Two-step hyperentanglement purification with the quantum-state-joining method

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Hyperentanglement is a promising resource in quantum information processing, especially for increasing the channel capacity of long-distance quantum communication. Hyperentanglement purification is an important method to obtain high-fidelity nonlocal hyperentangled states from mixed hyperentangled states in long-distance quantum communication process with noisy channels. Here, we present a two-step hyperentanglement purification protocol for nonlocal mixed hyperentangled states with polarization bit-flip errors and spatial-mode phase-flip errors, resorting to the polarization-spatial phase-check quantum nondemolition detector and the quantum-state-joining method (QSJM). With QSJM, it can preserve the states that are discarded in the previous hyperentanglement purification protocols. It has the advantage of a high efficiency and it is useful for improving the entanglement of photon systems with several degrees of freedom in long-distance high-capacity quantum communication.

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I. INTRODUCTION

Entanglement is an essential quantum resource in quantum information processing and it improves the methods of manipulating and transmitting information in quantum computation and quantum communication [1]. The maximally entangled photon system, used as quantum channel, is a core resource in quantum communication [2–13]. With maximally entangled photon pairs, the two remote parties in quantum communication, say Alice and Bob, can implement faithful teleportation [2]. That is, they teleport the unknown quantum state of a single particle without moving the particle itself [2]. Also, Alice can transmit two or more bits of information to Bob by moving only a photon forth and back with quantum dense coding protocols based on a maximally entangled photon pair [3, 4]. They can create a private key with maximally entangled photon pairs, resorting to quantum key distribution protocols [5–7], even in the case with a collective-noise optical-fiber channel [8]. With quantum secure direct communication protocols based on entangled photon pairs [8, 10], Alice and Bob can transmit their secret message directly as well, without creating a private key in advance. With entanglement-based quantum secret sharing protocols [11, 13], the parities in quantum communication can share some private keys with a potentially dishonest party.

Hyperentanglement, defined as the entanglement in several degrees of freedom (DOFs) of a quantum system [14–16], has attracted much attention for improving the power of quantum computation (e.g., hyperparallel photonic quantum computation [17, 18]) and increasing the channel capacity of long-distance quantum communication, such as photonic superdense coding with the polarization-orbital-angular-momentum hyperentanglement [19], quantum teleportation with photon pairs entangled in two DOFs [20], entanglement swapping based on photonic spatial-polarization hyperentanglement [20, 21], hyperentangled Bell-state analysis [20–24], hyperentanglement concentration [25, 26], and so on. Hyperentanglement has also been used to assist the long-distance quantum communication in one DOF of photon systems, such as quantum repeater [27], complete Bell-state analysis [28–32], and deterministic entanglement purification [32, 33] on the polarizations of photon pairs. In long-distance quantum communication protocols, the photon signals can only be transmitted no more than several hundreds of kilometers over an optical fiber or a free space, and quantum repeaters are required in this condition. The (hyper-)entangled photon systems are always produced locally, and they inevitably suffer from environment noise in their distribution process and storage process in quantum communication. In general, the maximally (hyper-)entangled photon states may decay into less entangled pure states or even into mixed states, which decreases the fidelity and the security of long-distance quantum communication protocols.

Entanglement purification is an interesting passive way to suppress the effect of the environment noise in quantum communication process. Entanglement purification is used to distill some quantum systems in high-fidelity entangled states from those in less entangled mixed states [32–52]. In 1996, Bennett et al. [37] introduced the first entanglement purification protocol (EPP) for quantum systems in a Werner state with quantum controlled-not (CNOT) gates. Since this pioneering work, some interesting EPPs were proposed for improving the polarization entanglement of photon systems with either nonlinear optical elements [32, 37, 40] or linear optical elements [33–35, 47, 49]. In 2001, Pan et al. [47] presented a polarization EPP with linear optics. In 2002, Simon and Pan proposed a polarization EPP assisted by hyperentanglement...
with the available parametric down-conversion (PDC) resource [48], which was demonstrated by Pan et al. [49] in 2003. In 2008, Sheng et al. [39] proposed an efficient polarization EPP with hyperentanglement based on a PDC resource, resorting to cross-Kerr nonlinearity. In 2010, Sheng et al. introduced the concept of deterministic entanglement purification for two-photon systems with hyperentanglement [32]. Subsequently, some interesting deterministic EPPs were proposed [32–36]. The EPPs for multipartite entangled photon systems in mixed polarization Greenberger-Horne-Zeilinger (GHZ) states have also been investigated with nonlinear optical elements in the past few years [40–41]. In 2013, a hyperentanglement purification protocol (hyper-EPP) was proposed for two-photon systems in mixed polarization-spatial hyperentangled Bell states with polarization bit-flip errors and spatial-mode bit-flip errors resorting to nonlinear optical elements [44]. However, the previous hyper-EPP for photon systems in mixed polarization-spatial hyperentangled Bell states is implemented by purifying the polarization states and the spatial-mode states of photon pairs independently. In 2013, Vitelli et al. demonstrated experimentally the quantum-state-joining process with linear optical elements, which can combine the two-dimensional quantum states of two input photons into an output single photon.

In this paper, we present a two-step high-efficiency hyper-EPP for nonlocal photon systems in mixed polarization-spatial hyperentangled states, resorting to the polarization-spatial phase check quantum nonde-molition detectors (P-S-QNDs) and the quantum-state-joining method (QSJM). Our P-S-QND is implemented with the hybrid CNOT gate [43] based on the giant optical circular birefringence (GOCB) of the quantum-dot (QD) spin inside a double-sided optical microcavity (a QD-cavity system). And our QSJM is also constructed with the GOCB of a double-sided QD-cavity system. With our QSJM, the mixed hyperentangled states with only one DOF in the preserving condition can be combined into the mixed hyperentangled states with both two DOFs in the preserving condition, and the efficiency of our two-step hyper-EPP can be improved largely by preserving the states that are discarded in the previous hyper-EPP, compared with the hyper-EPP without QSJM. It is very useful for improving the entanglement of photon systems with several DOFs in long-distance high-capacity quantum communication.

This paper is organized as follows: In Sec. II A we introduce the GOCB of a quantum-dot spin inside double-sided optical microcavity. Based on GOCB, we give the ways for our QSJM and our P-S-QND in Secs. II B and II C, respectively. In Sec III we introduce our two-step high-efficiency hyper-EPP for nonlocal two-photon systems with QSJM. A high-efficiency hyper-EPP for hyperentangled three-photon GHZ states with QSJM is given in Sec. IV. A discussion and a summary are given in Sec. V. In Appendix A we give the detailed calculation for the fidelities and efficiencies of P-S-QND and QSJM.

II. QUANTUM-STATE-JOINING METHOD AND POLARIZATION-SPATIAL PHASE-CHECK QND

A. Giant optical circular birefringence of a quantum-dot electron spin in a double-sided optical resonant microcavity

The QSJM and P-S-QND in our proposal are both constructed with the optical property of a singly charged QD [e.g., a self-assembled In(Ga)As QD or GaAs interfacial QD] embedded in a double-sided optical resonant microcavity, as shown in Fig. I(a). The two distributed Bragg reflectors of the double-sided optical resonant microcavity are partially reflective and low loss for on-resonance transmission, which support both the two polarization modes of photon systems [51]. If an excess electron is injected into the QD, the singly charged QD has the optical resonance with the creation of a negatively charged exciton $X^-$, which consists of two antiparallel electron spins bound to one hole [55], and it shows spin-dependent transitions with circularly polarized lights according to Pauli’s exclusion principle [56] (shown in Fig. 1 (b)). If the excess electron is in the spin state $|\uparrow\rangle$, a circularly polarized photon with the spin $S_z = +1$ is absorbed to create the negatively charged exciton $X^-$ in the state $|\uparrow\downarrow\rangle$. While for the excess electron-spin state $|\downarrow\rangle$, a circularly polarized photon with spin $S_z = -1$ is absorbed to create the negatively charged exciton $X^-$ in the state $|\downarrow\uparrow\rangle$. Here $|\uparrow\rangle (|\downarrow\rangle)$ represents the heavy-hole spin $|+\frac{3}{2}\rangle (|\mp\frac{1}{2}\rangle)$, and $|\uparrow\downarrow\rangle$ represents the excess electron spin $|+\frac{3}{2}\rangle (-\frac{1}{2}\rangle$).

