Conserved currents of the three-reggeon interaction.

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Abstract

We consider an extension of Lipatov’s conjecture about the deep relation between amplitudes in the high-energy limit of QCD and XXX Heisenberg chains with non-compact spins.

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1 Introduction

In Regge theory the high-energy asymptotics of the hadron-hadron scattering amplitudes is determined by singularities of partial waves in the complex angular momentum plane. It was observed [1] that the regularity of quantum mechanics, which relates high-energy scattering amplitudes to the singularities of the partial waves in the complex angular momentum plane is valid for quantum field theory as well. Namely, in Regge kinematics

\[ s \gg -t \sim M^2 \tag{1} \]

where \( M \) is a characteristic hadronic mass scale, the hadron-hadronic scattering amplitude \( A(s,t) \)

\[ A(s,t) = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dJ}{2\pi i} \left( \frac{s}{M^2} \right)^J f(J,t) \tag{2} \]

is governed by singularities of the partial waves \( f(J,t) \), by Regge poles and Regge cuts. Among the Regge singularities there is one with vacuum quantum numbers, the so-called Pomeron, which provides the dominant contribution to the scattering amplitude. The bootstrap conjecture has been proposed, according to which all particle-like excitations correspond to some Regge singularity and are related to each other via unitarity of the S-matrix and sum rules.

However the programme of building the axiomatic quantum field theory from assumptions of only unitarity and analyticity of the S-matrix failed, because Regge theory itself does not allow to calculate the positions of these singularities. Now, QCD as a theory of strong interaction is called to describe the Regge behavior of the scattering amplitudes [2]. V.N. Gribov proposed the conjecture, that reggeons form new collective excitations and QCD in the high-energy limit can be replaced by an effective reggeon theory [3]. This has been confirmed in series of works, initiated by L.N.Lipatov [4]. It was shown that in leading logarithmic approximation (LLA), which is the natural approximation in the Regge limit of QCD, \( A(s,t) \) can be expressed as a sum of Feynman diagrams describing the multiple exchange of reggeized gluons in the \( t \)-channel. The perturbative expressions for corresponding Feynman diagrams including large logarithmic factors \( \alpha^s \log^n s \) \((m=n,n-1,...)\) have to be resummed to all orders in \( \alpha_s \), because bare gluons and quarks are not a good approximation in the Regge limit. The leading contribution \((m=n)\) comes from ladder diagrams, corresponding to exchange of \( n \)-reggeons in \( t \)-channel. Being built from an infinite number of perturbative gluons, the reggeons carry the quantum numbers of the gluon and become a new collective excitation in the Regge limit \((1.1)\). It is well known that the leading logarithmic approximation results in an asymptotics, which violates the Froissart bound. Unitarity is restored by taking into account sub-leading contributions as well. In the generalised leading logarithmic approximation (GLLA) some minimal set of non-leading terms is included to restore unitarity [5]. The interaction of the reggeons is determined by LLA.

In LLA the dominant contribution to the partonic scattering amplitude comes from the soft gluons and this leads to the gluon reggezation property. It can be shown that infrared divergences are cancelled for colourless external states due gauge invariance.

Of course the distribution of partons inside the hadron are described by the nonperturbative wave function of the hadron. The non-perturbative effects can be taken into account in the approach of constructing the high-energy effective action [5]. However, the perturbative investigation of high-energy QCD makes sense as a first approximation to begin with. In particular results obtained in the BFKL pomeron approximation are in a good agreement with experimental data of semi-hard processes and especially of deep-inelastic scattering at small \( x \). Then, some information about non-perturbative corrections can be extracted from analyzing the behaviour of the perturbative series in the infrared region [6].
2 High-energy QCD as an integrable model

As mentioned above, the high-energy asymptotics of the scattering amplitude in leading logarithmic approximation is determined by the contribution of diagrams, describing two reggeon exchange in the $t$-channel. It is a result of summing up an infinite number of the famous ladder diagrams \[4\] and corresponds to the exchange of the Pomeron. Contributions of diagrams with three (Odderon) and more reggeized gluons can be considered as higher corrections. Pomeron contribution corresponds to elasical scattering. Diagrams with Odderon exchange describe processes with the exchange of negative charge parity.

The dominating contribution in LLA comes from the multi-Regge kinematics:

\[ s = (p_A + p_B)^2 \approx 2p_A \cdot p_B \]
\[ s_i = (k_i + k_{i+1})^2 \approx 2k_i \cdot k_{i-1} \]
\[ i = 1, \ldots, n+1, \quad k_0 = p_A, \quad k_i = q_{i+1} - q_i, \quad k_{n+1} = p_B \]
\[ s \gg s_i \gg |q_i|^2 \]
\[ s_1 s_2 \cdots s_{n+1} = s \prod_{i=1}^{n} (-k_{\perp i}^2) \]
\[ k_1 \cdot p_A \ll k_2 \cdot p_A \ll \cdots \ll k_n \cdot p_A \]
\[ k_1 \cdot p_B \gg k_2 \cdot p_B \gg \cdots \gg k_n \cdot p_B \]

where $k_{\perp}$ is defined by the Sudakov decomposition

\[
k^\mu = \frac{k \cdot p_A}{p_A \cdot p_B} p_B^\mu + \frac{k \cdot p_B}{p_A \cdot p_B} p_A^\mu + k^\mu
\]

Owing to this decomposition the scattering amplitudes in the Regge limit exhibit the remarkable separation of the longitudinal and transverse directions with respect to the plane, spanned by the momenta of the initial particles.

In the generalized leading logarithmic approximation the interaction between reggeons is elastic and pairwise. We shall restrict ourselves to the case where the number of reggeons in the $t$-channel $N$ is conserved. For a given $N$ the reggeon Green function $f_{\{i_k\}}$ satisfies the Bethe-Salpeter-like equation \[7\]

\[ \omega f_{\{i_k\}} = f_{\{i_k\}}^{(0)} + \sum_{i<j} \mathcal{H}_{\{i_k\},\{j_k\}} f_{\{j_k\}} \]

The set $\{i_k\} = (i_1, \ldots, i_r)$ labels the reggeons, $i = G$ stands for gluon and $i = F$ or $i = \bar{F}$ stand for reggeized quarks of corresponding helicity. The partial wave $f$ carries also an index $\alpha_i$, labelling the gauge group representation of the corresponding reggeons and depends in their transverse momentum $k_{\perp i}$ or their impact parameter $x_i$.

