Vacuum Plane Waves in 4+1 D
and
Exact solutions to Einstein’s Equations in 3+1 D

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March 1, 2022

Abstract

In this paper we derive homogeneous vacuum plane-wave solutions to Einstein’s field equations in 4+1 dimensions. The solutions come in five different types of which three generalise the vacuum plane-wave solutions in 3+1 dimensions to the 4+1 dimensional case. By doing a Kaluza-Klein reduction we obtain solutions to the Einstein-Maxwell equations in 3+1 dimensions. The solutions generalise the vacuum plane-wave spacetimes of Bianchi class B to the non-vacuum case and describe spatially homogeneous spacetimes containing an extremely tilted fluid. Also, using a similar reduction we obtain 3+1 dimensional solutions to the Einstein equations with a scalar field.

1 Introduction

Recently a full list of spatially homogeneous spacetimes in 4+1 dimensions was given [1]. Also, for the spacetimes with simply transitive spatial hypersurfaces, all the equations of motion were written down. In this paper we will use these equations and solve them for an important class of spacetimes: vacuum plane-wave spacetimes possessing a covariantly constant null Killing vector. These spacetimes – which have been studied in the literature lately (see e.g. [2,3]) – have many interesting properties. For example, the covariantly constant null vector $k^\nu$ satisfies

$$C_{\alpha\beta\mu\nu}k^\nu = 0, \quad k^\nu k_\nu = 0,$$

(1)

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where $C_{\alpha\beta\mu\nu}$ is the Weyl tensor; thus they are also algebraically special.\footnote{The 4+1 dimensional plane-wave are also algebraically special; of type 1111 according to De Smet’s scheme \cite{DeSmet}. However, it should be noted that this classification needs further refinement due to some pathologies.}

In 3+1 dimensional cosmology algebraically special solutions usually play a particular role; they are equilibrium points, or sometimes even attractors, for more general classes of solutions \cite{DeSmet,DeSmet2,DeSmet3}. In particular, the vacuum plane-wave solutions of dimension 3+1 are attractors for non-tilted\footnote{For tilted fluids the situation is still unsettled, but for a recent work on the futures of tilted Bianchi models, see \cite{Batic2019}.} non-inflationary models of Bianchi class B. Hence, general solutions of Einstein’s equations of Bianchi class B containing non-tilted non-inflationary perfect fluids can be approximated with vacuum plane-wave solutions at late times. For higher dimensional models the situation is still unsettled as no stability analysis has been done to date. However, if plane-wave solutions in higher dimensions play the same role as they do in 3+1 dimensional models then that would definitely increase the physical relevance of spacetimes of plane-wave type. This paper will provide a first step in such an analysis; we will identify them and describe them within the framework set in \cite{DeSmet}. A stability analysis for 4+1 dimensional vacuum plane-waves should then be straightforward (although it might be somewhat lengthy).

A different motivation for considering higher dimensional models comes from string theory \cite{string1,string2}. String theory requires higher dimensional models of spacetime and plane-waves spacetimes are perhaps the simplest spacetimes not being maximally symmetric. Yet, despite the fact that they are not maximally symmetric, the plane-wave spacetimes still admit supersymmetry which makes them so interesting for string theorists. Furthermore, the fact that all spacetimes have a plane wave as a limit \cite{DeSmet} has caught many theorists attention lately due to its implications for the AdS/CFT correspondence \cite{AdS/CFT}.

According to Kaluza-Klein theory \cite{Kaluza,Klein1,Klein2,Klein3}, the assumption of an extra small spatial dimension has an interesting consequence. From a 3+1 dimensional perspective two new fields will arise; a two-form field $F$ – which can be interpreted as an electromagnetic field tensor – and a scalar field $\phi$. The scalar essentially measures the size of the extra dimension – which has to be really small in order not to be detected experimentally – and couples to the kinetic part of the field $F$. This field is often called the dilaton and in the special case where the dilaton is constant, the theory reduces to ordinary Einstein-Maxwell theory.\footnote{It has also recently been suggested that higher dimensional models may explain the acceleration of the universe observed at the present time \cite{AcceleratingUniverse}.}

In this paper we show that this reduction is highly effective when it comes to generating solutions to 3+1 Einstein gravity. For Bianchi type models containing a tilted perfect fluid, there are not many exact self-similar solutions known. However, by using insight and knowledge of plane-wave solutions we can obtain vacuum plane-wave solutions in 4+1 dimensions. Upon reduction these solutions turn out to correspond to Bianchi type models of class B containing an extremely tilted fluid. The solutions describe plane-wave solutions in 3+1 dimensions containing a null electromagnetic field \cite{nullElectMagnetField}. Even though these solutions have been found before, the reduction procedure provides us with a geometrical interpretation of their nature; these plane-wave spacetimes with a
null electromagnetic field are nothing but vacuum plane-waves in one dimension higher. By studying the Einstein-Maxwell equations, this interpretation is well hidden and therefore not easily recognizable. Similarly, we will also consider a different special case of the Kaluza-Klein reduction. Assuming that the electromagnetic field vanishes, the Kaluza-Klein reduction results in Einstein gravity with a scalar field. The 3+1 dimensional scalar field solutions are conformally equivalent to Bianchi class B spacetimes.

This paper is organised as follows. In section 2 we review some properties of homogeneous plane-wave spacetimes and use these to simplify the equations of motion. In section 3 we list all the solutions obtained which describes vacuum plane-wave solutions in 4+1 dimensions. Then in section 4 we do a Kaluza-Klein reduction of the spacetime. This leads to exact solutions for the Einstein-Maxwell equations describing plane-wave spacetimes of Bianchi class B with an extremely tilted fluid and scalar field spacetimes conformally equivalent to Bianchi type B spacetimes.

