The circular Kardar-Parisi-Zhang equation as an inflating, self-avoiding ring polymer

Silvia N. Santalla,1 Javier Rodríguez-Laguna,2,3 and Rodolfo Cuerno3

1Departamento de Física and Grupo Interdisciplinar de Sistemas Complexos (GISC), Universidad Carlos III de Madrid, Leganés, Spain
2ICFO–Institut de Ciències Fotòniques, Castelldefels, Spain
3Departamento de Matemáticas and GISC, Universidad Carlos III de Madrid, Leganés, Spain

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We consider the Kardar-Parisi-Zhang (KPZ) equation for a circular interface in two dimensions, unconstrained by the standard small-slopes and no-overhang approximations. Numerical simulations using an adaptive scheme allow us to elucidate the complete time evolution as a crossover between a short-time regime with the interface fluctuations of a self-avoiding ring or 2D vesicle, and a long-time regime governed by the Tracy-Widom distribution expected for this geometry. For small noise amplitudes, scaling behavior is only of the latter type. Large noise is also seen to renormalize the bare physical parameters of the ring, akin to analogous parameter renormalization for equilibrium 3D membranes. Our results bear particular importance on the relation between relevant universality classes of scale-invariant systems in two dimensions.

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Recently, statistical fluctuations are revealing interesting features for a number of one-dimensional systems confined to circular geometries. For instance, for semiflexible polymers of a fixed length, like constrained DNA rings, the closure condition influences the scaling, shape, and transport behavior [1]. Topology is actually expected to play a key role in a large number of related biophysical processes, such as e.g. translocation in nanochannels or nanopores [2, 3], or knot localization [4]. In particular, circular DNA molecules in two dimensions have been experimentally found [1] to be well described as pressurized vesicles [3], their scaling behavior depending on the geometry [6]. Thus, for deflated rings (negative pressure difference, Δp), fluctuations are in the universality class of lattice animals, while for Δp = 0, statistics are those of a ring self-avoiding walk (SAW) [1]. The latter is important as a paradigmatic model of polymers [7] and because the SAW is believed to constitute a conformally invariant system in two dimensions [8].

For planar rings evolving far from equilibrium, the closure condition is also proving non-trivial, as recently observed in experiments with droplets of turbulent liquid crystals [9], for the edge of a drying colloidal suspension [10, 11], and for many more systems, from epitaxy to bacterial growth [12]. Thus, as proposed in [13], the probability distribution function (pdf) of the height fluctuations for interfaces which, like many of these, belong to the Kardar-Parisi-Zhang (KPZ) universality class [14, 15], depends on the global curvature. The eponymous equation [16], which is the prime representative for these systems, is a continuum model for the evolution of a rough interface between a (stable, e.g. solid) phase that grows at the expense of a(n unstable, e.g. vapor) phase,

\[ \partial_t h = v + \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\mathbf{x}, t) \]

where \( h(\mathbf{x}, t) \) is the height field above substrate position \( \mathbf{x} \in \mathbb{R}^d \) at time \( t \), \( \eta \) is Gaussian white noise, \( v \) is the growth speed for a flat interface, and \( \nu > 0 \), \( \lambda \) are additional parameters. On the one hand, from the point of view of the theory of stochastic processes, the KPZ equation features a remarkable example of a time crossover [17, 18] between the two main universality classes of kinetic roughening [11, 12], namely, the Edwards-Wilkinson (EW) class at short times, and the KPZ class at long times. Experimentally [3], however, while such a crossover may have been seen in \( d = 1 \) for interfaces with a null global curvature, it has not for the circular geometry case. On the other hand, for such ring-shaped interfaces height statistics are indeed distinctively described [13] by the Tracy-Widom (TW) pdf for the largest-eigenvalue of large random matrices in the Gaussian unitary ensemble (GUE), as recently supported by exact solutions of the KPZ equation on an infinite substrate and a wedge initial condition (but without explicit closure), or for related systems [13]. This actually occurs with a remarkable degree of universality [17, 20], as the same pdf, critical exponents, and limiting processes apply to discrete models, continuum equations [21], and experiments [3]. Hence, in two dimensions (2D) both pressurized vesicles and the KPZ equation not only demonstrate the non-trivial role of geometry, as a part of the universality class and related renormalization-group fixed point [3], in and out of equilibrium, respectively.