The $r$-reggeon contribution to the partial wave is obtained by contracting the $r$-reggeon Green function with the parton distribution functions of the scattered particles. The angular momentum is

\[ J = 1 + \omega - \frac{r_f}{2} \]

where $r_f$ is the number of exchanged fermions. The pairwise interaction of the reggeons is described by the Hamiltonian

\[
\mathcal{H}_{\{i_1, \ldots, i_r\}} f_{\{i_1, \ldots, i_r\}} = \frac{g^2}{(2\pi)^3} \int dk'_i dk'_j \delta(k_i + k_j - k'_j - k'_i) [(T_i \otimes T_j) \mathcal{H}_{i_1, \ldots, i_r} f_{i_1, \ldots, i_r}] + \]

\[(T_i \otimes T_j) \mathcal{G}_{i_1, \ldots, i_r} f_{i_1, \ldots, i_r}\]
The first term in the square bracket corresponds to the interaction via an $s$-channel gluon:

$$
(T_i \otimes T_j) \mathcal{H}_{(\alpha_k \cdot \alpha_k)} = \prod_{k \neq i, j} \delta_{\alpha_k, \alpha_k} (T^{\alpha})_{\alpha_i \alpha_j} (T^{\alpha})_{\alpha_j \alpha_i}
$$

(7)

and the second one corresponds to the interaction via an $s$-channel fermion

$$
(T_i \otimes T_j) \mathcal{G}_{(\alpha_k \cdot \alpha_k)} = \prod_{k \neq i, j} \delta_{\alpha_k, \alpha_k} (T^{\alpha^{\prime}})_{\alpha_i \alpha_j} (T^{\alpha^{\prime}})_{\alpha_j \alpha_i}
$$

(8)

with the sum over the fermion colour states $\alpha$. The overall group state in the $t$-channel has to be the gauge singlet.

While the longitudinal part of the scattering amplitude is extracted as a kinematical factor, the transverse dynamical part can be described by simple Feynmann rules, corresponding to the multi-regge effective action [9]. These graphical rules allow the simple derivation of the interaction kernels, entering the equation (6). As operators in impact space these kernels take the following form

$$
\mathcal{H}_{GG} = H_G + H^*_G
$$

(9)

$$
\mathcal{H}_{FF}^{\omega} = H_F^{\omega} + P_{12} H_F^{(\omega)} P_{12}
$$

$$
\mathcal{H}_{FG} = H_G + \mathcal{P}_{12} H_F^{*} \mathcal{P}_{12}
$$

$$
\mathcal{G}_{FG} = (x_{12}^* \partial_{12}^*)^{-1}
$$

where

$$
H_G = -2 \psi(1) + \partial_1^{-1} \log x_{12} \partial_1 + \partial_2^{-1} \log x_{12} \partial_2 + \log \partial_1 \partial_2
$$

(10)

$$
H_F^{\omega} = -2 \psi(1) + \partial_1^{-1+\omega/2} \log x_{12} \partial_1^{-1-\omega/2} + \partial_2^{-\omega/2} \log x_{12} \partial_2^{\omega/2} + \log \partial_1 \partial_2
$$

and $\psi(1) = -\gamma$ is the Euler number. $\mathcal{P}_{12}$ represents the operator permuting the reggeons 1 and 2.

L.N.Lipatov [8] applied the quantum inverse scattering method to solve equations for wave functions of compound states of $n$ reggeized gluons. The eigenvalue problem related to the operators [10] arises, because the position of the singularities in $\omega$ of the $t$-channel partial wave $f_{1k}$ is determined by their eigenvalues.

Lipatov proposed to diagonalize the $N$ reggeon problem by establishing the correspondence between the operators [10] and the XXX Heisenberg model. He has noticed that the operator $H_G$ has two equivalent representations:

$$
H_G = \partial_1^{-1} \log x_{12} \partial_1 + \partial_2^{-1} \log x_{12} \partial_2 + \log \partial_1 \partial_2 - 2 \psi(1) = x_{12} \log \partial_1 \partial_2 x_{12}^{-1} + 2 \log x_{12} - 2 \psi(1)
$$

(11)

and therefore the transposed operator can be represented in two ways as follows:

$$
(H_G)^T = \partial_1 \log x_{12} \partial_1^{-1} + \partial_2 \log x_{12} \partial_2^{-1} + \log \partial_1 \partial_2 - 2 \psi(1) = \partial_1 \partial_2 H_G (\partial_1 \partial_2)^{-1}.
$$

(12)

On the other hand we have

$$
(H_G)^T = x_{12} \log \partial_1 \partial_2 x_{12}^{-1} + 2 \log x_{12} - 2 \psi(1) = x_{12}^{-1} H_G x_{12}^2
$$

(13)

Then we can deduce

$$
[H_G; C^{00}] = 0, \quad C^{00} = x_{12}^2 \partial_1 \partial_2
$$

(14)
This equation expresses the fact of conformal invariance of \(H_G\). Indeed, this operator coincides with Casimir operator of \(SL(2)\) of zero conformal weights \(C^{00}\). Then it is reasonable to denote \(H_G\) as \(H^{00}\).

The hamiltonian

\[
H_{123} = H_{12}^{00} + H_{23}^{00} + H_{31}^{00} = \partial_1^{-1} \log x_{31} x_{12} \partial_1 + \partial_2^{-1} \log x_{12} x_{23} \partial_2 + \partial_3^{-1} \log x_{23} x_{31} \partial_3 + 2 \log \partial_1 \partial_2 \partial_3 - 6 \psi(1) = x_{12} \log \partial_1 x_{12}^{-1} + x_{23} \log \partial_2 x_{23}^{-1} + x_{31} \log \partial_3 x_{31}^{-1} + 2 \log x_{12} x_{23} x_{31} + 6 \psi(1)
\]
corresponds to the exchange of three reggeized gluons (Fig.1).

The transposition gives

\[
(H_{123})^T = \partial_1 \partial_2 \partial_3 H_{123} (\partial_1 \partial_2 \partial_3)^{-1} = (x_{12} x_{23} x_{31})^{-1} H_{123} x_{12} x_{23} x_{31},
\]
i.e. it commutes with the operator \(A_3\):

\[
[H_G; A_3] = 0, \quad A_3 = x_{12} x_{23} x_{31} \partial_1 \partial_2 \partial_3
\]

Notice, that operator \(A_3\) is the commutator of partial Casimir operators of the chain links:

\[
A_3 = [C_{00}^{00}, C_{31}^{00}] = [C_{23}^{00}, C_{12}^{00}] = [C_{31}^{00}, C_{23}^{00}]
\]

Therefore eq. (17) is a consequence of (18) and the Jacobi identity. It is easy to check, that the \(N\)-reggeon hamiltonian

\[
H_N = \sum_{i=1}^{N} H_{i,i+1}, \quad H_{N,N+1} \equiv H_{N,1}
\]
commutes with

\[
A_N = \prod_{i=1}^{N-1} x_{i,i+1} (\prod_{j=1}^{N} \partial_j)
\]

It can be checked also, that \(H_N\) commutes with \(C_N\),

\[
C_N = \sum_{1 \leq i < j \leq N} x_{ij}^2 \partial_i \partial_j.
\]

The eigenvalue problem:

\[
\tilde{A}\psi_\Delta(x_1, x_2, x_3) = \tilde{a}\psi_\Delta(x_1, x_2, x_3), \quad \tilde{C}\psi_\Delta(x_1, x_2, x_3) = \Delta(1 - \Delta)\psi_\Delta(x_1, x_2, x_3),
\]
can be considered instead of the original eigenvalue problem:

\[
\tilde{H}_{123}\psi_\Delta(x_1, x_2, x_3) = \tilde{E}_{123}\psi_\Delta(x_1, x_2, x_3),
\]
which looks much more complicated.

