Minimum Cuts in Directed Graphs via Partial Sparsification

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Abstract—We give an algorithm to find a minimum cut in an edge-weighted directed graph with \( n \) vertices and \( m \) edges in \( \tilde{O}(n \cdot \max\{m^{3/2}, n\}) \) time. This improves on the 30 year old bound of \( O(nm) \) obtained by Hao and Orlin for this problem. Using similar techniques, we also obtain \( \tilde{O}(n^{3/2}/\epsilon^2) \)-time \((1 + \epsilon)\)-approximation algorithms for both the minimum edge and minimum vertex cuts in directed graphs, for any fixed \( \epsilon \). Before our work, no \((1 + \epsilon)\)-approximation algorithm better than the exact runtime of \( O(nm) \) is known for either problem.

Our algorithms follow a two-step template. In the first step, we employ a partial sparsification of the input graph to preserve a critical subset of cut values approximately. In the second step, we design algorithms to find the (edge/vertex) mincut among the preserved cuts from the first step. For edge mincut, we give a new reduction to \( \tilde{O}(\min\{n, m^{1/3}, \sqrt{n}\}) \) calls of any maxflow subroutine, via packing arborescences in the sparsifier. For vertex mincut, we develop new local flow algorithms to identify small unbalanced cuts in the sparsified graph.

I. INTRODUCTION

The minimum cut (or mincut) problem is one of the most widely studied problems in graph algorithms. In (edge-)weighted\(^1\) directed graphs (or digraphs), a mincut is a bipartition of the vertices into two non-empty sets \((S, V \setminus S)\) so that the total weight of edges from \( S \) to \( V \setminus S \) is minimized. This problem can be solved by solving the \( s\)-mincut problem (also called rooted mincut), where for a given root vertex \( s \), we want to find the minimum weight cut \((S, V \setminus S)\) such that \( s \in S \). (We call such cuts minimum \( s \)-cuts or \( s \)-mincuts.) This is because the mincut can be computed as the minimum between two \( s \)-mincuts for an arbitrary vertex \( s \): one with the original edge directions in the input digraph, and the other with the edge directions reversed.

A simple algorithm for \( s \)-mincut (and thus mincut) on an \( m \)-edge, \( n \)-vertex digraph is to use \( n - 1 \) maxflow calls to obtain the minimum \( s \)-\( t \) cut for every vertex \( t \neq s \) in the graph, and return the minimum among these. A beautiful result of Hao and Orlin [16] showed that these maxflow calls can be amortized to match the running time of a single maxflow call, provided one uses the push-relabel maxflow algorithm [15]. This leads to an overall running time of \( O(nm) \). Since their work, better maxflow algorithms have been designed, but the amortization does not work for these algorithms. As a consequence, the Hao-Orlin bound remains the best known for the directed mincut problem almost 30 years after their work.

A. Our Results

In this paper, we can solve the \( s \)-mincut—thus the directed mincut problem—by essentially reducing it to \( O(\sqrt{n}) \) maxflow calls. At first glance, this is worse than the Hao-Orlin algorithm that only uses a single maxflow call. But crucially, while the Hao-Orlin algorithm is restricted to a specific maxflow subroutine and therefore cannot take advantage of faster, more recent maxflow algorithms, our new algorithm treats the maxflow subroutine as a black box, thereby allowing the use of any maxflow algorithm. Using state of the art maxflow algorithms that run in \( \tilde{O}(m + n^{3/2}) \) time [28], this already improves on the Hao-Orlin bound. Using some additional ideas, we further reduce to \( \tilde{O}(\min\{n^{1/3}, \sqrt{n}\}) \) maxflow calls, which yields our eventual running time of \( \tilde{O}(nm^{2/3} + n^2) \):

**Theorem I.1.** There is a randomized Monte Carlo algorithm that solves \( s \)-mincut (and therefore directed mincut) whp in \( \tilde{O}(nm^{2/3} + n^2) \) time on an \( n \)-vertex, \( m \)-edge (edge-weighted) directed graph.

In fact, our reduction in general implies a running time bound of \( \tilde{O}(\min\{mk, n^2\} + \frac{n}{k} \cdot F(m, n)) \), where \( k \) is
a parameter that we can choose and $F(m, n)$ is the time complexity of maxflow (see Theorem I.1).

Our techniques also yield fast approximations for the mincut problem in directed graphs. In particular, for any $\epsilon \in (0, 1)$, we can find a $(1+\epsilon)$-approximate mincut in $O(n^2/\epsilon^2)$ time:

**Theorem I.2.** For any $\epsilon \in (0, 1)$, there is a randomized Monte Carlo algorithm that finds a $(1+\epsilon)$-approximate $s$-mincut (and therefore directed mincut) w.h.p in $O(n^2/\epsilon^2)$ time on an $n$-vertex (edge-weighted) directed graph.

Finally, we consider vertex-weighted digraphs. A vertex cut in a digraph is defined as a tri-partition of vertices into sets $(L, X, R)$ such that there is no edge from $L$ to $R$. (In other words, removing the vertices in $X$ results in a digraph where the directed cut $(L, R)$ is empty.) A minimum vertex cut (or vertex mincut) is a vertex cut $(L, X, R)$ that minimizes the sum of weights of vertices in $X$. We give an algorithm to find a $(1+\epsilon)$-approximate vertex mincut in $O(n^2/\epsilon^2)$ time:

**Theorem I.3.** For any $\epsilon \in (0, 1)$, there is a randomized Monte Carlo algorithm that finds a $(1+\epsilon)$-approximate minimum vertex $s$-cut and the minimum global vertex cut w.h.p in $O(n^2/\epsilon^2)$ time on an $n$-vertex (vertex-weighted) directed graph.

To the best of our knowledge, before this work, the fastest algorithms for $(1+\epsilon)$-approximation of mincuts in edge or vertex weighted directed graphs were the respective exact algorithms themselves, which obtained a running time of $O(mn)$ [16], [17]. Our approximation results establish a separation between the best exact and $(1+\epsilon)$-approximation algorithms for both edge and vertex mincut problems in directed graphs.

Our results are the first to break the $O(mn)$ barrier for directed mincut problems in general, weighted digraphs. For all values of $m$ except when $m = n^{1+o(1)}$, this is immediate from the above theorems. If $m = n^{1+o(1)}$, we can also break the $O(mn)$ barrier by employing the recent $O(m^{1.5-1/32})$-time max-flow algorithm of [12] to obtain $O(m^{1-o(1)})$-time algorithms for all problems in Theorems I.1 to I.3.

**Related Work:** The directed (edge) mincut problem has been studied over several decades. Early work focused on unweighted graphs [6], [27] eventually resulting in an $O(mn)$-time algorithm due to Mansour and Schieber [21]. This was matched (up to log factors) in weighted graphs by Hao and Orlin [16], whose result we improve in this paper. For unweighted graphs (and graphs with small integer weights), the current record is a recent $O(n^2)$-time algorithm due to Chekuri and Quanrud [3]. A similar story has unfolded for directed vertex mincuts. Early work again focused on unweighted graphs [25], [6], [4], [11] until the work of Henzinger, Rao, and Gabow [17] who obtained an $O(mn)$-time algorithm for weighted graphs. The current best for directed vertex mincut in unweighted graphs is an $O(mn^{11/12+o(1)})$-time algorithm due to Li et al. [19]. Faster algorithms are known when the mincut size is small and for $(1+\epsilon)$-approximations in unweighted digraphs [24], [8], [3].

