THE BERGMAN KERNEL ON THE INTERSECTION 
OF TWO BALLS IN $\mathbb{C}^2$

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Abstract. We obtain an asymptotic expansion and some regularity results for the Bergman kernel on the intersection of two balls in $\mathbb{C}^2$.

1. Introduction

Let $B_1 = \{(z_1, z_2); |z_1|^2 + |z_2|^2 < 1\}, B_2 = \{(z_1, z_2); |z_1 - a_1|^2 + |z_2 - a_2|^2 < r^2\}$ be two balls in $\mathbb{C}^2$ such that $\partial B_1, \partial B_2$ intersect real transversally. There exist two complex tangent points $p, q \in \partial B_1 \cap \partial B_2$.

In this paper we obtain an asymptotic expansion for the Bergman kernel of $\Omega := \partial B_1 \cap \partial B_2$. (The Bergman kernel function $K_\Omega$ on $\Omega \times \Omega$ is characterized by the conditions that $K_\Omega(z, \cdot)$ is holomorphic and square-integrable for all $z$, and that $\int_{\Omega} K_\Omega(z, \zeta) f(\zeta) \, dV_\zeta = f(z)$ for all holomorphic square-integrable $f$. $K_\Omega$ satisfies $\overline{K_\Omega(z, \zeta)} = K_\Omega(\zeta, z)$.)

A generic description of our main result runs as follows.

**THEOREM 1.1.** i) For each $\zeta \in \Omega$ the Bergman kernel function $K_\Omega(z, \zeta)$ is holomorphic in a neighborhood of $\Omega \setminus \{p, q\}$.

ii) For $z$ near a complex tangent point $q$ we have an asymptotic expansion of the form

$$K_\Omega(z, \zeta) \sim \sum_j \langle z - q, T a \rangle^{n_j} \langle z - q, a \rangle^{\gamma_j} P_j (\log \langle z - q, a \rangle, \zeta): \tag{1.1}$$

here $a = (a_1, a_2)$; $T a = (\overline{a_2}, -\overline{a_1})$; $\cdot, \cdot$ denotes the hermitian inner product in $\mathbb{C}^n$; the $n_j$ are nonnegative integers; the $\gamma_j$ are (possibly complex) exponents lying in the half-plane Re $\gamma_j > -1$.
\[ \frac{u_j}{2}; \text{ and the } P_j(\log(z - q, a), \zeta) \text{ are polynomials in } \log(z - q, a) \text{ with coefficients varying anti-holomorphically with } \zeta. \]

See Theorem 4.1 below for a more precise description of the expansion; up to a change of coordinates it is equivalent to the simpler version found in (4.6).

The regularity properties (in the Sobolev sense) of the functions \( K_\Omega(\cdot, \zeta), \zeta \in \Omega, \) will be determined by the pattern of exponents above; details are given in §6 below.

The location and geometry of the complex tangent points are discussed in §2. §3 sets up a change of coordinates serving to reveal the symmetries of \( \Omega. \) In §4 we represent the Bergman kernel function of (an image of) \( \Omega \) as a sum of integrals, and we explain how the residue calculus can be used to extract the desired asymptotic expansions, modulo estimates provided in §5.

2. GEOMETRY OF COMPLEX TANGENT POINTS

We set \( \rho^2 = |a_1|^2 + |a_2|^2. \)

In order for \( \partial B_1 \) and \( \partial B_2 \) to intersect non-trivially and real transversally we must have

\[ |1 - \rho| < r < 1 + \rho. \tag{2.1} \]

Complex tangent points of \( \partial B_1 \cap \partial B_2 \) will occur when the radius vectors for the two spheres are \( \mathbb{C} \)-dependent. Since the difference of these two vectors is simply the vector \( a \) joining the two centers, it follows that both radii are \( \mathbb{C} \)-multiples of \( a. \) In particular, if \( z = (z_1, z_2) \) is a complex tangent point then \( z \) is a multiple of \( a. \) Assuming for simplicity that \( a_1 \neq 0, \) the complex tangent points of \( \partial B_1 \cap \partial B_2 \) are contained in \( \left\{ \left( z_1, \frac{a_2}{a_1} z_1 \right); \ z_1 \in \mathbb{C} \right\}. \)

We shall show that there exist precisely two complex tangent points. Let \( \left( z_1, \frac{a_2}{a_1} z_1 \right) \in \partial B_1 \cap \partial B_2 \) be a complex tangent point. Then
\[ |z_1|^2 + \left| \frac{a_2}{a_1} \right|^2 |z_1|^2 = 1 \]

\[ |z_1 - a_1|^2 + \left| \frac{a_2}{a_1} z_1 - a_2 \right|^2 = r^2 \]

\[ \left| \frac{z_1}{a_1} \right| = \frac{1}{\rho} \]

\[ \left| \frac{z_1}{a_1} - 1 \right| = \frac{r}{\rho}. \]

We can write \( \frac{z_1}{a_1} = \frac{1}{\rho} e^{i\chi} \), \(-\pi \leq \chi < \pi\). From the last equation of the above system we obtain

\[ r^2 = 1 + \rho^2 - 2\rho \cos \chi. \]

Since \( \cos \chi = \frac{1 + \rho^2 - r^2}{2\rho} \in (-1, 1) \) (by (2.1)), there are precisely two possible values of \( \chi \) in \((-\pi, \pi)\), \( \chi \neq 0 \). Hence, there exist precisely two complex tangent points.

Let \( \theta \) denote the angle between the two radii; thus \( z - a = re^{i\theta}z \). In particular, \( z_1 - a_1 = re^{i\theta}z_1 \) so that

\[ e^{i\chi} - 1 = \frac{z_1}{a_1} - 1 = re^{i\theta} \frac{z_1}{a_1} = \frac{r}{\rho} e^{i(\theta + \chi)} \]  

(2.2)

and thus \( 1 - \rho e^{-i\chi} = re^{i\theta} \); taking real and imaginary parts we find that

\[ 1 - r \cos \theta = \rho \cos \chi \]

\[ r \sin \theta = \rho \sin \chi. \]  

(2.3)

Note that the values of \( \chi \) and \( \theta \) at \( q \) can be taken to be the negatives of the corresponding values at \( p \).

The parameters \( r \) and \( \theta \) have an interpretation extending to more general situations. Consider smooth real hypersurfaces \( M_1 \) and \( M_2 \) in \( \mathbb{C}^2 \) intersecting real-transversally with a complex tangency at \( z \in M_1 \cap M_2 \). Then for suitable \( \theta \), the rotation \( R_\theta \) of \( T_z \mathbb{C}^2 \) given by multiplication by \( e^{i\theta} \) will map \( T_z M_1 \) to \( T_z M_2 \) and \( T_z M_1/H_z \) to \( T_z M_2/H_z \), where \( H_z = T_z M_j \cap T_z M_2 \) is the maximal complex subspace of both \( M_j \).

The Levi-form \( \mathcal{L}_j \) of \( M_j \) at \( z \) is a hermitian \( T_z M_1/H_z \)-valued form on \( H_z \); it can be defined by the equation

\[ \mathcal{L}_j(X_z) \equiv [X, R_{\pi/2}X]_z \mod H_z \]
for all smooth vector fields on $M_j$ with values in the maximal complex subspace of $TM_j$. Since $\dim \mathbb{C} \mathcal{H}_z = 1$, if $\mathcal{L}_2$ is non-degenerate then there is $r \in \mathbb{R}$ so that $\mathcal{L}_1 = rR_\theta \mathcal{L}_2$. The parameters $r, \theta$ defined this way match the ones already defined in the special case of spheres.

3. Projective transformation

Let

\[
p := \left( p_1, \frac{a_2}{a_1} p_1 \right), \quad p_1 = \frac{1}{\rho} e^{i\chi},
\]
\[
q := \left( q_1, \frac{a_2}{a_1} q_1 \right), \quad q_1 = \frac{1}{\rho} e^{-i\chi},
\]

$-\pi < \chi < \pi$, $\chi \neq 0$ be the two complex tangent points.

We view the balls as embedded in $\mathbb{C}P^2$. If $(z_1 : z_2 : z_3)$ are the homogeneous coordinates in $\mathbb{C}P^2$ then the equations of the two balls will become

\[
|z_1|^2 + |z_2|^2 - |z_3|^2 < 0
\]

\[
|z_1 - a_1 z_3|^2 + |z_2 - a_2 z_3|^2 - r^2 |z_3|^2 < 0.
\]

Let $f : \mathbb{C}P^2 \to \mathbb{C}P^2$ be a projective transformation such that

\[
f(0 : 0 : 1) = \left( q_1 : \frac{a_2}{a_1} q_1 : 1 \right)
\]
\[
f(0 : 1 : 0) = \left( p_1 : \frac{a_2}{a_1} p_1 : 1 \right).
\]

Let us assume that

\[
f(w_1 : w_2 : w_3) = (z_1 : z_2 : z_3)
\]

where

\[
z_1 = a_{11} w_1 + a_{12} w_2 + a_{13} w_3
\]
\[
z_2 = a_{21} w_1 + a_{22} w_2 + a_{23} w_3
\]
\[
z_3 = a_{31} w_1 + a_{32} w_2 + a_{33} w_3. \quad (3.1)
\]

Due to the above constraints we see that the matrix $A$ of the transformation $f$ is
\[
\begin{pmatrix}
a_{11} & \lambda p_1 & \mu q_1 \\
a_{21} & \lambda \frac{a_2}{a_1} p_1 & \mu \frac{a_2}{a_1} q_1 \\
a_{31} & \frac{1}{\lambda} & \frac{1}{\mu}
\end{pmatrix}.
\]

The equation of the first ball shall be transformed under \( f \) to

\[
(|a_{11}|^2 + |a_{21}|^2 - |a_{31}|^2)|w_1|^2
\]

\[
+ 2 \text{Re} \left[ \lambda \left\{ p_1 \left( a_{11} + a_{21} \left( \frac{a_2}{a_1} \right) \right) - a_{31} \right\} w_1 \overline{w_2} \right]
\]

\[
+ 2 \text{Re} \left[ \mu \left\{ q_1 \left( \overline{a}_{11} + \overline{a}_{21} \frac{a_2}{a_1} \right) - \overline{a}_{31} \right\} w_3 \overline{w_1} \right]
\]

\[
+ 2 \text{Re} \left[ \lambda \overline{p_1} \left( p_1 \overline{q_1} + \left| \frac{a_2}{a_1} \right|^2 p_1 \overline{q_1} - 1 \right) w_2 \overline{w_3} \right]
\]

\[
< 0.
\]

