Optimal Selection of Transaction Costs in a Dynamic Principal-Agent Problem

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Abstract

Environments with fixed adjustment costs such as transaction costs or ‘menu costs’ are widespread within financial systems. In economic systems, the presence of fixed minimal adjustment costs produces adjustment stickiness so that agents must choose a sequence of points at which time to perform actions. This paper performs an analysis of the effect of transaction costs on agent behaviour and in doing so, for the first time studies incentive-distortion theory within an optimal stochastic impulse control model. The setup consists of an agent that maximises their utility criterion by performing a sequence of purchases of some consumable good whilst facing transaction costs for each purchase and a Principal that chooses the fixed transaction cost faced by the agent. This results in a Principal-Agent model in which the agent uses impulse controls to perform adjustments on a (jump-)diffusion process. We analyse the effect of varying the transaction cost on the agent’s purchasing activity and address the question of which fixed value of the transaction cost the Principal must choose to induce a desired behaviour from the agent. In particular, we show that with an appropriate choice of transaction cost, the agent’s preferences can be sufficiently distorted by the Principal so that the agent finds it optimal to maximise the Principal’s objective even when the agent’s liquidity is unobserved by the Principal.

Keywords: Impulse control, Principal-Agent model, transaction costs, optimal stochastic control, verification theorem, implementability, inverse optimal control.

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Introduction

There are numerous environments in which financial agents incur fixed or minimal costs when adjusting their financial positions; trading environments with transaction costs, real options pricing and real estate and large-scale infrastructure investing are a few important examples. However, despite the fundamental relevance in theoretical finance and economic theory, the task of modelling minimally bounded adjustment costs within a dynamic Principal-Agent model, mechanism design or generally multiplayer model with informational asymmetries has as of yet, received no analytic treatment.

In this paper, we analyse the effect of transaction costs within a dynamic Principal-Agent model. In this environment, an agent makes purchases of some costly good over some time horizon. Each time the agent performs a purchase, the agent incurs at least some fixed minimal cost (e.g. a transaction cost) which is chosen in advanced by a Principal. The cost of each purchase is drawn from the agent’s liquidity which is modelled by a jump-diffusion process and is observed only by the agent. When the agent’s liquidity process hits 0, the process is terminated as at this point the agent goes bankrupt. Therefore, since the agent’s purchases incur fixed minimal costs, the agent performs a sequence of discrete purchases (possibly of varying size) in order to maximise their utility over the horizon of the problem. Since the Principal gets to choose the fixed value of the transaction cost, the Principal aims to choose a transaction cost that induces a specific consumption behaviour from the agent. Since the agent cannot perform its purchases in a continuous fashion, we model the agent’s problem as an impulse control which allows us to study optimal control problems in which each action incurs some fixed cost.

Overview

The aim of this analysis is twofold: the first objective is to study the effect of introducing a transaction cost on the agent’s consumption policy and the relationship between the agent’s policy and the transaction cost. The second objective of the paper is to fully determine the value of the transaction cost that induces an agent policy that is desirable for the Principal. Thus in the latter case, the choice of transaction cost serves to condition the agent’s preferences so that the timing, magnitude (and total number) of the agent’s investment adjustments coincide with the Principal’s objectives. The analysis of the paper is performed with sufficient generality to allow for the Principal to be uninformed about the agent’s preferences and cash-flow process. Nonetheless, the Principal can transfer wealth to (or from) the agent at the point of the agent’s investment adjustments in order to induce desirable changes in the agent’s purchasing policy.

The analysis of the paper is selected with appeal to investigate financial environments with transaction costs and in which the optimal choice of transaction cost is unknown. The study of public-private partnerships (e.g. employment initiatives or capital investments within), trading with transaction costs and central authorities that seek to condition the behaviour of players in a given financial environment are some examples.

Background

Consider firstly the example of a single irreversible investment for a firm that privately observes the demand process. In order to maximise its overall profit, the firm strategically selects a profit-maximising time to enter the market. Secondly consider the case of a firm that wishes to adjust its production capacity according its observations of market (demand) fluctuations in order to maximise its cumulative profits. For the firm, increasing production capacity involves paying investment costs which include fixed costs with which the increases in production yield additional firm revenue. In this case, to maximise overall profits the firm selects some $optimal\ sequence$ of capital adjustments implemented over the firm’s time horizon.

In the case of the single irreversible firm investment, it is widely known that the optimal firm strategy is to delay investment beyond the point at which the expected returns of investment becomes positive - from the agent’s perspective, the late entry of investment results in a socially inefficient outcome \cite{DP08}. Similarly, in multiple production capacity case the firm’s decision process relating to profit maximising production capital levels often also produce socially inefficient outcomes.
In both cases, it is therefore natural to ask whether it is possible for an (uninformed) central planner to sufficiently modify the firm’s preferences so that the firm’s investment decisions produce socially efficient outcomes. The case of a single irreversible firm investment (with asymmetric information) was analysed in [SK12] in which it was shown that a regulator can induce socially efficient entry decisions through the use of a posted-price mechanism.

In particular, in [SK12] it is shown that by performing a transfer of wealth at the point of an agent’s decision, a central authority or Principal who does not observe the state of the world can sufficiently distort an informed agent’s preferences in an optimal stopping problem so that the agent’s decision to stop the process coincides with the Principal’s optimal stopping time.

Presently however, the literature concerning multiple sequential investment analysis has been primarily limited to entrance and exit problems within environments of complete information (see for example [DZ00]). Thus, the important case of Principal-Agent models with multiple sequential investments has thus far not been studied.

Theoretical Framework

The appropriate modelling framework for multiple sequential investment problem in environments of future uncertainty is optimal stochastic control theory. In stochastic control theory, the inclusion of fixed minimal control costs induces a form of system modifications enacted by the agent or controller known as impulse control. Impulse control models are optimal control problems in which the cost of control is bounded below so that modifying the system dynamics incurs at least, some fixed minimum cost. In impulse control models, the dynamics of the system are modified through a sequence of discrete actions or bursts chosen at times that the agent chooses to apply the control policy. This distinguishes impulse control models from the classical (continuous) optimal control models in which players are assumed to continuously make infinitesimally fine adjustments for which the associated costs can be made arbitrarily small.

Given the discrete nature of the modifying actions of impulse controls, impulse control models represent appropriate modelling frameworks for financial environments with transaction costs, liquidity risks and economic environments in which players face fixed adjustment costs (e.g. ‘menu costs’). More generally, impulse control models are suitable for describing systems in which the dynamics are modified by sequences of discrete, timed actions.

We refer the reader to [BL82] as a general reference to impulse control theory and to [VLVP07; PS10] for articles on applications. Additionally, matters relating to the application of impulse control models within finance have been surveyed extensively in [Kor99].

Literature

Current modelling methods of multiplayer interactions with asymmetric information with multiple ($N > 2$) adjustments are modelled by stochastic differential games with player controls restricted to those belonging to an absolutely continuous class of controls (e.g. [Car07; CR09; CS10]). In particular, the restriction to absolutely continuous controls implies players modify their positions by performing infinitesimally fine adjustments throughout the horizon of the problem. This renders models with absolutely continuous controls unsuitable for describing behaviour in systems with fixed minimal costs since continuous adjustment would result in immediate ruin.

Contribution

This paper addresses the absence of models that analyse strategic interactions within environments with fixed minimal costs. Our main result is to determine the value of the transaction cost that induces the Principal’s desired consumption policy to be executed by the agent. We also conduct a brief analysis of the transaction cost parameter and the solution to the agent’s optimal control policy.

The results of the paper also lead to a solution to the following inverse impulse control problem:

\[ \text{Stochastic differential games represent the multiplayer generalisation of stochastic control theory.} \]
Let $X$ be a one-dimensional diffusion and let the agent’s impulse control problem be given by the following problem:

$$
\sup_{u \in \mathcal{U}} \mathbb{E} \left[ \int_{t}^{\tau_0} h(s, X^t_{s \wedge \tau_0, u}) + \sum_{j \geq 1} c(X^t_{\tau_j}, z_j) + \phi(\tau_0, X^{t_{\tau_0}, u}) \right],
$$

where the control policy takes the form $u(s) = \sum_{j \geq 1} z_j \cdot 1_{\{\tau_j < \tau_0\}}(s)$, $s \in [t, \tau_0]$, $z \in \mathcal{Z}$ and $\{\tau_j\}_{j \in \mathbb{N}}$ are $\mathcal{F}$-measurable stopping times. The functions $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the running cost and terminal payoff functions (resp.) and $c : \mathbb{R} \times \mathcal{Z} \to \mathbb{R}$ is an intervention cost function for some given set of admissible interventions $\mathcal{Z}$. The set $\mathcal{U}$ is an admissible control set.

Let $D \equiv \{ X \in \mathcal{S} : X^t_{\tau_0, u} < x^* \}$ be a given continuation region, that is a region in which the agent finds it optimal to execute no interventions and suppose the optimal intervention magnitude $\hat{z}$ is given by $\hat{z} = \hat{x} - x^*$ for some real-valued constant $\hat{x}$. Let the parameters $\lambda \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ by the proportional cost and fixed cost parts respectively so that an impulse execution of magnitude $z$ incurs a cost $(1 + \lambda)z + \kappa$. The inverse impulse control problem is to determine the value of $\kappa$ and $\lambda$ for any given pair $(\hat{x}, x^*)$.

