On degenerating finite element tetrahedral partitions

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Abstract
Degenerating tetrahedral partitions show up quite often in modern finite element analysis. Actually the commonly used maximum angle condition allows some types of element degeneracies. Also, mesh generators and various adaptive procedures may easily produce degenerating mesh elements. Finally, complicated forms of computational domains (e.g. along with a priori known solution layers, etc) may demand the usage of elements of various degenerating shapes. In this paper, we show that the maximum angle condition presents a threshold property in interpolation theory, as the interpolation error may grow (or at least does not decay) if this condition is violated (which does not necessarily imply that FEM error grows). We also demonstrate that the popular red refinements, if done inappropriately, may lead to degenerating partitions which break the maximum angle condition. Finally, we prove that not all tetrahedral elements from a family of tetrahedral partitions are badly shaped when the discretization parameter tends to zero.

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1 Introduction

The classical FEM convergence theory (and backgrounding interpolation results) is based on some regularity properties of employed families of simplicial partitions [5]. The most common property is what we call the minimum angle condition, where any degeneracy of simplicial elements is not allowed, and a weaker condition—known as the maximum angle condition, which allows some degeneracies of simplicial elements [1, 6, 7, 21, 30]. However, real mesh generators (along with popular refinement techniques and adaptive strategies) can easily produce degenerating simplicial elements that do not satisfy the maximum angle condition. See Fig. 1 for some examples of degenerate tetrahedra.

In addition, narrow and flat elements are almost unavoidable in covering thin slots, layers, gaps or strips of different materials or to approximate functions that change more rapidly in one direction than in another direction. Shrinking elements are generally employed in FE-approximation of convection–diffusion problems with interior and boundary layers. Such elements may be easily generated in various adaptive algorithms, say if the \( n \)-section algorithms (see e.g. [19]) are employed.

In what follows, we deal with simplicial partitions of a bounded polygonal or polyhedral domain.

By Cea’s lemma [5], small interpolation error implies small discretization error, but not vice versa. Therefore, geometric conditions on partitions for convergence of interpolations and the FEM for elliptic problems boundary value problems are expected.

![Fig. 1](classification-degenerate-tetrahedra.png)  

Fig. 1 Classification of degenerate (narrow and flat) tetrahedra from [4, 8]
to be different in general. In this context, using several examples, we demonstrate in this paper that the maximum angle condition presents a certain threshold for interpolation. On the other hand, it is not necessary for FEM convergence, even though it is often used for both. Thus, earlier—in [12, 23] we showed that the FEM for elliptic problems may still converge while the maximum angle condition is violated.

Rippa in [32] asks the question: What are good triangles for linear interpolation of a quadratic function \( v \)? The conclusion is that triangles in triangulations used should be long (short) in directions, along which the magnitude of the second derivatives of \( v \) is small (large). Huang [13] introduces a monitor function giving a complete control of the size, shape, and orientation of elements to reduce the interpolation error. This procedure has many applications not only in the convergence theory of the finite element method, but also in the modeling of problems having boundary layers, shock waves, sharp interfaces, etc., see [13].

Let us recall that the interpolation error on degenerate linear elements need not converge to zero, in general, as \( h \to 0 \). We will show it by the following illustrative two-dimensional counter-example.

**Example 1.1** Denote by \( \Omega = \left[ -\frac{1}{2}, \frac{1}{2} \right] \times [0, 1] \) the unit square. Divide its horizontal sides into \( 3^k \) equal parts and its vertical sides into \( 3^{2k} \) equal parts for \( k = 0, 1, 2, \ldots \) In this way we can construct uniform partitions of \( \Omega \) consisting of congruent rectangles. Now divide each rectangle by both diagonals into four triangles. We see that one half of such generated triangles is obtuse (cf. Fig. 2), their maximum angle converges to \( \pi \) as \( k \to \infty \), and their diameter is equal to

\[
h = 3^{-k}, \quad k = 0, 1, 2, \ldots
\]

Denote these triangular partitions by \( T_h \) and let

\[
\mathcal{O}_h = \{ T \in T_h \mid T \text{ is obtuse and lies in the strip } \left[ -\frac{h}{2}, \frac{h}{2} \right] \times [0, 1] \}.
\]

Consider the function

\[
v(x_1, x_2) = |x_1|^{3/2}
\]

and its linear interpolant \( \pi_h v \) defined in the usual way \( (\pi_h v)|_T = \pi_T v \) for all triangles \( T \in T_h \), where \( \pi_T v \) is a linear function having the same values at vertices of \( T \) as \( v \). Using the standard Sobolev space notation, we find that

\[
\| v - \pi_h v \|_{1, \Omega}^2 \geq \frac{\partial}{\partial x_2} (v - \pi_h v)^2 \bigg|_{0, \Omega} = \frac{\partial (\pi_h v)}{\partial x_2} \bigg|_{0, \Omega}^2 = \sum_{T \in T_h} \frac{\partial (\pi_T v)}{\partial x_2} \bigg|_{0, T}^2
\]

\[
\geq \sum_{T \in \mathcal{O}_h} \frac{\partial (\pi_T v)}{\partial x_2} \bigg|_{0, T}^2 = \sum_{T \in \mathcal{O}_h} \text{vol}_2 T \left( \frac{\partial (\pi_T v)}{\partial x_2} \right)^2
\]

\[
= \frac{2}{h^2} \cdot \frac{h^3}{4} \cdot \frac{1}{2h} = \frac{1}{4} \quad \text{for } k = 1, 2, \ldots.
\]

(1)
where $2h^{-2}$ is the number of triangles in $\mathcal{O}_h$, the vertical altitude of each obtuse triangle is $\frac{1}{2}h^2$, $\text{vol}_2 T = \frac{1}{2}h^3$, and for the derivative over $T \in \mathcal{O}_h$ we have

$$\left| \frac{\partial (\pi_T v)}{\partial x_2} \right| = \left( \frac{\frac{1}{2}h}{\frac{1}{2}h^2} \right)^{3/2} = \frac{1}{\sqrt{2}h}.$$ (2)

Hence, by (1) the global interpolation error does not vanish in the Sobolev $H^1(\Omega)$-norm as $h \to 0$. We were inspired by the idea of Ženíšek from [36, pp. 365–366], where a lower bound of the interpolation error is considered only over one degenerating triangle. Ženíšek’s example became well-known after it was quoted in [34, p. 138] and [33, p. 91].

**Remark 1.1** The function $v$ in Example 1.1 is not from the Sobolev space $H^2(\Omega)$. However, this example can be easily modified to the polynomial $v(x_1, x_2) = x_1^2$ if the vertical step $h^2$ in Fig. 2 is replaced by $h^3$. Then (2) changes to $|\partial (\pi_T v)/\partial x_2| = 1/(2h)$ and the norm in (1) diverges

$$\|v - \pi_h v\|_{1, \Omega}^2 \geq \frac{2}{h^3} \cdot \frac{h^4}{4} \cdot \frac{1}{4h^2} = \frac{1}{8h} \to \infty$$ (3)
as $h \to 0$. This example also demonstrates that the behavior of the interpolation error may essentially depend on the rate of degeneracy of used elements.

