On the Leray-Hopf Extension Condition for the Steady-State Navier–Stokes Problem in Multiply-Connected Bounded Domains

Giovanni P. Galdi

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Abstract

Employing the approach of A. Takeshita [Pacific J. Math., 157 (1993), 151–158], we give an elementary proof of the invalidity of the Leray-Hopf Extension Condition for certain multiply connected bounded domains of \( \mathbb{R}^n \), \( n = 2, 3 \), whenever the flow through the different components of the boundary is non-zero. Our proof is alternative to and, to an extent, more direct than the recent one proposed by J.G. Heywood [J. Math. Fluid Mech. 13 (2011), 449–457].

Keywords. Navier–Stokes equations, Non-homogeneous boundary conditions, Multiply connected domains

MSC (2000) 35Q30, 35Q35, 30E25

1 Introduction

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \), \( n = 2, 3 \). As is well known, the Leray-Hopf Extension Condition is related to the solvability of the following Navier–Stokes equations

\[
\begin{align*}
\nu \Delta v &= v \cdot \nabla v + \nabla p + f \quad \text{in } \Omega, \\
\text{div} v &= 0
\end{align*}
\]

(1)

under prescribed non-homogeneous boundary conditions

\[ v = v_* \quad \text{at } \partial \Omega. \]

(2)

Here, as customary, \( v, p \) and \( \nu > 0 \) denote velocity and pressure fields, and kinematic viscosity of the liquid, respectively, while \( f \) is representative of a body force possibly acting on it. Moreover, \( v_* \) is a given distribution of velocity at the boundary \( \partial \Omega \), which, by \( \{1\} \) and the Gauss theorem must satisfy the compatibility condition

\[
\sum_{k=1}^{N} \int_{\Gamma_k} v_* \cdot n := \sum_{k=1}^{N} \Phi_k = 0,
\]

(3)
where $\Gamma_k$, $k = 1, \cdots, N$, are the connected components of $\partial \Omega$, and $n$ is its unit outer normal. From the physical viewpoint, $\Phi_k$ is the (mass) flow-rate through the portion $\Gamma_k$ of the boundary. To fix the ideas, we assume that $\Gamma_i$, $i = 1, 2, \cdots, N - 1$, are all surrounded by $\Gamma_N$ and lie outside of each other.

The existence of a (weak, in principle) solution to the problem (1)–(3) is readily established (e.g., by Galerkin method or by Leray-Schauder theory), provided we are able to show (formally, at least) that the velocity field of the searched solution satisfies the a priori bound

$$\|\nabla v\|_2^2 \leq C,$$  \hspace{1cm} (4)

where $\| \cdot \|_2$ is the $L^2(\Omega)$-norm and $C$, here and in the following, denotes a constant depending at most on $\Omega$, $f$, $v_*$, and $\nu$; see [7, Chapter IX] for details.

One way of attempting to prove (4) is to extend the boundary data $v_*$ to $\Omega$ by a solenoidal function $V$, and introduce the new velocity field $u := V - v_*$. Clearly, $v_*$ satisfies a bound of the type (4) if and only if $u$ does. Now, writing (1) in terms of $u$, dot-multiplying both sides of the resulting equation by $u$, integrating by parts over $\Omega$ and using the fact that $u$ is solenoidal and that vanishes at $\partial \Omega$, we formally show the following relation:

$$\nu \|\nabla u\|_2^2 = - (u \cdot \nabla V, u) - (V \cdot \nabla u, u) - \nu (\nabla V, \nabla u) + (f, u),$$  \hspace{1cm} (5)

where we have adopted the standard notation

$$(a, b) = \int_{\Omega} a_i b_i, \quad a, b \in \mathbb{R}^n; \quad (A, B) = \int_{\Omega} A_{ij} B_{ij}, \quad A, B \in \mathbb{R}^{n \times n}.$$

Thus, assuming $V \in W^{1,2}(\Omega)$ and using Cauchy–Schwartz and classical embedding inequalities, from (5) we show

$$\nu \|\nabla u\|_2^2 \leq - (u \cdot \nabla V, u) + C.$$  \hspace{1cm} (6)

