Simulating the Microstructure of Financial Markets

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Abstract. In the beginning of exchange based trading, floor trading was the most widespread form of trading. In the course of the introduction and the progress in information technology, trading processes were adapted to the computational infrastructure at the international financial markets and electronic exchanges were created. These fully electronic exchanges are the starting point for recent agent-based models in econophysics, in which the explicit structure of electronic order books is integrated. The electronic order book structure builds the underlying framework of financial markets which is also contained in the recently introduced realistic Order Book Model [T. Preis et al., Europhys. Lett. 75, 510 (2006), T. Preis et al., Phys. Rev. E 76, 016108 (2007)]. This model provides the possibility to generate the stylized facts of financial markets with a very limited set of rules. This model is described and analyzed in detail. Using this model, it is possible to obtain short-term anti-correlated price time series. Furthermore, simple profitability aspects of the market participants can be reproduced. A nontrivial Hurst exponent can be obtained based on the introduction of a market trend, which leads to an anti-persistent scaling behavior of price changes on short time scales, a persistent scaling behavior on medium time scales, and a diffusive regime on long time scales. A coupling of the order placement depth to the prevailing market trend, which is identified to be a key variable in the Order Book Model, is able to reproduce fat-tailed price change distributions.

1. Introduction
As a first-order approximation it is often assumed that price dynamics of financial markets obey random walk statistics. However, this rough approximation neglects the real nature of financial market time series, and a number of empirical deviations between financial market data sets and models consisting of random walk behavior have been observed [1]\textsuperscript{5}. From a physicist’s point of view financial markets show a behavior just like a complex system in physics with non-Gaussian price increments. The price process of an exchange traded asset can be seen as the result of superimposed trading decisions of market participants. However, in context of modeling the “complex particle system” financial market with agent-based models using Monte Carlo techniques, market microstructure has a huge importance as well. “Market micro structure

\textsuperscript{4} Supplementary information can be found on http://www.tobiaspreis.de

\textsuperscript{5} In computer science applications and diverse interdisciplinary science fields such as computational physics [2] or quantitative finance[3], the computational power requirements led recently to general purpose computing on graphics processing units (GPU).
is the study of the process and outcomes of exchanging assets under explicit trading rules”, as can be found in [4]. However, these trading rules changed in history. Thus, a historical outline is presented in the next section, in order to highlight the transformation from the beginning of floor trading to fully electronic market places in recent years.

1.1. Tulip mania in the United Provinces
A classic example for exchange based trading can be found in the famous tulip mania [5, 6, 7], which took place in the first part of the 17th century, in which tulips became a subject of speculation. This tulip mania had serious impact on the social life in the area which is known as the Netherlands today. The tulip mania is one of the first huge speculation bubbles in history. Tulips, which were originally only located in Asian countries, were introduced in Europe in the middle of the 16th century. In 1560, first tulips were transported from Istanbul, the court of the Ottoman Empire, to Vienna. There, botanist Carolus Clusius was responsible for the imperial botanical garden of Emperor Maximilian II, where he cultivated these plants in 1573 after importing 1,500 seeds. The tulip cultivation in the United Provinces, which are today known as the Netherlands, started then in 1593 after Carolus Clusius had come from Vienna and when he was able to breed tulips under the special, not ideal climate conditions in the Netherlands. This exotic flower started to enjoy great popularity and in the following time tulips became more and more the state of a status symbol. This fact triggered an avalanche and started a competition for possession of the rarest tulips. In the beginning the largest part of market participants at this “tulip market” can be found in the upper class of society. Tulips with lines and flames on the petals were particularly popular. However, this effect was often caused by a special virus, which infects only tulips and is known as the Tulip Breaking potyvirus.

This development can be seen as the starting point of a speculation bubble. In the following time, the competition for possession of tulips escalated and very high price levels were reached when the demands for tulip bulbs exceeded the offers. In 1623, a bulb of a very rare and preferred tulip could change hands for a thousand Dutch florins. This has to be seen in relation to the financial background — the average income per year was only about 150 Dutch florins. In the beginning, bulbs were only traded during the planting season. Later with increasing demand, bulbs could change hands during the whole year with the consequence, that bulbs were sold without the knowledge by the buyer of the exact tulip appearance. So, tulip trading became a speculation business. This aspect operated as a catalyzer for paintings in order to give impressions of the future appearance of a tulip to interested clients.

In the 1630s, the situation had come to a head. One reason was, that not only transactions with cash settlement were recorded. Tulips were also traded via option contracts. An option contract is a financial derivative and one can differentiate between call and put option. Buying a call option provides the right to buy a specified quantity of an underlying – in this case tulip bulbs – at a set price at some time on or before expiration of the contract, while buying a put option provides the right to sell the underlying at the specified price, the so-called strike price. These financial instruments make it possible to hedge price risks and allow producers to sell future crops. On the other hand, consumers can lock cheap buying prices for the future. As instrument of speculation, option contracts, which are traded at derivatives markets today, exhibit a high risk, as a large leverage effect is reachable with comparative small trading accounts. These Dutch option contracts were the first derivative instruments in history. And the increased use of option contracts in the Dutch tulip bulb trading can be regarded as a first evidence for a starting crisis. A high grade of debt of some market participants could be observed, which is one important ingredient for the formation of a speculation bubble. The prices of bulbs escalated with a factor of 50 in the years from 1634 till 1637 [7]. In Amsterdam, e.g., a mansion was sold for three tulip bulbs. And a transaction of 40 bulbs for 100,000 Dutch florins was observed and documented in 1635. A sale of a very rare tulip bulb – the Semper Augustus – was recorded
with a price of 6,000 Dutch florins. These transactions have to be seen in comparison with the prevailing consumer price level in the United Provinces at that time. A ton of butter cost roughly 100 Dutch florins. This example clarifies the speculative nature of tulip bulb trading. In modern risk management, financial derivatives are reasonable instruments in order to control risks of a portfolio, which is a composition of various assets. In the Dutch tulip bulb crisis, however, option contracts were used in order to clear bottlenecks.