In section 4, we perform an analysis of the effect of changes to the parameter $\lambda$ on the quantities $(\hat{x}, x^*)$. We also determine the values of the fixed cost parameters $\lambda$ and $\kappa$ s.t.

Lastly, as a corollary to the above, the paper also yields two results relating to optimal stochastic impulse control theory: it is shown that the solutions to two distinct optimal impulse control problems can be made to be identical after a transformation that acts purely on the intervention cost function.

Organisation

In section 1, we give a description of the problem and highlight the connection to optimal stochastic control theory with impulse control. In section 2 we give some definitions central to the apparatus of the impulse control and Principal-Agent problem. We additionally state some known results which we shall make use of in the main analysis (respectively). In section 3, we give the statement of the main results of the paper which is immediately followed by some known results which we shall make use of in the main analysis (respectively). In section 4, we perform an analysis of the effect of changes to the parameter $\lambda$ on the quantities $(\hat{x}, x^*)$. We also determine the values of the fixed cost parameters $\lambda$ and $\kappa$ s.t.
space(\(\Omega,\mathbb{P},\mathcal{F}\)) = \{\mathcal{F}_s\}_{s \in [t,\tau_0])}. We assume that \(N\) and \(B\) are independent. \(\sigma: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\) is a diffusion coefficient and \(S \in \mathbb{R}\) is the state space. We assume that the functions \(\mu: [t,\tau_0] \times \mathbb{R} \to \mathbb{R}, \sigma: [t,\tau_0] \times \mathbb{R} \to \mathbb{R}\) and \(\gamma: \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) satisfy the usual assumptions so as to ensure the existence of \(\mathbb{F}\) (see [SV08] for exhaustive discussions).

At any time, the agent may make a purchase which incurs some fixed minimal cost, hence the agent makes purchases over a sequence of \(\mathcal{F}_\tau\)-measurable times. The sizes of the purchases are \(\{z_k\}_{k \in \mathbb{Z}}\) and the sequence of times of the agent’s purchases is given by \(\{\tau_k(\omega)\}_{k \in \mathbb{N}}\) an increasing sequence of \(\mathcal{F}_\tau\)-measurable discretionary stopping times where \(\tau_k, s_k) \in \mathcal{U}\) where \(\mathcal{Z}\) is the set of feasible agent purchases and \(\mathcal{T} \subseteq \mathbb{R}^+\) is a set of \(\mathcal{F}_\tau\)-measurable discretionary stopping times. Thus the double sequence \((\tau, \mathcal{Z}) \equiv \sum_{j \in \mathbb{N}} z_j, 1_{\{\tau_1, < \infty\}} \in \mathcal{U}\) is the agent’s control.

In this environment, the Principal chooses a transaction cost which consists of a fixed cost \(\kappa \in \mathbb{R}\) and a marginal cost parameter \(\lambda \in \mathbb{R}\) which is proportional to the size of the agent’s purchase both of which are incurred by the agent at the point of each purchase. We denote by \(S \subseteq \mathbb{R}\) the state space.

We define the time of bankruptcy as the point at which the agent’s cash-flow process goes below 0 at which point the agent exits the market. In particular the bankruptcy time is \(\tau_S := \inf\{s > t; X_{s,t}^x(t,\tau, Z) \leq 0\}\).

The agent’s cash-flow process is therefore affected sequentially at the points of purchases performed by the agent and is described by a stochastic process \(X_s = X(s, \omega) \in \mathbb{R}^+ \times \Omega\) on \((\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq t}, \mathbb{P}_0)\) that obeys the following expression:

\[
X_s = x + \int_t^{s \wedge \tau_S} \Gamma X_{t,x}^{s,x,(\tau,\mathcal{Z})} dr - \sum_{j \geq 1} ((1 + \lambda) z_j + \kappa) \cdot 1_{\{\tau_j < \infty\}} + \int_t^{s \wedge \tau_S} \sigma X_{t,x}^{s,x,(\tau,\mathcal{Z})} d\mathcal{B} + \int_t^{s \wedge \tau_S} \int \gamma(X_{t,x}^{s,x,(\tau,\mathcal{Z})}, z) \tilde{N}(dr, dz), \quad X_t = x
\]

\(\forall (s, x) \in \mathbb{R}^+ \times \mathbb{R}, (\tau, \mathcal{Z}) \in \mathcal{U}, \mathbb{P}\)-a.s. and where \(\kappa > 0, \lambda\) are fixed constants which we shall refer to as the fixed part of the transaction cost and proportional part of the transaction cost respectively.

The aim of the agent is to maximise their purchases before bankruptcy. The agent’s payoff function \(\Pi(\tau,\mathcal{Z}) : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}\) is therefore given by the following expression:

**Agent Payoff Function**

\[
\Pi(\tau,\mathcal{Z})(t, x) = \mathbb{E}\left[\int_t^{\tau_S} e^{-\delta r} U(X_{r,x}^{s,x,(\tau,\mathcal{Z})}) dr + \sum_{j \geq 1} e^{-\delta \tau_j} c(X_{\tau_j,x}^{s,x,(\tau,\mathcal{Z})}, z_j) \cdot 1_{\{\tau_j < \tau_S\}}\right],
\]

\(\forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}, (\tau, \mathcal{Z}) \in \mathcal{U}, \mathbb{P}\)-a.s. where \((t, x)\) is the initial point, \(U: \mathbb{R} \to \mathbb{R}\) is some utility function (we shall later specialise to the case in which \(U\) is a power utility function). The reward term \(c\) is given by \(c(., z_j) = z_j\) which is endowed to the agent at each purchase.

Similarly, the Principal has a payoff function \(Q(\tau,\mathcal{Z}) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\) which is composed of a running gain function \(W: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}\), a purchase gain function \(c_p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\).

Hence, if the agent performs purchases according to the control \((\tau, \mathcal{Z}) \equiv [\tau_j, z_j]_{j \in \mathbb{N}} \in \mathcal{U}\) then the Principal’s payoff function is given by the following:

**Principal Payoff Function**

\[
Q(\tau,\mathcal{Z})(t, x) = \mathbb{E}\left[\int_t^{\tau_S} W(r, X_{r,x}^{s,x,(\tau,\mathcal{Z})}) dr + \sum_{j \geq 1} e^{-\rho \tau_j} c_p(X_{\tau_j,x}^{s,x,(\tau,\mathcal{Z})}, z_j) \cdot 1_{\{\tau_j < \tau_S\}}\right],
\]

\(\forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}, (\tau, \mathcal{Z}) \in \mathcal{U}, \mathbb{P}\)-a.s. where \(\rho \in (0, 1)\) is the Principal’s discounting factor. We will in particular assume that the Principal purchase gain function \(c_p\) is given by \(c_p(\tau_j, z_j) = \lambda_p z_j + c_p \tau_j + \alpha_p\) where \(\lambda_p, c_p, \alpha_p \in \mathbb{R}^+\) are constants.

The problem faced by the Principal is to determine the parameters \((\lambda, \kappa) \in \mathbb{R} \times \mathbb{R}\) which induce agent purchases at the times and by the magnitudes that the Principal would like (i.e. that coincide with the policy that maximises (3)).

The agent’s problem is an optimal stochastic control problem, in particular, the agent’s problem is to find a sequence of selected magnitudes or an impulse control that alters the agent’s liquidity process in such a way that maximises the agent’s state-dependent payoff. Since the controls incur fixed minimal adjustment costs, the appropriate model of optimal
Throughout the script we adopt the following standard notation (e.g. [CG14; MDG10; KS07]):

**Notation**

Let $\Omega$ be a bounded open set on $\mathbb{R}^{p+1}$. Then we denote by: $\bar{\Omega}$ - The closure of the set $\Omega$.

$Q(s, x; R) = (s', x') \in \mathbb{R}^{p+1}: \max |s' - s|^{1/2}, |x' - x| < R, s', s < s$.

$\partial \Omega$ - the parabolic boundary $\Omega$ i.e. the set of points $(s, x) \in \mathcal{S}$ s.th. $R > 0, Q(s, x; R) \not\subset \Omega$.

$C^{1, 2}([t, \tau_S], \Omega) = \{h \in C^{1, 2}(\Omega) : \partial_x h, \partial_x x, h \in C(\Omega)\}$, where $\partial_x$ and $\partial_{x, x}$ denote the temporal differential operator and second spatial differential operator respectively.

$\nabla \phi = (\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_p})$ - The gradient operator acting on some function $\phi \in C^1([t, \tau_S] \times \mathbb{R}^p)$.

$C^d([a, b]; U)$ - The set of càdlàg functions that map $[a, b] \mapsto U$ for some set $U \subseteq \mathbb{R}^p$.

$\| \cdot \|$ - The Euclidean norm to which $\mathbb{Z}$ is the set of admissible impulse values. Indeed, if we suppose that an impulse is determined by some admissible policy $u \in \mathcal{U}$, then the cost of control is bounded from below. In impulse control models, the task of determining the optimal control policy now consists of finding both the optimal sequence of times to apply the control policy, in addition to determining the optimal control magnitudes.

