A Note on Optimal Design of Multiphase Elastic Structures

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Abstract The paper describes the first exact results in optimal design of the three-phase elastic structures. Two elastic materials, the “strong” and the “weak” one, are laid out with void in a given design domain so that the compliance plus the cost of a structure is minimized. As in the classical problem of two-phase optimal design, subdomains filled with pure phases and their mixtures with optimal microgeometries are included in optimal layouts. Here, we consider a special case when the relative cost of the “strong” and “weak” material depends on their Young’s moduli which leads to the nonuniqueness of these layouts. Several numerical examples are discussed.

Keywords multiphase optimal design · optimal composites · optimal elastic design · structural optimization

1 Introduction

Multiphase elastic structures designed for minimal compliance are made from composites with microgeometries of maximal stiffness. In the classical problem of two-phase optimal design, the best mixtures are first- or second-rank laminates; in the latter the “strong” material envelopes the “weak” one. Concentration of the “strong” phase in a laminate grows with the intensity of the average stress and the anisotropy of a microstructure follows the anisotropy of a stress tensor, see (Lurie and Cherkaev, 1982; Gibiansky and Cherkaev, 1987; Bendsøe and Kikuchi, 1988; Cherkaev, 2000). The same pattern is also observed in the topology optimization problem when the “weak” material degenerates to void, see (Allaire, 2002; Bendsøe and Sigmund, 2003).

Optimal multiphase composites are much less investigated; notice the pioneering contribution of Gibiansky and Sigmund (2000) and Sigmund (2000), see also (Albin et al., 2007; Cherkaev and Zhang, 2011; Cherkaev and Dzierżanowski, 2013) for continuation and extensions. In the present paper we take another step forward; we discuss the first exact generalization of both classical problems mentioned above. We consider a two-dimensional linearly elastic structure made from three phases - two materials and a void - and we minimize its compliance assuming that: (i) the amount of each phase in a structure is a priori given, and (ii) the relative cost of materials depend on their Young’s moduli.

2 The problem

2.1 Notation

For simplicity we assume that the materials (nondegenerate phases) are isotropic with Poisson coefficients...
equal to zero. It follows that bulk and shear moduli of each material are equal. However, the results can be easily generalized to arbitrary well-ordered phases, that is to say when both the bulk and shear modulus of one phase is greater than the corresponding modulus of the other.

Let $K_1$ and $K_2$, $K_1 < K_2$, denote material compliances (inverses of Young’s moduli); the compliance of void is infinite. Define

$$K(x) = \begin{cases} K_1 & \text{if } x \in \Omega_1, \\ K_2 & \text{if } x \in \Omega_2, \\ +\infty & \text{if } x \in \Omega_3, \end{cases}$$

where $\Omega_1$, $\Omega_2$ and $\Omega_3 = \Omega_{void}$ are disjoint subdomains in a bounded domain $\Omega$ occupied by material 1 (strong phase), material 2 (weak phase) and void (degenerate phase) respectively.

The equilibrium conditions and constitutive equations of linearized elasticity are

$$\nabla \cdot \tau = 0 \quad \text{in} \quad \Omega, \quad \tau n = f \quad \text{on} \quad \partial \Omega_f, \quad \tau = \tau^T$$

(2)

$$2K\tau = \nabla u + \nabla u^T, \quad u = u_0 \quad \text{on} \quad \partial \Omega_u,$$

(3)

where $\tau$ is the $2 \times 2$ tensor stress field, $u$ is the vector displacement field, $f$ denotes traction on the boundary component $\partial \Omega_f$, $u_0$ stands for a trace of displacement field on $\partial \Omega_u$, and $n$ is the normal to the boundary.

The stress energy accumulated in the $i$-th material, $i = 1, 2$, is given by

$$2W_i(\tau) = K_i \text{Tr}(\tau^2),$$

(4)

and for void (phase 3) it is assumed that

$$W_3(\tau) = \begin{cases} 0 \quad \text{if } \tau = 0, \\ +\infty \quad \text{otherwise}. \end{cases}$$

(5)

**Remark:**

All stress tensors introduced in the sequel have common eigenbasis $\mathcal{E} = \{e_1 \otimes e_1, e_1 \otimes e_1\}$. It is thus convenient to write $\sigma = (\sigma_1, \sigma_1)$, i.e. to identify the eigenvalues of a given stress tensor with the coordinates of a point in $\mathbb{R}^2$ with basis $\mathcal{E}$.