This bubble boosted the trading also in other parts of the Dutch society. In order to speculate in these upcoming markets, Dutch citizens of all ranks invested large parts of their wealth in tulips. As a result, some speculators made very large profits in that time and other people lost their whole wealth or even more. The speculation bubble burst in 1637. The inflated prices for tulip bulbs could no longer be realized by the tulip traders. The demand was falling which resulted in an upcoming panic. Everybody had the intention to sell, nobody bought tulip bulbs. The prices had reduplicated in January. And in February a top was reached. As in every bubble, there were also here winners and losers. Some people were holding open option contracts for selling tulips at prices, which were very high in relation to the current market value. The counterpart, who had the opposite position, had the duty to buy these bulbs for these utopian prices. Thousands of Dutch people were financially ruined.

1.2. History of exchange based trading

After this short outline of the first speculation bubble in history, one has to state, that the trading in the Dutch tulip mania was unorganized and a kind of a over-the-counter market (OTC). Contracts were concluded individually by the parties.

In the historical development of exchange based trading one can differentiate between spot markets and derivatives markets. Whereas spot market contracts, e.g. shares of a stock corporation, have to be fulfilled immediately, contracts on derivatives markets are fulfilled on a set date in the future, as already described in Section 1.1 for the option contracts. However, not only options can be traded at derivatives exchanges. Also an other standard group of contracts is available, future contracts. A future contract contains the right and the duty to buy or sell a specified quantity of an underlying at a set price at the expiration time of the contract.

First evidence of exchange based trading can be found in the tulip bulb bubble as mentioned before. At that time, options became important for economic processes for the first time. However, not only options on tulip bulbs were traded. Also contracts of the East India Company, which was founded in 1602, changed hands [8]. In Germany, options were tradeable for the first time in 1970. However, in the United Kingdom and in the United States of America (USA), options were traded already in the 18th century. Despite of the fact, that the following outline is not exhaustive, the beginning of standardized futures trading should be mentioned, which took place at the Chicago Board of Trade (CBOT) in 1865. Additionally, despite of an established OTC market, the trading of exchange based stock options in the USA was possible at the Chicago Board Options Exchange (CBOE) since 1973. In Europe, the European Options Exchange (EOE) was founded in Amsterdam in the year 1978. And after teething troubles the EOE was established as the primary market for options in Europe in the early 1980s. This position was lost not until when the Deutsche Terminbörse (DTB) had been created in 1990 in Germany. In 1998, then the merger of DTB and Swiss Options and Futures Exchange (SOFFEX) led to a world leading derivative market, the European Exchange (EUREX), which was a pure electronic trading platform in contrast to established exchanges in the USA, which used a...

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6 Over-the-counter markets (OTC) differ from exchange based trading, as they are not regulated. Furthermore, the fulfilling of OTC market contracts is not guaranteed by a central clearing house and so, the risk exists, that the contract counterpart is not able to fulfill an arranged contract.

7 In the Federal Republic of Germany, spot market contracts will be cleared within at most two exchange trading days.
floor trading concept. There, traders exchange assets on the floor in a physical way. EUREX bought additionally the International Securities Exchange (ISE) in 2007, in order to play a pioneering task in the upcoming exchange consolidation process. However, this step has to be seen in context with power shifts in the large exchange companies all over the world. Also activities of hedge funds influenced this evolution, which were holding large stock packages of the Deutsche Börse AG\(^8\) and the London Stock Exchange (LSE). However, the initial intention to force on a merger between both exchanges miscarried. Instead, a merger between the two spot markets New York Stock Exchange (NYSE) and Euronext was established and the NYSE Euronext was born – a large competitor of Deutsche Börse AG and LSE. The NYSE as a large spot market with a long tradition was founded in 1792. Up to March 2006, almost the whole trading on NYSE was handled on the floor. Then, with the acquisition of the electronic trading platform Archipelago, at NYSE the electronic era started. This process was enforced after fusion with Euronext. Today, a clear tendency to electronic market places with transparency can be observed. As could be read in September 2007\(^9\), the NYSE will close two more trading rooms on the floor. Simultaneously, one has to state that roughly one third of the trading volume at NYSE is generated by program trading\(^9\) in the meantime. Program trading, which means the execution of automatic trading decisions generated by computer algorithms, can be realized on full-electronic trading platforms. Beside NYSE, XETRA – the electronic spot market platform of Deutsche Börse AG – should be mentioned, which replaced the old IBIS system in 1997 and started also a great success story.

A very impressive example for the change from floor trading to electronic trading can be found in trading of the Euro-Bund future\(^10\) (FGBL), which is a fixed income derivative. In the beginning of the 1990s, the FGBL contract was traded at the young DTB and also at the 1982 founded London International Financial Futures Exchange (LIFFE), which is today part of NYSE Euronext. At that time, LIFFE was a pure floor trading exchange and the FGBL was favorably traded at LIFFE. However, an empirical comparison between these two exchanges benefited the DTB, even though the transaction volume was lower than the one at LIFFE. One reason for the success of DTB was the higher level of transparency. The order book of the DTB, which stores offers and demands of all market participants and matches orders against each other, was distributed partly to the market members. Thus, each trader had an impression of the market depth. The advantage of anonymity boosted the electronic trading. With an increasing liquidity the DTB was able to concentrate the whole trading volume of the Euro-Bund future in its own trading system. This was one milestone for the triumphal procession of full-electronic exchange trading.