The equation (21) expresses the conformal invariance of the \(N\) reggeon system and suggests the connection with the isotropic \(sl(2)\) Heisenberg model with \(N\) cites and cyclic boundary conditions, because \(C_N\) can be identified with Casimir operator of this symmetry algebra.
Lipatov has related to each site of the Heisenberg chain the following Lax operator

\[ L_i = \begin{pmatrix} \lambda + S_i^0 & S_i^- \\ -S_i^+ & \lambda - S_i^0 \end{pmatrix} \]  

(24)

where the \( SL(2) \) spin operators \( S_i^a \)

\[ [S_i^0, S_j^\pm] = \pm \delta_{ij} S_i^\pm, \quad [S_i^+, S_j^-] = -2\delta_{ij} S_i^0 \]  

(25)

are represented as differential operators

\[ S_i^+ = x_i^2 \partial_i + 2\Delta_i x_i, \quad S_i^0 = x_i \partial_i + \Delta_i, \quad S_i^- = \partial_i \]  

(26)

and \( \lambda \) is the spectral parameter. The Lax operator (21) satisfies the Yang-Baxter equation. Therefore the trace of the monodromy matrix

\[ T(\lambda) = \prod_{i=1}^{N} L_i(\lambda) \]  

(27)

is the generating function for the set of \( N \) mutually commuting differential operators \( Q_k \):

\[ Q_k = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \partial_{i_1} \partial_{i_2} \ldots \partial_{i_k} \]  

(28)

Now it is plausible enough that all operators \( Q_k \) commute with the Hamiltonian and the latter can be represented as a function of \( Q_k \). Explicit calculations for low \( N \) cases and additional mathematical arguments [9] confirm this. Moreover, the relation between the high-energy QCD kernels and the \( XXX \) Heisenberg spin chains turns out to be much deeper. Namely, this connection can be extended to the case where fermions are incorporated. The ideas of conformal symmetry in Regge asymptotics [10] have been developed in application to the fermion exchange in [11].

### 3 Review of integrable chains

Let us review the main facts of the theory of integrable systems, concerning the \( XXX \) Heisenberg magnet. In our review we shall follow Sklyanin’s work [12]. The phenomenon of the integrability of quantum systems can be understood by means of their relation to the linear ones via separation of variables. Namely, the quantum system is integrable, if its non-linear equations of motion can be represented as the zero-curvature conditions of some integrable linear system [13]. Physically this means, that the interaction of such systems reduces to the elastic scattering and the only result of it consists in the exchange of quantum numbers (momenta etc.) of the scattered particles. Accordingly, the \( S \)-matrix of the theory is factorized into the product of blocks, corresponding to \( 2 \rightarrow 2 \) scattering and also \( 1 \rightarrow 1 \) in the presence of a boundary [14].

A set of (annihilation) operators \( Z_{a}(\lambda) \) satisfying the Zamolodchikov algebra: has been proposed for an algebraic description of the factorizable scattering.

\[ Z_a(\lambda) Z_b(\mu) = S_{ab,cd}(\lambda - \mu) Z_d(\mu) Z_c(\lambda) \]  

(29)

where \( S \) is \( n^2 \times n^2 \) matrix. The consistency condition of this system, which follows from the associativity property of the triple product \( Z_{a_1}(\lambda_1) Z_{a_2}(\lambda_2) Z_{a_3}(\lambda_3) \), that is the Yang-Baxter equation for the \( S \)-matrix:

\[ S_{a_j a_k}(\lambda_j - \lambda_k) S_{a_j a_l}(\lambda_j - \lambda_l) S_{a_k a_l}(\lambda_k - \lambda_l) = S_{a_k a_l}(\lambda_k - \lambda_l) S_{a_j a_l}(\lambda_j - \lambda_l) S_{a_j a_k}(\lambda_j - \lambda_k). \]  

(30)
Extending this algebra by adding n conjugated (creation) operators $Z^\dagger_n(\mu)$, one gets the Zamolodchikov - Faddeev algebra:

$$Z_a(\lambda)Z^\dagger_b(\mu) = \delta_{ab}\delta(\lambda - \mu) + Z^\dagger_c(\mu)\hat{S}_{ac, bd}(\lambda - \mu)Z_d(\lambda), \quad (31)$$

or in matrix notations:

$$A(\lambda) \otimes A(\mu) \equiv A_1(\lambda)A_2(\mu) = S_{12}(\lambda - \mu)A_2(\mu)A_1(\lambda), \quad (32)$$

where $A(\lambda)$ and $A^\dagger(\mu)$ are the column $(Z_1(\lambda), ..., Z_n(\lambda))^t$ and the row $(Z^\dagger_1(\mu), ..., Z^\dagger_n(\mu))$ correspondingly, subscripts refer to the corresponding isotopic spaces $\mathbb{C}^n \otimes \mathbb{C}^m \equiv V_1V_2$ and $S_{21} = \mathcal{P}S_{12}\mathcal{P}$, $\hat{S}_{12} = \mathcal{P}S_{12}$, $\mathcal{P}$ is the permutation operator in $\mathbb{C}^m \otimes \mathbb{C}^n$. The complete scattering matrix $S(\{\lambda_k\})$ of the M particle is factorized then into the ordered product of $M(M - 1)/2$ two-particle $S$-matrices [24]. For example the $S$-matrix of the $j$-th particle on the other $M - 1$ particles is given by $t(\lambda_j; \{\lambda_m\})$, i.e. the particular value of the transfer matrix for $\lambda = \lambda_j$:

$$t(\lambda; \{\lambda_m\}) = tr_aT(\lambda; \{\lambda_m\}) = tr_a\prod_k S_{ak}(\lambda - \lambda_k), \quad (33)$$

The trace in this expression is taken over the auxiliary space $V_a$, while the transfer matrix acts in the quantum space $\otimes_{k=1}^M V_k$. In the framework of the quantum inverse scattering method (QISM) [16, 12] instead of the original non-linear problem the auxiliary linear one is considered:

$$\frac{d}{dx}T(\lambda, x) = L(\lambda, x)T(\lambda, x) \quad (34)$$

or

$$T(n + 1, \lambda) = L_{n+1}(\lambda)T(n, \lambda)$$

in discrete case.

This is the Lax operator of the QISM [17]. The solution of (34):

$$T(\lambda, x) = P\exp(\int^x L(\lambda, y)dy), \quad (35)$$

$$T(n, \lambda) = L_n(\lambda)L_{n-1}(\lambda)...L_1(\lambda)$$

in discrete case, defines the monodromy matrix $T(\lambda)$.

