A FAMILY OF FUSION SYSTEMS RELATED TO THE GROUPS
\( \text{Sp}_4(p^a) \) AND \( \text{G}_2(p^a) \)

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Abstract. A family of exotic fusion systems generalizing the group fusion systems on Sylow \( p \)-subgroups of \( \text{G}_2(p^a) \) and \( \text{Sp}_4(p^a) \) is constructed.

1. Introduction

In this paper we will construct an infinite series of exotic fusion systems. More precisely for each prime \( p \geq 5 \) we build an exotic fusion system on a \( p \)-group which contains an extraspecial \( p \)-group of order \( p^{p-2} \) of index \( p \) (see Proposition 3.5). The catalyst for this construction came from the authors’ investigation of groups \( G \) that contain a subgroup \( H \) which is an automorphism group of a simple group of Lie type in characteristic \( p \), such that \( |G : H| \) is coprime to \( p \) [5]. In [5, Chapter 16] we apply the results of this article to extend the main theorem of [7] to groups of rank two with some exceptions related to the fact that the fusion systems constructed in this article are exotic. A thorough discussion of fusion systems is presented in [2].

Our construction of the exotic fusion systems develops in two phases. First for an arbitrary finite field \( F \), we define a group \( P \) that generalizes the structure of the normalizer of a root subgroup in \( \text{G}_2(F) \) and \( \text{PSp}_4(F) \) and show that a certain amalgam exists if and only if \( F \) has prime order \( p \) and \( O_p(P) \) is extraspecial of order \( p^{p-2} \). In the second phase, we show that the fusion system determined by the free amalgamated product of the amalgam is saturated and exotic. The smallest of the amalgams is for \( p = 5 \) on a Sylow 5-subgroup of \( \text{Sp}_4(5) \) and has the sporadic group \( \text{Co}_1 \) as a completion; however the fusion systems do not coincide.

2. The amalgams

Let \( F \) be a finite field of characteristic \( p > 0 \), \( F[X, Y] \) be the polynomial algebra in two commuting variables and \( V_m \) the \((m+1)\)-dimensional subspace of \( F[X, Y] \) consisting of homogeneous polynomials of degree \( p - 1 \geq m \geq 1 \). Set \( L = F^\times \times \text{GL}_2(F) \). Then, for \((t, \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right)) \in L\), we define an action of this element on \( V_m \) via

\[
X^aY^b \cdot (t, \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right)) = t(aX + \beta Y)^\alpha(\gamma X + \delta Y)^b
\]

where \( a + b = m \). Since \( m < p \), \( V_m \) is an irreducible \( FL \)-module [3].

Define a bilinear function \( \beta_m \) on \( V_m \) by setting

\[
\beta_m(X^aY^b, X^cY^d) = \begin{cases} 
0 & \text{if } a \neq d \\
\left( \frac{-1}{m} \right)^a & \text{if } a = d
\end{cases}
\]

and extending bilinearly.

Lemma 2.1. The following hold:
(i) $\beta_m$ is alternating if and only if $m$ is odd.
(ii) $\beta_m$ is non-degenerate.
(iii) $\beta_m$ is preserved up to scalars by $L$ and the scale factor of an element $(t, A) \in L$ is $t^2(\det A)^m$.

Proof. Since the matrix associated with $\beta_m$ has zeros everywhere other than on the anti-diagonal, $\beta_m$ is non-degenerate and alternating if and only if $m$ is odd.

Let $G = \{(1, A) \mid A \in \text{SL}_2(\mathbb{F})\}$. We will show that $\beta_m$ is $G$-invariant. For this exercise we forget the first factor of the elements of $G$ and simply work with matrices. We also suppress $\beta_m$. It suffices to prove that the form is invariant under a set of generators of $G$. So observe that $G = \{(\lambda X, \lambda^{-1}Y), (0 1 0), (1 1) \mid \lambda \in \mathbb{F}\}$.

Suppose that $\lambda \in \mathbb{F}$. Then

$$(X^aY^b (\lambda X, \lambda^{-1}Y), X^cY^d (\lambda X, \lambda^{-1}Y)) = ((\lambda X)^a(\lambda^{-1}Y)^b, (\lambda X)^c(\lambda^{-1}Y)^d) = \lambda^{a-b+c-d}(X^aY^b, X^cY^d).$$

Since $(X^aY^b, X^cY^d)$ is only non-zero when $a = d$ (so $b = c$), the form is invariant under these elements. We have

$$(X^aY^b (-1 0 1), X^cY^d (-1 0 1)) = (Y^a(-X)^b, Y^c(-X)^d) = (-1)^{b+d}(X^bY^a, X^dY^c).$$

The last term is non-zero if and only if $c = b$ (so $a = d$). Hence the final term is

$$(-1)^{b+d}(X^bY^a, X^dY^c) = \frac{(-1)^{a+b}}{m} = \frac{(-1)^{a}}{m} = (X^aY^b, X^cY^d)$$

as required.

