Efficiently Sampling Multiplicative Attribute Graphs
Using a Ball-Dropping Process

Abstract
We introduce a novel and efficient sampling algorithm for the Multiplicative Attribute Graph Model (MAGM - Kim & Leskovec (2010)). Our algorithm is strictly more efficient than the algorithm proposed by Yun & Vishwanathan (2012), in the sense that our method extends the best time complexity guarantee of their algorithm to a larger fraction of parameter space. Both in theory and in empirical evaluation on sparse graphs, our new algorithm outperforms the previous one.

To design our algorithm, we first define a stochastic ball-dropping process (BDP). Although a special case of this process was introduced as an efficient approximate sampling algorithm for the Kronecker Product Graph Model (KPGM - Leskovec et al. (2010)), neither why such an approximation works nor what is the actual distribution this process is sampling from has been addressed so far to the best of our knowledge.

Our rigorous treatment of the BDP enables us to clarify the rational behind a BDP approximation of KPGM, and design an efficient sampling algorithm for the MAGM.

1. Introduction
In this paper we are concerned with statistical models on graphs. The scalability of the model’s inference and sampling algorithm is becoming a critical issue especially for sparse graphs, as more and more graph data is becoming available. For instance, one can easily crawl a graph with millions of nodes in few days from Twitter.

In this regard, the Kronecker Product Graph Model (KPGM) of Leskovec et al. (2010) is particularly attractive. In contrast to traditional models such as Exponential Random Graph Model (ERGM) of Robins et al. (2007) or Latent Factor Model of Hoff (2009) which cannot scale beyond graphs with thousands of nodes, both inference in and sampling from a KPGM scale to graphs with millions of nodes.

However, the model has recently been criticized to be not very realistic, both in theory (Seshadhri et al., 2011) and in practice (Moreno & Neville, 2009). This is actually not very surprising, as the KPGM is clearly under-parametrized; usually only four parameters are used to fit a graph with millions of nodes.

In order to enrich the expressive power of the model Kim & Leskovec (2010) recently proposed a generalization of KPGM, which is named Multiplicative Attribute Graph Model (MAGM). The advantage of MAGM over KPGM has been argued from both theoretical (Kim & Leskovec, 2010) and empirical (Kim & Leskovec, 2011) perspectives.

No matter how attractive such a generalization is in terms of modeling, we still need to ask does the new model have efficient algorithms for inference and sampling? The inference part of this question was studied by Kim & Leskovec (2011), while sampling part was partially addressed by Yun & Vishwanathan (2012). In this paper, we further investigate the sampling issue.

It is straightforward to sample a graph from a MAGM in $\Theta(n^2)$ time, where $n$ is the number of nodes. Of course, such a naive algorithm does not scale to large graphs. Therefore, Yun & Vishwanathan (2012) suggested an algorithm which first samples $O((\log_2 n)^2)$ graphs from a KPGM and quilts relevant parts of the sampled graphs together to generate a single sample from the MAGM. Since approximate sampling from KPGM takes expected $O(e_K \log_2 n)$ time, where $e_K$ is the expected number of edges in the KPGM, thequilting algorithm runs in $O((\log_2 n)^3 e_K)$ time with high probability. The unsatisfactory aspect of the approach of Yun & Vishwanathan (2012), however, is that the complexity bound holds only when certain technical
conditions are met.

On the other hand, for the most commonly used parameter settings (see Section 4.5) our algorithm runs in $O \left( (\log_2 n)^3 (e_K + e_M) \right)$ time with high probability, where $e_M$ is the expected number of edges in the MAGM. When the technical conditions of Yun & Vishwanathan (2012) are met, then $e_M = e_K$.

Therefore, our method extends the best time complexity of Yun & Vishwanathan (2012) to a larger fraction of parameter space. Not only is our algorithm theoretically more interesting, we also show that it empirically outperforms the previous algorithm in sampling sparse graphs.

To design our algorithm, we first define a stochastic ball-dropping process (BDP) (Charakbarati et al. 2004, Groom et al. 2010) and Gleich & Owen (To appear). Although a special case of BDP was already introduced as an approximate sampling algorithm for KPGM (Leskovec et al., 2010), to the best of our knowledge neither why such an approximation works nor what is the actual distribution this process is sampling from has been addressed so far.

Our rigorous treatment of these problems enables us to clarify the rational behind a BDP approximation of KPGM (Section 3), and design an efficient sampling algorithm for MAGM (Section 4). We let BDP to propose candidate edges, and then reject some of them with certain probability to match the actual MAGM. This is the classic accept-reject sampling scheme for sampling distributions. The main technical challenge which we address in this paper is to show that the proposal distribution compactly bounds the target distribution, so that we can guarantee the efficiency of the algorithm.

2. Notation and Preliminaries

We use upper-case letters for matrices (e.g., $A$). Sets are denoted by upper-case calligraphic letters (e.g., $E$). We use Greek symbols for parameters (e.g., $\mu$), and integers are denoted in lower-case (e.g., $a, b, i, j$).

