Massless and massive representations in the spinor technique

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The technique [1–7] for representing spinors and the definition of the discrete symmetries [8] is used to illustrate on a toy model [9–12] properties of massless and massive spinors states, in the first and the second quantized picture. Since in this toy model the number of the starting massless representations is well defined as well as the origin of masses and charges in $d = (3+1)$ space, this contribution might help to clarify the problem about Dirac, Weyl and Majorana kinds of representations [13–16] in physically more interesting cases.

I. INTRODUCTION

We illustrate on a toy model, defined in the refs. [9–12], massless and massive, positive and negative energy solutions of the equations of motion for spinors (fermions), and study the properties of particles and the corresponding antiparticles states, by taking into account the definition of the discrete symmetry operators ($C_N$, $P_N$ and $T_N$) in the second quantized picture, designed in the paper [8] for higher dimensional spaces. For the second quantized picture we use, like in the paper [8], the concept of the Dirac sea, since it offers a nice physical understanding.

We use this toy model, in which we start with one spinor Weyl representation and assume the $M^{5+1}$ manifold to break into $M^{3+1} \times$ an almost $S^2$ sphere due to the zweibein and spin connection fields in $d = (5, 6)$, since in this toy model the origin of the charge is the spin or the total angular momentum in extra $d = (3+1)$ dimensions ($d > (3 + 1)$) and the mass originates either in the dynamics in the higher dimensions or in the vacuum expectation values of the scalar fields, which are the gauge fields with the scalar index with respect to $d = (3 + 1)$, and therefore well defined, making discussions about the representations of spinors well defined and transparent. The illustration makes evident the fact that without knowing the action which leads to massless and massive solutions, the analyse of the Dirac equation which already assumes the mass and do not pay attention on the origin of the charge might not help a lot.

We comment also the representations in $d = (7 + 1)$ and $d = (13 + 1)$. In this last case one Weyl representation of massless states contains left handed weak charged quarks and leptons as well as right handed weak chargeless quarks and leptons, that is all the family members postulated
by the standard model, with right handed neutrinos included, while the simple starting Lagrange density for spinors and gauge fields offers the explanation for the standard model as the low energy effective theory of this spin-charge-family theory of one of us (S.N.M.B.) [5–7, 17–20], with the families included.

Assuming the Lorentz invariance and causality, the theorem of CPT is generally valid, while the $C, P$ and $T$ symmetries, separately or in pairs, depend on effective theories. In our toy model [9–12], like in all Kaluza-Klein kind of models in even dimensional spaces (in odd dimensional spaces there is no mass protection mechanism and they do not lead effectively to observable phenomena, that is to almost massless quarks and leptons), besides $\mathbb{C}_N \times \mathbb{P}_N \times T_N$, only $\mathbb{C}_N \times \mathbb{P}_N$ and $T_N$ are good symmetries, since only these symmetries operate among the eigenstates of the equations of motion belonging to the starting spinor representation. The Clifford odd operators, like $\gamma^0, \gamma^0\gamma^5$ (we use $\Gamma^{(3+1)}$ instead of $\gamma^5$) in the refs. [13–16], namely transform the solutions into another Weyl representation, which only makes sense if the equations of motion contain the operators which connect both representations.

The representations in the spinor technique [1–7, 18] makes our illustration of free as well as interacting particles and antiparticles, massless and massive in $d = (3 + 1)$, with their discrete symmetries included, easier to follow.

We do not study in this paper the families of spinors. Families, their number is in even dimensional spaces equal to $2^{d^2-1}$ (and therefore in $d = (5 + 1)$ equal to 4 in our toy model) would not clarify much the point of this paper, since they form equivalent representations with respect to the here presented states [24]. One Weyl representation in an even dimensional space contains $2^{d^2-1}$ states, the same number as there is the number of families, which is in the case of our toy model in $d = (5 + 1)$, with $\mathcal{M}^{3+1} \times$ an almost $S^2$ sphere, equal to 4.

Following the refs. [8–12] we present in sect. II the action for a spinor (fermion) in $d = (5 + 1)$, leading to the Weyl equations which manifest mass protected massless with the spin in $(5,6)$ dimension as the charge in $d = (3+1)$ state and the series of states with the total angular moments in $d = (5,6)$ as the charges (II B). We look also for the massive chargeless solutions of the Weyl equations, for masses of which are responsible, by the assumption, nonzero vacuum expectation values of the scalar fields [17–19, 21] - spin connections and zweibeins with the scalar index with respect to $d = (3+1)$ (subsect. II D), which are the gauge fields of $S^{56}$, playing the role of the higgs in the standard model and carrying in this toy model case only the hyper charge $Y$, determined by the operator $S^{56}$. The particle, antiparticle and Majorana states (II F) are presented.

In subsect. II A the discrete symmetry operators defined in the ref. [8] are presented.
In subsect. II E the representations are commented from the point of view of the usual Dirac ones \[23\].

We conclude our paper with discussions (sect. III) on the (starting) Weyl representation of the toy model and the corresponding the first and the second quantized states as they manifest in \(d = (3 + 1)\). We extend discussions also on the cases in \(d = (7 + 1)\) and \(d = (13 + 1)\) and comment in all these cases the action of the Clifford odd operators on the states.

II. THE TOY MODEL WITH THE MANIFOLD \(M^{5+1}\) BROKEN INTO \(M^{3+1} \times \text{AN ALMOST } S^2\) AND THE REPRESENTATIONS

We make a choice of the action for massless (Weyl) spinors \[9\] living on the manifold \(M^{1+5}\)

\[
S = \int d^d x \mathcal{L}_W,
\]

\[
\mathcal{L}_W = \bar{\psi} E \gamma^0 \gamma^a f_{a} p_\alpha + \frac{1}{2E} [p_\alpha, f^a_\alpha E] - \frac{1}{2} S_{cd} \omega_{cd} \psi
\]

\[
= \frac{1}{2} \bar{\Psi} E \gamma^0 (\gamma^m p_0 m + \gamma^s p_0 s) \Psi + h.c.,
\]

\[
p_0 m = p_m - S^{56} \omega_{56m},
\]

\[
p_0 s = f_{\sigma}^s p_\sigma + \frac{1}{2E} [p_\sigma, f^\sigma_s E] - S^{56} \omega_{56s},
\]

\[
m = (0, 1, 2, 3), \quad s = (5, 6).
\]

(1)

\(f^a_\alpha\) are vielbeins and \(\omega_{cd}\) spin connection fields, the gauge fields of the moments \(p_\alpha\) and \(S^{ab}\), respectively. We take flat \((3 + 1)\) space: \(f^\mu_m = \delta^\mu_m\), \(\mu = ((0), (1), (2), (3))\), \(m = (0, 1, 2, 3)\). In our toy model the manifold \(M^{5+1}\) breaks into \(M^{3+1} \times \text{(an almost) } S^2\). \(S^{56}\) then manifests in \(d = (3+1)\) as the (Kaluza-Klein) charge, \(\omega_{56m}\) as the corresponding vector and \(\omega_{56s}\) as the corresponding scalar gauge fields. Not paying attention to the family quantum numbers, we left in Eq. (1) out all the terms which carry family quantum numbers \[25\] and determine correspondingly interaction among different families.

The Weyl equation of the action (Eq. (1)) can be written as follows

\[
(\gamma^m p_0^m + (+) p_{0+}^m + (-) p_{0-}^m) \psi = 0,
\]

\[
p_{0 \pm} = p_0^{\mp} \pm i p_0^\sigma,
\]

\[
p_{00} = f^\sigma_s (p_\sigma - \frac{1}{2} S^{ab} \omega_{ab}) + \frac{1}{2E} [p_\sigma, f^\sigma_s E] - S^{56} \omega_{56s},
\]

\[
(\pm) = \frac{1}{2} (\gamma^5 \pm i \gamma^6).
\]

(2)
The explanation how does the technique work can be found in the refs. [1–7], the short version, taken from the ref. [8, 17–19], is in the appendix.

