GRAVITATIONAL LENSING EFFECT ON COSMIC MICROWAVE BACKGROUND ANISOTROPIES: A POWER SPECTRUM APPROACH

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ABSTRACT

The effect of gravitational lensing on cosmic microwave background (CMB) anisotropies is investigated using the power spectrum approach. The lensing effect can be calculated in any cosmological model by specifying the evolution of gravitational potential. Previous work on this subject is generalized to a non-flat universe and to a nonlinear evolution regime. Gravitational lensing cannot change the gross distribution of CMB anisotropies, but it may redistribute the power and smooth the sharp features in the CMB power spectrum. The magnitude of this effect is estimated using observational constraints on the power spectrum of gravitational potential from galaxy and cluster surveys and also using the limits on correlated ellipticities in distant galaxies. For realistic CMB power spectra the effect on CMB multipole moments is less than a few percent on degree angular scales, but gradually increases towards smaller scales. On arcminute angular scales the acoustic oscillation peaks may be partially or completely smoothed out because of the gravitational lensing.

Subject headings: gravitational lenses, cosmic microwave background — cosmology: large-scale structure of the universe

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1. Introduction

The effect of gravitational lensing on the cosmic microwave background anisotropies has been studied in the past by several groups (Kashlinsky 1988; Blanchard & Schneider 1987; Cole & Efstathiou 1989; Sasaki 1989; Tomita & Watanabe 1989; Linder 1990a,b; Cayón, Martínez-González & Sanz 1993a,b; Bassett et al. 1994; Fukushige et al. 1994). Using different approaches these authors came to very different conclusions about the importance of the effect. Cole & Efstathiou (1989) used a nonlinear CDM model and found a small effect on CMB. Cayón et al. (1993a) used a linear model and found an appreciable effect on arcminute angular scales for some models. On degree angular scales they also found a negligible effect. On the other hand, using different approaches such as the Dyer-Roeder distance-redshift relation or simplified N-body simulations Bassett et al. (1994) and Fukushige et al. (1994) found a significant effect even on degree angular scales.

There are several shortcomings of these studies that do not allow one to draw a firm conclusion on the importance of the lensing effect on CMB. First, the studies are based on a particular cosmological model and the results could change significantly if the model is changed. While some groups (e.g. Cole & Efstathiou 1989; Cayón et al. 1993a,b) attempted to assess this uncertainty by presenting results for different viable cosmological models, others (e.g. Bassett et al. 1994; Fukushige et al. 1994) used models that do not allow a direct comparison with existing observational constraints and thus may not even be realistic models. A second shortcoming of previous studies is that they do not fully include the evolution of large-scale structure in their models. While Cole & Efstathiou (1989) calculated the effect only at late epochs when the matter is in the nonlinear regime, Cayón et al. (1993a,b) only included the linear evolution, whereas Bassett et al. (1994) and Fukushige et al. (1994) neglected any evolution at all and assumed that the universe did not change from a certain redshift until today.

The purpose of this paper is to provide a more realistic answer on the importance of the effect by using observational constraints on large-scale structure distribution and properly including its evolution. The method used is based on the power spectrum approach in linearized gravity and is similar to the one used by Linder (1990a,b), Kaiser (1992) and Cayón et al. (1993a,b). An equivalent method based on optical scalars has been developed by Gunn (1967) and extended by Blandford & Jaroszynski (1981). Present work differs from previous studies in that I also include the nonlinear effects by modeling the power spectrum evolution in the nonlinear regime. By comparing the nonlinear calculation to the linear approximation one can identify the angular scale where the nonlinear effects become important. I also extend the calculation to the case of an open (or closed) universe and correct some erroneous expressions in the literature, all of which allows to calculate the
lensing effect in any standard cosmological model (i.e. in any model based on a weakly perturbed metric in a universe that is homogeneous and isotropic on large scales). The estimate of the lensing effect on the CMB is based on the observational constraints on the power spectrum and on the ellipticity correlations of distant galaxies, which enables to assess its magnitude in our universe. The results are presented in terms of the CMB anisotropy power spectrum, which allows one to discuss the effect independent of the observational strategy. In the conclusion section I discuss the possible sources of discrepancy between present results and some of the previous work on this subject.

2. Formalism

In this section I review the formalism to compute the gravitational lens effect on a pair of propagating photons separated by an angle \( \theta \) at the observer’s position. The starting point is a perturbed Robertson-Walker model with small-amplitude scalar metric fluctuations. In the longitudinal gauge (Bardeen 1980) one can write the line element as (adopting c=1)

\[
\begin{aligned}
ds^2 &= a^2(\tau) \left[ -(1 + 2\phi) d\tau^2 + (1 - 2\phi) [d\chi^2 + \sin^2 K \chi (d\theta^2 + \sin^2 \theta d\phi^2)] \right],
\end{aligned}
\]

where the metric is expressed with comoving spherical coordinates and conformal time \( \tau \) and \( a(\tau) \) denotes the expansion factor. I defined

\[
\sin_K \chi \equiv \begin{cases} 
K^{-1/2} \sin K^{1/2} \chi, & K > 0 \\
\chi, & K = 0 \\
(-K)^{-1/2} \sinh(-K)^{1/2} \chi, & K < 0
\end{cases}
\]

