ON DOUBLE POISSON STRUCTURES ON COMMUTATIVE ALGEBRAS

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ABSTRACT. Double Poisson structures (à la Van den Bergh) on commutative algebras are considered. The main result shows that there are no non-trivial such structures on polynomial algebras of Krull dimension greater than one. For an arbitrary commutative algebra \( A \), this places significant restrictions on possible double Poisson structures. Exotic double Poisson structures are exhibited by the case of the polynomial algebra on a single generator, previously considered by Van den Bergh.

1. INTRODUCTION

The notion of a double Poisson structure on an associative \( R \)-algebra (for \( R \) a commutative unital ring) was introduced by Van den Bergh as a form of non-commutative Poisson structure [VdB08]; the structure is defined by a double bracket, which is an \( R \)-linear map \( A^{\otimes 2} \to A^{\otimes 2} \) satisfying antisymmetry and non-commutative derivation conditions. For a double Poisson structure, the double bracket also satisfies the double Jacobi relation (see Sections 2 and 3).

The naïve relationship with (commutative) Poisson structures is as follows: composing with the multiplication map of \( A \) gives a bracket \( A \otimes A \to A \) which induces a Poisson structure on the abelianization of \( A \). More generally, a double Poisson structure induces a Poisson structure on the associated representation schemes [VdB08]. There is a related, but weaker, version of non-commutative Poisson structure, due to Crawley-Boevey [CB11]; this is sufficient to induce a Poisson structure on the representation schemes.

The notion of double Poisson structure is very rigid; nevertheless, interesting examples are known, for example those related to non-commutative symplectic structures. Moreover, a classification of certain double Poisson structures on free associative algebras (tensor algebras) has been given in small rank [ORS13, Sok13]; however, a double Poisson structure on a non-commutative algebra does not in general induce a double Poisson on its abelianization.

It is natural to consider what happens when the algebra \( A \) is already commutative. For example, Van den Bergh stated a classification of (homogeneous) double Poisson structures on the polynomial algebra \( \mathbb{k}[t] \) over a field: up to scalar, there are only two non-trivial (homogeneous) structures. A proof of the corresponding result (over a commutative ring \( R \) on which squaring is injective) is given here as Proposition A.1, also giving the (essentially unique) non-homogeneous example. This provides important exotic examples.

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For higher Krull dimension the situation is more dramatic; the following is Theorem [5.7] below:

**Theorem 1.** Let $R$ be a commutative ring on which the squaring map $x \mapsto x^2$ is injective, then there is no non-trivial double Poisson structure on $A := R[t_1, \ldots, t_d]$ for $d \geq 2$.

The result is a simple consequence of a general structure result on multi-derivations (see Theorem [4.3]); these multi-derivations (defined in Section 2) correspond to the $n$-brackets of Van den Bergh, except that the ‘anti-equivariance’ condition with respect to the action of the cyclic group $\mathbb{Z}/n$ is not imposed.

This highlights the fact that, on commutative algebras, the axioms of a double Poisson structure are highly restrictive and provides further evidence that the notion of double Poisson structure should be relaxed, considering weaker structures such as Crawley-Boevey’s non-commutative Poisson structures.

Other authors have observed that it is useful to relax the axioms of double Poisson algebras (see [Art15], for example); it is however desirable (from the computational viewpoint) to retain the multi-derivation property, so the general structure result, Theorem [4.3] applies in this setting. Corollary [4.7] shows that, in the polynomial case (of Krull dimension greater than one), this relaxation is not sufficient to be able to construct non-trivial non-commutative Poisson structures (in the sense of [CB11]).

Section 6 considers the general case of double Poisson structures on a commutative algebra. These are either standard, arising from double brackets on polynomial algebras, or are exotic. The results for polynomial algebras give a reasonable understanding of the standard double Poisson structures; the exotic case is illustrated by the results for $\mathbb{K}[t]$, as indicated above. Further consequences will be considered elsewhere.

Various other lines of investigation are possible. For instance, the work of Berest, Ramadoss et al. [BKR13, BCER12] suggests that double Poisson structures for algebras should be studied in the derived setting.

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## 2. Multi-derivations

Fix a commutative unital ring $R$ and a unital, associative $R$-algebra $A$; all tensor products are taken over $R$. For $2 \leq n \in \mathbb{N}$, the symmetric group $\mathfrak{S}_n$ acts by permutations on the tensor product $A^\otimes n$ ($\sigma(a_1 \otimes \cdots \otimes a_n) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$) and, hence, by conjugation on $\text{Hom}_R(A^\otimes n, A^\otimes n)$ via $\varphi \mapsto \sigma \circ \varphi := \sigma \circ \varphi \circ \sigma^{-1}$, so that a linear map $\varphi$ is $\mathfrak{S}_n$-equivariant if and only if it is fixed under this action.

The group $\mathbb{Z}/n$ is considered as a subgroup of $\mathfrak{S}_n$, hence the above action restricts to $\mathbb{Z}/n$.

