On (2-d)-Kernels in Two Generalizations of the Petersen Graph

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Abstract: A subset \( J \) is a (2-d)-kernel of a graph \( G \) if \( J \) is independent and 2-dominating simultaneously. In this paper, we consider two different generalizations of the Petersen graph and we give complete characterizations of these graphs which have (2-d)-kernel. Moreover, we determine the number of (2-d)-kernels of these graphs as well as their lower and upper kernel number. The property that each of the considered generalizations of the Petersen graph has a symmetric structure is useful in finding (2-d)-kernels in these graphs.

Keywords: domination; independence; (2-d)-kernel; generalized Petersen graphs

1. Introduction

In general, we use the standard terminology and notation of graph theory (see [1]). Let \( G \) be an undirected, connected, and simple graph with the vertex set \( V(G) \) and the edge set \( E(G) \). The order of the graph \( G \) is the number of vertices in \( G \). The size of the graph \( G \) is its number of edges. By \( P_n, n \geq 1 \) and \( C_n, n \geq 3 \), we mean a path and a cycle of order \( n \), respectively.

Let \( G = (V, E) \) and \( G' = (V', E') \) be two graphs. If \( V' \subseteq V \) and \( E' \subseteq E \), then \( G' \) is a subgraph of \( G \), written as \( G' \subseteq G \). If \( G' \subseteq G \) and \( G' \) contain all the edges \( xy \in E \) with \( x, y \in V \), then \( G' \) is an induced subgraph of \( G \) and we write \( G' := (V')_G \). Graphs \( G \) and \( G' \) are called isomorphic, and denoted by \( G \cong G' \), if there exists a bijection \( \phi : V \rightarrow V' \) with \( xy \in E \Leftrightarrow \phi(x)\phi(y) \in E' \) for all \( x, y \in V \). The complement of the graph \( G \) is a graph \( \overline{G} \) such that \( V(\overline{G}) = V(G) \) and two distinct vertices of \( \overline{G} \) are adjacent if and only if they are not adjacent in \( G \). A graph \( G \) is called bipartite if \( V(G) \) admits a partition into two classes such that every edge has its ends in different classes.

A subset \( D \subseteq V(G) \) is a dominating set of \( G \) if each vertex of \( G \) not belonging to \( D \) is adjacent to at least one vertex of \( D \). A subset \( S \subseteq V(G) \) is called an independent set of \( G \) if no two vertices of \( S \) are adjacent in \( G \). A subset \( J \) being independent and dominating is a kernel of \( G \).

The concept of kernels was initiated in 1953 by von Neumann and Morgenstern in digraphs with regard to game theory (see [2]). One of the pioneers studying the kernels in digraphs was C. Berge (see [3–5]). In literature, we can find many types and generalizations of kernels in digraphs (for results and applications, see, for example, [6–11]). The problem of the existence of kernels in undirected graphs is trivial because every maximal independent set is a kernel. Currently, distinct kind of kernels in undirected graphs are being studied quite intensively and many papers are available. For results and application, see, for example, [12–18]. Among many types of kernels in undirected graphs, there are kernels related to multiple domination, introduced by Fink and Jacobson in [19]. Let \( p \geq 1 \) be an integer. A subset \( S \) is said to be \( p \)-dominating if every vertex outside \( S \) has at least \( p \) neighbors in \( S \). If \( p = 1 \), then we obtain a dominating set in the classical sense. If \( p = 2 \), we get a 2-dominating set. A set which is 2-dominating and independent is named a 2-dominating kernel ((2-d)-kernel in short). The concept of (2-d)-kernels was introduced by...
A. Włoch in [20]. Some properties of $(2-d)$-kernels were studied in [21–24]. In particular, in [23], it was proved that the problem of the existence of $(2-d)$-kernels is $NP$-complete for general graphs. In [25], Nagy extended the concept of $(2-d)$-kernels to $k$-dominating kernels. He considered a $k$-dominating set instead of the 2-dominating set, which he called $k$-dominating independent sets. Some properties of these sets were studied in [26,27]. The number of $(2-d)$-kernels in the graph $G$ is denoted by $\sigma(G)$. Let $G$ be a graph with the $(2-d)$-kernel. The minimum cardinality of the $(2-d)$-kernel of $G$ is called a lower $(2-d)$-kernel number and denoted by $\gamma(G)$. The maximum cardinality of the $(2-d)$-kernel of $G$ is called an upper $(2-d)$-kernel number and is denoted by $\Gamma(G)$.

