Quantum Phase
Transitions in quasi-one dimensional systems
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Quantum Phase Transitions in Quasi-one-dimensional Systems

Among the various systems, one dimensional (1D) and quasi-one dimensional (quasi-1D) systems are a fantastic playground for quantum phase transitions (QPTs), with rather unique properties. There are various reasons for that special behavior.

First, purely 1D systems are rather unique. Contrary to their higher-dimensional counterparts [1], interactions play a major role since in 1D particles cannot avoid the effects of interactions. This transforms any individual motion of the particles into a collective one. In addition to these very strong interaction effects, in 1D the quantum and thermal fluctuations are pushed to a maximum, and prevent the breaking of continuous symmetries, making simple mean-field physics inapplicable. The combination of these two effects leads to a very special universality class for interacting quantum systems, known as Luttinger liquids (LLs) [2].

I will not review here all the aspects of LL physics since many such reviews exist, but refer the reader in particular to [3] for a complete description of this area of physics, with the same notations as the ones used in the present chapter. For what concerns us, the important point is that we take the LL to be in a critical phase, in which correlations decrease, at zero temperature, as power laws of space and time. This makes the system extremely fragile to external perturbations and leads to a host of QPTs. Examples of such perturbations are the effects of a lattice, which leads to a Mott transition, and disorder that leads to localized phases such as Anderson localization or the Bose glass. Each of these transitions is characterized by a quantum critical point (QCP) that can be computed from LL theory. The 1D nature of LLs has other consequences: the excitations can fractionalize. In particular, an excitation such as adding an electron can split into several collective excitations, such as one carrying spin but no charge, called a spinon and one carrying charge but no spin, called a holon. I will not dwell on this physics of purely 1D systems that is now well characterized and refer the reader to [3] for this aspect of QCPs in purely 1D systems.

Although purely 1D physics and QPTs are by now rather well under control from the theoretical point of view, there is a category of perturbations that is still at the frontier of our theoretical knowledge. These are the perturbations that are produced by the coupling of several 1D systems. Then, when one
parameter, for example the temperature or the inter-chain coupling, is varied, the system crosses over from a 1D situation with exotic LL physics, to the more conventional high dimensional one. How one can reconcile such different physical limits, for example recombining the spinons and holons to re-form an electron, to perform such a dimensional crossover is a very challenging and still open question. Such questions are not only important on the theoretical side but have direct applications to experimental systems such as organic \cite{4} or inorganic \cite{5} superconductors, spin chains and ladders \cite{6, 7, 8} and cold atomic systems \cite{9} (see also Chap. ???) which provide realizations of such coupled 1D systems.

Quasi-1D systems thus leads to their own interesting sets of QCPs, and these are the ones on which I will focus in the present chapter. I will start by examining the simple case of coupled spin chains and ladders, then move to the case of bosons, and finally deal with the more complicated and still largely open case of fermions.

1.1 Spins: From Luttinger Liquids to Bose-Einstein Condensates

The simplest example of coupled 1D system is provided by coupled spin systems (see e.g. \cite{6, 7} for experimental systems). In addition to their own intrinsic interest and their direct experimental realization, they will also serve to illustrate several important concepts that will be directly transposed with increasing complexity to the case of bosons and fermions.

Coupling chains starting from 1D is a highly nontrivial process. Going from one spin chain to two, called the spin ladder problem, already leads to nontrivial physics. Indeed, although spin-1/2 systems are gapless the coupling of two spin-1/2 chains leads to the formation of a spin gap, similar to the Haldane gap that occurs for integer spins \cite{10}. I will not discuss this physics in details since it is by now well established and covered in several textbooks and refer the reader to the literature on the subject \cite{8, 3}.

Here, I consider the case when an infinite number of low dimensional units are coupled. As can be readily understood the physics will depend crucially on the fact that the systems that got coupled are already in a critical state (such as spin-1/2 chains) or whether they have a gap (such as spin dimers, spin 1 chains or two legs ladders). These two cases are the prototypes of QCP in coupled 1D systems and we will examine them separately.
1.1.1 Coupled Spin-1/2 Chains

An isolated spin-1/2 chain is described by a LL. As can be expected in 1D, no long range order can exist. However, the spin-spin correlation functions decay as a power law, at zero temperature, indicating the presence of quasi-long range order. Focusing on the case of the antiferromagnetic exchange, which is the natural realization in condensed matter systems, spin-spin correlations decay as

\[ \langle S^+(x)S^-(0) \rangle \propto (-1)^x \left( \frac{1}{x} \right)^{1/(2K)}, \]
\[ \langle S^z(x)S^z(0) \rangle \propto (-1)^x \left( \frac{1}{x} \right)^{2K}, \]  

(1.1)

where \( K \) is the LL parameter and depends only on the spin exchange anisotropy between the \( XY \) and \( Z \) plane, \( J_Z/J_{XY} \). For an isotropic Heisenberg interaction \( K = 1/2 \), both correlations decrease as \( 1/r \), up to logarithmic corrections.

Temperature cuts this power-law decrease and transforms it into an exponential decay of the correlation beyond a scale of order \( u \beta \), where \( \beta \) is the inverse temperature and \( u \) the velocity of spin excitations.