Curvature \( K \) can be expressed using the present density parameter \( \Omega_0 \) and the present Hubble parameter \( H_0 \) as \( K = (\Omega_0 - 1) H_0^2 \). The total density \( \Omega(\tau) \) is in general time dependent and can have contributions from mass density \( \Omega_m(\tau) \) or vacuum energy density \( \Omega_v(\tau) \), \( \Omega(\tau) = \Omega_m(\tau) + \Omega_v(\tau) \). The metric perturbation \( \phi \) can be interpreted as the Newtonian potential since, neglecting the contributions from wavelengths larger than the Hubble distance, it obeys the cosmological Poisson equation

\[
\nabla^2 \phi = \frac{3}{2} H^2 \Omega_m(\tau) a^2 \delta,
\]

where \( \delta \) is the mass density fluctuation. Statistical properties of the potential on scales small compared to the curvature scale can be described with its Fourier transform \( \phi(k, \tau) \).
where $\phi(\vec{x}, \tau) = \int d^3k \phi(\vec{k}, \tau) e^{i\vec{k} \cdot \vec{x}}$. Its ensemble mean and variance are $\langle \phi(\vec{k}, \tau) \rangle = 0$ and $\langle \phi(\vec{k}, \tau) \phi^*(\vec{k}', \tau) \rangle = P_\phi(k, \tau) \delta^3(\vec{k} - \vec{k}')$, where $P_\phi(k, \tau)$ is the power spectrum of the potential at time $\tau$.

A photon propagating through the universe will be deflected by the mass concentrations along its path. The rate of change in the photon direction $\vec{n}$ is given by the photon geodesic equation, which applied to the metric in equation [1] gives

$$\frac{d\vec{n}}{dl} = -2\vec{n} \times (\vec{n} \times \vec{\nabla} \phi) \equiv -2\vec{\nabla}_\perp \phi, \quad (4)$$

where the symbol $\vec{\nabla}_\perp \phi$ denotes the transverse derivative of the potential and $l$ is the comoving path length along the photon geodesic. Gravitational potential $\phi$ can be viewed as providing a force deflecting the photons while they propagate through the unperturbed space-time, described by a 3-sphere (closed universe), 3-hyperboloid (open universe) or Euclidean space (flat universe). Because the only observable photon direction is that at the observer’s position it is convenient to propagate photons relative to their final direction (i.e. backwards in time). Gravitational lensing is not expected to lead to large deflection angles (e.g. Linder 1990; Seljak 1994) and one can replace the transverse derivatives in equation (4) with the transverse derivatives with respect to the observed direction of the photon or with respect to any other direction that has a small angular separation with the photon. In this spherical plane approximation the observed direction of the photon can be described with a two-dimensional angle $\vec{\theta}$ with respect to the origin. Moreover, the null geodesic condition for photons gives $d\tau \approx -d\chi$ neglecting corrections of the order $O(\phi)$. Even when metric perturbations are present, one can continue to parametrize the geodesic with the unperturbed comoving radial distance $\chi$ or the conformal time $\tau$, which are related through $\chi = \tau_0 - \tau$, where $\tau_0$ is the conformal time today.

The total deflection angle between the photon source at the last-scattering surface$^1$ and the observer is given by

$$\delta\vec{\alpha} = -2 \int_0^{\chi_{rec}} \vec{\nabla}_\perp \phi d\chi. \quad (5)$$

Similarly, the photon angular excursion on the last-scattering surface relative to its observed value is given by

$$\delta\vec{\theta} = -2 \int_0^{\chi_{rec}} W(\chi, \chi_{rec}) \vec{\nabla}_\perp \phi d\chi,$$

$^1$I assume throughout the paper that CMB anisotropies are generated at a recombination time $\tau_{rec} \approx 0$ and therefore $\chi_{rec} \approx \tau_0$. This is a good approximation for all the models where the CMB fluctuations are generated at a high redshift.
\[ W(\chi, \chi_{\text{rec}}) = \frac{\sin K (\chi_{\text{rec}} - \chi)}{\sin K \chi_{\text{rec}}}. \]  

(6)

In a flat universe the latter simplifies to \( W(\chi, \chi_{\text{rec}}) = (1 - \chi/\chi_{\text{rec}}) \). Note that it is \( \delta \vec{\theta} \) that is relevant for the discussion of lensing effects on CMB, because one is interested in the angular excursion of a photon on the CMB last-scattering surface and not in the change in its direction. Some of the previous work on this subject used \( \delta \vec{\alpha} \) instead of \( \delta \vec{\theta} \) (e.g. Cayón et al. 1993a,b). As shown by Seljak (1994), in a flat \( \Omega_m = 1 \) linear theory this leads to a factor of \( 10^{1/2} \) overestimate of the relative dispersion between two photons. In the following I will restrict the discussion to \( \delta \vec{\theta} \).

