A HEAT EQUATION ON A QUATERNIONIC CONTACT MANIFOLD

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Abstract. A quaternionic contact (qc) heat equation and the corresponding qc energy functional are introduced. It is shown that the qc energy functional is monotone non-increasing along the qc heat equation on a compact qc manifold provided certain positivity conditions are satisfied.

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1. INTRODUCTION

We introduce a quaternionic contact (qc) heat equation and the corresponding qc energy functional. The purpose of this paper is to show that the qc energy functional is monotone non-increasing along the qc heat equation on a compact qc manifold provided certain positivity conditions are satisfied. In dimensions at least eleven the positivity condition coincides with the Lichnerowicz-type positivity condition used in [5, 6] to derive a sharp lower bound for the first eigenvalue of the sub-Laplacian an a compact qc manifold. In dimension seven, in addition, we need to assume the positivity of the introduced in [7] P-function.

It is well known that the sphere at infinity of a non-compact symmetric space of rank one carries a natural Carnot-Carathéodory structure, see [10, 11]. A quaternionic contact (qc) structure, [1], appears naturally as the conformal boundary at infinity of the quaternionic...
hyperbolic space. Following Biquard, a quaternionic contact structure (qc structure) on a real (4n+3)-dimensional manifold $M$ is a codimension three distribution $H$ (the horizontal distribution) locally given as the kernel of a $\mathbb{R}^3$-valued one-form $\eta = (\eta_1, \eta_2, \eta_3)$, such that the three two-forms $d\eta|_H$ are the fundamental forms of a quaternionic Hermitian structure on $H$. In other words, a quaternionic contact (qc) manifold $(M, g, Q)$ is a $4n+3$-dimensional manifold $M$ with a codimension three distribution $H$ equipped with an $Sp(n)Sp(1)$ structure. Explicitly, $H$ is the kernel of a local 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ together with a compatible Riemannian metric $g$ and a rank-three bundle $Q$ consisting of endomorphisms of $H$ locally generated by three almost complex structures $I_1, I_2, I_3$ on $H$ satisfying the identities of the imaginary unit quaternions.

On a qc manifold one can associate a linear connection with torsion preserving the qc structure, see [1], which is called the Biquard connection. One defines the horizontal Ricci-type tensor with the trace of the curvature of the Biquard connection, called the qc Ricci tensor. This is a symmetric tensor whose trace-free part is determined by the torsion endomorphism of the Biquard connection [4] while the trace part is determined by the scalar curvature of the qc-Ricci tensor, called the qc-scalar curvature.

Let $(M, g, Q)$ be a compact qc manifold. We consider the qc heat equation

$$\frac{\partial}{\partial t} u = -\Delta u,$$

where $u(x, t): M \times [0, +\infty) \to \mathbb{R}$ is smooth function and $\Delta: \mathcal{F}(M) \to \mathcal{F}(M)$ is the sub-Laplacian on $M$. From now on, $u$ will be a positive solution of (1.1). We introduce the functions $\varphi = -\ln u$ and $F = u^\alpha$, where $\alpha \in \mathbb{R}, \alpha \neq 0, \frac{1}{2}$. The energy functional for (1.1) is defined by

$$\mathcal{F}(\varphi) = \int_M |\nabla \varphi|^2 e^{-\varphi} Vol_\eta.$$

Our main result follows.

**Theorem 1.1.** Let $(M, g, Q)$ be a compact $4n+3$-dimensional quaternionic contact manifold and the Lichnerowicz type condition (1.3) holds, $L(X, X) \geq 0$ for any $X \in \Gamma(H)$.

i) If $n > 1$ then the energy functional (1.2) is monotone non-increasing along the qc heat equation (1.1).

ii) In the case $n = 1$ suppose in addition that the $P-$function of any $F^{1\over 2\alpha}$, corresponding to a (positive) solution $u$ of (1.1) is non-negative. Then the energy functional (1.2) is monotone non-increasing along the qc heat equation (1.1).

The Lichnerowicz type assumption cf. (2.2), (2.6),

$$(1.3) \quad L(X, X) = Ric(X, X) + \frac{2(4n+5)}{2n+1} T^0(X, X) + \frac{6(2n^2 + 5n - 1)}{(n-1)(2n+1)} U(X, X)$$

$$= 2(n+2) Sg(X, X) + \frac{4n^2 + 14n + 12}{2n+1} T^0(X, X) + \frac{4(n+2)^2(2n-1)}{(n-1)(2n+1)} U(X, X) \geq k_0 g(X, X),$$

(the third term in the left-hand side is dropped if $n = 1$) yields a sharp lower bound of the first eigenvalue of the sub-Laplacian when $n > 1$ [5] while for $n = 1$ one needs additional assumption expressed in terms of the positivity of the $P$-function defined in [6] to achieve the
validity of the same lower bound [6]. The $P$-function of a smooth function $f$ is defined with the help of the Biquard connection, the qc-scalar curvature and the $Sp(n)Sp(1)$-components of the torsion tensor see (2.8) below.

