Investigation on the Accuracy of Approximate Solutions Obtained by Perturbation Method for Galloping Equation of Iced Transmission Lines

1. Introduction

As an important energy transmission channel, high-voltage transmission lines are related to national social and economic development as well as people’s lives closely, so it is necessary to ensure the effective and normal operation of high-voltage transmission lines [1]. The galloping of iced transmission lines has always been a hot topic and iced transmission lines study is of great application value [2, 3]. As is known to us, the cross section of transmission lines would change from circular cross section to crescent or D-shape cross section under complex climatic conditions such as snow-rime, downburst, typhoon, and hurricane [3–5]. And, when iced transmission lines with noncircular cross section are subjected to horizontal wind loads, the iced transmission lines would be galloping. And the galloping of...
iced transmission lines is a self-excited vibration with low frequency and large amplitude [6], which may cause flashover, short circuit, frequent tripping, broken strand or wire, and other accidents [7, 8]. In order to investigate the galloping of iced transmission lines, as early as 1932, Den Hartog [9] proposed the vertical galloping mechanism and found that the horizontal vibration amplitude of iced transmission lines was much smaller than the vertical amplitude, indicating that galloping mainly occurred in the vertical direction. Based on this concept, only the galloping in the vertical direction for iced transmission lines is considered in this paper.

In fact, the span length of transmission lines is much larger than the diameter of the transmission lines, so the transmission lines belong to the cable element structure. And since the transmission lines belong to the cable element structure, the partial differential galloping equation of transmission lines is also derived based on the cable element structure [10]. The partial differential vibration equation of cable element structure can be transformed into ordinary differential vibration equation by Galerkin method [11, 12]; then, Runge-Kutta method, averaging method, and multiple scales method can be used to solve the ordinary differential vibration equation. By solving the vibration equation, the vibration characteristics of cable element structure can be studied. In addition, vibration characteristics of cable element structure can be studied through experiments [13, 14]. However, for studying the nonlinear vibration characteristics of cable element structure, the perturbation method is used mostly. For example, Luongo [15] used the multiple scales method to obtain the periodic and quasiperiodic solutions of the suspension cable and studied the influences of turbulences on structural characteristics. Nielsen used averaging method to obtain the approximate solutions of cable subjected to superharmonics excitation and discussed the influences about combined harmonics and superharmonics effect on vibration characteristics of cable [16]. For the nonlinear dynamic behavior of a string-beam coupled system subjected to parametric and external excitation, the case of 1:2 internal resonance between the modes of the beam and string—principal parametric resonance-1/2 subharmonic resonance for the beam and primary resonance for the string—is considered in [17], and Zhang [18, 19] analyzed global bifurcation and chaos by the techniques of phase portrait, waveform, and Poincaré map. The global perturbation method is employed to analyze the global bifurcations and chaotic dynamics of the string-beam coupled system in [20]. Yao [21] introduced how to employ the extended Melnikov method to analyze the Shilnikov-type multipulse homoclinic bifurcations and chaotic dynamics of high-dimensional nonlinear systems in engineering applications. Based on the averaging equation obtained by the perturbation method, the nonlinear dynamics of the string-beam coupled system is studied in [17–21]. In addition, Luongo [22] used the first-order multiple scales method to analyze the amplitude-frequency function of cable under the external excitation and obtained a bifurcation diagram. Luongo [23, 24] established a model of cable with two DOFs considering the effects of static swing and dynamic torsion and used second-order multiple scales method to solve the corresponding nonlinear ordinary differential vibration equation of the model. Benedettini [25] and Srinil [26] used second-order multiple scales method to analyze the linear stability of stay-cable under main external resonance and also studied the stability of stay-cable under 1:1 internal resonance. Based on second-order multiple scales method, Yao [27] studied the vibration characteristic of nonlinear oscillations system with internal resonance of 1:1 and 2:1. Benedettini [28] also used the fourth-order multiple scales method to solve the nonlinear partial differential equation of shallow cable under weak periodic excitation and analyzed the multivalued effect of the amplitude-frequency function of shallow cable.

The above research papers mainly used the averaging method and the first-order, second-order, third-order, and fourth-order multiple scales methods to solve the nonlinear ordinary differential equation. These methods all fall under the category of perturbation method, which is also a commonly used method for solving galloping equation of iced transmission lines. However, few scholars have studied the influences of perturbation method on the accuracy of approximate solutions of galloping equation of transmission lines. Based on this concept, this paper systematically analyzed the accuracy of different-order’s multiple scales method. Firstly, the galloping governing equation of iced transmission line under wind excitation is obtained by physical modeling. The quadratic and cubic nonlinear restoring force and the Rayleigh damping in the governing equation have constituted the hysteretic nonlinear restoring force, which is widely representative. Next, the approximate analytical solution of the governing equation is obtained by using the averaging method and the different-order’s multiple scales method. Finally, the accuracy of the analytical solution is obtained by comparing the analytical solution with the numerical solution. Through the above analysis, this paper studied the influence of quadratic restoring force \((q^2)\) on the galloping system of transmission lines for the first time and found that the quadratic restoring force makes the vibration center of the system shift and the magnitude of the restoring force is directly related to drift of the system. It is also found in this paper that the existence of quadratic restoring force also has affected the use of the perturbation method. The approximate analytical solution of the averaging method cannot reflect the drift of the vibration center of the system, but the approximate analytical solution of the multiple scales method (first-order, second-order, third-order, and fourth-order) can reflect the drift. In addition, this paper also analyzed the accuracy of different-order’s multiple scales method for solving weakly nonlinear systems. As the condition of the system nonlinearity (damping force and restoring force) increasing with the changing of physical and aerodynamic parameters, it is found that the accuracy of the fourth-order multiple scales method is higher than that of the lower-order multiple scales method. From the theoretical point of view, the conclusion of this paper can give a reasonable explanation for the phenomenon of the drift of vibration center of the system. In practical application, the conclusion of this paper can guide
researchers and engineers to use reasonable perturbation method to study weak nonlinear system. More importantly, the conclusions obtained in this paper would be helpful to the solutions of galloping equation of iced transmission lines and could also give some references to practical engineering.

2. Dynamic Model of Iced Transmission Lines

Most of high-voltage transmission lines in plain areas can be equivalent to equal-height transmission lines. Therefore, single-span equal-height transmission lines with two fixed supports are considered in this paper. The mechanical model of single-span iced transmission lines is established as shown in Figure 1, and it is considered that the wind is along z-axis direction as shown in Figure 2, which is the most dangerous condition. The transmission lines are assumed to be homogenous and linearly elastic with negligible torsion, bending, and shear stiffness. The initial static equilibrium configuration of the transmission lines under gravity is \( \xi_1 \), and when the static equilibrium configuration of the transmission lines changes under the action of other external loads, the corresponding equilibrium configuration becomes \( \xi_2 \) (dynamic equilibrium configuration).

Figure 1 is the model of single-span equal-height transmission lines, and, in Figure 1, \( u(s,t) \) and \( v(s,t) \) are the displacements measured from the dynamic equilibrium configuration in \( x \)-axis and \( y \)-axis directions, respectively; and the associated static equilibrium configuration of transmission lines can be described through the parabola \( y = 4d[s/L-(s/L)^2] \) [29], in which \( s \) is the curvilinear abscissa, \( l \) is the span length, and \( d \) is the sag.

In order to facilitate the study of the aerodynamic loads of iced transmission lines, it is assumed that the ice is changeless along the transmission lines and the iced shape is a common crescent iced shape.

Figure 2 is the model of cross section of iced transmission lines, Figure (a) is the physical model, and Figure (b) is the force analysis model. In Figure 2, \( O_1 z_2 \) is the axis of symmetry of the model of cross section, \( O_1 z_1 \) is the direction in which the horizontal wind acts on, \( O_1 \zeta \) is the horizontal axis, \( \alpha \) is wind attack angle, \( \alpha_0 \) is initial wind attack angle, \( \alpha_1 \) is relative wind attack angle, \( U \) is horizontal wind velocity, \( U_0 \) is relatively wind velocity, and \( v \) is the vertical galloping velocity.

