SOME REMARKS CONCERNING SYMMETRY-BREAKING FOR THE GINZBURG-LANDAU EQUATION

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Abstract. The correlation term, introduced in [13] to describe the interaction between very far apart vortices, governs symmetry-breaking for the Ginzburg-Landau equation in \( \mathbb{R}^2 \) or bounded domains. It is a homogeneous function of degree \((-2)\), and then for \( \frac{2\pi}{N} \)-symmetric vortex configurations can be expressed in terms of the so-called correlation coefficient. Ovchin- nikov and Sigal [13] have computed it in few cases and conjectured its value to be an integer multiple of \( \frac{\pi}{4} \). We will disprove this conjecture by showing that the correlation coefficient always vanishes, and will discuss some of its consequences.

Keywords: Ginzburg-Landau equation, Symmetry-breaking, correlation term

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1. Introduction

The Ginzburg-Landau theory is a very popular model in superconductivity [6]. Stationary states are described by complex-valued solutions \( u \) of the planar equation

\[-\Delta u = k^2 u (1 - |u|^2),\]

where \( k > 0 \) is the Ginzburg-Landau parameter. The condensate wave function \( u \) describes the superconductive regime in the sample by simply interpreting \( |u|^2 \) as the density of Cooper electrons pairs. The zeroes of \( u \), where the normal state is restored, are called vortices. The parameter \( k \) depends on the physical properties of the material and distinguishes between Type I superconductors \( k < \frac{1}{\sqrt{2}} \) (in this normalization of constants) and Type II superconductors \( k > \frac{1}{\sqrt{2}} \).

In the entire plane \( \mathbb{R}^2 \) the parameter \( k \) does not play any role, as we can reduce to the case \( k = 1 \) by simply changing \( u \) into \( u(\frac{x}{k}) \). Supplemented by the correct asymptotic behavior at infinity, the Ginzburg-Landau equation now reads as

\[
\begin{aligned}
-\Delta U &= U (1 - |U|^2) \quad \text{in } \mathbb{R}^2 \\
|U| &\to 1 \text{ as } |x| \to \infty.
\end{aligned}
\]

The condition \( |U| \to 1 \) as \( |x| \to \infty \) allows to define the (topological) degree \( \deg U \) of \( U \) as the winding number of \( U \) at \( \infty \):

\[
\deg U = \frac{1}{2\pi} \int_{|x|=R} d(\arg U),
\]

where \( R > 0 \) is chosen large so that \( |U| \geq \frac{1}{2} \) in \( \mathbb{R}^2 \setminus B_R(0) \). Given \( n \in \mathbb{Z} \), the only known solution of (1.1) with \( \deg U = n \) is the “radially symmetric” one \( U_n(x) = S_n(|x|)(\frac{x}{|x|})^n \) (in complex notations with \( x \in \mathbb{C} \)), where \( S_n \) is the solution of the following ODE:

\[
\begin{cases}
\frac{\dot{S}_n}{r} + \frac{1}{r} S_n - \frac{n^2}{r^2} S_n + S_n(1 - S_n^2) = 0 & \text{in } (0, +\infty) \\
S_n(0) = 0, & \lim_{r \to +\infty} S_n = 1.
\end{cases}
\]
Existence and uniqueness of $S_n$ is shown in [11]. Moreover, the solution $U_n$ is stable for $|n| \leq 1$ and unstable for $|n| > 1$ [11]. When $n = \pm 1$, the solution $U_{\pm 1}$ is unique, modulo translations and rotations, in the class of functions $U$ with deg $U = \pm 1$ and $\int_{\mathbb{R}^2}(|U|^2 - 1)^2 dx < +\infty$ [10].

One of the open problems (Problem 1) – that Brezis-Merle-Rivi`ere raise out in [3] – concerns the existence of solutions $U$ of (1.1) with conf $U$ = $n$, $|n| > 1$, which are not “radially symmetric” around any point. So far there is no rigorous answer, but a strategy to find them has been proposed in [12]. Formally, a solution $U$ of (1.1) is a critical point of the functional

$$\mathcal{E}(|\Psi|) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \Psi|^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} (|\Psi|^2 - 1)^2 dx.$$ 

Since $\mathcal{E}(|\Psi|) = +\infty$ for any $C^1$–map $\Psi$ so that $|\Psi| \rightarrow 1$ as $|x| \rightarrow +\infty$ and $\text{deg} (\Psi) \neq 0$, Ovchinnikov and Sigal [11] have proposed to correct $\mathcal{E}$ into

$$\mathcal{E}_{\text{ren}}(|\Psi|) = \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla \Psi|^2 - \frac{(\text{deg} \Psi)^2}{|x|^2} \chi + \frac{1}{4} (|\Psi|^2 - 1)^2 \right) dx,$$

