Families of spinors in $d = (1 + 5)$ with zweibein and two kinds of spin connection fields on an almost $S^2$

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We studied [1, 2] properties of spinors in a toy model in $d = (1 + 5)$ as a step towards realistic Kaluza-Klein (like) theories in non compact spaces. $\mathcal{M}^{(5+1)}$ was assumed to break to an infinite disc with a zweibein which makes a disc curved on $S^2$ and with a spin connection field which allows on such a sphere only one massless spinor state. This time we are taking into account families of spinors interacting with several spin connection fields, as required for this toy model by the spin-charge-family theory [4–6]. We are studying possible masslessness of families of spinors: Spinors regroup into subgroups of an even number of families.

I. INTRODUCTION

The spin-charge-family theory [4, 5], proposed by N.S. Mankoč Borštnik, is offering an explanation for the appearance of families of fermions in any dimension. Starting in $d = (13 + 1)$ with a simple action for massless fermions interacting with the gravitational interaction only – that is with the vielbeins and two kinds of the spin connection fields – the theory manifests effectively at low energies the so far observed properties of fermions and bosons, explaining the assumptions of the standard model, among them the appearance of families, the Higgs scalar and Yukawa couplings, and predicting the fourth family, a stable fifth family (explaining the existence of the dark matter) and several scalar fields, which could be observed at the LHC.

A simple toy model [1–3], which includes families as in the spin-charge-family theory [4, 5], is expected to help to better understand properties of families of spinors at observable energies. This contribution is the continuation of the ref. [3], where families were already included, and so were two kinds of the spin connection fields. We make this time a small step further in understanding properties of families of spinors, allowing the influence of several spin connection fields. We start with a massless spinor in a flat manifold $\mathcal{M}^{(5+1)}$, which breaks into $\mathcal{M}^{(3+1)}$ times an infinite disc. The vielbein on the disc curves the disc into (almost) a sphere $S^2$

$$e^s_\sigma = f^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f^s_\sigma = f \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(1)
with
\[
\begin{align*}
f &= 1 + \left( \frac{\rho}{2 \rho_0} \right)^2 = \frac{2}{1 + \cos \vartheta}, \\
x^{(5)} &= \rho \cos \phi, \quad x^{(6)} = \rho \sin \phi, \quad E = f^{-2}.
\end{align*}
\]

The angle \( \vartheta \) is the ordinary azimuthal angle on a sphere. The last relation follows from
\[
\begin{aligned}
ds^2 &= e_s e_t dx^s dx^t = f^{-2}(d\rho^2 + \rho^2 d\phi^2). \\
\text{We use indices } s,t &= 5,6 \text{ to describe the flat index in the space of an infinite plane, and } \sigma, \tau = (5),(6), \text{ to describe the Einstein index. Rotations around the axis through the two poles of a sphere are described by the angle } \phi, \text{ while } \rho = 2\rho_0 \sqrt{\frac{1 - \cos \vartheta}{1 + \cos \vartheta}}.
\end{aligned}
\]
The volume of this non-compact sphere is finite, equal to \( V = \pi (2 \rho_0)^2 \). The symmetry of \( S^2 \) is a symmetry of \( U(1) \) group. We look for chiral fermions on this sphere, that is the fermions of only one handedness and accordingly mass protected, without including any extra fundamental gauge fields to the action from Eq. (5). We study the influence of several spin connection fields on properties of families.

We take into account that there are two kinds of the \( \gamma \) operators, beside the Dirac \( \gamma^a \) also \( \tilde{\gamma}^a \) introduced in [4, 5, 7]. Correspondingly the covariant momentum of spinor is
\[
\begin{align*}
p_{0a} &= f^a_\sigma p_\sigma + \frac{1}{2E} \{ p_\sigma, f^a_\sigma E \} - \frac{1}{2} S^\sigma_{\alpha \beta \gamma} \omega_{\alpha \beta \gamma} - \frac{1}{2} \tilde{S}^\sigma_{\alpha \beta \gamma} \tilde{\omega}_{\alpha \beta \gamma}, \\
S^{ab} &= \frac{i}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a), \\
\tilde{S}^{ab} &= \frac{i}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a),
\end{align*}
\]

with \( E = \det(e^a_\alpha) \) and with vielbeins \( f^a_\sigma \), the gauge fields of the infinitesimal generators of translation, and with the two kinds of the spin connection fields: i. \( \omega_{\alpha \beta \gamma} \), the gauge fields of \( S^{ab} \) and ii. \( \tilde{\omega}_{\alpha \beta \gamma} \), the gauge fields of \( \tilde{S}^{ab} \).

