Characterizing multipartite entanglement classes via higher-dimensional embeddings

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Abstract
Witness operators are a central tool to detect entanglement or to distinguish among the different entanglement classes of multiparticle systems, which can be defined using stochastic local operations and classical communication (SLOCC). We show a one-to-one correspondence between general SLOCC witnesses and a class of entanglement witnesses in an extended Hilbert space. This relation can be used to derive SLOCC witnesses from criteria for full separability of quantum states; moreover, given SLOCC witnesses can be viewed as entanglement witnesses. As applications of this relation we discuss the calculation of overlaps between different SLOCC classes and the SLOCC classification in $2 \times 3 \times 3$-dimensional systems.

Keywords: multiparticle entanglement, entanglement witnesses, SLOCC classes

(Some figures may appear in colour only in the online journal)

1. Introduction

Entanglement is considered to be an important resource for applications in quantum information processing, making its characterization essential for the field [1, 2]. This includes its quantification and the development of tools to distinguish between different classes of entanglement. In general, entanglement is a resource if the parties are spatially separated and therefore the allowed operations are restricted to local operations assisted by classical
communication (LOCC). It can neither be generated nor increased by LOCC transformations. Hence, convertibility via LOCC imposes a partial order on the entanglement of the states, and this order has been studied in detail [3–8].

For multipartite systems the classification via LOCC is, however, even for pure states very difficult, so one may consider a coarse grained classification. This can be done using the notion of stochastic local operations assisted by classical communication (SLOCC). By definition, an SLOCC class is formed by those pure states that can be converted into each other via local operations and classical communication with non-zero probability of success [9]. SLOCC classes and their transformations have been characterized for small system sizes and symmetric states [9–18] and it has been shown that for multipartite systems there are finitely many SLOCC classes for tripartite systems with local dimensions of up to $2 \times 3 \times m$ and infinitely many otherwise [19].

Another important problem in entanglement theory is the separability problem, i.e. the task to decide whether a given quantum state is entangled or separable. Even though several criteria have been found which can decide separability in many instances [1, 2, 20–25], the question whether a general multipartite mixed state is entangled or not, remains highly non-trivial. In fact, if the separability problem is formulated as a weak membership problem, it has been proven to be computationally NP-hard [26, 27] in the dimension of the system.

One method to certify entanglement uses entanglement witnesses [2, 28, 29]. An entanglement witness is a Hermitian operator which has a positive expectation value for all separable states but gives a negative value for at least one entangled state. In opposition to other criteria, one main advantage of witnesses lies in the fact that no complete knowledge of the state is necessary and one just has to measure the witness observable. A special type of witnesses are projector-based witnesses of the form $W = \lambda \mathbb{1} - |\psi\rangle\langle\psi|$, with $\lambda$ being the maximal squared overlap between the entangled state $|\psi\rangle$ and the set of all product states. Such projector based witnesses can also be used to distinguish between different SLOCC classes [30, 31]. In that case, $\lambda$ is the maximal squared overlap between a given state $|\psi\rangle$ in SLOCC class $S_{|\psi\rangle}$ and the set of all states within another SLOCC class $S_{|\phi\rangle}$. If a negative expectation value of $W$ is measured, the considered state $\rho$ cannot be within the convex hull of $S_{|\phi\rangle}$ or lower entanglement classes. In this context one should note that such statements require an understanding of the hierarchic structure of SLOCC classes, in the sense that some classes are contained in others [30, 31].

In this paper we establish an one-to-one correspondence between general SLOCC witnesses for multipartite systems and a class of entanglement witnesses in a higher-dimensional system, built by two copies of the original one. This extends the results of [32] from the bipartite setting to the multipartite one and provides at the same time a simpler proof. The equivalence provides not only a deeper insight in the structure of SLOCC classes but enables to construct whole sets of entanglement witnesses for high-dimensional systems from the SLOCC structure of lower dimensions and vice versa. As such, from the solution for one problem, the solution to the related one readily follows.

The paper is organized as follows. In section 2 we briefly review the notion of SLOCC operations, entanglement witnesses and SLOCC witnesses. Section 3 states the main result of our work, the one-to-one correspondence among certain entanglement- and SLOCC witnesses. Furthermore, as optimizing the overlap $\lambda$ between SLOCC classes is in general a hard problem and as such often not feasible analytically, a possible relaxation of the set of separable states to states with positive partial transpose is discussed. Section 4 focuses on systems consisting of one qubit and two qutrits. Using numerical optimization, we find the maximal overlaps between all pairs of representative states of one SLOCC class and arbitrary states of another SLOCC class. The implications of these results for the hierarchic structure of SLOCC classes are then discussed. Section 5 concludes the paper and provides an outlook.
2. Preliminaries

In this section the basic notions and definitions are briefly reviewed. We start with the notion of SLOCC equivalence of two states and then move on to the definition of entanglement witnesses. Finally, we will relate both concepts by recapitulating the notion of witness operators that are able to separate between different SLOCC classes.

2.1. SLOCC classes

As mentioned before two pure states are within the same SLOCC class if one can convert them into each other via LOCC with a non-zero probability of success. It can be shown that this implies the following definition [9].

**Definition 1.** Two $N$-partite pure quantum states $|\psi\rangle, |\varphi\rangle$ are called equivalent under SLOCC if there are $N$ matrices $\{A_i\}$ $\det(A_i) \neq 0$ such that

$$
|\varphi\rangle = \bigotimes_{i=1}^{N} A_i |\psi\rangle \quad \text{and consequently} \quad |\psi\rangle = \bigotimes_{i=1}^{N} A_i^{-1} |\varphi\rangle.
$$

That is, an SLOCC class or SLOCC orbit includes all states that are related by local, invertible operators. To extend this definition to mixed states one defines the class $S|\psi\rangle$ with the representative $|\psi\rangle$ as those mixed states that can be built as convex combinations of pure states within the SLOCC orbit of $|\psi\rangle$ and of all pure states that can be approximated arbitrarily close by states within this orbit [30, 31].

2.2. Entanglement witness

An hermitean operator that can be used to distinguish between different classes of entanglement is called a witness operator. Recall that a mixed state that can be written as a convex combination of product states of the form $|\psi_s\rangle = |A\rangle |B\rangle \cdots |N\rangle$ is called fully separable, and states which are not of this form are entangled [1, 2]. A witness operator that can certify entanglement has to fulfill the following properties [28, 29]:

**Definition 2.** A hermitean operator $W$ is an entanglement witness if

(i) $\text{tr}(\rho W) \geq 0$ for all separable states $\rho$,

(ii) $\text{tr}(\rho W) < 0$ for at least one state $\rho$.

holds.

