JACOBI STRUCTURES ON REAL TWO- AND THREE-DIMENSIONAL LIE GROUPS AND THEIR JACOBI–LIE SYSTEMS

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Using the adjoint representations of Lie algebras, we classify all Jacobi structures on real two- and three-dimensional Lie groups. We also study Jacobi–Lie systems on these real low-dimensional Lie groups and illustrate our results with examples of Jacobi–Lie Hamiltonian systems on some real two- and three-dimensional Lie groups.

Keywords: Lie group, Jacobi structure, Lie system, Jacobi–Lie system

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1. Introduction

The analysis of Lie systems dates back to the end of the 19th century, when Vessiot [1], Guldberg [2], and Lie [3] pioneered the study of systems of first-order ordinary differential equations with a superposition rule [3], in other words, a map expressing its general solution in terms of a general finite family of particular solutions and some constants. Systems of first-order differential equations admitting a superposition rule are called Lie systems.

A Lie system is a system of \( t \)-dependent first-order ordinary differential equations that describes the integral curves of a \( t \)-dependent vector field taking values in a finite-dimensional Lie algebra of vector fields, called a Vessiot–Guldberg Lie algebra [3], [4].

In recent years, much attention has been given to Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a geometric structure [4]–[13]. In [14], a special case of Lie systems on Jacobi manifolds, called Jacobi–Lie systems, that admit a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a Jacobi structure was presented, and the authors classified Jacobi–Lie systems on the real line and the plane. In our previous work [15], we studied Jacobi–Lie Hamiltonian systems on real low-dimensional Jacobi–Lie groups and their Lie symmetries.

Here, we present a method for classifying Jacobi structures on a Lie group using the adjoint representation of the Lie algebra and classify all Jacobi structures on real two- and three-dimensional Lie groups. Using these Jacobi structures, we also obtain some examples of Jacobi–Lie Hamiltonian systems on these Lie groups.

The plan of this paper is as follows. In Sec. 2, we briefly review some formulations related to a Lie system and a Jacobi–Lie Hamiltonian system. In Sec. 3, we describe a convenient method for generating Jacobi

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structures on low-dimensional Lie groups using the adjoint representations, and we find such structures on real two- and three-dimensional Lie groups. In Sec. 4, using the results in [14], [16], we obtain Jacobi–Lie systems on real two- and three-dimensional Lie groups.

2. A brief review of Lie systems and Jacobi–Lie Hamiltonian systems

For completeness of the presentation, we briefly describe Lie systems [13] and Jacobi–Lie Hamiltonian systems (see [14] for a review).

2.1. A time-dependent vector field and a Lie system. For two given subsets \( a \) and \( b \) of a Lie algebra \( \mathfrak{g} \), we introduce the vector space \( \langle a,b \rangle \) spanned by the Lie brackets between elements of \( a \) and \( b \) and let \( \text{Lie}(a) \) denote the smallest Lie subalgebra of \( (\mathfrak{g},[\cdot,\cdot]) \) containing \( a \).

A time-dependent vector field on a manifold \( M \) is a continuous map \( X: \mathbb{R} \times M \to TM \) such that \( X(t,x) \in T_xM \) for each \( (t,x) \in \mathbb{R} \times M \). In other words, each time-dependent vector field amounts to a family of vector fields \( \{X_t\}_{t \in \mathbb{R}} \), where the map \( X_t: M \to TM, \ t \mapsto X_t(x) = X(t,x) \), is a vector field on \( M \), and vice versa [16].

An integral curve of a time-dependent vector field is an integral curve \( \gamma: \mathbb{R} \to \mathbb{R} \times M, t \mapsto (t,x(t)) \), of the suspension of the time-dependent vector field [17]:

\[
\dot{X}: \mathbb{R} \times M \to T(\mathbb{R} \times M) \cong T\mathbb{R} \oplus TM, \quad (t,x) \mapsto \frac{\partial}{\partial t} + X(t,x).
\]

For every integral curve \( \gamma \), we have \( dx(t)/dt = (X \circ \gamma)(t) \). The smallest Lie subalgebra of a time-dependent vector field \( X \) on \( M \) is the smallest real Lie algebra containing the vector fields \( \{X_t\}_{t \in \mathbb{R}} \), i.e., \( \mathfrak{g}^X = \text{Lie}(\{X_t\}_{t \in \mathbb{R}}) \).

Definition 1. A Lie system is a time-dependent vector field \( X \) on \( M \) whose algebra \( \text{Lie}(\{X_t\}_{t \in \mathbb{R}}) \) is finite-dimensional [13].

Definition 2. A superposition rule depending on \( n \) particular solutions of a time-dependent vector field \( X \) on \( M \) is a function \( \Gamma: M^n \times M \to M, (x_{(1)}, \ldots, x_{(n)}; \theta) \mapsto x \), such that the general solution \( x(t) \) of \( X \) can be brought into the form \( x(t) = \Gamma(x_{(1)}(t), \ldots, x_{(n)}(t); \theta) \), where \( x_{(1)}(t), \ldots, x_{(n)}(t) \) is a general family of particular solutions and \( \theta = (\theta_1, \ldots, \theta_m) \) is a point of \( M \) to be related to the initial conditions [3].

Theorem 1 [3], [18]. A time-dependent vector field \( X \) on \( M \) admits a superposition rule if and only if it has the form \( X(t,x) = \sum_{i=1}^{r} a_i(t)X_i(x) \) for a certain family \( a_1(t), \ldots, a_r(t) \) of time-dependent functions and a family of vector fields \( X_1, \ldots, X_r \) on \( M \) spanning an \( r \)-dimensional real Lie algebra.

2.2. Jacobi–Lie Hamiltonian systems. The concept of a Jacobi manifold was introduced by Lichnerowicz [19] and Kirillov [20]. The Jacobi manifolds that we discuss here are Lichnerowicz Jacobi manifolds, also known as a local Lie algebra structure on \( C^\infty(M,\mathbb{R}) \), introduced in [20]. Jacobi–Lie Hamiltonian systems were also presented in [14].

Definition 3. A Jacobi manifold is a triple \( (M, \Lambda, E) \) with \( \Lambda \in \Gamma(\wedge^2 TM) \) and \( E \in \Gamma(TM) \) satisfying

\[
[[\Lambda, \Lambda]] = 2E \wedge \Lambda, \quad [[E, \Lambda]] = 0,
\]

where \( [[\cdot,\cdot]] \) denotes the Schouten–Nijenhuis bracket (see [21] for details).
Definition 4. A vector field $X_f$ on a Jacobi manifold $(M, \Lambda, E)$ is said to be Hamiltonian if it can be written in the form

$$X_f = [[\Lambda, f]] + fE = \Lambda^\#(df) + fE$$

for some $f \in C^\infty(M)$, which is called the Hamiltonian. Here, the bivector $\Lambda$ induces a unique bundle morphism $\Lambda^\# : TM^* \to TM$ such that $\Lambda^\#(\alpha)(\beta) = \Lambda(\alpha, \beta)$ for every $\alpha, \beta \in TM^*$ [14].

