STRONG SOLUTIONS OF SDE’S WITH GENERALIZED DRIFT AND MULTIDIMENSIONAL FRACTIONAL BROWNIAN INITIAL NOISE

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ABSTRACT. In this paper we prove the existence of strong solutions to a SDE with a generalized drift driven by a multidimensional fractional Brownian motion for small Hurst parameters \( H < \frac{1}{2} \). Here the generalized drift is given as the local time of the unknown solution process, which can be considered an extension of the concept of a skew Brownian motion to the case of fractional Brownian motion. Our approach for the construction of strong solutions is new and relies on techniques from Malliavin calculus combined with a "local time variational calculus" argument.

1. INTRODUCTION

Consider the \( d \)-dimensional stochastic differential equation (SDE)

\[
X_t^x = x + \alpha L_t(X^x) \cdot 1_d + B^H_t, \quad 0 \leq t \leq T, \ x \in \mathbb{R}^d,
\]

(1.1)

where the driving noise \( B^H \) of this equation is a \( d \)-dimensional fractional Brownian motion, whose components are given by one-dimensional independent fractional Brownian motions with a Hurst parameter \( H \in (0, 1/2) \), and where \( \alpha \in \mathbb{R} \) is a constant and \( 1_d \) is the vector in \( \mathbb{R}^d \) with entries given by 1. Further, \( L_t(X^x) \) is the (existing) local time at zero of \( X^x \), which can be formally written as

\[
L_t(X^x) = \int_0^t \delta_0(X^x_s)ds,
\]

where \( \delta_0 \) denotes the Dirac delta function in 0.

We also assume that \( B^H \) is defined on a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \).

We recall here for \( d = 1 \) and Hurst parameter \( H \in (0, 1) \) that \( B^H_t, 0 \leq t \leq T \) is a centered Gaussian process with covariance structure \( R_H(t, s) \) given by

\[
R_H(t, s) = E[B^H_t B^H_s] = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).
\]

For \( H = \frac{1}{2} \) the fractional Brownian motion \( B^H \) coincides with the Brownian motion. Moreover, \( B^H \) has a version with \((H - \varepsilon)\)-Hölder continuous paths for all \( \varepsilon \in (0, H) \) and is the only stationary Gaussian process having the self-similarity property, that is

\[
\{ B^H_{\gamma t} \}_{t \geq 0} = \{ \gamma^H B^H_t \}_{t \geq 0}
\]

in law for all \( \gamma > 0 \). Finally, we mention that for \( H \neq \frac{1}{2} \) the fractional Brownian motion is neither a Markov process nor a (weak) semimartingale. The latter properties however
complicate the study of SDE’s driven by $B^H$ and in fact call for the development of new construction techniques of solutions of such equations beyond the classical Markovian framework. For further information about the fractional Brownian motion, the reader may consult e.g. [47] and the references therein.

In this paper we want to analyze for small Hurst parameters $H \in (0, 1/2)$ strong solutions $X^x$ to the SDE (1.1), that is solutions to (1.1), which are adapted to a $P$-augmented filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ generated by $B^H$. Let us mention here that solutions to (1.1) can be considered a generalization of the concept of a skew Brownian motion to the case of a fractional Brownian motion. The skew Brownian motion, which was first studied in the 1970ties in [27] and [55] and which has applications to e.g. astrophysics, geophysics or more recently to the simulation of diffusion processes with discontinuous coefficients (see e.g. [60], [33], [18]), is a solution to the SDE

$$X^x_t = x + (2p - 1) L_t(X^x) + B_t, \quad 0 \leq t \leq T, \quad x \in \mathbb{R},$$

where $B$ is a one-dimensional Brownian motion, $L_t(X^x)$ the local time at zero of $X^x$ and $p$ a parameter, which stands for the probability of positive excursions of $X^x$.

It was shown in [26] that the SDE (1.2) has a unique strong solution if and only if $p \in [0, 1]$. The approach used by the latter authors relies on a one-to-one transformation of (1.2) into a SDE without drift and the symmetric Itô-Tanaka formula. An extension of the latter result to SDE’s of the type

$$dX_t = \sigma(X_t) dB_t + \int_{\mathbb{R}} \nu(dx) dL^x_t(X)$$

was given in the work [32] under fairly general conditions on the coefficient $\sigma$ and the measure $\nu$, where the author also proves that strong solutions to (1.3) can be obtained through a limit of sequences of solutions to classical Itô-SDE’s by using the comparison theorem.

We remark here that the Walsh Brownian motion [55] also provides a natural extension of the skew Brownian motion, which is a diffusion process on rays in $\mathbb{R}^2$ originating in zero and which exhibits the behaviour of a Brownian motion on each of those rays. A further generalization of the latter process is the spider martingale, which has been used in the literature for the study of Brownian filtrations [59].

Other important generalizations of the skew Brownian motion to the multidimensional case in connection with weak solutions were studied in [50] and [8]: Using PDE techniques, Portenko in [50] gives a construction of a unique solution process associated with an infinitesimal generator with a singular drift coefficient, which is concentrated on some smooth hypersurface.

On the other hand Bass and Chen in [8] analyze (unique) weak solutions of equations of the form

$$dX_t = dA_t + dB_t,$$

where $B$ is a $d$-dimensional Brownian motion and $A_t$ a process, which is obtained from limits of the form

$$\lim_{n \to \infty} \int_0^t b_n(X_s) ds$$
in the sense of probability uniformly over time $t$ for functions $b_n : \mathbb{R}^d \to \mathbb{R}^d$. Here the $i$th components of $A_t$ are bounded variation processes, which correspond to signed measures in the Kato class $K_{d-1}$. The method of the authors for the construction of unique weak solutions of such equations is based on the construction of a certain resolvent family on the space $C_b(\mathbb{R}^d)$ in connection with the properties of the Kato class $K_{d-1}$.

In this context we also mention the paper [21] on SDE’s with distributional drift coefficients. As for a general overview of various construction techniques with respect to the skew Brownian motion and related processes based e.g. on the theory of Dirichlet forms or martingale problems, the reader is referred to [34]. See also the book [48].

The objective of this paper is the construction of strong solutions to the multidimensional SDE (1.1) with fractional Brownian noise initial data for small Hurst parameters $H < \frac{1}{2}$, where the generalized drift is given by the local time of the unknown process. Note that in contrast to [26] in the case of a skew Brownian motion we obtain in this article the existence of strong solutions to (1.1) for all parameters $\alpha \in \mathbb{R}$.

Since the fractional Brownian motion is neither a Markov process nor a semimartingale, if $H \neq \frac{1}{2}$, the methods of the above mentioned authors cannot be (directly) used for the construction of strong solutions in our setting. In fact, our construction technique considerably differs from those in the literature in the Wiener case. More specifically, we approximate the Dirac delta function in zero by means of functions $\varphi_\varepsilon$ for $\varepsilon \to 0$ given by

$$\varphi_\varepsilon(x) = \varepsilon^{-\frac{d}{2}} \varphi(\varepsilon^{-\frac{1}{2}} x), x \in \mathbb{R}^d,$$

where $\varphi$ is e.g. the $d$-dimensional standard Gaussian density. Then we prove that the sequence of strong solutions $X^n_t$ to the SDE’s

$$X^n_t = x + \int_0^t \alpha \varphi_\varepsilon(\frac{1}{n} X^n_s) \cdot 1_d ds + B_H^H$$

converges in $L^2(\Omega)$, strongly to a solution to (1.11) for $n \to \infty$. In showing this we employ a compactness criterion for sets in $L^2(\Omega)$ based on Malliavin calculus combined with a "local time variational calculus" argument. See [7] for the existence of strong solutions of SDE’s driven by $B_H$, $H < \frac{1}{2}$, when e.g. the drift coefficients $b$ belong to $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ or see [42] in the Wiener case. We also refer to a series of other papers in the Wiener and Lévy process case and in the Hilbert space setting based on that approach: [40], [25], [43], [20], [5], [6].

Here we also want to point out a recent work of Catellier, Gubinelli [9], which came to our attention, after having finalized our article. In their striking paper, which extends the results of Davie [14] to the case of a fractional Brownian noise, the authors study the problem, which fractional Brownian paths actually regularize solutions to SDE’s of the form

$$dX^x_t = b(X^x_t) dt + dB^H_t, X^x_0 = x \in \mathbb{R}^d$$

for all $H \in (0, 1)$. The (unique) solutions constructed in [9] are path by path with respect to time-dependent vector fields $b$ in the Besov-Hölder space $B^{\alpha+1}_{\infty, \infty}$, $\alpha \in \mathbb{R}$ and in the case of distributional vector fields solutions to the SDE’s, where the drift term is given by a non-linear Young type of integral based on an averaging operator. In proving existence and uniqueness results the authors use the Leray-Schauder-Tychonoff fixed point theorem.
and a comparison principle in connection with an average translation operator. Further, Lipschitz-regularity of the flow \((x \mapsto X^x_t)\) under certain conditions is shown.

We remark that our techniques are very different from those developed by Catellier, Gubinelli [9], which seem not to work in the case of the SDE (1.1) (private communication with one of the authors in [9]). Further, their methods fail in the case of SDE’s with vector fields \(b\), which are merely bounded and measurable.

Finally, we mention that the construction technique in this article may be also used for showing strong solutions of SDE’s with respect to generalized drifts in the sense of (1.4) based on Kato classes. The existence of strong solutions of such equations in the Wiener case is to the best of our knowledge still an open problem. See the work of Bass, Chen [8].

Our paper is organized as follows: In Section 2 we introduce the framework of our paper and recall in this context some basic facts from fractional calculus and Malliavin calculus for (fractional) Brownian noise. Further, in Section 3 we discuss an integration by parts formula based on a local time on a simplex, which we want to employ in connection with a compactness criterion from Malliavin calculus in Section 5. Section 4 is devoted to the study of the local time of the fractional Brownian motion and its properties. Finally, in Section 5 we prove the existence of a strong solution to (1.1) by using the results of the previous sections.

2. Framework

In this section we pass in review some theory on fractional calculus, Malliavin calculus for fractional Brownian noise and occupation measures which will be progressively used throughout the article. The reader might consult [39], [38] or [16] for a general theory on Malliavin calculus for Brownian motion and [47, Chapter 5] for fractional Brownian motion. For theory on occupation measures we refer to [22] or [28].

2.1. Fractional calculus. We start up here with some basic definitions and properties of fractional derivatives and integrals. For more information see [53] and [36].

Let \(a, b \in \mathbb{R}\) with \(a < b\). Let \(f \in L^p([a, b])\) with \(p \geq 1\) and \(\alpha > 0\). Introduce the left- and right-sided Riemann-Liouville fractional integrals by

\[
I^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha - 1} f(y) dy
\]

and

\[
I^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y - x)^{\alpha - 1} f(y) dy
\]

for almost all \(x \in [a, b]\) where \(\Gamma\) is the Gamma function.

Further, for a given integer \(p \geq 1\), let \(I^\alpha_{a^+} (L^p)\) (resp. \(I^\alpha_{b^-} (L^p)\)) be the image of \(L^p([a, b])\) of the operator \(I^\alpha_{a^+}\) (resp. \(I^\alpha_{b^-}\)). If \(f \in I^\alpha_{a^+} (L^p)\) (resp. \(f \in I^\alpha_{b^-} (L^p)\)) and \(0 < \alpha < 1\) then define the left- and right-sided Riemann-Liouville fractional derivatives by

\[
D^\alpha_{a^+} f(x) = \frac{d}{dx} I^{1-\alpha}_{a^+} f(x)
\]

\[
= \frac{1}{\Gamma(1 - \alpha)} \int_a^x f(y) (x - y)^{\alpha - 1} dy
\]
and
\[ D_0^{-\alpha} f(x) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y - x)^\alpha} dy. \]

The left- and right-sided derivatives of \( f \) defined as above can be represented as follows by
\[ D_0^{\alpha+} f(x) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right) \]
and
\[ D_0^{\alpha-} f(x) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy \right). \]

Finally, we see by construction that the following relations are valid
\[ I_{a+}^\alpha (D_0^{\alpha+} f) = f \]
for all \( f \in I_{a+}^\alpha (L^p) \) and
\[ D_0^{\alpha+} (I_{a+}^\alpha f) = f \]
for all \( f \in L^p([a, b]) \) and similarly for \( I_{b-}^\alpha \) and \( D_0^{\alpha-} \).

2.2. Shuffles. Let \( m \) and \( n \) be integers. We denote by \( S(m, n) \) the set of shuffle permutations, i.e. the set of permutations \( \sigma : \{1, \ldots, m + n\} \rightarrow \{1, \ldots, m + n\} \) such that
\( \sigma(1) < \cdots < \sigma(m) \) and \( \sigma(m + 1) < \cdots < \sigma(m + n) \).

The \( m \)-dimensional simplex is defined as
\[ \Delta^m_{\theta,t} := \{(s_m, \ldots, s_1) \in [0, T]^m : \theta < s_m < \cdots < s_1 < t\}. \]
The product of two simplices then is given by the following union
\[ \Delta^m_{\theta,t} \times \Delta^n_{\theta,t} = \bigcup_{\sigma \in S(m,n)} \{ (w_{m+n}, \ldots, w_1) \in [0, T]^{m+n} : \theta < w_{\sigma(m+n)} < \cdots < w_{\sigma(1)} < t \} \cup \mathcal{N}, \]
where the set \( \mathcal{N} \) has null Lebesgue measure. Thus, if \( f_i : [0, T] \rightarrow \mathbb{R}, i = 1, \ldots, m + n \) are integrable functions we obtain that
\[
\begin{align*}
\int_{\Delta^m_{\theta,t}} \prod_{j=1}^m f_j(s_j) ds_m \ldots ds_1 \int_{\Delta^n_{\theta,t}} \prod_{j=m+1}^{m+n} f_j(s_j) ds_{m+n} \ldots ds_{m+1} &= \sum_{\sigma \in S(m,n)} \int_{\Delta^m_{\theta,t}} \prod_{j=1}^{m+n} f_{\sigma(j)}(w_j) dw_{m+n} \ldots dw_1.
\end{align*}
\]

We hereby give a slight generalization of the above lemma, whose proof can be also found in [7]. This lemma will be used in Section 5. The reader may skip this lemma at first reading.

**Lemma 2.1.** Let \( n, p \) and \( k \) be integers, \( k \leq n \). Assume we have integrable functions \( f_j : [0, T] \rightarrow \mathbb{R}, j = 1, \ldots, n \) and \( g_i : [0, T] \rightarrow \mathbb{R}, i = 1, \ldots, p \). We may then write
\[
\begin{align*}
\int_{\Delta^m_{\theta,t}} f_1(s_1) \ldots f_k(s_k) & \int_{\Delta^p_{\theta,s_k}} g_1(r_1) \ldots g_p(r_p) dr_p \ldots dr_1 f_{k+1}(s_{k+1}) \ldots f_n(s_n) ds_n \ldots ds_1 \\
&= \sum_{\sigma \in A_{n,p}} \int_{\Delta^{n+p}_{\theta,t}} h_1^\sigma(w_1) \ldots h_{n+p}^\sigma(w_{n+p}) dw_{n+p} \ldots dw_1,
\end{align*}
\]
where \( h_i^q \in \{ f_j, g_i : 1 \leq j \leq n, 1 \leq i \leq p \} \). Here \( A_{n,p} \) is a subset of permutations of \( \{1, \ldots, n + p\} \) such that \( \#A_{n,p} \leq C^{n+p} \) for a constant \( C \geq 1 \), and we use the definition \( s_0 = 0 \).

**Proof.** The proof of the result is given by induction on \( n \). For \( n = 1 \) and \( k = 0 \) the result is trivial. For \( k = 1 \) we have

\[
\int_0^t f_1(s_1) \int_{\Delta_{n,s_1}^p} g_1(r_1) \ldots g_p(r_p) dr_p \ldots dr_1 ds_1 = \int_{\Delta_{n+1,t}^p} f_1(w_1) g_1(w_2) \ldots g_p(w_{p+1}) dw_{p+1} \ldots dw_1,
\]

where we have put \( w_1 = s_1, w_2 = r_1, \ldots, w_{p+1} = r_p \).

Assume the result holds for \( n \) and let us show that this implies that the result is true for \( n + 1 \). Either \( k = 0, 1 \) or \( 2 \leq k \leq n + 1 \). For \( k = 0 \) the result is trivial. For \( k = 1 \) we have

\[
\int_{\Delta_{n+1,t}^p} f_1(s_1) \int_{\Delta_{n,s_1}^p} g_1(r_1) \ldots g_p(r_p) dr_p \ldots dr_1 f_2(s_2) \ldots f_{n+1}(s_{n+1}) ds_{n+1} \ldots ds_1 = \int_0^t f_1(s_1) \left( \int_{\Delta_{n,s_1}^p} \int_{\Delta_{n-1,s_1}^p} g_1(r_1) \ldots g_p(r_p) dr_p \ldots dr_1 f_2(s_2) \ldots f_{n+1}(s_{n+1}) ds_{n+1} \ldots ds_2 \right) ds_1.
\]

The result follows from (2.1) coupled with \( \#S(n,p) = \frac{(n+p)!}{n!p!} \leq C^{n+p} \leq C^{(n+1)+p} \). For \( k \geq 2 \) we have from the induction hypothesis

\[
\int_{\Delta_{n+1,t}^p} f_1(s_1) \ldots f_k(s_k) \int_{\Delta_{n,k}^p} g_1(r_1) \ldots g_p(r_p) dr_p \ldots dr_1 f_{k+1}(s_{k+1}) \ldots f_{n+1}(s_{n+1}) ds_{n+1} \ldots ds_1 = \int_0^t f_1(s_1) \left( \int_{\Delta_{n,s_1}^p} \int_{\Delta_{n-1,s_1}^p} \ldots \right) \times f_{k+1}(s_{k+1}) \ldots f_{n+1}(s_{n+1}) ds_{n+1} \ldots ds_2 ds_1
\]

\[
= \sum_{\sigma \in A_{n,p}} \int_0^t f_1(s_1) \int_{\Delta_{n,s_1}^p} h_{n+1}^\sigma(w_1) \ldots h_{n+p}^\sigma(w_{n+p}) dw_{n+p} \ldots dw_1 ds_1
\]

where \( A_{n+1,p} \) is the set of permutations \( \tilde{\sigma} \) of \( \{1, \ldots, n + 1 + p\} \) such that \( \tilde{\sigma}(1) = 1 \) and \( \tilde{\sigma}(j + 1) = \sigma(j), j = 1, \ldots, n + p \) for some \( \sigma \in A_{n,p} \).

\[\square\]

**Remark 2.2.** We remark that the set \( A_{n,p} \) in the above Lemma also depends on \( k \) but we shall not make use of this fact.

