ABSTRACT

Gravitational instantons of Bianchi type IX space are constructed in Ashtekar’s canonical formalism. Instead of solving the self-duality condition, we fully solve the constraint on the “initial surface” and “Hamiltonian equations”. This formalism is applicable to the matter coupled system with cosmological constant.
1. Introduction

In Ashtekar’s canonical formalism of general relativity,\(^1\) gravity is described as the singular system but with only polynomial constraints. This bonus is paid with the expense of extending the canonical variables to complex ones. So the system is subject to so called reality condition which guarantees the equivalence with Einstein gravity. This reality condition, however, may spoil the nice feature of Ashtekar’s formalism.\(^2\) This condition comes from the fact that the (anti) self-duality complexizes spin connection in Lorenzian region.

In a quantum theory of gravity, gravitational instantons\(^3\) are expected to play essential roles as Yang-Mills instantons do in quantum chromo-dynamics.\(^4\) In Euclidean region, Ashtekar’s formalism is free from the reality condition and may make a breakthrough in quantum theory of gravity.\(^5\)

The purpose of the present paper is to give various four-dimensional gravitational instanton solutions in Ashtekar’s formalism. This article is the detailed and enriched explanation of our previous report.\(^6\) Explicitly, we examine Euclidean Bianchi type IX space because it is especially suitable to treat in Ashtekar’s formalism.\(^7\) We have not found the new solution of instanton. However, the procedures in solving Einstein equation are quite different from the conventional ones. That is, instead of solving the self-duality condition, we fully solve the constraints on the ‘initial surface’ and ‘Hamilton equations’. So this formalism is applicable to the matter coupled systems with cosmological constant.

This paper is organized as follows. In section two we give the general framework of Ashtekar’s formalism in homogeneous space with Euclidean signature, which is applied to the special case of Bianchi type IX space. In section three we consider the most simple solutions of Bianchi type IX. They give the vanishing \(SO(3)\) field strength. It is shown that they correspond to Eguchi-Hanson\(^8\) and Euclidean Taub-NUT\(^9\) solutions. Then we proceed to discuss the same Bianchi IX space with cosmological constant in section four. In this case the field strength cannot be zero from Hamiltonian constraint. However, we can find the very simple
solutions in the special cases. They are Euclidean de Sitter and Fubini-Study solutions.\cite{3} Furthermore, by adopting the general ansatz in field strength, we can derive the Taub-NUT-de Sitter metric.\cite{8} Section five is devoted to discussions. Throughout this paper we only consider four dimensional Euclidean Einstein equations of homogeneous universes.

2. General Framework

Euclidean homogeneous universe is characterized by the line element

\[ ds^2 = N^2 d\tau^2 + \delta_{ab} e_i^a e_j^b \sigma^i \sigma^j. \]  \hspace{1cm} (2.1)

Here \( e_i^a \) are triad depending only on Euclidean time \( \tau \). First alphabets \( a, b, \ldots \) (middle alphabets \( i, j, \ldots \)) indicate spatial part of internal indices (that of world indices). The symbols \( \sigma^i \) are left invariant one-forms which satisfy

\[ d\sigma^i = C^i_{jk} \sigma^j \wedge \sigma^k, \] \hspace{1cm} (2.2)

where \( C^i_{jk} \) is the structure constant.

Ashtekar's \( SO(3) \) field strength is defined by

\[ F^a = dA^a - \frac{\kappa}{2} \epsilon^a_{bc} A^b \wedge A^c, \] \hspace{1cm} (2.3)

where \( A^a \) is one form \( A^a \equiv A^a_i \sigma^i \) and \( \kappa = 8\pi G \). Eq.(2.3) comes from the self-dual Riemann curvature,

\[ ^+R^{0a} = d^+1\omega^0 - \epsilon^a_{bc} ^+\omega^0 \wedge ^+\omega^0 \] \hspace{1cm} (2.4)

with an identification

\[ A^a \equiv \frac{2}{\kappa} ^+\omega^0 \equiv \frac{2}{\kappa} (\omega^0 + \frac{1}{2} \epsilon^a_{bc} \omega^b \omega^c). \] \hspace{1cm} (2.5)