**Convention 1.2.**

a) We shall use $X,Y,Z,U$ to denote horizontal vector fields, i.e. $X,Y,Z,U \in H$.

b) $\{e_1, \ldots , e_{4n}\}$ denotes a local orthonormal basis of the horizontal space $H$.

c) The triple $(i,j,k)$ denotes any cyclic permutation of $(1,2,3)$.

d) $s$ will be any number from the set $\{1,2,3\}$, $s \in \{1,2,3\}$.

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2. Quaternionic contact manifolds

Quaternionic contact manifolds were introduced in [1]. We also refer to [4] and [8] for further results and background.

2.1. Quaternionic contact structures and the Biquard connection. A quaternionic contact (qc) manifold $(M,g,Q)$ is a $4n + 3$-dimensional manifold $M$ with a codimension three distribution $H$ equipped with an $Sp(n)Sp(1)$ structure. Explicitly, $H$ is the kernel of a local 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ together with a compatible Riemannian metric $g$ and a rank-three bundle $Q$ consisting of endomorphisms of $H$ locally generated by three almost complex structures $I_1, I_2, I_3$ on $H$ satisfying the identities of the imaginary unit quaternions. Thus, we have $I_1I_2 = -I_2I_1 = I_3$, $I_1I_2I_3 = -id|_H$ which are hermitian compatible with the metric $g(I_s,I_s) = g(.,.)$ and the following compatibility conditions hold $2g(I_sX,Y) = d\eta_s(X,Y)$.

On a qc manifold of dimension $(4n + 3) > 7$ with a fixed metric $g$ on $H$ there exists a canonical connection defined in [1]. Biquard also showed that there is a unique connection $\nabla$ with torsion $T$ and a unique supplementary subspace $V$ to $H$ in $TM$, such that:

i) $\nabla$ preserves the splitting $H \oplus V$ and the $Sp(n)Sp(1)$ structure on $H$, i.e., $\nabla g = 0, \nabla \sigma \in \Gamma(Q)$ for a section $\sigma \in \Gamma(Q)$, and its torsion on $H$ is given by $T(X,Y) = -[X,Y]|_V$;

ii) for $\xi \in V$, the endomorphism $T(\xi,.)|_H$ of $H$ lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$;

iii) the connection on $V$ is induced by the natural identification $\varphi$ of $V$ with $Q$, $\nabla \varphi = 0$.

When the dimension of $M$ is at least eleven [1] also described the supplementary vertical distribution $V$, which is (locally) generated by the so called Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ determined by

$$\eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \lrcorner d\eta_s)|_H = 0, \quad (\xi_s \lrcorner d\eta_k)|_H = -(\xi_k \lrcorner d\eta_s)|_H, \quad (\xi_s \lrcorner d\eta_k)|_H = -(\xi_k \lrcorner d\eta_s)|_H,$$

where $\lrcorner$ denotes the interior multiplication.

If the dimension of $M$ is seven Duchemin shows in [3] that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then the Biquard result holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1). This implies the existence of the connection with properties (i), (ii) and (iii) above.
The fundamental 2-forms $\omega_s$ of the quaternionic structure are defined by

$$2\omega_s|_H = d\eta_s|_H, \quad \xi \omega_s = 0, \quad \xi \in V.$$  

The torsion restricted to $H$ has the form $T(X, Y) = -[X, Y]|_V = 2\sum_{s=1}^{3} \omega_s(X, Y)\xi_s$.

2.2. **Invariant decompositions.** Any endomorphism $\Psi$ of $H$ can be decomposed with respect to the quaternionic structure $(\mathbb{Q}, g)$ uniquely into four $Sp(n)$-invariant parts $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-++} + \Psi^{--+}$, where $\Psi^{+++}$ commutes with all three $I_i$, $\Psi^{+--}$ commutes with $I_1$ and anti-commutes with the others two, etc. The two $Sp(n)Sp(1)$-invariant components are given by $\Psi_{[3]} = \Psi^{+++}$, $\Psi_{[-1]} = \Psi^{+--} + \Psi^{-++} + \Psi^{--+}$. These are the projections on the eigenspaces of the Casimir operator $\mathcal{Y} = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$, corresponding, respectively, to the eigenvalues 3 and -1, see [2]. Note here that each of the three 2-forms $\omega_s$ belongs to the [-1]-component, $\omega_s = \omega_{s[-1]}$ and constitute a basis of the Lie algebra $sp(1)$.