The $R$-module of double derivations $\text{Der}(A)$ is by definition the submodule

$$\text{Der}(A, A^\otimes 2) \subset \text{Hom}_R(A, A^\otimes 2)$$

of derivations, where $A^\otimes 2$ is equipped with the outer bimodule structure; explicitly $\psi \in \text{Hom}_R(A, A^\otimes 2)$ belongs to $\text{Der}(A)$ if and only if, for all $a, b \in A$, $\psi(ab) = (a \otimes 1)\psi(b) + \psi(a)(1 \otimes b)$, using the product in $A^\otimes 2$. (See [Gin05], for example.)
Example 2.1. The double derivation $d_A \in \mathcal{D}(A)$ is the $R$-linear map $a \mapsto a \otimes 1 - 1 \otimes a$. This induces the universal derivation, $A \to \Omega^n_A$, where the bimodule $\Omega^n_A$ of non-commutative differentials is identified as the kernel of the multiplication $A \otimes A \xrightarrow{\mu} A$.

Lemma 2.2. Let $A$ be a commutative $R$-algebra. Multiplication at the codomain $A^\otimes 2$ induces a morphism of $R$-modules:

$$\mathcal{D}(A) \otimes A^\otimes 2 \to \mathcal{D}(A).$$

In particular, the double derivation $d_A$ gives rise to the morphism of $R$-modules:

$$\Pi : A^\otimes 2 \to \mathcal{D}(A)$$

sending $\Theta \in A^\otimes 2$ to $a \mapsto (a \otimes 1 - 1 \otimes a)\Theta$.

Proof. Straightforward. □

Remark 2.3. This result does not require that $A$ is a commutative and corresponds to the usual $A$-bimodule structure on $\mathcal{D}(A)$ provided by the inner bimodule structure of $A^\otimes 2$. This formulation is given for ease of comparison with Lemma 2.7 (where commutativity is required).

By analogy with the case of double derivations, $\varphi \in \operatorname{Hom}_R(A^\otimes n, A^\otimes n)$ is said to be a derivation with respect to the last variable if, $\forall a,b \in A$ and $\forall \alpha \in A^{\otimes n-1}$:

$$\varphi(a \otimes ab) = (a \otimes 1^{\otimes n-1})\varphi(ab) + \varphi(ab)(1^{\otimes n-1} \otimes b),$$

using the product of $A^\otimes n$. This allows the following definition of multi-derivations, where the $\mathbb{Z}/n$-action is used to define the relevant bimodule structures.

Definition 2.4. For $2 \leq n \in \mathbb{N}$, the $R$-module of multi-derivations

$$\mathcal{M}(A^\otimes n, A^\otimes n) \subset \operatorname{Hom}_R(A^\otimes n, A^\otimes n)$$

is the submodule of morphisms $\varphi$ such that $\sigma \cdot \varphi$ is a derivation with respect to the last variable, for every $\sigma \in \mathbb{Z}/n$.

Let

$$\mathcal{M}(A^\otimes n, A^\otimes n)^{sgn} \subset \mathcal{M}(A^\otimes n, A^\otimes n)$$

denote the sub $R$-module of multi-derivations $\varphi$ such that $\sigma \cdot \varphi = (-1)^{\text{sgn}(\sigma)}\varphi$, $\forall \sigma \in \mathbb{Z}/n$.

The following is clear from the definition:

Lemma 2.5. The sub $R$-modules

$$\mathcal{M}(A^\otimes n, A^\otimes n)^{sgn} \subset \mathcal{M}(A^\otimes n, A^\otimes n) \subset \operatorname{Hom}_R(A^\otimes n, A^\otimes n)$$

are stable under the action of $\mathbb{Z}/n$.

Remark 2.6. For $2 \leq n \in \mathbb{N}$, $\mathcal{M}(A^\otimes n, A^\otimes n)^{sgn}$ is the $R$-module of $n$-brackets (in the terminology of [VdB08, Definition 2.2.1]). In particular, for $n = 2$, this gives the definition of a double bracket, namely an anti-symmetric bi-derivation and, for $n = 3$, triple brackets are multi-derivations which are cyclically invariant.

The following results provide analogues of Lemma 2.2.
Lemma 2.7. Let $A$ be a commutative $R$-algebra and $2 \leq n \in \mathbb{N}$. Multiplication in the codomain induces a morphism of $R[\mathbb{Z}/n]$-modules

$$\text{MDer}(A^\otimes n, A^\otimes n) \otimes A^\otimes n \rightarrow \text{MDer}(A^\otimes n, A^\otimes n),$$

where the left hand side is equipped with the diagonal $\mathbb{Z}/n$-action.

Proof. Straightforward. □

Proposition 2.8. Let $A$ be a commutative $R$-algebra and $2 \leq n \in \mathbb{N}$. The map $\varphi_n \in \text{Hom}_R(A^\otimes n, A^\otimes n)$ defined by

$$\varphi_n(a_1 \otimes \ldots \otimes a_n) := \prod_{\sigma \in \mathbb{Z}/n} \sigma(a_{\sigma(n)} \otimes 1^\otimes n - 1^\otimes n - a_{\sigma(n)})$$

(where the product is formed in $A^\otimes n$) is a $\mathbb{Z}/n$-equivariant multi-derivation (that is $\varphi_n \in \text{MDer}(A^\otimes n, A^\otimes n)_{\mathbb{Z}/n}$).