There are \(2^{d-1}\) (4 in our case of \(d = 6\)) basic spinor states [26], forming the fundamental representation of the group \(SO(5,1)\), the generators of which are \(S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)\)

\[
\begin{align*}
\Psi_1 &= (+i)(+)|\text{vac}>_{fam}, \\
\Psi_2 &= (+i)(-)|\text{vac}>_{fam}, \\
\Psi_3 &= (-i)(+)|\text{vac}>_{fam}, \\
\Psi_4 &= (-i)(-)|\text{vac}>_{fam},
\end{align*}
\]

with \(|\text{vac}>_{fam}\) defined so that these four states are nonzero and normalized: \(\Psi_i^\dagger\Psi_j = \delta_{ij}\). (This vacuum is not a second quantized vacuum.) All the basic states are eigen states of the Cartan subalgebra (of the Lorentz transformation Lie algebra), for which we take: \(S^{03}, S^{12}, S^{56}\), with the eigen values, which can be read from Eq. (3) if taking \(\frac{1}{2}\) times the numbers \(\pm i\) or \(\pm 1\) in the parentheses of nilpotents \((k)\) and projectors \([k]\): \(S_{ab} \sim (k), S_{ab} \prescript{ab}{}[k] = \frac{k}{2} \prescript{ab}{}[k]\). One notices that two are right (\(\Psi_1\) and \(\Psi_3\)) and two left (\(\Psi_2\) and \(\Psi_4\)) handed with respect to \(d = (3 + 1)\) (while all four carry the same, left, handedness with respect to \(d = (5 + 1)\)). The operator of handedness is defined in Eq. (11). The operator with an odd number of the Clifford algebra objects, like it is \(\gamma^0\Gamma^{(3+1)}\) (\(\gamma^0\gamma^5\) in usual notation) would transform any state into the state of the opposite handedness.

The following choice of the zweibein fields causes that the infinite surface \(d = (5,6)\) curls into an almost \(S^2\) (with one hole [9])

\[
\begin{align*}
e^{s\sigma} &= f^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, f^{s\sigma} s = f \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, f = 1 + \left(\frac{\rho}{2\rho_0}\right)^2, \\
E &= \det(e^s\sigma) = f^{-2}, e^s\sigma f^s\sigma = \delta^s_i, \\
x^{(5)} &= \rho \cos \phi, \quad x^{(6)} = \rho \sin \phi,
\end{align*}
\]

while \(d = (3 + 1)\) space stays flat (\(f^\mu_m = \delta^\mu_m\)). We choose the spin connection fields on this \(S^2\) as

\[
f^{s\sigma} s\omega_{st\sigma} = iF f e^{s\sigma} x^\sigma \left(\frac{\rho_0}{\rho}\right)^2, \quad 0 < 2F \leq 1, \quad s = 5, 6, \quad \sigma = (5,6),
\]

in order to guarantee that there manifests in \(d = (3 + 1)\) only one massless and correspondingly mass protected state [9], while the rest of states are all massive. There is the whole interval for the constant \(F\) (\(0 < 2F \leq 1\)), which fulfills the condition of only one massless state of the right handedness in \(d = (3 + 1)\), which is square integrable.
When requiring that the solutions of Eq. (2) have the angular moments in \( d = (3 + 1) \) \( (M^{56} = x^5p^6 - x^6p^5 + S^{56} = -i\frac{\partial}{\partial \rho} + S^{56}) \), we write the wave functions \( \psi^{(6)}_{n+1/2} \) for the choice of the coordinate system \( p^a = (p^0, 0, 0, p^3, p^5, p^6) \) as follows

\[
\psi^{(6)}_{n+1/2} = (A_n \begin{pmatrix} 03 & 12 & 56 \end{pmatrix} + B_{n+1} e^{i\phi} \begin{pmatrix} 03 & 12 & 56 \end{pmatrix}) e^{i\phi} e^{-i(p^0 x^0 - p^3 x^3)}.
\] (5)

Besides one massless \( (\psi^{(6)}_{1/2}) \) there is the whole series of massive solutions manifesting in \( d = (3 + 1) \) the (Kaluza-Klein) charge \( n + 1/2 \): \( M^{56} \psi^{(6)}_{n+1/2} = (n + 1/2) \psi^{(6)}_{n+1/2} \), and solve Eq. (2), provided that \( A_n \) and \( B_{n+1} \) are the solutions of the equations

\[
\begin{align*}
-if \left\{ \left( \frac{\partial}{\partial \rho} + \frac{n + 1}{\rho} \right) - \frac{1}{2} \left( 1 + 2F \right) \right\} B_{n+1} + mA_n &= 0, \\
-if \left\{ \left( \frac{\partial}{\partial \rho} - \frac{n}{\rho} \right) - \frac{1}{2} \left( 1 - 2F \right) \right\} A_n + mB_{n+1} &= 0.
\end{align*}
\] (6)

The massless positive energy solution with spin \( \frac{1}{2} \), left handed (Eq. (11) in \( d = (5 + 1) \), the charge in \( d = (3 + 1) \) equal to \( \frac{1}{2} \) and right handed with respect to \( \Gamma^{(3+1)} \) is equal to

\[
\psi^{(6)}_{1/2} = N_0 e^{-F+1/2} \begin{pmatrix} 03 & 12 & 56 \end{pmatrix} e^{-i(p^0 x^0 - p^3 x^3)}.
\] (7)

For the special choice of \( F = \frac{1}{2} \) (from the interval in Eq. (4) allowing only right handed square integrable massless states) the solution (Eq. (7)) simplifies to

\[
\psi^{(6)}_{1/2} = N_0 \begin{pmatrix} 03 & 12 & 56 \end{pmatrix} e^{-i(p^0 x^0 - p^3 x^3)}.
\] (8)

Massive solutions are in this special case expressible in terms of the associate Legendre function \( P^l_n(x) \), \( x = \frac{1-u^2}{1+u^2} \), \( u = \frac{\rho}{2\rho_0} \), where \( \rho_0 \) is the radius of (an almost) \( S^2 \), as follows

\[
\begin{align*}
A_n^{(l+1)} &= P^l_n, \\
B_{n+1}^{(l+1)} &= \frac{-i}{\rho_0 m} \sqrt{1-x^2} \left( \frac{d}{dx} + \frac{n}{1-x^2} \right) A_n^{(l+1)},
\end{align*}
\] (9)

with the masses \( \{27\} \) determined by \( (\rho_0 m)^2 = l(l+1) \) and \( l = 1, 2, 3, \ldots, 0 \leq n < l \).

We shall comment all the solutions, massless and massive, in the subsections of this section.

Let us summarize this section and its subsections: Starting with one Weyl massless representation and the action for a massless spinor in \( d = (5 + 1) \), we end up with one massless and a series of massive solutions in \( d = (3 + 1) \). All the solutions are, from the point of view of \( d = (3 + 1) \) distinguishable according to their charges and masses and also \( (p^0, p^1, p^2, p^3) \) and all belong to the starting left handed spinor representation of Eq. (3). (The operator of handedness is presented in Eq. (11).)
We check (II A) the discrete symmetries of the action (Eq.(1)) and of the Weyl equation (Eq.(2)), using the operators expressible with an even number of the Clifford algebra objects (even number of $\gamma^a$'s), so that the transformed states stay within the starting Weyl representation. Any Clifford odd operator would, namely, transform any state into a state of the opposite handedness in $d = (5 + 1)$, which would have meaning if there would be terms in the starting action (Eq.(1)) connecting states of different handedness, which is not the case.

From massless (subsect. II B) or massive (subsect. II C) positive energy solutions the negative ones follow by the application of $C_N \cdot \mathcal{P}_N^{(d-1)}$. The positive energy states, put on the top of the Dirac sea, represent particles. The antiparticle states follow from the particle ones either by emptying the negative energy state in the Dirac sea or directly from the particle state by the application of the operator $C_N \cdot \mathcal{P}_N^{(d-1)}$ and putting the obtained state on the top of the Dirac sea. Any antiparticle, massless or massive, carries the opposite charge than the corresponding particle.

Assuming that the scalar fields gain nonzero vacuum expectation values, the massless solutions no longer exist. The effective equations of motion (Eqs.(19, 20)) lead to two positive energy states, and two corresponding negative energy states, the holes in which represent antiparticle states, which are indistinguishable from the particle states and are indeed the Majorana particles. Not paying attention to the origin of the mass term any longer (Eq.(19)) (one of) the discrete symmetry operators have to be redefined.