**B. Our Techniques**

Our results are obtained by solving the $s$-mincut problem. Let us consider the edge-weighted case. Gabow [9] obtained a running time of $O(m\lambda)$ for this problem (assuming integer weights), where $\lambda$ is the size of an $s$-mincut. He did so via arborescence packing: Define an $s$-arborescence to be any spanning tree rooted at $s$ with edges pointing toward the leaves. In $O(m\lambda)$ time, Gabow’s algorithm computes $\lambda$ $s$-arborescences such that an edge of weight $w(e)$ is contained in at most $w(e)$ arborescences (this is called arborescence packing). ² Gabow’s algorithm is at least as fast as that of Hao and Orlin for unweighted simple graphs (since $\lambda \leq n - 1$, but can be much worse for weighted (or multi) graphs. Nevertheless, Karger [18] gave an interesting approach to use arborescence packing for the mincut problem even with edge weights, but only in undirected graphs. Karger’s algorithm had three main steps:

(a) sparsify the input graph $G$ to $H$ by random sampling of edges to reduce the mincut value in $H$ to $O(\log n)$ while guaranteeing that the mincut in $G$ is a $(1+\epsilon)$-approximate mincut in $H$,

(b) pack $O(\log n)$ $s$-arborescences³ in the sparsifier $H$, and

(c) find the minimum weight cut among those that have at most two edges in an arborescence using a dynamic program.

The last step is sufficient because the duality between cuts and arborescences ensures that an $s$-mincut, which is now a $(1+\epsilon)$-approximate mincut after sparsification, has at most two edges in at least one $s$-arborescence.⁴ Karger implements all these steps in $O(m)$ time to obtain an $O(m)$-time mincut algorithm in undirected graphs.

Unfortunately, steps (a) and (c) in Karger’s scheme are not valid in a directed graph. To begin with, directed

²Gabow actually constructs a directionless spanning tree packing, which is a relaxation of an arborescence packing, but we ignore this technical detail here since it is not relevant to our eventual algorithm.

³Since Karger’s algorithm considered undirected graphs, the arborescences are simply spanning trees.

⁴The duality implies that if the undirected mincut is $\lambda$, then we can pack $\lambda$ spanning trees where every edge appears in at most two arborescences.
graphs do not admit sparsifiers similar to Karger’s spar-
sifier: while Karger’s sparsifier approximately preserves
all cuts in an undirected graph (after some scaling), it
is well known that a sparsifier with a similar property
does not exist in directed graphs (see, e.g., [1]). This is
mainly because we cannot bound the number of mincuts
and approximate mincuts in directed graphs while we
can do so in undirected graphs.

Partial Sparsification: Since it is impossible for
a sparsifier to preserve all cuts in directed graphs,
it is natural to try to preserve a partial subset of
cuts. Suppose we were guaranteed that the s-mincut
\((S,V \setminus S)\) (recall that \(s \in S\)) is unbalanced in the
sense that \(|V \setminus S| \leq k\) for some parameter \(k\) that we
will fix later. Let us randomly sample edges to scale
down the value of the mincut to \(\tilde{O}(k)\). In an undirected
graph, as long as \(k = \Omega(\log n)\), all the cuts would
converge to their expected values. This is not true in
digraphs, but crucially, \(all the unbalanced cuts converge
to their expected values\) since there are only \(n^{O(k)}\) of
them. However, it is possible that some balanced cut
is (misleadingly) the new mincut of the sampled graph,
having been scaled down disproportionately by the ran-
don sampling. So, we overlay this sampled graph with a
star rooted at \(s\), and show that this sufficiently increases
the values of all balanced cuts, while only distorting
the unbalanced cuts slightly. After this overlay, we can
claim that we now have a digraph where \((S,V \setminus S)\) is
a \((1 + \epsilon)\)-approximate mincut. We use one additional
idea here. We show that in the sparsifier, every vertex
in \(V \setminus S\) has only \(\tilde{O}(k)\) incoming edges (note that each
dge can be weighted)—\(\tilde{O}(k)\) edges across the cut from
\(S\) and \(\leq k\) edges from within \(V \setminus S\). By contracting all
vertices with unweighted in-degree \(\geq \tilde{O}(k)\) into \(s\), we
reduce the number of edges in the digraph to \(\tilde{O}(nk)\).

But, what if our premise that the \(s\)-mincut is unbal-
anced does not hold? This case is actually simple. We
use a uniform random sample of \(\tilde{O}(\frac{n}{k})\) vertices, and
find \(s - t\) mincuts for all vertices \(t\) from the sample,
using \(\tilde{O}(\frac{n}{k})\) maxflow calls. It is easy to see that whp, the
sample hits \(V \setminus S\), and hence, the minimum weighted
cut among these \(s - t\) mincuts will reveal the \(s\)-mincut of
the graph.

Let us, therefore, return to the case where the \(s\)-
mincut is unbalanced. Recall that we have already
sparsified the graph to one that has only \(\tilde{O}(nk)\) edges,
and where the mincut has weight \(\tilde{O}(k)\). The next step
is to create a maximum packing of edge-disjoint \(s\)-
arborescences. Because the graph is weighted, instead
of using Gabow’s algorithm described above, we create a
(fractional) packing using a multiplicative weights
update procedure (e.g., [30]). By duality\(^5\), these ar-
borescences have the following property: if we sample
\(O(\log n)\) random \(s\)-arborescences, then whp there will
be at least one arborescence \(T\) such that there is exactly
one edge in \(T\) that goes from \(S\) to \(V \setminus S\). In this case, we
say that the cut \((S,V \setminus S)\) \(1\)-respects the arborescence
\(T\).

1-respecting cut algorithm: Our final task, there-
fore, is the following: given an arborescence, find the
minimum weight cut in the original graph among all
those that \(1\)-respect the arborescence \(T\). At first sight,
this may look similar to part (e) of Karger’s algo-

\(^5\)By duality, we have that if the directed mincut is \(\lambda\), then we
can pack \(\lambda\) arborescences where each edge appears in at most one
arborescence. Note that this is different from the case of undirected
graphs where each edge appears in at most two arborescences.
of our sparsifiers (beyond sparsity). We finally obtain $O(n^2/e^2)$ running time by balancing the two cases and calling the max flow algorithm by [28].

