A MODULARITY LIFTING THEOREM FOR WEIGHT TWO
HILBERT MODULAR FORMS

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Abstract. We prove a modularity lifting theorem for potentially Barsotti-Tate representations over totally real fields, generalising recent results of Kisin. Unfortunately, there was an error in the original version of this paper, meaning that we can only obtain a slightly weaker result in the case where the representations are potentially ordinary; an erratum has been added explaining this error.

1. Introduction

In [Kis04] Mark Kisin introduced a number of new techniques for proving modularity lifting theorems, and was able to prove a very general lifting theorem for potentially Barsotti-Tate representations over $\mathbb{Q}$. In [Kis05] this was generalised to the case of $p$-adic representations of the absolute Galois group of a totally real field in which $p$ splits completely. In this note, we further generalise this result to:

Theorem. Let $p > 2$, let $F$ be a totally real field in which $p$ is unramified, and let $E$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Let $\rho : G_F \to \text{GL}_2(\mathcal{O})$ be a continuous representation unramified outside of a finite set of primes, with determinant a finite order character times the $p$-adic cyclotomic character. Suppose that

1. $\rho$ is potentially Barsotti-Tate at each $v | p$.
2. $\overline{\rho}$ is modular.
3. $\overline{\rho}_{F(\zeta_p)}$ is absolutely irreducible.

Then $\rho$ is modular.

We emphasise that the techniques we use are entirely those of Kisin. Our only new contributions are some minor technical improvements; specifically, we are able to prove a more general connectedness result than Kisin for certain local deformation rings, and we replace an appeal to a result of Raynaud by a computation with Breuil modules with descent data.

The motivation for studying this problem was the work reported on in [Gee06], where we apply the main theorem of this paper to the conjectures of [BDJ05]. In these applications it is crucial to have a lifting theorem valid for $F$ in which $p$ is unramified, rather than just totally split.

2. Connected components

Firstly, we recall some definitions and theorems from [Kis04]. We make no attempt to put these results in context, and the interested reader should consult section 1 of [Kis04] for a more balanced perspective on this material.

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Let $p > 2$ be prime. Let $k$ be a finite extension of $\mathbb{F}_p$, of cardinality $q = p^r$, and let $W = W(k)$, $K_0 = W(k)[1/p]$. Let $K$ be a totally ramified extension of $K_0$ of degree $e$. We let $\mathcal{S} = W[[u]]$, equipped with a Frobenius map $\phi$ given by $u \mapsto u^p$, and the natural Frobenius on $W$. Fix an algebraic closure $\overline{K}$ of $K$, and fix a uniformiser $\pi$ of $K$. Let $E(u)$ denote the minimal polynomial of $\pi$ over $K_0$.

Let $(\text{Mod } / \mathcal{S})$ denote the category of $\mathcal{S}$-modules $\mathfrak{M}$ equipped with a $\phi$-semilinear map $\phi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $\phi^*(\mathfrak{M}) \to \mathfrak{M}$ is killed by $E(u)$. For any $\mathbb{Z}_p$-algebra $A$, set $\mathcal{S}_A = \mathcal{S} \otimes_{\mathbb{Z}_p} A$. Denote by $(\text{Mod } / \mathcal{S})_A$ the category of pairs $(\mathfrak{M}, \iota)$ where $\mathfrak{M}$ is in $(\text{Mod } / \mathcal{S})$ and $\iota : A \to \text{End}(\mathfrak{M})$ is a map of $\mathbb{Z}_p$-algebras.

We let $(\text{Mod } \mathfrak{F}_1 / \mathcal{S})_A$ denote the full subcategory of $(\text{Mod } / \mathcal{S})_A$ consisting of objects $\mathfrak{M}$ such that $\mathfrak{M}$ is a projective $\mathcal{S}_A$-module of finite rank.

