Adiabatic Invariant Treatment of a Collapsing Sphere of Quantized Dust

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Abstract

The semiclassical collapse of a sphere of quantized dust is studied. A Born-Oppenheimer decomposition is performed for the wave function of the system and the semiclassical limit is considered for the gravitational part. The method of adiabatic invariants for time dependent Hamiltonians is then employed to find (approximate) solutions to the quantum dust equations of motions. This allows us to obtain corrections to the adiabatic approximation of the dust states associated with the time evolution of the metric. The diverse non-adiabatic corrections are generally associated with particle (dust) creation and related fluctuations. The back-reaction due to the dominant contribution to particle creation is estimated and seen to slow-down the collapse.
1 Introduction

The canonical quantization of general relativistic isotropic systems carried out in suitably chosen variables leads to the whole dynamics being determined by the Hamiltonian constraint of the Arnowitt, Deser and Misner (ADM) construction. Such an approach is particularly useful if one wishes to study the semiclassical regime of a system of self gravitating matter and has been applied to the collapse of a sphere of classical dust with the associated scalar field being related to time [1]. In a previous paper [2], two of the present authors again applied the ADM formalism to a model describing the collapse of a sphere of homogeneous dust [3] leading to a black hole and examined how Hawking radiation arises within such an approach. In particular the Born-Oppenheimer (or adiabatic) approximation for the coupled matter-gravity system was consistently implemented according to the original formulation of Refs. [4, 5] (for alternative views see also [6, 7, 8] and for their comparison see [9]) by assuming the gravitational degree of freedom evolved slowly with respect to the matter degree of freedom. In this note we shall study corrections to this approximation.

The classical collapse of an isotropic homogeneous sphere of dust [3] has been studied extensively and is treated in numerous text books (e.g., [10]). A particularly interesting aspect of such a collapse is that the interior of the sphere is a three dimensional space of constant curvature whose radius depends on time (in the language of cosmological models it is a section of a Friedmann universe). Further, it has recently been shown that boundary effects are in general absent for classical fluids with a step-function discontinuity of the kind associated with such models [11] (see also [12] for a more specific treatment of the sphere of dust). The treatment of a scalar field in a cosmological context is known and for a particular regime one has that the homogeneous mode of the free scalar field is related to dust, that is a fluid with constant density and zero pressure [13, 14]. One may then naturally ask to what extent is this possible for the spherically symmetric collapse of a massive scalar field, that is what is the effect of the boundary of the sphere. Obviously if the radius of the sphere is infinite one must reproduce the cosmological models. Therefore, whatever the difference is between the two cases, it must vanish as the radius of the sphere tends to infinity. Since a free massive scalar field is localized within its Compton wavelength and just “feels” objects at such a distance, the boundary will only affect fields a Compton wavelength away and its net effect on the sphere of matter will be proportional to the ratio of the Compton wavelength of the scalar field to the radius of the sphere. The former ratio is expected to be small for sufficiently large spheres (thus 3-curvature is relatively small), a condition which has been previously noted and related to a breakdown of the adiabatic approximation for dust [2] or to the validity of the semiclassical (WKB) approximation [15].

Let us illustrate the above consideration by a simple model. In the comoving frame the dust particles are at rest and the adiabatic approximation implies that the radius of the sphere can be kept approximately fixed [2] on solving the matter equation of motion. Let us then consider a static spherically symmetric space-time metric

\[ ds^2 = -d\tau^2 + dr^2 + r^2 d\Omega^2, \] (1.1)

which of course corresponds to a flat 3-space and also agrees with the Schwarzschild vacuum far from the event horizon. The action for a spherically symmetric scalar field
Φ(r, τ) with inverse Compton wavelength μ will be given by

\[ S_Φ = 4\pi \int dτ \int r^2 dr \frac{1}{2} \left[ (\dot{Φ})^2 - (Φ')^2 - μ^2 Φ^2 - 2 V Φ^2 \right], \quad (1.2) \]

where \( \dot{} \equiv d/dr, \cdot \equiv d/dτ \) and we have introduced a potential V which is responsible for the spherical confinement of the scalar field and is zero inside the sphere (\( r < r_0 \)). The equation of motion for Φ is given by

\[ \ddot{Φ} - Φ'' - 2 ri Φ + µ^2 Φ + 2 V Φ = 0. \quad (1.3) \]

Since we seek an analogy for dust, we want a solution that is homogeneous for \( r < r_0 \) and zero for \( r \gg r_0 \) (no matter outside the sphere). Such a solution is given by

\[ Φ = \frac{φ(τ)}{r} f(r), \quad (1.4) \]

\[ \ddot{φ} + µ^2 φ = 0, \]

with \( \ddot{φ} + µ^2 φ = 0 \), since our solution is stationary, and

\[ f = \begin{cases} r & r < r_0 \\ r_0 e^{-µ(r-r_0)} & r > r_0 \end{cases}, \quad (1.5) \]

in which we have made the choice of \( f/r \) dimensionless (so that φ has the same dimensions as Φ) and any other dimensionless factor has been absorbed into φ. The corresponding potential is then for \( r > r_0 \)

\[ V = \frac{µ^2}{2}, \quad (1.6) \]

plus a Dirac delta singularity at \( r = r_0 \) which ensures the continuity of the derivatives of \( f \). One may second quantize Φ, by introducing creation (\( \hat{a}^\dagger \)) and destruction (\( \hat{a} \)) operators for the above solution, obtaining (\( m_φ \equiv \hbar/µ \) is the mass of a scalar quantum)

\[ \hat{Φ} \sim \sqrt{\frac{m_φ µ}{2}} \left[ \hat{a}^\dagger e^{-i µ τ} + \hat{a} e^{i µ τ} \right] \frac{f}{r}, \quad (1.7) \]

and subsequently evaluate quantities of physical interest for any state of dust quanta. One finds for the energy density [13, 14]

\[ \bar{ρ} = \frac{1}{2} \left[ (\dot{Φ})^2 + (Φ')^2 + μ^2 Φ^2 + 2 V Φ^2 \right] \]