**Summary:** To summarize, our algorithms distinguish between balanced and unbalanced mincuts, solving the former using maxflows on randomly sampled terminals. For unbalanced mincuts, we follow a two-step template. In the first step, we employ partial sparsification to preserve the values of unbalanced cuts approximately, while suppressing balanced cuts using an overlay. In the second step, we design algorithms to find the (edge/vertex) mincut among unbalanced cuts. For edge mincut, the sparsifier allows one to quickly obtain an arborescence that respects the directed mincut. From this arborescence, we obtain the exact minimum $F$ of the cut, i.e., $F = \min_{\partial^+U} \{w(e) : e \in \partial^+(U)\}$. Our goal is to compute the minimum $s$-t maxflow on a digraph with $n$ vertices and $m$ edges. The current record for this bound is $F(m, n) = \tilde{O}(m + n^{3/2})$ [28]. We emphasize that our directed mincut algorithm uses maxflow subroutines in a black box manner and therefore, any maxflow algorithm suffices. Correspondingly, we express our running times in terms of $F(m, n)$.

**Theorem II.1.** There is a Monte Carlo algorithm that finds a minimum $s$-cut $\epsilon$-whp in $\tilde{O}(\min\{mk, nk^2\} + F(m, n)\frac{n}{k})$ time, where $k$ is a parameter and $F(m, n)$ is the time complexity of $s$-t maxflow.

This section is devoted to prove Theorem II.1. If we set $k = m^{1/3} + n^{1/2}$ and use the $O(m + n^{3/2})$ max-flow algorithm, the time complexity becomes $O(nm^{2/3} + n^2)$, which establishes Theorem I.1. If we assume an hypothetical $\tilde{O}(m)$-time max-flow algorithm, then our result becomes $\tilde{O}(\min\{nm^{2/3}, mn^{1/2}\})$ for $k = \min\{m^{1/3}, n^{1/2}\}$.

We obtain Theorem II.1 via a new $s$-mincut algorithm. The algorithm considers the following two cases, computing a $s$-cut for each case and returning the minimum as its final output. The cases are split on $|T^*|$ by a threshold $k > 0$.

1) The first case aims to compute the correct mincut in the event that $|T^*| > k$. In this case, if we randomly sample $t \in V$, then with probability at least $1/k$, $t \in T^*$. Then $T^*$ can be obtained via the maxflow from $s$ to $t$. Repeating the sampling $O(\frac{n}{k} \log n)$ times, we obtain the minimum $s$-cut whp. The total running time for this case is $O(F(m, n)\frac{n}{k} \log n)$.

2) The second case is for the the event that $|T^*| \leq k$. Let $\lambda$ denote the value of the minimum rooted cut. By enumerating $O(\log n)$ powers of 2, we can obtain an estimate $\lambda$ such that $\lambda \leq \lambda \leq 2\lambda$. For each value of $\lambda$, we apply Lemma II.5 to sparsify the graph in the following manner. First, Lemma II.5 returns a set of vertices $V_\delta \subseteq V$ such that $s \in V_\delta$ and $T^* \subseteq V_\delta$ whp. In particular, one can safely contract any vertex $v \in V \setminus V_\delta$ into $s$ without affecting the minimum $s$-cut. We contract $G$ accordingly and, overloading notation, let $G$ denote the contracted graph with vertex set $V_\delta$ henceforth. Second, Lemma II.5 returns a graph $G_0 = (V_0, E_0)$ in which $T^*$ still induces an $(1 + \epsilon)$-approximate $s$-mincut, but the weight of the cut is now reduced to $O(k \log(n))$. We note that $G_0$ is not necessarily a subgraph of $G$. We then invoke Lemma II.6 from Section II-B to fractionally pack an approximately maximum amount of $O(k \log n)$ $s$-arborescences in $G_0$ in $\tilde{O}(m + \min\{mk, nk^2\})$ time. In a random sample of $O(\log n)$ $s$-arborescences from this packing, one of them will $\epsilon$-whp respect the $s$-mincut in $G$ (for appropriate $\lambda$) whp:

**Definition II.2.** A directed $s$-cut $(S, V \setminus S)$ $k$-respects an $s$-arborescence if there are at most $k$ edges in the arborescence from $S$ to $V \setminus S$.

Finally, for each of the $O(\log n)$ $s$-arborescences, the algorithm computes the minimum $s$-cut that 1-respects each arborescence. This algorithm is described in Algorithm 1 and proved in Theorem II.7 from Section II-C. It runs in $O((F(m, n) + m) \cdot \log n)$ time for each of the $O(\log n)$ arborescences.

Combining both cases, the total running time becomes $O(\min\{mk, nk^2\} + F(m, n)\frac{n}{k})$, which establishes Theorem II.1.

**Fast approximations:** The exact algorithm described above can be modified to produce a randomized $\tilde{O}(n^2/e^2)$-time approximation algorithm that computes a $(1 + \epsilon)$-approximate minimum $s$-cut (hence also the
global cut). With logarithmic overhead, we can obtain a parameter $k$ such that $k/2 \leq |T^*| \leq k$, where $\partial^-(T^*)$ is the minimum $s$-cut. We then follow the same steps as in the exact algorithm, except whenever we compute the max-flow, we compute it in the sparsifier produced by Lemma II.5 instead. Since the sparsifier has at most $O(nk/e^2)$ edges, we obtain a running time of the form $O(\min\{nk^2/e^2\} + F(nk/e^2, n)(n/k))$. For $F(m, n) = O(m + n^{1.5})$, this gives a running time of $O(n^2/e^2)$.

**Theorem II.3.** For $\epsilon \in (0, 1)$, an $(1 + \epsilon)$-approximate minimum $s$-cut (hence global minimum cut) can be computed in $O(n^2/e^2)$ time whp.

We remark that $O(n^2/e^2)$ can also be obtained by using a local connectivity algorithm similar to the approach for vertex mincuts in Section III, instead of via an arborescence packing. See [26] for details.

*Organization:* The rest of this section is organized as follows. The following subsections present each step of the algorithm described above. First, we establish the partial sparsification subroutine in Section II-A. Next, in Section II-B, we obtain an arborescence packing from which sampling yields an arborescence that is $1$-respected by the mincut. Finally, in Section II-C, we describe the algorithm to retrieve the mincut among those that $1$-respect a given arborescence.

**A. Partial Sparsification**

This section aims to reduce mincut value to $O(k)$ and edge size to $\min\{m, O(nk\log(n)/e^2)\}$ while keeping $\partial^+(S^*)$ a $(1 + \epsilon)$-approximate $s$-mincut for a constant $\epsilon > 0$ that we will fix later. Our algorithm in this stage has three steps. First, we use random sampling to discretize and scale down the expected value of all cuts such that the expected value of the mincut $\delta^+(S^*)$ becomes $O(k)$. We also claim that $\partial^+(S^*)$ remains an approximate mincut among all unbalanced cuts by using standard concentration inequalities. However, since the number of balanced cuts far exceeds that of unbalanced cuts, it might be the case that some balanced cut has now become much smaller in weight than all the unbalanced cuts. This would violate the requirement that $\partial^+(S^*)$ should be an approximate mincut in this new graph. This is where we need our second step, where we overlay a star on the sampled graph to raise the values of all balanced $s$-cuts above the expected value of $\partial^+(S^*)$ while only increasing the value of $\partial^+(S^*)$ by a small factor. The third step leverages the fact that, after scaling, the minimum edge weight is $1$, and the minimum cut is $O(k\log(n)/e^2)$. It follows that any vertex with in-degree at least a constant factor greater than $k\log(n)/e^2$ cannot be in the sink component, and can be safely contracted into the root without affecting the $s$-mincut.