Choose elements $\pi_n \in \overline{K}$ ($n \geq 0$) so that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. Let $K_\infty = \bigcup_{n \geq 1} K(\pi_n)$. Let $\mathcal{O}_\mathcal{E}$ be the $p$-adic completion of $\mathcal{S}[1/u]$. Let $\text{Rep}_{Z_p}(G_{K_\infty})$ denote the category of continuous representations of $G_{K_\infty}$ on finite $Z_p$-modules. Let $\Phi M_{\mathcal{O}_\mathcal{E}}$ denote the category of finite $\mathcal{O}_\mathcal{E}$-modules $M$ equipped with a $\phi$-semilinear map $M \to M$ such that the induced map $\phi^* M \to M$ is an isomorphism. Then there is a functor

$$T : \Phi M_{\mathcal{O}_\mathcal{E}} \to \text{Rep}_{Z_p}(G_{K_\infty})$$

which is in fact an equivalence of abelian categories (see section 1.1.12 of [Kis04]). Let $A$ be a finite $\mathbb{Z}_p$-algebra, and let $\text{Rep}'_A(G_{K_\infty})$ denote the category of continuous representations of $G_{K_\infty}$ on finite $A$-modules, and let $\text{Rep}_A(G_{K_\infty})$ denote the full subcategory of objects which are free as $A$-modules. Let $\Phi M_{\mathcal{O}_\mathcal{E}, A}$ denote the category whose objects are objects of $\Phi M_{\mathcal{O}_\mathcal{E}}$ equipped with an action of $A$.

**Lemma 2.1.** The functor $T$ above induces an equivalence of abelian categories

$$T_A : \Phi M_{\mathcal{O}_\mathcal{E}, A} \to \text{Rep}'_A(G_{K_\infty}).$$

The functor $T_A$ induces a functor

$$T_{\mathcal{O}_\mathcal{E}, A} : (\text{Mod } \mathfrak{F}_1 / \mathcal{S})_A \to \text{Rep}_A(G_{K_\infty}) ; \mathfrak{M} \mapsto T_A(\mathcal{O}_\mathcal{E} \otimes_{\mathcal{S}} \mathfrak{M}).$$

**Proof.** Lemmas 1.2.7 and 1.2.9 of [Kis04].

Fix $\mathbb{F}$ a finite extension of $\mathbb{F}_p$, and a continuous representation of $G_K$ on a 2-dimensional $\mathbb{F}$-vector space $V_\mathbb{F}$. We suppose that $V_\mathbb{F}$ is the generic fibre of a finite flat group scheme, and let $M_\mathbb{F}$ denote the preimage of $V_\mathbb{F}(-1)$ under the equivalence $T_\mathbb{F}$ of Lemma 2.1.

In fact, from now on we assume that the action of $G_K$ on $V_\mathbb{F}$ is trivial, that $k \subset \mathbb{F}$, and that $k \neq \mathbb{F}_p$. In applications we will reduce to this case by base change.

Recall from Corollary 2.1.13 of [Kis04] that we have a projective scheme $\mathcal{G}R_{V_\mathbb{F}, 0}$, such that for any finite extension $\mathbb{F}'$ of $\mathbb{F}$, the set of isomorphism classes of finite flat models of $V_\mathbb{F} = V_\mathbb{F} \otimes \mathbb{F}'$ is in natural bijection with $\mathcal{G}R_{V_\mathbb{F}, 0}(\mathbb{F}')$. We work below with the closed subscheme $\mathcal{G}R_{V_\mathbb{F}, 0}$ of $\mathcal{G}R_{V_\mathbb{F}, 0}$, defined in Lemma 2.4.3 of [Kis04], which parameterises isomorphism classes of finite flat models of $V_\mathbb{F}$, with cyclotomic determinant.

