LONG-TIME EXISTENCE FOR SEMILINEAR WAVE EQUATIONS WITH
THE INVERSE-SQUARE POTENTIAL

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Abstract. In this paper, we study the semilinear wave equations with the inverse-square potential. By transferring the original equation to a “fractional dimensional” wave equation and analyzing the properties of its fundamental solution, we establish a long-time existence result, for sufficiently small, spherically symmetric initial data. Together with the previously known blow-up result, we determine the critical exponent which divides the global existence and finite time blow-up. Moreover, the sharp lower bounds of the lifespan are obtained, except for certain borderline case. In addition, our technology allows us to handle an extreme case for the potential, which has hardly been discussed in literature.

1. Introduction

In this paper, we are interested in a kind of semilinear wave equations with the inverse-square potential and small, spherically symmetric initial data, which has the form

\begin{equation}
\begin{aligned}
\frac{\partial^2 U}{\partial t^2} - \Delta U + V r^{-2} U &= |U|^p, \\
U(0,x) &= \varepsilon U_0(r), \\
U_t(0,x) &= \varepsilon U_1(r);
\end{aligned}
\end{equation}

where \( p > 1, n \geq 2, 0 < \varepsilon \ll 1 \) and \( V \geq -(n-2)^2/4 \) is a constant. We will study the long-time existence and global solvability of (1.1). Specifically, setting \( T_\varepsilon \) to be the lifespan of the solution to (1.1), we want to know its relation with \( n, V, p \) and \( \varepsilon \).

When \( V = 0 \), this problem reduces to the well known Strauss conjecture, which has been extensively studied in a long history. See, e.g., [5], [17], [16], [20] and the references therein for more information. Let \( p_S(n) \) be the positive root of \( h_S(p;n) = 0 \), where

\[ h_S(p;n) := \frac{(n-1)p^2 - (n+1)p - 2}{2^{(p-1)/2(p-1)}}. \]

From the early researches, under some natural requirements of \((U_0, U_1)\), it is known that

\[
\begin{cases}
T_\varepsilon \approx \varepsilon^{\frac{2p(p-1)}{n-1}}, & \text{max}(1, \frac{2}{n-1}) < p < p_S; \\
\ln T_\varepsilon \approx -\varepsilon^{p(p-1)}, & p = p_S; \\
T_\varepsilon = \infty, & p > p_S.
\end{cases}
\]

Here and in what follows, we denote \( x \lesssim y \) and \( y \gtrsim x \) if \( x \leq Cy \) for some \( C > 0 \), independent of \( \varepsilon \), which may change from line to line. We also denote \( x \approx y \) if \( x \lesssim y \lesssim x \).

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When there exists a potential, i.e., \( V \neq 0 \), the problem becomes much more complicated. This is partly because that the inverse-square potential is in the same scaling as the wave operator, which means that it provides a comparable effect to the evolution of the solution. Meanwhile, the extra singularity at the origin also needs to be taken care of.

The elliptic operator \(-\Delta + V|x|^{-2}\) has been studied in several different equations related to physics and geometry, such as in heat equations (see, e.g., [18]), in quantum mechanics (see, e.g., [8]), in Schrödinger equations and wave equations. Among others, the Strichartz estimates for wave equations with the inverse square potential have been well-developed in many works. Such result was firstly developed in [14] for the wave equations with radial data. Shortly afterwards, the radial requirement was removed by [1]. A decade later, the Strichartz estimates with angular regularity were developed in [13]. Despite these results, we expect that these kind of estimates still have room to improve and generalize.

Turn back to the equation (1.1). Note that the initial data of (1.1) are spherically symmetric, which suggest that the solution \( U \) is also spherically symmetric. Let

\[
A := 2 + \sqrt{(n-2)^2 + 4V}, \quad u(t, r) := r^{\frac{n-A}{2}} U(t, x). 
\]

A formal calculation shows that \( u \) satisfies the equation

\[
\begin{cases}
\partial_t^2 u - \Delta_A u = \frac{r^{\frac{n-A}{2}}}{r^2} |u|^p, & (t, r) \in \mathbb{R}^2_+, \\
u(0, x) = \varepsilon r^{\frac{n-A}{2}} U_0(r), & u_t(0, x) = \varepsilon r^{\frac{n-A}{2}} U_1(r),
\end{cases}
\]

where \( \Delta_A := \partial_r^2 + (A - 1)r^{-1}\partial_r \). When \( A \in \mathbb{Z}_+ \), the operator \( \Delta_A \) agrees with the \( A \)-dimensional Laplace operator (for radial functions), from which we consider the parameter \( A \) as the spatial “dimension” for the equation after the transformation.

The blow-up result of (1.1) has been systematically considered in the previous paper [2] by the first author and his collaborators. Here we define

\[
h_F(p; n) := np - (n + 2)
\]

with \( p_F(n) \) be the root of \( h_F(p; n) = 0 \), and use abbreviations

\[
p_d = p_d(A) := \frac{2}{A - 1}, \quad p_F = p_F((n + A - 2)/2), \quad p_S = p_S(n),
\]

\[
h_S = h_S(p; n), \quad h_F = h_F(p; (n + A - 2)/2),
\]

if these do not lead to ambiguity. Then, under some requirements of initial data, there exists a constant \( C = C(p; n, A) \) such that when \( (3 - A)(A + n - 2) < 8 \), where \( p_d < p_F < p_S \), we have

\[
T\varepsilon \leq \begin{cases} C\varepsilon^{\frac{p-1}{p_F}}, & p \leq p_d; \\
C\varepsilon^{\frac{p(d-1)}{p_S}}, & p_d < p < p_S; \\
\exp(C\varepsilon^{-(p-1)}), & p = p_S.
\end{cases}
\]

When \( (3 - A)(A + n + 2) = 8 \), where \( p_d = p_F = p_S \), we have

\[
T\varepsilon \leq \begin{cases} C\varepsilon^{\frac{p-1}{p_F}}, & p < p_F; \\
\exp(C\varepsilon^{-(p-1)}), & p = p_F.
\end{cases}
\]
When $(3 - A)(A + n + 2) > 8$, where $p_m > p_F > p_S$, we have
\[
T_e \leq \begin{cases} 
C\varepsilon^{-\frac{n-1}{p}}, & p < p_F; \\
\exp(C\varepsilon^{-(p-1)}), & p = p_F.
\end{cases}
\]

This result suggests that two effects will impact the lifespan. For simplicity we call one Strauss effect and the other Fujita effect, since $p_S$ is the Strauss exponent and $p_F$ is the Fujita exponent. On the other hand, we remark that $p_F((n + A - 2)/2) = p_G((n + A)/2)$, where $p_G(n) = \frac{n+1}{n-1}$ is the Glassey exponent. The Glassey exponent appears in the wave equations with derivative nonlinearity $|\partial_t u|^p$, which suggests that there may exist some relation between the Glassey conjecture (see, e.g., [19]) and our problem.

For the existence part, there are also a few studies of (1.1). Using Strichartz estimates, the global existence result was shown in [14, 1] if
\[
p \geq \frac{n + 3}{n - 1}, \quad \frac{A - 2}{2} > \frac{n - 2}{2} - \frac{2}{p - 1} + \max \left\{ \frac{1}{2p}, \frac{1}{n+1(p-1)} \right\}.
\]

Later, the result was further extended in [13], where the global result in the radial case was obtained for $1 + \frac{4n}{(n+1)(n-1)} < p < \frac{n+3}{n-4}$,
\[
V > \max \left\{ \frac{1}{(n-1)^2} - \frac{(n-2)^2}{4}, \frac{n}{q_0(nq_0 - n + 2)} \cdot \frac{n - n - 2}{2(n - n)} \cdot \frac{n - n - 2}{2} \right\},
\]
\[
q_0 = \frac{(p-1)(n+1)}{2}, \quad r_0 = \frac{(n+1)(p-1)}{2p}.
\]

However, compared with the result of the problem without potential, in general, it seems that the sharp result for (1.1) could not be obtained by the Strichartz estimates without weight. On the other hand, there is also a gap between these results and the blow-up result we mentioned before.

Now, we are in a juncture to state our main results in this paper. Firstly, we give the definition of the solution, and see Section 2 for further discussions.

**Definition 1.** We call $U$ is a weak solution of (1.1) in $[0, T] \times \mathbb{R}^n$ if $U$ satisfies
\[
\int_0^T \int_{\mathbb{R}^n} |U|^p \Phi \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} U \left( \partial_t^2 - \Delta + \frac{V}{r^2} \right) \Phi \, dx \, dt - \varepsilon \int_{\mathbb{R}^n} (U_1 \Phi(0, x) - U_0 \partial_t \Phi(0, x)) \, dx,
\]
for any $\Phi(t, x) \in \left\{ r^{\frac{A-n}{2}} \varphi(t, x) : \varphi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n) \right\}$.

For convenience we introduce the notations
\[
p_m := \frac{n + 1}{n - 1}, \quad p_M := \begin{cases} 
\frac{n+1}{n-A} & n > A \\
\infty & n \leq A
\end{cases}, \quad p_L := \frac{n + A}{n - 1}, \quad p_{conf} := \frac{n + 3}{n - 1}.
\]

Then, we give the existence results for $A \in [2, 3]$.

**Theorem 1.1.** Set $2 \leq n, 2 \leq A \leq 3$ and $p_m < p < p_M$. Assume that the initial data satisfy
\[
\|r^{\frac{n-A+2}{3}} U_0'(r)\|_{L^{p_L}} + \|r^{\frac{n-A}{3}} U_0(r)\|_{L_0^{p_L}} + \|r^{\frac{n-A+2}{3}} U_1(r)\|_{L_0^{p_L}} < \infty,
\]
and supported in $[0,1)$, where $L^p_t$ stands for $L^p((0,\infty),dr)$. Then, there exists an $\varepsilon_0 > 0$ and a constant $c = c(p,n,A)$, such that for any $0 < \varepsilon < \varepsilon_0$, there is a weak solution $U$ of (1.1) in $[0,T_*] \times \mathbb{R}^n$ which satisfies

$$r \frac{n-A}{r} U \in L^\infty_{loc,x}([0,T_*) \times \mathbb{R}^n).$$

Where, when $(3 - A)(A + n + 2) < 8$, we have $p_d < p_F < p_S$, then

$$T_* = \begin{cases} 
\infty, & p > p_S; \\
\exp(\varepsilon p(1-p)), & p = p_S; \\
\exp(\varepsilon p(1-p)), & p = p_d; \\
\varepsilon^{\frac{p-1}{p-2}} r, & p < p_d; 
\end{cases}$$

(1.5)

When $(3 - A)(A + n + 2) = 8$, we have $p_d = p_F = p_S$, then

$$T_* = \begin{cases} 
\infty, & p > p_d; \\
\exp(\varepsilon p(1-p)), & p = p_d; \\
\varepsilon^{\frac{p-1}{p-2}} r, & p < p_d. 
\end{cases}$$

(1.6)

When $(3 - A)(A + n + 2) > 8$, we have $p_d > p_F > p_S$, then

$$T_* = \begin{cases} 
\infty, & p > p_S; \\
\exp(\varepsilon p(1-p)), & p = p_S; \\
\varepsilon^{\frac{p-1}{p-2}} r, & p < p_d. 
\end{cases}$$

(1.7)

Next, we give the existence results for $A \in [3, \infty)$.

**Theorem 1.2.** Set $2 \leq n$, $A \geq 3$ and $1 < p < p_{\text{con}}$ and define $T_*$ by

$$T_* = \begin{cases} 
\varepsilon^{\frac{p-1}{p-2}} r, & 1 < p < p_S; \\
\exp(\varepsilon p(1-p)), & p = p_S; \\
\infty, & p > p_S, 
\end{cases}$$

(1.8)

which is the same as (1.5) since that $p_d \leq 1$ when $A \geq 3$.

Assume that $1 < p \leq p_m$ and the initial data $(U_0, U_1)$ satisfy

$$\|r^{\frac{n-1}{2}} U_0(r)\|_{L^p_x} + \|r^{\frac{n-1}{2}} U_1(r)\|_{L^p_x} < \infty,$$

and supported in $[0,1)$. Then there exists an $\varepsilon_0 > 0$ and a constant $c$, such that for any $\varepsilon < \varepsilon_0$, (1.1) has a weak solution in $[0,T_*] \times \mathbb{R}^n$ verifying

$$\|(1 + t)^{-\frac{(n-1)p-n-1}{2p}} r^{\frac{n+1}{2}} U\|_{L^p_t L^p_x([0,T_*] \times \mathbb{R}^n)} < \infty,$$

with $T_*$ defined in (1.8).

Assume that $p_m \leq p < p_S$, the initial data satisfy (1.9) and

$$\|r^{\frac{n+1}{2}} U_0(r)\|_{L^\infty_x} + \|r^{\frac{n+1}{2}} U_1(r)\|_{L^\infty_x} < \infty,$$

with no compact support requirement. Then there exists an $\varepsilon_0 > 0$ and a constant $c$, such that for any $\varepsilon < \varepsilon_0$, (1.1) has a weak solution in $[0,T_*] \times \mathbb{R}^n$ verifying

$$\|t^{-\frac{(n-1)p-n-1}{2p}} r^{\frac{n+1}{2}} U\|_{L^p_t L^p_x([0,T_*] \times \mathbb{R}^n)} < \infty,$$

with $T_*$ defined in (1.8).
Assume that \( p = p_S \), and the initial data satisfy (1.9) and (1.10) for \( p = p_S \) as well as some \( p > p_S \). Then there exists an \( \varepsilon_0 > 0 \) and a constant \( c \), such that for any \( \varepsilon < \varepsilon_0 \), (1.1) has a weak solution in \([0, T_\ast] \times \mathbb{R}^n\) verifying

\[
\| r^{n+1} U \|_{L^2_{r,s} L^2_{t}([0,1] \times \mathbb{R}_+)} + \| r^{n+1} \frac{1}{r} U \|_{L^2_{r,s} L^2_{t}([1, T_\ast] \times \mathbb{R}_+)} < \infty,
\]

with \( T_\ast \) defined in (1.8).

Assume that \( p > p_S \) and the initial data satisfy

\[
\| r^{n-1} U_0(r) \|_{L^q} + \| r^{n+1} U_1(r) \|_{L^q} < \infty, \quad q := \frac{2(p-1)}{(n+3)-(n-1)p}.
\]

Then there exists an \( \varepsilon_0 > 0 \), such that for any \( \varepsilon < \varepsilon_0 \), (1.1) has a weak solution in \( \mathbb{R}_+ \times \mathbb{R}^n \) verifying

\[
\| r^{n+1} U \|_{L^2_{r} L^2_{t}([0,1] \times \mathbb{R}_+)} + \| r^{n+1} \frac{1}{r} U \|_{L^2_{r} L^2_{t}([1, T_\ast] \times \mathbb{R}_+)} < \infty.
\]

Remark 1.1. Here we use a graph with \( n \in [4, 8] \) as an example to describe the results we got.

The white area stands for the region that the solution is global, the light gray area \( (p > p_d) \) stands for the region that the Strauss effect plays role, the dark gray area \( (p < p_r) \) stands for the region that Fujita effect plays role, and the chessboard area stands for the region that we can not deal with due to the technical difficulty. When \( n \in [2, 3] \), we find \( \frac{3n-1}{n+1} \leq 2 \), which means that the dark gray area does not exist for \( p \geq p_m = \frac{n+1}{n+3} \). When \( n = 2 \) we have \( p_M = \infty \) for all \( A \geq 2 \). This means that the lower right chessboard area does not exist. When \( n \in [9, \infty) \), we find \( 1 + \frac{4}{n} > \frac{n+1}{n-2} \), which means that the dark gray area will be slightly blocked by the lower right chessboard area. Here we list these situations as the figures below.
Remark 1.2. The nonlinear term $|U|^p$ in (1.1) can be replaced by any $F_p(U)$ which satisfies

$$|F_p(U)| \lesssim |U|^p, \quad |F_p(U_1) - F_p(U_2)| \lesssim |U_1 - U_2| \max(|U_1|, |U_2|)^{p-1},$$

and typical examples include $F_p(U) = \pm |U|^p$ and $F_p(U) = \pm |U|^{p-1}U$. The only difference is that the constants in the result and proof need to be changed.

