Representation dimensions of triangular matrix algebras

Hongbo Yin, Shunhua Zhang

School of Mathematics, Shandong University, Jinan 250100, P. R. China

Abstract

Let $A$ be a finite dimensional hereditary algebra over an algebraically closed field $k$, $T_2(A) = \begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$ be the triangular matrix algebra and $A^{(1)} = \begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ be the duplicated algebra of $A$ respectively. We prove that $\text{rep.dim } T_2(A)$ is at most three if $A$ is Dynkin type and $\text{rep.dim } T_2(A)$ is at most four if $A$ is not Dynkin type. Let $T$ be a tilting $A$-module and $\mathcal{T} = T \oplus \mathcal{T}$ be a tilting $A^{(1)}$-module. We show that $\text{End}_{A^{(1)}} \mathcal{T}$ is representation finite if and only if the full subcategory $\{ (X, Y, f) \mid X \in \text{mod } A, Y \in \tau^{-1} \mathcal{F}(T_A) \cup \text{add } A \}$ of $\text{mod } T_2(A)$ is of finite type, where $\tau$ is the Auslander-Reiten translation and $\mathcal{F}(T_A)$ is the torsion-free class of $\text{mod } A$ associated with $T$. Moreover, we also prove that $\text{rep.dim } \text{End}_{A^{(1)}} \mathcal{T}$ is at most three if $A$ is Dynkin type.

Key words and phrases: Representation dimension, tilting module, finite type.

MSC(2000): 16E10, 16G10

1 Introduction

Representation dimension of Artin algebras was introduced by M. Auslander in [4], this concept gives a reasonable way of measuring how far an Artin algebra $\Lambda$ is from being
representation-finite. In particular, M. Auslander has shown that an Artin algebra is representation-finite if and only if its representation dimension is at most 2.

O. Iyama, in [14], proved that the representation dimension of an Artin algebra is always finite. Recently, Rouquier proved in [18] that the representation dimension of Artin algebras can be arbitrary large.

An interesting relationship between the representation dimension and the finitistic dimension conjecture has been shown by K. Igusa and G. Todorov [13], which is, if the representation dimension of an algebra is at most three, then its finitistic dimension is finite. Since then, many important algebras were proved to have representation dimension at most three. Such as tilted algebras, \( m \)-replicated algebras, quasi-tilted algebras etc., see [2][16][17] for details.

We follow this direction and investigate some special kinds of triangular matrix algebras with small representation dimensions.

Let \( A \) be a finite dimensional hereditary algebra over an algebraically closed field \( k \), \( T_2(A) = \begin{pmatrix} A & 0 \\ A & A \end{pmatrix} \) be the triangular matrix algebra and \( A^{(1)} = \begin{pmatrix} A & 0 \\ DA & A \end{pmatrix} \) be the duplicated algebra of \( A \) respectively.

The following theorems are the main results of this paper.

**Theorem 1.** Let \( A \) be a finite dimensional hereditary algebra over an algebraically closed field \( k \). Then \( \text{rep.dim} \ T_2(A) \leq 3 \) if \( A \) is Dynkin type and \( 3 \leq \text{rep.dim} \ T_2(A) \leq 4 \) if \( A \) is not Dynkin type.

**Remark.** Theorem 1 improves the well known result about representation dimension of \( T_2(A) \). According to [9], we know that \( \text{rep.dim} \ T_2(A) \leq \text{rep.dim} \ A + 2 \), which implies that \( \text{rep.dim} \ T_2(A) \leq 5 \) if \( A \) is a finite dimensional hereditary algebras over an algebraically closed field.

Tilting theory of duplicated algebra \( A^{(1)} \) has strong relationship with cluster tilting theory induced in [7], and it has been widely investigated in [1][15][20][21]. In this paper, we mainly investigate the representation type and representation dimension of endomorphism algebras of tilting modules over duplicated algebra \( A^{(1)} \).
Let $T$ be a basic tilting $A$-module and $\overline{T} = T \oplus P$ be a tilting $A^{(1)}$-module, where $P$ is the direct sum of all non-isomorphic indecomposable projective-injective $A^{(1)}$-modules.

**Theorem 2.** Take the notation as above. Then $\text{End}_{A^{(1)}} \overline{T}$ is representation finite if and only if the full subcategory $\{(X, Y, f) \mid X \in \mod A, Y \in \tau^{-1} \mathcal{F}(T_A) \cup \text{add } A\}$ of $\mod T_2(A)$ is of finite type, where $\tau$ is the Auslander-Reiten translation and $\mathcal{F}(T_A)$ is the torsion-free class associated with $T$.

**Theorem 3.** Take the notation as above and assume that $A$ is Dynkin type. Then $\text{rep.dim End}_{A^{(1)}} \overline{T} \leq 3$.

**Remark** We should mention that Theorem 1 can be obtained from Theorem 3 by taking $T = DA$. We prove them differently, which seems to be of independent interest.

This paper is arranged as follows. In Section 2, we collect definitions and basic facts needed for our research. Section 3 is devoted to the proof of Theorem 1, and in section 4, we prove Theorem 2 and Theorem 3.

## 2 Preliminaries

Let $\Lambda$ be a finite dimensional algebra over an algebraically closed field $k$. We denote by $\mod \Lambda$ the category of all finitely generated right $\Lambda$-modules and by $\ind \Lambda$ the full subcategory of $\mod \Lambda$ containing exactly one representative of each isomorphism class of indecomposable $\Lambda$-modules. We denote by $\text{pd } X$ (resp. $\text{id } X$) the projective (resp. injective) dimension of an $\Lambda$-module $X$ and by $\text{gl.dim } \Lambda$ the global dimension of $\Lambda$. Let $D = \text{Hom}_k(-, k)$ be the standard duality between $\mod \Lambda$ and $\mod \Lambda^{\text{op}}$, and $\tau_\Lambda$ be the Auslander-Reiten translation of $\Lambda$. The Auslander-Reiten quiver of $\Lambda$ is denoted by $\Gamma_\Lambda$.

