A Twisted Version of Controlled $K$-theory

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Abstract
There are a number of (co-)homology theories on coarse spaces. Controlled operator $K$-theory is by far the most popular one of them. Our approach is geometric. We study when does the Roe-algebra of a space restrict to a subspace. Then we show the Roe-algebra is a cosheaf on the coarse topology. A result is a Mayer–Vietoris exact sequence in the presence of a coarse cover. We compute examples as an application.

Keywords Coarse geometry · Operator $K$-theory · Roe-algebra · Cosheaf · Mayer–Vietoris

Mathematics Subject Classification 19K56 · 51F30

1 Introduction

The $K$-theory of the Roe-algebra is one of the most popular homological invariants on coarse metric spaces. Meanwhile a new cohomological invariant on coarse spaces recently appeared in [3] which studies sheaf cohomology on coarse spaces.

In this paper, we study the $K$-theory of the Roe-algebra of a proper metric space $X$ which is introduced in [4, Chapter 6.3]. Note that this theory does not appear as a derived functor as far as we know.

In [5] is studied a coarse excisive property on coarse spaces which we recall now. If $Y \subseteq X$ is a closed subspace then $C^*(Y, X)$ denotes the ideal in $C^*(X)$ which is the norm closure of operators with support near $Y$. Let $A, B \subseteq X$ be two closed subsets of a proper metric space which are $\omega$-excisive. Then
is a six-term Mayer–Vietoris exact sequence by [5, Sect. 5].

Note our approach can be compared with [8] where it was shown that a quotient $D(X)/C^*(X)$ is a sheaf on the underlying topological space of $X$.

The paper [3] introduced a Grothendieck topology $X_{ct}$ associated to a coarse space $X$. The underlying category of $X_{ct}$ is the poset of subsets of $X$ and the coverings are finite collections of subsets called coarse covers.

Theorem 13 shows if $X$ is a proper metric space then the association

$$ U \mapsto C^*(\bar{U})/\mathbb{K}(\mathcal{H}_U), $$

for every subset $U \subseteq X$ with restriction maps is a cosheaf on $X_{ct}$.

Note that in a general setting cosheaves with values in the category of abelian groups $\mathbb{Ab}$ do not give rise to a derived functor. In [1] is explained that the dual version of sheafification, cosheafification, does not work in general. Moreover, the category of $C^*$-algebras $CStar$ is not abelian.

Our result gives rise to new computational tools one of which is a new Mayer–Vietoris six-term exact sequence which is Corollary 14: If $U_1, U_2 \subseteq X$ are subsets of a proper metric space that coarsely cover a subspace $U \subseteq X$ then

$$ K_1(\hat{C}^*(U_1 \cap U_2)) \longrightarrow K_1(\hat{C}^*(U_1)) \oplus K_1(\hat{C}^*(U_2)) \longrightarrow K_1(\hat{C}^*(U)) $$

$$ K_0(\hat{C}^*(U)) \leftarrow K_0(\hat{C}^*(U_1)) \oplus K_0(\hat{C}^*(U_2)) \leftarrow K_0(\hat{C}^*(U_1 \cap U_2)) $$

is exact. Here $\hat{C}^*(Y) = C^*(\bar{Y})/\mathbb{K}(\mathcal{H}_Y)$ for $Y \subseteq X$ a subset.

The outline of this paper is as follows: The Chapter 2 discusses cosheaves on coarse spaces. The main part of the study is in Chapters 3 and 4 computes examples.

## 2 Cosheaves

If $X$ is a metric space, a subset $E \subseteq X \times X$ is called an entourage if

$$ \sup_{(x, y) \in E} d(x, y) < \infty. $$

A subset $B$ is bounded if $\sup_{x, y \in B} d(x, y) < \infty$. A map $\varphi : X \to Y$ between metric spaces is called coarse if $\varphi \times \varphi$ maps entourages to entourages and $\varphi^{-1}$ maps bounded
sets to bounded sets. Two maps \( \phi, \psi : X \to Y \) between metric spaces are close if \( \phi \times \psi \) maps the diagonal to an entourage. The coarse category consists of metric spaces as objects and coarse maps modulo close as morphisms.