In this paper, we consider two different generalizations of the Petersen graph. Various types of domination in the class of generalized Petersen graphs have been extensively studied in the literature (see [28–32]). Referring to this research, we will consider $(2-d)$-kernels for two different generalizations of the Petersen graph. We solve the problem of the existence of $(2-d)$-kernels, their number, and their cardinality in these graphs. Moreover, we determine a lower and an upper kernel number in these graphs. It is worth noting that each of presented generalizations of the Petersen graph has a symmetric structure. This property is useful in finding $(2-d)$-kernels in these graphs.

2. Main Results

In this section, we consider the problem of the existence of $(2-d)$-kernels in two different generalizations of the Petersen graph. In particular, we give complete characterizations of these generalizations, which have the $(2-d)$-kernel. We determine the number of $(2-d)$-kernels in these graphs as well as the lower and the upper $(2-d)$-kernel number.

In the further part of the paper, we will use green color to mark vertices belonging to the $(2-d)$-kernel, and red color to indicate vertices that cannot belong to it.

2.1. Generalized Petersen Graph

Let $n \geq 3, k < \frac{n}{2}$ be integers. The graph $P(n,k)$ is called the generalized Petersen graph, if $V(P(n,k)) = \bigcup_{i=0}^{n-1} \{u_i, v_i\}$ and $E(P(n,k)) = \bigcup_{i=0}^{n-1} \{u_iu_{i+1}, u_iv_i, v_iv_{i+k}\}$, where subscripts are reduced modulo $n$. These graphs were first defined by Watkins in [33]. Figure 1 shows generalized Petersen graphs $P(10,3)$, $P(5,2)$ and examples of $(2-d)$-kernels in these graphs.

![Figure 1](image.png)

Figure 1. Examples of $(2-d)$-kernels in $P(10,3)$ and $P(5,2)$.

We start with the problem of existence of $(2-d)$-kernels. At the beginning, we give a sufficient condition, emerging from the property of bipartite graphs. We have the following complete characterization of bipartite generalized Petersen graphs.

Proposition 1 ([34]). Let $n \geq 3, k < \frac{n}{2}$ be integers. The graph $P(n,k)$ is bipartite if and only if $n$ is even and $k$ is odd.
From this characterization we directly obtain the sufficient condition for the existence of \((2,d)\)-kernels.

**Proposition 2.** Let \(n \geq 3, k < \frac{n}{2}\) be integers. If \(n\) is even and \(k\) is odd, then the graph \(P(n,k)\) has at least two \((2,d)\)-kernels which are a partition of the vertex set.

**Proof.** Let \(n, k\) be as in the statement of the proposition. From Proposition 1, it follows that the graph \(P(n,k)\) is a bipartite graph. Thus, there exist two independent sets of vertices \(V_1, V_2\) that are a partition of the set \(V(P(n,k))\). Moreover, the graph \(P(n,k)\) is a 3-regular graph. Therefore, sets \(V_1, V_2\) are \((2,d)\)-kernels of the graph \(P(n,k)\).

Now, we improve the above proposition to obtain the complete characterization of the generalized Petersen graph having \((2,d)\)-kernel.

**Theorem 1.** Let \(n \geq 3, k < \frac{n}{2}\) be integers. The graph \(P(n,k)\) has a \((2,d)\)-kernel if and only if

1. \(n\) is even and \(k\) is odd
2. \(n \equiv 0 \pmod{5}\) and \(k \equiv 2 \pmod{5}\)
3. \(n \equiv 0 \pmod{5}\) and \(k \equiv 3 \pmod{5}\)

**Proof.** If \(n = 3, 4\), then the result is obvious. Let \(n \geq 5, k < \frac{n}{2}\) be integers. If \(n\) is even and \(k\) is odd, then by Proposition 2, (i) follows. Let \(n \equiv 0 \pmod{5}\), \(k \equiv j \pmod{5}\), \(j = 2, 3\). We will show that the set \(J = \{u_i; i \in \{0, 5, \ldots, n\}\} \cup \{u_{i+2}; i \in \{0, 5, \ldots, n\}\} \cup \{v_{i+3}; i \in \{0, 5, \ldots, n\}\} \cup \{v_{i+4}; i \in \{0, 5, \ldots, n\}\}\) is a \((2,d)\)-kernel of \(P(n,k)\). The independence of \(J\) follows from the definition of \(P(n,k)\). Let us assume that \(x \in V(P(n,k)) \setminus J\). Then, either \(x = u_s, s \in \{0, 1, \ldots, n-1\}, s \equiv a \pmod{5}\), \(a = 1, 3, 4\) or \(x = v_t, t \in \{0, 1, \ldots, n-1\}, t \equiv b \pmod{5}\), \(b = 0, 1, 2\). We consider two cases.