A directed graph is an ordered set $(V, E)$, where $V$ is the set of nodes $V = \{1, 2, \ldots, n\}$, and $E$ is the set of edges $E \subset V \times V$. We say that there is an edge from node $i$ to $j$ when $(i, j) \in E$. Furthermore, for each edge $(i, j) \in E$, $i$ and $j$ are called source and target node of the edge, respectively. Note that although we mainly discuss directed graphs in this paper, most of our ideas can be straightforwardly applied to the case of undirected graphs.

It is convenient to describe a graph in terms of its $n \times n$ adjacency matrix $A$ where the $(i, j)$-th entry $A_{ij}$ of $A$ denotes the number of edges from node $i$ to $j$. When there exists at most one edge between every $(i, j)$ pair, i.e., $A_{ij} \leq 1$ for all $i, j$, then we call it a simple graph. On the other hand if multiple edges are allowed then it is called a multi-graph. In either case, $|E|$, the number of edges in the graph, is equal to $\sum_{i,j=1}^{n} A_{ij}$.

The Kronecker multiplication of matrices is defined as follows (Bernstein, 2005).

**Definition 1** Given real matrices $X \in \mathbb{R}^{n \times m}$ and $Y \in \mathbb{R}^{p \times q}$, the Kronecker product $X \otimes Y \in \mathbb{R}^{np \times mq}$ is

$$X \otimes Y := \begin{bmatrix} X_{11}Y & X_{12}Y & \ldots & X_{1m}Y \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1}Y & X_{n2}Y & \ldots & X_{nm}Y \end{bmatrix}.$$  

The $k$-th Kronecker power $X^{(k)}$ is $\otimes_{i=1}^{k} X$.

2.1. Kronecker Product Graph Model (KPGM)

The Kronecker Product Graph Model (KPGM) of Leskovec et al. (2010) is usually parametrized by a $2 \times 2$ initiator matrix

$$\Theta := \begin{bmatrix} \theta_{00} & \theta_{01} \\ \theta_{10} & \theta_{11} \end{bmatrix},$$

with each $\theta_{ij} \in [0, 1]$, and additional size parameter $d \in \mathbb{Z}^+$. Using Kronecker multiplication, we construct a $2^d \times 2^d$ matrix $\Gamma$ from $\Theta$:

$$\Gamma := \Theta^{[d]} = \Theta \otimes \Theta \otimes \ldots \otimes \Theta,$$

$d$ times

$\Gamma$ is called an edge probability matrix, because under the KPGM the probability of observing an edge from node $i$ to node $j$ is simply $\Gamma_{ij}$ (see Figure 1). From an adjacency matrix point of view each $A_{ij}$ is an independent Bernoulli random variable with $P[A_{ij} = 1] = \Gamma_{ij}$.

Note that one can make the model more general by using multiple initiator matrices $\Theta^{(1)}, \Theta^{(2)}, \ldots, \Theta^{(d)}$ rather than a single matrix. In this case, the definition of edge probability matrix $\Gamma$ is modified to

$$\Gamma := \Theta^{(1)} \otimes \Theta^{(2)} \otimes \ldots \otimes \Theta^{(d)}.$$ 

In this paper we will adopt the more general setting (3). For notational convenience, we stack these initiator matrices to form the parameter array

$$\tilde{\Theta} := (\Theta^{(1)}, \Theta^{(2)}, \ldots, \Theta^{(d)}).$$
Also, $\theta_{ab}^{(k)}$ denotes $(a+1,b+1)$-th entry of $\Theta^{(k)}$. Given these parameters, the expected number of edges $e_K$ of KPGM can be calculated using

$$e_K = \sum_{i,j=1}^{n} \Gamma_{ij} = \prod_{k=1}^{d} \left( \sum_{0 \leq a,b \leq 1} \theta_{ab}^{(k)} \right).$$  

(5)

### 2.2. Multiplicative Attribute Graph Model (MAGM)

An alternative way to view KPGM is as follows: associate the $i$-th node with a bit-vector $b(i)$ of length $d$ such that $b_k(i)$ is the $k$-th digit of integer $(i - 1)$ in its binary representation. Then one can verify that the $(i,j)$-th entry of the edge probability matrix $\Gamma$ in (3) can be written as

$$\Gamma_{ij} = \prod_{k=1}^{d} \theta_{b_k(i) \ b_k(j)}^{(k)}.$$  

(6)

Under this interpretation, one may consider $b_k(i) = 1$ (resp. $b_k(i) = 0$) as denoting the presence (resp. absence) of the $k$-th attribute in node $i$. The factor $	heta_{b_k(i) \ b_k(j)}^{(k)}$ denotes the probability of an edge between nodes $i$ and $j$ based on the value of their $k$-th attribute. The attributes are assumed independent, and therefore the overall probability of an edge between $i$ and $j$ is just the product of $\theta_{b_k(i) \ b_k(j)}^{(k)}$.