Note, Eq. (5) is just the small-slope, single-valued approximation of a more general equation [22],

\[ \partial_t \mathbf{r} = (A_0 + A_1 K(\mathbf{r}) + A_2 \eta(\mathbf{r}, t)) \mathbf{u}_n, \]

where \( \mathbf{r}(t) \in \mathbb{R}^{d+1} \) gives the interface position, \( K(\mathbf{r}) \) is the local extrinsic surface curvature, \( \mathbf{u}_n \) is the normal direction pointing towards the unstable phase, and constants \( A_0, A_1, \) and \( A_2 \) relate to parameters in Eq. (5) in a simple way [22]. They account for, respectively, the...
average growth speed along the local normal direction, surface tension effects, and noise in the local growth velocity, precisely the physical mechanisms at play in the formulation of the KPZ equation as a continuum interface model \(^{(10)}\). However, those produced by Eq. \((6)\) are not constrained to small slopes or lack of overhangs \(^{(22)}\).

For a ring geometry and \(d = 1\), Eq. \((5)\) actually has to be given up in favor of Eq. \((6)\), since the closure condition hinders description of the interface profile by a single-valued function altogether. Alternative formulations to Eq. \((6)\) are available, see e.g. in \(^{(12)}\), although most include the neglect of overhangs, and/or additional simplifications. A natural question is then whether Eqs. \((5)\) and \((6)\) have the same dynamic scaling properties. Here we show that, for planar rings, this is not the case. Namely, while asymptotics are indeed of the expected TW-GUE type also for Eq. \((6)\), which implements explicitly a closure condition, the early times differ substantially as compared with Eq. \((5)\); now, for small noise amplitudes no scaling behavior other than KPZ is obtained, as in experiments \(^{(9)}\), while 2D-SAW universality is obtained at short times for large noise amplitude values \(A_{n}\). Such large fluctuations renormalize additional parameters like \(A_{1}\), in a form that is reminiscent of surface tension renormalization by non-equilibrium fluctuations, as experimentally assessed e.g. in \(^{(23)}\) for 3D active membranes. In parallel with the equilibrium behavior of 2D vesicles \(^{(3)}\) as a function of \(\Delta p\), the change from early-time SAW to late-time KPZ scaling behavior correlates with an evolution in time from a freely fluctuating ring to an average circular shape. Thus, the generalized KPZ equation \((6)\) also predicts a crossover to occur during the time evolution of the system between two equally celebrated universality classes under large noise conditions. From this point of view the experiments in \(^{(3)}\) correspond to a small-noise condition, while a prediction is provided for suitable large noise situations, which should be amenable to experimental verification.

We have performed numerical simulations of Eq. \((6)\) using planar rings of various initial radii \(R_{0}\) and center \((0,0)\) as initial conditions. We employ an adaptive algorithm as in \(^{(22,24)}\), which does not need to assume a single-valued polar function. The interface is represented by a chain of points (a “polymer”) \(\{P_{i}(t)\}_{i=1}^{N(t)} \subset \mathbb{R}^{2}\) defining a piecewise continuous curve which always leaves the stable phase on its left. The distance between them is forced to remain in an interval \([l_{\text{min}}, l_{\text{max}}]\), which is achieved by inserting or removing points dynamically. Interface properties, like curvature, are evaluated in a geometrically natural way \(^{(22)}\). Unavoidably, self-intersections occur along the evolution. We always remove the smaller interface component, eliminating both cavities and out-growths, thus implementing self-avoidance and rendering the dynamics irreversible. This approximation is akin to restricting dynamics to that of the active zone in growth systems \(^{(22)}\). Time updates are via an Euler-Maruyama scheme with spacing \(\Delta t\), sufficiently small that it does not appreciably influence results.