The first two steps described above are implemented in the next lemma, whose proof is deferred to the full version.

**Lemma II.4.** Let $G = (V, E)$ be a directed graph with positive edge weights. Let $s \in V$ be a fixed root vertex. Let $\epsilon \in (0, 1)$, let $\lambda > 0$, and let $k \in \mathbb{N}$ be given parameters. Suppose there is an $s$-mincut of the form $\partial^-(T^*)$, where $s \notin T^*$, $\lambda/2 \leq \delta_G(T^*) \leq 2\lambda$ and $|T^*| \leq k$. In randomized nearly linear time, one can compute a randomized directed and reweighted subgraph $G_0 = (V, E_0)$, where $V_0 \subseteq V$ and $s \in V_0$ with the following properties.

(i) $G_0$ has integral edge weights and the minimum $s$-cut has weight at most $O(k\log(n)/e^2)$.

(ii) $\partial_{G_0}^-(T^*)$ is a $(1 + \epsilon)$-approximate minimum $s$-cut in $G_0$.

(iii) Every $\alpha$-approximate minimum $s$-cut in $G_0$ induces an $(1 + \epsilon)\alpha$-approximate minimum $s$-cut in $G$.

The preceding lemma importantly allows us to reduce the weight of the minimum cut to roughly $k$, where $k$ is the number of the vertices in the sink component. However, it has not actually reduced the size in the graph, in terms of the number of edges. This is accomplished by our third step, which we formalize in the following lemma that reduces the number of edges to $O(nk)$.

**Lemma II.5.** Let $G = (V, E)$ be a directed graph with positive edge weights. Let $s \in V$ be a fixed root vertex. Let $\epsilon \in (0, 1)$, $\lambda > 0$, and $k \in \mathbb{N}$ be given parameters. Suppose there is a minimum $s$-cut of the form $\partial^-(T^*)$, where $s \notin T^*$, $\lambda/2 \leq \delta_G(T^*) \leq 2\lambda$ and $|T^*| \leq k$. In randomized nearly linear time, one can compute a randomized directed and edge-weighted graph $G_0 = (V_0, E_0)$, where $V_0 \subseteq V$ and $s \in V_0$.

(i) $G_0$ has integral edge weights and the $s$-mincut in $G_0$ has weight at most $O(k\log(n)/e^2)$.

(ii) $G_0$ has at most $\max\{m, O(nk\log(n)/e^2)\}$ edges.

(iii) $G_0$ is a subgraph of the graph obtained by contracting $V \setminus V_0$ into $s$ in $G$.

(iv) We have $T^* \subseteq V_0$, and $\partial_{G_0}^-(T^*)$ is an $(1 + \epsilon)$-approximate minimum $s$-cut in $G_0$.

(v) Every $(1 + \epsilon)$-approximate minimum $s$-cut in $G_0$ induces an $(1 + \epsilon)^2$-approximate minimum $s$-cut in $G$.

*Proof:* Consider the reweighted subgraph produced by Lemma II.4, which we denote by $G_1 = (V, E_1)$. We claim that every vertex $v \in T^*$ has unweighted in-degree at most $O(k\log(n)/e^2)$. Indeed, at most $k - 1$ of these edges are from other vertices in $T$, and the
remaining edges must be in \( \partial_G^-(T^*) \). But \( \partial_G^-(T^*) \) has at most \( \delta_{G_1}(T^*) = O(k \log(n)/\epsilon^2) \) edges by properties (i) and (ii) of Lemma II.4.

Let \( G_0 = (V_0, E_0) \) be the graph obtained from \( G_1 \) by contracting all vertices with unweighted in-degree \( \geq ck \log(n)/\epsilon^2 \) for a sufficiently large constant \( c \) that excludes all vertices in \( T^* \). It is easy to see that \( G_0 \) satisfies the claimed properties, particularly as contractions into \( s \) do not decrease the \( s \)-mincuts, and \( \partial_{G_0}^-(T^*) \) is preserved exactly as in \( G_1 \).

**B. Finding a 1-respecting Arborescence**

In this section, we assume that there is an unbalanced \( s \)-mincut and show how to obtain an \( s \)-arborescence that 1-respects the \( s \)-mincut. More formally, we prove the following:

**Lemma II.6.** Given weighted digraph \( G \) and a fixed root vertex \( s \), suppose the sink side of an \( s \)-mincut \( T^* \) has at most \( k \) vertices. In \( O(m \log n + \min\{mk \log^2 n, nk^2 \log^3 n\}) \) time, we can find \( O(\log n) \) \( s \)-arborescences on vertex set \( V_0 \supset T^* \), such that whp an \( s \)-mincut 1-respects at least one of them.

The idea of this lemma is as follows. First, we apply Lemma II.5 to our graph \( G \) and obtain the graph \( G_0 \). Whp, a minimum \( s \)-cut \( \partial^-(T^*) \) in \( G \) corresponds to a \((1 + \epsilon)\)-approximate minimum \( s \)-cut in \( G_0 \). It remains to find an arborescence in \( G_0 \) that 1-respects \( \partial^-(T^*) \). To do this, we employ a multiplicative weight update (MWU) framework. The algorithm begins by setting all edge weights to be uniform (say, weight 1). Then, we repeat the following for \( O(k \log(n)/\epsilon^2) \) rounds: in each round, we find a minimum weight arborescence in \( O(m) \) time and multiplicatively increase the weight of every edge in the arborescence. Using the fact that there is no duality gap between arborescence packing and mincut [5], [9], a standard MWU analysis implies that these arborescences that we found, after scaling, form a \((1 + \epsilon)\)-approximately optimal fractional arborescence packing. So our arborescence crosses \( T^* \) at most \((1 + O(\epsilon)) < 2 \) times on average. Thus, if we sample \( O(\log n) \) arborescences from this set, one of them will 1-respect \( T^* \) whp.\(^6\) To obtain the running time bound, we note that each iteration of the MWU framework requires us to find the minimum cost \( s \)-arborescence, for which an \( O(m) \)-time algorithm is known [10].

Since the argument above is a standard application of the MWU framework, we defer the detailed proof to the full version.

\(^6\)This should be compared with Karger’s mincut algorithm in the undirected case, where there is a factor 2 gap, and hence Karger can only guarantee a 2-respecting tree in the undirected case.

**C. MinCut Given 1-respecting Arborescence**

We propose an algorithm (Algorithm 1) that uses \( O(\log n) \) maxflow subroutines to find the minimum \( s \)-cut that 1-respects a given \( s \)-arborescence. The result is formally stated in Theorem II.7.