If we require that the \( w_1 \) direction be tangent to \( \partial B_1 \) at \((0 : 1 : 0)\) and at \((0 : 0 : 1)\) then

\[
\lambda \left\{ p_1 \left( a_{11} + a_{21} \left( \frac{a_2}{a_1} \right) \right) - a_{31} \right\} = 0
\]

\[
\mu \left\{ q_1 \left( \overline{a}_{11} + \overline{a}_{21} \frac{a_2}{a_1} \right) - \overline{a}_{31} \right\} = 0.
\]

Since \( \lambda, \mu \in \mathbb{C}^* \), \( p_1 \neq q_1 \) the above system yields

\[
a_{31} = 0
\]

\[
a_{11} = -a_{21} \left( \frac{a_2}{a_1} \right).
\]

The matrix \( A \) of the transformation \( f \) shall become

\[
\begin{pmatrix}
-a_{21} & \lambda p_1 & \mu q_1 \\
a_{21} & \lambda \frac{a_2}{a_1} p_1 & \mu \frac{a_2}{a_1} q_1 \\
0 & \frac{1}{\lambda} & \frac{1}{\mu}
\end{pmatrix}
\]

\( (3.2) \)

with \( \lambda, \mu \in \mathbb{C}^* \).
We shall normalize the coefficient $|a_{11}|^2 + |a_{21}|^2 - |a_{31}|^2 = \frac{\rho^2}{|a_{11}|^2} |a_{21}|^2$ of $|w_1|^2$ such that it equals 1. This will imply that

$$|a_{21}| = \frac{|a_1|}{\rho}. \quad (3.3)$$

We shall also choose $\lambda, \mu \in \mathbb{C}^*$ such that

$$\lambda \bar{\mu} \left\{ p_1 \bar{q}_1 \left( 1 + \left| \frac{a_2}{a_1} \right|^2 \right) - 1 \right\} = -1$$

or equivalently (using the fact that $\frac{p_1}{a_1} = \frac{1}{\rho} e^{i\chi}$, $\frac{q_1}{a_1} = \frac{1}{\rho} e^{-i\chi}$)

$$\lambda \bar{\mu} (e^{2i\chi} - 1) = -1. \quad (3.4)$$

We can choose

$$\lambda = \frac{1}{e^{i\chi} - 1} = \frac{e^{-i\frac{\chi}{2}}}{2i \sin \frac{\chi}{2}}, \quad \mu = \frac{-1}{e^{-i\chi} + 1} = \frac{-e^{i\frac{\chi}{2}}}{2 \cos \frac{\chi}{2}}. \quad (3.5)$$

Note: This is possible since $|\cos \chi| < 1$.

Thus, the first ball is now described by the equation

$$|w_1|^2 - 2 \Re w_2 \bar{w}_3 < 0. \quad (3.6)$$

The equation of the second ball is transformed under $f$ to

$$|a_{21}|^2 \left( 1 + \left| \frac{a_2}{a_1} \right|^2 \right) |w_1|^2$$

$$+ 2 \Re \left[ \lambda \bar{\mu} \left\{ (p_1 - a_1)(\bar{q}_1 - a_1) \left( 1 + \left| \frac{a_2}{a_1} \right|^2 \right) - r^2 \right\} w_2 \bar{w}_3 \right]$$

$$< 0.$$

From the previous normalization we have that $|a_{21}|^2 (1 + |\frac{a_2}{a_1}|^2) = 1$, so the equation of the second ball becomes

$$|w_1|^2 + 2 \Re \phi w_2 \bar{w}_3 < 0 \quad (3.7)$$

with

$$\phi = \lambda \bar{\mu} \left\{ (p_1 - a_1)(\bar{q}_1 - a_1) \left( 1 + \left| \frac{a_2}{a_1} \right|^2 \right) - r^2 \right\}$$

$$= \lambda \bar{\mu} \left\{ \rho^2 \left( \frac{p_1}{a_1} - 1 \right) \left( \frac{q_1}{a_1} - 1 \right) - r^2 \right\}.$$
Recalling from (2.2) that
\[
\frac{p_1}{a_1} - 1 = \frac{r}{\rho} e^{i(\theta + \chi)},
\]
\[
\frac{q_1}{a_1} - 1 = \frac{r}{\rho} e^{-i(\theta + \chi)},
\]
and using (3.4), we can rewrite \(\phi\) as
\[
\phi = -r^2 e^{2i(\theta + \chi)} - 1 = -r^2 e^{i\theta} \sin(\theta + \chi) / \sin \chi.
\]

Using the angle addition formula and the identities in (2.3) we find that
\[
\phi = -re^{i\theta}.
\]

Let \(B_1', B_2'\) be the preimages of \(B_1, B_2\) under the projective transformation \(f\) and standard normalizations. Let \(\Omega' = B_1' \cap B_2'\). Since \(w_3 \neq 0\) in \(\Omega'\) (in view of (3.6) and (3.7)), setting \(\tilde{w}_1 = \frac{w_1}{w_3}, \tilde{w}_2 = \frac{w_2}{w_3}\) we may summarize our work as follows.

**Proposition 3.1.** The inverse of the projective transformation induced by the matrix (3.2) subject to (3.3) and (3.5) maps \(\Omega\) to
\[
\hat{\Omega}' = \text{affine } \Omega' = \{ (\tilde{w}_1, \tilde{w}_2); |\tilde{w}_1|^2 < \min\{2 \text{ Re } \tilde{w}_2, -2 \text{ Re } \phi \tilde{w}_2\}\}.
\]

**Proof of Theorem 1.1, part (i).** The domain \(\hat{\Omega}'\) admits the \(\mathbb{R} \times S^1\) action
\[
(s, \theta) \cdot (\tilde{w}_1, \tilde{w}_2) = (se^{i\theta} \tilde{w}_1, s^2 \tilde{w}_2).
\]
This action pulls back to a \(\mathbb{R} \times S^1\) action on \(\Omega\).

The result now follows by application of [Ba1, proof of Theorem 3] to this action. (It also follows from later computations in this paper.) \(\square\)

For future reference we note that the inverse of the map (3.1) is given by
\[
\begin{align*}
  w_1 &= -\frac{z_1 a_2 - z_2 a_1}{a_1 a_21 (1 + |\frac{a_2}{a_1}|^2)} \\
  w_2 &= \frac{z_1 + (\frac{a_2}{a_1}) z_2 - q_1 (1 + |\frac{a_2}{a_1}|^2) z_3}{\lambda (p_1 - q_1) (1 + |\frac{a_2}{a_1}|^2)} \\
  w_3 &= \frac{z_1 + (\frac{a_2}{a_1}) z_2 - p_1 (1 + |\frac{a_2}{a_1}|^2) z_3}{-\mu (p_1 - q_1) (1 + |\frac{a_2}{a_1}|^2)}. 
\end{align*}
\]
4. The Bergman kernel of the intersection of two balls

Let $\Phi : \tilde{\Omega}' \to \mathbb{C}^2$ be the transformation defined by $\Phi(\tilde{w}_1, \tilde{w}_2) = (t_1, t_2)$ with

\[
\begin{align*}
t_1 &= \frac{1}{\sqrt{2}} \tilde{w}_1 \tilde{w}_2^{-\frac{1}{2}}, \quad t_2 = u + iv = \log \tilde{w}_2 \\
\tilde{w}_1 &= \sqrt{2} e^{t_2^2} t_1, \quad \tilde{w}_2 = e^{t_2}.
\end{align*}
\]

Then $D' := \Phi(\tilde{\Omega}')$ is defined by the inequality

\[
\{|t_1|^2 < \psi_{r,\theta}(v)\}.
\]

where $\psi_{r,\theta}(v) = \min\{\cos v, r \cos(v + \theta)\}$.

$D'$ is a Hartogs domain invariant under the rotations $(t_1, t_2) \mapsto (e^{i\alpha}t_1, t_2)$. By Fourier expansion the Bergman space $\mathcal{H}(D')$ (the space consisting of all square-integrable, holomorphic functions in $D'$) admits an orthogonal decomposition

\[
\mathcal{H}(D') = \bigoplus \mathcal{H}_j(D')
\]

where $\mathcal{H}_j(D')$ is the subspace consisting of all square-integrable, holomorphic functions $f$ in $D'$ that satisfy $f(e^{i\alpha}t_1, t_2) = e^{ij\alpha} f(t_1, t_2)$. Functions with this property are of the form $f(t_1, t_2) = t_1^j f_1(t_2)$, $f_1$ holomorphic in $t_2$. The Bergman kernel $K_{D'}(t, \tau)$ satisfies

\[
K_{D'}(t, \tau) = \sum_{j \geq 0} K_j(t, \tau),
\]

where $K_j(t, \tau)$ is the reproducing kernel for $\mathcal{H}_j(D')$.

Using an argument similar to the one in Section 1 of [Ba2] and noting that for $f, g$ holomorphic functions in $t_2$ we have

\[
\int_{D'} f(t_2) t_1^j \overline{g(t_2)} t_1^k dV = \begin{cases} 0 & j \neq k, \\ \pi \int_{v_{\text{min}}}^{v_{\text{max}}} f(t_2) \overline{g(t_2)} \psi_{r,\theta}^{j+1}(v) dA & j = k, \end{cases}
\]

we find that
\[ K_D((t_1, t_2), (\tau_1, \tau_2)) \]

\[ = \frac{1}{2\pi^2} \sum_{j \geq 0} t_1^j \tau_1^j (j + 1) \int_{-\infty}^{\infty} \frac{e^{i(t_2-\tau_2)(\xi)}}{\psi_{r,\theta}^{j+1}(v) e^{-2v\xi}} dv \]

\[ = \frac{1}{4\pi^2} \sum_{j \geq 0} t_1^j \tau_1^j (j + 1) \int_{-\infty}^{\infty} \frac{e^{i(t_2-\tau_2)(\xi)}}{\psi_{r,\theta}^{j+1}(v) e^{-v\xi}} dv. \]

(4.1)

To simplify notation we shall write from now on \((v_{\min}, v_{\max}) := J\) and \(\eta_{j+1}(v) := -(j + 1) \log \psi_{r,\theta}(v)\).

Let us assume for the moment that we can apply contour integration arguments to each one of the above integrals for appropriate \(t_2, \tau_2\). Then for \(h > 0\) we have:

\[ \int_{-\infty}^{\infty} \frac{e^{i(t_2-\tau_2)(\xi)}}{j \int e^{(-v\xi-\eta_{j+1}(v))}} dv = -2\pi i \sum_{-h < \text{Im} \xi < 0} \text{Res} \left( \frac{e^{i(t_2-\tau_2)(\xi)}}{F_{j+1}(\cdot)}, \xi \right) + \]

\[ + \int_{-\infty}^{\infty} \frac{e^{i(t_2-\tau_2)(-x-i\hbar)}}{F_{j+1}(-x-i\hbar)} dx \]

(4.2)

where

\[ F_{j+1}(\xi) := \int_{j} e^{-v\xi-\eta_{j+1}(v)} dv \]

(4.3)

\((\text{hence } F_{j+1}(-x-i\hbar) := \int_{j} e^{iv\hbar e^{vx-\eta_{j+1}(v)}} dv\).