Let $M$ be a $\Lambda$-module. We denote by $\text{add } M$ the subcategory of $\mod \Lambda$ whose objects are the direct summands of finite direct sums of $M$. A module $M$ is called a generator if all projective modules are in $\text{add } M$ and is called a cogenerator if all injective modules are in $\text{add } M$. We denote by $\text{rep.dim } \Lambda$ the representation dimension of $\Lambda$ which is defined by Auslander in [1] as following.

$$\text{rep.dim } \Lambda = \min \{ \text{gl.dim } \text{End}_\Lambda M \mid M \text{ is a generator – cogenerator for } \Lambda \}$$
The generator-cogenerator realizing the representation dimension is called the Auslander generator. The following well known lemma is the crucial tool to determine the upper bound of the representation dimension.

**Lemma 2.1.** \[4, 8, 19\]. Let \( A \) be an Artin algebra. \( M \) is a generator-cogenerator for \( \text{mod} \ A \). Then \( \text{gld} \ \text{End}_A M \leq n + 2 \) if and only if for each \( A \)-module \( X \) there is an exact sequence

\[
(*) \quad 0 \rightarrow M_n \rightarrow \cdots \rightarrow M_0 \rightarrow X \rightarrow 0
\]

with all \( M_i \) belongs to add \( M \), such that the induced sequence

\[
0 \rightarrow \text{Hom}_A(M, M_n) \rightarrow \cdots \rightarrow \text{Hom}_A(M, M_0) \rightarrow \text{Hom}_A(M, X) \rightarrow 0
\]

is exact.

**Remark.** The exact sequence of (\( * \)) in Lemma 2.1 is called an add \( M \)-resolution of \( X \).

Let \( C \) be an additive Krull-Schmit Hom-finite \( k \)-category, and \( \mathcal{X} \) a full subcategory of \( C \). We denote by \( \text{ind} \ \mathcal{X} \) the subcategory consisting of indecomposable objects of \( \mathcal{X} \) and we say \( \mathcal{X} \) is of finite type if \( \text{ind} \ \mathcal{X} \) is a finite set. Recall from \[5\], a map \( X' \rightarrow A \) with \( X' \in \mathcal{X} \) and \( A \in C \) is called a right \( \mathcal{X} \)-approximation of \( A \) if the induced map \( \text{Hom}(X, X') \rightarrow \text{Hom}(X, A) \) is an epimorphism for all \( X \in \mathcal{X} \). A map \( f : A \rightarrow B \) in category \( C \) is called right minimal, if for every \( g : A \rightarrow A \) such that \( fg = f \), the map \( g \) is an isomorphism. A right (left) approximation that is also a right (left) minimal map is called a minimal right (left) approximation of \( T \). The subcategory \( \mathcal{X} \) is called contravariantly finite if any object in \( C \) admits a (minimal) right \( \mathcal{X} \)-approximation. The notions of (minimal) left \( \mathcal{X} \)-approximation and covariantly finite subcategory can be defined dually.

A module \( T \in \text{mod} \ \Lambda \) is called a tilting module if the following conditions are satisfied:

1. \( \text{pd}_\Lambda T \leq 1 \);
2. \( \text{Ext}_\Lambda^1(T, T) = 0 \);
3. There is an exact sequence \( 0 \rightarrow \Lambda \rightarrow T_0 \rightarrow T_1 \rightarrow 0 \) with \( T_i \in \text{add} \ T \) for \( 0 \leq i \leq 1 \).

Let \( T \) be a tilting \( \Lambda \)-module and \( B = \text{End}_\Lambda T \). According to \[10\], \((\mathcal{F}(T), \mathcal{F}(T))\) is the torsion pair in \( \text{mod} \ \Lambda \) generated by \( T \), where \( \mathcal{F}(T) = T^\perp = \{ X \in \text{mod} \ \Lambda \mid \text{Ext}_\Lambda^1(T, X) = 0 \} \).
0 \rightleftharpoons \text{gen } T \text{ and } \mathcal{F}(T) = \{X \in \text{mod } A \mid \operatorname{Hom}\_A(T, X) = 0\}, \text{ the corresponding torsion pair in } B\text{-mod is } (\mathcal{X}(T), \mathcal{Y}(T)), \text{ where } \mathcal{X}(T) = \{X \in \text{mod } B \mid T \otimes_B X = 0\} \text{ and } \mathcal{Y}(T) = \{Y \in \text{mod } B \mid \text{Tor}^B_1(T, Y) = 0\}.

A torsion pair is called splitting if every indecomposable \(\Lambda\)-module either belongs to the torsion class or belongs to the torsion-free class. A tilting module \(T\) is called splitting if the corresponding torsion pair \((\mathcal{X}(T), \mathcal{Y}(T))\) is splitting, and \(T\) is called separating if the corresponding torsion pair \((\mathcal{X}(T), \mathcal{Y}(T))\) is splitting. Note that every tilting module of hereditary algebras is splitting.

**Lemma 2.2.** Let \(T\) be a tilting module of algebra \(\Lambda\). \(B = \operatorname{End}\_\Lambda T\).

(i) \(\operatorname{Hom}\_\Lambda(T, -) : \mathcal{F}(T) \to \mathcal{Y}(T)\) and \(- \otimes_B T : \mathcal{F}(T) \to \mathcal{X}(T)\) are equivalent functors.

(ii) \(T\) is splitting if and only if \(\operatorname{id} X = 1\) for every \(X \in \mathcal{F}(T)\).

(iii) \(T\) is separating if and only if \(\operatorname{pd} Y = 1\) for every \(Y \in \mathcal{X}(T)\).

Let \(\mathcal{T}\_\Lambda\) be the set of all basic tilting \(\Lambda\)-modules up to isomorphism. Recall from [11], the tilting quiver \(\mathcal{K}(\Lambda)\) of \(\Lambda\) is defined as the following. The vertices of \(\mathcal{K}(\Lambda)\) are the elements of \(\mathcal{T}\_\Lambda\). There is an arrow \(T' \to T\) in \(\mathcal{K}(\Lambda)\) if and only if \(T' = M \oplus X\) and \(T = M \oplus Y\) with \(X\) and \(Y\) indecomposable such that there is a short exact sequence \(0 \to X \xrightarrow{f} E \xrightarrow{g} Y \to 0\) such that \(f\) is a minimal left add \(M\)-approximation of \(X\) and that \(g\) is a minimal right add \(M\)-approximation of \(Y\).

