Recently spin chains have been proposed as potential channels for short distance quantum communications (See, for example, Refs. [1, 2]). The basic idea is to simply place the state to be transmitted at one end of a spin chain initially in its ground state, allow it to propagate for a specific amount of time, and then receive it at the other end. Generically, while propagating, the information will also inevitably disperse in the chain, and even when a transmission is considered complete (i.e., the state is considered to have been received with some fidelity/probability), some information of the state lingers in the channel. It is thus assumed that a reset of the spin chain to its ground state is made after each transmission [3]. If, on the other hand, a second transmission is performed through the channel without resetting, then the memory of the first transmission should affect the second transmission. A spin chain channel without resetting is thus an interesting physical model of a channel with memory [4].

In this paper, we show that a ferromagnetic spin chain used without resetting is a very different channel than those studied so far in the extensive literature of quantum channels with memory [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Firstly, the channels usually studied are those with the noise during multiple uses being correlated with each other [4, 5, 6, 7, 8, 10, 11], but being independent of the transferred states. In our model, however, the state transmitted during the first use modifies the type of noise during the second use. Secondly, the noise is most often assumed as Markovian correlated [5, 6, 7, 8, 10], while this is not the case for us. Thirdly, and most importantly, the channel noises in our case stem from a physical model described by a Hamiltonian. This should stimulate activity in calculating its capacities. To this end, we also introduce a memory parameter to quantify the amount of memory. This parameter depends on the distance between the Kraus operators of the second use of the channel with and without memory, so this method can be used to quantify the amount of memory for those channels that admit a description in terms of separate Kraus operators on different uses.

There is also a very important practical issue which motivates our work. The standard way of resetting the chain requires its interaction with a zero temperature environment [18] and this may open up unnecessary avenues for decoherence. Thus one either resets actively by performing a cooling sequence at the chain ends [2] or uses it several times without resetting which automatically raises the question of the effect of memory of one transmission on a subsequent transmission. Multiple usage of a chain of two spins has been studied in [19] to compute the rate of information transmission, but using the swap operators on both spins, a chain of length \( N = 2 \) removes the memory effects. We will compare and contrast our results for the ferromagnetic channel without resetting with some results that have emerged in the recent literature [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

Let us consider a communication system like that of fig. 1(a) which has a set of sender and receiver registers in both side of the spin chain. (a) Setup for state transfer and the classical capacity problem. (b) Setup for entanglement distribution and the quantum capacity problem.
The state of the register is transferred to the register $R_k$ (albeit with a certain fidelity), so the Kraus operators can be easily derived. We will write down the Kraus operators in a certain way (for simplicity and interpretation), though ours may not be the only way to write the Kraus operators for the channel. Two of the Kraus operators of the channel with the initial state $|\psi_1\rangle$ are as in (4) multiplied by the coefficient \( \sqrt{1-p_1^2}|f_{11}(\tau_1)|^2 \) and the others are some matrices that we shall soon introduce. Thus we can describe the effect of the channel with initial state $|\psi_1\rangle$ as a probabilistic effect, which means that with probability $q = \frac{1}{p_1^2} |f_{11}(\tau_1)|^2$ the channel affects the inputs like an amplitude damping channel with Kraus operators (4) and with the probability $(1-q)$ the effect of the channel is specified by the following Kraus operators:

\[
M'_m = \frac{1}{\sqrt{p_1-p_1q}} \begin{pmatrix} A_m \sqrt{1-r^2} e^{i\phi} & f_{m1}(\tau_2) r^r \\ 0 & 0 \end{pmatrix} B_{mN} \sqrt{1-r^2} e^{i\phi},
\]

\[
M'_N = \frac{1}{\sqrt{p_1-p_1q}} \begin{pmatrix} A_N \sqrt{1-r^2} e^{i\phi} & rf_{mN}(\tau_2) \\ 0 & 0 \end{pmatrix},
\]

