Representation dimension of \( m \)-replicated algebras*

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Dedicated to Professor Shaoxue Liu on the occasion of his eightieth birthday

Abstract \ Let \( A \) be a finite dimensional hereditary algebra over an algebraically closed field and \( A^{(m)} \) the \( m \)-replicated algebra of \( A \). We prove that the representation dimension of \( A^{(m)} \) is at most three, and that the dominant dimension of \( A^{(m)} \) is at least \( m \).

Key words: \( m \)-replicated algebras, representation dimension, dominant dimension

1 Introduction

Representation dimension of Artin algebras was introduced by M. Auslander in [A] to study the connection of arbitrary Artin algebras with representation finite Artin algebras. It gives a reasonable way to understand how far an Artin algebra is from being representation finite type by measuring the global dimension of all

MSC(2000): 16E10, 16G10

*Supported by the NSF of China (Grant No. 10771112).

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endomorphism rings of modules which are both generators and cogenerators. M. Auslander proved that an Artin algebra $A$ is representation finite if and only if its representation dimension is at most two.

An interesting relationship between the representation dimension and the finitistic dimension conjecture was shown by K. Igusa and G. Todorov [IT], which is, if the representation dimension of an algebra is at most three, then its finitistic dimension is finite. Furthermore, in [ZZ], A.Zhang and S.Zhang proved that if quasi-hereditary algebras have representation dimensions at most three, then the finitistic dimension conjecture holds. Recently, many important classes of algebras, such as tilted algebras, laura algebras, trivial extensions of iterated tilted algebras etc., have been shown to have representation dimensions at most three, see [APT, CP, EHIS, X].

Let $A$ be a hereditary algebra over an algebraically closed field $k$, and $A^{(m)}$ be the $m$-replicated algebra of $A$. This kinds of algebras give a one-to-one correspondence between the basic tilting $A^{(m)}$-modules and the basic tilting objects in $m$-cluster category $C_m(A)$, see [ABST1, ABST2]. This motivates further investigation on tilting modules about this kinds of algebras, see [LLZ,Z1,Z2] for details. In this paper, we prove that the representation dimension of $A^{(m)}$ is at most three, and that the dominant dimension of $A^{(m)}$ is at least $m$.

The following theorems is the main results of this paper.

**Theorem 1.** Let $A^{(m)}$ be the $m$-replicated algebra of a hereditary algebra $A$. Then its representation dimension is at most three.

**Theorem 2.** Let $A^{(m)}$ be the $m$-replicated algebra of a hereditary algebra $A$. Then its dominant dimension is at least $m$.

This paper is arranged as follows. In section 2, we fix the notations and collect necessary definitions and basic facts. Section 3 is devoted to the proof of our main results.
2 Preliminaries

Let $\Lambda$ be a basic connected algebra over an algebraically closed field $k$. We denote by $\text{mod } \Lambda$ the category of all finitely generated right $\Lambda$-modules and by $\text{ind } \Lambda$ a full subcategory of $\text{mod } \Lambda$ containing exactly one representative of each isomorphism class of indecomposable $\Lambda$-modules. The bounded derived category of $\text{mod } \Lambda$ is denoted by $D^b(\text{mod } \Lambda)$ and the shift functor by $[1]$. For a $\Lambda$-module $M$, we denote by $\text{add } M$ the full subcategory of $\text{mod } \Lambda$ whose objects are the direct summands of finite direct sums of copies of $M$ and by $\Omega^{-1}_\Lambda M$ the first cosyzygy which is the cokernel of an injective envelope $M \hookrightarrow I$. The projective dimension of $M$ is denoted by $\text{pd } M$, the global dimension of $\Lambda$ by $\text{gl.dim } \Lambda$ and the Auslander-Reiten translation of $\Lambda$ by $\tau_\Lambda$. $D = \text{Hom}_k(-, k)$ is the standard duality between $\text{mod } \Lambda$ and $\text{mod } \Lambda^{\text{op}}$. For further definitions and facts about $\text{mod } \Lambda$, we refer to [ARS, Rin].