As in section 2.4.4 of [Kis04], if $\mathbb{F}_{\text{sep}}$ is the residue field of $K_{0, \text{sep}}$, and $\sigma \in \text{Gal}(K_{0, \text{sep}})$, we denote by $e_\sigma \in k \otimes_{\mathbb{F}_p} \mathbb{F}_p'$ the idempotent which is 1 on the kernel of the map $1 \otimes e_\sigma : k \otimes_{\mathbb{F}_p} \mathbb{F}_p' \to \mathbb{F}_{\text{sep}}$ corresponding to $\sigma$, and 0 on the other maximal ideals of $k \otimes_{\mathbb{F}_p} \mathbb{F}_p'$.
Lemma 2.2. If $F'$ is a finite extension of $F$, then the elements of $\mathcal{GR}_{V,F}^{\nu,0}(F')$ naturally correspond to free $k \otimes_{F,F'} F[[u]]$-submodules $\mathcal{M}_{F'} \subset M_{F'} := M_{F} \otimes_{F} F'$ of rank 2 such that:

(1) $\mathcal{M}_{F'}$ is $\phi$-stable.
(2) For some (so any) choice of $k \otimes_{F,F'} F[[u]]$-basis for $\mathcal{M}_{F'}$, for each $\sigma$ the map $\phi : \epsilon \sigma M_{F'} \rightarrow \epsilon_{\sigma \phi^{-1}} M_{F'}$ has determinant $\alpha u^\epsilon$, $\alpha \in F[[u]]^\times$.

Proof. This follows just as in the proofs of Lemma 2.5.1 and Proposition 2.2.5 of [Kis04]. More precisely, the method of the proof of Proposition 2.2.5 of [Kis04] allows one to “decompose” the determinant condition into the condition that for each $\sigma$ we have

$$\dim_{F'}(\epsilon_{\sigma \phi^{-1}} M_{F'}/\phi(\epsilon \sigma M_{F'})) = e,$$

and then an identical argument to that in the proof of Lemma 2.5.1 [Kis04] shows that this condition is equivalent to the stated one.

We now number the elements of Gal$(K_0/IP)$ as $\sigma_1, \ldots, \sigma_r$, in such a way that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$ (where we identify $\sigma_{i+1}$ with $\sigma_1$). For any sublattice $\mathfrak{M}_F$ of $\text{(Mod}/\mathfrak{S})_F$ and any $(A_1, \ldots, A_r) \in M_2(F((u)))^r$, we write $\mathfrak{M}_F \sim A$ if there exist bases $\{e_1, e_2\}$ for $\epsilon \sigma_i M_{F'}$ such that

$$\phi \left( \begin{array}{c} e_1 \\ e_2 \end{array} \right) = A_i \left( \begin{array}{c} e_{i+1} \\ e_{i+1} \end{array} \right).$$

If we have fixed such a choice of basis, then for any $(B_1, \ldots, B_r) \in M_2(F((u)))^r$ we denote by $B\mathfrak{M}$ the module generated by $\left( B_i \left( \begin{array}{c} e_1 \\ e_2 \end{array} \right) \right)$, and consider $B\mathfrak{M}$ with respect to the basis given by these entries.

Proposition 2.3. Let $F'/F$ be a finite extension. Suppose that $x_1, x_2 \in \mathcal{GR}_{V,F}^{\nu,0}(F')$ and that the corresponding objects of $\text{(Mod}/\mathfrak{S})_{F'}$, $\mathfrak{M}_{F',1}$ and $\mathfrak{M}_{F',2}$ are both non-ordinary. Then (the images of) $x_1$ and $x_2$ both lie on the same connected component of $\mathcal{GR}_{V,F}^{\nu,0}(F')$.

Proof. Replacing $V_F$ by $F' \otimes_{F,F'} V_F$, we may assume that $F' = F$. Suppose that $\mathfrak{M}_{F',1} \sim A$. Then $\mathfrak{M}_{F',2} \sim B \cdot \mathfrak{M}_{F',1}$ for some $B \in \GL_2(k,(u)))^r$, and $\mathfrak{M}_{F',2} \sim (\phi(B_i) \cdot A_i \cdot B^{-1})$. Each $B_i$ is uniquely determined up to left multiplication by elements of $\text{GL}_2(F[[u]])$, so by the Iwasawa decomposition we may assume that each $B_i$ is upper triangular. By Lemma 2.2 $\det(\phi(B_i)) \det B^{-1} \in F[[u]]^\times$ for all $i$, which implies that $\det(B_i) \in F[[u]]^\times$ for all $i$, so that the diagonal elements $B_{ii}$ are $\mu_1 u^{-\alpha_i}$, $\mu_2 u^{\alpha_i}$ for $\mu_1, \mu_2 \in F[[u]]^\times$, $\alpha_i \in \mathbb{Z}$. Replacing $B_i$ with $\text{diag}(\mu_1^i, \mu_2^i \cdot B_i)$, we may assume that $B_i$ has diagonal entries $u^{-\alpha_i}$ and $u^{\alpha_i}$.