The input-output optical property of the double-sided QD-cavity system can be described by the Heisenberg equations of motion for the cavity field operator $\hat{a}_r$ and $X^-$ dipole operator $\hat{\sigma}_-$ in the interaction picture [57, 58],

$$\frac{d\hat{a}_r}{dt} = -i(\omega_c - \omega) + \kappa \hat{a}_r \hat{a} - g \hat{\sigma}_-, \quad -\sqrt{\kappa} \hat{a}_r \hat{\alpha}_n,$$

$$\frac{d\hat{\sigma}_-}{dt} = -i(\omega_{X^-} - \omega) + \frac{\kappa}{2} \hat{\sigma}_- - g \hat{\sigma}_r \hat{a}_r,$$

$$\hat{a}_r = \hat{a}_n + \sqrt{\kappa} \hat{\alpha}, \quad \hat{\alpha}_n = \hat{\alpha}_n + \sqrt{\kappa} \hat{a}_r, \quad \hat{\sigma}_- = \hat{\sigma}_- - g \hat{\sigma}_r \hat{a}_r,$$

where $\omega_c$, $\omega_{X^-}$, and $g$ are the frequencies of the cavity mode, the input photon, and $X^-$ transition, respectively. $\kappa$ is the cavity decay rate. $\alpha$ is the coupling strength of the negatively charged exciton $X^-$ and the cavity mode. $\gamma/2$ is the decay rate of $X^-$. $\kappa$ is the decay rate of the cavity field mode, and $\kappa_S/2$ is the decay rate of the cavity field mode to the cavity side leakage mode. $\hat{a}_n$, $\hat{\alpha}_n$, and $\hat{\sigma}_-$ are the input and the output field operators of the double-sided QD-cavity system, respectively.

In the weak excitation condition, $X^-$ is dominant in the ground state with $\langle \hat{\sigma}_- \rangle = -1$, and the reflection ($\hat{r}(\omega))$ and transmission ($\hat{t}(\omega)$) coefficients of the double-
the photon in the state $|\phi_{\text{c}}\rangle$ is transmitted through the QD-cavity system with a phase shift relative to the reflected photon (QD is decoupled to the cavity). For the electron spin state $|\downarrow\rangle$, the input circularly polarized photon $|R\rangle$ (|L\rangle) is transmitted through the QD-cavity system, and the input circularly polarized photon $|R\rangle$ (|L\rangle) is reflected by the QD-cavity system. That is, the reflection and the transmission rules of the photon polarization states can be summarized as follows [18, 58],

$$
|L_{r},i_2,\uparrow\rangle \rightarrow |L_{t},i_2,\uparrow\rangle, \quad |L_{t},i_1,\downarrow\rangle \rightarrow |R_{t},i_1,\downarrow\rangle,
$$

$$
|R_{r},i_2,\downarrow\rangle \rightarrow -|R_{t},i_2,\downarrow\rangle, \quad |L_{t},i_1,\uparrow\rangle \rightarrow -|L_{t},i_1,\uparrow\rangle,
$$

$$
|R_{t},i_1,\downarrow\rangle \rightarrow |R_{t},i_1,\downarrow\rangle, \quad |L_{t},i_2,\downarrow\rangle \rightarrow |R_{t},i_2,\downarrow\rangle. \quad (4)
$$

Here $i_1$ and $i_2$ ($i=a,b$) are the two spatial modes of the photon $i$ (shown in Fig. 2).

### B. Quantum-state-joining method

Our QSSM for photonic states introduced here is used to transfer the polarization state of the photon $A$ into the polarization state of the photon $B$ without disturbing the spatial-mode state of the photon $B$, resorting to the reflection-transmission optical property of the double-sided QD-cavity system, as shown in Fig. 2 (a). The initial states of the two photons $A$ and $B$ are

$$
|\phi_{A}\rangle = (\alpha_1 |R\rangle + \alpha_2 |L\rangle)_A (|\gamma_1|a_1\rangle + |\gamma_2|a_2\rangle),
$$

$$
|\phi_{B}\rangle = (|\beta_1|R\rangle + |\beta_2|L\rangle)_B (|\delta_1|b_1\rangle + |\delta_2|b_2\rangle). \quad (5)
$$

The excess electron spin in QD is prepared in the state $|\uparrow\rangle_{e} = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)_{e}$.

We put the photon $A$ into the circularly polarizing beam splitter CPBS$_1$, the wave plate $U_1$, QD, U$_2$, CPBS$_2$, and the half-wave plate $X$ in sequence, as shown in Fig. 2 (a). After the photon $A$ passes through the quantum circuit shown in Fig. 2 (a), the state of the quantum system composed of QD and the photon $A$ is transformed from $|\phi_{AE}\rangle_0$ to $|\phi_{AE}\rangle_1$. Here

$$
|\phi_{AE}\rangle_0 = |\uparrow\rangle_{e} \otimes |\phi_{A}\rangle,
$$

$$
|\phi_{AE}\rangle_1 = \frac{1}{\sqrt{2}}[(|R\rangle_A (|\alpha_1|\uparrow\rangle + |\alpha_2|\downarrow\rangle)_{e} + |L\rangle_A (|\alpha_2|\uparrow\rangle + |\alpha_1|\downarrow\rangle)_{e} + |\gamma_1|a_1\rangle + |\gamma_2|a_2\rangle]. \quad (6)
$$

By measuring the polarization state of the photon $A$ in the orthogonal basis $\{|R\rangle, |L\rangle\}$, the polarization state of the photon $A$ is transformed into the excess electron spin state in QD.

We assume the excess electron spin in QD is in the state $|\phi_{e}\rangle = (|\alpha_1|\uparrow\rangle + |\alpha_2|\downarrow\rangle)_{e}$ (the polarization state of the photon $A$ is projected into the state $|R\rangle$). The photon $B$ is put into the quantum circuit shown in Fig. 2 (a) after a Hadamard operation performed on the excess electron spin in QD, and the state of the system...
By measuring the excess electron spin $e$ in the orthogonal basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, the state of the excess electron spin $e \langle |\phi_e\rangle = (|\alpha_1\rangle |\uparrow\rangle + |\alpha_2\rangle |\downarrow\rangle)_{e}$ is transformed into the polarization state of the photon $B$ without disturbing its spatial-mode state. If the polarization state of the photon $A$ is projected into the state $|L\rangle$, a bit-flip operation $(\sigma_x^B = |R\rangle\langle L| + |L\rangle\langle R|)$ is performed on the polarization DOF of the photon $B$, and a phase-flip operation $(\sigma_y^B = |R\rangle\langle R| - |L\rangle\langle L|)$ is performed on the polarization DOF of the photon $B$ if the excess electron spin $e$ is projected into the state $|\downarrow\rangle_e$. After the conditional operations performed on the polarization DOF of the photon $B$, the state of the photon $B$ is changed to be $|\phi_B\rangle_f = (|\alpha_1\rangle |R\rangle + |\alpha_2\rangle |L\rangle)_{B}(\delta_1|b_1\rangle + \delta_2|b_2\rangle)$. This is the result of the QSJM for combining the polarization state of the photon $A$ and the spatial-mode state of the photon $B$ into an output single photon.

The QSJM can also be used to transfer the spatial-mode state of the photon $A$ into the polarization state of the photon $B$. This requires the photon $A$ pass through the quantum circuit shown in Fig. 2 (b), which can swap the polarization state and the spatial-mode state of the photon $A$. Subsequently, we can transfer the polarization state of the photon $A$ into the polarization state of the photon $B$ with the quantum circuit shown in Fig. 2 (a). Then the result of the QSJM for combining the spatial-mode state of the photon $A$ and the spatial-mode state of the photon $B$ into a single output photon is achieved.