We make a choice of the spin connection fields of the two kinds on the infinite disc as follows (assuming that there must be some fermion sources causing these spin connections)
\[
\begin{align*}
f^a_\sigma \omega_{\sigma \tau} &= i F_{56} f \varepsilon_{\sigma \tau} \frac{\varepsilon^a_{\sigma \tau} x^\sigma}{(\rho_0)^2} = -\frac{1}{2E} \{ p_{\sigma}, E f^a_\sigma \} - \varepsilon_{\sigma \tau} 4 F^5_{56}, \\
f^a_\sigma \tilde{\omega}_{\sigma \tau} &= i \tilde{F}_{56} f \varepsilon_{\sigma \tau} \frac{\varepsilon^a_{\sigma \tau} x^\sigma}{(\rho_0)^2} = -\frac{1}{2E} \{ p_{\sigma}, E f^a_\sigma \} - \varepsilon_{\sigma \tau} 4 \tilde{F}_{56}, \\
f^a_\sigma \tilde{\omega}_{mn} &= -\frac{1}{2E} \{ p_{\sigma}, E f^a_\sigma \} - 4 \tilde{F}_{mn} , \tilde{F}_{mn} = -\tilde{F}_{nm}, \\
&\quad \text{ for } s = 5,6, \quad \sigma = (5),(6).
\end{align*}
\]

We take the starting action in agreement with the \textit{spin-charge-family} theory for this toy model in \( d = (5+1) \), that is the action for a massless spinor \( (S_f) \) with the covariant momentum \( p_{0a} \) from Eq. (3) interacting with gravity only and for the vielbein and the two kinds of the spin connection
fields \( (S_b) \)

\[
S = S_b + S_f ,
\]

\[
S_b = \int d^d x \left( \alpha R + \alpha \tilde{R} \right), \quad L_f = \psi^\dagger \gamma^0 \gamma^a E p_0 a \psi .
\] (5)

The two Riemann scalars, \( R = R_{abcd} \eta^{ac} \eta^{bd} \) and \( \tilde{R} = \tilde{R}_{abcd} \eta^{ac} \eta^{bd} \), are determined by the Riemann tensors \( R_{abcd} = \frac{1}{2} f_{\alpha[a} f_{\beta b]} (\omega_{cd}^\beta, \omega - \omega_{cea} \omega_{d}^\epsilon_{,d}) \), \( \tilde{R}_{abcd} = \frac{1}{2} f^{\alpha[a} f^{\beta b]} (\tilde{\omega}_{cd}^\beta, \omega - \omega_{cea} \tilde{\omega}_{d}^\epsilon_{,d}) \), \( [a \ b] \) means that the anti-symmetrization must be performed over the two indices \( a \) and \( b \).

We assume no gravity in \( d = (3 + 1) \): \( f^\mu_m = \delta^\mu_m \) and \( \omega_{mn\mu} = 0 \) for \( m, n = (0, 1, 2, 3) \), \( \mu = (0, 1, 2, 3) \). Accordingly \( (a, b, \ldots) \) run in Eq. (5) only over \( s \in (5, 6) \). Taking into account the subgroup structure of the operators \( \tilde{S}^{mn} \)

\[
\tilde{N}^{\oplus} = \frac{1}{2} (\tilde{S}^{23} \pm i \tilde{S}^{01} , \tilde{S}^{31} \pm i \tilde{S}^{02} , \tilde{S}^{12} \pm i \tilde{S}^{03}) ,
\]