Definition 5. A Lie system is called a Jacobi–Lie system if it admits a Vessiot–Guldberg Lie algebra $g$ of Hamiltonian vector fields with respect to a Jacobi structure [14].

Definition 6. A Jacobi–Lie system is called a Jacobi–Lie Hamiltonian system $(M, \Lambda, E, f)$ if $X_{f_t}$ is a Hamiltonian vector field with a Hamiltonian function $f_t$ such that the algebra $\text{Lie} \{f_t\}_{t \in \mathbb{R}, \{\cdot, \cdot\}_{\Lambda, E}}$ is finite-dimensional. Here, $f : \mathbb{R} \times M \to M, (t, x) \mapsto f_t(x)$, is a $t$-dependent function for all $t \in \mathbb{R}$ [14].

3. Classification of Jacobi structures on real low-dimensional Lie groups

In this section, using noncoordinate bases, we translate the Jacobi structures on a Lie group into a Lie algebra. Next, applying the adjoint representation of Lie algebras and the vielbeins (see below) for the Lie group, we classify Jacobi structures on real two- and three-dimensional Lie groups.

3.1. Jacobi structures on manifolds. Let $x^\mu (\mu = 1, \ldots, \dim M)$ be local coordinates on a manifold $M$. For the Jacobi structure $(\Lambda, E)$ on $M$, we have the relations

$$\Lambda = \frac{1}{2} \Lambda^{\mu\nu} \partial_\mu \wedge \partial_\nu, \quad E = E^\mu \partial_\mu, \quad (4)$$

and the Jacobi bracket on $M$

$$\{f, g\}_{\Lambda, E} = \Lambda^{\mu\nu} \partial_\mu f \partial_\nu g + fE^\mu \partial_\mu g - gE^\mu \partial_\mu f, \quad f, g \in C^\infty(M), \quad (5)$$

where we use the Einstein summation convention and $\partial_\mu f$ means that we compute the usual derivative of $f$ with respect to $x^\mu$. Substituting Jacobi bracket (5) in the Jacobi identity, we obtain the relations

$$\Lambda^{\mu\rho} \partial_\rho \Lambda^{\nu\mu} + \Lambda^{\mu\rho} \partial_\rho \Lambda^{\nu\mu} + \Lambda^{\mu\rho} \partial_\rho \Lambda^{\nu\mu} + E^\rho \Lambda^{\mu\nu} + E^\mu \Lambda^{\nu\rho} + E^\nu \Lambda^{\rho\mu} = 0, \quad (6)$$

Equations (6) are called the Jacobi equations, which can also be obtained from (2). To obtain the general form of the Jacobi structures on a manifold $M$, we can calculate the general solution of the Jacobi equations [22]. Here, we consider the Jacobi structures on real low-dimensional Lie groups viewed as smooth manifolds.

3.2. Jacobi structures on real low-dimensional Lie algebras. Using noncoordinate bases, we now translate Eqs. (6) from a Lie group into a Lie algebra. In the coordinate basis, the tangent bundle $T_pM$ is spanned by $\{e_\mu\} = \{\partial_\mu\}$, and the cotangent bundle $T^*_pM$ is spanned by $\{dx^\mu\}$. We consider the linear combinations

$$\hat{e}_a = e_a^\mu \partial_\mu, \quad \hat{\theta}^a = e_\mu^a dx^\mu, \quad e_a^\mu \in GL(m, \mathbb{R}), \quad (7)$$

where the coefficients $e_a^\mu$ are called vierbeins if the space is four-dimensional and vielbeins if it is many-dimensional [23].
Definition 7. The bases \{\hat{e}_a\} and \{\hat{\theta}^a\} are called the noncoordinate bases [23].

In relations (7), \(\det e^\mu_\mu > 0\), i.e., \(\{\hat{e}_a\}\) is the frame of basis vectors obtained by an orientation-preserving \(\text{GL}(m, \mathbb{R})\) rotation\(^1\) of the basis \(\{e_\mu\}\). Using \(\langle \hat{e}_a, \hat{\theta}^b \rangle = \delta^b_a\), we obtain

\[
e^a_\mu e^\mu_a = \delta^a_\mu, \quad e^a_\mu e^\mu_b = \delta^a_b,
\]

where \(e^a_\mu\) is the inverse of \(e^\mu_a\) and the indices \(\mu, \nu, \ldots\) and \(a, b, \ldots\) are respectively related to the Lie group coordinates and the Lie algebra basis.

Taking \(\hat{e}_a = e^\mu_a \partial_\mu\) into account, we obtain \(\hat{e}_a = e^\rho_a e^\mu_\rho \hat{e}_\mu\), where the coefficient \(f^c_{ab}\) is related to the vielbein \(e^\mu_a\) by the Maurer–Cartan relation [23]

\[
f^c_{ab} = e^c_\nu (e^\mu_a \partial_\mu e^\nu_b - e^\mu_b \partial_\mu e^\nu_a).
\]

If the manifold \(M\) is a Lie group \(G\), then these coefficients are the structure constants of the corresponding Lie algebra \(g\).

Writing the Jacobi structure \((G, \Lambda, E)\) in terms of the noncoordinate basis, we obtain

\[
\Lambda^{\mu\nu} = e^\mu_a e^\nu_b \Lambda^{ab}, \quad E^\nu = e^\mu_a E^a, \quad E^a = e^\mu_a E^a,
\]

where the Jacobi structure constants \(\Lambda^{ab}\) and \(E^a\) are related to the Lie algebra; we assume that they are independent of the coordinates on the Lie group. Substituting (10) in (6) and using Maurer–Cartan relation (9), we obtain the relations

\[
f^{f}_{cd} \Lambda^{ab} \Lambda^{ce} + f^{f}_{cd} \Lambda^{ab} \Lambda^{cf} + f^{f}_{ad} \Lambda^{ab} \Lambda^{hf} + E^f \Lambda^{eh} + E^e \Lambda^{hf} + E^h \Lambda^{fe} = 0,
\]

\[
f^{d}_{ac} E^a \Lambda^{ce} + f^{d}_{ab} E^a \Lambda^{dh} = 0.
\]

It is quite difficult to obtain results working with the tensor form of Eqs. (11), and we therefore propose using the adjoint representations of Lie algebras

\[
f^{c}_{ab} = -(\chi^b_a)_c, \quad f^{c}_{ab} = -(\chi^c_{ab}),
\]

to write these equations in the matrix form. We can hence rewrite relations (11) as

\[
-(\Lambda^{ce}(\chi^b_a) + \Lambda \chi^c_a) + (\Lambda \chi^b_a) \Lambda^{be} + E^e \Lambda^{fb} + E^f \Lambda^{eh} + \Lambda^{fh} E^h = 0,
\]

\[
(\Lambda \chi_a - (\Lambda \chi_a)^t) E^a = 0,
\]

where the superscript \(t\) denotes transposed matrices.

To find general solutions of these equations, we use the Maple program. We present real two- and three-dimensional Lie algebras in Tables 1 and 2 in Appendix A and the Jacobi structures on these Lie algebra obtained based on solutions of the Jacobi equations in Tables 3 and 4 in Appendix B.