1. \(x = u_s\).
   If \(s \equiv 1 \pmod{5}\), then \(\{u_{s-1}, u_{s+1}\} \subseteq N(u_s)\) and \(u_{s-1}, u_{s+1} \in J\). If \(s \equiv 3 \pmod{5}\), then \(\{u_{s-1}, v_s\} \subseteq N(u_s)\) and \(u_{s-1}, v_s \in J\). If \(s \equiv 4 \pmod{5}\), then \(\{u_{s+1}, v_s\} \subseteq N(u_s)\) and \(u_{s+1}, v_s \in J\).

2. \(x = v_t\).
   Let \(t \equiv 0 \pmod{5}\). If \(k \equiv 2 \pmod{5}\), then \(\{u_t, v_{t-2}\} \subseteq N(v_t)\) and \(u_t, v_{t-2} \in J\). If \(k \equiv 3 \pmod{5}\), then \(\{u_t, v_{t+1}\} \subseteq N(v_t)\) and \(u_t, v_{t+1} \in J\). If \(k \equiv 1 \pmod{5}\), then \(\{v_{t-k}, v_{t+k}\} \subseteq N(v_t)\) and \(v_{t-k}, v_{t+k} \in J\), \(k \equiv 2 \pmod{5}\), \(j = 2, 3\). Let \(t \equiv 2 \pmod{5}\). If \(k \equiv 2 \pmod{5}\), then \(\{u_t, v_{t+2}\} \subseteq N(v_t)\) and \(u_t, v_{t+2} \in J\). If \(k \equiv 3 \pmod{5}\), then \(\{u_t, v_{t-3}\} \subseteq N(v_t)\) and \(u_t, v_{t-3} \in J\).

Summing up all the above cases we obtain that every vertex \(x \in V(P(n,k)) \setminus J\) is 2-dominated by \(J\). Hence, \(J\) is a \((2,d)\)-kernel of \(P(n,k)\).

Conversely, let \(n \geq 5, k < \frac{n}{2}\), \(i \in \{0, 1, \ldots, n-1\}\) be integers and let \(J\) be a \((2,d)\)-kernel of \(P(n,k)\). If \(u_i, u_{i+1}, u_{i+2} \notin J\), then the vertex \(u_{i+1}\) is not 2-dominated by \(J\). Thus, each connected component of the graph \(\bigcup_{i=0}^{n-1} u_i \setminus J\) is isomorphic to either \(P_1\) or \(P_2\). We will show that in the graph \(P(n,k)\) having a \((2,d)\)-kernel, the configurations of these paths \(P_1, P_2\) on the outer cycle, which are shown in the Figure 2 are forbidden.

![Figure 2](https://example.com/figure2.png)

**Figure 2.** Forbidden configurations of the paths \(P_1, P_2\) for the graph \(P(n,k)\) with the \((2,d)\)-kernel.

Let us consider the following cases.

1. First, we will prove that the configuration of the paths \(P_1, P_2\) shown on the left side of the Figure 2 is forbidden. Suppose that \(u_i, u_{i+3}, u_{i+6} \in J\) for some \(i\), as in Figure 3. Then,
$v_{i+1}, v_{i+2}, v_{i+4}, v_{i+5} \in J$; otherwise, vertices $u_{i+1}, u_{i+2}, u_{i+4}, u_{i+5}$ are not 2-dominated by $J$.

Therefore, for every $k$ vertices $v_{i+1+k}, v_{i+2+k}, v_{i+4+k}, v_{i+5+k} \notin J$.

**Figure 3.** The case when $u_i, u_{i+3}, u_{i+6} \in J$.

We have the next two possibilities.

1.1. $v_{i+3+k} \notin J$ for some $i$ (see Figure 4).

Since $v_{i+3+k} \notin J$, the vertex $u_{i+3+k} \in J$ and $u_{i+2+k}, u_{i+4+k} \notin J$. Then $v_{i+2+2k}, v_{i+3+2k}, v_{i+4+2k} \notin J$. This means that $u_{i+2+2k}, u_{i+3+2k}, u_{i+4+2k} \notin J$. Hence, the vertex $u_{i+3+2k}$ is not 2-dominated by $J$, a contradiction.

**Figure 4.** The case when $u_i, u_{i+3}, u_{i+6} \in J$ (the first subcase).

1.2. $v_{i+3+k} \in J$ for some $i$ (see Figure 5).

