HERMITIAN LIE ALGEBROIDS OVER ANALYTIC SPACES

ABHISHEK SARKAR

Abstract. We develop some properties of complex Riemannian geometry for complex algebraic varieties and study Hermitian metrics on analytic spaces. Then, we define Hermitian metrics on a holomorphic Lie algebroid and consider the associated characteristic foliation with a canonically induced inner product. Later, we examine hypercohomologies associated with leaf spaces, leaves and some $L$-invariant subspaces for the characteristic foliation $a(L)$ of a holomorphic Lie algebroid $a : \mathcal{L} \to T_X$ over a Hermitian manifold $X$. Additionally, we extend the notion equivariant de Rham cohomology in the analytic case.

1. Introduction

Riemannian manifolds are a classical concept in differential geometry, where a smooth manifold with metric or its tangent bundle with inner product is defined [Mor01]. In the context of complex geometry, as an analogue of Riemannian manifolds, we have two kind of objects for study, one is Hermitian manifolds [GH78] and another one is holomorphic Riemannian manifolds [LeB83,DZ09]. Here, we extend some of the theories of metrics on complex manifolds to metrics on analytic spaces or complex algebraic varieties. For that first we develop some important properties of complex Riemannian geometry for complex algebraic varieties and Hermitian metrics on analytic spaces, using the notions of complex Riemannian geometry [LeB83,DZ09], Riemannian Rinehart space [PVdV16b] and Hermitian manifolds. Here we consider an analytic subspace $Y$ of a Hermitian manifold $X$ and find the condition for which $Y$ inherits a metric from $X$. For that we need to consider the coherent sheaves of Lie-Rinehart algebras or Lie algebroids, namely the sheaf of logarithmic derivations $T_X(-\log Y) \subset T_X$ for $Y \subset X$ (where $T_X := Der_{\mathcal{O}_X}(\mathcal{O}_X)$ is the tangent sheaf of $X$) and the tangent sheaf $T_Y$ of $Y$ [MS22]. Note that, Lie algebroids in the algebro-geometric settings allows one to treat several geometric structures, such as Poission analytic spaces, singular foliations (or generalized involutive distributions) in a unified manner [Pym13,MS22]. More generally, to study calculus on smooth (i.e. non-singular) and singular geometric objects (known as algebraic spaces) from a common framework, we need to consider Lie algebroids in the generalized set up [Kap07,Pym13,Sch19,MS22].

We consider an analogue of the Levi-Civita connection for a Hermitian manifold $X$, provides a covariant connection on an analytic subspace $Y$, satisfying compatibility condition with the induced metric. For a complex manifold $X$, locally one gets the standard Hermitian metric and unique Levi-Civita connection (standard covariant derivative). We globalize these inner products and connections by considering a smooth partition of unity associated with an atlas of $X$. Thus, altogether it provides an analogue of Riemannian geometry over smooth manifolds, in the context of singular spaces by using algebro-geometric language. After that, we derive a relationship between Atiyah algebroid $\mathcal{A}_t(I)$ over a principal ideal sheaf

AMS Mathematics Subject Classification: 17B66, 32C35, 53C07.

Key words and phrases. Lie algebroids, tangent sheaf, Hermitian metrics, characteristic foliations, analytic de Rham cohomology.

The author acknowledges support from the institutes IIT Kanpur, ISI Bangalore, and IISER Pune, as the work was partially conducted at each of these institutes to completion.
I and sheaf of logarithmic derivations $\mathcal{T}_X(-\log Y)$ over the associated principal divisor $Y := V(I)$ (is the vanishing locus or zero set of the ideal sheaf $I$) for a Hermitian manifold $X$.

In [Bou11], the concept of inner products (or Riemannian metrics) on smooth vector bundles is extended to Riemannian metrics on (smooth) Lie algebroids [Mac05]. On the otherhand, we have Hermitian metrics on holomorphic vector bundles [GH78]. These ideas lead to define Hermitian metrics on holomorphic Lie algebroids and more generally for Lie algebroids over some analytic spaces, we refer it by Hermitian Lie algebroids. Then we consider leaves (or orbits) and $\mathcal{L}$-invariant subspaces associated to the characteristic foliation $\mathfrak{a}(\mathcal{L})$ of a Hermitian Lie algebroid $\mathfrak{a} : \mathcal{L} \to \mathcal{T}_X$, with induced inner product. As an example, we discuss about a standard Hermitian Lie algebroid structure on the cotangent sheaf $\mathcal{L} := \Omega^1_X$ of the 3-dimensional complex Euclidean space $X := \mathbb{C}^3$ equipped with a canonical Poisson manifold structure. Then we show that for some $\mathcal{L}$-invariant subspace $Y := V(I)$ where $\mathfrak{a}(\mathcal{L})(I) \subset I$ (following an example from [Pym13]), we have a canonically induced Hermitian Lie algebroid structure on its tangent sheaf.

In [Sch74] and [Hei73], the de Rham cohomologies for (smooth) leaf space and leaves associated to a smooth foliation on a Riemannian manifold has been studied. Here we present a counterpart in the complex geometry context, namely for singular foliations on a Hermitian manifold and discuss on its leaves or orbits. To study the cohomology theory we use associated hypercohomologies, which coincides with canonical singular cohomology for non-singular cases by using the analytic de Rham theorem. After that, as a continuation of the cohomology theory we consider equivariant analytic de Rham cohomology and extend it to define equivariant Lie algebroid cohomology. For that first we define $G$-analytic subspaces of a complex manifold with a complex Lie group $G$-action. Then we consider the hypercohomology of the associated orbit space. To consider this we follow the theory of Lie-Rinehart cohomology for quotient of singularities by finite group [EG10].

In Section 2, we consider complex Riemannian geometry of complex affine varieties and Hermitian metrics on analytic spaces. Then in Section 3, we define Hermitian metrics on holomorphic Lie algebroids and study the associated characteristic foliations with induced metrics. In Section 4, we consider analytic de Rham cohomologies for leaves, leaf space of a holomorphic foliation on a Hermitian manifold. In Section 5, we consider equivariant Lie algebroid cohomologies in the analytic geometry set up.

2. Riemannian geometry over topological ringed spaces

Riemannian geometry over a smooth manifold is a classical topic in differential geometry. For a smooth manifold we can consider an inner product on its tangent bundle (see [Swa62, Mor01, Wei13], from that a Riemannian metric appears. More generally, in [Bou11] Lie algebroid over a smooth manifold with an inner product is studied, known as Riemannian metrics on a Lie algebroid. In [PVdV16b], the authors studied Riemannian manifolds as Lie-Rinehart algebras, generalizes the concept of inner product on the $C^\infty(M)$-module $\mathfrak{X}(M)$ of vector fields on $M$ to general $R$-modules for an $K$-algebra $R$, which are related to $\text{Der}_R(K)$.

Here we consider an analogue of these notions, in the context of complex (analytic or algebraic) geometry. For that we consider special type of locally ringed spaces, which are known as topological ringed spaces [JV21] or algebraic spaces [MS22].

2.1. Complex Riemannian Geometry of complex affine varieties. In the articles [PVdV16b,Pes17], the authors have considered Riemannian Rinehart spaces and some of the associated quotient Rinehart spaces as an algebraic generalization of submanifold theory of Riemannian geometry. Here we recall the notion (quotient)
Riemannian Rinehart spaces and consider some of the key objects from it and describe relationships among them.

**Definition 2.1.** By a Rinehart space, we mean a dual pair \( \mathcal{A} \) whose primary and secondary \( \mathcal{O} \)-module denoted by \( \mathfrak{X} \) and \( \Omega \) respectively, and for which there exists a derivation \( d \in \text{Der}_\mathcal{O}(\mathcal{X}, \mathcal{O}) \), such that the associated map \( d \in \text{Hom}_\mathcal{O}(\mathfrak{X}, \text{Der}_\mathcal{O}(\mathcal{O})) \) turns \( \mathfrak{X} \) into a \((\mathfrak{X}, \mathcal{O})\)-Lie-Rinehart algebra (equivalently say, \((\mathcal{O}, \mathfrak{X})\) forms a Lie-Rinehart pair).

Furthermore, the following extra conditions are imposed:

- The \( \mathcal{O} \)-modules \( \mathfrak{X} \) and \( \Omega \) are finitely generated (or finite rank),
- The \( \mathcal{O} \)-module \( \Omega \) is spanned by the image of the map \( d : \mathcal{O} \to \mathcal{X} \),
- The pairing between \( \mathfrak{X} \) and \( \Omega \) is non-degenerate.

We denote the Rinehart space by \((\mathcal{O}, \mathfrak{X}, \Omega)\).

By a Riemannian metric on a Rinehart space \( \mathcal{A} := (\mathcal{O}, \mathfrak{X}, \Omega) \), we simply mean a symmetric, non-degenerate bilinear map \( \langle \cdot, \cdot \rangle : \mathfrak{X} \otimes_\mathcal{O} \mathfrak{X} \to \mathcal{O} \). We call the resulting pair a Riemannian Rinehart space.

Suppose \( \mathcal{A} := (\mathcal{O}, \mathfrak{X}, \Omega) \) is a Rinehart space and \( \mathcal{B} := (\mathcal{I}, \mathfrak{X}', \Omega') \) is an ideal subpair of \( \mathcal{A} \), i.e. \( \mathcal{I} \subset \mathcal{O} \) is an ideal, \( \mathfrak{X}' \) is a Lie ideal of \( \mathfrak{X} \) and \( \mathcal{d}(\mathcal{I}) \subset \Omega' \). Then the quotient pair \( \mathcal{A}/\mathcal{B} := (\mathcal{O}/\mathcal{I}, \mathfrak{X}/\mathfrak{X}', \Omega/\Omega') \) induces a canonical Rinehart space structure, known as the quotient Rinehart space of \( \mathcal{A} \) by \( \mathcal{B} \).

The quotient map \( \rho : \mathcal{A} \to \mathcal{A}/\mathcal{I} \) becomes a surjective morphism of Rinehart spaces.

In the following, we consider some of the key objects in the context of Riemannian Rinehart space and describe relationships among them.

Let \( \mathcal{O} \) be a \( \mathbb{C} \)-algebra and \( \mathcal{I} \) be a finitely presented ideal (prime ideal) of \( \mathcal{O} \). Consider the standard Rinehart space

\[ \mathcal{A} := (\mathcal{O}, \mathfrak{X} := \text{Der}_\mathbb{C}(\mathcal{O}), \Omega := \Omega_{\mathcal{O}/\mathbb{C}}) \]

where \( \text{Der}_\mathbb{C}(\mathcal{O}) \) is the space of \( \mathbb{C} \)-linear derivations over the algebra \( \mathcal{O} \) and \( \Omega_{\mathcal{O}/\mathbb{C}} \) is the space of \( \mathbb{K} \)-algebraic differentials of the algebra \( \mathcal{O} \) over \( \mathbb{C} \).

The induced quotient Rinehart space of \( \mathcal{A} \) associated with ideal \( \mathcal{I} \) is

\[ \mathcal{A}/\mathcal{I} := (\mathcal{O}/\mathcal{I}, \mathfrak{X}/\mathcal{I} := \text{Der}_\mathbb{C}(\mathcal{O}/\mathcal{I}), \Omega/\mathcal{I} := \Omega/\mathcal{I}/\langle \mathcal{d}g | g \in \mathcal{I} \rangle \).

For the case of polynomial algebra \( \mathcal{O} = \mathbb{C}[x_1, \ldots, x_n] \) for some \( n \in \mathbb{N} \), the spaces \( \mathcal{A} \) and \( \mathcal{A}/\mathcal{I} \) are associated with geometry of affine algebraic set (affine variety) \( Y := \mathcal{V}(\mathcal{I}) \) inside the affine space \( \mathfrak{X} := \mathbb{A}^n \), otherwise we can think it as geometry of affine scheme \( Y = \text{Spec}(\mathcal{O}/\mathcal{I}) \) associated with \( X = \text{Spec}(\mathcal{O}) \). We consider the canonical action of \( \mathbb{K} \)-algebraic differentials on derivations as a dual pair

\[ \langle \cdot, \cdot \rangle : \Omega \otimes_\mathcal{O} \mathfrak{X} \to \mathcal{O} \]
defined as \( \langle \omega, D \rangle \mapsto \langle \omega, D \rangle := w(D) \) for any \( \omega \in \Omega, \ D \in \mathfrak{X} \). If there is a inner product on space of derivations

\[ \langle \cdot, \cdot \rangle_X : \mathfrak{X} \otimes_\mathcal{O} \mathfrak{X} \to \mathcal{O} \]
given by \( \langle D_1, D_2 \rangle_X \) for any \( D_1, D_2 \in \mathfrak{X} \), then the triple \((\mathcal{O}, \mathfrak{X}, \Omega)\) forms a Riemannian Rinehart space. In particular, for the case of polynomial algebra, these operations are defined by

\[ \langle df, \sum_{i=1}^n f_i \partial_{x_i} \rangle = \sum_{i=1}^n f_i \frac{\partial f}{\partial x_i}, \quad \langle \sum_{i=1}^n f_i \partial_{x_i}, \sum_{i=1}^n g_i \partial_{x_i} \rangle_X = \sum_{i=1}^n f_i g_i \]
respectively, where \( \partial_{x_i} := \frac{\partial}{\partial x_i} \) and \( f, f_i, g_i \in \mathcal{O} \) (\( i = 1, \ldots, n \)) and \( df(D) = D(f) \) for \( D \in \mathfrak{X} \). Similar ideas are present in the above references by a more algebraic approach. Now we construct some of the important Rinehart spaces, for that first we need to consider some special \( \mathcal{O} \)-modules given as follows.
Consider the $\mathcal{O}$-module $\mathfrak{X}^T := \{D \in \text{Der}_C(\mathcal{O}) \mid D(\mathcal{I}) \subset \mathcal{I}\}$, consists with such derivations which induces derivations on $\mathcal{O}/\mathcal{I}$. In the case of polynomial algebra, it represents space of (algebraic) vector fields on $X := \mathbb{A}^n$ that are tangent to $Y := V(\mathcal{I})$.

