ON GRADED NIL-GOOD RINGS

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Abstract

In this paper we introduce and study the notion of a graded nil-good ring which is graded by a group. We investigate extensions of graded nil-good rings to graded group rings, Further, we discuss graded matrix ring extensions and trivial extensions of graded nil-good rings. Furthermore, we show that the class of graded rings which are nil-good and the class of graded nil-good rings are not comparable. Moreover, we discuss the question of when the nil-good property of the component, which corresponds to the identity element the grading group, implies that the whole graded ring is graded nil-good is also treated.

keywords: Graded rings and modules, nil-good rings, group rings, matrix rings, trivial ring extension.

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1 Introduction

In 1977, W.K Nicholson has introduced in [20] a new class of rings called clean rings whose every element can be written as a sum of an idempotent and a unit. Since then, many works have been done about rings in which elements can be written as a sum of two elements with certain properties. In particular, some authors have investigated rings in which elements can be written as a sum of a nilpotent element and an element with a certain property. As an example of these rings there is, nil-clean rings, fine rings and nil-good rings introduced respectively in [8], [4] and [7].

Some authors have given a graded versions of some of the previous class of rings, such as Graded nil-clean rings introduced in [12], and graded 2-nil-good rings introduced in [11]. In this work, we define and study graded nil-good rings as a graded version of nil-good rings introduced in [7]. In [7], a nil-good ring is defined as a ring whose every element is either nilpotent or a sum of a unit and a nilpotent. This class of rings is a generalization of the notion fine rings (see [4]), whose every nonzero element can be written as a sum of a unit and a nilpotent element.

In this paper, graded nil-good ring is defined as a group graded ring whose every homogeneous element is either nilpotent or can be written as a sum of a homogeneous unit and a homogeneous nilpotent. We first give some properties of graded nil-good rings which represent graded versions of results of nil-good rings. Moreover, we discuss when the graded group ring is graded nil-good. These lead to an interesting question of how the graded nil-good property of a group graded ring depends on the nil-good property of the component which corresponds to the identity element of the grading group. It is proved that nil-good property of the component corresponding to the identity element of the grading group does not imply the graded nil-good property of the whole graded ring in general. However, under some extra hypothesis, this implication becomes true. Finally, we give a sufficient condition for the graded matrix ring over a graded commutative nil-good ring to be graded nil-good.

2 Preliminaries

All rings are assumed to be associative with identity. If $R$ is a ring, then, $J(R)$ denotes the Jacobson radical of $R$, $U(R)$ is the multiplicative group of
units of $R$, and $\text{Nil}(R)$ denotes the set of nilpotent elements of $R$.

Next, we recall the notions of a group graded ring and module, and how the group ring over a group graded ring can be graded. For more details on graded rings theory, we refer to [14] and [19].

Let $R$ be a ring, $G$ a group with identity element $e$, and let $\{R_g\}_{g \in G}$ be a family of additive subgroups of $R$. $R$ is said to be $G$-graded if $R = \bigoplus_{g \in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The set $H = \bigcup_{g \in G} R_g$ is called the homogeneous part of $R$, elements of $H$ are called homogeneous, and subgroups $R_g$ ($g \in G$) are called components. If $a \in R_g$, then we say that $a$ has the degree $g$.

A right ideal (left, two-sided) $I$ of a $G$-graded ring $R = \bigoplus_{g \in G} R_g$ is called homogeneous or graded if $I = \bigoplus_{g \in G} I \cap R_g$. If $I$ is a two-sided homogeneous ideal, then $R/I$ is a $G$-graded ring with components $(R/I)_g = R_g/I \cap R_g$. A graded ring $R$ is graded-nil if every homogeneous element of $R$ is nilpotent, and a homogeneous ideal $I$ is called graded-nil if every homogeneous element of $I$ is nilpotent.

If $R = \bigoplus_{g \in G} R_g$ is a $G$-graded ring, then a $G$-graded $R$-module is an $R$-module $M$ such that $M = \bigoplus_{g \in G} M_g$, where $M_g$ are additive subgroups of $M$, and such that $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. A submodule $N$ of a $G$-graded $R$-module $M = \bigoplus_{g \in G} M_g$ is called homogeneous if $N = \bigoplus_{g \in G} N \cap M_g$.

