Non-CMC solutions of the Einstein constraint equations on asymptotically Euclidean manifolds

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Abstract
In this note we prove two existence theorems for the Einstein constraint equations on asymptotically Euclidean manifolds. The first is for arbitrary mean curvature functions with restrictions on the size of the transverse-traceless data and the non-gravitational field data, while the second assumes a near-CMC condition, with no other restrictions.

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1. Introduction
This paper capitalizes on several recent advances concerning the existence of solutions of the Einstein constraint equations using the conformal method. Using these new techniques we construct solutions of these equations, both in the vacuum setting and also with coupled non-gravitational fields, on asymptotically Euclidean (AE) manifolds. This is carried out under two separate sets of hypotheses: either the mean curvature function $\tau$ is arbitrary but the transverse-traceless part of the data $\sigma$ (and the non-gravitational field densities) are very small (‘far-from-CMC’ data) or else with smallness assumptions on $d\tau/\tau$ (the ‘near-CMC’ case). We do this by adapting the methods of Holst–Nagy–Tsogtgerel [HNT08] and Maxwell [Ma09].
We recall that a Riemannian manifold \((M, \hat{g})\) and a symmetric 2-tensor \(\hat{K}\) on \(M\) satisfy the Einstein constraint equations with non-gravitational energy density \(\hat{\rho}\) (a scalar function) and non-gravitational momentum density \(\hat{J}\) (a vector field) if
\[
|\hat{K}|^2 - (\text{tr}_g \hat{K})^2 = R(\hat{g}) - \hat{\rho}, \\
\text{div}_g \hat{K} - \nabla \text{tr}_g \hat{K} = \hat{J}.
\]
(1)

Note that we work here only with non-gravitational fields which require no further constraints; this is the case for fluid fields, for example. (These results easily extend to theories such as Einstein–Maxwell which do introduce extra constraints; for brevity, we do not treat such cases here.) The cosmological constant \(\Lambda\) vanishes because all data here is AE.

The conformal method generates an initial data set \((M, \hat{g}, \hat{K}, \hat{\rho}, \hat{J})\) satisfying the constraints (1) by first (freely) choosing the conformal data: a Riemannian manifold \((M, g)\), a symmetric tensor \(\sigma\) which is transverse (\(\text{div}_g \sigma = 0\)) and traceless (\(\text{tr}_g \sigma = 0\)) with respect to \(g\), a scalar function \(\tau\) (the mean curvature), a non-negative scalar function \(\rho\), and a vector field \(J\). One then seeks a positive scalar function \(\phi\) and vector field \(W\) which solve the conformal constraint equations

\[
\begin{align*}
(\text{i}) & \quad \Delta_g \phi - cnR_g \phi + cn[a\sigma + DW_j^i g^{i-j} \phi^{-N-1} - b_n \phi^2 \phi^{N-1} + cn \rho \phi^{N/2}] = 0, \\
(\text{ii}) & \quad \Delta_L W + \frac{n-1}{n} \phi^n \nabla \tau + J = 0.
\end{align*}
\]
(2)

Here \(R_g\) is the scalar curvature of \(g\), \(D\) is the conformal Killing operator acting on vector fields
\[
(DW)_{ij} := \nabla_j W_i + \nabla_i W_j - \frac{2}{n} (\text{div}_g W) g_{ij},
\]
\(\Delta_g\) is the scalar Laplacian, \(\Delta_L := -\text{div} \circ D\) is the vector Laplacian, and
\[
N := \frac{2n}{n-2}, \quad c_n := \frac{n-2}{4(n-1)}, \quad b_n := \frac{n-2}{4n}.
\]

If \((\phi, W)\) is a solution of (2), then the initial data set
\[
\hat{g} = \phi^{\frac{2n}{n-2}} g, \quad \hat{K}^{ij} = \phi^{-\frac{2n}{n-2}} (\sigma + DW^j_i) + \phi^{\frac{n}{n-1}} \tau J^{ij}, \quad \hat{\rho} = \phi^{-\frac{2}{n-1}} \hat{\rho}, \quad \hat{J} = \phi^{-N} J,
\]
satisfies the Einstein constraints (1).

For convenience below, if \(v\) is any positive function (in a suitable function space), then \(W(v)\) denotes the solution of equation (ii) of (2), where this function \(v\) replaces \(\phi\) in the second term. Similarly, we write the Lichnerowicz operator on the left of equation (i) of (2) as \(\mathcal{N}(\phi, W)\). Thus a solution \((\phi, W)\) of the coupled system (2) corresponds to a solution \(\phi\) of the single non-local equation \(\mathcal{N}(\phi, W(\phi)) = 0\).

These equations decouple when \(\tau\) is constant (the CMC case), and incisive results are then available; see [Is95, CBIY00, Ma05]. In the near-CMC regime, when \(\nabla \tau / \tau\) is assumed to be sufficiently small, there are also fairly comprehensive results for compact manifolds [IsMo96, AIChs, Ma09], while in the AE case [CBIY00], the results are somewhat less complete.

