Evidence estimation in finite and infinite mixture models and applications

Adrien Hairault\textsuperscript{1}, Christian P. Robert\textsuperscript{1,2} and Judith Rousseau\textsuperscript{3}
\textsuperscript{1}Université Paris Dauphine, \textsuperscript{2}University of Warwick, and \textsuperscript{3}University of Oxford

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Abstract

Estimating the model evidence - or marginal likelihood of the data - is a notoriously difficult task for finite and infinite mixture models and we reexamine here different Monte Carlo techniques advocated in the recent literature, as well as novel approaches based on Geyer (1994) reverse logistic regression technique, Chib (1995) algorithm, and Sequential Monte Carlo (SMC). Applications are numerous. In particular, testing for the number of components in a finite mixture model or against the fit of a finite mixture model for a given dataset has long been and still is an issue of much interest, albeit yet missing a fully satisfactory resolution. Using a Bayes factor to find the right number of components \( K \) in a finite mixture model is known to provide a consistent procedure. We furthermore establish the consistence of the Bayes factor when comparing a parametric family of finite mixtures against the nonparametric ‘strongly identifiable’ Dirichlet Process Mixture (DPM) model.

1 Introduction

Mixture models, defined as the convex combination of probability distributions, are of significant interest due to their convenient way of modelling heterogeneity in a given population. When some group structure is present in the data, Finite Mixture models (FM) arise as a natural tool to conduct a Bayesian clustering analysis. They have been successfully applied to computer vision (Stauffer and Grimson (1999)), document classification (Blei et al. (2003)) but also more generally to genetics, physics, or economics (see McLachlan et al. (2019) for a comprehensive review of applications). Alternatively, the Dirichlet Process Mixture model (DPM), or ‘infinite’ mixture model, first introduced by Ferguson (1983), has become one of the main tools in the field of Bayesian non-parametrics. Its range of applications is broad, from multivariate clustering (see, e.g., Crépet and Tressou (2011), Quintana and Iglesias (2003)) to Bayesian density estimation (Escobar and West (1995), Neal (1992), Rasmussen et al. (1999), Lo (1984)).

Bayesian model selection is primarily done by comparing the marginal likelihoods of the data (a.k.a model evidence) for competing models, defined as

\[
m(y) = \int_\Theta f(x|\theta)\pi(\theta)d\theta
\]

for some data \( y \), density function \( f \), prior \( \pi \), and parameter space \( \Theta \).
In the context of mixture modelling, being able to compute the evidence of a model is of great importance and has several applications of interest. We here present two of them.

First, the usual problem of statistical modelling of determining whether an $n$-sample $(y_1, \ldots, y_n)$ arises from a particular parametric family of distributions can be formalised as a goodness of fit test against a nonparametric alternative, like the DPM (Kamary et al. (2014)). Tokdar and Martin (2021) designs a particular DPM alternative for testing normality. The Bayes Factor was proven to be consistent for a point null hypothesis against a large class of nonparametric alternatives by Dass and Lee (2004) or Verdinelli and Wasserman (1998). Mcvinish et al. (2009) derive sufficient conditions on nonparametric distributions for which the consistence of the Bayes factor for testing a parametric family of distributions holds. We establish here (Theorem 4.3) the consistence of the Bayes factor comparing the parametric family of finite mixtures against the nonparametric location Dirichlet Process Mixture (DPM) model. Obviously, these results can only be of use in practice when one is able to produce a numerical approximation to marginal likelihoods and, subsequently, to the Bayes Factor.

Second, marginal likelihood can be used to find the right number of components $K$ in a finite mixture model. One way of addressing this challenge is indeed to perform model selection using the Bayes Factor by computing the marginal likelihood of the data for various values of $K$. From a clustering perspective, it can be regarded as finding the optimal number of groups $K$ in a population.

Evaluating the model evidence (1) is unfortunately a notoriously difficult task for mixture models. For finite mixtures, several Monte Carlo methods have been developed in order to estimate (1), among which Importance Sampling (IS) methods such as Bridge Sampling are popular (Frühwirth-Schnatter, 2004). Chib (1995) proposed another approach based on the Bayes identity which initial bias, as pointed out by Neal (1999), can be corrected (Lee and Robert, 2016). However, if one wishes to retrieve an estimator with a reasonable variance, those methods get very computationally expensive as $K$, the number of mixture components, increases (see Lee and Robert (2016)). Values of $K$ as small as 6 already represent a significant computational burden, stressing the need for new estimators more suited to real-life applications where $K$ need not be small.

While there exists an abundant literature and well-established methods for computing the evidence of parametric models, equivalent methods for non-parametric models are far from numerous. In fact, to our knowledge, only Basu and Chib (2003) directly address the issue of evidence estimation for the DPM by adapting the method of Chib (1995). SMC tools have been proposed by MacEachern et al. (1999) for beta-binomial Dirichlet Process mixtures. This idea was later generalized by Griffin (2017) and applied to a very particular kind of DPM by Tokdar and Martin (2021). However, their proposed SMC framework relies on the strong assumption that the concentration parameter $M$ of the DP is known and fixed. Neither Chib’s algorithm nor SMC appear to be widely used by practitioners. Hence, it is a common practice to use the DPM without considering alternative parametric or non-parametric models, although it may not always be appropriate. For instance, Miller and Harrison (2014) show that for data assumed to come from a finite mixture with an unknown number of components, the DPM posterior on the number of clusters in the observed data does not converge to the true number of components.

Using the partitioning structure induced by finite mixture models, we derive in Section 2
an unbiased estimator of the marginal likelihood of a conjugate finite mixture model which computational time does not grow exponentially with \( K \), as is the case with the popular Chib (1995) method, for instance. Based on Kong et al. (1994) sequential imputation algorithm, we also propose an effective Sequential Importance Sampling (SIS) strategy for estimating the evidence of conjugate finite mixtures. Given their simplicity and effectiveness, we believe these marginal likelihood estimators should greatly ease tasks like the selection of the number of components for finite mixture models, for instance. We also provide a comparative assessment of many different marginal likelihood estimators, and identify those that scale well in difficult scenarios where both the size of the data \( n \) and the number of components \( K \) are large, which to our knowledge has not been done before.

As for the DPM, we propose in Section 3 an alternative algorithm based on Reverse Logistic Regression (Geyer 1994) which does not require the concentration parameter \( M \) to be fixed but rather to follow a Gamma prior distribution. We show empirically that this method scales better with the amount of data than Basu and Chib (2003) does. We also provide a review and assessment of the ways to estimate the marginal likelihood of a non-parametric DPM model, which, to our knowledge, has not been done before. In a similar way to Section 2 we provide an empirical study of marginal likelihood estimators, including scenarios where \( n \) is large. We also empirically assess the behaviour of the Bayes Factor comparing a parametric family of finite mixtures against a nonparametric DPM alternative.

While the asymptotic behaviour of the marginal likelihood has been studied In Section 4 we study the asymptotic behavior of the marginal likelihood of the DPM for strongly identifiable emission distributions, when fitted to data assumed to arise from a finite location mixture. The obtained result implies the consistence of the Bayes Factor in this setting. While the asymptotic behaviour of a finite mixture is well-established (see for instance Chambaz and Rousseau (2008) or Drton and Plummer (2017)), equivalent results for the DPM and in particular an upper bound on the marginal likelihood, had not been derived before.

2 Evidence approximation for finite mixtures

Assume that a random i.i.d sample \( y = (y_1, \ldots, y_n) \) arises from a finite mixture of \( K \) distributions which densities \( f(\cdot|\theta_k), k = 1, \ldots, K \) are parametrized by a component-specific \( \theta_k \in \Theta \subset \mathbb{R}^d, d \geq 1 \). One can write the corresponding mixture likelihood function as

\[
p_K(y|\vartheta) = \prod_{i=1}^{n} \sum_{k=1}^{K} \varpi_k f(y_i|\theta_k)
\]  

(2)

where \( \vartheta = (\theta, \varpi), \theta = (\theta_1, \ldots, \theta_K) \in \Theta^K \subset \mathbb{R}^{d \times K}, \varpi = (\varpi_1, \ldots, \varpi_K) \in \Delta_{K-1} \), the \((K-1)\)-dimensional simplex.

Another way of specifying a finite mixture model is by introducing latent variables \( z = (z_1, \ldots, z_n) \in \{1, \ldots, K\}^n \) that indicate cluster membership of the individual observations \((y_1, \ldots, y_n)\). They
allow for a convenient generative construction of the finite mixture model.

\[ y_i | \theta, z_i \sim f(\cdot | \theta, z_i), \quad i = 1, \ldots, n \]

\[ z_1, \ldots, z_n | \pi \sim \text{Cat}(\pi_1, \ldots, \pi_K) \]

\[ \pi = (\pi_1, \ldots, \pi_K) \sim \mathcal{D}(\alpha) \]  

\[ \theta_1, \ldots, \theta_K \sim \mathcal{G} \]  

where \( \mathcal{G} \) is a prior on \( \theta \) while \( \mathcal{D}(\cdot) \) and \( \text{Cat}(\cdot) \) denote the Dirichlet and the categorical distributions, respectively.

2.1 Popular existing estimators of the marginal likelihood for finite mixtures

Since the seminal article by Tanner and Wong (1987), a common strategy for sampling from the posterior distribution of a finite mixture model \( \pi_K(\theta | y) \) takes advantage of the latent variable representation (3). Popular Gibbs sampling schemes have been developed for this purpose by Diebolt and Robert (1994), for example. However, efficient and reliable algorithms to estimate the marginal likelihood of such models, that scale with the number of mixture components \( K \), still need to be derived. One can define this quantity by

\[ m_K(y) := \int_{\Theta^K \times \Delta_{K-1}} p_K(y | \theta) \pi_K(d\theta) \]  

where \( \pi_K(\theta) = \mathcal{G}_0(\theta | \alpha) \). Integral (4) is unfortunately intractable and difficult to estimate, in particular because of the \( K! \) equiprobable modal configurations of the posterior distribution. This specific complexity is at the source of the development of \textit{ad hoc} estimation methods for finite mixture models. Below we describe the two most popular Monte-Carlo algorithms for marginal likelihood estimation of FM, namely Bridge Sampling (Frühwirth-Schnatter (2019)) and Chib’s algorithm (Chib (1995)) as well as its subsequent corrections as proposed by Berkhof et al. (2003) and Lee and Robert (2016). We explain why these algorithms suffer from the curse of dimensionality as \( K \) increases. Sequential Monte Carlo (SMC) is also described in its finite mixture version as suggested by Chopin (2002). We then propose another correction to Chib’s algorithm based on the partitioning structure implied by mixture models. We also adapt Kong et al. (1994)’s sequential imputation strategy to finite mixture model, which appears to be both robust to an increase in the number of mixture components \( K \) and the number of observations \( n \).

Chib’s algorithm. Introduced by Chib (1995), the core idea of this method relies on the simple Bayes identity

\[ m_K(y) = \frac{p_K(y | \theta_0) \pi_K(\theta_0)}{\pi_K(\theta_0 | y)} \]  

for some \( \theta_0 \in \Theta^K \times \Delta_{K-1} \).

To estimate the posterior density \( \pi_K(\theta_0 | y) \) that is not available in closed-form, Chib (1995) uses the output \( \{ \theta^{(t)}, z^{(t)} \}_{t=1}^T \) of a Gibbs sampler targeting the posterior distribution of \( (\theta, z) \) in a
Rao-Blackwellized estimator
\[ \hat{\pi}_K(\theta_0|y) = \frac{1}{T} \sum_{t=1}^{T} \pi_K(\theta_0|z^{(t)}, y) \] (6)

where \( \theta_0 \) typically is the Maximum A Posteriori (MAP) estimator. Note that the densities \( \pi_K(\cdot|z, y) \) are available in closed form for conjugate finite mixtures. The posterior density estimator (6), that is plugged into (5), is unfortunately biased in the case of poor mixing of the Monte Carlo Markov Chain producing the sequence \( \{\theta^{(t)}, z^{(t)}\}_{t=1}^{T} \), as pointed out by Neal (1999). Indeed, due to mixture posterior invariance to relabeling of components (see, e.g., Frühwirth-Schnatter (2006b)), there exists \( K! \) equally likely posterior modes. When the Gibbs sampler does not visit evenly every one of the \( K! \) modes, or in other words all \( K! \) permutations of the MAP estimator \( z_0 \), (6) is over-estimated, which in turn implies an underestimation of Chib’s marginal likelihood estimator. This ill-behavior of the Gibbs sampler is in practice very common.

For small values of \( K \), however, an easy fix has been suggested by Berkhof et al. (2003). It consists in summing over all permutations of \( z^{(t)} \) for \( t = 1, \ldots, T \) yielding the following estimator
\[ \tilde{\pi}_K(\theta_0|y) = \frac{1}{K!T} \sum_{t=1}^{T} \sum_{s \in S(\{1, \ldots, k\})} \pi_K(\theta_0|s(z^{(t)}), y) \] (7)

where \( S(\{1, \ldots, k\}) \) denotes the set of all permutations of \( \{1, \ldots, k\} \).

This estimator, although very accurate, cannot be computed in an appropriate time as soon as \( K \geq 5 \). A trick suggested by Marin and Robert (2008) is to sample at random 100 permutations for such values of \( K \) in order to keep computational time below a reasonable threshold. Unfortunately, 100 random permutations should not be enough to keep the good variance properties of Chib’s estimator when \( K \geq 8 \), as for such values, \( K! \gg 100 \).

