Tata lectures on overlap fermions *

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ABSTRACT: Overlap formalism deals with the construction of chiral gauge theories on the lattice. These set of lectures provide a pedagogical introduction to the subject with emphasis on chiral anomalies and gauge field topology. Subtleties associated with the generating functional for gauge theories coupled to chiral fermions are discussed.

KEYWORDS: Chiral gauge theories, Anomalies, Gauge field topology, Lattice gauge theories.

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1. Introduction

Chiral symmetry, chiral anomalies and gauge field topology are important in our understanding of theories that describe fundamental interactions. Since the focus of these lectures is on the lattice formulation of chiral gauge theories, we will focus on continuum Euclidean field theory and use the path integral representation. It is sufficient to consider two dimensional chiral fermions and couple them to an abelian gauge field to understand all the relevant concepts. We will therefore focus only on such theories. The extension to any other chiral gauge theory will be straightforward.
The full path integral will be broken up into two pieces:

$$Z = \int [dA] e^{-S_g(A)} \int [d\psi][d\bar{\psi}] e^{S_f(\psi,\bar{\psi},A)} = \int [dA] e^{-S_g(A) - W_f(A)}. \quad (1.1)$$

We first treat the Abelian gauge field as an external classical field and only consider the path integral over fermions. The second step is to take the resulting generating functional for the gauge field and combine it with the standard action for the gauge field and perform the path integral over the gauge fields. The conceptually difficult part is the computation of $W_f(A)$ and we will only concern ourselves with this part. Our aim is to obtain a correct lattice regularized version for $W_f(A)$. One can use standard numerical techniques to perform the integral over the lattice gauge field.

The organization of the lectures is as follows. We will start by describing the details of abelian gauge fields on a two dimensional lattice with periodic boundary conditions. This will help us understand the Hodge decomposition of the gauge fields into torons, representative in a gauge orbit, gauge transformations and a global topological piece.

We will take the continuum limit of the background gauge field and couple a Dirac fermion to it. We will be able to solve the eigenvalue problem using the Hodge decomposition. In particular, we will be able to show the presence of chiral zero modes when the background gauge field has a global topological piece. Formally, the path integral over fermions is the product over the eigenvalues as long as there are no zero modes and if one could properly regularize the Jacobian. This will lead us to the problem of fermions on the lattice as a possible regularization scheme.

We will explain the presence of lattice doublers and the solution of Wilson. But this will not help us deal with chiral fermions on the lattice. In order to understand the subtleties associated with regulating a chiral fermion, we will first focus on continuum Pauli-Villars regularization (we do not consider dimensional regularization since we are ultimately interested in the non-perturbative lattice regularization) and show the need for an infinite number of Pauli-Villars fields to regulate a single chiral fermion. We will only show that all diagrams are regulated but we will show that the idea can be extended to the lattice and therefore outside perturbation theory. This is called the overlap formalism.

We will describe the overlap formalism in detail. We will derive the formula for the generating function associated with a chiral fermion. We will derive the massless overlap Dirac operator for the case of vector like theories. We will show how the overlap formalism realizes gauge field topology.

Our discussion of chiral anomalies will start with the discussion of consistent and covariant current in the context of the overlap formalism. We will show that the covariant current is unambiguously defined but the consistent current is not. We will show that there is an unambiguous curvature obtained from the difference of the two currents. Vanishing curvature would imply an anomaly free chiral gauge theory in the continuum. We will use torons to demonstrate cancellation of anomalies in a chiral gauge theory. We will use specific gauge field backgrounds to compute the covariant anomaly. We will also demonstrate gauge field topology on the lattice. All these will be numerical computations to help us
understand how to work with overlap fermions and to also understand lattice artifacts in the context of overlap fermions.

2. Abelian gauge fields in two dimensions

Consider a two dimensional $L_1 \times L_2$ lattice with $(n_1, n_2)$; $0 \leq n_1 < L_1$ and $0 \leq n_2 < L_2$ labeling the sites on the lattice. Let $U_1(n_1, n_2)$ and $U_2(n_1, n_2)$ be the link variables on the links connecting $(n_1, n_2)$ with $(n_1 + 1, n_2)$ and $(n_1, n_2 + 1)$ respectively. The link variables are $U(1)$ values phases.

Under a local gauge transformation,

$$U_1(n_1, n_2) \rightarrow U_1'(n_1, n_2) = g^\dagger(n_1, n_2)U_1(n_1, n_2)g(n_1 + 1, n_2);$$
$$U_2(n_1, n_2) \rightarrow U_2'(n_1, n_2) = g^\dagger(n_1, n_2)U_2(n_1, n_2)g(n_1, n_2 + 1),$$

(2.1)

where $g(n_1, n_2) \in U(1)$ satisfy periodic boundary conditions. Define the plaquette variable as

$$e^{iE(n_1, n_2)} = U_1(n_1, n_2)U_2(n_1 + 1, n_2)U_1^\dagger(n_1, n_2 + 1)U_2^\dagger(n_1, n_2); \quad -\pi < E(n_1, n_2) \leq \pi;$$

(2.2)

and it is easy to show that it is invariant under a local gauge transformation. Let

$$e^{i\pi h_1} = \prod_{n_1=0}^{L_1-1} U_1(1, n_1); \quad e^{i\pi h_2} = \prod_{n_2=0}^{L_2-1} U_2(0, n_2); \quad -1 < h_1, h_2 \leq 1.$$  

(2.3)

These two additional variables are called torons and are also invariant under local gauge transformations. The gauge action on the lattice is

$$S_g = -\beta \sum_{n_1, n_2} \cos E(n_1, n_2).$$

(2.4)

In order to take the continuum limit on a torus of fixed physical size, $l_1 \times l_2$, we will take $\beta, L_1, L_2 \rightarrow \infty$, keeping $\frac{l_1}{\sqrt{\beta}} = l_1$ and $\frac{l_2}{\sqrt{\beta}} = l_2$ fixed.

2.1 Hodge decomposition

Since a constant $g(n_1, n_2)$ does not change the link variables, we set $g(0, 0) = 1$. We can choose $g(n_1, 0)$; $0 < n_1 < L_1$ and $g(0, n_2)$; $0 < n_2 < L_2$ such that

$$U_1'(n_1, 0) = e^{i\pi \frac{h_1}{L_1}}; \quad 0 \leq n_1 < L_1$$

(2.5)

$$U_2'(0, n_2) = e^{i\pi \frac{h_2}{L_2}}; \quad 0 \leq n_2 < L_2$$

(2.6)

Next, we choose the remaining $g(n_1, n_2)$ such that

$$U_1'(n_1, n_2) = e^{i\pi \frac{h_1}{L_1}}; \quad 0 \leq n_1 < L_1 - 1; \quad 0 < n_2 < L_2.$$  

(2.7)

It is now clear that

$$U_2'(n_1, n_2) = e^{i\sum_{k_1=0}^{n_1-1} E(k_1, n_2)} e^{i\pi \frac{h_2}{L_2}}; \quad 0 < n_1 < L_1; \quad 0 \leq n_2 < L_2,$$  

(2.8)
\[ U'_1(L_1 - 1, n_2) = e^{-i \sum_{k_2=0}^{n_2-1} E(k_1,k_2)} e^{i \frac{2 \pi h_1}{L_1}}; \quad 0 < n_2 < L_2. \]  

(2.9)

Since the above procedure did not require \( E(L_1 - 1, L_2 - 1) \), we conclude that all plaquettes variables are not independent variables and they satisfies the condition

\[ \sum_{k_1=0}^{L_1-1} \sum_{k_2=0}^{L_2-1} E(k_1, k_2) = 2\pi Q \]  

(2.10)

where \( Q \) is an integer, referred to as the topological charge.

Let us now assume we are given the gauge invariant degrees of freedom, namely, 

- \((L_1L_2)\) electric flux degrees of freedom \( E(n_1, n_2) \) that satisfy (2.10);
- and the two toron variables, \( h_1 \) and \( h_2 \).

The full set of gauge fields consistent with the above data are

\[
U_1(n_1, n_2) = e^{ \frac{2 \pi h_1}{L_1} } U'_1(n_1, n_2) e^{-i \chi(n_1,n_2)} e^{[\phi(n_1,n_2-1)-\phi(n_1,n_2)] e^{i \chi(n_1,n_2+1)}}
\]

\[
U_2(n_1, n_2) = e^{ \frac{2 \pi h_2}{L_2} } U'_2(n_1, n_2) e^{-i \chi(n_1,n_2)} e^{[\phi(n_1,n_2)-\phi(n_1-1,n_2)] e^{i \chi(n_1,n_2+1)}}
\]

(2.11)

where \( \phi(n_1,n_2) \) is the solution to

\[
\Box \phi(n_1,n_2) = -4 \phi(n_1,n_2) + \phi(n_1-1,n_2) + \phi(n_1+1,n_2) + \phi(n_1,n_2-1) + \phi(n_1,n_2+1) - E(n_1,n_2) - \frac{2\pi Q}{L_1L_2};
\]

(2.12)

and

\[
U'_1(n_1, n_2) = e^{-i \frac{2 \pi Q}{L_1} n_1}; \quad 0 \leq n_1 < L_1; \quad 0 \leq n_2 < L_2;
\]

\[
U'_2(n_1, n_2) = \begin{cases} e^{i \frac{2 \pi Q}{L_2} n_2} & \text{if } 0 \leq n_1 < L_1 \text{ and } 0 \leq n_2 < L_2 - 1 \\ 1 & \text{if } 0 \leq n_1 < L_1 \text{ and } n_2 = L_2 \end{cases}
\]

(2.13)

is the topological part of the gauge field.

In the continuum limit,

\[ U_\mu(n_1, n_2) \rightarrow e^{ \frac{2 \pi}{\beta} A_\mu(x_1,x_2) }; \quad x_\mu = \frac{n_\mu}{\sqrt{\beta}}. \]

(2.14)

In the continuum limit, (2.13), leads to a discontinuous function for \( A_\mu(x_1, x_2) \) and it is natural to move the discontinuity to a gauge transformation that is not periodic on the torus. The gauge field in the continuum limit is written as

\[
A_1(x_1, x_2) = \frac{\pi h_1}{l_1} + \partial_1 \chi(x_1, x_2) - \partial_2 \phi(x_1, x_2) - \frac{2\pi Q}{L_1L_2} x_2
\]

\[
A_2(x_1, x_2) = \frac{\pi h_2}{l_2} + \partial_2 \chi(x_1, x_2) + \partial_1 \phi(x_1, x_2).
\]

(2.15)

Clearly, \( A_1(x_1, 0) \neq A_1(x_1, l_2) \), but

\[ A_1(x_1, l_2) = A_1(x_1, 0) - \partial_1 \frac{2\pi Q}{l_1} x_1, \]

(2.16)
referred to as a gauge transformation on the torus with non-trivial winding. The continuum limit of (2.12) is
\[ \square \phi(x_1, x_2) = (\partial_1^2 + \partial_2^2) \phi(x_1, x_2) = e(x_1, x_2) - \frac{2\pi Q}{l_1 l_2} \] (2.17)
where $\beta E(n_1, n_2) \rightarrow e(x_1, x_2)$ is the continuum limit of the electric field.