Its entries are the new variables (the quantum scattering data), which commutation relations are defined by

$$\sum_{j_1, j_2=1}^n R_{i_1i_2, j_1j_2}(\lambda - \mu)T_{j_1k_1}(\lambda)T_{j_2k_2}(\mu) = \sum_{j_1, j_2=1}^n T_{i_2j_2}(\mu)T_{i_1j_1}(\lambda)R_{j_1j_2, k_1k_2}(\lambda - \mu). \quad (36)$$

We see that integrable systems are specified by the $R$-matrix, which acts on $\mathbb{C}^m \otimes \mathbb{C}^n$ and satisfies the Yang-Baxter equation

$$\sum_{j_1, j_2, j_3=1}^n R_{i_1i_2, j_1j_2}(\lambda)R_{j_1j_3, k_1k_3}(\lambda + \mu)R_{j_2j_3, k_2k_3}(\mu) = \sum_{j_1, j_2, j_3=1}^n R_{i_2i_3, j_2j_3}(\mu)R_{i_1j_3, k_1k_3}(\lambda + \mu)R_{j_1j_2, k_1k_2}(\lambda). \quad (37)$$
In general the $R$-matrix depends on the spectral parameter $\lambda$ and other parameters. Although there is no complete mathematical theory of the Yang-Baxter equation, a variety of solutions are known as well as different fields of their application. They are classified by the Lie algebra, its irreducible representations, and the spectral parameter dependence: rational, trigonometric and elliptic ones \textsuperscript{13}. Given a solution $R(\lambda)$ one can define the quadratic algebra $T_R$ of $n \times n$ matrix elements $T_{ij}$, which is generated by eq. \textsuperscript{33}. The associative algebra $T_R$ realizes the representation space of a quantum integrable system. The commutative integrals of motion are $t(\lambda) = \text{tr} T(\lambda)$, which follows from \textsuperscript{13}, taking the trace of $T_1 T_2 = R_{12}^{-1} T_2 T_1 R_{12}$. The algebra $T_R$ possesses the commutative property: if $T_1(\lambda)$ and $T_2(\lambda)$ are two representations of $T_R$ in the quantum spaces $V_1$ and $V_2$, then the matrix

$$T_{ik}(\lambda) = T_{1,ik}(\lambda)T_{2,jk}(\lambda)$$

is a representation of $T_R$ in the tensor product $V_1 \otimes V_2$. This property allows to represent $T(\lambda)$ as a product of elementary representations, the so called Lax operators. It follows from $SL(2)$ symmetry of the $R$-matrix that an arbitrary constant $d \times d$ matrix $K$ provides the simplest representation of the algebra $T_R$. This algebra has a central element, the quantum determinant of $T(\lambda)$:

$$\Delta(\lambda) \equiv \det_q T(\lambda) = D(\lambda + \eta/2)A(\lambda - \eta/2) - B(\lambda - \eta/2)C(\lambda + \eta/2) =$$

$$A(\lambda - \eta/2)D(\lambda + \eta/2) - C(\lambda - \eta/2)C(\lambda + \eta/2) =$$

$$A(\lambda + \eta/2)D(\lambda - \eta/2) - B(\lambda + \eta/2)C(\lambda - \eta/2) =$$

$$D(\lambda + \eta/2)A(\lambda - \eta/2) - C(\lambda + \eta/2)B(\lambda - \eta/2) =$$

which has the following remarkable properties

$$\det_q T_1(\lambda) T_2(\lambda) = \det_q T_1(\lambda) \det_q T_2(\lambda)$$

and

$$\det_q K = \det K.$$  

The next representation is given by Lax operator, mentioned above, which takes the especially simple form for the XXX spin chain:

$$L(\lambda) = \lambda + \eta \sum_{\alpha=1}^{3} S_{\alpha} \sigma_{\alpha} = \begin{pmatrix} \lambda + \eta S^0 & \eta S^- \\ -\eta S^+ & \lambda - \eta S^0 \end{pmatrix}$$

where operators $S_{\alpha}$ belonging to some irreducible representation of $sl(2)$ have commutation relation \textsuperscript{22}. Note that the $R$-matrix itself can be choosen as a Lax operator, if the auxiliary space is two-dimensional. We have

$$\det_q L(\lambda) = \lambda^2 - \eta^2(C + 1/4), \quad C = (S^0)^2 - \frac{1}{2}(S^+ S^- + S^- S^+).$$

Since the $L(\lambda)$-operator, being the elementary representation of $T_R$, satisfies the Yang-Baxter relation and the $R$-matrix depends only on the difference of the spectral parameters, the shift $L(\lambda) \rightarrow L(\lambda - \omega)$ defines an automorphism in $T_R$:

$$R(\lambda - \omega_1 + S_1^0 \sigma_{\alpha}) (\lambda - \omega_2 + S_2^0 \sigma_{\beta}) (\lambda - \omega_1 + S_1^0 \sigma_{\alpha}) R =$$

Separating the terms, linear in $\lambda$ in this equation one deduces that the $R$-matrix is $SL(2)$-invariant

$$[R; S_1^0 + S_2^0] = 0$$
and depends only on difference $\omega_{12} = \omega_1 - \omega_2$. The $SL(2)$-invariance implies that the $R$-matrix has to have the form

$$R = \sum \rho_j(\omega_{12}) P_j,$$

where $P_j$ are the projectors corresponding to the decomposition of the tensor product of two initial representations into the sum of irreducible representations labelled by spin $j$.

Furthermore the part of eq. (44), which contains no $\lambda$ gives for $\rho_j(\omega_{12})$ the recurrence relation:

$$\rho_{j+1}(\omega_{12}) = \frac{\omega_{12} + \eta(j + 1)}{\omega_{12} - \eta(j + 1)} \rho_j(\omega_{12}),$$

which determines $R$ up to a scalar factor.

Particular solutions of the Y-B equation have properties, which are important for different applications, but which are not necessarily valid for a given solution:

- regularity
  $$R(0) = \rho(0)^{1/2} P_{12}$$

- P-symmetry
  $$\mathcal{P} R_{12}(\lambda) \mathcal{P} \equiv R_{21}(\lambda) = R_{12}(\lambda)$$

- T-symmetry
  $$R_{12}^{t}(\lambda) = R_{12}(\lambda)$$

- unitarity
  $$R_{12}(\lambda) = R_{21}(-\lambda) = \rho(\lambda) I$$

- crossing symmetry
  $$R_{12}(\lambda) = V(1) R_{12}^t(-\lambda - \eta)V^{-1}(1)$$

- quasiclassical property
  $$R(\lambda, \eta) = I + \eta r(\lambda) + O(\eta^2),$$

Here the superscript $t$ denotes matrix transposition, $r(\lambda)$ is the classical $R$-matrix, $\rho(\lambda)$ is an even scalar function, $\eta$ is the crossing parameter and $V$ determines the crossing matrix $M \equiv V^t V = M^t$. The quasiclassical property gives rise to the direct connection of the quantum model to the corresponding classical one. Many $R$-matrices have only the combined $PT$-symmetry: $R_{12}^t(\lambda) = R_{21}(\lambda)$. The regularity is used to extract from $t(\lambda)$ the local integrals of motion.

Thus the general solution of (14) is given by

$$T(\lambda, \overline{\omega}) = KL = L_1(\lambda - \omega_1) \ldots L_2(\lambda - \omega_2) L_N(\lambda - \omega_N) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},$$

because any permutation of the multipliers gives the equivalent result in the algebra $\mathcal{T}_R$. Notice, that the $L$-operator is noting else than a $R$-matrix, acting in auxiliary and quantum spaces $\mathcal{C}^2 \otimes \mathcal{V}_i$: $L_i(\lambda) \equiv R_{ai}(\lambda)$.