Two photons \( A \) and \( B \) observed with an angular separation \( \theta \) have a different angular separation when emitted from the source position. Its mean is equal to the unperturbed value, while the dispersion is given by (Seljak 1994),

\[
\sigma(\theta) = 2^{-1/2} \left\langle \left[ \delta \vec{\theta}^A - \delta \vec{\theta}^B \right]^2 \right\rangle^{1/2}_\theta = \left[ C_{\text{gl}}(0) - C_{\text{gl}}(\theta) \right]^{1/2}
\]

\[
C_{\text{gl}}(\theta) = 16\pi^2 \int_0^{\infty} k^3 dk \int_0^{\chi_{\text{rec}}} P_\phi(k, \tau = \tau_0 - \chi) W^2(\chi, \chi_{\text{rec}}) J_0(k\theta \sin K \chi) d\chi,
\]  

(7)

where \( \langle \rangle_\theta \) denotes the ensemble average performed over all pairs of photons with a fixed observed angular separation \( \theta \) and \( J_0(x) \) is the Bessel function of order 0. The derivation of equation (4) is based on Limber’s equation in Fourier space (e.g. Kaiser 1992), which assumes that the dominant scales contributing to the dispersion are much smaller than the photon travel distance. This condition is satisfied for sources at cosmological distances. No assumption on the power spectrum has been made and equation (6) can be used both in the linear and in the non-linear regime, both of which can be described by a time evolution of the power spectrum \( P_\phi(k, \tau) \). I neglected the Poisson contribution arising from the discrete nature of galaxies, which is only important if \( \Omega_{m0} < 0.1 \) (Blandford & Jaroszynski 1981, Cole & Efstathiou 1989). An important approximation needed to derive equation (7) is that the potential sampled by the perturbed photon geodesic can be replaced with the potential along the unperturbed geodesic. This limits its applicability to the regime where \( \sigma(\theta)/\theta \ll 1 \). When \( \sigma(\theta)/\theta \) becomes of order unity the above assumption is no longer valid and the separation between the photons may grow significantly faster than what the model would predict. This point will be discussed again in the last section.

It is useful to give a physical understanding of the lensing effect on two nearby photons. For simplicity I restrict the discussion to scattering on a single scale \( k^{-1} \) and to a flat space. In real space \( k^{-1} \) is a correlation length and determines the scale on which regions become uncorrelated. For sufficiently small angles the photons separated by distance \( \chi \theta \) propagating through a region of size \( k^{-1} \) are coherently deflected (assuming \( \chi \theta < k^{-1} \)).
The change in the relative angle between the two photons crossing this region is given by
\[ \delta \theta_1 \approx \chi \theta \nabla_\perp (2k^{-1} \nabla_\perp \phi) \approx 2k \chi \theta \phi. \]
The photons pass through \( N \approx k \chi_{\text{rec}} \) uncorrelated regions and the total rms deflection angle between the photons grows in a random walk fashion, \( \delta \theta = N^{1/2} \delta \theta_1 \) or \( \sigma(\theta)/\theta \propto (k \chi_{\text{rec}})^{3/2} \phi \). Adding the contributions from different modes one reproduces, numerical factors aside, the small angle limit of equation 6. For large separation angles the scattering is incoherent (\( \chi \theta > k^{-1} \)) and each photon is deflected by \( (k \chi_{\text{rec}})^{1/2} \phi \), independently of \( \theta \), implying \( \sigma(\theta)/\theta \to 0 \). This asymptotic behavior is confirmed by the numerical results presented in the next section.

Once \( \sigma(\theta) \) is known as a function of \( \theta \) it is straightforward to calculate the lensing effect on the CMB fluctuations. The effect is most easily expressed in terms of the temperature anisotropy correlation function \( C(\theta) = \langle \frac{\Delta T}{T} A \Delta T / T_B \rangle \). Using the two-dimensional formalism and isotropic approximation presented in the Appendix one obtains the modified correlation function \( \tilde{C}(\theta) \),
\[
\tilde{C}(\theta) = \frac{1}{\sigma^2(\theta)} \int_0^\pi \beta d\beta C(\beta)e^{-(\beta^2+\theta^2)/2\sigma^2(\theta)} I_0 \left[ \frac{\theta \beta}{\sigma^2(\theta)} \right],
\]
where \( I_0 \) is the modified Bessel function of order 0. This equation is strictly valid only for gaussian fluctuations, but should give a reasonable estimate of the effect even when this condition is not satisfied. Note that the effect of lensing is to integrate the correlation function with approximately a gaussian centered at \( \theta \) with dispersion \( \sigma(\theta) \), as can be seen using the asymptotic expansion of \( I_0 \) combined with the exponential in equation 8. Thus, lensing acts as a filter smoothing out the sharp features in the correlation function. For lensing to be important the correlation function at \( \theta \) must be changing rapidly on a scale \( \sigma(\theta) \).

