CHARACTERISTIC NUMBERS OF ELLIPTIC CURVES WITH FIXED J-INVariant

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Abstract. We solve the problem of counting elliptic curves with fixed j-invariant in projective space with tangency conditions. This is equivalent to counting rational nodal curves with condition on the node of the image. The solution is given in the form of effective recursions. We give explicit formulas when the dimension of the ambient projective space is at most 5. Many numerical examples are provided. A C++ program implementing all of the recursions is available upon request.

1. Introduction

Characteristic numbers of curves in projective spaces is a classical problem in algebraic geometry: how many curves in $\mathbb{P}^r$ of given degree and genus that pass through a general set of linear subspaces, and are tangent to a general set of hyperplanes? Presented in this form, the problem seems almost unattackable, as not much is known even in the case of genus two space curves. However, the cases of genus zero and genus one space curves are well understood. Incidence-only (meaning no tangency condition is considered) characteristic numbers of rational plane curves were first computed by Kontsevich, see [FP]. The method was to pull back the WDVV equation on $\overline{\mathcal{M}}_{0,4}$ onto the moduli space of stable maps $\overline{\mathcal{M}}_{0,n}(2,d)$ to obtain a recursion counting rational plane curves. The same method works equally well for rational space curves. In [P1], Lemma 2.3.1, it was shown that the tangency divisor is numerically equivalent to a linear combination of the incident divisor and boundary divisors on $\overline{\mathcal{M}}_{0,n}(r,d)$. Hence one can write down a recursion computing full characteristic numbers of rational space curves.

In genus one, there are at least two counting problems. One could try to obtain enumeration of genus one curves with generic j− invariant, or of genus one curves with fixed j− invariant. This note will deal with the latter. Incidence-only characteristic numbers for genus one space curves with fixed-j invariant have been computed in [I] and [Z]. In this note, recursions computing all characteristic numbers will be provided. In case of incidence-only numbers, we obtain an algebraic solution that works over any closed field of zero characteristic, in contrast to the analytic method in [I] and [Z]. The results in this note will also be used to compute characteristic numbers of elliptic space curves in an upcoming paper by the author.

All the recursions are based on our algorithm counting rational two nodal reducible curves. These are projective curves having two rational smooth component intersecting at two points (or with a choice of two intersection points in the case of plane curves). Counting these curves is in turn based on an algorithm counting rational curves, now with an additional type of
conditions: special tangent conditions. This will be defined in Section 2. We work out in detail the algorithm counting rational curves with special tangent conditions in ambient space of dimension at most 5. For dimension 6 or higher, the numbers could in theory be expressed as intersections of tautological classes on a blowup of $\mathcal{M}_{0,1}(r, d)$, but this is much less implementable.

We use the following results to obtain our recursions. We use the WDVV equation on $\overline{\mathcal{M}}_{0,n}(r, d)$. We use the divisor theory on $\overline{\mathcal{M}}_{0,n}(r, d)$ as developed in [P1]. We do not use any outside input, and our method for incidence-only characteristic numbers is different from those in [I], [Z].

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2. Definitions and Notations

2.1. The moduli space of stable maps of genus 0 in $\mathbb{P}^r$. As usual, $\overline{\mathcal{M}}_{0,n}(r, d)$ will denote the Kontsevich compactification of the moduli space of genus zero curves with $n$ marked points of degree $d$ in $\mathbb{P}^r$. We will also be using the notation $\overline{\mathcal{M}}_{0,S}(r, d)$ where the markings are indexed by a set $S$. The following are Weil divisors on $\overline{\mathcal{M}}_{0,S}(r, d)$:

- The divisor $(U || V)$ of $\overline{\mathcal{M}}_{0,S}(r, d)$ is the closure in $\overline{\mathcal{M}}_{0,S}(r, d)$ of the locus of curves with two components such that $U \cup V = S$ is a partition of the marked points over the two components.
- The divisor $(d_1, d_2)$ is the closure in $\overline{\mathcal{M}}_{0,S}(r, d)$ of the locus of curves with two components, such that $d_1 + d_2 = d$ is the degree partition over the two components.
- The divisor $(U, d_1 || V, d_2)$ is the closure in $\overline{\mathcal{M}}_{0,S}(r, d)$ of the locus of curves with two components, where $U \cup V = S$ and $d_1 + d_2 = d$ are the partitions of markings and degree over the two components respectively.

2.2. The constraints and the ordering of constraints. We will be concerned with the number of curves passing through a constraint. Each constraint is denoted by a $(r+1)-$tuple $\Delta$ as follows:

(i) $\Delta(0)$ is the number of hyperplanes that the curves need to be tangent to.
(ii) For $0 < i \leq r$, $\Delta(i)$ is the number of subspaces of codimension $i$ that the curves need to pass through.
(iii) If the curves in consideration have a node and we place a condition on the node, that is the node has to belong to a general codimension $k$ linear subspace, then $\Delta$ has $r+2$ elements and the last element, $\Delta(r+1)$, is $k$.

Note that because in general a curve of degree $d$ will always intersect a hyperplane at $d$ points, introducing an incident condition with a hyperplane essentially means multiplying...
the cycle class cut out by other conditions by \( d \). For example, if we ask how many genus zero curves of degree 4 in \( \mathbb{P}^3 \) that pass through the constraint \( \Delta = (1, 2, 3, 4, 0)(\Delta(1) = 2) \), that means we ask how many genus zero curves of degree 4 pass through three lines, four points, are tangent to one hyperplane, and then multiply that answer by \( 4^2 \). We will also refer to \( \Delta \) as a set of linear spaces, hence we can say, pick a space \( p \) in \( \Delta \).

We consider the following ordering on the set of constraints, in order to prove that our algorithm will terminate later on. Let \( r(\Delta) = -\sum_{i \geq 1}^{\text{ir}} \Delta[i] \cdot i^2 \), and this will be our rank function. We compare two constraints \( \Delta, \Delta' \) using the following criteria, whose priority are in the following order:

- If \( \Delta(0) = \Delta'(0) \) and \( \Delta \) has fewer non-hyperplane elements than \( \Delta' \) does, then \( \Delta < \Delta' \).
- If \( \Delta(0) > \Delta'(0) \) then \( \Delta < \Delta' \).
- If \( r(\Delta) < r(\Delta') \) then \( \Delta < \Delta' \).

Informally speaking, characteristic numbers where the constraints are more spread out at two ends are computed first in the recursion. We write \( \Delta = \Delta_1 \Delta_2 \) if \( \Delta = \Delta_1 + \Delta_2 \) as a partition of the set of linear spaces in \( \Delta \).

2.3. The stacks \( \mathcal{R}, \mathcal{N}, \mathcal{RR}, \mathcal{RR}_2 \). We list the following definitions of stacks of stable maps that will occur in our recursions.

1) Let \( \mathcal{R}(r, d) \) be the usual moduli space of genus zero stable maps \( \overline{M}_{0,0}(r, d) \).

2) Let \( \mathcal{N}(r, d) \) be the closure in \( \overline{M}_{0,\{A,B\}}(r, d) \) of the locus of maps of smooth rational curves \( \gamma \) such that \( \gamma(A) = \gamma(B) \). Informally, \( \mathcal{N}(r, d) \) parametrizes degree \( d \) rational nodal curves in \( \mathbb{P}^r \).

3) For \( d_1, d_2 > 0 \), let \( \mathcal{RR}(r, d_1, d_2) \) be \( \overline{M}_{0,\{C\}}(r, d_1) \times \overline{M}_{0,\{C\}}(r, d_2) \) where the fibre product is taken over evaluation maps \( ev_C \) to \( \mathbb{P}^r \).

4) Similarly we can define \( \mathcal{NR}(r, d_1, d_2) \) (see figure 1).

5) For \( d_1, d_2 > 0 \), let \( \mathcal{RR}_2(r, d_1, d_2) \) be the closure in \( \overline{M}_{0,\{A,C\}}(r, d_1) \times \mathbb{P} \overline{M}_{0,\{B,C\}}(r, d_2) \) (the projections are evaluation maps \( e_{C'} \)) of the locus of maps \( \gamma \) such that \( \gamma(A) = \gamma(B) \). We call maps in this family rational two-nodal reducible curves.
2.4. **Special Tangent Condition.** It is necessary to understand the enumerative geometry of rational curves, now considering extra conditions of the form: there is a fixed marked point \( A \) on the curve, and the projective tangent line at \( A \) passes through a given codimension 2 linear subspace \( M \). The corresponding (Weil) divisor is denoted by \( W^M_A \). When there is no need to consider any particular codimension 2 subspace \( M \), we will only write \( W_A \). We would also need to consider the case where there is a condition on \( A \), which means it could be specified to lie on a certain linear subspace. By characteristic numbers of rational space curves with special tangent conditions, we mean the numbers of rational space curves having a marked point \( A \) that satisfy the following conditions:

- Pass through various linear spaces and are tangent to various hyperplanes.
- The tangent line at \( A \) to the curve passes through various codimension 2 linear spaces.
- The point \( A \) may or may not lie on a given linear space.

![Fig 2. A curve with a special tangent condition](image)

2.5. **Stacks of stable maps with constraints.** Let \( \mathcal{F} \) be a maps of curves into \( \mathbb{P}^r \). For a constraint \( \Delta \), we define \((\mathcal{F}, \Delta)\) be the closure in \( \mathcal{F} \) of the locus of maps that satisfy the constraint \( \Delta \). If the stack of maps \( \mathcal{F} \) has two marked points \( A \) and \( B \), we define \((\mathcal{F}, \mathcal{L}^u_A \mathcal{L}^v_B)\) to be the closure in \( \mathcal{F} \) of the locus of maps \( \gamma \) such that \( \gamma(A) \) lies on \( u \) general hyperplanes, and that \( \gamma(B) \) lies on \( v \) general hyperplanes.

If \( \mathcal{F} \) has one marked point \( A \) then we define \((\mathcal{F}, \mathcal{L}^u_A W^v_A)\) to be the closure of maps \( \gamma \) such that \( \gamma(A) \) lies on \( u \) general hyperplanes, and that the image of \( \gamma \) is smooth at \( \gamma(A) \) and the tangent line to the image of \( \gamma \) at \( \gamma(A) \) passes through \( v \) general codimension 2 subspaces (\( v \) special tangent conditions).

If a stack \( \mathcal{F} \) is supported on a finite number of points then we denote \( \# \mathcal{F} \) to be the stack-theoretic length of \( \mathcal{F} \).

If \( \mathcal{F} \) is a closed substack of the stacks \( \mathcal{NR}, \mathcal{RR} \) then we denote \((\mathcal{F}, \Gamma_1, \Gamma_2, k)\) to be the closure in \( \mathcal{F} \) of the locus of maps \( \gamma \) such that the restriction of \( \gamma \) on the \( i \)-th component...
satisfies constraint $\Gamma_i$ and that $\gamma(C)$ lies on $k$ general hyperplanes. We use the notation $(\mathcal{F}, \Delta, k)$ if we don’t want to distinguish the conditions on each component.