Then, $u_{i+3+k} \in J$ and $u_{i+2+k}, u_{i+4+k} \in J$; otherwise, they are not 2-dominated by $J$. Because $J$ is an independent set, $u_{i+1+k}, u_{i+5+k} \notin J$. Moreover, $u_{i+k}, u_{i+6+k} \in J$ to 2-dominate $u_{i+1+k}, u_{i+5+k}$. Hence, $v_{i+k}, v_{i+6+k} \notin J$. To 2-dominate $v_{i+1+k}, v_{i+5+k}$, we must have $v_{i+1+2k}, v_{i+5+2k} \in J$. Moreover, $u_{i+1+2k}, u_{i+5+2k} \in J$ and $u_{i+2+2k}, u_{i+4+2k} \notin J$. Next, $u_{i+2+2k}, u_{i+4+2k} \in J$ to 2-dominate $u_{i+1+2k}, u_{i+5+2k}$ and $v_{i+2+2k}, v_{i+4+2k} \notin J$. Thus, $v_{i+2+3k}, v_{i+3+3k}, v_{i+4+3k} \in J$ to 2-dominate $v_{i+2+2k}, v_{i+3+2k}, v_{i+4+2k}$. Therefore, $u_{i+2+3k}, u_{i+3+3k}, u_{i+4+3k} \notin J$. This means that $u_{i+3+3k}$ is not 2-dominated, a contradiction.

**Figure 5.** The case when $u_i, u_{i+3}, u_{i+6} \in J$ (the second subcase).

Hence, for each $n$ and $k$, it is not possible that the vertices $u_i, u_{i+3}, u_{i+6}$ belong to a $(2,d)$-kernel of $P(n,k)$.

2. Now, we will prove that the configuration of the paths $P_1, P_2$ shown on the right side of the Figure 2 is forbidden. Suppose that $u_i, u_{i+2}, u_{i+4}, u_{i+7} \in J$ for some $i$, as in Figure 6.

Then, $v_{i+5}, v_{i+6}$, which causes $v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+7} \notin J$.

**Figure 6.** The case when $u_i, u_{i+2}, u_{i+4}, u_{i+7} \in J$.

We consider four subcases.

2.1. $v_{i+1}, v_{i+3} \notin J$ for some $i$ (see Figure 7).

Then, $v_{i+1-k}, v_{i+3-k}, v_{i+1+k}, v_{i+3+k} \in J$. Since $v_{i+2}$ must be 2-dominated, so $v_{i+2-k} \in J$ or $v_{i+2+k} \in J$. Without loss of generality, assume that $v_{i+2-k} \in J$. Thus, $u_{i+1+k}, u_{i+2+k}, u_{i+3+k} \notin J$. Hence, the vertex $u_{i+2+k}$ is not 2-dominated, a contradiction.

**Figure 7.** The case when $u_i, u_{i+2}, u_{i+4}, u_{i+7} \in J$ (the first subcase).
2.2. \( v_{i+1} \notin J \) and \( v_{i+3} \in J \) for some \( i \) (see Figure 8).

Then, \( v_{i+3+k}, v_{i+5+k}, v_{i+6+k} \notin J \). Since \( v_{i+7} \) must be 2-dominated, we obtain that \( v_{i+7-k} \in J \) or \( v_{i+7+k} \in J \). Without loss of generality, assume that \( v_{i+7+k} \in J \). Thus, \( u_{i+7+k} \notin J \) and \( u_{i+6+k} \in J \). Because \( J \) is an independent set and \( u_{i+6+k} \in J \), \( u_{i+5+k} \notin J \). Therefore, \( u_{i+4+k} \notin J \) which causes \( u_{i+3+k}, u_{i+4+k} \notin J \). Moreover, \( u_{i+3+2k}, u_{i+4+2k}, u_{i+5+2k} \notin J \) and finally \( u_{i+3+3k}, u_{i+4+2k}, u_{i+5+2k} \notin J \). Hence, the vertex \( u_{i+4+2k} \) is not 2-dominated, a contradiction.

![Figure 8. The case when \( u_i, u_{i+2}, u_{i+4}, u_{i+7} \in J \) (the second subcase).](image)

2.3. \( v_{i+1} \in J \) and \( v_{i+3} \notin J \) for some \( i \) (see Figure 9).

