Unsolvability of the Halting Problem in Quantum Dynamics

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It is shown that the halting problem cannot be solved consistently in both the Schrödinger and Heisenberg pictures of quantum dynamics. The existence of the halting machine, which is assumed from quantum theory, leads into a contradiction when we consider the case when the observer’s reference frame is the system that is to be evolved in both pictures. We then show that in order to include the evolution of observer’s reference frame in a physically sensible way, the Heisenberg picture with time going backwards yields a correct description.

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With the construction of universal quantum Turing machine, Deutsch proposed a quantum version of the halting problem first proved by Alan Turing in 1936. In recent years, a lot of interest has been focused on quantum computation, and the discussion of the halting problem using a quantum computer has also received attention. Myers argued that due to entanglement between a halt qubit and a system, it may be difficult to measure the halt qubit, which may spoil the computation. Subsequent discussions on the halting problem with a quantum computer have mainly focused on the superposition and entanglement of the halt qubit.

In this paper, we approach the halting scheme differently and use two pictures of quantum dynamics, i.e., the Schrödinger and Heisenberg pictures. Schrödinger’s wave mechanics and Heisenberg’s matrix mechanics were formulated in the early twentieth century and have been considered to be equivalent, i.e., two different ways of describing the same physical phenomenon that we observe. Therefore, in order to consider a halting scheme for a quantum system, we need to examine whether the procedure is consistent in both the Schrödinger and Heisenberg pictures. We will give an example in quantum dynamics that shows this cannot be achieved. We will then argue that it is the Heisenberg picture, rather than both pictures, that yields the correct description that not only does not run into the inconsistency shown through the halting scheme but also is physically sensible.

In order to discuss the halting problem, we first wish to define notations to be used. In particular we will follow a similar notation used in such that it is convenient in both Schrödinger and Heisenberg pictures. A qubit, a basic unit of quantum information, is a two-level quantum system written as \( |\theta, \phi \rangle \). A unit vector \( \hat{v} \), where \( i = x, y, z \) (also see for a general transformation of a single qubit in a Bloch sphere). We will write the transformation of \( \hat{v} \) under the unitary operation \( U \) as \( \hat{v}' = U \hat{v} U^\dagger \), implying the unitary transformation is applied to the corresponding \( \sigma_i \).

For the remainder of this paper, we will treat the two vectors \( \hat{v} \) and \( \hat{e} \) \( \sigma_j \) by \( U^\dagger \sigma_j U \) for \( e_j \), which we represent as \( \hat{e}' = U^\dagger \hat{e} U \). As an example, we again consider the case when \( U \) is a rotation about \( y \)-axis by \( \alpha \) as follows \( \hat{e}' \equiv U^\dagger U \) \( \sigma_j \) is transformed into \( \hat{v}' = \langle x, y, z \rangle = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta) \), pointing \( (\vartheta, \varphi) \) direction, the observable would be \( \hat{e} \cdot \hat{\sigma} \). In the Heisenberg picture of quantum theory, it is the unit basis vector \( \hat{e} \) that is transformed (p243). Using a similar transformation rule as in \( \hat{v} \), a unitary transformation of the observable in the basis vector notation can be obtained by applying \( U^\dagger \) to the \( \sigma_j \) by \( U^\dagger \sigma_j U \) for \( e_j \) which we represent as \( \hat{e}' = U^\dagger \hat{e} U \).

