Solvability of the asymmetric Bingham fluid equations

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Anderson Luis Albuquerque de Araujo  
Departamento de Matemática,  
Universidade Federal de Viçosa, Viçosa, MG, Brasil  
Email: anderson.araujo@ufv.br

Nikolai V. Chemetov  
Universidade de Lisboa, Edifício C6, 1 Piso, Campo Grande,  
1749-016 Lisboa, Portugal  
Email: nvchemetov@fc.ul.pt, nvchemetov@gmail.com

Marcelo M. Santos  
Departamento de Matemática,  
IMECC-Instituto de Matemática, Estatística e Computação Científica,  
UNICAMP-Universidade Estadual de Campinas, Campinas, SP, Brazil  
Email: msantos@ime.unicamp.br

Abstract

In this work, we investigate the asymmetric Bingham fluid equations. The asymmetric fluid of Bingham includes symmetric and anti-symmetric stresses with such stresses appearing as an elastic response to the micro-rotational deformations of grains in a complex fluid. We show the global-in-time solvability of a weak solution for three dimensional boundary value problem with Navier boundary conditions of the asymmetric Bingham fluid equations.

Keywords: asymmetric fluids, Bingham plastic fluids, yield stress, global existence, weak solution.

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1 Introduction

The Newtonian fluid obeys the constitutive relations that the deviation stress tensor is a linear function of the stress rate-of-strain tensor and if
the fluid is isotropic then the stress and rate-of strain tensors are symmetric. Many of fluids can not be described by the Newtonian constitutive relations, such as slurries, animal blood, mud (mixtures of water, clay), viscous polymers, polymeric suspensions. These fluids does not commence to flow till the applied stress attains a certain optimum magnitude, called the \( \textit{yield stress} \ \tau^* \), after of that they behave as a Newtonian fluid. An example is toothpaste, which will not be extruded until a certain pressure is applied to the tube. Then it is pushed out as a solid plug. The physical reason for such behaviour is that the fluid contains particles, such as toothpaste, paints, clay, or large molecules, such as polymers, animal blood, which have an interaction, creating a weak rigid structure. Therefore a certain stress is required to break this weak rigid structure. As soon as the structure has been broken, the particles move with the fluid under viscous forces. The particles will associate again if the stress is removed. Such behaviour was firstly presented in an experimental study by Bingham \[3\], where he proposed its mathematical model. Later on these type of fluids have been called as Bingham plastic fluids. The Bingham plastic fluid behaves as a rigid body at low stresses but flows as a viscous fluid at high stress. Nowadays it is used as a mathematical model of mud flows in drilling engineering, heavy oil, lava (being a mix with melting snow, stones), in the handling of slurries, waxy crude oils. Recent examples concerns the propane flow within the hydro-fracture \[20\]. Significant efforts of the study of the Bingham plastic fluids have been done by Oldroyd \[14\], Mossolov, Miasnikov \[13\], Glowinski, Wachs \[11\], Papanastasiou \[15\] and many others scientists.

The Newtonian flow in the Bingham fluid (after the load greater than the \( \textit{yield stress} \ \tau^* \)) has a drastic limitation, since it does not account the behaviour of the fluid, that contains the particles. Most of above mentioned fluid systems contain rigid, randomly oriented particles, irregularly shaped particles (drops in emulsions), branched and entangled molecules in case of polymeric systems, or loosely formed clusters of particles in suspensions, etc. The particles may shrink and expand or change their shape, they may \textit{rotate}, independently of the rotation of the fluid. To describe accurately the behaviour of such fluids a so-called \textit{asymmetric continuum theory} \[22\] (or \textit{micropolar theory} \[7\], \[9\]) has been developed that ignores the deformation of the particles but takes into account geometry, intrinsic motion of material particles. This theory is a significant and a simple generalization of the classical Navier-Stokes model, that describe the Newtonian fluids. Only one new vector field, called as the \textit{angular velocity} field, of rotation of particles is introduced. As a consequence, only one equation is added, that represents the conservation of the angular momentum. The asymmetric/micropolar fluids belong to the class of fluids with non-symmetric stress tensor. This class of fluids is more general than the classical Newtonian fluids.

Shelukhin, Růžička \[21\] suggested a mathematical model that describe the behaviour of an asymmetric/micropolar viscous fluid of Bingham. In
the paper \cite{19} the global solvability for the mathematical model of \cite{21} has been demonstrated in a special case of one dimensional flow. Later on in \cite{18} the authors have obtained the solvability of the stationary solution for this model already for three dimensional case, but in a particular case when the stress tensor does not have non-symmetric part.

The main objective of the current work is to correct the model suggested in \cite{21} and to show the well-posedness of a modified model.

The paper is organized as follows:

- First, in the section 2 we describe the model, proposed in \cite{21}, and modify this model.
- In section 3 we explain the main idea of the modification in the Shelukhin-Růžička model and collect some technical results that are used in our main result, related with the proof of global-in-time solvability result for the modified model. In particular, we introduce a potential for the viscous part of the stress tensor and characterizes completely its sub-differential (see Proposition 3.7).
- In section 4 we formulate boundary-value problem and a global-in-time existence theorem 4.1 for the modified model.
- In section 5 we introduce an approximated problem (27), depending on a regularization index \( n \in \mathbb{N} \), and show the solvability of this approximated problem (27) by Schauder fixed point theorem. Also we derive a priori estimates for the solution of (27), which are independent on \( n \);
- Finally, in section 6 we prove Theorem 4.1 applying the Lions-Aubin compactness theorem and a priori estimates of the section 5.

\section{Model of asymmetric Bingham fluids}

In what follows we explain the mathematical model of asymmetric Bingham fluid that was proposed in the article \cite{21}. For simplicity of consideration in this article we consider a particular case when the angular velocity field is zero.

For any matrix \( X \in \mathbb{R}^{3 \times 3} \) we define the symmetric and asymmetric parts

\[
X_s = \frac{1}{2} (X + X^T) \quad \text{and} \quad X_a = \frac{1}{2} (X - X^T) \tag{1}
\]
with the adjoint matrix $X^T$ having the property $(X^T)_{ij} = x_{ji}$. Also we denote
\[ X^d = X - \frac{\text{tr} X}{3} I \quad \text{with} \quad \text{tr} X = \sum_{i=1}^{3} x_{ii}. \]

For any matrices $X, Y \in \mathbb{R}^{3 \times 3}$ the scalar product $X : Y$ and the modulus $|X|$ of $X$ are defined by
\[ X : Y = \sum_{i,j=1}^{3} x_{ij} y_{ij}, \quad |X| = (X : X)^{1/2}. \tag{2} \]

Let $v = v(x, t)$ be the velocity of the mass center of the material point $(\xi, t)$ for an asymmetric Bingham fluid. We denote the rate of strain tensor
\[ B = B(v) = \frac{\partial v}{\partial x}, \quad (\frac{\partial v}{\partial x})_{ij} = \frac{\partial v_i}{\partial x_j}, \tag{3} \]
and introduce the matrix
\[ B_0 = B_s + \kappa B_a, \quad \kappa = \frac{2\mu_1}{\mu_2} \tag{4} \]
where
\[ B_s = \left( \frac{\partial v}{\partial x} \right)_s \quad \text{and} \quad B_a = \left( \frac{\partial v}{\partial x} \right)_a \tag{4} \]
are the symmetric and asymmetric parts of $B = B(v)$, respectively. The positive constants $\mu_i$ are viscosities of the asymmetric Bingham fluid. An instant stress state of the fluid is described by the Cauchy stress tensor $T = -pI + S$, where $p$ and $S$ are the pressure and the viscous part of the stress tensor. In [19] the viscous part $S$ of the stress tensor of the fluid was suggested to be expressed as
\[ S = \begin{cases} 2\mu_1 B_0 + \tau_s B_0, & B_0 \neq 0 \\ S_{\text{plug}}, & B_0 = 0 \end{cases} \tag{5} \]
for some tensor $S_{\text{plug}} \in \mathbb{R}^{3 \times 3}$, such that $|S_{\text{plug}}| \leq \tau_s$. The positive constant $\tau_s$ is the yield stress.

