Augmented Polynomial Symplectomorphisms and Quantization

Alexei Kanel-Belov∗1, Andrey Elishev†2,3, and Jie-Tai Yu‡1

1College of Mathematics and Statistics, Shenzhen University, Shenzhen, 518061, China
2Laboratory of Advanced Combinatorics and Network Applications, Moscow Institute of Physics and Technology, Dolgoprudny, Moscow Region, 141700, Russia
3Department of Innovations and High Technology, Moscow Institute of Physics and Technology, Dolgoprudny, Moscow Region, 141700, Russia

Abstract

We introduce a certain augmented (or quantized) version of the Weyl algebra and of its classical counterpart, the commutative Poisson algebra. We observe that the group of augmented symplectomorphisms – that is, automorphisms of the augmented Poisson algebra – possesses a dense tame subgroup (in power series topology) in complete analogy with the non-augmented case and that a canonical lifting to automorphisms of the augmented Weyl algebra exists. The canonicity allows us to provide a proof of a conjecture of Kontsevich on the Weyl algebra automorphisms. The main idea of the proof of canonical approximation and lifting is the study of singularities of curves of augmented automorphisms and their images under Ind-morphisms.

1 Introduction

Let $W_n, K$ denote the $n$-th Weyl algebra over a field $K$, which is by definition the quotient of the free associative algebra

$$\mathbb{K}\langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle$$

by the two-sided ideal generated by elements

$$b_i a_j - a_j b_i - \delta_{ij}, \quad a_i a_j - a_j a_i, \quad b_i b_j - b_j b_i.$$ 

Let also $P_n, K$ denote the commutative polynomial algebra $\mathbb{K}[x_1, \ldots, x_{2n}]$ equipped with the standard Poisson bracket

$$\{x_i, x_j\} = \omega_{ij} \equiv \delta_{i,n+j} - \delta_{i+n,j}$$

($\delta_{ij}$ is the Kronecker symbol). We will call $P_n, K$ the (commutative) Poisson algebra. The Poisson algebra $P_n$ is the classical counterpart of the Weyl algebra $W_n$ with respect to the deformation quantization. It is known that in the case of the polynomial algebra, the deformation quantization admits a natural reverse procedure,
which we will call anti-quantization, that allows one to construct classical objects (that is, objects in $P_n$ or those defined by subsets of $P_n$) from quantized ones. The correspondence takes a rather elaborate form in characteristic zero.

This paper is dedicated to the study of the following conjecture, proposed by Kontsevich.

**Conjecture 1.1.** Whenever $\text{char } K = 0$, one has the following (canonical) isomorphism

$$\text{Aut } W_{n,K} \simeq \text{Aut } P_{n,K}.$$

The functor $\text{Aut}$ returns the group of automorphisms of a $K$-algebra. Henceforth we call the elements of $\text{Aut } P_{n,K}$ polynomial symplectomorphisms, keeping in mind the fact that this group is in natural one-to-one correspondence with the group of polynomial automorphisms of the affine space $A_{K}^{2n}$ which preserve the standard symplectic structure. Conjecture 1.1 then states that in characteristic zero, the group of Weyl algebra automorphisms is naturally isomorphic to the group of polynomial symplectomorphisms.

In the work [1], various approaches to Conjecture 1.1 along with its generalizations were considered. In particular, a group homomorphism

$$\Phi : \text{Aut } W_{n,C} \rightarrow \text{Aut } P_{n,C}$$

was constructed and its properties were analyzed. The morphism $\Phi$ is a candidate for the isomorphism between the automorphism groups over $C$ and is in fact identical on the subgroups of so-called tame automorphisms. More precisely, a theorem established in [1] asserts that the morphism $\Phi$ induces an isomorphism of tame subgroups

$$\Phi : \text{TAut } W_{n,C} \sim \rightarrow \text{TAut } P_{n,C}.$$

The construction of $\Phi$ goes back to Tsuchimoto [22] and relies on reduction of the Weyl algebra to positive characteristic. The essence of the latter procedure, which we will describe in the next section along with the definition of $\Phi$, is the representation of the ground field $C$ as a reduced direct product, modulo a fixed non-principal ultrafilter $U$ on the index set, of algebraically closed fields $F_p$ of positive characteristic $p$

$$C \simeq \left( \prod_p F_p \right) / U$$

with $p$ running along an $U$-unbounded sequence of primes. This rather unusual technique allows one to use the fact that in characteristic $p$ the algebra

$$W_{n,F_p} \simeq F_p[x_1, \ldots, x_n, d_1, \ldots, d_n]$$

is Azumaya over its center; the center itself is the (commutative) polynomial algebra generated by the $p$-th powers of algebra generators:

$$F_p[x_1^p, \ldots, x_n^p, d_1^p, \ldots, d_n^p]$$

and – crucially – possesses a Poisson bracket induced from the commutator in $W_n$. The endomorphisms of $W_n$ also preserve the center and so can be restricted to this Poisson algebra to produce symplectic polynomial mappings; those can then be re-assembled from the ultraproduct decomposition and returned to characteristic zero, and the procedure is manifestly homomorphic. Note that this construction (specifically the ultraproduct decomposition of $C$) requires the ground field to be algebraically closed, which, along with the assumption of Conjecture 1.1 that
char $\mathbb{K} = 0$ allows one to set $\mathbb{K} = \mathbb{C}$ without loss of generality. The approach to Conjecture involving the morphism $\Phi$, therefore, excludes the case of the rational numbers as the ground field.

The main result of the present paper is as follows.

**Main Theorem.** *The homomorphism*

$$
\Phi : \text{Aut} W_n,\mathbb{C} \to \text{Aut} P_n,\mathbb{C}
$$

*defined previously is a group isomorphism.*

In order to prove that $\Phi$ is an isomorphism, one may try and construct its inverse. To do so, one needs to find a way to lift polynomial symplectomorphisms to Weyl algebra automorphisms over $\mathbb{C}$. A viable approach to this lifting problem is made possible by the tame isomorphism property of $\Phi$ as stated above; indeed, tame symplectomorphisms are lifted unambiguously, therefore if one could find a suitable topology on $\text{Aut} P_n,\mathbb{C}$, such that $T\text{Aut} P_n,\mathbb{C}$ were a dense subgroup, one would represent arbitrary polynomial symplectomorphisms as limits of sequences of tame symplectomorphisms and then make the limits of the pre-images under $\Phi$ of those sequences into automorphisms of $W_n,\mathbb{C}$. Such a topology does in fact exist and is the so-called formal power series topology (or, as we sometimes refer to it, augmentation topology) introduced in the classical work of Anick [4]. A symplectic version of Anick’s approximation results also holds, as we have shown in our recent work [5], so that the path to symplectomorphism lifting is seemingly clear.

The idea of approximation by tame symplectomorphisms is a natural approach to the lifting problem. However, one will encounter difficulties if one is to proceed with approximation in a straightforward manner. Namely, in approximating a symplectomorphism by tame symplectomorphisms, one cannot ensure that two distinct tame sequences converging to the same limit will lift to sequences also converging to the same limit. We refer to this issue as the non-canonicity of approximation. The introduction of augmented algebras (that is, algebras whose Poisson structure is deformed by the addition of central, or Planck, variables) provides a way to resolve the non-canonicity.

The other apparent issue with lifting by means of tame approximation is the fact that the lifted limit is an automorphism of the power series completion of $W_n$, so that in order to complete the lifting one must find a way to truncate the resulting series. The modification of the lifting procedure – the augmentation of algebras – accounts for that drawback as well. Instead of lifting to a family of associative (Weyl) algebras with star product parameterized by Planck’s constant $h$, we introduce auxiliary central (Planck) variables $h_i$, distort the Poisson structure of $P_n$ to account for the modified commutator in $W_n$ and consider the augmented versions of $W_n$ and $P_n$, which we denote by $W_n^h$ and $P_n^h$, respectively. The automorphism groups of the new algebras are Ind-varieties analogous to Aut $W_n$ and Aut $P_n$, and a proper extension of the notion of tameness exists. Also the main ingredients of the lifting procedure – that is, the homomorphism $\Phi$, the induced isomorphism of tame subgroups, and the power series topology of Aut $P_n^h$ together with tame approximation – continue to be valid, after appropriate modifications. This allows one to conduct the lifting of elements of Aut $P_n^h$ in a way parallel to the non-augmented setting. Once the lifting morphism is defined, one needs to prove a specialization of $h_i$ exists such that the respective varieties become Aut $W_n$ and Aut $P_n$. The lifting procedure allows for control of the degree of $h_i$ (which, by design, play a role analogous to $h$ in the handling of the lifted limit) and, unlike the non-augmented case, is canonical (i.e. independent of the choice of the converging tame sequence) near the identity automorphism.
Dodd [14] has established a number of deep results of homological nature which imply Conjecture [1.1]. His argument (which is also a lifting construction) relies on the theory of holonomic $D$-modules and properties of the so-called $p$-support, defined by Kontsevich in [7]. Several interesting properties of $p$-support are due to Bitoun [15]; also cf. Van den Bergh [16] for elegant proofs.