The inter-chain coupling introduces a term of the form

\[ H_\perp = J_\perp \sum_{\langle \mu \nu \rangle} \int dx S^z_\mu(x) \cdot S^z_\nu(x), \]

(1.2)

where \( \langle \mu, \nu \rangle \) denotes two neighboring chains \( \mu \) and \( \nu \). Because the spin is an object that admits a good classical limit, one can analyze the physics of such a term in a mean-field approximation by assuming that the spin on each chain acquires an average value, for example in the \( Z \) direction. This allows one to decouple (1.2) and transform it to an effective Hamiltonian corresponding to a self-consistent staggered magnetic field applied on a single chain \( H_\perp \simeq J_\perp \sum_{\nu} h_{\text{eff}} \int dx (-1)^x S^z_\nu(x) \). Using the standard bosonization representation of the spins, the Hamiltonian then becomes a sine-Gordon Hamiltonian whose sine term represents the effects of the effective staggered field [11]. The physics of such a Hamiltonian is well known, and there are two phases. First, there is a critical phase, where one recovers the massless excitations. This corresponds to the high temperature phase where the chains are essentially decoupled. Second, there is a massive phase where the cosine is relevant* and acquires an average value. This means that \( \langle (-1)^x S^z(x) \rangle \) is now non-zero, which signals true long range order in the system. The system thus exhibits a genuine phase transition as a function of the temperature towards an ordered state that would correspond to anisotropic antiferromagnetic three dimensional (3D) behavior. The critical temperature can be analyzed by using scaling analysis of the inter-chain coupling. Using (1.1) leads to the renormalization flow of

*In the standard language of renormalization theory, terms are defined as relevant when they do not tend to zero under a renormalization transformation, and irrelevant otherwise.
the inter-chain exchange:

\[
\frac{dJ_{XY}}{dl} = J_{XY}(2 - \frac{1}{2K}), \quad (1.3) \\
\frac{dJ_Z}{dl} = J_Z(2 - 2K), \quad (1.4)
\]

where \(l\) describes the renormalization of the bandwidth of the systems \(\Lambda(l) = \Lambda_0 e^{-l}\) and \(\Lambda_0\) is the bare bandwidth. One sees that one of the couplings is always relevant regardless of the value of \(K\) and that one has always an ordered state (for a non frustrated inter-ladder coupling) at low enough temperature.

However, the critical nature of the 1D systems leads to a strong renormalization of the critical temperature with respect to a naive mean-field approximation; the latter would result in \(T_c \sim J_\perp\). Instead, strong 1D fluctuations lead to

\[
T_{cXY}^c = J_\parallel \left( \frac{J_\perp}{J_\parallel} \right)^{1/(2-1/2K)}, \quad T_{cZ}^c = J_\parallel \left( \frac{J_\perp}{J_\parallel} \right)^{1/(2-2K)}, \quad (1.5)
\]

as can be deduced directly from (1.3), since the critical temperature follows from \(l^* = \log(\Lambda_0/T_c)\) for which the running coupling is \(J_\perp(l^*) \sim J_\parallel\). The strong 1D fluctuations thus have a large effect on the critical temperature and create a wide regime where the system is dominated by 1D fluctuations, as indicated on Fig. 1.1. Although the temperature scale is strongly affected, the critical behavior still corresponds to that of the higher-dimensional case. However, the quasi-1D nature of the problem has strong consequences for the existence of extra modes of excitations in comparison to what happens for a more isotropic system [11]. We will come back to this point when discussing bosons, where these modes can be more simply understood.

### 1.1.2 Dimer or Ladder Coupling

A much more complex behavior occurs when the objects that become coupled have a gap in their spectrum, a gap that is in competition with the presence of the inter-chain coupling (1.2). In this case one can expect a real QPT to occur in which the system goes from a low-dimensional gapped situation, to a higher-dimensional ungapped one. This transition is called generically a deconfinement transition, since the system changes both its effective dimensionality and the nature of its spectrum at the same time. This particular type of QCP manifests itself in several types of systems and we will examine it for spins, bosons and fermions.

The case of spin is the simplest. To illustrate the nature of this QCP let us consider first the case of a system made of dimers, weakly coupled by (1.2). In this case each dimer has a gap between a singlet state and the three triplet states. The gap is of order \(J_\delta\), the dimer spin exchange. Since there is a gap in the spin excitation spectrum, the dimer is robust to the inter-dimer
FIGURE 1.1
(a) Coupled 1D chains. The inter-chain coupling $J_{\perp}$ (dashed line) is much weaker than the intra-chain one $J_{\parallel}$ (full line). This leads to properties linked to the Luttinger liquid ones of the 1D chains. (b) Depending on the Luttinger liquid parameter $K$, the critical temperature is a power law of the inter-chain coupling $J_{\perp}$, since the coupling is strongly renormalized by 1D fluctuations. The exponent is either smaller than one (full line), or when the fluctuations increase, larger than one (dashed line). If $K$ is below (or above depending on the correlations, see text) a certain value, fluctuations are small enough and lead to an ordered state as soon as some inter-chain coupling is introduced. However, if the fluctuations are large enough, ordering is suppressed unless the inter-chain coupling reaches a critical value (dashed-dotted line). Note that in this case usually another correlation orders since several instabilities are in competition. (c) Because of the strong 1D fluctuations, the mean-field temperature can be quite different from the actual critical temperature to an ordered state.

exchange coupling and the ground state in the case $J_{\perp} \ll J_d$ is made of essentially uncoupled dimers. In this case we are considering a cluster of zero-dimensional objects coupled by the inter-dimer coupling. If we now place the system in a magnetic field the dimer gap reduces and ultimately the lowest triplet state reaches the level of the singlet one as depicted in Fig. 1.2. In this case the inter-dimer coupling is able to delocalize the triplets and lead to a transition where the system will go from a set of essentially uncoupled zero-dimensional objects to an essentially 3D antiferromagnet. Quite remarkably, this deconfinement transition can be analyzed by mapping the singlet-lowest triplet onto a hard-core boson. The system is thus equivalent to a set of hard-core bosons, the density of which is controlled by the magnetic field.†