For later convenience we will consider the action with the cosmological constant.
The Jacobson-Smolin's action\textsuperscript{[10]}
\[
S = \frac{1}{\kappa} \int (R - \Lambda) \sqrt{g} d^4 x
\]  
(2.6)
is transformed to
\[
S = \int d^4 x \left[ \bar{e}_a^i \dot{A}_i^a + D_i \bar{e}_a^i A_0^a + N^j \bar{e}_a^i F_{ji}^a + \frac{1}{2} \bar{N} \{ \epsilon_a^{\ b c} \bar{e}_b^i \bar{e}_c^j F_{ij}^a - \frac{2}{\kappa} \det(\bar{e}_a^i) \Lambda \} \right]
\]  
(2.7)
by the (3+1) decomposition;
\[
e^{00} = 1/N, \quad e^{i0} = N^i/N, \quad e^{0a} = 0, \quad e^{ia} = (3)\bar{e}^{ia}.
\]  
(2.8)
Here
\[
\bar{N} = N/\det(e_i^a), \quad \text{and} \quad \bar{e}_a^i = \det(e_i^a)\bar{e}_a^i.
\]
Dot denotes derivative with respect to $\tau$. From Eq.(2.7) we know that $\bar{e}_a^i$ and $A_i^a$ are canonical partners and $A_0^a$ are multipliers for the internal $SO(3)$ gauge rotation. Thus, we assume $A_i^a$ are depending only on $\tau$ as $\bar{e}_a^i$. Therefore, eq. (2.3) is expressed as
\[
F_{ij}^a = 2A_k^a C_{ij}^k - \kappa \epsilon_{bc}^a \bar{e}_b^i \bar{e}_c^j.
\]  
(2.9)
The whole first class constraints are expressed as polynomial forms:
- Hamiltonian constraint: $\epsilon_a^{\ b c} F_{ij}^a \bar{e}_b^i \bar{e}_c^j - \frac{2}{\kappa} \det(\bar{e}_a^i) \Lambda = 0$,  
(2.10)
- Momentum constraint: $F_{ij}^a \bar{e}_a^j = 0$,  
(2.11)
- Gauss’ law constraint: $D_i \bar{e}_a^i = 2C_{ji}^j \bar{e}_a^j - \kappa \epsilon_{ab}^c A_i^b \bar{e}_a^c = 0$.  
(2.12)
Time developments of new variables are given from the Poisson bracket with Hamiltonian;
\[
\dot{\bar{e}}_a^i = \bar{N} (C_{jk}^i \epsilon_a^{\ b c} \bar{e}_b^j \bar{e}_c^k - \kappa \bar{e}_a^i \bar{e}_b^j A_j^b + \kappa \bar{e}_b^i A_j^b \bar{e}_a^j),
\]  
(2.13)
\[
\dot{A}_i^a = -\bar{N} (F_{ij}^b \epsilon_b^{\ ac} \bar{e}_c^j - \frac{1}{\kappa} \tilde{\Delta}_i^a \Lambda),
\]  
(2.14)
where $\tilde{\Delta}_i^a$ is $(i, a)$ element of the cofactor of $\det(\bar{e}_a^i)$. 

In the subsequent sections we will deal with the Bianchi type IX space. That is

\[ C_{jk}^i = \epsilon_{jk}^i, \tag{2.15} \]

where \( \epsilon_{jk}^i \) is fully antisymmetric tensor with \( \epsilon_{23}^1 = 1 \). In this case Ashtekar’s formulation takes especially a simple form and is solved explicitly. Hereafter, we take \( \kappa = 2 \) unit.

3. Solutions without cosmological constant

In this and the subsequent sections, we consider exact Euclidean solutions of Einstein equation in the Bianchi type IX space. In this section we consider the case with \( \Lambda = 0 \).

