Abstract. This article addresses the problem of enumerating the tilings of a plane by lozenges, under the restriction that these tilings be doubly periodic. Kasteleyn’s Pfaffian method is applied to compute the generating function of those permutations. The monomials of this function represent the different types of tilings, grouping them according to the number of lozenges in each orientation. We present an alternative approach to compute these types. Finally, two additional classes of tilings are proposed as open enumeration problems.

1 Introduction

We consider the tiling of the plane by equilateral triangles, assuming that the vertices of those are the points of the lattice \( \Lambda_0 \) spanned by \( \mathbf{u} = (1,0), \mathbf{v} = (1/2, \sqrt{3}/2) \).

By merging two adjacent triangles of this tiling we obtain a rhombus or lozenge. It is obvious then that the plane can also be tiled with such lozenges. These tiles may take three orientations, which we denote by the notation set in Figure 2.

There is a rich literature devoted to studying the different lozenge arrangements that tile certain bounded regions of the plane (see [10, 2, 4, 9] and the references therein). The simplest scenario appears in the case of a convex hexagon.
drawn over the grid of Figure 1 with pairs of opposite sides of the same length (a semiregular hexagon). Its lozenge tilings are associated by an appealing bijection with the plane partitions whose parts are bounded by the lengths of the hexagon sides. We recall that the solid Young diagram of plane partition is an arrangement of unit cubes located in the positive octant of $\mathbb{R}^3$ and satisfying that if there is a cube at position $(i, j, k)$, $a \leq i$, $b \leq j$, and $c \leq k$, then position $(a, b, c)$ is neither empty.

A classical formula (proved in [11]) states that the number of plane partitions fitting in a box of sides $a, b$, and $c$ is:

$$
\prod_{i=0}^{a-1} \prod_{j=0}^{b-1} \prod_{k=0}^{c-1} \frac{i + j + k + 2}{i + j + k + 1}.
$$

Throughout this article, we employ bold letters to denote vectors $a = (a_1, a_2)$, considered as columns when using matrix notation: $a = (a_1, a_2)^t$. We are interested in lozenge tilings of the whole plane which are doubly periodic. More explicitly, let

$$
B = \begin{bmatrix}
a_1 & b_2 \\
a_2 & b_2
\end{bmatrix} \in \mathbb{Z}^{2 \times 2}
$$

be a 2-rank matrix and consider the sublattice

$$
\Lambda = [\mathcal{U}[\mathcal{V}]B\mathbb{Z}^2 \subseteq \Lambda_0.
$$

The tilings we are interested of are invariable under the translations by $a_1 u + a_2 v$ and $b_1 u + b_2 v$; or equivalently, by any element of $\Lambda$. Dealing with infinite tilings, unlike in the finite regions case, we do not have a boundary to “support” our arguments.

We will define the type of a tiling as the number of lozenges in each orientation it consists of. Note that, as can be checked using the argument exposed in [3]; in the mentioned case of a semiregular hexagon with sides $a, b$, and $c$, all its lozenge tilings have a constant number of lozenges in each orientation ($ab$, $ac$, and $bc$).

As we have said, a lozenge tiling is built merging pairs of triangles under certain restrictions. Therefore, lozenge tilings correspond to perfect matchings
in an associated graph (see Figure \[8\]). Perfect matchings enumeration is a rich field of research (see [13] for an illustrating survey), stimulated by several problems in the domain of Physics and Chemistry.

Kasteleyn developed a method for enumerating the perfect matchings of a planar graph by means of a Pfaffian computation [6, 7, 8]. In the first of these references, the problem is also solved for a non-planar graph, which can be embedded in a torus. This is accomplished by computing a linear combination of four Pfaffians. In Section 3 we explain how this method applies to counting doubly periodic lozenge tilings. The output is a polynomial generating function \(Z(L,D,R) \in \mathbb{Z}[L,D,R]\), where the coefficient in \(L^iD^jR^k\) equals the number of tilings of type \((i,j,k)\).

In Section 4 we introduce a different approach to compute the different types of doubly periodic tilings modulo a given period \(\Lambda\). We prove that the pair of height increments in the (infinite analogue to the) solid Young diagram through a pair of vectors spanning \(\Lambda\) characterises the type of the tiling. As a consequence, the possible types correspond to lattice points in a certain triangle associated to a lattice basis \(B\), in the sense of (1). The vertices of this triangle correspond to the three uniform tilings (those with all of its lozenges arranged in a constant orientation).