1.3. The Gold Fixing in London

However, even if electronic trading has mostly displaced floor trading and manual trading processes, some relics can still be observed today. Naturally, some exchanges indulge in floor trading for public relation reasons. The Deutsche Börse AG hangs on to floor trading. Recently, the accouterment was even modernized in order to present the general public a manifestation of financial markets. A daily trading process, which can be classified as a traditional institution, is the Gold Fixing in London\(^4\). The first Gold Fixing took place on September 12th, 1919 at 11.00 am in London in physical presence of the original five founding members N. M. Rothschild & Sons, Mocatta & Goldschmid, Samuel Montagu & Co., Pixley & Abell, and Sharps & Wilkins. The gold fixing is executed twice a day. At that time, the five members came together. Since

\(^8\) EUREX is a joint venture of Deutsche Börse AG and Swiss Exchange (SWX).

\(^9\) Detailed information can be found in the news releases of NYSE Euronext on the website [www.nyse.com](http://www.nyse.com).

\(^10\) The underlying of the Euro-Bund future contract is a certain debt security issued by the Federal Republic of Germany with a remaining term of 8.5 to 10.5 years.
2004, a dedicated conference call is used\textsuperscript{11}. In the beginning of each fixing, the Chairman of the meeting, who is one of the members, announces an opening price. The other four members have then the possibility to contact their customers and include their orders into their decision. Then, each member has to declare if the member institution is a buyer or a seller at the announced price level. In the case, that there are buyers and sellers, additionally the quantity offered and demanded at this price is asked in units of bars. If only buyers or only seller exist at the announced price, or if the difference between offered and demanded bars at a price level is larger then 50 bars, then the procedure starts again and the price is moved. If a price finally fulfills the requirements, the chairman announces that the price is fixed. In order to have the possibility to contact clients, each member has a verbal flag. As long as a flag is raised, a price cannot be declared fixed. However, this verbal flag has physical roots. Before the dedicated conference call was established, each member had a small Union Flag on the desk for this purpose. Since 1919, the gold price fixing provided market participants with the opportunity to buy and sell gold at a specific price. The London Gold Fixing is still today a benchmark for trading with gold. However, as seen above, former procedures were adapted to progress in information technology. One has to state, that there exists full-electronic trading of gold today. On the New York Mercantile Exchange (NYMEX) founded in 1882 and on the New York Commodities Exchange (COMEX), which is a sub division of NYMEX today, gold can be traded continuously.

1.4. Econophysics – Physics meets Finance

In the remainder of this Section, in a short outline, the milestones of modeling financial markets should be presented, especially in context of physicists’ contributions. Some decades ago, M. F. M. Osborne \cite{11,12} and B. B. Mandelbrot \cite{13,14} started to apply methods and also models from statistical physics to problems arising in the examination of the behavior of financial markets, and thus they became founders of a new research field, which is nowadays often called econophysics. They discovered that the price fluctuations at financial markets do not obey random walk statistics \cite{15,16,17}, as assumed in many economic theories. However, the modeling of financial market time series by a diffusive stochastic process dates already back to L. Bachelier \cite{18} in 1900. Geometric Brownian motion \cite{19,20,11}, which is a variant of the Brownian motion and which was independently investigated by L. Bachelier and A. Einstein \cite{21}, has become the standard mathematical model, and the theory of fair prices of option contracts \cite{22,23} is based on it. With this theory, it was possible to find a fair value for options for the first time – a huge progress since option trading in the Dutch tulip bulb crisis. However, also a sophisticated model does not protect against losses. Two originators of the theory, namely Black and Merton, founded the hedge fund Long Term Capital Management (LTCM) in 1994 in order to exploit advantage derived from the theory. However, after an initially enormous success, the fund lost more than 4 billion dollars in 1998 during the Russian Financial crisis. It is an often found fact, that in extreme events, models lose their stability and sometimes their validation.

The insights found by B. B. Mandelbrot were based on time series of very limited length. With the progress in information technology, trading processes at the international exchanges were adapted to the available IT infrastructure and full electronic exchanges like EUREX were created. The trading at full electronic exchanges can produce time series data down to centiseconds base and thus nowadays a huge quantity of historical financial market time series is available. By using these new possibilities, the work of B. B. Mandelbrot could be confirmed. Noticing discrepancies between economic theory and financial market reality, physicists started approaching the complex “multi particle systems” of financial markets and economy in general with a large variety of physical methods. Recent examples can be found in \cite{24,25}. An early contribution to this enterprise was provided by P. Bak \cite{26}. His very simple agent based model

\textsuperscript{11} More information can be found on the website http://www.goldfixing.com.
uses the reaction diffusion process \( A + B \to 0 \) and contains elements of trader imitation and feedback mechanisms. Nowadays, physicists are interested in understanding the process of price formation in more detail, as former models were not able to reproduce the behavior of financial markets completely. In this context, the model of S. Maslov [27, 28] has to be mentioned. He published an alternative model, in which two different types of orders were introduced. Further agent based market models [29, 30, 31, 32, 33, 34, 35, 36] were developed with the aim to reproduce and interpret empirical stylized facts of financial market time series. Especially a model of J. D. Farmer et al. [37, 38] has to be mentioned. In this statistical model, the continuous double auction is examined analytically and numerically.

In the following Sections, the Order Book Model, which was introduced in [39] and analyzed in detail in [40], is presented in combination with the introduction of the order book structure. This model can be used in order to obtain a “mechanistic understanding” of the price formation process. Some extensions are given leading to empirical stylized facts, which can be observed on real financial markets.