With a quantum system and a halt qubit, Deutsch introduced a quantum version of the halting problem wherein the completion of every valid quantum algorithm through a unitary process applied to the quantum system is accompanied by the change in a halt qubit to 1 that remains 0 otherwise. We will assume such a halting machine exists and will argue that this assumption leads into a contradiction. With the introduced notations, we will consider one particular case of the halting machine,
Let us take an example of a one-dimensional line as shown in Fig. 2. In order to claim a dot, which is lying on the line, is either on the right or on the left, there should be a reference point. For example, with respect to the origin or with respect to +3, one may say the dot is on the left or on the right. Instead of looking at the line from outside, suppose there is an observer being confined to the one-dimensional line facing into the paper as shown in (C) of Fig. 2. The observer measures or perceives whether the dot is on the right or on the left. Depending on where the observer is sitting, the outcome of the measurement, i.e., either on the right or on the left, will change. In this case, we note that the observer him or herself is serving the role of the reference point. Therefore when the observer makes a measurement and gets a result that the dot is on the right or on the left, this implies that with respect to his or her reference frame of the position on the line, the dot is on the right or on the left. Let us apply the same logic to the case of a single qubit in a Bloch sphere. When an observer measures a qubit in a certain direction, say in \( \hat{n} \), the outcome of the measurement is either +1 or -1. The eigenvalue obtained is with respect to the measurement direction \( \hat{n} \). It is noted that \( \hat{n} \) is playing a similar role as the reference point in the case of the one-dimensional line example. We also note that the measurement outcome of +1 or -1 is the perception experienced by the observer. That is, it is the observer who obtains the outcome +1 or -1. Therefore, the outcome should be meaningful with respect to the observer’s certain reference frame. Because we already know that the eigenvalue outcome +1 or -1 is meaningful with respect to the measurement direction \( \hat{n} \), it leads us to consider the observer’s reference frame as \( \hat{n} \) for our single qubit measurement case. Using the unit vector notations we previously defined, we propose the following:

**Postulate I:** Given a unit vector \( \hat{\mathbf{v}} \), an observer’s reference frame is identified with a basis unit vector \( \hat{\mathbf{e}} \).

With this postulate, two pictures of quantum theory can have a natural physical realization between an observer and a system. Fig. 1 shows that, in the Schrödinger picture, the observer’s reference frame, represented by the unit basis vector \( \hat{\mathbf{e}} \), stays still while the state vector is rotated clockwise by \( \alpha \), and the Heisenberg picture shows the unit vector stays still and the observer’s coordinate is rotated counterclockwise by \( \alpha \). In both cases, the observer would observe exactly the same phenomenon.

It should be noted that we are not using a notion of detector or apparatus in the place of an observer. According to our postulate, for a given unit vector, the observer’s reference frame is represented with a unit basis vector in a Bloch sphere. However, it was shown that a finite dimensional detector cannot encode an arbitrary unitary transformation whereas, according to our postulate, the observer’s identified coordinate unit basis vector represents an arbitrary measurement basis for a given qubit. Therefore, we do not use the term detector or an apparatus to replace an observer. If one wants to
We now consider the halting machine in \( U \) with one particular input. That is, when the input state to be transformed is the Alice’s unit basis vector, i.e., \( \hat{v}_s = \hat{e}_s \). Note that we are treating \( \hat{v} \) and \( \hat{e} \) on a equal footing. In the Schrödinger picture, the evolution is, \( \hat{e}_s \rightarrow U_\delta \hat{e}_s U_\delta^\dagger \equiv \hat{e}_s'' \), and Alice also transforms \( \hat{v}_h \rightarrow -\hat{v}_h \). We now consider the same procedure in the Heisenberg picture. In this case, the unit basis vector \( \hat{e}_s \), is transformed as \( \hat{e}_s \rightarrow U_\delta \hat{e}_s U_\delta^\dagger \equiv \hat{e}_s'' \) and \( \hat{e}_h \rightarrow -\hat{e}_h \). Note that \( \hat{e}_s'' \neq \hat{e}_s'' \) unless \( \hat{e}_s = \pm \hat{n} \) or \( \delta = k\pi \) where \( k = 0, 1, 2, \ldots \). For the example of a system with an observer and the halting machine studied in the previous paragraph, the vector \( \hat{v}_s \) has evolved, with respect to \( \hat{e}_s \), into the same output in both Schrödinger and Heisenberg pictures. Similarly, with respect to \( \hat{e}_h \), the halt qubit, \( \hat{v}_h \), halted in both pictures. However, in the case with \( \hat{e}_s \) as an input we just considered, while the halt qubit \( \hat{v}_h \) halted on both occasions with respect to \( \hat{e}_h \), the vector that is being evolved, i.e., \( \hat{e}_s \), turned out as two generally different outputs in two pictures. This contradicts our assumption about the halting machine in \( U \) because the machine should yield an output that is a rotation of the input by \( \delta \) about a \( \hat{n} \)-axis and is unique.