Bridge Sampling. Bridge sampling is a very popular generalisation of Importance Sampling introduced by Meng and Wong (1996). It relies on identity (8) that holds for all positive function \( \alpha(\theta) \) such that \( \int_{\Theta} \alpha(\theta)q_K(\theta)\pi_K(\theta|y)d\theta > 0 \), where \( q_K(\theta) \) is an importance density aiming at approximating the posterior distribution \( \pi_K(\theta|y) \).

\[ \int_{\Theta} \alpha(\theta)q_K(\theta)\pi_K(\theta|y)d\theta = \int_{\Theta} \alpha(\theta)q_K(\theta)\frac{p_K(y|\theta)\pi_K(\theta)}{m_K(y)}d\theta \] (8)

which gives the Bridge Sampling estimator of the marginal likelihood

\[ m_K(y) = \frac{\mathbb{E}_{q_K(\theta)}(\alpha(\theta)p_K(y|\theta)\pi_K(\theta))}{\mathbb{E}_{\pi_K(\theta|y)}(\alpha(\theta)q_K(\theta))} \]

Assuming one can obtain \( T_1 \) i.i.d draws \( \theta_1^1, \ldots, \theta_1^{T_1} \) from \( q_K(\theta) \) and \( T_2 \) (auto-correlated) MCMC samples \( \theta_2^1, \ldots, \theta_2^{T_2} \) from \( \pi_K(\theta|y) \) respectively, Meng and Wong (1996) derive the optimal choice for \( \alpha(\theta) \) as
\[ \alpha(\theta) = (N_1 q_K(\theta) + N_2^* \pi_K(\theta | y))^{-1} \]

with \( N_2^* \) the Effective Sample Size (ESS) of the MCMC sample from the posterior. As this optimal choice of \( \alpha(\theta) \) involves the unknown marginal likelihood \( m_K(y) \), the BS estimator is written recursively as

\[ \hat{m}_{K,n}^{BS}(y) = \frac{\sum_{t=1}^{T_1} \frac{p_K(y | \theta_t^1) \pi_K(\theta_t^1)}{N_1 q_K(\theta_t^1) + N_2^* \pi_K(\theta_t^1 | y)}}{\sum_{t=1}^{T_2} \frac{q_K(\theta_t^2)}{N_1 q_K(\theta_t^2) + N_2^* \pi_K(\theta_t^2 | y)}} \]

where \( m_{K,n}^{BS} \) is usually a simple IS estimator with \( q(\theta) \) as importance distribution. As \( n \) grows, one can retrieve the optimal BS estimator \( m_{K,n}^{BS}(y) = m_{K,n}^{BS}(y) \).

Fahrwirth-Schnatter (2019) provides a comprehensive review on how to successfully apply the Bridge Sampling framework to finite mixtures. As can be expected, the difficulty lies in deriving a good importance density \( q_K(\theta) \) that yields a good approximation to the notoriously complex posterior distribution of a finite mixture model. In the same vein as Berkhof et al. (2003)’s, Fahrwirth-Schnatter (2019) proposes the following choice for \( q(\theta) \):

\[ q(\theta) = \frac{1}{K! T_0} \sum_{t=1}^{T_0} \sum_{s \in \sigma(\{1, \ldots, K\})} \pi_K(\theta | s(\tilde{z}^{(t)}), y) \]

where \( T_0 \ll T_1 \) and \( \{\tilde{z}^{(t)}\}_{t=1}^{T_0} \) is drawn with replacement from the Markov chain targeting the posterior \( \{\tilde{z}^{(t)}\}_{t=1}^{T_1} \). Although \( T_0 \) is a smaller number than \( T_1 \), it is clear that this estimator suffers from the same type of computational shortcomings as Chib’s symmetrized estimator. For a given number of observations \( n \), computing estimator (9) comes at the cost of evaluating \( T_0 \times K! \times (T_1 + T_2) \) times the augmented posterior \( \pi_K(\theta | z, y) \) distribution.

Although both Chib’s and Bridge Sampling approaches are well-established and known to provide reliable estimators of the marginal likelihood of finite mixtures, their computational time grows exponentially with the number of components \( K \).

**Sequential Monte-Carlo (SMC).** Formalised by Del Moral et al. (2006), SMC is based on Sequential Importance Sampling (SIS). By sequentially sampling from a collection of distributions \( \{\pi_t\}_{t=0}^{T} \) where \( \pi_0 \) is usually the prior distribution and \( \pi_T \) is the posterior distribution of interest, the core idea of SMC is to effectively create a bridge from the prior to the posterior distribution. At each step \( t = 1, \ldots, T - 1 \) of the algorithm, weighted samples \( \{\varphi_t^{(i)}, w_t^{(i)}\}_{i=1}^{N} \) from \( \pi_t \) are reweighted through an importance sampling step so that their distribution becomes approximately \( \pi_{t+1} \).

\[ w_{t+1}^{(i)} = \frac{\pi_{t+1}(\varphi_t^{(i)})}{\pi_t(\varphi_t^{(i)})} \]

Equation (10) above stresses the importance of choosing successive distributions \( \pi_t \) and \( \pi_{t+1} \) as not being too dissimilar in order to ensure good variance properties. One common strategy is to choose the sequence of distributions \( \{\pi_t\}_{t=0}^{T} \) to be a tempered version of the posterior.
distribution. More precisely, a tempering sequence \((\lambda_t)\) is leading to the ‘tempered’ posterior distributions

\[
\pi_t(\theta) := \pi_t(\theta | y) \propto \pi_K(\theta) p_K(y | \theta)^{\lambda_t},
\]

where

\[
\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_T = 1
\]

Based on readily available estimates of the Effective Sample Size (ESS), Buchholz et al. (2021) gives a fully adaptive method to choose the tempering sequence \((\lambda_t)\). Another alternative is to use data-tempered posterior distributions in which batches of data are sequentially incorporated into the posterior. This approach is closely related to the Sequential Importance Sampling (SIS) strategy we suggest in the next sub-section. Therefore, we use the tempering strategy given in (11) so that both methods are described in this article.

A multinomial resampling step is usually implemented in order to select the particles with relatively higher weights (10). This unfortunately can cause particle degeneracy that occurs when only a few particles \(\{\theta_i^t\}\) with a large weight are selected during the resampling step. To compensate for this frequent shortcoming, a mutation step is added to the algorithm in which each particle is moved through an MCMC \(\pi_{t+1}\)-invariant kernel \(K_t\). In practice, kernel \(K_t\) is applied to each particles \(M\) number of times to ensure good mixing, where \(M\) is a user-defined number. Although SMC algorithms are known to be powerful in that they are robust to the curse of dimensionality, all these tuning steps make SMC a tedious method to implement. Following the successful application of SMC to finite mixture models of Chopin (2002), we make use of a random walk Metropolis-Hasting kernel.

As shown in Del Moral et al. (2006), an unbiased estimate of the marginal likelihood \(m_K(y)\) is given as a by-product of SMC as

\[
\hat{m}_{K}^{SMC}(y) = \prod_{t=0}^{T} \frac{1}{N} \sum_{i=1}^{N} u^{(i)}_t
\]
Algorithm 1 Tempered SMC

**Input**: Number of particles $N$, prior distribution $\pi_K$, Markov kernels $K_t$ that are $\pi_t$ invariant, where $\pi_t(\theta|y) \propto \pi_K(\theta)p_K(y|\theta)^{\lambda_t}$

**Initialisation**: $t = 0, \lambda_0 = 0$

while $\lambda_t < 1$
  
  if $t=0$ then
  - for $i = 1, \ldots, N$ do
    - Sample $\tilde{\theta}_0^{(i)} \sim \pi_K$
  
  else
    - for $i = 1, \ldots, N$ do
      - Mutation. Move particles $\tilde{\theta}_t^{(i)} \sim K_t(\tilde{\theta}_t^{(i)} - 1, d\theta)$ (Repeat $M$ times)
  
  Find the next temperature $\lambda_{t+1} > \lambda_t$ adaptively using the current state of the particles.

  **Reweighting.** for $i = 1, \ldots, N$ do
  - $w_t^{(i)} = p_K(y|\theta_t^{(i)})^{\lambda_{t+1} - \lambda_t}$

  **Resampling.** Resample with replacement from $\{(\theta_t^{(i)}, w_t^{(i)})\}_{i=1}^N$ to obtain a new sample with equal weights $\{\tilde{\theta}_t^{(i)} \sim \frac{1}{N}\}$

  $t = t + 1$

**Output**: $\hat{m}_{SMC}^K(y) = \prod_t \frac{1}{N} \sum_{i=1}^N w_t^{(i)}$

2.2 Proposed estimators

In this section, we present two novel estimators of the marginal likelihood for conjugate finite mixture models. One of them is inspired by Chib’s method and takes advantage of the partitioning of the data induced by such models. It provides efficient and robust results as well as a dramatic reduction in computational time even for $K > 5$. The second one is an application of Kong et al. (1994)’s sequential imputation algorithm, which is surprisingly not widely used as a solution to the problem of marginal likelihood estimation for finite mixtures.

**Chib’s estimator on the partition.** Partitioning comes as a natural by-product of a classical Gibbs Sampler for finite mixture models, when the additional cluster membership latent variable $z$ is introduced. Indeed, if the output of such an MCMC algorithm is $\{(\theta^{(i)}, z^{(i)})\}_{i=1}^T$, we denote by $C(z^{(i)})$ the partition on $[n] = \{1, \ldots, n\}$ induced by $z^{(i)} = (z_1^{(i)}, \ldots, z_n^{(i)})$, for all $t = 1, \ldots, T$.

For example, if $n = 4$ and $z = (1, 2, 1, 3)$, then the corresponding partition is $C(z) = \{\{y_1, y_3\}, \{y_4\}, \varnothing, \{y_2\}\}$, i.e observations $y_1$ and $y_3$ are in the same cluster whereas $y_2$ and $y_4$ are the only members of their respective cluster.

The core idea of the ChibPartition estimator that we propose in this article, is to apply the marginal likelihood identity (5) to an estimator of the MAP partition. This reads

$$p(y) = \frac{p_K(y|C^0)\pi_K(C^0)}{\pi_K(C^0|y)}, \quad C^0 \in \mathcal{P}_K([n]) \tag{13}$$
where we denote by \( P_K([n]) \) the number of partitions of \( \{1, \ldots, n\} \) with at most \( K \) parts. Note that the induced prior on the partitions \( \pi_K(C) \) is available in closed form provided that a Dirichlet prior is used for the weights \( \varpi \). It can indeed be written as

\[
\pi_K(C(z)) = \frac{K!}{(K-K^+)!} \pi_K(z) = \frac{K!}{(K-K^+)!} \int \pi_K(z|\pi)d\mathcal{D}(\pi|\alpha) = \frac{K!}{(K-K^+)!} \frac{\Gamma\left(\sum_{j=1}^{K} \alpha_j\right)}{\Gamma\left(\alpha_j\right)} \prod_{j=1}^{K} \frac{\Gamma(n_j + \alpha_j)}{\Gamma(n_j)} \tag{14}
\]

where \( n_j = \sum_{i=1}^{n} 1\{z_i = j\}, \) i.e. the number of observations assigned to cluster \( j \) for all \( j = 1, \ldots, K \), and \( K^+ = \sum_{j=1}^{K} 1\{n_j > 0\} \), which is the number of non-empty clusters implied by a given allocation vector \( z \).

Now for a conjugate prior \( G_0 \) on \( \theta \), the likelihood of a partition is available in closed form and given by

\[
p(y|C(z)) = p(y|z) = \prod_{j=1}^{K} \int_{\Theta} \prod_{i:z_i=k} p(y_i|\theta)G_0(d\theta) := \prod_{j=1}^{K} m(C_k(z)) \tag{15}
\]

with the convention that \( m(\varnothing) = 1 \).

The posterior density of a partition \( C \) is unfortunately not available in closed form. However, we can estimate it with the following simple Monte Carlo estimator readily computable from the Gibbs sampler output.

\[
\hat{\pi}_K(C^0|y) = \frac{1}{T} \sum_{t=1}^{T} 1\{C^0 = C(z^{(t)})\} \tag{16}
\]

for some sample \( \{z^{(t)}\} \) distributed according to the posterior distribution and where we use \( \hat{=} \) to denote an equivalence relation between two partitions. For example,

\[
\{\{y_1, y_3\}, \{y_2\}, \{y_4\}\} \hat{=} \{\{y_2\}, \{y_4\}, \{y_1, y_3\}\}
\]
as both partitions imply the same partitioning of the data, up to a permutation of the clusters’ index. From a computational point of view, note that comparing two partitions with the equivalence relation \( \hat{=} \) is a simple \( O(n \log n) \) operation. Below is given the pseudocode of the ChibPartition estimator.

Possible shortcomings of Algorithm 2 stem from the cardinality of the set \( \mathcal{P}_K([n]) \) potentially making the estimation of the posterior density difficult. Note that there exists no simple expression giving the exact value of \( |\mathcal{P}_K([n])| \) for all \( k \) and \( n \) but one can write

\[
|\mathcal{P}_K([n])| = \sum_{k=1}^{K} S(n, k) := \sum_{k=1}^{K} \frac{1}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} (k - l)^n \tag{17}
\]

where \( S(n, k) \) are the Stirling numbers of the second kind, counting the number of ways to
Algorithm 2 ChibPartition estimator

Input: \((z^{(t)})_{t=1}^T\) from an MCMC at stationarity targeting \(\pi_K(z|y)\)

for \(t = 1, \ldots, T\) do

Compute \(\tilde{\pi}(C(z^{(t)}))|y) = p_K(y|C(z^{(t)}))\pi_K(C(z^{(t)}))\) using (15) and (14)

\[ C^0 = \operatorname{argmax}_{t=1,\ldots,T} \{\tilde{\pi}(C(z^{(t)}))|y)\} \]

Set \(\hat{\pi}(C^0|y) = 0\)

for \(t = 1, \ldots, T\) do

Compute \(\hat{\pi}(C^0|y) = \hat{\pi}(C^0|y) + 1_{\{C^0 = C(z^{(t)})\}}/T\)

Output: \(m_K(y) = p_K(y|C^0)\pi_K(C^0)/\hat{\pi}(C^0|y)\)

partition a set of \(n\) distinguishable objects into \(k\) nonempty subsets (see Graham et al. (1989) for example). The number of possible partitions given by (17) is potentially very large. For instance, when using a finite mixture with \(K = 8\) components on \(n = 82\) data points, \(|\mathcal{P}_K([n])| \approx 2.80 \times 10^{69}\). It is therefore crucial to choose \(C^0\) to be the estimated MAP or any other high-posterior probability partition to retrieve good variance for estimator (16). As simulations in Section 2.3 tend to show, such a strategy appears to be sufficient to ensure a robust estimator of the posterior density.