Let $C$ be a full subcategory of $\text{mod } \Lambda$, $C_M \in C$ and $\varphi : C_M \longrightarrow M$ with $M \in \text{mod } \Lambda$. The morphism $\varphi$ is a right $C$-approximation of $M$ if the induced morphism $\text{Hom}(C, C_M) \longrightarrow \text{Hom}(C, M)$ is surjective for any $C \in C$. A minimal right $C$-approximation of $M$ is a right $C$-approximation which is also a right minimal morphism, i.e., its restriction to any nonzero summand is nonzero. The subcategory $C$ is called contravariantly finite if any module $M \in \text{mod } \Lambda$ admits a (minimal) right $C$-approximation. If $C$ is also closed under extensions and $K$ is the kernel of a minimal right $C$-approximation of $M$, then Wakamatsu’s Lemma states that $\text{Ext}^1_C(L, K) = 0$ for all $L \in C$ (see [W]). The notions of (minimal) left $C$-approximation and of covariantly finite subcategory are dually defined. It is well known that add $M$ is both a contravariantly finite subcategory and a covariantly finite subcategory.

Let $M$ and $N$ be two indecomposable $\Lambda$-modules. A path from $M$ to $N$ in $\text{ind } \Lambda$ is a sequence of non-zero morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = N$$

with all $M_i$ in $\text{ind } \Lambda$. Following [Rin], we denote the existence of such a path by $M \preceq N$. We say that $M$ is a predecessor of $N$ (or that $N$ is a successor of $M$).
More generally, if $S_1$ and $S_2$ are two sets of modules, we write $S_1 \leq S_2$ if every module in $S_2$ has a predecessor in $S_1$, every module in $S_1$ has a successor in $S_2$, no module in $S_2$ has a successor in $S_1$ and no module in $S_1$ has a predecessor in $S_2$. The notation $S_1 < S_2$ stands for $S_1 \leq S_2$ and $S_1 \cap S_2 \neq \emptyset$.

Let $\hat{\Lambda}$ be the repetitive algebra of $\Lambda$. Then $\hat{\Lambda}$ is the infinite matrix algebra

$$\hat{\Lambda} = \begin{pmatrix} \cdots & 0 \\ \Lambda_{i-1} & \Lambda_i \\ Q_i & \Lambda_i \\ Q_{i+1} & \Lambda_{i+1} \\ 0 & \cdots \end{pmatrix}$$

where matrices have only finitely many non-zero coefficients, $\Lambda_i = \Lambda$ and $Q_i = _\Lambda D\Lambda_\Lambda$ for all $i \in \mathbb{Z}$, all the remaining coefficients are zero and multiplication is induced from the canonical isomorphisms $\Lambda \otimes_\Lambda D\Lambda \cong _\Lambda D\Lambda_\Lambda \cong D\Lambda \otimes_\Lambda \Lambda$ and the zero morphism $D\Lambda \otimes_\Lambda D\Lambda \to 0$ (see [HW]). From [H], we have the following important connection between $\hat{\Lambda}$ and $\mathcal{D}^b(\text{mod } \Lambda)$.

**Lemma 2.1.** Let $\Lambda$ be of finite global dimension, then the derived category $\mathcal{D}^b(\text{mod } \Lambda)$ is equivalent, as a triangulated category, to the stable module category $\text{mod } \hat{\Lambda}$.

The right repetitive algebra $\Lambda'$ of $\Lambda$, introduced in [ABM], is the quotient of $\hat{\Lambda}$ defined as follows:

$$\Lambda' = \begin{pmatrix} \Lambda_0 & 0 \\ Q_1 & \Lambda_1 \\ Q_2 & \Lambda_2 \\ 0 & \cdots \end{pmatrix}$$

where, as above, $\Lambda_i = \Lambda$ and $Q_i = _\Lambda D\Lambda_\Lambda$ for all $i \geq 0$. We denote by $\Sigma_0$ the set of all non-isomorphic indecomposable projective $\Lambda$-modules and set $\Sigma_k = \Omega^{-k}_{\Lambda'} \Sigma_0$ for $k \geq 0$.