where $\chi$ is a smooth cut–off function with $\chi = 0$ when $|x| \leq R$ and $\chi = 1$ when $|x| \geq R + R^2$, and $R >> 1$ is given. Given a vortex configuration $(\underline{a}, \underline{n}) = (a_1, \ldots, a_K, n_1, \ldots, n_K)$, a $C^1$–map $\Psi$ so that $|\Psi| \rightarrow 1$ as $|x| \rightarrow +\infty$ has vortex configuration $(\underline{a}, \underline{n})$ if $a_1, \ldots, a_K$ are the only zeroes of $\Psi$ with local indices $n_1, \ldots, n_K$, denoted for short as conf $\Psi = (\underline{a}, \underline{n})$. Given $n_0$, Ovchinnikov and Sigal [12] introduce the “intervortex energy” $E$ given by

$$E(\underline{a}) = \inf \{ \mathcal{E}_{\text{ren}}(\Psi) : \text{conf } \Psi = (\underline{a}, \underline{n}) \},$$

and conjecture that $\underline{a}_0$ is a critical point of $E$ if and only if there is a minimizer $U$ for $E(\underline{a}_0)$, yielding to a solution of (1.1) with conf $U = (\underline{a}_0, \underline{n}_0)$ which is not “radially symmetric” around any point by construction. Letting $d_{\underline{a}} = \min_{i \neq j} |a_i - a_j|$, the following asymptotic expression is established [12]:

$$E(\underline{a}) = \sum_{j=1}^{K} \mathcal{E}_{\text{ren}}(U_{n_j}) + H(\frac{\underline{a}}{R}) + \text{Rem}$$

(1.2)

with Rem = $O(d_{\underline{a}}^{-2})$ as $d_{\underline{a}} \rightarrow +\infty$, where $H(\underline{a}) = -\pi \sum_{i \neq j} n_i n_j \ln |a_i - a_j|$ is the energy of the vortex pairs interactions. When $\nabla H(\underline{a}) = 0$, the estimate in (1.2) improves up to Rem = $O(d_{\underline{a}}^{-2})$.

When $\nabla H(\underline{a}) = 0$ (a so-called forceless vortex configuration), by choosing refined test functions the asymptotic expression (1.2) is improved [13] in the form of the following upper bound:

$$E(\underline{a}) \leq \sum_{j=1}^{K} \mathcal{E}_{\text{ren}}(U_{n_j}) + H(\frac{\underline{a}}{R}) - A(\underline{a}) + \text{Rem}$$

(1.3)

with Rem = $O(d_{\underline{a}}^{-2} + R^{-2})$ as $d_{\underline{a}} \rightarrow +\infty$, where the correlation term $A(\underline{a})$ is a homogeneous function of degree $(-2)$ given as

$$A(\underline{a}) = \frac{1}{4} \int_{\mathbb{R}^2} \left( \sum_{j=1}^{K} |\nabla \varphi_j|^4 - \sum_{j=1}^{K} |\nabla \varphi_j|^4 \right),$$

with $\varphi_j(x) = n_j \theta(x - a_j), j = 1, \ldots, K,$ and $\theta(x)$ the polar angle of $x \in \mathbb{R}^2$.

To push further the analysis, in [13] the attention is restricted to symmetric vortex configurations in order to reduce the number of independent variables in $E(\underline{a})$. In particular, the simplest $\frac{2\pi}{N}$–symmetric vortex configurations $(\underline{a}, \underline{n})$ (which are invariant under $\frac{2\pi}{N}$–rotations...
and reflections w.r.t. the real axis) have the form: $a_0 = 0$, $a_1, \ldots, a_N$ are the vertices of a regular $N$-polygon with $a_1 = 1$ and $n_1 = \cdots = n_N = m$. We impose also the forceless condition $\nabla H(\theta) = 0$, which simply reads as $n_0 = -\frac{N-1}{2}m$. Since $|a_1| = \cdots = |a_N|$, the only variable is the size $a = |a_1|$ of the polygon, and then the intervortex energy will be in the form $E(a)$. Since $A(\theta)$ is homogeneous of degree $-2$, we have that $A(\theta) = \frac{A_0}{a^2}$, where

$$A_0 := A(1, e^{2\pi i/N}, \ldots, e^{2\pi i(N-1)/N})$$

is the correlation coefficient for given $n_0 = -\frac{N-1}{2}m$ and $n_1 = \cdots = n_N = m$. In [13] the existence of c.p.'s of $E(a)$ is shown for the cases $(N, m) = (2, 2)$ and $(N, m) = (4, 2)$ by comparing $E(a)$ for a small and large, and using the positive sign of $A_0$ (the correlation coefficient has value $8\pi$ and $80\pi$, respectively). It is also conjectured [13] that the correlation coefficient has values which are integer multiples of $\frac{\pi}{2}$. With a long but tricky computation, in the next section we will disprove such a conjecture by showing

**Theorem 1.1.** The correlation coefficient in (1.4) always vanishes: $A_0 = 0$, for all $N \geq 2$ and $m \in \mathbb{Z}$.