Our approach is different from Dodd’s and focuses mainly on the topological properties of automorphism ind-varieties. This technique is in line with the philosophy of Shafarevich and his school.

The schemes we consider in the paper are given by normalizations of schemes corresponding to varieties $\text{Aut}^{\leq N}$. Transition to normal schemes is required for the establishment of the canonicity of the tame approximation, by means of the investigation of the behavior of singularities of curves of augmented automorphisms. This critical property is not present in the non-augmented case; thus the introduction of augmented algebras is justified.

It may be noted that in our prior work [23] we presented a way to handle the lifting to the $\hbar$-parameterized family of Weyl algebras without the augmentation of Poisson structure. This method (of truncating the power series in the formal limit) requires further tuning, yet we believe it to be ultimately viable as well.

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2 Automorphism groups and the homomorphism $\Phi$

We begin by making a few precise definitions of the varieties involved and then proceed to explain the construction of the morphism $\Phi$.

2.1 Basic definitions

The Weyl algebra $W_n, \mathbb{C}$ is a free $\mathbb{C}$-module; once the choice of generators is specified, we fix the basis of the $\mathbb{C}$-module consisting of monomials of the form

$$x_1^{k_1} \cdots x_n^{k_n} d_1^{l_1} \cdots d_n^{l_n} \equiv x^K d^L$$

($K$ and $L$ are multi-indices) thus imposing an ordering of generators that requires all $x$’s to be written ahead of all $d$’s.

**Definition 2.1.** We set every generator $x$ and $d$ to have degree one, and if $f$ is an element of $W_n$, its degree $\deg f$ is defined as that of its highest-degree monomial (this definition does not depend on the ordering we fix due to the form of the commutation relations).

**Definition 2.2.** If $f$ is an element of $W_n$, its height is defined to be the degree of its smallest-degree monomial, in the ordering we have fixed:

$$\text{ht } f = \inf \{ \deg x^K d^L : x^K d^L \text{ is in } f \text{ with non-zero coefficient} \}.$$
The height is obviously sensitive to the ordering of generators we work with. The notions of degree and height are identical for the polynomial algebra $P_n$ with the added benefit of indifference toward the ordering.

**Definition 2.3.** If

$$(f_1, \ldots, f_m)$$

is a finite set of polynomials (in $P_n$ or in $W_n$), its degree is defined to be the largest value of $\deg f_k$, while its height is the smallest value of $\text{ht} f_k$, when $k = 1, \ldots, m$.

Any $\mathbb{C}$-endomorphism $\varphi$ of $W_n$ (or $P_n$) is identified with the set of images

$$(\varphi(x_1), \ldots, \varphi(x_n), \varphi(d_1), \ldots, \varphi(d_n))$$

of the algebra generators. The degree and the height of $\varphi$ are then defined as above.

The group $\text{Aut} W_{n,\mathbb{C}}$ admits a filtration by subsets

$$\text{Aut}^{\leq N} W_{n,\mathbb{C}} = \{ \varphi \in \text{Aut} W_{n,\mathbb{C}} : \deg \varphi \leq N \}.$$

An identical definition holds for $\text{Aut} P_{n,\mathbb{C}}$.

The sets $\text{Aut}^{\leq N} W_{n,\mathbb{C}}$ are in fact affine algebraic sets. Indeed, any element $\varphi$ of $\text{Aut}^{\leq N} W_{n,\mathbb{C}}$ is identified with a set of $2n$ polynomials as above. These in turn are identified with an array of their coefficients, which together serve as coordinates of a point in an affine space of sufficiently large dimension. The requirement for $\varphi$ to be an automorphism imposes constraints on these coordinates which obviously have the form of polynomial equations. The same is true for $\text{Aut}^{\leq N} P_{n,\mathbb{C}}$.

The sets $\text{Aut}^{\leq N} W_{n,\mathbb{C}}$ are connected by means of the obvious embeddings

$$\text{Aut}^{\leq N} W_{n,\mathbb{C}} \rightarrow \text{Aut}^{\leq N+1} W_{n,\mathbb{C}}$$

which are Zariski-closed; the direct limit of the inductive system of these mappings is of course the entire group $\text{Aut} W_{n,\mathbb{C}}$. The same holds for $\text{Aut} P_{n,\mathbb{C}}$.

In the sequel, it will be essential for us to work not with the varieties $\text{Aut}^{\leq N}$ (and corresponding schemes), but rather with their **normalized versions**. For convenience, we agree from now on to mean by $\text{Aut}^{\leq N}$ the normalization of the appropriate automorphism variety. As it turns out (cf. Theorem 2.20), the mappings that define the conjectured isomorphism $\Phi$ are actually morphisms of normalized schemes. This property is critical to the construction of the inverse by means of approximation.

The introduction of the augmented algebras, which are the subject of the next subsection, is also indispensable to the approximation: the idea of approximating arbitrary symplectomorphisms by sequences of tame symplectomorphisms, followed by lifting (of the approximating sequence) to automorphisms of the Weyl algebra, is only natural. Approximation in the non-augmented setting, however, suffers from the fatal flaw of being non-canonical: different tame symplectomorphism sequences, while converging to the same symplectomorphism, correspond (by tame subgroup isomorphism) to Weyl algebra automorphism sequences whose limits may be distinct. In the augmented case, however, the mapping $\Phi$ is continuous at the identity automorphism in the power series topology – a result we will establish in the main body of the proof – which guarantees canonicity. Transition to the normalized varieties is still required, as only in that case can we prove that $\Phi^h$ is a morphism, thus
opening the way to study the behavior of singularities of curves of automorphisms and their images under $\Phi$.

The power series topology is defined as follows. Let us for definiteness consider $P_n$ (whose generators we briefly refer to as $z_1, \ldots, z_{2n}$), and let

$$I = (z_1, \ldots, z_{2n})$$

be the ideal spanned by all its generators (we call it the **augmentation ideal**).

**Definition 2.4.** For any positive integer $N$, define the subgroups of $\text{Aut} P_{n,\mathbb{C}}$:

$$H_N = \{ \varphi \in \text{Aut} P_{n,\mathbb{C}} : \varphi(z_i) \equiv z_i \ (\text{mod } I^N) \}.$$ 

The elements of $H_N$ are automorphisms which are identity modulo terms of height at least $N$. This specifies a proper system of neighborhoods of the neutral element of $\text{Aut} P_{n,\mathbb{C}}$ and therefore defines a topology, which we refer to as the **augmentation topology** or, alternatively, **power series topology** (due to its being effectively the power series topology of $P_n$ induced by $I^N$). An analogous definition is made for $W_n$.

**Definition 2.5.** The rank of an endomorphism, $\text{rank}(\varphi)$, is defined as the height

$$\text{ht}(\varphi - \text{Id})$$

where the difference between endomorphisms is an endomorphism obtained by taking the difference between the images of the generators.

The rank measures how close the endomorphism is to the identity morphism. If $\varphi$ has rank $N$, then $\varphi$ is identity modulo $I^N$ as defined above.

### 2.2 Tame automorphisms

Suppose first that $\mathbb{K}[x_1, \ldots, x_n]$ is the polynomial algebra over a field $\mathbb{K}$, and let $\varphi$ be an automorphism of this algebra.

**Definition 2.6.** We call $\varphi$ an elementary automorphism, if it is of the form

$$\varphi = (x_1, \ldots, x_{k-1}, ax_k + f(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), x_{k+1}, \ldots, x_n)$$

with $a \in \mathbb{K}^\times$.

Observe that linear invertible changes of variables – that is, transformations of the form

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)A, \ A \in \text{GL}(n, \mathbb{K})$$

are realized as compositions of elementary automorphisms.

**Definition 2.7.** A tame automorphism is, by definition, an element of the subgroup $\text{TAut} \mathbb{K}[x_1, \ldots, x_n]$ generated by all elementary automorphisms. Automorphisms that are not tame are called wild.

All automorphisms of $\mathbb{K}[x, y]$ are tame [9][10]; whether all automorphisms of $\mathbb{K}[x_1, \ldots, x_n]$ (with $n$ even) are tame is unknown, but for $n = 3$ the celebrated Nagata’s automorphism is an example of an automorphism which is not tame [18][19].