### 2.2 Fermion zero modes

The massless Dirac operator coupled to the abelian gauge field is given by
\[ \mathcal{D} = \sum_{\mu=1}^{2} \sigma_\mu (\partial_\mu + iA_\mu). \] (2.18)

We will use the chiral representation for the Pauli matrices, namely,
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = -i\sigma_1\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (2.19)

The chiral symmetry is due to the identity,
\[ \mathcal{D}\sigma_3 = -\sigma_3 \mathcal{D}, \] (2.20)

since eigenvalues of $\mathcal{D}$ comes in $\pm i\lambda$ pairs. In addition, the Dirac operator always has exactly $|Q|$ zero modes with definite chirality (eigenvectors of $\sigma_3$). These zero modes are robust and are present for all gauge fields with a fixed $Q$ since
\[ \mathcal{D} = f^* e^{-\sigma_3 \phi} \mathcal{D}_Q f e^{-\sigma_3 \phi}, \] (2.21)

where
\[ f = e^{i\pi h_1 x_1 l_1 + i\pi h_2 x_2 l_2 + i\chi}, \] (2.22)

and
\[ \mathcal{D}_Q = \begin{pmatrix} 0 & C_Q \\ -C_Q^\dagger & 0 \end{pmatrix}; \quad C_Q = \partial_1 - i\partial_2 - ibx_2; \quad b = \frac{2\pi Q}{l_1 l_2}. \] (2.23)

To solve for the zero modes of $\mathcal{D}_Q$, note that
\[ -\mathcal{D}_Q^2 = \begin{pmatrix} C_Q C_Q^\dagger & 0 \\ 0 & C_Q^\dagger C_Q \end{pmatrix} = \begin{pmatrix} K - b & 0 \\ 0 & K + b \end{pmatrix}, \] (2.24)

where
\[ K = -\partial_1^2 - \partial_2^2 + 2ibx_2\partial_1 + b^2 x_2^2. \] (2.25)

In order to make the Dirac equation gauge covariant in the presence of non-trivial winding as given in (2.16), we are interested in solutions to
\[ K \pm b\psi = 0, \] (2.26)

with the fermion wave-functions satisfying the boundary conditions
\[ \psi(x_1 + l_1, x_2) = \psi(x_1, x_2); \quad \psi(x_1, x_2 + l_2) = e^{\frac{2\pi Q}{l_1} x_1}\psi(x_1, x_2), \] (2.27)
on the torus. Solutions have to be of form
\[ \psi(x_1, x_2) = \sum_{p=-\infty}^{\infty} a_p e^{i \frac{2\pi p}{L_1} x_1} e^{-\frac{\pi |Q|}{L_1} (x_2 - pL_2)} \] (2.28)
where \( h(y) \) is the solution to the Harmonic Oscillator,
\[ (-\partial_y^2 + b^2 y^2 + b) h(y) = 0. \] (2.29)
The normalizable solution is
\[ h(y) = e^{-|b| y^2}, \] (2.30)
and has positive chirality if \( Q > 0 \) and negative chirality if \( Q < 0 \). Our solutions are therefore of the form,
\[ \psi(x_1, x_2) = \sum_{p=-\infty}^{\infty} a_p e^{i \frac{2\pi p}{L_1} x_1} e^{-\frac{\pi |Q|}{L_2} (x_2 - \frac{pl_2}{l_1})^2}. \] (2.31)
Since
\[ \psi(x_1, x_2 + l_2) = \sum_{p=-\infty}^{\infty} a_{p+Q} e^{i \frac{2\pi (p+Q)}{L_1} x_1} e^{-\frac{\pi |Q|}{L_2} (x_2 - \frac{pl_2}{l_1})^2}, \] (2.32)
we see that
\[ a_{p+Q} = a_p, \] (2.33)
for all \( p \) in order for the solution to satisfy the boundary conditions in (2.27). Therefore, we have exactly \( |Q| \) zero modes with definite chirality. The orthogonal set of solutions are
\[ \psi_k(z_1, z_2) = \frac{1}{\sqrt{L_1 L_2}} \sum_{p=-\infty}^{\infty} e^{i 2\pi (p+Q)|z_1|} e^{-\pi |Q| |z_2 - \frac{k}{L_2}|^2} \] (3.1)
for \( k = 0, \cdots, |Q| - 1 \) and \( z_{\mu} = \frac{x_{\mu}}{l_{\mu}}, \tau = \frac{x_2}{l_2}. \)

The zero modes define an index since \( C_Q^\dagger \) has \( Q \) zero modes and \( C_Q \) has none for \( Q > 0 \) and vice-versa. It is not possible to realize this using a finite matrix for \( C_Q \). The need for an infinite number of degrees of freedom to properly realize a single chiral fermions becomes clear and we will see this to be the case even at the level of continuum perturbation theory.

3. Wilson Fermions

Before, we proceed with the continuum regularization of chiral fermions, we briefly digress to understand a well known problem with realizing fermionic degrees of freedom on the lattice.

Since \( E(n_1, n_2) \) and \( \phi(n_1, n_2) \) are periodic functions, we can write them in terms of their Fourier components:

\[ E(n_1, n_2) = \frac{2\pi Q}{L_1 L_2} = \frac{1}{L_1 L_2} \sum_{p_1=-\left[\frac{L_1-1}{2}\right]}^{\left[\frac{L_1-1}{2}\right]} \sum_{p_2=-\left[\frac{L_2-1}{2}\right]}^{\left[\frac{L_2-1}{2}\right]} \tilde{E}(p_1, p_2) e^{i \frac{2\pi p_1 n_1}{L_1} + i \frac{2\pi p_2 n_2}{L_2}}, \] (3.1)
and
\[
\phi(n_1, n_2) = \frac{1}{L_1 L_2} \sum_{p_1 = -\left\lfloor \frac{L_1 - 1}{2} \right\rfloor}^{\left\lceil \frac{L_1 - 1}{2} \right\rceil} \sum_{p_2 = -\left\lfloor \frac{L_2 - 1}{2} \right\rfloor}^{\left\lceil \frac{L_2 - 1}{2} \right\rceil} \tilde{\phi}(p_1, p_2) e^{i \frac{2\pi p_1 n_1}{L_1} + i \frac{2\pi p_2 n_2}{L_2}}. \tag{3.2}
\]

(2.10) implies that \( \tilde{E}(0, 0) = 0 \) and (2.12) becomes
\[
\tilde{\phi}(p_1, p_2) = \begin{cases} 
0 & \text{if } p_1 = p_2 = 0 \\
-\frac{\tilde{E}(p_1, p_2)}{4 \sin^2 \left( \frac{\pi p_1}{L_1} \right) + \sin^2 \left( \frac{\pi p_2}{L_2} \right)} & \text{otherwise}
\end{cases}. \tag{3.3}
\]

In the continuum limit, \( p_\mu \), take on all integer values and since \( \beta \tilde{E}(p_1, p_2) \to \tilde{c}(p_1, p_2) \), we see that (3.3) correctly goes to the continuum limit of the solutions to (2.17) for all finite values of \( p_\mu \). The ultraviolet behavior is modified by the lattice regulator but the denominator in (3.3) which is the propagator of a Klein-Gordon field remains a monotonic function of the momenta. In particular, the only pole in the propagator in a Brillouin zone, \( p_\mu \in \left[ -\left\lfloor \frac{L_\mu - 1}{2} \right\rfloor, \left\lceil \frac{L_\mu - 1}{2} \right\rceil \right] \) is at \( p_1 = p_2 = 0 \).

Let \( \psi(n_1, n_2) \) be a Dirac field in two dimensions. In analogy with (2.12), we can write the Dirac operator as
\[
\slashed{D} \psi(n_1, n_2) = \sigma_1 [\psi(n_1 + 1, n_2) - \psi(n_1 - 1, n_2)] + \sigma_2 [\psi(n_1, n_2 + 1) - \psi(n_1, n_2 - 1)], \tag{3.4}
\]
which is
\[
\slashed{D} \tilde{\psi}(p_1, p_2) = 2i \left[ \sigma_1 \sin \left( \frac{2\pi p_1}{L_1} \right) + \sigma_2 \sin \left( \frac{2\pi p_2}{L_2} \right) \right] \tilde{\psi}(p_1, p_2) \tag{3.5}
\]
after a Fourier transform. Contrary to the case of Klein-Gordon field, the propagator of a Dirac field is not a monotonic function of the momenta and has additional zeros at the boundaries of the Brillouin zero when \( L_\mu \to \infty \). Therefore, we will end up with four massless Dirac fermions in the continuum limit of a two dimensional theory. This is referred to as the “doubling phenomenon”.

The number of particles do not change if we add a mass term but we could give masses to the particles at the boundary of the Brillouin zone keeping the fermion massless at the center of the Brillouin zone. If the masses for the particles at the boundary become infinite in the continuum limit, we will only have one massless fermion in the continuum limit and we could treat the infinitely massive fermions as regulator fields. In fact the simplest choice for a momentum dependent mass term is \( -\Box \psi \). The massless Wilson-Dirac operator is
\[
\slashed{D}_w = \slashed{D} - \Box, \tag{3.6}
\]
and will describe a single massless Dirac fermion in the continuum limit as can be seen by taking the limit of \( \sqrt{\beta} \slashed{D}_w \) in momentum space for momenta inside the Brillouin zone and for momenta in the boundary of the Brillouin zone.

Whereas the Wilson-Dirac operator solves the problems of doublers, adding a mass term breaks the chiral symmetry, namely,
\[
\sigma_3 \slashed{D} \sigma_3 = -\slashed{D}. \tag{3.7}
\]
In particular, it does not solve the problem when we want to write down a theory with Weyl fermions, namely,

$$\mathcal{C} = \mathcal{D} P_+ = P_- \mathcal{D},$$  \hspace{1cm} (3.8)

where

$$P_\pm = \frac{1 \pm \sigma_3}{2}. \hspace{1cm} (3.9)$$

In order to understand the solution to the problem, we will need to review chiral anomalies and fermion zero modes in a gauge field background with non-zero topological charge.