\[ \sim m_φ µ \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \left[ (µ^2 + V) \frac{f^2}{r^2} + \frac{1}{2} \left( \frac{f}{r} \right)^2 \right], \]

\[ \sim m_φ µ^3 \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \begin{cases} 1 & r < r_0 \\ 2 \frac{r^2}{µ^2} e^{-2µ(r-r_0)} \left( 1 + \frac{1}{2µr} + \frac{1}{4µ^2 r^2} \right) & r > r_0 \end{cases}. \quad (1.8) \]
which corresponds to a constant density inside the sphere and an exponentially decreasing density with range $\sim 1/\mu$ outside, as desired. Lastly one may evaluate the pressure $P$ \cite{13, 14} obtaining

$$P = \frac{1}{2} \left( \langle \dot{\Phi}^2 \rangle - \frac{1}{3} \langle \dot{\Phi}' \rangle^2 - \mu^2 \Phi^2 - 2 V \Phi^2 \right)$$

$$\sim -m_\phi \mu^3 \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \left\{ \begin{array}{ll} 0 & r < r_0 \\ \frac{2}{3} \frac{r_0^2}{r^2} e^{-2\mu(r-r_0)} \left( 1 + \frac{1}{2\mu r} + \frac{1}{4\mu^2 r^2} \right) & r > r_0 \end{array} \right. \quad (1.9)$$

which is zero inside the sphere and its magnitude decreases exponentially outside.

We may at this point, as in cosmological models, obtain an effective action for $\phi(\tau)$ by substituting the particular spatial function $f(r)$, Eq. (1.5), in the original Eq. (1.2) and integrating over the spatial coordinate $r$:

$$S_\Phi \sim 4\pi \int d\tau \int r^2 dr \frac{1}{2} \left[ (\phi'^2 - \mu^2 \phi^2) \frac{f^2}{r^2} - \phi^2 \left( \frac{f}{r} \right)^2 - 2V \phi^2 \frac{f^2}{r^2} \right]$$

$$\sim \frac{4\pi r_0^3}{3} \int d\tau \left[ \frac{1}{2} (\phi'^2 - \mu^2 \phi^2) + O \left( \frac{1}{\mu r_0} \right) \right] \equiv S_\phi + O \left( \frac{1}{\mu r_0} \right). \quad (1.10)$$

Thus, unless one has $r_0 \sim 1/\mu$, the edge effects are negligible as we heuristically indicated before. Of course in the above simple model we have not considered all possible modes for the scalar field, but only the mode corresponding to dust and we have seen that for such a mode upon quantization one obtains, for any state, pressureless dust inside the sphere \cite{2}.

In section 2 we briefly review the effective action used to describe the collapsing sphere of dust along the lines suggested above (which will allow us to maintain the Robertson-Walker form of the metric) and the results obtained in Ref. \cite{2}, in order to prepare the ground for the study of the back-reaction on the geometry induced by quantum evolution (collapse) of the dust. This will require relaxing the adiabatic approximation and is the aim of the present note.

In section 3 the method of adiabatic invariants \cite{16, 17} is illustrated and is applied to solve the matter Schrödinger equation with a time dependent Hamiltonian obtained through the Born-Oppenheimer decomposition of the matter-gravity wave function and in the semiclassical limit for gravitation. After the identification of the parameter describing the adiabatic limit and the associated states, coherent states for the matter wave function having correct classical limits are constructed and used to determine the expectation value of the diverse quantities of physical interest. Since the collapse is adiabatic at some initial time, from the expressions obtained we estimate and compare the subsequent deviations from the adiabatic approximation and see the effect of back-reaction on gravitation. Lastly in section 4 our results are summarized and discussed.

We use units for which $c = 1, \kappa \equiv 8\pi G$, the Planck length is then $\ell_p \equiv \sqrt{\kappa \hbar}$ and the Planck mass is $m_p = \hbar/\ell_p$. 

3
2 Semiclassical collapse of a sphere of dust

As mentioned in the Introduction, in order to describe the evolution of a collapsing sphere of homogeneous dust in vacuum it is convenient to consider a time-dependent scalar field $\phi$ confined inside a spherical portion of a Robertson-Walker space-time (the interior of the sphere) with line element

$$ds^2 = K^2(\eta) \left[ -d\eta^2 + \frac{d\rho^2}{1 - \epsilon \rho^2} + \rho^2 d\Omega^2 \right], \quad 0 \leq \rho \leq \rho_0 ,$$

(2.1)

where $\epsilon = 0, \pm 1$. The radius of the sphere then follows from the matching condition with the external Schwarzschild metric and is given by $r_0 = \rho_0 K$ [2, 10]. The corresponding total effective action (gravity plus matter) is given by

$$S = \frac{1}{2} \int d\tau \left[ -\frac{1}{\kappa} \left( K \dot{K}^2 - \epsilon K \right) + K^3 \left( \dot{\phi}^2 - \mu^2 \phi^2 \right) \right],$$

(2.2)

where $d\tau = K d\eta$ is the proper time of an observer comoving with the dust.

The Hamiltonian obtained from the action in Eq. (2.2) is

$$H = -\frac{1}{2} \left( \frac{\pi_K^2}{K} + \frac{\epsilon}{\kappa} K \right) + \frac{1}{2} \left( \frac{\pi_\phi^2}{K^3} + \mu^2 \phi^2 K^3 \right) \equiv H_G + H_M ,$$

(2.3)

where $\pi_K = -K \dot{K}/\kappa$ and $\pi_\phi = K^3 \dot{\phi}$. Canonical quantization leads to the Wheeler-DeWitt equation,

$$\frac{1}{2} \left[ \frac{\hbar^2}{K^2} \frac{\partial^2}{\partial K^2} - \frac{\epsilon}{\kappa} K - \frac{\hbar^2}{K^3} \frac{\partial^2}{\partial \phi^2} + \mu^2 \phi^2 K^3 \right] \Psi(K, \phi) = 0 ,$$