2. Market Structure and Order Book Model

As described in the introduction, electronic exchanges are the prevailing form of trading nowadays. Upon this background, the Order Book Model [39, 40], which is based on structures found at real electronic financial markets, is defined in its basic form in this Section. Aim is the accurate reproduction of structure and the mechanisms of an order book at real financial markets, as shown in Fig. 1. The function of an order book is to store offers and demands of the market participants. In the simulations, only one order book is used, in which one individual asset is traded. This asset can be, e.g., a share of a stock company, a loan, or a derivative product. There are also various types of orders at real financial markets. In the Order Book Model, only the two most important types are implemented, namely limit orders and market orders. Limit orders are executed only at a specific price or a price, which is better for the trader, who placed this limit order, whereas market orders are executed immediately against the best available limit order. Limit orders are added chronologically, which realizes a price time priority based matching algorithm. Thus, at a given price level with more than one limit order, the limit order, which was inserted first, has execution priority. Limit and market orders are performed by agents in the Order Book Model, which also distinguishes between liquidity providers and liquidity takers. These two groups of agents differ in the types of orders they are permitted to submit. On the one hand, \( N_A \) liquidity providers transmit only limit orders. In the case of a limit sell order, a liquidity provider offers an asset for sale at a set limit price or a higher price. Analogously, a limit buy order indicates a demand for buying an asset and the order is executed at a set limit price or any better price for the liquidity provider. Let \( p_a \) be the so-called best ask price, which is the lowest price level of all limit sell prices in the order book, and analogously \( p_b \) the so-called best bid price, being the highest price level for which at least one limit buy order is stored in the order book. In the Order Book Model, limit orders are placed around the midpoint \( p_m = \frac{p_a + p_b}{2} \) with a rate \( \alpha \), i.e., \( \alpha \cdot N_A \) new limit orders are submitted per time step. Let \( q_{\text{provider}} \) be the probability with which a limit order is a limit buy order. Thus, with probability \( 1 - q_{\text{provider}} \), the limit order to be placed is a limit sell order. The liquidity provider, which can be identified as so-called market maker, supplies liquidity to the market in order to exploit the non-zero spread \( s = p_a - p_b \) for earning money: these market participants intend, e.g., to sell an asset at price \( p_a \) or higher and then to buy it back at price \( p_b \) or lower. Thus, they have earned at least the spread \( s \). As seen in this example, so-called short sales are allowed, i.e., it is allowed for agents to sell assets even if they do not possess them.

On the other hand, there are \( N_A \) liquidity takers, who transmit only market orders. These market orders are submitted with rate \( \mu \), i.e., \( \mu \cdot N_A \) market orders are inserted per time step into the order book. A market order will be immediately executed after arrival. A market sell order
Figure 1. Market micro structure of electronic order books: Limit buy (white squares) and limit sell orders (black squares) are added chronologically to the appropriate discrete price level $p$. In this example, a new sell order is placed at price $p_0$, at which there is already a buy order stored in the order book. This limit sell order, which can be executed immediately against the buy order on the bid side, is a so-called crossing limit order and so the order degenerates to a market order. With the execution of the two orders at price $p_0$, a trade is performed at price $p_0$, which is then called last traded price. The spread $s$, which is the difference between the best ask and the best bid price, was two ticks before the arrival of the crossing limit sell order at price level $p_0$ and increases to three ticks after this trade, as shown in Fig. 2.

is executed at price level $p_0$, a market buy order at price level $p_a$. A market order of a liquidity taker is a market buy order with probability $q_{\text{taker}}$ and a market sell order with probability $1 - q_{\text{taker}}$. In the basic version of the Order Book Model, the simple case $q_{\text{provider}} = q_{\text{taker}} = \frac{1}{2}$ is applied. Thus, all orders will be produced symmetrically around the midpoint. In practice, it is possible, that the limit price of a limit sell order is lower than the current best bid price and the limit price of a limit buy order is higher than the current best ask price. Such so-called crossing
Figure 2. Market microstructure of electronic order books: This figure shows the order book after the execution in Fig. 1. The new spread is three ticks.

limit orders degenerate to market orders and thus, they are executed immediately. In the Order Book Model, only pure limit orders will be used. Limit orders, which are stored in the order book, can also expire or can be deleted. In the model, this canceling is realized in the way that each stored order is deleted with probability $\delta$ per time unit. As there are overall $2N_A$ agents in the multi-agent system, each Monte Carlo step (MCS) consists of $2N_A$ moves. In each move, one agent is randomly selected and can perform one action according to the probability rates. If the chosen agent is a liquidity provider, then a limit order with probability $\alpha$ is submitted by this agent. On the other hand, if the selected agent is a liquidity taker, then a market order with probability $\mu$ is placed in the order book and will be immediately executed. Orders in the Order Book Model have the constant order volume 1. Thus, it is possible only to buy or sell one asset unit, e.g. one share, with a single order.

Based on these simple rules, first an unrealistic independent identically distributed (IID) order placement depth is applied. This is realized in the way that limit buy orders are entered on each
price level in the interval of \([p_a - 1 - \text{int}_a; p_a - 1]\) with the same probability, and accordingly, limit sell orders are transmitted uniformly distributed in the interval of \([p_b + 1; p_b + 1 + \text{int}_b]\). Already with this definition of the Order Book Model, profits and losses of the agents can be analyzed. And one can find, that the averaged wealth value of liquidity takers and liquidity providers drifts apart linearly in time \([40]\). If comparing these results with real financial markets, it has to be stated that liquidity takers are disadvantaged compared to liquidity providers systematically. The distinction in the Order Book Model between the two types of orders reflects the two types of orders. In real financial markets, each market participant is not restricted to one order type.

In the next step, a more realistic order placement depth will be integrated in the Order Book Model. The order book depth of real financial markets can be described by a log-normal distribution \([28]\). And, to take this into account the IID limit order placement is replaced by an exponentially distributed order placement depth. Thus, for placing a limit order, the limit price \(p_l\) is determined for a limit buy order through

\[ p_l = p_a - 1 - \eta \]

and for a limit sell order according to

\[ p_l = p_b + 1 + \eta \]

whereby \(\eta\) is an exponentially distributed integer random number created by \(\eta = [-\lambda \cdot \ln(x)]\) with \(x\) being a uniformly distributed random number in the interval \([0; 1)\) and \(\lfloor z \rfloor\) denoting the integer part of \(z\). With this construction, the submission process of limit orders has the tendency to reduce the gap between best bid price and best ask price. Also, crossing limit orders are avoided, as the price of a limit buy order cannot be equal or larger than the best ask price and the price of a limit sell order cannot be equal or lower than the best bid price.