The Multiplicative Attribute Graph Model (MAGM) of Kim & Leskovec (2010) is also obtained by associating a bit-vector $f(i)$ with a node $i$. However, $f(i)$ need not be the binary representation of $(i - 1)$ as was the case in the KPGM. In fact, the number of nodes $n$ need not even be equal to $2^d$. We simply assume that $f_k(i)$ is a Bernoulli random variable with $P[f_k(i) = 1] = \mu^{(k)}$. In addition to $\Theta$ defined in (4), the model now has additional parameters $\tilde{\mu} := (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(d)})$, and the $(i,j)$-th entry of the edge probability matrix $\Psi$ is written as

$$\Psi_{ij} = \prod_{k=1}^{d} \theta_{f_k(i) \ f_k(j)}^{(k)}.$$  

(7)

The expected number of edges under this model will be denoted $e_M$, and can be calculated using

$$e_M = n^2 \cdot \prod_{k=1}^{d} \left( \sum_{0 \leq a,b \leq 1} \mu^{a+b} (1 - \mu)^{2-a-b} \theta_{ab}^{(k)} \right).$$  

(8)

Note that when $\mu^{(1)} = \mu^{(2)} = \cdots = \mu^{(d)} = 0.5$, we have $e_M = e_K$ (see Figure 4).

### 3. Ball-Dropping Process (BDP)

A naïve but exact method of sampling from KPGM is to generate every entry of adjacency matrix $A$ individually. Of course, such an approach requires $\Theta (n^2)$ computation and does not scale to large graphs. Alternatively, Leskovec et al. (2010) suggest the following stochastic process as an approximate but efficient sampling algorithm (see Figure 1):

- First, sample the number of edges $|E|$ from a Poisson distribution with parameter $e_K$.
- The problem of sampling each individual edge is then converted to the problem of locating the position of a “ball” which will be dropped on a $2^d \times 2^d$ grid. The probability of the ball being located at coordinate $(i,j)$ is proportional to $\Gamma_{ij}$. This problem can be solved in $O(d)$ time by employing a divide-and-conquer strategy (Leskovec et al., 2010). See Figure 1 for a graphical illustration, and Algorithm 1 in Appendix B for the pseudo-code.

If a graph is sampled from the above process, however, there is a nonzero probability that the same pair of nodes is sampled multiple times. Therefore, the process generates multi-graphs while the sample space of KPGM is simple graphs. The above generative process is called a ball-dropping process (BDP), in order to distinguish it from the KPGM distribution. Of course, the two are closely related. We show the following theorem which characterizes the distribution of BDP and clarifies the connection between the two.

**Theorem 2 (Distribution of BDP)** If a multigraph $G$ is sampled from a BDP with parameters $\Theta$ and $d$, then $A_{ij}$ follows an independent Poisson distribution with rate parameter $\Gamma_{ij}$ defined by (3).

**Proof** See Appendix A.1.
distribution with the same parameter $p$. Then, using the Taylor expansion
\[
P[X = 0] = \exp(-p) = (1 - p) + O(p^2)
\]
\[
= P[Y = 0] + O(p^2),
\]
In practice we are interested in large sparse graphs, therefore most $\Gamma_{ij}$ values are close to zero, and the Poisson distribution provides a good approximation.

In fact, this property of the Poisson distribution is often used in statistical modeling of sparse graphs to make both analysis tractable and computation more efficient (see e.g., Karrer & Newman (2011)).

### 3.1. Two Observations

Note that $\exp(-p) \geq 1 - p$ and consequently the probability of an edge not being sampled is higher in the BDP than in the KPGM. Consequently, the BDP generates sparser graphs than exact sampling from KPGM. Leskovec et al. (2010) observed this and recommend sampling extra edges to compensate for this effect. Our analysis shows why this phenomenon occurs.

As the BDP is characterized by a Poisson distribution instead of the Bernoulli, it only requires non-negativity of its parameters. Therefore, for a BDP we do not need to enforce the constraint that every $\theta_{ab}^{(k)}$ parameter is bounded by 1. This extra bit of generality will be found useful in the next section.

### 4. Sampling Algorithm

In the MAGM, each entry $A_{ij}$ of the adjacency matrix $A$ follows a Bernoulli distribution with parameter $\Psi_{ij}$. To efficiently sample graphs from the model, again we approximate $A_{ij}$ by a Poisson distribution with the same parameter $\Psi_{ij}$, as discussed in Section 3.