If $n = 1$ then the space of symmetric endomorphisms commuting with all $I_s$ is 1-dimensional, i.e., the $[3]$-component of any symmetric endomorphism $\Psi$ on $H$ is proportional to the identity, $\Psi_{[3]} = -\frac{1}{4}T^0_\xi Id|_H$.

2.3. **The torsion tensor.** The torsion endomorphism $T_\xi = T(\xi, \cdot) : H \to H, \quad \xi \in V$ will be decomposed into its symmetric part $T^0_\xi$ and skew-symmetric part $b_\xi, T_\xi = T^0_\xi + b_\xi$. Biquard showed in [1] that the torsion $T_\xi$ is completely trace-free, $tr T_\xi = tr T_\xi \circ I_s = 0$, its symmetric part has the properties $T^0_\xi I_i = -I_i T^0_\xi I_2(T^0_\xi)^{+++} = I_1(T^0_\xi)^{+--}, \quad I_3(T^0_\xi)^{-++} = I_2(T^0_\xi)^{+++}, \quad I_1(T^0_\xi)^{-++} = I_3(T^0_\xi)^{+++}$. The skew-symmetric part can be represented as $b_\xi = I_1 U$, where $U$ is a traceless symmetric (1,1)-tensor on $H$ which commutes with $I_1, I_2, I_3$. Therefore we have $T_\xi = T^0_\xi + I_1 U$. When $n = 1$ the tensor $U$ vanishes identically, $U = 0$, and the torsion is a symmetric tensor, $T_\xi = T^0_\xi$.

The two $Sp(n)Sp(1)$-invariant trace-free symmetric 2-tensors on $H$

$$T^0(X, Y) = g((T^0_\xi I_1 + T^0_\xi I_2 + T^0_\xi I_3)X, Y) \quad \text{and} \quad U(X, Y) = g(uX, Y)$$

were introduced in [4] and enjoy the properties

$$T^0(X, Y) + T^0(I_1X, I_1Y) + T^0(I_2X, I_2Y) + T^0(I_3X, I_3Y) = 0,$$

$$U(X, Y) = U(I_1X, I_1Y) = U(I_2X, I_2Y) = U(I_3X, I_3Y).$$

From [8, Proposition 2.3] we have

$$4T^0(\xi_s, I_sX, Y) = T^0(X, Y) - T^0(I_sX, I_sY),$$

hence, taking into account (2.4) it follows

$$T(\xi_s, I_sX, Y) = T^0(\xi_s, I_sX, Y) + g(I_suI_sX, Y)$$

$$= \frac{1}{4}[T^0(X, Y) - T^0(I_sX, I_sY)] - U(X, Y).$$
2.4. **Torsion and curvature.** Let \( R = [\nabla, \nabla] - \nabla_{[\cdot, \cdot]} \) be the curvature tensor of \( \nabla \) and the dimension is \( 4n + 3 \). We denote the curvature tensor of type \((0,4)\) and the torsion tensor of type \((0,3)\) by the same letter, \( R(A, B, C, D) := g(R(A, B)C, D), \quad T(A, B, C) := g(T(A, B), C), \) \( A, B, C, D \in \Gamma(TM) \). The qc-Ricci tensor \( \text{Ric} \), normalized qc-scalar curvature \( S \) of the Biquard connection are defined, respectively, by the following formulas (cf. Convention 1.3), \( \text{Ric}(A, B) = \sum_{b=1}^{4n} R(e_b, A, B, e_b), \quad 8n(n + 2)S = \sum_{a,b=1}^{4n} R(e_b, e_a, e_a, e_b). \)

The qc-Ricci tensor and the normalized qc-scalar curvature are determined by the torsion of the Biquard connection as follows [4]

\[
\begin{align*}
\text{Ric}(X, Y) &= (2n + 2)T^0(X, Y) + (4n + 10)U(X, Y) + 2(n + 2)Sg(X, Y), \\
T(\xi, \eta) &= -S\xi - [\xi, \eta]_H, \quad S = -g(T(\xi_1, \xi_2), \xi_3).
\end{align*}
\]

Note that for \( n = 1 \) the above formulas hold with \( U = 0 \).

Any 3-Sasakian manifold has zero torsion endomorphism, and the converse is true if in addition the qc-scalar curvature is a positive constant [4].

