MONOIDAL FUNCTOR CATEGORIES AND GRAPHIC FOURIER TRANSFORMS

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Abstract. This article represents a preliminary attempt to link Kan extensions, and some of their further developments, to Fourier theory and quantum algebra through ∗-autonomous monoidal categories and related structures.

INTRODUCTION

In Part 1 of this article we use ∗-autonomous symmetric monoidal categories \( \mathcal{V} \) (in the sense of [1]) and their resulting functor categories \( \mathcal{P}(\mathcal{A}) = [\mathcal{A}, \mathcal{V}] \), to describe aspects of the “graphic” upper and lower convolutions of functors into \( \mathcal{V} \), and their transforms.

Thus the obvious notion of “Fourier” transform (via Kan extension) of a functor in a monoidal functor category \( [\mathcal{A}, \mathcal{V}] \) is reiterated in §1.3 below, where the transform of the convolution product of two such functors is seen to be the (often pointwise) tensor product of their transforms.

The basic notion of multiplicative kernel is defined in §1.2, this being the main source of multiplicative functors. Examples are described, with a brief account of the association scheme example at the end of §1.2.

In §1.3 we look at transforms of functors (called analytic functors after A. Joyal [12]) in the context of what we call the Joyal-Wiener category. This category of transforms is, under simple conditions, a monoidal category equivalent to the original monoidal domain \( [\mathcal{A}, \mathcal{V}] \); of course, in Joyal’s case, it is also equivalent to the category of all Joyal-analytic functors and weakly cartesian maps between them.

In general, the transform of a functor behaves like a classical Fourier transform, and there is usually a corresponding inversion process. For instance it is possible (see §2.1) that a transformation functor can have a tentative left inverse which then leads to the construction of the required (two-sided) inversion data.

These transforms also generalize the Fourier transforms of Hopf algebras (as discussed in [4] for example) which extend directly from Hopf algebras in \( \mathcal{V} \) to the many-object Hopf algebroids of [10]; some mention of this is made in §2.1. Several other types of examples are described in §2.2.

The original convolution construction on a functor category of the form \( [\mathcal{A}, \mathcal{V}] \) for a small promonoidal structure \( \mathcal{A} \), may be found in [6] and [7], where such categories are viewed in much the same light as function algebras. It is emphasised here that all the categorical concepts used below, such as “category”, “functor”, “natural transformation”, etc., are \( \mathcal{V} \)-enriched (in the sense of [13]) over the given

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symmetric monoidal closed base category $\mathcal{V}$, unless otherwise mentioned in the text.

1. Multiplicative kernels

1.1. Upper and lower convolution. Let $\mathcal{V} = (\mathcal{V}_0, I, \otimes, [-, -], (-)^*)$ be a complete (hence cocomplete) $*$-autonomous symmetric monoidal closed category (in the sense of [1]) with $I$ as dualising object. Recall that $[X, Y] \cong (X \otimes Y^*)^*$, and if an object $Z$ has a dual $Z^\vee$ in $\mathcal{V}$, then $Z^\vee \cong [Z, I] = Z^*$ in which case $[Z, X] \cong Z^* \otimes X$ and $(Z \otimes X)^* \cong Z^* \otimes X^*$ for all $X$ in $\mathcal{V}$.

If $(\mathcal{A}, p, j)$ is a small promonoidal category over $\mathcal{V}$, then the upper convolution of $f$ and $g$ in the functor category $[\mathcal{A}, \mathcal{V}]$ is defined in [6] as

$$f \star g = \int^{ab} f(a) \otimes g(b) \otimes p(a, b, -)$$

and $I = j$. If $(\mathcal{A}^{op}, p, j)$ is a small promonoidal category, then the lower convolution of $f$ and $g$ in $[\mathcal{A}, \mathcal{V}]$ is defined as

$$f \sharp g = \left( \int^{ab} f(a)^* \otimes g(b)^* \otimes p(a, b, -))^* \right)$$

and $L = j^*$. (See [8] and [16] for example.)

Both products yield associative and unital monoidal structures on $[\mathcal{A}, \mathcal{V}]$; the upper product preserves $\mathcal{V}$-colimits in each variable, while the lower product preserves $\mathcal{V}$-limits in each variable. The upper product $f \star g$ in $[\mathcal{A}, \mathcal{V}]$ using $p$ on $\mathcal{A}$ gives $f \star g$ on

$$[\mathcal{A}, \mathcal{V}]^{op} = [\mathcal{A}^{op}, \mathcal{V}^{op}]$$

which transforms under the equivalence $\mathcal{V}^{op} \simeq \mathcal{V}$ to the lower product $f^* \sharp g^*$ in $[\mathcal{A}^{op}, \mathcal{V}]$ using the same $p$ on $\mathcal{A}$, since

$$(f \star g)^* = \left( \int^{ab} f(a) \otimes g(b) \otimes p(a, b, -))^* \right)$$

$$\cong \left( \int^{ab} f(a)^* \otimes g(b)^* \otimes p(a, b, -))^* \right)$$

$$= f^* \sharp g^*.$$

An antipode $S$ on a (promonoidal) category $(\mathcal{A}, p, j)$ is a functor

$$S : \mathcal{A}^{op} \longrightarrow \mathcal{A}$$

such that $S^{op} \dashv S$ with $S^2 \cong 1$. Then the resulting data $q(a, b, c) = p(Sa, Sb, Sc)$ and $k(c) = j(Sc)$ become part of an obvious promonoidal structure on $\mathcal{A}^{op}$.

If the set of data $(\mathcal{A}, p, j, S)$ is an $S$-autonomous category, in the sense that the cyclic condition

$$p(a, b, c) \cong p(b, c, Sa)$$
holds naturally in \( a, b, c \in \mathcal{A} \), then the derived set \((\mathcal{A}^{\text{op}}, q, k, S^{\text{op}})\) is \( S^{\text{op}} \)-autonomous since
\[
p(Sa, Sb, S^2c) \cong p(Sb, Sc, S^2a);
\]
that is
\[
q(a, b, Sc) \cong q(b, c, Sa).
\]

**Theorem.** If \((\mathcal{A}, p, j, S)\) is a \( S \)-autonomous promonoidal category and \((\mathcal{V}, I, \otimes, (-)^\ast)\) is the \( \ast \)-autonomous base category, then the upper convolution structure on \([\mathcal{A}, \mathcal{V}]\) is \( \ast \)-autonomous under the antipode defined by \( f^\ast(a) = f(Sa)^\ast \) for \( f \) in \([\mathcal{A}, \mathcal{V}]\).