Beside the role of $A_0$ in symmetry-breaking phenomena for (1.1) in $\mathbb{R}^2$, as already discussed, we will also explain its connection with the Ginzburg-Landau equation

$$\begin{cases}
-\Delta u = k^2 u (1 - |u|^2) & \text{in } \Omega \\
u = g & \text{on } \partial \Omega
\end{cases}$$

(1.5)

on a bounded domain $\Omega$ for strongly Type II superconductors $k \to +\infty$, where $g : \partial \Omega \to S^1$ is a smooth map.

The energy functional for (1.5)

$$E_k(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{k^2}{4} \int_{\Omega} (1 - |u|^2)^2$$

has always a minimizer $\tilde{u}_k$ in the space $H = \{u \in H^1(\Omega, \mathbb{C}) : u = g$ on $\partial \Omega\}$. When $d = \deg g \neq 0$, by [2][13][16] we know that on simply connected domains $\tilde{u}_k$ has exactly $|d|$ simple zeroes $a_1, \ldots, a_{|d|}$ for $k$ large, where $(a_1, \ldots, a_{|d|})$ is a critical point for a suitable “renormalized energy” $W(a_1, \ldots, a_{|d|})$. The symmetry-breaking phenomenon here takes place, driven by an external mechanism like the boundary condition that forces the confinement of vortices in some equilibrium configuration. A similar result does hold [2] on star-shaped domains for any solutions sequence $u_k$ of (1.5). Near any vortex $a_i$, the function $u(\frac{x}{k} + a_i)$ behaves like $U_{a_i}(x)$.

Once the asymptotic behavior is well understood, a natural question concerns the construction of such solutions for any given c.p. $(a_1, \ldots, a_K)$ of $W$, and a positive answer has been given by a heat-flow approach [8][9], by topological methods [1] and by perturbative methods [5][13] in case $n_1 = \cdots = n_K = \pm 1$. In [14], page 12, it is presented as an open problem to know whether or not there are solutions having vortices collapsing as $k \to \infty$, the simplest situation being problem (1.6) on the unit ball $B$ with boundary value $g_0 = \frac{\pi^2}{|x|^2}$:

$$\begin{cases}
-\Delta u = k^2 u (1 - |u|^2) & \text{in } B \\
u = g_0 & \text{on } \partial B.
\end{cases}$$

It is conjectured the existence of solutions to (1.6) having a vortex of degree $-1$ at the origin $a_0 = 0$ and three vortices of degrees $+1$ at the vertices $ia_j$, $a_j = e^{2\pi i(j-1)}$ for $j = 1, 2, 3$, of a
small \((l << 1)\) equilateral triangle centered at 0. This vortex configuration is \(\frac{2\pi}{3}\)-symmetric, forceless and has “renormalized energy”

\[
W(l) = -6\pi \ln 3 - 6\pi \ln(1 - l^6) + O(l^9), \quad l > 0.
\]  

(1.7)

In collaboration with J. Wei, we were working on this problem. Inspired by [5], we were aiming to use a reduction argument of Lyapunov-Schmidt type, starting from the approximating solutions \(U_k\) for (1.6) given by

\[
U_k(x) = e^{i\varphi_k(x)}U_{-1}(kx) \prod_{j=1}^{3} U_1 \left(k(x - le^{\frac{2\pi i}{3}(j - 1)})\right)
\]

with \(l \to 0\) and \(lk \to +\infty\), where the function \(\varphi_k\) is an harmonic function so that \(U_k|_{\partial B} = g_0\). The interaction due to the collapsing of three vortices onto 0 gives at main order a term \((lk)^{-2}\) with the plus sign, i.e. for some \(J_0 > 0\) there holds the energy expansion

\[
E_k(U_k) = 4\pi \ln k + I + \frac{1}{2}W(l) + J_0(lk)^{-2} + o((lk)^{-2})
\]

\[
= 4\pi \ln k + I - 3\pi \ln 3 + 3\pi l^6 + J_0(lk)^{-2} + o \left(l^6 + (lk)^{-2}\right),
\]  

(1.8)

in view of (1.7). The aim is to construct a solution \(u_k\) in the form \(U_k[\eta(1 + \psi) + (1 - \eta)e^{\psi}]\), where \(\psi = \psi(k)\) is a remainder term small in a weighted \(L^\infty(B)\)-norm and \(l = l(k)\) as \(k \to +\infty\). The function \(\eta\) is a smooth cut-off function with \(\eta = 1\) in \(\bigcup_{j=0}^{3} B_{1/k}(l_a_j)\) and \(\eta \equiv 0\) in \(B \setminus \bigcup_{j=0}^{3} B_{2/k}(l_a_j)\). The function \(\psi = \psi(k)\) is found thanks to the solvability theory (up to a finite-dimensional kernel) of the linearized operator for (1.6) at \(U_k\) as \(l \to 0\) and \(lk \to +\infty\), and by the Lyapunov-Schmidt reduction the existence of \(l(k)\) follows as a c.p. of

\[
\tilde{E}_k := E_k(U_k[\eta(1 + \psi(k)) + (1 - \eta)e^{\psi(k)}]).
\]

If \(U_k\) is sufficiently good as an approximating solution of (1.6), we have that \(E_k = E_k(U_k) + o((lk)^{-2})\). Since \(3\pi l^6 + J_0(lk)^{-2}\) has always a minimum point of order \(k^{-\frac{1}{2}}\) as \(k \to +\infty\), by (1.8) we get the existence of \(l = l(k)\) in view of the persistence of minimum points under small perturbations.