The system has twofold \( SO(3) \) invariance. One comes from the Ashtekar’s field strength, that is, the invariance with respect to the internal coordinates. Another is due to the spatial property of Bianchi type IX universe, i.e., the invariance with respect to the world coordinates. These invariances allow twofold rotations

\[ T_a^b \epsilon^i_a T^{ij}_i = \epsilon^j_b, \]

then we can always set

\[ \tilde{\epsilon}^i_a = \text{diag} \,(X(\tau), Y(\tau), Z(\tau)). \tag{3.1} \]

The most simple solution of Eqs. (2.10) and (2.11) is the pure gauge solution

\[ \gamma F_{ij} \equiv F_{ij}^a \tau_a = 0, \tag{3.2} \]

where \( \tau_a \) is the generator of \( SO(3) \). This solution is not trivial in contrast to the Yang-Mills theory and does not imply a flat metric. Hereafter, we recognize
\( \gamma = 1, 2, 3 \) as \( x, y, z \), respectively. From Eq.(2.3), Eq.(3.2) takes the form

\[
\frac{1}{2} \gamma F_{xy} = \gamma A_z - \gamma A_x \times \gamma A_y = 0
\]

(cyclic permutation with respect to \( x, y \) and \( z \)),

where \( \gamma A_x = A^a_x \epsilon_a \) and the symbol \( \times \) means vector product in the internal space.

The solutions to Eq.(3.3) are

(i) \( \gamma A_x = \gamma A_y = \gamma A_z = 0 \) \hspace{1cm} (3.4)

or

(ii) \( \gamma A_x, \gamma A_y, \gamma A_z \) are orthonormal. \hspace{1cm} (3.5)

The Gauss' law constraint in the Bianchi type IX space is reduced to

\[
\gamma A_i \times \gamma e^i = 0
\]

Namely, \( \gamma A_i \) is parallel to \( \gamma e^i \) in the case (ii).

Equations of motion for \( A_i^a \) are automatically satisfied due to the vanishing field strength. So we may concentrate on the dynamical equations of \( \epsilon^i_a \). We will discuss the solutions (i) and (ii) separately in the remaining part of this section.

3-(i) Vanishing \( \gamma A_i \)

Substituting the diagonal form (3.1) into the equation of motion (2.13), we get

\[
\dot{X} = 2\gamma Y Z, \quad \dot{Y} = 2\gamma Z X, \quad \dot{Z} = 2\gamma X Y. \hspace{1cm} (3.6)
\]

In order to solve Eq.(3.6) we fix lapse function \( \gamma N \) as

\[
\gamma N = \frac{1}{4XYZ}. \hspace{1cm} (3.7)
\]
So $\dot{X}$ satisfies $X\dot{X} = 1/2$ etc. and therefore

$$
X = \sqrt{\tau - C_1}, \quad Y = \sqrt{\tau - C_2}, \quad Z = \sqrt{\tau - C_3}, \quad (3.8)
$$

where $C_1, C_2$ and $C_3$ are arbitrary constants. The line element is

$$
ds^2 = N^2 d\tau^2 + \frac{YZ}{X}\sigma_x^2 + \frac{ZX}{Y}\sigma_y^2 + \frac{XY}{Z}\sigma_z^2. \quad (3.9)
$$

Here the original lapse function $N$ is

$$
N^2 \equiv \tilde{N}^2 (\det e_i^a)^2 = \left(\frac{1}{4XYZ}\right)^2 XYZ. \quad (3.10)
$$

Thus from Eqs. (3.8), (3.9) and (3.10) we obtain the metric form given by Belinskii et al.$^{[11]}$

$$
ds^2 = \frac{d\tau^2}{16\sqrt{(\tau - C_1)(\tau - C_2)(\tau - C_3)}} + \sqrt{\frac{(\tau - C_2)(\tau - C_3)}{(\tau - C_1)}} \sigma_x^2 \\
+ \sqrt{\frac{(\tau - C_3)(\tau - C_1)}{(\tau - C_2)}} \sigma_y^2 + \sqrt{\frac{(\tau - C_1)(\tau - C_2)}{(\tau - C_3)}} \sigma_z^2. \quad (3.11)
$$

Let us introduce new coordinate $r$ defined by $r^4 = \tau$, and assume axial symmetry, i.e., $C_1 = C_2 = a^4, C_3 = 0$. Then we get

$$
ds^2 = \{1 - \left(\frac{a}{r}\right)^4\}^{-1} dr^2 + r^2 \left[\sigma_x^2 + \sigma_y^2 + \{1 - \left(\frac{a}{r}\right)^4\} \sigma_z^2\right]. \quad (3.12)
$$