If $\mathcal{F}$ is a closed substack of $\mathcal{R}_2(r, d_1, d_2)$ then we denote $(\mathcal{F}, \Gamma_1, \Gamma_2, k, l)$ to be the closure in $\mathcal{F}$ of the locus of maps $\gamma$ such that the restriction of $\gamma$ on the $i$--th component satisfies constraint $\Gamma_i$ and that $\gamma(C)$ lies on $l$ general hyperplanes, and that $\gamma(A) = \gamma(B)$ lies on $k$ general hyperplanes. Similarly, we use the notation $(\mathcal{F}, \Delta, k, l)$ if we don’t want to distinguish the conditions on each component.

Note that for maps of reducible source curves, tangency condition include the case where the image of the node lies on the tangency hyperplane, as the intersection multiplicity is 2 in this case.

3. COUNTING ONE-NODAL REDUCIBLE CURVES IN $\mathbb{P}^r$

In this section we discuss how to count maps with reducible source curves.

**Proposition 3.1.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two families of stable maps with marked point $C$. Let $\Gamma_1$ and $\Gamma_2$ be two constraints. Then we have

$$\#(\mathcal{F}_1 \times_{ev_C} \mathcal{F}_2, \Gamma_1, \Gamma_2, k) = \#(\mathcal{F}_1, \Delta'_1) \cdot \#(\mathcal{F}_2, \Delta'_2)$$

where $\Delta'_i$ are determined as follows. Let $e_1$ be the dimension of the pushforward under $ev_C$ of $(\mathcal{F}_1, \Gamma_1)$ into $\mathbb{P}^r$. Let $e_2$ be the dimension of the pushforward under $ev_C$ of $(\mathcal{F}_2, \Gamma_2)$ into $\mathbb{P}^r$. Then $\Delta'_i$ is obtained from $\Gamma_i$ by adding a subspace of codimension $e_i$.

**Proof.** Let $\alpha_i$ be the class of $ev_C^*(\mathcal{F}_i, \Gamma_i)$ in the Chow ring of $\mathbb{P}^r$. Let $h$ be the class of a subspace of codimension $k$. Then $\#(\mathcal{F}_1 \times_{ev_C} \mathcal{F}_2, \Gamma_1, \Gamma_2, k)$ is equal to the intersection product $\alpha_1 \cdot \alpha_2 \cdot h$ which is $\text{deg}(\alpha_1) \cdot \text{deg}(\alpha_2)$. To compute $\text{deg}(\alpha_i)$, we intersect $\alpha_i$ with a subspace of codimension $e_i$, thus

$$\text{deg}(\alpha_i) = \#(\mathcal{F}_i, \Delta'_i)$$

which proves the proposition. $\square$

The following lemma is useful because it allow us to express the tangency condition on maps of reducible curves in terms of tangency conditions on maps of each component and condition on the node.

**Lemma 3.2.** Let $\mathcal{X}_1, \mathcal{X}_2$ be stacks of stable maps into $\mathbb{P}^r$. Assume each map in each family carries at least one marked point $C$. Let $\mathcal{X} = \mathcal{X}_1 \times_{ev_C} \mathcal{X}_2$. Let $\mathcal{T}$ be the tangency divisor-tangent on $\mathcal{X}$, and $\mathcal{T}_i$ be the pull-back of the tangency divisor on the $i$--th component. Then on $\mathcal{X}$ we have this divisorial equation: $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + 2\mathcal{L}_C$.

**Proof.** Let $\mathcal{C}$ be a general curve in $\mathcal{X}$. $\mathcal{C}$ has the following description. There is a family of nodal curves over $\mathcal{C}$, $\pi : S \to \mathcal{C}$ such that $S$ is the union of two families of nodal curves $\mathcal{X}_1, \mathcal{X}_2$ along a section $s : \mathcal{C} \to S$. The section $s$ represents the marked point $C$ of each family. There is also a map $\mu : S \to \mathbb{P}^r$ such that the restriction of $\mu$ on each fiber is an element
(a map) of $\mathcal{X}_1 \times_{ev_c} \mathcal{X}_2$. Now choose a general hyperplane $H$ in $\mathbb{P}^r$. Then the restriction of the tangency divisor $\mathcal{T}$ on $\mathcal{C}$ is the branched divisor of the map $\pi : \mu^{-1}(H) = \mathcal{D} \to \mathcal{C}$. This map is a $d_1 + d_2$ sheet covering of $\mathcal{C}$. The ramification points of this map come from three sources:

- The ramification points on $\mu^{-1}(H)|_{X_1}$.
- The ramification points on $\mu^{-1}(H)|_{X_2}$.
- The intersections $\mu^{-1}(H) \cap s$.

The first two sources contribute to the pull backs $\mathcal{T}_1 \cdot \mathcal{C}$ and $\mathcal{T}_2 \cdot \mathcal{C}$ respectively. The intersections points $\mu^{-1}(H) \cap s$ correspond precisely to the maps $\gamma$ with $\gamma(C) \in H$. These points are the nodes of the curve $\mathcal{D}$, because through each of them, there are two branches: one from $\mu^{-1}(H)|_{X_1}$, one from $\mu^{-1}(H)|_{X_2}$. If $P \in \mathcal{D}$ is one of such points, then the branched divisor of $\pi$ contains $\pi(P)$ with multiplicity 2. Thus we have $\mathcal{T} \cdot \mathcal{C} = \mathcal{T}_1 \cdot \mathcal{C} + \mathcal{T}_2 \cdot \mathcal{C} + 2L_C \cdot \mathcal{C}$. □

Using the lemma, we can “expand” the tangency conditions on $\mathcal{F}_1 \times_{ev_c} \mathcal{F}_2$ until we have tangency conditions only on each individual component.

**Proposition 3.3.** Let $\Delta$ be a constraint and let $\Delta_l$ be the constraint obtained from $\Delta$ by removing $l$ tangency conditions. Then we have the following equality:

$$\#(\mathcal{F}_1 \times_{ev_c} \mathcal{F}_2, \Delta, k) = \sum_{l=0}^{\Delta(0)} 2^l \binom{\Delta(0)}{l} \sum_{\Gamma_1, \Gamma_2 = \Delta_l} \#(\mathcal{F}_1 \times_{ev_c} \mathcal{F}_2, \Gamma_1, \Gamma_2, k + l).$$

**Proof.** There are $\binom{n}{l}$ ways to remove $l$ tangency conditions. Doing this results in a codimension $k + l$ condition on the node (the image of $C$), and the multiplicity is $2^l$. □

Applying the proposition to the family $\mathcal{R}\mathcal{R}_2(r, d_1, d_2)$ we have:

**Corollary 3.4.**

$$\#(\mathcal{R}\mathcal{R}_2(r, d_1, d_2), \Delta, k, k') = \sum_{l=0}^{\Delta(0)} 2^l \binom{\Delta(0)}{l} \sum_{\Gamma_1, \Gamma_2 = \Delta_l} \#(\mathcal{R}\mathcal{R}_2(r, d_1, d_2), \Gamma_1, \Gamma_2, k, k' + l).$$

### 4. Counting Rational Space Curves With Special Tangent Conditions

In this section, we will describe the algorithm counting rational space curves with special tangent conditions in $\mathbb{P}^r$. Let $X = \overline{\mathcal{M}}_{n, \{A\}}(r, d)$ throughout this section. Following the notation in [P1] let $\mathcal{H}$ be the incident divisor (incident to a codimension 2 subspace), and let $\mathcal{K}^A_j$ be the boundary divisor of $\overline{\mathcal{M}}_{n, \{A\}}(r, d)$ whose points represent reducible curves in which the component containing $A$ is mapped with degree $j$. The main difficulty when we have multiple special tangent conditions is excess intersection: any special tangent divisor $\mathcal{W}^M_A$ passes through the locus of maps $\gamma$ where $\gamma(A)$ is not a smooth point of the image.
However, we have the following result that helps us reduce the number of special tangent divisors in our computation.

**Proposition 4.1.** Any characteristic number of rational curves with \( l \geq r - 1 \) special tangent conditions is expressible in terms of characteristic numbers of rational curves with at most \( r - 2 \) special tangent conditions.

Proof of this statement will be given in section 5. \( \square \)

Thus, we only need to care about excess intersection locus in codimension at most \( r - 2 \). The following proposition lists all components of this locus.

**Proposition 4.2.** Let \( S_n \) be the closure of locus of maps \( \gamma \) in \( \mathcal{X} \) such that the source curve has \( n + 1 \) components, and the component containing \( A \), called the principal component, is incident with \( n \) other components. Moreover, \( \gamma \) contracts the principal component. Then \( S_2, \ldots, S_{r-2} \) are the components of codimension at most \( r - 2 \) of the excess intersection locus of the special tangent divisors. Furthermore, \( S_n \) contributes to the excess intersection only if there are at least \( 2n - 2 \) special tangent conditions. In particular, only \( S_n \)'s with \( 2n \leq r \) are relevant in counting curves with special tangent conditions.

**Proof.** Let \( \gamma \) be a map in \( \mathcal{X} \) such that \( \gamma(A) \) is not a smooth point of its image. If \( \gamma \) does not contract the component of the source curve containing \( A \) then \( \gamma(A) \) is at least a nodal singularity. Maps of this type vary in a family of codimension at least \( r - 1 \). Thus if \( \gamma \) belongs to a component (of the excess locus) of codimension at most \( r - 2 \), \( \gamma \) must contract the component of the source curve that contains \( A \). For a multi-index \( I(d, n) = (d_1, \ldots, d_n) \) with \( \sum_i d_i = d \), let \( V_{I(d, n)} \) be \( \prod_i \overline{M}_{0,\{A\}}(r, d_i) \) where the product is taken over the evaluation maps \( ev_A \). It is easy to see that each component of \( S_n \) is a finite quotient of a \( \overline{M}_{0,n+1} \times V_{I(d, n)} \), where \( \overline{M}_{0,n+1} \) is the moduli space of genus zero stable curves with \( n + 1 \) marked points. Now \( \overline{M}_{0,n+1} \) is of dimension \( n - 2 \), which means the "enumerative codimension" of \( S_n \) is \( n - 2 \) less than its codimension, hence is \( 2n - 2 \). Since we will only need to count rational curves with at most \( r - 2 \) special tangent conditions, only \( S_n \) in which \( 2n - 2 \leq r - 2 \), or equivalently \( 2n \leq r \), is relevant. \( \square \)

![Fig 3. S₄](image)

We will blow up \( S_n \)'s in order to discount the excess contribution. The above proposition provides us with an useful guideline. In \( \mathbb{P}^3 \), no blowup is needed. One blowup of \( S_2 \) is needed.
for $\mathbb{P}^4$ and $\mathbb{P}^5$. More generally, we need one more blowup for each increase by two in the dimension of the ambient space. In the rest of this section, we provide explicit formula for the cases $\mathbb{P}^3, \mathbb{P}^4, \mathbb{P}^5$, which only requires at most one blowup as expect.

**Case 1:** Counting rational curves with one special tangent condition in $\mathbb{P}^r, r \geq 3$.

We can express the special tangent divisor as linear combinations of boundary divisors and incident divisors, as shown in the following lemma.

