A low-energy perturbation theory is developed from the nonperturbative framework of covariant Loop Quantum Gravity (LQG) [1]. In the formulation, a spin foam amplitude \(A(\mathcal{K})\) is defined on a given simplicial manifold \(\mathcal{K}\) for the transition of boundary quantum 3-geometries (spin-network states in LQG) [25]. The spin foam amplitude sums over the history of spin-networks, and suggests a foam-like quantum space-time structure.

In this paper, a low-energy perturbation theory is developed from the nonperturbative framework of LQG. The perturbation theory explains how classical gravity emerges from the group-theoretic spinfoam formulation, and provides the high-energy (high curvature) and quantum corrections. Importantly, the perturbation theory developed here shows for the first time that covariant LQG produce the high curvature corrections, which modifies the UV behavior of Einstein gravity. And it is the first time that a systematic way is developed to compute the high curvature corrections from a full LQG framework.

The discussion here focuses on the Lorentzian spinfoam amplitude proposed by Engle-Pereira-Rovelli-Livine (EPRL) [2]. The nonperturbative construction of EPRL spinfoam amplitude is purely (quantum-)group-theoretic. As one of the representations [3], the EPRL spinfoam amplitude reads

\[
A(\mathcal{K}) = \sum_{\mathbf{f}} d_{\mathbf{f}} \text{tr} \left[ \prod_{e \in \mathcal{K}} P^\text{inv}_e \right] \tag{1}
\]

where \(g_{\text{ve}}^e\) is a \(\text{SL}(2, \mathbb{C})\) group variable associated with each dual half-edge. \(z_{\mathbf{f}}\) is a 2-component spinor. The spinfoam action \(S\) given by

\[
S \left[ J_f, g_{\text{ve}}, z_{\mathbf{f}} \right] = \sum_{(e,f)} \left[ J_f V_f \left[ g_{\text{ve}}, z_{\mathbf{f}} \right] + i y J_f K_f \left[ g_{\text{ve}}, z_{\mathbf{f}} \right] \right] \tag{3}
\]

where the short-hand notations \(V_f\) and \(K_f\) are defined by

\[
V_f = \ln \left[ (Z_{\text{vef}}, Z_{\text{vef}})^2 (Z_{\text{vef}}, Z_{\text{vef}})^{-1} (Z_{\text{vef}}, Z_{\text{vef}})^{-1} \right]
\]

\[
K_f = \ln \left[ (Z_{\text{vef}}, Z_{\text{vef}}) (Z_{\text{vef}}, Z_{\text{vef}}) \right] \tag{4}
\]

with \(Z_{\text{vef}} = g_{\text{ve}}^e z_{\mathbf{f}}, \ y \in \mathbb{R}\) is the Barbero-Immirzi parameter.

