The dynamical analysis of the modified rossler system

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Abstract. The synthesis of novel chaotic systems is a modern branch of nonlinear dynamics since deterministic chaos properties can be successfully applied in various engineering and scientific problems. In this paper we investigate changes in the dynamics of the modified Rossler system after applying coordinates transformation to the original model. We perform the bifurcation analysis of the obtained model and experimentally show that its behavior differs with the behavior of the prototype. We study the finite-difference schemes obtained for considered chaotic systems and find their similarity in simulation with different integration steps. We show that discretization effects are the source of the small differences between the two models. The obtained results can be used in theoretical nonlinear dynamics, nonlinear systems simulation, development of communication and control systems.

1. Introduction
Chaotic dynamics can be observed in electrical circuits [1–4], liquid or gas convection [5], motion trajectories of celestial bodies [6], a human heartbeat [7] and many others. Identification and improvement of mathematical models of such processes is an important task. On the other hand, sensitivity to initial values and topological mixing allow to efficiently apply chaotic systems in control, security, and communication systems [8–10]. Thus, the synthesis of chaotic systems with the required properties is an actual topic of nonlinear science. In [11] methods for obtaining symmetric discrete maps with an adaptive phase space were proposed. Furthermore, methods of controlling the number of equilibria and a Lyapunov exponent value were considered in [12–14]. Recently, Nepomuceno et al. proposed an easily implementable digital model with chaotic behavior [15]. Such studies usually thoroughly describe the technique of synthesizing new mathematical objects from a well-studied system leaving in the background the questions of the developed models adequacy and the influence of a discrete operator. Nevertheless, it was repeatedly shown that even the order of performing mathematical operations can significantly affect its dynamics while a chaotic system is simulated [16].

In our study we experimentally compare the behavior of the well-known chaotic Rossler system with its modified version obtained using an affine coordinates transformation [12]. We expect, that the applied synthesis method will not introduce significant changes in the system dynamics excluding the discretization effects impact.

The paper is organized as follows. In Section II we describe a mathematical model of the modified Rossler system. Bifurcation analysis and the Largest Lyapunov exponent calculation are carried out in Section III. Section IV shows experimental study of numerical integration methods applied to considered systems. Finally, some conclusions are given in Section VI.

2. The double-Rossler chaotic system
In [12] X. Wang and G. Chen proposed the technique for constructing system with any number of equilibria. The proposed method is based on a coordinate transformation that allows to obtain the phase space symmetry relative to a chosen axis. The number of equilibrium points is controlled by the rotation angle. Let us consider well-known Rossler chaotic system [17]
where \(a, b, c\) are nonlinearity parameters. The Rossler system has one saddle-focus equilibrium point with 2D unstable manifold and another one saddle-focus equilibrium point with 1D unstable manifold (Fig. 1).

Rewrite system (1) in terms of \(u, v\) and \(w\) as follows:

\[
\begin{align*}
\dot{u} &= -u - w; \\
\dot{v} &= u + av; \\
\dot{z} &= b + w(u - c).
\end{align*}
\]

(2)

Consider the following simple coordinate transformation:

\[
\begin{align*}
\dot{u} &= x^2 - y^2; \\
\dot{v} &= 2xy; \\
\dot{w} &= z.
\end{align*}
\]

(3)

Transformation (3) can add a \(z\)-axis rotation symmetry \(R_z(\pi)\) to the original system (1) because for each \((u, v, w)\) there are two points \((\pm x, \pm y, \pm z)\) corresponding to \((u, v, w)\).

After the transformation (3), the system becomes

\[
\begin{align*}
\dot{x} &= -x^3y - zx - y^3 + 2ax^2y; \\
\dot{y} &= xy^2 + zy + x^3 + 2ax^2y; \\
\dot{z} &= b + z(x^2 - y^2 - c);
\end{align*}
\]

(4)
We call the resulting system as the *double-Rossler system* since it possesses two pairs of equilibrium points symmetrical by the $z$-axis (Fig. 2). As in the original model, one equilibrium point is located in the middle of the attractor and is a saddle-focus with an unstable 2D manifold. Second equilibria is outside of the region of the attractor and is saddle-focus equilibrium point with 1D unstable manifold.