Since both terms in (6) are quadratic in $u$, from this relation it is not clear how to get a bound on $V u$ of the type (4), unless we make the obvious assumption that the viscosity $\nu$ is “sufficiently large” compared to the magnitude of $\nabla V$, or equivalently, of $v_*$ in suitable trace norm. However, such a restriction can be avoided whenever $v_*$ obeys the Leray–Hopf Extension Condition [11, p. 38], [9, p. 772], namely, for any $\varepsilon > 0$, there exists a solenoidal extension, $V_{\varepsilon} \in W^{1,2}(\Omega)$, of $v_*$ such that

$$- (u \cdot \nabla V_{\varepsilon}, u) \leq \varepsilon \|\nabla u\|_2^2,$$ \hspace{1cm} (EC)

for all solenoidal vector functions $u \in W^{1,2}_0(\Omega)$. It is then obvious that the validity of (EC) along with (6) furnishes the desired uniform bound for $u$, without imposing any restriction on the magnitude of $\nu > 0$.

1We employ standard notation for Lebesgue, Sobolev and trace spaces; see e.g. [1].

2Summation convention over repeated indeces applies.

3This condition on $V$ is certainly satisfied if $v_* \in W^{1/2,2}(\partial \Omega)$ and $\Omega$ is Lipschitz.

4See footnote 2.

5Notice that (EC) is weaker than the so-called “Leray Inequality”, the latter consisting in replacing the left-hand side of (EC) with $| (u \cdot \nabla V_{\varepsilon}, u) |$. The validity of Leray’s Inequality is originally studied, and disproved under certain conditions, in [13] and, more recently, in [3].
The validity of (EC) has been investigated by many authors, beginning with the cited pioneering works of J. Leray and E. Hopf, who showed that (EC) certainly holds provided \( v_\ast \) satisfies a condition stronger than (3), namely, that 
\[
\Phi_k = 0, \text{ for each } k = 1, \ldots, N.
\]
More recently, a proof of (EC) under the general assumption (3) was given in the two-dimensional case by L.I. Sazonov [14], and, independently, by H. Fujita [5], provided, however \( \Omega, v_\ast, \) and \( u \) satisfy suitable symmetry hypotheses; see also [12, 4]. As a result, existence to problem (1)–(3) follows on condition that also \( f \) is prescribed in an appropriate class of symmetric functions. The method of Fujita was successively extended by V.V. Pukhnachev [13] to cover the three-dimensional case, again under appropriate symmetry assumptions.

The fact that (EC) may not be true unless some restrictions are imposed, was already clear after the work of A. Takeshita [15, Section 3] and the present author [6, pp. 22–23], where it was shown that even in the simplest case when \( \Omega \) is an annulus \( A \), (EC) fails in general. More precisely, denoting by \( \Gamma_2 \) and \( \Gamma_1 \) the outer and inner concentric circles bounding \( A \), one proves that (EC) cannot hold at least when the flow-rate, \( \Phi := \Phi_2 = -\Phi_1 \) through \( \Gamma_2 \) is strictly negative (inflow condition). A similar result remains valid also in the case when \( \Omega \) is a spherical shell, as stated in [15, p. 157] and clearly worked out in [3].

The counterexamples mentioned above require \( \Phi < 0 \). The case \( \Phi > 0 \) (outflow condition) presented, presumably, more difficulty and, as a result, the question of whether (EC) holds under the latter assumption on the flow-rate remained apparently open for several years. Quite recently, in [8], J.G. Heywood finally provided very interesting ideas on how to show the invalidity of (EC) also for the case \( \Phi > 0 \). This is achieved by using appropriate functions \( u \) in (EC), that he names “U-tube test functions.”

Objective of this note is to give a direct and elementary proof of the invalidity of (EC) when \( \Omega \) is an annulus (see Section 2) or a spherical shell (see Section 3), and \( \Phi > 0 \). Our proof uses Takeshita’s approach—which allows us to replace in (EC) the extension \( V_\varepsilon \) with its integral average over all possible rotations—in conjunction with an appropriate choice of the function \( u \).