Finally we consider

$$(X^aY^b (1 0 1), X^cY^d (1 0 1)) = (X^a(X + Y)^b, X^c(X + Y)^d)$$

$$= \sum_{j=0}^{b} \binom{b}{j} X^{a+j} Y^{b-j} \sum_{k=0}^{d} \binom{d}{k} X^{c+(d-k)} Y^k$$

$$= \sum_{j=0, a+j=k}^{b} (-1)^{a+j} \binom{b}{j} \binom{d}{k} \binom{m}{a+j} \frac{b!d!(a+j)!(m-a-j)!}{(b-j)!j!(d-k)!k!m!}$$

$$= \sum_{j=0, a+j=k}^{b} (-1)^{a+j} \frac{b!d!}{j!(d-k)!m!} \sum_{j=0}^{d-a} (-1)^{a+j} \frac{(d-a)!}{j!(d-a-j)!}$$

$$= \frac{b!d!}{m!(d-a)!} \sum_{j=0}^{d-a} (-1)^{a+j} \binom{d}{j}.$$
Now the final term here is zero unless \( d = a \) in which case
\[
\frac{b! a!}{m!(d-a)!} \sum_{j=0}^{d-a} (-1)^{a+j} \binom{d}{j} = (-1)^a \frac{(m-a)!a!}{m!} = (-1)^a \binom{a}{m}
\]
as required to show that the form is invariant. This establishes (ii).

Given (ii), to prove (iii), we note that the matrix \((t, (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}))\) scales the form by \(t^2 \det \lambda m\).

From now on suppose that both \( p \) and \( m \) are odd. The construction of the group which will turn out to be \( O_p(P) \) only requires that \( \beta_m \) is a non-degenerate alternating form. Set
\[
Q = V_m \times \mathbb{F}^+
\]
and define a binary operation on \( Q \) by
\[
(v, y)(w, z) = (v + w, y + z + \beta_m(v, w))
\]
for \((v, y), (w, z) \in Q\). Then, as \( \beta_m \) is alternating, \( \beta_m(v, v) = 0 \) and so \( Q \) is a group.

**Lemma 2.2.** The following statements hold.

(i) If \((v, y) \in Q\), then \(C_Q((v, y)) = \{(w, z) \mid w \in v^\perp, z \in \mathbb{F}\}\).

(ii) The \( p \)-group \( Q \) is special with
\[
Z(Q) = \{(0, \lambda) \mid \lambda \in \mathbb{F}\} = Q' = \Phi(Q).
\]

**Proof.** Let \((w, z) \in C_Q((v, y))\). Then
\[
(w, z)(v, y) = (w + v, z + y + \beta_m(w, v))
\]
and
\[
(v, y)(w, z) = (v + w, y + z + \beta_m(v, w)).
\]

Since \( \beta_m \) is alternating, we see that these two equation are equal if and only if \( \beta_m(v, w) = 0 \). Thus (i) holds and, as \( \beta_m \) is non-degenerate, we have \( Z(Q) = \{(0, \lambda) \mid \lambda \in \mathbb{F}\} \).

Plainly \( Q/Z(Q) \) is abelian of exponent \( p \). Hence to prove (ii), we just need to show that \( Q' \geq Z(Q) \). So we calculate
\[
[(v, y), (w, z)] = (v, y)(-w, -z)(v, y)(w, z)
= (v - w, -y - z + \beta_m(y, z))(v + w, y + z + \beta_m(z, y))
= (0, 2\beta(y, z))
\]
Thus (ii) follows as \( p \) is odd. \( \square \)

For \((t, A) \in L \) and \((v, z) \in Q \) define
\[
(v, z)^{(t, A)} = (tv \cdot A, t^2(\det A)^m z).
\]
Notice that
\[
((v, y)(w, z))^{(t, A)} = (v + w, y + z + \beta_m(v, w))^{(t, A)}
= (tv + w \cdot A, t^2(\det A)^m(y + z + \beta_m(v, w))
= (tv \cdot A + tw \cdot A, t^2(\det A)^m y + t^2(\det A)^m z + \beta_m(tv \cdot A, tw \cdot A))
= (tv \cdot A, t^2(\det A)^m y)(tw \cdot A, t^2(\det A)^m z)
= (v, y)^{(t, A)}(w, z)^{(t, A)}.
\]
Therefore, \( L \) acts on \( Q \).

We define the following subgroups of \( L \):

\[
B_0 = \mathbb{F}^x \times \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{F}^x, \gamma \in \mathbb{F} \right\}
\]

and

\[
S_0 = \{1\} \times \left\{ \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \mid \gamma \in \mathbb{F} \right\}.
\]

Next we form the semidirect product of \( Q \) and \( L \) and some subgroups

\[
P = P(m, \mathbb{F}) = LQ;
\]

\[
B = B_0Q; \text{ and}
\]

\[
S = S_0Q.
\]

Plainly \( B = N_P(S) \).