The most striking recent advances, stemming from the paper of Holst, Nagy and Tsogtgerel [HNT08], later refined and simplified by Maxwell [Ma09], treat the case where \(\tau\) varies with no restrictions; these results are still perturbative in a different sense because they require \(\sigma, \rho,\) and \(J\) to be very small, hence are a special case of the general far-from-CMC case. These arguments rely on the Yamabe positivity of the underlying conformal class \([g]\) and on various smallness conditions to construct barriers. This particular far-from-CMC scenario has now been worked out in several settings. The original papers treat the case where \(M\) is closed; the more recent papers of Holst, Meier and Tsogtgerel [HMS13] and Dilts [Di13]
handle the case where $M$ is a manifold with boundary, considering a wide range of boundary conditions; finally, Leach [Le12] has dealt with the case where $M$ is complete with cylindrical ends. In the present paper we continue this line of research and prove an existence result in this far-from-CMC case for manifolds with AE ends, which is one of the standard and most important settings in relativity. The new issue to be faced here is the way that barriers must be constructed near infinity. This is similar to what must be done in the cylindrical case, but the argument here is simpler than in [Le12]. We also determine the precise asymptotics of solutions. Using very similar techniques, we also prove a near-CMC theorem for AE data, which overlaps with [CBIY00], though the proof here is considerably simpler.

We now state our main results. The metric $g$, as well as the rest of the fields comprising the initial data, are taken to lie in standard weighted Sobolev or Hölder spaces defined for AE tensor fields. The precise definitions of these spaces are given in the next section. The far-from-CMC result holds for data in either type of space, while the near-CMC result requires data to lie in weighted Hölder spaces. This near-CMC result is somewhat more quantitative than the ones appearing in [CBIY00], which has motivated its inclusion here.

**Theorem 1.1** (Far-from-CMC). Suppose that $(M, g)$ is a $W_2^{2,p}$ AE metric with positive Yamabe invariant, where $p > n$ and $\gamma \in (2 - n, 0)$, and set $\delta = \gamma / 2$. Fix data $\tau \in W_2^{2,p}$, $\sigma \in L_2^{2,1}$, non-negative $\rho \in L_2^{2,2}$, and $J \in L^p$, and assume that $|\sigma|_{2,1}, \|\rho\|_{2,2}$, and $|J|_{2,p}$ are sufficiently small (depending on $\tau$, $g$ and $n$). Then corresponding to each end $E_j$, $j = 1, \ldots, k$, of $M$ there is an open interval $I_j \subset (0, \infty)$ such that if $A_j \subset I_j$ for all $j$, then there exists a unique solution $(\phi, W)$ to (2) with $W \in W_2^{2,p}$, $W \in W_2^{2,p}$, $\phi > 0$ and $\phi - A_j \in W_2^{2,p}$ on $E_j$.

The analogous conclusion holds if $(M, g)$ is a $C^{2,\alpha}$ AE metric with positive Yamabe invariant, assuming that $\tau \in C_2^{1,\alpha}$, $\sigma \in C_2^{0,\alpha}$, $0 \leq \rho \in C_2^{0,\alpha}$ and $J \in C_2^{0,\alpha}$, with the norms of $\tau, \rho$ and $J$ suitably small. These solutions $(\phi, W)$ satisfy $W \in C_2^{2,\alpha}$ and $\phi - A_j \in C_2^{0,\alpha}$ on $E_j$.

A simpler way to express the asymptotic condition on $\phi$ is that $\phi - \omega \in W_2^{2,p}$, or $\phi - \omega \in C_2^{2,\alpha}$, where $\omega = \omega(A_1, \ldots, A_k)$ is a fixed strictly positive smooth function on $M$ such that $\omega \rightarrow A_j$ at infinity in $E_j$. We show in lemma 4.1 below that there is a unique harmonic function with these properties, so we assume for convenience that $\Delta_2 \omega = 0$. Since $(M, g)$ is AE, there is a coordinate system $x$ on each end $E_j$ so that $g \sim |dx|^2$. The solution metric $\hat{g} = \phi^2 g$ is asymptotic on $E_j$ to $A_2^{2,\alpha}(dx)^2$, but this is again AE since we can rescale the coordinate system by setting $y = A_2^{2,\alpha}$. To state the next result, fix a strictly positive smooth function $r$ on $M$ which equals the radial distance $|x|$ on each end $E_j$.

**Theorem 1.2** (Near-CMC). Let $(M, g)$ be AE of class $C^{2,\alpha}$, as in theorem 1.1, with $\gamma, \delta$ as above. Assume as well that $\tau \in C_2^{1,\alpha}$ and $\tau^2 - B \rho^{2\delta} \| \tau \|_{2,\alpha} \| \rho \|_{2,\alpha} > 0$ for some $B > 0$ depending on the conformal data and $|\tau|_{2,\alpha}$. Then there are intervals $I_j \subset (0, \infty)$ such that if $A_j \subset I_j$, then there exists a unique solution $(\phi, W)$ to (2) with $W \in C_2^{2,\alpha}$, $\phi > 0$ and $\phi - \omega \in C_2^{2,\alpha}$, where $\omega$ is the same harmonic function depending on the $A_j$ as above.