Then, \( v_{i+3+k}, v_{i+5+k}, v_{i+6+k}, v_{i+3+k} \notin J \). Since \( v_{i+4} \) must be 2-dominated, \( v_{i+4-k} \in J \) or \( v_{i+4+k} \in J \). Without loss of generality, assume that \( v_{i+4+k} \in J \). Thus, \( u_{i+4+k} \notin J \). Moreover, \( u_{i+2+k}, u_{i+5+k} \notin J \) which causes \( u_{i+1+k}, u_{i+6+k}, u_{i+2+k} \notin J \). To 2-dominate \( u_{i+1+k} \), we must have \( u_{i+k} \in J \). Then, \( v_{i+k} \notin J \) and \( v_{i+2+k}, v_{i+1+k}, v_{i+2+k} \notin J \). From the independence of the set \( J \), we get that \( u_{i+2k}, u_{i+1+2k}, u_{i+2+2k} \notin J \). Hence, the vertex \( u_{i+1+2k} \) is not 2-dominated, a contradiction.

![Figure 9. The case when \( u_i, u_{i+2}, u_{i+4}, u_{i+7} \in J \) (the third subcase).](image)

2.4. \( v_{i+1}, v_{i+3} \in J \) for some \( i \).

Proving analogously as in subcase 2.3., we obtain a contradiction with the assumption that \( J \) is a \((2-d)\)-kernel. Therefore, for each \( n \) and \( k \), it is not possible that the vertices \( u_i, u_{i+2}, u_{i+4}, u_{i+7} \) belong to a \((2-d)\)-kernel of \( P(n, k) \).

Hence, for the graph with the \((2-d)\)-kernel, the configurations of \( P_1, P_2 \) shown in the Figure 10 are the only ones that may be possible. Now, we will show that they are indeed possible.

![Figure 10. Possible configurations of the paths \( P_1, P_2 \) for the graph \( P(n, k) \) with the \((2-d)\)-kernel.](image)

3. Suppose that \( u_i, u_{i+2}, u_{i+4} \notin J \) for some \( i \), as in Figure 11. Then, \( u_{i+1}, u_{i+3}, v_i, v_{i+2}, v_{i+4} \notin J \).

![Figure 11. The case when \( u_i, u_{i+2}, u_{i+4} \notin J \).](image)

We consider four subcases.

3.1. \( v_{i+2}, v_{i+3} \notin J \) for some \( i \) (see Figure 12).

Since \( v_{i+2} \) must be 2-dominated, we obtain that \( v_{i+2+k} \in J \) or \( v_{i+2-k} \in J \). Without loss of generality, assume that \( v_{i+2+k} \in J \). Moreover, \( v_{i+1+k}, v_{i+3+k} \in J \) and \( u_{i+1+k}, u_{i+2+k}, u_{i+3+k} \notin J \). Hence, the vertex \( u_{i+2+k} \) is not 2-dominated, a contradiction.
Then, \( v_{i+3} \notin J \) and \( v_{i+4} \notin J \). Since \( v_{i+2} \) must be 2-dominated, \( v_{i+2+k} \in J \) or \( v_{i+2-k} \in J \). Without loss of generality, assume that \( v_{i+2+k} \in J \). Thus, \( u_{i+1+k}, u_{i+2+k} \notin J \) and \( u_{i+k}, u_{i+3+k} \in J \), which causes \( v_{i+k}, u_{i+4+k} \notin J \) and \( v_{i+4+k} \in J \). Moreover, \( v_{i+1+2k}, v_{i+2+2k}, v_{i+3+2k} \notin J \), \( v_{i+2k} \in J \), \( u_{i+2k} \notin J \), \( u_{i+1+2k} \in J \), \( u_{i+2+2k} \notin J \), \( u_{i+3+2k} \in J \) and \( u_{i+4+2k} \notin J \). Finally, \( v_{i+2+3k}, v_{i+3+3k}, v_{i+4+3k} \notin J \) and \( u_{i+2+3k}, u_{i+3+3k}, u_{i+4+3k} \notin J \). Hence, the vertex \( u_{i+3+3k} \) is not 2-dominated, a contradiction.

**Figure 13.** The case when \( u_{i}, u_{i+2}, u_{i+4} \in J \) (the second subcase).

3.3. \( v_{i+1} \notin J \) and \( v_{i+3} \notin J \) for some \( i \).

Proving analogously as in Section 3.2., we obtain a contradiction with the assumption that \( J \) is a \((2d)\)-kernel.

3.4. \( v_{i+1}, v_{i+3} \in J \) for some \( i \) (see Figure 14).