2.5. **The Ricci identities.** We use repeatedly the Ricci identities of order two and three, see also [8]. Let \( f \) be a smooth function on the qc manifold \( M \) with horizontal gradient \( \nabla f \) defined by \( g(\nabla f, X) = df(X) \). The sub-Laplacian of \( f \) is \( \triangle f = -\sum_{a=1}^{4n} \nabla^2 f(e_a, e_a) \). We have the following Ricci identities (see e.g. [4, 9])

\[
\begin{align*}
\nabla^2 f(X, Y) - \nabla^2 f(Y, X) &= -2 \sum_{s=1}^{3} \omega_s(X, Y)df(\xi_s), \\
\nabla^2 f(X, \xi_s) - \nabla^2 f(\xi_s, X) &= T(\xi_s, X, \nabla f), \\
\nabla^3 f(X, Y, Z) - \nabla^3 f(Y, X, Z) &= -R(X, Y, Z, \nabla f) - 2 \sum_{s=1}^{3} \omega_s(X, Y)\nabla^2 f(\xi_s, Z).
\end{align*}
\]

We also need the qc-Bochner formula [5, (4.1)]

\[
\begin{align*}
\frac{1}{2} \triangle |\nabla f|^2 &= |\nabla^2 f|^2 - g(\nabla(\triangle f), \nabla f) + 2(n + 2)S|\nabla f|^2 + 2(n + 2)T^0(\nabla f, \nabla f) \\
&\quad + 2(2n + 2)U(\nabla f, \nabla f) + 4 \sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s \nabla f).
\end{align*}
\]

2.6. **The horizontal divergence theorem.** Let \( (M, g, \mathbb{Q}) \) be a qc manifold of dimension \( 4n + 3 \geq 7 \). For a fixed local 1-form \( \eta \) and a fixed \( s \in \{1, 2, 3\} \) the form \( \text{Vol}_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_s^{2n} \) is a locally defined volume form. Note that \( \text{Vol}_\eta \) is independent of \( s \) as well as the local one forms \( \eta_1, \eta_2, \eta_3 \). Hence, it is a globally defined volume form. The (horizontal) divergence of a horizontal vector field/one-form \( \sigma \in \Lambda^1(H) \), defined by \( \nabla^* \sigma = -tr|_H \nabla \sigma = -\nabla \sigma(e_a, e_a) \) supplies the integration by parts formula, [4], see also [12],

\[
\int_M (\nabla^* \sigma) \text{Vol}_\eta = 0.
\]
2.7. The P-form. We recall the definition of the P-form from [6]. Let \((M, g, \mathbb{Q})\) be a compact quaternionic contact manifold of dimension \(4n + 3\) and \(f\) a smooth function on \(M\).

For a smooth function \(f\) on \(M\) the \(P\)-form \(P \equiv P_f \equiv P[f]\) on \(M\) is defined by [6]

\[
P_f(X) = \nabla^3 f(X, e_b, e_b) + \sum_{t=1}^{3} \nabla^3 f(I_t X, e_b, I_t e_b) - 4n S\, df(X) + 4n T^0(X, \nabla f)
\]

\[(2.8)\]

\[
P_f(X) = \nabla^3 f(X, e_b, e_b) + \sum_{t=1}^{3} \nabla^3 f(I_t X, e_b, I_t e_b) - \frac{8n(n-2)}{n-1} U(X, \nabla f), \quad \text{if } n > 1,
\]

\[
P_f(X) = \nabla^3 f(X, e_b, e_b) + \sum_{t=1}^{3} \nabla^3 f(I_t X, e_b, I_t e_b) - 4S\, df(X) + 4T^0(X, \nabla f), \quad \text{if } n = 1.
\]

The \(C\)-operator is the fourth-order differential operator independent of \(f\) defined by

\[
Cf = -\nabla^* P_f = (\nabla e_a P_f)(e_a).
\]

We say that the \(P\)-function of \(f\) is non-negative if its integral exists and is non-positive

\[
(2.9) \int_M f \cdot C f \, Vol_\eta = - \int_M P_f(\nabla f) \, Vol_\eta \geq 0.
\]

If (2.9) holds for any smooth function of compact support we say that the \(C\)-operator is non-negative. It turns out that the \(C\)-operator is non-negative on any compact qc manifold of dimension at least eleven [6].

One of the key identities which relates the \(P\)-function and the qc Bochner formula (2.7) on a compact manifolds is the next identity, (dropping the last term when \(n = 1\), [6, (3.4)]

\[
\int_M \sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s \nabla f) \, Vol_\eta = \int_M \left[ - \frac{1}{4n} P_f(\nabla f) - \frac{1}{4n} (\Delta f)^2 - S |\nabla f|^2 + \frac{(n+1)}{n-1} U(\nabla f, \nabla f) \right] \, Vol_\eta.
\]

3. The QC heat equation and its energy functional

The next lemma is crucial for the proof of our main result.