A homogeneous right ideal $M$ of a graded ring $R$ is said to be graded-maximal right ideal if it is contained in no other proper homogeneous right ideal of $R$. A ring $R$ is graded-local if it has a unique graded-maximal right ideal.

The graded Jacobson radical $J^g(R)$ of a $G$-graded ring $R$ is defined to be the intersection of all graded-maximal right ideals of $R$. Moreover, $J^g(R)$ is a homogeneous two-sided ideal (see for instance [19 Proposition 2.9.1]).

Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring. According to [18], we have that the group ring $R[G]$ is $G$-graded with the $g$-component $(R[G])_g = \sum_{h \in G} R_{gh}^{-1}h$ and with the multiplication defined by $(r_gg')(r_hh') = r_{gh}(h^{-1}g/\text{hh'})$, where $g, g', h, h' \in G$ and $r_g \in R_g$, $r_h \in R_h$.

All the group rings in this paper, if observed as graded rings, are assumed to be graded in one of the above described ways.

Let $A$ be a commutative ring, $E$ an $A$-module and $R := A \times E$ the set of pairs $(a, e)$ with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. $R$ is called the trivial ring extension of $A$ by $E$. Considerable work has been concerned with trivial ring extensions of commutative rings. Part of it has been summarized in Glaz’s book [9] and Huckaba’s book [10].
In this paper, we consider the same construction for noncommutative rings. In [2], it has been proved that if \( R = A \propto E \) where \( A \) is a commutative ring, then \( U(R) = U(A) \propto E \) and \( \text{Nil}(R) = \text{Nil}(A) \propto E \). We can check easily that these properties hold true even if \( A \) is a noncommutative ring.

Let \( A \) be a \( G \)-graded ring and \( E \) a \( G \)-graded \( A \)-module. According to [2, Section 3], the trivial ring extension \( R = A \propto E \) is \( G \)-graded where \( R_g = A_g \oplus E_g \).

If \( R \) is a \( G \)-graded ring and \( n \) a natural number, then the matrix ring \( M_n(R) \) can be seen as a \( G \)-graded ring in the following manner. Let \( \sigma = (g_1, \ldots, g_n) \in G^n \), \( \lambda \in G \) and \( M_n(R)_\lambda(\sigma) = (a_{ij})_{n \times n} \), where \( a_{ij} \in R_{g_i \lambda g_j^{-1}} \), \( i, j \in \{1, \ldots, n\} \). Then, \( M_n(R) = \bigoplus_{\lambda \in G} M_n(R)_\lambda(\sigma) \) is a \( G \)-graded ring with respect to usual matrix addition and multiplication. This ring is denoted by \( M_n(R)(\sigma) \).

\[
R_\lambda \ R_\lambda \ \ldots \ R_\lambda \\
R_\lambda \ R_\lambda \ \ldots \ R_\lambda \\
\vdots \ \vdots \ \ldots \ \vdots \\
R_\lambda \ R_\lambda \ \ldots \ R_\lambda
\]

Note that if \( \sigma = (e, e, \ldots, e) \in G^n \), then \( M_n(R)_\lambda(\sigma) = \begin{pmatrix} R_\lambda & R_\lambda & \ldots & R_\lambda \\ R_\lambda & R_\lambda & \ldots & R_\lambda \\ \vdots & \vdots & \ldots & \vdots \\ R_\lambda & R_\lambda & \ldots & R_\lambda \end{pmatrix} \).

3 Grade nil-good rings

Let \( G \) be a group with identity \( e \).

**Definition 3.1.** A homogeneous element of a \( G \)-graded ring is said to be graded nil-good if it is either nilpotent or it can be written as a sum of a homogeneous unit and a homogeneous nilpotent. A \( G \)-graded ring is said to be graded nil-good if every of its homogeneous elements is graded nil-good.

**Example 3.1.** Let \( G = \{e, g\} \) be a cyclic group of order 2 and \( R := \mathbb{Z}_2[X]/(X^2) \). We have that \( R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 X \) is a \( G \)-graded ring. Since \( \mathbb{Z}_2 \) is a nil-good ring (see [2, Example 1]) and every element of \( \mathbb{Z}_2 X \) is nilpotent, then \( R \) is a graded nil-good ring.