When the well-known maximum angle condition for tetrahedral elements (see (8)–(9) below) is satisfied, then for any sufficiently smooth function $v$ its linear interpolant converges to $v$ as the discretization parameter tends to zero, see e.g. [3, 14–16, 21, 27]. However, note that the maximum angle condition is not necessary to achieve the optimal convergence rate when simplical (tetrahedral) finite elements are used to solve elliptic problems. This condition is only sufficient. In fact, finite element
approximations may converge to the true solution even though some (or even quite many) dihedral angles of simplicial elements tend to $\pi$ (see Fig. 2) and the interpolation error tends to infinity, see e.g. [12, 24, 29].

For a given tetrahedron $T$ let

$$h_T = \text{diam } T. \quad (4)$$

Further,

$$r_T = \frac{3 \text{vol}_3 T}{\text{vol}_2 \partial T} \quad (5)$$

is the radius of the inscribed ball of $T$, where $\text{vol}_3 T$ is the three-dimensional volume of $T$, $\partial T$ denotes the boundary of $T$, $\text{vol}_2 \partial T$ is the total surface area, and $r_T$ is called the inradius of $T$.

Throughout this paper, a face-to-face tetrahedral partition of a bounded polyhedral domain $\Omega$ is denoted by $\mathcal{T}_h$, where the standard discretization parameter $h$ satisfies $h = \max_{T \in \mathcal{T}_h} h_T$. In what follows, we shall consider a family $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$ of face-to-face tetrahedral partitions $\mathcal{T}_h$ of a bounded polyhedral domain. Its existence is proved in [20].

**Definition 1.1** A family $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$ of partitions into tetrahedra is said to be regular if there exists a constant $\varkappa > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $T \in \mathcal{T}_h$ we have

$$\varkappa h_T \leq r_T. \quad (6)$$

**Definition 1.2** Let $T$ be an arbitrary tetrahedron. Then the ratio

$$\sigma_T = \frac{h_T}{r_T} \quad (7)$$

is called a measure of degeneracy of $T$.

Clearly, if the family $\mathcal{F}$ is regular, then

$$\sigma_T \leq \varkappa^{-1}$$

for all $T \in \mathcal{T}_h$ and all partitions $\mathcal{T}_h$ from $\mathcal{F}$. By [3] this definition is equivalent to the well-known minimum angle condition, i.e., when all angles of triangular faces and all dihedral angles of all tetrahedra are bounded from below by a fixed positive constant. Note that there exist several other similar definitions of shape measures of tetrahedral elements in the literature (see e.g. [3, 4, 8, 25, 26]).

**Remark 1.2** Consider now the well-known Sommerville tetrahedron with vertices

$$A = (-1, 0, 0) \top, \ B = (1, 0, 0) \top, \ C = (0, -1, 1) \top, \ D = (0, 1, 1) \top.$$
Its diameter is 2, the volume is $2/3$, and the surface area $4\sqrt{2}$. Hence, by (5) and (7) we find that the corresponding measure of degeneracy is $\sigma_T = 4\sqrt{2}$, cf. [37, p. 550]. The smallest possible measure of degeneracy $2\sqrt{6}$ is attained for the regular tetrahedron.

**Remark 1.3** For $h \in (0, 1)$ consider a needle tetrahedron with vertices

$$A = (0, 0, 0)\top, \quad B = (h, 0, 0)\top, \quad C = (0, h^2, 0)\top, \quad D = (0, 0, h^2)\top.$$  

We easily find that the corresponding measure of degeneracy tends to $\infty$ as $h \to 0$. The same observation is true for all tetrahedra from Fig. 1.

In Sect. 2, we prove that the global interpolation error does not converge to zero also for special degenerate tetrahedral partitions that do not satisfy the maximum angle condition.

In Sect. 3, we will prove that the measure of degeneracy of tetrahedra that are generated by red refinements may tend to $\infty$. Therefore, there is not a finite number of classes of similar tetrahedra, in general, see Corollary 3.2 and Figs. 7 and 8. The red refinement algorithm has a lot of applications in multigrid methods. Nested refinements of tetrahedra are necessary for hierarchical-basis methods and domain decomposition methods, as well. Also when using adaptive methods, red refinements (and suitable post-refinements) should be done carefully so that optimal approximation properties of resulting tetrahedral elements are preserved [2, 11, 21, 22].

In Sect. 4, we show that families of tetrahedral partitions cannot contain only badly shaped tetrahedra. Finally, in Sect. 5, we present some conclusion remarks and open problems.

## 2 The global interpolation error may not decay for some degenerating tetrahedral elements

For any $v \in C(\overline{\Omega})$ the linear interpolant $\pi_h v$ over $\overline{\Omega}$ is defined in the usual way $(\pi_h v)|_T = \pi_T v$ for all tetrahedra $T \in T_h$, where $\pi_T v$ is a linear function having the same values at vertices of $T$ as $v$.

Let a family $\mathcal{F} = \{T_h\}_{h \to 0}$ of tetrahedral partitions be given. If it is regular (see (6)), then we get the optimal interpolation order [5]:

$$\|v - \pi_h v\|_{1, \Omega} \leq C h |v|_{2, \Omega} \quad \forall v \in H^2(\Omega).$$

However, the optimal interpolation order $O(h)$ of linear tetrahedral elements is still preserved under the following weaker condition (see [3]): there exists a constant $\gamma_0 < \pi$ such that for any tetrahedron $T \in T_h$ and any $T_h \in \mathcal{F}$ we have

$$\gamma_T \leq \gamma_0$$

and

$$\varphi_T \leq \gamma_0,$$
where $\gamma_T$ is the maximum angle of all triangular faces of the tetrahedron $T$ and $\varphi_T$ is the maximum dihedral angle between faces of $T$. We say that the family $\mathcal{F}$ satisfies the maximum angle condition if (8) and (9) hold simultaneously.

According to [21], the two conditions (8) and (9) are independent. Indeed, for the spike tetrahedron from Fig. 1 condition (8) is violated for some faces, but (9) holds. By this statement we shall mean more precisely from now on that there exists a family of partitions containing spike tetrahedra such that condition (8) is violated and (9) holds. We shall construct such a family in Example 2.1. Similarly, for the cap and sliver tetrahedra from Fig. 1, the maximum angle condition (8) holds for any triangular face, whereas several dihedral angles between faces converge to $\pi$, cf. Example 2.2. For the spindle, spear, and spade tetrahedron from Fig. 1 none of the conditions (8) and (9) is satisfied.