A close examination of (6) and (7) reveals that KPGM and MAGM are very related. The only difference is that in the case of KPGM the $i$-th node is mapped to the bit vector corresponding to $(i - 1)$ while in the case of MAGM it is mapped to an integer $c_i$ (not necessarily $(i - 1)$) whose bit vector representation is $f(i)$. We will call $c_i$ the color of node $i$ in the sequel\(^2\). The concept of color clarifies the connection between KPGM and MAGM through the following equality
\[
\Psi_{ij} = \Gamma_{c_i c_j}.
\] \[(9)\]

#### 4.1. Problem Transformation

Let $\mathcal{V}_c$ be the set of nodes with color $0 \leq c \leq n - 1$
\[
\mathcal{V}_c := \{i : c_i = c\}.
\]

Instead of sampling the adjacency matrix $A$ directly, we will first generate another matrix $B$, with $B_{cc'}$ defined as
\[
B_{cc'} := \sum_{i \in \mathcal{V}_c} \sum_{j \in \mathcal{V}_{c'}} A_{ij}.
\] \[(11)\]

In other words, $B_{cc'}$ is the number of edges from nodes with color $c$ to nodes with color $c'$. It is easy to verify that each $B_{cc'}$ is a sum of Poisson random variables and hence also follows Poisson distribution (Chapter 13, DasGupta (2011)). Let $\Lambda_{cc'}$ be the rate parameter of the Poisson distribution in $B_{cc'}$, which can be calculated from (9) and (11)
\[
\Lambda_{cc'} = |\mathcal{V}_c| \cdot |\mathcal{V}_{c'}| \cdot \Gamma_{cc'}.
\] \[(12)\]

Given matrix $B$, it is easy to sample the adjacency matrix $A$. Uniformly sampling $B_{cc'}$-number of $(i, j)$ pairs in $\mathcal{V}_c \times \mathcal{V}_{c'}$ for each nonzero entry $B_{cc'}$ of $B$, and incrementing $A_{ij}$ by 1 for each sampled pair will sample $A$ conditioned on $B$. An argument similar to the proof of Theorem 2 can be used to show the validity of such an operation.

\(^2\)Yun & Vishwanathan (2012) call it attribute configuration, but in our setting we think color conveys the idea better.
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Figure 2. (a) Poisson parameter matrix $\Lambda$ of target distribution $B$. (b) Parameter matrix $\Lambda'$ of proposal distribution $B'$. Each entry of $\Lambda'$ must be larger than the corresponding entry in $\Lambda$ for $B'$ to be a valid proposal. (c) The acceptance ratio is obtained by Hadamard (element-wise) division of $\Lambda$ by $\Lambda'$. The acceptance ratio is high when the gap between $\Lambda$ and $\Lambda'$ is small. In all three figures darker cells imply higher values and a white cell denotes a zero value. Parameters $\Theta = (0.7, 0.85; 0.85, 0.9)$, $d = 3$ and $\mu = 0.7$ was used for these plots.

That said, now the question is how to efficiently sample $B$. In Section 4.4, we will efficiently construct another $n \times n$ random matrix $B'$, with $B'_{cc'}$ following an independent Poisson distribution with parameter $\Lambda'_{cc'}$. $B'$ will bound $B$, in the sense that for any $c$ and $c'$ (see Figure 2),

$$\Lambda_{cc'} \leq \Lambda'_{cc'}.$$  \hspace{1cm} (13)

For each nonzero value of $B'_{cc'}$, sampling from the binomial distribution of size $B'_{cc'}$ with parameter $\frac{\Lambda_{cc'}}{\Lambda'_{cc'}}$ will generate a valid $B_{cc'}$. As a filtered Poisson process is still a Poisson process with an adjusted parameter value, this step remains valid (Chapter 13.3, DasGupta [2011]).

To summarize, we will first generate $B'$, use $B'$ to sample $B$, and then convert $B$ to $A$. The time complexity of the algorithm is dominated by the generation of $B'$. See Algorithm 2 of Appendix B for the pseudo-code.

Note that the relation between $B$ and $B'$ is similar to that between target and proposal distribution in accept-reject sampling. While $B$ is the target distribution we want to sample, we first generate a proposal $B'$ and correct each entry $B'_{cc'}$ using acceptance ratio $\frac{\Lambda_{cc'}}{\Lambda'_{cc'}}$. Just like it is important to find a good proposal distribution which compactly bounds the target distribution in accept-reject sampling, we need $B'$ which compactly bounds $B$. The remainder of this section is devoted to show how this can be done.

4.2. Simple Illustrative Proposal

To illustrate the idea behind our construction of $B'$, let us first construct a simple but non-optimal proposal. Let $m$ be the maximum number of nodes with the same color

$$m := \max_{0 \leq c \leq n-1} |V_c|.$$  \hspace{1cm} (14)

Using the notation in (4), if one generates a random matrix $B'$ from BDP with the parameter $\Theta'$ with each component $\Theta_{(k)}$ defined as

$$\Theta_{(k)} := (m)^{2/d} \begin{bmatrix} \theta_{00}^{(k)} & \theta_{01}^{(k)} \\ \theta_{10}^{(k)} & \theta_{11}^{(k)} \end{bmatrix},$$  \hspace{1cm} (15)

then, by calculation we have $\Lambda'_{cc'} = m^2 \Gamma_{cc'}$. From definition (12) and (14), it is obvious that (13) holds

$$\Lambda_{cc'} = |V_c| \cdot |V_{c'}| \cdot \Gamma_{cc'} \leq m^2 \cdot \Gamma_{cc'} = \Lambda'_{cc'},$$  \hspace{1cm} (16)

and hence $B'$ is a valid proposal for $B$.

