A CRANK-BASED APPROACH TO THE THEORY OF 3-CORE PARTITIONS

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Abstract. This note is concerned with the set of integral solutions of the equation \( x^2 + 3y^2 = 12n + 4 \), where \( n \) is a positive integer. We will describe a parametrization of this set using the 3-core partitions of \( n \). In particular we construct a crank using the action of a suitable subgroup of the isometric group of the plane that we connect with the unit group of the ring of Eisenstein integers. We also show that the process goes in the reverse direction: from the solutions of the equation and the crank, we can describe the 3-core partitions of \( n \). As a consequence we describe an explicit bijection between 3-core partitions and ideals of the ring of Eisenstein integers, explaining a result of G. Han and K. Ono obtained using modular forms.

1. Introduction

Let \( t \) be a positive integer. The set of \( t \)-core partitions is a subset of the set of integer partitions which is important in the representation theory of the symmetric groups \( S_n \). Indeed, the famous Nakayama conjecture (since proven) says that, when \( t \) is a prime number, \( t \)-core partitions label the \( t \)-blocks of the symmetric groups. In particular, defect zero characters of the symmetric group are labeled by \( t \)-core partitions of \( n \). On the other hand, \( t \)-core partitions also appear crucially in number theory. For example, F. Garvan, D. Kim and D. Stanton give in [8] a very elegant proof of Ramanujan’s congruences by constructing cranks using \( t \)-core partitions. Following Dyson [6], a crank over a finite set \( A \) is a map \( c : A \to B \) to another set \( B \), such that all fibers of \( c \) have the same cardinality. In this paper, we consider the equation over integers

\[
x^2 + 3y^2 = k,
\]

where \( k \) is a nonnegative integer. Denote by \( \mathcal{U}_k \) the set of tuples \( (x, y) \in \mathbb{Z}^2 \) that are solution of (1), and set \( u(k) = |\mathcal{U}_k| \). Equation (1) was first considered by Fermat who conjectured in 1654 [3, p. 8] that if \( k \) is a prime number congruent to 1 modulo 3, then \( u(k) \neq 0 \). This conjecture was proved by Euler [5, p. 12] in 1759.

In this note, we consider Equation (1) for \( k = 12n + 4 \), where \( n \) is a nonnegative integer. We will construct a crank \( c : \mathcal{U}_{12n+4} \to (\mathbb{Z}/3\mathbb{Z})^2 \) connected with the set \( \mathcal{E}_n \) of 3-core partitions of \( n \). In particular, we obtain a parametrization of the elements of \( \mathcal{U}_{12n+4} \) that depends on \( \mathcal{E}_n \). Moreover, this process goes in the reverse direction. From the set of solutions of (1), we can recover \( \mathcal{E}_n \) as a fiber of \( c \). In fact, such a connection is not surprising. Indeed, in 2014, N.D. Baruah and K. Nath showed in [2, Theorem 3.1] using the theory of modular forms and Ramanujan’s

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theta function identities that
\[ u(12n + 4) = 6a_3(n), \]
where \( a_3(n) = |\mathcal{E}_n| \). Note that as a consequence of our work, we obtain a new proof of this equality.

Recall that a partition \( \lambda \) of \( n \) is a non-increasing sequence of positive integers summing to \( n \). The integer \( n \) is then called the size of \( \lambda \), also denoted by \( |\lambda| \). Let \( \mathcal{P} \) be the set of partitions of integers, and \( \mathcal{P}_n \) be the one of partitions of \( n \). To any \( \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P} \), we associate its Young diagram. This is a collection of boxes arranged in left- and top-aligned rows so that the number of boxes in its \( j \)th row is \( \lambda_j \).

The hook associated to a box \( b \) of the Young diagram of \( \lambda \) is the set of the boxes of the diagram to the right and below \( b \). The number of boxes appearing in a hook is called its hooklength. We recall that a \( 3 \)-core partition is a partition with no hooklength equal to \( 3 \). We denote by \( \mathcal{E} \) the set of \( 3 \)-core partitions. Furthermore, as already mentioned, we write \( \mathcal{E}_n \) for the set of \( \lambda \in \mathcal{E} \) of size \( n \). Recall that A. Granville and K. Ono showed in [9] that every natural number has a \( t \)-core partition whenever \( t \geq 4 \); see also Remark 4.3.

The construction in this paper is based on the characteristic vectors approach for the \( 3 \)-core partitions. In [8], Garvan, Kim and Stanton proved that \( \mathcal{E} \) is in bijection with the set \( \mathcal{C} = \{(c_0, c_1, c_2) \in \mathbb{Z}^3 \mid c_0 + c_1 + c_2 = 0\} \).

Recently, the two authors proposed a new approach based on the Frobenius symbol of a partition [7] to interpret this bijection using abaci theory. Note that abaci methods were previously used in [15] by K. Ono and L. Sze to connect \( 4 \)-core partitions and class group structure of the integer rings of the imaginary quadratic fields \( \mathbb{Q}(-8n - 5) \). Such links were also established in many cases; see for example [1], [3] and [14].

Now, we present more precisely our strategy. For any nonnegative integer \( n \), we define a crank \( c : U_{12n+4} \to (\mathbb{Z}/3\mathbb{Z})^2 \). To prove that \( c \) is a crank, we will describe a natural action of \( \mathbb{Z}/6\mathbb{Z} \) over \( U_{12n+4} \) and show that it permutes the fibers of \( c \); see Theorem 2.1. On the other hand, we will construct an injective map between \( \mathcal{E}_n \) and \( U_{12n+4} \) (see Lemma 3.1), and we show that its image is a fiber of \( c \). In particular, this is a set of representatives for the \( \mathbb{Z}/6\mathbb{Z} \)-action. We then obtain a way to parametrize the elements of \( U_{12n+4} \). See Theorem 3.2. Conversely, we can recover \( \mathcal{E}_n \) from \( U_{12n+4} \). See Remark 3.4.

In [12], we give the main result of the paper. Let \( K = \mathbb{Q}(\sqrt{-3}) \). In 2011, G. Han and K. Ono connect \( 3 \)-core partitions and ideals of the ring of Eisenstein integers [11, Theorem 1.4] by proving an identity between modular forms. Ono has asked for a direct explanation for this phenomenon. We will see that our construction allows to describe \( \mathcal{E}_n \) using this ring. In particular, we obtain in Theorem 4.1 an explicit bijection between the set of \( 3 \)-core partitions and the ideals of the ring of Eisenstein integers of norm \( 3n + 1 \). This explains the result of Han and Ono.