Then, by [AI], the $m$—replicated algebra $\Lambda^{(m)}$ of $\Lambda$ is defined as the quotient of
the right repetitive algebra $\Lambda^\prime$, hence of the repetitive algebra $\hat{\Lambda}$, which is,

$$
\Lambda^{(m)} = \begin{pmatrix}
\Lambda_0 & 0 \\
Q_1 & \Lambda_1 \\
Q_2 & \Lambda_2 \\
& \ddots \\
0 & Q_m & \Lambda_m
\end{pmatrix}.
$$

If $m = 1$, then $\Lambda(1)$ is the duplicated algebra of $\Lambda$ (see [ABST1]). Also from [AI], we have that

$$m + \text{gl.dim } \Lambda \leq \text{gl.dim } \Lambda^{(m)} \leq (m + 1) \text{gl.dim } \Lambda + m.$$

The next lemma is from [ABST2].

**Lemma 2.2.** Let $A$ be hereditary. Then

1. The standard embeddings $\text{ind } A_i \hookrightarrow \text{ind } A^{(m)}$ (where $0 \leq i \leq m$) and $\text{ind } A^{(m)} \hookrightarrow \text{ind } A^\prime$ are full, exact, preserve indecomposable modules, almost split sequences and irreducible morphisms.

2. Let $M$ be an indecomposable $A^\prime$-module which is not projective and $k \geq 1$. Then the followings are equivalent:
   - (a) $\text{pd } M = k$,
   - (b) there exists $N \in \text{ind } A$ such that $M \cong \tau_{A^\prime}^{-1} \Omega_{A^\prime}^{(k-1)} N$,
   - (c) $\Sigma_{k-1} < M \leq \Sigma_k$.

3. Let $M$ be an indecomposable $A^{(m)}$-module which is not in $\text{ind } A = \text{ind } A_0$. Then its projective cover in $\text{mod } A^{(m)}$ is projective-injective and coincides with its projective cover in $\text{mod } A^\prime$.

4. Let $M$ be an $A^{(m)}$-module having all projective-injective indecomposable modules as direct summands. For $A^{(m)}$-module $X$, if $X$ has a projective-injective projective cover, then a minimal right add $M$-approximation of $X$ is an epimorphism.

We refer to [A] for the original definition, and we would rather use the following characterisation, which was proved to be equivalent to the original one in [A].
Definition 2.3. Let \( \Lambda \) be a non-semisimple Artin algebra. The representation dimension \( \text{rep}.\text{dim} \Lambda \) of \( \Lambda \) is the infimum of the global dimensions of the algebras \( \text{End} \ M \), where \( M \) is a generator and a cogenerator of \( \text{mod} \ \Lambda \).

The next lemma in [A, CP, EHIS, X] is well known.

Lemma 2.4. Let \( \Lambda \) be a non-semisimple Artin algebra, \( n \) be a positive integer, and \( M \) be a generator-cogenerator of \( \text{mod} \ \Lambda \). Then \( \text{gl}.\text{dim} \ \text{End} \ M \leq n + 1 \) if and only if for each \( \Lambda \)-module \( X \), there exists an exact sequence

\[
0 \to M_n \to \cdots \to M_1 \to X \to 0
\]

with \( M_i \) in \( \text{add} \ M \) for all \( i \), such that the induced sequence

\[
0 \to \text{Hom}_\Lambda(L, M_n) \to \cdots \to \text{Hom}_\Lambda(L, M_1) \to \text{Hom}_\Lambda(L, X) \to 0
\]

is exact for all \( L \) in \( \text{add} \ M \). In particular, \( \text{rep}.\text{dim} \ \Lambda \leq n + 1 \).