\[
M'_{N+1} = \frac{1}{\sqrt{p_1-p_1q}} \begin{pmatrix} 0 & 0 \\ \sqrt{\sum_{k_1,k_2} |B_{k_1,k_2}|^2} & 0 \end{pmatrix},
\]

where the index $m$ goes from 1 to $N-1$, $\sum_{k_1,k_2}$ goes from 1 to $N-1$, $\sum_{k_1,k_2} = \sum_{k_1=1}^{N-1} \sum_{k_2=k_1+1}^{N-1}$ and $A_m = \sum_{n=2}^{N-1} f_{mn}(\tau_2) f_{mN}(\tau_1)$. $B_{k_1,k_2} = \sum_{n=2}^{N-1} f_{k_1,k_2,Nn}(\tau_2) f_{nN}(\tau_1)$ is the two excitation amplitude transition with $f_{pq,nm} = \langle pq | e^{-iHt} | nm \rangle$, and $|nm\rangle$ means all the spins of the channel are in $|0\rangle$ except the sites $n$ and $m$. Notice that $B_{k_1,k_2}$ includes physical interaction (scattering) between the first and second state.

In order to get a complete description of the channel for the second use we know that with probability $p_0$ the state of the channel is $|0\rangle$ (the spin chain is an amplitude damping channel) and with probability $p_1,q$ the state of the chain is $|\psi_1\rangle$ but acts as an amplitude damping channel. Thus with total probability $p_0+p_1 q$ the spin chain is an amplitude damping channel, otherwise with the probability $p_1 (1-q)$ the channel is in the state $|\psi_1\rangle$ and its effect is specified by the Kraus operators (4). Therefore, we have

\[
\rho_{R_2}(2) = (p_0 + p_1q) \xi_{AD}(\rho_{S_2}(0)) + (p_1 - p_1q) \xi_{M_{em}}(\rho_{S_2}(0)),
\]

where $\xi_{AD}$ is the amplitude damping evolution (4) and $\xi_{M_{em}}$ is the evolution with Kraus operators (4).

If we consider the memory as a deviation of the channel effect from the memoryless case, then to find a distance between the two evolutions we can consider the distance between the Kraus operators in the two cases. Thus, to quantify this deviation, the following memory parameter is suggested:

\[
\Delta = (p_1-p_1q) \text{Tr} \left\{ \sum_{m=1}^{N+1} (M'_m - M_m)(M'_m - M_m)^\dagger \right\}.
\]

Notice that we have multiplied the summation of the distances in Eq. (5) by $p_1 - p_1 q$ which is the probability that this evo-
lution takes place. By substituting the exact form of the operators in (6) for the case $\tau_1 = \tau_2 = \tau$, we arrive at
\[
\Delta/2 = (1-r^2)(1-|f_{11}|^2-|f_{N1}|^2)+(r-\sqrt{p_1-p_2})^2. \tag{7}
\]
It is clear that the memory parameter is dependent on the first input of the channel as well as the channel parameter $\tau$. The largest deviation from the memoryless case is given for $r = 0$, corresponding to the transmission of $|1\rangle$ on the first use. In this case the maximum of $\Delta$ is $4(1-|f_{N1}|^2-|f_{11}|^2)$. For $|f_{N1}(\tau_1)| = 1$ we have perfect transfer, and for $|f_{11}(\tau_1)| = 1$ the first state is swapped out by the sender into $S_2$.