4. Continuum Pauli-Villars regularization

Let $$\psi_\pm(x) = P_\pm \psi(x)$$ and $$\bar{\psi}_\pm(x) = \bar{\psi}(x) P_\mp$$ denote a left(+) or right(-) handed fermion and anti-fermion at the Euclidean point $$x = (x_1, x_2)$$. It will be useful to define $$\partial = \partial_1 + i \partial_2$$ and $$\bar{\partial} = \partial_1 - i \partial_2$$. It will also be useful to define, $$A = A_1 + i A_2$$ and $$\bar{A} = A_1 - i A_2$$. The fermion action for a left handed field of charge $$q_+$$ is

$$S_{+f}(\bar{\psi}_+, \psi_+, A) = \int d^2 x \bar{\psi}_+ \partial \psi_+ + iq_+ \int d^2 x \bar{\psi}_+ A \psi_+ \hspace{1cm} (4.1)$$

and the fermion action for a right handed field of charge $$q_-$$ is

$$S_{-f}(\bar{\psi}_-, \psi_-, A) = \int d^2 x \bar{\psi}_- \partial \psi_- + iq_- \int d^2 x \bar{\psi}_- \bar{A} \psi_- \hspace{1cm} (4.2)$$

Using the Hodge decomposition in (2.15) and setting $$h_\mu = 0$$ and $$Q = 0$$ for the moment, we can write

$$A = \partial \xi; \hspace{1cm} \xi = \chi + i \phi. \hspace{1cm} (4.3)$$

Our aim is to compute the path integral over the fermion fields for a fixed background $$\xi$$. Consider the result of the path integral over left handed fermions:

$$e^{-W_+(A)} = \int [d\bar{\psi}] [d\psi] e^{S_{+f}(\bar{\psi}_+, \psi_+, A)}. \hspace{1cm} (4.4)$$

The change of variables,

$$\psi_+ \rightarrow \psi'_+ = e^{i \xi} \psi_+; \hspace{0.5cm} \bar{\psi}_+ \rightarrow \bar{\psi}'_+ = e^{-i \xi} \bar{\psi}_+; \hspace{0.5cm} \psi_- \rightarrow \psi'_- = e^{i \xi} \psi_-; \hspace{0.5cm} \bar{\psi}_- \rightarrow \bar{\psi}'_- = e^{-i \xi} \bar{\psi}_- \hspace{1cm} (4.5)$$

formally decouples the fermion action from the gauge field. Any dependence on $$\xi$$ due to the integration over the fermion fields should arise from the Jacobian. The Jacobian associated with such a transformation is not unity since there are divergences associated with the path integral.

To see the above argument in standard perturbation theory of (4.4), we write

$$\psi(x) = \frac{1}{\sqrt{l_1 l_2}} \sum_{p_\mu = -\infty}^{\infty} \psi(p) e^{\frac{2\pi p_1 x_1}{l_1} + \frac{2\pi p_2 x_2}{l_2}} \hspace{1cm} \text{and} \hspace{1cm} \bar{\psi}(x) = \frac{1}{\sqrt{l_1 l_2}} \sum_{p_\mu = -\infty}^{\infty} \bar{\psi}(p) e^{-\frac{2\pi p_1 x_1}{l_1} - \frac{2\pi p_2 x_2}{l_2}}$$

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\[ A(x) = \sum_{p_{\mu} = -\infty}^{\infty} \tilde{A}(p)e^{i \frac{2\pi p_{x_1}}{l_1} + i \frac{2\pi p_{x_2}}{l_2}}. \quad (4.6) \]

Note that (4.3) implies that \( \tilde{A}_\mu(0) = 0 \).

The free fermion propagator is

\[ \langle \tilde{\psi}^+ (q) \tilde{\psi}^+ (p) \rangle = -\frac{\delta_{pq}}{P}, \quad P = 2\pi i \left( \frac{p_1}{l_1} + i \frac{p_2}{l_2} \right), \quad (4.7) \]

and the interaction vertex is

\[ S_I = iq_+ \sum_{p_\mu, q_\mu = -\infty}^{\infty} \tilde{\psi}^+ (p + q) \sigma_\mu \tilde{A}_\mu(p) \tilde{\psi}^+ (q). \quad (4.8) \]

The contributions to \( W_+ (A_\mu) \) correspond to a single fermion loop with several insertions of \( A_\mu \):

\[
W_+ (A_\mu) = \sum_{k=1}^{\infty} (-iq_+)^k \sum_{p_\mu = -\infty}^{\infty} \left[ \prod_{j=1}^{k} \tilde{A}(p') \right] \sum_{r_\mu = -\infty}^{\infty} \frac{1}{R R + \mathcal{P}^1 R + \mathcal{P}^1 + \mathcal{P}^2 \cdot \cdot \cdot \mathcal{P}^{k-1}} \delta \left( \sum_{j=1}^{k} p' \right). \quad (4.9)
\]

There is no contribution from \( k = 1 \) since \( \tilde{A}(0) = 0 \). For \( k > 2 \), the sum over \( r_\mu \) will be convergent and we can rearrange the sum as follows. We can write each term as a sum over simple poles and the sum of the residues will be zero. We can shift the sum over \( r_\mu \), such that all poles are at zero. Since the sum over residues is zero, the sum will be zero. There is no contribution to \( W_+ (A_\mu) \) from \( k > 2 \).

In order to compute the contribution from \( k = 2 \), we will use a Pauli-Villars regularization that acts independently on the left and right handed fields and fully regulates the combined contribution from the left and right handed fields provided the full set of fields satisfy the anomaly cancellation condition:

\[
\sum_{i=1}^{n_+} (q^+_i)^2 = \sum_{j=1}^{n_-} (q^-_j)^2. \quad (4.10)
\]

If \( n_+ = n_- \) and \( q^+_i = q^-_i \) for all \( i \), we have a QCD-like theory and this can be regulated in the standard manner using a finite number of Pauli-Villars fields. If the left handed field content is not the same as the right handed field content, we need to have Pauli-Villars fields associated with a single left or right handed field. Since Pauli-Villars fields have a mass and have twice the number of degrees of freedom compared to a single left or right handed field, a finite number of Pauli-Villars fields is not sufficient to regulate a single
component. We therefore assume that the regulated path integral associated with (4.4) is of the form
\[ e^{-W_+(A_\mu)} = \int [d\psi][d\bar{\psi}] e^{\int d^2x \bar{\psi} \left[ \sigma_\mu (\partial_\mu + iq_+ A_\mu) + P_+ M + P_- M^\dagger \right] \psi}, \] (4.11)
where \( \psi \) represents an infinite set of Dirac fields, \( \psi_n, n = 0, \cdots, \infty \). Every Dirac field couples to the same gauge field and carries the same charge.
Under parity, \( x_1 \to -x_1 \) and \( x_2 \to x_2 \),
\[ A_1(x_1, x_2) \to -A_1(-x_1, x_2); \quad A_2(x_1, x_2) \to A_2(-x_1, x_2), \] (4.12)
and
\[ \psi(x_1, x_2) \to \sigma_2 \bar{\psi}(-x_1, x_2); \quad \bar{\psi}(x_1, x_2) \to \bar{\psi}(-x_1, x_2) \sigma_2; \] (4.13)
and we see that the action for a left handed field in (4.1) goes into the action for a right handed field in (4.2) with the same charge. We also see that the regularized action in (4.11) goes into an action with \( M \to M^\dagger \). We will require \( M \) to be an infinite dimensional complex mass matrix such that \( M \) has one zero mode and \( M^\dagger \) has no zero mode in order to correctly describe a single left handed fermion. Under parity, the zero mode shifts from one chirality to other as expected.

### 4.1 Free fermion propagator:

The free fermion part of the action in (4.11) is
\[ S_q = \int d^2x (\bar{\psi}_+(x) \bar{\psi}_-(x)) \left[ \frac{\partial}{M} M^\dagger \right] \left( \psi_+(x) \psi_-(x) \right), \] (4.14)
which is
\[ S_q = \sum_{p_\mu = -\infty}^{\infty} (\bar{\psi}_+(p) \bar{\psi}_-(p)) \left( P_1 + iP_2 \right) \left( M \right) \left( P_1 - iP_2 \right) \left( \bar{\psi}_+(p) \bar{\psi}_+(p) \right); \quad P_\mu = 2\pi i p_\mu \] (4.15)
using (4.6). The interaction vertex in momentum space is
\[ S_I = iq_+ \sum_{p_\mu = -\infty}^{\infty} \left[ \bar{\psi}_+(p + q) \bar{\psi}_+(q) \bar{\psi}_-(p + q) \bar{\psi}_-(q) \right], \] (4.16)
using (4.6).

The free fermion propagator is
\[ \langle \bar{\psi}_+(p) \bar{\psi}_+(p) \rangle = \frac{P_1 - iP_2}{|P|^2 + M^\dagger M}, \quad \langle \bar{\psi}_-(p) \bar{\psi}_-(p) \rangle = \frac{P_1 + iP_2}{|P|^2 + M^\dagger M}, \quad \langle \bar{\psi}_+(p) \bar{\psi}_-(p) \rangle = \frac{1}{|P|^2 + M^\dagger M}, \quad \langle \bar{\psi}_-(p) \bar{\psi}_+(p) \rangle = \frac{-M^\dagger}{|P|^2 + M^\dagger M}, \] (4.17)
where \( |P|^2 = -P_1^2 - P_2^2 \).

As a specific example, let us take the infinite dimensional space to be labelled by a discrete index, \( i = 0, 1, \cdots, \infty \). Let
\[ M_{jk} = \Lambda k \delta_{j, k-1}; \quad M^\dagger_{jk} = \Lambda j \delta_{j, k+1}; \] (4.18)
where $\Lambda$ is the regulator mass to be taken to $\infty$. The zero mode of $M$ is

$$v_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}; \quad Mv = 0,$$

(4.19)

and there is no zero mode for $M^\dagger$. Furthermore,

$$(MM^\dagger)_{jk} = \Lambda^2(j+1)^2\delta_{jk}; \quad (M^\dagger M)_{jk} = \Lambda^2j^2\delta_{jk}.$$  

(4.20)

Explicit expressions for the propagators are

$$G^{++}_{jj}(p) = \frac{P_1 - iP_2}{P^2 + \Lambda^2j^2}; \quad G^{--}_{jj}(p) = \frac{P_1 + iP_2}{P^2 + \Lambda^2(j+1)^2};$$

$$G^{+++}_{j(j+1)}(p) = \frac{\Lambda(j+1)}{P^2 + \Lambda^2(j+1)^2}; \quad G^{++-}_{j(j+1)}(p) = \frac{\Lambda(j+1)}{P^2 + \Lambda^2(j+1)^2};$$

(4.21)

for $j = 0, \cdots, \infty$. The only massless particle is one with positive chirality and all other poles in the propagator correspond to the regulator poles.