These results show that, for any fixed choice of the conformal data $(M, g, \sigma, \tau, \rho, J)$ satisfying the hypotheses of either of these two theorems, there is a $k$-dimensional family of solutions (where $k$ is the number of ends), parametrized by the product of the intervals $I_j$ for far-from-CMC data, and the product of intervals $I_j$ for near-CMC data. This non-uniqueness leads one to enquire about the full extent of these families of solutions. Unfortunately, neither the necessary analysis of the linearizations of the operators in (2), nor the a priori estimates.
for the solutions, is clear at this time, so we do not yet have more definitive results on the full family of solutions in either case.

The primary task in proving these theorems is to establish the existence of upper and lower barriers for equations (2). After discussing AE manifolds and function spaces in section 2, and then reviewing the mapping properties of the scalar and vector Laplacian operators on AE manifolds in section 3, we construct these barriers in section 4. A standard fixed point theorem is then used in section 5 to prove theorems 1.1 and 1.2.

2. Asymptotically Euclidean manifolds

Let \((M, g)\) be an AE manifold. This means that \(M\) is a complete manifold such that for some compact set \(K \subset M\), the complement \(M \setminus K\) has finitely many components, \(E_1, \ldots, E_k\), where each \(E_j\) is diffeomorphic to the exterior of a ball in a Euclidean space, \(E_j \cong \mathbb{R}^n \setminus B_R(0)\). These are called the ends of \(M\). On each of these ends, the metric \(g\) is asymptotic to the Euclidean metric.

To make this more precise, recall that a function \(u \in W^{k, p}_\delta(M)\) if

\[
\sum_{|\beta| \leq k} ||r^{-\delta - \frac{n}{2} + |\beta|} \partial^\beta u||_{L^p} < \infty.
\]

Here, as above, \(r\) is a smooth positive function on \(M\) which agrees with the radial function \(|x|\) on each end. Similarly, we say that \(u \in C^{k, \alpha}_\delta(M)\) if

\[
|\nabla^j u| \leq C r^{\delta - j}, \quad \text{and} \quad \sup_{B} |\nabla^k u|_{B, \alpha} \leq C r^{\delta - k}
\]

for each \(j \leq k\). The supremum in the second term is over all balls \(B \subset M\) of unit radius, and \([\cdot]_{B, \alpha}\) denotes the Hölder seminorm on that ball. To extend any of these spaces and norms to tensors, as needed in the characterization of the decay of the metric \(g\) above, we require the corresponding regularity and decay for each component with respect to a constant frame in the background Euclidean metric.

Thus we say that \(g\) is AE of class \(W^{k, p}_\gamma\), for some \(\gamma < 0\), if in some fixed Euclidean coordinate system for that end,

\[
g|_{E_j} - g_{\text{Eucl}} \in W^{k, p}_\gamma,
\]

and similarly, \(g\) is AE of class \(C^{k, \alpha}_\gamma\) if this difference lies in \(C^{k, \alpha}_\gamma\). The regularity of the tensor field \(\hat{K}\) and the scalar and vector fields \(\hat{\rho}\) and \(\hat{J}\) are defined analogously. We refer the reader to [Ba86] for a survey of the well-known properties of these weighted Sobolev spaces, and to the appendix in [CMP12] for further discussion and references for these weighted Hölder spaces.

We single out one fact about the weighted Sobolev spaces which we use repeatedly: if \(p > n\), and \(w \in W^{k, p}_\delta\) for any \(\delta \in \mathbb{R}\), then

\[
|w| \leq r^\delta ||w||_{W^{k, p}}.
\]

In the following, we always assume that \((M, g)\) is AE of class \(W^{2, p}_\gamma\) with \(p > n\) and \(2(2 - n) < \gamma < 0\), or else is AE of class \(C^{2, \alpha}_\gamma\), for the same range of \(\gamma\). We also always assume that \(\delta = \gamma / 2\), so \(2 - n < \delta < 0\). All results below have obvious modifications if we assume that \(g\) is AE of class \(W^{k, p}_\gamma\) with \(k > 1 + n/p\), or of class \(C^{k, \alpha}_\gamma\) with \(k > 2\), and if we assume as well that the other components of the data have the correspondingly higher regularity.
3. Mapping properties of the scalar and vector Laplacians

The mapping properties of elliptic operators on \( \text{AE} \) spaces is now classical, going back at least to [Mc79], but see also [Ma05], [CBIY00] and the appendix in [CMP12]. We record a few such results pertaining to the solvability of the inhomogeneous linear equation

\[ Pu = f, \]

where \( P \) is either the conformal Laplacian \( \Delta_{c} - c_{n}R \) or the vector Laplacian \( \Delta_{\nabla} \).