We recall the definition of triangular matrix algebra from [6]. Let \(A\) and \(B\) be finite dimensional algebras over \(k\) and \(A M_B\) be an \(A\)-\(B\)-bimodule. \(\Lambda = \begin{pmatrix} B & 0 \\ M & A \end{pmatrix}\) is called a triangular matrix algebra, its elements are \(\begin{pmatrix} b & 0 \\ m & a \end{pmatrix}\) where \(b \in B, a \in A, m \in M\), and its addition and multiplication are given by the usual matrix operation.

**Remark.** There are two special kinds of triangular matrix algebras. One is \(T_2(A) = \begin{pmatrix} A & 0 \\ A & A \end{pmatrix}\) and the other is \(A^{(1)} = \begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}\) which is also called the duplicated algebra of \(A\).
It is well known that the module category of \( \Lambda = \begin{pmatrix} B & 0 \\ M & A \end{pmatrix} \) is equivalent to the category \( \text{rep}(AM_B) \), called the representation of the bimodule \( AM_B \), see [3, Appendix A 2.7]. The objects of \( \text{rep}(AM_B) \) are triples \((X, Y, f)\), where \( X \) is an \( A \)-module, \( Y \) is a \( B \)-module and \( f : X \otimes_A M \rightarrow Y \) is a \( B \)-module morphism. The morphism between \((X_1, Y_1, f_1)\) and \((X_2, Y_2, f_2)\) is a pair \((x, y)\) makes the following diagram commutative.

\[
\begin{array}{ccc}
X_1 \otimes_A M & \xrightarrow{x \otimes_A M} & X_2 \otimes_A M \\
\downarrow f_1 & & \downarrow f_2 \\
Y_1 & \xrightarrow{y} & Y_2
\end{array}
\]

Using the adjoint isomorphism between \(- \otimes M\) and \( \text{Hom}(M, -) \), the category \( \text{rep}(AM_B) \) can also be described as follows. Its objects are triples \((X, Y, f)\), where \( X \) is an \( A \)-module, \( Y \) is a \( B \)-module and \( f : X \rightarrow \text{Hom}_B(M, Y) \) is an \( A \)-module morphism. The morphism between \((X_1, Y_1, f_1)\) and \((X_2, Y_2, f_2)\) is a pair \((x, y)\) makes the following diagram commutative.

\[
\begin{array}{ccc}
X_1 & \xrightarrow{x} & X_2 \\
\downarrow f_1 & & \downarrow f_2 \\
\text{Hom}_B(M, Y_1) & \xrightarrow{\text{Hom}(M, y)} & \text{Hom}_B(M, Y_2)
\end{array}
\]

In the following, we will freely use the two descriptions as the modules of the triangular matrix algebras.

The indecomposable projective \( A \)-modules are isomorphic to objects of the form \((0, Q, 0)\) where \( Q \) is an indecomposable projective \( B \)-module and \((P, P \otimes_A M, \text{id})\) where \( P \) is an indecomposable \( A \)-module, \( \text{id} \) is the identity map. Dually, the indecomposable injective \( A \)-modules are isomorphic to objects of the form \((I, 0, 0)\) where \( I \) is an indecomposable injective \( A \)-module and \((\text{Hom}_B(M, J), J, \text{id})\) where \( J \) is an indecomposable injective \( B \)-module. See [6, III, Proposition 2.5] for details.

Let \( C \) be an additive Krull-Schmit Hom-finite \( k \)-category. We define \( \text{Mor}\ C \) to be the morphism category of \( C \) whose objects are triples \((X, Y, f)\) where \( X, Y \in C, f \in \text{Hom}_C(X, Y) \) and the morphism between \((X_1, Y_1, f_1)\) and \((X_2, Y_2, f_2)\) is a pair \((x, y)\) makes the following diagram commutative.
We also use $X \xrightarrow{f} Y$ to denote the objects of $\text{Mor} \mathcal{C}$. It is easy to see that $\text{Mor} \mathcal{C}$ is also a Krull-Schmidt category. In particular, $\text{Mor} \text{mod} A$ is equivalent to $\text{mod} T_2(A)$.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two full subcategories of $\mathcal{C}$. We denote by $\mathcal{X}_1 \cup \mathcal{X}_2$ the full subcategory of $\mathcal{C}$ consisting of the objects which either belong to $\mathcal{X}_1$ or belong to $\mathcal{X}_2$ and $\mathcal{X}_1 \setminus \mathcal{X}_2$ the full subcategory of $\mathcal{C}$ consisting of the objects belonging to $\mathcal{X}_1$ and not belonging to $\mathcal{X}_2$. If $\mathcal{C} = \text{mod} \Lambda$ is a module category, then we denote by $\tau \mathcal{C}_1$ (resp. $\tau^{-1} \mathcal{C}_1$) the full subcategory of $\mathcal{C}$ whose objects are obtained from $\mathcal{C}_1$ by the once action of $\tau$ (resp. $\tau^{-1}$).

Let $\Lambda$ be an Artin algebra of finite representation type and $M_1, M_2, \ldots, M_n$ be a complete set of non-isomorphic indecomposable $\Lambda$-modules. According to Auslander in [6], $M = M_1 \oplus \cdots \oplus M_n$ is called an additive generator of $\text{mod} \Lambda$ and $\Gamma_M = \text{End}_A (M)^{op}$ is said to be the Auslander algebra of $\Lambda$. It is well known that $\Gamma_M$ and $T_2(\Lambda)$ have the same representation type.

Let $\mathcal{C}$ be an additive Krull-Schmidt Hom-finite $k$-category of finite type and $M$ be an object of $\mathcal{C}$. $M$ is said to be an additive generator of $\mathcal{C}$ if every indecomposable object of $\mathcal{C}$ is a direct summand of $M$, then $\Gamma_M = \text{End}_\mathcal{C} (M)^{op}$ is said to be the Auslander algebra of $\mathcal{C}$.

**Remark.** If $\mathcal{C} = \text{mod} \Lambda$ for some representation-finite algebra $\Lambda$, then the Auslander algebra of $\mathcal{C}$ is the same as the original definition of Auslander algebra of $\Lambda$.

The following proposition is similar with Proposition 5.8 in [6, p.215].