Therefore, we have shown that the existence of the halting machine that is assumed from quantum theory leads into a contradiction when we consider the input of the unit basis vector \( \hat{e}_s \) (for simplicity, we will omit the subscript \( s \) from now on), which is transformed into two generally different outputs in Schrödinger and Heisenberg pictures. However, not only can the halting problem not be solved consistently in both pictures, but also the evolution of the unit basis vector \( \hat{e} \) is physically sensible in neither of the two pictures in quantum dynamics. With our first postulate, we were able to impose a physical meaning on the Schrödinger and Heisenberg pictures of quantum theory. That is, in case of the Schrödinger picture, the system is evolving while an observer’s reference frame is intact and, for the Heisenberg’s picture, an ob-

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**FIG. 2:** For (A), it is not possible to claim the black dot is on the right or on the left. In (B), we may say, with respect to the flag in +3, the dot is on the left. If we assume there is an observer living and sitting at +3 while facing into the paper (i.e., the same direction as the reader of this paper) as in (C) and if the observer measures and obtains the result that the dot is on the left, then it is the observer who is serving the role of the flag in (B), i.e., as a reference point.

include an apparatus or detector, we may consider the state, i.e., \( \hat{v} \), to be a larger system that includes a qubit and an apparatus and the coordinate vector for an observer would also be represented by the same larger basis vector. However, in this paper, we only consider the simplest possible case of a single qubit.

Let us now consider a system with an observer and the halting machine defined with the evolution in \( U \). That is, we are considering a closed system consisted of a quantum state, represented by the unit vector \( \hat{v}_s = (v_x, v_y, v_z) \), an observer, whom we call Alice, represented by the reference frame \( \hat{e}_s = (e_x, e_y, e_z) \) introduced above, and a halt qubit \( \hat{v}_h \) along with Alice’s reference frame for the halt qubit defined as \( \hat{e}_h \equiv (0, 0, 1) \). Alice is to transform the unit vector \( \hat{v}_s \) by \( \delta \) about an arbitrary \( \hat{n} = (n_x, n_y, n_z) \)-axis with \( U_\delta = \cos(\delta/2) I - i \sin(\delta/2) \hat{n} \cdot \hat{\sigma} \) and also applies \( \sigma_z \) on a halt qubit such that \( \hat{v}_h \rightarrow -\hat{v}_h \). If Alice were to measure the evolved vector state, the expectation value would be \( \hat{e}_s \cdot (U_\delta \hat{v}_s U_\delta^\dagger) \). Next, we wish to consider the same procedure in the Heisenberg picture. In the Schrödinger picture we discussed above, the unitary evolution was performed on \( \hat{v}_s \). Therefore, in the Heisenberg picture, the \( U_\delta \) transforms the basic vector \( \hat{e}_s \) into \( U_\delta^\dagger \hat{e}_s U_\delta \) where \( U_\delta^\dagger = \cos(\delta/2) I + i \sin(\delta/2) (\hat{n} \cdot \hat{\sigma}) \) and the observable for the halt qubit, i.e., \( \hat{e}_h \), is transformed into \( -\hat{e}_h \). It yields the expectation value of \( (U_\delta^\dagger \hat{e}_s U_\delta) \cdot \hat{v}_s \) which is equal to the expectation value in the Schrödinger picture, \( \hat{e}_s \cdot (U_\delta \hat{v}_s U_\delta^\dagger) \).
server’s coordinate is evolving and the system is staying still. The equivalence of these two pictures comes from the fact that the observer would observe the same phenomenon and would not be able to tell the difference between them. For example, an observer applying a unitary operation to a qubit is experiencing a unitary evolution being applied to the qubit and this experience is the same in both pictures. But when it is the observer’s reference frame that is evolving, it is difficult to imagine how an observer could observe or experience it. As shown in (a) of Fig. 3, let us assume that initially vector $\hat{e}$ is pointing $z$-direction and with the unitary operation of rotation about $y$-axis, $\hat{e}$ evolves under