Finally, we have the momentum balance law describing the motion of asymmetric fluid of Bingham
\[ \rho \dot{v} = \text{div} T + \rho f, \tag{6} \]
where $\rho$ is the density and $f$ is the mass force vector.
Remark 2.1. Let us introduce the potential

\[ V(X) = \mu_1 |X|^2 + \tau_\ast |X|, \quad \forall \, X \in \mathbb{R}^{3 \times 3}. \]

Then the constitutive law (5) for the asymmetric Bingham fluid can be formulated as \( S \in \partial V(B_0) \). Let us remind that this inclusion is equivalent to the variational inequality

\[ V(X) - V(B_0) \geq S : (X - B_0), \quad \forall \, X \in \mathbb{R}^{3 \times 3}. \tag{7} \]

Nevertheless that the relation (5) describes a plug zone in the Bingham fluid, there exists a significant restriction in such modelling. As we mentioned in Introduction, in the articles [18], [19] the solvability of the model (5)-(6) was shown only in the case when the asymmetric part is not present in the tensor \( S \). As it is well known, one of the principal approach for the study of problems with inclusions is the theory of monotone operators, that was developed by Duvaut, Lions [8]. The major difficulty in the study of Shelukhin-Růžička model [21] consists in the presence of the term \( S : B_0 \) in the inequality (7). The asymmetric part \( B_a \) of \( B_0 \) does not permit to apply the theory of monotone operators.

In the following we present our modification in the above described model (5)-(6) in such way that permits to apply the theory of monotone operators. Moreover, in our model the viscous part \( S \) will be a more general then in the mathematical model of [21]. For vectors functions \( v \in \mathbb{R}^3 \) we consider \( B = B(v) \) introduced in (3). For such defined tensor \( B \) we introduce the following tensors

\[ B_\mu = 2\mu_1 |B_s|^{p-2}B_s + \mu_2 |B_a|^{p-2}B_a, \quad B_\nu = |B_s|^{\frac{p-2}{2}}B_s + \nu|B_a|^{\frac{p-2}{2}}B_a, \]

\[ B_{\nu,2} = |B_s|^{p-2}B_s + \nu^2|B_a|^{p-2}B_a, \tag{8} \]

where \( p \geq 2, \mu_1, \mu_2 \) are the viscosities of the viscoplastic fluid of Bingham and \( \tau_\ast \) is a so-called plug parameter. Let us define the viscous part \( S \) of asymmetric fluid of Bingham by

\[ S = \begin{cases} 
B_\mu + \tilde{\tau}_\ast \frac{B_{\nu,2}}{|B_\nu |^{\frac{2}{p-2}}}, & B_\nu \neq 0, \\
S_{plug}, & B_\nu = 0,
\end{cases} \tag{9} \]

for some tensor \( S_{plug} = S_{plug}(x, t) \in \mathbb{R}^{3 \times 3} \), which fulfils the restriction \(|S_{plug}| \leq \tau_\ast \). Here we denote

\[ \tilde{\tau}_\ast = \frac{\tau_\ast}{\max(1, \nu^{2/p})}. \tag{10} \]

The major explanation of this modification is based on Proposition 3.7 proved in the following section.
3 Some useful results

In what follows the following algebraic result will be very useful.

**Lemma 3.1.** The space of the matrices endowed with the dot product \((2)\) is the direct sum of the spaces of symmetric matrices and anti-symmetric matrices. More precisely, for any matrices \(X, Y \in \mathbb{R}^{3 \times 3}\), we have

\[
X_s : Y = X_s : Y_s, \quad X_a : Y = X_a : Y_a, \quad X_s : Y_a = X_a : Y_s = 0,
\]

\[
X_s : X = |X_s|^2, \quad X_a : X = |X_a|^2,
\]

with the notation introduced in \((1)\).

**Proof.** The coefficients of the matrices \(X_s\) and \(X_a\) are equal to \(x_s^{ij} = \frac{x_{ij} + x_{ji}}{2}\) and \(x_a^{ij} = \frac{x_{ij} - x_{ji}}{2}\). Then \(x_s^{ij} = x_s^{ji}, x_a^{ij} = -x_a^{ji},\)

\[
X_s : Y = \sum_{i,j=1}^{3} x_s^{ij} y_{ij} = \frac{1}{2} \left[ \sum_{i,j=1}^{3} x_s^{ij} y_{ij} + \sum_{i,j=1}^{3} x_s^{ji} y_{ji} \right] = \sum_{i,j=1}^{3} x_s^{ij} \frac{1}{2} (y_{ij} + y_{ji}) = \sum_{i,j=1}^{3} x_s^{ij} \frac{1}{2} y_{ij} = X_s : Y_s
\]

and

\[
X_a : Y = \sum_{i,j=1}^{3} x_a^{ij} y_{ij} = \frac{1}{2} \left[ \sum_{i,j=1}^{3} x_a^{ij} y_{ij} + \sum_{i,j=1}^{3} x_a^{ji} y_{ji} \right] = \frac{1}{2} \left[ \sum_{i,j=1}^{3} x_a^{ij} y_{ij} - \sum_{i,j=1}^{3} x_a^{ji} y_{ji} \right] = \sum_{i,j=1}^{3} x_a^{ij} \frac{1}{2} (y_{ij} - y_{ji}) = \sum_{i,j=1}^{3} x_a^{ij} y_{ij} = X_a : Y_a.
\]

Moreover, we have

\[
\left( \frac{X + X^T}{2} \right) : \left( \frac{Y + Y^T}{2} \right) = \frac{1}{4} \sum_{i,j=1}^{3} (x_{ij} \pm x_{ji})(y_{ij} \mp y_{ji}) = \frac{1}{4} \sum_{i,j=1}^{3} (x_{ij} y_{ij} - x_{ji} y_{ji}) = 0.
\]

Before proceeding let us recall two basic theorems on convex analysis.

**Theorem 3.2.** (see [16, Theorem 23.1] or [2, Proposition 17.2]) Let \(f\) be a convex function from \(\mathbb{R}^n\) to \([-\infty, +\infty]\), and let \(x\) be a point where \(f\) is finite. Then, for each \(y \in \mathbb{R}^n\) there exists the one-sided directional derivative of \(f\) at \(x\) with respect to the vector \(y\), i.e.

\[
f'(x; y) = \lim_{\lambda \to 0^+} \frac{f(x + \lambda y) - f(x)}{\lambda}.
\]

In fact, the difference quotient \((f(x + \lambda y) - f(x))/\lambda\) is a non-decreasing function of \(\lambda > 0\), so that

\[
f'(x; y) = \inf_{\lambda > 0} (f(x + \lambda y) - f(x))/\lambda.
\]
Remark 3.3. For any function \( f : \mathbb{R}^n \to \mathbb{R}^m \) positively homogeneous of order 1, we have
\[
f'(0; y) = f(y), \quad \forall y \in \mathbb{R}^n.
\]
Indeed, in this case one has
\[
f'(0; y) = \lim_{\lambda \to 0} \frac{f(\lambda y) - f(0)}{\lambda} = \lim_{\lambda \to 0} \frac{\lambda f(y)}{\lambda} = f(y).
\]

Let us remember the concept of sub-differential.

Definition 3.4. A vector \( x^* \) is said to be a sub-gradient of \( f \) at \( x \) if
\[
f(y) \geq f(x) + (x^*, y), \quad \forall y \in \mathbb{R}^n.
\]
The set of all sub-gradients of \( f \) at \( x \) is called sub-differential of \( f \) at \( x \) and is denoted by \( \partial f(x) \).

Theorem 3.5. (see [16, Theorem 23.2] or [2, Proposition 17.7]) Let \( f \) be a convex function, and let \( x \) be a point where \( f \) is finite. Then \( x^* \) is a sub-gradient of \( f \) at \( x \) if and only if
\[
f'(x; y) \geq (x^*, y), \quad \forall y \in \mathbb{R}^n.
\]

Let us also remark the following fact.

Remark 3.6. Let \( \| x \|_{l^p(\mathbb{R}^n)} = \left( |x_1|^p + \cdots + |x_n|^p \right)^{1/p}, \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n, \)
de note the \( l^p \) norm in \( \mathbb{R}^n \). Then the \( l^p \) norm is decreasing with respect to \( p \in [1, \infty] \).