To compare the quality of transmission we can compare the average fidelities. The average fidelity in the $k$th use of the channel is $F_{av}(k) = \int F(k)d\Omega$ where $F(k) = tr(\rho_{S_k}(0)\rho_{R_k}(k))$ is the fidelity of the $k$th transmission and the integration performed over the surface of the Bloch sphere for all pure input states $\rho_{S_k}(0)$. The total description of the channel in the second use, Eq. (5) helps to compute the average fidelity for the second transmission. It is easy to show that,
\[
F_{av}(2) = (p_0 + p_1 q)F_{av}(1) + \frac{1-r^2}{6}\sum_{m=1}^{N-1}2Re(A_mB^*_m)-\frac{(1-r^2)|A_N|^2}{6} + \frac{2(1-r^2)}{3}(1-|f_{11}|^2-|f_{N1}|^2), \tag{8}
\]
where $F_{av}(1) = \frac{1}{2} + \frac{f_{N1} + f_{11}^*}{6} + \frac{|f_{N1}|^2}{6}$ is the average fidelity for memoryless case, and we have used the identity that $\sum_{m=1}^{N-1}|A_m|^2 = \sum_{k_0}^{N-1}\sum_{k_{-1}}^{N}B_{k_0,k_{-1}}^2 = 1 - |f_{N1}(\tau_1)|^2 - |f_{N1}(\tau_1)|^2$ to simplify the final result. In fig. 2(a) the average fidelities for the second use of the channel has been plotted for equal time evolutions $\tau_1 = \tau_2 = \tau$ (setting $J = 1$). In this figure the average fidelity for the memoryless case has been compared with the case where the state $|1\rangle$ has been transferred in the first use and with the case of average inputs in the first transmission. When the average fidelity of the first transmission has a peak, which means almost perfect transmission, the next transmission is also good. In non-optimal times when the first transmission is not good the memory effect can improve the quality of transmission. In fig. 2(b) the parameter $1 - \Delta/4$ (we have used this parameter instead of $\Delta$ just for simplicity) and the average fidelity for the second transmission after sending the state $|1\rangle$ in the first use, have been plotted together. When $1 - \Delta/4$ take its minimum it means that the amount of the memory in the channel is high, so the average fidelity in the second transmission has a low value because the state of the channel is highly mixed and there is information from the previous transmission in it. In the other case when the parameter $1 - \Delta/4$ has a peak it means that after the first transmission the channel has been nearly reset to the initial ground state. But in this case the average fidelity for the second transmission is not necessarily high because the average fidelity also depends from the time evolution $\tau$. For example in fig. 2(b) for $\tau \simeq 4.6$ the memory has a low value but the average fidelity is not high because of the non-optimal $\tau$. In this non-optimal time, $|f_{11}|$ has a large value, which means that the information is packed in the first spin and swapped out to the sender register, so the chain reset to its ground state. The same happens for the second transmission, so that the average fidelity is low.

Another problem that can be compared for different uses of the channel is the entanglement distribution. In this case the sender registers are a set of pair registers like fig. 1(b). Dual registers $S'_kS_k$ $(k=1,2)$ contain a maximally entangled state. In the first transmission the state $S_1$ is transferred to the register $R_1$ to create an entangled pair (not necessarily maximal) between $S'_1R_1$. In the second transmission, without resetting the chain, the state of $S_2$ is transferred to the register $R_2$ to create the entanglement between $S'_2R_2$. In fig. 2(c) the concurrence as a measure of entanglement [20] for the states $\rho_{S'_1R_1}$ (memoryless) and $\rho_{S'_1R_1}$ (memory case) has been plotted. It shows that the effect of memory is always destructive. The peaks of entanglement are located at times where nearly perfect transmission happens.

Let us now discuss the dependence of the fidelity on $\Delta$. As shown above the quality of state transmission in the second use of the channel depends on the time evolution $\tau_1$ as well. We chose a range of $3.3 \leq \tau_1 \leq 3.9$ such that the memory parameter is increasing for the case that the state $|1\rangle$ is transferred in the first use. For each value of $\tau_1$ we have compared the maximum average fidelity in a long range of $\tau_2$. In fig. 3 we have plotted this maximum value of the average fidelity $F_{av}$ in the second transmission versus $\Delta$. Figure 3 is very interesting because it shows that the average fidelity is decreasing when $\Delta$ is increased. This shows that the remaining probability amplitude in the chain has a destructive effect on the quality of transfer in the second use of the channel.