We recall [3, Definition 45]:

**Definition 1** (coarse cover) If \( X \) is a metric space and \( U \subseteq X \) a subset then a finite family of subsets \( U_1, \ldots, U_n \subseteq U \) is said to coarsely cover \( U \) if for every entourage \( E \subseteq X^2 \) there is a bounded set \( B \subseteq X \) such that

\[
U^2 \cap \left( \bigcup_i U_i^2 \right)^c \cap E \subseteq B^2.
\]

Coarse covers determine a Grothendieck topology \( X_{ct} \) associated to a metric space \( X \). If \( f : X \to Y \) is a coarse map between metric spaces then there is a morphism of Grothendieck topologies \( f^{-1} : Y_{ct} \to X_{ct} \).

**Definition 2** (precosheaf) A precosheaf on \( X_{ct} \) with values in a category \( C \) is a covariant functor \( \text{Cat}(X_{ct}) \to C \).

**Definition 3** (cosheaf) Let \( C \) be a category with finite limits and colimits. A precosheaf \( \mathcal{F} \) on \( X_{ct} \) with values in \( C \) is a cosheaf on \( X_{ct} \) with values in \( C \) if for every coarse cover \( \{U_i \to U\}_i \) there is a coequalizer diagram:

\[
\bigoplus_{ij} \mathcal{F}(U_i \cap U_j) \rightrightarrows \bigoplus_i \mathcal{F}(U_i) \to \mathcal{F}(U). \tag{1}
\]

Here the two arrows on the left side relate to the following 2 diagrams:

\[
\bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \longrightarrow \bigoplus_i \mathcal{F}(U_i) \quad \uparrow \quad \uparrow
\]

\[
\mathcal{F}(U_i \cap U_j) \longrightarrow \mathcal{F}(U_i)
\]

and

\[
\bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \longrightarrow \bigoplus_j \mathcal{F}(U_j) \quad \uparrow \quad \uparrow
\]

\[
\mathcal{F}(U_i \cap U_j) \longrightarrow \mathcal{F}(U_j)
\]

where \( \bigoplus \) denotes the coproduct over the index set.

**Notation 4** If we write

- \( \sum_i a_i \in \bigoplus_i \mathcal{F}(U_i) \) then \( a_i \) is supposed to be in \( \mathcal{F}(U_i) \)
- \( \sum_{ij} b_{ij} \in \bigoplus_{ij} \mathcal{F}(U_i \cap U_j) \) then \( b_{ij} \) is supposed to be in \( \mathcal{F}(U_i \cup U_j) \)
Proposition 5 If \( \mathcal{F} \) is a precosheaf on \( X_{ct} \) with values in a category \( C \) with finite limits and colimits and for every coarse cover \( \{ U_i \rightarrow U \}_i \)

1. and every \( a \in \mathcal{F}(U) \) there is some \( \sum_i a_i \in \bigoplus_i \mathcal{F}(U_i) \) such that \( \sum_i a_i|_U = a \)

2. and for every \( \sum_i a_i \in \bigoplus_i \mathcal{F}(U_i) \) such that \( \sum_i a_i|_U = 0 \) there is some \( \sum_{ij} b_{ij} \in \bigoplus_{ij} \mathcal{F}(U_i \cap U_j) \) such that \( \sum_{ij} b_{ij} |_{U_i} = a_i \) for every \( i \).

then \( \mathcal{F} \) is a cosheaf.

Proof We have to prove that conditions (1) and (2) are equivalent to exactness of the diagram 1. Call the right map \( \beta \) and the left map \( \alpha \). Then exactness at \( \mathcal{F}(U) \) means the map \( \alpha \) is surjective. That is condition (1).

Now \( \text{im}(\beta) \subseteq \ker(\alpha) \) always holds. If \( \sum_i a_i \in \bigoplus_i \mathcal{F}(U_i) \) then \( \sum_i a_i|_U = 0 \) is equivalent to \( \sum a_i \in \ker(\alpha) \). If \( \ker(\alpha) \subseteq \text{im}(\beta) \) then there exists some \( \sum_{ij} b_{ij} \in \bigoplus_{ij} \mathcal{F}(U_i \cap U_j) \) with \( \sum_{ij} (\sum_j b_{ij} - b_{ji}) |_{U_i} = \beta(\sum_{ij} b_{ij}) = \sum_i a_i \). This is condition 2. \( \square \)

Remark 6 Denote by \( \text{CStar} \) the category of \( C^* \)-algebras. According to [6], all finite limits and finite colimits exist in \( \text{CStar} \).

