The Hamiltonian structure of a coupled system derived from a supersymmetric breaking of Super KdV equations

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Abstract

A supersymmetric breaking procedure for $N = 1$ Super KdV, using a Clifford algebra, is implemented. Dirac’s method for the determination of constraints is used to obtain the Hamiltonian structure, via a Lagrangian, for the resulting solitonic system of coupled Korteweg-de Vries type system. It is shown that the Hamiltonian obtained by this procedure is bounded from below and in that sense represents a model which is physically admissible.

Keywords: supersymmetric models, integrable systems, conservation laws, nonlinear dynamics of solitons, partial differential equations

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1 Introduction

The coupled systems which are extensions of the Korteweg-de Vries (KdV) equation arise in several physical problems and have interesting properties. Such is the case of the coupled Ito system [1], which describes the interaction of two internal long waves. Among the properties of this system are the existence of multisolitonic solutions which may be obtained using the bilinear Hirota method [2,3], the symmetries and conserved quantities
as well as the existence of the Painlevé property, Lax pair and Bäcklund transformations \[5\]. The supersymmetric KdV (SKdV) \[6, 7\] system is described by coupled systems of partial differential equations in terms of bosonic and fermionic fields. Such extensions of KdV equation have infinite local and non-local conserved quantities \[8, 6, 7, 9, 10, 11, 12\], Lax pairs and at least one hamiltonian structure. Recently, using a bosonization approach \[13\], exact solutions for the \(N = 1\) (SKdV) \[14, 15\] and for the supersymmetric Ito equation \[16\] have been obtained. The bosonization approach avoids to deal directly with Grassmann valued fields, not suitable for many practical purposes, for example, in the search for new solutions and in the analysis of the stability of solitonic solutions.

An important aspect related to those systems is the supersymmetry breaking and the resulting coupled equations derived from that procedure. We consider here a supersymmetric breaking implemented by replacing the Grassmann algebra by a Clifford algebra. This scheme was already used in several works, see for example \[17\].

In that sense we present in this paper a coupled Korteweg-de Vries system, with fields valued on a Clifford algebra, which has four local conserved quantities and solitonic solutions, and obtain its hamiltonian structure, with the consequent Poisson brackets between the respective fields. For any Lagrangian system one can always apply the Dirac’s method for analyzing the constraints of the theory and obtain from it the hamiltonian structure of the system \[18\]. This approach was used in \[19, 20\] to obtain the, previously known, first and second hamiltonian structures of KdV equation.

We also use the Dirac’s method to obtain the hamiltonian structure of the coupled system obtained from the supersymmetric breaking scheme. We start obtaining a Lagrangian formulation for the system then derive the hamiltonian via a Legendre transformation and apply Dirac’s method to obtain the constraints in the phase space. It turns out that they are second class constraints. We thus use the Dirac’s brackets to obtain the Poisson structure of the system in the constrained phase space. We finally prove that the emerging hamiltonian is bounded from below and this property grants a physically admissible content.

\section{Supersymmetry breaking procedure}

We denote by \(u(x, t)\) and \(\xi(x, t)\) the fields describing \(N = 1\) SKdV equations \[6\], taking values at the even and odd part of a Grassmann algebra respectively. The SKdV are

\begin{align}
  u_t &= -u'''' + 6uu' - 3\xi\xi'' \\
  \xi_t &= -\xi'''' + 3(\xi u)'.
\end{align} \quad (1)
This system of partial differential equations have infinite local conserved charges as well as infinite non-local conserved charges [10, 11, 12]. The first few of them are

\[H_2 = \int_{-\infty}^{+\infty} \xi \, dx, \]
\[H_1 = \int_{-\infty}^{+\infty} u \, dx, \]
\[H_3 = \int_{-\infty}^{+\infty} (u^2 - \xi \xi') \, dx, \]
\[H_5 = \int_{-\infty}^{+\infty} (2u^3 + (u')^2 - \xi'\xi'' - 4u\xi\xi') \, dx, \]

which we give explicitly in order to compare with the conserved charges of the system with broken supersymmetry we will consider.

\[H_1\] and \[H_3\] become manifestly positive self-adjoint operators in the quantum formulation of the theory. The fields \(u\) and \(\xi\) may be expanded in terms of the Virasoro generators of a superconformal algebra which may be realized in terms of an oscillator algebra [7, 21, 22]. Using normal ordering the expressions of \(H_1\) and \(H_2\) are manifestly positive. In particular \(H_1\) is the hamiltonian of a supersymmetric harmonic oscillator.