**Proposition 3.1.** If \((M, g)\) is \( \text{AE} \) of class \( W^{2,p}_{\gamma} \), then

\[ P : W^{2,p}_{\gamma} \longrightarrow L^{p}_{\gamma-2} \quad (4) \]

is Fredholm of index zero, and there exists a constant \( C > 0 \) such that

\[ \| \psi \|_{W^{2,p}_{\gamma}} \leq C \| P \psi \|_{L^{p}_{\gamma-2}}, \]

for all \( \psi \in W^{2,p}_{\gamma} \). The map (4) is an isomorphism if and only if \( P \) has no nullspace in \( W^{2,p}_{\gamma} \). For \( P = \Delta_{c} - c_{n}R \), this is the case provided the Yamabe invariant \( \mathcal{Y}(|g|) \) is positive, while if \( P = \Delta_{\nabla} \), this holds assuming that \((M, g)\) admits no global conformal Killing fields. Under this isomorphism condition, the a priori estimate above can be strengthened to

\[ \| \psi \|_{W^{2,p}_{\gamma}} \leq C \| P \psi \|_{L^{p}_{\gamma-2}}. \]

Similarly, if \((M, g)\) is \( \text{AE} \) of class \( C^{2,\alpha}_{\gamma} \), then

\[ P : C^{2,\alpha}_{\gamma} \longrightarrow C^{0,\alpha}_{\gamma-2} \quad (6) \]

is Fredholm of index zero, with a corresponding a priori estimate. If \( P \) has no nullspace in \( C^{2,\alpha}_{\gamma} \), then there exists a constant \( C > 0 \) such that

\[ \| \psi \|_{C^{2,\alpha}_{\gamma}} \leq C \| P \psi \|_{C^{0,\alpha}_{\gamma-2}}. \]

We record two useful corollaries.

**Proposition 3.2.** Assume that \( \mathcal{Y}(|g|) \) is positive and let \( P \) be the conformal Laplacian \( \Delta_{c} - c_{n}R \). If \( f = r^{\gamma'} - \tilde{f} \in L^{p}_{\gamma'-2} \) for \( \gamma' < \gamma \), then there is a unique solution \( w \) to \( Pw = f \) with \( w = c_{\gamma} r^{\gamma'} + \tilde{w} \), \( c_{\gamma} = (\gamma'^{2} + (n - 2)\gamma) \), and \( \tilde{w} \in W^{2,p}_{\gamma} \) where \( \gamma' = \max(\gamma', 2\gamma) \) if this number is greater than \( 2 - n \) (or else \( \gamma' = (2 - n) \)).

Similarly, if \( \tilde{f} \in C^{0,\alpha}_{\gamma-2} \), then this unique solution \( w \) decomposes as \( c_{\gamma} r^{\gamma'} + \tilde{w} \), with \( \tilde{w} \in C^{2,\alpha}_{\gamma-2} \).

**Proof.** Write \( w = c_{\gamma} r^{\gamma'} + \tilde{w} \) and let \( \tilde{g} \) be a \( W^{2,p} \) metric which agrees with \( g \) away from the ends but is exactly Euclidean on each \( E_{j} \). Then we must solve

\[ (\Delta_{c} - c_{n}R)\tilde{w} = \tilde{f} + (r^{\gamma' - 2} - c_{\gamma} (\Delta_{c} - R_{c})) r^{\gamma'} - c_{\gamma} ((\Delta_{c} - c_{n}R) - (\Delta_{\tilde{g}} - c_{n}R_{\tilde{g}})) r^{\gamma'}. \]

The second term on the right is \( L^{p}_{\gamma'-2} \) with compact support, while the third term lies in \( L^{p}_{\gamma'-2} \), so the entire right hand side is \( L^{p}_{\gamma'-2} \). Since the Yamabe positivity of \( g \) rules out the kernel of \( P \), the result follows from proposition 3.1.

The proof in the Hölder setting is the same. \( \square \)

The second of these is a slight weakening of a standard Schauder estimate (in this weighted setting) which is adequate for our purposes.

**Proposition 3.3.** If \((M, g)\) is \( \text{AE} \) and has no conformal Killing fields, and if \( f \in L^{p}_{\gamma-2} \), then the unique solution \( W \in W^{2,p}_{\gamma} \) to \( \Delta_{\nabla} W = f \) satisfies

\[ \| D W \|_{L^{p}_{\gamma-2}} \leq C_{1} r^{\delta} \| f \|_{L^{p}_{\gamma-2}}, \]

(7)
Proof. Combining (5) and (3), we get
\[ r^{1-4}|DW| \leq \|DW\|_{L^2} \leq C_1 \|DW\|_{W^{1,4}} \leq C_1 \|W\|_{W^{1,4}} \leq C_1 \|f\|_{L^2}, \]
which gives (7).
\[ \square \]

4. Barriers

We begin by recalling the notion of global sub- and supersolutions. The function \( \phi_+ \) is called a global supersolution for (2) if \( \mathcal{N}(\phi_+, W(\phi)) \leq 0 \) whenever \( 0 < \phi \leq \phi_+ \). Similarly, \( \phi_- \) is called a global subsolution if \( \mathcal{N}(\phi_-, W(\phi)) \geq 0 \) whenever \( \phi \in L^p \) and \( \phi_- < \phi \).

In the first main result of this section we construct global supersolutions, allowing the mean curvature to be arbitrary but requiring that the other data be quite small.