Hence, $W$ witnesses the non-membership with respect to the convex set of separable states. If $\text{tr}(\rho W) < 0$ for some state $\rho$, then $W$ is said to detect $\rho$. A special class of witness operators are projector-based witnesses. Their construction is based on the maximal value $\lambda$ of the squared overlap between a given entangled state $|\psi\rangle$ with the set of all product states $\{|\psi_s\rangle\}$. More precisely, $W = \lambda I - |\psi\rangle\langle\psi|$ with $|\psi\rangle$ being some entangled state and $\lambda = \sup_{\{|\psi_s\rangle\}} |\langle\psi|\psi_s\rangle|^2$ is a valid entanglement witness [2]. We stress, however, that this is not the most general way to construct witnesses. Other construction methods make use of various separability criteria or physical quantities like Hamiltonians in spin models [33] or structure factors [34].
2.3. SLOCC witness

The concept of entanglement witnesses can be generalized to SLOCC witnesses. An SLOCC witness is an operator from which one can conclude that a state $\rho$ is not in the SLOCC class $S_|\psi\rangle$ [30, 31].

Definition 3. A hermitean operator $W$ is an SLOCC witness for the class $S_|\psi\rangle$ if

\begin{equation}
(\text{i}) \quad \text{tr}(|\eta\rangle\langle\eta|W) \geq 0 \quad \text{for all pure states} \quad |\eta\rangle \in S_|\psi\rangle,
\end{equation}

\begin{equation}
(\text{ii}) \quad \text{tr}(\rho W) < 0 \quad \text{for at least one state} \quad \rho,
\end{equation}

holds.

Thus $W$ detects for $\text{tr}(\rho W) < 0$ states $\rho$ that are not within $S_|\psi\rangle$. Note that it suffices to check positivity on all pure states $|\eta\rangle$ in the set of mixed states $S_|\psi\rangle$, as these form the extreme points of this set. Also if one considers $|\psi\rangle = |A\rangle|B\rangle \cdots |N\rangle$, then the set of all SLOCC equivalent states are just all product states and the SLOCC witness is just a usual entanglement witness.

One can construct an SLOCC witness via

\begin{equation}
W = \lambda \mathbb{I} - |\varphi\rangle\langle\varphi|,
\end{equation}

where $\lambda$ denotes the maximal squared overlap between all pure states $|\eta\rangle$ in the SLOCC class $S_|\psi\rangle$ and the representative state $|\varphi\rangle$ of SLOCC class $S_|\psi\rangle$, i.e. $\lambda = \sup_{|\eta\rangle} |\langle\varphi|\eta\rangle|^2$. Our main result, however, does not assume this type of witness and is valid for general SLOCC witnesses.

A special class of SLOCC witnesses are those verifying the Schmidt rank of a given bipartite state. Note that the Schmidt rank is the only SLOCC invariant for bipartite systems, and a one-to-one correspondence between Schmidt number witnesses and entanglement witnesses in an extended Hilbert space has been found [32]. In the next section we will show that in fact there is a one-to-one correspondence between SLOCC- and entanglement witnesses for arbitrary multipartite systems.

3. One-to-one correspondence between SLOCC- and entanglement witnesses

In the following we will show how to establish a one-to-one correspondence between SLOCC witnesses and certain entanglement witnesses within a higher-dimensional Hilbert space for arbitrary multipartite systems. In order to improve readability, our method will be presented for the case of tripartite systems, however, the generalization to more parties is straightforward. Then, we will discuss one possibility to use this correspondence to derive an SLOCC witnesses from separability criteria.

3.1. The correspondence between the two witnesses

Let us start with formulating the problem. Consider the pure state $|\psi\rangle$, which is a representative state of the SLOCC class $S_|\psi\rangle$. Then all pure states, $|\eta\rangle$, within the SLOCC orbit of $|\psi\rangle$ can be reached by applying local invertible operators $A, B$ and $C$, that is $|\eta\rangle = A \otimes B \otimes C |\psi\rangle$. Here, one has to take care that $|\eta\rangle$ is normalized; so, if considering general matrices $A, B, C$, one has to renormalize the state. The aim will be to maximize the overlap between a given state $|\varphi\rangle$ and a pure state $|\eta\rangle$ within $S_|\psi\rangle$. 


which is the main step for constructing the projector-based witness. Stated differently, the quantity of interest is the minimal value $\lambda > 0$, such that

$$
\sup_{A,B,C} \frac{|\langle \varphi| \eta \rangle|^2}{\|A \otimes B \otimes C|\psi\rangle\|} = \lambda
$$

(5)

It can easily be seen that this is true if and only if

$$
\lambda \langle \psi| A^\dagger A \otimes B^\dagger B \otimes C^\dagger C|\psi\rangle \\
- \langle \psi| A^\dagger \otimes B^\dagger C^\dagger |\varphi\rangle \langle \varphi| A \otimes B \otimes C|\psi\rangle \geq 0
$$

(7)

holds. One can then define a witness operator $W = \lambda \mathbb{I} - |\varphi\rangle\langle\varphi|$ which, with the definition of $|\eta\rangle$ from before, satisfies:

$$
\langle \eta| W |\eta\rangle \geq 0.
$$

(8)

Note that in the formulation of equations (7) and (8) the normalization of $|\eta\rangle = A \otimes B \otimes C|\psi\rangle$ is irrelevant, this trick has already been used in [35].

The key idea to establish the connection is the following: in order to prove that $W$ is an SLOCC witness, one has to minimize in equation (7) over all matrices $A, B, C$, which do not have any constraint anymore. A matrix like $A$ acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ can be seen as a vector on the two-copy system $\mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2}$. Then, the remaining optimization is the same as optimizing over all product states in the higher-dimensional system and requesting that the resulting value is always positive. Consequently, the SLOCC witness $W$ corresponds to a usual witness $\tilde{W}$ on the higher-dimensional system. More precisely, as stated in the following theorem, one can show that if equation (8) holds, then the operator $\tilde{W} = W \otimes |\psi^*\rangle\langle\psi^*|$ is positive on all separable states $|\xi_{\text{sep}}\rangle$ and vice versa. Here and in the following $^*$ denotes complex conjugation in a product basis.