The corresponding quantum determinant is

$$\Delta(\lambda) = \det_q T(\lambda) = \det K \prod_{i=1}^{N} ((\lambda - \omega_i)^2 - \eta^2 (\mathcal{C}_i + 1/4))$$
Now, it follows from

\[ R_{12}(\lambda - \mu)T^{(1)}(\lambda, \overline{\omega})T^{(2)}(\mu, \overline{\omega}) = T^{(2)}(\mu, \overline{\omega})T^{(1)}(\lambda, \overline{\omega})R_{12}(\lambda - \mu) \]  

(50)

that

\[ [t(\lambda, \overline{\omega}); t(\lambda, \overline{\omega})] = 0, \]

(51)

where \( t(\lambda, \overline{\omega}) = \text{tr}T(\lambda, \overline{\omega}) \). The trace is taken over the auxiliary space.

Among the integrals of motion (51), we look for local ones, i.e. quantities \( H^{(k)} \) \( k = 1, 2, 3, \ldots \),

which can be expressed as the sum of local operators,

\[ H^{(k)} = \sum_{i=1}^{N} H^{(k)}_{i, i-1,...,i-k+1} \]

(52)

The periodicity, \( N + 1 \equiv 1 \), is supposed. The local densities \( H^{(k)}_{i, i-1,...,i-k+1} \) should involve only \( k \) adjacent spins \( S_i, S_{i+1}, ..., S_{i-k+1} \). An important case when such local integrals exist is that of the homogeneous spin chain, corresponding to equal spins \( \Delta \equiv \Delta \) and zero shifts \( \overline{\omega} = 0 \). It has the important property of translational invariance. The corresponding \( R \)-matrix is regular. The similarity transformation

\[ US_i^a U^{-1} = S_{i+1}^a, \quad US_N U^{-1} = K_1 S_1 K_1^{-1}, \]

(53)

where \( K \) permutes the boundary matrix \( K \) and with the Lax operator \( L_1: KL_1(\lambda) = K^{-1}L(\lambda)KK \).

This transformation generalizes the ordinary translation for the periodic chain (\( K = 1 \)) to the twisted periodic boundary condition, specified by the matrix \( K \) and \( U^N \neq 1 \) in contrast to the case \( K = 1 \), when operator \( U \) takes the especially simple form: \( U = P_{12} P_{23} ... P_{N-1N} \). The unitarity of \( U \) allows to represent it in exponential form

\[ U = e^{iP}, \]

(54)

where operator \( P \) has the physical meaning of the total momentum of the chain. The hamiltonian of the model then acquires the form

\[ H = \frac{d}{d\lambda} t(\lambda) |_{\lambda=0} = \sum_{i=1}^{N} \frac{d}{d\lambda} P_{i,i+1} R_{i,i+1}(\lambda) |_{\lambda=0}. \]

(55)

Faddeev and Korchemsky have shown \[9\] that the \( N \)-reggeon Hamiltonian \( [13] \), corresponding to the homogeneous chain can be obtained in this manner. Unfortunately, the \( R \)-matrix, corresponding to the inhomogeneous chain (see below) possesses no regularity property and the corresponding Hamiltonian cannot be related to the derivative of the transfer-matrix in a simple way.

The analysis of the integrable systems is modified upon imposing boundary conditions different from the periodic ones. This case is related to the factorizable scattering of particles with internal degrees of freedom on a half-line \[18\]. The algebraic description involves additionally a boundary operator \( B \) into the ZF-algebra \[13\]:

\[ Z_a(\lambda) B = K_{ab}(\lambda) Z_b(-\lambda) B \]

(56)

Then the two particles factorizability gives rise the reflection equation (compare \[29\]):

\[ S_{12}(\lambda - \mu)K_1(\lambda)S_{21}(\lambda + \mu)K_2(\mu) = K_2(\mu)S_{12}(\lambda + \mu)K_1(\lambda)S_{21}(\lambda - \mu) \]

(57)

in addition to the Yang-Baxter equation \([30]\). The reflection matrix has the same properties as the \( R \)-matrix, regularity: \( K(0) = I \); unitarity: \( K(\lambda) K(-\lambda) = I \), T-symmetry: \( K_t(\lambda) = K(\lambda) \); the crossing symmetry is more elaborated and it involves the \( S \)-matrix as well \[19\].
Then the boundary operator $B$ can be constructed by

$$B = \exp\left( \int \phi(\lambda) d\lambda \right)$$

from the combination,

$$\phi(\lambda) = Z_a(-\lambda) K_{ab}(\lambda) Z_b(-\lambda),$$

which is a "local" field $\phi(\lambda)$: $[\phi(\lambda); \phi(\mu)] = 0$. Due to the $SL(2)$-symmetry of the $S$-matrix, the corresponding $K$-matrix can be transformed $K \to K' = G K G^{-1}$ with arbitrary $G$ and the general solution of the reflection equation [57] for the rational case, which we are interested in, is

$$K(\lambda) = \xi I + \lambda E, \quad E^2 = I.$$  \hfill (60)

The reflection equation has an important covariance property: if $T(\lambda)$ and $K(\lambda)$ satisfy the relations (87) and (57) then $K'(\lambda) = T(\lambda)K(\lambda)T(-\lambda)^{-1}$ is also a solution of (81), provided the entries of $K(\lambda)$ and $T(\lambda)$ commute, $[K_{ab}(\lambda), T_{cd}(\lambda)] = 0$. The proof follows easily by the substitution of $K'(\lambda)$ into (81) and by using the fundamental Y.B. relation in the different form

$$T_{(2)}^{-1}(-\mu) R_{12}(\lambda + \mu) T_{(1)}(\lambda) = T_{(1)}^{-1}(\lambda) R_{12}(\lambda + \mu) T_{(2)}(-\mu).$$

If the matrix $T(\lambda)$ is constructed as an ordered product of $N$ independent Lax operators, then $K'(\lambda)$ can be interpreted as the monodromy matrix of $N$ site lattice model with a boundary interaction described by the operator valued entries of the matrix $K(\lambda)$. It is called Sklyanin's monodromy matrix. The corresponding transfer matrix is defined as the trace

$$t(\lambda) = \text{tr}\bar{K}(\lambda) T(\lambda) K(\lambda) T^{-1}(-\lambda),$$

where the matrix $\bar{K}(\lambda)$ is any solution of (57), corresponding to the other boundary, is commutative [12]

$$[t(\lambda), t(\mu)] = 0.$$  \hfill (58)

In the context of the Heisenberg chain equation (67) takes the form:

$$R_{12}(\lambda - \mu) K_{(1)}^-(\lambda) R_{12}^{t(1) t(2)}(\lambda + \mu) K_{(2)}^-(\mu) =$$

$$K_{(2)}^-(\mu) R_{12}(\lambda + \mu) K_{(1)}^-(\lambda) R_{12}^{t(1) t(2)}(\lambda - \mu),$$

$$R_{12}(-\lambda + \mu)(K_{(1)}^{+})^{t(1)} (\lambda) M_{(1)}^{-1} R_{12}^{t(1) t(2)}(-\lambda - \mu - 2\eta) M_{(1)} (K_{(2)}^{+})^{t(2)} (\mu) =$$

$$(K_{(2)}^{+})^{t(2)} (\mu) M_{(1)} R_{12}(-\lambda - \mu - 2\eta) M_{(1)}^{-1} (K_{(1)}^{+})^{t(1)} (\lambda) R_{12}^{t(1) t(2)}(-\lambda + \mu)$$