Consider the $\mathcal{O}$-module $\mathfrak{X}^\perp := \{D \in \text{Der}_C(\mathcal{O}) \mid \langle D, \mathfrak{X}^T \rangle \subset \mathcal{I}\}$, consists with derivations whose elements represent vector fields normal to $Y := V(\mathcal{I})$ in $X := \mathbb{A}^n$ in the case of polynomial algebra.

Consider the $\mathcal{O}$-module $\Omega^\perp := \{w \in \Omega \mid w = f \ dg \text{ for some } f \in \mathcal{O}, g \in \mathcal{I}\}$ consists with some special differential 1-forms. Thus, we can express it represents space of (algebraic) vector fields on $X$ derivations which induces derivations on $\mathcal{O}$ consisting with such differential 1-forms.

Consider the $\mathcal{O}$-module $\Omega^0 := \{D \in \text{Der}_C(\mathcal{O}) \mid \langle \Omega, D \rangle \subset \mathcal{I}\}$, consists with derivations such that in the case of polynomial algebra $\mathcal{O} = \mathbb{C}[x_1, \ldots, x_n]$ each of this derivations on $\mathcal{O}$ (or vector fields over $X = \mathbb{A}^n$) induces a zero derivation on $\mathcal{O}/\mathcal{I}$ (or vector field vanishes on $Y$).

Consider the $\mathcal{O}$-module $\Omega^0 := \{w \in \Omega \mid \langle w, \mathfrak{X}\rangle \subset \mathcal{I}\}$, it represents Kähler differentials that vanishes on $Y = V(\mathcal{I})$ for the polynomial algebra $\mathcal{O}$.

Consider the quotient map $\rho : \mathcal{O} \to \mathcal{O}/\mathcal{I}$, a surjective $\mathbb{C}$-algebra homomorphism, provide the homomorphism of Lie-Rinehart algebras

$$\rho_* : \mathfrak{X}^T \to \mathfrak{X}_I$$

defined by $D \mapsto \tilde{D}$. It is well defined because if we take $[f] = [g]$ that is $f - g \in \mathcal{I}$ for some $f, g \in \mathcal{O}$ then $D(f - g) \in \mathcal{I}$ for any $D \in \mathfrak{X}^T$ which implies $\tilde{D}([f]) = [D(f)] = [D(g)] = \tilde{D}([g])$. Note that, if $D \in \mathfrak{X}^0, D' \in \mathfrak{X}^T, f \in \mathcal{O}$ then $[D, D'](f) = D(D'(f)) - D'(D(f)) \in \mathcal{I}$. Thus, we have the Lie ideal

$$\ker \rho_* = \{D \in \text{Der}_C(\mathcal{O}) \mid D(\mathcal{O}) \subset \mathcal{I}\} = \mathfrak{X}^0 \subset \mathfrak{X}^T.$$

Note that the differential $d : \mathcal{O} \to \Omega$ provides the Lie-Rinehart homomorphism

$$d : \mathfrak{X}^T \to \mathfrak{X}$$

defined as $\tilde{D}D(f) = d(f)(D)$ for any $f \in \mathcal{O}$ and $D \in \mathfrak{X}^T$. Thus $\mathfrak{A}^T := (\mathfrak{X}, \mathfrak{X}^T, \Omega)$ is a Rinehart subspace of $\mathfrak{A}$.

The differential $d : \mathcal{O} \to \Omega$ induces $\tilde{d} := d|_{\mathcal{I}} : \mathcal{I} \to \Omega^\perp$, which provides a natural $(\mathbb{C}, \mathcal{I})$-Lie-Rinehart algebra homomorphism $\tilde{d} : \mathfrak{X}^T \to \text{Der}_C(\mathcal{I})$ (if $f \in \mathcal{O}, g \in \mathcal{I}$ and $D \in \mathfrak{X}^T$ then $\langle D, dg, D' \rangle = \langle dg, f \ D' \rangle = f \ D(g) \in \mathcal{I}$). Thus we get a Rinehart space $(\mathcal{I}, \mathfrak{X}^T, \Omega^\perp)$.

**Remark 2.2.** The above described Rinehart spaces can be considered as Riemannian Rinehart spaces by considering the induced structure coming from a Riemannian Rinehart space structure on $(\mathcal{O}, \mathfrak{X}, \Omega)$.

Most of the notions up to now followed from the paper [PVdV16b]. All of these concepts form a local model in the context of algebraic geometry. Globalize these ideas using sheaf theory provides an analogue of Riemannian geometry of algebraic variety (and more generally for schemes). We consider the corresponding global version in the complex geometry context.

From now we use these kind of ideas to do complex Riemannian geometry and Hermitian geometry for analytic spaces. It help to generalize submanifold theory of a Riemannian manifold in classical differential geometry.

**Remark 2.3.** In [DZ00, LeB83, PVdV16a], the authors have considered holomorphic Riemannian manifold as a complex manifold $X$ endowed with a holomorphic Riemannian metric. A holomorphic Riemannian metric on a complex manifold $X$ is a holomorphic section of the tensor bundle $\otimes^2 T^*X$ (i.e., a holomorphic covariant 2-tensor) which is symmetric and non-degenerate, in otherwords it is a holomorphic field of non degenerate complex quadratic forms on the holomorphic tangent bundle $TX$. For example, consider $X = \mathbb{C}^n$ with the global flat holomorphic Riemannian metric $\sum_{j=1}^n (dz_j)^{\otimes 2}$ (similar to the inner product $\mathbb{D}$ where $\{z_j\}_{j=1}^n$ is the standard coordinate system and take any quotient of $\mathbb{C}^n$ by a lattice with induced metric.
2.2. Hermitian metrics on analytic spaces. By following the concepts of Hermitian metric on a complex vector bundle [GH78] and Riemannian manifolds as Lie-Rinehart algebras [PVdV16b,Pes17], we consider an analogue of Hermitian metric in the context of complex geometry over analytic spaces.

Let \( \{(U_i, \phi_i)\} \) be a holomorphic atlas of a complex manifold \( X \) (dimension of \( X \) is \( n \)), i.e. \( \phi_i : U_i \to \mathbb{V} \) is a biholomorphism with some proper open subset \( V \) of \( \mathbb{C}^n \). Then we can induce an inner product locally on the holomorphic tangent bundle \( TX \) from the standard inner product on \( \mathbb{C}^n \), and globalize it by using a \( C^\infty \)-partition of unity associated to the open cover of \( X \). This provides us a \( \mathbb{C} \)-sesquilinear, conjugate-symmetric, positive non degenerate map

\[
h_x : T_xX \times T_xX \to \mathbb{C}
\]
on each holomorphic tangent space \( T_xX \) (at \( x \in X \)) of \( X \), varies smoothly, can be viewed as pointwise positive definite Hermitian inner products

\[
\langle \cdot, \cdot \rangle_x : T_xX \otimes_{\mathbb{C}} T_xX \to \mathbb{C},
\]
where \( \langle v, c w \rangle := \bar{c} h_x(v, w) \) for \( v, w \in T_xX \), induces a \( C^\infty \)-map

\[
\langle \cdot, \cdot \rangle : T_xX \to (TX \otimes TX)^* \text{ given by } x \mapsto \langle \cdot, \cdot \rangle_x
\]
for any \( x \in X \), known as positive-definite Hermitian form on \( TX \). Note that \( T_X := \mathcal{D}er_X(\mathcal{O}_X) \) (sheaf of derivations over the sheaf of holomorphic functions \( \mathcal{O}_X \)), isomorphic to the sheaf of sections \( \mathfrak{X}_X := \Gamma TX \) of the holomorphic tangent bundle over \( X \) (i.e. sheaf of holomorphic vector fields on \( X \)), known as tangent sheaf of \( X \). Thus, on an open set \( U_i \) associated with a chart, we get \( C^\infty_X(U_i) \)-module

\[
\langle \cdot, \cdot \rangle_{U_i} : T_X(U_i) \otimes_{\mathcal{O}_X(U_i)} \overline{T_X(U_i)} \to C^\infty_X(U_i),
\]
the \( C^\infty_X(U_i) \)-module \( \overline{T_X(U_i)} \) has a standard \( \mathcal{O}_X(U_i) \)-module structure, given by \( (f, D) \mapsto \bar{f} D \) for any \( f \in \mathcal{O}_X(U_i) \) and \( D \in T_X(U_i) \), \( i \in \mathbb{N} \) which is positive definite Hermitian form. Now, by using a \( C^\infty \)-partition of unity \( \{f_i\}_{i \in \mathbb{N}} \) for the open cover \( \{U_i\}_{i \in \mathbb{N}} \), we can form a Hermitian inner product on \( X \) as

\[
\langle \cdot, \cdot \rangle_X := \sum_i f_i \langle \cdot, \cdot \rangle_{U_i}
\]
i.e., in the global situation by considering respective sheaves, we get

\[
\langle \cdot, \cdot \rangle_X : T_X \otimes_{\mathcal{O}_X} \overline{T_X} \to C^\infty_X
\]
given by \( \langle D, D' \rangle_X = \sum_{i \in \mathbb{N}} f_i (D|_{U_i}, D'|_{U_i})_{U_i} \), for sections \( D, D' \) are in \( T_X \cong \mathfrak{X}_X \).
The notions \( \overline{T_X}, \overline{T_X(U_i)} \) and \( \overline{T_X} \) are complex conjugation of \( T_X, T_X(U_i) \) and \( T_X \) respectively (i.e., as \( \mathbb{C} \)-vector space, \( \mathcal{O}_X(U_i) \), and \( \mathcal{O}_X \) module respectively). An inner product on the sheaf of holomorphic vector fields \( \mathfrak{X}_X \) of a complex manifold \( X \) is called a Hermitian metric on \( X \), and the complex manifold with a Hermitian metric is known as a Hermitian manifold.

Remark 2.4. On a chart \( (U, (x_1, \ldots, x_n)) \) of a complex manifold, a Hermitian metric is given by \( \sum_{i=1}^n h_{ij} \, dx_i \otimes \overline{dx_j} \) where \( h_{ij}(x) = \langle \partial_{x_i}, \partial_{x_j} \rangle_x \) is determined by the Hermitian inner product on \( T_xX \). The standard Hermitian inner product on an open subset \( U \) of \( \mathbb{C}^n \) is given by \( \sum_{i=1}^n f_i \, \partial_{x_i}, \sum_{j=1}^n g_j \, \partial_{x_j} \rangle_U = \sum_{i=1}^n f_i \, \overline{g_i} \), for all \( f_i, g_i \in \mathcal{O}_{\mathbb{C}^n}(U) \) (\( i = 1, \ldots, n \)).

Note 2.5. The concept of a Hermitian metric on a complex manifold, i.e. a Hermitian inner product on its tangent bundle extends to the idea of considering a Hermitian metric on a complex vector bundle, by considering smoothly varying fiber-wise Hermitian inner product (see [GH78] for details).
Note 2.6. We extend the idea of complex conjugate of a complex vector bundle on a complex manifold $X$ (see [GH78]) by considering complex conjugate on its sheaf of sections of $\mathcal{O}_X$-modules and we use the sheaf of $\mathbb{C}$-valued smooth functions $C^\infty_X$ on $X$ by considering underlying smooth structure of the complex manifold.

Analogously, we define Hermitian metric on a holomorphic vector bundle in the algebrao-geometric approach, by considering positive-definite Hermitian forms on the associated sheaf of sections.

Recall the definition of a Lie algebroid in the algebrao-geometric language (see [Pym13], [Bry17], [MS22] for details).

**Definition 2.7.** A Lie algebroid $\mathcal{L}$ over an analytic space $(X, \mathcal{O}_X)$ is a coherent sheaf of $C^\infty_X$-Lie-Rinehart algebras.

That is $\mathcal{L}$ is a coherent $\mathcal{O}_X$-module and a $C^\infty_X$-Lie algebra equipped with a morphism of $\mathcal{O}_X$-modules $a : (\mathcal{L}, [\cdot, \cdot]) \to (\text{Der}_C(\mathcal{O}_X), [\cdot, \cdot])$, called the anchor map. The map $a$ is also a morphism of $C^\infty_X$-Lie algebras and satisfying the Leibniz rule: $[D, f D'] = f [D, D'] + a(D)(f) D'$ for sections $f \in \mathcal{O}_X$ and $D, D' \in \mathcal{L}$.

We denote this Lie algebroid as $(\mathcal{L}, [\cdot, \cdot], a)$ or simply by $\mathcal{L}$.

