Neutral meson properties in hot and magnetized quark matter

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Abstract. In this work we evaluate the $\pi^0$ pole-mass in the RPA approximation at finite magnetic field and temperature in the NJL SU(2) model. To this end, we employ an alternative version of the Magnetic Field Independent Regularization based on the Riemann-Hurwitz zeta function. To employ this formalism, we present a set of equations applied to the gap equation and pseudo-scalar polarization loop at the mean field approximation and random phase approximation.

1. Introduction
The importance of understanding the behavior of the pole-mass of neutral mesons at finite magnetic field and $T = 0$ in the context of the NJL SU(2) model has been the subject of some previous studies[1, 2, 3, 4]. Also, the importance of adopting an appropriate regularization scheme in NJL-type models has been calling the attention [5, 6, 7, 8, 9] in recent years. In a magnetized medium, it has been clearly demonstrated the importance of the regularization scheme, where the choice of some prescriptions may give rise to spurious solutions [6, 7]. These problems can be avoided if one chooses regularization schemes where the explicit separation of the vacuum and magnetized medium contributions are done. Recently, these methods have been baptized as MFIR (magnetic field independent regularization) in ref [6].

In this work, we wish to show an alternative way to explore this separation, introducing the formalism of ref. [10] where a study of a magnetized relativistic electron gas was made in terms of the Hurwitz-Riemann zeta function. Some works studying Bose-Einstein condensation of relativistic fermions at finite magnetic field have applied some similar ideas [11, 12]. We apply this formalism for the regularization of the quark mass gap equation and polarization loop integral within the SU(2) Nambu-Jona-Lasinio model in the mean field approximation in a hot and magnetized medium.

2. Formalism
2.1. zMFIR - a regularization scheme based on the Hurwitz-Riemann zeta function
The Grand canonical potential and the thermodynamical properties, in general, can be derived in a variety of effective models. We can start with the general structure of this quantity, which is:
\[ I = \sum_{f} \sum_{s=\pm1} \int \frac{d^3p}{(2\pi)^3} f(E_f), \quad E_f = \sqrt{p^2 + M_f^2}, \quad (1) \]

We can make use for the following prescription, to pass from non-magnetized to magnetized system

\[ I_f(0) = I_f(B = 0) = \sum_{s=\pm1} \int \frac{d^3p}{(2\pi)^3} f(E_f) \rightarrow I_f(B) = \beta_f \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_3}{(2\pi)^3} f(E_n), \quad (2) \]

\[ E_f = \sqrt{p^2 + M_f^2} \rightarrow E_n = \sqrt{p_3^2 + M_f^2 + 2\beta_f n}, \]

The key ingredient in this formalism is the non-normalized density of states, that will allow us to formulate the new regularization scheme, given by:

\[ g_f(E, B) = \frac{\beta_f}{(2\pi)^2} \sum_{n=0}^{\infty} g_n \int_{-\infty}^{\infty} dp_3 \delta(E - E_n), \quad (3) \]

where the integral \( I_f(B) \) can be rewrited as:

\[ I_f(B) = \int_{M_f}^{\infty} dE \ g_f(E, B) \ f(E). \quad (4) \]

Following the analytical procedures of [10, 13], we can obtain after some straightforward steps:

\[ g_f(E, B) = E \left( \frac{2\beta_f}{(2\pi)^2} \right)^{1/2} \left\{ 2 \left[ \zeta\left( \frac{1}{2}, \{q_E\} \right) - \zeta\left( \frac{1}{2}, q_E + 1 \right) \right] - \frac{1}{q_E^{1/2}} \right\}, \quad (5) \]

where \( \{q_E\} \equiv q_E - [q_E] \) is the fractional part of \( q_E \) and \( \zeta(x, y) \) is the Riemann-Hurwitz zeta function.