### 4.2 Computation of $W_+(A_\mu)$:

The spin-statistics is assigned as follows to successfully regulate the theory:

$$\bar{\psi}^+; \quad \psi^+ : \begin{cases} \text{fermion for even } j \\ \text{boson for odd } j \end{cases};$$

$$\bar{\psi}^-; \quad \psi^- : \begin{cases} \text{fermion for odd } j \\ \text{boson for even } j \end{cases}.$$

(4.22)

Only the $k = 2$ term contributes to (4.9) which can be written as

$$W_+(A_\mu) = q^2 \sum_{p_\mu = -\infty}^{\infty} \sum_{q_\mu = 0}^{\infty} (-1)^jG^{++}_{jj}(q - \frac{p}{2})G^{+++}_{jj}(q + \frac{p}{2})$$

$$+ \int A(p)A(-p)\sum_{q_\mu = 0}^{\infty} \sum_{j = 0}^{\infty} (-1)^jG^{--}_{jj}(q - \frac{p}{2})G^{++-}_{jj}(q + \frac{p}{2})$$

$$+ \int A(p)A(-p)\sum_{q_\mu = 0}^{\infty} \sum_{j = 0}^{\infty} (-1)^jG^{+-}_{jj}(q - \frac{p}{2})G^{++-}_{jj}(q + \frac{p}{2})$$

$$+ \int A(p)A(-p)\sum_{q_\mu = 0}^{\infty} \sum_{j = 0}^{\infty} (-1)^jG^{+-}_{jj}(q - \frac{p}{2})G^{++-}_{jj}(q + \frac{p}{2})$$

(4.23)

The sum over the internal momenta, $q_\mu$, convergent term by term in the expressions in the last two lines of (4.23). We focus on the first two terms and write the intermediate result before we perform the sum over $p_\mu$ and $q_\mu$ as

$$q^2 \left[ \int A_1(p)A_1(-p) + A_2(p)A_2(-p) \left[(Q_1 - \frac{P}{2})(Q_1 + \frac{P}{2}) - (Q_2 - \frac{P}{2})(Q_2 + \frac{P}{2}) \right] \right]$$

− 11 −
\[
\sum_{j=-\infty}^{\infty} \left( \frac{(-1)^j}{(Q - \frac{p}{\Lambda})^2 + \Lambda^2 j^2} \right)
\]
\[
+ i q^2 \left\{ \left[ \tilde{A}_2(p) \tilde{A}_1(-p) - \tilde{A}_1(p) \tilde{A}_2(-p) \right] \left[ (Q - \frac{p}{\Lambda}) (Q + \frac{p}{\Lambda}) \right] + \left[ \tilde{A}_1(p) \tilde{A}_2(-p) + \tilde{A}_2(p) \tilde{A}_1(-p) \right] \left[ (Q - \frac{p}{\Lambda}) (Q + \frac{p}{\Lambda}) \right] \right\}
\]
\[
= \left[ (Q - \frac{p}{\Lambda})^2 \right] \left( \frac{1}{(Q + \frac{p}{\Lambda})^2} \right) \tag{4.24}
\]

If we repeat the calculation for a regulated right handed fermion, we would have interchanged \( M \) and \( M^\dagger \) in (4.11) and we would have also interchanged the statistics in (4.22). Working through the intermediate steps, we would conclude that the we would have obtained the complex conjugate of the expression in (4.24) along with \( q_+ \rightarrow q_- \). If we satisfy, (4.10), then the resulting expression would be real. Since
\[
\sum_{j=-\infty}^{\infty} \frac{(-1)^j}{R^2 + \Lambda^2 j^2} = \frac{\pi}{2} \frac{1}{|R|} \sinh \left( \frac{\pi |R|}{\Lambda} \right), \tag{4.25}
\]
goestos zero exponentially fast as \( |R| \rightarrow \infty \), the sum over \( q_\mu \) will converge and result in a finite expression for \( W(A_\mu) \) for the anomaly free chiral gauge theory. Since each chiral fermion has its own infinite tower of Pauli-Villars regulator fields, the chiral symmetry that acts on each chiral fermion is still a symmetry of the regulated theory.

5. Overlap formalism

Our aim is to extend this idea to define a non-perturbative regularization of anomaly free chiral gauge theories. It is best to think of the infinite set of fields as an extra dimension and view the fermion fields as \( \psi(x, s) \) and \( \overline{\psi}(x, s) \). Our example in the previous section corresponds to
\[
M = \frac{\Lambda}{2} \left( \partial_s + s \right) \sqrt{-\partial_s^2 + s^2 - 1}. \tag{5.1}
\]
Let us view \( s \) as time and consider the many body Hamiltonian associated with (4.11). The Hamiltonian will be quadratic in the fermion (boson) creation and annihilation operator and will conserve particle number. The path integral is regulated because the ground state energy was arranged to be zero for our choice of the set of fermions and bosons. The ground state as \( s \rightarrow \infty \) is different from the ground state as \( s \rightarrow -\infty \) due to the \( s \) dependent operator \( M \). The result of the path integral is simply an overlap of these two ground states.

We make the above idea explicit by considering the following set up. We choose
\[
M = \partial_s + m(s); \quad m(s) = \begin{cases} 
\Lambda & \text{for } s > 0 \\
-m & \text{for } s \leq 0
\end{cases}; \quad m, \Lambda > 0. \tag{5.2}
\]
Clearly, $M$ has a zero mode in the space of normalizable functions, namely,

$$
\phi(s) = \begin{cases} 
e^{-s} & \text{for } s \geq 0 \\
\ne^{ms} & \text{for } s \leq 0
\end{cases} \tag{5.3}
$$

but $M^\dagger$ does not have a zero mode. In order to deal with spin-statistics, we first set $s \in [-l_s, l_s]$ with the aim of taking it to infinite at the end. Let the fermions with the above $M$ operator be referred to as $a$-type. We consider two additional sets of fermions with $s \in [-\frac{l_s}{2}, \frac{l_s}{2}]$ and

$$
M = \partial_s + m(s); \quad m(s) = -m, \tag{5.4}
$$

for one set ($b$-type) and

$$
M = \partial_s + m(s); \quad m(s) = \Lambda, \tag{5.5}
$$

for the other set ($c$-type).

The action in (4.11) for a given $M$ can be written as

$$
S_f = \int d^2x ds \bar{\psi} \left[ \sum_{\mu=1}^{3} \sigma_{\mu} (\partial_\mu + i q_+ A_\mu(x)) + m(s) \right] \psi. \tag{5.6}
$$

where the fields, $\bar{\psi}$ and $\psi$ are one of three types of fermions depending upon the choice of $m(s)$. In all three cases, we have a three dimensional operator with a specific three dimensional gauge field background that is obtained from the two dimensional gauge field background ($A_1(x); A_2(x); A_3 = 0$) and with mass term, namely, $m(s)$. It is best to use the second quantized picture to work out the result of the path integral since there is no gauge field in the third direction and the gauge fields in the two directions do not depend on the third direction. The two many body Hamiltonians associated with the path integrals with $s$ playing the role of time are

$$
H^\pm = a^\dagger H^\pm a; \quad H^\pm = \sigma_3 \left[ \sum_{\mu=1}^{2} \sigma_\mu (\partial_\mu + i A_\mu) + m^\pm \right]; \quad m^+ = \Lambda; \quad m^- = -m. \tag{5.7}
$$

Let $\Lambda_\pm$ and $|\pm\rangle$ be the highest eigenvalues and the corresponding eigenvectors of $H_\pm$. The $a$ type see both Hamiltonians, $b$ sees only $H^-$ and $c$ sees only $H^+$. Our aim is to take $l_s$ to infinity with free boundary conditions in the two ends but with the condition that $a$ type sees $|+\rangle$ as the boundary state on the positive $s$ side and $|-\rangle$ as the boundary state on the negative $s$ side;

$b$ type sees $|-\rangle$ as the boundary state on both ends;

$c$ type sees $|+\rangle$ as the boundary state on both ends.

Let $e^{W_a,b,c(A_\mu)}$ be the result of the path integral over the $a, b, c$ type fermions respectively. Then,

$$
e^{W_a(A_\mu)} = e^{l_s (\Lambda_+ + \Lambda_-)} (-|+\rangle); \quad e^{W_b(A_\mu)} = e^{l_s \Lambda_-}; \quad e^{W_c(A_\mu)} = e^{l_s \Lambda_+}. \tag{5.8}
$$

We set the regulated determinant of a single left handed Weyl fermion to

$$
e^{W_+(A_\mu)} = \lim_{l_s \to \infty} \frac{e^{W_a(A_\mu)}}{e^{W_b(A_\mu)} e^{W_b(A_\mu)}} = \langle -|+\rangle \tag{5.9}
$$
and this is the overlap formula. Since we divided by the result of the integral for $b$ and $c$ fermions, their statistics is opposite to that of the $a$ type fermions. Their purpose is the same as the Pauli-Villars regulator fields in the continuum formalism. They subtract the contributions coming from the non-zero modes of $M^\dagger M$. It turns out we can make one more simplification and take $\Lambda \to \infty$ since it does not affect our zero mode condition nor does it affect the dependence of the gauge field in an physical way as we will see.

The overlap formula trivially extends to any even dimensional chiral gauge theory. We will show in what follows that the overlap formula has all the correct continuum properties desired of a well regulated chiral determinant.

A lattice realization of the overlap formula is simple since $H^-$ is the Hermitian massive Dirac operator and we can replace it by

$$H_w = \sigma_3(\bar{\psi}_w - m)$$

using (3.6). We will assume this to be the case from now on and assume that $H^+ = \sigma_3 (\Lambda \to \infty)$ and $H^- = H_w$. We will also assume that the background gauge field are given by the lattice link variables, $U_{\mu}(x)$.