It is customary to present different models of CMB anisotropies in terms of the power spectrum, given by the multipole moments \( C_l \). These are obtained by the Legendre transform of correlation function, \( C_l = 2\pi \int_0^\pi \sin \theta C(\theta) P_l(\cos \theta) d\theta \), where \( P_l(x) \) is the Legendre polynomial of order \( l \). The lensing effect on the \( C_l \) multipoles can be efficiently calculated using the Gauss-Legendre integration of \( \tilde{C}(\theta) \) in equation 8. One can estimate the effect on \( C_l \) by assuming \( \epsilon = \sigma(\theta)/\theta \) is a constant (Bond 1995). While this is not true in general (figure 4), one may hope to use this approximation if \( \sigma(\theta)/\theta \) is sufficiently slowly changing with \( \theta \) and is small, so that only correlations over a narrow range of \( \theta \) are mixed by lensing. In this case \( \epsilon \) should be determined by its value at \( \theta \propto l^{-1} \) (Bond 1995). From equation 8 follows in the \( \epsilon \ll 1 \) limit
\[
\tilde{C}_l = \int_0^\infty C_{l'} \frac{dl'}{\sqrt{2\pi \epsilon l'}} e^{-(l-l')^2/2(\epsilon l')^2}. \]
Lensing thus smoothes the spectrum of \( C_l \) with a gaussian of relative width \( \epsilon \), similar to the effect on the correlation function.
One way to calculate $\sigma(\theta)$ is to use the observational constraints on the power spectrum from the large-scale structure observations, carefully including the effects of the evolution in a given cosmological model. This approach will be explored in the next section. A somewhat less model-dependent estimate can be obtained from the observational constraints on correlated distortions of distant galaxy images. This can be described by $p(\theta)$, the average polarization within a circular aperture of radius $\theta$, which describes the correlations in the ellipticities of galaxy images as a function of angle $\theta$. It is related to the power spectrum using a $\Omega \neq 1$ generalization of the expression given by Blandford et al. (1991) and Kaiser (1992),

$$p^2(\theta) = 16\pi^2 \int_0^\infty k^5 dk \int_0^{\chi_\text{g}} P_\phi[k, \tau = \tau_0 - \chi] W^2(\chi, \chi_\text{g}) \sin^2 K \frac{[2J_1(k\theta \sin K \chi)][k\theta \sin K \chi]}{k^3} d\chi,$$

where I assumed for simplicity that all the galaxies lie at the same source position $\chi_\text{g}$. Using a small argument Taylor expansion of Bessel functions in equations 7 and 10 one obtains a simple scaling between $\sigma(\theta)$ and $p(\theta)$ for a flat $\Omega_{m0} = 1$ universe in the linear regime, independent of the power spectrum on small angular scales, $\sigma(\theta)/\theta = 2^{-1}p(\theta)(\chi_{\text{rec}}/\chi_\text{g})^{3/2}$. Nonlinear effects and $\Omega_{m0} < 1$ make the low redshift contributions more important relative to the case above, which decreases $\sigma(\theta)/\theta$ derived from $p(\theta)$. Numerical evaluation confirms this prediction and so the scaling above can be used to give an upper limit on $\sigma(\theta)/\theta$ from the observational limits on $p(\theta)$ on arcminute scales.

3. Estimate of the Lensing Effect in Our Universe

To compute the lensing effect one needs to specify the power spectrum of potential as a function of scale and time. In linear regime the time dependence of density perturbation in a CDM dominated universe obeys the well known growing mode solution. For the particular case of $\Omega_{m0} = 1$ universe the potential does not change in time and lensing contributions at early times are as important as those at late times. For the nonlinear evolution of the power spectrum one can either use N-body simulation results or adopt a semianalytic approximation for it. Here I adopted the approximation given by Hamilton et al. (1991), generalized to $\Omega \neq 1$ by Peacock & Dodds (1994) and to the density power spectra with slopes $n < -1$ by Mo, Jain & White (1995). This prescription is based on an educated guess of what the evolution of the density correlation function should be in the nonlinear regime. Although not exact, it agrees well with the results of N-body simulations (see Peacock & Dodds 1994 and Mo et al. 1995 for a detailed discussion of its applicability) and should give a good estimate of the nonlinear power spectrum in the regime where
dissipative baryonic processes can be neglected. The linear to nonlinear mapping is most easily expressed using the mass density variance $\Delta^2(k)$, which is related to the potential power spectrum via $\Delta^2(k) = 16\pi k^7 P_\phi(k)/9\Omega_m(\tau)^2 H^4 a^4$. The relation between the linear and nonlinear power spectrum is given by

$$\Delta^2(k_{nl}) = f_{nl}[\Delta^2(k_l)]; \quad k_l = [1 + \Delta^2(k_{nl})]^{-1/3} k_{nl}$$

$$f_{nl}(x) = \frac{1 + 0.2 \beta x + (Ax)^{\alpha \beta}}{1 + ([Ax]^{\alpha} g^3(\Omega_m, \Omega_v, a)/(11.68 x^{1/2}))^{\beta}}^{1/\beta},$$

(11)

where $A = 0.84 [g(\Omega_m, \Omega_v, a)]^{0.2}, \alpha = 2/g(\Omega_m, \Omega_v, a)$ and $\beta = 2g(\Omega_m, \Omega_v, a)$. The linear growth factor $g(\Omega_m, \Omega_v, a)$ can be approximated with a few percent accuracy as (Lahav et al. 1991; Carroll, Press & Turner 1992)

$$g(\Omega_m, \Omega_v, a) \approx \frac{5\Omega_m}{2 [X(1 + [\Omega_m/aX]^{0.6}) - a^2 \Omega_v/(\Omega_m + 2a)]}$$

$$X = 1 + \Omega_m(a^{-1} - 1) + \Omega_v(a^2 - 1).$$

(12)