**Theorem 4.** Consider the operator $W$ on the tripartite space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and the operator $\tilde{W} = W \otimes |\psi^*\rangle\langle\psi^*|$ on the two-copy space $\mathcal{H} \otimes \mathcal{H}$. Then, $W$ is an SLOCC witness for the class $S_{\langle\psi\rangle}$, if and only if the operator $\tilde{W}$ is an entanglement witness with respect to the split $(A_1: A_2 | B_1: B_2 | C_1: C_2)$:

$$
\langle \eta| W |\eta\rangle \geq 0 \iff \langle \xi_{\text{sep}}| \tilde{W}|\xi_{\text{sep}}\rangle \geq 0,
$$

(9)

where $|\xi_{\text{sep}}\rangle$ are product states within the two-copy system, that is they are of the form $|\xi_{\text{sep}}\rangle = |\alpha_{A_1}\rangle \otimes |\beta_{B_1}\rangle \otimes |\gamma_{C_1}\rangle$ and $|\eta\rangle \in S_{\langle\psi\rangle}$.

**Proof.** The ‘only if’ part (‘$\Rightarrow$’) of the proof can be shown as follows:

One can always write the witness operator $W$ in its eigenbasis $W = \sum_n \kappa_n |\alpha^{(n)}\rangle \langle \alpha^{(n)}|$ and therefore

$$
\langle \eta| W |\eta\rangle = \sum_n \kappa_n |\langle \varphi| A^\dagger \otimes B^\dagger \otimes C^\dagger |\alpha^{(n)}\rangle|^2 \geq 0.
$$

(10)

Moreover, it holds that

$$
\langle \psi| A^\dagger \otimes B^\dagger \otimes C^\dagger |\alpha^{(n)}\rangle = \text{tr}(A^\dagger \otimes B^\dagger \otimes C^\dagger |\alpha^{(n)}\rangle \langle \varphi|).
$$

(11)
We consider a single summand in equation (10) and use the following representation of the SLOCC operations $A$, $B$, and $C$ and the state $|\psi\rangle$. We write $A = \sum_{ij} A_{ij} |i\rangle \langle j|$, $B = \sum_{ij} B_{ij} |i\rangle \langle j|$, $C = \sum_{ij} C_{ij} |i\rangle \langle j|$. \( |\alpha^{(n)}\rangle = \sum_{kk'} \alpha^{(n)}_{kk'} |kk'\rangle |kk'\rangle \) and $|\psi\rangle = \sum_{ll'} \psi_{ll'} |ll'\rangle |ll'\rangle$. Then we have
\[
\text{tr}(A^\dagger \otimes B^\dagger \otimes C^\dagger |\alpha^{(n)}\rangle \langle \psi|) = \sum\limits_{ij\ddot{ij}\dddot{ij}\dddot{l}l} A_{ij}^* B_{ij}^* C_{ij}^* \alpha^{(n)}_{ij} \psi_{ll'}, \\
\equiv \langle A_{12} \otimes B_{12} \otimes C_{12} |\alpha^{(n)}_1, \psi_2^* \rangle,
\]
(12)
where the indices 1 and 2 indicate now the copies of the system and we use ket-vectors like $|\dot{ij}\rangle$ on the two-copy Hilbert space of each particle $Y \in \{A, B, C\}$. In the same way we obtain:
\[
\text{tr}(|\psi\rangle \langle \alpha^{(n)}| A \otimes B \otimes C) \equiv \langle \alpha^{(n)}_1, \psi_2^* | A_{12} \otimes B_{12} \otimes C_{12} \rangle.
\]
(13)
Thus equation (10) can be written as
\[
\langle A_{12} \otimes B_{12} \otimes C_{12} |V_1 \otimes |\psi^*\rangle\langle \psi^* | A_{12} \otimes B_{12} \otimes C_{12} \rangle \geq 0.
\]
(14)
So far, the vectors $|Y_{12}\rangle$ with $Y \in \{A, B, C\}$ are not entirely arbitrary, as the operators $A, B$ and $C$ are invertible. However, as any non-invertible matrix can be approximated arbitrarily well by invertible matrices and the expression under consideration is continuous, the positivity condition in equation (14) holds for any vectors $|Y_{12}\rangle$. Let us finally note that it is straightforward to see that if $W$ is not positive semidefinite then $\tilde{W}$ is not positive semidefinite as well. This completes the ‘only if’ part of the proof.

The ‘if’ part of theorem (‘$\\Rightarrow$’) follows from the fact that equation (14) for all $|Y_{12}\rangle$ implies equation (10); moreover, $\tilde{W} = W \otimes |\psi^*\rangle\langle \psi^* |$ being not positive semidefinite implies that $W$ is not positive semidefinite.

In order to start the discussion, we first note that statement of the theorem clearly holds for any number of parties, the proof can be directly generalized. Also, we note that the complex conjugation $|\psi^*\rangle$ is relevant, as there are instances where $|\psi^*\rangle$ and $|\psi\rangle$ are not equivalent under SLOCC [7, 36].

Second, we compare the theorem with known results. The theorem presents a generalization of the main result from [32] from the bipartite to the multipartite case. The SLOCC classes in the bipartite case are characterized by the Schmidt number and the Schmidt witnesses considered in [32] are just the SLOCC witnesses for the bipartite case. A similar connection for the special case of bipartite witnesses for Schmidt number one has also been discussed in [37]. Furthermore, for the multipartite case, where the Schmidt number classification is a coarse graining of the SLOCC classification, a connection between Schmidt witnesses and entanglement witnesses has been proved in [38]. This connection, however, is not equivalent to ours, as the dimension of the enlarged space in [38] is in general larger.

Third, theorem 4 provides the possibility to consider the problem of maximizing the overlap of two states under SLOCC from a different perspective. That is, by solving the problem of finding the minimal value of $\lambda$, for which $W = (\lambda I - |\varphi\rangle\langle \varphi|) \otimes |\psi^*\rangle\langle \psi^* |$ is an entanglement witness for full separability one can determine the value of the maximal overlap between $|\varphi\rangle$ and $|\psi\rangle$ under SLOCC operations. In order to provide a concrete application of theorem 4, we derived the maximal squared overlap between an $N$-qubit GHZ state
\[ |\text{GHZ} \rangle = \frac{1}{\sqrt{2}} (|00\cdots0 \rangle + |11\cdots1 \rangle) \]  

(15)

and the SLOCC class of the \(N\)-qubit W state

\[ |W \rangle = \frac{1}{\sqrt{N}} (|10\cdots0 \rangle + |01\cdots0 \rangle + \cdots + |00\cdots1 \rangle) \]  

(16)

using the relation derived above in the appendix. The resulting value is 3/4 for \(N = 3\) (numerically already known from [30]) and 1/2 for \(N \geq 4\) (for four-qubit states this value has been already found in [31]). It should be noted that there is an asymmetry: while the SLOCC class of the three-qubit W state can approximate the GHZ state only to a certain degree, one can find arbitrarily close to the W state a state in the SLOCC orbit of the GHZ state [30].