In the last section, we give some consequence of our approach. First, we connect the numbers of \( 3 \)-core partitions of \( n \) and of \( kn + \frac{k-1}{2} \) for any integer \( k \) not divisible by \( p \) and such that the \( 3 \)-valuation at all prime with residue 2 modulo 3 is even. See Formula (13). This generalizes results of D. Hirschhorn, and J. A. Sellers [12],...
also proved by another method by Baruah and Nath in [2]. Then we will prove that the new equality
\[ a_3(3n^2 + (3^{k+1} + 2)n + 3^k) = a_3(n)a_3(n + 3^k) \]  
(3)
holds for any integers \( n \geq 1 \) and \( k \geq 0 \). Furthermore, using the crank \( c \) of [2], we construct an explicit bijection that explains it. This example illustrates the advantage of the methods developed in the present work. Finally, we discuss a question of Han [10, 5.2].

2. Construction of a crank

For any symmetric matrix \( A \in M_2(\mathbb{R}) \), we denote its corresponding orthogonal groups by \( O_2(A) \), that is
\[ O_2(A) = \{ Q \in GL_2(\mathbb{R}) \mid Q^T AQ = A \}, \]
where \( Q^T \) denotes the transposed matrix of \( Q \). When \( A = I \) is the identity matrix, we simply write \( O_2(\mathbb{R}) \) for \( O_2(I) \).

Let \( k \) be a nonnegative integer. In this section, we consider the sets
\[ A_k = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = k \} \quad \text{and} \quad B_k = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 3y^2 = k \}. \]

Note that we can interpret the sets \( A_k \) and \( B_k \) using matrices. For that, we write \( D \in M_2(\mathbb{R}) \) for the diagonal matrix with entries 1 and 3. By abuse of notation, we identify the elements of \( \mathbb{R}^2 \) with column vectors. Then we have \( X \in A_k \) if and only if \( X^T X = k \). Similarly, \( Y \in B_k \) if and only if \( Y^T D Y = k \). Moreover, we remark that \( O_2(\mathbb{R}) \) and \( O_2(D) \) act respectively on \( A_k \) and \( B_k \) by left multiplication.

Now, we denote by \( P \) the diagonal matrix with entries 1 and \( \sqrt{3} \). In particular, \( P^T P = D \), and, by elementary bilinear algebra results, conjugation and multiplication by \( P^{-1} \) induce respectively a group isomorphism between \( O_2(\mathbb{R}) \) and \( O_2(D) \) and a bijection between \( A_k \) and \( B_k \). We remark that \( U_k \) is the subset of \( B_k \) consisting of points with integral coordinates.

**Theorem 2.1.** Assume \( k = 12n + 4 \) for a nonnegative integer \( n \). Then there is a cyclic subgroup \( G \) of \( O_2(D) \) of order 6 that acts freely on \( U_k \). Moreover, \( G \) acts on the fibers of the map
\[ \epsilon : U_k \longrightarrow (\mathbb{Z}/3\mathbb{Z})^2, \quad (x, y) \mapsto (\bar{x}, \bar{2x} - 2y), \]
where \( \bar{m} \) denotes the residue class of \( m \) in \( \mathbb{Z}/3\mathbb{Z} \).

**Proof.** The orthogonal group is well-known in dimension 2. It consists of rotations and orthogonal symmetry. It follows that
\[ O_2(D) = P^{-1}O_2(\mathbb{R})P \]
\[ = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} \cos \theta & -\varepsilon \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix} \\ | \theta \in \mathbb{R}, \varepsilon \in \{-1, 1\} \right\} \]
\[ = \left\{ \begin{pmatrix} \cos \theta & -\varepsilon \sqrt{3} \sin \theta \\ \sqrt{3} \sin \theta & \cos \theta \end{pmatrix} \\ | \theta \in \mathbb{R}, \varepsilon \in \{-1, 1\} \right\}, \]
where \( P \) is defined as above. Now, consider the element \( R \) of \( O_2(D) \) associated with the parameters \( \theta = \pi/3 \) and \( \varepsilon = 1 \) in the above description, that is
\[ R = \frac{1}{2} \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}. \]
This is the conjugate by $P^{-1}$ of the rotation matrix $R_{\pi/3}$ of angle $\pi/3$ of $O_2(\mathbb{R})$. In particular, $R$ has order 6.

We set $G = \langle R \rangle$. Then $G$ acts on $B_k$ by left multiplication. Let $(x, y) \in B_k$. Write $X = (x \, y)^T$. Denote by $H$ the $G$-stabilizer of $(x, y)$. It is a subgroup of $G$ of order 1, 2, 3 or 6. The order of $H$ is not 6, because $(x, y) \neq (0, 0)$. We now exclude the orders 2 and 3. Suppose $H$ has order 2. Since $G$ is cyclic, it only has one subgroup of order 2, hence $H = \langle R^3 \rangle$, and $R^3X = -X$. It follows that $X = -X$, that is $X = 0$ which is not possible. Suppose $H$ has order 3. Then $H = \langle R^2 \rangle$. The condition $R^2X = X$ gives the system

$$ \begin{cases} -3x - 3y = 0 \\ x - 3y = 0 \end{cases}.$$ 

Here again, $X = 0$. It follows that $H$ is trivial and the $G$-action is free. Each $G$-orbit then has size 6.

As a subgroup of $O_2(D)$, $G$ acts naturally on $B_k$ by left multiplication. We now check that this action stabilizes $U_k$. Let $(x, y) \in U_k$. Then $x$ and $y$ are integers and $x^2 + 3y^2 = k$. Furthermore, since $k$ is even by assumption, we deduce that $x^2 \equiv y^2 \mod 2$, and $x$ and $y$ have the same parity. Thus, the coordinates of $RX$ gives the system

$$ R \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x - 3y \\ x + y \end{pmatrix} $$(4)

are integers, as required.

Now, we study the residue modulo 3 of the coordinates of a vector of $U_k$ under the action $R$. First, we observe that if $x = 2x'$ is even, then $x' \equiv 2x \mod 3$, and if $x = 2x' + 1$ is an odd number, then $x' \equiv 2(x - 1) \mod 3$.