**Lemma 4.3.** The following equality holds in the group $A^1(\mathcal{X}) \otimes \mathbb{Q}$, for $r > 2$:

$$W_A = 2L_A + \psi_A$$

where $\psi_A$ is the psi-class. In particular, we have

$$W_A = \left(2 - \frac{2}{d}\right)L_A + \frac{1}{d^2}H + \sum_{j=1}^{<d} \frac{(d-j)^2}{d^2}K^{A,j}$$

**Proof.** We use the method as described in [P1], intersecting the two sides of the equations with a general curve $C$ in $\mathcal{X}$. Let $\gamma$ denote the image of $C$ under the evaluation map $ev_A$. Let $M$ be the codimension 2 subspace in $\mathbb{P}^r$ corresponding to the special tangent condition $W_A$. Because $C$ is a general curve, we can assume $\gamma$ is smooth. Let $L$ be a general line in $\mathbb{P}^r$, and let $\pi_M : \mathbb{P}^r - M \rightarrow L$ be the projection onto $L$ from $M$. Let $\phi_A$ be the line bundle on $\gamma$ described as follows. For each point $p \in \gamma$, $ev_A^{-1}(p)$ is a map $\alpha \in C$. The fibre of $\phi_A$ over $p$ is then the tangent vector to the image of $\alpha$ at $\alpha(A)$. Let $R$ be the zero scheme of the bundle map $\phi_A \rightarrow \pi_M^*(T_L)$, with $T_L$ being the tangent bundle of $L$. Geometrically, $R$ represents the locus pf points $p \in \gamma$, such that the map $ev_A^{-1}(p)$ satisfies special tangent condition with respect to the subspace $M$. Thus $\deg R = R \cap [\gamma] = C \cap W_A$.

We have $\deg R = -c_1(\phi_A) + \deg(\pi_M|_C)c_1(T_L)$.

Now $c_1(T_L) = 2[\text{class of a point}]$, and $\deg(\pi_M|_C) = \deg \gamma = L_A \cap C$. The pullback of $\phi_A$ by $ev_A$ is isomorphic to the line bundle on $C$ obtained by attaching to each map the tangent vector at $A$ to the source curve. Hence $-c_1(\phi_A) \cap \gamma = -c_1(ev_A^*(\phi_A)) \cap C = \psi_A \cap C$ is the usual psi-class. In short, we have $W_A = 2L_A + \psi_A$.

The second equality follows from the fact that $\psi_A = -\pi_*(s_2^2)$ on $\overline{M}_{0,\{A\}}(r,d)$ and Lemma 2.2.2 in [P1].

**Case 2:** Counting rational curves with two special tangent conditions in $\mathbb{P}^r, r \geq 4$.

Let $\pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ the blowup of $\mathcal{X}$ along $S_2$. Let $S_2^j$ be the component of $S_2$ with degree partition $(j,0,d-j)$, and let $E_2^j$ be the corresponding exceptional divisor. We have that $S_2^j$ is a $\mathbb{Z}_2$-quotient of $\mathcal{R}\mathcal{R}(j,d-j)$. A general element $E_2^j$ has following geometric interpretation: it is a pair $(\gamma,l)$ where $\gamma$ is a map in $\mathcal{R}\mathcal{R}(j,d-j)$, and $l$ is a line in $\mathbb{P}^r$. $l$ must lie
on the plane \((l_1, l_2)\) where \(l_i\) is the projective tangent line to the image (under \(\gamma\)) of the \(i\)-th component at the image (under \(\gamma\)) of \(A\) (here we use \(A\) to denote the node of the family \(\mathcal{R}(j, d - j)\), instead of using \(C\) as in the definition in Section 2.2, but this does not change anything). For each divisor \(\mathcal{D}\) of \(\mathcal{X}\), let \(\bar{\mathcal{D}}\) be its proper transformation. The next lemma allows us to compute the class \(\pi_*(\bar{\mathcal{W}}^2_A)\)

**Lemma 4.4.** The following equality holds in \(A^2(\mathcal{X}) \otimes \mathbb{Q}\):

\[
\pi_*(\bar{\mathcal{W}}^2_A) = \left(2 - \frac{2}{d}\right) \mathcal{W}_A \mathcal{L}_A + \frac{1}{d^2} \mathcal{W}_A \mathcal{H} + \sum_{j=1}^{j<d} \frac{(j - d)^2}{d^2} \pi_*(\bar{\mathcal{W}}_A \bar{\mathcal{K}}^{A,j}) + \sum_{j=1}^{j\leq d/2} \frac{2j^2 - 2jd}{d^2} S^j_2
\]

The class \(\pi_*(\bar{\mathcal{W}}_A \bar{\mathcal{K}}^{A,j})\) is the class of the closure of the locus of maps with reducible source curves, where the restriction onto the component containing \(A\) satisfies one special tangent condition.

Counting maps in \(\pi_*(\bar{\mathcal{W}}_A \bar{\mathcal{K}}^{A,j})\) is doable by Lemma 4.1 and results in section 3. Counting maps in \(S^j_2\) is equivalent to counting maps in \(\mathcal{R}(j, d - j)\), which is also doable by results in section 3.

**Proof.** We pull back the main equation of Lemma 4.3:

\[
\pi^*\mathcal{W}_A = \left(2 - \frac{2}{d}\right) \tilde{\mathcal{L}}_A + \frac{1}{d^2} \tilde{\mathcal{H}} + \sum_{j=1}^{j<d} \frac{(d - j)^2}{d^2} \pi^*\bar{\mathcal{K}}^{A,j}
\]

\[
\pi^*\mathcal{W}_A = \bar{\mathcal{W}}_A + \sum_j E^j_2 \text{ and } \pi^*\bar{\mathcal{K}}^{A,j} = \bar{\mathcal{K}}^{A,j} + m_j E^j_2 \text{ where } m_j \text{ is } 1 \text{ if } j \neq d - j \text{ and } 2 \text{ if } j = d - j.
\]

Rearranging the terms, we have

\[
\bar{\mathcal{W}}_A = \left(2 - \frac{2}{d}\right) \tilde{\mathcal{L}}_A + \frac{1}{d^2} \tilde{\mathcal{H}} + \sum_{j=1}^{j<d} \frac{(d - j)^2}{d^2} \bar{\mathcal{K}}^{A,j} + \sum_{j=1}^{j\leq d/2} \frac{2j^2 - 2jd}{d^2} E^j_2
\]

Now it is obvious that \(\pi_*(\bar{\mathcal{W}}_A E^j_2) = S^j_2\). Multiply the above equation with \(\bar{\mathcal{W}}_A\) and pushforward yields the desired equation. \(\square\)

Using Lemma 4.4, we can reduce a counting problem involving two special tangent conditions into various counting problems involving at most one special tangent condition.

**Case 3:** Counting rational curves with three special tangent conditions in \(\mathbb{P}^r, r \geq 5\).

View \(\mathcal{R}(j, d - j)\) as \(\overline{\mathcal{M}}_{0,\{A\}}(r, j) \times_{ev_A} \overline{\mathcal{M}}_{0,\{A\}}(r, d - j)\). Let \(\mathcal{W}^{(i)}\) be the pullback of the special tangent divisor of the \(i\)-th factor. Let \(p : \mathcal{R}(j, d - j) \to S^j_2\) be the natural projection. We have the following lemma.
Lemma 4.5. The following equality holds in $A^3(\mathcal{X}) \otimes \mathbb{Q}$:

$$\pi_*(\tilde{W}_A^3) = \left(2 - \frac{2}{d}\right) \pi_*(\tilde{W}_A^2) \mathcal{L}_A + \frac{1}{d^2} \pi_*(\tilde{W}_A)^2 \mathcal{H} + \sum_{j=1}^{j<d/2} \frac{(j - d)^2}{d^2} \pi_*(\tilde{W}_A^2 \tilde{K}^{A,j})$$

$$+ \sum_{j=1}^{j<d/2} \frac{2j^2 - 2jd}{d^2} \pi_*(\tilde{W}_A E_2^j)$$

$\pi_*(\tilde{W}_A^2 \tilde{K}^{A,j})$ is the closure in $\mathcal{X}$ of the locus of maps with reducible source curves, where the restriction of the map on the component containing $A$ satisfies two special tangent conditions. Counting maps in this locus is doable by Lemma 4.4 and results in section 3. Furthermore, for any constraint $\Delta$ we have

$$(\pi_*(\tilde{W}_A^2 E_2^j), \Delta) = (\mathcal{W}^{(1)} + \mathcal{W}^{(2)}, \Delta)$$

if both sides are finite.

Proof. Only the last equality needs proving. Because the constraint $\Delta$ cuts out a one-dimensional family on $\mathcal{R}\mathcal{R}(j, d-j)$, proving the equality is an intersection theory problem on a $\mathbb{P}^1$-bundle over a curve. We reformulate the problem as follows. Let $\mathcal{F}_1$ be a one-dimensional family of projective rational curves of degree $j$ with a marked point $A$. We associated with $\mathcal{F}_1$ the line bundle $l_1$ which is the line bundle of the projective tangent lines at $A$. Similarly, we have $\mathcal{F}_2$ and $l_2$, where curves in $\mathcal{F}_2$ have degree $d-j$. Let $\mathcal{C} = \mathcal{F}_1 \times_{ev_A} \mathcal{F}_2$, which is a curve ($\mathcal{F}_i$’s are chosen so that $\mathcal{C}$ is not empty). Let $\mathcal{P}$ be the projectivization of $l_1 \oplus l_2$. Thus $\pi : \mathcal{P} \to \mathcal{C}$ is a rank-one projective bundle. A general element of $\mathcal{P}$ is a pair of curve-line $(\gamma, l)$ with $\gamma \in \mathcal{C}$ and $l \subset (l_1, l_2)$. Let $\mathcal{W}$ be the divisor on $\mathcal{P}$ define as follows. For a general codimension 2 subspace $M \in \mathbb{P}^r$, a pair $(\gamma, l) \in \mathcal{P}$ is in $\mathcal{W}$ if and only if $l \subset M$. We have a natural inclusion $\mathcal{F}_i = P(l_i) \subset \mathcal{P}$, with $P(l_i)$ being the projectivization of the line bundle $l_i$. Let $\mathcal{D}$ be the canonical line bundle on $\mathcal{P}$, and let $\mathcal{G}$ be the pullback of a point $\pi^{(-1)}(p)$ for any $p \in \mathcal{C}$. With this reformulation, the equality that we need to prove becomes

$$\mathcal{W}^2 = \mathcal{W}\mathcal{F}_1 + \mathcal{W}\mathcal{F}_2$$

Let $a_i = -c_1(l_i) \cdot C$. We have

$$\mathcal{F}_1 = \mathcal{D} + \pi^*(c_1(\phi_2) \cap C) = \mathcal{D} - a_2\mathcal{G}$$

hence

$$\deg(\mathcal{F}_1^2) = \deg(\pi^*(\mathcal{D}^2 - 2a_2\mathcal{G} + a_2^2\mathcal{G}^2)) = \deg(s_1(F) \cap C) - 2a_2 = a_1 + a_2 - 2a_2 = a_1 - a_2$$

which means that $\mathcal{F}_1^2 = a_1 - a_2$ as $F_1^2$ is of dimension 0 in the Chow ring of $\mathcal{P}$. Similarly $\mathcal{F}_2^2 = a_2 - a_1$, thus $F_1^2 + F_2^2 = 0$. Now let $\mathcal{W} = a\mathcal{F}_1 + b\mathcal{G}$. Then we have $\mathcal{W}\mathcal{G} = 1 = a(\mathcal{F}\mathcal{G}) \Rightarrow a = 1$. Now we have $\mathcal{W}\mathcal{F}_1 = \mathcal{F}_1^2 + b \Rightarrow b = \mathcal{W}\mathcal{F}_1 - \mathcal{F}_1^2$. That leads to $\mathcal{W}^2 = 2\mathcal{W}\mathcal{F}_1 - \mathcal{F}_1^2$. Similarly $\mathcal{W}^2 = 2\mathcal{W}\mathcal{F}_2 - \mathcal{F}_2^2$. Add the two equalities together we have

$$\mathcal{W}^2 = \frac{1}{2}(2\mathcal{W}\mathcal{F}_1 + 2\mathcal{W}\mathcal{F}_2 - \mathcal{F}_1^2 - \mathcal{F}_2^2) = \mathcal{W}\mathcal{F}_1 + \mathcal{W}\mathcal{F}_2.$$
Using Lemma 4.5, we can reduce a counting problem involving three special tangent conditions into various counting problem involving at most 2 special tangent conditions.