Unfortunately, this is not the case. Pushing further the analysis, we were able to identify the leading term \(\psi_0 = \psi_0(k)\) of \(\psi = \psi(k)\), and compute its contribution into the energy expansion, yielding to a correction in the form:

\[
\tilde{E}_k = 4\pi \ln k + I + \frac{1}{2}W(l) + J_1(lk)^{-2} + o((lk)^{-2}).
\]  

(1.9)

By (1.7) and (1.9) a c.p. \(l(k)\) of \(\tilde{E}_k\) always exists provided \(J_1 > 0\). First numerically, and then rigorously, we were disappointed to find that \(J_1 = 0\).

Later on, we realized that \(-J_1\) is exactly the correlation coefficient \(A_0\) in (1.4) (with \(N = 3\) and \(m = 1\)) introduced by Ovchinnikov and Sigal [13]. If \(u\) is a solution of (1.6) with vortices \(a_0 = 0\) and \(l_a j, a_j = e^{\frac{2\pi i}{3}(j - 1)}\) for \(j = 1, 2, 3\), with \(n_0 = -1\) and \(n_1 = n_2 = n_3 = 1\), then the function \(U(x) = u(\frac{x}{k})\) does solve

\[
\begin{cases}
-\Delta U = U(1 - |U|^2) & \text{in } B_k \\
U = g_0 & \text{on } \partial B_k
\end{cases}
\]

(1.10)

with vortices \(a_0\) and \(lka_j\) of vorticities \(n_0 = -1, n_1 = n_2 = n_3 = 1\). Since (1.1) and (1.10) formally coincide when \(k = +\infty\), it is natural to find a correlation term in the energy expansion
\[ \tilde{E}_k \text{ in the form } -\frac{\Delta u}{\alpha} = J_1(\lambda k)^{-2}, \text{ where } a = \lambda k \text{ is the modulus of the } \lambda ka_j \text{'s for } j = 1, 2, 3. \] Even more and not surprisingly, the function \( \tilde{U}_k(x) \), where \( U_k[\eta(1 + \psi_0(k)) + (1 - \eta)e^{\psi_0(k)}] \) is a very good approximating solution for (1.6) which improves the approximation rate of \( U_k \), does coincide with the refined test functions used by Ovchinnikov and Sigal [13] to get the improved upper bound (1.3). In conclusion, the vanishing of the correlation coefficient \( A_0 \) does not support any conjecture concerning symmetry-breaking phenomena for (1.1) or the existence of collapsing vortices for (1.6) when \( k \to +\infty \). Higher-order expansions would be needed in their study.

2. The Correlation Coefficient

Let \( N \geq 2 \). Let \( a_j = e^{\frac{2\pi i(j-1)}{N}}, j = 1, \ldots, N \), be the \( N \)-roots of unity, and set \( n_j = m \in \mathbb{Z} \) for all \( j = 1, \ldots, N \), \( a_0 = 0 \) and \( n_0 = -\frac{N-1}{2}m \). We aim to compute the correlation coefficient \( A_0 = A_0(m) \) given in (1.4). Since (in complex notation) \( \nabla \theta(x) = |x|^{-2}(-x_2, x_1) \) has the same modulus as \( \bar{x}/|x|^2 \), the correlation coefficient takes the form

\[
A_0 = \frac{1}{4} \int_{\mathbb{R}^2} \left[ \sum_{j=0}^{N} \frac{n_j}{x-a_j} \right]^4 - \sum_{j=0}^{N} \left| \frac{n_j}{x-a_j} \right|^4 \right]. \tag{2.1}
\]

Since the integer \( m \) comes out as \( m^4 \) from the expression (2.1), we have that \( A_0(m) = m^4A_0(1) \). Hereafter, we will assume \( m = 1 \) and simply denote \( A_0(1) \) as \( A_0 \).

Let us first notice that \( A_0 \) is not well-defined without further specifications, because the integral function in (2.1) is not integrable near the points \( a_j, j = 0, \ldots, N \). Recall that the \( N \)-roots of unity \( a_1, \ldots, a_N \) do satisfy the following symmetry properties:

\[
\sum_{j=1}^{N} a_j^l = 0 \quad \forall |l| \leq N, \ l \neq 0, \tag{2.2}
\]

as it can be easily deduced by the relation \( x^N - 1 = \prod_{j=1}^{N} (x - a_j) \). A first application of (2.2) is the validity of

\[
\sum_{j=1}^{N} \frac{1}{x-a_j} = \sum_{j=1}^{N} \frac{x^{N-1} + a_jx^{N-2} + \cdots + a_j^{N-1}}{x^N - 1} = \frac{Nx^{N-1}}{x^N - 1}, \tag{2.3}
\]