(2.4)

where we have chosen a suitable operator ordering in the gravitational kinetic term [2]. One now expresses $\Psi$ in the factorized form $\Psi(K, \phi) = \chi(K, \phi)$ which, after multiplying on the l.h.s. of Eq. (2.4) by $\chi^* \chi$ and integrating over the matter degrees of freedom, leads to an equation for the gravitational part [2],

$$\frac{1}{2} \left[ \frac{\hbar^2}{K^2} \frac{\partial^2}{\partial K^2} - \frac{\epsilon}{\kappa} K \right] \langle \chi | \chi \rangle \langle \chi | \chi \rangle \frac{\partial}{\partial K} \left( 1 - \frac{\langle \chi | \chi \rangle}{\langle \chi | \chi \rangle} \right) \langle \chi | \chi \rangle \frac{\partial}{\partial K} \langle \chi | \chi \rangle \psi$$

$$= -\frac{\kappa \hbar^2}{K^2} \langle \tilde{\chi} | \frac{\partial^2}{\partial K^2} | \tilde{\chi} \rangle \psi \equiv -\frac{\kappa \hbar^2}{2} \left( \langle \chi | \chi \rangle \frac{\partial}{\partial K} \langle \chi | \chi \rangle \right) \psi ,$$

(2.5)

where we have defined a scalar product

$$\langle \chi | \chi \rangle \equiv \int d\phi \chi^*(\phi, K) \chi(\phi, K) ,$$

(2.6)

and we have set

$$\psi = e^{-i \int K' A(K') dK'} \tilde{\psi} \quad \chi = e^{+i \int K' A(K') dK'} \tilde{\chi} ,$$

(2.7)
with
\[ A \equiv -i \frac{1}{\langle \chi | \chi \rangle} \langle \chi | \frac{\partial}{\partial K} | \chi \rangle \equiv -i \left\langle \frac{\partial}{\partial K} \right\rangle. \tag{2.8} \]

If we now multiply Eq. (2.5) by $\tilde{\psi}$ and subtract it from Eq. (2.4) we obtain [2]
\[ \tilde{\psi} K \left[ \hat{H}_M - \langle \hat{H}_M \rangle \right] \tilde{\chi} + \kappa h^2 \left( \frac{\partial^2}{\partial K^2} \right) \frac{\partial \tilde{\psi}}{\partial K} = \kappa h^2 \tilde{\psi} \left[ \left\langle \frac{\partial^2}{\partial K^2} \right\rangle - \frac{2}{2} \left\{ \left\langle \frac{\partial}{\partial K} \right\rangle \right\} \right] \tilde{\chi}, \tag{2.9} \]
which is the equation for the matter (scalar field) wave function.

On neglecting the r.h.s. of Eq. (2.5) one may introduce time [19, 4, 5] by writing a semiclassical (WKB) approximation for the wave function $\tilde{\psi}$:
\[ \tilde{\psi} \simeq \left( \frac{\partial S_{\text{eff}}}{\partial K} \right)^{-1/2} e^{i \frac{1}{\hbar} S_{\text{eff}}}, \tag{2.10} \]
where $S_{\text{eff}}$ is the effective action satisfying the Hamilton-Jacobi equation associated with the l.h.s. of Eq. (2.5),
\[ S_{\text{eff}} = -\frac{1}{2\kappa} \int d\tau \left[ K_c \dot{K}_c^2 - \left( \epsilon K_c - 2\kappa \langle \hat{H}_M \rangle \right) \right], \tag{2.11} \]
and $\langle \hat{H}_M \rangle$ is now evaluated for $K = K_c$ which is where $\tilde{\psi}$, Eq. (2.10), has support. One may then define a (proper) time variable,
\[ \frac{\partial \tilde{\psi}}{\partial K} \frac{\partial}{\partial \tau} \simeq \frac{i}{\hbar} \frac{1}{\hbar} \left( \frac{\partial S_{\text{eff}}}{\partial K} \right)^{-1} \frac{\partial^2 S_{\text{eff}}}{\partial K^2} \frac{\partial \tilde{\psi}}{\partial K} \frac{\partial}{\partial \tau} \]
\[ \equiv \frac{i K \tilde{\psi}}{\kappa h} \frac{\partial}{\partial \tau} - \frac{1}{2} \left( \frac{\partial S_{\text{eff}}}{\partial K} \right)^{-1} \frac{\partial^2 S_{\text{eff}}}{\partial K^2} \frac{\partial \tilde{\psi}}{\partial K} \frac{\partial}{\partial \tau}. \tag{2.12} \]

Further if the r.h.s. of Eq. (2.9) and the second term in the r.h.s. of Eq. (2.12) are negligible, one obtains the Schrödinger equation,
\[ i\hbar \frac{\partial \chi_s}{\partial \tau} = \frac{1}{2} \left[ -\frac{h^2}{K_c^3} \frac{\partial^2}{\partial \phi^2} + \mu^2 K_c^3 \phi^2 \right] \chi_s = \hat{H}_M \chi_s, \tag{2.13} \]
where we have scaled the dynamical phase
\[ \chi_s \equiv \tilde{\chi} \exp \left\{ -\frac{i}{\hbar} \int^\tau \langle \hat{H}_M (\tau') \rangle d\tau' \right\}, \tag{2.14} \]
and omitted $\tilde{\psi}$ while setting $K = K_c$, which is where the semiclassical gravitational wave-function has support.

In Ref. [2] Eq. (2.13) was solved by making the adiabatic approximation $\dot{K}_c/K_c \ll 1$ and evaluating $\langle \hat{H}_M \rangle$ for such solutions. Eq. (2.11) then becomes the action of the Oppenheimer-Snyder model [3],
\[ S_{\text{cl}} = -\frac{1}{2\kappa} \int d\tau \left[ K_{cl} \dot{K}_{cl}^2 - \left( \epsilon K_{cl} - 2 K_0 \right) \right], \tag{2.15} \]
with $K_0$ constant and $K_{cl}$ is one of the classical trajectories

$$
\begin{align*}
K_{cl} &= K_0 \partial_\eta h_\epsilon(\eta) \\
\tau &= K_0 h_\epsilon(\eta)
\end{align*}
$$

The matching condition between the inner metric and the external Schwarzschild metric then gives for the mass parameter of the sphere

$$
M/\kappa = K_0 \rho_0^3/\kappa = \langle \hat{H}_M \rangle \rho_0^3 = N m_\phi \rho_0^3 ,
$$

where $N = \langle \hat{H}_M \rangle / m_\phi$ is the (constant) number of dust particles.