Finally, our result reflects that the separability problem as well as the problem of deciding whether two tripartite states are within the same SLOCC class are both computationally highly non-trivial. In fact, they were shown to be NP-hard [26, 27, 39].

In the following section we will discuss a relaxation of witness condition to be positive on all separable states. Instead one can consider the condition that \(\tilde{W}\) should be positive on states having a positive partial transpose (PPT) for any bipartition.

3.2. Using entanglement criteria for the witness construction

In general, it can be very difficult to find an analytical solution for the minimal value of \(\lambda\) such that the expectation value of \(\tilde{W} = (\lambda I - |\varphi \rangle \langle \varphi |) \otimes |\psi^* \rangle \langle \psi^* |\) is positive on all product states \(|\xi_{\text{sep}} \rangle\). To circumvent this problem, one can try to broaden the restrictions on the set of states on which \(\tilde{W}\) is positive in a way that the new set naturally includes the original set of separable states.

One potential way to do that uses the criterion of the positivity of the partial transpose (PPT), as the set of separable states is a subset of the states which are PPT [20]. More precisely, one can demand that \(\tilde{W}\) is positive on the set of states which are PPT with respect to all subsystems in the considered bipartite splittings, i.e.

\[
\text{tr}(\rho_{A_1;B_1;C_1} \tilde{W}) \geq 0 \\
\text{for all } \rho_{A_1;B_1;C_1} \text{ with } : \rho^Y_{T_i} \geq 0, \ Y = \{A, B, C\}.
\]  

(17)

Although the set of PPT states is known to include PPT entangled states, this relaxation of the initial conditions offers an advantage, as we are able to formulate the problem of determining \(\lambda\) as a semi-definite program (SDP) and as such provides a way for an exact result [40]. For a given \(\lambda\) one can consider the optimization problem

\[
\begin{align*}
\text{minimize :} & \text{tr}(\rho \tilde{W}) \\
\text{subject to :} & \rho \geq 0, \\
& \rho_{T_i}^Y \geq 0 \text{ for } i = A, B, C, \\
& \text{tr}(\rho) = 1.
\end{align*}
\]  

(18)

Such optimization problems can be solved with standard computer algebra systems. If the obtained value in equation (18) is non-negative, the initial operator \(W = \lambda I - |\varphi \rangle \langle \varphi |\) was an SLOCC witness, so \(\lambda\) is an upper bound on the maximal overlap.

To give an example, one may use this optimization for obtaining an upper bound on the overlap between the four-qubit cluster state and the SLOCC orbit of the four-qubit GHZ state.
or vice versa. In all the interesting examples, however, one obtains only the trivial bound $\lambda = 1$. This finds a natural explanation: if $\lambda$ is the exact maximal overlap, then the witness $\tilde{W}$ detects some entangled states which are PPT with respect to any bipartition. Consequently, relaxing the positivity on separable states to positivity on PPT state is a rather wasteful approximation in our case, and the resulting estimate on $\lambda$ is also wasteful.

The key observation is that given two pure bipartite states, $|\phi\rangle$ and $|\psi^*\rangle$ in a $d_1 \times d_1$ and $d_2 \times d_2$ system, respectively, the total state

```
| \psi_0 \rangle | \psi_1 \rangle | \psi_2 \rangle | \psi_3 \rangle | \psi_4 \rangle | \psi_5 \rangle | \psi_6 \rangle | \psi_7 \rangle | \psi_8 \rangle | \psi_9 \rangle | \psi_{10} \rangle | \psi_{11} \rangle | \psi_{12} \rangle | \psi_{13} \rangle | \psi_{14} \rangle | \psi_{15} \rangle | \psi_{16} \rangle | \psi_{17} \rangle \\
| \psi_0 \rangle | 1 | 2/3 | 2/3 | 2/3 | 3/4 | 3/4 | 0.5625 | 3/4 | 3/4 | 0.65
| \psi_1 \rangle | 3/4 | 1 | 2/3 | 2/3 | 2/3 | 3/4 | 3/4 | 0.5433 | 0.7 | 3/4 | 0.6129
| \psi_2 \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_3 \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_4 \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_5 \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_6 \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_7 \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_8 \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_9 \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_{10} \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_{11} \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_{12} \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_{13} \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_{14} \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_{15} \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_{16} \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
| \psi_{17} \rangle | 1 | 1 | 1 | 2/3 | 2/3 | 0.875 | 3/4 | 3/4 | 3/4 | 3/4 | 0.7252
```

Table 1. This table shows the numerical values for the maximal squared overlap between $|\psi_i\rangle$ (column) and the SLOCC orbit of $|\psi_j\rangle$ (row). See text for further details.
one can construct a specific splitting. This can be easily understood using our result and the results of [32], as in this case representatives and SLOCC classes where discussed above, typically also detects PPT entangled states.

The unnormalized representative states of the fully entangled SLOCC classes within a system are [19]:

\[ |\psi_0\rangle = |000\rangle + |111\rangle, \]
\[ |\psi_1\rangle = |000\rangle + |011\rangle + |101\rangle, \]
\[ |\psi_2\rangle = |000\rangle + |011\rangle + |102\rangle, \]
\[ |\psi_3\rangle = |000\rangle + |011\rangle + |120\rangle, \]
\[ |\psi_4\rangle = |000\rangle + |011\rangle + |122\rangle, \]
\[ |\psi_5\rangle = |000\rangle + |011\rangle + |101\rangle + |102\rangle + |120\rangle, \]
\[ |\psi_6\rangle = |000\rangle + |011\rangle + |110\rangle + |121\rangle, \]
\[ |\psi_7\rangle = |000\rangle + |011\rangle + |102\rangle + |120\rangle, \]
\[ |\psi_8\rangle = |000\rangle + |011\rangle + |112\rangle, \]
\[ |\psi_9\rangle = |000\rangle + |011\rangle + |110\rangle + |121\rangle, \]
\[ |\psi_{10}\rangle = |000\rangle + |011\rangle + |022\rangle + |101\rangle, \]
\[ |\psi_{11}\rangle = |000\rangle + |011\rangle + |022\rangle + |101\rangle + |112\rangle, \]
\[ |\psi_{12}\rangle = |000\rangle + |011\rangle + |110\rangle + |121\rangle, \]
\[ |\psi_{13}\rangle = |000\rangle + |011\rangle + |112\rangle + |120\rangle, \]
\[ |\psi_{14}\rangle = |000\rangle + |011\rangle + |100\rangle + |122\rangle, \]
\[ |\psi_{15}\rangle = |000\rangle + |011\rangle + |022\rangle + |101\rangle, \]
\[ |\psi_{16}\rangle = |000\rangle + |011\rangle + |022\rangle + |101\rangle + |112\rangle. \] (20)