In this paper, we study the effect of the fixed cost parameters $(\lambda, \kappa)$ associated to the agent’s impulse control on the agent’s consumption pattern. A central aim of this paper is determine the pair $(\lambda, \kappa)$ and the conditions under which a desirable control policy is induced, that is determining the transaction cost that leads to the agent finding it optimal to exercise a control that maximises the Principal’s objective $\mathbb{E}_S$.

Throughout the script we adopt the following standard notation (e.g. [CG14; MDG10; KS07]):

**Standing Assumptions**

**A.1. Lipschitz Continuity**

We assume the Lipschitzianity of the functions $U$ and $W$ that is, we assume the existence of real-valued constants $c_U, c_W > 0$ s.th. $\forall s \in [t, \tau_S], \forall s \in \mathbb{R}^p$ we have for $R \in \{U, W\}$:

$$|R(s, x) + R(s, y)| \leq c_R|x - y|.$$  

**A.2. Minimally Bounded Costs**

We also assume that there exist constants $\lambda_c, \lambda_{cp} > 0$ s.th. $\inf_{z \in \mathcal{Z}} c(\cdot, z) \geq \lambda_c$, $\inf_{z \in \mathcal{Z}} c_p(\cdot, z) \geq \lambda_{cp}$.

**Controlled State Process**

The process $X$ which describes the agent’s liquidity process is influenced by impulse controls $u \in \mathcal{U}$ exercised by the agent where $u(s) = \sum_{j \geq 1} z_j \cdot 1\{\tau_1 \leq \tau_j\}(s) \in [t, \tau_S]$. The impulses $\{z_j\}_{j \in \mathbb{N}} \in \mathcal{Z} \subseteq \mathcal{S} \subseteq \mathbb{R}$ where $t \leq \tau_1 < \tau_2 < \ldots < \tau_S$ and where $\mathcal{S} \subseteq \mathbb{R}$ is a given set so that an impulse control policy is given by the following double sequence:

$$u = (\tau_1, \tau_2, \ldots, z_1, z_2, \ldots) \in \mathcal{U} \text{ where } (\tau_k, z_k)_{k \in \mathbb{N}} \in [t, \tau_S] \times \mathcal{Z}.$$  

We assume $\mathcal{U} \subseteq \mathbb{R}$ is a convex cone which is the set of control actions for the agent and $\mathcal{Z}$ is the set of admissible impulse values. Indeed, if we suppose that an impulse $\zeta \in \mathcal{Z}$ determined by some admissible policy $w$ is applied at some $\mathcal{F}$-measurable stopping time
τ when the state is \( x' = X_t^{t_,x_0,\cdot}(\tau) \), then the state immediately jumps from \( x' = X_t^{t_,x_0,w} \) to \( X_t^{t_,x_0,w} = \Gamma(x', \zeta) \) where \( \Gamma : \mathbb{R} \times \mathcal{Z} \rightarrow \mathbb{R} \) is called the impulse response function and \( (t, x_0) \in [t, \tau_S] \times \mathbb{R} \).

2 Preliminaries

**Definition 2.1.** For a given agent reward function \( c \), the agent and Principal have value functions given by the following expressions (resp.) \( \forall x \in [t, \tau_S] \times \mathbb{R} : \)

\[
v_A(x) = \sup_{u \in \mathcal{U}} \Pi^{c,w}(t,x), \quad v_P(x) = \sup_{u \in \mathcal{U}} Q^{c,w}(t,x).
\]

(6)

The above expressions define the agent and Principal impulse control problems (resp.). Where it will not cause confusion we will occasionally write \( v_A(x) \equiv v(x) \) for \( x \in [t, \tau_S] \times \mathbb{R} \) for the agent value function.

**Definition 2.2.** [Implementability] We say that the agent reward function \( c : \mathbb{R} \times \mathcal{Z} \rightarrow \mathbb{R} \) and transaction cost parameters \( \lambda \in \mathbb{R} \) and \( \kappa \in \mathbb{R} \) implement an impulse control policy \( u^* \in \mathcal{U} \) if \( \forall x \equiv (t, x_0) \in [t, \tau_S] \times \mathbb{R} \), \( u^* \in \mathcal{U} \) the following condition is satisfied:

\[
\Pi^{c,(u^*)}(x) \geq \Pi^{c,(u)}(x).
\]

(7)

where the controls \( u^* = [\tau^*_j, z^*_j]_{j \geq 1} \in \mathcal{U} \) and \( u' = [\tau'_j, z'_j]_{j \geq 1} \in \mathcal{U} \).

Indeed, with reference to the Principal’s problem (6), (7) is equivalent to the following:

Find some cost function \( c \) that consists of cost parameters \( \lambda \) and \( \kappa \) s.t.h.:

\[
\Pi^{(u^*)}(x) = v_A(x), \quad Q^{(u^*)}(x) = v_P(x).
\]

(8)

and hence the implementability condition asserts the optimality of the policy \( u^* \in \mathcal{U} \) for the agent, given the transaction costs \( c \).

Therefore, to analyse the Principal’s problem it suffices to characterise transaction cost parameters \( \lambda \) and \( \kappa \) and the conditions on the Principal’s policy for which the agent always finds it optimal to enact the prefixed impulse control policy \( u^* \in \mathcal{U} \) (so that the inequality in (7) is satisfied).

We now give some definitions which are central:

**Definition 2.3.** Let \( u \) be an impulse control policy. We say that an impulse control is admissible on \( [t, \tau_S] \) if the number of impulse interventions is finite \( \mathbb{P} \)-a.s, that is to say we have that \( \mathbb{E}[\mu_t(t, \tau_S)(u)] < \infty \).

We shall hereon use the symbol \( \mathcal{U} \) to denote the set of admissible controls.

**Definition 2.4.** Suppose we denote the space of measurable functions by \( \mathcal{H} \), suppose also that the function \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) with \( \phi \in \mathcal{H} \). Let \( \tau \in [t, \tau_S] \) be some \( \mathcal{F}_\tau \)-measurable stopping time. We define the agent (non-local) intervention operator \( \mathcal{M} : \mathcal{H} \rightarrow \mathcal{H} \) by the following expression:

\[
\mathcal{M}[\phi] := \sup_{z \in \mathcal{Z}} \left[ \phi(\Gamma(X_t^{t_,x_0,\cdot}, z)) + c(X_t^{t_,x_0,\cdot}, z) \right] \mathbf{1}_{\{\tau \leq \tau_S\}},
\]

(9)

where \( \Gamma : \mathbb{R} \times \mathcal{Z} \rightarrow \mathbb{R} \) is the impulse response function defined earlier.

The following theorem characterises the optimal policy in optimal stochastic control problems:

**Theorem 2.5.** Verification Theorem for Optimal Impulse Control [KS07]

Let \( X \) be a stochastic process \( X_t = X(s, \omega) \in \mathbb{R}^+ \times \Omega \) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_\omega) \). Let \( \tau \) be some \( \mathcal{F}_\tau \)-measurable stopping time and denote by \( \hat{X}(\tau) = X(\tau^-) + \Delta_N X(\tau) \) where \( \Delta_N X(\tau) \) denotes a jump at some \( \mathcal{F}_\tau \)-measurable time \( \tau \) due to \( \hat{X} \). Denote also by \( \Delta_\tau \phi(X(\tau)) := \phi(\Gamma(X(\tau^-), z)) - \phi(X(\tau^-)) + \Delta_N X(\tau) \) where \( \tau \in [t, \tau_S] \) and \( z \in \mathcal{Z} \) is some \( \mathcal{F}_\tau \)-measurable stopping time and intervention (resp.).

In the following we will use the shorthand and denote by \( X(s) \equiv (s, X) \forall (s, X) \in [t, \tau_S] \times \mathbb{R} \) and \( \phi \equiv \phi(X, \cdot) \equiv \phi(X) \forall X \in S \).

Suppose also that there exists a function \( \phi \in C^1([t_0, \tau_S], S) \cap C([t_0, \tau_S], \hat{S}) \) s.t.h. for all \( s \in [t, \tau_S] \):

i. \( \phi \geq \mathcal{M}\phi \) in \( S \setminus \partial D \) where the continuation region \( D \) is defined by: \( D = \{ X \in S, s \in [t, \tau_S]; \phi(X) > \mathcal{M}\phi(X) \} \).

ii. \( \frac{\partial \phi}{\partial s} + \mathcal{L}\phi(X_{\cdot}^s(s)) + U(X_{\cdot}^s(s)) \leq 0 X \in S \setminus \partial D \).
are made. The two regions are characterised by the following expressions:

\[ \phi(x) = \Pi^{(\tau, z)}(x) = \sup_{(r, Z) \in U} \Pi^{(\tau, z)}(x) \] (10)

for all \((t, x) \in [t, \tau_S] \times \mathbb{R}\).

Theorem 2.5 has some technical conditions which we relegate to the appendix. A proof of the theorem is reported as Theorem 7.2 in [kS07].