Then, \( v_{i+1+k}, v_{i+3+k} \notin J \). First, we will show that \( v_{i+k} \) and \( v_{i-k} \) must belong to a \((2d)\)-kernel \( J \). Suppose on contrary that \( v_{i+k} \notin J \). Since \( v_{i+k} \) must be 2-dominated, \( u_{i+k} \in J \). Thus, \( u_{i+1+k} \notin J \) and \( u_{i+2+k} \in J \). Moreover, \( v_{i+1+2k}, v_{i+1+2k}, v_{i+2+2k} \in J \) and \( u_{i+2+2k}, u_{i+1+2k}, u_{i+2+2k} \in J \). Hence, the vertex \( u_{i+1+2k} \) is not 2-dominated, a contradiction.

**Figure 14.** The case when \( u_{i}, u_{i+2}, u_{i+4} \in J \) (the fourth subcase).

This means that \( v_{i+k}, v_{i-k} \in J \) and also \( v_{i+2+k}, v_{i+4+k}, u_{i+1+k}, u_{i+3+k} \) belong to a \((2d)\)-kernel (see Figure 15).

**Figure 15.** The case when \( u_{i}, u_{i+2}, u_{i+4} \in J \) implies that \( v_{i+k}, v_{i+2+k}, v_{i+4+k}, u_{i+1+k}, u_{i+3+k} \in J \).

Hence, \( n \) must be even, and from the definition of \( P(n, k) \), we conclude that \( k \) must be odd, which proves (i).

4. Suppose that \( u_{i}, u_{i+2}, u_{i+5} \in J \) for some \( i \). Then, \( u_{i+1}, u_{i+3}, u_{i+4}, v_{i}, v_{i+2}, v_{i+5} \notin J \). Since \( u_{i+3}, u_{i+4} \) must be 2-dominated, \( v_{i+3}, v_{i+4} \notin J \). First, we prove that \( v_{i+1} \notin J \). Suppose on contrary that \( v_{i+1} \in J \), as in Figure 16. Then, \( v_{i+1+k}, v_{i+3+k}, v_{i+4+k} \notin J \). Since \( v_{i+1+k} \) must be 2-dominated, \( v_{i+k} \in J \) or \( v_{i+k} \in J \). Without loss of generality, assume that \( v_{i+k} \in J \). Thus, \( u_{i+k} \notin J \), \( u_{i+1+k} \in J \) and \( u_{i+2+k} \notin J \). Moreover, \( u_{i+3+k} \in J \), \( u_{i+4+k} \notin J \), \( u_{i+5+k} \in J \), \( v_{i+5+k} \notin J \), and \( v_{i+2+k} \in J \). Proving analogously as in Section 3.3., we obtain a contradiction with the assumption that \( J \) is a \((2d)\)-kernel.

**Figure 16.** The case when \( u_{i}, u_{i+2}, u_{i+5}, v_{i+1} \in J \).
Hence, \( v_{i+1} \notin J \) (see Figure 17). Moreover, \( v_{i+1+k}, v_{i+3+k}, v_{i+4+k} \notin J \).

We consider two subcases.

\[ +k \]

Figure 17. The case when \( u_i, u_{i+2}, u_{i+5} \in J \) and \( v_{i+1} \notin J \).

4.1. \( v_{i+2+k} \notin J \) for some \( i \) (see Figure 18).

Then, \( u_{i+2+k} \in J, u_{i+3+k} \notin J, u_{i+4+k} \in J, u_{i+5+k} \notin J \), and \( v_{i+1+k} \notin J \). Moreover, \( v_{i+k} \in J \) and \( u_{i+k} \notin J \); otherwise, we obtain the same configuration as in Section 3.3.

\[ +k \]

Figure 18. The case when \( u_i, u_{i+2}, u_{i+5} \in J \) (the first subcase).

Hence, \( n \) must be divisible by 5, and from the definition of \( P(n,k) \), we conclude that \( k \equiv 2 \pmod{5} \), which proves (ii).

4.2. \( v_{i+2+k} \in J \) for some \( i \) (see Figure 19).

Then, \( u_{i+2+k} \notin J, u_{i+k}, u_{i+3+k} \in J \) and \( v_{i+k}, u_{i+4+k} \notin J \). Moreover, \( u_{i+5+k} \in J \) and \( v_{i+5+k} \notin J \).

\[ +k \]

Figure 19. The case when \( u_i, u_{i+2}, u_{i+5} \in J \) (the second subcase).

Hence, \( n \) must be divisible by 5, and from the definition of \( P(n,k) \), we conclude that \( k \equiv 3 \pmod{5} \), which proves (iii), which ends the proof.