### 5.1 Phase of the overlap

The highest states of $\mathcal{H}^\pm$ are obtained by filling the positive eigenstates of $H^\pm$ respectively. For $U_{\mu}(x) = 1$, $H_w$ will have equal number of positive and negative eigenvalues and this will remain true in perturbation theory. For definiteness, let us set the size of the Hamiltonians to be $2n$. and set $\psi^+_k$, $k = 1, \cdots, n$, as the positive eigenstates of $H^\pm$ respectively. Then,

$$\langle -|+ \rangle = \det O; \quad O_{jk} = [\psi^+_j \psi^-_k]. \quad (5.11)$$

Since $H_+ = \sigma_3$, it is diagonal with

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

being the set of eigenvectors with positive eigenvalues and negative eigenvalues respectively. Let

$$H_w X = X \Lambda,$$

with

$$X = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}; \quad \Lambda = \text{diag}(\lambda^+_1, \cdots, \lambda^+_n, -\lambda^-_1, \cdots, -\lambda^-_n), \quad (5.13)$$

and $\lambda^\pm_i > 0$ for all $i$.

Then,

$$\langle -|+ \rangle = \det \alpha,$$

for the chiral determinant of left handed fermions.

Under a gauge transformation, $H_w$ will transform according to a unitary transformation, $V$, of the form

$$V(x, y) = e^{i\chi(x)} \delta(x - y); \quad H_w \to H_w^g = V^\dagger H_w V. \quad (5.15)$$
Under this gauge transformation, we can set

\[ X^g = \begin{pmatrix} V^\dagger_\alpha & V^\dagger_\gamma \\ V^\dagger_\beta & V^\dagger_\delta \end{pmatrix} \]  

(5.16)

as the eigenvectors of \( H_w^g \) with the same set of eigenvalues as \( H_w \). Since \( \int \chi(x) d^2x = 0 \), it is clear that we have

\[ \langle -|+ \rangle = \langle -|+ \rangle^g \]  

(5.17)

and we have a result that is formally gauge invariant. But this result is ambiguous since the choice \( X^g \) we made is not unique. In particular, we could have an arbitrary gauge field dependent phase that multiplies the overlap formula. Therefore, we have properly defined only the absolute value of the chiral determinant and the imaginary part of \( W_+(A_\mu) \) is not defined. The root of this problem can be seen in the continuum Pauli-Villars analysis. Only the real part was properly regularized in (4.24) and the imaginary part was only formally regularized in the case when anomalies cancelled. Now, we have a formula that is finite for all gauge field backgrounds. The only way out for the overlap formula is to have a phase that is ambiguous.

We can try to choose a phase in perturbation theory. If we solve for all eigenvalues for \( H_w \) when \( A_\mu = 0 \), we can do standard perturbation theory to find the eigenvectors in powers of \( A_\mu(x) \). The problem will be that the overlap of perturbed eigenvectors with the unperturbed ones will have an undetermined phase. The standard approach in quantum mechanics is to follow the Wigner-Brillouin phase choice, namely, one assumes that the overlap of the perturbed eigenvectors with the unperturbed ones is real and positive. If we call \( |-\rangle_0 \) as the unperturbed state, our overlap formula becomes

\[ e^{i W_+^{WB}(A_\mu)} = \frac{\langle -| - \rangle_0}{\langle -| - \rangle} \langle -|+ \rangle. \]  

(5.18)

There is no reason for this to be gauge invariant and we need to see if it the case or not. In particular, one needs to compute the variation of \( W_+^{WB}(A_\mu) \) under a gauge transformation and find out if it zero or not. If we find it to be non-zero, we still do not have an unambiguous statement about gauge invariance. In order to perform a careful analysis of this problem, one needs to study the variation of \( W_+(A_\mu) \) with respect to \( A_\mu \) and separate the current into an ambiguous and an unambiguous piece. This split of the current is the difference between consistent and covariant anomaly. An important result in the overlap formalism is the ability to find a covariant current that is unambiguous and therefore unambiguously identify the covariant anomaly. We will discuss other aspects of the overlap formula before we start our discussion of consistent and covariant anomalies. This is due to the simple fact that the overlap formula for a vector like theory is unambiguously defined.

### 5.2 Gauge field topology

The overlap formula in (5.9) will be exactly zero if the fermion number of the two ground states are not the same. The ground state, \( |+\rangle \), will be half filled. The number of positive
and negative eigenvalues of \( H_w \) need not be the same for all background gauge fields. To see this, note that we can write \( H_w \) in the form

\[
H_w = \begin{pmatrix} B - m & C \\ C^\dagger & -B + m \end{pmatrix}
\]  

(5.19)

where \( C = \frac{1 + \sigma_3}{2} \mathcal{D} \) is the chiral Dirac operator and \( B \) is the wilson term, \(-\Box\). Note that \( B \) is a positive definite Hermitian matrix. If

\[
H_w \begin{pmatrix} u \\ v \end{pmatrix} = 0; \quad u^\dagger u + v^\dagger v = 1
\]  

(5.20)

then

\[
(B - m)u + Cv = 0; \quad C^\dagger u - (B - m)v = 0.
\]  

(5.21)

Therefore it follows that

\[
u^\dagger Bu + v^\dagger Bv = m,
\]  

(5.22)

which can be satisfied provided \( m > 0 \). This also tells us that nothing was lost in sending \( \Lambda \to \infty \) in (5.2).

Consider a continuous evolution from \( U_\mu^Q = 1 \) to \( U_\mu^Q \) in (2.13) for \( Q = \pm 1 \). We expect an eigenvalue of \( H_w \) to cross zero at some point in this evolution. The overlap will be exactly zero after the eigenvalue crosses since the ground state, \(|-\rangle\), will have one less or more fermion number compared to \(|+\rangle\). One will need to insert a fermion creation or annihilation operator between the two ground states to make it non-zero and this is this expectation value of a single fermion is the zero mode in the presence of non-zero topological charge discussed in section 2.2. In essence we have a definition of the index on the lattice: Let \( n_+ \) and \( n_- \) be the number of positive and negative eigenvalues of \( H_w \) for a given gauge field configuration. The fermionic index is

\[
Q_f = \frac{1}{2}(n_+ - n_-).
\]  

(5.23)

5.3 Generating functional

In order to obtain a formula for a right handed fermion in a manner similar to (5.14) for left handed fermions, we choose

\[
M = \partial_s + m(s); \quad m(s) = \begin{cases} -m & \text{for } s > 0 \\ \Lambda & \text{for } s \leq 0 \end{cases}; \quad m, \Lambda > 0.
\]  

(5.24)

Clearly, \( M \) does not have a zero mode but \( M^\dagger \) has a zero mode. The effect of such a change is to interchange \( \mathcal{H}^+ \) with \( \mathcal{H}^- \). The result is

\[
\langle +| - \rangle = \det \alpha^\dagger.
\]  

(5.25)

for the chiral determinant of right handed fermions.

It is important to note that we need separate copies of the many body Hamiltonians, one for each left handed fermion and one for each right handed fermion. Let us therefore, refer to \( a_{RL} \) and \( a^\dagger_{RL} \) as the annihilation and creation operators of the right and left handed
fermions respectively. Let us label the corresponding many body Hamiltonians by $H^\pm_{R,L}$ and the ground sates by $|\pm\rangle_{R,L}$. Let

$$b_{R,L} = X^\dagger a_{R,L}$$

(5.26)

and it follows that $b_{R,L}, b^\dagger_{R,L}$ also obey canonical anti-commutation relations. Let us split

$$b_{R,L} = (u_{R,L}^\dagger d_{R,L}^\dagger)$$

(5.27)

into the $P_\pm$ pieces. Let $|0\rangle$ be the vacuum state that is annihilated by all the destruction operators. Then the highest state of $H^-_{R,L}$ is

$$|\mp\rangle_{R,L} = u_{R,L}^\dagger u_{R,L}^\dagger_{R,L,n} \cdots u_{R,L}^\dagger_{R,L,2} u_{R,L}^\dagger_{R,L,1} |0\rangle$$

(5.28)

and the highest state of $H^+_{R,L}$ is

$$|\pm\rangle_{R,L} = u_{R,L}^\dagger u_{R,L}^\dagger_{R,L,n} \cdots u_{R,L}^\dagger_{R,L,2} u_{R,L}^\dagger_{R,L,1} |0\rangle.$$  (5.29)

The generating functional of a left handed fermion is given by

$$Z_L(\bar{\xi}_L, \xi_L) = L \langle - | e^{\xi_L d_L^\dagger + \bar{\xi}_L u_L} | + \rangle_L$$

(5.30)

and

$$Z_R(\bar{\xi}_R, \xi_R) = R \langle + | e^{\xi_R u_R^\dagger + \bar{\xi}_R d_R} | - \rangle_R$$

(5.31)

for right handed fermions. The sources, $\bar{\xi}_R$, $\bar{\xi}_L$, $\xi_R$ and $\xi_L$ are all Grassmann variables and they anti-commute with the fermionic operators. Note that the generating functional has the following properties.

1. It does not depend on the ordering of the operators since the two terms in the exponent commute with each other in both factors.

2. It is clear that $d_L^\dagger$ and $u_L$ are the propagating degrees of freedom in the first factor since $d_L^\dagger |+\rangle$ and $u_L^\dagger |+\rangle$ are both zero. The converse holds for the second factor.