From Theorem 2.5 we observe that the state space \(S\) splits into two regions, one region in which the agent performs no purchases which we shall refer to as the continuation region and another region which we shall refer to as intervention region which upon entry by agent’s cash-flow process, the agent makes an immediate purchase, we therefore have the following result:

**Corollary 2.6.** The sample space splits into two regions that represent a region in which the agent performs an immediate purchase \(I_1\), and a region \(I_2\) in which no agent purchases are made. The the two regions are characterised by the following expressions:

\[ I_1 = \{ x \in [t, \tau_S] \times \mathbb{R} : V(x) = MV(x), LV(x) + U(x) \geq 0 \} \]
\[ I_2 = \{ x \in [t, \tau_S] \times \mathbb{R} : V(x) > MV(x); LV(x) + U(x) = 0 \} \]

**Remark 2.7** Let us denote by \(D\) the region \(D = \{ (s, X) \in [t, \tau_S] \times \mathbb{R} : v(X(s)) > \mathcal{M}[v(X(s))] \}\) so that \(D\) represents the region in which the agent finds an immediate intervention suboptimal. Since \(S \subseteq \mathbb{R}\), given \(s \in [t, T]\), we can infer the existence of a value \(x^*(s) \in S\) for which \(OD = \{ X = x^*(s)|X, x^* \in S \} \) - that is to say that the agent performs an intervention as soon as the cash-flow process \(X\) attains a value \(x^*\), hence we shall hereon refer to the value \(x^*\) as the agent’s intervention threshold.

## 3 Main Results

We now present the main results of the paper; we postpone the proof of the results until section 4.

**Theorem 3.1.** Let \(x^*\) be the Principal’s target for the agent’s intervention threshold and define \(\hat{x} = \hat{z} + x^*\) where \(\hat{z}\) is the fixed optimal purchase magnitude. Then the agent adopts the Principal’s target for the pair \((\hat{x}, x^*)\) whenever the transaction cost parameters \((\lambda, \kappa)\) are set to the following expressions:

\[ \lambda(\hat{x}, x^*) = \frac{G(\hat{x}, x^*)}{b \ln(\frac{\hat{x}}{x^*})} \frac{l_1 z_1^{l_1-1} + l_2 z_2^{l_2-1}}{l_1 z_1^{l_1} + l_2 z_2^{l_2}} - 1 \] (11)
\[ \kappa(\hat{x}, x^*) = z - \frac{z_1 z_2^{l_1} + l_1 z_1^{l_1-1} + l_2 z_2^{l_2-1}}{l_1 z_1^{l_1} + l_2 z_2^{l_2}} - \frac{G(\hat{x}, x^*)}{b \ln(\frac{\hat{x}}{x^*})} \frac{l_1 z_1^{l_1-1} + l_2 z_2^{l_2-1}}{l_1 z_1^{l_1} + l_2 z_2^{l_2}} - z. \] (12)

where \(G(\hat{x}, x^*) := \hat{x}x^* \ln(\hat{z}/x^*), b := \epsilon \delta^{-1}\) and \(\hat{z}^m := \hat{x}^m - x^m\) for \(m \in \mathbb{N}\).

If the proportional part \(\lambda\) is fixed, then the value of the fixed part \(\kappa\) is given by the following:

\[ \kappa(\hat{x}, x^*, \lambda) = \kappa = l_1^{-1} \left[ z_2^{l_2-1} - (1 + \lambda)b \frac{z_2^{l_2}}{x^* \hat{z}} \right] (\hat{z}^{l_1-1} z_1^{l_2-1} - \hat{z}^{l_1-1} z_1^{l_2-1})^{-1} z_1^{l_1} \]
\[ + l_2^{-1} \left[ z_1^{l_1-1} - (1 + \lambda)b \frac{z_1^{l_1}}{x^* \hat{z}} \right] (\hat{z}^{l_2-1} z_2^{l_1-1} - \hat{z}^{l_2-1} z_2^{l_1-1})^{-1} z_2^{l_2} - z + b(1 + \lambda) \ln(\frac{\hat{x}}{x^*}). \] (13)

When the agent’s liquidity process contains no jumps \((\gamma(z) \equiv 0 \text{ in } \Xi)\), the fixed parameters \(l_1\) and \(l_2\) are given by:

\[ l_1 = \Gamma - \frac{1}{2} \sigma^2 - \frac{\sqrt{(\Gamma - \frac{1}{2})^2 - 2 \sigma^2 \delta}}{\sigma^2}, \quad l_2 = \frac{1}{2} \sigma^2 - \Gamma + \sqrt{(\Gamma - \frac{1}{2})^2 - 2 \sigma^2 \delta}. \] (14)

For the general case \((\gamma(z) \not\equiv 0 \text{ in } \Xi)\), the constants \(l_1\) and \(l_2\) are solutions to the equation:

\[ h(l) = 0 \] (15)
Theorem 3.1 says that if the Principal imposes the a transaction cost with proportional part and fixed part given by (11) and (12) respectively, then the agent’s continuation region is given by \( D = \{ x < x^* | x, x^* \in \mathbb{R} \} \) i.e. the agent will make a purchase whenever the agent’s cash-flow attains the value \( x^* \). Moreover, the agent’s purchase times are \( \tau_j = \inf \{ s > \tau_j; X_s - h(s) > x^* \} \) and \( \tau_S \) and the agent’s purchases will have size given by \( \hat{z} = \hat{x} - x^* \).

The first result of the theorem relates to the case when the Principal is free to choose the value of the proportional part of the transaction cost parameter \( \lambda \) and the fixed part of the transaction cost parameter \( \kappa \). The second result relates to the case when the Principal is free to choose the value of the fixed part of the transaction cost parameter \( \kappa \) but the proportional cost parameter \( \lambda \) is exogenous and fixed.

**Proposition 3.2.** Let the values \( l_1 \) and \( l_2 \) be as in Theorem 3.1 and suppose the initial fixed and proportional costs are given by \( \kappa_0 \) and \( \lambda_0 \) respectively. Suppose now that the fixed and proportional costs undergo the transformations \( \kappa_0 \to \kappa_1 \) and \( \lambda_0 \to \lambda_1 \), then the agent’s intervention threshold and consumption magnitude attain the values \( x^*_1 = x^*_0 + h_1 \) and \( \hat{x}_1 = \hat{x}_0 + h_1 \) whenever the values \( \lambda_1 \) and \( \kappa_1 \) are given by the following expressions:

\[
\lambda_1 = \frac{m_{-1} m_1}{b} \frac{(l_1 M_0^{1-1} + l_2 M_0^{2-1})}{l_1 M_0^{1-1} + l_2 M_0^{2-1}} - 1, \tag{17}
\]

\[
\kappa_1 = M_0 \frac{M_0^{11-1} + M_0^{12-1}}{l_1 M_0^{11-1} + l_2 M_0^{12-1}} \ln \left( \frac{m_{-1} + h_1}{m_1 + h_1} \right) \frac{(m_{-1} + h_1)(m_1 + h_1)}{(m_{-1} + h_1)(m_1 + h_1)} \left[ \frac{l_1 M_0^{11-1} + l_2 M_0^{12-1}}{l_1 M_0^{11-1} + l_2 M_0^{12-1}} \right], \tag{18}
\]

where \( M^k := (m_{-1} + h_1)^k - (m_1 + h_1)^k \) where \( m_{-1} < m_{-1} \) and \( m_1, m_{-1} \) are the solutions to the equations:

\[
Q_1(m, \kappa_0, \lambda_0) = 0, \tag{19}
\]

\[
Q_2(m, \kappa_0, \lambda_0) = 0, \tag{20}
\]

where \( Q_1 \) and \( Q_2 \) are given by

\[
Q_1(m, c, d) := \frac{b}{m_1 m_2} \frac{l_1 M_0^{11-1} + l_2 M_0^{12-1}}{l_1 M_0^{11-1} + l_2 M_0^{12-1}} \frac{1}{1 + d}, \tag{21}
\]

\[
Q_2(m, c, d) := \frac{bM}{m_1 m_2} \frac{M_0^{11} + M_0^{12}}{l_1 M_0^{11-1} + l_2 M_0^{12-1}} \frac{M + c}{1 + d} + b \ln \left( \frac{m_1}{m_2} \right). \tag{22}
\]

**Proposition 3.2** says that a shift of size \( h_1 \) and \( h_1 \) in the agent intervention threshold and consumption magnitudes (respectively) can be induced whenever the fixed and proportional costs are made to be the values \( \kappa_1 \) and \( \lambda_1 \) of the proposition. Here, interestingly the initial agent intervention threshold and consumption magnitude do not feature in any of the equations that determine the values \( \kappa_1 \) and \( \lambda_1 \), hence the only required data are the target parameters \( (x^*_1, \hat{x}_1) \) and the observables \( (\kappa_0, \lambda_0) \).