**Proof.** We have the natural isomorphisms
\[
[f, g](c) = \int_{ab} [f(a) \otimes p(c, a, b), g(b)] \text{ by definition of } [-, -] \text{ in } [\mathcal{A}, \mathcal{V}],
\]
\[
\cong \int_{ab} (f(a) \otimes p(c, a, b) \otimes g(b)^\ast)^\ast \text{ since } \mathcal{V} \text{ is } \ast \text{-autonomous monoidal},
\]
\[
\cong (\int_{ab} f(a) \otimes g(b)^\ast \otimes p(c, a, b))^\ast,
\]
\[
\cong (\int_{ab} f(a) \otimes g(Sb)^\ast \otimes p(c, a, Sb))^\ast \text{ since } S^2 \cong 1,
\]
\[
\cong (\int_{ab} f(a) \otimes g^\ast(b) \otimes p(a, b, Sc))^\ast
\]
since \( p(c, a, Sb) = p(a, b, Sc) \) because \((\mathcal{A}, p, j, S)\) is \( S \)-autonomous,
\[
= (f \ast_p g^\ast)^\ast \text{ by definition of } \ast_p \text{ on } [\mathcal{A}, \mathcal{V}].
\]
\[
\Box
\]

The upper convolution \( f \ast_p g \) is then related to the lower convolution \( f \ast_q g \) on the functor category \([\mathcal{A}, \mathcal{V}]\) using \( q(a, b, c) = p(Sa, Sb, Sc) \) on \( \mathcal{A}^{\text{op}} \), the latter being naturally isomorphic to the (\( \ast \)-autonomous) product
\[
(f^\ast \ast_p g^\ast)^\ast
\]
on \([\mathcal{A}, \mathcal{V}]\). This follows from the convolution calculation
\[
(f^\ast \ast_p g^\ast)^\ast(c) = \left( \int_{ab} f(Sa)^\ast \otimes g(Sb)^\ast \otimes p(a, b, Sc) \right)^\ast \text{ by definition of } \ast_p,
\]
\[
\cong \left( \int_{ab} (f(a)^\ast \otimes g(b)^\ast \otimes p(Sa, Sb, Sc))^\ast \right)^\ast \text{ using } S^2 \cong 1,
\]
\[
\cong \left( \int_{ab} f(a)^\ast \otimes g(b)^\ast \otimes q(a, b, c) \right)^\ast \text{ by definition of } q \text{ in } \mathcal{A}^{\text{op}},
\]
\[
= f(a) \ast_q g \text{ by definition of } \ast_q.
\]

**Note.** In the sequel we shall not go into the particular \( \ast \)-autonomous aspects of the theory in any great detail.

**1.2. Multiplicative kernels.** For the given base category \( \mathcal{V} \), a multiplicative kernel from a promonoidal \( \mathcal{A} \) to another promonoidal \( \mathcal{E} \) is a \( \mathcal{V} \)-functor
\[
K : \mathcal{A}^{\text{op}} \otimes \mathcal{E} \longrightarrow \mathcal{V}
\]
together with two natural structure isomorphisms
\[
\int^{yz} K(a, y) \otimes K(b, z) \otimes p(y, z, x) \cong \int^c K(c, x) \otimes p(a, b, c) \quad \text{and}
\]
\[
j(x) \cong \int^c K(c, x) \otimes j(c).
\]

A \(\mathcal{V}\)-natural transformation
\[
\sigma : H \longrightarrow K : \mathcal{A}^{\text{op}} \otimes \mathcal{X} \longrightarrow \mathcal{V}
\]
is called multiplicative if it commutes with the structure isomorphisms of \(H\) and \(K\).

A simple calculation with coends shows that the “module composite” of two multiplicative kernels is again a multiplicative kernel and, for the given \(\mathcal{V}\), this leads to a monoidal bicategory, in the sense of [10], with promonoidal \(\mathcal{V}\)-categories as the objects (0-cells), multiplicative kernels \(K : \mathcal{A}^{\text{op}} \otimes \mathcal{X} \longrightarrow \mathcal{V}\) as the 1-cells \(\mathcal{A} \longrightarrow \mathcal{X}\), and the multiplicative \(\mathcal{V}\)-natural transformations \(\sigma\) as the 2-cells.

Here are a few routine examples:

**Examples.**

(a) In the case where \(\mathcal{A}\) is monoidal and \(\mathcal{X}\) is comonoidal, so that
\[
p(a, b, c) = \mathcal{A}(a \otimes b, c)
\]
\[
j(c) = \mathcal{A}(I, c)
\]
and
\[
p(y, z, x) = \mathcal{X}(y, x) \otimes \mathcal{X}(z, x)
\]
\[
j(x) = I
\]
the structure isomorphisms for a multiplicative \(K : \mathcal{A}^{\text{op}} \otimes \mathcal{X} \longrightarrow \mathcal{V}\) reduce to isomorphisms
\[
K(a, x) \otimes K(b, x) \cong K(a \otimes b, x)
\]
\[
I \cong K(I, x)
\]
by the Yoneda lemma.

(b) If \(\mathcal{A}\) and \(\mathcal{X}\) are both monoidal then a \(\mathcal{V}\)-functor \(\phi : \mathcal{A} \longrightarrow \mathcal{X}\) is multiplicative (i.e.,
\[
\phi(a) \otimes \phi(b) \cong \phi(a \otimes b)
\]
\[
I \cong \phi(I)
\]
if and only if the module
\[
K(a, x) = \mathcal{X}(\phi(a), x)
\]
is a multiplicative kernel, again by Yoneda.

(c) For any \(\mathcal{A}\) and \(\mathcal{X}\) promonoidal and \(\mathcal{V}\)-functor \(\psi : \mathcal{X} \longrightarrow \mathcal{A}\) the conditions for the module
\[
K(a, x) = \mathcal{A}(a, \psi(x))
\]
to be a multiplicative kernel are precisely the conditions for restriction along \( \psi \) to be a multiplicative functor:

\[
[\psi, 1]: [\mathcal{A}, V] \rightarrow [\mathcal{X}, V]
\]

when \([\mathcal{A}, V]\) and \([\mathcal{X}, V]\) are given their upper convolution tensor products.

(d) If \( \mathcal{A} = I \)(the identity \( V \)-category), then \( K: \mathcal{X} \rightarrow V \) is a multiplicative kernel if and only if \( K^{-1}(1) \subset M \) is a \( R \)-submodule. In two trivial cases, take \( V = (0 \leq 1) \) (cartesian closed) and \( \mathcal{A} = 1 \). First let \( \mathcal{X} = M \) be a module over a ring \( R \), and define

\[
p(x, y, z) = \begin{cases} 1 & \text{iff } z = rx + sy \text{ for some } r, s \in R \\ 0 & \text{else.} \end{cases}
\]

Then \( K: M \rightarrow (0 \leq 1) \) is a multiplicative kernel if and only if there exists a natural “structure” isomorphism

\[
M_a \circ M_b \cong \int c p(a, b, c) \otimes M_c.
\]

Again this follows directly from the Yoneda lemma applied to the multiplicative kernel criteria for this particular example.