In [3] we prove the following interpolation theorem for linear elements, where the standard Sobolev space notation is used.

**Theorem 2.1** Let $\mathcal{F}$ be a family of tetrahedral partitions satisfying the maximum angle condition (8)–(9). Then there exists a constant $C > 0$ such that for any $T_h \in \mathcal{F}$ and any tetrahedron $T \in T_h$ we have

$$\|v - \pi_T v\|_{1,\infty,T} \leq C h_T |v|_{2,\infty,T} \quad \forall v \in C^2(T),$$

where $\pi_T$ is the standard Lagrange linear interpolant.

By (10), $\|v - \pi_T v\|_{1,\infty,T} \leq C h |v|_{2,\infty,\Omega}$ for all $T \in T_h$ satisfying the maximum angle condition, and thus we get a similar error estimate as in Theorem 2.1 also for the Hilbert $H^1(\Omega)$-norm,

$$\|v - \pi_T v\|_{1,\Omega} \leq \tilde{C} \|v - \pi_T v\|_{1,\infty,\Omega} \leq C \tilde{C} h |v|_{2,\infty,\Omega} \quad \forall v \in C^2(\Omega),$$

where $\tilde{C} > 0$ is a constant characterizing the topological imbedding $W^1_\infty(\Omega) \subset H^1(\Omega)$.

We see that the maximum angle condition (8)–(9) is satisfied for the needle, splinter, and wedge tetrahedron as $h \to 0$, see Fig. 1. In this case, we get the optimal interpolation order of linear tetrahedral elements, even though the measure of degeneracy (7) of all such elements tends to infinity.

Let us point out that John L. Synge [35, p. 212] showed that the constant $C$ in the upper bound (10) tends to infinity for a similar two-dimensional case if the maximum angle of triangles tends to $\pi$. However, this does not imply that the interpolation error tends to infinity. For this purpose one has to derive a suitable lower bound as in (3).

The maximum angle condition (8)–(9) represents only a sufficient condition guaranteeing the optimal interpolation order. Now we demonstrate that if one of the two angle conditions does not hold, then the global interpolation error need not converge to zero as $h \to 0$. First we show that for the spike tetrahedron, which violates condition (8) and satisfies (9), To this end we modify Example 1.1 as follows.

**Example 2.1** Denote by $\overline{\Omega} = \left[-\frac{1}{2}, \frac{1}{2}\right] \times [0, 1] \times [0, 1]$ the unit cube. We divide its four edges parallel with $x_1$-axis into $3^k$ equal parts and the remaining eight edges into...
3 equal parts for \( k = 0, 1, 2, \ldots \). In this way we can construct a family of uniform partitions of \( \Omega \) into blocks, i.e., each partition consists of congruent blocks. Let \( G \) be the center of gravity of a given block. Now we divide it into six pyramids whose common vertex is \( G \). Each pyramid will be divided into two tetrahedra using that diagonal of its rectangular base that has a positive slope with respect to the coordinate axes \( x_1, x_2, \text{ and } x_3 \). In this way the block will be divided into eight spike tetrahedra and four needle tetrahedra. One spike tetrahedron \( ABCG \) inside the block is sketched in Fig. 3. Denote such tetrahedral face-to-face partitions by \( T_h \), where

\[
T_h = \{ T \in T_h \mid T \text{ has a horizontal face and } T \subset \left[ \frac{\pm h}{2}, \frac{h}{2} \right] \times [0, 1] \times [0, 1] \}. 
\]

To investigate dihedral angles of \( T \in \mathcal{O}_h \), it is enough to consider the tetrahedron from Fig. 3, since the other cases can be treated similarly. All dihedral angles can be calculated by means of four outward unit normals to \( \partial T \):

\[
n_1 = \frac{1}{\sqrt{1 + h^2}} (h, 0, 1)^T, \quad n_2 = \frac{1}{\sqrt{1 + h^2}} (-h, 1, 0)^T, \quad n_3 = \frac{1}{\sqrt{2}} (0, -1, 1)^T, 
\]

and \( n_4 = (0, 0, -1)^T \). Their mutual scalar products are always less than \( \sqrt{2}/2 \) and the biggest (and the only obtuse) dihedral angle \( \varphi_T \) is at the edge \( BG \). We have

\[
- \cos \varphi_T = \cos (\pi - \varphi_T) = \|n_1\| \|n_3\| \cos (\pi - \varphi_T) = n_1^T n_3 = \frac{1}{\sqrt{1 + h^2}} < \frac{\sqrt{2}}{2}.
\]

Hence, \( \cos \varphi_T > -\sqrt{2}/2 \) and thus \( \varphi_T < \frac{3}{4} \pi \). On the other hand, it is obvious that \( \gamma_T = \angle AGC \), and it tends to \( \pi \) as \( h \to 0 \). More precisely, for any \( \gamma_0 < \pi_0 \) there exists a triangulation \( T_h \) and \( T \in T_h \) such that \( \gamma_T > \gamma_0 \). Thus, condition (9) holds and condition (8) does not.
Consider the function
\[ v(x_1, x_2, x_3) = |x_1|^{3/2} \]
and its linear interpolant \( \pi_h v \). Then similarly to (1) we have
\[
\| v - \pi_h v \|_{1, \Omega}^2 \geq \left| \frac{\partial (\pi_h v)}{\partial x_3} \right|_{0, \Omega}^2 \geq \sum_{T \in \mathcal{O}_h} \text{vol}_3 T \left| \frac{\partial (\pi_T v)}{\partial x_3} \right|^2
\]
\[
= \frac{4}{h^4} \cdot \frac{h^5}{12} \cdot \frac{1}{2h} = \frac{1}{6} \quad \text{for } k = 1, 2, \ldots,
\]
where \( 4h^{-4} \) is the number of (spike) tetrahedra in \( \mathcal{O}_h \), the area of their horizontal bases is \( \frac{1}{2} h^3 \) and the vertical altitude is \( \frac{1}{2} h^2 \), \( \text{vol}_3 T = \frac{1}{12} h^5 \), and for the constant derivative with respect to \( x_3 \) we have (cf. (2))
\[
\left| \frac{\partial (\pi_T v)}{\partial x_3} \right| = \left( \frac{1}{2} h \right)^{3/2} = \frac{1}{\sqrt{2}h}.
\]
Hence, the interpolation error does not vanish in the Sobolev \( H^1(\Omega) \)-norm as \( h \to 0 \).

Now we show that for the cap tetrahedron, which satisfies (8) and violates the second condition (9), the global interpolation error need not converge to zero. Remark 1.1 can be easily modified to Examples 2.1 and 2.2.