**Proposition 3.3.** The marginal rates of change in \( \hat{x} \) and \( x^* \) w.r.t. \( \lambda \) and \( \kappa \) are given by the following expressions:

\[
\frac{\partial \hat{x}}{\partial \lambda} = [f_1(\hat{x}, x^*)]^{-1}, \tag{23}
\]

\[
\frac{\partial x^*}{\partial \lambda} = [f_2(\hat{x}, x^*)]^{-1},
\]

\[
\frac{\partial \hat{x}}{\partial \kappa} = [f_3(\hat{x}, x^*)]^{-1},
\]

\[
\frac{\partial x^*}{\partial \kappa} = [f_4(\hat{x}, x^*)]^{-1}.
\]

where the parameters \( l_1 \) and \( l_2 \) are solutions to the equation (15) and the functions \( f_1, f_2, f_3 \) and \( f_4 \) are given by (79) - (80).

**Proposition 3.3** therefore evaluates the change in the intervention threshold and consumption magnitudes due to a marginal change in the cost parameters \( \lambda \) and \( \kappa \).
The following corollary follows directly from Theorem 3.1 and relates two general (complete information) stochastic impulse control problems.

**Corollary 3.4.** Let $X$ be a stochastic process $X_s = X(s, \omega) \in \mathbb{R}^+ \times \Omega$ on $(\Omega, F, (F_s)_{s \geq t}, \mathbb{P}_0)$ that evolves according to (11).

Consider the following pair of impulse control problems:

I. Find $u_1^* = [\tau^1_l, z^1_l]_{l \in \mathbb{N}} \in \mathcal{U}$ and $\phi_1 \in \mathcal{H}$ s.t $\phi_1 \in \mathcal{H}$ s.th. $\forall x \equiv (t, x_0) \in [t, \infty) \times \mathbb{R}$:

$$\phi_1(x) = j_1^{(u_1)}(x) = \sup_{u_1 \in \mathcal{U}} j_1^{(u_1)}(x),$$

II. Find $u_2^* = [\tau^2_l, z^2_l]_{l \in \mathbb{N}} \in \mathcal{U}$ and $\phi_2 \in \mathcal{H}$ s.t $\forall x \in [t, \infty) \times \mathbb{R}$:

$$\phi_2 = j_2^{(u_2)}(x) = \sup_{u_2 \in \mathcal{U}} j_2^{(u_2)}(x),$$

for the payoff functionals for problem I and II are given by the following expressions:

$$j_1^{(u_1)}(x) = \mathbb{E}^x\left[\int_t^{\tau_S} \alpha e^{-\beta t} \ln(X_s) ds + \sum_{j \geq 1} (\lambda_j z_j + \kappa_j) \cdot 1_{\{\tau_j < \tau_S\}} + \Psi_1(\tau_S, X_{\tau_S}^{(t,x_0,u_1)}) \right],$$

$$j_2^{(u_2)}(x) = \mathbb{E}^x\left[\int_t^{\tau_S} F(s, X_s) ds + \sum_{j \geq 1} l_j(\tau_{\tau_j}^l, x_j) \cdot 1_{\{\tau_j < \tau_S\}} + \Psi_2(\tau_S, X_{\tau_S}^{(t,x_0,u_2)}) \right],$$

where $\alpha, \beta \in \mathbb{R}$ and $F, l_j, \Psi_1, \Psi_2$ are bounded Lipschitz continuous functions. Suppose also that the controlled process (with interventions) evolves according to (2). Then if $u^*_2 \in \text{argsup}_{u_2 \in \mathcal{U}} j_2^{(u_2)}(x)$, then $u_1^* = u^*_2$ whenever

$$\lambda_j = \frac{G(\hat{x}, x^*) l_1 z_j^{l_1-1} + l_2 z_j^{l_2-1} - 1}{\rho \ln(\frac{\hat{x}}{x^*}) z_j^{l_1} + l_2 z_j^{l_2}},$$

$$\kappa_j = z_j - \frac{G(\hat{x}, x^*) l_1 z_j^{l_1-1} + l_2 z_j^{l_2-1}}{l_1 z_j^{l_1} + l_2 z_j^{l_2}} - z_j,$$

for all $x \in \mathbb{R}$ and $z \in \mathbb{Z}$ where $\rho := \alpha \delta^{-1}, \hat{x}_2 = x_2^* - z_2^*, G(\hat{x}, x^*) := \hat{x}_2 \cdot \ln(\frac{x_2^*}{x^*})$, and $z_m := x_m - x_m^*$ for $m \in \mathbb{N}$ and the constants $l_1$ and $l_2$ are solutions to the equation $m(l) = 0$ where $m : \mathbb{R} \rightarrow \mathbb{R}$ is the function:

$$m(l) = \frac{1}{2} \sigma^2(l - 1) + \Gamma - \delta + \int_{\mathbb{R}} \left(1 + \gamma(z)\right)^l - 1 - \gamma(z)\right) \nu(dz).$$

The parameter $x_2^*$ is the parabolic boundary of the continuation region for problem II - that is to say, given some continuation region for player II, each $x_2^*$ is of the form $x_2^* = \{X \in \mathcal{S} : X_s^{(t,x_0,u_2) \in \partial D}\}$ and $x^* := \text{argsup}_{x \in \mathbb{Z}} \{\phi_2(\Gamma(X(\tau_k^l), z)) + l_2(X(\tau_k^l), z)\}$ quantifies the player II optimal intervention magnitude.

Corollary 3.4 says that impulse control problem I has the same optimal control policy solution as that of problem II whenever player I’s intervention cost function has a proportional cost and fixed cost given by the $\lambda$ and $\kappa$ respectively.

Hence, in the language of Section 2, Corollary 3.4 therefore characterises the cost parameters which ensure that the agent finds it optimal to intervene according to the Principal’s optimal intervention policy.

### 4 Main Analysis

We begin by proving Theorem 3.1 which is proved by showing that given the cost function parameters $\lambda$ and $\kappa$ defined in (11) - (12), it is optimal for the agent to execute the sequence of interventions that maximises the Principal’s payoff $Q$.

We seek to characterise the cost function parameters $\lambda$ and $\kappa$ which implement the Principal’s control policy. The sets $\mathcal{S}, \mathcal{Z}$ and $\mathcal{U}$ are the state space, the feasible impulse interventions and the set of admissible controls (resp.) as before.
Suppose that the agent makes purchases according to the policy $(\tau_k, z_k)_{k\geq 1} \equiv (\tau, Z) \in \mathcal{U}$, hence the agent’s payoff function (without transfers) is given by the expression:

$$\Pi(\tau, Z)(t, x) = \mathbb{E}\left[ \int_t^{T_s} e^{-\delta r} U(X_{t,r}^{\tau, Z}) dr + \sum_{j \geq 1} e^{-\delta \tau_j} z_j \cdot 1_{\{\tau_j < \infty\}} \right],$$

Denote $(\tau^*, Z^*) \in \mathcal{U}$ the following:

$$\Pi(\tau^*, Z^*)(s, x) = \sup_{(\tau, Z) \in \mathcal{U}} \Pi(\tau, Z)(s, x),$$

∀ $(s, x) \in \mathbb{R}^+ \times \mathbb{R}$, so that given some cost function $c$, the agent’s optimal purchase strategy is given by $(\tau^*, Z^*) \in \mathcal{U}$.

Recall that the state process obeys the following expression:

$$X_s = x + \int_t^{s \wedge T_s} \Gamma X_{t,r}^{\tau, Z} dr - \sum_{j \geq 1} ((1 + \lambda) z_j + \kappa) \cdot 1_{\{\tau_j < \infty\}} + \int_t^{s \wedge \rho} \sigma X_{t,r}^{\tau, Z} dB + \int_t^{s} \int \gamma(X_{t,r}^{\tau, Z}, z)\tilde{N}(dr, dz), \quad X_t = x$$ (29)

∀ $(s, x) \in \mathbb{R}^+ \times \mathbb{R}, (\tau, Z) \in \mathcal{U}, \mathbb{P}$-a.s.. We now specialise to the case in which the agent’s utility function $U$ is given by:

$$U(x) = \epsilon \ln(x),$$ (30)

for some constant $\epsilon \in \mathbb{R}\setminus 0$ so that $U$ can be viewed as a limiting case of the CRRA utility function: $U(x) = \frac{e^{\frac{\gamma - 1}{\gamma} \ln(x)}}{\frac{\gamma - 1}{\gamma}}$ when $\gamma \to 1$.