By also defining an antipode

\[
T : B^{op} \rightarrow B
\]
on \( B \), one can incorporate the notion of transpose matrix into this setting by defining

\[
M_a^T(x, y) = M_a(Ty, Tx).
\]

Finally, in the case of an association scheme, one has that (for \( V = \text{Set} \)) the category \( B^{op} \otimes B \) corresponds to the cartesian product \( X \times X \) of a set \( X \) with itself, while \( \mathcal{A} \) is the discrete category with objects the members of the given partition of \( X \times X \), the cardinals of the respective promultiplication values \( p(a, b, c) \) being the structure constants of the association scheme, and \( j \) being represented by the identity relation on \( X \). Here \((\mathcal{A}, p, j)\) has \( p(a, b, c) \cong p(b^*, a^*, c^*) \) under the antipode \( Sa = a^* \) (the reverse relation on \( a \)), while \( T \) is just the identity function on \( X \).
1.3. **Transforms and analytic functors.** Given \( K \) as before, define the (\( K \)-)transform of \( f \) in \([\mathcal{A}, \mathcal{V}]\) as the (left) Kan extension
\[
\mathcal{K}(f)(x) = \int_a K(a, x) \otimes f(a),
\]
and similarly the dual transform of \( h \) in \([\mathcal{A}^{\text{op}}, \mathcal{V}]\) is defined as the (right) Kan extension
\[
\mathcal{K}^\vee(h)(x) = \int_a [K(a, x), h(a)]
\]
— especially used here when \( \mathcal{V} \) is \(*\)-autonomous, in which case we have
\[
\mathcal{K}^\vee(h) = \int_a (K(a, x) \otimes h(a)^*)^* = K(h)^*.
\]

**Theorem.** If \( K \) is multiplicative, then

(i) \( \mathcal{K} \) preserves upper convolution, and

(ii) \( \mathcal{K}^\vee \) preserves lower convolution.

**Proof.**

(i) \[
\mathcal{K}(f \triangleright g) = \mathcal{K}(\int_{ab} f(a) \otimes g(b) \otimes p(a, b, -))
\]
\[
= \int_c K(c, -) \otimes \int_{ab} f(a) \otimes g(b) \otimes p(a, b, c)
\]
\[
\cong \int_{ab} \int_{yz} K(a, y) \otimes K(b, z) \otimes p(y, z, -) \otimes f(a) \otimes g(b)
\]
\[
\cong \int_{ab} \int_{yz} K(a, y) \otimes f(a) \otimes K(b, z) \otimes g(b) \otimes p(y, z, -)
\]
\[
= \int_{yz} (\int_{a} K(a, y) \otimes f(a)) \otimes (\int_{b} K(b, z) \otimes g(b)) \otimes p(y, z, -)
\]
\[
= \mathcal{K}(f) \triangleright \mathcal{K}(g).
\]

(ii) For functors \( h = f^* \) and \( k = g^* \) in \([\mathcal{A}^{\text{op}}, \mathcal{V}]\), we have
\[
\mathcal{K}^\vee(h \triangleright k) = \mathcal{K}((h \triangleright k)^*)^*
\]
\[
\cong \mathcal{K}(f \triangleright g)^* \quad \text{since } h \triangleright k = f^* \triangleright g^* \cong (f \triangleright g)^*,
\]
\[
\cong (\mathcal{K}(f) \triangleright \mathcal{K}(g))^* \quad \text{since } \mathcal{K} \text{ preserves } \triangleright \text{ by (i),}
\]
\[
\cong \mathcal{K}(f)^* \triangleright \mathcal{K}(g)^*
\]
\[
\cong \mathcal{K}(h)^* \triangleright \mathcal{K}(k)^*
\]
\[
= \mathcal{K}^\vee(h) \triangleright \mathcal{K}^\vee(k).
\]
\(\square\)
Each \( \mathcal{V} \)-functor \( K : \mathcal{A}^{\text{op}} \otimes \mathcal{X} \to \mathcal{V} \) yields the standard Kan adjunction

\[
(\epsilon, \eta) : K \dashv K : [\mathcal{X}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}],
\]

where

\[
K(f)(a) = \int_x [K(a, x), f(x)].
\]

In the case where \( K \) is a multiplicative kernel we call \( K \) an “analytic” or “Fourier” transformation if and only if \( \eta \) is an equaliser (i.e., a regular monomorphism) in \([\mathcal{A}, \mathcal{V}]\).

Note. Many of the examples in Part 2 have \( K \) fully faithful which means \( \eta \) is an isomorphism, i.e.,

\[
[f, g] \cong [K(f), K(g)].
\]

Then, if \( \mathcal{V} \) is \(*\)-autonomous, there results a dual isomorphism for

\[
\langle f, g \rangle = \int_a f(a)^* \otimes g(a),
\]

namely

\[
\langle f, g \rangle \cong \langle K(f), K(g) \rangle.
\]

This follows from

\[
\langle f, g \rangle^* = (\int_a f(a)^* \otimes g(a))^*
\]

\[
= \int_a (g(a) \otimes f(a))^*
\]

\[
= \int_a [g(a), f(a)]
\]

\[
= \int_x [K(g)(x), K(f)(x)]
\]

since \( K \) is fully faithful,

\[
= \int_x (K(g)(x) \otimes K(f)(x))^*
\]

\[
= (\int_x K(f)(x)^* \otimes K(g)(x))^*
\]

\[
= \langle K(f), K(g) \rangle^*
\]

which is a type of Parseval relation.

Returning to the case where \( \eta \) is merely a regular monomorphism, we have that

\[
f \xrightarrow{\eta} K(f) \xrightarrow{\eta K} K K(f)
\]

is an equaliser diagram in \([\mathcal{A}, \mathcal{V}]\) (see [2] and [13] for example).

For each such kernel \( K \) one can construct a “Joyal-Wiener” category, here denoted \( \text{Joy}(K) \), as follows. A map

\[
\alpha : K(f) \to K(g)
\]

in \([\mathcal{X}, \mathcal{V}]\) is called regular when

\[
K K(\alpha) K(\eta) = K(\eta) \alpha;
\]
then using the equalizer hypothesis on \( \eta \), each such regular \( \alpha \) equals \( K(\beta) \) for a unique \( \beta : f \rightarrow g \) in \([\mathcal{A}, \mathcal{V}]\). With this in mind, the \( \mathcal{V} \)-category \( \text{Joy}(K) \) is defined to be that subcategory of the Kleisli category \([\mathcal{A}, \mathcal{V}]_T\) for the monad

\[
T = (\overline{K}, K\overline{K}, \eta),
\]

with the same objects as \([\mathcal{A}, \mathcal{V}]_T \subset [\mathcal{X}, \mathcal{V}]\) but with the equaliser equations

\[
\text{Joy}(K)(f, g) \rightarrow [\mathcal{X}, \mathcal{V}](K(f), K(g)) \rightarrow [\mathcal{X}, \mathcal{V}](K\overline{K}(f), K\overline{K}(g))
\]

in \( \mathcal{V} \) defining its respective (\( \mathcal{V} \)-enriched) homs.

As a result, we obtain the usual Kleisli factorisation

\[
\xymatrix{[\mathcal{A}, \mathcal{V}] \ar[r] & [\mathcal{X}, \mathcal{V}] \ar[r] & [\mathcal{X}, \mathcal{V}]_T \ar @/_1pc/[l]_\mathcal{K} \ar @/^1pc/[l]^{\overline{K}_T}
}
\]

where \( \overline{K}_T \) is conservative, and \( \text{Joy}(K) \) is a \( \mathcal{V} \)-category with an equivalence

\[
[\mathcal{A}, \mathcal{V}] \xrightarrow{\cong} \text{Joy}(K)
\]

and a conservative embedding into \([\mathcal{X}, \mathcal{V}]\).