This is nothing but Eguchi-Hanson metric.$^{[8]}$

3-(ii) Orthonormal $\gamma A_i$
In the case of orthonormal solution, the discussions of the previous subsection are valid with the modification of Eq.(3.6) to

\[
\begin{align*}
\dot{X} &= 2\mathcal{N}\{YZ - X(Y + Z)\}, \\
\dot{Y} &= 2\mathcal{N}\{ZX - Y(Z + X)\}, \\
\dot{Z} &= 2\mathcal{N}\{XY - Z(X + Y)\}.
\end{align*}
\]

(3.13)

We assume axial symmetry \(X = Y\). Then Eq.(3.13) is reduced to

\[
\begin{align*}
\dot{X} &= -2\mathcal{N}X^2, \\
\dot{Z} &= 2\mathcal{N}\{X^2 - 2ZX\}.
\end{align*}
\]

(3.14)

If we fix lapse function \(\mathcal{N}\) as

\[
\mathcal{N} = \frac{1}{2X},
\]

Eq.(3.14) is integrated to

\[
X = Ce^{-\tau}, \quad Z = De^{-2\tau} + Ce^{-\tau},
\]

(3.15)

where \(C\) and \(D\) are integration constants. The original laps is

\[
N^2 = \mathcal{N}^2X^2Z = \frac{Z}{4}.
\]

(3.16)

Eqs.(3.9), (3.15) and (3.16) give the line element

\[
ds^2 = \frac{Z}{4}d\tau^2 + Z(\sigma_x^2 + \sigma_y^2) + \frac{X^2}{Z}\sigma_z^2.
\]

(3.17)

When \(D = 0\) the metric is flat. When \(D \neq 0\) we introduce new coordinate \(r\) as

\[
Z = De^{-2\tau} + Ce^{-\tau} = \text{sgn}(D)(r^2 - M^2),
\]

(3.18)

where \(\text{sgn}(D)\) is the signature of \(D\) and \(M^2 = C^2/4|D|\). Then \(X\) becomes

\[
X = 2\text{sgn}(D)M(\pm r - M)
\]
from Eqs.(3.15) and (3.18). Thus we obtain the line element

\[ ds^2 = \pm \left[ \frac{1}{4(r-M)} dr^2 + (r^2 - M^2) \{ \sigma_x^2 + \sigma_y^2 \} + 4M^2 \frac{r-M}{r+M} \sigma_z^2 \right]. \tag{3.19} \]

This is the Euclidean Taub-NUT metric.

4. Solution with cosmological constant

Anti self-duality has solved the ‘Euclidean’ initial value problems and we have obtained Eguchi-Hanson (E-H) and Euclidean Taub-NUT (T-N) solutions. As indicated by Eguchi and Hanson,\(^8\) E-H and T-N solutions compose the triplet together with Fubini-Study (F-S) solution. The last one is the solution to the Einstein gravity with cosmological constant. The presence of cosmological constant prevents the anti duality to be the solution to Hamiltonian constraint. In this section we consider gravitational instantons with cosmological constant known as Taub-NUT-de Sitter solution.

Hamiltonian constraint (2.10) suggests that \( F_{ij}^a \) is related by \( \tilde{e}^i_a \) algebraically. So we assume

\[ F_{ij}^a = M^{ab} \tilde{e}_b^k \epsilon_{kij}. \tag{4.1} \]

From momentum constraint (2.11), \( M^{ab} \) is symmetric. For diagonal \( \tilde{e}_a^i \),

\[ \tilde{e}_a^i = \text{diag}(X(\tau), Y(\tau), Z(\tau)), \tag{4.2} \]

Hamiltonian constraint leads

\[ M_a^a = \frac{\Lambda}{2}. \tag{4.3} \]

From the equation of motion (2.13), it is obvious that the diagonal form of \( A_i^a \),

\[ A_i^a = \text{diag}(\alpha(\tau), \beta(\tau), \gamma(\tau)), \tag{4.4} \]

guarantees \( \tilde{e}_a^i \) to evolve retaining its diagonal form. Eqs.(4.2) and (4.4) satisfy momentum and Gauss’ law constraint trivially in the Bianchi IX space. For the
diagonal $A_i^a$ of Eq.(4.4), non-vanishing components of $SO(3)$ field strength become

\[ F_{xy}^3 = 2(\gamma - \alpha \beta), \]
\[ F_{yz}^1 = 2(\alpha - \beta \gamma), \]
\[ F_{zx}^2 = 2(\beta - \gamma \alpha). \]  