Let us consider the results of the dynamical analysis of the obtained model in comparison with the original Rossler system.

3. **Dynamical analysis**

3.1. **Bifurcation analysis**

Bifurcation analysis of the chaotic Rossler system usually is carried out with fixed values $a = 0.2$ and $b = 0.2$ while $c$ is varied. Within $2.6 \leq c \leq 4.2$, the system have a stable limit cycle. In this interval there is a period doubling. Then, after $c = 4.2$ chaotic behavior can be observed.

In this study to plot the bifurcation diagrams we simulate the considered systems on the time interval $t = 700$ sec using the semi-explicit numerical integration $SED$-method [18] with integration step $h = 0.01$. We use variable $z$ to construct the Poincaré sections. The bifurcation diagrams for both considered systems are shown in Fig. 3.

One can see that the behavior of the double-Rossler model is similar to the original system while increasing parameter $c$. The most significant difference can be noticed in the so-called windows of bifurcation diagrams, corresponding to the intervals $[5.3; 5.6]$, $[6.7; 6.9]$ and $[7.8; 8.2]$. This mismatch is also confirmed by the values of the Largest Lyapunov exponent (LLE). In the original model LLE tends to a negative value near $c = 8$ which corresponds to the harmonic mode of oscillation. In the double-Rossler system in this interval a chaotic mode is observed. It can be assumed that a variation of models behavior is related to the appearance of the effects from the discretization method. Let us consider methods for obtaining finite difference schemes of the investigated chaotic models and compare them.

![Figure 3. Bifurcation diagrams and LLE for Rossler and double-Rossler systems](image)

3.2. **Analysis of finite difference schemes**

The study of discretization methods is performed using so-called step bifurcation diagrams. If we choose the integration step as the nonlinearity parameter, the diagram is called a step-diagram or $h$-diagram. We consider the common techniques of integration ordinary differential equations including explicit and implicit midpoint methods and the semi-implicit $SED$-method. The experimental results are shown in Fig. 4–6.
The simulation of investigated systems using the explicit midpoint method shows the differences between two considered models. For the double-Rossler system chaotic modes appear in the interval [0.1; 0.15] in contrast with the original system. In diagrams obtained for the implicit midpoint method and the SED-method the windows of bifurcation diagrams are shifted towards an integration step increasing. Thus, one can conclude that the possible source of the different behavior of the Rossler models is the discretization method. Furthermore, the data type can significantly affect the discrete model due rounding the result of a calculation. Since the equations after the coordinate transformation are more complex than the original Rossler system, then it is worth using high precision floating-point data types and high-order integration methods to achieve the required model adequacy.

Figure 4. $h$-diagrams of explicit midpoint method for Rossler and double-Rossler systems

Figure 5. $h$-diagrams of implicit midpoint method for Rossler and double-Rossler systems

Figure 6. $h$-diagrams of semi-implicit CD-method for Rossler and double-Rossler systems
4. Conclusions
In this paper we considered the modified Rossler system obtained using coordinates transformation of the original model. We show that using the same integration method for discretization of differential equations yields similar, but still different results. The additional investigation was carried out to reveal the reasons of such behavior. Using the step-diagrams, we explicitly show that the main source of misbehavior is the applied discrete operators. Thus, one should be extremely careful when constructing novel chaotic systems using coordinate transformation approach.

The topic of our further studies will be the investigation of various models of chaotic systems obtained through coordinate transformations and the detailed study of numerical method impact. The most interesting case to investigate are conservative chaotic systems that are significantly different from dissipative counterparts. We will study the various oscillations modes arising in such systems and the property of preserving the phase volume during long-term simulations.

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