3. The generating functional is invariant under global chiral transformations:

$$\xi_R \rightarrow e^{i\varphi_R} \xi_R; \quad \bar{\xi}_R \rightarrow \bar{\xi}_R e^{-i\varphi_R}; \quad \xi_L \rightarrow e^{i\varphi_L} \xi_L; \quad \bar{\xi}_L \rightarrow \bar{\xi}_L e^{-i\varphi_L}.$$  (5.32)

4. The propagator for right-handed and left-handed fermions are

$$G^{ij}_L = \frac{L \langle - | d^\dagger_{L,j} u_{L,i} | + \rangle_L}{L \langle - | + \rangle_L}; \quad G^{ij}_R = \frac{R \langle + | u^\dagger_{R,j} d_{R,i} | - \rangle_R}{L \langle + | - \rangle_R}$$

(5.33)

and they obey the relation

$$G^\dagger_R = G_L.$$  (5.34)

Since the propagators of left and right handed fermions above are related by hermitian conjugation as opposed to anti-hermitian conjugation our definitions are for the hermitian Dirac operator obtained by a multiplication of the conventional anti-hermitian Dirac operator by $\sigma_3$. 

\[\text{– 17 –}\]
5.3.1 Right handed fermions:

We will now show that

\[ Z_R(\bar{\xi}_R, \xi_R) = \left[ e^{\bar{\xi}_R \beta \alpha^{-1} \xi_R \text{ det } \alpha^\dagger} \right] \]  

(5.35)

We start by noting that we can write

\[ \bar{\xi}_R d_R + \xi_R u_R^\dagger = Q_R^- + Q_R^+ \]  

(5.36)

where

\[ Q_R^+ = \bar{\xi}_R(\delta^{-1})^\dagger d_R + \xi_R u_R^\dagger \alpha^{-1}; \quad Q_R^- = -\bar{\xi}_R(\gamma \delta^{-1})^\dagger u_R - \xi_R d_R^\dagger \beta \alpha^{-1}, \]  

(5.37)

and we have used (5.13), (5.26) and (5.27). Since we can also write \( Q_R^+ \) as

\[ Q_R^+ = \bar{\xi}_R d_R + \xi_R u_R^\dagger - Q_R^- \]  

(5.38)

it follows that

\[ \left[ Q_R^-, Q_R^+ \right] = -\bar{\xi}_R \left( \beta \alpha^{-1} - (\gamma \delta^{-1})^\dagger \right) \xi_R, \]  

(5.39)

and

\[ e^{\bar{\xi}_R d_R + u_R^\dagger \xi_R} = e^{Q_R^-} e^{\frac{1}{2} \bar{\xi}_R(\beta \alpha^{-1} - (\gamma \delta^{-1})^\dagger) \xi_R}. \]  

(5.40)

We have used the identity,

\[ e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \]  

(5.41)

when \([A,B]\) is a c-number. Since

\[ R\langle + | e^{Q_R^-} = R\langle + |; \quad e^{Q_R^+}|-R = |-R, \]  

(5.42)

it follows that

\[ R\langle + | e^{\bar{\xi}_R d_R + u_R^\dagger \xi_R}|-R = e^{\frac{1}{2} \bar{\xi}_R(\beta \alpha^{-1} - (\gamma \delta^{-1})^\dagger) \xi_R} R\langle + |-R = e^{\bar{\xi}_R \beta \alpha^{-1} \xi_R \text{ det } \alpha^\dagger}, \]  

(5.43)

and we have used \( X^\dagger X = 1 \) to show that

\[ (\gamma \delta^{-1})^\dagger + \beta \alpha^{-1} = 0. \]  

(5.44)

5.3.2 Left handed fermions:

We will now show that

\[ Z_L(\bar{\xi}_L, \xi_L) = \left[ e^{\bar{\xi}_L [\beta \alpha^{-1}]^\dagger \xi_L \text{ det } \alpha} \right]. \]  

(5.45)

In the same manner as before, we can write

\[ \xi_L d_L^\dagger + \bar{\xi}_L u_L = Q_L^- + Q_L^+ \]  

(5.46)

where

\[ Q_L^+ = \bar{\xi}_L(\alpha^{-1})^\dagger u_L^\dagger + \xi_L d_L^\dagger \delta^{-1}; \quad Q_L^- = -\bar{\xi}_L(\beta \alpha^{-1})^\dagger d_L - \xi_L u_L^\dagger \gamma \delta^{-1}, \]  

(5.47)

and we have used (5.13), (5.26) and (5.27). Since we can also write \( Q_L^+ \) as

\[ Q_L^+ = \xi_L d_L^\dagger + \bar{\xi}_L u_L - Q_L^- \]  

(5.48)
it follows that
\[
[Q_L^+, Q_L^-] = -\xi_L \left( (\beta^{-1})^\dagger - \gamma \delta^{-1} \right) \xi_L, \tag{5.49}
\]
and
\[
e^\xi_L d_L^+ + \xi_L u_L = e^{Q_L^+} Q_L^- \frac{1}{2} \xi_L (\beta^{-1})^\dagger - \gamma \delta^{-1}) \xi_L. \tag{5.50}
\]
Since
\[
L\langle - | e^{Q_L^-} L \langle - | + \rangle_L = | + \rangle_L, \tag{5.51}
\]
it follows that
\[
L\langle - | e^{\xi_L d_L^+ + \xi_L u_L} | + \rangle_L = e^{\frac{1}{2} \xi_L ((\beta^{-1})^\dagger - \gamma \delta^{-1}) \xi_L L\langle - | + \rangle_L = e^{\xi_L (\beta^{-1})^\dagger \xi_L \det \alpha}. \tag{5.52}
\]

5.4 The massless overlap Dirac operator

A vector like gauge theory is obtained by pairing a left handed fermion with a right handed
fermion with the same charge (same representation of the gauge group). The fermion
determinant is real and positive since
\[
R\langle - | + \rangle_R L\langle + | - \rangle_L = \det \alpha \det \alpha^\dagger. \tag{5.53}
\]
The phase choice for \(| + \rangle_R L\) are tied together since they are the ground states of identical
many body operators. The same is true for \(| - \rangle_R L\). Therefore, the generating functional
is unambiguous and does not depend upon the phase choice present in the unitary
matrix, \(U\), that diagonalizes \(H_w\).

In practice, one can avoid exact diagonalization of \(H_w\) which is needed for the compu-
tation of \(X\) since one has an overlap-Dirac operator for vector like theories. Consider the
unitary operator,
\[
V = \sigma_3 \epsilon(H_w). \tag{5.54}
\]
It follows from (5.12) and (5.13) that
\[
\frac{1 + V}{2} X = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}; \quad \frac{1 - V}{2} X = \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix}, \tag{5.55}
\]
and therefore
\[
\frac{1 - V}{1 + V} = \begin{pmatrix} 0 & - (\beta^{-1})^\dagger \\ \delta \beta^{-1} & 0 \end{pmatrix}. \tag{5.56}
\]
Since
\[
\det X = \frac{\det \alpha}{\det \delta^\dagger}; \tag{5.57}
\]
we have the identity,
\[
\det \alpha \det \alpha^\dagger = \det \delta \det \delta^\dagger, \tag{5.58}
\]
and therefore,
\[
\det \frac{1 + V}{2} = \det \alpha \det \alpha^\dagger. \tag{5.59}
\]
We can use (5.56) and (5.59) along with an efficient implementation of \(V\) to compute the
generating functional. Note that the operator appearing in the determinant in (5.59) is
not identical to the operator used to compute the propagator in (5.56).
6. Chiral anomalies

The chiral determinant defined in (5.9) is a function of the background gauge field \( U_\mu(x) \). Let us combine \( \mu \) and \( x \) into one index, \( \alpha \), and refer to the background gauge field as \( \xi_\alpha \). Variation of the \( W^+ \) with respect to \( \xi_\alpha \) gives us the current. Since \(|\pm\rangle\) does not depend on \( \xi \) (we are assuming \( \Lambda \rightarrow \infty \)), the variation with respect to \( \xi \) comes solely from \(|-\rangle\). Even though \( H^- \) transforms covariantly under a gauge transformation, there is an arbitrary phase associated with \(|-\rangle\) as discussed in section 5.1 and it need not transform covariantly. Following standard notation, we will define

\[
j^{\text{cons}}(\xi) = \partial_\alpha W^+(\xi) = \frac{\langle \partial_\alpha - |- \rangle \rangle}{\langle -|+ \rangle d\xi_\alpha}
\]

as the consistent current. This current is an exact one form since

\[
dj^{\text{cons}} = 0
\]

but it will depend upon the phase of \(|-\rangle\) and therefore ambiguous.

It will be useful to look at the current in terms of our \( \phi(x) \) (gauge invariant) and \( \chi(x) \) (gauge transformation) variables. Setting \( Q = 0 \) and \( h_\mu = 0 \),

\[
j^{\text{cons}} = \frac{\partial \ln\langle -|+ \rangle}{\partial A_\mu(x)} d[\partial_\mu \chi(x)] + \frac{\partial \ln\langle -|+ \rangle}{\partial A_\mu(x)} d[\epsilon_{\nu\mu} \partial_\nu \phi(x)]
\]

\[
= -\partial_\mu \left[ \frac{\partial \ln\langle -|+ \rangle}{\partial A_\mu(x)} \right] d\chi(x) - \epsilon_{\nu\mu} \partial_\nu \left[ \frac{\partial \ln\langle -|+ \rangle}{\partial A_\mu(x)} \right] d\phi(x).
\]

If the chiral determinant is gauge invariant, the first term will be zero and the second term will only depend upon \( \phi(x) \). The difficulty arises when there is an anomaly. In this case, the first term will not be zero. Furthermore, both terms could depend upon \( \chi(x) \). In order to understand the problem of anomalous gauge theories within the context of the overlap formula, we will show that we can write

\[
j^{\text{cons}}(\xi) = j^{\text{cov}}(\xi) + \Delta j(\xi)
\]

where \( j^{\text{cov}}(\xi) \) transforms covariantly and is unambiguous independent of whether the underlying theory is anomalous or not. Even though, \( \Delta j(\xi) \) will depend on the phase choice,

\[
d\Delta j = (\partial_\alpha \Delta j_\beta - \partial_\beta \Delta j_\alpha) d\xi_\alpha d\xi_\beta = F_{\alpha\beta} d\xi_\alpha d\xi_\beta,
\]

will be unambiguous.

If anomalies cancel, we expect \( d\Delta j = 0 \), implying that \( \Delta j \) is exact. If so, we can redefine the phase of \(|-\rangle\) and get rid of \( \Delta j \) making the consistent current equal to the covariant current. The covariant current will not have an anomaly associated with it and the chiral determinant will be gauge invariant.

The situation on the lattice most likely requires a fine tuning of \( H_w \) in the following sense. Given a continuum gauge field background, we can restrict it to a finite lattice and compute \( d\Delta j \). For the fixed continuum gauge field, \( d\Delta j \) will go to zero as we take the
continuum limit on the lattice. But, lattice spacing effects might result in a non-zero \( d\Delta j \) and there is no phase choice that will make the chiral determinant gauge invariant on the lattice. It is possible that a finely tuned variation of \( H_w \) gives a zero \( d\Delta j \). In the absence of such a lattice Hamiltonian, we either have to argue that the non-zero \( d\Delta j \) on the lattice will not affect continuum physics or we have to take the continuum limit with a fixed gauge field background and then perform the path integral over gauge fields.