In particular, $\varphi_n$ together with the map of Lemma 2.7 induce a $\mathbb{Z}/n$-equivariant map:

$$\Pi_n : A^\otimes n \rightarrow \text{MDer}(A^\otimes n, A^\otimes n).$$

Proof. That $\varphi_n$ is $\mathbb{Z}/n$-equivariant is clear from the construction. The proof that it is a multi-derivation is analogous to the proof that $d_A$ is a double derivation. The final statement is then an immediate consequence of Lemma 2.7. □

Remark 2.9. For $n = 2$, $\varphi_2(x \otimes y) = (x \otimes 1 - 1 \otimes x)(y \otimes 1 - 1 \otimes y)$ is clearly invariant under exchange of $x$ and $y$. This reflects the fact that, when $A$ is commutative, $A^\otimes 2$ has a canonical bimodule structure given by the algebra structure.

For $n = 3$, $\varphi_{a,b,c} := \varphi_3(a \otimes b \otimes c)$ is given explicitly by:

$$\varphi_{a,b,c} = (c \otimes 1 \otimes 1 - 1 \otimes 1 \otimes c)(1 \otimes 1 \otimes b - 1 \otimes b \otimes 1)(1 \otimes a \otimes 1 - a \otimes 1 \otimes 1)$$

$$= ac \otimes b \otimes 1 - ac \otimes 1 \otimes b - c \otimes ab \otimes 1 + c \otimes a \otimes b - a \otimes b \otimes c$$

$$+ a \otimes 1 \otimes bc + 1 \otimes ab \otimes c - 1 \otimes a \otimes bc,$$

where the terms have been arranged using the left lexicographical order for the partial order corresponding to the number of terms in a monomial in $a, b, c$ of $A$. (Observe that there is a unique term of maximal lexicographical order, namely $ac \otimes b \otimes 1$.)

The expression for $\varphi_3(a \otimes b \otimes c)$ is normalized (up to sign) by the choice of the bimodule structure of $A^\otimes 3$, corresponding to the factor $(c \otimes 1 \otimes 1 - 1 \otimes 1 \otimes c)$. Although $\varphi_3$ is invariant under the action of $\mathbb{Z}/3$, $\varphi_3$ evaluated on $b \otimes c \otimes a$ clearly gives a different expression, contrary to the behaviour for $n = 2$.

2.1. Graded algebras. When $A$ is an $\mathbb{Z}$-graded $R$-algebra it is natural to consider the graded components of multi-derivations.

Remark 2.10. The grading is not taken into account in the symmetric monoidal structure on graded $R$-modules.

Lemma 2.11. For $A$ a $\mathbb{Z}$-graded $R$-algebra which is finitely-generated as a graded algebra and $2 \leq n \in \mathbb{N}$, there is a $R[\mathbb{Z}/n]$-equivariant decomposition into homogeneous components:

$$\text{MDer}(A^\otimes n, A^\otimes n) \cong \bigoplus_t \text{MDer}(A^\otimes n, A^\otimes n)^t$$

where $\text{MDer}(A^\otimes n, A^\otimes n)^t$ is the submodule of morphisms of degree $t$. 
In particular, any element $\varphi \in \mathcal{M} \text{Der}(A^\otimes n, A^\otimes n)$ can be written in terms of homogeneous components $\varphi = \sum_{t \in \mathbb{Z}} \varphi^t$, where $\varphi^t = 0$ for $|t| \gg 0$.

Proof. The multi-derivation property and the fact that $A$ is assumed to be finitely-generated implies that an element $\varphi \in \mathcal{M} \text{Der}(A^\otimes n, A^\otimes n)$ is determined by its restriction to $(V)^\otimes n$ for $V$ a finitely-generated graded $R$-submodule of $A$, so that $(V)^\otimes n$ is a finitely-generated $R$-module. The proof is then straightforward. \qed

Notation 2.12. For $A, \varphi \neq 0$ as in Lemma 2.11, write $\varphi^{\text{min}}$ and $\varphi^{\text{max}}$ respectively for the non-trivial homogeneous components of minimal and maximal degrees.

3. Recollections on double Poisson algebras

Definition 3.1. For $\varphi \in \text{Hom}_R(A^\otimes 2, A^\otimes 2)$, the double Jacobiator $\text{Jac} \varphi$ is

$$
\text{Jac} \varphi := \sum_{\sigma \in \mathbb{Z}/3} \sigma \cdot ((\varphi \otimes 1_A) \circ (1_A \otimes \varphi)) \in \text{Hom}_R(A^\otimes 3, A^\otimes 3)_{\mathbb{Z}/3}.
$$

A basic fact is the following (using the terminology of $n$-brackets recalled in Remark 2.6).

Proposition 3.2. [VdB08, Proposition 2.3.1] If $\varphi$ is a double bracket on $A$, then $\text{Jac} \varphi$ is a triple bracket.

Definition 3.3. [VdB08] A double Poisson structure on $A$ is a double bracket $\{ \cdot , \cdot \} : A \otimes A \rightarrow A \otimes A$, $a \otimes b \mapsto \{ a, b \}$, such that the double Jacobiator $\{ \cdot , \cdot \} := \text{Jac} \{ \cdot , \cdot \}$ is zero (the double Jacobi relation).

