WEIGHTED ESTIMATES FOR BELTRAMI EQUATIONS

Albert Clop and Victor Cruz

Universitat Autònoma de Barcelona, Departament de Matemàtiques
08193 Bellaterra, Barcelona, Catalonia; albertcp@mat.uab.cat

Universidad Tecnológica de la Mixteca, Instituto de Física y Matemáticas
69000 Huajuapan de León, Oaxaca, México; victorcruz@mixteco.utm.mx

Abstract. We obtain a priori estimates in $L^p(\omega)$ for the generalized Beltrami equation, provided that the coefficients are compactly supported $VMO$ functions with the expected ellipticity condition, and the weight $\omega$ lies in the Muckenhoupt class $A_p$. As an application, we obtain improved regularity for the jacobian of certain quasiconformal mappings.

1. Introduction

In this paper, we consider the inhomogeneous, Beltrami equation

$$(1) \quad \partial f(z) - \mu(z) \partial f(z) - \nu(z) \overline{\partial f(z)} = g(z), \quad \text{a.e. } z \in \mathbb{C},$$

where $\mu, \nu$ are $L^\infty(\mathbb{C}; \mathbb{C})$ functions such that $\| |\mu| + |\nu| \|_\infty \leq k < 1$, and $g$ is a measurable, $\mathbb{C}$-valued function. The derivatives $\partial f, \overline{\partial f}$ are understood in the distributional sense. In the work [3], the $L^p$ theory of such equation was developed. More precisely, it was shown that if $1 + k < p < 1 + \frac{1}{k}$ and $g \in L^p(\mathbb{C})$ then (1) has a solution $f$, unique modulo additive constants, whose differential $Df$ belongs to $L^p(\mathbb{C})$, and furthermore, the estimate

$$(2) \quad \| Df \|_{L^p(\mathbb{C})} \leq C \| g \|_{L^p(\mathbb{C})}$$

holds for some constant $C = C(k, p) > 0$. For other values of $p$, (1) the claim may fail in general. However, in the previous work [9], Iwaniec proved that if $\mu \in VMO(\mathbb{C})$, then for any $1 < p < \infty$ and any $g \in L^p(\mathbb{C})$ one can find exactly one solution $f$ to the $\mathbb{C}$-linear equation

$$\bar{\partial} f(z) - \mu(z) \partial f(z) = g(z)$$

with $Df \in L^p(\mathbb{C})$. In particular, (2) holds whenever $p \in (1, \infty)$. Recently, Koski [11] has extended this result to the generalized equation (1). For results in other spaces of functions, see [5].

In this paper, we deal with weighted spaces, and so we assume $g \in L^p(\omega)$, $1 < p < \infty$. Here $\omega$ is a measurable function, and $\omega > 0$ at almost every point. By checking the particular case $\mu = \nu = 0$, one sees that, for a weighted version of the estimate (2) to hold, the Muckenhoupt condition $\omega \in A_p$ is necessary. It turns out that, for compactly supported $\mu \in VMO$, this condition is also sufficient.

**Theorem 1.** Let $1 < p < \infty$. Let $\mu$ be a compactly supported function in $VMO(\mathbb{C})$, such that $\| \mu \|_\infty < 1$, and let $\omega \in A_p$. Then, the equation

$$\bar{\partial} f(z) - \mu(z) \partial f(z) = g(z)$$

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has, for \( g \in L^p(\omega) \), a solution \( f \) with \( Df \in L^p(\omega) \), which is unique up to an additive constant. Moreover, one has
\[
\|Df\|_{L^p(\omega)} \leq C \|g\|_{L^p(\omega)}
\]
for some \( C > 0 \) depending on \( \mu \), \( p \) and \([\omega]_{A_p}\).

The proof copies the scheme of [9]. In particular, our main tool is the following compactness Theorem, which extends a classical result of Uchiyama [18] about commutators of Calderón–Zygmund singular integral operators and \( VMO \) functions.

**Theorem 2.** Let \( T \) be a Calderón–Zygmund singular integral operator. Let \( \omega \in A_p \) with \( 1 < p < \infty \), and let \( b \in VMO(\mathbb{R}^n) \). The commutator \([b, T]: L^p(\omega) \to L^p(\omega)\) is compact.

Theorem 2 is obtained from a sufficient condition for compactness in \( L^p(\omega) \). When \( \omega = 1 \), this sufficient condition reduces to the classical Frechet–Kolmogorov compactness criterion. Theorem 1 is then obtained from Theorem 2 by letting \( T \) be the Beurling–Ahlfors singular integral operator.

A counterpart to Theorem 1 for the generalized Beltrami equation,
\[
(3) \quad \bar{\partial}f(z) - \mu(z) \partial f(z) - \nu(z) \overline{\partial f(z)} = g(z),
\]
can also be obtained under the ellipticity condition \( ||\mu| + |\nu||_{\infty} \leq k < 1 \) and the \( VMO \) smoothness of the coefficients (see Theorem 8 below). Theorem 2 is again the main ingredient. However, for (3) the argument in Theorem 1 needs to be modified, because the involved operators are not \( \mathbb{C} \)-linear, but only \( \mathbb{R} \)-linear. In other words, \( \mathbb{C} \)-linearity is not essential. See also [11].

It turns out that any linear, elliptic, divergence type equation can be reduced to equation (3) (see e.g. [2, Theorem 16.1.6]). Therefore the following result is no surprise.

**Corollary 3.** Let \( K \geq 1 \). Let \( A: \mathbb{R}^2 \to \mathbb{R}^{2 \times 2} \) be a matrix-valued function, satisfying the ellipticity condition
\[
\frac{1}{K} \leq v^t A(z) v \leq K, \quad \text{whenever } v \in \mathbb{R}^2, \ |v| = 1,
\]
at almost every point \( z \in \mathbb{R}^2 \), and such that \( A - \text{Id} \) has compactly supported \( VMO \) entries. Let \( p \in (1, \infty) \) be fixed, and \( \omega \in A_p \). For any \( g \in L^p(\omega) \), the equation
\[
\text{div}(A(z) \nabla u) = \text{div}(g)
\]
has a solution \( u \) with \( \nabla u \in L^p(\omega) \), unique up to an additive constant, and such that
\[
\|\nabla u\|_{L^p(\omega)} \leq C \|g\|_{L^p(\omega)}
\]
for some constant \( C = C(A, \omega, p) \).

Other applications of Theorem 1 are found in connection to planar \( K \)-quasiconformal mappings. Remember that a \( W_{\text{loc}}^{1,2} \) homeomorphism \( \phi: \Omega \to \Omega' \) between domains \( \Omega, \Omega' \subseteq \mathbb{C} \) is called \( K \)-quasiconformal if
\[
|\bar{\partial}\phi(z)| \leq \frac{K - 1}{K + 1} |\partial\phi(z)| \quad \text{for a.e. } z \in \Omega.
\]
In general, jacobians of \( K \)-quasiconformal maps are Muckenhoupt weights belonging to the class \( A_p \) for any \( p > K \) (see [2, Theorem 13.4.2 ], or also [3]), and this is sharp.

As a consequence of Theorem 1, we obtain the following improvement.
Corollary 4. Let $\mu \in VMO$ be compactly supported, such that $\|\mu\|_\infty < 1$, and let $\phi: \mathbb{C} \to \mathbb{C}$ be a quasiconformal solution of
\[
\overline{\partial}\phi(z) - \mu(z) \partial\phi(z) = 0.
\]
Then, for every $1 < p < \infty$ there exists a constant $C = C(p) \geq 1$ such that the estimate
\[
(4) \quad \left( \int_D |J(z, \phi)|^p \, dz \right)^{\frac{1}{p}} \leq C_p \int_D |J(z, \phi)| \, dz,
\]
holds for every disk $D \subset \mathbb{C}$.

By quasiconformality, the above result is equivalent to say that the inverse mapping $\phi^{-1}$ has jacobian determinant $J(\cdot, \phi^{-1}) \in A_p$ for every $p > 1$. In turn, Johnson and Neugebauer [10] proved that this is equivalent to the fact that the composition with $\phi^{-1}$ quantitatively preserves the Muckenhoupt class $A_2$, and this is what we actually prove. The above Corollary improves the results in [9], which assert that $J(\cdot, \phi) \in L^p_{loc}$ for every finite $p > 1$. Note also that general $K$-quasiconformal maps need not satisfy the estimate (4) if $p \geq \frac{K}{K-1}$ [3].

The paper is structured as follows. In Section 2 we prove Theorem 2. In Section 3 we prove Theorem 1 and its counterpart for the generalized Beltrami equation. In Section 4 we study some applications. By $C$ we denote a positive constant that may change at each occurrence. $B(x, r)$ denotes the open ball with center $x$ and radius $r$, and $2B$ means the open ball concentric with $B$ and having double radius.

2. Compactness of commutators

By singular integral operator $T$, we mean a linear operator on $L^p(\mathbb{R}^n)$ that can be written as
\[
Tf(x) = \int_{\mathbb{R}^n} f(y) K(x, y) \, dy.
\]
Here $K: \mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\} \to \mathbb{C}$ obeys the bounds
\[
\begin{align*}
(1) \quad |K(x, y)| &\leq \frac{C_1}{|x-y|^n}, \\
(2) \quad |K(x, y) - K(x, y')| &\leq C_2 \frac{|y-y'|}{|x-y'|^{n-1}} \text{ whenever } |x-y| \geq 2|y-y'|, \\
(3) \quad |K(x, y) - K(x', y)| &\leq C_3 \frac{|x-x'|}{|x-y'|^{n-1}} \text{ whenever } |x-y| \geq 2|x-x'|.
\end{align*}
\]
One then calls $\|T\|_{CZ} = \max\{C_1, C_2, C_3\}$ the Calderón–Zygmund constant of $T$. Given a singular integral operator $T$, we define the truncated singular integral as
\[
T_\epsilon f(x) = \int_{|x-y| \geq \epsilon} K(x, y) f(y) \, dy
\]
and the maximal singular integral by the relationship
\[
T_* f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)|.
\]
As usually, we denote $\int_E f(x) \, dx = \frac{1}{|E|} \int_E f(x) \, dx$. A weight is a function $\omega \in L^1_{loc}(\mathbb{R}^n)$ such that $\omega(x) > 0$ almost everywhere. A weight $\omega$ is said to belong to the Muckenhoupt class $A_p$, $1 < p < \infty$, if
\[
[\omega]_{A_p} := \sup \left( \int_Q \omega(x) \, dx \right) \left( \int_Q \omega(x)^{-\frac{p'}{p}} \, dx \right)^{\frac{p}{p'}} < \infty.
\]
where the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \), and where \( \frac{1}{p} + \frac{1}{p'} = 1 \). One may equivalently consider balls instead of cubes. By \( L^p(\omega) \) we denote the set of measurable functions \( f \) that satisfy

\[
\|f\|_{L^p(\omega)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right)^{\frac{1}{p}} < \infty.
\]

(6)

The quantity \( \|f\|_{L^p(\omega)} \) defines a complete norm in \( L^p(\omega) \). It is well know that if \( T \) is a Calderón–Zygmund operator, then \( T \) and also \( T_\ast \) are bounded in \( L^p(\omega) \) whenever \( \omega \in A_p \) (see for instance [7, Cap. IV, Theorems 3.1 and 3.6]). Also the Hardy–Littlewood maximal operator \( M \) is bounded in \( L^p(\omega) \). For more about \( A_p \) classes and weighted spaces \( L^p(\omega) \), we refer the reader to [7].