Basing on the proof of Theorem 1, the following corollaries are obtained. They concern the number of \( (2-d) \)-kernels in the generalized Petersen graph as well as the lower and upper \( (2-d) \)-kernel numbers. By a rotation of configurations shown on Figure 10, condition (i) of Theorem 1 gives two \( (2-d) \)-kernels in generalized Petersen graph and conditions (ii) and (iii) give five \( (2-d) \)-kernels. Therefore, if \( n \) and \( k \) satisfy more than one of these conditions, we obtain more \( (2-d) \)-kernels. Moreover, the proof of the Theorem 1 presents the constructions of the \( (2-d) \)-kernels in the generalized Petersen graph \( P(n,k) \). Figure 20 shows the smallest and the largest \( (2-d) \)-kernel in the graph \( P(20,7) \).

\[ P(20,7) \]

\[ P(20,7) \]

Figure 20. The largest (left side) and the smallest (right side) \( (2-d) \)-kernel in the graph \( P(20,7) \).
**Theorem 2.** Let \( n \geq 5 \) be an integer. Let \( n \equiv 0 \) (mod 10) and \( k \equiv a \) (mod 10), \( a = 3,7 \), then

\[
\gamma(P(n,k)) = \frac{4}{5}n \quad \text{and} \quad \Gamma(P(n,k)) = n.
\]

**Corollary 2.** Let \( n \geq 3, k < \frac{n}{2} \) be integers. If \( n \equiv 0 \) (mod 10) and \( k \equiv a \) (mod 10), \( a = 3,7 \), then

\[
\gamma(P(n,k)) = \frac{4}{5}n \quad \text{and} \quad \Gamma(P(n,k)) = n.
\]

**Corollary 3.** Let \( n \geq 3, k < \frac{n}{2} \) be integers. If \( n \equiv 5 \) (mod 10) and \( k \equiv a \) (mod 5), \( a = 2,3 \) or \( n \equiv 0 \) (mod 10) and \( k \equiv a \) (mod 10), \( a = 2,8 \), then

\[
\gamma(P(n,k)) = \Gamma(P(n,k)) = \frac{4}{5}n.
\]

**Corollary 4.** Let \( n \geq 3, k < \frac{n}{2} \) be integers. If \( n \equiv 0 \) (mod 10) and \( k \equiv a \) (mod 10), \( a = 1,5,9 \) or \( n \) is even, \( n \equiv 0 \) (mod 10) and \( k \) is odd, then

\[
\gamma(P(n,k)) = \Gamma(P(n,k)) = n.
\]

The above corollaries characterize all possible graphs \( P(n,k) \), which have the \( (2-d) \)-kernel.

### 2.2. The Second Generalization of the Petersen Graph

Now, we consider another generalization of the Petersen graph. Let \( n \geq 5 \) be an integer. Let \( C_n \) be a cycle and \( \overline{C_n} \) its complement such that \( V(C_n) = \{x_1, x_2, \ldots, x_n\} \), \( V(\overline{C_n}) = \{x_1^2, x_2^2, \ldots, x_n^2\} \) with the numbering of vertices in the natural order. Let \( G(n) \) be the graph such that \( V(G(n)) = V(C_n) \cup V(\overline{C_n}) \) and \( E(G(n)) = E(C_n) \cup E(\overline{C_n}) \cup \{x_ix_i^2; i \in \{1,2,\ldots,n\}\} \). Figure 21 shows an example of a \( (2-d) \)-kernel in \( G(13) \). It is easy to check that if \( n = 5 \), then \( G(5) \) is isomorphic to the Petersen graph.

![Figure 21](image-url)

**Figure 21.** An example of a \( (2-d) \)-kernel in \( G(13) \).

The next Theorem shows a complete characterization of graphs \( G(n) \) with the \( (2-d) \)-kernel.

**Theorem 2.** Let \( n \geq 5 \) be integer. The graph \( G(n) \) has a \( (2-d) \)-kernel if and only if \( n \) is odd.