We end this section with some examples.

**Example 4.6.** How many conics in \( \mathbb{P}^3 \) passing through 3 points, that have a marked point \( A \) which must lie on a fixed line \( M \), and that the tangent line at \( A \) to the curve passes through a fixed line \( L \)? The answer is 1.

**Proof.** Because the three points that the conic passes through determine its plane \( H \), this problem reduces to an enumerative problem in \( \mathbb{P}^2 \): how many conics in \( \mathbb{P}^2 \) that pass through 3 points and is tangent to a line at a fixed point? The answer is therefore 1. Now we will compute this number in a different way, using Lemma 4.3. Let \( \Delta = (0,0,0,3) \), and \( \Delta' = (0,0,1,3) \). We need to compute \( \#((\mathcal{M}_{0,\{A\}}(3,2), \Delta), \mathcal{L}_A^2 W_A) \). On \( \mathcal{M}_{0,\{A\}}(3,2) \), there is one boundary divisor, \( \mathcal{K} = (0,1 \| \{A\},1), \) which parametrize pair of lines intersecting at one point, and the marked point \( A \) is on one of them. Using lemma 4.3 we have

\[
\mathcal{W}_A = \mathcal{L}_A + \frac{\mathcal{H}}{4} + \frac{\mathcal{K}}{4}
\]

Thus

\[
\#((\mathcal{M}_{0,\{A\}}(3,2), \Delta), \mathcal{L}_A^2 W_A) = \#((\mathcal{M}_{0,\{A\}}(3,2), \Delta), \mathcal{L}_A^3) + \frac{1}{4} \#((\mathcal{M}_{0,\{A\}}(3,2), \Delta'), \mathcal{L}_A^2) + \frac{1}{4} \#((\mathcal{K}, \Delta), \mathcal{L}_A^2)
\]

\[
= 0 + \frac{1}{4} + \frac{1}{4} 3 = 1.
\]

The first ”#” term of the right hand side is the number of conics in \( \mathbb{P}^r \) passing through 4 points. The second ”#” term is the number of conics in \( \mathbb{P}^r \) passing through 3 points and 2 lines. The last ”#” term is the number of pair of lines in \( \mathbb{P}^r \) with one common point, that pass through 3 points, and that the component with the marked point \( A \) intersect a line at \( A \). \( \square \)

**Example 4.7.** There are 2 conics in \( \mathbb{P}^4 \) satisfying the following conditions. The conics pass through 3 points and a plane, and there is a marked point \( A \) on the curve, the projective tangent line at which passes through 2 other planes.

**Proof.** Again, the three point conditions determine the plane \( H \) for the conics. Thus in fact we have a plane curve counting problem. The conics must pass through 4 points (the plane condition now become point condition), and the tangent line at \( A \) must pass through 2 other points on the plane \( H \). Thus the problem is equivalent to counting plane conics through 4 points and tangent to 1 line, thus the answer is two. We must show that

\[
\#((\mathcal{M}_{0,A}(4,2), \Delta), \mathcal{W}_A^2) = 2
\]
with $\Delta = (0, 1, 0, 0, 3)$. From the proof of Lemma 4.4 we have
\[
\widetilde{W}_A = \bar{L}_A + \frac{\bar{H}}{4} + \frac{\bar{K}^{A,1}}{4} - \frac{E_j^2}{2}
\]
Multiply the equation with $\widetilde{W}_A$, pushforward and integrate against $(\mathcal{M}_{0,(4,2)}, \Delta)$ we have
\[
\#(\mathcal{M}_{0, A}(4,2), \Delta), \mathcal{W}_A^2) = \#((\mathcal{M}_{0, A}(4,2), \Delta), \mathcal{W}_A \mathcal{L}_A) + \frac{1}{4}((\mathcal{M}_{0, A}(4,2), \Delta'), \mathcal{W}_A) \\
+ \frac{1}{4}#((\mathcal{K}^{A,1}, \Delta), \mathcal{W}_A) - \frac{1}{2}#(E_j^2, \Delta) \\
= 3 + \frac{2}{4} + 0 - \frac{3}{2} = 2
\]
where $\Delta' = (0, 2, 0, 0, 3)$. We list below several numbers of curves with special tangent conditions in $\mathbb{P}^3, \mathbb{P}^4, \mathbb{P}^5$. The special class $(a, b)$ means the marked point as a codimension $a$ condition and there are $b$ special tangent conditions.

| Degree | Condition | Special Classes | Numbers |
|--------|-----------|-----------------|---------|
| Cubic  | (1, 2, 3) | (3, 1)          | 34      |
| Cubic  | (4, 2, 2) | (2, 1)          | 4736    |
| Quartic| (7, 2, 3) | (1, 1)          | 35131904|
| Quintic| (4, 4, 6) | (0, 1)          | 280111872|
| Quintic| (2, 2, 7) | (2, 1)          | 352176  |
| Sextic | (3, 4, 7) | (3, 1)          | 340403776|

Table 1. Some enumerative numbers with special class in $\mathbb{P}^3$

| Degree | Condition | Special Classes | Numbers |
|--------|-----------|-----------------|---------|
| Conic  | (1, 1, 2, 1) | (1, 2)          | 38      |
| Cubic  | (2, 1, 1, 3) | (1, 2)          | 980     |
| Quartic| (2, 2, 1, 4) | (2, 2)          | 37792   |
| Quintic| (3, 3, 1, 5) | (2, 2)          | 31565232|
| Sextic | (3, 3, 4, 5) | (1, 2)          | 49679646304|

Table 2. Some enumerative numbers with special classes in $\mathbb{P}^4$

| Degree | Condition | Special Classes | Numbers |
|--------|-----------|-----------------|---------|
| Conic  | (1, 1, 1, 0, 2) | (0, 3)          | 20      |
| Cubic  | (1, 1, 1, 2, 2) | (0, 3)          | 1240    |
| Quartic| (2, 3, 1, 2, 2) | (3, 3)          | 1181400 |
| Quintic| (2, 2, 3, 4, 2) | (0, 3)          | 1654232816|

Table 3. Some enumerative numbers with special classes in $\mathbb{P}^5$
5. Counting curves in $\mathcal{R}_2(r, d_1, d_2)$

First we need a result about the Chow ring of $\text{Bl}_D(\mathbb{P}^r \times \mathbb{P}^r)$, which is the blowup of $\mathbb{P}^r \times \mathbb{P}^r$ along the diagonal. For details of the derivation, we refer the readers to [N2].

**Proposition 5.1.** The Chow ring of $\text{Bl}_D(\mathbb{P}^r \times \mathbb{P}^r)$ is generated by $h, k$, the hyperplane class of the first and second factor, and the exceptional divisor $e$ with the following relations:

\[
\begin{align*}
  h^r + 1 &= k^r + 1 = 0, \\
  he &= ke, \\
  e^r &= \sum_{i < r} (-1)^{i-1}(r+1)h^i e^{r-i} + \sum_{i \geq 0} h^i k^{r-i}.
\end{align*}
\]

**Example.** The following are the third relation in the case $r = 1, 2, 3, 4$:

\[
\begin{align*}
  e &= h + k, \\
  e^2 &= 3he - (h^2 + hk + k^2), \\
  e^3 &= 4he^2 - 6h^2e + (h^3 + h^2k + hk^2 + k^3), \\
  e^4 &= 5he^3 - 10h^2e^2 + 5h^3e - (h^4 + h^3k + h^2k^2 + hk^3 + k^4).
\end{align*}
\]

Recall that $\mathcal{R}_2(r, d_1, d_2)$ is a substack of $\overline{\mathcal{M}}_{0,\{A,C\}}(r, d_1) \times_{\text{ev}_C} \overline{\mathcal{M}}_{0,\{B,C\}}(r, d_2)$ of maps $\gamma$ such that $\gamma(A) = \gamma(B)$. We rephrase the problem of counting maps in $\mathcal{R}_2(r, d_1, d_2)$ as follows:

*Given two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of maps of rational curves with two marked points $A, C$. How many times a map $\gamma_1$ from $\mathcal{F}_1$ and a map $\gamma_2$ from $\mathcal{F}_2$ intersect in such a way that:

- $\gamma_1(A) = \gamma_2(A)$ and $\gamma_1(C) = \gamma_2(C)$.
- $\gamma_1(A)$ lies on a fixed linear space of codimension $p$.
- $\gamma_1(C)$ lies on a fixed linear space of codimension $q$.*

We consider the evaluation map

$$
\text{ev}_{AC} : \mathcal{F}_i \longrightarrow (\mathbb{P}^r \times \mathbb{P}^r)
$$

Let $T_i$ be the closure in $\text{Bl}_D(\mathbb{P}^r \times \mathbb{P}^r)$ of $\text{ev}_{AC}(\mathcal{F}_i)$. Let $h, k$ be the hyperplane classes of the first and second factor in $\text{Bl}_D(\mathbb{P}^r \times \mathbb{P}^r)$. Then the answer to our enumerative problem above is the intersection number

$$
T_1 T_2 h^p k^q
$$

where the product is evaluated in the Chow ring of $\text{Bl}_D(\mathbb{P}^r \times \mathbb{P}^r)$. ($T_i$ parametrizes ordered pair of points on the curves in $\mathcal{F}_i$. The blowup is to prevent us from counting in the case where two points run into each other).
To count maps in $\mathcal{R}_2(r, d_1, d_2)$ satisfying the constraint $(Δ, p, q)$, we first consider all the partitions $Δ = Γ_1 Γ_2$, and for each such partition, assign constraint $Γ_i$ to the $i$-th component. If $Δ(0) ≠ 0$, meaning if there are tangency conditions, we also have to distribute the tangency conditions over each component first, in the sense of Proposition 3.3. Then the constraint $Γ_1$ cuts out a family $F_1$ on $\overline{\mathcal{M}}_{0,\{A,C\}}(r, d_1)$. Similarly, $γ_2$ cuts out a family $F_2$ on $\overline{\mathcal{M}}_{0,\{A,C\}}(r, d_2)$. Let $T_i$ be the closure of $ev_{AC}(F_i)$ in $Bl_D(\mathbb{P}^r × \mathbb{P}^r)$. Then we calculate the product

$$T_1 T_2 h^p k^q$$

in the Chow ring $A^*(Bl_D(\mathbb{P}^r × \mathbb{P}^r))$. Then we take the sum over all partitions $Δ = Γ_1 Γ_2$ to get the number of maps $#(\mathcal{R}_2(r, d_1, d_2), Δ, p, q)$. We need a result to calculate the classes of $T_i$ in $A^*(Bl_D(\mathbb{P}^r × \mathbb{P}^r))$. The following lemma is useful:

**Lemma 5.2.** Let $F$ be a family of stable maps in $\overline{\mathcal{M}}_{0,\{A,C\}}(r, d)$ such that $A, C$ moves freely, that is, the forgetful map $\overline{\mathcal{M}}_{0,\{A,C\}}(r, d) → \overline{\mathcal{M}}_{0,0}(r, d)$ has fibre dimension 2. Let $T$ be the closure in $Bl_D(\mathbb{P}^r × \mathbb{P}^r)$ of the image of $F$ under the evaluation map $ev_{AC} : F → \mathbb{P}^r × \mathbb{P}^r$. Let $G$ be the family of stable maps in $\overline{\mathcal{M}}_{0,\{A\}}(r, d)$ that is the image of $F$ under the forgetful morphism $\overline{\mathcal{M}}_{0,\{A,C\}}(r, d) → \overline{\mathcal{M}}_{0,\{A\}}(r, d)$. Assume $\dim T ≤ 2r$. Then we have

- For $m, n$ such that $m + n = \dim T$ :
  $$Th^m k^n = #(F, L_A^m L_C^n).$$

- For $m$ such that $m + 1 = \dim T$ :
  $$Th^m e = #(G, L_A^m).$$

- For $m, n$ such that $m + n = \dim T$ , we have
  $$Th^m e(h + k - e)^{n-1} = #(G, L_A^m W_A(n-1)).$$

**Proof.** The first equality is trivial. The number $Th^m k^n$ is the number of maps $γ ∈ F$ such that $γ(A)$ belongs to $h$ hyperplanes, and that $γ(C)$ belongs to $k$ hyperplanes. That is precisely the number $#(F, L_A^m L_C^n)$. The second equality follows from the fact that multiplying with $e$ is the same as replacing the family $F$ by the family $G$.

Now we prove the third equality. Let

$$[x_0 : x_1 : ⋯ : x_n] × [y_0 : y_1 : ⋯ : y_n]$$

be a homogeneous coordinate system of $\mathbb{P}^r × \mathbb{P}^r$. Let $H$ be the hypersurface $x_0 y_n = x_n y_0$ in $\mathbb{P}^r × \mathbb{P}^r$. $H$ contains $D$ with multiplicity one and $T = h + k$ in $A^*(\mathbb{P}^r × \mathbb{P}^r)$, hence the proper transformation $\tilde{H}$ of $H$ in $Bl_D(\mathbb{P}^r × \mathbb{P}^r)$ satisfies

$$\tilde{H} = h + k - e.$$
entire curve $f_x$ onto $x$. The intersection $H \cap P_x$ is a hyperplane in $P_x$ which is the span of $x$ and the codimension 2 subspace $x_0 = y_0 = 0$. Then for a point $y \in T$ with $\pi(y) = x$, we have $y \in T \cap e \cap \tilde{H}$ iff $f_x$, as a curve in the projective space $P_x$ is tangent to $H_x$ at $x$. Thus intersecting with $\tilde{H}$ (after intersecting with $e$) has the effect of imposing one special tangent condition on the family $\mathcal{G}$. It follows that intersecting with $n - 1$ instances of $\tilde{T}$ has the effect of imposing $n - 1$ special tangent conditions. □

![Diagram](image)

Fig 4.

Now we have enough to be able compute the class of $T = ev_{AC_*} (\mathcal{F})$ in $A^*(Bl_D(\mathbb{P}^r \times \mathbb{P}^r))$. The formal statement of that fact is the following proposition, whose proof is trivial.

**Proposition 5.3.** Let $T \in A^*(Bl_D(\mathbb{P}^r \times \mathbb{P}^r))$ be a class of codimension $d, 0 \leq d \leq 2r$. Then the following intersection products determine $T$ :

- $Th^m k^n$ with $0 \leq m \leq r, 0 \leq n \leq r$.
- $Th^m e(h + k - e)^n$ with $0 \leq m \leq r, 0 \leq n \leq r - 2$.
- $Th^{d-1} e$.

with $m, n$ appropriately chosen so that the intersection number is well-defined.

The reason the power $n$ of $h + k - e$ is at most $r - 2$ is because $e'$ is expressible as polynomials in $h$ and $k$, so we never need to multiply $T$ with a power of $e$ that is more than $r - 1$, in order to determine $T$. □

In particular, if we know all characteristic numbers of rational curves with at most $r - 2$ special tangent conditions, then that is enough to count maps in $\mathcal{R}\mathcal{R}_2(d_1, d_2)$.

**Proof of Proposition 4.1.** If the number of special tangent conditions $l$ is greater than $2r - 2$, then the number is 0 because the tangent line at $\gamma(A)$ can pass through at most
2r − 2 general codimension 2 subspaces. Now assume l ≤ 2r − 2. Let Δ be the constraint (beside the special tangent conditions). Let $\mathcal{F}$ be $(\overline{\mathcal{M}}_{0,\{A,C\}}(r, d), \Delta)$ and $T$ be the closure in $Bl_D(\mathbb{P}^r \times \mathbb{P}^r)$ of the image of $\mathcal{F}$ under $ev_{AC}$. We have $\dim T < 2r$. If we know all the characteristic numbers with at most $r − 2$ special tangent conditions, then Proposition 5.3 shows that we can determine $T$. Then the characteristic number with constraint $\Delta$ (and $\mathcal{L}_A^m$) and $l$ special tangent conditions is the intersection number $Th^m e(h + k − e)^l$.

□

We end the section with some examples.

**Example 5.4.** How many pair of lines $(L_1, L_2)$ in $\mathbb{P}^3$ such that they intersect twice, and that each of them passes through 3 lines? The answer is 0.

The answer is obvious because two distinct lines can never intersect twice. But our algorithm does not know that. Let $\Delta = (0, 0, 3, 0)$. We need to compute

$$\frac{1}{2} \#(\mathcal{R}_{\Delta}(3, 1, 1), \Delta, \Delta).$$

The factor 1/2 accounts for the fact that the statement of the problem does not distinguish the two intersection points. Let $\mathcal{F}_i$ be the family of the lines $L_i$ with a choice of two marked points $A, C$ on them. Let $T_i$ be the pushforward of $\mathcal{F}_i$ under the evaluation maps $ev_{AC} : \mathcal{F}_i \to Bl_D(\mathbb{P}^r \times \mathbb{P}^r)$. $T_i$ is three dimensional, so we can assume

$$T_1 = \alpha(h^3 + k^3) + \beta(h^2k + hk^2) + \gamma eh^2 + \mu e^2h.$$

The coefficients of $h^3$ and $k^3$ must be the same due to symmetry. Similarly the coefficients of $h^2k$ and $hk^2$ must be the same.

$$\alpha = \alpha h^3 k^3 = T_1 k^3 = \#(\overline{\mathcal{M}}_{0,\{A,C\}}(3, 1), \Delta, \mathcal{L}_A^3) = 0$$

$$\beta = \beta h^3 k^3 = T_1 k^3 = \#(\overline{\mathcal{M}}_{0,\{A,C\}}(3, 1), \Delta, \mathcal{L}_A^2 \mathcal{L}_C) = 2$$

$$\mu = \mu h^3 e^3 = T_1 h^2 e = \#(\overline{\mathcal{M}}_{0,\{A\}}(3, 1), \Delta, \mathcal{L}_A^2) = 2$$

Computation of $\gamma$ is a little bit lengthier. First we have

$$\gamma = \gamma h^3 k^3 = T_1 h e^2 - \mu e^4 h^2 = (2T_1 e^2 - T_1 h e(h + k - e)) - 4\mu$$

$$= -2\mu - T_1 h e(h + k - e).$$

Now $T_1 h e(h + k - e) = \#(\overline{\mathcal{M}}_{0,\{A\}}(3, 1), \Delta, \mathcal{L}_A \mathcal{W}_A)$ is the number of lines with a marked point $A$ in $\mathbb{P}^3$ that pass through 3 lines, such that $A$ lies on a fixed plane, and such that the tangent line at $A$ passes through a general line. This number is the same as the number of lines passing through 4 general lines in $\mathbb{P}^r$, which is 2. Thus $\gamma = -2\mu - T_1 h k(h + k - e) = -4 - 2 = -6$. Therefore

$$T_1 = 2(h^2k + hk^2) - 6h^2e + 2he^2$$

Obviously $T_1 = T_2$, so after a bit of algebra we have

$$T_1 T_2 = (2(h^2k + hk^2) - 6h^2e + 2he^2)^2 = 0.$$

□
Example 5.5. How many pair of conics-twisted cubics in \( \mathbb{P}^5 \) intersecting at two nodes, with the first node being on a fixed hyperplane and the second node being on a fixed 3-space, such that the conic passes through one 3-space, one general plane, one general line, one general point, and the cubic passes through two general 3-spaces, one general plane, one general line, two general points? The answer is 956.

Let \( \Gamma_1 = (0, 0, 1, 1, 1, 0) \) and \( \Gamma_2 = (0, 0, 2, 1, 1, 2, 0) \). We need to compute

\[
\#(\mathcal{R}\mathcal{R}_2(5, 2, 3), \Gamma_1, \Gamma_2, 1, 2).
\]

Let \( \mathcal{F}_1 \) be a family of lines conics in \( \mathbb{P}^5 \) with a choice of two marked points \( A, C \) on them, such that the conics satisfy \( \Gamma_1 \). Let \( \mathcal{F}_2 \) be the a family of twisted cubics in \( \mathbb{P}^5 \) with a choice of two marked points \( A, C \) on them, such that the cubics satisfy \( \Gamma_2 \). Let \( T_i \) be the pushforward of \( \mathcal{F}_i \) under \( ev_{AC} \), onto the Chow ring \( A^*(\text{Bl}_D(\mathbb{P}^5 \times \mathbb{P}^5)) \). The we need to compute the intersection product \( hh^2T_1T_2 \).