### 2.3. Fractional Brownian motion

Denote by \( B^H = \{ B_t^H, t \in [0, T] \} \) a \( d \)-dimensional fractional Brownian motion with Hurst parameter \( H \in (0, 1/2) \). So \( B^H \) is a centered
Gaussian process with covariance structure
\[(R_H(t, s))_{i,j} := E[B^H_{t,\phi}B^H_{s,\phi}] = \delta_{ij} \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H}\right), \quad i, j = 1, \ldots, d,\]
where \(\delta_{ij}\) is one, if \(i = j\), or zero else. Observe that \(E[|B^H - B^H_s|^2] = d|t - s|^{2H}\) and hence \(B^H_t\) has stationary increments and Hölder continuous trajectories of index \(H - \varepsilon\) for all \(\varepsilon \in (0, H)\). Observe that the increments of \(B^H, H \in (0, 1/2)\) are not independent. As a matter of fact, this process does not satisfy the Markov property, either. Another obstacle one is faced with is that \(B^H\) is not a semimartingale, see e.g. [47, Proposition 5.1.1].

We give an abridged survey on how to construct fractional Brownian motion via an isometry. We will do it in one dimension inasmuch as we will treat the multidimensional case componentwise. See [47] for further details.

Let \(\mathcal{E}\) be the set of step functions on \([0, T]\) and let \(\mathcal{H}\) be the Hilbert space given by the closure of \(\mathcal{E}\) with respect to the inner product
\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).
\]
The mapping \(1_{[0,t]} \mapsto B_t\) has an extension to an isometry between \(\mathcal{H}\) and the Gaussian subspace of \(L^2(\Omega)\) associated with \(B^H\). We denote the isometry by \(\varphi \mapsto B^H(\varphi)\). Let us recall the following result (see [47, Proposition 5.1.3]) which gives an integral representation of \(R_H(t, s)\) when \(H < 1/2\):

**Proposition 2.3.** Let \(H < 1/2\). The kernel
\[
K_H(t, s) = c_H \left[ \left( \frac{t}{s} \right)^{\frac{H}{2}} (t - s)^{H - \frac{1}{2}} + \left( \frac{1}{2} - H \right) \frac{1}{2} - H \int_s^t u^{H - \frac{3}{2}} (u - s)^{H - \frac{3}{2}} du \right],
\]
where \(c_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+1/2)}}\) being \(\beta\) the Beta function, satisfies
\[
R_H(t, s) = \int_0^{t\wedge s} K_H(t, u)K_H(s, u)du. \tag{2.2}
\]

The kernel \(K_H\) also has the following representation by means of fractional derivatives
\[
K_H(t, s) = c_H \Gamma \left( H + \frac{1}{2} \right) s^{\frac{1}{2} - H} \left( D_{t-}^{\frac{1}{2} - H} u^{H - \frac{1}{2}} \right) (s).
\]
Consider now the linear operator \(K_H^* : \mathcal{E} \to L^2([0, T])\) defined by
\[
(K_H^* \varphi)(s) = K_H(T, s)\varphi(s) + \int_s^T (\varphi(t) - \varphi(s)) \frac{\partial K_H}{\partial t}(t, s)dt
\]
for every \(\varphi \in \mathcal{E}\). We see that \((K_H^* 1_{[0,t]})(s) = K_H(t, s)1_{[0,t]}(s)\), then from this fact and \((\text{2.2})\) one can conclude that \(K_H^\ast\) is an isometry between \(\mathcal{E}\) and \(L^2([0, T])\) which extends to the Hilbert space \(\mathcal{H}\). See e.g. [15] and [1] and the references therein.

For a given \(\varphi \in \mathcal{H}\) one proves that \(K_H^\ast\) can be represented in terms of fractional derivatives in the following ways
\[
(K_H^\ast \varphi)(s) = c_H \Gamma \left( H + \frac{1}{2} \right) s^{\frac{1}{2} - H} \left( D_{t-}^{\frac{1}{2} - H} u^{H - \frac{1}{2}} \varphi(u) \right) (s)
\]
and

\[
(K_H^* \varphi)(s) = c_H \Gamma \left( H + \frac{1}{2} \right) \left( D_{T-}^{1-H} \varphi(s) \right)(s) + c_H \left( \frac{1}{2} - H \right) \int_s^T \varphi(t)(t-s)^{H-\frac{1}{2}} \left( 1 - \left( \frac{t}{s} \right)^{H-\frac{1}{2}} \right) \, dt.
\]

One finds that \( \mathcal{H} = I_{\mathcal{T}}^{\frac{1}{2}-H}(L^2) \) (see [15] and [1 Proposition 6]).

Using the fact that \( K_H^* \) is an isometry from \( \mathcal{H} \) into \( L^2([0, T]) \) the \( d \)-dimensional process \( W = \{ W_t, t \in [0, T] \} \) defined by

\[
W_t := B^H((K_H^*)^{-1}(1_{[0, t]}))
\]

is a Wiener process and the process \( B^H \) can be represented as follows

\[
B_t^H = \int_0^t K_H(t, s) dW_s, \quad (2.4)
\]

see [11].

We also need to introduce the concept of fractional Brownian motion associated with a filtration.

**Definition 2.4.** Let \( \mathcal{G} = \{ \mathcal{G}_t \}_{t \in [0, T]} \) be a right-continuous increasing family of \( \sigma \)-algebras on \( (\Omega, \mathcal{F}, P) \) such that \( \mathcal{G}_0 \) contains the null sets. A fractional Brownian motion \( B^H \) is called a \( \mathcal{G} \)-fractional Brownian motion if the process \( W \) defined by (2.3) is a \( \mathcal{G} \)-Brownian motion.

In what follows, we will denote by \( W \) a standard Wiener process on a given probability space \( (\Omega, \mathcal{A}, P) \) equipped with the natural filtration \( \mathcal{F} = \{ \mathcal{F}_t \}_{t \in [0, T]} \) which is generated by \( W \) and augmented by all \( P \)-null sets, we shall denote by \( B := B^H \) the fractional Brownian motion with Hurst parameter \( H \in (0, 1/2) \) given by the representation (2.4).

In this paper, we want to use a version of Girsanov’s theorem for fractional Brownian motion which is due to [15 Theorem 4.9]. Here we recall the version given in [45 Theorem 2]. However, we first need the definition of an isomorphism \( K_H \) from \( L^2([0, T]) \) onto \( I_{0+}^{H+\frac{1}{2}}(L^2) \) associated with the kernel \( K_H(t, s) \) in terms of the fractional integrals as follows, see [15 Theorem 2.1]

\[
(K_H \varphi)(s) = I_{0+}^{2H} s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} \varphi, \quad \varphi \in L^2([0, T]).
\]

It follows from this and the properties of the Riemann-Liouville fractional integrals and derivatives that the inverse of \( K_H \) takes the form

\[
(K_H^{-1} \varphi)(s) = s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0+}^{2H} \varphi(s), \quad \varphi \in I_{0+}^{H+\frac{1}{2}}(L^2).
\]

The latter implies that if \( \varphi \) is absolutely continuous, see [45], one has

\[
(K_H^{-1} \varphi)(s) = s^{H-\frac{1}{2}} I_{0+}^{1-H} s^{\frac{1}{2}-H} \varphi'(s). \quad (2.5)
\]

**Theorem 2.5** (Girsanov’s theorem for fBm). Let \( u = \{ u_t, t \in [0, T] \} \) be an \( \mathcal{F} \)-adapted process with integrable trajectories and set \( \overline{B}_t^H = B_t^H + \int_0^t u_s \, ds, \quad t \in [0, T] \). Assume that...
Then the shifted process $\tilde{B}^H$ under the new probability $H$

**Remark**

where $^*$ helps substitute for the lack of independent increments of the underlying noise.

which was shown by [49] for general Gaussian vector fields. The latter property will be a

exists a positive finite constant $C > 0$ (depending on $m$) such that

for any $t \in [0, T], 0 < r < t$ and for $i = 1, \ldots, d$,

$$
\text{Var} \left[ B_t^{H,i} \left\{ B_s^{H,i} : |t - s| \geq r \right\} \right] \geq K r^{2H}.
$$

(2.7)

3. AN INTEGRATION BY PARTS FORMULA

In this section we recall an integration by parts formula, which is essentially based on the local time of the Gaussian process $B^H$. The whole content as well as the proofs can be found in [7].

Let $m$ be an integer and let $f : [0, T]^m \times (\mathbb{R}^d)^m \rightarrow \mathbb{R}$ be a function of the form

$$
f(s, z) = \prod_{j=1}^{m} f_j(s_j, z_j), \quad s = (s_1, \ldots, s_m) \in [0, T]^m, \quad z = (z_1, \ldots, z_m) \in (\mathbb{R}^d)^m,
$$

(3.1)

where $f_j : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}, j = 1, \ldots, m$ are smooth functions with compact support. Further, let $\varphi : [0, T]^m \rightarrow \mathbb{R}$ be a function of the form

$$
\varphi(s) = \prod_{j=1}^{m} \varphi_j(s_j), \quad s \in [0, T]^m;
$$

(3.2)

where $\varphi_j : [0, T] \rightarrow \mathbb{R}, j = 1, \ldots, m$ are integrable functions.
Next, denote by $\alpha_j$ a multiindex and $D^{\alpha_j}$ its corresponding differential operator. For $\alpha = (\alpha_1, \ldots, \alpha_m)$ considered an element of $\mathbb{N}_0^{d \times m}$ so that $|\alpha| := \sum_{j=1}^m \sum_{l=1}^d \alpha_j^{(l)}$, we write

$$D^{\alpha} f(s, z) = \prod_{j=1}^m D^{\alpha_j} f_j(s_j, z_j).$$

In this section we aim at deriving an integration by parts formula of the form

$$\int_{\Delta_m \theta, t} D^{\alpha} f(s, B_s) ds = \int (\mathbb{R}^d)^m \Lambda^f_{\alpha}(\theta, t, z) dz,$$  \hspace{1cm} (3.3)

for a suitable random field $\Lambda^f_{\alpha}$, where $\Delta_m \theta, t$ is the $m$-dimensional simplex as defined in Section 2.2 and $B_s = (B_{s1}, \ldots, B_{sm})$ on that simplex. More specifically, we have that

$$\Lambda^f_{\alpha}(\theta, t, z) = (2\pi)^{-dm} \int (\mathbb{R}^d)^m \int_{\Delta_m \theta, t} \prod_{j=1}^m f_j(s_j, z_j) (-iu_j)^{\alpha_j} \exp\{-i\langle u_j, B_{sj} - z_j \rangle\} ds du.$$  \hspace{1cm} (3.4)

Let us start by defining $\Lambda^f_{\alpha}(\theta, t, z)$ as above and show that it is a well-defined element of $L^2(\Omega)$.

To this end, we need the following notation: Given $(s, z) = (s_1, \ldots, s_m, z_1 \ldots, z_m) \in [0, T]^m \times (\mathbb{R}^d)^m$ and a shuffle $\sigma \in S(m, m)$ we write

$$f_\sigma(s, z) := \prod_{j=1}^{2m} f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})$$

and

$$x_\sigma(s) := \prod_{j=1}^{2m} x_{[\sigma(j)]}(s_j),$$

where $[j]$ is equal to $j$ if $1 \leq j \leq m$ and $j - m$ if $m + 1 \leq j \leq 2m$.

For integers $k \geq 0$ let us define the expressions

$$\Psi^f_k(\theta, t, z) = \prod_{l=1}^d \sqrt{2(2|\alpha(l)|)!} \sum_{\sigma \in S(m, m)} \int_{\Delta_m \theta, t} f_\sigma(s, z) \prod_{j=1}^{2m} \frac{1}{|s_j - s_{j-1}|^{H(d+2\sum_{l=1}^d \alpha^{(l)}_{[\sigma(j)]})}} ds_1 \ldots ds_{2m}$$

respectively,

$$\Psi^x_k(\theta, t) = \prod_{l=1}^d \sqrt{2(2|\alpha(l)|)!} \sum_{\sigma \in S(m, m)} \int_{\Delta_m \theta, t} x_\sigma(s) \prod_{j=1}^{2m} \frac{1}{|s_j - s_{j-1}|^{H(d+2\sum_{l=1}^d \alpha^{(l)}_{[\sigma(j)]})}} ds_1 \ldots ds_{2m}.$$  \hspace{1cm} (3.4)

**Theorem 3.1.** Suppose that $\Psi^f_k(\theta, t, z), \Psi^x_k(\theta, t) < \infty$. Then, defining $\Lambda^f_{\alpha}(\theta, t, z)$ as in (3.4) gives a random variable in $L^2(\Omega)$ and there exists a universal constant $C = C(T, H, d) > 0$ such that

$$E[|\Lambda^f_{\alpha}(\theta, t, z)|^2] \leq C^{m+|\alpha|} \Psi^f_k(\theta, t, z).$$  \hspace{1cm} (3.5)
Moreover, we have
\[
\left| E \left[ \int_{(\mathbb{R}^d)^m} \Lambda_0^f(\theta, t, z) dz \right] \right| \leq C_{m/2+|\alpha|/2} \prod_{j=1}^{m} \| f_j \|_{L^1(\mathbb{R}^d; L^\infty([0,T]))} (\Psi_k^2(\theta, t))^{1/2}. \tag{3.6}
\]

**Proof.** For notational convenience we consider \( \theta = 0 \) and set \( \Lambda_0^f(t, z) = \Lambda_0^f(0, t, z) \).

For an integrable function \( g : (\mathbb{R}^d)^m \to \mathbb{C} \) we can write
\[
\left| \int_{(\mathbb{R}^d)^m} g(u_1, \ldots, u_m) du_1 \ldots du_m \right|^2
\]
\[
= \int_{(\mathbb{R}^d)^m} g(u_1, \ldots, u_m) du_1 \ldots du_m \int_{(\mathbb{R}^d)^m} \overline{g(u_{m+1}, \ldots, u_{2m})} du_{m+1} \ldots du_{2m}
\]
\[
= \int_{(\mathbb{R}^d)^m} g(u_1, \ldots, u_m) du_1 \ldots du_m (-1)^{dn} \int_{(\mathbb{R}^d)^m} g(-u_{m+1}, \ldots, -u_{2m}) du_{m+1} \ldots du_{2m},
\]
where we used the change of variables \((u_{m+1}, \ldots, u_{2m}) \mapsto (-u_{m+1}, \ldots, -u_{2m})\) in the third equality.

This gives
\[
\left| \Lambda_0^f(\theta, t, z) \right|^2
\]
\[
= (2\pi)^{-2dn} (-1)^{dm} \int_{(\mathbb{R}^d)^2m} \prod_{j=1}^{2m} f_j(s_j, z_j) (-iu_j)^{\alpha_j} e^{-i\langle u_j, B_j - z_j \rangle} ds_1 \ldots ds_{2m}
\]
\[
\times \int_{s_1}^{s_2} \prod_{j=m+1}^{2m} f_j(s_j, z_j) (-iu_j)^{\alpha_j} e^{-i\langle u_j, B_j - z_j \rangle} ds_{m+1} \ldots ds_{2m} du_1 \ldots du_{2m}
\]
\[
= (2\pi)^{-2dn} (-1)^{dm} \sum_{\sigma \in S(m, m)} \int_{(\mathbb{R}^d)^2m} \left( \prod_{j=1}^{m} e^{-i\langle z_j, u_j + u_{j+m} \rangle} \right)
\]
\[
\times \int_{s_1}^{s_2} \prod_{j=1}^{2m} u_{\sigma(j)}^{\alpha_{\sigma(j)}} \exp \left\{ -\sum_{j=1}^{2m} \langle u_{\sigma(j)}, B_j \rangle \right\} ds_1 \ldots ds_{2m} du_1 \ldots du_{2m}
\]
where we used (2.1) in the last step.

Taking the expectation on both sides yields
\[
E \left[ \left| \Lambda_0^f(\theta, t, z) \right|^2 \right] \tag{3.7}
\]
\[
= (2\pi)^{-2dn} (-1)^{dm} \sum_{\sigma \in S(m, m)} \int_{(\mathbb{R}^d)^2m} \left( \prod_{j=1}^{m} e^{-i\langle z_j, u_j + u_{j+m} \rangle} \right)
\]
\[
\times \int_{s_1}^{s_2} \prod_{j=1}^{2m} u_{\sigma(j)}^{\alpha_{\sigma(j)}} \exp \left\{ -\frac{1}{2} Var \left\{ \sum_{j=1}^{2m} \langle u_{\sigma(j)}, B_j \rangle \right\} \right\} ds_1 \ldots ds_{2m} du_1 \ldots du_{2m}
\]
\[
= (2\pi)^{-2dn} (-1)^{dm} \sum_{\sigma \in S(m, m)} \int_{(\mathbb{R}^d)^2m} \left( \prod_{j=1}^{m} e^{-i\langle z_j, u_j + u_{j+m} \rangle} \right)
\[
\times \int_{\Delta^{2m,0}} f_\sigma(s, z) \prod_{j=1}^{2m} u^{\alpha_{\sigma(j)}} \exp \left\{ -\frac{1}{2} \sum_{l=1}^{d} \text{Var} \left[ \sum_{j=1}^{2m} u^{(l)}_{\sigma(j)} B_{s_j}^{(l)} \right] \right\} ds_1 \ldots ds_{2m} du^{(1)}_1 \ldots du^{(1)}_{2m} \\
\ldots du^{(d)}_1 \ldots du^{(d)}_{2m} \\
= (2\pi)^{-2dm} (-1)^{dm} \sum_{\sigma \in S(m,m)} \left( \prod_{j=1}^{m} e^{-i(z_j u_j + u_{j+m})} \right) \times \int_{\Delta^{2m,0}} f_\sigma(s, z) \prod_{j=1}^{2m} u^{\alpha_{\sigma(j)}} \prod_{l=1}^{d} \exp \left\{ -\frac{1}{2} \left( (u^{(l)}_{\sigma(j)})_{1 \leq j \leq 2m} \right)^T Q \left( (u^{(l)}_{\sigma(j)})_{1 \leq j \leq 2m} \right) \right\} ds_1 \ldots ds_{2m} \\\ndu^{(1)}_{\sigma(1)} \ldots du^{(1)}_{\sigma(2m)} \ldots du^{(d)}_{\sigma(1)} \ldots du^{(d)}_{\sigma(2m)},
\]

where

\[
Q = Q(s) := \left( E[B^{(1)}_{s_1} B^{(1)}_{s_2}] \right)_{1 \leq i,j \leq 2m}.
\]

Further, we see that

\[
\int_{\Delta^{2m,0}} |f_\sigma(s, z)| \left( \prod_{j=1}^{2m} \prod_{l=1}^{d} |u^{(l)}_{\sigma(j)}|^{\alpha_{\sigma(j)}} \right) ds_1 \ldots ds_{2m} \\\ndu^{(1)}_{\sigma(1)} \ldots du^{(1)}_{\sigma(2m)} \ldots du^{(d)}_{\sigma(1)} \ldots du^{(d)}_{\sigma(2m)} = \int_{\Delta^{2m,0}} |f_\sigma(s, z)| \left( \prod_{j=1}^{2m} \prod_{l=1}^{d} |u^{(l)}_{\sigma(j)}|^{\alpha_{\sigma(j)}} \right) ds_1 \ldots ds_{2m} \\\xdots du^{(1)}_{2m} \ldots du^{(d)}_{2m} ds_1 \ldots ds_{2m} = \int_{\Delta^{2m,0}} |f_\sigma(s, z)| \prod_{l=1}^{d} \left( \prod_{j=1}^{2m} |u^{(l)}_{\sigma(j)}|^{\alpha_{\sigma(j)}} \right) \exp \left\{ -\frac{1}{2} \left( Qu^{(l)}, u^{(l)} \right) \right\} du^{(l)}_{1} \ldots du^{(l)}_{2m} ds_1 \ldots ds_{2m},
\]

where

\[
u^{(l)} := (u^{(l)}_{j})_{1 \leq j \leq 2m}.
\]

We have that

\[
\int_{\mathbb{R}^{2m}} \left( \prod_{j=1}^{2m} |u^{(l)}_{j}|^{\alpha_{\sigma(j)}} \right) \exp \left\{ -\frac{1}{2} \left( Qu^{(l)}, u^{(l)} \right) \right\} du^{(l)}_{1} \ldots du^{(l)}_{2m} = \frac{1}{(\det Q)^{1/2}} \int_{\mathbb{R}^{2m}} \left( \prod_{j=1}^{2m} \left| \langle Q^{-1/2}u^{(l)}, e_j \rangle \right|^{\alpha_{\sigma(j)}} \right) \exp \left\{ -\frac{1}{2} \left( u^{(l)}, u^{(l)} \right) \right\} du^{(l)}_{1} \ldots du^{(l)}_{2m},
\]

where \(e_i, i = 1, \ldots, 2m\) is the standard ONB of \(\mathbb{R}^{2m}\).