**Theorem II.7.** Consider a directed graph \( G = (V, E) \) with polynomially bounded edge weights \( w_e > 0 \). Let \( s \in V \) be a fixed root vertex and \( S \ni s \) be the source side of a fixed \( s \)-mincut. Given an \( s \)-arborescence \( T \) with \(|T \cap \partial^+(S)| = 1\), Algorithm 1 outputs a \( s \)-mincut of \( G \) in time \( O((F(m,n) + m) \cdot \log n) \).

We first give some intuition for Algorithm 1. Because \( s \in S \), if we could find a vertex \( t \in S \), then computing the \( s-t \) mincut using one maxflow call would yield a global mincut of \( G \). However, we cannot afford to run one maxflow between \( s \) and every other vertex in \( G \). Instead, we carefully partition the vertices into \( \ell = O(\log n) \) sets \( (C_i)_{i=1}^\ell \). We show that for each \( C_i \), we can modify the graph appropriately so that it allows us to (roughly speaking) compute the maximum flow between \( s \) and every vertex \( c \in C_i \) using one maxflow call.

More specifically, Algorithm 1 has two stages. In the first stage, we compute a centroid decomposition of \( T \). Recall that a centroid of \( T \) is a vertex whose removal disconnects \( T \) into subtrees with at most \( n/2 \) vertices. This process is done recursively, starting with the root \( s \) of \( T \). Let \( P_1 \) denote the subtrees resulting from the removal of \( s \) from \( T \). In each subsequent step \( i \), we compute the set \( C_i \) of the centroids of the subtrees in \( P_i \). Then we remove the centroids and add the resulting subtrees to \( P_{i+1} \). This process continues until no vertices remain.

In the second stage, for each layer \( i \), we construct a directed graph \( G_i \) and perform one maxflow computation on \( G_i \). The maxflow computation on \( G_i \) would yield candidate cuts for every vertex in \( C_i \), and after computing the appropriate maximum flow across every layer, we output the minimum candidate cut as the minimum cut of \( G \). The details are presented in Algorithm 1.
We defer the proofs of Lemmas II.8 and II.9, and first use them to prove Theorem II.7.

Proof of Theorem II.7: We first prove the correctness of Algorithm 1.

Because $C_0 = \{s\}$ and $s \in S$, and the $C_i$’s form a disjoint partition of $V$, there must be a layer $i$ such that for the first time, we have a centroid $u \in C_i$ that belongs to $\overline{S}$. By Lemma II.8, we know that $\overline{S}$ must be contained in exactly one subtree $U \in P_i$, and hence $u$ must be the centroid of $U$. In summary, we have $u \in \overline{S}$ and $\overline{S} \subseteq U$.

Consider the graph $G_i$ constructed for layer $i$. By Lemma II.9, based on the flow $f_i^*$, we can recover the value of the minimum cut from $u$ to $u'$. Because $\overline{S} \subseteq U$ and $u \in \overline{S}$, the cut $(S, \overline{S})$ is one possible cut that separates $\overline{S}$ and $u$. Therefore, the flow $f_i^*$ is equal to the $s$-mincut value in $G$.

In addition, the candidate cut value for any other centroid $u'$ of a subtree $U' \in P_i$ must be at least the mincut value between $s$ and $u'$. This is because the additional restriction that the cut has to separate $\overline{U'}$ from $u'$ can only make the mincut value larger, and the value of this cut in $G_i$ is equal to the value of the same cut in $G$. Therefore, the minimum candidate cut value in all $\ell$ layers must be equal to the $s$-mincut value of $G$.

Now we analyze the running time of Algorithm 1. We can find the centroid of an $n$-node tree in time $O(n)$ (see e.g., [22]). The total number of layers $\ell = O(\log n)$ because removing the centroids reduces the size of the subtrees by at least a factor of 2. Thus, the running time of Stage I of Algorithm 1 is $O(n \log n)$. In Stage II, we can construct each $G_i$ in $O(m)$ time and every $G_i$ has $O(m)$ edges. Since there are $O(\log n)$ layers and the maximum flow computations take a total of $O(MF(m, n) \cdot \log n)$ time, the overall runtime is $O(n \log n + (MF(m, n) + m) \cdot \log n) = O((MF(m, n) + m) \log n)$.

Before proving Lemmas II.8 and II.9 we first prove the following lemma.

Lemma II.10. If $x$ and $y$ are vertices in $\overline{S}$, then every vertex on the (undirected) path from $x$ to $y$ in the arborescence $T$ also belongs to $\overline{S}$.

Proof: Consider the lowest common ancestor $z$ of $x$ and $y$. Because there is a directed path from $z$ to $x$ and a directed path from $z$ to $y$, we must have $z \in \overline{S}$. Otherwise, there are at least two edges in $T$ that go from $S$ to $\overline{S}$.

Because $s \in S$ and $z \in \overline{S}$, there is already an edge in $T$ (on the path from $s$ to $z$) that goes from $S$ to $\overline{S}$. Consequently, all other edges in $T$ cannot go from $S$ to $\overline{S}$, which means the entire path from $z$ to $x$ (and similarly $z$ to $y$) must be in $\overline{S}$.
Recall that Lemma II.8 states that if all the centroids in previous layers are in $S$, then $\overline{S}$ is contained in exactly one subtree $U$ in the current layer $i$.

**Proof of Lemma II.8:** For contradiction, suppose that there exist distinct subtrees $U_1$ and $U_2$ in $P_i$ and vertices $x, y \in \overline{S}$ such that $x \in U_1$ and $y \in U_2$.

By Lemma II.10, any vertex on the (undirected) path from $x$ to $y$ also belongs to $\overline{S}$. Consider the first time that $x$ and $y$ are separated into different subtrees. This must have happened because some vertex on the path from $x$ to $y$ is removed. However, the set of vertices removed at this point of the algorithm is precisely $\bigcup_{0 \leq j < i} C_j$, but our hypothesis assumes that none of them are in $\overline{S}$. This leads to a contradiction and therefore $\overline{S}$ is contained in exactly one subtree of $P_i$.

It follows immediately that at most one centroid $u \in C_i$ can be in $\overline{S}$.

Next we prove Lemma II.9, which states that the maximum flow between $s$ and $t_i$ in the modified graph $G_i$ allows one to simultaneously compute a candidate mincut value for each vertex $u \in C_i$.

**Proof of Lemma II.9:** First observe that the maxflow computation from $s$ to $t_i$ in $G_i$ can be viewed as multiple independent maxflow computations. The reason is that, for any two subtrees $U_1, U_2 \subseteq P_i$, there are only edges that go from $s$ into $U_1$ and from $U_1$ to $t_i$ (similarly for $U_2$), but there are no edges that go between $U_1$ and $U_2$.

The above observation allows us to focus on one subtree $U \subseteq P_i$. Consider the procedure that we produce $G_i$ from $G$ in Steps 12 to 14 of Algorithm 1. The edges with both ends in $U$ are intact (the edge set $E_1$). If we contract all vertices outside of $U$ into $s$, then all edges that enter $U$ would start from $s$, which is precisely the effect of removing cross-subtree edges and adding the edges in $E_2$. One final infinity-capacity edge $(u, t_i) \in E_3$ connects the centroid of $U$ to the super sink $t_i$.