†Zero boson density means that all dimers are in the singlet state, while one boson per site
FIGURE 1.2
(a) If one of the exchanges $J_d$ (thick line) is larger than the others $J$ (dashed line) then one has a system made of coupled dimers. Because a dimer goes into a singlet state (oval shape), the ground state of such system is made of decoupled singlets. (b) One can map such a system onto a system of hard-core bosons, the presence of a boson denoting a triplet state on the dimer, and its absence a singlet. Because of spin exchange one has an equivalent system of bosons hopping on a lattice with a kinetic energy given by the inter-dimer magnetic exchange $J$. In addition to the hard-core constraints bosons have nearest-neighbor interactions. (c) Application of a magnetic field lowers the energy of one of the triplet states (thick dot) compared to the singlet one (small dot). Because the triplet disperses, there is a band of triplet excitations (triplons). Changing the magnetic field allows one to fill this band of triplons which are the hard-core bosons of (b). The field $H_{c1}$ corresponds to the first triplon entering the system, while the field $H_{c2}$ is a filled band of triplons. Such a system thus provides an excellent venue in which to study interacting bosons on a lattice, since the density of bosons can be controlled directly by the magnetic field, and measured by the magnetization along the field direction. (d) this system has a quantum phase transition at $H_{c1}$; a similar transition exists at $H_{c2}$, not shown here. The triplons exhibit Bose-Einstein condensation (BEC), which corresponds in spin language to antiferromagnetic order in the direction perpendicular to the magnetic field. “Gap” and “QC” denote the gapped state in which there are no triplons and the quantum critical state, respectively.
QPT in $Q1D$ systems

When the first triplets enter the system at a critical field $h_{c1}$ the bosons are extremely dilute; thus their hard-core interaction is not felt very strongly. The phase transition is thus Bose-Einstein condensation [12]. The QCP at $T = 0$ corresponds to the point where the chemical potential is such that a finite density of bosons starts to appear. This has several interesting consequences for the nature of the phase diagram and in particular allows one to predict features such as the critical temperature which behaves as

$$T_c \propto (h - h_{c1})^{2/d},$$

(1.6)
as well as non-monotonous temperature dependence of the magnetization of the system. Since its original prediction, these behaviors have been studied and observed in several compounds with both 3D [13, 14, 15] and bi-dimensional structure [16]. For a review on these aspects I refer the reader to [17].

A similar class of deconfinement transitions occurs when the objects that are coupled have a 1D structure. For example, when one deals with a spin one chain [18] or a two leg spin ladder [12]. Both these structures are characterized by a gap. Application of a magnetic field allows one to break the gap and to study the transition to the 3D behavior. In this case one gets a very interesting behavior which is depicted in Fig. 1.3. If one is far from the critical field $H_{c1}$ the chemical potential of the excitations in the 1D chain is high and in particular larger than the inter-chain coupling. One is thus dealing essentially with the situation depicted in the previous section, of weakly coupled 1D LL, and one can study the transition to the ordered state. Dimers present several advantages to study this phase transition since the singlet is extremely robust to external perturbations, such as dipolar interactions. Such interactions would break the spin rotation symmetry in the $XY$ plane and thus in the boson mapping break the phase $U(1)$ symmetry. Recently, very nice experimental realizations of ladder systems have been analyzed [19, 20, 21, 22]. These analyses have allowed for a quantitative test of the predictions of the LL and of the generic scenarios described above for the transition to the ordered state.

When getting closer to the field $H_{c1}$ one cannot consider that the system is made of coupled LLs since the temperature is getting larger than the distance to the bottom of the band, and one has to consider the 1D quantum critical behavior. Such a situation, although more complex, can still be analyzed by various techniques [23]. In a similar way, when lowering the temperature one has to consider an additional crossover where the temperature becomes smaller than the inter-ladder coupling $J_\perp$. This corresponds to going from a 1D quantum critical regime of weakly coupled ladders to the 3D one of coupled dimers, as described in Sec. 1.1.1. The resulting physical behavior is thus quite complex and largely not understood, despite the analysis of several

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means that each dimer is fully polarized.
quantities, such as the NMR response. Other ways to control such a phase transition include applying pressure instead of the magnetic field; this changes the ratio $J_\perp/J_\parallel$ and thus makes the system more 3D, as described in Chap. ??.

We proceed to consider this type of transition in more detail for the related but different case of itinerant bosons and fermions.

**1.2 Bosons: From Mott Insulators to Superfluids**

Consider first the case of coupled bosonic chains. In principle, such a system is very close to the problem of coupled spin chains, since a spin 1/2 can be represented by a hard-core boson. However, the absence of the hard-core constraint, and of the nearest neighbor interaction that corresponds to the $J_\perp \sum_i S_i^z S_{i+1}^z$, lead in practice to quite different regimes than for coupled spin chains. Nevertheless, most of the techniques and concepts that we used for spin chains will be directly useful for coupled bosonic chains.

Although in principle one can realize coupled bosonic systems in condensed matter, e.g. using Josephson junction arrays [24], it is relatively difficult to obtain a good realization. Recently, cold atomic systems in optical lattices have provided a remarkable and very controlled realization on which many of the aspects discussed below can be tested in experiments, as discussed in Chap. ??.

For the case of bosons the coupling between the chains comes mostly from the single-particle hopping from one chain $\mu$ to the neighboring one $\nu$. This term in the Hamiltonian takes the form

$$H_\perp = -t_\perp \sum_{\langle \nu, \mu \rangle} \int dx \psi_\mu^\dagger(x) \psi_\nu(x). \quad (1.7)$$

Along the chains, interacting bosons can be represented either by a continuum theory or directly on a lattice by a Bose-Hubbard model. In 1D both these cases can be mapped to a LL description for the low energy properties [3, 25, 26]. The corresponding Hamiltonian is

$$H^0 = \frac{1}{2\pi} \int dx \left[ \frac{u}{K} \pi \Pi(x) + u K (\nabla \theta)^2 \right], \quad (1.8)$$

where $\theta$ is the superfluid phase determined by $\psi(x) = \rho(x)^{1/2} e^{i\theta(x)}$. The field $\Pi(x)$ is canonically conjugate to $\theta$ and is associated with density fluctuations. The long-wavelength density fluctuations can indeed be represented as $\rho(x) = \rho_0 - \pi^{-1} \nabla \phi(x)$ and $\pi \Pi(x) = \nabla \phi(x)$. Also, $u$ is the velocity of the sound waves in the 1D systems, while $K$ is the LL exponent which depends on the microscopic interactions. For bosons with a contact interaction, $K = \infty$ for
non-interacting bosons, while $K \to 1^+$ when the contact interaction becomes infinite. The latter is the Tonks-Girardeau limit.