2 Spatially homogeneous spacetimes

Our focus will be on spatially homogeneous spacetimes, thus we will assume that there exist a group \( G \) acting simply transitive on the spatial hypersurfaces \( \Sigma_t \). Following a previous paper \( \text{[1]} \), which was based on a procedure by Ellis and MacCallum \( \text{[21]} \) in the 3+1 dimensional case, we write the spacetime (at least locally) as a warped product

\[
M = \Sigma_t \times \mathbb{R}.
\] (2)

The time-direction \( u \) can be chosen to be orthogonal to the surfaces of transitivity. Choosing a spatial vierbein\(^6\) \( e_a \), we can form an orthonormal frame in spacetime \( \{e_a\} = \{u, e_a\} \). A particular useful choice of frame, is to choose \( \{e_a\} \) to be a left invariant frame. If the surface of transitivity is spanned by the Killing vectors \( \xi^b \), then a left invariant frame is obtained by Lie transport

\[
\mathcal{L}_{\xi^b} e_a = [\xi^b, e_a] = 0.
\] (3)

Note that we also have

\[
[\xi_b, u] = 0.
\] (4)

The corresponding dual one-forms \( \omega^\mu \) obey

\[
d\omega^k = -\frac{1}{2} C^k_{ij} \omega^i \wedge \omega^j
\] (5)

where the structure constants \( C^k_{ij} \) are constants on each orbit of transitivity.

We define the volume expansion tensor of the unit normal \( u \), equivalently the extrinsic curvature tensor \( \theta^\mu_\nu \) of the spatially homogeneous hypersurfaces, by

\[
u_{\mu;\nu} = \theta^\mu_\nu.
\] (6)

\(^5\)For a different approach to the simply transitive models, see \( \text{[22, 23]} \).
\(^6\)We will use the notation where Latin indices run over the spatial hypersurfaces while Greek indices run over the full spacetime manifold.
One can easily check that this tensor is symmetric and purely spatial. We split the expansion tensor into a trace part and a trace-free part

$$\theta_{ab} = \frac{1}{4} \theta h_{ab} + \sigma_{ab},$$

(7)

where $h_{ab}$ is the metric on the four-surfaces $\Sigma_t$, $\theta = \theta^\mu_\mu$ is the volume expansion scalar, and $\sigma_{ab}$ is the shear tensor.

Using this orthonormal frame formalism, one can derive all the equations of motion. These equations are derived in [1] and listed in the generic case.

The structure constants $C^{ik}_{ij}$ determine, via their Lie algebra, the isometry group of the spacetime. Hence, using one of the four-dimensional real Lie algebras, we can construct a spatially homogeneous spacetime [1]. For the plane-wave spacetimes, the most relevant Lie Algebras turn out to be $A_{4,n}$ where $n = 2, 3, 4, 5, 6$. The reason for this is, as we shall see later, that they are all non-unimodular rank 0 Lie algebras. Rank 0 means that these algebras have a 3 dimensional abelian Lie subalgebra. Thus they have three linearly independent vectors $X_A$, $A = 1, 2, 3$ for which all the commutators vanish identically

$$[X_A, X_B] = 0.$$  

(8)

This enables us to write the structure constants as a matrix

$$C^A_{B4} = \Theta^A_B.$$  

(9)

For all of the rank 0 Lie algebras, except for $A_{4,6}^{pq}$, this matrix can, with an appropriate orientation of the orthogonal frame, be put onto an upper-triangular form. In Table 1 the matrix $\Theta^A_B$ (in the particular gauge mentioned) are given for the types $A_{4,2}$ to $A_{4,5}$. The various invariant properties of this matrix determines the group type. The type $A_{4,6}^{pq}$ has two complex conjugate eigenvalues for the matrix $\Theta^A_B$, but can always be put onto the form

$$A_{4,6}^{pq}: \quad \Theta_A^B = \begin{bmatrix} \Theta_1^1 & \Theta_1^2 & \Theta_1^3 \\ 0 & \Theta_2^2 & \Theta_2^3 \\ 0 & \Theta_3^2 & \Theta_3^3 \end{bmatrix}, \quad \Theta_A^A = 3a$$

(10)

where $p\Theta_1^1 = q\Theta_2^2$ and $(\Theta_2^2)^2 = -q^2\Theta_3^2\Theta_3^3$.

The Lie algebra is unimodular if $\text{Tr}(\Theta_A^B) = 0$, and non-unimodular if $\text{Tr}(\Theta_A^B) \neq 0$. For non-unimodular Lie algebras we can define the vector

$$a_i \equiv \frac{1}{3} C^j_{ji},$$

(11)

which, according to the contracted Jacobi identity, obeys

$$a_i C^i_{jk} = 0.$$  

(12)

We will see that there exists only non-trivial plane-wave solutions for non-unimodular Lie algebras.

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7 We will use the notation in Patera et al [24, 25] for the enumeration of the 4-dimensional Lie algebras. See also MacCallum’s report [26].

8 A Lie group where every left-invariant vector field preserves volume is called unimodular.
2.1 Tracking down the plane waves

Homogeneous plane-wave spacetimes possessing a covariantly constant null Killing vector are spaces with higher symmetry than usual. We can use this fact to simplify the equations of motion and find a large number of plane-wave solutions. In fact, the technique leads us to three-parameter families of plane-wave solutions for each of the non-unimodular Lie algebras of rank 0 discussed in the previous section.