which implies that the integral function in (2.1) near 0 has the form

\[
\left| \sum_{j=0}^{N} \frac{n_j}{x-a_j} \right|^4 - \sum_{j=0}^{N} \left| \frac{n_j}{x-a_j} \right|^4 = -\frac{N(N-1)^3}{2} \text{Re} \left( \frac{x^N}{(x^N - 1)|x|^4} \right) + O(1) \tag{2.4}
\]
and is not integrable at 0 when $N = 2$. Similarly, setting $\alpha_k(x) = -\frac{N-1}{2x} + \sum_{j=1}^{N} \frac{1}{x - a_j}$ for $k = 1, \ldots, N$, near $a_k$ we have that

$$|\sum_{j=0}^{N} \frac{n_j}{x - a_j}|^4 - |\sum_{j=0}^{N} \frac{n_j}{x - a_j}|^4 = \frac{4}{|x - a_k|^4} \text{Re}[(x - a_k)\alpha_k(x)] + \frac{2}{|x - a_k|^2} |\alpha_k(x)|^2$$  \hspace{1cm} (2.5)

$$+ \left(2 \text{Re}\left(\frac{x - a_k)\alpha_k(x)}{|x - a_k|^2} + |\alpha_k(x)|^2\right)^2 - \frac{(N - 1)^4}{16|x|^4} - \sum_{j=1}^{N} \frac{1}{|x - a_j|^4}.$$  

The function $\alpha_k$ cannot be computed explicitly, but we know that

$$\alpha_k(a_k) = -\frac{N - 1}{2a_k} + \sum_{j=1, j \neq k}^{N} \frac{1}{a_k - a_j} = a_k^{N-1} \left(-\frac{N - 1}{2} + \sum_{j=2}^{N} \frac{1}{1 - a_j}\right)$$  \hspace{1cm} (2.6)

$$= a_k^{N-1} \left(-\frac{N - 1}{2} + \sum_{j=2}^{N} \frac{1 - \cos \frac{2\pi(j-1)}{N} + i \sin \frac{2\pi(j-1)}{N}}{2(1 - \cos \frac{2\pi(j-1)}{N})}\right)$$

$$= ia_k^{N-1} \sum_{j=2}^{N} \frac{\sin \frac{2\pi(j-1)}{N}}{2(1 - \cos \frac{2\pi(j-1)}{N})} = 0$$

in view of $\{a_j a_k^{N-1} : j = 1, \ldots, N, j \neq k\} = \{a_2, \ldots, a_N\}$ and the symmetry of $\{a_1, \ldots, a_N\}$ under reflections w.r.t. the real axis. By inserting (2.6) into (2.5) we deduce that the integral in (2.1) near $a_k$ has the form

$$|\sum_{j=0}^{N} \frac{n_j}{x - a_j}|^4 - |\sum_{j=0}^{N} \frac{n_j}{x - a_j}|^4 = \frac{4}{|x - a_k|^4} \text{Re}[\alpha'_k(x - a_k)^2] + O\left(\frac{1}{|x - a_k|}\right)$$  \hspace{1cm} (2.7)

and is not integrable at $a_k$ when $\alpha'_k(a_k) \neq 0$. Since the (possible) singular term in (2.4), (2.7) has vanishing integrals on circles, the meaning of $A_0$ is in terms of a principal value:

$$A_0 = \frac{1}{4} \lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus \cup_{k=0}^{N} B_\epsilon(a_k)} \left[\sum_{j=0}^{N} \frac{n_j}{x - a_j}|^4 - \sum_{j=0}^{N} |\frac{n_j}{x - a_j}|^4\right].$$  \hspace{1cm} (2.8)

We would like to compute $A_0$ in polar coordinates, even tough the set $\mathbb{R}^2 \setminus \cup_{k=0}^{N} B_\epsilon(a_k)$ is not radially symmetric. The key idea is to make the integral function in (2.8) integrable near any $a_j, j = 1, \ldots, N$, by adding suitable singular terms, in such a way that the integral in (2.8) will have to be computed just on the radially symmetric set $\mathbb{R}^2 \setminus B_\epsilon(a_0)$. To this aim, it is crucial to compute $\alpha'_k(a_k)$. Arguing as before, we get that

$$\alpha'_k(a_k) = -\frac{N - 1}{2a_k} - \sum_{j=1, j \neq k}^{N} \frac{1}{(a_k - a_j)^2} = a_k^{N-2} \left(-\frac{N - 1}{2} - \sum_{j=2}^{N} \frac{1}{(1 - a_j)^2}\right)$$

$$= a_k^{N-2} \left(-\frac{N - 1}{2} - \sum_{j=2}^{N} \frac{(1 - \cos \frac{2\pi(j-1)}{N})^2 - \sin^2 \frac{2\pi(j-1)}{N}}{4(1 - \cos \frac{2\pi(j-1)}{N})^2}\right)$$