Finally, in Section 5 we propose two ways of grouping together significantly similar tilings, whose enumeration remains, up to our knowledge, an open problem. Before all, let us fix the notation and definitions we will need:

### 2 Definitions

There are two types of triangles in Figure 1: upwards and downwards-pointing. Any lozenge contains a triangle of each type. In this article, we refer to generic elements of these classes by the symbols \(\triangle\) and \(\triangledown\). We identify both sets of triangles with \(\Lambda_0\) by means of the following convention: a point in \(\Lambda_0\) represents the right-most upwards-pointing and downwards-pointing triangles which have that point as vertex.

Let \(\Lambda\) be defined by Equation (1). We refer to the index of \(\Lambda\) in \(\Lambda_0\) simply
as the index of $\Lambda$:
$$[\Lambda_0 : \Lambda] = |\det B| = 2(\text{vol } \Lambda)/\sqrt{3}.$$ 
In order to define a $\Lambda$-periodic tiling, we need to decide with which of the three adjacent $\nabla$ is merged every $\triangle$ in a fundamental set whose size is the index of $\Lambda$. The following map returns the difference $\nabla - \triangle$ within a lozenge on its orientation as input.

$$\xi : \{L, D, R\} \to \Lambda_0$$

\begin{align*}
L & \rightarrow v - u \\
D & \rightarrow 0 \\
R & \rightarrow v
\end{align*}

**Definition 1** Let $\Lambda \subseteq \Lambda_0$ be a 2-rank lattice. We define a $\Lambda$-periodic tiling as a map

$$\tau : \Lambda_0 \to \{L, D, R\}$$

satisfying the following two axioms:

i) **Compatibility:**

$$\forall x \in \Lambda_0 \forall y \in \Lambda, \ \tau(x) = \tau(x + y).$$

ii) **Tiling:** The following map is bijective:

$$\hat{\tau} : \Lambda_0 \to \Lambda_0$$

$$x \mapsto x + \xi(\tau(x)).$$

We use the notation $T_\Lambda$ for the set of $\Lambda$-periodic tilings.

Note that the second axiom is equivalent to (see Figure 5):

$$\forall x \in \Lambda_0, \text{ exactly one of the following conditions is satisfied:}$$

$$\tau(x) = R, \ \tau(x + u) = L, \ \tau(x + v) = D.$$

![Figure 5: Axiom ii') Exactly one of the three possible lozenges occurs.](image)

As we have already mentioned, it is enough to define $\tau$ on a representative of each class modulo $\Lambda$. We consider the induced mapping:

$$\hat{\tau} : \Lambda_0/\Lambda \to \{L, D, R\}$$
and define the type of a Λ-periodic tiling as:

\[ t(\tau, \Lambda) = (\#\tau^{-1}(L), \#\tau^{-1}(D), \#\tau^{-1}(R)) \in \mathbb{N}^3. \]

Infinite lozenge tilings can be represented by an infinite analogue to a solid Young diagram, namely, the complement of a subset of \( \mathbb{Z}^3 \) closed under addition of elements in the semigroup \( \mathbb{N}^3 \). In other words, a staircase diagram in three dimensions; with two possible identifications. As Figure 6 shows, there are two lozenge tilings of the unit hexagon. One of them is prominent (a solid cube), and the other one is a “hole”, limited by three walls. In the rest of this article, we consider that the left design of Figure 6 is the solid cube.

![Figure 6: An outside and an inside corner.](image)

With this identification, the vertices of a lozenge tiling can be labelled by three-dimensional coordinates and given a height function, defined by the sum of these coordinates. The application of height labels in this context dates back to [1, 15] and has been useful in the study of tilings of bounded regions.

An edge in the tiling is associated with a coordinates and a height increment:

|   | F   | f   |
|---|-----|-----|
| u | (−1, 0, 0) | −1  |
| −u | (+1, 0, 0) | +1  |
| v | (0, 0, +1) | +1  |
| −v | (0, 0, −1) | −1  |
| u − v | (0, +1, 0) | +1  |
| v − u | (0, −1, 0) | −1  |

Note that this effect of an edge on the height of a vertex is only valid is the edge does appear in the lozenge tiling. For instance, if two vertices \( x \) and \( x + u \) are not connected by \( u \), then \( \tau(x) = D \) and they are connected by the concatenation \( (v, u − v) \). Therefore, the coordinates increment is not \((−1, 0, 0)\), but \((0, +1, +1)\). In general, if the height increment of an existing edge is \( h \), the effect of the same non-occurring edge is \( −2h \).