**Lemma 3.1.** Let \((M, g, \mathbb{Q})\) be a compact \(4n + 3\)-dimensional quaternionic contact manifold. Then the next formula holds

\[
\alpha^2 \frac{d}{dt} \mathcal{F}(\varphi) = \frac{4\alpha}{3(1-2\alpha)} \int_M F^{\frac{1}{2\alpha}-2}(\Delta F)^2 \, Vol_\eta
\]

\[
\quad + \frac{48n\alpha^2 - 2(16n - 3)\alpha - 3}{12(2n+1)\alpha^2} \int_M F^{\frac{1}{2\alpha}-4} |\nabla F|^4 \, Vol_\eta + \frac{4(3 - 4\alpha)^2}{(2n+1)(1-2\alpha)} \int_M P F^{\frac{1}{2\alpha}-\frac{3}{2}}(\nabla F, \nabla F) \, Vol_\eta
\]

\[
\quad - \frac{2n(3 - 4\alpha)}{3(n+2)(1-2\alpha)} \int_M F^{\frac{1}{2\alpha}-2} L(\nabla F, \nabla F) \, Vol_\eta - \frac{4n(3 - 4\alpha)}{3(2n+1)(1-2\alpha)} \int_M F^{\frac{1}{2\alpha}-2} p(F) \, Vol_\eta.
\]
In the formula (3.1), \( P_{\frac{1}{2n}}(\nabla F_{\frac{1}{2n}}) \) is the \( P \)-function defined in (2.8) of \( F_{\frac{1}{2n}} \), \( L(\nabla F, \nabla F) \) is the left-hand side of the Lichnerowicz’ type assumption (1.3) with \( X := \nabla F \) and

\[
p(F) \overset{\text{def}}{=} |\nabla^2 F|^2 - \frac{1}{4n} (\Delta F)^2 - \frac{1}{4n} \sum_{s=1}^{3} [g(\nabla^2 F, \omega_s)]^2
\]

is a non-negative function on \( M \).

### 3.1. Proof of Lemma 3.1.

The next relation between the sub-Laplacians of \( u \) and \( \varphi \) holds

\[
(3.2) \quad \Delta u = -\frac{\Delta \varphi + |\nabla \varphi|^2}{e^\varphi},
\]

which follows easily by the definitions of \( \Delta \) and \( \varphi \). We get the formula

\[
(3.3) \quad \frac{\partial}{\partial t} \varphi = -\Delta \varphi - |\nabla \varphi|^2,
\]

as a simply consequence of the definition of \( \varphi \), (1.1) and (3.2). Further, the next chain of equalities holds

\[
(3.4) \quad \frac{d}{dt} \mathcal{F}(\varphi) = \frac{d}{dt} \int_M \left( -\Delta \varphi - \frac{\partial}{\partial t} \varphi \right) u \mathrm{Vol}_\eta = - \frac{d}{dt} \int_M \Delta \varphi u \mathrm{Vol}_\eta + \frac{d}{dt} \int_M \left( \frac{\partial}{\partial t} u \right) \mathrm{Vol}_\eta
\]

\[
= - \int_M \left( \frac{\partial}{\partial t} \varphi \right) u + \Delta \varphi \frac{\partial}{\partial t} u \right) \mathrm{Vol}_\eta = - \int_M \left( \frac{\partial}{\partial t} \varphi - \Delta \varphi \right) \Delta u \mathrm{Vol}_\eta
\]

\[
= \int_M e^{-\varphi} \left[ -2(\Delta \varphi)^2 - 3\Delta \varphi |\nabla \varphi|^2 - |\nabla \varphi|^4 \right] \mathrm{Vol}_\eta,
\]

where we used (3.3) for the first equality, the definition of \( \varphi \) for the second one, (1.1) and the divergence theorem for the third equality. Finally, we took into account the self-adjointness of the sub-Laplacian to obtain the fourth equality and (3.2), (3.3) for the last one.