We will see below in Corollary 5.4 that the union of the zero sets of the \(F_{j+1}\) is finite (counting multiplicity) in each strip \(-h \leq \text{Im} \xi \leq 0\).

In particular, for all but a discrete set of \(h\), we have

\[ F_{j+1}(-x-i\hbar) \neq 0 \text{ for all } x \in \mathbb{R}, \ j \geq 0 \]

(4.4)

so that the final integrand in (4.2) does not suffer a vanishing denominator.

In §5 we shall show that the use of the residue theorem above is valid, and in Proposition 6.1 we show that when the residue expansions (4.2) are substituted into (4.1), the sum of integrals is an error term of
magnitude $O(e^{t_2 h/2})$ as $\text{Re} t_2 \to -\infty$, uniformly as $\tau$ ranges over any compact subset of $D'$.

Using the last remark and applying the transformation formula

$$K_{\hat{\Omega}'}(\tilde{w}, \tilde{\omega}) = \frac{1}{2(\tilde{w}_2 \tilde{\omega}_2)^{\frac{1}{2}}} K_{D'}(\Phi(\tilde{w}), \Phi(\tilde{\omega})) \quad (4.5)$$

for the Bergman kernel we obtain:

$$K_{\hat{\Omega}'}((\tilde{w}_1, \tilde{w}_2), (\tilde{\omega}_1, \tilde{\omega}_2))$$

$$= \frac{1}{4\pi i (\tilde{w}_2 \tilde{\omega}_2)^{\frac{1}{2}}} \sum_{j \geq 0} \sum_{-h < \text{Im} \xi < 0} \frac{j+1}{(\tilde{w}_1 \tilde{\omega}_1)^j}{\frac{1}{(\tilde{w}_2 \tilde{\omega}_2)^{\frac{j+1}{2}}}} \text{Res} \left( \frac{e^{i(\log \tilde{w}_2 - \log \tilde{\omega}_2)(\tau)}}{F_{j+1}(\xi)} \right) + O\left(\frac{h-3}{\tilde{w}_2}^2\right)$$

as $\tilde{w} \to 0$ in $\hat{\Omega}'$, uniformly as $\tilde{\omega}$ ranges over any compact subset of $\hat{\Omega}'$.

If the zeroes of $F_{j+1}(\xi)$ in the strip $-h < \text{Im} \xi < 0$ are all simple, the expansion may be written in the form

$$K_{\hat{\Omega}'}((\tilde{w}_1, \tilde{w}_2), (\tilde{\omega}_1, \tilde{\omega}_2))$$

$$= \frac{1}{(\tilde{w}_2 \tilde{\omega}_2)^{\frac{1}{2}}} \sum_{j \geq 0} \sum_{k} c_{j,k} \left(\frac{\tilde{w}_1 \tilde{\omega}_1}{\tilde{w}_2 \tilde{\omega}_2}\right)^{\frac{j+1}{2}} \left(\frac{\tilde{w}_2}{\tilde{\omega}_2}\right)^{\frac{j+1}{2} + \xi_{j,k}}$$

$$+ O\left(\frac{h-3}{\tilde{w}_2}^2\right),$$

where $\xi_{j,k}$ is an enumeration of zeros in the strip $-h < \text{Im} \xi < 0$, and the $c_{j,k}$ are constants that arise from the residue principle.

In the general case, if $F_{j+1}(\xi)$ has a root of multiplicity $m_k$ at $\xi_{j,k}$ then our expansion looks like:

$$K_{\hat{\Omega}'}((\tilde{w}_1, \tilde{w}_2), (\tilde{\omega}_1, \tilde{\omega}_2))$$

$$= \frac{1}{(\tilde{w}_2 \tilde{\omega}_2)^{\frac{1}{2}}} \sum_{j \geq 0} \sum_{k} \frac{(\tilde{w}_1 \tilde{\omega}_1)^j}{(\tilde{w}_2 \tilde{\omega}_2)^{\frac{1}{2}}} \left(\frac{\tilde{w}_2}{\tilde{\omega}_2}\right)^{\frac{j+1}{2} + \xi_{j,k}} P_{j,k} \left(\log \frac{\tilde{w}_2}{\tilde{\omega}_2}\right) +$$

$$+ O\left(\frac{h-3}{\tilde{w}_2}^2\right), \quad (4.6)$$

where $P_{j,k}$ is a polynomial of degree $m_k - 1$. 
We are only a step away from our original goal of obtaining the asymptotic expansion for the Bergman kernel on the intersection of the two balls in $\mathbb{C}^2$. Using (3.8) and recalling that

$$\tilde{w}_1 = \frac{w_1}{w_3}, \quad \tilde{w}_2 = \frac{w_2}{w_3},$$

we can rewrite $\tilde{w}_1$, $\tilde{w}_2$ as functions of $z_1, z_2, z_3$. Setting $z_3 = 1$ we can obtain a biholomorphic map

$$\Psi : B_1 \cap B_2 \to \hat{\Omega}'$$

$$(z_1, z_2) \mapsto (\tilde{w}_1, \tilde{w}_2)$$

where

$$\tilde{w}_1 = \frac{2ir\mu \sin \theta}{a_1a_{21}(1 + |a_2|^2)} \left(\langle z - q, a \rangle - 2ir\sin \theta\right)$$

$$\tilde{w}_2 = -\frac{r}{\lambda} \left(\langle z - q, a \rangle - 2ir\sin \theta\right).$$

Applying the transformation formula for the Bergman kernel we obtain our main result.

**THEOREM 4.1.** Let $h > 0$ satisfy (4.4). Then

$$K_{\Omega}(z, \zeta)$$

$$= \sum \sum_{j} \sum_{-h < \Im \xi < 0} \frac{\langle z - q, a \rangle^j}{\langle \zeta - q, a \rangle^{\frac{j}{2} + \frac{1}{2}} \langle z - q, a \rangle^{\frac{j}{2} + \frac{1}{2}} \langle \zeta - q, a \rangle^{\frac{j}{2} + \frac{1}{2}}} \cdot \frac{\langle \zeta - q, a \rangle - 2ir\sin \theta}{\langle z - q, a \rangle - 2ir\sin \theta} \cdot \frac{\langle z - q, a \rangle^{2}}{\langle \zeta - q, a \rangle^{2}} \cdot \frac{\langle \zeta - q, a \rangle^{2}}{\langle z - q, a \rangle^{2}}.$$

$$\cdot P_{j,k} \left( \log \frac{\langle z - q, a \rangle}{\langle \zeta - q, a \rangle} \frac{\langle \zeta - q, a \rangle - 2ir\sin \theta}{\langle z - q, a \rangle - 2ir\sin \theta} \right) + O \left( \langle z - q, a \rangle^{-\frac{h}{2}} \right) \quad (4.7)$$

as $z \to q$, uniformly as $\zeta$ ranges over any compact subset of $\Omega$; here

- $q$ is a complex tangent point of the intersection of the two balls;
- $a = (a_1, a_2)$;
• \( T a := (a_2, -a_1) \);
• \( \{\xi_{j,k}\} \) is an enumeration of the zeroes of \( F_{j+1} \) in the strip \( -h < \text{Im} \xi < 0 \);

and

• \( P_{j,k} \) is a polynomial of degree one less than the multiplicity of \( F_{j+1} \) at \( \xi_{j,k} \).

For brevity we set

\[
J := (v_{\min}, v_{\max}) = \left\{ \left(-\frac{\pi}{2}, \frac{\pi}{2} - \theta\right), \quad \theta > 0 \right\} \cup \left\{ \left(-\frac{\pi}{2} - \theta, \frac{\pi}{2}\right), \quad \theta < 0 \right\}.
\] (4.8)

**Lemma 4.2.** \( F_{j+1} (\xi) \) does not vanish in the strip \(|\text{Im} \xi| \leq \frac{\pi}{|J|}\).

**Proof.** Let \( v^* \) denote the midpoint of \( J \). Then

\[
\text{Re } e^{v^* \xi} F_{j+1} (\xi) = \int_J \text{Re } e^{(v^* - v)\xi - \eta_{j+1}(v)} \, dv > 0
\]

(since the integrand is positive).

\[ \square \]

**Proof of Theorem 1.1.** The expansion (1.1) is obtained by expanding the non-logarithmic factors in (4.7) in powers of \( \langle z - q, T a \rangle \) or \( \langle z - q, a \rangle \). The assertion about the location of the \( \gamma_j \) follows from Lemma 4.2. \( \square \)

If we make a change of coordinates that sends

\[
(0,0) \mapsto (q_1, q_2) \quad (1,0) \mapsto (p_1, p_2)
\]

then in the new coordinates the Bergman kernel will look like

\[
K((\zeta_1, \zeta_2); (\zeta_1', \zeta_2')) \sim \frac{1}{(\zeta_1(\zeta_1 - 1)\zeta_1'((\zeta_1'-1))^2} \sum_{j,k} \left( \frac{\zeta_2^2\zeta_2'^2}{(\zeta_1(\zeta_1 - 1)\zeta_1'((\zeta_1'-1)))} \right)^{\frac{1}{2}} \left( \frac{\zeta_1 \zeta_1' - 1}{\zeta_1 \zeta_1'} \right)^{\frac{i}{2}} P_{j,k} \left( \log \frac{\zeta_1 ((\zeta_1'-1))}{(\zeta_1'-1) \zeta_1} \right)
\]
in a neighborhood of \((0,0)\).

The results from [Ba3] can be used to provide a more operator-theoretic approach to obtaining such asymptotic expansions.
We conclude this section with some remarks on the location of the zeroes \( \xi_{j,k} \) in special cases.

First, in the special case (not otherwise allowed in this paper) \( \theta = 0 \) where \( \Omega \) is a ball we have

\[
F_{j+1}(\xi) = \begin{cases} 
2(j + 1)! \cosh(\pi \xi/2) & j + 1 \text{ odd;} \\
(1 + \xi^2)(9 + \xi^2) \cdots ((j + 1)^2 + \xi^2), & j + 1 \text{ even.}
\end{cases}
\]

In particular, the zeroes of each \( F_{j+1} \) are simple.