To relate this to the work of Joyal [12], we now consider the special case where \( \mathcal{A} \) is promonoidal and \( \mathcal{X} = \mathcal{V} \) itself (monoidal under \( \otimes \)). Denote a fixed kernel \( E \)

by

\[
X(a) = E(a, X)
\]

and think of the coend

\[
F(X) = \int_a f(a) \otimes X(a)
\]

as being an “E-analytic” functor of \( X \in \mathcal{V} \) with “coefficients” \( f \) in \([\mathcal{A}, \mathcal{V}]\). Thus we obtain obvious types of extensions of \( A \), Joyal’s original notion of analytic functor [12]. That is, if we take \( \mathcal{A} \) to be the free \( \mathcal{V} \)-category on the groupoid \{finite sets and bijections\} and let \( E(a, X) = \bigotimes^a X \), then an \( E \)-analytic functor \( F : \mathcal{V} \rightarrow \mathcal{V} \) takes the form

\[
F(X) = \int_a f(a) \otimes X(a)
\]

\[
\cong \sum_a f(a) \otimes_{\text{Aut}(a)} \bigotimes^a X.
\]

Note that, for \( \mathcal{V} = \text{Set} \) and \( \mathcal{V} = \text{Vect}_k \), the units \( \eta_f \) of the adjuction \( E \dashv \overline{E} \) are equalisers in \([\mathcal{A}, \mathcal{V}]\) because the diagram

\[
\xymatrix{f(b) \ar[r]^-{\eta_{f,b}} & \int_a [E(b, X), \int_a E(a, X) \otimes f(a)] \ar[d]^\gamma \\
\int_a \mathcal{A}(a, b) \otimes f(a) \ar[u] & \int_a E(a, b) \otimes f(a) \ar[r]^-{f \circ m \otimes 1} & \int_a E(a, b) \otimes f(a) \ar[d]^t}
\]
commutes, where $t$ is the composite map
\[
\int_x [E(b, x), \int^a E(a, x) \otimes f(a)] \overset{\text{proj}}{\longrightarrow} [E(b, b), \int^a E(a, b) \otimes f(a)]
\]
and $\int^a m \otimes 1$ is a monomorphism in $\mathcal{V}$ since
\[
m : x \rightarrow E(a, b)
\]
is a cartesian natural monomorphism ($x$ being a groupoid). Hence
\[
[A, V] \cong \text{Joy}(E)
\]
so that, for $\mathcal{V} = \text{Set}$, $\text{Joy}(E)$ is equivalent to the category of all Joyal-analytic functors on $\text{Set}$ and weakly cartesian maps between them, as one might expect.

Finally, even in the case of general $\mathcal{V}$, one can define the convolution product of two $E$-analytic functors
\[
F(X) = \int^a f(a) \otimes X(a) \quad \text{and} \quad G(X) = \int^a g(a) \otimes X(a)
\]
by
\[
F * G(X) = \int^{ab} f(a) \otimes g(b) \otimes \int^c p(a, b, c) \otimes X(c)
\]
\[
\cong \int^c (\int^{ab} f(a) \otimes g(b) \otimes p(a, b, c)) \otimes X(c).
\]
The Hadamard product $[\mathcal{X}]$ can be defined also as
\[
F \times G(X) = \int^a (f(a) \otimes g(a)) \otimes X(a)
\]
if $\mathcal{A}$ has a comonoidal structure as well as its initial promonoidal structure. In fact, if $K : \mathcal{A}^{op} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ is multiplicative with respect to these two structures, then the $K$-transform of $F$, defined by
\[
\overline{K}(F)(X) = \int^b (\int^a f(a) \otimes K(a, b)) \otimes X(b),
\]
has the property
\[
\overline{K}(F * G)(X) \cong (\overline{K}(F) \times \overline{K}(G))(X).
\]

2. **Examples of transformation functors**

2.1. **Examples where $\overline{K}$ is fully faithful.** In this section we provide explicit inverses to various examples of $\mathcal{V}$-functors of the form
\[
\overline{K} : [\mathcal{A}, \mathcal{V}] \longrightarrow [\mathcal{X}, \mathcal{V}].
\]
In these examples, $\mathcal{A}$ may be a large $\mathcal{V}$-category so that the functor category $[\mathcal{A}, \mathcal{V}]$ does not exist in the $\mathcal{V}$-universe; however the underlying "category" $[\mathcal{A}, \mathcal{V}]_0$ makes some sense.

The method is to firstly find a left inverse $\Gamma$ to $\overline{K}$, then show that $\Gamma_0$ is faithful when restricted to the full image of $\overline{K}_0$ in $[\mathcal{A}, \mathcal{V}]_0$. For convenience, we shall here suppose that the unit $I \in \mathcal{V}_0$ generates $\mathcal{V}_0$, so that $\Gamma$ is then $\mathcal{V}$-faithful on the
full image of \( \overline{K} \); by virtue of this monomorphism, the full image of \( \overline{K} \) is usually a genuine \( \mathcal{V} \)-category. Thus, since

\[
\begin{array}{ccc}
[f, g] & \xrightarrow{\overline{K}} & [\overline{K}(f), \overline{K}(g)] \\
\cong & \nearrow \swarrow & [\Gamma \overline{K}(f), \Gamma \overline{K}(g)]
\end{array}
\]

commutes, one has that \( \Gamma \) is an isomorphism, hence that \( \overline{K} \) is fully faithful. In other words

\[ [\mathcal{A}, \mathcal{V}] \cong \text{Joy}(K) \]

where \( \text{Joy}(K) \subset [\mathcal{X}, \mathcal{V}] \) is a full embedding, not merely a conservative one.

In this part we shall suppose that \( \mathcal{V} \) is a complete and cocomplete symmetric monoidal closed category. Of course the upper transformation \( \overline{K} \) has a corresponding lower \( \overline{K}^\vee \) if \( \mathcal{V} \) happens to be \( \ast \)-autonomous also, in which case

\[ \overline{K}^\vee(h) \cong (\overline{K}^\vee)(h^*)^* \]

so that \( \Gamma \overline{K} \cong 1 \) if and only if

\[ \Gamma^\vee \overline{K}^\vee(h) = \Gamma(\overline{K}^\vee(h^*)^*) \cong h \]

for all \( h \) in \( [\mathcal{A}^{\text{op}}, \mathcal{V}] \).

**Example 1.** One of the simplest examples of a Fourier transform is that obtained from a Hopf algebroid in the sense of \([10]\). First \( \mathcal{A} \) is given the promonoidal structure

\[ p(a, b, c) = \mathcal{A}(a, Sb) \otimes \mathcal{A}(b, c) \quad \text{and} \quad j(a) = I, \]

where \( S \) denotes the antipode of the algebroid, so that convolution on \( [\mathcal{A}, \mathcal{V}] \) becomes

\[
f \ast g(c) = \int_{a, b} f(a) \otimes g(b) \otimes p(a, b, c) \\
\cong \int_b^a \int_b^a f(a) \otimes \mathcal{A}(a, Sb) \otimes g(b) \otimes \mathcal{A}(b, c) \\
\cong \int_b^a f(Sb) \otimes g(b) \otimes \mathcal{A}(b, c)
\]

on applying the Yoneda lemma.