Practically, we apply the background field method to Eq.(2) and consider the perturbations of the spinfoam variables around a given background configuration (spinfoam data \((J_f, g_{\text{ve}}, z_{\mathbf{f}})\) on \(\mathcal{K}\)) [26]. The perturbative expansion is performed in the semiclassical and low-energy regime. Such a regime can be specified in the following way: The existing semiclassical results suggest that the semiclassical geometry emerges from spinfoam is discrete with a (triangle-area) spacetime scale \(\alpha_f\) comes from \(h \to 0\) and implies the semiclassicality. \(\alpha_f \ll L^2\) implies the low-energy approximation, since it requires that the mean wavelength of the gravitational fluctuation is much larger than the lattice scale. Adapting Eq.(5) to the spinfoam formulation, \(\ell_p^2 \ll \alpha_f\) can be implemented by \(J_f \gg 1\) for all \(f\), while \(\alpha_f \ll L^2\) means that the deficit angle \(|\Theta_f| \ll 1\) for all \(f\), because \(|\Theta_f| \sim \alpha_f/L^2[1 + o(\alpha_f/L^2)]\) [18]. In the following, the perturbative analysis of spinfoam amplitude \(A(\mathcal{K})\) is performed with respect to a certain background spinfoam configuration in the semiclassical low-energy regime Eq.(5). The
The expansion parameters $A_{\mathbf{f}}$ employed to study the asymptotic behavior of the partial amplitude $W[\mathbf{f}]$ in the large-$J$ regime. The partial amplitude $A_{\mathbf{f}}(\mathcal{K})$ has been defined by collecting the $(g, z)$-integrals in Eq. (2). The spins $f_j \equiv \lambda_j$ is large for all $f_j$, where $\lambda \gg 1$ is the mean value of $J_f$. By the linearity of $S[\mathbf{f}; g_{\mathbf{f}}, z_{\mathbf{f}}]$ in $J_f$, the stationary phase analysis is employed to study the asymptotic behavior of the partial amplitude $A_{\mathbf{f}}(\mathcal{K})$ as $J_f$ uniformly large. Such an analysis has been developed in [6] in the asymptotics, the leading contribution of $A_{\mathbf{f}}(\mathcal{K})$ comes from the spinfoam critical configurations, i.e. the solutions of $\mathcal{K} = 0$ and $\partial_{\mathbf{f}} S = 0$. It turns out that each critical configuration is interpreted as a certain type of geometry on $\mathcal{K}$. Moreover the critical configurations also know if the manifold is oriented and time-oriented [6]. As a result, the critical configurations are classified according to their geometrical interpretations and the information about orientations:

|       | $\mathcal{V}_f$ | $\mathcal{K}_f$ |
|-------|----------------|----------------|
| Lorentz Time-Oriented | 0 | $\varepsilon \text{ sgn}(V_{\mathbf{f}}(\Theta_{\mathbf{f}}))$ |
| Lorentz Time-Unoriented | $i\pi$ | $\varepsilon \text{ sgn}(V_{\mathbf{f}}(\Theta_{\mathbf{f}}))$ |
| Euclidean | $ie \text{ sgn}(V_{\mathbf{f}}(\Theta_{\mathbf{f}}^2 + \pi n_{\mathbf{f}}))$ | 0 |
| Vector | $i\Theta_{\mathbf{f}}$ | 0 |

The first 2 classes of critical geometries on $\mathcal{K}$. Each critical configuration $(\mathbf{f}; g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}})$ in the first 2 classes is equivalent to a set of geometrical data $(\varepsilon, E(t), 0)$ [6] with $\varepsilon = \pm 1$. $E(t)$ is a cotetrad on $\mathcal{K}$ (the edge-vectors satisfying some conditions), up to a overall sign $\varepsilon$, in each 4-simplex. $E(t)$ determines the oriented volume $V_{\mathbf{f}}(\mathcal{K}) = \det(E(t))$. The local spacetime orientation is defined by $\text{sgn}(V_{\mathbf{f}})$. $E(t)$ also determines uniquely a spin connection $\Omega_{\varepsilon} \in \text{SO}(1,3)$ along each dual edge $e$. The critical configuration gives a locally time-oriented spacetime if the corresponding spin connection along a closed loop $\Omega_{\varepsilon} = \prod_{e \in \mathcal{E}} \Omega_{\varepsilon} \in \text{SO}(1,3)$. Additionally, the last 2 classes of critical configurations give the Euclidean simplicial geometry and degenerate vector geometry on $\mathcal{K}$. It turns out that $\mathcal{V}_f$ and $\mathcal{K}_f$ defined in Eq. (6) take different values in each class of critical configurations, as is shown in the above table. Here $\Theta_{\mathbf{f}}(\Theta_{\mathbf{f}})$ denotes the Lorentzian (Euclidean) deficit angle, $\Phi_{\mathbf{f}}$ denotes the vector-geometry angle, and $n_{\mathbf{f}} \in \{0, 1\}$.