We first show the following sufficient condition for compactness in \( L^p(\omega) \), \( \omega \in A_p \). Remember that a metric space \( X \) is **totally bounded** if for every \( \epsilon > 0 \) there exists a finite number of open balls of radius \( \epsilon \) whose union is the space \( X \). In addition, a metric space is compact if and only if it is complete and totally bounded.

**Theorem 5.** Let \( p \in (1, \infty) \), \( \omega \in A_p \), and let \( \mathcal{F} \subset L^p(\omega) \). Then \( \mathcal{F} \) is totally bounded if it satisfies the next three conditions:

1. \( \mathcal{F} \) is uniformly bounded, i.e. \( \sup_{f \in \mathcal{F}} \|f\|_{L^p(\omega)} < \infty \).
2. \( \mathcal{F} \) is uniformly equicontinuous, i.e. \( \sup_{f \in \mathcal{F}} \|f(\cdot) - f(\cdot + h)\|_{L^p(\omega)} \xrightarrow{h \to 0} 0 \).
3. \( \mathcal{F} \) uniformly vanishes at infinity, i.e. \( \sup_{f \in \mathcal{F}} \|f - \chi_{Q(0,R)}f\|_{L^p(\omega)} \xrightarrow{R \to \infty} 0 \), where \( Q(0,R) \) is the cube with center at the origin and sidelength \( 2R \).

Let us emphasize that Theorem 5 is a strong sufficient condition for compactness in \( L^p(\omega) \), because for a general weight \( \omega \in A_p \) the space \( L^p(\omega) \) is not invariant under translations. Theorem 5 is proved by adapting the arguments in [8]. In particular, the following result (which can be found in [8, Lemma 1]) is essential.

**Lemma 6.** Let \( X \) be a metric space. Suppose that for every \( \epsilon > 0 \) one can find a number \( \delta > 0 \), a metric space \( W \) and an mapping \( \Phi: X \to W \) such that \( \Phi(X) \) is totally bounded, and the implication

\[
d(\Phi(x), \Phi(y)) < \delta \quad \Rightarrow \quad d(x, y) < \epsilon
\]

holds for any \( x, y \in X \). Then \( X \) is totally bounded.

**Proof of Theorem 5.** Suppose that the family \( \mathcal{F} \) satisfies the three conditions of Theorem 5, and let \( \epsilon > 0 \) be fixed. According to the third assumption on \( \mathcal{F} \), we can choose a positive quantity \( R > 0 \) such that

\[
(7) \quad \sup_{f \in \mathcal{F}} \|f - \chi_{Q(0,R)}f\|_{L^p(\omega)} < \frac{\epsilon}{4}.
\]

Let us also find \( \rho > 0 \) small enough so that

\[
(8) \quad \sup_{h \in Q(0,2\rho)} \left( \sup_{f \in \mathcal{F}} \|f(\cdot) - f(\cdot + h)\|_{L^p(\omega)} \right) < \frac{\epsilon}{2^{2+n/p}}.
\]

Such a \( \rho \) exists due to the equicontinuity assumption on \( \mathcal{F} \). Now, let us choose \( N \) cubes \( Q_1, \ldots, Q_N \) with sidelength \( 2\rho \), having pairwise disjoint interiors, and such that

\[
(9) \quad Q(0,R) \subset \bigcup_i Q_i.
\]
Define
\[
\Phi f(x) = \begin{cases} 
\int_{Q_i} f(z) \, dz, & x \in Q_i, i = 1, \ldots, N, \\
0, & \text{otherwise.}
\end{cases}
\]
Since functions in \( L^p(\omega) \) are locally integrable, \( \Phi f \) is well defined for any \( f \in \mathcal{F} \). Moreover,
\[
\int_{\mathbb{R}^n} |\Phi f(x)|^p \omega(x) \, dx = \sum_{i=1}^{N} \left( \int_{Q_i} f(z) \, dz \right)^p \int_{Q_i} \omega(x) \, dx 
\leq \sum_{i=1}^{N} \left( \int_{Q_i} |f(z)|^p \omega(z) \, dz \right) \left( \int_{Q_i} \omega(z) \, dz \right)^p \int_{Q_i} \omega(x) \, dx 
\leq [\omega] A_p \| f \|^p_{L^p(\omega)}.
\]
In particular, \( \Phi : L^p(\omega) \to L^p(\omega) \) is a bounded operator. As \( \mathcal{F} \) is bounded, then \( \Phi(\mathcal{F}) \) is a bounded subset of a finite dimensional Banach space, and hence \( \Phi(\mathcal{F}) \) is totally bounded.

On the other hand, by (7) and (9) one gets that
\[
\| f \chi_{\mathbb{R}^n \cup_i Q_i} \|_{L^p(\omega)} \leq \| f \chi_{\mathbb{R}^n \cup_i Q_i, (0, R)} \|_{L^p(\omega)} < \epsilon^p \frac{1}{4},
\]
for any \( f \in \mathcal{F} \). Also, by Jensen’s inequality,
\[
\| f \chi_{\cup_i Q_i} - \Phi f \|^p_{L^p(\omega)} = \sum_{i=1}^{N} \left( \int_{Q_i} f(z) - \int_{Q_i} f(z) \omega(x) \, dx \right)^p 
\leq \sum_{i=1}^{N} \left( \int_{Q_i} f(z) - \int_{Q_i} f(z) \omega(x) \, dx \right)^p \omega(x) \, dx.
\]
Now, if \( x, z \in Q_i \), then \( z - x = h \in Q(0, 2\rho) \). Therefore, after a change of coordinates,
\[
\| f \chi_{\cup_i Q_i} - \Phi f \|^p_{L^p(\omega)} \leq \sum_{i=1}^{N} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q(0, 2\rho)} |f(x) - f(x + h)|^p \, dh \omega(x) \, dx 
= \frac{1}{|Q(0, \rho)|} \int_{Q(0, 2\rho)} \sum_{i=1}^{N} \int_{Q_i} |f(x) - f(x + h)|^p \omega(x) \, dx \, dh 
\leq \frac{1}{|Q(0, \rho)|} \int_{Q(0, 2\rho)} \int_{\mathbb{R}^n} |f(x) - f(x + h)|^p \omega(x) \, dx \, dh 
= 2^n \int_{Q(0, 2\rho)} \| f(\cdot) - f(\cdot + h) \|^p_{L^p(\omega)} \, dh 
\leq 2^n \sup_{h \in Q(0, 2\rho)} \left( \sup_{f \in \mathcal{F}} \| f(\cdot) - f(\cdot + h) \|^p_{L^p(\omega)} \right) < \left( \frac{\epsilon}{4} \right)^p.
\]
Summarizing,
\[
\| f - \Phi f \|_{L^p(\omega)} \leq \| f \chi_{\mathbb{R}^n \cup_i Q_i} \|_{L^p(\omega)} + \| f \chi_{\cup_i Q_i} - \Phi f \|_{L^p(\omega)} < \frac{\epsilon}{2},
\]
for any \( f \in \mathcal{F} \). Hence
\[
\|f\|_{L^p(\omega)} < \frac{\epsilon}{2} + \|\Phi f\|_{L^p(\omega)}, \quad \text{whenever } f \in \mathcal{F}.
\]
Since \( \Phi \) is linear, this means that
\[
\|f - g\|_{L^p(\omega)} < \frac{\epsilon}{2} + \|\Phi f - \Phi g\|_{L^p(\omega)}, \quad \text{whenever } f, g \in \mathcal{F}.
\]
Set \( \delta = \epsilon/2 \). The above inequality says that if \( f, g \in \mathcal{F} \) are such that \( d(\Phi f, \Phi g) < \delta \), then \( d(f, g) < \epsilon \). By the previous Lemma, it follows that \( \mathcal{F} \) is totally bounded. \( \square \)

In order to prove Theorem 2, we will first reduce ourselves to smooth symbols \( b \). Let us recall that commutators \( C_b = [b, T] \) with \( b \in BMO(\mathbb{R}^n) \) are continuous in \( L^p(\omega) \) [15, Theorem 2.3]. Moreover, in [13, Theorem 1] the following estimate is shown,
\[
\|C_b f\|_{L^p(\omega)} \leq C \|b\|_* \|M^2 f\|_{L^p(\omega)},
\]
where \( \|b\|_* \) denotes the \( BMO \) norm of \( b \), and the constant \( C \) may depend on \( \omega \), but not on \( b \). Now, by the boundedness of the Hardy–Littlewood operator \( M \) on \( L^p(\omega) \), we obtain
\[
\|C_b f\|_{L^p(\omega)} \leq C \|b\|_* \|f\|_{L^p(\omega)}.
\]
Since by assumption \( b \in VMO(\mathbb{R}^n) \), we can approximate the function \( b \) by functions \( b_j \in C_c^\infty(\mathbb{R}^n) \) in the \( BMO \) norm, and thus
\[
\|C_{b_j} f - C_{b_j} f\|_{L^p(\omega)} = \|C_{b-b_j} f\|_{L^p(\omega)} \leq C \|b - b_j\|_* \|f\|_{L^p(\omega)}.
\]
In particular, the commutators with smooth symbol \( C_{b_j} \) converge to \( C_b \) in the operator norm of \( L^p(\omega) \). Therefore it suffices to prove compactness for the commutator with smooth symbol.