We refer to [ARS] for the following definition.

Definition 2.5. Let \( \Lambda \) be a non-semisimple Artin algebra. The dominant dimension \( \text{dom}.\text{dim} \ \Lambda \) of \( \Lambda \) is the supremum of all \( n \in \mathbb{N} \) having the property that if

\[
0 \to \Lambda \to I_1 \to I_2 \to \cdots \to I_n \to \cdots
\]

is the minimal injective resolution of \( \Lambda \), the \( T_j \) is projective for all \( j \leq n \).

3 The representation dimension of \( A^{(m)} \)

In this section, we will prove our main result.

Let \( \Lambda \) be a, not necessarily hereditary, finite dimensional algebra and \( \hat{\Lambda} \) the repetitive algebra of \( \Lambda \). For \( \hat{\Lambda} \)-modules \( X, Y \), we define

\[
\varphi : \text{Hom}_{\hat{\Lambda}}(Y, \Omega_{\hat{\Lambda}}^{-1}X) \to \text{Ext}^1_{\hat{\Lambda}}(Y, X)
\]
as follows: for $h \in \text{Hom}_{\hat{\Lambda}}(Y, \Omega^{-1}_{\hat{\Lambda}}X)$, we have the following commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \rightarrow & X & \rightarrow & E & \rightarrow & Y & \rightarrow & 0 \\
\| & & \downarrow & & \downarrow h & & \downarrow & \\
0 & \rightarrow & X & \rightarrow & I_X & \rightarrow & \Omega^{-1}_{\hat{\Lambda}}X & \rightarrow & 0,
\end{array}
$$

where the right square is a pullback and $I_X$ is the injective envelope of $X$ and also the projective cover of $\Omega^{-1}_{\hat{\Lambda}}X$. Define $\varphi(h)$ as the short exact sequence

$$
0 \rightarrow X \rightarrow E \rightarrow Y \rightarrow 0.
$$

By the uniqueness of pullback, $\varphi$ is well defined and an epimorphism. The next lemma shows that $\varphi$ can induce an isomorphism between $\text{Hom}_{\hat{\Lambda}}(Y, \Omega^{-1}_{\hat{\Lambda}}X)$ and $\text{Ext}^1_{\hat{\Lambda}}(Y, X)$.

**Lemma 3.1.** Let $h \in \text{Hom}_{\hat{\Lambda}}(Y, \Omega^{-1}_{\hat{\Lambda}}X)$. Then $h$ factors through projective-injective $\hat{\Lambda}$-modules if and only if $\varphi(h)$ is zero in $\text{Ext}^1_{\hat{\Lambda}}(Y, X)$.

**Proof.** If $\varphi(h)$ is zero in $\text{Ext}^1_{\hat{\Lambda}}(Y, X)$, we can write $\varphi(h)$ as

$$
0 \rightarrow X \rightarrow X \oplus Y \xrightarrow{f} Y \rightarrow 0,
$$

where $f$ is a retraction. Then the following commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \rightarrow & X & \rightarrow & X \oplus Y & \xrightarrow{f} & Y & \rightarrow & 0 \\
\| & & \downarrow g & & \downarrow \frac{f}{f'} & & \downarrow h & \\
0 & \rightarrow & X & \rightarrow & I_X & \xrightarrow{\pi} & \Omega^{-1}_{\hat{\Lambda}}X & \rightarrow & 0
\end{array}
$$

yields that $h = \pi gf'$, which implies that $h$ can factor through a projective-injective $\hat{\Lambda}$-module.