$$= a_k^{N-2} \sum_{j=2}^{N} \frac{1}{2(1 - \cos \frac{2\pi(j-1)}{N})} = a_k^{N-2} \sum_{j=2}^{N} \frac{1}{|1 - a_j|^2}.$$  \hspace{1cm} (2.9)
Since there holds $\sum_{j=1}^{N-1} a_k^j = \sum_{j=2}^N a_j = -1$ for all $k = 2, \ldots, N$ in view of (2.2), we have that

$$\prod_{j=2}^N (z - a_j) = \frac{z^N - 1}{z - 1} = \sum_{p=0}^{N-1} z^p, \quad \prod_{j=2 \atop j \neq k}^N (z - a_j) = \sum_{p=0}^{N-1} z^p \sum_{l=0}^{N-2} a_k^l,$$

and then

$$\prod_{j=2}^N (1 - a_j) = N, \quad \prod_{j=2 \atop j \neq k}^N (1 - a_j) = \sum_{l=0}^{N-2} (N - l - 1)a_k^l. \quad (2.10)$$

By (2.10) we get that

$$\beta_N := \sqrt{\frac{4}{N^2}} \sum_{j=2}^N |1 - a_j|^2 = \sqrt{\frac{4}{N^2}} \prod_{k=2 \atop k \neq j}^N |1 - a_k|^2 = \sqrt{\frac{4}{N^2}} \sum_{j=2}^N \sum_{l=0}^{N-2} (N - l - 1)(N - p - 1)a_j^{l-p}$$

$$= \frac{4}{N^2} \left( N - 1 \right) \sum_{l=1}^{N-1} l^2 - \frac{4}{N^2} \sum_{l=1}^{N-1} l^p = \frac{4}{N^2} \left( N - 1 \right) \sum_{l=1}^{N-1} l^2 - \frac{4}{N^2} \left( \sum_{l=1}^{N-1} l \right)^2 = 2(N-1)(2N-1) - (N-1)^2$$

$$= \frac{N^2 - 1}{3}$$

in view of (2.2). Since by (2.9) $\alpha_k'(a_k) = \frac{d \alpha_k}{d a_k}$, by (2.7) we have that

$$|\sum_{j=0}^N \frac{n_j}{x - a_j}|^4 - |\sum_{j=0}^N \frac{n_j}{x - a_j}|^4 - \sum_{j=1}^N Re[\frac{\beta_N a_j^2}{(x - a_j)^2(1 + |x - a_j|^2)}] \in L^1(\mathbb{R}^2 \setminus \{0\}).$$

Since

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus \{0\}} \frac{a_j^2}{(x - a_j)^2(1 + |x - a_j|^2)} = \lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\epsilon(a_j)} \frac{a_j^2}{(x - a_j)^2(1 + |x - a_j|^2)} = 0,$$

we can re-write $A_0$ as

$$A_0 = \frac{1}{4} \lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\epsilon(0)} \left[ \frac{(N + 1)x^N + (N - 1)}{2x(x^N - 1)} + \frac{(N - 1)4}{16|x|^4} - \sum_{j=1}^N \frac{1}{|x - a_j|^4} \right]$$

$$- \sum_{j=1}^N Re[\frac{\beta_N a_j^2}{(x - a_j)^2(1 + |x - a_j|^2)}]$$

$$= \frac{1}{4} \lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\epsilon(0) \cup \{1 - \epsilon \leq |x| \leq \frac{1}{1 - \epsilon}\}} \left[ \frac{(N + 1)x^N + (N - 1)}{2x(x^N - 1)} + \frac{(N - 1)4}{16|x|^4} - \sum_{j=1}^N \frac{1}{|x - a_j|^4} \right]$$

$$- \frac{1}{4} \Re \left[ \lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\epsilon(0) \cup \{1 - \epsilon \leq |x| \leq \frac{1}{1 - \epsilon}\}} \sum_{j=1}^N \frac{\beta_N a_j^2}{(x - a_j)^2(1 + |x - a_j|^2)} \right] = \frac{1}{4} - \frac{1}{4} \Pi \quad (2.11)$$

in view of (2.3).
As far as I, let us write the following Taylor expansions: for $|x| < 1$ there hold

$$\frac{(N + 1)x^N + (N - 1)}{(1 - x^N)^2} = \left( (N - 1)^2 + 2(N^2 - 1)x^N + (N + 1)^2x^{2N} \right) \sum_{k \geq 0} (k + 1)x^k \Rightarrow (N - 1)^2 \sum_{k \geq 1} 4N(kN - 1)x^{kN} = \sum_{k \geq 0} c_kx^{kN}$$

(2.12)

and

$$\frac{(N - 1)x^N + (N + 1)}{(1 - x^N)^2} = \left( (N + 1)^2 + 2(N^2 - 1)x^N + (N - 1)^2x^{2N} \right) \sum_{k \geq 0} (k + 1)x^k \Rightarrow (N + 1)^2 \sum_{k \geq 1} 4N(kN + 1)x^{kN} = \sum_{k \geq 0} d_kx^{kN},$$