Besides \(H_1\) and \(H_3\) we are also interested in \(H_2\) and \(H_5\). \(H_2\) will be relevant in a stability analysis and \(H_5\) is very important because stationary points of \(H_5\) subject to \(H_3\) give rise to solitonic solutions. Hence we would like to break supersymmetry in a way to have a positive \(H_3\) and conserved charges analogous to \(H_2\), \(H_1\), \(H_5\). It turns out that \(H_5\) is the hamiltonian of the new system obtained by breaking supersymmetry.

To break supersymmetry we consider the fields \(u\) and \(\xi\) to take values on a Clifford algebra instead of being Grassmann algebra valued. We thus take \(u\) to be a real valued field while \(\xi\) to be an expansion in terms of an odd number of the generators \(e_i\), \(i = 1, \ldots\) of the Clifford algebra:

\[\xi = \sum_{i=1}^{\infty} \varphi_i e_i + \sum_{ijk} \varphi_{ijk} e_i e_j e_k + \cdots \]  

where

\[e_i e_j + e_j e_i = -2\delta_{ij} \]  

and \(\varphi_i, \varphi_{ijk}, \ldots\) are real valued functions. We define \(\bar{\xi} = \sum_{i=1}^{\infty} \varphi_i \bar{e}_i + \sum_{ijk} \varphi_{ijk} \bar{e}_i \bar{e}_j \bar{e}_k + \cdots\) where \(\bar{e}_i = -e_i\). We denote as in superfield notation the body of the expansion those terms associated with the identity generator and the soul the remaining ones. Consequently the body of \(\xi\bar{\xi}\), denoted by \(\mathcal{P}(\xi\bar{\xi})\), is equal to \(\sum_i \varphi_i^2 + \sum_{ijk} \varphi_{ijk}^2 + \cdots\) In what follows, without lost of generality, we rewrite \(\sum_{i=1}^{\infty} \varphi_i^2 + \sum_{ijk} \varphi_{ijk}^2\) simply as \(\mathcal{P}(\xi\bar{\xi}) = \Sigma_i \varphi_i^2\).

We now propose the following system of partial differential equations which has the required properties as discussed before,

\[u_t = -u''' - uu' - \frac{1}{2}(\mathcal{P}(\xi\bar{\xi}))' \]
\[\xi_t = -\xi''' - \frac{1}{2}(\xi u)' \]  

This system with a change of sign in the third term of the right hand member of the first equation of system (5) was introduced from a different point of view in [25, 26]. On
the other hand the system (5) in the particular case of only two components is included in the classification given in [27].

The system (5) has the following conserved charges

$$
\begin{align*}
\hat{H}_1 &= \int_{-\infty}^{+\infty} \xi dx, \\
\hat{H}_2 &= \int_{-\infty}^{+\infty} u dx, \\
V &\equiv H_3 = \int_{-\infty}^{+\infty} \left( u^2 + \mathcal{P}(\xi \bar{\xi}) \right) dx, \\
M &\equiv H_5 = \int_{-\infty}^{+\infty} \left( -\frac{1}{3} u^3 - \frac{1}{2} u \mathcal{P}(\xi \bar{\xi}) + (u')^2 + \mathcal{P}(\xi' \bar{\xi'}) \right) dx.
\end{align*}
$$

(6)

It is interesting to remark that the following non-local conserved charge of Super KdV [12] is also a non-local conserved charge for the system (5), in terms of the Clifford algebra valued field $\xi$,

$$
\int_{-\infty}^{\infty} \xi(x) \int_{-\infty}^{x} \xi(s) ds dx.
$$

However the non-local conserved charges of Super KdV in [10] are not conserved by the system (5). For example,

$$
\int_{-\infty}^{\infty} u(x) \int_{-\infty}^{x} \xi(s) ds dx
$$

is not conserved by (5).

We notice that

$$
\int_{-\infty}^{\infty} \mathcal{P}(\xi \bar{\xi}) dx = \sum_{i=1}^{\infty} \varphi_i^2 = ||\xi||_L^2
$$

(7)

hence $V$ is manifestly positive definite, one of the properties we were looking forward to have. The system has solitonic solutions, for example: $u(x,t) \equiv \phi(x,t) = 3C \frac{1}{\cosh(z)}$, $z \equiv \frac{1}{2} C^2 (x - (1 + C)t + a) \ (a \in \mathbb{R})$, where $\phi(x,t)$ is the one-soliton solution of KdV equation, together with $\xi(x,t) = 0$ is a one soliton solution of the new system. In the same way the multi-solitonic solutions of KdV together with $\xi(x,t) = 0$ are solutions of the new system.