**Remark 3.1.** Let \( R = \bigoplus_{g \in G} R_g \) be a \( G \)-graded ring. If \( a = u + n \) where \( u \) (resp. \( n \)) is a homogeneous unit (resp. nilpotent), then \( a, u \) and \( n \) are all of the same degree. Indeed, assume that the degree of \( a \) is \( g \). If we suppose that the degree of \( u \) is not \( g \) we will have \( a = 0 \) or \( a = n \). Both cases lead to a contradiction. Hence, the degree of \( u \) is \( g \), and the degree of \( n \) is also \( g \).
Proposition 3.1. Let $A$ be a nil-good ring. Then, the Laurent polynomial ring $R := \bigoplus_{n \in \mathbb{Z}} AX^n$ (with $R_0 = A$) is a $\mathbb{Z}$-graded nil-good ring.

Proof. Let $aX^n$ ($n \in \mathbb{Z}$) be a homogeneous element of $R$. Since $A$ is nil-good, then $a = u + b$ where $u \in U(A) \cup \{0\}$ and $b \in \text{Nil}(A)$, and so $aX^n = uX^n + bX^n$, since $uX^n \in U(R) \cup \{0\}$ and $bX^n \in \text{Nil}(R)$ and both are homogeneous, then $R$ is graded nil-good.

Example 3.2. According to the previous proposition, the $\mathbb{Z}$-graded domain $R = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}_2 X^n$ is graded nil-good.

Let us notice that $R$ is not a nil-good ring since $1 + X$ is not nil-good element. Indeed, if we suppose that $1 + X$ is nil-good. Since $R$ is a domain then $\text{Nil}(R) = (0)$, hence, $1 + X$ must be a unit in $R$. According to [17, page 1], we have that $1 + X$ is a homogeneous which is a contradiction.

Next example shows that the class of graded rings which are nil-good and the class of graded nil-good rings are not comparable.

Example 3.3. Let $G = \{e, g\}$ be a cyclic group of order 2, and $R := M_2(\mathbb{Z}_2)$. Thus, $R = \begin{pmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_2 \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & 0 \end{pmatrix}$ is a $G$-graded ring. $R$ is not a graded nil-good ring since the homogeneous element $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R_e$ is not graded nil-good.

But $R$ is a nil-good ring (see [17, Example 2]), hence a nil-good ring is not necessarily a graded nil-good ring. According to Example 3.2, we deduce that the class of graded rings which are nil-good and the class of graded nil-good rings are not comparable.

Proposition 3.2. 

1. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded nil-good ring. Then, $R_e$ is a nil-good ring.

2. Let $R$ be a $G$-graded commutative ring. If $R$ is graded nil-good, then every homogeneous element of $R$ is either unit or nilpotent.

3. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded nil-good ring. If $U(R) = U(R_e)$, then every $x \in R_g$ ($g \neq e$) is nilpotent.

Proof. 

1. Let $a \in R_e$ be a non nilpotent element. Since $R$ is graded nil-good, then $a = u + n$ where $u \in U(R) \cap R_g$ and $n \in \text{Nil}(R) \cap R_h$. 

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for some \( g, h \in G \). Suppose that \( g \neq e \), in this case we will have \( a \) is nilpotent which is a contradiction. Hence, \( g = e \), and so \( u, n \in R_e \). Thus, \( R_e \) is a nil-good ring.

2. Since \( R \) is a commutative ring, then by [3, page 10] we have that \( U(R) + \text{Nil}(R) = U(R) \). Hence, if \( R \) is commutative graded nil-good, then every homogeneous element of \( R \) is either nilpotent or unit.

3. Let \( x \in R_g \) where \( g \neq e \). Since \( R \) is graded nil-good, then \( x = u + n \) where \( u \in U(R_e) \cup \{0\} \) and \( n \) is homogeneous nilpotent element of \( R \). Suppose that \( u \neq 0 \), by comparing degrees we have that \( x = 0 \) or \( x = n \), this implies that \( u = -n \) or \( u = 0 \) which is a contradiction since \( u \in U(R_e) \). Hence, \( u = 0 \) and so \( x = n \in \text{Nil}(R) \).

\[ \blacksquare \]

**Proposition 3.3.** If \( R \) is a \( G \)-graded nil-good ring with \( U(R) = \{1\} \), then \( R = R_e \cong \mathbb{Z}_2 \).