A set of representative snapshots are shown in Fig. 1 for \(R_{0} = 10\), \(A_{0} = 0.01\), \(A_{1} = 0.01\), \(A_{n} = 1\), and different times. Qualitatively, the ring can be seen to undergo two different regimes: (I) for \(t \lesssim 100\) it fluctuates without significant growth while, its shape becoming less and less circular; (II) for \(t \gtrsim 400\), the ring grows steadily, progressively recovering an average circular shape. In order to interpret these observations, we can consider the deterministic case, i.e., \(A_{n} = 0\). Rings with smaller \(R_{0}\) than a certain threshold shrink, since the constant average velocity \(A_{0}\) is not able to compensate for the effect of surface tension \(A_{1}\). On the other hand, for \(R_{0} \gtrsim A_{1}/A_{0}\), the ring grows very slowly at first, and with velocity \(A_{0}\) for longer times.

From Eq. \((6)\), a simplified evolution equation can be derived for the average ring radius \(R(t)\),

\[
\frac{dR(t)}{dt} = \tilde{A}_{0} + \frac{\tilde{A}_{1}}{R(t)},
\]

where local variations in the normal velocity are neglected and the total ring length \(L(t)\) is approximated by that of the average circle, see details in \(^{(25)}\). Here, \(A_{0,1}\) have values that will in general differ from their “bare” counterparts, \(A_{0,1}\), due to noise-induced renormalization. Fig. 2 shows \(R(t)\) for the same parameter choice of Fig. 1. Regimes I and II are clearly distinguished in the growth rate. Remarkably, the numerical \(\tilde{R}(t)\) fits the exact solution of Eq. \((12)\) for \(\tilde{A}_{0} \approx 0.026 > 0.01 = A_{0}\) and \(\tilde{A}_{1} \approx 0.1 > 0.01 = A_{1}\). For small noise amplitudes, no such noise renormalization occurs, see \(^{(25)}\). Thinking of \(A_{0}\) as a pressure difference that attempts to “inflate” the ring \(^{(3)}\), the effect of \(A_{0}\) can be thought of as a fluctuation-induced pressure boost \(\tilde{A}_{0}\). Alongside, \(\tilde{A}_{1}\) becomes an enhanced surface tension, due to the noisy dynamics. Similar fits are obtained for a wide range of

![FIG. 1: (Color online) (a) Interface evolution for \(R_{0} = 10\), \(A_{0} = 0.01\), \(A_{1} = 0.01\), and \(A_{n} = 1\), with \(\Delta t = 0.1\), \(l_{\text{min}} = 0.1\), and \(l_{\text{max}} = 1\). Curves for times \(t = 2, 20, 150, 1000\), and 5000, inner to outer. (b) Rescaled view to ease comparison. All units are arbitrary.](image-url)
bare parameters.

Fig. 2 (a) also depicts the numerical evolution of the actual \( L(t) \) (for a space cut-off \( l_{\text{min}} \)). Very early in regime I, while the average radius remains almost constant, this length increases due to fluctuations. In regime II, when \( R(t) \) grows steadily, \( L(t) \) actually becomes proportional to it. This behavior is appreciated in Fig. 2(b), where the \( L(t) \) is plotted vs. \( R(t) \). There is a threshold total length, proportional to \( R_0 \), below which no radial growth occurs, and above which both measures become proportional. As seen in Fig. 1 prior to regime II noise basically induces loss of the initial circular symmetry. In Fig. 2 (c) we quantify this effect by plotting the asymmetry parameter \( \Sigma(t) = \langle S_{G1}^2 / S_{G2}^2 \rangle \), i.e. the ratio of the smallest to largest eigenvalues, \( S_{G1}^2, S_{G2}^2 \), of the gyration tensor \( S \). This is frequently used to assess polymer classes in terms of self-avoidance, dimensionality, rigidity, etc. [1, 6, 27, 28]. In our case, \( \Sigma(t) \) decreases with time, reaches a minimum value \( \Sigma(t \approx 400) \approx 0.68 \), and increases back, approaching the characteristic value of a circular swollen polymer in regime II. Other measures of anisotropy lead to the same conclusion, see e.g. Fig. 2 (c) for the quadrupole moment, \( Q^2 = \langle |x^2 - y^2| \rangle / \langle r^2 \rangle \), where \( x, y, \) and \( r \) are relative to the CM of the \( \{ P_i \}_{t=1}^{N(t)} \) point distribution.