**Example 2.8.** Let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf of a complex manifold $(X, \mathcal{O}_X)$. The sheaf of logarithmic derivations $\mathcal{T}_X(\log \mathcal{I}) := \{ D \in \mathcal{T}_X \mid D(\mathcal{I}) \subset \mathcal{I} \}$ for an analytic space $(Y := V(\mathcal{I}), \mathcal{O}_Y = \mathcal{O}_X/\mathcal{I})$ in $(X, \mathcal{O}_X)$ has a canonical Lie algebroid structure. Also, the tangent sheaf $\mathcal{T}_X := \text{Der}_C(\mathcal{O}_X)$ of $X$ and $\mathcal{T}_Y := \text{Der}_C(\mathcal{O}_Y)$ of $Y$ has the standard Lie algebroid structure (see [Pym13], [MS22]).

We have quotient map $\rho : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = \mathcal{O}_Y$ given by $f \mapsto [f]$, a surjective $\mathbb{C}$-algebra homomorphism. Consider the canonical surjective map $\hat{\rho} : \mathcal{T}_X(\log \mathcal{I}) \to \mathcal{T}_Y$ defined by $\hat{\rho}(D) = \hat{D}$ where $\hat{D}([f]) = [D(f)]$ (since $D(\mathcal{I}) \subset \mathcal{I}$ for all $D \in \mathcal{T}_X(\log \mathcal{I})$, the assignment given by $\hat{\rho}$ is well defined). Thus, $\ker \hat{\rho} = \mathcal{I}\mathcal{T}_X := \mathcal{T}_X^0$.

**Remark 2.9.** A locally free Lie algebroid over a complex manifold is equivalent to a holomorphic Lie algebroid [Pym13], where a holomorphic Lie algebroid is defined as an analogue of smooth Lie algebroids [Mac05] in the holomorphic category. These are the classical cases, where we have Lie-Rinehart algebra structures on the space of sections of a smooth or holomorphic vector bundles [MS22].

We consider Hermitian metrics on Lie algebroids over analytic spaces, as a generalization of Hermitian manifold. For that first we prove some results related to Hermitian manifold. Then we define the analogue for Lie algebroids over analytic spaces with Hermitian inner product and consider some of its special cases.

**Theorem 2.10.** Let $X$ be a Hermitian manifold and $U$ an open set contained in some chart of $X$. Let $D \in \mathcal{T}_X(U)$ such that $\langle D, D' \rangle \in \mathcal{I}(U)$ for all $D' \in \mathcal{T}_X(U)$. Then $D \in \mathcal{T}_X^0(U)$, and moreover for any global section $\hat{D}$ of $\mathcal{T}_X$ if locally this result holds, then it implies that $\hat{D} \in \mathcal{T}_X^0(X)$.

**Proof.** Let $x_1, \ldots, x_n$ be the co-ordinate functions on the chart $U$. Then $D = \sum_{i=1}^n g_i \partial_{x_i}$ for some $g_1, \ldots, g_n \in \mathcal{O}_X(U)$. Take $D' = \partial_{x_i}$ and thus we get $g_j = \langle \sum_{i=1}^n g_i \partial_{x_i}, \partial_{x_j} \rangle \in \mathcal{I}(U)$, $j = 1, \ldots, n$. Hence, $D(\mathcal{O}_X(U)) \subset \mathcal{I}(U)$ then $g_j = D(x_j) \in \mathcal{I}(U)$ and if $D \in \mathcal{I}(U)\mathcal{T}_X(U)$ then $D(f) = \sum_{i=1}^n g_i \partial_{x_i}(f) \in \mathcal{I}(U)$ for all $f \in \mathcal{O}_X(U)$.

Let $\hat{D} \in \mathcal{T}_X(X)$. Then for each point $x \in X$ there exist an open set $U_x$ contained in some chart such that the stalk of $\hat{D}$ at $x$ say $D_x$ satisfies the previous condition, then $D_x \in \mathcal{T}_X^0(U_x)$. Therefore, $\hat{D} \in \mathcal{T}_X^0(X)$. \hfill $\square$

Since $\mathcal{I} \subset \mathcal{O}_X$ is a principal ideal sheaf (that is $Y$ become a hypersurface singularities), for each point $x \in X$ there is an open neighborhood $U$ (contained in
some chart) containing \( x \) such that \( \mathcal{I}(U) \) is generated by some \( g \in \mathcal{O}_X(U) \), i.e. \( \mathcal{I}(U) = \langle g \rangle \). Then for any \( D \in \mathcal{T}_X(U) \), define
\[
D^T := D - \langle D, \nabla g \rangle \nabla g
\]
where
\[
\nabla g = \sum_{j=1}^{n} \partial_{x_j}(g) \partial_{x_j} \in \mathcal{N}_{Y/X}(U)
\]
is the gradient of \( g \in \mathcal{O}_X(U) \) and \( \mathcal{N}_{Y/X} \) is the sheaf of normals on \( Y \) inside \( X \). To show \( \mathcal{D}^T = D - \langle D, \nabla g \rangle \nabla g \in \mathcal{T}_X(-\log Y)(U) \), we need to check \( \mathcal{D}^T(fg) \in \mathcal{I}(U) \) for any \( f \in \mathcal{O}_X(U) \). Notice that we have the following identity
\[
\mathcal{D}^T(fg) = D(fg) - D(g) \nabla (fg) = (D + D(g)\nabla g)g + f \nabla g(g).
\]
In the above equation the first term is in the ideal \( \langle g \rangle = \mathcal{I}(U) \) and the second term also is in \( \mathcal{I}(U) \) if the condition
\[
1 - \langle \nabla g, \nabla g \rangle \in \mathcal{I}(U)
\]
holds. In the special case of complex submanifold \( Y \) of \( X \), of codimension 1, we choose an atlas of \( X \) such that \( Y \) is locally represented as \( \{x_1 = 0\} \) for some coordinate function \( x_i \) in some chart of \( X \). In that case we have the identity
\[
\langle \nabla g, \nabla g \rangle = (\nabla g(g) = \sum_{j=1}^{n} \partial_{x_j}(g) \cdot \partial_{x_j}(g) = 1
\]
(putting \( g = x_i \)) and so the required condition holds. But in general, the condition
\[
\text{does not hold for hypersurface singularities. Like when we consider the normal crossing divisor } Y = V(xy) \text{ in } X = \mathbb{C}^2, \text{ we face difficulties. In this case the principal ideal sheaf is } \mathcal{I} = \langle xy \rangle \text{ and } \mathcal{O}_X \text{ is the sheaf of holomorphic functions. Thus, the sheaf of logarithmic derivations is } \mathcal{T}_X(-\log Y) = \langle \{x_1, y \partial_y \} \rangle \text{ and the sheaf of normals } \mathcal{N}_{Y/X} = \langle y \partial_y + x \partial_x \rangle. \text{ Since here } g = xy, \text{ we get } \langle \nabla g, \nabla g \rangle = (\nabla g(g) = x^2 + y^2. \text{ Now, } 1 - \langle \nabla g, \nabla g \rangle = 1 - \nabla g(g) = 1 - (x^2 + y^2) \notin \langle \{xy\} \rangle.
\]
Therefore if the condition holds, we get the Lie-Rinehart algebra homomorphisms
\[
\mathcal{T}_X(U) \xrightarrow{\rho_*} \mathcal{T}_X(-\log Y)(U) \xrightarrow{\rho} \mathcal{T}_Y(Y \cap U) \text{ defined by } D \mapsto D^T \mapsto D^T := D.
\]
Hence, in each of the charts we have these morphisms which are compatible with any open subsets of a chart. Induced morphisms provides stalkwise morphisms of Lie-Rinehart algebras, sheafifying it we get the homomorphisms of Lie algebroids (coherent sheaf of Lie-Rinehart algebras as described in \([\text{Pym13}]\)).

**Note:** we use the fact that \( \mathcal{I} \subset \mathcal{O}_X \subset C^\infty_X \) holds for a complex manifold \( X \). Thus we can consider the quotient sheaf \( C^\infty_Y := C^\infty_X / \mathcal{I} \) on \( Y \). We use the restriction map of \( \rho \) from \( C^\infty_X \) to \( C^\infty_Y / \mathcal{I} \), to define inner product on \( \mathcal{T}_Y \).

**Theorem 2.11.** Let \( X \) be a Hermitian manifold and \( Y \) an analytic subspace of \( X \) satisfying the condition \([\text{Pym13}]\). Then \( Y \) inherits from \( X \), an \( \mathcal{O}_Y \)-bi-linear form which is locally determined by the equation: for any \( D, D' \in \mathcal{T}_Y(U) \), \( (D, D')_Y \in \mathcal{U} := \rho(U)(\langle D, D' \rangle_U) \) for some \( D, D' \in \mathcal{T}_X(U) \) where \( U \) is an open set contained in a chart of \( X \), such that \( \pi(U)(D) = \rho_*U(D^T) = D \) and \( \pi(U)(D') = \rho_*U(D'^T) = D' \) holds.
Proof. Well definedness: Let \( D, D'' \in \mathcal{T}_X(U) \) such that \( \pi(U)(D) = \pi(U)(D') = \pi(U)(D'') = \bar{D} = \rho_*U(D^{\mathbb{T}}) = \rho_*U(D'') \). Then \( \pi(U)(D - D'') = 0 \), i.e. \( D - D'' \in \mathcal{T}_X(U) \) and hence \( (D - D'')_{U'} \in \mathcal{I}(U) \). Thus, \( \rho(U)((D - D'')_{U'}) = 0 \) or \( \rho(U)((D, D'')_{U'}) = \rho(U)((D', D'')_{U'}) \). Therefore, for an open set \( U_i \) associated with a chart we have the induced \( \mathcal{O}_Y|_{Y \cap U_i} \)-bilinear form \( \langle \cdot, \cdot \rangle_{Y \cap U_i} \) on \( Y \cap U_i \) as defined in the statement. Now using the \( C^\infty \)-partition of unity \( \{ f_i \} \) of \( Y \), we can form the induced \( \mathcal{O}_Y \)-bilinear form \( \langle \cdot, \cdot \rangle_Y \) on \( Y \) as

\[
\langle \cdot, \cdot \rangle_Y := \sum_i f_i |_{Y \cap U_i} \langle \cdot, \cdot \rangle_{Y \cap U_i}
\]

induces a sheaf homomorphism of \( \mathcal{O}_Y \)-modules

\[
\langle \cdot, \cdot \rangle_Y : \mathcal{T}_Y \otimes_{\mathcal{O}_Y} \mathcal{T}_Y \to \mathcal{O}_Y.
\]

Note: This is a global counterpart of a result that appeared in [PVdV 16b, Pes17].

Remark 2.12. Conjugate-symmetry of this induced bilinear form on \( \mathcal{T}_Y \) appears from the conjugate-symmetry of the inner product on \( \mathcal{T}_X \) as follows

\[
\langle D, D' \rangle_{Y \cap U} := \rho(U)(D_{U'}) \rho(U)(D'_{U'}) = \rho(U)(\bar{D}_{U'}) \rho(U)(\bar{D}')_{U'} = \bar{\langle D, D' \rangle}_{Y \cap U}.
\]

But positive non degeneracy property (or positive definiteness) is not induced naturally: if for any \( D \in \mathcal{T}_Y(Y \cap U) \) such that \( \langle D, D' \rangle_{Y \cap U} = 0 \) holds for all \( D' \in \mathcal{T}_Y(Y \cap U) \), then \( \langle D, D' \rangle_{Y \cap U} = 0 \) for all \( D' \in \mathcal{T}_X(-\log Y)(U) \). This does not imply \( D \in \mathcal{T}_X(U) \)

always, but here it holds using condition (4) and thus \( D = 0 \). In that case we get an inner product structure on \( \mathcal{T}_Y \), we call it induced metric on \( Y \).

Remark 2.13. For a Hermitian manifold \( X \), by considering the restriction of the Hermitian inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{T}_X \) to \( \mathcal{T}_X(-\log Y) \) where \( Y \) is an analytic subspace of \( X \) satisfying condition (4), provides a Hermitian inner product on \( \mathcal{T}_X(-\log Y) \).