In lower dimension, the weighted $L^\infty$ norm estimate, which firstly appeared in [7], is very useful to prove the long-time existence result. In [9], the authors showed long-time existence results for a two-dimensional wave system, where they use a trick that they take different weights in different zones. In this paper, we further develop such method, adapted for the wave equations with potential, and finally show the long-time existence result for $A \in [2, 3]$. On the one hand, our result is sharp in general, in the sense that our lower bound of the lifespan has the same order as the upper bound estimates as appeared in the blow-up results, except for the borderline case $p = p_d$. On the other hand, we notice that $A = 2$, which means $V = -(n-2)^2/4$, is an extreme case to the operator $-\Delta + Vr^{-2}$. In this case, the operator is still non-negative but not positive any more, which makes the implementation of the classical energy method more difficult. However, our approach could handle this extreme case as well as the usual case that $V > -(n-2)^2/4$, without any additional difficulties.

In higher dimension ($n \geq 3$), it is well known that the weighted Strichartz estimates is a helpful tool for the Strauss conjecture, particularly for the high dimensional case (see, e.g., [11], [5], and [17]). In this paper, we adapt the approach of [11] to the fractional dimension $\alpha \geq 3$, and give the long-time existence result for $A \in [3, \infty)$, which gives the sharp lower bound of the lifespan.

After comparing all the results we knew, we find that the determination of the exact lifespan can be considered by a competition between the Strauss effect and the Fujita effect. When $p > p_d$, the Strauss effect is stronger, where the final result is only determined by the Strauss exponent. When $p < p_d$, the Fujita effect is stronger, the final result is only determined by the Fujita exponent.

Compared with the results for the problem without potential, it seems that the requirement $p > p_m$ in Theorem 1.1 is only a technical restriction. Also, compared with the result of 2-dimensional Strauss conjecture with $p = 2$ (though $2 \leq p_m(2) = 3$), we expect that the both the lower bound and the upper bound of lifespan for $p = p_d$ can be further improved.

On the other hand, it will be interesting to investigate the problem with non-radial data, as well as the problem with more general potential functions. It is known that when $n = 3, 4$ and the potential function is of short range, the similar long-time existence results (including non-radial case) as in Theorem 1.2 are available in [12]. When the potential function has asymptotic behavior $Vr^{-2}$ with $V > 0$, the subcritical blow-up result ($p < \max(p_S, p_F)$) was recently obtained in [10]. The existence theory for the corresponding problem remains largely open.

Another interesting problem is whether or not the Cauchy problem (1.1) admits global solutions for initial data with lowest possible regularity. For example, when $V = 0$ and $2 \leq n \leq 4$, such a result is available for $p \in (p_S, p_{conf})$ with small, spherically symmetric data in the scale-invariant space $H^{s_c} \times H^{s_c-1}$ space ($s_c = n/2 - 2/(p-1)$). See [15, 6, 4] for more discussion. Here we should remark that the global result in Theorem 1.2 reaches the lowest regularity requirement (in the sense of scale invariance, though not in $H^s$ space), but Theorem 1.1 still has room for improvement.
The rest of the paper is organized as follows. In Section 2 we give a detailed discussion of (1.2) and its solution. In Section 3, we restrict $A \in [2, 3]$ and show the long-time existence of the solution by weighted $L^\infty$ norm estimate. In Section 4, we move to situation $A \in [3, \infty)$ and establish the long-time existence result through weighted Strichartz type estimate.

2. Some preparations

In this section, we transfer (1.1) into the equivalent equation (1.2), and explain the rationality of Definition 1. After that, we show the formula of the solution and analyze the properties of this solution.

2.1. The definition of weak solution. As we said before, after introducing $u(t, r) := r^{n-A}U(t, x)$, a formal calculation shows that $u$ satisfied the equation (1.2). We pause here and consider its linear form equation

\begin{equation}
\begin{cases}
\partial_t^2 u - \Delta_A u = F(t, r), & r \in \mathbb{R}_+ \\
u(0, r) = f(r), & u_t(0, r) = g(r),
\end{cases}
\end{equation}

with $f$, $g$, $F$ good enough. When $A \in \mathbb{Z}_+$, equation (2.1) can be considered as an $A$-dimensional spherically symmetric wave equation, where $u(t, r)$ is a classical solution in $[0, T]$ if $u(t, r)$ satisfies (2.1) and $u(t, |x|) \in C^2([0, T] \times \mathbb{R}^A)$. Thus, for general situation, we should say $u$ is a classical solution of (2.1) in $[0, T]$ if $u \in C^2([0, T] \times \mathbb{R}_+)$, $\partial_t u(t, 0) = 0$ and $u$ satisfies (2.1) point wise.

Here we give a quick proof to show that such classical solution is unique. When $f, g, F = 0$, multiplying $r^{A-1}u_t$ to both sides of (2.1) and integrating them in $\Omega := \{(t, r) : 0 < t < T; 0 < r < R + T - t\}$, we see

$$0 = \frac{1}{2} \int_0^R r^{A-1} (u_t^2 + u_r^2) \, dr \left|_{t=T} \right. + \frac{1}{2\sqrt{2}} \int_{t+R}^{t+r+T + R} \int_{t < s < T} r^{A-1}(u_t - u_r)^2 \, d\sigma_{t,r}.$$ 

This gives the uniqueness. After the discussion of the classical solution, we naturally say $u$ is the weak solution of (2.1) if $u$ satisfies

\begin{equation}
\int_0^T \int_0^\infty F \varphi r^{A-1} \, dr \, dt = \int_0^T \int_0^\infty u(\partial_t^2 - \Delta_A) \varphi r^{A-1} \, dr \, dt - \int_0^\infty (g\varphi(0, r) - f\varphi(0, r)) r^{A-1} \, dr,
\end{equation}

for any $\varphi(t, r) \in C_0^\infty((-\infty, T) \times \mathbb{R})$ with $\partial_t^{1+2k}\varphi(t, 0) = 0$ for any $k \in \mathbb{N}_0$. Also, set $u = r^{\frac{n-A}{2}}U$, $\varphi = r^{\frac{n-A}{2}}\Phi$, $f = \varepsilon r^{\frac{n-A}{2}}U_0$, $g = \varepsilon r^{\frac{n-A}{2}}U_1$ and $F = r^{\frac{(A-n)(2n-A)}{2}+\frac{A}{2}}|u|^p$, it is obvious that $u$ satisfies (2.2) is equivalent to that $U$ satisfies (1.3) with $\Phi(t, x) \in \left\{ r^{\frac{n-A}{2}}\varphi(t, x) : \varphi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n) \right\}$. That’s the reason we use Definition 1 as the definition of weak solution of (1.1).

2.2. The formula of classical solution. In this section we are going to give the formula of solution to (2.1). We denote by $u_g$, $u_f$ and $u_F$ the solution of (2.1) with only $g \neq 0$, $f \neq 0$ and $F \neq 0$, respectively.
Lemma 2.1. Assume that $f, g, F$ are smooth enough. Then, the classical solution of (2.1) is $u = u_f + u_g + u_F$ with

$$u_g = r^{\frac{1-A}{2}} \int_0^{t+r} \rho^{\frac{d-1}{2}} g(\rho) I_A(\mu) \, d\rho, \quad \mu = \frac{r^2 + \rho^2 - t^2}{2r\rho},$$

$$u_f = r^{\frac{1-A}{2}} \partial_t \int_0^{t+r} \rho^{\frac{d-1}{2}} g(\rho) I_A(\mu) \, d\rho, \quad \mu = \frac{r^2 + \rho^2 - t^2}{2r\rho},$$

$$u_F = r^{\frac{1-A}{2}} \int_0^{t} \rho^{\frac{d-1}{2}} F(s, \rho) I_A(\mu) \, d\rho \, ds, \quad \mu = \frac{r^2 + \rho^2 - (t-s)^2}{2r\rho},$$

$$I_A(\mu) := \frac{2^{\frac{1-A}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \int_{-1}^{1} \chi_+^{\frac{1-A}{2}} (\lambda - \mu) \sqrt{1 - \lambda^2} \, d\lambda.$$

Remark 2.1. Here $\chi_+^\alpha$ is a distribution, which has the expression

$$\chi_+^\alpha(x) = \begin{cases} 0, & x < 0, \alpha > -1, \\ \frac{\alpha}{\Gamma(\alpha + 1)} x^\alpha, & x > 0, \alpha > -1, \\ \frac{d}{dx} \chi_+^{\alpha+1}(x), & \alpha \leq -1, \end{cases}$$

with $\Gamma$ the Gamma function and $\frac{d}{dx}$ the weak derivative.

Remark 2.2. Consider $\mu = \frac{r^2 + \rho^2 - t^2}{2r\rho}$. When $r > t$, we have

$$\mu|_{0 \leq \rho < t-r} > \mu|_{\rho = t-r} = 1 > \mu|_{t-r < \rho < t+r} > 0 < \mu|_{\rho = t+r} = 1 < \mu|_{t+r < \rho},$$

and when $r < t$, we have

$$\mu|_{0 \leq \rho < t-r} < \mu|_{\rho = t-r} = -1 < \mu|_{t-r < \rho < t+r} < \mu|_{\rho = t+r} = 1 < \mu|_{t+r < \rho}.$$

By Lemma 2.2 below, the $\int_0^{t+r}$ in the formula of $u_g$ and $u_f$ in Lemma 2.1 can be replaced by $\int_{\max(0,t-r)}^{t+r}$, and $\int_0^{t-s+r}$ in $u_F$ can be replaced by $\int_{\max(0,r+s-t)}^{t+s-r}$.

To show Lemma 2.1, we need to explore some properties of $I_A(\mu)$.

Lemma 2.2. For $A > 1$ and $I_A(\mu)$ defined in Lemma 2.1, we have

$$I_A(1-) = \frac{1}{2}, \quad I_A(\mu) = 0 \text{ for } \mu > 1.$$  

Moreover, for $A \notin 1 + 2\mathbb{Z}_+$ with some constants $C_0$ and $C_1$ depending on $A$, we have

$$|\partial^m\mu I_A(\mu)| \lesssim (1 - \mu)^{-\frac{d}{2} - m}, \quad \mu \leq -2, m = 0, 1;$$

$$I_A(\mu) = C_0 \ln |1 + \mu| + O(1), \quad -2 < \mu < 1;$$

$$\partial^m\mu I_A(\mu) = C_1 (1 + \mu)^{-1} + O(|\ln |1 + \mu|| + 1), \quad -2 < \mu < 1.$$

On the other hand, for $A \in 1 + 2\mathbb{Z}_+$, we have

$$I_A(\mu) = 0, \quad \mu < -1;$$

$$|\partial^m\mu I_A(\mu)| \lesssim 1, \quad -1 < \mu < 1, \quad m = 0, 1.$$
2.3. Proof of Lemma 2.1. Here we only show the proof of $u = u_g$ with $f = F = 0$, the other formulas can be demonstrated by a direct calculation and Duhamel’s principle. Without loss of generality we only deal with the case $A \notin \mathbb{Z}_+$. 

Part 1: Alternative expressions of $u_g$. 

Before the proof, we give an alternative expressions of $u_g$ constructed in Lemma 2.1. We first introduce a change of the variables

$$(\rho, \lambda) = \left(\sqrt{r^2 + \rho^2 - 2r\rho\lambda}, \frac{r - \rho\lambda}{\sqrt{r^2 + \rho^2 - 2r\rho\lambda}}\right)$$

and

$$\leftrightarrow (\tilde{\rho}, \tilde{\lambda}) = \left(\sqrt{r^2 + \rho^2 - 2r\rho\lambda}, \frac{r - \rho\lambda}{\sqrt{r^2 + \rho^2 - 2r\rho\lambda}}\right).$$

A direct calculation shows that the map $(\rho, \lambda) \mapsto (\tilde{\rho}, \tilde{\lambda})$ satisfies the relation

$$\frac{d(\rho, \lambda)}{d(\tilde{\rho}, \tilde{\lambda})} = \frac{\tilde{\rho}^2}{\rho^2 + \tilde{\rho}^2 - 2r\tilde{\rho}\tilde{\lambda}} = \frac{\tilde{\rho}^2}{\rho^2}, \quad \rho^2(1 - \lambda^2) = \tilde{\rho}^2(1 - \tilde{\lambda}^2),$$

and is a bijection from $(0, \infty) \times (-1, 1)$ to itself. For $A = 1 + 2k + 2\theta$ with $k \in \mathbb{N}_0$ and $\theta \in (0, 1)$, we substitute $I_A(\mu)$ into $u_g$. Noticing $X_+^\alpha$ is a homogeneous distribution of degree $\alpha$, we find

$$u_g = \frac{1}{\Gamma(k + \theta)} \int_0^\infty \int_{-1}^1 g(\rho) X_{+}^{k+\alpha} \left(t^2 - r^2 + 2r\rho\lambda\right) \rho^{A-1} \sqrt{1 - \lambda^2} \, d\lambda \, d\rho.$$ 

Set $\tilde{\rho} = t\sigma$ and $\tilde{\lambda} = \lambda$. Considering the definition of $X_{+}^{k-\theta}$ we finally reach

$$u_g = \frac{1}{\Gamma(k + \theta)} \left(\frac{\partial_k}{2t}\right)^k \left(t^{1+2k} \int_{-1}^1 \int_0^\infty g\left(\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}\right) (1 - \sigma^2)^\theta \, d\lambda \, d\sigma\right).$$

(2.9)

Part 2: Differentiability, boundary requirement and initial requirement.

Now we begin the proof. Firstly, using the expression we just obtained, we can easily check that $u \in C^2(\mathbb{R}^+_1)$ while $g \in C_0^\infty((0, \infty))$. We can also calculate that

$$\partial_{t\sigma}u = \frac{1}{\Gamma(k + \theta)} \left(\frac{\partial_k}{2t}\right)^k \left(t^{1+2k} \int_{-1}^1 \int_0^\infty \frac{r - t\sigma\lambda}{\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}} \, g'(\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}) (1 - \sigma^2)^\theta \, d\lambda \, d\sigma\right).$$
Let \( r = 0 \). Since the integrand is an odd function of \( \lambda \), such \( u \) satisfies the boundary requirement.

To check the initial conditions we temporarily use the original expression in Lemma 2.1. Using Lemma 2.2 we know \( I_A(\mu) = 0 \) when \( \mu > 1 \), which happens when \( \rho < r-t \) with \( r > t \). Then for any \( r > t > 0 \), we have

\[
\frac{u(t,r)}{r} = \frac{t^{r-A}}{t} \int_{r-t}^{r+t} \rho \frac{\Lambda_1}{t} g(\rho) I_A(\mu) d\rho
\]

\[
u(t,r) = \frac{t^{r-A}}{t} \left( (t+r) \frac{\Lambda_1}{t} g(t+r) + (r-t) \frac{\Lambda_1}{t} g(r-t) \right) I_A(1-)
\]

\[+ \frac{r^{r-A}}{r} \int_{r-t}^{r+t} \rho \frac{\Lambda_1}{r} g(\rho) \frac{-t}{\rho} I_A(\mu) d\rho.\]

Let \( t \to 0 \). Using Lemma 2.2 again we find \( u(0,r) = 0 \) and \( u_t(0,r) = g(r) \).