\[
\tilde{N}^{\ominus} = \tilde{N}^{\ominus 1} \pm i \tilde{N}^{\ominus 2} , \quad \tilde{N}^{\odot} = \tilde{N}^{\odot 1} \pm i \tilde{N}^{\odot 2} ,
\] (6)

we can rewrite the \( \frac{1}{2} \tilde{S}^{cd} \omega_{cda} \) part of the covariant momentum (Eq. 3) as follows

\[
- \frac{1}{2} f \tilde{S}^{mn} \tilde{\omega}_{mn\pm} = \sum_i \tilde{N}^{\odot i} \tilde{A}_\pm^{\odot i} + \sum_i \tilde{N}^{\ominus i} \tilde{A}_\pm^{\ominus i} = \tilde{N}^{\ominus 1} \tilde{A}_\pm^{\ominus 1} + \tilde{N}^{\ominus 2} \tilde{A}_\pm^{\ominus 2} + \tilde{N}^{\ominus 3} \tilde{A}_\pm^{\ominus 3} + \tilde{N}^{\odot 1} \tilde{A}_\pm^{\odot 1} + \tilde{N}^{\odot 2} \tilde{A}_\pm^{\odot 2} + \tilde{N}^{\odot 3} \tilde{A}_\pm^{\odot 3} ,
\]

\[
\tilde{\omega}_{mn\pm} = \tilde{\omega}_{mn5} \mp i \tilde{\omega}_{mn6} .
\] (7)

The notation was used

\[
f^\sigma_s \tilde{A}_\sigma = - f^\sigma_s \{ (\omega_{23s} \mp i \tilde{\omega}_{01s}) , (\omega_{31s} \mp i \tilde{\omega}_{02s}) , (\omega_{12s} \mp i \tilde{\omega}_{03s}) \} = \delta^\sigma_s \frac{1}{2E} (p_\sigma , Ef) \}
\]

\[
\tilde{A}_\sigma^{\oplus 1} = \frac{1}{2} (\tilde{A}_s^{\oplus 1} \mp i \tilde{A}_s^{\oplus 2}) , \quad \tilde{A}_\sigma^{\ominus 1} = \frac{1}{2} (\tilde{A}_s^{\ominus 1} \mp i \tilde{A}_s^{\ominus 2}) ,
\]

\[
\tilde{F}^{\ominus 1} = (\tilde{F}^{23} \mp ^{02}) - i (\mp \tilde{F}^{31} + \tilde{F}^{01}) , \quad \tilde{F}^{\ominus 3} = (\tilde{F}^{12} - i \tilde{F}^{03}) ,
\]

\[
\tilde{F}^{\ominus 0} = (\tilde{F}^{23} + \tilde{F}^{02}) + i (\mp \tilde{F}^{31} + \tilde{F}^{01}) , \quad \tilde{F}^{\ominus 1} = (\tilde{F}^{12} + i \tilde{F}^{03}) ,
\]

\[
\sigma = (5, 6) , \quad s = (5, 6) ,
\] (8)

with \( \omega_{abc} \) and \( \tilde{\omega}_{abc} \) defined in Eq. (4).

We study intervals within which the parameters of both kinds of the spin connection fields \( (F_{56} , \tilde{F}_{56} , \tilde{F}^{\oplus 1} , \tilde{F}^{\ominus 3} , \tilde{F}^{\odot 3} , \tilde{F}^{\ominus 3}) \) allow massless solutions of the equation

\[
\{ E \gamma^0 \gamma^m p_m + Ef \gamma^0 \gamma^a \delta^s_s (p_0 a + \frac{1}{2E} (p_\sigma , Ef) ) \} \psi = 0 , \quad \text{with} \quad p_0 a = p_a - \frac{1}{2} S^{st} \omega_{st\sigma} - \frac{1}{2} S^{ab} \tilde{\omega}_{ab\sigma} ,
\] (9)
for one or several families of spinors. To solve Eq. (9) we must tell more about the appearance of families of spinors.