From Figure 2, the following can be obtained:

\[
\tan(\alpha) = \frac{\dot{v}}{U} = \alpha. \tag{1}
\]

The relative wind acts on the iced transmission lines, resulting in an air drag \( F_D \) along the relative wind direction and an upward air lift \( F_L \) perpendicular to the relative wind direction; therefore, the following can be obtained:

\[
F_y = F_L \cos(\alpha) - F_D \sin(\alpha). \tag{2}
\]

Considering small deformation, that is, \( \sin(\alpha) = \alpha \), \( \cos(\alpha) = 1 \), (2) can be simplified as follows:

\[
F_y = F_L - \alpha F_D. \tag{3}
\]

According to fluid-induced vibration theory, the expressions of \( F_L \) and \( F_D \) can be listed [30]:

\[
\begin{align*}
F_L &= \frac{1}{2} \rho U^2 D C_L, \tag{4a} \\
F_D &= \frac{1}{2} \rho U^2 D C_D, \tag{4b}
\end{align*}
\]

where \( C_L \) is the coefficient of aerodynamic lift, \( C_D \) is the coefficient of aerodynamic drag, \( \rho \) is the air density, and \( D \) is the diameter of transmission lines.

Substituting (4a) and (4b) into (3), based on Taylor’ law, the aerodynamic coefficients in \( y \)-axis direction can be fitted with a cubic curve:

\[
C_f = A' \alpha + B' \alpha^3 + C' \alpha^5 + C_0, \tag{5}
\]

where \( A', B', C', \) and \( C_0 \) are aerodynamic coefficients related to the aerodynamic loads.

According to (2)–(4a) and (4b), the following can be obtained:

\[
F_y = \frac{1}{2} \rho U^2 D C_f. \tag{6}
\]

According to (5)–(6), the following can be obtained:

\[
F_y = \ddot{a} \dot{v} + \ddot{b} v^2 + \ddot{c} v^2 + \ddot{d}, \tag{7}
\]

where

\[
\begin{align*}
\ddot{a} &= \rho U DA' \\
\ddot{b} &= \rho DB' \\
\ddot{c} &= \rho DC' \\
\ddot{d} &= \frac{\rho U^2 D C_0}{2}
\end{align*}
\]

The initial wind attack angle and icing shape of transmission lines remain unchanged, and the quasistatic assumption is adopted; then, \( A', B', C' \), and \( C_0 \) are constants. Since the density \( \rho \) and the diameter \( D \) are also constants, the coefficient \( \ddot{c} \) is a constant. The fitting coefficient \( \ddot{c} \) does not change with the wind velocity \( U \), so coefficient \( \ddot{c} \) is also a constant term. Therefore, the coefficient \( \ddot{c} \) of aerodynamic load is not considered in this paper.
addition, the fitting coefficient \( \bar{d} \) does not change with the vertical wind velocity \( (\dot{v}) \), and the force of aerodynamic coefficient \( \bar{d} \) is a constant. Then, the coefficient \( \bar{d} \) has nothing to do with the generation of transmission line galloping. Similarly, the coefficient \( \bar{d} \) of aerodynamic load is not considered in this paper.

Den Hartog [9] found that the horizontal vibration amplitude of iced transmission lines was much smaller than the vertical amplitude, indicating that galloping mainly occurred in the vertical direction. Based on this concept, only the galloping in the vertical direction for iced transmission lines is considered in this paper. And according to [28–30], the governing equation in the vertical direction of iced transmission lines is

\[
\begin{align*}
H \ddot{v} + ES \left( y' + \dot{v}' \right) \int_0^l \left[ \frac{y' \dot{v}' + \dot{v}'^2}{2} \right] dx' + F_y - \mu \ddot{v} = m \dddot{v},
\end{align*}
\]

where \( H \) is the tension of the transmission lines, \( E \) is Young’s modulus of the transmission lines, and \( S \) is the cross-sectional area of transmission lines. \( \dot{v}' \) is the first derivative of the vertical motion function with respect to \( x \), \( y' \) is the first derivative of the parabolic equation with respect to \( x \), \( \ddot{v} \) and \( \dddot{v} \) are the first derivative and second derivative of the vertical motion function with respect to time \( t \), respectively, \( \mu \) represents the structural damping, and \( m \) represents the self-weight per unit unstretched length.

### 3. Numerical Solutions

The displacement \( v(x, t) \) in (9) can be written as

\[
v(x, t) = f(x)q(t),
\]

where \( f(x) \) is the mode shapes and \( q(t) \) is the time function.

Based on Galerkin method [31], substituting (10) into (9) can obtain the nonlinear ordinary differential equation:

\[
\ddot{q} + \omega^2 q + c_1 \dot{q}^2 + c_2 \dot{q}^4 + (\mu - c_3) \dot{q} + c_4 q^4 = 0.
\]

Equation (11) represents the galloping equation of transmission lines under the action of horizontal wind, and the parameters in (11) are

\[
\begin{align*}
\omega^2 &= \frac{1}{m I_m} I_0, \\
c_1 &= \frac{12 ES I_1}{m l I_m} , \\
c_2 &= \frac{ES I_2}{2 ml I_m} , \\
\mu^* &= \frac{\mu}{m} , \\
c_3 &= \frac{\bar{a}}{m} , \\
c_4 &= \frac{\bar{b}}{m} I_b / I_m ,
\end{align*}
\]

where

\[
\begin{align*}
I_0 &= H \int_0^l f^1 \cdot f' dx + 64 \frac{d^2 ES}{l^5} \left( \int_0^l f dx \right)^2 , \\
I_1 &= \int_0^l f^2 dx \int_0^l f dx , \\
I_2 &= \left( \int_0^l f^2 dx \right)^2 , \\
I_m &= \int_0^l f^2 dx , \\
I_b &= \int_0^l f^4 dx.
\end{align*}
\]

Equation (11) contains the first-order and third-order nonlinear damping terms, the second-order and the third-order nonlinear restoring force term of \( q \). The coefficient of the aerodynamic loads in the system includes the first-order and third-order terms, which is consistent with damping term of the Rayleigh equation (Rayleigh damping). The aerodynamic loads are presented in the form of the Rayleigh damping in the nonlinear galloping equation. Moreover, the nonlinear restoring force term is consistent with the Duffing
equation. Therefore, (11) can be regarded as the combined form of the Duffing equation and the Rayleigh equation.

For the physical parameters and aerodynamic parameters of iced transmission lines, refer to [32] shown in Table 1.

Substituting the physical parameters and aerodynamic parameters in Table 1 into (11) and using the Runge-Kutta function [33–36] in MATLAB, we can obtain numerical solutions of the galloping equation. Figure 3 is the displacement response curve obtained by numerical method, Figure 3(a) is the time history of displacement curve obtained by numerical method, Figure 3(b) is the time history of velocity curve obtained by numerical method, and Figure 3(c) is the phase diagram obtained by numerical method.

It can be seen from Figure 3 that vibration amplitude and vibration velocity would increase with time gradually and vibration amplitude and vibration velocity tend to be stable after 850 s. The vibration amplitude and vibration velocity generate a stable limit cycle in Figure 3(c). The maximal vibration amplitude $q = 0.57$ m (0.51 m) in the vertical direction, which indicates that the vibration center of system is not symmetric about the equilibrium point. And the non-linearity of the transmission lines leads to the drift of vibration center in the process of galloping.