We need the next two identities:

\[
(3.5) \quad |\nabla \varphi|^2 = \alpha^{-2} F^{-2} |\nabla F|^2, \quad \Delta \varphi = -\alpha^{-1} \left( F^{-2} |\nabla F|^2 + F^{-1} \Delta F \right),
\]

which, substituted into (3.4), give

\[
(3.6) \quad \alpha^2 \frac{d}{dt} \mathcal{F}(\varphi) = -2 \int_M F_{\frac{1}{\alpha}}^{-2}(\Delta F)^2 \mathrm{Vol}_\eta
\]

\[
+ (3 - 4\alpha)\alpha^{-1} \int_M F_{\frac{1}{\alpha}}^{-3} \Delta F |\nabla F|^2 \mathrm{Vol}_\eta + (-1 + 3\alpha - 2\alpha^2)\alpha^{-2} \int_M F_{\frac{1}{\alpha}}^{-4} |\nabla F|^4 \mathrm{Vol}_\eta.
\]

Next, we consider the (horizontal) vector field \( F_{\frac{1}{\alpha}}^{-2} |\nabla F|^2 \), in order to deal with the term \( \int_M F_{\frac{1}{\alpha}}^{-3} \Delta F |\nabla F|^2 \mathrm{Vol}_\eta \) in (3.6). We get by some standard calculations, using the divergence formula,

\[
(3.7) \quad 0 = - \int_M \nabla^* \left( F_{\frac{1}{\alpha}}^{-2} |\nabla F|^2 \right) \mathrm{Vol}_\eta
\]

\[
= \int_M g \left( \nabla(F_{\frac{1}{\alpha}}^{-2} \Delta F), \nabla F \right) \mathrm{Vol}_\eta - \int_M F_{\frac{1}{\alpha}}^{-2} \Delta F \nabla^* \nabla F \mathrm{Vol}_\eta
\]

\[
= \int_M F_{\frac{1}{\alpha}}^{-2} g \left( \nabla(\Delta F), \nabla F \right) \mathrm{Vol}_\eta + \left( \frac{1}{\alpha} - 2 \right) \int_M F_{\frac{1}{\alpha}}^{-3} \Delta F |\nabla F|^2 \mathrm{Vol}_\eta - \int_M F_{\frac{1}{\alpha}}^{-2}(\Delta F)^2 \mathrm{Vol}_\eta.
\]
Integrate the qc-Bochner formula (2.7) over the compact $M$ and use (3.7) to get

$$\frac{1}{\alpha} - 2 \int_M F^{\frac{1}{\alpha} - 3} \Delta F |\nabla F|^2 Vol_{\eta}$$

$$= \int_M F^{\frac{1}{\alpha} - 2} \left[ -\frac{1}{2} \Delta |\nabla F|^2 - |\nabla^2 F|^2 - 2(n + 2)S|\nabla F|^2 - 2(n + 2)T^0(\nabla F, \nabla F)$$

$$- 2(2n + 2)U(\nabla F, \nabla F) - 4 \sum_{s=1}^3 \nabla^2 F(\xi_s, I_s \nabla F) + (\Delta F)^2 \right] Vol_{\eta}.$$  

The next step is to find some suitable representations of the two terms $\int_M F^{\frac{1}{\alpha} - 2} \Delta |\nabla F|^2 Vol_{\eta}$ and $\int_M F^{\frac{1}{\alpha} - 2} \sum_{s=1}^3 \nabla^2 F(\xi_s, I_s \nabla F) Vol_{\eta}$. To deal with the first, we consider the (horizontal) vector field $F^{\frac{1}{\alpha} - 2} \nabla |\nabla F|^2$. We obtain the next sequence of equalities, using the divergence formula and some standard calculations:

$$\frac{1}{\alpha} - 2 \int_M \nabla^* \left( F^{\frac{1}{\alpha} - 2} \nabla |\nabla F|^2 \right) Vol_{\eta}$$

$$= \left( \frac{1}{\alpha} - 2 \right) \int_M F^{\frac{1}{\alpha} - 3} g(\nabla F, \nabla |\nabla F|^2) Vol_{\eta} - \int_M F^{\frac{1}{\alpha} - 2} \Delta |\nabla F|^2 Vol_{\eta}$$

$$= \left( \frac{1}{\alpha} - 2 \right) \int_M F^{\frac{1}{\alpha} - 3} |\nabla F|^2 \Delta F Vol_{\eta} - \left( \frac{1}{\alpha} - 2 \right) \left( \frac{1}{\alpha} - 3 \right) \int_M F^{\frac{1}{\alpha} - 4} |\nabla F|^4 Vol_{\eta}$$

$$- \int_M F^{\frac{1}{\alpha} - 2} \Delta |\nabla F|^2 Vol_{\eta}.$$  

To get the third equality in (3.9) we used the identity

$$0 = \int_M \nabla^* \left( F^{\frac{1}{\alpha} - 3} |\nabla F|^2 \nabla F \right) Vol_{\eta} = - \int_M F^{\frac{1}{\alpha} - 3} |\nabla F|^2 \Delta F Vol_{\eta}$$