II. SOLUTIONS OF EQUATIONS OF MOTION FOR FAMILIES OF SPINORS

We first briefly explain, following the refs. [1–3, 5, 6], the appearance of families in our toy model, using what is called the technique [7]. There are $2^{d/2-1} = 4$ families in our toy model, each family with $2^{d/2-1} = 4$ members. In the technique [7] the states are defined as a product of nilpotents and projectors

$$ab(±i) = \frac{1}{2}(\gamma^a ± \gamma^b), \quad [±i] = \frac{1}{2}(1 ± \gamma^a \gamma^b),$$

for $\eta^{aa} \eta^{bb} = -1,$

$$ab(±) = \frac{1}{2}(\gamma^a ± i\gamma^b), \quad [±] = \frac{1}{2}(1 ± i\gamma^a \gamma^b),$$

for $\eta^{aa} \eta^{bb} = 1,$

$$ab(±i) = \frac{1}{2}(\gamma^a ± i\gamma^b), \quad [±i] = \frac{1}{2}(1 ± i\gamma^a \gamma^b),$$

which are the eigen vectors of $S^{ab}$ as well as of $\tilde{S}^{ab}$ as follows

$$S^{ab}(k) = \frac{k}{2} \frac{ab}{k}[ab], \quad \tilde{S}^{ab}(k) = \frac{k}{2} \frac{ab}{k}[ab],$$

with the properties that $\gamma^a$ transform $ab(k)$ into $[−k]$, while $\tilde{\gamma}^a$ transform $ab(k)$ into $[k]$.

$$\gamma^a ab(k) = \eta^{aa} ab[−k], \quad \gamma^b ab(k) = -ik ab[−k], \quad \gamma^a ab[k] = (−k), \quad \gamma^b ab[k] = -ik ab(−k),$$

$$\tilde{\gamma}^a ab(k) = -i\eta^{aa} ab[k], \quad \tilde{\gamma}^b ab(k) = -k ab[k], \quad \tilde{\gamma}^a ab[k] = i ab[k], \quad \tilde{\gamma}^b ab[k] = -k \eta^{aa} ab[k].$$

After making a choice of the Cartan subalgebra, for which we take: $(S^{03}, S^{12}, S^{56})$ and $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, the four spinor families, each with four vectors, which are eigen vectors of the chosen Cartan subalgebra with the eigen values from Eq. (11), can be found in [3] written as four times four products of projectors $[k]$ and nilpotents $(k)$. We present here only one member of each family, the one with even values of $S^{56}$ and $S^{12}$ both equal to $\frac{1}{2}$, and of left handedness with respect to $d = (5 + 1)$ and $d = (3 + 1)$

$$\varphi^{1I}_1 = (+) (+i)(+) \psi_0, \quad \varphi^{1I}_1 = (+) (+i)(+) \psi_0,$$

$$\varphi^{1II}_1 = [+] (+i)[+] \psi_0, \quad \varphi^{1II}_1 = [+] (+i)[+] \psi_0,$$

$$\varphi^{1III}_1 = [+] (+i)[+] \psi_0, \quad \varphi^{1III}_1 = [+] (+i)[+] \psi_0,$$

where $\psi_0$ is a vacuum for the spinor state. One can reach from the first member $\varphi^{1I}_1$ of the first family the same family member of all the other families by the application of $\tilde{S}^{ab}$. The reader
can create the remaining three members of each family by applying the generators $S^{ab}$ on the
presented member $\psi_0$. The rest of the members of, say, the first family are: $\varphi_2^I = +[i][+] \psi_0$, $\varphi_2^I = -[i][+] \psi_0$. If we write the operators of handedness in $d = (1 + 5)$
as $\Gamma^{(1+5)} = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6$ ($= 2^3 i 5^{03} S^{12} S^{56}$), in $d = (1 + 3)$ as $\Gamma^{(1+3)} = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3$ ($= 2^2 i 5^{03} S^{12}$)
and in the two dimensional space as $\Gamma^{(2)} = i \gamma_5 \gamma_6$ ($= 2 S^{56}$), we find that all the states of all the
families are left handed with respect to $\Gamma^{(1+5)}$, with the eigen value $-1$, the first two states of the
first family, and correspondingly the first two states of any family, are right handed and the second
two states are left handed with respect to $\Gamma^{(2)}$, with the eigen values 1 and $-1$, respectively, while
the first two are left handed and the second two right handed with respect to $\Gamma^{(1+3)}$ with the eigen
values $-1$ and 1, respectively. Having the rotational symmetry around the axis perpendicular to
the plane of the fifth and the sixth dimension we require that $\psi^{(6)}$ is the eigen function of the total
angular momentum operator $M^{56} = x^5 p^6 - x^6 p^5 + S^{56} = -\frac{i \partial}{\partial \theta} + S^{56}$