Remark 3.4. For $A$, $\{ \cdot , \cdot \}$ a double Poisson algebra, the bracket

$$
\{ \cdot , \cdot \} : A^\otimes 2 \rightarrow A
$$

defined as the composite of $\{ \cdot , \cdot \}$ with the product of $A$ is a left Leibniz algebra, by [VdB08, Corollary 2.4.4]. Moreover, [VdB08, Proposition 1.4] implies that, if $A$ is commutative, $\{ \cdot , \cdot \}$ defines a Poisson algebra structure on $A$.

3.1. Graded algebras. As in Section 2.1, let $A$ be a $\mathbb{Z}$-graded $R$-algebra which is finitely-generated as a graded algebra.

Definition 3.5. A double Poisson structure $\{ \cdot , \cdot \}$ on the graded algebra $A$ is homogeneous if $\{ \cdot , \cdot \} = \{ \cdot , \cdot \}^t$ for some $t \in \mathbb{Z}$.

As in Notation 2.12, the following notation is adopted:

Notation 3.6. For $A$ as above and $\{ \cdot , \cdot \}$ a double Poisson structure on $A$, write $\{ \cdot , \cdot \}^{\text{min}}$ and $\{ \cdot , \cdot \}^{\text{max}}$ for the components of minimal (respectively maximal) degree.

Lemma 3.7. For $A$, $\{ \cdot , \cdot \}$ as above, $\{ \cdot , \cdot \}^{\text{min}}$ and $\{ \cdot , \cdot \}^{\text{max}}$ define homogeneous double Poisson structures on $A$.

Proof. Straightforward. \qed
4. Multi-derivations for polynomial algebras

In this section, $A$ is taken to be the polynomial algebra $R[t_1, \ldots, t_d]$, where $d \geq 2$ and $R$ is a commutative unital ring. Hence $A^{\otimes 2}$ is a polynomial algebra on $2d$ generators, and the elements $t_i \otimes 1 - 1 \otimes t_i$ are algebraically independent and can be extended to a set of algebra generators of $A^{\otimes 2}$. In particular, the elements $t_i \otimes 1 - 1 \otimes t_i$ are regular elements of $A^{\otimes 2}$.

A key observation is the following:

**Lemma 4.1.** Let $A$ be the polynomial algebra $R[t_1, \ldots, t_d]$, where $d \geq 2$. The morphism of $R$-modules of Lemma 2.2

$$
\Pi : A^{\otimes 2} \rightarrow \text{Der}(A)
$$

is an isomorphism.

**Proof.** It is straightforward to see that $\Pi$ is a monomorphism of $R$-modules, hence it suffices to check surjectivity. Consider $\varphi \in \text{Der}(A)$ and $x, y \in A$, the commutativity relation $xy = yx$ gives in $A^{\otimes 2}$

$$
\varphi(xy) = (x \otimes 1)\varphi(y) + \varphi(x)(1 \otimes y) = (y \otimes 1)\varphi(x) + \varphi(y)(1 \otimes x),
$$

thus

(1) $$
(x \otimes 1 - 1 \otimes x)\varphi(y) = (y \otimes 1 - 1 \otimes y)\varphi(x).
$$

The double derivation property of $\varphi$ implies that $\varphi$ is determined by its restriction to the $R$-module generated by the $t_i$ and, by $R$-linearity, by the elements $\varphi(t_i)$, $i \in \{1, \ldots, d\}$. Taking $x = t_i$ and $y = t_j$ for $i \neq j$ (recall that $d \geq 2$ by hypothesis), equation (1) implies that

$$
\varphi(t_\alpha) = (t_\alpha \otimes 1 - 1 \otimes t_\alpha)\Theta
$$

for $\alpha \in \{i, j\}$ and for some $\Theta \in A^{\otimes 2}$. Hence this equation holds for all $\alpha \in \{1, \ldots, d\}$, showing that $\varphi = \Pi(\Theta)$, as required. \[\square\]

**Remark 4.2.** The above argument extends to treat the map

$$
\Pi_M : M \rightarrow \text{Der}(A, M),
$$

when $M$ is a free $A^{\otimes 2}$-module (the module structure giving the $A$-bimodule structure).

**Theorem 4.3.** Let $A$ be the polynomial algebra $R[t_1, \ldots, t_d]$, where $d \geq 2$. For $2 \leq n \in \mathbb{N}$ the morphism of $R$-modules of Proposition 2.6

$$
\Pi_n : A^{\otimes n} \rightarrow \text{MDer}(A^{\otimes n}, A^{\otimes n}),
$$

is an isomorphism of $R[\mathbb{Z}/n]$-modules.

**Proof.** It clearly suffices to prove that $\Pi_n$ is an isomorphism of $R$-modules. It is straightforward to check that $\Pi_n$ is a monomorphism, thus it suffices to show that $\Pi_n$ is surjective.

First consider the case $n = 2$ and take $\varphi \in \text{MDer}(A^{\otimes 2}, A^{\otimes 2})$; then, for fixed $a \in A$, the map $\varphi(a \otimes -) : A \rightarrow A^{\otimes 2}$ belongs to $\text{Der}(A)$, hence Lemma 4.1 implies that

$$
\varphi(a \otimes b) = (b \otimes 1 - 1 \otimes b)\Theta_a
$$

for some $\Theta_a \in A^{\otimes 2}$ that is independent of $b$. 
Now take $b = t_1$, so that $b \otimes 1 - 1 \otimes b$ is a regular element of $A^{\otimes 2}$. It follows that $a \mapsto \Theta_a$ defines a double derivation of $\text{Der}(A)$. Again by Lemma 4.1, $\Theta_a$ can be written as $(a \otimes 1 - 1 \otimes a)\Theta$, for some $\Theta \in A^{\otimes 2}$ that is independent of $a$, so that

$$\varphi(a \otimes b) = (a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b)\Theta$$

for any $a, b \in A$, as required.