as a state on a \(d_1d_2 \times d_1d_2\)-system is PPT, but typically entangled. This holds for nearly arbitrary choices for \(|\phi\rangle\) and \(|\psi^*\rangle\) and small values of \(p\) [41]. Note that states of the form given in equation (19) lead to \(\text{tr}[\lambda (I - |\phi\rangle\langle \phi|) \otimes (|\psi^*\rangle\langle \psi^*|)\sigma] < 0\) for any \(\lambda < 1\), so they are detected by the witness \(W\). Hence, the relaxation to states that are PPT does, for general \(|\phi\rangle\) and \(|\psi\rangle\) not allow to determine possible non-trivial values of \(\lambda\) for which \(W\) is an entanglement witness.

We mention that in [41] operators of the form \((\lambda I - |\phi\rangle\langle \phi|) \otimes (|\psi^*\rangle\langle \psi^*|)\) with an appropriate choice of \(\lambda\) have been shown to be bipartite entanglement witnesses for the case where the Schmidt rank of \(|\psi^*\rangle\) is smaller than the Schmidt rank of \(|\phi\rangle\) for the considered bipartite splitting. This can be easily understood using our result and the results of [32], as in this case \(|\phi\rangle\) and \(|\psi\rangle\) are in different bipartite SLOCC classes and \(|\phi\rangle\) cannot be approximated arbitrarily close by a state in the SLOCC class of \(|\psi\rangle\).

Finally, we add that considering other relaxations of the set of separable states can provide a way to estimate the maximal SLOCC overlap using an SDP. Here, other positive maps besides the transposition, such as the Choi map [1], or the SDP approach of [24] seems feasible.

4. SLOCC overlaps for \(2 \times 3 \times 3\) systems

Systems consisting of one qubit, one qutrit and one system of arbitrary dimension mark the last cases, which still have a finite number of SLOCC classes [19], and for general systems the number of SLOCC classes is infinite [10]. For one qubit and two qutrits there are 17 different classes with 12 of these being truly tripartite entangled and six of them containing entangled states with maximal Schmidt rank across the bipartitions [13, 19]. Finding the maximal overlap of the representative states of the different classes not only indicates towards an hierarchy among them, but, as shown in section 3, gives insight in the entanglement properties of states in an enlarged two-copy system. In fact, one can then construct entanglement witnesses, \(W\) which detect entanglement within states of dimension \(4 \times 6 \times 6\). Thus, for all pairs of representatives and SLOCC classes where \(\lambda < 1\) one can construct a specific \(W\) which, as discussed above, typically also detects PPT entangled states.

The unnormalized representative states of the fully entangled SLOCC classes within a \(2 \times 3 \times 3\) system are [19]:

\[ \sigma = \frac{1 - p}{(d_1 - 1)(d_2 - 1)} \left( (I_1 - |\phi\rangle\langle \phi|) \otimes (I_2 - |\psi^*\rangle\langle \psi^*|) \right) + p|\phi\rangle\langle \phi| \otimes |\psi^*\rangle\langle \psi^*|. \] (19)
One can compute the overlap between one of these states and the SLOCC orbit of another state via direct optimization. As for the GHZ class and the W state, it can happen that one class can approximate one state arbitrarily well, so we set the overlap to one, if the numerical obtained value approximates this with a numerical precision of $10^{-12}$. Note that an exact value of one is impossible, as the SLOCC classes are proven to be different.

The values of the numerical maximization of the SLOCC overlap for the different SLOCC classes with respect to the representative states from above is given in table 1. They should be interpreted as follows: for the overlaps between $|\psi_6\rangle$ and $|\psi_7\rangle$ two different values are given. The value $\lambda = 1$ means that the SLOCC orbit of state $|\psi_6\rangle$ approximates $|\psi_7\rangle$ arbitrarily well. The value $\lambda = 3/4$ means that the SLOCC orbit of $|\psi_7\rangle$ cannot approximate $|\psi_6\rangle$ so well, only an overlap of $\lambda = 3/4$ can be reached. This implies that $W = \frac{3}{4} \times 1 - |\psi_6\rangle\langle\psi_6|$ is an SLOCC witness, discriminating $|\psi_6\rangle$ from the SLOCC orbit of $|\psi_7\rangle$. Note that $|\psi_6\rangle$ and $|\psi_7\rangle$ are essentially the three-qubit GHZ- and W states encountered above.

This also has consequences for the classification of mixed states, see figure 1. For a mixed state, one may ask whether it can be written as a convex combination of pure states within some SLOCC class. If a state can be written as such a convex combination of states from the orbit of $|\psi_7\rangle$, it can also be written with states from the orbit of $|\psi_6\rangle$, as the latter can approximate the former arbitrarily well. Consequently, there is an inclusion relation for the mixed states, as depicted in figure 1.

5. Conclusions

For arbitrary numbers of parties and local dimensions we showed a one-to-one correspondence between an operator $W$ able to distinguish between different SLOCC classes of a system and another operator $\tilde{W}$ that detects entanglement in a two-copy system. This correspondence thereby enables us to directly transfer a solution for one problem to the other. Though the relaxation to PPT states in order to construct the entanglement witness did not prove to be helpful for reasons stated in section 3, it very well might be that other possible relaxations on the set of separable states will give more insight and a good approximation for an upper bound on the maximal overlap. As an concrete application of the presented relation we derived the maximal overlap between the $N$-qubit GHZ state and states within the $N$-qubit W class. The calculations in section 4 for the qubit-qutrit–qutrit system do not only indicate a hierarchy among the SLOCC classes but also provides us with the option to construct a whole set of entanglement witnesses for the doubled system of dimensions $4 \times 6 \times 6$.