Here we consider the perturbations of the spinfoam variables around a critical configuration $(\mathbf{f}; g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}})$ in the 1st class, which corresponds to the globally oriented and time-oriented Lorentzian simplicial geometry with $\text{sgn}(V_{\mathbf{f}}) = 1, \tilde{\varepsilon} = -1$ globally. It turns out that Einstein gravity is recovered from the perturbations around such a background. The background deficit angles $|\Theta_{\mathbf{f}}| \ll 1$ since we are interested in the low-energy perturbations. The background spins $\tilde{\lambda}_f = \lambda_f$ with $\lambda \gg 1$ for the semiclassical approximation.

The partial amplitude can be written as $A_{\mathbf{f}}(\mathcal{K}) = \exp(iW[\mathbf{f}])$, where $W[\mathbf{f}]$ is an effective action obtained by integrating out the $(g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}})$-variables in Eq. (2). $W[\mathbf{f}]$ is computed in a neighborhood at the background $(\mathbf{f}; g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}})$, by generalizing the method of computing effective action to the case of a complex action [13] (sometimes called almost-analytic machinery).

$$W[\mathbf{f}] = S[\mathbf{f}; g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}}] + \cdots$$

where $\cdots$ stands for the subleading contributions of $\mathcal{O}(1/\lambda)$. $S[\mathbf{f}; g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}}]$ is the analytic continuation of the action $S[\mathbf{f}; g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}}]$ in a complex neighborhood at $(\mathbf{f}; g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}})$. The leading contribution of $W[\mathbf{f}]$ is given by evaluating $S$ at the solution $(g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}})$ of $\partial_{\mathbf{f}} S = 0$. In a neighborhood of spins at $\mathbf{f}$, the real part of $S[\mathbf{f}; Z(\mathbf{f})]$ is nonvanishing and negative unless $J_f = J_f^*$, where $Z(\mathbf{f})$ reduces to the real value $g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}}$.

The leading contribution $S[\mathbf{f}; Z(\mathbf{f})]$ can be analyzed by Taylor expansion in perturbations $\varepsilon_f = f - f^*$:

$$S = i \left[ \sum_f \gamma_f \tilde{\Theta}_f + \sum_f \gamma_f \tilde{\Theta}_f \tilde{\varepsilon}_f + \sum_{f, f'} W_{f, f'} \varepsilon_{f'} + o\left(\varepsilon^2\right) \right]$$

The computations of the above coefficients at different orders are given in [10]. $W_{f, f'}$ is local in the sense that it vanishes unless $f, f'$ belong to the same tetrahedron $e$.

The above result is for the partial amplitude $A_{\mathbf{f}}(\mathcal{K})$. In order to compute $A(\mathcal{K})$, we implement the sum over perturbations $\varepsilon_f$ inside a neighborhood at $J_f$. The spinfoam amplitude is written as $A(\mathcal{K}) = \sum_{f} e^{iW[\mathbf{f}] + \varepsilon_f} W_{f, f'} \varepsilon_{f'} + o(\varepsilon^2)$. The Poisson resummation formula can be applied to the sum over the perturbations $\varepsilon_f$, which results in the following perturbative expression for $A(\mathcal{K})$:

$$e^{\varepsilon f J_f} \tilde{\Theta}_f \sum_{k_f \in \varepsilon f} \int \left[ d\varepsilon_f \right] e^{i\left[ \sum_f \left( \tilde{\Theta}_f - 4nk_f \varepsilon_f \right) + \sum_f W_{f, f'} \varepsilon_{f'} + o(\varepsilon^2) \right]}$$

where again $\cdots$ stands for the subleading contributions in $1/\lambda$.

The above discussion considers the large-$J$ regime for the spinfoam amplitude for the semiclassical approximation. Now we implement the low-energy approximation. The low-energy regime is achieved when the background configuration $(\mathbf{f}; g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}})$ is such that $\tilde{\Theta}_f \ll 1$. Firstly let’s consider the integrals with $k_f \neq 0$ in Eq. (7) and apply the stationary phase analysis as $\lambda \gg 1$. The equation of motion from $S[\mathbf{f}; Z(\mathbf{f})]$ is given by

$$0 = \partial_{\mathbf{f}} S[\mathbf{f}; Z(\mathbf{f})] = \partial_{\mathbf{f}} Z(\mathbf{f})$$

