Research Article

On Semi-c-Periodic Functions

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1. Introduction

The notion of periodicity plays a fundamental role in mathematics. A continuous function \( f: I \longrightarrow E \), where \( E \) is a topological space and \( I = \mathbb{R} \) or \( I = [0, \infty) \), is said to be periodic if and only if there exists a real number \( \omega > 0 \) such that \( f(x + \omega) = f(x) \) for all \( x \in I \). The notion of periodicity has recently been reconsidered by Alvarez et al. [1], who proposed the following notion: a continuous function \( f: \tilde{I} \longrightarrow E \), where \( E \) is a complex Banach space, is said to be \((\omega, c)\)-periodic \((\omega > 0, c \in \mathbb{C} \setminus \{0\})\) if and only if \( f(x + \omega) = cf(x) \) for all \( x \in \tilde{I} \). Due to ([1], Proposition 2.2), we know that a continuous function \( f: I \longrightarrow E \) is \((\omega, c)\)-periodic if and only if the function \( g(\cdot) = e^{-(\cdot)/\omega}f(\cdot) \) is periodic and \( g(x + \omega) = g(x) \) for all \( x \in I \); here, \( e^{-(\cdot)/\omega} \) denotes the principal branch of the exponential function (see also the research articles [2, 3] by Alvarez et al., the conference paper [4] by Pinto, where the idea for introduction of \((\omega, c)\)-periodic functions was presented for the first time, and [5, 6] for some generalizations of the concept of \((\omega, c)\)-periodicity).

In the sequel, by \( E \) we denote a complex Banach space equipped with the norm \( \| \cdot \| \); \( C(I; E) \) denotes the vector space consisting of all continuous functions \( f: I \longrightarrow E \). A function \( f \in C(I; E) \) is said to be \( c\)-periodic \((c \in \mathbb{C} \setminus \{0\})\) if and only if there exists a real number \( \omega > 0 \) such that the function \( f(\cdot) \) is \((\omega, c)\)-periodic. The class of \( c\)-periodic functions extends two important classes of functions:

1. The class of antiperiodic functions, i.e., the class of \((-1)\)-periodic functions: in this case, any positive real number \( \omega > 0 \) satisfying \( f(x + \omega) = -f(x) \), \( x \in I \), is said to be an antiperiod of \( f(\cdot) \). Any antiperiodic function is periodic, since we can apply the above functional equality twice in order to see that \( f(x + 2\omega) = -f(x) \) for all \( x \in I \).

2. The class \((\omega, k)\)-periodic functions \((\omega > 0, k \in \mathbb{R})\), i.e., the class of continuous functions \( f: I \longrightarrow E \) satisfying \( f(x + \omega) = e^{ik\omega}f(x) \) for all \( x \in I \). The number \( \omega \) is usually called Bloch period of \( f(\cdot) \), the number \( k \) is usually called the Bloch wave vector or Floquet exponent of \( f(\cdot) \), and in the case that \( k\omega = \pi \), the class of \((\omega, k)\)-periodic functions is equal to the class of antiperiodic functions having the number \( \omega \) an antiperiod. If the function \( f(\cdot) \) is Bloch \((\omega, k)\)-periodic, then we inductively obtain \( f(x + m\omega) = e^{imk\omega}f(x) \) for all \( x \in I \) and \( m \in \mathbb{N} \), so that the function \( f(\cdot) \) must be periodic provided that \( k\omega \in \mathbb{Q} \), but, if \( k\omega \notin \mathbb{Q} \), then the function \( f(\cdot) \) need not be periodic as the
following simple counterexample shows: the function
\[ f(x) := e^{ix} + e^{i(\sqrt{2} - 1)x}, \quad x \in \mathbb{R}, \]
is Bloch \((\omega, k)\)-periodic with \(\omega = 2\pi + \sqrt{2}\pi\) and \(k = \sqrt{2} - 1\) but not periodic. In ([7], Remark 1), we have recently observed that any Bloch \((\omega, k)\)-periodic function must be almost periodic (see also the research articles [8] by Hasler and [9] by Hasler and Guérékata, where it has been noted that the Bloch \((\omega, k)\)-periodic functions are unavoidable in condensed matter and solid state physics).