**Lemma 2.3.** Suppose that \( m < p \). Then the following hold:

(i) \( C_L(Q) = \{ (\mu^{-m}, \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}) \mid \mu \in \mathbb{F}^x \} \);

(ii) \( C_Q(S_0) = \langle (\mu X^m, \lambda) \mid \lambda, \mu \in \mathbb{F} \rangle ; \) and

(iii) \( C_{Q/Z(Q)}(S_0) = C_{Q/Z(Q)}(s) = C_Q(S_0)/Z(Q) \) for all \( s \in S_0^\# \).

**Proof.** Obviously \( C_L(Q) \leq Z(L) \) and so the elements are of type \( (t, \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}) \) Now the action on \((X^m, 0)\) gives \((tm^mX^m, 0)\). Hence \( t = \mu^{-m} \).

As \( Q/Z(Q) \) is an irreducible \( L \)-module, we have with \([9]\) that \( C_{Q/Z(Q)}(S_0) \) is 1-dimensional and so the same applies for all \( 1 \neq s \in S_0 \). Obviously \( S_0 \) centralizes \((X^m, 0)\), which implies (iii). As \([Z(Q), S_0] = 1\), also (ii) follows. \( \square \)

So we have constructed the group \( P \) of our amalgam. Next we will construct \( K \), which is an extension of the natural module by \( \text{SL}_3(\mathbb{F}) \). Hence we will consider \( K \) as the subgroup of \( \text{SL}_3(\mathbb{F}) \) consisting of the matrices of the form

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \alpha & \beta \\
0 & \gamma & \delta
\end{pmatrix}
\]

with determinant 1. Let

\[
C = \left\{ \begin{pmatrix}
1 & 0 & 0 \\
0 & \alpha & \theta \\
0 & \delta & \epsilon
\end{pmatrix} \mid \alpha, \delta, \epsilon \in \mathbb{F}, \theta \in \mathbb{F}^x \right\},
\]

\[
D = \left\{ \begin{pmatrix}
0 & 1 & 0 \\
\delta & 1 & 0 \\
\epsilon & 0 & 1
\end{pmatrix} \mid \alpha, \delta, \epsilon \in \mathbb{F} \right\}
\]

and \( W = O_p(K) \). So

\[
W = \left\{ \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & \theta
\end{pmatrix} \mid \alpha, \delta \in \mathbb{F} \right\}.
\]

Then we have

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & \delta
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & \theta
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \alpha & \epsilon \\
0 & \delta & \epsilon
\end{pmatrix}.
\]

Our objective is to determine under what conditions \( P/C_L(Q) \) has a subgroup, which we call \( W_0 \), such that \( N_{P/Z(P)}(W_0Z(P)/Z(P)) \) is isomorphic to \( C \) in such a way that \( W_0 \) maps to \( W \). Suppose that \( W_0 \) is such a subgroup, and let \( C_0 \) be the preimage of \( N_P(W_0) \). Obviously \( Z(Q) = Z(S) \leq W_0 \) and, after conjugation in \( P \), we may assume that \( N_S(W_0) \in \text{Syl}_p(C_0) \). Suppose that \( W_0 \leq Q \). Then, as \( Q \) is special by Lemma 2.2(ii), \( N_S(W_0) = Q \) and so \( m = 1 \). Furthermore, \( W_0 = \{(u, z) \mid u \in w^k\} \) where \( x \) is an arbitrary member of \( W_0 \setminus Z(Q) \). But then \( S \)
normalizes $W_0$, which is impossible as $S \not\leq C_0$. As $Z(Q) \leq W_0$, $N_S(W_0)$ contains $R = \langle (X^m, \lambda) \mid \lambda \in \mathbb{F} \rangle$ which is the preimage of $C_{Q/Z(Q)}(S)$. Let $Q_0 = C_Q(R)$. Then $W_0Q_0/Q_0$ is normalized by $C_0Q_0/Q_0 \leq B/Q_0$. Since $C_0$ acts irreducibly on $W_0/Z(Q)$, $|W_0Q_0/Q_0| = p^a$ and, since $B$ normalizes $Q$ and $Q_0S$, $Q_0W$ is diagonal to these subgroups. We intend to determine the elements of $B$ which are candidates for the diagonal elements of $C_0$. Now acting on the $S_0Q_0/Q_0$, elements of the form $d = (t, (\lambda \mu))$ normalize $\langle \lambda Y^m \mid \lambda \in \mathbb{F} \rangle + Q_0$ and $Q_0S_0$. Furthermore, $d$ acts by scaling $Y^m$ by $t \mu^m$ and mapping $(\lambda, \theta, \mu \gamma)X^m$ to $(\lambda_{\mu^{-1}}, \theta, \mu \gamma)X^m$. Assume that $d \in C_0$ is in the image of an element of $C$ which acts on $D$ as $\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta^{-1}\end{array}\right)$. Then the entries in $d$ must satisfy