**Proof.** Since \( 1 + \text{Nil}(R) \subseteq U(R) \), then \( \text{Nil}(R) = \{0\} \). On the other hand, since \( U(R) = U(R_e) \) then by Proposition 3.2 (3), each \( a \in R_g \) is nilpotent where \( g \neq e \). Since, \( \text{Nil}(R) = 0 \) then every \( a = 0 \) for all \( a \in R_e \) where \( g \neq e \). Hence, \( R_g = 0 \) for all \( g \neq e \), and so \( R = R_e \). On the other hand, \( R_e \) is nil-good by proposition 3.2, therefore by [7, Proposition 2.3] we have that \( R_e \cong \mathbb{Z}_2 \), which completes the proof.

\[ \blacksquare \]

In [7] it is proved that \( R \) is a nil-good ring if and only if \( R/I \) is nil-good, whenever \( I \) is a nil ideal of \( R \). Here we have the following result.

**Theorem 3.1.** Let \( R \) be a \( G \)-graded ring and \( I \) a graded-nil ideal of \( R \). Then \( R \) is a graded nil-good ring if and only if \( R/I \) is graded nil-good.

**Proof.** If \( R \) is graded nil-good, then \( R/I \) is also graded nil-good as a graded homomorphic image if \( R \).

Conversely, let \( R/I \) be a graded nil-good ring and \( x \in R_g \) where \( g \in G \). We have two cases:

- **Case 1:** \( x \) is nilpotent in \( R/I \). Since \( I \) is graded-nil, we have \( x \) is also nilpotent in \( R \).
- **Case 2:** \( x = \bar{u} + \bar{n} \), where \( \bar{u} \) is homogeneous unit of \( R/I \) and \( n \) is nilpotent homogeneous element of \( R \) (case 1). Since \( I \) is graded-nil ideal, it is contained
Proposition 3.4. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded nil-good ring of finite support, then $J^g(R)$ is a graded-nil ideal.

Proof. Since $R$ is graded nil-good, then $R_e$ is a nil-good ring. According to [7, Proposition 2.5] applied to the ring $R_e$, $J(R_e)$ is nil. Now, Corollary 2.9.3 in [19] implies that $J(R_e) = J^g(R_e)$. Therefore, if $a \in J^g(R_e)$ then it is nilpotent. On the other hand, if $a \in J^g(R) \cap R_e$ where $g \neq e$. Since the support of $R$ is finite then by [19, Corollary 2.9.4] $J^g(R) \subseteq J(R)$, hence $a \in J(R)$. Now, suppose that $a$ is not nilpotent, then $a = n + u$ where $n \in \text{Nil}(R)$ and $u \in U(R)$ (since $R$ is graded nil-good). Therefore, $a = -u(1 - u^{-1}a)$. Hence, $a$ is a unit of $R$ which is a contradiction since $a \in J(R)$. Finally, $a$ is nilpotent and so $J^g(R)$ is a graded-nil ideal. 

Corollary 3.1. Let $R$ be a $G$-graded ring such of finite support. Then $R$ is graded nil-good if and only if $J^g(R)$ is graded-nil and $R/J^g(R)$ is graded nil.

Remark 3.2. By Theorem 3.1, if $J^g(R)$ is graded-nil and $R/J^g(R)$ is graded nil-good, then $R$ is graded nil-good for any cardinality of the support of $R$.

Lemma 3.1. Let $R = \bigoplus_{g \in G} R_g$ be a commutative $G$-graded nil-good ring. Then $J^g(R)$ is a graded-nil ideal.

Proof. Since $R$ is commutative graded nil-good, then by Proposition 3.2 (2), every homogeneous element of $R$ is either nilpotent or unit. Since $J^g(R)$ is a proper ideal of $R$, then every homogeneous element of $J^g(R)$ is nilpotent. Hence, $J^g(R)$ is graded-nil.

Corollary 3.2. Let $R$ be a $G$-graded commutative ring. Then $R$ is graded nil-good if and only if $J^g(R)$ is graded-nil and $R/J^g(R)$ is graded nil-good.