(2.13)

where $c_k = \max\{4N(kN - 1), (N - 1)^2\}$ and $d_k = \max\{4N(kN + 1), (N + 1)^2\}$. Letting $\epsilon > 0$ small, by (2.12)-(2.13) we have that in polar coordinates (w.r.t. the origin) I writes as

$$I = \int_{1-\epsilon}^{1} \rho^2 \int_{0}^{2\pi} d\theta \left[ \frac{1}{16\rho^4} \sum_{k \geq 0} c_k\rho^{kN}e^{ikN\theta} - \frac{(N - 1)^4}{16\rho^4} - \sum_{j=1}^{N} \sum_{k \geq 0} (k + 1)a_j^{k(N-1)}\rho^k e^{ik\theta} \right]$$

$$+ \int_{\pi}^{\pi+2\pi} \rho^2 \int_{0}^{2\pi} d\theta \left[ \frac{1}{16\rho^4} \sum_{k \geq 0} d_k\rho^{-kN}e^{-ikN\theta} - \frac{(N - 1)^4}{16\rho^4} - \frac{1}{\rho^4} \sum_{j=1}^{N} \sum_{k \geq 0} (k + 1)a_j\rho^{-k}e^{-ik\theta} \right]$$

$$+ o_{c}(1)$$

with $o_{c}(1) \to 0$ as $\epsilon \to 0$, in view of

$$|x - a_j|^{-4} = |a_j^{N-1}x - 1|^{-4} = |\sum_{k \geq 0} (k + 1)a_j^{k(N-1)}x^k|^{-2}, \quad |1 - a_j|^{-4} = |\sum_{k \geq 0} (k + 1)a_j^kx^k|^{-2}$$

for $|x| < 1$. By the Parseval’s Theorem we get that

$$I = 2\pi \int_{0}^{1-\epsilon} \left[ \frac{1}{16} \sum_{k \geq 1} |c_k|^2 \rho^{2kN - 3} - N \sum_{k \geq 0} (k + 1)^2 \rho^{2k+1} \right] d\rho$$

$$+ 2\pi \int_{\pi}^{\pi+2\pi} \left[ \frac{1}{16} \sum_{k \geq 1} |d_k|^2 \rho^{-2kN - 3} + \frac{(N + 1)^4 - (N - 1)^4}{16\rho^3} - \sum_{k \geq 0} (k + 1)^2 \rho^{-2k-3} \right] d\rho$$

$$+ o_{c}(1)$$

$$= 2\pi N \int_{0}^{1-\epsilon} \left[ N \sum_{k \geq 0} (kN + N - 1)^2 \rho^{2kN + 2N - 3} - \sum_{k \geq 0} (k + 1)^2 \rho^{2k+1} \right] d\rho$$

$$+ 2\pi N \int_{\pi}^{\pi+2\pi} \left[ N \sum_{k \geq 0} (kN + N + 1)^2 \rho^{-2kN - 2N - 3} - \sum_{k \geq 0} (k + 1)^2 \rho^{-2k-3} \right] d\rho$$

$$+ o_{c}(1) = 2\pi N \int_{0}^{1-\epsilon} \left[ N \sum_{k \geq 0} (kN + N - 1)^2 \rho^{2kN + 2N - 3} + N \sum_{k \geq 0} (kN + N + 1)^2 \rho^{2kN + 2N + 1}$$

$$- 2 \sum_{k \geq 0} (k + 1)^2 \rho^{2k+1} \right] d\rho + N(N^2 + 1)\frac{\pi}{2} + o_{c}(1)$$
as $\epsilon \to 0$. We compute now the integrals and let $\epsilon \to 0$ to end up with
\[ I = 2\pi N \left[ \frac{N}{2} \sum_{k \geq 0} (kN + N - 1)\rho^{2kN+2N^2-2} + \frac{N}{2} \sum_{k \geq 0} (kN + N + 1)\rho^{2kN+2N^2+2} - \sum_{k \geq 0} (k + 1)\rho^{2k+2} \right] \bigg|_0^1 + N(N^2 + 1)\frac{\pi}{2}. \]

Denoting the function inside brackets as $f(\rho)$, we need now to determine the explicit expression of $f(\rho)$ for $\rho < 1$:
\[
f(\rho) = \frac{N^2}{2}\rho^{2N^2-2}(1 + \rho^4) \sum_{k \geq 0} (k + 1)(\rho^{2N})^k - \frac{N^2}{2}\rho^{2N^2-2}(1 - \rho^4) \sum_{k \geq 0} (\rho^{2N})^k - \rho^2 \sum_{k \geq 0} (k + 1)(\rho^2)^k 
\]
\[ = \frac{N^2}{2}\rho^{2N^2-2} \frac{1 + \rho^4}{(1 - \rho^{2N})^2} - \frac{N^2}{2}\rho^{2N^2-2} \frac{1 - \rho^4}{(1 - \rho^{2N})^2} - \rho^2 \frac{\sum_{j = 0}^{N-1} \rho^{2j}}{(1 - \rho^{2N})^2}, \]