We now investigate the time complexity of sampling $B'$. Since BDP with parameter $\Theta$ generates $e_K$ number of edges in expectation, $B'$ will generate $m^2 \cdot e_K$ edges in expectation because its BDP parameter is $\Theta' = m^{2/d} \Theta$. As sampling each edge takes $O(d)$ time, the overall time complexity is $O(d \cdot m^2 \cdot e_K)$.

If $\mu^{(1)} = \mu^{(2)} = \cdots = \mu^{(d)} = 0.5$ and $n = 2^d$, Yun & Vishwanathan (2012) showed that $m \leq \log_2 n$ with high probability. Therefore, the overall time complexity of sampling is $O(d \cdot (\log_2 n)^2 \cdot e_K)$.

Roughly speaking, the quilting algorithm of Yun & Vishwanathan (2012) always uses the same $B'$ irrespective of $\mu^{(k)}$’s. When $\mu^{(k)}$’s are not exactly equal to 0.5, $m$ is no longer bounded by $\log_2 n$. To resolve this problem Yun & Vishwanathan (2012) suggest some heuristics. Instead, we construct a more careful proposal which adapts to values of $\mu^{(k)}$.

4.3. Partitioning Colors

To develop a better proposal $B'$, we define quantities similar to $m$ but bounded by $\log_2 n$ with high probability for general $\mu^{(k)}$ values. To do this, we first partition colors into a set of frequent colors $F$ and infrequent colors $I$.

$$F := \{ c : E[|V_c|] \geq 1 \}, \hspace{1cm} (17)$$

$$I := \{ c : E[|V_c|] < 1 \} \setminus F. \hspace{1cm} (18)$$

The rational behind this partitioning is as follows: When $E[|V_c|] \geq 1$, the variance is smaller than that of the mean thus $Var[|V_c|] \leq E[|V_c|]$. On the other hand, when $E[|V_c|] < 1$, then the variance is greater than that of the mean thus $Var[|V_c|] > E[|V_c|]$. Therefore, the frequencies of colors in $F$ and those
Finally, we construct the proposal distribution. The matrix \( B' \) is the sum of four different BDP matrices
\[
B' = B^{(F,F)} + B^{(F,I)} + B^{(I,F)} + B^{(I,I)}.
\]

Intuitively, \( B^{(F,F)} \) concentrates on covering entries of \( B \) between frequent colors, while \( B^{(I,I)} \) spreads out its parameters to ensure that every entry of \( B \) is properly covered. On the other hand, \( B^{(F,I)} \) and \( B^{(I,F)} \) covers entries between a frequent color and other colors. Figure 3 visualizes the effect of each component.

For \( A, B \in \{F, I\} \), let \( \tilde{\Theta}^{(A)} \) and \( d \) be parameters of BDP \( B^{(A,B)} \). Following notation in (4) again, the \( k \)-th component of these matrices are defined as
\[
\Theta^{(F,F)(k)} := (n m_{F,F})^{\frac{1}{2}} 
\end{equation}
\[
\Theta^{(F,I)(k)} := (n m_{F,I} m_{I,F})^{\frac{1}{2}} 
\end{equation}
\[
\Theta^{(I,F)(k)} := (n m_{I} m_{F})^{\frac{1}{2}} 
\end{equation}
\[
\Theta^{(I,I)(k)} := (m_{I})^{\frac{1}{2}} 
\end{equation}


The following theorem proves that \( B' \) is a valid proposal. That is, \( B' \) bounds \( B \) in the sense discussed in Section 4.1, and therefore given \( B' \) we can sample \( B \).

**Theorem 4 (Validity of Proposal)** For any \( c \) and \( c' \) such that \( 0 \leq c, c' \leq n - 1 \), we have
\[
A_{cc'} \leq A_{cc''}. \tag{22}
\]

**Proof** See Appendix A.3.

Also see Algorithm 2 of Appendix B for the pseudo-code of the overall algorithm.