Hence the agent’s payoff function $\Pi(\tau, Z) : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ is given by:

$$\Pi(\tau, Z)(t, x) = \mathbb{E}\left[ \int_t^{T_s} e^{-\delta r} \epsilon \ln(X_{t,r}^{\tau, Z}) dr + \sum_{j \geq 1} e^{-\delta \tau_j} z_j \cdot 1_{\{\tau_j < \infty\}} \right],$$

Hence, given some test function $\phi \in \mathcal{C}^{(1,2)}([t, T], \mathbb{R})$, the generator $\mathcal{L}$ for (29) is given by the following expression (c.f. (11))

$$\mathcal{L}\phi(x) = \Gamma_x \frac{\partial \phi}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} + \int_{\mathbb{R}} \left\{ \phi(s, x(1 + \gamma(z)) - \phi(s, x) - x\gamma(z) \frac{\partial \phi}{\partial x} \right\} \nu(dz).$$ (31)

By (iii) of Theorem 2.5 we have that on $\mathcal{D}$ the following expression holds:

$$U(s, x) + \frac{\partial \phi}{\partial s} + \mathcal{L}\phi = 0.$$ (32)

Hence, using (31) and by (32) we have that:

$$0 = e^{-\delta s} \epsilon \ln(x) + \frac{\partial \phi}{\partial s} + \Gamma_x \frac{\partial \phi}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \phi}{\partial x^2} + \int_{\mathbb{R}} \left\{ \phi(s, x(1 + \gamma(z)) - \phi(s, x) - x\gamma(z) \frac{\partial \phi}{\partial x} \right\} \nu(dz).$$ (33)

Let us try the following ansatz for the candidate function for $\phi$:

$$\phi = \phi_a + \phi_b,$$ (34)

where

$$\phi_a(s, x) = e^{-\delta s} ax^l, \quad \phi_b(s, x) = e^{-\delta s} (b \ln(x) + c).$$ (35)

for some constants $a, b, c \in \mathbb{R}$.

We firstly seek to ascertain the values of the constants $a, b$ and $c$ hence, inserting the expression for $\phi$ into (33) we find that:

$$h_a + h_b = 0$$ (36)

where the functions $h_a$ and $h_b$ are given by:

$$h_a(l) = \frac{1}{2} \sigma^2 l(l - 1) + l\Gamma - \delta + \int_{\mathbb{R}} \left\{ (1 + \gamma(z))^l - 1 - l\gamma(z) \right\} \nu(dz)$$ (37)

$$h_b(x) = \epsilon \ln(x) - \delta(b \ln(x) + c) + b\Gamma - \frac{1}{2} \sigma^2 b + \int_{\mathbb{R}} \left\{ b \ln(1 + \gamma(z)) - b\gamma(z) \right\} \nu(dz).$$ (38)
from which we find that the equation \( h_b(x) = 0 \) is solved by the following values for \( b \) and \( c \):

\[
b = \epsilon \delta^{-1},
\]
\[
c = \epsilon \left( \Gamma - \frac{1}{2} \sigma^2 \right) + \epsilon \int_{\mathbb{R}} \left\{ \ln(1 + \gamma(z)) - \gamma(z) \right\} \nu(dz).
\]

Let us make a brief excursion to discuss the case when the process \( \mathbb{I} \) contains no jumps i.e. when \( \gamma \equiv 0 \). In this case, we readily observe that (37) now reduces to the following expression:

\[
h_{a,0} := \frac{1}{2} \sigma^2 l^2 + l \left( \Gamma - \frac{1}{2} \sigma^2 \right) - \delta.
\]

After some simple algebra, we can deduce that in this case, we observe that there exists two solutions to the equation \( h_{a,0}(l) = 0 \), namely \( l_1 \) and \( l_2 \) where \( l_1 \) and \( l_2 \) are given by:

\[
l_1 = -\Gamma - \frac{1}{2} \sigma^2 + \sqrt{\left( \Gamma - \frac{1}{2} \sigma^2 \right)^2 - 2 \sigma^2 \delta},
\]
\[
l_2 = \frac{1}{2} \sigma^2 - \Gamma + \sqrt{\left( \Gamma - \frac{1}{2} \sigma^2 \right)^2 - 2 \sigma^2 \delta}.
\]

Let us now return to the case when the process \( \mathbb{I} \) contains jumps. Using (37), we now make the following observations:

\[
\lim_{m \to \infty} h_a(m) = +\infty, \quad h_a(m)^{m=0} = -\delta.
\]

Hence we deduce the existence of values \( l_1, l_2 \) s.th.:

\[
h(l_1) = h(l_2) = 0.
\]

W.log. let us assume that \( l_1 < l_2 \), since \( \forall l, z \) we have that: \( \{(1 + \gamma(z))l - 1 - l\gamma(z)\} \nu(dz) > 0 \) we find that:

\[
|l_1| > l_1.
\]

and

\[
l_1 < 0 < l_2,
\]

We therefore find that the function \( \phi \) is given by the following (c.f. (34)):

\[
\phi(s, x) = e^{-\delta s [a_1 x^l_1 + a_2 x^l_2 + b \ln(x)]} + c,
\]

where \( a_1 \) and \( a_2 \) are a pair of as of yet, undetermined constants and \( b \) and \( c \) are given by (39) - (40).

Our ansatz for the continuation region \( \mathcal{D} \) is that it takes the form:

\[
\mathcal{D} = \{ x < x^* | x, x^* \in \mathbb{R} \}.
\]

We now seek to determine the value of \( x^* \) and characterise the optimal intervention magnitude \( \hat{z} \).

Now by Corollary 2.6 we find that for all \( x_1 \geq x^* \) we have:

\[
\phi(x) = \mathcal{M} \phi(x) = \sup_{z \in \mathbb{Z}} \{ \phi(x - \kappa - (1 + \lambda)z + z) \}.
\]

We wish to determine the value \( z \) that maximises (49), hence let us now define the function \( G \) by the following expression:

\[
G(z) = \phi(x - \kappa - (1 + \lambda)z) + z,
\]

\( \forall z \in \mathbb{Z}, x \in \mathbb{R} \).

Our task now is to evaluate the maxima of (49) from which we readily observe that the first order condition for the maximum of \( G \) is given by:

\[
\phi'(x - \kappa - (1 + \lambda)\hat{z}) = \frac{1}{1 + \lambda}.
\]

Let us now consider a unique point \( \hat{x} \in (0, x^*) \) then:

\[
\phi'(\hat{x}) = \frac{1}{1 + \lambda}.
\]
Using (50), we now observe that the following expression holds $\forall x \in \mathbb{R}$:

$$x^* - \kappa - (1 + \lambda) \dot{x} = \dot{x}. \tag{53}$$

We now find that

$$\dot{\varepsilon}(x) = \frac{x - \dot{x} - \kappa}{1 + \lambda}. \tag{54}$$

We therefore deduce that $\phi$ is given by the following expression $\forall x \in \mathbb{R}$:

$$\phi(x) = \phi(\dot{x}) + \dot{\varepsilon}. \tag{55}$$

Using (49), we now observe that the following expression holds $\forall x \in \mathbb{R}$:

$$\phi'(\dot{x}) = \frac{1}{1 + \lambda} \tag{56}$$

$$\phi'(x^*) = \frac{1}{1 + \lambda} \tag{57}$$

$$\phi(x^*) - \phi(\dot{x}) = \frac{x^* - \dot{x} - \kappa}{1 + \lambda}. \tag{58}$$

In the following analysis, it is useful to separate the analysis into two cases starting with the case in which the transaction cost $\lambda$ is fixed (case I) and then analysing a second case in which the principal is free to choose the value of $\lambda$ (case II).

**Case I**

Inserting (47) into (50) - (53) and by the high contact principle\(^2\), we arrive at the following system of equations:

i. $$a_1 l_1 \dot{x}^{l_1} + a_2 l_2 \dot{x}^{l_2} + \frac{b}{x} = \frac{1}{1 + \lambda} \tag{59}$$

ii. $$a_1 l_1 x^{l_1} - a_2 l_2 x^{l_2} + \frac{b}{x} = \frac{1}{1 + \lambda} \tag{60}$$

iii. $$a_1 (x^{l_1} - \dot{x}^{l_1}) + a_2 (x^{l_2} - \dot{x}^{l_2}) = \frac{x^* - \dot{x} - \kappa}{1 + \lambda} + b \ln \left(\frac{\dot{x}}{x}\right) \tag{61}$$

where $b := \epsilon \delta^{-1}$.

The system of 3 equations (i) - (iii) contains 3 unknowns ($\kappa, a_1, a_2$), hence we can solve for the three unknown parameters (see [appendix]).

Using (i) - (iii) to solve for $a_1$ and $a_2$ we find that:

$$a_1 = \frac{1}{l_1} \left[ \frac{b \dot{x}^{l_2}}{x^{l_1}} - \frac{z^{l_2 - 1}}{1 + \lambda} (z^{l_2 - 1} - \dot{x}^{l_1 - 1} z^{l_1 - 1})^{-1} \right] \tag{62}$$

$$a_2 = \frac{1}{l_2} \left[ \frac{b \dot{x}^{l_2}}{x^{l_1}} - \frac{z^{l_1 - 1}}{1 + \lambda} (z^{l_1 - 1} - \dot{x}^{l_2 - 1} z^{l_2 - 1})^{-1} \right] \tag{63}$$

where $b := \epsilon \delta^{-1}$ and $z^m := \dot{x}^m - x^m$ for $m \in \mathbb{N}$.