C. Polarization-spatial phase-check QND

Now, we introduce the construction of the polarization-spatial phase-check QND (P-S-QND), which is used to distinguish the hyperentangled Bell states with relative phase “0” from those with relative phase “π” in both the polarization and the spatial-mode DOFs. Our P-S-QND is implemented with the hybrid CNOT gate introduced in our previous work [13], as shown in Fig. 3 (a). The 16 polarization-spatial hyperentangled Bell states are defined as $|\phi_k\rangle_{AB} = |\phi_k\rangle^P_{AB} \otimes |\phi_1\rangle^S_{AB}$ ($k = 1, 2, 3, 4$), where

\[
|\phi_1\rangle^P_{AB} = \frac{1}{\sqrt{2}}(|RR\rangle + |LL\rangle)_{AB},
|\phi_2\rangle^P_{AB} = \frac{1}{\sqrt{2}}(|RR\rangle - |LL\rangle)_{AB},
|\phi_3\rangle^P_{AB} = \frac{1}{\sqrt{2}}(|RL\rangle + |LR\rangle)_{AB},
|\phi_4\rangle^P_{AB} = \frac{1}{\sqrt{2}}(|RL\rangle - |LR\rangle)_{AB},
|\phi_1\rangle^S_{AB} = \frac{1}{\sqrt{2}}(|a_1b_1\rangle + |a_2b_2\rangle),
|\phi_2\rangle^S_{AB} = \frac{1}{\sqrt{2}}(|a_1b_1\rangle - |a_2b_2\rangle),
|\phi_3\rangle^S_{AB} = \frac{1}{\sqrt{2}}(|a_1b_2\rangle + |a_2b_1\rangle),
|\phi_4\rangle^S_{AB} = \frac{1}{\sqrt{2}}(|a_1b_2\rangle - |a_2b_1\rangle),
\]
Here the subscript $AB$ represents a photon pair $AB$, and the superscripts $P$ and $S$ represent the polarization and the spatial-mode DOFs of a photon system, respectively.

\[
|\phi_{4}\rangle_{AB}^{P} = \frac{1}{\sqrt{2}}(a_1b_2 - a_2b_1).
\]

The initial states of the excess electron spins $e_1$ in QD$_1$ and $e_2$ in QD$_2$ are $|+\rangle_{e_1} = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)_{e_1}$ and $|+\rangle_{e_2} = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)_{e_2}$ respectively. If we have the photons $A$ and $B$ pass through the quantum circuit of the hybrid CNOT gate shown in Fig. 3 (a) in sequence, the state of the system composed of the photons $AB$ and the excess electron spins $e_1e_2$ (in QD$_1$ and QD$_2$, respectively) is transformed into

\[
|\phi_{k_1k_2}\rangle_{AB}|+\rangle_{e_1}|+\rangle_{e_2} \rightarrow |\phi_{k_1k_2}\rangle_{AB}|+\rangle_{e_1}|-\rangle_{e_2},
\]

\[
|\phi_{k_1k_2}\rangle_{AB}|+\rangle_{e_1}|+\rangle_{e_2} \rightarrow |\phi_{k_1k_2}\rangle_{AB}|+\rangle_{e_1}|+\rangle_{e_2},
\]

\[
|\phi_{k_1k_2}\rangle_{AB}|+\rangle_{e_1}|+\rangle_{e_2} \rightarrow |\phi_{k_1k_2}\rangle_{AB}|+\rangle_{e_1}|+\rangle_{e_2},
\]

\[
|\phi_{k_1k_2}\rangle_{AB}|+\rangle_{e_1}|+\rangle_{e_2} \rightarrow |\phi_{k_1k_2}\rangle_{AB}|+\rangle_{e_1}|+\rangle_{e_2},
\]

where $k_1(k_1) = 1, 3$, $k_2(l_2) = 2, 4$, and $|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$. By measuring the spin states of the excess electrons $e_1$ and $e_2$ in the orthogonal basis $\{|+\rangle, |-\rangle\}$, we can distinguish the hyperentangled Bell states with the relative phase “0” from those with the relative phase “π” in both the polarization and the spatial-mode DOFs. If the excess electron spin $e_1$ ($e_2$) is projected into the state $|+\rangle_{e_1}$ ($|\downarrow\rangle_{e_2}$), the polarization (spatial-mode) state of the hyperentangled Bell state has the relative phase “0”. Otherwise the polarization (spatial-mode) state of the hyperentangled Bell state has the relative phase “π”.

If we perform Hadamard operations on both the spatial-mode and the polarization DOFs of the photons $A$ and $B$ before (and after) we put them into the quantum circuit of our P-S-QND as shown in Fig. 3 (b), the odd-parity modes and the even-parity modes of the hyperentangled Bell states can be distinguished in both the polarization and the spatial-mode DOFs, which is the result of the polarization-spatial parity-check QND.

**III. TWO-STEP HYPER-EPP FOR MIXED HYPERENTANGLED BELL STATES WITH THE QUANTUM-STATE-JOINING METHOD**

In this section, we introduce our two-step hyperentanglement purification for mixed hyperentangled Bell states with polarization bit-flip errors and spatial-mode phase-flip errors, resorting to our P-S-QND and our QSJM introduced in Sec. III. The principle of our two-step hyper-EPP with QSJM is shown in Fig. 4. It includes two steps in each round of the hyper-EPP process, and they are discussed in detail as follows.

**A. The first step of our two-step hyper-EPP with P-S-QNDs**

In this step, two identical two-photon systems in a mixed hyperentangled Bell state are required. They are described as follows:

\[
\rho_{AB} = [F_1|\phi_1^{P}_{AB}\rangle\langle\phi_1| + (1 - F_1)|\phi_3^{P}_{AB}\rangle\langle\phi_3|] \otimes [F_2|\phi_1^{S}_{AB}\rangle\langle\phi_1| + (1 - F_2)|\phi_2^{S}_{AB}\rangle\langle\phi_2|],
\]

\[
\rho_{CD} = [F_1|\phi_1^{P}_{CD}\rangle\langle\phi_1| + (1 - F_1)|\phi_3^{P}_{CD}\rangle\langle\phi_3|] \otimes [F_2|\phi_1^{S}_{CD}\rangle\langle\phi_1| + (1 - F_2)|\phi_2^{S}_{CD}\rangle\langle\phi_2|].
\]

Here the subscripts $AB$ and $CD$ represent two photon pairs shared by the two remote parties, called Alice and Bob, respectively. The two photons $A$ and $C$ belong to Alice, and the two photons $B$ and $D$ belong to Bob. $F_1$ and $F_2$ represent the probabilities of the states $|\phi_1^{P}_{AB}\rangle$ ($|\phi_1^{P}_{CD}\rangle$) and $|\phi_1^{S}_{AB}\rangle$ ($|\phi_1^{S}_{CD}\rangle$) in the mixed states, respectively.

The initial state of the four-photon system $ABCD$ is $\rho_0 = \rho_{AB} \otimes \rho_{CD}$. It can be viewed as the mixture of 16 maximally hyperentangled pure states. That is, it is the mixture of the four pure states in the polarization DOF $|\phi_1^{P}_{AB}\rangle \otimes |\phi_1^{P}_{CD}\rangle$ with the probability $F_1^2$, $|\phi_1^{P}_{AB}\rangle \otimes |\phi_3^{P}_{CD}\rangle$ with the probability $F_1(1 - F_1)$, $|\phi_3^{P}_{AB}\rangle \otimes |\phi_1^{P}_{CD}\rangle$ with the probability $F_1(1 - F_1)$, and $|\phi_3^{P}_{AB}\rangle \otimes |\phi_3^{P}_{CD}\rangle$ with the probability $(1 - F_1)^2$, respectively; and it is the
spatial-mode DOFs of the photon pairs and P-S-QNDs on both the polarization and the spatial-mode phase-flip errors is shown in Fig. 4 (a), which probability (1 − Δ) 2

Alice and Bob measure the excess electron spins in QDs

resorting to the QSJM. Bob performs the same operations as Alice by replacing the photons A and C with the photons B and D, respectively. (b) Schematic diagram of the second step of our hyper-EPP for mixed hyperentangled Bell states with the photons 1′ and 2′, respectively. |Φ1⟩AB, |Φ2⟩AB, |Φ3⟩AB, and |Φ4⟩AB are distinguished from the other two mixed hyperentangled Bell states with the probability Δ2.

The first step of our hyper-EPP for mixed hyperentangled Bell states with polarization bit-flip errors and spatial-mode phase-flip errors is shown in Fig 4(a), which requires Alice and Bob perform the Hadamard operations and P-S-QNDs on both the polarization and the spatial-mode DOFs of the photon pairs AC and BD in sequence. After performing HPS on the polarization and the spatial-mode DOFs of the photon pairs AC and BD, Alice and Bob measure the excess electron spins in QDs to read out the results of the parity modes of the photon pairs AC and BD in both the polarization and the spatial-mode DOFs.

1) If the two photon pairs AC and BD are in the same polarization parity mode and the same spatial-mode parity mode (both in either the even-parity mode or the odd-parity mode), the polarization pure states |φ1⟩AB ⊗ |φ1⟩CD and |φ3⟩AB ⊗ |φ3⟩CD are distinguished from the other two polarization pure states, and the spatial-mode pure states |φ1⟩AB ⊗ |φ1⟩CD and |φ2⟩AB ⊗ |φ2⟩CD are distinguished from the other two spatial-mode pure states. After the measurement on the excess electron spins in P-S-QNDs, the polarization DOF of the four-photon system is projected into a mixed state composed of the states |Φ1⟩P (or |Φ2⟩P) and |Φ3⟩P (or |Φ4⟩P), and the spatial-mode DOF of the four-photon system is projected into a mixed state composed of the states |Φ1⟩S (or |Φ2⟩S) and |Φ3⟩S (or |Φ4⟩S).