### 4. The Approximate Solutions of Averaging Method

The approximate solutions can be easily got by averaging method and it is more convenient to calculate the partial differential galloping equation. Quoting solution form of small parameter in [16], (11) can be written as

$$\ddot{q} + \omega^2 q = -\epsilon (c_1 \dot{q}^2 + c_2 q^3 + \mu q + c_3 \dot{q}^3),$$  \hspace{1cm} (14)

where $\epsilon$ is a small but finite parameter. Because (14) can be seen as a weakly nonlinear autonomous system, the vibration solutions of (14) are as follows:

$$q = \bar{a} \cos (\omega t + \theta), \quad (15a)$$

$$\dot{q} = -\overline{\omega} \sin (\omega t + \theta), \quad (15b)$$

where $\bar{a}(t)$ is an amplitude function and $\theta(t)$ is a phase function.

The amplitude function can be obtained by averaging method, which is

$$\bar{a} = \sqrt{\frac{1}{\left(\left(1/\alpha_0^2\right) + (3\omega^2 c_4/4(c_3 + \mu))\right) e^{(\epsilon \tau)} e^\epsilon - \left(3\omega^2 c_4/4(c_3 + \mu)\right)}}. \quad (16)$$

In the same way, the phase function can also be obtained by averaging method

$$\theta = \theta_0 + \frac{3c_2 \epsilon t}{8\left(\left(1/\alpha_0^2\right) + (3\omega^2 c_4/4(c_3 + \mu))\right) e^{(\epsilon \tau)} e^\epsilon - \left(3\omega^2 c_4/4(c_3 + \mu)\right)}.$$

(17)

According to (15a) and (15b)-(17), the approximate solutions of (14) are

$$q = \sqrt{\left(\left(1/\alpha_0^2\right) + (3\omega^2 c_4/4(c_3 + \mu))\right) e^{(\epsilon \tau)} e^\epsilon - \left(3\omega^2 c_4/4(c_3 + \mu)\right)} \cos \left(\frac{\omega t + \frac{3c_2 \epsilon t}{8\left(\left(1/\alpha_0^2\right) + (3\omega^2 c_4/4(c_3 + \mu))\right) e^{(\epsilon \tau)} e^\epsilon - \left(3\omega^2 c_4/4(c_3 + \mu)\right)}}{\left(\left(1/\alpha_0^2\right) + (3\omega^2 c_4/4(c_3 + \mu))\right) e^{(\epsilon \tau)} e^\epsilon - \left(3\omega^2 c_4/4(c_3 + \mu)\right)}\right).$$

(18)

In (18), $a_0$ represents the initial disturbance of the transmission lines. Taking the derivative of (18) with respect to $t$, the following can be obtained:

$$\dot{q} = \left(p_0 + n_0 a_0^2\right) \bar{a} \cos (\omega t + \theta) - \left(\omega + g_0 a_0^2\right) \bar{a} \sin (\omega t + \theta).$$

(19a)
where

\[ p_0 = -\varepsilon (c_3 + \mu) \frac{c_4}{2}, \]
\[ n_0 = -\frac{3\varepsilon \omega^2 c_1}{8}, \]
\[ g_0 = \frac{3\varepsilon c_2}{8\omega}. \]

In order to more intuitively reflect the accuracy of the approximate solutions of averaging method, Figure 4 is the comparison curves of numerical solutions and approximate solutions of averaging method.

As shown in Figure 4, there are some differences in value of amplitudes \( q \) between two curves. For example, the phase diagram in Figure 4 does not overlap, and the vibration centers of the two curves do not overlap either. The point with coordinates \((0, 0)\) in Figure 4 represents the vibration center of the numerical solutions. To this end, the curve of the approximate solutions of the averaging method is symmetrical about the coordinate center, and the phase diagram of the numerical method is not symmetrical about the coordinate center. In other words, the approximate solutions of the averaging method do not reflect the drift of the vibration center of the transmission lines.

5. Approximate Solutions of Multiple Scales Method

5.1. Fourth-Order Multiple Scales Method. The multiple scales method is a commonly used to solve galloping equation of iced transmission lines. According to (1)–(10), the nonlinear galloping equation of the transmission lines under steady wind velocity is established, that is, (11):

\[ \ddot{q} + \omega^2 q + c_1 \dot{q}^2 + c_2 q^3 + (\mu^* - c_3) \dot{q} + c_4 \dot{q}^3 = 0, \]

where, due to the inclusion of quadratic and cubic nonlinear terms, the aerodynamic coefficient of the aerodynamic loads

\[ \text{Figure 3: Displacement response curve obtained by numerical method. (a) Time history of displacement curve. (b) Time history of velocity curve. (c) Phase diagram.} \]
in the system also includes the first-order and third-order terms. It is generally considered that the damping and nonlinear terms have little effect on the system. To this end, the solution form of small parameter is

\[ q = \bar{q}, \quad \dot{q} = \varepsilon \dot{\bar{q}}, \quad \mu \dot{\bar{q}} = \varepsilon \mu \bar{q}, \quad \ddot{\bar{q}}^3 = \varepsilon^3 \bar{q}^3 \]

(21)

where \( \bar{q}, \bar{\mu}, \) and \( \ddot{\bar{q}} \) are new variables of the system.

In order to make the equation easy to express, \( q, \mu, \) and \( \ddot{q} \) are still used to represent \( \bar{q}, \bar{\mu}, \) and \( \ddot{\bar{q}} \). Substituting (21) into (20), we can obtain

\[ \dddot{q} + \omega^2 q + \varepsilon(c_1 q^2 + c_3 \dot{q}) + \varepsilon^2(c_2 \dot{q}^3 + \mu \dot{q}) + \varepsilon^4 c_4 \ddot{q}^3 = 0. \]

(22)

The fourth-order approximate solutions have been discussed, and set \( q \) as

\[ q = q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + \varepsilon^3 q_3 + \varepsilon^4 q_4. \]

(23)

A new independent time variable \( T_n = \varepsilon^n t(n = 0, 1, 2, 3, 4) \) was obtained by multiple scales method, in which \( n \) represents the order of dividing the time variable. The larger the value of \( n \), the greater the amount of calculation and the complexity. Then, let the partial differential operator symbol for deriving \( t \) be

\[ DT = D_0 T_0 + \varepsilon D_1 T_1 + \varepsilon^2 D_2 T_2 + \varepsilon^3 D_3 T_3 + \varepsilon^4 D_4 T_4 \]

(24)

Substituting (23) and (24) into (22) and equating coefficients of like powers of \( \varepsilon^n \) \((n = 0, 1, 2, 3, \) and \( 4) \) lead to the following linear ordinary equations, respectively:

\[ D_0^2 q_0 + \omega^2 q_0 = 0, \]  

(25a)

\[ D_0^2 q_1 + \omega^2 q_1 = -2D_0 D_1 q_0 - c_1 q_0^2, \]  

(25b)

\[ D_0^2 q_2 + \omega^2 q_2 = -\left[D_0^2 q_0 + 2D_0 D_2 q_0 + 2D_0 D_1 q_1 + 2c_1 q_0 q_1 + c_3 q_0^3 + (\mu - c_3)D_0 q_0 \right], \]  

(25c)

\[ D_0^2 q_3 + \omega^2 q_3 = -\left[D_0^2 q_1 + 2(D_0 D_3 q_0 + D_1 D_2 q_0 + D_2 D_1 q_1 + D_0 D_2 q_1 + D_1 D_3 q_2 + c_1 (q_1^2 + 2q_0 q_2)) + 3c_2 q_0^2 q_1 + (\mu + c_3)(D_0 q_1 + D_1 q_0) \right], \]  

(25d)

\[ D_0^2 q_4 + \omega^2 q_4 = \left[D_0^2 q_2 + D_2^2 q_0 + 2(D_0 D_4 q_0 + D_1 D_3 q_0 + D_2 D_2 q_0 + D_1 D_4 q_1 + D_2 D_3 q_1 + D_3 D_2 q_1 + D_0 D_4 q_2 + D_1 D_5 q_2 + D_2 D_4 q_2 + D_3 D_3 q_2 + 2c_1 (q_1 q_2 + q_0 q_3) \right], \]  