This is easy to prove: given \( p_1 \leq p_2 \) in \( [1, \infty] \) and \( x = (x_1, \cdots, x_n) \neq 0 \), let \( y_i = |x_i|/\|x\|_{p_2} \). Then \( |y_i| \leq 1 \), so
\[
|y_i|^{p_1} \geq |y_i|^{p_2}, \quad \|y\|_{p_1} \geq 1,
\]
and, consequently, \( \|x\|_{l^p_1} \geq \|x\|_{l^p_2} \).

The following results are used in a crucial way in the proof of our Theorem 4.1.

Proposition 3.7. For given \( p \geq 2 \) let us introduce the potential
\[
V(X) = \frac{2\mu_1}{p} |X_s|^p + \frac{\mu_2}{p} |X_a|^p + \hat{\tau}_s|X_s|^{\frac{p-2}{2}} X_s + \nu|X_a|^{\frac{p-2}{2}} X_a
\]  
(11)
for any matrix $X \in \mathbb{R}^{3 \times 3}$. The potential $V$ is convex, differentiable at any $X \neq 0$ (equivalently, $X_{i\nu} \neq 0$) in $\mathbb{R}^{3 \times 3}$, with

\[ DV(X) = X_{\mu} + \hat{\tau}_s \frac{X_{\nu^2}}{|X_{\nu}|^\frac{2(p-1)}{p}}, \]

the matrices $X_{\mu}$, $X_{\nu}$, $X_{\nu^2}$ are defined in $\mathbb{S}$ (instead of $B$ we substitute $X$). Moreover:

(a) $B_{r_p}(0) \subset (\hat{\tau}_s)^{-1}\partial V(0) \subset B_{\max\{1, \nu^2\}}(0)$, where $B_{r_p}(0)$ is the closed ball of a radius $r_p$ at the center 0 and

\[ r_p = \nu^2/(1 + (\nu^2)^\frac{p-2}{2}); \]

(b) let $q = p/(p - 1)$, then

\[ (\hat{\tau}_s)^{-1}\partial V(0) = \{ S \in \mathbb{R}^{3 \times 3}: |S|_q + \nu^{2(1-q)}|S|_{\nu}^q \leq 1 \}. \]

**Proof.** We write $V(X) = U(X) + \hat{\tau}_s W(X)$ with

\[ U(X) = \frac{2\mu_1}{p} |X_s|^p + \frac{\mu_2}{p} |X_{\alpha}|^p, \quad W(X) = |||X_s|_{\nu}^\frac{p-2}{2} X_s + \nu |X_{\alpha}|_{\nu}^\frac{p-2}{2} X_{\alpha}|^\frac{2}{p}. \]

To see that $V$ is convex, we first notice that the functions

\[ X \mapsto |X_s|^p, \quad X \mapsto |X_{\alpha}|^p \]

are convex since they are a composition of the convex function $t \mapsto |t|^\frac{p}{2}$ with quadratic functions $X \mapsto |X_s|^2$, $X \mapsto |X_{\alpha}|^2$. In addition, we can check that the function $W$ is convex using that any norm is convex and the fact that

\[ W(X) = \sqrt[\frac{2}{p}]{|X_s|^p + \nu^2 |X_{\alpha}|^p} = ||(|X_s|, \nu^\frac{2}{p} |X_{\alpha}|)||_p(\mathbb{R}^2) \]

is also a norm. Thus, $V$ is convex because it is a linear combination of convex functions.

The function $V$ is differentiable at any $X \in \mathbb{R}^{3 \times 3}\setminus\{0\}$ due to the chain rule, and we can compute $DV(X)$ differentiating directly $U$ and $W$ with respect to the standard variables $x_{ij}$ in $\mathbb{R}^{3 \times 3}$, or, alternatively, using the chain rule, one has

\[ DV(X) = 2\mu_1 |X_s|^p X + \mu_2 |X_{\alpha}|^p X + \hat{\tau}_s DW(X) \]

\[ = X_{\mu} + \hat{\tau}_s (|X_s|^p + \nu^2 |X_{\alpha}|^p)^\frac{1}{2} (|X_s|_{\nu}^p X_s + \nu^2 |X_{\alpha}|_{\nu}^p X_{\alpha}) \]

\[ = X_{\mu} + \hat{\tau}_s |X_{\mu}|_{\nu}^p X_{\nu}^p. \tag{12} \]

Now, the function $U$ is differentiable also at $X = 0$ and $DU(0) = 0$. Then, the sub-differential $\partial V(0)$ is equal to $\hat{\tau}_s \partial W(0)$. Let us show the items (a) and (b) in the statement of the Proposition).
By Remark 3.3,
\[ W'(0; Y) = W(Y) \quad \text{for any } Y \in \mathbb{R}^{3 \times 3}. \]

Then, by Theorem 3.5,
\[ S \in \partial W(0) = \hat{\tau}_s^{-1} \partial V(0) \iff W(Y) \geq S : Y, \quad \forall Y \in \mathbb{R}^{3 \times 3}. \]

(13)

Taking in the inequality \( Y = S \) and using the Remark 3.6, we obtain
\[ |S| \leq \|([S_s], \nu^{2/p}|S_a|)\|_{p'(\mathbb{R}^2)} \leq \max\{1, \nu^{2/p}\} \|([S_s], |S_a|)\|_{p'(\mathbb{R}^2)} \leq \max\{1, \nu^{2/p}\} |S|, \]

hence we have proved the claim \( \partial W(0) \subset B_{\max\{1, \nu^{2/p}\}}(0) \) of item (a) of the Proposition.

Now let us show the claim \( B_{r_p}(0) \subset \partial W(0) = \hat{\tau}_s^{-1} \partial V(0) \) (14)
of item (a). Let us consider arbitrary matrix \( S \in \mathbb{R}^{3 \times 3} \) satisfying the property
\[ |S||Y| \leq W(Y), \quad \forall Y \in \mathbb{R}^{3 \times 3}. \]

(15)

Then, by the Cauchy-Schwarz inequality, we have also
\[ S : Y \leq W(Y), \quad \forall Y \in \mathbb{R}^{3 \times 3}. \]

By Theorem 3.5 we conclude that any \( S \in \mathbb{R}^{3 \times 3} \), satisfying the property (15), belongs to \( \partial W(0) \). On the other hand, by the positive homogeneity of \( W \) the property (15) is equivalent to
\[ |S| \leq \min_Y \{W(Y) : Y \in \mathbb{R}^{3 \times 3} \text{ with } |Y| = 1\}. \]

Let us demonstrate that this minimum is equal to \( r_p \), that gives the claim (14). The exact value of \( r_p \) is defined in the statement of item (a) in the Proposition. Since \( 1 = |Y|^2 = |Y_s|^2 + |Y_a|^2 \), writing \( t = |Y|^2_a \), we have \( |Y_s|^2 = 1 - t \) and
\[ W(Y) = W(t) = \sqrt{(1-t)^{\frac{p}{2}} + \nu^2 t^{\frac{p}{2}}}, \quad t \in [0, 1]. \]

By a straightforward computation, we obtain that the minimum of the function
\[ \alpha(t) = (1-t)^{\frac{p}{2}} + \nu^2 t^{\frac{p}{2}} \quad \text{in the interval } [0, 1] \]
is \( (r_p)^p \), and it is attained at \( t_* = 1/(1 + \nu^{-2/p}) \). Thus, we have proved the claim (a).
To show claim (b) we follow a similar reasoning as above, and with the help of the Hölder inequality. Let \( S \in \partial W(0) \). Then, by Theorem 3.5 we have
\[
S : Y \leq W(Y), \quad \forall Y \in \mathbb{R}^{3 \times 3}.
\]
If we take in this inequality \( Y \), having
\[
Y_s = |S_s|^q S_s \quad \text{and} \quad Y_a = \nu^{-\frac{2q}{p}} |S_a|^q S_a,
\]
we obtain
\[
|S_s|^q + \nu^{-\frac{2q}{p}} |S_a|^q \leq \sqrt{|S_s|^q + \nu^{-2q+2} |S_a|^q}.
\]
This implies that
\[
|S_s|^q + \nu^{2(1-q)} |S_a|^q \leq 1,
\]
because \(-2q/p = 2(1-q)\) and \( p \geq 2 \). Reciprocally, if
\[
|S_s|^q + \nu^{2(1-q)} |S_a|^q \leq 1
\]
then, for all \( Y \in \mathbb{R}^{3 \times 3} \), using the Hölder inequality, we have
\[
S : Y = S_s : Y_s + (\nu^{-2/p} S_a) : (\nu^{2/p} Y_a) \\
\leq \sqrt{|S_s|^q + \nu^{-2q/p} |S_s|^q} \sqrt{|Y_s|^p + \nu^2 |Y_a|^p} \leq W(Y).
\]
Then, again by Theorem 3.5 we obtain that \( S \in \partial W(0) \).