$$+ \int_M F^{\frac{1}{\alpha} - 3} g(\nabla F, \nabla |\nabla F|^2) Vol_{\eta} + \left( \frac{1}{\alpha} - 3 \right) \int_M F^{\frac{1}{\alpha} - 4} |\nabla F|^4 Vol_{\eta}$$

in order to take an appropriate representation of the term $\int_M F^{\frac{1}{\alpha} - 3} g(\nabla F, \nabla |\nabla F|^2) Vol_{\eta}$. To handle the term $\int_M F^{\frac{1}{\alpha} - 2} \sum_{s=1}^3 \nabla^2 F(\xi_s, I_s \nabla F) Vol_{\eta}$ we use the next formula [5, (3.12)]

$$\int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) Vol_{\eta} = - \int_M \left[ 4n \sum_{s=1}^3 (df(\xi_s))^2 + \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] Vol_{\eta}.$$  

Set $f := F^{\frac{1}{\alpha}}$ into (3.10) to get after some calculations that

$$\int_M F^{\frac{1}{\alpha} - 2} \sum_{s=1}^3 \nabla^2 F(\xi_s, I_s \nabla F) Vol_{\eta}$$

$$= - \int_M F^{\frac{1}{\alpha} - 2} \left[ 4n \sum_{s=1}^3 (dF(\xi_s))^2 + \sum_{s=1}^3 T(\xi_s, I_s \nabla F, \nabla F) \right] Vol_{\eta}.$$  

Now, we substitute (3.9), (3.11) in (3.8) and use the properties of the torsion tensor (2.3), (2.5) to obtain the identity
Substitute the right-hand side of (3.10) into (2.10) one obtains for $f := F^\frac{1}{4\alpha}$ the formula

(3.13) 

$$-4n \int_M F_\alpha^{\frac{1}{\alpha} - 2} \sum_{s=1}^3 \left( dF(\xi_s) \right)^2 \text{Vol}_\eta$$

$$= \int_M \left[ -\frac{\alpha^2}{n} P_{F^\frac{1}{4\alpha}}(\nabla F^{\frac{1}{4\alpha}}) - \frac{1}{4n} F^{\frac{1}{\alpha} - 2}(\Delta F)^2 + \frac{1}{2n} \left( \frac{1}{2\alpha} - 1 \right) F^{\frac{1}{\alpha} - 3} \Delta F|\nabla F|^2 - \frac{1}{4n} \left( \frac{1}{2\alpha} - 1 \right)^2 F^{\frac{1}{\alpha} - 4}|\nabla F|^4 - F^{\frac{1}{\alpha} - 2} \left( S|\nabla F|^2 - T^0(\nabla F, \nabla F) + \frac{2(n-2)}{n-1} U(\nabla F, \nabla F) \right) \right] \text{Vol}_\eta.$$ 

It follows from the inequalities [5, (4.6), (4.7)] the next representation of the norm of the horizontal Hessian:

(3.14) 

$$|\nabla^2 F|^2 = \frac{1}{4n}(\Delta F)^2 + \frac{1}{4n} \sum_{s=1}^3 \left[ g(\nabla^2 F, \omega_s) \right]^2 + p(F),$$

where $p(F)$ is a non-negative function on $M$.

Now, a substitution of (3.13) and (3.14) in (3.12) give the identity

(3.15) 

$$\int_M F_\alpha^{\frac{1}{\alpha} - 3} \Delta F|\nabla F|^2 \text{Vol}_\eta = \frac{8\alpha^3}{(3n+2)(1-2\alpha)} \int_M P_{F^\frac{1}{4\alpha}}(\nabla F^{\frac{1}{4\alpha}}) \text{Vol}_\eta$$

$$+ \frac{2n+1 - 2(3n+1)\alpha}{2(3n+2)\alpha} \int_M F^{\frac{1}{\alpha} - 4}|\nabla F|^4 \text{Vol}_\eta + \frac{(3+4n)\alpha}{2(3n+2)(1-2\alpha)} \int_M F^{\frac{1}{\alpha} - 2}(\Delta F)^2 \text{Vol}_\eta$$

$$- \frac{2n\alpha}{(3n+2)(1-2\alpha)} \int_M F^{\frac{1}{\alpha} - 2} \left[ 2nS|\nabla F|^2 + 2(n+2)T^0(\nabla F, \nabla F) + \frac{4n(n+1)}{n-1} U(\nabla F, \nabla F) \right] \text{Vol}_\eta$$

$$- \frac{2n\alpha}{(3n+2)(1-2\alpha)} \int_M F^{\frac{1}{\alpha} - 2} \left[ \frac{1}{4n} \sum_{s=1}^3 \left[ g(\nabla^2 F, \omega_s) \right]^2 + p(F) \right] \text{Vol}_\eta.$$ 