(25e)
where $D_k$ represents the partial derivative of $T_i$; the solutions of (25a) can be written in this form
\[
q_0 = A(T_1, T_2, T_3, T_4)e^{i\omega T_0} + \bar{A}(T_1, T_2, T_3, T_4)e^{-i\omega T_0},
\]
(26)
where $A$ can be set as
\[
A(T_1, T_2, T_3, T_4) = \frac{1}{2a_4(T_1, T_2, T_3, T_4)} \cdot \exp[i\beta_4(T_1, T_2, T_3, T_4)],
\]
(27)
where $a_i(t)$ and $\beta_i(t)$, respectively, represent the amplitude function and phase function of the system ($i = 1, 2, 3, 4$). Substituting (26) into (25a)-(25b), the following can be obtained:
\[
D_3A = 0,
\]
where
\[
q_3 = \frac{c_1}{9\omega^3} \left(2c_3 - 2\mu - 4c_4^2\right)A^2 e^{2i\omega t} + \left(\frac{59c_1^4}{54\omega^6} - \frac{31c_1c_2}{12\omega^4}\right)A^4 e^{2i\omega t} + \left(\frac{c_1^3}{54\omega^6} + \frac{c_1c_2}{12\omega^4}\right)A^4 e^{4i\omega t} + \left(\frac{5c_1c_2}{9\omega^4} - \frac{19c_1^4}{9\omega^6}\right)A^2 e^{3i\omega t} + cc.
\]
(30a)
Substituting (26) and (28a)-(30b) into (25e), we obtain
\[
D_4A = -\left(\frac{485c_1^4}{108\omega^6} + \frac{15c_2^2}{16\omega^3} - \frac{173c_1^4}{12\omega^5}\right)iA^3 e^{-i\omega t} + \left(\frac{4\mu^2c_2}{9\omega^5} + \frac{2\mu\omega c_4}{9\omega^4} + \frac{3\omega^2c_4}{2}\right)A^3 e^{-i\omega t} - \left(\frac{\mu^2}{8\omega^2} + \frac{c_3}{4\omega} + \frac{c_4}{8\omega}\right)A + cc.
\]
(31)
Combining (24) and (27)-(31) to obtain the averaging equation about amplitude and phase of the system,
\[
\dot{a}_4 = -\frac{\epsilon^2}{2} (\mu + c_3)a + \left(\frac{c_1^3}{9\omega^4} + \frac{c_2^3}{18\omega^3} + \frac{\mu \omega^2 c_3}{18\omega^4} - \frac{3\omega^2 c_4}{8}\right)\epsilon^4 a^3,
\]
(32a)
\[
\dot{\beta}_4 = \frac{\epsilon^2}{8} \left(\frac{3c_2}{8\omega^2} - \frac{5c_1^2}{12\omega^4}\right) a^2 - \epsilon^4 \left(\frac{485c_1^4}{1728\omega^7} + \frac{15c_2^2}{256\omega^4} - \frac{173c_1^4}{192\omega^5}\right) a^4 - \epsilon^4 \left(\frac{\mu^2}{8\omega^2} + \frac{c_3}{4\omega} + \frac{c_4}{8\omega}\right).
\]
(32b)
After integrating (32a) and (32b) over time, the following can be obtained:
\[
a_4 = \sqrt{\frac{1}{\left(\frac{1}{1/a_0^2} + (n/m)\right)e^{-2nt} - (n/m)}},
\]
(33a)
\[
\beta_4 = \beta_0 - \frac{jhm^2}{2n}\ln\left(\frac{m}{a_0^2}e^{-2mt} + ne^{-2mt} - n\right) + \frac{jhm^2}{4n}\left(\frac{m}{a_0^2}e^{-2mt} + ne^{-2mt} - n\right)^2 - \frac{jhm^2}{n^3}t - kt
\]- \frac{jhm^2}{2n} \left(1 + \frac{m}{n\omega_0^2} \left[\frac{m}{a_0^2} + n\right]\left(\frac{m}{a_0^2}e^{-2mt} + ne^{-2mt} - n\right)\right)^{-1}. 
\]
(33b)
In (33a)-(33b), \( \beta_0 \) represents the integral constant, and other parameters are
\[
m = -\frac{\epsilon^2}{2} (\mu + c_1),
\]
\[
k = \epsilon^4 \left( \frac{\mu^2}{8\omega} + \frac{c_3}{4\omega} + \frac{c_1^2}{8\omega} \right),
\]
\[
n = \left( \frac{c_1 c_3}{9\omega^3} - \frac{c_1^2 c_3}{18\omega^4} + \frac{\mu c_1^2}{18\omega^4} - \frac{3\omega^2 c_4}{8} \right) \epsilon^4,
\]
\[
j = \epsilon^4 \left( \frac{3c_2}{8\omega} - \frac{5c_1^2}{12\omega^3} \right),
\]
\[
h = \left( \frac{485c_4}{1728\omega^5} + \frac{15c_2^2}{256\omega^3} - \frac{173c_2^2 c_4}{192\omega^5} \right) \epsilon^4.
\]
Combining (23), (26), (27), and (33a)–(33c) to obtain the expression of zero-order amplitude,
\[
q^0 = a_4 \cos \psi_4 + \epsilon \frac{c_1}{6\omega} a_4^2 \cos 2\psi_4 + \epsilon^2 \left( \frac{c_1}{48\omega^3} + \frac{c_2}{32\omega^3} \right) a_4^4 \cos 3\psi_4
\]
\[
+ \epsilon^3 \left[ -\frac{c_1}{9\omega^3} (\mu + c_3) a_4^2 \cos 2\psi_4 + \left( \frac{5c_1}{432\omega^3} - \frac{31c_1 c_2}{96\omega^5} \right) a_4^4 \cos 2\psi_4 \right.
\]
\[
\left. + \epsilon^3 \left( \frac{c_1}{432\omega^3} + \frac{c_1 c_2}{96\omega^5} \right) a_4^4 \cos 4\psi_4 + \left( \frac{5c_1 c_2}{8\omega^5} - \frac{19c_3}{72\omega^7} \right) a_4^4 \right]
\]
where \( \psi_4 = \omega t + \beta_4 \).

Equations (34), (36), and (37), respectively, correspond to approximate solutions of zero-order (\( q^0 \)), second-order (\( q^1 \)), and third-order (\( q^2 \)) amplitude of multiple scales method. In order to obtain the accuracy of (34), (36), and (37), the time history displacement curves and phase diagram about the numerical solutions and the approximate solutions of multiple scales method are compared. Figure 5 is the comparison curves about numerical solutions and the approximate solutions of multiple scales method.

As shown in Figure 5, the time history displacement curve obtained by multiple scales method has the same phenomenon as that obtained by averaging method, which has shown that the expressions of zero-order amplitude cannot accurately describe the vibration amplitude of the nonlinear galloping equation. However, the expressions of second-order amplitude and the expressions of third-order amplitude are very accurate in describing the vibration amplitude of nonlinear galloping equation.