**Proof.** Let \( n \geq 5 \) be odd. We will show that \( J = \{x_i^2, x_i^3, x_i, x_4, x_6, \ldots, x_{n-1}\} \) is the \( (2-d) \)-kernel of the graph \( G(n) \). The independence of \( J \) is obvious. It is sufficient to show that \( J \)
is a 2-dominated set. By the definition of the graph $G(n)$, we can assume that $x_{n+1} = x_1$. Suppose that $y \in V(G(n)) \setminus J$. Hence, $y \in V(C_n)$ or $y \in V(C_n^c)$. Let $y \in V(C_n)$. Thus $y = x_k$, $k \in \{2, 3, 5, \ldots, n\}$. If $x_k \notin J$, then there exist vertices $x_{k-1}, x_{k+1} \in J$ adjacent to $x_k$. If $x_k \in J$, then $k = 2$ or $k = 3$. For $k = 2$, the vertex $x_2$ is adjacent to $x_1, x_2 \in J$. Moreover, if $k = 3$, then the vertex $x_3$ is adjacent to $x_4, x_3 \in J$. Hence, every vertex from the set $V(C_n)$ is 2-dominated by the set $J$. Let now $y \in V(C_n^c)$. Thus $y = x_k'$, $k \in \{1, 4, 5, \ldots, n\}$. Then, the vertex $x_k'$, $k \in \{5, 6, \ldots, n\}$ is adjacent to $x_4, x_5^2 \in J$. If $k = 1$, then $x_1'x_1, x_1'x_5^2 \in E(G(n))$. Moreover, for $k = 4$ the vertex $x_k'$ is adjacent to $x_4, x_5^2$. Therefore, vertices from the set $V(C_n)$ are 2-dominated by $J$ and hence $J$ is a $(2,d)$-kernel of $G(n)$.

Conversely, suppose that a graph $G(n)$ has a $(2,d)$-kernel $J$. We will show that $n$ is odd. By the definition of the graph $G(n)$, we obtain that $J \cap V(C_n) \neq \emptyset$. Otherwise, vertices from the set $V(C_n)$ are not 2-dominated by the set $J$. Let $x_k \notin J$. Then either $x_k^2 \notin J$ or $x_k^2 \notin J$. Otherwise, $x_k^2$ or $x_k^2$ is not 2-dominated. Hence, $|J \cap V(C_n)| = 2$. Without loss of generality assume that $x_k^2 \notin J$. This means that $x_k^2, i \in \{4, 5, \ldots, n-1\}$ is 2-dominated by $J$ and $x_k^2, x_k^2$ are dominated by $J$. Let $J^* = J \setminus \{x_k^2, x_k^2\}$. Then, $J^* \subset V(C)$. Since $J$ is the $(2,d)$-kernel, $x_3, x_k \in J^*$. Therefore, the graph $(\{x_3, x_k, \ldots, x_n\})_{G(n)} \cong P_{n-2}$ must have a $(2,d)$-kernel to 2-dominate vertices from $V(C_n) \setminus J^*$. This means that $n$ must be odd. Thus, $J^* = \{x_3, x_5, \ldots, x_n\}$, which ends the proof.

Finally, it turns out that if a graph $G(n)$ has $(2,d)$-kernels, then the number of $(2,d)$-kernels depends linearly on the number of vertices. Moreover, each $(2,d)$-kernel of $G(n)$ has the same cardinality.

**Corollary 5.** If $n \geq 5$ is odd, then $\sigma(G(n)) = n$ and

$$
\gamma(2,d)(G(n)) = \Gamma(2,d)(G(n)) = \left\lceil \frac{n}{2} \right\rceil + 2.
$$

**Proof.** Let $n \geq 5$ be odd. From the construction of a $(2,d)$-kernel described in the proof of Theorem 2, we conclude that exactly two not adjacent vertices from the set $V(C_n) \subset V(G(n))$ belong to a $(2,d)$-kernel. The selection of these two vertices will determine the $(2,d)$-kernel in $G(n)$. Since two not adjacent vertices can be chosen on $n$ ways, $\sigma(G(n)) = n$. Moreover, from the construction of $(2,d)$-kernels in $G(n)$, it follows that all of them have the same cardinality. Hence, $\gamma(2,d)(G(n)) = \Gamma(2,d)(G(n)) = \left\lceil \frac{n}{2} \right\rceil + 2$, which ends the proof.

### 3. Concluding Remarks

In this paper, we considered two different generalizations of the Petersen graph, and we discussed the problem of the existence of $(2,d)$-kernels in these graphs. In particular, we determined the number of $(2,d)$-kernels in these graphs and their lower and upper $(2,d)$-kernel number. The generalized Petersen graphs considered in this paper are special cases of $I$-graphs (see, for example, [35]). The $I$-graph $I(n,j,k)$ is a graph with a vertex set $V(I(n,j,k)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and an edge set $E(I(n,j,k)) = \{u_iu_{i+j}, u_iv_i, v_{i+k}; i \in \{1, 2, \ldots, n\}\}$, where subscripts are reduced modulo $n$. Because $P(n,k) = I(n,1,k)$, the results obtained could be a starting point to studying and counting $(2,d)$-kernels in $I$-graphs. It could also be interesting to investigate the number of $(2,d)$-kernels in other generalizations of generalized Petersen graphs. For more generalizations, see, for example, [36].

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