Separatrix chaos: new approach to the theoretical treatment

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We develop a new approach to the theoretical treatment of the separatrix chaos, using a special analysis of the separatrix map. The approach allows us to describe boundaries of the separatrix chaotic layer in the Poincaré section and transport within the layer. We show that the maximum which the width of the layer in energy takes as the perturbation frequency varies is much larger than the perturbation amplitude, in contrast to predictions by earlier theories suggesting that the maximum width is of the order of the amplitude. The approach has also allowed us to develop the self-consistent theory of the earlier discovered (PRL 90, 174101 (2003)) drastic facilitation of the onset of global chaos between adjacent separatrices. Simulations agree with the theory.

Keywords: Hamiltonian chaos, separatrix map, nonlinear resonance
1. Introduction

Even a weak perturbation of an integrable system possessing a separatrix results in the onset of chaotic motion inside a layer\(^1\)\(^-\)\(^5\) which we shall further call as the separatrix chaotic layer (SL). The separatrix chaos plays a fundamental role for the Hamiltonian chaos, being also relevant to various applications.\(^1\)\(^-\)\(^5\) The boundaries of the SL in the Poincaré section may be easily found numerically.\(^3\) However, it is also important, both from the theoretical and practical points of view, to be able to theoretically calculate them and describe transport within the SL.

One of the most powerful theoretical tools for the SL study is the separatrix map (SM), introduced in\(^6\) for the nearly integrable systems with the 3/2 degrees of freedom and called sometimes\(^5\) as the Zaslavsky separatrix map. It may also be generalized for systems with more degrees of freedom and for strongly non-integrable systems (see\(^5\) for the most recent major review). We shall further consider the case of the 3/2 degrees of freedom but the generalization of our method for other cases may be done too.

One of the most interesting for physical applications relevant quantities is the width of the SL in energy.\(^1\)\(^-\)\(^4\),\(^6\)\(^-\)\(^11\) There are various heuristic criteria\(^1\)\(^-\)\(^4\) based on the separatrix map and various conjectures. The width by these criteria does not depend on the angle and, as a function of a perturbation frequency \(\omega_f\), possesses a maximum at \(\omega_f\) of the order of the eigenfrequency in the stable state \(\omega_0\) while the maximum itself is of the order of the perturbation amplitude \(h\). However, the work\(^10\) has demonstrated in simulations for double-separatrix systems that the maximum width may be much larger as the SL absorbs one or two nonlinear resonances. The recent work\(^11\) has proved this, developing a new method for the analysis of the separatrix map. The method is of a general validity, as shown in the present work. We show that the maximum width occurs at the frequency which is typically smaller than \(\omega_0\) by the logarithmic factor \(\ln(1/h)\) while the maximum width is typically much larger than \(h\) - either by a numerical factor or by the logarithmic factor (apart from the adiabatic divergence in certain class of systems\(^7\)). Besides, the method allows to describe major statistical properties of transport within the SL.

Note that there were various mathematical works considering the SL in rather different contexts (see\(^5\) for the review). In particular, they analysed the SL width in normal coordinates. However, to the best of our knowledge, these works do not specify the relation between the normal coordinates and variables conventional in physics (e.g. energy-angle or coordinate-momentum). Besides, these works just estimate the width from above and
below while our method allows to carry out an accurate calculation of the width in energy and, moreover, of the SL boundaries in the Poincaré section. Finally and most importantly, the methods described in the literature do not make a resolution between the resonance frequency range and other frequency ranges while our method shows that the SLs in these ranges drastically differ from each other, as confirmed by simulations.

Below, we describe the basic ideas of our method (Sec. 2), review the results of its application to the double-separatrix case (Sec. 3) and present rough estimates for the single-separatrix case (Sec. 4).