Secondly, let \( \mathcal{X} \) be \( \mathcal{A} \) also; then the pointwise tensor on \( [\mathcal{X}, \mathcal{V}] \) is the usual

\[ f \otimes g(c) = f(c) \otimes g(c). \]

If \( K : \mathcal{A}^{\text{op}} \otimes \mathcal{X} \longrightarrow \mathcal{V} \) is taken to be the hom functor of \( \mathcal{A} \), we then get an isomorphism

\[
\overline{K} : [\mathcal{A}, \mathcal{V}] \cong [\mathcal{X}, \mathcal{V}] \quad \text{with} \quad \overline{K}(f \ast g) \cong f \otimes g
\]

because, for such a Hopf algebroid, the so-called “Fourier” natural isomorphism

\[
\mathcal{A}(a, Sb) \otimes \mathcal{A}(b, c) \cong \mathcal{A}(a, c) \otimes \mathcal{A}(b, c)
\]
is always available. In the particular case where \( \mathcal{A} \) is the single Hopf algebra \( H \), the Fourier isomorphism is the composite

\[
\begin{array}{ccc}
H \otimes H & \xrightarrow{\cong} & H \otimes H \\
\downarrow 1 \otimes \delta & & \downarrow \mu \otimes 1 \\
H \otimes H \otimes H & \xrightarrow{1 \otimes S \otimes 1} & H \otimes H \otimes H
\end{array}
\]

with inverse \((\mu \otimes 1)(1 \otimes \delta)\); see [1] §2.3.

**Example 2.** Given any (small) promonoidal \( \mathcal{V} \)-category \((\mathcal{A}, p, j)\), let \( \mathcal{A} \) be \( \mathcal{A} \) itself,

\[
K = p : \mathcal{A}^{\text{op}} \otimes \mathcal{A}^{\text{op}} \otimes \mathcal{A} \rightarrow \mathcal{V}
\]

and \( \mathcal{K} = \mathcal{A}^{\text{op}} \otimes \mathcal{A} \). Then

\[
\overline{K} : [\mathcal{A}, \mathcal{V}] \rightarrow [\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]
\]

is given by

\[
\overline{K}(f) = \int_a f(a) \otimes p(a, -, -).
\]

Since \( f \ast g = \int x y f(x) \otimes g(y) \otimes p(x, y, -) \), we obtain

\[
\overline{K}(f \ast g)(u, v) = \int_{x y z} f(x) \otimes g(y) \otimes p(x, y, z) \otimes p(z, u, v)
\]

\[
\cong \int_z \int_x f(x) \otimes p(x, z, v) \otimes \int_y g(y) \otimes p(y, u, z)
\]

\[
= \int_z \overline{K}(f)(z, v) \otimes \overline{K}(g)(u, z)
\]

\[
= (\overline{K}(f) \circ \overline{K}(g))(u, v),
\]

so \( K \) is multiplicative. A tentative \( \Gamma \) for this \( \overline{K} \) is given by

\[
\Gamma(F)(b) = \int_a F(a, b) \otimes j(a)
\]

since

\[
\Gamma \overline{K}(f)(b) = \int_a \overline{K}(f)(a, b) \otimes j(a)
\]

\[
= \int_a \int_x f(x) \otimes p(x, a, b) \otimes j(a)
\]

\[= \int_x f(x) \otimes \mathcal{A}(x, b) \quad \text{since } p \ast j = \mathcal{A}(-, -)
\]

\[= f(b) \quad \text{by Yoneda}.
\]

**Proposition.** If \( \mathcal{V} = \text{Vect}_k \) and \( j(d) \) is finite dimensional for all \( d \), then \( \Gamma \) is \( \mathcal{V} \)-faithful on all of \([\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]\) if \( j : \mathcal{A} \rightarrow \mathcal{V} \) is \( \mathcal{V} \)-faithful and monomorphisms split in \([\mathcal{A}, \mathcal{V}]\).
Proof. We require $F \to [j, F * j]$ to be monic for all $F$ in $[\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]$. But
\[ [j, F * j](d, c) = [j(d), F * j(c)] \]
\[ = [j(d), \int^x F(x, c) \otimes j(x)] \]
\[ \cong \int^x F(x, c) \otimes [j(d), j(x)] \]
and
\[ F(d, c) \cong \int^x F(x, c) \otimes \mathcal{A}(d, x) \quad \text{(Yoneda)} \]
\[ \cong \int^x F(x, c) \otimes [j(d), j(x)] \]
if $j$ is $\mathcal{V}$-faithful. \qed

Example 3. Let $(\mathcal{A}, p, j)$ be a small promonoidal $\mathcal{V}$-category. A functor
\[ \phi : \mathcal{A}^{\text{op}} \to \mathcal{V} \]
is called multiplicative if there are natural isomorphisms
\[ \int^c \phi(c) \otimes p(a, b, c) \cong \phi(a) \otimes \phi(b) \]
\[ \int^c \phi(c) \otimes j(c) \cong I \]
(in the sense of [9]). A natural transformation between two such functors is multiplicative if it commutes with these isomorphisms.

The (naturally) comonoidal $\mathcal{V}$-category
\[ \mathcal{X} = \mathcal{M}(\mathcal{A}^{\text{op}}, \mathcal{V}) \]
is defined to be the free $\mathcal{V}$-category in the ordinary category of all multiplicative functors from $\mathcal{A}^{\text{op}}$ to $\mathcal{V}$, and multiplicative natural transformations between them. The kernel
\[ K : \mathcal{A}^{\text{op}} \otimes \mathcal{M}(\mathcal{A}^{\text{op}}, \mathcal{V}) \to \mathcal{V} \]
is given by evaluation $K(a, \phi) = \phi(a)$ so that
\[ K(f)(\phi) = \int^a f(a) \otimes \phi(a). \]

If each representable functor $\mathcal{A}(-, a)$ is multiplicative, we can define a functor
\[ \Gamma : [\mathcal{M}(\mathcal{A}^{\text{op}}, \mathcal{V}), \mathcal{V}] \to [\mathcal{A}, \mathcal{V}] \]
by
\[ \Gamma(F)(d) = F(\mathcal{A}(-, d)), \]
so that $\Gamma K \cong 1$ by the Yoneda lemma. But this $\Gamma$ is faithful on the full subcategory of $[\mathcal{M}(\mathcal{A}^{\text{op}}, \mathcal{V}), \mathcal{V}]$ consisting of those functors $F$ which preserve the Yoneda colimit
\[ \phi \cong \int^a \phi(a) \otimes \mathcal{A}(-, a). \]
Moreover, each $K(f)$ is clearly an $F$ with this property, so that $\overline{K}$ is a full embedding.