### 6.1 Splitting the current

We split \( |\partial_\alpha - \rangle \) into

\[
|\partial_\alpha - \rangle = |\partial_\alpha - \rangle_\perp + |-\rangle\langle-|\partial_\alpha - \rangle.
\]

(6.6)

where

\[
\langle-|\partial_\alpha - \rangle_\perp = 0
\]

(6.7)

since \( \langle-|\rangle = 1 \). We will show that

\[
j_{\alpha}^{\text{cov}} = \frac{\langle \partial_\alpha - |+\rangle}{\langle-|+\rangle},
\]

(6.8)

transforms covariantly and is unambiguous. Then,

\[
\Delta j_\alpha = \langle \partial_\alpha - |-\rangle,
\]

(6.9)

is the difference between the consistent and covariant current and it will be ambiguous.

A local gauge transformation is given by (2.1) and therefore we can write

\[
\delta \xi_\alpha^g = \delta \xi_\beta \left[D^{-1}(g)\right]_{\beta\alpha}.
\]

(6.10)

Under a gauge transformation, \( H^- \), will transform as

\[
H^-(\xi^g) = G^\dagger(g)H^- (\xi) G(g),
\]

(6.11)

where \( G \) is a unitary operator on the many body space since each \( U_\mu(x) \) transforms according to (2.1) and therefore \( H_w \) undergoes a unitary transformation.

Let us write,

\[
H^-(\xi + \delta \xi) - H^- (\xi) = \delta \xi_\alpha R_\alpha(\xi).
\]

(6.12)

Then

\[
H^-(\xi^g + \delta \xi^g) - H^- (\xi^g) = \delta \xi^g_\alpha R_\alpha(\xi^g) = \delta \xi_\beta \left[D^{-1}(g)\right]_{\beta\alpha} R_\alpha(\xi^g),
\]

(6.13)

where we have used (6.10). But, using (6.11) and (6.12),

\[
H^-(\xi^g + \delta \xi^g) - H^- (\xi^g) = G^\dagger(g) \left[H^- (\xi + \delta \xi) - H^- (\xi) \right] G(g) = \delta \xi_\beta G^\dagger(g) R_\beta(\xi) G(g).
\]

(6.14)

Combining the two equations above,

\[
R_\alpha(\xi^g) = [D(g)]_{\alpha\beta} G^\dagger(g) R_\beta(\xi) G(g).
\]

(6.15)
Using standard perturbation theory,
\[
|\partial_\alpha-\rangle_\perp(\xi) = \frac{1}{\mathcal{H}_-^{\perp}(\xi) - E_0(\xi)} \left[\langle -|R_\alpha(\xi)|-\rangle - R_\alpha(\xi)|-\rangle\right]; \quad \mathcal{H}_-^{\perp}(\xi)|-\rangle = E_0(\xi)|-\rangle. \tag{6.16}
\]

Under a gauge transformation,
\[
|\partial_\alpha-\rangle_\perp^g(\xi) = e^{i\phi(\xi,g)}D_\alpha^g\alpha(\xi,g)|\partial_\alpha-\rangle_\perp(\xi), \tag{6.17}
\]
follows from (6.11) and \(\phi(\xi,g)\) is an arbitrary phase that can depend on the background gauge field, \(\xi\), and the gauge transformation, \(g\). Since \(\mathcal{H}_+\) does not change under a gauge tranformation, it follows that
\[
G(g)|+\rangle = |+\rangle. \tag{6.18}
\]

A straigh forward calculation using (6.15) and (6.17) results in
\[
|\partial_\alpha-\rangle_\perp^g(\xi) = e^{i\phi(\xi,g)\mathcal{D}_\alpha_\beta}(g)\mathcal{G}_1^\dagger(g)|\partial_\beta-\rangle_\perp(\xi), \tag{6.19}
\]
and therefore
\[
\left[j^{\text{cov}}_\alpha\right]^g = \left[j^{\text{cov}}_\beta\right] \left[D^{-1}(g)\right]_{\beta\alpha}, \tag{6.20}
\]
using (6.18) showing that it transforms covariantly. In addition, the arbitrary phase factor, \(\phi(\xi,g)\), disappears and the covariant current does not depend on the phase choice.

Following (6.3), we can compute the covariant anomaly by considering the variation with respect to \(\chi\) in (6.8) as opposed to \(\partial_\mu\chi\). Since the phase choice does not matter, we can follow the steps in section 5.1. The covariant anomaly is given by
\[
A_{\text{cov}} = 1 - \text{det} \left( \alpha^\dagger V^\dagger \alpha + \beta^\dagger V^\dagger \beta \right), \tag{6.21}
\]
where we have used \(\text{det} V = 1\). Assuming \(\chi(x)\) is infinitesimal, we get
\[
A_{\text{cov}} = i \sum_x \chi(x) \sum_i \left( \alpha_i^\dagger \alpha_{xi} + \beta_i^\dagger \beta_{xi} \right) = i \frac{1}{2} \sum_x \chi(x) \text{tr} \epsilon_{xx}, \tag{6.22}
\]
where the little trace is over the two spin components. Since the fermionic index, \(Q_f\), defined in (5.23) can be written as \(\sum_x \text{tr} \epsilon_{xx}\), the connection between anomaly and topology follows.

In order to show that \(\mathcal{F}_{\alpha\beta}\) as defined in (6.5) is also independent of the phase choice, we start by noting that
\[
\mathcal{F}_{\alpha\beta} = \langle \partial_\alpha - |\partial_\beta-\rangle - \langle \partial_\beta - |\partial_\alpha-\rangle. \tag{6.23}
\]
Let us define the projector
\[
P = |\partial_\alpha-\rangle\langle -|; \quad P^2 = P, \tag{6.24}
\]
Clearly, \(P\) does not depend on the phase choice of the ground state. A simple computation shows that
\[
\mathcal{F}_{\alpha\beta} = \text{Tr} \left[ (\partial_\beta P) P (\partial_\alpha P) - (\partial_\alpha P) P (\partial_\beta P) \right], \tag{6.25}
\]
implying that it is unabiguos.
6.2 Sample numerical calculations

We demonstrate the presence of $\mathcal{F}$ in the continuum if the theory is anomalous and show that the cancellation is not always exact on the lattice even if the continuum theory has no anomalies. We also demonstrate the ambiguous nature of the consistent current and $\Delta j$ analytically by computing these quantities in lattice perturbation theory. Finally we show the presence of topological zero modes in the presence of gauge fields that carry a topological change. We choose to perform a numerical calculation since it will help lay out the details of an overlap calculation on the lattice. We will set $L_1 = L_2$ in what follows. In order to not have a chirally biased presentation, we will work with right handed fermions in the subsection.

6.2.1 $\mathcal{F}$ in anomalous and anomaly free cases

We will set $\phi = \chi = 0$ and also restrict ourselves to $Q = 0$. For the case of,

$$U_{\mu}(n_1, n_2) = e^{i \frac{\pi h_{\mu}}{L}}; \quad \mu = 1, 2$$  \hspace{1cm} (6.26)

the eigenfunctions of $H_w$ come in pairs of the form

$$\frac{1}{\sqrt{2\mu(\mu - \alpha)}} \begin{pmatrix} \beta & \mu - \alpha \\ \mu - \alpha & -\beta^* \end{pmatrix},$$  \hspace{1cm} (6.27)

with eigenvalues $\mu$ and $-\mu$ where

$$\begin{align*}
\mu^2 &= \alpha^2 + \beta \beta^*; \\
\alpha &= 2 \sum_{\mu} \sin^2 \left( \frac{\pi}{L} \left( p_{\mu} + \frac{h_{\mu}}{2} \right) \right) - m; \\
\beta &= i \left\{ \sin \left[ \frac{2\pi}{L} \left( p_1 + \frac{h_1}{2} \right) \right] - i \sin \left[ \frac{2\pi}{L} \left( p_2 + \frac{h_2}{2} \right) \right] \right\}; \\
&\quad \frac{L}{2} \leq p_{\mu} < \frac{L}{2}.
\end{align*}$$  \hspace{1cm} (6.28)

The chiral determinant is therefore,

$$e^{W-(h_{\mu})} = \prod_{p_{\mu}} \frac{\beta}{\sqrt{2\mu(\mu - \alpha)}}.$$  \hspace{1cm} (6.29)

Consider the continuum limit, $L \to \infty$, keeping $m > 0$ fixed. For the case of $p_{\mu} \approx 0$, we can write

$$\begin{align*}
\alpha &\approx 2 \frac{\pi^2}{L^2} \sum_{\mu} \left( p_{\mu} + \frac{h_{\mu}}{2} \right)^2 - m; \\
\beta &\approx i \frac{2\pi}{L} \left\{ \left( p_1 + \frac{h_1}{2} \right) - i \left( p_2 + \frac{h_2}{2} \right) \right\}.
\end{align*}$$  \hspace{1cm} (6.30)

Therefore, $\mu$ leads off as $m$ and $\sqrt{2\mu(\mu - \alpha)}$ leads off as $2m$ implying that the contribution to the chiral determinant from these modes is proportional to $\left( p_1 + \frac{h_1}{2} \right) + i \left( p_2 + \frac{h_2}{2} \right)$. For the case of $p_1 \approx \frac{L}{2}$ and $p_2 \approx 0$, we can write

$$\begin{align*}
\alpha &\approx 2 \frac{\pi^2}{L^2} \sum_{\mu} \left( p_{\mu} + \frac{h_{\mu}}{2} \right)^2 + 2 - m; \\
\beta &\approx -i \frac{2\pi}{L} \left\{ \left( p_1 + \frac{h_1}{2} \right) + i \left( p_2 + \frac{h_2}{2} \right) \right\}.
\end{align*}$$  \hspace{1cm} (6.31)
Let us assume that $0 < m < 2$. Then,

$$\mu \approx (2 - m) \left[ 1 + \frac{3 - m}{2(2 - m)^2} \frac{4\pi^2}{L^2} \sum_{\mu} \left( p_{\mu} + \frac{h_{\mu}}{2} \right)^2 \right], \quad (6.32)$$

and therefore, $\sqrt{\mu(\mu - \alpha)}$ leads off as $\frac{2\pi}{L} \sqrt{\sum_{\mu} \left( p_{\mu} + \frac{h_{\mu}}{2} \right)^2}$. The contribution from these modes to the chiral determinant is therefore of order one. If $m > 2$, the contribution from these modes to the chiral determinant would be that of a fermion with negative chirality. In order for the overlap to properly reproduce a single chiral fermion. The mometa associated with the physical chiral fermion is a ball centered around $p_1 = p_2 = 0$ that does not touch the boundary of the first Brillouin zone.