Finally we investigate whether the memory effect (taking equal evolution times $\tau_1 = \tau_2 = \tau$ for simplicity) can enhance either the quantum capacity or the single-shot classical
capacity which are both known for the memoryless (amplitude damping) channel [3]. As we will show below, such enhancement is indeed possible, and can be demonstrated even without explicitly calculating the capacities. We compare the Holevo bound for a special equiprobable bipartite input states in memory channel with the classical capacity of separable input states in memoryless channels [3]. Assume that all the four possible equiprobable classical input data are encoded into a special kind of input states,

$$\begin{align*}
|\phi_1(\theta)\rangle &= \cos \theta |++\rangle + \sin \theta |--\rangle \\
|\phi_2(\theta)\rangle &= \sin \theta |++\rangle - \cos \theta |--\rangle \\
|\phi_3(\theta)\rangle &= \cos \theta |+-\rangle + \sin \theta |-+\rangle \\
|\phi_4(\theta)\rangle &= \sin \theta |+-\rangle - \cos \theta |-+\rangle,
\end{align*}$$

(9)

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and all these states vary from separable states $\theta=0$ to the maximally entangled one $\theta = \pi/4$. In fig. 1(a) one can prepare any of states $|\phi_i\rangle$ in registers $S_1$ and $S_2$ and by applying the operator $W(2)W(1)$ this state is received as the state $\theta_i$ in registers $R_1$ and $R_2$. The Holevo bound for input states (9) per use is $C(\tau, \theta) = 1/(2 \sum_{i=1}^4 p_i S(\phi_i))$, where $p_i = 1/4$ and $S(\rho) = -tr\rho \ln \rho$ is the Von Neumann entropy of the state $\rho$. To find the optimal input states one can maximize $C(\tau, \theta)$ over the parameter $\theta$. Surprisingly, the maximum $C_{max}(\tau) = max_{\theta} C(\tau, \theta)$ is always achieved by separable states ($\theta = 0$). In fig. 4(a) we have plotted the $C_{max}(\tau)$ and also the real capacity of memoryless channel with separable input states [3] in terms of $\tau$. The memory helps to increase the classical capacity in non-optimal times. These results for spin chains are analogous to those of memory dephasing channel [17].

The coherent information as a lower bound for quantum capacity is $I = S(\xi(\rho)) - S(\xi(|\phi\rangle\langle\phi|))$, where $\rho$ is the input and $|\phi\rangle$ is a purification of $\rho$. In fig. 1(b) consider two maximally entangled states in registers $S_1 S_3$ and $S_2 S_4$ so the states of unprimed sender registers are $\rho_{S_1 S_2} = I/2 \otimes I/2$. These two states are transferred through the chain by $W(2)W(1)$ and we can consider two maximally entangled states in registers $S_1 S_3$ and $S_2 S_4$ as a purification of transferred states. In fig. 4(b) we compare the quantum capacity of [3] with the coherent information per use in our model. We see that though the effect is small, there are certain memory channels (i.e., certain $\tau$) for which even a lower bound to the true quantum capacity exceeds the memoryless quantum capacity.

In conclusion, we have given a characterization of the behavior of a spin chain without resetting. It provides an interesting example of a quantum memory channel, where the memory of the state transmitted during the first use produces a qualitatively different channel in the second use.

We have found the relevant Kraus operators for this model and we have introduced a parameter to quantify the amount of memory in the channel which has broader applicability even outside the domain of spin chain channels.

We have shown that the memory effect can enable one to exceed the known classical capacity for separable inputs and the quantum capacity of the memoryless channel. Our study might pave the way for the computation of the full capacities of such a spin chain channel with memory.

S.B. is supported by the EPSRC, through which a part of the stay of A.B. at UCL is funded, the QIP IRC (GR/S82176/01), the Wolfson Foundation, and the Royal Society of Science and Technology. A.B. thanks the British Council in Iran for financial support. D.B. and S.M. are grateful to S.B. for the hospitality at UCL.

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