Another reduction in the proof of Theorem 2 will be made by slightly modifying the singular integral operator \( T \). This technique comes from Krantz and Li [12]. More precisely, for every \( \eta > 0 \) small enough, let us take a continuous function \( K_\eta \) defined on \( \mathbb{R}^n \times \mathbb{R}^n \), taking values in \( \mathbb{R} \) or \( \mathbb{C} \), and such that:
\begin{enumerate}
\item \( K_\eta(x, y) = K(x, y) \) if \( |x - y| \geq \eta \),
\item \( |K_\eta(x, y)| \leq \frac{C_0}{|x - y|^{n+\eta}} \) for \( \frac{\eta}{2} < |x - y| < \eta \),
\item \( K_\eta(x, y) = 0 \) if \( |x - y| \leq \frac{\eta}{2} \),
\end{enumerate}
where \( C_0 \) is independent of \( \eta \). Due to the growth properties of \( K \), it is not restrictive to suppose that condition 2 holds for all \( x, y \in \mathbb{R}^n \). Now, let
\[
T_\eta f(x) = \int_{\mathbb{R}^n} K_\eta(x, y) f(y) \, dy,
\]
and let us also denote
\[
C_\eta^b f(x) = [b, T_\eta] f(x) = \int_{\mathbb{R}^n} \{b(x) - b(y)\} K_\eta(x, y) f(y) \, dy.
\]
We now prove that the commutators \( C_\eta^b \) approximate \( C_b \) in the operator norm.

**Lemma 7.** Let \( b \in C_c^\infty(\mathbb{R}^n) \). There exists a constant \( C = C(n, C_0) \) such that
\[
|C_b f(x) - C_\eta^b f(x)| \leq C \eta \|\nabla b\|_\infty M f(x) \quad \text{almost everywhere},
\]
for every \( \eta > 0 \). As a consequence,
\[
\lim_{\eta \to 0} \|C_\eta^b - C_b\|_{L^p(\omega) \to L^p(\omega)} = 0
\]
whenever \( \omega \in A_p \) and \( 1 < p < \infty \).

**Proof.** Let \( f \in L^p(\omega) \). For every \( x \in \mathbb{R}^n \) we have

\[
C_b f(x) - C_b^\eta f(x) = \int_{|x-y|<\eta} (b(x) - b(y))K(x, y) f(y) \, dy \\
- \int_{\frac{\eta}{2} \leq |x-y| \leq \eta} (b(x) - b(y))K^\eta(x, y) f(y) \, dy \\
= I_1(x) + I_2(x).
\]

Using the smoothness of \( b \) and the size estimates of \( K^\eta \), we have that

\[
|I_1(x)| \leq \int_{|x-y|<\eta} |b(y) - b(x)||K(x, y)||f(y)| \, dy \\
\leq C_0 \|\nabla b\|_\infty \sum_{j=0}^{\infty} \int_{\frac{\eta}{2^{j+1}} \leq |x-y| \leq \frac{\eta}{2^{j}}} \frac{|f(y)|}{|x-y|^{n-1}} \, dy \\
\leq 2^n C_0 \|\nabla b\|_\infty \sum_{j=0}^{\infty} \eta \frac{|B(0, 1)|}{2^{j+1}} \int_{|x-y|<\frac{\eta}{2^{j}}} |f(y)| \, dy \\
\leq \eta 2^n C_0 \|\nabla b\|_\infty |B(0, 1)| \ M f(x)
\]

for almost every \( x \). For the other term, similarly

\[
|I_2(x)| \leq \eta \|\nabla b\|_\infty \int_{\frac{\eta}{2} < |x-y| < \eta} |K^\eta(x, y)||f(y)| \, dy \\
\leq \eta C_0 \|\nabla b\|_\infty \int_{\frac{\eta}{2} < |x-y| < \eta} \frac{|f(y)|}{|x-y|^n} \, dy \\
\leq \eta 2^n C_0 \|\nabla b\|_\infty |B(0, 1)| \int_{|x-y|<\eta} |f(y)| \, dy \\
\leq \eta 2^n C_0 \|\nabla b\|_\infty |B(0, 1)| \ M f(x).
\]

Therefore, the pointwise estimate follows. Now, the boundedness of \( M \) in \( L^p(\omega) \) for any \( A_p \) weight \( \omega \) implies that

\[
\|C_b f - C_b^\eta f\|_{L^p(\omega)} \leq C \eta \|\nabla b\|_\infty \|M f\|_{L^p(\omega)} \leq C \eta \|\nabla b\|_\infty \|f\|_{L^p(\omega)} \to 0,
\]
as \( \eta \to 0 \). This finishes the proof of Lemma 7. \( \square \)

We are now ready to conclude the proof of Theorem 2. From now on, \( \eta > 0 \) and \( b \in C^1_c(\mathbb{R}^n) \) are fixed, and we have to prove that the commutator \( C_b^\eta = [b, T^\eta] \) is compact. Thus, the constants that will appear may depend on \( b \) and \( \eta \).

We denote \( \mathcal{F} = \{C_b^\eta f; f \in L^p(\omega), \|f\|_{L^p(\omega)} \leq 1\} \). Then \( \mathcal{F} \) is uniformly bounded, because \( C_b^\eta \) is a bounded operator on \( L^p(\omega) \). To prove the uniform equicontinuity of \( \mathcal{F} \), we must see that

\[
\lim_{h \to 0} \sup_{f \in \mathcal{F}} \|C_b^\eta f(\cdot) - C_b^\eta f(\cdot + h)\|_{L^p(\omega)} = 0.
\]
To do this, let us write

\[ C_b^n f(x) - C_b^n f(x + h) = (b(x) - b(x + h)) \int_{\mathbb{R}^n} K^n(x, y) f(y) \, dy \]

\[ + \int_{\mathbb{R}^n} (b(x + h) - b(y))(K^n(x, y) - K^n(x + h, y)) f(y) \, dy \]

\[ = \int_{\mathbb{R}^n} I_1(x, y, h) \, dy + \int_{\mathbb{R}^n} I_2(x, y, h) \, dy. \]

For \( I_1(x, y, h) \), using the regularity of the function \( b \) and the definition of the operator \( T_\ast \),

\[
\left| \int_{\mathbb{R}^n} I_1(x, y, h) \, dy \right|
\leq \|\nabla b\|_\infty |h| \left( \int_{|x-y|>\frac{\eta}{2}} (K^n(x, y) - K^n(x, h)) f(y) \, dy + \int_{|x-y|>\frac{\eta}{2}} K^n(x, y) f(y) \, dy \right)
\leq \|\nabla b\|_\infty |h| \left( \int_{|x-y|>\frac{\eta}{2}} |K^n(x, y) - K^n(x, h)| |f(y)| \, dy + T_\ast f(x) \right)
\leq \|\nabla b\|_\infty |h| (CM_f(x) + T_\ast f(x))
\]

for some constant \( C > 0 \) that may depend on \( \eta \), but not on \( h \). Therefore

\[
(13) \quad \left( \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} I_1(x, y, h) \, dy \right|^p \omega(x) \, dx \right)^{\frac{1}{p}} \leq C |h| \|f\|_{L^p(\omega)},
\]

for \( C \) independent of \( f \) and \( h \). Here we used the boundedness of \( M \) and \( T_\ast \) on \( L^p(\omega) \) (see \[7, Chap. IV, Th. 3.6\]). We will divide the integral of \( I_2(x, y, h) \) into three regions:

\[
A = \left\{ y \in \mathbb{R}^n : |x-y| > \frac{\eta}{2}, |x + h - y| > \frac{\eta}{2} \right\},
\]

\[
B = \left\{ y \in \mathbb{R}^n : |x-y| > \frac{\eta}{2}, |x + h - y| < \frac{\eta}{2} \right\},
\]

\[
C = \left\{ y \in \mathbb{R}^n : |x-y| < \frac{\eta}{2}, |x + h - y| > \frac{\eta}{2} \right\}.
\]

Note that \( I_2(x, y, h) = 0 \) for \( y \in \mathbb{R}^n \setminus A \cup B \cup C \). Now, for the integral over \( A \), we use the smoothness of \( b \) and \( K^n \),

\[
\left| \int_{A} I_2(x, y, h) \, dy \right| \leq C \|\nabla b\|_\infty |h| \int_{|x-y|>\frac{\eta}{2}} \frac{|f(y)|}{|x-y|^{n+1}} \, dy
\leq C \|\nabla b\|_\infty \frac{|h|}{\eta} \sum_{j=0}^{\infty} 2^{-j} \int_{|x-y|<2^{j+2}} |f(y)| \, dy \leq C \|\nabla b\|_\infty \frac{|h|}{\eta} M_f(x),
\]

thus

\[
\left( \int_{\mathbb{R}^n} \left| \int_{A} I_2(x, y, h) \, dy \right|^p \omega(x) \, dx \right)^{\frac{1}{p}} \leq C |h| \|f\|_{L^p(\omega)},
\]

for some constant \( C \) that may depend on \( \eta \) and \( b \), but not on \( h \). In particular, the term on the right hand side goes to 0 as \( |h| \to 0 \).
The integrals of $I_2(x, y, h)$ over $B$ and $C$ are symmetric, so we only give the details once. For the integral over the set $B$, let us assume that $|h|$ is very small. We can first choose $R_0 > \eta/2 + |h|$ such that $b$ vanishes outside the ball $B_0 = B(0, R_0)$. It then follows that $b(x + h)$ has support in $2B_0$. Then, since $B \subset B(x, |h| + \eta/2)$, we have for $|x| < 3R_0$ that $B \subset 4B_0$ and therefore

$$
\left| \int_B I_2(x, y, h) \, dy \right| \leq C_0 \| \nabla b \|_{\infty} \int_{B \cap 4B_0} \frac{|x + h - y|}{|x - y|^n} |f(y)| \, dy
$$

$$
\leq C_0 \| \nabla b \|_{\infty} \int_{B \cap 4B_0} \frac{|f(y)|}{|x - y|^{n-1}} \, dy
$$

$$
\leq C_0 \| \nabla b \|_{\infty} (2/\eta)^{n-1} \int_{B \cap 4B_0} |f(y)| \omega(y)^{\frac{2}{p'}} \omega(y)^{-\frac{1}{p'}} \, dy
$$

$$
\leq C_0 \| \nabla b \|_{\infty} (2/\eta)^{n-1} \| f \|_{L^p(\omega)} \left( \int_{B \cap 4B_0} \omega(y)^{-\frac{1}{p'}} \, dy \right)^{\frac{1}{p'}}
$$

whence

$$
\int_{3B_0} \left| \int_B I_2(x, y, h) \, dy \right|^p \omega(x) \, dx \leq C \| f \|_{L^p(\omega)}^p \left( \int_{3B_0} \omega(x) \, dx \right) \left( \int_{B \cap 4B_0} \omega(y)^{-\frac{1}{p'}} \, dy \right)^{\frac{p}{p'}}
$$

for some constant $C$ that might depend on $\eta$, but not on $h$. If, instead, we have $|x| \geq 3R_0$, then $b(x + h) = 0$ (because $|h| < R_0$ so that $|x + h| > 2R_0$). Note also that for $y \in B$ one has $|x| \leq C|x - y|$ where $C$ depends only on $\eta$. Therefore

$$
\left| \int_B I_2(x, y, h) \, dy \right| \leq C \| b \|_{\infty} \int_{B \cap 4B_0} \frac{|f(y)|}{|x - y|^n} \, dy \leq C \| b \|_{\infty} \frac{|f(y)|}{|x|^n} \int_{B \cap 4B_0} \, dy
$$

$$
\leq C \| b \|_{\infty} \| f \|_{L^p(\omega)} \left( \int_{B \cap 4B_0} \omega(y)^{-\frac{1}{p'}} \, dy \right)^{\frac{1}{p'}}.
$$