Mapping in equation (11) can be improved by allowing for the variation in the shape of the power spectrum. One can introduce an effective index of the density power spectrum $n_{eff} = d\ln[k^4 P_\phi(k)]/d\ln k$ at a wavevector $k$ defined such that the rms mass fluctuations averaged within a sphere of size $k^{-1}$ is unity ($\sigma_{k^{-1}} = 1$). For $n_{eff} = 0$ the mapping above gives reliable results, while for $n_{eff} < -1$ there are substantial deviations from the N-body simulations. As shown by Mo et al. (1995), one can improve this by replacing $\Delta^2$ by $\Delta^2/B(n_{eff})$, where

$$B(n) = 0.795 \left[ \frac{\Gamma \left( \frac{17+n}{10+2n} \right)}{\Gamma \left( \frac{11+n}{10+2n} \right)} \right]^{-(5+n)}.$$  

(13)

In this case a better fitting formula for the $\Omega_m = 1$ case is given by (Mo et al. 1995)

$$f_{nl}(x) = x \left( \frac{1 + 2x^2 - 0.6x^3 - 1.5x^{7/2} + 1x^4}{1 + 0.0037x^3} \right)^{1/2}.$$  

(14)

The system of equations presented above can be used to calculate $\sigma(\theta)/\theta$ for most cosmological models of current interest (one exception being the models with massive neutrinos on small scales where neutrino free streaming is important). To obtain an estimate of the lensing effect in our universe I will use observational constraints on the power spectrum, as compiled by Peacock & Dodds (1994). The power spectrum can be parametrized with a CDM type linear transfer function (Bardeen et al. 1986) with two free parameters, the amplitude $\sigma_8$, determined by the mass fluctuation averaged within a sphere of radius $8h^{-1}$Mpc and the shape parameter $\Omega_m h$, determined by the turnover position.
in the power spectrum (\( h \) is the present day Hubble parameter in units of 100km/s/Mpc). For wavevectors between \( 10^{-2} \) and \( 1 \) hMpc\(^{-1} \) all the galaxy and cluster surveys are in a reasonable agreement with a CDM type linear power spectrum with \( \Omega_{m0} h \approx 0.25 \) (Peacock & Dodds 1994; da Costa et al. 1994). For normalization I will adopt \( \sigma_8 = 0.8 \), which is close to the normalization obtained by Peacock & Dodds (1994) and by White, Efstathiou & Frenk (1993) using the cluster abundances normalization over most of the interesting range of \( \Omega_{m0} \). This normalization also agrees with the COBE normalization for the currently favored \( \Omega_{m0} = 0.4, h = 0.65 \) case, both in the open universe model (Sorski et al. 1993) and in the cosmological constant dominated model with a modest tilt (Stompor, Górski & Banday 1995). The adopted power spectrum is likely to be within a factor of two of the real power spectrum on the arcminute scales and larger.

Given the linear power spectrum and its nonlinear evolution one can compute \( \epsilon = \sigma(\theta) / \theta \) as a function of \( \theta \). In figure 1 the results are presented for the power spectrum discussed above in flat \( \Omega_{m0} = 1 \), flat low \( \Omega_{m0} \) model and open low \( \Omega_{m0} \) model, all of which are known to phenomenologically agree with most of the large-scale structure observations. The thick curves give the result of a full nonlinear calculation, while the thin curves show the corresponding linear case. One can see that while in the linear case \( \sigma(\theta) / \theta \) approaches to a constant for small \( \theta \), it continues to increase in the nonlinear case. Therefore in the real universe one cannot define a typical coherence angle, which was used by previous studies (e.g. Sasaki 1989; Linder 1990a; Cayón et al. 1993a,b) and the approximation \( \epsilon = \text{const} \) is not valid on any scale. The results can only be used on angular scales above a few arcseconds, where \( \sigma(\theta) / \theta \ll 1 \) and where the nonlinear mapping (based on the evolution of collisionless matter) gives reliable estimates.

As seen in figure 1 in the linear regime the lensing effect on CMB decreases with \( \Omega_{m0} \). This is mainly due to the linear decrease of potential with \( \Omega_m \) in the Poisson equation, partly offset by the longer travel distance, larger growth factor ratio \( g(\Omega_m, \Omega_v, a) / g(\Omega_{m0}, \Omega_{v0}, a_0) \) and in the nonlinear regime by larger nonlinear effects in the low \( \Omega_{m0} \) models. The latter is more important because the scales that are nonlinear today became nonlinear earlier than the corresponding scales in an \( \Omega_{m0} = 1 \) universe. In a low \( \Omega_{m0} \) model the universe changes from \( \Omega_{m0} \approx 1 \) to \( \Omega_{m0} \ll 1 \) earlier than in a cosmological constant model with the same matter density and in addition the relation between the conformal time and angular distance changes, all of which leads to a larger lensing effect on very small scales. The value of \( \sigma(\theta) / \theta \) linearly increases with \( \sigma_8 \) in the linear regime, but grows faster than that in the nonlinear regime. While the value of \( \sigma_8 \) is still somewhat uncertain, it is unlikely that \( \sigma_8 \) is much bigger than 1 even in an open model and the curves on figure 1 should indicate the range of the lensing effect in our universe. To investigate the sensitivity of the effect to the shape of the power spectrum I compared the flat model above to the standard CDM model.
Fig. 1.— $\sigma(\theta)/\theta$ versus $\theta$ for 3 different values of $\Omega_{m0}$ and $\Omega_{r0}$ using the power spectrum with $\Omega_{m0} = 0.25$ and $\sigma_8 = 0.8$. Thick lines are the result of a full nonlinear calculation, while the thin lines give the corresponding linear case. Also indicated are the 90% c.l. upper limits from ellipticity correlations of distant galaxies, as derived from observations by Fahlman et al. (1994) (A) and Mould et al. (1994) (B).