$$M^{56} \psi^{(6)} = (m + \frac{1}{2}) \psi^{(6)} .$$ (14)

Accordingly we write, when taking into account Eq. (13), the most general wave function $\psi^{(6)}$
obeying Eq. (9) in $d = (1 + 5)$ as

$$\psi^{(6)} = N \sum_{i = I, II, III, IV} (A_i^I (+) \psi_{(+)\alpha}^{(4i)} + B_i^{I+} e^{i\phi} [-i] \psi_{(-)\alpha}^{(1i)}) e^{im\theta} \psi^{(6)} .$$ (15)

where $A_i^I$ and $B_i^I$ depend on $x^\sigma$, while $\psi_{(+)\alpha}^{(4i)}$ and $\psi_{(-)\alpha}^{(1i)}$ determine the spin and the coordinate
dependent parts of the wave function $\psi^{(6)}$ in $d = (3 + 1)$ in accordance with the definition in
Eq. (13), for example,

$$\psi_{(+)\alpha}^{(4I)} = \alpha_{+}^{I} (i) [+] \psi_{(+)\alpha}^{(4i)} + \beta_{+}^{I} [-i] \psi_{(-)\alpha}^{(1i)} ,$$

$$\psi_{(-)\alpha}^{(4I)} = \alpha_{-}^{I} [-i] (+) + \beta_{-}^{I} (+) [-i] .$$ (16)

$(+) = (i) = (+)$, for $i = I, II$ and $(+) = [+]$ for $i = III, IV$, while $[-] = [-]$ for $i = I, II$ and

$M^{-1} \psi^{(6)} = 0$ for $i = III, IV$. Using $\psi^{(6)}$ in Eq. (9) and separating dynamics in $(1 + 3)$ and on $S^2$, the
following relations follow, from which we recognize the mass term $m^I$: $\frac{\alpha_{+}^{I}}{\alpha_{-}^{I}} (p_0 - p^3) - \frac{\beta_{+}^{I}}{\beta_{-}^{I}} (p_1 - i p^2) = m^I$, $\frac{\beta_{+}^{I}}{\beta_{-}^{I}} (p_0 + p^3) - \frac{\alpha_{+}^{I}}{\alpha_{-}^{I}} (p_1 + i p^2) = m^I$, $\frac{\alpha_{+}^{I}}{\alpha_{-}^{I}} (p_0 + p^3) + \frac{\beta_{+}^{I}}{\beta_{-}^{I}} (p_1 - i p^2) = m^I$, $\frac{\beta_{+}^{I}}{\beta_{-}^{I}} (p_0 - p^3) + \frac{\alpha_{+}^{I}}{\alpha_{-}^{I}} (p_1 + i p^2) = m^I$. (One notices that for massless solutions $(m^I = 0)$ $\psi_{(+)\alpha}^{(4i)}$ and $\psi_{(-)\alpha}^{(1i)}$, for each $i = I, II, III, IV$, decouple.)

For a spinor with the momentum $p^m = (p_0, 0, 0, p^3)$ in $d = (3 + 1)$ the spin and coordinate
dependent parts for four families is: $\psi_{(+)\alpha}^{(4I)} = \alpha (+) (+) + \psi_{(+)\alpha}^{(4i)} = \alpha (+) [+i] (+) + \psi_{(+)\alpha}^{(4i)} = \alpha (+i) (+) + \psi_{(+)\alpha}^{(4i)} = \alpha (+i) [+i] .}$
Taking the above derivation into account (Eqs. (15, 2, 4, 16, 6, 7, 8)) the equation of motion for spinors follows [3] from the action (5)

\[ i \psi \left\{ \left( \frac{\partial}{\partial \rho} + \frac{1}{2} f \left( 1 - 2 F_{56} - 2 \tilde{F}_{56} - 2 \tilde{F}^{(3)} \right) \right) + \frac{1}{2} \frac{\partial f}{\partial \rho} \right\} \psi^{(6)} = 0. \]  