Thus, $M^{ab}$ has only diagonal components. Then Eq.(4.1) is reduced to

\[ F_{ij}^a = \mu^a \epsilon^k_a e_{kij} \quad (a : \text{no summation}), \]  

with

\[ \sum_a \mu^a = \frac{\Lambda}{2}. \]  

The ansatz Eq.(4.6) with Eq.(4.7) is an extension of the Ashtekar-Renteln or the Samuel ansatz.[12]

Here we, furthermore, assume axial symmetry $X = Y$. The invariant line element reads

\[ ds^2 = N^2 d\tau^2 + Z(\sigma_x^2 + \sigma_y^2) + \frac{X^2}{Z} \sigma_z^2, \]

where the laps function is given by

\[ N^2 = \mathcal{N} X^2 Z. \]  

The equation of motion (2.13) leads us to $\alpha = \beta$, then

\[ \dot{X} = 2\mathcal{N} X \{Z(1 - \gamma) - \alpha X\}, \]
\[ \dot{Z} = 2\mathcal{N} X(X - 2\alpha Z). \]  

Similarly, Eq.(2.14) requires

\[ \dot{\alpha} = 2\mathcal{N} \alpha(1 - \gamma) Z, \]
\[ \dot{\gamma} = 2\mathcal{N} \frac{X^2}{Z}(\gamma - \alpha^2). \]  

In Eq.(4.11) use has been made of Eq.(2.10).
From Eqs. (4.5) and (4.6), we obtain
\[ 2(\gamma - \alpha^2) = \mu^3 Z, \]
\[ 2\alpha(1 - \gamma) = \mu^1 X \] (4.12)
with \(2\mu^1 + \mu^3 = \Lambda/2\).

Compatibility of the equations of motion for \(X, Z, \alpha\) and \(\beta\) gives the evolution equations for \(\mu^a\)
\[ \dot{\mu}^1 = -2\mathcal{N}\alpha X(\mu^3 - \mu^1), \]
\[ \dot{\mu}^3 = 4\mathcal{N}\alpha X(\mu^3 - \mu^1). \] (4.13)
Since non-degenerate metric requires \(X \neq 0\) then \(\mu^a\) can be constant when \(\alpha = 0\) or \(\mu^1 = \mu^3\). Otherwise, Eq.(4.13) forces \(\mu^a\) to be also functions of \(\tau\). We discuss these cases separately.

4-(i) \(\alpha = 0\) (\(\mu^1 = 0\)) case

In \(\alpha = 0\) case, from Eqs.(4.7) and (4.12), we get \(\mu^1 = 0, \mu^3 = \Lambda/2\) then
\[ 4\gamma = \Lambda Z. \] (4.14)

We find the equations of motion for \(X, Z\) from Eqs.(4.10) and (4.14),
\[ \dot{X} = 2\mathcal{N}XZ(1 - \frac{\Lambda}{4}Z), \]
\[ \dot{Z} = 2\mathcal{N}X^2. \] (4.15)
By taking the laps function as
\[ \mathcal{N} = \frac{1}{2X^2}, \] (4.16)
Eq.(4.15) is easily solved as follows
\[ Z = \tau - \tau_0, \]
\[ X^2 = -\frac{\Lambda}{6}(\tau - \tau_0)^3 + (\tau - \tau_0)^2 + C, \] (4.17)
where $\tau_0$ and $C$ are integration constants. Using Eq.(4.14) we get

$$\alpha = 0, \quad \gamma = \frac{\Lambda}{4}(\tau - \tau_0).$$  (4.18)

From Eq.(4.8) with Eqs.(4.9) and (4.16), we obtain the metric in the form

$$ds^2 = \frac{Z}{4X^2}d\tau^2 + Z(\sigma_x^2 + \sigma_y^2) + \frac{X^2}{Z}\sigma_z^2,$$  (4.19)

where $X, Z$ are given by Eq.(4.17). In $\Lambda = 0$ limit, Eq.(4.18) shows $A_i^a = 0$. Therefore, this solution is an extended one of E-H solution to $\Lambda \neq 0$ case.