**Example 2.2** Let \( \overline{\Omega} \) be the triangular prism with vertices
\[
\left( -\frac{1}{2}, -\frac{\sqrt{3}}{6}, 0 \right)^T, \quad \left( \frac{1}{2}, -\frac{\sqrt{3}}{6}, 0 \right)^T, \quad \left( 0, \frac{\sqrt{3}}{3}, 0 \right)^T,
\]
\[
\left( -\frac{1}{2}, -\frac{\sqrt{3}}{6}, 1 \right)^T, \quad \left( \frac{1}{2}, -\frac{\sqrt{3}}{6}, 1 \right)^T, \quad \left( 0, \frac{\sqrt{3}}{3}, 1 \right)^T,
\]
i.e., all edges have length 1 and the point \((0, 0, 0)\) is the center of gravity of the triangular base. For \( k = 0, 1, 2 \ldots \) divide its triangular faces uniformly into equilateral triangles (see Fig. 4) all of whose sides have lengths
\[
h = 3^{-k}.
\]
Divide vertical edges into \( 3^{2k} \) equal parts. In this way we can construct a family of uniform partitions of \( \overline{\Omega} \) into degenerating triangular prisms.

Now we divide each rectangular face by any of its diagonal into two triangles. Taking the convex hull of the center of gravity \( G \) of a given prism and its surface triangles, we obtain 8 tetrahedra. Two of them are cap tetrahedra and the other six are spike tetrahedra. In this way we obtain a face-to-face tetrahedral partition \( T_h \). Let
\[
\mathcal{O}_h = \{ T \in T_h \mid T \text{ has a horizontal face and } T \subset \left[ -\frac{h}{2}, \frac{h}{2} \right] \times \left[ 0, \frac{\sqrt{3}}{2} \right] \times [0, 1] \}.
\]
We see that all triangular faces of $T \in \mathcal{O}_h$ are acute, i.e. $\gamma_T \leq \pi/2$, whereas $\varphi_T \to \pi$ as $h \to 0$.

Consider again the function

$$v(x_1, x_2, x_3) = |x_1|^{3/2}$$

and its linear interpolant $\pi_h v$. Let $T \in \mathcal{O}_h$ be an arbitrary tetrahedron and let $E$ be the center of gravity of its horizontal base $ABC$, where $C$ lies in the plane $x_1 = 0$. Clearly, $E$ is an orthogonal projection of $G$ (see Fig. 5). Since

$$v(C) = (\pi_T v)(C) = 0$$

and $\pi_T v$ is a linear function, we find that

$$(\pi_T v)(E) = (\pi_T v)(\frac{1}{3}(A + B + C)) = \frac{1}{3}((\pi_T v)(A) + (\pi_T v)(B)) = \frac{2}{3} \left(\frac{1}{2} h\right)^{3/2}. $$

Hence, for the constant derivative with respect to $x_3$ we have

$$\left| \frac{\partial (\pi_T v)}{\partial x_3} \right| = \left| \frac{(\pi_T v)(G) - (\pi_T v)(E)}{|GE|} \right| = \frac{2}{3} \left(\frac{1}{2} h\right)^{3/2} = \frac{2}{3} \cdot \frac{1}{\sqrt{2} h},$$

where $(\pi_T v)(G) = v(G) = 0$ and the vertical altitude of $T$ is $\frac{1}{2} h^2$. Now, similarly to (1) we obtain

$$\|v - \pi_h v\|_{1, \Omega}^2 \geq \left| \frac{\partial (\pi_h v)}{\partial x_3} \right|_{0, \Omega}^2 \geq \sum_{T \in \mathcal{O}_h} \text{vol}_3 T \left| \frac{\partial (\pi_T v)}{\partial x_3} \right|^2 = \frac{2}{h^3} \cdot \frac{\sqrt{3} h^4}{24} \cdot \frac{2}{9 h} = \frac{\sqrt{3}}{54}$$

for $k = 1, 2, \ldots,$

where $2h^{-3}$ is the number of (cap) tetrahedra in $\mathcal{O}_h$, the area of their horizontal bases is $\frac{\sqrt{3}}{4} h^2$, and $\text{vol}_3 T = \frac{\sqrt{3}}{24} h^4$. Hence, the interpolant $\pi_h v$ does not converge to $v$ in the $H^1(\Omega)$-norm as $h \to 0$.\[ Springer\]
A similar procedure can be done for a sliver tetrahedron inserted into the block $h \times h \times h^2$, see Fig. 1. A method for treating degenerate tetrahedra that do not satisfy simultaneously the conditions (8) and (9) is sketched in Sect. 5.

Finally note that for the $L^2(\Omega)$-norm we have (see [5, p. 118, 120])

$$\|v - \pi_h v\|_{0,\Omega} \leq C h^2 |v|_{2,\Omega} \quad \forall v \in H^2(\Omega)$$

without any geometric restrictions on elements. However, finite element error estimates in the $L^2(\Omega)$-norm rely on the Aubin-Nitsche trick [5, p. 137], which requires the optimal approximation property to hold in $H^1(\Omega)$. This is backwards from interpolation, where apparently $L^2$-estimates can hold, but $H^1$-estimates fail. Note that optimal error estimates for the test H1 approximation hold without the maximum angle condition.

### 3 Degenerating tetrahedral red refinements

Red refinements of planar triangular partitions are done by midlines. In this way each triangle is divided into four congruent subtriangles which are similar to the original triangle, i.e., their shapes obviously do not degenerate. In red refinements of tetrahedral partitions, we also use midlines of triangular faces (cf. Fig. 6 below). However, in contrast to the two-dimensional case, not every tetrahedron can be subdivided, in general, into smaller congruent tetrahedra that are all similar to the original tetrahedron. In fact, only the Sommerville tetrahedron can be subdivided into 8 similar and congruent tetrahedra, see [17, 20, 28, 37]. Moreover, there is no uniqueness in the red refinement strategy, even though all successive partitions are always nested.

Let $ABCD$ be an arbitrary tetrahedron. Denote by $M_1, M_2, M_3, M_4, M_5, \text{ and } M_6$ the midpoints of its edges $AB, BC, AC, CD, AD, \text{ and } BD$, respectively. In Fig. 6, we observe the red refinement of a tetrahedron into eight subtetrahedra (see [20]):
Fig. 6 Red refinement of a tetrahedron $ABCD$ with the interior octahedron $M_1M_2M_3M_4M_5M_6$

$AM_1M_3M_5, M_1BM_2M_6, M_3M_2CM_4, M_5M_6M_4D,$
$M_6M_1M_3M_5, M_1M_3M_2M_6, M_3M_2M_6M_4, M_5M_6M_4M_3.$

Note that the octahedron $M_1M_2M_3M_4M_5M_6$ can be subdivided into 4 tetrahedra (= 2 pyramids), whose common edge can be $M_1M_4$ or $M_2M_5$ or $M_3M_6$ (see Fig. 6 for the last case). Thus we have three different ways to subdivide the tetrahedron $ABCD$ into 8 tetrahedra so that all faces of $ABCD$ are divided by midlines.