**Proposition 2.3.** Let $\mathcal{C}$ be an additive Krull-Schmidt Hom-finite $k$-category of finite type and $M$ an additive generator of $\mathcal{C}$. Then the Auslander algebra $\Gamma_M$ of $\mathcal{C}$ is of finite representation type if and only if the category $\text{Mor} \mathcal{C}$ is of finite type.

Throughout this paper, the notations will be fixed as above. We refer to [3, 6] for the other concepts of representation theory of Artin algebras.
3 Representation dimension of triangular matrix algebras

In this section, we assume that $A$ is a finite dimensional hereditary algebra over an algebraically closed field $k$. We will give a bound of the representation dimension of $T_2(A)$ by using Proposition 2.3, and then prove Theorem 1.

**Theorem 3.1.** Let $A$ be a hereditary algebra of Dynkin type, then $\text{rep.dim} \ T_2(A) \leq 3$.

**Proof** Note that $A$ is an additive generator of $\text{add} \ A$ and that $\text{End}_A A = A^{op}$ is of finite representation type. Then $\text{Mor} (\text{add} \ A)$ is of finite type by Proposition 2.3. By the same argument we know that $\text{Mor} (\text{add} \ DA)$ is also of finite type.

Let $(P_1, P_2, f)$ and $(I_1, I_2, g)$ be additive generators of $\text{Mor} (\text{add} \ A)$ and $\text{Mor} \text{ add} \ DA$ respectively and let $M = (P_1, P_2, f) \oplus (I_1, I_2, g)$. Then $M$ is a generator-cogenerator for $\text{mod} \ T_2(A)$.

We claim that $\text{gl.dim} \ End_{T_2(A)} M \leq 3$.

In fact, let $(X, Y, h)$ be an indecomposable $T_2(A)$-module. We may assume that $(X, Y, h)$ does not belong to $\text{add} \ M$.

**Case I.** Assume that $X$ and $Y$ have no non-zero injective direct summand. Then there is a minimal right $\text{add} \ M$-approximation $(\alpha_1, \alpha_2) : (P_1', P_2', f') \to (X, Y, h)$ of $(X, Y, h)$ with $(P_1', P_2', f')$ belongs to $\text{Mor} \ A$ since there is no non-zero homomorphism from $(I_1, I_2, g)$ to $(X, Y, h)$. Then we have the following commutative diagram.

\[
\begin{array}{ccc}
0 & \to & P_1'' \\
& \beta_1 \swarrow & \downarrow \alpha_1 \\
P_1' & \to & X \\
& \downarrow f' & \downarrow h \\
0 & \to & Y \\
& \beta_2 \swarrow & \downarrow \alpha_2 \\
P_2'' & \to & 0
\end{array}
\]

(3.1)

with exact rows. This is a short exact sequence of $T_2(A)$-modules, and it follows that $P_1''$ and $P_2''$ are projective $A$ modules since $A$ is hereditary and $P_1', P_2'$ are projective. In particular, $(P_1', P_2', f')$ belongs to $\text{add} \ M$ and the diagram of (3.1) is an add $M$-resolution of $(X, Y, h)$.

**Case II.** $X$ or $Y$ have non-zero injective direct summand, that is, $(X, Y, h)$ is of the form $h = \begin{pmatrix} h_1 & 0 \\ h_2 & h_3 \end{pmatrix} : X_1 \oplus I_1' \to Y_1 \oplus I_2'$, where $X_1$ and $Y_1$ both have no non-zero injective direct summand, $I_1'$ and $I_2'$ are injective $A$-modules (may be zero).
Let $i_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : I'_1 \to X_1 \oplus I'_1$ and $i_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : I'_2 \to Y_1 \oplus I'_2$. It is easy to see that

\[
\begin{array}{c}
I'_1 \xrightarrow{i_1} X_1 \oplus I'_1 \\
\downarrow h_3 \\
I'_2 \xrightarrow{i_2} Y_1 \oplus I'_2
\end{array}
\]

is a minimal right add$(I_1, I_2, g)$-approximation of $(X, Y, h)$.

Let

\[
\begin{array}{c}
P'_1 \xrightarrow{\alpha_1} X \\
\downarrow f' \\
P'_2 \xrightarrow{\alpha_2} Y
\end{array}
\]

be a minimal right add$(P_1, P_2, f)$-approximation of $(X, Y, h)$. It is epimorphism since all the indecomposable projective $T_2(A)$-modules belong to add$(P_1, P_2, f)$. Then $(P'_1, P'_2, f') \oplus (I'_1, I'_2, h_3)$ is a right add $M$-approximation of $(X, Y, h)$.

We have the following commutative diagram.

\[
\begin{array}{c}
0 \to \text{Ker} \pi_1 \to P'_1 \oplus I'_1 \xrightarrow{\pi_1} X \to 0 \\
\downarrow \varphi \bigg| \begin{pmatrix} f' & 0 \\ 0 & h_3 \end{pmatrix} \bigg| \downarrow h \\
0 \to \text{Ker} \pi_2 \to P'_2 \oplus I'_2 \xrightarrow{\pi_2} Y \to 0
\end{array}
\] (3.2)

with $\pi_1 = (\alpha_1, i_1)$ and $\pi_2 = (\alpha_2, i_2)$. Now we need to determine Ker $\pi_1$ and Ker $\pi_2$.

Consider the pull-back of $(\alpha_1, i_1)$:

\[
\begin{array}{c}
D \xrightarrow{d_2} I'_1 \\
\downarrow d_1 \\
P'_1 \xrightarrow{\alpha_1} X
\end{array}
\]

which implies that Ker $\pi_1 \simeq D$. Note that a pull-back diagram is also a push-out diagram if and only if $(\alpha_1, i_1)$ is epimorphism see [12, exercise 6.7]. Hence, the above diagram is also a push-out of $(d_1, d_1)$. In particular, we have that $d_2$ is monomorphism because $i_1$ is and Ker $\pi_1$ is projective. Then Ker $\pi_2$ is also projective by the same argument. Hence $(\text{Ker} \pi_1, \text{Ker} \pi_2, \varphi)$ belongs to add $M$ and (3.2) is an add $M$-resolution of $(X, Y, h)$.