For $n > 2$, the above argument is modified in the obvious way, by appealing to Remark 4.2. For example, given $\varphi \in M\text{Der}(A^{\otimes n}, A^{\otimes n})$, fix $\alpha \in A^{\otimes n-1}$ and consider the map $\varphi(\alpha \otimes -)$ as belonging to $\text{Der}(A, A^{\otimes n})$, where $A^{\otimes n}$ is the free bimodule with respect to the outer bimodule structure. As above, one deduces that

$$\varphi(\alpha \otimes b) = (b \otimes 1^{\otimes n-1} - 1^{\otimes n-1} \otimes b)\Theta_\alpha$$

where $\Theta_\alpha$ is independent of $b$. The argument is then repeated recursively, starting as above by analysing $\Theta_\alpha$, at each step reducing the number of dependencies. \(\square\)

**Remark 4.4.** The argument for the case $n = 2$ (and, by extension, the general case) depends on the fact that each $t_i \otimes 1 - 1 \otimes t_i$ is a regular element. Clearly the argument fails in general for $A$ an arbitrary commutative ring; even the injectivity of $\Pi_n$ need not hold.

**Example 4.5.** For $A = \mathbb{k}[t]$, with $\mathbb{k}$ a field, there is a double bracket defined by

$$t \otimes t \mapsto t \otimes 1 - 1 \otimes t$$

(see Section A). This is clearly not in the image of $\Pi_2$.

**Remark 4.6.** For the free associative algebra $T(V)$ on a free $R$-module $V$ and $2 \leq n \in \mathbb{N}$, any morphism $V^{\otimes n} \to T(V)^{\otimes n}$ extends uniquely to an element of $M\text{Der}(T(V)^{\otimes n}, T(V)^{\otimes n})$ (and clearly every multi-derivation is determined by its restriction to $V^{\otimes n}$). The corresponding result is false in the commutative case; Theorem 4.3 provides an analogous (but much stronger) result.

For $A$ a commutative $R$-algebra, the multiplication $\mu : A^{\otimes 2} \to A$ induces an $R$-linear map $\text{Hom}_R(A^{\otimes 2}, A^{\otimes 2}) \to \text{Hom}_R(A^{\otimes 2}, A)$ which restricts to a map $M\text{Der}(A^{\otimes 2}, A^{\otimes 2}) \to \text{Hom}_R(A^{\otimes 2}, A)$.

**Corollary 4.7.** Let $A$ be the polynomial algebra $R[t_1, \ldots, t_d]$, where $d \geq 2$. Then the morphism of $R$-modules

$$M\text{Der}(A^{\otimes 2}, A^{\otimes 2}) \to \text{Hom}_R(A^{\otimes 2}, A)$$

is trivial.

**Proof.** By inspection, the map $\varphi_2$ is sent to zero, whence the result, by Theorem 4.3, using the definition of $\Pi_2$. \(\square\)

**Remark 4.8.** Corollary 4.7 shows that no Poisson structure on $R[t_1, \ldots, t_d]$ is induced by a multi-derivation. This shows that the weakening of the notion of double Poisson structure proposed in [Art15] does not provide further non-trivial examples of non-commutative Poisson structures (in the sense of [CB11]).
5. Double Poisson structures on polynomial algebras

Let \( R \) be a commutative unital ring and \( A \) be a commutative \( R \)-algebra.

Notation 5.1. Write

(1) \( \Lambda^2(A) \subset A^\otimes 2 \) for the sub \( R \)-module of anti-commutative elements (namely the kernel of \( \text{id} + \tau : A^\otimes 2 \to A^\otimes 2 \), where \( \tau \) transposes the tensor factors);

(2) \( (A^\otimes 3)^{Z/3} \subset A^\otimes 3 \) for the sub \( R \)-module of cyclically invariant elements.

Remark 5.2. In characteristic two the above does not give the usual definition of \( \Lambda^2(A) \).

Proposition 5.3. Let \( A = R[t_1, \ldots, t_d] \), where \( d \geq 2 \).

(1) The isomorphism \( \Pi_2 : A^\otimes 2 \xrightarrow{\cong} \text{MDer}(A^\otimes 2, A^\otimes 2) \) restricts to an isomorphism of \( R \)-modules

\[ A^2(A) \cong \text{MDer}(A^\otimes 2, A^\otimes 2)^{\text{sgn}}, \]

where \( \text{MDer}(A^\otimes 2, A^\otimes 2)^{\text{sgn}} \) is the \( R \)-module of double brackets on \( A \).