3 Roe–Calkin Algebra

This exposition uses notation from [4, Chapter 6] which is a standard reference for \( K \)-theory of the Roe-algebra. We recall a few of the definitions:

Let \( X \) be a proper metric space. A presentation \( \rho : C_0(X) \to \mathcal{B}(\mathcal{H}_X) \) of \( C_0(X) \) on a separable Hilbert space \( \mathcal{H}_X \) is called an \( X \)-module if it is non-degenerate and ample. The support of a vector \( v \in \mathcal{H}_X \) is the complement in \( X \) of the union of all open subsets \( U \subseteq X \) such that \( \rho(f)v = 0 \) for all \( f \in C_0(X) \). An operator \( T \in \mathcal{B}(\mathcal{H}_X) \) is called locally compact on \( X \) if \( \rho(f)T \) and \( T\rho(f) \) are compact operators for all \( f \in C_0(X) \). The support of an operator \( T \in \mathcal{B}(\mathcal{H}_X, \mathcal{H}_Y) \) is the complement in \( Y \times X \) of the union of all open subsets \( U \times V \subseteq Y \times X \) such that \( \rho(f)T\rho(g) = 0 \) for every \( f \in C_0(U) \) and \( g \in C_0(V) \). An operator \( T \in \mathcal{B}(\mathcal{H}_X) \) is said to be controlled if \( \text{supp}(T) \) is an entourage. The Roe algebra \( C^*(X) \) is the norm closure of the algebra of locally compact, controlled operators on \( \mathcal{H}_X \). If \( C_0(X) \) is represented by an \( X \)-module then the \( K \)-theory of the Roe-algebra is a functor on coarse proper metric spaces. If \( \varphi : X \to Y \) is a coarse map between metric spaces then a bounded operator \( V : \mathcal{H}_X \to \mathcal{H}_Y \) covers \( \varphi \) if the two maps \( \pi_1 \) and \( \varphi \circ \pi_2 \) from \( \text{supp}(V) \subseteq Y \times X \) are close.

Lemma 7 If \( X \) is a proper metric space and \( Y \subseteq X \) is a closed subspace then

- The subset \( I(Y) = \{ f \in C_0(X) : f|_Y = 0 \} \) is an ideal of \( C_0(X) \) and we have \( C_0(Y) = C_0(X)/I(Y) \)

- There exists a sub-Hilbert space \( \mathcal{H}_Y \subseteq \mathcal{H}_X \) and a non-degenerate representation \( \rho_Y : C_0(Y) \to \mathcal{B}(\mathcal{H}_Y) \) that is a natural restriction of a non-degenerate representation \( \rho_X : C_0(X) \to \mathcal{B}(\mathcal{H}_X) \).

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The inclusion $i_Y : \mathcal{H}_Y \to \mathcal{H}_X$ covers the inclusion $i : Y \to X$.

**Proof**
- This one follows by Gelfand duality.
- We define $\mathcal{H}_{I(Y)} = \rho_X(I(Y))\mathcal{H}_X$. Then
  \[ \mathcal{H}_X = \mathcal{H}_{I(Y)} \oplus \mathcal{H}_{I(Y)}^\perp \]
  is the direct sum of reducing subspaces for $\rho_X(C_0(X))$. We define
  \[ \mathcal{H}_Y = \mathcal{H}_{I(Y)}^\perp \]
  and a representation of $C_0(Y)$ on $\mathcal{H}_Y$ by
  \[ \rho_Y([a]) = \rho_X(a)|_{\mathcal{H}_Y} \]
  for every $[a] \in C_0(Y)$. Note that $\rho_X(\cdot)|_{\mathcal{H}_Y}$ annihilates $I(Y)$ so this is well defined.
- Note that the support of $i_Y$ is
  \[ \text{supp}(i_Y) = \Delta_Y \subseteq X \times Y \]

\[ \square \]

**Remark 8** Note that we cannot conclude the following: If the representation $\rho_X : C_0(X) \to \mathbb{B}(\mathcal{H}_X)$ is ample and $Y \subseteq X$ is a closed subspace then the induced representation $\rho_Y : C_0(Y) \to \mathbb{B}(\mathcal{H}_Y)$ is ample. Thus if we want to restrict representations to subspaces, we have to check the ample property each time.