The notion of almost periodicity was introduced by Harald Bohr, a younger brother of Nobel Prize winner Niels Bohr, around 1925 and later generalized by many other mathematicians. In [10], we have analyzed the following generalization of the notion of almost periodicity, called \(c\)-almost periodic \((c \in \mathbb{C} \setminus \{0\})\): let \(f: I \to E\) be a continuous function, and let a number \(\epsilon > 0\) be given. We call a number \(\tau > 0\ an (\epsilon, c)\)-period for \(f()\) if and only if \(||f(x + \tau) - cf(x)|| \leq \epsilon\) for all \(x \in I\); by \(\Theta_{\epsilon}(f, c)\) we denote the set consisting of all \((\epsilon, c)\)-periods for \(f()\). It is said that \(f()\) is \(c\)-almost periodic if and only if for each \(\epsilon > 0\) the set \(\Theta_{\epsilon}(f, c)\) is relatively dense in \([0, \infty)\), which means that for each \(\epsilon > 0\) there exists a finite real number \(I > 0\) such that any subinterval \(I'\) of \([0, \infty)\) of length \(I\) meets \(\Theta_{\epsilon}(f, c)\).

Any \(c\)-periodic function is \(c\)-almost periodic and any \(c\)-almost periodic function is almost periodic ([10]); if \(c = 1\), resp. \(c = -1\), then we also say that the function \(f()\) is \(c\)-almost periodic, resp. \(c\)-antiperiodic (for the primary source of information about almost periodic functions and their applications, we refer the reader to the research monographs by Besicovitch [11], Diagana [12], Fink [13], Guérékata [14], Kostić [15], and Zaidman [16]).

In [10], besides the class of \(c\)-almost periodic functions, we have introduced and analyzed the classes of \(c\)-uniformly recurrent functions, semi-\(c\)-periodic functions, and their Stepanov generalizations, where \(c \in \mathbb{C}\) and \(|c| = 1\) (the classes of semiperiodic functions and semi-antiperiodic functions, i.e., the classes of semi-1-periodic functions and semi-\((-1)\)-periodic functions, have been previously considered by Andres and Pennequin in [17], the research article of invaluable importance for us, and Chauouchi et al. in [7]; the notion of semi-Bloch \(k\)-periodicity, where \(k \in \mathbb{R}\), has been also analyzed in [7], but it differs from the notion of semi-\(c\)-periodicity analyzed in [10] and this paper). If \(|c| = 1\), then we know that a function \(f \in C(I; E)\) is \(c\)-periodic if and only if there exists a sequence \((f_n)\) of \(c\)-periodic functions in \(C(I; E)\) such that \(\lim_{n \to \infty} f_n(x) = f(x)\) uniformly in \(I\); in this case, a \(c\)-periodic function need not be \(c\)-periodic [10]. For example, we have the following (see ([17], Example 1), ([7], Example 4 and Example 5), and ([10], Example 2.16)): let \(p\) and \(q\) be odd natural numbers such that \(p - 1 \equiv 0 (\mod q)\), and let \(c = e^{i \pi p/q}\). The function
\[ f(x) := \sum_{n=1}^{\infty} \frac{e^{i (nx/(2q n + 1))}}{n^2}, \quad x \in \mathbb{R}, \]
is semi-\(c\)-periodic because it is a uniform limit of \([\pi \cdot (1 + 2q) \ldots (1 + 2Nq)]\)-periodic functions
\[ f_N(x) := \sum_{n=1}^{N} \frac{e^{i (nx/(2q n + 1))}}{n^2}, \quad x \in \mathbb{R} \ (N \in \mathbb{N}). \]