One easily recognizes that, due to the break of \( \mathcal{M}^{(5+1)} \) into \( \mathcal{M}^{(3+1)} \times \) an infinite disc, which concerns (by our assumption) \( S^{ab} \) and \( \tilde{S}^{ab} \) sector, there are two times two coupled families: The first and the second, and the third and the fourth, while the first and the second remain decoupled from the third and the fourth. We end up with two decoupled groups of equations of motion [3] (Eqs. (16.26-16.33)) (which all depend on the parameters \( F_{56} \) and \( \tilde{F}_{56} \)):

i. The equations for the first and the second family

\[ -i f \left\{ \left( \frac{\partial}{\partial \rho} - \frac{n}{\rho} \right) + \frac{1}{2} \frac{\partial f}{\partial \rho} \left( 1 - 2 F_{56} - 2 \tilde{F}_{56} - 2 \tilde{F}^{(3)} \right) \right\} A_{n+1}^I + \gamma^0 \gamma^5 m \psi^{(6)} = 0, \]  

ii. The equations for the third and the fourth family

\[ \psi^{(6)} = 0. \]
A general solution is any superposition of these two. Similarly is true for \((\phi_{I,II})\) from the first one, when knowing massless solutions of the first group of families. It follows although depending on different parameters of the spin connection fields, can be treated in an equivalent way. Let us therefore study massless solutions of the first group of equations of motion.

For \(m = 0\) the equations for \(A^n_I\) and \(A^n_{II}\) in Eq. (19) decouple from those for \(B^{I,I}_{n+1}\) and \(B^{II}_{n+1}\). We get for massless solutions \(\{13\}\)

\[
\begin{align*}
A^n_I &= a_\pm \rho^n f^{\pm}(1-2F_{56}-2\tilde{F}_{56}) f^{\pm}\sqrt{(F^{\oplus 3})^2 + F^{\oplus 3} F_{\oplus 3}}, \\
A^n_{II} &= \pm \sqrt{(\tilde{F}^{\oplus 3})^2 + \tilde{F}^{\oplus 3} \tilde{F}_{\oplus 3} - \tilde{F}^{\oplus 3}} \quad A^n_{II}, \\
B^{I,II}_{n+1} &= b_\pm \rho^{-n-1} f^{\pm}(1+2F_{56}+2\tilde{F}_{56}) f^{\pm}\sqrt{(F^{\oplus 3})^2 + F^{\oplus 3} F_{\oplus 3}}, \\
B^{I,II}_{n+1} &= \pm \sqrt{(\tilde{F}^{\oplus 3})^2 + \tilde{F}^{\oplus 3} \tilde{F}_{\oplus 3} - \tilde{F}^{\oplus 3}} \quad B^{I,II}_{n+1},
\end{align*}
\]

\(n\) is a positive integer. The solutions \((A^n_{I}, A^n_{II})\) and \((A^n_{I}, A^n_{II})\) are two independent solutions, a general solution is any superposition of these two. Similarly is true for \((B^{I,II}_{n+1}, B^{II}_{n+1})\).

In the massless case also \(A^n_{I,II}^{\pm}\) decouple from \(B^{I,II}_{n+1}\).