We now show that $x_1$ and $x_2$ are connected by a chain of rational curves, using the following lemma:

Lemma 2.4. Suppose that $(N_i)$ are nilpotent elements of $M_2(F((u)))$ such that $\mathfrak{M}_{F',2} = (1 + N) \cdot \mathfrak{M}_{F',1}$. If $\phi(N_i) AN_i^\text{ad} \in M_2(F[[u]])$ for all $i$, then there is a map $\mathbb{P}^1 \rightarrow \mathcal{GR}_{V,F}^{\nu,0}$ sending $0$ to $x_1$ and $1$ to $x_2$.

Proof. Exactly as in the proof of Lemma 2.5.7 of [Kis04].
In fact, we will only apply this lemma in situations where all but one of the \( N_i \) are zero, so that the condition of the lemma is automatically satisfied.

**Lemma 2.5.** With respect to some basis, \( \phi : M_{\mathcal{F}} \to M_{\mathcal{F}} \) is given by \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

*Proof.* This is immediate from the definition of \( M_{\mathcal{F}} \) (recall that we have assumed that the action of \( G_K \) on \( V_{\mathcal{F}} \) is trivial). \( \square \)

Let \( v_1, v_2 \) be a basis as in the lemma, and let \( \mathcal{M}_{\mathcal{F}} \) be the sub-\( k \otimes_{\mathbb{F}_p} \mathbb{F}[[u]] \)-module generated by \( u^{e/(p-1)} v_1 \) and \( v_2 \) (note that the assumption that the action of \( G_K \) on \( V_{\mathcal{F}} \) is trivial guarantees that \( e \neq (p-1) \)). Then \( \mathcal{M}_{\mathcal{F}} \) corresponds to an object of \( \mathcal{G}_R_{V_{\mathcal{F}},0}(\mathbb{F}') \), and \( \mathcal{M}_{\mathcal{F}} \sim (A_i) \) where each \( A_i = \begin{pmatrix} u^i & 0 \\ 0 & 1 \end{pmatrix} \), so that \( \mathcal{M}_{\mathcal{F}} \) is ordinary.

Furthermore, every object of \( \mathcal{G}_R_{V_{\mathcal{F}},0}(\mathbb{F}') \) is given by \( B \cdot \mathcal{M}_{\mathcal{F}} \) for some \( B = (B_i) \), where \( B_i = \begin{pmatrix} u^{-i} & v_i \\ 0 & 1 \end{pmatrix} \), and \( \phi(B_i)A_iB_{i+1}^{-1} \in M_{\mathcal{A}}(\mathbb{F}[[u]]) \) for all \( i \). Examining the diagonal entries of \( \phi(B_i)A_iB_{i+1}^{-1} \), we see that we must have \( e \geq p\alpha_i - a_{i+1} \geq 0 \) for all \( i \).

**Lemma 2.6.** We have \( e/(p-1) \geq a_i \geq 0 \) for all \( i \). Furthermore, if any \( a_i = 0 \) then all \( a_i = 0 \); and if any \( a_i = e/(p-1) \), then all \( a_i = e/(p-1) \).