Conversely, assume that $h$ can factor through a projective-injective $\hat{\Lambda}$-module. Let $\varphi(h)$ be the short exact sequence

$$
0 \rightarrow X \rightarrow E \rightarrow Y \rightarrow 0.
$$

Therefore we have, by [H], a triangle in the stable module category $\text{mod} \hat{\Lambda}$

$$
X \rightarrow E \rightarrow Y \xrightarrow{h} \Omega^{-1}_{\hat{\Lambda}}X,
$$
where \( h \) is zero in \( \text{Hom}_{\hat{A}}(Y, \Omega^{-1}_A X) \), by our assumption. Therefore this triangle is split, and thus \( X \) and \( Y \) are both direct summands of \( E \), which implies \( \varphi(h) \) is zero. This completes the proof. \( \square \)

Assume from now on that \( A \) is a hereditary algebra, and that \( A' \) is the right repetitive algebra of \( A \) and \( A^{(m)} \) is the \( m \)-replicated algebra.

**Lemma 3.2.** Let \( X \in \text{ind } A \), and \( \alpha \in \text{Hom}_{\hat{A}}(\Omega^{-i}_A A, \Omega^{-j}_A X) \). If \( i < j \), then \( \alpha \) factors through a projective-injective \( \hat{A} \)-module.

**Proof.** The statement follows easily from that

\[
\text{Hom}_{\hat{A}}(\Omega^{-i}_A A, \Omega^{-j}_A X) \\
\cong \text{Hom}_{D^b(\text{mod } A)}(A[i], X[j]) \\
= \text{Hom}_{D^b(\text{mod } A)}(A, X[j - i]) \\
= \text{Ext}^{j-i}_A(A, X) \\
= 0.
\]

\( \square \)

Now let \( \text{gl.dim } A^{(m)} = t \). Recall that \( m + 1 \leq t \leq 2m + 1 \), \( \Sigma_0 \) is the set of all non-isomorphic indecomposable projective \( A \)-modules and that \( \Sigma_k = \Omega^{-k}_A \Sigma_0 \).

By Lemma 2.4 and the Auslander-Reiten quiver of \( A^{(m)} \), \( \Sigma_{t-1} \cap \text{ind } A_m \neq \emptyset \) or \( \tau_{A^{(m)}}^{-1} \Sigma_{t-1} \cap \text{ind } A_m \neq \emptyset \). We denote by \( U_k \) the direct sum of all the indecomposable modules in \( \Sigma_k \cap \text{ind } A^{(m)} \) and by \( P \) the direct sum of all the projective-injective indecomposable \( A^{(m)} \)-modules.

**Theorem 3.3.** Let \( M = A \oplus DA_m \oplus P \oplus \bigoplus_{k=1}^{t-1} U_k \). Then \( \text{gl.dim } \text{End}_{A^{(m)}}(M) \leq 3 \). In particular, \( \text{rep.dim } A^{(m)} \leq 3 \).

**Proof.** By Lemma 2.4, it suffices to find, for each indecomposable \( A^{(m)} \)-module \( X \), a short exact sequence

\[
0 \rightarrow M_2 \rightarrow M_1 \rightarrow X \rightarrow 0
\]

with \( M_1, M_2 \in \text{add } M \), such that the induced sequence

\[
0 \rightarrow \text{Hom}_{A^{(m)}}(L, M_2) \rightarrow \text{Hom}_{A^{(m)}}(L, M_1) \rightarrow \text{Hom}_{A^{(m)}}(L, X) \rightarrow 0
\]
is exact for all $L \in \text{add}\ M$.

It is clear that we can assume that $X$ is not in $\text{add}\ M$. Suppose firstly that $\Sigma_0 < X < \Sigma_1$. Consider the minimal projective resolution

$$0 \longrightarrow P_2 \longrightarrow P_1 \xrightarrow{f} X \longrightarrow 0.$$ 

Since $X \in \text{ind}\ A$ and $\text{Hom}_{A^{(m)}}(DA_m \oplus P \oplus \bigoplus_{k=1}^{t-1} U_k, X) = 0$, it is easy to see that $f$ is a right $\text{add}\ M$-approximation of $X$. Therefore the induced sequence

$$0 \longrightarrow \text{Hom}_{A^{(m)}}(L, P_2) \longrightarrow \text{Hom}_{A^{(m)}}(L, P_1) \longrightarrow \text{Hom}_{A^{(m)}}(L, X) \longrightarrow 0$$

is exact for all $L \in \text{add}\ M$.