We note that in the classification of these Jacobi structures, some of the structures are equivalent, and we must therefore define an equivalence relation and use the following theorem.

\(^1\)The coordinate basis \(\{e_\mu\} = \{\partial_\mu\}\) is changed to the noncoordinate basis \(\{\hat{e}_a\}\) by multiplications of vielbeins (i.e., \(e^\mu_a\)) is an \(m \times m\) real transformation matrix belonging to \(\text{GL}(m, \mathbb{R})\) [23].
Theorem 2. The Jacobi structures \((\Lambda, E)\) and \((\Lambda', E')\) are equivalent if there exists \(A \in \text{Aut}(\mathfrak{g})\) (the automorphism group of the Lie algebra \(\mathfrak{g}\)) such that

\[ \Lambda = A^t \Lambda A, \quad E^c = E'^b A_b^c. \]  

Proof. By the definition of an automorphism \(A : \mathfrak{g} \rightarrow \mathfrak{g}\) of the Lie algebra \(\mathfrak{g}\) with the basis \(\{X_a\}\) and the structure constants \(f_{ab}^c\), we have \(AX_a = A_a^b X_b\), where \(A_a^b\) satisfies the relation

\[ A_a^b f_{kl}^{m} A_l^b = f_{ab}^c A_c^m. \]  

Applying \((12)_i n(15)\), we obtain the matrix relations

\[ A^m_n A^t = \gamma^m_n A^m_c, \quad A\chi_1 A_b^t = \chi_b A, \]  

where \(A\) is the matrix form of \(A_a^b\).

Substituting \((14)_i n(13)\), using \((16)\), and multiplying the left-hand side of Eq. \((16)\) by \((A^t)^{-1})_s^f\) and the right-hand side by \((A^{-1})_e^t(A^{-1})_h^p\), we obtain the relation

\[ -(\Lambda'^{ab} (A_a^t \Lambda') + \Lambda'^{ab} A') + (\Lambda'_\chi_b \Lambda'^d b + E'^b \Lambda')^{su} + E'^b \Lambda'^b u + \Lambda'^{ab} E'^u = 0. \]  

Hence, \((\Lambda, E)\) and \((\Lambda', E')\) are solutions of Eqs. \((13)\), and they are equivalent.

In the same way, substituting \((14)\) in the second relation in \((13)\), we obtain

\[ (A^t \Lambda' \Lambda_a - (A^t \Lambda' \Lambda_a)') E'^b A_b^s = 0. \]  

Using the second relation in \((16)\) and multiplying the left-hand side of Eq. \((17)\) by \((A^t)^{-1}\) and the right-hand side by \(A^{-1}\), we then obtain

\[ (\Lambda'_\chi_b - (\Lambda'_\chi_b)) E'^b = 0. \]  

Hence, \(E\) and \(E'\) are equivalent, and the Jacobi structures \((\Lambda, E)\) and \((\Lambda', E')\) are consequently equivalent.

We use the Patera–Winternitz classification [24] of two-dimensional Lie algebras (see Tables 1 and 3) and the Landau–Lifshitz classification [25] of three-dimensional Lie algebras (see Tables 2 and 4). To illustrate our method, we consider the three-dimensional Lie algebra III in the Bianchi classification.

3.3. An example for Lie algebra III. We illustrate all details of the method for classifying Jacobi structures on real low-dimensional Lie algebras with the example of Lie algebra III with the nonzero commutators \([X_1, X_2] = -(X_2 + X_3)\) and \([X_1, X_3] = -(X_2 + X_3)\). We write the matrix form of the bivector field \(\Lambda\) and Reeb vector field \(E\):

\[ \Lambda = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} \\ -\lambda_{12} & 0 & \lambda_{23} \\ -\lambda_{13} & -\lambda_{23} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \]  

where \(\lambda_{ij}\) and \(e_i\) are arbitrary real constants. Using \((12)\), we obtain the adjoint representations of Lie algebra III

\[ \chi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \chi_3 = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]  

and

\[ \gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]  

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Substituting (19) and (20) in (13), we obtain Λ and E for the Lie algebra.

One solution has the form

\[
\Lambda = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{13} \\
-\lambda_{12} & 0 & \lambda_{23} \\
-\lambda_{13} & -\lambda_{23} & 0
\end{pmatrix}, \quad E = \begin{pmatrix}
0 \\
-\lambda_{13} + \lambda_{12} \\
\lambda_{13} - \lambda_{12}
\end{pmatrix}.
\]

Using Theorem 2, we show that the Jacobi structure \((\Lambda, E)\) contains the following equivalence classes.

Applying the automorphism group of Lie algebra III [26] (also see Table 5)

\[
A = \begin{pmatrix}
1 & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & a_{23} & a_{22}
\end{pmatrix}
\]

and substituting this automorphism in the relations \(\Lambda' = A^t \Lambda A\) and \(E' = EA\), we obtain

\[
A = \begin{pmatrix}
1 & \frac{\lambda_{23}}{\lambda_{12}^2 - \lambda_{13}^2} & b \\
0 & \frac{\lambda_{12}}{\lambda_{12}^2 - \lambda_{13}^2} & \frac{\lambda_{12}^2 - \lambda_{13}^2}{\lambda_{12}^2 - \lambda_{13}^2} \\
0 & \frac{\lambda_{12}^2 - \lambda_{13}^2}{\lambda_{12}^2 - \lambda_{13}^2} & \frac{\lambda_{12}^2 - \lambda_{13}^2}{\lambda_{12}^2 - \lambda_{13}^2}
\end{pmatrix}, \quad \det A = -\frac{1}{(\lambda_{12} - \lambda_{13})(\lambda_{12} + \lambda_{13})}.
\]

Because \(\det A \neq 0\) and is independent of \(\lambda_{23}\), this parameter can take any value. Moreover, we also have \(\lambda_{12} \neq \pm \lambda_{13}\).

First, if \(\lambda_{12} = 0 \neq \pm \lambda_{13}\), then the Jacobi structure \((\Lambda, E)\) is classified as

\[
\Lambda' = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad E' = \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix}.
\]

Second, if \(\lambda_{12} = -\lambda_{13} \neq 0\), then

\[
A = \begin{pmatrix}
1 & \frac{-b\lambda_{13}^2 - 2c\lambda_{13}\lambda_{23} + \lambda_{23}}{\lambda_{13}^2} & b \\
0 & \frac{\lambda_{13}^2}{c} & \frac{c\lambda_{13} - 1}{\lambda_{13}} \\
0 & \frac{c\lambda_{13} - 1}{\lambda_{13}} & c
\end{pmatrix}, \quad \det A = \frac{2c\lambda_{13} - 1}{\lambda_{13}^2} \neq 0,
\]

and the Jacobi structure \((\Lambda, E)\) is hence classified as

\[
\Lambda'' = \begin{pmatrix}
0 & -1 & 1 \\
1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad E'' = \begin{pmatrix}
0 \\
-2 \\
2
\end{pmatrix}.
\]

Third, if \(\lambda_{12} = \lambda_{13}\), then the Jacobi structure \((\Lambda, E)\) is classified as a Poisson structure. Therefore, the Jacobi structure \((\Lambda, E)\) is the disjoint union of the equivalence classes \((\Lambda', E')\) and \((\Lambda'', E'')\) and the Poisson structure.
We similarly determined all the Jacobi structures for real two- and three-dimensional Lie algebras. The results are listed in Tables 3 and 4.