We also get that

\[
\int_{\mathbb{R}^{2m}} \left( \prod_{j=1}^{2m} \left| \langle Q^{-1/2}u^{(l)}, e_j \rangle \right|^{\alpha_{\sigma(j)}} \right) \exp \left\{ -\frac{1}{2} \left( u^{(l)}, u^{(l)} \right) \right\} du^{(l)}_{1} \ldots du^{(l)}_{2m}
\]
\[
= (2\pi)^m E[\prod_{j=1}^{2m} \langle Q^{-1/2} Z, e_j \rangle^{a_{(l)}[\sigma(j)]}],
\]
where
\[
Z \sim \mathcal{N}(O, I_{2m \times 2m}).
\]
We know from Lemma A.6, which is a type of Brascamp-Lieb inequality that
\[
E[\prod_{j=1}^{2m} \langle Q^{-1/2} Z, e_j \rangle^{a_{(l)}[\sigma(j)]}] \leq \sqrt{\text{perm}(\sum)} = \left( \sum_{\pi \in S_2|\alpha]} 2|\alpha_{(l)}| \prod_i a_{i\pi(i)} \right),
\]
where \( \text{perm}(\sum) \) is the permanent of the covariance matrix \( \sum = (a_{ij}) \) of the Gaussian random vector
\[
\langle Q^{-1/2} Z, e_1 \rangle, \ldots, \langle Q^{-1/2} Z, e_{1} \rangle, \ldots, \langle Q^{-1/2} Z, e_{2} \rangle, \ldots, \langle Q^{-1/2} Z, e_{2m} \rangle, \ldots, \langle Q^{-1/2} Z, e_{2m} \rangle,
\]
\( |\alpha_{(l)}| := \sum_{j=1}^{m} \alpha_{j(1)} \) and where \( S_n \) stands for the permutation group of size \( n \).

In addition, using an upper bound for the permanent of positive semidefinite matrices (see [3]) or direct computations we get that
\[
\text{perm}(\sum) = \sum_{\pi \in S_2|\alpha]} 2|\alpha_{(l)}| \prod_i a_{i\pi(i)} \leq (2|\alpha_{(l)}|)! \prod_i a_{ii}. \tag{3.9}
\]

Let now \( i \in \left\{ \sum_{r=1}^{j-1} \alpha_{[\sigma]}^{(1)} + 1, \alpha_{[\sigma(j)]}^{(1)} \right\} \) for some arbitrary fixed \( j \in \{1, \ldots, 2m\} \). Then
\[
a_{ii} = E[\langle Q^{-1/2} Z, e_j \rangle \langle Q^{-1/2} Z, e_j \rangle].
\]

Further using substitution, we also have that
\[
E[\langle Q^{-1/2} Z, e_j \rangle \langle Q^{-1/2} Z, e_j \rangle] = (\det Q)^{1/2} \frac{1}{(2\pi)^m} \int_{\mathbb{R}^{2m}} \langle u, e_j \rangle^2 \exp(-\frac{1}{2} \langle Qu, u \rangle) du_1 \ldots du_{2m}
\]
\[
= (\det Q)^{1/2} \frac{1}{(2\pi)^m} \int_{\mathbb{R}^{2m}} u_j^2 \exp(-\frac{1}{2} \langle Qu, u \rangle) du_1 \ldots du_{2m}
\]

We now want to use Lemma A.7

Then we get that
\[
\int_{\mathbb{R}^{2m}} u_j^2 \exp(-\frac{1}{2} \langle Qu, u \rangle) du_1 \ldots du_m
\]
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\[
= \frac{(2\pi)^{(2m-1)/2}}{(\det Q)^{1/2}} \int_{\mathbb{R}} v^2 \exp\left(-\frac{1}{2}v^2\right)dv \frac{1}{\sigma_j^2}
\]

\[
= \frac{(2\pi)^{m}}{(\det Q)^{1/2} \sigma_j^2},
\]

where \(\sigma_j^2 := \text{Var}[B_{s_j}^H | B_s^H, \ldots, B_{s_{2m}}^H \text{ without } B_{s_j}^H].\)

We now want to use strong local non-determinism of the form (see (2.7)): For all \(t \in [0, T], 0 < r < t:\)

\[
\text{Var}[B_t^H | B_s^H, |t - s| \geq r] \geq Kr^{2H}.
\]

The latter implies that

\[
(\det Q(s))^{1/2} \geq K^{(2m-1)/2} |s_1|^H |s_2 - s_1|^H \ldots |s_{2m} - s_{2m-1}|^H
\]

as well as

\[
\sigma_j^2 \geq K \min\{|s_j - s_{j-1}|^{2H}, |s_{j+1} - s_j|^{2H}\}.
\]

Thus

\[
\prod_{j=1}^{2m} \frac{\sigma_j^{-2\alpha^{(1)}_{\sigma(j)}}}{\sigma_j^2} \leq K^{-2m} \prod_{j=1}^{2m} \min\{|s_j - s_{j-1}|^{2H\alpha^{(1)}_{\sigma(j)}}, |s_{j+1} - s_j|^{2H\alpha^{(1)}_{\sigma(j)}}\} \leq C^{m+|\alpha^{(1)}|} \prod_{j=1}^{2m} \frac{1}{|s_j - s_{j-1}|^{4H\alpha^{(1)}_{\sigma(j)}}}
\]

for a constant \(C\) only depending on \(H\) and \(T\).

Hence, it follows from (3.9) that

\[
\text{perm}(\sum) \leq (2 |\alpha^{(1)}|)! \prod_{i=1}^{2|\alpha^{(1)}|} a_{ii}
\]

\[
\leq (2 |\alpha^{(1)}|)! \prod_{j=1}^{2m} ((\det Q)^{1/2} \frac{(2\pi)^m}{(\det Q)^{1/2} \sigma_j^2})^{\alpha^{(1)}_{\sigma(j)}} \leq (2 |\alpha^{(1)}|)! C^{m+|\alpha^{(1)}|} \prod_{j=1}^{2m} \frac{1}{|s_j - s_{j-1}|^{4H\alpha^{(1)}_{\sigma(j)}}}.
\]

So

\[
E\left[\prod_{j=1}^{2m} |\langle Q^{-1/2}Z, e_j \rangle|^{\alpha^{(1)}_{\sigma(j)}\sigma(j)}\right] \leq \sqrt{\text{perm}(\sum)} \leq \sqrt{(2 |\alpha^{(1)}|)! C^{m+|\alpha^{(1)}|} \prod_{j=1}^{2m} \frac{1}{|s_j - s_{j-1}|^{2H\alpha^{(1)}_{\sigma(j)}}}}.
\]

Therefore we obtain from (3.7) and (3.8) that

\[
E[|\Lambda_d^{(1)}(\theta, t, z)|^2]
\]
one obtains that

\[ \int_{|f_\sigma(s,z)|} \prod_{j=1}^{2m} |u_j^{(l)}|^{\alpha_{\sigma(j)}} \exp \left\{ -\frac{1}{2} \langle Q^{(l)} u^{(l)}, u^{(l)} \rangle \right\} \, du_1^{(l)} \ldots du_{2m}^{(l)} \, ds_1 \ldots ds_{2m} \]

\[ \leq M^m \int_{\Delta_{0,t}^m} |f_\sigma(s,z)| \prod_{j=1}^{2m} \sqrt{2 |\alpha(j)|! C_m + |\alpha(j)|} \prod_{j=1}^{2m} \frac{1}{|s_j - s_{j-1}|^{2H_{\alpha_{\sigma(j)}}}} \, ds_1 \ldots ds_{2m} \]

\[ = M^m C^{d+|\alpha|} \prod_{j=1}^{2m} \sqrt{2 |\alpha(j)| !} \int_{\Delta_{0,t}^m} |f_\sigma(s,z)| \prod_{j=1}^{2m} \frac{1}{|s_j - s_{j-1}|^{H(d+2 \sum_{l=1}^d \alpha_{\sigma(j)})}} \, ds_1 \ldots ds_{2m} \]

for a constant \( M \) depending on \( d \).

Finally, we show estimate (3.6). Using the inequality (3.5), we find that

\[ \left| E \left[ \int_{(\mathbb{R}^d)^m} \Lambda_{\alpha}^{sf}(\theta, t, z) \, dz \right] \right| \]

\[ \leq \int_{(\mathbb{R}^d)^m} \left( E \left[ |\Lambda_{\alpha}^{sf}(\theta, t, z)|^2 \right] \right)^{1/2} \, dz \leq C^{m/2 + |\alpha|/2} \int_{(\mathbb{R}^d)^m} \left( \Psi_k^{sf}(\theta, t, z) \right)^{1/2} \, dz. \]

Taking the supremum over \([0, T]\) for each function \( f_j \), i.e.

\[ \left| f_{\sigma(j)}(s_j, z_{\sigma(j)}) \right| \leq \sup_{s_j \in [0, T]} \left| f_{\sigma(j)}(s_j, z_{\sigma(j)}) \right|, \quad j = 1, \ldots, 2m \]

one obtains that

\[ \left| E \left[ \int_{(\mathbb{R}^d)^m} \Lambda_{\alpha}^{sf}(\theta, t, z) \, dz \right] \right| \]

\[ \leq C^{m+|\alpha|} \max_{\sigma \in S(m, m)} \int_{(\mathbb{R}^d)^m} \left( \prod_{l=1}^{2m} \left| f_{\sigma(j)}(\cdot, z_{\sigma(j)}) \right| L^\infty([0, T]) \right)^{1/2} \, dz \]

\[ \times \left( \prod_{l=1}^{d} \sqrt{2 |\alpha(l)| !} \sum_{\sigma \in S(m, m)} \int_{\Delta_{0,t}^m} \left| \kappa_{\alpha}(s) \right| \prod_{j=1}^{2m} \frac{1}{|s_j - s_{j-1}|^{H(d+2 \sum_{l=1}^d \alpha_{\sigma(j)})}} \, ds_1 \ldots ds_{2m} \right)^{1/2} \]

\[ = C^{m+|\alpha|} \max_{\sigma \in S(m, m)} \int_{(\mathbb{R}^d)^m} \left( \prod_{l=1}^{2m} \left| f_{\sigma(j)}(\cdot, z_{\sigma(j)}) \right| L^\infty([0, T]) \right)^{1/2} \, dz \cdot (\Psi_k^{sf}(\theta, t))^{1/2} \]

\[ = C^{m+|\alpha|} \int_{(\mathbb{R}^d)^m} \prod_{j=1}^{m} \left| f_j(\cdot, z_j) \right| L^\infty([0, T]) \, dz \cdot (\Psi_k^{sf}(\theta, t))^{1/2} \]

\[ = C^{m+|\alpha|} \prod_{j=1}^{m} \left| f_j(\cdot, z_j) \right| L^1(\mathbb{R}^d, L^\infty([0, T])) \cdot (\Psi_k^{sf}(\theta, t))^{1/2}. \]

The next result is a key estimate which shows why fractional Brownian motion regularizes (1.1). It rests in fact on the earlier integration by parts formula. This estimate is given in more explicit terms when the function \( \kappa \) is chosen to be

\[ \kappa_j(s) = (K_H(s, \theta) - K_H(s, \theta'))^{\epsilon_j}, \quad \theta < s < t \]
and,
\[ \varpi_j(s) = (K_H(s, \theta))^{\varepsilon_j}, \quad \theta < s < t \]
for every \( j = 1, \ldots, m \) with \( (\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m \). It will be made clear why these choices are important in the forthcoming section.

**Proposition 3.2.** Let \( B^H, H \in (0,1/2) \) be a standard \( d \)-dimensional fractional Brownian motion and functions \( f \) and \( \varpi \) as in (3.1), respectively as in (3.2). Let \( 0, \theta < t \in [0, T] \) with \( \theta < t \).

\[ \varpi_j(s) = (K_H(s, \theta) - K_H(s, \theta))^\varepsilon_j, \theta < s < t \]
for every \( j = 1, \ldots, m \) with \( (\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m \) for \( \theta, \theta t \in [0, T] \) with \( \theta < t \). Let \( \alpha \in (\mathbb{N}_0^d)^m \) be a multi-index. If
\[
H < \frac{\frac{1}{4} - \gamma}{(d + 2 \sum_{l=1}^d \alpha_l)}
\]
for all \( j \), where \( \gamma \in (0, H) \) is sufficiently small, then there exists a universal constant \( C \) (depending on \( H, T \) and \( d \), but independent of \( m \), \( \{f_i\}_{i=1}^m \) and \( \alpha \)) such that for any \( \theta, t \in [0, T] \) with \( \theta < t \) we have
\[
\left| E \int_{\Delta_{\theta,t}^m} \left( \prod_{j=1}^m D^{\alpha_j} f_j(s_j, B^H_{s_j}) \varpi_j(s_j) \right) ds \right|
\leq C^{m+|\alpha|} \prod_{j=1}^m \|f_j(\cdot, z_j)\|_{L^1(\mathbb{R}^d, L^\infty([0,T]))} \gamma \sum_{j=1}^m \varepsilon_j \theta^{(H-\frac{1}{2})} \sum_{j=1}^m \varepsilon_j
\times \prod_{l=1}^d (2 |\alpha_l|)! \left( \frac{1}{4}(t - \theta) - H(16d + 4 |\alpha|) + 2(H - \frac{1}{2}) \sum_{j=1}^m \varepsilon_j + m \right)^{1/2}.
\]

**Proof.** By definition of \( \Lambda^{\varpi, f}_{\alpha} \) (3.4) it immediately follows that the integral in our proposition can be expressed as
\[
\int_{\Delta_{\theta,t}^m} \left( \prod_{j=1}^m D^{\alpha_j} f_j(s_j, B^H_{s_j}) \varpi_j(s_j) \right) ds = \int_{\mathbb{R}^{dm}} \Lambda^{\varpi, f}_{\alpha}(\theta, t, z) dz.
\]
Taking expectation and using Theorem 3.1 we obtain
\[
\left| E \int_{\Delta_{\theta,t}^m} \left( \prod_{j=1}^m D^{\alpha_j} f_j(s_j, B^H_{s_j}) \varpi_j(s_j) \right) ds \right| \leq C^{m+|\alpha|} \prod_{j=1}^m \|f_j(\cdot, z_j)\|_{L^1(\mathbb{R}^d, L^\infty([0,T])))} \Psi^\varpi_{\alpha}(\theta, t)^{1/2},
\]
where in this situation
\[
\Psi^\varpi_{\alpha}(\theta, t) := \prod_{l=1}^d \sqrt{(2 |\alpha_l|)!} \sum_{\sigma \in S(m)} \times \int_{\Delta_{\theta,t}^m} \left( \prod_{j=1}^{2m} (K_H(s_j, \theta) - K_H(s_j, \theta))^\varepsilon_{\sigma(j)} \right) \frac{1}{|s_j - s_{j-1}|^{H(2d + \sum_{l=1}^d \alpha_l)}} ds_1 ... ds_{2m}.
\]
We want to apply Lemma [A.8]. For this, we need that \(-H(d + 2 \sum_{l=1}^{d} \alpha^{(1)}_{[\sigma(j)]}) + (H - \frac{1}{2} - \gamma) \varepsilon_{[\sigma(j)]} > -1\) for all \(j = 1, \ldots, 2m\). The worst case is, when \(\varepsilon_{[\sigma(j)]} = 1\) for all \(j\). So \(H < \frac{\frac{1}{2} - \gamma}{(d+1+2 \sum_{l=1}^{d} \alpha^{(1)}_{[\sigma(j)]})}\) for all \(j\). Hence, we have

\[
\Psi^{\varphi}_{k}(\theta, t) \leq \sum_{\alpha \in S(m, m)} \left( \frac{\theta - \theta r}{\theta \theta r} \right)^{\gamma \sum_{j=1}^{m} \varepsilon_{[\sigma(j)]}} \theta^{(H - \frac{1}{2} - \gamma) \sum_{j=1}^{m} \varepsilon_{[\sigma(j)]}}
\times \prod_{l=1}^{d} \left( 2 \alpha^{(l)} \right)! \Pi_{\gamma}(2m)(t - \theta)^{-H(2md + 4|\alpha|) + (H - \frac{1}{2} - \gamma) \sum_{j=1}^{m} \varepsilon_{[\sigma(j)]} + 2m},
\]

where \(\Pi_{\gamma}(m)\) is defined as in Lemma [A.8]. The latter can be bounded above as follows

\[
\Pi_{\gamma}(2m) \leq \frac{\prod_{j=1}^{2m} \Gamma(1 - H(d + 2 \sum_{l=1}^{d} \alpha^{(1)}_{[\sigma(j)]}))}{\Gamma(-H(2md + 4|\alpha|) + (H - \frac{1}{2} - \gamma) \sum_{j=1}^{m} \varepsilon_{[\sigma(j)]} + 2m)}.
\]