Theorem 4.1 (Far-from-CMC global supersolution). Let \((M, g)\) be AE of class \( W^{2, p}_c \) with positive Yamabe invariant, \( \mathcal{Y}(g) > 0 \). If \( \|\sigma\|_{L^2} \), \( \|J\|_{L^2} \), and \( \|\rho\|_{L^2} \) are sufficiently small, then there exists a global supersolution \( \phi_+ > 0 \) with \( \phi_+ - \eta \in W^{2, p}_c \) for some constant \( \eta > 0 \).

Similarly, if \((M, g)\) is AE of class \( C^{2, \alpha}_c \) with positive Yamabe invariant, and if the corresponding Hölder norms of \( \sigma, J \frac{\partial}{\partial \gamma} \), and \( \rho \) are sufficiently small, then there exists a global supersolution \( \phi_+ \) with \( \phi_+ - \eta \in C^{2, \alpha}_\gamma \) for some \( \eta > 0 \).

The main ideas used in this proof are similar to those used in the compact case, but there are new issues arising in the construction of the supersolution on each end. Because the proofs in the Sobolev and Hölder settings are identical, we present only the former.

Proof. Choose a smooth, positive function \( F \) which equals \( r^{-\gamma} \) outside a compact set (recall that \( \gamma \) indexes the asymptotic behavior of the AE metric). By proposition 3.1, there exists a (unique) \( \Psi = c_1 + \Psi \), with \( \Psi \in W^{2, p}_c \) such that
\[ (\Delta - c_0 R) \Psi = -F + c_0 R, \]
or equivalently
\[ (\Delta - c_0 R)(1 + \Psi) = -F. \]
Note that, by the maximum principle, \( 1 + \Psi > 0 \).

Now set \( \phi_+ = \eta(1 + \Psi) \), where the constant \( \eta > 0 \) is to be chosen below. We claim that, for appropriate \( \eta \), \( \phi_+ \) is a global supersolution. To verify this, we first note that from (7), with \( f = \frac{\alpha - 1}{\pi} \phi^\gamma \nabla \phi + J \), we have
\[ \|DW\|_{L^2} \leq C r^{1-4} (\|\sigma\|_{L^2} \|\phi\|_{L^2} + \|J\|_{L^2}), \]
and hence
\[ |\sigma + DW|^2 \leq C r^{2\alpha - 2} (\|\sigma\|_{L^2} \|\phi\|_{L^2} + \|J\|_{L^2}^2). \]

Since \( \Psi \) decays at the precise rate \( r^\gamma \) (and is strictly positive), then deleting subscripts denoting the norms for simplicity, we calculate
\[ \mathcal{N}(\phi_+, W(\phi)) \leq -\eta F + r^{2\alpha - 2} (C_1 \eta^{N-1} + C_2 \eta^{-N-1} (|\sigma|^2 + |J|^2) + C_3 \eta^{-\frac{\gamma}{2}} |\rho|)), \]
The constants \( C_1 \), \( C_2 \) and \( C_3 \) depend only on \( F \) and the dimension \( n \). Since \( 2\delta - 2 = \gamma - 2 < 0 \) and \( N = 1 \), we first choose \( \eta \) sufficiently small so that
\[ -\frac{1}{2} \eta F + C_1 \eta^{N-1} r^{2\alpha - 2} < 0, \]
and then choose \( |\sigma|, |J| \) and \( |\rho| \) sufficiently small (depending on \( C_1, F, n \) and \( \eta \)), so that
\[ -\frac{1}{2} \eta F + r^{2\alpha - 2} (C_2 \eta^{N-1} (|\sigma|^2 + |J|^2) + C_3 \eta^{-\frac{\gamma}{2}} |\rho|)) < 0 \]
as well. This proves that \( \phi_+ \) is a global supersolution. \( \square \)
Theorem 4.1 supplies one key part of the proof of theorem 1.1. The corresponding step in the proof of theorem 1.2 is contained in the following:

**Theorem 4.2 (Near-CMC global supersolution).** Let \((M, g)\) be AE of class \(C^{2,\alpha}_\gamma\) and fix \(\rho \in C^{0,\alpha}_{23-2}\), \(\rho > 0\), \(J \in C^{0,\alpha}_{43-2}\) and \(\sigma \in C^{0,\alpha}_{43-1}\). Then there exists a constant \(B > 0\), depending on the conformal data and \(\|\tau\|_{C^{4,\alpha}_{3-1}}\), such that if \(\tau \in C^{1,\alpha}_{3-1}\) satisfies \(\tau^2 - B\|\tau\|^2_{C^{2,\alpha}_{3-2}} \geq 0\), then there exists a global supersolution \(\phi_\tau > 0\) for (2) such that \(\phi_\tau - \eta \in C^{2,\alpha}_\gamma\) for some constant \(\eta \gg 0\).

**Remark 4.1.** The hypothesis \(\tau^2 - Br^{23-2}\|\tau\|^2_{C^{2,\alpha}_{3-4}} > 0\) is precisely where the use of Hölder rather than Sobolev data is important for AE data. Indeed, if \(\tau\) satisfies this inequality, then in particular, \(\tau \geq C\rho^{\gamma-2}\), so the norm of \(\tau\) in \(L^4_{\rho}\) is necessarily infinite.