As a result of applying the exponential order placement rule, a log-normally distributed order book depth profile is obtained, as shown in Fig. 3. As order book depth profile, one describes the cumulative order volumes of the orders at each price level of the order book. In Fig. 4, an exemplary price sequence of \(10^6\) MCS is shown. The price history of an Order Book Model simulation is used in order to analyze the most important properties of price time series.

One important time series characteristic can be measured by the Hurst exponent, which is shown in Fig. 5 for various agent numbers \(N_A\). Generally the Hurst exponent \(H(q)\) can be calculated by the relationship

\[ \langle |p(t + \Delta \tau) - p(t)|^q \rangle^{1/q} \propto \Delta \tau^{H(q)}, \]  

as defined for example in \([41]\). If it is not mentioned otherwise \(H(\Delta \tau) \equiv H(\Delta \tau, q = 2)\) is used. Mainly, the Hurst exponent provides a measurement of the scaling behavior of a process. \(H < 0.5\) indicates an antipersistent behavior, i.e., that after a movement, a reversion tendency can be found. A Hurst exponent larger than 0.5 can be found for persistent processes. There the probability for continuation of a movement is larger than for a reversion. The random walk process exhibits \(H = 1/2\). For comparison, the Hurst exponent of the random walk, which is constantly \(1/2\) for all time lags, is shown in Fig. 5, too. On short time scales, the price process provides an antipersistent behavior, which is a consequence of the order book structure. On long time scales, the price process converges towards random walk properties. The antipersistence on short time scales, which can also be found in real financial market data, is a consequence of the short time anticorrelation. The autocorrelation of the dedicated price changes time series \(\delta p(t) = p(t + 1) - p(t)\) shows the same behavior as can be found in real financial markets. A negative autocorrelation exists for the time lag \(\delta \tau = 1\). Thus, a positive price change succeeds...
Figure 3. For an exponential order placement depth and with the parameter selections $\alpha = 0.15$, $\mu = 0.025$, $\delta = 0.025$, $\lambda_0 = 100$, $N_A = 250$, and $q_{\text{provider}} = q_{\text{taker}} = 0.5$, one obtains the shown order book depth. The results were averaged over $10^4$ MCS. A fit with the log-normal distribution function $\phi_{\text{LN}}(p-p_m) = a/((p-p_m) \exp\{-[\ln(p-p_m)-b]^2/c\}$ leads to the parameters $a = 527 \pm 1$, $b = 4.483 \pm 0.004$, and $c = 1.79 \pm 0.01$.

A negative price change with large probability and vice versa. This is a consequence of the “mechanics” of the order book. As there are constant order rates and equal probabilities for buy and sell orders, an executed market order of any kind increases the probability that the next last traded price will be anti-correlated with the previous price. An other most important stylized fact of financial market time series is the property, that price change distributions show non-Gaussian shapes. In fact, the probability for large price changes in any direction is larger than predicted by the Gaussian distribution. The price change distributions for this basic implementation of the Order Book Model are shown in Fig. 6. These distributions exhibit no fat tails. However, they can rather well be approximated by a Gaussian distribution.

3. Parameter space

In the previous section, results for a specific parameter selection were presented. However, in the Order Book Model, a large number of essential control parameters can be found. And in order to apply the Order Book Model to a large variety of situations, it is necessary to approach the parameter space of the Order Book Model. The important order rates – the limit order rate $\alpha$ and the market order rate $\mu$ – and also the characteristic order entry depth $\lambda_0$ can be tuned. It could be found that the order book depth profile is of crucial importance. If the density of order book depth is too high, then the price process has the tendency to freezing. If the density is too low, then randomly consecutive market orders of the same kind, i.e. either only market buy orders or only market sell orders, can empty the contrary side of the order book, which
For an exponential order placement depth and for the parameter selection $\alpha = 0.15$, $\mu = 0.025$, $\delta = 0.025$, $\lambda_0 = 100$, $N_A = 125$, and $q_{\text{provider}} = q_{\text{taker}} = 0.5$, an exemplary price sequence of $10^6$ MCS is shown.

was previously filled with limit orders. For this reason, the order book depth profile is analyzed quantitatively in dependence of the two most important parameters $\lambda_0$ and $\mu$ with fixed $\alpha$, $\delta$, and $N_A$. After a short transient time of few thousands MCS, the order book depth is stationary and log-normally distributed according to

$$P_{\text{LN}}(x) = A \frac{1}{Sx\sqrt{2\pi}} \exp \left( -\frac{(\ln x - M)^2}{2S^2} \right)$$

(4)

with the parameters $A$, $S^2$, and $M$. In Fig. 7, one can find the relation between market order rate $\mu$ and scaling factor $A$ of the log-normal distribution for various values of $\lambda_0$. However, this prefactor $A$ can be theoretically obtained, too. $A$ determines the integral over the log-normal distribution function and thus corresponds to the total number of limit orders stored in the order book. The number of orders in the order book at time step $t + 1$ can be calculated iteratively by the various order rates $\alpha$, $\delta$, $\mu$, the number of orders $N(t)$ at the previous time step $t$, and by the number of agents $N_A$ acting in the system, through

$$N(t + 1) = N(t) + \alpha N_A - (N(t) + \alpha N_A) \delta - \mu N_A.$$  

(5)

In equilibrium state,

$$\frac{N_{\text{eq}}}{N_A} = \alpha \left( \frac{1}{\delta} - 1 \right) - \frac{\mu}{\delta}.$$  

(6)
Figure 5. For the same parameter selections as in Fig. 4, the Hurst exponent is shown for various numbers of agents $N_A$, averaged over 50 simulation runs each.

can be obtained for the number of stored orders per liquidity provider. One can additionally define an effective limit order rate $\alpha^* = \alpha (1 - \delta)$. Then, the expression

$$\frac{N_{eq}}{N_A} \delta = \alpha^* - \mu$$

provides, that a stable order book is only then achieved, if the conditions $\alpha^* > \mu$ and $\delta > 0$ are fulfilled. When fitting the order book depth to the log-normal distribution, only one half of the orders is considered, leading to an expression for $N_{eq}/2$ of $1462.5 - 10^4 \mu$, calculated by the values used in the order book simulations. This expression is given additionally in Fig. 7 as a solid line. It can be found that the theoretical considerations fit to the simulation results on medium and large values of the market order rate $\mu$. A linear decrease of $A$ for medium and large values of $\mu$ has to be stated. And this behavior is independent of the parameter $\lambda_0$. The deviation between theory and simulation found for small $\mu$ results from the fact that the order book depth does not approach a log-normal distribution for very small market order rates $\mu$. Then, a fit to the log-normal distribution function is of course no longer valid.