Observe that \(\sum_{j=1}^{m} \varepsilon_{[\sigma(j)]} = 2 \sum_{j=1}^{m} \varepsilon_{j}\). Therefore, we have that

\[
\left( \Psi^{\varphi}_{k}(\theta, t) \right)^{1/2} \leq C^{m} \left( \frac{\theta - \theta r}{\theta \theta r} \right)^{\gamma \sum_{j=1}^{m} \varepsilon_{j}} \theta^{(H - \frac{1}{2} - \gamma) \sum_{j=1}^{m} \varepsilon_{j}}
\times \left( \prod_{l=1}^{d} \left( 2 \alpha^{(l)} \right)! \right)^{1/2} (t - \theta)^{-H(2md + 4|\alpha|) + (H - \frac{1}{2} - \gamma) \sum_{j=1}^{m} \varepsilon_{j} + m}
\times \frac{\prod_{j=1}^{2m} \Gamma(1 - H(d + 2 \sum_{l=1}^{d} \alpha^{(1)}_{[\sigma(j)]}))}{\Gamma(-H(2md + 4|\alpha|) + 2(H - \frac{1}{2} - \gamma) \sum_{j=1}^{m} \varepsilon_{j} + 2m)^{1/2}},
\]

where we used \(\prod_{j=1}^{2m} \Gamma(1 - H(d + 2 \sum_{l=1}^{d} \alpha^{(1)}_{[\sigma(j)]})) \leq C^{m}\) for a large enough constant \(C > 0\) and \(\sqrt{a_{1} + \ldots + a_{m}} \leq \sqrt{a_{1}} + \ldots + \sqrt{a_{m}}\) for arbitrary non-negative numbers \(a_{1}, \ldots, a_{m}\).

\[
\square
\]

Proposition 3.3. Let \(B^{H}, H \in (0, 1/2)\) be a standard \(d\)-dimensional fractional Brownian motion and functions \(f\) and \(\varphi\) as in (3.1), respectively as in (3.2). Let \(\theta, t \in [0, T]\) with \(\theta < t\) and

\[
\varphi_{j}(s) = (K_{H}(s, \theta))^{\varepsilon_{j}}, \theta < s < t
\]

for every \(j = 1, \ldots, m\) with \((\varepsilon_{1}, \ldots, \varepsilon_{m}) \in \{0, 1\}^{m}\) for \(\theta, \theta t \in [0, T]\) with \(\theta t < \theta\). Let \(\alpha \in (\mathbb{N}^{d})^{m}\) be a multi-index. If

\[
H < \frac{\frac{1}{2} - \gamma}{(d + 2 \sum_{l=1}^{d} \alpha^{(1)}_{[\sigma(j)]})}
\]

for all \(j\), where \(\gamma \in (0, H)\) is sufficiently small, then there exists a universal constant \(C\) (depending on \(H, T\) and \(d\), but independent of \(m, \{f_{j}\}_{j=1, \ldots, m}\) and \(\alpha\)) such that for any \(\theta, t \in [0, T]\) with \(\theta < t\) we have

\[
\left| E \int_{\Delta_{0,t}^{m}} \left( \prod_{j=1}^{m} D^{\alpha_{j}} f_{j}(s, B_{s}^{H}) \varphi_{j}(s) \right) ds \right|
\]

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\[
\leq C^{m+|\alpha|} \prod_{j=1}^{m} \left\| f_j (\cdot, z_j) \right\|_{L^1 (\mathbb{R}^d; L^\infty ([0,T]))} \left( \frac{\theta - \theta \tau}{\theta \theta^\tau} \right)^{\gamma \sum_{j=1}^{m} \varepsilon_j} \theta^{(H - \frac{1}{2} - \gamma) \sum_{j=1}^{m} \varepsilon_j + m} \\
\times \left( \prod_{l=1}^{d} (2 |\alpha^{(l)}|)! \right)^{1/4} (t - \theta)^{-H (md + 2 |\alpha|) - (H - \frac{1}{2} - \gamma) \sum_{j=1}^{m} \varepsilon_j + m} \\
\Gamma (-H (2md + 4 |\alpha|) + 2(H - \frac{1}{2} - \gamma) \sum_{j=1}^{m} \varepsilon_j + 2m)^{1/2}.
\]

\textbf{Proof.} The proof is similar to the previous proposition. \qed

\textbf{Remark 3.4.} We mention that

\[
\prod_{l=1}^{d} (2 |\alpha^{(l)}|)! \leq (2 |\alpha|)! C^{d|\alpha|}
\]

for a constant \(C\) depending on \(d\). Later on in the paper, when we deal with the existence of strong solutions, we will consider the case

\[
\alpha^{(l)}_j \in \{0, 1\} \text{ for all } j, l
\]

with \(|\alpha| = m\).

4. LOCAL TIMES OF A FRACTIONAL BROWNIAN MOTION AND PROPERTIES

One can define, heuristically, the local time \(L_t^x (B^H)\) of \(B^H\) at \(x \in \mathbb{R}^d\) by

\[
L_t^x (B^H) = \int_0^t \delta_x (B^H_s) ds.
\]

It is known that \(L_t^x (B^H)\) exists and is jointly continuous in \((t, x)\) as long as \(Hd < 1\). See e.g. [49] and the references therein. Moreover, by the self-similarity property of the fBm one has that \(L_t^x (B^H) \overset{\text{law}}{=} t^{1-Hd} L_{t^H}^x (B^H)\) and, in particular

\[
L_t^0 (B^H) \overset{\text{law}}{=} t^{1-Hd} L_{t^H}^0 (B^H).
\]

The rigorous construction of \(L_t^x (B^H)\) involves approximating the Dirac delta function by an approximate unity. It is convenient to consider the Gaussian approximation of unity

\[
\varphi_\varepsilon (x) = \varphi \left( \frac{x}{\varepsilon} \right), \quad \varepsilon > 0,
\]

for every \(x \in \mathbb{R}^d\) where \(\varphi\) is the \(d\)-dimensional standard Gaussian density. Then, we can define the smoothed local times

\[
L_t^x (B^H, \varepsilon) = \int_0^t \varphi_\varepsilon (B^H_s - x) ds
\]

and construct \(L_t^x (B^H)\) as the limit when \(\varepsilon\) tends to zero in \(L^2 (\Omega)\). Note that, using the Fourier transform, one can write \(\varphi_\varepsilon (x)\) as follows

\[
\varphi_\varepsilon (x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp \left( i \langle \xi, x \rangle_{\mathbb{R}^d} - \frac{|\xi|^2_{\mathbb{R}^d}}{2} \right) d\xi.
\]
The previous expression allows us to write

\[ L_t^x (B^H, \varepsilon) = \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} \exp \left( i \langle \xi, B^H_s - x \rangle_{\mathbb{R}^d} - \varepsilon \frac{\|\xi\|_{\mathbb{R}^d}^2}{2} \right) d\xi ds, \]

and

\[ \mathbb{E} \left[ L_t^x (B^H, \varepsilon)^m \right] = \frac{m!}{(2\pi)^{md}} \int_{T_m(0,t)} \int_{\mathbb{R}^{md}} \mathbb{E} \left[ \exp \left( i \sum_{j=1}^m \langle \xi_j, B^H_{s_j} \rangle_{\mathbb{R}^d} \right) \right] \times \exp \left( -\varepsilon \sum_{j=1}^m \langle \xi_j, x \rangle_{\mathbb{R}^d} \right) d\xi ds, \quad (4.1) \]

where \( \bar{\xi} = (\xi_1, \ldots, \xi_m) = (\xi^1_1, \ldots, \xi^d_1, \ldots, \xi^1_m, \ldots, \xi^d_m) \in \mathbb{R}^{md} \) and \( s = (s_1, \ldots, s_m) \in T_m(0,t) = \{0 \leq s_1 < s_2 < \cdots < s \leq t\} \). Next, note that

\[ \mathbb{E} \left[ \exp \left( i \sum_{j=1}^m \langle \xi_j, B^H_{s_j} \rangle_{\mathbb{R}^d} \right) \right] = \exp \left( -\frac{1}{2} \text{Var} \left[ \sum_{j=1}^m \sum_{k=1}^d \xi^k_j B^H_{s_j} \right] \right) \]

\[ = \exp \left( -\frac{1}{2} \sum_{k=1}^d \text{Var} \left[ \sum_{j=1}^m \xi^k_s B^H_{s_j} \right] \right) \]

\[ = \exp \left( -\frac{1}{2} \sum_{k=1}^d \langle \xi^k, Q(s)\xi^k \rangle_{\mathbb{R}^m} \right), \]

where \( \xi^k = (\xi^1, \ldots, \xi^k) \) and \( Q(s) \) is the covariance matrix of the vector \( (B^H_{s_1}, \ldots, B^H_{s_m}) \). Rearranging the terms in the second exponential in equation (4.1) we can write

\[ \mathbb{E} \left[ L_t^x (B^H, \varepsilon)^m \right] = \frac{m!}{(2\pi)^{md}} \int_{T_m(0,t)} \int_{\mathbb{R}^{md}} \exp \left( -\frac{1}{2} \sum_{k=1}^d \langle \xi^k, Q(s)\xi^k \rangle_{\mathbb{R}^m} + \varepsilon \frac{\|\xi^k\|_{\mathbb{R}^m}^2}{2} \right) \times \exp \left( -i \sum_{j=1}^m \langle \xi_j, x \rangle_{\mathbb{R}^d} \right) d\xi ds \]

\[ \leq \frac{m!}{(2\pi)^{md}} \int_{T_m(0,t)} \left( \int_{\mathbb{R}^m} \exp \left( -\frac{1}{2} \langle \xi^1, Q(s)\xi^1 \rangle_{\mathbb{R}^m} - \varepsilon \frac{\|\xi^1\|_{\mathbb{R}^m}^2}{2} \right) d\xi^1 \right)^d ds \]

\[ \leq \frac{m!}{(2\pi)^{md}} \int_{T_m(0,t)} \left( \int_{\mathbb{R}^m} \exp \left( -\frac{1}{2} \langle \xi^1, Q(s)\xi^1 \rangle \right) d\xi^1 \right)^d ds \]

\[ = \frac{m!}{(2\pi)^{md}} \int_{T_m(0,t)} (\det Q(s))^{-\frac{d}{2}} ds \Delta m. \]

Hence, by dominated convergence, we can conclude that \( \mathbb{E} \left[ L_t^x (B^H, \varepsilon)^m \right] \) converges when \( \varepsilon \) tends to zero as long as \( \alpha_m < \infty \). If \( \alpha_2 < \infty \), then one can similarly show that

\[ \lim_{\varepsilon_1,\varepsilon_2 \to 0^+} \mathbb{E} \left[ L_t^x (B^H, \varepsilon_1) L_t^x (B^H, \varepsilon_2) \right] \]
exists, which yields the convergence in $L^2(\Omega)$ of $L^\epsilon_t \left(B^H, \varepsilon \right)$. If $\alpha_m < \infty$ for all $m \geq 1$ one can deduce the convergence in $L^p(\Omega), p \geq 2$ of $L^\epsilon_t \left(B^H, \varepsilon \right)$.

The following well known result can be found in Anderson [4, p. 42].

**Lemma 4.1.** Let $(X_1, \ldots, X_m)$ be a mean-zero Gaussian random vector. Then,

$$\det(\text{Cov} \left[X_1, \ldots, X_m \right]) = \text{Var} \left[X_1 | X_2 \right] \text{Var} \left[X_2 | X_1 \right] \cdots \text{Var} \left[X_m | X_{m-1}, \ldots, X_1 \right].$$

Another useful elementary result is:

**Lemma 4.2.** Let $X$ be a square integrable random variable and $G_1 \subset G_2$ be two $\sigma$-algebras. Then,

$$\text{Var} \left[X | G_1 \right] \geq \text{Var} \left[X | G_2 \right].$$

Combining Lemmas 4.1 and (2.7) we get that

$$\det Q(s) = \text{Var} \left[B_{s_1}^{H,1} \right] \text{Var} \left[B_{s_2}^{H,1} \right] \cdots \text{Var} \left[B_{s_m}^{H,1} | B_{s_{m-1}}^{H,1} \ldots, B_{s_1}^{H,1} \right]$$

and, therefore,

$$\int_{T_m(0,t)} (\det Q(s))^{-\frac{d}{2}} ds \leq K^{\frac{d}{2}(1-m)} \int_{T_m(0,t)} s_1^{-Hd} (s_2 - s_1)^{-Hd} \cdots (s_m - s_{m-1})^{-Hd} ds$$

$$= K^{\frac{d}{2}(1-m)} \left( \prod_{j=1}^m B(j (1 - Hd), 1 - Hd) \right) t^{m(1-Hd)} < \infty,$$

if $Hd < 1$. Finally, we have proved the bound

$$\mathbb{E} \left[ \left| L^\epsilon_t \left(B^H \right) \right|^m \right] \leq \frac{m!}{(2\pi)^{\frac{d}{2}}} K^{\frac{d}{2}(1-m)} \left( \prod_{j=1}^m B(j (1 - Hd), 1 - Hd) \right) t^{m(1-Hd)} \quad (4.2)$$

**Remark 4.3.** We just have checked that if $Hd < 1$ then $L^\epsilon_t \left(B^H \right)$ exists and has moments of all orders. By checking that $\sum_{m \geq 1} \frac{\alpha_m}{m!} < \infty$, one can deduce that $L^\epsilon_t \left(B^H \right)$ has exponential moments or all orders. Furthermore, one can also show the existence of exponential moments of $L^\epsilon_t \left(B^H \right)^2$ by doing similar computations as before. However, one may also use Theorem 4.4 below to show that the exponential moments are finite.

Chen et al. [10] proved the following result on large deviations for local times of fractional Brownian motion, which we won’t use in our paper but which is of independent interest:

**Theorem 4.4.** Let $B^H$ be a standard fractional Brownian motion with Hurst index $H$ such that $Hd < 1$. Then the limit

$$\lim_{a \to \infty} a^{-\frac{d}{2}} \log \mathbb{P} \left( L^0_t(B^H) \geq a \right) = -\theta(H,d),$$

exists and $\theta(H,d)$ satisfies the following bounds

$$\left( \frac{\pi \epsilon^2 H}{2} \right)^{\frac{d}{2}} \theta(H,d) \leq \theta(H,d) \leq (2\pi)^{\frac{d}{2}} \theta(H,d),$$
where $c_H$ is given by and
\[
\theta_0(\lambda) = \lambda \left( \frac{(1 - \lambda)^{1 - \lambda}}{\Gamma(1 - \lambda)} \right)^{1/\lambda}.
\]

5. Existence of strong solutions

As outlined in the introduction the object of study is a generalised SDE with additive $d$-dimensional fractional Brownian noise $B^H$ with Hurst parameter $H \in (0, 1/2)$, i.e.
\[
X^x_t = x + \alpha L_t(X^x) \cdot 1_d + B^H_t, 0 \leq t \leq T, x \in \mathbb{R}^d,
\]
where $L_t(X^x), t \in [0, T]$ is a stochastic process of bounded variation which arises from taking the limit
\[
L_t(X^x) := \lim_{\varepsilon \to 0} \int_0^t \varphi_\varepsilon(X^x_s) ds,
\]
in probability, where $\varphi_\varepsilon$ are probability densities approximating $\delta_0$, denoting $\delta_0$ the Dirac delta generalized function with total mass at 0. We will consider
\[
\varphi_\varepsilon(x) = \varepsilon^{-d/2} \varphi(\varepsilon^{-1/2}x), \quad \varepsilon > 0,
\]
where $\varphi$ is the $d$-dimensional standard Gaussian density function.

Hereunder, we establish the main result of this section.

**Theorem 5.1.** If $H < 1/(2(2 + d))$, $d \geq 1$ there exists a continuous strong solution $X^x = \{X^x_t, t \in [0, T], x \in \mathbb{R}^d\}$ of equation (5.1) for all $\alpha$. Moreover, for every $t \in [0, T]$, $X_t$ is Malliavin differentiable in the direction of the Brownian motion $W$ in (2.3).

**Proposition 5.2.** Retain the conditions of Theorem 5.1. Let $Y^x \cdot$ be another solution to the SDE (5.1). Suppose that the Doleans-Dade exponentials
\[
\mathcal{E}(\int_0^T -K_H^{-1}(\int_0^t \varphi_\varepsilon(Y^x_u)1_d du)^s du) dW_s), \varepsilon > 0
\]
converge in $L^p(\Omega)$ for $\varepsilon \to 0$ for all $p \geq 1$, where $\varphi_\varepsilon$ is the approximation of the Dirac delta $\delta_0$ in (5.2) and $*$ denotes transposition. Then strong uniqueness holds for such solutions.

In particular, this is the case, if e.g. uniqueness in law is satisfied.

The proof of Theorem (5.1) essentially consists of four steps:

1. In the first step, we construct a weak solution $X$ to (5.1) by using the version of Girsanov’s theorem for the fractional Brownian motion, that is we consider a probability space $(\Omega, \mathcal{F}, P)$ on which a fractional Brownian motion $B^H$ and a process $X^x$ are defined such that (5.1) holds. However, a priori the solution is not a measurable functional of the driving noise, that is $X^x$ is not adapted to the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ generated by $B^H$.

2. In the next step, we approximate the generalized drift coefficient $\delta_0$ by the Gaussian kernels $\varphi_\varepsilon$. Using classical Picard iteration, we know that for each smooth coefficient $\varphi_\varepsilon, \varepsilon > 0$, there exists unique strong solution $X^x_\varepsilon$ to the SDE
\[
dX^x_t = \alpha \varphi_\varepsilon(X^x_t) \cdot 1_d dt + dB^H_t, \quad 0 \leq t \leq T, \quad X^x_0 = x \in \mathbb{R}^d.
\]

The proof of Theorem (5.1) essentially consists of four steps:
Then we prove that for each $t \in [0, T]$ the family $\{X^\varepsilon_t\}_{\varepsilon>0}$ converges weakly as $\varepsilon \searrow 0$ to the conditional expectation $E[X_t|\mathcal{F}_t]$ in the space $L^2(\Omega, \mathcal{F}_t)$ of square integrable, $\mathcal{F}_t$-measurable random variables.