Compared to the other corrections of Chib (1995)’s original method that have been proposed in the past such as Berkhof et al. (2003)’s Chib Permutation Estimator (‘ChibPermut’ hereafter) or Marin and Robert (2008)’s Chib Random Permutation Estimator (‘ChibRandPermut’ hereafter), our proposed solution has a considerably lower computational time. Note that those three methods only differ in the way they estimate the posterior density function at a given point in the parameters’ space. Therefore it is enough to compare the computational cost of this estimation as is given by Table 1.

| Complexity of posterior estimation |
|-----------------------------------|
| ChibPermut                        | \(\mathcal{O}(TK!)\)         |
| ChibRandPermut                    | \(\mathcal{O}(TR)\)          |
| ChibPartition                     | \(\mathcal{O}(T)\)           |

Table 1: Computational complexity of estimating the posterior density function for three corrections of Chib (1995)’s original marginal likelihood estimator: \(T\) is the length of the Markov Chain targeting the posterior distribution, \(K\) is the number of mixture components, and \(R\) is the number of random permutations chosen for the ChibRandPermut estimator.

Thanks to the ergodicity of the Gibbs output and using the auto-correlation consistent variance estimator given by Newey and West (1986), an estimate of the variance of (16) is given by

\[
\hat{V}(\tilde{\pi}_K(C^0|y)) = \frac{1}{T} \left( \sigma_0 + 2 \sum_{s=1}^{q} \left( 1 - \frac{s}{q+1} \right) \sigma_s \right)
\]
where
\[
\sigma_s = \frac{1}{T} \sum_{t=s+1}^{T} \left( 1_{\{C^0 \neq C(z^{(t)})\}} - \hat{\pi}_K(C^0|y) \right)^2
\]
for \( q \) large enough.

The Delta method immediately yields the following estimator for the variance of the marginal likelihood ChibPartition estimator.
\[
\hat{V}(\log \hat{m}_K(y)) = \hat{V}(\log \hat{\pi}_K(C^0|y)) = (\hat{\pi}_K(C^0|y))^2 \hat{V}(\hat{\pi}_K(C^0|y)).
\]

### Sequential Importance Sampling (SIS).
Kong et al. (1994) addresses the issue of missing data problems by sequential imputation, using a latent variable \( z \), representing the missing part of the data. The authors show that for a particular choice of importance distribution \( \pi^\ast \), a Sequential Importance Sampling (SIS) procedure yields a direct estimator of the marginal likelihood \( m(y) \). This is applicable whenever one can sample easily from distributions \( p(z_i|y_{1:i}, z_{1:i-1}) \) for all \( i \geq 2 \) where \( z_{1:i} = (z_1, \ldots, z_i) \) and whenever the prequential predictive densities \( p(y_i|y_{1:i-1}, z_{1:i-1}) \) are available in closed form for all \( i \geq 2 \). Indeed, let us define \( \pi^\ast \) as an approximation to the posterior distribution in which the latent variable is imputed sequentially,
\[
\pi^\ast(z_1, \ldots, z_n|y) = p(z_1|y_1) \prod_{i=2}^{n} p(z_i|y_{1:i}, z_{1:i-1})
\]
Then,
\[
\frac{\pi(z_1, \ldots, z_n|y)}{\pi^\ast(z_1, \ldots, z_n|y)} = \frac{p(z, y)}{p(y)} \frac{p(y_1)}{p(z_1, y_1)} \frac{p(z_1, y_1, y_2)}{p(z_1, y_1, z_2, y_2)} \cdots \frac{p(z_{1:n-1}, y)}{p(z, y)} = \frac{w(z, y)}{p(y)}
\]
where \( w(z, y) := p(y_1) \prod_{i=2}^{n} p(y_i|z_{1:i-1}, y_{1:i-1}) \).

Hence, if \( \{z^{(t)}\}_{t=1}^{T} \) are drawn sequentially from \( \pi^\ast \), the following quantity is an unbiased estimator of the marginal likelihood of the data \( p(y) \)
\[
m_{K}^{SIS}(y) = \frac{1}{T} \sum_{i=1}^{T} w(z^{(t)}, y)
\]
(18)

The link to the latent cluster membership variable \( z \) defined in (3) is direct and it is straightforward to derive the necessary quantities to apply the idea of Kong et al. (1994) to mixture models, as also noticed by Carvalho et al. (2010). Despite the well-known efficiency of such samplers, note that SIS, and, more generally, SMC approaches, are not as popular as Chib’s algorithm or Bridge sampling, the latter being the favoured method of the STAN community. However, for conjugate models, sampling from \( \pi^\ast \) and evaluating \( w(z, y) \) is simple (see details in Appendix A).

As remarked by Irwin et al. (1994), a simple application of the Delta method yields the
following estimator for the standard deviation of $\hat{m}^{SIS}_K(y)$.

$$\hat{sd}(\log \hat{m}^{SIS}_K(y)) = \frac{1}{\sqrt{M}} \hat{sd}(w)$$

where $\hat{sd}(w)$ denotes the sample standard deviation of $\{w(z^{(t)}, y)\}_t$.

**Algorithm 3 SIS**

**Input**: Number of iterations $T$

for $t=1,\ldots,T$ do

Sample $z_1^{(t)} \sim Cat(\alpha_1/\sum_{j=1}^K \alpha_j, \ldots, \alpha_K/\sum_{j=1}^K \alpha_j)$

Compute $p(y_1) = m(\{y_1\})$

Set $w^{(t)} \leftarrow p(y_1)$

for $i=2,\ldots,n$ do

for $k=1,\ldots,K$ do

Compute $\gamma_k = n_k(z_1^{(t)} \cup \{y_i\}) + \alpha_k / \sum_{j=1}^K n_j(z_1^{(t)} \cup \{y_i\}) + \alpha_j$

Sample $z_i^{(t)} \sim Cat(\gamma_1/\sum_{k=1}^K \gamma_k, \ldots, \gamma_K/\sum_{k=1}^K \gamma_k)$

Set $w^{(t)} \leftarrow w^{(t)} \times \gamma_k / \sum_{k=1}^K \gamma_k$

Return $1/T \sum_{t=1}^T w^{(t)}$

### 2.3 Simulation study

#### 2.3.1 Experiment 1: Galaxy data.

The first experiment we design aims at assessing the relative performance of our suggested ChibPartition and SIS algorithms in a basic setting. As is usually done in the mixture modelling community, we use the benchmark galaxies data set that contains the radial velocity of 82 galaxies, was introduced by Postman et al. [1986]. A long series of articles addressing the issue of evidence computation in mixture models usually evaluate their results on this data (see e.g. Chib [1995], Richardson and Green [1997], Lee and Robert [2016], Frühwirth-Schnatter [2019]).

A conjugate mixture of normal distributions is implemented. We use the conditionally conjugate normal-inverse gamma prior for the location and scale parameters $\theta = (\mu, \sigma^2)$ as described in Frühwirth-Schnatter [2006a] that yields for all $k = 1,\ldots,K$

$$\sigma_k^2 \sim \Gamma^{-1}(a_0, b_0)$$
$$\mu_k | \sigma_k^2 \sim \mathcal{N}(\mu_0, \sigma_k^2 / \lambda_0)$$

where $\Gamma^{-1}$ is the inverse gamma distribution in the shape and scale parametrization. The hyperparameters $(a_0, b_0, \mu_0, \lambda_0)$ are derived empirically following recommendation from Raftery [1996]: $a_0 = 1.28, b_0 = 0.36(\bar{y}^2 - \bar{y}^2), \mu_0 = \bar{y}, 1/\lambda_0 = (y_{max} - y_{min})/2.6$. Note that this choice of prior ensures that $\pi_K(\theta | y, z) = p(\mu, \sigma_k^2 | y, z)$ is available in closed form, which is a prerequisite to most of the algorithms we wish to implement.
Table 2: Average time for $K = 5$ required to get one repetition with a precision similar to the boxplot of Figure 1. ChibPartition is computed with $T = 95000$ iterations after a burn-in of 5000. SIS is iterated 6000 times (i.e. $T = 6000$). Adaptive SMC has 10000 particles and the mutation MCMC is iterated 10 times. For Bridge Sampling, $T_1 = 1000$ after a burn-in period of 1000, $T_2 = 1000$ and $T_0 = 100$

| Method       | Time in seconds |
|--------------|-----------------|
| ChibPartition| 425.22          |
| SIS          | 472.87          |
| SMC          | 7539.92         |
| BS           | 57983.62        |

Figure 1 shows boxplots of the estimates of the marginal likelihood given by the different methods described in the previous section for an increasing number of mixture components $K$. In each scenario, the simple arithmetic mean estimator computed with a very large number of simulations is included. It is defined as

$$\hat{m}^{AM}_K(y) = \frac{1}{T} \sum_{t=1}^{T} p_k(y|\vartheta^{(t)})$$  \hspace{1cm} (20)

given a sample $\{\vartheta\}_{t=1}^T$ from the prior $\pi_K(\vartheta)$. This prohibitively time-consuming estimator is here solely for benchmark purposes and used as a reference for the other methods. The harmonic mean estimator is also included in this comparative study. It is defined as

$$\hat{m}^{HM}_K(y) = \left( \frac{1}{T} \sum_{t=1}^{T} 1/p_k(y|\vartheta^{(t)}) \right)^{-1}$$ \hspace{1cm} (21)

given a sample $\{\vartheta\}_{t=1}^T$ from the posterior $\pi_K(\vartheta|y)$.

Except for the arithmetic mean, all other algorithms are allocated as much time as required for them to converge, provided this time is reasonable. For instance, Bridge Sampling and the fully-permuted Chib’s estimator (ChibPerm) are not included for $K = 6$ and $K = 8$ as they fail to converge in a time comparable to the other methods. Hence, Figure 1 only provides insights about which of the methods are able to provide reliable estimates for an increasing number of mixture components $K$.

For $K = 3$, although most estimates agree on a common value for $\hat{m}(x)$, Chib’s method is almost exactly off by a factor log 3! $\approx 1.79$, which is the consequence of an almost complete lack of label switching during the Gibbs sampling step, as discussed earlier. The sum over all permutations produced by the fully-permuted Chib’s estimator (ChibPerm) makes up for this bias. We see that as $K$ grows, all the classical methods except for adaptive SMC fail to estimate the marginal likelihood while our candidates ChibPartition and SIS are consistently pointing to the reference value given by the arithmetic mean estimator. For $K = 5$, Table 2 gives the time taken by each of the four successful algorithms (ChibPartition, adaptive SMC, SIS, and bridge sampling) to yield one estimate of the marginal likelihood. The time given corresponds to the average computational time over 20 repetitions of each estimator on one core Intel(R) Xeon(R) CPU E5-2630 v4 @ 2.20GHz. As is expected theoretically, one can observe the great decrease
Figure 1: Boxplot of the marginal likelihood estimators 20 repetitions each. Dashed line: mean of the Arithmetic Mean estimator. The tuning parameters are given in Table E.1 in Appendix.
in computational time offered by ChibPartition and SIS with respect to Bridge Sampling and the fully permuted Chib’s estimator. Although it is not done in this experiment, note that it is straightforward to parallelize the embarrassingly parallel SIS algorithm and thus to further reduce its computational time. Figure 3 shows the evolution of the Mean Squared Error (MSE) as a function of time for the 5 component-mixture model on the Galaxy data. Note that methods SIS and SMC are not implemented in their parallelized version. Despite this, SIS clearly outperforms the other algorithms, closely followed by ChibPartition.

Figure 2 gives boxplots of the marginal likelihoods estimators given by SIS for different values of $K$ and indicates that a mixture of 5 components is best supported by the galaxy data, for our choice of prior.

2.3.2 Experiment 2 : Synthetic data, $n = 1000$ and $n = 2000$.

For more realistic applications, our goal is to find which algorithms scale well as both $K$ and $n$ get large. To our knowledge, no earlier work has been conducted towards identifying reliable estimators for this kind of challenging scenarios.

We generate two data sets of respectively 1000 and 2000 observations from a six-component mixture of Gaussians. Given the results of the previous experiments in which all but three methods failed to converge in a reasonable time, we only consider ChibPartition, SIS, SMC and Bridge sampling for this more difficult scenario. As in the previous experiment, the conditionally conjugate prior (19) is chosen. The tuning parameters used in Figure 4 and given in Table 4 are chosen so that the running times are comparable.

For the most simple scenario ($n = 1000$, $K = 3$), ChibPartition is showing a very large variance, probably due to the high cardinality of the set of partitions $\mathcal{P}_3([1000])$ compared to the number of iterations of the Gibbs sampler. Adaptive SMC is showing good results until $n = 2000$ and $K = 13$ where its variance becomes very large. On the other hand, SIS is consistently the
Figure 3: Galaxy data. MSE vs time with regression lines.
best estimator in terms of variance, whereas adaptive SMC breaks down in the most demanding experimental setting.

The good performance of Bridge sampling should be noted when \( n \) is large and \( K = 3 \), although it cannot converge in a decent time when \( K = 13 \) and is hence not displayed in this more difficult scenario.

3 Evidence approximation for Dirichlet Process Mixtures

The Dirichlet Process Mixture model arises naturally as the limit when \( K \to \infty \) of the finite mixture model with a Dirichlet prior on the weights \( \mathcal{D}(M/K, \ldots, M/K) \) for some \( M > 0 \). The limiting distribution is called the Dirichlet Process Mixture model (DPM) and is characterized by
\[ y_i | z_i, \theta \overset{i.i.d.}{\sim} f(y_i | \theta_{z_i}), \ i = 1, \ldots, n \]  
\[ p(z_i = k) = \varpi_k, \ k = 1, 2, \ldots \]  
\[ \varpi_1, \varpi_2, \ldots \sim \text{GEM}(M) \]  
\[ M \sim \pi(M) \]  
\[ \theta_1, \theta_2, \ldots \overset{i.i.d.}{\sim} G_0 \]  

where \( \text{GEM} \) denotes the stick-breaking representation for the distribution of the weights \( (\varpi_k)_k \) as described by Sethuraman (1994) and given by

\[ \varpi_k = v_k \prod_{i=1}^{k-1} (1 - v_i) \]

for \( \{v_i\}_i \) an infinite sequence following the \( \text{Beta}(1, M) \) distribution ensuring that \( \sum_{i=1}^\infty \varpi_k = 1 \) a.s.