(2) The isomorphism \( \Pi_3 : A^\otimes 3 \xrightarrow{\cong} \text{MDer}(A^\otimes 3, A^\otimes 3) \) restricts to an isomorphism of \( R \)-modules

\[ (A^\otimes 3)^{Z/3} \cong \text{MDer}(A^\otimes 3, A^\otimes 3)^{\text{sgn}}, \]

where \( \text{MDer}(A^\otimes 3, A^\otimes 3)^{\text{sgn}} \) is the \( R \)-module of triple brackets on \( A \).

Proof. An immediate consequence of the definitions and Theorem 4.3, using the \( R[[\mathbb{Z}/n]] \)-equivariance in the cases \( n = 2 \) and \( n = 3 \). \( \square \)

Remark 5.4. Explicitly, for \( \Psi \in \Lambda^2(A) \subset A^\otimes 2 \) (so that \( \tau \Psi = -\Psi \)), the associated double bracket is

\[ \{ \{ a, b \} \} \Psi = (a \otimes 1 - 1 \otimes b)(b \otimes 1 - 1 \otimes a) \Psi. \]

Notation 5.5. Let \( -23 \) denote the \( R \)-linear map \( A^\otimes 2 \to A^\otimes 3 \), \( a \otimes b \mapsto 1 \otimes a \otimes b \), and \( -13 \) the map \( a \otimes b \mapsto a \otimes 1 \otimes b \).

Proposition 5.6. Let \( A = R[t_1, \ldots, t_d] \), where \( d \geq 2 \). Under the isomorphisms of Proposition 5.3, the set map induced by the double Jacobiator

\[ \{ \text{double brackets on } A \} \xrightarrow{-\otimes} \{ \text{triple brackets on } A \} \]

(cf Proposition 3.2) identifies with the (non-linear) map

\[ \mathcal{J} : \Lambda^2(A) \to (A^\otimes 3)^{Z/3} \]

\[ \Psi \mapsto \sum_{\sigma \in \mathbb{Z}/3} \sigma \cdot (\Psi_{13}\Psi_{23}) \]

where the product \( \Psi_{13}\Psi_{23} \) is formed in \( A^\otimes 3 \).

Proof. By Proposition 5.3 it suffices to identify \( -\otimes \{ \}, \{ \} \) in the image of \( \Pi_3 \). It is clear that the expression must be a \( \mathbb{Z}/3 \)-invariant quadratic expression in \( \Psi \). The result follows by direct calculation, using the anti-symmetry \( \tau \Psi = -\Psi \). (The calculation may be simplified by using the proof of \[\text{[VdB08, Proposition 2.3.1]}\].) \( \square \)

Theorem 5.7. Let \( R \) be a commutative ring on which the squaring map \( x \mapsto x^2 \) is injective, then there is no non-trivial double Poisson structure on \( A := R[t_1, \ldots, t_d] \) for \( d \geq 2 \).
Proof. It follows from the identification given in Remark 2.9 that, for any \( i, j, k \in \{1, \ldots, d\} \), the element \( \varphi_3(t_i, t_j, t_k) \) is a regular element of \( A^\otimes 3 \) (this does not require the hypothesis upon \( R \)). Hence, by Proposition 5.6 to prove the result it suffices to show that \( \Psi \in A^\otimes 2 \) is zero if and only if \( \sum_{\sigma \in \mathbb{Z}/3} \sigma \cdot (\Psi_{13}\Psi_{23}) \in A^\otimes 3 \) is zero (anti-symmetry of \( \Psi \) plays no role here).

The latter fact is seen by exploiting the natural grading of \( A \) (placing the generators in degree one, so the grading coincides with the length grading), together with the induced left lexicographical ordering on \( A^\otimes 2 \) and \( A^\otimes 3 \). Namely, if \( \Psi \) is non-zero, the terms of maximal lexicographical order in \( \Psi \) contribute to a non-zero term of maximal lexicographical order in \( \sum_{\sigma \in \mathbb{Z}/3} \sigma \cdot (\Psi_{13}\Psi_{23}) \) (cf. Remark 2.9).

Explicitly, writing \( \Psi = \sum_m \alpha_m \otimes m \) in terms of the monomial basis of \( A \), one considers the contributions

\[
(\alpha_m)^2 \otimes m \otimes m
\]

in \( \Psi_{13}\Psi_{23} \) to the terms of maximal lexicographical order in \( \sum_{\sigma \in \mathbb{Z}/3} \sigma \cdot (\Psi_{13}\Psi_{23}) \).

Finally, the hypothesis upon \( R \) implies that \( (\alpha_m)^2 \otimes m \otimes m \) is zero if and only if \( \alpha_m \) is zero. \( \square \)

6. **Double Poisson structures on commutative algebras**

In this section, \( A \) denotes a commutative \( R \)-algebra.

Definition 6.1. A double bracket on \( A \) is standard if it lies in the image of the morphism of \( R \)-modules

\[
\Pi_2 : \Lambda^2(A) \to \mathrm{MDer}(A^\otimes 2, A^\otimes 2)_{\mathrm{sgn}}
\]

induced by \( \Pi_2 \) (as in Proposition 5.3) and is exotic otherwise, so that the \( R \)-module of exotic double brackets is the cokernel of the above morphism.

Remark 6.2.

(1) Exotic double brackets exist: cf. Example 4.5. However, these cannot be classified easily (cf. the case \( A = R[t] \) in Section 4).