It follows by use of the argument principle that in any strip \( -h \leq \text{Im} \xi \leq 0 \), all the \( F_{j+1} \) have simple zeroes provided that \( |\theta| \) does not exceed \( \theta_0(h) \).

Let us now consider the case \( r = 1, \theta \neq 0 \). In this case we have

\[
F_1(\xi) = \begin{cases} 
\frac{(1+ie^{-\pi \theta/2}\xi+(\xi^2))}{(1-ie^{-\pi \theta/2}\xi+(\xi^2))} \cdot e^{\pi \xi^2}, & \theta > 0 \\
\frac{(1+ie^{\pi \theta/2}\xi+(\xi^2))}{(1-ie^{\pi \theta/2}\xi+(\xi^2))} \cdot e^{-\pi \xi^2}, & \theta < 0.
\end{cases}
\]

The zeroes of \( F_1 \) are given by

\[
i\xi_{0,n} = \begin{cases} 
-1 + \frac{2(n+1)\pi}{\pi - |\theta|}, & n \text{ odd;} \\
1 + \frac{2n\pi}{\pi - |\theta|}, & n \text{ even.}
\end{cases}
\quad (4.9)
\]

Using repeated integration by parts, the general form of the \( F_{j+1} \) for \( r = 1 \) can be given as follows.

(i) If \( j + 1 \) is odd, then

\[
F_{j+1}(\xi) = Ce^{\theta \xi} \frac{\cosh((\pi - \theta)\xi/2) - P_\theta^j(\xi)}{(1 + \xi^2)(9 + \xi^2) \cdots ((j + 1)^2 + \xi^2)}
\]

where \( P_\theta^j(\xi) \) is an even polynomial in \( \xi \), of degree \( j \), whose coefficients depend on \( \theta \) and \( C \) is some absolute constant.

(ii) If \( j + 1 \) is even, then

\[
F_{j+1}(\xi) = C' e^{(\pi - \theta)\xi} \frac{\sinh((\pi - \theta)\xi/2) - Q_\theta^j(\xi)}{\xi(4 + \xi^2)(16 + \xi^2) \cdots ((j + 1)^2 + \xi^2)}
\]
where $Q^\theta_j(\xi)$ is an odd polynomial in $\xi$, of degree $j$, whose coefficients depend on $\theta$ and $C'$ is some absolute constant.

The above formulas indicate that the zeroes lying off the imaginary axis will occur in pairs $\xi, -\xi$; the pair will always lead to terms of equal strength (as far as it concerns estimates) so both terms should be considered together.

We cannot provide an explicit formula for the first zero of $F_2$, but for $r = 1, \theta \sim 0$ the first zero can be approximated by

$$i\xi_{1,1} = 4 + \frac{8}{\pi}|\theta| + O(\theta^2).$$

Similarly, for $r = 1, \theta = \pi - \epsilon, \epsilon \sim 0^+$, we find by rescaling $\xi$ that the conjugate pair $i\xi_{1,1}, i\xi_{1,2}$ can be approximated by

$$\frac{14.995\ldots \pm i5.537\ldots}{\epsilon} + O(1).$$

For general values of $\theta$ the roots can at least be explored numerically.

Figure 1 shows a portion of the root pattern for $r = 1, \theta = \frac{\pi}{2}$. The arrows connect zeroes of $F_j$ to nearby zeroes of $F_{j+1}$. The zeroes of $F_1$ lie on the imaginary axis at $(8m \pm 1)i$. Passing to $F_2$, the first four zeroes (at least) are converted into two root pairs, symmetric across the imaginary axis. As $j$ increases further, the roots move down and out.
When $r > 1$ we can use as before repeated integration by parts to compute $F_{j+1}$ – however there won’t be such nice formulas as above. $F_{j+1}$ can be written as a finite sum of exponential functions

$$
\sum_j e^{a_j \frac{\xi^2}{2}} P_j(\xi)
$$

where the $P_j(\xi)$ are rational functions in $\xi$, whose coefficients depend on $\theta, r$.

When $r > 1$, $\theta \sim 0$ we can get the following information

$$
i\xi_0, k = 1 + 2k + \frac{2r k (k + 1)}{(r - 1)\pi} \theta |\theta| + O(\theta^3)
i\xi_1, k = 2 + 2k + O(\theta^3),
$$

while when $r > 1$, $\theta = \pi - \epsilon$, $\epsilon \sim 0^+$ we can rescale as before to obtain

$$
i\xi_0, k = \frac{a_k}{\epsilon} + O(1)i\xi_1, k = \frac{b_k}{\epsilon} + O(1).
$$

For $\theta$ close enough to zero, Proposition 5.1 below can be combined with a Rouché’s theorem argument to show that $\xi_{0,1}$ will have the largest imaginary part among roots in the lower half-plane and thus will provide the key to regularity properties of the kernel function. (See §6 below.)

5. ESTIMATES FOR FOURIER-LAPLACE TRANSFORMS OF LOG-CONVEX FUNCTIONS

We would like to use Residue Calculus to obtain asymptotic expansions for every $j$ for the integrals

$$
\int_{-\infty}^{\infty} e^{i((\xi_2 - \tau_2)\frac{\xi^2}{2})} d\xi
$$

here $F_{j+1}$ is the function defined in (4.3), in which $\eta_{j+1}(v)$ is the piecewise $C^2$, strictly convex function $-(j + 1) \log \psi_{r,0}(v)$. 
For convenience we will focus on the case $\theta > 0$. Also, we assume that $(t_1, t_2) \in D', (\tau_1, \tau_2) \in D'$; it follows that

$$\text{Im}(t_2 - \tau_2) \in (-\pi, \pi - 2\theta). \quad (5.2)$$

We are interested in the behavior of the above integrals as $\text{Re } t_2 \to -\infty$. To get started we shall need lower bounds for $|F_{j+1}(x - ih)| := \left| \int_J e^{ih} e^{x - \eta_{j+1}(v)} dv \right|$. 

Recall that

$$\eta_{j+1}(v) = \begin{cases} -(j + 1) \log \cos v & -\frac{\pi}{2} < v \leq v_0 \\ -(j + 1) \log[r \cos(v + \theta)] & v_0 \leq v < \frac{\pi}{2} - \theta \end{cases},$$

where $v_0 = \arctan \left[ \frac{r \cos \theta - 1}{r \sin \theta} \right]$, i.e. $v_0$ is the point in $J$ where $\cos v$ and $r \cos(v + \theta)$ intersect.

For convenience, we set $\eta_{j+1}(v) = +\infty$ for $v \notin J$.

We are going to use the Legendre transform

$$\tilde{\eta}_{j+1}(\xi) := \max\{v\xi - \eta_{j+1}(v); v \in J\}$$

of $\eta_{j+1}$.

At the points where $\eta_{j+1}$ is differentiable we can compute the Legendre transform using differential calculus (see [Hö], pages 16-19).

We have:

$$\tilde{\eta}_{j+1}(\xi) = \begin{cases} \xi \arctan \frac{\xi}{j+1} - \frac{j+1}{2} \log(1 + (\frac{\xi}{j+1})^2) & \text{for } \xi < (j + 1) \tan v_0; \\ v_0 \xi + (j + 1) \log \cos v_0 & \text{for } (j + 1) \tan v_0 \leq \xi \leq (j + 1) \tan(v_0 + \theta); \\ \xi(\arctan \frac{\xi}{j+1} - \theta) - \frac{j+1}{2} \log(1 + (\frac{\xi}{j+1})^2) + \log r^{j+1} & \text{for } \xi > (j + 1) \tan(v_0 + \theta). \end{cases} \quad (5.3)$$

Since $\eta_{j+1}$ is strictly convex, there is a unique $\mu_{j+1}(\xi)$ satisfying

$$\tilde{\eta}_{j+1}(\xi) = \xi \mu_{j+1}(\xi) - \eta_{j+1}(\mu_{j+1}(\xi));$$
in fact,
\[ \mu_{j+1}(\xi) = \begin{cases} \arctan \frac{\xi}{j+1} & \xi < (j+1) \tan v_0 \\ v_0 & (j+1) \tan v_0 \leq \xi \leq (j+1) \tan(v_0 + \theta) \\ \arctan \frac{\xi}{j+1} - \theta & \xi > (j+1) \tan(v_0 + \theta). \end{cases} \] (5.4)

We shall show the following:

**PROPOSITION 5.1.** Let \( \tilde{\eta}_{j+1} := \max \{ v \xi - \eta(v); v \in J \} \) denote the Legendre transform of \( \eta_{j+1} \). Then

i) For every \( h_0 > 0 \), there exists a positive constant \( L = L(h_0) \) such that for \( j \geq j_0(h_0, |J|) \) sufficiently large, \( 0 \leq h \leq h_0 \), and \( x < (j+1) \tan v_0 \) or \( x > (j+1) \tan(v_0 + \theta) \), we have

\[ |F_{j+1}(-x - ih)| \geq \frac{L}{\sqrt{\eta''_{j+1}(\mu_{j+1}(x))}} e^{\tilde{\eta}_{j+1}(x)}. \] (5.5)

ii) For every \( h_0 > 0 \), there exists a positive constant \( K = K(h_0) \) such that for \( j \geq j_0(h_0, |J|) \) sufficiently large, \( 0 \leq h \leq h_0 \), and \( (j+1) \tan v_0 \leq x \leq (j+1) \tan(v_0 + \theta) \), we have:

\[ |F_{j+1}(-x - ih)| \geq \frac{K}{j+1} e^{\tilde{\eta}_{j+1}(x)}. \]

**Remark:** For any \( j \), i) still holds if we take \( |x| > x_0(j, h_0, |J|) \) sufficiently large.

**LEMMA 5.2.** There is a constant \( C_1 > 1 \) independent of \( v \) and \( j \) so that

\[ C_1^{-1} \leq \frac{\eta''_{j+1}(v)}{j+1} (\text{dist}(v, \partial J))^2 \leq C_1 \] (5.6)

for \( v \in J \setminus \{v_0\} \).

**Proof.** This follows easily from

\[ \eta''_{j+1}(v) = \begin{cases} (j+1) \sec^2 v, & -\frac{\pi}{2} < v < v_0; \\ (j+1) \sec^2(v + \theta), & v_0 < v < \frac{\pi}{2} - \theta. \end{cases} \] (5.7)
**Lemma 5.3.** There exists a constant $C_2 \geq 1$ independent of $v^\sharp, j$ such that

$$\max\{\eta''_{j+1}(v) ; |v - v^\sharp| \leq \frac{1}{2} \text{dist}(v^\sharp, \partial J), v \neq v_0\} \leq C_2 \min\{\eta''_{j+1}(v) ; |v - v^\sharp| \leq \frac{1}{2} \text{dist}(v^\sharp, \partial J), v \neq v_0\}$$

for all $v^\sharp \in J$.