with $\Omega_{m0} = 0.5$. The relative difference between the two models at $\theta \sim 1'$ only depends on the power spectrum amplitude, because the dominant scales there are similar to the scales that contribute to the $\sigma_8$ normalization. The inverse wavenumber that makes a dominant contribution to $\sigma(\theta)$ is approximately $0.5 \, h \, \text{Mpc}$ for $\theta = 1'$ and $0.05 \, h \, \text{Mpc}$ at $\theta = 1^\circ$ and above. On larger angular scales the differences in $\sigma(\theta)/\theta$ between the different spectral shapes increase (with standard CDM model having less power and thus smaller $\sigma(\theta)/\theta$ for a given $\sigma_8$ normalization), but the overall effect is decreasing and becomes rather small on degree angular scales independent of the model.

In figure 1 the 90% c.l. upper limits on $\sigma(\theta)/\theta$ are also indicated, as derived from Mould et al. (1994) and Fahlman et al. (1994) limits on the correlated ellipticities. Both groups report a null detection of average ellipticity within a $4.8'$ and $2.76'$ radius aperture, respectively, with a sensitivity of about 1%. Adopting median redshifts of $z = 0.9$ and $z = 0.7$ gives radial distances $0.27$ and $0.23$ times the comoving distance to the horizon, respectively. For the two surveys one obtains upper limits that are comparable to the power spectrum estimates, which gives additional confidence that the effect was not severely underestimated. A general conclusion that can be derived from these results is that $\sigma(\theta)/\theta$ is less than 20% on scales above $1'$ and less than 5% on scales larger than $1^\circ$. 
Fig. 2.— CMB anisotropy power spectrum \( l(l+1)C_l \) versus \( l \) with lensing (dashed lines) and without lensing (solid lines). Upper curves are for adiabatic CDM model with \( h = 0.5 \), \( \Omega_{m0} = 0.4 \) and \( \Omega_{v0} = 0.6 \), lower curves are for adiabatic CDM model with \( h = 0.5 \), \( \Omega_{m0} = 1 \) and \( \Omega_{v0} = 0 \). Both models are normalized to COBE. Lensing smoothes the sharp features in the power spectrum, but leaves the overall shape unchanged. The two models show a typical range of the lensing effect on CMB.

Figure 2 shows the lensing effect on the CMB fluctuation power spectrum for the models discussed in the previous paragraph. The CMB multipole moments were obtained from a numerical integration of perturbed Einstein, Boltzmann and fluid equations (Bode & Bertschinger 1995). These models exhibit characteristic acoustic oscillations (Doppler peaks) and suppression on small scale due to the diffusion (Silk) damping. Lensing induces very little gross change in the power spectrum of CMB. However, the peaks of acoustic oscillations are smoothed because of lensing and on smaller angular scales they can be completely erased. This occurs both because \( \sigma(\theta)/\theta \) increases and because the relative width of the oscillations becomes narrower towards the smaller angular scales. Observational sensitivity to this effect depends on the particular experimental setup, but for most experiments the window functions are relatively broad in \( l \)-space and consequently the effect is diluted.

Currently popular models predict that primary CMB anisotropies are only important above the Silk damping scale of order of a few arcminutes. As shown in figure 2, on these scales the linear contributions to the dispersion \( \sigma(\theta)/\theta \) are still dominant and nonlinear effects are negligible. In fact, using only the linear theory evolution of power spectrum gives results indistinguishable from the fully nonlinear calculation in the regime of interest.
Fig. 3.— Comparison between several approximations for calculating the lensing effect on the CMB anisotropies in the COBE normalized CDM model with $h = 0.5$ and $\Omega_{m0}h = 0.5$. Both nonlinear and linear isotropic approximations give results that are almost indistinguishable from the fully nonlinear and nonisotropic calculation over this angular range, while $\epsilon = \text{const}$ approximation gives reliable results only over a limited range of $l$ and cannot be used for an accurate calculation of the lensing effect.

($l < 2000$, see figure 3). These angular scales are thus unaffected by the uncertainties of the nonlinear evolution and are also the scales where the large-scale observations place the best constraints on the power spectrum. The approximation given in equation 8 gives reasonable results only if one adopts $\epsilon \approx 4\pi l^{-1}$, which is significantly larger angle than expected (Bond 1995) and again implies that nonlinear effects are not important until very large $l$. Even then the agreement is only approximate (figure 3) and limited to $l < 1000$. In general it is better to use the isotropic approximation in equation 8 together with the Gauss-Legendre integration to calculate the lensing effect on the multipole moments, as it is not significantly harder to compute than the approximation given in equation 8.