Let us emphasize that Eq. (2.17) introduces the size of the ball in the model. If, as suggested in the introduction (as well as in Refs. [2, 15]), boundary effects are negligible, the above is the only difference with respect to cosmological models where there is no exterior metric. Further the adiabatic approximation amounts to $\langle \hat{H}_M \rangle = K_0 \rho_0^3$ being constant, that is the number of dust particles is constant, although the frequency $\omega_{c} = K_{cl} \langle \hat{H}_M \rangle$ of the corresponding Schrödinger state in the conformal time $\eta$ is not [2] (of course in the proper time the frequency $= \langle \hat{H}_M \rangle$ is constant).

In Ref. [2] it was subsequently verified that all the approximations were consistent for the collapse of the sphere up to its horizon radius $r_\mu = 2M$ if the Compton wavelength of the scalar field is much smaller than the Schwarzschild radius of the sphere,

$$
2M \gg \frac{1}{\mu} ,
$$

which is precisely the condition $\mu r_0 > 2\mu M \gg 1$ that one needs in order to neglect edge effects [15] (see Eq. (1.10)).

In the following section we shall obtain solutions to Eq. (2.13) without making the adiabatic approximation, thus allowing for a change in the number of dust particles, and shall estimate the corresponding corrected classical trajectories $K_{c}$ [20].

### 3 Adiabatic invariants and quantized matter

A suitable method for the study of time dependent quantum systems is that of adiabatic invariants. In particular given a time dependent Hamiltonian $\hat{H}_M(\tau)$, a Hermitian operator $\hat{I}(\tau)$ is called an adiabatic invariant if it satisfies [16, 17]

$$
i\hbar \frac{\partial \hat{I}(\tau)}{\partial \tau} + [\hat{I}(\tau), \hat{H}_M(\tau)] = 0 .
$$

The general solutions to the Schrödinger equation,

$$
i\hbar \frac{\partial \chi_s(\tau)}{\partial \tau} = \hat{H}_M(\tau) \chi_s(\tau)
$$

are then given by

$$
| \chi(\tau) \rangle_{Is} = \sum_n C_n e^{i \varphi_n(\tau)} | n, \tau \rangle_I ,
$$
where $|n,\tau\rangle_I$ is an eigenvector of $\hat{I}(\tau)$ with time-independent eigenvalue $\lambda_n$ and the $\{C_n\}$ are complex coefficients. The phase $\varphi_n$ is given by

$$\varphi_n(\tau) = \frac{i}{\hbar} \int_{\tau_0}^{\tau} I(n,\tau') |\hbar \partial_{\tau'} + i \tilde{H}_M(\tau')| n,\tau' \rangle_I d\tau', \quad (3.4)$$

and is the sum of the geometrical phase whose associated connection is given in Eq. (2.8) and the dynamical phase displayed in Eq. (2.14).

The Hamiltonian in Eq. (2.13) corresponds to a harmonic oscillator of fixed frequency $\mu$ and variable mass $K_c^3$. In this case it is useful to introduce the following linear (non-hermitian) invariant

$$\hat{I}_b(\tau) \equiv e^{i\Theta(\tau)} \hat{b}(\tau), \quad (3.5)$$

where the phase $\Theta$ is given by

$$\Theta(\tau) = \int_{\tau_0}^{\tau} \frac{d\tau'}{K_c^3(\tau') x^2(\tau')}, \quad (3.6)$$

and the operator $\hat{b}$ by

$$\hat{b}(\tau) \equiv \frac{1}{\sqrt{2} \hbar} \left[ \frac{\hat{\phi}}{x} + i \left( x \hat{\pi}_\phi - \dot{x} K_c^3 \hat{\phi} \right) \right]. \quad (3.7)$$

The function $x = x(\tau)$ is to be determined as a solution of the nonlinear equation

$$\ddot{x} + \frac{3K_c}{K_c^3} \dot{x} + \mu^2 x = \frac{1}{K_c^6} \frac{1}{x^3}, \quad (3.8)$$

with suitable initial conditions.

The system admits an invariant ground state (vacuum) $|0,\tau\rangle_b$ defined by

$$\hat{I}_b(\tau) |0,\tau\rangle_b = 0, \quad (3.9)$$

and one can define an invariant basis of states $\mathcal{B} = \{|n,\tau\rangle_b\}$ with a Fock space

$$|n,\tau\rangle_{bs} \equiv \frac{(|\hat{b}^\dagger|^n)}{\sqrt{n!}} |0,\tau\rangle_{bs} = e^{-in\Theta} \frac{(|\hat{b}^\dagger|^n)}{\sqrt{n!}} |0,\tau\rangle_b = e^{i\varphi_0(n)} |0,\tau\rangle_b = e^{i\varphi_n(\tau)} |n,\tau\rangle_b, \quad (3.10)$$

where $\varphi_0$ may be replaced by $-\Theta/2$ and

$$\hat{b} |n,\tau\rangle_b = \sqrt{n} |n-1,\tau\rangle_b \quad (3.11)$$

$$\hat{b}^\dagger |n,\tau\rangle_b = \sqrt{n+1} |n+1,\tau\rangle_b .$$
Thus, since \([\hat{b}, \hat{b}^\dagger] = 1\), in the following we will refer to \(\hat{b}\) and its Hermitian conjugate \(\hat{b}^\dagger\) as the \textit{invariant annihilation} and \textit{creation} operators, and introduce the hermitian quadratic invariant

\[
\hat{I}_c \equiv \hbar \left( \hat{b}^\dagger \hat{b} + \frac{1}{2} \right) = \hbar \left( \hat{b}^\dagger \hat{b} + \frac{1}{2} \right),
\]

with \(\hat{b}^\dagger \hat{b}\) the \textit{invariant number} operator.