**Definition 2** Let \( \Lambda \subseteq \Lambda_0 \) be a 2-rank lattice and \( \tau \) an \( \Lambda \)-periodic tiling. We define a path in \( \tau \) as a succession \( (x_i)_{i=0}^N \subseteq \Lambda_0 \) satisfying

\[ x_{i+1} - x_i \in \{ \pm u, \pm v, \pm (u - v) \} \]

and:
Each path $p = (x_i)_{i=0}^N$ in $\tau$ can be associated with a height increment, in the following way:

$$h_{\tau}(p) = \sum_{i=0}^{N-1} f(x_{i+1} - x_i).$$

For a given tiling $\tau$, the height increment of a path only depends on its extreme points. Therefore, setting $h_{\tau}(0) = 0$, we can associate a height to every point in $\Lambda_0$, defining a mapping $h_{\tau}$ over $\Lambda_0$. As we usually treat elements of $\Lambda_0$ by its coordinates on basis $(u|v)$, we will employ the following change of coordinates:

$$e_{\tau}: \mathbb{Z}^2 \rightarrow \mathbb{Z}$$

$$a \rightarrow h_{\tau}([u|v]a).$$

Note that as a lozenge tiling is the projection of an pile of cubes along lines parallel to vector $(1, 1, 1)$, the height $h_{\tau}(x)$ of a point (together with its projected position in the plane) determines its three-dimensional coordinates, which by analogy we may denote $H_{\tau}(x)$. 

![Figure 7: The h function.](image-url)
**Proposition 3** Let $\Lambda \subseteq \Lambda_0$ be a 2-rank lattice, $\tau$ a $\Lambda$-periodic lattice and $x = \lambda_1 u + \lambda_2 v \in \Lambda_0$. Then,

$$H_\tau(x) = (-\lambda_1, 0, \lambda_2) + \frac{h_\tau(x) - \lambda_2 + \lambda_1}{3}(1, 1, 1).$$

**Proof.** The maybe improper “path” $(0, u, 2u, \ldots, \lambda_1 u, \lambda_1 u + v, \ldots, x)$ would give a height label $h = \lambda_2 - \lambda_1$ and coordinates label $H = (-\lambda_1, 0, \lambda_2)$. Any wrong step can be replaced by a concatenation of two, changing the height by $\pm 3$ and the coordinates by $\pm (1, 1, 1)$.

Note that if $\tau$ and $\tau'$ are tilings, we have $H_\tau(x) - H_{\tau'}(x) \in \mathbb{Z}((1, 1, 1))$ and $h_\tau(x) - h_{\tau'}(x) \in (3)$, for every $x \in \Lambda_0$.

Let $B = [a|b]$ be a basis of $\Lambda$ with respect to $\Lambda_0$ (see Equation (1)). We say that the pair $\delta(\tau,B) = (e_\tau(a),e_\tau(b))$ composed of the height increment of the basic vectors is the fingerprint of $\tau$ in basis $B$. We note that for $x \in \Lambda_0$ and $y \in \Lambda$,

$$h_\tau(x + y) = h_\tau(x) + h_\tau(y),$$

and therefore,

$$h_\tau(x + [u|v]B(\lambda_1, \lambda_2)^t) = h_\tau(x) + \lambda_1 e_\tau(a) + \lambda_2 e_\tau(b), \ \forall \lambda_1, \lambda_2 \in \mathbb{Z}.$$  

In other words, the fingerprint of a tiling determines the height of every point in the lattice $\Lambda$.

### 3 The Permanent-Determinant method

In this article we focus the problem of enumerating the $\Lambda$-periodic tilings, for a given full-rank sublattice of $\Lambda_0$. Indeed, these tilings correspond to perfect matchings in a certain “honeycomb-like” graph (see Figure 8), obtained as follows. We consider two sets of representatives of $\Lambda_0/\Lambda$ such that the set of associated $\triangle$ and $\triangledown$ triangles is connected. This way, from $\Lambda$ we build (in principle not in a unique way) a graph whose vertices are the triangles in a fundamental region: $V = \Lambda_0/\Lambda \times \Lambda_0/\Lambda$. An edge (weighted if desired by L,D,R) joins a pair of triangles which can be merged in a lozenge.
The Hafnian of a symmetric square matrix of order $2n$ is defined by:

$$\text{Hf}(A) = \sum_{m \in U_n} \prod_{\{i,j\} \in m} A_{i,j},$$

where $U_n = \{\{i_1,j_1\}, \ldots, \{i_n,j_n\}\} \cup \cup_{1 \leq i \leq n} \{i_l,j_l\} = \{1, \ldots, 2n\}$ is the set of the $(2n)!/(n!2^n)$ (unordered) matchings of the set $\{1, \ldots, 2n\}$. The number of perfect matchings in an undirected graph with an even number of vertices (the problem keeps no interest if the number of vertices is odd) equals the Hafnian of its adjacency matrix, but the computation of a Hafnian is unfortunately a \#P-complete problem [16]. The Pfaffian is a related, but tractable, function defined over antisymmetric matrices:

$$\text{Pf}(A') = \sum_{m \in U_n} (-1)^{\sigma(m)} \prod_{\{i,j\} \in m} A'_{i,j},$$

where the parity $\sigma(m)$ of a matching $m = \{\{i_1,j_1\}, \ldots, \{i_n,j_n\}\}$, written in such a way that $i_l < j_l$, for $1 \leq l \leq n$, is that of the permutation

$$\begin{pmatrix}
1 & 2 & 3 & 4 & \cdots & 2n - 1 & 2n \\
i_1 & j_1 & i_2 & j_2 & \cdots & i_n & j_n
\end{pmatrix}.$$

It can be shown that, for every antisymmetric matrix $A'$ of even order, we have $\det(A') = \text{Pf}(A')^2$. This result also holds for matrices of odd order, defining their Pfaffian as zero.

Given a directed graph, the Pfaffian of its adjacency matrix $A'$ counts its perfect matchings, but some of them affected by a negative sign, leading to an (in principle) meaningless sum. Kasteleyn proved (see [7]) that every planar graph can be oriented in such a way that $\text{Hf}(A) = \pm \text{Pf}(A')$, counting therefore the perfect matchings. This Pfaffian orientation can be achieved requiring that in every face, the number of border edges in clockwise direction is odd.

However, the graph of Figure 8 is not planar. In general, the graphs we are interested in can be embedded in a torus. In [6], the perfect matching enumeration problem is solved for the rectangular lattice on a torus, by computing a linear combination of four Pfaffians. In a more general fashion, it is stated in [8] that the problem can be solved with $4^g$ Pfaffians for any graph drawn in a surface of genus $g$. This statement is proved in [5]; and independently in [14], which contains a general method that uses $2^2 - \chi$ Pfaffians for a graph embedded in a surface of Euler characteristic $\chi$, improving therefore the previous statement for non-orientable surfaces. Tesler method [14] starts drawing the graph in a surface represented by a polygon with pasted borders, distinguishing between edges contained in the interior of the polygon and those crossing its borders. If some of the interior edges form a cycle enclosing the rest, a crossing orientation is given to the graph according to the following rule (R4):

- The set of interior edges is given a Pfaffian orientation, as a planar graph.
• Each of the remaining edges is oriented in such a way that any face it forms with the interior edges has an odd number of clockwise edges as well.

Before proceeding, let us remark that the graphs we consider are bipartite (△ may only match ▽, and conversely). For bipartite graphs, the Permanent-Determinant variant, introduced in [12], simplifies the Hafnian-Pfaffian method. Indeed, if $G$ and $\tilde{G}$ are, respectively, an undirected bipartite and a directed bipartite graph, let $A$ and $A'$ denote their adjacency matrices, and $B$ and $B'$ their bipartite adjacency matrices (whose columns and rows represent black and white vertices, respectively). Then, $\text{Hf}(A) = \text{Per}(B)$ and $\text{Pf}(A') = \text{det}(B')$.

Let $\Lambda$ be a full-rank sublattice $\Lambda$ of $\Lambda_0$. Firstly, we compute the (unique) matrix $B = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in \mathbb{Z}^{2 \times 2}$ such that $a > 0$, $0 \leq c < b$, and $\Lambda$ is generated by the columns of $[u|v]B$ (see Equation (1)). We choose sets of △ and ▽ representatives formed in both cases by $b$ rows of $a$ triangles, in the way depicted in Figure 8. More explicitly:

$\triangle: \{iv + ju \mid 0 \leq i < b, 0 \leq j < a\},$

$\nabla: \{(i+1)v + ju \mid 0 \leq i < b, 0 \leq j < a\}.$

We label the edges of the graph with L,D, or R, depending of the orientation of the corresponding lozenge. Then, the bipartite adjacency matrix of the undirected graph, identifying rows with △ and columns with ▽, presents the following decomposition in $b \times b$ square blocks of order $a$:

$$M = \begin{bmatrix} X & Z \\ Z & X \end{bmatrix},$$

where $Z = D \cdot \text{Id}_a$,

$$X = \begin{bmatrix} R & L \\ L & R \\ \vdots & \vdots \\ L & R \end{bmatrix}, \text{ and } Z' = \begin{bmatrix} D \cdot \text{Id}_{a-c} \\ \end{bmatrix}. $$

In the extreme case $a = 1$, block $X$ equals $[R + L]$; and when $b = 1$, the block decomposition of $M$ collapses to $[X + Z']$. Now, in order to define a suitable orientation, we start by the edges which do not cross the border of the rectangle, orienting by the rule $\triangle \rightarrow \nabla$ the D and R edges, and conversely the L edges, as is shown in the left side of Figure 9.