**Lemma 3.1** Let $ABCD$ be an arbitrary tetrahedron and let $M_i$ be midpoints of its edges as sketched in Fig. 6. Then the distance between midpoints of two opposite edges, e.g., $M_1 \in AB$ and $M_4 \in CD$ is given by

$$|M_1M_4| = \frac{1}{2}\sqrt{|AC|^2 + |AD|^2 + |BC|^2 + |BD|^2 - |AB|^2 - |CD|^2}. \quad (12)$$

**Proof** Let $M$ be the midpoint of the side $AB$ of a general triangle $ABC$. First, we shall employ the Cosine theorem for the triangles $AMC$ and $BMC$. Thus, we find that

$$|AC|^2 = |AM|^2 + |CM|^2 - 2|AM||CM|\cos \mu, \quad |BC|^2 = |BM|^2 + |CM|^2 + 2|BM||CM|\cos \mu,$$

where $\mu = \angle AMC$. Since $|AM| = |BM|$, we can eliminate $\cos \mu$ and after some manipulations we find the well-known Stewart formula for the length of the median $|CM|$. 

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\[ |CM|^2 = \frac{1}{4} (2|AC|^2 + 2|BC|^2 - |AB|). \]  

(13)

Now applying formula (13) three times successively for the triangles (see Fig. 6)

- \( ABM_4 \) with \( M_1 \) being the midpoint of \( AB \),
- \( AC \) \( D \) with \( M_4 \) being the midpoint of \( CD \),
- \( BCD \) with \( M_4 \) being the midpoint of \( CD \),

we obtain

\[
4|M_1M_4|^2 = 2|AM_4|^2 + 2|BM_4|^2 - |AB|^2
= \frac{1}{2} [(2|AC|^2 + 2|AD|^2 - |CD|^2) + (2|BC|^2 + 2|BD|^2
- |CD|^2) - 2|AB|^2]
= |AC|^2 + |AD|^2 + |BC|^2 + |BD|^2 - |AB|^2 - |CD|^2.
\]

\[ \square \]

**Example 3.1** For the regular tetrahedron \( ABCD \) with \( |AB| = 1 \), the distance between two midpoints of opposite edges is by (12) equal to \( \frac{1}{2} \sqrt{2} \). Therefore, after dividing \( ABCD \) into eight tetrahedra, the longest edge of each interior tetrahedron has length \( \frac{1}{2} \sqrt{2} \) and the other edges have length \( \frac{1}{2} \).

Now we similarly divide each interior tetrahedron into 8 smaller subtetrahedra by connecting the midpoint of the longest edge and its opposite edge of length \( \frac{1}{2} \sqrt{2} \). Their distance is by (12) equal to

\[
\frac{1}{2} \sqrt{\frac{1}{2}} (1 + 1 + 1 + 1 - (\sqrt{2})^2) = \frac{1}{4}.
\]

In this way the interior suboctahedron is divided by the shortest spatial diagonal into two regular subtetrahedra whose edges have length \( \frac{1}{4} \) and two subtetrahedra whose one edge has length \( \frac{1}{4} \sqrt{2} \) and the other edges of the resulting subtetrahedra have length \( \frac{1}{4} \). Now we can repeat this process recursively and obtain only two types of tetrahedra, i.e., the corresponding family is regular.

**Remark 3.1** The partitions of Example 3.1 are, in fact, well-known in crystallography, since the three-dimensional space can be alternately tessellated by regular tetrahedra and regular octahedra which form a lattice of diamonds. When dividing the inner octahedra always by the longest diagonal (\( M_1M_4 \) or \( M_2M_5 \) or \( M_3M_6 \)) during red refinements, we get special tetrahedra described recursively by the following definition, cf. [37, p. 545]. In Theorem 3.1, we show that they degenerate inappropriately. So we should avoid such a choice of randomly selected spatial diagonals, since the measure of degeneracy of resulting tetrahedra may tend to infinity.

**Definition 3.1** Let \( A_0B_0C_0D_0 \) be the regular tetrahedron. For \( k \in \{0, 1, 2, \ldots \} \) define recursively the following midpoints

\[
A_{k+1} = \frac{1}{2} (B_k + C_k), \quad B_{k+1} = \frac{1}{2} (A_k + C_k), \quad C_{k+1} = \frac{1}{2} (A_k + B_k),
\]  

(14)
Fig. 7 Growth dynamics of the Zhang tetrahedra for \( k = 0, 1, 2, \ldots, 5 \). For a better visualization they are scaled

and

\[
D_{k+1} = \begin{cases}
\frac{1}{2}(A_k + D_k) & \text{if } k \text{ is odd}, \\
\frac{1}{2}(B_k + D_k) & \text{if } k \text{ is even.}
\end{cases}
\] (15)

Then the sequence \( A_k B_k C_k D_k \) for \( k = 1, 2, \ldots, \) is named the sequence of the Zhang tetrahedra, see [37] and Fig. 7. They are called in the same way also under any translation, rotation, reflection, and scaling.

Lemma 3.2 Consider the sequence of the Zhang tetrahedra such that \( |A_0 B_0| = 1 \). Then for any \( k \in \{2, 3, 4, \ldots\} \) the lengths of edges \( A_k B_k, A_k C_k, B_k C_k, A_k D_k, C_k D_k, B_k D_k \) multiplied by the scaling factor \( 2^k \) are

\[
\begin{align*}
&\{1, 1, 1, \sqrt{c_k}, \sqrt{c_{k-1}}, \sqrt{c_{k-2}}\} \text{ if } k \text{ is even and } \\
&\{1, 1, 1, \sqrt{c_{k-2}}, \sqrt{c_{k-1}}, \sqrt{c_k}\} \text{ if } k \text{ is odd,}
\end{align*}
\]

where

\[
c_{2j} = j^2 + j + 1 \quad \text{and} \quad c_{2j+1} = j^2 + 2j + 2 \quad \text{for } j = 0, 1, 2, \ldots
\] (16)

Proof By definition (14) we immediately see that \( A_k B_k C_k \) are equilateral triangles with \( |A_k B_k| = 2^{-k} \) for all \( k = 0, 1, 2, \ldots \), cf. Figs. 7 and 8.

The longest edge of the tetrahedron \( A_1 B_1 C_1 D_1 \) is \( |B_1 D_1| = \frac{1}{2}\sqrt{2} \) and the other have length \( \frac{1}{2} \) (cf. Example 3.1). Similarly, for the next Zhang tetrahedron \( A_2 B_2 C_2 D_2 \) we get \( |A_2 D_2| = \frac{1}{4}\sqrt{3}, |C_2 D_2| = \frac{1}{4}\sqrt{2} \), and the remaining edges have length \( \frac{1}{4} \).
Hence, the first relation in (16) is satisfied for \( k = 2j = 2 \) and we have \( c_0 = 1, c_1 = 2, c_2 = 3 \).