Using relations (10), we can similarly transform the obtained structures into the Jacobi structures $\Lambda'$ and $E'$ on the Lie group. To find these Jacobi structures, we must determine the vielbein $e^a_{\mu}$ for the Lie groups, and for this, we must calculate the left-invariant 1-form on the Lie group:

$$g^{-1} dg = e^a_{\mu} X_a dx^\mu, \quad g \in G,$$

where $\{X_a\}$ are generators of the Lie group. All left-invariant 1-forms were previously obtained in [26], [27].

We next obtain the inverse of the vielbein $e^a_{\mu}$ (i.e., $e_{\mu}^a$) for Lie group III:

$$e_{\mu}^a = \begin{pmatrix}
1 & 0 & 0 \\
-x_2 - x_3 & 1 & 0 \\
-x_2 - x_3 & 0 & 1
\end{pmatrix}.$$

Consequently, substituting (24) and (28) in (10), we can calculate the Jacobi structures $\Lambda'$ and $E'$ on Lie group III:

$$\Lambda' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -x_2 - x_3 \\ -1 & x_2 + x_3 & 0 \end{pmatrix}, \quad E' = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Moreover, substituting (26) and (28) in (10), we can calculate the other Jacobi structures $\Lambda''$ and $E''$ on Lie group III:

$$\Lambda'' = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -2x_2 - 2x_3 \\ -1 & 2x_2 + 2x_3 & 0 \end{pmatrix}, \quad E'' = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}.$$

4. Jacobi–Lie Hamiltonian systems on real two- and three-dimensional Lie groups

In this section, we show how our results can be illustrated by some important examples of Jacobi–Lie Hamiltonian systems.

**Example 1.** We consider the two-dimensional real Lie group $A_2$ with the local coordinates $\{x_1, x_2\}$. One of the Jacobi structures on the Lie algebra $A_2$ has the form (see Table 3)

$$\Lambda = \begin{pmatrix} 0 & \lambda_{12} \\ -\lambda_{12} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $\lambda_{12} \in \mathbb{R} - \{0\}$. The inverse of the vielbein $e^a_{\mu}$ for the Lie group $A_2$ is obtained as

$$e_{\mu}^a = \begin{pmatrix} e^{x_2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Substituting (31) and (32) in (10), we can calculate the Jacobi structures $\Lambda$ and $E$ on the Lie group $A_2$:

$$\Lambda = \begin{pmatrix} 0 & e^{x_2} \lambda_{12} \\ -e^{x_2} \lambda_{12} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} e^{x_2} \\ 0 \end{pmatrix}.$$
We can directly show that $[[\mathbf{A}, \mathbf{A}]] = 2\mathbf{E} \wedge \mathbf{A}$ and $[[\mathbf{E}, \mathbf{A}]] = 0$ and $(\mathbf{A}_2, \mathbf{A}, \mathbf{E})$ is therefore a Jacobi manifold. Using (33) and (3), we now obtain the Hamiltonian vector fields

$$X_1^H = (-\lambda_{12} + x_2)e^{x_2}\partial_{x_1},$$

$$X_2^H = \frac{((-x_2 + 1)\lambda_{12}^2 + (x_2^2 - x_2)\lambda_{12} + x_2^3)x_1}{(x_2 - \lambda_{12})^2}\partial_{x_1} + \frac{\lambda_{12}x_2}{x_2 - \lambda_{12}}\partial_{x_2},$$

which span the Lie algebra $A_2$ with the nonzero commutators $[X_1^H, X_2^H] = X_1^H$. On $A_2$, we consider the system

$$\frac{d\alpha_2}{dt} = \sum_{i=1}^{2} a_i(t)X_i^H(\alpha_2), \quad \alpha_2 \in A_2, \quad (34)$$

with arbitrary functions $a_i(t)$. The associated time-dependent vector field $X^{A_2} = \sum_{i=1}^{2} a_i(t)X_i^H$ is a Lie system because $X^{A_2}$ takes values in the Lie algebra $A_2$. Moreover, the vector fields $X_i^H$ and $X_2^H$ are Hamiltonian with respect to $(A_2, \mathbf{A}, \mathbf{E})$ with the respective Hamiltonians $f_1 = x_2$ and $f_2 = e^{x_2}x_2x_1/(x_2 - \lambda_{12})$. Consequently, $(A_2, \mathbf{A}, \mathbf{E}, X^{A_2})$ is a Jacobi–Lie system.

In addition, it is easy to see that $\{f_1, f_2\}_{\mathbf{A}, \mathbf{E}} = f_1$. Therefore, $X^{A_2}$ admits a Jacobi–Lie Hamiltonian system $(A_2, \mathbf{A}, \mathbf{E}, f)$.

**Example 2.** We now consider the three-dimensional real Lie group $\Pi$ with the local coordinates $\{x_1, x_2, x_3\}$. One of the Jacobi structures on Lie algebra $\Pi$ has the form

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (35)$$

The inverse of the vielbein $e^\mu_\nu$ for Lie group $\Pi$ is obtained as

$$e^\mu_\nu = \begin{pmatrix} 1 & x_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (36)$$

Substituting (35) and (36) in (10), we can calculate the Jacobi structures $\Lambda$ and $E$ on Lie group $\Pi$:

$$\Lambda = \begin{pmatrix} 0 & 0 & x_3 \\ 0 & 0 & 1 \\ -x_3 & -1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (37)$$

It is easy to prove that $[[\mathbf{A}, \mathbf{A}]] = 2\partial x_1 \wedge \partial x_2 \wedge \partial x_3 = 2\mathbf{E} \wedge \mathbf{A}$ and $[[\mathbf{E}, \mathbf{A}]] = 0$. Therefore, $(\Pi, \mathbf{A}, \mathbf{E})$ is a Jacobi manifold.

Using (37) and (3), we obtain the Hamiltonian vector fields

$$X_1^H = \frac{1}{x_2}\partial_{x_1} - \frac{1}{x_2^2}\partial_{x_3}, \quad X_2^H = x_2\partial_{x_1} + \partial_{x_3},$$

$$X_3^H = \frac{x_1}{2x_2^2}\partial_{x_1} - \frac{1}{2x_2}\partial_{x_2} - \frac{x_1}{x_2^2}\partial_{x_3},$$

$$X_4^H = \frac{x_1}{x_2^2}\partial_{x_1} - \frac{1}{2x_2}\partial_{x_2} - \frac{x_1}{x_2^2}\partial_{x_3},$$
which span Lie algebra II with the nonzero commutators \([X_2^H, X_3^H] = X_1^H\).