The plane-wave spacetime possesses three commuting Killing vectors $X_A$ spanning the wave front. Thus in a suitable chosen frame, the three left-invariant vectors $e_A$ will also be commuting. The fourth left-invariant vector $e_4$, has to be orthogonal to $e_A$ by the Jacobi identity and parallel to the vector $a_i$. We transform the orthonormal frame into a frame with two null vectors by the transformation

$$
\begin{bmatrix}
e_+ \\
e_-
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1
\end{bmatrix} \begin{bmatrix}e_0 \\
\eta^+
\end{bmatrix}, \quad \begin{bmatrix}\eta^- \\
\omega^0
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1
\end{bmatrix} \begin{bmatrix}e_4 \\
\omega^4
\end{bmatrix}.
$$

(13)

In this frame the metric can be written

$$
\text{ds}^2 = 2\eta^+\eta^- + \delta_{AB}\omega^A\omega^B
$$

(14)

For the plane-wave spacetime there does also exist a null Killing vector which is orthogonal to the three commuting Killing vectors spanning the wave front. Hence, $\xi = e^\chi e_-$, where $\chi$ is a scalar. The wave front is spanned by the three Killing vectors $X_A$, so $\xi$ must be constant along the wave front. This implies that $X_A(\chi) = 0$. Furthermore, we require that the null Killing vector must commute with the left-invariant vectors $e_A$; i.e.

$$
[\xi, e_A] = 0.
$$

(15)

Inserting the transformation eq. (13) implies the following restrictions on the structure constants of the orthonormal frame:

$$
C_{0A}^{\mu} = C_{4A}^{\mu}.
$$

(16)

Table 1: Structure constants for the Lie algebras $A_{4,2}$ to $A_{4,5}$. 

| Type  | Restrictions on $\Theta^A_B$ |
|-------|-----------------------------|
| $A_{4,5}$  | $p = 1$, $\Theta^1_1 > 0$, $p\Theta^1_1 = \Theta^2_2$, $q\Theta^1_1 = \Theta^3_3$ |
|        | $\Theta^1_2 = 0$ |
|        | $\Theta^2_3 = 0$ |
|        | $\Theta^1_2 = \Theta^2_3 = \Theta^1_3 = 0$ |
| $A_{4,2}$  | $p = 1$ |
|        | $\Theta^2_2 = \Theta^3_2 \neq 0$, $\Theta^1_1 = p\Theta^2_2$ |
|        | $\Theta^1_2 = 0$ |
| $A_{4,3}$  | $\Theta^2_2 = \Theta^3_2 = 0$, $\Theta^1_1 > 0$ |
| $A_{4,4}$  | $\Theta^1_1 = \Theta^2_2 = \Theta^3_3 = 0$ |

$$
\Theta^A_B = \begin{bmatrix}
\Theta^1_1 & \Theta^1_2 & \Theta^1_3 \\
0 & \Theta^2_2 & \Theta^3_3 \\
0 & 0 & \Theta^3_3
\end{bmatrix}, \quad \Theta^A_A = 3a.
$$
The requirement that the Killing vector $\xi$ is covariantly constant, together with the orthogonality condition imply that

$$C^0_{\ 0A} = C^0_{\ 4A} = C^4_{\ 0A} = C^4_{\ 4A} = C^4_{\ AB} = C^0_{\ AB} = 0.$$ \hspace{1cm} (17)

Hence, we get the following relation between the volume expansion tensor and the spatial structure constants

$$-\theta^4_B + \Omega^A_B = C^A_{\ 4B},$$ \hspace{1cm} (18)

Here, $\Omega_{ab}$ is the angular velocity in the $ab$-plane of a Fermi-propagated axis with respect to the triad $e_a$. By taking the trace of the above equation, we get

$$\theta - \theta^4_4 = 3a.$$ \hspace{1cm} (19)

We also have

$$C^4_{\ B4} = 0, \quad a_i = a\delta^4_i,$$ \hspace{1cm} (20)

so the only non-zero commutators are therefore

$$C^a_{\ 04}, \quad C^4_{\ B4} \equiv \Theta^A_{\ B}, \quad \Theta^4_4 = 3a.$$ \hspace{1cm} (21)

Thus we can restrict ourselves to study the types of rank 0 which we discussed in the previous section. Combining eqs. (18) and (21), the expansion tensor, the rotation tensor and the structure constants have to relate via

$$\theta^A_B - \Omega^A_B = \Theta^A_B.$$ \hspace{1cm} (22)

In solving the equations of motion, we have put $\Theta^A_B$ onto an upper-triangular form as explained earlier. This can be done for all the rank 0 types, except when $\Theta^A_B$ has two complex conjugate eigenvalues. When $\Theta^A_B$ is of this form, the rotation tensor $\Omega^A_B$ can easily be expressed in terms of the shear components and thus eliminated from the equations of motion.

