BELL POLYNOMIALS AND k-GENERALIZED DYCK PATHS

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Abstract. A k-generalized Dyck path of length n is a lattice path from (0, 0) to (n, 0) in the plane integer lattice \( \mathbb{Z} \times \mathbb{Z} \) consisting of horizontal-steps \((k, 0)\) for a given integer \( k \geq 0 \), up-steps \((1, 1)\), and down-steps \((1, -1)\), which never passes below the \( x \)-axis. The present paper studies three kinds of statistics on \( k \)-generalized Dyck paths: ”number of \( u \)-segments”, ”number of internal \( u \)-segments” and ”number of \((u, h)\)-segments”. The Lagrange inversion formula is used to represent the generating function for the number of \( k \)-generalized Dyck paths according to the statistics as a sum of the partial Bell polynomials or the potential polynomials. Many important special cases are considered leading to several surprising observations. Moreover, enumeration results related to \( u \)-segments and \((u, h)\)-segments are also established, which produce many new combinatorial identities, and specially, two new expressions for Catalan numbers.

Keywords: Bell polynomials, Potential polynomials, \( k \)-paths, Catalan numbers

2000 Mathematics Subject Classification: Primary 05A05, 05A15; Secondary 05C90

1. Introduction

Let \( \mathcal{L}_{n,k} \) denote the set of lattice paths of length \( n \) from \((0, 0)\) to \((n, 0)\) in the plane integer lattice \( \mathbb{Z} \times \mathbb{Z} \) consisting of horizontal-steps \( h = (k, 0) \) for a given integer \( k \geq 0 \), up-steps \( u = (1, 1) \), and down-steps \( d = (1, -1) \). Let \( \mathcal{L}^{m,j}_{n,k} \) be the set of lattice paths in \( \mathcal{L}_{n,k} \) with \( m \) up-steps and \( j \) horizontal-steps. Let \( L \) be any lattice path in \( \mathcal{L}^{m,j}_{n,k} \). A \( u \)-segment of \( L \) is a maximal sequence of consecutive up-steps in \( L \). Define \( \alpha_i(L) \) to be the number of \( u \)-segments of length \( i \) in \( L \) and call \( L \) having the \( u \)-segments of type \( 1^{\alpha_1(L)}2^{\alpha_2(L)}\ldots \). Let \( \mathcal{L}^{m,j}_{n,k,r} \) be the subset of lattice paths in \( \mathcal{L}^{m,j}_{n,k} \) with \( r \) \( u \)-segments.

A \( k \)-generalized Dyck path or \( k \)-path (for short) of length \( n \) is a lattice path in \( \mathcal{L}_{n,k} \) which never passes below the \( x \)-axis. By our notation, a Dyck path is a 0-path, a Motzkin path is a 1-path and a Schröder path is a 2-path. Let \( \mathcal{P}^{m,j}_{n,k} \) denote the set of \( k \)-paths of length \( n \) (i.e. \( n = 2m + kj \)) with \( m \) up-steps and \( j \) horizontal-steps and let \( \mathcal{Q}^{m,j}_{n,k} \) be the subset of \( k \)-paths in \( \mathcal{P}^{m,j}_{n,k} \) with no horizontal-step at \( x \)-axis. Define \( \mathcal{Q}^{m,j}_{n,k,r} \) \( (\mathcal{Q}^{m,j}_{n,k,r}) \) to be the subset of \( k \)-paths in \( \mathcal{P}^{m,j}_{n,k} \) \( (\mathcal{Q}^{m,j}_{n,k,r}) \) with \( r \) \( u \)-segments.

In [9], we study two kinds of statistics on Dyck paths: ”number of \( u \)-segments” and ”number of internal \( u \)-segments”. In this paper, we consider these two statistics together with ”number of \((u, h)\)-segments” in the more extensive setting of \( k \)-paths. In order to do this, we present two necessary tools: Lagrange inversion formula and the potential polynomials.

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Lagrange Inversion Formula \[^{10}\.\] If \( f(x) = \sum_{n \geq 1} f_n x^n \) with \( f_1 \neq 0 \), then the coefficients of the composition inverse \( g(x) \) of \( f(x) \) (namely, \( f(g(x)) = g(f(x)) = x \)) can be given by
\[
[x^n]g(x) = \frac{1}{n}[x^{n-1}]{\frac{x}{f(x)}}^n.
\]

More generally, for any formal power series \( \Phi(x) \),
\[
[x^n]\Phi(g(x)) = \frac{1}{n}[x^{n-1}]\Phi^\prime(x){\frac{x}{f(x)}}^n,
\]
for all \( n \geq 1 \), where \( \Phi^\prime(x) \) is the derivative of \( \Phi(x) \) with respect to \( x \).

The Potential Polynomials \[^{5} pp. 141,157\]. The potential polynomials \( P_n^{(\lambda)} \) are defined for each complex number \( \lambda \) by
\[
1 + \sum_{n \geq 1} P_n^{(\lambda)} \frac{x^n}{n!} = \left(1 + \sum_{n \geq 1} f_n \frac{x^n}{n!}\right)^{\lambda},
\]
which can be represented by Bell polynomials
\[
P_n^{(\lambda)} = P_n^{(\lambda)}(f_1,f_2,f_3,\ldots) = \sum_{1 \leq k \leq n} \binom{\lambda}{k} k! B_{n,k}(f_1,f_2,f_3,\ldots),
\]
where \( B_{n,i}(x_1,x_2,\ldots) \) is the partial Bell polynomial on the variables \( \{x_j\}_{j \geq 1} \) (see \[^{2}\.\]).

In this paper, with the Lagrange inversion formula, we can represent the generating functions for the number of \( k \)-paths according to our statistics (see Sections 2-4) as a sum of partial Bell polynomials or the potential polynomials. For example,
\[
\sum_{P \in \mathcal{P}_{n,k}^{m,j}} \prod_{i \geq 1} t_i^{\alpha_i(P)} = \frac{1}{m+1} \binom{m+j}{m} \sum_{r=0}^{m} \binom{m+j+1}{r} \frac{r!}{m!} B_{m,r}(1!t_1,2!t_2,\ldots),
\]
\[
\sum_{Q \in \mathcal{Q}_{n,k}^{m,j}} \prod_{i \geq 1} t_i^{\alpha_i(Q)} = \frac{m}{(m+j+1)(m+j)} \binom{m+j}{j} P_{m}^{(m+j+1)} \frac{1}{m!} (1!t_1,2!t_2,\ldots).
\]

We consider a number of important special cases. These lead to several surprising results. Moreover, enumeration results related to \( u \)-segments and \((u,h)\)-segments are also established in Section 5, producing many new combinatorial identities and in particular the following two new expressions for the Catalan numbers:
\[
\sum_{p=0}^{[n/2]} \frac{1}{2p+1} \binom{3p}{p} \binom{n+p}{3p} = \frac{1}{n+1} \binom{2n}{n}, \quad (n \geq 0)
\]
\[
\sum_{p=0}^{[(n-1)/2]} \frac{1}{2p+1} \binom{3p+1}{p+1} \binom{n+p}{3p+1} = \frac{1}{n+1} \binom{2n}{n}, \quad (n \geq 1).
\]