Proposition 3.5. Let $R$ be a $G$-graded-local ring of finite support. Then, $R$ is graded nil-good if and only if $J^g(R)$ is a graded-nil ideal.

Proof. The necessity follows directly from Lemma 3.1. To prove the sufficiency, given a homogeneous element $r$, we have that $r \in U(R)$ or $r \in J^g(R)$. In the second case, $r$ will be nilpotent since $J^g(R)$ is graded-nil. Hence, every homogeneous element is either nilpotent or unit. Thus, $R$ is graded nil-good.
4 Extensions of graded nil-good rings

4.1 Group rings and trivial ring extensions

In this subsection we investigate graded nil-good property of graded group rings and trivial ring extensions. However, we establish some sufficient conditions for a group ring to be graded nil-good.

First, we investigate the trivial ring extensions of graded nil-good and nil-good rings.

**Theorem 4.1.** Let $A$ be a $G$-graded ring, let $E$ be a $G$-graded $A$-module, and let $R := A \ltimes E$ be the trivial ring extension of $A$ by $E$. Then, $A$ is graded nil-good if and only if so is $R$.

**Proof.** $\Leftarrow$ Assume that $R$ is graded nil-good. By [2, Theorem 3.1], we have that $A \cong \frac{R}{(0 \times E)}$. Hence, $A$ is graded nil-good as a graded homomorphic image of $R$.

$\Rightarrow$ Assume that $A$ is graded nil-good. By [2, Theorem 3.2] we have that $\text{Nil}(R) = \text{Nil}(A) \ltimes E$, and according to [2, Theorem 3.7], we have that $U(R) = U(A) \ltimes E$. Now, let $(a, e)$ be a homogeneous element of $R$. Since $A$ is graded nil-good, then either $a \in \text{Nil}(A)$ or $a = u + n$ where $u \in U(A)$ and $n \in \text{Nil}(A)$. Now, if $a \in \text{Nil}(A)$, then $(a, e) \in \text{Nil}(R)$. In the second case, we have that $(a, e) = (u, e) + (n, 0)$. Since $(u, e) \in U(R)$ and $(n, 0) \in \text{Nil}(R)$, then $(a, e)$ is graded nil-good. Finally, $R$ is graded nil-good.

$\blacksquare$

**Theorem 4.2.** Let $A$ be a ring, let $E$ be an $A$-module, and let $R := A \ltimes E$ be the trivial ring extension of $A$ by $E$. Then, $R$ is nil-good if and only if so is $A$.

**Proof.** The same as the proof of Theorem 4.1.

$\blacksquare$

Now, we are able to give a class of rings which are graded nil-good but not nil-good.

**Example 4.1.** Let $A$ be a graded nil-good ring that is not nil-good (e.g Example 3.2), let $E$ be any graded $A$-module, and let $R := A \ltimes E$. According to Theorem 4.1 and Theorem 4.2, $R$ is a graded nil-good ring but not a nil-good ring.
Next we deal with the graded nil-good property of graded group rings. We recall that if $G$ is a group, and $H$ a normal subgroup of $G$, then a $G$-graded ring $R = \bigoplus_{g \in G} R_g$ can be viewed as $G/H$-graded ring $R = \bigoplus_{C \in G/H} R_C$ where $R_C = \bigoplus_{x \in C} R_x$ (see, for instance, [14, 19]).

**Theorem 4.3.** Let $R$ be a $G$-graded ring, where $G$ is a locally finite $p$-group, and let $H$ be a normal subgroup of $G$. Also, let us assume that $p$ is nilpotent in $R$. If $R$ is graded nil-good as a $G/H$-graded ring, then $R[H]$ is graded nil-good as a $G/H$-graded ring.

**Proof.** Following the proof of Theorem 2.3 in [21], we may assume that $H$ is a finite $p$-group. According to [19, page 180], the augmentation mapping $R[H] \rightarrow R$, given by $\sum_{h \in H} r_h h \mapsto \sum_{h \in H} r_h$, where $R[H]$ is considered as a $G/H$-graded ring, is degree-preserving. Therefore, the kernel of the augmentation mapping, that is the augmentation ideal $\Delta(R[H])$, is homogeneous. This means that $R[H]/\Delta(R[H])$ is a $G/H$-graded ring. Moreover, $R[H]/\Delta(R[H])$ and $R$ are isomorphic as a $G/H$-graded rings. Hence, $R[H]/\Delta(R[H])$ is graded nil-good. Now, since $p$ is nilpotent, by [6, Theorem 9] we have that $\Delta(R[H])$ is nilpotent, and therefore graded-nil. Applying Theorem 3.1 completes the proof.