3 Vacuum plane-wave solutions

Using the above procedure, the equations of motion reduce drastically, and the question of finding plane-wave solutions basically reduces to solving a very simple set of differential equations. First of all, the 5D Jacobi identity and the $R_{0a}$-equations (see eqs. (22) and (29) in [1]) imply the constraints $\theta^4_A = \Omega^4_A = 0$. The shear equations, using the constraint equations, all reduces to the same form,

$$\dot{\sigma}_{AB} + \theta\sigma_{AB} - 3a\sigma_{AB} = 0.$$ \hspace{1cm} (23)

The equation for $a$ is

$$\dot{a} + \frac{1}{3}a = 0.$$ \hspace{1cm} (24)

Hence, in principle, there are only two types of equations that needs to be solved, subject to the constraint equations.
In the following the solutions for the rank 0 non-unimodular Lie algebras are given. The metrics correspond to vacuum plane-wave solutions where the gravitational wave propagates in the \( w \)-direction. They all possess the covariantly constant null Killing vector
\[
\xi = e^{-t+w} \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial w} \right). 
\] (25)

These spacetimes are also self-similar; hence, they possess a homothetic vector field. This homothetic vector field is given by
\[
\xi_H = \frac{\partial}{\partial t} - \frac{\partial}{\partial w} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} 
\] (26)
as can be easily verified.

### 3.1 Type \( A_{4,5}^{pq} \)

Let \( \beta_+, \beta_-, Q_1, Q_2, Q_3 \) be free parameters such that
\[
8(\beta_+^2 + \beta_-^2) + \frac{2}{3}(Q_1^2 + Q_2^2 + Q_3^2) \leq 1. 
\] (27)

A five parameter set of plane-wave solutions can be given by (in upper triangular form)
\[
ds^2 = e^{2t}(-dt^2 + dw^2) + e^{2s(w+t)}
\times \left[ e^{-4\beta_+(w+t)} \left( dx + \frac{Q_1}{P_1} e^{P_1(w+t)} dy + \frac{Q_1 Q_3 + P_3 Q_2}{P_3 P_2} e^{P_2(w+t)} dz \right)^2 
+ e^{2(\beta_+ + \sqrt{3}\beta_-)(w+t)} \left( dy + \frac{Q_3}{P_3} e^{P_3(w+t)} dz \right)^2 
+ e^{2(\beta_+ - \sqrt{3}\beta_-)(w+t)} dz^2 \right] 
\] (28)

where
\[
s(1-s) = 2(\beta_+^2 + \beta_-^2) + \frac{1}{6}(Q_1^2 + Q_2^2 + Q_3^2) 
\]
\[
P_1 = 3\beta_+ + \sqrt{3}\beta_- 
\]
\[
P_2 = 3\beta_+ - \sqrt{3}\beta_- 
\]
\[
P_3 = -2\sqrt{3}\beta_- 
\] (29)

The group parameters are related to these parameters as follows
\[
p = \frac{s + (\beta_+ + \sqrt{3}\beta_-)}{s + (\beta_+ - \sqrt{3}\beta_-)} 
\]
\[
q = \frac{s - 2\beta_+}{s + (\beta_+ - \sqrt{3}\beta_-)}. 
\] (30)

The requirement that \( p \neq 1 \) leads to \( P_3 \neq 0 \) (see Table 1). Similarly, \( q \neq 1 \) and \( p \neq q \) lead to \( P_2 \neq 0 \) and \( P_1 \neq 0 \) respectively. If \( P_3 = 0 \) then we have to set \( Q_3 = 0 \) first.
Hence, two of the parameters define what group type the spacetime belongs to. For each group type we have a three-parameter family of plane-wave solutions (roughly given by the parameters $Q_1$, $Q_2$ and $Q_3$).

In the unimodular limit (where the trace of the commutators is zero) we have $s \to 0$. The only way to obtain $s = 0$ is when $\beta_\pm = Q_1 = 0$. Thus these plane-wave solutions approaches the Minkowski spacetime in this limit.

We can recover the $\text{VI}_h \oplus \mathbb{R}$ plane waves by requiring $q = 0$. Thus we set $s = 2\beta_+$ and the metric simplifies to

\[
\begin{align*}
\text{ds}^2 &= e^{2t}(-dt^2 + dw^2) + \\
&+ \left(dx + \frac{Q_1}{P_1}e^{P_1(w+t)}dy + \frac{Q_1Q_3 + P_3Q_2}{P_3P_2}e^{P_2(w+t)}dz\right)^2 \\
&+ e^{(3s+\sqrt{3}\beta_\pm)(w+t)}\left(dy + \frac{Q_3}{P_3}e^{P_3(w+t)}dz\right)^2 \\
&+ e^{(3s-\sqrt{3}\beta_\pm)(w+t)}dz^2.
\end{align*}
\]

Hence, in general we can have $\text{VI}_h \oplus \mathbb{R}$ plane-wave solutions where the extra dimension is tilted. This is exactly what will lead to the electromagnetic field when we reduce this metric to the 3+1 case. To recover the $\text{VI}_h \oplus \mathbb{R}$ plane waves where the extra dimension is trivially added, we must put $Q_1 = Q_2 = 0$.