where $M$ is crossing matrix, defined above. In practice, if $K^-(\lambda)$ is a solution of (61) then $K^+(\lambda) = (K^-(\lambda - \eta))^t M$ is a solution of (62). The eq. (57) has an important covariance property: if $T(\lambda, \tilde{\omega})$ and $K_{\pm}(\lambda)$ satisfies the relations (57) and (61), (62) then Sklyanin's monodromy matrix:

$$U(\lambda, \tilde{\omega}) = T(\lambda, \tilde{\omega}) K^-(\lambda) \tilde{T}(\lambda, \tilde{\omega}),$$

where $\tilde{T}(\lambda, \tilde{\omega}) = R_{N_0}(\lambda - \omega N) \ldots R_{2a}(\lambda - \omega_2) R_{1a}(\lambda - \omega_1)$, (cr. with (58)), satisfies the relation

$$R_{12}(\lambda - \mu) U_{(1)}(\lambda, \tilde{\omega}) R_{12}^{t(1) t(2)}(\lambda + \mu) U_{(2)}(\mu, \tilde{\omega}) = U_{(2)}(\mu, \tilde{\omega}) R_{12}(\lambda + \mu) U_{(1)}(\lambda, \tilde{\omega}) R_{12}^{t(1) t(2)}(\lambda - \mu).$$

Indeed, we note that unitarity and crossing symmetry together imply the relation

$$M_{(1)} R_{12}^{t(2)}(-\lambda - \eta) M_{(1)}^{-1} R_{12}^{t(1)}(\lambda - \eta) = \rho(\lambda).$$

Futhermore, we see that unitarity implies $T(\lambda, \tilde{\omega}) \tilde{T}(\lambda, \tilde{\omega}) = \prod \rho(\lambda - \omega_i)$. Therefore, up to a scalar factor, $\tilde{T}(\lambda, \tilde{\omega})$ is the inverse of $T(\lambda, \tilde{\omega})$. 

The commutativity of the transfer matrix \( t(\lambda, \vec{\omega}) \) implies integrability of the open quantum spin chain with the Hamiltonian \([12]\):

\[
H = \sum_{i=1}^{N-1} H_{ii+1} + 1/2(K_+^{(1)})^T + \frac{\text{tr}_0 K_+^{(0)} H_{N0}}{\text{tr} K_+^{(0)}}
\]

(66)

whose two-site terms are given by

\[
H_{ii+1} = \frac{d}{d\lambda} \mathcal{P}_{ii+1} R_{ii+1}(\lambda)|_{\lambda=0}
\]

(67)
in the standard fashion.

### 4 Closed XXX Heisenberg chain

There are two ways of including fermions. The first corresponds to considering closed Heisenberg chains with different spins, i.e. contain the operators \( \mathcal{G}_F \) and \( H_F \) together with \( H_G \). This case arises in amplitudes with the exchange of two adjoint fermions and one gluon. The same Hamiltonian also describes the exchange of three fermions in the fundamental representation of \( SU(3) \) if not all three have the same helicity. In the first case the Regge singularity is near \( j = 0 \) and in the second near \( j = -1/2 \). The second way of including fermions will be considered in the next section.

We consider the conformally covariant operator obtained from \( H_F^\omega \) by substituting \( \omega = 0 \),

\[
H_{\omega F}^0 = \partial_1^{-1} \log x_{12} \partial_1 + \log x_{12} + \log \partial_1 \partial_2 - 2\psi(1) = x_{12} \log \partial_1 x_{12}^{-1} + \log \partial_2 + 2 \log x_{12} - 2\psi(1)
\]

(68)

with \( \Delta_1 = 0, \Delta_2 = 1/2 \) and the conjugated operator

\[
H_{\omega F}^\dagger = \partial_2^{-1} \log x_{12} \partial_2 + \log x_{12} + \log \partial_1 \partial_2 - 2\psi(1) = x_{12} \log \partial_2 x_{12}^{-1} + \log \partial_1 + 2 \log x_{12} - 2\psi(1)
\]

(69)

with \( \Delta_1 = 1/2, \Delta_2 = 0 \). We have also \( \bar{H}_F \) which should be denoted by

\[
H_{\omega F}^{\bar{\dagger}} = 2 \log x_{12} + \log \partial_1 \partial_2 = x_{12}^{-1} H_{\omega F}^{00} x_{12}
\]

(70)

with the weights \( \Delta_1 = \Delta_2 = \frac{1}{2} \). We have used the identity

\[
(x_{12} \partial_2)^{-1} = \partial_2^{-1} \log x_{12} \partial_2 - \log x_{12}
\]

The operator \( H_{\omega F}^{\bar{\dagger}} \) is selfconjugated.

\[
(H_{\omega F}^{\bar{\dagger}})^T = H_{\omega F}^{\bar{\dagger}}
\]

(71)

and for \( H_{\omega F}^{\dagger} \) we have

\[
(H_{\omega F}^{\dagger})^T = \partial_1 \log x_{12} \partial_1^{-1} + \log x_{12} + \log \partial_1 \partial_2 - 2\psi(1) = \partial_1 H_{\omega F}^{\dagger} \partial_1^{-1} = x_{12}^{-1} \log \partial_1 x_{12} + \log \partial_2 + 2 \log x_{12} - 2\psi(1) = \mathcal{P}_{12} x_{12}^{-1} H_{\omega F}^{01} \mathcal{P}_{12}
\]

(72)

Therefore,

\[
[H_{\omega F}^{01}, \mathcal{P}_{12} x_{12} \partial_1] = 0
\]

(73)

Taking into account that

\[
(\mathcal{P}_{12} x_{12} \partial_1)^2 = \mathcal{P}_{12} x_{12} \partial_1 \mathcal{P}_{12} x_{12} \partial_1 = x_{21} \partial_2 x_{12} \partial_1 = -C_{0\frac{1}{2}}^{0\frac{1}{2}}
\]

(74)
and comparing with the general expression
\[ C^{\Delta_1 \Delta_2} = (\vec{S}_1 + \vec{S}_2)^2 = x_{12}^2 \partial_1 \partial_2 + 2x_{12}(\Delta_1 \partial_2 - \Delta_2 \partial_1) + (\Delta_1 + \Delta_2)(1 - \Delta_1 - \Delta_2) \] (75)
we can conclude that indeed under conformal transformations the operator \( H^{0_2} \) transforms covariantly with weights 0, \( \frac{1}{2} \).
Consider first the homogeneous closed chain, which consists out of three fermions.