The particle annihilation and creation operators \(\hat{a}\) and \(\hat{a}^\dagger\) are defined by

\[
\hat{a}(\tau) \equiv \sqrt{\frac{\mu K_c^3}{2\hbar}} \left[ \hat{\phi} + i \frac{\hat{\pi}_\phi}{\mu K_c^3} \right],
\]

and the Hamiltonian operator is then

\[
\hat{H}_M(\tau) = \hbar \mu \left[ \hat{N} + \frac{1}{2} \right],
\]

where \(\hat{N} \equiv \hat{a}^\dagger \hat{a}\) is the \textit{particle number} operator which counts the number of quanta of the scalar field \(\phi\). One also has a corresponding vacuum \(|0, \tau\rangle_a\) defined by \(\hat{a} |0, \tau\rangle_a = 0\) and a complete set of eigenstates \(A = \{|n, \tau\rangle_a\}\),

\[
|n, \tau\rangle_a \equiv \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0, \tau\rangle_a,
\]

such that

\[
\hat{a} |n, \tau\rangle_a = \sqrt{n} |n-1, \tau\rangle_a
\]

\[
\hat{a}^\dagger |n, \tau\rangle_a = \sqrt{n+1} |n+1, \tau\rangle_a,
\]

and \([\hat{a}, \hat{a}^\dagger] = 1\).

The two different Fock space basis are related. Indeed one has

\[
\hat{\phi} = \sqrt{\frac{\mu}{2}} x (\hat{b} + \hat{b}^\dagger)
\]

\[
\hat{\pi}_\phi = \sqrt{\frac{\mu}{2}} \left[ \frac{\dot{x}}{x} (\hat{b}^\dagger - \hat{b}) + \dot{x} K_c^3 (\hat{b}^\dagger + \hat{b}^\dagger) \right],
\]

from which

\[
\begin{align*}
\hat{a} &= B^* \hat{b} + A^* \hat{b}^\dagger \\
\hat{a}^\dagger &= B \hat{b}^\dagger + A \hat{b},
\end{align*}
\]

where

\[
A(\tau) = \frac{1}{2} \sqrt{\mu K_c^3} \left( x - \frac{1}{\mu K_c^3 x} - i \frac{\dot{x}}{\mu} \right)
\]

\[
B(\tau) = \frac{1}{2} \sqrt{\mu K_c^3} \left( x + \frac{1}{\mu K_c^3 x} - i \frac{\dot{x}}{\mu} \right),
\]
are the Bogoliubov coefficients. From the above one has an inverse relation,

\[
\begin{align*}
\hat{b} &= B\hat{a} - A^{\ast}\hat{a}^{\dagger} \\
\hat{b}^{\dagger} &= B^{\ast}\hat{a}^{\dagger} - A\hat{a}.
\end{align*}
\tag{3.20}
\]

The two basis \(A\) and \(B\) will coincide in the adiabatic limit for which the derivatives of \(xK_{c}^{3/2}\) are small. Let us suppose that for \(\tau = \tau_0\) the two sets \(\{n, \tau_0\}_a\) and \(\{n, \tau_0\}_b\) coincide. This is achieved if

\[
\hat{b}(\tau_0) = \hat{a}(\tau_0) \Rightarrow \mu \hat{I}_c(\tau_0) = \hat{H}_M(\tau_0),
\tag{3.21}
\]

which leads to the following initial conditions for the function \(x\)

\[
\begin{align*}
x(\tau_0) &= \left[\mu K_{c}^{3}(\tau_0)\right]^{-1/2} \\
x'(\tau_0) &= 0.
\end{align*}
\tag{3.22}
\]

In particular, the second condition in Eq. (3.22) guarantees that the state of the system satisfies adiabaticity in the limit \(\tau \to \tau_0\).

In order to study deviations from adiabaticity it is convenient to introduce \(\sigma = (\mu K_{c}^{3})^{1/2} x\) whereupon Eq. (3.8) becomes

\[
\frac{1}{\mu^{2}}\ddot{\sigma} + \left(1 - \frac{3\tilde{K}_{c}}{2\mu^{2}K_{c}} - \frac{3\tilde{K}_{c}^{2}}{4\mu^{2}K_{c}^{2}}\right)\sigma \equiv \frac{1}{\mu^{2}}\ddot{\sigma} + \Omega^{2}\sigma = \frac{1}{\sigma^3}.
\tag{3.23}
\]

As a first approximation [2], which we shall return to when we consider the effect of back-reaction, we set \(K_c \simeq K_{cl}\) which amounts to taking \(\langle \tilde{H}_M \rangle\) as constant and equal to \(K_0/\kappa\). This may be assumed to occur for the time \(\tau_0\) for which the adiabatic approximation is valid (\(\hat{a}\) and \(\hat{b}\) coincide). One then obtains

\[
\Omega = \left[1 + \frac{3\epsilon\delta^{2}}{4(\partial_{\eta}h_{\epsilon})^{2}}\right]^{1/2} \simeq \left[1 + \frac{3\epsilon\delta^{2}}{8(\partial_{\eta}h_{\epsilon})^{2}}\right],
\tag{3.24}
\]

where \(\delta = (\mu K_0)^{-1}\). In the above, \(\delta\) plays the role of an adiabaticity parameter which connects the departure from adiabaticity to the time as the collapse proceeds \((\tau < \tau_0)\). Indeed the collapse is adiabatic for \(\delta \to 0\), in agreement with Eq. (2.18). For the case \(\epsilon = 0\), one has \(\Omega = 1\) and the exact solutions of the non-linear Eq. (3.23) (with \(K_c = K_{cl}\)) can be obtained from the expressions displayed below by setting \(\epsilon = 0\). For a general value of \(\epsilon (= 0, \pm 1)\) the solutions to Eq. (3.23) can be expanded in positive powers of \(\delta^2\) [21],

\[
\sigma = \sum_{n=0}^{\infty} \delta^{2n} \sigma_n,
\tag{3.25}
\]

where \(\sigma_n\) may be expressed in terms of \(\Omega\) and its derivatives with respect to \(\tau\). To the lowest order in \(\delta^2\) one has

\[
\sigma = \Omega^{-1/2} + \mathcal{O}(\delta^4) \simeq 1 - \frac{3\epsilon}{16\mu^{2}K_{cl}^{2}}.
\tag{3.26}
\]
Further, given a particular solution \( \tilde{\sigma} \) for \( \sigma \) (and correspondingly for \( x \)), the general solution to the classical equation for \( \phi \) obtained from Eq. (2.2),

\[
\ddot{\phi}_c + 3 \frac{K_c}{\dot{K}_c} \dot{\phi}_c + \mu^2 \phi = 0 ,
\]  