### 3.2 Type $A_{4,2}^\mathbb{P}$

Let $\beta_+$, $Q_1$, $Q_2$, $Q_3$ be free parameters such that

\[
8\beta_+^2 + \frac{2}{3}(Q_1^2 + Q_2^2 + Q_3^2) \leq 1.
\]

A four parameter set of plane-wave solutions can be given by (in upper triangular form)

\[
\begin{align*}
\text{ds}^2 &= e^{2t}(-dt^2 + dw^2) + e^{2s(w+t)} \\
&\times \left[e^{-4\beta_+(w+t)}\left(dx + \frac{Q_1}{P_1}e^{3\beta_+(w+t)}dy + [A + B(w + t)]e^{3\beta_+(w+t)}dz\right)^2 \\
&+ e^{2\beta_+(w+t)}(dy + Q_3(w + t)dz)^2 \\
&+ e^{2\beta_+(w+t)}dz^2\right]
\end{align*}
\]

where $s$ and $P_i$ are given in eq. (29) with $\beta_- = 0$, and

\[
\begin{align*}
A &= \frac{3\beta_+Q_2 - Q_1Q_3}{9\beta_+^2} \\
B &= \frac{Q_1Q_3}{3\beta_+}.
\end{align*}
\]

The group parameter is given by

\[
p = \frac{s - 2\beta_+}{s + \beta_+}.
\]

In the limit where $p \to 0$ we end up in the decomposable case $\text{IV} \oplus \mathbb{R}$. Also in this case the plane-wave solutions describe in general a tilted extra dimension.
3.3 Type $A_{4,3}$

Plane-wave solutions for the Lie algebra type $A_{4,3}$ can be obtained by taking the $p \to \infty$ limit of $A_{p,2}^4$. In this limit we get $\beta_+ = -s$ and thus the metric can be written as

$$ds^2 = e^{2t}(-dt^2 + dw^2) + e^{6s(w+t)} \left( dx + \frac{Q_1}{P_1} e^{-3s(w+t)} dy + [A + B(w + t)] e^{-3s(w+t)} dz \right)^2 + (dy + Q_3(w + t) dz)^2 + dz^2$$

where

$$s = \frac{1}{6} \left( 1 \pm \sqrt{1 - 2(Q_1^2 + Q_2^2 + Q_3^2)} \right),$$

and $A, B$ are given in eq. (34) with $\beta_+ = -s$.

3.4 Type $A_{4,4}$

The parameters $Q_i$ are free parameters and $s$ is given in eq. (37) with $\beta_+ = 0$. There is a three-parameter set of plane-wave solutions given by

$$ds^2 = e^{2t}(-dt^2 + dw^2) + e^{2s(w+t)}$$

$$\times \left[ \left( dx + Q_1(w + t) dy + (w + t) \left[ Q_2 + \frac{Q_3}{2} (w + t) \right] dz \right)^2 + (dy + Q_3(w + t) dz)^2 + dz^2 \right].$$

In the limit where the trace of the structure constants go to zero, $a \to 0$, and thus $s \to 0$. Hence, we have $Q_i = 0$ and we recover flat five-dimensional Minkowski space. $Q_i = 0$ yields also the possibility $s = 1$. In that case the above metric reduces to the five-dimensional Milne universe.

3.5 Type $A_{4,6}^{pq}$

Let $\beta_+, \omega, \beta, Q_1, Q_2$ be free parameters and let $s$ be given by

$$s(1 - s) = 2\beta_+^2 + \frac{2}{3} \omega^2 \sinh^2 2\beta + \frac{1}{6}(Q_1^2 + Q_2^2).$$

Define also the two one-forms

$$\omega^2 = \cos[\omega(w + t)] dy - \sin[\omega(w + t)] dz$$

$$\omega^3 = \sin[\omega(w + t)] dy + \cos[\omega(w + t)] dz.$$  

The plane-wave solutions of type $A_{4,6}^{pq}$ can now be written

$$ds^2 = e^{2t}(-dt^2 + dw^2) + e^{2s(w+t)}$$

$$\times \left[ e^{-4\beta_+(w+t)} \left( dx + e^{3\beta_+(w+t)} \left( q_1 e^{-\beta_+} \omega^3 - q_2 e^{\beta_+} \omega^2 \right) \right)^2 + e^{2\beta_+(w+t)} \left( e^{-2\beta_+} (\omega^2)^2 + e^{2\beta_+} (\omega^3)^2 \right) \right].$$
where
\[ q_1 = \frac{Q_1 \omega + 3 \beta_+ Q_2 e^{2\beta}}{\omega^2 + 9 \beta_+^2} \]
\[ q_2 = \frac{Q_2 \omega - 3 \beta_- Q_1 e^{-2\beta}}{\omega^2 + 9 \beta_-^2}. \] (42)

The group parameters are related to these constants via
\[ p = \frac{\beta_+ (s - 2 \beta_+)}{\omega (s + \beta_+)} \]
\[ q = \frac{\beta_+}{\omega}. \] (43)

In the VII\(_h \oplus \mathbb{R}\) limit, we have \(s = 2 \beta_+\). The spacetime has in general a tilted extra dimension and we will recover the VII\(_h\) plane waves only if we simultaneously set \(Q_1 = Q_2 = 0\).

### 4 Kaluza-Klein reduction

Kaluza-Klein theory considers a five-dimensional spacetime which possesses (at least) one spatial Killing vector \(\xi\). If, say, this Killing vector has the corresponding left-invariant form \(dx\), when the metric can be written as
\[ ds^2 = g_{\mu\nu} \omega^\mu \omega^\nu + \phi^2 (dx + A_\mu \omega^\mu)^2. \] (44)

The Kaluza-Klein reduction can now be performed if we identify the space under a translation in the \(\xi\)-direction. The fifth dimension is assumed to be small and its size will in general define the electromagnetic coupling constant. Also, the field
\[ A = A_\mu \omega^\mu \] (45)

can be interpreted as a vector potential for an electromagnetic field. Geometrically this vector potential arises whenever the Killing vector \(\xi\) is not orthogonal to the reduced four-dimensional spacetime. This vector potential is related to the electromagnetic field strength, \(F\), in the usual manner,
\[ F = dA. \] (46)

In general we will have a non-constant dilaton field \(\phi\) which couples to the electromagnetic field in a way not standard in Einstein gravity. However, we will consider two special cases, both of which can be interpreted within the standard Einstein gravity.