Similarly, the group of tame symplectomorphisms $\text{TAut} P_{n,\mathbb{K}}$ is defined as the subgroup of those tame automorphisms of $\mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n]$ that preserve the
Poisson bracket. If \( \varphi \) is an elementary automorphism of \( \mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n] \), then in order to preserve the symplectic structure, it clearly must be either a linear symplectic change of generators:

\[
(x_1, \ldots, x_n, p_1, \ldots, p_n) \mapsto (x_1, \ldots, x_n, p_1, \ldots, p_n)A
\]

with \( A \in \text{Sp}(2n, \mathbb{K}) \) a symplectic matrix, or an elementary transformation of one of two following types:

\[
(x_1, \ldots, x_{k-1}, x_k + f(p_1, \ldots, p_n), x_{k+1}, \ldots, x_n, p_1, \ldots, p_n)
\]

and

\[
(x_1, \ldots, x_n, p_1, \ldots, p_{k-1}, p_k + g(x_1, \ldots, x_n), p_{k+1}, \ldots, p_n).
\]

The subgroup of tame symplectomorphisms \( \text{TAut}_{P_n, \mathbb{K}} \) is the group generated by such elementary symplectomorphisms.

The definition of the group \( \text{TAut}_{W_n, \mathbb{K}} \) of tame automorphisms of the Weyl algebra mirrors that of tame symplectomorphisms, with the commuting generators \( p_i \) replaced by \( d_i \). We also note that in all cases, we do not include in our definition of automorphism those mappings that allow the images \( \varphi(x) \) to contain a non-zero free part (an element in the span of the unit). This omittance is evidently not significant to our present discussion. In particular, the polynomials that define the elementary automorphisms as above have zero free term.

The notion of tame automorphisms provides an excellent tool for approximating arbitrary polynomial automorphisms. Classical results in this regard were established by Anick \cite{Anick}. More specifically, let \( \varphi \in \text{Aut} \mathbb{K}[x_1, \ldots, x_n] \) be an automorphism of the polynomial algebra.

**Definition 2.8.** We say that \( \varphi \) is approximated by tame automorphisms if there is a sequence

\[
\psi_1, \psi_2, \ldots, \psi_k, \ldots
\]

of tame automorphisms such that

\[
\text{ht}((\psi_k^{-1} \circ \varphi)(x_i) - x_i) \geq k
\]

for \( 1 \leq i \leq n \) and all \( k \) sufficiently large. In other words, an automorphism is approximated by a sequence of tame automorphisms if there is such a sequence that converges to this automorphism in power series topology.

Observe that any tame automorphism \( \psi \) is approximated by itself – that is, by a stationary sequence \( \psi_k = \psi \).

The following theorem is true.

**Theorem 2.9.** Let \( \varphi = (\varphi(x_1), \ldots, \varphi(x_n)) \) be an automorphism of the polynomial algebra \( \mathbb{K}[x_1, \ldots, x_n] \) over a field \( \mathbb{K} \) of characteristic zero, such that its Jacobian

\[
\text{J}(\varphi) = \det \left[ \frac{\partial \varphi(x_i)}{\partial x_j} \right]
\]

is equal to 1. Then there exists a sequence \( \{ \psi_k \} \subset \text{TAut}_{\mathbb{K}[x_1, \ldots, x_n]} \) of tame automorphisms approximating \( \varphi \).

As it turns out, this theorem of Anick has an analogue in the symplectic setting, which we have proved recently. The theorem is a crucial point to the construction of the inverse homomorphism in Conjecture \cite{Conjecture} and we state it here.
Theorem 2.10. Let $\sigma = (\sigma(x_1), \ldots, \sigma(x_n), \sigma(p_1), \ldots, \sigma(p_n))$ be a symplectomorphism of $\mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n]$ with unit Jacobian. Then there exists a sequence $\{\tau_k\} \subset \text{TAut } P_n(\mathbb{K})$ of tame symplectomorphisms approximating $\sigma$.

The proof is detailed in [5]. The theorem states that the subgroup of tame symplectomorphisms is dense in augmentation topology in the group of all polynomial symplectomorphisms.

We now turn to the definition of the candidate homomorphism for Conjecture 1.1.

2.3 The homomorphism $\Phi$

As mentioned in the introduction, the homomorphism

$$\Phi : \text{Aut } W_n, \mathbb{C} \to \text{Aut } P_n, \mathbb{C}$$

is constructed by means of a certain ultraproduct decomposition. The few items needed are hereby presented. The detailed discussion and proofs can be found in [22] or in [3].

Let $\mathcal{U} \subset 2^\mathbb{N}$ be a fixed non-principal ultrafilter on the set of positive integers (which is the index set for the sequences and the product defined below). The ultrafilter $\mathcal{U}$ induces an equivalence relation on the set of all countable sequences of prime numbers: if $(p_m) = \{p_1, \ldots, p_m, \ldots\}$ and $(q_m)$ are two sequences, they are equivalent modulo $\mathcal{U}$ if the set

$$\{m : p_m = q_m\}$$

is in $\mathcal{U}$. Since $\mathcal{U}$ is non-principal, the equivalence relation partitions the set of all prime number sequences into equivalence classes which will be of two types: classes that contain a (necessarily unique) stationary sequence and classes whose every representative is unbounded. The classes $[p]$ of the second kind are referred to as infinite primes. The name is in agreement with the notion of prime element in the ring $^{+}\mathbb{Z}$ of hyperintegers, which can be obtained as the quotient of the direct product of a countable set of copies of $\mathbb{Z}$ modulo the minimal prime ideal

$$(\mathcal{U}) = \{(a_m) \in \prod_{m \in \mathbb{N}} \mathbb{Z} : \text{the set } \{m : a_m = 0\} \text{ is in } \mathcal{U}\}.$$

The fact that $(\mathcal{U})$ is a minimal prime ideal is true as long as the components in the direct product are integral domains. If all the components are fields, then $(\mathcal{U})$ is also maximal, since the product of fields is always von Neumann regular. Note also that taking the quotient by $(\mathcal{U})$ is the same as partitioning into classes modulo $\mathcal{U}$ in the sense of the equivalence relation defined above.

Let $(p_m)$ be a prime number sequence which defines an infinite prime $[p]$ with respect to the fixed ultrafilter $\mathcal{U}$. Consider, for each $p_m$ in the sequence, the algebraically closed field $\mathbb{F}_{p_m}$ of characteristic $p_m$. The following lemma is true.

Lemma 2.11. The ultraproduct

$$\left( \prod_{m \in \mathbb{N}} \mathbb{F}_{p_m} \right) / \mathcal{U}$$

\footnote{This statement is an easy consequence of the ultrafilter properties.}
has cardinality of the continuum and is an algebraically closed field of characteristic zero.

The algebraic closedness and characteristic zero are straightforward. The cardinality is a little less straightforward, the argument may be found in [22] or in [3]. As a corollary of the well-known Steinitz’s theorem, we then have

**Lemma 2.12.**

\[
\left( \prod_{m \in \mathbb{N}} \mathbb{F}_{p^m} \right) / \mathcal{U} \simeq \mathbb{C}.
\]

This lemma constitutes the reduction of the ground field modulo infinit prime: the scalars are represented as modulo \( \mathcal{U} \) classes of sequences \( (a_m) \), with (most of) the elements \( a_m \) being algebraic over \( \mathbb{Z}_{p^m} \).

The next item concerns the properties of the Weyl algebra \( W_{n,K} \) in positive characteristic. We have already mentioned that if \( \text{char} \ K = p > 0 \), then the center

\[ C(W_{n,K}) \simeq K[z_1, \ldots, z_{2n}], \]

where \( z_i \) are \( p \)-th powers of the generators of \( W_n \) (the proof is an easy exercise). Next, the following important property holds.

**Lemma 2.13.** If \( \varphi \) is an endomorphism of \( W_{n,K} \) (with \( \text{char} \ K = p \)), then

\[ \varphi(C(W_{n,K})) \subset C(W_{n,K}) \]

– that is, the endomorphism \( \varphi \) induces an endomorphism of the center.

An elegant proof can be found in [21] (Lemma 4).

Next we define the Poisson bracket on the center of \( W_{n,K} \). Set for definiteness \( K = \mathbb{F}_p \). Then the center is given by

\[ \mathbb{F}_p [x_1^p, \ldots, x_n^p, d_1^p, \ldots, d_n^p] \]

(\( x_i \) and \( d_j \) are Weyl algebra generators) and therefore contains as a subalgebra the algebra

\[ \mathbb{Z}_p [x_1^p, \ldots, x_n^p, d_1^p, \ldots, d_n^p] \]

of polynomials over \( \mathbb{Z}_p \). Now, for any two elements \( a, b \in \mathbb{Z}_p [x_1^p, \ldots, x_n^p, d_1^p, \ldots, d_n^p] \), define

\[ \{a, b\} = -\rho \left( \frac{\rho^{-1}(a) \rho^{-1}(b)}{p} \right), \]

Here

\[ \rho : W_{n,Z} \to W_{n,Z_p} \]

is the (usual) modulo \( p \) reduction of the Weyl algebra over \( Z \), and \( a_0 \) and \( b_0 \) are elements of \( W_{n,Z} \) in the pre-images \( \rho^{-1}(a_0) \) and \( \rho^{-1}(b_0) \), respectively. It can be checked that the bracket is well defined, and the bracket admits a straightforward extension to the entire center. It is also bilinear, takes values in the center, and satisfies the required Leibnitz and Jacobi identities. Finally, it is standard in the sense that

\[ \{d_i^p, x_j^p\} = \delta_{ij}. \]

As the Poisson bracket on the center is induced by the Weyl algebra commutator, the following lemma holds.
Lemma 2.14. If $\varphi$ is an endomorphism of $W_{n,F_p}$, then the induced endomorphism $\varphi^c$ of the center $C(W_{n,F_p})$ preserves the Poisson bracket.