**Proof.** This follows easily from Lemma 5.2, setting $C_2 = 9C_2^2$.

**Proof of Proposition 5.1, part (i).** We start by noting that

$$|F_{j+1}(-x - ih)| \geq \left| \int_{\mu_{j+1}(x) - h_0^{-1}}^{\mu_{j+1}(x) + h_0^{-1}} e^{\mu_{j+1}(x) - \eta_j + 1(v)} dv \right| - \int_{|v - \mu_{j+1}(x)| > h_0^{-1}} e^{\mu_{j+1}(x) - \eta_j + 1(v)} dv.$$

There is no significant cancellation in the first term of the right hand side of the above inequality; in fact we have

$$\left| \int_{\mu_{j+1}(x) - h_0^{-1}}^{\mu_{j+1}(x) + h_0^{-1}} e^{\mu_{j+1}(x) - \eta_j + 1(v)} dv \right| \geq \Re \int_{\mu_{j+1}(x) - h_0^{-1}}^{\mu_{j+1}(x) + h_0^{-1}} e^{i(v - \mu_{j+1}(x)) h} e^{\mu_{j+1}(x) - \eta_j + 1(v)} dv$$

$$\geq \frac{1}{2} \int_{\mu_{j+1}(x) - h_0^{-1}}^{\mu_{j+1}(x) + h_0^{-1}} e^{\mu_{j+1}(x) - \eta_j + 1(v)} dv$$

since $\cos \alpha > \frac{1}{2}$ for $|\alpha| \leq 1$.

Let $\epsilon = \epsilon(x, h_0, j) > 0$. We will determine an explicit value of $\epsilon$ later on, but we will require that

$$\epsilon(x, h_0, j) \leq \frac{1}{2} \text{dist}(\mu_{j+1}(x), \partial J) \quad (5.8)$$

and

$$\epsilon(x, h_0, j) \leq \frac{1}{2h_0} \quad (5.9)$$

Thus in particular
\[
|F_{j+1}(-x - ih)| \geq \frac{1}{2} \int_{\mu_{j+1}(x) - h_0^{-1}}^{\mu_{j+1}(x) + h_0^{-1}} e^{vx - \eta_{j+1}(v)} dv - \\
\quad \int_{v - \mu_{j+1}(x) > h_0^{-1}} e^{vx - \eta_{j+1}(v)} dv \\
\geq \frac{1}{2} \int_{\mu_{j+1}(x) - \epsilon}^{\mu_{j+1}(x) + \epsilon} e^{vx - \eta_{j+1}(v)} dv - \\
\quad \int_{v - \mu_{j+1}(x) > h_0^{-1}} e^{vx - \eta_{j+1}(v)} dv.
\]

The integrand is strictly increasing for \( v \leq \mu_{j+1}(x) \) and strictly decreasing for \( v \geq \mu_{j+1}(x) \); hence

\[
|F_{j+1}(-x - ih)| \geq \frac{\epsilon}{2} \left( e^{\Theta_{j+1}(\mu_{j+1}(x) - \epsilon)} + e^{\Theta_{j+1}(\mu_{j+1}(x) + \epsilon)} \right) - \\
\quad |J| \left( e^{\Theta_{j+1}(\mu_{j+1}(x) - h_0^{-1})} + e^{\Theta_{j+1}(\mu_{j+1}(x) + h_0^{-1})} \right) \quad (5.10)
\]

where

\[
\Theta_{j+1}(v) := vx - \eta_{j+1}(v). \quad (5.11)
\]

Let us work for the moment with the case that

\[ v_0 \notin (\mu_{j+1}(x) - h_0^{-1}, \mu_{j+1}(x) + h_0^{-1}) \]

We shall estimate \( \Theta_{j+1} \) using the integral form of Taylor’s theorem at \( \mu_{j+1}(x) \). We have

\[
\Theta_{j+1}(\mu_{j+1}(x) - \epsilon) = \Theta_{j+1}(\mu_{j+1}(x)) + \\
\quad \left. \int_{\mu_{j+1}(x) - \epsilon}^{\mu_{j+1}(x)} \Theta''_{j+1}(t) (t - (\mu_{j+1}(x) - \epsilon)) \right) dt \\
\Theta_{j+1}(\mu_{j+1}(x) + \epsilon) = \Theta_{j+1}(\mu_{j+1}(x)) + \\
\quad \left. \int_{\mu_{j+1}(x) + \epsilon}^{\mu_{j+1}(x)} \Theta''_{j+1}(t) ( \mu_{j+1}(x) + \epsilon - t) \right) dt \quad (5.12a)
\]

\[
\Theta_{j+1}(\mu_{j+1}(x) + \epsilon) = \Theta_{j+1}(\mu_{j+1}(x)) + \\
\quad \left. \int_{\mu_{j+1}(x) + \epsilon}^{\mu_{j+1}(x)} \Theta''_{j+1}(t) ( \mu_{j+1}(x) + \epsilon - t) \right) dt \\
\]

where

\[
\Theta''_{j+1}(t) = \frac{\partial^2}{\partial v^2} \left( vx - \eta_{j+1}(v) \right)
\]
\[ \Theta_{j+1}(\mu_{j+1}(x) - h_0^{-1}) = \Theta_{j+1}(\mu_{j+1}(x)) + \]
\[ \int_{\mu_{j+1}(x) - h_0^{-1}}^{\mu_{j+1}(x)} \Theta''_{j+1}(t) \left( t - (\mu_{j+1}(x) - h_0^{-1}) \right) dt \]
\[ \Theta_{j+1}(\mu_{j+1}(x) + h_0^{-1}) = \Theta_{j+1}(\mu_{j+1}(x)) + \]
\[ \int_{\mu_{j+1}(x) + h_0^{-1}}^{\mu_{j+1}(x)} \Theta''_{j+1}(t) \left( \mu_{j+1}(x) + h_0^{-1} - t \right) dt. \]

Using Lemma 5.3 to estimate the integrands in (5.12 a,b) we obtain the following estimates:
\[ \Theta_{j+1}(\mu_{j+1}(x) - \epsilon) \geq \Theta_{j+1}(\mu_{j+1}(x)) - \frac{C_2\epsilon^2}{2} \eta''_{j+1}(\mu_{j+1}(x)) \] (5.13a)
\[ \Theta_{j+1}(\mu_{j+1}(x) + \epsilon) \geq \Theta_{j+1}(\mu_{j+1}(x)) - \frac{C_2\epsilon^2}{2} \eta''_{j+1}(\mu_{j+1}(x)). \] (5.13b)

Reducing the intervals of integration in (5.12) (c) and (d) to \([\mu_{j+1}(x) - \epsilon, \mu_{j+1}(x)]\) and \([\mu_{j+1}(x), \mu_{j+1}(x) + \epsilon]\), respectively, and invoking Lemma 5.3 and the assumption (5.9) we see that the integrands are bounded above by \(-\frac{\eta''_{j+1}(\mu_{j+1}(x))}{2C_2h_0}\). This yields
\[ \Theta_{j+1}(\mu_{j+1}(x) - h_0^{-1}) \leq \Theta_{j+1}(\mu_{j+1}(x)) - \frac{\epsilon}{2C_2h_0} \eta''_{j+1}(\mu_{j+1}(x)) \] (5.13c)
\[ \Theta_{j+1}(\mu_{j+1}(x) + h_0^{-1}) \leq \Theta_{j+1}(\mu_{j+1}(x)) - \frac{\epsilon}{2C_2h_0} \eta''_{j+1}(\mu_{j+1}(x)). \] (5.13d)

(If \(\mu_{j+1}(x) - h_0^{-1}\) or \(\mu_{j+1}(x) + h_0^{-1}\) lands outside of \(J\) then the corresponding estimate holds by default.)

If \(v_0 \in (\mu_{j+1}(x) - h_0^{-1}, \mu_{j+1}(x))\) then (5.12c) must be adjusted by inclusion of the term
\[ \left( \Theta'_{j+1}(v_0) - \Theta'_{j+1}(v_0) \right) \left( v_0 - (\mu_{j+1}(x) - h_0^{-1}) \right) = \left( -\eta'_{j+1}(v_0) + \eta'_{j+1}(v_0) \right) \left( v_0 - (\mu_{j+1}(x) - h_0^{-1}) \right) \leq 0. \]
Thus (5.13c), still holds, as well as (5.13b) and (5.13d), though (5.13a) may fail.

Similar considerations apply to the case where \(v_0\) lies in the interval \((\mu_{j+1}(x), \mu_{j+1}(x) + h_0^{-1})\).
Applying all of this to (5.10) (and dropping the term for which we have no positive lower bound) we obtain:

\[ |F_{j+1}(-x - ih)| \]
\[ \geq e^{\Theta_{j+1}(\mu_{j+1}(x))} \left( \frac{\epsilon}{2} e^{-\frac{C_2 h^0}{2} \eta''_{j+1}(\mu_{j+1}(x))} - 2|J| e^{-\frac{C_2 h^0}{2} \eta''_{j+1}(\mu_{j+1}(x))} \right) \]
\[ = e^{\tilde{\eta}_{j+1}(x)} \left( \frac{\epsilon}{2} e^{-\frac{C_2 h^0}{2} \eta''_{j+1}(\mu_{j+1}(x))} - 2|J| e^{-\frac{C_2 h^0}{2} \eta''_{j+1}(\mu_{j+1}(x))} \right) . \] (5.14)

We shall choose \( \epsilon \) such that

\[ 2|J| e^{-\frac{C_2 h^0}{2} \eta''_{j+1}(\mu_{j+1}(x))} \leq \frac{\epsilon}{4} e^{-\frac{C_2 h^0}{2} \eta''_{j+1}(\mu_{j+1}(x))} , \]
i.e.

\[ \log \left( \frac{8|J|}{\epsilon} + \frac{C_2 h^0}{2} \eta''_{j+1}(\mu_{j+1}(x)) \right) \leq \frac{1}{2} \eta''_{j+1}(\mu_{j+1}(x)) . \] (5.15)

Then

\[ |F_{j+1}(-x - ih)| \geq \frac{\epsilon}{4} e^{-\frac{C_2 h^0}{2} \eta''_{j+1}(\mu_{j+1}(x))} e^{\tilde{\eta}_{j+1}(x)} . \] (5.16)

It remains to choose \( \epsilon \) satisfying (5.8), (5.9), (5.15). We can choose

\[ \epsilon := \frac{C_3}{\sqrt{\eta''_{j+1}(\mu_{j+1}(x))}} \] (5.17)
with \( C_3 = C_3(h_0) > 0 \) independent of \( x \) and \( j \).