The corresponding Hamiltonian is
\[ H^{1_{23}}_{123} = H^{1_2}_{32} + H^{1_3}_{21} + H^{1_1}_{13} = 2 \log x_{12}x_{23}x_{31} + 2 \log \partial_1 \partial_2 \partial_3. \] (76)
Conjugation arguments as described above give nothing for \( H^{111} \), while the commutators of particular Casimir operators provide us with two conserved currents:
\[ D^{(3)}_{123} = x_{12}x_{23}x_{31} \partial_1 \partial_2 \partial_3 + 1/2(x_{23}(x_{31} - x_{12}) \partial_2 \partial_3 + x_{12}(x_{23} - x_{31}) \partial_1 \partial_3 + x_{23}(x_{12} - x_{23}) \partial_3 \partial_1) - 1/2(x_{12} \partial_3 + x_{23} \partial_1 + x_{31} \partial_2). \] (77)
is a third order differential operators and
\[ D^{(2)}_{123} = (x_{12}^2 \partial_1 \partial_2 + x_{23}^2 \partial_2 \partial_3 + x_{31}^2 \partial_3 \partial_1 - x_{31} \partial_3 + x_{12} \partial_2) + (x_{31} - x_{12}) \partial_1 + (x_{12} - x_{23}) \partial_2 + (x_{23} - x_{31}) \partial_3, \] (78)
is a second order one. The latter reflects the conformal symmetry of the system. The relations
\[ [H^{111}_{123}, D^{(2)}_{123}] = 0 \] (79)
and
\[ [H^{111}_{123}, D^{(3)}_{123}] = 0 \] (80)
can be checked by direct calculations. These conserved currents can be obtained also in more regular way. They appear as coefficients in front of \( \lambda^1 \) and \( \lambda^0 \) in the monodromy matrix expansion \( t(\lambda) \), where
\[ t(\lambda) = tr(L^{1/2}_1(\lambda)L^{1/2}_2(\lambda)L^{1/2}_3(\lambda)) = D^{(3)}_{123} + \lambda D^{(2)}_{123} + \lambda^3 + 1/4 \] (81)
and the Lax operators \( L^\Delta_i(\lambda) \) are defined in (23) with \( \Delta_i = 1/2 \).
Let us consider now the inhomogeneous closed chain, which consists of
\[ \text{one fermion and two gluons.} \]

The corresponding Hamiltonian is
\[ \tilde{H}^{00}_{123} = H^{0_2}_{32} + H^{0_1}_{21} + H^{0_1}_{13} = 2 \log x_{12}x_{23}x_{31} + 2 \log \partial_1 \partial_2 \partial_3 + \partial_3^{-1}(x_{23}^{-1} - x_{31}^{-1}) + \partial_2^{-1}(x_{12}^{-1} - x_{23}^{-1}). \] (82)
Transposition arguments do not work here. However, for closed chains with three sites there is always a third order differential operator, commuting with operator of total spin of chain $C$. Indeed, the result for the commutator

$$[C_{12}^{\Delta_1\Delta_2}; C_{23}^{\Delta_3\Delta_4}] = [x_{12}^2 \partial_1 \partial_2 + 2x_{12}(\Delta_1 \partial_2 - \Delta_2 \partial_1); x_{31}^2 \partial_3 \partial_1 + 2x_{31}(\Delta_3 \partial_1 - \Delta_1 \partial_3)] =$$

$$x_{12}x_{23}x_{31} \partial_1 \partial_2 \partial_3 + \Delta_3 x_{12}(x_{23} - x_{31}) \partial_1 \partial_2 + \Delta_2 x_{31}(x_{12} - x_{23}) \partial_3 \partial_1 + \Delta_1 x_{23}(x_{31} - x_{12}) \partial_2 \partial_3$$

$$-2(\Delta_2 \Delta_3 x_{23} \partial_1 + \Delta_3 \Delta_1 x_{31} \partial_2 + \Delta_1 \Delta_2 x_{23} \partial_3) \equiv D_{\Delta_1\Delta_2\Delta_3}^{(3)}$$

is symmetric under cyclic permutation of $(123)$. Therefore its commutator with $D_{\Delta_1\Delta_2\Delta_3}^{(2)} = C_{12}^{\Delta_1\Delta_2} + C_{23}^{\Delta_2\Delta_3} + C_{31}^{\Delta_3\Delta_1}$ vanishes due to the Jacoby identity. However, the hamiltonian $\hat{H}_{123}$ commuting with $D_{\Delta_1\Delta_2\Delta_3}^{(2)}$ does not commute with $D_{\Delta_1\Delta_2\Delta_3}^{(3)}$:

$$[\hat{H}_{123}, D_{\Delta_1\Delta_2\Delta_3}^{(3)}] =$$

$$[(2 \log x_{12}x_{23}x_{31} + 2 \log \partial_1 \partial_2 \partial_3 + \partial_3^{-1}(x_{23}^{-1} - x_{31}^{-1}) + \partial_2^{-1}(x_{12}^{-1} - x_{23}^{-1})),$$

$$(x_{12}x_{23}x_{31} \partial_1 \partial_2 \partial_3 + 1/2(x_{12}^2 - x_{31}^2) \partial_2 \partial_3)] = 1/2 \partial_3^{-1} x_{12}x_{31}^2 \partial_2 - 1/2 \partial_2^{-1} x_{31}^2 x_{12}^2 \partial_3.$$

These operators $D^{(2)}, D^{(3)}$ also appear as coefficients in front of $\lambda^1$ and $\lambda^0$ in the monodromy matrix expansion $t(\lambda)$, where

$$t(\lambda) = \text{tr}(L_{1}^{1/2}(\lambda)L_{0}(\lambda)L_{3}^{0}(\lambda)) = D^{(3)} + \lambda D^{(2)} + \lambda^2(2\lambda - 1).$$

The next inhomogeneous chain is the one with one gluon

and two fermions:

The corresponding hamiltonian commutes with the Casimir operator

$$[(x_{12} \partial_1 x_{12} \partial_2 + x_{23} \partial_3 x_{23} \partial_2 + x_{31} \partial_1 x_{31} \partial_3); (2 \log x_{12}x_{23}x_{31} + 2 \log \partial_1 \partial_2 \partial_3 + \partial_2^{-1}(x_{12}^{-1} - x_{23}^{-1}))] = 0,$$

and does not commute with the next current

$$[(x_{23} \partial_3 x_{31} \partial_1 x_{12} \partial_2 - 1/2 x_{31} \partial_1 \partial_3 x_{31}); (2 \log x_{12}x_{23}x_{31} + 2 \log \partial_1 \partial_2 \partial_3 + \partial_2^{-1}(x_{12}^{-1} - x_{23}^{-1}))] =$$

$$1/2 \partial_2^{-1} x_{31}(x_{23}^2 \partial_1 - x_{12}^{-2} \partial_3)x_{31}.$$

The same happens for the longer closed chains. The hamiltonians for that chains commute only with the first current, the Casimir operator. This means that in order to describe the integrable model, the corresponding hamiltonian should be modified. Indeed, the inhomogeneous chains (fig. 3,4) have to be considered as the chains with impurity. Probably, the expression for $H^{0\Delta_2}$ has to be changed slightly.

5 XXX Heisenberg chain with open boundary

The second way of including fermions is to build an open chain with the fermions (now in the fundamental gauge group representation) at the ends. This open chain corresponds to the Regge exchange with meson quantum numbers in the $t$-channel.
Let us consider the hamiltonian corresponding to the following chain

$$H_{123}^{00} = H_{32}^{00} + H_{21}^{01} =$$

$$\log x_{12} + \partial_2^{-1} \log x_{32}x_{21} \partial_2 + \log \partial_1 \partial_2^2 \partial_3 + \partial_3^{-1} \log x_{32} \partial_3 =$$

$$\log \partial_1 + x_{32} \log \partial_2 \partial_3 x_{32}^{-1} + x_{21} \log \partial_2 x_{21}^{-1} + 2 \log x_{21} x_{32},$$

The transposed Hamiltonian has form

$$(H_{123}^{00})^T = \partial_2 \partial_3 H_{123} \partial_2 \partial_3 = P_{123}(x_{32} x_{21})^{-1} H_{123} x_{32} x_{21} P_{123}. \quad (88)$$

Then $H$ commutes with

$$A_{123}^{00} = P_{123} x_{32} x_{21} \partial_2 \partial_3 P_{123}. \quad (89)$$

The permutation operator $P_{123}$ maps the sites 1, 2, 3 of the chain into the sites 3, 2, 1 correspondingly.