Given the relevance of fluctuations in these dynamics, we assess them in Fig. 3 (a), where we show the time evolution of the global interface roughness, \( W_{\text{CM}}(t) = \langle (P(t) - R_{\text{CM}}(t))^2 \rangle^{1/2} \), with \( R_{\text{CM}}(t) \) being the position of the CM [29]. Scaling behavior \( W(t) \sim t^{\beta} \) holds, with \( \beta \approx 1/3 \), both in regimes I and II. As standard for kinetic roughening systems in a circular geometry, \( W(t) \) does not saturate due to the non-interrupted growth of the system size [30]. Moreover, as indicated in Fig. 2 (c), during regime II the quadrupole moment \( Q \) decays as \( t^{-2/3} \), which follows if we estimate \( Q \) as the ratio of the radial

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**FIG. 2:** (Color online) Evolution of interface shape for the case shown in Fig. 1 (a) Interface length (above) and circle length \( 2\pi R(t) \) (below), with \( R(t) \) the fitted radius, vs. time. For long times, both are linear in \( t \). Dashed line: fit to a renormalized deterministic growth. (b) Interface length vs. radius for different initial radii, \( R_0 = 10, 20, 30, 40 \), and 50 (growing upwards). (c) Anisotropy vs. time: \( \Sigma(+) \), left vertical axis, and quadrupole moment \( Q(x) \), right vertical axis. The slope of the straight line, \(-0.667\), corresponds to a fit for long times. Units are arbitrary.

**FIG. 3:** (Color online) Evolution of the interface fluctuations for the case shown in Fig. 1 (a) Global roughness (+) and standard deviation of \( R(t) \) (×) vs. time. Both straight lines have slopes 1/3. (b) \( W(t) \) for decreasing noise amplitude, as in legend, top (same data as in main panel) to bottom. (c) Local roughness vs. window size, for \( t = 2, 20, 1200, 8000, \) and 14000, bottom to top. Straight lines have slopes as in the legend. (d) Correlation length vs. time. Straight lines have slopes 1/2 (lower left corner) and 2/3 (upper right corner). All units are arbitrary.
fluctuations to the average radius, $t^{3-1} \approx t^{-2/3}$. We also measure the local roughness $w(l, t)$, namely, the interface fluctuations (restricted to windows of size $l$) around a fitting circular arc, which is drawn with respect to the CM. Data are shown in Fig. 3(c) as functions of $t$, for several times. Scaling behavior ensues, $w(l) \sim l^\alpha$, provided that, as in the standard Family-Vicsek (FV) Ansatz [14], the window size $l$ is smaller than a correlation length $\xi(t)$, which itself grows as $\xi(t) \sim t^{1/2}$. The FV scaling relation $z = \alpha/\beta$ holds for exponent values $(\alpha, z)$ which are $(2/3, 2)$ in regime I, and $(1/2, 3/2)$ in regime II, see Fig. 3(d). Indeed, in both cases $\beta = 1/3$, as implied by $W(t)$. Hence, the fractal dimension, $D_F = 2 - \alpha$ [14], changes from $4/3$ in regime I to $3/2$ in regime II. Overall, the evolution is from kinetic roughening in the SAW class (regime I), for which $s_{\text{SAW}} = 2/3$ [7] and $\beta_{\text{SAW}} = 1/3$ [31, 32], to asymptotic KPZ scaling in regime II, for which $s_{\text{KPZ}} = 1/2$ and $\beta_{\text{KPZ}} = 1/3$ [14]. If the noise amplitude decreases significantly ($A_n \lesssim 0.01$), the roughness remains constant in regime I, namely the SAW stage disappears, the only measurable scaling behavior being the KPZ asymptotics in regime II, as in the experiments for circular geometry [3]. See Fig. 3(b).