Further, we shall proceed by induction. First let (16) hold for some even integer \( k = 2j \geq 2 \). Then by (12), (15), and (16) for the spatial diagonal \( B_{k+1}D_{k+1} \) we get

\[
2^{k+1}|B_{k+1}D_{k+1}| = \sqrt{1 + c_k + 1 + c_{k-1} - 1 - c_{k-2}}
\]

\[
= \sqrt{1 + j^2 + j + 1 + 1 + (j-1)^2 + 2(j-1) + 2 - 1 - (j-1)^2 - (j-1) - 1}
\]

\[
= \sqrt{j^2 + 2j + 2} = \sqrt{c_{k+1}}.
\]

Moreover, according to (14) and (15), we find that the edge \( A_{k+1}D_{k+1} \) is the midline parallel to \( C_kD_k \) (cf. Fig. 6), i.e.,

\[
2^{k+1}|A_{k+1}D_{k+1}| = 2^k|C_kD_k| = \sqrt{c_{k-1}}.
\]

Analogously, we get the remaining relation

\[
2^{k+1}|C_{k+1}D_{k+1}| = 2^k|A_kD_k| = \sqrt{c_k}.
\]

The proof for an odd \( k = 2j + 1 \geq 3 \) proceeds analogously. \( \square \)

**Remark 3.2** By (12) and the sharp inequalities \( c_{k-2} < c_{k-1} < c_k \) we find that the remaining two spatial diagonals are shorter than that in (17) for any even \( k \geq 2 \),

\[
2^{k+1}|A_{k+1}U_{k+1}| = \sqrt{1 + c_{k-1} + 1 + c_{k-2} - 1 - c_{k-1}}
\]

\[
< 2^{k+1}|C_{k+1}V_{k+1}| = \sqrt{1 + c_k + 1 + c_{k-2} - 1 - c_{k-1}}
\]

\[
< 2^{k+1}|B_{k+1}D_{k+1}|,
\]

where \( U_{k+1} = \frac{1}{2}(A_k + D_k) \) and \( V_{k+1} = \frac{1}{2}(C_k + D_k) \) are midpoints (see [18]). Similarly we can investigate the case when \( k \geq 3 \) is odd. Consequently, in terms of selection of the diagonals of octahedra we should choose (to avoid degeneracies) the longest diagonal (cf. Remark 3.1). By (4) and (16), the discretization parameter for even \( k \) tends to zero, i.e.,

\[
h_{T_k} = \text{diam } T_k = |A_kD_k| = \frac{\sqrt{c_k}}{2^k} = \frac{\sqrt{j^2 + j + 1}}{2^{2j}} \to 0 \quad \text{as } k = 2j \to \infty.
\]
The case \( k = 2j + 1 \) proceeds similarly. Zhang tetrahedra thus form an infinite sequence.

**Lemma 3.3** Let \( T \) be an arbitrary tetrahedron and let \( r \) be its inradius. Then

\[
r < \rho_i, \quad i = 1, 2, 3, 4,
\]

where \( \rho_i \) are radii of inscribed circles of the four triangular faces \( F_i \) of \( T \).

**Proof** Let \( i \in \{1, 2, 3, 4\} \) be given and let \( O \) be the center of the inscribed ball \( B \) of \( T \). Consider a plane \( P \) passing through \( O \) and parallel with \( F_i \). Then the circle \( B \cap P \) has the radius \( r \) and is contained in the triangle \( T \cap P \). Hence,

\[
r \leq \rho,
\]

where \( \rho \) is the radius of the inscribed circle to \( T \cap P \). Since the triangle \( T \cap P \) is similar to \( F_i \), but smaller than \( F_i \), we get \( \rho < \rho_i \). From this and (19) we get (18). \( \square \)

**Theorem 3.1** The sequence of the Zhang tetrahedra violates both inequalities (8) and (9) of the maximum angle condition.

**Proof** (1) It is enough to assume that \( k \) is even, \( k = 2(j + 1) \), and \( j = 0, 1, 2 \ldots \). Consider the triangular face \( A_k B_k D_k \) of the Zhang tetrahedron \( A_k B_k C_k D_k \) as sketched in Fig. 8. Denote by \( \beta_k \) the angle \( \angle A_k B_k D_k \). Then by the Cosine theorem, Definition 3.1, and Lemma 3.2 we obtain

\[
(j + 1)^2 + (j + 1) + 1 = (j^2 + j + 1) + 1 - 2\sqrt{j^2 + j + 1} \cos \beta_k.
\]

Hence,

\[
\cos \beta_k = -\frac{2j + 1}{2\sqrt{j^2 + j + 1}} < -\frac{2j + 1}{2\sqrt{j^2 + 2j + 1}} = -\frac{2j + 1}{2j + 2}.
\]

(20)

Since the inverse function \( \arccos \) is decreasing, by (20) we find that

\[
\beta_k \geq \arccos\left(-\frac{2j + 1}{2j + 2}\right) \rightarrow \arccos(-1) = \pi
\]

(21)

as \( k \rightarrow \infty \). Hence, condition (8) does not hold.

2) Now we show that the dihedral angle \( \varphi_k \) at the edge \( B_k C_k \) also converges to \( \pi \) as \( k \rightarrow \infty \). It is again enough to consider only \( k \) even. By the recurrence formulae (14) and (15), we obtain the explicit expression of vertices

\[
A_k = 2^{-k}\left(-\frac{1}{2}, -\frac{\sqrt{3}}{6}, 0\right)^\top, \quad B_k = 2^{-k}\left(\frac{1}{2}, -\frac{\sqrt{3}}{6}, 0\right)^\top, \\
C_k = 2^{-k}\left(0, \frac{\sqrt{3}}{3}, 0\right)^\top, \quad D_k = 2^{-k}\left(\frac{k}{2}, 0, \frac{\sqrt{6}}{3}\right)^\top.
\]

(22)
From this we can derive that
\[ n_1 = \frac{1}{\sqrt{4 + 2^{2k-3}}} (\sqrt{3}, 1, -2^{k-2}\sqrt{2})^\top \quad \text{and} \quad n_4 = (0, 0, -1)^\top \]
are the outward unit normals of the faces \( A_k B_k D_k \) and \( A_k B_k C_k \) of \( T_k \), respectively. Hence,
\[ -\cos \varphi_k = \cos(\pi - \varphi_k) = \|n_1\| \|n_4\| \cos(\pi - \varphi_k) = n_1^\top n_4 = \frac{\sqrt{2^{2k-3}}}{\sqrt{4 + 2^{2k-3}}}. \]
Consequently, \( \cos \varphi_k \to -1 \) and \( \varphi_k \to \pi \) as \( k \to \infty \), i.e., (9) is violated.