As for spins, the properties will be crucially dependent on whether the 1D bosonic system is gapless or not. In the continuum, the full description of the system is indeed given by (1.8). In the presence of a lattice or a periodic potential one should take into account oscillations of the density with periodicity $2\pi \rho_0$, of the form $\delta \rho(x) \propto \cos(2\pi \rho_0 x - 2\phi(x))$. These oscillations, when commensurate with the period of the lattice, e.g. with one boson per site, can lead to a Mott-insulating phase for the bosons [27, 28, 3, 26]. In this case the full 1D Hamiltonian becomes

$$H = H^0 - g \int dx \cos(2\phi(x)), \quad (1.9)$$

where $g$ is a constant proportional to the lattice strength for small lattices or the interaction for large ones [3]. Such a term becomes relevant for $K < 2$ and leads to an ordered phase $\phi(x)$. Since $\phi$ is locked by the cosine term, this corresponds to frozen density fluctuations, and thus a Mott-insulating phase with an integer number of bosons per site. Since $\phi$ and $\theta$ are conjugate variables this implies that superfluid correlations decrease exponentially and that the quasi-long-range superfluid order is destroyed. Let us examine both these cases, with and without the commensurate term.

### 1.2.1 Coupled Superfluid: Dimensional Crossover

the case in the absence of the lattice, or when the lattice is irrelevant, is very similar to the coupled spin chains examined in Sec. 1.2. Each 1D chain is critical with a quasi-long-range superfluid order, since with (1.8) the superfluid correlations decay as a power-law:

$$\langle \psi(x)\psi^\dagger(0) \rangle \propto x^{-1/2K}. \quad (1.10)$$

One can treat the inter-chain coupling in the mean-field approximation, since boson single-particle operators can have a mean-field value:

$$H_\perp = -t_\perp \sum_{(\mu\nu)} \int dx \left[ (\psi^\dagger_\mu(x))\psi_\nu(x) + h.c. \right] \to -\Delta \int dx \cos(\theta(x)), \quad (1.11)$$

where $\Delta = 2z\rho_0^{1/2}t_\perp \langle \psi^\dagger(x) \rangle$, and $z$ is the coordination of the lattice. Thus one finds a sine-Gordon Hamiltonian in the superfluid phase that can freeze the phase $\theta$ and lead to long range superfluid order [29, 30]. Note that without the mean-field approximation the interaction term in phase language becomes

$$H_\perp = -2t_\perp \rho_0 \sum_{(x,\mu)} \int dx \cos(\theta_\mu(x) - \theta_\mu(x)). \quad (1.12)$$
Given the quadratic form of (1.8), and taking time as an extra classical dimension, one can immediately map this problem onto coupled $XY$ planes. For a 3D system one would thus be in the universality class of the five-dimensional $XY$ model, justifying the use of the mean-field approximation.‡ As for the problem of spins, 1D fluctuations strongly renormalize the critical temperature as compared to the naive mean-field value $T_c \sim t_\perp$. The scaling is very similar to that of Sec. 1.2 [30].

An interesting effect can be seen when looking at fluctuations around the ground state in the low temperature superfluid phase. As can be readily seen by performing a random-phase-approximation (RPA) treatment [30] of the Hamiltonian $H^{1D} + H_\perp$, two eigenmodes exist. One is the standard phase mode, where the amplitude of the order parameter is essentially fixed but $\theta_\nu(x)$ slowly varies in space and from chain to chain. The energy of this mode goes to zero, since this is the standard Goldstone mode of the superfluid. However, another eigenmode exists, corresponding to a change in amplitude of the order parameter, and thus also $\rho(x) = \rho_0 + \delta \rho(x)$. Not surprisingly, this mode is dispersing above a finite energy $E_0$ but exists as a sharply defined mode, in a way very similar to plasmons in charged systems. Such a mode would not appear in a more isotropic superfluid, as can be readily seen by solving the Gross-Pitaevskii equation [31]. This is one clear-cut case where the higher-dimensional system still shows some traces of its 1D origin and displays qualitative differences as compared to an isotropic system. Such modes have also been observed close to Mott transitions in isotropic systems [32, 30].

1.2.2 Coupled Mott Chains: Deconfinement Transition

As for spins, the situation is much more interesting and complex when the 1D chains are in the Mott-insulating phase. In this case it is clear that there is a competition between the Mott term (1.9), caused by the periodic potential along the chains that prefers order in the phase $\phi$ controlling the density, and the inter-chain Josephson term (1.12) that prefers to order the superfluid phase $\theta$. This is the bosonic equivalent of the competition between the spin gap and the transverse magnetic order that existed for the spin chains, and was discussed in Sec. 1.2.1. This competition leads to a deconfinement QPT.

In contrast to the case of spins, where the 1D gap was closed by changing the magnetization, in the case of bosons one stays at a commensurate density and the critical point is reached by changing the strength of the inter-chain hopping. This is very similar to the question of the application of pressure in the case of the spins. A similar transition to the one studied in Sec. 1.2.1 could also occur in the case of bosons. It corresponds to the application of a chemical potential taking the system away from the commensurate point.

‡The mean-field approximation is exact in the limit that the number of nearest neighbors approaches infinity.
In this case the 1D system is described by a commensurate-incommensurate phase transition [3] and the universality class of the deconfinement transition is different. We confine our present discussion to the commensurate case; see [30] for the incommensurate one.