*Proof.* Suppose that \( a_j \leq 0 \). Then \( p\alpha_j \geq a_{j+1} \), so \( a_{j+1} \leq 0 \). Thus \( a_i \leq 0 \) for all \( i \). But adding the inequalities gives \( (p-1)(a_1 + \cdots + a_r) \geq 0 \), so in fact \( a_1 = \cdots = a_r = 0 \). The other half of the lemma follows in a similar fashion. \( \square \)

Note that the ordinary objects are precisely those with all \( a_i = 0 \) or all \( a_i = e/(p-1) \). We now show that there is a chain of rational curves linking any non-ordinary point to the point corresponding to \( C \cdot \mathcal{M}_{\mathcal{F}} \), where \( C_i = \begin{pmatrix} u^i & 0 \\ 0 & 1 \end{pmatrix} \).

Choose a non-ordinary point \( D \cdot \mathcal{M}_{\mathcal{F}}, D_i = \begin{pmatrix} u^{-i} & w_i \\ 0 & u^i \end{pmatrix} \). We claim that there is a chain of rational curves linking this to the point \( D' \cdot \mathcal{M}_{\mathcal{F}}, D'_i = \begin{pmatrix} u^{-i} & 0 \\ 0 & u^i \end{pmatrix} \). Clearly, it suffices to demonstrate that there is a rational curve from \( D \cdot \mathcal{M}_{\mathcal{F}} \) to the point \( D^j \cdot \mathcal{M}_{\mathcal{F}} \), where

\[
D_i^j = \begin{cases} D_i, & i \neq j \\ \begin{pmatrix} u^{-i} & 0 \\ 0 & u^i \end{pmatrix}, & i = j. \end{cases}
\]

But this is easy; just apply Lemma 2.4 with \( N = (N_i) \),

\[
N_i = \begin{cases} 0, & i \neq j \\ 0 - w_j u^{-i} & i = j. \end{cases}
\]

It now suffices to show that there is a chain of rational curves linking \( D' \cdot \mathcal{M}_{\mathcal{F}} \) to \( C \cdot \mathcal{M}_{\mathcal{F}} \). Suppose that \( D'' \cdot \mathcal{M}_{\mathcal{F}} \) also corresponds to a point of \( \mathcal{G}_R_{V_{\mathcal{F}},0}(\mathbb{F}) \), where for some \( j \) we have

\[
D_i'' = \begin{cases} D_i', & i \neq j \\ \begin{pmatrix} u^{-i} & 0 \\ 0 & u^i \end{pmatrix}, & i = j. \end{cases}
\]

Then we claim that there is a rational curve linking \( D' \cdot \mathcal{M}_{\mathcal{F}} \) and \( D'' \cdot \mathcal{M}_{\mathcal{F}} \). Note that \( D'' = ED' \), where

\[
E_i = \begin{cases} 1, & i \neq j \\ \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}, & i = j. \end{cases}
\]
Since \((u^{-1} 0) = (0 \ 1_{2a})(u^{-1} -u)\), and \((0 \ 1_{2a}) \in \text{GL}_2(\mathbb{F}[u])\), we can apply Lemma 2.3 with

\[
N_i = \begin{cases} 
0, & i \neq j \\
\begin{pmatrix} 1 & -u \\
u^{-1} & -1 \end{pmatrix}, & i = j.
\end{cases}
\]

Proposition 2.7 now follows from:

**Lemma 2.7.** If \(e/(p-1) > a_i > 0\) and \(e \geq pa_i - a_{i+1} \geq 0\) for all \(i\), and not all the \(a_i\) are equal to 1, then for some \(j\) we can define

\[
a'_i = \begin{cases} 
a_i, & i \neq j \\
a_j - 1, & i = j
\end{cases}
\]

and we have \(e \geq pa'_i - a'_{i+1} \geq 0\) for all \(i\).