where $\partial_{\mathbf{f}} S[\mathbf{f}; Z(\mathbf{f})] = 0$ because $Z(\mathbf{f})$ is the solution of $\partial_{\mathbf{f}} S = 0$. $\partial_{\mathbf{f}} S = \partial_{\mathbf{f}} \tilde{\varepsilon} = \partial_{\mathbf{f}} \varepsilon = 0$. The condition $\mathcal{R}[\mathbf{f}; Z(\mathbf{f})] = 0$ implies the perturbation $\varepsilon_f = 0$ where $Z(\mathbf{f})$ reduces to $g_{\mathbf{f}}, z_{\mathbf{f}}, \tilde{\varepsilon}_{\mathbf{f}}$. Taking into account both the equations of motion and $\mathcal{R}[\mathbf{f}; Z(\mathbf{f})]$ is the result that $\gamma_f = 0$ for $k_f \neq 0$, which cannot be satisfied in the low-energy regime where $|\tilde{\Theta}_f| \ll 1$ (with $\gamma \sim o(1)$).
or less). As a result, all the integrals with \( k \neq 0 \) in Eq.\(^\text{(8)}\) are exponentially decaying, according to the principle of stationary phase analysis [\(\text{[17]}\)].

We thus focus on the integral with \( k_f = 0 \) in Eq.\(^\text{(8)}\):

\[
\int [d\gamma] e^{i\left(\sum f \gamma f \theta_f + \sum f, f' \gamma f, f' \theta_f + o(\theta_f)\right)} = 1.
\]

We denote by \(|\Theta| \ll 1\) the mean value of the background deficit angle \( \Theta_f = \Theta \Delta f \). The 2d space of \((\lambda, \Theta)\) may be viewed as the parameter space for our perturbation theory, where the semiclassical and low-energy regime is located in \( \lambda \gg 1, |\Theta| \ll 1 \). Now a new parameter is defined by \( \beta := \lambda \Theta \), or a coordinate transformation is defined from \((\lambda, \Theta)\) to \((\lambda, \beta)\), where \( \beta \) is treated independent of \( \lambda \). Then Eq.\(^\text{(10)}\) reads

\[
\int [d\gamma] e^{i\left(\sum f \gamma f \theta_f + \sum f, f' \gamma f, f' \theta_f + o(\theta_f)\right)} = 1.
\]

Again the stationary phase analysis is applied as \( \lambda \gg 1 \). We find \( \gamma_f = 0 \) is a solution of both \( \delta \gamma_f \left[ \sum f, f' \lambda_f, f' \gamma f, f' \theta f + o(\theta_f) \right] = 0 \) and \( R \left[ \sum f, f' \lambda_f, f' \gamma f, f' \theta f + o(\theta_f) \right] = 0 \). Note that \( R \left[ \sum f, f' \lambda_f, f' \gamma f, f' \theta f + o(\theta_f) \right] = R \delta \), since we \( \gamma f, \hat{\gamma} f, \hat{\beta} f \) is purely imaginary. The standard stationary phase formula [\(\text{[17]}\)] leads to the following result from Eq.\(^\text{(11)}\) in the neighborhood of the background spins \( \gamma_f \) (\( \gamma_f = 0 \)):

\[
\sum_{n=0}^{\infty} (1/\lambda)^n \delta \left[ \sum_{f} \gamma f \theta_f + \sum_{f, f'} \gamma f, f' \theta f + o(\theta_f) \right] \delta_{n,r} = \sum_{n=0}^{\infty} \sum_{r=0}^{2n} (\gamma' \theta' / \lambda^{n-r}) f_{n,r} \quad \text{[12]}
\]