**Lemma 9** If $X$ is a proper metric space,
- $B \subseteq X$ is a compact subset and $T \in C^*(X)$ is an operator with
  \[ \text{supp} T \subseteq B^2 \]
  then $T$ is a compact operator;
- The converse does not hold. If $T \in C^*(X)$ is a compact operator then there does not necessarily exist a bounded set $B \subseteq X^2$ such that $\text{supp} T \subseteq B^2$;
- The $C^*$-algebra of compact operators $\mathbb{K}(\mathcal{H}_X)$ is an ideal in $C^*(X)$.

**Proof**
- Suppose there is a non-degenerate representation $\rho : C_0(X) \to \mathbb{B}(\mathcal{H}_X)$. For every $f \in C_0(B^c), g \in C_0(X)$ the equations $\rho(f)T\rho(g) = 0$ and $\rho(g)T\rho(f) = 0$ hold. This implies $T(I(B)) = 0$ and $\text{im } T \cap I(B) = 0$. Thus $T : \mathcal{H}_B \to \mathcal{H}_B$ is the same map. Thus $T \in C^*(B)$ already. Now $T$ is locally compact, $B$ is compact thus $T$ is a compact operator.
- Note the set of ghost operators as defined in [9, Definition 1.2] contains the compact operators. The space $X$ has property A if and only if every ghost operator is compact by [9, Theorem 1.3]. Not all of them have bounded support. The paper...
[2] shows that there exists a metric space \( O \) with non-compact ghost projections in \( \mathcal{B}(L_2(O)) \). This implies that \( O \) does not coarsely embed into Hilbert space. More precisely if \( X \) is the space \( \mathbb{Z} \), the Hilbert space \( \mathcal{H}_X := \ell^2(X) \otimes \ell^2(\mathbb{N}) \) the standard ample \( X \)-module, \( D \) is a diagonal operator with coefficients in \( C_0(X) \) and \( e \) is a finite rank projection then \( T := D \otimes e \in C^*(X, \mathcal{H}_X) \) is compact and has support the full diagonal which cannot be contained in \( B \times B \) with \( B \) bounded.

This is already [7, Lemma 4.12]. For the convenience of the reader, we recall the proof: Let \( K \subseteq X \) be a set and \( v \in \mathcal{H}_X \) be a vector with \( \text{supp } v \subseteq K \). Then for every \( f \in I(K) \), we obtain \( \rho(f)v = 0 \). Now a vector in \( \mathcal{H}_{I(K)} \) can be written as \( \rho(f^*)w \) where \( f \in I(K) \), \( w \in \mathcal{H}_X \). Then

\[
\langle v, \rho(f)w \rangle = \langle \rho(f^*)v, w \rangle = 0
\]

Thus \( v \in \mathcal{H}_K \). If on the other hand \( v' \in \mathcal{H}_K \) is any vector then

\[
0 = \langle \rho(f)w, v' \rangle = \langle w, \rho(f^*)v' \rangle
\]

for every \( f \in I(K) \), \( w \in \mathcal{H}_X \). This implies \( \rho(f^*)v' = 0 \) for every \( f \in I(K) \). Thus \( \text{supp } v' \subseteq K \). Since \( X \) can be written as a union of bounded sets, \( X = \bigcup B_i \) with \( B_i \) bounded for every \( i \), the vectors with compact support form an orthonormal basis of \( \mathcal{H}_X \).

A finite rank operator \( T \) with respect to this basis belongs to \( C^*(X) \): First of all, \( T \) is locally compact since it is compact. We can write

\[
T : h \mapsto \sum_{i=1}^n \alpha_i \langle h, v_i \rangle u_i
\]

here \( \alpha_i \geq 0 \) and \( v_i, u_i \) are vectors with compact support \( \text{supp } v_i \subseteq B_i \), \( \text{supp } u_i \subseteq A_i \) for \( 1 \leq i \leq n \). Let \( v \in \mathcal{H}_X \) be a vector. If \( g \in I(B_i) \) and \( f \in C_0(X) \) are two functions then

\[
\alpha_i \langle \rho(g)v, v_i \rangle \rho(f)u_i = \alpha_i \langle v, \rho(g^*)v_i \rangle \rho(f)u_i = 0
\]

Now let \( g \in C_0(X), f \in I(A_i) \) be functions. Then \( \alpha_i \langle \rho(g)v, v_i \rangle \rho(f)u_i = 0 \) since \( \text{supp } u_i \subseteq A_i \). Thus \( \text{supp } T \subseteq \bigcup_{i=1}^n A_i \times B_i \) is controlled.