The progressive dominance of radial fluctuations can be appreciated in Fig. 3(a), where we plot the time evolution for the standard deviation of the random variable $R(t)$. Although this quantity grows fast with $t$, numerically it remains much smaller than $W(t)$ until onset of regime II, after which both remain proportional. Actually, we can further explore radial fluctuations in the asymptotic KPZ regime. Thus, following Práhofer and Spohn [13], we rewrite

$$R(t) \approx \rho_0 + V t + \Gamma t^\beta \chi,$$

where $\rho_0$, $V$ and $\Gamma$ are constants, $\beta = \beta_{\text{KPZ}}$, and $\chi$ is a random variable with zero average and unit variance, whose probability distribution is stationary and corresponds to the GUE-TW distribution [4, 11, 13, 19]. We have collected the instantaneous radii data for $17$ different times in the range $t \in [700, 1500]$, i.e. well within regime II, for $2500$ noise realizations, in order to check this conjecture. Results are shown in Fig. 3(a), where we plot the probability distribution of $\chi$, obtained following the procedure described in [33], and compare it with the analytical result [17]. Moreover, we have measured the third and fourth cumulants of this $\chi$ distribution, i.e. the skewness and kurtosis, which are, respectively, $\langle \chi^3 \rangle_c \approx 0.233$ and $\langle \chi^4 \rangle_c \approx 0.0733$. The theoretical values for the TW-GUE distribution are $0.224$ for the skewness and $0.093$ for the kurtosis, which are close enough. For comparison [13], for the TW-GOE distribution the skewness is $0.293$ and the kurtosis $0.165$, both being zero for the Gaussian distribution. Fig. 3(b) shows the time evolution of the cumulants of $R(t)$ towards the TW-GUE values. We must remark the negative sign that we obtain for parameter $\Gamma$ in Eq. 4, implying a negative skewness for $R(t)$. Physically, this is due to the fact (data not shown) that, in regime II, the number of cavities removed per unit length and unit time by the self-intersection removal condition is smaller than the number of removed outgrowths.

In summary, while for relatively small noise, perhaps as the experimentally studied case [3], only KPZ scaling is obtained, for large noise intensities Eq. 4 predicts a circular interface to cross over in time between an early-time SAW regime, in which it behaves as a freely fluctuating ring “polymer”, and the late-time regime controlled by KPZ fluctuations in the presence of non-zero average curvature. For small times, the local driving does not suffice to counteract fluctuations, so that the average circular shape smears out, interactions among interface points being controlled by surface tension (note the dynamic exponent indeed is $z = 2$ in regime I). Since $\xi(t)$ increases while $R(t)$ stays almost constant, eventually the system becomes fully correlated. From that time on, the increasing length needs to be accommodated in the finite area enclosed by the initial radius, and the intersections removal mechanism becomes relevant, smoothing out the interface. Because of the average (convex) circular geometry, cavities are removed more frequently than outgrowths, and the interface starts to grow leading to the expected KPZ regime, with TW-GUE characteristics.

Our results conspicuously connect the celebrated 2D SAW and KPZ universality classes, both of which underscore the importance of geometrical constraints for scaling behavior, in and out-of-equilibrium. Crucially,
the transition in time between them can only be elucidated through the existence of overhangs, which eludes other continuum models of kinetic roughening. Hence, the large noise regime I of Eq. (16) might constitute a scaling limit for 2D ring SAW [3], while providing an efficient algorithmic procedure to generate them [2]. Alternative connections between the KPZ and SAW classes are available, namely, between iso-height lines of the 2+1 dimensional (3D) KPZ equation and 2D SAW-related formulations [33]. In general, the conformation and dynamics of circular polymers is still a subject of considerable interest [34]. Current experimental capabilities reach even down to single molecule experiments [1], so that one might speculate on the possibility to observe a dynamical transition of the type elucidated here in appropriate non-equilibrium, 2D constrained settings [37].