It should be emphasized that once the result is established for these special domains, the invalidity of (EC) can be extended to more general domains, even multiply connected, whenever for each “interior” connected component \( \Gamma_i \) of \( \partial \Omega, i = 1, 2, \ldots, N - 1 \), there is a circumference (spherical surface) completely contained in \( \Omega \), and that surrounds only \( \Gamma_i \). Actually, combining the results of [8] with ours, no restrictions need to be imposed on the sign of \( \Phi_k \), provided, of course, (3) is satisfied. This generalization can be obtained by following exactly the same argument of [3, Corollary 1], and it is stated in Theorem 1 in Section 4.

We wish to end this introductory section with a final observation. A different way of proving the a priori estimate (4), again suggested by J. Leray [11, p. 28 and 38], is to use a contradiction argument. By this argument one shows that (4) is true (and so existence to (1)–(2) is proved under the general condition (3))

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6In this regard, see [8, Section 8].
provided the following requirements on the pair \((w, \pi)\) (in a suitable function class) are incompatible

\[
\begin{align*}
\begin{cases}
\mathbf{w} \cdot \nabla \mathbf{w} = \nabla \pi \\
\text{div} \mathbf{w} = 0
\end{cases}
\end{align*}
\] in \(\Omega\),

\[
\mathbf{w} = 0 \quad \text{on } \partial \Omega,
\]

\[-(\mathbf{w} \cdot \nabla V, \mathbf{w}) = \nu.\tag{7}\]

Here \((w, \pi)\) are limits (in appropriate topology) of certain normalized sequence \(s\) of solutions to (1)–(2), while \(V\) is a given extension of the boundary data \(v_\ast\).

It is then interesting to notice that if (7) has a solution, then (EC) cannot be true. In fact, writing

\[
V_\varepsilon = V + (V_\varepsilon - V),
\]

from (7) we find

\[
-(\mathbf{w} \cdot \nabla V_\varepsilon, \mathbf{w}) = \nu + (\mathbf{w} \cdot \nabla w, V_\varepsilon - V) = \nu + (\nabla \pi, V_\varepsilon - V) = \nu,
\]

where, in the last step, we have used that \(V_\varepsilon - V\) is solenoidal and vanishes at \(\partial \Omega\). Consequently, admitting (EC) would imply \(\nu \leq \varepsilon \|\nabla w\|_2^2\) for all \(\varepsilon > 0\), namely, \(\nu = 0\). These considerations suggest that the contradiction argument could be a weaker requirement than the validity of (EC), and that it might lead to the proof of the a priori estimate (4) under more general assumptions than those of symmetry requested by the use of (EC). This fact was already hinted by C.J. Amick [2], but only recently was it fully confirmed by M.V. Korobkov, K. Pileckas, and R. Russo [10] who showed that (7) are indeed incompatible when \(\Omega\) is a doubly-connected, two-dimensional (Lipschitz) domain, under the sole assumption that the flow-rate satisfies the inflow condition.

2 The Case \(\Omega\) an Annulus

We follow and specialize the approach of [15]. Let \(\Omega := \{x \in \mathbb{R}^2 : R_1 < |x| < R_2\}, \quad R_1 > 0, \quad \Gamma_i := \{x \in \mathbb{R}^2 : |x| = R_i\}, \quad i = 1, 2\). Moreover, set

\[
\Phi := \int_{\Gamma_2} v_\ast \cdot n = -\int_{\Gamma_1} v_\ast \cdot n,
\]

and assume \(\Phi > 0\). We want to show that the validity of (EC) then leads to a contradiction. For \(x \in \Omega\), we put

\[
y = \mathcal{R}_\varphi \cdot x
\]

with \(\mathcal{R}_\varphi \in \text{SO}(2)\) rotation matrix of angle \(\varphi \in [0, 2\pi]\), and define the average of \(V_\varepsilon\):

\[
\mathcal{A}(V_\varepsilon)(y) := \frac{1}{2\pi} \int_{0}^{2\pi} \mathcal{R}_\varphi \cdot V_\varepsilon(\mathcal{R}_\varphi^T \cdot y) \, d\varphi,
\]