Here

\[
|Φ1⟩P = \frac{1}{\sqrt{2}}(|RRRR⟩ + |LLLL⟩)_{ABCD},
\]

\[
|Φ2⟩P = \frac{1}{\sqrt{2}}(|RRLL⟩ + |LRLR⟩)_{ABCD},
\]

\[
|Φ3⟩P = \frac{1}{\sqrt{2}}(|RLRL⟩ + |LRLR⟩)_{ABCD},
\]

\[
|Φ4⟩P = \frac{1}{\sqrt{2}}(|RRRR⟩ + |LLLL⟩)_{ABCD},
\]

\[
|Φ1⟩S = \frac{1}{\sqrt{2}}(|a_1b_1c_1d_1⟩ + |a_2b_2c_2d_2⟩),
\]

\[
|Φ2⟩S = \frac{1}{\sqrt{2}}(|a_1b_1c_2d_2⟩ + |a_2b_2c_1d_1⟩),
\]

\[
|Φ3⟩S = \frac{1}{\sqrt{2}}(|a_1b_2c_1d_2⟩ + |a_2b_1c_2d_1⟩),
\]

\[
|Φ4⟩S = \frac{1}{\sqrt{2}}(|a_1b_2c_2d_1⟩ + |a_2b_1c_1d_2⟩).
\]

The state |Φ2⟩P can be transformed into |Φ1⟩P by performing polarization bit-flip operations (σ′ P) on the photons C and D. In a similar way, the states |Φ4⟩P, |Φ2⟩S, and |Φ4⟩S can be transformed into the states |Φ3⟩P, |Φ1⟩S, and |Φ3⟩S, respectively. Subsequently, Alice and Bob perform the Hadamard operations on both the polarization and the spatial-mode DOFs of the photons C and D, and the states |Φ1⟩P, |Φ3⟩P, |Φ1⟩S, and |Φ3⟩S are transformed into the states |Φ1⟩P, |Φ3⟩P, |Φ1⟩S, and |Φ3⟩S, respectively. Here

\[
|Φ1⟩P = \frac{1}{\sqrt{2}}[(|RR⟩ + |LL⟩)_{AB}(|RR⟩ + |LL⟩)_{CD},
\]

\[
|Φ3⟩P = \frac{1}{\sqrt{2}}[(|RL⟩ + |LR⟩)_{AB}(|RR⟩ + |LL⟩)_{CD},
\]

\[
|Φ1⟩S = \frac{1}{\sqrt{2}}[(|a_1b_1⟩ + |a_2b_2⟩)(|c_1d_1⟩ + |c_2d_2⟩),
\]

\[
+ (|a_1b_1⟩ - |a_2b_2⟩)(|c_1d_2⟩ + |c_2d_1⟩),
\]

\[
- (|a_1b_1⟩ - |a_2b_2⟩)(|c_1d_2⟩ + |c_2d_1⟩),
\]

\[
+ (|a_1b_1⟩ + |a_2b_2⟩)(|c_1d_1⟩ + |c_2d_2⟩)\].
\]
\[
|\Phi_S^{'})_S = \frac{1}{2\sqrt{2}}[(a_1b_2) + |a_2b_1|(|c_1d_1| - |c_2d_2|)] 
+ (-|a_1b_2| + |a_2b_1|)(|c_2d_1| - |c_1d_2|).
\]

At last, the photons C and D are detected as shown in Fig. 3 (a). If the two clicked photon detectors of the photons C and D are in the even-parity spatial mode (even-parity polarization mode), the spatial-mode (polarization) DOF of the two-photon system AB is projected into the state $|\phi_1^S_{AB} \otimes |\phi_3^S_{CD}|$ or $|\phi_2^S_{AB} \otimes |\phi_3^S_{CD}|$. If the outcome of the two clicked detectors is in the odd-parity spatial mode (odd-parity polarization mode), a phase-flip operation $\sigma^S_x = |b_1\rangle\langle b_1| - |b_2\rangle\langle b_2|$ (or $\sigma^S_x$) is performed on the required spatial mode (polarization) DOF of the photon B is required to obtain the state $|\phi_1^S_{AB} \otimes |\phi_3^S_{CD}|$ or $|\phi_1^S_{AB} \otimes |\phi_3^S_{CD}|$. (2) If the two photon pairs AC and BD are in the different polarization parity modes and the different spatial-mode pure states, respectively. After the first round of our hyper-EPP process with QSJM, the polarization DOF of the four-photon system is projected into a mixed state composed of the states $|\Phi_5\rangle_S$ (or $|\Phi_6\rangle_S$) and $|\Phi_7\rangle_S$ (or $|\Phi_8\rangle_S$). In this case, the second step is required in our hyper-EPP with QSJM.

\[
|\Phi_5\rangle_S = \frac{1}{\sqrt{2}}(|RRRL⟩ + |LLRR⟩)_{ABCD},
|\Phi_6\rangle_P = \frac{1}{\sqrt{2}}(|RRRL⟩ + |LLRL⟩)_{ABCD},
|\Phi_7\rangle_P = \frac{1}{\sqrt{2}}(|RLRR⟩ + |LRLL⟩)_{ABCD},
|\Phi_8\rangle_P = \frac{1}{\sqrt{2}}(|RLLL⟩ + |LRRR⟩)_{ABCD},
|\Phi_5\rangle_S = \frac{1}{\sqrt{2}}(|a_1b_1c_2d_2⟩ + |a_2b_2c_1d_1⟩),
|\Phi_6\rangle_S = \frac{1}{\sqrt{2}}(|a_1b_2c_2d_1⟩ + |a_2b_1c_1d_2⟩),
|\Phi_7\rangle_S = \frac{1}{\sqrt{2}}(|a_1b_1c_1d_2⟩ + |a_2b_2c_2d_1⟩),
|\Phi_8\rangle_S = \frac{1}{\sqrt{2}}(|a_1b_2c_2d_2⟩ + |a_2b_1c_1d_1⟩).
\]

As Alice and Bob cannot identify which one of the photon pairs AB and CD has polarization bit-flip error (spatial-mode phase-flip error), the two photon pairs have to be discarded in this case.

(3) If the two photon pairs AC and BD are in the same polarization parity mode and the different spatial-mode parity modes, the polarization pure states $|\phi_1^S_{AB} \otimes |\phi_3^S_{CD}|$ and $|\phi_1^S_{AB} \otimes |\phi_3^S_{CD}|$ are distinguished from the other two polarization pure states, and the spatial-mode pure states $|\phi_1^S_{AB} \otimes |\phi_3^S_{CD}|$ and $|\phi_2^S_{AB} \otimes |\phi_3^S_{CD}|$ are distinguished from the other two spatial-mode pure states. After the measurement on the extra electron spins in P-S-QNDS, the polarization DOF of the four-photon system is projected into a mixed state composed of the states $|\Phi_3\rangle_S$ (or $|\Phi_4\rangle_S$), and the spatial-mode DOF of the four-photon system is projected into a mixed state composed of the states $|\Phi_5\rangle_S$ (or $|\Phi_6\rangle_S$) and $|\Phi_7\rangle_S$ (or $|\Phi_8\rangle_S$). In this case, the second step is required in our hyper-EPP with QSJM.

\[\rho_{AB} = \left[F_1^S|\phi_1^S_{AB} \otimes |\phi_3^S_{CD}| + (1 - F_1^S)|\phi_2^S_{AB} \otimes |\phi_3^S_{CD}|\right]
\]

B. The second step of our two-step hyper-EPP with QSJM

In this step, we show that the efficiency of the hyper-EPP is improved with QSJM by preserving the case (3) or (4) in the first step, as shown in Fig. 3 (b).

Suppose that there are four identical photon pairs AB, CD, A’B’, and C’D’ shared by Alice and Bob. The photons ACA’C’ belong to Alice, and the photons BDB’D’ belong to Bob. The same operations are performed on the four-photon systems A’BCD and A’B’C’D’ by Alice and Bob in the first step. If the four-photon systems A’BCD and A’B’C’D’ are projected into the states in the cases (3) and (4) in the first step, respectively, Alice and Bob can use the QSJM (introduced in Sec. 11) to transform the polarization state of the four-photon system A’B’C’D’ into the polarization state of the four-photon system A’BCD. Then the preserving condition of the case (1) in the first step is achieved. After performing Hadamard operations and detections on both the polarization and the spatial-mode DOFs of the photons CD and performing conditional phase-flip operation $\sigma^S_x$ (or $\sigma^S_x$) on the photon B, the spatial-mode states $|\phi_1^S_{AB}\rangle$ and $|\phi_3^S_{AB}\rangle$ (polarization states $|\phi_1^S_{AB}\rangle$ and $|\phi_3^S_{AB}\rangle$) are obtained.