Now we show the auxiliary result that explains the definition of \( \hat{\tau}_s \) by (10) in (9).

**Corollary 3.8.** For any matrix \( B \in \mathbb{R}^{3 \times 3} \setminus \{0\} \) one has the estimate
\[
\frac{|B_s|^2}{\sqrt{|B_{2(p-1)}|^2}} \leq \max\{1, \nu^{\frac{2}{p}}\}. \tag{16}
\]

**Proof.** As the derivative of the functional \( W(X) \) has been calculated in (12) and equals to
\[
DW(B) = \frac{B_s^2}{\sqrt{|B_{(2p-1)}|^2}} \quad \text{at any} \quad B \neq 0.
\]

Now we claim that \( DW(B) \in \partial W(0) \), and thus the estimate (16) shall follow from this fact, by (a) of Proposition 3.7 since
\[
\partial W(0) = (\hat{\tau}_s)^{-1} \partial V(0) \subset B_{\max\{1, \nu^{\frac{2}{p}}\}}(0).
\]
Accounting (13) we have to show the claim
\[
DW(B) : Y \leq W(Y) \quad \text{for all} \quad Y \in \mathbb{R}^{3 \times 3}.
\]
By Theorem 3.2 we have that the directional derivative
\[ DW(B) : Y = \inf_{\lambda > 0} \lambda^{-1} (W(B + \lambda Y) - W(B)). \]
In addition, \(W\) is a norm by Remark 3.6, then
\[ W(B + \lambda Y) \leq W(B) + \lambda W(Y). \]
Summing up, we obtain our above claim. ■

Remark 3.9. By Proposition 3.7, the relation (9) is equivalent to the variational inequality
\[ V(X) - V(B) \geq S : (X - B), \quad \forall X \in \mathbb{R}^{3 \times 3}. \]

Next we present two technical results we shall use latter.

**Lemma 3.10.** Let \( W = W(X) \) be a positive convex function on \( X \in \mathbb{R}^{3 \times 3} \).
Then, for any natural \( n \), the approximated function
\[ W_n(X) = \sqrt[p]{(W(X))^p + n^{-1}} \]
is also convex with respect of the parameter \( X \in \mathbb{R}^{3 \times 3} \).

**Proof.** Note that the function \( \varphi(z) = \sqrt[p]{z^p + n^{-1}} \) is monotone increasing and convex function for \( z \geq 0 \). Therefore applying the definition of convex function, we easily derive that the composition \( W_n(X) = \varphi(W(X)) \) is also convex with respect of the parameter \( X \in \mathbb{R}^{3 \times 3} \). ■

**Lemma 3.11.** Let \( n \) be an arbitrary natural number. We consider the convex potential
\[ V^n(X) = \frac{2\mu_1}{p} |X_s|^p + \frac{\mu_2}{p} |X_a|^p + \tilde{\tau}_s \sqrt{||X_s|^2 + |X_a|^2}^p X_s + \nu |X_a| \tilde{\tau}_a X_a |^2 + n^{-1}, \]
defined for arbitrary \( X \in \mathbb{R}^{3 \times 3} \). Let
\[ S^n = B_\mu + \tilde{\tau}_s \sqrt{(|B_\nu|^2 + n^{-1} p^{-1})^p}. \]
Then, for any given \( B \in \mathbb{R}^{3 \times 3} \), we have \( \frac{\partial V^n}{\partial X}(B) = S^n \), i.e.
\[ V^n(X) - V^n(B) \geq S^n : (X - B), \quad \forall X \in \mathbb{R}^{3 \times 3}. \]

**Proof.** Straightforward computation. ■
4 Statement of the problem

Let us consider the motion of an asymmetric Bingham fluid, assuming that the fluid is incompressible. For simplicity of considerations we admit that the density $\rho$ is equal to 1 and neglect the mass force vector $f$. Then the flow equations (6) for the velocity $v$ in a bounded domain $\Omega \subset \mathbb{R}^3$ with the boundary $\Gamma$ are

$$v_t + (v \cdot \nabla) v = \text{div} \ T, \quad \text{div} \ v = 0 \quad \text{in} \quad \Omega_T = (0, T) \times \Omega, \quad (17)$$

where $T = -p I + S$ and $S$ satisfies the constitutive law (9) with the relations (3), (4), (8). We add to this system the initial data

$$v|_{t=0} = v_0 \quad \text{in} \quad \Omega. \quad (18)$$

The system (17) is mostly supplemented with the usual Dirichlet boundary condition. The Dirichlet condition implies the adherence of fluid particles to the boundary. For the motion of Bingham fluids (such as the extrusion of the toothpaste from the tube, the mud flows in drilling engineering, the propane flow within the hydro-fracture, etc.) it is more natural to study the system (17) with slip type boundary conditions, permitting the slippage of the fluid against the boundary. To describe accurately physical phenomena we consider homogeneous Navier slip boundary conditions

$$v \cdot n = 0, \quad [T n + \alpha v] \cdot \tau = 0 \quad \text{on} \quad \Gamma_T = (0, T) \times \Gamma, \quad (19)$$

where $\alpha$ is a positive friction coefficient. For the discussion of the Navier slip boundary conditions we refer to the articles [4]-[6].

Let us introduce some notations to formulate our main result. We denote by $(\cdot, \cdot)$ the dot product in $L^2(\Omega)$. Also we define the spaces

$$H = \{v \in L^2(\Omega) : \text{div} \ v = 0 \quad \text{in} \quad D'(\Omega), \quad v \cdot n = 0 \quad \text{in} \quad H^{-1/2}(\Gamma)\},$$

$$V = \{v \in H^1(\Omega) : \text{div} \ v = 0 \quad \text{a.e. in} \quad \Omega, \quad v \cdot n = 0 \quad \text{in} \quad H^{1/2}(\Gamma)\},$$

$$V_p = \{v \in V : |\nabla v|^p \in L^1(\Omega)\}. \quad (20)$$

The space $V_p$ is endowed with the norm $\|v\|_{V_p} = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^p(\Omega)}$.

The main objective of our article is to show the well-posedness of the system (17)-(19) for unknown functions $v$ and $S$. This result of the well-posedness is formulated in the following theorem, in which we also define the concept of the weak solution for the system (17)-(19). This concept is a direct consequence of the equations (17), the boundary conditions (19) and the integral equality

$$- \int_{\Omega} \text{div} \ T \cdot \varphi \ dx = - \int_{\Gamma} (T n) \cdot \varphi \ d\gamma + \int_{\Omega} T : \frac{\partial \varphi}{\partial x} \ dx, \quad (21)$$
which is valid for any \((3 \times 3)\)-matrix function \(T \in H^1(\Omega)\) and any 3D-vector function \(\varphi \in H^1(\Omega)\).

**Theorem 4.1.** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^3\) with a \(C^1\)–smooth boundary \(\Gamma\), \(p \geq 2\), a given real number, \(v_0 \in H\) and \(\alpha \in L^2(0,T; L^\infty(\Omega))\).

Then there exists a function \(v\) and a \((3 \times 3)\)–matrix function \(S\), such that

\[
v \in L^\infty(0,T; H) \cap L^2(0,T; V_p), \quad v_t \in L^2(0,T; V'_p) \tag{23}
\]

and the pair \((v, S)\) is a weak solution of the system \((17)-(19)\), satisfying the integral equality

\[
\int_\Omega \left[ v \partial_t \varphi + (v \otimes v - S) : \frac{\partial \varphi}{\partial x} \right] dx dt + \int_\Omega v_0 \varphi(0) dx = \int_{\Gamma_T} \alpha(v \cdot \varphi) d\gamma dt \tag{24}
\]

for any test function \(\varphi \in H^1(0,T; V_p)\) such that \(\varphi(\cdot, T) = 0\).