Since the finite rank operators are dense in the compact operators \( \mathbb{K}(\mathcal{H}_X) \), we obtain the inclusion \( \mathbb{K}(\mathcal{H}_X) \subseteq C^*(X) \). Since the composition with a compact operator yields a compact operator, the subset \( \mathbb{K}(\mathcal{H}_X) \) is an ideal in \( C^*(X) \).

\[\square\]

**Definition 10** Let \( X \) be a proper metric space then

\[
\hat{C}^*(X) = C^*(X)/\mathbb{K}(\mathcal{H}_X)
\]
where $\mathbb{K}(\mathcal{H}_X)$ denotes the compact operators of $\mathcal{B}(\mathcal{H}_X)$ is called the Roe–Calkin algebra of $X$.

We want to assign a $\text{C}^*$-algebra to every subset $U \subseteq X$.

**Remark 11** If $U \subseteq X$ is a subset of a proper metric space then the inclusion $U \to \bar{U}$ is coarsely surjective which means that there is some $R \geq 0$ such that every point of $\bar{U}$ lies in an $R$-neighborhood of $U$. We define

$$\hat{\text{C}}^*(U) := \hat{\text{C}}^*(\bar{U})$$

This way we can use Lemma 7 to restrict representations and elements of the Roe–Calkin algebra to subspaces.

**Lemma 12** If $Y \subseteq X$ is a closed subspace and $i_Y : \mathcal{H}_Y \to \mathcal{H}_X$ the inclusion operator of Lemma 7 then

- the operator

$$\text{Ad}(i_Y) : \text{C}^*(Y) \to \text{C}^*(X)$$

$$T \mapsto i_Y T i_Y^*$$

is well defined and maps compact operators to compact operators.

- Then the induced operator on quotients

$$\hat{\text{Ad}}(i_Y) : \hat{\text{C}}^*(Y) \to \hat{\text{C}}^*(X)$$

is the dual version of a restriction map, which means $U \mapsto \hat{\text{C}}^*(U)$ is a precosheaf on $X$.

**Proof**

- $i_Y$ covers the inclusion the other statement follows since composition with compact operators gives a compact operator.
- The assignment is a covariant functor.

$\square$

**Theorem 13** If $X$ is a proper metric space, then the assignment

$$U \mapsto \hat{\text{C}}^*(U)$$

for every subspace $U \subseteq X$ is a cosheaf with values in $\text{CSt}ar$.

**Proof** Let $U_1, \ldots, U_n \subseteq U$ be subsets that coarsely cover $U \subseteq X$ and $V_i : \mathcal{H}_{U_i} \to \mathcal{H}_U$ and $V_{ij} : \mathcal{H}_{U_{ij}} \to \mathcal{H}_{U_i}$ the corresponding inclusion operators for $i, j = 1, \ldots, n$.

Let $T \in \text{C}^*(U)$ be a locally compact controlled operator. We need to construct $T_i \in \text{C}^*(U_i)$ such that

$$\sum_i V_i T_i V_i^* = T$$
modulo compacts. Denote by $E = \text{supp}(T)$ the support of $T$ in $U$. Define

$$T_i := V_i^*TV_i - \sum_{j=1}^{i-1} V_{ij}V_{ij}^*T_jV_{ij}$$

then the $T_i$ are locally compact controlled operators, thus elements in $C^*(U_i)$. We now show that $T - \sum_i V_i T_i V_i^* = 0$ on $U_1 \times U_1 \cup \cdots \cup U_n \times U_n$ which shows the result since $E \cap (U_1^2 \times \cdots \times U_n^2)^c$ is bounded. Let $(x, y) \in (U_1^2 \times U_n^2) \cap \text{supp} T$ be a point and choose $k$ minimal with $(x, y) \in U_k \times U_k$. Then $T_k(x) = T(x) = y$. We now show $(x, y) \notin \text{supp} T_i$ for $i \neq k$. For $i = 1, \ldots, k - 1$, this is clear since $k$ is minimal. We now use induction on $i = k + 1, \ldots, n$: If $i = k + 1$ then $(x, y) \in \text{supp}(V_{k+1}^*T V_{k+1})$ exactly when $(x, y) \in \text{supp}(V_{k+1, k} V_{k+1, k+1}^* T_k V_{k, k+1}^* V_{k+1, k+1})$. Thus $(x, y) \notin \text{supp} T_{k+1}$. If the statement holds for $k + 1, \ldots, i$ then $(x, y) \in \text{supp}(V_i^*TV_i+1)$ exactly when $(x, y) \in \text{supp}(V_{i+1, k} V_{i+1, k+1}^* T_k V_{k, i+1} V_{k+1, i+1}^* V_{i+1, k+1})$. Thus $(x, y) \notin \text{supp} T_{i+1}$. This implies $\sum T_i|_U = T$, axiom 1).