where \(V_\varepsilon \in W^{1,2}(\Omega)\) is a solenoidal extension of \(v_\ast\) for which (EC) is supposed to hold, and \(^T\) denotes transpose. Taking into account the properties of \(V_\varepsilon\),
and the proper orthogonality of $\mathcal{R}_\phi$, one at once shows that
\[
\text{div} \mathcal{A}(\mathbf{V}_\varepsilon)(y) = 0 \quad y \in \Omega,
\]
\[
\int_{\Gamma_2} \mathcal{A}(\mathbf{V}_\varepsilon) \cdot n = -\int_{\Gamma_1} \mathcal{A}(\mathbf{V}_\varepsilon) \cdot n = \Phi. \tag{9}
\]
Furthermore, by construction, $\mathcal{A}(\mathbf{V}_\varepsilon)$ is invariant under rotation. Therefore, observing that, denoted by $(r, \theta)$ a system of polar coordinates with the origin at $x = 0$, the corresponding base vectors $\{e_r, e_\theta\}$ are both invariant, we infer
\[
\mathcal{A}(\mathbf{V}_\varepsilon) = v_1(r)e_r + v(r)e_\theta.
\]
However, $\mathcal{A}(\mathbf{V}_\varepsilon)$ must satisfy (9), so that we conclude
\[
\mathcal{A}(\mathbf{V}_\varepsilon) = \frac{1}{2\pi} \Phi e_r + v(r)e_\theta. \tag{10}
\]
It is now straightforward to prove that since $\mathbf{V}_\varepsilon$ satisfies (EC), also $\mathcal{A}(\mathbf{V}_\varepsilon)$ does. In fact, following [15], by Fubini theorem and (8), for all solenoidal $u \in W^{1,2}_0(\Omega)$ we have
\[
\int_{\Omega} u \cdot \nabla_y(\mathcal{A}(\mathbf{V}_\varepsilon)) \cdot u \, dy = \frac{1}{2\pi} \int_0^{2\pi} \left( \int_{\Omega} u \cdot \nabla_y(\mathcal{R}_\phi \cdot \mathbf{V}_\varepsilon(\mathcal{R}_\phi^\top \cdot y)) \cdot u \, dy \right) d\varphi
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \left( \int_{\Omega} (\mathcal{R}_\phi^\top \cdot u) \cdot \nabla_x \mathbf{V}_\varepsilon \cdot (\mathcal{R}_\phi^\top \cdot u) \, dx \right) d\varphi.
\]
Clearly, $\mathcal{R}_\phi^\top \cdot u \in W^{1,2}_0(\Omega)$ and is solenoidal, and $\|\nabla(\mathcal{R}_\phi^\top \cdot u)\|_2 = \|\nabla u\|_2$, so that from (EC) we deduce
\[
-\frac{1}{2\pi} \int_0^{2\pi} \left( \int_{\Omega} (\mathcal{R}_\phi^\top \cdot u) \cdot \nabla_x \mathbf{V}_\varepsilon \cdot (\mathcal{R}_\phi^\top \cdot u) \, dx \right) d\varphi \leq \frac{\varepsilon}{2\pi} \int_0^{2\pi} \|\nabla u\|_2^2 d\varphi = \varepsilon \|\nabla u\|_2^2.
\]
Combining the latter with (11) we thus obtain the desired inequality, namely,
\[
-\int_{\Omega} u \cdot \nabla(\mathcal{A}(\mathbf{V}_\varepsilon)) \cdot u \, dy \leq \varepsilon \|\nabla u\|_2^2, \tag{12}
\]
for all solenoidal $u \in W^{1,2}_0(\Omega)$. Denoting by $\mathcal{D}[\mathcal{A}(\mathbf{V}_\varepsilon)]$ the symmetric part of $\nabla \mathcal{A}(\mathbf{V}_\varepsilon)$, we show, on the one hand,
\[
\int_{\Omega} u \cdot \nabla(\mathcal{A}(\mathbf{V}_\varepsilon)) \cdot u \, dy = \int_{\Omega} u \cdot \mathcal{D}[\mathcal{A}(\mathbf{V}_\varepsilon)] \cdot u \, dy
\]
and, on the other hand, from (10),
\[
\mathcal{D}[\mathcal{A}(\mathbf{V}_\varepsilon)] = \begin{pmatrix}
-\frac{\Phi}{\pi^2} & v'(r) - \frac{1}{r} v(r) \\
v'(r) - \frac{1}{r} v(r) & \frac{\Phi}{\pi^2}
\end{pmatrix}.
\]
As a result, (12) becomes
\[
\frac{\Phi}{\pi} \int_{\Omega} \frac{u_r^2 - u_\theta^2}{r^2} + \int_{\Omega} (v'(r) - \frac{1}{r} v(r)) u_r u_\theta \leq \varepsilon \|\nabla u\|_2^2, \tag{13}
\]
for all solenoidal \( u \in W_0^{1,2}(\Omega) \), and where \( u_r \) and \( u_\theta \) denote the polar components of \( u \). We now choose \( u = (u_r, u_\theta) \) where
\[
u_r = \frac{m R_2 - R_1}{r} \left\{ \cos \left( \frac{2\pi(r - R_1)}{R_2 - R_1} \right) - 1 \right\} \cos(m\theta) := U(r)[m \cos(m\theta)]
\]
\[
u_\theta = \sin \left[ \frac{2\pi(r - R_1)}{R_2 - R_1} \right] \sin(m\theta) := W(r)[\sin(m\theta)],
\tag{14}
\]
with \( m \) an integer that will be specified further on. It is obvious that \( u \in W_0^{1,2}(\Omega) \), as well as, by taking into account that \( (rU)' = -W \), that \( \text{div} \ u = 0 \) in \( \Omega \). Moreover, by a direct computation we show (with \( \rho = R_2/R_1 \))
\[
\int_{\Omega} \frac{u_r^2 - u_\theta^2}{r^2} = m^2 F_1(\rho) - F_2(\rho),
\tag{15}
\]
where
\[
F_1(\rho) := \frac{(\rho - 1)^3}{4\pi} \int_0^1 \frac{1}{[(\rho - 1)z + 1]^3} \{\cos(2\pi z) - 1\}^2 dz,
\]
\[
F_2(\rho) := \pi(\rho - 1) \int_0^1 \frac{1}{(\rho - 1)z + 1} \sin^2(2\pi z) dz.
\]
Furthermore, setting \( G(r) := (v'(r) - \frac{1}{r} v(r))U(r)W(r) \), we get
\[
\int_{\Omega} (v'(r) - \frac{1}{r} v(r)) u_r u_\theta = m \int_{R_1}^{R_2} rG(r) \, dr \int_0^{2\pi} \sin(m\theta) \cos(m\theta) \, d\theta = 0. \tag{16}
\]
Thus, by fixing \( m \) sufficiently large so that \( \kappa := m^2 F_1(\rho) - F_2(\rho) > 0 \), from (13) and (16) we conclude
\[
\kappa \Phi \leq \varepsilon \|\nabla u\|_2^2,
\]
which, by the arbitrariness of \( \varepsilon > 0 \), and the assumption \( \Phi > 0 \) furnishes a contradiction. As originally showed by A. Takeshita [15], a similar result also holds if \( \Phi < 0 \). Actually, it is enough to choose in (13) instead of the field (14), the following one
\[
u_r \equiv 0, \quad u_\theta = f(r)
\]
with \( f(r) \) any sufficiently smooth function satisfying \( f(R_1) = f(R_2) = 0 \), in which case the left-hand side of (13) becomes \(-2\Phi \int_{R_1}^{R_2} rf^2(r) \, dr\).