Let $X^T = (x, y) \in U_k$. Write $(a, b)$ for the coordinates of $RX$. As remarked above, $x$ and $y$ have the same parity. Assume first that $x = 2x'$ and $y = 2y'$ are both even. Then (4) gives

$$ a = x' - 3y' \equiv 2x \mod 3 \quad \text{and} \quad b = x' + y' \equiv 2(x + y) \mod 3.$$ 

Assume now that $x = 2x' + 1$ and $y = 2y' + 1$ are odd. Then

$$ a = x' - 3y' - 1 \equiv 2(x - 1) - 1 \equiv 2x \mod 3$$

and

$$ b = x' + y' + 1 \equiv 2(x - 1) + 2(y - 1) + 1 \equiv 2(x + y) \mod 3.$$ 

In any cases, we obtain

$$ a \equiv 2x \mod 3 \quad \text{and} \quad b \equiv 2(x + y) \mod 3, $$ (5)

which does not depend on the parity of $x$ and $y$. Note that

$$ \epsilon(x, y) = (\overline{x}, \overline{y}). $$

Since $(x, y) \in U_k$, we have $x^2 \equiv 1 \mod 3$ because $k \equiv 1 \mod 3$ by assumption. Hence, $x \equiv \pm 1 \mod 3$ and there is no condition on the residue modulo 3 of $y$. The possible residue classes modulo 3 for the coordinates of $(x, y)$ are elements of

$$ \{ (\overline{x}, \overline{y}), (\overline{x}, \overline{3 - y}), (\overline{x}, \overline{3 - y}), (\overline{x}, \overline{y}), (\overline{x}, \overline{y}) \}. $$ (6)

Now, using (5), we compute the residue modulo 3 of the coordinates of $RX$ for a vector $X$ whose coordinates has residue in (6). We summarize the result in the following graph. The vertices are labeled by the residues modulo 3 of the coordinates of the vector and an arrow between two vertices represents a multiplication by $R$. 

\begin{center}
\begin{tikzpicture}
\end{center}
We already remark that the \( \mathbb{Z}/3\mathbb{Z} \)-residue vector of \((x, y)\) lies in \( (6) \). Then by the graph, the elements \( c(R^i(x, y)) \) for \( 0 \leq i \leq 5 \) are exactly the ones of the set \( (6) \). In particular, the image of \( c \) is the set \( (6) \) and multiplication by \( R \) permutes cyclically the fibers of \( c \).

\[ \square \]

**Remark 2.2.** Let \( k = 12n + 4 \) be a positive integer.

(i) In the proof the theorem, we only use the fact that \( k \) is even and has residue 1 modulo 3. By Chinese Remainder Theorem, this condition is equivalent to \( k \equiv 4 \mod 6 \). Suppose that \( k = 6q + 4 \) with \( q \) an odd integer. Then there is \( m \in \mathbb{Z} \) such that \( q = 2m + 1 \), and \( k = 12m + 10 \). Assume \( (x, y) \in U_k \).

Then \((x, y)\) satisfies equation (11), and by reducing the equality modulo 12, we find that the equation \( x^2 + 3y^2 = 10 \) has a solution over \( \mathbb{Z}/12\mathbb{Z} \). However, an exhaustive computation in \( \mathbb{Z}/12\mathbb{Z} \) shows that this equation has no solution. Hence, \( U_k = \emptyset \), and we only have to consider integers satisfying the assumption of the theorem.

(ii) Note that the group \( G \) constructed in Theorem 2.1 does not lie in the orthogonal group over \( \mathbb{Z} \) of \( D \). This is a subgroup of matrices over \( \mathbb{R} \) that stabilizes the integral vectors of \( B_k \).

(iii) Since \( G \) permutes cyclically the fibers of \( c \), each fiber has the same cardinality. Hence, \( c \) is a crank for \( U_k \) for any \( k \) a positive even integer. In particular, this proves that

\[ |U_k| \equiv 0 \mod 6. \]

(iv) Each elements of \( (13) \) appear only once in all \( G \)-orbits. Hence, each \( G \)-orbits contain only one vector whose coordinates have \( \mathbb{Z}/3\mathbb{Z} \)-residue \((1, 1)\).

3. **Connection with 3-core partitions**

First, we roughly recall how to connect the set \( C \) of characteristic vectors given in (2) and the Frobenius symbol of 3-core partitions. For more details, we refer to [4, §3.2]. To any partition \( \lambda \), we attach bijectively two sets of nonnegative integers \( A_\lambda \) and \( L_\lambda \) of the same size, respectively called the set of arms and of legs of \( \lambda \). Geometrically, these sets can be interpreted on the Young diagram of \( \lambda \): an arm of \( \lambda \) attached to a diagonal box \( d \) of the diagram is the number of boxes in the same row and to the right of \( d \). Similarly, the leg corresponding to \( d \) is the one in the same column and below \( d \). This information is encoded into the Frobenius symbol of \( \lambda \), denoted by \( \mathfrak{F}(\lambda) = (L_\lambda \mid A_\lambda) \).

There is another geometric interpretation of the Frobenius symbol of \( \lambda \) in terms of a pointed \( t \)-abacus, where \( t \) is a positive integer. We recall it only for \( t = 3 \).
A pointed 3-abacus is an abacus with three runners labeled 0, 1 and 2 equipped with a fence. On each runner, slots both over and under the fence are labeled by the set of nonnegative integers. In each slot, a white or black bead is drawn so that the number of black beads over the fence is finite and is equal to the one of white beads under the fence. The pointed 3-abacus of \( \lambda \) is then obtained as follows. Let \( 0 \leq i \leq 2 \) and \( q \) be a positive integer. Then \( 3q + r \in \mathcal{A}_\lambda \) if and only if the pointed 3-abacus of \( \lambda \) has a black bead over the fence at position \( q \) on the runner \( i \). Similarly, it has a white bead under the fence at position \( q \) on the runner \( 2 - r \) if and only if \( 3q + r \) is a leg of \( \lambda \).

Let \( \lambda \) be a 3-core partition, and \( \mathcal{T} \) its pointed 3-abacus. For \( 0 \leq i \leq 2 \), if the \( i \)-th-runner of \( \mathcal{T} \) has no white bead under the fence, we write \( c_i \) for the number of black beads over the fence. Otherwise, we write \(-c_i\) for the number of white beads under the fence of the \( i \)-th-runner of \( \mathcal{T} \). Then we set \( c_\lambda = (c_0, c_1, c_2) \). This tuple is called the characteristic vector of \( \lambda \). In [1] §3.3, it is proved that the map

\[
\varphi : \mathcal{E} \rightarrow \mathcal{C}, \quad \lambda \mapsto c_\lambda
\]

is a well-defined bijection. In the following, we identify \( \mathbb{Z}^2 \) with \( \mathcal{C} \) using the map \( \mathbb{Z}^2 \rightarrow \mathcal{C}, (x, y) \mapsto (-x - y, x, y) \). By this abuse of notation, we also will write \( c_\lambda = (x, y) \) for the characteristic vector of \( \lambda \).