5.2. Third-Order Multiple Scales Method. In order to analyze the accuracy of approximate solutions obtained by multiple scales method of the solution form of third-order small parameter, let \( \epsilon^4 c_3 q^3 \) in (22) be \( c_3 q^3 \); then
\[
\ddot{q} + \omega^2 q + \epsilon (c_1 q^3 + c_1 \dot{q}) + \epsilon^2 (c_2 q^3 + \mu \dot{q}) + \epsilon^3 c_3 q^3 = 0.
\]
Similarly, using the multiple scales method to solve (38), the following can be obtained:
\[
a_3 = \frac{1}{\left[ \left( 1/a_0^2 \right) + (n_3/m_3) \right] e^{-2m_3 t} - (n_3/m_3)}
\]
\[
\beta_3 = \beta_0 - \frac{m_1 k c_2^2}{n_3} t - \left( \frac{\epsilon^2}{8a_0^2} c_2^3 + \frac{\epsilon m c_3}{4\omega} \right) t
\]
\[
- \frac{k c_2^3}{2n_3} \ln \left( \frac{m_1 e^{-2m_3 t} + n_3 e^{-2m_3 t} - n_3}{m_1} \right).
\]
Figure 5: Comparison curves of numerical solutions and approximate solutions of multiple scales method. (a) Zero-order expression of amplitude ($q(\varepsilon^0)$). (b) Second-order expression of amplitude ($q(\varepsilon^2)$). (c) Third-order expression of amplitude ($q(\varepsilon^3)$).
where \( k_3 = (3c_2/8\omega) - (5c_1^2/12\omega^3); m_3 = - (\varepsilon(c_3/2) + \varepsilon^3(\mu/2) + \varepsilon^3(c_3^2/32\omega^5)); n_3 = \varepsilon^3((1/18\omega^6)c_0^2 c_3 - (3/8)c_4\omega^2), \)

The approximate solutions of multiple scales method of the solution form of third-order small parameter (\( T_0, T_1, T_2, \) and \( T_3 \)) are

\[
q(\varepsilon^0, \varepsilon^1, \varepsilon^2) = a_2 \cdot \cos(\psi_3) + \varepsilon \left[ \frac{c_1}{6\omega^2} \cdot \cos(2\psi_3) - \frac{c_1}{2\omega^2} \right] + \varepsilon^2 \left[ \frac{c_1^2}{128\omega^6} \cdot \cos(2\psi_3) + \frac{5c_1^2}{576\omega^6} \cdot \cos(3\psi_3) + \frac{c_1}{9\omega^2} \cdot \sin(2\psi_3) \right]
\]

where \( \psi_3 = \omega t + \beta_3. \)

5.3. Second-Order Multiple Scales Method. In order to analyze the accuracy of approximate solutions obtained by multiple scales method of the solution form of second-order small parameter, let \( \varepsilon^2 c_4^2 q^2 \) \( \varepsilon^0 c_4 q^2 \) \( \varepsilon^2 c_4 q^2 \)

\[
\bar{q} + \omega^2 \bar{q} + \varepsilon (c_1 q^2 + c_3 \bar{q}) + \varepsilon^2 (c_2 q^3 + \mu \bar{q}) + \varepsilon^3 c_4 q^3 = 0.
\]

Similarly, using the multiple scales method to solve (46), the following can be obtained:

\[
q(\varepsilon^0, \varepsilon^1, \varepsilon^2) = a_2 \cdot \cos(\psi_3) + \varepsilon \left[ \frac{1}{6\omega^2} \cdot c_1 a_2^2 \cdot \cos(2\psi_3) - \frac{1}{2\omega^2} \cdot c_1 a_2^2 \right] + \varepsilon^2 \left[ \frac{c_1}{32\omega^2} \cdot \cos(2\psi_3) + \frac{c_1}{48\omega^2} \cdot \cos(3\psi_3) + \frac{c_1}{9\omega^2} \cdot \sin(2\psi_3) \right],
\]

where \( \psi_2 = \omega t + \beta_2. \)

5.4. First-Order Multiple Scales Method. In order to analyze the accuracy of approximate solutions obtained by multiple scales method of the solution form of first-order small parameter, \( q(\varepsilon^0, \varepsilon^1, \varepsilon^2) \) \( \varepsilon \)

\[
\bar{q} + \omega^2 \bar{q} + \varepsilon (c_1 q^2 + c_3 \bar{q}) + \varepsilon^2 (c_2 q^3 + \mu \bar{q}) + \varepsilon^3 c_4 q^3 = 0.
\]

Similarly, using the multiple scales method to solve (44), the following can be obtained:

\[
a_1 = \sqrt{\frac{1}{[[1/a_0^2] + (n_1/m_1)] e^{-2m_1 t} - (n_1/m_1)]}},
\]

\[
\beta_1 = \beta_0 - \frac{m_1 k_2 \varepsilon}{n_2} \ln \left( \frac{m_1}{a_0^2} e^{-2m_1 t} + n_2 e^{-2m_1 t} - n_1 \right),
\]

where \( m_1 = (-\varepsilon^2 \mu + \varepsilon^0 c_2)/2); n_1 = (-3/8)\varepsilon^2 \omega^2 c_4; k_2 = (3c_2/8\omega) - (5c_1^2/12\omega^3), \)

The approximate solutions of multiple scales method of the solution form of first-order small parameter (\( T_0, T_1 \)) are
6. Influences of Parameter on the Approximate Solutions

Under the condition that the physical parameters and aerodynamic parameters remain constant, Sections 4 and 5.1 discuss the accuracy of the approximate solutions of the nonlinear galloping equation by the perturbation method. This section studies whether the changes of physical parameters and aerodynamic coefficients would cause the nonlinear changing (damping and restoring force) of the system, so as to affect the accuracy of the analytical solution of the perturbation method. And the major contents of this section are mainly the following:

1. The change of tension and sag will cause the change of parameters in the nonlinear galloping equation.
2. The change of Young’s modulus will cause the change of parameters in the nonlinear galloping equation.
3. The change of wind velocity will cause the change of parameters in the nonlinear galloping equation.

6.1. The Influences of Tension on the Approximate Solutions. For flexible cable structures, the relationship between tension and sag is \( d = \frac{mgL^2}{8H} \) [41], where tension \( (H) \) is inversely proportional to sag \( (d) \) and tension \( (H) \) is proportional to span \( (L) \). The iced transmission lines are a type of flexible cable structure. The greater the tension is, the greater the structural stiffness will be. As the tension of iced transmission lines changing, the natural frequency term \( \omega^2 \) and the coefficient \( c_1 \) of quadratic term will also change. Table 2 compares the accuracy of approximate solutions of perturbation method under different tension.

In Table 2, firstly, substituting the tension \( H = 20 000, 30 000, 40 000, 50 000, \) and \( 60 000 \) N into (12)-(13), the corresponding parameters \( (\delta) \) are obtained; secondly, substituting these parameters \( (\delta) \) into (11), then the numerical solution in Table 2 is obtained by solving (11) with Runge-Kutta method in MATLAB; finally, substituting the parameters \( (\delta) \) into (18), (46), (43), and (37), the approximate analytical solution in Table 2 is obtained. It is worth noting that, in Table 2, the amplitude represents the mean value of the largest absolute value of the vibration amplitude \( (\pm) \) after the system is stable; the drift represents the difference-value between the vibration center of the system and the equilibrium point of the system; the error represents the difference between the approximate analytical solution and the numerical solution. And the definition of the amplitude, drift, and error in Tables 3-6 is consistent with Table 2, respectively.

As shown in Table 2, with the increasing in tension, the coefficient \( c_1 \) of the quadratic term in (11) would decrease. With the decreasing in the quadratic term coefficient \( c_1 \), the drift of vibration center of the transmission lines would continue to decrease. Therefore, the quadratic restoring force term has a direct effect on the drift of vibration center of the system. In addition, the error of the approximate solutions of the averaging method also continues to decrease, which shows that the existence of the quadratic restoring force term is directly related to the accuracy of approximate solutions of the averaging method.

It can be seen in Table 2 that the approximate solutions of the first-order, second-order, and fourth-order expression of amplitude of multiple scales method have basically the same accuracy for the nonlinear galloping equation. Under the same parameter conditions, increasing the tension of the transmission lines would reduce the vibration amplitude of the transmission lines.

6.2. The Influence of Young’s Modulus on the Approximate Solutions. Young’s modulus is a structural parameter of iced transmission lines. The greater Young modulus is, the greater the stiffness of the transmission lines will be. The natural frequency term \( \omega^2 \), quadratic term coefficient \( c_1 \), and cubic term coefficient \( c_2 \) in (11) also change with the changing of Young’s modulus of iced transmission lines. Table 3 compares the accuracy of approximate solutions of perturbation method under different Young’s modulus.