Now, we can separate \( g_f(E, B) = g_f(E) + \bar{g}_f(E, B) \), where \( g_f(E) \) is the non-magnetic contribution:

\[ g_f(E) = \frac{E \sqrt{E^2 - M_f^2}}{\pi^2}, \quad (6) \]

and \( \bar{g}_f(E, B) \) is the purely magnetic contribution:

\[ \bar{g}_f(E, B) = E \left( \frac{2\beta_f}{2\pi^2} \right)^{1/2} \left[ \zeta\left( \frac{1}{2}, \{q_E\} \right) - \zeta\left( \frac{1}{2}, q_E \right) - 2q_E^{-1/2} + \frac{1}{2q_E^{1/2}} \right]. \quad (7) \]

Now, we can rewrite the integral in terms of these two contributions \( I_f(B) = I_f(0) + \tilde{I}_f(B) \), where from eq.(6) we can obtain the vacuum contribution of the model and the pure magnetic contribution. Applying these results in eq.(4), and defining:

\[ \tilde{\mathcal{H}}_{1/2}(q_E) = \left[ \zeta\left( \frac{1}{2}, \{q_E\} \right) - \zeta\left( \frac{1}{2}, q_E \right) - 2q_E^{-1/2} + \frac{1}{2q_E^{1/2}} \right]. \quad (8) \]
we can obtain the gap equation for the magnetized NJL model, using $f(E_\alpha) = (p_\alpha^2 + M^2 + 2\beta p_e)\alpha^{-1/2}$ applied to eq.(2) by [13]:

$$\frac{M - m}{2MG} = I_{vac} + I_G(T, \mu) + I(B) + I(B, T) \, .$$

(9)

where each quantity is given by

$$I_G = \frac{N_c}{\pi^2} \left[ \Lambda \epsilon_A - M^2 \ln \left( \frac{\Lambda + \epsilon_A}{M} \right) \right],$$

$$I_G(T, \mu) = -2N_cN_f \int \frac{d^3p}{(2\pi)^3} \frac{n(E) + \tilde{n}(E)}{E},$$

$$I_G(B) = N_c \sum_{f=u,d} \int_0^\infty dq_E \frac{(2\beta_f)^{3/2}}{(2\pi)^2} \tilde{H}_{1/2}(qE) \frac{1}{E(qE)},$$

$$I_G(B, T, \mu) = -N_c \sum_{f=u,d} \int_0^\infty dq_E \frac{(2\beta_f)^{3/2}}{(2\pi)^2} \tilde{H}_{1/2}(qE) \left[ \frac{n(E(qE)) + \pi(E(qE))}{E(qE)} \right].$$

(10)

(11)

in the last expression, $n(x)$ is the Fermi-Dirac distribution and $E(qE) = \sqrt{M_f^2 + 2\beta_f qE}$.

2.2. **zMFIR applied to the $\pi^0$ pole mass in a magnetized medium**

Making use of the usual RPA approximation, and selecting the quantum numbers associated to the neutral pion, one can show that the pole-mass of the $\pi^0$ meson is given by the relation:

$$1 - 2G\Pi_{\mu\nu}(k^2)|_{k^2=m_{\pi^0}^2} = 0 \, .$$

(12)

where one obtains

$$m_{\pi^0}^2 = -\frac{m^2}{M} \frac{1}{4GN_cN_f I(m_{\pi^0}^2)} \, .$$

(13)

where in previous equation $k = (k_0 = m_{\pi^0}, \vec{k} = \vec{0})$ and after using the Matsubara formalism in the latter integral, one obtains:

$$I(k_0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E(k_0^2 - 4E^2)} \left[ 1 - n(E) - \tilde{n}(E) \right] \, ,$$

(14)

To use the alternative zMFIR scheme, we apply the formalism presented in the last section for the polarization integral at finite magnetic field and temperatures, and one obtains [13]:

$$I(k_0^2, B, T) = I_{vac}(k_0^2) + I(k_0^2, B) + I_{T,\mu}(k_0^2) + I_{T,\mu}(k_0^2, B) \, ,$$