For positive integer \( n \), we denote by \( \mathcal{C}_n \) for the elements of \( \mathcal{C} \) corresponding to 3-core partitions of size \( n \).

**Lemma 3.1.** Let \( n \) be a positive integer. Then the map

\[
\vartheta : \mathcal{C}_n \rightarrow \mathcal{U}_{12n+4}, \quad (x, y) \mapsto (6x + 3y + 1, 3y + 1)
\]

is injective.

**Proof.** For any 3-core partition \( \lambda \) with characteristic vector \( c_\lambda = (x, y) \in \mathbb{Z}^2 \), recall that gives \(|\lambda| = 3x^2 + 3xy + 3y^2 + x + 2y\). See for example [1] Corollary 3.20. However,

\[
3x^2 + 3xy + 3y^2 + x + 2y = \frac{1}{12}(6x + 3y + 1)^2 + \frac{1}{4}(3y + 1)^2 + \frac{1}{3}
\]

Furthermore, \( \lambda \in \mathcal{E}_n \) if and only if \(|\lambda| = n \) if and only if

\[
(6x + 3y + 1)^2 + 3(3y + 1)^2 = 12n + 4.
\]

In particular, \((6x + 3y + 1, 3y + 1) \in \mathcal{U}_{12n+4}\), and the map \( \vartheta \) is well-defined. It is clearly injective. \( \square \)

The next result connects \( \mathcal{E}_n \) and \( \mathcal{U}_{12n+4} \). This gives a way to parameterize the solutions of Equation (1).

**Theorem 3.2.** Let \( n \) be a nonnegative integer. Then

\[
\mathcal{U}_{12n+4} = \{(6x + 3y + 1, 3y + 1), (3x - 3y - 1, 3x + 3y + 1), (-3x - 6y - 2, 3x), (-6x - 3y - 1, -3y - 1), (-3x + 3y + 1, -3x - 3y - 1), (3x + 6y + 2, -3x) \mid (x, y) \in \mathcal{C}_n \}.
\]

**Proof.** Write \( R \) for the matrix of \( O_2(D) \) described in the proof of Theorem 2.1. Since \( 12n + 4 \) is even, the group \( G = \langle R \rangle \) acts on \( \mathcal{U}_{12n+4} \), and each \( G \)-orbits has size 6. Let \( \vartheta : \mathcal{C}_n \rightarrow \mathcal{U}_{12n+4} \) be the injective map given in Lemma 3.1. First, we will prove that \( \text{Im}(\vartheta) \) is the set

\[
\mathcal{F}_n = \{(u, v) \in \mathcal{U}_{12n+4} \mid u \equiv 1 \mod 3 \text{ and } v \equiv 1 \mod 3 \}.
\]
It is clear that $\text{Im}(\vartheta) \subseteq \mathcal{F}_n$ by definition of $\vartheta$. Conversely, let $(u, v) \in \mathcal{F}_n$. Write $z \in \mathbb{Z}$ such that $u = z + v$. We have

$$12n + 4 = u^2 + 3v^2 = (z + v)^2 + 3v^2 = z^2 + 2zv + 4v^2.$$ \hspace{1cm} (7)

Considering this equality modulo 2, we obtain that $z^2 \equiv 0 \mod 2$, hence $z$ is even. On the other hand, $z + v$ and $v$ have the same residue modulo 3 since $(z + v, v) \in \mathcal{F}_n$. This implies that $z \equiv 0 \mod 3$. Hence, $z$ is divisible by 6, and there is $x \in \mathbb{Z}$ such that $z = 6x$. Furthermore, $v \equiv 1 \mod 3$. Thus, there is $y \in \mathbb{Z}$ such that $v = 3y + 1$, and

$$\vartheta(x, y) = (6x + 3y + 1, 3y + 1) = (z + v, v) = (u, v),$$

proving that $\mathcal{F}_n \subseteq \text{Im}(\vartheta)$. Hence,

$$\text{Im}(\vartheta) = \mathcal{F}_n.$$ \hspace{1cm} (8)

However, we proved in Theorem 2.1 that $\mathcal{F}_n$ is a set of representatives of the $G$-orbits on $U_{12n+4}$. Note that for any $X^T = (u, v) \in U_{12n+4}$,

$$R X = \frac{1}{2} \begin{pmatrix} u - 3v \\ u + v \end{pmatrix}, \quad R^2 X = \frac{1}{2} \begin{pmatrix} -u - 3v \\ u - v \end{pmatrix}, \quad R^3 X = -X,$$

$$R^4 X = \frac{1}{2} \begin{pmatrix} -u + 3v \\ -u - v \end{pmatrix}, \quad R^5 X = \frac{1}{2} \begin{pmatrix} u + 3v \\ -u + v \end{pmatrix}.$$ \hspace{1cm} (7)

Finally, we conclude by computing the orbit of $\vartheta(x, y)$ for $(x, y) \in C_n$ using these relations.

**Remark 3.3.** For any $(u, v) \in U_{12n+4}$, we write $G \cdot (u, v)$ for its $G$-orbit. By Theorem 3.2 we have

$$u(12n + 4) = |U_{12n+4}| = \sum_{(x, y) \in C_n} |G \cdot \vartheta(x, y)| = 6|C_n| = 6|E_n|.$$ \hspace{1cm} (9)

In particular, this proves [[2, Theorem 3.1]].

**Remark 3.4.** By Theorem 2.1 and Theorem 3.2 the fiber of $(\mathbf{1}, \mathbf{1})$ with respect to the crank $c$ is the set $C_n$. Then we obtain a way to recover $E_n$ from $U_{12n+4}$.