$$t \mu^m = \lambda \mu^{-1} = \theta^{-1}.$$  

Furthermore, using Equation (1) and $(X^m)^d = t\lambda^mX_m$, we additionally require:

$$t\lambda^m = \theta^2.$$  

There is also a third equation forced by the action of $d$ on $Z(Q)$, but it turns out that this is dependent on the former two equations. The first equality in Equation (2) yields $t = \lambda \mu^{-m-1}$ and then combining Equation (2) and (3) gives us

$$1 = \theta^{-2}\theta^2 = t^2\mu^2m\lambda^m = \lambda^m\mu^{-2m}\lambda^{-m} = (\lambda\mu^{-1})^{m+3} = \theta^{-(m+3)}.$$  

Since $\theta$ is an arbitrary element of $\mathbb{F}^\times$, it follows that every element of $\mathbb{F}^\times$ is an $(m+3)$rd root of unity. Thus, $\mathbb{F}$ is finite and letting $p^a = |\mathbb{F}|$, we have $p^a - 1$ divides $m + 3$. Since $m < p$ and $m$ is odd, we deduce that $a = 1$ and either $p = 3$ and $m = 1$ or $p - 1 = m + 3$. Assume that $p = 3$ and $m = 1$. Let

$$U = \langle (1, (\frac{1}{\gamma} \mu)) \mid (\gamma, \mu) \in \mathbb{F} \rangle.$$  

Then $U$ has index 9 in $S$ and $U$ contains no non-trivial normal subgroups of $S$. Hence $S$ is isomorphic to a Sylow 3-subgroup of $A\ell(9)$. In the Sylow 3-subgroup of $A\ell(9)$ we can show that every elementary abelian subgroup of order 9 is contained in the extraspecial subgroup of order 27 or has centralizer of order 27. Hence this case does not occur.

**Proposition 2.4.** Suppose that $P/C_L(Q)$ has a $p$-subgroup $W_0$ such that $N_{P/Z(P)}(W_0Z(P)/Z(P))$ is isomorphic to $C$ in such a way that $W_0$ maps to $W$. Then $\mathbb{F}$ has prime order $p$ and $p = m + 4$.

So from now on suppose that $p = m + 4$ and $\mathbb{F}$ has order $p$. In particular, by Lemma 2.3 $Q$ is extraspecial of order $p^{m+3}$ and of exponent $p$ and $P$ is isomorphic to a subgroup of $p^{m+3}$: $\text{Sp}_{p-3}(p)$ which is isomorphic to a subgroup of $\text{Sp}_{p-1}(p)$. In this representation the Jordan form of a $p$-element of $S$ has no blocks of size $p$ and consequently $S$ has exponent $p$. We now explicitly show that when $p = m + 4$, then $C$ is isomorphic to a subgroup of $P/C_L(Q)$. To do this, we first write down a candidate for $W_0$ and then determine its normalizer.

Define

$$w(\alpha) = \sum_{j=0}^{m} \frac{\alpha^{j+1}}{(j+1)} \binom{m}{j} X^j Y^{m-j} \in V_m$$

and set

$$W_0 = \langle (1, (\frac{1}{\gamma} \mu))(w(\gamma), \delta) \mid \gamma, \delta \in \mathbb{F} \rangle \leq S.$$
To calculate explicitly in the extraspecial group $Q$, we need the following facts:

**Lemma 2.5.** Suppose that $\lambda, \mu \in \mathbb{F}$. Then

(i) $\beta_m(\lambda X^m, w(\mu)) = -\lambda \mu$; and

(ii) $\beta_m(w(\lambda), w(\mu)) = \frac{(\lambda - \mu)^{m+2} - \lambda^{m+2} + \mu^{m+2}}{(m+1)(m+2)}$.

*Proof.* For the first part, as the coefficient of $Y^m$ in $w(\mu)$ is $\alpha$, we have

$$\beta_m(\lambda X^m, w(\mu)) = \beta_m(\lambda X^m, \mu Y^m) = -\lambda \mu.$$  

For part (ii), we calculate

$$\beta_m(w(\lambda), w(\mu)) = \beta_m(\beta_m^{m+1}(\lambda^{m+1} X^{m+1}Y^{m-j} - \sum_{j=0}^{m} \frac{\lambda^{m-j+1}}{(j+1)} X^j Y^{m-j}))$$

$$= \sum_{j=0}^{m} \frac{\lambda^{j+1}}{(j+1)} \sum_{j=0}^{m} \frac{\mu^{j+1}}{(j+1)} X^j Y^{m-j}$$

$$= \sum_{j=0}^{m} \frac{\lambda^{j+1}}{(j+1)} (-\mu)^{m+2-(j+1)} \frac{m+2}{j+1}$$

$$= \frac{(\lambda - \mu)^{m+2} - \lambda^{m+2} + \mu^{m+2}}{(m+1)(m+2)}.$$

□

**Lemma 2.6.** Let $a, b, \lambda, \mu \in \mathbb{F}$ with $a$ and $b$ non-zero. Then

$$w(\lambda)^{(1, (a \mu \ b))} = \frac{b^{m+1}}{a} \left( w\left( \frac{a \lambda + \mu}{b} \right) - w\left( \frac{\mu}{b} \right) \right).$$