(3) Further, it is well known, see e.g. [47], that for each $t \in [0, T]$ the strong solution $X^\varepsilon_t$, $\varepsilon > 0$, is Malliavin differentiable, and that the Malliavin derivative $D_sX^\varepsilon_t$, $0 \leq s \leq t$, with respect to $W$ in (2.3) solves the equation

$$D_sX^\varepsilon_t = K_H(t, s)I_d + \int_s^t \alpha \varphi'_\varepsilon(X^\varepsilon_u) \cdot 1_d D_sX^\varepsilon_u du,$$

where $\varphi'_\varepsilon$ denotes the Jacobian of $\varphi_\varepsilon$. Using a compactness criterion based on Malliavin calculus (see Appendix A) we then show that for every $t \in [0, T]$ the set of random variables $\{X^\varepsilon_t\}_{\varepsilon>0}$ is relatively compact in $L^2(\Omega)$, which enables us to conclude that $X^\varepsilon_t$ converges strongly as $\varepsilon \searrow 0$ in $L^2(\Omega; \mathcal{F}_t)$ to $E[X_t|\mathcal{F}_t]$. As a consequence of the compactness criterion we also observe that $E[X_t|\mathcal{F}_t]$ is Malliavin differentiable.

(4) Finally, we prove that $E[X_t|\mathcal{F}_t] = X_t$, which entails that $X_t$ is $\mathcal{F}_t$-measurable and thus a strong solution on our specific probability space, on which we assumed our weak solution.

Let us first have a look at step 1 of our programme, that is we want to construct weak solutions of (5.1) by using Girsanov’s theorem. Let $(\Omega, \mathcal{A}, \tilde{P})$ be some given probability space which carries a $d$-dimensional fractional Brownian motion $\tilde{B}^H$ with Hurst parameter $H \in (0, 1/2)$ and set $X^\xi_t := x + \int_0^t \delta_x(X^\xi_u)du$, $t \in [0, T]$, $x \in \mathbb{R}^d$. Set $\theta_t := (K_H^{-1}(\int_0^t \delta_0(X^\xi_u)du)1_d)(t)$ and consider the Doléans-Dade exponential

$$\xi_t := \exp \left\{ \int_0^t \theta_s^T dW_s - \frac{1}{2} \int_0^t \theta_s^T \theta_s ds \right\}, \quad t \in [0, T].$$

formally.

If we were allowed to implement Girsanov’s theorem in this setting we would arrive at the conclusion that the process

$$B^H_t := X^\xi_t - x - \int_0^t \delta_x(X^\xi_u)du1_d$$

$$= \tilde{B}^H_t - \int_0^t \delta_0(B^H_s)du1_d$$

is a fractional Brownian motion on $(\Omega, \mathcal{A}, P)$ with Hurst parameter $H \in (0, 1/2)$, where $\frac{d\tilde{P}}{dP} = \xi_T$. Hence, because of (5.5), the couple $(X^\xi, B^H)$ will be a weak solution of (5.1) on $(\Omega, \mathcal{A}, P)$.

Therefore, in what follows we show that the requirements of Theorem 2.5 are accomplished.

Lemma 5.3. Let $x \in \mathbb{R}^d$. If $H < \frac{1}{2(1+d)}$ then

$$\sup_{\varepsilon>0} E[\exp(\mu \int_0^T (K_H^{-1}(\int_0^t \varphi_{x,\varepsilon}(B^H_u)(du))^2 dt)] < \infty.$$
Moreover, where
\[
\varphi_{x,\varepsilon}(B_u^H) = \frac{1}{(2\pi \varepsilon)^{\frac{d}{2}}} \exp(-\frac{|B_u^H - x|^2}{2\varepsilon}).
\]

**Proof.** In order to prove Lemma 5.3, we can write
\[
K_H^{-1}\left( \int_0^1 \varphi_{x,\varepsilon}(B_r^H)dr \right)(t) = t^{\frac{1}{2}-H(1+d)} \int_0^1 \gamma_{-\frac{1}{2}-H,\frac{1}{2}-H}(1,u)\varphi_{xt^{-u},\varepsilon}(t)(B_u^H)du,
\]
where
\[
\gamma_{\alpha,\beta}(t, u) = (t-u)^\alpha u^\beta.
\]
Using the self-similarity of the fBm we can write
\[
K_H^{-1}\left( \int_0^1 \varphi_{x,\varepsilon}(B_r^H)dr \right)(t) \xrightarrow{\text{law}} t^{\frac{1}{2}-H(1+d)} \int_0^1 \gamma_{-\frac{1}{2}-H,\frac{1}{2}-H}(1,u)\varphi_{xt^{-u},\varepsilon}(t)(B_u^H)du,
\]
where \(\varepsilon(t) := \varepsilon t^{-2H}\), and hence
\[
K_H^{-1}\left( \int_0^1 \varphi_{x,\varepsilon}(B_r^H)dr \right)(t)^{2m} \xrightarrow{\text{law}} t^{2m(\frac{1}{2}-H(1+d))}\int_0^1 \gamma_{-\frac{1}{2}-H,\frac{1}{2}-H}(1,u)\varphi_{xt^{-u},\varepsilon}(t)(B_u^H)du^{2m}
\]
\[
= t^{2m(\frac{1}{2}-H(1+d))}(2m)! \int_{\mathcal{T}_m(0,1)} \prod_{j=1}^{2m} \gamma_{-\frac{1}{2}-H,\frac{1}{2}-H}(1,u_j)\varphi_{xt^{-u},\varepsilon}(t)(B_{u_j}^H)du,
\]
where \(\mathcal{T}_n(0, s) = \{0 \leq u_1 < u_2 < \ldots < u_n \leq s\}\) and
\[
\varphi_{xt^{-u},\varepsilon}(t)(B_{u_j}^H) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(i \langle \xi, B_{u_j}^H - xt^{-H} \rangle_{\mathbb{R}^d}) - \varepsilon(t) \xi^2_{\mathbb{R}^d} d\xi.
\]
Then
\[
E[\left( \int_0^T \varphi_{x,\varepsilon}(B_t^H)dt \right)^{2m}] \leq T^{m-1} \int_0^T E[K_H^{-1}\left( \int_0^1 \varphi_{x,\varepsilon}(B_r^H)dr \right)^{2m} dt]
\]
\[
= T^{m-1} \int_0^T t^{2m(\frac{1}{2}-H(1+d))}(2m)! \int_{\mathcal{T}_m(0,1)} \prod_{j=1}^{2m} \gamma_{-\frac{1}{2}-H,\frac{1}{2}-H}(1,u_j) E[\prod_{j=1}^{2m} \varphi_{xt^{-u},\varepsilon}(t)(B_{u_j}^H)] dudt.
\]
Moreover,
\[
E[\prod_{j=1}^{2m} \varphi_{xt^{-u},\varepsilon}(t)(B_{u_j}^H)]
\]
\[
= \frac{1}{(2\pi)^{2dm}} \int_{\mathbb{R}^d} \exp(i \langle \xi_j, B_{u_j}^H - xt^{-H} \rangle_{\mathbb{R}^d}) - \varepsilon(t) \xi_j^2_{\mathbb{R}^d} d\xi_j]
\]
\[
\frac{1}{(2\pi)^{2dm}} \int_{\mathbb{R}^{2dm}} \mathbb{E}[\exp(i \sum_{j=1}^{2m} \langle \xi_j, B_{u_j}^H \rangle_{\mathbb{R}^d})] \times \exp(-i \sum_{j=1}^{2m} \langle \xi_j, xt^{-H} \rangle_{\mathbb{R}^d}) \times \exp(-\frac{\varepsilon(t)}{2} \sum_{j=1}^{2m} |\xi_j|^2_{\mathbb{R}^d})d\xi_1...d\xi_{2m}.
\]

Next note that
\[
\mathbb{E}[\exp(i \sum_{j=1}^{2m} \langle \xi_j, B_{u_j}^H \rangle_{\mathbb{R}^d})] = \exp(-\frac{1}{2} \text{Var} \left[ \sum_{j=1}^{2m} \sum_{k=1}^{d} \xi_j \xi_k B_{u_j}^{H,k} \right]) = \exp(-\frac{1}{2} \sum_{k=1}^{d} \text{Var} \left[ \sum_{j=1}^{2m} \xi_j \xi_k B_{u_j}^{H,k} \right]) = \exp(-\frac{1}{2} \sum_{k=1}^{d} \langle \xi_k, Q(u)\xi_k \rangle_{\mathbb{R}^{2m}}),
\]

where
\[Q(u) = \text{Cov}(B_{u_1}^{H,1}, ..., B_{u_{2m}}^{H,1}).\]

Hence,
\[
\mathbb{E}\left[ \prod_{j=1}^{2m} \varphi_{xt^{-H},\varepsilon(t)}(B_{u_j}^H) \right] \leq \frac{1}{(2\pi)^{2dm}} \int_{\mathbb{R}^{2dm}} \exp(-\frac{1}{2} \sum_{k=1}^{d} \langle \xi_k, Q(u)\xi_k \rangle_{\mathbb{R}^{2m}})d\xi_1...d\xi_{2m} = \frac{1}{(2\pi)^{2dm}} \int_{\mathbb{R}^{2dm}} \exp(-\frac{1}{2} \sum_{k=1}^{d} \langle \xi_k, Q(u)\xi_k \rangle_{\mathbb{R}^{2m}})d\xi_1...d\xi_{2m} = \frac{1}{(2\pi)^{dm}} \int_{\mathbb{R}^{2m}} \exp(-\frac{1}{2} \sum_{k=1}^{d} \langle \xi_k, Q(u)\xi_k \rangle_{\mathbb{R}^{2m}})d\xi_1...d\xi_{2m} \leq \frac{1}{(2\pi)^{dm}}(\det Q(u))^{-\frac{d}{2}}.
\]

Using the last estimate, we get that
\[
\mathbb{E}\left[ \left( K_H^{-1} \left( \int_0^T \varphi_{x,\varepsilon} (B_u^H) du(t) \right)^2 dt \right)^m \right]
\]
Proposition 5.4. Let \( d \in C \) for all \( p \) with respect to the Radon-Nikodym-derivative \( X \) is a fractional Brownian motion with Hurst parameter \( H \).

Proof. Without loss of generality let \( I \in C, \mu \) for constants \( C \leq I \leq C \) such that

\[
\begin{align*}
&\leq \frac{T^{m-1}}{(2\pi)^{dm}} T^{2m(\frac{1}{2} - H(1+d))} \\
&\times (2m)! \int_{T^{m}(0,1)}^{2m} \left( \prod_{j=1}^{2m} \gamma_{\frac{1}{2} - H, \frac{1}{2} - H}(1, u_j) \right) (\det Q(u))^{-\frac{d}{2}} du \\
&\leq \frac{T^{m-1}}{(2\pi)^{dm}} T^{2m(\frac{1}{2} - H(1+d))} \\
&\times C_{H,d}^{m} (m!)^{2H(1+d)},
\end{align*}
\]

where the last bound is due to Lemma A.5 for a constant \( C_{H,d} \) only depending on \( H \) and \( d \). So the result follows.

\[ \square \]

**Proposition 5.4.** Let \( x \in \mathbb{R}^d \) and \( H < \frac{1}{2(1+d)} \). Then there exists a \( X \in L^p(\Omega) \) such that

\[
\mathcal{E}(\int_0^T K_H^{-1}(\int_0^t \varphi_{x,1/n}(B_u^H)1_d u)^*(s) dW_s) \rightarrow_{n \to \infty} X \in L^p(\Omega)
\]

for all \( p \geq 1 \). Furthermore,

\[
B_t^H - L_t^\phi(B^H)1_d, 0 \leq t \leq T
\]

is a fractional Brownian motion with Hurst parameter \( H \) under the change of measure with respect to the Radon-Nikodym-derivative \( X \).

**Proof.** Without loss of generality let \( p = 1 \). Then using \(|e^x - e^y| \leq |x - y| e^{x+y}, \) Hölder’s inequality, the supermartingale property of Doleans-Dade exponentials we get in connection with the previous Lemma that

\[
\begin{align*}
&\mathbb{E}\left[ \mathcal{E}(\int_0^T K_H^{-1}(\int_0^t \varphi_{x,1/n}(B_u^H)1_d u)^*(s) dW_s) - \mathcal{E}(\int_0^T K_H^{-1}(\int_0^t \varphi_{x,1/r}(B_u^H)1_d u)^*(s) dW_s) \right] \\
&\leq C(I_1 + I_2) \mathbb{E},
\end{align*}
\]

where

\[
I_1 := \mathbb{E}[\int_0^T \left| K_H^{-1}(\int_0^t \varphi_{x,1/n}(B_u^H)1_d u)^*(s) - K_H^{-1}(\int_0^t \varphi_{1/r}(B_u^H)1_d u)^*(s) \right|^2 ds]^{1/2},
\]

\[
I_2 := \mathbb{E}[\int_0^T \left| K_H^{-1}(\int_0^t \varphi_{x,1/n}(B_u^H)1_d u)^*(s) \right|^2 ds - \int_0^T \left| K_H^{-1}(\int_0^t \varphi_{1/r}(B_u^H)1_d u)^*(s) \right|^2 ds]^{1/2}
\]

\[
E := \mathbb{E}[\exp\{\mu_1 \int_0^t \left| K_H^{-1}(\int_0^t \varphi_{x,1/n}(B_u^H)1_d u)^*(s) \right|^2 ds\}^{1/4}]
\]

\[
\cdot \mathbb{E}[\exp\{\mu_2 \int_0^t \left| K_H^{-1}(\int_0^t \varphi_{1/r}(B_u^H)1_d u)^*(s) \right|^2 ds\}^{1/4}
\]

for constants \( C, \mu_1, \mu_2 \geq 0 \).
Now, let us have a look at the proof of the previous Lemma and adopt the notation therein. In the sequel we omit $1_d$. Then we obtain for $m = 1$ by using the self-similarity of the fBm in a similar way (but under expectation) that

$$E\left[ K_H^{-1}(\int_0^t \varphi_{\varepsilon_1}(B_u^H)du)^* (t) \right]^2 \mid K_H^{-1}(\int_0^t \varphi_{\varepsilon_2}(B_u^H)du)^* (t) \right]^2$$

$$= E[\left( \varepsilon_{2m} \varphi_{\varepsilon_1}(B_u^H)du \right) (2m)!]^2 \int_{T_{2m}(0,1)}^{2m} \prod_{j=1}^2 \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, u_j) \varphi_{x-t, \varepsilon_1(t)}(B_u^H)du$$

$$\times \int_{T_{2m}(0,1)}^{2m} \prod_{j=1}^2 \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, u_j) \varphi_{x-t, \varepsilon_2(t)}(B_u^H)du],$$

where $\varepsilon_i(t) = \varepsilon_i t^{-2H}$, $i = 1, 2$. Using shuffling (see Section 2.2), we get that

$$E\left[ K_H^{-1}(\int_0^t \varphi_{\varepsilon_1}(B_u^H)du)^* (t) \right]^2 \mid K_H^{-1}(\int_0^t \varphi_{\varepsilon_2}(B_u^H)du)^* (t) \right]^2$$

$$= E[\left( \varepsilon_{2m} \varphi_{\varepsilon_1}(B_u^H)du \right) (2m)!]^2$$

$$\times \sum_{\sigma \in S(2m,2m)} \prod_{j=1}^{4m} f_{\sigma(j)}(u_j)du,$$

where $f_j(s) := \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, s) \varphi_{x-t, \varepsilon_1(t)}(B_u^H)$, if $j = 1, ..., 2m$ and $\gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, s) \varphi_{x-t, \varepsilon_2(t)}(B_u^H)$, if $j = 2m + 1, ..., 4m$. Without loss of generality, consider the case

$$\prod_{j=1}^{2m} f_{\sigma(j)}(u_j)$$

$$= \prod_{j=1}^{2m} \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, u_j) \varphi_{x-t, \varepsilon_1(t)}(B_u^H)$$

$$\times \prod_{j=2m+1}^{4m} \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, u_j) \varphi_{x-t, \varepsilon_2(t)}(B_u^H).$$

Then

$$E[(\varepsilon_{2m} \varphi_{\varepsilon_1}(B_u^H)du) (2m)!]^2 \int_{T_{2m}(0,1)}^{4m} \prod_{j=1}^{4m} f_{\sigma(j)}(u_j)du]$$

$$= (\varepsilon_{2m} \varphi_{\varepsilon_1}(B_u^H)du) (2m)!]^2$$

$$\times \int_{T_{2m}(0,1)}^{2m} \prod_{j=1}^{2m} \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, u_j) \prod_{j=2m+1}^{4m} \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, u_j)$$

$$\times E[\prod_{j=1}^{2m} \varphi_{x-t, \varepsilon_1(t)}(B_u^H) \prod_{j=2m+1}^{4m} \varphi_{x-t, \varepsilon_2(t)}(B_u^H)]du$$
Hence, using dominated convergence in connection with Lemma A.5, we see that

\[ \int_{T_{4m}(0,1)} 4m \prod_{j=1}^{4m} \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, u_j) \]

\[ \times \mathbb{E}\left[ \prod_{j=1}^{2m} \int_{\mathbb{R}^d} \exp(i \langle \xi_j, B_{u_j}^H - xt^{-H}\rangle_{\mathbb{R}^d} - \varepsilon_1(t) \frac{\langle \xi_j \rangle_{\mathbb{R}^d}^2}{2}) \) d\xi_j \right] \]

\[ \times \prod_{j=2m+1}^{4m} \int_{\mathbb{R}^d} \exp(i \langle \xi_j, B_{u_j}^H - xt^{-H}\rangle_{\mathbb{R}^d} - \varepsilon_2(t) \frac{\langle \xi_j \rangle_{\mathbb{R}^d}^2}{2}) \) d\xi_j \right] d\mathbf{u} \]

\[ = \left( i^{2m}(\frac{1}{2}-H(1+d)) (2m)! \right)^2 \]

\[ \times \int_{T_{4m}(0,1)} 4m \prod_{j=1}^{4m} \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, u_j) \]

\[ \times \frac{1}{(2\pi)^{4md}} \int_{\mathbb{R}^{4md}} \mathbb{E}[\exp(i \sum_{j=1}^{4m} \langle \xi_j, B_{u_j}^H \rangle_{\mathbb{R}^d})] \]

\[ \times \exp(-i \sum_{j=1}^{4m} \langle \xi_j, xt^{-H} \rangle_{\mathbb{R}^d}) \exp(-\frac{\varepsilon_1(t)}{2} \sum_{j=1}^{2m} \langle \xi_j \rangle_{\mathbb{R}^d}^2 - \frac{\varepsilon_2(t)}{2} \sum_{j=1}^{2m} \langle \xi_j \rangle_{\mathbb{R}^d}^2) \)

\[ d\xi_1 \cdots d\xi_{4m} d\mathbf{u} \]