Covariant connection: For a locally free \( \mathcal{O}_X \)-module \( \mathcal{E} \) of rank \( r \) (or a holomorphic vector bundle of rank \( r \)) over a complex manifold \( (X, \mathcal{O}_X) \) for some \( r \in \mathbb{N} \), on each charts we have the standard covariant derivative as an \( \mathcal{O}_X(U_i) \)-linear map

\[
\nabla^{U_i} : \mathcal{T}_X(U_i) \to \text{End}_{\mathcal{E}}(\mathcal{E}(U_i))
\]

(associated with an open set \( U_i \) of some chart) satisfies the Leibniz rule

\[
\nabla^{U_i}(f s) = f \nabla^{U_i}s + df \otimes_{\mathcal{O}_X(U_i)} s
\]

i.e., \( \nabla^{U_i}_D(f s) := \nabla^{U_i}_D(f)(s) = f \nabla^{U_i}_D(s) + D'(f)s \),

for sections \( f \in \mathcal{O}_X(U_i), D' \in \mathcal{T}_X(U_i) \) and \( s \in \mathcal{E}(U_i) \). A covariant connection can be defined by the following local descriptions (see [Mor01])

\[
\nabla^{U_i}(\sum_{i=1}^r g_i s_i) = \sum_{i=1}^r D(g_i)s_i
\]

where \( D \in \mathcal{T}_X(U_i) \) and \( \{ s_1, \ldots, s_r \} \) is a basis of \( \mathcal{E}(U_i) \) (without loss of generality we are considering a common open cover \( \{ U_i \} \) of \( X \) where both \( \mathcal{T}_X(U_i) \) and \( \mathcal{E}(U_i) \)

are free \( \mathcal{O}_X(U_i) \)-modules of finite ranks). Now by using the \( C^\infty \)-partitions of unity on this open cover of \( X \) we can form a sheaf homomorphism of \( C^\infty \)-modules

\[
\nabla : \mathcal{T}_X \to \mathfrak{d} \text{End}_{\mathcal{E}}(\mathcal{E})
\]

is defined by

\[
\nabla := \sum_i f_i \nabla^{U_i}, \text{ i.e. } \nabla_D(s) = \sum_i f_i \nabla^{U_i}_D(s)
\]

for sections \( D \in \mathcal{T}_X \) and \( s \in \mathcal{E} \). In the special case \( \mathcal{E} = \mathcal{T}_X \), we say that a covariant derivative \( \nabla \) is symmetric if for any open subset \( U \) of \( X \) we have

\[
\nabla(U)D'D' - \nabla(U)D = [D, D']_U,
\]

for all \( D, D' \in \mathcal{T}_X(U) \), and compatible with the inner product \( \langle \cdot, \cdot \rangle \), i.e. for sections \( D, D', D'' \in \mathcal{T}_X(U) \) on an open subset \( U \) of \( X \) we have

\[
\langle D(D', D'') \rangle_U = \langle \nabla^{U}_D D', D'' \rangle_U + \langle D', \nabla^{U}_D D'' \rangle_U.
\]
We call the covariant derivative connection as a Levi-Civita connection, if it is both symmetric and compatible with the inner product given on the tangent bundle. The standard covariant derivative on $X$ is a Levi-Civita connection.

Remark 2.14. If $F \subset TX$ is an involutive subbundle of the tangent bundle of a Hermitian manifold $X$, then the corresponding normal bundle of $F$ in $TX$ provides a locally free sheaf of $\mathcal{O}_X$-module (by its sheaf of sections). Thus it has a canonical covariant connection as described above. Later in Section 4 we use this concept to define generalized Bott connection.

Remark 2.15. The concepts of covariant connections extends to Lie algebroid connection in (smooth real) differential and complex geometric settings (see [Fer12, Pym13]). Consider the Lie algebroid $\mathcal{L} = TX(-\log Y)$ for an analytic subspace $Y$ in $X$. If a $\mathcal{L}$-connection exists, it is called a logarithmic connection which is a meromorphic connection with poles along the divisor $Y$ [Sar22].

Theorem 2.16. Let $X$ be a Hermitian manifold with a Levi-Civita connection $\nabla$ and $Y$ an analytic space with the induced metric. Then $Y$ inherits a Levi-Civita connection $\nabla$ from the connection $\nabla$ on $X$, determined by the equation:

$$\nabla^Y_{D'}(D') = \pi(U)(\nabla^X_{D'}D')$$

where $D = \pi(U)(D), D' = \pi(U)(D')$ and $U$ is an open set of $X$.

Proof. Well definedness: Let $\pi(U)(D) = \bar{D} = \pi(U)(D_1)$. Then $D'' := D - D_1 \in T_X(U)$, hence $\nabla^X_{D''}D' \in T_X(U)$ (since $D'' = \sum_{i=1}^k g_i D_i$ for some $g_1, \ldots, g_k \in \mathcal{I}(U)$, thus $\nabla^X_{D''}D' = \sum_{i=1}^k g_i \nabla^X_{D_i}D' \in \mathcal{I}(U)T_X(U)$). Therefore, $\pi(U)(\nabla^X_{D''}D') = 0$ and since we have

$$\nabla^X_{D''}D' = \pi(U)(\nabla^X_{D''}D') = \pi(U)(\nabla^X_{D_1}D')$$

thus $\pi(U)(\nabla^Y_{D'}D') = \pi(U)(\nabla^Y_{D_1}D')$. Now, let $\pi(U)(D') = \bar{D}' = \pi(U)(D'_1)$. Then $D'' := D' - D'_1 \in T_X(U)$, i.e. $D'' = \sum_{i=1}^k g_i D_i$ for some $g_1, \ldots, g_k \in \mathcal{I}(U)$. Without loss of generality we can take $D \in T_X(-\log Y)(U)$ because $\pi(U)(D) = \pi(U)(D')$ for every $D \in T_X(U)$ and hence $\nabla^X_{D''}D'' = \sum_{i=1}^k g_i D_i \in \mathcal{I}(U)T_X(U) = T_X(U)$. Thus, $\pi(U)(\nabla^X_{D''}D'') = 0$ and since we have $\nabla^X_{D''}D'' = \pi(U)(\nabla^Y_{D''}D'') = \pi(U)(\nabla^Y_{D_2}D') = \pi(U)(\nabla^Y_{D'}D')$. Hence, the induced connection $\nabla$ on $Y$ is well defined.

Symmetry:

$$\nabla^Y_{D'}(\bar{D}) - \nabla^Y_{\bar{D}}(D') = \pi(U)(\nabla^X_{D'}D' - \nabla^X_{\bar{D}}D)$$

$$= \pi(U)(\{D, D'\}_U)$$

$$= \pi(U)(\{D, D'\}_U)$$

$$= \{\bar{D}, \bar{D}'\}_Y \cap U.$$

Compatibility:

$$\nabla^Y_{D'}(\bar{D}) = \pi(U)(\nabla^X_{D'}D')$$

Note: This is a global counterpart of a result that appeared in [PVdV16, Pes17].

Corollary 2.17. If an analytic subspace $Y$ of a Hermitian manifold $X$ satisfies condition i.e. Remark 2.14, then there exist an induced inner product and a Levi-civita connection on $T_Y$ (generalizes the theory of Riemannian submanifold).
2.3. Relationship between Atiyah algebroid and logarithmic derivations. Let $Y$ be a nonsingular divisor (submanifold) of a complex manifold $X$ of codimension 1. Thus, there exists an open cover $\{U_{\alpha}\}_{\alpha}$ such that the divisor $Y = V(\mathcal{I})$ (vanishing set of an principal ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$) where $\mathcal{I}(U_{\alpha}) = \langle f_\alpha \rangle$ and $X = N_Y$ (total space of the line bundle, namely the normal bundle of $Y$ in $X$). Let $\mathcal{N}_Y$ be the associated sheaf of sections of the line bundle $N_Y \to Y$ (see [MS22]).

For a (coherent) $\mathcal{O}_X$-module $\mathcal{E}$, the Atiyah algebroid $\mathcal{A}(\mathcal{E})$ consists of the sheaf of first order differential operators on $\mathcal{E}$ [Pym13, Bru17, Sar22], i.e.

$$\mathcal{A}(\mathcal{E}) = \{ D \in \mathfrak{End}_{\mathcal{O}_X}(\mathcal{E}) | D(f) = f D(s) + \sigma_D(f)s \text{ for a unique } \sigma_D \in \mathcal{T}_X, $$

where $f \in \mathcal{O}_X$ and $s \in \mathcal{E} \}$, with the anchor map defined by

$$a : \mathcal{A}(\mathcal{E}) \to \mathcal{T}_X \text{ where } D \mapsto \sigma_D$$

and the Lie bracket is the commutator bracket (forms a Lie algebroid). Hence, we get the short exact sequence (s.e.s.) of Lie algebroids over $(Y, \mathcal{O}_Y)$ as follows

$$0 \to \mathfrak{End}_{\mathcal{O}_Y}(\mathcal{N}_Y) \to \mathcal{A}(\mathcal{N}_Y) \xrightarrow{\delta} \mathcal{T}_Y \to 0.$$ 

Here $\mathcal{I} \subset \mathcal{O}_X$ is a locally free $\mathcal{O}_X$-module of rank 1. Thus, we have the isomorphism

$$(5) \quad \mathcal{A}(\mathcal{N}_Y) \cong \mathfrak{End}_{\mathcal{O}_Y}(\mathcal{N}_Y) \oplus \mathcal{T}_Y$$

as $\mathcal{O}_Y$-modules, see [Tor17] for related details.

More generally, if $\mathcal{I}$ is a principal ideal sheaf but not necessarily locally free then $Y$ turns out to be a hypersurface with singularities (also it is known as a principal divisor). By considering a Hermitian inner product $\langle \cdot, \cdot \rangle : \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{T}_X \to \mathcal{C}^\infty$ on $\mathcal{T}_X$, we get the $\mathcal{O}_X$-module

$$(6) \quad \mathcal{N}_{Y/X} := \{ D \in \mathcal{T}_X | \langle \mathcal{T}_X(-\log Y), D \rangle \in \mathcal{I} \}.$$ 

It is called the sheaf of vector fields on $X$ (or derivations of $\mathcal{O}_X$) that are normal to $Y$ and the restriction of $\mathcal{N}_{Y/X}$ on $Y$ is the sheaf $\mathcal{N}_Y$.

Here, we consider an analogues result of the above mentioned isomorphism in the context of analytic spaces.

**Theorem 2.18.** For a (principal) free divisor $Y$ of a Hermitian manifold $X$ (i.e. $\mathcal{T}_X(-\log Y)$ is locally free $\mathcal{O}_X$-module) we get the canonical isomorphism

$$\mathcal{A}(\mathcal{N}_{Y/X}) \cong \mathcal{T}_X(-\log Y) \oplus \mathcal{I}.$$ 

**Proof.** Since $Y$ is a principal divisor, associated to the open cover $\{U_{\alpha}\}_{\alpha}$ of $X$ (as mentioned in the above case) we get

$$\mathcal{N}_{Y/X}|_{U_{\alpha}} = \langle \nabla f_{\alpha} \rangle,$$

where $\nabla f_{\alpha} = \sum_{i=1}^n \partial_{x_i}(f_{\alpha}) \partial_{x_i}$, for all $\alpha \in I$. Thus, $\nabla f_{\alpha}$ is determined by $\{ \partial_{x_i}(f_{\alpha}) \}_{i=1}^n$, fully depends on the $f_{\alpha}$. An element of $\mathcal{N}_{Y/X}(U_{\alpha})$ depends on an element of $\mathcal{I}(U_{\alpha})$ and we get an isomorphism between $\mathcal{I}$ and $\mathcal{N}_{Y/X}$ (locally defined as $gf_{\alpha} \mapsto g\nabla f_{\alpha}$ for any coefficient $g \in \mathcal{O}_X(U_{\alpha})$) of $\mathcal{O}_X$-modules. Thus,

$$\mathcal{A}(\mathcal{N}_{Y/X}) \cong \mathcal{A}(\mathcal{I}).$$

Thus, we find local sections $D \in \mathfrak{End}_{\mathcal{O}_X}(\mathcal{I}(U_{\alpha}))$ satisfying the property

$$D(g f_{\alpha}) = g D(f_{\alpha}) + \sigma_D(g) f_{\alpha},$$

for $g \in \mathcal{O}_X(U_{\alpha})$. Our claim is $D \in \mathcal{A}(\mathcal{I})$ if and only if $D \in \mathcal{T}_X(-\log Y)(U_{\alpha})$ or $D \in \mathcal{I}(U_{\alpha})$. For any $D \in \mathcal{T}_X(-\log Y)(U_{\alpha})$ we get

$$D(g f_{\alpha}) = g D(f_{\alpha}) + D(g) f_{\alpha} \in \mathcal{I}(U_{\alpha}),$$

and for any $h \in \mathcal{I}(U_{\alpha})$ (note that $\mathfrak{End}_{\mathcal{O}_X(U_{\alpha})}(\mathcal{I}(U_{\alpha})) \cong \mathcal{I}(U_{\alpha})$ canonically) we get

$$h(g f_{\alpha}) = g h f_{\alpha} + 0.$$ 

Therefore, in the global case, we get the above required isomorphism. \qed
Corollary 2.19. In general, we have the s.e.s. of $\mathcal{O}_X$-modules
\[ 0 \to \mathcal{I} \to \mathcal{A}(\mathcal{N}_{Y/X}) \xrightarrow{\sigma} \mathcal{T}_X(-\log Y) \to 0, \]
for any principal divisor $Y := V(\mathcal{I})$ of a Hermitian manifold $X$.

Remark 2.20. We have two interrelated short exact sequences
\begin{enumerate}
  \item $0 \to \mathcal{T}_X(-\log Y) \to \mathcal{T}_X \xrightarrow{\iota} \mathcal{N}_{Y/X} \to 0$ (of coherent $\mathcal{O}_X$-modules), and
  \item $0 \to \mathcal{T}_X^0 \to \mathcal{T}_X(-\log Y) \xrightarrow{\tilde{\iota}} \mathcal{T}_Y \to 0$ (of Lie algebroids over analytic spaces).
\end{enumerate}
Here $D^\perp := \{D, \nabla f\} \nabla f$, for any $D \in \mathcal{T}_X(U_a)$, $[D] := \tilde{\rho}(D')$ is the equivalence class given by the canonical map associated with an element $D' \in \mathcal{T}_X(-\log Y)(U_a)$ and $\mathcal{T}_X^0 = IT_X$ is the space of vector fields that vanishes on $Y$ (or derivations that sends $\mathcal{O}_X$ to $\mathcal{I}$). The underlying algebraic concepts can be found in Section 2.7.