On II, we consider the system

\[
\frac{d\beta}{dt} = \sum_{i=1}^{3} a_i(t) X_i^H(\beta), \quad \beta \in \Pi,
\]

with arbitrary functions \(a_i(t)\). The associated time-dependent vector field \(X^\Pi = \sum_{i=1}^{3} a_i(t) X_i^H\) is a Lie system because it takes values in Lie algebra II. Moreover, the vector fields \(X_1^H\), \(X_2^H\), and \(X_3^H\) are Hamiltonian with respect to \((\Pi, \mathbf{A}, \mathbf{E})\) with the respective Hamiltonian functions \(f_1 = 1/x^2\), \(f_2 = x^2\), and \(f_3 = (x_2 x_3 + x_1)/2x_2^2\). Consequently, \((\Pi, \mathbf{A}, \mathbf{E}, X^\Pi)\) is a Jacobi–Lie system.

In addition, we have the equality \(\{f_2, f_3\}_{\mathbf{A}, \mathbf{E}} = f_1\). Therefore, \((\Pi, \mathbf{A}, \mathbf{E}, f)\), where \(f = \sum_{i=1}^{3} a_i(t) f_i\), is a Jacobi–Lie Hamiltonian system for the vector field \(X^\Pi\).

**Example 3.** We now consider the three-dimensional real Lie group III with the local coordinates \(\{x_1, x_2, x_3\}\). One of the Jacobi structures on Lie algebra III has the form

\[
\Lambda = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad E = \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix}.
\]

(39)

The inverse of the vielbein \(e^a_\mu\) for Lie group III is obtained as

\[
e^a_\mu = \begin{pmatrix}
1 & 0 & 0 \\
-x_2 - x_3 & 1 & 0 \\
-x_2 - x_3 & 0 & 1
\end{pmatrix}.
\]

(40)

Substituting (39) and (40) in (10), we can calculate the Jacobi structures \(\mathbf{A}\) and \(\mathbf{E}\) on the Lie group:

\[
\mathbf{A} = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & -x_2 - x_3 \\
-1 & x_2 + x_3 & 0
\end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix}.
\]

(41)

It is easy to verify that \([\mathbf{A}, \mathbf{A}] = 2\partial x_1 \wedge \partial x_2 \wedge \partial x_3 = 2\mathbf{E} \wedge \mathbf{A}\) and \([\mathbf{E}, \mathbf{A}] = 0\). Consequently, \((\Pi, \mathbf{A}, \mathbf{E})\) is a Jacobi manifold.

Using (41) and (3), we obtain the Hamiltonian vector fields

\[
X_1^H = -\partial x_2 + \partial x_3, \quad X_2^H = -x_1 \partial x_2 + (1 + x_1) \partial x_3, \\
X_3^H = -\partial x_1 - x_1 \partial x_2 + (1 + x_1) \partial x_3,
\]

which span Lie algebra II with the nonzero commutators \([X_2^H, X_3^H] = X_1^H\).

On III, we consider the system

\[
\frac{d\gamma}{dt} = \sum_{i=1}^{3} a_i(t) X_i^H(\gamma), \quad \gamma \in \Pi,
\]

(42)

with arbitrary functions \(a_i(t)\). The associated \(t\)-dependent vector field \(X^{\Pi} = \sum_{i=1}^{3} a_i(t) X_i^H\) is a Lie system because it takes values in Lie algebra III. Moreover, the vector fields \(X_1^H\), \(X_2^H\), and \(X_3^H\) are Hamiltonian.
with respect to \((\mathbf{III}, \mathbf{A}, \mathbf{E})\) with the respective Hamiltonian functions \(f_1 = 1\), \(f_2 = x_1\), and \(f_3 = x_1 + x_2 + x_3\). Therefore, \((\mathbf{III}, \mathbf{A}, \mathbf{E}, \mathbf{X}^{\mathbf{III}})\) is a Jacobi–Lie system.

In addition, we have the equality \(\{f_2, f_3\}_{\mathbf{A}, \mathbf{E}} = f_1\). Therefore, \((\mathbf{III}, \mathbf{A}, \mathbf{E}, f)\), where \(f = \sum_{i=1}^{3} a_i(t)f_i\), is a Jacobi–Lie Hamiltonian system for the vector field \(\mathbf{X}^{\mathbf{III}}\).

Another equivalence class of Jacobi structures on Lie algebra \(\mathbf{III}\) has the form

\[
\Lambda' = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E' = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}. \tag{43}
\]

Substituting (43) and (3) in (10), we can calculate the Jacobi structures \(\Lambda'\) and \(E'\) on the Lie group:

\[
\Lambda' = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -2x_2 - 2x_3 \\ -1 & 2x_2 + 2x_3 & 0 \end{pmatrix}, \quad E' = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}. \tag{44}
\]

It is easy to verify that \(\|[\Lambda', \Lambda']\| = 0 = 2E' \wedge \Lambda'\) and \(\|[\Lambda', \Lambda']\| = 0\). Consequently, \((\mathbf{III}, \Lambda', \mathbf{E})\) is a Jacobi manifold.

Using (44) and (3), we see that the vector fields \(X_1, X_2,\) and \(X_3\) cannot form a basis for Lie group \(\mathbf{III}\). Hence, we cannot discuss the Lie system.

**Example 4.** We now consider the three-dimensional real Lie group \(\mathbf{IV}\) with the local coordinates \(\{x_1, x_2, x_3\}\). One of the Jacobi structures on Lie algebra \(\mathbf{IV}\) has the form

\[
\Lambda' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{45}
\]

The inverse of the vielbein \(e^a{}_{\mu}\) for Lie group \(\mathbf{IV}\) is obtained as

\[
e^a{}_{\mu} = \begin{pmatrix} 1 & 0 & 0 \\ -x_2 & 1 & 0 \\ x_2 - x_3 & 0 & 1 \end{pmatrix}. \tag{46}
\]

Substituting (45) and (46) in (10), we can calculate the Jacobi structures \(\Lambda'\) and \(E'\) on the Lie group:

\[
\Lambda' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -x_2 + x_3 \\ 0 & x_2 - x_3 & 0 \end{pmatrix}, \quad E' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{47}
\]

It is easy to verify that \(\|[\Lambda', \Lambda']\| = 2\partial x_1 \wedge \partial x_2 \wedge \partial x_3 = 2E' \wedge \Lambda'\) and \(\|[\Lambda', \Lambda']\| = 0\). Hence, \((\mathbf{IV}, \Lambda', \mathbf{E})\) is a Jacobi manifold.