A. Discrete symmetry operators

To discuss representations of particle and antiparticle states we must define the discrete symmetry operators in the second quantized picture. The ref. [8] proposes the definition of the discrete symmetries operators for the Kaluza-Klein kind of theories, for the first and the second quantized picture, so that the total angular moments in higher dimensions manifest as charges in $d = (3 + 1)$. We shall use, as in the ref. [8], the Dirac sea second quantized picture to make discussions transparent.

The ref. [8] proposes the following discrete symmetry operators

$$C_N = \prod_{\gamma^m, m=0}^{3} \gamma^m \Gamma^{(3+1)} K I_{x^6, x^8, \ldots, x^d},$$

$$T_N = \prod_{\gamma^m, m=1}^{3} \gamma^m \Gamma^{(3+1)} K I_{x^0} I_{x^5, \ldots, x^{d-1}},$$

$$\mathcal{P}_N^{(d-1)} = \gamma^0 \Gamma^{(3+1)} \Gamma^{(d)} I_{x^3}. \quad (10)$$
The operator of handedness in even \(d\) dimensional spaces is defined as
\[
\Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}} \gamma^a),
\]
with products of \(\gamma^a\) in ascending order. We choose \(\gamma^0, \gamma^1\) real, \(\gamma^2\) imaginary, \(\gamma^3\) real, \(\gamma^5\) imaginary, \(\gamma^6\) real, alternating imaginary and real up to \(\gamma^d\) real. Operators \(I\) operate as follows:
\[
I_{x^0} x^0 = -x^0; \quad I_{x^a} x^a = -x^a; \quad I_{x^0} \vec{x} = -\vec{x}; \quad I_{x^a} \vec{x} = (x^0, -x^1, -x^2, -x^3, x^5, x^6, \ldots, x^d); \quad I_{x^5, x^7, \ldots, x^{d-1}} (x^0, x^1, x^2, x^3, x^5, x^6, x^7, x^8, \ldots, x^{d-1}, x^d)
\]
and
\[
C_N \text{ transforms the state, put on the top of the Dirac sea, into the corresponding negative energy state in the Dirac sea.}
\]
We need the operator, we name \([8, 17, 22]\) it \(C_N\), which transforms the starting single particle state on the top of the Dirac sea into the negative energy state and then empties this negative energy state. This hole in the Dirac sea is the antiparticle state put on the top of the Dirac sea. Both, a particle and its antiparticle state (both put on the top of the Dirac sea), must solve the Weyl equations of motion.

This \(C_N\) is defined as a product of the operator \([17, 22]\) "emptying", (which is really an useful operator, although it is somewhat difficult to imagine it, since it is making transformations into a complete different Fock space)
\[
"emptying" = \prod_{\Re \gamma^a} \gamma^a K = (-)^{\frac{d}{2}+1} \prod_{\Im \gamma^a} \gamma^a \Gamma^{(d)} K,
\]
and \(C_N\)
\[
C_N = \prod_{\Re \gamma^a, a=0}^d \gamma^a K \prod_{\Im \gamma^m, m=0}^3 \gamma^m \Gamma^{(3+1)} K I_{x^5, x^7, \ldots, x^d}
\]
\[
= \prod_{\Re \gamma^s, s=5}^d \gamma^s I_{x^5, x^7, \ldots, x^d}.
\]

Let us present also the second quantized notation, following the notation in the ref. \([8]\). Let \(\Psi_p^\dagger (\Psi_p)\) be the creation operator creating a fermion in the state \(\Psi_p\) and let \(\Psi_p^\dagger (\vec{x})\) be the second quantized field creating a fermion at position \(\vec{x}\). Then
\[
\{ \Psi_p^\dagger (\Psi_p) \} = \int \Psi_p^\dagger (\vec{x}) \Psi_p (\vec{x}) d^{(d-1)} x \} |\text{vac}\rangle
\]
so that the antiparticle state becomes
\[
\{ C_N \Psi_p^\dagger (\Psi_p) \} = \int \Psi_p (\vec{x}) (C_N \Psi_p (\vec{x})) d^{(d-1)} x \} |\text{vac}\rangle.
\]
The antiparticle operator $\Psi^\dagger_a[\Psi_p]$, to the corresponding particle creation operator, can also be written as

$$\Psi^\dagger_a[\Psi_p]|\text{vac}\rangle = C_N \Psi^\dagger_p[\Psi_p]|\text{vac}\rangle = \int \Psi^\dagger_a(\vec{x}) (C_N \Psi_p(\vec{x})) d^{(d-1)}x |\text{vac}\rangle,$$

$$C_H = "emptying" \cdot C_H.$$ (14)

The equations of motion for our toy model (Eqs. (2,4), and correspondingly the solutions (Eq. 5)) manifest the discrete symmetries $C_N \cdot P_N$, $C_N \cdot P_N \cdot T_N$, and $C_N \cdot P_N \cdot T_N$, with the operators presented in Eqs. (10, 13). Both, $C_N \cdot P_N \cdot \Psi(6)$ and $C_N \cdot P_N \cdot \Psi(6)$ (13) solve the equations of motion, provided that $\omega_{56m}(x^0, \vec{x}_3)$ is a real field. The field $\omega_{56m}(x^0, \vec{x}_3)$ transforms under $C_N \cdot P_N$ and $C_N \cdot P_N$ to $-\omega_{56m}(x^0, -\vec{x}_3)$, like the $U(1)$ field must [23]. We shall comment in the subsect. II C that $F$ in Eq. (4) transforms into $-F$ for either $C_N \cdot P_N$ or $C_N \cdot P_N$, as well as for $C_N \cdot P_N \cdot T_N$.

Let us summarize: The starting action (1) and the corresponding Weyl equation (2) manifest discrete symmetries from Eqs. (10,13). We comment only the Clifford even operators, which keep the transformed states within the starting spinor representation.

B. Massless solutions of the Weyl equation with charges

In Eq. (7) the massless solution with the spin $\frac{1}{2}$ is presented, solving Eq. (2) for a toy model with the vielbeins and spin connection fields presented in Eqs. (4). For $0 < F \leq \frac{1}{2}$ the spinor state is massless, mass protected, and represents, when put on the top of the Dirac sea, a free massless charged particle. To simplify the discussions we shall choose $F = \frac{1}{2}$. The state of Eq. (8) solves the Weyl equation

$$(-2i S^{03} p^0 = p^3) \psi,$$ (15)

where the coordinate system in $d = (3 + 1)$ was chosen, to simplify the discussions, so that $p^m = (p^0, 0, 0, |p^3|)$.

In Table II (taken from the papers [8, 17, 22]) all the solutions of the Weyl equation are represented. The first two lines represent the two positive energy solutions of Eq. (15) with the spin $\pm \frac{1}{2}$, both carrying the charge $\frac{1}{2}$, both right handed with respect to $d = (3+1)$ and correspondingly mass protected.

There are two additional positive energy solutions (the third and the fourth line of the table), which manifest indeed the holes in the Dirac sea of the two negative energy solutions, presented in
| $\psi^{(6)}_{i,j}$ | positive energy state | $\gamma^0 \gamma^a |p|$, $\gamma^a |p|$ | $(-2i S_{03})$ | $\Gamma^{(3+1)}$ | $S^{56}$ | $2p^2 S^{12}$ |
|---|---|---|---|---|---|---|
| $\psi^{(6)}_{\frac{1}{2},+}$ | $03 \begin{pmatrix} 12 \\ 56 \end{pmatrix}$ | $e^{-i p^0 |x^0| + i p^3 |x^3|}$ | $+1$ | $+1$ | $+1$ | $\frac{i}{2}$ | $1$ |
| $\psi^{(6)}_{\frac{1}{2},-}$ | $03 \begin{pmatrix} 12 \\ 56 \end{pmatrix}$ | $e^{-i p^0 |x^0| - i p^3 |x^3|}$ | $+1$ | $-1$ | $+1$ | $\frac{i}{2}$ | $1$ |
| $\psi^{(6)}_{\frac{1}{2},+}$ | $03 \begin{pmatrix} 12 \\ 56 \end{pmatrix}$ | $e^{-i p^0 |x^0| + i p^3 |x^3|}$ | $+1$ | $-1$ | $-1$ | $-\frac{i}{2}$ | $-1$ |
| $\psi^{(6)}_{\frac{1}{2},-}$ | $03 \begin{pmatrix} 12 \\ 56 \end{pmatrix}$ | $e^{-i p^0 |x^0| - i p^3 |x^3|}$ | $+1$ | $+1$ | $-1$ | $-\frac{i}{2}$ | $-1$ |