2. Basic ideas

Consider any 1D Hamiltonian system possessing at least one separatrix. Let us add a weak time-periodic perturbation,

\[ H = H_0(p, q) + hV(p, q, t), \quad V(p, q, t + 2\pi/\omega_f) = V(p, q, t), \quad h \ll 1. \quad (1) \]

The motion near any of the separatrices may be approximated by the separatrix map (SM).\(^1\text{-}^6,\text{11}\) The map slightly differs for different types of separatrix. Our method applies to all types but, to be concrete, we consider in this section only the separatrix with a single saddle and two loops (like in a double-well potential system). Then the SM reads\(^11\) (cf. also\(^1\text{-}^6\)):

\[ E_{i+1} = E_i + \sigma_i h \epsilon \sin(\varphi_i), \]
\[ \varphi_{i+1} = \varphi_i + \frac{\omega_f \pi (3 - \text{sign}(E_{i+1} - E_s))}{2 \omega (E_{i+1})}, \]
\[ \sigma_{i+1} = \sigma_i \text{sign}(E_s - E_{i+1}), \]
\[ \epsilon \equiv \epsilon(\omega_f) = -\text{sign}(\partial H_0/\partial p|_{t \rightarrow -\infty}) \int_{-\infty}^{\infty} dt \, \partial H_0/\partial p|_{E_s} \sin(\omega_f t), \]
\[ E_i \equiv H_0(p, q)|_{t_i}, \quad \varphi_i \equiv \omega_f t_i, \quad \sigma_i \equiv \text{sign}(\partial H_0/\partial p|_{t_i}), \]

where \(E_s\) is the separatrix energy. The quantity \(\epsilon\) is often called as the Melnikov\(^2\) or Poincaré-Melnikov\(^5\) integral. The quantity \(\delta \equiv h|\epsilon|\) is sometimes called separatrix split.\(^3\) For the sake of simplicity, let absolute values of all parameters of \(H_0\) and of \(V\) be \(\sim 1\). Then \(|\epsilon| \sim 1\) too, if \(\omega_f \sim 1\).

Consider two most general ideas of the method.

1. The SM trajectory that includes any state with \(E = E_s\) is chaotic since the angle of this state is not correlated with the angle of the state at the preceding step of the map, due to the divergence of \(\omega^{-1}(E \rightarrow E_s)\).
2. The frequency of eigenoscillation as a function of energy is proportional to the reciprocal of the logarithmic factor:

$$\omega(E) = \frac{a \pi \omega_0}{\ln(\Delta H|E - E_s|)}, \quad a = \frac{3 - \text{sign}(E - E_s)}{2},$$

(3)

$$|E - E_s| \ll \Delta H \sim E_s - E_{st}^{(1)} \sim E_s - E_{st}^{(2)}$$

($E_{st}^{(1,2)}$ are energies of the stable states).

Given that the argument of the logarithm is large in the relevant range of $E$, the function $\omega(E)$ is nearly constant at a rather significant variation of the argument. Therefore, as the SM maps the state $(E_0 = E_s, \varphi_0)$ onto the state with $E = E_1 \equiv E_s + \sigma_0 h \epsilon \sin(\varphi_0)$, the value of $\omega(E)$ for the given $\text{sign}(\sigma_0 \epsilon \sin(\varphi_0))$ is nearly the same for most of angles $\varphi_0$ (except the close vicinity of $\pi$ multiples), namely

$$\omega(E) = \omega_r^{(\pm)} = \omega(E_s \pm h) \quad \text{for} \quad \text{sign}(\sigma_0 \epsilon \sin(\varphi_0)) = \pm 1.$$  

(4)

Moreover, even if the deviation of the trajectory of the SM from the separatrix further increases/decreases, $\omega(E)$ remains close to $\omega_r^{(\pm)}$ provided the deviation is not too large/small, namely if $|\ln((E - E_s)/h)| \ll \ln(\Delta H/h)$. If $\omega_f \approx \omega_r^{(\pm)}$, then the evolution of the map (2) may be regular-like for a long time until the energy returns back to the close vicinity of the separatrix, where the correlation of angle is lost again. Such a behavior is especially pronounced if the perturbation frequency is close to $\omega_r^{(+)}$ or $\omega_r^{(-)}$ or to one of their multiples of a not too high order: the resonance between the perturbation and eigenoscillation gives rise to an accumulation of the energy gain for many steps of the SM, which results in the deviation of $E$ from $E_s$ greatly exceeding the separatrix split $\delta \sim h$. As a function of $\omega_f$, the largest (along the SL boundary) deviation from the separatrix takes its maximum at the frequencies slightly exceeding $\omega_r^{(+)}$ and $\omega_r^{(-)}$, for the upper and lower boundaries of the SL respectively. This corresponds to the absorption of the nonlinear resonance by the SL.