Using the above notation, if \( P \) is a realisation of the Dirichlet Process with concentration parameter \( M \) and base measure \( G_0 \), denoted \( P \sim DP(M, G_0) \), then

\[ P = \sum_{i=1}^\infty \varpi_i \delta_{\theta_i} \]

where \( \varpi_1, \varpi_2, \ldots \) follow the stick-breaking distribution \( \text{GEM} \) and \( \theta_1, \theta_2, \ldots \) are i.i.d realisations from \( G_0 \). Hence, the density of a DPM can be written as the infinite mixture

\[ f_P(y) = p(y|P) = \prod_{i=1}^n \int f(y_i | \theta) dP(\theta) = \prod_{i=1}^n \sum_{i=1}^\infty \varpi_i f(y_i | \theta_i) \]  

One can derive a useful representation of the DPM model in terms of cluster allocations in the case where the mixture kernel \( f \) and the base measure of the Dirichlet Process \( G_0 \) are conjugate. Using the Pólya Urn scheme (Blackwell and MacQueen (1973)) one can write

\[ p(z_i = k | z_1, \ldots, z_{i-1}, M) = \frac{M}{M + i} \cdot \frac{\delta_{k = \max_{1 \leq j \leq i-1} (z_j)} + 1}{M + i - 1} \sum_{j=1}^{i-1} \delta_{k = z_j} \]  

and thanks to the exchangeability of the sequence \( z = (z_1, \ldots, z_n) \) one can derive the induced joint prior distribution on the cluster allocations

\[ \pi(z | M) = \frac{\Gamma(M)}{\Gamma(M + n)} M^{K_+} \prod_{j=1}^{K_+} \Gamma(n_j) \]

where \( K_+ \) is the number of distinct values among \( (z_1, \ldots, z_n) \) and for all \( j = 1, \ldots, K_+ \), \( n_j \) is the number of \( z_i \)'s equal to \( j \). This representation in terms of cluster allocations also allows for
a convenient form of \( p(y|z_1, \ldots, z_n, M) \) as
\[
p(y|z, M) = \prod_{j=1}^{K_+} \int \prod_{i: z_i = j} f(y_i|\theta_j) dG_0(\theta_j)
\] (26)

Obviously this expression of the likelihood function is only useful when an analytic form for (26) exists, which is the case when \( G_0 \) and \( f \) are conjugate, which we shall assume hereafter. Then one can derive the posterior distribution of \((z_1, \ldots, z_n, M)\) up to a constant as
\[
\pi(z, M|y) \propto p(y|z, M) \pi(z|M) \pi(M)
\] (27)

Neal (2000) gives an extensive review of Gibbs sampling strategies targeting the posterior of a Dirichlet Process mixture (27).

While literature is abundant on how to derive efficient estimators of the marginal likelihood of a parametric model, as we discussed in the previous section, equivalent methods for non-parametric models such as the DPM model are far from numerous. To our knowledge, only Basu and Chib (2003) address the issue of evidence computation for the DPM in the conjugate case by extending the well-established method of Chib (1995). It is to be noted that it is difficult to find examples of application in literature of this non-parametric version of Chib’s algorithm. There exists an SMC framework summarized by Griffin (2017) that is applicable if \( M \) is assumed to be known. This is not our approach in this article since, as illustrated on Figure 3 of Tokdar and Martin (2021), this parameter can have a decisive influence on the marginal likelihood and thus on the Bayes Factor.

In this section we aim at assessing the method proposed in Basu and Chib (2003) and deriving alternative ways to compute the marginal likelihood of a DPM.

3.1 Existing algorithm

Chib’s algorithm. Basu and Chib (2003) adapt Chib’s algorithm from Chib (1995) to the conjugate Dirichlet Process model using Bayes identity
\[
\hat{\pi}(M^*|y) = \frac{L(y|M^*, G_0) \pi(M^*)}{\hat{\pi}(M^*|y)}
\] (28)

where \( M^* \) is some point in \((0, \infty)\) and \( L(y|M^*, G_0) \) is the integrated likelihood with respect to the Dirichlet Process \( DP(M, G_0) \),
\[
L(y|M, G_0) = \int f_p(y) dDP(P|M, G_0)
\] (29)

Just like for its finite counterpart, Chib’s algorithm for the DPM uses a Rao-blackwellised estimator of the posterior density with the introduction of a latent variable \( \eta \) in the Gibbs sampler, as suggested by Escobar and West (1995),
\[
\hat{\pi}(M^*|y) = \sum_{t=1}^{T} \pi(M^*|y, \eta^{(t)}, K_+^{(t)})
\] (30)

19
where $K_+$ is the number of non-empty clusters implied by the allocation vector $z$. The full conditional distribution on $M$ is available in closed form provided $M$ follows a Gamma distribution $\Gamma(a, b)$ a priori and is given by the mixture

$$M|y, \eta, K_+ \sim \omega \Gamma(a + K_+, b - \log(\eta)) + (1 - \omega)\Gamma(a + K_+ - 1, b - \log(\eta))$$

where $\omega = \frac{(a + K_+ - 1)}{\{n(b - \log(\eta)) + a + K_+ - 1\}}$ and $\eta|y, M \sim \Beta(M + 1, n)$.

Unlike the finite mixture model, the likelihood ordinate (29) is defined by an intractable integral which must be estimated. Following Kong et al. (1994), the authors propose a Sequential Importance Sampling (SIS) scheme using the following importance distribution where for any vector $x \in \mathbb{R}^n$ we define $x_1: i := (x_1, \ldots, x_i)$, for all $i = 1, \ldots, n$.

$$\pi^*(z_1, \ldots, z_n|M) = \prod_{i=1}^n \pi(z_i|y_1:i, z_1:i-1, M) \quad (31)$$

which is available in closed form in the conjugate case. It can be easily derived that the final importance weight can be expressed as

$$w_n(z_1, \ldots, z_n) = \frac{\pi(z_1, \ldots, z_n|M)}{\pi^*(z_1, \ldots, z_n|M)} = \frac{p(y_1|z_1, G_0) \prod_{i=2}^n p(y_i|y_1:i-1, z_{1:i}, G_0)}{L(y|M^*, G_0)}$$

Hence, given a sequence $\{z^{(t)}\}_{t=1}^T$ generated sequentially from (31), one can derive as a by-product of SIS

$$\hat{L}(y|M^*, G_0) = \frac{1}{T} \sum_{t=1}^T p(y_1|z_1^{(t)}, G_0) \prod_{i=2}^n p(y_i|y_1:i-1, z_{1:i-1}^{(t)}, G_0) \quad (32)$$

Algorithm 4 gives the details of implementation of this variant of Chib’s algorithm.

### 3.2 Proposed algorithm

The adapted version of Chib’s algorithm proposed by Basu and Chib (2003) is very efficient provided a sensible choice is made for the value of $M^*$ in equation (30). However, it is difficult to identify a candidate value with high posterior probability since $p(y|M)$ is difficult to compute for all $M$. We here propose another approach related to Bridge Sampling that we believe to be more stable.

**Reverse Logistic Regression.** Introduced by Geyer (1994), Reverse Logistic Regression (RLR) is a biased IS-related method. It requires $T_1$ and $T_2$ i.i.d observations from two distributions $\pi_1 = \tilde{\pi}_1/c_1$, the adversarial distribution for which the normalizing constant $c_1$ is known, and $\pi_2 = \tilde{\pi}_2/c_2$, the distribution of interest, only known up to a normalizing constant $c_2$. The main idea of Reverse Logistic Regression is to ignore from which distribution each observation stems (i.e. ‘forgetting’ the labels). By doing so, observations can be assumed to arise from a mixture model which mixture weights are proportional to the normalizing constants $c_1$ and $c_2$. The idea of Geyer (1994) is then to perform some ‘reverse’ logistic regression, in the sense that the response variable is not random while the predictors are, in order to retrieve an estimator.
Assume that \( \pi_2 \) is the posterior distribution. Note that simpler choices for \( \pi_1 \) and \( \pi_2 \) are respectively i.i.d samples from \( \pi(z, M) \) for instance. We here apply this idea to the DPM by considering as an importance distribution \( \pi_1(z, M) := \pi^*(z|y, M) \pi(M) \), where \( \pi^* \) is defined as in (31), and \( \pi_2(z, M) = \pi_2(z, M)/c_2 := \pi(z, M|y)/m(y) \) is the posterior distribution. Note that simpler choices for \( \pi_1 \) exists, in particular for the non-conjugate case, such as the prior distribution \( \pi(z, M) \) for instance.

We can therefore estimate the marginal likelihood \( c_2 := \hat{m}(y) \) can be estimated by finding the maximum of the log quasi-likelihood

\[
\ell(c_2) = \sum_{i=1}^{T_1} \log \gamma_1(z^{(1,i)}, M^{(1,i)}) + \sum_{j=1}^{T_2} \log \gamma_2(z^{(2,j)}, M^{(2,j)})
\]

(33)

where

\[
\gamma_1(z, M) = \frac{T_1}{T_1 + T_2} \pi_1(z, M) + \frac{T_2}{T_1 + T_2} \pi_2(z, M)/c_2
\]

and

\[
\gamma_2(z, M) = 1 - \gamma_1(z, M)
\]
Chen and Shao (1997) show that Reverse Logistic Regression is essentially equivalent to optimal Bridge sampling. Therefore, standard results about Bridge Sampling as given in Frühwirth-Schnatter (2004) can be applied to derive the standard error of the simulated marginal likelihood \( \hat{m}^{RLR}_{DP}(y) \).

\section*{Algorithm 5 RLR algorithm for the DPM - SIS instrumental distribution}

\textbf{Input} : \( \{z^{(1,t)}\}_{t=1}^{T_1}, \{M^{(1,t)}\}_{t=1}^{T_1} \sim \pi^*(z|M)y, M \) and \( \{z^{(2,t)}\}_{t=1}^{T_2}, \{M^{(2,t)}\}_{t=1}^{T_2} \) from a Markov Chain at stationarity targeting \( \pi_K(z,M|y) \) for \( t = 1, \ldots, T_1 \) do

Compute \( \pi_1(z^{(1,t)}, M^{(1,t)}) = \pi^*(z^{(1,t)}|M^{(1,t)}, y)\pi(M^{(1,t)}) \) (this step can be avoided if done alongside the simulation of \( \{z^{(1,t)}\}_{t=1}^{T_1}, \{M^{(1,t)}\}_{t=1}^{T_1} \) )

Compute \( \tilde{\pi}_2(z^{(1,t)}, M^{(1,t)}) = p(y|z^{(1,t)}, M^{(1,t)})\pi(z^{(1,t)}|M^{(1,t)})\pi(M^{(1,t)}) \) as in (27)

for \( t = 1, \ldots, T_2 \) do

Compute \( \pi_1(z^{(2,t)}, M^{(2,t)}) = \pi^*(z^{(2,t)}|M^{(2,t)}, y)\pi(M^{(2,t)}) \)

Compute \( \tilde{\pi}_2(z^{(2,t)}, M^{(2,t)}) = p(y|z^{(2,t)}, M^{(2,t)})\pi(z^{(2,t)}|M^{(2,t)})\pi(M^{(2,t)}) \) as in (27)

Optimize function \( \ell(c_2) \) defined in (33)

\textbf{Output} : \( \hat{m}^{RLR}_{DP}(y) = \arg \max \ell(c_2) \)

\subsection*{3.3 Simulation study}

\subsubsection*{3.3.1 Experiment 3 : Galaxy data.}

In this section, we evaluate the relative performances of Basu and Chib (2003)’s algorithm with other competing alternatives which, to our knowledge, has never been done. To do so, we once again start by using the galaxy data and consider a Dirichlet Process mixture of normal distributions with unknown location and scale parameters and use the conditionally conjugate Normal-Inverse Gamma prior \( G_0 \) as defined in (19).

Note that, while, for finite mixtures, model complexity is a function of the number of mixture components \( K \), most difficulties arise in the DPM for growing numbers of observations \( n \). This is partly due to the fact that \( K = O(M \log(n)) \). Hence we design three different experiments with respectively 6, 36 and 82 observations from the Galaxy data as shown on Figure 5. One can note in particular that simple estimators such as the Arithmetic Mean and Harmonic Mean estimators, which show rather good results for a small data size, fail to converge as soon as the amount of data becomes moderately large. Since we are not able to make the Arithmetic Mean estimator converge (i.e to make its variance low) as in the finite mixture experiments of Section 2.3, we use Reverse Logistic Regression where the prior is used as the adversarial distribution (hereafter RLR-Prior) as a reference value. This method is indeed independent of both Basu and Chib (2003)’s algorithm and RLR with a SIS adversarial distribution since it does not depend on the sequential imputation scheme (31). It is interesting to note that RLR-Prior yields rather satisfactory results given the simplicity of the adversarial distribution, compared with
the Arithmetic Mean estimator that also relies on prior samples and suffers from pathological variance. For non-conjugate Dirichlet Process Mixture models, this rather inexpensive estimator could be a good, easy-to-implement first approach to the marginal likelihood estimation problem.

Figure 6 shows the decrease of the MSE as a function of time and shows that for a given allocated time, the error produced by Chib’s estimator is greater than that of RLR-SIS by about a factor $\exp(0.3) \approx 1.35$.

### 3.3.2 Experiment 4: Synthetic data, $n = 1000$.

We now assess the scalability of both methods on a synthetic data set of 1000 observations, arising from a 6-component mixture of normal distributions. Although we cannot compute analytically the true value of the marginal likelihood, we do know that both Chib’s algorithm and RLR-SIS are consistent, as illustrated in the previous section. That is, provided we do not observe a large simulation-related variance, the estimator can be trusted. The tuning parameters used for the experiment, as well as the run time are given in the caption of Figure 7. The run time corresponds to the total time required to compute 20 repetitions of each estimator on 10 cores, each core being an Intel(R) Xeon(R) CPU E5-2630 v4 @ 2.20GHz. Note that no parallelization of the embarrassingly parallel SIS step was done (which would decrease the total run time of both algorithms dramatically, but by the same factor).

On Figure 7, it is clear that Chib’s estimator suffers from a large variance, which in turn translates into a downward bias on the log-scale. On the other hand, RLR-SIS exhibits much more stability. Note that although more iterations and computational time are allocated to Chib’s estimator, this does not seem to be enough to correct Chib’s pathological variance while RLR-SIS yields a more accurate estimate of the marginal likelihood within a much lower run time.