Since \( \eta''_{j+1}(\mu_{j+1}(x)) \) is bounded below uniformly in \( j \), (5.9) will hold if \( C_3 \) is small enough.

Lemma 5.2 shows that (5.8) is also guaranteed for small \( C_3 \).

Condition (5.15) now reads

\[ \log \left( \frac{8|J|}{C_3} \sqrt{\eta''_{j+1}(\mu_{j+1}(x))} \right) + \frac{C_2 h^0}{2} \leq \frac{C_3}{2} \sqrt{\eta''_{j+1}(\mu_{j+1}(x))} . \]

This will hold provided that \( \eta''_{j+1}(\mu_{j+1}(x)) \) exceeds some absolute constant \( M \).

Consulting (5.7) we see that \( \eta''_{j+1}(\mu_{j+1}(x)) > M \) in the following cases:

- for any fixed \( j \) provided that \( x \) is large enough;
- for all \( x \) provided that \( j \) is large enough.

Combining (5.17) with (5.16) we see that in these cases we have (5.5) with \( L = \frac{C_4}{4} e^{-C_2 C_3/2} \).
Proof of Proposition 5.1, part (ii). We explain where the proof of part (i) must be modified.

In the current case we have \( \mu_{j+1}(x) = v_0 \), so \( \Theta_{j+1} \) will not be differentiable at \( \mu_{j+1}(x) \), but the one-sided derivatives \( \Theta'_{j+1}(v_0+) \) and \( \Theta'_{j+1}(v_0-) \) will exist.

Recalling (5.11) we have

\[
\Theta'_{j+1}(v_0+) - \Theta'_{j+1}(v_0-) = -(j + 1)B,
\]

where \( B := \eta''_0(v_0+) - \eta''_0(v_0-) > 0 \).

Since \( \Theta_{j+1} \) has a maximum at \( v_0 \), we must have

\[
\Theta'_{j+1}(v_0+) \leq 0 \leq \Theta'_{j+1}(v_0-). \tag{5.18}
\]

Combining (5.11) and (5.18) we find that

\[
-(j + 1)B \leq \Theta'_{j+1}(v_0+) \leq 0 \leq \Theta'_{j+1}(v_0-) \leq (j + 1)B. \tag{5.19}
\]

The Taylor expansions (5.12) must be modified by inclusion on the right-hand side of the terms \(-\Theta'_{j+1}(v_0-)\epsilon, \Theta'_{j+1}(v_0+)\epsilon, -\Theta'_{j+1}(v_0-)h_0^{-1}, \) and \( \Theta'_{j+1}(v_0+)h_0^{-1} \), respectively.

Focusing on the latter two expansions, we see that the new terms are negative. If we restrict the integrals to \([v_0 - \min\{h_0^{-1}, \frac{1}{2}\text{dist}(v_0, \partial J)\}, v_0] \) and \([v_0, v_0 + \min\{h_0^{-1}, \frac{1}{2}\text{dist}(v_0, \partial J)\}] \), respectively, we obtain modified versions of (5.13 c,d) taking the following form:

\[
\Theta_{j+1}(\mu_{j+1}(x) - h_0^{-1}) \leq \Theta_{j+1}(\mu_{j+1}(x)) - (j + 1)C_4 \tag{5.20a}
\]

\[
\Theta_{j+1}(\mu_{j+1}(x) + h_0^{-1}) \leq \Theta_{j+1}(\mu_{j+1}(x)) - (j + 1)C_4. \tag{5.20b}
\]

where \( C_4 \) depends on \( h_0 \) but not on \( j \) or \( x \).

To obtain modified versions of (5.13 a,b) we apply the inequalities (5.19) to the new first derivative terms to obtain the following:

\[
\Theta_{j+1}(\mu_{j+1}(x) - \epsilon) \geq \Theta_{j+1}(\mu_{j+1}(x)) - (j + 1)B\epsilon - (j + 1)C_5\epsilon^2 \tag{5.20a}
\]

\[
\Theta_{j+1}(\mu_{j+1}(x) + \epsilon) \geq \Theta_{j+1}(\mu_{j+1}(x)) - (j + 1)B\epsilon - (j + 1)C_5\epsilon^2. \tag{5.20b}
\]

The inequality (5.13) is now modified to read

\[
\log \frac{4|J|}{\epsilon} + (j + 1)B\epsilon + (j + 1)C_5\epsilon^2 \leq (j + 1)C_4. \tag{5.21}
\]

If we now set \( \epsilon = \frac{C_6}{|J|} \) \((C_6 > 0 \) depends on \( h_0 \) but not on \( x \) or \( j \)) we find that (5.8) and (5.9) will hold if \( C_6 \) is chosen small enough, while (5.21) holds for all large enough \( j \).
Under these conditions we obtain the following modified form of (5.16):

\[ |F_{j+1}(-x - ih)| \geq \frac{\epsilon}{2} e^{-(j+1)(B\epsilon+C_5\epsilon^2)} e^{\tilde{\eta}_{j+1}(x)} \]

\[ \geq \frac{K}{j+1} e^{\tilde{\eta}_{j+1}(x)}, \]

with \( K = \frac{C_6}{2} e^{-BC_6-C_5C_6}. \)

\[ \square \]

**Corollary 5.4.** The union of the zero sets of the \( F_{j+1} \) is finite (counting multiplicity) in any strip \(-h_0 \leq \text{Im} \xi \leq 0\).

**Proof.** By part (ii) of Proposition 5.1, the \( F_{j+1} \) are zero-free when \( j \geq j_0(h_0, |J|) \). But by the remark in the statement of Proposition 5.1, the zero set of \( F_{j+1} \) in our strip is finite for \( 0 \leq j < j_0(h_0, |J|) \).

From Proposition 5.1 we see that the union of the zero sets of the \( F_j \) contains finitely many points in each strip \(-h_0 \leq \text{Im} \xi \leq 0\). Thus in particular, for all but a discrete set of \( h > 0 \) the \( F_j \) are all non-vanishing on the line \( \text{Im} \xi = -h \). We assume for the rest of this section that \( h \) has been chosen with the property.

Returning to the integrals (5.1), we see from Proposition 5.1, (5.3) and (5.2) that the integrand decays exponentially on the strip \(-h \leq \text{Im} \xi \leq 0\). Thus we may apply the residue theorem on this strip as indicated in §4. We still need upper bounds for the shifted integrals

\[ \int_{-\infty}^{\infty} e^{i(t_2 - \tau_2) \frac{(-x - ih)}{2}} \frac{dx}{F_{j+1}(-x - ih)}. \]

Setting \( b := \text{Im}(t_2 - \tau_2) \in (-\pi, \pi - 2\theta) \), we have

\[ \int_{-\infty}^{\infty} e^{i(t_2 - \tau_2) \frac{(-x - ih)}{2}} \frac{dx}{F_{j+1}(-x - ih)} \leq e^{h \text{Re}(t_2 - \tau_2)/2} \int_{-\infty}^{\infty} e^{x^2/2} \frac{dx}{|F_{j+1}(-x - ih)|}. \]

(5.22)
From Proposition 5.1 together with (5.3), (5.4), (5.7) we have

$$e^{\frac{x^2}{2}} |F_{j+1}(-x - ih)| \leq \begin{cases} \frac{1}{2} \sqrt{(1 + j) \left(1 + \frac{x^2}{(1+j)^2}\right)} e^{\frac{x^2}{2} - \tilde{\eta}_{j+1}(x)} & \text{for } x \notin [(j+1) \tan v_0, (j+1) \tan(v_0 + \theta)]; \\ \frac{j+1}{K} e^{\frac{x^2}{2} - \tilde{\eta}_{j+1}(x)} & \text{for } x \in [(j+1) \tan v_0, (j+1) \tan(v_0 + \theta)] \end{cases}$$

for $j$ large.

Set

$$H := -\inf\{\tilde{\eta}_1(x); x \in \mathbb{R}\} + 1.$$

(5.23)

**Lemma 5.5.** For all $\delta \in (0, 1)$ there is $M = M(\delta)$ independent of $j, x$ so that

$$e^{\frac{x^2}{2}} |F_{j+1}(-x - ih)| \leq Me^{\frac{x^2}{2} + (\delta - 1)\tilde{\eta}_{j+1}(x)+\delta(j+1)H}.$$

**Proof.** Setting $\alpha = \frac{x}{j+1}$ and recalling that $\tilde{\eta}_{j+1}(\zeta) = (j+1)\tilde{\eta}_1 \left(\frac{\zeta}{j+1}\right)$ we must choose $M$ so that

$$\frac{j+1}{K} \leq Me^{(j+1)\delta(\tilde{\eta}_1(\alpha)+H)} \text{ for } \alpha \in [\tan v_0, \tan(v_0 + \theta)],$$

$$\frac{\sqrt{j+1}}{L} \sqrt{1 + \alpha^2} \leq Me^{(j+1)\delta(\tilde{\eta}_1(\alpha)+H)} \text{ for } \alpha \notin [\tan v_0, \tan(v_0 + \theta)].$$

It will suffice to choose $M$ so that

$$\left(\frac{1}{L^2} + \frac{1}{K}\right) (j+1) \leq Me^{(j+1)\delta(\tilde{\eta}_1(\alpha)+H)} \quad (5.24)$$

and

$$1 + \alpha^2 \leq Me^{(j+1)\delta(\tilde{\eta}_1(\alpha)+H)} \quad (5.25)$$

hold for all $\alpha$. That (5.24) is possible follows from the fact that the right-hand side exceeds $Me^{(j+1)\delta}$. Similarly, (5.25) is possible since the right-hand side exceeds $Me^{\delta(\tilde{\eta}_1(\alpha)+H)}$, which grows exponentially with $\alpha$. \hfill \Box

**Lemma 5.6.** Let $X$ be a compact subset of $J$. Then there are $Q > 0, \gamma = \gamma(X) > 0$ independent of $j$ so that

$$\tilde{\eta}_{j+1}(x - (j+1)Q) \leq \tilde{\eta}_{j+1}(x) - \gamma$$

$$\tilde{\eta}_{j+1}(x + (j+1)Q) \geq \tilde{\eta}_{j+1}(x) + \gamma$$
when \( \tilde{\eta}_{j+1}(x) \in X \).

Proof. Again setting \( \alpha = \frac{x}{j+1} \) we are reduced to the case \( j = 0 \). Picking \( Q > \tan(v_0 + \theta) - \tan v_0 \), our claim follows easily from the fact that \( \tilde{\eta}_1 : \mathbb{R} \to J \) is a continuous non-decreasing surjective function which is strictly increasing off of the interval \([\tan v_0, \tan(v_0 + \theta)]\).