(3.27)
is [22]

\[
\phi_c = \frac{\tilde{\sigma}}{\sqrt{\mu}} K_c^{-\frac{3}{2}} (D \cos \Theta + E \sin \Theta) ,
\]  

(3.28)

where \( D \) and \( E \) are constants and \( \Theta \) is given in Eq. (3.6). We observe that this solution, obtained here through the adiabatic expansion of Ref. [21], actually coincides with the WKB-type solution of chapter 3.5 of Ref. [23]. In fact, the non-linear equation (3.103) in [23] can be recovered from our Eq. (3.23) by defining \( W = 1/\sigma^2 \) and the vacuum corresponding to \( \sigma \) as given above is the second adiabatic order vacuum in the terminology of Ref. [23].

The introduction of the eigenstates of adiabatic invariants allows one to consider the effect of particle production due to the variation of the metric on the evolution of matter. In order to see this, it is convenient to introduce coherent states in the \( \hat{b} \) modes,

\[
\begin{align*}
| \alpha, \tau \rangle_{bs} &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n, \tau \rangle_b ,
\end{align*}
\]  

(3.29)

where \( \alpha = u + iv \) is an arbitrary constant. We note that the modulus squared of the coefficient of \( | n, \tau \rangle_b \) in the above equation satisfies a Poisson distribution with a maximum at \( n = |\alpha|^2 \), average value \( |\alpha|^2 \) and standard deviation \( |\alpha| \). If for some time \( \tau_o \) the adiabatic approximation is valid then the above Eq. (3.29) will also be a coherent state in the \( A \) basis. However, subsequently it will correspond to a squeezed state in the \( A \) basis, since the \( B \) basis is related to the \( A \) basis through a Bogolubov transformation Eq. (3.20).

One may evaluate the expectation value of the diverse quantities of physical interest with respect to the above state. On defining \( \langle \hat{O} \rangle_{bs} \equiv \langle \alpha, \tau | \hat{O} | \alpha, \tau \rangle_{bs} \) one obtains

\[
\phi_c \equiv \langle \hat{\phi} \rangle_{bs} = \sqrt{2} \hbar |\alpha| x \cos(\Theta - \beta) , \quad \tan \beta = v/u ,
\]  

(3.30)
in agreement with Eq. (3.28). Similarly for the momentum \( \pi_{\phi} \) one obtains

\[
\pi_{\phi,c} \equiv \langle \hat{\pi}_{\phi} \rangle_{bs} = \sqrt{2} \hbar |\alpha| \left[ K_c^2 \dot{x} \cos(\Theta - \beta) - \frac{1}{x} \sin(\Theta - \beta) \right] .
\]  

(3.31)

Further,

\[
\langle (\Delta \hat{\phi})^2 \rangle_{bs} \equiv \langle (\hat{\phi} - \phi_c)^2 \rangle_{bs} = \frac{\hbar}{2} x^2
\]

(3.32)

\[
\langle (\Delta \hat{\pi}_\phi)^2 \rangle_{bs} \equiv \langle (\hat{\pi}_\phi - \pi_{\phi,c})^2 \rangle_{bs} = \frac{\hbar}{2} \left( \frac{1}{x^2} + K_c^6 \dot{x}^2 \right) ,
\]

both of which are independent of \( \alpha \) and lead to

\[
\langle (\Delta \hat{\phi})^2 \rangle_{bs}^{1/2} \langle (\Delta \hat{\pi}_\phi)^2 \rangle_{bs}^{1/2} = \frac{\hbar}{2} \sqrt{1 + K_c^6 x^2 \dot{x}^2} .
\]  

(3.33)
Again this result is independent of $\alpha$ and we note that the uncertainty relation above is a minimum in the adiabatic approximation when the two Fock spaces $A$ and $B$ coincide. One also has

$$\langle \hat{I}_c \rangle_{bs} = \hbar \left( |\alpha|^2 + \frac{1}{2} \right)$$

(3.34)

$$\langle \hat{H}_M(\tau) \rangle_{bs} = \frac{1}{2 K_c^3} \left[ \langle \hat{\pi}^2_\phi \rangle_{bs} + \mu^2 K_c^6 \langle \hat{\phi}^2 \rangle_{bs} \right]$$

$$= \frac{1}{2 K_c^3} \left[ \frac{\hbar}{2} \left( \hat{x}^2 K_c^6 + \frac{1}{x^2} + \mu^2 x^2 K_c^6 \right) + \langle \hat{\pi}_\phi \rangle_{bs}^2 + \mu^2 K_c^6 \langle \hat{\phi} \rangle_{bs}^2 \right]$$

(3.35)

and, of course, in the adiabatic approximation,

$$\mu \langle \hat{I}_c \rangle_{bs} \simeq \langle \hat{H}_M(\tau) \rangle_{bs}.$$  (3.36)

It is of particular interest to examine the behaviour of the above quantities when one has (small) deviations from the adiabatic approximation. This may be achieved by considering corrections of $O(\delta^2)$ to the adiabatic approximation and will allow us to see the effect of matter not following gravitation adiabatically and the corresponding back-reaction on gravitation. In particular one has

$$\phi_c \simeq |\alpha| \left[ \frac{2 \hbar}{\mu K_0 (\partial_\eta h_\epsilon)} \right]^{1/2} \left[ 1 - \frac{3 \epsilon \delta^2}{16 \mu^2 (\partial_\eta h_\epsilon)^2} \right] \cos(\Theta - \beta),$$