In the commensurate case there are several ways to analyze the deconfinement transition. In the mean-field approximation the system is described by a double sine-Gordon Hamiltonian. This Hamiltonian has a set of remarkable properties that have been looked at in various contexts [33]. The critical point can be crudely obtained via renormalization of the two relevant operators and fixing the phase of the operator that first reaches strong coupling. A more sophisticated analysis can be found in Ref. [30]. In particular, the universality class of the transition can be shown to be that of the \((d+1)\)-dimensional XY model [28]. Indeed the operator \(\cos(2\phi)\) is nothing but the vortex creation operator for excitations of the phase \(\theta\). Each chain can thus be mapped onto a discrete XY Hamiltonian of the form \(H = J \sum_{i,j} \cos(\theta_i - \theta_j)\) and the inter-chain coupling has a similar form but with a different coefficient. A schematic representation of the phase diagram is shown in Fig. 1.4. As for the spins, there is a deconfinement transition between an essentially 1D insulating phase, where the bosons are in a Mott state, and an anisotropic 3D superfluid phase. In the Mott phase, there is a gap towards excitations, and the density is well ordered, i.e. one particle per site. The anisotropic superfluid phase is gapless and is similar to the one that was discussed in the absence of a lattice in the previous section. Such physics can be probed in cold atomic systems in systems made of coupled bosonic tubes such as [34].

\subsection*{1.3 Fermions: Dimensional Crossover and Deconfinement}

Let us finally move to the very challenging problem of coupled 1D fermionic chains. As for bosons, the system is described by a Hamiltonian

\[
H = \sum_{\alpha} H_{1D}^\alpha - t_{\perp} \sum_{(\alpha,\beta)} \int dx \psi_{\alpha}^\dagger(x) \psi_{\beta}(x).
\]

However, there is a very important difference between the fermionic case and the two previous sections. Indeed, for fermions the single-particle operator cannot have an average value: \(\langle \psi_{\alpha} \rangle = 0\). We cannot treat the inter-chain coupling by treating the single-particle operator in a mean-field approximation as we did before. It is thus difficult to find theoretical tools to tackle this problem on the analytical side. Similarly, on the numerical side one cannot use the efficient methods of the 1D world, such as the density matrix renormalization group of Chap. ???. One has to use the arsenal of higher-dimensional Monte
Carlo methods, which can suffer from the sign problem, as described in several chapters of Part ??.

Analytically, one can use a renormalization technique similar to the one introduced in Secs. 1.2.1-1.2.2 to study the relevance of the inter-chain hopping, as we will discuss in more detail below. Unfortunately, it will only yield information about whether the inter-chain coupling is relevant, not what the strong-coupling fixed point actually is. To understand this physics, and replace the mean-field treatment used in the two previous cases, two approximate methods have proven useful for fermions, as summarized in Fig. 1.5. The first and simplest one is to treat the inter-chain coupling in RPA [35]. This leads to

\[ G(k, k_\perp, \omega) = \left( G_{1D}^{-1}(k, \omega) - t_\perp \cos(k_\perp) \right)^{-1}. \]  

(1.14)

RPA has the advantage of being very simple. However, it neglects all feedback of the inter-chain hopping on the 1D properties themselves, which is clearly a very brutal approximation. A better approximation is provided by an extension of dynamical mean-field theory (DMFT) [36], where one treats all the chains but one as an external self-consistent bath into which the particles can jump [37, 38]. This is detailed in Fig. 1.5. From a more formal point of view one could view it naively as performing a mean-field approximation on the second-order term in the hopping in the action

\[ S'_\perp = t_\perp^2 \sum_{\mu, \nu} \int dr dr' \psi_{\mu}^\dagger(r) \psi_{\nu}(r) \psi_{\nu}(r') \psi_{\mu}^\dagger(r') \] 

\[ \rightarrow t_\perp^2 \sum_{\mu, \nu} \int dr dr' \psi_{\mu}^\dagger(r) \langle \psi_{\nu}(r) \psi_{\nu}^\dagger(r') \rangle \psi_{\mu}(r'), \]  

(1.15)

where \( r = (x, \tau) \) are the space-time coordinates. One thus has an effective intra-chain kinetic energy, nonlocal in space and time, whose amplitude is controlled by the single-particle Green’s function on another chain \( G_{\nu}(r, r') \). Such a Green’s function must thus be determined self-consistently, and one has to solve an effective 1D Hamiltonian with a modified kinetic energy. Because the energy now depends self-consistently on the results of the inter-chain tunneling, there is a direct feedback of the inter-chain tunneling on the 1D features, contrary to the case of the RPA. Although still imperfect this is an improvement which should allow one to obtain several of the features more accurately.

As for bosons, let us examine the two cases depending on whether the 1D system is critical (LL) or gapped (typically a Mott insulator). In the first case the inter-chain hopping leads simply to a dimensional crossover between 1D and higher-dimensional behavior, while in the second case a deconfinement transition occurs.
1.3.1 Dimensional Crossover

If the 1D system is in a critical LL state, the inter-chain hopping has in general a strong influence on it. In a similar way as for the bosons, one can estimate the relevance of the single-particle hopping by a simple scaling analysis. If the single-particle Green’s function decreases as

\[ G^{1D}(r, t) \propto r^{-\left[ K_\rho + K_\rho^{-1} + 2\right] / 4} t^{-\left[ K_\rho + K_\rho^{-1} + 2\right] / 4}, \]  

(1.16)

where \( K_\rho \) is the charge LL parameter [3], then the perpendicular hopping obeys the renormalization equation coming from the second order expansion in the inter-chain hopping [39, 40]:

\[ \frac{\partial t_\perp}{\partial t} = t_\perp \left[ 2 - \frac{1}{4} \left( K_\rho + K_\rho^{-1} + 2\right) \right]. \]  

(1.17)

Thus when \( K_\rho + K_\rho^{-1} > 6 \) the inter-chain hopping is irrelevant. The intra-chain interactions are enough to prevent the coherent hopping since a single-particle excitation must be reconstructed for the electron to be able to hop from one chain to the next. Note that this implies rather strong, as well as finite range interactions [3]. Indeed, for a purely local interaction such as the one coming from a Hubbard model, \( 1/2 < K_\rho < 2 \) and thus the inter-chain hopping would always be relevant. The fact that the inter-chain hopping is irrelevant does not mean that there is no coupling at all between the chains; it just means that single-particle excitations cannot propagate coherently between them. One must then go to second order in the inter-chain hopping. To second order, the inter-chain hopping generates both particle-hole coupling, i.e. either density-density or spin-spin, or particle-particle coupling, i.e. Josephson. One of these couplings can become relevant and lead to an ordered state. The couplings can be treated by mean-field theory, as explained in Secs. and in more detail in Ref. [3].