By using the assumption that each $\phi$ is multiplicative the kernel $K$ can be seen to be multiplicative.
Example 4. Let $\mathcal{C}$ be a monoidal $\mathcal{V}$-category and let $\mathcal{A} \subset \mathcal{C}$ be a small Cauchy dense promonoidal $\mathcal{V}$-subcategory. Let $\mathcal{X} = \mathcal{C} \otimes \mathcal{V}$ with the product monoidal structure from $\mathcal{C}$ and $\mathcal{V}$, and let

$$K : \mathcal{A}^{\text{op}} \otimes (\mathcal{C} \otimes \mathcal{V}) \longrightarrow \mathcal{V}$$

be given by

$$K(a, c, x) = \mathcal{C}(a, c) \otimes x.$$

Then

$$K(f)(c, x) = \int_a f(a) \otimes \mathcal{C}(a, c) \otimes x$$
and

$$\Gamma(F)(a) = F(a, I)$$

since

$$\Gamma K(f)(b) = K(f)(b, I)$$

$$= \int_a f(a) \otimes \mathcal{A}(a, b) \otimes I$$

$$\cong \int_a f(a) \otimes \mathcal{A}(a, b)$$

$$\cong f(b).$$

Further application of the Yoneda lemma, and the calculus of coends, shows that $K$ is multiplicative because

$$\int^a K(a, z, x) \otimes p(b, c, a) \cong \int^a (\mathcal{C}(a, z) \otimes x) \otimes \mathcal{C}(b \otimes c, a)$$

since $p(b, c, a) = \mathcal{C}(b \otimes c, a)$ on $\mathcal{A}$,

$$\cong \mathcal{C}(b \otimes c, z) \otimes x$$

since $\mathcal{A} \subset \mathcal{C}$ is Cauchy dense,

$$\cong \int^z \mathcal{C}(b, z') \otimes x' \otimes (\mathcal{C}(c, z'') \otimes x'') \otimes (\mathcal{C}(z' \otimes z'', z) \otimes [x' \otimes x''] \otimes x),$$

$$= \int^z \mathcal{C}(b, z') \otimes K(c, z'', x'') \otimes p((z', x'), (z'', x''), (z, x))$$
as required. Moreover, $K$ lands in the full subcategory of $[\mathcal{C} \otimes \mathcal{V}, \mathcal{V}]$ consisting of those $F$ for which the canonical map

$$x \otimes F(a, I) \rightarrow F(a, x)$$
is an isomorphism, on which $\Gamma$ is clearly faithful.

Example 5. Suppose $\mathcal{V}$ is $*$-autonomous and $\mathcal{A}$ is a small monoidal $\mathcal{V}$-category with a given natural isomorphism $\mathcal{A}(a, b) \cong \mathcal{A}(b, a)^{*}$. Let $\mathcal{X} = [\mathcal{A}, \mathcal{V}]^{\text{op}}$ (monoidal under convolution) and define

$$K : \mathcal{A}^{\text{op}} \otimes [\mathcal{A}, \mathcal{V}]^{\text{op}} \longrightarrow \mathcal{V}$$

by $K(a, g) = g(a)^{\ast}$. Then

$$\mathcal{K}(f)(g) = \int^a f(a) \otimes g(a)^{*} = (g, f)$$
so that \( \Gamma(F)(a) = F(\mathcal{A}(a, -)) \) since
\[
\Gamma K(f)(b) = \int^a f(a) \otimes \mathcal{A}(b, a)^*
\leq \int^a f(a) \otimes \mathcal{A}(a, b)
\leq f(b).
\]

To show that \( K \) is multiplicative, we first prove:

**Lemma.** \( g(a)^* \cong [\mathcal{A}, \mathcal{V}](\mathcal{A}(a, -), g)^* \cong [\mathcal{A}, \mathcal{V}](g, \mathcal{A}(a, -)) \) for all \( g \) in \([\mathcal{A}, \mathcal{V}]\).

**Proof.**
\[
\int_b [g(b), \mathcal{A}(a, b)] = \int_b (g(b) \otimes \mathcal{A}(a, b))^*
\leq (\int_b g(b) \otimes \mathcal{A}(b, a))^*
\leq g(a)^* \text{ by Yoneda.}
\]

**Proposition.** \( K \) is multiplicative.

**Proof.** We require a natural isomorphism
\[
\int^g \int^h K(a, g) \otimes K(b, h) \otimes [\mathcal{A}, \mathcal{V}](k, g \ast h) \cong \int^c K(c, k) \otimes p(a, b, c).
\]
By the lemma, the left side is isomorphic to
\[
\int^g \int^h [\mathcal{A}, \mathcal{V}](g, \mathcal{A}(a, -)) \otimes [\mathcal{A}, \mathcal{V}](h, \mathcal{A}(b, -)) \otimes [\mathcal{A}, \mathcal{V}](k, g \ast h)
\cong [\mathcal{A}, \mathcal{V}](k, \mathcal{A}(a, -) \ast \mathcal{A}(b, -)) \text{ by Yoneda,}
\cong [\mathcal{A}, \mathcal{V}](\mathcal{A}(a \otimes b, -), k)^* \text{ by the Lemma,}
\cong k(a \otimes b)^*
\cong \int^c K(c, k) \otimes p(a, b, c),
\]
as required.

Also \( \Gamma \) is faithful on the full image of \( K \) since
\[
\Gamma K(f)(g)^* \cong \int_a [f(a), g(a)]
\cong \int_a [\Gamma K(f)(\mathcal{A}(a, -)), g(a)].
\]

**Example 6.** Let \( \mathcal{V} = \text{Vect}_k \) for a fixed field \( k \), and let \( \mathcal{X} \) be a finite promonoidal \( \mathcal{V} \)-category. Take
\[
\mathcal{A} = [\mathcal{X}, \text{Vect}]^{\text{op}} \text{ (which is monoidal under convolution)}
\]
and let
\[
K : \mathcal{A}^{\text{op}} \otimes \mathcal{X} \to \mathcal{V} \text{ be evaluation.}
\]
Then
\[ K(f)(x) = \int_x^a f(a) \otimes a(x) \cong f(\mathcal{X}(x, -)), \]
and we choose
\[ \Gamma(F)(a) = \int_x [a(x), F(x)]. \]
Then
\[ \Gamma K(f)(b) = \int_x [b(x), \int_x a(x) \otimes f(a)] \]
\[ \cong \int_x \int_x b(x)^* \otimes a(x) \otimes f(a) \quad \text{since} \ \int_x \ \text{is left exact}, \]
\[ \cong \int_x \mathcal{A}(b, a) \otimes f(a) \]
\[ \cong f(b) \quad \text{by Yoneda}. \]
Thus \( \Gamma \) is faithful on the full image of \( K \) since
\[ \Gamma K(f)(\mathcal{X}(x, -)) \cong f(\mathcal{X}(x, -)) \]
\[ \cong Kf(x); \]
moreover, \( K \) is multiplicative by Yoneda, as required.

**Example 7.** Let \( \mathcal{C} \) be a small braided compact closed category, let \( \mathcal{A} = \mathcal{C}^{\text{op}} \) with the monoidal structure induced from \( \mathcal{C} \), and let \( \mathcal{X} \) be a (finite) Cauchy dense full subcategory of \( \mathcal{A} \) (cf. [11] and [15]).