Suppose $T_i \in C^*(U_i)$ are elements with

$$\sum_i V_i T_i V_i^* = 0$$

modulo compacts. Denote by $V_{ijk} : \mathcal{H}_{U_i \cap U_j \cap U_k} \to \mathcal{H}_{U_i \cap U_j}$ the covering isometry operator associated to the inclusion $U_i \cap U_j \cap U_k \to U_i \cap U_j$. Define for $1 \leq i < j \leq n$:

$$T_{ij} := V_{ij}^*T_i V_{ij} + \sum_{k<i} V_{ijk}V_{ikj}^*T_{ki}V_{klj}V_{ljk}^* - \sum_{i<k<j} V_{ijk}V_{ikj}^*T_{ik}V_{ikj}V_{ljk}^*.$$  

Using $V_i V_{ij} = V_j V_{ji}$ and combinatorical information, we can show

$$\sum_j (T_{ij} - T_{ji})|_{U_i} = \sum_{i<j} T_{ij}|_{U_i} - \sum_{j>i} T_{ji}|_{U_i} = T_i$$

modulo compacts. Thus axiom 2) of a cosheaf holds. \hfill \Box

4 Computing Examples

**Corollary 14** If $U_1, U_2$ coarsely cover a subset $U$ of a proper metric space $X$ then there is a six-term Mayer–Vietoris exact sequence

$$K_1(\hat{C}^*(U_1 \cap U_2)) \to K_1(\hat{C}^*(U_1)) \oplus K_1(\hat{C}^*(U_2)) \to K_1(\hat{C}^*(U))$$

$$K_0(\hat{C}^*(U)) \leftarrow K_0(\hat{C}^*(U_1)) \oplus K_0(\hat{C}^*(U_2)) \leftarrow K_0(\hat{C}^*(U_1 \cap U_2))$$
Proof If \( A \subseteq X \) is a subset define \( C^*(A, X) \) to be the \( C^* \)-algebra generated by all locally compact operators with finite propagation on \( \mathcal{H}_X \) whose support is contained in \( E[A] \times E[A] \) for some entourage \( E \subseteq X \times X \). It can be observed that \( C^*(A, X) \) forms an ideal in \( C^*(X) \). The inclusion \( C^*(A) \to C^*(A, X) \) induces an isomorphism on the \( K \)-theory of the algebras obtained by modding out \( \mathbb{K}((\mathcal{H}_X)) \). That is because we have a commuting diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & \mathbb{K}((\mathcal{H}_X)) & \to & C^*(A) & \to & \hat{C}^*(A) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{K}((\mathcal{H}_X)) & \to & C^*(A, X) & \to & C^*(A, X)/\mathbb{K}((\mathcal{H}_X)) & \to & 0
\end{array}
\]

where the left vertical arrow is an isomorphism, the middle vertical arrow induces an isomorphism in \( K \)-theory by [4, Proposition 6.4.7]. By the five lemma, the right vertical arrow induces an isomorphism in \( K \)-theory.

We define \( I_1 := C^*(U_1, X)/\mathbb{K}((\mathcal{H}_X)), I_2 := C^*(U_2, X)/\mathbb{K}((\mathcal{H}_X)) \) and \( I_{12} = C^*(U_1 \cap U_2, X)/\mathbb{K}((\mathcal{H}_X)) \). They are the smallest ideals containing \( \hat{C}^*(U_1), \hat{C}^*(U_2) \) and \( C^*(U_1 \cap U_2) \), respectively. Now Theorem 13 and the first cosheaf axiom imply \( \hat{C}^*(U) = \hat{C}^*(U_1) + \hat{C}^*(U_2) \) and the second cosheaf axiom implies \( \hat{C}^*(U_1) \cap \hat{C}^*(U_2) = \hat{C}^*(U_1 \cap U_2) \). This implies \( I_1 + I_2 = C^*(U, X)/\mathbb{K}((\mathcal{H}_X)) \) and \( I_1 \cap I_2 = I_{12} \). With those properties and [4, Exercise 4.10.21], the six-term exact sequence in \( K \)-theory is obtained. \( \square \)