Using (47) and (3), we obtain the Hamiltonian vector fields

\[
X_1^H = \partial x_2 + \partial x_3,
\]

\[
X_2^H = -\frac{e^{-x_3}}{x_2 - x_3} \partial x_1 - e^{-x_3}(x_3 + \log(-x_2 + x_3))(-1 + x_2 - x_3) \partial x_2 + 
+ e^{-x_3}(x_3 + \log(-x_2 + x_3)) \partial x_3,
\]

\[
X_3^H = -e^{-x_3}(-1 + x_2 - x_3) \partial x_2 + e^{-x_3} \partial x_3,
\]

\[1402\]
which span Lie algebra IV with the nonzero commutators

\[ [X^H_1, X^H_2] = -X^H_2 + X^H_3, \quad [X^H_1, X^H_3] = -X^H_3. \]

On IV, we consider the system

\[ \frac{d\delta}{dt} = \sum_{i=1}^{3} a_i(t) X^H_i(\delta), \quad \delta \in \text{IV}, \]

with arbitrary functions \( a_i(t) \). The associated time-dependent vector field \( X^{\text{IV}} = \sum_{i=1}^{3} a_i(t) X^H_i \) is a Lie system because it takes values in Lie algebra IV. Moreover, the vector fields \( X^H_1, X^H_2, \) and \( X^H_3 \) are Hamiltonian with respect to \((\text{IV}, \Lambda', E')\) with the respective Hamiltonian functions \( f_1 = 1, \quad f_2 = e^{-z}(\log(-y+z)+1+z), \) and \( f_3 = e^{-z} \). Consequently, \((\text{IV}, \Lambda', E', X^{\text{IV}})\) is a Jacobi–Lie system.

In addition, we have \( \{f_1, f_2\}_{\Lambda', E'} = -f_2 + f_3 \) and \( \{f_1, f_3\}_{\Lambda', E'} = -f_3 \). Therefore, \((\text{IV}, \Lambda', E', f)\), where \( f = \sum_{i=1}^{3} a_i(t)f_i \), is a Jacobi–Lie Hamiltonian system for the vector field \( X^{\text{IV}} \).

Another equivalence class of Jacobi structures on Lie algebra IV has the form \( \Lambda = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \) and \( E = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

Substituting (46) and (49) in (10), we can calculate the Jacobi structures \( A \) and \( E \) on the Lie group:

\[ A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -x_2 \\ -1 & x_2 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

It is easy to verify that \([A, A] = 0 = 2E \wedge A\) and \([E, A] = 0\). Hence, \((\text{IV}, A, E)\) is a Jacobi manifold.

Using (50) and (3), we see that the vector fields \( X_1, X_2, \) and \( X_3 \) cannot form a basis for Lie group IV. Hence, we cannot discuss the Lie system.

**Example 5.** We now consider the three-dimensional real Lie group VI\(_0\) with the local coordinates \( \{x_1, x_2, x_3\} \). One of the Jacobi structures on Lie algebra VI\(_0\) has the form

\[ A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

The inverse of the vielbein \( e^\alpha_\mu \) for Lie group VI\(_0\) is obtained as

\[ e^\alpha_\mu = \begin{pmatrix} \cosh x_3 & \sinh x_3 & 0 \\ \sinh x_3 & \cosh x_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Substituting (51) and (52) in (10), we can calculate the Jacobi structures \( A \) and \( E \) on the Lie group:

\[ A = \begin{pmatrix} 0 & 0 & \sinh x_3 \\ 0 & 0 & \cosh x_3 \\ -\sinh x_3 & -\cosh x_3 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} \cosh x_3 \\ \sinh x_3 \end{pmatrix}. \]
It is easy to prove that $[[A, A]] = 2\partial x_1 \wedge \partial x_2 \wedge \partial x_3 = 2E \wedge A$ and $[[E, A]] = 0$. Hence, $(II, A, E)$ is a Jacobi manifold.

Using (53) and (3), we obtain the Hamiltonian vector fields
\[
X_1^H = \cosh(x_3)\partial_{x_1} + \sinh(x_3)\partial_{x_2},
\]
\[
X_2^H = (-\sinh x_3 + x_3 \cos x_3)\partial_{x_1} + (\cosh x_3 + x_3 \sinh x_3)\partial_{x_2},
\]
\[
X_3^H = -x_2\partial_{x_1} - x_1\partial_{x_2} - \partial_{x_3},
\]
which span Lie algebra $II$ with the nonzero commutator $[X_2^H, X_3^H] = X_1^H$.

On $VI_0$, we consider the system
\[
\frac{d\xi}{dt} = \sum_{i=1}^3 a_i(t)X_i^H(\zeta), \quad \zeta \in VI_0,
\]
with arbitrary functions $a_i(t)$. The associated time-dependent vector field $X^{VI_0} = \sum_{i=1}^3 a_i(t)X_i^H$ is a Lie system because it takes values in Lie algebra $VI_0$. In addition, the vector fields $X_1^H$, $X_2^H$, and $X_3^H$ are Hamiltonian with respect to $(VI_0, A, E)$ with the respective Hamiltonian functions $f_1 = 1$, $f_2 = x_3$, and $f_3 = x_1 \sinh x_3 - x_2 \cosh x_3$. Therefore, $(VI_0, A, E, X^{VI_0})$ is a Jacobi–Lie system.

In addition, the functions $f_1$, $f_2$, and $f_3$ satisfy $\{f_2, f_3\}_{A_2, E_2} = f_1$. Therefore, $(VI_0, A, E, f)$, where $f = \sum_{i=1}^3 a_i(t)f_i$ is a Jacobi–Lie Hamiltonian system for vector field $X^{VI_0}$.

**Example 6.** We now consider the three-dimensional real Lie group $VII_0$ with the local coordinates $\{x_1, x_2, x_3\}$. One of the Jacobi structures on Lie algebra $VII_0$ has the form
\[
\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]
(55)
The inverse of the vielbein $e^\alpha_\mu$ for Lie group $VII_0$ is obtained as
\[
e^\alpha_\mu = \begin{pmatrix} \cos x_3 & \sin x_3 & 0 \\ -\sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
(56)
Substituting (55) and (56) in (10), we can calculate the Jacobi structures $A$ and $E$ on the Lie group:
\[
A = \begin{pmatrix} 0 & 0 & \sin x_3 \\ 0 & 0 & \cos x_3 \\ -\sin x_3 & -\cos x_3 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} \cos x_3 \\ -\sin x_3 \\ 0 \end{pmatrix}.
\]
(57)
It is easy to prove that $[[A, A]] = 2\partial x_1 \wedge \partial x_2 \wedge \partial x_3 = 2E \wedge A$ and $[[E, A]] = 0$. Hence, $(II, A, E)$ is a Jacobi manifold.

Using (57) and (3), we obtain the Hamiltonian vector fields
\[
X_1^H = \cos(x_3)\partial_{x_1} - \sin(x_3)\partial_{x_2},
\]
\[
X_2^H = (-\sin x_3 + x_3 \cos x_3)\partial_{x_1} - (\cos x_3 + x_3 \sin x_3)\partial_{x_2},
\]
\[
X_3^H = ((-2x_1x_3 + x_2)\cos^2 x_3 + (2x_3x_3 + x_1)\cos x_3 \sin x_3 - x_3 x_3^2 + x_1 x_3 - 2x_2)\partial_{x_1} +
\]
\[
+ \left((2x_2x_3 + x_1)\cos^2 x_3 + 2 \left(x_1 x_3 - \frac{x_2}{2}\right) \sin x_3 \cos x_3 + x_1 x_3^2 - x_2 x_3 + x_1\right)\partial_{x_2} - (x_3^2 + 1)\partial_{x_3},
\]
which span Lie algebra $\text{VII}_0$ with the nonzero commutators $[X_i^\text{H}, X_j^\text{H}] = -X_j^\text{H}$ and $[X_i^\text{H}, X_j^\text{H}] = X_i^\text{H}$.