$$U = e^{-i\sigma_y t/2}$$

in the Schrödinger picture. And the final state of $\hat{e}$ would be rotated by $\chi$ after time $t$, which we write as $\chi(t)$. The difficulty with this evolution is that in order to experience the unitary evolution, Alice needs to be in another reference frame, say $\chi'(t)$. However, $\hat{e}$ itself is Alice’s reference frame and there cannot be another reference frame. Similarly, in the Heisenberg picture, $\hat{e}$ evolves under

$$U^\dagger = e^{i\sigma_y t/2}$$

As shown in Fig. 3, the vector is being rotated counterclockwise and is in $-\chi(t)$. In this case, for the observer in the reference frame of $-\chi(t)$, there needs to be additional vector in $\chi'(t)$ in order for Alice to experience the evolution of $\hat{e}$. Again, this is not possible because $-\chi(t)$ is not only Alice’s reference frame but also the system vector. Therefore, in order to have a satisfactory picture of Alice observing her own reference frame’s evolution, Alice needs another reference frame or another vector.

Therefore, it is not possible for either picture to be the correct way to describe the observer’s experience of the evolution of $\hat{e}$. Because $\hat{e}$ is serving the role of both what the observer experiences and the observer’s own reference frame, we need a picture such that the evolution of $\hat{e}$ is neither of them yet yields the observer’s experience of $\hat{e}$’s evolution. In order to resolve the dilemma discussed above and to determine the correct description for the observer’s experience, we introduce our second postulate as follows:

**Postulate II: What an observer observes or experiences must be time forwarding.**

Note that we are only postulating that the observer’s experience is time forwarding and not necessarily the whole system, i.e., including the physical system and the observer, is time forwarding.

Let us re-consider the evolution of $\hat{e}$ under the Heisenberg picture. Note that for the unitary operation in [4], it is possible to change the signs of $t$ and $\sigma_y$ while keeping the whole unitary operator the same, that is

$$U^\dagger = e^{-i\sigma_y (-t)/2}$$

This corresponds to the vector evolving to $\chi$ while $t$ is going to the minus direction compared to the previous Heisenberg case wherein the vector evolved to $-\chi$ with time going forward. In this case, we note that the observer cannot be in the reference frame $\chi(-t)$ because from the second postulate, we assumed what the observer observes or experiences is only time forwarding. If Alice is in the reference frame that is moving backward in time, she would observe everything going backward in time. However, from the assumption we made with the second postulate, this is not possible. We may consider the same trick with Schrödinger picture evolution, that is, by putting minus signs for both time and $\sigma_y$. But in this case, it still requires an additional observer’s reference frame because the observer who is in the reference frame with time forwarding would simply observe $\chi$ in $+t$. This is similar to the way an electron in the negative energy would appear as a positron in the positive energy to an observer who is also in the positive energy. Therefore, in the Schrödinger picture, this new view still requires an additional reference frame and is not satisfactory.

Therefore, with two postulates, in order to have a satisfactory description of experiencing the evolution of $\hat{e}$ as well as of $\hat{v}$, we are forced to conclude that the quantum evolution follows according to the Heisenberg picture, not the Schrödinger picture, with time going backwards as shown in [5]. Moreover, it leads us to abandon the general picture having the observer being in a certain reference frame evolving in time and observing the other vector. In other words, the more familiar picture of the observer being in the reference frame that is evolving forward in time should be abandoned, and the observer should be identified as what is being observed, i.e., $\chi$, and its association with time, $t$.

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