2. "\( u \)-segments" statistics in \( k \)-paths

We start this section by studying the generating function for the number of \( k \)-paths of length \( n \) according to the statistics \( \alpha_1, \alpha_2, \ldots \), that is,
\[
P(x,z; t) = P(x,z;t_1,t_2,\ldots) = \sum_{m,j \geq 0} x^m z^j \sum_{P \in \mathcal{P}_{n,k}^{m,j}} \prod_{i \geq 1} t_i^{\alpha_i(P)}.
\]
Proposition 2.1. The ordinary generating function $P(x, z; t)$ is given by

\begin{equation}
(2.1) \quad P(x, z; t) = 1 + zP(x, z; t) + \sum_{j \geq 1} t_j x^j P^j(x, z; t) + z \sum_{j \geq 1} t_j x^j P^{j+1}(x, z; t).
\end{equation}

Proof. Note that $P(x, z; t)$ can be written as $P(x, z; t) = 1 + zP(x, z; t) + \sum_{j \geq 1} P_j(x, z; t)$, where $P_j(x, z; t)$ is the generating function for the number of $k$-paths with initial $u$-segment of length $j$ according to the statistics $\alpha_1, \alpha_2, \ldots$. An equation for $P_j(x, z; t)$ is obtained from the first return decomposition of a $k$-path starting with a $u$-segment of length $j$: either $P = u^j P(1) dP(2) d \ldots dP(j)$ or $P = u^j h P(1) dP(2) d \ldots dP(j+1)$, where $P(1), \ldots, P(j)$ are $k$-paths, see Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{First return decomposition of a $k$-path starting with exactly $j$ up-steps.}
\end{figure}

Thus $P_j(x, z; t) = t_j x^j P^j(x, z; t) + z t_j x^j P^{j+1}(x, z; t)$. Hence, $P(x, z; t)$ satisfies $P(x, z; t) = 1 + zP(x, z; t) + \sum_{j \geq 1} t_j x^j P^j(x, z; t) + z \sum_{j \geq 1} t_j x^j P^{j+1}(x, z; t)$, as required.

Define $y = y(x, z; t) = xP(x, z; t)$ and $T(x) = 1 + \sum_{j \geq 1} t_j x^j$. Then (2.1) reduces to $y = (x + zy)T(y)$. Let $y^* = y(x, z; t)$, then we have $y^* = x(1 + zy^*)T(y^*)$.

Theorem 2.2. For any integers $n, m \geq 1$ and $k, j \geq 0$,

\begin{equation}
\sum_{P \in \mathcal{P}_{n,k}} \prod_{i \geq 1} t_{i}^{\alpha_i(P)} = \frac{1}{m+1} \left( \begin{array}{c} m+j \\ m \end{array} \right) \sum_{r=0}^{m} \left( \begin{array}{c} m+j+1 \\ r \end{array} \right) \frac{r!}{m!} B_{m,r}(1!t_1, 2!t_2, \ldots).
\end{equation}

Proof. Using (1.2) and (1.3), we obtain

\begin{align*}
\sum_{P \in \mathcal{P}_{n,k}} \prod_{i \geq 1} t_{i}^{\alpha_i(P)} &= [x^{m+j+1} z^j] y^* = \frac{[x^{m+j+1} z^j]}{m+j+1} \{ (1 + zx)T(x) \}^{m+j+1} \\
&= \frac{1}{m+j+1} \left( \begin{array}{c} m+j+1 \\ j \end{array} \right) [x^{m}] T(x)^{m+j+1} \\
&= \frac{1}{m+j+1} \left( \begin{array}{c} m+j+1 \\ j \end{array} \right) P_{m}^{(m+j+1)} \frac{(1!t_1, 2!t_2, \ldots)}{m!} \\
&= \frac{1}{m+1} \left( \begin{array}{c} m+j \\ m \end{array} \right) \sum_{r=0}^{m} \left( \begin{array}{c} m+j+1 \\ r \end{array} \right) \frac{r!}{m!} B_{m,r}(1!t_1, 2!t_2, \ldots),
\end{align*}

as claimed.

Replace $t_i$ by $qt_i$ in Theorem 2.2 and note that

\begin{equation}
(2.2) \quad B_{m,r}(t_1, t_2, \cdots) = \sum_{\kappa_m(r)} \frac{m!}{r_1! r_2! \cdots r_m!} \left( \frac{t_1}{1!} \right)^{r_1} \left( \frac{t_2}{2!} \right)^{r_2} \cdots \left( \frac{t_m}{m!} \right)^{r_m},
\end{equation}

\begin{equation}
(2.3) \quad B_{m,r}(1!qt_1, 2!qt_2, \cdots) = q^r B_{m,r}(1!t_1, 2!t_2, \cdots),
\end{equation}

where the summation $\kappa_m(r)$ is for all the nonnegative integer solutions of $r_1 + r_2 + \cdots + r_m = r$ and $r_1 + 2r_2 + \cdots + mr_m = m$, we have
Corollary 2.3. For any integers $n, m, r \geq 1$ and $k, j \geq 0$, there holds
\[
\sum_{P \in \mathcal{P}_{n,k,r}} \prod_{i \geq 1} t_i^{a_i(P)} = \frac{1}{m+1} \binom{m+j}{m} \binom{m+j+1}{j} r! m! B_{m,r}(1!t_1, 2!t_2, \ldots).
\]

By comparing the coefficient of $t_1^1 t_2^2 \cdots t_m^m$ in Corollary 2.3, one can obtain that

Corollary 2.4. The number of $k$-paths in $\mathcal{P}_{n,k,r}$ with $u$-segments of type $1^r 2^r \cdots m^r$ is
\[
\frac{1}{m+1} \binom{m+j}{m} \binom{m+j+1}{j} (r_1, r_2, \ldots, r_m).
\]
Specially, the number of Dyck paths of length $2m$ with $u$-segments of type $1^r 2^r \cdots m^r$ is
\[
\frac{1}{m+1} \binom{m+1}{r} (r_1, r_2, \ldots, r_m), \text{ (the case } k = 0 \text{ implies } j = 0).
\]

2.1. Applications. In what follows we consider many special cases of $T(x)$. These produce several interesting results, as described in Examples 2.5-2.14. We also obtain several identities involving Stirling numbers of the first (second) kind, idempotent numbers and other combinatorial sequences.