**Example 3.5.** We consider the equation $x^2 + 3y^2 = 448$ over $\mathbb{Z}^2$. First, we remark that $448 = 12 \cdot 37 + 4$. Then we have to describe $E_{37}$. On the other hand, $37 = 4 \times 9 + 1$. By [4, Corollary 5.3], $E_9$ and $E_{37}$ are in bijection. Now, we remark by looking at its character table that the symmetric group $S_9$ has exactly two defect zero characters labeled by the partitions

$$\lambda = (5, 3, 1) \quad \text{and} \quad \mu = (3, 2^2, 1^2).$$

These are the 3-core partitions of 9. Furthermore, we have $\mathfrak{F}(\lambda) = (2, 0 \mid 1, 4)$ and $\mathfrak{F}(\mu) = \mathfrak{F}(\lambda^*) = (4, 1 \mid 0, 2)$, and the corresponding pointed 3-abacus are

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Remark 3.6. Consider the map
\[ \vartheta(-4, 1) = (-20, 4) \quad \text{and} \quad \vartheta(4, -3) = (16, -8). \]
We conclude with Theorem 3.2 that
\[ U_{448} = \{ (-20, 4), (-16, -8), (4, -12), (20, -4), (16, 8), (-4, 12), \]
\[ (-16, 8), (20, 4), (4, 12), (-16, 8), (-20, -4), (-4, -12) \}. \]

**Remark 3.6.** Consider the map
\[ \text{conj} : U_{12n+4} \rightarrow U_{12n+4}, \ (u, v) \mapsto \left( \frac{1}{2}(-u + 3v), \frac{1}{2}(u + v) \right). \]
This map is well-defined, because for \((u, v) \in U_{12n+4}\), we have \(\frac{1}{2}(-u + 3v) \in \mathbb{Z}\) and \(\frac{1}{2}(u + v) \in \mathbb{Z}\) since \(u \equiv v \mod 2\), and
\[ \left( \frac{1}{2}(-u + 3v) \right)^2 + \left( \frac{1}{2}(u + v) \right)^2 = u^2 + 3v^2 = 12n + 4. \]
Now, for a partition \(\lambda\) of \(n\), we write \(\lambda^*\) for its conjugate partition. By [4, Proposition 3.12], if \(c_\lambda = (x, y)\), then \(c_{\lambda^*} = (-x, x + y)\). Furthermore, for any 3-core partition \(\lambda\), we have
\[ \text{conj} \circ \vartheta(c_\lambda) = \vartheta(c_{\lambda^*}). \]
Hence, the image under \(\vartheta\) of the set of self-conjugate 3-cores of \(n\) consisting of the vectors \((u, v) \in U_{12n+4}\) such that \((u, v)\) lies in the 1-eigenspace of
\[ S = \frac{1}{2} \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}. \]
This is the set of \((u, u) \in \mathbb{Z}^2\) such that \(u \equiv 1 \mod 3\) and \(u^2 = 3n + 1\). In particular, there is at most one self-conjugate 3-core partition of \(n\). Let \((u, u)\) be such an element and \(k\) be any integer not divisible by 3. Let \(\varepsilon \in \{-1, 1\}\) be such that \(\varepsilon k \equiv 1 \mod 3\). Then \(\varepsilon ku \equiv 1 \mod 3\) and
\[ (\varepsilon ku)^2 = 3 \left( k^2n + \frac{k^2 - 1}{3} \right) + 1. \]
It follows that the map \(\vartheta^{-1}(u, u) \mapsto \vartheta^{-1}(\varepsilon ku, \varepsilon ku)\) from the set of self-conjugate 3-core partitions of \(n\) into the ones of 3-core partitions of \(k^2n + \frac{k^2 - 1}{3}\) is well-defined and bijective. Hence,
\[ \text{asc}_3(n) = \text{asc}_3 \left( k^2n + \frac{k^2 - 1}{3} \right), \]
where \(\text{asc}_3(n)\) denotes the number of self-conjugate 3-core partitions of \(n\). This generalizes [2, Theorem 3.6].

Note that the group generated by \(R\) and \(S\) is a dihedral group of order 12.
4. Connection with the ring of Eisenstein integers

Theorem 3.2 shows that we can recover $E_n$ from $U_{12n+4}$. See also Remark 3.4. In this section, we give another way to determine $U_{12n+4}$ using the ring of Eisenstein integers $\mathcal{R} = \mathbb{Z}[j]$, where $j = e^{2\pi i/3}$. We write $\mathcal{N} : \mathcal{R} \rightarrow \mathbb{N}$, $z \mapsto z\overline{z}$ for the norm of $\mathcal{R}$, where $\overline{z}$ denotes the complex conjugation. Many results here are standard, but we recall them for the convenience of the reader. Let $u, v \in \mathbb{Z}$. We have

\[ \frac{u^2 + \sqrt{3}v}{2} = \frac{u}{2} + \frac{v\sqrt{3}}{2}i, \]

thus $\frac{u^2 + \sqrt{3}v}{2} \in \mathcal{R}$ if and only if $u$ and $v$ have the same parity. Note also that, in this case, we have

\[ \mathcal{N}\left(\frac{u^2 + \sqrt{3}v}{2}\right) = \frac{1}{4}(u^2 + 3v^2). \]

However, as remarked in the proof of Theorem 2.1, each element of $U_{12n+4}$ has this property, hence the map

\[ \rho : U_{12n+4} \rightarrow \mathcal{R}, \quad (x, y) \mapsto \frac{x + y\sqrt{3}i}{2}, \]

is well-defined. Furthermore, $(u, v) \in U_{12n+4}$ if and only if $\mathcal{N}(\rho(u, v)) = 3n + 1$.

Write $\Pi_1$ and $\Pi_2$ for the sets of prime numbers with residue 1 and 2 modulo 3, respectively. For any integer $k$, we denote by $\nu_p(k)$ the $p$-adic valuation of $k$. By [13, Proposition 9.1.4], each $p \in \Pi_2$ is irreducible in $\mathcal{R}$, and for $p \in \Pi_1$, there is an irreducible element $x_p \in \mathcal{R}$ such that $p = \mathcal{N}(x_p)$. Furthermore, we define

\[ V_n = \prod_{p \in \Pi_1} \{0 \leq j_p \leq \nu_p(3n+1)\}, \]

and for any $j = (j_p | p \in \Pi_1) \in V_n$, we set

\[ x_j = \prod_{p \in \Pi_1} x_p^{j_p} \overline{x_p}^{\nu_p(3n+1)-j_p}. \]

**Theorem 4.1.** The set $U_{12n+4}$ is nonempty if and only if $\nu_p(3n+1)$ is even for all $p \in \Pi_2$. In this case, we set

\[ q_n = \prod_{p \in \Pi_2} p^{\frac{1}{2}\nu_p(3n+1)} \quad \text{and} \quad y_j = x_j q_n \quad \text{for } j \in V_n. \]

Then, for any $j \in V_n$, there are integers $u_j$ and $v_j$ with residue 1 modulo 3 such that

\[ y_j = \frac{1}{2} \left( u_j + v_j i\sqrt{3} \right), \]

and

\[ C_n = \{ \vartheta^{-1}(u_j, v_j) | j \in V_n \}, \]

where $\vartheta : C_n \rightarrow U_{12n+4}$ is the map defined in Lemma 3.1.