*Proof.*

$$w(\lambda)^{(1, (a \mu \ b))} = \left( \sum_{j=0}^{m} \frac{\lambda^{j+1}}{(j+1)} (a X)^j (\mu X + b Y)^{m-j} \right)^{(1, (a \mu \ b))}$$

$$= \sum_{j=0}^{m} \frac{\lambda^{j+1}}{(j+1)} (a X)^j (\mu X + b Y)^{m-j}$$

$$= \sum_{j=0}^{m} \frac{\lambda^{j+1}}{(j+1)} (a X)^j \left( \sum_{k=0}^{m-j} \frac{m-j}{k} (\mu X)^k (b Y)^{m-j-k} \right).$$
We now determine the coefficient of $X^eY^{m-e}$:

\[
\sum_{f=0}^{e} \frac{\lambda^{f+1} a^f \mu^{e-f} b^{m-e}}{f+1} (m-f) \left( \frac{m-f}{e-f} \right) \mu^{e-f} b^{m-e} = \sum_{f=0}^{e} \frac{\lambda^{f+1} a^f \mu^{e-f} b^{m-e}}{f+1} \left( \frac{e}{e} \right) \left( \frac{m}{f+1} \right) = b^{m-e} a(e+1) \sum_{f=0}^{e} \lambda^{f+1} a^f \mu^{e-f} b^{m-e} \left( \frac{m}{f+1} \right) = b^{m+1} a(e+1) \left( \frac{a \lambda + \mu}{b} \right)^{e+1} \left( \frac{\mu}{b} \right)^{e+1}.
\]

Therefore,

\[
w(\lambda)^{(1, \frac{a}{\lambda}, \frac{0}{b})} = b^{m+1} a \left( w \left( \frac{a \lambda + \mu}{b} \right) - w \left( \frac{\mu}{b} \right) \right),
\]

as claimed.

One of the nice consequences of Lemma 2.6 is that $W_0$ is a subgroup of $S$. By the discussion before Proposition 2.4, we have $N_P(W_0)Q/Q$ is isomorphic to a subgroup of

\[
\{ (\frac{a}{b^{m+1}}, \frac{a}{b^{e}}) \mid a, b \in F \}.
\]

On the other hand, Lemma 2.6 shows that elements of the form $(\frac{a}{b^{m+1}}, (\frac{a}{b^{e}})) (0, 0)$ normalize $W_0$.

**Lemma 2.7.** We have

\[
N_P(W_0) = \{ (\frac{a}{b^{m+1}}, (\frac{a}{b^{e}}))(w(\lambda) + \tau X^m, \theta) \mid a, b \in F^\times, \lambda, \tau, \theta \in F \}.
\]

In a moment we shall write down a homomorphism from $N_P(W_0)$ onto $C$. To check that this is a homomorphism the following remark is helpful. (Note that it uses $p = m + 4$.)

**Lemma 2.8.** Suppose that

\[
x = (\frac{a}{b^{m+1}}, (\frac{a}{b^{e}}))(w(\lambda) + \tau X^m, \theta)
\]

and

\[
y = (\frac{c}{d^{m+1}}, (\frac{c}{d^{e}}))(w(\mu) + \sigma X^m, \phi)
\]

are elements of $N_P(W)$. Then

\[
xy = \left( \frac{ac}{bd} \right)^{(m+1)} (\frac{ac}{bd} + \mu) (w(\frac{cb\lambda + bd\mu}{bd}) + (\frac{e^{m+1}}{e^{m+1}} + \tau + \sigma) X^m,
\]

\[
\frac{c^{m+2}}{d^{m+2}} \theta + \phi + \left( \frac{(\frac{a}{d})^{m+2} - (\frac{c}{d}^{m+2} + \mu)^{m+2} + \mu^{m+2}}{(m+1)(m+2)} \right) - \frac{c^{m+1}}{d^{m+1}} \tau \mu + \frac{c \lambda \sigma}{d}.
\]

Now a straightforward calculation using Lemma 2.8 shows that the map $\Theta$ defined by

\[
(ab^{-m-1}, (\frac{a}{b}, \frac{0}{b}))(w(\lambda) + \tau X^m, \theta) \mapsto \left( \frac{1}{b^{(m+1)}}, \frac{1}{b^{(m+1)(m+2)}} \right)
\]

is a homomorphism from $N_P(W_0)$ onto $C$. To check that this is a homomorphism the following remark is helpful. (Note that it uses $p = m + 4$.)
is a surjective homomorphism from $N_P(W_0)$ to $C$ with kernel $C_L(Q)$.