#### 4.1 Exact solutions to the Einstein-Maxwell equations

We have already a set of vacuum solutions in 4+1 dimensions. These must correspond to a set of solutions in 3+1 dimensions to the Einstein-Maxwell-dilatonic equations of motion; according to the Kaluza-Klein theory, it is only a matter of reinterpreting the solutions. In general we will obtain a dilatonic field
when we are compactifying along the $\partial\omega$ field. Hence, they are solutions to the Einstein-Maxwell equations. 

By inspection of the vacuum plane-wave solutions in 4+1 dimensions, we readily see that the constraint $\phi = constant$ can be fulfilled by requiring

$$s = 2\beta_+$$  \hspace{1cm} (47)

when we are compactifying along the $\frac{\partial}{\partial x}$ Killing vector. The compactification can in principle be along any of the Killing vectors spanning the wave-front\(^9\), but we see that up to permutations all the non-trivial solutions with $\phi = constant$ can be obtained this way. Hence, there is no loss of generality to assume that we compactify along $\frac{\partial}{\partial x}$.

Apart from the requirement $s = 2\beta_+$, there are no further restrictions on the parameters. Performing the reduction on $A_{\mu\nu}^{9}$, $A_{\mu\nu}^{11}$, and $A_{\mu\nu}^{9,11}$ (the others lead to trivial spacetimes) yield a Bianchi type IV, VI\(_h\), and VII\(_h\) spacetime, respectively. The spaces possess an electromagnetic two-form field (renaming the $w$ direction as $x^1$) given by

$$F = dA = e^{-t}(\omega^0 + \omega^1) \wedge (Q_1\omega^2 + Q_2\omega^3).$$ \hspace{1cm} (48)

Here, we have introduced an orthonormal frame $\omega^\mu$, so that

$$ds^2 = \eta_{\mu\nu}\omega^\mu \omega^\nu,$$ \hspace{1cm} (49)

and $\eta_{\mu\nu}$ is the Minkowski metric. The solutions are

$$IV:\begin{align*}
\omega^0 &= e^t dt \\
\omega^1 &= e^t dw \\
\omega^2 &= e^{(w+t)}[dy + Q_3(w+t)dz] \\
\omega^3 &= e^{(w+t)}dz \\
s(1-s) &= \frac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2)
\end{align*} \hspace{1cm} (50)

$$VI_h:\begin{align*}
\omega^0 &= e^{(s+b)(w+t)}[dy - \frac{Q_3}{2b}e^{-2b(w+t)}dz] \\
\omega^1 &= e^{(s-b)(w+t)}dz \\
s(1-s) &= b^2 + \frac{1}{4}(Q_1^2 + Q_2^2 + Q_3^2)
\end{align*} \hspace{1cm} (51)

$$VII_h:\begin{align*}
\omega^0 &= e^{(w+t)}e^{-\beta}\{\cos\omega(w+t)dy - \sin\omega(w+t)dz\} \\
\omega^1 &= e^{(w+t)}e^{\beta}\{\sin\omega(w+t)dy + \cos\omega(w+t)dz\} \\
s(1-s) &= \omega^2 \sinh^2 2\beta + \frac{1}{4}(Q_1^2 + Q_2^2).
\end{align*} \hspace{1cm} (52)

Note that all these solutions are self-similar, since they all possess the homothety

$$\xi_H = \frac{\partial}{\partial t} - \frac{\partial}{\partial w} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \hspace{1cm} (53)$$

\(^9\)We also have to require that the action of the Killing vector acts freely on spacetime, otherwise the resulting spacetime will have orbifold-singlarities.
One can readily verify that the source-free Maxwell equations
\[ \text{d} F = 0, \quad \text{d}^\dagger F = 0, \quad (54) \]
and the Einstein field equations are satisfied (for the specific choice of constants $16\pi G = c = c = 1$).

The type IV, VI$_h$, and VII$_h$ metrics are the same as those in [20] and generalise Harvey and Tsoubelis’ [27], Collins’ [28], and Lukash’s [29] vacuum plane-wave solutions, respectively.\(^{10}\) The source is an electromagnetic field which is of a very particular type. The electric and magnetic fields are (in the orthonormal frame)
\[ E_i = e^{-t}(0, -Q_1, -Q_2), \quad B_i = e^{-t}(0, -Q_2, Q_1). \quad (55) \]
Thus this is a null field where all invariants composed of the two-form field vanish\(^{12}\)
\[ F_{\mu\nu} F^{\mu\nu} = F_{\mu\nu} (\ast F)^{\mu\nu} = 0. \quad (56) \]
The energy density of the field is
\[ \rho_{EM} = \frac{1}{2}(E^2 + B^2) = e^{-2t}(Q_1^2 + Q_2^2). \quad (57) \]
These solutions describes plane-wave spacetimes with a plane-wave electromagnetic field propagating in it. Generally they are non-vacuum plane-wave solutions of Bianchi types IV, VI$_h$ and VII$_h$. They interpolate between a one-parameter\(^{13}\) family of electro-magneto-vacuum spacetimes of type V and the well known vacuum plane-wave solutions of type IV, VI$_h$ and VII$_h$.