**Remark 3.3** From the proof of Theorem 3.1 and Fig. 8 we observe that the Zhang tetrahedra look like needles but they are not listed in Fig. 1, since the angle \( \angle A_k B_k D_k \) tends to \( \pi \) as \( k \to \infty \). By the Cosine theorem and Definition 3.1, we can similarly derive that the angles \( \angle A_k C_k D_k \) and \( \angle C_k B_k D_k \) tend to \( 2\pi/3 \).

**Example 3.2** Consider again the regular tetrahedron with vertices
\[
A_0 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{6}, 0\right)^\top, \quad B_0 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{6}, 0\right)^\top, \\
C_0 = \left(0, \frac{\sqrt{3}}{3}, 0\right)^\top, \quad D_0 = \left(0, 0, \frac{\sqrt{6}}{3}\right)^\top
\]
and the corresponding sequence of the Zhang tetrahedra defined by (14)–(15). From (23) and (14) we observe that the equilateral triangular faces \( A_k B_k C_k \) are in the \( x_1x_2 \)-plane for all \( k = 0, 1, 2, \ldots \). Their common center of gravity \( G = (0, 0, 0) \) is independent of \( k \). Now, by induction we can explicitly derive the following coordinates of vertices: if \( k \) is odd then
\[
A_k = 2^{-k} \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, 0\right)^\top, \quad B_k = 2^{-k} \left(-\frac{1}{2}, \frac{\sqrt{3}}{6}, 0\right)^\top, \\
C_k = 2^{-k} \left(0, -\frac{\sqrt{3}}{3}, 0\right)^\top, \quad D_k = 2^{-k} \left(\frac{k}{2}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}\right)^\top,
\]
and if \( k \) is even then (22) is valid.

By (22) and (24) we get
\[
\text{vol}_3 T_k = 2^{-k} \frac{1}{3} \text{vol}_2 (A_k B_k C_k) \frac{\sqrt{6}}{3} = 2^{-3k-2} \frac{\sqrt{2}}{3}.
\]

Now we shall deal with a measure of degeneracy (see Definition 1.2) for tetrahedra from a general family \( \mathcal{F} = \{ T_h \}_{h \to 0} \) of partitions. We will keep the same notation as in the maximum angle condition (8)–(9).

**Theorem 3.2** Let \( \gamma_T \to \pi \) or \( \varphi_T \to \pi \). Then the measure of degeneracy of \( T \) tends to \( \infty \).
Proof (1) Let there exist a sequence \( \{T_k\}_{k=1}^{\infty} \) such that each \( T_k \) belongs to some tetrahedral partition \( T_h \) from a family \( \mathcal{F} \) and let

\[ \gamma_k := \gamma_{T_k} \to \pi \quad \text{as} \quad k \to \infty. \]

Let \( a_k \leq b_k \leq c_k \) be sides of a triangle with the maximum angle \( \gamma_k \) and let \( \rho_k \) be the corresponding radius of its inscribed circle. The area of this triangle can be expressed in the following two ways

\[ \frac{1}{2} a_k b_k \sin \gamma_k = \rho_k \frac{a_k + b_k + c_k}{2}. \]

From this, (7), and Lemma 3.3 we have

\[ \sigma_{T_k} = \frac{h_{T_k}}{r_{T_k}} > \frac{h_{T_k}}{\rho_k} \geq \frac{c_k}{\rho_k} = \frac{c_k(a_k + b_k + c_k)}{a_k b_k \sin \gamma_k} \geq \frac{c_k^2}{a_k b_k \sin \gamma_k} \geq \frac{1}{\sin \gamma_k} \to \infty \]

as \( k \to \infty \).

2) Let there exist a sequence \( \{T_k\}_{k=1}^{\infty} \) such that each \( T_k \) belongs to some tetrahedral partition \( T_h \) from a family \( \mathcal{F} \) and let

\[ \varphi_k := \varphi_{T_k} \to \pi \quad \text{as} \quad k \to \infty. \]

Without loss of generality we may assume that the maximum dihedral angle \( \varphi_k \) is at the edge \( B_k C_k \) of the tetrahedron \( T_k = A_k B_k C_k D_k \) (cf. Figure 7). Let \( z_k \) be the length of the spatial altitude of \( T_k \) on the face \( Z_k = A_k B_k C_k \). By (7) and (5) we have

\[ \sigma_{T_k} = \frac{h_{T_k}}{r_{T_k}} = \frac{h_{T_k} \text{vol}_2 \partial T_k}{3 \text{vol}_3 T_k} > \frac{h_{T_k} \text{vol}_2 Z_k}{z_k \text{vol}_2 Z_k} = \frac{h_{T_k}}{z_k} \geq \frac{1}{\sin(\pi - \varphi_k)} = \frac{1}{\sin \varphi_k} \to \infty \]

as \( k \to \infty \), where the last inequality is due to the fact that the longest edge need not be opposite to the largest dihedral angle (see [3, Sect. 8.7]). \( \square \)

Corollary 3.1 The measure of degeneracy of the Zhang tetrahedra tends to \( \infty \) as \( k \to \infty \).

The proof follows immediately from Theorems 3.1 and 3.2. Contrary to red refinements of triangular partitions, we have the following result:

Corollary 3.2 The algorithm (14)–(15) does not produce a finite number of classes of similar tetrahedra.

Remark 3.4 In Definition 3.1, the initial tetrahedron was regular. For the case of a general tetrahedron it is enough to consider an affine one-to-one mapping between these two tetrahedra and similar degeneracy effects appear as those in Theorem 3.1. In practical calculations, the shortest spatial diagonal \( M_1 M_4, M_2 M_5, \) or \( M_3 M_6 \) of the inner octahedron is usually chosen, since then the resulting tetrahedra do not degenerate.
when $h \to 0$. This was proved by Zhang [37]. Also Liu and Joe [26] present a red-type refinement algorithm that yields only a finite number of classes of similar tetrahedra. An uncontrolled random choice of spatial diagonals is not recommended, since any choice of the longest diagonal during red refinements will produce a new term in the sequence of Zhang tetrahedra, see Remark 3.1.

4 Families of tetrahedral partitions cannot contain only badly shaped tetrahedra

In the previous section, we have seen that red refinements may lead to sequences of tetrahedral partitions containing degenerating elements violating the maximum angle condition. It is a natural question whether or not such badly shaped elements prevent convergence of the finite element method or not. This question remains unanswered even in the planar case, however there are two relevant results which imply that this is a difficult question. First, there is the Babuška-Aziz counterexample to FEM convergence which consider triangulations such as in Fig. 9, where all interior elements quickly degenerate, cf. [29]. On the other hand, the paper [10] gives a construction of a sequence of partitions containing many elements violating the maximal angle condition, yet displaying optimal convergence of the FEM. The construction is such that a sequence of meshes satisfying the maximal angle condition is refined by subdividing each triangle into arbitrarily ‘bad’ sub-elements. The natural question thus arises whether one can use this procedure to produce a sequence of partitions where all interior elements degenerate, similarly as in the Babuška-Aziz counterexample. This question was answered negatively in [23, p. 134], where it is shown that a ‘nice’ triangle cannot be partitioned into ‘badly shaped’ triangles only. Here ‘nice’ can be taken to mean ‘having maximal angle smaller than $\frac{2}{3}\pi$’ and ‘badly shaped’ means ‘having...
maximal angle larger than $\frac{2}{3}\pi$. In particular one cannot take e.g. a right triangle and subdivide it only into triangles with maximal angles close to $\pi$. This prevents using the strategy from [10] to construct sequences of degenerating partitions containing only badly shaped elements on which the FEM would display optimal convergence.