The \((3 \times 3)\)–matrix function \(S \in L^{p/(p-1)}(\Omega_T)\) fulfils the relation \((9)\).

Moreover, if

\[
p \geq \frac{7 + \sqrt{19}}{5} \approx 2.272,
\]

then the solution \((v, S)\) is unique.

## 5 Construction of the approximation problem

In this section we consider an approximated problem for the system \((17)-(19)\) and solve this approximated problem applying the Faedo-Galerkin method and the Schauder fixed point argument (see for instance [1]).

Since the space \(V_p\) is separable, it is the span of a countable set of linearly independent functions \(\{e_k\}_{k=1}^\infty\). More precisely, we can choose this set as the eigenfunctions for the following non-linear Stokes type equations with Navier boundary conditions:

\[
\begin{cases}
-\text{div}(T(e_k)) = \lambda_k e_k, \quad \text{div } e_k = 0 \quad \text{in } \Omega, \\
e_k \cdot n = 0, \quad (T(e_k)n + \alpha e_k) \cdot \tau = 0 \quad \text{on } \Gamma
\end{cases}
\]
with $T(e_k) = -p_k I + |\nabla e_k|^{p-2} \nabla e_k$. The solvability of this problem follows from the spectral theory $[10]$. This theory permits to construct this set $\{e_k\}_{k=1}^{\infty}$ as an orthogonal basis for $V_p$ and an orthonormal basis for $H$.

We can consider the subspace $V_p^n = \text{span} \{e_1, \ldots, e_n\}$ of $V_p$, for any fixed natural $n$. Let us define the vector function

$$v^n(t) = \sum_{k=1}^{n} c_k^{(n)}(t) e_k, \quad c_k^{(n)}(t) \in \mathbb{R},$$

(26)

as the solution of the approximate system

$$\begin{cases}
\int_{\Omega} [\partial_t v^n e_k + (v^n \cdot \nabla) v^n e_k + S^n : \frac{\partial v^n}{\partial x}] \, dx + \int_{\Gamma} \alpha (v^n \cdot \tau)(e_k \cdot \tau) \, d\gamma \, dt = 0, \\
\forall k = 1, 2, \ldots, n, \\
v^n(0) = v^n_0.
\end{cases}
$$

(27)

The $(3 \times 3)$–matrix functions $T^n, S^n$ are prescribed by the relations

$$T^n = -p^n I + S^n, \quad S^n = B^n_{\mu} \frac{B^n_{\nu}}{\sqrt{(|B^n_{\nu}|^2 + n-1)p-1}},$$

(28)

and the matrix functions $B^n = B(v^n), B^n_{\mu}, B^n_{\nu}, B^n_{\nu^2}$ are calculated through the formulas $[3], [1], [8]$. The function $v^n_0$ is the orthogonal projection of $v_0 \in H$ into the space $V_p^n$. Note that the system (26)-(28) is a weak formulation of the problem

$$\begin{cases}
\partial_t v^n + (v^n \cdot \nabla) v^n = \text{div} T^n, \quad \text{div} v^n = 0, \quad \text{in} \; \Omega_T, \\
v^n \cdot \mathbf{n} = 0, \quad [T^n n + \alpha v^n] : \tau = 0 \quad \text{on} \; \Gamma_T, \\
v^n |_{t=0} = v^n_0 \quad \text{in} \; \Omega.
\end{cases}
$$

Next we prove that the approximated problem (24) is solvable.

**Lemma 5.1.** Let us assume that the data $v_0, \alpha$ satisfy the conditions (22). Then there exists a solution $v^n \in L^\infty(0,T;V_p)$ of the system (26)-(28), satisfying the a priori estimate

$$\int_{\Omega} |v^n|^2 \, dx + \int_0^T \left[ \int_{\Omega} |\partial_x v^n|^p \, dx + \int_{\Gamma} \alpha |v^n|^2 \, d\gamma \right] \, dt \leq A, \quad t \in [0,T],$$

(29)

where $A$ is a constant independent on $n$. More precisely, $A$ depends only on the data $v_0, \mu, \nu$.

**Proof.** The system (24) is a system of $n$ ordinary differential equations of the first order, which can be written in the form

$$\frac{dc_k^{(n)}}{dt} = F(c_k^{(n)}), \quad t \in (0, T),$$
for the vector function \( \mathbf{c}^{(n)}(t) = (c_1^{(n)}(t), \ldots, c_n^{(n)}(t)) \) with \( c_k^{(n)} \) introduced in [26]. We can solve system using the Schauder fixed point theorem.

For an arbitrary \( \hat{\mathbf{c}}^{(n)} \in C([0, T]) \), we define

\[
\hat{\mathbf{v}}^n(t) = \sum_{k=1}^{n} \hat{c}_k^{(n)}(t) \mathbf{e}_k \quad \text{and} \quad \hat{\tau}_s(\hat{\mathbf{c}}^{(n)}) = \frac{1}{\sqrt{\left| B_0^n(\mathbf{v}^n)^2 + n^{-1}p^{-1} \right|}}.
\]

Let \( \mathbf{c}^{(n)} \) be an unknown, such that the vector function

\[
\mathbf{v}^n(t) = \sum_{k=1}^{n} c_k^{(n)}(t) \mathbf{e}_k
\]

solves the system

\[
\left\{ \begin{array}{l}
\int_{\Omega} \left[ \partial_t \mathbf{v}^n \mathbf{e}_k + (\hat{\mathbf{v}}^n \cdot \nabla) \mathbf{v}^n \mathbf{e}_k + \hat{S}^n : \partial_x \mathbf{v}^n \right] d\mathbf{x} + \int_{\Gamma} \alpha (\mathbf{v}^n \cdot \tau)(\mathbf{e}_k \cdot \tau) d\gamma dt = 0, \\
\forall k = 1, 2, \ldots, n, \\
\mathbf{v}^n(0) = \mathbf{v}_0^n
\end{array} \right.
\]

with the \((3 \times 3)\)-matrix function

\[
\hat{S}^n = B_\mu^n + \hat{\tau}_s(\hat{\mathbf{c}}^{(n)})B_\nu^n,
\]

and the matrix functions \( B^n = B(v^n), B_\mu^n, B_\nu^n, B_\nu^2, \) given by the formulas \(4, 8\) and \(15\). From the theory of ordinary differential equations, it follows that the linear system \((30)-(31)\), of \( n \) ordinary linear differential equations, has an unique solution \( \mathbf{c}^{(n)} \in C^1([0, T]) \). Therefore, we can consider the operator \( K : C([0, T]) \rightarrow C([0, T]) \) defined as

\[
\mathbf{c}^{(n)} = K \left( \hat{\mathbf{c}}^{(n)} \right).
\]

The solvability of the system \((26)-(28)\) will be shown if we demonstrate that this operator \( K \) has a fixed point, which we shall do by the Schauder fixed point theorem. Thus, we have to prove that this operator is compact on a bounded convex subset \( M \) of \( C([0, T]) \).

First, let us deduce a priori estimates for \( \mathbf{c}^{(n)} \). We multiply \((31)_1\) by \( c_k^{(n)} \) and take the sum on the index \( k = 1, \ldots, n \). Then the integration over the time interval \((0, t)\) gives

\[
\frac{1}{2} \int_{\Omega} |\mathbf{v}^n|^2 d\mathbf{x} + \int_{0}^{t} \left[ \int_{\Omega} \hat{S}^n : \partial_x \mathbf{v}^n d\mathbf{x} + \int_{\Gamma} \alpha |\mathbf{v}^n|^2 d\gamma \right] dt = \frac{1}{2} \int_{\Omega} |\mathbf{v}_0^n|^2 d\mathbf{x}.
\]

Lemma \(5.1\) and the definition \(49\) of \( \hat{S}^n \) imply

\[
B_\mu^n : \partial_x \mathbf{v}^n = 2\mu_1 |B_s^n|^p + \mu_2 |B_a^n|^p, \quad B_\nu^n : \partial_x \mathbf{v}^n = |B_s^n|^p + \nu^2 |B_a^n|^p.
\]
and
\[ \hat{S}^n : \frac{\partial \nu^n}{\partial x} = 2\mu_1 |B_s^n|^p + \mu_2 |B_a^n|^p + \hat{\tau}_s(\hat{c}^{(n)}) \left( |B_s^n|^p + \nu^2 |B_a^n|^p \right). \]

Therefore, we deduce the inequality
\[
\frac{1}{2} \int_{\Omega} |v^n|^2 \, dx + \int_0^T \int_{\Omega} (2\mu_1 |B_s^n|^p + \mu_2 |B_a^n|^p) \, dx + \int_\Gamma \alpha |v^n|^2 \, d\gamma \\
\leq \frac{1}{2} \int_{\Omega} |v_0^n|^2 \, dx,
\]
which gives the apriori estimate (29).