A second important parameter, which is influenced by changes of the parameters $\lambda_0$ and $\mu$, is the center point $M$ of the log-normal distribution. The larger the market order rate $\mu$, the more limit orders can be removed from the bid side and the ask side of the order book by order matching. The more limit orders, however, are removed from the inner part of the log-normal distributions at both sides of the order book, the more the center point $M$ is departing from the midpoint $p_m$ being between the two distributions. This is illustrated in Fig. 8. This phenomenon can be found for $\mu > 0.01$. Deviations from the log-normal behavior are considered, e.g., for small values of $\mu$. In this case, the distribution of the order book depth resembles the exponentially
Figure 6. Distributions of price changes for various $\Delta \tau$ for the same parameter values as in Fig. 4. Again, the results were averaged over 50 simulation runs each. For $\Delta \tau = 200$, the fit parameters for the Gaussian distribution function $\phi(\Delta p) = a \cdot \exp(-b \cdot \Delta p^2)$ are given by $a = 2.122 \times 10^{-2} \pm 4 \times 10^{-5}$ and $b = 1.388 \times 10^{-3} \pm 10^{-6}$.

distributed order placement depth used for submitting the limit order. The reason for this observation is that the number of market orders is not sufficient in order to remove the limit orders at best bid price and best ask price. In the extreme case of vanishing $\mu$, the order book freezes. Best bid price and best ask price become constant in time and the price jumps between best bid price and best ask price, if there are still market orders. Another extreme situation occurs by large market order rates $\mu$. Then, the increased number of submitted market orders increases the probability that one side of the order book is completely cleared and then trading will stop.

4. Augmentation of the Order Book Model

The previous version of the Order Book Model is able to reproduce the short time antipersistence and the random walk behavior on long time scales. After the investigation of the parameter space, it is obvious, that some extensions of the basic version of the Order Book Model are required in order to reproduce further stylized facts of financial markets. So far, a persistent behavior on medium time scales cannot be reproduced. In the basic version of the model, always a symmetry between the buy and the sell probability is applied by $q_{\text{provider}} = q_{\text{taker}} = 1/2$. This setting is in accordance with the assumption that financial markets fulfill the properties of a stationary behavior. However, a stationary state cannot be found at real financial markets. In fact, asymmetries are indicated by variable order rates, also on intra-day time scales. In short time trend phases, an increased buy or increased sell probability causes the movement of price
in one direction. Therefore, the symmetry $q_{\text{provider}} = 1 - q_{\text{provider}} = q_{\text{taker}} = 1 - q_{\text{taker}} = 1/2$ will now be broken in a way that $q_{\text{taker}}$ shall be changed in time but still have an average value of $1/2$, in order to protect the order book stability. With $q_{\text{provider}} = 1/2$, the buy and sell probability of liquidity providers stays in equilibrium. With this method, the market in the simulation of the Order Book Model is “deflected” from its stationary state. One possible ansatz for the realization of such a market deflection is a deterministic symmetry disturbance. However, it can be shown [40], that the deterministic perturbation is not able to produce a more realistic price behavior. But it can be stated, that it is possible to change the behavior of the Hurst exponent especially on medium time scales. As deterministic perturbations with a discrete spectrum for the returning time of $q_{\text{taker}}$ to the start value $1/2$ lead to oscillations of the Hurst exponent, a stochastic perturbation with continuous return time distributions is introduced in the next section.

4.1. Feedback random walk
A first stochastic perturbation is realized by a bounded random walk approach [39]. For this ansatz, the value of $q_{\text{taker}}$ is changed in time by a bounded random walk with reflecting boundary conditions and with an average value of $1/2$. The buy and sell probability of liquidity providers stays constant through $q_{\text{provider}} = 1/2$. This extension of the Order Book Model leads to an antipersistent price behavior on short time scales, a persistent price behavior on medium time scales, and a random walk behavior on long time scales. One critical aspect is, that the maximum value of $H(\Delta \tau)$ is too large. As described in [13, 14, 42], only maximum values of approximately 0.6 should be expected. However, the shape of the price change distributions is dramatically

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**Figure 7.** Fit parameter $A$ as a function of the market order rate $\mu$ for various values of the order placement depth $\lambda_0$. 
changed. A bimodal shape can be observed, which is in conflict with real financial markets. This behavior is caused by the constant residence distribution of a bounded random walk [40]. For this background, a sophisticated solution is a feedback random walk (FRW) – a random walk with increased probability for returning to the mean value. The FRW is again only applied to $q_{\text{taker}}$; $q_{\text{provider}}$ stays constant at 1/2. In the following, the FRW process is described in detail. In the beginning, $q_{\text{taker}}$ starts at the mean value of 1/2. Then, with proceeding time, $q_{\text{taker}}$ is incremented and decremented by a value of $\Delta S$ in each Monte Carlo step. The probability for drifting to the start and average value of 1/2 is given by $1/2 + |q_{\text{taker}}(t) - 1/2|$. Thus, the probability for departing from the start and mean value of 1/2 is given by $1/2 - |q_{\text{taker}}(t) - 1/2|$. With these rules, the FRW tends to return to its start and average value $\langle q_{\text{taker}} \rangle = 1/2$. This FRW process obeys a continuous return time spectrum, which is qualitatively comparable to that of the bounded random walk. However, the residence distribution of the probability $q_{\text{taker}}$ exhibits an almost Gaussian shape for the FRW, whereas the bounded random walk exhibits uniformly distributed residence.