**Proof.** First, we have

\[ 3n + 1 = \prod_{p \in \Pi_1} p^{\nu_p(3n+1)} \cdot \prod_{p \in \Pi_2} p^{\nu_p(3n+1)} \]

\[ = \prod_{p \in \Pi_1} p^{\nu_p(3n+1)} \cdot \overline{x_p}^{\nu_p(3n+1)} \cdot \prod_{p \in \Pi_2} p^{\nu_p(3n+1)}, \]
that gives a factorization into irreducible elements in \( \mathcal{R} \). Now, we note that there is \( z = \frac{5}{2} + \frac{1}{2}i\sqrt{3} \in \mathcal{R} \) such that \( 3n + 1 = \mathbb{N}(z) = z\overline{z} \) if and only if \( a^2 + 3b^2 = 12n + 4 \) if and only if \((a, b) \in U_{12n+4} \). Assume there is \( z \in \mathcal{R} \) such that \( 3n + 1 = \mathbb{N}(z) \).

By [13 Proposition 1.4.2], \( \mathcal{R} \) is a Euclidean domain. Hence, for any \( p \in \Pi_3 \) such that \( \nu_p(3n+1) \neq 0 \), \( p \) divides \( z \) or \( \overline{z} \) in \( \mathcal{R} \). We can assume without loss of generality that \( p \) divides \( z \). Then \( p = \overline{p} \) divides \( \overline{z} \), proving that \( p^\nu \) divides \( 3n+1 \) in \( \mathbb{Z} \). It follows that \( \nu_p(3n+1) \) is even. Furthermore, by the uniqueness (up to a unit element) of the decomposition into irreducible elements in a Euclidean ring, we deduce first that \( q_n \) divides \( z \), and then that there are \( \xi \in \mathcal{R}^\times \) and integers \( 0 \leq j_p \leq \nu_p(3n+1) \) and \( 0 \leq j'_p \leq \nu_p(3n+1) \) for any \( p \in \Pi_1 \), such that

\[
z = \xi q_n \prod_{p \in \Pi_1} x_p^{j_p} \overline{x}_p^{j'_p}.
\]

Hence,

\[
3n + 1 = z\overline{z} = q_n^2 \prod_{p \in \Pi_1} x_p^{j_p+j'_p} \overline{x}_p^{j_p+j'_p}.
\]

By the uniqueness of the decomposition, we deduce that \( j_p + j'_p = \nu_p(3n+1) \). It follows that

\[
z = \xi q_n \prod_{p \in \Pi_1} x_p^{j_p} \overline{x}_p^{
u_p(3n+1)-j_p} = \xi q_n x_j = \xi y_j,
\]

with \( j = (j_p \mid p \in \Pi_1) \). Conversely, any factorization in \( \mathcal{R} \) of \( 3n + 1 \) of the form \( z\overline{z} \) is as in (12) for some \( \xi \in \mathcal{R}^\times \) and \( j \in \mathcal{V}_n \).

Now, by [13 Proposition 9.1.1], \( \mathcal{R}^\times \) has order 6 and \( \mathcal{R}^\times = (j+1) \). Note that \( \mathcal{R}^\times \) acts on \( \rho(U_{12n+4}) \) by multiplication since the elements of \( \mathcal{R}^\times \) have norm 1. On the other hand, if \( z = \frac{1}{2}(a + bi\sqrt{3}) \) for \( a, b \in \mathbb{Z} \), then

\[
(j+1)z = \frac{1}{2} \left( \frac{1}{2}(3b) + \frac{1}{2}(a + bi) \right).
\]

Comparing with (11), we see that \( (R) \) acts on \( \mathcal{C}_n \) like \( \mathcal{R}^\times \) on \( \rho(U_{12n+4}) \). In particular, by Theorem [3.2] for any \( j \in \mathcal{V}_n \), there is \( \xi \in \mathcal{R}^\times \) such that \( \xi q_n x_j = \frac{1}{2}(u_j + v_ji) \) satisfies \( u_j \equiv v_j \equiv 1 \mod 3 \). Then \( \mathcal{O}(\mathcal{C}_n) = \{ (u_j, v_j) \mid j \in \mathcal{V}_n \} \) by Theorem [3.2] as required.

**Remark 4.2.** By Theorem [3.2] and Theorem [11], the description of the set of 3-core partitions of \( n \) is reduced to

(i) The determination of the prime decomposition in \( \mathbb{Z} \) of \( 3n + 1 \).

(ii) The decomposition of the prime \( p \in \Pi_1 \) dividing \( 3n + 1 \) into irreducible elements in \( \mathbb{Z}[j] \). This factorization is of the form \( z\overline{z} \) for irreducible elements \( z \) and \( \overline{z} \) of \( \mathbb{Z}[j] \).

**Remark 4.3.** By Theorem [3.2] we have

\[
|\mathcal{E}_n| = |\mathcal{C}_n| = |\{ (u_j, v_j) \mid j \in \mathcal{V}_n \}| = |\mathcal{V}_n| = \prod_{p \in \Pi_1} (\nu_p(3n+1) + 1),
\]

and we recover a well-known result first proven by Granville and Ono [9], and also found in [12 Theorem 6] and [2 Corollary 3.3].