**Part 3:** Differential equation requirement.

Finally, we need to check that \( u \) satisfies (2.1). By a calculation trick

\[
\partial_t^2 \left( \frac{\partial_t}{T} \right)^k t^{1+2k} = \left( \frac{\partial_t}{T} \right)^{k+1} t^{2k+2} \partial_t,
\]

(see e.g. [3, Lemma 2 in Section 2.4]) we calculate that

\[
\partial_t^2 u = \frac{2}{\Gamma(k + \theta) \Gamma(1 - \theta)} \left( \frac{\partial_t}{2t} \right)^{k+1} (t^{2k+2} w_1),
\]

\[
w_1 := \int_0^1 \int_{-1}^1 \frac{t^2 \sigma^2 - r \sigma \lambda}{\sqrt{t^2 + t^2 \sigma^2 - 2rt \sigma \lambda}} g'(\sqrt{t^2 + t^2 \sigma^2 - 2rt \sigma \lambda}) \sigma^{A-1} \sqrt{1 - \lambda^2} d \lambda d \sigma.
\]

On the other hand, a similar process as that deduced (2.9) also shows

\[
u = \frac{1}{\Gamma(k + \theta) \Gamma(2 - \theta)} \left( \frac{\partial_t}{2t} \right)^{k+1} (t^{2k+2} \tilde{w}_2),
\]

\[
\tilde{w}_2 := \int_0^1 \int_{-1}^1 \frac{g(\sqrt{t^2 + t^2 \sigma^2 - 2rt \sigma \lambda})}{(1 - \sigma^2)^{\beta-1}} \sigma^{A-1} \sqrt{1 - \lambda^2} d \lambda d \sigma.
\]

Then, we see

\[
\partial_t \tilde{w}_2 = t \int_0^1 \int_{-1}^1 \frac{r - t \sigma \lambda}{\sqrt{t^2 + t^2 \sigma^2 - 2rt \sigma \lambda}} g'(\sqrt{t^2 + t^2 \sigma^2 - 2rt \sigma \lambda}) \sigma^{A-1} \sqrt{1 - \lambda^2} \sigma^{A-3} \sqrt{1 - \lambda^2} d \lambda d \sigma
\]

\[= - \int_0^1 \int_{-1}^1 g(\sqrt{t^2 + t^2 \sigma^2 - 2rt \sigma \lambda}) \sigma^{A-1} \sqrt{1 - \lambda^2} \sigma^{A-3} \sqrt{1 - \lambda^2} d \lambda d \sigma
\]

\[- \int_0^1 \int_{-1}^1 g(\sqrt{t^2 + t^2 \sigma^2 - 2rt \sigma \lambda}) \sigma^{A-1} \sqrt{1 - \lambda^2} \sigma^{A-3} \sqrt{1 - \lambda^2} d \lambda d \sigma.
\]

Using integration by parts, we get

\[
\partial_t \tilde{w}_2 = -2(1 - \theta) \int_0^1 \int_{-1}^1 \lambda g(\sqrt{t^2 + t^2 \sigma^2 - 2rt \sigma \lambda}) \sigma^{A} \sqrt{1 - \lambda^2} \sigma^{A-3} \sqrt{1 - \lambda^2} d \lambda d \sigma.
\]
Thus we have
\[ \partial_r u = \frac{2}{\Gamma(k + \theta) \Gamma(1 - \theta)} \left( \frac{\partial}{\partial t} \right)^{k+1} \left( t^{2k+2} w_2 \right) \]
\[ w_2 := - \int_0^1 \int_{-1}^1 \lambda g(\sqrt{r^2 + t^2 \sigma^2} - 2rt\sigma\lambda) \sigma^A \sqrt{1 - \lambda^2 A^3} \, d\lambda \, d\sigma. \]

Taking the derivative again, we also have
\[ \partial_t^2 u = \frac{2}{\Gamma(k + \theta) \Gamma(1 - \theta)} \left( \frac{\partial}{\partial t} \right)^{k+1} \left( t^{2k+2} w_3 \right) \]
\[ w_3 := \int_0^1 \int_{-1}^1 \frac{t \sigma^2 \lambda^2 - r \sigma \lambda \cdot g'(\sqrt{r^2 + t^2 \sigma^2} - 2rt\sigma\lambda)}{(1 - \sigma^2)^\theta} \sigma^{A-1} \sqrt{1 - \lambda^2 A^3} \, d\lambda \, d\sigma. \]

Gluing \( \partial_t^2 u, \partial_r^2 u \) and \( \partial_r u \) together, we finally calculate
\[ (\partial_t^2 - \partial_r^2 - (A - 1)r^{-1}\partial_r) u = \frac{2r^{\theta-1}}{\Gamma(k + \theta) \Gamma(1 - \theta)} \left( \frac{\partial}{\partial t} \right)^{k+1} \left( t^{2k+2}(rw_1 - rw_3 - (A - 1)w_2) \right) \]
where
\[ rw_1 - rw_3 = - \int_0^1 \int_{-1}^1 \partial_\lambda g(\sqrt{r^2 + t^2 \sigma^2} - 2rt\sigma\lambda) \sigma^A \sqrt{1 - \lambda^2 A^3} \, d\lambda \, d\sigma \]
\[ = (A - 1)w_2. \]

This finishes the proof.

2.4. **Proof of Lemma 2.2.** We begin with the second half of (2.3), it is trivial since \( \lambda^{\frac{1-\theta}{2}} (\lambda - \mu) = 0 \) for \( \lambda \in [-1, 1] \) and \( \mu > 1 \). As for other results, we need to divide \( A \) into two cases.

**Part 1:** \( A \) is not odd.

We begin with the case that \( A \) is not odd, i.e. \( A = 1 + 2k + 2\theta \) with \( k \in \mathbb{N}_0 \) and \( 0 < \theta < 1 \). By definition we see that when \( \mu \leq -2 \) and \( \lambda \in [-1, 1] \), we have
\[ \partial_\mu^m \lambda^{\frac{1-A}{2}} (\lambda - \mu) \approx (1 - \mu)^{\frac{1-A}{2} - m}, \]
which gives (2.4). When \(-1 < \mu < 1\), \( I_\lambda \) has the formula
\[ I_\lambda(\mu) = \frac{2^{-k-\theta}}{\Gamma(k + \theta)} \int_{-1}^1 \partial_\lambda^k \lambda^{\theta}(\lambda - \mu) (1 - \lambda^2)^{k+\theta-1} \, d\lambda \]
\[ = \frac{2^{-k-\theta}}{\Gamma(k + \theta)} \int_{-1}^1 \lambda^{\theta}(\lambda - \mu) (-\partial_\lambda)^k (1 - \lambda^2)^{k+\theta-1} \, d\lambda \]
\[ = \frac{2^{-k-\theta}}{\Gamma(k + \theta) \Gamma(1 - \theta)} \int_{\mu}^1 (\lambda - \mu)^{-\theta} \sum_{j=0}^{[k/2]} C_{j,k,\theta} \lambda^{k-2j} (1 - \lambda^2)^{j+\theta-1} \, d\lambda \]

with some constants \( C_{j,k,\theta} \). Here \([a]\) stands for the integer part of \( a \).

**Part 1.1:** \( \mu \) close to \( 1^- \).
Firstly we let $\mu$ close to $1^-$. Introducing $\lambda = (1 - \mu) \sigma + \mu$, we have

$$
\int_{\mu}^{1} (\lambda - \mu)^{-\theta} \lambda^{k-2j} (1 - \lambda^2)^{j+\theta-1} \, d\lambda
$$

$$
= (1 - \mu)^j \int_{0}^{1} \sigma^{-\theta} (1 - \sigma)^{j+\theta-1} (\sigma + \mu - \mu \sigma)^{k-2j} (\sigma + \mu - \mu \sigma + 1)^{j+\theta-1} \, d\sigma.
$$

Let $\mu \to 1^-$. Using dominated convergence theorem, we find the limit is nonzero only if $j = 0$, where

$$
\lim_{\mu \to 1^-} \int_{0}^{1} \sigma^{-\theta} (1 - \sigma)^{\theta-1} (\sigma + \mu - \mu \sigma)^{k} (\sigma + \mu - \mu \sigma + 1)^{\theta-1} \, d\sigma
$$

$$
= 2^{\theta-1} \int_{0}^{1} \sigma^{-\theta} (1 - \sigma)^{\theta-1} \, d\sigma.
$$

Now, we calculate

$$
C_{0,k,\theta} = \begin{cases} 
1, & k = 0; \\
2^{k}(k+\theta-1)(k+\theta-2)\cdots, & k > 0,
\end{cases}
$$

then

$$
\lim_{\mu \to 1^-} I_{\lambda}(\mu) = \frac{1}{2} \frac{1}{\Gamma(\theta)\Gamma(1-\theta)} \int_{0}^{1} \sigma^{-\theta} (1 - \sigma)^{\theta-1} \, d\sigma = \frac{1}{2}.
$$

This finishes the first half of (2.3) for non odd $A$.

For derivative, we calculate

$$
\partial_{\mu} \left( (1 - \mu)^j \int_{0}^{1} \sigma^{-\theta} (1 - \sigma)^{j+\theta-1} (\sigma + \mu - \mu \sigma)^{k-2j} (\sigma + \mu - \mu \sigma + 1)^{j+\theta-1} \, d\sigma \right)
$$

$$
= j(1 - \mu)^{j-1} \int_{0}^{1} \sigma^{-\theta} (1 - \sigma)^{j+\theta-1} (\sigma + \mu - \mu \sigma)^{k-2j} (\sigma + \mu - \mu \sigma + 1)^{j+\theta-1} \, d\sigma
$$

$$
+ (k - 2j)(1 - \mu)^{j} \int_{0}^{1} \sigma^{-\theta} (1 - \sigma)^{j+\theta} (\sigma + \mu - \mu \sigma)^{k-2j-1} (\sigma + \mu - \mu \sigma + 1)^{j+\theta-1} \, d\sigma
$$

$$
+ (j + \theta - 1) \int_{0}^{1} \sigma^{-\theta} (1 - \sigma)^{j+\theta} (\sigma + \mu - \mu \sigma)^{k-2j} (\sigma + \mu - \mu \sigma + 1)^{j+\theta-2} \, d\sigma,
$$

with no singularity in all these integrals. This means $\partial_{\mu} I_{\lambda}(\mu) = O(1)$ for $\mu$ close to $1^-$, which corroborates with (2.6).

**Part 1.2:** $\mu$ close to $-1^+$.

Then we let $\mu$ close to $-1^+$, without loss of generality we assume $\mu < -1/2$, then

$$
\int_{\mu}^{1} (\lambda - \mu)^{-\theta} \lambda^{k-2j} (1 - \lambda^2)^{j+\theta-1} \, d\lambda
$$

$$
= \int_{\mu}^{0} (\lambda - \mu)^{-\theta} \lambda^{k-2j} (1 - \lambda^2)^{j+\theta-1} \, d\lambda + \int_{0}^{1} (\lambda - \mu)^{-\theta} \lambda^{k-2j} (1 - \lambda^2)^{j+\theta-1} \, d\lambda
$$

$$
= \int_{\mu}^{0} (\lambda - \mu)^{-\theta} (1 + \lambda)^{j+\theta-1} h(\lambda) \, d\lambda + O(1),
$$

where $h(\lambda)$ is bounded for $\lambda < 0$. Thus we have

$$
\int_{\mu}^{1} (\lambda - \mu)^{-\theta} \lambda^{k-2j} (1 - \lambda^2)^{j+\theta-1} \, d\lambda
$$

$$
= \int_{\mu}^{0} (\lambda - \mu)^{-\theta} (1 + \lambda)^{j+\theta-1} h(\lambda) \, d\lambda + O(1),
$$

with no singularity in all these integrals. This means $\partial_{\mu} I_{\lambda}(\mu) = O(1)$ for $\mu$ close to $-1^-$, which corroborates with (2.6).
where \( h(\lambda) := \lambda^{k-2j}(1-\lambda^{j+\theta-1}) \) satisfying \( h(\lambda) \in C^\infty([-1,0]) \). For the first integral, we split it to

\[
\int_\mu^0 (\lambda - \mu)^{-\theta}(1 + \lambda)^{j+\theta-1}(h(\lambda) - h(-1)) \, d\lambda + h(-1) \int_\mu^0 (1 + \lambda)^{j-1} \, d\lambda \\
+ h(-1) \left( \int_\mu^{1+2\mu} + \int_1^{1+2\mu} \right) ((\lambda - \mu)^{-\theta} - (1 + \lambda)^{-\theta}) (1 + \lambda)^{j+\theta-1} \, d\lambda \\
= J_1 + J_2 + J_3 + J_4.
\]

Using the mean value theorem, it is easy to find that

\[
|J_1| \leq \int_\mu^0 (\lambda - \mu)^{-\theta}(1 + \lambda)^{j+\theta} \, d\lambda \lesssim 1, \\
J_2 = C_j \ln(1 + \mu) + O(1), \\
|J_3| \lesssim (1 + \mu)^{j+\theta-1} \int_\mu^{1+2\mu} (\lambda - \mu)^{-\theta} - (1 + \lambda)^{-\theta} \, d\lambda \lesssim (1 + \mu)^j \lesssim 1, \\
|J_4| \lesssim (1 + \mu) \int_1^{1+2\mu} (1 + \lambda)^{j-2} \, d\lambda \lesssim 1.
\]

Adding together, we find

\[
\int_\mu^1 (\lambda - \mu)^{-\theta} \lambda^{k-2j}(1 - \lambda^{j+\theta-1}) \, d\lambda = C_j \ln(1 + \mu) + O(1),
\]

which gives (2.5) for \(-1 < \mu\).

As for the derivative, we introduce the change of variable \( \lambda = \sigma(1 + \mu) - 1 \), then

\[
\int_\mu^1 (\lambda - \mu)^{-\theta} \lambda^{k-2j}(1 - \lambda^{j+\theta-1}) \, d\lambda \\
= (1 + \mu)^j \int_0^{(1+\mu)^{-1}} (\sigma - 1)^{-\theta} \sigma^{j+\theta-1} h(\sigma(1 + \mu) - 1) \, d\sigma \\
+ \int_0^1 (\lambda - \mu)^{-\theta} \lambda^{k-2j}(1 - \lambda^{j+\theta-1}) \, d\lambda.
\]

Taking derivative and splitting it similarly as above, we also find

\[
\partial_\mu \int_\mu^1 (\lambda - \mu)^{-\theta} \lambda^{k-2j}(1 - \lambda^{j+\theta-1}) \, d\lambda = C_j' (1 + \mu)^{-1} + O(\ln(1 + \mu) + 1),
\]

which gives (2.6) for \(-1 < \mu\).

**Part 1.3:** \( \mu \) close to \(-1\).

To get another part of (2.5), we only need to control \( I_A(\mu) - I_A(-2 - \mu) \) for \(-3/2 < \mu < -1\). Here, for \(-2 \leq \mu < -1\), \( I_A \) has the formula

\[
I_A(\mu) = \frac{2^{-k-\theta}}{\Gamma(k+\theta) \Gamma(1-\theta)} \int_{1-\mu}^1 (\lambda - \mu)^{-\theta} \sum_{j=0}^{[k/2]} C_{j,k,\theta} \lambda^{k-2j}(1 - \lambda^{j+\theta-1}) \, d\lambda.
\]
Thus, to show (2.5), we only need to estimate

\[\int_{-1}^{1} (\lambda - \mu)^{-\theta} \lambda^{k-2j} (1 - \lambda^2)^{j+\theta-1} \ d\lambda - \int_{-\mu}^{\mu} (\lambda + \mu + 2)^{-\theta} \lambda^{k-2j} (1 - \lambda^2)^{j+\theta-1} \ d\lambda\]

\[= \int_{-1}^{-\mu} (\lambda - \mu)^{-\theta} (1 + \lambda)^{j+\theta-1} h(\lambda) \ d\lambda\]

\[+ \left( \int_{-\mu}^{-2\mu} + \int_{-2\mu}^{0} \right) (\lambda - \mu)^{-\theta} - (\lambda + \mu + 2)^{-\theta} (1 + \lambda)^{j+\theta-1} h(\lambda) \ d\lambda\]

\[+ \int_{0}^{1} ((\lambda - \mu)^{-\theta} - (\lambda + \mu + 2)^{-\theta}) \lambda^{k-2j} (1 - \lambda^2)^{j+\theta-1} \ d\lambda\]

\[\equiv J_1 + J_2 + J_3 + J_4.\]

Here we have

\[|J_1| \lesssim (-1 - \mu)^{-\theta} \int_{-1}^{-\mu} (1 + \lambda)^{j+\theta-1} |h(\lambda)| \ d\lambda \lesssim 1,\]

\[|J_2| \lesssim (-1 - \mu)^{j+\theta-1} \int_{-\mu}^{-2\mu} (\lambda - \mu)^{-\theta} - (\lambda + \mu + 2)^{-\theta} \ d\lambda \lesssim (-1 - \mu)^j \lesssim 1,\]

\[|J_3| \lesssim (-1 - \mu) \int_{-2\mu}^{0} (1 + \lambda)^{j-2} \ d\lambda \lesssim 1,\]

\[|J_4| \lesssim \int_{0}^{1} (1 - \lambda)^{j+\theta-1} \ d\lambda \lesssim 1.\]

In summary, we finish the proof of (2.5).

As for the derivative, we introduce the change of variable \( \lambda = \mu - \sigma(1 - \mu) \) for \( J_2 \).

A similar approach as above we find \( |\partial_{\mu}(J_1 + J_2 + J_3 + J_4)| \lesssim |\ln (1 + \mu)| + 1 \). This finishes the proof of (2.6).