Also, we have
\[
|B_{\mu_1}^n|^2 = 4\mu_1^2 |B_s^n|^{2(p-1)} + \mu_2^2 |B_a^n|^{2(p-1)}, \quad |B_{\mu_2}^n|^2 = |B_s^n|^{2(p-1)} + \nu^4 |B_a^n|^{2(p-1)},
\]
by Lemma 3.1. Since
\[
\hat{\tau}_s(\hat{c}^{(n)}) \leq \frac{1}{\sqrt{n}^{p-1}}, \quad \text{then} \quad \hat{S}^n \leq |B_{\mu_1}^n| + \frac{1}{\sqrt{n}^{p-1}} |B_{\mu_2}^n|,
\]
and using (34), we obtain
\[
||\hat{S}^n||_{L^p/(p-1)(\Omega_T)} \leq C(n)
\]
where the constant \( C(n) \) depends only on \( n \).

Let us define the bounded convex set
\[
M = \left\{ c^{(n)} \in C([0, T]) : ||c^{(n)}|| \leq A \right\},
\]
where the norm \( ||c^{(n)}||^2 = \max_{t \in [0, T]} \sum_{k=1}^n (c_k^{(n)}(t))^2 \) is defined on the space \( C([0, T]) \) and the constant \( A \) is prescribed in (29).

Let us assume that \( \hat{c}^{(n)} \in M \). Since \( \{e_j\}_{j=1}^{\infty} \) is the orthonormal basis for the space \( H \), then the equality (31) can be written as
\[
\frac{dc_k^{(n)}}{dt} = -\int_{\Omega} \left( \hat{\nabla} \cdot \nabla \right) v^n e_k + \hat{S}^n : \frac{\partial e_k}{\partial x} \, dx - \int_{\Gamma} \alpha (v^n \cdot \tau)(e_k \cdot \tau) \, d\gamma \, dt
\]
for any \( k = 1, \ldots, n \). Since \( V_p \subset H^1(\Omega) \), then the Sobolev continuous embedding \( H^1(\Omega) \hookrightarrow L^6(\Omega) \cap L^2(\Gamma) \) and the Holder inequality imply
\[
\left| \frac{dc_k^{(n)}}{dt} \right| \leq \{ ||\hat{\nabla}v^n||_{L^4(\Omega)} ||\nabla v^n||_{L^2(\Omega)} + ||\sqrt{\alpha}||_{L^\infty(\Omega)} ||\sqrt{\alpha}v^n||_{L^2(\Gamma)} \} ||e_k||_{H^1(\Omega)} \]
\[ + ||\hat{S}^n||_{L^p/(p-1)(\Omega)} ||\frac{\partial e_k}{\partial x}||_{L^p(\Omega)}, \]
(36)
that is
\[
\left|\frac{dc^{(n)}_k}{dt}\right| \leq C(n)\left\{ \|\hat{v}^n\|_{L^4(\Omega)}\|\nabla v^n\|_{L^2(\Omega)} + \|\sqrt{\alpha}v^n\|_{L^2(\Gamma)} + \|\hat{S}^n\|_{L^{p/(p-1)}(\Omega)} \right\}. \tag{37}
\]

Let us recall the Gagliardo–Nirenberg-Sobolev inequality (see [23])
\[
\int_0^T \left( \|\hat{v}^n\|_{L^4(\Omega)}\|\nabla v^n\|_{L^2(\Omega)} \right)^{8/7} dt \leq C \int_0^T \left( \|\hat{v}^n\|_{L^{7/2}(\Omega)}^{1/4}\|\nabla v^n\|_{L^2(\Omega)}^{7/4} \right)^{8/7} dt
\leq C\|\hat{v}^n\|_{L^\infty(0,T;L^2(\Omega))}\|v^n\|_{L^2(0,T;H^1(\Omega))^*}^{2/7}. \tag{38}
\]

Therefore, the integration of the inequality (37) over the time interval \([t, t+h]\subset[0, T]\), the Hölder inequality and \(\hat{c}^{(n)}\in M\), plus the estimates (29), (38), imply
\[
|c_k^{(n)}(t+h) - c_k^{(n)}(t)| \leq \int_t^{t+h} \left|\frac{dc^{(n)}_k}{dt}\right| dt \leq C(n)h^{\frac{1}{q}} \quad \text{with} \quad q = \min\left\{\frac{1}{8}, \frac{1}{2}, \frac{1}{p}\right\}
\]
for each \(k = 1, \ldots, n\). Therefore the operator \(K : M \to M\) is compact by the Arzela-Ascoli theorem.

The continuity of \(K\) is a direct consequence of the theorem on continuous dependence of the solution of the Cauchy problem (30)-(31) with respect to the coefficients \(\hat{c}^{(n)}\). Therefore, the operator \(K\) fulfills the conditions of the Schauder fixed point theorem, which implies the existence of a fixed point of \(K\), and gives the solution of the system (26)-(28).

\[\blacksquare\]

\textbf{Lemma 5.2.} We assume that the data \(v_0, \alpha\) fulfil the conditions (22). Then there exists a solution
\[
v^n \in L^\infty(0, T; H) \cap L^2(0, T; V_p)
\]
of the system (26)-(28) that satisfies the following estimates
\[
\int_\Omega |v^n|^2 dx + \int_0^T \left[ \int_\Omega \left| \frac{\partial v^n}{\partial x} \right|^p dx + \int_\Gamma |\alpha| v^n|^2 \ d\gamma \right] dt \leq C, \quad t \in [0, T], \tag{39}
\]
\[
\|S^n\|_{L^{p/(p-1)}(\Omega_T)} \leq C \tag{40}
\]
and
\[
\|\partial_t v^n\|_{L^{8/7}(0, T; V^*_p)} \leq C. \tag{41}
\]

Here and below \(C\) is a positive constant that does not depend on \(n\), but may depend on \(v_0\) and \(\alpha\).
Proof. Let us note that in Lemma 5.1 we already shown that the solution \( v^n \in L^\infty(0,T;V^n) \) of (26)-(28) satisfies (39). Moreover, from the definition (28) of \( S^n \), we have

\[ |S^n| \leq |B^s_n| + \tau_* \]

This is a direct consequence of the estimate (16). Hence we obtain the a priori estimate (40) from (39) and the equality

\[ |B^s_n|^2 = 4\mu_1^2|B^s_n|^{2(p-1)} + \mu_2^2|B^a_n|^{2(p-1)} \]

Let us consider a subspace \( V^n_p \) of \( V^n \), defined in Lemma 5.1. Let \( P_n \) be the orthogonal projection of \( V^n \) onto \( V^n_p \). Let \( \phi \in H^1(0,T;V^n_p) \) be an arbitrary function. The first equality of (27) is linear with respect of the functions \( e_k, k = 1, \ldots, n \), then we have

\[
\begin{cases}
\int_{\Omega} \left[ \partial_t v^n(P_n\phi) + (v^n \cdot \nabla) v^n (P_n\phi) + S^n : \frac{\partial (P_n\phi)}{\partial x} \right] \, dx \\
\quad + \int_{\Gamma} \alpha(v^n \cdot (P_n\phi)) \, d\gamma \, dt = 0,
\end{cases}
\]

since \( \{e_j\}_{j=1}^\infty \) is the orthogonal basis for the space \( V^n_p \).