(15)

where, we can identify each quantity as:

$$I_{vac}(k_0^2) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E(k_0^2 - 4E^2)} \, ,$$

(16)

$$I(k_0^2, B) = -\sum_{f=u,d} \frac{(2\beta_f)^{1/2}}{32\pi^2} \int_0^\infty dx xH_{1/2}(x) \frac{1}{E_f(x^2 - x_0^2)} \, ,$$

(17)

$$I_{T,\mu}(k_0^2, B) = \sum_{f=u,d} \frac{(2\beta_f)^{1/2}}{32\pi^2} \int_0^\infty dx xH_{1/2}(x) \frac{n(E) + \tilde{n}(E)}{E_f(x^2 - x_0^2)} \, ,$$

(18)

where $x_0^2 = (k_0^2/4 - M^2)/(2\beta_f)$, $x_f \equiv M^2/(2\beta_f)$ and $I_{T,\mu}(k_0^2)$ is given by equation (14). Of course, the latter expressions also have to be interpreted as Cauchy principal values when $x_0^2$ is greater than zero or, equivalently, when $k_0 > 2M$. 

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3. Numerical Results

The parameters used in this work are $\Lambda = 664.3$ MeV, $m_0 = 5.0$ MeV and $G = \frac{G_0}{N_c}$ [15]. In the figures 1 and 2 we show the numerical equivalence between the MFIR and zMFIR formalism, using as an example the evaluation of the gap equation at $T = 0$ and at $T \neq 0$. Also, in the figures 3 and 4 we evaluate the normalized pressure $P_N = P(eB, T) - P(eB, 0)$ in both formalisms (more details see Ref [13]).

![Figure 1](image1.png)  
Figure 1: The effective quark mass as a function of magnetic field at $T = 0$.  

![Figure 2](image2.png)  
Figure 2: Effective quark mass as a function of temperature for different values of magnetic field.  

![Figure 3](image3.png)  
Figure 3: The normalized Pressure at $T = 0$ as a function of magnetic field.  

![Figure 4](image4.png)  
Figure 4: Normalized pressure $P_N$ as a function of temperature for different values of magnetic field.  

In Fig. 5 we show the results for the effective quark mass and the pole-mass of neutral meson $\pi_0$ at finite temperature and the $eB = 0$. At low temperatures the effective quark masses are, in a very good approximation, the same as the calculated in the vacuum, i.e, $M(T) \approx M(0)$. In this phase, the neutral meson is the pseudo-goldstone boson, that has a finite mass $m_\pi \approx 135$ MeV. At the pseudo-critical temperature, the effective quark masses becomes almost the current quark mass $M \approx m_0$. In this region when the temperature increases we achieve $m_\pi = 2M$ which defines the Mott temperature [14]. At this point the equations (14) should be interpreted as its Principal Value, when $p = \sqrt{\frac{m_\pi^2}{4} - M^2}$ [16, 17]. As can be seen from our results, the neutral meson becomes a thermal excitation.
The effective quark and pion masses at finite temperature at \( eB = 0.1 \text{GeV}^2 \) can be seen in the Fig. 6. The pole-mass of the neutral collective excitations are showed in the same figure as well. At low temperatures, the magnetic field enhances the chiral condensate and the effective quark masses becomes stronger (magnetic catalysis \[18\]). When the temperature is greater than the pseudo-critical temperature, the chiral symmetry partially restores and the effective quark masses becomes weaker. At high enough temperatures, i.e., \( T > T_{\text{Mott}} \) the neutral meson enters in the Wigner-Weyl phase. In this phase, the neutral meson is a thermal excitation with a finite decay width, but the increase of the magnetic field causes the thermal excitation to become more energetic when compared with the zero magnetic field case. The dimensional reduction \[18, 19\] play a main role in the “jump” of the thermal excitation at \( T_{\text{Mott}} \). At temperatures above the Mott dissociation, \( T > T_{\text{Mott}} \), the thermal energy is not sufficient to excite all possible states in the phase space due the dimensional reduction caused by the strong magnetic field. The resonant \( q - \bar{q} \) pair in this case has less states to occupy. Hence, the “jump” in the resonant mass is just the \( \pi_0 \) mass going to its lowest possible energy state, when all other states are not accessible anymore. In Fig. 7 we sketch the results with \( eB = 0.2 \text{GeV}^2 \) and as expected the previous qualitative analysis still can be used, however, the thermal excitations beyond \( T_{\text{Mott}} \) becomes even more energetic.