(2) The restriction to \( \Lambda^2(A) \) is not severe. For example, if 2 is invertible in \( R \), the inclusion \( \Lambda^2(A) \to A^\otimes 2 \) admits the retract \( x \otimes y \mapsto \frac{1}{2}(x \otimes y - y \otimes x) \).

Observe that the set map \( \mathfrak{J} : \Lambda^2(A) \to (A^\otimes 3)^{Z/3} \) of Proposition 5.6 can be defined for any commutative algebra \( A \).

Theorem 6.3. Let \( \Psi \in \Lambda^2(A) \) and consider the associated (standard) double bracket \( \{\}, \}_\Psi := \Pi_2(\Psi) \). Then:

1. the associated bracket \( \{\}, \}_\Psi : A^\otimes 2 \to A \) is trivial;
2. \( \{\}, \}_\Psi \) defines a double Poisson structure on \( A \) if and only if \( \Pi_3(\mathfrak{J}(\Psi)) \) is zero in \( \mathrm{MDer}(A^\otimes 3, A^\otimes 3)_{\mathrm{sgn}} \).

Proof. The first statement follows as for Corollary 4.7. The fact that \( \{\}, \}_\Psi \) is a standard double bracket implies that the calculation of Proposition 5.6 is universal, the only difference being that the triple brackets on \( A \) cannot be identified with \( (A^\otimes 3)^{Z/3} \) via \( \Pi_3 \). \( \square \)

Remark 6.4. Theorem 6.3 provides a recipe for constructing examples of non-trivial double Poisson structures on commutative algebras: for any \( \Psi \in \Lambda^2(R[t_i]) \) (\( R[t_i] \) a polynomial algebra) it suffices to pass to a quotient \( A \) of \( R[t_i] \) for which \( \Pi_3(\mathfrak{J}(\Psi)) \)
is trivial in $\text{MDer}(A^{\otimes 3}, A^{\otimes 3})\text{sgn}$. Note that, in all cases, the associated bracket (as in Corollary 4.7) is trivial.

**Appendix A. Double Poisson structures on $R[t]$**

In [VdB08, Example 2.3.3], Van den Bergh stated a classification of the (homogeneous) double Poisson structures on the polynomial algebra $k[t]$, for $k$ a field. A proof over a more general ring, also considering non-homogeneous structures, is given here.

**Proposition A.1.** Let $R$ be a commutative ring on which $x \mapsto x^2$ is injective, then the only homogeneous double Poisson structures on $A := R[t]$ are scalar multiples of the double Poisson brackets determined by
\[
\{\{t, t\}\}^1 = t \otimes 1 - 1 \otimes t,
\]
\[
\{\{t, t\}\}^3 = t^2 \otimes t - t \otimes t^2 = (t \otimes 1 - 1 \otimes t)(t \otimes t),
\]
where the suffix corresponds to the degree of the element $\{\{t, t\}\}$.

In general, for $\lambda, \mu, \nu \in R$,
\[
\{\{t, t\}\} = \lambda \{\{t, t\}\}^1 + \mu(t^2 \otimes 1 - 1 \otimes t^2) + \nu\{\{t, t\}\}^3
\]
defines a double Poisson structure if and only if $\lambda \nu - \mu^2 = 0$ and any double Poisson structure on $A$ is of this form.

**Proof.** A double Poisson structure on $R[t]$ is determined by $\{\{t, t\}\}$. It is straightforward to verify that $\{\{t, t\}\}^1$ and $\{\{t, t\}\}^3$ define homogeneous double Poisson structures on $R[t]$.

The derivation property (using induction upon $n \geq 1$) implies that
\[
\{\{t, t^n\}\} = \left( \sum_{i+j=n-1} t^i \otimes t^j \right) \{\{t, t\}\},
\]
where the product is formed in the algebra $A^{\otimes 2}$.

Anti-symmetry implies that a homogeneous double bracket $\{\{., .\}\}$ is an $R$-linear combination of terms of the form $(t^{N-i} \otimes t^i - t^i \otimes t^{N-i})$, for $N$ corresponding to the homogeneous degree and $0 \leq i < N/2$.

First consider the case where $\{\{t, t\}\}^N = \lambda(t^{N-i} \otimes t^i - t^i \otimes t^{N-i})$ for some $i$; this is subdivided into two cases:

1. $\{\{t, t\}\}^N = \lambda(t^{a+1} \otimes t^a - t^a \otimes t^{a+1})$, for $a \in \mathbb{N}$ (so that $N = 2a + 1$) and $\lambda \in R$. The cases $a \in \{0, 1\}$ correspond to the two cases given above, hence suppose that $a > 1$ (which implies that $2a > a + 1$).

   Consider the coefficient of $t^{2a} \otimes t^{a+1} \otimes t^a$ in $\{\{t, t, t\}\}$. Write $\Phi$ for the element $(\{\{t, t\}\} \otimes \text{id}) \circ (\text{id} \otimes \{\{t, t\}\})(t \otimes t \otimes t)$. Thus the double Jacobiator $\{\{t, t, t\}\}$ is the sum of the cyclic permutations of $\Phi$. Hence it is necessary to consider the coefficients of $t^{2a} \otimes t^{a+1} \otimes t^a$, $t^a \otimes t^{2a} \otimes t^{a+1}$ and $t^{a+1} \otimes t^a \otimes t^{2a}$ in $\Phi$. The coefficient of the first is zero (the two contributions cancel) and the second has coefficient $-\lambda^2$; the hypothesis on $a$ ensures that the third cannot occur. Thus $\{\{t, t\}\}^N = 0$ implies that $-\lambda^2 = 0$, so that $\{\{t, t\}\}^N = 0$.