Combining the above discussion with Proposition 2.4 yields

**Theorem 2.9.** Suppose that $p$ is an odd prime and $m \leq p-1$ is also odd. Set $P = P(m,F) = LQ$. Then $P/C_L(Q)$ has a $p$-subgroup $W_0$ such that $N_P/Z(P)(W_0Z(P)/Z(P))$ is isomorphic to $C$ in such a way that $W_0$ maps to $W$ if and only if $F$ has prime order $p$ and $p = m + 4$.

□

Using the homomorphism $\Theta$ we can build the free amalgamated product

$$G = P/C_L(Q) \ast_C K.$$

3. THE FUSION SYSTEMS

Saturated fusion systems were designed by Puig to capture the $p$-local properties of defect groups of $p$-blocks in representation theory. Given a group $X$ and $p$-subgroup $T$, the fusion system $\mathcal{F}_T(X)$ is a category with objects the subgroups of $T$ and, for objects $P$ and $Q$, the morphisms from $P$ to $Q$, $\text{Hom}_{\mathcal{F}_T(X)}(P,Q)$, are the conjugation maps $c_x$ where $x \in X$ and $P^x \leq Q$. If $X$ is a finite group and $T \in \text{Syl}_p(X)$, then $\mathcal{F}_T(X)$ is saturated. An exotic fusion system is a saturated fusion system which is not $\mathcal{F}_T(X)$ for any finite group $X$ with $T \in \text{Syl}_p(X)$. For an extensive introduction to fusion systems we recommend Craven’s book [2].

In this section we construct a fusion system from the free amalgamated product $G = P/C_L(Q) \ast_C K$. Set $P_1 = P/C_L(Q)$ and identify $P_1$ and $K$ with their images in $G$, set $C = P_1 \cap K$ and $D = O_p(C)$. We identify subgroups of $S$ with their images in the quotient $P_1$. Thus $D = S \cap K$, $Q = O_p(P_1)$ and $W = O_p(K)$. We intend to show that $\mathcal{F} = \mathcal{F}_{S}(G)$ is an exotic saturated fusion system. We see that $\mathcal{F}$ contains two sub-fusion systems $\mathcal{F}_{S}(P_1)$, $\mathcal{F}_{D}(K)$ and since $P_1$ and $K$ are finite groups with Sylow $p$-subgroups $S$ and $D$ respectively, these fusion systems are saturated. A finite $p$-subgroup $T$ of an infinite group $X$ is a Sylow $p$-subgroup of $X$ provided every finite $p$-subgroup of $X$ is $X$-conjugate to a subgroup of $T$.

**Lemma 3.1.** We have that $S$ is a Sylow $p$-subgroup of $G$ and

$$\mathcal{F} = \langle \mathcal{F}_{S}(P_1), \mathcal{F}_{D}(K) \rangle$$

is the smallest fusion system on $S$ which contains both $\mathcal{F}_{S}(P_1)$ and $\mathcal{F}_{D}(K)$.

**Proof.** This follows with [6, Theorem 1]. □

The free amalgamated product $G$ determines a graph $\Gamma$. This graph has vertices all the cosets of $P_1$ in $G$ and all the cosets of $K$ in $G$. Two vertices are adjacent precisely when they have non-empty intersection. By [3, Theorem 7], $\Gamma$ is a tree and, by construction, $G$ acts transitively on the edges of $\Gamma$ and has two orbits on the vertices of $\Gamma$. Moreover, $\Gamma$ is bi-partite. We let $\alpha = P_1$ and $\beta = K$ (vertices of $\Gamma$) and note that $(\alpha, \beta)$ is an edge. For $\gamma \in \Gamma$, $\Gamma(\gamma)$ denotes the set of neighbours of $\gamma$ in $\Gamma$. The stabiliser $G_\gamma$ of $\gamma$ in $G$ is $G$-conjugate to either $P_1$ or $K$ and, especially, $G_{\alpha} = P_1$ and $G_{\beta} = K$. Finally, we note that $G_\gamma$ operates transitively on $\Gamma(\gamma)$. For a subgroup $X$ of $G$ denote by $\Gamma^X$ the subgraph of $\Gamma$ fixed vertex wise by $X$. Since $\Gamma$ is a tree, so is $\Gamma^X$.

In the proof of part (iii) of the next lemma, we need to consider centric subgroups of $\mathcal{F}$. These are subgroups $T$ of $S$ such that $C_{S}(T\alpha) = Z(T\alpha)$ for all $\alpha \in \text{Hom}_{\mathcal{F}_{S}(G)}(T,S)$. Note that centric subgroups have order at least $p^2$. 