Our main result, Theorem 1, states that the following phenomenon occurs in case \(|c| \neq 1\): if \((f_n)\) is a sequence of \(c\)-periodic functions and \(\lim_{n \to \infty} f_n(x) = f(x)\) uniformly in \(I\), then \(f()\) is \(c\)-periodic. Therefore, in this case, any concept of semi-\(c\)-periodicity introduced below coincides with the concept of \(c\)-periodicity (more precisely, in this paper, we analyze the concepts of semi-\(c\)-periodicity of type \(i(i)\), where \(i = 1, 2\) and \(c \in \mathbb{C} \setminus \{0\}\); if \(|c| = 1\), all these concepts are equivalent and reduced to the concept of semi-\(c\)-periodicity, while in case \(|c| \neq 1\), all these concepts are equivalent and reduced to the concept of \(c\)-periodicity).

For any function \(f \in C(I; E)\), we set \(\|f\|_{\infty} := \sup_{x \in I} \|f(x)\|\). The notion of \(c\)-uniform recurrence plays an important role in the proof of our main result [10].

**Definition 1.** A continuous function \(f: I \to E\) is said to be \(c\)-uniformly recurrent \((c \in \mathbb{C} \setminus \{0\})\) if and only if there exists a strictly increasing sequence \((\alpha_n)\) of positive real numbers such that \(\lim_{n \to \infty} \alpha_n = +\infty\) and
\[ \lim_{n \to \infty} \|f(\cdot + \alpha_n) - cf(\cdot)\|_{\infty} = 0. \]

The space consisting of all \(c\)-uniformly recurrent functions from the interval \(I\) into \(E\) will be denoted by \(UR_c(I; E)\). If \(c = 1\), resp. \(c = -1\), then we also say that the function \(f()\) is uniformly recurrent, resp. uniformly antirecurrent.

Although the notion of uniform recurrency was analyzed already by Bohr in his landmark paper [18] (1924), the precise definition of a uniformly recurrent function was firstly given by Harroux and Souplet [19] in 2004, who proved that the function \(f: \mathbb{R} \to \mathbb{R}\), given by
\[ f(x) := \sum_{n=1}^{\infty} \frac{\sin^2 \left(\frac{x}{2^n}\right)}{n}, \quad x \in \mathbb{R}, \]
is unbounded, Lipschitz continuous and uniformly recurrent; moreover, we have that \(f()\) is \(c\)-uniformly recurrent if and only if \(|c| = 1\) (see [10], Example 2.19(i)). The first example of a uniformly antirecurrent function has recently been constructed in ([10], Example 2.20), where we have proved that the function \(g: \mathbb{R} \to \mathbb{R}\), given by
\[ g(x) := \sin x \sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \left(\frac{x}{2^n}\right), \quad x \in \mathbb{R}, \]
is unbounded, Lipschitz continuous and uniformly antirecurrent. Any \(c\)-almost periodic function is \(c\)-uniformly recurrent, while the converse statement does not hold in general.

For completeness, we will include all details of the proof of the following auxiliary lemma from [10].
**Lemma 1** (A). Suppose that \( f \in UR_c(I; E) \) and \( c \in \mathbb{C} \setminus \{0\} \) satisfies \(|c| \neq 1\). Then, \( f \equiv 0 \).

**Proof.** Without loss of generality, we may assume that \( I = [0, \infty) \). Suppose to the contrary that there exists \( x_0 \geq 0 \) such that \( f(x_0) \neq 0 \). Inductively, (4) implies

\[
|c|^k - \frac{|c|^k - 1}{n(|c| - 1)} \leq \|f(x)\| \leq |c|^k - \frac{|c|^k - 1}{n(|c| - 1)}
\]