This implies that

$$
\int_{\mathbb{R}^n \setminus 3B_0} \left| \int_B I_2(x, y, h) \, dy \right|^p \omega(x) \, dx
$$

$$
\leq C \| b \|_{\infty}^p \| f \|_{L^p(\omega)}^p \left( \int_{\mathbb{R}^n \setminus 3B_0} \frac{\omega(x)}{|x|^{np}} \, dx \right) \left( \int_{B \cap 4B_0} \omega(y)^{-\frac{1}{p'}} \, dy \right)^{\frac{p}{p'}}.
$$

Summarizing,

$$
\int_{\mathbb{R}^n} \left| \int_B I_2(x, y, h) \, dy \right|^p \omega(x) \, dx
$$

$$
\leq C \| f \|_{L^p(\omega)}^p \left( \int_{B \cap 4B_0} \omega(y)^{-\frac{1}{p'}} \, dy \right)^{\frac{p}{p'}} \left( \int_{3B_0} \omega(x) \, dx + \int_{\mathbb{R}^n \setminus 3B_0} \frac{\omega(x)}{|x|^{np}} \, dx \right).
$$

After proving that

$$
\int_{|x| > 3R_0} \frac{\omega(x)}{|x|^{np}} \, dx < \infty,
$$

the left hand side of (14) will converge to 0 as $|h| \to 0$ since $|B| \to 0$ as $|h| \to 0$. To prove the above claim, let us choose $q < p$ such that $\omega \in A_q$ [7, Theorem 2.6,
Ch. IV. For such \( q \), we have

\[
\int_{|x|>R} \frac{\omega(x)}{|x|^p} \, dx = \sum_{j=1}^{\infty} \int_{2^{j-1} < |x| < 2^j} \frac{\omega(x)}{|x|^p} \, dx \leq \sum_{j=1}^{\infty} (2^{j-1}R)^{-np} \omega(B(0, 2^j R)).
\]

By [7, Lemma 2.2], we have

\[
\int_{|x|>R} \frac{\omega(x)}{|x|^p} \, dx \leq \sum_{j=1}^{\infty} (2^{j-1}R)^{-np} (2^j R)^n \omega(B(0, 1)) = C \frac{R^m}{R^{m-q}} < \infty
\]
as desired. The equicontinuity of \( \mathfrak{F} \) follows.

Finally, we show the decay at infinity of the elements of \( \mathfrak{F} \). Let \( x \) be such that \( |x| > R > R_0 \). Then, \( x \not\in \text{supp } b \), and

\[
|C^p_b f(x)| = \left| \int_{\mathbb{R}^n} (b(x) - b(y)) K^p(x, y)f(y) \, dy \right| \leq C_0 \|b\|_\infty \int_{\text{supp } b} \frac{|f(y)|}{|x-y|^n} \, dy
\]

\[
\leq \frac{C \|b\|_\infty}{|x|^n} \int_{\text{supp } b} |f(y)| \, dy \leq \frac{C \|b\|_\infty}{|x|^n} \|f\|_{L^p(\omega)} \left( \int_{\text{supp } b} \omega(y)^{-\frac{n'}{p'}} \, dy \right) \frac{1}{p'}
\]
whence

\[
\left( \int_{|x|>R} |C^p_b f(x)|^p \omega(x) \, dx \right)^\frac{1}{p} \leq C \|b\|_\infty \|f\|_{L^p(\omega)} \left( \int_{|x|>R} \frac{\omega(x)}{|x|^p} \, dx \right)^\frac{1}{p}.
\]
The right hand side above converges to 0 as \( R \to \infty \), due to (15). By Theorem 5, \( \mathfrak{F} \) is totally bounded. Theorem 2 follows.

### 3. A priori estimates for Beltrami equations

We first prove Theorem 1. To do this, let us remember that the Beurling–Ahlfors singular integral operator is defined by the following principal value

\[
\mathcal{B} f(z) = -\frac{1}{\pi} P.V. \int \frac{f(w)}{(z-w)^2} \, dw.
\]

This operator can be seen as the formal \( \partial \) derivative of the Cauchy transform,

\[
C f(z) = \frac{1}{\pi} \int \frac{f(w)}{z-w} \, dw.
\]

At the frequency side, \( \mathcal{B} \) corresponds to the Fourier multiplier \( m(\xi) = \frac{\xi}{\xi²} \), so that \( \mathcal{B} \) is an isometry in \( L^2(\mathbb{C}) \). Moreover, this Fourier representation also explains the important relation

\[
\mathcal{B}(\overline{\partial} f) = \partial f
\]
for smooth enough functions \( f \). By \( \mathcal{B}^* \) we mean the singular integral operator obtained by simply conjugating the kernel of \( \mathcal{B} \), that is,

\[
\mathcal{B}^* f(z) = -\frac{1}{\pi} P.V. \int \frac{f(w)}{(\bar{z} - \bar{w})^2} \, dw.
\]

Note that \( \mathcal{B}^* \) has Fourier multiplier \( m^*(\xi) = \frac{\xi}{\xi} \). Thus,

\[
\mathcal{B}\mathcal{B}^* = \mathcal{B}^*\mathcal{B} = \text{Id}.
\]
In other words, $B^*$ is the $L^2$-inverse of $B$. It also appears as the $C$-linear adjoint of $B$,

$$\int_C B f(z) \overline{g(z)} \, dz = \int_C f(z) \overline{B^* g(z)} \, dz.$$ 

The complex conjugate operator $\overline{B}$ is the composition of $B$ with the complex conjugation operator $C f = \overline{f}$, that is,

$$\overline{B}(f) = C B(f) = \overline{B(f)}.$$ 

It then follows that

$$\overline{B} = C B = B^* C.$$ 

Note that $B$ and $B^*$ are $C$-linear operators, while $\overline{B}$ is only $R$-linear. See [2, Chapter 4] for more about the Beurling–Ahlfors transform.

Proof of Theorem 1. We follow Iwaniec’s idea [9, pp. 42–43]. For every $N = 1, 2, \ldots$, let

$$P_N = \text{Id} + \mu B + \cdots + (\mu B)^N.$$ 

Then

$$(\text{Id} - \mu B) P_{N-1} = P_{N-1} (\text{Id} - \mu B) = \text{Id} - \mu^N B^N + K_N,$$ 

where $K_N = \mu^N B^N - (\mu B)^N$. Each $K_N$ consists of a finite sum of operators that contain the commutator $[\mu, B]$ as a factor. Thus, by Theorem 2, each $K_N$ is compact in $L^p(\omega)$. On the other hand, the $N$-th iterate $B^N$ of the Beurling transform is another convolution-type Calderón–Zygmund operator, whose kernel is

$$b_N(z) = \frac{(-1)^N N z^{N-1}}{\pi} z^{N+1}$$ 

(see for instance [16, p. 73]). Arguing as in [6, Lemma 7.9 & Theorem 7.11], one sees that the operator norm $\|B^N\|_{L^p(\omega)}$ depends linearly on both the unweighted norm $\|B^N\|_{L^p(\mathbb{R}^n)}$ and the Calderón–Zygmund constant $\|B^N\|_{CZ}$. Since both quantities are bounded by a constant multiple of $N^2$, one immediately sees that

$$(17) \quad \|B^N\|_{L^p(\omega)} \leq C N^2,$$ 

with constant $C$ that depends on $[\omega]_{A_p}$, but not on $N$. As a consequence,

$$\|\mu^N B^N f\|_{L^p(\omega)} \leq C N^2 \|\mu\|_{A_p}^N \|f\|_{L^p(\omega)},$$ 

and therefore, for large enough $N$, the operator $\text{Id} - \mu^N B^N$ is invertible. This, together with (16), says that $\text{Id} - \mu B$ is an Fredholm operator. Now apply the index theory to $\text{Id} - \mu B$. The continuous deformation $\text{Id} - t \mu B$, $0 \leq t \leq 1$, is a homotopy from the identity operator to $\text{Id} - \mu B$. By the homotopical invariance of Index,

$$\text{Index} (\text{Id} - \mu B) = \text{Index} (\text{Id}) = 0.$$ 

Since injective operators with 0 index are onto, for the invertibility of $\text{Id} - \mu B$ it just remains to show that it is injective. So let $f \in L^p(\omega)$ be such that $f = \mu B f$. Then $f$ has compact support. Now, since belonging to $A_p$ is an open-ended condition (see
there exists $\delta > 0$ such that $p - \delta > 1$ and $\omega \in A_{p-\delta}$. Then

$$\omega^{-\frac{1}{p-1}} \in L^1_{\text{loc}}(\mathbb{C}).$$

Taking $\epsilon = \frac{\delta}{p-\delta}$, we obtain

$$\int_C |f(x)|^{1+\epsilon} \, dx \leq \left( \int_{\text{supp } f} |f(x)|^p \omega(x) \, dx \right)^{\frac{1+\epsilon}{p}} \left( \int_{\text{supp } f} \omega(x)^{-\frac{1+\epsilon}{p-1-\epsilon}} \, dx \right)^{\frac{p-(1+\epsilon)}{p}} < \infty,$$

therefore $f \in L^{1+\epsilon}(\mathbb{C})$. But $\text{Id} - \mu \mathcal{B}$ is injective on $L^p(\mathbb{C})$, $1 < p < \infty$, when $\mu \in \text{VMO}(\mathbb{C})$, by Iwaniec’s Theorem. Hence, $f \equiv 0$.