**Part 2: \( A \) is odd.**

Next, we consider the case \( A = 1 + 2k \) with \( k \in \mathbb{Z}_+ \). In this case we have

\[\text{supp} \lambda^{1-A} (\lambda) = \text{supp} \delta^{(k-1)}(x) = \{0\},\]

which gives (2.7). On the other hand, when \(-1 < \mu < 1\), we have

\[I_A(\mu) = \frac{2^{-k}}{\Gamma(k)} \int_{-1}^{1} \delta(\lambda - \mu) (1 - \lambda^2)^{k-1} \ d\lambda\]

\[= \frac{2^{-k}}{\Gamma(k)} \int_{-1}^{1} \delta(\lambda - \mu) (-\partial_\lambda)^{k-1} (1 - \lambda^2)^{k-1} \ d\lambda\]

\[= \frac{2^{-k}}{\Gamma(k)} \sum_{j=0}^{[(k-1)/2]} C_{j,k} \mu^{k-1-2j}(1 - \mu^2)^j.\]

This means there is no singularity both for \( I_A(\mu) \) and its derivates, which lead to (2.8).

Here we also find \( C_{0,k} = 2^{k-1}(k-1)! \), which implies the first half of (2.3) for odd \( A \).

Now we finish the proof of Lemma 2.2.

2.5. **Additional discussion of weak solutions.** In light of the fact that the framework we take is slightly different from the usual one, we will discuss a bit more of the weak
solution. We will show that when
\begin{equation}
(2.10) \quad r^{A-1} f \in L^1_{loc,r}, \quad r^{A-1} g \in L^1_{loc,r}, \quad r^{A-1} F \in L^1_{loc,t,r}, \quad r^{A-1} u \in L^1_{loc,t,r}
\end{equation}
with \( u \) calculated by Lemma 2.1, then (2.2) holds.

To show this result, we divide \( u \) to \( u_g \), \( u_f \) and \( u_F \). We begin with \( u_g \) part. Noticing \( I_A(\mu) = 0 \) while \( \mu > 1 \), in this case we have
\[
\int_0^T \int_0^\infty u \cdot (\partial_t^2 - \Delta_A) \varphi r^{A-1} \, d\rho \, dt = \int_0^T \int_0^\infty \int_0^r r^{A-1} \rho^{A-1} g(\rho) I_A \left( \frac{\rho^2}{2r} \right) \varphi(\rho) \, d\rho \, dr \, dt.
\]
Set \( t = T - s \), swap \( r \) and \( \rho \) then exchange the order of integration. It goes to
\[
\int_0^\infty \int_0^T \int_0^r r^{A-1} \rho^{A-1} g(r) I_A \left( \frac{\rho^2}{2r} \right) \varphi(T - s, \rho) \, d\rho \, ds \, dr.
\]
Here \( \varphi(T - s, \rho) \) has zero initial data at \( s = 0 \) and regular enough, by expression of \( u_F \) deduced in Lemma 2.1, we know
\[
r^{A-1} \int_0^T \int_0^r \rho^{A-1} I_A \left( \frac{\rho^2}{2r} \right) \varphi(T - s, \rho) \, d\rho \, ds = \varphi(T - s, r)|_{s = T} = \varphi(0, r).
\]
This gives (2.2) with \( f = F = 0 \). The proof of \( u_f \) and \( u_F \) parts is similar, we leave them to the interested reader.

3. Long-time existence for \( A \in [2, 3] \)

In this section, we will consider the case \( A \in [2, 3] \), and show the proof of Theorem 1.1. Without loss of generality we assume \( n \neq A \), otherwise \( V = 0 \) then (1.1) reduced to the equation of Strauss conjecture.

By the discussion in the last section, we begin to study the equation (1.2) and (2.1).

3.1. Estimate for homogeneous solution. In this subsection, we will give an estimate of the homogeneous solution to (2.1).

**Lemma 3.1.** Let \( A \in [2, 3] \), \( n \geq 2 \) and assume \( \text{supp}(f, g) \subset [0, 1) \). We have
\[
|u_f + u_g| \lesssim (t + r)^{1 + A/2} \left( \|\rho g(\rho)\|_{L^\infty_\rho} + \|f(\rho)\|_{L^\infty_\rho} + \|\rho f'(\rho)\|_{L^\infty_\rho} \right).
\]
Here and throughout the paper, \( (a) \) stands for \( \sqrt{|a|^2 + 4} \).

**Proof of Lemma 3.1.** Here we define
\[
\Omega_0 := \{ \rho : 0 < \rho < t - r \}, \quad \Omega_1 := \{ \rho : |t - r| < \rho < \min(1, t + r) \},
\]
with \( \Omega_0 = 0 \) when \( t < r \).

**Part 1:** Estimate of \( u_g \) with \( A \in [2, 3] \).
Firstly we consider \( u_g \) with \( A \in [2, 3] \). For \( u_g \), by Lemma 2.1 and (2.3) we have
\[
u_g(t, r) = r^{1 + A/2} \left( \int_{\Omega_0} + \int_{\Omega_1} \right) I_A(\mu) \rho^{1 + A/2} g(\rho) \, d\rho \equiv J_0 + J_1.
\]
Also, for \( A \in [2, 3] \) we have
\[
|\ln |1 + \mu|| \lesssim |1 + \mu|^{\frac{1 + A}{2}} \lesssim |1 + \mu|^{\frac{1 + A}{2}}, \quad -2 < \mu < 1.
\]
Part 1.1: $t + r \leq 4$.
In this part, we have $\langle t + r \rangle \approx \langle t - r \rangle \approx 1$ and $r \lesssim 1$. In the region of $\Omega_0$ where $\mu < -1$, by (2.4) and (2.5) we see

$$|I_A(\mu)| \lesssim (1 + \mu)^{\frac{4-\mu}{2}} = \left(\frac{2r \rho}{(t + r + \rho)(t - r - \rho)}\right)^{\frac{4-\mu}{2}} \lesssim r^{\frac{4-\mu}{2}}(t - r - \rho)^{\frac{1-\mu}{2}}.$$

Then we have

$$|J_0| \lesssim \int_0^{t-r} \langle t - r - \rho \rangle^{\frac{1-\mu}{2}} \rho^{\frac{4-\mu}{2}} \rho |g(\rho)| \, d\rho \lesssim \left\| \rho g(\rho) \right\|_{L^\infty} \lesssim \langle t + r \rangle^{\frac{1-\mu}{2}} \langle t - r \rangle^{\frac{1-\mu}{2}} \left\| \rho g(\rho) \right\|_{L^\infty}.$$

In the region of $\Omega_1$ where $\mu > -1$, by (2.5) we see

$$|I_A(\mu)| \lesssim (1 + \mu)^{\frac{4-\mu}{2}} = \left(\frac{2r \rho}{(t + r + \rho)(r + \rho - t)}\right)^{\frac{4-\mu}{2}} \lesssim \rho^{\frac{4-\mu}{2}}(r + \rho - t)^{\frac{1-\mu}{2}}.$$

Thus we get

$$|J_1| \lesssim \int_{t-r}^{t+r} \langle r + \rho - t \rangle^{\frac{4-\mu}{2}} \rho^{\frac{4-\mu}{2}} \rho |g(\rho)| \, d\rho \lesssim \left\| \rho g(\rho) \right\|_{L^\infty} \lesssim \langle t + r \rangle^{\frac{1-\mu}{2}} \langle t - r \rangle^{\frac{1-\mu}{2}} \left\| \rho g(\rho) \right\|_{L^\infty}.$$

Part 1.2: $t + r \geq 4, -1 \leq t - r \leq 2$.
In this part, we have $\langle t - r \rangle \approx 1$ and $r \approx \langle t + r \rangle$. In the region of $\Omega_0$ where $\mu < -1$, we have

$$|I_A(\mu)| \lesssim (1 + \mu)^{\frac{4-\mu}{2}} = \left(\frac{2r \rho}{(t + r + \rho)(t - r - \rho)}\right)^{\frac{4-\mu}{2}} \lesssim r^{\frac{4-\mu}{2}}(t - r - \rho)^{\frac{1-\mu}{2}}.$$

Then we get

$$|J_0| \lesssim \int_0^{t-r} \langle t - r - \rho \rangle^{\frac{1-\mu}{2}} \rho^{\frac{4-\mu}{2}} \rho g(\rho) \, d\rho \lesssim \langle t + r \rangle^{\frac{1-\mu}{2}} \langle t - r \rangle^{\frac{1-\mu}{2}} \left\| \rho g(\rho) \right\|_{L^\infty}.$$

In the region of $\Omega_1$ where $\mu > -1$, we also have

$$|I_A(\mu)| \lesssim (1 + \mu)^{\frac{4-\mu}{2}} = \left(\frac{2r \rho}{(t + r + \rho)(r + \rho - t)}\right)^{\frac{4-\mu}{2}} \lesssim \rho^{\frac{4-\mu}{2}}(r + \rho - t)^{\frac{1-\mu}{2}}.$$

Then we see

$$|J_1| \lesssim \int_{t-r}^{t+r} \langle r + \rho - t \rangle^{\frac{4-\mu}{2}} \rho^{\frac{4-\mu}{2}} |g(\rho)| \, d\rho \lesssim \langle t + r \rangle^{\frac{1-\mu}{2}} \langle t - r \rangle^{\frac{1-\mu}{2}} \left\| \rho g(\rho) \right\|_{L^\infty}.$$

Part 1.3: $t + r \geq 4, t - r \geq 2$.
In this part, we have $t + r \gtrsim \langle t + r \rangle$ and $t - r - 1 \gtrsim \langle t - r \rangle$. Here $\Omega_1 = \emptyset$ so we only need to consider $\Omega_0$ where $\mu < -1$. Again we see

$$|I_A(\mu)| \lesssim (1 + \mu)^{\frac{4-\mu}{2}} = \left(\frac{2r \rho}{(t + r + \rho)(t - r - \rho)}\right)^{\frac{4-\mu}{2}} \lesssim r^{\frac{4-\mu}{2}} \rho^{\frac{4-\mu}{2}}(t - r - \rho)^{\frac{1-\mu}{2}}.$$
Then we have
\[ |u_g| \lesssim (t + r)^{1/2} \langle t - r \rangle^{1/2} \int_0^1 \rho^{A-1} |g(\rho)| \, d\rho \]
\[ \lesssim (t + r)^{1/2} \langle t - r \rangle^{1/2} \|\rho g(\rho)\|_{L^\infty}. \]

In summary, we finish the estimate of \( u_g \) when \( A < 3 \).

**Part 2:** Estimate of \( u_f \) with \( A \in [2,3) \).

Next we consider \( u_f \). For simplicity let \( u_g = \varphi \) stand for \( u_g \) with \( g = \varphi \). By Lemma 2.1 and the expression of \( u_g \) (2.9), we know
\[ u_f = \partial_t (u_g = f) \]
\[ = \frac{1}{\Gamma(\frac{A-1}{2})} \frac{1}{\Gamma(\frac{A-3}{2})} \partial_t \left( t \int_0^1 \int_{-1}^1 \frac{f(\sqrt{r^2 + t^2\sigma^2 - 2tr\sigma\lambda})}{(1 - \sigma^2)^{\frac{3-A}{2}}} \sigma^{A-1} \sqrt{1 - \lambda^2}^{A-3} \, d\lambda \, d\sigma \right) \]
\[ \lesssim t \int_0^1 \int_{-1}^1 \frac{|t\sigma^2 - r\sigma\lambda|}{\sqrt{r^2 + t^2\sigma^2 - 2tr\sigma\lambda}} \, f'(\sqrt{r^2 + t^2\sigma^2 - 2tr\sigma\lambda}) \sigma^{A-1} \sqrt{1 - \lambda^2}^{A-3} \, d\lambda \, d\sigma \]
\[ + \int_0^1 \int_{-1}^1 \frac{|f(\sqrt{r^2 + t^2\sigma^2 - 2tr\sigma\lambda})|}{(1 - \sigma^2)^{\frac{3-A}{2}}} \sigma^{A-1} \sqrt{1 - \lambda^2}^{A-3} \, d\lambda \, d\sigma \]
\[ \equiv H_1 + H_2. \]

Since \( A \in [2,3) \), we can easily find that \( H_2 \lesssim \|f\|_{L^\infty} \). Meanwhile, we find that
\[ H_2 \approx t^{-1} u_g = |f| \lesssim t^{-1} (t + r)^{-\frac{A}{2}} \langle t - r \rangle^{-\frac{A}{2}} \|f(\rho)\|_{L^\infty}. \]

Adding together, we finish the estimate of \( H_2 \). Next we consider \( H_1 \). Noticing that when \( \lambda \in [-1,1] \), we have
\[ \left( \frac{t\sigma^2 - r\sigma\lambda}{\sqrt{r^2 + t^2\sigma^2 - 2tr\sigma\lambda}} \right)^2 = \frac{r^2\sigma^2 + t^2\sigma^4 - 2rt\sigma^3\lambda}{r^2 + t^2\sigma^2 - 2tr\sigma\lambda} \leq \sigma^2 \leq 1. \]

Then we calculate
\[ H_1 \lesssim u_g = |f| \lesssim (t + r)^{-\frac{A}{2}} \langle t - r \rangle^{-\frac{A}{2}} \|\rho f'(\rho)\|_{L^\infty}. \]

Adding together, we finish the estimate of \( u_f \) when \( A < 3 \).

**Part 3:** Estimate of \( u_g \) and \( u_f \) for \( A = 3 \).

When \( A = 3 \), the estimate for \( u_g \) is similar, where we use (2.7) and (2.8) instead of (2.4) and (2.5). As for \( u_f \), we easily calculate
\[ u_f = \frac{(t + r)f(t + r) - (t - r)f(|t - r|)}{2r} \]
with \( \text{supp } u_f \subset \{(t,r) : |t - r| \leq 1\} \). When \( r < t \), by mean value theorem we see
\[ |u_f| = \left| \frac{(t + r)f(t + r) - (t - r)f(t - r)}{2r} \right| \leq \|\partial_\rho (\rho f(\rho))\|_{L^\infty}. \]

When \( t < r \), we see
\[ |u_f| \leq |f(t + r)| + |f(r - t)| \lesssim \|f(\rho)\|_{L^\infty}. \]

In summary, we obtain the desired estimate of \( u_g \) and \( u_f \) for \( A = 3 \), and finish the proof of Lemma 3.1. \( \square \)
3.2. Estimate for non-homogeneous solution. In this subsection, we will omit the initial data and give the estimate of solution to the nonlinear equation (1.2). For simplicity, we shift the time variable and consider the equation

\begin{align}
(\partial_t^2 - \partial_r^2 - (A - 1)r^{-1}\partial_r) v(t, r) = r^{(A-n)p+n-A/2}G(t, r) \\
v(4, r) = 0, \quad v_t(4, r) = 0.
\end{align}

**Lemma 3.2.** Define \(\Omega := \{ (t, r) \in [4, \infty) \times \mathbb{R}_+ : t > r + 2 \} \) and \(\Lambda(t, r) := \{ (s, \rho) \in \Omega : s + \rho < t + r, s - \rho < t - r \} \). We will show that, if \(v\) solves the equation (3.1) with \(\supp v \subset \Omega\), then for any \((t, r) \in \Omega\) and \(k = 1, 2, 3\) we have

\begin{equation}
\langle t + r \rangle^{\frac{A-1}{2}} |v| \lesssim N_k(t - r)\|\omega_k^pG\|_{L^p_{\rho, \rho}(\Lambda)}.
\end{equation}

Here

\[\omega_k(t, r) := \langle t + r \rangle^{\frac{A-1}{2}} \beta_k(t - r), \quad k = 1, 2, 3;\]

\[\beta_k(t - r) := \begin{cases} 
(t - r)^{\frac{A-1}{2}}, & k = 1, \text{ for } p_m < p < p_M, \\
(t - r)^{(n-1)p-n-1}, & k = 2, \text{ for } p_m < p < p_t, \\
(t - r)^{\frac{A-1}{2} (\ln(t - r))^{-1}}, & k = 3, \text{ for } p = p_t > p_d;
\end{cases}\]

\[N_1(t - r) := \begin{cases} 
(t - r)^{\frac{1-n}{2}}, & p > \max(p_d, p_t) \text{ or } p = p_d > p_t \text{ or } p_F < p < p_d, \\
(t - r)^{\frac{1-n}{2}} \ln(t - r), & p = p_t > p_d \text{ or } p = p_F < p_d, \\
(t - r)^{(1-n)p+1}, & p_t > p > p_d, \\
(t - r)^{\frac{1-n}{2} p + n + 1}, & p < \min(p_d, p_F), \\
(t - r)^{\frac{1-n}{2} p + n + 1} \ln(t - r), & p = p_d < p_t, \\
(t - r)^{\frac{1-n}{2} (\ln(t - r))^2}, & p = p_d = p_t;
\end{cases}\]

\[N_2(t - r) := \begin{cases} 
(t - r)^{(1-n)p+1}, & p_S < p < p_t, \\
(t - r)^{(1-n)p+1} \ln(t - r), & p = p_S < p_t, \\
(t - r)^{(1-n)p+1} p^2 + 2p + 1, & p_m < p < \min(p_S, p_t);
\end{cases}\]

\[N_3(t - r) := \langle t + r \rangle^{\frac{1-n}{2}} \ln(t - r), \quad p = p_t > p_d.\]