### 3.3.3 Experiment 5: Testing a finite mixture against a DPM.

In this experiment, we test whether the Bayes factor converges to infinity under the null hypothesis that data arises from a finite mixture with $K_0$ components, when fitting a DPM. That is, we want to check whether

$$ BF := \frac{m_{K_0}(y)}{m_{DPM}(y)} \overset{n \to \infty}{\longrightarrow} \infty $$

To do so, 100 data sets $y$ are generated from $P_{K_0}$, where $P_{K_0}$ is a finite mixture of $K_0$ normal components. The Bayes Factor is then computed for successive values of $n$ and the resulting ‘Bayes Factor paths’ are displayed on Figure 8. On the left hand side, $K_0 = 1$ and the datasets simply arise from a normal distribution with mean $\mu = 0$ and scale $\sigma = 2$. On the right hand side, $K_0 = 3$ with mean parameters $\mu = (-3, 4, 12)$, scales $\sigma = (2, 2, 2)$ and weights $\varpi = (0.3, 0.2, 0.5)$. Empirically, we can see that the Bayes Factor appears to converge to infinity, in both scenarios as it seems that $P_{K_0}(BF > 1) \longrightarrow 1$.

In the next section we formalize this empirical study by proving the consistence of the Bayes Factor for a particular class of ‘strongly identifiable’ Dirichlet Process Mixture models under a finite mixture null hypothesis.
Figure 5: Galaxy data. Boxplots of the different methods (20 repetitions each) for different values of \( n \). The dashed line corresponds to the mean of the RLR-prior estimator. The choice of tuning parameters is given in Table 5 in Appendix.
Figure 6: Galaxy data. MSE vs time with regression lines.

Figure 7: Boxplots of the marginal likelihoods estimates, 20 repetitions each. Dirichlet process mixture model, synthetic data, \( n = 1000 \). Chib left hand side: Chib : \( T_1 = 10^4 \), \( burnIn = 10^3 \), \( T_2 = 600 \), Run time : 04:09:51. Chib right hand side \( T_1 = 10^5 \), \( burnIn = 10^4 \), \( T_2 = 2000 \), Run time : 34:01:18. RLR-SIS : \( T_1 = 10^4 \), \( burnIn = 10^3 \), \( T_2 = 600 \), Run time : 06:23:12.
Figure 8: log-Bayes Factor paths for 100 datasets as \( n \) increases. Left-hand side: normal distribution. Right-hand side: 3-component mixture of normal distributions. The dashed line is the line \( \log(BF) = 0 \).

4 Asymptotic behaviour of the marginal likelihood associated to the Dirichlet process mixture

In this section we study the asymptotic behaviour of the marginal likelihood \( m_{DP}(y) = \int f_{P}(y) d\pi(P) \) when \( \pi \) is a \( DP(M,G_0) \) and \( y = (y_1, \cdots, y_n) \) with \( y_i \sim f_0 \), a probability density on \( \mathbb{R}^d \).

Understanding the behaviour of \( m_{DP}(y) \) corresponds to determining asymptotic lower and upper bounds for \( m_{DP}(y) \). Determining lower bounds on marginal densities is typically done using the technique of Ghosal et al. (2000), Lemma 8.1 for instance, where it is used to derive posterior concentration rates under a Dirichlet Process mixture model. There is now a large literature on posterior contraction rates in Dirichlet process mixture models, see for instance Ghosal and Van Der Vaart (2007), Kruijer et al. (2010), Shen et al. (2013), and Scricciolo (2014) in which a lower bound on \( m_{DP}(y) \) is derived.

The difficult part when assessing evidence in this setting stands in obtaining an upper bound on \( m_{DP}(y) \), since it requires a refined understanding on neighbourhoods of \( f_0 \). In the following section we concentrate on deriving such an upper bound when \( f_0 \in \bigcup_{K \in \mathbb{N}} \mathcal{M}_K \), where \( \mathcal{M}_K \) denotes the model of mixtures with \( K \) densities \( f_\theta(y) \). Indeed an important application of such an upper bound is in the goodness of fit test (or test for the number of components)

\[
H_0 : f_0 \in \mathcal{M}_K, \quad \text{versus} \quad H_1 : f_0 \notin \mathcal{M}_K,
\]

to prove that the Bayes factor is consistent under the null:

\[
\frac{m_K(y)}{m_{DP}(y)} \to \infty, \quad \text{in probability under } f_0
\]

Obtaining such an upper bound is of interest even outside the context of testing, since it is a way to understand the behaviour of credible regions in infinite dimensional models (see, e.g., Rousseau and Szabo (2020)), together with proving a lower bound on posterior contraction rates.
In this section we assume that $P_0 = \sum_{j=1}^{k_0} \pi_j^0 \delta_{\theta_j^0}$ where $\pi_j^0 > 0$ for all $j$ and $\theta_j^0 \neq \theta_i^0$ for all $i \neq j$.

We also consider the following regularity assumptions on the distribution $f(y|\theta)$.

**Assumption A1** [Regularity] For all $y \in \mathcal{Y} \subset \mathbb{R}^d$, the function $\theta \rightarrow f(y|\theta)$ is twice continuously differentiable and there exist $H_1 \in L^2(\mathbb{R}^d)$, $H_2 \in L^1(\mathbb{R}^d)$, $H_3 \in L^1(\mathbb{R}^d)$, and $\delta_0, \delta > 0$ such that for all $j = 1, \ldots, k_0$,

$$
\sup_{\theta \in \Theta} \|\nabla f(y|\theta)\| \leq H_1(y), \quad \sup_{\|\theta - \theta_j^0\| \leq \delta_0} \|D^2 f(y|\theta)\| \leq H_2(y),
$$

$$
|D^2 f(y|\theta) - D^2 f(y|\theta_j^0)| \leq H_3(y)\|\theta - \theta_j^0\|^{\delta}, \quad \forall \|\theta - \theta_0\| \leq \delta_0,
$$

(34)

We then consider two types of mixture models: location mixtures and strongly identifiable mixtures. The latter are defined by the following assumption: Denote by $S^+_d$ the set of symmetrical semi-definite matrices of dimension $d$.

**Assumption A2** [Strong identifiability] For all $\epsilon > 0$ and all $\nu_0$ non-null measure on $A_0 = [\bigcup_{j=1}^{k_0} B(\theta_j^0, \epsilon)]^c$ satisfying $\nu_0(A_0) \leq 1$ and all $\alpha_0 \geq 0$, $\alpha_j \in \mathbb{R}$, $\beta_j \in \mathbb{R}^d$ and $\gamma_j \in S^+_d$, $j = 1, \ldots, k_0$,

$$
\alpha_0 f_{\nu_0}(y) + \sum_{j=1}^{k_0} (\alpha_j f_{\nu_0} + \beta_j \nabla f_{\nu_0} + \gamma_j D^2 f_{\nu_0} \gamma_j) = 0 \quad \Leftrightarrow \quad \alpha_0 = \alpha_j = 0, \quad \beta_j = \gamma_j = 0, \quad \forall j
$$

**Assumption A3** For all $x$, $f_\Theta(x)$ goes to 0 at $\bar{\Theta} \cap \Theta^c$, where $\bar{\Theta}$ is the closure of $\Theta$ and

$$
\sup_\Theta \|f_\Theta(x)\|_\infty < \infty
$$

The cornerstone of the proof of Theorem 4.3 is the existence and boundedness of the density function of the Dirichlet Process random mean defined as $\mu(P) := \int \theta dP(\theta)$ for $P$ distributed from $DP(M, G_0)$. Feigin and Tweedie (1989) show that the following assumption is a necessary and sufficient condition for the mean $\mu(P)$ to exist.

**Assumption A4** The Dirichlet Process base measure $G_0$ is such that

$$
\int_\Theta \log(1 + \|\theta\|_\infty) dG_0(\theta) < \infty
$$

**Remark 4.1.** A sufficient condition for Assumption A4 to be true is that distribution $G_0$ has a finite first moment, which includes Gaussian distributions for instance. Note that although the Cauchy distribution has no finite expectation, it does verify Assumption A4.

**Remark 4.2.** Let $\varphi$ denote the characteristic function of $G_0$. Since for all $(\varpi_1, \varpi_2, \ldots)$ following the stick-breaking distribution and for all $z \in \mathbb{R}^d$,

$$
\prod_{j=1}^{\infty} \varphi(\varpi_j z) = E_\Theta \left[ \exp \left( iz^T \sum_{j=1}^{\infty} \varpi_j \theta_j \right) \right] = E_\Theta \left[ \exp (iz^T \mu(P)) \right]
$$
then, $\prod_{j=1}^{\infty} \varphi(\varpi j z)$ exists almost surely if $\mu(P)$ exists almost surely. Hence, Assumption A4 implies the existence of $\prod_{j=1}^{\infty} \varphi(\varpi j z)$.

The next Theorem shows that under Assumptions A1, A2, A3 and A4 and if $f_{P_0}$ is a mixture with $k_0$ components, then $m_{DP}(y)$ is bounded from above by $o(n^{-(k_0-1+d k_0+1)/2})$ for some $t > 0$:

**Theorem 4.3.** Assume that $y_1, \cdots, y_n$ are iid $f_{P_0}$ with $P_0 = \sum_{j=1}^{k_0} \varpi_j \delta_{\theta_j}$ such that Assumptions A1, A2, A3, A4 are satisfied. Consider a Dirichlet process prior on $\tilde{P}$: $P \sim DP(M, G_0)$ with $G_0$ satisfying $G_0(\|\theta\| > u) \leq e^{-a_0 u^\tau}$ for some $a_0, \tau > 0$ when $u$ is large enough. Then there exists $t > 0$ such that for all $\epsilon > 0$

$$P_{f_0} \left( m_{DP}(y) > n^{-(k_0-1+d k_0+1)/2} \right) = o(1)$$

Moreover there exists $q \geq 0$ such that

$$\Pi_{DP} \left( \|f_0 - f_p\|_1 \leq \frac{(\log n)^q}{\sqrt{n}} |y| \right) = 1 + o_{P_{f_0}}(1).$$

**Remark 4.4.** Exploiting the proof of Proposition 3.10 of Gassiat and Van Handel (2014) (see Proposition 3.12) Assumption A2 is established in location mixtures, i.e. for models in the form

$$f_\theta(y) = f(y - \theta)$$

and Theorem 4.3 is valid under Assumptions A1, A3, A4.

**Remark 4.5.** The posterior concentration in $L_1$ distance given in (36) has been obtained for Dirichlet Process mixtures when the kernel $f_\theta(x) = f(x - \theta)$ and when $f$ is analytic, see for instance Ghosal and Van Der Vaart (2007). To the best of our knowledge it has not been extended to more general mixtures satisfying Assumption A2.

### 4.2 Technical lemmas

The following lemmas provide an expression for the characteristic function $\psi$ of $\mu(P)$ and subsequently establish that it is integrable. This fact is at the core of the proof of Theorem 4.3.

**Lemma 4.6.** (Yamato (1984)) Under Assumption A4, the characteristic function $\psi$ of $\mu(P)$ can be written as

$$\psi(z) = E \left[ \prod_{j=1}^{\infty} \varphi(\varpi j z) \right], \ z \in \mathbb{R}^d$$

**Lemma 4.7.** For all $d \geq 1$, if $P = \sum_{j=1}^{\infty} \varpi_j \delta_{\theta_j} \sim DP(M, G_0)$

$$E_P \left[ \left( \sum_{j=1}^{d+1} \varpi_j^2 \right)^{-\frac{d}{2}} \right] < \infty$$

**Proof:** See Appendix.
Lemma 4.8. If $P = \sum_{j=1}^{\infty} \omega_j \delta_{\theta_j} \sim DP(M, G_0)$ where $G_0$ has a density with respect to Lebesgue measure and verifies
\[
\int |\varphi(u)| du = C_{G_0} < \infty,
\]
with $\varphi$ the characteristic function of $G_0$ and $C_{G_0}$ a constant, then under Assumption A4 $\mu(P)$ has a bounded density with respect to the Lebesgue measure.

Proof of Lemma 4.8. Using Lemma 4.6, the characteristic function $\psi$ of $\mu(P)$ is equal to
\[
\psi(z) = \mathbb{E} \prod_{j=1}^{\infty} \varphi(\omega_j z), \quad P = \sum_j \omega_j \delta_{\theta_j}
\]
where $\varphi$ is the characteristic function of $G_0$. Let $p_{(1)} = \max_j \omega_j$ and note that, in particular, $|\psi(z)| \leq \mathbb{E}|\varphi(p_{(1)} z)|$. Hence,
\[
\int |\psi(z)| dz \leq \mathbb{E}_{p_{(1)}} \left[ \frac{1}{p_{(1)}} \int |\varphi(u)| du \right] \leq C_{G_0} \mathbb{E}_{p_{(1)}} \left[ \left( \sum_{j=1}^{r} \omega_j^2 \right)^{-d/2} \right], \text{ for some } r > 0.
\]
Using Lemma 4.7 with $r = d + 1$, we obtain the result. \qed

5 Conclusion and prospects

This article addressed the difficult problem of evidence estimation for both finite and Dirichlet Process mixture models. For finite mixtures, we identified two methods, namely ChibPartition and SIS, that scale with the number of mixture components and/or with the number of observations, as opposed to classical algorithms. Furthermore, the adaptive SMC algorithm that we have presented shows good performance and can be used in the non-conjugate case. For the DPM, we noticed that there does not seem to exist any reference method widely used by practitioners. We benchmarked Basu and Chib (2003)’s approach with other algorithms, including a method based on Geyer (1994)’s reverse logistic regression that appears reliable and scales better with the number of observations $n$.

An immediate application are goodness-of-fit tests that compare a parametric null to a non-parametric alternative. We formalized this procedure theoretically by establishing the consistency of the Bayes Factor for this kind of scenario, when the nonparametric alternative is a ‘strongly-identifiable’ Dirichlet Process mixture model, which includes location mixture kernels.