Finally, since $\text{Id} - \mu \mathcal{B} : L^p(\omega) \to L^p(\omega)$ is linear, bounded, and invertible, it then follows that it has a bounded inverse, so the inequality

$$\|g\|_{L^p(\omega)} \leq C \|(\text{Id} - \mu \mathcal{B})g\|_{L^p(\omega)}$$

holds for every $g \in L^p(\omega)$. Here the constant $C > 0$ depends only on the $L^p(\omega)$ norm of $\text{Id} - \mu \mathcal{B}$, and therefore on $p$, $k$ and $[\omega]_{A_p}$, but not on $g$. As a consequence, given $g \in L^p(\omega)$, and setting

$$f := C(\text{Id} - \mu \mathcal{B})^{-1}g,$$

we immediately see that $f$ satisfies $\overline{\partial}f - \mu \partial f = g$. Moreover, since $\omega \in A_p$,

$$\|Df\|_{L^p(\omega)} \leq \|\partial f\|_{L^p(\omega)} + \|\overline{\partial} f\|_{L^p(\omega)} = \|\mathcal{B}(\text{Id} - \mu \mathcal{B})^{-1}g\|_{L^p(\omega)} + \|(\text{Id} - \mu \mathcal{B})^{-1}g\|_{L^p(\omega)} \leq C\|g\|_{L^p(\omega)},$$

where still $C$ depends only on $p$, $k$ and $[\omega]_{A_p}$.

For the uniqueness, let us choose two solutions $f_1$, $f_2$ to the inhomogeneous equation. The difference $F = f_1 - f_2$ defines a solution to the homogeneous equation $\overline{\partial}F - \mu \partial F = 0$. Moreover, one has that $DF \in L^p(\omega)$ and, arguing as before, one sees that $DF \in L^{1+\epsilon}(\mathbb{C})$. In particular, this says that $(I - \mu \mathcal{B})(\overline{\partial} F) = 0$. But for $\mu \in \text{VMO}(\mathbb{C})$, it follows from Iwaniec’s Theorem that $\text{Id} - \mu \mathcal{B}$ is injective in $L^p(\mathbb{C})$ for any $1 < p < \infty$, whence $\overline{\partial} F = 0$. Thus $DF = 0$ and so $F$ is a constant.

The $\mathcal{C}$-linear Beltrami equation is a particular case of the following one,

$$\overline{\partial} f(z) - \mu(z) \partial f(z) - \nu(z) \overline{\partial f(z)} = g(z),$$

which we will refer to as the generalized Beltrami equation. It is well known that, in the plane, any linear, elliptic system, with two unknowns and two first-order equations on the derivatives, reduces to the above equation (modulo complex conjugation), whence the interest in understanding it is very big. An especially interesting example is obtained by setting $\mu = 0$, when one obtains the so-called conjugate Beltrami equation,

$$\overline{\partial} f(z) - \nu(z) \overline{\partial f(z)} = g(z).$$

A direct adaptation of the above proof immediately drives the problem towards the commutator $[\nu, \mathcal{B}]$. Unfortunately, as an operator from $L^p(\omega)$ onto itself, such commutator is not compact in general, even when $\omega = 1$. To show this, let us choose

$$\nu = i \nu_0 \chi_D + \nu_1 \chi_{\mathbb{C}\setminus D},$$

with $\nu_0, \nu_1 \in \mathcal{C}(\mathbb{C})$. Then
where the constant $\nu_0 \in \mathbb{R}$ and the function $\nu_1$ are chosen so that $\nu$ is continuous on $C$, compactly supported in $2\mathbb{D}$, with $\|\nu\|_{\infty} < 1$. Let us also consider

$$E = \{ f \in L^p; \| f \|_{L^p} \leq 1, \text{supp}(f) \subset \mathbb{D} \},$$

which is a bounded subset of $L^p$. For every $f \in E$, one has

$$\nu \bar{B}(f) - \bar{B}(\nu f) = \chi_{\mathbb{D}} i\nu_0 \bar{B}(f) + \chi_{C \setminus \mathbb{D}} \nu_1 \bar{B}(f) - \bar{B}(i\nu_0 f)$$

$$= \chi_{\mathbb{D}} i\nu_0 \bar{B}(f) + \chi_{C \setminus \mathbb{D}} \nu_1 \bar{B}(f) + i\nu_0 \bar{B}(f) - \bar{B}(i\nu_0 f)$$

$$= \chi_{\mathbb{D}} 2i\nu_0 \bar{B}(f) + \chi_{C \setminus \mathbb{D}} (i\nu_0 + \nu_1) \bar{B}(f).$$

In view of this relation, and since $\bar{B}$ is not compact, we have just cooked a concrete example of function $\nu \in VMO$ for which the commutator $[\nu, \bar{B}]$ is not compact. Nevertheless, it turns out that still a priori estimates hold, even for the generalized equation.

**Theorem 8.** Let $1 < p < \infty$, $\omega \in A_p$, and let $\mu, \nu \in VMO(C)$ be compactly supported, such that $\|\|\mu\| + \|\nu\|_{\infty} < 1$. Let $g \in L^p(\omega)$. Then the equation

$$\bar{\partial} f(z) - \mu(z) \partial f(z) - \nu(z) \bar{\partial} f(z) = g(z)$$

has a solution $f$ with $Df \in L^p(\omega)$ and

$$\|Df\|_{L^p(\omega)} \leq C \|g\|_{L^p(\omega)}.$$

This solution is unique, modulo an additive constant.

A previous proof for the above result has been shown in [11] for the constant weight $\omega = 1$. For the weighted counterpart, the arguments are based on a Neumann series argument similar to that in [11], with some minor modification. We write it here for completeness. The following Lemma will be needed.

**Lemma 9.** Let $\mu, \nu \in L^\infty(C)$ be measurable, bounded with compact support, such that $\|\|\mu\| + \|\nu\|_{\infty} < 1$. If $1 < p < \infty$ and $p' = \frac{p}{p-1}$, then the following statements are equivalent:

1. The operator $\text{Id} - \mu B - \nu \bar{B}: L^p(C) \to L^p(C)$ is bijective.
2. The operator $\text{Id} - \nu \bar{B}': \bar{B}'(C) \to L^p(C)$ is bijective.

**Proof.** When $\nu = 0$, the above result is well known, and follows as an easy consequence of the fact that, with respect to the dual pairing

$$\langle f, g \rangle = \int_C f(z) \bar{g}(z) \, dz,$$

the operator $\text{Id} - \mu B: L^p(C) \to L^p(C)$ has precisely $\text{Id} - B' \bar{\mu}: L^{p'}(C) \to L^p(C)$ as its $C$-linear adjoint. Unfortunately, when $\nu$ does not identically vanish, $\mathbb{R}$-linear operators do not have an adjoint with respect to this dual pairing. An alternative proof can be found in [11]. We will think the space of $C$-valued $L^p$ functions $L^p(C)$ as an $\mathbb{R}$-linear space,

$$L^p(C) = L^p_{\mathbb{R}}(C) \oplus L^p_{\mathbb{R}}(C),$$

by means of the obvious identification $u + iv = (u, v)$. According to this product structure, every bounded $\mathbb{R}$-linear operator $T: L^p_{\mathbb{R}}(C) \oplus L^p_{\mathbb{R}}(C) \to L^p_{\mathbb{R}}(C) \oplus L^p_{\mathbb{R}}(C)$ has an obvious matrix representation

$$T(u + iv) = T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$
where every $T_{ij}: L^p_{\mathbb{R}}(\mathbb{C}) \to L^p_{\mathbb{R}}(\mathbb{C})$ is bounded. Similarly, bounded linear functionals $U: L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C}) \to \mathbb{R}$ are represented by

$$U\left(\begin{array}{c} u \\ v \end{array}\right) = \left(\begin{array}{c} U_1 \\ U_2 \end{array}\right) \left(\begin{array}{c} u \\ v \end{array}\right),$$

where every $U_j: L^p_{\mathbb{R}}(\mathbb{C}) \to \mathbb{R}$ is bounded. By the Riesz Representation Theorem, we get that $L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ has precisely $L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ as its topological dual space. In fact, we have an $\mathbb{R}$-bilinear dual pairing,

$$\left\langle \left(\begin{array}{c} u \\ v \end{array}\right), \left(\begin{array}{c} u' \\ v' \end{array}\right) \right\rangle = \int u(z) u'(z) \, dz + \int v(z) v'(z) \, dz,$$

whenever $(u, v) \in L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ and $(u', v') \in L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$, and which is nothing but the real part of (19). Under this new dual pairing, every $\mathbb{R}$-linear operator $T: L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C}) \to L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ can be associated another operator

$$T': L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C}) \to L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C}),$$

called the $\mathbb{R}$-adjoint operator of $T$, defined by the common rule

$$\left\langle \left(\begin{array}{c} u \\ v \end{array}\right), T'\left(\begin{array}{c} u' \\ v' \end{array}\right) \right\rangle = \left\langle T\left(\begin{array}{c} u \\ v \end{array}\right), \left(\begin{array}{c} u' \\ v' \end{array}\right) \right\rangle.$$