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Appendix. Maximal squared overlap between the GHZ-state and states in the W-class

In this appendix we will provide an example of how the relation among SLOCC witnesses and entanglement witnesses can be employed and compute the maximal squared overlap between the GHZ-state of $N$-qubits, $|\text{GHZ}_N\rangle = 1/\sqrt{2}(|00\ldots0\rangle + |11\ldots1\rangle)$, and
a normalized $N$-qubit state in the W-class (with representative $|W_N⟩$ = $1/\sqrt{N}(|00⋯01⟩ + |010⋯00⟩ + ⋯ + |0⋯010⟩ + |0⋯01⟩)$). We show that for $N=3$ the maximal squared overlap is given by $\frac{1}{4}$ (see also [30]), whereas for $N = 4$ it is given by $\frac{1}{2}$. For 4-qubit states this value has been already found in [31].

In order to do so we consider

$$\mathcal{W}_N = (\lambda_N |1⟩⟨1| - |GHZ_N⟩⟨GHZ_N|) \otimes |W_N⟩⟨W_N|$$  \hspace{1cm} (A.1)

and show that it is an entanglement witness (for 2$N$-qubit states) with respect to the splitting $\bigotimes U_{i} (|00⋯0⟩ + |10⋯0⟩ + ⋯ + |01⋯0⟩ + |00⋯1⟩)$ with $x_0 \geq 0$, $x_i > 0$ for $i \in \{1, \ldots, N\}$ and $U_i$ unitary [9]. Note that we do not impose that the states are normalized. Equivalently, one can write it as $U_1 D_1 \otimes U_2 D_2 \otimes \cdots \otimes U_{N-2} D_{N-2} \otimes U_{N-1} D_{N-1} \otimes U_N D_N \langle W_N|$ where $D_i = \text{diag}(1, \overline{x}_i)$ with $\overline{x}_i = x_i/x_N > 0$ and

$$g_{N-1} = \left(\begin{array}{ccc} x_N & x_0 & x_{N-1} \\ 0 & x_N & \end{array}\right)$$ \hspace{1cm} (A.2)

For the local unitaries on the qubits we will use the parametrization $U_i = U_{ph}(\gamma_i)X(\alpha_i)U_{ph}(\beta_i)$ with $X(\delta) = e^{i\delta X}$ and $U_{ph}(\delta) = \text{diag}(1, e^{i\delta})$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$. In order to simplify our argumentation we will use the symmetry that $\bigotimes U_{ph}(\delta)|W_N⟩ = e^{i\delta}|W_N⟩$ and choose $\beta_N = 0$, $\beta_i = \beta_i - \beta_N$ for $i \in \{1, \ldots, N-2\}$ and $x_i = x_N e^{-i\delta_N}$ for $j = 0, N-1$. Furthermore, using for the GHZ state the symmetry that $U_{ph}(\delta_1) \otimes U_{ph}(\delta_2) \otimes \cdots \otimes U_{ph}(\delta_{N-2}) \otimes U_{ph}(−\sum_{i\in\mathbb{Z}} \delta_i) \otimes U_{ph}(\delta_N)|GHZ_N⟩ = |GHZ_N⟩$ where here and in the following $I_0 = \{1, 2, \ldots, N-2, N\}$ one can easily see that when computing the maximal SLOCC overlap between the GHZ state and a W class state one can equivalently choose $\gamma_i = 0$ for $i \in I_0$ and $\gamma_{N-1} = \sum_{i=1}^{\infty} \gamma_i$.

We will now make use of the fact that $⟨\eta| (\lambda_N |1⟩⟨1| - |GHZ_N⟩⟨GHZ_N|)|\eta⟩ \geq 0$ for $|\eta⟩ = A \otimes B \otimes \cdots \otimes Z|W_N⟩$ if and only if $⟨\xi_{\text{SEP}}| (\lambda_N |1⟩⟨1| - |GHZ_N⟩⟨GHZ_N|)|\xi_{\text{SEP}}⟩ \geq 0$ for $|\xi_{\text{SEP}}⟩ = |A⟩ \otimes |B⟩ \otimes \cdots \otimes |Z⟩$ with $|I_{\text{SE}}⟩ = |I⟩ \otimes |I⟩ \otimes |I⟩ \otimes |I⟩$ for $i \in I_0$ and $|\phi_{N-1}⟩ = (U_{N-2} \otimes \mathbb{I} \otimes \mathbb{1})|\phi⟩$. As before the expectation value of $\mathcal{W}_N$ for states with some separable $|\phi⟩$ can be approximated arbitrarily close by the expectation value for a state $|\xi_{\text{SEP}}⟩$ for which all $|\phi⟩$ are entangled. Note that $⟨\xi_{\text{SEP}}|\mathcal{W}_N|\xi_{\text{SEP}}⟩ \geq 0$ for all $|\xi_{\text{SEP}}⟩$ as defined above iff the operator $\tilde{\mathcal{W}}_N ≡ ⟨\xi_{\text{SEP}}|\mathcal{W}_N|\xi_{\text{SEP}}⟩ \geq 0$ is positive semidefinite for all $\xi_{\text{SEP}} = \otimes_{i\in\mathbb{Z}} |\phi⟩$ with $|\phi⟩$ as defined above. This is due to the fact that the parameters of $|\xi_{\text{SEP}}⟩$ and $|\phi⟩$ can be chosen independently and $|\phi⟩$ is an arbitrary state.

One obtains for the respective terms of $\tilde{\mathcal{W}}_N$ that
\[
\langle \text{SEP} | \mathbb{I}_1 \otimes (|W_N\rangle\langle W_N|)_2 | \text{SEP} \rangle = \frac{1}{N^2} \mathbb{I}_{\Gamma_1} \otimes \mathbb{I}_{\Gamma_2} + \sum_{i=1}^{N-2} \tilde{x}_i^2 \langle 0|0 \rangle_{\Gamma_1}. \tag{A.3}
\]

where \( \Gamma \) refers to party \( N - 1 \). The other term can be written as

\[
\langle \text{SEP} | (|\text{GHZ}_N\rangle\langle \text{GHZ}_N|)_1 \otimes (|W_N\rangle\langle W_N|)_2 | \text{SEP} \rangle = \langle \varphi \rangle_{\Gamma_1, \Gamma_2},
\]