Assume now $\Sigma_i < X < \Sigma_{i+1}$, $1 \leq i \leq t-2$. Then $X$ is not in $\text{ind}\ A$ and by Lemma 2.2(2), $X$ has a projective-injective projective cover. It follows from Lemma 2.2(4) that a minimal right $\text{add}\ (U_i \oplus P)$-approximation of $X$ is an epimorphism. Consider the following short exact sequence

$$0 \longrightarrow K \longrightarrow M_1 \xrightarrow{g} X \longrightarrow 0, \quad (*)$$

where $g$ is a minimal right $\text{add}\ (U_i \oplus P)$-approximation of $X$ and $K$ is the kernel of $g$. By Lemma 2.2(2), there is an indecomposable $A$-module $Y$ such that $X \cong \Omega^{-i}_{A^{(m)}} Y$. Then $g$ is also a minimal right $\text{add}\ M$-approximation of $X$ by Lemma 3.2 and the fact that $\text{Hom}_{A^{(m)}}(DA_m \oplus \bigoplus_{i<j}^{t-2} U_j, X) = 0$. Since the short exact sequence $(*)$ is not split, $K$ is not projective-injective. Clearly, $K \leq \Sigma_i$. Let $N$ be a non-projective-injective indecomposable direct summand of $K$ and assume that $\Sigma_i < N < \Sigma_{i+1}$, for some $0 \leq l \leq i - 1$. Then there exists, by Lemma 2.2(2), an indecomposable $A$-module $N'$ such that $N \cong \Omega^{-l}_{A^{(m)}} N'$. It follows from Wakamatsu’s Lemma that $\text{Ext}^1_{A^{(m)}}(\Omega^{-l}_{A^{(m)}} A, K') = 0$. Then we have that

$$0 = \text{Ext}^1_{A^{(m)}}(\Omega^{-l-1}_{A^{(m)}} A, K')$$

$$\cong \text{Ext}^1_{\hat{A}}(\Omega^{-l-1}_{\hat{A}} A, K) \quad (1)$$

$$\cong \text{Hom}_{\hat{A}}(\Omega^{-l-1}_{\hat{A}} A, \Omega^{-1}_{\hat{A}} K) \quad (2)$$

$$\cong \text{Hom}_{\mathcal{D}^{(\text{mod}\ A)}}(A[l + 1], K[1]) \quad (3)$$

$$\cong \text{Hom}_{\mathcal{D}^{(\text{mod}\ A)}}(A, K[-l]) \quad (4)$$
where (1) follows from Lemma 2.2(1), (2) is from Lemma 3.1, (3) follows from Lemma 2.1 and (4) holds because \([1]\) is a selfequivalence of \(\mathcal{D}^b(\text{mod} \, A)\). In particular,

\[
\text{Hom}_{\mathcal{D}^b(\text{mod} \, A)}(A, N[-l]) \cong \text{Hom}_A(A, N') = 0,
\]

and hence \(N' = N = 0\), which implies that \(K \in \text{add}( \oplus_{0 \leq j \leq i} U_j \oplus P)\) and hence \(K \in \text{add} M\). Since \(g\) is a right add \(M\)-approximation of \(X\), the induced sequence

\[
0 \rightarrow \text{Hom}_A(m)(L, K) \rightarrow \text{Hom}_A(m)(L, M_1) \rightarrow \text{Hom}_A(m)(L, X) \rightarrow 0
\]

is exact for all \(L \in \text{add} M\).