| $\psi^{(6)}_{i,j}$ | negative energy state | $\gamma^0 \gamma^a |p|$, $\gamma^a |p|$ | $(-2i S_{03})$ | $\Gamma^{(3+1)}$ | $S^{56}$ | $2p^2 S^{12}$ |
|---|---|---|---|---|---|---|
| $\psi^{(6)}_{\frac{1}{2},+}$ | $03 \begin{pmatrix} 12 \\ 56 \end{pmatrix}$ | $e^{i p^0 |x^0| + i p^3 |x^3|}$ | $-1$ | $+1$ | $-1$ | $\frac{i}{2}$ | $-1$ |
| $\psi^{(6)}_{\frac{1}{2},-}$ | $03 \begin{pmatrix} 12 \\ 56 \end{pmatrix}$ | $e^{i p^0 |x^0| - i p^3 |x^3|}$ | $-1$ | $-1$ | $+1$ | $\frac{i}{2}$ | $-1$ |

| $\psi^{(6)}_{\frac{1}{2},+}$ | $03 \begin{pmatrix} 12 \\ 56 \end{pmatrix}$ | $e^{i p^0 |x^0| + i p^3 |x^3|}$ | $-1$ | $-1$ | $+1$ | $\frac{i}{2}$ | $-1$ |
| $\psi^{(6)}_{\frac{1}{2},-}$ | $03 \begin{pmatrix} 12 \\ 56 \end{pmatrix}$ | $e^{i p^0 |x^0| - i p^3 |x^3|}$ | $-1$ | $+1$ | $-1$ | $-\frac{i}{2}$ | $-1$ |

TABLE I: Two positive energy states of the charge $\frac{1}{2}$ (index $i$), the right handed (with respect to $d = (3+1)$) and with the spin (determined by the index $j$) $\frac{1}{2}$, the first line, and $\frac{1}{2}$, the second line, representing particles when put on the top of the Dirac sea. The operator $\mathcal{C}_N : \mathcal{P}_{N}^{(d-1)} = \gamma^0 \gamma^5 \mathbf{K} \mathbf{I}_{\hat{3}} \mathbf{I}_{x^6}$ transforms these two states into the negative energy state in the Dirac sea, presented in the last two lines of the table; the first line into the fifth one and the second into the sixth one. The remaining two positive energy states with the charge $-\frac{1}{2}$ represent holes in the Dirac sea, the third line corresponds to the fifth and the fourth line to the sixth one. These are the two antiparticle states, put on the top of the Dirac sea, which follow also directly from the starting particle state by the application of $\mathcal{C}_N : \mathcal{P}_{N}^{(d-1)} = \gamma^0 \gamma^5 \mathbf{K} \mathbf{I}_{\hat{3}} \mathbf{I}_{x^6}$. The coordinate system in $d = (3+1)$ is chosen so that $p^m = (p^0, 0, 0, p^3)$. $\Gamma^{(5+1)} = -1$ and $\Gamma^{(d-1)+1}$ define the handedness in $d = (5+1)$-dimensional space-time, $S^{56}$ defines the charge in $d = (3+1)$, $2p^2 S^{12}$ defines the helicity. Nilpotents $(k)$ and projectors $(k)$ operate on the vacuum state $|\text{vac}>_{fam}$ not written in the table. Table is partly taken from [8, 17, 22].

the fifth and the sixth line in Table II These two positive energy solutions - the two holes in the Dirac sea - represent the corresponding antiparticle state to the two starting state.

There is the Clifford even discrete symmetry operator $\mathcal{C}_N : \mathcal{P}_{N}^{(d-1)}$, Eqs. [10][13] (the operator with the even number of $\gamma^a$'s), which transforms the two positive energy solutions (presented in the first two lines in Table I) into the corresponding two negative energy solutions (the first line transformed into the fifth line and the second line into the sixth line), keeping all the states - particles and antiparticles - within the starting Weyl representation of Eq. (3). For $d = (5+1)$ the operator $\mathcal{C}_N : \mathcal{P}_{N}^{(d-1)}$ equals to $\gamma^0 \gamma^2 \mathbf{K} \mathbf{I}_{\hat{3}} \mathbf{I}_{x^6}$.

The two antiparticle states of positive energy follow also directly from the particle states by the application of the operator $\mathcal{C}_N : \mathcal{P}_{N}^{(d-1)} = \gamma^0 \gamma^5 \mathbf{K} \mathbf{I}_{\hat{3}} \mathbf{I}_{x^6}$.
The states, presented in Table I, are the solutions of the Weyl equations

\[(\Gamma^{(3+1)} \frac{p^0}{|p^0|} = 2\vec{\gamma} \cdot \vec{S} \psi).\]  

(16)

for the choice \((0,0,p^3)\). Here \(S = (S^{23}, S^{31}, S^{12})\), \(S^{ab} = \frac{i}{2} (\gamma^a\gamma^b - \gamma^b\gamma^a)\), and \(\Gamma^{(d-1)+1}\) (in usual notation is for \(d = (3+1) \) named \(\gamma^5\)) determines handedness for fermions in any \(d\). For \(d = (5+1)\), \(\Gamma^{(5+1)} = \prod_a \gamma^a\) in ascending order, equal also to \(\Gamma^{(3+1)} \cdot (-2S^{56})\).

For the choice \(p^m = (p^1, p^2, p^3)\) and the spin \(\pm \frac{1}{2}\) are the solutions presented in Eq. (17), the two particle states of positive energy are in the first two lines, the corresponding particle states of the negative energy are in the last two lines, while the corresponding antiparticle states, representing the hole in the negative energy states of the Dirac sea are written in the third and the fourth line.

As in the simplified case also in this general case the negative energy states and the antiparticle states follow from the positive energy states by the application of \(C_N \cdot \mathcal{P}_N\) and \(C_N \cdot \mathcal{P}_N\), respectively.

particle states

\[p^0 = |p^0|,\]

\[\psi^{(6)}_{\frac{3}{2}, \frac{1}{2}} (\vec{p}) = \left( \begin{array}{c} 03 \\ +i \\ 12 \\ + \end{array} \right) \left( \begin{array}{c} 12 \\ 56 \\ \frac{p^1 + i p^2}{|p^0| + |p^3|} \\ \frac{[\vec{p}] - 12}{[\vec{p}] - 12} \\ \frac{[\vec{p}] + 12}{[\vec{p}] + 12} \end{array} \right) e^{-i(p^0\cdot x^0 - \vec{p} \cdot \vec{x})},\]

antiparticle states

\[\psi^{(6)}_{\frac{1}{2}, -\frac{1}{2}} (\vec{p}) = \left( \begin{array}{c} 03 \\ -i \\ 12 \\ + \end{array} \right) \left( \begin{array}{c} 12 \\ 56 \\ \frac{p^1 - i p^2}{|p^0| + |p^3|} \\ \frac{[\vec{p}] - 12}{[\vec{p}] - 12} \\ \frac{[\vec{p}] + 12}{[\vec{p}] + 12} \end{array} \right) e^{-i(p^0\cdot x^0 + \vec{p} \cdot \vec{x})},\]

\[\psi^{(6)}_{\frac{1}{2}, -\frac{1}{2}} (\vec{p}) = \left( \begin{array}{c} 03 \\ +i \\ 12 \\ - \end{array} \right) \left( \begin{array}{c} 12 \\ 56 \\ \frac{p^1 - i p^2}{|p^0| + |p^3|} \\ \frac{[\vec{p}] + 12}{[\vec{p}] + 12} \\ \frac{[\vec{p}] - 12}{[\vec{p}] - 12} \end{array} \right) e^{-i(p^0\cdot x^0 - \vec{p} \cdot \vec{x})},\]

states in the Dirac sea

\[p^0 = -|p^0|,\]

\[\psi^{(6)}_{\frac{1}{2}, -\frac{1}{2}} (\vec{p}) = \left( \begin{array}{c} 03 \\ -i \\ 12 \\ - \end{array} \right) \left( \begin{array}{c} 12 \\ 56 \\ \frac{p^1 + i p^2}{|p^0| + |p^3|} \\ \frac{[\vec{p}] + 12}{[\vec{p}] + 12} \\ \frac{[\vec{p}] - 12}{[\vec{p}] - 12} \end{array} \right) e^{i(p^0\cdot x^0 + \vec{p} \cdot \vec{x})},\]

\[\psi^{(6)}_{\frac{1}{2}, -\frac{1}{2}} (\vec{p}) = \left( \begin{array}{c} 03 \\ -i \\ 12 \\ - \end{array} \right) \left( \begin{array}{c} 12 \\ 56 \\ \frac{p^1 - i p^2}{|p^0| + |p^3|} \\ \frac{[\vec{p}] - 12}{[\vec{p}] - 12} \\ \frac{[\vec{p}] + 12}{[\vec{p}] + 12} \end{array} \right) e^{i(p^0\cdot x^0 - \vec{p} \cdot \vec{x})},\]