As it was done in (36), we obtain

\[
|\partial_t v^n, \phi|_{L^2(\Omega)} \leq \{ ||v^n||_{L^4(\Omega)} ||\nabla v^n||_{L^2(\Omega)} + ||\sqrt{\alpha} ||_{L^\infty(\Omega)} ||\sqrt{\alpha} v^n||_{L^2(\Gamma)} \} ||P_n\phi||_{H^1(\Omega)} + ||S^n||_{L^p/(p-1)(\Omega)} ||\frac{\partial (P_n\phi)}{\partial x}||_{L^p(\Omega)},
\]

then

\[
||\partial_t v^n||_{V^n_p} = \sup_{\phi \in V^n_p} \{ |(\partial_t v^n, \phi)|_{L^2(\Omega)} : ||\phi||_{V^n_p} = 1 \}
\leq C \left\{ ||v^n||_{L^4(\Omega)} ||\nabla v^n||_{L^2(\Omega)} + ||\sqrt{\alpha} v^n||_{L^2(\Gamma)} + ||S^n||_{L^p/(p-1)(\Omega)} \right\},
\]

since the norm \( ||\phi||_{V^n_p} = ||\phi||_{L^2(\Omega)} + ||\phi||_{L^p(\Omega)} \) on the space \( V^n_p \) of the continuous operator \( P_n \) is less or equal than 1.

Therefore this last inequality (43), the a priori estimates (39) - (40) and the Gagliardo–Nirenberg–Sobolev inequality (38), written for \( v^n \) instead of \( \hat{v}^n \) imply the estimate (41).

\[ \blacksquare \]
6 Limit transition

We have that $V_p \subset H^1(\Omega)$, then the embedding result $H^1(\Omega) \hookrightarrow L^2(\Gamma)$ and the estimates \([39] - [41]\) imply the existence of sub-sequences, such that

\[
\begin{align*}
&v^n \rightharpoonup v \quad \text{*-weakly in } L^\infty(0,T;L^2(\Omega)), \\
v^n \rightharpoonup v \quad \text{weakly in } L^2(0,T;V_p \cap L^2(\Gamma)), \\
&S^n \rightharpoonup S \quad \text{weakly in } L^2(\Omega_T).
\end{align*}
\]

Let us remember the Aubin-Lions-Simon compactness result \([17, 23]\).

**Lemma 6.1.** Let $X_0$, $X$ and $X_1$ be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that $X_0$, $X_1$ are reflexive, $X_0$ is compactly embedded in $X$, and that $X$ is continuously embedded in $X_1$. Let

\[
W = \left\{ v \in L^2(0,T;X_0), \quad \partial_t v \in L^{8/7}(0,T;X_1) \right\}.
\]

Then the embedding of $W$ into $L^2(0,T;X)$ is compact.

Therefore the estimates \([39] - [41]\), the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and Lemma 6.1 give that

\[
v^n \to v \quad \text{strongly in } L^2(\Omega_T) \quad \text{and a.e. in } \Omega_T.
\]

Hence applying the convergences \((44)-(45)\) in \((42)\), we deduce that the limit functions $v$, $S$ fulfill the integral equality

\[
\int_{\Omega_T} \left[ v \partial_t \varphi + (v \otimes v - S) : \frac{\partial \varphi}{\partial X} \right] \, dx \, dt + \int_{\Omega} v_0 \varphi(0) \, dx = \int_{\Gamma_T} \alpha(v \cdot \varphi) \, d\gamma \, dt
\]

for any function $\varphi \in H^1(0,T;V_p)$, $\varphi(\cdot,T) = 0$.

In what follows we use the approach of the theory of variational inequalities \([8, 10]\) to demonstrate the relation \((9)\), that ends the proof of Theorem 4.1. For a fixed natural $n$ we consider the convex potential

\[
V^n = V^n(X), \forall X \in \mathbb{R}^{3 \times 3},
\]

introduced in Lemma 3.11. By this Lemma we have

\[
S^n = \frac{\partial V^n}{\partial X}(B^n) \quad \text{with } B^n = \frac{\partial v^n}{\partial X}
\]

and

\[
V^n(X) - V^n(B^n) \geq S^n : (X - B^n) \quad \text{a.e. in } \Omega_T, \quad \forall X \in \mathbb{R}^{3 \times 3}.
\]
Let us denote \( \Omega_r = (0, r) \times \Omega \) for arbitrary \( r \in (0, T) \), the equality \([33]\) can be written as
\[
\frac{1}{2} \int_{\Omega} \left[ |v^n|^2(r) - |v^0_0|^2 \right] \, dx + \int_0^r \int_{\Gamma} \alpha |v^n|^2 \, d\gamma \, dt = - \int_{\Omega_r} \left[ S^n : \frac{\partial v^n}{\partial x} \right] \, dx \, dt.
\] (49)

If we substitute (49) in (48), then the lower semi-continuity property of convex functional with respect of weak convergence gives
\[
\int_{\Omega_r} V(X) - V(B) \, dx \, dt \geq \lim_{n \to \infty} \inf_{\Omega_r} \int_{\Omega_r} V^n(X) - V^n(B^n) \, dx \, dt
\]
\[
\geq \lim_{n \to \infty} \inf_{\Omega_r} \int_{\Omega_r} S^n : X \, dx \, dt + \frac{1}{2} \int_{\Omega} \left[ |v^n|^2(r) - |v^0_0|^2 \right] \, dx + \int_0^r \int_{\Gamma} \alpha |v^n|^2 \, d\gamma \, dt
\]
\[
\geq \int_{\Omega_r} S : X \, dx \, dt + \frac{1}{2} \int_{\Omega} \left[ |v|^2(r) - |v_0|^2 \right] \, dx + \int_0^r \int_{\Gamma} \alpha |v|^2 \, d\gamma \, dt
\]
by use the convergences (44). Hence for any matrix function \( X \in L^2(\Omega_T) \) we have the inequality
\[
\int_{\Omega_r} V(X) - V(B) \, dx \, dt \geq \int_{\Omega_r} S : (X - B) \, dx \, dt
\] (50)

Let us take \( \varphi = v(1 - sgn^\varepsilon_+(t - r)) \) in the equality \([24]\) for a fixed \( r \in (0, T) \), where
\[
sgn^\varepsilon_+(t) = \begin{cases} 
0, & \text{if } t < 0; \\
t/\varepsilon, & \text{if } 0 \leq t < \varepsilon; \\
1, & \text{if } \varepsilon \leq t.
\end{cases}
\]

In the obtained equality the limit transition on \( \varepsilon \to 0 \) implies
\[
\frac{1}{2} \int_{\Omega} \left[ |v|^2(r) - |v_0|^2 \right] \, dx + \int_0^r \int_{\Gamma} \alpha |v|^2 \, d\gamma \, dt = - \int_{\Omega_r} \left[ S : \frac{\partial v}{\partial x} \right] \, dx \, dt.
\] (51)

Substituting (51) in (50), we derive
\[
\int_{\Omega_r} V(X) - V(B) \, dx \, dt \geq \int_{\Omega_r} S : (X - B) \, dx \, dt \quad \text{with } B = \frac{\partial V}{\partial x}
\]
Since the matrix function \(X \in L^2(\Omega_T)\) is arbitrary, we can choose in this inequality \(X = B + \varepsilon Z\) for any positive \(\varepsilon\) and any matrix function \(Z \in L^2(\Omega_T)\), which gives

\[
\lim_{\varepsilon \to 0^+} \int_{\Omega_r} \frac{V(B + \varepsilon Z) - V(B)}{\varepsilon} \, dx \, dt \geq \int_{\Omega_r} S : Z \, dx \, dt.
\]

So, passing the limit over the sign of integration, we obtain

\[
\int_{\Omega_r} V'(B; Z) \, dx \, dt \geq \int_{\Omega_r} S : Z \, dx \, dt,
\]

for any matrix function \(Z \in L^2(\Omega_T)\). Now, since \(V'(B; Z)\) and \(S : Z\) are positively homogeneous with respect to \(Z\), this implies indeed in

\[
\int_{\Omega_r} V'(B; Z) \xi \, dx \, dt \geq \int_{\Omega_r} (S : Z) \xi \, dx \, dt,
\]

for any positive function \(\xi \in L^\infty(\Omega_T)\) and any matrix function \(Z \in L^2(\Omega_T)\). Therefore, \(V'(B; Z) \geq S : Z\), for any matrix function \(Z \in L^2(\Omega_T)\), and thus, \(S(x, t) \in \partial V(B(x, t))\) by Theorem 3.5 and \(S\) has the form (9) by Proposition 3.7. In particular, if \(B = 0\), we have \(|S(x, t)| \leq \tau_\ast\) by (a) of Proposition 3.7.