One can easily write down massless solutions of the second group of two families, decoupled from the first one, when knowing massless solutions of the first group of families. It follows

\[
\begin{align*}
A^n_{II} &= a_\pm \rho^n f^{\pm}(1-2F_{56}+2\tilde{F}_{56}) f^{\pm}\sqrt{(F^{\oplus 3})^2 + F^{\oplus 3} F_{\oplus 3}}, \\
A^n_{III} &= \pm \sqrt{(\tilde{F}^{\oplus 3})^2 + \tilde{F}^{\oplus 3} \tilde{F}_{\oplus 3} - \tilde{F}^{\oplus 3}} \quad A^n_{III}, \\
B^{I,II}_{n+1} &= b_\pm \rho^{-n-1} f^{\pm}(1+2F_{56}+2\tilde{F}_{56}) f^{\pm}\sqrt{(F^{\oplus 3})^2 + F^{\oplus 3} F_{\oplus 3}}, \\
B^{I,II}_{n+1} &= \pm \sqrt{(\tilde{F}^{\oplus 3})^2 + \tilde{F}^{\oplus 3} \tilde{F}_{\oplus 3} - \tilde{F}^{\oplus 3}} \quad B^{I,II}_{n+1},
\end{align*}
\]
\( n \) is a positive integer.

Requiring that only normalizable (square integrable) solutions are acceptable

\[
2\pi \int_0^\infty E \rho d\rho (A^i_n A^i_n + B^i_n B^i_n) < \infty ,
\]

\((22)\)

\( i \in \{I, II, III, IV\} , \) one finds that \( A^i_n \) and \( B^i_n \) are normalizable \[1, 2, 13\] under the following conditions

\[
A^{I,II}_n : -1 < n < 2(F_{56} + \tilde{F}_{56} \pm \sqrt{(\tilde{F}^{\oplus 3})^2 + \tilde{F}^{\oplus \ominus} \tilde{F}^{\oplus \ominus}}),
\]

\[
B^{I,II}_n : 2(F_{56} - \tilde{F}_{56} \pm \sqrt{(\tilde{F}^{\oplus 3})^2 + \tilde{F}^{\oplus \ominus} \tilde{F}^{\oplus \ominus}}) < n < 1 ,
\]

\[
A^{III,IV}_n : -1 < n < 2(F_{56} - \tilde{F}_{56} \pm \sqrt{(\tilde{F}^{\oplus 3})^2 + \tilde{F}^{\oplus \ominus} \tilde{F}^{\oplus \ominus}}),
\]

\[
B^{III,IV}_n : 2(F_{56} + \tilde{F}_{56} \pm \sqrt{(\tilde{F}^{\oplus 3})^2 + \tilde{F}^{\oplus \ominus} \tilde{F}^{\oplus \ominus}}) < n < 1 .
\]

\((23)\)

One immediately sees that for \( F_{56} = 0 = \tilde{F}_{56} \) there is no solution for the zweibein from Eq. \((2)\). Let us first assume that \( \tilde{F}^{\oplus i} = 0 ; i \in \{1, 2, 3\} \). Eq. \((23)\) tells us that the strengths \( F_{56}, \tilde{F}_{56} \) of the spin connection fields \( (\omega_{56\sigma} \text{ and } \tilde{\omega}_{56\sigma}) \) can make a choice between the massless solutions \( (A^{I,II}_n, A^{III,IV}_n) \) and \( (B^{I,II}_n, B^{III,IV}_n) \):

For

\[
0 < 2(F_{56} + \tilde{F}_{56}) \leq 1 , \quad \tilde{F}_{56} < F_{56}
\]

\((24)\)

there exist four massless left handed solutions with respect to \((1 + 3)\). For

\[
0 < 2(F_{56} + \tilde{F}_{56}) \leq 1 , \quad \tilde{F}_{56} = F_{56}
\]

\((25)\)

the only massless solution are the two left handed spinors with respect to \((1 + 3)\)

\[
\psi^{(6, I, II) n=0}_{\frac{1}{2}} = \mathcal{N}_0 \begin{pmatrix} - F_{56} - \tilde{F}_{56} + 1/2 \end{pmatrix}^{56} \psi^{(4, I, II)}_{(+)}.
\]

\((26)\)

The solutions \(\psi^{(6, I, II) n=0}_{\frac{1}{2}}\) are the eigen functions of \( M^{56} \) with the eigen value \( 1/2 \). Since no right handed massless solutions are allowed, the left handed ones are mass protected. For the particular choice \( 2(F_{56} + \tilde{F}_{56}) = 1 \) the spin connection fields \(-S^{56}_{\omega_{56\sigma}} - \tilde{S}^{56}_{\tilde{\omega}_{56\sigma}}\) compensate the term \( \frac{1}{2\mathbb{F}} \{ p_\sigma, E f \} \) and the left handed spinor with respect to \( d = (1 + 3) \) becomes a constant with respect to \( \rho \) and \( \phi \). To make one of these two states massive, one can try to include terms like \( \tilde{F}^{\oplus i} \).