In order to apply Rule (R4) for orienting the rest of the edges, we would need that those already oriented are enclosed in a cycle, condition that is not fulfilled
in our case. However, we can slightly modify (R4) taking into account those D-edges which, joined to some interior edges, form a cycle enclosing an odd number of vertices (no R- or L-edges present this pathology under our construction). Then, we get a crossing orientation if we orient the border crossing edges as follows (see Figure 9):

- L-edges: $\triangle \rightarrow \blacklozenge$.
- D-edges:
  - $\text{Id}_{a-c}$: $\triangle \rightarrow \blacklozenge$, if $b$ is odd; and conversely otherwise.
  - $\text{Id}_c$: $\blacklozenge \rightarrow \triangle$, if $b$ is odd; and conversely otherwise.

Marking edges crossing the diagonal border with $\omega_1$ and those crossing the horizontal one with $\omega_2$, the block structure of the bipartite adjacency matrix $M'$ of this directed graph remains as in $M$, substituting the blocks by: $Z = D \text{Id}_a$:

$$X = \begin{bmatrix} R & \omega_1 L \\ -L & R \\ \vdots & \vdots \\ -L & R \end{bmatrix}, \text{ if } a > 1, \text{ and } X = [R + \omega_1 L] \text{ otherwise;}$$

$$Z' = \begin{bmatrix} (R - L)^b \omega_1 \omega_2 \text{Id}_c \\ (-1)^b \omega_1 \omega_2 \text{Id}_c \end{bmatrix}.$$

We obtain, as a corollary of [14, Theorem 5.2]:

**Theorem 4** Let $\Lambda$ be a full-rank sublattice of $\Lambda_0$. If $g(\omega_1, \omega_2) = \det(M') \in \mathbb{Z}[L, D, R][\omega_1, \omega_2]$, where $M'$ is defined above, the generating function of the $\Lambda$-periodic tilings is:

$$Z(L, D, R) = \frac{1}{2} (g(1, 1) + g(1, -1) + g(-1, 1) - g(-1, -1)).$$

It would be interesting to derive a “closed formula” for this generating function, or at least for the number of $\Lambda$-periodic tilings $Z(1,1,1)$. 

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**Figure 9: Crossing orientation**
4 Different tiling types

The three constant mappings \( \tau_L \), \( \tau_D \), and \( \tau_R \) are \( \Lambda \)-periodic tilings for any sublattice \( \Lambda \subseteq \Lambda_0 \). The identification with piles of cubes gives, for each of these tilings, a plane (orthogonal to the \( Z \), \( X \), and \( Y \) axes respectively; see Figure 11 for the latter two).

Let \( B \in \mathbb{Z}^{2 \times 2} \) be a 2-rank matrix. We call fundamental triangle to the triangle whose vertices are the fingerprints in \( B \) of the constant tilings. With \( B = [a | b] \), we have:

\[
\begin{align*}
\delta(\tau_L, B) &= (-a_1 - 2a_2, -b_1 - 2b_2), \\
\delta(\tau_D, B) &= (2a_1 + a_2, 2b_1 + b_2), \\
\delta(\tau_R, B) &= (-a_1 + a_2, -b_1 + b_2).
\end{align*}
\]

Therefore, the area of the fundamental triangle equals \((9/2) \det B = (9/2)[\Lambda_0 : \Lambda]\). Note that the vertices of the triangle are in the lattice \((3\mathbb{Z})^2\), possibly shifted (see Figure 10). Taking as example the lattice \( \Lambda = [u | v] \mathbb{Z}^2 \), where

\[
B = \begin{bmatrix} 2 & -2 \\ 2 & 4 \end{bmatrix},
\]

the fundamental triangle is defined by the points:

\[
\delta(\tau_L, B) = (-6, -6), \; \delta(\tau_D, B) = (6, 0), \; \delta(\tau_R, B) = (0, 6).
\]

Figure 10: Types of \( \Lambda \)-periodic tilings.