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Lemma 3.2. If \( X \leq S \) has order at least \( p^2 \), then \( \Gamma^X \) is finite. Furthermore,

(i) \( X \) does not fix a path of length 5 with middle vertex a coset of \( P_1 \).
(ii) \( N_G(X) \) is conjugate to a subgroup of either \( P_1 \) or \( K \);
(iii) \( \mathcal{F} \) is a saturated fusion system.

Proof. Assume that \( X \) is a \( p \)-subgroup of \( G \) of order at least \( p^2 \). Assume the path \( \pi = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \) of length 5 is in \( \Gamma^X \) with \( \alpha_3 \) a coset of \( P_1 \). Then, as \( G \) acts edge transitively on \( \Gamma \), we may conjugate \( X \) inside \( \Gamma \) so that \( \alpha_2 = \beta \) and \( \alpha_3 = \alpha \). Then \( X \leq G_\alpha \cap G_\beta = P_1 \cap K = C \). Thus \( X \leq D \) as \( X \) is a \( p \)-group.

Now \( X \) also fixes \( \alpha_1 \) and so, as any two Sylow \( p \)-subgroups of \( K \) intersect in \( W \) and \( X \) has order at least \( p^2 \), we have that \( W = X \). Similarly, \( X = O_p(G_\alpha) \) and consequently \( X = W = O_p(G_\beta) = O_p(G_\alpha) \). Now \( G_\alpha \) acts transitively on \( \Gamma(\alpha) \) and so there exists \( g \in G \) so that \( G_{\alpha^g} = G_\beta \). But then \( g \) normalizes \( W \). Since, by Lemma 3.4, \( G_\beta = K \geq C = N_{P_1}(W) = N_{G_\alpha}(W) \), we see that \( \beta = \alpha_4 \) and this is a contradiction. This proves (i). Using (i), we see that \( X \) fixes no paths of length 6 and so \( |\Gamma^X| \) is finite. In particular, if \( X \) is a centric subgroup in \( \mathcal{F} \) then \( |\Gamma^X| \) is finite.

Since \( \Gamma^X \) is finite and \( \Gamma \) is bipartite, we now see with [8] Corollary, page 20 that \( N_G(X) \) fixes a vertex of \( \Gamma \). This proves (ii).

Finally, application of [1] Corollary 3.4] yields that \( \mathcal{F} \) is saturated and this is (iii).

A subgroup \( X \) of \( S \) is fully \( \mathcal{F} \)-normalized provided \( |N_S(X)| \geq |N_S(X_\alpha)| \) for all \( \alpha \in \text{Hom}_\mathcal{F}(X, S) \).

Lemma 3.3. We have \( W \) is centric and fully \( \mathcal{F} \)-normalized.

Proof. Suppose that \( \alpha \in \text{Hom}_\mathcal{F}(W, S) \) and \( |N_S(W_\alpha)| > |N_S(W)| \). We have that \( \alpha = c_g \) for some \( g \in G \) with \( W^g \leq S \). Since \( |N_S(W^g)| > p^3 \), \( P_1 \) is the unique vertex of \( \Gamma \) fixed by \( N_S(W^g) \). Since \( W^g \) fixes \( Kg \) and \( Kgh \) for all \( h \in N_S(W^g) \), we infer that \( W^g \) fixes a path of length at least 5 with \( P \) at the middle vertex, this contradicts Lemma 3.2(i). Hence \( W \) is fully \( \mathcal{F} \)-normalized. Since \( |N_S(W)| = p^3 \), it also follows that \( W \) is centric.

Lemma 3.4. The fusion system \( \mathcal{F} = \mathcal{F}_S(G) \) is exotic.

Proof. Suppose \( \mathcal{F} \) is not exotic and let \( H \) be a finite group with Sylow \( p \)-subgroup \( S \) such that \( \mathcal{F}_S(H) = \mathcal{F}_S(G) \). Then there exists subgroups \( Q_0 \) and \( W_0 \) of \( S \) such that \( \text{Aut}_H(Q_0) = \text{Aut}_\mathcal{F}(Q) \) and \( \text{Aut}_H(W_0) = \text{Aut}_\mathcal{F}(W) \). Let \( K_0 = N_H(W_0) \) and \( P_0 = N_H(Q_0) \).

We may assume \( O_{p'}(H) = 1 \). Let \( N \) be a minimal normal subgroup of \( H \). Then, as \( p \) divides \( |N| \) and \( Z(S) \) has order \( p \), \( Z(S) \leq N \). The action of \( K_0 \) implies \( W = \langle Z(S)^{K_0} \rangle \leq N \). Hence \( Z_2(S) \leq [Q, S] \leq N \) and finally \( S = QW = \langle Z_2(S)^{K_1} \rangle W \leq N \). Hence \( N = O_{p'}(H) \). Since \( \langle S^{P_1} \rangle \) is not a \( p \)-group, \( N \) is a direct product of isomorphic non-abelian simple groups. Moreover, as \( |Z(S)| = p \), \( N \) is a simple group and \( C_H(N) = 1 \). Therefore \( H \) is an almost simple group. We now consider the finite simple groups as given by the classification theorem.