The phase of the eigenfunctions in (6.27) are arbitrary. If one used the Wigner-Brillouin phase choice, then the result is

$$e^{W_{WB}(h_\mu)} = e^{\frac{\pi}{2} \beta(h_k - h) \frac{\partial(h_\mu + i)}{\eta(i)}}, \quad h = h_1 + ih_2, \quad (6.33)$$

in the continuum. We will focus on $j^\text{cov}_{\alpha}$ and $F_{\alpha\beta}$ that do not depend on the phase choice.

We have two degrees of freedoms, namely, $h_1$ and $h_2$. We can compute $F_{\alpha\beta}$ using (6.25).

The projector, $P$, has a block diagonal form with one $2 \times 2$ block for each $p_\mu$ given by

$$P_p(h) = \frac{1}{2} - \frac{1}{2m} [\alpha\sigma_3 + \beta_1 \sigma_1 - \beta_2 \sigma_2] = \frac{1}{2} + \frac{1}{2} \bar{w}_p(h) \cdot \vec{\sigma}, \quad (6.34)$$

where $\beta_\mu = \frac{2\pi}{L} \left( \frac{h_\mu}{2} \right)$ and the components of the unit vector, $\hat{w}_k(h)$ are $w^1_p(h) = \frac{\beta_3}{\mu}$, $w^2_p(h) = -\frac{\beta_2}{\mu}$ and $w^3_p(h) = \frac{\beta_1}{\mu}$. The only non-zero component is

$$F_{12}(h) = \frac{i}{2} \sum_p w_p \cdot \left( \frac{\partial\bar{w}_p}{\partial h_1} \wedge \frac{\partial\bar{w}_p}{\partial h_2} \right). \quad (6.35)$$

In the continuum limit, $L \to \infty$, the sum over $p$ becomes an integral over a torus and the result will become independent of $h_\mu$. The integrand is just the area element on the surface of a sphere therefore the result of the integral is just a count of the number of times the map from the torus wraps around the sphere. This will depend on the sign on $m$ and the result is zero wrapping for $m < 0$ and one wrapping for $0 < m < 2$. This is illustrated in fig. 1 obtained for a fixed $L$ by performing the sum in (6.35) for one arbitrary choice of $h_\mu$.

The result clearly shows the contribution from one chiral fermion centered around the ball at $p_\mu = 0$ for $0 < m < 2$. Since the region around $p_1 = \frac{L}{2}, p_2 = 0$ and $p_1 = 0, p_2 = \frac{L}{2}$ also contribute for $2 < m < 4$ with opposite chirality as that of the region around $p_\mu = 0$, we obtain a result corresponding to one Dirac fermion ($F_{12} = 0$) and one fermion of negative chirality. If $m > 4$, the region around $p_\mu = \frac{L}{2}$ contribute with the same chirality as that of the region around $p_\mu = 0$ resulting in a zero result due to two Dirac fermions.

The result for $F_{12}$ in the continuum limit is proportional to $q^2$ and opposite in sign for fermions with negative chirality. Therefore, we expect it to be zero for a chiral gauge theory with one right handed fermion of charge $q_+ = 2$ and four right handed fermions of
charge $q_- = 1$. But, we do not expect the cancellation to be exact for finite $L$. Therefore, even though we have chiral fermions on the lattice we do not have exact cancellation of anomalies for a chiral gauge theory that is anomaly free in the continuum. Furthermore, the non-cancellation of the anomalies on the lattice is a property of the lattice Hamiltonian alone since the computation of $F_{12}$ only depends on the Hamiltonian and is independent of the phase choice of the ground states. The non-cancellation of the anomalies is related to the dependence of $F_{12}$ on $h_\mu$ as can be seen for $q = 2$ and $L = 8$ in fig. 2. For the special case of gauge fields that only contain $h_\mu$, we can achieve exact cancellation of anomalies by the following judicious choice: Let the $q_+ = 2$ fermion be on a $L^2$ lattice and obey periodic boundary conditions in both directions. Let the four $q_- = 1$ fermions be on a $(L/2)^2$ lattice and obey four different boundary conditions, namely: AA, AP, PA, PP, where A is anti-periodic and P is periodic and the pair of symbols denote the boundary conditions in the two directions. Even though, each fermionic component results in a $F_{12}$ that depends on $h_\mu$, the 1112 model is anomaly free on the lattice for this set of gauge fields as can be seen in fig. 2. This can be traced back to an identity obeyed by the theta function appearing in (6.33).

6.2.2 Consistent and covariant anomalies

As discussed in section 4, there is only a quadratic contribution to $W_-(A_\mu)$. We may
Figure 2: A plot of $F_{12}$ in (6.35) with $h_1 = 0.37$ as a function of $h_2$ for the various components of the 1112 model.

Therefore write the general expression in momentum space as

$$ W_-(A_\mu) = \sum_p \left[ -\phi(p)\phi(-p)f(p) + i\phi(p)\chi(-p)g(p) \right]. \tag{6.36} $$

We could compute $f(p)$ and $g(p)$ using lattice perturbation theory but our aim here is to perform a numerical computation. We start by choosing a gauge field. We set $Q = 0$ so that we are in the perturbative sector. We set $h_1 = h_2 = \frac{1}{2}$ in (2.11) which is equivalent to choosing anti-periodic boundary conditions. This avoids a zero eigenvalue for the free massless Dirac operator which would have resulted in a zero for the chiral determinant. We set $\chi = 0$ for our base gauge field configuration. We choose $\phi$ to have a fixed momentum, namely,

$$ \phi(n_1, n_2) = \phi_0 \cos \left[ \frac{2\pi}{L}(p_1 n_1 + p_2 n_2) \right]. \tag{6.37} $$

If we choose a value for $p_\mu$ and $\phi_0 = 0.1$ and set $m = 1$, we have all the ingredients to explicitly write down $H_w$ on a finite $L \times L$ lattice. We can compute $W_-(A_\mu)$ using (5.9) but it would depend on the phase choice for $|\rangle$. But the real part of $W_-(A_\mu)$ does not depend on the phase choice and a computation using several different choices for $p_\mu$ and $\phi$ will show that

$$ \lim_{L \to \infty} \text{Re}[W_-(\phi_0, p)] = -\frac{\pi}{2}p^2 \phi_0^2. \tag{6.38} $$
A sample computation for $p_1 = 1$, $p_2 = 0$, $\phi_0 = 0.1$ as a function of $L$ is illustrated in Fig. 3.

![Graph](image)

**Figure 3:** A plot of the real part of $W_-(A_\mu)$ as a function of $L$ for a specific choice of the background gauge field given by (6.37).

The imaginary part of $W_-(A_\mu)$ will depend upon on the phase choice of $|\rangle$. One possible choice is to pick the phase of $|\rangle$ such that the imaginary part is zero for all $\phi$ when $\chi = 0$. Since we can obtain the eigenstate for any $\chi$ by a gauge transformation from $\phi = 0$, we can use this to set the phase of $|\rangle$ for all $\chi$. Due to the nature of $|\rangle$, it is easy to see that the imaginary part will be zero for all $\phi$ and $\phi$. Another possible choice is the Wigner-Brillouin phase choice discussed in section 5.1. This will not result in the imaginary part of $W^{WB}(A_\mu)$ being zero for all $A_\mu$. But, we can compute the consistent and covariant anomalies by computing these currents by varying $\chi$ for a fixed $\phi$ as discussed in (6.3). We choose $\chi$ to have the same momentum as $\phi$ to obtain a non-zero anomaly. The results are shown in Fig. 4 and Fig. 5. The covariant anomaly is non-zero and independent of the phase choice as expected but the consistent anomaly depends on the phase choice. The presence of a covariant anomaly unambiguously shows that the continuum theory is anomalous. One can proceed like in the previous section and study the 1112 model. For smooth backgrounds like the one considered here, the model will look anomaly free on the lattice. But, this will not be the case for an arbitrary choice of $\phi$ and $\chi$. The special choice of boundary conditions and lattices prescribed in the previous subsection will not render
the lattice theory anomaly free for an arbitrary $\phi$ and $\chi$.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{A plot of the consistent anomaly as a function of $L$ for a specific choice of the background gauge field given by (6.37).}
\end{figure}

### 6.2.3 Topology on the lattice

The gauge field configuration given in (2.13) has a topological charge equal to $Q$ for integer values of $Q$. But, the configuration is well defined on the lattice for any real value of $Q$ and smoothly interpolates between different topological sectors. This is clearly a notion that is not present in the continuum since (2.16) can only be defined for integer values of $Q$. Furthermore, we also know that $Q$ has to be an integer in (2.10). We can compute $Q_f$ as defined in (5.23) for a choice of $Q$. We fix $Q = 1$ and plot $Q_f$ as a function of $m$ for two different values of $L$ in fig. 6. We make two remarks. The index does not switch from $Q_f = 0$ to $Q_f = 1$ exactly at $m = 0$ for $L = 5$. This is a lattice artifact which goes away in the continuum limit. The index for $2 < m < 4$ is that of a fermion with negative chirality since we have one Dirac fermion and one fermion of negative chirality in this region. The index is zero for $m > 4$ since we have two Dirac fermions. Treating $Q$ as a real number in (2.13), we plot $Q_f$ vs $Q$ for $L = 8$ and $m = 1$ in fig. 7. Clearly, $Q_f$ has to be an integer and the switch occurs for a $Q$ close to half integer. The lattice field configuration close to switch does not have a continuum analog.
Figure 5: A plot of the covariant anomaly as a function of $L$ for a specific choice of the background gauge field given by (6.37).

We can take the uniform $Q = -1$ background on a $L^2$ lattice and compute $\text{tr} \epsilon_{xx}$ to obtain the anomaly as per (6.22). We will find this to be uniform in $x$ and equal to $-\frac{1}{L^2}$ such that the sum over the lattice gives the correct value of $Q$. We can add a $\phi$ component to the uniform background and we can compute the local distribution of topological charge. There is a single zero mode in the continuum given by (2.34) when $Q = -1$. We can compute the unnormalized zero mode on the lattice by computing

$$\psi(x) = L\langle -|u_L(x)|+\rangle_L$$

as per the generating functional in (5.30) where $u_L(x)$ stands for the operator at the location $x$. We plot $\frac{\psi(x,x)}{\psi(0,0)}$, the diagonal part of the zero mode, in fig. 8 and compare it with the corresponding function in the continuum.

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The above set of references serve only as a guide to further understand the topics covered in these lectures. The lectures themselves did not cover all developments pertaining to overlap fermions. Papers cited by the above list of references will provide a further understanding of lattice chiral gauge theories and citations of the above references lead to further developments.