Summary the above discussions, we know that gl.dim $\text{End}_{T_2(A)}M \leq 3$, which forces that rep.dim $T_2(A) \leq 3$. The proof is completed.

\[\square\]

**Theorem 3.2.** Let $A$ be a hereditary algebra of Euclidean or wild type. Then $3 \leq \text{rep.dim} T_2(A) \leq 4$.
The first inequality is obvious since $T_2(A)$ is representation infinite when $A$ is not Dynkin type.

For the second inequality, we choose $M = T_2(A) \oplus DT_2(A)$ as a generator-cogenerator for $\text{mod } T_2(A)$. Note that $\text{gld } T_2(A) \leq \text{gld } A + 1 = 2$, see for example [6] Proposition 2.6, p 78] for details.

Let $(X,Y,h)$ be a $T_2(A)$-module. We may assume that $(X,Y,h)$ does not belong to $\text{add } M$.

If both $X$ and $Y$ have no non-zero injective direct summand, then a projective resolution of $(X,Y,h)$ also is its $\text{add } M$-resolution which obviously has length at most 2.

If $X$ or $Y$ have non-zero injective direct summand, then $(X,Y,h)$ is of the form

\[
\begin{pmatrix}
  h_1 & 0 \\
  h_2 & h_3
\end{pmatrix}
\]

$X_1 \oplus I'_1 \rightarrow Y_1 \oplus I'_2$, where $X_1$ and $Y_1$ both have no non-zero injective direct summand, $I'_1$ and $I'_2$ are injective $A$ modules (may be zero).

Let $i_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : I'_1 \rightarrow X_1 \oplus I'_1$ and $i_3 = \begin{pmatrix} 0 \\ h_3 \end{pmatrix} : I'_1 \rightarrow Y_1 \oplus I'_2$.

Then

\[
\begin{array}{ccccccc}
  I'_1 & \xrightarrow{i_1} & X_1 \oplus I'_1 \\
  \downarrow & & \downarrow \\
  I'_1 & \xrightarrow{i_3} & Y_1 \oplus I'_2
\end{array}
\]

is a minimal right $\text{add } DT_2(A)$-approximation of $(X,Y,h)$, and let

\[
\begin{array}{cccc}
P'_1 & \xrightarrow{\alpha_1} & X & \xrightarrow{0} \\
\downarrow & & \downarrow h \\
P'_2 & \xrightarrow{\alpha_2} & Y & \xrightarrow{0}
\end{array}
\]

is a minimal right $\text{add } T_2(A)$-approximation of $(X,Y,h)$. Then we have a commutative diagram similar to (3.2)

\[
\begin{array}{cccc}
  0 & \xrightarrow{\varphi} & \text{Ker } \pi_1 & \xrightarrow{P'_1 \oplus I'_1 } \pi_1 \xrightarrow{X} \xrightarrow{0} \\
  \downarrow & & \downarrow (f' \ 0 \\ 0 \ 1) & \downarrow h \\
  0 & \xrightarrow{\text{Ker } \pi_2} & \xrightarrow{P'_2 \oplus I'_1 } \pi_2 \xrightarrow{Y} \xrightarrow{0}
\end{array}
\]

(3.3)

By the same argument as in Theorem 3.1, we know $\text{Ker } \pi_1$ is projective. Let $P''_2$ be the projective cover of $\text{Ker } \pi_2$. Then $(\text{Ker } \pi_1, \text{Ker } \pi_1, 1) \oplus (0, P''_2, 0)$ is a right $\text{add } T_2(A)$-approximation of $(\text{Ker } \pi_1, \text{Ker } \pi_2, \varphi)$, and we have the following commutative diagram with exact rows:
where $P_3$ is a projective $A$-module. According to diagrams (3.3) and (3.4), we get a right add $M$-resolution of $(X, Y, h)$ of length at most 2, hence $\text{rep.dim } T_2(A) \leq 4$. This completes the proof.

4 Endomorphism algebras of tilting modules of duplicated algebras

Let $A$ be a hereditary algebra and $T$ be a tilting right $A$-module. Let $B = \text{End}_A T$. Then $B T_A$ is a $B$-$A$-bimodule. Let $A^{(1)}$ be the duplicated algebra of $A$ and $\overline{P}$ be the direct sum of all non-isomorphic indecomposable projective-injective $A^{(1)}$-modules. Then $T = T \oplus \overline{P}$ is a tilting right $A^{(1)}$-module. We will prove Theorem 2 and Theorem 3 in this section.

Note that $\overline{P}$ and $T$ regarded as $A^{(1)}$-modules can be written as $\overline{P} = (A, DA, \text{id})$ and $T = (0, T, 0)$. Then we have follows.

$$\text{Hom}_{A^{(1)}}((0, T, 0), \overline{P}) = \text{Hom}_{A}(T, DA) = DT$$

$$\text{End}_{A^{(1)}} T = \begin{pmatrix} \text{End}_A T & 0 \\ \text{Hom}_{A^{(1)}}((0, T, 0), \overline{P}) & \text{End}_{A^{(1)}} \overline{P} \end{pmatrix} = \begin{pmatrix} B & 0 \\ DT & A \end{pmatrix}.$$

**Theorem 4.1.** Take the notations as above. Then $\text{End } T$ is representation finite if and only if the full subcategory $\{ (X, Y, f) \mid X \in \text{mod } A, Y \in \tau^{-1} F(T_A) \cup \text{add } A \}$ of $\text{mod } T_2(A)$ is of finite type.

In order to prove the theorem, we need following lemmas.

**Lemma 4.2.** Let $A$, $T$ and $B$ be as above. Then $DT$ is a separating convex tilting right $B$-module.

**Proof** It is shown in [10] that $\text{ind add}(DT)$ is a complete slice in $\text{mod } B$. Hence $DT$ is a convex tilting right $B$-module, and $DT$ is a splitting tilting left $A$-module since $A$ is hereditary. Then it follows that $DT$ is also a separating tilting right $B$-module. $\square$
Lemma 4.3. Assume $A$, $T$ and $B$ be as above. Let $(\mathcal{F}(T), \mathcal{F}(T))$ and $(\mathcal{X}(DT), \mathcal{Y}(DT))$ be the torsion pairs in mod $A$ corresponding to $T$ and $DT$ respectively, and let $(\mathcal{F}(DT), \mathcal{F}(DT))$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ be the torsion pairs in mod $B$ corresponding to $DT$ and $T$ respectively. Then we have the following.