and then by the l'Hôpital’s rule we get that
\[
4N^2 f(1) = 2 \lim_{\rho \to 1} \frac{N(N - 1)\rho^{N-1} + N(N + 1)\rho^{N+1} - 2\rho(\sum_{j = 0}^{N-1} \rho^j)^2 + N\rho^{2N-1} - N\rho^{2N+1}}{(1 - \rho)^2} 
\]
\[ = \lim_{\rho \to 1} \frac{N^2(N - 2)\rho^{N-2} - N^2(N + 2)\rho^N + 2(\sum_{j = 0}^{N-1} \rho^j)^2 + 4\rho(\sum_{j = 0}^{N-1} \rho^j)(\sum_{j = 0}^{N-2} (j + 1)\rho^j)}{1 - \rho} 
\]
\[ + N \lim_{\rho \to 1} \frac{(2N + 1)\rho^{2N} - (2N - 1)\rho^{2N-2} - \rho^{N+2} - \rho^N}{1 - \rho} = -\frac{N^2(N^2 + 5)}{3}. \]

In conclusion, for $I$ we get the value
\[ I = \frac{\pi}{3} N(N^2 - 1). \quad (2.14) \]

**Remark 2.1.** In [13] the value of $A_0$ was computed neglecting the term $II$ in [2.11]. By [2.11] notice that $\frac{\mu}{\pi} I = \frac{\pi}{12} m^4 N(N^2 - 1)$ does coincide with $8\pi$ when $(N, m) = (2, 2)$ and $80\pi$ when $(N, m) = (4, 2)$, in agreement with the computations in [13].

As far as $II$, let us compute in polar coordinates the value of
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}^n \setminus B(0, \epsilon) \cup \{1 - \epsilon \leq |\mathbf{a}_j| \leq \epsilon \}} \frac{\mathbf{a}_j^2}{(x - \mathbf{a}_j)^2(1 + |x - \mathbf{a}_j|^2)} = \lim_{\epsilon \to 0} \int_{(0,1) \cup (\frac{1}{\epsilon}, +\infty)} \rho \Gamma(\rho) d\rho, 
\]

where the function $\Gamma(\rho)$ is defined in the following way:
\[
\Gamma(\rho) = \sum_{j = 1}^{N} \int_{0}^{2\pi} \frac{(\rho e^{i\theta} - \mathbf{a}_j)^2(2 + \rho^2 - a_j \rho e^{-i\theta} - a_j^{N-1} \rho e^{i\theta})}{(\rho w - a_j)^2(w^2 - 2\frac{\rho^2}{w} a_j w + a_j^2)} d\theta 
\]
\[ = \frac{i}{\rho} \sum_{j = 1}^{N} a_j^3 \int_{\gamma} \frac{dw}{(\rho w - a_j)^2(w^2 - 2\frac{\rho^2}{w} a_j w + a_j^2)}. \]
with $\gamma$ the counterclockwise unit circle around the origin. Since

$$w^2 - \frac{2 + \rho^2}{\rho}a_j w + a_j^2 = \left(w - \frac{2 + \rho^2}{2\rho}a_j\right)^2 + a_j^2 \left(1 - \frac{2 + \rho^2}{2\rho}\right),$$

observe that $w^2 - \frac{2 + \rho^2}{\rho}a_j w + a_j^2$ vanishes at $\rho a_j$, with

$$\rho_{\pm} = \frac{2 + \rho^2}{2\rho} \pm \sqrt{\left(\frac{2 + \rho^2}{2\rho}\right)^2 - 1}$$

satisfying $\rho_- < 1 < \rho_+$ in view of $\frac{2 + \rho^2}{2\rho} > \sqrt{2}$. Since

$$\frac{1}{w^2 - \frac{2 + \rho^2}{\rho}a_j w + a_j^2} \frac{d}{d\rho} \left(\frac{a_j}{\rho}\right) = a_j^{N-3} \rho^5,$$

by the Cauchy’s residue Theorem the function $\Gamma(\rho)$ can now be computed explicitly as

$$\Gamma(\rho) = \frac{i}{\rho^3} \sum_{j=1}^{N} a_j^3 \int_{\gamma} \frac{dw}{(w - \frac{a_j}{\rho})^2(w - \rho_-a_j)(w - \rho_+a_j)} = 2\pi N \left\{ \begin{array}{ll} (pp_- - 1)^{-2}(pp_+ - pp_-)^{-1} & \text{if } \rho < 1 \\ (pp_- - 1)^{-2}(pp_+ - pp_-)^{-1} - \rho^2 & \text{if } \rho > 1. \end{array} \right.$$
Finally, inserting (2.14) and (2.17) into (2.11) we get that the correlation coefficient vanishes: \( A_0 = 0 \). Then, there holds \( A_0(m) = 0 \) for all \( m \in \mathbb{Z} \), as claimed.

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