Recall that \( p \geq 5 \), \( |S| = p^{p-1} \) and \( S \) is of exponent \( p \) and is not abelian. In particular, \( H \) is not an alternating group.

Suppose that \( H \) is a Lie type group in characteristic \( p \). Then, by the Borel -Tits Theorem, \( K \) is contained in some parabolic subgroup \( L \) of \( H \) and \( W_0 \leq O_p(K_0) \leq
$O_p(L)$. As $W_0$ is centric by Lemma 3.3 (i), $W_0 \geq Z(O_p(L))$. Since $W_0$ is fully $\mathcal{F}$-normalized, $W_0$ is not normal in $L$ and so $Z(O_p(L)) < W_0$, contrary to $K_0 < L$ and $W_0$ being a minimal normal subgroup of $K_0$. Hence $H$ is not of Lie type in characteristic $p$.

Assume that $H$ is of Lie type in characteristic $r \neq p$. If $p$ does not divide $|Z(H)|$, where $\hat{H}$ is the universal version of $H$, then application of [4, Theorem 4.10.3(e)] to $W_0$ shows $p = 5$ and $H = E_8(r^a)$. Since $|S| = 5^4$ and $5^5$ divides the order of $E_8(r^a)$, this is impossible. So we may assume that $p$ divides $|Z(H)|$. In particular $H \cong PSL_n(r^a)$ or $PSU_n(r^a)$ and $p$ divides $(r^a - 1, n)$ in the first case and $(r^a + 1, n)$ in the second. Therefore $S$ contains a toral subgroup of order $p^{n-2} \geq p^{p-2}$, as $p$ divides $n$. Since $Q_0$ is extraspecial of order $p^{p-2}$ and $|S| = p^{p-1}$, and the largest abelian subgroup of $Q_0$ is of order $p^{(p-1)/2}$, we infer that $(p - 1)/2 + 1 \geq p - 2$, and then $p = 5$. Thus $H$ contains a subgroup $L \cong 5^3.Sym(5)$. Since this does not embed in $P_1$, this is impossible.

Finally suppose that $H$ is a sporadic simple group. Inspection of the lists in [4, Table 5.3] shows the either $H = Co_1$ and $p = 5$ or $H = F_1$ and $p = 7$. In the first case we again observe a subgroup $5^3.PGO_3(5)$, which does not exist in $\mathcal{F}$. In the second case $Aut_H(Q) \cong 6.\text{Alt}(7)$ which is impossible.

We conclude that $\mathcal{F}$ is exotic. 

We summarize the above result in a way that we can easily cite it in [5].

**Proposition 3.5.** Let $p \geq 5$ be a prime. Set $P_1 = QL$, where $Q$ is extraspecial of order $p^{p-2}$, $L \cong GL_2(p)$ and $L'$ induces the irreducible module of homogeneous polynomials in $X, Y$ of degree $p - 4$ on $Q/Z(Q)$. Then for $S$ a Sylow $p$-subgroup of $QL$, there is an exotic fusion system on $S$ which extends $\mathcal{F}_{QL}(S)$.

We close the paper by remarking that (when $m = p - 4$) the amalgam $B/C_L(Q)*_C K$ also provides an exotic fusion system. This example has exactly one class of essential subgroups.

**References**

[1] M. Clelland, Chr. Parker, Two families of exotic fusion systems, J. Algebra, 323, 2010, 287 - 304.

[2] David A. Craven, The theory of fusion systems. An algebraic approach. Cambridge Studies in Advanced Mathematics, 131. Cambridge University Press, Cambridge, 2011.

[3] R. Brauer, C. Nesbitt, On the modular characters of groups, Ann. Math. 42, 1941, 556-590.

[4] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups.Number 3, Part I Chapter A, Almost simple $K$-groups, Math. Survey Monogr. vol 40.3, Amer. Math. Soc. 1998.

[5] Christopher Parker, Gerald Pientka, Andreas Seidel, Gernot Stroth, Groups which are almost groups of Lie type in characteristic $p$, manuscript.

[6] G.R. Robinson. Amalgams, blocks, weights. fusion systems and simple groups, J. Algebra 314, 2007, 912 –923.

[7] R. Salarian, G. Stroth, Existence of strongly $p$-embedded subgroups, Comm. in Algebra, to appear.

[8] Serre, Jean-Pierre. Trees. Translated from the French by John Stillwell. Springer-Verlag, Berlin-New York, 1980.

[9] S. Smith, Irreducible modules and parabolic subgroups, J. Algebra 75, 1982, 286–289.

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