Note: As in the case of complex Riemannian geometry (discussed in Section 2.1), consider the normal crossing divisor $Y = V(\langle xy \rangle)$ in the complex manifold $X = \mathbb{C}^2$ with standard holomorphic Riemannian metric. Then, we have $\mathcal{T}_X(-\log Y) = \langle \{x\partial_x, y\partial_y\} \rangle$ and $\mathcal{N}_{Y/X} = \langle \{y\partial_x + x\partial_y\} \rangle$. This is different from the usual Hermitian metric. If we take the restrictions of these sheaves on the divisor $Y$ (that is $\mathcal{T}_Y$ and $\mathcal{N}_Y$ respectively) and consider their stalks then both are of rank 1 at $Y \setminus \{(0,0)\}$. The ranks changes at the singular point $0 := (0,0)$ where $\mathcal{T}_{Y,0}$ is an $\mathcal{O}_{Y,0}$-module of rank 2 and $\mathcal{N}_{Y,0}$ is an $\mathcal{O}_{Y,0}$-module of rank 0.

3. HERMITIAN METRICS ON LIE ALGEBROIDS

For a Riemannian manifold we have a symmetric, positive definite bilinear form on its tangent bundle (or inner product on the space of vector fields). In the paper [Bou11], it is generalized from tangent bundle to Lie algebroids over smooth manifolds, named as Riemannian metrics on Lie algebroids. Here we consider its counter-part in complex geometry context.

Definition 3.1. Let $(L, a, \langle \cdot, \cdot \rangle)$ be a holomorphic Lie algebroid over a complex manifold $X$. It is said to be a Hermitian Lie algebroid if there exist a Hermitian metric on the underlying holomorphic vector bundle $L$.

More generally, for a Lie algebroid $(L, a, \langle \cdot, \cdot \rangle)$ over a complex manifold $(X, \mathcal{O}_X)$, a Hermitian inner product on $L$ is a conjugate-symmetric, positive non-degenerate sesquilinear form (or Hermitian form) given by a homomorphism of sheaf of $\mathcal{O}_X^\infty$-modules (by extending the notations defined in Section 2.2)
\[ \langle \cdot, \cdot \rangle : L \otimes_{\mathcal{O}_X} \overline{\mathcal{L}} \to C_X^\infty, \]
where $\overline{\mathcal{L}}$ is the conjugation of the $\mathcal{O}_X$-module $\mathcal{L}$. Then $\mathcal{L}$ is said to be a Hermitian Lie algebroid over $(X, \mathcal{O}_X)$.

Example 3.2. (1) Let $X$ be a complex manifold. Then the tangent sheaf $\mathcal{T}_X$ of $(X, \mathcal{O}_X)$ induces a Hermitian Lie algebroid structure (see Section 2.2).

(2) Also, we can consider Hermitian metric on the tangent sheaf $\mathcal{L} = \mathcal{T}_Y$ of an analytic subspace $(Y, \mathcal{O}_Y)$ if it satisfies condition (3), i.e. if Remark 2.13 holds.

Remark 3.3. Smooth and holomorphic Lie algebroids are viewed as locally free Lie algebroids of finite rank in the generalized context [MS22].

A Hermitian metric $\langle \cdot, \cdot \rangle$ on a holomorphic Lie algebroid $a : L \to TX$ can be viewed as pointwise $\mathbb{C}$-sesquilinear conjugate-symmetric, positive non-degenerate maps $\langle \cdot, \cdot \rangle_x$ varies smoothly on $X$ and the metric $\langle \cdot, \cdot \rangle : X \to (L \otimes \overline{L})^\ast$ is given by the smooth map $x \mapsto \langle \cdot, \cdot \rangle_x$. This is an analogue of the definition given in [Bou11] for Riemannian metrics on Lie algebroids in the complex geometry context.

Definition 3.4. A singular (regular) foliation $\mathcal{F}$ on a complex manifold $X$ is a coherent (locally free of finite rank) $(\mathbb{C}_X, \mathcal{O}_X)$-Lie-Rinehart subalgebra of $\mathfrak{X}_X$ or $\mathcal{T}_X$. It provides a generalized involutive (analytic) distribution on $X$ and vice-versa.
In otherwords, a singular (regular) foliation $F$ on a complex manifold $X$ is a subsheaf of $\mathcal{O}_X$-modules consists with holomorphic vector fields which are (a) closed under the $\mathbb{C}_X$-Lie bracket (i.e., an involutive subsheaf of $\mathfrak{X}_X$ or $\mathcal{T}_X$), (b) locally finitely generated (free) over $\mathcal{O}_X$ (i.e., a coherent $\mathcal{O}_X$-module).

**Remark 3.5.** A regular foliation is the sheaf of sections of a holomorphic Lie subalgebroid of the Lie algebroid $TX$ of a complex manifold $X$.

**Example 3.6.** For a Lie algebroid $(\mathcal{L}, a, [, , ])$ over a complex manifold $(X, \mathcal{O}_X)$, the image $F := a(\mathcal{L}) \subset \mathcal{T}_X$ of the anchor map forms a singular foliation.

**Remark 3.7.** In classical differential geometric setup, a foliation over a smooth manifold $X$ is an involutive subbundle $F \to X$ of the tangent bundle $TX \to X$, induces a Frobenious distribution. Its sheaf of sections $\Gamma_F$ provides a regular foliation in the smooth context, as defined above.

**Definition 3.8.** An integral submanifold of a foliation $F$ (in both real smooth or complex analytic cases) is an immersed submanifold $Z \subset X$ (where $X$ is smooth or complex manifold respectively) with the property that for all $p \in Z$,

$$T_pZ = \text{Span}\{D(p) \in T_pX \mid D \in F\}.$$ 

A leaf (or orbit) is a maximal connected integral submanifold.

**Theorem 3.9.** (1) (Stefan-Sussman’s theorem) If $F$ is a singular foliation on a real smooth manifold $X$ then there is a leaf through every point in $X$ (see [Fer02]).

(2) (Nagano’s theorem) If $F$ is a singular analytic distribution (foliation) on a complex manifold $X$ then $X$ is partitioned into leaves (see [Pym13]).

**Note 3.10.** In general, it is useful to consider analytic subspaces $Y \subset X$ that are union of leaves of the foliation $a(\mathcal{L})$ for a Lie algebroid $\mathcal{L}$. These are $\mathcal{L}$-invariant subspaces, appear to study all of the vector fields coming from $\mathcal{L}$ that are tangent to $Y$. More explicitly, if a closed complex analytic subspace $Y := V(\mathcal{I})$ of some ideal sheaf $\mathcal{I}$ satisfies $a(\mathcal{L})(\mathcal{I}) \subset \mathcal{I}$, then $Y$ is called a $\mathcal{L}$-invariant subspace $[Pym13]$.

**Note 3.11.** Frobenius distributions and Stefan-Sussmann distributions are represents integrable regular and singular foliations respectively over a smooth manifold $(X, C^\infty)$. These corresponds with a Lie algebroids over $(X, C^\infty)$, namedly generalized involutive subsheaf of $C^\infty$-modules of the tangent sheaf $\mathcal{T}_X$. In complex geometry its analogue provides a special kind of holomorphic foliation, which forms a Lie algebroid over a complex manifold $(X, \mathcal{O}_X)$.

3.1. **Characteristic foliation with induced inner product.** Let $(\mathcal{L}, [, , ], a)$ be a holomorphic Lie algebroid over a complex manifold $X$ $[Pym13]$. Thus its image of the anchor map $a : L \to TX$ defines a singular foliation of $X$. Consider the sheaf of holomorphic sections $\Gamma_L$ of the map $p : L \to X$, denote it by $\mathcal{L}$. Since the sheaf homomorphism $a$ is an $\mathcal{O}_X$-linear and $\mathbb{C}_X$-Lie algebra homomorphism, for any two sections $D_1, D_2 \in a(\mathcal{L})$ and a section $f \in \mathcal{O}_X$ we have $fD_1 = a(fD_1)$ and $[D_1, D_2] = a([D_1, D_2])$ are sections of $a(\mathcal{L})$ where $D_1, D_2$ are some preimage of $D_1, D_2$ respectively. Thus the image $\mathcal{Fm}(a)$ or $a(\mathcal{L})$ of the anchor map forms a coherent involutive subsheaf of the tangent sheaf $\mathcal{T}_X$ defined by the sheafification of the presheaf $U \mapsto \text{Im}(a(U))$, called characteristic foliation $[Bou11]$ or orbit foliation $[Fer02]$ of $L$. In the complex geometry context, we use Nagano’s theorem on the integrability of singular analytic distributions (Theorem 6.1).

We denote by $L_Z$ the restriction of the holomorphic Lie algebroid $L$ to a leaf $Z$. One can deduce easily that the bracket $[,]$ induces a bracket on the space of holomorphic sections of $p_Z : L_Z \to Z$ where $p : L \to X$ is the projection and $p|_Z := p_Z$, and hence a transitive Lie algebroid structure.
Let \( \langle \cdot, \cdot \rangle \) be a Hermitian metric on the Lie algebroid \( a : L \to TX \). Then for any leaf \( Z \) of the characteristic foliation and for any \( x \in Z \), the \( \mathbb{C} \)-inner product space
\[
L_x = G^0_x \oplus G^T_x,
\]
where \( L_x = p^{-1}(x) \) and \( G^T_x \) is the orthogonal complement to \( G^0_x = \ker(a_x) \) with respect to \( \langle \cdot, \cdot \rangle_x \), where \( a_x \) is the fiberwise map at \( x \). Now the restriction of the map \( a_x \) to \( G^T_x \) is an isomorphism into \( T_xZ \) and hence induces a scalar product on \( T_xZ \), is given by
\[
\langle v, v' \rangle_{T_x Z} := \langle b, b' \rangle_x,
\]
where \( b, b' \in G^T_x \) with \( a_x(b) = v \) and \( a_x(b') = v' \), which varies smoothly on \( Z \). Thus \( \langle \cdot, \cdot \rangle \) induces a Hermitian metric \( \langle \cdot, \cdot \rangle_Z \) on \( Z \). Fix a leaf \( Z \) and consider \( p_Z : L_Z \to Z \).

We have
\[
L_Z = G^0_Z \oplus G^T_Z
\]
and we call the elements of \( \Gamma(G^0_Z) \) vertical sections and the elements of \( \Gamma(G^T_Z) \) horizontal sections. Thus, we get a short exact sequence of Hermitian Lie algebroids
\[
0 \to G^0_Z \to L_Z \to TZ \to 0
\]
is formally identical to a Hermitian submersion.

The characteristic foliation of a holomorphic Lie algebroid \( a : L \to TX \) is associated with a system of first order partial differential equations, provides a family of leaves as integral solutions and forms \( L \)-invariant subspaces (see Note 3.10). For a \( \mathcal{L} \) is \( L \)-invariant subspace \( Y \) in \( X \), the foliation \( a(\mathcal{L}) \) is a Lie subalgebroid of \( T_X(-\log Y) \). Our aim is to show that there is a Hermitian metric on \( Y \) (or Hermitian inner product on \( T_Y \)) induces from the Hermitian metric on the Lie algebroid.

For that first notice that \( \mathcal{L} = \mathcal{Ker}(a) \oplus (\mathcal{Ker}(a))^\perp \) as \( \mathcal{O}_X \)-modules, where the \( \mathcal{O}_X \)-module \( (\mathcal{Ker}(a))^\perp \) is the sheafification of the presheaf
\[
U \mapsto \{ D \in \mathcal{L}(U) \mid \langle \text{Ker}(a(U)), D \rangle = 0 \}.
\]
Thus, for any section \( D \in \mathcal{L} \) (i.e. \( D \in \mathcal{L} \)), we have \( D_1 \in \mathcal{Ker}(a) \) and \( D_2 \in (\mathcal{Ker}(a))^\perp \) with \( D = D_1 + D_2 \) or \( D = (D_1, D_2) \). In this situation, we get \( a(D) = a(D_2) \), implies \( a(\mathcal{L}) = a(\mathcal{Ker}(a))^\perp \).

**Theorem 3.12.** A \( \mathcal{L} \)-invariant subspace \( Y \) of a Hermitian Lie algebroid \( \mathcal{L} \) canonically induces a Hermitian metric on \( Y \) if it satisfies \( a(\mathcal{L}) = T_X(-\log Y) \).

**Proof.** Let \( D, D' \in a(\mathcal{L}) = T_X(-\log Y) \), then we define an inner product on \( a(\mathcal{L}) \) as
\[
\langle D, D' \rangle_{a(\mathcal{L})} := \langle \tilde{D}, \tilde{D}' \rangle
\]
where \( \tilde{D}, \tilde{D}' \in (\mathcal{Ker}(a))^\perp \subset \mathcal{L} \) such that \( a(\tilde{D}) = D \), \( a(\tilde{D}') = D' \) (without loss of generality). To show that the map is well defined, consider \( \tilde{D}, \tilde{D} \in \mathcal{L} \) such that \( a(\tilde{D}) = D = a(\tilde{D}') \) and \( \tilde{D}' \in (\mathcal{Ker}(a))^\perp \) with \( a(\tilde{D}') = D' \). Thus, \( a(\tilde{D} - \tilde{D}') = 0 \), implies \( \tilde{D} - \tilde{D}' \in \mathcal{Ker}(a) \), and hence \( \langle \tilde{D} - \tilde{D}', \tilde{D}' \rangle = 0 \), implies \( \langle \tilde{D}, \tilde{D}' \rangle = \langle \tilde{D}', \tilde{D}' \rangle \).