Note also that the energy-momentum tensor is given by
\[ T_{\mu\nu} = \rho_{EM} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (58) \]
and hence, describes a universe containing a fluid with extreme tilt. According to an observer comoving with the homogeneous hypersurfaces, there will be a stream of photons in the e$_1$-direction. The momentum-flow is therefore null, and thus describes a fluid with extreme tilt.

We know that the vacuum plane-wave solutions in 3+1 dimensions are attractor solutions for a large class of spacetimes containing non-tilted perfect fluids \(^{17}\). However, what happens for spacetimes containing a tilted fluid is still uncertain. A recent work indicates that tilted fluids stiffer than radiation

\(^{10}\) Thanks to Professor MacCallum for bringing this paper to my attention.

\(^{11}\) Note that all these solutions are given in [30], but they are all given in the more general Brinkman form,
\[ ds^2 = 2dudv + H(u, x^i)du^2 + dx^i dx_i. \]
In the paper [20], Araujo and Skea used these exact solutions in Brinkman form and identified the ones admitting a simply transitive group of isometries.

\(^{12}\) $F_{\mu\nu} F^{\mu\nu} = 0$ is, in fact, a direct consequence of the requirement $\phi =$ constant.

\(^{13}\) These solutions are axisymmetric and hence one of the parameters can be gauged away by a rotation.
may be important for the late time behaviour of Bianchi cosmologies class B [9].
What is certainly the case is that these solutions provide us with examples of type IV and VII\(_h\) spacetimes which do not asymptote a vacuum plane-wave at late times. Hence, the vacuum solutions of Lukash cannot be global attractors not even within the Einstein-Maxwell field equations.

### 4.2 Exact solutions to the Einstein equations with a scalar field

There is another case where it is possible to interpret the solutions after the reduction procedure classically. Consider the case \(\phi\) non-constant. Let \(ds^2_5\) be a 5 dimensional vacuum solution and \(ds^2_4\) be the reduced 4 dimensional metric; i.e.

\[
\begin{align*}
    ds^2_5 &= ds^2_4 + \phi^2(dx + A_\mu \omega^\mu)^2 .
\end{align*}
\]

Then, if \(A = 0\), the 4 dimensional conformally related metric

\[
\begin{align*}
    ds^2_4 &= \phi ds^2_4 ,
\end{align*}
\]

will satisfy the 4D Einstein equations with a free scalar field. Alternatively, the scalar field can be thought of as a stiff fluid with equation of state \(\rho = \phi\).

Again one get three cases where the metrics are all conformally related to Bianchi type IV, VI\(_h\), and VII\(_h\) metrics. However, in general the conformal factor breaks the spatial homogeneity so the solutions are no longer spatially homogeneous. In all the cases, the requirement \(A = 0\) implies \(Q_1 = Q_2 = 0\). Using an orthonormal frame \(\omega^\mu\), the solutions are

- **CIV** :

\[
\begin{align*}
    \omega^0 &= e^{\Omega(w+t)+t}dt \\
    \omega^1 &= e^{\Omega(w+t)+t}dw \\
    \omega^2 &= e^{\Omega(2w+t)}[dy + Q_3(w+t)dz] \\
    \omega^3 &= e^{\Omega(2w+t)}dz \\
    s(1 + 2\Omega - s) &= 3\Omega^2 + \frac{1}{4}Q_3^2
\end{align*}
\]

- **CVI\(_h\)** :

\[
\begin{align*}
    \omega^0 &= e^{\Omega(w+t)+t}dt \\
    \omega^1 &= e^{\Omega(w+t)+t}dw \\
    \omega^2 &= e^{(s+b)(w+t)}[dy - \frac{Q_3}{2b}e^{-2b(w+t)}dz] \\
    \omega^3 &= e^{(s-b)(w+t)}dz \\
    s(1 + 2\Omega - s) &= 3\Omega^2 + b^2 + \frac{1}{4}Q_3^2
\end{align*}
\]

- **CVII\(_h\)** :

\[
\begin{align*}
    \omega^0 &= e^{\Omega(w+t)+t}dt \\
    \omega^1 &= e^{\Omega(w+t)+t}dw \\
    \omega^2 &= e^{\Omega(2w+t)}e^{-\beta} \{\cos[\omega(w+t)]dy - \sin[\omega(w+t)]dz\} \\
    \omega^3 &= e^{\Omega(2w+t)}e^\beta \{\sin[\omega(w+t)]dy + \cos[\omega(w+t)]dz\} \\
    s(1 + 2\Omega - s) &= 3\Omega^2 + \omega^2 \sinh^2 2\beta.
\end{align*}
\]

The scalar field is given by \(\varphi = -2\sqrt{3}\Omega(w + t)\), and the energy-momentum tensor has the same form as eq. \ref{eq:55}. Hence, the scalar field is extremely tilted.

These solutions can be interpreted as stiff fluid solutions which can be generated using the generation techniques described in Theorem 10.2 in [30]. They
reduce to the ordinary vacuum plane wave when $\Omega = 0$, and this is the only case they are spatially homogeneous. There are, however, always an abelian $\mathbb{R}^3$ acting simply transitive on the null hypersurfaces $w + t = \text{constant}$. The metrics are all conformally related to Bianchi types of class B as indicated in eq. (63).

What role these solutions may have for the evolution of generic cosmological models is unknown to the author.