(i) $\mathcal{F}(DT) = \mathcal{X}(T) \cup \text{add } DT$, $\mathcal{F}(DT) = \mathcal{Y}(T) \setminus \text{add } DT$;

(ii) $\mathcal{Y}(DT) = \tau^{-1} \mathcal{F}(T) \cup \text{add } A$, $\mathcal{X}(DT) = \tau^{-1}(\mathcal{F}(T) \setminus \text{add } DA)$.

Proof (i) According to Lemma 4.2, $DT$ is a separating convex tilting right $B$-module. We have the following.

\[ \text{ind } \mathcal{F}(DT) = \{ M \in \text{ind } B \mid \text{Hom}_{B}(DT, M) \neq 0 \}, \]

\[ \text{ind } \mathcal{F}(DT) = \{ M \in \text{ind } B \mid \text{Hom}_{B}(DT, M) = 0 \}. \]

Let $M$ be an indecomposable $B$-module. Then either $M \in \mathcal{Y}(T)$ or $M \in \mathcal{X}(T)$ since $T$ is splitting.

If $M \in \mathcal{Y}(T)$, then $M = \text{Hom}_{A}(T, N)$ for some $N \in \text{ind } \mathcal{F}(T)$,

\[ \text{Hom}_{B}(DT, M) = \text{Hom}_{B}(\text{Hom}_{A}(T, DA), \text{Hom}_{A}(T, N)) = \text{Hom}_{A}(DA, N). \]

In this situation, $M \in \mathcal{F}(DT)$ if and only if $N \in \text{add } DA$, which equivalent to $M \in \text{add } DT$.

If $M \in \mathcal{X}(T)$, then $M = \text{Ext}_{A}^{1}(T, N)$ for some $N \in \text{ind } \mathcal{F}(T)$. Applying $\text{Hom}_{A}(T, -)$ to an injective resolution $0 \rightarrow N \rightarrow I_{1} \rightarrow I_{2} \rightarrow 0$ of $N$, we get the following exact sequence

\[ 0 \rightarrow \text{Hom}_{A}(T, N) \rightarrow \text{Hom}_{A}(T, I_{1}) \rightarrow \text{Hom}_{A}(T, I_{2}) \rightarrow \text{Ext}_{A}^{1}(T, N) \rightarrow 0. \]

Therefore, $\text{Hom}_{B}(\text{Hom}_{A}(T, I_{2}), \text{Ext}_{A}^{1}(T, N)) \neq 0$.

On the other hand, by using $DT = \text{Hom}_{A}(T, DA)$ we have

\[ \text{Hom}_{B}(DT, M) = \text{Hom}_{B}(\text{Hom}_{A}(T, DA), \text{Ext}_{A}^{1}(T, N)) \neq 0, \]

hence $\mathcal{F}(DT) = \mathcal{X}(T) \cup \text{add } DT$. Since $DT$ is separating, we have that $\mathcal{F}(DT) = \mathcal{Y}(T) \setminus \text{add } DT$.

(ii) We have $\mathcal{Y}(DT) = \{ \text{Hom}_{B}(DT, M) \mid M \in \mathcal{F}(DT) \},$ and by (i), $\mathcal{Y}(DT) = \{ \text{Hom}_{B}(DT, M) \mid M \in \mathcal{X}(T) \cup \text{add } DT \}$.

If $M$ is an indecomposable $B$-module which belongs to $\text{add } DT$, then $\text{Hom}_{B}(DT, M) \in \text{add } A.$

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If $M \in \text{ind } \mathcal{X}(T)$, then $M = \text{Ext}_A^1(T, N)$ for some $N \in \text{ind } \mathcal{F}(T)$. Then we have

\[
\text{Hom}_B(DT, M) = \text{Hom}_B(\text{Hom}_A(T, DA), \text{Ext}_A^1(T, N)) = \text{Ext}_A^1(DA, N) = D\text{Hom}_A(\tau^{-1}N, DA) = \tau^{-1}N
\]

By using the equivalence of $\mathcal{X}(T)$ and $\mathcal{F}(T)$, we have \{Hom$_B(DT, M) \mid M \in \mathcal{X}(T)\} = \tau^{-1}\mathcal{F}(T)$. Hence, $\mathcal{Y}(DT) = \tau^{-1}\mathcal{F}(T) \cup \text{add } A$.

Finally, we determine $\mathcal{X}(DT)$. According to Lemma 2.2 and by using (i), we know that $\mathcal{X}(DT) = \{\text{Ext}_B^1(DT, M) \mid M \in \mathcal{F}(DT)\} = \{\text{Ext}_B^1(DT, M) \mid M \in \mathcal{Y}(T) \setminus \text{add } DT\}$.

Let $M$ be an indecomposable $A$-module in $\mathcal{X}(DT)$. Then $M = \text{Hom}_A(T, N)$ for some $N \in \text{ind } \mathcal{F}(T)$ with $N \notin \text{add } DA$.

Then we have

\[
\text{Ext}_B^1(DT, M) = \text{Ext}_B^1(DT, \text{Hom}_A(T, N)) = \text{Ext}_A^1(DT \otimes_B T, N) = \text{Ext}_A^1(DA, N) = D\text{Hom}_A(\tau^{-1}N, DA) = \text{Hom}_A(A, \tau^{-1}N) = \tau^{-1}N
\]

Hence, $\mathcal{X}(DT) = \tau^{-1}(\mathcal{F}(T) \setminus \text{add } DA)$. The proof is completed. \qed

**Proof of Theorem 4.1.** If $A$ is not Dynkin type, then both $\{(X, Y, f) \mid X \in \text{mod } A, Y \in \tau^{-1}\mathcal{F}(T_A) \cup \text{add } A\}$ and $\text{End}_{A(1)} \mathcal{T}$ are of infinite type. Hence without loss of generality, we can assume that $A$ is Dynkin type.