\( \delta \) is a differential operator of order \( 2n \) (in \( \partial \gamma_f \)) where all the interactions from the Lagrangian are encoded (see [\(\text{[17]}\)] for a general expression). Applying the differential operator \( \delta \) to \( \psi^{\beta \gamma} \gamma f, \hat{\gamma} f, \hat{\beta} f \), gives the power-counting result in Eq.\(^\text{(12)}\). The coefficients \( f_{n,r} \) are functions of \( \lambda \) and \( (\gamma_f, \hat{\gamma}_f, \hat{\beta}_f) \), which are regular as \( \lambda \to \infty \) [\(\text{[27]}\)]. Inserting Eq.\(^\text{(12)}\) to Eq.\(^\text{(8)}\) and recalling \( \beta = \lambda \Theta \), the following expansion for \( A(\lambda) \) is obtained:

\[
A(\lambda) \sim \int [d\gamma] e^{i\sum f \gamma f \theta_f} \sum_{n=0}^{\infty} \sum_{r=0}^{2n} (\gamma' \theta' / \lambda^{n-r}) f_{n,r} \quad \text{[13]}
\]

where the exponentially decaying contributions have been neglected. We can read from the above result an effective action \( i\lambda_{\text{eff}}(\gamma_f, \hat{\gamma}_f, \hat{\beta}_f) \) by expressing \( A(\lambda) \sim \exp i\lambda_{\text{eff}} \), where the effective action at the background \((\gamma_f, \hat{\gamma}_f, \hat{\beta}_f) \) is an expansion w.r.t. \( \Theta \) and \( \lambda^{-1} \):

\[
i\lambda_{\text{eff}} = \lambda \left[ \gamma f \sum f \gamma f \hat{\gamma} f + \frac{\gamma^2}{4} \sum f, f' W_{f, f'}^{-1} \hat{\gamma} f, \hat{\beta} f + o(\gamma^3 \theta^3, \lambda^{-1}) \right] \quad \text{[14]}
\]

The coefficient \( W_{f, f'}^{-1} \) is the inverse of \( W_{f, f'} \) in Eq.\(^\text{(7)}\). \( W_{f, f'} \) is nonzero only when \( f, f' \) belong to the same tetrahedron \( e \):

\[
W_{f, f'} = \frac{2(1 + 2i \gamma - 4 \gamma^2 - 2i \gamma^3)}{5 + 2i \gamma} \hat{n}_{f, f'}^{-1} \hat{n}_{f, f'} \quad \text{[15]}
\]

where \( \hat{n}_{f, f'} = \sum f \gamma f \hat{\epsilon}_f \hat{n}_{f, f'} \). Here the unit 3-vector \( \hat{n}_{f, f'} \) determined by \((\gamma_f, \hat{\gamma}_f, \hat{\beta}_f) \) is the normal vector of the triangle \( f \) in the frame of the tetrahedron \( e \) [\(\text{[6]}\), \(\text{[16]}\)]. Although \( W_{f, f'} \) is local in \( f, f' \), the inverse \( W_{f, f'}^{-1} \) is nonlocal in general, it may be nonzero for far away \( f, f' \). So the \( \gamma^2 \theta^2 \) term is a nonlocal curvature correction in \( \lambda_{\text{eff}} \). Moreover there is a systematic way developed in [\(\text{[16]}\)] to compute in principle all the \( \gamma \theta^2 \) corrections.

There are several remarks for the effective action Eq.\(^\text{(14)}\).

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**Low-energy effective action as curvature expansion:** The terms \( \lambda (\gamma \theta)^3 \in \lambda \) are understood as the high-energy correction to the leading order \( i \lambda \sum f \gamma f \hat{\gamma} f \), since \(|\Theta| \ll 1\) implements the low-energy approximation [\(\text{[28]}\)]. Therefore as a power-series of \( \Theta \), \( i\lambda_{\text{eff}} \) is understood as a low-energy effective action from covariant LQG. The deficit angle \( \Theta \sim \alpha \lambda \), where \( \alpha \) is the mean (area) spacing of the lattice given by the background data \((\hat{\gamma}_f, \hat{\beta}_f) \). \( R \) is the mean curvature of the background. Thus the effective action \( i\lambda_{\text{eff}} \) can be viewed as a curvature expansion, where the high-energy corrections are given by \( \alpha^2 \gamma^2 \theta^2 + \alpha^3 \gamma^3 \theta^2 + \cdots \) with \( \alpha \) being the (effective) coupling constant of the high-derivative interactions.