2.2 The optimization problem

Consider the optimization problem: among all divisions of $\Omega$ into disjoint subdomains $\Omega_i$, $i = 1, 2, 3$, with $|\Omega_i|$ a priori given, choose the one that minimizes structural compliance. This problem can be formulated in a variational form

$$I = \inf \left\{ \int_{\Omega} F(\tau) + \int_{\partial \Omega_f} f \cdot u \, ds \mid \tau \text{ as in (2)} \right\}$$

(6)

where

$$F(\tau) = \begin{cases} 0 & \text{if } \tau = 0, \\ \min \{ \Phi_1(\tau), \Phi_2(\tau) \} & \text{otherwise.} \end{cases}$$

(7)

with $\Phi_i(\tau) = 2W_i(\tau) + \gamma_i$, $i = 1, 2$, denoting the energy well of $i$-th material. Here $\gamma_i$ stand for the Lagrange multipliers related to the restrictions on $|\Omega_i|$ set above. They can be understood as “costs” of materials. Here we set $0 < \gamma_2 < \gamma_1$.

The integrand $F$ is not a convex function of $\tau$, hence has no solution in the sense that optimal division of $\Omega$ into three disjoint subdomains occupied by pure phases does not exist in general. Consequently, the original optimization problem needs relaxation by allowing arbitrary microstructural mixtures of pure phases in optimal design; effective constitutive properties of thus obtained composite materials are determined by the homogenization theory, see [Cherkaev, 2000].

Technically, relaxation results in replacing $F$ with its quasiconvex envelope $QF$, see [Dacorogna, 2008], that is the stress energy density of the optimal (stiffest) composite material. Due to the local character of homogenization, formulae for $QF(\tau)$ and the properties of related optimal microstructures can be determined independently at each point in the design domain $\Omega$.

In the problem considered here, $QF$ is locally supported by the energy wells $\Phi_i$, $i = 1, 2$, and $\tau^{(3)} = 0$ in phase 3 (void). Volume fractions of phases in the locally optimal microstructure, $m_i$, $i = 1, 2, 3$, depend on the average stress $\tau$ and the relation between $\gamma_1$ and $\gamma_2$. They satisfy $m_1 + m_2 + m_3 = 1$.

3 Study of the quasiconvex envelope

3.1 High cost of material 2

Consider the case when the cost $\gamma_2$ is high and material 2 is not used in the optimal design at all, i.e. $m_2^{opt} = 0$. As in the topology optimization problem, the quasiconvex envelope is supported by the energy well $\Phi_1$ and $W_3(0) = 0$, see [Cherkaev, 2000; Allaire, 2002]. For $\tau = (\tau_1, \tau_1)$ we thus have $QF = QF_{13}$.

$$QF_{13}(\tau) = \begin{cases} 2K_1[\xi_0(|\tau_1| + |\tau_1|) - |\tau_1|] & \text{if } |\tau_1| + |\tau_1| \leq \xi_0, \\ K_1(\tau_1^2 + \tau_1^2) + \gamma_1 & \text{otherwise,} \end{cases}$$

(8)

and $\xi_0 = \sqrt{\gamma_1/K_1}$.

For $|\tau_1| + |\tau_1| \leq \xi_0$, optimal composites are made from phase 1 (strong material) and phase 3 (void),
they take a form of second-rank orthogonal laminates $L(13, 1)$. Optimal volume fraction of phase 1 in $L(13, 1)$ is given by $m_{1}^{\text{opt}} = (|\tau_1| + |\tau_{31}|)/\xi_0$. Stresses inside of each phase of $L(13, 1)$ satisfy the pointwise relations

$$|\tau_1^{(1)}| + |\tau_1^{(1)}| = \xi_0 \text{ in } \Omega_1, \quad \tau_1^{(3)} = \tau_{31}^{(3)} = 0 \text{ in } \Omega_3. \quad (9)$$

3.2 Special cost of material 2

Assume now that the cost $\gamma_2$ decreases to a special value $\gamma_2 = \gamma_A$, $\gamma_A = 2K_1 \gamma_1$ for which the energy well $\Phi_2$ touches $QF_{13}$. Indeed, one may check that $2W_2(\tau) + \gamma_A = QF_{13}(\tau)$ for $\tau = \xi_i$, $i = 1, \ldots, 4$, where

$$\xi_1 = (\xi, \xi), \quad \xi_2 = (-\xi, -\xi), \quad \xi_3 = (-\xi, \xi), \quad \xi_4 = (\xi, -\xi), \quad (11)$$

and

$$\xi = \frac{K_1}{K_1 + K_2} \xi_0. \quad (12)$$

Note that $\xi_1, \xi_2$ define pure spherical stress and $\xi_3, \xi_4$ are pure deviators.