After the first round of our hyper-EPP process with these two steps, the state of the photon pair AB is transformed into

\[\rho_{AB} = \left|F_1^S|\phi_1^S_{AB} \otimes |\phi_3^S_{CD}| + (1 - F_1^S)|\phi_2^S_{AB} \otimes |\phi_3^S_{CD}|\right|
\]
\[ \otimes [F'_2] (\phi_1)_{AB}^S \langle \phi_1 | + (1 - F'_2) (\phi_3)_{AB}^S \langle \phi_3 |]. \] (16)

Here \( F'_1 = \frac{F_1^2}{[F_1^2 + (1 - F_1)^2]}, \quad F'_2 = \frac{F_2^2}{[F_2^2 + (1 - F_2)^2]}, \) and \( F_i > 1/2 \) (\( i = 1, 2 \)). The fidelity of the state \( (\phi_1)_{AB}^S \langle \phi_1 | \) in Eq. (16) is \( F'_1 = \frac{F_1^2}{F_2^2 + (1 - F_2)^2}\). The state \( (\phi_3)_{AB}^S \langle \phi_3 | \) can be transformed into \( (\phi_2)_{AB}^S \) with the Hadamard operations on the spatial-mode DOF of the photons \( A \) and \( B \). With the iteration of our hyper-EPP process, the fidelity of the two-photon state can be improved (shown in Fig. 4 for the cases with \( F_1 = F_2 \)).

![FIG. 5: (Color online) The fidelity and efficiency of our hyper-EPP for mixed hyperentangled Bell states. \( n \) is the iteration number of the hyper-EPP process. \( Y_o \) and \( Y \) represent the efficiencies of the first round of the hyper-EPP process without and with QSJM, respectively. The parameters of the mixed hyperentangled Bell state are chosen as \( F_1 = F_2 \).](image)

The efficiency of an EPP is defined as the probability of obtaining a high-fidelity entangled photon system from a pair of photon systems transmitted over a noisy channel without photon loss. Our hyper-EPP is constructed to purify the mixed hyperentangled Bell states with spatial-mode phase-flip errors and polarization bit-flip errors. In our previous work, we have introduced a hyper-EPP for the mixed hyperentangled Bell states with bit-flip errors in both the spatial-mode and the polarization DOFs, which only preserves the case (1) in the first step with the probability of

\[ F' = [F_1^2 + (1 - F_1)^2] \times [F_2^2 + (1 - F_2)^2] \] (17)

(in the first round of the hyper-EPP process), and the efficiency is

\[ Y_o = [F_1^2 + (1 - F_1)^2] \times [F_2^2 + (1 - F_2)^2]. \] (18)

In the present hyper-EPP, the case (3) or (4) in the first step can also be preserved with the QSJM, and the efficiency of the first round of the hyper-EPP process is

\[ Y = [F_1^2 + (1 - F_1)^2 + F_2 (1 - F_2)] \times [F_2^2 + (1 - F_2)^2]. \] (19)

Here \( F_1 > F_2 \). The efficiency of the hyper-EPP is improved resorting to the QSJM (shown in Fig. 4 for the cases with \( F_1 = F_2 \)).

IV. HYPER-EPP FOR HYPERENTANGLED GHZ STATES WITH QSJM

In this section, we generalize our hyper-EPP for mixed hyperentangled GHZ states of three-photon systems. There are eight polarization GHZ states and eight spatial-mode GHZ states, and they are described as follows:

\[ |\psi_{0}^+\rangle_P = \frac{1}{\sqrt{2}}(|RRR\rangle \pm |LLL\rangle)_{ABC}, \]

\[ |\psi_{1}^+\rangle_P = \frac{1}{\sqrt{2}}(|RRL\rangle \pm |LRL\rangle)_{ABC}, \]

\[ |\psi_{2}^+\rangle_P = \frac{1}{\sqrt{2}}(|RLR\rangle \pm |LRR\rangle)_{ABC}, \]

\[ |\psi_{3}^+\rangle_P = \frac{1}{\sqrt{2}}(|LRR\rangle \pm |RLL\rangle)_{ABC}, \]

\[ |\psi_{0}^+\rangle_S = \frac{1}{\sqrt{2}}(|a_1 b_1 c_1 \rangle \pm |a_2 b_2 c_2 \rangle), \]

\[ |\psi_{1}^+\rangle_S = \frac{1}{\sqrt{2}}(|a_1 b_1 c_2 \rangle \pm |a_2 b_2 c_1 \rangle), \]

\[ |\psi_{2}^+\rangle_S = \frac{1}{\sqrt{2}}(|a_1 b_2 c_1 \rangle \pm |a_2 b_1 c_2 \rangle), \]

\[ |\psi_{3}^+\rangle_S = \frac{1}{\sqrt{2}}(|a_2 b_1 c_1 \rangle \pm |a_1 b_2 c_2 \rangle). \] (20)

Here the subscript \( ABC \) represents the photons obtained by the remote users Alice, Bob, and Charlie, respectively. Suppose that the original state of the photon system \( ABC \) is \( |\psi_{0}^+\rangle_P \otimes |\psi_{0}^+\rangle_S \). If there is a polarization bit-flip error on the original state of the three-photon system after the transmission over a noisy channel, the polarization state of the three-photon system will become \( |\psi_{1}^+\rangle_P \), \( |\psi_{2}^+\rangle_P \), or \( |\psi_{3}^+\rangle_P \). While a spatial-mode phase-flip error is more likely to be taken place on the original state of the three-photon system [44, 59], and the spatial-mode state of the three-photon system will become \( |\psi_{0}^+\rangle_S \).

After the photon system is transmitted over the noisy channel with polarization bit-flip error and spatial-mode phase-flip error, its state becomes

\[
\rho = \langle F_0 | \psi_{0}^+\rangle_P \langle \psi_{0}^+ | + F_1 | \psi_{1}^+\rangle_P \langle \psi_{1}^+ | \\
+ F_2 | \psi_{2}^+\rangle_P \langle \psi_{2}^+ | + F_3 | \psi_{3}^+\rangle_P \langle \psi_{3}^+ |angle \otimes \langle F_0 | \psi_{0}^+\rangle_S \langle \psi_{0}^+ | + F_1 | \psi_{1}^+\rangle_S \langle \psi_{1}^+ |. \] (21)
Here $F_0$ and $P_0$ are the probabilities of the states $|\psi_1^+\rangle_p$ and $|\psi_0^-\rangle_s$ in the mixed states, respectively, and they satisfy the relations $F_0 + F_1 + F_2 + F_3 = 1$ and $P_0 + P_1 = 1$. In order to obtain the high fidelity entangled three-photon systems, the three remote users have to perform hyper-EPP on the three-photon systems, which requires three remote users to divide their photon systems into many groups with a pair of three-photon systems in each group. The photons in each group are labeled as $ABCA'ABCA''C'$, where $ABC$ and $A'B'C'$ represent two identical three-photon systems in the same mixed hyperentangled GHZ state. As the phase-flip error of the three-photon GHZ state can not be transformed into the bit-flip error with Hadamard operations, we discuss the principles of the EPPs for the polarization states and the spatial-mode states of the three-photon systems in Sec. [IVA] and Sec. [IVB] respectively. The hyper-EPP for mixed hyperentangled GHZ states with QSJM is introduced in Sec. [IVC].

A. EPP for polarization GHZ states

The polarization state of the system $ABCA'ABCA''C'$ can be viewed as the mixture of 16 pure states $|\psi_1^+\rangle_p \otimes |\psi_1^+\rangle_p$ with the probability of $F_i F_j$ ($i, j = 0, 1, 2, 3$). Alice, Bob, and Charlie perform polarization parity-check QNDs on their photon pairs $AA'$, $BB'$, and $CC'$ respectively, and they pick up the groups with the results of the three polarization parity-check QNDs are ambiguous about which one of the three-photon systems $ABC$ and $A'B'C'$ is in other conditions, the group of the three photon systems is ambiguous about which one of the three-photon systems $ABC$ and $A'B'C'$ has polarization bit-flip error, and it will be discussed in the hyper-EPP with QSJM in Sec. [IVC].

\[ |\Psi_0^+\rangle_s = \frac{1}{\sqrt{2}} (|RRR\rangle|LLL\rangle + |LLL\rangle|RRR\rangle)_{ABCA'ABCA''C'}. \]

With the polarization bit-flip operations $(\sigma^F_s)$ on the three photons $A'B'C'$, the polarization state $|\Psi_0^+\rangle_s$ can be transformed into $|\Psi_1^-\rangle_s (i = 0, 1, 2, 3)$. After performing Hardmann operations and detections on the polarization DOF of the three photons $A'B'C'$ and performing conditional phase-flip operation $\sigma^F_s$ on the photon $A$, the states $|\psi_0^+\rangle_p, |\psi_1^+\rangle_p, |\psi_2^+\rangle_p, \text{and } |\psi_3^+\rangle_p$ are obtained with the probabilities $F_0, F_1, F_2, \text{and } F_3$, respectively. If the results of the three polarization parity-check QNDs are in other conditions, the group of the three photon systems is ambiguous about which one of the three-photon systems $ABC$ and $A'B'C'$ has polarization bit-flip error, and it will be discussed in the hyper-EPP with QSJM in Sec. [IVC].