Example 2.5. Let $T(x) = 1 + q(e^x - 1) = (1 - q) + qe^x$, that is, $t_i = q/i!$ for all $i \geq 1$. Note that $B_{n,1}(q, q, q, \cdots) = S(n, i) q^i [5$, pp.135], where $S(n, i)$ is the Stirling number of the second kind. Then Theorem 2.2 gives
\[
\sum_{P \in \mathcal{P}_{n,k,r}} \prod_{i \geq 1} \frac{m!}{(i!)^a_i(P)} = \frac{1}{m+j+1} \binom{m+j+1}{j} \sum_{i=0}^m \binom{m+j+1}{i} i! S(m, i) q^i = \frac{1}{m+j+1} \binom{m+j+1}{j} \sum_{i=0}^m \binom{m+j+1}{i} i^m q^i (1-q)^{m+j-i+1},
\]
which leads to
\[
\sum_{P \in \mathcal{P}_{n,k,r}} \prod_{i \geq 1} \frac{m!}{(i!)^a_i(P)} = \frac{1}{m+j+1} \binom{m+j+1}{j} \binom{m+j+1}{r} r! S(m, r).
\]

Example 2.6. Let $T(x) = 1 + qxe^x$ which is equivalent to $t_i = q/(i-1)!$ for all $i \geq 1$. Note that $B_{m,1}(q, 2q, 3q, \cdots) = \binom{m+1}{1} i^{m-i} q^i$, which are called the idempotent numbers [5, pp.135] when $q = 1$. By Corollary 2.3, we have
\[
\sum_{P \in \mathcal{P}_{n,k,r}} \prod_{i \geq 1} \frac{m!}{(i!)^a_i(P)} = \frac{1}{m+j+1} \binom{m+j+1}{j} \binom{m+j+1}{r} r! S(m, r).
\]

Example 2.7. If $T(x) = (e^x - 1)/x$, then $t_i = 1/(i+1)!$ for all $i \geq 1$. It is well known that the Stirling numbers of the second kind satisfy $(\frac{x-1}{x})^k/k! = \sum_{m \geq 0} S(m+k,k)x^m/(m+k)!$. Thus, Theorem 2.2 leads to
\[
\sum_{P \in \mathcal{P}_{n,k,r}} \prod_{i \geq 1} \frac{1}{(i+1)!} a_i(P) = \frac{1}{(2m+j+1)!} \binom{m+j+1}{j} S(2m+j+1, m+j+1).
\]

Example 2.8. If $T(x) = \frac{1}{x} \ln \frac{1}{1-x}$, then $t_i = 1/(i+1)$ for all $i \geq 1$. It is well known that the Stirling numbers of the first kind $s(n, i)$ satisfy $(\frac{1}{x} \ln \frac{1}{1-x})^k/k! = \sum_{m \geq 0} |s(m+k,k)|x^m/(m+k)!$. Thus, Theorem 2.2 leads to
\[
\sum_{P \in \mathcal{P}_{n,k,r}} \prod_{i \geq 1} \frac{1}{(i+1)!} a_i(P) = \frac{1}{(2m+j+1)!} \binom{m+j+1}{j} |s(2m+j+1, m+j+1)|.
Example 2.9. If \( T(x) = 1 + q \ln \frac{1}{1-x} \), then \( t_i = q/i \) for all \( i \geq 1 \). Using the fact that \( B_{n,i}(0!q,1!q,2!q,\cdots) = |s(n,i)|q^i \) [5 pp.135] together with Corollary 2.3, we have
\[
\sum_{P \in \Psi_{n,k}^{m,j}} \prod_{i \geq 1} \left\{ \frac{1}{i} \right\}^{\alpha_i(P)} = \frac{1}{m+1} \binom{m+j}{m} \binom{m+j+1}{r} \frac{r!}{m!} |s(m,r)|.
\]

Example 2.10. If \( T(x) = 1/(1-x)^{\lambda} \), then \( t_i = \binom{\lambda+i-1}{i} \) for all \( i \geq 1 \), where \( \lambda \) is an indeterminant. So, Theorem 2.2 leads to
\[
\sum_{P \in \Psi_{n,k}^{m,j}} \prod_{i \geq 1} \left( \lambda + i - 1 \right)^{\alpha_i(P)} = \frac{1}{m+1} \binom{m+j}{m} \binom{\lambda(m+j+1)+m-1}{m},
\]
which generates that when \( \lambda = 1 \) the set \( \Psi_{n,k}^{m,j} \) is counted by \( \frac{1}{m+1} \binom{m+j}{m} \binom{2m+j}{m} \), in particular, \( \Psi_{n,k}^{m,m} \) is counted by \( \frac{1}{m+1} \binom{2m}{m} \binom{3m}{m} \).

Example 2.11. Let \( T(x) = 1 + x + x^2 + \cdots + x^r \), that is, \( t_i = 1 \) for \( 1 \leq i \leq r \) and \( t_i = 0 \) for all \( i \geq r + 1 \). Then Theorem 2.2 gives
\[
\sum_{P \in \Psi_{n,k}^{m,j}} \prod_{i \geq r+1} \left( \lambda + i - 1 \right)^{\alpha_i(P)} = \frac{1}{m+1} \binom{m+j}{m} \sum_{i=0}^{m+1} (-1)^i \binom{m+j+1}{i} \binom{2m+j-(r+1)i}{m+j},
\]
which implies that the number of \( k \)-paths \( P \) of length \( 2n \) with no \( u \)-segments of length greater than \( r \) is given by
\[
\frac{1}{m+1} \binom{m+j}{m} \sum_{i=0}^{m+1} (-1)^i \binom{m+j+1}{i} \binom{2m+j-(r+1)i}{m+j}.
\]

Example 2.12. Let \( T(x) = \frac{1}{1-x} + (q-1)x^r \), that is, \( t_i = 1 \) for all \( i \geq 1 \) except for \( i = r \) and \( t_r = q \). Then Theorem 2.2 gives
\[
\sum_{P \in \Psi_{n,k}^{m,j}} q^{\alpha_r(P)} = \frac{1}{m+1} \binom{m+j}{m} \sum_{i=0}^{m} \binom{m+j+1}{i} \binom{2m+j-(r+1)i}{m+j-i} (q-1)^i,
\]
which implies that the number of \( k \)-paths \( P \) of length \( n \) with exactly \( p \) \( u \)-segments of length \( r \) (namely \( \alpha_r(P) = p \)) is given by \( \frac{1}{m+1} \binom{m+j}{m} \sum_{i=0}^{m} (-1)^{i-p} \binom{m+j+1}{m+j-i} (q-1)^i \binom{2m+j-(r+1)i}{m+j-i} (i)^p \).

Example 2.13. If \( T(x) = 1 + \frac{qxr^r}{1-x^r} = \frac{1 + (q-1)x^r}{1-x^r} \), then \( t_i = q \) if \( i \equiv 0 \mod r \) and \( 0 \) otherwise. Thus, Theorem 2.2 leads to
\[
\sum_{P \in \Psi_{2rm+k}^{m,j}} \prod_{i \equiv 0 \mod r} \left\{ \frac{1}{i} \right\}^{\alpha_i(P)} = \frac{1}{rm+1} \binom{rm+j}{rm} \sum_{i=1}^{m} \binom{rm+j+1}{m-i} q^i = \frac{1}{rm+1} \binom{rm+j}{rm} \sum_{i=0}^{m} \binom{rm+j+1}{m-i} (r^{m+j-i}) (q-1)^i.
\]
which produces the following results. The number of \( k \)-paths in \( \Psi_{2rm+k}^{m,j} \) such that the length of any \( u \)-segment is a multiple of \( r \) (i.e., the case \( q=1 \)) is given by \( \frac{1}{rm+1} \binom{rm+j}{rm} \binom{r^{m+j}}{m} \) (by Vandermonde convolution). More precisely, the number of \( k \)-paths in \( \Psi_{2rm+k}^{m,j} \) with exactly \( i \) \( u \)-segments such that the length of any \( u \)-segment is a multiple of \( k \) is given by
\[
\frac{1}{rm+1} \binom{rm+j}{rm} \binom{r^{m+j+1}}{m-i} \frac{r^{m+j+1-i}}{p} \binom{r^{m+j+1}}{m-p} (p)^i.
\]
Example 2.14. Let $T(x)$ be the generating function $f'(x)$, where $f(x)$ is the generating function for complete $p$-ary plane trees (see, for instance, [8] and [7] pp.112-113), which satisfies the relation $f(x) = 1 + xf^p(x)$. By the Lagrange inversion formula (1.2), we can deduce $t_i = \frac{r}{p^i+1} {p+i \choose i}$. Then Theorem 2.2 leads to

$$
\sum_{P \in P_{n,k,r}} \prod_{i \geq 1} \left( \frac{r}{p^i+1} \right)^{\alpha_i(P)} = \frac{1}{(m+1)!} \frac{(m+j+1)!}{(m+j+1)j + m!} \left( \frac{m+j}{m} \right)^{r+j+1}\frac{m}{m+j+1!}
$$