Then \( \mathcal{X} \) is promonoidal with respect to the structure induced by \( \mathcal{A} \), namely
\[ p(x, y, z) = \mathcal{C}(z, x \otimes y) \]
\[ j(z) = \mathcal{C}(z, I); \]
the associative and unital axioms follow from the Cauchy density of \( \mathcal{X} \subset \mathcal{A} \).
Moreover, here the unit of the \( K \dashv K \) adjunction, where \( K(a, x) = \mathcal{C}(x, a) \), is an isomorphism since
\[ \eta_f : f(a) \longrightarrow \int_x [K(a, x), \int_x^b K(b, x) \otimes f(b)] \]
gives
\[ f(a) \longrightarrow \int_x [\mathcal{C}(x, a), \int_x^b \mathcal{C}(x, b) \otimes f(b)] \]
\[ \cong \int_x [\mathcal{C}(x, a) \otimes f(x)] \]
by the Yoneda lemma, which becomes the isomorphism
\[ f(a) \cong \int_x [\mathcal{C}(x, a), f(x)] \]
by the Cauchy density of \( \mathcal{X} \subset \mathcal{A} \).
Also, $K$ is multiplicative because the kernel $K$ satisfies the conditions

$$
\int^{xy} K(a, x) \otimes K(b, y) \otimes p(x, y, z) = \int^{xy} C(x, a) \otimes C(y, b) \otimes C(z, x \otimes y)
$$

$$\cong C(z, a \otimes b) \quad \text{by Cauchy density of } \mathcal{X} \subset \mathcal{A},$$

while

$$
\int^c K(c, z) \otimes C(c, a \otimes b) = \int^c C(z, c) \otimes C(c, a \otimes b)
$$

$$\cong C(z, a \otimes b)$$

by the Yoneda lemma applied to $c \in C$, and

$$j(x) \cong \int^c C(x, c) \otimes j(c) = \int^c K(c, x) \otimes j(c)$$

for the same reason.

2.2. **When $K$ is conservative.** Sometimes $K$ is only conservative; i.e., the unit of the $K \dashv K$ adjunction is merely an equaliser, not an isomorphism. Then, as pointed out earlier, we obtain an equivalence

$$[\mathcal{A}, \mathcal{V}] \simeq \text{Joy}(K)$$

with a conservative multiplicative embedding of $\text{Joy}(K)$ into $[\mathcal{X}, \mathcal{V}]$. In fact, examples are readily available if we assume that all strong monomorphisms in $\mathcal{V}$ are equalisers, and that the canonical identity map

$$\text{id} : I \longrightarrow \int_x [K(a, x), K(a, x)]$$

is a strong monomorphism, preserved by the $\otimes$ in $\mathcal{V}$, for all $a \in \mathcal{A}$: In addition, we only need that each coprojection

$$K(a, x) \otimes f(a) \longrightarrow \int^b K(b, x) \otimes f(b)$$

is a strong monomorphism in $\mathcal{V}$ and each $K(a, x)$ has a dual in $\mathcal{V}$. This follows from the consideration of the diagram

$$
\begin{array}{ccc}
\int_x [K(a, x), K(a, x)] \otimes f(a) & \xrightarrow{\eta_f} & \int_x K(a, x)^* \otimes K(a, x) \otimes f(a) \\
\downarrow \text{id} \otimes 1 & & \downarrow \cong \\
\int_x [K(a, x), f(b)] & \longrightarrow & \int_x K(a, x)^* \otimes K(a, x) \otimes f(a)
\end{array}
$$

which commutes by observing that both legs are natural in $f \in [\mathcal{A}, \mathcal{V}]$, and then $f$ can be replaced by a representable functor and the Yoneda lemma employed.
Example 8. The coprojections

\[ K(a, x) \otimes f(a) \longrightarrow \int^b K(b, x) \otimes f(b) \]

are coretractions if

\[ \mathcal{A}(a, b) \cong \mathcal{A}(b, a)^* \]

naturally in \( a, b \in \mathcal{A} \), where we suppose that the hom-spaces of \( \mathcal{A} \) have duals in \( \mathcal{V} \). This result is fairly immediate from the Yoneda lemma.

Example 9 (cf. §2.1 Example 2). Let \( \mathcal{C} \) be a cocomplete monoidal category for which each functor of the form \( - \otimes C : \mathcal{C} \rightarrow \mathcal{C} \) preserves colimits, and suppose that there exists a full embedding \( N : \mathcal{A}^{op} \hookrightarrow \mathcal{C} \) where \( \{Nx; x \in \mathcal{A}\} \) is a small dense set of projectives in \( \mathcal{C} \). Suppose also that \( I = NI \) for some \( I \in \mathcal{A} \).

Define the functor \( U : \mathcal{C} \longrightarrow [\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{V}] \) by

\[ U(C)(x, y) = \mathcal{C}(Ny, Nx \otimes C). \]

If \( [\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{V}] \) is given the monoidal structure of bimodule composition, then \( U \) becomes a conservative multiplicative functor and there is also an induced monoidal equivalence

\[ \overline{N} : \mathcal{C} \simeq [\mathcal{A}, \mathcal{V}] \]

given by \( \overline{N}(C)(x) = \mathcal{C}(Nx, C) \), where \( \mathcal{A} \) has the promonoidal structure for which

\[ p : \mathcal{A}^{op} \otimes \mathcal{A}^{op} \otimes \mathcal{A} \longrightarrow \mathcal{V} \]

corresponds to \( UN : \mathcal{A}^{op} \longrightarrow [\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{V}] \)
and

\[ j = \mathcal{A}(I, -) : \mathcal{A} \longrightarrow \mathcal{V}, \]

and where \( [\mathcal{A}, \mathcal{V}] \) has the resulting convolution monoidal structure.

Example 10 (\( \mathcal{V} = \text{commutative monoids in Set}; \text{see [17]} \)). Let \( G \) be a finite group, let \( \mathcal{A} = \text{Span}(G\text{-Set}_f) \), and let \( \mathcal{X} = \text{Rep}_f(G) \) (for a fixed field) viewed as \( \mathcal{V} \)-categories. Define the kernel

\[ K : \mathcal{A}^{op} \otimes \mathcal{X} \longrightarrow \mathcal{V} \]

by \( K(a, x) = \mathcal{X}(Fa, x) \) where

\[ F : \mathcal{A} \longrightarrow \mathcal{X} = \text{Rep}_f(G) \]

is the extension of the canonical functor

\[ F : G\text{-Set}_f \longrightarrow \mathcal{X}, \]

defined by “summing over the fibres” (see [17] §10 for example), and define the functor

\[ U : \mathcal{X} \longrightarrow \text{Span}(G\text{-Set}) \]

by \( U(x) \) is the large induced \( G \)-Set of the representation \( x : G \longrightarrow \text{Vect}_f \) in \( \mathcal{X} \).