An interesting research avenue would be to identify scalable evidence estimation techniques for non-conjugate Dirichlet Process mixtures. The RLR-Prior algorithm that we introduced can be a good first approach but suffers from a large variance when the number of observation increases. As to the asymptotics of the marginal likelihood under a DPM, it would be interesting to extend Theorem 4.3 to the location-scale Dirichlet Process mixture model.
References

Basu, S. and Chib, S. (2003). Marginal likelihood and Bayes factors for Dirichlet process mixture models. *Journal of the American Statistical Association* **98** 224–235.

Berkhof, J., Van Mechelen, I. and Gelman, A. (2003). A Bayesian approach to the selection and testing of mixture models. *Statistica Sinica* **42** 3–442.

Blackwell, D. and MacQueen, J. B. (1973). Ferguson distributions via Pólya urn schemes. *The Annals of Statistics* **1** 353–355.

Blei, D. M., Ng, A. Y. and Jordan, M. I. (2003). Latent dirichlet allocation. *the Journal of machine Learning research* **3** 993–1022.

Buchholz, A., Chopin, N. and Jacob, P. E. (2021). Adaptive tuning of hamiltonian monte carlo within sequential monte carlo. *Bayesian Analysis* **1** 1–27.

Carvalho, C. M., Lopes, H. F., Polson, N. G. and Taddy, M. A. (2010). Particle learning for general mixtures. *Bayesian Analysis* **5** 709–740.

Castillo, I. (2008). Lower bounds for posterior rates with Gaussian process priors. *Electronic Journal of Statistics* **2** 1281–1299.

Chambaz, A. and Rousseau, J. (2008). Bounds for Bayesian order identification with application to mixtures. *The Annals of Statistics* **36** 938–962.

Chen, M.-H. and Shao, Q.-M. (1997). On Monte Carlo methods for estimating ratios of normalizing constants. *The Annals of Statistics* **25** 1563–1594.

Chib, S. (1995). Marginal likelihood from the Gibbs output. *Journal of the American Statistical Association* **90** 1313–1321.

Chopin, N. (2002). A sequential particle filter method for static models. *Biometrika* **89** 539–552.

Crépet, A. and Tressou, J. (2011). Bayesian nonparametric model with clustering individual co-exposure to pesticides found in the French diet. *Bayesian analysis* **6** 127–144.

Dass, S. C. and Lee, J. (2004). A note on the consistency of Bayes factors for testing point null versus non-parametric alternatives. *Journal of statistical planning and inference* **119** 143–152.

Del Moral, P., Doucet, A. and Jasra, A. (2006). Sequential monte carlo samplers. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **68** 411–436.

Diebolt, J. and Robert, C. P. (1994). Estimation of finite mixture distributions through Bayesian sampling. *Journal of the Royal Statistical Society: Series B (Methodological)* **56** 363–375.

Drton, M. and Plummer, M. (2017). A Bayesian information criterion for singular models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **79** 323–380.

Escobar, M. D. and West, M. (1995). Bayesian density estimation and inference using mixtures. *Journal of the American Statistical Association* **90** 577–588.
Feigin, P. D. and Tweedie, R. L. (1989). Linear functionals and Markov chains associated with Dirichlet processes. In *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 105. Cambridge University Press.

Ferguson, T. S. (1983). Bayesian density estimation by mixtures of normal distributions. In *Recent advances in statistics*. Elsevier, 287–302.

Frühwirth-Schnatter, S. (2004). Estimating marginal likelihoods for mixture and Markov switching models using bridge sampling techniques. *The Econometrics Journal* 7 143–167.

Frühwirth-Schnatter, S. (2006a). *Finite mixture and Markov switching models*. Springer Science & Business Media.

Frühwirth-Schnatter, S. (2006b). Practical Bayesian Inference for a Finite Mixture Model with Known Number of Components. *Finite Mixture and Markov Switching Models* 57–98.

Frühwirth-Schnatter, S. (2019). Keeping the balance—Bridge sampling for marginal likelihood estimation in finite mixture, mixture of experts and Markov mixture models. *Brazilian Journal of Probability and Statistics* 33 706–733.

Gassiat, E. and Van Handel, R. (2014). The local geometry of finite mixtures. *Transactions of the American Mathematical Society* 366 1047–1072.

Geyer, C. J. (1994). Estimating normalizing constants and reweighting mixtures.

Ghosal, S., Ghosh, J. K. and Van Der Vaart, A. W. (2000). Convergence rates of posterior distributions. *Annals of Statistics* 500–531.

Ghosal, S. and Van Der Vaart, A. (2007). Posterior convergence rates of Dirichlet mixtures at smooth densities. *The Annals of Statistics* 697–723.

Ghosal, S. and Van Der Vaart, A. W. (2001). Entropies and rates of convergence for maximum likelihood and Bayes estimation for mixtures of normal densities. *The Annals of Statistics* 29 1233–1263.

Graham, R. L., Knuth, D. E., Patashnik, O. and Liu, S. (1989). *Concrete mathematics: a foundation for computer science*, vol. 3. American Institute of Physics.

Griffin, J. (2017). Sequential Monte Carlo Methods for Normalized Random Measure with Independent Increments Mixtures. *Statistics and Computing* 27 131–145.

Irwin, M., Cox, N. and Kong, A. (1994). Sequential imputation for multilocus linkage analysis. *Proceedings of the National Academy of Sciences* 91 11684–11688.

Kamary, K., Mengersen, K., Robert, C. P. and Rousseau, J. (2014). Testing hypotheses via a mixture estimation model. *arXiv preprint arXiv:1412.2044*.

Kong, A., Liu, J. S. and Wong, W. H. (1994). Sequential imputations and Bayesian missing data problems. *Journal of the American statistical association* 89 278–288.

Kruijver, W., Rousseau, J. and Van Der Vaart, A. (2010). Adaptive Bayesian density estimation with location-scale mixtures. *Electronic Journal of Statistics* 4 1225–1257.
Lee, J. E. and Robert, C. P. (2016). Importance sampling schemes for evidence approximation in mixture models. *Bayesian Analysis* **11** 573–597.

Lo, A. Y. (1984). On a class of Bayesian nonparametric estimates: I. Density estimates. *The Annals of Statistics* 351–357.

MacEachern, S. N., Clyde, M. and Liu, J. S. (1999). Sequential importance sampling for nonparametric Bayes models: The next generation. *Canadian Journal of Statistics* **27** 251–267.

Marin, J.-M. and Robert, C. (2008). Approximating the marginal likelihood in mixture models. *arXiv preprint arXiv:0804.2414*.

McLachlan, G. J., Lee, S. X. and Rathnayake, S. I. (2019). Finite mixture models. *Annual review of statistics and its application* **6** 355–378.

McVinish, R., Rousseau, J. and Mengersen, K. (2009). Bayesian goodness of fit testing with mixtures of triangular distributions. *Scandinavian Journal of Statistics* **36** 337–354.

Meng, X.-L. and Wong, W. H. (1996). Simulating ratios of normalizing constants via a simple identity: a theoretical exploration. *Statistica Sinica* 831–860.

Miller, J. W. and Harrison, M. T. (2014). Inconsistency of Pitman-Yor process mixtures for the number of components. *The Journal of Machine Learning Research* **15** 3333–3370.

Neal, R. M. (1999). Erroneous Results in Marginal Likelihood from the Gibbs Output. URL [https://www.cs.utoronto.ca/~radford/ftp/chib-letter.pdf](https://www.cs.utoronto.ca/~radford/ftp/chib-letter.pdf).

Neal, R. M. (2000). Markov chain sampling methods for Dirichlet process mixture models. *Journal of computational and graphical statistics* **9** 249–265.

Newey, W. K. and West, K. D. (1986). A simple, positive semi-definite, heteroskedasticity and autocorrelation-consistent covariance matrix .

Postman, M., Huchra, J. P. and Geller, M. J. (1986). Probes of large-scale structure in the Corona Borealis region. *The Astronomical Journal* **92** 1238–1247.

Quintana, F. A. and Iglesias, P. L. (2003). Bayesian clustering and product partition models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **65** 557–574.

Quintana, F. A. and Newton, M. A. (2000). Computational aspects of nonparametric Bayesian analysis with applications to the modeling of multiple binary sequences. *Journal of Computational and Graphical Statistics* **9** 711–737.

Raftery, A. E. (1996). Hypothesis testing and model 163–188.

Rasmussen, C. E. et al. (1999). The infinite Gaussian mixture model. In *NIPS*, vol. 12.
A SIS sampling strategy for mixture models

The likelihood of a given partition of the data $C(z) = \{C_1(z), \ldots, C_K(z)\}$ is given by

$$P(y|C(z)) = \prod_{k=1}^{K} p(y|C_k(z)) = \prod_{k=1}^{K} \int_{\Theta} \prod_{z_i=k} p(y_i|\theta)G_0(d\theta) := \prod_{k=1}^{K} m(C_k(z)) \quad (39)$$

For conjugate finite mixtures, this integral is easily computable in closed form.

Notice that the parameters $\theta$ and the weights $\eta$ are independent a posteriori and their respective augmented posteriors are given by

$$P(d\theta_k|y, z) = \frac{\prod_{z_i=k} p(y_i|\theta_k)G_0(d\theta_k)}{\int_{\Theta} \prod_{z_i=k} p(y_i|\theta)G_0(d\theta)} = \frac{\prod_{z_i=k} p(y_i|\theta_k)G_0(d\theta_k)}{m(C_k(z))} \quad (40)$$

and

$$P(\eta|z) = D(n_1(z) + \alpha_1, \ldots, n_K(z) + \alpha_K) \quad (41)$$
where \( n_k(z) = \sum_{i=1}^{m} 1_{\{z_i = k\}} \), \( k = 1, \ldots, K \), provided the weights \( \eta \) are given a Dirichlet prior \( \mathcal{D}(\alpha_1, \ldots, \alpha_K) \).

The weights \( w(z, y) \) can be derived by noticing that the posterior predictive distribution for all \( y_i \), where \( i \geq 2 \) is given by

\[
p(y_i|z_{1:i-1}, y_{1:i-1}) = \int \int p(y_i|\eta, \theta) \prod_{k=1}^{K} P(d\theta_k|y_{1:i-1}, z_{1:i-1}) P(d\eta|z_{1:i-1})
\]

\[
= \sum_{k=1}^{K} \int \int \eta_k p(y_i|\theta_k) p(\theta_k|y_{1:i-1}, z_{1:i-1}) d\theta P(d\eta|z_{1:i-1})
\]

\[
= \sum_{k=1}^{K} \frac{m(C_k(z_{1:i-1}) \cup \{y_i\})}{m(C_k(z_{1:i-1}))} \int \eta_k P(d\eta|z_{1:i-1})
\]

\[
= \sum_{k=1}^{K} \frac{m(C_k(z_{1:i-1}) \cup \{y_i\})}{m(C_k(z_{1:i-1}))} \frac{n_k(z_{1:i-1}) + \alpha_k}{\sum_{k=1}^{K} n_k(z_{1:i-1}) + \alpha_k}
\]

The expression above determines completely the quantity of interest \( w(z, y) \).

The allocation \( z_i \) where \( i \geq 2 \) should be drawn sequentially from the categorical distribution

\[
p(z_i = k|y_{1:i}, z_{1:i-1}) \propto \frac{m(C_k(z_{1:i-1}) \cup \{y_i\})}{m(C_k(z_{1:i-1}))} \frac{n_k(z_{1:i-1}) + \alpha_k}{\sum_{k=1}^{K} n_k(z_{1:i-1}) + \alpha_k}, \text{ for } k = 1, \ldots, K
\]

**B Additional lemma**

**Lemma B.1.** Let \( P = \sum_{i=1}^{\infty} \omega_i \delta_{\theta_i} \) be a realisation of the Dirichlet Process. Define

\[
\Delta(P) = \int_{A_0} f_\theta dP(\theta) + \sum_{j=1}^{k_0} [p_j - \bar{p}_j] f_{\theta_j} + \int_{B(\theta_j, \epsilon)} (\theta - \theta_j)^T \nabla f_{\theta_j} dP(\theta)
\]

\[
+ \frac{1}{2} \int_{B(\theta_j, \epsilon)} (\theta - \theta_j)^T D^2 f_{\theta_j} (\theta - \theta_j) dP(\theta)
\]

(42)

where for all \( j = 1, \ldots, k_0 \),

\[
p_j = \sum_{\nu: \theta_\nu \in B(\theta_j, \epsilon)} \omega_i
\]

(43)

for some \( \epsilon > 0 \). Then there exists a constant \( c(f_0) \) depending only on \( f_0 \) such that

\[
\|\Delta(P)\|_1 \geq c(f_0) \left[ \int_{A_0} P(A_0) + \sum_{j=1}^{k_0} |p_j - \bar{p}_j| + \int_{B(\theta_j, \epsilon)} (\theta - \theta_j)^T dP(\theta) \right]
\]

\[
+ \int_{B(\theta_j, \epsilon)} \|\theta - \theta_j\|^2 dP(\theta)
\]

(44)

\[:= c(f_0)N(P)\]
Proof.

