \left( tg\sqrt{2(2N-1)} \right) (a^\dagger)^2, \quad b_{32} = -i \frac{\sin \left( tg\sqrt{2(2N-1)} \right)}{\sqrt{2N-1}} a^\dagger,$$

$$b_{33} = \frac{N-1 + N \cos \left( tg\sqrt{2(2N-1)} \right)}{2N-1}.$$

We leave the remainder to the readers.

References

[1] M. Tavis and F. W. Cummings: Exact Solution for an N–Molecule–Radiation–Field Hamiltonian, Phys. Rev. 170(1968), 379.

[2] L. Allen and J. H. Eberly: Optical Resonance and Two–Level Atoms, Wiley, New York, 1975.
Note added in the text
After submitting this paper Pablo P. Munhoz kindly informed us of the paper
M.S. Kim, J. Lee, D. Ahn, and P.L. Knight: Entanglement induced by a single-mode heat environment, Phys. Rev. A 65, 040101 (2002), quant-ph/0109052

and suggested seeing the formula (4). We have checked the agreement between the formula (4) and our result, so our result is not new. However, how to derive the formula is not written in the paper and our method seems much simpler (or easy to understand), so our paper is still valuable.

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