The statements \(\tau^2 - B\|\tau\|^2_{C^{2,\alpha}_{3-4}} \geq 0\), which in particular imposes a lower bound on the decay of \(\tau\) and requires that \(\tau\) never vanishes, and that \(B\) is a constant depending on \(\|\tau\|_{C^{4,\alpha}_{3-1}}\) present competing, and not necessarily compatible, requirements. When \(M\) is compact, then for the corresponding condition phrased in terms of Sobolev spaces, as in [HNT08], it seems to be the case that for certain metrics \(g\) there are no functions \(\tau\) which satisfy this inequality. The situation is less clear when \(M\) is AE, though it is likely that the condition may be vacuous for certain choices of the other data.

**Proof.** If \(\tau \neq 0\), then \(\tau^2 \geq C\rho^{23-2} = C\tau^{\gamma-2}\) for some \(C > 0\), and so since \(R \geq -C\tau^{\gamma-2}\), there exists \(\kappa > 0\) such that \(c_nR + \kappa b_n\tau^2 \geq 0\). It is now possible to choose \(u = 1 + \tilde{u}, v = 1 + \hat{v}\) with \(\tilde{u}, \hat{v} \in C^{2,\alpha}_\gamma\) and so that

\[
(\Delta - c_nR)u = b_n\kappa \tau^2 u,
\]

\[
\nabla(u^2\nabla v) - b_n\tau^2 v = -c_n(\rho + |\sigma|^2).
\]

Indeed, the first of these is equivalent to

\[
(\Delta - (c_nR + b_n\kappa \tau^2))\hat{u} = c_nR + b_n\kappa \tau^2 \in C^{0,\alpha}_{\gamma-2},
\]

so there is a unique solution in \(C^{2,\alpha}_\gamma\). The second equation is solvable by similar considerations. Since \(u, v \to 1\), there is a finite Harnack constant

\[
\lambda := \frac{\sup_{\Omega} uv}{\inf_{\Omega} uv} < \infty.
\]

We now estimate \(\lambda\) in terms of the norm of \(\tau\) as follows. Since

\[
\frac{\sup_{\Omega} uv}{\inf_{\Omega} uv} \leq \left(\frac{\sup_{\Omega} u}{\inf_{\Omega} u}\right) \left(\frac{\sup_{\Omega} v}{\inf_{\Omega} v}\right)
\]

it suffices to estimate the Harnack constants for each equation separately. We first estimate \(u\) and use this result to show that the second equation is uniformly elliptic, and then estimate \(v\). Assume that \(\|\tau\|_{C^{4,\alpha}_{3-1}} \leq \mu\). Since \(c_nR + b_n\kappa \tau^2 \geq 0\), the function \(u\) is subharmonic, so \(\lim_{|x| \to \infty} u = \sup u = 1\). Hence it suffices to find an a priori lower bound \(\inf u \geq c > 0\), where \(c\) depends only on \(\mu\). Choose \(\xi \in C^{0,\alpha}_{\gamma-2}\) so that \(c_nR + b_n\kappa \tau^2 \leq \xi\) whenever \(\|\tau\|_{C^{4,\alpha}_{3-1}} \leq \mu\) as above. Let \(\tilde{u}\) be the unique solution of \((\Delta - \xi)\tilde{u} = 0, \tilde{u} \to 1\). Then it is clear that the solution \(u\) associated to any such \(\tau\) is bounded below by \(\inf \tilde{u} > 0\). A similar sub- and supersolution argument then allows one to bound \(\sup v / \inf v\) in terms of \(\mu\).

Now set \(\phi_\tau = \eta uv\), where \(\eta\) is chosen below, and calculate

\[
u(\Delta - c_nR)\phi_\tau = \eta u(\nabla(v\nabla u + u\nabla v) - c_nRuv)
\]

\[
= \eta u(\Delta - c_nR)u + \eta v(\nabla^2 u) = \eta(b_n\kappa \tau^2 v(1 + \kappa u^2) - c_n(\rho + |\sigma|^2)),
\]

7
which then gives
\[
u N(\phi_+, W(\phi)) = \eta(b_\alpha \tau^2 v + b_\alpha \tau^2 \kappa u^2 v - c_\alpha (\rho + |\sigma|^2)) - b_\alpha \tau^2 (\eta v)^N - N^{-1} u^N + c_\alpha (\rho (\eta v)^{-N/2} u^{-N/2+1})
\]
\[
\leq b_\alpha \tau^2 (-(\eta v)^{-N-1} u^N + \eta v + \eta \kappa u^2 v) - c_\alpha (\rho + |\sigma|^2)
\]
\[+ 2c_\alpha (|\sigma|^2 + |DW|^2 (\eta v)^{-N-1} u^{-N} + c_\alpha \rho (\eta v)^{-N/2} u^{-N/2+1}).\]