Thus, the surjective \( \mathbb{C}_X \)-algebra homomorphism \( \rho : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{L} \) and the surjective Lie algebroid homomorphism \( \hat{\rho} : T_X(-\log Y) \to T_Y \) canonically induces a Hermitian inner product on \( T_Y \) as
\[
\langle D, D' \rangle_Y := \rho(\langle \hat{D}, \hat{D}' \rangle),
\]
where \( \hat{D}, \hat{D}' \in T_Y \) such that \( a(\hat{D}) = D \), \( a(\hat{D}') = D' \) (see Theorem 2.11 for details).

**Corollary 3.13.** In the above mentioned situation, we get a short exact sequence of Hermitian Lie algebroids over \( (X, \mathcal{O}_X) \) as
\[
0 \to \mathcal{Ker}(a) \hookrightarrow \mathcal{L} \xrightarrow{\rho} \mathcal{Ker}(a)^\perp \to 0.
\]

**Remark 3.14.** Let \( Y = V(\mathcal{I}) \) be an hypersurface with isolated singularities of \( X \). Consider the stalks \( T_{X, x}(-\log Y) \) of the Lie algebroid \( T_X(-\log Y) \) of sheaf of holomorphic vector fields on \( X \) that are tangent to \( Y \) (or derivations of \( \mathcal{O}_X \) that
preserves the ideal sheaf \( \mathcal{I} \). This Lie algebroid associated with the characteristic foliation and the corresponding stalks change its rank if it is not regular. Thus leaves (or orbits) of this foliation are of different dimensions and forms L-invariant subspace \( Y \) as a singular analytic space.

We extend the notion of covariant connection \( \nabla \) in the context of holomorphic Lie algebroids. Then we define an analogue of Levi-Civita connection for a Hermitian Lie algebroids.

Let \( \mathcal{L} \) be a holomorphic Lie algebroid over a complex manifold \( (X, \mathcal{O}_X) \). Since \( \mathcal{L} \) is a locally free \( \mathcal{O}_X \)-module of some finite rank, say \( n \), there exist an open cover \( \mathcal{U} := \{ U_i \}_{i \in \mathbb{N}} \) of \( X \) such that \( \mathcal{L}|_{U_i} \) is a free \( \mathcal{O}_X|_{U_i} \)-module of rank \( n \), for each \( i \in \mathbb{N} \).

Consider a \( C^\infty \)-module homomorphism

\[
\nabla : \mathcal{L} \to \mathfrak{A}(\mathcal{L})
\]

such that the restriction map \( \nabla|_{U_i} \) is locally determined by an \( \mathcal{O}_X|_{U_i} \)-module homomorphism \( \nabla^{U_i} : \mathcal{L}(U_i) \to \mathfrak{A}(\mathcal{L}(U_i)) \), for each \( i \in \mathbb{N} \), and globalizing by a partition of unity of \( \mathcal{U} \). In this context, we call it as a \( \mathcal{L} \)-connection on \( \mathcal{L} \).

For example, consider the \( \mathcal{L} \)-connection \( \nabla := \sum_i f_i \nabla^{U_i} \), i.e. \( \nabla_D(s) = \sum_i f_i \nabla^{U_i}_D(s) \), for \( D, s \in \mathcal{L}(U_i) \) where \( \{ f_i \}_{i \in \mathbb{N}} \) is a smooth partition of unity for the open cover of \( X \). The map \( \nabla_D^{U_i} \) is defined by \( \nabla_D^{U_i} = \sum_{j=1}^n g_j D^j \) where \( \{ D^1, \ldots, D^n \} \) is a basis of \( \mathcal{L}(U_i) \), for each \( U_i \in \mathcal{U} \).

An important fact about Hermitian metrics on Lie algebroids is the existence of an analogue of the Levi-Civita connection (see [Bon11]). If \( \langle \cdot, \cdot \rangle \) is a Hermitian metric on a Lie algebroid \( L : X \to X \), then the Koszul formula

\[
2\langle \tilde{\nabla}_D D', D'' \rangle = \langle a(D)(D', D'') + a(D')(D, D'') - a(D'')(D, D') \rangle
\]

holds for sections \( D, D', D'' \) in \( \mathcal{L} := \Gamma(L) \), defines a linear \( \mathcal{L} \)-connection on \( \mathcal{L} \) which is characterized by the two properties (on each space of sections) as follows

(i) \( \tilde{\nabla} \) is compatible with the metric: \( \langle a(D)(D', D'') \rangle = \langle \tilde{\nabla}_D D', D'' \rangle + \langle D', \tilde{\nabla}_D D'' \rangle \),

(ii) \( \tilde{\nabla} \) is torsion free or symmetric: \( \tilde{\nabla}_D D' - \tilde{\nabla}_D D = [D, D'] \).

We call \( \tilde{\nabla} \) is a Levi-Civita \( \mathcal{L} \)-connection associated to the Hermitian metric \( \langle \cdot, \cdot \rangle \). If \( \nabla \) be a \( TX \)-connection on \( L \) then there are two obvious linear \( \mathcal{L} \)-connection

\[
\nabla^0_D D' = \nabla_a(D) D' \quad \text{and} \quad \nabla^1_D D' = \nabla_a(D') D + [D, D'],
\]

which are same for the standard case of the Lie algebroid \( TX \) with the Levi-Civita connection.

**Proposition 3.15.** A \( \mathcal{L} \)-invariant subspace \( Y \) of a Hermitian Lie algebroid \( \mathcal{L} \), canonically induces a \( \mathcal{L} \)-Levi-civita connection with the induced metric on \( Y \).

In a system of coordinates \( (x_1, \ldots, x_n) \) over a trivializing neighborhood \( U \) of \( X \) where \( L \) admits a basis of local sections \( \{ \tilde{D}_1, \ldots, \tilde{D}_r \} \) the Levi-Civita connection is determined by the following identity

\[
\tilde{\nabla}_{\tilde{D}_i} \tilde{D}_j = \sum_{k=1}^r \Gamma^k_{ij} \tilde{D}_k,
\]

where \( \Gamma^k_{ij} \) are the Christoffel’s symbols determined by \( a(\tilde{D}_s) = \sum_{i=1}^n b^s_i \partial_{x_i}, [\tilde{D}_s, \tilde{D}_i] = \sum_{u=1}^r c^s_{iu} \tilde{D}_u, g_{ij} = \langle \tilde{D}_i, \tilde{D}_j \rangle \) and the inverse matrix \( (g^{ij}) \) of \( (g_{ij}) \).

**Example 3.16.** We now consider the special case associated with the nilpotent cone \( \mathbb{P}^n_{MS} [Pym13, MS22] \). It is well known that the cotangent bundle over a Poisson manifold has a natural Lie algebroid structure, in the classical set up [Mac05]. The associated sheaf of sections provides a Lie algebroid in the general context.

Let us consider the Poisson manifold \( \mathbb{C}^3 \) with the Poisson algebra structure on \( (\mathcal{O}_{\mathbb{C}^3}, \{ \cdot, \cdot \}) \) induced by the Lie algebra \( \{ \mathfrak{g}_2(\mathbb{C}), \{ \cdot, \cdot \} \} [Pym13, MS22] \). One may describe this nilpotent cone through a generalized involutive distribution generated
Then the associated Hamiltonian vector fields are given by
\[ D_x = 2y\partial_y - 2z\partial_z, \quad D_y = -2y\partial_x + x\partial_z, \quad D_z = 2z\partial_x - x\partial_y. \]
We consider the system of first order homogeneous partial differential equations
\[ D_x(f) = D_y(f) = D_z(f) = 0. \]
Then we find solution of the above system is \( f(x, y, z) = x^2 + 4yz \). It describes a family of level sets \( f^{-1}(c) \), is parametrized by \( c \in \mathbb{C} \). For each nonzero values of \( c \) we get a 2-dimensional submanifold. Now for \( c = 0 \), \( Y := f^{-1}(0) \) is the vanishing set of the ideal sheaf \( \mathcal{I} := \langle x^2 + 4yz \rangle \subset \mathcal{O}_X \), known as a nilpotent cone.

It yields a Lie algebroid structure on the cotangent sheaf \( \Omega^1_{\mathbb{C}^3} \) where the anchor map \( \alpha : \Omega^1_{\mathbb{C}^3} \to T_{\mathbb{C}^3} \) is locally given by \( f dg \to f \{ g, \cdot \} \), and the Lie algebra structure is locally given by \( \langle df, dg \rangle = \Sigma_{a(df)}(dg) - \Sigma_{a(dg)}(df) - d\langle f, g \rangle \) for any \( f, g \in \mathcal{O}_{\mathbb{C}^3}(U) \), where \( U \) is an open set in \( \mathbb{C}^3 \). The sheaf of sections \( \Omega^1_{\mathbb{C}^3} \) of the cotangent bundle \( T^*\mathbb{C}^3 \) (a Lie algebroid in the classical set up), provides the Lie algebroid \( \mathcal{A}_m(\mathfrak{a}) \) in the general set up. It is a generalized involutive subsheaf of the tangent sheaf \( T_{\mathbb{C}^3} \). As a Lie algebroid the \( \Omega^1_{\mathbb{C}^3} \)-invariant subspaces \( \mathcal{P}_{\mathbb{P}^3}[13] \) are the hypersurface singularities, one of them is the nilpotent cone \( Y \) associated with the principal ideal sheaf \( \mathcal{I} = \langle x^2 + 4yz \rangle \). The associated integrable system of first order homogeneous PDE’s for the Hamiltonian vector fields is given as follows
\[ a(dx) = \langle x, \cdot \rangle = D_x, \quad a(dy) = \langle y, \cdot \rangle = D_y, \quad a(dz) = \langle z, \cdot \rangle = D_z, \]
corresponds to the characteristic foliation \( \mathcal{I} = \mathcal{T}_{\mathbb{C}^3}(-\log Y) \).

Consider the standard Hermitian metric (see \cite{PVdV16}) on the Lie algebroid \( \Omega^1_{\mathbb{C}^3} \), given as
\[ \langle dx, dx \rangle = \langle dy, dy \rangle = \langle dz, dz \rangle = 1; \quad \langle dx, dy \rangle = \langle dx, dz \rangle = \langle dy, dz \rangle = 0. \]
This provides an \( \mathcal{O}_{\mathbb{C}^3} \)-bilinear form on the characteristic foliation \( \mathcal{T}_{\mathbb{C}^3}(-\log Y) \), given by
\[ \langle D_x, D_x \rangle = \langle D_y, D_y \rangle = \langle D_z, D_z \rangle = 1; \quad \langle D_x, D_y \rangle = \langle D_x, D_z \rangle = \langle D_y, D_z \rangle = 0. \]
Since \( \mathcal{T}_{\mathbb{C}^3}(-\log Y) \) is the \( \mathcal{O}_{\mathbb{C}^3} \)-module generated by \( D_x, D_y, D_z \), thus any section of it is of the form \( D = fD_x + gD_y + hD_z \) for some \( f, g, h \in \mathcal{O}_{\mathbb{C}^3} \). To show that the induced inner product is positive, non degenerate, notice that \( \langle D, D_x \rangle = \langle D, D_y \rangle = \langle D, D_z \rangle = 0 \) implies \( f = g = h = 0 \), i.e. \( D = 0 \) and \( \langle D, D \rangle = |f|^2 + |g|^2 + |h|^2 = 0 \) implies \( D = 0 \). Hence, \( \mathcal{T}_{\mathbb{C}^3}(-\log Y) \) is a Hermitian Lie algebroid over the complex manifold \( (\mathbb{C}^3, \mathcal{O}_{\mathbb{C}^3}) \) with the induced Hermitian inner product.

Now, we consider the standard inner product \( \langle \cdot, \cdot \rangle_{\mathbb{C}^3} \) of \( \mathcal{T}_{\mathbb{C}^3} \) (described in Remark \[2.7\]) and take its restriction on \( \mathcal{T}_{\mathbb{C}^3}(-\log Y) \), to show that it will differ from the above described inner product. Thus, from the equations \[6.10\] we get the followings:
\[ \langle D_x, D_x \rangle_{\mathbb{C}^3} = 4|y|^2 + |z|^2, \quad \langle D_y, D_y \rangle_{\mathbb{C}^3} = |x|^2 + 4|y|^2, \quad \langle D_z, D_z \rangle_{\mathbb{C}^3} = |x|^2 + 4|z|^2, \]
\[ \langle D_x, D_y \rangle_{\mathbb{C}^3} = -2z\bar{x}, \quad \langle D_x, D_z \rangle_{\mathbb{C}^3} = -2y\bar{x}, \quad \langle D_y, D_z \rangle_{\mathbb{C}^3} = -4y\bar{z}. \]

Therefore, the induced inner product on the characteristic foliation \( \mathcal{I} = \mathcal{T}_{\mathbb{C}^3}(-\log Y) \) differs from the standard inner product on \( \mathcal{T}_{\mathbb{C}^3}(-\log Y) \subset \mathcal{T}_{\mathbb{C}^3} \).