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**SUPPLEMENTAL MATERIAL**

The covariant KPZ equation

The Kardar-Parisi-Zhang (KPZ) equation describes the dynamics of the interface between a 2D unstable phase and a stable phase [16]. The interface is represented as the graph of a single-valued function \( h(x,t) \), i.e.: a height field, in the small-slopes approximation. Its dynamics follows a non-linear stochastic partial differential equation (eq. (1) of the main text):

\[
\partial_t h = v + \nu(\partial^2 h) + \frac{\lambda}{2} (\partial_x h)^2 + \sqrt{D} \eta \tag{5}
\]

where \( v, \lambda, \nu, \) and \( D \) are parameters quantifying (average and excess-) growth, surface tension, and noise intensity, with \( \eta \) being a random field with unit variance, white in space and time. The study of the KPZ equation has allowed to elucidate the KPZ universality class, which encompasses many different models, both discrete and continuous.

The KPZ equation relies on approximations including the assumption of small slopes and the absence of overhangs. Moreover, by construction it cannot describe explicitly systems, like circular interfaces, which cannot be specified by the graph of a single-valued function. There have been attempts to study an intrinsic-geometry version of the KPZ equation, free from such assumptions, which have found difficulties due to the strong non-linearity of the equation; on the other hand, phase-field studies are too expensive computationally to provide a thorough analysis of the scaling properties, see a brief overview in [22].

A solution to all these shortcomings was proposed in [22], in which a covariant form of the KPZ equation is put forward (eq. (2) of the main text):

\[
\partial_t r = (A_0 + A_1 K(r) + A_n \eta(r,t)) u_n. \tag{6}
\]

where \( A_0 \) and \( A_1 \) refer to growth speed and surface tension, while \( A_n \) provides the noise amplitude. Here, \( K(r) \) stands for the local extrinsic curvature and \( u_n \) is the local normal direction, pointing towards the non-aggregated phase. The long time behavior of equation (5) in band geometry with periodic boundary conditions was studied in [22], where ample numerical evidence was given that it falls into the KPZ universality class, with \( \alpha = 1/2 \) and \( \beta = 1/3 \).

The main advantage of equation (6) is that it can be studied in circular geometry directly as it stands. Its covariance properties are even stronger, since any change in the background metric can be straightforwardly absorbed by the equation.

In order to obtain a well-defined dynamical system for an interface, equation (6) must be complemented with a prescription for the treatment of intersections. As in [22] and [24], whenever a self-intersection appears in the interface, we have chosen to remove the smaller component, whether it is a cavity or an outgrowth. Mathematically, intersection removal is thus the price one needs to pay in order to have a simply connected interface. Physically, by comparison with more realistic (e.g. stochastic moving boundary) KPZ-related growth models, of which equation (5) and similar models are effective descriptions, its role for the morphological evolution seems to be quite marginal even for parameter conditions in which voids and bubbles occur, see e.g. [38]. Indeed, with respect to the kinetic roughening behavior, the main role in the morphological evolution is played by the envelope of the so-called active growth zone [14], that corresponds in our case to the simply connected interface that we keep track of after self-intersection removal. Our choice is akin to the standard procedure by which the full interface dynamics of, e.g., discrete growth models that lead to bubbles and overhangs due to bulk vacancies [15] is traded for that of such an envelope.

**Simulation technique**

Most approaches to continuous models of circular growth describe the interface via a (single-valued) polar function \( R(\theta) \), thus preventing the creation of radial overhangs. Our approach does not suffer from this constraint either [22].