(3.37)

where now

$$\Theta \simeq \frac{1}{\delta} \int_{\eta_0(\gamma_0)}^{\eta} d\zeta \partial_\zeta h_\epsilon \left[ 1 + \frac{3 \epsilon \delta^2}{4 (\partial_\zeta h_\epsilon)^2} \right]^{1/2}.$$  (3.38)

Further,

$$\pi_{\phi,c} \simeq -\sqrt{\frac{2 \hbar}{\delta}} |\alpha| K_0 (\partial_\eta h_\epsilon)^{3/2} \left\{ \left[ 1 + \frac{3 \epsilon \delta^2}{16 (\partial_\eta h_\epsilon)^2} \right] \sin(\Theta - \beta) \right.$$ 

$$+ \frac{3}{2} \frac{\delta \partial_\eta^2 h_\epsilon}{(\partial_\eta h_\epsilon)^2} \cos(\Theta - \beta) \right\},$$

(3.39)

and

$$\langle (\Delta \hat{\phi})^2 \rangle_{bs} \simeq \frac{\hbar \delta}{2 K_0^2 (\partial_\eta h_\epsilon)^3} \left[ 1 - \frac{3 \epsilon \delta}{8 (\partial_\eta h_\epsilon)^2} \right]$$

(3.40)

$$\langle (\Delta \hat{\pi}_\phi)^2 \rangle_{bs} \simeq \frac{\hbar}{2 \delta} K_0^2 (\partial_\eta h_\epsilon)^3 \left\{ 1 + \frac{3 \delta^2}{4 (\partial_\eta h_\epsilon)^2} \left[ \frac{\epsilon}{2} + 3 \frac{(\partial_\eta^2 h_\epsilon)^2}{(\partial_\eta h_\epsilon)^2} \right] \right\}$$

(3.41)

$$\langle (\Delta \hat{\phi})^{1/2} \rangle_{bs} \langle (\Delta \hat{\pi}_\phi)^{1/2} \rangle_{bs} \simeq \frac{\hbar}{2} \left[ 1 + \frac{9 \delta^2 (\partial_\eta^2 h_\epsilon)^2}{(\partial_\eta h_\epsilon)^4} \right]^{1/2}.$$  (3.42)
Also, the expectation value of the matter Hamiltonian is given by
\[
\langle \hat{H}_M(\tau) \rangle_{bs} \simeq \frac{\hbar \mu}{2} + \hbar \mu |\alpha|^2 \left\{ \left[ \sin(\Theta - \beta) + \frac{3}{2} \delta \frac{\partial^2 \hat{c}}{\partial \eta \partial \hat{c}} \cos(\Theta - \beta) \right]^2 + \cos^2(\Theta - \beta) \right\}
\]
\[
= \hbar \mu \left( |\alpha|^2 + \frac{1}{2} \right) + 3 \hbar \mu |\alpha|^2 \delta \frac{\partial^2 \hat{c}}{\partial \eta \partial \hat{c}} \sin(\Theta - \beta) \cos(\Theta - \beta)
\]
\[
= \frac{K_0}{\kappa} + \Delta \langle \hat{H}_M \rangle_{bs},
\]
(3.43)
where only the first term $K_0/\kappa$ on the r.h.s. of Eq. (3.43) survives in the adiabatic approximation and the second term is associated with particle production.

In order to better understand our results, in particular Eq. (3.42) and Eq. (3.43), we may consider $\eta^2$ small and $\gg \delta$ corresponding to $\mu^{-1} \ll \tau \ll \tau_0$, which implies that we are almost adiabatic (we only consider small deviations from adiabaticity). It is then straightforward to see that, in such a case, from Eq. (3.42),
\[
\langle (\Delta \hat{\phi})^2 \rangle_{bs}^{1/2} \langle (\Delta \hat{\pi}_\phi)^2 \rangle_{bs}^{1/2} \simeq \frac{\hbar}{2} \left[ 1 + \mathcal{O} \left( \frac{\delta^2}{\eta^3} \right) \right],
\]
(3.44)
which of course shows that as $\eta$ becomes smaller (during the collapse), matter becomes less and less classical. On the other hand, from Eq. (3.43) one obtains
\[
\Delta \langle \hat{H}_M \rangle_{bs} \simeq \hbar \mu |\alpha|^2 \mathcal{O} \left( \frac{\delta}{\eta^3} \right),
\]
(3.45)
again showing that, as $\eta$ becomes small, one has an increasing production of matter. Clearly such a production will induce a back-reaction on gravitation and $K_c$ will change [24] from $K_{cl}$ to $K_{cl} + \Delta K_c$, where $\Delta K_c$ is $K_0 \mathcal{O}(\delta/\eta^3)$. Since we are considering an exterior Schwarzschild metric and the time at which the horizon is crossed by the last shell of matter is given by $2M = \rho_0 K_c(\eta_H)$, the presence of the additional term $\Delta K_c$ will lead to $\rho_0 K_{cl}(\eta_H) \simeq 2M - \rho_0 K_0 \mathcal{O}(\delta/\eta^3)$. This of course implies that the horizon is crossed at a later time (smaller value of $\eta$) and that the back-reaction (matter production) slows down the collapse. The effect of particle creation on a Friedmann-like collapse and the possible avoidance of the cosmological singularity due to quantum effects have been studied in Ref. [14].