Using Lemma 5.2 and Proposition 5.3, we can find the classes of \( T_i \) to be:

\[
T_1 = 2h^4 + 6h^3k + 8h^2k^2 + 6hk^3 + 2k^4 - 42h^3e + 29h^2e^2 - 9he^3 + e^4
\]
\[
T_2 = 45h^3 + 88h^2k + 88hk^2 + 45k^3 - 308h^2e + 140he^2 - 23e^3
\]

Using proposition 5.1, we can calculate the product:

\[
(2h^4 + 6h^3k + 8h^2k^2 + 6hk^3 + 2k^4 - 42h^3e + 29h^2e^2 - 9he^3 + e^4)
\times
(45h^3 + 88h^2k + 88hk^2 + 45k^3 - 308h^2e + 140he^2 - 23e^3)hh^2 = 956.
\]

Some numbers;

| Degree  | Degree  | Constraint | Constraint | Nodes | Number |
|---------|---------|------------|------------|-------|--------|
| Conic   | Conic   | (2, 3, 1)  | (2, 3, 1)  | (0, 0) | 3360  |
| Conic   | Cubic   | (2, 3, 1)  | (3, 4, 1)  | (1, 1) | 614656|
| Line    | Quadratic | (0, 1, 0)  | (3, 4, 3)  | (2, 2) | 570752|
| Cubic   | Cubic   | (3, 3, 2)  | (1, 4, 2)  | (0, 3) | 963360|
| Conic   | Quadratic | (3, 3, 1)  | (0, 6, 4)  | (0, 0) | 2253312|

Table 4. Some enumerative numbers of pair of rational curves in \( \mathbb{P}^3 \)

| Degree  | Degree  | Constraint | Constraint | Nodes | Number |
|---------|---------|------------|------------|-------|--------|
| Conic   | Conic   | (1, 1, 2, 1) | (0, 0, 0, 3) | (0, 0) | 4     |
| Conic   | Cubic   | (1, 2, 1, 1) | (1, 1, 2, 2) | (1, 2) | 4816  |
| Line    | Conic   | (0, 1, 1, 0) | (1, 1, 1, 1) | (1, 2) | 18    |
| Cubic   | Cubic   | (3, 1, 0, 3) | (3, 1, 0, 3) | (1, 1) | 2297664|

Table 5. Some enumerative numbers of pair of rational curves in \( \mathbb{P}^4 \)
First we gave a recursion counting incidence-only characteristic numbers of rational nodal curves (with condition on the node) in $\mathbb{P}^r$.

**Theorem 6.1.** Let $\Delta$ be a constraint that $\Delta(0) = 0$. Let $k = \Delta(r + 1)$. Choose a subspace $u$ in $\Delta$ which is not a hyperplane, such that the dimension of $u$ is largest possible. Then choose any two other subspaces $s, t$ in $\Delta$. The following constraints are derived from $\Delta$:

1) $\Delta_0$ by removing $u, s, t$ from $\Delta$.
2) $\Delta_1$ is derived from $\Delta_0$, by replacing $p$ and $s$ with $p \cap s$.
3) $\Delta_2$ is derived from $\Delta_0$, by replacing $q$ and $t$ with $q \cap t$.
4) $\Delta_3$ is derived from $\Delta_0$, by replacing $s$ and $t$ with $s \cap t$.

If $\Gamma$ is a set of linear spaces, and $a$ and $b$ are two linear spaces, denote $\Gamma^{(a,b)}$ the set obtained from $\Gamma$ by adding $a$ and $b$. Then the following formula holds:

$$
\#(\mathcal{N}(r, d), \Delta) = - \sum_{d_1 + d_2 = d} \binom{\Delta}{\Gamma_1} \#(\mathcal{N}\mathcal{R}(r, d_1, d_2), \Gamma_1^{(s,t)}, \Gamma_2^{(p,q)}, 0) - \sum_{d_1 + d_2 = d} \binom{\Delta}{\Gamma_1} \#(\mathcal{N}\mathcal{R}(r, d_1, d_2), \Gamma_1^{(p,q)}, \Gamma_2^{(s,t)}, 0) - 2 \sum_{d_1 + d_2 = d} \binom{\Delta}{\Gamma_1} \#(\mathcal{R}\mathcal{R}_2(r, d_1, d_2), \Gamma_1^{(p,q)}, \Gamma_2^{(s,t)}, k, 0) + \sum_{d_1 + d_2 = d} \binom{\Delta}{\Gamma_1} \#(\mathcal{N}\mathcal{R}(r, d_1, d_2), \Gamma_1^{(q,t)}, \Gamma_2^{(p,s)}, 0) + \sum_{d_1 + d_2 = d} \binom{\Delta}{\Gamma_1} \#(\mathcal{N}\mathcal{R}(r, d_1, d_2), \Gamma_1^{(p,s)}, \Gamma_2^{(q,t)}, 0) + 2 \sum_{d_1 + d_2 = d} \binom{\Delta}{\Gamma_1} \#(\mathcal{R}\mathcal{R}_2(r, d_1, d_2), \Gamma_1^{(p,s)}, \Gamma_2^{(q,t)}, k, 0) - \#(\mathcal{N}(r, d), \Delta_3) + \#(\mathcal{N}(r, d), \Delta_1) + \#(\mathcal{N}(r, d), \Delta_2).
$$

| Degree | Degree | Constraint | Constraint | Nodes | Number |
|--------|--------|------------|------------|-------|--------|
| Conic  | Conic  | (0, 0, 0, 2, 1) | (0, 0, 0, 2, 1) | (1, 1) | 2      |
| Conic  | Cubic  | (1, 0, 1, 0, 2) | (1, 1, 0, 1, 3) | (0, 0) | 144    |
| Line   | Quartic| (0, 0, 0, 0, 1) | (2, 0, 0, 2, 3) | (1, 3) | 844    |
| Cubic  | Cubic  | (3, 4, 1, 1, 1) | (2, 1, 1, 2, 1) | (1, 2) | 1027324928 |

Table 6. Some enumerative numbers of pair of rational curves in $\mathbb{P}^5$. 
Furthermore, $\Delta_1, \Delta_2, \Delta_3$ are all of lower rank than that of $\Delta$. Here $\binom{n}{\beta} = \prod_{i=0}^{n} \binom{n(i)}{\beta(i)}$ for any two tuples $\alpha, \beta$ having the same length.

**Proof.** Let $S$ be a set of markings that is in one-to-one correspondence $\mu : \Delta_0 \to S$ with the linear spaces in $\Delta_0$. Let $X$ be the moduli space $\mathcal{M}_{0,\{A,B\}} \cup S(r,d)$, and let $\mathcal{N}^{(S)}(r,d)$ be the closure in $X$ of the locus of maps $\gamma$ such that $\gamma(A) = \gamma(B)$. Let $Y$ be the closure in $\mathcal{N}^{(S)}$ of the locus of maps $\gamma$ such that $\gamma(\mu(m)) \in m$ for all $m \in \Delta_0$. Because $\#(\mathcal{N}(r,d), \Delta)$ is finite, $Y$ is one-dimensional. We consider two equivalent divisors on $X$:

$$\{(\mu(p), \mu(q)) \mid \{\mu(s), \mu(t)\}\} = \{(\mu(p), \mu(s)) \mid \{\mu(q), \mu(t)\}\}.$$ Let $\mathcal{K}_1 = \{(\mu(p), \mu(q)) \mid \{\mu(s), \mu(t)\}\}$, and let $\mathcal{K}_2 = \{(\mu(p), \mu(s)) \mid \{\mu(q), \mu(t)\}\}$. Then we have

$$\#(Y \cap \mathcal{K}_1) = \#(Y \cap \mathcal{K}_2).$$

Let us analyze the left-hand side of the equation. Let $\gamma$ be a general point of $Y \cap \mathcal{K}_1$. Then $\gamma$ is a stable map whose source curve has two components $C_1, C_2$ joined at a node, such that $\mu(p), \mu(q) \in C_1$ and $\mu(s), \mu(t) \in C_2$. There are several cases to consider:

- $\deg \gamma|_{C_1} = 0$. If only $A$ or $C$ is on $C_1$ then by dimension counting we have that this case has no contribution. If both $A, C$ are on $C_1$ then the image curve has a cusp, on which we impose condition like those we impose on $p, q$. By dimension count again, we also have that the case has no contribution. The quick reason is that if a map contracted a component containing at least 4 special points (marked or nodes), then the dimension of the family of image curves is less than the dimension of the family of maps, therefore is enumeratively irrelevant. Now if $A, B \in C_2$, $\gamma|_{C_2}$ is a rational nodal curve and satisfies the constraint $\Delta$ (but these conditions are marked). The contribution to $\#(Y \cap \mathcal{K}_1)$ in this case is $\#(\mathcal{N}(r,d), \Delta)$.

- $\deg \gamma|_{C_2} = 0$. Arguing similarly, we have that the contribution to $\#(Y \cap \mathcal{K}_1)$ is $\#(\mathcal{N}(r,d), \Delta_3)$

- $\gamma$ has positive degree $d$, component $C_i$. There are three subcases:

  - $A, B \in C_1$: In this case, $\gamma|_{C_1}$ is a rational nodal curve and $\gamma|_{C_2}$ is a rational curve. The contribution in this case is

  $$\sum_{d_1 + d_2 = 0} \#(\mathcal{N}\mathcal{R}(r,d_1,d_2), \Gamma_1^{(p,q)}, \Gamma_2^{(s,t)}, 0).$$

  - $A, B \in C_2$: The contribution is

  $$\sum_{d_1 + d_2 = d} \#(\mathcal{N}\mathcal{R}(r,d_1,d_2), \Gamma_1^{(s,t)}, \Gamma_2^{(p,q)}, 0).$$

  - $A \in C_1, B \in C_2$ or vice versa. In this case the image of $\gamma$ is a curve having two components that intersect twice at distinguished points. The contribution is therefore

  $$2 \sum_{d_1 + d_2 = d} \#(\mathcal{R}\mathcal{R}(r,d_1,d_2), \Gamma_1^{(p,q)}, \Gamma_2^{(s,t)}, k, 0).$$
We can analyze $\mathcal{Y} \cap \mathcal{K}_2$ in the same way and after rearranging the terms, we derive the equation in the statement of the theorem.

It is now possible to use the results so far to compute the characteristic number of rational nodal curves.

**Theorem 6.2.** Let $\Delta$ be a constraint such that $\Delta(0) > 0$. Let $\Delta(r + 1) = k$ Let $\Delta''$ be the constraint obtained from $\Delta$ by removing a tangency hyperplane. Let $\Delta'$ be the constraint obtained from $\Delta''$ by adding an incident codimension 2 subspace. Then we have the following equality, provided that the left hand side is finite.

\[
\#(\mathcal{N}(r, d), \Delta) = \frac{d-1}{d} \#(\mathcal{N}(r, d), \Delta') + \sum_{d_1+d_2=d} \left( \#(\mathcal{N}\mathcal{R}(r, d_1, d_2), \Delta'') + \#(\mathcal{R}\mathcal{R}_2(r, d_1, d_2), \Delta'', k, 0) \right).
\]

**Warning :** if $\Delta(0) \neq 0$ then those summands above involving reducible curves contain (twice) the case where the node is mapped to a tangency hyperplane. Also, in computing those summands, one needs to consider all possible splitting of constraints over two components (see Proposition 3.3 and Corollary 3.4).