Let

\[ \tilde{\Delta}(P) = \frac{\Delta(P)}{N(P)} \]

\[ = \left\| \alpha_0 \int_{A_0} f_\theta(x) d\nu_0(x) + \sum_{j=1}^{k_0} \alpha_j f_\theta^j(x) + \beta_j^T \nabla f_\theta^j(x) + \frac{1}{2} \text{tr}(D^2 f_\theta^j(x) \gamma_j) \right\|_1 \]

where

\[ \alpha_0 = \frac{P(A_0)}{N(P)}, \quad \nu_0(d\theta) = \frac{P(d\theta)1_{A_0}}{P(A_0)}, \quad \alpha_j = \frac{p_j - p_j^0}{N(P)} \]

\[ \beta_j = \frac{1}{N(P)} \int_{B(\theta_0^j, \epsilon)} (\theta - \theta_0^j) dP(\theta), \quad \gamma_j = \frac{1}{2N(P)} \int_{B(\theta_0^j, \epsilon)} (\theta - \theta_0^j)(\theta - \theta_0^j)^T dP(\theta) \]

and assume that (44) does not hold. Then there exists a sequence \( \alpha^m, \beta^m, \gamma^m, \nu^m_0 \) along which \( \tilde{\Delta}(P) \) goes to zero. Note that by construction \( \text{tr}(\gamma_j) = \int_{B(\theta_0^j, \epsilon)} \|\theta - \theta_0^j\|^2 dP(\theta) \) and that \( \alpha^m, \beta^m, \gamma^m \) belong to a compact set so that there exists a sub-sequence which is convergent to some value \( \alpha, \beta, \gamma \). Similarly, \( \nu^m_0 \) is a sequence of measure with mass bounded by 1, so it converges vaguely to a sub-probability measure \( \nu_0 \) on \( A_0 \) along a subsequence (also denoted \( \nu^m_0 \))

and since for all \( x \), \( f_\theta(x) \) is continuous in \( \theta \) and converges to 0 on the boundary of \( \Theta \),

\[ \int_{A_0} f_\theta(x) d\nu_0^m \rightarrow \int_{A_0} f_\theta(x) d\nu_0, \quad \forall x \]

Now \( \int_{A_0} f_\theta(x) d\nu_0^m \leq \sup_\Theta \|f_\theta(\cdot)\|_\infty \) hence on any compact subset \( B \) of \( \mathbb{R}^d \),

\[ \|1_B \int_{A_0} f_\theta(x)(d\nu_0^m - d\mu_0)(\theta)\|_1 = o(1) \]

so that for all compact \( B \), at the limit, for all \( x \in B \)

\[ \alpha_0 \int_{A_0} f_\theta(x) d\nu_0(\theta) + \sum_{j=1}^{k_0} \alpha_j f_\theta^j(x) + \beta_j^T \nabla f_\theta^j(x) + \frac{1}{2} \text{tr}(D^2 f_\theta^j(x) \gamma_j) = 0 \]

since the relation is true for all \( B \) it is true for all \( x \in \mathbb{R}^d \). The strong identifiability assumption \( \textbf{A2} \) implies that \( \alpha = 0, \beta = 0 \) and \( \gamma = 0 \), which is not possible since \( \sum_j |\alpha_j| + |\beta_j| + |\gamma_j| = 1 \).

Hence (44) is valid.
C Proof of Lemma 4.7

We first notice that

\[
\begin{align*}
V_1 &= \varpi_1 \\
V_2(1 - V_1) &= \varpi_2 \\
&\vdots \\
V_{d+1}(1 - V_d) \ldots (1 - V_1) &= \varpi_{d+1}
\end{align*}
\]

\[
\Leftrightarrow \begin{align*}
V_1 &= \varpi_1 \\
V_2 &= \varpi_2 \\
&\vdots \\
V_{d+1} &= \frac{\varpi_{d+1}}{\sum_i \varpi_i}
\end{align*}
\] (45)

where \( V_i \overset{i.i.d.}{\sim} B(1, M) \). Hence,

\[
I = \mathbb{E} \left[ \frac{1}{(\sum_{j=1}^{d+1} \varpi_j)^2} \right]
= \left( \frac{\Gamma(M)}{\Gamma(1 + M)} \right)^{d+1} \int_{[0,1]^{d+1}} \frac{(1 - V_1)^{M-1} \ldots (1 - V_{d+1})^{M-1}}{(V_1^2 + \ldots + V_{d+1}^2(1 - V_d)^2 \ldots (1 - V_1)^2)^{d/2}} dV_1 \ldots dV_{d+1}
\]

Let \( B_\varepsilon = \{(V_1, \ldots, V_{d+1}) : \exists i \in \{1, \ldots, d+1\}, \varpi_i > \varepsilon\} \) and fix \( \varepsilon = 1/(2(d+1)) \). Then,

\[
I \leq \left( \frac{\Gamma(M)}{\Gamma(1 + M)} \right)^{d+1} \left( C \varepsilon^{-d} + \int_{[0,1]^{d+1} \cap B_\varepsilon} \frac{(1 - V_1)^{M-1} \ldots (1 - V_{d+1})^{M-1}}{(V_1^2 + \ldots + V_{d+1}^2(1 - V_d)^2 \ldots (1 - V_1)^2)^{d/2}} dV_1 \ldots dV_{d+1} \right)
\]

for some constant \( C > 0 \). We then use the change of variable given by (45), which Jacobian is

\[
\det(J) = \frac{1}{1 - \varpi_1} \times \frac{1}{1 - \varpi_1 - \varpi_2} \times \ldots \times \frac{1}{1 - \sum_i \varpi_i}
\]

Hence,

\[
I \leq \left( \frac{\Gamma(M)}{\Gamma(1 + M)} \right)^{d+1} \left( C \varepsilon^{-d} + \int_{\Delta_{d+1} \cap B_\varepsilon} \frac{1}{(\sum_{j=1}^{d+1} \varpi_j)^2} \times \frac{(1 - \sum_{j=1}^{d+1} \varpi_j)^{M-1}}{(1 - \varpi_1) \ldots (1 - \sum_{j=1}^{d} \varpi_j)} d\varpi_1 \ldots d\varpi_{d+1} \right)
\]

where \( \Delta_{d+1} = \{q \in \mathbb{R}^{d+1} : 0 \leq \varpi_j \leq 1, \forall j \text{ and } \sum_j \varpi_j \leq 1\} \).

\[
I \leq \left( \frac{\Gamma(M)}{\Gamma(1 + M)} \right)^{d+1} \left( C \varepsilon^{-d} + 2^d \int_{\Delta_{d+1} \cap B_\varepsilon} \frac{(1 - \sum_{j=1}^{d+1} \varpi_j)^{M-1}}{(\sum_{j=1}^{d+1} \varpi_j^2)^{d/2}} d\varpi_1 \ldots d\varpi_{d+1} \right)
\]
If $M \geq 1$

$$I \leq \left( \frac{\Gamma(M)}{\Gamma(1 + M)} \right)^{d+1} \left( \varepsilon^{-d}C + 2^d \int_{\Delta_{d+1}} \frac{1}{(\sum_{j=1}^{d+1} \omega_j^2)^{\frac{d}{2}}} \, d\omega_1 \ldots d\omega_{d+1} \right)$$

If $M < 1$

$$I \leq \left( \frac{\Gamma(M)}{\Gamma(1 + M)} \right)^{d+1} \left( \varepsilon^{-d}C + 2^{d+1-M} \int_{\Delta_{d+1}} \frac{1}{(\sum_{j=1}^{d+1} \omega_j^2)^{\frac{d}{2}}} \, d\omega_1 \ldots d\omega_{d+1} \right)$$

Let $J = \int_{\Delta_{d+1}} 1/(\sum_{j=1}^{d+1} \omega_j^2)^{\frac{d}{2}} d\omega_1 \ldots d\omega_{d+1}$. We shall prove that $J$ is finite.

Using the hyperspherical change of variable

$$\left\{ \begin{array}{l}
\omega_1 = r \cos \theta_1 \\
\omega_2 = r \sin \theta_1 \cos \theta_2 \\
\vdots \\
\omega_d = r \sin \theta_1 \ldots \cos \theta_d \\
\omega_{d+1} = r \sin \theta_1 \ldots \sin \theta_d
\end{array} \right. $$

where $\theta_1, \ldots, \theta_{d-1} \in [0, \pi]$ and $\theta_d \in [0, 2\pi)$. The Jacobian of this transformation is bounded by $r^d$, and noticing that $r = \left( \sum_{j=1}^{d+1} \omega_j^2 \right)^{\frac{1}{2}}$, we get

$$J \leq \int_0^1 \int_0^{2\pi} \cdots \int_0^{2\pi} r^d \, d\theta_1 \ldots d\theta_d \, dr$$

Hence,

$$J \leq 2\pi \left( \frac{\pi}{2} \right)^{d-1}$$

and

$$\mathbb{E} \left[ \frac{1}{(\sum_{j=1}^{d+1} \omega_j^2)^{\frac{d}{2}}} \right] < \infty$$

D Proof of Theorem 4.3

To prove Theorem 4.3 we prove that the associated posterior concentrates in $L_1$ norm at the rate $(\log n)^q / \sqrt{n}$ for some $q > 0$ and then we bound from above $\Pi(P : \|f_P - f_0\|_1 \leq (\log n)^q / \sqrt{n})$.

More precisely we first prove that

$$\mathbb{P}_{f_0} \left( \int_{\|f_P - f_0\|_1 > \delta_n} e^{\ell_n(P) - \ell_n(f_0)} d\Pi(P) > n^{-D(k_0) - t} \right) = o(1) \quad (46)$$

where $\delta_n = (\log n)^q / \sqrt{n}$, $D(k_0) = (k_0 - 1 + dk_0)/2$ and $\ell_n(P) = \log f_P(y)$ is the log-likelihood.
Then, to bound from above \( \int \| f_p - f_0 \|_1 \leq \delta_n \) we use a simple Markov inequality:

\[
\mathbb{P}_{f_0} \left( \int \| f_p - f_0 \|_1 > n^{-D(k_0) - \ell} \right) \leq n^{D(k_0) + 1} \mathbb{P}(\| f_p - f_0 \|_1 \leq \delta_n) \tag{47}
\]

and we can conclude the proof as soon as we bound \( \mathbb{P}(\| f_p - f_0 \|_1 \leq \delta_n) \). This is the most difficult part of the proof and is done in Section D.2.

### D.1 Proof of \( L_1 \)-concentration rate (46)

Throughout the proof \( C \) denotes a generic constant depending only on \( f_0 \). Let \( \eta > 0 \) be arbitrarily small, and

\[
\mathcal{F}_n = \{ f_p; \ P(\Theta_n^\epsilon) \leq n \delta_n \}, \quad \Theta_n = \{ ||\theta|| \leq (\log n)^a \},
\]

for some \( a > 0 \). To prove (46), we first prove that the covering number of each slice \( \mathcal{F}_{n, \ell} = \{ f_p \in \mathcal{F}_n; \ ||f_p - f_0||_1 \in (\ell \delta_n, (\ell + 1) \delta_n) \} \) by \( L_1 \) balls of radius \( \zeta \ell \delta_n \) is bounded by

\[
\mathcal{N}_{n, \ell} \lesssim e^{C(\log n)^{ad} \log(1/n)}. \tag{48}
\]

Using Lemma B.1, we have that for all \( f_p \in \mathcal{F}_{n, \ell} \), if \( \ell \delta_n \leq \epsilon_0 \) for some \( \epsilon_0 > 0 \) small enough,

\[
P(A_0) + \sum_{j=1}^{k_0} \left[ |p_j - p_j^0| + ||\mu(p_j) - \theta_j^0|| + \int_{B(\theta_j^0, \epsilon)} ||\theta - \theta_j^0||^2 dP(\theta) \right] \leq (\ell + 1) \delta_n / c(f_0)
\]

where \( \mu(P) := \int_{B(\theta_j^0, \epsilon)} \theta dP(\theta) \).

Let \( \xi > 0 \), we consider a covering of \( A_{0, n} = A_0 \cap \Theta_n \) with balls \( U_{i,0} \) with center \( \vartheta_i \) and radius \( \xi \) and \( N_0(\xi) \), the number of such balls, is bounded by \( (C|\Theta_n|/\xi)^d \). Define \( U_{1,0} = U_{1,0} \) and
\[ U_{i,0} = U_{i,0} \setminus \cup_{j=1}^{i-1} U_{j,0} \]. Consider \( P, P' \) such that \( f_P, f'_P \in \mathcal{F}_{n,s} \),

\[
\int \left| \int_{A_0} f_\theta(x)(dP - dP')(\theta) \right| dx \leq \int \left| \sum_{i=1}^{N_0(\xi)} \left\{ \int_{U_{i,0}} f_\theta(x) - f_{\theta_i}(x)(dP - dP')(\theta) \right\} \right| \\
+ \int \left| \sum_{i=1}^{N_0(\xi)} \left\{ \int_{U_{i,0}} f_{\theta_i}(x)(dP - dP')(\theta) \right\} \right| \\
+ \int_{U_{i,0}} \left| f_\theta(x) - f_{\theta_i}(x) \right| dP(\theta) \\
+ \int_{U_{i,0}} \left| f_\theta(x) - f_{\theta_i}(x) \right| dP'(\theta) + f_{\theta_i}(x) \int_{U_{i,0}} d(P - P')(\theta) \right| \\
\leq \left| H_1(x) \right| \| P(A_0) + P'(A_0) \| + \sum_{i=1}^{N_0(\xi)} \left| P(U_{i,0}) - P'(U_{i,0}) \right| \\
\leq C\xi(\ell + 1)\delta_n + \sum_{i=1}^{N_0(\xi)} \left| P(U_{i,0}) - P'(U_{i,0}) \right| \\
= \left( \int_{A_0} f_\theta(\cdot)(dP - dP')(\theta) \right) \leq C\xi\ell\delta_n. \tag{49}
\]

Hence if for all \( i \leq N_0(\xi) \), \( |P(U_{i,0}) - P'(U_{i,0})| \leq \xi/N_0(\xi) \),

\[
\left\| \int_{A_0} f_\theta(\cdot)(dP - dP')(\theta) \right\|_1 \leq C\xi\ell\delta_n. \tag{50}
\]

Moreover for each \( j = 1, \ldots, k_0 \),

\[
\int_{B(\theta_j^{0,\epsilon}, \epsilon)} f_\theta(x)(dP - dP')(\theta) = \int_{B(\theta_j^{0,\epsilon}, \epsilon)} \left\{ f_{\theta_j^{0}}(x) + (\theta - \theta_j^{0})^T \nabla f_{\theta_j^{0}}(x) + \frac{1}{2} (\theta - \theta_j^{0})^T D^2 f_{\theta_j^{0}}(x)(\theta - \theta_j^{0}) \right\} \| dP - dP'(\theta) \right| \\
+ \mathcal{O} \left( \| \theta - \theta_j^{0} \|^2 \right) \\
= \int_{B(\theta_j^{0,\epsilon}, \epsilon)} \left\{ f_{\theta_j^{0}}(x) + (\theta - \theta_j^{0})^T \nabla f_{\theta_j^{0}}(x) \right\} (p_j - p_j^{0}) \left| \right| + \mid \mu(P_j) - \mu(P_j^{0}) \mid ^T \nabla f_{\theta_j^{0}}(x) \right| \\
+ \frac{1}{2} \text{tr} \left( D^2 f_{\theta_j^{0}}(x) \int (\theta - \theta_j^{0})(\theta - \theta_j^{0})^T d(P_j - P_j^{0})(\theta) \right) \right| \\
+ \mathcal{O} \left( \int \| \theta - \theta_j^{0} \|^2 \right) \\
\text{where } p \text{ and } p' \text{ are defined as in } (43).
\]

Hence if \( |p_j - p_j^{0}| \leq \xi(\ell + 1)\delta_n \), \( |\mu(P_j) - \mu(P_j^{0})| \leq \xi(\ell + 1)\delta_n \) and

\[
\left\| \int (\theta - \theta_j^{0})(\theta - \theta_j^{0})^T d(P_j - P_j^{0})(\theta) \right\|_F \leq \xi(\ell + 1)\delta_n.
\]
\[
\left\| \int_{B(\theta_j^\epsilon,c)} f_\theta(\cdot)(dP - dP') (\cdot) \right\|_1 \\
\leq \xi (\ell + 1) \delta_n \left( 1 + \| \theta_j^\epsilon \|_1 H_1(\cdot) + \frac{\| H_2(\cdot) \|_1}{2} \right) + O(\epsilon^\delta (\ell + 1) \delta_n)
\]
\[
\leq C(\epsilon^\delta + \xi \ell \delta_n).
\]
So that \( \| f_P - f_{P'} \|_1 \leq C \xi \ell \delta_n \).