In general, for any two points \( \delta_1, \delta_2 \in \mathbb{Z}^2 \) such that \( \delta_1 - \delta_2 \in (3\mathbb{Z})^2 \), there exists a third point \( \delta_3 = -(\delta_1 + \delta_2) \) defining with the former the fundamental triangle associated to a basis.
Next result identifies the type of a tiling with a point of the fundamental triangle. In the following, when speaking of points in the fundamental triangle, we restrict ourselves to those in $\delta(\tau_D, B) + (3\mathbb{Z})^2$.

**Theorem 5** Let $\Lambda \subseteq \Lambda_0$ be a 2-rank lattice and $\tau$ a $\Lambda$-periodic tiling. Writing $(L, D, R) = t(\tau, \Lambda)$ for the type of $\tau$, its fingerprint in a base $B$ of $\Lambda$ has triangular coordinates proportional to $(L, D, R)$:

$$
\delta(\tau, B) = \frac{1}{|\Lambda_0 : \Lambda|} (L\delta(\tau_L, B) + D\delta(\tau_D, B) + R\delta(\tau_R, B)).
$$

**Proof.** Let $B = [a | b]$ be the considered basis. We denote by $o_1$ and $o_2$ the orders in the group $\Lambda_0 / \Lambda$ of $u$ and $v$, respectively, and consider the following two representations of $\Lambda_0 / \Lambda$:

|   |   |   |   |
|---|---|---|---|
| 0 | $u$ | $2u$ | $(o_1 - 1)u$ |
| $x_1$ | $x_1 + u$ | $x_1 + 2u$ | $x_1 + (o_1 - 1)u$ |
| ... | ... | ... | ... |
| $x_l - 1$ | $x_l - 1 + u$ | $x_l - 1 + 2u$ | $x_l - 1 + (o_1 - 1)u$ |

|   |   |   |   |
|---|---|---|---|
| 0 | $v$ | $2v$ | $(o_2 - 1)v$ |
| $y_1$ | $y_1 + v$ | $y_1 + 2v$ | $y_1 + (o_2 - 1)v$ |
| ... | ... | ... | ... |
| $y_{l_2 - 1}$ | $y_{l_2 - 1} + v$ | $y_{l_2 - 1} + 2v$ | $y_{l_2 - 1} + (o_2 - 1)v$ |

where $l_i = [\Lambda_0 : \Lambda] / o_i$. For $i \in \{1, 2\}, 0 \leq j < l_i$, let $L_j^1, D_j^1,$ and $R_j^1$ be the number of elements in the $j$th row of the $i$th table whose image by $\tau$ is $L, D,$ and $R$, respectively. We have:

$$L_j^1 + D_j^1 + R_j^1 = o_i, \forall i, \forall j.$$

There exist constants $\kappa_1 = (\kappa_1^1, \kappa_1^2, \kappa_1^3) = H_\tau(o_1u), \kappa_2 = (\kappa_2^1, \kappa_2^2, \kappa_2^3) = H_\tau(o_2v)$ such that

$$\kappa_1^1 = -L_j^1 - R_j^1, \quad \kappa_1^2 = \kappa_1^3 = D_j^1,$$

$$\kappa_2^1 = \kappa_2^2 = -L_j^2, \quad \kappa_2^3 = D_j^2 + R_j^2.$$

Therefore, the numbers $D_j^1, L_j^2, L_j^1 + R_j^1,$ and $D_j^2 + R_j^2$ are independent of index $j$. We consider the matrix

$$P = B^{-1} \begin{bmatrix} o_1 \\ o_2 \end{bmatrix} = \begin{bmatrix} b_2l_1 & -b_1l_2 \\ -a_2l_1 & a_1l_2 \end{bmatrix}.$$

Writing $k_i = \kappa_i^1 + \kappa_i^2 + \kappa_i^3$, we have $(k_1, k_2) = (e_\tau(a), e_\tau(b))P$, and therefore,

$$\begin{cases} e_\tau(a) = a_1k_1/o_1 + a_2k_2/o_2 \\ e_\tau(b) = b_1k_1/o_1 + b_2k_2/o_2, \end{cases}$$
\[ [\Lambda_0 : \Lambda] \varepsilon_{\tau}(a) = \left( a_1 l_1(2D_j^1 - L_j^1 - R_j^1) + a_2 l_2(D_j^2 - 2L_j^2 + R_j^2) \right) = a_1(2D - L - R) + a_2(D - 2L + R), \]
\[ [\Lambda_0 : \Lambda] \varepsilon_{\tau}(b) = \left( b_1 l_1(2D_j^1 - L_j^1 - R_j^1) + b_2 l_2(D_j^2 - 2L_j^2 + R_j^2) \right) = b_1(2D + L + R) + b_2(D - 2L + R). \]

The result follows easily. □

For instance, the tiling shown in Figure 4 is \( \Lambda \)-periodic where \( \Lambda \) is defined in the example from Figure 10. Its type is \((2,2,8)\); and its fingerprint in basis \([0,1,3]\).