4. The analytic de Rham cohomology for Characteristic Foliation

Here we consider characteristic foliation mentioned in Section \[5.1\] associated with a (holomorphic) Lie algebroid over a Hermitian manifold in the context of complex geometry and study cohomology theory on the induced leaf space, leaves (and some union of leaves). This is our model object for singular analytic foliation on the Hermitian manifold. We consider an analogue of de Rham cohomology of the leaf.
Cohomology of leaves (or orbits).

4.2. some sense, the sheaf of algebras of holomorphic differential forms on the leaf space $X/\mathcal{F}$. Recall that the leaf space $X/\mathcal{F}$ is a topological space obtained by identifying each leaf of $\mathcal{F}$ to a point. The cochain complex of $\mathcal{O}_X$-modules $\Omega^\bullet_{\mathcal{F}}$ is, in some sense, the sheaf of algebras of holomorphic differential forms on the leaf space $X/\mathcal{F}$ and the cohomology ring $H^\bullet_{\mathcal{F}}(X)$ is the analytic de Rham cohomology of $X/\mathcal{F}$.

4.2. Cohomology of leaves (or orbits). Consider the normal sheaf to a foliation $\mathcal{F}$ in the Hermitian Lie algebroid $\mathcal{T}_X$, denote it by $\nu$ and thus $\nu = \mathcal{T}_X/\mathcal{F}$. If $\mathcal{F}$ is regular then there is a canonical connection $\nabla$ on $\nu$ which is flat along $\mathcal{F}$ (sheafifying the smooth Bott connection $\mathcal{T}_X$ and considering its analogue in holomorphic context), i.e. $\nu$ is a $\mathcal{T}_X$-module and the associated cotangent sheaf to the foliation forms a cochain complex of sheaves

$$\Omega^\bullet_{\mathcal{F}}(\nu) = (\wedge_{\mathcal{O}_X}^\bullet (\mathcal{F}^*) \otimes \mathcal{O}_X \nu, \tilde{d})$$

where $\wedge_{\mathcal{O}_X}^\bullet (\mathcal{F}^*) \otimes \mathcal{O}_X \nu$ is the sheafification of the presheaf

$$U \mapsto \text{Hom}_{\mathcal{O}_X(U)}(\wedge_{\mathcal{O}_X(U)}^\bullet \mathcal{F}(U), \nu(U)) = \wedge_{\mathcal{O}_X(U)}^\bullet (\mathcal{F}(U))^* \otimes \mathcal{O}_X(U) \nu(U)$$

(since here $\mathcal{F}$ and $\nu$ both are locally free $\mathcal{O}_X$-module of finite rank) and the differential $\tilde{d}$ is the Chevalley-Eilenberg-de Rham differential $\mathcal{Sar22}$ for the Lie algebroid $\mathcal{F} \subset \mathcal{T}_X$ with coefficient in the $\mathcal{F}$-module $(\nu, \nabla|_{\mathcal{F}})$.

If the characteristic foliation of a holomorphic Lie algebroid $\mathcal{L}$ is of the form $\mathcal{T}_X(-logY)$ for some $\mathcal{L}$-invariant subspace $Y \subset X$, then the cochain complex is the Lie algebroid complex of $\mathcal{T}_X(-logY)$ with coefficients in the $\mathcal{O}_X$-module $\nu = \mathcal{N}_{Y/X}$. Specifically, for a free divisor $Y$ (i.e. $\mathcal{T}_X(-logY)$ is locally free $\mathcal{O}_X$-module), this complex consists of logarithmic differential forms $\Omega^\bullet_{\mathcal{F}}(logY) \otimes \mathcal{O}_X \mathcal{N}_{Y/X}$ (meromorphic forms on $X$ with poles along the divisor $Y$) $\mathcal{Sar22}$.

When $\mathcal{F}$ is a regular foliation, the restriction $\mathcal{F}|_Z$ and $\nu|_Z$ on a leaf $Z$ are sheaf of sections of the tangent bundle and the normal bundle of $Z$ respectively. By extending the analytic de Rham theorem, locally using complex geometric analogue of the case as described in $\mathcal{Hei73}$, we get the following result.
Proposition 4.2. The hypercohomology of the cochain complex of $\mathcal{O}_X$-modules $(\wedge^*_\mathcal{O}_X(T^*_X) \otimes_{\mathcal{O}_X} \nu[Z, d])$ is isomorphic (as graded $\mathbb{C}$-vector spaces) to the singular cohomology $H^*(Z, \mathbb{C}^\nu)$ where $q = \dim \nu$.

5. EQUIVARIANT LIE ALGEBROID COHOMOLOGY

In [EG10], Lie-Rinehart cohomology for quotients of singularities by finite groups (affine algebraic geometry settings) has been studied. We consider an analogue of it for smooth and analytic cases.

**Geometry of Orbit spaces:** Let $X$ be a smooth manifold carrying an action of a Lie group $G$. Then the Lie group action can be expressed as a group homomorphism

$$\phi : G \to \text{Aut}_\mathbb{R}(C^\infty(X))$$

where $\phi_g(f) = g \cdot f$ and $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for all $x \in X$. Then $\phi$ naturally induces a group action of $G$ on $\text{Der}_\mathbb{R}(C^\infty(X))$ (conjugate action), given by

$$g \cdot D = g D g^{-1}$$

for all $g \in G$ and $D \in \text{Der}_\mathbb{R}(C^\infty(X))$.

Thus, the induced Lie algebra homomorphism

$$\phi : \mathfrak{g} \to \text{Der}_\mathbb{R}(C^\infty(X))$$

is the pushforward along the diffeomorphism. See [Mor01] for details.

The hypercohomology of the cochain complex of $\mathcal{O}_X$-modules

$$(\wedge^*_\mathcal{O}_X(T^*_X) \otimes_{\mathcal{O}_X} \nu[Z, d])$$

is isomorphic (as graded $\mathbb{C}$-vector spaces) to the singular cohomology $H^*(Z, \mathbb{C}^\nu)$ where $q = \dim \nu$.

In [EG10], Lie-Rinehart cohomology for quotients of singularities by finite groups (affine algebraic geometry settings) has been studied. We consider an analogue of it for smooth and analytic cases.

**Geometry of Orbit spaces:** Let $X$ be a smooth manifold carrying an action of a Lie group $G$. Then the Lie group action can be expressed as a group homomorphism

$$\phi : G \to \text{Aut}_\mathbb{R}(C^\infty(X))$$

where $\phi_g(f) = g \cdot f$ and $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for all $x \in X$. Then $\phi$ naturally induces a group action of $G$ on $\text{Der}_\mathbb{R}(C^\infty(X))$ (conjugate action), given by

$$g \cdot D = g D g^{-1}$$

for all $g \in G$ and $D \in \text{Der}_\mathbb{R}(C^\infty(X))$.

Thus, the induced Lie algebra homomorphism

$$\phi : \mathfrak{g} \to \text{Der}_\mathbb{R}(C^\infty(X))$$

is the pushforward along the diffeomorphism. See [Mor01] for details.

where $(\psi_t)_*$ is the pushforward along the diffeomorphism. See [Mor01] for details.
Similarly, for a Lie group action of $G$ on $X$, for each point $g \in G$ we have the diffeomorphism $\phi_g : X \to X$, induces an isomorphism $\phi_g : \mathfrak{X}(X) \to \mathfrak{X}(X)$ of $C^\infty(X)$-modules defined by the Lie derivative of a vector field along the flow given by $\phi_g$ is
\[
\phi_g(D_1)(x) = (\phi_g)_*(D_1)_{\phi_g^{-1}(x)} =: g \cdot (D_1)_{\phi_g^{-1}(x)},
\]
where $(\phi_g)_*$ is the pushforward of vector fields appears by the differential map of $\phi_g$.

But when go from the $C^\infty$ (smooth) real geometry to complex geometry and algebraic geometry, this type of information of space of global sections does not work, there we need to deal with all local sections for which we use classical sheaf theoretic language accordingly.

5.1. **Complex Lie group action on an analytic space:** Let $(X = \mathbb{C}^n, \mathcal{O}_X)$ be the standard complex manifold with its sheaf of holomorphic functions and $(Y = V(\mathcal{I}), \mathcal{O}_Y = \mathcal{O}_X/\mathcal{I})$ be an analytic space induced by a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Denote the group of all biholomorphisms on $X$ by $\text{biholo}(X)$ and the group of all homeomorphisms on $Y$ by $\text{homeo}(Y)$.

Let $G$ be a complex Lie group. Consider a holomorphic $G$-action on $X$, i.e. a group homomorphism $\phi : G \to \text{biholo}(X)$. The subspace $Y$ is said to be a $G$-invariant subspace of $X$ if the map $\phi$ provides a group homomorphism $\phi : G \to \text{homeo}(Y)$ defined by $g \mapsto \phi_g := \phi_g|_Y$, for all $g \in G$. We call $(Y, \mathcal{O}_Y)$ a $G$-analytic subspace of $X$ if in addition it satisfies the following criteria.

Let $U$ be an open set of $X$, then $V := Y \cap U$ is an open set of $Y$. The $G$ action on $Y$, provides a $G$ action on $\mathcal{O}_Y$ as
\[
G \times \mathcal{O}_Y(V) \to \mathcal{O}_Y(G \cdot V),
\]
\[
(g, f) \mapsto g \cdot f \text{ where } (g \cdot f)(y) = f(g^{-1} \cdot y),
\]
compatible with restrictions. Note that $G \cdot V = \cup_{g \in G}(g \cdot V)$ is an open set in $Y$, since the homeomorphism $\phi_g : Y \to Y$ produces open sets $g \cdot V = \phi_g(V)$ for every $g \in G$ and arbitrary union of open sets is open. This kind of group action canonically induces an $\mathcal{O}_Y - G$ module structure on $\mathcal{D}_{\mathbb{C}_Y}(\mathcal{O}_Y)$ as
\[
G \times \mathcal{D}_{\mathbb{C}_Y}(\mathcal{O}_Y(V)) \to \mathcal{D}_{\mathbb{C}_Y}(\mathcal{O}_Y(G \cdot V)),
\]
\[
(g, D)(f) := g \cdot (D(g^{-1} \cdot f)),
\]
and this action is compatible with the restriction morphism $\text{rest}_{V/W}$, for all $g \in G, D \in \mathcal{D}_{\mathbb{C}_Y}(\mathcal{O}_Y(V))$ and for all open sets $W \subset V \subset Y$ (since for $g \in G, f \in \mathcal{O}_Y(g \cdot V)$ we get $g^{-1} \cdot f \in \mathcal{O}_Y(V)$ defined as $(g^{-1} \cdot f)(g^{-1} \cdot y) = f(y))$. Thus, there is a canonical homomorphism of sheaf of Lie algebras
\[
\tilde{\phi} : \mathfrak{g}_Y \to \mathcal{D}_{\mathbb{C}_Y}(\mathcal{O}_Y)
\]
defined by the exponential map, where $\mathfrak{g}_Y$ and $\mathbb{C}_Y$ are the constant sheaves on $Y$ with stalks $\mathfrak{g}$ (the $\mathbb{C}$-Lie algebra of $G$) and $\mathbb{C}$ respectively.

Also, we get natural $\mathcal{O}_Y - G$ module structure on $\Omega_Y^1$ defined naturally on local sections as $(g \cdot \omega)(D) = g \cdot (\omega(g^{-1} \cdot D))$, for all $g \in G, D \in \mathcal{D}_{\mathbb{C}_Y}(\mathcal{O}_Y(V)), \omega \in \Omega_Y^1(V)$ and for all open set $V \subset Y$.

**Remark 5.1.** *This process works for algebraic varieties instead of analytic spaces, for which we need to replace the standard Euclidean space $\mathbb{C}^n$ by the affine space $\mathbb{A}^n$ and sheaf of holomorphic functions replaced by sheaf of regular functions and biholomorphisms replaced by biregular maps. But, the (equivalent) criteria of the $\mathcal{O}_Y - G$ module structure on $\mathcal{D}_{\mathbb{C}_Y}(\mathcal{O}_Y)$ does not holds here, since the exponential map is analytic but not a polynomial map (or regular map).*

5.2. **Equivariant Lie algebroid cohomology:** For a real $C^\infty$ manifold $X$ with a Lie group $G$ action, the cohomology of the homotopy quotient or the homotopy orbit space $EG \times_G X$ (i.e. the orbit space $(EG \times X)/G$ of the free $G$-action), is known as
the equivariant de Rham cohomology. It is given by the cohomology of the cochain complex formed by equivariant differential forms \( \Omega^*_G(X) := (\text{Sym}(\mathfrak{g}^*) \otimes_R \Omega^*(X))^G \) (i.e., a \( G \)-invariant element of \( \text{Sym}(\mathfrak{g}^*) \otimes_R \Omega^*(X) \)) together with the equivariant exterior derivative \( d_g : \Omega^*_G(X) \to \Omega^{*+1}_G(X) \) given as follows
\[
d_g(\alpha(D)) = d(\alpha(D)) - i_D(\alpha(D)),
\]
where the equivariant differential form \( \alpha \) is viewed as the polynomial map \( \alpha : \mathfrak{g} \to \Omega^*(X) \) defined by \( \alpha(\text{Ad}(g) \ D) = g \alpha(D) \), here \( \text{Ad} \) is the adjoint map, \( d \) is the de Rham differential and \( i_D \) is the contraction operator with respect to the fundamental vector field \( D \in \mathfrak{X}(X) \) associated to \( D \in \mathfrak{g} \). Suppose \( G \) is a compact, connected Lie group, and \( X \) is a \( G \)-manifold. Then there is a canonical isomorphism,
\[
H^*_G(X; \mathbb{R}) := H^*(EG \times_G X; \mathbb{R}) \cong H^*(\Omega^*_G(X)).
\]
A \( G \)-manifold \( X \) is said to be equivariantly formal if
\[
H^*_G(X; \mathbb{R}) \cong (\text{Sym}(\mathfrak{g}^*))^G \otimes_R H^*_\text{dR}(X),
\]
as \( (\text{Sym}(\mathfrak{g}^*))^G \)-module, where \( \text{Sym}(\mathfrak{g}^*) \) is the symmetric algebra of the dual Lie algebra \( \mathfrak{g}^* \) of the Lie group \( G \) and \( H^*_\text{dR}(X) \) is the de Rham cohomology ring of the smooth manifold \( X \).