Let us introduce $r$ instead of $\tau$ by

$$Z = \tau - \tau_0 = r^2,$$

then we get

$$ds^2 = \left\{1 - \left(\frac{a}{r}\right)^4 - \frac{\Lambda}{6}r^2\right\}^{-1} dr^2 + r^2 \left[\sigma_x^2 + \sigma_y^2 + \left(1 - \left(\frac{a}{r}\right)^4 - \frac{\Lambda}{6}r^2\right)\sigma_z^2\right],$$  (4.20)

where we have set $C = -a^4$.

When $C = 0$ the metric (4.19) is Fubini-Study metric. To see this explicitly, we introduce another coordinate $\rho$ by

$$Z = \frac{\rho^2}{1 + \frac{4\Lambda}{3}\rho^2}.$$  (4.21)

and set $C = 0$. Then it follows that

$$X^2 = \frac{\rho^4}{(1 + \frac{4\Lambda}{3}\rho^2)^3}.$$  (4.22)

Temporal part is

$$N^2d\tau^2 = N^2X^2Zd\tau^2 = \frac{d\rho^2}{(1 + \frac{\Lambda}{6}\rho^2)^2}.$$
Therefore, the invariant line element turns out to be

\[ ds^2 = \frac{1}{(1 + \frac{\Lambda}{6} \rho^2)^2} (d\rho^2 + \rho^2 \sigma_z^2) + \frac{\rho^2}{1 + \frac{\Lambda}{6} \rho^2} (\sigma_x^2 + \sigma_y^2) \]  \hspace{1cm} (4.23)

from Eq.(4.8). This is the Fubini-Study metric in conventional form.

4-(ii) \( \mu^1 = \mu^3 \) case

In this case, \( \mu^a = \frac{\Lambda}{6} \) then (4.1) becomes\(^6\)

\[ F_{ij}^a = \frac{\Lambda}{3!} \epsilon_{ijk} \tilde{e}^{ka}. \]  \hspace{1cm} (4.24)

This case is discussed by some authors.\(^{12}\) Equation (4.24) together with Eq.(4.5) gives

\[ \gamma - \alpha^2 = \frac{\Lambda}{12} Z, \]
\[ \alpha(1 - \gamma) = \frac{\Lambda}{12} X. \]  \hspace{1cm} (4.25)

Equation of motions (4.10) and (4.11) under ansatz (4.25) are

\[ \dot{X} = 2\mathcal{N} X \{Z(1 - \gamma) - \alpha X\}, \]
\[ \dot{Z} = 2\mathcal{N} X (X - 2\alpha Z), \]  \hspace{1cm} (4.26)

and

\[ \dot{\alpha} = \mathcal{N} \frac{\Lambda}{6} X Z, \]
\[ \dot{\gamma} = \mathcal{N} \frac{\Lambda}{6} X^2. \]  \hspace{1cm} (4.27)

So, fixing the gauge of \( \mathcal{N} \) as

\[ \mathcal{N} = \frac{1}{2X^2}, \]  \hspace{1cm} (4.28)
we can solve Eq.(4.27). First, $\gamma$ is immediately given by

$$\gamma = \frac{\Lambda}{12}(\tau - \tau_0). \quad (4.29)$$

As for $\alpha$, $\dot{\alpha}$ is reduced to

$$\dot{\alpha} = \frac{\Lambda}{12} \frac{\gamma - \alpha^2}{\alpha(1 - \gamma)} \quad (4.30)$$

by use of Eqs.(4.27) and (4.25). Substituting Eq.(4.29) into Eq.(4.30) we obtain the solution $\alpha$ in the form

$$\alpha^2 = 2\left(\frac{\Lambda}{12}(\tau - \tau_0) - 1\right) + 1 + \frac{12}{\Lambda} C \left(\frac{\Lambda}{12}(\tau - \tau_0) - 1\right)^2, \quad (4.31)$$

where $C$ is an arbitrary constant. We can set $\tau_0$ as $\frac{\Lambda}{12}\tau_0 = -1$ without loss of generality. Then

$$\alpha^2 = 1 + \frac{\Lambda}{12}(2\tau + C\tau^2),$$

$$\gamma = 1 + \frac{\Lambda}{12}\tau. \quad (4.32)$$

In $\Lambda = 0$ limit, $\alpha = \gamma = 1$. Thus the solution for 4-(ii) is an extended one of the Taub-NUT solution to $\Lambda \neq 0$ case.