According to Figure 10, there are (at most) seven nonconstant types of \( \Lambda \)-periodic tilings. Four of them involve lozenges in the three different orientations. Let us prove that every point \( \delta \) in the fundamental triangle represents at least one tiling.

**Theorem 6** Let \( B = [a \mid b] \) be a 2-rank integer matrix and \( \Lambda = [u \mid v]BZ^2 \). The set of types \( \{t(\tau, \Lambda) \mid \tau \in T_\Lambda \} \) coincides with the intersection of the fundamental triangle with \( \delta(\tau_D, B) + (3\mathbb{Z})^2 \).

**Proof.** As a corollary of Theorem 5, every type lies in that intersection. For the converse inclusion, let \( \delta \) be a point in the fundamental triangle. According to Proposition 3, the “skeleton” of any tiling with fingerprint \( \delta \) is determined. We claim that there is a tiling \( \tau \) with the “skeleton” determined by \( \delta \). More formally, there is a \( \Lambda \)-periodic tiling \( \tau \) such that the three-dimensional coordinates of the points \( a_1u + a_2v \) and \( b_1u + b_2v \) are, respectively,

\[
\Delta(\tau, B) = \left( \frac{1}{3}(\delta_1 - 2a_1 - a_2, \delta_1 + a_1 - a_2, \delta_1 + a_1 + 2a_2) , \right.
\]
\[ \left. \frac{1}{3}(\delta_2 - 2b_1 - b_2, \delta_2 + b_1 - b_2, \delta_2 + b_1 + 2b_2) \right). \]

A particular tiling can be formed just by placing a cube under every point of that skeleton and filling the position whose coordinates are not bigger component-wise. For instance, applying this process on the skeleton \((-2,0,2), (1,-1,3)\) we get the tiling depicted in Figure 4. We just need to show that for every \( x, y \in \Lambda, H_+(x) - H_+(y) \notin (\mathbb{N} \setminus \{0\})^3 \). In other words, that no point in the skeleton is “hidden”. It is sufficient to check that the normal to the plane defined by the two components of \( \Delta(\tau, B) \) lies in the cone \( \{ \pm (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0 \} \). This condition holds, because of Theorem 5 and the fact that the cone is a convex set. □

The number of points in the border of the fundamental triangle are:

\[ \tau_D, \tau_L : \gcd(a_1 + a_2, b_1 + b_2) + 1 \]
\[ \tau_L, \tau_R : \gcd(a_2, b_2) + 1 \]
\[ \tau_R, \tau_D : \gcd(a_1, b_1) + 1 \]

By Pick’s Theorem, the number of interior points in the fundamental triangle equals

\[ \frac{1}{2} |\Lambda_0 : \Lambda| - \frac{1}{2}(\gcd(a_1, b_1) + \gcd(a_2, b_2) + \gcd(a_1 + a_2, b_1 + b_2)) + 1 \]
and the number of monomials in the generating function $Z(L, D, R)$ is
\[
\frac{1}{2}[\Lambda_0 : \Lambda] + \frac{1}{2}(\gcd(a_1, b_1) + \gcd(a_2, b_2) + \gcd(a_1 + a_2, b_1 + b_2)) + 1.
\]

5 Grouping similar tilings

The coefficients in the monomials corresponding to borders of the triangle (i.e. types involving just one or two lozenge orientations) are easily determined. Let us show the case of tiling types with no “L” lozenge. Put $d = \gcd(a_1, b_1)$. There are $d + 1$ points in the edge limited by $\delta(\tau_D, B)$ and $\delta(\tau_R, B)$. These are:
\[
\delta_i = \frac{i}{d}\delta(\tau_D, B) + \frac{d - i}{d}\delta(\tau_R, B), \quad i = 0, \ldots, d.
\]

According with Theorem 5, a $\Lambda$-periodic tiling $\tau$ such that $\delta(\tau, B) = \delta_i$ has type $(i[\Lambda_0 : \Lambda]/d, 0, (d - i)[\Lambda_0 : \Lambda]/d)$. This kind of tilings are constant on the lines directed by $v$ and can be enumerated as follows:

- Compute a triangular form of $B$:

\[
\begin{bmatrix}
  d & 0 \\
  * & *
\end{bmatrix}.
\]

- Select $i$ elements $x \in \{0, u, 2u, \ldots, (d - 1)u\}$, and define $\tau(x) = D$ for them. For the rest, set $\tau(x) = R$.