B. EPP for spatial-mode GHZ states

The spatial-mode state of the system $ABCA'ABCA''C'$ can be viewed as the mixture of 4 pure states $|\psi_0^+\rangle_s \otimes |\psi_0^+\rangle_s, |\psi_0^+\rangle_s \otimes |\psi_0^-\rangle_s, |\psi_0^-\rangle_s \otimes |\psi_0^+\rangle_s, \text{and } |\psi_0^-\rangle_s \otimes |\psi_0^-\rangle_s$ with the probabilities of $P_0^2, P_0 P_1, P_0 P_1, \text{and } P_1^2$, respectively. With spatial-mode Hadamard operations on the three photons, the states $|\psi_0^+\rangle_s$ and $|\psi_0^-\rangle_s$ are transformed into states $|\psi^+\rangle_s$ and $|\psi^-\rangle_s$, respectively. Here

\[ |\psi^+\rangle_s = \frac{1}{2} (|a_1 b_1 c_1\rangle + |a_1 b_2 c_2\rangle + |a_2 b_1 c_2\rangle + |a_2 b_2 c_1\rangle), \]

\[ |\psi^-\rangle_s = \frac{1}{2} (|a_1 b_1 c_2\rangle + |a_1 b_2 c_1\rangle + |a_2 b_2 c_2\rangle + |a_2 b_1 c_1\rangle). \]

Alice, Bob, and Charlie perform spatial-mode parity-check QNDs on their photon pairs $AA'$, $BB'$, and $CC'$ respectively, and they pick up the groups with the odd number of photon pairs in the even-parity spatial mode. If the three photon pairs are all in the even-parity spatial mode, the spatial-mode DOF of the six-photon system is projected into a mixed state composed of two pure states $|\Psi_0^s\rangle_s$ and $|\Psi_1^s\rangle_s$. Here

\[ |\Psi_0^s\rangle_s = \frac{1}{2} (|a_1 b_1 c_1\rangle |a'_1 b'_1 c'_1\rangle + |a_1 b_2 c_2\rangle |a'_1 b'_2 c'_2\rangle + |a_2 b_1 c_2\rangle |a'_2 b'_1 c'_2\rangle + |a_2 b_2 c_1\rangle |a'_2 b'_2 c'_1\rangle), \]

\[ |\Psi_1^s\rangle_s = \frac{1}{2} (|a_1 b_1 c_2\rangle |a'_1 b'_1 c'_2\rangle + |a_1 b_2 c_1\rangle |a'_1 b'_2 c'_1\rangle + |a_2 b_2 c_2\rangle |a'_2 b'_2 c'_2\rangle + |a_2 b_1 c_1\rangle |a'_2 b'_1 c'_1\rangle). \]

If the photon pairs $AA'$ and $BB'$ are in the odd-parity spatial mode and the photon pair $CC'$ is in the even-parity spatial mode, the spatial-mode DOF of the six-photon system is projected into a mixed state composed of two pure states $|\Psi_0^s\rangle_s$ and $|\Psi_1^s\rangle_s$. Here

\[ |\Psi_0^s\rangle_s = \frac{1}{2} (|a_1 b_1 c_1\rangle |a'_1 b'_1 c'_1\rangle + |a_2 b_2 c_1\rangle |a'_1 b'_1 c'_1\rangle + |a_2 b_1 c_1\rangle |a'_2 b'_2 c'_2\rangle + |a_2 b_2 c_2\rangle |a'_2 b'_2 c'_1\rangle). \]
If the photon pairs $AA'$ and $CC'$ are in the odd-parity spatial mode and the photon pair $BB'$ is in the even-parity spatial mode, the spatial-mode DOF of the six-photon system is projected into a mixed state composed of two pure states $|\Psi_2\rangle_S$ and $|\Psi_3\rangle_S$. Here

$$|\Psi_2\rangle_S = \frac{1}{2}((|a_1b_1c_1\rangle|a'_1b'_1c'_1\rangle + |a_2b_2c_1\rangle|a'_2b'_2c'_2\rangle + |a_1b_2c_1\rangle|a'_2b'_1c'_2\rangle + |a_2b_1c_1\rangle|a'_2b'_1c'_1\rangle),$$

$$|\Psi_3\rangle_S = \frac{1}{2}((|a_1b_1c_1\rangle|a'_1b'_1c'_1\rangle + |a_2b_2c_1\rangle|a'_2b'_2c'_2\rangle + |a_1b_2c_1\rangle|a'_2b'_1c'_2\rangle + |a_2b_1c_1\rangle|a'_2b'_1c'_1\rangle).$$

(26)

If the photon pairs $BB'$ and $CC'$ are in the odd-parity spatial mode and the photon pair $AA'$ is in the even-parity spatial mode, the spatial-mode DOF of the six-photon system will project into a mixed state composed of two pure states $|\Psi_4\rangle_S$ and $|\Psi_5\rangle_S$. Here

$$|\Psi_4\rangle_S = \frac{1}{2}((|a_1b_1c_1\rangle|a'_1b'_1c'_1\rangle + |a_2b_2c_1\rangle|a'_2b'_2c'_2\rangle + |a_1b_2c_1\rangle|a'_1b'_2c'_1\rangle + |a_2b_1c_1\rangle|a'_1b'_1c'_2\rangle),$$

$$|\Psi_5\rangle_S = \frac{1}{2}((|a_1b_1c_1\rangle|a'_1b'_1c'_1\rangle + |a_2b_2c_1\rangle|a'_2b'_2c'_2\rangle + |a_1b_2c_1\rangle|a'_1b'_2c'_1\rangle + |a_2b_1c_1\rangle|a'_1b'_1c'_2\rangle).$$

(27)

With the spatial-mode bit-flip operations ($\sigma_z^S$) on the two photons of the three-photon system $A'B'C'$, the spatial-mode states $|\Psi_0\rangle_S$ (or $|\Psi_2\rangle_S$, $|\Psi_4\rangle_S$) and $|\Psi_1\rangle_S$ (or $|\Psi_3\rangle_S$, $|\Psi_5\rangle_S$) can be transformed into $|\Psi_0\rangle_S$ and $|\Psi_1\rangle_S$. After performing Hadamard operations and detections on the spatial-mode DOF of the three photons $A'B'C'$ and performing conditional phase-flip operation $\sigma_z^S$ on the photon $A$, the states $|\varphi^+\rangle_S$ and $|\varphi^-\rangle_S$ are obtained with the probabilities $P_2$ and $P_1$, respectively. If the results of the three spatial-mode parity-check QNDs are in other conditions, the group of the three-photon systems is ambiguous about which one of the three-photon systems $ABC$ and $A'B'C'$ has the spatial-mode phase-flip error, and it will be discussed in the hyper-EPP with QSJM in Sec. IV C. The states $|\varphi^+\rangle_S$ and $|\varphi^-\rangle_S$ can be transformed into $|\psi_0^+\rangle_S$ and $|\psi_0^-\rangle_S$ by performing spatial-mode Hadamard operations on the three-photon systems.

C. Hyper-EPP for hyperentangled GHZ states with QSJM

The hyper-EPP for mixed hyperentangled GHZ states with QSJM can also be implemented with the quantum circuit shown in Fig. 4. Alice, Bob, and Charlie first perform the same operations (shown in Fig. 4(a)) on their photon pairs $AA'$, $BB'$, and $CC'$ respectively. If the results of the P-S-QNDs show that there are three photon pairs in the even-(odd-) parity polarization mode and an odd number of photon pairs in the even-parity spatial mode, the groups of the three photon systems are preserved. If there are three photon pairs in the even-(odd-) parity polarization mode and an even number of photon pairs in the even-parity spatial mode, the polarization state of this group can be transformed into the polarization state of another group which has three photon pairs not all in the even-(odd-) parity polarization mode and an even number of photon pairs in the even-parity spatial mode, resorting to the QSJM shown in Fig. 4(b). And the groups of the three photon systems have to be discarded, if the results of the P-S-QNDs show that the group has three photon pairs not all in the even-(odd-) parity polarization mode and an odd number of photon pairs in the even-parity spatial mode.