**Corollary 4.1.** Let $R$ be a $G$-graded ring, where $G$ is a locally finite $2$-group, and let $H$ be a normal subgroup of $G$. Also, let us assume that $R_e$ is a nil-clean ring. If $R$ is graded nil-good as a $G/H$-graded ring, then $R[H]$ is graded nil-good as a $G/H$-graded ring.

**Proof.** Since $R_e$ is by assumption nil-clean, we have that $2$ is nilpotent by [8, Proposition 3.14]. Now, the previous theorem completes the proof.

**Corollary 4.2.** Let $R$ be a $G$-graded ring, where $G$ is a locally finite $p$-group. Also, let us assume that $p$ is nilpotent in $R$. If $R$ is graded nil-good, then $R[G]$ is graded nil-good.

**Proof.** The same as the proof of Theorem 4.2.

**Theorem 4.4.** Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring such that the units and nilpotents of $R$ are all homogeneous. If $R[G]$ is graded nil-good, then $R$ is graded nil-good.
Proof. Since $R[G]$ is graded nil-good, $(R[G])_e$ is nil-good. According to [13, Proposition 2.1 (4)], the mapping $f : R \rightarrow (R[G])_e$, given by $f(\sum_{g \in G} r_g) = \sum_{g \in G} r_g g^{-1}$ is a ring isomorphism. Therefore, $R$ is nil-good. On the other hand, all the units and nilpotents of $R$ are homogeneous, hence $R$ is graded nil-good.

\[ \blacksquare \]

**Remark 4.1.** Let $R$ be a $G$-graded ring where $G$ is a locally finite $p$-group and $p$ is nilpotent in $R$, and the units and nilpotents of $R$ are homogeneous. We assume that $(R[G])_e$ is a nil-good ring. It is clear that $R$ is also nil-good (since $R \cong (R[G])_e$). By assumption, $R$ is a graded nil-good ring. Hence, by Corollary 4.2 $R[G]$ will be a graded nil-good ring.

Previous remark yields to the question of when the following implication holds true:

$$R_e \text{ is nil-good } \Rightarrow R = \bigoplus_{g \in G} R_g \text{ is graded nil-good.}$$

The following example shows that the above implication does not hold in general.

**Example 4.2.** Let $R := \mathbb{Z}_2[X]$. $R$ is $\mathbb{Z}$-graded with $R_i = \mathbb{Z}_2 X^i$ if $i \geq 0$ and $R_i = 0$ if $i < 0$. Then $R_0 = \mathbb{Z}_2$ is a nil-good ring (see [7, Example 1]), but $R$ is not graded nil-good since $X$ is not graded nil-good.

We continue by giving some sufficient conditions for the above implication to be true. Let us first recall the definition of $PI$-ring.

**Definition 4.1 ([13]).** A ring $R$ is a $PI$-ring if there is, for some natural integer $n$ an element $P$ of $\mathbb{Z}[X_1, \ldots, X_n]$ such that for all $(r_1, \ldots, r_n) \in R^n$ we have that $P(r_1, \ldots, r_n) = 0$.

**Theorem 4.5.** Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded $PI$-ring without unity which is Jacobson radical (i.e. $J(R) = R$). If $R_e$ is nil-good, then $R$ is graded nil-good.

**Proof.** According to [7, Proposition 2.5], $J(R_e)$ is nil. Now, by [16, Theorem 3] we have that $J(R)$ is nil since $R$ is by assumption $PI$. On the other hand, $R$ is by assumption Jacobson radical ring. Hence, $R = J(R)$ is a nil ring. In particular, every homogeneous element is nilpotent. Therefore, $R$ is graded nil-good.

\[ \blacksquare \]
Theorem 4.6. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded PI-ring which is graded-local, and let $G$ be a finite group such that the order of $G$ is a unit in $R$. Assume that $R_g R_{g^{-1}} = 0$ for every $g \in G \setminus \{e\}$, then, if $R_e$ is nil-good, $R$ is graded nil-good.