If $T$ is a $\mathbb{C}$-linear operator, then $T'$ is the same as the $\mathbb{C}$-adjoint $T^*$ (i.e. the adjoint with respect to (19)) so in particular for the Beurling–Ahlfors transform $\mathcal{B}$ we have an $\mathbb{R}$-adjoint $\mathcal{B}'$, and moreover $\mathcal{B}^* = \mathcal{B}'$. Similarly, the pointwise multiplication by $\mu$ and $\nu$ are also $\mathbb{C}$-linear operators. Thus their $\mathbb{R}$-adjoints $\mu'$, $\nu'$ agree with their respective complex conjugates. Symbolically, $\mu' = \overline{\mu}$ and $\nu' = \overline{\nu}$. In contrast, general $\mathbb{R}$-linear operators need not have a $\mathbb{C}$-adjoint. For example, for the complex conjugation,

$$\mathbb{C} = \left(\begin{array}{cc} \text{Id} & 0 \\ 0 & -\text{Id} \end{array}\right)$$

one simply has $\mathbb{C}' = \mathbb{C}$. Putting all these things together, one easily sees that

$$(\text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}})' = (\text{Id} - \mu \mathcal{B} - \nu \mathcal{C}\overline{\mathcal{B}})' = \text{Id} - (\mu \mathcal{B})' - (\nu \mathcal{C}\overline{\mathcal{B}})'$$

$$= \text{Id} - \mathcal{B}'\mu' - \mathcal{C}\overline{\mathcal{B}}'\overline{\nu'} = \text{Id} - \mathcal{B}'\overline{\mu} - \mathcal{C}\overline{\mathcal{B}}\overline{\nu}$$

$$= \mathcal{B}^* (\text{Id} - \overline{\nu}\mathcal{B}^* - \mathcal{C}\overline{\mathcal{B}}^*) \mathcal{B} = \mathcal{B}^* (\text{Id} - \overline{\nu}\mathcal{B}^* - \nu \mathcal{C}^* \mathcal{B}^*) \mathcal{B},$$

where we used the fact that $\mathcal{B}^* \mathcal{B} = \mathcal{B} \mathcal{B}^* = \text{Id}$. As a consequence, and using that both $\mathcal{B}$ and $\mathcal{B}^*$ are bijective in $L^p(\mathbb{C})$, we obtain that the bijectivity of the operator $\text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}}$ in $L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ is equivalent to that of $\text{Id} - \overline{\nu}\mathcal{B}^* - \nu \mathcal{C}^* \mathcal{B}^*$ in the dual space $L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$. Similarly, one proves that

$$(\text{Id} - \mu \mathcal{B}^* - \nu \mathcal{C}^* \mathcal{B}^*)' = \mathcal{B}(\text{Id} - \overline{\nu}\mathcal{B} - \nu \overline{\mathcal{B}})\mathcal{B}^*.$$  

Hence, the bijectivity of $\text{Id} - \mu \mathcal{B}^* - \nu \mathcal{C}^* \mathcal{B}^*$ in $L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ is equivalent to the bijectivity of $\text{Id} - \overline{\nu}\mathcal{B} - \nu \overline{\mathcal{B}}$ in $L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$.

\textbf{Lemma 10.} If $1 < p < \infty$, $\omega \in A_p$, $\mu, \nu \in VMO$ have compact support, and $|||\mu| + |\nu|||_{\infty} \leq k < 1$, then the operators

$$\text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}} \quad \text{and} \quad \text{Id} - \mu \mathcal{B}^* - \nu \overline{\mathcal{B}}^*$$

are bijective.
are Fredholm operators in $L^p(\omega)$.

Proof. We will show the claim for the operator $\text{Id} - \mu B - \nu B$. For $\text{Id} - \mu B - \nu B$, the proof follows similarly. It will be more convenient for us to write $B = CB$. As in the proof of Theorem 1, we set

$$P_N = \sum_{j=0}^{N} (\mu B + \nu CB)^j.$$  

Then

$$(\text{Id} - \mu B - \nu CB) \circ P_{N-1} = \text{Id} - (\mu B + \nu CB)^N,$$

$$(P_{N-1} \circ (\text{Id} - \mu B + \nu CB)) = \text{Id} - (\mu B + \nu CB)^N.$$  

We will show that

$$(\mu B + \nu CB)^N = R_N + K_N$$

where $K_N$ is a compact operator, and $R_N$ is a bounded, linear operator such that

$$\|R_N f\|_{L^p(\omega)} \leq C k^N N^3 \|f\|_{L^p(\omega)}.$$  

Then, the Fredholm property follows immediately. To prove (20), let us write, for any two operators $T_1, T_2$,

$$(T_1 + T_2)^N = \sum_{\sigma \in \{1,2\}^N} T_{\sigma},$$

where $\sigma \in \{1,2\}^N$ means that $\sigma = (\sigma(1), \ldots, \sigma(N))$ and $\sigma(j) \in \{1,2\}$ for all $j = 1, \ldots, N$, and

$$T_{\sigma} = T_{\sigma(1)}T_{\sigma(2)}\ldots T_{\sigma(N)}.$$  

By choosing $T_1 = \mu B$ and $T_2 = \nu CB$, one sees that every $T_{\sigma(j)}$ can be written as

$$T_{\sigma(j)} = M_{\sigma(j)}C_{\sigma(j)}B$$

being $M_1 = \mu$, $M_2 = \nu$, $C_1 = \text{Id}$ and $C_2 = C$. Thus

$$T_{\sigma} = M_{\sigma(1)}C_{\sigma(1)}B M_{\sigma(2)}C_{\sigma(2)}B \ldots M_{\sigma(N)}C_{\sigma(N)}B.$$  

Our main task consists of rewriting $T_{\sigma}$ as

$$(21) \quad T_{\sigma} = M_{\sigma(1)}C_{\sigma(1)}M_{\sigma(2)}C_{\sigma(2)}\ldots M_{\sigma(N)}C_{\sigma(N)} B_{\sigma} + K_{\sigma}.$$  

for some compact operator $K_{\sigma}$ and some bounded operator $B_{\sigma} \in \{B, B^*\}^N$. If this is possible, then one gets that

$$(T_1 + T_2)^N = \sum_{\sigma \in \{1,2\}^N} M_{\sigma(1)}C_{\sigma(1)}M_{\sigma(2)}C_{\sigma(2)}\ldots M_{\sigma(N)}C_{\sigma(N)} B_{\sigma} + \sum_{\sigma \in \{1,2\}^N} K_{\sigma} = R_N + K_N.$$  

It is clear that $K_N$ is compact (it is a finite sum of compact operators). Moreover, from $B_{\sigma} \in \{B, B^*\}^N$, one has

$$|B_{\sigma} f(z)| \leq \sum_{j=1}^{N} |B^n f(z)| + \sum_{j=1}^{N} |(B^*)^n f(z)|.$$  

Thus
\[
|R_N f(z)| \leq \sum_{\sigma \in \{1, 2\}^N} |M_{\sigma(1)} C_{\sigma(1)} \ldots M_{\sigma(N)} C_{\sigma(N)} B_{\sigma} f(z)|
\]
\[
\leq \sum_{\sigma \in \{1, 2\}^N} |M_{\sigma(1)}(z)| \ldots |M_{\sigma(N)}(z)| \left( \sum_{n=1}^N |B^n f(z)| + \sum_{j=1}^N |(B^*)^n f(z)| \right)
\]
\[
= \left( \sum_{n=1}^N |B^n f(z)| + \sum_{j=1}^N |(B^*)^n f(z)| \right) . (|M_1(z)| + |M_2(z)|)^N
\]

Now, since \( \|B^j f\|_{L^p(\omega)} \leq C_\omega j^2 \|f\|_{L^p(\omega)} \) (and similarly for \((B^*)^n\), see (17)), one gets that
\[
\|R_N f\|_{L^p(\omega)} \leq ||M_1|| + |M_2|^{N} C_\omega \left( \sum_{j=1}^N j^2 \right) \|f\|_{L^p(\omega)} = C k^N N^3 \|f\|_{L^p(\omega)}
\]

and so (20) follows from the representation (21). To prove that representation (21) can be found, we need the help of Theorem 2, according to which the differences
\[
K_j = B M_{\sigma(j)} - M_{\sigma(j)} B
\]
are again compact, because \( B \) is com-

Now, by reminding that \( C B = B^* C \),
we have that \( B C_{\sigma(j+1)} = C_{\sigma(j+1)} B_j \) for some \( B_j \in \{B, B^*\} \). Thus
\[
T_\sigma = M_{\sigma(1)} C_{\sigma(1)} M_{\sigma(2)} C_{\sigma(2)} B \ldots M_{\sigma(N)} C_{\sigma(N)} B
\]
\[
= M_{\sigma(1)} C_{\sigma(1)} M_{\sigma(2)} C_{\sigma(2)} M_{\sigma(3)} \ldots B C_{\sigma(N)} B + K_\sigma
\]
where all the factors containing \( K_j \) are included in \( K_\sigma \). In particular, \( K_\sigma \) is compact.

Now, one can start again. On one hand, the differences \( B_j M_{\sigma(j+2)} - M_{\sigma(j+2)} B_j \) are again compact, because \( B_j \in \{B, B^*\} \) and \( M_{\sigma(j+2)} \in VMO \). Moreover, the composition \( B_j C_{\sigma(j+2)} \) can be written as \( C_{\sigma(j+2)} \tilde{B}_j \), where \( \tilde{B}_j \) need not be the same as \( B_j \) but still \( \tilde{B}_j \in \{B, B^*\} \). So, with a little abuse of notation, and after repeating this algorithm a total of \( N - 1 \) times, one obtains (21). The claim follows. \( \square \)

**Proof of Theorem 8.** The equation we want to solve can be rewritten, at least formally, in the following terms
\[
(\Id - \mu B - \nu \tilde{B})(\tilde{f}) = g,
\]
so that we need to understand the \( \textbf{R} \)-linear operator \( T = \Id - \mu \tilde{B} - \nu \tilde{B} \). By Lemma 10, we know that \( T \) is a Fredholm operator in \( L^p(\omega), 1 < p < \infty \). Now, we prove that it is also injective. Indeed, if
\[
T(h) = 0
\]
for some \( h \in L^p(\omega) \) and \( \omega \in A_p \), it then follows that
\[
h = \mu B(h) + \nu \tilde{B}(h)
\]
so that \( h \) has compact support, and thus \( h \in L^{1+}(\textbf{C}) \) for some \( \epsilon > 0 \) (arguing as in (18)). We are then reduced to show that
\[
T : L^{1+\epsilon}(\textbf{C}) \to L^{1+\epsilon}(\textbf{C}) \quad \text{is injective}.
\]
Let us first see how the proof finishes. Injectivity of $T$ in $L^{1+\epsilon}(\mathbb{C})$ gives us that $h = 0$. Therefore, $T$ is injective also in $L^p(\omega)$. Being as well Fredholm, it is also surjective, so by the open map Theorem it has a bounded inverse $T^{-1} : L^p(\omega) \rightarrow L^p(\omega)$. As a consequence, given any $g \in L^p(\omega)$, the function
\[
 f = CT^{-1}(g)
\]
is well defined, and has derivatives in $L^p(\omega)$ satisfying the estimate
\[
\|Df\|_{L^p(\omega)} \leq \|\partial f\|_{L^p(\omega)} + \|\overline{\partial f}\|_{L^p(\omega)} = \|\mathcal{B}T^{-1}(g)\|_{L^p(\omega)} + \|T^{-1}(g)\|_{L^p(\omega)} \leq (C+1)\|T^{-1}(g)\|_{L^p(\omega)} \leq C\|g\|_{L^p(\omega)},
\]
because $\omega \in A_p$. Moreover, we see that $f$ solves the inhomogeneous equation
\[
 \overline{\partial}f(z) - \mu(z)\partial f(z) - \nu(z)\overline{\partial f(z)} = g(z).
\]
Finally, if there were two such solutions $f_1$, $f_2$, then their difference $F = f_1 - f_2$ solves the homogeneous equation, and also $DF \in L^p(\omega)$. Thus
\[
 T(\overline{\partial}F) = 0.
\]
By the injectivity of $T$ we get that $\overline{\partial}F = 0$, and from $DF \in L^p(\omega)$ we get that $\partial F = 0$, whence $F$ must be a constant.