**4.5. Time Complexity**

As it takes \( \Theta(d) \) time to generate each edge in BDP, let us calculate the expected number of edges \( B' \) will generate. The following quantities similar to (5) and (8) will be found useful
\[
e_{MK} = n \prod_{k=1}^{d} \left( \sum_{0 \leq a,b \leq 1} \mu^a (1 - \mu)^{1-a} \theta_{ab}^{(k)} \right), \tag{23}
\]
\[
e_{KM} = n \prod_{k=1}^{d} \left( \sum_{0 \leq a,b \leq 1} \mu^b (1 - \mu)^{1-b} \theta_{ab}^{(k)} \right). \tag{24}
\]

In general, \( e_{MK} \) and \( e_{KM} \) are not necessarily lower or upper bounded by \( e_M \) or \( e_K \). However, for many of known parameter values for KPGM and MAGM, especially those considered in Kim & Leskovec (2010) and Yun & Vishwanathan (2012), we empirically observe that they are indeed between \( e_M \) and \( e_K \).

\[
\min \{e_M, e_K\} \leq e_{MK}, e_{KM} \leq \max \{e_M, e_K\}. \tag{25}
\]

see Figure 4 for a graphical illustration.

From straightforward calculation, \( B^{(F,F)}, B^{(F,I)}, B^{(I,F)} \) and \( B^{(I,I)} \) generates \( m_{F,F} e_M, m_{F,I} m_{I,F} e_M, \) and \( m_{I} m_{F} e_M, m_{I} m_{F} e_M, \) respectively.
Theorem 3, the overall time complexity is $O \left( d \cdot (\log n)^2 \right)$ with high probability. When (25) holds, it can be further simplified to $O \left( d \cdot (\log n)^2 \cdot (e_K + e_M) \right)$. Note that $d$ is also usually chosen to be $d \leq \log n$. This implies that the time complexity of the whole algorithm is almost linear in the number of expected edges in MAGM and an equivalent KPGM.

Note that the time complexity of algorithm in Yun & Vishwanathan (2012) is at least $O(d \cdot e_K)$ and attains the best guarantee of $O(d (\log n)^2 e_K)$ when $e_M = e_K$. When (25) holds, therefore, our algorithm is at least as efficient as their algorithm.

4.6. Combining two Algorithms

Note that one can combine our algorithm and the algorithm of Yun & Vishwanathan (2012) to get improved performance. For both algorithms, it only takes $O(nd)$ time to estimate the expected running time. Thus one can always select the best algorithm for a given set of parameter values.

5. Experiments

We empirically evaluated the efficiency and scalability of our sampling algorithm. Our experiments are designed to answer the following questions: 1) How does our algorithm scale as a function of $e_M$, the expected number of edges in the graph? 2) What is the advantage of using our algorithm compared to that of Yun & Vishwanathan (2012)?

Our algorithm is implemented in C++ and will be made available for download from http://anonymous. For quilting algorithm, we used the original implementation of Yun & Vishwanathan (2012) which is also written in C++ and compiled with the same options. All experiments are run on a machine with a 2.1 GHz processor running Linux.

Following Yun & Vishwanathan (2012), we uniformly set $n = 2^d$, and used the same $\Theta$ matrices and $\mu$ values at all levels: i.e., $\Theta = \Theta^{(1)} = \Theta^{(2)} = \cdots = \Theta^{(d)}$ and $\mu = \mu^{(1)} = \cdots = \mu^{(d)}$. Furthermore, we experimented with the following $\Theta$ matrices used by Kim & Leskovec (2010) and Moreno & Neville (2009) to model real world graphs:

$$
\Theta_1 = \begin{bmatrix} 0.15 & 0.7 \\ 0.7 & 0.85 \end{bmatrix} \quad \text{and} \quad \Theta_2 = \begin{bmatrix} 0.35 & 0.52 \\ 0.52 & 0.95 \end{bmatrix}.
$$

Figure 5 shows the running time of our algorithm vs

\[ \text{(a) } \mu = 0.2 \]

\[ \text{(b) } \mu = 0.3 \]

\[ \text{(c) } \mu = 0.5 \]

\[ \text{(d) } \mu = 0.7 \]

\[ \text{(e) } \mu = 0.9 \]

Figure 5. Comparison of running time (in seconds) of our algorithm vs the quilting algorithm of Yun & Vishwanathan (2012) as a function of expected number of edges $e_M$ for two different values of $\Theta$ and five values of $\mu$. 
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(a) $\mu \leq 0.5$

(b) General Value of $\mu$

Figure 6. Comparison of running time (in seconds) of our algorithm vs the quilting algorithm of Yun & Vishwanathan (2012) as a function of $\mu$ for two different values of $\Theta$ and $n = 2^{17}$.

Yun & Vishwanathan (2012) as a function of expected number of edges $e_{M}$. Each experiment was repeated ten times to obtain error bars. As our algorithm has theoretical time complexity guarantee, irrespective of $\mu$ the running time is almost linear in $e_{M}$. On the other hand, Yun & Vishwanathan (2012) shows superb performance when dealing with relatively dense graphs ($\mu > 0.5$), but when dealing with sparser graphs ($\mu < 0.5$) our algorithm outperforms.