Proof. According to [7, Proposition 2.5], we have that $J(R_e)$ is nil. Moreover, by assumption $R_e / J(R_e)$ is nil-good. Now, according to [5, Theorem 4.4] we have that $J^g(R) = J(R)$. On the other hand, [16, Theorem 3] implies that $J(R)$ is nil, and hence $J^g(R)$ is graded-nil. Let $H$ be the homogeneous part of $R / J^g(R)$. According to the proof of Theorem 3.27 in [12], we have that $H \cong R_e / J(R_e)$, this means that every homogeneous element of $R / J^g(R)$ is graded nil-good. Hence, $R / J^g(R)$ is graded nil-good. Finally, according the Theorem 3.1, $R$ is graded nil-good. \hfill \blacksquare

Theorem 4.7. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring of finite support, where $G$ is a torsion free group. Also, let $R$ be a semiprimary ring with $R_e$ local nil-good. Then, $R$ is graded nil-good.

Proof. Since $R_e$ is nil-good, by [7, Proposition 2.8] we have that $R_e / J(R_e)$ is nil-good too. By [19, Proposition 9.6.4], we have that $J^g(R) = J(R)$ and that $R / J(R) = R_e / J(R_e)$. Therefore, $R / J(R)$ is graded nil-good. On the other hand, $R$ is a semiprimary ring, hence, $J(R)$ is nil. According to Theorem 3.1, $R$ is graded nil-good. \hfill \blacksquare

4.2 Matrix rings

In [11, Corollary 4.2], it’s been shown that if $R$ is a $G$-graded 2-good ring, then $M_n(R)(\sigma)$ is graded 2-good for every natural number $n$ and for every $\sigma \in G^n$. Also, in [12, Theorem 3.18] it is proved that a matrix ring over a graded clean ring is also graded clean.

In this subsection, we try to obtain a similar result for graded nil-good rings. However, the author in [11, Corollary 4.3] has found a similar result concerning graded 2-nil-good rings which are crossed product.

The example below shows that if $R$ is a $G$-graded nil-good ring then $M_n(R)(\sigma)$ is not necessarily graded nil-good for every natural number $n$ and for every $\sigma \in G^n$. 

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Example 4.3. Let $G = \{e, g\}$ be a group of order two, and let $R := \mathbb{Z}_2 \rtimes \mathbb{Z}_2$. We have that $R = (\mathbb{Z}_2 \rtimes 0) \oplus (0 \rtimes \mathbb{Z}_2)$ is a $G$-graded ring with $R_e = \mathbb{Z}_2 \rtimes 0$ and $R_g = 0 \rtimes \mathbb{Z}_2$. Homogeneous elements of $R$ are $(0,0)$, $(1,0)$ and $(0,1)$. We have that, $(1,0) \in U(R)$ and $(0,1) \in \text{Nil}(R)$, and so $R$ is graded nil-good. Let $\sigma = (e,e)$. We claim that $M_2(R)(\sigma)$ is not graded nil-good. Indeed, it is easy to show that the homogeneous element $\begin{pmatrix} (1,0) & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)_e(\sigma)$ is not graded nil-good. In fact, if we suppose it is graded nil-good, we obtain that $(1,0)$ is nilpotent which is a contradiction since it is a unit of $R$.

Now, we give a sufficient condition for the matrix ring to be graded nil-good. We first define matrix 'in good form'.

Definition 4.2 ([4]). Let $R$ be a ring and $n \geq 2$. We say that a matrix \begin{pmatrix} A & \beta \\ \gamma & d \end{pmatrix} \in M_n(R)$ is 'in good form' if $A \in M_{n-1}(R)$ is nonzero and $d \in R$ is also nonzero.

Next, we give a graded version of the notion fine rings introduced in [4].

Definition 4.3. Let $R$ be a $G$-graded ring. $R$ is said to be graded fine if every nonzero homogeneous element of $R$ can be written as a sum of a unit and a nilpotent. Clearly, every graded fine ring is graded nil-good.