**Proof.** Suppose not. Then for each \(i\), either \(pa_i - (a_i - 1) > e\), or \(p(a_i - 1) - a_{i+1} < 0\); that is, either \(pa_i - a_i = e\), or \(p-1 \geq pa_i - a_{i+1} \geq 0\). Comparing the statements for \(i, i+1\), we see that either \(pa_i - a_{i+1} = e\) for all \(i\), or \(p-1 \geq pa_i - a_{i+1} \geq 0\) for all \(i\). In the former case we have \(a_i = e/(p-1)\) for all \(i\), a contradiction. In the latter case, summing the inequalities gives \(r(p-1) \geq (p-1)(a_1 + \cdots + a_r) \geq (r+1)(p-1)\), a contradiction.

\[\square\]

### 3. Modularity lifting theorems

The results of section 2 can easily be combined with the machinery of [Kis04] to yield modularity lifting theorems. For example, we have the following:

**Theorem 3.1.** Let \(p > 2\), let \(F\) be a totally real field, and let \(E\) be a finite extension of \(\mathbb{Q}_p\) with ring of integers \(\mathcal{O}\). Let \(\rho : G_F \to \text{GL}_2(\mathcal{O})\) be a continuous representation unramified outside of a finite set of primes, with determinant a finite order character times the \(p\)-adic cyclotomic character. Suppose that

1. \(\rho\) is potentially Barsotti-Tate at each \(v\)|\(p\).
2. There exists a Hilbert modular form \(f\) of parallel weight 2 over \(F\) such that \(\rho_f \sim \rho\), and for each \(v\)|\(p\), \(\rho\) is potentially ordinary at \(v\) if and only if \(\rho_f\) is.
3. \(\rho|_{F(\zeta_p)}\) is absolutely irreducible, and if \(p > 3\) then \([F(\zeta_p) : F] > 3\).

Then \(\rho\) is modular.

**Proof.** The proof of this theorem is almost identical to the proof of Theorem 3.5.5 of [Kis04]. Indeed, the only changes needed are to replace property (iii) of the field \(F'\) chosen there by “(iii) If \(v|p\) then \(\mathcal{P}_F G_F\) is trivial, and the residue field at \(v\) is not \(F_p\)”, and to note that Theorem 3.4.11 of [Kis04] is still valid in the context in which we need it, by Proposition 2.5.

\[\square\]

For the applications to mod \(p\) Hilbert modular forms in [Gee06] it is important not to have to assume that \(\rho\) is potentially ordinary at \(v\) if and only if \(\rho_f\) is. Fortunately, in [Gee06] it is only necessary to work with totally real fields in which \(p\) is unramified, and in that case we are able, following [Kis05], to remove this assumption.

**Theorem 3.2.** Let \(p > 2\), let \(F\) be a totally real field in which \(p\) is unramified, and let \(E\) be a finite extension of \(\mathbb{Q}_p\) with ring of integers \(\mathcal{O}\). Let \(\rho : G_F \to \text{GL}_2(\mathcal{O})\) be a continuous representation unramified outside of a finite set of primes, with
determinant a finite order character times the $p$-adic cyclotomic character. Suppose that

1. $\rho$ is potentially Barsotti-Tate at each $v|p$.
2. $\overline{\rho}$ is modular.
3. $\overline{\rho}_{F(\zeta_p)}$ is absolutely irreducible.

Then $\rho$ is modular.

Proof. Firstly, note that by a standard result (see e.g. [BDJ05]) we have $\overline{\rho} \sim \overline{\rho}_f$, where $f$ is a form of parallel weight 2. We now follow the proof of Corollary 2.13 of [Kis05]. Let $S'$ denote the set of $v|p$ such that $\rho_{|G_v}$ is potentially ordinary. After applying Lemma 3.3 below, we may assume that $\overline{\rho} \sim \overline{\rho}_f$, where $f$ is a form of parallel weight 2, and $\rho_f$ is potentially ordinary and potentially Barsotti-Tate for all $v \in S'$.