After substituting (62) - (63) into (iii) we readily obtain the expression for the fixed cost parameter $\kappa$:

$$\kappa(\dot{x}, x^*, \lambda) = l_1^{-1} \left[ z^{l_2 - 1} - (1 + \lambda) b \frac{z^{l_2}}{x^{l_1} \dot{x}} (z^{l_1 - 1} - \dot{x}^{l_1 - 1} z^{l_1 - 1}) z^{l_1} \right]$$

$$+ l_2^{-1} \left[ z^{l_1 - 1} - (1 + \lambda) b \frac{z^{l_1}}{x^{l_2} \dot{x}} (z^{l_2 - 1} - \dot{x}^{l_2 - 1} z^{l_2 - 1}) z^{l_2} - z + b(1 + \lambda) \ln \left(\frac{\dot{x}}{x}\right) \right]. \tag{64}$$

Hence, given a pair of target cost parameters $(\dot{x}^*, \dot{x}) \in \mathbb{R} \times \mathbb{R}$, we see that any optimal control for the agent’s intervention threshold becomes $\dot{x}^*$ and optimal consumption magnitude becomes $z = x^* - \dot{x}$.

**Case II**

In order to identify the parameters $\kappa$ and $\lambda$ we set $a_1 = a_2$ in (47), then substituting into (50) - (53) we arrive at the following system of equations:

i. $$a_1 l_1 \dot{x}^{l_1} + a_2 l_2 \dot{x}^{l_2} + \frac{b}{x} = \frac{1}{1 + \lambda} \tag{65}$$

\(^2\)The high contact principle is a condition that asserts the continuity of the value function at the boundary of the continuation region see [LS05], [LS02] for further details.
ii. \[ a(l_1 x^{1i} - \hat{x}^{1i} + x^{2i} - \hat{x}^{2i}) = \frac{\hat{x}^{1i} + \hat{x}^{2i}}{1+\lambda} + b \ln \left( \frac{x}{\hat{x}} \right) \]

where the constant \( b \) is given in (49).

The system \((i)-(iii)\) of 3 equations now consists of 3 unknowns \((\kappa, \lambda, a)\), hence we can solve for the three unknown parameters. Now, eliminating the constant \( a \) from the system \((i)-(iii)\) yields the following expressions for the cost parameters:

\[ M = \frac{G(\hat{x}, x^*) - 1}{\ln(\frac{x}{\hat{x}})} \]

By inverting the procedure, we can further deduce that using equations \((i)-(iii)\), we can derive the values \( m_1, m_{-1} \) and \( H \) s.t.h.:

\[ x^* = m_1 \]
\[ \hat{x} = m_{-1} \]

where \( m_1, m_{-1} \) are solutions to the system of equations:

\[ Q(m, \kappa, \lambda) = 0 \]

for \( Q \) given by

\[ Q_1(m, c, d) := \frac{b}{m_1 m_2} \frac{l_1 M^{1i} + l_2 M^{2i}}{l_1 M^{1i} + l_2 M^{2i} - 1} - \frac{1}{1 + d} \]

\[ Q_2(m, c, d) := \frac{bM}{m_1 m_2} \frac{M^{1i} + M^{2i}}{M^{1i}} - \frac{M + c}{1 + d} + b \ln \left( \frac{m_1}{m_2} \right) \]

where \( M := m_1 - m_2 \).

To complete the proof of Theorem 3.1, we first start with the given set of target parameters \( \{x^*_i, \hat{x}_i\} \) and the observed parameters \( \{\hat{\lambda}_0, \hat{\kappa}_0\} \), using \((i)-(iii)\), we ascertain the following expressions for the parameters \( \lambda_1 \) and \( \kappa_1 \):

\[ \lambda_1 = \frac{m_{-1} m_1}{b} \frac{(l_1 M^{1i}_0 + l_2 M^{2i}_0)}{(l_1 M^{1i}_0 + l_2 M^{2i}_0 - 1)} - 1 \]

\[ \kappa_1 = M_0 \frac{M^{1i}_0 + M^{2i}_0}{l_1 M^{1i}_0 + l_2 M^{2i}_0} - \ln \left( \frac{m_{-1} + h_{-1}}{m_1 + h_1} \right) - (m_{-1} + h_{-1}) (m_1 + h_1) \left[ \frac{(l_1 M^{1i}_0 + l_2 M^{2i}_0)}{(l_1 M^{1i}_0 + l_2 M^{2i}_0)} \right] - M_0 \]

where \( M_0 := m_{-1} - m_1 \) and \( m_1, m_{-1} \) the solutions to the system of equations:

\[ Q(m, \kappa_0, \lambda_0) = 0 \]

Though it is not possible to obtain a closed analytic solution to \((64)-(65)\), the values \( m_1 \) and \( m_{-1} \) can be approximated using numerical methods.

**Proof of Proposition 3.3**

To prove Proposition 3.3, we differentiate \((i)-(iii)\) w.r.t. \( \hat{x} \) and \( x^* \) respectively and plugging in \((64)\) and \((65)\), we now observe that \( \frac{\partial \hat{x}}{\partial \lambda}, \frac{\partial x^*}{\partial \lambda}, \frac{\partial \hat{x}}{\partial \kappa}, \frac{\partial x^*}{\partial \kappa} \) are given by the following expressions:

\[ \frac{\partial \hat{x}}{\partial \lambda} = [f_1(\hat{x}, x^*)]^{-1} \]

\[ \frac{\partial x^*}{\partial \lambda} = [f_2(\hat{x}, x^*)]^{-1} \]

\[ \frac{\partial \hat{x}}{\partial \kappa} = [f_3(\hat{x}, x^*)]^{-1} \]

\[ \frac{\partial x^*}{\partial \kappa} = [f_4(\hat{x}, x^*)]^{-1} \]
where the functions $f_1, f_2, f_3, f_4$ are given by:

$$f_1(\hat{x}, x^*) = \frac{x^*}{b} \left( \frac{l_1 x^{l_1-1} + l_2 z^{l_2-1}}{l_1 z^{l_1} + l_2 z^{l_2}} \right) \left( 1 - \hat{x} \frac{\hat{\xi}^{l_1-1} + \hat{y}^{l_2-1}}{l_1 z^{l_1} + l_2 z^{l_2}} \right)$$

(76)

$$f_2(\hat{x}, x^*) = \frac{\hat{x}}{b} \left( \frac{l_1 x^{l_1-1} + l_2 z^{l_2-1}}{l_1 z^{l_1} + l_2 z^{l_2}} \right) \left( 1 + x^* \frac{\hat{\xi}^{l_1-1} + \hat{y}^{l_2-1}}{l_1 z^{l_1} + l_2 z^{l_2}} \right)$$

(77)

$$f_3(\hat{x}, x^*) = (\hat{x} - G) \left( \frac{l_1 x^{l_1-2} - l_2 z^{l_2-2}}{l_1 z^{l_1} + l_2 z^{l_2}} \right) + \frac{z^{l_1} + z^{l_2}}{l_1 z^{l_1} + l_2 z^{l_2}} \left( 1 - \frac{l_1 \hat{\xi}^{l_1-1} - l_2 \hat{y}^{l_2-1}}{l_1 z^{l_1} + l_2 z^{l_2}} \right)$$

(78)

$$f_4(\hat{x}, x^*) = (G - x^*) \left( \frac{l_1 x^{l_1-2} - l_2 z^{l_2-2}}{l_1 z^{l_1} + l_2 z^{l_2}} \right) - \frac{z^{l_1} + z^{l_2}}{l_1 z^{l_1} + l_2 z^{l_2}} \left( 1 - \frac{l_1 \hat{\xi}^{l_1-1} - l_2 \hat{y}^{l_2-1}}{l_1 z^{l_1} + l_2 z^{l_2}} \right)$$

(79)

**Proof of Corollary 3.4**

To prove Corollary 3.4, we firstly consider a control solution to the problem (25) (the solution can be obtained using Theorem 2.5). Denote the solution by $u^*_2 \in U$ so that $u^*_2 \in \arg \sup_{u \in \mathcal{U}} J_2^*(t, x)$ and $u^*_2 = \{\tau^*_2, z^*_2\}_{j \geq 1} \in \mathcal{U}$ and the sets $\{\tau^*_2\}_{j \in \mathbb{N}}$ and $\{z^*_2\}_{j \in \mathbb{N}}$ are sequences of $\mathcal{F}_\tau$-measurable intervention times and intervention magnitudes respectively. Then by Corollary 2.6 there exist constants $\hat{x}_2 \in \mathbb{R}$ and $x^*_2 \in \mathbb{R}$ s.th $\hat{\tau}_j = \inf\{s > \tau_j : X^{-u}(s) > x^*_2 \} \wedge \tau_j$ and $\hat{x} = \hat{x}_2 - x^*_2$, hence by setting $x^* = x^*_2$ and $\hat{x} = \hat{x}_2$ in Theorem 3.1 we immediately deduce the result after applying the theorem.

## 5 Conclusion

In this paper we performed an analysis of the effects of imposing transaction cost on consumption behaviour. The inclusion of a transaction cost precludes agent behaviour policies in which the agent makes purchases continuously, hence in the model studied in this paper, the agent’s behaviour is modelled using impulse control. In particular, we studied the effect of the transaction cost parameters on the consumption policy for an agent whose utility is given by a power utility function. The results of the paper provide a full characterisation of the parameters of transaction costs that sufficiently distorts the incentive of a rational agent so that the agent finds it optimal to adopt a consumption pattern that maximises the Principal’s objective. Indeed, this paper describes for the first time, a Principal-Agent model with impulse control. Although the results of the paper are studied within the context of a liquidity-consumption problem, the results are broadly applicable. Indeed, as described in Corollary 3.4, the results can be applied to any pair of impulse control problems so that the cost parameters can be fixed so as to change the optimal control to match that of some other external objective function.