**Remark 3.1.** By the definition of \(\beta_k\), we can easily find that for any \(\xi, \eta > 2\) with \(\xi/\eta \in (1/2, 2)\), we have

\[\beta_k(\xi) \approx \beta_k(\eta).\]

Also if \(2 < \eta_1 < \eta_2\), we have

\[\beta_k(\eta_1) \lesssim \beta_k(\eta_2).\]

**Proof of Lemma 3.2.** Using Lemma 2.1, we find

\[v = r^{\frac{1-n}{2}} \int_{\Lambda} I_A(\mu) \rho^{\frac{(A-n)p+n-1}{2}} G(s, \rho) \, d\rho \, ds,\]

with \(\mu := \frac{r^2 + s^2 - (t-s)^2}{2r}\). To reach (3.2), we calculate

\[|\langle t + r \rangle^{\frac{A-1}{2}} v| \leq r^{\frac{1-n}{2}} \langle t + r \rangle^{\frac{A-1}{2}} \|\omega_k^pG\|_{L^p_{\rho, \rho}(\Lambda)} \int_{\Lambda} |I_A(\mu)| \rho^{\frac{(A-n)p+n-1}{2}} \omega_k^{-p} \, d\rho \, ds.\]
Thus, we only need to show that

\[(3.3) \quad J_{ij;k} := r^{\frac{n-2}{2}} (t+r)^{\frac{A-1}{2}} \int_{\Lambda_{ij}} |I_\Lambda(\mu)| \rho^{\frac{(A-n)p+n-1}{2}} \omega_k^{-p} \ d \rho \, d s \lesssim N_k(t-r)\]

for \(i = 1, 2, 3, \ j = 1, 2\) with

\[
\begin{align*}
\Lambda_{11} & := \{(s, \rho) \in \Lambda : s + \rho \in (t-r, t+r), \rho \leq s/2\}; \\
\Lambda_{12} & := \{(s, \rho) \in \Lambda : s + \rho \in (t-r, t+r), \rho \geq s/2\}; \\
\Lambda_{21} & := \{(s, \rho) \in \Lambda : s + \rho \in \left(\frac{t-r}{2}, t-r\right), \rho \leq s/2\}; \\
\Lambda_{22} & := \{(s, \rho) \in \Lambda : s + \rho \in \left(\frac{t-r}{2}, t-r\right), \rho \geq s/2\}; \\
\Lambda_{31} & := \{(s, \rho) \in \Lambda : s + \rho \in \left(3, \frac{t-r}{2}\right), \rho \leq s/2\}; \\
\Lambda_{32} & := \{(s, \rho) \in \Lambda : s + \rho \in \left(3, \frac{t-r}{2}\right), \rho \geq s/2\}.
\end{align*}
\]

It’s easy to find that

\[(3.4) \quad \begin{cases} 
  s + \rho \leq 3(s-\rho) \leq 3(s+\rho), & (s, \rho) \in \Lambda_{11} \cup \Lambda_{21} \cup \Lambda_{31}, \\
  s + \rho \leq 3\rho \leq 3(s + \rho), & (s, \rho) \in \Lambda_{12} \cup \Lambda_{22} \cup \Lambda_{32}.
\end{cases}\]

Then, a quick calculation shows

\[(3.5) \quad \begin{cases} 
  t-r \leq s + \rho \leq \min\{t+r, 3(t-r)\}, & (t-r)/3 \leq s - \rho \leq t-r, \quad (s, \rho) \in \Lambda_{11}, \\
  t-r \leq s + \rho \leq t+r, & 2 \leq s - \rho \leq \min\{(t+r)/3, t-r\}, \quad (s, \rho) \in \Lambda_{12}, \\
  (t-r)/2 \leq s + \rho \leq t-r, & (s + \rho)/3 \leq s - \rho \leq s + \rho, \quad (s, \rho) \in \Lambda_{21}, \\
  (t-r)/2 \leq s + \rho \leq t-r, & 2 \leq s - \rho \leq (t-r)/3, \quad (s, \rho) \in \Lambda_{22}, \\
  4 \leq s + \rho \leq (t-r)/2, & (s + \rho)/3 \leq s - \rho \leq s + \rho, \quad (s, \rho) \in \Lambda_{31}, \\
  6 \leq s + \rho \leq (t-r)/2, & 2 \leq s - \rho \leq (s + \rho)/3, \quad (s, \rho) \in \Lambda_{32}.
\end{cases}\]

From now on, we introduce \(\xi := s + \rho\) and \(\eta := s - \rho\). We will always adopt (3.4) and (3.5) in each region.

**Part 1:** Preparation for \(A \in [2,3]\) with \(r \leq t/2\).
We will firstly consider $A \in [2, 3)$, notice that $\mu = \frac{r^2 + \rho^2 - (t-s)^2}{\rho} < -1$ when $s + \rho < t - r$, and $\mu > -1$ when $s + \rho > t - r$. In this part we have

\[ t + r \leq 3(t - r) \leq 3(t + r). \]

In the region of $\Lambda_{11}$, using (2.5) we have

\[ |I_A(\mu)| \lesssim (1 + \mu) \frac{A-1}{2} = \left( \frac{2r \rho}{(r + \rho + t - s)(r + \rho - t + s)} \right)^{\frac{A-1}{2}} \lesssim \left( \frac{\rho}{r + \rho - t + s} \right)^{\frac{A-1}{2}}. \]

Then we find

\[ J_{11} \lesssim r^{\frac{1}{2} - A}(t - r)^{\frac{A-1}{2}} \int_{\Lambda_{11}} \rho^{\frac{(A-n)p + n - A + 2}{2}} (s + \rho)^{\frac{A-1}{2} - p} \beta(s + \rho)^{-p}(r + \rho + t - s)^{\frac{A-1}{2}} \, d\rho \, ds \]

\[ \lesssim r^{\frac{1}{2} - A}(t - r)^{\frac{1}{2}(1-n)p + n + 2} \beta(t - r)^{-p} \int_{(t-r)/2}^{t-r} (\xi - (t-r))^{-\frac{1}{2}} d\xi \]

\[ \lesssim (t - r)^{\frac{1}{2}(1-n)p + n + 2} \beta(t - r)^{-p}, \]

where we noticed $\frac{(A-n)p + n - A + 2}{2} > -1$ while $p < p_M$. In the region of $\Lambda_{12}$, we also have (3.6). Then we find

\[ J_{12} \lesssim r^{\frac{1}{2} - A}(t - r)^{\frac{A-1}{2}} \int_{\Lambda_{12}} (s + \rho)^{\frac{(1-n)p + n - A + 2}{2}} \beta(s - \rho)^{-p}(r + \rho + t - s)^{\frac{A-1}{2}} \, d\rho \, ds \]

\[ \lesssim r^{\frac{1}{2} - A}(t - r)^{\frac{1}{2}(1-n)p + n + 1} \beta(t - r)^{-p} \int_{2(t-r)/3}^{t-r} (\xi - (t-r))^{-\frac{1}{2}} d\xi \int_{t-r}^{(t-r)/2} \beta(\eta)^{-p} \, d\eta \]

\[ \lesssim (t - r)^{\frac{1}{2}(1-n)p + n + 1} \beta(t - r)^{-p}. \]

In the region of $\Lambda_{21}$, using (2.4) and (2.5) we have

(3.7)

\[ |I_A(\mu)| \lesssim (-1 - \mu) \frac{1}{2} = \left( \frac{2r \rho}{(t + r - s + \rho)(t - r - s - \rho)} \right)^{\frac{A-1}{2}} \lesssim \left( \frac{r}{t - r - s - \rho} \right)^{\frac{A-1}{2}}. \]

Then we find

\[ J_{21} \lesssim (t - r)^{\frac{A-1}{2}} \int_{\Lambda_{21}} \rho^{\frac{(A-n)p + n - 1}{2}} (s + \rho)^{\frac{1}{2}(A-1)p} \beta(s + \rho)^{-p}(t - r - s - \rho)^{\frac{A-1}{2}} \, d\rho \, ds \]

\[ \lesssim (t - r)^{-\frac{1}{2}(1-n)p + n + 1} \beta(t - r)^{-p} \int_{(t-r)/2}^{(t-r)/3} (\xi - (t-r))^{-\frac{1}{2}} d\eta \, d\xi \]

\[ \lesssim (t - r)^{\frac{1}{2}(1-n)p + n + 3} \beta(t - r)^{-p}, \]
where we require $p < p_M$ to ensure that $\frac{(A-n)+n-1}{2} > -1$. In the region of $\Lambda_{22}$, we still have (3.7). Then we find

$$J_{22} \lesssim (t-r)^{\frac{A+1}{2}} \int_{\Lambda_{22}} \langle s + \rho \rangle^{\frac{(1-n)+n-1}{2}} \beta(s-\rho)^{-p} (t - r - s - \rho)^{\frac{A+1}{2}} d\rho d s$$

$$\lesssim (t-r)^{\frac{(1-n)+n+1}{2}} \int_{(t-r)/2}^{t-r} (t - r - \xi)^{\frac{A+1}{2}} d\xi \int_{2}^{t-r/3} \beta(\eta)^{-p} d \eta d \xi$$

In the region of $\Lambda_{31}$ we have (3.8)

$$|A(\mu)| \lesssim (1 - \mu)^{\frac{A+1}{2}} = \left( \frac{2r}{(t + r - s + \rho)(t - r - s - \rho)} \right)^{\frac{A+1}{2}} \lesssim \left( \frac{r}{(t-r)^2} \right)^{\frac{A+1}{2}}.$$  

Then we find

$$J_{31} \lesssim (t-r)^{\frac{A+1}{2}} \int_{\Lambda_{31}} \rho^{\frac{(A-n)+n+1}{2}} \langle s + \rho \rangle^{\frac{(1-n)+n+1}{2}} \beta(s+\rho)^{-p} (t-r)^{-A+1} d\rho d s$$

$$\lesssim (t-r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{\frac{\xi}{3}}^{\xi} \langle \xi - \eta \rangle^{\frac{(A-n)+n+1}{2}} \langle \xi \rangle^{\frac{A+1}{2}} \beta(\xi)^{-p} d \eta d \xi$$

where we noticed $\frac{(A-n)+n+1}{2} > -1$ while $p < p_M$. In the region of $\Lambda_{32}$ we still have (3.8). Then we find

$$J_{32} \lesssim (t-r)^{\frac{A+1}{2}} \int_{\Lambda_{32}} \langle s + \rho \rangle^{\frac{(1-n)+n+1}{2}} \beta(s-\rho)^{-p} (t-r)^{-A+1} d\rho d s$$

$$\lesssim (t-r)^{\frac{1-A}{2}} \int_{6}^{(t-r)/2} \int_{2}^{\frac{\xi}{3}} \langle \xi \rangle^{\frac{(A-n)+n+1}{2}} \beta(\xi)^{-p} d \eta d \xi.$$  

Part 2: Preparation for $A \in [2, 3]$ with $r \geq t/2$.  

In this part we have $t + r \leq 3r \leq 3(t + r)$.  

In the region of $\Lambda_{11}$, we have (3.6). Then we find

$$J'_{11} \lesssim \int_{\Lambda_{11}} \rho^{\frac{(A-n)+n-n+1}{2}} \langle s + \rho \rangle^{\frac{(1-n)+n-1}{2}} \beta(s+\rho)^{-p} (r + \rho - t + s)^{\frac{A+3}{2}} d\rho d s$$

$$\lesssim (t-r)^{\frac{(1-n)+n+1}{2}} \beta(t-r)^{-p} \int_{t-r}^{3(t-r)} \int_{(t-r)/3}^{\xi} \langle \xi - \eta \rangle^{\frac{(A-n)+n+1}{2}} \langle \xi \rangle^{\frac{A+3}{2}} d \eta d \xi$$

$$\lesssim (t-r)^{\frac{(1-n)+n+3}{2}} \beta(t-r)^{-p} \int_{t-r}^{3(t-r)} \langle \xi - (t-r) \rangle^{\frac{A-3}{2}} d \xi$$

$$\lesssim (t-r)^{\frac{(1-n)+n+3}{2}} \beta(t-r)^{-p}.$$

where we require \( p > p_m \) so that \( \frac{(1 - n) p + n + 1}{2} < -1 \). In the region of \( \Lambda_{22} \) we have (3.9)

\[
|I_A(\mu)| \lesssim (1 - \mu)^{\frac{1 - A}{2}} = \left( \frac{2r \rho}{(t + r - s + \rho)(t - r - s - \rho)} \right)^{\frac{A - 1}{2}} \lesssim \left( \frac{\rho}{t - r - s - \rho} \right)^{\frac{A - 1}{2}}.
\]

Then we find

\[
J_{21}' \lesssim \int_{\Lambda_{21}} \frac{(A - n)p + n + A - 2}{2}(s + \rho)^{\frac{(1 - n)p + n + A - 2}{2}} \beta(s - \rho)^{-p}(t - r - s - \rho)^{\frac{1 - A}{2}} \ d \rho \ d s
\]

\[
\lesssim (t - r)^{\frac{(1 - A)p}{2}} \beta(t - r)^{-p} \int_{(r - \xi)/2}^{t - r} (\xi - \eta)^{\frac{(1 - n)p + n + A - 2}{2}} (t - r - \xi)^{\frac{1 - A}{2}} \ d \eta \ d \xi
\]

\[
\lesssim (t - r)^{\frac{(1 - n)p + n + A + 3}{2}} \beta(t - r)^{-p},
\]

where \( \frac{(A - n)p + n + A - 2}{2} > -1 \) since \( p > p_M \). In the region of \( \Lambda_{22} \), we have (3.9). Then we find

\[
J_{22}' \lesssim \int_{\Lambda_{22}} (s + \rho)^{\frac{(1 - n)p + n + A - 2}{2}} \beta(s - \rho)^{-p}(t - r - s - \rho)^{\frac{1 - A}{2}} \ d \rho \ d s
\]

\[
\lesssim (t - r)^{\frac{(1 - A)p}{2}} \int_{(r - \xi)/2}^{t - r} (t - r - \xi)^{\frac{1 - A}{2}} \ d \xi
\]

\[
\lesssim (t - r)^{\frac{(1 - n)p + n + 1}{2}} \beta(t - r)^{-p} \ d \eta.
\]

In the region of \( \Lambda_{31} \), we have (3.10)

\[
|I_A(\mu)| \lesssim (1 - \mu)^{\frac{1 - A}{2}} = \left( \frac{2r \rho}{(t + r - s + \rho)(t - r - s - \rho)} \right)^{\frac{A - 1}{2}} \lesssim \left( \frac{\rho}{t - r} \right)^{\frac{A - 1}{2}}.
\]
Then we find
\[ J'_{31} \lesssim \int_{\Lambda_{31}} \rho^{(A-n)p+n+A-2} (s + \rho)^{(1-A)p} \frac{1}{2} \beta(s + \rho)^{-p} (t - r)^{\frac{1-A}{2}} \, d \rho \, ds \]
\[ \lesssim (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{\xi/3}^{\xi} (\xi - \eta) \left( \frac{(A-n)p+n+A-2}{2} \right) \frac{1}{2} \beta(\xi)^{-p} \, d \eta \, d \xi \]
\[ \lesssim (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{2}^{\xi/3} (\xi - \eta) \left( \frac{(1-n)p+n+A-2}{2} \right) \frac{1}{2} \beta(\xi)^{-p} \, d \eta \, d \xi, \]
where \( \frac{(A-n)p+n+A-2}{2} > -1 \) since \( p < p_M \). In the region of \( \Lambda_{32} \), we have (3.10). Then we find
\[ J'_{32} \lesssim \int_{\Lambda_{32}} (s + \rho)^{\frac{(1-n)p+n+A-2}{2}} \beta(s - \rho)^{-\nu} (t - r)^{\frac{1-A}{2}} \, d \rho \, ds \]
\[ \lesssim (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{2}^{\xi/3} (\xi - \eta) \left( \frac{(1-n)p+n+A-2}{2} \right) \frac{1}{2} \beta(\eta)^{-p} \, d \eta \, d \xi. \]