The system (5) is not invariant under supersymmetric transformations, as expected. Moreover, there isn’t a conserved charge of dimension 7, that is there is no analogue of $H_7$ as in SKdV or KdV systems. The mechanism has not only broken supersymmetry but also the symmetries related to $H_7$ and probably to all higher local higher dimensional ones. There remain, however, (6) as conserved charges of the system.

In the following section we will consider an extension of the system (5) which depends on a parameter $\lambda$. For $\lambda = 1$ we will recover system (5). We will prove that the system is Lagrangian and we will derive the Hamiltonian structure of it.

## 3 The Hamiltonian structure

We introduce the fields $w$ and $\eta_i$ defined by

$$
u = w' \quad \text{and} \quad \varphi_i = \eta_i'.
$$

(8)
The following Lagrangian formulated in terms of $w$ and $\eta_i$

$$S(w, \eta_i) = \int dx dt \left[ \frac{1}{2} w' \partial_t w + \frac{1}{6} (w')^3 - \frac{1}{2} (w'')^2 + \frac{1}{4} \lambda w' (\eta_i')^2 + \frac{1}{2} \eta_i' \partial_t \eta_i - \frac{1}{2} (\eta_i'')^2 \right],$$  \hspace{1cm} (9)

where a repeated index $i$ implies summation on that index, yields under variations of $w$ and $\eta_i$ the system of equations

$$\partial_t u = -u''' - \frac{1}{2} (u^2)' - \frac{1}{4} \lambda (\phi_i^2)',$$
$$\partial_t \phi_i = -\phi_i''' - \frac{1}{2} \lambda (u \phi_i'),$$ \hspace{1cm} (10)

When $\lambda = 1$ it reduces to (5), the only case for which the dynamical equation for the field $\xi$ is the same as the equation for the odd Grassmann field in the Super KdV equations. It has the following local conserved charges

$$\tilde{H}_1 = \int_{-\infty}^{+\infty} \xi dx,$$
$$\tilde{H}_1 = \int_{-\infty}^{+\infty} u dx,$$
$$\tilde{H}_3 = \int_{-\infty}^{+\infty} (u^2 + \mathcal{P}(\xi \bar{\xi})) dx,$$
$$\tilde{H}_5 = \int_{-\infty}^{+\infty} \left( -\frac{1}{3} u^3 - \frac{1}{2} \lambda u \mathcal{P}(\xi \bar{\xi}) + (u')^2 + \mathcal{P}(\xi' \bar{\xi}') \right) dx.$$ \hspace{1cm} (11)

as well as the non-local conserved charge

$$\int_{-\infty}^{\infty} \varphi_i(x) \int_{-\infty}^{x} \varphi_i(s) ds dx.$$ \hspace{1cm} (12)

We will now show that $\tilde{H}_5$ is the Hamiltonian of the system (10). We also remark that $\tilde{H}_3$ is equal to the $L_2$ norm of the Clifford algebra valued fields,

$$\tilde{H}_3 = \| (u, \xi) \|_{L_2}^2 \geq 0.$$ \hspace{1cm} (13)

In order to construct the Hamiltonian of system (10) we introduce the conjugate momenta to $(w, \eta_i)$. They will be denoted by $(p, \sigma_i)$. We have

$$p := \frac{\partial \mathcal{L}}{\partial (\partial_t w)} = \frac{1}{2} w' = \frac{1}{2} u$$
$$\sigma_i := \frac{\partial \mathcal{L}}{\partial (\partial_t \eta_i)} = \frac{1}{2} \eta_i' = \frac{1}{2} \phi_i.$$ \hspace{1cm} (14)

Since from (14) we cannot obtain $\partial_t w$ and $\partial_t \eta_i$ in terms of their conjugate momenta, then these equations are primary constraints on the phase space \([15]\).

In order to obtain the Hamiltonian of the system we perform a Legendre transformation

$$H = \langle p \partial_t w + \sigma_i \partial_t \eta_i - \mathcal{L} \rangle_x$$ \hspace{1cm} (15)

where $\langle \rangle_x$ denotes integration on the whole real line using $x$ as the integration variable.
We obtain
\[ H = \frac{1}{2} \tilde{H}_5, \quad (16) \]
where \( \tilde{H}_5 \) is given in equation (11).