Using the duality theorem \( \tilde{\eta} = \eta \) (see \([\text{H"o, Thm. 1.3.3}]\)) we have

\[
\sup \left\{ \frac{b}{2} - (1 - \delta)\tilde{\eta}_{j+1}(x) ; x \in \mathbb{R} \right\} = (1 - \delta) \sup \left\{ \frac{b}{2(1 - \delta)} - \tilde{\eta}_{j+1}(x) ; x \in \mathbb{R} \right\} = (1 - \delta)\tilde{\eta}_{j+1}\left( \frac{b}{2(1 - \delta)} \right) = (1 - \delta)\eta_{j+1}\left( \frac{b}{2(1 - \delta)} \right).
\]

Since \( x\frac{b}{2} - \tilde{\eta}_{j+1}(x) \to -\infty \) as \( |x| \to +\infty \) and \( \tilde{\eta}_{j+1} \) is \( C^1(\mathbb{R}) \) we can find an \( x_0 \) so that \( x_0\frac{b}{2} - (1 - \delta)\tilde{\eta}_{j+1}(x_0) = (1 - \delta)\eta_{j+1}\left( \frac{b}{2(1 - \delta)} \right) \). But then \( \frac{dx}{dx} (x\frac{b}{2} - (1 - \delta)\tilde{\eta}_{j+1}(x)) \) vanishes at \( x = x_0 \), so we can deduce from Lemma \( 5, 6 \) and the integral form of Taylor’s theorem that

\[
x\frac{b}{2} - (1 - \delta)\tilde{\eta}_{j+1}(x) \leq \begin{cases} (1 - \delta)\eta_{j+1}\left( \frac{b}{2(1 - \delta)} \right) + (1 - \delta)\gamma(x - (x_0 - (j + 1)Q)) & \text{for } x \leq x_0 - (j + 1)Q; \\ (1 - \delta)\eta_{j+1}\left( \frac{b}{2(1 - \delta)} \right) - (1 - \delta)\gamma(x - (x_0 + (j + 1)Q)) & \text{for } x \geq x_0 + (j + 1)Q. \end{cases}
\]

provided that \( \tilde{\eta}_{j+1}(x_0) = \frac{b}{2(1 - \delta)} \in X \).
Combining this with Lemma 5.5 we have

\[ \int_{-\infty}^{\infty} e^{\frac{b}{2} x} \frac{1}{|F_{j+1}(-x - ih)|} \, dx \]  

(5.26)

\[ \leq M e^{(1 - \delta) \eta_{j+1} \left( \frac{b}{2(1 - \delta)} \right) + \delta(j+1)H} \int_{-\infty}^{x_0 - (j+1)Q} e^{(1 - \delta) \gamma(x - (x_0 - (j+1)Q))} \, dx + \]

\[ + M e^{(1 - \delta) \eta_{j+1} \left( \frac{b}{2(1 - \delta)} \right) + \delta(j+1)H} \int_{x_0 - (j+1)Q}^{x_0 + (j+1)Q} \, dx + \]

\[ + M e^{(1 - \delta) \eta_{j+1} \left( \frac{b}{2(1 - \delta)} \right) + \delta(j+1)H} \int_{x_0 + (j+1)Q}^{\infty} e^{(1 - \delta) \gamma(x - (x_0 + (j+1)Q))} \, dx \]

\[ = M \left( 2(j + 1)Q + \frac{2}{(1 - \delta) \gamma} \right) e^{(1 - \delta) \eta_{j+1} \left( \frac{b}{2(1 - \delta)} \right) + \delta(j+1)H}. \]

If \( \frac{b}{2} \) is restricted to a compact subset \( X \) of \( J \) then \( \frac{b}{2(1 - \delta)} \) is restricted to a slightly larger compact subset \( X' \) provided that \( \delta \leq \delta_0(X) \).

Combining (5.26) with (5.22), we have proved the following.

**PROPOSITION 5.7.** For \( X \) a compact subset of \( J \) there are \( R = R(X) > 0, \delta_0(X) > 0 \) so that

\[ \left| \int_{\mathbb{R} - ih} e^{i(t_2 - \tau_2) \xi} \frac{\xi}{F_{j+1}(\xi)} \, d\xi \right| \leq R \cdot (j + 1) e^{(1 - \delta) \eta_{j+1} \left( \frac{b}{2(1 - \delta)} \right) + \delta(j+1)H} \]

when \( \frac{b}{2} = \frac{1}{2} \text{Im}(t_2 - \tau_2) \in X \) and \( 0 < \delta < \delta_0(X) \).

With harder work one can obtain the following sharper estimates in Proposition 5.7. For \( X \) a compact subset of \( J \) there exists \( R' = R'(X) > 0 \) so that

\[ \left| \int_{\mathbb{R} - ih} e^{i(t_2 - \tau_2) \xi} \frac{\xi}{F_{j+1}(\xi)} \, d\xi \right| \leq R' \cdot (j + 1)^2 e^{\eta_{j+1} \left( \frac{b}{2} \right)} \]

when \( \frac{b}{2} = \frac{1}{2} \text{Im}(t_2 - \tau_2) \in X \).
6. The asymptotic formula and regularity for the Bergman kernel in $\Omega'\backslash\Omega$

Let $\eta(b^2) := \eta_1(b^2) = -\log \psi_{r,\vartheta}(b^2)$.

Suppose that we choose $t = (t_1, t_2)$, $(\tau_1, \tau_2) \in D'$ such that

$$|t_1|^2 \leq e^{-\eta(u)}$$
$$|\tau_1|^2 < e^{-\eta(v)}.$$

for $u = \text{Im } t_2 \in J$, $v = \text{Im } \tau_2 \in J$.

Set $b = u + v$ as before. By the convexity of $\eta$ we have:

$$|t_1| |\tau_1| e^{\eta(b^2)} \leq |t_1| |\tau_1| e^{(1-\delta)\eta(b^2)} < 1.$$  \hspace{1cm} (6.1)

Choose $\delta > 0$ small so that

$$|t_1| |\tau_1| e^{(1-\delta)\eta(b^2)} + \delta H < 1,$$

where $H$ is defined in (5.23).

Consider the series:

$$\sum_{j \geq 0} j + 1 \frac{4\pi^2}{(t_1 \overline{\tau_1})^j} \int_{\mathbb{R}} e^{i(t_2 - \tau_2) \frac{x - x + ih}{2}} dx.$$  \hspace{1cm} (6.2)

Setting $t_2 - \overline{\tau_2} := a + ib$ and invoking Proposition 5.7, the absolute value of the above series can be majorized by

$$R e^{a \frac{b}{2}} \sum_{j \geq 0} (j + 1)^2 |t_1 \overline{\tau_1}|^j e^{(j+1)(1-\delta)\eta(b^2)}.$$  \hspace{1cm} (6.3)

Applying the root test and taking into account (6.1) we have

$$\limsup \sqrt[j]{(j + 1)^2 |t_1 \overline{\tau_1}|^j e^{(j+1)(1-\delta)\eta(b^2)}} = |t_1 \overline{\tau_1}| e^{(1-\delta)\eta(b^2)} + \delta H < 1.$$  \hspace{1cm} (6.4)

Thus we have proved the following:
PROPOSITION 6.1. The series \((6.2)\) is \(O(e^{h\Re t_2/2})\) as \(\Im t_2 \to -\infty\), uniformly as \(\tau\) ranges over any compact subset of \(D'\).

Positive regularity results

Choose \(h > 0\) so that the strip \(-h \leq \Im \xi \leq 0\) is zero-free for all \(F_{j+1}(\xi)\).

Then for appropriate \(t, \tau\) we have:

\[
K_{D'}(t, \tau) = \sum_{j \geq 0} j + 1 \frac{(t_1 \overline{t_1})^j}{4\pi^2} \int_{\mathbb{R}} e^{i(t_2 - \tau_2)(-x - ih)}/F_{j+1}(-x - ih)\,dx = O(e^{h\Re t_2/2}).
\]

Let us fix \(\tau \in D'\). A careful inspection of our earlier work shows that we may differentiate the above formula with respect to \(t_1, t_2\) to obtain:

\[
D_{t_1} K_{D'}(t, \tau) = \sum_{j \geq l} \frac{(j+1)(j-1)\cdots(j-l+1)}{4\pi^2} (t_1 \overline{t_1})^{j-l} \int_{\mathbb{R}} e^{i(t_2 - \tau_2)(-x - ih)}/F_{j+1}(-x - ih)\,dx
\]

and

\[
D_{t_2} K_{D'}(t, \tau) = \sum_{j \geq 0} \frac{j + 1}{4\pi^2} (t_1 \overline{t_1})^j \int_{\mathbb{R}} e^{i(t_2 - \tau_2)(-x - ih)}/F_{j+1}(-x - ih)\,dx
\]

with corresponding formulae for mixed partials.

Arguing as in Proposition [1.1] we find that each partial derivative satisfies

\[
D_{t_1} D_{t_2} K_{D'}(t, \tau) = O(e^{h\Re t_2/2}) \quad \text{as} \quad \Im t_2 \to -\infty \quad (6.3)
\]

Let us fix a point \(\tilde{\omega} \in \hat{\Omega}'\). Using the transformation formula (1.5) for the Bergman kernel and differentiating with respect to \(\tilde{w}_1, \tilde{w}_2\) we obtain:
\[
\frac{\partial}{\partial \tilde{w}_1} K_{\tilde{t}}(\tilde{w}, \tilde{\omega}) = \frac{1}{2(\tilde{w}_2 \tilde{\omega}_2)^{\frac{3}{2}}} \frac{\partial}{\partial t_1} K_{D'}(\Phi(\tilde{w}), \Phi(\tilde{\omega})) \frac{\partial t_1}{\partial \tilde{w}_1}
\]
\[
\frac{\partial}{\partial \tilde{w}_2} K_{\tilde{t}}(\tilde{w}, \tilde{\omega}) = -\frac{3}{4(\tilde{w}_2)^{\frac{3}{2}}} K_{D'}(\Phi(\tilde{w}), \Phi(\tilde{\omega})) + \frac{1}{2(\tilde{w}_2 \tilde{\omega}_2)^{\frac{3}{2}}} \left( \frac{\partial t_1}{\partial \tilde{w}_2} \frac{\partial}{\partial \tilde{t}_1} K_{D'}(\Phi(\tilde{w}), \Phi(\tilde{\omega})) \right) + \frac{1}{2(\tilde{w}_2 \tilde{\omega}_2)^{\frac{3}{2}}} \left( \frac{\partial t_2}{\partial \tilde{w}_2} \frac{\partial}{\partial \tilde{t}_2} K_{D'}(\Phi(\tilde{w}), \Phi(\tilde{\omega})) \right).
\]

But
\[
\frac{\partial t_1}{\partial \tilde{w}_1} = \frac{1}{\sqrt{2 \tilde{w}_2}}, \quad \frac{\partial t_1}{\partial \tilde{w}_2} = -\frac{1}{2\sqrt{2 \tilde{w}_2}}, \quad \frac{\partial t_2}{\partial \tilde{w}_2} = \frac{1}{\tilde{w}_2}.
\]

Combining the transformation laws with (6.3) and the fact that \(|\tilde{w}_1| \leq |\sqrt{\tilde{w}_2}|\) in \(\hat{\Omega}'\) we see that
\[
|\nabla_{\tilde{w}} K_{\tilde{t}}(\tilde{w}, \tilde{\omega})| \leq O \left( \frac{h^{-\frac{5}{2}}}{\tilde{w}_2^3} \right)
\]
and, more generally,
\[
|\nabla_{\tilde{w}}^k K_{\tilde{t}}(\tilde{w}, \tilde{\omega})| \leq O \left( \frac{h^{-\frac{3}{2} - 2k}}{\tilde{w}_2^{\frac{3}{2}}} \right) \tag{6.4}
\]
for \(\tilde{w}\) near 0.