We may now make a solvable base change so that the hypotheses on $F$ in Theorem 3.1 are still satisfied, and in addition $[F : \mathbb{Q}]$ is even, and at every place $v|p$ $f$ is either unramified or special of conductor 1. By our choice of $f$, $\rho_f|G_v$ is Barsotti-Tate and ordinary at each place $v \in S'$. In order to apply Theorem 3.1 we need to check that we can replace $f$ by a form $f'$ such that $\overline{\rho} \sim \overline{\rho}_{f'}$, and $\rho_{f'}$ is Barsotti-Tate at all $v|p$ and is ordinary if and only if $\rho$ is. That is, we wish to choose $f'$ so that $\rho_{f'}$ is Barsotti-Tate and ordinary at all places $v \in S'$, and is Barsotti-Tate and non-ordinary at all other places dividing $p$. The existence of such an $f'$ follows at once from the proof of Theorem 3.5.7 of [Kis04]. The theorem then follows from Theorem 3.1.

Lemma 3.3. Let $F$ be a totally real field in which $p$ is unramified, and $S'$ a set of places of $F$ dividing $p$. Let $f$ be a Hilbert modular cusp form over $F$ of parallel weight 2, with $\overline{\rho}_f$ absolutely irreducible, and suppose that for $v \in S'$ $\overline{\rho}_{f_v}|G_{F_v}$ is the reduction of a potentially Barsotti-Tate representation of $G_{F_v}$ which is also potentially ordinary.

Then there is a Hilbert modular cusp form $f'$ over $F$ of parallel weight 2 with $\overline{\rho}_{f'} \sim \overline{\rho}_{f}$, and such that for all $v \in S'$, $\rho_{f'}$ becomes ordinary and Barsotti-Tate over some finite extension of $F_v$. 

Proof. We follow the proof of Lemma 2.14 of [Kis05], indicating only the modifications that need to be made. Replacing the appeal to [CDT99] with one to Proposition 1.1 of [Dia05], the proof of Lemma 2.14 of [Kis05] shows that we can find $f'$ such that $\overline{\rho}_{f'} \sim \overline{\rho}_f$, and such that for all $v \in S'$, $\rho_{f'}$ becomes Barsotti-Tate over $F_v(\zeta_{q_0})$, where $q_0$ is the degree of the residue field of $F$ at $v$. Furthermore, we can assume that the type of $\rho_{f'}|G_{F_v}$ is $\tilde{\omega}_1 \oplus \tilde{\omega}_2$, where $\overline{\rho}_{f'}|G_{F_v} \sim (\omega_1, \omega_2)$, where $\chi$ is the cyclotomic character, and a tilde denotes the Teichmuller lift. Let $\overline{G}$ denote the $p$-divisible group over $\mathcal{O}_{F_v}(\zeta_{q_0})$ corresponding to $\rho_{f'}|_{F_v(\zeta_{q_0})}$. Then by a scheme-theoretic closure argument, $\overline{G}[p]$ fits into a short exact sequence

$$0 \rightarrow \overline{G}_1 \rightarrow \overline{G}[p] \rightarrow \overline{G}_2 \rightarrow 0.$$ 

The information about the type then determines the descent data on the Breuil modules corresponding to $\overline{G}_1$ and $\overline{G}_2$. We will be done if we can show that $\overline{G}_1$ is multiplicative and $\overline{G}_2$ is étale. However, by the hypothesis on $S'$ we can write down a multiplicative group scheme $\mathcal{G}_2'$ with the same descent data and generic fibre as $\overline{G}_1$. Then Lemma 3.3 below shows that $\mathcal{G}_1$ is indeed multiplicative. The same argument shows that $\mathcal{G}_2'$ is étale. 

□
Lemma 3.4. Let $k$ be a finite field of characteristic $p$, and let $L = W(k)[1/p]$. Fix $\pi = (-p)^{1/(q^d-1)}$ where $d = [k : \mathbb{F}_p]$, and let $K = L(\pi)$. Let $E$ be a finite field containing $k$. Let $G$ and $G'$ be finite flat rank one $E$-module schemes over $O_K$ with generic fibre descent data to $L$. Suppose that the generic fibres of $G$ and $G'$ are isomorphic as $G_L$-representations, and that $G$ and $G'$ have the same descent data. Then $G$ and $G'$ are isomorphic.