If \( K_\rho + K_\rho^{-1} < 6 \), called moderate interactions, the inter-chain hopping is a relevant perturbation. There will thus exist an energy scale below which the system will crossover from 1D to higher-dimensional behavior. Note that this is a simple crossover and that no phase transition occurs here. The 1D nature of the system strongly affects the scale at which the inter-chain hopping acts. This scale is roughly determined by the condition \( t_\perp (t^*) = 1 \). Thus one finds the crossover at

\[ E_{\text{cross}} \propto E_F \left( \frac{t_\perp}{E_F} \right)^{2/(2 - 2\zeta)}, \]  

(1.18)

where \( \zeta = \left[ K_\rho + K_\rho^{-1} + 2\right] / 4 \) is the single-particle correlation exponent. As we saw for spins, interactions considerably lower the crossover scale and reinforce the range of validity of the 1D regime. The properties of the resulting

\[ \text{e.g. temperature, frequency determined via probes such as optical conductivity, energy determined via probes such as STM, etc. For more details on how to probe this crossover see [41, 42].} \]
low-temperature phase is still a largely open question. In particular, how much this phase remembers the effects of strong correlations coming from the high energy 1D physics is important to determine. Some elements of response can be obtained via the various mean-field approximations mentioned above. Quantities strongly depending on the transverse directions are very interesting but also very difficult to compute. This includes the Hall effect and the transverse conductivity. In particular, the latter can be an indication of the dimensional crossover transition temperature since the absence in the 1D regime or the presence in the higher-dimensional regime of well formed single-particle excitations will lead to very different temperature dependence. This is the case e.g. for organics superconductors [43] and for inorganic compounds [5].

1.3.2 Deconfinement Transition

The situation is particularly difficult for fermions when the 1D phase is gapped. The expected phase diagram is schematically indicated in Fig. 1.6. Let us examine in more detail the features of such a transition. Although the generic shape of the $T-t_\perp$ diagram reminds us of what is to be expected for a generic QPT, the order and even the number of transitions are not known with certainty. In some cases, both from the RPA and from Ch-DMFT, two different transitions are expected to occur.

The physics of the massive (Mott) phase is relatively clear. The effective gap is reduced by the additional kinetic energy provided by the inter-chain hopping. Such an effect is well described by the Ch-DMFT approximation, which shows a reduction of the 1D gap as $t_\perp$ increases. In principle, one needs the transverse directions to be on a non-bipartite lattice. Otherwise, the Fermi surface remains nested despite the inter-chain hopping and the gap does not vanish. This is specially important for the organic compounds. At a certain critical value of the inter-chain hopping, one expects to break the Mott-insulating phase and recover a higher-dimensional metal. How this transition occurs is still unclear. Within RPA and Ch-DMFT approximations, one goes through an intermediate phase where pockets appear as depicted in Fig. 1.6. At larger values of the inter-chain hopping the pockets merge and one expects to recover an open Fermi surface. This scenario is established for spinless particles [44]. Whether it survives for fermions with spins is still an open question.

Another important open question is to get an accurate description of the properties of the higher-dimensional metal, and in particular whether one gets back a Fermi liquid or whether there is a serious influence of the strong correlations that existed in the 1D part of the phase diagram. Even if one recovers a Fermi liquid, as is the case with the Ch-DMFT method for example, there is clearly a strong variation of the lifetime and quasiparticle weight along the Fermi surface (hot spots). How to reliably compute such effects is a considerable challenge.
Finally, let us point out that such transitions are important for a host of quasi-1D systems. Deconfinement transitions are investigated in organics [42, 45] but the inter-chain hopping also clearly plays a crucial role in systems such as purple bronze [5]. Both these systems still have a poorly understood superconducting phase in the higher-dimensional regime. How much such a phase is influenced by the 1D nature of the material still remains to be determined. Cold atomic systems are now allowing us to realize quasi-1D structures of fermions as well, and will undoubtedly provide an excellent experimental realization in which to study these problems.

1.4 Conclusions and Perspectives

1D and quasi-1D systems are a paradise for QPTs. Due to the intrinsic critical nature of interacting quantum systems, a pure 1D system can present a set of instabilities, the simplest one being associated with the occurrence of one or several gaps in the systems. This occurs in the case of the Mott transition in 1D, among many other examples.

Another important class of QPTs is driven by the inter-chain coupling between 1D systems. This is an especially important case given the direct experimental relevance for several realizations, ranging from naturally occurring materials to cold atoms in optical lattices. In this case two main classes exist. If the 1D chain is gapless, one usually finds a dimensional crossover between a high temperature or high energy 1D regime and a low temperature one dominated by the inter-chain coupling. For spins or bosons, the low-temperature state is usually ordered. Although this state is mostly an anisotropic version of the 3D one, it can still retain some special features coming from the quasi-1D character. The case of fermions is more complex and the low-temperature phase is a higher-dimensional metal. If the 1D chains are gapped one finds a deconfinement transition where the system goes at zero temperature from a 1D gapped state to a high dimensional ordered or gapless one, the latter occurring in particular for fermions. The nature of this transition in fermionic systems is largely not understood and constitutes a very challenging research field.

Although we have some of the tools and some understanding of such transitions many crucial questions remain. First, in contrast to the case of a purely 1D system for which we have a whole arsenal of analytical and numerical tools to tackle the questions of such transitions, the quasi-1D case is much more difficult. Most of the numerical techniques are becoming very inefficient, either due to their intrinsic limitations, such as the sign problem for the fermions, or simply the large anisotropy of the system that makes even well-controlled methods difficult to apply. Clearly some new techniques are needed. On the
analytical side it is difficult to go beyond mean-field theory, and thus to compute some of the correlation functions, especially those involving directly the transverse degrees of freedom. Going through the critical regime is also quite challenging, even if we have a good idea of the various phases. of special importance are the transverse transport, the Hall or Nernst effect, and the propagation of some transverse modes.