The last item is the ultraproduct of Weyl algebras. Just as with fields, the ultrafilter $\mathcal{U}$ induces a congruence on the direct product of Weyl algebras, so that one may define the algebra

$$A_n(\mathcal{U}, [p]) = \left( \prod_{m \in \mathbb{N}} A_{n,F_{p_m}} \right) / \mathcal{U}.$$  

As the coefficients of elements of $A_n(\mathcal{U}, [p])$ take values in $\left( \prod_{m \in \mathbb{N}} F_{p_m} \right) / \mathcal{U}$, the algebra $A_n(\mathcal{U}, [p])$ is a $\mathbb{C}$-algebra, which contains the Weyl algebra $W_{n,C}$ as a proper subalgebra (of sums of monomials of globally bounded degree). However, unlike $W_{n,C}$, the ultraproduct algebra has a huge center given, obviously, by the ultraproduct of centers $C(W_{n,F_{p_m}})$. Therefore:

Lemma 2.15. There is an injection $\mathbb{C}[z_1, \ldots, z_{2n}] \to C(A_n(\mathcal{U}, [p])).$

The homomorphism $\Phi$ is constructed in the following way. Given an endomorphism $\varphi$ of $W_{n,C}$, one represents its coefficients (that is, the coefficients of the images $\varphi(x_i)$ and $\varphi(d_j)$ in the standard $\mathbb{C}$-basis) as elements of the reduced direct product $\left( \prod_{m \in \mathbb{N}} F_{p_m} \right) / \mathcal{U}$ and constructs from them an array of endomorphisms $\varphi_{p_m}^c$ of $W_{n,F_{p_m}}$ (as there are only finitely many constraints on the coefficients of $\varphi$, this reconstruction will be possible for almost all – in the sense of $\mathcal{U}$ – indices $m$). Then one restricts to the center to obtain $\varphi_{p_m}^c$. Since these will be endomorphisms of bounded degree, they will give an endomorphism

$$C(A_n(\mathcal{U}, [p])) \to C(A_n(\mathcal{U}, [p]))$$

of the center of the ultraproduct algebra which will map its subalgebra $\mathbb{C}[z_1, \ldots, z_{2n}]$ to itself.

One could stop here and denote the resulting symplectomorphism of $\mathbb{C}[z_1, \ldots, z_{2n}]$ by $\Phi(\varphi)$; we will, however, slightly twist the morphism in order to get rid of the explicit dependence of the choice of infinite prime $[p]$. Note that in characteristic $p$ the correspondence $\varphi \mapsto \varphi^c$, when $\varphi$ is a linear change of variables (given by a symplectic matrix $A = (a_{ij})$), produces a symplectomorphism whose matrix is $A^c = (a_{ij}^p)$. If one applied the inverse Frobenius automorphism to the scalars (takes the $p$-th root), one would recover a symplectomorphism given by exactly the same set of coordinates (entries of the matrix in this case) as that of the Weyl algebra morphism. Now, any automorphism of the base field induces an automorphism of the algebra, so this added procedure does not ruin the homomorphic property of the correspondence.

Combined in the ultraproduct, the (inverse) Frobenius automorphisms give rise to an automorphism of $\mathbb{C}$, which again induces an automorphism of the algebras. The entire correspondence now consists of the following steps: start with a Weyl algebra automorphism, take it to the ultraproduct, restrict to the center by $\varphi \mapsto \varphi^c$ (for almost all $p_m$), twist by inverse Frobenius, reassemble as a $\mathbb{C}$-symplectomorphisms. Denote the resulting homomorphism by $\Phi(\varphi)$. By Lemma \[2.13\], $\Phi(\varphi)$ preserves the Poisson bracket – in other words, it is a symplectic map. To complete the construction, we state the following lemma, which is proved as part of the main proposition of \[22\].
Lemma 2.16. The resulting polynomial endomorphism $\Phi(\varphi)$ is an automorphism if and only if $\varphi$ is an automorphism.

This concludes the definition of the homomorphism $\Phi$ which is our candidate for Conjecture 1.1.

Some of the properties of $\Phi$ may be established right away, without the need for a complicated development such as approximation and lifting. For the proof of the next proposition, cf. [1].

Proposition 2.17. $\Phi$ is injective.

Surjectivity of $\Phi$ would imply Conjecture 1.1; there seems to be no direct way to obtain this result (in a manner similar to some of the proofs of [21], for instance). Constructing an inverse homomorphism via approximation and lifting of symplectomorphisms was viewed as a more viable approach.

It is interesting to note another conjecture in connection to the morphism $\Phi$, or rather to the way it is constructed.

Conjecture 2.18. The homomorphism $\Phi$ is independent of the choice of infinite prime $[p]$.

The conjecture is not vacuous, for while we did get rid of the explicit dependence of the coefficients on $[p]$ by inverse Frobenius twist, the construction of $\Phi$ via a reduced direct product decomposition allows for an implicit dependence in the following way. Observe that at the stage of decomposed automorphism $\varphi$ (with induced positive characteristic automorphisms $\varphi_{p_m}$ for almost all $p_m$), the place $p_m$ (i.e. the value $p_m$ at index $m$) could have an influence on which monomials are not zero in the images $\varphi_{p_m}(x_i)$, and while the highest-degree monomials do not change (which follows easily if one recalls what $\varphi \mapsto \varphi^c$ is for every $p$ and keeps in mind the commutation relations), it cannot be readily obtained that the rest of the monomials are fixed. Any direct approach to nail down these lower-degree places seems to be insufficient, therefore an indirect method of handling the places, by means of discovering certain rigid properties of $\Phi$, has to be implemented. This is what has been done by us recently, and Conjecture 2.18 is indeed positive. The conjecture is the subject of our work [3], and the proof (specifically its later stage) ultimately uses the properties of the same augmented versions of $W_n$ and $P_n$ we present in this paper.

We finish this subsection by formulating the crucial theorem which allows $\Phi$ to be used in the subsequent development of lifting. It first appeared and was proved in [1].

Theorem 2.19. The homomorphism $\Phi$ induces an isomorphism of the tame subgroups:

$$\Phi : \text{TAut } W_n, C \xrightarrow{\sim} \text{TAut } P_n, C.$$ 

The theorem makes lifting of tame symplectomorphisms possible at once, and the approximation as developed in [5] allows lifting of arbitrary symplectomorphisms to automorphisms of power series completion of $W_n$, so that a local version of Conjecture 1.1 holds.

To conclude this subsection, we state the following theorem.

Theorem 2.20. The mappings

$$\Phi_N : \text{Aut}^{\leq N} W_n, C \to \text{Aut}^{\leq N} P_n, C$$

induced by $\Phi$ are morphisms of (normalized) algebraic varieties.
The proof can be found in [3]. This theorem has an exact (and central to our approach) analogue in the setting of the quantized algebras $W_n$ and $P_n$, which we state for the reference in the next subsection.

2.4 Augmented Weyl algebra structure

We now define the augmented analogues of the algebras $W_n$ and $P_n$.

**Definition 2.21.** We define the augmented $n$-th Weyl algebra, which we denote by $W_n^h$ (or by $W_{n,\mathbb{R}}^h$ when indicating the ground field), as the quotient of the free associative algebra over $3n$ variables

$$\mathbb{K}\langle a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n \rangle$$

by the ideal spanned by the following elements

$$b_i a_j - a_i b_j - \delta_{ij} c_i,$$
$$a_i c_j - a_j c_i,$$
$$b_i c_j - b_j c_i,$$
$$a_i a_j - a_j a_i,$$
$$b_i b_j - b_j b_i,$$
$$c_i c_j - c_j c_i.$$

The algebra $W_n^h$ is generated by $x_i$, $d_i$ and $h_i$, $i = 1, \ldots, n$, which fulfill

$$[d_i, x_j] = h_i \delta_{ij}, \quad [d_i, h_j] = 0, \quad [x_i, h_j] = 0$$

(in addition to mutual commutativity within the respective generator subsets).

To account for the distortion of the commutator in $W_n$, we introduce the commutative variables (which we also call $h_i$; no confusion should arise, for nowhere in the sequel do the generators of $W_n^h$ and $P_n^h$ appear at the same time) to the polynomial algebra and define the (augmented) bilinear bracket:

$$\{ p_i, x_j \} = h_i \delta_{ij}.$$

**Definition 2.22.** The augmented Poisson algebra $P_{n,\mathbb{C}}^h$ is defined as the polynomial algebra of $3n$ variables

$$\mathbb{C}[x_1, \ldots, x_n, p_1, \ldots, p_n, h_1, \ldots, h_n]$$

with the additional bracket defined above.