Remark 15 Now for every proper metric space there is a short exact sequence

\[
0 \to \mathbb{K}((\mathcal{H}_X)) \to C^*(X) \to \hat{C}^*(X) \to 0
\]

which induces a 6-term sequence in \( K \)-theory:

\[
\begin{array}{cccccc}
K_0(\mathbb{K}((\mathcal{H}_X))) & \to & K_0(C^*(X)) & \to & K_0(\hat{C}^*(X)) & \\
& \uparrow & & \downarrow & & \\
K_1(\hat{C}^*(X)) & \leftarrow & K_1(C^*(X)) & \leftarrow & K_1(\mathbb{K}((\mathcal{H}_X))) &
\end{array}
\]

If \( X \) is flasque [10] then

\[
K_i(\hat{C}^*(X)) = \begin{cases} 
0 & i = 0 \\
\mathbb{Z} & i = 1
\end{cases}
\]

Remark 16 Note that the result of Corollary 14 is applicable when computing controlled \( K \)-theory if the property ample is preserved by restricting the representation of \( U \) to the representations of \( U_1, U_2 \).

If \( X \) is a Riemannian manifold then fixing a volume form \( \nu \) the Hilbert space \( \mathcal{H}_X = L^2(X, \nu) \otimes \ell^2 \) is an ample \( X \)-module with \( \ell^2 \) the standard separable Hilbert space and
\[ \rho_X : C_0(X) \rightarrow \mathbb{B}(\mathcal{H}_X) \] trivial on the second factor. In Example 17, Example 18, we will use this canonical representation on \( \mathbb{R}, \mathbb{R}^2 \) and certain subspaces of them without mentioning it. In those cases, the property ample is preserved by restricting \( \mathbb{R} \) to \( \mathbb{R}_{\geq 0} \), the space \( \mathbb{R}^2 \) to \( V_1, V_2 \) and \( V_1 \cap V_2 \) to \( U_1, U_2 \), respectively.

**Example 17** (\( \mathbb{R} \)) Now \( \mathbb{R} \) is the coarse disjoint union of two copies of \( \mathbb{R}_{\geq 0} \) which is a flasque space. By Corollary 14, there is an isomorphism

\[
K_i(\hat{C}^*(\mathbb{R})) = \begin{cases}
0 & i = 0 \\
\mathbb{Z} \oplus \mathbb{Z} & i = 1
\end{cases}
\]

Applying Remark 15, we can compare

\[
K_i(C^*(\mathbb{R})) = \begin{cases}
0 & i = 0 \\
\mathbb{Z} & i = 1
\end{cases}
\]

in [4, Theorem 6.4.10], if it matches our computation. And indeed it does.

**Example 18** (\( \mathbb{R}^2 \)) We coarsely cover \( \mathbb{R}^2 \) with \( V_1 = \mathbb{R}_{\geq 0} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{R}_{\geq 0} \) and \( V_2 = \mathbb{R}_{<0} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{R}_{<0} \). The space \( V_1 \cap V_2 \) is coarsely covered by \( U_1 = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \) and \( U_2 = \mathbb{R}_{<0} \times \mathbb{R}_{<0} \). The second cover and Corollary 14 gives

\[
K_i(\hat{C}^*(V_1 \cap V_2)) = \begin{cases}
0 & i = 0 \\
\mathbb{Z} \oplus \mathbb{Z} & i = 1
\end{cases}
\]

Since \( V_1, V_2 \) are coarsely equivalent to flasque spaces, the first cover and Corollary 14 imply that \( K_0(\hat{C}^*(\mathbb{R}^2)) \) and \( K_1(\hat{C}^*(\mathbb{R}^2)) \) have the same (free abelian) rank. Translating back using Remark 15, the groups

\[
K_i(C^*(\mathbb{R}^2)) = \begin{cases}
\mathbb{Z} & i = 0 \\
0 & i = 1
\end{cases}
\]

of [4, Theorem 6.4.10] also fit in the exact sequence.

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