Some other topics are directly related to these issues and present very challenging topics in themselves. The self-consistent dynamical mean-field approximation of coupled LL replaces this problem by that of a LL in equilibrium with an external bath. The bath has drastic consequences on the critical properties of the LL. This type of problem also occurs directly, either due to the presence of external electrons or noise. How to tackle such questions is certainly one of the frontiers of our knowledge of 1D systems. Finally, all the examples of QCPs examined in this chapter were based on a LL or a gapped phase as the description of the 1D system. However, there are now several identified 1D cases in which one has to go beyond the LL paradigm to describe the physics of the systems. Understanding the physics of such non-Luttinger liquids is a challenge in itself, much as the understanding of non-Fermi liquids is. How to go from the standard LL behavior to a non-LL one, or what happens when non-LLs are coupled in a 3D lattice is a totally uncharted territory.

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[1] G. D. Mahan, *Many Particle Physics* (Plenum, New York, 1981).
[2] F. D. M. Haldane, J. Phys. C 12, 4791 (1979); F. D. M. Haldane, J. Phys. C 14, 2585 (1981); F. D. M. Haldane, Phys. Rev. Lett. 45, 1358 (1980).
[3] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, Oxford, 2004).
[4] Chem. Rev. 104, (2004), special issue on Molecular Conductors.
[5] F. Wang *et al.*, Physical Review Letters 103, 136401 (2009); J. Hager *et al.*, Physical Review Letters 95, 186402 (2005); X. Xu and *et al.*, Physical Review Letters 102, 206602 (2009), and refs. therein.
[6] D. A. Tennant, R. A. Cowley, S. E. Nagler, and A. M. Tsvelik, Phys. Rev. B 52, 13368 (1995).
[7] B. Lake, D. A. Tennant, and S. E. Nagler, Phys. Rev. Lett. 85, 832 (2000).
[8] E. Dagotto and T. M. Rice, Science 271, 5249 (1996); E. Dagotto, Rep. Prog. Phys. 62, 1525 (1999).
[9] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
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[10] F. D. M. Haldane, Phys. Rev. Lett. 50, 1153 (1983); J. P. Renard et al., Europhys. Lett. 3, 945 (1987).

[11] H. J. Schulz, Phys. Rev. Lett. 77, 2790 (1996); M. Bocquet, F. H. L. Essler, A. M. Tsvelik, and A. O. Gogolin, Phys. Rev. B 64, 094425 (2001).

[12] T. Giamarchi and A. M. Tsvelik, Phys. Rev. B 59, 11398 (1999).

[13] T. Nikuni, M. Oshikawa, A. Oosawa, and H. Tanaka, Phys. Rev. Lett. 84, 5868 (2000).

[14] C. Ruegg et al., Nature (London) 423, 62 (2003).

[15] M. Matsumoto, B. Normand, T. M. Rice, and M. Sigrist, Phys. Rev. B 69, 54423 (2004).

[16] S. E. Sebastian et al., Nature 441, 617 (2006).

[17] T. Giamarchi, C. Ruegg, and O. Tchernyshyov, Nature Physics 4, 198 (2008).

[18] I. Affleck, Phys. Rev. B 41, 6697 (1990).

[19] B. C. Watson et al., Phys. Rev. Lett. 86, 5168 (2001).

[20] M. Klanjsek et al., Phys. Rev. Lett. 101, 137207 (2008).

[21] C. Ruegg et al., Phys. Rev. Lett. 101, 247202 (2008).

[22] B. Thielemann et al., Phys. Rev. B 79, 020408(R) (2008).9.

[23] E. Orignac, R. Citro, and T. Giamarchi, Phys. Rev. B 75, 140403(R) (2007).

[24] R. Fazio and H. van der Zant, Phys. Rep. 355, 235 (2001).

[25] M. A. Cazalilla, J. Phys. B 37, S1 (2003).

[26] T. Giamarchi, in Lectures on the Physics of Highly Correlated Electron Systems X (AIP conference proceedings, 2006), p. 94, arXiv:cond-mat/0605472.

[27] F. D. M. Haldane, Phys. Rev. Lett. 47, 1840 (1981).

[28] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B 40, 546 (1989).

[29] K. B. Efetov and A. I. Larkin, Sov. Phys. JETP 42, 390 (1975).

[30] A. F. Ho, M. A. Cazalilla, and T. Giamarchi, Phys. Rev. Lett. 92, 130405 (2004); M. A. Cazalilla, A. F. Ho, and T. Giamarchi, New. J. of Phys. 8, 158 (2006).

[31] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Clarendon Press, Oxford, 2003).
[32] S. D. Huber, B. Theiler, E. Altman, and G. Blatter, Physical Review Letters 100, 050404 (2008).

[33] J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B 16, 1217 (1977); L. P. Kadanoff, J. Phys. A 11, 1399 (1978); T. Giamarchi and H. J. Schulz, J. Phys. (Paris) 5, 819 (1988).

[34] T. Stöferle et al., Phys. Rev. Lett. 92, 130403 (2004).

[35] F. H. L. Essler and A. M. Tsvelik, Phys. Rev. B 65, 115117 (2002).

[36] A. Georges, G. Kotliar, W. Krauth, and M. J. Rozenberg, Rev. Mod. Phys. 68, (1996).

[37] E. Arrigoni, Phys. Rev. Lett. 83, 128 (1999); A. Georges, T. Giamarchi, and N. Sandler, Phys. Rev. B 61, 16393 (2000); E. Arrigoni, Phys. Rev. B 61, 7909 (2000); S. Biermann, A. Georges, A. Lichtenstein, and T. Giamarchi, Phys. Rev. Lett. 87, 276405 (2001); C. Berthod, T. Giamarchi, S. Biermann, and A. Georges, Physical Review Letters 97, 136401 (2006).