An interesting avenue for future research is the effect of transaction costs on general impulse control problems in addition to specialised Principal objectives such as risk-minimisation and regime-dependant behaviour.
6 Appendix

**Technical Conditions for Theorem 2.5**

i. $\partial D$ is a Lipschitz surface - that is to say that $\partial D$ is locally the graph of a Lipschitz continuous function.

ii. $\phi \in C^{1,2}([t_0, T], (S \setminus \partial D))$ with locally bounded derivatives.

iii. $E\left[\int_t^T 1_{\partial D}(X^{s,u}(s))ds\right] = 0 \ \forall X \in S, \forall u \in U$.

iv. $X^{s,u}(\tau_S) \in \partial S$ $P$-a.s. on $\{\tau_S < \infty\}$ and $\phi(X^{s,u}(s)) \to G(X^{s,u}(\tau_S)) \cdot 1_{\{\tau_S < \infty\}}$ as $s \to \tau_S$ $P$-a.s., $\forall X \in S, \forall u \in U$.

v. The sets $\{\phi^{-1}(X^{s,u}(\tau_m)); \tau_m \in [t, \tau_S], \forall m \in \mathbb{N}\}$ are uniformly integrable $\forall X \in S, u \in U$.

vi. $E[|\phi(X^{s,u}(\tau_m))|] + |\phi(X^{s,u}(\rho_j))| + \int_t^T |L\phi(X^{s,u}(s))|ds < \infty, \forall \tau_m, \rho_j \in [t, \tau_S], u \in U$.

vii. $\hat{z}_k \in \text{argmin}_{z\in\mathbb{Z}}\{\phi(\Gamma(X(\tau_k-), z)) + c(\tau_k, z)\}$ is a Borel Measurable selection $\forall X \in S$.

**Analysis of constants in Theorem 3.1.**

We firstly prove the results of case 1. Recall equations (3) - (5):

\begin{align*}
  a_1l_1\hat{x}_l^{i-1} + a_2l_2\hat{x}_l^{i-1} + \frac{b}{\hat{x}} &= \frac{1}{1+\lambda} \quad (80) \\
  a_1l_1x^{s_l^{i-1}} + a_2l_2x^{s_l^{i-1}} + \frac{b}{x^s} &= \frac{1}{1+\lambda} \quad (81) \\
  a_1(x^{s_l^{i-1}} - \hat{x}_l^{i-1}) + a_2(x^{s_l^{i-2}} - \hat{x}_l^{i-2}) &= \frac{x^s - \hat{x} - \kappa}{1 + \lambda} + b \ln \left(\frac{\hat{x}}{x^s}\right) \quad (82)
\end{align*}

Multiplying (80) and (81) by $x^{s_l^{i-1}}$ and $\hat{x}_l^{i-1}$ respectively gives:

\begin{align*}
  a_1l_1(\hat{x}x^s)^{i-1} + a_2l_2\hat{x}_l^{i-1} - x^{s_l^{i-1}} &= \frac{x^{s_l^{i-1}}}{1+\lambda} \quad (83) \\
  a_1l_1(x^s\hat{x})^{i-1} + a_2l_2x^{s_l^{i-1}} - \hat{x}_l^{i-1} &= \frac{\hat{x}_l^{i-1}}{1+\lambda} \quad (84)
\end{align*}

Deducting (83) from (84) gives:

\begin{align*}
  a_2l_2(\hat{x}_l^{i-1} - x^{s_l^{i-1}} - x^{s_l^{i-1}}) + b\frac{x^{s_l^{i-1}} - \hat{x}_l^{i-1}}{x^s \hat{x}} &= \frac{x^{s_l^{i-1}} - \hat{x}_l^{i-1}}{1+\lambda} \quad (85)
\end{align*}

Adding and subtracting $\hat{x}_l^{i-1} - \hat{x}_l^{i-1}$ to the LHS of (85) and after performing some simple manipulations we obtain:

\begin{align*}
  a_2l_2(\hat{x}_l^{i-1} - x^{s_l^{i-1}} - x^{s_l^{i-1}}) &= \hat{x}_l^{i-1} - \hat{x}_l^{i-1} \quad (86)
\end{align*}

from which we readily obtain:

\begin{align*}
  a_2 &= l_2^{-1}\left[\frac{\hat{x}_l^{i-1}}{1+\lambda} - \frac{x^{s_l^{i-1}}}{x^s \hat{x}}\right] \quad (87)
\end{align*}

Using analogous steps we obtain the following expression for $a_1$:

\begin{align*}
  a_1 &= l_1^{-1}\left[\frac{x^{s_l^{i-1}}}{1+\lambda} - \frac{\hat{x}_l^{i-1}}{x^s \hat{x}}\right] \quad (88)
\end{align*}

After substituting $a_1$ and $a_2$ into (5) and multiplying by $1 + \lambda$ we find that:

\begin{align*}
  l_1^{-1}\left[\hat{x}_l^{i-1} - (1 + \lambda)b\frac{x^{s_l^{i-1}}}{x^s \hat{x}}\right] \left(\hat{x}_l^{i-1} - \hat{x}_l^{i-1} - \hat{x}_l^{i-1} - \hat{x}_l^{i-1}\right)^{-1} \quad (89)
\end{align*}
After which we find that:

\[
\kappa = l_1^{-1} \left[ z^{l_1 - 1} - (1 + \lambda) \frac{z^{l_2}}{x^+ x} \right] (\hat{x}^{l_1 - 1} z^{l_1 - 1} - \hat{x}^{l_2 - 1} z^{l_1 - 2}) z^{l_1}
\]   
\[+ l_2^{-1} \left[ z^{l_1 - 1} - (1 + \lambda) \frac{z^{l_1}}{x^+ x} \right] (\hat{x}^{l_2 - 1} z^{l_1 - 1} - \hat{x}^{l_1 - 1} z^{l_2 - 1}) z^{l_2}
\[+ b(1 + \lambda) \ln \left( \frac{\hat{x}}{x^+} \right) \] (90)

which is the desired result.

Lastly substituting (100) and (96) into (99) gives:

\[a(l_1 \hat{x}^{l_1 - 1} + l_2 \hat{x}^{l_2 - 1}) + \frac{b}{x^+} = \frac{1}{1 + \lambda} \] (91)

\[a(l_1 x^{*l_1 - 1} + l_2 x^{*l_2 - 1}) + \frac{b}{x^+} = \frac{1}{1 + \lambda} \] (92)

\[a(x^{*l_1} - \hat{x}^{l_1} + x^{*l_2} - \hat{x}^{l_2}) = \frac{x^* - \hat{x} - \kappa}{1 + \lambda} + b \ln \left( \frac{\hat{x}}{x^+} \right) \] (93)

We firstly observe that subtracting (92) from (91) yields the following:

\[a(l_1 z^{l_1 - 1} + l_2 z^{l_2 - 1}) = \frac{b z}{x^+ x^+}, \] (94)

After which we straightforwardly deduce that the constant \(a\) is given by the following expression:

\[a = \frac{b z}{x^+ (l_1 z^{l_1 - 1} + l_2 z^{l_2 - 1})}. \] (95)

We now observe that after multiplying (91) and (92) by \(\hat{x}\) and \(x^*\) respectively then subtracting the result we find:

\[a(l_1 z^{l_1} + l_2 z^{l_2}) = \frac{z}{1 + \lambda} \] (96)

After substituting the expression for \(a\) (95) into (96) and dividing through by \(z\) we find that:

\[\frac{b(l_1 z^{l_1} + l_2 z^{l_2})}{x^+ (l_1 z^{l_1 - 1} + l_2 z^{l_2 - 1})} = \frac{1}{1 + \lambda}, \] (97)

from which we straightforwardly deduce that

\[\lambda = \frac{x^*}{b} \left( l_1 z^{l_1 - 1} + l_2 z^{l_2 - 1} \right) - 1 \] (98)

which is the stated result.

Lastly, to derive the expression for \(\kappa\) we appeal to (93), indeed multiplying (93) by \((1 + \lambda)\) and rearranging immediately yields:

\[\kappa = a(1 + \lambda)(z^{l_1} + z^{l_2}) - b \ln \left( \frac{\hat{x}}{x^+} \right) (1 + \lambda) - z \] (99)

Now, inverting (97) gives:

\[1 + \lambda = \frac{x^* (l_1 z^{l_1 - 1} + l_2 z^{l_2 - 1})}{b(l_1 z^{l_1} + l_2 z^{l_2})} \] (100)

Lastly substituting (100) and (96) into (99) gives:

\[\kappa = z \frac{z^{l_1} + z^{l_2}}{l_1 z^{l_1} + l_2 z^{l_2}} - \ln \left( \frac{\hat{x}}{x^+} \right) x^+ \left[ \frac{l_1 z^{l_1 - 1} + l_2 z^{l_2 - 1}}{l_1 z^{l_1} + l_2 z^{l_2}} \right] - z, \] (101)

from which we deduce the result.
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