**Proof.** We have the following equality of divisors on $\overline{\mathcal{M}}_{0,\{A,B\}}(r, d)$

\[
T = \frac{d-1}{d} \mathcal{H} + \sum_{d>0}^{j\leq d/2} \frac{j(d-j)}{d} (j, d-j).
\]

For a proof of this see [P1], Lemma 2.3.1. Thus

\[
\#(\mathcal{N}(r, d), \Delta) = \#((\mathcal{N}(r, d), \Delta''), T)
\]

\[
= \frac{d-1}{d} \#((\mathcal{N}(r, d), \Delta''), \mathcal{H}) + \sum_{j>0}^{j\leq d/2} \#(\mathcal{N}(r, d) \cap (j, d-j), \Delta'').
\]

Now we will analyze $\#(\mathcal{N}(r, d) \cap (j, d-j), \Delta'')$. A general map $\gamma \in \mathcal{N}(r, d) \cap (j, d-j)$ has two-component source curve. There are two cases:

- $A, B$ belong to a same component. The contribution is $\#(\mathcal{N}\mathcal{R}(j, d-j), \Delta'') + \#(\mathcal{N}\mathcal{R}(d-j, j), \Delta'')$ if $j < d-j$ depending on whether $A, B$ are in the component of lower or higher degree. If $j = d-j$, the contribution is just $\#(\mathcal{N}\mathcal{R}(j, d-j), \Delta'')$.
- $A, B$ belong to different components. The contribution is $2\#(\mathcal{R}\mathcal{R}_2(j, d-j), \Delta'', k, 0)$ if $j < d-j$ and is $\#(\mathcal{R}\mathcal{R}_2(j, d-j), \Delta'', k, 0)$ if $j = d-j$.

Sum up all possibilities, we derive the formula in Theorem 6.2. \qed
Calculation of \(#(\mathcal{R} \mathcal{R}_2(r, d_1, d_2), \Delta'', k, 0)\) should make use of Corollary 3.4. One point worth mentioning when counting rational nodal curves with tangency conditions and with condition on the node is that maps with degree 2 do contribute enumeratively. Rational nodal curves with degree two are rational degree two covers of \(\mathbb{P}^1\) with a marked point specified as the node. For these maps, having a hyperplane passing through the branched points count as tangency.

From characteristic number of rational nodal curves, it is easy to get characteristic number of rational nodal curves. Let \(m = \Delta(0)\), and \(\Delta_i\) be the constraint received by removing \(i\) tangency conditions and replace them by a codimension \(i\) on the node. Then we have the number of elliptic curves with fixed \(j\)– invariant, with \(j\) generic, of degree \(d\) in \(\mathbb{P}^r\) satisfying constraint \(\Delta\) denoted \(#(\mathcal{J}(r, d), \Delta)\), is:

\[
#(\mathcal{J}(r, d), \Delta) = \sum_{i=0}^{m} 2^i \binom{n}{i} #(\mathcal{N}(r, d), \Delta_i). 
\]

Now we give several numerical examples. We recover all previously known numbers in literature. The characteristic numbers of plane nodal cubics were computed in [A]. The characteristic numbers of elliptic plane curves with fixed \(j\)– invariant were computed in [V2]. Characteristic numbers of rational plane cubics in \(\mathbb{P}^3\) were computed in [HMX]. Let \(N, N_l, N_p\) be the family of rational nodal curves, rational nodal curves with the node on a fixed line, rational nodal curves with the node on a fixed point. Similarly, we denote \(N_s, N_b, N_f\) for the same family with the node on a fixed plane, a fixed 3–space, or a fixed 4–space. The following tables list the characteristic numbers of such families and of elliptic curves with fixed \(j\)– invariant (denoted by \(\mathcal{J}\)). Below are tables of characteristic numbers of such families of low degree (2, 3, 4, 5). In some tables, we put some point conditions so that the numbers are small enough to fit in the table. The only other conditions are tangency, and top incident condition. For example, in the table for quartics in \(\mathbb{P}^4\), the curves must pass through 2 points, the other conditions are combination of tangency and incident to planes.

| \# tang | \(N\) | \(N_l\) | \(N_p\) | \(\mathcal{J}\) |
|--------|------|------|------|--------|
| 0      | 0    | 0    | 0    | 0      |
| 1      | 0    | 0    | 0    | 0      |
| 2      | 0    | 2    | 1    | 0      |
| 3      | 0    | 3    | 3/2  | 12     |
| 4      | 0    | 3/2  |      | 48     |
| 5      | 0    |      |      | 75     |

Table 7. Plane conics.
### Table 8. Plane cubics.

| # tang | $N$ | $N_t$ | $N_p$ | $\mathcal{J}$ |
|--------|-----|-------|-------|--------------|
| 0      | 12  | 6     | 1     | 12           |
| 1      | 36  | 22    | 4     | 48           |
| 2      | 100 | 80    | 16    | 192          |
| 3      | 240 | 240   | 52    | 768          |
| 4      | 480 | 604   | 142   | 2784         |
| 5      | 712 | 1046  | 256   | 8832         |
| 6      | 756 | 1212  | 304   | 21828        |
| 7      | 600 | 1000  |       | 39072        |
| 8      | 400 |       |       | 50448        |

### Table 9. Plane quartics.

| # tang | $N$    | $N_t$  | $N_p$  | $\mathcal{J}$ |
|--------|--------|--------|--------|--------------|
| 0      | 1860   | 768    | 96     | 1860         |
| 1      | 6552   | 2952   | 384    | 8088         |
| 2      | 21600  | 10712  | 1448   | 33792        |
| 3      | 65328  | 35616  | 4992   | 134208       |
| 4      | 178272 | 106752 | 15516  | 497952       |
| 5      | 429120 | 281348 | 42416  | 1696320      |
| 6      | 886632 | 633972 | 99024  | 5193768      |
| 7      | 1515960| 1166352| 187248 | 13954512     |
| 8      | 2097648| 1705856| 279152 | 31849968     |
| 9      | 2350752| 1986672| 329496 | 60019872     |
| 10     | 2184480| 1893528|       | 92165280     |
| 11     | 1745712|        |       | 115892448    |

### Table 10. Conics in $\mathbb{P}^3$.

| # tang | $N$ | $N_s$ | $N_t$ | $N_p$ | $\mathcal{J}$ |
|--------|-----|-------|-------|-------|--------------|
| 0      | 0   | 0     | 0     | 0     | 0            |
| 1      | 0   | 0     | 0     | 0     | 0            |
| 2      | 0   | 16    | 8     | 2     | 0            |
| 3      | 0   | 24    | 12    | 3     | 96           |
| 4      | 0   | 20    | 10    | $7/2$ | 384          |
| 5      | 0   | 10    | 5     |       | 840          |
| 6      | 0   | 5     |       |       | 1200         |
| 7      | 0   |       |       |       | 1470         |
| # tang | N   | $N_s$ | $N_l$ | $N_p$ | $\mathcal{J}$           |
|-------|-----|-------|-------|-------|--------------------------|
| 0     | 12960 | 5040  | 904   | 72    | 12960                    |
| 1     | 29520 | 13120 | 2512  | 216   | 39600                    |
| 2     | 61120 | 32048 | 6568  | 612   | 117216                   |
| 3     | 109632| 64608 | 13904 | 1384  | 332640                   |
| 4     | 167616| 107072| 23904 | 2524  | 849024                   |
| 5     | 214400| 144960| 33304 | 3732  | 1890240                  |
| 6     | 230240| 162760| 38432 | 4656  | 3625440                  |
| 7     | 211200| 155288| 37808 | 5112  | 5994096                  |
| 8     | 170192| 130048| 32864 | 5424  | 8631120                  |
| 9     | 124176| 98352 | 25664 | 11038224 | 12875520                |
| 10    | 85440 | 70880 |       |       | 14422080                 |

Table 11. Cubics in $\mathbb{P}^3$.

| # tang | N   | $N_s$ | $N_l$ | $N_p$ | $\mathcal{J}$           |
|-------|-----|-------|-------|-------|--------------------------|
| 0     | 247191840 | 61582704 | 7487280 | 402216 | 247191840               |
| 1     | 519424512 | 138566640 | 17469840 | 975192 | 642589920               |
| 2     | 1034619648 | 295896480 | 38636160 | 2242512 | 1618835328             |
| 3     | 1932171072 | 588656160 | 79348512 | 4785408 | 3920405760          |
| 4     | 3353134848 | 1079389056 | 149728320 | 9378160 | 9020858112            |
| 5     | 5361957120 | 1808973504 | 257515200 | 16752296 | 19509189120         |
| 6     | 7841572992 | 2752793920 | 401264800 | 27140752 | 39298619520        |
| 7     | 10431095808 | 3788712880 | 564734880 | 39830752 | 73227372288        |
| 8     | 12599060192 | 4716456320 | 718744512 | 53161088 | 125665152480      |
| 9     | 13851211968 | 5333385216 | 831757440 | 65099040 | 198307833792      |
| 10    | 13948252800 | 5522229504 | 883153920 | 74131776 | 288227491200     |
| 11    | 12986719872 | 5292561600 | 870495360 | 79929312 | 387635041920     |
| 12    | 11309818368 | 4757882880 | 807883200 | 84550992 | 486058242048     |
| 13    | 9330496512 | 4070594880 | 715629312 | 57423507200 | 648194719872   |
| 14    | 7394421888 | 3381893376 |       |       | 71549059080       |
| 15    | 5703866880 |       |       |       |                |

Table 12. Quartics in $\mathbb{P}^3$.  

| # tang | $N$    | $N_s$   | $N_t$   | $N_p$   | $J$    |
|-------|--------|---------|---------|---------|--------|
| 0     | 2987074368 | 597069288 | 59293632 | 2757288 | 2987074368 |
| 1     | 6654861504  | 1393675584 | 142403568 | 6890568  | 7849000080 |
| 2     | 14302171008 | 3141287760 | 330349200 | 16691344 | 20114047872 |
| 3     | 29534616768 | 6800411520 | 736077600 | 387843208 | 631585386720 |
| 4     | 58394890752 | 14081928256 | 1569037056 | 87466348 | 1538700870672 |
| 5     | 110164217088 | 27795971008 | 3189343752 | 188200508 | 281761911168 |
| 6     | 197654921184 | 52144209544 | 6165495488 | 387843208 | 631585386720 |
| 7     | 336286484448 | 92755042440 | 11312688400 | 765476504 | 1358700870672 |
| 8     | 541376364848 | 156271230640 | 19684719200 | 1449944208 | 2800306366128 |
| 9     | 823917940992 | 249556959696 | 32520764016 | 2653490208 | 5526457857888 |
| 10    | 1186459103808 | 379132252128 | 51221741472 | 4769939328 | 10455705197568 |
| 11    | 1621483284864 | 552185368704 | 77488852608 | 74888852608 | 19030887269760 |
| 12    | 2114474172288 | 783085854720 | 111405535872 | 58098921777408 | 33559605535872 |
| 13    | 2648546358528 |                 |                 |         |              |

Table 13. Quintics in $\mathbb{P}^3$, passing through 3 points.