Theorem [3.2] can also be useful to find the set of 3-core partitions of an integer \( n \); a hard problem in general.
Example 4.4. We determine the 3-core partitions of $n = 100$. First, consider the equation $x^2 + 3y^2 = 4 \cdot 301$. Then describe $\mathcal{U}_{1204}$ using Theorem [4.4]. We remark that $301 = 7 \times 43$, and 7 and 43 lie in $\Pi_1$. However, 7 = $N(2 + i\sqrt{3})$ and $43 = N(4 + 3i\sqrt{3})$. Set $x_3 = 2 + i\sqrt{3}$ and $x_{43} = 4 + 3i\sqrt{3}$, and consider
\[
\alpha_1 = \overline{x_3 x_{43}} = \frac{1}{2}(-2 - 20i\sqrt{3}), \quad \alpha_2 = \overline{x_7 x_{43}} = \frac{1}{2}(34 - 4i\sqrt{3}),
\]
\[
\alpha_3 = \overline{x_7 x_{43}} = \frac{1}{2}(34 + 4i\sqrt{3}) \quad \text{and} \quad \alpha_4 = \overline{x_7 x_{43}} = \frac{1}{2}(-2 + 20i\sqrt{3}).
\]
Thus, $y_{0,0} = \alpha_1$ and $y_{0,1} = \alpha_3$, and
\[
y_{1,0} = (1 + j)^2\alpha_2 = \frac{1}{2}(-11 + 19i\sqrt{3}) \quad \text{and} \quad y_{1,1} = (1 + j)^2\alpha_4 = \frac{1}{2}(-29 - 11i\sqrt{3}).
\]
Thus,
\[
\mathcal{F}_{100} = \{(−2, −20), (−11, 19), (−29, −11), (34, 4)\}.
\]
Now, as in the proof of Theorem 3.19 we obtain
\[
\varphi(3, −7) = (−2, −20), \quad \varphi(−5, 6) = (−11, 19), \quad \varphi(−3, −4) = (−29, −11)
\]
and
\[
\varphi(5, 1) = (34, 4).
\]
Hence,
\[
\mathcal{C}_{100} = \{(4, 3, −7), (−1, −5, 6), (7, −3, −4), (−6, 5, 1)\}.
\]
Let $\lambda$ be the 3-core partition with characteristic vector $(4, 3, −7)$. Using [4] Lemma 3.19, we obtain
\[
\mathfrak{f}(\lambda) = (18, 15, 12, 9, 6, 3, 0 | 0, 1, 3, 4, 6, 7, 9),
\]
and we deduce that
\[
\lambda = (10, 9^2, 8^2, 7^2, 6^2, 5^2, 4^2, 3^2, 2^2, 1^2).
\]
Similarly, we find that the 3-core partitions of 100 with characteristic vectors $(-1, -5, 6), (7, -3, -4)$ and $(-6, 5, 1)$ are respectively
\[
(18, 16, 14, 12, 10, 8, 6, 4, 3^2, 2^2, 1^2), \quad (19, 17, 15, 13, 11, 9, 7, 5, 3, 1) \quad \text{and} \quad (14, 12, 10, 8, 7^2, 6^2, 5^2, 4^2, 3^2, 2^2, 1^2).
\]

5. Applications

5.1. Generalization of Hirschhorn-Sellers formula. Let $n$ be a positive integer. Let $k$ be a positive integer not divisible by 3. Write $m$ be for the product of prime factors of $k$ with residue 2 modulo 3. Assume $m$ is a square. Then
\[
a_3\left(kn + \frac{k - 1}{3}\right) = a_3(n) \prod_{p \in \Pi_1} \frac{\nu_p(k) + \nu_p(3n + 1) + 1}{\nu_p(3n + 1) + 1},
\]
(13)
where $a_3(n) = |\mathcal{E}_n|$.

Indeed, we first observe that $m \equiv 1 \mod 3$ since $m$ is a square, and it follows that $k \equiv 1 \mod 3$. Set $N = kn + \frac{1}{3}(k - 1)$. We have
\[
3N + 1 = k(3n + 1) = \prod_{p \in \Pi_1} p^{\nu_p(k) + \nu_p(3n + 1)} m'm,
\]
where $m'$ is the product (with multiplicity) of the prime factors with residue 2 modulo 3 of $3n + 1$. By assumption, $mm'$ is a square if and only if $m'$ is. Hence,
Theorem 4.1 implies that \( a_3(n) = 0 \) if and only if \( a_3(N) = 0 \). Assume \( a_3(n) \neq 0 \).

Now, again by Theorem 4.1, we obtain

\[
a_3(N) = \prod_{p \in \mathbb{P}_1} (\nu_p(k) + \nu_p(3n+1) + 1) = \prod_{p \in \mathbb{P}_1} \frac{\nu_p(k) + \nu_p(3n+1) + 1}{\nu_p(3n+1) + 1} a_3(n),
\]

as required.

**Remark 5.1.**

(i) When \( k \) is a square not divisible by 3 and with no prime factors with residue 1 modulo 3, then

\[
a_3 \left( kn + \frac{k-1}{3} \right) = a_3(n).
\]

This generalizes [12, Corollary 8] and [2, Theorem 5].

(ii) Assume \( k \) is not divisible by 3 and has no prime factors congruent to 1 modulo 3. In [4, Theorem 4.1], we prove the injectivity of the map

\[
C_n \rightarrow C_{k^2n+\frac{4}{3}(k^2-1)}, \quad (x, y) \mapsto (\varepsilon kx, \varepsilon(ky + q)),
\]

where \( \varepsilon \in \{-1, 1\} \) and \( q \in \mathbb{N} \) are such that \( k = 3q + \varepsilon \). By (i), this map is bijective.

### 5.2. An amazing equality.

In this part, we derive \([3]\) from Theorem 4.1. Then using the crank \( \varepsilon \) of \([2]\) and the map \( \vartheta \) of Lemma 4.1, we construct an explicit bijection that explains this equality.

Let \( n \) and \( m \) be two positive integers such that \( 3n + 1 \) and \( 3m + 1 \) are coprime. Then the prime decomposition of \((3n+1)(3m+1)\) is the “concatenation” of the one of \(3n+1\) with the one of \(3m+1\). Since \((3n+1)(3m+1) = 3(3mn + m + n) + 1\), Theorem 4.1 gives

\[
a_3(3mn + m + n) = a_3(n)a_3(m).
\]

Assume \( m = n + 3^k \) for some \( k \in \mathbb{N} \). If \( d \) divides \( 3n + 1 \) and \( 3m + 1 \), then \( 3n + 1 \equiv 0 \equiv 3m + 1 \mod d \). Hence, \( n \equiv m \mod d \) since \( d \) and \( 3 \) are coprime, and \( 3^k \equiv 0 \mod d \). Using again that \( d \) and \( 3 \) are coprime, we deduce that \( d = 1 \). The integers \( 3n + 1 \) and \( 3m + 1 \) are then coprime, and Equality 3 holds.