**2-parameter expansion:** There are two parameters involved in the expression of effective action \( i\lambda_{\text{eff}} \), i.e. \( \lambda \gg 1 \) and \( \Theta \ll 1 \) (or \( \alpha \) with dimension \(-2\)). \( 1/\lambda \) counts the quantum corrections, while \( \Theta \) (or \( \alpha \)) counts the high-energy corrections. The two expansion parameters implement the semiclassical low-energy regime \( \Theta \ll 1 \approx \alpha \ll L^2 \).

**Restriction of \( \Theta \):** The effective action \( i\lambda_{\text{eff}} \) has a negative real part, which is contained in the terms of higher-curvature [\(\text{[16]}\)], i.e. \( R[i\lambda_{\text{eff}}] = \lambda \sum f (1/w^2) \gamma f \hat{\gamma} f + o(\gamma^3 \lambda \hat{\gamma}^3) \cdots \leq 0 \) where \( \cdots \) stands for the terms suppressed by 1/\( \lambda \). This negative real part on the exponential would have given an exponentially decaying factor in \( A(\lambda) \) if \( \lambda \Theta \) was of \( o(1) \), which is not our case because of \( \Theta \ll 1 \). The non-decaying \( A(\lambda) \) requires that \( R[i\lambda_{\text{eff}}] \) doesn’t scale to be large by \( \lambda \gg 1 \), which results in a nontrivial bound of the deficit angle \( \Theta \), i.e.

\[
\Theta | \leq \gamma^{-1} \lambda^{-1} \quad \text{[16]}
\]

The situation is illustrated in FIG[1]. The red region in FIG[1] illustrates the space (in the coordinates \( \lambda \) and \( \Theta \)) of background configurations \((\hat{\gamma}_f, \hat{\beta}_f) \), which validates the 2-parameter expansion of the effective action \( i\lambda_{\text{eff}} \). If \( \Theta \) is beyond the bound Eq.\(^\text{(16)}\), where the approximation Eq.\(^\text{(12)}\) is invalid, the integral Eq.\(^\text{(10)}\) is exponentially decaying as \( \lambda \gg 1 \) by the same argument for \( k \neq 0 \) integrals. Thus the red region in FIG[1] illustrates the semiclassical low-energy effective degrees of freedom from the above approximation.

**Einstein-Hilbert action:** After the restriction Eq.\(^\text{(16)}\), the leading contribution in \( i\lambda_{\text{eff}} \) is \( i\lambda \sum f \gamma f \hat{\gamma} f \), which is the Regge action of GR as a functional of the edge-lengths determined by \((\hat{\gamma}_f, \hat{\beta}_f) \) (by identifying \( \gamma \lambda f = \alpha^2 f L^{-2} \) to be the area of the triangle \( f \) in Planck unit). Moreover given that that \( \Theta \sim \alpha_f / L^2 \ll 1 \) [\(\text{[20]}\)] and that \( i\lambda_{\text{eff}} \) is a power-series in \( \alpha_f \), the leading order contribution is essentially the Einstein-Hilbert action on a smooth manifold \( M \), i.e. as a functional (see e.g.
The corrections of higher order in curvature (in deficit angle) modifies the Einstein (Regge) gravity in high-energy regime. It is interesting to further investigate these high-curvature terms predicted from covariant LQG, in order to see if LQG can provide a UV-completion of perturbative Einstein gravity. The origin of high-curvature terms is the sum over non-Regge-like spins (the spins that cannot be viewed as Regge areas) in spinfoam amplitude. The non-Regge-like spins are the extra UV degrees of freedom in addition to GR predicted by LQG. Their dynamics may be studied via the action Eq. (7) to see if they regulate Einstein gravity at UV.

Finally we remark that the analysis beyond the Einstein sector \( R_E \) can also be carried out. There exists different other sectors, well-separated from \( R_E \), where the similar analysis results in the leading order effective actions different from Einstein gravity. We refer to [14,16] for detailed discussions.