Figure 6 shows the dependence of running time on $\mu$ more clearly. In our parameter setting, the number of expected edges is an increasing function of $\mu$ (see Figure 4 for $d = 1$). As the time complexity of our algorithm depends on $e_{M}$, the running time of our algorithm increases accordingly as $\mu$ increases. In the case of quilting algorithm, however, the running time is almost symmetric with respect to $\mu = 0.5$. Thus, when $\mu < 0.5$ it is relatively inefficient, compared to when $\mu \geq 0.5$.

6. Conclusion

We introduced a novel and efficient sampling algorithm for the MAGM. The run-time of our algorithm depends on $e_{K}$ and $e_{M}$. For sparse graphs, which are primarily of interest in applications, the value of $e_{M}$ is well bounded, and our method is able to outperform the quilting algorithm. However, when $\mu$ is greater than 0.5, MAGM produces dense graphs. In this case the heuristics of Yun & Vishwanathan (2012) work well in practice. One can combine the two algorithms to produce a fast hybrid algorithm. Theoretical investigation of the quilting algorithm and its heuristics may provide more insights into improving both algorithms.

For the parameter settings we studied the corresponding KPGM graphs are sparse and can be sampled efficiently. However, for some values of $\Theta$ the corresponding KPGM graphs can become dense and difficult to sample. Removing dependency of time complexity on $e_{K}$ remains an open question, and a focus of our future research.

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A. Technical Proofs (not included in 8 page limit)

A.1. Proof of Theorem 2

**Proof** By conditioning on the number of edges $|\mathcal{E}|$, the probability mass function can be written as

$$P[A] = P[|\mathcal{E}|] \cdot P[A | |\mathcal{E}|].$$

Recall that the marginal distribution of $|\mathcal{E}|$ follows Poisson distribution with rate parameter $\lambda_K$. Using (5) and the definition of a Poisson probability mass function,

$$P[|\mathcal{E}|] = \exp \left( -n \sum_{i,j=1}^{n} \Gamma_{ij} \right) \frac{(\sum_{i,j=1}^{n} \Gamma_{ij})^{|\mathcal{E}|}}{|\mathcal{E}|!}.$$  \hspace{1cm} (27)

On the other hand, the conditional distribution of $A$ given $|\mathcal{E}|$ is defined by the multinomial distribution, and its probability mass function is given by

$$P[A | |\mathcal{E}|] = \binom{|\mathcal{E}|}{A_{1,1},A_{1,2},\ldots,A_{n,n}} \prod_{i,j=1}^{n} \left( \frac{\Gamma_{ij}}{\sum_{i,j=1}^{n} \Gamma_{ij}} \right)^{A_{ij}}.$$ \hspace{1cm} (28)

where $\binom{|\mathcal{E}|}{A_{1,1},A_{1,2},\ldots,A_{n,n}}$ is the multinomial coefficient. By definition $|E| := \sum_{i,j=1}^{n} A_{ij}$ and after some simple algebra, we have

$$P[A] = \prod_{i,j=1}^{n} \exp (-\Gamma_{ij}) \frac{\Gamma_{ij}^{A_{ij}}}{A_{ij}!}. \hspace{1cm} (29)$$

By the factorization theorem, every $A_{ij}$ is independent of each other. Furthermore, $A_{ij}$ follows a Poisson distribution with rate parameter $\Gamma_{ij}$. \hfill \Box

A.2. Proof of Theorem 3

**Proof** For $c \in \mathcal{F}$, we apply the multiplicative form of Hoeffding-Chernoff inequality (Chapter 35.1, DasGupta (2011)) to get

$$P[|V_c| \geq \log_2 n \cdot \mathbb{E}[|V_c|]] \leq \left( \frac{\exp (\log_2 n - 1)}{\log_2 n} \right)^{|\mathbb{E}[V_c]|} \hspace{1cm} (30)$$

$$\leq \left( \frac{\exp (\log_2 n - 1)}{\log_2 n \log_2 n} \right)^{|\mathbb{E}[V_c]|}.$$ \hspace{1cm} (31)
for large enough \( n \). Then using the union bound

\[
P \left[ \bigcup_{c \in \mathcal{C}} |\mathcal{V}_c| \geq \log_2 n \cdot \mathbb{E} [|\mathcal{V}_c|] \right] \leq \sum_{c \in \mathcal{C}} P \left[ |\mathcal{V}_c| \geq \log_2 n \cdot \mathbb{E} [|\mathcal{V}_c|] \right]
\]

(32)

\[
\leq n \cdot \left( \frac{\exp(\log_2 n - 1)}{(\log_2 n)^{\log_2 n}} \right) \to 0
\]

(33)

as \( n \to \infty \). For \( c \in \mathcal{C} \), on the other hand, we apply the additive form of Hoeffding-Chernoff inequality:

\[
P \left[ |\mathcal{V}_c| \geq \log_2 n \right] \leq \left( \frac{\mathbb{E} [|\mathcal{V}_c|]}{\log_2 n} \right)^{\log_2 n} \left( \frac{1 - \mathbb{E} [|\mathcal{V}_c|]}{\log_2 n} \right)^{-\log_2 n}
\]