(17)

We do not pay attention on the normalization.

Let us summarize: We presented massless solutions of the Weyl equation (Eqs.(24)) for the positive and the negative energies. The positive energy states, put on the top of the Dirac sea, represent particles. The negative energy states can be found also from the positive energy states by the application of \(C_N \cdot \mathcal{P}_N^{(d-1)} = \gamma^0 \gamma^2 K I_{x^3} I_{x^6}\). Antiparticle states follow from the particle ones.
by either emptying the negative energy state in the Dirac sea or directly from the particle states by the application of the operator $C_N \cdot \mathcal{P}^{(d-1)}_N = \gamma^0 \gamma^5 I_{\bar{x}_3} I_{x^6}$ and putting these states on the top of the Dirac sea. There are, for each choice of the four momentum $(|p^0|, p^1, p^2, p^3)$ two positive and two negative energy states, while the two antiparticle states represent the hole in the Dirac sea. The antiparticle states, having opposite charges than the corresponding particle states, have obviously also different handedness $\Gamma^{3+1}$ but still left handedness with respect to $\gamma_5^{3+1}$ - than the particle states.

C. Massive solutions of the Weyl equation with charges

Since the discrete symmetries of Eqs.\ ([10], [13]) are the symmetries of the equations of motion (2), also the massive states, presented in Eq.\ ([5]), manifest these symmetries. As discussed in the ref. [8], and can easily be checked, the state with the charge $(n + \frac{1}{2})$ and spin $\frac{1}{2}$, $\psi^{(6)}_{n+\frac{1}{2}, \frac{1}{2}}$, which has for $F = \frac{1}{2}$ the mass $(m\rho_0)^2 = l(l+1)$, $l = 1, 2, 3, \ldots; 0 \leq n \leq l$, transforms under $C_N \cdot \mathcal{P}^{(d-1)}_N$ into its anti-particle state $\psi^{(6)}_{-n-\frac{1}{2}, \frac{1}{2}}$ of the same spin and mass and opposite charge

$$C_N \cdot \mathcal{P}^{(d-1)}_N \psi^{(6)}_{n+\frac{1}{2}, \frac{1}{2}} = (B_{n+1} + i(+) (+) + A_n e^{i\phi} [-i(+) (+)] \cdot e^{-i(n+1)\phi} e^{-i(p_0 x_0 + p^3 x^3)} \cdot 18)$$

This massive antiparticle state (if put on the top of the Dirac sea, representing the hole in the Dirac sea in the negative energy state $C_N \cdot \mathcal{P}^{(d-1)}_N \psi^{(6)}_{n+1/2, \frac{1}{2}}$) solves the Weyl equation (6) with $F \rightarrow -F$, $B_{n+1} = A_{-n-1}$ and $A_n = B_{-n}$.

The massive particles carry the charges $(n + \frac{1}{2})$ and the masses $(m\rho_0)^2 = l(l+1)$, $l = 1, 2, 3, \ldots; 0 \leq n \leq l$, which depend on the radius of $S^2$. The corresponding antiparticles carry the same mass (determined by $l$) and the opposite charge $(-n - \frac{1}{2})$.

Let us summarize: There are two positive energy states for each mass $(m\rho_0)^2 = l(l+1)$ and $(|p^0|, p^1, p^2, p^3)$ and each charge $n + \frac{1}{2}$ with the spin $\pm \frac{1}{2}$, which, if put on the top of the Dirac sea, represent the particles. There are two corresponding negative energy states and there are the corresponding two antiparticle states of different handedness $\Gamma^{3+1}$ (in the usual notation $\gamma^5$), which, put on the top of the Dirac sea, represent the holes in the Dirac sea.

D. Massive chargeless solutions of the Weyl equation

Let us now assume that the scalar fields, the gauge fields of $S^{56}$, that is $f_s^\sigma \omega_{56\sigma}$, with $s = (5, 6)$ and $\sigma = ((5), (6))$, gain non zero vacuum expectation values. These two scalar fields are the analogy
to the complex higgs scalar of the standard model: Higgs in the standard model carries the weak
and the hyper charge, while our scalar fields carry only the "hyper" charge $S^{56}$. The charge, which
is the spin in $d = (5, 6)$, is after the scalar fields gain nonzero vacuum expectation values no longer
the conserved quantity.

In this case we replace in the Weyl equation (2) the quantities $p_{0\pm}$ with their vacuum expectation
values $< p_{0\pm} >$, so that the equations of motion read

$$< p_{0\pm} >= m_{\pm} ,$$

$$\gamma^m p_m + (\pm) m_{\mp} (m_{\pm})^{(6)} \psi(6) = 0.$$  

(19)

To simplify, the coordinate system in $d = (3 + 1)$ with $\vec{p} = 0$ is chosen. Then Eq. (19) reads

$$\{ p_0 + \gamma^0 (\pm) m_{\mp} (m_{\pm})^{(6)} \} \psi^{(6)} = 0 ,$$  

(20)

The two positive and the two negative energy solutions with the spin in $d = (3+1) \pm \frac{1}{2}$, the nonzero
mass $m$ manifesting in $d = (3+1)$ (with no conserved charge) obeying Eq. (20), are as follows

$$\psi^{(6)}_{\frac{1}{2},m} = \left( \frac{03}{12} \frac{56}{12} \frac{56}{56} \frac{12}{m_{\mp}} \right) \left( \frac{12}{56} \frac{56}{12} \frac{12}{56} \frac{12}{56} \right) e^{-imx} ,$$

$$\psi^{(6)}_{\frac{3}{2},m} = \left( \frac{03}{12} \frac{56}{12} \frac{56}{56} \frac{12}{m_{\mp}} \right) \left( \frac{12}{56} \frac{56}{12} \frac{12}{56} \frac{12}{56} \right) e^{imx} ,$$

$$\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)} \psi^{(6)}_{\frac{1}{2},m} = \left( \frac{03}{12} \frac{56}{12} \frac{56}{56} \frac{12}{m_{\mp}} \right) \left( \frac{12}{56} \frac{56}{12} \frac{12}{56} \frac{12}{56} \right) e^{imx} ,$$

$$\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)} \psi^{(6)}_{\frac{3}{2},m} = \left( \frac{03}{12} \frac{56}{12} \frac{56}{56} \frac{12}{m_{\mp}} \right) \left( \frac{12}{56} \frac{56}{12} \frac{12}{56} \frac{12}{56} \right) e^{imx} ,$$

$$m^2 = m_{\mp} m_{\pm} , \quad m_{\mp} = -m_{\pm} , \quad (p_0)^2 = m^2.$$  

(21)

There is no massless state any longer.

Let us notice that $(\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)}) p_{0\pm} (\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)})^{-1} = -p_{0\pm}$ (provided that $\omega_{56\pm} = \omega_{56\pm}$).