Our numerical method of simulation will be the same as in [22]. The interface is represented by a chain of points (a “polymer”), \( \{P_i\}_{i=1}^N \), defining a piecewise continuous curve which always leaves the solid region on its left. The distance between them is forced to remain in a certain interval \([l_{\text{min}}, l_{\text{max}}]\), which is achieved by inserting or removing points dynamically. All the geometric elements are obtained in a natural way. Thus, we define the tangent line at \( P_i \) as the segment joining \( P_{i-1} \) and \( P_{i+1} \), and the normal vector is the unit vector orthogonal to it pointing outwards. The extrinsic curvature at \( P_i \) is defined as the inverse of the radius of the circumsphere which passes through \( P_{i-1}, P_i \) and \( P_{i+1} \), which is a discrete approximation to the osculating circle. This extrinsic curvature is signed: it is defined to be positive when \( P_i \) is at the left side of the tangent line. In other terms, when the interface is locally convex.

Numerical integration of the stochastic partial differential equation was performed with an Euler-Maruyama algorithm. In all cases, the initial condition will be a circular interface with small radius \( R_0 \), centered at the origin \((0,0)\).

Our numerical scheme does not suffer from instabilities from the non-linear character of the equation. An important aspect of the algorithm is the detection of self-intersections, for which we have employed the technique described in [22]. Our self-intersection removal al-
algorithm works in three stages: (1) mark the pairs of segments which are candidates for intersection by evaluating whether the cartesian boxes which contain them overlap, (2) for the candidate pairs, find out whether they do possess an intersection point, (3) if the intersection point exists, mark it as a new point of the interface, detect the smaller component (in length) and remove it.

For the results described in the main text, we have run 2500 samples of equation (7), starting with a circle with $R_0 = 10$. The values of the parameters are $A_0 = 0.01$, $A_1 = 0.01$ and $A_n = 1$, with $T_{max} = 1600$, $∆t = 0.1$, $l_{min} = 0.1$ and $l_{max} = 1$. Results have been checked to ensure the continuum limit has been suitably approximated. A small number of samples (50) were allowed to continue up to $T_{max} = 16000$.

Measurement procedure

Let us consider the set of points $\{P_i\}_{i=1}^N$ that determines the interface at a given time. There are several possible approaches in order to measure the roughness of the interface:

- Fit to a circumference with center at the origin. The average radius will be given by $R_0^2 = \langle |P_i - (0,0)|^2 \rangle$, where $\langle \cdot \rangle$ denotes average over all realizations of the interface for a given time. The roughness $W$ is given by the average deviation from the fitting circumference: $W_0 = \langle (R_i - R_0)^2 \rangle^{1/2}$, i.e.: the error of this fit.

- Fit to a circumference with adjustable center, which we may interpret as the center of mass (CM). The average radius, $R_{CM}$ will always be smaller than $R_0$. The roughness is again given by $W_{CM} = \langle (R_i - R_{CM})^2 \rangle^{1/2}$, and it is smaller than in the fixed-center case in all cases.

- The best shape to fit the interface need not be a circle. We may also try a fit to an ellipse. In this case, the output of the fitting procedure is the center of mass, the two principal radii, $R_a$ and $R_b$ (the average radius corresponding to $R = \sqrt{R_a R_b}$) and the angle which the largest radius makes with a fixed direction $θ$.

Since this last fit to ellipses is extremely expensive from a computational point of view, we have established the following compromise:

- Average radius from the origin $R_0$ and from the center of mass, $R_{CM}$.

- Root-mean-square distance of the CM from the origin, $D_{CM}$.

- Anisotropy of the interface: we measure the quadrupole moment: $Q^2 = \frac{\langle |x^2 - y^2| \rangle}{\langle r^2 \rangle}$, \hspace{1cm} (7)

where $x$, $y$ and $r$ are always relative to the CM. For a completely isotropic model, like ours, this is a measure of the eccentricity of the fitting ellipse. Additionally, by defining the gyration tensor $S_{ij}$, \hspace{1cm} (8)

$$S_{ij} = \langle x_i x_j \rangle,$$

again with $x_i$ and $x_j$ defined with respect to the CM, another measurement that we have performed is the ratio of its smallest to largest eigenvalues, $S_{ij}^2 / S_{ee}^2$, namely, the so-called asymmetry parameter $Σ = \langle S_{ij}^2 / S_{ee}^2 \rangle$. \hspace{1cm} (9)

All these values are referred to a single snapshot. A second averaging procedure is required over all the ensemble of realizations of the solution to the stochastic differential equation (7).