We now prove the injectivity of $T$ in $L^p(\mathbb{C})$, $1 < p < \infty$. First, if $p \geq 2$ and $h \in L^p(\mathbb{C})$ is such that $T(h) = 0$, then $h$ has compact support, whence $h \in L^2(\mathbb{C})$. But $\mathcal{B}, \overline{\mathcal{B}}$ are isometries in $L^2(\mathbb{C})$, whence
\[
 \|h\|_2 \leq k\|\mathcal{B}h\|_2 = k\|f\|_2
\]
and thus $h = 0$, as desired. For $p < 2$, we recall from Lemma 9 that the bijectivity of $T$ in $L^p(\mathbb{C})$ is equivalent to that of $T' = \text{Id} - \overline{\mu}\mathcal{B} - \nu\overline{\mathcal{B}}$ in the dual space $L^p(\mathbb{C})$. For this, note that the injectivity of $T'$ in $L^p(\mathbb{C})$ follows as above (since $p' \geq 2$). Note also that, by Lemma 10 we know that $T'$ is a Fredholm operator in $L^{p'}(\mathbb{C})$, since $\overline{\mu}$ and $\nu$ are compactly supported $VMO$ functions. The claim follows. \hfill \Box

4. Applications

We start this section by recalling that if $\mu, \nu \in L^\infty(\mathbb{C})$ are compactly supported with $\|\mu\|_\infty + \|\nu\|_\infty \leq k < 1$ then the equation
\[
 \overline{\partial}\phi(z) - \mu(z)\partial\phi(z) - \nu(z)\overline{\partial\phi(z)} = 0
\]
admits a unique homeomorphic $W^{1,2}_{\text{loc}}(\mathbb{C})$ solution $\phi : \mathbb{C} \rightarrow \mathbb{C}$ such that $|\phi(z) - z| \rightarrow 0$ as $|z| \rightarrow \infty$. We call it the principal solution, and it defines a global $K$-quasiconformal map, $K = \frac{1+k}{1-k}$. See the monograph [1].

Applications of Theorem 1 are based in the following change of variables lemma, which is already proved in [3, Lemma 14]. We rewrite it here for completeness.

Lemma 11. Given a compactly supported function $\mu \in L^\infty(\mathbb{C})$ such that $\|\mu\|_\infty \leq k < 1$, let $\phi$ denote the principal solution to the equation
\[
 \overline{\partial}\phi(z) - \mu(z)\partial\phi(z) = 0.
\]
For a fixed weight $\omega$, let us define
\[
 \eta(\zeta) = \omega(\phi^{-1}(\zeta))J(\zeta, \phi^{-1})^{1-\frac{p}{2}}.
\]
The following statements are equivalent:
(a) For every $h \in L^p(\omega)$, the inhomogeneous equation
\begin{equation}
\partial f(z) - \mu(z) \partial f(z) = h(z)
\end{equation}
has a solution $f$ with $Df \in L^p(\omega)$ and
\begin{equation}
\|Df\|_{L^p(\omega)} \leq C_1 \|h\|_{L^p(\omega)}.
\end{equation}
(b) For every $\tilde{h} \in L^p(\eta)$, the equation
\begin{equation}
\partial g(\zeta) = \tilde{h}(\zeta)
\end{equation}
has a solution $g$ with $Dg \in L^p(\eta)$ and
\begin{equation}
\|Dg\|_{L^p(\eta)} \leq C_2 \|\tilde{h}\|_{L^p(\eta)}.
\end{equation}

Proof. Let us first assume that (b) holds. To get (a), we have to find a solution $f$ of (22) such that $Df \in L^p(\omega)$ with the estimate (23). To this end, we make in (22) the change of coordinates $g = f \circ \phi^{-1}$. We obtain for $g$ the following equation
\begin{equation}
\partial g(\zeta) = \tilde{h}(\zeta),
\end{equation}
where $\zeta = \phi(z)$ and
\begin{equation}
\tilde{h}(\zeta) = h(z) \frac{\partial \phi(z)}{J(z, \phi)}.
\end{equation}

In order to apply the assumption (b), we must check that $\tilde{h} \in L^p(\eta)$. However,
\begin{align*}
\|\tilde{h}\|_{L^p(\eta)}^p &= \int |\tilde{h}(\zeta)|^p \eta(\zeta) \, d\zeta = \int |\tilde{h}(\phi(z))|^p \omega(z) J(z, \phi)^{\frac{p}{2}} dz \\
&= \int |h(z)|^p \frac{\omega(z)}{(1 - |\mu(z)|^2)^{\frac{p}{2}}} \, dz \leq \frac{1}{(1 - k^2)^{\frac{p}{2}}} \|h\|_{L^p(\omega)}^p.
\end{align*}
Since $\tilde{h} \in L^p(\eta)$, (b) applies, and a solution $g$ to (26) can be found with the estimate
\begin{equation}
\|Dg\|_{L^p(\eta)} \leq C_2 \|\tilde{h}\|_{L^p(\eta)} \leq \frac{C_2}{(1 - k^2)^{\frac{p}{2}}} \|h\|_{L^p(\omega)}.
\end{equation}

With such a $g$, the function $f = g \circ \phi$ is well defined, and
\begin{align*}
\int |Df(z)|^p \omega(z) \, dz &= \int |Dg(\phi(z))| D\phi(z)^p \omega(z) \, dz \\
&= \int |Dg(\zeta)| D\phi(\phi^{-1}(\zeta))|^p \omega(\phi^{-1}(\zeta)) J(\zeta, \phi^{-1}) \, d\zeta \\
&\leq \left(\frac{1 + k}{1 - k}\right)^{\frac{p}{2}} \int |Dg(\zeta)|^p J(\phi^{-1}(\zeta), \phi)^{\frac{p}{2}} \omega(\phi^{-1}(\zeta)) J(\zeta, \phi^{-1}) \, d\zeta \\
&= \left(\frac{1 + k}{1 - k}\right)^{\frac{p}{2}} \int |Dg(\zeta)|^p \eta(\zeta) \, d\zeta.
\end{align*}
due to the $1 + k/k$-quasiconformality of $\phi$. Moreover, $f$ satisfies the desired equation, and so (a) follows, with constant $C_1 = \frac{C_2}{1 - k^2}$.

To show that (a) implies (b), for a given $\tilde{h} \in L^p(\eta)$ we have to find a solution of (24) satisfying the estimate (25). Since this is a $\partial\bar{\partial}$-equation, this could be done by simply convolving $\tilde{h}$ with the Cauchy kernel $\frac{1}{\pi z}$, but then the desired estimate for the solution $g$ cannot be obtained in this way, because at this point the weight $\eta$ is...
not known to belong to $A_p$. So we will proceed in a different manner. Namely, let $\hat{h} \in L^p(\eta)$ be fixed, and set $h(z) = \hat{h}(\phi(z)) \partial\phi(z) (1 - |\mu(z)|^2)$. Then

$$\int |h(z)|^p \omega(z) \, dz = \int |\hat{h}(\zeta)|^p (1 - |\mu(\phi^{-1}(\zeta))|^2)^{p/2} \eta(\zeta) \, d\zeta \leq \int |\hat{h}(\zeta)|^p \eta(\zeta) \, d\zeta,$$

and so $h \in L^p(\omega)$. By (a), the equation

$$\overline{\partial}f(z) - \mu(z) \partial f(z) = h(z)$$

has a unique solution $f$ with $Df \in L^p(\omega)$, and furthermore $\|Df\|_{L^p(\omega)} \leq C_1 \|\hat{h}\|_{L^p(\eta)}$.

Now we simply set $g = f \circ \phi^{-1}$. By the chain rule, one gets that $\overline{\partial}g = \hat{h}$, and

$$\int |Dg(\zeta)|^p \eta(\zeta) \, d\zeta = \int |Dg(\phi^{-1}(z))|^p J(z, \phi^{-1}) \eta(\phi^{-1}(z)) \, dz$$

$$= \int |D\mu(\phi^{-1}(z))| J(z, \phi^{-1}) \eta(\phi^{-1}(z)) \, dz$$

$$\leq \left( \frac{1 + k}{1 - k} \right)^{\frac{p}{2}} \int |Df(z)|^p J(z, \phi^{-1}) \eta(\phi^{-1}(z)) \, dz$$

$$= \left( \frac{1 + k}{1 - k} \right)^{\frac{p}{2}} \int |Df(z)|^p \omega(z) \, dz.$$ 

Thus, $\|Dg\|_{L^p(\eta)} \leq C_2 \|\hat{h}\|_{L^p(\eta)}$ with $C_2 = \left( \frac{1 + k}{1 - k} \right)^{\frac{p}{2}} C_1$, and (b) follows. \hfill \square

According to the previous Lemma, a priori estimates for $\overline{\partial} - \mu \partial$ in $L^p(\omega)$ are equivalent to a priori estimates for $\overline{\partial}$ in $L^p(\eta)$. However, by Theorem 1, if $\omega$ is taken in $A_p$, the first statement holds, at least, when $\mu$ is compactly supported and belongs to $VMO$. We then obtain the following consequence.

**Corollary 12.** Let $\mu \in VMO$ be compactly supported, such that $\|\mu\|_\infty < 1$, and let $\phi$ be the principal solution of

$$\overline{\partial}\phi(z) - \mu(z) \partial\phi(z) = 0.$$

If $1 < p < \infty$ and $\omega \in A_p$, then the weight

$$\eta(z) = \omega(\phi^{-1}(z)) J(z, \phi^{-1})^{1-p/2}$$

belongs to $A_p$. Moreover, its $A_p$ constant $[\eta]_{A_p}$ can be bounded in terms of $\mu$, $p$ and $[\omega]_{A_p}$.