Accepting the expectation values $< p_{0\pm} >$ as the "mass" term $m_{\pm}$ (indeed $m^2 = m_{\mp} m_{\pm}$) it
then follows that $(\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)}) m_{\pm} (\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)})^{-1} = -m_{\pm} = m_{\mp}$, requiring that both, $m_{\pm}$, are
imaginary, in accordance with Eq. (21). Since also $\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)} \gamma^0 (\pm) (\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)})^{-1} = \gamma^0 (\pm)$,
$\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)}$ stays as the symmetry of the effective equations of motion.

Let us check also the discrete symmetry operator $\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)}$. One finds that $\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)} \gamma^0 (\pm)
(\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)})^{-1} = \gamma^0 (\mp)$ and that $\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)} p_{0\pm} (\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)})^{-1} = p_{0\mp}$, while $\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)} m_{\pm}
(\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)})^{-1} = m_{\pm}$. This means that the effective Weyl equation (Eq.20) is not invariant with
respect to $\mathbb{C}N \cdot \mathcal{P}_N^{(d-1)}$. For this effective Weyl equation the operator "emptying" must be therefore changed. Let us multiply it with $-i \Gamma^{(6)} \Gamma^{(3+1)}$. The operator $\Gamma^{(6)}$ is a number, which is $-1$ in the case of our starting representation, Eq. (3), while $\Gamma^{(3+1)}$ distinguishes among these four states, having the alternating values $\pm 1$, respectively. One finds that $-i \Gamma^{(6)} \Gamma^{(3+1)} = \gamma^5 \gamma^6 \frac{56}{56} (=-2iS^{56})$.

Let us now check how does the equation of motion $\{\gamma^0 \gamma^m p_m + \gamma^0 ((+ \ m_+\ (-) \ m_-)) \} \psi^{(6)} = 0$ transform with respect to this new proposed symmetry

\[
-\frac{1}{i} \Gamma^{(6)} \Gamma^{(3+1)} \mathbb{C}N \cdot \mathcal{P}_N^{(d-1)} \psi^{(6)}_{\frac{1}{2},m} = ((-i) \frac{03}{03} \frac{12}{12} \frac{56}{56} \frac{1}{1} + m_+ \frac{m_+}{m_+}) \frac{03}{03} \frac{12}{12} \frac{56}{56} \frac{1}{1} \frac{1}{1} e^{-imx^0},
\]

\[
-\frac{1}{i} \Gamma^{(6)} \Gamma^{(3+1)} \mathbb{C}N \cdot \mathcal{P}_N^{(d-1)} \psi^{(6)}_{-\frac{1}{2},m} = ((-i) \frac{03}{03} \frac{12}{12} \frac{56}{56} \frac{1}{1} - i \frac{m_+}{m_+} \frac{m_+}{m_+}) \frac{03}{03} \frac{12}{12} \frac{56}{56} \frac{1}{1} \frac{1}{1} e^{-imx^0},
\]

\[
m^2 = m_+ m_-, \quad m_+ = -m_-, \quad (p_0)^2 = m^2,
\]

which means that $-\frac{1}{i} \Gamma^{(6)} \Gamma^{(3+1)} \mathbb{C}N \cdot \mathcal{P}_N^{(d-1)} \psi^{(6)}_{\frac{1}{2},m} = \frac{i m_+}{m_+} \psi^{(6)}_{\frac{1}{2},m}$. The two positive solutions of the effective Weyl equations (Eq. (20)) representing particle carrying no charge are indistinguishable from the two positive energy solutions for the corresponding two antiparticles, which are indeed the two holes in the Dirac sea.

One sees that the other two superposition, $\psi^{(6)}_{\frac{1}{2},m} = \frac{03}{03} \frac{12}{12} \frac{56}{56} \frac{1}{1} \frac{1}{1} + m_+ \frac{m_+}{m_+} \frac{m_+}{m_+}$ and $\psi^{(6)}_{-\frac{1}{2},m} = \frac{03}{03} \frac{12}{12} \frac{56}{56} \frac{1}{1} \frac{1}{1} - i \frac{m_+}{m_+} \frac{m_+}{m_+} e^{-imx^0}$ are not solutions of the Weyl equation and so are not also the negative energy states, obtained by the application of $\mathbb{C}N \cdot \mathcal{P}_N^{(d-1)}$ on these two states.

In this discussion only one family is assumed. To obtain true masses of spinors one must take into account the families and also the loop corrections in all orders, to which also the dynamical scalar and vector gauge fields contribute.
Let us summarize: In our effective equations of motion (Eq. (19)) the vacuum expectation values of the scalar fields are assumed. Correspondingly no massless solutions exist any longer. We are left instead with the two positive energy states (representing particles if put on the top of the Dirac sea) and the two corresponding negative energy states, the holes in which represent (put on the top of the Dirac sea) the antiparticle states (Eq. (23)). The antiparticle states are indistinguishable from the particle states and are indeed the Majorana particles. For the effective equations of motion the discrete symmetry "emptying" has to be changed, so that the operator which transform the particle state into the antiparticle state is $-i\Gamma^{(6)}\Gamma^{(3+1)}C_N\cdot\mathcal{P}_N^{(d-1)}$.

All these states solve the Weyl equation (20).

E. Dirac representation of spinor states

Let us connect our positive and negative energy solutions for the particular choice of the coordinate system ($p^m = (m,0,0,0)$) and then for the general choice of the coordinate system $p^m = (p^0, p^1, p^2, p^3)$, solving the Weyl equation (Eq. (19))

$$p^0 = |p^0|, $$
$$\psi^{(6)}_{\frac{1}{2},m}(\vec{p}) = N(\vec{p}, m) \{ p^1 + ip^2 \left[ \begin{array}{c} 03 \\ 12 \\ 56 \\ m_+ \end{array} \right] |p^0| + |p^3| + m \left[ \begin{array}{c} 03 \\ 12 \\ 56 \\ -m_+ \end{array} \right] e^{-i(p^0)x^0 - \vec{p}\cdot \vec{x}} \right) = \frac{p^1 + ip^2}{|p^0| + |p^3| + m} \left[ \begin{array}{c} 03 \\ 12 \\ 56 \\ m_+ \end{array} \right] \left( (i^0)(+)(+) - \frac{m_+}{m_+} [-i](+)[-] \right) e^{-i(p^0)x^0 - \vec{p}\cdot \vec{x}}, $$

$$p^0 = -|p^0|, $$
$$\psi^{(6)}_{-\frac{1}{2},m}(\vec{p}) = N(\vec{p}, m) \left\{ p^1 - ip^2 \left[ \begin{array}{c} 03 \\ 12 \\ 56 \\ m_+ \end{array} \right] |p^0| + |p^3| + m \left[ \begin{array}{c} 03 \\ 12 \\ 56 \\ -m_+ \end{array} \right] e^{i(p^0)x^0 + \vec{p}\cdot \vec{x}} \right) = \frac{p^1 - ip^2}{|p^0| + |p^3| + m} \left[ \begin{array}{c} 03 \\ 12 \\ 56 \\ m_+ \end{array} \right] \left( (i^0)(+)(+) - \frac{m_+}{m_+} [-i](+)[-] \right) e^{i(p^0)x^0 + \vec{p}\cdot \vec{x}}, \quad \text{(24)}$$

with the usual notation in the textbooks [23].

The two positive and the two negative solutions of Eq. (21), $(\psi^{(6)}_{\frac{1}{2},m}, \psi^{(6)}_{-\frac{1}{2},m})$ with $p^0 = |p^0|$ and $i\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)} \psi^{(6)}_{\frac{1}{2},m}, -i\mathcal{C}_N \cdot \mathcal{P}_N^{(d-1)} \psi^{(6)}_{-\frac{1}{2},m}$ with $p^0 = -|p^0|$), respectively, if we do not pay attention
on normalization, represent the four states, usually written as
\[
\begin{pmatrix}
\varphi \\
\frac{\sigma \cdot p}{|p|+m} \varphi
\end{pmatrix} e^{-i(p^0 x^0 - p \cdot x)}, \quad \varphi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \varphi = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
\[
\begin{pmatrix}
\frac{\sigma \cdot p}{|p|+m} \chi \\
\chi
\end{pmatrix} e^{i(p^0 x^0 - p \cdot x)}, \quad \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
with \( \frac{\sigma}{2} = (S^{23}, S^{31}, S^{12}) \). In Eq. (21) \( p^0 = m \) and \( \vec{p} = 0 \), while the general case (Eq. (24)) corresponds to \( p^m = (p^0, p^1, p^2, p^3) \).