**Part 3:** Estimate for \( A \in [2, 3) \).

Turning to the proof of (3.3), we will only present the estimate of \( J_{32} \), and the other terms can be estimated in a similar manner. At first, for \( J_{32;1} \) with \( p_m < p < p_M \), we have
\[ J_{32;1} \lesssim (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{2}^{\xi/3} (\xi - \eta) \left( \frac{(1-n)p+n+A-2}{2} \right) \frac{1}{2} \beta(\eta)^{-p} \, d \eta \, d \xi. \]

When \( p > p_d = \frac{2}{(A - 1)} \), it is easy to see that
\[ J_{32;1} \lesssim (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{2}^{\xi/3} (\xi - \eta) \left( \frac{(1-n)p+n+A-2}{2} \right) \frac{1}{2} \beta(\eta)^{-p} \, d \eta \, d \xi \lesssim N_1(t - r). \]

Similarly, when \( p \leq p_d \), we have
\[ J_{32;1} \lesssim \begin{cases} (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{2}^{\xi/3} (\xi - \eta) \left( \frac{(1-n)p+n+A-2}{2} \right) \frac{1}{2} \beta(\eta)^{-p} \ln \xi \, d \xi, & p = p_d, \\ (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{2}^{\xi/3} (\xi - \eta) \left( \frac{(1-n)p+n+A-2}{2} \right) \frac{1}{2} \beta(\eta)^{-p} \, d \eta \, d \xi, & p < p_d, \end{cases} \]
which are controlled by \( N_1(t - r) \) and this finishes the proof of (3.2) with \( k = 1 \). For \( J_{32;2} \) with \( p_m < p < p_t \), we have
\[ J_{32;2} \lesssim (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{2}^{\xi/3} (\xi - \eta) \left( \frac{(1-n)p+n+A-2}{2} \right) \frac{1}{2} \beta(\eta)^{-\nu} \, d \eta \, d \xi \]
\[ \lesssim (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{2}^{\xi/3} (\xi - \eta) \left( \frac{(1-n)p+n+A-2}{2} \right) \frac{1}{2} \beta(\eta)^{-\nu} \, d \eta \, d \xi \lesssim N_2(t - r). \]

Finally, for \( J_{32;3} \) with \( p = p_t > p_d \), we have
\[ J_{32;3} \lesssim (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} \int_{2}^{\xi/3} (\xi - \eta) \left( \frac{(1-A)p}{2} \right) \beta(\eta)^{-\nu} \, d \eta \, d \xi \]
\[ \lesssim (t - r)^{\frac{1-A}{2}} \int_{1}^{(t-r)/2} (\xi)^{-1} \, d \xi \lesssim (t - r)^{\frac{1-A}{2}} \ln(t - r) = N_3(t - r). \]

In conclusion, this completes the proof for \( A \in [2, 3) \).

**Part 4:** Estimate for \( A = 3 \).

The case \( A = 3 \) is much simpler, thanks to (2.7), we only need to consider \( \Lambda_{11} \) and \( \Lambda_{12} \). By (2.8) and a similar approach as above we get the desired estimate. \qed
3.3. **Long-time existence.** In this subsection, we will construct a Cauchy sequence to approximate the desired solution. We set \( u_{-1} = 0 \) and let \( u_{j+1} \) be the solution of the equation

\[
\begin{aligned}
\partial_t^2 u_{j+1} - \Delta_A u_{j+1} &= r^{\frac{(A-n)p+n-A}{2}} |u_j|^p, \quad r \in \mathbb{R}_+,
\quad u_{j+1}(0, x) = \varepsilon r^{\frac{n-A}{2}} U_0(r), 
\quad \partial_t u_{j+1}(0, x) = \varepsilon r^{\frac{n-A}{2}} U_1(r).
\end{aligned}
\]

By Lemma 3.1 and Lemma 3.2, noticing that for any \( p > 1 \), we have

\[
||a|^p - |b|^p| \lesssim |a - b| \max(|a|, |b|)^{p-1},
\]

then we see

\[
(t + r)^{\frac{1}{4}} |u_{j+1}| \leq \varepsilon C_0 (t - r) \frac{1}{4} A \Psi + C_0 N_k(t - r) \|\omega_k u_j\|_{L_{r,p}^p(\Omega)}^p,
\]

\[
(t + r)^{\frac{1}{4}} |u_j - u_{j-1}| \leq C_0 N_k(t - r) \|\omega_k(u_j - u_{j-1})\|_{L_{r,p}^p(\Omega)} \max_{l \in \{j-1, j\}} \|\omega_k u_l\|_{L_{r,p}^p(\Omega)}^p - 1,
\]

with \( k = 1, 2, 3 \) and \( C_0 \) large enough. Here

\[
\Psi := \|r^{\frac{n-A}{2}} U_0'(r)\|_{L_r^p} + \|r^{\frac{n-A}{2}} U_0(r)\|_{L_r^p} + \|r^{\frac{n-A}{2}+2} U_1(r)\|_{L_r^p},
\]

\[
\Lambda(t, r) := \{(s, \rho) \in \Omega : s + \rho < t + r, s - \rho < t - r\},
\]

\[
\Omega := \{(t, r) \in \mathbb{R}_+^2 : t > r - 1\}.
\]

To prove Theorem 1.1, we need to separate \( p \in (p_m, p_M) \) into more parts rather than that in (1.5), (1.6) or (1.7). For the reader’s convenience, we list them as below. When \( (3 - A)(A + n + 2) < 8 \), we have \( p_d < p_F < p_S < p_t \). Then the proof for \( p < p_d \) will be found in Part 4, \( p = p_d \) in Part 6, \( p_d < p < p_S \) in Part 8, \( p = p_S \) in Part 9, \( p_S < p < p_t \) in Part 3, \( p = p_t \) in Part 2 and \( p > p_t \) in Part 1. When \( (3 - A)(A + n + 2) > 8 \), we have \( p_d = p_F = p_S = p_t \). The proof for \( p < p_d \) will be found in Part 4, \( p = p_d \) in Part 7 and \( p > p_d \) in Part 1. Finally when \( (3 - A)(A + n + 2) > 8 \), we have \( p_d > p_F > p_S > p_t \). The proof for \( p < p_F \) will be found in Part 4, \( p = p_F \) in Part 5 and \( p > p_F \) in Part 1.

Now, we are prepared to give the proofs for each part.

**Part 1:** \( \max(p_t, p_F) < p \).

In this part, we choose \( k = 1 \). For \( (t, r) \in \Omega \) we find

\[
\omega_1 |u_{j+1}| \leq \varepsilon C_0 \Psi + C_0 \|\omega_1 u_j\|_{L_{r,p}^p(\Omega)}^p,
\]

\[
\omega_1 |u_{j+1} - u_j| \leq C_0 \|\omega_1 (u_j - u_{j-1})\|_{L_{r,p}^p(\Omega)} \max_{l \in \{j-1, j\}} \|\omega_1 u_l\|_{L_{r,p}^p(\Omega)}^{p-1}.
\]

Taking the \( L_{r,p}^p(\Omega) \) norm on both sides and we get

\[
\|\omega_1 u_{j+1}\|_{L_{r,p}^p(\Omega)} \leq \varepsilon C_0 \Psi + C_0 \|\omega_1 u_j\|_{L_{r,p}^p(\Omega)}^p,
\]

\[
\|\omega_1 (u_{j+1} - u_j)\|_{L_{r,p}^p(\Omega)} \leq C_0 \|\omega_1 (u_j - u_{j-1})\|_{L_{r,p}^p(\Omega)} \max_{l \in \{j-1, j\}} \|\omega_1 u_l\|_{L_{r,p}^p(\Omega)}^{p-1}.
\]

For any \( \varepsilon > 0 \) satisfying \( (2\varepsilon C_0) < \varepsilon \Psi \), we find

\[
\|\omega_1 u_j\|_{L_{r,p}^p(\Omega)} \leq 2\varepsilon C_0 \Psi
\]

holds for any \( j \) since \( u_{-1} = 0 \). Meanwhile, it also gives us

\[
\|\omega_1 (u_{j+1} - u_j)\|_{L_{r,p}^p(\Omega)} \leq C_0 (2\varepsilon C_0)^{p-1} \|\omega_1 (u_j - u_{j-1})\|_{L_{r,p}^p(\Omega)}
\]

\[
\leq \frac{1}{2} \|\omega_1 (u_j - u_{j-1})\|_{L_{r,p}^p(\Omega)}.
\]
This means \( \{ u_j \} \) is a Cauchy sequence in weighted \( L^\infty \) norm. Set the limit as \( u \). It is easy to check \( u )\) and \( F = r^{(\frac{(d-1)p+1}{2})} u^p \) satisfy (2.10) while \( p < p_F \). Thus we get the desired global weak solution.

**Part 2:** \( p_S < p_t = p \).

In this part we take \( k = 3 \). For \((t, r) \in \Omega\) we find

\[
\begin{align*}
\omega_3 |u_j+1| &\leq \varepsilon C_0 (t-r)^{-1} \Psi + C_0 \| \omega_3 u_j \|_{L^\infty_{t,r}(\Lambda)}, \\
\omega_3 |u_j+1 - u_j| &\leq C_0 \| \omega_3 (u_j - u_j-1) \|_{L^\infty_{t,r}(\Lambda)} \max_{t \in (j, j-1)} \| \omega_3 u_l \|_{L_{t,r}^{p-1}(\Lambda)}.
\end{align*}
\]

Noticing \((t-r) \geq 2\), by a similar process as above we get the desired global solution.

**Part 3:** \( p_S < p < p_t \).

In this part we take \( k = 2 \). For \((t, r) \in \Omega\) we find

\[
\begin{align*}
\omega_2 |u_j+1| &\leq \varepsilon C_0 (t-r)^{-\frac{(n-1)p-(n-A)}{2}} \Psi + C_0 \| \omega_2 u_j \|_{L^\infty_{t,r}(\Lambda)}, \\
\omega_2 |u_j+1 - u_j| &\leq C_0 \| \omega_2 (u_j - u_j-1) \|_{L^\infty_{t,r}(\Lambda)} \max_{t \in (j, j-1)} \| \omega_2 u_l \|_{L_{t,r}^{p-1}(\Lambda)}.
\end{align*}
\]

Here \( \frac{(n-1)p-(n-A)}{2} \leq 0 \) since \( p < p_t \), by a similar process again we get the desired global solution.

**Part 4:** \( p < \min(p_d, p_F) \).

In this case, we choose \( T_* = T_*(\varepsilon) \) which satisfies \( \varepsilon p^{-1} T_* \geq \frac{(n-A+2)(p+n-A+2)^2}{4} = a \) with \( a \) to be fixed later. Here we define \( \Omega_* = \Omega \cap \{ t : t < T_* \} \) and choose \( k = 1 \). Then for \((t, r) \in \Omega_*\) we find

\[
\begin{align*}
\omega_1 |u_j+1| &\leq \varepsilon C_0 \Psi + C_0 (t-r)^{\frac{(n-A+2)(p+n-A+2)}{2}} \| \omega_1 u_j \|_{L^\infty_{t,r}(\Lambda)}, \\
\omega_1 |u_j+1 - u_j| &\leq C_0 (t-r)^{\frac{(n-A+2)(p+n-A+2)}{2}} \| \omega_1 (u_j - u_j-1) \|_{L^\infty_{t,r}(\Lambda)} \\
&\quad \times \max_{t \in (j, j-1)} \| \omega_1 u_l \|_{L_{t,r}^{p-1}(\Lambda)}.
\end{align*}
\]

Taking the \( L^\infty_{t,r}(\Omega_*) \) norm on both sides, when \( T_* \geq 3 \) we get

\[
\begin{align*}
\| \omega_1 u_j+1 \|_{L^\infty_{t,r}(\Omega_*)} &\leq \varepsilon C_1 \Psi + C_1 T_*^{\frac{(n-A+2)(p+n-A+2)}{2}} \| \omega_1 u_j \|_{L^\infty_{t,r}(\Omega_*)}, \\
\| \omega_1 (u_j+1 - u_j) \|_{L^\infty_{t,r}(\Omega_*)} &\leq C_1 T_*^{\frac{(n-A+2)(p+n-A+2)}{2}} \| \omega_1 (u_j - u_j-1) \|_{L^\infty_{t,r}(\Omega_*)} \\
&\quad \times \max_{t \in (j, j-1)} \| \omega_1 u_l \|_{L_{t,r}^{p-1}(\Omega_*)},
\end{align*}
\]

with some \( C_1 \) large enough. Considering \( \varepsilon \) such that \( T_*(\varepsilon) \geq 3 \) and defining \( a = (2C_1)^{-p} \Psi^{-1} \), we conclude that

\[
\begin{align*}
\| \omega_1 u_j \|_{L^\infty_{t,r}(\Omega_*)} &\leq 2 \varepsilon C_1 \Psi, \\
\| \omega_1 (u_j+1 - u_j) \|_{L^\infty_{t,r}(\Omega_*)} &\leq \frac{1}{2} \| \omega_1 (u_j - u_j-1) \|_{L^\infty_{t,r}(\Omega_*)}
\end{align*}
\]

for any \( j \), which are sufficient to get the desired solution.

**Part 5:** \( p = p_F < p_d \).
In this case, we choose $T_*$ which satisfies $\varepsilon^{p-1} \ln T_* = a$ with $a$ to be fixed later, $\Omega_*$ as above and $k = 1$. For $(t, r) \in \Omega_*$ we find

$$
\omega_1 |u_{j+1}| \leq C_0 \Psi + C_0 \ln(t - r) \|\omega_1 u_j\|_{L_{\infty, p}^\alpha(A)},
$$
$$
\omega_1 |u_{j+1} - u_j| \leq C_0 \ln(t - r) \|\omega_1 (u_j - u_{j-1})\|_{L_{\infty, p}^\alpha(A)} \max_{l \in \{j, j-1\}} \|\omega_1 u_l\|_{L_{\infty, p}^\alpha(A)}^{p-1}.
$$

Similar as above, taking $\varepsilon$ such that $T_*(\varepsilon) \geq 3$, defining $a := (2C_1)^{-p} \Psi_1^{-p}$ with $C_1$ large enough, we get the Cauchy sequence $\{u_j\}$ and the desired solution.

**Part 6:** $p = p_d < p_F$.

In this case, we choose $T_*$ which satisfies

$$
\varepsilon^{p-1} T_*^{-\frac{(-n-A+2)p+n+A+2}{2}} \ln T_* = a
$$

and $\Omega_*$ same as above. Taking $k = 1$, for $(t, r) \in \Omega_*$ we find

$$
\omega_1 |u_{j+1}| \leq C_0 \Psi + C_0 \ln(t - r) \|\omega_1 u_j\|_{L_{\infty, p}^\alpha(A)},
$$
$$
\omega_1 |u_{j+1} - u_j| \leq C_0 \ln(t - r) \|\omega_1 (u_j - u_{j-1})\|_{L_{\infty, p}^\alpha(A)} \max_{l \in \{j, j-1\}} \|\omega_1 u_l\|_{L_{\infty, p}^\alpha(A)}^{p-1} \varepsilon
$$

Choosing $\varepsilon$ such that $T_*(\varepsilon) \geq 3$, $a = (2C_1)^{-p} \Psi_1^{-p}$ with $C_1$ large enough, we get the Cauchy sequence $\{u_j\}$ and the desired solution. To finish the proof of Theorem 1.1 for this part, we introduce the following claim and postpone its proof to the end of this section.

**Claim 3.3.** Assume that $T_*$ satisfies $\varepsilon^{p-1} T_*^{-\frac{(-n-A+2)p+n+A+2}{2}} \ln T_* = a$ with some constant $a$, then there exists two constant $c_1, c_2$ such that $c_1 \varepsilon^{\frac{p-1}{p+1}} \ln \varepsilon \ln^{\frac{p+1}{p+2}} T_* \leq c_2 \varepsilon^{\frac{p-1}{p+1}} \ln \varepsilon \ln^{\frac{p+1}{p+2}}$ for $\varepsilon$ small enough.