2.2. A combinatorial proof of Corollary 2.3 Let $\hat{P}_{n,k,r}^{m,j}$ be the set of lattice paths $P^* = PD$ such that there is one colored down-step in $P^*$, where $P \in \hat{P}_{n,k,r}^{m,j}$. To give a bijective proof of Corollary 2.3 we need the following lemma.

Lemma 2.15. There exists a bijection $\phi$ between the sets $\hat{P}_{n,k,r}^{m,j}$ and $L_{n,k,r}^{m,j}$ such that $P^* \in \hat{P}_{n,k,r}^{m,j}$ has the same type of $u$-segments as $\phi(P^*) \in L_{n,k,r}^{m,j}$.

Proof. Any $P^* \in \hat{P}_{n,k,r}^{m,j}$ can be uniquely partitioned into $P^* = P_1dQ_1$, where $P_1, Q_1$ are lattice paths and $d$ is the colored down-step. Define $\phi(P^*) = Q_1P_1$, then it is easy to verify that $\phi(P^*) \in L_{n,k,r}^{m,j}$ and note that the length of any $u$-segment in $\phi(P^*)$ is the same as in $P^*$.

Conversely, for any $L \in L_{n,k,r}^{m,j}$, we can find the leftmost point which has the lowest ordinate, then $L$ can be uniquely partitioned into two parts in this sense, namely, $L = L_1L_2$. Define $\phi^{-1}(L) = L_2dL_1$, where the $d$ is the colored down-step, then it is easily to verify that $\phi^{-1}(L) \in \hat{P}_{n,k,r}^{m,j}$ which has the same type of $u$-segments with $L$.

Hence $\phi$ is indeed a bijection between the sets $\hat{P}_{n,k,r}^{m,j}$ and $L_{n,k,r}^{m,j}$, which preserves the type of $u$-segments not changed. \hfill $\square$

An ordered partition $B_1, B_2, \ldots, B_r$ of $[m] = \{1, 2, \ldots, m\}$ into $r$ blocks is a partition of $[m]$ such that the $r$ blocks as well as the elements of each block are ordered.

Now we can give a bijective proof of Corollary 2.3.

Proof. For any ordered partition $B_1, B_2, \ldots, B_r$ of $[m]$ into $r$ blocks, regard each block $B_i$ as a labeled $u$-segment $U_i$ for $1 \leq i \leq r$. For $m$ down-steps and $j$ horizontal-steps, there are $\binom{m+j}{j}$ ways to obtain $(d,h)$-words of length $m + j$ on $\{d,h\}$ with $m$ $d$’s and $j$ $h$’s. Then we can insert the labeled $u$-segments $U_1, U_2, \ldots, U_r$ orderly into the $m + j + 1$ positions (repetition is not allowed) of any $(d,h)$-word of length $m + j$, which can produce $\binom{m+j}{m} r! \binom{m^j}{r}$ labeled lattice paths in $L_{n,k,r}^{m,j}$. Note that $r!B_{m,r}(1!t_1, 2!t_2, \ldots)$ is just the generating function for ordered partitions $B_1, B_2, \ldots, B_r$ of $[m]$ into $r$ blocks such that each block $B_p$ is weighted by $t_i$ with $i = |B_p|$ for $1 \leq p \leq r$. So $\binom{m+j}{m} r! \binom{m^j}{r}$ $r!B_{m,r}(1!t_1, 2!t_2, \ldots)$ is the generating function for the labeled lattice paths in $L_{n,k,r}^{m,j}$ such that each $u$-segment of length $i$ is weighted by $t_i$.

However, by Lemma 2.15 any $k$-path $P \in \hat{P}_{n,k,r}^{m,j}$ can lead to $m$ labeled $k$-paths, and $P^* = PD \in \hat{P}_{n,k,r}^{m,j}$ can generate $m + 1$ lattice paths in $L_{n,k,r}^{m,j}$ and vice versa. Hence $\frac{1}{m+j+1} \binom{m+j}{m} r! \binom{m^j}{r}$ $m!B_{m,r}(1!t_1, 2!t_2, \ldots)$ is the generating function of $k$-paths in $\hat{P}_{n,k,r}^{m,j}$ such that each $u$-segment of length $i$ is weighted by $t_i$, which makes the proof complete. \hfill $\square$

3. "Internal $u$-segments" statistics in $k$-paths

An internal $u$-segment of a $k$-path $P$ is a $u$-segment between two steps such as $dd$, $hh$, $hd$, $dh$, i.e., all $u$-segments except for the first one are internal $u$-segments. Define $\beta_i(P)$ to be the number internal $u$-segments of length $r$ in a $k$-path $P$. We start this section by studying
Using (1.2) and (1.3), we obtain as claimed.

The ordinary generating function for the number of \( n \)-paths of length \( n \) according to the statistics \( \beta_1, \beta_2, \ldots \), that is,

\[
F(x, z; t) = F(x, z; t_1, t_2, \ldots) = \sum_{m,j\geq 0} x^m z^j \prod_{P \in \mathcal{P}_{n,k}} \prod_{i \geq 1} t_i^{\beta(P)},
\]

which can be represented as follows in terms of the generating function \( P(x, z; t) \).

**Proposition 3.1.** The ordinary generating function \( F(x, z; t) \) is given by

\[
\frac{1 + zP(x, z; t)}{1 - xP(x, z; t)} = 1 + zP(x, z; t) + \sum_{j \geq 1} x^j P^j(x, z; t) + z \sum_{j \geq 1} x^j P^{j+1}(x, z; t).
\]

**Proof.** An equation for \( F(x, z; t) \) is obtained from the decomposition of a \( k \)-path: either

\[
P = hP', \quad P = u^j dP^{(j)} \cdot dP^{(j-1)} \cdots dP^{(2)} dP^{(1)}, \quad \text{or} \quad P = u^j hP^{(j+1)} \cdot dP^{(j)} \cdots dP^{(2)} dP^{(1)}
\]

for some \( j \geq 1 \), where \( P', P^{(1)}, \ldots, P^{(j+1)} \) are \( k \)-paths. Then \( F(x, z; t) \) satisfies the equation \( F(x, z; t) = 1 + zP(x, z; t) + \sum_{j \geq 1} x^j P^j(x, z; t) + z \sum_{j \geq 1} x^j P^{j+1}(x, z; t) \), as required. \( \square \)