Following the Dirac approach to obtain the Hamiltonian structure of equations (10), we consider the canonical Hamiltonian
\[ \tilde{H} = H + \left\{ \Lambda \left( p - \frac{1}{2} w' \right) + \Lambda_i \left( \sigma_i - \frac{1}{2} \eta'_i \right) \right\}_x, \quad (17) \]
where \( \Lambda \) and \( \Lambda_i \) are Lagrange multipliers.

The conservation of \( p - \frac{1}{2} w' \) yields
\[ \left\{ p - \frac{1}{2} w', \tilde{H} \right\} = -w'w'' - \frac{\lambda}{4} (\eta'_i)^2' - \Lambda = 0 \quad (18) \]
and the conservation of \( \sigma_i - \frac{1}{2} \eta'_i = 0 \) implies
\[ \left\{ \sigma_i - \frac{1}{2} \eta'_i, \tilde{H} \right\} = - \frac{\lambda}{2} (w' \eta'_i)' - \eta''''_i - \Lambda_i = 0, \quad (19) \]
where \( \{,\} \) denotes the Poisson bracket on the original unconstrained phase space.

(18) and (19) determine the Lagrange multipliers \( \Lambda \) and \( \Lambda_i \) respectively. Hence the Dirac procedure ends up with these equations. There are no more contraints in the phase space.

It turns out that both constraints (14) are second class ones.

In fact, if we denote
\[ v := p - \frac{1}{2} w', \quad v_i := \sigma_i - \frac{1}{2} \eta'_i \quad (20) \]
and \( v_I := (v, v_i) \), we then have
\[ \{ v_I(x), v_J(x') \} = -\delta_{IJ} \partial_x \delta(x - x'). \quad (21) \]

The Poisson structure of the constrained Hamiltonian is then determined by the Dirac brackets [18]. For any two functionals on the phase space \( F \) and \( G \), the Dirac bracket is defined as
\[ \{ F, G \}_{DB} := \{ F, G \} - \langle \{ F, v_I(x') \} \{ v_I(x'), v_J(x'') \}^{-1} \cdot \{ v_J(x''), G \} \rangle_{x''}, \quad (22) \]
where
\[ \langle \{ v_I(x'), v_J(x'') \}^{-1} g(x'') \rangle_{x''} = -\delta_{IJ} \int_{-\infty}^{x'} g(\tilde{x}) d\tilde{x}. \quad (23) \]
We then have
\[
\begin{align*}
\{u(x), u(y)\}_{DB} &= \partial_x \delta(x, y), \\
\{\varphi_i(x), \varphi_j(y)\}_{DB} &= \delta_{ij} \partial_x \delta(x, y), \\
\{u(x), \varphi_i(y)\}_{DB} &= 0.
\end{align*}
\]
Consequently,
\[
\begin{align*}
\partial_t u &= \{u, H\}_{DB} = -\frac{1}{2} (u^2)' - u'' - \frac{\lambda}{4} (\varphi_i^2)' \\
\partial_t \varphi_i &= \{\varphi_i, H\}_{DB} = -\varphi_i''' - \frac{\lambda}{2} (u \varphi_i)',
\end{align*}
\] (24)

where \(H\) is given by (16) and can be directly expressed in terms of \(u\) and \(\xi\). Notice that \(\{u, \tilde{H}\}_{DB} = \{u, H\}_{DB}\) and the same happens for any function of \(u\) and \(\varphi_i\).

We have then derived the Poisson structure of the Hamiltonian system. It follows directly from the existence of a Lagrangian for the dynamical system. The Dirac procedure determines the constraints of the phase space together with the Poisson structure. The final Poisson structure is obtained from the Dirac bracket (which satisfies the properties of a Poisson bracket, in particular the Jacobi identity).

4 Properties of the Hamiltonian

An important property of the Hamiltonian of a physical system is that it is bounded from below. The Hamiltonian of the Super KdV system satisfies that property, in fact it is positive. We now show that the Hamiltonian \(H\) of the dynamical system arising from the breaking of supersymmetry indeed has also this property.

We consider
\[
\tilde{H}_3 + \tilde{H}_5 = \|(u, \xi)\|_{H_1}^2 + \int_{-\infty}^{+\infty} \left( -\frac{1}{3} u^3 - \frac{\lambda}{2} u \mathcal{P}(\xi \bar{\xi}) \right) dx
\]
where the Sobolev norm \(\|\cdot\|_{H_1}\) is defined by
\[
\|(u, \xi)\|_{H_1}^2 := \int_{-\infty}^{+\infty} \left[ u^2 + \mathcal{P}(\xi \bar{\xi}) + u^2 + \mathcal{P}(\xi' \bar{\xi}') \right] dx.
\]
We also noticed that
\[
\tilde{H}_3 = \|(u, \xi)\|_{L^2}^2
\]
where \(\|\cdot\|_{L^2}\) is the \(L^2\) norm.