The two positive energy antiparticle states, put on the top of the Dirac sea, obtained from the two positive energy states in Eq. (24) by the application of \(-i C \Gamma (6) \Gamma (3+1) \) (or by emptying the corresponding negative energy states), are the same as the starting two particle states, if \( \vec{p} \) is replaced by \(-\vec{p} \), in the exponent \( e^{-i|p|^0 + i\vec{p} \cdot \vec{x}} \) into \( e^{-i|p|^0 - i\vec{p} \cdot \vec{x}} \), and in all the coefficients of the wave functions \((p^1, p^2, |p^3|) \) go to \((-p^1, -p^2, -|p^3|)\).

Let us summarize: The two particle and two antiparticle states with spin either \( \frac{1}{2} \) or \(-\frac{1}{2} \) are for \( \vec{p} = 0 \) in \( d = (3 + 1) \) undistinguishable, due to the fact that there are no conserved charges. For \( \vec{p} \neq 0 \), the particle and antiparticle state distinguish in the sign of \( \vec{p} \). This means that in Eq. (25) \( \vec{p} \) for particles must be replaced by \(-\vec{p} \). This means again that the particle states are indistinguishable from the antiparticle ones.

F. Majorana spinors

In the case that there is no conserved charge the Majorana particle, the state of which (put on the top of the Dirac sea) is the sum of the particle and the corresponding antiparticle state \( \psi_{\pm}^{(6)\text{majorana}} \), reads for \( \vec{p} = 0 \)
\[
\psi_{\pm}^{(6)\text{majorana}} = \frac{1}{\sqrt{2}} \left( \psi_{\pm}^{(6)\text{m}} + (\pm) \left( i \right) \Gamma^{(6)} \Gamma^{(3+1)} \psi_{\pm}^{(6)\text{m}} (\vec{p} = 0) \right),
\]
\[
(\pm), \quad \text{if} \quad m_+ = (\mp) i m,
\]
is obviously equal to the particle and the antiparticle state at the same time, which is also true for any \( \vec{p} \), with the mass
\[
- \langle \psi_{\pm}^{(6)\text{majorana}} | \gamma^0 \left( \begin{pmatrix} 56 \\ m_+ \end{pmatrix} + \begin{pmatrix} 56 \\ m_- \end{pmatrix} \right) | \psi_{\pm}^{(6)\text{majorana}} \rangle = m
\]
Let us summarize: The Majorana particle is, in this case of no charge, the particle and the antiparticle at the same time.

III. COMMENTS AND CONCLUSIONS

This paper is written to stress some well known properties of quantum states. The properties of particle and antiparticle states are determined by the action: Starting with a well defined action, which demonstrates in \( d = (3 + 1) \) the charges and the masses, so that the origin of both is well understood, the discrete symmetry operators are (although model dependent) well defined. Changing the starting action into the effective one, the discrete symmetries will usually need a redefinition. The action as well the corresponding Weyl equations for massless particle (and antiparticle) states contain an even number of the Clifford operators - \( \gamma^a \) matrices, as it do also the Weyl equations for massless states in \( d = (3 + 1) \). The Clifford odd operators transform one Weyl representation into another one. Analysing properties of states obtained from the solutions of the Weyl equation with respect to the Clifford odd operators makes sense, if the starting action connects even and odd representations. The mass term, appearing with the Clifford odd operator in the Dirac equation, is the effective one. The origin of the mass is unknown.

We demonstrate in this paper on a toy model that whatever properties of particles and antiparticles are studied, the action is needed, the solutions of which are the states under consideration, if one wants to understand properties of states.

We discuss degrees of freedom of particles and antiparticles, starting with a well defined representation. We take the toy model in which the \( M^{5+1} \) manifold breaks into \( M^{3+1} \times \) an almost \( S^2 \) sphere due to the zweibein and the spin connection fields in \( d = (5, 6) \). We look for the solutions of the Weyl equations within one Weyl representation in \( d = (5 + 1) \) and study the degrees of freedom and symmetries, which massless and massive particles and antiparticles manifest in \( d = (3 + 1) \).

The toy model [9–12] is chosen to make discussions as transparent as possible. The massless and mass protected spinors manifest the spin in \( d = (5, 6) \) as the (Kaluza-Klein) charge in \( d = (3 + 1) \), while a spinor with the total angular momentum equal to \( n + \frac{1}{2} \), \( 0 < n \leq l, l, 0, 1, 2, \ldots \), manifest \( n + \frac{1}{2} \) as the charge and carry the nonzero mass \((m\rho_0)^2 = l(l+1)\), \( \rho_0 \) is the radius of an almost \( S^2 \) with respect to \( d = (3 + 1) \). The spinors states differ in masses and charges. We also treat the case when the spin connections in with the indices (5,6), and correspondingly scalars with respect to \( d = (3 + 1) \), gain non zero vacuum expectation values. In this case there are no massless particles any longer, as well as no conserved charges massive chargeless particles in \( d = (3 + 1) \) behave as
the Majorana particles.

We use the technique \cite{1-7} for representing spinors, which makes the illustration transparent. We use the concept of the Dirac sea to treat the second quantized picture, what enables a nice physical understanding. The discrete symmetry operators for the first and the second quantized picture are taken from the ref. \cite{8, 17}.

We conclude that in all the studied cases, in the case of the massless (with $U(1)$) charged states in the case of the massive (with $U(1)$) charged states and in the massive chargeless case, there are two positive and two negative energy solutions of the equations of motion. All the states, the particle states, the negative energy states in the Dirac sea, as well as the antiparticle states, which are the holes in the Dirac sea, solve the equations of motion. Either the negative energy states or the positive energy states are obtainable also directly from the particle states by the application of the discrete symmetry operators \cite{8, 17, 22} $C_N \cdot P_N^{(d-1)}$ and $C_N \cdot P_N^{(d-1)}$, respectively.

As long as the charges are conserved quantities, the antiparticle states distinguish from the particle states in charges and masses, as expected. The antiparticle states are of the opposite handedness in $d = (3+1)$ as the particle states, they both are of the same handedness in $d = (5+1)$.

In the case that the vacuum expectation values of the scalar fields, the analogue of the higgs in the standard model carrying in the toy model case the hyper charge only (it is the $U(1)$ charge of the integer value), causing that no charge is conserved and that massless solutions become massive, the antiparticle states coincide with the particle states, representing the Majorana particles.

If one assumes $d = (7 + 1)$ instead of $d = (5 + 1)$ and lets all the scalar (with the indices $(5, 6, 7, 8)$) spin connection fields to gain nonzero vacuum expectation values, it is still true that there are two positive (with spin $\pm \frac{1}{2}$) and two negative (again with spin $\pm \frac{1}{2}$) solutions of the Weyl equations of motion, but in this case the antiparticle states distinguish from the particle states \cite{28} rather then $C_{(N)} \times P_{N}^{(d-1)}$, which is the case in $d = 2(2n + 1)$ and is correspondingly a good discrete symmetry of the system.

The higher is the dimension the more charges are available. Taking $d = (13 + 1)$, the action of Eq. (1) has in the starting Weyl representation all the particle and antiparticle states as required by the standard model, with the right handed neutrinos added. The right handed quarks and leptons, weak chargeless, carry the additional $SU(2)$ charge which after the break of the starting symmetries manifest the hyper charge, while the left handed ones are weak charged and carry no $SU(2)$ charge of the second kind and consequently no hyper charge. The quarks distinguish from the leptons besides in the colour charge also in the "fermion" number, which is $\frac{1}{6}$ for quarks and $-\frac{1}{2}$ for leptons, while the antiquark and the antilepton states, appearing in the same Weyl representation,
carry the opposite weak, colour, additional $SU(2)$ charge and the "fermion" quantum number. Correspondingly is the "fermion" charge equal to zero for particles and antiparticles separately.