5 Conclusion

We have found 5 classes of homogeneous vacuum plane-wave solutions in 4+1 dimensions. For each non-unimodular Lie algebra of rank 0, there is a three-parameter family of solutions to the Einstein field equations. An open question which we have not addressed here, but nonetheless would be very interesting to answer, is: Are these solutions the late time attractors for a more general set of models? We know that the 3+1 dimensional vacuum plane waves are late-time attractors for non-tilted perfect fluid cosmologies of Bianchi class B [5]. Therefore we might wonder if the 4+1 dimensional plane waves plays the same role in 4+1 dimensional cosmology? To this date, no such stability analysis has been done mostly because the lack of understanding of spatially homogeneous cosmological models of dimension 4+1. However, by the previous work [11] and this work, such a stability analysis seems to be within reach. Such an analysis will hopefully give us some hints whether 3+1 dimensional is special or if it has the same features as higher dimensional cosmology.

By doing a Kaluza-Klein reduction, we showed explicitly that some of these solutions correspond to the exact solutions of the Einstein-Maxwell equations found by Araujo and Skea [20] using a different type of analysis. These solutions describe spacetimes containing an extremely tilted fluid. The benefit of the Kaluza-Klein theory is therefore twofold; not only can we obtain exact solutions in 3+1 dimensions, but in doing so we also give a geometrical explanation of their nature. The obtained solutions of the Einstein-Maxwell equations are nothing but vacuum plane-waves in one dimension higher. Amazingly, this sets the 4+1 dimensional cosmology in an incredible context. The study of 4+1 dimensional cosmology may help us to understand 3+1 dimensional cosmology. In this context the plane-wave spacetimes in homogeneous form have a great advantage with respect to the more commonly used Brinkman form.

Also 3+1 dimensional solutions with a massless scalar field were derived using the Kaluza-Klein reduction. These solutions were not spatially homogeneous in general but they possessed three commuting Killing vectors acting simply transitively on null hypersurfaces. These are solutions that can be obtained by well known generation techniques [30].

Notwithstanding, this may indicate that cosmology in 3+1 dimensions may simplify and be a lot more comprehensible by increasing the dimensionality by one. Hence, the study of 4+1 dimensional models may be more down-to-Earth than we originally might have thought.
Acknowledgments

The author deeply acknowledges the insightful and useful comments made by S.T.C. Siklos, J.D. Barrow and A.A. Coley. Thanks also to James Lucietti for reading through the manuscript and making useful comments. This work was funded by the Research Council of Norway and an Isaac Newton Studentship.

References

[1] S. Hervik, *Class. Quant. Grav.* 19 (2002) 5409 (Preprint: gr-qc/0207079)
[2] M. Blau and M. O’Loughlin, hep-th/0212135
[3] D. Marolf and S.F. Ross, *Class. Quant. Grav.* 19 (2002) 6289
[4] P.-J. De Smet, *Class. Quant. Grav.* 19 (2002) 4877
[5] S.T.C. Siklos, *J. Phys. A: Math. Gen.* 14 (1981) 395
[6] J.D. Barrow and D.H. Sonoda, *Gen. Rel. Grav.* 17 (1985) 409
[7] J.D. Barrow and D.H. Sonoda, *Phys. Reports* 139 (1986) 1
[8] C.G. Hewitt and J. Wainwright in *Dynamical Systems in Cosmology*, eds: J. Wainwright and G.F.R. Ellis, Cambridge University Press (1997)
[9] J.D. Barrow and S. Hervik, *Class. Quantum Grav.* 20 (2003) PageNo.
[10] J. Polchinsky, *String Theory*, Cambridge University Press (1998)
[11] C.V. Johnson, *D-Branes*, Cambridge University Press (2003)
[12] R. Penrose, *Any space-time has a plane wave as a limit* in *Differential Geometry and Relativity*, Reitil, Dordrecht (1976)
[13] D. Berenstein, J. Maldacena, and H. Nastase, JHEP 0204 (2002) 013
[14] P.K. Townsend and M.N.R. Wohlfarth, hep-th/0303097
[15] T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Phys. Math.* K1 (1921) 966
[16] O. Klein, *Z. Phys.* 37 (1926) 875
[17] O. Klein, *Nature* 118 (1926) 516
[18] T. Appelquist, A. Chodos, and P.G.O. Freund, editors, *Modern Kaluza-Klein Theories*, Addison-Wesley (1987)
[19] C. Duval, G.W. Gibbons, and P. Horvathy, *Phys.Rev.* D43 (1991) 3907
[20] M.E. Araujo and J.E.F. Skea, *Class. Quantum Grav.* 5 (1988) 1073
[21] G.F.R. Ellis and M.A.H. MacCallum, *Comm. Math. Phys.* 12 (1969) 108
[22] T. Christodoulakis, G. O. Papadopoulos and A. Dimakis, *J.Phys.* A36 (2003) 427-442
[23] T. Christodoulakis, E. Korfiatis and G. O. Papadopoulos, Commun. Math. Phys. 226 (2002) 377

[24] J. Patera, R.T. Sharp and P. Winternitz, J. Math. Phys. 17 (1976) 986

[25] J. Patera and P. Winternitz, J. Math. Phys. 18 (1977) 1449

[26] M.A.H. MacCallum, *On the enumeration of the real four-dimensional Lie algebras* In A.L. Harvey, editor, On Einsteins’s Path: essays in honor of Engelbert Schucking, pages 299-317. Springer Verlag, New York (1999)

[27] A. Harvey and D. Tsoubelis, Phys. Rev. D15 (1977) 2734

[28] C.B. Collins, *PhD Thesis*, University of Cambridge (1972)

[29] V. Lukash, Sov. Phys. JETP 40 (1975) 792

[30] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein’s Field Equations, 2nd Ed*. Cambridge University Press (2003)