We then have
\[
\tilde{H}_3 + \tilde{H}_5 \geq \|(u, \xi)\|_{H_1}^2 - \frac{m}{2} \int_{-\infty}^{+\infty} |u| \left( u^2 + \mathcal{P}(\xi \bar{\xi}) \right) dx
\]
where \(m = \max(1, |\lambda|)\).

We now use the bound
\[
\sup |u| \leq \frac{\|u\|_{H_1}}{\sqrt{2}} \leq \frac{\|(u, \xi)\|_{H_1}}{\sqrt{2}},
\]
to obtain
\[
\tilde{H}_3 + \tilde{H}_5 \geq \|(u, \xi)\|_{H_1}^2 - \frac{m}{2\sqrt{2}} \|(u, \xi)\|_{H_1} \|(u, \xi)\|_{L^2}.
\]
Consequently
\[
\tilde{H}_3 + \tilde{H}_5 + \left(\frac{m}{4\sqrt{2}}\right)^2 \tilde{H}_3 \geq \left(\|(u, \xi)\|_{H_1} - \frac{m}{4\sqrt{2}} \|(u, \xi)\|_{L^2}\right)^2 \geq 0.
\]
Finally
\[
\tilde{H}_5 \geq - \left(1 + \left(\frac{m}{4\sqrt{2}}\right)^2\right) \tilde{H}_3,
\]
hence for a normalized state satisfying \(\|(u, \xi)\|_{L^2} = 1\) we have
\[
\tilde{H}_5 \geq - \left(1 + \left(\frac{m}{4\sqrt{2}}\right)^2\right).
\]
The Hamiltonian is then bounded from below in the space of normalized \(L_2\) configurations and it is thus physically admissible.

We notice that is important to have \(\tilde{H}_3 = \|(u, \xi)\|_{L_2}^2\) in order to conclude this property of the Hamiltonian. The dynamical system (10) for any value of \(\lambda\) has then a Hamiltonian structure which is physically admissible.

For the particular value \(\lambda = 2\), the system (10) is invariant under the Galileo transformations
\[
t \rightarrow \hat{t} = t \\
x \rightarrow \hat{x} = x + ct \\
u \rightarrow \hat{u} = u + c \\
\phi_i \rightarrow \hat{\phi}_i = \phi_i.
\]
However for this particular value of \(\lambda\) the system decouples into independent KdV equations.

The dynamical system (10) has multi-solitonic solutions corresponding to \(u\) a multi-soliton of KdV and \(\xi = 0\). It is well known that these solutions are Liapunov stable for the KdV equation [23, 24]. The same stability problem for the system (10) cannot be deduced straightforwardly from it since small initial perturbations of \(\xi\) may grow to produce instabilities in the system. A detailed analysis of this problem is presented in [28], where we show that the solitonic solutions of the system (5) has nice stability properties. On the other side, it is straightforward to show that the solution \(u = \xi = 0\) corresponding to the minimum of \(\tilde{H}_3\) is \(L^2\) stable. In fact, given \(\epsilon\), since \(\tilde{H}_3 = \|(u, \xi)\|_{L^2}\) is conserved under the dynamics of system (10), if initially the perturbation satisfies \(\|(u, \xi)\|_{L^2} < \delta < \epsilon\) then is also bounded by \(\epsilon\) for any \(t > 0\).
5 Conclusions

We discussed a solitonic KdV coupled system defined in terms of a Clifford algebra. It was derived from a supersymmetric breaking of $N = 1$ SKdV equation. This procedure not only breaks the supersymmetry but also the symmetries related to the higher order local conserved quantities of Super KdV equations. Only a finite number of local conserved quantities remain valid. Nevertheless these conserved quantities are enough to ensure nice stability properties of the solitonic solutions [28]. The coupled system has also non-local conserved quantities, a non trivial one is explicitly shown in the paper. It is exactly the same conserved quantity that appears in the SKdV $N = 1$ system.

We obtained the hamiltonian structure and the consequent Poisson bracket of the system, using the Dirac’s method for analyzing the constraints. We also proved that the emerging hamiltonian is bounded from below and in that sense it has a physically admissible content.

We believe that the procedure followed in this work can be used in several of the known KdV coupled systems.

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