Without solving Eq.(4.26), we get $X, Z$ immediately from Eqs.(4.32) and (4.25) in the form

$$Z = -(\tau + C\tau^2)$$

$$X^2 = \left\{1 + \frac{\Lambda}{12}(2\tau + C\tau^2)\right\}\tau^2. \quad (4.33)$$

From Eq.(4.8) with Eqs.(4.9) and (4.28), we obtain

$$ds^2 = \frac{Z}{4X^2}d\tau^2 + Z(\sigma_x^2 + \sigma_y^2) + \frac{X^2}{Z}\sigma_z^2, \quad (4.34)$$

where $X, Z$ are given by Eq.(4.33).
We introduce new coordinate $\rho$ instead of $\tau$ by

$$Z = -(\tau + C\tau^2) = \begin{cases} \rho^2 & \text{for } C = 0 \\ \text{sgn}(C)(L^2 - \rho^2) & \text{for } C \neq 0 , \end{cases}$$

where $L^2 \equiv \frac{1}{4|C|}$. Then

$$X^2 = \begin{cases} \rho^4(1 - \frac{\Lambda}{\rho^2}) & \text{for } C = 0 \\ 4L^2\{(\rho - L)^2 + \text{sgn}(C)\frac{\Lambda}{12}\{-3L^4 + 8L^3\rho - 6L^2\rho^2 + \rho^4\} \} & \text{for } C \neq 0 , \end{cases}$$

and

$$N^2d\tau^2 = \begin{cases} (1 - \frac{\Lambda}{\rho^2})^{-1}d\rho^2 & \text{for } C = 0 \\ L^2\frac{\text{sgn}(C)(L^2 - \rho^2)}{X^2}d\rho^2 & \text{for } C \neq 0 \end{cases}$$

In the case of $C = 0$, the metric is Fubini-Study in the form of Eq.(4.20) with $a = 0$.

On the other hand, when $C \neq 0$, the result is a special case of the Taub-NUT-de Sitter metric. This fact will be presented in the next subsection.

Eq.(4.33) requires that the metric becomes isotropic when $C = \Lambda/12$. It is nothing but the de Ditter ($C > 0$) or anti-de Sitter ($C < 0$) solution.

4-(iii) $\mu^a \neq \text{const. case}$

Fixing the gauge of $\mathcal{N}$ as

$$\mathcal{N} = \frac{1}{\alpha X} \quad (4.35)$$

we can solve Eq.(4.13) in the form

$$\mu^1 = \frac{\Lambda}{6}\{\mu_0e^{6\tau} + 1\},$$

$$\mu^3 = \frac{\Lambda}{6}\{-2\mu_0e^{6\tau} + 1\}, \quad (4.36)$$

where $\mu_0$ is an arbitrary non-vanishing constant. Since the absolute value of $\mu_0$ can be removed by a translation of $\tau$, we may only note the signature of $\mu_0$. 

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Together with Eqs.(4.12) and (4.35), Eq.(4.11) requires

\[ \frac{1}{2}(\alpha^2) = 2\frac{\mu_1}{\mu^3}(\gamma - \alpha^2), \]
\[ \dot{\gamma} = 2\frac{\mu^3}{\mu_1}(1 - \gamma). \]  

(4.37)

Substituting the solutions of \( \mu_1 \) and \( \mu_3 \) into it, Eq.(4.37) is easily integrated to

\[ \gamma = \frac{CA}{12} \{ e^{4\tau} \pm e^{-2\tau} \} + 1, \]
\[ \alpha^2 = \frac{CA}{12} \{ -e^{4\tau} \pm 2e^{-2\tau} \} + 1 + \frac{DA}{12} \{ 2e^{2\tau} \mp e^{-4\tau} \}. \]  

(4.38)

Here \( C \) and \( D \) are integral constants and the double sign corresponds to the signature of \( \mu_0 \). Thus having solved \( \mu^a \) and \( A_i^a \), we can derive the explicit form of \( \tilde{e}^i_a \) from Eq.(4.12),

\[ X^2 = (Ce^{-2\tau})^2 \left\{ \frac{CA}{12} \{ -e^{4\tau} \pm 2e^{-2\tau} \} + 1 + \frac{DA}{12} \{ 2e^{2\tau} \mp e^{-4\tau} \} \right\}, \]
\[ Z = \pm \{ -Ce^{-2\tau} + De^{-4\tau} \}. \]  