In Table 3, Young’s modulus \( E = 9 560.7, 47 803.3, 239 017, \) and \( 478 033 \) N/m². With the increase in Young’s modulus, it can be seen from Table 3 that the drift of vibration center of the system increases first and then decreases. And the error of the averaging method increases first and then decreases, which also shows that the drift of vibration center of the system is directly related to the accuracy of the averaging method.

It can be seen from Table 3 that when Young’s modulus increases from \( E = 9 560.7 \) N/m² to \( E = 478 033 \) N/m², the error of different-order multiple scales method increases from 0.516% to 2.033%. Therefore, when only the elastic modulus of the system is changed, the accuracy of the first-order, second-order, and fourth-order multiscale methods is basically the same. As Young’s modulus increases, the vibration amplitude of the iced transmission lines would decrease.

In order to verify the influence of Young’s modulus on the drift of vibration center of the system, Table 4 compares the drift of vibration center under different Young’s modulus. The natural frequency term \( \omega^2 \), the quadratic term coefficient \( c_1 \), and the cubic term coefficient \( c_2 \) of Young’s modulus \( E = 9 561–478 033 \) N/m² are introduced into (11); then the drift of vibration center is obtained by Runge-Kutta function in MATLAB.
### Table 2: The accuracy comparison of perturbation methods under different tension.

| Tension sag (N/m²) | Numerical solutions drift amplitude (°) | Averaging method amplitude (°) | First-order multiple scales q (ε₀^2) | Second-order multiple scales q (ε₀^2, ε₀, and ε₀^3) | Fourth-order multiple scales q (ε₀^2) |
|------------------|----------------------------------------|-------------------------------|-------------------------------------|-----------------------------------------------------|-------------------------------------|
| 20 000 N         | (-0.415 m, 0.372 m)                    | (-0.390 m, 0.387 m)          | (-0.410 m, 0.368 m)                | (-0.410 m, 0.367 m)                                 | (-0.412 m, 0.366 m)                |
| 3.209 m          | 0.022 m                                | 4.925%                        | 1.141%                              | 1.260%                                              | 1.230%                              |
| 30 000 N         | (-0.452 m, 0.411 m)                    | (-0.428 m, 0.429 m)          | (-0.450 m, 0.409 m)                | (-0.449 m, 0.409 m)                                 | (-0.449 m, 0.410 m)                |
| 1.539 m          | 0.021 m                                | 4.881%                        | 0.551%                              | 0.690%                                              | 0.534%                              |
| 40 000 N         | (-0.427 m, 0.400 m)                    | (-0.415 m, 0.415 m)          | (-0.429 m, 0.402 m)                | (-0.429 m, 0.402 m)                                 | (-0.429 m, 0.403 m)                |
| 1.154 m          | 0.014 m                                | 3.180%                        | 0.433%                              | 0.417%                                              | 0.592%                              |
| 50 000 N         | (-0.395 m, 0.377 m)                    | (-0.387 m, 0.390 m)          | (-0.401 m, 0.377 m)                | (-0.398 m, 0.379 m)                                 | (-0.396 m, 0.382 m)                |
| 0.924 m          | 0.009 m                                | 2.813%                        | 0.768%                              | 0.675%                                              | 0.786%                              |
| 60 000 N         | (-0.365 m, 0.354 m)                    | (-0.363 m, 0.362 m)          | (-0.368 m, 0.357 m)                | (-0.368 m, 0.357 m)                                 | (-0.368 m, 0.358 m)                |
| 0.770 m          | 0.006 m                                | 1.303%                        | 0.802%                              | 0.884%                                              | 0.891%                              |

### Table 3: The accuracy comparison of perturbation methods under different Young’s modulus.

| Young’s modulus (N/m²) | Numerical solutions drift | Averaging method error | First-order multiple scales q (ε₀^2) | Second-order multiple scales q (ε₀^2, ε₀, and ε₀^3) | Fourth-order multiple scales q (ε₀^2) |
|------------------------|---------------------------|------------------------|-------------------------------------|-----------------------------------------------------|-------------------------------------|
| 9 560.7                | (-0.509 m, 0.494 m)       | (-0.503 m, 0.504 m)    | (-0.503 m, 0.496 m)                | (-0.513 m, 0.495 m, 0.516 m)                         | (-0.511 m, 0.494 m)                |
| 47 803.3               | (-0.452 m, 0.408 m)       | (-0.429 m, 0.422 m)    | (-0.450 m, 0.409 m)                | (-0.449 m, 0.409 m)                                 | (-0.449 m, 0.410 m)                |
| 239 017                | (-0.300 m, 0.264 m)       | (-0.276 m, 0.276 m)    | (-0.296 m, 0.261 m)                | (-0.296 m, 0.260 m)                                 | (-0.295 m, 0.260 m)                |
| 478 033                | (-0.226 m, 0.203 m)       | (-0.210 m, 0.212 m)    | (-0.222 m, 0.200 m)                | (-0.223 m, 0.198 m)                                 | (-0.222 m, 0.198 m)                |

### Table 4: The vibration center drift of the system under different Young’s modulus.

| Young’s modulus (N/m²) | Amplitude drift | Amplitude drift | Amplitude drift | Amplitude drift |
|------------------------|----------------|----------------|----------------|----------------|
| 9 561 N/m²             | 32 507 N/m²    | 78 399 N/m²    | 170 183 N/m²   | 239 021 N/m²   |
| 0.008 m                | (-0.050 m, 0.494 m) | (-0.047 m, 0.438 m) | (-0.041 m, 0.371 m) | (-0.039 m, 0.295 m) |
| 32 507 N/m²            | 0.017 m        | 0.023 m        | 0.022 m        | 0.019 m        |
| 78 399 N/m²            | (-0.282 m, 0.247 m) | (-0.267 m, 0.236 m) | (-0.253 m, 0.223 m) | (-0.229 m, 0.205 m) |
| 170 183 N/m²           | 0.015 m        | 0.012 m        | 0.011 m        | 0.011 m        |

### Table 5: Comparison of the accuracy of q (ε₀^2), q (ε₀^3), q (ε₀^4), and q (ε₀^5) under different wind velocity.

| Wind velocity (m/s) | Numerical solutions drift | Approximate solutions q (ε₀^2) | Approximate solutions q (ε₀^3) | Approximate solutions q (ε₀^4) | Approximate solutions q (ε₀^5) |
|---------------------|---------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 4                   | (-0.214 m, 0.205 m)       | (-0.212 m, 0.212 m)           | (-0.217 m, 0.207 m)           | (-0.217 m, 0.207 m)           | (-0.217 m, 0.208 m)           |
| 8                   | (-0.452 m, 0.408 m)       | (-0.429 m, 0.430 m)           | (-0.450 m, 0.409 m)           | (-0.450 m, 0.410 m)           | (-0.451 m, 0.411 m)           |
| 12                  | (-0.700 m, 0.608 m)       | (-0.647 m, 0.647 m)           | (-0.693 m, 0.601 m)           | (-0.695 m, 0.602 m)           | (-0.697 m, 0.604 m)           |
| 16                  | (-0.960 m, 0.796 m)       | (-0.866 m, 0.867 m)           | (-0.949 m, 0.784 m)           | (-0.952 m, 0.787 m)           | (-0.956 m, 0.791 m)           |
| 20                  | (-1.237 m, 0.978 m)       | (-1.087 m, 1.088 m)           | (-1.217 m, 0.959 m)           | (-1.222 m, 0.961 m)           | (-1.229 m, 0.973 m)           |
| 24                  | (-1.535 m, 1.163 m)       | (-1.321 m, 1.321 m)           | (-1.512 m, 1.132 m)           | (-1.521 m, 1.139 m)           | (-1.526 m, 1.156 m)           |
| 28                  | (-1.834 m, 1.333 m)       | (-1.551 m, 1.557 m)           | (-1.822 m, 1.290 m)           | (-1.834 m, 1.309 m)           | (-1.839 m, 1.328 m)           |
| 32                  | (-2.115 m, 1.487 m)       | (-1.776 m, 1.779 m)           | (-2.126 m, 1.431 m)           | (-2.149 m, 1.460 m)           | (-2.154 m, 1.490 m)           |
| Mean error          | 0                          | 10.299%                      | 1.586%                        | 1.243%                        | 0.639%                        |
Table 6: The accuracy comparison of perturbation methods under different wind velocity.