provided that \( k \in \mathbb{N} \) and \( x \in [k\alpha_n, (k + 1)\alpha_n] \). Consider now case \(|c| < 1\). Let \( 0 < \varepsilon < c \|f(x_0)\| \). Then, (7) yields that there exist integers \( k_0 \in \mathbb{N} \) and \( n \in \mathbb{N} \) such that for each \( k \geq k_0 \), we have \( \|f(x)\| \leq (\varepsilon/2) \). \( x \in [k\alpha_n, (k + 1)\alpha_n] \). Then, the contradiction is obvious because for each \( m \in \mathbb{N} \) with \( m > n \), there exists \( k \in \mathbb{N} \) such that \( x_0 + \alpha_m \in [k\alpha_n, (k + 1)\alpha_n] \), and therefore \( \|f(x_0 + \alpha_m)\| \geq |c|\|f(x_0)\| - \varepsilon \). \( m \rightarrow +\infty \). Consider now case \(|c| > 1\); let \( n \in \mathbb{N} \) be such that \( \|f(x_0)\| > (1/(n(|c| - 1))) \) and \( M = \max_{x \in [0, 2\alpha_n]} \|f(x)\| \geq 0 \). Then, for each \( m \in \mathbb{N} \) with \( m > n \), there exists \( k \in \mathbb{N} \) such that \( \alpha_m \in [(k - 1)\alpha_n, k\alpha_n] \), and therefore \( \|f(x + \alpha_m)\| \leq 1 + |c|M \). \( x \in [0, 2\alpha_n] \). On the other hand, we obtain inductively from (4) that

\[
\|f(x_0 + \alpha_m)\| \geq |c|^k \left[ \|f(x_0)\| - \frac{1}{n(|c| - 1)} \right]
\]

which immediately yields a contradiction. \( \square \)

**2. Semi-c-Periodic Functions**

Set \( \mathbb{S} = \mathbb{N} \) if \( I = [0, \infty) \), and \( \mathbb{S} = \mathbb{Z} \) if \( I = \mathbb{R} \). In this paper, we introduce and analyze the following notion with \( c \in \mathbb{C} \setminus \{0\} \).

**Definition 2.** Let \( f \in C(I; E) \).

(i) It is said that \( f(\cdot) \) is semi-c-periodic of type 1 if and only if

\[
\forall \varepsilon > 0 \exists m > 0 \forall x \in \mathbb{S} \forall x \in I \quad \|f(x + mw) - c^m f(x)\| \leq \varepsilon.
\]

(ii) It is said that \( f(\cdot) \) is semi-c-periodic of type 2 if and only if

\[
\forall \varepsilon > 0 \exists m > 0 \forall x \in \mathbb{S} \forall x \in I \quad \|c^{-m} f(x + mw) - f(x)\| \leq \varepsilon.
\]

The space of all semi-c-periodic functions of type \( i \) will be denoted by \( \mathcal{D}' \mathcal{P}_{c,i} (I; E) \), \( i = 1, 2 \).

**Definition 3.** Let \( f \in C(I; E) \).

(i) It is said that \( f(\cdot) \) is semi-c-periodic of type 1, if and only if \( \varepsilon > 0 \exists m > 0 \forall x \in \mathbb{N} \forall x \in I \) \( \|f(x + mw) - c^m f(x)\| \leq \varepsilon \).

(ii) It is said that \( f(\cdot) \) is semi-c-periodic of type 2, if and only if \( \varepsilon > 0 \exists m > 0 \forall x \in \mathbb{N} \forall x \in I \) \( \|c^{-m} f(x + mw) - f(x)\| \leq \varepsilon \).

The notion of semi-c-periodicity of type 1 has been introduced in ([10], Definition 2.4), where it has been simply called semi-c-periodicity. Due to ([10], Proposition 2.5), we have that the notion of a semi-c-periodicity of type \( i (i_1) \), where \( i = 1, 2 \), is equivalent with the notion of semi-c-periodicity introduced there, provided that \( |c| = 1 \).

Now we will focus our attention to the general case \( c \in \mathbb{C} \setminus \{0\} \). We will first state the following:

**Lemma 2** (B).

(i) If \(|c| \geq 1 \) and \( f : I \longrightarrow E \) is semi-c-periodic of type 1, then \( f(\cdot) \) is semi-c-periodic of type 2.

(ii) If \(|c| \leq 1 \) and \( f : I \longrightarrow E \) is semi-c-periodic of type 2, then \( f(\cdot) \) is semi-c-periodic of type 1.