If applying the FRW mechanism in the Order Book Model simulations, one obtains the behavior of the Hurst exponent as shown in Fig. 9. The Hurst exponent $H(\Delta \tau)$ was averaged over 50 simulation runs each consisting of $10^6$ Monte Carlo steps (MCS). As already found in the basic version of the Order Book Model an antipersistent behavior on short time scales and a diffusive regime on long time scales is obtained. Additionally a persistent regime on medium time scales can be found. The maximum value of the Hurst exponent increases with increasing agent numbers $N_A$. In Fig. 10, the price change distributions for the FRW approach are presented, which are almost Gaussian shaped. Thus, also this extension of the Order Book

![Figure 8. Fit parameter $M$ as a function of the market order rate $\mu$ for various values of the order placement depth $\lambda_0$.](image-url)
Model fails in producing fat tailed price change distributions. However, the fact, that $H > 0.5$ is obtained on medium time scales, can be interpreted in the context of autocorrelations. The Hurst exponent is often mentioned in order to characterize the behavior of time series and stochastic processes. Oftentimes a connection to the autocorrelation property is drawn. In literature, it is widely assumed, that $H \neq 1/2$ implies long time correlations. However, it was recently shown theoretically in [43] that $H \neq 1/2$ implies not necessarily long time correlations. The Order Book Model supports this statement, as a persistent regime can be found on medium time scales in combination with no non-vanishing autocorrelations on these time scales [40]. The autocorrelation function of price changes indicates a negative autocorrelation for time lag $\Delta \tau = 1$ in agreement with the antipersistent behavior on short time scales [39]. The autocorrelation functions for the quadratic price changes and for the absolute price changes are positive on short time scales and converge roughly exponentially towards zero [40]. This is in agreement with the so-called volatility clustering found in real financial market data as analyzed in detail in [44, 45, 46]. A posteriori the volatility can be simply calculated by the standard deviation of the values in the price time series.

4.2. Dynamic order placement depth
In order to motivate a further extension of the Order Book Model, one has to discuss risks existing for the market participants. A liquidity provider has the intention to earn money by placing a limit buy order at or near the best bid price and by placing a limit sell order at or near the best ask price. Under beneficial conditions, both orders are executed in a short time-frame.
However, it is also possible, that only one order, e.g., the limit sell order is matched against a market buy order. If after the execution, a trend with increasing prices is established, then the liquidity provider has only a small probability to close his position by getting an execution for the limit buy order stored in the order book. Instead, the liquidity provider is losing money during the time in which the market is rising. Concerning this aspect it follows, that liquidity providers can reduce their risk by adapting their limit order placement to the prevailing market conditions. In trendless market phases, in which only small price fluctuations are to be expected, liquidity providers can place their limit orders close to the midpoint, in order to be able to participate in these movements. However, in strong trend phases, the risk of the liquidity providers to possess positions, which are orientated against the prevailing market trend increases. Therefore, it is a consequence, that a liquidity provider adapts the characteristic order placement depth in order to take into account changing conditions. In the Order Book Model, the strength of a trend is related to the deviation of the variable \( q_{\text{taker}} \) from the symmetric case \( q_{\text{taker}} = 1/2 \). Therefore, the constant order placement depth \( \lambda_0 \) is replaced by:

\[
\lambda (t) = \lambda_0 \left( 1 + \frac{|q_{\text{taker}} (t) - \frac{1}{2}|}{\sqrt{\langle (q_{\text{taker}} (t) - \frac{1}{2})^2 \rangle}} \cdot C_{\lambda} \right)
\]

This extension of the Order Book model corresponds to \( C_{\lambda} = 0 \) in the original version of the Order Book Model. In Fig. 11 and Fig. 12, the Hurst exponent and the price change distributions
**Figure 11.** Dynamic order placement depth: Hurst exponent $H(\Delta \tau)$ for various values of $N_A$ with $\lambda_0 = 100$, $C_\lambda = 10$, $\alpha = 0.15$, $\mu = 0.025$, $\delta = 0.025$, and $\Delta S = 0.001$.

are shown for coupling the order placement depth $\lambda$ to the prevailing trend. For the calculation, an average is taken of 50 independent simulation runs. Each simulation run lasts $10^6$ MCS. With the dynamic order placement depth, the Order Book Model can reproduce a persistent scaling behavior on medium time scales caused by the FRW mechanism and is also able to exhibit fat-tailed price change distributions. The origin of fat-tailed price change distributions is a widely discussed problem in econophysics. Often a truncated Lévy distribution \[16, 17, 47\] is suggested in order to approximate fat-tailed price change distributions, which can be found at real financial markets indeed. A Lévy stable distribution has the following properties: On the one hand, a Lévy stable distribution is scale invariant, on the other hand, it exhibits an infinite variance \[16\]. In contrast, a truncated Lévy distribution (TLD) \[48, 49\] exhibits a finite variance and shows scaling behavior in a large, but finite interval. As a second approach, also the possibility of power law tails in the price change distributions are widely discussed \[16\]. For example, it is shown in \[50\] for the stock market index S&P 500, that the price change distributions for time lags $\Delta \tau \leq 4$ days show a power-law behavior with an exponent $\alpha \approx 3$. And this exponent is outside the stable Lévy regime, which can be found in the range $0 < \alpha_L < 2$. For larger time lags, a slow convergence to random walk statistics is found. For this background, a TLD \[49\] will be used for fitting fat-tailed price changing distributions, which are obtained by the Order Book Model with dynamic order placement depth. The characteristic function of the TLD is given by

$$\Lambda_{\alpha_L, C_1, l} (f_n) = \exp \left( c_0 - c_1 \frac{(f_n^2 + 1/l^2)^{\alpha_L/2}}{\cos(\pi \alpha_L/2)} \cos (\alpha_L \arctan(|l| f_n)) \right)$$  \hspace{1cm} (9)
Figure 12. Dynamic order placement depth: Distributions of price changes for various values of $\Delta \tau$ with the same parameter values as in Fig. 11. A Gaussian approximation $\phi(\Delta p) = a \cdot \exp(-b \cdot \Delta p^2)$ for $\Delta \tau = 200$ with the parameters $a = 2.54 \times 10^{-3} \pm 10^{-5}$ and $b = (2.80 \pm 0.02) \times 10^{-5}$ strongly underestimates the probability for large price changes.