[38] S. Biermann, A. Georges, T. Giamarchi, and A. Lichtenstein, in Strongly correlated Fermions and Bosons in low dimensional disordered systems, edited by I. V. Lerner et al. (Kluwer Academic Publishers, Dordrecht, 2002), p. 81, cond-mat/0201542.

[39] S. Brazovskii and V. Yakovenko, J. de Phys. (Paris) Lett. 46, L111 (1985); C. Bourbonnais and L. G. Caron, Physica (Utrecht) 143B, 450 (1986); X. G. Wen, Phys. Rev. B 42, 6623 (1990); C. Bourbonnais and L. G. Caron, Int. J. Mod. Phys. B 5, 103 (1991); V. M. Yakovenko, JETP Lett. 56, 510 (1992); D. Boies, C. Bourbonnais, and A.-M. S. Tremblay, Phys. Rev. Lett. 74, 698 (1995).

[40] H. J. Schulz, in Correlated Fermions and Transport in Mesoscopic Systems, edited by T. Martin, G. Montambaux, and J. Tran Thanh Van (Editions frontières, Gif sur Yvette, France, 1996), p. 81.

[41] T. Giamarchi, Chem. Rev. 104, 5037 (2004).

[42] C. Bourbonnais and D. Jerome, in Physics of Organic Superconductors and Conductors, edited by A. G. Lebed (Springer, Heidelberg, 2008), p. 357, T. Giamarchi, ibid, p. 719.

[43] J. Moser et al., Eur. Phys. J. B 1, 39 (1998); M. Dressel et al., Phys. Rev. B 71, 75104 (2005); W. Henderson et al., Eur. Phys. J. B 11, 365 (1999).

[44] C. Berthod, T. Giamarchi, S. Biermann, and A. Georges, Phys. Rev. Lett. 97, 136401 (2006).

[45] M. H. A. Pashkin, M. Dressel and C. A. Kuntscher, arXiv:0909.4795.
FIGURE 1.3

(a) Coupled ladders correspond to a hierarchy of coupling where the leg coupling $J_\parallel$ (solid line) is smaller than the rung coupling $J_d$ (thick line), but larger than the inter-ladder coupling $J'$ (dashed line). In the same way as in Fig. 1.1 this leaves room for 1D fluctuations and Luttinger liquid physics to modify the behavior as compared to the more isotropic case of Fig. 1.2. (b) As a result, a LL regime exists between $H_{c1}$ and $H_{c2}$. In this regime a good physical description of the triplons is to consider that they behave as spinless fermions. The regime 3$d$ where the spins order antiferromagnetically is the equivalent of the BEC regime of Fig. 1.2. (c) As long as the coupling $J'$ is larger than the chemical potential $H - H_{c1}$ one can consider the system as made of coupled 1D systems and 1D fluctuations play a major role. On the contrary, when $J' > (H - H_{c1})$ one must consider a 3D system from the start since there is no room for 1D fluctuations to take place. One thus returns to the case of Fig. 1.2. (d) As a result there is a more complex crossover regime when the field $H$ gets close to $H_{c1}$. C-Lad denotes coupled ladders and BEC is the Bose-Einstein condensation of Fig. 1.2.
FIGURE 1.4
Phase diagram for quasi-1D bosons on a lattice (at $T = 0$): $g$ is the intra-chain periodic potential responsible for the Mott transition for commensurate filling; $t_\perp$ is the inter-chain kinetic energy or Josephson coupling; and $K$ is the Luttinger parameter that depends on the intra-chain interactions ($K = \infty$ corresponds to free bosons). The thick solid line is the boundary between a 1D Mott insulator and a quasi-ordered 1D superfluid. For very small $g$ the Mott phase occurs for $K < 2$. The thick dashed line indicates how the extra kinetic energy provided by the inter-chain coupling weakens the Mott state. The green lines are the deconfinement transition between a Mott insulator and an anisotropic 3D superfluid.
FIGURE 1.5
The two main approximations used to tackle the inter-chain coupling for fermions. RPA: the fermions hops but essentially never comes back to the original chain. The properties of a single chain are thus not affected at all by the inter-chain hopping. In particular, the Mott gap is strictly independent of $t_\perp$. Ch-DMFT: all the chains but one are treated as a self-consistent bath. The 1D Green’s function thus depends approximately on the inter-chain hopping. This corresponds to the approximation of taking the 1D self-energy independent of the transverse momentum $k_\perp$ but potentially dependent on the frequency and momentum along the chains.
(a) Phase diagram for the deconfinement transition of quasi-1D coupled fermionic chains. There is a quantum critical point (dot) that separates a Mott insulator from a higher-dimensional metal (HDM). For small inter-chain hopping $t_\perp < t_c$ there is a crossover scale, the renormalized Mott gap $\Delta$. For temperatures $T > \Delta$ the system behaves as a LL, while it acts as a Mott insulator for smaller temperatures. On the metallic side there is a coherence scale $T^*$ scaling with the inter-chain hopping that separates the LL regime from the higher-dimensional metal in which coherent hopping between the chains occurs. The nature of the deconfinement transition, and whether there is a unique transition or a more complex scenario such as two consecutive transitions, are still largely open questions. (b) The effective Mott gap $\Delta$ as predicted both by the RPA and the Ch-DMFT approximation. (c) With the current approximations there would be three different phases, leading to different Fermi surfaces. (i) At small gap one has a Mott insulator with only zeros of the Green’s function. (ii) At intermediate gaps the inter-chain hopping induces an indirect doping, leading to pockets on the Fermi surface. (iii) For larger hopping the gap closes and the pockets join to give back the open Fermi surface of a quasi-1D metal. Note that the Fermi parameter varies strongly on such a Fermi surface, reminiscent of the 1D character. In particular, there would be hot spots (dots). The scenario with two transitions can be established for spinless fermions with the Ch-DMFT approximation. Whether such a scenario survives with fermions with spins is still an open question. In the RPA the gap is not destroyed by the inter-chain hopping, leading to a rigid-band scenario. As a consequence, the pockets depicted in (c) never close in such an approximation.