The discrete symmetry operators of Eq. (10,13) have the desired properties also in the case of $d = (13 + 1)$ (as they are in all even dimensional spaces). All the antiparticle states are reachable from the particle ones by the application of the operator $\mathcal{C}_N \times \mathcal{P}_N^{(d-1)}$. (The operator of "emptying" is, although not easy to imagine, since it operate among two completely different Fock spaces, very useful). All the negative energy states are reachable by the application of $\mathcal{C}_N \times \mathcal{P}_N^{(d-1)}$. $\mathcal{C}_N \times \mathcal{P}_N^{(d-1)}$ is the conserved quantity, unless the families are taken into account and the scalar fields gain nonzero vacuum expectation values, offering $\frac{(n-1)(n-2)}{2}$ complex phases. The scalar fields with the space index $(7,8)$ carry the $SU(2)$ weak charge $(\pm \frac{1}{2})$ and the hyper charge $(\pm \frac{1}{2})$ like in the standard model.

Although in $d = (3 + 1)$ there exist the left and the right handed solutions (an example is presented Eq. (17)), one can never come from one to another solution by the application of an Clifford odd operator. The change of the handedness in $d = (3 + 1)$ is always accompanied by the change of a spin in higher dimensions (representing a charge in $d = (3 + 1)$.

We conclude from this discussions that analysing the Dirac states in $d = (3 + 1)$ without having a model behind, telling where do the effective masses and charges originate, might be misleading. We hope that our discussions in this paper will help to clarify the matter.

Appendix: The technique for representing spinors [8, 17, 18], a shortened version of the one presented in [17, 18].

The technique [1–8, 18] can be used to construct a spinor basis for any dimension $d$ and any signature in an easy and transparent way. Equipped with the graphic presentation of basic states, the technique offers an elegant way to see all the quantum numbers of states with respect to the Lorentz groups, as well as transformation properties of the states under any Clifford algebra object.

The objects $\gamma^a$ have properties $\{\gamma^a, \gamma^b\} = 2\eta^{ab} I$, for any $d$, even or odd. $I$ is the unit element in the Clifford algebra.

The Clifford algebra objects $S^{ab}$ close the algebra of the Lorentz group $S^{ab} := (i/4)(\gamma^a \gamma^b - \gamma^b \gamma^a)$, $\{S^{ab}, S^{cd}\}_- = i(\eta^{ad} S^{bc} + \eta^{bc} S^{ad} - \eta^{ac} S^{bd} - \eta^{bd} S^{ac})$. The “Hermiticity” property for $\gamma^a$’s: $\gamma^a \dagger = \eta^{aa} \gamma^a$ is assumed in order that $\gamma^a$ are formally unitary, i.e. $\gamma^a \gamma^a = I$.

The Cartan subalgebra of the algebra is chosen in even dimensional spaces as follows: $S^{03}, S^{12}, S^{56}, \ldots, S^{d-1}d$, if $d = 2n \geq 4$. 

The choice for the Cartan subalgebra in \( d > 4 \) is straightforward. It is useful to define one of the Casimir operators of the Lorentz group - the handedness \( \Gamma (\{ \Gamma , S^{a b} \}_- = 0 \) in any \( d \), for even dimensional spaces it follows: \( \Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{a a}} \gamma^a) \), if \( d = 2n \). The product of \( \gamma^a \)'s in the ascending order with respect to the index \( a \): \( \gamma^0 \gamma^1 \cdots \gamma^d \) is understood. It follows for any choice of the signature \( \eta^{a a} \) that \( \Gamma^1 = \Gamma \), \( \Gamma^2 = I \). For \( d \) even the handedness anticommutes with the Clifford algebra objects \( \gamma^a (\{ \gamma^a , \Gamma \}_+ = 0) \).

To make the technique simple the graphic presentation is introduced

\[
\begin{align*}
\frac{ab}{(k)} &:= \frac{1}{2} (\gamma^a + \frac{\eta^{a a}}{ik} \gamma^b), & \frac{ab}{[k]} &:= \frac{1}{2} (1 + \frac{i}{k} \gamma^a \gamma^b),
\end{align*}
\]  

(A.1)

where \( k^2 = \eta^{a a} \eta^{a b} \). One can easily check by taking into account the Clifford algebra relation and the definition of \( S^{a b} \) that if one multiplies from the left hand side by \( \gamma \), \( S^{a b} \) the Clifford algebra objects \( [k] \) and \( [k] \), it follows that

\[
\begin{align*}
S^{a b} (k) &= \frac{1}{2} k [k] , & S^{a b} [k] &= \frac{1}{2} k [k] ,
\end{align*}
\]  

(A.2)

which means that we get the same objects back multiplied by the constant \( \frac{1}{2} k \). This also means that when \( (k) \) and \( [k] \) act from the left hand side on a vacuum state \( | \psi_0 \rangle \) the obtained states are the eigenvectors of \( S^{a b} \). One can further recognize that \( \gamma^a \) transform \( (k) \) into \( [k] \), never to \( [k] \):

\[
\begin{align*}
\gamma^a_{(k)} &= \eta^{a a} [k], & \gamma^b_{(k)} &= -ik [k], & \gamma^a_{[k]} &= (k), & \gamma^b_{[k]} &= -ik \eta^{a a} (k).
\end{align*}
\]  

(A.3)

Let us deduce some useful relations

\[
\begin{align*}
\frac{ab}{(k)(k)} &= 0 , & \frac{ab}{(k) (-k)} &= \eta^{a a} \frac{ab}{[k]} , & \frac{ab}{[k] [k]} &= \frac{ab}{(k)}, & \frac{ab}{[k] (-k)} &= 0 ,
\end{align*}
\]  

(A.4)

Taking into account the above equations it is easy to find a Weyl spinor irreducible representation for \( d \)-dimensional space.

For \( d \) even we simply make a starting state as a product of \( d/2 \), let us say, only nilpotents \( (k) \), one for each \( S^{a b} \) of the Cartan subalgebra elements, applying it on an (unimportant) vacuum state. Then the generators \( S^{a b} \), which do not belong to the Cartan subalgebra, being applied on the starting state from the left, generate all the members of one Weyl spinor.

\[
\begin{align*}
(k_{0 d})(k_{12})(k_{35}) \cdots (k_{d - 1} d - 2) \psi_0 \\
[-k_{0 d}][-k_{12}](k_{35}) \cdots (k_{d - 1} d - 2) \psi_0 \\
\vdots \\
(k_{0 d})[-k_{12}][-k_{35}] \cdots (k_{d - 1} d - 2) \psi_0 \\
\vdots
\end{align*}
\]  

(A.5)
All the states have the handedness $\Gamma$, since $\{\Gamma, S^a_{\text{ub}}\} = 0$. States, belonging to one multiplet with respect to the group $SO(q, d-q)$, that is to one irreducible representation of spinors (one Weyl spinor), can have any phase. We made a choice of the simplest one, taking all phases equal to one.

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[24] The *spin-charge-family* theory of one of us (S.N.M.B.) offers the explanation not only for the appearing of charges and gauge vector and scalar fields but also for the appearance of families.

[25] The *spin-charge-family* theory assumes the presence of the fields $\tilde{\omega}_{abc}$, the gauge fields of the generators $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^{a}\tilde{\gamma}^{b} - \tilde{\gamma}^{b}\tilde{\gamma}^{a})$, with $\{\tilde{\gamma}^{a}, \gamma^{b}\} = 0$. This second kind of the Clifford algebra objects form obviously the equivalent representations with respect to the Dirac one. There are only two kinds of gamma operators: the Dirac one corresponding to the left multiplication of any Clifford algebra object and this second one corresponding to the right multiplication of the same Clifford algebra object.

[26] According to the *spin-charge-family* theory there is the same number of families as there are spinor states within one family representation: $2^{\frac{d}{2} - 1}$.

[27] In the case that $d = (5, 6)$ is a compact $S^2$ sphere these massive solutions would make infinite spectrum with quantum numbers $(l, n)$, $l$ defining in $d = (3 + 1)$ the mass and $n + \frac{1}{2}$ the Kaluza-Klein charge. In the case of an almost $S^2$ the spectrum start to stop when the energy approaches the strengths of the source which causes the vielbein leading to an almost $S^2$.

[28] One must pay attention that in even dimensional cases with $d = 4n$ the operator $C_{(N)}$ has an even number of $\gamma^{a}$'s.