### 3 The case \( \Omega \) a Spherical Shell

In this case \( \Omega := \{ x \in \mathbb{R}^3 : R_1 < |x| < R_2 \}, R_1 > 0 \), \( \Gamma_i := \{ x \in \mathbb{R}^3 : |x| = R_i \}, i = 1, 2, \)
\[
\Phi := \int_{\Gamma_2} v_s \cdot n = - \int_{\Gamma_1} v_s \cdot n,
\]
and assume $\Phi > 0$. Again following the strategy of [15], the proof, in its first part is basically the same as in the case of the two-dimensional annulus. The only change being that the generic rotation matrix is now an element $\mathbf{R}_{\alpha_1, \alpha_2, \alpha_3}$ of $SO(3)$ characterized by the Euler angles $\alpha_i$, $i = 1, 2, 3$. Consequently, (8) takes the form $y = \mathbf{R}_{\alpha_1, \alpha_2, \alpha_3} \cdot x$, and the average $A(V_\varepsilon)$ becomes

$$A(V_\varepsilon)(y) = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi} \mathbf{R}_{\alpha_1, \alpha_2, \alpha_3} \cdot V_\varepsilon(\mathbf{R}_\alpha^T \cdot y) \sin \alpha_3 \, d\alpha_1 d\alpha_2 d\alpha_3.$$ 