Therefore, the maximum $s$-$t_i$ flow $f^*_U$ computes the maximum flow between $U$ and $u \in U$ simultaneously for all $U \subseteq P_i$, whose value is reflected on the edge $(u, t_i)$. It follows from the maxflow mincut theorem that the flow on edge $(u, t_i)$ is equal to the mincut value between $U$ and $u$ in $G$ (i.e., the minimum value $w(A, \overline{A})$ among all $A \subseteq V$ with $\overline{U} \subseteq A$ and $u \in \overline{A}$).

**III. MINIMUM CUT ALGORITHMS IN VERTEX-WEIGHTED DIRECTED GRAPHS**

In this section we present the approximation algorithm for the minimum rooted and global vertex cut. Similar to Section II, the main focus is on rooted cuts, and the algorithm is presented in three main parts. All three parts are parameterized by values $\kappa > 0$ and $k \in \mathbb{N}$ that, in principle, are meant to be constant factor estimates for the weight and the number of vertices in the sink component of the minimum rooted vertex cut. The first part, in Section III-A, presents the sparsification lemma that reduces the number of edges to roughly $nk$ and the rooted mincut to roughly $k$ in a graph with integer weights. This sparsifier is used in the remaining two parts. The second part, in Section III-B, gives a roughly $nk^2$ time approximation algorithm for the minimum rooted cut via a new local flow algorithm. The third part, in Section III-C, gives a roughly $n^2 + n^{2.5}/k$ time approximation algorithm via sampling and $(s,t)$-flow (as with minimum edge cuts before). Finally, in Section III-D, we balance terms to obtain the claimed running time for rooted cut. The rooted vertex mincut algorithm then leads to a global vertex mincut algorithm via an argument due to [17] (with some modifications).

**A. Partial Sparsification**

The first part is a sparsification lemma that preserves rooted vertex cuts where the number of vertices in the sink component is below some given parameter. It is similar in spirit to Lemma II.5, but with some necessary changes as we are now preserving the vertex mincut rather than edge mincut. We give a brief overview of the algorithm, highlighting in particular the differences from the partial edge cut sparsifier. The proof and algorithmic details are deferred to the full version.

At a high level, the following sparsifier for vertex cuts randomly samples the vertex weights so that the weights are integral, and the weight of the minimum vertex cut becomes $O(k \log(n)/\epsilon^2)$. Similar to the partial edge sparsifier, this rounding is calibrated to preserve $s$-cuts with (roughly) $k$ or fewer vertices in the sink components. To pad the weight of vertex $s$-cuts with large sink components, we add an weighted auxiliary vertex on a short directed path between $s$ and each vertex (as opposed to just adding an edge from $s$, as we did for edge cuts). If a sampled weight of a vertex $v$ is $0$, we cannot simply drop the vertex from the graph (in the way we can drop weight $0$ edges) since the vertex may be in the sink component of the min $r$-cut. Instead we remove all outgoing edges from $v$. Also, when we detect that a vertex $v$ cannot be in the sink component (by a similar counting argument as before), rather than contract $v$ into $s$ (which may effect the min vertex $s$-cut), we replace all of the incoming edges to $v$ with a single edge from $s$. The culmination of these modifications is a similar net effect as for edge cuts: a graph with $O(nk \log(n)/\epsilon^2)$ that preserves the sink component of the minimum vertex $s$-cut. That said, the following bounds are more detailed than the bounds for preserving the edge cut in Lemma II.5. These additional properties play a critical role in the customized local flow algorithms presented later.
In the following, let $N^+(v \mid G)$ denote the set of out-neighbors of $v$ in the graph $G$. We omit $G$ and simply write $N^+(v)$ when $G$ can be inferred from the context.

**Lemma III.1.** Let $G = (V, E)$ be a directed graph with positive vertex weights. Let $s \in V$ be a fixed vertex. Let $k, \kappa \geq 0$ be given parameters. Let $V' = V \setminus \{(s) \cup N^+(s)\}$. In randomized linear time, one can compute a randomized directed and vertex-weighted graph $G_0 = (V_0, E_0)$, and a scaling factor $\tau > 0$, with the following properties.

(i) $s \in V_0$.
(ii) Let $V_0' = V_0 \setminus \{(s) \cup N^+(s \mid G_0)\}$. We have $V_0' = V'$.
(iii) $G_0$ has integer vertex weights between 0 and $O(k \log(n)/\epsilon^2)$. 
(iv) Every vertex $v \in V_0$ has at most $O(k \log(n)/\epsilon^2)$ incoming edges.
(v) Every vertex $v$ with weight 0 has no outgoing edges.
(vi) With high probability, for all $S \subseteq V'$, the weight of the vertex in-cut induced by $S$ in $G_0$ (up to scaling by $\tau$) is at least the minimum of the $(1 - \epsilon)$ times the weight of the induced vertex in-cut in $G$ or $c \epsilon s$ (for any desired constant $c > 1$), and at most $(1 + \epsilon)$ times its weight in $G$ plus $c \epsilon s |S|/k$.
(vii) With high probability, for all $S \subseteq V'$ such that $|S| \leq k$ and the weight of the induced vertex in-cut is $\leq O(\kappa)$, we have $S \subseteq V_0'$. (That is, $S$ is still the sink component of an $s$-cut in $G_0$.)

In particular, if the minimum vertex $s$-cut has weight $\Theta(\kappa)$, and the sink component of a minimum vertex $s$-cut has at most $k$ vertices, then with high probability $G_0$ preserves the minimum vertex $s$-cut up to a $(1 + O(\epsilon))$-multiplicative factor.

As stated above, the proof is deferred to the full version.

**B. Rooted vertex mincut for small sink components**

This section presents an approximation algorithm for rooted vertex mincut for the particular setting where the sink component is small. In particular, we are given an upper bound $k$ on the number of vertices in the sink component, and want to obtain running times of the form $n \text{poly}(k)$. When a similar situation arose previously for small integer capacities in [3], [3] modified a local algorithm from [8] which works well for unweighted graphs. Here, while Lemma III.1 produces relatively sparse graphs with integral vertex capacities, the vertex capacities imply that the algorithm from [3], [8] would take roughly $nk^3/\epsilon^5$ time. This section develops an alternative algorithm that is inspired by these local algorithms for (global and rooted) vertex cuts, but reduces the dependency on $k$ to $k^2$. Compared to [8], [3], the algorithm here is designed to take full advantage of the properties of the graph produced by Lemma III.1. These modifications have some tangible benefits. First, it improves the dependency on $k$ and $\epsilon$. Second, the local subroutine here is deterministic whereas before they were randomized. Third and last, as suggested by the better running time and the determinism, the version presented here is arguably simpler and more direct than the previous algorithms (for this setting).