Proof. This follows from a direct computation using Breuil modules with descent data. Specifically, it follows at once from Example A.3.3 of [Sav06], which computes the generic fibre of any finite flat rank one $E$-module scheme over $O_K$ with generic fibre descent data to $L$. □

4. Erratum

Unfortunately, the proof of Theorem 3.2 (and thus the main Theorem stated in the introduction to the paper) is incomplete; in particular, the proof of Lemma 3.3 is incorrect. Specifically, one cannot automatically assume that the type of $\rho_f$ is $\tilde{\omega}_1 \oplus \tilde{\omega}_2$; to do so is to make a rather strong assumption about the Serre weights of $\overline{\rho}_f$. In addition one cannot conclude that the type determines the descent data on $G_1$ and $G_2$, at least in the case where $\overline{\rho}_f|_{G_{F_v}}$ is split; either $G_1$ can correspond to $\tilde{\omega}_1$ and $G_2$ to $\tilde{\omega}_2$, or vice versa.

As a consequence, we are only able to obtain a slightly weaker modularity lifting theorem; the most general result we can obtain is:

Theorem 4.1. Let $p > 2$, let $F$ be a totally real field, and let $E$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $O$. Let $\rho : G_F \to \text{GL}_2(O)$ be a continuous representation unramified outside of a finite set of primes, with determinant a finite order character times the $p$-adic cyclotomic character. Suppose that

1. $\rho$ is potentially Barsotti-Tate at each $v|p$.
2. There exists a Hilbert modular form $f$ of parallel weight 2 over $F$ such that $\overline{\rho}_f \sim \overline{\tau}$, and for each $v|p$, if $\rho$ is potentially ordinary at $v$ then so is $\rho_f$.
3. $\overline{\rho}_f|_{G_{F_v}}$ is absolutely irreducible, and if $p = 5$ and the projective image of $\overline{\rho}$ is isomorphic to $\text{PGL}_2(\mathbb{F}_5)$ then $[F(\zeta_5) : F] = 4$.

Then $\rho$ is modular.

Proof. The proof is extremely similar to that of Theorem 3.1. Hypothesis (3) has been weakened because of a corresponding weakening of (3.2.3)(3) in the final version of [Kis04]. As for (2), we need only check that after making a base change, we may assume that at each place dividing $p$, $\rho$ is potentially ordinary if and only if $\rho_f$ is. This is easily achieved by employing Lemma 3.1.5 of [Kis04] at each place where $\rho$ is not potentially ordinary. □

Additionally, we would like to thank Fred Diamond and Florian Herzig for independently bringing to our attention a minor error in the proof of Proposition 2.3. The points $D'$ constructed in the proof are not necessarily points on $\mathcal{G}_R_{V_r,0}$. However, their only use is in showing that the points $D$ and $D'$ lie on the same component, and this in fact follows immediately from an application of Lemma 2.4 with $N = (N_i)$,

$$N_i = \begin{pmatrix} 0 & -w_i u_i^{-b_i} \\ 0 & 0 \end{pmatrix}.$$
References

[BDJ05] Kevin Buzzard, Fred Diamond, and Frazer Jarvis, On Serre’s conjecture for mod $l$ Galois representations over totally real fields, in preparation, 2005.

[CDT99] Brian Conrad, Fred Diamond, and Richard Taylor, Modularity of certain potentially Barsotti-Tate Galois representations, J. Amer. Math. Soc. 12 (1999), no. 2, 521–567.

[Dia05] Fred Diamond, A correspondence between representations of local Galois groups and Lie-type groups, to appear in L-functions and Galois representations (Durham 2004), 2005.

[Gee06] Toby Gee, On the weights of mod $p$ Hilbert modular forms, 2006.

[Kis04] Mark Kisin, Moduli of finite flat group schemes, and modularity, 2004.

[Kis05] ———, Modularity for some geometric Galois representations, to appear in L-functions and Galois representations (Durham 2004), 2005.

[Sav06] David Savitt, Breuil modules for Raynaud schemes, appendix to [Gee06], 2006.

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