Then we have a canonical natural transformation

\[ \tau : \mathcal{X}(Fb, Fa) \longrightarrow \text{Span}(G\text{-Set})(b, UFa) \]
in \( \mathcal{V} \), and a natural coretraction
\[
\beta : a \longrightarrow UF(a)
\]
in \( \text{Span}(G\text{-Set}) \). The fact that \( \overline{K} \) is conservative now follows from commutativity of the diagram

\[
\begin{align*}
f(a) \xrightarrow{\eta_f} & \int_x \left[ \mathcal{X}(Fa, x), \int^b \mathcal{X}(Fb, x) \otimes f(b) \right] \\
\cong & \int^b \mathcal{X}(Fb, Fa) \otimes f(b) \\
\cong & \int^b \mathcal{X}(Fb, Fa) \otimes f(b) \\
& \downarrow \rho \otimes 1 \\
\int^b \mathcal{A}(b, a) \otimes f(b) \xrightarrow{\rho} & \int^b \text{Span}(G\text{-Set})(b, UFa) \otimes f(b)
\end{align*}
\]

where \( \rho \) is the coretraction induced by \( \beta \), and the isomorphisms shown in the diagram are again by the Yoneda lemma.

The main result of [9] applies to show that \( \overline{K} \) is multiplicative.

**Example 11 (cf. §1.2).** As mentioned in §1.2(c), an arbitrary \( \mathcal{V} \)-functor
\[
\psi : \mathcal{X} \longrightarrow \mathcal{A},
\]
with \( \mathcal{A} \) and \( \mathcal{X} \) small promonoidal \( \mathcal{V} \)-categories, induces a multiplicative functor
\[
\overline{K} = [\psi, 1] : [\mathcal{A}, \mathcal{V}] \longrightarrow [\mathcal{X}, \mathcal{V}]
\]
between the respective convolutions if and only if the kernel functor on \( \mathcal{A}^{\text{op}} \otimes \mathcal{X} \) given by
\[
K(a, x) = \mathcal{A}(a, \psi(x))
\]
is multiplicative.

Of course, \( \overline{K} = [\psi, 1] \) is conservative if \( \psi \) is a surjection on object sets.

**Example 12.** In Example 2 of §2.1 we have that
\[
\overline{K} : [\mathcal{A}, \mathcal{V}] \longrightarrow [\mathcal{A}^{\text{op}} \otimes \mathcal{A}, \mathcal{V}]
\]
is conservative if \( (\mathcal{A}, p, j) \) is a closed category; i.e., if
\[
p(a, x, y) = \mathcal{A}(a, [x, y]) \quad \text{and} \quad j(y) = \mathcal{A}(I, y).
\]
In fact, here \( \eta_f \) is a coretraction since
\[
\eta_f : f(a) \longrightarrow \int_{xy} [p(a, x, y), \int^b p(b, x, y) \otimes f(b)] \\
\cong \int_{xy} [\mathcal{A}(a, [x, y]), f([x, y])] \quad \text{by Yoneda},
\]
where
\[
\begin{align*}
\int_{x<y} [\mathcal{A}(a, [x, y]), f([x, y])]
\end{align*}
\]
commutes.

APPENDIX

The term “graphic” applied to coends, convolution products, transforms, etc., in the above context is intended to relate especially to base categories like \( \mathcal{V} = \text{Set} \) and \( \mathcal{V} = \text{Vect}_k \), where there is a definite notion of finite graph. In such cases we call a given \( \mathcal{V} \)-subgraph \( \mathcal{X} \) of the \( \mathcal{V} \)-category \( \mathcal{A} \) a finite \( \mathcal{V} \)-graph if \( \text{ob}(\mathcal{X}) \) is finite and each \( \mathcal{V} \)-object \( \mathcal{X}(x, y) \) of edges is finite (or finite dimensional).

Then, if the category \( \mathcal{A} \) is the directed union of all its finite \( \mathcal{V} \)-subgraphs \( \mathcal{X}_\phi \) (or some convenient subset of these), there results a canonical isomorphism
\[
\colim \int^x T\phi(x, x) \cong \int^a T(a, a),
\]
where \( T \phi \) denotes the restriction of each \( \mathcal{V} \)-functor
\[
T : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \to \mathcal{V}
\]
to the \( \mathcal{V} \)-graph \( \mathcal{X}_\phi^{\text{op}} \otimes \mathcal{X}_\phi \), and where each finite “coend” \( \int^x T\phi(x, x) \) over \( x \in \mathcal{X}_\phi \) is computed in the same way as the usual coend of a functor over any (small) category. In particular, any finite \( \mathcal{V} \)-limit which commutes with each finite \( \int^x \), also commutes with their filtered colimit \( \int^a \) over \( a \in \mathcal{A} \).

However, we note that for many practical purposes the finite “coend” \( \int^x \) can be replaced by the (usual) coend over the corresponding full subcategory of \( \mathcal{A} \) determined by \( \text{ob}(\mathcal{X}) \).

REFERENCES

[1] M. Barr, *-Autonomous categories (with appendix by Po Hsiang Chu), Lecture Notes in Mathematics 752 (Springer-Verlag, Berlin 1979).
[2] M. Barr and C. Wells, Toposes, Triples and Theories, Springer-Verlag, 1985. Also Reprints in Theory Appl. Categories 12 (2005).
[3] R. A. Bailey, Association schemes: Designed experiments, algebra and combinatorics, Cambridge University Press, 2004.
[4] B. Bakalov, A. D’Andrea and V. G. Kac, Theory of finite pseudoalgebras, Adv. Math. 162 (2001) pp. 1–140.
[5] F. Bergeron, G. Labelle and P. Leroux, Combinatorial species and tree-like structures, Cambridge University Press, 1998.
[6] B. J. Day, On closed categories of functors, Lecture Notes in Mathematics 137 (Springer-Verlag, Berlin 1970), pp. 1–38.
[7] B. J. Day, On closed categories of functors II, Lecture Notes in Mathematics 420 (Springer-Verlag, Berlin 1974), pp. 20–54.
[8] B. J. Day, *-Autonomous categories in quantum theory, arXiv: math.QA/0605037, 2006.
[9] B. J. Day and R. H. Street, Kan extensions along promonoidal functors, Theory Appl. Categories 1 (1995) pp. 72–77.
[10] B. J. Day and R. H. Street, Monoidal bicategories and Hopf algebroids, Adv. Math. 129 (1997) pp. 99–157.
[11] R. Häring-Oldenburg, Reconstruction of weak quasi-Hopf algebras, J. Alg. 194 (1997) pp. 14–35.
[12] A. Joyal, Foncteurs analytiques et espèces de structures, Lecture Notes in Mathematics 1234 (Springer-Verlag, Berlin 1986) pp. 126–159.
[13] G. M. Kelly, Basic concepts of enriched category theory, LMS Lecture Note Series 64, Cambridge University Press 1982. Also Reprints in Theory Appl. Categories 10 (2005).
[14] G. M. Kelly and A. J. Power, Adjunctions whose counits are coequalisers, and presentations of finitary enriched monads, J. Pure Appl. Algebra 89 (1993) pp 163–179.
[15] G. Moore and N. Seiberg, Classical and quantum conformal field theory, Comm. Math. Phys. 123 (1989) pp. 177–254.
[16] H. Narayanan, Submodular functions and electrical networks, ADM, North Holland (Elsevier Science B.V.) 54 (1997).
[17] E. Padchadcharan and R. Street, Mackey functors on compact closed categories, (preprint, Macquarie University, 2006) submitted for publication.

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