with

\[
|\varphi\rangle_{\Gamma_1, \Gamma_2} = \frac{1}{\sqrt{2N}} \left\{ \sum_{j \in I_0} |\psi(\alpha_j)\rangle e^{-i\beta_j} \prod_{k \in \mathcal{E} \setminus \{j\}} \cos(\alpha_k) \right| 0 \rangle_{\Gamma_1} + \sum_{j \in I_0} |\psi(\alpha_j)\rangle e^{-i\beta_j} \prod_{k \in \mathcal{E} \setminus \{j\}} (-i \sin(\alpha_k)) \right| 1 \rangle_{\Gamma_1} \otimes |0 \rangle_{\Gamma_2} + \prod_{j \in I_0} \cos(\alpha_j) |0 \rangle_{\Gamma_1} + \prod_{j \in I_0} (-i \sin(\alpha_j)) |1 \rangle_{\Gamma_1} \otimes |1 \rangle_{\Gamma_2} \right\}
\]

\[
\equiv |\varphi_0\rangle_{\Gamma_1, |0 \rangle_{\Gamma_2}} + |\varphi_1\rangle_{\Gamma_1, |1 \rangle_{\Gamma_2}}. \tag{A.4}
\]

Hence, we have that \( \bar{W}_N = \frac{N}{N^2} \mathbb{I}_{\Gamma_1} \otimes \mathbb{I}_{\Gamma_2} + \sum_{i=1}^{N-2} \tilde{x}_i^2 \langle 0|0 \rangle_{\Gamma_1} = \langle \varphi \rangle_{\Gamma_1, \Gamma_2} \). Defining \( \mu = ||\varphi_0|| \) and \( \nu = ||\varphi_1|| \) we can write \( |\varphi\rangle = \mu |\Phi_0\rangle_{\Gamma_1, |0 \rangle_{\Gamma_2}} + \nu |\Phi_1\rangle_{\Gamma_1, |1 \rangle_{\Gamma_2}} \) where \( ||\Phi_i|| = 1 \).

We construct now the following orthonormal basis:

\[
|\psi_0\rangle = \frac{\mu}{\sqrt{\mu^2 + \nu^2}} |\Phi_0\rangle_{\Gamma_1, |0 \rangle_{\Gamma_2}}, \tag{A.5}
\]

\[
|\psi_1\rangle = \frac{\nu}{\sqrt{\mu^2 + \nu^2}} |\Phi_0\rangle_{\Gamma_1, |0 \rangle_{\Gamma_2}} - \frac{\mu}{\sqrt{\mu^2 + \nu^2}} |\Phi_1\rangle_{\Gamma_1, |1 \rangle_{\Gamma_2}}, \tag{A.6}
\]

\[
|\psi_2\rangle = |\Phi_0\rangle_{\Gamma_1, |0 \rangle_{\Gamma_2}}, \tag{A.7}
\]

\[
|\psi_3\rangle = |\Phi_1\rangle_{\Gamma_1, |1 \rangle_{\Gamma_2}}, \tag{A.8}
\]

where \( |\Phi_i\rangle |\Phi_i^+\rangle = 0 \) for \( i \in \{0, 1\} \).

It can be easily seen that \( \bar{W}_N = \sum_{j=0}^{1} \lambda_j |\psi_j\rangle \langle \psi_j| + \frac{\lambda_N}{N} \left( 1 + \sum_{i=1}^{N-2} \tilde{x}_i^2 \right) |\psi_2\rangle \langle \psi_2| + \frac{\lambda_N}{N} |\psi_3\rangle \langle \psi_3| \) with

\[
\Lambda = \left( \begin{array}{cc}
\frac{\lambda_N}{N} & \sum_{i=1}^{N-2} \frac{x_i^2 \mu^2}{\mu^2 + \nu^2} \\
\sum_{i=1}^{N-2} \frac{x_i^2 \mu^2}{\mu^2 + \nu^2} & \frac{\lambda_N}{N} (1 + \sum_{i=1}^{N-2} x_i^2 \mu^2)
\end{array} \right).
\tag{A.9}
\]

Note that as we consider the case \( \lambda_N > 0 \) (otherwise \( \bar{W}_N < 0 \) which implies that it cannot be an entanglement witness) and as \( \tilde{x}_i \in \mathbb{R} \) we have that \( \bar{W}_N \geq 0 \) iff \( \Lambda \succeq 0 \). In order to determine for which values of \( \lambda_N \) the matrix \( \Lambda \) is a positive semidefinite matrix we impose that \( \text{tr}(\Lambda) \geq 0 \) and \( \det(\Lambda) \geq 0 \). It can be easily seen that \( \det(\Lambda) \geq 0 \) implies \( \text{tr}(\Lambda) \geq 0 \) and one straightforwardly obtains that \( \Lambda \succeq 0 \) iff \( \frac{\lambda_N}{N} \geq \frac{\mu^2}{\sum_{i=1}^{N-2} x_i^2} + \nu^2 \). Hence, the minimal \( \lambda_N \) for which \( \bar{W}_N \) is an entanglement witness is given by

\[
\lambda_N^C = \sup_{\tilde{x}_i, \alpha_i, \beta_i \in \mathbb{R}} N \left( \frac{\mu^2}{\sum_{i=1}^{N-2} x_i^2} + \nu^2 \right). \tag{A.10}
\]

One can easily derive from equation (A.4) that
\[ \mu^2 = \frac{1}{2N} \left[ \sum_{j \in I} \sin(\alpha_j) \tilde{x}_j e^{-i\delta_j} \prod_{k \in I \setminus \{j\}} \cos(\alpha_k) \right]^2 + \left| \sum_{j \in I} \cos(\alpha_j) \tilde{x}_j e^{-i\delta_j} \prod_{k \in I \setminus \{j\}} \sin(\alpha_k) \right|^2 \] (A.11)

and

\[ \nu^2 = \frac{1}{2N} \prod_{j \in I} \cos^2(\alpha_j) + \prod_{j \in I} \sin^2(\alpha_j). \] (A.12)

Note that as \( \sum_j a_j \leq \sum_j |a_j| \) for any complex numbers \( a_j \) (and as any possible pair of values of \( |\sin(\delta)| \) and \( |\cos(\delta)| \) is attained for \( \delta \in [0, \pi/2] \) and \( \sin(\delta) \geq 0 \) and \( \cos(\delta) \geq 0 \) for this parameter range) one obtains that the supremum in equation (A.10) is attained for \( \beta_i = 0 \) and \( \alpha_i \in [0, \pi/2] \).