with scaling factors

$$c_0 = \frac{l^{-\alpha_L}}{\cos(\pi \alpha_L / 2)}$$

(10)

and $c_1$. For the reason, that only the characteristic function of the TLD is given in an analytic form, a discrete Fourier transformation of the price change distributions is necessary, in order to analyze the obtained simulation data. After applying this method, one can compare the Lévy exponent $\alpha_L$, which was measured for real financial data time series and which takes values in the range of $\approx 1.4 - 1.5$ [48, 52]. For this background, the parameter selection $C_\lambda = 1$ seems to be the best approximation [40]. If one couples the order placement depth to the prevailing trend, then fat-tailed price change distributions can be observed. Furthermore, this feature is independent of the persistent behavior on medium time scales. The persistence could be obtained by the FRW and so by imposing non-stationary increments on the price process. On the other hand, it is also possible to generate fat-tailed price change distributions without $H > 1/2$ for medium time scales, such that these two properties are truly independent.

5. Summary
In the beginning of exchange based trading, floor trading was a widespread form of trading. The tulip bulb crisis in the United Provinces and the London Gold Fixing are exemplarily discussed being famous examples in the history of exchange based trading. In the course of the introduction and the progress in information technology, electronic exchanges were created.
Based on the market microstructure of modern electronic financial markets, the Order Book Model is presented, which was introduced in [39] as a multi-agent based system for modeling financial markets and which was analyzed in detail in [40]. It can be shown, that liquidity providers have a systematic advantage by the possibility of transmitting limit orders, compared to the group of the liquidity takers. This version of the Order Book Model is able to reproduce very simple facts of financial markets. With an exponentially distributed order placement depth, a log-normally distributed order book depth is obtained. The corresponding price time series possess an antipersistent price behavior on short time scales. On medium and long time scales, the Hurst exponent converges towards a random walk behavior. The price change distributions obey an almost Gaussian shape. This basic realization of the Order Book Model reflects a stationary market due to a symmetry, reached by static and identical buy and sell probabilities. However, when one additionally introduces a symmetry-breaking extension by the feedback random walk, the Order Book Model is displaced from its stationary state, in order to implement market trends on short time scales. With this perturbation, a persistent price behavior on medium time scales is obtained. Furthermore, a coupling of the characteristic order placement depth to the prevailing market trend leads to widened price change distributions. They can be fitted by a truncated Lévy distribution. With all these extensions of the Order Book Model, it is possible to demonstrate that the generation of a nontrivial Hurst exponent $H \neq 1/2$ is independent of the generation of fat-tailed price change distributions.

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References

[1] T. Preis, W. Paul, J. J. Schneider: Europhysics Letters 82, 68005 (2008)
[2] T. Preis, P. Virnau, W. Paul, J. J. Schneider: Journal of Computational Physics 228, 4468 (2009)
[3] T. Preis, P. Virnau, W. Paul, J. J. Schneider: New Journal of Physics (2009)
[4] M. O’Hara: Market Microstructure Theory (Blackwell Publishing, Malden 2007)
[5] P. M. Garber: The Journal of Political Economy 97, 535 (1997)
[6] P. M. Garber: Famous First Bubbles: The Fundamentals of Early Manias (MIT Press, Cambridge 2000)
[7] A. Goldgar: Tulipmania: Money, Honor, and Knowledge in the Dutch Golden Age (University of Chicago Press, Chicago 2007)
[8] S. Trautmann: Investitionen. Bewertung, Auswahl und Risikomanagement (Springer, Berlin 2006)
[9] Deutsche Börse Group: Geschäftsbericht 2005 (Deutsche Börse Group, Frankfurt 2006)
[10] N. Kuls: Das New Yorker Börsenparkett schrumpft auf die Hälfte. In: Frankfurter Allgemeine Zeitung (Frankfurt, 27th September 2007) p 23
[11] M. F. M. Osborne: Operations Research 7, 145 (1959)
[12] M. F. M. Osborne: The Stock Market and Finance from a Physicist’s Viewpoint (Crossgar Press, Minneapolis 1995)
[13] B. Mandelbrot: The Journal of Business 36, 394 (1963)
[14] B. Mandelbrot: The Journal of Business 37, 393 (1964)
[15] J.-P. Bouchaud, M. Potters: Theory of Financial Risks - From Statistical Physics to Risk Management (Cambridge University Press, Cambridge 2000)
[16] R. N. Mantegna, H. E. Stanley: An Introduction to Econophysics - Correlations and Complexity in Finance (Cambridge University Press, Cambridge 2000)
[17] W. Paul, J. Baschnagel: Stochastic Processes: From Physics to Finance (Springer, Heidelberg 2000)
[18] L. Bachelier: Ph.D. thesis, Faculté des Sciences de Paris, Paris (1900)
[19] K. Itô: Proceedings of the Imperial Academy Tokyo 20, 519 (1944)
[20] P. A. Samuelson, Industrial Management Review 6, 13 (1965)
[21] A. Einstein: Annalen der Physik 17, 549 (1905)
[22] F. Black, M. Scholes: Journal of Political Economy 81, 637 (1973)