So

\[
\mathbb{E}\left[ \left( i^{2m}(\frac{1}{2}-H(1+d)) (2m)! \right)^2 \right] \int_{T_{4m}(0,1)} 4m \prod_{j=1}^{4m} f_{\sigma(j)}(u_j) d\mathbf{u} \]

\[ = \left( i^{2m}(\frac{1}{2}-H(1+d)) (2m)! \right)^2 \int_{T_{4m}(0,1)} 4m \prod_{j=1}^{4m} \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, u_j) \]

\[ \times \frac{1}{(2\pi)^{4md}} \int_{\mathbb{R}^{4md}} \exp(-\frac{1}{2} \sum_{k=1}^{d} \langle \xi^k, Q(u)\xi^k \rangle_{\mathbb{R}^{4md}}) \]

\[ \times \exp(-i \sum_{j=1}^{4m} \langle \xi_j, xt^{-H} \rangle_{\mathbb{R}^d}) \exp(-\frac{\varepsilon_1(t)}{2} \sum_{j=1}^{2m} \langle \xi_j \rangle_{\mathbb{R}^d}^2 - \frac{\varepsilon_2(t)}{2} \sum_{j=1}^{2m} \langle \xi_j \rangle_{\mathbb{R}^d}^2) \)

\[ d\xi_1 \cdots d\xi_{4m} d\mathbf{u} \]

Hence, using dominated convergence in connection with Lemma A.5, we see that

\[
\int_0^T \mathbb{E}\left[ \left( i^{2m}(\frac{1}{2}-H(1+d)) (2m)! \right)^2 \right] \int_{T_{4m}(0,1)} 4m \prod_{j=1}^{4m} f_{\sigma(j)}(u_j) d\mathbf{u} dt \]

\[ \rightarrow \int_0^T \left( i^{2m}(\frac{1}{2}-H(1+d)) (2m)! \right)^2 \int_{T_{4m}(0,1)} 4m \prod_{j=1}^{4m} \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, u_j) \]

\[ \times \frac{1}{(2\pi)^{4md}} \int_{\mathbb{R}^{4md}} \exp(-\frac{1}{2} \sum_{k=1}^{d} \langle \xi^k, Q(u)\xi^k \rangle_{\mathbb{R}^{4md}}) \]
\[ \exp(-i \sum_{j=1}^{4m} \langle \xi_j, xt^{-H} \rangle_{\mathbb{R}^d}) d\xi_1 ... d\xi_{4m} dt \]

for \( \varepsilon_1, \varepsilon_2 \searrow 0 \). For other \( \sigma \in S(2m, 2m) \), we obtain similar limit values. In summary, we find (by also considering the case \( \varepsilon_1 = \varepsilon_2 \)) that

\[ E[\int_0^T \left| K_H^{-1}(\int_0^s \varphi_{\varepsilon_1}(B^H_u) du) - K_H^{-1}(\int_0^s \varphi_{\varepsilon_2}(B^H_u) du) \right|^2 ds]^2] \rightarrow 0 \]

for \( \varepsilon_1, \varepsilon_2 \searrow 0 \). Thus

\[ I_2 = I_2(n, r) \rightarrow 0 \text{ for } n, r \rightarrow \infty. \]

Similarly, we have that

\[ I_1 = I_1(n, r) \rightarrow 0 \text{ for } n, r \rightarrow \infty. \]

Since \( E = E(n, r) \) is uniformly bounded with respect to \( n, r \) because of the previous Lemma 5.3, we obtain the convergence of the Radon-Nikodym derivatives to a \( X \) in \( L^p(\Omega) \) for \( p = 1 \). The second statement of the Lemma follows by using characteristic functions with dominated convergence. \( \square \)

Henceforth, we confine ourselves to the filtered probability space \((\Omega, \mathcal{F}, P, \mathcal{F}) = \{\mathcal{F}_t\}_{t \in [0, T]}\) which carries the weak solution \((X^x, B^H)\) of (5.1).

We now turn to the second step of our procedure.

**Lemma 5.5.** Suppose that \( H < \frac{1}{2(1+d)} \) and let \( \{\varphi_\varepsilon\}_{\varepsilon > 0} \) be defined as

\[ \varphi_\varepsilon(y) = \varphi_{\varepsilon,x}(y) = \varepsilon^{-\frac{1}{2}} \varphi(\varepsilon^{-\frac{1}{2}}(y - x)), \varepsilon > 0, \]

where \( \varphi \) is the \( d \)-dimensional standard normal density. Denote by \( X^{x,\varepsilon}_t = \{X^{x,\varepsilon}_t, t \in [0, T]\} \) the corresponding solutions of (5.1), if we replace \( \delta_x \) by \( \varphi_{\varepsilon,x}(y), \varepsilon > 0 \). Then for every \( t \in [0, T] \) and bounded continuous function \( \eta : \mathbb{R}^d \rightarrow \mathbb{R} \) we have that

\[ \eta(X^{x,\varepsilon}_t) \Rightarrow \eta(X^x_t) \text{ for } t \in [0, T]. \]

weakly in \( L^2(\Omega, \mathcal{F}, P) \).

**Proof.** Without loss of generality let \( x = 0 \). We mention that

\[ \Sigma_t := \{\exp\{\sum_{j=1}^{k} \langle \alpha_j, B^{H}_{t_j} - B^{H}_{t_{j-1}} \rangle\} : \{\alpha_j\}_{j=1}^{k} \subset \mathbb{R}^d, 0 = t_0 < ... < t_k = t, k \geq 1\} \]

is a total subset of \( L^2(\Omega, \mathcal{F}, P) \). Denote \( X^{x,\varepsilon}_t \) by \( X^n_t \) for \( \varepsilon = 1/n \). Then using Girsanov's theorem, we find that

\[ E[\eta(X^n_t) \exp\{\sum_{j=1}^{k} \langle \alpha_j, B^{H}_{t_j} - B^{H}_{t_{j-1}} \rangle\}] \]

\[ = E[\eta(X^n_t) \exp\{\sum_{j=1}^{k} \langle \alpha_j, X^n_{t_j} - X^n_{t_{j-1}} - \int_{t_{j-1}}^{t_j} \varphi_{1/n}(X^n_s)\mathbf{1}_d ds \rangle\}] \]
\[ E[\eta(B_t^H)] \exp\left\{ \sum_{j=1}^{k} \left( \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H - \int_{t_{j-1}}^{t_j} \varphi_{1/n}(B_s^H) 1 ds \right) \right\} \]

\[ \cdot \mathcal{E}\left( \int_0^t K_H^{-1}(\int_0^t \varphi_{1/n}(B_u^H) 1 du)_{*(s)}(s) dW_s \right). \]

On the other hand, we obtain by \(|e^x - e^y| \leq |x - y| e^{x+y}\), Hölder’s inequality, the supermartingale property of Doleans-Dade exponentials and the proof of Proposition 5.4 that

\[
\begin{align*}
&\left| E[\eta(B_t^H)] \exp\left\{ \sum_{j=1}^{k} \left( \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H - \int_{t_{j-1}}^{t_j} \varphi_{1/n}(B_s^H) 1 ds \right) \right\} \right|
\cdot \mathcal{E}\left( \int_0^t K_H^{-1}(\int_0^t \varphi_{1/n}(B_u^H) 1 du)_{*(s)}(s) dW_s \right)
- \eta(B_t^H) \exp\left\{ \sum_{j=1}^{k} \left( \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H - \int_{t_{j-1}}^{t_j} \delta_0(B_s^H) ds 1 \right) \right\}
\cdot X \right|
\leq C(I_1 + I_2 + I_3) E,
\end{align*}
\]

where

\[
I_1 := E\left[ \left( \sum_{j=1}^{k} \left( \alpha_j, \int_{t_{j-1}}^{t_j} \delta_0(B_s^H) ds 1 - \int_{t_{j-1}}^{t_j} \varphi_{1/n}(B_s^H) 1 ds \right) \right)^2 \right]^{1/2},
\]

\[
I_2 := \lim_{r \to \infty} E\left[ \left\| K_H^{-1}(\int_0^t \varphi_{1/n}(B_u^H) 1 du)_{*(s)}(s) - K_H^{-1}(\int_0^t \varphi_{1/r}(B_u^H) 1 du)_{*(s)}(s) \right\| ds \right]^{1/2},
\]

\[
I_3 := \lim_{r \to \infty} E\left[ \left\| K_H^{-1}(\int_0^t \varphi_{1/n}(B_u^H) 1 du)_{*(s)}(s) \right\|^2 ds - \int_0^t \left\| K_H^{-1}(\int_0^t \varphi_{1/r}(B_u^H) 1 du)_{*(s)}(s) \right\|^2 ds \right]^{1/2}
\]

and

\[
E := \sup_{r \geq 1} E\left[ \exp\left\{ \sum_{j=1}^{k} \left( \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H - \int_{t_{j-1}}^{t_j} \varphi_{1/n}(B_s^H) 1 ds \right) \right\} \right]^{1/8}
\]

\[
\cdot E\left[ \exp\left\{ \sum_{j=1}^{k} \left( \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H - \int_{t_{j-1}}^{t_j} \delta_0(B_s^H) ds 1 \right) \right\} \right]^{1/8}
\]

\[
\cdot E\left[ \exp\left\{ \mu_1 \int_0^t \left\| K_H^{-1}(\int_0^t \varphi_{1/n}(B_u^H) 1 du)_{*(s)}(s) \right\| ds \right\} \right]^{1/8}
\]

\[
\cdot E\left[ \exp\left\{ \mu_2 \int_0^t \left\| K_H^{-1}(\int_0^t \varphi_{1/r}(B_u^H) 1 du)_{*(s)}(s) \right\| ds \right\} \right]^{1/16}
\]

for constants \(C, \mu_1, \mu_2 > 0\).

By inspecting the proof of Proposition 5.4 once again, we know that

\[
I_3 = I_3(n) \to 0 \text{ for } n \to \infty.
\]
and
\[ I_2 = I_2(n) \to 0 \text{ for } n \to \infty. \]

Since \( L_t^\varepsilon(B^H, \varepsilon) \) converges to \( L_t(B^H) \) in \( L^p(\Omega) \) for all \( p \geq 1 \), we also conclude that
\[ I_1 = I_1(n) \to 0 \text{ for } n \to \infty. \]

On the other hand, we obtain from [4.2], Theorem [4.4] and Lemma [5.3] that
\[ E = E(n) \leq K \]
for all \( n \), where \( K \) is a constant. So we see that
\[
E[\eta(X_t^n)] \exp\left\{ \sum_{j=1}^k \left\langle \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H \right\rangle \right\} \to \\
E[\eta(B_t^H)] \exp\left\{ \sum_{j=1}^k \left\langle \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H - \int_{t_{j-1}}^{t_j} \delta_0(B_s^H)1 ds \right\rangle \right\} \cdot X \\
= E[E[\eta(X_t)] | {\mathcal F}_t] \exp\left\{ \sum_{j=1}^k \left\langle \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H \right\rangle \right\}
\]
for \( n \to \infty \), which completes the proof. \( \square \)

We continue with the third step of our scheme. This is the most challenging part. For notational convenience let us from now on assume that \( \alpha = 1 \) in (5.1) and that \( \varphi_\varepsilon \) stands for the Jacobian of \( \varphi_\varepsilon 1_d \). The following result is based on a compactness criterion for subsets of \( L^2(\Omega) \) which is summarised in the Appendix.

**Lemma 5.6.** Let \( \{\varphi_\varepsilon\}_{\varepsilon > 0} \) the family of Gaussian kernels approximating Dirac’s delta function \( \delta_0 \) in the sense of (5.3). Fix \( t \in [0, T] \) and denote by \( X_t^\varepsilon \) the corresponding solutions of (5.1) if we replace \( L_t(X^\varepsilon) \) by \( \int_0^t \varphi_\varepsilon(X_s^\varepsilon)ds \), \( \varepsilon > 0 \). Then there exists a \( \beta \in (0, 1/2) \) such that
\[
\sup_{\varepsilon > 0} \int_0^t \int_0^t E[\| D_\theta X_t^\varepsilon - D_{\theta'} X_t^\varepsilon \|^2 | \theta' - \theta |^{1+2\beta}] d\theta' d\theta < \infty
\]
and
\[
\sup_{\varepsilon > 0} \| D_\theta X_t^\varepsilon \|_{L^2(\Omega \times [0, T], \mathbb{R}^{d \times d})} < \infty. \tag{5.6}
\]

**Proof.** Fix \( t \in [0, T] \) and take \( \theta, \theta' > 0 \) such that \( 0 < \theta' < \theta < t \). Using the chain rule for the Malliavin derivative, see [47], Proposition 1.2.3, we have
\[
D_\theta X_t^\varepsilon = K_H(t, \theta) I_d + \int_\theta^t \varphi_\varepsilon'(X_s^\varepsilon) D_\theta X_s^\varepsilon ds
\]
\( P \)-a.s. for all \( 0 \leq \theta \leq t \) where \( \varphi_\varepsilon'(z) = \left( \frac{\partial}{\partial z_j} \varphi_\varepsilon(z) \right)_{i,j=1,...,d} \) denotes the Jacobian matrix of \( \varphi_\varepsilon \) and \( I_d \) the identity matrix in \( \mathbb{R}^{d \times d} \). Thus we have
\[
D_\theta X_t^\varepsilon - D_\theta X_t^{\varepsilon'} = K_H(t, \theta') I_d - K_H(t, \theta) I_d
\]
Using Picard iteration applied to the above equation we may write

\[
D_\theta X^\varepsilon_t - D_\theta X^\varepsilon_0 = K_H(t, \theta')I_d - K_H(t, \theta)I_d \\
+ \sum_{m=1}^{\infty} \int_{\Delta_{\theta,d,j}^m} \prod_{j=1}^{m} \varphi^\varepsilon(X^\varepsilon_{s_j}) (K_H(s_m, \theta')I_d - K_H(s_m, \theta)I_d) ds_m \cdots ds_1 \\
+ \left( I_d + \sum_{m=1}^{\infty} \int_{\Delta_{\theta,d,j}^m} \prod_{j=1}^{m} \varphi^\varepsilon(X^\varepsilon_{s_j}) ds_m \cdots ds_1 \right) (D_{\theta'}X^\varepsilon_0 - K_H(\theta, \theta')I_d).
\]

On the other hand, observe that one may again write

\[
D_{\theta'}X^\varepsilon_0 - K_H(\theta, \theta')I_d = \sum_{m=1}^{\infty} \int_{\Delta_{\theta',\theta,j}^m} \prod_{j=1}^{m} \varphi^\varepsilon(X^\varepsilon_{s_j}) (K_H(s_m, \theta')I_d - K_H(s_m, \theta)I_d) ds_m \cdots ds_1.
\]

Altogether, we can write

\[
D_{\theta'}X^\varepsilon_t - D_\theta X^\varepsilon_t = I_1(\theta', \theta) + I_2^\varepsilon(\theta', \theta) + I_3^\varepsilon(\theta', \theta),
\]

where

\[
I_1(\theta', \theta) := K_H(t, \theta')I_d - K_H(t, \theta)I_d \\
I_2^\varepsilon(\theta', \theta) := \sum_{m=1}^{\infty} \int_{\Delta_{\theta,d,j}^m} \prod_{j=1}^{m} \varphi^\varepsilon(X^\varepsilon_{s_j}) (K_H(s_m, \theta')I_d - K_H(s_m, \theta)I_d) ds_m \cdots ds_1 \\
I_3^\varepsilon(\theta', \theta) := \left( I_d + \sum_{m=1}^{\infty} \int_{\Delta_{\theta,d,j}^m} \prod_{j=1}^{m} \varphi^\varepsilon(X^\varepsilon_{s_j}) ds_m \cdots ds_1 \right) \times \left( \sum_{m=1}^{\infty} \int_{\Delta_{\theta',\theta,j}^m} \prod_{j=1}^{m} \varphi^\varepsilon(X^\varepsilon_{s_j}) (K_H(s_m, \theta')I_d) ds_m \cdots ds_1. \right).
\]

It follows from Lemma A.4 that

\[
\int_0^t \int_0^t \frac{||I_1(\theta', \theta)||^2_{L^2(\Omega)}}{|\theta' - \theta|^{1+2\beta}} d\theta d\theta' = \int_0^t \int_0^t \frac{|K_H(t, \theta') - K_H(t, \theta)|^2}{|\theta' - \theta|^{1+2\beta}} d\theta d\theta' < \infty \tag{5.7}
\]

for a suitably small \( \beta \in (0, 1/2) \).
Let us continue with the term $I_2^*(\theta', \theta)$. Then Girsanov's theorem, Cauchy-Schwarz inequality and Lemma 5.3 imply

$$E[\|I_2^*(\theta', \theta)\|^2] \leq C E \left[ \left\| \sum_{m=1}^{\infty} \prod_{j=1}^{m} \varphi_{\varepsilon}(x + B_{s_j}^H) (K_H(s_m, \theta') - K_H(s_m, \theta)) ds_m \cdot \cdots ds_1 \right\|^4 \right]^{1/2},$$

where $C > 0$ is an upperbound from Lemma 5.3.