The description of the regular-like parts of the chaotic trajectory in the case close to the resonance may be done within the resonance approximation. The explicit matching between the SM and the resonance approximation is carried out in.\(^{11}\) The resonance approximation is done in terms of slow variables, action $I \equiv I(E)$ (note that $dI/dE = \omega^{-1}(E)$) and slow angle $\tilde{\psi} \equiv \psi - \omega_f t \equiv \psi - \varphi$, by means of the resonance Hamiltonian $\tilde{H}(I, \tilde{\psi})$.\(^{1-4,9-11}\)
Fig. 1. Schematic illustrations to the formation of the SL upper boundary. The separatrix energy $E_s$ corresponds to the lower boundary of the box. The GSS curve is shown by the dashed line. Solid lines show examples of resonant trajectories (RTs) overlapping the GSS curve. The SL boundary (thick solid line) is formed by: (a) the RT \textit{tangent} to the GSS curve, or (b) the upper part of the \textit{self-intersecting} RT (resonant separatrix).

Fig. 1 schematically illustrates the formation of the upper boundary of the SL in the Poincaré section presented in the $E - \tilde{\psi}$ variables. The chaotic trajectory jumps from the separatrix onto the curve which we call the upper \textit{generalized separatrix split} (GSS) curve, within an even $\pi$ interval:

$$E_{GSS}^{(+)}(\tilde{\psi}) = E_s + \delta |\sin(\tilde{\psi})|, \quad \delta \equiv h|\epsilon|, \quad \tilde{\psi} \in [\pi + 2\pi n, 2\pi + 2\pi n], \quad n = 0, \pm 1, \pm 2, ...$$

The GSS curve relates to the SM equation for $E$ with $E_0 = E_s$. The relevance of just the even $\pi$ intervals of $\tilde{\psi}$ is a consequence of a necessarily positive sign of $E_1 - E_0$ as far as we consider the upper boundary of the SL. Then the system follows the trajectory of the resonance Hamiltonian $\tilde{H}$ ($E$ necessarily increases initially because $\tilde{\psi}(0)$ is within an even $\pi$ interval), until it again reaches the GSS curve (necessarily in an odd $\pi$ interval, where $E$ decreases). After that, the system jumps onto the separatrix, where the angle correlation is lost. There is a statistical distribution of regular-like parts of the chaotic trajectory corresponding to the homogeneous distribution of initial angles. The boundary of the SL is formed by the trajectory of the resonance Hamiltonian (RT) starting from the GSS curve at such initial angle that provides for the deviation of the RT from the separatrix to be larger than that for any other initial angle. From the topological point of view, there may be two different situations: either the relevant RT is \textit{tangent} to the GSS curve (Fig. 1(a)) or it intersects it while being a self-intersecting trajectory i.e. a \textit{separatrix} of the resonance Hamiltonian (Fig.
1(b)). In the latter case, the outer boundary is formed by the outer part of the separatrix (in Fig. 1(b), relating to the upper boundary, this part lies above the saddle $s$; an example of the boundary formed by the separatrix of a different type is shown in Fig. 4(a)).

The boundary strongly depends on the angle. The maximal/minimal deviation of $E$ from $E_s$ is much larger/smaller than $\delta$.

Fig. 2. The potential $U(q) = (0.2 - \sin(q))^2/2$, the separatrices and the eigenfrequency $\omega(E)$ of the unperturbed system $H_0 = p^2/2 + U(q)$, in (a), (b) and (c), respectively.

Fig. 3. (a). The diagram indicating (shading) the perturbation parameters range for which global chaos exists in the perturbed system, $H = H_0(p,q) - h q \cos(\omega_f t)$. (b). The comparison of the major minimum of the diagram with the lowest-order theory (dashed lines) and the theory allowing for higher-order corrections (solid lines).

3. Application to the double-separatrix chaos

It has been found in\textsuperscript{10} that, if the unperturbed Hamiltonian $H_0$ possesses more than one separatrix (cf. Fig. 2) while the perturbation is time-periodic, the onset of global chaos in between the adjacent separatrices possesses a remarkable feature: it is greatly facilitated if the perturbation frequency $\omega_f$ is close to certain frequencies: the perturbation amplitude $h$ required for the chaos onset is much smaller for such frequencies than for others (Fig. 3). This is related to the characteristic shape of $\omega(E)$ in between the separatrices: $\omega(E)$ approaches a rectangular form in the asymptotic limit of
a small separation between the separatrices (cf. Fig. 2(c)). If \( \omega_f \) is slightly smaller than the local maximum \( \omega_m \) of \( \omega(E) \), then there are two nonlinear resonances that are very wide in energy: they may simultaneously overlap each other and the separatrix chaotic layers, that occurs at the value of \( h \) which is logarithmically smaller than a typical value required for the chaos onset when \( \omega_f \) lies beyond the immediate vicinity of \( \omega_m \).