After the first round of this hyper-EPP process with the two steps, the state of the three-photon system $ABC$ is transformed into

$$\rho = (F_0|\psi_0^+\rangle_0^S \langle \psi_0^+ | + F_1|\psi_1^+\rangle_0^S \langle \psi_1^+ | + F_2|\psi_2^+\rangle_0^S \langle \psi_2^+ | + F_3|\psi_3^+\rangle_0^S \langle \psi_3^+ |) \otimes (P_0|\psi_0^+\rangle_1^S \langle \psi_0^+ | + P_1|\psi_1^+\rangle_1^S \langle \psi_1^+ |).$$

(29)
Here

\[
F'_0 = \frac{F_0^2}{F_0^2 + F_1^2 + F_2^2 + F_3^2},
\]

\[
F'_1 = \frac{F_1^2}{F_0^2 + F_1^2 + F_2^2 + F_3^2},
\]

\[
F'_2 = \frac{F_2^2}{F_0^2 + F_1^2 + F_2^2 + F_3^2},
\]

\[
F'_3 = \frac{F_3^2}{F_0^2 + F_1^2 + F_2^2 + F_3^2},
\]

\[
P'_0 = \frac{P_0^2}{P_0^2 + P_1^2},
\]

\[
P'_1 = \frac{P_1^2}{P_0^2 + P_1^2}.
\] (30)

If \( P_0 > 1/2 \), the fidelity is \( P'_0 > P_0 \). And the fidelity is \( F'_0 > F_0 \), if \( F_0 \) satisfies the relation

\[
F_0 > \frac{1}{4} \left[ 3 - 2F_1 - 2F_2 - \sqrt{1 + 4(F_1 + F_2) - 12(F_1^2 + F_2^2) - 8F_1F_2} \right]. \] (31)

The fidelity of \( |\psi_0^+\rangle_P \otimes |\psi_0^+\rangle_S \) in Eq. (29) is \( F' = F'_0 \times P'_0 \).

With the iteration of our hyper-EPP process, the fidelity of this three-photon state can be improved (shown in Fig. 6) for the cases with \( F_1 = F_2 = F_3 \) and \( P_0 = F_0 \).

In the hyper-EPP without QSJM, the groups of three-photon systems can only be preserved with the probability of \( (F_0^2 + F_1^2 + F_2^2 + F_3^2)(P_0^2 + P_1^2) \) in the first round of the hyper-EPP process, and the efficiency is

\[
Y_0 = (F_0^2 + F_1^2 + F_2^2 + F_3^2)(P_0^2 + P_1^2). \] (32)

While in this hyper-EPP with QSJM, the probability of the preserved three-photon systems in the first round of the hyper-EPP process can be increased to

\[
F' = \min\{(F_0^2 + F_1^2 + F_2^2 + F_3^2), (P_0^2 + P_1^2)\}, \] (33)

and the efficiency is

\[
Y = \min\{(F_0^2 + F_1^2 + F_2^2 + F_3^2 + F_0F_1 + F_0F_2 + F_0F_3 + F_1F_2 + F_1F_3 + F_2F_3)(P_0^2 + P_1^2),
\]

\[
(F_0^2 + F_1^2 + F_2^2 + F_3^2)(P_0^2 + P_1^2 + P_0P_1)\}. \] (34)

Figure 7 shows the efficiencies of the hyper-EPPs (in the first round) with and without QSJM (for the cases with \( F_1 = F_2 = F_3 \) and \( P_0 = F_0 \)), and it is obvious that the efficiency of the hyper-EPP with QSJM is improved largely, compared with the one without QSJM.

V. DISCUSSION AND SUMMARY

The transmission and reflection rule of the double-sided QD-cavity system is the key element for the P-S-QND and the QSJM, and it may be not perfect because of decoherence and dephasing. The electron spin decoherence may reduce the fidelity of the proposal, but this effect can be suppressed by extending electron coherence time to \( \mu s \) using spin echo techniques (using single-photon pluses to play the role of the \( \pi \) pulse), which is longer than the cavity photon lifetime (\( \sim 10 ns \)) and the time interval of the input photons (\( \sim 10 \) ns). The exciton dephasing, including optical dephasing and hole spin dephasing, can also reduce the fidelity in a few percent. The hole spin coherence time is three times shorter than the cavity photon lifetime \( \sim 10\mu s \), and the optical coherence time of exciton (\( \sim 100 \) ps) is ten times longer than the cavity photon lifetime \( \sim 1 \) ns, so the effect of the exciton dephasing may be decreased. The heavy-light hole mixing may also reduce the fidelity by a few percent, but this effect can be decreased by improving the shape, size, and type of QDs. The fine structure splitting can be immune for charged exciton due to the quenched exchange interaction. The Hadamard operation, which is used to transform electron-spin states \( |\uparrow\rangle \) and \( |\downarrow\rangle \) to \( |+\rangle \) and \( |-\rangle \), respectively, can be implemented by nanosecond electron-spin resonance microwave pulses or picosecond optical pulses.

FIG. 7: (Color online) Fidelity and efficiency of our P-S-QND (for even-parity mode) vs the coupling strength and cavity side leakage rate with \( \gamma = 0.1\kappa \).

In the resonant condition (\( \omega_c = \omega_X = \omega_0 = \omega \)), the
fidelity of our proposal is mainly reduced by the cavity side leak and cavity coupling strength (discussed in the Appendix A). The fidelities and the efficiencies of the P-S-QND and the QSJM are shown in Fig.7 and Fig.8 with the coupling strength and cavity side leak, respectively. Both these two operations work efficiently with the strong coupling strength and the low side leakage and cavity loss rate \((\kappa_s/\kappa)\). The strong coupling strength \((g/(\kappa + \kappa_s)) \simeq 0.5\) has been observed in a \(d = 1.5 \mu m\) micropillar microcavity \[68\] with a quality factor of \(Q \sim 8800\). By improving the sample designs, growth, and fabrication \[69\], the coupling strength \((g/(\kappa + \kappa_s)) \simeq 2.4\) \((Q \sim 4 \times 10^4)\) has been achieved \[70\] in \(d = 1.5 \mu m\) micropillar microcavity. The fidelities and the efficiencies of the two operations are \(F_p = 87.5\%\), \(\eta_p = 55\%\), \(F_j = 80\%\), and \(\eta_j = 64\%\) in the case \((g/(\kappa + \kappa_s)) \simeq 0.5\) and \(\kappa_s/\kappa \simeq 0.3\). While they are \(F_p = 91.3\%\), \(\eta_p = 60\%\), \(F_j = 84\%\), and \(\eta_j = 68.5\%\) for the coupling strength \((g/(\kappa + \kappa_s)) \simeq 2.4\) with \(\kappa_s/\kappa \simeq 0.3\), and \(F_p = 100\%\), \(\eta_p = 96.6\%\), \(F_j = 99\%\), and \(\eta_j = 97.4\%\) for the side leakage and cavity loss rate \(\kappa_s/\kappa \simeq 0\) \((g/(\kappa + \kappa_s)) \simeq 2.4\). The fidelities and the efficiencies of the two operations are mainly reduced by the weak coupling strength and the high cavity side leakage. In experiment, the quality factor of a micropillar microcavity is dominated by the side leakage and cavity loss rate \(\kappa_s/\kappa\). The side leakage and cavity loss rate \(\kappa_s/\kappa \simeq 0.7\) has been achieved in \(d = 1.5 \mu m\) micropillar with the quality factor of \(Q \sim 1.7 \times 10^4\) \((g/(\kappa + \kappa_s)) \simeq 1\) by thinning down the top mirrors \[60\]. In this case, the fidelities and the efficiencies of the two operations are \(F_p = 75.4\%\), \(\eta_p = 39.2\%\), \(F_j = 66\%\), and \(\eta_j = 50\%\), which means high efficiency operations in experiment are required to get a stronger coupling strength with a lower side leakage in micropillars.

With the P-S-QND and the QSJM, we construct a two-step hyper-EPP for the mixed hyperentangled Bell states with polarization bit-flip errors and spatial-mode phase-flip errors. In the hyper-EPP without QSJM, the photonic states with only one DOF in the preserving condition have to be discarded. While in the present hyper-EPP, the photonic state with only polarization (spatial-mode) DOF in the preserving condition can transform its polarization (spatial-mode) state to another photonic state with the spatial-polarization (polarization) DOF in the preserving condition, which can preserve more high-fidelity entangled photon pairs. That is, the efficiency of our two-step hyper-EPP can be improved by using our QSJM. Moreover, we generalize our hyper-EPP for mixed hyperentangled GHZ states with polarization bit-flip errors and spatial-mode phase-flip errors, which is more complicated than the one for mixed hyperentangled Bell states. It shows that our hyper-EPP with QSJM can be used to improve the efficiency of the hyper-EPP for mixed multiphoton hyperentangled states as well.