| Wind velocity (m/s) | Averaging method error | First-order multiple scales \( q(\epsilon^0) \) and \( \epsilon^1 \) | Second-order multiple scales \( q(\epsilon^0), \epsilon^1, \epsilon^2, \) and \( \epsilon^3 \) | Third-order multiple scales \( q(\epsilon^0), \epsilon^1, \epsilon^2, \epsilon^3, \) and \( \epsilon^4 \) | Fourth-order multiple scales \( q(\epsilon^0) \) |
|---------------------|------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 4                   | (−0.210 m, 0.207 m)    | (−0.217 m, 0.207 m)            | (−0.217 m, 0.207 m)            | (−0.218 m, 0.207 m)            | (−0.217 m, 0.208 m)            |
|                     | 0.210 m) 2.228%        | 1.428%                         | 1.439%                         | 1.405%                         | 1.591%                         |
| 8                   | (−0.429 m, 0.410 m)    | (−0.449 m, 0.409 m)            | (−0.449 m, 0.409 m)            | (−0.450 m, 0.407 m)            | (−0.451 m, 0.411 m)            |
|                     | 0.428 m) 4.404%        | 0.499%                         | 0.552%                         | 0.668%                         | 0.201%                         |
| 12                  | (−0.648 m, 0.599 m)    | (−0.695 m, 0.599 m)            | (−0.693 m, 0.597 m)            | (−0.693 m, 0.597 m)            | (−0.697 m, 0.604 m)            |
|                     | 0.646 m) 6.851%        | 1.169%                         | 1.223%                         | 1.425%                         | 0.541%                         |
| 16                  | (−0.863 m, 0.786 m)    | (−0.948 m, 0.785 m)            | (−0.946 m, 0.779 m)            | (−0.956 m, 0.791 m)            | (−0.956 m, 0.791 m)            |
|                     | 0.862 m) 9.126%        | 1.169%                         | 1.300%                         | 1.800%                         | 0.539%                         |
| 20                  | (−1.081 m, 0.961 m)    | (−1.216 m, 0.957 m)            | (−1.207 m, 0.951 m)            | (−1.229 m, 0.973 m)            | (−1.229 m, 0.973 m)            |
|                     | 1.080 m) 11.529%       | 1.762%                         | 2.078%                         | 2.608%                         | 0.608%                         |
| 24                  | (−1.305 m, 1.139 m)    | (−1.507 m, 1.129 m)            | (−1.502 m, 1.129 m)            | (−1.481 m, 1.116 m)            | (−1.526 m, 1.156 m)            |
|                     | 1.305 m) 13.586%       | 1.955%                         | 2.563%                         | 3.780%                         | 0.621%                         |
| 28                  | (−1.529 m, 1.303 m)    | (−1.818 m, 1.303 m)            | (−1.805 m, 1.289 m)            | (−1.765 m, 1.284 m)            | (−1.839 m, 1.328 m)            |
|                     | 1.533 m) 15.844%       | 1.534%                         | 2.430%                         | 3.702%                         | 0.285%                         |
| 30                  | (−1.634 m, 1.365 m)    | (−1.960 m, 1.355 m)            | (−1.947 m, 1.355 m)            | (−1.897 m, 1.357 m)            | (−1.987 m, 1.411 m)            |
|                     | 1.636 m) 16.724%       | 1.966%                         | 2.641%                         | 3.849%                         | 0.351%                         |
| 32                  | (−1.745 m, 1.447 m)    | (−2.117 m, 1.447 m)            | (−2.098 m, 1.437 m)            | (−2.035 m, 1.429 m)            | (−2.154 m, 1.490 m)            |
|                     | 1.743 m) 17.321%       | 1.412%                         | 2.105%                         | 3.837%                         | 1.012%                         |
| Mean error          | 10.846%                | 1.303%                         | 1.815%                         | 2.563%                         | 0.639%                         |

With the increase in Young’s modulus from \( E = 9 561 \text{ N/m}^2 \) to \( E = 78 399 \text{ N/m}^2 \), the vibration center drift first increases from 0.008 m to 0.023 m. When Young’s modulus increases from \( E = 78 399 \text{ N/m}^2 \) to \( E = 478 033 \text{ N/m}^2 \), the drift decreases by 0.011 m. In addition, when Young’s modulus \( E = 9 560.7 \text{ N/m}^2 \) increases to \( E = 478 033 \text{ N/m}^2 \), however, the amplitude of iced transmission line decreases from (−0.509 m, 0.494 m) to (−0.226 m, 0.203 m).

As Young’s modulus gradually increases, the structural stiffness also increases. Under the combined contribution of the quadratic coefficient \( c_1 \) and the cubic coefficient \( c_2 \), the vibration amplitude of the iced transmission line decreases gradually, and the vibration center drift of the system increases first and decreases later. Under the same parameter conditions, increasing Young’s modulus of the transmission lines is beneficial to reducing the vibration amplitude of transmission lines.

6.3. The Relationship between the Coefficient and the Drift.

The galloping of iced transmission lines is affected by the transmission line’s own structure, the shape of the transmission line’s ice-cover shape, and wind loads. In general, the vibration amplitude and vibration velocity gradually increase with the increase in wind velocity.

In order to determine whether the increase in wind velocity affects the accuracy of the approximate solutions of multiple scales methods, the following sections would discuss the influence of different wind velocity on the accuracy of the multiple scales method and averaging method. Figure 6(a) shows the time history displacement diagram when the wind velocity is 8 m/s. The vibration amplitude of the transmission line increases slowly with time, until the amplitude gradually stabilizes at 1 050 s. And when the transmission lines are galloping, the vibration amplitude is 0.412 m and −0.452 m. The wind velocity in Figure 6(b) is 10 m/s, the galloping time of transmission lines is reduced to 650 s, and the galloping amplitude is increased to 0.796 m and −0.960 m. As the wind velocity increases, the time for the transmission line to reach galloping is shortened, and the amplitude of the galloping is increased.

Through the comparison of different tensions in Table 2 and different Young’s modulus in Table 3, it is found that the value of the quadratic coefficient is related to the drift of the vibration center. Then, in order to clarify the relationship between the coefficient of the quadratic term and the drift of the vibration center, let the coefficient of the quadratic term in (11) be equal to 0. And Figures 6(c) and 6(d) correspond to wind velocities \( U = 8 \text{ m/s} \) and \( U = 10 \text{ m/s} \), respectively. After comparing Figures 6(a) and 6(c), it can be found that the existence of the quadratic coefficient makes the vibration amplitude of the negative axis larger in Figure 6(a); that is, the phenomenon of vibration center drift occurs. Figures 6(b) and 6(d) also reveal the same phenomenon. To this end, it is easy to derive that the existence of quadratic restoring force makes the vibration center of the system drift.

6.4. The Effect of Wind Velocity on the Accuracy of \( q(\epsilon^0) \), \( q(\epsilon^1) \), \( q(\epsilon^2) \), and \( q(\epsilon^3) \).