2. $\{\{t, t\}\}^N = \lambda(t^{N-a} \otimes t^a - t^a \otimes t^{N-a})$, with $N - a > a + 1$. Consider the coefficient of $t^{2(N-a-1)} \otimes t^{a+1} \otimes t^a$ in $\{\{t, t, t\}\}$. In this case, the coefficient of $t^{2(N-a-1)} \otimes t^{a+1} \otimes t^a$ in $\Phi$ is $\lambda^2$. If $2(N - a - 1) > N - a$ then the coefficients of $t^a \otimes t^{2(N-a-1)} \otimes t^{a+1}$ and $t^{a+1} \otimes t^a \otimes t^{2(N-a-1)}$ in $\Phi$ are both
trivial. Hence (in this case) the condition $\{\cdot, \cdot\} = 0$ implies that $\lambda^2 = 0$ and again $\{\cdot, \cdot\}^N = 0$.

The inequality $2(N - a - 1) > N - a$ is equivalent to $N > a + 2$, since $N > 2a + 1$, by hypothesis, this is satisfied if $a \geq 1$ or if $a = 0$ and $N > 2$.

In the remaining case, $N = 2$ and $a = 0$, it can be checked directly that $\{\cdot, \cdot\}^N = 0$.

To complete the proof, one considers the case where $\{t, t\}^N$ has at least two non-trivial coefficients with respect to the basis $\{t^{N-i} \otimes t^i : 0 \leq i < N/2\}$.

Thus one can write

$$\{t, t\}^N = \lambda(t^{N-a} \otimes t^a - t^a \otimes t^{N-a}) + \mu(t^{N-b} \otimes t^b - t^b \otimes t^{N-b})$$

$$+ \sum_{b < k < N/2} \nu_k(t^{N-k} \otimes t^k - t^k \otimes t^{N-k})$$

where $\lambda \neq 0$, $\mu \neq 0$ and $0 \leq a < b < N/2$ (hence $N > 2$).

Consider the coefficient of $t^{2N-b-a-1} \otimes t^b \otimes t^a$

in $\{t, t, t\}$. As above using the notation $\Phi$,

(1) the coefficient of $t^{2N-b-a-1} \otimes t^b \otimes t^a$ in $\Phi$ is $\lambda^2 + \lambda \mu$;

(2) the coefficient of $t^a \otimes t^{2N-b-a-1} \otimes t^b$ in $\Phi$ is $-\lambda \mu$ (the sign arises from antisymmetry);

(3) the term $t^b \otimes t^a \otimes t^{2N-b-a-1}$ cannot arise in $\Phi$, since $2N - b - a - 1 > N - a$ (the difference is $N - 1 - b$ and the latter is positive by the hypotheses).

It follows that the coefficient of $t^{2N-b-a-1} \otimes t^b \otimes t^a$ in $\{t, t, t\}$ is $\lambda^2$, thus $\lambda = 0$, contradicting the hypothesis that $\lambda \neq 0$.

Finally, consider the non-homogeneous case. Here, by Lemma 3.7 the only non-trivial possibility is

$$\{t, t\} = \lambda(t \otimes 1 - 1 \otimes t) + \mu(t^2 \otimes 1 - 1 \otimes t^2) + \nu(t^2 \otimes t - t \otimes t^2)$$

where, if $\mu \neq 0$, then both $\lambda$ and $\nu$ are non zero.

The associated double Jacobiator $\{t, t, t\}$ in principle has terms in degrees 1, 2, 3, 4 and 5; since $\{\cdot, \cdot\}^N$ and $\{\cdot, \cdot\}^2$ give double Poisson structures, the terms in degrees 1 and 5 vanish (as already observed in Lemma 3.7). A straightforward calculation also shows that the terms in degrees 2 and 4 vanish.

Finally, one finds that

$$\{t, t, t\} = (\lambda \nu - \mu^2)(1 \otimes t \otimes t^2 - 1 \otimes t^2 \otimes t)$$

where $1 \otimes t \otimes t^2$ and $1 \otimes t^2 \otimes t$ denote the respective $\mathbb{Z}/3$-orbit sums. Hence the double bracket defines a double Poisson structure if and only if $\lambda \nu = \mu^2$. 

□

Remark A.2.

(1) The transformation given by [VdB08, Example 2.3.3] associated to the change of variables $t \mapsto t^{-1}$ (after extending to $k[t^{\pm 1}]$) acts by $\lambda \mapsto -\nu$, $\nu \mapsto -\lambda$ and $\mu \mapsto -\mu$, as expected.

(2) Over a field $k$, up to scalar multiplication and the action of $k^*$, considered as automorphisms of $k[t]$ via $\alpha : t \mapsto \alpha t$, this gives the single non-homogeneous example

$$\{t, t\} := (t \otimes 1 - 1 \otimes t) + (t^2 \otimes 1 - 1 \otimes t^2) + (t^2 \otimes t - t \otimes t^2).$$
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