In summary, we have investigated the possibility of improving the efficiency of the hyper-EPP for mixed hyperentangled states with polarization bit-flip errors and spatial-mode phase-flip errors, resorting to the P-S-QND and the QSJM that are constructed with the transmission-reflection rule of double-sided QD-cavity systems. The present two-step hyper-EPP can improve the efficiency largely by preserving the states that are discarded in the hyper-EPP without QSJM. We have analyzed the experimental feasibility of the P-S-QND and the QSJM, and it shows that they can work efficiently in the strong coupling regime with low cavity side leak. The present two-step hyper-EPP with our QSJM can be generalized to improve the efficiency of the hyper-EPP for mixed multi-photon hyperentangled states, and it is useful for improving the entanglement of photon systems with several DOFs in long-distance high-capacity quantum communication.

![Fig. 8: (Color online) Fidelity and efficiency of our QSJM (in the present hyper-EPP) vs the coupling strength and cavity side leakage rate with \(\gamma = 0.1\kappa\).](image)

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Appendix A: Fidelities and efficiencies of P-S-QND and QSJM

In the resonant condition ($\omega_c = \omega_{\chi} = \omega_0 = \omega$), the fidelity of our proposal is mainly reduced by the cavity side leak and cavity coupling strength. And the transmission and reflection rule in Eq. (1) is reduced to

\[
\begin{align*}
|R^\uparrow, i_2, \uparrow\rangle &\rightarrow |r|L^\uparrow, i_2, \uparrow\rangle - |t||L^\uparrow, i_1, \uparrow\rangle, \\
|L^\downarrow, i_1, \uparrow\rangle &\rightarrow |r||R^\downarrow, i_1, \uparrow\rangle - |t||L^\downarrow, i_2, \uparrow\rangle, \\
|R^\downarrow, i_1, \downarrow\rangle &\rightarrow |r|L^\downarrow, i_1, \downarrow\rangle - |t||R^\downarrow, i_2, \downarrow\rangle, \\
|L^\uparrow, i_2, \downarrow\rangle &\rightarrow |r||R^\uparrow, i_2, \downarrow\rangle - |t||L^\uparrow, i_1, \downarrow\rangle, \\
|R^\uparrow, i_2, \downarrow\rangle &\rightarrow -|t_0||R^\uparrow, i_2, \uparrow\rangle + |r_0||L^\uparrow, i_1, \uparrow\rangle, \\
|L^\downarrow, i_2, \uparrow\rangle &\rightarrow -|t_0||L^\downarrow, i_1, \uparrow\rangle + |r_0||R^\downarrow, i_2, \uparrow\rangle, \\
|R^\downarrow, i_2, \downarrow\rangle &\rightarrow -|t_0||R^\downarrow, i_2, \downarrow\rangle + |r_0||L^\downarrow, i_2, \downarrow\rangle, \\
|L^\uparrow, i_1, \downarrow\rangle &\rightarrow -|t_0||L^\uparrow, i_1, \downarrow\rangle + |r_0||R^\uparrow, i_1, \downarrow\rangle.
\end{align*}
\] (A1)

The fidelity of a quantum information process is defined as $F = \langle \psi_f | \psi_i \rangle^2$, where $|\psi_f\rangle$ is the ideal final state of the system after the quantum information processing, and $|\psi_i\rangle$ is the final state of the system by considering experimental environment. The efficiency of a photonic quantum information process is defined as the probability of the photons to be detected after the quantum information processing in the practical experimental environment. The fidelity and the efficiency of our P-S-QND (for even-parity mode) can be described as

\[
F_p = \frac{\left(\sum_{i=1}^{4} m_i^2\right)^2}{4 \sum_{i=1}^{3} m_i^2},
\]

\[
\eta_p = \frac{1}{16} (|r|^2 + |t|^2 + |t_0|^2 + |r_0|^2)^4.
\] (A2)

The fidelity and the efficiency of our QSJM (in the present hyper-EPP process) can be described as

\[
F_j = \frac{(|r| + |t| + |r_0| + |t_0|)^2(|r| + |t_0|)^2}{\sum_{k=1}^{4} n_k^2 (|r| + |t| + |r_0| + |t_0|)^2},
\]

\[
\eta_j = \frac{1}{8} (|r|^2 + |t|^2 + |t_0|^2 + |r_0|^2)^3.
\] (A3)

Here

\[
\begin{align*}
m_1 &= |(t|^2 + |r|^2)|t_0|^2 + |(t_0|^2 + |r_0|^2)|r_0|^2, \\
m_2 &= |(t|^2 + |r|^2)|t_0|^2 + |(t_0|^2 + |r_0|^2)|t_0|^2, \\
m_3 &= |(t|^2 + |r|^2)|t_0|^2 + |(t_0|^2 + |r_0|^2)|t_0|^2, \\
m_4 &= |(t|^2 + |r|^2)|t_0|^2 + |(t_0|^2 + |r_0|^2)|t_0|^2, \\
m_5 &= 2(|t_0|^2 + |r|^2)|t_0|^2 + |(t_0|^2 + |r_0|^2)|t_0|^2, \\
m_6 &= |4|t_0|^2|t|^2(|r_0|^2 + |r|^2), \\
m_7 &= |4|t_0|^2|t|^2(|r_0|^2 + |r|^2), \\
m_8 &= |4|t_0|^2|t|^2(|r_0|^2 + |r|^2), \\
n_1 &= |(t + |r|)|r| + (|t| - |t_0|)|t_0| + (|t_0| + |r_0|)|r_0| \\
&- (t_0 + |r_0|)|r_0|, \\
n_2 &= |(t + |r|)|r| + (|t| - |t_0|)|t_0| + (|t_0| + |r_0|)|r_0| \\
&- (t_0 + |r_0|)|r_0|, \\
n_3 &= (|t_0| + |r_0|)|t_0| + (|t| - |t_0|)|t_0| + (|t_0| + |r_0|)|r_0| \\
&- (|t| - |t_0|)|t_0|, \\
n_4 &= (|t_0| + |r_0|)|t_0| + (|t| - |t_0|)|t_0| + (|t_0| + |r_0|)|r_0| \\
&- (|t| - |t_0|)|t_0|, \\
n_5 &= |(t + |r|)|t_0| + (|t| - |t_0|)|t_0| + (|t_0| + |r_0|)|r_0| \\
&- (t_0 + |r_0|)|r_0|, \\
n_6 &= |(t + |r|)|t_0| + (|t| - |t_0|)|t_0| + (|t_0| + |r_0|)|r_0| \\
&- (t_0 + |r_0|)|r_0|, \\
n_7 &= |(t_0| + |r_0|)|t_0| - (|t_0| - |t_0|)|t_0| + (|t| + |r|)|r_0| \\
&- (|t| - |t_0|)|t_0|, \\
n_8 &= (|t_0| + |r_0|)|t_0| - (|t_0| - |t_0|)|t_0| + (|t| + |r|)|r_0| \\
&- (|t| - |t_0|)|t_0|.
\] (A4)

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[3] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[4] X. S. Liu, G. L. Long, D. M. Tong, and F. Li, Phys. Rev. A 65, 022304 (2002).
[5] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. 74, 145 (2002).
[6] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[7] C. H. Bennett, G. Brassard, and N. D. Mermin, Phys. Rev. Lett. 68, 557 (1992).
[8] X. H. Li, F. G. Deng, and H. Y. Zhou, Phys. Rev. A 78, 022321 (2008).
[9] G. L. Long and X. S. Liu, Phys. Rev. A 65, 032302 (2002).
[10] F. G. Deng, G. L. Long, and X. S. Liu, Phys. Rev. A 68, 042317 (2003).
[11] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A 59, 1829 (1999).
[12] A. Karlsson, M. Koashi, and N. Imoto, Phys. Rev. A 59, 162 (1999).
[13] L. Xiao, G. L. Long, F. G. Deng, and J. W. Pan, Phys. Rev. A 69, 052307 (2004).
[14] J. T. Barreiro, N. K. Langford, N. A. Peters, and P. G. Kwiat, Phys. Rev. Lett. 95, 260501 (2005).
[15] W. B. Gao, C. Y. Lu, X. C. Yao, P. Xu, O. Gühne, A. Goebel, Y. A. Chen, C. Z. Peng, Z. B. Chen, and J. W. Pan, Nature Phys. 6, 331 (2010).