Using the semi-heuristic approach, the lowest-order asymptotic theory based on the resonance Hamiltonian analysis was developed \(^{10} \) for the minima of the global chaos boundary \( h_{gc}(\omega_f) \). The theory was compared to results of computer simulations for the given example (for a moderately small separation: Fig. 2). The value of \( \omega_f \) in the minimum was well described by the lowest-order formula but the discrepancy for \( h_{gc} \) in the minimum was nearly 50\%. Besides, it was unclear how the overlap of resonances with the chaotic layers occurred and why even a small excess of \( h \) over \( h_{gc} \) resulted in the onset of chaos in a large area of the Poincaré section despite that chaotic layers associated with the nonlinear resonances were exponentially narrow for \( h \leq h_{gc} \). These problems have been resolved in our recent work \(^{11} \) developing the method similar to that described in the previous section. The agreement between the theory and simulations has greatly improved (Fig. 3(b)).

Our present work generalizes the above method for any separatrix (see Sec. 2 above and Sec. 4 below). The general validity of the method is brightly demonstrated by Fig. 4, that shows the direct comparison of the theoretical SL boundaries and the SLs generated by computer. Though the layers are still related to the system with two separatrices, the given perturbation amplitude is so small that the layers are well separated from each other and therefore the presence of a second separatrix does not play a significant role for any of the layers. The lower and upper layers demonstrate two characteristic situations: the lower layer has not yet absorbed the relevant nonlinear resonance (though the closeness to the resonance gives rise to a rather strong increase of the layer width) while the upper layer has absorbed the resonance so that its maximal width greatly exceeds that of the lower layer and there is a large island of stability in the layer.

4. Single-separatrix layer: estimates of the largest width

As already mentioned, the SL width in energy takes its maximum at \( \omega_f \approx \omega_f^{(+/-)} \). The rigorous treatment for the single-separatrix case may be done similar to the double-separatrix case.\(^{11} \) It will be done elsewhere while, here, we present rough estimates for the width.
Fig. 4. The separatrix chaotic layers (shaded) in the plane of action $I$ and slow angle $\psi$ for the system exploited in Fig. 3, for $h = 0.003$ and $\omega_f = 0.401$, as described by our theory. The dashed lines represent the relevant GSS curves. The solid lines represent relevant trajectories of the resonance Hamiltonian $\tilde{H}$: two solid lines with the saddles represent separatrices of the nonlinear resonances, two other solid lines are the trajectories of $\tilde{H}$ which are tangent to the lower and upper GSS curves respectively. (b). Comparison of the layers obtained from computer simulations (dots) with the theoretically calculated boundaries (solid lines) shown in the box (a).

Let us transform from $p - q$ to action $I \equiv I(E)$ and angle $\psi$ and expand $V$ in Eq. (1) into the double Fourier series (in $t$ and $\psi$):$^1$

$$V = \frac{1}{2} \sum_{k, l} V_{k, l} e^{i(k\psi - l\omega_f t)} + c.c. \quad (6)$$

Let us single out the term with the maximum absolute value of $V_{k, l}$ (typically, it corresponds to $k = l = 1$) and denote it as $V_0 e^{i\theta}$:

$$V_0(I) = \max_{k, l} |V_{k, l}| \quad (7)$$

The maximum width of the SL corresponds to the perturbation frequency at which the SL has just absorbed the widest nonlinear resonance (cf.$^{11}$). From the rigorous results for the double-separatrix case,$^{11}$ we may assume that the SL width is dominated by the width of the nonlinear resonance then, i.e. the width of the resonance separatrix in energy $\Delta E$ greatly exceeds the minimal separation in energy between the resonance separatrix and $E_s$. Obviously, this assumption should be verified in the end.