Now, let $(X, Y, g)$ be any indecomposable $\text{End}_{A(1)} \mathcal{T}$-module. According to Lemma 4.2, $DT$ is a separating convex tilting $B$-module, hence $Y$ can be written as $Y = M \oplus N$ where $M \in \mathcal{F}(DT)$ and $N \in \mathcal{F}(DT)$. By Lemma 4.3 (i) Hom$_B(DT, N) = 0$, and there are two kinds of indecomposable $\text{End}_{A(1)} \mathcal{T}$-modules.

(1) $N \neq 0$. Then $(0, N, 0)$ is a direct summand of $(X, M \oplus N, g)$, it forces that $(X, M \oplus N, g) = (0, N, 0)$ and $N$ is indecomposable. Note that the number of indecomposable $\text{End}_{A(1)} \mathcal{T}$ modules of this kind is finite since $A$ is Dynkin type.

(2) $N = 0$. Then $(X, M \oplus N, g) = (X, M, g)$. Hence $\text{End}_{A(1)} \mathcal{T}$ is representation finite if and only if there are finite number of indecomposable modules of the second type.
We write the full subcategory of the second kind of modules by \( \text{mod}_2 \text{End}_{A(1)} \overline{T} \). We define a Functor \( F : (X, M, g) \rightarrow (X, \text{Hom}_B(DT, M), g') \). By Lemma 4.3 (ii) we know that \( F \) is an equivalence between the \( \text{mod}_2 \text{End}_{A(1)} \overline{T} \) and \( \{ (X, Y, f) \in \text{mod} T_2(A) | X \in \text{mod} A, Y \in \text{add} A \cup \tau^{-1} F(T) \} \). This completes the proof of the theorem. \( \square \)

**Example.** Let \( A \) be the path algebra of the quiver

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\]

the Aulander-Retein quiver of \( A \) is as follows.

Consider the APR-tilting module \( T \) of \( A \) whose indecomposable direct summand is denoted by \( \bullet \) in the Auslander-Reiten quiver. \( \mathcal{F}(T) \) has one indecomposable module which is the simple projective \( A \)-module. It is easy to see that \( \tau^{-1} \mathcal{F}(T) \cup \text{add} A \) is of finite type and its Auslander algebra \( \Gamma \) is given by the quiver

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\]

with the relation that the sum of the three roads with length two in the middle mesh equals zero. Note that \( \Gamma \) is representation infinite, hence by Theorem 4.1 and Proposition 2.3, \( \text{End}_{A(1)} \overline{T} \) is also representation infinite.

**Corollary 4.4.** Let \( A \) be a finite dimensional hereditary algebra over an algebraically closed field \( k \) and let \( T_1 \) and \( T_2 \) be two (basic) tilting module of \( A \). Assume that there is a path from \( T_1 \) to \( T_2 \) in the tilting quiver \( \mathcal{X}(A) \), if \( \text{End}_{A(1)} \overline{T_1} \) is representation infinite, then \( \text{End}_{A(1)} \overline{T_2} \) is also representation infinite.
Proof It follow from the fact that $\mathcal{T}(T_1) \subseteq \mathcal{T}(T_2)$ since there is a path from $T_1$ to $T_2$. \qed

Remark. The converse of Corollary 4.4 is not true, that is, there exists tilting modules $T_1$ and $T_2$ with a path between them in the tilting quiver such that $\text{End}_{A(t)} T_1$ is representation finite, but $\text{End}_{A(t)} T_2$ is representation infinite. See the example above, $A$ is representation finite while the unique APR-tilting module is representation infinite and there is an arrow from $A$ to the APR-module.

In the rest part of this section, we investigate the representation dimension of Endomorphism algebras of tilting modules over duplicated algebras. The following lemma is useful in our research.

Lemma 4.5. Let $A$ be a finite dimensional hereditary algebra over an algebraically closed field $k$ and $T$ be a tilting $A$-module. Assume that $f : T_1 \to X$ is a right $T$-approximation of $X$, then $Ker f \in \text{add } T$.

Proof $\text{Hom}_A(T, X) = \text{Hom}_A(T, tX)$, hence $f$ is in fact the composition of $T_1 \xrightarrow{f} tX \to X$. Note $T_1 \xrightarrow{f} tX$ is the $T$-approximation of $tX$. Because $tX \in \mathcal{T}(T)$, $tX$ is generated by $T$. So, $T_1 \xrightarrow{f} tX \to 0$ is epimorphism. Applying $\text{Hom}_A(T, -)$ to the exact sequence

$$0 \to \text{Ker} f \to T_1 \to tX \to 0 \quad (4.1)$$

We get

$$\text{Hom}_A(T, T_1) \xrightarrow{(T,f)} \text{Hom}_A(T, tX) \to \text{Ext}^1_A(T, \text{Ker} f) \to \text{Ext}^1_A(T, T_1) = 0$$

Because $\text{Hom}_A(T, f)$ is epimorphism, we have $\text{Ext}^1_A(T, \text{Ker} f) = 0$. So, $\text{Ker} f \in \mathcal{T}(T)$.

Let $U$ be any module in $\mathcal{T}(T)$. Applying $\text{Hom}(-, U)$ to (4.1) we get

$$0 = \text{Ext}^1_A(T_1, U) \to \text{Ext}^1_A(\text{Ker} f, U) \to \text{Ext}^2_A(tX, U) = 0$$

So, $\text{Ext}^1_A(\text{Ker} f, U) = 0$, i.e. $\text{Ker} f$ is $\text{Ext}$-projective in $\mathcal{T}(T)$. We know that $\text{Ker} f$ belongs to $\text{add } T$. This completes the proof. \qed

The following lemma is taken from [2, Proposition 2.2] which will be used later.

Lemma 4.6. Let $A$ be an Artin algebra. $M = T \oplus N$ is an $A$-module. $X$ is generated by $M$ and $0 \to K \to M \to X \to 0$ is a minimal $M$-resolution of $X$. If $N = DA$ and $T$ is a convex tilting of module, then $K \in \text{add } T$. 

Now, we can prove Theorem 3 promised in introduction.

**Theorem 4.7.** Let $A$ be a hereditary algebra of Dynkin type over an algebraically closed field $k$, and $T$ be a tilting $A$-module. Let $\mathcal{T} = T \oplus \mathcal{T}$ be a tilting $A^{(1)}$-module as above. Then $\text{rep.dim } \text{End}_{A^{(1)}} \mathcal{T} \leq 3$.