4. Discussion

The lensing effect on the primary CMB anisotropies can be calculated for any cosmological model with a specified evolution of gravitational potential power spectrum. This formalism was applied to several currently popular models, which best fits the
observational data on large scales. The results indicate that the gravitational lensing does not significantly affect the CMB power spectrum on degree scales and larger, but becomes gradually more important towards the smaller scales. Lensing redistributes the power in the angular correlation function and the amplitude of the effect depends on the smoothness of the underlying CMB spectrum. For standard adiabatic models the acoustic oscillation peaks are rather prominent even at small angular scales (beyond $l \approx 1000$) and lensing may completely erase this structure.

Recently, two groups claimed that the gravitational lensing effect on CMB has been severely underestimated in previous calculations and that it importantly changes the CMB pattern even on degree angular scales. Bassett et al. (1994) assume a model in which photons propagate through a homogeneous universe with a density smaller than its mean density to account for the fact that some of the mass resides in dense clumps. Using the Dyer-Roeder distance-redshift relation in such a universe they obtain an increase in angular separation between the two photons relative to its unperturbed value. Similarly, Fukushige et al. (1994) assume a model in which the universe is populated by a number of massive clumps embedded in a large empty void. Here the angular separation between two photons is additionally increased with every passage of the photons by a clump, because the closer photon is always deflected more than the more distant one. This leads to an exponential growth of the angular separation until it reaches the mean projected separation between the clumps. One problem with these models is that they cannot be applied to the large (supercluster) scales, where the density fluctuations are small, given that in these models the mass density in a box on scales smaller than the mean distance between the clumps is either zero or very large. Observational data on large scales suggest that density fluctuations are close to gaussian and both underdensities and overdensities have to be included for a proper description of light propagation. The effect of underdensities is to decrease the angular separation between the two photons and this leads to a random walk growth of rms deviation between them. As shown in this paper the lensing effect on CMB on arcminute scales and larger is dominated by the linear regime, where underdensities and overdensities play equivalent roles. Numerical studies of light propagation in realistic models (Jaroszynski et al. 1991) confirm that there are no large distortions in the relative photon trajectories present for most lines of sight, at least on scales above their resolution scale of a few arcseconds. Another problem with the above models is that flux conservation requires that the angular distance between the photons (as defined in a homogeneous universe with the same density parameter) remains on average unchanged, implying that exponential growth in separation between photons passing on the same side of a clump is balanced by a strong focusing of photons that pass on the opposite sides of the clump. This would lead to multiple images (strong lensing), but it is known observationally that such
situations are rare in our universe, especially on very large scales, such as superclusters. It is nevertheless possible that in a highly nonlinear regime (i.e. on very small scales) our universe could be well approximated by the models discussed by Fuku shige et al. (1994) and Bassett et al. (1994). In such a regime the models presented in this paper should predict relative fluctuations larger than unity and their predictions would become unreliable, because the assumption that the potential deflecting the photons can be calculated along the unperturbed paths would not be satisfied. As long as $\sigma(\theta)/\theta$ remains small this is not the case and for the density fluctuations as measured in our universe this condition is satisfied at least on angular scales above a few arcseconds.

Although gravitational lensing is of small significance for the present day experiments, mostly sensitive to the degree angular scales, it may become relevant for the future experiments that will probe smaller angular scales with a much higher sensitivity and sky coverage. Gravitational lensing effect will be especially important for the high precision determination of cosmological parameters planned for the next generation of experiments. The uncertainties caused by the gravitational lensing should be included in the modelling of extraction of cosmological parameters from the CMB measurements. The formalism developed in this paper allows to calculate the lensing effect on the CMB for any specified cosmological model and can be included as a postprocessor to the standard calculations of the CMB multipole moments.

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A. Appendix

Calculating the gravitational lensing effect on the CMB can be cumbersome in general, but it simplifies considerably if only small angular scales are considered and if the fluctuations in relative separation between the two photons can be considered gaussian. The first assumption is not very restrictive, since one does not expect the lensing effect to be important on large angular scales. The second assumption should really limit the validity of the calculation to the linear scales only, where the prediction of most models that the initial fluctuations are gaussian guarantees its validity. In reality its validity extends beyond that to the quasi-linear scales, because the relative fluctuations are obtained by a projection of a 3-dimensional distribution over a broad radial window function and are in general much more gaussian than the 3-d distribution of the gravitational potential derivative itself.
In the spherical plane approximation one can write the temperature anisotropies \( \Delta T (\vec{\theta}) \) in terms of its Fourier transform,

\[
\frac{\Delta T}{T} (\vec{\theta}) = \int d^2 \vec{l} e^{-i \vec{l} \cdot \vec{\theta}} \Delta T (\vec{l}).
\]