Note that we have the representation

(3.16) 

$$2nS|\nabla F|^2 + 2(n+2)T^0(\nabla F, \nabla F) + \frac{4n(n+1)}{n-1} U(\nabla F, \nabla F)$$

$$= -S|\nabla F|^2 + T^0(\nabla F, \nabla F) - \frac{2(n-2)}{n-1} U(\nabla F, \nabla F) - \frac{2n+1}{2(n+2)} L(\nabla F, \nabla F).$$

Moreover, we obtain from the formula [7, (4.12)]

$$\int_M \left[ -S|\nabla f|^2 + T^0(\nabla f, \nabla f) - \frac{2(n-2)}{n-1} U(\nabla f, \nabla f) \right] \text{Vol}_\eta = \int_M \left[ \frac{1}{4n} P_f(\nabla f) + \frac{1}{4n}(\Delta f)^2 \right] \text{Vol}_\eta.$$
\[
-\frac{1}{4n} \sum_{s=1}^{3} [g(\nabla^2 f, \omega_s)]^2 \] Vol_{\eta}
\]

with \( f := F^{\frac{n}{2}} \) the next identity:

\[
\begin{align*}
(3.17) \quad & \int_M F^{\frac{n}{2} - 2} \left[ -S|\nabla F|^2 + T^0(\nabla F, \nabla F) - \frac{2(n - 2)}{n - 1} U(\nabla F, \nabla F) \right] Vol_{\eta} \\
& = \int_M \left\{ \frac{1}{4n} \left[ F^{\frac{n}{2} - 2}(\Delta F)^2 - 2\left( \frac{1}{2\alpha} - 1 \right) F^{\frac{1}{2} - 3} \Delta F |\nabla F|^2 + \left( \frac{1}{2\alpha} - 1 \right)^2 F^{\frac{1}{2} - 4} |\nabla F|^4 \right] \\
& \quad + \frac{\alpha^2}{n} F^{\frac{1}{n}} (\nabla F^{\frac{n}{4}}) - \frac{1}{4n} F^{\frac{n}{2} - 2} \sum_{s=1}^{3} [g(\nabla^2 F, \omega_s)]^2 \right\} Vol_{\eta}.
\end{align*}
\]

Taking into account (3.16) and (3.17) in (3.15), we get after some simple calculations

\[
(3.18) \quad \frac{3(2n + 1)}{2} \int_M F^{\frac{n}{2} - 3} \Delta F |\nabla F|^2 Vol_{\eta} = \frac{8n + 3 - 6(4n + 1)\alpha}{8\alpha} \int_M F^{\frac{1}{2} - 4} |\nabla F|^4 Vol_{\eta} \\
+ \frac{(2n + 1)\alpha}{1 - 2\alpha} \int_M F^{\frac{1}{2} - 2}(\Delta F)^2 Vol_{\eta} + \frac{6\alpha^3}{1 - 2\alpha} \int_M F^{\frac{1}{n}} (\nabla F^{\frac{n}{4}}) Vol_{\eta} \\
- \frac{2n\alpha}{1 - 2\alpha} \int_M F^{\frac{1}{2} - 2} \sum_{s=1}^{3} [g(\nabla^2 F, \omega_s)]^2 \] Vol_{\eta},
\]

which is the needed representation of the term \( \int_M F^{\frac{n}{2} - 3} \Delta F |\nabla F|^2 Vol_{\eta} \).

Finally, we substitute (3.18) into (3.6) to obtain (3.1). This ends the proof of Lemma 3.1.

3.2. Proofs of Theorem 1.1. The polynomial \( h_n(\alpha) \defeq 48n\alpha^2 - 2(16n - 3)\alpha - 3 \) that appears in the right-hand side of (3.1) is non-positive for \( \alpha \in \left[ \frac{16n - 3 - \sqrt{256n^2 + 48n + 9}}{48n}, \frac{16n - 3 + \sqrt{256n^2 + 48n + 9}}{48n} \right] \).

If we choose \( \alpha \in \left[ \frac{16n - 3 - \sqrt{256n^2 + 48n + 9}}{48n}, 0 \right) \) and suppose that the conditions (i) and (ii) of Theorem 1.1 hold, it is easy to see that any summand in the right-hand side of (3.1) is non-positive, which proofs Theorem 1.1.

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