By (10) and the inequality \(\phi \leq \phi_+\), we have
\[
|DW|^2 \leq 2^{2N} ((\sup sup \phi)^N |d\tau|)^2 + |J|_{\gamma_2}^2 
\]
\[
\leq 2C_2^2 r^{2N} (\eta^{2N} (\sup sup \eta)^N |d\tau|)^2 + |J|_{\gamma_2}^2
\]
\[
\leq 2C_2^2 r^{2N} (\eta^{2N} \lambda^{2N} |\eta^{2N} (\eta v)^N + |J|_{\gamma_2}^2);\]

the constant \(C\) is exactly the one appearing in (10). This leads to the estimate
\[
u N(\phi_+, W(\phi)) \leq c_\alpha (|\sigma|^2 (-(\eta + 2\eta^{N-1} (\eta v)^{-N+1} u) + c_\alpha (\rho + |\sigma|^2)
\]
\[
+ \eta^{N-1} \left(-\frac{b_\alpha}{2} \tau^2 + C_1 \lambda^{2N} |d\tau|_{\gamma_2}^2 r^{2N-2} \right) u (\eta v)^{N-1}
\]
\[
+ \left(-\frac{b_\alpha}{2} \eta^{N-1} \tau^2 (\eta v)^{-N+1} u + \eta b_\alpha \tau^2 \eta^{N-1} (uv)^{-N+1} \right)
\]
\[
+ C_2 \eta^{N-1} (uv)^{-N+1} u (\eta v)^{N-2} |\eta v|_{\gamma_2}^2.\]

All five terms here can be made negative using the hypotheses of the theorem and by making \(\eta\) sufficiently large. The only one that needs comment is the third parenthetical summand. The constant \(C_1\) here depends only on the metric \(g\) and the dimension, so to make this term negative we must require that
\[
\frac{2}{b_\alpha} C_1 \lambda^{2N} r^{2N-2} ||d\tau||^2 \leq \tau^2,
\]
or in other words, that the constant \(B\) in the hypothesis satisfies \(B \geq (2C_1/b_\alpha) \lambda^{2N}\). We have shown that the right hand side in this last inequality has a bound depending on \(|d\tau|_{\gamma_2}^2\), which means that it is possible to choose a suitable constant \(B\) which depends on \(n, C_1\) and this norm of \(\tau\).

We now turn to the construction of a global subsolution. This turns out to be the same for both the far-from-CMC and near-CMC cases.

**Theorem 4.3** (Global subsolution). Let \((M, g, \sigma, \tau, \rho, J)\) be a set of conformal data satisfying the hypotheses of either theorem 1.1 or 1.2. Let \(\psi \in W_0^{1,p}\), respectively \(\psi \in C_0^{1,p}\), be chosen so that \(1 + \psi > 0\) and \(\tilde{g} = (1 + \psi)^{N-2} g\) has scalar curvature \(R_{\tilde{g}} = -\frac{\alpha^2}{\lambda^2} \tau^2\) (see the remark below). Then \(\alpha (1 + \psi)\) is a global subsolution for any \(0 < \alpha \leq 1\).

**Proof.** Having defined \(\psi\) in this way, let \(\phi_- = \alpha (1 + \psi)\). Then
\[
\mathcal{N}(\phi_-, W(\phi)) = b_\alpha \tau^2 (1 + \psi)^{N-1} (\alpha - \alpha^{N-1})
\]
\[+ |\sigma + DW(\phi)|^2 (\alpha (1 + \psi))^{-N-1} + c_\alpha (\alpha (1 + \psi))^{-N/2} \geq 0,
\]
as required. Note that this inequality holds regardless of whether or not \(\phi \geq \phi_-\).

**Remark 4.2.** It is known, both on compact manifolds [Ma05] and in this AE setting [DiGI14], that the prescribed scalar curvature problem in the hypothesis here is solvable if and only if the Lichnerowicz equation itself has a solution. The solvability of this problem on the AE spaces considered in this paper is proved in [CBIY00, section 7].
Remark 4.3. In the proof of theorem 4.1, we could have also have taken \( \phi_+ = \beta (\Psi + \omega) \), where \( \omega \) is the harmonic function which tends towards any specified constant values \( B_j > 0 \) on each end \( E_j \), where
\[
(\Delta - c_p R)(\Psi + \omega) = -F.
\]
Similarly, in the proof of theorem 4.2 we could have let \( u = w + \omega \) be a solution to
\[
(\Delta - (c_p R + b_p \kappa \tau^2))(w + \omega) = 0.
\]
This function \( u \) tends to \( B_j \) on \( E_j \). If the function \( v \) is defined as in that proof but using this new \( u \), then \( \phi_+ = \eta (uv) \) is a supersolution when \( \eta \gg 0 \).

We recall briefly that such harmonic functions do exist.

Lemma 4.1. Fix constants \( B_j > 0 \), \( j = 1, \ldots, k \). Then there exists a unique harmonic function \( \omega \) such that \( \omega \to B_j \) on \( E_j \).