On $\text{VII}_0$, we consider the system

$$\frac{d\eta}{dt} = \sum_{i=1}^{3} a_i(t) X_i^\text{H}(\eta), \quad \eta \in \text{VII}_0,$$

(58)

with arbitrary non-autonomous functions $a_i(t)$. The associated nonautonomous vector field $X^\text{VII}_0 = \sum_{i=1}^{3} a_i(t) X_i^\text{H}$ is a Lie system because it takes values in Lie algebra $\text{VII}_0$. Moreover, the vector fields $X_1^\text{H}$, $X_2^\text{H}$ and $X_3^\text{H}$ are Hamiltonian with respect to $(\text{VII}_0, \Lambda, \text{E})$ with the respective Hamiltonian functions $f_1 = 1$, $f_2 = x_3$, and $f_3 = (-x_2x_3^2 - x_1x_3 - x_2) \cos x_3 - \sin x_3(x_1x_3^2 - x_2x_3 + x_1)$. Therefore, $(\text{VII}_0, \Lambda, \text{E}, X^\text{VII}_0)$ is a Jacobi–Lie system.

In addition, we see that $\{f_1, f_3\}_\Lambda, \text{E} = -f_2$ and $\{f_2, f_3\}_\Lambda, \text{E} = f_1$. Therefore, $(\text{VII}_0, \Lambda, \text{E}, f)$, where $f = \sum_{i=1}^{3} a_i(t)f_i$, is a Jacobi–Lie Hamiltonian system for the vector field $X^\text{VII}_0$.

Appendix A: Real two- and three-dimensional Lie algebras

| Lie algebra | Commutation relations |
|-------------|----------------------|
| $A_1$       | $[X_i, X_j] = 0$     |
| $A_2$       | $[X_1, X_2] = X_1$   |

Real two-dimensional Lie algebras.

| Lie algebra | Commutation relations | Comments     |
|-------------|----------------------|--------------|
| I           | $[X_i, X_j] = 0$     |              |
| II          | $[X_2, X_3] = X_1$   |              |
| III         | $[X_1, X_3] = -(X_2 + X_3)$, $[X_1, X_2] = -(X_2 + X_3)$ |              |
| IV          | $[X_1, X_3] = -X_3$, $[X_1, X_2] = -(X_2 - X_3)$ |              |
| V           | $[X_1, X_3] = -X_3$, $[X_1, X_2] = -X_2$ |              |
| VI$_0$      | $[X_2, X_3] = X_1$, $[X_1, X_3] = X_2$ |              |
| VI$_a$      | $[X_1, X_3] = -(X_2 + aX_3)$, $[X_1, X_2] = -(aX_2 + X_3)$ | $a \in \mathbb{R} - \{1\}$, $a > 0$ |
| VII$_0$     | $[X_2, X_3] = X_1$, $[X_1, X_3] = -X_2$ |              |
| VII$_a$     | $[X_1, X_3] = -(X_2 + aX_3)$, $[X_1, X_2] = -(aX_2 - X_3)$ | $a \in \mathbb{R}$, $a > 0$ |
| VII         | $[X_2, X_3] = X_1$, $[X_1, X_3] = -X_2$, $[X_1, X_2] = -X_3$ |              |
| IX          | $[X_2, X_3] = X_1$, $[X_1, X_3] = -X_2$, $[X_1, X_2] = X_3$ |              |

Real three-dimensional Lie algebras.
Appendix B: Jacobi structures on two- and three-dimensional Lie algebras and equivalence classes

### Table 3

| Jacobi structure on Lie algebra $A_1$ | Equivalence classes | Comments |
|--------------------------------------|---------------------|----------|
| $\Lambda = \lambda_1 \partial_{x_1} \land \partial_{x_2}$, $E = -e_1 \partial_{x_1} - e_2 \partial_{x_2}$ | $\Lambda = \partial_{x_1} \land \partial_{x_2}, E = -\partial_{x_1}$ |          |

### Table 4

| Jacobi structure on Lie algebra I | Equivalence classes | Comments |
|----------------------------------|---------------------|----------|
| $A_1 = -\frac{e_1 \lambda_3 + e_2 \lambda_3}{e_3} \partial_{x_1} \land \partial_{x_2} + \lambda_3 \partial_{x_1} \land \partial_{x_2} + \lambda_2 \partial_{x_2} \land \partial_{x_3}$, $E_1 = -e_1 \partial_{x_1} - e_2 \partial_{x_2} - e_3 \partial_{x_3}$ | $\Lambda = \partial_{x_1} \land \partial_{x_2} + \partial_{x_2} \land \partial_{x_3}$, $E = -\partial_{x_3}$ |          |

| Jacobi structure on Lie algebra II | Equivalence classes | Comments |
|-----------------------------------|---------------------|----------|
| $A_2 = \lambda_2 \partial_{x_1} \land \partial_{x_2} + \lambda_3 \partial_{x_1} \land \partial_{x_3}$, $E_2 = -e_1 \partial_{x_1} - e_2 \partial_{x_2} - e_3 \partial_{x_3}$ | $\Lambda = \partial_{x_1} \land \partial_{x_2}, E = -\partial_{x_2}$ |          |

| Jacobi structure on Lie algebra II | Equivalence classes | Comments |
|-----------------------------------|---------------------|----------|
| $A_3 = \lambda_2 \partial_{x_1} \land \partial_{x_2} + \lambda_2 \partial_{x_1} \land \partial_{x_3}$, $E_3 = -e_1 \partial_{x_1}$ | $\Lambda = \partial_{x_1} \land \partial_{x_2} + \partial_{x_2} \land \partial_{x_3}$, $E = -\partial_{x_1}$ |          |

Jacobi structures on two-dimensional Lie algebras and equivalence classes.

Jacobi structures on three-dimensional Lie algebras and equivalence classes.
| Jacobi structures on Lie algebra IV | Equivalence classes | Comments |
|-----------------------------------|---------------------|----------|
| $\Lambda_1 = 0$, $E_1 = -e_1 \partial_x - e_2 \partial_y - e_3 \partial_z$ | $\Lambda = 0$, $E = -e_1 \partial_x$ | $e_1 \in \mathbb{R} - \{0\}$ |
| $\Lambda_2 = \lambda_1 \partial_{x_1} + \lambda_2 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\Lambda = \partial_{x_1} + \partial_{x_2}$, $E = \partial_{x_3}$ | $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$ |
| $\Lambda_3 = e_{12} \partial_{x_1} + \lambda_3 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\Lambda = \lambda_3 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\lambda_3 = 0$ |
| $\Lambda_4 = -e_2 \partial_{x_2} - e_3 \partial_{x_3}$ | $\Lambda = \lambda_3 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\lambda_3 = 0$ |