**Part 7:** $p = p_d = p_F$.

In this case, we choose $T_*$ which satisfies $\varepsilon^{p-1} (\ln T_*)^2 = a$ and $\Omega_*$ same as above. Taking $k = 1$, for $(t, r) \in \Omega_*$ we find

$$
\omega_1 |u_{j+1}| \leq C_0 \Psi + C_0 (\ln(t - r))^2 \|\omega_1 u_j\|_{L_{\infty, p}^\alpha(A)},
$$
$$
\omega_1 |u_{j+1} - u_j| \leq C_0 (\ln(t - r))^2 \|\omega_1 (u_j - u_{j-1})\|_{L_{\infty, p}^\alpha(A)} \max_{l \in \{j, j-1\}} \|\omega_1 u_l\|_{L_{\infty, p}^\alpha(A)}^{p-1} \varepsilon
$$

Choosing $\varepsilon$ such that $T_*(\varepsilon) \geq 3$, $a = (2C_1)^{-p} \Psi_1^{-p}$ and $C_1$ large enough, we get the Cauchy sequence $\{u_j\}$ and finish the proof.

**Part 8:** $p_d < p < p_S$.

In this case, we choose $T_*$ which satisfies $\varepsilon^{p(p-1)} T_*^{\frac{(-n-1)n^2+(n+1)p+2}{p}} = a$ with $a$ to be fixed later and $\Omega_*$ as above. Moreover, we separate the region $\Omega_*$ to

$$
\Omega_{s1} := \Omega_\ast \cap \{(t, r) : (t - r) \leq (bc)^{\frac{2(p-1)}{(-n-A+2)p+n+A+2}}\},
$$
$$
\Omega_{s2} := \Omega_\ast \cap \{(t, r) : (t - r) \geq (bc)^{\frac{2(p-1)}{(-n-A+2)p+n+A+2}}\},
$$

with $b$ to be fixed later. Firstly we take $k = 1$, for $(t, r) \in \Omega_{s1}$ we find

$$
\omega_1 |u_{j+1}| \leq C_0 \Psi + C_0 (t - r) \|\omega_1 u_j\|_{L_{\infty, p}^\alpha(A)},
$$
$$
\omega_1 |u_{j+1} - u_j| \leq C_0 (t - r) \|\omega_1 (u_j - u_{j-1})\|_{L_{\infty, p}^\alpha(A)} \max_{l \in \{j, j-1\}} \|\omega_1 u_l\|_{L_{\infty, p}^\alpha(A)}^{p-1} \varepsilon
$$

and
Taking the $L^p_{\text{loc}}(\Omega_{+1})$ norm on both sides we get
\[
\|\omega_1 u_{j+1}\|_{L^p_{\text{loc}}(\Omega_{+1})} \leq \varepsilon C_0 \Psi + C_0 (\varepsilon \Psi)^{1-p}\|\omega_1 u_j\|_{L^p_{\text{loc}}(\Omega_{+1})},
\]
\[
\|\omega_1 (u_{j+1} - u_j)\|_{L^p_{\text{loc}}(\Omega_{+1})} \leq C_0 (\varepsilon \Psi)^{1-p}\|\omega_1 (u_j - u_{j-1})\|_{L^p_{\text{loc}}(\Omega_{+1})},
\]
holds for any $j$. This means $\{u_j\}$ is a Cauchy sequence in weighted $L^\infty$ norm on $\Omega_{+1}$ region. On the other hand, for $(t, r) \in \Omega_{+2}$, we separate $\Lambda$ into $\Lambda_1 := \Lambda \cap \Omega_{+1}$ and $\Lambda_2 := \Lambda \cap \Omega_{+2}$. Because of the linearity of function, we can separate $u_{j+1}(t, r)$ to two terms which determined by the nonlinear terms supported in $\Lambda_1$ and $\Lambda_2$ respectively. Taking $k = 1$ in the former region and $k = 2$ in the latter region, for $(t, r) \in \Omega_{+2}$ we find
\[
\omega_2 |u_{j+1}| \leq \varepsilon C_0 (t - r) \frac{(n-1)p-n-4}{2} \Psi + C_0 \|\omega_1 u_j\|_{L^p_{\text{loc}}(\Lambda_1)} + C_0 (t - r) \frac{(1-n)p+(n+1)p+2}{2} \|\omega_2 u_j\|_{L^p_{\text{loc}}(\Lambda_2)}
\]
\[
\omega_2 |u_{j+1} - u_j| \leq C_0 \|\omega_1 (u_j - u_{j-1})\|_{L^p_{\text{loc}}(\Lambda_1)} \max_{t \in \{j, j-1\}} \|\omega_1 u_j\|_{L^p_{\text{loc}}(\Lambda_1)} + C_0 (t - r) \frac{(1-n)p+(n+1)p+2}{2} \|\omega_2 (u_j - u_{j-1})\|_{L^p_{\text{loc}}(\Lambda_2)} \max_{t \in \{j, j-1\}} \|\omega_2 u_j\|_{L^p_{\text{loc}}(\Lambda_2)}
\]
Taking the $L^p_{\text{loc}}(\Omega_{+2})$ norm on both sides, noticing $(t - r) \geq (\varepsilon \Psi)^{1-p}$ for $(t, r) \in \Omega_{+2}$ and $(2C_0)^{p-1}\Psi^{1-p} = 1$, and using the estimate in $\Omega_{+1}$, when $T_* \geq 3$ we get
\[
\|\omega_2 u_{j+1}\|_{L^p_{\text{loc}}(\Omega_{+2})} \leq C_1 (\varepsilon \Psi)^p + C_1 T_* \frac{(1-n)p+(n+1)p+2}{2} \|\omega_2 u_j\|_{L^p_{\text{loc}}(\Omega_{+2})},
\]
\[
\|\omega_2 (u_{j+1} - u_j)\|_{L^p_{\text{loc}}(\Omega_{+2})} \leq C_1 (\varepsilon \Psi)^{p-1}\|\omega_1 (u_j - u_{j-1})\|_{L^p_{\text{loc}}(\Omega_{+1})} + C_1 T_* \frac{(1-n)p+(n+1)p+2}{2} \|\omega_2 (u_j - u_{j-1})\|_{L^p_{\text{loc}}(\Omega_{+2})} \max_{t \in \{j, j-1\}} \|\omega_2 u_j\|_{L^p_{\text{loc}}(\Omega_{+2})},
\]
with some $C_1$ large enough. Choosing $\varepsilon$ such that $T_* (\varepsilon) \geq 3$ and $a = (2C_1)^{p-1}\Psi^{1-p}$, we find
\[
\|\omega_2 u_{j+1}\|_{L^p_{\text{loc}}(\Omega_{+2})} \leq 2C_1 (\varepsilon \Psi)^p,
\]
\[
\|\omega_2 (u_{j+1} - u_j)\|_{L^p_{\text{loc}}(\Omega_{+2})} \leq C_1 (\varepsilon \Psi)^{p-1}\|\omega_1 (u_j - u_{j-1})\|_{L^p_{\text{loc}}(\Omega_{+1})} + \frac{1}{2} \|\omega_2 (u_j - u_{j-1})\|_{L^p_{\text{loc}}(\Omega_{+2})},
\]
holds for any $j$. Now, since we already know $\{u_j\}$ is a Cauchy sequence on $\Omega_{+1}$ region, we can also know $\{u_j\}$ is a Cauchy sequence in weighted $L^\infty$ norm on $\Omega_{+2}$ region. In summary, we get the desired solution.

Part 9: $p_d < p = ps$. 

In this part, we choose \( T_* \) which satisfies \( \varepsilon^{p(p-1)} \ln T_* = a \) and \( \Omega_{s+1}, \Omega_{s+2} \) as above with \( a, b \) to be fixed latter. Similarly, for \( (t, r) \in \Omega_1 \), we find
\[
\omega_1 |u_{j+1} - u_j| \leq C_0 |t - r|^{\frac{(n-p)A}{p}} \|u_j\|_{L^\infty_{x,r}(\Lambda)}^{p}.
\]
\[
\omega_1 |u_{j+1} - u_j| \leq C_0 |t - r|^{\frac{(n-p)A}{p}} \|u_{j+1} - u_{j-1}\|_{L^\infty_{x,r}(\Lambda)} \max_{l \in \{j, j-1\}} \|u_j\|_{L^\infty_{x,r}(\Lambda)}^{p-1}.
\]
For \( (t, r) \in \Omega_{s+2} \), we find
\[
\omega_2 |u_{j+1} - u_j| \leq C_0 |t - r|^{\frac{(n-p)A}{p}} \|u_j\|_{L^\infty_{x,r}(\Lambda)}^{p}.
\]
\[
\omega_2 |u_{j+1} - u_j| \leq C_0 |t - r| \|u_{j+1} - u_{j-1}\|_{L^\infty_{x,r}(\Lambda)} \max_{l \in \{j, j-1\}} \|u_j\|_{L^\infty_{x,r}(\Lambda)}^{p-1}.
\]
Taking \( \delta \) satisfying \( (2C_1)^p + \varepsilon \leq 1 \), choosing \( \varepsilon \) such that \( T_* (\varepsilon) \geq 3 \), \( a = (2C_1)^p \varepsilon^{p(p-1)} \) and \( C_1 \) large enough, we get the Cauchy sequence \{\( u_j \)\} and the desired solution.

Before the end of this section, we show the proof of Claim 3.3. Following (3.12), we can easily find
\[
\varepsilon^{\frac{2(p-1)}{p(p-1)}} \leq T_* \leq \varepsilon^{\frac{2(p-1)}{p(p-1)}}
\]
for any \( \varepsilon \) small enough. This suggests us to define
\[
S(\varepsilon) := \varepsilon^{-\frac{2(p-1)}{p(p-1)}}, \quad \varepsilon^\delta \leq S \leq 1.
\]
Then, (3.12) goes to
\[
a = S^{-\frac{h_F(p)}{2}} \ln \left( \varepsilon^{\frac{2(p-1)}{p(p-1)}}, S \right) \approx S^{-\frac{h_F(p)}{2}} |\ln \varepsilon|,
\]
and then
\[
S \approx |\ln \varepsilon|^{-\frac{2}{p(p-1)}}, \quad T_* \approx \varepsilon^{\frac{2(p-1)}{p(p-1)}}, \ln \varepsilon|^{-\frac{2}{p(p-1)}},
\]
which finishes the proof.

4. Long-time existence for \( A \in [3, \infty) \)

In this section, we will consider the case \( A \in [3, \infty) \), and show the proof of Theorem 1.2. Again, we only need to consider the equation (1.2).

4.1. Estimate for linear solution. In this subsection, we will construct some prior estimates of the solution to the linear equation (2.1). Firstly we give the following argument.

**Lemma 4.1.** Let \( u \) be the solution of (2.1). We have
\[
\|r^{\frac{A-1}{p}} u\|_{L^q_{t,r} L^p} \leq \|r^{\frac{A+1}{2}} g\|_{L^q_{t,r} L^p} + \|r^{\frac{A+1}{2}} f\|_{L^q_{t,r} L^p} + \|r^{\frac{A+1}{2}} F\|_{L^q_{t,r} L^p},
\]
for any \( 1 < q < \infty \), and
\[
\|r^{\frac{A-1}{p} - \alpha} u\|_{L^q_{t,r} L^p} \leq \|r^{\frac{A+1}{2}} g\|_{L^q_{t,r} L^p} + \|r^{\frac{A+1}{2}} f\|_{L^q_{t,r} L^p} + \|r^{\frac{A+1}{2}} F\|_{L^q_{t,r} L^p},
\]
for any \( 1 < q < \infty \), and
provided that
\begin{equation}
1 < \frac{q}{p} < \frac{\sigma}{p} < \infty, \quad \alpha p = 1 - \frac{p}{q} + \frac{p}{\sigma}.
\end{equation}

We also have a modification result.

**Lemma 4.2.** Let \( u \) be the solution of (2.1). If \( 1 < p \leq \frac{(n+1)}{(n-1)} \), then
\begin{equation}
(T + 1)^{\frac{(n-1)p-n-1}{2p}} \left\| u(T, r) \right\|_{L^p(r < T+1)} \leq \left\| r^{\frac{A-1}{2}} g \right\|_{L^\infty_p} + \left\| r^{\frac{A-1}{2} + \frac{\delta}{p}} f \right\|_{L^\infty_p} + \left\| r^{\frac{A-1}{2} + \frac{\delta}{p}} F \right\|_{L^1_p L^1_t(t < T)}.
\end{equation}

Also, if \( \frac{(n+1)}{(n-1)} \leq p \leq p_S \), then
\begin{equation}
T^{\frac{(n-1)p-n-1}{2p}} \left\| u(T, r) \right\|_{L^p} \leq \left\| r^{\frac{A-1}{2}} g \right\|_{L^\infty_p} + \left\| r^{\frac{A-1}{2} + \frac{\delta}{p}} f \right\|_{L^\infty_p} + \left\| r^{\frac{A-1}{2} + \frac{\delta}{p}} F \right\|_{L^1_p L^1_t(t < T)} + \left\| r^{\frac{A-1}{2}} g \right\|_{L^\infty_p} + \left\| r^{\frac{A-1}{2} + \frac{\delta}{p}} f \right\|_{L^\infty_p} + \left\| r^{\frac{A-1}{2} + \frac{\delta}{p}} F \right\|_{L^1_p L^1_t(t < T/4)}.
\end{equation}

**Proof of Lemma 4.1.** When \( A \in \mathbb{Z}_+ \), this result is exactly the same with the Theorem 4.7 of [11] since we can take \( \kappa \) in Theorem 4.7 arbitrary close to 2. So we only deal with the non-integer case.

Here we introduce \( \delta < \frac{(A-3)}{2} \) which will be fixed later. Firstly we consider \( u = u_g \). Using Lemma 2.1 and Lemma 2.2 with \( \mu = \frac{r^2 + \rho^2 - t^2}{2r} \), we have
\[
\int_{|t-r|}^{t+r} (1 + \mu)^{-\delta} \rho^{\frac{A-1}{2}} |g(\rho)| \, d\rho + \int_{0 < \rho < t-r} \frac{t}{2} \rho^{\frac{A-1}{2}} |g(\rho)| \, d\rho + \int_{0 < \rho < t-r} (1 - \mu)^{-\delta} \rho^{\frac{A-1}{2}} |g(\rho)| \, d\rho
\]
\[
\leq \int_{|t-r|}^{t+r} (1 + \mu)^{-\delta} \rho^{\frac{A+1}{2}} |g(\rho)| \, d\rho + \int_{0 < \rho < t-r} (1 - \mu)^{-1} (1 + \mu)^{-\delta} \rho^{\frac{A+1}{2}} |g(\rho)| \, d\rho,
\]
where the second integral does not appear for \( t < r \). Using Proposition 2.7 and Proposition 4.4 in [11], we obtain the estimate as we desired.
Next we consider \( u = u_f \). Here for simplicity we denote \( h(\rho) := \rho^{\Delta - 1} f(\rho) \). Using Lemma 2.1 and Lemma 2.2 again with \( \mu = \frac{r^2 + \rho^2 - t^2}{2r\rho} \), for \( r < t \) we find

\[
\begin{align*}
    r^{\frac{\Delta - 1}{2}} u_f &= \frac{1}{2} h(t + r) - \text{P.V.} \int_0^t \frac{t}{r^2} I_\nu(\mu) h(\rho) \, d\rho, \\
    r^{\frac{\Delta - 1}{2}} |u_f| &\lesssim \frac{1}{2} |h(t + r)| + \text{P.V.} \int_0^t \frac{t}{r^2} (1 + \mu)^{-1} h(\rho) \, d\rho + \int_{\frac{r^2 + \rho^2 - t^2}{2r\rho} < \mu} \frac{t}{r^2} |1 - \mu|^{-\frac{1}{2}} |h(\rho)| \, d\rho \\
    &\lesssim \frac{1}{2} |h(t + r)| + \text{P.V.} \int_0^t \frac{t}{r^2} (1 + \mu)^{-1} h(\rho) \, d\rho + \int_0^{t-r} \frac{t}{r^2} |1 - \mu|^{-1} |h(\rho)| \, d\rho + \int_0^{t+r} \frac{t}{r^2} |1 + \mu|^{-1} |h(\rho)| \, d\rho + \int_{t-r}^{t+r} \frac{t}{r^2} |1 + \mu|^{-1} |h(\rho)| \, d\rho \\
    &\equiv K_1 + K_2 + K_3 + K_4 + K_5.
\end{align*}
\]

It’s easy to find that

\[
\| K_1 \|_{L_t^\infty L_x^2} \lesssim \| h \|_{L_x^2}.
\]