Strictly speaking, $V_0$ strongly depends on $I$ in the relevant range of $I$, end the rigorous treatment of the nonlinear resonance is complicated (cf.$^{9–11}$). But for a rough estimate of the resonance width, it is sufficient to use a simple Chirikov approximation of the resonance Hamiltonian.
\(\hat{H}(I, \dot{\psi})_{1-4,9,12}\) which reduces to the auxiliary pendulum dynamics. The width of the corresponding resonance separatrix in energy is expressed as

\[
\Delta E \approx \sqrt{\frac{8\hbar V_0 \omega_f}{|d\omega/dE|}}
\]  

(8)

The Chirikov approximation assumes that \(|d\omega/dE| \approx \text{const}\) within the range of energies relevant to the resonance separatrix. In our case, it is not so since the quantity \(|d\omega/dE| \approx \omega_f^2/(a\pi\omega_0 |E - E_s|)\) strongly varies within the relevant range of energies. However, we may still use Eq. (8) for the rough estimate, putting in it \(|E - E_s| \sim \Delta E\). Then we obtain for \(\Delta E\) the following rough asymptotic equation

\[
\Delta E \sim V_0(E = E_s \pm \Delta E)\hbar \ln(1/\hbar), \quad \hbar \to 0,
\]  

(9)

where we took into account that the relevant \(\omega_f\) is close to \(\omega_r^{(+/-)} \sim \omega_0/\ln(1/\hbar)\) and omitted numerical factors.

The asymptotic solution of Eq. (9) essentially depends on the function \(V_0(x)\). In this context, all perturbed systems may be divided in two classes.

1. The separatrix of the unperturbed system has more than one saddle while the relevant coefficient \(\tilde{V}_l \equiv \tilde{V}_l(E, \psi)\) in the Fourier expansion (in time) of the perturbation \(V\) possesses different values in adjacent saddles. An archetypal example is a pendulum subject to a dipole time-periodic perturbation.\(^1,7\) If \(E\) is close to \(E_s\), then the system stays mostly near one of the saddles, so that \(\tilde{V}_l\) depends on \(\psi\) nearly in a piece-wise manner: it oscillates between the values corresponding to the adjacent saddles. Therefore, \(V_0\) (which is the absolute value of the relevant coefficient in the Fourier expansion of \(\tilde{V}_l\) in \(\psi\)) approaches in the asymptotic limit \(h \to 0\) some non-zero constant. As follows from Eq. (8), \(\Delta E\) is logarithmically large:

\[
\Delta E \sim h \ln(1/\hbar) \gg h, \quad h \to 0.
\]  

(10)

This estimate agrees with the rigorous result and the result of simulations in the case considered in.\(^1^1\)

2. Either the separatrix has a single saddle (like for a double-well potential system\(^5,8\)) or the separatrix has more than one saddle while the perturbation possesses identically equal values at different saddles. Archetypal examples are a pendulum in a wave with the same spatial period\(^1-4\) and a pendulum with the oscillating suspension point.\(^5\) Then \(\tilde{V}_l(E \to E_s, \psi)\)
keeps nearly one and the same value for most of the period of $\psi$ (as it stays most of the period near the saddle/s): it significantly differs from this value only during a small part of the period, which is $\sim \omega(E)/\omega_0$. Hence, $V_0(E = E_s \pm \Delta E) \sim 1/\ln(1/\Delta E)$, so that the solution of Eq. (9) is:

$$\Delta E \sim h, \quad h \to 0.$$  \hspace{1cm} (11)

This means that the asymptotic functional dependence of the resonance width is the same as that of the SL width in frequency ranges beyond the resonance. So, the functional dependence of the SL in the resonance range is of the same type as beyond it, being $\propto h$. At the same time, a ratio between the resonance width and $h$ may still be a large number. Both these conclusions are in agreement with computer simulations. Thus, for the archetypal example of the Duffing oscillator subject to the dipole time-periodic perturbation,\textsuperscript{5,8} the ratio $\Delta E/h$ approaches in the limit $h \to 0$ the constant value approximately equal to 20: see Fig. 3(b) in.\textsuperscript{8} For another archetypal example, namely a pendulum in the wave of the same spatial period,\textsuperscript{1–4} our recent simulations for the parameters exploited in\textsuperscript{1,3,4} have shown that $\Delta E/h \xrightarrow{h \to 0} \text{const} \approx 50$.

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