The notion of algebra automorphism is the same for the augmented versions. A tame (in particular, elementary) automorphism of $P_{n,\mathbb{C}}^h$ is a tame (elementary) automorphism of $\mathbb{C}[x_1, \ldots, x_n, p_1, \ldots, p_n, h_1, \ldots, h_n]$ which preserves the augmented bracket. Tame automorphisms of $W_n^h$ are defined similarly by using the concept from the non-augmented case.

The groups $\text{Aut} W_{n,\mathbb{C}}$, $\text{Aut} P_{n,\mathbb{C}}$, $\text{TAut} W_{n,\mathbb{C}}$, and $\text{TAut} P_{n,\mathbb{C}}$ are realised as subgroups of $\text{Aut} W_n^h$, $\text{Aut} P_n^h$, $\text{TAut} W_n^h$, and $\text{TAut} P_n^h$, respectively. Note that the augmented algebras admit tame automorphisms which have no analogue in the non-augmented case. For instance, the following linear change is admissible for $P_{n,\mathbb{C}}^h$ (we denote temporarily by $x$ the row of generators $(x_1, \ldots, x_n)$):

$$x \mapsto \Lambda x, \quad p \mapsto \Lambda p, \quad h \mapsto \Lambda^2 h$$
with $\lambda$ a non-singular matrix.

In the case of $W_h^n$, the additional variables $h_i$ are central, therefore the properties of the Weyl algebra in characteristic $p$ will translate to the quantized case. The steps in the construction of the homomorphism $\Phi$ can then be repeated without major modifications. This will produce a group homomorphism

$$\Phi^h : \text{Aut} W_h^n, \mathbb{C} \to \text{Aut} P_h^n, \mathbb{C},$$

which also induces an isomorphism of the tame subgroups, in full analogy with Theorem 2.19. This result provides the basis for the lifting procedure.

We are inclined to refer to elements of $\text{Aut} P_h^n, \mathbb{C}$ as augmented polynomial symplectomorphisms, hence the name of the paper, although the label is somewhat inaccurate. The bracket in the definition of $P_h^n$ has no apparent geometric meaning. The introduction of the algebras $W_n$ and $P_n$ is but a device to facilitate the canonical lifting of (non-augmented) symplectomorphisms and then perform the truncation of power series in the lifted limit to arrive at an automorphism of $W_n$.

The analogue of Theorem 2.20 is straightforward.

**Theorem 2.23.** The mappings

$$\Phi^h_N : \text{Aut}^{\leq N} W_h^n, \mathbb{C} \to \text{Aut}^{\leq N} P_h^n, \mathbb{C}$$

are morphisms of (normalized) algebraic varieties.

We also observe the following

**Corollary 2.24.** The mappings of Theorems 2.20 and 2.23 are closed morphisms.

**Proof.** This property may be justified by the following line of reasoning. For the moment, let $\varphi(t)$ denote a one-parameter family (a curve) of automorphisms of $W_n, \mathbb{C}$. Suppose the curve approaches infinity at $t = t_0$. Let $a(t)$ be the coefficient in the set defining $\varphi(t)$ such that

$$a(t) \simeq (t - t_0)^{-k}, \ k > 0$$

provides the leading coefficient in the asymptotic expansion near $t_0$. Then, by construction of $\Phi$, the curve

$$\Phi(\varphi(t))$$

will have the same asymptotic behavior near $t_0$. Therefore, the morphism $\Phi$ maps curves approaching infinity to curves approaching infinity. This means that $\Phi$ is closed. The augmented case $\Phi^h$ is analogous. 

---

3 Approximation and lifting

We state the propositions necessary to establishing the approximation theorems analogous to [5]. The proof of the majority of technical results is similar to the proofs presented in [5] (in fact, they are identical modulo change in ranges of summation in some expressions, and definition of certain homogeneous modules).

The approximation theorems are established almost identically to [5], if one observes that in this setting, the new variables $h_i$ can be treated to a certain extent as parameters. This requires the assumption that $h_i h_j^{-1}$ are not large (in the sense of
their absolute value in the function field), so that if one, say, restricts \{h_1, \ldots, h_n\} to a non-singular curve by the assignment
\[ h_i = \lambda(t)h_i, \]
no singularities should appear. In this case, composition of the target automorphism \( \varphi \) on the right with (the inverses) of tame automorphisms approximating it allows for elimination of terms of successively ascending degree in the images \( \varphi(x_i) \) of generators.

Briefly, we have the following. Let \( \varphi \) be an augmented polynomial symplectomorphism (i.e. an automorphism of \( P^h_{n,\mathbb{C}} \)).

**Lemma 3.1.** There is a linear transformation \( \varphi_L \) such that its composition \( \varphi_L \circ \varphi \) fulfills
\[
\text{ht}(\varphi_L \circ \varphi(x_i) - x_i) \geq 2, \quad \text{ht}(\varphi_L \circ \varphi(p_i) - p_i) \geq 2, \quad \text{ht}(\varphi_L \circ \varphi(h_i) - h_i) \geq 2.
\]

This lemma combs the linear terms in the automorphism. The next lemma provides an inductive step in the successive processing (by composing with tame automorphisms as in Definition 2.8) of higher-degree terms.

**Lemma 3.2.** Let \( \psi \) be an automorphism \( P^h_{n,\mathbb{C}} \) such that
\[
\varphi(x_i) = x_i + U_i, \quad \varphi(p_i) = p_i + V_i, \quad \varphi(h_i) = h_i + W_i
\]
and \( U_i, V_i, W_i \) are of height at least \( k \). Then there exists a tame automorphism \( \psi_k \) such that the polynomials \( \tilde{U}_i = (\varphi^{-1}_k \circ \varphi)(x_i) - x_i, \quad \tilde{V}_i = (\varphi^{-1}_k \circ \varphi)(p_i) - p_i \) and \( \tilde{W}_i = (\varphi^{-1}_k \circ \varphi)(h_i) - h_i \) are of height at least \( k + 1 \).

The next proposition asserts the existence of tame sequences such that the coefficients of tame automorphisms comprising it are sufficiently well-behaved.

**Proposition 3.3.** Let \( \varphi \) be an automorphism of \( P^h_{n,\mathbb{C}} \) and let \( O_{\varphi} \) be the local ring of \( \text{Aut} P^h_{n,\mathbb{C}} \) with maximal ideal \( m \). Then there exists a sequence of tame automorphisms \( \{\varphi_k\} \) which converges to \( \varphi \) in augmentation topology, such that the coefficients of \( \varphi_k \) converge to coefficients of \( \varphi \) in \( m \)-adic topology.

The lifting of tame augmented symplectomorphisms is performed by means of \( (\Phi^h|_{\text{TAut}})^{-1} \). If \( \{\varphi_k\} \) is a sequence of tame automorphisms which converges to a fixed automorphism \( \varphi \) (and belongs to a subclass of sequences from Proposition 3.3), then we construct a sequence
\[
\{\sigma_k\}, \quad \sigma_k = (\Phi^h|_{\text{TAut}})^{-1}(\varphi_k)
\]
of tame automorphisms of \( W^h_{n,\mathbb{C}} \). Its limit consists of images of generators of \( W^h_n \), however these are now power series in \( x_i, d_i \) and \( h_i \), as in the non-augmented case. The point of introducing the augmentation is that now this lifted limit’s degree structure does not depend on the choice of the approximating sequence (we therefore refer to this lifting as canonical).

The next theorem is central to our approach.

**Theorem 3.4.** The morphism \( \Phi^h \) is continuous in power series topology.
The proof of this fact utilizes a technique we make frequent use of in our research and in this paper in particular – that of extracting properties of objects of interest by examining the behavior of singularities of various one-parameter subsets (or curves, in the setting of varieties) of points. The following elementary lemma justifies most of our deductions.

**Lemma 3.5.** Let

\[ \Phi : X \to Y \]

be a morphism of affine algebraic sets, and let \( \Lambda(t) \) be a curve (more simply, a one-parameter family of points) in \( X \). Suppose that \( \Lambda(t) \) does not tend to infinity as \( t \to 0 \). Then the image \( \Phi \Lambda(t) \) under \( \Phi \) also does not tend to infinity as \( t \to 0 \).