A different problem is posed by the morphology curves, which show the roughness at each scale, $w(l)$. In \cite{22} a technique was developed to obtain the roughness at a given length, by using movable windows of width $l$. All points within a window are fitted to a straight line, and their deviations from that line constitute the local roughness. In order to transfer this scheme to the circular geometry setup, we have to take into account that the natural fit is not to a straight line, but to an arc of circumference. This problem can be circumvented using the following “rectification” procedure:

- For every point $P_i = (x_i, y_i)$, the polar coordinates are found from the CM: $P_i = (x_{CM} + R_i \cos(θ_i), y_{CM} + R_i \sin(θ_i))$.

- We apply a rectification mapping: $P_i \rightarrow \tilde{P}_i = (R_i θ_i, R_i - R_{CM})$. In other terms: each point is projected on the fitting circle. The new $x$ coordinate is the arc-length in this circle. The new $y$ coordinate is its distance to the fitting circle.

Once the new points are found, $\{\tilde{P}_i\}$, the previous algorithms apply in a straightforward way.

Effective dynamics of the average radius

We next write down an approximate model for the time evolution of the average interface radius $R(t)$. We first
integrate Eq. (6) over the full length of the evolving curve at a fixed time
\[ \oint (\partial_t \mathbf{r} \cdot \mathbf{u}_n) \, ds = A_0 L - A_1 K + H, \] (10)
where \( L(t) \) is the total curve length, \( K(t) \) its total curvature, and \( H(t) \) is the integral of the noise contribution. Next we average over realizations of the zero-average noise \( \eta \), to obtain
\[ 2\pi R \frac{dR}{dt} = 2\pi A_0 R - 2\pi A_1, \] (11)
where we have made additional assumptions:

- We simplify \( \partial_t \mathbf{r} \cdot \mathbf{u}_n = \frac{dR}{dt} \). Hence, we are neglecting local changes in the normal velocity along the curve.

- In principle, the factor multiplying \( A_0 \) in the first term on the right hand side would be \( \langle L \rangle \). We approximate it by the circle length in order to have a differential equation which is closed in \( R(t) \).

Thus we finally get
\[ \frac{dR(t)}{dt} = \dot{A}_0 - \frac{\dot{A}_1}{R(t)}, \] (12)
which is equation (3) of the main text, provided \( \dot{A}_0 = A_0 \) and \( \dot{A}_1 = A_1 \). Note, an equation of the latter form is the simplest one would expect starting from (6). Thus, e.g. \( A_1/R(t) = A_1 K(t) \) for the deterministic case of a circle. Note this surface-tension term contributes to the global interface velocity, in contrast with the case of band geometry. Eq. (12) can be exactly solved as an implicit function for \( R(t) \), which reads
\[ R(t) - a \ln \left( \frac{R(t) + a}{R_0 + a} \right) = R_0 + A_0 t, \] (13)
where \( R(t = 0) = R_0 \) and \( a = \dot{A}_1/\dot{A}_0 \). This is the exact solution employed in the fit shown in the main text. As discussed there, the excellent fit suggests equation as an accurate description for \( R(t) \), provided one allows for parameter renormalization whereby \( \dot{A}_0 \neq A_0 \) and \( \dot{A}_1 \neq A_1 \) in general. Actually, such a change in parameter values is induced by the noise. Thus, in Fig. 5 we see that, for sufficiently small but non-zero noise amplitudes \( A_n < 1 \), the evolution of the radius \( R(t) \) cannot be distinguished from the one obtained in the deterministic limit of Eq. (6).