On the other, we introduce the well known \textit{Hardy-Littlewood} inequality

\[
\| |y|^{-\alpha} f(x + y) \|_{L_t^\infty L_x^p(\mathbb{R}^2)} \lesssim \| f \|_{L_t^q(\mathbb{R})},
\]

with (4.3). Now, taking \( f(x) = |h(x)| \chi_{|0,\infty)}(x) \), we also find

\[
\| r^{-\alpha} K_1 \|_{L_t^\infty L_x^p} \lesssim \| h \|_{L_x^2},
\]

provided that (4.3). Meanwhile, adopting Proposition 2.5 in [11] to deal with \( K_4 \), and adopting Proposition 4.4 in [11] for \( K_3 \) and \( K_5 \), we find both of them have the same control as \( K_1 \). As for \( K_2 \), noticing

\[
\frac{t}{r^2} (1 + \mu)^{-1} = \frac{1}{r + \rho - t} - \frac{1}{t + r + \rho},
\]

we can control \( K_2 \) by

\[
\begin{align*}
    K_2 &\lesssim \text{P.V.} \int_0^\infty \frac{1}{\rho + r - t} h(\rho) \, d\rho + \int_0^\infty \frac{1}{\rho + r - t} |h(\rho)| \, d\rho \\
    &\quad + \int_{\rho + r - t}^{t + r} |h(\rho)| \, d\rho \\
    &\equiv K_{2,1}(t - r) + K_{2,2}(t, r) + K_{2,3}(t + r).
\end{align*}
\]

For \( K_{2,1} \), we introduce the estimate of \textit{Hilbert}-transform

\[
\left\| \text{P.V.} \int \frac{1}{x - y} f(y) \, dy \right\|_{L_x^2} \lesssim \| f \|_{L_x^2}
\]

with \( 1 < q < \infty \). Taking \( f(x) = h(x) \chi_{|0,\infty)}(x) \), we find

\[
\| K_{2,1}(t - r) \|_{L_t^\infty L_x^q} = \| K_{2,1}(t) \|_{L_x^q} \lesssim \| h \|_{L_x^2}.
\]
Also, using Hardy-Littlewood inequality again, we get the dominate of $K_{2,1}(t-r)$ same as that of $K_1$ provided that $(4.3)$. As for $K_{2,3}$, we introduce the Hardy-Littlewood maximal inequality

$$
\left\| \sup_{y>0} \frac{1}{2y} \int_{\frac{y}{2}}^{y+\frac{y}{2}} f(z) \, dz \right\|_{L_2^q} \lesssim \|f\|_{L^q_t}
$$

with $1 < q < \infty$. Taking $f(x) = h(x) \chi_{[0,\infty)}(x)$, we find

$$
\|K_{2,3}(t+r)\|_{L_\infty^q \rightarrow L_2^p} = \|K_{2,3}(r)\|_{L_\infty^q} \lesssim \|h\|_{L_2^q}.
$$

Using Hardy-Littlewood inequality again, we get the dominate of $K_{2,3}(t+r)$ same as that of $K_1$ provided that $(4.3)$. Finally, for $K_{2,2}(t,r)$, we have

$$
K_{2,2}(t,r) = \int_r^\infty \frac{1}{\rho + r} |h(\rho + t)| \, d\rho \leq \int_r^\infty \rho^{-1} |h(\rho + t)| \, d\rho.
$$

Then, using Hardy’s inequality we find

$$
\|K_{2,2}\|_{L_\infty^q \rightarrow L_2^p} \lesssim \|h\|_{L_2^q}.
$$

On the other hand, for any $G(t,r)$ with $\|G\|_{L_\infty^{q'} \rightarrow L_2^p} \leq 1$ we see

$$
\int_0^\infty \int_0^\infty r^{-\alpha} K_{2,2}(t,r) G(t,r) \, dr \, dt
$$

$$
= \int_0^\infty \int_t^\infty \int_0^{\rho - t} r^{-\alpha} |h(\rho)| G(t,r) \, dr \, d\rho \, dt
$$

$$
\lesssim \int_0^\infty \int_t^\infty |h(\rho)| \left\| \frac{r^{-\alpha}}{\rho + r - t} \right\|_{L_\infty^q(0,\rho-t)} \|G\|_{L_\infty^{q'}} \, d\rho \, dt
$$

$$
\approx \int_t^\infty \int_t^\infty |h(\rho)| |\rho - t|^{-\alpha - 1 + \frac{1}{p'}} \|G\|_{L_\infty^{q'}} \, d\rho \, dt
$$

$$
\lesssim \int_t^\infty |h(\rho)| |\rho - t|^{-\alpha - 1 + \frac{1}{p'}} \, d\rho \|G\|_{L_\infty^q}
$$

$$
\lesssim \|h\|_{L_\infty^q},
$$

where in the last step we use the Hardy-Littlewood inequality. Now, we find

$$
\|r^{-\alpha} K_{2,2}\|_{L_\infty^q \rightarrow L_\infty^p} \lesssim \sup_{\|G\|_{L_\infty^{q'} \rightarrow L_2^p} \leq 1} \left\langle r^{-\alpha} K_{2,2}, G \right\rangle \lesssim \|h\|_{L_\infty^q}.
$$

Mixing these results, we obtain the estimate for $r < t$ part. For $r > t$, we have

$$
r^{\frac{\alpha-1}{2}} u_f = \frac{1}{2} h(t + r) + \frac{1}{2} h(r - t) - \int_{r-t}^{t+r} \frac{t}{r \rho} I'_A(\mu) h(\rho) \, d\rho
$$

$$
eq K'_1 + K'_2 + K'_3.
$$

The estimate of $K'_1$ and $K'_2$ is the same as that of $K_1$ in $r < t$ part, and the estimate of $K'_3$ is the same as that of $K_4$. Adding all together, we finish the proof of $u_f$ part.
Finally, we consider $u = u_F$. Using Lemma 2.1 and Lemma 2.2 again with $\mu = \frac{r^2 + p^2 - (r - s)^2}{2r \rho}$, similarly we have
\[
\begin{align*}
r^{\frac{4-1}{2}} u_F &= \int_0^1 \int_{t-r+s}^{t+s} I_A(\mu) \rho^{\frac{4-1}{2}} F(s, \rho) \, d\rho \, ds \\
r^{\frac{4-1}{2}} |u_F| &\lesssim \int_0^1 \int_{t-r+s}^{t+s} (1 + \mu)^{-\delta} \rho^{-1} |G(s, \rho)| \, d\rho \, ds \\
&\quad + \iint_{0 < \rho \leq \frac{r}{t-s-r}} (1 - \mu)^{-1} |1 + \mu|^{-\delta} \rho^{-1} |G(s, \rho)| \, d\rho \, ds
\end{align*}
\]

where $G(s, \rho) := \rho^{\frac{4-1}{2}} F(s, \rho)$ and the second integral does not appear for $t - s < r$. Here we choose $\delta$ small enough. Using Proposition 4.5 in [11], we obtain the desired estimate for $u_F$. Now, we finish the proof of Lemma 4.1. \hfill \square

**Proof of Lemma 4.2.** The proof of Lemma 4.2 is almost the same with that of Theorem 6.4 in [11]. Thus, we only give a sketch of the proof. The estimate (4.4) and part of estimate (4.5) are direct consequence of (4.1) with $q = p$. We only need to show the estimate of $T \leq \rho - n + \frac{1}{2p} + 1 \parallel \rho^{\frac{n-1}{2p}} u(T, r) \parallel_{L_t^\infty(r < T/4)}$. To dominate $u_f$, we separate $f = f_0 + f_1$ with $f_0 = \chi_{[0, T/4]} f$. Then, $u_f = f_1$ depends on $f_1(\rho)$ with $T/4 < \rho < 5T/4$. So, we obtain the weight of $T$ by extracting the weight of $\rho$. For $u_f = f_0$, we find that $\rho, r < T/4$ in the expression of $u_f = f_0$ where $|1 \pm \mu|^{-1} \approx r \rho / T^2$. Then we get the desired estimate by a direct calculation.

The estimate of $u_g$ is similar to that of $u_f$. Finally for $u_F$, we separate the integral in $u_F$ into three parts: \{r \geq (T - s)/4\}, \{r, \rho \leq (T - s)/4\} and \{r \leq (T - s)/4 \leq \rho\}. Then we get the estimate by a similar discussion. \hfill \square

### 4.2. Long-time existence for $1 < p < p_{\text{conf}}$

In this subsection, we will give the proof of Theorem 1.2. The main process of proof is almost the same as that of [11, Theorem 5.1, Theorem 6.1 and Theorem 6.3]. So we only prove global existence in $p_S < p < p_{\text{conf}}$ and long-time existence in $p_m \leq p < p_S$ to show such processes fit our frame.

**Part 1:** Proof of $p_S < p < p_{\text{conf}}$

Similar to the last section, we will construct a Cauchy sequence to approach the weak solution. We set $u_{-1} = 0$ and let $u_{j+1}$ be the solution of the equation (3.11).

We are going to use the estimate (4.2), where we set
\[
q = \frac{2(p - 1)}{(n + 3) - (n - 1)p}, \quad \sigma = pq, \quad \alpha = \frac{(n - 1)p - n - 1}{2p}.
\]

Here $p < p_{\text{conf}}$ so $0 < q < \infty$ and $p > p_S$ so $q > p$. Then, we conclude
\[
\begin{align*}
\left\| r^{\frac{(A-n)p+n+1}{2p}} u_{j+1} \right\|_{L_t^{q}L_x^p} &\leq \varepsilon C_0 \Psi + C_0 \left\| r^{\frac{(A-n)p+n+1}{2}} |u_j|^p \right\|_{L_t^1 L_x^1} \\
&\leq \varepsilon C_0 \Psi + C_0 \left\| r^{\frac{(A-n)p+n+1}{2}} u_j \right\|_{L_t^{q}L_x^p}^{p-1} \left\| r^{\frac{n-1}{2p}} U_1 \right\|_{L_x^p} + \left\| r^{\frac{n-1}{2p}} U_0 \right\|_{L_x^p}
\end{align*}
\]

for some $C_0$ large enough. Then, for any $\varepsilon$ satisfies $(2\varepsilon C_0)^p < \varepsilon \Psi$ we find
\[
\left\| r^{\frac{(A-n)p+n+1}{2p}} u_j \right\|_{L_t^{q}L_x^p} \leq 2\varepsilon C_0 \Psi
\]
holds for any $j \geq 0$. By this result and (4.2) we also find
\[ \left\| r \frac{(A_n)_{j+1} + 1}{2p} (u_{j+1} - u_j) \right\|_{L^p_t L^p_x} \]
\[ \leq C_0 \left\| r \frac{(A_n)_{j+1} + 1}{2p} \left( |u_j|^p - |u_{j-1}|^p \right) \right\|_{L^1_t L^1_x} \]
\[ \leq C_1 \left\| r \frac{(A_n)_{j+1} + 1}{2p} (u_j - u_{j-1}) \right\|_{L^p_t L^p_x} \]
\[ \max_{k \in \{j, j-1\}} \left\| r \frac{(A_n)_{j+1} + 1}{2p} u_k \right\|_{L^p_t L^p_x} \]
\[ \leq C_1 (2\varepsilon C_0 \Psi)^{p-1} \left\| r \frac{(A_n)_{j+1} + 1}{2p} (u_j - u_{j-1}) \right\|_{L^p_t L^p_x} \]
with some $C_1$ large enough. Now, for any $0 < \varepsilon$ with $C_1 (2\varepsilon C_0 \Psi)^{p-1} < 1/2$, we find
\[ \{u_j\} \text{ is a Cauchy sequence in its space.} \]
Set the limit as $u$. Using (4.1) we can also find
\[ r \frac{A_n}{2} \|u\|_{L^p_t L^p_x} \leq 2\varepsilon C_2 \Psi \]
with some $C_2$. To check it is the weak solution of (1.2) indeed, we need to show (2.10). For any compact set $K_t \times K_r \subset \mathbb{R}^2$, we find
\[ \left\| r \frac{(A_n)_{j+1} + 1}{2p} - u_j \right\|_{L^p_t L^p_x} \leq C(K_t) \left\| r \frac{(A_n)_{j+1} + 1}{2p} - u_j \right\|_{L^p_t L^p_x} \]
\[ \leq (2\varepsilon C_0 \Psi)^{p-1} C(K_t) \Psi \]
\[ \leq (2\varepsilon C_1) \Psi \Psi C(K_t) \Psi \leq \infty. \]
This finishes the proof.

**Part 2**: Proof of $p_m \leq p < p_S$.

Next we consider $(n+1)/(n-1) < p < p_S$. Set
\[ A_j(T) := T \frac{(A_n)_{j+1} + 1}{2p} u_j(T, r) \]
using (4.5) we see
\[ A_{j+1}(T) \leq \varepsilon C_0 \Psi + C_0 T^{\frac{1}{2}} \left\| r \frac{(A_n)_{j+1} + 1}{2p} |u_j|^p \right\|_{L^p_t L^p_x(T/4 < t < T)} \]
\[ + C_0 \left\| r \frac{(A_n)_{j+1} + 1}{2p} |u_j|^p \right\|_{L^p_t L^p_x(t < T/4)} \]
\[ \leq \varepsilon C_0 \Psi + C_0 T^{\frac{1}{2}} \left\| r \frac{(A_n)_{j+1} + 1}{2p} A_j(t) \right\|_{L^p_t L^p_x(T/4 < t < T)} \]
\[ + C_0 \left\| r \frac{(A_n)_{j+1} + 1}{2p} A_j(t) \right\|_{L^p_t L^p_x(t < T/4)} \]
for some $C_0$ large enough and
\[ \Psi := \left\| r \frac{n+1}{2p} + \frac{1}{2} U_1 \right\|_{L^p_t} + \left\| r \frac{n+1}{2p} + \frac{1}{2} U_0 \right\|_{L^p_t} + \left\| r \frac{n+1}{2p} U_1 \right\|_{L^p_t} + \left\| r \frac{n+1}{2p} U_0 \right\|_{L^p_t}. \]
When $\sup_{0 \leq T \leq T_*} A_j(T) \leq 2\varepsilon C_0 \Psi$ holds for some $j$ and $T_*$ defined in (1.8) with $c$ to be fixed later, we find
\[ \sup_{0 \leq T \leq T_*} A_{j+1}(T) \leq \varepsilon C_0 \Psi + C_1 (2\varepsilon C_0 \Psi)^{P T_*} \]
\[ \leq 2\varepsilon C_0 \Psi + \varepsilon C_1 (2\varepsilon C_0 \Psi)^{\frac{P c}{2p}} \]
with some $C_1$. Choosing $c$ small enough such that $C_1 (2\varepsilon C_0 \Psi)^{\frac{P c}{2p}} \leq C_0 \Psi$ and noticing $A_{-1}(T) \equiv 0$, we find such estimate holds for any $j$. A similar manner also
shows
\[
\sup_{0 \leq T \leq T_0} B_j(T) \leq \frac{1}{2} \sup_{0 \leq T \leq T_0} B_j(T),
\]
\[
B_j(T) := T^{\frac{(n-1)p-n-1}{2p}} \left\| t^{\frac{(A-n)p+n+1}{2p}} \left( u_j - u_j - 1 \right)(T, r) \right\|_{L^p_r}.
\]

Then, we get the desired solution \( u \) as the limit of \( \{u_j \} \). Now we check (2.10). For any compact set \( K_t \times K_r \subset \mathbb{R}^3_+ \), we find
\[
\left\| t^{(n-A)p-n-1} |u|^p \right\|_{L^1_t(K_t \times K_r)} \leq t^{\frac{(n-1)p-n-1}{2p}} \left\| t^{\frac{(A-n)p+n+1}{2p}} u \right\|_{L^\infty_t L^p_r}^p \times t^{\frac{(1-n)p+n+1}{2p}} \left\| t^{\frac{A-3}{2p}} \right\|_{L^1_t L^p_r(K_t \times K_r)}^p,
\]
\[
\left\| t^{A-1} u \right\|_{L^1_t(K_t \times K_r)} \leq t^{\frac{(n-1)p-n-1}{2p}} \left\| t^{\frac{(A-n)p+n+1}{2p}} u \right\|_{L^\infty_t L^p_r}^p \times t^{\frac{(1-n)p+n+1}{2p}} \left\| t^{\frac{(A-2)p-n-1}{2p}} \right\|_{L^1_t L^p_r(K_t \times K_r)}^p.
\]

Noticing \( \frac{(n-A-2)p-n-1}{2p} > \frac{A-3}{2p} \geq 0 \) and \( \frac{(1-n)p+n+1}{2p} > -1 \) due to \( p < p_S < p_{\text{conf}} \), we find both of the above two terms are finite. This finishes the proof.

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