To demonstrate continuity of \( \Phi^h \), it is enough to show that \( \Phi^h \) fulfills the Cauchy property for the system of neighborhoods of the identity. For \( \text{Aut } P^h_{n,\mathbb{C}} \), these neighborhoods are

\[ H_N = \{ \varphi \in \text{Aut } P^h_{n,\mathbb{C}} : \varphi(\xi) \equiv \xi \pmod{I^N} \} \]

as in Definition 2.4. We also define

\[ G_N = \{ \varphi \in \text{Aut } W^h_{n,\mathbb{C}} : \varphi(\xi) \equiv \xi \pmod{I^N}, \} \]

to be the system of neighborhoods of the identity in \( \text{Aut } W^h_{n,\mathbb{C}} \). In the sequel, we will also need the larger sets of automorphisms which are homothety modulo \( I^N \). These are defined as

\[ \tilde{H}_N = \{ \varphi \in \text{Aut } P^h_{n,\mathbb{C}} : \varphi(\xi) \equiv c\xi \pmod{I^N}, c \in \mathbb{C}^\times \} \]

and

\[ \tilde{G}_N = \{ \varphi \in \text{Aut } W^h_{n,\mathbb{C}} : \varphi(\xi) \equiv c\xi \pmod{I^N}, c \in \mathbb{C}^\times \} \]

for \( P^h_n \) and \( W^h_n \), respectively.

The statement of Theorem 3.4 now reduces to demonstrating that

\[ \Phi^h(G_N) \subseteq H_N, \]

which in turn can be shown by manipulating parametric sets of generator substitutions with the help of Lemma 3.5. The proof below essentially copies that of similar statements in [3] and [25].

The one-parameter family of substitutions \( \Lambda(t) \) that we use is given by linear changes of variables acting on generators \( x_i \) and \( p_i \) of \( P^h_{n,\mathbb{C}} \). The linear changes will be represented by invertible matrices \( \Lambda \) whose non-diagonal part is skew-symmetric; the curve is as follows:

\[ x \mapsto \Lambda x, \; p \mapsto \Lambda p, \; h \mapsto \Lambda^2 h \]

(with \( x \), \( p \) and \( h \) denoting the column-vectors of the respective generators). The substitutions in this one-parameter family induce tame automorphisms of \( P^h_{n,\mathbb{C}} \) and as such are preserved by the morphism \( \Phi^h \).

If \( \varphi \) is an automorphism of the \( h \)-augmented Poisson algebra \( P^h_{n,\mathbb{C}} \), one can conjugate it with a fixed one-parameter family \( \Lambda(t) \) of linear substitutions to produce another one-parameter family

\[ \Lambda(t) \varphi \Lambda(t)^{-1} \]

of algebra automorphisms. The idea of the proof is that the behavior of such objects – and their \( \Phi^h \)-images – near singularities of \( \Lambda(t) \), tells us much about the way
\( \Phi^h \) respects the degree structure of the images under automorphisms of algebra generators.

Suppose that, as \( t \) tends to zero, the \( i \)-th eigenvalue of \( \Lambda(t) \) also tends to zero as \( t^{k_i} \), \( k_i \in \mathbb{N} \). Such a family will always exist, since the constraints imposed on the entries of \( \Lambda \) are rather loose – something which is not the case for the usual Poisson algebra \( P_{n, \mathbb{C}} \), where, for instance, the diagonal \( \Lambda \) is already constrained by

\[
  x \mapsto \lambda x, \quad p \mapsto \lambda^{-1} p.
\]

Let \( \{k_i, i = 1, \ldots, n\} \) be the set of degrees of singularity of eigenvalues of \( \Lambda(t) \) at zero. For every pair \( (i, j) \), there exists a positive integer \( m \) such that

\[
  \text{either } k_i m \leq k_j \text{ or } k_j m \leq k_i.
\]

We will call the largest such \( m \) the order of \( \Lambda(t) \) at \( t = 0 \). As \( k_i \) are all set to be positive integer, the order equals the integer part of \( \frac{k_{\text{max}}}{k_{\text{min}}} \).

The following lemma summarizes the situations described above.

**Lemma 3.6.** Let \( \varphi \) be a fixed automorphism of \( P_{n, \mathbb{C}}^h \). The curve \( \Lambda(t) \varphi \Lambda(t)^{-1} \) has no singularity at zero for any \( \Lambda(t) \) of order \( \leq N \) if and only if \( \varphi \in \hat{H}_N \) (where \( \hat{H}_N \) is defined above).

**Proof.** Suppose \( \varphi \in \hat{H}_N \). Then the action of \( \Lambda(t) \varphi \Lambda(t)^{-1} \) upon any generator \( \xi \) (\( x_i, p_i \) or \( h_i \)) is given by the expression

\[
  \Lambda(t) \varphi \Lambda(t)^{-1}(\xi) = c\xi + t^{-2k_i} \sum_{l_1 + \cdots + l_n = N} a_{l_1 \ldots l_n} t^{(2k_i l_1 + \cdots + k_i l_n)} P_{i,n}(\xi) + S_i(t, \xi),
\]

where \( c \) denotes the homothety ratio of the linear part of \( \varphi \) and \( S_i \) is a polynomial in \( x_i, p_i \) and \( h_i \) of height greater than \( N \). One sees that for any choice of \( l_1, \ldots, l_{2n} \) in the sum, the expression

\[
  k_i l_1 + \cdots + k_n l_n - k_i \geq k_{\text{min}} \sum l_i - k_i = k_{\text{min}} N - k_i \geq 0
\]

for every \( i \), so whenever \( t \) goes to zero, the coefficient will not blow up to infinity. The same argument applies to higher-degree monomials within \( S_i \).

To prove the other direction, we proceed by contraposition: assuming \( \varphi \notin \hat{H}_N \), we must show the existence of a curve \( \Lambda(t) \) such that the conjugation of \( \varphi \) by \( \Lambda(t) \) produces a singularity at zero.

Suppose first that the linear part \( \bar{\varphi} \) of \( \varphi \) is not a scalar matrix. Then, without loss of generality it is not a diagonal matrix; as such it has a non-zero entry in position \( (i, j) \). Consider a diagonal matrix \( \Lambda(t) = D(t) \) such that it has \( t^{k_j} \) at entry \( (r, r) \) for every \( r \neq j \), while at the remaining main diagonal entry \( (j, j) \) it has \( t^{k_i} \). Then \( D(t) \bar{\varphi} D^{-1}(t) \) has entry \( (i, j) \) with the coefficient \( t^{k_i - k_j} \) and if \( k_j > k_i \) it has a singularity at \( t = 0 \).

Let also \( k_i < 2k_j \). Then the non-linear part of \( \varphi \) does not produce singularities and cannot compensate the singularity of the linear part. This case is thus processed.

Now suppose that the linear part \( \bar{\varphi} \) is scalar. Then the linear part of \( \Lambda(t) \varphi \Lambda(t)^{-1} \) does not possess a singularity, therefore one needs to look at the smallest non-linear term. Let \( \varphi \in \hat{H}_N \setminus \hat{H}_{N+1} \). Changing variables if necessary, we may assume that

\[
  \varphi(x_1) = \alpha x_1 + \beta x_2^N + S,
\]
with \( \text{ht}(S) > N \).

Let \( \Lambda(t) = D(t) \) be a diagonal matrix of the form \((t^{k_1}, t^{k_2}, t^{k_1}, \ldots, t^{k_1})\) and let \((N + 1) \cdot k_2 > k_1 > N \cdot k_2 \). Then in \( \Lambda(t)^{-1} \varphi \Lambda(t) \) the term \( \beta x_N^N \) will be transformed into \( \beta x_N^N t^{Nk_2 - k_1} \), and all other terms are multiplied by \( t^{lk_2 + sk_1 - k_1} \) with \((l, s) \neq (1, 0)\) and \( l, s > 0 \). In this case \( l k_2 + s k_1 - k_1 > 0 \). This concludes the second case, and with it the proof of the lemma.

We can now complete the proof of Theorem 3.4. Suppose that for some \( N \), the image \( \Phi^h(G_N) \) is not contained in \( H_N \). Then there exists an automorphism \( \varphi \) of \( W_{n,C}^h \) which is identity modulo \( I_N \) and whose image \( \Phi^h(\varphi) \) has terms of degree strictly between 1 and \( N \). Then, by Lemma 3.6, there exists a family of linear substitutions \( \Lambda(t) \), such that the curve

\[
\Lambda(t) \Phi^h(\varphi) \Lambda(t)^{-1}
\]

admits a singularity of order \( N \) at \( t = 0 \). On the other hand, as \( \Phi^h \) preserves tame automorphisms, the curve \( \Lambda(t) \Phi^h(\varphi) \Lambda(t)^{-1} \) is the image

\[
\Phi^h(\Lambda(t) \varphi \Lambda(t)^{-1})
\]

of the curve \( \Lambda(t) \varphi \Lambda(t)^{-1} \) which, by an obvious analogue of Lemma 3.6 for \( \text{Aut} W_{n,C}^h \), has no singularity of order \( N \) at \( t = 0 \). This yields a contradiction with Lemma 3.5, as \( \Phi^h \) is a morphism by Theorem 2.23. Theorem 3.4 is proved.

### 4 Canonicity and power series truncation

The most important consequence of the continuity of \( \Phi^h \) is the canonicity of deformed automorphism lifting, formulated as the following elementary corollary.