Let $\| \cdot \|$ denote the matrix norm in $\mathbb{R}^{d \times d}$ such that $\|A\| = \sum_{i,j=1}^{d} |a_{ij}|$ for a matrix $A = \{a_{ij}\}_{i,j=1,\ldots,d}$, then taking this matrix norm and expectation we have

$$E[\|I_2^*(\theta', \theta)\|^2] \leq C \left( \sum_{m=1}^{\infty} \sum_{i,j=1}^{d} \sum_{l_1,\ldots,l_{m-1}=1}^{d} \left\| \int_{\Delta_{s_j}^m} \frac{\partial}{\partial x_{l_1}} \varphi_{\varepsilon}^{(i)}(x + B_{s_1}^H) \frac{\partial}{\partial x_{l_2}} \varphi_{\varepsilon}^{(l_1)}(x + B_{s_2}^H) \cdots \right. \left. \frac{\partial}{\partial x_{l_{m-1}}} \varphi_{\varepsilon}^{(l_{m-1})}(x + B_{s_m}^H) (K_H(s_m, \theta') - K_H(s_m, \theta)) ds_m \cdot \cdots ds_1 \right\|_{L^4(\Omega, \mathbb{R})} \right)^2.$$

Now we concentrate on the expression

$$J_2^*(\theta', \theta) := \int_{\Delta_{s_j}^m} \frac{\partial}{\partial x_{l_1}} \varphi_{\varepsilon}^{(i)}(x + B_{s_1}^H) \cdots \frac{\partial}{\partial x_{l_{m-1}}} \varphi_{\varepsilon}^{(l_{m-1})}(x + B_{s_m}^H) (K_H(s_m, \theta') - K_H(s_m, \theta)) ds.$$

(5.8)

Then, shuffling $J_2^*(\theta', \theta)$ as shown in (2.1), one can write $(J_2^*(\theta', \theta))^2$ as a sum of at most $2^{2m}$ summands of length $2m$ of the form

$$\int_{\Delta_{s_j}^{2m}} g_1^\varepsilon(B_{s_1}^H) \cdots g_{2m}^\varepsilon(B_{s_{2m}}^H) ds_{2m} \cdot \cdots ds_1,$$

(5.9)

where for each $l = 1, \ldots, 2m$,

$$g_l^\varepsilon(B^H) \in \left\{ \frac{\partial}{\partial x_{j}} \varphi_{\varepsilon}^{(i)}(x + B^H), \frac{\partial}{\partial x_{j}} \varphi_{\varepsilon}^{(l_{i})}(x + B^H) (K_H(\cdot, \theta') - K_H(\cdot, \theta)) \right\}, i,j = 1, \ldots, d.$$

Repeating this argument once again, we find that $J_2^*(\theta', \theta)^4$ can be expressed as a sum of, at most, $2^{4m}$ summands of length $4m$ of the form

$$\int_{\Delta_{s_j}^{4m}} g_1^\varepsilon(B_{s_1}^H) \cdots g_{4m}^\varepsilon(B_{s_{4m}}^H) ds_{4m} \cdot \cdots ds_1,$$

(5.10)

where for each $l = 1, \ldots, 4m$,

$$g_l^\varepsilon(B^H) \in \left\{ \frac{\partial}{\partial x_{j}} \varphi_{\varepsilon}^{(i)}(x + B^H), \frac{\partial}{\partial x_{j}} \varphi_{\varepsilon}^{(l_{i})}(x + B^H) (K_H(\cdot, \theta') - K_H(\cdot, \theta)) \right\}, i,j = 1, \ldots, d.$$

It is important to note that the function $(K_H(\cdot, \theta') - K_H(\cdot, \theta))$ appears only once in term (5.8) and hence only four times in term (5.10). So there are indices $j_1, \ldots, j_4 \in$
\{1,\ldots,4m\} such that we can write (5.10) as
\[
\int_{\Delta_{\theta,t}^n} \left( \prod_{j=1}^{4m} g^i_j(B_{s_j}) \right) \prod_{i=1}^4 (K_H(s_j, \theta') - K_H(s_j, \theta)) \, ds_{4m} \cdots ds_1,
\]
where
\[
g^i_j(B^H) \in \left\{ \frac{\partial}{\partial x_j} \phi^{(i)}(x + B^H), \, i, j = 1, \ldots, d \right\}, \quad l = 1, \ldots, 4m.
\]

The latter enables us to use the estimate from Proposition 3.2 with \(\sum_{j=1}^{4m} \varepsilon_j = 4\), \(\sum_{l=1}^d \alpha_{[\sigma(j)]}^{(1)} = 1\) for all \(j\), \(|\alpha| = 4m\) and Remark 3.4. Thus we obtain that
\[
E(J_2')^4 \leq \left( \frac{\theta - \theta'}{\theta \theta'} \right)^{4\gamma} \theta^{2(H - \frac{1}{2} - \gamma)} C^{4m} \| \varphi \|_{L^2(\mathbb{R}^d)} A^\gamma_{m}(H, d, |t - \theta|)
\]
whenever \(H < \frac{1}{2(2+d)}\) and \(\gamma \in (0, H)\), where
\[
A^\gamma_{m}(H, d, |t - \theta|) := \frac{(8m)!}{\Gamma(-H(d+2))8m + 8(H - \frac{1}{2} - \gamma)} + 8m)^{1/2}.
\]

Note that \(\| \varphi \|_{L^1(\mathbb{R}^d)} = 1\).

Altogether, we see that
\[
E \left[ \| I_2' (\theta', \theta) \|^2 \right] \leq \left( \frac{\theta - \theta'}{\theta \theta'} \right)^{2\gamma} \theta^{2(H - \frac{1}{2} - \gamma)} \left( \sum_{m=1}^\infty d^{m+1} C^m \| \varphi \|_{L^1(\mathbb{R}^d)} A^\gamma_{m}(H, d, |T|) \right)^{1/4}.
\]

So we can find a constant \(C > 0\) such that
\[
\sup_{\varepsilon > 0} E \left[ \| I_2' (\theta', \theta) \|^2 \right] \leq C \left( \frac{\theta - \theta'}{\theta \theta'} \right)^{2\gamma} \theta^{2(H - \frac{1}{2} - \gamma)}
\]
for \(\gamma \in (0, H)\) provided that \(H < \frac{1}{2(2+d)}\). It is easy to see that we can choose \(\gamma \in (0, H)\) such that there is a suitably small \(\beta \in (0, 1/2)\), \(0 < \beta < \gamma < H < 1/2\) so that it follows from the proof of Lemma A.4 that
\[
\int_0^t \int_0^t \left| \frac{\theta - \theta'}{\theta \theta'} \right|^{2\gamma} \left| \theta - \theta' \right|^{2(H - \frac{1}{2} - \gamma)} \, d\theta' \, d\theta < \infty, \tag{5.11}
\]
for every \(t \in (0, T]\).

We now turn to the term \(I_3' (\theta', \theta)\). Observe that term \(I_3' (\theta', \theta)\) is the product of two terms, where the first one will simply be bounded uniformly in \(\theta, t \in [0, T]\) under expectation. This can be shown by following meticulously the same steps as we did for \(I_2' (\theta', \theta)\) and observing that in virtue of Proposition 3.3 with \(\varepsilon_j = 0\) for all \(j\) the singularity in \(\theta\) vanishes.

Again Girsanov’s theorem, Cauchy-Schwarz inequality several times and Lemma 5.3 lead to
\[
E[\| I_3' (\theta', \theta) \|^2] \leq C \left\| I_d + \sum_{m=1}^\infty \int_{\Delta_{\theta,t}^n} \prod_{j=1}^m \varphi'^i_j(B_{s_j}) \, ds_{m} \cdots ds_1 \right\|_{L^2(\Omega, \mathbb{R}^d)}^2
\]
where \( C > 0 \) denotes an upperbound obtained from Lemma 5.3.

Again, we have

\[
E[\|I_3(\theta', \theta)\|^2] \leq C \left( 1 + \sum_{m=1}^{\infty} \sum_{i=1}^{d} \sum_{j=1}^{d} \left\| \int_{\Delta^m_{\theta', \theta}} \frac{\partial}{\partial x_j} \varphi^{(i)}_{\epsilon}(x + B^H_{s_1}) \right\|_{L^2(\Omega, \mathbb{R})} \right)^2 \times \left( \sum_{m=1}^{\infty} \sum_{i=1}^{d} \sum_{j=1}^{d} \left\| \int_{\Delta^m_{\theta', \theta}} \frac{\partial}{\partial x_j} \varphi^{(l_{m-1})}_{\epsilon}(x + B^H_{s_m}) K_H(s_m, \theta') ds_m \right\|_{L^4(\Omega, \mathbb{R})} \right)^2.
\]

Using exactly the same reasoning as for \( I_3(\theta', \theta) \) we see that the first factor can be bounded by some finite constant \( C \) depending on \( H, d, T \), i.e.

\[
E[\|I_3(\theta', \theta)\|^2] \leq C \left( \sum_{m=1}^{\infty} \sum_{i=1}^{d} \sum_{j=1}^{d} \left\| \int_{\Delta^m_{\theta', \theta}} \frac{\partial}{\partial x_j} \varphi^{(i)}_{\epsilon}(x + B^H_{s_1}) \right\|_{L^2(\Omega, \mathbb{R})} \right)^2 \times \left( \sum_{m=1}^{\infty} \sum_{i=1}^{d} \sum_{j=1}^{d} \left\| \int_{\Delta^m_{\theta', \theta}} \frac{\partial}{\partial x_j} \varphi^{(l_{m-1})}_{\epsilon}(x + B^H_{s_m}) K_H(s_m, \theta') ds_m \right\|_{L^4(\Omega, \mathbb{R})} \right)^2.
\]

As before, we pay attention to

\[
J_3^\epsilon(\theta', \theta) := \int_{\Delta^m_{\theta', \theta}} \varphi^{(i)}_{\epsilon}(x + B^H_{s_1}) \cdots \varphi^{(l_{m-1})}_{\epsilon}(x + B^H_{s_m}) K_H(s_m, \theta') ds_m \cdots ds_1.
\]

(5.12)

We can express \((J_3^\epsilon(\theta', \theta))^4\) as a sum of, at most, \(2^{8m}\) summands of length \(4m\) of the form

\[
\int_{\Delta^m_{\theta', \theta}} g^1_4(B^H_{s_1}) \cdots g^4_4(B^H_{s_m}) ds_4 \cdots ds_1,
\]

(5.13)

where for each \( l = 1, \ldots, 4m,\)

\[
g^l_4(B^H_{s_l}) \in \left\{ \frac{\partial}{\partial x_j} \varphi^{(i)}_{\epsilon}(x + B^H_{s_1}), \frac{\partial}{\partial x_j} \varphi^{(i)}_{\epsilon}(x + B^H_{s_1}) K_H(\cdot, \theta'), i, j = 1, \ldots, d \right\},
\]

where the factor \( K_H(\cdot, \theta') \) is repeated four times in the integrand of (5.13). Now we can simply apply Proposition 3.3 with \( \sum_{j=1}^{4m} \epsilon_j = 4, \sum_{i=1}^{d} \alpha^{(1)}_{\epsilon_j} = 1 \) for all \( j, |\alpha| = 4m \) and Remark 3.4 in order to get

\[
E[(J_3^\epsilon(\theta', \theta))^4] \leq \theta^4(4^{-\frac{1}{2}}) C^{4m} \|\varphi^{(4m)}_{\epsilon}\|_{L^2(\mathbb{R}^d)}^4 \alpha^0_m(H, d, |\theta - \theta'|),
\]

where \( \alpha^0_m(H, d, |\theta - \theta'|) \) is a suitable upperbound obtained from Lemma 5.3.
whenever $H < \frac{1}{2(2+d)}$ where $A_m^0(H, d, |\theta - \theta'|)$ is defined as in (5) by inserting $\gamma = 0$.

As a result,
\[
E[\|I_3^3(\theta', \theta)\|^2] \leq \theta^{2(H - \frac{1}{2})} \left( \sum_{m=1}^{\infty} d^{m+1} C_m \left\| \varphi_{\epsilon_{m,\epsilon}}(\theta, \theta') \right\|_{L^1(\mathbb{R}^d)}^m \right)^2.
\]

Since the exponent of $|\theta - \theta'|$ appearing in $A_m^0(H, d, |\theta - \theta'|)$ is strictly positive by assumption, we can find a small enough $\delta > 0$ and a constant $C := C_{H,d,T} > 0$ such that
\[
\sup_{\epsilon > 0} E[\|I_3^3(\theta', \theta)\|^2] \leq C|\theta|^{2(H - \frac{1}{2})}|\theta - \theta'|^\delta
\]
provided $H < \frac{1}{2(2+d)}$. Then again, it is easy to see that we can choose $\beta \in (0, 1/2)$ small enough so that it follows from the proof of Lemma A.4 that
\[
\int_0^t \int_0^t |\theta|^{2(H - \frac{1}{2})}|\theta - \theta'|^{\epsilon - 1 - 2\beta} d\theta' d\theta < \infty,
\]
for every $t \in [0, T]$.

Altogether, taking a suitable $\beta$ so that (5.11), (5.14) and (5.14) are finite, we have
\[
\sup_{\epsilon > 0} \int_0^t \int_0^t E[\|D_\theta X^\epsilon_{\theta'} - D_\theta X^\epsilon_{\theta}\|^2] |\theta - \theta'|^{\epsilon - 1 - 2\beta} d\theta' d\theta < \infty.
\]

Similar computations show that
\[
\sup_{\epsilon > 0} \|D_\theta X^\epsilon_{\theta}\|_{L^2(\Omega \times [0,T] \times \mathbb{R}^d)} < \infty.
\]

**Corollary 5.7.** Let $\{X^\epsilon_t\}_{\epsilon > 0}$ be the family of approximating solutions of (5.1) in the sense of (5.3). Then for every $t \in [0, T]$ and bounded continuous function $h : \mathbb{R}^d \to \mathbb{R}$ we have
\[
h(X^n_t) \xrightarrow{n \to \infty} h(E[X_t | \mathcal{F}_t])
\]
strongly in $L^2(\Omega; \mathcal{F}_t)$. In addition, $E[X_t | \mathcal{F}_t]$ is Malliavin differentiable for every $t \in [0, T]$.

**Proof.** This is an immediate consequence of the relative compactness from Lemma 5.6 and by Lemma 5.5 we can identify the limit as being $E[X_t | \mathcal{F}_t]$ then the convergence holds for any bounded continuous functions as well. The Malliavin differentiability of $E[X_t | \mathcal{F}_t]$ is shown by taking $h = I_d$ and estimate (5.6) together with [47, Proposition 1.2.3].

Finally, we can prove the main result of this section.

**Proof of Theorem 5.1.** It remains to prove that $X_t$ is $\mathcal{F}_t$-measurable for every $t \in [0, T]$. It follows that there exists a strong solution in the usual sense that is Malliavin differentiable. Indeed, let $h$ be a globally Lipschitz continuous function, then by Corollary 5.7 we have that
\[
\varphi(X^{1/n}_t) \to \varphi(E[X_t | \mathcal{F}_t]), \quad P - a.s.
\]
as $n \to \infty$. 
On the other hand, by Lemma 5.5 we also have
\[ h(X_t^{1/n}) \to E[\varphi(X_t)|\mathcal{F}_t] \]
weakly in \( L^2(\Omega; \mathcal{F}_t) \) as \( n \to \infty \). By the uniqueness of the limit we immediately have
\[ h(E[X_t|\mathcal{F}_t]) = E[h(X_t)|\mathcal{F}_t], \quad P - a.s. \]
which implies that \( X_t \) is \( \mathcal{F}_t \)-measurable for every \( t \in [0, T] \).

Let us finally show that our strong solution has a continuous modification. We observe
\[ E[|X_t^x - X_s^x|^m] \leq C_{d,m}(E[(\int_s^t \delta_0(X_u^x)du)^m] + E[|B_t^H - B_s^H|^m]) \]
\[ \leq C_{d,m}(E[(\int_s^t \delta_0(X_u^x)du)^m] + |t-s|^{mH}). \]

On the other hand, we have
\[ E[(\int_s^t \delta_0(Y_u^x)du)^m] \leq E[(\int_s^t \delta_0(B_u^H + x)du)^{2m}]^{1/2} E[X^x_2]^{1/2}, \]
where \( X \) is the Radon-Nikodym derivative as constructed in Proposition 5.4. Further, we know from (4.2) for a similar estimate that
\[ E[(\int_s^t \delta_0(B_u^H + x)du)^{2m}]^{1/2} \leq C_{d,m,H} |t-s|^{m(1-Hd)} \]
So
\[ E[|X_t^x - X_s^x|^m] \leq C(|t-s|^{m(1-Hd)} + |t-s|^{mH}), s \leq t, m \geq 1, \]
which entails by Kolmogorov’s Lemma the existence of a continuous modification of \( X^x \).

\textbf{Proof of Proposition 5.2.} Denote by \( Y \) the \( L^p \)-limit of the Doleans-Dade exponentials. Using characteristic functions combined with Novikov’s condition, we see that
\[ Y_t^x = B_t^H + L_t(Y^x)1_d \]
is a fractional Brownian motion under a change of measure with respect to the density \( Y \). The latter enables us to proceed similarly to arguments in the proof of Lemma 5.5 and to verify that
\[ E[Y_t^x \exp\{\sum_{j=1}^k (\alpha_j, B^H_{t_j} - B^H_{t_{j-1}})\}] = E[X_t^x \exp\{\sum_{j=1}^k (\alpha_j, B^H_{t_j} - B^H_{t_{j-1}})\}] \]
for all \( \{\alpha_j\}_{j=1}^k \subset \mathbb{R}^d, 0 = t_0 < ... < t_k = t, k \geq 1 \), where \( X^x \) denotes the constructed strong solution of our main theorem. This allows us conclude that both solutions must coincide a.e. \( \square \)
Appendix A. Technical Results

The following result which is due to [13, Theorem 1] provides a compactness criterion for subsets of $L^2(\Omega)$ using Malliavin calculus.

**Theorem A.1.** Let $\{\Omega, A, P; H\}$ be a Gaussian probability space, that is $(\Omega, A, P)$ is a probability space and $H$ a separable closed subspace of Gaussian random variables of $L^2(\Omega)$, which generate the $\sigma$-field $A$. Denote by $D$ the derivative operator acting on elementary smooth random variables in the sense that

$$D(f(h_1, \ldots, h_n)) = \sum_{i=1}^n \partial_i f(h_1, \ldots, h_n)h_i, \quad h_i \in H, \ f \in C^\infty_b(\mathbb{R}^n).$$

Further let $\mathbb{D}^{1,2}$ be the closure of the family of elementary smooth random variables with respect to the norm

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|DF\|_{L^2(\Omega; H)}.$$

Assume that $C$ is a self-adjoint compact operator on $H$ with dense image. Then for any $c > 0$ the set

$$G = \{G \in \mathbb{D}^{1,2} : \|G\|_{L^2(\Omega)} + \|C^{-1}DG\|_{L^2(\Omega; H)} \leq c\}$$

is relatively compact in $L^2(\Omega)$.

In order to formulate compactness criteria useful for our purposes, we need the following technical result which also can be found in [13].

**Lemma A.2.** Let $v_s, s \geq 0$ be the Haar basis of $L^2([0, T])$. For any $0 < \alpha < 1/2$ define the operator $A_\alpha$ on $L^2([0, T])$ by

$$A_\alpha v_s = 2^{k\alpha} v_{s}, \text{ if } s = 2^k + j$$

for $k \geq 0, 0 \leq j \leq 2^k$ and

$$A_\alpha 1 = 1.$$

Then for all $\beta$ with $\alpha < \beta < (1/2)$, there exists a constant $c_1$ such that

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0,T])} + \left( \int_0^T \int_0^T \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} \, dt \, dt' \right)^{1/2} \right\}.$$ 

A direct consequence of Theorem A.1 and Lemma A.2 is now the following compactness criteria.