Let us keep \( \tilde{F}^{\oplus i} = 0 \ i \in \{1, 2, 3\} \) and \( F_{56} = \tilde{F}_{56} \), while we take \( \tilde{F}^{\oplus 3}, \tilde{F}^{\oplus \ominus} \) non zero. Now it is still true that due to the conditions in Eq. \((23)\) there are no massless solutions determined by \( A^{III,VI} \) and \( B^{III,VI} \). There is now only one massless and mass protected family for \( F_{56} = \tilde{F}_{56} \). In
this case the solutions $A_0^{I-}$ and $A_0^{II-}$ are related

$$A_0^{I-} = \mathcal{N}_0 \pm \frac{1}{2} f_{56}^2 \sqrt{(F \hat{\circ} \Box)^2 + (\hat{F} \circ \hat{\Box})},$$

$$A_0^{II-} = - \frac{\sqrt{(F \hat{\circ} \Box)^2 + (\hat{F} \circ \hat{\Box}) + \hat{F} \circ \hat{\Box}}}{\hat{F} \circ \hat{\Box}} A_0^{I-}. \quad (27)$$

There exists, however, one additional massless state, with $A_0^{I+}$ related to $A_0^{II+}$ and $B_0^{I+}$ related to $B_0^{II+}$, which fulfil Eq. (23). But since we have left and right handed massless solution present, it is not mass protected any longer.

One can make a choice as well that none of solutions would be massless.

Let us conclude this section by recognizing that for $\hat{F} \circ \hat{\Box} = 0$ and $\hat{F} \circ \hat{\Box} = 0$ all the families decouple. There is then the choice of the parameters $(\hat{F}_5, \hat{\mathcal{F}}, \hat{\mathcal{F}} \circ \hat{\Box}, \hat{F} \circ \hat{\Box})$ which determine how many massless and mass protected families exist, if any.

### III. CONCLUSIONS AND DISCUSSIONS

We make in this contribution a small step further with respect to the ref. [3] in understanding the existence of massless and mass protected spinors in non compact spaces in the presence of families of spinors after breaking symmetries. We take a toy model in $\mathcal{M}^{5+1}$, which breaks into $\mathcal{M}^{3+1} \times$ an infinite disc curled into an almost $S^2$ under the influence of the zweibein. Following the spin-charge-family theory we have in this toy model four families. We study properties of families when allowing that besides the spin connection field - the gauge field of $S^{\mu t} = \frac{i}{4} (\gamma^s \gamma^t - \gamma^t \gamma^s)$ - also the gauge fields of $\tilde{S}^{\mu t} = \frac{i}{4} (\tilde{\gamma}^s \tilde{\gamma}^t - \tilde{\gamma}^t \tilde{\gamma}^s)$ determine families properties (as suggested by the spin-charge-family theory).

We simplify our study by assuming the same radial dependence of all the spin connection fields, the one used in studies without families, allowing here that only strengths of the field vary within some intervals. We take these strengths as parameters, which are allowed to change.

We found that the choices of the parameters allows within some intervals four, two or none massless and mass protected spinors.

Families like obviously to be massless and mass protected in even numbers.

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[14] $f^a_\alpha$ are inverted vielbeins to $e^\alpha_a$ with the properties $e^\alpha_a f^a_b = \delta^\alpha_b$, $e^\alpha_a f^\beta_a = \delta^\beta_\alpha$. Latin indices $a, b, ..., m, n, ..., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, ..., \mu, \nu, ..., \sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, \alpha$), from the middle of both the alphabets the observed dimensions 0, 1, 2, 3 ($m, n, \mu, \nu, ..$), indices from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = \text{diag}\{1, -1, -1, ..., -1\}$. 