**Proposition 4.1.** Let \( \varphi \) be an automorphism of \( P_{n,C}^h \) and let

\[
\varphi_1, \ldots, \varphi_k, \ldots
\]

and

\[
\varphi'_1, \ldots, \varphi'_k, \ldots
\]

be two sequences of tame automorphisms which converge to \( \varphi \). Then, the lifted sequences

\[
\{(\Phi^h_{|\text{TAut}})^{-1}(\varphi_k)\} \quad \text{and} \quad \{(\Phi^h_{|\text{TAut}})^{-1}(\varphi'_k)\}
\]

converge to the same automorphism of the power series completion of \( W_{n,C}^h \), i.e. one must have

\[
(\Phi^h_{|\text{TAut}})^{-1}(\varphi_k) \circ (\Phi^h_{|\text{TAut}})^{-1}(\varphi'_k) \equiv \text{Id (mod } I_N(k))
\]

with \( N(k) \to \infty \) as \( k \to \infty \).

The above proposition constitutes a crucial property of the lifting not present in the non-augmented case, in the way that it fixes the lifted limit independent of the selected approximating sequence. This allows for a truncation of the resulting power series. In order to demonstrate that, we will need some preparation.

Firstly, we observe that the lifting procedure may be slightly modified in the following sense. Given the augmentation variables \( \{h_1, \ldots, h_n\} \) and the corresponding augmented algebras \( P_{n,C}^h \) and \( W_{n,C}^h \), we may add to the ground ring all expressions
of the form $h_i h_j^{-1}$ as scalars of degree zero (in the augmentation topology) and consider the algebras over $\mathbb{C}[\{h_i h_j^{-1}\}]$. The notion of tame automorphism subgroup is extended naturally to $\mathbb{C}[\{h_i h_j^{-1}\}]$-algebras, and the closure in power series topology of the tame subgroup is the whole automorphism group, as before. 

As the elements $h_i h_j^{-1}$ carry degree zero in the augmentation topology, the analogue of Theorem 3.4 holds (the proof can be repeated in this case), therefore the lifting, as the formal limit of the lifted tame sequence, is still well defined.

The point of this extension will be made clear in the proof of the following theorem.

**Theorem 4.2.** Let $\varphi \in \text{Aut} \, P^h_n,\mathbb{C}$ and let $\{\varphi_k\}$ be a sequence of tame automorphisms converging to $\varphi$. Denote by

$$\Theta^h(\varphi)$$

the limit of the lifted sequence $\{ (\Phi^h|_{\text{TAut}})^{-1}(\varphi_k) \}$. Then the monomials in the power series

$$\Theta^h(\varphi)(x_i) \text{ and } \Theta(\varphi)(d_i)$$

are of globally bounded degree in $x_i$ and $d_i$.

**Proof.** The proof of Theorem 4.2 utilizes a technique similar to the one we used in Theorem 4.4. The slight difference with the prior instance involves the extension by $h_i h_j^{-1}$.

Suppose $\varphi$ is such that $\Theta^h(\varphi)$ is a true power series in $x_i$ and $d_i$. Let $k = \text{rank}(\varphi)$. The objective is to demonstrate that such a $\varphi$ gives rise to a curve of automorphisms regular at $t = 0$ whose lift is singular at $t = 0$, contradicting the continuity of $\Phi^h$.

To be more precise, the curve will lie in the space of automorphisms over a larger number of variables: we are going to add auxiliary variables and make the lift singular with respect to them.

Let $\tau$ be the automorphism of $P^h_n$ whose action on the generators is given by the dilation

$$x_i \mapsto tx_i, \quad p_i \mapsto tp_i, \quad h_i \mapsto t^2 h_i$$

(with $t$ a parameter, so that $\tau$ is in fact a curve in $\text{Aut} \, P^h_n,\mathbb{C}$). Then one can form the automorphism

$$\tilde{\varphi} = \tau^{-1} \circ \varphi \circ \tau;$$

obviously $\text{rank}(\tilde{\varphi}) = k$. The curve $\tilde{\varphi}$ will have a pole at $t = 0$ of order $l = k - 1$.

For every triple $(x_i, p_i, h_i)$ (connected by means of the augmented Poisson identity), we add a triple of auxiliary variables $(u_i, v_i, h_i')$ (also such that $\{v_i, u_i\} = h_i$) and extend the action of the automorphism $\varphi$ to these variables by requiring it to act identically on them. Denote by $\varphi_e$ the resulting automorphism of $P^h_{2n}$. The parameter-dependent automorphism $\tau$ is defined for $P^h_{2n}$ in an obvious way, and the curve

$$\tilde{\varphi}_e = \tau^{-1} \circ \varphi_e \circ \tau$$

is formed as before, with the pole at zero of order $k - 1$.

Define the following tame automorphism $\psi$ of $P^h_{2n}$: for every pair of triples $(x_i, p_i, h_i)$ and $(u_i, v_i, h_i')$, the map $\psi$ is identical on $h_i$ and $h_i'$, while

$$(x_i, p_i, u_i, v_i) \mapsto (A x_i + B_i u_i, C_i p_i + D_i v_i, A_i' x_i + B_i' u_i, C_i' p_i + D_i' v_i)$$

2The proof of the fact that $\text{TAut} \, P^h_n,\mathbb{C}$ is dense in $\text{Aut} \, P^h_n,\mathbb{C}$ found in [3] admits a straightforward adaptation to this case as well.
where the coefficients $A_i, \ldots, D_i'$ take values in $\mathbb{C}[\{h_i h_j^{-1}\}]$ and are chosen in such a way that the mapping is invertible and preserves the augmented Poisson bracket. These coefficients can indeed be so chosen, for instance one can take

$$(x, p) \mapsto \left( \frac{1}{\sqrt{2}}(x + \frac{h}{h'}u), \frac{1}{\sqrt{2}}(p + v) \right)$$

(omitting the index $i$ to demonstrate the idea). To allow for such transforms (given by tame automorphisms that admit lifting) was the point of the extension of the ring of scalars defined above.

Moreover, one may parameterize the coefficients to obtain a curve

$$\psi = \psi_t$$

which can be made to be identity modulo terms proportional to $t^N$ with $N$ arbitrarily large, and such that the curve

$$\tilde{\varphi}^{-1}_e \circ \psi_t \circ \tilde{\varphi}_e$$

(where $\tilde{\varphi}_e$ carries the homothety parameter $t$) is regular at $t = 0$ (as $N$ can be taken to be much larger than the rank of $\varphi$).

On the other hand, since the action of $\Phi^h$ (and $\Theta^h$) on tame automorphisms is identical and $\Theta^h(\varphi)$ is a true power series by assumption, the lifted curve

$$\Theta^h(\tilde{\varphi}^{-1}_e \circ \psi_t \circ \tilde{\varphi}_e)$$

(where we denoted by $\Theta^h$ the lifting with respect to the extended ring of scalars – by the remark preceding this theorem, this lifting is well defined and canonical) will have a singularity at $t = 0$, which is a contradiction. Theorem 4.2 is proved. □

Theorem 4.2 allows one to dispose of power series in the following way. The homomorphism

$$\Phi^h : \text{Aut}_{\mathbb{C}} W_{n,h} \to \text{Aut}_{\mathbb{C}} P_{n,h},$$

by virtue of Theorem 4.2, induces a $\mathbb{C}((h_1, \ldots, h_n))$-morphism

$$\overline{\Phi}^h : \overline{X}^h \to \overline{Y}^h$$

where Ind-varieties $X^h$ and $Y^h$ consist of automorphisms of $\mathbb{C}((h_1, \ldots, h_n))$-algebras, obtained from $W_{n,h}$ and $P_{n,h}$ respectively, by regarding the augmentation central variables $h_i$ as scalar coefficients and preserving the commutation relations. As before, the map $\overline{\Phi}^h$ is a collection of maps $(\overline{\Phi}^h)^N$ with domains and codomains given by the strata $(X^h)^N$ and $(Y^h)^N$ of automorphisms of degree $\leq N$ (these are affine $\mathbb{C}((h_1, \ldots, h_n))$-varieties).

As a consequence of Theorems 2.21 and 2.23, one has

Corollary 4.3. The maps $(\overline{\Phi}^h)^N$ are morphisms.

On the other hand, the lifting morphism $\Theta^h$ induces a map

$$\overline{\Theta}^h : \overline{Z}^h \to \overline{X}^h$$

where $\overline{Z}^h$ is the neighborhood of the point in $\overline{Y}^h$ corresponding to the identity automorphism, such that the lifting (equivalently, the approximation procedure) is well defined for its points. By construction, the composition $\overline{\Phi}^h \circ \overline{\Theta}^h$ is the identity map. The composition $\overline{\Theta}^h \circ \overline{\Phi}^h$, however, makes sense when $\overline{\Phi}^h$ is restricted to the image of $\overline{\Theta}^h$. To prove that the specialization of $h_i$ is justified, we must extend the domain of $\overline{\Theta}^h$. 19
4.1 The image of $\Phi^h$

We now establish two main propositions.