Theorem 4.8. Let $R$ be a $G$-graded commutative ring. Assume that $G$ is finite and $1 = u + v$ where $u, v \in U(R_e)$. If $R$ is graded nil-good, then $M_2(R)(\sigma)$ is graded nil-good where $\sigma = (e,e)$.

Proof. Since $R$ is commutative, by Proposition 3.2 (2) we have that every nonzero homogeneous element of $\overline{R} = R/J^9(R)$ is unit. Let $M$ be a homogeneous nonzero matrix of $M_2(\overline{R})(\sigma)$. According to [4, Corollary 3.7], $M$ is similar to a matrix in good form. Hence, there exist a matrix $V$ invertible and a matrix $A$ in good form of $M_2(\overline{R})(\sigma)$ such that $M = VAV^{-1}$. Following the proof of [4, Corollary 3.7], we deduce that $V$ is invertible in $M_2(\overline{R}_e)$ and $A$ is homogeneous. Now, since $A$ is in good form we have that $A = \begin{pmatrix} u_1 & b \\ c & u_2 \end{pmatrix} = \begin{pmatrix} u_1 & b \\ 0 & u_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} = U + N$ where $u_1, u_2 \in U(\overline{R})$ is a graded nil-good decomposition of $A$. Hence, $M = VUV^{-1} + VNV^{-1}$, since $VUV^{-1}$ is invertible and $VNV^{-1}$ is nilpotent and both are homogeneous,
we deduce that $M$ is graded fine. Thus, $M_2(R)(\sigma) \cong M_2(R)/M_2(J^g(R))$ is graded fine, hence graded nil-good.

On the other hand, $M_2(J^g(R)) \cong J^g(M_2(R))$. Moreover, by [7, Theorem 2.19] we have that $M_2(R)_e(\sigma) = M_2(R_e)$ is nil good since $R_e$ is nil-good, this implies that $J\left(M_2(R)_e(\sigma)\right)$ is nil (by [7, Proposition 2.5]). We know by Amitsur-Levitski theorem (see [1]) that $M_2(R)$ is PI-ring. Now according to [16, Theorem 3] we have that $J(M_2(R))$ is nil. Since $G$ is finite, by [5, Theorem 4.4] we obtain $J^g(M_2(R)) \subseteq J(M_2(R))$. Hence, $J^g(M_2(R))$ is graded-nil.
Finally, by Theorem 3.1 $M_2(R)(\sigma)$ is graded nil-good.

\[\square\]

**Corollary 4.3.** Let $R$ be a $G$-graded commutative ring. Assume that $G$ is finite and $1 = u + v$ where $u, v \in U(R_e)$. If $R$ is graded nil-good and $n$ a natural number, then $M_n(R)(\sigma)$ is graded nil-good where $\sigma = (e, \ldots, e) \in G^n$.

**Proof.** Using mathematical induction on $n$, we will prove that $M_n(R)(\sigma)$ is graded fine where $R = R/J^g(R)$.

For $n = 2$, see the proof of Theorem 4.7.

Now, assume that $n \geq 2$ and the claim holds for $M_n(R)(\sigma)$. According to [4, Proposition 3.9], every nonzero homogeneous matrix of $M_{n+1}(R)(\sigma)$ is similar to a homogeneous matrix in good form. Moreover, the change of basis matrix has to be from $M_{n+1}(R_e)$.

Let $A \in M_{n+1}(R)(\sigma)$ be a homogeneous matrix in good form. We have that $A = \begin{pmatrix} M & \beta \\ \gamma & d \end{pmatrix}$ where $M \in M_n(R)$ nonzero and $d \in U(R)$. Since by assumption $M_n(R)(\sigma)$ is graded fine, then $M = U + N$ where $U$ (resp. $N$) is an invertible (resp. a nilpotent) matrix of $M_n(R)$. Hence, $A = \begin{pmatrix} U & \beta \\ 0 & d \end{pmatrix} + \begin{pmatrix} N & 0 \\ \gamma & 0 \end{pmatrix}$ is a graded fine decomposition of $A$. Thus, we deduce that $M_n(R)(\sigma)$ is graded fine. Now, following the second part of the proof of Theorem 4.7, we obtain that $M_n(R)(\sigma)$ is graded nil-good.

\[\square\]

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