(34)

\[
\leq n \left( \frac{1}{\log_2 n} \cdot \frac{1 - \log_2 n/n}{1 - 1/n} \right)^{\log_2 n} .
\]

(35)

Using union bound again,

\[
P \left[ \bigcup_{c \in \mathcal{C}} |\mathcal{V}_c| \geq \log_2 n \right] \to 0 .
\]

(36)

\[\blacksquare\]

### B. Pseudo-Code of Algorithms

**Algorithm 1** Description of Ball-Dropping Process

Function BDP

- **Input:** parameter \( \tilde{\Theta} \)
- **Output:** set of edges \( \mathcal{E} \)

\[\mathcal{E} \leftarrow \emptyset\]

\[e_k \leftarrow \prod_{k=1}^{d} \left( \theta_{0(k)} + \theta_{1(k)} + \theta_{11(k)} \right)\]

Generate \( X \sim \text{Poisson}(e_k) \).

for \( x = 1 \) to \( X \)

- \( S_{\text{start}}, T_{\text{start}} \leftarrow 1 \)

  for \( k \leftarrow 1 \) to \( d \)

  - Sample \( (a, b) \sim \mathcal{U}_{\mathcal{F}} \)
  - \( S_{\text{start}} \leftarrow S_{\text{start}} + \frac{ab}{2} \)
  - \( T_{\text{start}} \leftarrow T_{\text{start}} + \frac{b}{2} \)
  - \( S_{\text{end}} \leftarrow S_{\text{end}} - \left( 1 - a \right) n \)
  - \( T_{\text{end}} \leftarrow T_{\text{end}} - \left( 1 - b \right) n \)

end for

- # We have \( S_{\text{start}} = S_{\text{end}}, T_{\text{start}} = T_{\text{end}} \)

- \( \mathcal{E} \leftarrow \mathcal{E} \cup \{ (S_{\text{start}}, T_{\text{start}}) \} \)

end for

\[\blacksquare\]

### A.3. Proof of Theorem 4

**Proof** Let \( \Lambda^{(A, B)} \) be rate parameter matrix of \( B^{(A, B)} \).

From the definition (3),

\[
\Lambda^{(A, B)} := \Theta^{(A, B)(1)} \otimes \Theta^{(A, B)(2)} \otimes \cdots \otimes \Theta^{(A, B)(d)}.
\]

(37)

From (21) and (37), it is easy to verify that

\[
\Lambda_{cc'}^{(F, F)} = (m_{F}) \cdot \mathbb{E} [|\mathcal{V}_c|] \cdot \mathbb{E} [|\mathcal{V}_{c'}|] \cdot \Gamma_{cc'} \quad \text{for} \quad c \in \mathcal{F}, c' \in \mathcal{F},
\]

\[
\Lambda_{cc'}^{(I, F)} = m_{I} \cdot (m_{F} \cdot \mathbb{E} [|\mathcal{V}_c|] \cdot \Gamma_{cc'}) \quad \text{for} \quad c \in \mathcal{F}, c' \in \mathcal{I},
\]

\[
\Lambda_{cc'}^{(I, F)} = m_{I} \cdot (m_{F} \cdot \mathbb{E} [|\mathcal{V}_c|] \cdot \Gamma_{cc'}) \quad \text{for} \quad c \in \mathcal{I}, c' \in \mathcal{F},
\]

\[
\Lambda_{cc'}^{(I, I)} = (m_{I})^2 \Gamma_{cc'} \quad \text{for} \quad c \in \mathcal{I}, c' \in \mathcal{I}.
\]

(38)

Using (12) and (19) obtains

\[
\Lambda_{cc'} \leq \Lambda_{cc'}^{(A, B)} \leq \Lambda_{cc'},
\]

for any \( A, B \in \{ \mathcal{F}, \mathcal{I} \} \), \( c \in A \), and \( c' \in B \).
Algorithm 2 BDP Sampler of MAGM

Input: parameters $\tilde{\Theta}, \tilde{\mu}$
Output: set of edges $E$

$E \leftarrow \emptyset$

for $A$ in $\{F, I\}$ do
  for $B$ in $\{F, I\}$ do
    for $(c, c')$ in $\text{BDP}(\tilde{\Theta}(A, B))$ do
      if $c \in A$ and $c' \in B$ then
        Generate $u \sim \text{Uniform}(0, 1)$.
        if $u \leq \frac{\Lambda_{ij}}{\Lambda_{ij}}$ then
          Sample $i$ uniformly from $V_c$.
          Sample $j$ uniformly from $V_{c'}$.
          $E \leftarrow E \cup \{(i, j)\}$.
        end if
      end if
    end for
  end for
end for