In the sequel we will show the uniqueness result. First let us show that the norms

\[
\|v\|_{V_p} = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^p(\Omega)}, \quad \|v\|_{W^1_p} = \|v\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)}
\]

are equivalent. It is enough to show the following result.

**Lemma 6.2.** There exists a constant \(C\), such that

\[
\|v\|_{W^1_p} \leq C\|v\|_{V_p}, \quad \forall v \in W^1_p(\Omega).
\]

**Proof.** Assume that the affirmation of Lemma is not true, then for any \(n \in \mathbb{N}\) there exists a vector function \(v_n \in W^1_p(\Omega)\), such that \(\|v_n\|_{W^1_p(\Omega)} > n\|v_n\|_{V_p}\). Let us define \(\tilde{v}_n = \frac{v_n}{\|v_n\|_{W^1_p(\Omega)}}\), that fulfills

\[
\|\tilde{v}_n\|_{W^1_p(\Omega)} = 1, \quad \|\tilde{v}_n\|_{V_p} < \frac{1}{n}
\]

Hence \(\{\tilde{v}_n\}_{n=1}^\infty\) is compact in \(L^p(\Omega)\), such that there exists a strongly convergent subsequence \(\tilde{v}_{n'}\), to some \(\tilde{v} \in L^p(\Omega)\). This subsequence \(\tilde{v}_{n'}\) is a strongly convergent to \(\tilde{v}\) in the space \(L^2(\Omega)\) by the inequality

\[
\|v\|_{L^2(\Omega)} \leq C\|v\|_{L^p(\Omega)}, \quad \forall v \in L^p(\Omega).
\]
Also from (53) we have that
\[
\lim_{n \to \infty} \| \nabla \tilde{v}_n \|_{L^p(\Omega)} = 0, \quad \lim_{n \to \infty} \| \tilde{v}_n \|_{L^p(\Omega)} = 1 \quad \text{and} \quad \lim_{n \to \infty} \| \tilde{v}_n \|_{L^2(\Omega)} = 0,
\]
that implies
\[
\| \tilde{v} \|_{L^p(\Omega)} = 1 \quad \text{and} \quad \| \tilde{v} \|_{L^2(\Omega)} = 0,
\]
which is impossible. Hence (52) is true.

Since for any \( \beta \in [0, 1] \), and any \( a, b \geq 0 \), we have \((a + b)\beta \leq a^\beta + b^\beta\), then Lemma 6.2 and the well known interpolation inequality \( \| v \|_{L^r(\Omega)} \leq C \| v \|_{L^2(\Omega)} \| \nabla v \|_{L^p(\Omega)} \), valid for any \( v \in W^1_p(\Omega) \) (see e.g. of [1, Lemma 2.2]), give the following interpolation result.

**Lemma 6.3.** There exists a positive constant \( C \), such that
\[
\| v \|_{L^r(\Omega)} \leq C (\| v \|_{L^2(\Omega)}^{1-\beta} \| \nabla v \|_{L^p(\Omega)}^{\beta} + \| v \|_{L^2(\Omega)}), \quad \forall v \in V_p, \tag{54}
\]
where \( \beta = \left( \frac{1}{2} - \frac{1}{r} \right) / \left( \frac{5}{6} - \frac{1}{p} \right) \).

Now we are able to prove the uniqueness result. Let us denote the difference of two functions \( f_1 \) and \( f_2 \) by \( \tilde{f} \), i.e. \( \tilde{f} = f_1 - f_2 \). Let us admit the existence of two different solutions \( v_1, v_2 \) with respective tensors \( S_1, S_2 \), satisfying the relation (9). By (24) the difference \( \nabla \) fulfils the equality
\[
\int_{\Omega_T} \left[ \nabla \partial_t \varphi + (\nabla \otimes \nabla - S) : \frac{\partial \varphi}{\partial x} \right] dx dt = \int_{\Gamma_T} \alpha (\nabla \cdot \varphi) d\gamma dt \tag{55}
\]
for any \( \varphi \in H^1(0, T; V_p) \), such that \( \varphi(\cdot, T) = 0 \) in \( \Omega \).

It is easily to check that
\[
\int_{\Omega} \left[ (\nabla \otimes \nabla) : \frac{\partial \nabla}{\partial x} \right] dx = - \int_{\Omega} \left[ (\nabla \otimes \nabla) : \frac{\partial \nabla}{\partial x} \right] dx.
\]

Also there exists a constant \( \tilde{C} > 0 \), depending only on \( \mu_1, \mu_2 \) and \( p \), such that
\[
C(||B_s||^p + ||B_a||^p) \leq \tilde{B}_\mu : \frac{\partial \nabla}{\partial x} \leq \tilde{S} : \frac{\partial \nabla}{\partial x},
\]
being a consequence of the monotonicity of the second term in the relation (9) and the inequality (2.5) of Lemma 1.19, shown in [12].

Let us fix an arbitrary \( r \in (0, T) \) and take \( \varphi = \nabla(1 - \text{sgn}_\varepsilon(t - r)) \) in (55). Then the limit transition on \( \varepsilon \to 0 \) in the obtained equality and the
Hölder inequality gives that

\[
\frac{1}{2} \int_\Omega |v|^2(r) \, dx + \tilde{C} \int_{\Omega_r} |\nabla v|^p \, dx \, dt \\
\leq \frac{1}{2} \int_\Omega |v|^2(r) \, dx + \int_{\Omega_r} \left[ S \cdot \frac{\partial v}{\partial x} \right] \, dx \, dt + \int_{\Gamma_T} \alpha |\nabla|^2 \, d\gamma \, dt
\]
\[
= \int_{\Omega_r} (\nabla \otimes \nabla) : \frac{\partial v}{\partial x} \, dx \, dt. \tag{56}
\]

In the sequel we follow the ideas presented in Theorem 3.2 of [23] and Theorem 4.29 of [12]. By Hölder’s inequality and Lemma 6.4, the right hand side of (56) is estimated as

\[
\left| \int_{\Omega} (\nabla \otimes \nabla) : \frac{\partial v}{\partial x} \, dx \right| \leq C \||\nabla v_2||_{L^p(\Omega)} |\nabla|^2 \frac{2p}{2p-2} (\Omega)
\]
\[
\leq C \||\nabla v_2||_{L^p(\Omega)} \left( \frac{5p-6}{(5p-6)^\frac{3}{3}} \right) \left( \frac{3}{3} \right) |\nabla|^2 \frac{6}{(5p-6)} \left( \frac{5p-6}{2p} \right) + |\nabla|^2 \frac{2}{L^2(\Omega)}
\]
\[
\leq \varepsilon \left| \frac{\partial v}{\partial x} \right|^p_{L^p(\Omega)} + C\varepsilon \frac{\left( 5p-6 \right)}{\left( 5p-6 \right)^\frac{3}{3}} |\nabla|^2 \frac{6}{(5p-6)} |\nabla|^2 \frac{2}{L^2(\Omega)}
\]
\[
+C \||\nabla v_2||_{L^p(\Omega)} |\nabla|^2 \frac{2}{L^2(\Omega)}, \tag{57}
\]

where the \( \varepsilon \)-version of Young’s inequality has been used in the last inequality. Hence taking \( \varepsilon = \tilde{C} \), we obtain

\[
z(t) \leq C + C \int_0^t f(s) z(s) \, ds \quad \text{with} \quad z(t) = \int_\Omega |\nabla|^2 \, dx, \quad f(s) = \||\nabla v_2||_{L^p(\Omega)} \frac{2p-6}{5p-6}.
\]

Therefore, if \( p \geq \frac{5p-6}{5p-6} \), that is \( p \geq \frac{7+\sqrt{19}}{5} \approx 2.272 \), then applying the Gronwall inequality we obtain \( z(t) = 0 \) a.e. in \((0, T)\) and we deduce the global-in-time uniqueness result. \( \blacksquare \)

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