Since for all \( j \), when \( f_{P'} \in \mathcal{F}_{n,\ell} \), \( \mu(P_j) \in B(\theta_j^0, C \ell \delta_n) \), the covering number for \( \{ \mu(P_j) ; f_P \in \mathcal{F}_{n,\ell} \} \) by balls of radius is \( \xi \ell \delta_n \) is bounded by \( (C \xi)^{-d} \) and similarly the covering numbers for the terms \( \int (\theta - \theta_j^0)(\theta - \theta_j^0)^T dP_j (\cdot) \) and \( p_j \) are bounded respectively by \( (C \xi)^{-d(d+1)/2} \) and \( (C \xi)^{-1} \).

This implies that by choosing \( \xi \) and small enough,

\[
\mathcal{N}_{n,\ell} \lesssim \xi^{-k_0(1+d+d(d+1)/2)} (N_0(\xi)/\xi)_n
\]
\[
\leq \exp \left( (C|\Theta_n|/\xi)^d \log(|\Theta_n|/\xi) + \log(1/\xi) \right)
\]
\[
\leq \exp \left( (C|\Theta_n|/\xi)^d (\log(|\Theta_n|) + \log(1/\xi)) \right)
\]
\[
\leq \exp \left( (C(\log n)^a/\xi)^d (\log((\log n)^a) + \log(1/\xi)) \right)
\]

and since \( |\Theta_n| \leq (\log n)^a \) for some \( a > 0 \), \( (48) \) holds.

Now if \( \ell \delta_n > \epsilon_0 \), we use a similar decomposition as in \( (50) \) but with \( \Theta_n \) instead of \( A_0 \), which leads to a covering number bounded by

\[
\log \mathcal{N}_{n,\ell} \lesssim (\log n)^{ad} \log(1/\epsilon).
\]

Thus similarly to Theorem 7.1 of Ghosal et al. (2000), there exists \( (L_1) \) tests \( \phi_{n,\ell} \) such that

\[
\mathbb{E}_0(\phi_n) \leq \sum_{\ell \geq \ell_0} \mathcal{N}_{n,\ell} e^{-c_1 n \ell^2 \delta_n^2} = o(1)
\]

\[
\sup_{\| f_{P'} - f_0 \|_1 > \epsilon_0 \delta_n, \in \mathcal{F}_n} \mathbb{E}_{f_P}(1 - \phi_n) \leq e^{-c_1 n \ell^2 \delta_n^2}
\]

if \( \ell_0 \) is large enough and \( 2q > ad \). Moreover from Ghosal et al. (2000),

\[
\Pi(P(\Theta_n^\epsilon) > \epsilon \delta_n) \leq G_0(\| \theta \|_1 (\log n)^a) \leq n^{-(D(k_0) + 2t)}
\]

by choosing \( a \tau > 1 \) large enough.

We finally obtain that

\[
P_{f_0} \left( \int_{\| f_{P'} - f_0 \|_1 > \delta_n} e^{\ell_n(\cdot)(P) - \ell_n(P_0)} d\Pi(P) > n^{-D(k_0) - t} \right) \leq n^{-t} + e^{-c_1 \ell_n^2 \delta_n^2 n^{D(k_0)+t}} = o(1)
\]

for some \( c_1 > 0 \) if \( q \) is large enough and \( (46) \) is proved.
### D.2 Upper bound on $\Pi(\|f_P - f_0\|_1 \leq \delta_n)$.

In this section we prove that

$$
\Pi(\|f_P - f_0\|_1 \leq \delta_n) = o(n^{-D(k_0)-1}).
$$

(51)

for some $t > 0$. Using a similar idea to Lemma 3.8 of Gassiat and Van Handel (2014), we choose $\epsilon \leq \min\{\|\theta_i^0 - \theta_j^0\|/4; i \neq j \leq k_0\}$, so that the balls $B(\theta_i^0, \epsilon)$, $i \leq k_0$ are disjoint. We then write with $A_0 = (\cup_{i=1}^{k_0} B(\theta_i^0,\epsilon))^c$ and $p_j = P(B(\theta_j^0, \epsilon))$

$$
\|f_0 - f_P\|_1 = \left\| \int_{A_0} f_0 dP(\theta) + \sum_{j=1}^{k_0} [p_j - p_j^0] f_{\theta_j^0} + \int_{B(\theta_j^0, \epsilon)} (\theta - \theta_j^0)^T \nabla f_{\theta_j^0} dP(\theta) \right\|
$$

$$
+ \frac{1}{2} \int_{B(\theta_j^0, \epsilon)} (\theta - \theta_j^0)^T D^2 f_{\theta_j^0} (\theta - \theta_j^0) dP(\theta) + R
$$

$$
:= \|\Delta_n(P) + R\|_1
$$

where

$$
\|R\|_1 \leq \sum_{j=1}^{k_0} \int_{B(\theta_j^0, \epsilon)} \sup_{\|\theta' - \theta_j^0\| \leq \|\theta - \theta_j^0\|} \left\| (\theta - \theta_j^0)^T \left[ D^2 f_{\theta_j^0} - D^2 f_{\theta_j^0} \right] (\theta - \theta_j^0) \right\| dP(\theta)
$$

$$
\leq d \sum_{j=1}^{k_0} \|H_3\|_1 \int_{B(\theta_j^0, \epsilon)} \|\theta - \theta_j^0\|^{2+\delta} dP(\theta).
$$

By Lemma B.1 there exists $c(f_0)$ such that

$$
\|\Delta_n(P)\|_1 \geq c(f_0) \left[ P(A_0) + \sum_{j=1}^{k_0} [p_j - p_j^0] + \int_{B(\theta_j^0, \epsilon)} (\theta - \theta_j^0)^T dP(\theta) \right]
$$

$$
+ \int_{B(\theta_j^0, \epsilon)} \|\theta - \theta_j^0\|^2 dP(\theta)
$$

$$
:= c(f_0) N_n(P)
$$

(52)

Then if $\|f_0 - f_P\|_1 \leq \delta_n$, then

$$
N_n(P)c(f_0) \leq \delta_n + d \sum_{j=1}^{k_0} \|H_3\|_1 \int_{B(\theta_j^0, \epsilon)} \|\theta - \theta_j^0\|^{2+\delta} dP(\theta)
$$

and choosing $\epsilon$ small enough

$$
N_n(P) \leq \frac{2\delta_n}{c(f_0)} := \delta'_n.
$$

We now bound from above $\Pi(N_n(P) \leq \delta'_n)$. Consider $P_j = P|_{B(\theta_j^0, \epsilon)}/p_j$ and $p_0 = P(A_0)$, then under the Dirichlet Process prior $(p_0, \cdots, p_{k_0}); P_1, \cdots, P_{k_0}$ are mutually independent and
\( P_j \sim DP(MG_0(\cdot \cap B(\theta^i, \epsilon))) \). Hence
\[
\Pi(N_n(P) \leq \delta_n' \leq \delta_n^M \cdot \| \int \phi \cdot P_j(\theta) d\theta \| \leq \delta_n')
\]

Let \( \Pi_j \) be the marginal prior distribution of \( \int \phi \cdot P_j(\theta) d\theta \), then
\[
\Pi_j(B(\theta^i_j, \delta_n')) \leq \sup_{\|x - \theta_j^i\| \leq \delta_n'} \pi_j(x) \leq \delta_n'^d
\]
as soon as \( \Pi_j \) has a bounded density \( \pi_j \). Using Lemma 4.8 we know that \( \Pi_j \) has a bounded density \( \pi_j \) and (51) is proved.

E Choice of tuning parameters for the numerical experiments

E.1 Figure 1

| Tuning parameters |  |  |  |  |
|-------------------|---|---|---|---|
|                   |  |  |  |  |
| Chib              |  |  |  |  |
| \( K = 3 \)       | \( T = 10^5 \) | \( T = 10^5 \) | \( T = 2 \cdot 10^5 \) | \( T = 3 \cdot 10^5 \) |
| \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 2 \cdot 10^4 \) | \( burnIn = 3 \cdot 10^4 \) |
| Chib Permutation  |  |  |  |  |
| \( K = 5 \)       | \( T = 10^5 \) | \( T = 10^5 \) | \( T = 10^5 \) | \( T = 10^5 \) |
| \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | - | - |
| Adaptive SMC      |  |  |  |  |
| \( N = 10000 \)   | \( N = 10000 \) | \( N = 10000 \) | \( N = 10000 \) | \( N = 10000 \) |
| \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 10^4 \) |
| Bridge Sampling   |  |  |  |  |
| \( T_1 = 12 \cdot 10^3 \) | \( T_1 = 12 \cdot 10^3 \) | \( T_2 = 12 \cdot 10^3 \) | \( T_2 = 12 \cdot 10^3 \) | \( T_0 = 100 \) |
| \( T_2 = 12 \cdot 10^4 \) | \( T_2 = 12 \cdot 10^4 \) | \( T_0 = 100 \) | \( T_0 = 100 \) | \( burnIn = 5000 \) |
| \( burnIn = 5000 \) | \( burnIn = 5000 \) | \( burnIn = 5000 \) | \( burnIn = 5000 \) | \( burnIn = 5000 \) |
| ChibPartition     |  |  |  |  |
| \( T = 10^5 \)    | \( T = 10^5 \) | \( T = 2 \cdot 10^5 \) | \( T = 3 \cdot 10^5 \) | \( T = 3 \cdot 10^5 \) |
| \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 10^4 \) |
| SIS               |  |  |  |  |
| \( T = 10^4 \)    | \( T = 6 \cdot 10^3 \) | \( T = 7 \cdot 10^3 \) | \( T = 4 \cdot 10^4 \) | \( T = 10^4 \) |
| \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 10^4 \) | \( burnIn = 10^4 \) |
| Arithmetic Mean   |  |  |  |  |
| \( T = 3 \cdot 10^6 \) | \( T = 3 \cdot 10^6 \) | \( T = 3 \cdot 10^6 \) | \( T = 3 \cdot 10^6 \) | \( T = 4 \cdot 10^6 \) |

Table 3: Choice of tuning parameters for Figure 1
### E.2 Figure 4

| $n$ = 1000 | $n$ = 2000 |
|---|---|
| $K = 3$ | $K = 13$ | $K = 3$ | $K = 13$ |

- **Adaptive SMC**
  - $N = 2 \cdot 10^4$
  - $M = 10$
  - $n = 2 \cdot 10^4$
  - $N = 2 \cdot 10^4$
  - $M = 10$

- **ChibPartition**
  - $T = 10^5$
  - $burnIn = 10^4$

- **Bridge Sampling**
  - $T_1 = 12 \cdot 10^3$
  - $T_2 = 12 \cdot 10^3$
  - $T_0 = 100$
  - $burnIn = 5000$

- **SIS**
  - $T = 2 \cdot 10^3$
  - $T = 2 \cdot 10^3$
  - $T = 10^4$
  - $T = 10^4$

**Table 4**: Choice of tuning parameters for Figure 4

### E.3 Figure 5

| $n$ = 6 | $n$ = 36 | $n$ = 82 |
|---|---|---|

- **Chib**
  - $T_1 = 3 \cdot 10^4$
  - $T_1 = 5 \cdot 10^4$
  - $T_1 = 10^5$
  - $burnIn = 2 \cdot 10^3$
  - $burnIn = 5 \cdot 10^3$
  - $burnIn = 10^4$
  - $T_2 = 2 \cdot 10^3$
  - $T_2 = 2 \cdot 10^3$
  - $T_2 = 2 \cdot 10^3$

- **RLR-SIS**
  - $T_1 = 3 \cdot 10^4$
  - $T_1 = 5 \cdot 10^4$
  - $T_1 = 10^5$
  - $burnIn = 2 \cdot 10^3$
  - $burnIn = 5 \cdot 10^3$
  - $burnIn = 10^4$
  - $T_2 = 2 \cdot 10^3$
  - $T_2 = 2 \cdot 10^3$
  - $T_2 = 2 \cdot 10^3$

- **RLR-Prior**
  - $T_1 = 3 \cdot 10^4$
  - $T_1 = 5 \cdot 10^4$
  - $T_1 = 10^5$
  - $burnIn = 2 \cdot 10^3$
  - $burnIn = 5 \cdot 10^3$
  - $burnIn = 10^4$
  - $T_2 = 2.8 \cdot 10^4$
  - $T_2 = 4.5 \cdot 10^4$
  - $T_2 = 9 \cdot 10^4$

- **Harmonic Mean**
  - $T = 2 \cdot 10^4$
  - $T_1 = 5 \cdot 10^4$
  - $T_1 = 10^5$
  - $burnIn = 2 \cdot 10^3$
  - $burnIn = 5 \cdot 10^3$
  - $burnIn = 10^4$

- **Arithmetic Mean**
  - $T = 3 \cdot 10^4$
  - $T = 2 \cdot 10^5$
  - $T = 5 \cdot 10^5$

**Table 5**: Choice of tuning parameters for Figure 5