Proof. Let \( \tilde{\omega} = \sum \chi_j B_j \), where \( \chi_j \) is a cutoff function which equals 1 on \( E_j \) and vanishes outside a neighborhood of this end. Then \( \Delta \tilde{\omega} \in L^p_{q-2} \), and there exists a unique \( \hat{\omega} \in W^{2,p} \) with \( \Delta \hat{\omega} = \Delta \tilde{\omega} \). Then \( \omega = \tilde{\omega} - \hat{\omega} \) has the desired properties. Its uniqueness follows from the maximum principle.

Remark 4.3 and lemma 4.3 immediately give the following.

Corollary 4.1. Let either the assumptions of theorem 4.1 or 4.2 hold. For any \( B_j > 0 \), \( 1 \leq i \leq k \), there exists a global supersolution \( \phi_+ \) such that \( \phi_+ \to \eta B_i \) on \( E_i \). In the setting of theorem 4.1, \( \eta \) is chosen sufficiently small, while in theorem 4.2, \( \eta \) must be chosen quite large.

Since \( \phi_- \to \alpha \) and \( \phi_+ \to \eta B_i \) at infinity, and since both are strictly positive, we can choose \( \alpha \) sufficiently small so that \( \phi_- < \phi_+ \) everywhere.

5. Fixed point theorem and proof of the main results

Just as for the analogous far-from-CMC results on closed manifolds [HNT08] and [Ma09], once the existence of global sub- and supersolutions has been established, the existence of solutions \( (\phi, W) \) to (2) with \( \phi \) tending to suitably prescribed asymptotic constants on each end is obtained using the Schauder fixed point theorem. This proof is quite similar to the one for closed manifolds, so we only sketch it here.

Theorem 5.1. Fix an \( \Lambda \) conformal data set satisfying the hypotheses of either theorem 1.1 or 1.2. Then there are intervals \( I_j \subset (0, \infty) \), \( j = 1, \ldots, k \), such that if \( A_j \in I_j \), then there exists a solution \( (\phi, W) \) to (2), with \( \phi_- \leq \phi \leq \phi_+ \) and \( \phi - \omega \in W^{2,p} \), respectively \( \phi - \omega \in C^{2,\alpha} \).

Proof. As before, we work in the Sobolev space setting, since the Hölder space setting is almost identical. Let \( C^0_+ \) denote the set of strictly positive bounded functions on \( M \). (In the Hölder setting, this is replaced by \( C^{0,\alpha} \)). If \( \phi \in C^0_+ \), then by proposition 3.1, the vector field \( W(\phi) \in W^{2,p} \) is well-defined. Next, define \( I_j = [a_j^-, a_j^+] \) where \( a_j^\pm \) is the limit of \( \phi_\pm \) on \( E_j \) (it follows from the constructions of \( \phi_+ \) and \( \phi_- \) in section 4 that these limits exist). Choose any values \( A_j \in I_j \), and let \( \omega \) be the harmonic function with these limiting values. The space of such harmonic functions is identified with the product \( \mathcal{I} = \prod I_j \subset \mathbb{R}^k \). For any \( W \in W^{2,p} \) and \( A \in \mathcal{I} \), the proof in [CBIY00, appendix B] carries over directly to produce the unique solution \( \phi = T(W) = T(W, A) \) to \( \mathcal{N}(\phi, W) = 0 \) with \( \phi - \omega \in W^{2,p} \). Write \( \mathcal{N}(\phi, W) = 0 \) as
\[ L\phi = f(x, W, \phi), \]  
where \( L = (\Delta - c_\mu R_g - \mu) \) for some constant \( \mu \) which is chosen so large that \( \phi \mapsto f(x, W, \phi) \) is monotone decreasing when \( \phi_- \leq \phi \leq \phi_+ \). Now follow the usual monotone iteration scheme, solving for a sequence of functions \( \phi_\ell = \omega + \psi_\ell, \psi_\ell \in W^{2,p}_\delta \),

where

\[ L\phi_1 = f(x, W, \phi_-), \quad \text{and} \quad L\phi_{\ell+1} = f(x, W, \phi_\ell), \quad \ell \geq 1. \]

We see, as in the compact case, that the \( \phi_\ell \) converge to a solution \( \phi = \omega + \psi, \psi \in W^{2,p}_\delta \).

Still fixing \( A \in I \), let \( S \) denote the compact inclusion \( \mathbb{R}^k \oplus W^{2,p}_\delta \hookrightarrow C^0 \). A solution \( (\phi, W) \) to (2) corresponds to a fixed point of the mapping \( Q = S \circ T \circ W \). The continuity of \( W \) and \( S \) are obvious, while the continuity of \( T \) follows from the implicit function theorem.

Define the bounded convex set \( S := \{ \phi \in C^0_+ : \phi_- \leq \phi \leq \phi_+ \} \). By construction, \( Q \) maps \( S \) to itself, and hence \( Q(S) \) is relatively compact. Denote by \( \mathcal{H} \) its closed convex hull. Thus \( \mathcal{H} \subset S \), and \( Q : \mathcal{H} \rightarrow \mathcal{H} \). By the Schauder fixed point theorem, \( \mathcal{H} \) contains a fixed point \( \phi \) of \( Q \). Standard estimates imply that \( \phi \) and \( W(\phi) \) both have the desired regularity.

\[ \square \]

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