For $\gamma_2 = \gamma_A$ the optimal material is nonunique: one can use either pure phase 2, $L(13, 1)$, or a mixture of them. If the average stress $\tau$ in the composite is represented by any of points in $QF_{13}$, i.e. if $|\tau_1| = |\tau_{31}| = \xi$, then the cost of $L(13, 1)$ is equal to the cost of pure phase 2 that is $\gamma_1 m_{1}^{\text{opt}} = \gamma_A$.

The quasiconvex envelope is still given by $QF_{13}$ but now $QF_{13}$ is additionally supported by $\Phi_2$. Hence, for $|\tau_1| + |\tau_{31}| \leq \xi_0$, generally anisotropic optimal microstructures $L(13, 1)$ can be replaced with another composite that contains a fraction of material 2. In order to do this, one needs to fulfill the sufficient optimality conditions, that is to find a microstructure in which $\tau^{(1)}$ and $\tau^{(3)}$ (stresses in phases 1 and 3) satisfy $QF_{13}$ and $\tau^{(2)}$ (stress in phase 2) is given by any of the tensors described by $QF_{13}$ and $QF_{13}$. Stresses in phases must correspond to the average stress by $\sum_{i=1}^{3} m_i \tau^{(i)} = \tau$. In addition, the stress field in a microstructure defined as $\tau_{\text{micro}} = \tau^{(i)}$ in $i$-th phase, $i = 1, 2, 3$, must satisfy the equilibrium equation: $\nabla \cdot \tau_{\text{micro}} = 0$.

Optimal structures that satisfy all the above conditions are shown in Fig. 1(b). They are laminates of a rank whose geometric parameters are different in different regions of the average stress $\tau$. These parameters are found by the technique used previously in (Albin et al., 2007; Cherkaev and Zhang, 2011) and (Cherkaev and Dzierzakowski, 2013). Roughly speaking, they are determined by solving the equations of two types: (i) formulae for average

\[ \text{Fig. 1 a Three regions of optimality; b Cartoons of optimal microstructures in respective regions. All geometries represent laminates of a rank; the mixing of phases is hierarchical in scales. White color denotes phase 1, grey – phase 2 and black – phase 3; c Contour plot of } m_2^{\text{max}}, m_3^{\text{max}} = 1 \text{ at the points } (\pm \xi, \pm \xi), (\pm \xi, \pm \xi) \text{ where all regions of optimality meet. Coordinates of these points are determined by the eigenbasis of the stress tensor, see the Remark in Sec. 2.1.} \]
stresses in two neighboring phases, and (ii) continuity of normal stress component on the interface between these phases.

The maximum amount $m^{\text{max}}$ of material 2 that can be used to construct a laminate replacing $L(13, 1)$ depends on the relation between the eigenvalues $\tau_I$, $\tau_{II}$ of the average stress $\tau$ and $\xi$ introduced in (12). The non-trivial values $m^{\text{max}} > 0$ correspond to subregions

$$R_1: \xi < |\tau_I|, \xi < |\tau_{II}|, m^{\text{max}} = \frac{\xi_0 - |\tau_I| - |\tau_{II}|}{\xi_0 - 2\xi},$$

$$R_{2.1}: \xi < |\tau_I|, \xi > |\tau_{II}|, m^{\text{max}} = \frac{\xi_0 - |\tau_I| - |\tau_{II}|}{\xi_0 - \xi - |\tau_{II}|},$$

$$R_{2.2}: \xi > |\tau_I|, \xi < |\tau_{II}|, m^{\text{max}} = \frac{\xi_0 - |\tau_I| - |\tau_{II}|}{\xi_0 - |\tau_I| - \xi},$$

$$R_3: \xi < |\tau_I|, \xi < |\tau_{II}|, m^{\text{max}} = \frac{|\tau_I|}{\xi_0^2},$$

(13)

of the composite region bounded by $|\tau_I| + |\tau_{II}| \leq \xi_0$, see Fig. 1(a). The contour plot of $m^{\text{max}}$ in this region is shown in Fig. 1(c). Note that $m^{\text{max}}$ depends on the relative cost of materials 1 and 2 through (10) and (12).