**Theorem 3.2.** For any integers \( k, j \geq 0, n, m \geq 1 \),

\[
\sum_{P \in \mathcal{P}_{n,k}} \prod_{i \geq 1} t_i^{\beta(P)} = \sum_{m,j=0}^{\infty} \left[ \frac{m-p}{m+j} \binom{m+j}{j} + \frac{(m+j-1)(m-j)}{m+j} \right] P_p^{(m+j)}(1! t_1, 2! t_2, \cdots) = \sum_{m,j=0}^{\infty} \left[ \frac{m-p}{m+j} \binom{m+j}{j} + \frac{(m+j-1)(m-j)}{m+j} \right] P_p^{(m+j)}(1! t_1, 2! t_2, \cdots).
\]

**Proof.** Using (2.2) and (3.3), we obtain

\[
\sum_{P \in \mathcal{P}_{n,k}} \prod_{i \geq 1} t_i^{\beta(P)} = \left[ x^m z^j \right] \frac{1 + zP(x, z; t)}{1 - xP(x, z; t)} = \left[ x^m z^j \right] \frac{1 + zy^s}{1 - y^s}.
\]

\[
= \frac{1}{m+j} \left[ x^m z^{j-1} \right] \left( \frac{1 + z(xr)}{1 - x} \right) = \frac{1}{m+j} \left[ x^m \right] \frac{T(x)^{m+j}}{(1-x)^2} = \frac{1}{m+j} \sum_{p=0}^{m} \frac{m-p}{m+j} P_p^{(m+j)}(1! t_1, 2! t_2, \cdots) + \frac{1}{m+j} \sum_{j=1}^{m} \frac{m-j}{m+j} P_p^{(m+j)}(1! t_1, 2! t_2, \cdots)
\]

\[
= \sum_{p=0}^{m} \left[ \frac{m-p}{m+j} \binom{m+j}{j} + \frac{m-j}{m+j} \binom{m+j-1}{j-1} \right] P_p^{(m+j)}(1! t_1, 2! t_2, \cdots) = \sum_{p=0}^{m} \left[ \frac{m-p}{m+j} \binom{m+j}{j} + \frac{m-j}{m+j} \binom{m+j-1}{j-1} \right] P_p^{(m+j)}(1! t_1, 2! t_2, \cdots).
\]

as claimed. \( \square \)
4. "u-segments" and "internal u-segments" statistics in k-paths without a horizontal-step on the x-axis

4.1. u-segments statistics. We start this subsection by studying the generating function for the number of k-paths in \( Q_{n,k}^{m,j} \) according to the statistics \( a_1, a_2, \ldots \), that is,

\[
Q(x, z; t) = Q(x, z; t_1, t_2, \ldots) = \sum_{m,j \geq 0} x^m z^j \sum_{Q \in Q_{n,k}^{m,j}} \prod_{i=1}^{\ell_i(Q)} t_i^{a_i(Q)}.
\]

Proposition 4.1. The ordinary generating function \( Q(x, z; t) \) is given by

\[
Q(x, z; t) = \frac{P(x, z; t)}{1 + z P(x, z; t)} = T(y).
\]

Proof. Note that \( Q(x, z; t) \) can be written as \( Q(x, z; t) = 1 + \sum_{j \geq 1} Q_p(x, z; t) \), where \( Q_p(x, z; t) \) is the generating function for the number of k-paths starting with \( p \) up-steps and without a horizontal-step on the x-axis according to the statistics \( a_1, a_2, \ldots \). An equation for \( Q_p(x, z; t) \) is obtained from the first return decomposition of a k-path starting with a u-segment of length \( p \). Either \( P = u^p d P^{(p-1)}d P^{(p-2)} \ldots P^{(1)}d P^* \) or \( P = u^p h P^{(p)} d P^{(p-1)} \ldots P^{(1)}d P^* \), where \( P^{(1)}, \ldots, P^{(p)} \) are k-paths and \( P^* \) is a k-path without a horizontal-step on the x-axis. Thus \( Q_p(x, z; t) = t_p x^p P^{p-1}(x, z; t) Q(x, z; t) + z t_p x^p P^p(x, z; t) Q(x, z; t) \) and \( Q(x, z; t) \) satisfies the equation \( Q(x, z; t) = 1 + Q(x, z; t) \sum_{p \geq 1} t_p x^p P^{p-1}(x, z; t) + z \sum_{p \geq 1} t_p x^p P^p(x, z; t) \). Hence, by Proposition 4.1, we obtain the desired result. \( \square \)

Theorem 4.2. For any integers \( k, j \geq 0, n, m \geq 1 \),

\[
\sum_{Q \in Q_{n,k}^{m,j} \ell_i(Q)} t_i^{a_i(Q)} = \frac{m}{(m+j+1)(m+j)} \binom{m}{j} \frac{P^{(m+j+1)}(1! t_1, 2! t_2, \ldots)}{m!}
\]

\[
= \frac{m}{(m+j+1)(m+j)} \binom{m+j}{j} \sum_{r=0}^{m+j} \binom{m+j+1}{r} \frac{r!}{m!} B_{m,r}(1! t_1, 2! t_2, \ldots).
\]

Proof. Using (1.2) and (1.3), we obtain

\[
\sum_{Q \in Q_{n,k}^{m,j} \ell_i(Q)} t_i^{a_i(Q)} = [x^m z^j] Q(x, z; t) = [x^m z^j] T(y^*)
\]

\[
= \frac{1}{m+j} \binom{m+j-1}{j} \{T(x)\}' \{1 + zx\} T(x)^{m+j}
\]

\[
= \frac{1}{m+j} \binom{m+j}{j} \{T(x)^{m-1}\}' \{T(x)\}^{m+j}
\]

\[
= \frac{1}{m+j} \binom{m+j}{j} \frac{1}{m+j+1} \{T(x)^{m+j+1}\}'
\]

\[
= \frac{m}{(m+j+1)(m+j)} \binom{m+j}{j} \{T(x)^{m+j+1}\}
\]

\[
= \frac{m}{(m+j+1)(m+j)} \binom{m+j}{j} \frac{P^{(m+j+1)}(1! t_1, 2! t_2, \ldots)}{m!}
\]

\[
= \frac{m}{(m+j+1)(m+j)} \binom{m+j}{j} \sum_{r=0}^{m+j} \binom{m+j+1}{r} \frac{r!}{m!} B_{m,r}(1! t_1, 2! t_2, \ldots),
\]

which completes the proof. \( \square \)
Replacing $t_i$ by $qt_i$ in Theorem 4.2 and using (2.3) and (2.4), we have

**Corollary 4.3.** For any integers $n, m, r \geq 1$ and $k, j \geq 0$, there holds

$$
\sum_{Q \in \mathcal{Q}_{n,k}^{m,j}} \prod_{i \geq 1} t_i^{\alpha_i(Q)} = \frac{m}{(m+j+1)(m+j)} \binom{m+j+1}{j} \binom{m+j}{r} \frac{r!}{m!} B_{m,r}(1t_1, 2t_2, \cdots),
$$

which implies that the number of $k$-paths with no horizontal-step on the $x$-axis and with $u$-segments of type $1^r 2^{r_2} \cdots m^{r_m}$ is

$$
\left( \frac{m}{(m+j+1)(m+j)} \binom{m+j+1}{j} \binom{m+j}{r} \frac{r!}{m!} \right) H_{m,r}(1t_1, 2t_2, \cdots).
$$

**Remark 4.4.** Note that from Theorem 2.3 and 4.2 the ratio of $\sum_{P \in \mathcal{P}_{n,k}^{m,j}} \prod_{i \geq 1} t_i^{\alpha_i(P)}$ to $\sum_{Q \in \mathcal{Q}_{n,k}^{m,j}} \prod_{i \geq 1} t_i^{\alpha_i(Q)}$ is $\frac{(m+1)m}{(m+j+1)(m+j)}$. For the sake of conciseness, we omit many examples as done in Section 2. However, we may ask whether there is a combinatorial interpretation for this relation.