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[1] A. Perez. Living Rev. Relativity 16 (2013) 3
[2] J. Engle, et al. Nucl. Phys. B799 (2008) 136
[3] B. Bahr, et al. Class. Quant. Grav. 28 (2011) 105003
[4] M. Han. J. Math. Phys. 52 (2011) 072501 [arXiv:1012.4216]
W. Fairbairn, C. Meusburger. J. Math. Phys. 53 (2012) 022501
[5] M. Han. Phys. Rev. D 84 (2011) 064010 [arXiv:1105.2212]
[6] M. Han, T. Krajewski. [arXiv:1304.5626]
[7] A. Mikovic, M. Vojinovic. J. Phys.: Conf. Ser. 360 (2012) 012049; Class. Quant. Grav. 28 (2011) 225004
[8] A. Mikovic. [arXiv:1302.5564]
[9] J. W. Barrett, et al. Class. Quant. Grav. 27 (2010) 165009
[10] F. Conrady, L. Freidel. Phys. Rev. D78 (2008) 104023
[11] M. Han, M. Zhang, Class. Quantum Grav. 29 (2012) 165004 [arXiv:1109.0500]; Class. Quantum Grav. 30 (2013) 165012 [arXiv:1109.0499]
[12] H. Sahlmann, et al. Nucl. Phys. B606 (2001) 401
[13] E. Magliaro, C. Perini. Europhys. Lett. 95 (2011) 30007
[14] M. Han. Phys. Rev. D 88 (2013) 044051 [arXiv:1304.5628]
[15] A. Melin, J. Sjöstrand. Lect. Notes Math. 459 (1975) 120-223
[16] M. Han. [arXiv:1304.5627]
[17] L. Hörmander. The analysis of linear partial differential operators I. Springer-Verlag (1990)
[18] G. Feinberg, et al. Nucl. Phys. B245 (1984) 343
[19] E. Bianchi, Y. Ding. Phys. Rev. D 86 (2012) 104040
[20] E. Bianchi, et al. Nucl. Phys. B822 (2009) 245-269
[21] C. Rovelli, M. Zhang. Class. Quantum Grav. 28 (2011) 175010
[22] F. Hellmann, W. Kaminski. [arXiv:1210.5276]
V. Bonzom. Phys. Rev. D80 (2009) 064028
[23] D. Oriti. PoS(QG-Ph)030 (2007)
[24] C. Rovelli and M. Smerlak. Class. Quantum Grav. 29 (2012) 055004
[25] The spinfoam amplitude $A$ is a $\mathcal{H}$-valued function on the space of simplicial manifolds, where $\mathcal{H}$ is the boundary Hilbert space and $\mathcal{H} = \mathbb{C}$ if the manifold has no boundary.
[26] See [7, 8] for an early study of spinfoam amplitude through effective action.
[27] If the $\lambda^{-1}$-corrections are neglected, the $\gamma\Theta$-expansion of $I_{\text{eff}}$ is analytic in a neighborhood at $\gamma\Theta = 0$, by the analyticity of the spinfoam action [16].
[28] The terms linear to $\gamma\Theta$ are suppressed by $\lambda^{-3}$ except the leading Regge action.
[29] It is an open question about the interpretation for the regime of too large spins, which seems to give a too large lattice spacing scale $\alpha_f = \lambda L^2$, semiclassically, and contradict the observation of smooth spacetime. In order to remove the regime, a spin cutoff may be introduced via q-deformation [4], which produce a relatively large bare cosmological constant [5].
[30] Given $(j_f, \tilde{g}_v, \tilde{z}_v)$ with nontrivial mean curvature radius, in order to obtain Eq. (15), a large triangulation is needed. e.g. If the size of $K$ measured by $(j_f, \tilde{g}_v, \tilde{z}_v)$ is of the same order as the curvature radius $L$, the number of simplices is at least of the order $N \sim L^3/\alpha^2 \gg 1$. 