Now we will generalize this surprising result into the three-dimensional space. Let $V$ be an arbitrary vertex of an arbitrary tetrahedron $T$. Consider a unit sphere with center at $V$ and extend the three adjacent edges passing through $V$ beyond $T$, if they are shorter than 1. The area $\varepsilon$ of the spherical triangle on the unit sphere whose vertices are the intersection points of the three edges with the sphere is called the solid (or trihedral) angle, see e.g. [9]. It is measured in steradians. The following relationships between dihedral and trihedral angles is well known, see e.g [31, p. 83]. The solid angle $\varepsilon$ is equal to

$$\varepsilon = \alpha_1 + \alpha_2 + \alpha_3 - \pi,$$

where $\alpha_1$, $\alpha_2$, $\alpha_3$ are angles of the spherical triangle that are at the same time the dihedral angles at the corresponding edges. The solid angle $\varepsilon$ is sometimes also called the spherical excess. Now we show that if $\varepsilon \rightarrow 2\pi$, then $\alpha_i \rightarrow \pi$ for every $i = 1, 2, 3$.

**Lemma 4.1** Let $\delta > 0$ be arbitrary. If one solid angle of an arbitrary tetrahedron is greater than $2\pi - \delta$, then all three corresponding dihedral angles are greater than $\pi - \delta$.

**Proof** Let $\delta > 0$ be given and let a solid angle $\varepsilon$ satisfy $\varepsilon > 2\pi - \delta$. Then by (25) we see that

$$\alpha_1 + \alpha_2 + \alpha_3 = \varepsilon + \pi > 3\pi - \delta.$$

Since each $\alpha_i < \pi$ for every $i = 1, 2, 3$, we get that $\alpha_1 > 3\pi - \delta - \alpha_2 - \alpha_3 > \pi - \delta$. Similarly we obtain that $\alpha_i > \pi - \delta$ for $i = 2, 3$.

**Lemma 4.2** The sum of all four solid angles of an arbitrary tetrahedron is less than $2\pi$.

**Proof** Let $\alpha_1, \ldots, \alpha_6$ be dihedral angles of a given tetrahedron and let $\varepsilon_1, \ldots, \varepsilon_4$ be its solid angles. Then by formula for the spherical excess (25) we have

$$\sum_{i=1}^{4} \varepsilon_i = 2 \sum_{j=1}^{6} \alpha_j - 4\pi < 6\pi - 4\pi = 2\pi,$$

since the sum of all dihedral angles is less than $3\pi$, see [10].

From Lemma 4.1 we see that the upper bound $2\pi$ given in Lemma 4.2 cannot be reduced for the cap tetrahedron (see Fig. 1). Moreover, any tetrahedron may have only one large solid angle close to $2\pi$. Let us still note that for the sliver tetrahedron from Fig. 1 all solid angles tend to 0. Its faces are acute triangles and some dihedral angles tend to $\pi$ like for the cap tetrahedron.
Consider a family $\mathcal{F} = \{T_h\}_{h \to 0}$ of tetrahedral partitions. Assume that for any $\delta > 0$ there exists $T_h \in \mathcal{F}$ and $T \in T_h$ such that its maximum solid angle $\varepsilon$ satisfies $\varepsilon > 2\pi - \delta$. Tetrahedra whose maximum solid angle tends to $2\pi$ will be called badly shaped.

**Theorem 4.1** An arbitrary tetrahedron cannot be partitioned into only badly shaped tetrahedra.

**Proof** Assume to the contrary that there exists $h_0 > 0$ such that for any given $h \in (0, h_0)$ all tetrahedra $T \in T_h$ are badly shaped. Denote by $t$ the number of tetrahedra in some fixed partition $T_h$ with $h \in (0, h_0)$. Then the sum of all solid angles around all vertices we have

$$4\pi v_I + 2\pi v_B + s < 2\pi t,$$

where $v_I$ is the number of vertices in the interior of $T$, $v_B$ is the number of vertices in the interior of faces of $T$, $s$ is the number of the other solid angles at vertices that lie on edges of $T$, and the strict inequality follows from Lemma 4.2.

If each tetrahedron is badly behaving, then we have exactly $t$ large solid angles close to $2\pi$. Therefore, each interior vertex can be the vertex of at most two large solid angles and each boundary vertex can be the vertex of at most one large solid angle. Thus, the following inequality holds

$$t \leq 2v_I + v_B.$$

However, from (26) we get an opposite inequality

$$2v_I + v_B < t.$$

Hence, the result follows.

Thus we have extended the planar results of [23] to the three-dimensional case. These results along with the Babuška-Aziz counterexample demonstrate the difficulty of analyzing FEM convergence in the case of sequences of partitions violating the maximum angle condition, such as those produced by the red refinement procedure of Sect. 3.

### 5 Concluding remarks

In contrast to triangular partitions, the red refinement strategy is not uniquely determined for tetrahedral partitions. We showed that this may lead to families of partitions with degenerating tetrahedra which violate the maximum angle condition. Therefore, case has to be given to the choice of the red refinement strategy. The maximum angle condition is satisfied for needle, splinter, and wedge tetrahedra, see Fig. 1, and these elements preserve optimal interpolation properties. However, in Example 2.1 (resp.
2.2) we showed that the linear interpolant may lose its optimal interpolation order over families of partitions containing spike (resp. cap) tetrahedra.

Analogous examples can be constructed for other degenerate tetrahedra. For instance, stretching the diamond lattice from Remark 3.1 in the vertical $x_3$-direction, we get elongated octahedra and splinter tetrahedra. Each octahedron can be subdivided into 4 spear tetrahedra (see Fig. 1) that satisfy none of the conditions (8) and (9) as $h \to 0$.

Similarly compressing the diamond lattice in the $x_2$-direction, we obtain almost flat octahedra, each of which can be subdivided into 4 spade tetrahedra (see Fig. 1). They do not satisfy simultaneously (8) and (9) as $h \to 0$. In both cases, one can prove that the optimal interpolation properties are again violated.

Thus, a natural question arises: Whether some other projection operators could produce better approximation properties than the Lagrange linear interpolant, e.g., the best $H^1$-approximation (the Ritz projection) or higher order interpolants. This will be the topic of our further research.

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