Again, by the properties of $V_\varepsilon$ and the proper orthogonality of the rotation we show that the average satisfies (9). Moreover, the invariance of $A(V_\varepsilon)$ under the action of $SO(3)$ along with (9) implies that

$$A(V_\varepsilon) = \Phi \frac{4}{r^2} e_r,$$

where $\{e_r, e_\chi, e_\theta\}$ is the base of a system of spherical coordinates $(r, \chi, \theta)$ with the origin at $x = 0$; see [15, p. 157] for details. Next, proceeding verbatim as in Section 2, we prove that $A(V_\varepsilon)$ must satisfy (12), which by taking into account that this time

$$\{D[A(V_\varepsilon)]\}_{ij} = \Phi \frac{4}{r^2} \left( -\frac{3}{2} u_i u_j - \delta_{ij} \right),$$

is equivalent to the following

$$\frac{\Phi}{4\pi} \int_\Omega \frac{2u_i^2 - u_\chi^2 - u_\theta^2}{r^3} \leq \varepsilon \|\nabla u\|^2_2,$$

(17)

for all solenoidal $u \in W^{1,2}_0(\Omega)$. In order to show that (17) leads to a contradiction, we choose

$$u_r = m \tilde{U}(r) \cos(m \theta) \sin \chi, \quad u_\chi \equiv 0, \quad u_\theta = \frac{1}{r} W(r) \sin(m \theta) \sin^2 \chi,$$

(18)

where $\tilde{U} = U/r$, and the functions $U$ and $W$ are defined in [15]. It is easily proved that the vector $u$ with components given in (18) is solenoidal and is in $W^{1,2}_0(\Omega)$, and therefore can be replaced in (17). Since by a direct calculation we show that

$$\int_\Omega \frac{2u_i^2 - u_\chi^2 - u_\theta^2}{r^3} = m^2 G_1(\rho) - G_2(\rho)$$

where $G_i$, $i = 1, 2$, are positive functions of $\rho = R_2/R_1$, taking $m$ sufficiently large and using the arbitrariness of $\varepsilon$, we show that (17) is incompatible with the assumption $\Phi > 0$. One can show the incompatibility of (17) also with the alternative assumption $\Phi < 0$, by using in (17) an appropriate function $u$ different from (18). This has been shown in [3] by the choice $u = f(r) \sin \chi e_\phi$, with $f$ sufficiently smooth and satisfying $f(R_1) = f(R_2) = 0$, in which case the left-hand side of (17) becomes $-\frac{\Phi}{r^2} \int_{R_2}^{R_1} r^2 f(r)dr$. 7
4 The case $\Omega$ Multiply-Connected

We now assume that $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a multiply-connected Lipschitz domain of the type defined in the Introduction. Furthermore, following [3], we suppose that for each connected component $\Gamma_i$ of $\partial \Omega$, $i = 1, 2, \ldots, N - 1$, there is a circumference (spherical surface) completely contained in $\Omega$, and surrounding only the component $\Gamma_i$.

Combining the results of the previous two sections with Remarks 1 and 2, and the argument of [3, Theorem 1 and Corollary 1], we can show the following general result, whose proof follows exactly the same lines of [3, Corollary 1], and, consequently, will be omitted.

**Theorem 4.1** Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain satisfying the assumptions mentioned above. Moreover, let $v_* \in W^{1/2, 2}(\partial \Omega)$ obey the compatibility condition (3). Then the Leray-Hopf Extension Condition holds for $v_*$ (if and only if $\Phi_k = 0$ for all $k = 1, 2, \ldots, N$).

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