**Proof.** Under the above assumptions, by Theorem 1, we know that if $h \in L^p(\omega)$, then the equation $\overline{\partial}f - \mu \partial f = h$ can be found a solution $f$ with $Df \in L^p(\omega)$ and such that $\|Df\|_{L^p(\omega)} \leq C_0 \|h\|_{L^p(\omega)}$, for some constant $C_0 > 0$ depending on $k, p$ and $[\omega]_{A_p}$. Equivalently, by Lemma 11, for every $\hat{h} \in L^p(\eta)$ we can find a solution $g$ of the inhomogeneous Cauchy–Riemann equation

$$\overline{\partial}g = \hat{h},$$

with $Dg \in L^p(\eta)$ and in such a way that the estimate

$$\|Dg\|_{L^p(\eta)} \leq C \|\hat{h}\|_{L^p(\eta)}$$
holds for some constant $C$ depending on $C_0$, $k$ and $p$. Now, let us choose $\varphi \in C_0^\infty (C)$ and set $\tilde{h} = \partial \varphi$. Then of course $g = \varphi$ and $\partial \varphi = B (\partial \varphi)$, and the above inequality says that
\[
||| \partial \varphi || + | \partial \varphi ||_{L^p (\eta)} \leq C ||| \partial \varphi ||_{L^p (\eta)},
\]
whence the estimate
\[
(27) \quad ||| B (\psi) ||_{L^p (\eta)} \leq (C^p - 1) \frac{1}{\delta} ||| \psi ||_{L^p (\eta)}
\]
holds for any $\psi \in D^* = \{ \psi \in C_c^\infty (C) : \int \psi = 0 \}$. It turns out that $D^*$ is a dense subclass of $L^p (\eta)$, provided that $\eta \in L^1_{loc}$ is a positive function with infinite mass. But this is actually the case. Indeed, one has
\[
\int_{D(0,R)} \eta (\zeta) \, d\zeta = \int_{\phi^{-1} (D(0,R))} \omega (z) J (z, \phi) \frac{\mu}{z} \, dz.
\]
Above, the integral on the right hand side certainly grows to infinite as $R \to \infty$. Otherwise, one would have that $J (\cdot, \phi)^{\frac{1}{2}} \in L^p (\omega)$. But $\phi$ is a principal quasiconformal map, hence $J (z, \phi) = 1 + O (1/|z|^2)$ as $|z| \to \infty$. Thus for large enough $N > M > 0$,
\[
\int_{M<|z|<N} J (z, \phi) \frac{\omega (z)}{z} \, dz \geq C \int_{M<|z|<N} \omega (z) \, dz
\]
and the last integral above blows up as $N \to \infty$, because $\omega$ is an $A_p$ weight.

Therefore, the estimate (27) holds for all $\psi$ in $L^p (\eta)$. By [17, Ch. V, Proposition 7], this implies that $\eta \in A_p$, and moreover, $[\eta]_{A_p}$ depends only on the constant $(C^p - 1) \frac{1}{\delta}$, that is, on $k$, $p$ and $[\omega]_{A_p}$. \hfill \Box

The above Corollary is especially interesting in two particular cases. First, for the constant weight $\omega = 1$ the above result says that
\[
J (\cdot, \phi^{-1})^{1-p/2} \in A_p, \quad 1 < p < \infty.
\]
Without the VMO assumption, this is only true for the smaller range $1 + \| \mu \|_\infty < p < 1 + \frac{1}{\| \mu \|_{\infty}}$ (see e.g. [2, Theorem 13.4.2]). Secondly, by setting $p = 2$ in Corollary 12 we get the following.

\textbf{Corollary 13.} Let $\mu \in \text{VMO}$ be compactly supported, and assume that $\| \mu \|_{\infty} < 1$. Let $\phi$ be the principal solution of
\[
\overline{\partial} \phi (z) - \mu (z) \partial \phi (z) = 0.
\]
Then, for every $\omega \in A_2$ one has $\omega \circ \phi^{-1} \in A_2$.

The above result drives us to the problem of finding what homeomorphisms $\phi$ preserve the $A_p$ classes under composition with $\phi^{-1}$. Note that preserving $A_p$ forces also the preservation of the space $BMO$ of functions with bounded mean oscillation, and thus such homeomorphisms $\phi$ must be quasiconformal [14]. However, at level of Muckenhoupt weights, the question is deeper. As an example, simply consider the weight
\[
\omega (z) = \frac{1}{|z|^\alpha},
\]
and its composition with the inverse of a radial stretching $\phi (z) = z |z|^{K-1}$. It is clear that the values of $p$ for which $A_p$ contains $\omega$ and $\omega \circ \phi^{-1}$ are not the same, whence preservation of $A_p$ requires something else. This question was solved by Johnson and Neugebauer [10] as follows.
Theorem 14. Let $\phi: \mathbb{C} \to \mathbb{C}$ be $K$-quasiconformal. Then, the following statements are equivalent:

1. If $\omega \in A_2$ then $\omega \circ \phi^{-1} \in A_2$ quantitatively.
2. For a fixed $p \in (1, \infty)$, if $\omega \in A_p$ then $\omega \circ \phi^{-1} \in A_p$ quantitatively.
3. $J(\cdot, \phi^{-1}) \in A_p$ for every $p \in (1, \infty)$.

It follows from Corollary 13 and Theorem 14 that, if $\mu \in VMO$ is compactly supported, $\|\mu\|_\infty \leq k < 1$ and $\phi$ is the principal solution to the $C$-linear equation $\overline{\partial} \phi = \mu \partial \phi$, then

$$J(\cdot, \phi^{-1}) \in A_p, \quad \text{for every } p > 1.$$ 

By quasisymmetry, the $A_p$ condition (5) for $J(\cdot, \phi^{-1})$ also holds if $D$ is quasidisk. But then, after a change of coordinates, one gets for any disk $D'$ and $D = \phi(D')$ that

$$\left( \int_D J(\cdot, \phi^{-1}) \right)^{p-1} = \left( \int_{D'} J(\cdot, \phi) \right)^{-1} \left( \int_{D'} J(\cdot, \phi) \right)^{\frac{1}{p'}},$$

where $p' = \frac{p}{p-1}$. As a consequence, we get that $J(\cdot, \phi)$ satisfies the reverse Hölder estimate (4) for any $1 < p' < \infty$. This shows Corollary 4.

It is not clear to the authors what is the role of $C$-linearity in the above results. In other words, there seems to be no reason for Theorem 13 to fail if one replaces the $C$-linear equation by the generalized one, while maintaining the ellipticity, compact support and smoothness on the coefficients. Thus one may ask what is the class of weights $\omega > 0$ for which the estimate

$$\|Df\|_{L^2(\omega)} \leq C \|\overline{\partial} f - \mu \partial f - \nu \overline{\partial f}\|_{L^2(\omega)}$$

holds for any $f \in C_0^\infty(\mathbb{C})$. The following result, which is a counterpart of Lemma 11, explains this class contains $A_p$.

Lemma 15. To each pair $\mu, \nu \in L^\infty(\mathbb{C})$ of compactly supported functions with $\|\mu\| + \|\nu\|_\infty \leq k < 1$, let us associate, on one hand, the principal solution $\phi$ to the equation

$$\overline{\partial} \phi(z) - \mu(z) \partial \phi(z) - \nu(z) \overline{\partial \phi(z)} = 0,$$

and on the other, the function $\lambda$ defined by $\lambda \circ \phi = \frac{-2i\nu}{1-|\mu|^2 + |\nu|^2}$. For a fixed weight $\omega$, let us define

$$\eta(\zeta) = \omega(\phi^{-1}(\zeta)) J(\zeta, \phi^{-1})^{1-\frac{2}{p}}.$$

The following statements are equivalent:

(a) For every $h \in L^p(\omega)$, the equation

$$\overline{\partial} f(z) - \mu(z) \partial f(z) - \nu(z) \overline{\partial f(z)} = h(z)$$

has a solution $f$ with $Df \in L^p(\omega)$ and $\|Df\|_{L^p(\omega)} \leq C \|h\|_{L^p(\omega)}$.

(b) For every $\tilde{h} \in L^p(\eta)$, the equation

$$\overline{\partial} g(\zeta) - \lambda(\zeta) \operatorname{Im}(\partial g(\zeta)) = \tilde{h}(\zeta)$$

has a solution $g$ with $Dg \in L^p(\eta)$ and $\|Dg\|_{L^p(\eta)} \leq C \|\tilde{h}\|_{L^p(\eta)}$.

Although the proof requires quite tedious calculations, it follows the scheme of Lemma 11, and thus we omit it. From this Lemma, the following one is a natural question to ask.
Question 16. Let $\omega \in L^1_{\text{loc}}(C)$ be such that $\omega(z) > 0$ almost everywhere, and let $\lambda \in L^\infty_{\text{loc}}(C)$ be a compactly supported $\text{VMO}$ function, such that $\|\lambda\|_{\infty} < 1$. If the estimate

$$\|Df\|_{L^p(\omega)} \leq C \|\overline{\partial} f - \lambda \text{Im}(\partial f)\|_{L^p(\omega)}$$

holds for every $f \in C^\infty_0$, is it true that $\omega \in A_2$?

What we actually want is to find planar, elliptic, first order differential operators, different from the $\partial$, that can be used to characterize the Muckenhoupt classes $A_p$. In this direction, an affirmative answer to Question 16 would allow us to characterize $A_2$ weights as follows: given $\mu, \nu \in \text{VMO}$ uniformly elliptic and compactly supported, a positive a.e. function $\omega \in L^1_{\text{loc}}$ is an $A_2$ weight if and only if there is a constant $C \geq 1$ such that

$$\|Df\|_{L^2(\omega)} \leq C \|\overline{\partial} f - \mu \partial f - \nu \overline{\partial} f\|_{L^2(\omega)}, \quad \text{for every } f \in C^\infty_0(C).$$

Note that if one assumes $\|\mu\| + |\nu|_{\infty} < \epsilon$ for some $\epsilon > 0$, then (28) implies that

$$\|\partial f\|_{L^2(\omega)}^2 + \|\overline{\partial} f\|_{L^2(\omega)}^2 \leq C \|\overline{\partial} f\|_{L^2(\omega)}^2 + C \epsilon \|\partial f\|_{L^2(\omega)}.$$  

In particular, if for some reason $\epsilon < \frac{1}{C}$ then one gets

$$\|\partial f\|_{L^2(\omega)} \leq \frac{C - 1}{1 - C\epsilon} \|\overline{\partial} f\|_{L^2(\omega)}.$$  

From the above estimate, weighted bounds for $B$ easily follow, and so in this case such a characterization holds.

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