**Proof.** If \( x \in I, \omega > 0, m \in \mathbb{N} \) and \(|c| \geq 1 \), then we have

\[
\|f(x + mw) - c^m f(x)\| \leq \varepsilon \Rightarrow \|c^{-m} f(x + mw) - f(x)\| \leq \varepsilon,
\]

which implies (i); the proof of (ii) is similar. \( \square \)

The argumentation contained in the proofs of ([17], Lemma 1 and Theorem 1) can be repeated verbatim in order to see that the following important lemma holds true.

**Lemma 3** (C). Suppose that \(|c| \leq 1 \), resp. \(|c| \geq 1 \), and \( f : [0, \infty) \longrightarrow E \) is semi-c-periodic of type 1, resp. 2. Then, there exists a sequence \( (f_n : [0, \infty) \longrightarrow E)_{n \in \mathbb{N}} \) of c-periodic functions which converges uniformly to \( f(\cdot) \).

Now we are able to state and prove our main result.

**Theorem 1.** Let \(|c| \neq 1, i \in \{1, 2\} \) and \( f : [0, \infty) \longrightarrow E \). Then, \( f(\cdot) \) is c-periodic if and only if \( f(\cdot) \) is semi-c-periodic of type \( i (i_1) \).

**Proof.** Suppose that the function \( f(\cdot) \) is (\( w, c \))-periodic. Then, we have \( f(x + mw) = c^m f(x), x \in I, m \in \mathbb{S}, \) so that \( f(\cdot) \) is automatically semi-c-periodic of type \( i (i_1) \). To prove the converse statement, let us observe that any semi-c-periodic of type \( i \) is clearly semi-c-periodic of type \( i_1 \). Suppose first that \(|c| > 1 \). Due to Lemma 2 B(i), it suffices to show that if \( f(\cdot) \) is semi-c-periodic of type 2, then \( f(\cdot) \) is c-periodic. Assume first \( I = [0, \infty) \). Using Lemma C, we get the existence of a sequence \( (f_n : [0, \infty) \longrightarrow E)_{n \in \mathbb{N}} \) of c-periodic functions which...
converges uniformly to \( f(\cdot) \). Let \( f_n(x + \omega_n) = cf_n(x) \), \( x \geq 0 \) for some sequence \((\omega_n)\) of positive real numbers. Consider first case that \((\omega_n)\) is bounded. Then, there exists a strictly increasing sequence \((n_k)\) of positive integers and a number \( \omega_0 \geq 0 \) such that \( \lim_{k \to \infty} \omega_{n_k} = \omega \). Let \( \varepsilon > 0 \) be given. Then, there exists an integer \( k_0 \in \mathbb{N} \) such that \( \| f(x) - f_n(x) \| \leq \varepsilon (1 + 2 + 2|c|^{-1}) \) for all real numbers \( x \geq 0 \) and all integers \( k \geq k_0 \). Furthermore, we have
\[
\| c^{-1} f(x + \omega_n) - f(x) \| \leq \| c^{-1} f(x + \omega_n) - c^{-1} f_n(x + \omega_n) \| + \| c^{-1} f_n(x + \omega_n) - f_n(x) \| + \| f_n(x) - f(x) \| \\
= \| c^{-1} f(x + \omega_n) - c^{-1} f_n(x + \omega_n) \| + \| f_n(x) - f(x) \| \leq 2(1 + |c|^{-1}) \epsilon, \]
(14)
for all real numbers \( x \geq 0 \) and all integers \( k \geq k_0 \). Letting \( k \to +\infty \), we get \( f(x + \omega) = cf(x) \) for all \( x \geq 0 \). If \( \omega > 0 \), the above yields that \( f(\cdot) \) is \((\omega, c)\)-periodic while the assumption \( \omega = 0 \) yields \( f \equiv c f(\cdot) \) for any \( \omega \geq 0 \). In any case, \( f(\cdot) \) is \((\omega, c)\)-periodic. Suppose now that \((\omega_n)\) is unbounded. Then, with the same notation as above, we may assume that \( \lim_{n \to \infty} \omega_{n_k} = +\infty \). Using the same computation, it follows that \( \lim_{n \to \infty} \| c^{-1} f(x + \omega_n) - f(x) \| = 0 \), so that \( f \in UR_c((0,\infty); E) \). Due to Lemma 1 A, we get \( f(\cdot) \equiv 0 \). Assume now \( I = \mathbb{R} \). By the foregoing arguments, we know that there exists \( \omega > 0 \) such that \( f(x + \omega) = cf(x) \) for all \( x \geq 0 \). Let \( x < 0 \) and \( \epsilon > 0 \) be fixed. Since \( f(\cdot) \) is semi-\((\omega, c)\)-periodic, there exists \( \omega_0 > 0 \) such that \( \| c^{-m} f(x + \omega + m\omega_0) - f(x + \omega) \| \leq \epsilon \) and \( \| c^{-m} f(x + m\omega_0) - cf(x) \| \leq \epsilon \) for all \( m \in \mathbb{N} \). For all sufficiently large integers \( m \in \mathbb{N} \), we have \( x + m\omega_0 > 0 \) so that \( c^{-m} f(x + \omega + m\omega_0) = c^{-1} f(x + \omega) \), and therefore \( \| f(x + \omega) - cf(x) \| \leq 2\epsilon \). Since \( \epsilon > 0 \) was arbitrary, we get \( f(x + \omega) = cf(x) \), which completes the proof in case \( |c| > 1 \). Suppose now that \( |c| < 1 \). Due to Lemma 2(ii), it suffices to show that if \( f(\cdot) \) is semi-\((\omega, 1)\)-periodic, then \( f(\cdot) \) is \((\omega, c)\)-periodic. But, then we can apply Lemma 3 again and the similar arguments as above to complete the whole proof.