| # tang | $N$    | $N_b$   | $N_s$   | $N_t$   | $N_p$   | $J$    |
|-------|--------|---------|---------|---------|---------|--------|
| 0     | 7833840  | 2565720 | 468935  | 52140   | 2865   | 7833840 |
| 1     | 14708400 | 5294270 | 1017980 | 119400  | 6984   | 19839840 |
| 2     | 25085900 | 10073080 | 2038520 | 252192  | 15720  | 48138720 |
| 3     | 37705920 | 16296840 | 3416336 | 440272  | 28924  | 110777280 |
| 4     | 49732080 | 22491008 | 4833312 | 644504  | 44470  | 232897920 |
| 5     | 57643520 | 26854560 | 5889580 | 812540  | 59250  | 439941120 |
| 6     | 59232320 | 28240140 | 6319450 | 906690  | 70854  | 745702080 |
| 7     | 54660200 | 26636130 | 6095150 | 916962  | 78360  | 1141405440 |
| 8     | 45993500 | 22938610 | 5383586 | 858012  | 82584  | 1593774300 |
| 9     | 35861700 | 18337518 | 4423952 | 755184  | 85440  | 2055201960 |
| 10    | 26323500 | 138098900 | 342200 | 626640  | 87360  | 2480472300 |
| 11    | 18497240 | 9949360 | 2513120 | 480480  | 2841879120 |
| 12    | 12649200 | 6978480 | 1786880 | 3137555760 |
| 13    | 8510880  | 4808480 |         |         | 3385230720 |
| 14    | 5673920  |         |         |         | 3589051200 |

Table 14. Cubics in $\mathbb{P}^4$. 24
| # tang | $N$ | $N_b$ | $N_s$ | $N_l$ | $N_p$ | $\mathcal{I}$ |
|--------|-----|-------|-------|-------|-------|-------------|
| 0      | 264271032 | 61079694 | 8388348 | 749421 | 34860 | 264271032 |
| 1      | 493716948 | 120918936 | 17290038 | 1630488 | 81252 | 615876336 |
| 2      | 878434848 | 228232116 | 33980664 | 20905076 | 1429930 | 2189197336 |
| 3      | 1479817080 | 405964896 | 62797160 | 6629800 | 383672 | 306268560 |
| 4      | 2353692768 | 589968256 | 944430080 | 12151512 | 761888 | 6469681248 |
| 5      | 3530480992 | 815950592 | 1236374720 | 1429930 | 12151512 | 306268560 |
| 6      | 4995675728 | 1249818704 | 1834065600 | 1429930 | 12151512 | 6469681248 |
| 7      | 6680908448 | 1660225184 | 2490337920 | 1429930 | 12151512 | 6469681248 |
| 8      | 8472417440 | 2092994208 | 3139491200 | 1429930 | 12151512 | 6469681248 |
| 9      | 10234272948 | 2586854592 | 3935781600 | 1429930 | 12151512 | 6469681248 |
| 10     | 11836475952 | 3170518784 | 4753354720 | 1429930 | 12151512 | 6469681248 |
| 11     | 13167563808 | 3951890512 | 5927834704 | 1429930 | 12151512 | 6469681248 |
| 12     | 14112721248 | 4533584288 | 6800376384 | 1429930 | 12151512 | 6469681248 |
| 13     | 14531107200 | 5066370880 | 7289161120 | 1429930 | 12151512 | 6469681248 |

Table 15. Quartics in $\mathbb{P}^4$ passing through 2 points.

| # tang | $N$ | $N_b$ | $N_s$ | $N_l$ | $N_p$ | $\mathcal{I}$ |
|--------|-----|-------|-------|-------|-------|-------------|
| 0      | 5264130996 | 960390870 | 105886953 | 7801695 | 311311 | 5264130996 |
| 1      | 10335707556 | 1973618742 | 224710598 | 17371678 | 742316 | 12256489296 |
| 2      | 19791788388 | 3960252460 | 465840460 | 37911496 | 1746624 | 28109811168 |
| 3      | 36896035320 | 7737537944 | 940326944 | 80796848 | 4041128 | 63416490816 |
| 4      | 66880583024 | 14699954352 | 1845469104 | 167905648 | 9189708 | 140521932288 |
| 5      | 117792292576 | 27145486560 | 3519654728 | 340028520 | 20558296 | 305497218816 |
| 6      | 201506364736 | 48745168872 | 6523861268 | 670681448 | 45308086 | 651327035136 |
| 7      | 334871977648 | 85223104580 | 11759484440 | 1287078386 | 98524384 | 1362231952128 |
| 8      | 540951986840 | 145379939744 | 20637848154 | 2397410108 | 211715288 | 2797819372056 |
| 9      | 850242885024 | 242702404542 | 35332114224 | 4312424928 | 5652591017568 |
| 10     | 1301286873156 | 397849014300 | 59181220928 | 11257978051236 |
| 11     | 1938666465816 | 641728301752 | 963518793600 |
| 12     | 250469121008 | 736446174336 | 115267028672 |

Table 16. Quintics in $\mathbb{P}^4$ passing through 4 points.
Table 17. Cubics in $\mathbb{P}^5$

| # tang | $N$   | $N_f$   | $N_b$   | $N_s$   | $N_t$ | $N_p$ | $\mathcal{J}$ |
|--------|-------|---------|---------|---------|-------|-------|-------------|
| 0      | 3580435656 | 1034759292 | 189136374 | 24039939 | 2009982 | 85745 | 3580435656 |
| 1      | 5820250128 | 1803057816 | 343203840 | 45424176 | 3974516 | 178640 | 7889768712 |
| 2      | 8641680264 | 2888520852 | 572163144 | 78755588 | 7205344 | 341240 | 1661045704 |
| 3      | 11507535984 | 4048138080 | 1179603568 | 157723006 | 18576208 | 1267280 | 319397674176 |
| 4      | 13759570272 | 4992894416 | 1036797728 | 172833416 | 14773856 | 764324 | 6136232732 |
| 5      | 14867247680 | 5502189760 | 1161050240 | 172833416 | 17554792 | 954832 | 104391383040 |
| 6      | 14650427520 | 5502894720 | 1179603568 | 180279708 | 19079772 | 1102606 | 163351745280 |
| 7      | 13303631040 | 5066847184 | 1104900496 | 174051444 | 19343536 | 1204100 | 236503108800 |
| 8      | 11252393152 | 4350397184 | 967029476 | 157723006 | 18576208 | 1267280 | 319397674176 |
| 9      | 8959119120  | 3522421644 | 799569876 | 135605388 | 17095224 | 1305896 | 405992118672 |
| 10     | 6782773704  | 2715749316 | 629998440 | 111418656 | 15173120 | 1331840 | 490193697672 |
| 11     | 4929887760  | 2011043040 | 476256768 | 87775688 | 12973792 | 1349216 | 567210910536 |
| 12     | 347645440   | 1442366496 | 347592224 | 66354624 | 10586880 | 1360832 | 634363027200 |
| 13     | 2392303152  | 1010425424 | 246675392 | 48224736 | 738716078016 |
| 14     | 1624181888  | 696607744  | 171653932 | 34118336 | 738716078016 |
| 15     | 1092498624  | 474968055040 | 171653932 | 34118336 | 738716078016 |
| 16     | 730705920   | 321392512  | 81125866768 | 1305896 | 236503108800 |
| 17     | 487137280   | 838048055040 | 838048055040 | 838048055040 | 838048055040 |

Table 18. Quartics in $\mathbb{P}^5$ passing through 3 points.

| # tang | $N$   | $N_f$   | $N_b$   | $N_s$   | $N_t$ | $N_p$ | $\mathcal{J}$ |
|--------|-------|---------|---------|---------|-------|-------|-------------|
| 0      | 17793468  | 4315338  | 675729   | 82815   | 7629  | 408  | 17793468 |
| 1      | 33892524  | 8728578  | 1428506  | 187086  | 1122  | 42523200 |
| 2      | 61915284  | 16962956 | 2898296  | 406116  | 3012  | 99532512 |
| 3      | 108109320 | 31398264 | 5580216  | 834384  | 100788 | 7728  | 227691648 |
| 4      | 180450912 | 55599884 | 10188624 | 1618620 | 214248 | 384363027200 |
| 5      | 288477120 | 93327232 | 17697268 | 2968056 | 429304 | 44638 | 1099292256 |
| 6      | 442955328 | 151262244 | 29385528 | 5155156 | 807974 | 101692 | 2318653056 |
| 7      | 655304328 | 237174048 | 46930448 | 8512992 | 1413096 | 4771225200 |
| 8      | 936129552 | 361876128 | 72589134 | 13497600 | 9605588880 |
| 9      | 1291589856 | 539604810 | 109323720 | 18969484704 |
| 10     | 1716845652 | 788940756 | 36822211764 |
| 11     | 2184938712 | 838048055040 | 838048055040 | 838048055040 | 838048055040 |

Table 18. Quartics in $\mathbb{P}^5$ passing through 3 points.

REFERENCES

[A] P. Aluffi *The enumerative geometry of plane cubics II: nodal and cuspidal cubics*, Math. Annalen 289 (1991), 543-572.

[F] W. Fulton *Intersection Theory*, Second Edition, Springer 1996.

[FP] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, preprint 1996, alg-geom/9608011.
X. Hernandez, J. M. Miret, *The characteristic numbers of cuspidal plane cubics in $\mathbb{P}^3$*, Bull. Belg. Math. Soc. Simon Stevin, 10 (2003) No. 1, 115–124.

X. Hernandez, J. M. Miret and S. Xambo-Descamps, *Computing the characteristic numbers of the variety of nodal plane cubics in $\mathbb{P}^3$*, J. Symb. Comp. 42 (2007) 192–202.

E. Ionel, *Genus-one enumerative invariants in $\mathbb{P}^n$ with fixed $j$-invariant*, Duke Math. J. 94 (2) (1998) 279–324.

E. Getzler, *Intersection theory on $\overline{M}_{1,4}$ and elliptic Gromov-Witten invariants*, J. Amer. Math. Soc. 10 No. 4 (1997) 973–998.

D. Nguyen, *Doctoral thesis at Stanford University*, in preparation.

R. Pandharipande, *Intersection of $\mathbb{Q}$-divisors on Kontsevich’s moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ and enumerative geometry*, Trans. Amer. Math. Soc. 351 (1999), 1481-1505.

R. Pandharipande, *A note on elliptic plane curves with fixed $j$-invariant*, Proc. Amer. Math. Soc., 125, No. 12, 3471–3479.

R. Vakil, *The enumerative geometry of rational and elliptic plane curves in projective space*, J. Reine Angew. Math. (Crelle’s Journal), 529 (2000), 101–153.

R. Vakil, *Recursions for characteristic numbers of genus one plane curves*, Arkiv for Matematik, 39 (2001), no. 1, 157–180.

R. Vakil, A. Zinger, *A desingularization of the main component of the moduli space of genus-one stable maps to projective space*, Geom. Topol. 12 (2008), no. 1, 1-95.

A. Zinger, *Enumeration of one-nodal rational curves in projective spaces*, Topology 43 (2004) 793–829.