![Figure 6](image6.png)  
**Figure 6:** Effective quark mass and \( \pi_0 \) pole-mass as a function of the temperature at \( eB = 0.1 \text{GeV}^2 \).

![Figure 7](image7.png)  
**Figure 7:** Effective quark mass and \( \pi_0 \) pole-mass as a function of the temperature at \( eB = 0.2 \text{GeV}^2 \).
4. Conclusions
We have investigated some properties of the magnetized NJL SU(2) model at finite temperature in the mean field approximation. The regularization of the model was performed by using a new procedure based on the Riemann-Hurwitz zeta function. We have shown that the new formalism is equivalent to the usual MFIR one. We derived the gap equation and the pseudo-scalar polarization loop through this formalism. As an direct application of the formalism, we have evaluated the $\pi^0$ pole-mass within the standard random phase approximation. Although both the MFIR and the zMFIR are completely equivalent, the new method has shown to be more appropriate for the study of the neutral meson properties.

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[1] Avancini S S, Tavares W R, and Pinto M B 2016 Phys. Rev. D 93 014010
[2] Avancini S S, Farias R L S, Benghi Pinto M, Tavares W R and Timóteo V S 2017 Phys. Lett. B767 247-252
[3] Farias R L S, Avancini S S, Pinto M B, Tavares W R and Timóteo V S 2017 Int. J. Mod. Phys. Conf. Ser. 45 1760060
[4] Gómez D G, Izzo Villafae M F and Scoccola N N 2018, Phys. Rev. D 97 034025
[5] Kohyama H, Kimura D and Inagaki T 2015 Nucl. Phys. B 896 682
[6] Allen P G, Grunfeld A G and Scoccola N N 2015 Phys. Rev. D 92 074041
[7] Duarte D C, Allen P G, Farias R L S, Manso P H A, Ramos R O and Scoccola N N 2016 Phys. Rev. D 93(2) 025017
[8] Farias R L S, Duarte D C, Krein G and Ramos R O, Phys. Rev. D 94(7) 074011
[9] Duarte D C, Farias R L S and Ramos R O, Regularization issues for a cold and dense quark matter model in $\beta$ equilibrium, Preprint arXiv:1811.10598 [hep-ph]
[10] Dib C O and Espinosa O 2001 Nucl. Phys. B 612 492
[11] Feng B, Hou D F and Ren H C 2015 Phys. Rev. D 92 065011
[12] Feng B, Hou D F, Ren H C and Wu P 2016 Phys. Rev. D 93, 085019
[13] Avancini S S, Farias R L S and Tavares W R 2018 Neutral meson properties in hot and magnetized quark matter: a new magnetic field independent regularization scheme applied to NJL-type model Preprint arXiv:1812.00945 [hep-ph]
[14] Wergieluk A, Blaschke D, Kalinovsky Y L and Friesen A V 2013 Phys. Part. Nucl. Lett. 10 660
[15] Buballa M 2005 Phys. Rep. 407 205-376
[16] Florkowski W and Friman A V 1994 Acta Physica Polonica B 25 49
[17] Asakawa M and Yazaki K 1989 Nucl. Phys. A 504 668
[18] Miransky V A and Shovkovy I A 2015 Phys. Rep. 576 1
[19] Gusynin V P, Miranski V A and Shovkovy I A Nucl. Phys. B 462 249 (1996).