| Jacobi structures on Lie algebra V | Equivalence classes | Comments |
|-----------------------------------|---------------------|----------|
| $\Lambda_1 = 0$, $E_1 = -e_1 \partial_x - e_2 \partial_y - e_3 \partial_z$ | $\Lambda = 0$, $E = -e_1 \partial_x$ | $e_1 \in \mathbb{R} - \{0\}$ |
| $\Lambda_2 = (e_2 \lambda_3 + e_3) \partial_{x_1} + \lambda_2 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\Lambda = \lambda_3 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\lambda_3 = 0$ |
| $\Lambda_3 = -e_2 \partial_{x_2} - e_3 \partial_{x_3}$ | $\Lambda = \lambda_3 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\lambda_3 = 0$ |

| Jacobi structures on Lie algebra VI$_0$ | Equivalence classes | Comments |
|----------------------------------------|---------------------|----------|
| $\Lambda_1 = 0$, $E_1 = -e_1 \partial_x - e_2 \partial_y - e_3 \partial_z$ | $\Lambda = 0$, $E = -e_1 \partial_x$ | $e_1 \in \mathbb{R} - \{0\}$ |
| $\Lambda_2 = \lambda_3 \partial_{x_1} + \lambda_2 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\Lambda = \lambda_3 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\lambda_3 = 0$ |
| $\Lambda_3 = -e_2 \partial_{x_2} - e_3 \partial_{x_3}$ | $\Lambda = \lambda_3 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\lambda_3 = 0$ |

| Jacobi structures on Lie algebra VI$_a$ | Equivalence classes | Comments |
|----------------------------------------|---------------------|----------|
| $\Lambda_1 = 0$, $E_1 = -e_1 \partial_x - e_2 \partial_y - e_3 \partial_z$ | $\Lambda = 0$, $E = -e_1 \partial_x$ | $e_1 \in \mathbb{R} - \{0\}$ |
| $\Lambda_2 = \lambda_1 \partial_{x_1} + \lambda_2 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\Lambda = \lambda_1 \partial_{x_1} + \lambda_2 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\lambda_2 = 0$, $\lambda_3 = 0$ |
| $\Lambda_3 = -e_2 \partial_{x_2} - e_3 \partial_{x_3}$ | $\Lambda = \lambda_3 \partial_{x_2} + \lambda_3 \partial_{x_3}$ | $\lambda_3 = 0$ |

| Jacobi structures on three-dimensional Lie algebras and equivalence classes. | | |

Table 4 cont.
Appendix C: Automorphism groups of real low-dimensional Lie algebras

| Jacobi structures on Lie algebra VII | Equivalence classes | Comments |
|-------------------------------------|---------------------|----------|
| $A_1 = 0$, $E_1 = -e_1 \partial x_1 - e_2 \partial x_2 - e_3 \partial x_3$ | $\Lambda = 0$, $E = -e_3 \partial x_3$ | $e_3 \in \mathbb{R} - \{0\}$ |
| $A_2 = \lambda_1 \partial x_1 \wedge \partial x_2 + \lambda_3 \partial x_1 \wedge \partial x_3 + \lambda_2 \partial x_2 \wedge \partial x_3$, $E_2 = \lambda_2 \partial x_1 - \lambda_1 \partial x_2$ | $\Lambda = \partial x_2 \wedge \partial x_3$, $E = \partial x_1$ | $\lambda_1^2 + \lambda_2^2 \neq 0$ |
| $A_3 = \lambda_1 \partial x_1 \wedge \partial x_2$, $E_3 = -e_1 \partial x_1 - e_2 \partial x_2$ | $\Lambda = \lambda_1 \partial x_1 \wedge \partial x_2$, $E = -\partial x_2$ | $\lambda_{12} \in \mathbb{R} - \{0\}$ |

| Jacobi structures on Lie algebra VIIa | Equivalence classes | Comments |
|--------------------------------------|---------------------|----------|
| $A_1 = 0$, $E_1 = -e_1 \partial x_1 - e_2 \partial x_2 - e_3 \partial x_3$ | $\Lambda = 0$, $E = -e_1 \partial x_1$ | $e_1 \in \mathbb{R} - \{0\}$ |
| $A_2 = \lambda_2 \partial x_2 \wedge \partial x_3$, $E_2 = -e_2 \partial x_2 - e_3 \partial x_3$ | $\Lambda = \lambda_2 \partial x_2 \wedge \partial x_3$, $E = -\partial x_3$ | $\lambda_{23} \in \mathbb{R} - \{0\}$ |

Jacobi structures on three-dimensional Lie algebras and equivalence classes.

Table 5

| Lie Algebra | Automorphism groups | Comments |
|-------------|---------------------|----------|
| $A_1$ | $GL(2, \mathbb{R})$ | $a_{11} \in \mathbb{R} - \{0\}$ |
| $A_2$ | $\begin{pmatrix} a_{11} & 0 \\ a_{21} & 1 \end{pmatrix}$ | $a_{11} \in \mathbb{R} - \{0\}$ |
| $I$ | $GL(3, \mathbb{R})$ | |
| $\Pi$ | $\begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ | $a_{ij} \in \mathbb{R}$ $(i = 1, 2, j = 1, 2, 3)$ $a_{22}a_{33} \neq a_{23}a_{32}$ |
| $\Pi$, VIa | $\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$ | $a_{ij} \in \mathbb{R}$ $(i = 1, 2, j = 2, 3)$ $a_{22} \neq \pm a_{23}$ |
| IV | $\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{22} \end{pmatrix}$ | $a_{12}, a_{13}, a_{23} \in \mathbb{R}$, $a_{22} \in \mathbb{R} - \{0\}$ |
| V | $\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$ | $a_{ij} \in \mathbb{R}$ $(i = 1, 2, 3, j = 2, 3)$ $a_{22}a_{33} \neq a_{23}a_{32}$ |
| VIa | $\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{11} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}$, $\begin{pmatrix} a_{11} & a_{12} & 0 \\ -a_{12} & -a_{11} & 0 \\ a_{31} & a_{32} & -1 \end{pmatrix}$ | $a_{ij} \in \mathbb{R}$ $(i = 1, 3, j = 1, 2)$ $a_{11} \neq \pm a_{12}$ |

Automorphism groups of real two- and three-dimensional Lie algebras (also see [26]).
Table 5 cont.

| VII₀ | \[ \begin{pmatrix} a_{11} & a_{12} & 0 \\ -a_{12} & a_{11} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & -a_{11} & 0 \\ a_{31} & a_{32} & -1 \end{pmatrix} \] | \( a_{ij} \in \mathbb{R} \ (i = 1, 3, j = 1, 2) \) \( a_{11}^2 + a_{12}^2 \neq 0 \) |
| VII₀ | \[ \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & -a_{23} \\ 0 & a_{23} & a_{22} \end{pmatrix}, \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & -a_{23} \\ 0 & a_{23} & a_{22} \end{pmatrix} \] | \( a_{ij} \in \mathbb{R} \ (i = 1, 2, j = 2, 3) \) \( a_{22}^2 + a_{23}^2 \neq 0 \) |
| VIII | \( SL(2, \mathbb{R}) \) |
| IX | \( SO(3) \) |

Automorphism groups of real two- and three-dimensional Lie algebras (also see [26]).

Conflicts of interest. The authors declare no conflicts of interest.

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