Corollary 1. Let \( c \in C \setminus \{0\} \), let \( i \in \{1, 2\} \), and let \( f(\cdot) \) be semi-\((\omega, c)\)-periodic of type \( i \) (i.). Then, there exists two finite real constants \( M > 0 \) and \( \omega > 0 \) such that \( \| f(x) \| \leq M|c|^{(i/s)/\omega} \), \( t \in I \).

Using ([10], Theorem 2.14) and the proof of Theorem 1, we may deduce the following corollaries.

Corollary 2. Let \( f \in C(I; E) \) and \( c \in C \setminus \{0\} \). Then, \( f(\cdot) \) is semi-\((\omega, c)\)-periodic if and only if there exists a sequence \((f_n)\) of \((\omega, c)\)-periodic functions in \( C(I; E) \) such that \( \lim_{n \to \infty} f_n(x) = f(x) \) uniformly in \( I \).

Corollary 3. Let \( f \in C(I; E) \) and \( |c| \neq 1 \). If \( (f_n) \) is a sequence of \((\omega, c)\)-periodic functions and \( \lim_{n \to \infty} f_n(x) = f(x) \) uniformly in \( I \), then \( f(\cdot) \) is \((\omega, c)\)-periodic.

3. Conclusions

In this paper, the authors have studied the class of semi-\((\omega, c)\)-periodic functions with values in Banach spaces. In the case that \( c \) is a nonzero complex number whose absolute value is not equal to 1, the authors have proved that the notion of semi-\((\omega, c)\)-periodicity is equivalent with the notion of \((\omega, c)\)-periodicity. For further information concerning Stepanov semi-\((\omega, c)\)-periodic functions, composition principles for (Stepanov) semi-\((\omega, c)\)-periodic functions, and related applications to the abstract semilinear Volterra integrodifferential equations in Banach spaces, the reader may consult the forthcoming research monograph [20].

Data Availability

The data that support the findings of this study are available at https://www.researchgate.net/publication/342068071_SEMI-c-PERIODIC_FUNCTIONS_AND_APPLICATIONS (an extended version of the paper).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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