Finally, if \(\Sigma_{t-1} < X < DA_m\), by the same argument as above, we consider the short exact sequence

\[
0 \rightarrow K' \rightarrow M'_1 \xrightarrow{h} X \rightarrow 0,
\]

where \(h\) is a minimal right add \((U_{t-1} \oplus P)\)-approximation of \(X\). Then we get that \(K' \in \text{add}( \oplus_{0 \leq j \leq t-1} U_j \oplus P)\) and hence \(K' \in \text{add} M\). So the induced sequence

\[
0 \rightarrow \text{Hom}_A(m)(L, K') \rightarrow \text{Hom}_A(m)(L, M'_1) \rightarrow \text{Hom}_A(m)(L, X) \rightarrow 0
\]

is exact for all \(L \in \text{add} M\). This completes the proof. \(\square\)

**Example 3.4.** We now give an example with the case of \(m = 1\). Let \(Q\) be the quiver \(1 \xleftarrow{1} 2\), and \(Q^{(1)}\) be the quiver \(1 \xleftarrow{2} 2 \xleftarrow{1'} 2'\). Let \(A^{(1)} = kQ^{(1)}/I\) be the duplicated algebra of \(kQ\). The indecomposable projective-injective \(A^{(1)}\)-modules are \(P'_1 = \frac{1'}{22}\) and \(P'_2 = \frac{2'}{112}\), which are represented by their Loewy series.

We take a generator-cogenerator \(M\) of \(\text{mod} \, A^{(1)}\) as the following,

\[
M = 1 \oplus \frac{2}{11} \oplus 2' \oplus \frac{2'}{11} \oplus \frac{1'}{22} \oplus \frac{1'}{2} \oplus \frac{2'}{22} \oplus \frac{1'}{22} \oplus \frac{1'}{2} \oplus \frac{2'}{22} \oplus \frac{2'}{2} \oplus \frac{2'}{2} \oplus \frac{2'}{2} \oplus \frac{2'}{2} \oplus 1' \oplus 1' \oplus 1' \oplus 1' \oplus 1' \oplus 1' \oplus 1' \oplus 1'.
\]

According to Theorem 3.3, we have that \(\text{gl.dim} \text{End}_{A^{(1)}} M = 3\), and that \(\text{rep.dim} \, A^{(1)} = 3\).

If we take another generator-cogenerator \(M_0\) of \(\text{mod} \, A^{(1)}\) as the following,
\[ M_0 = 1 \oplus 2' \oplus 2' \oplus 1' \oplus 2' \oplus 1' \],

then we have that gl.dim\text{End}_{A^{(1)}} M_0 = 5.

**Remark.** The above example shows that the global dimension of endomorphism algebra of a generator-cogenerator can be really bigger than the representation dimension.

**Theorem 3.5** Let \( A^{(m)} \) be the \( m \)-replicated algebra of a hereditary algebra of \( A \). Then the dominant dimension of \( A^{(m)} \) is at least \( m \).

**Proof.** Assume that gl.dim \( A^{(m)} = t \). Then \( m + 1 \leq t \leq 2m + 1 \). For each indecomposable projective \( A \)-module \( P \), we take its minimal injective resolution:

\[
0 \rightarrow P \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots \rightarrow I_{t-2} \rightarrow I_{t-1} \rightarrow \cdots \rightarrow 0.
\]

By the AR-quiver of \( A^{(m)} \), we have \( P, \Omega_{A^{(m)}}^{-1} P, \cdots, \Omega_{A^{(m)}}^{-(t-2)} P \) are not in \text{ind} \( A_m \) and thus \( I_1, I_2, \cdots, I_{t-1} \) are all projective-injective \( A^{(m)} \). Therefore we have a long exact sequence

\[
0 \rightarrow A^{(m)} \rightarrow N_1 \rightarrow \cdots \rightarrow N_{t-1} \rightarrow \cdots \rightarrow 0
\]

with \( N_1, \cdots, N_{t-1} \) projective-injective, which implies dom.dim \( A^{(m)} \geq t - 1 \geq m \). This finishes the proof. \( \square \)

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