**Lemma III.2.** Let $G = (V, E)$ be a directed graph with positive vertex weights. Let $s \in V$ be a fixed root vertex. Let $\epsilon \in (0, 1)$, $\kappa > 0$ and $k \in \mathbb{N}$ be given parameters. There is a randomized linear time Monte Carlo algorithm that, with high probability, produces a deterministic data structure that supports the following query.

For $t \in V'$ defined $V = \{(s) \cup N^+(s)\}$, let $\kappa_{t,k}$ denote the weight of the minimum $(s,t)$-vertex cut such that the sink component has at most $k$ vertices. Given $t \in V'$, deterministically in $O(k^3 \log^2(n)/\epsilon^4)$ time, the data structure either (a) returns the sink component of a minimum $(s,t)$-vertex cut of weight at most $(1 + \epsilon)\kappa_{t,k}$, or (b) declares that $\kappa_{t,k} > \kappa$.

**Proof:** Given $s$, $\kappa$, $k$, and $\epsilon$, let $\epsilon' = c \epsilon$ for a sufficiently small constant $c > 0$. We first apply Lemma III.1 to $G$ with root $s$ and parameters $\kappa$, $k$, and $\epsilon'$. This produces a vertex capacitated graph $G_0 = (V_0, E_0)$ with $V \subseteq V_0$. We highlight the features that we leverage. All new vertices (in $V_0 \setminus V$) are in $N^+(s \mid G_0)$; that is, $V'$ equals $V_0' = V_0 \setminus \{(s) \cup N^+(s \mid G_0)\}$. Put alternatively, none of the new vertices is in the sink component of any $s$-cut. The vertex weights are integers between 0 and $O(k \log(n)/\epsilon^2)$. Every vertex has unweighted in-degree at most $O(k \log(n)/\epsilon^2)$. Every vertex with weight 0 has no outgoing edges.

With high probability, we have the following guarantees on the vertex $s$-cuts of $G_0$. The vertex weights in $G_0$ are scaled so that a weight of $\kappa$ in $G$ corresponds to weight $O(k \log(n)/\epsilon^2)$ in $G_0$. Modulo scaling, every vertex $s$-cut in $G_0$ has weight no less than the minimum of its weight in $G$ and $2\epsilon$. Additionally, modulo scaling, for every vertex $s$-cut in $G$ with capacity at most $\kappa$ and at most $k$ vertices in the sink component, the corresponding vertex cut in $G_0$ has weight at most $c_0 \kappa k$ additive factor larger than in $G$, for any desired constant $c_0 > 0$. We consider the algorithm to fail if the cuts are not preserved in the sense described above.

Given $t \in V$, the data structure will search for a small $(s,t)$-cut in $G_0$ via a customized, edge-capacitated flow algorithm. This algorithm may or may not return the sink component of $(s,t)$-cut. If the search does return
a sink component, and the corresponding vertex in-cut in $G_0$ has weight that, upon rescaling back to the scale of the input graph $G$, is at most $(1 + \varepsilon/2)\kappa$, the data structure returns it. Otherwise the data structure indicates that $s_{t,k} > \kappa$.

Proceeding with the flow algorithm, let $G_{\text{rev}}$ be the reverse of $G_0$, and let $G_{\text{split}}$ be the standard “split-graph” of $G_{\text{rev}}$ modeling vertex capacities with edge capacities. We recall that the split graph splits each vertex $v$ into an auxiliary “in-vertex” $v^-$ and an auxiliary “out-vertex” $v^+$.

For each $v$ there is a new edge $(v^-,v^+)$ with capacity equal to the vertex capacity of $v$. Each edge $(u,v)$ is replaced with an edge $(u^+,v^-)$ with capacity equal to the vertex capacity of $v$. Every $(s,t)$-edge cut in $G_0$ maps to a $(t^+,s^-)$-edge cut in $G_{\text{rev}}$ with the same capacity. Any $(t^+,s^-)$-edge capacitated cut maps to a $(s,t)$-vertex cut in $G_0$ (with negligible overhead in the running time). Now, recall that for each $v \in V'$, the sparsification procedures introduce an auxiliary path $(s,a_v,s^-)$ where $a_v$ is given weight $\Theta(\kappa \kappa/k)$. It is convenient to replace the corresponding auxiliary path $(v^+,a^-_v,v^+_n,s^-)$ in $G_{\text{rev}}$ with a single edge $(v^+,s^-)$ with capacity equal to the weight of $a_v$. This does not effect the minimum $(t^+,s^-)$-edge cut for any $t \in V'$. This adjustment can be easily made within the allotted preprocessing time.

In this graph, given $t \in V'$, we run a specialization of the Ford-Fulkerson algorithm [7] that either computes a minimum $(t^+,s^-)$-cut or concludes that the minimum $(t^+,s^-)$-cut is at least $O(k \log(n)/\epsilon^2)$ (which corresponds to $O(\kappa)$ in $G_{\text{rev}}$). The Ford-Fulkerson algorithm observes that since every vertex initially has unweighted out-degree at most $O(k \log(n)/\epsilon^2)$ in $G_{\text{rev}}$ (reversing the upper bound on the unweighted in-degrees in $G_0$), and the flow algorithm updates the residual graph along at most $O(k \log(n)/\epsilon^2)$ paths before terminating, the maximum unweighted out-degree over all vertices never exceeds $O(k \log(n)/\epsilon^2)$. We specialize the Ford-Fulkerson framework to take advantage of the auxiliary $(v^+,s^-)$ edges. Call an outer-vertex $v^+$ saturated if the auxiliary edge $(v^+,s^-)$ is saturated; that is, if $(v^+,s^-)$ is not in the residual graph. Call an inner-vertex $v^-$ saturated if the edge $(v^-,v^+)$ is saturated and $v^+$ is not saturated. (A vertex $v^+$ or $v^-$ is called unsaturated if it is not saturated.) We modify the search for an augmenting path to effectively end when we first visit an unsaturated vertex $v^+$ or an unsaturated $v^-$. If we visit an unsaturated $v^-$, then we automatically complete a path to $s^-$ via $v^+$. If we find an unsaturated $v^+$, then we automatically complete a path to $s^-$ via the edge $(v^+,s^-)$. It remains to bound the running time of this search. We first bound the number of saturated $v^+$.

Claim 1. There are at most $O(k/\epsilon)$ saturated $v^+'s$. Indeed, each saturated $v^+$ implies $O(\log(n)/\epsilon)$ units of flow along $(v^+,s^-)$, and the flow is bounded above $O(k \log(n)/\epsilon^2)$.

Note that Claim 1 also implies there are at most $O(k/\epsilon)$ $v^-$'s such that $v^+$ is saturated. The next claim bounds the total out-degree of saturated $v^-$'s.

Claim 2. The sum of out-degrees of saturated $v^-$'s is at most the amount of flow routed to $s^-$.

Indeed, the out-degree of a $v^-$ in the residual graph is bounded above by the amount of flow through $(v^-,v^+)$, since initially $(v^-,v^+)$ is the only outgoing edge from $v^-$. Recall that if $v^-$ is saturated, then by definition $v^+$ is unsaturated. As long as $v^+$ is unsaturated, each unit of flow through $(v^-,v^+)$ goes directly to $s^-$ via the edge $(v^+,s^-)$, and can be charged to the total flow.