In obtaining the above results we have neglected the r.h.s. of Eqs. (2.5)-(2.9) which are associated with fluctuations in the particle production due to the variation of the metric and the term $\mathcal{O}(\hbar)$ in our introduction of time in Eq. (2.12). Let us estimate these effects and compare them with particle production. For this purpose, order of magnitude estimates of the r.h.s. are sufficient. They can be obtained by just evaluating the r.h.s. for a state $| n = N, \tau \rangle_b$ such that $N = |\alpha|^2$, since it is for this value that the (modulus) coefficients of the expansion in Eq. (3.29) are peaked and corresponds to the amplitude of the classical oscillations (see Eq. (3.30)). For the r.h.s. of Eq. (2.5) one obtains
\[
\kappa \hbar^2 \langle N | \frac{\partial}{\partial N} (1 - | N \rangle \langle N |) \frac{\partial}{\partial N} | N \rangle = \frac{K}{K_2} \sum_{L \neq N} \langle N | \hat{H}_M | L \rangle \langle L | \hat{H}_M | N \rangle
\]
\[
\simeq K_{cl} \Delta \langle \hat{H}_M \rangle_{bs} \mathcal{O} \left( \frac{\delta}{\eta^3} \right),
\]
(3.46)
whereas for the modulus of the r.h.s. of Eq. (2.9) one has
\[
\left| \frac{\hbar^2 \kappa}{2} \left[ \left( \frac{\partial^2}{\partial K^2} - \langle N | \frac{\partial^2}{\partial K^2} | N \rangle \right) - 2\langle N | \frac{\partial}{\partial K} | N \rangle \left( \frac{\partial}{\partial K} - \langle N | \frac{\partial}{\partial K} | N \rangle \right) \right] | N \rangle \right|
\]
\[
= \frac{\hbar^2 \kappa}{2} \sum_{L \neq N} \frac{| L \rangle}{\hbar(N - L)} \left[ \langle L | \frac{\partial^2}{\partial K^2} | N \rangle + \sum_{P \neq N, L} \hbar(L - P) \langle L | \hat{H}_M | P \rangle \langle P | \hat{H}_M | N \rangle \right] \right|
\]
\[
\simeq \kappa_{cl} \Delta \langle \hat{H}_M \rangle_{ba} \mathcal{O} \left( \frac{\delta}{\eta^2} \right). \tag{3.47}
\]

Further, the corrections of \( \mathcal{O}(\hbar) \) to Eq. (2.12) lead to an additional term for the Hamilton-Jacobi equation associated with Eq. (2.5),
\[
\frac{\kappa \hbar^2}{2} \left[ \frac{3}{4} \frac{\dot{\pi}_K}{K^2 \pi_K^2} - \frac{\pi_K}{2 K^2 \pi_K} + \frac{\tilde{K} \pi_K}{2 K^3 \pi_K} \right] \simeq \frac{1}{|\alpha|^2} K_{cl} \Delta \langle \hat{H}_M \rangle_{ba} \mathcal{O} \left( \frac{\delta}{\eta^2} \right), \tag{3.48}
\]
whereas the modulus of the corresponding contribution to the r.h.s. of Eq. (2.9) is (omitting \( \tilde{\psi} \))
\[
\kappa \hbar^2 \left| \frac{\dot{\pi}_K}{2 K \pi_K} \left( \frac{\partial}{\partial K} - \langle N | \frac{\partial}{\partial K} | N \rangle \right) | N \rangle \right| \simeq K_{cl} \Delta \langle \hat{H}_M \rangle_{ba} \mathcal{O} \left( \frac{\delta}{\eta^2} \right), \tag{3.49}
\]
which is again of higher order with respect to \( K_{cl} \Delta \langle \hat{H}_M \rangle_{ba} \).

The above results Eqs. (3.46)-(3.47) are associated with particle production due to the metric variations. In particular we see from Eq. (3.44) that, as the collapse proceeds the evolution of matter becomes less and less classical and one has the production of particles (Eq. (3.45)) which, as a consequence, slows down the collapse (back-reaction). Further one has lower order contributions, Eqs. (3.46)-(3.47), which are associated with fluctuations in particle productions and affect both the evolution of gravitation and matter and corrections to the classical gravitational motion (see Eq. (3.48)).

4 Conclusions

The matter-gravity system lends itself to a study analogous to that employed in molecular-dynamics where one also has two mass (or time) scales. We previously applied this analogy to the collapse of a sphere of dust in the adiabatic approximation, with the matter equation of motion being solved while freezing the gravitational degrees of freedom. In this note we have generalized such a treatment, to allow for the time dependence induced in the matter Hamiltonian by the time variation of the gravitational (metric) degree of freedom.

Time has been introduced by considering the semiclassical (WKB) limit for gravitation while the matter equation of motion was solved by neglecting fluctuations and employing the method of adiabatic invariants. Such invariants, in contrast with the Hamiltonian, have time-independent eigenvalues. In particular, in our case, which corresponds to a harmonic oscillator with time dependent mass, on freezing the gravitational degree of freedom one reproduces the adiabatic results [2]. The adiabatic invariants are then immediately related to the usual harmonic oscillator (particle) creation and destruction operators and its Hamiltonian.
In general the (non-hermitian) adiabatic invariants are related to the usual creation and destruction operators through a Bogolubov transformation and therefore do not correspond to the latter. In analogy with the particle destruction and creation operators, however, one can construct coherent states of the adiabatic invariants. We have also evaluated the expectation values of diverse quantities of physical interest with respect to such states and shown, for example, that the expectation value of the scalar field operator is a solution to the classical equation of motion.

In order to obtain some estimate of the corrections to the adiabatic approximation we have determined the diverse quantities to lowest order in \( \delta \) (the ratio of the Compton wavelength of the scalar particle to the Schwarzschild radius of the black hole) and for small enough times \( \eta \) (nonetheless satisfying \( \eta^3 > \delta \)). This has shown that the dominant term \( (O(\delta/\eta^3)) \) comes from particle creation in the average matter Hamiltonian, whereas the corrections due to the fluctuations in the number of produced particles, in the uncertainty principle or in the semiclassical approximation are of higher order \( (O(\delta^2/\eta^6)) \). To this order of approximation (see Eq. (3.45)) there is no distinction between the different three geometries \( (\epsilon = 0, \pm 1) \).

Clearly the production of matter will influence the collapse and one may consider the associated back-reaction. As we have mentioned, the leading term is of order \( O(\delta/\eta^3) \). It comes from the expectation value of the matter Hamiltonian with respect to the adiabatic invariant coherent states and we have seen that it leads to a slowing down of the collapse, as expected. It would be of interest to also consider the effect of matter being radiated away, which would need the extension of our results to a Vaidya, rather than a Schwarzschild metric.

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