(4.39)

Eq.(4.8) together with Eqs.(4.9) and (4.35) requires the invariant line element to be

\[ ds^2 = Z\frac{\alpha^2}{\alpha^2}d\tau^2 + Z(\sigma_x^2 + \sigma_y^2) + \frac{X^2}{Z}\sigma_z^2. \]  

(4.40)

\( X, Z \) and \( \alpha \) in Eq.(4.40) are given by Eqs.(4.39) and (4.38).

We introduce new coordinate \( \rho \) in place of \( \tau \) by

\[ Z = \pm \{ -Ce^{-2\tau} + De^{-4\tau} \} = \begin{cases} \rho^2 & \text{for } D = 0 \\ \pm \text{sgn}(D)(\rho^2 - L^2) & \text{for } D \neq 0 \end{cases} \]

where \( L^2 \equiv \frac{C^2}{4|D|} \). The range of \( \rho \) should be chosen so that \( Z \) is positive. Thus,

\[ N^2d\tau^2 = \frac{Z}{\alpha^2}d\tau^2 = \begin{cases} \{1 - \frac{CA}{12} \frac{C^2}{\rho^4} - \frac{A}{6}\rho^2\}^{-1}d\rho^2 & \text{for } D = 0 \\ \pm L^2\frac{\text{sgn}(D)(\rho^2 - L^2)}{X^2}d\rho^2 & \text{for } D \neq 0 \end{cases} \]  

(4.41)
and

\[ X^2 = \begin{cases} 
\rho^4\{1 - \frac{C\Lambda C^2}{12} - \frac{\Lambda}{6}\rho^2}\quad &\text{for } D = 0 \\
4L^2\left[\rho^2 - (2 + \frac{C^3}{4L^2})\rho + L^2 \\
+ \text{sgn}(D)\frac{\Lambda}{4}(L^4 + \frac{2}{3}L^3\rho + 2L^2\rho^2 - \frac{1}{3}\rho^4)\right] &\text{for } D \neq 0 
\end{cases} \] (4.42)

In the case of \( D = 0 \), the line element becomes the extended E-H metric Eq.(4.20). On the other hand, in the case of \( D \neq 0 \), we obtain

\[ ds^2 = \pm \left[ \frac{\rho^2 - L^2}{4\Delta}d\rho^2 + (\rho^2 - L^2)(\sigma_x^2 + \sigma_y^2) + \frac{4L^2\Delta}{\rho^2 - L^2}\sigma_z^2 \right]. \] (4.43)

Here \( \Delta \) is defined by

\[ \Delta \equiv \rho^2 - 2M\rho + L^2 \pm \frac{\Lambda}{4}(L^4 + 2L^2\rho^2 - \frac{1}{3}\rho^4) \]

with

\[ M \equiv \frac{1}{3}\Lambda L^2 + 1 + \frac{C^3}{8L^2}. \]

This is nothing but the Taub-NUT-de Sitter metric.

5. Discussion

In this article we have discussed about the explicit solutions of Euclidean Einstein equation of Bianchi type IX universe. The equations we have treated are the non linear ordinary differential equations with many dependent variables. As we have mentioned the non-linearity of the equations are drastically reduced in the Ashtekar formalism.

Further development beyond this article we may look in two routes. One is to study the homogeneous space other than Bianchi type IX universe. In Bianchi type IX case, invariance groups of the field strength and of the considering space are
both $SO(3)$. This coincidence brings about the additional simplicity. However, the loss of this simplicity does not change the fundamental framework mentioned above and our formulation seems to be applicable to the various types of homogeneous universes. Development in another route may be more substantial, which is to extend independent variables to two, three and finally four. (As a direct extension we may consider multi instanton extension of the solutions obtained in this article.)

In this route we are confronted with the integrable non-linear sciences. In this well established region we have already the very powerful tools and concepts such as Lax pair, Painlevé properties, inverse scattering method and Hirota's direct method and so on.

It is very interesting to ask what the Ashtekar formalism can add to this fertile field.

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