The correlation function including lensing is given by

\[
\tilde{C} (\theta) = \langle \frac{\Delta T}{T} (\vec{\theta}^A + \delta \vec{\theta}^A) \frac{\Delta T}{T} (\vec{\theta}^B + \delta \vec{\theta}^B) \rangle_{\vec{\theta}^A,\vec{\theta}^B=\cos \theta} = \\
\int d^2 \vec{l} \int d^2 \vec{l}' e^{-i \vec{l} \cdot \vec{\theta}^A + i \vec{l}' \cdot \vec{\theta}^B} \langle e^{-i \vec{l} \cdot \delta \vec{\theta}^A + i \vec{l}' \cdot \delta \vec{\theta}^B} \frac{\Delta T}{T} (\vec{l}) \Delta T^* (\vec{l}') \rangle,
\]

where \( \delta \vec{\theta}^A \) and \( \delta \vec{\theta}^B \) are the angular excursions of the two photons that are separated at the observer’s position by an angle \( \theta \). In equation (A2) one has to average both over the intrinsic temperature anisotropies \( \Delta T (\vec{l}) \) and over the lensing fluctuations \( \delta \vec{\theta} \). The first averaging gives the angular power spectrum of CMB,

\[
\left\langle \frac{\Delta T}{T} (\vec{l}) \frac{\Delta T}{T} (\vec{l}') \right\rangle = C_l \frac{\Delta^2 (\vec{l} - \vec{l}')}{(2\pi)^2},
\]

while the second gives the characteristic function of a gaussian field \( \delta \vec{\theta}^A - \delta \vec{\theta}^B \),

\[
\langle e^{-i \vec{l} \cdot (\delta \vec{\theta}^A - \delta \vec{\theta}^B)} \rangle = e^{-<|\vec{l} \cdot (\delta \vec{\theta}^A - \delta \vec{\theta}^B)|^2>/2}.
\]

Performing the ensemble averaging over the lensing fluctuations \( \delta \vec{\theta} \) in the equation above with the help of Limber’s equation (Kaiser 1992) leads to the lensed correlation function

\[
\tilde{C} (\theta) = (2\pi)^{-2} \int_0^\infty l d l \int_0^{2\pi} d \varphi_l \exp \left\{ \frac{-l^2}{2} [\sigma^2 (\theta) - \cos (2\varphi_l) C_{gl,2}(\theta)] - il \theta \cos (\varphi_l) \right\},
\]

where \( \sigma (\theta) \) is the rms dispersion in the angular positions of the two photons defined in equation (5) and \( C_{gl,2}(\theta) \) can be obtained from \( C_{gl}(\theta) \) in equation (6) by replacing \( J_0 \) with \( J_2 \). Assuming \( l^2 C_{gl,2}(\theta) \ll 1 \) (which follows from assuming \( \sigma (\theta)/\theta \ll 1 \), together with \( l \sim \theta^{-1} \) and \( C_{gl,2}(\theta) < \sigma^2 (\theta) \)) one may Taylor expand the exponential in equation (A5) and integrate over \( \varphi_l \). This leads to

\[
\tilde{C} (\theta) = (2\pi)^{-1} \int_0^\infty l d l e^{-\sigma^2 (\theta) l^2 / 2} C_l \left[ J_0 (l \theta) + \frac{l^2}{2} C_{gl,2}(\theta) J_2 (l \theta) \right].
\]

One approximation often used in the literature is to keep only the isotropic term \( J_0 (l \theta) \) in equation (A6), which gives the dominant contribution to the lensing effect. In the following I will assume this approximation, because it gives a good agreement with the
exact calculation, as shown in figure 3. With this approximation the lensed correlation function becomes

\[ \tilde{C}(\theta) = (2\pi)^{-1} \int_0^\infty l \, dl e^{-\sigma^2(\theta)l^2/2} C_I J_0(l\theta). \]  \hspace{1cm} (A7)

This equation is essentially the same as the Wilson & Silk (1981) expression for the correlation function observed with an instrument that has a gaussian beam profile, the only difference being that in the present case the dispersion \( \sigma(\theta) \) depends on the angular separation \( \theta \). After another Fourier transform and using equation 6.615 from Gradshteyn & Ryzhik (1965) one obtains equation 3. Alternatively, one can also express the lensing effect directly in terms of the CMB power spectrum,

\[ \tilde{C}_l = \int_0^\pi d\theta \int_0^\infty l' \, dl' e^{-\sigma^2(\theta)l'^2/2} C_I J_0(l\theta) J_0(l'\theta). \]  \hspace{1cm} (A8)

The above expression can be further simplified if one assumes that \( \epsilon = \sigma(\theta)/\theta \) is a constant. Equation (A8) then reduces to

\[ \tilde{C}_l = \int_0^\infty \frac{l' \, dl'}{(\epsilon l')^2} C_I J_0 \left( \frac{l}{\epsilon^2 l'} \right) \exp \left[ \frac{l^2 + l'^2}{2(\epsilon l')^2} \right], \]  \hspace{1cm} (A9)

where \( J_0(x) \) is the modified Bessel function of order 0. Further assuming \( \epsilon \ll 1 \) one can asymptotically expand the modified Bessel function \( J_0(x) \) and use \( l' \approx l \) everywhere except in the exponential. This finally leads to equation 8, which is similar to the expression given by Bond (1995).

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