We now apply the above two claims to bound the total running time for each search, as follows.

Claim 3. Every (modified) search for an augmenting path traverses at most $O(k^2 \log(n)/\epsilon^2)$ edges.

We first observe that every vertex visited in the search, except the unsaturated vertex terminating the search, is either (a) a saturated $v^-$, (b) a saturated $v^+$, or (c) an unsaturated $v^-$ such that $v^+$ is saturated. We will upper bound the number of edges traversed in each iteration based on the type of vertex at the initial point of that edge. First, the amount of time spent exploring edges leaving (a) a saturated $v^-$ is, by Claim 2, at most the size of the flow at that point, which is at most $O(k \log(n)/\epsilon^2)$. Second, consider the time spent traversing edges leaving either (b) a saturated $v^+$ or (c) an unsaturated $v^-$ such that $v^+$ is saturated. By Claim 1, there are at most $O(k/\epsilon)$ such vertices, and each has out-degree at most $O(k \log(n)/\epsilon^2)$. Thus we spend $O(k^2 \log(n)/\epsilon^2)$ time traversing such edges. All together, we obtain an upper bound of $O(k^2 \log(n)/\epsilon^2)$ total edges per search.

Claim 3 also bounds the running time for each iteration. The algorithm runs for at most $O(k \log(n)/\epsilon^2)$ iterations before either finding an $(t^+,s^-)$-cut or concluding that the weight of the minimum $(t^+,s^-)$-cut, rescaled to the input scale of $G$, is at least a constant factor greater than $\kappa$. The total running time follows.

We now present the overall algorithm for finding vertex $s$-cuts with small sink components. The algorithm combines Lemma III.2 with randomly sampling for a vertex $t$ in the sink component of an approximately minimum $s$-cut. In the following, we let $\deg^+(s)$ denote the unweighted out-degree of $s$ in $G$.

Lemma III.3. Let $G = (V,E)$ be a directed graph with positive vertex weights. Let $s \in V$ be a fixed root vertex. Let $\epsilon \in (0,1)$, $\kappa > 0$ and $k \in \mathbb{N}$ be given parameters. There is a randomized algorithm that
runs in $O(m + (n - \deg^+(s))k^2 \log^3(n)/\epsilon^4)$ time and has the following guarantee. If there is a vertex $s$-cut of capacity at most $\kappa$ and where the sink component has at most $k$ vertices, then with high probability, the algorithm returns a vertex $(s,t)$-cut of capacity at most $(1 + \epsilon)\kappa$.

Proof: Let $T^*$ be the sink component of the minimum vertex $s$-cut subject to $|T^*| \leq k$. Assume the capacity of the vertex in-cut of $T^*$ is at most $\kappa$ (since otherwise the algorithm makes no guarantees). Let $V^* = V \setminus (\{s\} \cup N^+(s))$ and note that $|V^*| = n - 1 - \deg^+(s)$.

Suppose we had a factor-2 overestimate $\ell \in [\lceil |T^*| \rceil, 2|T^*|]$ of the number of vertices in $T^*$. We apply Lemma III.2 with upper bounds $\kappa$ on the size of the cut and $\ell$ on the number of vertices in the sink component, which returns a data structure that, with high probability, is correct for all queries. Let us assume the data structure is correct (and otherwise the algorithm fails). We randomly sample $O((n - \deg^+(s))\log(n)/\ell)$ vertices from $V^*$. For each sampled vertex $t$, we query the data structure from Lemma III.2. Observe that if $t \in T^*$, then the query for $t$ returns an $s$-cut with capacity at most $(1 + \epsilon)\kappa$. With high probability we sample at least one vertex from $T^*$, which produces the desired $s$-cut. By Lemma III.2, the total running time to serve all queries is $O(m + (n - \deg^+(s))\ell^2 \log^3(n)/\epsilon^4)$.

A factor-2 overestimate $\ell$ can be obtained by enumerating powers of 2 between 1 and $2k$, and the running time is dominated by the maximum choice of $\ell$. ■

C. Rooted vertex mincut for large sink components

The third and final part (before the overall algorithm) is an approximation for the rooted vertex cut that is well-suited for large sink components.

Lemma III.4. Let $G = (V, E)$ be a directed graph with positive vertex weights. Let $s \in V$ be a fixed root vertex. Let $\epsilon \in (0, 1)$, $\kappa > 0$, and $k \in \mathbb{N}$ be given parameters. There is a randomized algorithm that runs in $\tilde{O}(m + (n - \deg^+(s))(n/\epsilon^2 + n^{1.5}/k))$ time and has the following guarantee. If there is a vertex $s$-cut of capacity at most $\kappa$ and where the sink component has at most $k$ vertices, then with high probability, the algorithm returns a vertex $(s,t)$-cut of capacity at most $(1 + \epsilon)\kappa$.

Proof: Let $T^*$ be the sink component of the minimum $s$-cut subject to $|T^*| \leq k$. We assume the capacity of the $s$-cut induced by $T^*$ is at most $\kappa$. (Otherwise the output is not well-defined.) Let $V^* = V \setminus (\{s\} \cup N^+(s))$ and note that $|V^*| < n - \deg^+(s)$.

We apply Lemma III.1 to produce the graph $G_0$. Lemma III.1 succeeds with high probability and for the rest of the proof we assume it was successful. (Otherwise the algorithm fails.) We sample $O((n - \deg^+(s))\log(n)/k)$ vertices $t \in V'$. For each sampled $t$, we compute the minimum $(s,t)$-vertex cut in $G_0$. With high probability, some $t$ will be drawn from the sink component of the true minimum $s$-cut, in which case the minimum $(s,t)$-cut in $G_0$ gives an $(1 + \epsilon)$-approximate $s$-cut in $G$ (by Lemma III.1). We use the $\tilde{O}(m + n^{1.5})$ time vertex-capacitated flow algorithm [29]. By Lemma III.1, we have $m = O(nk\log(n)/\epsilon^2)$. This gives the total running time.

D. Approximating the rooted and global vertex mincut

Next we use the algorithm for rooted vertex mincut to obtain an algorithm for global vertex mincut and establish Corollary III.6. [17] showed that running times of the form $(n - \deg^+(s))T$ for rooted mincut from a root $s$ imply a randomized $nT$ expected time algorithm for global vertex mincut. Theorem III.5 gives a $\tilde{O}(m + n(n - \deg^+(s))/\epsilon^2)$ running time, so some modifications have to be made to address the additional $O(m)$ additive factor. This establishes the remaining part of I.3. The proof is deferred to the full version.

Theorem III.5. Let $\epsilon \in (0, 1)$, let $G = (V, E)$ be a directed graph with polynomially bounded vertex weights, and let $s \in V$ be a fixed root. A $(1 + \epsilon)$-approximate minimum vertex $s$-cut can be computed with high probability in $\tilde{O}(m + (n - \deg^+(s))/\epsilon^2)$ randomized time.

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