We will in the following distinguish between \( N = 3 \) and \( N \geq 4 \) and first discuss the case \( N = 3 \). Inserting the corresponding expressions for \( \mu^2 \) and \( \nu^2 \) in equation (A.10) and using \( \beta_1 = \beta_2 = 0 \) one straightforwardly obtains that

\[ \lambda_3^\mathcal{C} = \sup_{x, \alpha_1, \alpha_3 \in \mathbb{R}} \frac{1}{2} \left[ 1 + \frac{x}{1 + x^2} \sin(2\alpha_1) \sin(2\alpha_3) \right]. \] (A.13)

It is easy to see that therefore the supremum is obtained for \( \alpha_1 = \alpha_3 = \pi/4 \) and \( x = 1 \) which implies that \( \lambda_3^\mathcal{C} = \frac{3}{4} \). Hence, if \( \lambda_3 < 1 \) as for \( \lambda_3 \geq 1 \) the operator \( \hat{\mathcal{W}}_3 \) is positive semidefinite. However, it should be noted that \( \hat{\mathcal{W}}_3 \) is only an entanglement witness if \( \lambda_3 < 1 \) as for \( \lambda_3 \geq 1 \) the operator \( \hat{\mathcal{W}}_3 \) is positive semidefinite and there exists no state that is detected. A state that attains the maximum overlap of 3/4 is given by \( |\pm\rangle \rangle = 1/\sqrt{3}(|++\rangle + |--\rangle + |\cdots\rangle) \) with \(|\pm\rangle \rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle) \). Using \( \lambda_3 = 3/4, \beta_1 = \beta_2 = 0, x = 1 \) and \( \alpha_1 = \alpha_3 = \pi/4 \) the remaining parameters for a state in the W class that attains the maximum can be obtained by calculating the eigenvector of \( \hat{\mathcal{W}}_3 \) for the eigenvalue 0. Note that in order to obtain the state presented here symmetries of the GHZ and W state have been used.

We will proceed with \( N \geq 4 \) and will use that the supremum is attained for \( \beta_i = 0 \). Note that then \( \frac{\mu^2}{\sum_{j \in \mathcal{N}} \nu^2} \) can be equivalently written as

\[ (\tilde{\nu}_0 \cdot \tilde{\nu}_1)^2 + (\tilde{\nu}_0 \cdot \tilde{\nu}_2)^2, \] (A.14)

where

\[ \tilde{\nu}_0 = \frac{1}{\sqrt{\sum_{i \in \mathcal{N}} \tilde{x}_i^2}} (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{N-2}, \tilde{x}_N) \] (A.15)

\[ \tilde{\nu}_1 = (y_1, \ldots, y_{N-2}, y_N) \text{ with } y_j = \frac{1}{\sqrt{2N}} \sin(\alpha_j) \prod_{k \in \mathcal{N} \setminus \{j\}} \cos(\alpha_k) \] (A.16)

\[ \tilde{\nu}_2 = (z_1, \ldots, z_{N-2}, z_N) \text{ with } z_j = \frac{1}{\sqrt{2N}} \cos(\alpha_j) \prod_{k \in \mathcal{N} \setminus \{j\}} \sin(\alpha_k). \] (A.17)

Hence, one obtains

\[ \lambda_3^\mathcal{C} = \sup_{\tilde{x}, \alpha_i \in \mathbb{R}} N[(\tilde{\nu}_0 \cdot \tilde{\nu}_1)^2 + (\tilde{\nu}_0 \cdot \tilde{\nu}_2)^2 + \nu^2] \leq \sup_{\alpha_i \in \mathbb{R}} N[(\tilde{\nu}_1)^2 + (\tilde{\nu}_2)^2 + \nu^2] \] (A.18)
\[
\lambda^S_{ij} \leq \sup_{\alpha_i \in \mathbb{R}} \frac{1}{2} \left( \sum_{j \in \mathcal{I}_0} \cos^2(\alpha_j) \prod_{k \in \mathcal{I}_0 \setminus \{j\}} \sin^2(\alpha_k) + \sum_{j \in \mathcal{I}_0} \sin^2(\alpha_j) \prod_{k \in \mathcal{I}_0 \setminus \{j\}} \cos^2(\alpha_k) \right)
\]
\[
+ \prod_{j \in \mathcal{I}_0} \cos^2(\alpha_j) + \prod_{j \in \mathcal{I}_0} \sin^2(\alpha_j)
\]
\[
= \sup_{\alpha_i \in \mathbb{R}} \frac{1}{2} \left( \sum_{j \in \mathcal{I}_0 \setminus \{N\}} \cos^2(\alpha_j) \prod_{k \in \mathcal{I}_0 \setminus \{j\}} \sin^2(\alpha_k) + \sum_{j \in \mathcal{I}_0} \sin^2(\alpha_j) \prod_{k \in \mathcal{I}_0 \setminus \{j\}} \cos^2(\alpha_k) \right)
\]
\[
+ \prod_{j \in \mathcal{I}_0 \setminus \{N\}} \cos^2(\alpha_j) + \prod_{j \in \mathcal{I}_0 \setminus \{N\}} \sin^2(\alpha_j)
\]
\[
\leq \sup_{\alpha_i \in \mathbb{R}} \frac{1}{2} \left( \sum_{j \in \{1,2,3\}} \cos^2(\alpha_j) \prod_{k \in \{1,2,3\}, k \neq j} \sin^2(\alpha_k) + \sum_{j \in \{1,2,3\}} \sin^2(\alpha_j) \prod_{k \in \{1,2,3\}, k \neq j} \cos^2(\alpha_k) \right)
\]
\[
+ \prod_{j \in \{1,2,3\}} \cos^2(\alpha_j) + \prod_{j \in \{1,2,3\}} \sin^2(\alpha_j)
\]
\[
\leq \frac{1}{2}.
\]
(A.19)

Note that for the second inequality we used that \(0 \leq \cos^2(\alpha_i) \leq 1\) and \(0 \leq \sin^2(\alpha_i) \leq 1\) and then repeatedly applied the same argumentation. Note further that the upper bound obtained in the last line is equal to 1/2 independent of the value of the parameters \(\alpha_i\) for \(i \in \{1,2,3\}\). As the state \((00 \ldots 0)\) which can be approximated arbitrarily close by a state in the W class has a squared overlap with the GHZ state of 1/2 we also have that \(\lambda^W_{ij} \geq 1/2\). Hence, one obtains \(\lambda^W_{ij} = 1/2\) for \(N \geq 4\). Note that this is also the maximal squared overlap between the GHZ state and an arbitrary separable state.

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