Now, we assume \( a_3(n) \) and \( a_3(m) \) are non-zero. For any positive integer \( t \), and any \( j \in V_t \), we write \( x_j^{(t)} \) and \( y_j^{(t)} \) for the elements defined in \([10]\) and \([11]\). Set \( N = 3n^2 + (3^k + 1)(m + 3^k) \). Since \( 3n + 1 \) and \( 3m + 1 \) are coprime, we obtain

\[
\mathcal{V}_N = \mathcal{V}_n \times \mathcal{V}_m, \quad x_j^{(N)} = x_j^{(n)} y_j^{(m)} \quad \text{and} \quad qN = q_n q_m,
\]

where \( j = (j_n, j_m) \). Hence \( y_j^{(N)} = y_j^{(n)} y_j^{(m)} \), and any factorizations of \( 3N + 1 = z\nu \) satisfies \( z = z_n z_m \) and \( z_m \) are equal up to a unit of \( \mathbb{Z}[j] \) to \( y_j^{(n)} \) and \( y_j^{(m)} \), respectively. However, by Theorem 4.1, these elements are equal (up to a unit element) to \( \rho \circ \vartheta(x, y) \) and \( \rho \circ \vartheta(x', y') \) for some \( (x, y) \in \mathcal{C}_n \) and \( (x', y') \in \mathcal{C}_m \), where \( \rho \) is the map defined in \([5]\). Furthermore,

\[
\rho \circ \vartheta(x, y) \rho \circ \vartheta(x', y') = (X, Y),
\]

where \( X = \frac{1}{2}(-1 + 3x - 3y + 3x' - 3y' + 18x x' + 9y y' + 9y x' - 9y y') \) and \( Y = \frac{1}{2}(9yy' + 1 + 9yx' + 3y' + 3x + 9y x' + 3x' + 3y) \). We remark that \( \varepsilon(2X, 2Y) = (2, 1) \).
By the graph page\cite{1} we deduce that the element of $U_{2N+4}$ which lies in the image of $\vartheta$ and on the $(R)$-orbit of $(2X, 2Y)$ is $R^{-1}(2X, 2Y) = (6a + 3b + 1, 3b + 1)$, where

$$a = x + x' + 3xx' + 3yy' + 3yx' \quad \text{and} \quad b = y + y' - 3xx' + 3yy'. $$

This proves that the map $\alpha : C_n \times C_{n+3^k} \to C_{3n^2 + (3^k+1)n + 3^k}$ is defined for any $(x, y) \in C_n$ and any $(x', y') \in C_{n+3^k}$ by

$$\alpha((x, y), (x', y')) = (x + x' + 3xx' + 3xy' + 3yx', y + y' - 3xx' + 3yy')$$

is a bijection.

5.3. Remarks on a question of Han. In \cite[5.2]{10}, Han conjectured a criteria characterizing integers with no 3-cores. We now discuss this question.

(i) First, we remark that if $N = 4^m n + \frac{1}{3}(10 \cdot 4^{m-1} - 1)$ for some integers $n \geq 0$ and $m \geq 1$, then

$$3N + 1 = 4^m - 1(12n + 10) = 2^{m-1}(6n + 5).$$

However, $6n + 5$ is odd, hence $\nu_2(3N + 1) = 2m - 1$ is odd. Since $2 \in \Pi_2$, the first part of Theorem 4.1 gives $a_3(N) = 0$. This proves the point (i) of \cite[5.2]{10}.

(ii) Now, we focus on the point (ii) of \cite[5.2]{10}, asserting that if there are integers $n \geq 0$, $m \geq 1$, $k \geq 1$ with $m \not\equiv 2k - 1 \mod (6k - 1)$ such that

$$N = (6k - 1)^2n + (6k - 1)m + 4k - 1, \quad (14)$$

then $a_3(N) = 0$. Consider the case $N = 58$. We have $a_3(58) = 2$. We can see that by noting that $3N + 1 = 5^2 \cdot 7$ and by applying Theorem 4.1. However,

$$58 = (6k - 1)m + 4k - 1$$

for $m = 1$ and $k = 6$. Since $m \not\equiv 2k - 1 \mod 6k - 1$, 58 satisfies the previous assumptions, but $a_3(58) \neq 0$, giving a counterexample of the statement.

Note that

$$3N + 1 = 3(6k - 1)^2n + 3(6k - 1)m + 12k - 2$$

$$= (6k - 1)(3n(6k - 1) + 3m + 2).$$

If we additionally assume that $\nu_p(6k - 1)$ is odd whenever the prime $p$ has residue 2 modulo 3, then, for such a $p$, we have $6k - 1 \equiv 0 \mod p$, and $2k - 1 \equiv -4k \mod p$. Suppose $p$ divides $(3n(6k - 1) + 3m + 2)$. Then $3m + 2 \equiv 0 \mod p$, and $3m \equiv -2 \mod p$. Multiplying by $2k$, we obtain

$$m \equiv -4k \equiv 2k - 1 \mod p,$$

which is a contradiction. Hence, $\nu_p(3N + 1)$ is odd and $p \in \Pi_2$. Theorem 4.1 again gives $a_3(N) = 0$.

(iii) Let $N \in \mathbb{N}$ be such that $a_3(N) = 0$. By Theorem 4.1, there is $p \in \Pi_2$ such that $\nu_p(3N + 1)$ is odd. In fact, we can assume that $p \neq 2$, because there are at least two distinct prime numbers with this property since $3N + 1 \equiv 1 \mod 3$. Thus, $p^{\nu_p(3N+1)} \equiv -1 \mod 6$, and $p^{\nu_p(3N+1)} \geq 5$. It follows that there exists $k \geq 1$ such that $p^{\nu_p(3N+1)} = 6k - 1$. Let now $q \in \mathbb{N}$ be such that $3N + 1 = p^{\nu_p(3N+1)}q$. Then $q$ is not divisible by $p$. Note that we must have $q \equiv 2 \mod 3$ since $3N + 1 \equiv 1 \mod 3$. Assume $q > 2$. Then $q \geq 5$, and
there is $m \geq 1$ such that $q = 3m + 2$. Furthermore, $q \not\equiv 0 \pmod{p}$. The same computation as above gives that $m \not\equiv 2k - 1 \pmod{p}$, and

$$N = \frac{(6k - 1)(3m + 2) - 1}{3} = (6k - 1)m + 4k - 1.$$ 

Thus $N$ is as in (ii) with $n = 0$. Finally, assume that $q = 2$. We have $3N + 1 = 2(6k - 1) = 2(6(k - 1) + 5) = 4 \cdot 3(k - 1) + 10$. Hence,

$$N = 4(k - 1) + \frac{1}{3}(10 \cdot 4^0 - 1),$$

and $N$ is as in (i) for $n = k - 1$ and $m = 1$.

Hence, the point (iii) of [10, 5.2] holds if we use the new assumptions introduced in (ii).

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