The above results follows from the following facts. By the Milnor’s construction of the universal \( G \)-bundle \( G \to EG \to BG := EG/G \), the product space \( EG \times X \) is homotopy equivalent to \( X \) and the diagonal \( G \)-action on \( EG \times X \) is a free action. Consider the orbit space \( EG \times_G X := (EG \times X)/G \) of the free \( G \)-action as the homotopy quotient. Note that the orbit space \( X/G \) is homotopy equivalent to the homotopy quotient \( EG \times_G X \) if the action of \( G \) on \( X \) is a free action. Considering the associated \( X \)-bundle on \( BG \), the Borel’s fibration \( X \to EG \times_G X \to BG \), and using the associated spectral sequence, we get the above results.

A Lie group \( G \)-action on a smooth Lie algebroid \( a : L \to TX \) is given by a Lie algebra homomorphism \( \rho : \mathfrak{g} \to \Gamma(L) \) such that the map \( \Gamma(a) \circ \rho : \mathfrak{g} \to \mathfrak{X}(X) \) provides a \( G \)-action on \( X \). Consider the graded vector space
\[
\mathcal{A}^* = \text{Sym}^*(\mathfrak{g}^*) \otimes \Gamma(\wedge^* L^*),
\]
with the equivariant differential \( d_g : \mathcal{A}^* \to \mathcal{A}^{*+1} \) by setting
\[
d_g(\mathcal{P} \otimes \omega)(\xi) = \mathcal{P}(\xi)(d_L(\omega) - i_{\rho(\xi)}(\omega)),
\]
where \( \mathcal{P} \in \text{Sym}^*(\mathfrak{g}^*) \) and \( \omega \in \Gamma(\wedge^* L^*) \), \( d_L \) is the Lie algebroid differential, \( i_{\rho(\xi)} \) is the contraction operator with respect to \( \rho(\xi) \) and both sides have evaluated on an element \( \xi \in \mathfrak{g} \). Then the cohomology of the chain complex \( (\mathcal{A}^*_g, d_g) \) where \( \mathcal{A}^*_g := \text{Ker} \ d_g^2 \) is denoted by \( H^*_\text{co}(L) \) and called the equivariant cohomology of the pair \((L, \rho)\) (see [BCRR09] for details).

In the following part we extend the above ideas in the complex algebrao-geometric settings to consider equivariant logarithmic de Rham cohomology.

For a \( G \)-analytic space \( Y \) in \( X \), we can canonically induce the cochain complex of sheaves of logarithmic differential forms \([\text{MS22}, \text{Sr22}]\)
\[
(\Omega^*_{\text{X}}(\text{log}Y) \otimes_{\mathcal{O}_X} \mathcal{N}_{Y/X}, d)
\]
(with poles along the divisor \( Y \) and coefficient in the normal sheaf \( \mathcal{N}_{Y/X} \) of \( Y \) in \( X \)). If \( Y \) is a (complex) submanifold, then we show this cochain complex of sheaves is \( G \)-equivariant, i.e. \( d \circ (g \cdot) = (g \cdot) \circ d \) for each \( g \in G \). In [EG10], it has been shown that for a module \((M, \nabla)\) over the Lie-Rinehart algebra \( \text{Der}_G(R) \), if the connection \( \nabla \) is \( G \)-invariant, i.e. \( g \cdot \nabla = \nabla \) for all \( g \in G \) holds then \( G \) acts on the Lie-Rinehart cohomology ring \( H^*(\text{Der}_G(R), M) \), i.e. \( g \cdot d = d \cdot g \) for any \( g \in G \). Here, we consider the Lie-Rinehart algebra \( \mathfrak{X}^T \) (usually denoted by \( \text{Der}_G((-\text{log}L)) \)) as a local model and replace a module \((M, \nabla)\) by space of sections of a vector bundle \( E \) with a covariant connection \( \nabla \). The \( G \)-action for the covariant connection \( \nabla \) on
$E$ is defined as

$$(g \cdot \nabla)_D(s) = g \cdot \nabla_{g^{-1} \cdot D}(g^{-1} \cdot s)$$

for any $g \in G$ and sections $D \in T_X(U)$ and $s \in \Gamma(E|_U)$, $U$ is an open set in $X$.

**Theorem 5.2.** The standard covariant connection on a (smooth or holomorphic) vector bundle over a (real smooth or complex analytic) $G$-manifold is $G$-invariant.

**Proof.** Let $(X, \mathcal{O}_X)$ be a smooth manifold or complex manifold with a Lie group $G$-action and $E \to X$ be a vector bundle of rank $r$. Then the sheaf of sections $\Gamma(E)$ is a locally free $\mathcal{O}_X$-module of rank $r$. Thus there is a local basis $\{s_1, \ldots, s_r\}$ of $\Gamma(E)$ or $\Gamma(E|_U) = \{\{s_1, \ldots, s_r\}\}$ for some open set $U \subset X$ where $E|_U \to U$ is a trivial bundle. Then the standard covariant connection locally defined as

$$\nabla_D(\sum_{i=1}^r f_is_i) = \sum_{i=1}^r D(f_i)s_i$$

where $D \in T_X(U)$ and $f_i \in \mathcal{O}_X(U)$, $i = 1, \ldots, r$. For any $g \in G$, we need to show $(g \cdot \nabla)_D(s) = \nabla_D(s)$ holds, for all $D \in T_X(U)$ and $s \in \Gamma(E|_U)$. Note that there exist $f_1, \ldots, f_r$ in $\mathcal{O}_X(U)$ such that $s = \sum_{i=1}^r f_is_i$.

Now, $g^{-1} \cdot (\sum_{i=1}^r f_is_i) = \sum_{i=1}^r (g^{-1} \cdot f_i) \cdot (g^{-1} \cdot s_i)$. Thus, we get

$$(g \cdot \nabla)_D(s) = g \cdot \nabla_{g^{-1} \cdot D}(\sum_{i=1}^r (g^{-1} \cdot f_i) \cdot (g^{-1} \cdot s_i))$$

$$= g \cdot (\sum_{i=1}^r ((g^{-1} \cdot D)(g^{-1} \cdot f_i)) \cdot (g^{-1} \cdot s_i))$$

$$= g \cdot (\sum_{i=1}^r (g^{-1} \cdot D(g^{-1} \cdot f_i)) \cdot (g^{-1} \cdot s_i))$$

$$= g \cdot (\sum_{i=1}^r (g^{-1} \cdot D(f_i)) \cdot (g^{-1} \cdot s_i))$$

$$= g \cdot (\sum_{i=1}^r g^{-1} \cdot (D(f_i) \cdot s_i))$$

$$= \sum_{i=1}^r D(f_i) \cdot s_i = \nabla_D(\sum_{i=1}^r f_is_i) = \nabla_D(s).$$

\[\square\]

**Corollary 5.3.** If a free divisor $Y$ is a $G$-analytic subspace of $X$, then the locally free $\mathcal{O}_X$-modules $T_X(-logY)$ and $N_{Y/X}$ have canonical $G$-invariant connections.

**Corollary 5.4.** The Lie algebroid cohomology $\mathbb{H}^\bullet(X, \Omega^\bullet_X(-logY) \otimes \mathcal{O}_X N_{Y/X})$ for a free divisor $Y$ with a $G$-analytic space structure, is $G$-equivariant.

By using concepts of equivariant Lie algebroid cohomology [BCRR09] and holomorphic Lie algebroid cohomology [BMRTT], we define the equivariant Lie algebroid complex for the Lie algebroid $T_X(-logY)$ as

$$(Sym(g_X) \otimes \mathcal{O}_X \Omega^\bullet_X(-logY), \tilde{d})$$

call it by equivariant logarithmic de Rham complex and then consider its hypercohomology.

**Acknowledgment.** I wholeheartedly express my deepest gratitude to my PhD thesis supervisor Dr. Ashis Mandal for his immense support and guidance. Also, I would like to thanks Prof. Mainak Poddar for some of his valuable comments.
References

[Bou11] Mohamed Boucetta, Riemannian geometry of Lie algebroids, J. Egyptian Math. Soc. 19 (2011), no. 1-2, 57–70.

[Bru17] Ugo Bruzzo, Lie algebroid cohomology as a derived functor, J. Algebra 483 (2017), 245–261.

[BCRR09] U. Bruststo, L. Chirio, P. Rossi, and V. N. Rubtsov, Equivariant cohomology and localization for Lie algebroids, Funktsional. Anal. i Prilozhen. 43 (2009), no. 1, 22–36.

[BMRT15] Ugo Bruzzo, Igor Mencattini, Vladimir N. Rubtsov, and Pietro Tortella, Nonabelian holomorphic Lie algebroid extensions, Internat. J. Math. 26 (2015), no. 5, 1550040, 26.

[DZ09] Sorin Dumitrescu and Abdelghani Zeghib, Global rigidity of holomorphic Riemannian metrics on compact complex 3-manifolds, Math. Ann. 345 (2009), no. 1, 53–81.

[EG10] Eirvind Eriksen and Trond Stolen Gustavsen, Equivariant Lie-Rinehart cohomology, Proc. Est. Acad. Sci. 59 (2010), no. 4, 294–300. MR 2752971

[Fer02] Rui Loja Fernandes, Lie algebroids, holonomy and characteristic classes, Adv. Math. 170 (2002), no. 1, 119–179.

[GH78] Phillip Griffiths and Joseph Harris, Principles of algebraic geometry, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978.

[Hei73] James L. Heitsch, A cohomology for foliated manifolds, Bull. Amer. Math. Soc. 79 (1973), no. 6, 1283–1285 (1974).

[Kap07] Mikhail Kapranov, Free Lie algebroids and the space of paths, Selecta Math. (N.S.) 13 (2007), no. 2, 277–319.

[LeB83] Claude LeBrun, Spaces of complex null geodesics in complex-Riemannian geometry, Trans. Amer. Math. Soc. 278 (1983), no. 1, 209–231.

[Mac05] Kirill C. H. Mackenzie, General theory of Lie groupoids and Lie algebroids, London Mathematical Society Lecture Note Series, vol. 213, Cambridge University Press, Cambridge, 2005.

[Mor01] Shigeyuki Morita, Geometry of differential forms, Translations of Mathematical Monographs, vol. 201, American Mathematical Society, Providence, RI, 2001, Translated from the two-volume Japanese original (1997, 1998) by Teruko Nagase and Katsumi Nomizu, Iwanami Series in Modern Mathematics.

[MS22] A. Mandal and A. Sarkar, On Lie algebroid over algebraic spaces, Communications in Algebra (2022). DOI: 10.1080/00927872.2022.2139971

[Pes17] Victor Pessers, Extensions of submanifold theory to non-real settings, with applications, [arXiv:1801.00371v1 [math.DG]] (2017).

[PVdV16a] Victor Pessers and Joeri Van der Veken, On holomorphic Riemannian geometry and submanifolds of Wick-related spaces, J. Geom. Phys. 104 (2016), 163–174.

[PVdV16b] Victor Pessers and Joeri Van der Veken, Riemannian manifolds as Lie-Rinehart algebras, Int. J. Geom. Methods Mod. Phys. 13 (2016), no. suppl., 1641003, 23.

[JV21] Joel Vilatero, On sheaves of Lie-Rinehart algebras, [arXiv:2010.15463v2 [math.DG]]

[Pym13] Brent Pym, Poisson Structures and Lie Algebroids in Complex Geometry, ProQuest LLC, Ann Arbor, MI, 2013, Thesis (Ph.D.)–University of Toronto (Canada).

[Sar22] A. Sarkar, Cohomology of Lie algebroid over algebraic spaces, [arXiv:2111.01738v5 [math.DG]] (2022).

[Sch74] Gerald W. Schwarz, On the de Rham cohomology of the leaf space of a foliation, Topology 13 (1974), 185–187.

[Sch19] T. Schedler, Deformation of algebras in noncommutative geometry., [arXiv:1212.0914v3 [math.RA]] (2019).

[Swa62] Richard G. Swan, Vector bundles and projective modules, Trans. Amer. Math. Soc. 105 (1962), 264–277.

[Tor17] Pietro Tortella, Representations of Atiyah algebroids and logarithmic connections, Int. Math. Res. Not. IMRN (2017), no. 1, 29–46.

[Wei13] Charles A. Weibel, The K-book, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013, An introduction to algebraic K-theory.

Abhishek Sarkar
Department of Mathematics,
Indian Institute of Science Education and Research Pune, India.
e-mail: abhisheksarkar49@gmail.ac.in