### 4.2. Internal $u$-segments statistics

In this subsection, we study the generating function for the number of $k$-paths in $\mathcal{Q}_{n,k}^{m,j}$ according to the statistics $\beta_1, \beta_2, \ldots$, that is,

$$H(x, z; t) = H(x, z; t_1, t_2, \ldots) = \sum_{m,j \geq 0} x^m z^j \sum_{Q \in \mathcal{Q}_{n,k}^{m,j}} \prod_{i \geq 1} t_i^{\beta_i(Q)}.
$$

**Proposition 4.5.** The ordinary generating function $H(x, z; t)$ is given by

$$(4.2)\quad H(x, z; t) = \frac{1 - xP(x, z; t)}{1 - xP(x, z; t) - zP(x, z; t)} = \frac{1 - y}{1 - y - x - zy}.$$

**Proof.** Note that $H(x, z; t)$ can be written as $H(x, z; t) = 1 + \sum_{p \geq 1} H_p(x, z; t)$, where $H_p(x, z; t)$ is the generating function for the number of $k$-paths starting with $p$ up-steps and without a horizontal-step on the $x$-axis according to the statistics $\beta_1, \beta_2, \ldots$. An equation for $H_p(x, z; t)$ is obtained from the first return decomposition of a $k$-path starting with a $u$-segment of length $p$: either $P = P^{dP(p-1)} dP(p-2) \cdots dP(1) dP^* $ or $P = w^p h P^{dP(p-1)} dP(1) dP^* $, where $P^{(1)}, \ldots, P^{(p)}$ are k-paths and $P^*$ is a k-path with no horizontal-step on the $x$-axis.

Thus $H_p(x, z; t) = x^p P^{p-1}(x, z; t) H(x, z; t) + z x^p P^p(x, z; t) H(x, z; t)$. Hence, $H(x, z; t)$ satisfies the equation $H(x, z; t) = 1 + \left( \sum_{p \geq 1} x^p P^{p-1}(x, z; t) + z \sum_{p \geq 1} x^p P^p(x, z; t) \right) H(x, z; t)$, a simplification reduces this to the required expression. \hfill $\square$

**Theorem 4.6.** For any integers $n, m, k, j \geq 0$, $m + j \geq 1$,

$$
\sum_{Q \in \mathcal{Q}_{n,k}^{m,j}} \prod_{i \geq 1} t_i^{\beta_i(Q)} = \sum_{i=0}^{m} \frac{i}{m+j-i} \binom{m+j-1}{j} \sum_{r=0}^{m-i-1} \binom{r+i}{r} \frac{P^{(m+j-i)}_{m-i-r-1}(1t_1, 2t_2, \cdots)}{(m-i-r-1)!} \cdot
$$

$$
+ \sum_{i=0}^{m} \frac{i}{m+j-i} \binom{m+j-1}{j-1} \sum_{r=0}^{m-i} \binom{r+i}{r} \frac{P^{(m+j-i)}_{m-i-r}(1t_1, 2t_2, \cdots)}{(m-i-r)!}.
$$
Proof. Using \([1,2]\) and \((1.3)\), we obtain

\[
\sum_{Q \in \Omega_{n,k}^m} \prod_{i \geq 1} t_i^{j(Q)} = [x^{m+j}z^j]Q(x,z) = [x^{m+j}z^j] \left( \frac{1 - y^r}{1 - y^s - x(1 + y^r)} \right)
\]

\[
= [x^{m+j}z^j] \sum_{i \geq 0} \left\{ x \left( \frac{1 + y^r}{1 - y^s} \right) \right\}^i = \sum_{i \geq 0} [x^{m+j-i}z^j] \left\{ \frac{1 + y^r}{1 - y^s} \right\}^i
\]

\[
= \sum_{i \geq 0} \frac{i}{m + j - 1} [x^{m+j-i-1}z^j] \left\{ \frac{(1 + z)(1 + zx)^{m+j-1}}{(1 - x)^{i+1}} \right\} \{T(x)\}^{m+j-i}
\]

\[
= \sum_{i \geq 0} \frac{i}{m + j - 1} \left( \frac{m + j - 1}{j} \right) \sum_{r = 0}^{m-1} \frac{(r + i) P_{m-i-r-1}(1!t_1, 2!t_2, \ldots)}{(m - i - r - 1)!}
\]

\[
+ \sum_{i \geq 0} \frac{i}{m + j - 1} \left( \frac{m + j - 1}{j - 1} \right) \sum_{r = 0}^{m-1} \frac{(r + i) P_{m-i-r-1}(1!t_1, 2!t_2, \ldots)}{(m - i - r)!},
\]

which completes the proof. \(\square\)

5. Statistics \((u, h)\)-segments and \(u\)-segments in \(k\)-paths

A \((u, h)\)-segment in a \(k\)-path is a maximum segment composed of up-steps and horizontal-steps. An \emph{internal \((u, h)\)-segment} in a \(k\)-path is a \((u, h)\)-segment between two down steps. Let \(\mathcal{P}_{n,k}^{m,j}\) denote the subset of \(\mathcal{P}_{n,k}^{m,j}\) such that (i) each internal \((u, h)\)-segment has length equal to a multiple of \(k\); (ii) the first \((u, h)\)-segment has length \(\equiv \ell \pmod{k}\) for \(0 \leq \ell \leq k - 1\). We note that the case \(j = 0\) is studied in \([9]\).