**Proposition 4.4.** The image of the restriction of $\Phi^h$ to $\Theta^h(Z^h)$ is dense in (the Zariski topology of) $Y^h$.

**Proposition 4.5.** $\Phi^h$ is a closed morphism.

To prove Proposition 4.4, we need to show that the image contains a Zariski-dense subset of $Y^h$. Such a subset is given by the set $(Z^h)_k = \{ \varphi \in Y^h : \varphi \equiv \text{Id} \pmod{I^k} \}$ ($I$ is the augmentation ideal in the commutative $\mathbb{C}((h_1, \ldots, h_n))$-algebra, $k > 1$ fixed). The set $(Z^h)_k$ is Zariski-dense due to the fact that it is an $m$-adic neighborhood of a point corresponding to $m$ and as such contains within itself a Zariski-open (and therefore Zariski-dense) base neighborhood of that point.

Proposition 4.5, on the other hand, is essentially a reformulation of Corollary 2.24.

As a consequence of the two, we have

**Corollary 4.6.** The image $\text{Im}(\Phi^h|_{\Theta^h(Z^h)})$ is the union of the irreducible components of points of $Z^h$ (where $Z^h$ is the domain of the map $\Theta^h$ induced by the lifting).

4.2 Extension of the lifting map and specialization of Planck variables

We now extend the domain of $\Theta^h$ to account for points outside of $Z^h$. This is accomplished by the following procedure. Let $\varphi$ be an automorphism in $Y^h \setminus Z^h$. We introduce a pair of auxiliary variables $u$, $v$ (for Poisson algebra) and $\hat{u}$, $\hat{v}$ for Weyl algebra (such that $\{v, u\} = 1$ and $[\hat{v}, \hat{u}] = 1$ and commutators with the initial variables are zero). Let $\varphi_a$ denote the extension of $\varphi$ to the new variables defined by setting

$$\varphi_a(u) = u, \quad \varphi_a(v) = v.$$

Fix $i \in \{1, \ldots, n\}$ (the number of the generator). Consider, for $\lambda \in \mathbb{C}$, the following (tame) automorphism

$$\psi_\lambda : u \mapsto u + \lambda x_i, \quad p_i \mapsto p_i - \lambda v$$

(other generators unchanged). If $\varphi$ is as before, it gives rise to the automorphism

$$\varphi_\lambda = \varphi_a^{-1} \circ \psi_\lambda \circ \varphi_a.$$

This automorphism is of degree greater than or equal to $\varphi$ and lies in the domain $Z^h$ of the lifting map (or rather, in the analogue of $Z^h$ defined for the Ind-variety obtained after extension by $u$ and $v$) – the latter assertion following from the fact that $\varphi_\lambda$ is the identity map when $\lambda = 0$. Therefore, there exists an automorphism $\hat{\varphi}$ of the (Weyl) algebra extended by $\hat{u}$, $\hat{v}$, such that its image under $\Phi^h$ is $\varphi_\lambda$.

The action of $\varphi_\lambda$ is given by

$$u \mapsto u + \lambda \varphi(x_i),$$
and the lifting is given by
\[ \hat{u} \mapsto \hat{u} + \mu P_i(x_1, \ldots, x_n, d_1, \ldots, d_n) \]
where \( \mu \) and \( \lambda \) are connected by the non-standard Frobenius, and \( P_i(x_1, \ldots, x_n, d_1, \ldots, d_n) \) is a polynomial differential operator.

Set
\[ \hat{\phi}(x_i) = P_i(x_1, \ldots, x_n, d_1, \ldots, d_n), \]
perform the procedure for all \( i \) and then switch the roles of \( x_i \) and \( d_i \) (which will yield polynomials \( Q_i(x_1, \ldots, x_n, d_1, \ldots, d_n) \)).

**Proposition 4.7.** The set of images
\[ (\hat{\phi}(x_1), \ldots, \hat{\phi}(x_n), \hat{\phi}(d_1), \ldots, \hat{\phi}(d_n)) \]
is a well-defined Weyl algebra automorphism.

**Proof.** One needs to verify the commutation relations. Once that is done, the invertibility of the endomorphism will follow the known result ([1, 22]) that states that \( \Phi \) takes automorphisms to automorphisms.

Let \( \phi \) be as before and let
\[ (\hat{\phi}(x_1), \ldots, \hat{\phi}(x_n), \hat{\phi}(d_1), \ldots, \hat{\phi}(d_n)) \]
be the resulting lifted mapping. To arrive at the expressions for the commutators, we start with \( \phi \), add Poisson pairs of variables \((u_1, u_2, v_1, v_2)\) and twist \( \phi \) with appropriate tame automorphisms, chosen in such a way that the commutator of the images of \( \hat{u}_1 \) and \( \hat{u}_2 \) under the lift of the twisted symplectomorphism will contain the commutator of the images under \( \hat{\phi} \). This can always be done. For instance, if we wish to prove that \([\hat{\phi}(x_1), \hat{\phi}(x_2)]=0\), we pick the tame automorphism
\[ \psi : u_1 \mapsto u_1 + x_1, \ u_2 \mapsto u_2 + x_2, \ p_1 \mapsto p_1 - v_1, \ p_2 \mapsto p_2 - v_2 \]
and twist
\[ \phi_t = \psi^{-1} \circ \psi \circ \phi. \]
The lift is \( \rho = \Theta^h(\phi_t) \). One has now
\[ 0 = [\rho(u_1), \rho(u_2)] = [\hat{u}_1, \hat{u}_2] + [\hat{\phi}(x_1), \hat{\phi}(x_2)] + [\hat{\phi}(x_1), \hat{\phi}(x_2)] \]
as desired. For the pair \((x_1, d_1)\), we consider the following two elementary automorphisms
\[ \psi_1 : u_1 \mapsto u_1 + x_1, \ p_1 \mapsto p_1 - v_1, \]
\[ \psi_2 : u_2 \mapsto u_2 + y_1, \ x_1 \mapsto x_1 - v_2 \]
and take \( \psi = \psi_2 \circ \psi_1 \). Proceeding as before, we twist it by \( \phi \) and take
\[ \phi_t = \psi^{-1} \circ \psi \circ \phi. \]
The desired result is again achieved by evaluating the commutator \([\rho(u_1), \rho(u_2)]\), with \( \rho \) the lift of \( \phi_t \).

We can thus extend the lifting map \( \Theta^h \) from \( Z^h \) to the whole \( Y^h \). Evidently, the composition \( \Theta^h \circ \Phi^h \) is then defined for every point in \( X^h \) and is the identity map by construction.
This completes the proof of the Main Theorem, as one may now safely specialize to \( h_i = 1 \). The Ind-varieties \( X^h \) and \( Y^h \) become \( \text{Aut} W_{n,C} \) and \( \text{Aut} P_{n,C} \), respectively, and the morphism \( \Phi \) has a well-defined inverse.

One notable consequence of Propositions 4.4-4.7 is that the lifting of augmented symplectomorphisms is polynomial in \( h_1, \ldots, h_n \). Precisely, we have the following result.

**Proposition 4.8.** Let \( \Theta^h \) be the lifting map defined in Theorem 4.2. Then the images of the generators of \( P^h_{n,C} \) are polynomial in \( h_i \). It follows that

\[
\Theta^h : \text{Aut} P^h_{n,C} \to \text{Aut} W^h_{n,C}
\]

is a well-defined lifting map which is the inverse of \( \Phi^h \).

**Proof.** Given an element \( \varphi \in \text{Aut} P^h_{n,C} \), the specialization shows that we may define a set of symplectomorphisms \( \{ \varphi_{h_1, \ldots, h_n} \} \) parameterized by \( h_i \); the coordinate functions \( a_I \) of elements of this set are polynomial in \( h_i \).

The lifting by \( \Theta^h \) produces a family of Weyl algebra automorphisms \( \hat{\varphi}_{h_1, \ldots, h_n} \) whose coefficients \( \hat{a}_I \) are power series in \( h_i \). By the specialization argument, together with Theorems 2.20 and 2.23 these functions of \( h_i \) are algebraic. As

\[
\Phi^h \circ \Theta^h = \text{Id}
\]

and \( \Phi^h \) is injective, the functions \( \hat{a}_I \) cannot have branch points.

Finally, the functions \( \hat{a}_I \) cannot have poles at points with finite values of \( h_1, \ldots, h_n \); otherwise the automorphism corresponding to the pole must give – under the morphism \( \Phi^h \) – a symplectomorphism whose coefficients also have a pole at that point, which is never the case for finite \( h_1, \ldots, h_n \).

Thus the functions \( \hat{a}_I(h_1, \ldots, h_n) \) are algebraic without finite poles or branching points and therefore must be polynomials. The statement of the proposition follows. \( \square \)

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