The Rayleigh damping coefficients \( c_1 \) and \( c_2 \) in (11) would change with the wind velocity. Under the condition of different wind velocity, the accuracy of \( q(\epsilon^0) \), \( q(\epsilon^1) \), \( q(\epsilon^2) \), and \( q(\epsilon^3) \) is studied in Table 5. \( q(\epsilon^0) \), \( q(\epsilon^1) \), \( q(\epsilon^2) \), and \( q(\epsilon^3) \) represent zero-order equation (34), first-order equation (35), second-order equation (36), and third-order equation (37), and all of them are the approximate analytical solutions of multiple scales method. And the wind velocity \( U = 4–32 \text{ m/s} \) is shown in Table 5.

It can be seen from Table 5 that the drift of the vibration center changes from 0.005 m of 4 m/s to 0.314 m of 32 m/s. Although the changing of wind velocity does not lead to a changing in value of quadratic term coefficient \( c_1 \), the drift of
vibration center would increase with the increase in wind velocity. It is shown that when there is quadratic restoring force in the system, the drift of vibration center would also change with wind velocity change.

In addition, the error of the expressions of zero-order amplitude increases from 2.293% of wind velocity $U = 4$ m/s to 17.822% of wind velocity $U = 32$ m/s; the error of the expressions of first-order amplitude is less than 2.145%; the error of the expressions of second-order amplitude is less than 1.790%; the error of the expressions of third-order amplitude is generally less than 1.012%. As the wind velocity increases, the nonlinearity and Rayleigh damping of the system gradually strengthen, and the accuracy of the approximate solutions of multiple scales method tends to decrease overall. In the interval of 4–32 m/s wind velocity, the expressions of first-order amplitude correct the drift of the vibration center of the transmission lines. And the mean error of the expressions of zero-order amplitude is 10.299%, which has been corrected to 1.586% by the expressions of first-order amplitude.

From the above analysis, it can be seen that, with the increase in wind velocity, the vibration center drift of iced transmission lines would increase gradually in Table 5. The greater the drift is, the greater the error of the approximate solutions will be. The approximate solutions of expressions of zero-order amplitude of multiple scales method are not suitable to describe the galloping process with high wind velocity. However, the expressions of first-order amplitude of multiple scales method can correct the error of expressions of zero-order amplitude.

The expressions of second-order amplitude reduce the mean error of the expressions of first-order amplitude from 1.586% to 1.243%. The expressions of third-order amplitude reduce the mean error of the expressions of second-order amplitude from 1.243% to 0.639%. Therefore, with the increase in the order of expression of amplitude, the error of approximate solutions by multiple scales method has been corrected; that is, the high-order approximate solutions of multiple scales method can correct the error of low-order approximate solutions.
6.5. The Effect of Wind Velocity on the Accuracy of \( q, q (\varepsilon^0) \) and \( q (\varepsilon^1) \), \( q (\varepsilon^0, \varepsilon^1) \), \( q (\varepsilon^0, \varepsilon^1, \varepsilon^2) \), \( q (\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3) \), \( q (\varepsilon^0, \varepsilon^1, \varepsilon^2) \), \( q (\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3) \) and \( q (\varepsilon^3) \). Section 5.4 has discussed the effect of wind velocity on the accuracy of \( q (\varepsilon^0) \), \( q (\varepsilon^1) \), \( q (\varepsilon^2) \), and \( q (\varepsilon^3) \) with the same solution form of small parameter. Therefore, this section will discuss the effect of wind velocity on the accuracy of the approximate solutions obtained by the perturbation method with different solution form of small parameter. And \( q, q (\varepsilon^0 \) and \( \varepsilon^1 \), \( q (\varepsilon^0, \varepsilon^1, \varepsilon^2), q (\varepsilon^1, \varepsilon^2, \varepsilon^3), q (\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3), \) and \( q (\varepsilon^3) \) respectively, represent the approximate solutions of different solution form of small parameter. Equations (18), (46), (43), (40), and (37) are the averaging method \( q \) and first-order multiple scales method \( q (\varepsilon^0, \varepsilon^1) \), second-order multiple scales method \( q (\varepsilon^0, \varepsilon^1, \varepsilon^2) \), third-order multiple scales method \( q (\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3) \), and fourth-order multiple scales method \( q (\varepsilon^3) \) approximate solutions of different solution form of small parameter, respectively.

Table 6 compares the accuracy of \( q, q (\varepsilon^0 \) and \( \varepsilon^1 \), \( q (\varepsilon^0, \varepsilon^1, \varepsilon^2), q (\varepsilon^1, \varepsilon^2, \varepsilon^3), q (\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3), \) and \( q (\varepsilon^3) \) under different wind velocity. The wind velocity \( U = 4 \)–\( 32 \) m/s is shown in Table 6. The error in Table 6 was obtained by comparing the approximate solutions in Table 6 with the numerical solutions in Table 5.

As the wind velocity increases, the nonlinearity and Rayleigh damping of the iced transmission line structure gradually increase. The error of averaging method becomes larger (from 2.228% of 4 m/s to 17.321% of 32 m/s). The error of \( q (\varepsilon^0 \) and \( \varepsilon^1 \) is less than 1.966%; the error of \( q (\varepsilon^0, \varepsilon^1, \varepsilon^2) \) is gradually increased from 1.439% of the wind velocity \( U = 4 \) m/s to 2.641% of the wind velocity \( U = 30 \) m/s. The error of \( q (\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3) \) is gradually increased from 1.405% of the wind velocity \( U = 4 \) m/s to 3.837% of the wind velocity \( U = 32 \) m/s. The error of \( q (\varepsilon^3) \) is only less than 1.600%, and the error is only less than 0.630% when the wind velocity is 8–30 m/s, which indicates that the accuracy of the fourth-order multiple scales perturbation method is better than that of the low-order multiple scales perturbation method when the range of wind velocity is 8–30 m/s. By comparing the mean error of multiple scales method, it can also be known that the accuracy of the first-order and fourth-order of multiple scales method of the solutions form of small parameter is more precise for galloping equation. By comparing the approximate solutions of the first-order multiple scales method and fourth-order multiple scales method, it is found that the higher the order is, the more difficult the solutions and the more complex the expression will be. Therefore, when the wind velocity ranges from 4 to 32 m/s and the accuracy requirements are not high, the multiple scales method of first-order solutions form is more suitable for calculation and analysis; when the accuracy requirements are higher, the multiple scales method of fourth-order solutions form is better than the multiple scales method of first-order solutions form.

7. Conclusions

The accuracy of approximate solutions of perturbation method for nonlinear galloping equation of iced transmission lines with small sag is studied in this paper. Then, the approximate solutions obtained by perturbation method are compared with the numerical method. The following conclusions can be listed:

(1) For the iced transmission lines excited by wind loads, the existence of the quadratic nonlinear restoring force term in the ordinary differential galloping equation would cause the drift of vibration center as the transmission line gallops. The greater the coefficient of the quadratic nonlinear restoring force term is, the greater the drift of the vibration center will be.

(2) For galloping equations with quadratic nonlinear restoring force terms, the error between approximate solutions of the averaging method and numerical solutions is 17.321% when the wind velocity \( U = 32 \) m/s. Therefore, it is not suitable to use the averaging method to solve the vibration equation where the system contains the quadratic term.

(3) When the wind range is 4–32 m/s, the mean error between the expressions of zero-order amplitude and numerical solutions is 10.299%, which has been corrected to 1.586% by the expressions of first-order amplitude. With the same solution form of small parameter, the approximate solutions of high-order multiple scales method can correct the error of the approximate solutions of low-order multiple scales method.

(4) With the enhancement of structural nonlinearity and Rayleigh damping, the error between the approximate solutions of multiple scales method and numerical solutions would increase. Using the multiple scales method of solution form of first-order small parameter, the mean error between approximate solutions and numerical solutions is less than 1.303% when the range of wind velocity is 4–32 m/s; and the mean error between approximate solutions of fourth-order multiple scales method and numerical solutions is less than 0.639%.

Data Availability

The authors declare that the data are available regarding the publication of this paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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