3.3 Low cost of material 2

Here we briefly outline the change in the boundaries of the composite region when the cost $\gamma_2$ further decreases and the well $\Phi_2$ penetrates through the quasiconvex envelope $QF_{13}$. For $\gamma_B \leq \gamma_2 \leq \gamma_A, \gamma_B = (K_1/K_2)\gamma_1$, the isolated zones appear around the points (11), see Fig. 2(a). In these zones the pure phase 2 is optimal and they expand into the subregions $R_i, i = 1, \ldots, 4$, described above.

When the cost further lowers, $\gamma_2 \leq \gamma_B$, the zones of optimality of pure material 2 merge. The remaining part of the composite region splits into two disconnected subdomains, see Fig. 2(b). In these subdomains, optimal composites are second rank laminates: $L(12, 1)$ or $L(23, 2)$. Three-material composites are not optimal for $\gamma_2 \leq \gamma_B$.

Finally, when the cost of material 2 decreases to zero, $\gamma_2 = 0$, phase 3 (void) disappears from optimal design. In this case, only second rank laminates $L(12, 1)$ are optimal. They can degenerate into simple laminates and pure materials for certain values of $\tau_I, \tau_{II}$, Fig. 2(c).

We intend to consider the details of this topic in our further research.

4 Results

To illustrate the discussion in Sec. 3.2 we computed two standard examples of optimal designs: a cantilever and

![Fig. 2 Boundaries of the composite region (top right quarter) for decreasing values of $\gamma_2$: a and a2 $\gamma_B \leq \gamma_2 \leq \gamma_A$; b $\gamma_2 \leq \gamma_B$; c $\gamma_2 = 0$. Grey areas represent the domain in which pure material 2 is optimal. Dotted lines show the boundaries of the composite region for $\gamma_2 = \gamma_A$, see Fig. 1(a). Coordinate system in all plots is determined by the eigenbasis of the stress tensor, see the Remark in Sec. 2.1]
a bridge. First, the topology optimization problem was solved for a given design domain $\Omega$ using the code in [Dzierżanowski, 2012]. In this way, the distribution of material 1 in $\Omega$ was found. Next, we determined the eigenvalues of the stress tensor at each $x \in \Omega$, using this we computed the maximal amount of material 2 that can be used to replace a microstructure equivalent to the optimal $L(13, 1)$.

Optimal microstructures according to regions in Fig. 1(a) and the dependence of the results on the relative cost between materials 1 and 2 are denoted in Figs. 3, 4.

5 Comments and Conclusions

1. The “strong” material 1 tends to be placed close to the supports and the loading, while the “weak” phase 2 tends to concentrate in the regions where the stress tensor is closer to isotropic. At the free boundary, the normal stress is zero and therefore phase 2 is not present.
2. Phases 1 and 2 tend to be mixed. As in the material-void design, the regions of pure material 1 alternate with the composite zones forming “ribs”. This provides the structural anisotropy and additional stiffness in the direction of maximal stress.
3. We observe the “almost void” regions in the designs of the bridge and cantilever. We remark that the optimal design does not allow the pure void regions, because the energy density must be nonzero everywhere due to optimality conditions. However, these conditions do not explain the sharp increase of the stress and the stiffness at some curves. We do not have a satisfactory explanation of this phenomenon.
4. Comparing two- and three-phase designs, we observe a larger variety of the microstructures in the latter. For example, a second interior arc from material 2 is formed in the optimal bridge. We expect that this variety will be even more visible in the general situation, for lower cost of material 2, $\gamma_2 < \gamma_A$.
5. The three-phase optimal layout is not unique. For certain values of the average stress, one has the option to replace a $L(13, 1)$ microstructure with pure material 2, or to include material 2 into the composition. This is often preferable because the structures with filler material 2 are more stable and therefore have a better response to the variations in loading.
Fig. 4 Optimal design of a bridge; a Design domain, load and supports; b (two-phase design) Optimal distribution of material 1; c–e (three-phase design, high relative cost $\gamma_A/\gamma_1$, see (10)) Optimal microstructure regions; updated material 1 distribution; material 2 distribution; f–h (three-phase design, low relative cost $\gamma_A/\gamma_1$, see (10)) Optimal microstructure regions; updated material 1 distribution; material 2 distribution. Black color in c and f denotes pure material 1 zone, other colors correspond to those in Fig.1(a).

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