**Proof** Let $(P_1, P_2, f)$ be the additive generator of Mor (add $A$). We can assume that $P_2 = \text{Hom}(DT, DT_2)$ with $DT_2 \in \text{add } DT$. Then $(P_1, DT_2, f)$ contains all the indecomposable projective $T_2(A)$-modules of the form $(P, DT, id)$.

Let $M = (P_1, DT_2, f) \oplus (T, DB, id) \oplus (0, DB \oplus DT, 0) \oplus (T, 0, 0) \oplus (0, B, 0)$ with $B = \text{End}_A T$. Then $M$ is a generator-cogenerator of $\text{mod } \text{End}_{A^{(1)}} \mathcal{T}$.

Let $(X, Y, f)$ be an indecomposable $\text{End}_{A^{(1)}} \mathcal{T}$-module. It is easy to see that there are three kinds of indecomposable $\text{End}_{A^{(1)}} \mathcal{T}$-modules.

1. The first case is $(X, Y, f)$ such that $X \neq 0$ and $Y \neq 0 \in \mathcal{T}(DT)$.

We claim that there is no morphism from $(DA, 0, 0)$ to $(X, Y, f)$.

In fact, we assume by contrary that $(g, 0) \neq 0$ is a morphism from $(DA, 0, 0)$ to $(X, Y, f)$, then $\text{Im } g$ is an injective direct summand of $X$ and also belongs to $\text{Ker } f$. It follows that $(\text{Im } g, 0, 0)$ is a direct summand of $(X, Y, f)$ which contradicts with the assumption of $(X, Y, f)$.

Let $M_1 \rightarrow (X, Y, f) \rightarrow 0$ be a minimal add $M$-approximation of $(X, Y, f)$. Then

$M_1 = (P', DT'', f') \oplus (T_1, DB_1, id) \oplus (0, DB_2 \oplus DT'', 0) \oplus (T_3, 0, 0)$. We get an exact sequence

$$
\begin{array}{ccccccc}
0 & \rightarrow & \text{Ker } \pi_1 & \rightarrow & P' \oplus T_1 \oplus T_3 & \rightarrow & X & \rightarrow & 0 \\
& & h \downarrow & & \pi_1 \downarrow & & \downarrow f & \\
0 & \rightarrow & \text{Ker } \pi_2 & \rightarrow & DT' \oplus DB_1 \oplus DB_2 \oplus DT'' & \rightarrow & Y & \rightarrow & 0
\end{array}
$$

We should mention that the lower row in above commutative diagram should be the sequence under functor $\text{Hom}_B(DT, -)$. But it doesn’t matter since $\text{Hom}_B(DT, -)$ is an equivalence between $\mathcal{F}(DT)$ and $\mathcal{Y}(DT)$.

Let $\pi_1 = (p, t)$ with $p : P' \rightarrow X$ and $t : T_1 \oplus T_2 \rightarrow X$ is an add $T$-approximation of $X$. According to Lemma 4.5, $\text{Ker } t \in \text{add } T$.

We consider the following pull-back diagram

$$
\begin{array}{ccccccc}
0 & \rightarrow & \text{Ker } t & \rightarrow & \text{Ker } \pi_1 & q \rightarrow & P' \\
& & \downarrow & & \downarrow p & & \\
0 & \rightarrow & \text{Ker } t & \rightarrow & T_1 \oplus T_2 & \rightarrow & X
\end{array}
$$
Note that $\text{Im } q = \mathcal{P}''$ is projective since $A$ is hereditary. Let $\text{Ker } t = T_4$. Then $\text{Ker } \pi_1 = \mathcal{P}'' \oplus T_4$.

We claim that $\pi_2: DT' \oplus DB_1 \oplus DB_2 \oplus DT'' \rightarrow Y$ is an minimal add $(DB \oplus DT)$-approximation of $Y$ and $\pi_2$ is epimorphism.

The reason is that $M_1 \rightarrow (X, Y, f) \rightarrow 0$ is minimal and $(0, DB \oplus DT, 0)$ is a direct summand of $M$. It follows that $\pi_2$ is an minimal add $(DB \oplus DT)$-approximation of $Y$. Note that $Y \in \mathcal{F}(DT)$ which implies that $Y$ is generated by $DT$, hence $\pi_2$ is epimorphism. By using Lemma 4.6, we know that $\text{Ker } \pi_2 \in \text{add } M$.

Let $\text{Ker } \pi_2 = DT_5$. Then $DT_5 \subset DT' \oplus DT''$, hence $(\text{Ker } \pi_1, \text{Ker } \pi_2, h) = (\mathcal{P}''', DT_5, h) \oplus (T_4, 0, 0)$ belongs to $\text{add } M$.

(2) The second case is $(X, Y, f) = (X, 0, 0)$ with $X \in \text{ind } A$. If $X \in \text{add } DA$, then $0 \rightarrow (X, 0, 0) \rightarrow (X, 0, 0) \rightarrow 0$ is an add $M$-resolution of $(X, 0, 0)$. If $X$ is not injective, then by using the same proof as in (1), we can obtain an add $M$-resolution of $(X, 0, 0)$ with the length at most one.

(3) The third case is $(0, Y, 0)$ with $Y \in \text{ind } B$. According to [2], we know that $\text{rep.dim } B \leq 3$ and $B \oplus DB \oplus DT$ is an Auslander generator for $\text{mod } B$.

Assume that $0 \rightarrow N_2 \rightarrow N_1 \rightarrow Y \rightarrow 0$ is an add $(B \oplus DB \oplus DT)$-resolution of $Y$. It is easy to check that there is no non-zero morphism from $(P_1, DT_2, f) \oplus (T, DB, id) \oplus (DA, 0, 0)$ to $(0, Y, 0)$, hence $0 \rightarrow (0, N_2, 0) \rightarrow (0, N_1, 0) \rightarrow (0, Y, 0) \rightarrow 0$ is an add $M$-resolution of $(0, Y, 0)$.

Summary the above discussions, we have shown that any indecomposable $\text{End}_{A^{(1)}} \mathcal{T}$-module admits an add $M$-resolution with the length at most one. Hence $\text{rep.dim } \text{End}_{A^{(1)}} \mathcal{T} \leq 3$. This completes the proof. $\square$

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