Let \(\tilde{\omega}\) be a point inside \(\hat{\Omega}'\). We are interested in the regularity of \(K_{\tilde{t}}(\cdot, \tilde{w}, \tilde{\omega})\) near the complex tangent point \((0, 0)\). Since \(\Omega'\) is Lipschitz and \(K(\cdot, \tilde{\omega})\) is harmonic we shall estimate the Sobolev \(L_p^{k+\epsilon}\) norm and Besov \(B_{k+\epsilon}^{1/p}\) norm of \(K(\cdot, \tilde{\omega})\) in a neighborhood of the complex tangent point using the following theorems by Jerison-Kenig [JK]:

**Theorem 6.2.** A) Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^n\). Let \(\delta(x)\) be the distance of \(x\) from the boundary of \(\Omega\). Define \(\nabla^k u\) as the vector of all \(k^{th}\) order derivatives of a function \(u\). Suppose that \(u\) is a harmonic function in \(\Omega\). Let \(0 \leq \epsilon \leq 1\), let \(k\) be a nonnegative integer, and let \(1 < p < \infty\). Then the following are equivalent:
B) Suppose that \( u \) is a harmonic function in \( \Omega \). Let \( 0 < \epsilon < 1 \), let \( k \) be a nonnegative integer, and let \( 1 \leq p \leq \infty \). Then the following are equivalent:

i) \( u \) belongs to \( B_{k+\epsilon}^p(\Omega) \),

ii) \( \delta^{1-\epsilon}|\nabla^{k+1}u| + |\nabla^k u| + |u| \) belongs to \( L^p(\Omega) \).

To apply these results to \( \hat{\Omega}' \) we use the following facts (valid for \( 1 < p < \infty \)):

(a) \( \tilde{w}_1^d \tilde{w}_2^d \in L^p \) (in a neighborhood of the complex tangent point \((0,0)\)) if and only if \( \text{Re} \, d + \frac{k}{2} + \frac{3}{p} > 0 \), for \( k \in \mathbb{N} \), \( d \in \mathbb{C} \);

(b) \( \delta^{1-\sigma}|\nabla^{k+1}u| + |\nabla^k u| + |u| \) belongs to \( L^p(\Omega) \) (in a neighborhood of the point \((0,0)\)) if and only if \( \text{Re} \, d + \frac{k}{2} + \frac{3}{p} + 1 > \sigma \), for all \( 0 \leq \sigma \leq 1 \), \( k \in \mathbb{N} \), \( d \in \mathbb{C} \);

here \( \delta = \delta(\tilde{w}, \partial \hat{\Omega}') \) is the distance to the boundary of \( \hat{\Omega}' \).

Let \( 1 < p < \infty \). Suppose that \( \frac{h-3}{2} + \frac{3}{p} > 0 \). Using (b) above in combination with Theorem 6.2(A) and the inequality (6.4) we have the following.

**Proposition 6.3.** Let \( 1 < p < \infty \). If \( \frac{h-3}{2} + \frac{3}{p} > 0 \) and \( \tilde{\omega} \in \hat{\Omega}' \) then \( K_{\hat{\Omega}'}(\cdot, \tilde{\omega}) \in L^p_s \) in a neighborhood of the complex tangent point \((0,0)\) for \( 0 \leq s < \frac{h-3}{2} + \frac{3}{p} \).

Note that for \( p = 2 \) we have \( K_{\hat{\Omega}'}(\cdot, \tilde{\omega}) \in L^2_s \) for \( 0 \leq s < \frac{h}{2} \).

When \( r = 1 \), \( \theta \sim 0 \) and positive then (by (4.9)) \( h \) can be chosen to be any positive number smaller than \( 3 + \frac{4\theta}{\pi - \theta} \).

When \( r = 1 \), \( \theta = \pi - \epsilon \), \( \epsilon \sim 0 \) and positive, \( h \) can be chosen to be very large since the very first zero has imaginary part \( -\frac{4\pi}{\epsilon} + 1 \). In this case, \( K_{\hat{\Omega}'} \) will be very regular.

The map \( \Psi : B_1 \cap B_2 \to \hat{\Omega}' \) defined in (6.7) is a diffeomorphism in a neighborhood of the complex tangent point \( q \). Using the local diffeomorphism invariance of Sobolev spaces (see for example [T], Chap.
XI, §2) we can conclude that $K_{B_1 \cap B_2} \in L^p_s$ in a neighborhood of $q$ for $0 \leq s < \frac{h-3}{2} + \frac{3}{p}$.

**Negative regularity results**

We begin by investigating the regularity of terms $\tilde{w}_1^j \tilde{w}_2^\alpha$, assuming $j \in \mathbb{N}_0 \cup \{0\}$, $\alpha \notin \mathbb{N}_0 \cup \{0\}$. We claim that $\tilde{w}_1^j \tilde{w}_2^\alpha \notin L^p_s$ for $s \geq \max\{0, \text{Re}a + \frac{i}{2} + \frac{3}{p}\}$. To see this, write $s = l + \sigma$ with $l \in \mathbb{N}_0 \cup \{0\}$, $0 \leq \sigma \leq 1$. Then

$$
\delta^{1-\sigma} (\frac{\partial}{\partial \sigma})^{l+1} (\tilde{w}_1^j \tilde{w}_2^\alpha) \notin L^p
$$

showing that $\tilde{w}_1^j \tilde{w}_2^\alpha$ indeed fails to lie in $L^p_s$.

In particular, $\tilde{w}_1^j \tilde{w}_2^{\frac{i \xi_j k}{2} - \frac{j+3}{2}} \notin L^p_s$ for $s \geq \max\{0, -\frac{\text{Im} \xi_j k - 3}{2} + \frac{3}{p}\}$, $\frac{i \xi_j k}{2} - \frac{j+3}{2} \notin \mathbb{N} \cup \{0\}$. When $p = 2$, our condition on $s$ simplifies to $s \geq \max\{0, -\frac{\text{Im} \xi_j k}{2}\}$.

Consider the special case $r = 1, \theta \sim 0$ and positive. (The root pattern in this case was discussed at the end of §II.) Suppose we choose $h$ such that the first zero of $F_1$ lies in the strip $-h < \text{Im} \xi < 0$, but no other residues lie in the closed strip $-h \leq \text{Im} \xi \leq 0$. Then the asymptotic formula for the Bergman kernel $K_{\hat{\Omega'}}$ will read

$$
K_{\hat{\Omega'}}(\tilde{w}, \tilde{\omega}) = C \tilde{w}_2^{\frac{i \xi_0}{2} - \frac{3}{2}} + O \left( \tilde{w}_2^{\frac{b-3}{2}} \right),
$$

with $C \neq 0$ for most $\tilde{\omega}$.

Taking into account that $\frac{i \xi_0}{2} = \frac{3}{2} + \frac{2\theta}{\pi - \theta}$ we see that

$$
\tilde{w}_2^{\frac{i \xi_0}{2} - \frac{3}{2}} \notin L^2_s \text{ for } s \geq \frac{3}{2} + \frac{2\theta}{\pi - \theta}
$$

Arguing as above the error term is in $L^2_s$ for $s < \frac{h}{2}$. Thus $K_{\hat{\Omega'}}(\tilde{w}, \tilde{\omega}) \notin L^2_s$ for $s \geq \frac{3}{2} + \frac{2\theta}{\pi - \theta}$.

It follows that for every $s > \frac{3}{2}$ we can choose $\hat{\Omega}'$ and $\tilde{\omega} \in \hat{\Omega}'$ so that $K_{\hat{\Omega'}}(\cdot, \tilde{\omega}) \notin L^2_s$.

More generally, we have the following for general $r, \theta$.

**PROPOSITION 6.4.** If the strip $-h_* \leq \text{Im} \xi \leq 0$ contains

a) a zero $\xi_j, k$ of $F_{j+1}$ with $\frac{i \xi_j k}{2} - \frac{j+3}{2} \notin \mathbb{N} \cup \{0\}$

or

b) a multiple zero of some $F_{j+1}$

and $\frac{h-3}{2} + \frac{3}{p} \geq 0$ then for most $\tilde{\omega} \in \hat{\Omega}'$ we have $K_{\hat{\Omega'}}(\cdot, \tilde{\omega}) \notin L^p_{\frac{h-3}{2} + \frac{3}{p}}$. 
In proving this result, we work on a strip $-h \leq \Im \xi \leq 0$ with $h$ a little bit larger than $h_\ast$.

Applying the same reasoning with $p = \infty$ and applying part (B) rather than part (A) of Theorem 6.2 we find that $K_{\tilde{\Omega}'}(\cdot, \tilde{\omega}) \notin B_s^\infty$ for $s > \frac{h_\ast - 3}{2}$. Recall that for $0 < \epsilon < 1$ the function space $B_s^\infty$ coincides with the usual Hölder class of order $\epsilon$.

Returning to the special case $r = 1, \theta \sim 0$ and positive, we find that for every positive $\epsilon$, we can choose $\tilde{\Omega}'$ and $\tilde{\omega} \in \tilde{\Omega}'$ so that $K_{\tilde{\Omega}'}(\cdot, \tilde{\omega})$ fails to be Hölder of order $\epsilon$.

As before, a change of variable argument allows us to transfer all of these conclusions to the behavior of $K_\Omega(\cdot, \zeta)$ near a complex tangent point $q \in \partial \Omega$.

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