Therefore, the number of tilings of type $(i[\Lambda_0 : \Lambda]/d, 0, (d - i)[\Lambda_0 : \Lambda]/d)$ is $\binom{d}{i}$ and there are $2^d$ tilings whose type is in the considered edge.

Continuing with the example from Figure 10, let us enumerate the possible tilings in with no “L” lozenge. We need to choose between $D$ and $R$ for $\tau(0)$ and $\tau(u)$. We get the four tilings depicted in Figure 11.

Indeed, the two tilings of type $\delta_1 = (0, 6, 6)$ (with fingerprint (3,3)) are the same modulo a shift. In general, we might also be interested in enumerating classes of tilings, identifying those which only differ on a shift. This is, defining the following equivalence relation on the set of $\Lambda$-periodic tilings $T_{\Lambda}$:
\[
\tau S \rho \iff \exists s \in \Lambda_0 \forall x \in \Lambda_0 : \tau(x) = \rho(x + s),
\]
we need to count elements of $T_{\Lambda}/S$. In the degenerated case of a triangle border, the number of classes associated to the point $\Delta_i$ is
\[
\frac{1}{d} \sum_{k \mid (i, d)} \varphi(k) \binom{d/k}{i/k},
\]
the number of necklaces with $d$ beans, $i$ of them coloured. The total number of classes of tilings modulo shifts in an edge is
\[
\frac{1}{d} \sum_{k \mid d} \varphi(k) 2^{d/k}.
\]
In this way, we can define a generating function $Z_1(L, D, R) \in \mathbb{Z}[L, D, R]$ of $T_\Lambda/S$, analogue to $Z(L, D, R)$, with the same set of monomials indeed.

Let us consider now an involution which associates pairs of tilings with the same type. It is easily derived from Definition 1 that a $\Lambda$-periodic tiling can be defined through a bijection $\tilde{\tau}$ in $\Lambda_0$ such that $\tilde{\tau}(x) - x \in \{0, v - u, v\}$ and $\tilde{\tau}(x + y) = \tilde{\tau}(x) + y$, for all $x \in \Lambda_0$, $y \in \Lambda$. We define then the involution $I$ as follows:

$$(I \tilde{\tau})(x) = -\tilde{\tau}^{-1}(-x).$$

This operation is compatible with the relation $S$ defined above and it is indeed more natural to consider $I$ acting on $T_\Lambda/S$. In Section 2, we set the convention that the left image in Figure 6 represents a solic cube. Considering the inverse convention corresponds to looking at the pile of cubes “from behind”. This change of viewpoint is encoded by the involution $I$. Another interpretation arises from rotating $180^\circ$ the plane representation of the tiling.

We may also find redundant to compute as different tilings related by this involution. This allows another simplification in the set $T_\Lambda/S$, defining a coarser partition. In the degenerated case, the cardinality of the new quotient is the number of reversible necklaces with $d$ beads, $i$ of them coloured, but as before, the computation of the generating function $Z_2(L, D, R)$ seems a more difficult problem.

As we have seen, for types lying in the border of the fundamental triangle these functions correspond to well-known combinatorics formulas. However, we
are not able to efficiently compute them in the more interesting case of points interior to the triangle.

Let $\tau$ be a $\Lambda$-periodic tiling containing at least one lozenge in each orientations (i.e., no component in its type is zero). This is equivalent to the condition of having an inner corner (see Figure 6) in the tiling. We call flip to a transformation of one tiling into another by changing an inner corner into a solic cube, or vice versa. It is clear that this operation keeps the type of a tiling.

For example, a flip in the dotted quoin in the tiling from Figure 4 (indeed, that tiling has only a quoin and an inner corner) gives the tiling depicted in Figure 12 which has two quoins and two inner corners.

![Figure 12: $\delta(\tau, B) = (0, 3)$](image)

It follows from the proof of Theorem 6 that starting with a $\Lambda$-periodic tiling, one may obtain all the tilings with the same type by means of flips. Perhaps this consideration is useful in the task on determining $Z_1, Z_2$. On the other hand, it is likely that the Hafnian-Pfaffian method, which has proved useful in enumerating symmetry classes of tilings (see [10]), facilitates the evaluation of those functions.

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