**Corollary A.3.** Let a sequence of $\mathcal{F}_T$-measurable random variables $X_n \in \mathbb{D}^{1,2}, \ n = 1, 2, \ldots,$ be such that there exists a constant $C > 0$ with

$$\sup_n E[|X_n|^2] \leq C,$$

$$\sup_n E\left[\|D_1 X_n\|_{L^2([0,T])}^2\right] \leq C$$

and there exists a $\beta \in (0, 1/2)$ such that

$$\sup_n \int_0^T \int_0^T E\left[\|D_1 X_n - D_1 V_n\|^2\right] \frac{dt \, dt'}{|t - t'|^{1+2\beta}} < \infty.$$
such that
\[
\int_0^t \int_0^t \frac{|K_H(t, \theta') - K_H(t, \theta)|^2}{|\theta' - \theta|^{1+2\beta}} d\theta d\theta' < \infty. \tag{A.1}
\]

**Proof.** Let \( \theta, \theta' \in [0, t] \), \( \theta' < \theta \) be fixed. Write
\[
K_H(t, \theta) - K_H(t, \theta') = c_H \left[ f_1(\theta) - f_1(\theta') + \left( \frac{1}{2} - H \right) (g_1(\theta) - g_1(\theta')) \right],
\]
where \( f_1(\theta) := \left( \frac{1}{\theta} \right)^{H-\frac{1}{2}} (t-\theta)^{H-\frac{1}{2}} \) and \( g_1(\theta) := \int_0^t \frac{f_1(\theta)}{u} du, \theta \in [0, t] \).

We will proceed to estimating \( K_H(t, \theta) - K_H(t, \theta') \). First, observe the following fact,
\[
\frac{y^{-\alpha} - x^{-\alpha}}{(x-y)^\gamma} \leq C y^{-\alpha-\gamma}
\]
for every \( 0 < y < x < \infty \) and \( \alpha := \left( \frac{1}{2} - H \right) \in (0, 1/2) \) and \( \gamma < \frac{1}{2} - \alpha \). This implies
\[
f_1(\theta) - f_1(\theta') = \left( \frac{t}{\theta} (t-\theta) \right)^{H-\frac{1}{2}} - \left( \frac{t}{\theta'} (t-\theta') \right)^{H-\frac{1}{2}}
\leq C \left( \frac{t}{\theta} (t-\theta) \right)^{H-\frac{1}{2} - \gamma} t^{2\gamma} \frac{(\theta - \theta')^\gamma}{(\theta')^\gamma}
\leq C \frac{(\theta - \theta')^\gamma}{(\theta')^\gamma} (t-\theta)^{H-\frac{1}{2} - \gamma}
\leq C \frac{(\theta - \theta')^\gamma}{(\theta')^\gamma} \theta^{H-\frac{1}{2} - \gamma}(t-\theta)^{H-\frac{1}{2} - \gamma}.
\]

Further,
\[
g_1(\theta) - g_1(\theta') = \int_\theta^t \frac{f_u(\theta) - f_u(\theta')}{u} du - \int_\theta^{\theta'} \frac{f_u(\theta')}{u} du
\leq \int_\theta^t \frac{f_u(\theta) - f_u(\theta')}{u} du
\leq C \frac{(\theta - \theta')^\gamma}{(\theta')^\gamma} \int_\theta^t \frac{(u-\theta)^{H-\frac{1}{2} - \gamma}}{u} du
\leq C \frac{(\theta - \theta')^\gamma}{(\theta')^\gamma} \theta^{H-\frac{1}{2} - \gamma} \int_1^\infty \frac{(u-1)^{H-\frac{1}{2} - \gamma}}{u} du
\leq C \frac{(\theta - \theta')^\gamma}{(\theta')^\gamma} \theta^{H-\frac{1}{2} - \gamma}
\leq C \frac{(\theta - \theta')^\gamma}{(\theta')^\gamma} \theta^{H-\frac{1}{2} - \gamma}(t-\theta)^{H-\frac{1}{2} - \gamma}.
\]
As a result, we have for every $\gamma \in (0, H), \ 0 < \theta' < \theta < t < T$,
\[
(K_H(t, \theta) - K_H(t, \theta'))^2 \leq C_{H,T} (\theta - \theta')^{2\gamma} \theta^{2H-1-2\gamma}(t - \theta)^{2H-1-2\gamma},
\]
for some constant $C_{H,T} > 0$ depending only on $H$ and $T$.

Thus
\[
\int_0^t \int_0^{\theta} \frac{(K_H(t, \theta) - K_H(t, \theta'))^2}{|\theta - \theta'|^{1+2\beta}} d\theta' d\theta \\
\leq C \int_0^t \int_0^{\theta} \frac{|\theta - \theta'|^{1-2\beta+2\gamma}}{(\theta')^{2\gamma}} \theta^{2H-1-2\gamma}(t - \theta)^{2H-1-2\gamma} d\theta' d\theta \\
= C \int_0^t \theta^{2H-1-4\gamma}(t - \theta)^{2H-1-2\gamma} \int_0^{\theta} |\theta - \theta'|^{1-2\beta+2\gamma} (\theta')^{-2\gamma} d\theta' d\theta \\
= C \int_0^t \theta^{2H-1-4\gamma} (t - \theta)^{2H-1-2\gamma} \Gamma(-2\beta + 2\gamma) \Gamma(-2\beta + 1) \theta^{-2\beta} d\theta \\
\leq C \int_0^t \theta^{2H-1-4\gamma} (t - \theta)^{2H-1-2\gamma} d\theta \\
= C \frac{\Gamma(2H-2\gamma) \Gamma(2H-4\gamma-2\beta)}{\Gamma(4H-6\gamma-2\beta)} t^{4H-6\gamma-2\beta-1} < \infty,
\]
for appropriately chosen small $\gamma$ and $\beta$.

On the other hand, we have that
\[
\int_0^t \int_0^{\theta} \frac{(\tilde{K}_H(t, \theta) - \tilde{K}_H(t, \theta'))^2}{|\theta - \theta'|^{1+2\beta}} d\theta' d\theta \\
\leq C \int_0^t \theta^{2H-1-4\gamma} (t - \theta)^{2H-1-2\gamma} \int_0^{\theta} |\theta - \theta'|^{1-2\beta+2\gamma} (\theta')^{-2\gamma} d\theta' d\theta \\
\leq C \int_0^t \theta^{2H-1-6\gamma} (t - \theta)^{2H-1-2\gamma} \int_0^{\theta} |\theta - \theta'|^{1-2\beta+2\gamma} d\theta' d\theta \\
= C \int_0^t \theta^{2H-1-6\gamma} (t - \theta)^{2H-1-2\beta} d\theta \\
\leq Ct^{4H-6\gamma-2\beta-1}.
\]

Hence
\[
\int_0^t \int_0^{\theta} \frac{(\tilde{K}_H(t, \theta) - \tilde{K}_H(t, \theta'))^2}{|\theta - \theta'|^{1+2\beta}} d\theta' d\theta < \infty.
\]

\[\square\]

Lemma A.5. If $H < \frac{1}{2(1+d)}$ we have that
\[
I := (2m)! \int_{\mathbb{T}_{2m}(0,1)} \prod_{j=1}^{2m} \gamma^{-\frac{1}{2} - H, \frac{1}{2} - H} (1, u_j) \left( \det \text{Cov} \left( B_{u_1}^{H,1}, \ldots, B_{u_{2m}}^{H,1} \right) \right)^{-\frac{d}{2}} du \leq C_{H,d}^m (m!)^{2H(1+d)},
\]
for some constant $C_{H,d}$ depending only on $H$ and $d$. 
Proof. We have that

\[ I = (2m)! \int_{T_{2m}(0,1)} \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( u_j \right) \left( \text{det Cov} \left( B_{u_1}^{H,1}, \ldots, B_{u_{2m}}^{H,1} \right) \right)^{-\frac{d}{2}} du \]

\[ \leq (2m)! \int_{T_{2m}(0,1)} \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( u_j \right) \left( u_j - u_{j-1} \right)^{-Hd} du \]

\[ \leq (2m)! \int_0^1 \int_0^{u_{2m}} \cdots \int_0^{u_3} \left( \gamma_H \left( 1, u_2 \right) \prod_{j=3}^{2m} \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( u_j \right) \left( u_j - u_{j-1} \right)^{-Hd} \right) \]

\[ \times \left( \int_0^{u_2} \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( u_1 \right) u_1^{-Hd} \left( u_2 - u_1 \right)^{-Hd} du_1 \right) du_2 \cdots du_{2m}. \]

The inner integral can be bounded by

\[ \int_0^{u_2} \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( u_1 \right) u_1^{-Hd} \left( u_2 - u_1 \right)^{-Hd} du_1 \]

\[ = u_2^{\frac{3}{2} - 2Hd - H} \int_0^1 \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( 1, u_1 \right) \left( 1 - u_2 u_1 \right)^{-\frac{1}{2} - H} du_1 \]

\[ \leq u_2^{\frac{3}{2} - 2Hd - H} \int_0^1 \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( 1, u_1 \right) du_1 \]

\[ = u_2^{\frac{3}{2} - 2Hd - H} B \left( \frac{1}{2} - H \left( 1 + d \right), \frac{3}{2} - H \left( 1 + d \right) \right), \]

where we have used that \( (1 - u_2 u_1)^{-\frac{1}{2} - H} \leq (1 - u_1)^{-\frac{1}{2} - H} \). Hence,

\[ I \leq (2m)! \int_0^1 \int_0^{u_{2m}} \cdots \int_0^{u_3} \left( \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( u_1 \right) \prod_{j=3}^{2m} \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( u_j \right) \left( u_j - u_{j-1} \right)^{-Hd} \right) \]

\[ \times \left( \int_0^{u_3} \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( u_2 \right) \left( u_3 - u_2 \right)^{-Hd} du_2 \right) du_3 \cdots du_{2m} \]

\[ \times B \left( \frac{1}{2} - H \left( 1 + d \right), \frac{3}{2} - H \left( 1 + d \right) \right). \]

The inner integral can be bounded by

\[ \int_0^{u_3} \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( u_2 \right) \left( u_3 - u_2 \right)^{-Hd} du_2 \]

\[ = u_3^{\frac{3}{2} - 3Hd - 2H} \int_0^1 \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( 1, u_2 \right) \left( 1 - u_3 u_2 \right)^{-\frac{1}{2} - H} du_2 \]

\[ \leq u_3^{\frac{3}{2} - 3Hd - 2H} \int_0^1 \gamma_{\gamma - \frac{1}{2} - H, \frac{1}{2} - H} \left( 1, u_2 \right) du_2 \]

\[ = u_3^{\frac{3}{2} - 3Hd - 2H} B \left( \frac{1}{2} - H \left( 1 + d \right), 3 - 2H \left( 1 + d \right) \right). \]
and we get
\[
I \leq (2m)! \prod_{j=1}^{2m} \mathcal{B}\left(\frac{1}{2} - H (1 + d), j \left(\frac{3}{2} - H (1 + d)\right)\right)
\]
\[
= (2m)! \prod_{j=1}^{2m} \frac{\Gamma\left(\frac{1}{2} - H (1 + d)\right) \Gamma\left(j\left(\frac{3}{2} - H (1 + d)\right)\right)}{\Gamma\left(\frac{1}{2} - H (1 + d) + j\left(\frac{3}{2} - H (1 + d)\right)\right)}
\]
\[
= \left(\Gamma\left(\frac{1}{2} - H (1 + d)\right)\right)^{2m} \frac{\Gamma\left(\frac{3}{2} - H (1 + d)\right) (2m)!}{\Gamma\left(\frac{1}{2} - H (1 + d) + 2m\left(\frac{3}{2} - H (1 + d)\right)\right)}
\]
\[
\times \prod_{j=1}^{2m-1} \frac{\Gamma\left(1 + \frac{1}{2} - H (1 + d) + j\left(\frac{3}{2} - H (1 + d)\right)\right)}{\Gamma\left(\frac{1}{2} - H (1 + d) + j\left(\frac{3}{2} - H (1 + d)\right)\right)}
\]
\[
= \left(\Gamma\left(\frac{1}{2} - H (1 + d)\right)\right)^{2m} \frac{\Gamma\left(\frac{3}{2} - H (1 + d)\right) (2m)!}{\Gamma\left(\frac{1}{2} - H (1 + d) + 2m\left(\frac{3}{2} - H (1 + d)\right)\right)}
\]
\[
\times \prod_{j=1}^{2m-1} \left(\frac{1}{2} - H (1 + d) + j\left(\frac{3}{2} - H (1 + d)\right)\right)
\]
\[
= \left(\Gamma\left(\frac{1}{2} - H (1 + d)\right)\right)^{2m} \frac{\Gamma\left(\frac{3}{2} - H (1 + d)\right) \Gamma\left(2m + 1\right)}{\Gamma\left(-\left(\frac{1}{2} + H (1 + d)\right) + m\left(3 - 2H (1 + d)\right) + 1\right)}
\]
\[
\times \left(\frac{2}{3 - 2H (1 + d)} + 2m + 1\right)^{2m-1} \frac{\Gamma\left(-\left(\frac{1}{2} + H (1 + d)\right) + m\left(3 - 2H (1 + d)\right) + 1\right)}{\Gamma\left(\frac{4\left(1 - H (1 + d)\right)}{3 - 2H (1 + d)}\right)}.
\]

Next, taking into account the following asymptotics, see Wendel [56],
\[
\Gamma (m + \lambda) \sim m^\lambda \Gamma (m),
\]
\[
\Gamma (\lambda m + 1) \sim \lambda^{\frac{1}{2}} \left(2\pi\right)^{-\frac{1}{2}} \lambda^{\frac{1}{2}} m^{-\frac{1}{2}} (m!)^{\lambda},
\]
we get that
\[
\Gamma (2m + 1) \sim 2^m \left(2\pi\right)^{-\frac{1}{2}} 4^m m^{-\frac{1}{2}} (m!)^2,
\]
\[
\Gamma \left(-\left(\frac{1}{2} + H (1 + d)\right) + m\left(3 - 2H (1 + d)\right) + 1\right) \sim (m\left(3 - 2H (1 + d)\right) + 1)^{-\left(\frac{1}{2} + H (1 + d)\right)}
\]
\[
\times \Gamma \left(m\left(3 - 2H (1 + d)\right) + 1\right).
\]
Lemma A.6. Assume that Lemma A.6.

\[ \Gamma \left( \frac{2}{3 - 2H(1 + d)} + 2m + 1 \right) \sim C_{H,d} K_{H,d}^m (m) \frac{2}{(m!)^{3 - 2H(1 + d)}}, \]

which yields

\[ \frac{\Gamma (2m + 1) \Gamma \left( \frac{2}{3 - 2H(1 + d)} + 2m + 1 \right)}{\Gamma \left( \frac{2}{3 - 2H(1 + d)} + 2m + 1 \right) \Gamma (2m + 1)} \sim C_{H,d}'' (K_{H,d}'')^m (m) \frac{1 + 2H(1 + d)}{2H(1 + d)} (m!)^{2H(1 + d)}, \]

and the result follows.

The next auxiliary result can be found in [35].

**Lemma A.7.** Assume that \( X_1, \ldots, X_n \) are real centered jointly Gaussian random variables, and \( \Sigma = (E[X_j X_k])_{1 \leq j, k \leq n} \) is the covariance matrix, then

\[ E[|X_1| \ldots |X_n|] \leq \sqrt{\text{perm}(\Sigma)}, \]

where \( \text{perm}(A) \) is the permanent of a matrix \( A = (a_{ij})_{1 \leq i, j \leq n} \) defined by

\[ \text{perm}(A) = \sum_{\pi \in S_n} \prod_{j=1}^n a_{j,\pi(j)} \]

for the symmetric group \( S_n \).

The next result corresponds to Lemma 3.19 in [11]:

**Lemma A.8.** Let \( Z_1, \ldots, Z_n \) be mean zero Gaussian variables which are linearly independent. Then for any measurable function \( g : \mathbb{R} \to \mathbb{R}_+ \) we have that

\[ \int_{\mathbb{R}^n} g(v_1) \exp \left( -\frac{1}{2} \text{Var}[\sum_{j=1}^n v_j Z_j] \right) dv_1 \ldots dv_n = \frac{(2\pi)^{(n-1)/2}}{(\det \text{Cov}(Z_1, \ldots, Z_n))^{1/2}} \int_{\mathbb{R}} g \left( \frac{v}{\sigma_1} \right) \exp \left( -\frac{1}{2} v^2 \right) dv, \]

where \( \sigma_1^2 := \text{Var}[Z_1 | Z_2, \ldots, Z_n] \).

The following Lemma is Lemma A.5 in [7]:

**Lemma A.9.** Let \( H \in (0, 1/2), \theta, t \in [0, T], \theta < t \) and \( (\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m \) be fixed. Assume \( w_j + (H - \frac{1}{2} - \gamma) \varepsilon_j > -1 \) for all \( j = 1, \ldots, m \). Then exists a finite constant \( C = C(H,T) > 0 \) such that

\[ \int_{\Delta_{\theta, t}^m} \prod_{j=1}^m (K_H(s_j, \theta) - K_H(s_j, \theta'))^{\varepsilon_j} |s_j - s_{j-1}|^{w_j} ds \]

\[ \leq C^m \left( \frac{\theta - \theta'}{\theta \theta'} \right)^{\gamma \sum_{j=1}^m \varepsilon_j} \left( H - \frac{1}{2} - \gamma \right)^{\sum_{j=1}^m \varepsilon_j} \Pi_\gamma(m) (t - \theta) \sum_{j=1}^m w_j + (H - \frac{1}{2} - \gamma) \sum_{j=1}^m \varepsilon_j + m \]

for \( \gamma \in (0, H) \), where

\[ \Pi_\gamma(m) := \prod_{j=1}^{m-1} \frac{\Gamma \left( \sum_{l=1}^j w_l + (H - \frac{1}{2} - \gamma) \sum_{l=1}^j \varepsilon_l + j \right) \Gamma (w_{j+1} + 1)}{\Gamma \left( \sum_{l=1}^{j+1} w_l + (H - \frac{1}{2} - \gamma) \sum_{l=1}^j \varepsilon_l + j + 1 \right)}. \]
Observe that if $\varepsilon_j = 0$ for all $j = 1, \ldots, m$ we obtain the classical formula.

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