**Theorem 5.1.** The number \(\bar{P}_{r,k,\ell}\) of \(k\)-paths of length \(n = kr + 2\ell\) satisfying the conditions (i) and (ii) is

\[
\bar{P}_{r,k,\ell} = \sum_{p=0}^{\lfloor r/2 \rfloor} \frac{\ell + 1}{kp + \ell + 1} \binom{(k+1)p + \ell}{p} \binom{r + 2(k-1)p + 2\ell}{r - 2p}.
\]

**Proof.** Note that for any path \(P \in \mathcal{P}_{n,k}^{m,j,\ell}\), by deleting all the \(j\) horizontal-steps, we obtain a Dyck path in \(\mathcal{P}_{2m,k}^{m,0}\). Conversely, any Dyck path in \(\mathcal{P}_{2m,k}^{m,0}\) goes through \(2m + 1\) integer points, if we insert the \(j\) horizontal-steps into any integer point (repetitions are allowed), then we get \(\binom{2m+j}{j}\) \(k\)-paths in \(\mathcal{P}_{n,k}^{m,j,\ell}\). However, the set \(\mathcal{P}_{2m,k}^{m,0}\) is counted by \(\frac{\ell + 1}{m+1}(m+1)^p\), which has
been proved in [9]. Hence we have
\[ \tilde{\mathcal{P}}_{r,k,\ell} = \sum_{2m+kj=n,m=kp+\ell} |\mathfrak{P}_{n,k}^{m,j,\ell}| \]
\[ = \sum_{2m+kj=n,m=kp+\ell} \frac{\ell+1}{m+1} \binom{m+p}{j} \binom{2m+j}{p} \]
\[ = \sum_{p=0}^{[r/2]} \frac{\ell+1}{kp+\ell+1} \binom{(k+1)p+\ell}{p} \binom{r+(k-1)p+\ell}{r-2p}, \]

as required. \( \square \)

Let \( \mathfrak{P}_{n,k}^{m,j,\ell} \) denote the subset of \( \mathfrak{P}_{n,k}^{m,j} \) such that (iii) each internal u-segment has length equal to a multiple of \( k \); (iv) the first u-segment has length \( \equiv \ell \mod k \) for \( 0 \leq \ell \leq k-1 \).

**Theorem 5.2.** The number \( \tilde{\mathcal{P}}_{r,k,\ell} \) of \( k \)-paths of length \( n = kr + 2\ell \) satisfying conditions (iii) and (iv) is
\[ \tilde{\mathcal{P}}_{r,k,\ell} = \sum_{p=0}^{[r/2]} \frac{\ell+1}{kp+\ell+1} \binom{(k+1)p+\ell}{p} \binom{r+(k-1)p+\ell}{r-2p}. \]

**Proof.** Note that for any path \( P \in \mathfrak{P}_{2m}^{m,j,\ell} \), by deleting all the \( j \) horizontal-steps, we obtain a Dyck path in \( \mathfrak{P}_{2m}^{m,0,\ell} \). Conversely, let \( D \) be a Dyck path from \( \mathfrak{P}_{2m}^{m,0,\ell} \), where \( m = kp + \ell \) for some \( p \geq 0 \). It can be shown that \( D \) has \( m + p + 1 \) proper integer points, where a proper integer point is a point where we may insert a horizontal step without violating the properties (iii) and (iv). By inserting \( j \) horizontal steps into these points (repetitions are allowed) we get \( \binom{m+p+j}{j} \) \( k \)-paths in \( \mathfrak{P}_{2m}^{m,j,\ell} \). Note that the set \( \mathfrak{P}_{2m}^{m,0,\ell} = \mathfrak{P}_{2m}^{0,0,\ell} \) is counted by \( \frac{\ell+1}{m+1} \binom{m+p}{p} \). Hence we have
\[ \tilde{\mathcal{P}}_{r,k,\ell} = \sum_{2m+kj=n,m=kp+\ell} |\mathfrak{P}_{n,k}^{m,j,\ell}| \]
\[ = \sum_{2m+kj=n,m=kp+\ell} \frac{\ell+1}{m+1} \binom{m+p}{j} \binom{m+p+j}{p} \]
\[ = \sum_{p=0}^{[r/2]} \frac{\ell+1}{kp+\ell+1} \binom{(k+1)p+\ell}{p} \binom{r+(k-1)p+\ell}{r-2p}, \]

as required. \( \square \)

It should be noted that \( \sum_{p\geq0} \frac{\ell+1}{kp+\ell+1} \binom{(k+1)p+\ell}{p} x^p = f(x)^{\ell+1}, \) where \( f(x) \) is the generating function for \( (k+1)\)-ary plane trees, and which satisfies the relation \( f(x) = 1 + xf(x)^{k+1} \).

Then it is easy to prove that the generating functions for \( \tilde{\mathcal{P}}_{r,k,\ell} \) and for \( \tilde{\mathcal{P}}_{r,k,\ell} \) are respectively

(5.1) \[ \tilde{P}_{k,\ell}(x) = \sum_{r\geq0} \tilde{\mathcal{P}}_{r,k,\ell} x^r = \frac{1}{(1-x)^{2\ell+1}} f\left( \frac{x^2}{(1-x)^{2k}} \right)^{\ell+1}, \]
(5.2) \[ \tilde{P}_{k,\ell}(x) = \sum_{r\geq0} \tilde{\mathcal{P}}_{r,k,\ell} x^r = \frac{1}{(1-x)^{\ell+1}} f\left( \frac{x^2}{(1-x)^{k+1}} \right)^{\ell+1}. \]
Replacing $x$ by $\frac{x}{1+x}$ in (5.1) and (5.2), one can deduce that
\begin{align}
\frac{1}{(1+x)^{2\ell+1}} \tilde{P}_{k,\ell}(x) = f(x^2(1+x)^{2k-2})^{\ell+1}, \\
\frac{1}{(1+x)^{\ell+1}} \tilde{P}_{k,\ell}(x) = f(x^2(1+x)^{k-1})^{\ell+1}.
\end{align}

Comparing the coefficient of $x^n$ in both sides of (5.3) and (5.4), one can deduce the following consequence:

**Corollary 5.3.** For any integers $n, k, \ell \geq 0$, there hold
\begin{align}
\sum_{p=0}^{n} (-1)^{n-p} \binom{n+2\ell}{p+2\ell} \tilde{P}_{p,k,\ell} &= \sum_{p=0}^{\lfloor\frac{n}{2}\rfloor} \left( \frac{\ell+1}{kp+\ell+1} \binom{k+1+p+\ell}{p} \binom{2(k-1)p}{n-2p} \right), \\
\sum_{p=0}^{n} (-1)^{n-p} \binom{n+\ell}{p+\ell} \tilde{P}_{p,k,\ell} &= \sum_{p=0}^{\lfloor\frac{n}{2}\rfloor} \left( \frac{\ell+1}{kp+\ell+1} \binom{k+1+p+\ell}{p} \binom{(k-1)p}{n-2p} \right).
\end{align}

Using the generalized Lagrange inversion formula obtained in [6], from (5.1), we have
\begin{align*}
\frac{\ell+1}{kn+\ell+1} \binom{(k+1)n+\ell}{n} &= [x^n] f(x)^{\ell+1} \\
&= [w^n] \left\{ (1-x)^{2\ell+1} \tilde{P}_{k,\ell}(x) \right\}_{w=\frac{x}{1-x}^{2n}} \\
&= \frac{1}{2n} \left[ (1-t)^{2n-1} (1-t)^{2kn+2\ell+1} \tilde{P}_{k,\ell}(t) \right] \\
&= \frac{2\ell+1}{2n} \left[ (1-t)^{2n-1} (1-t)^{2kn+2\ell+1} \tilde{P}_{k,\ell}(t) + \frac{1}{2n} \left[ (1-t)^{2n-1} (1-t)^{2kn+2\ell+1} \right] \frac{d}{dt} \tilde{P}_{k,\ell}(t) \right] \\
&= \sum_{p=0}^{2n-1} (-1)^p \binom{2kn+2\ell+1}{2n-p-1} \frac{2\ell+1}{2n} \tilde{P}_{p,k,\ell} + \sum_{p=0}^{2n} (-1)^p \left( \frac{2kn+2\ell+1}{2n-p} \right) \frac{p}{2n} \tilde{P}_{p,k,\ell} \\
&= \sum_{p=0}^{2n} (-1)^p \frac{kp+2\ell+1}{2n+2\ell+1} \left( \frac{2kn+2\ell+1}{2n-p} \right) \tilde{P}_{p,k,\ell}.
\end{align*}