In a similar way the conserved operator of the highest order in the derivatives can be obtained for the open chain with $N - 2$ gluonic operators and fermions of opposite spin at the ends:

$$H_{12...N}^{00...0} = \partial_2^{-1} \log x_{21} \partial_2 + \log x_{21} + \log \partial_2 \partial_1 + \sum_{i=2}^{N-1} H_{ii+1}^{00} \quad (90)$$

commutes with the charge operator

$$A_{12...N} = P_{12...N} x_{NN} ... x_{21} \partial_N \partial_{N-1} ... \partial_2. \quad (91)$$

For simplicity we put the reflection matrices $K^\pm$ equal to unity. From eq. (56) one can see that this implies the transfer matrix to be an even function of the spectral parameter. In our case we obtain up to insignificant c-number terms

$$t(\lambda) = (4\lambda^2 - 1)(D^{(4)} - \omega_2 D^{(3)}_{\frac{1}{2}00} - (\lambda^2 - \omega_2^2) D^{(2)}_{\frac{1}{2}00} + (\lambda^2 - \omega_2^2)^2), \quad (92)$$

where $D^{(3)}_{\frac{1}{2}00}$ and $D^{(2)}_{\frac{1}{2}00}$ are the same as in (38) for the closed chain and

$$D^{(4)}_{\frac{1}{2}00} = (x_{12}^2 x_{23} \partial_1 \partial_2 + x_{12} x_{23} x_{12} - x_{23}) \partial_1 + x_{12} x_{23}^2 \partial_2 + x_{23} (x_{12} - x_{23}) \partial_2 \partial_3.$$

This coincides with the square of the operator (75), obtained by transposition arguments.

We put here $\omega_2^3 = \omega_2^2 = \omega_2^1 + 3/4$ in order to achieve that the second order operator $D^{(2)}$ coincides with $C^{01/2}$. Now the operator $D^{(2)}$ commutes with hamiltonian. Thus, for the open chain (Fig. 5), described by the hamiltonian $H_{123}$ the eigenvalue problem like eq. (67) can be replaced by the one with the same Casimir operator and with the fourth order differential operator $D^{(4)}$ instead of $D^{(3)} = \tilde{A}$ as for the closed chain.

So we see that for open chains we are able to find conserved currents and its number is just enough to solve the eigenvalue problem for QCD reggeon interactions. Moreover, for
the open chain Fig. 5 with one fermion inside again everything is all right. The hamiltonian

\[ \mathcal{H}_{3_{112}} = \mathcal{H}_{3_{11}} + \mathcal{H}_{3_{22}} = 2 \log x_{12} x_{31} + 2 \log \partial_2^2 \partial_3 + \partial_2^{-1} \log x_{12} - \partial_3^{-1} \log x_{31} \]  

(93)

t(\lambda) = 4\lambda(\lambda + 1)x_{31} \partial_1 x_{12} \partial_1 x_{31} \partial_2 + 4\lambda^2(\lambda + 1)^2(x_{23} \partial_2 \partial_3 + x_{31} \partial_3 x_{31} \partial_3 + x_{12} \partial_1 x_{12} \partial_2)  

(94)

The case of the open chain with two fermions and one gluon can be considered as well.

For the chain

\[
\begin{array}{c}
\circ \quad 3 \\
1 \\
2 \\
\circ
\end{array}
\]

we have two conserved currents:

\[
[x_{12} x_{23} \partial_1 \partial_2 x_{23} x_{23} \partial_2; (2 \log x_{12} x_{23} + \log \partial_1 \partial_2 \partial_2 \partial_3 + \partial_2^{-1}(x_{12}^{-1} - x_{23}^{-1}))] = 0
\]

(95)

\[
[(x_{12} \partial_1 x_{12} \partial_2 + x_{23} \partial_2 x_{23} \partial_2 + x_{31} \partial_3 x_{31}; (2 \log x_{12} x_{23} + \log \partial_1 \partial_2 \partial_2 \partial_3 + \partial_2^{-1}(x_{12}^{-1} - x_{23}^{-1}))] = 0
\]

(96)

Another chain with two fermions and one gluon is:

\[
\begin{array}{c}
\circ \quad 3 \\
1 \\
2 \\
\circ
\end{array}
\]

We have following relations for this chain

\[
[x_{12} \partial_2 x_{23} \partial_3 x_{23} \partial_2 x_{12} \partial_1; (2 \log x_{12} x_{23} + \log \partial_1 \partial_2 \partial_2 \partial_3 - \partial_1^{-1} x_{12}^{-1})] = 0
\]

(97)

\[
[(x_{12} \partial_2 x_{12} \partial_1 + x_{23} \partial_2 \partial_3 x_{23} + x_{31} \partial_3 x_{31} \partial_1); (2 \log x_{12} x_{23} + \log \partial_1 \partial_2 \partial_2 \partial_3 - \partial_1^{-1}(x_{12}^{-1})] = 0
\]

(98)

For completeness let us consider also

the open chain with three fermions

\[
\begin{array}{c}
\circ \\
3 \\
1 \\
2 \\
\circ
\end{array}
\]

we can deduce following relations:

\[
[(\partial_1 x_{12} \partial_2 x_{23} \partial_3 x_{23} \partial_2 x_{12} + x_{12} \partial_2 x_{23} \partial_3 x_{23} \partial_2 x_{12} \partial_1; (2 \log x_{12} x_{23} + \log \partial_1 \partial_2 \partial_2 \partial_3)] = 0
\]

(99)

\[
[(x_{12} \partial_1 x_{12} + x_{23} \partial_3 x_{23} + x_{31} \partial_3 x_{31} \partial_1); (2 \log x_{12} x_{23} + \log \partial_1 \partial_2 \partial_2 \partial_3)] = 0
\]

(100)

Thus, all possible open chains with three sites corresponding to gluonic or fermionic reggeons are integrable. The conserved currents can be derived from the transfer matrix with boundaries represented by unity reflection matrices. The hamiltonians are constructed from the QCD kernels given above and commute with these two conserved currents given by the second and fourth order differential operators respectively. We believe that integrability holds for the longer open chains too.
6 Conclusions

The examples considered above, except for the difficulties with the inhomogeneous closed chains, show the validity of Lipatov’s conjecture about the deep connection between the kernels of high-energy QCD scattering amplitudes and the exactly solvable two-dimensional models in the cases with fermions as well. In the pure gluonic this connection context has already been used for the solution of the Odderon problem [20]. The case of open chain differs from the closed chain case by the higher order of non-trivial conserved charge; the fourth order appears instead of the third.

Recently it has been shown that the considered integrable structures appear also in hard (exclusive and deep-inelastic) scattering [21], corresponding to a different limiting region of high energy scattering amplitudes in QCD.

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