Similarly, from (5.2), we have
\begin{align*}
\frac{\ell+1}{kn+\ell+1} \binom{(k+1)n+\ell}{n} &= [x^n] f(x)^{\ell+1} \\
&= [w^n] \left\{ (1-x)^{\ell+1} \tilde{P}_{k,\ell}(x) \right\}_{w=\frac{x}{1-x}^{2n}} \\
&= \frac{1}{2n} \left[ (1-t)^{2n-1} (1-t)^{k+1+n} \tilde{P}_{k,\ell}(t) \right] \\
&= \sum_{p=0}^{2n} (-1)^p \frac{P(k+1)+2(\ell+1)}{2n(k+1)+2(\ell+1)} \left( \frac{n(k+1)+\ell+1}{2n-p} \right) \tilde{P}_{p,k,\ell}.
\end{align*}

Hence we obtain the next corollary:
Corollary 5.4. For any integers \( n, k, \ell \geq 0 \), it holds that
\[
\frac{\ell + 1}{kn + \ell + 1} \binom{(k+1)n + \ell}{n} = \sum_{p=0}^{2n} (-1)^p \frac{kp + 2\ell + 1}{2kn + 2\ell + 1} \binom{2kn + 2\ell + 1}{2n - p} \tilde{P}_{p,k,\ell},
\]
\[
\frac{\ell + 1}{kn + \ell + 1} \binom{(k+1)n + \ell}{n} = \sum_{p=0}^{2n} (-1)^p \frac{p(k+1) + 2(\ell + 1)}{2n(k+1) + 2(\ell + 1)} \binom{n(k+1) + \ell + 1}{2n - p} \tilde{P}_{p,k,\ell}.
\]

We consider below several special cases, leading to several interesting results.

Example 5.5. If \( k = 1 \) and \( \ell = 0 \) in (5.5), then \( f(x) = \frac{1-\sqrt{1-4x}}{2x} = C(x) \), which is the generating function for the Catalan numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \). Hence we have
\[
\tilde{P}_{1,0}(x) = \frac{1}{1-x} C\left(x^2 \right) \bigg/ (1-x)^2 = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2},
\]
which is the generating function \( M(x) \) for the Motzkin numbers \( M_n \). Then Theorem 5.2 together with (5.5) and (5.6) generates the well-known identities (see [1, 3])
\[
\sum_{p=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{2p} C_p = M_n,
\]
\[
\sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} M_p = \begin{cases} C_r & \text{if } n = 2r, \\
0 & \text{otherwise.}
\end{cases}
\]

Example 5.6. If \( k = 2 \) and \( \ell = 0 \) in (5.5), then \( f(x) = 1 + xf(x)^3 = \sum_{n \geq 0} \frac{1}{2n+1} \binom{3n}{n} x^n \), which is the generating function for complete 3-ary plane trees. So we have
\[
\tilde{P}_{2,0}(x) = \frac{1}{1-x} x f\left(x^2 \right) \bigg/ (1-x)^3 = \frac{1}{1-x} x f\left(x^2 \right) \bigg/ (1-x)^3 = \frac{1}{1-x} (1 + x^2 \tilde{P}_{2,0}(x)^3).
\]
If we let \( y = x \tilde{P}_{2,0}(x) \), it follows that \( y = x(1+y)^{-1} \). From this,
\[
\tilde{P}_{2,0}(x) = C(x) = \frac{1-\sqrt{1-4x}}{2x}.
\]
Similarly, when \( k = 2 \) and \( \ell = 1 \) in (5.5), we have
\[
\tilde{P}_{2,1}(x) = \tilde{P}_{2,0}(x)^2 = C(x)^2 = \frac{1 - 2x - \sqrt{1 - 4x}}{2x^2} = \frac{C(x) - 1}{x}.
\]

Hence we obtain the following statement:

Corollary 5.7. The number of 2-paths (i.e. Schröder paths) of length \( 2n \) such that all \( u \)-segments have even length is the Catalan number \( C_n \) for \( n \geq 0 \) and the number of 2-paths of length \( 2n + 2 \) such that all internal \( u \)-segments have even length and the first \( u \)-segment has odd length is the Catalan number \( C_{n+1} \) for \( n \geq 0 \).

Here is a simple bijective proof. For any Schröder path \( S \) of length \( 2n \) such that all \( u \)-segments have even length, replace each \( h \) step by \( uv \) steps, then we get a Dyck path of length \( 2n \). On the other hand, any Dyck path \( D \) of length \( 2n \) can be decomposed uniquely into \( D = u^{i_1}d^{j_1}u^{i_2}d^{j_2} \cdots u^{i_k}d^{j_k} \), where \( i, j \geq 1 \). Now replace a sub-path \( u^{i}d^{j} \) by \( u^{i-1}hd^{j-1} \) if \( i \) is odd, and do nothing if \( i \) is even. Then we get a desired Schröder path \( S \). A similar argument proves the second claim in Corollary 5.7.

Theorem 5.2 and Example 5.6 give rise to two new expressions for the Catalan numbers,
Corollary 5.8. For any integer $n \geq 0$,
\[
\sum_{p=0}^{\lfloor n/2 \rfloor} \frac{1}{2p+1} \binom{3p}{p} \binom{n+p}{3p} = \frac{1}{n+1} \binom{2n}{n}, \quad (n \geq 0),
\]
\[
\sum_{p=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2p+1} \binom{3p+1}{p+1} \binom{n+p}{3p+1} = \frac{1}{n+1} \binom{2n}{n}, \quad (n \geq 1).
\]

Remark 5.10. Corollary 5.9.

Example 5.6 together with (5.5) and (5.6) leads to several new identities involving Catalan numbers.

Corollary 5.9. For any integer $n \geq 0$,
\begin{align*}
\sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} C_p &= \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{1}{2p+1} \binom{3p}{p} \binom{p}{n-2p}, \\
\sum_{p=0}^{n} (-1)^{n-p} \binom{n+1}{p+1} C_{p+1} &= \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{1}{2p+1} \binom{3p+1}{p+1} \binom{p}{n-2p}, \\
\sum_{p=0}^{2n} (-1)^p \frac{3p+2}{2n+2p+2} \binom{3n}{n+p} C_p &= \frac{1}{2n+1} \binom{3n}{n}, \\
\sum_{p=0}^{2n} (-1)^p \frac{3p+4}{2n+2p+4} \binom{3n+1}{n+p+1} C_{p+1} &= \frac{1}{2n+1} \binom{3n+1}{n+1}.
\end{align*}

Acknowledgements. The authors are deeply grateful to the two anonymous referees for valuable suggestions on an earlier version of this paper which makes it more readable. Thanks also to Simone Severini for helpful comments. The second author is supported by (NSFC10726021) The National Science Foundation of China.

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