Variations on intra-theoretical logical pluralism: internal versus external consequence

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Abstract Intra-theoretical logical pluralism is a form of meaning-invariant pluralism about logic, articulated recently by Hjortland (Australas J Philos 91(2):355–373, 2013). This version of pluralism relies on it being possible to define several distinct notions of provability relative to the same logical calculus. The present paper picks up and explores this theme: How can a single logical calculus express several different consequence relations? The main hypothesis articulated here is that the divide between the internal and external consequence relations in Gentzen systems generates a form of intra-theoretical logical pluralism.

Keywords Logical pluralism · Intra-theoretical pluralism · Logical consequence · Internal consequence · External consequence

1 Prelude

Logical pluralism is the view that there is more than one correct logic. In truth, it is a large family of views, drawing on different sources, with various characteristics and different levels of plausibility, all of which grant that there is a plurality of correct logics (Cook 2010; Russell 2014). For the logical pluralist, one and the same argument can be valid according to one logic, invalid according to another, while both logics get things right. Double negation elimination, which allows inferring $A$ from $\neg\neg A$, is a valid (form of) argument in classical logic but it is invalid in intuitionist logic. By pluralist lights, neither the classical nor the intuitionist logician err in making these judgements, although they each err when rejecting the other’s assessment of the (in)validity of double negation elimination. So pluralism is,
essentially, a view about logical consequence relations (hereafter simply ‘consequence relations’ or ‘crs’). Logical pluralists hold that there are several, different but equally correct, accounts of logical consequence. Some versions of pluralism, notably that defended by Beall and Restall (2000, 2006), are conjoint with the thesis that the meaning of the logical constants does not change from one consequence relation to another.\(^1\) A sub-species of meaning-invariant logical pluralism, articulated by Hjortland (2013) and dubbed intra-theoretical, springs from the possibility of associating different consequence relations to a given calculus.

In the present paper, I pick up this theme and investigate the extent to which the divergence between the internal and external consequence relations of some logical calculi generates a similar sort of intra-theoretical pluralism. I will explain all this in due course; the first step is to introduce intra-theoretical pluralism in its context.

2 The theme: intra-theoretical pluralism

Meaning-invariant pluralists hold that ‘different accounts of logical consequence need not be restricted to relationships between languages’ but ‘can also occur within languages’ (Restall 2002, 443). Thus several different and equally correct consequence relations can co-exist even if the logical constants underlying them have the same meaning. Beall and Restall have articulated this view against the background of a model-theoretic approach to logical consequence, at whose centre lies the Generalised Tarski Thesis:

**Proposition 1** (GTT) An argument is valid, if in every case, in which the premises are true, the conclusion is true.

These cases may be Tarkian models, possible worlds, incomplete and/or inconsistent situations, constructions in the intuitionist’s sense, etc. Since the ‘settled core’ of ‘follows from’, consisting of necessity, normativity, and formality, does not single out a particular selection of cases, there are many equally correct accounts of logical consequence—or so the argument goes.

When recast in proof-theoretic guise, as in Restall (2014), meaning-invariant pluralism relies on the claim that the rules characterising the logical constants in sequent calculi are identical even if their context-parts are different.\(^2\) For instance,

\(^1\) This is in contradistinction with what we may call historical pluralism, i.e., a range of views expressed or suggested by philosophers like Tarski (2002), Carnap (1959) or Quine (1970), which are quite naturally interpreted as requiring meaning variance in order to obtain a plurality of consequence relations.

\(^2\) The contexts are, roughly, the collections of parametrised formula occurrences in a rule application, i.e., those occurrences that are neither used nor produced by an application of the rule. As per usual, Roman capitals from the beginning of the alphabet range over formulae, while those from the end of the alphabet range over sets of formulae and, in the schematic formulation of the rules, over contexts (which need not be sets).
the rules for negation could be formulated so as to yield the behaviour of classical negation:

\[
\frac{A, X : Y}{X : Y, \neg A} \quad \frac{X : Y, A}{\neg A, X : Y} \neg L
\]

The same rules, but with the succedents restricted to at most one formula occurrence:

\[
\frac{A, X :}{X : \neg A} \quad \frac{X : A}{\neg A, X :} \neg L
\]

govern the behaviour of intuitionistic negation. Antecedents that contain at most one formula occurrence make negation behave like dual-intuitionistic negation (Urbas 1996).

Recall that on the broadly inferentialist perspective that underscores proof-theoretic approaches to logic, the meaning of a logical constant is seen as determined by the rules that stipulate how to use it in proofs (Dummett 1991; Prawitz 1965). Hence, meaning-invariance would follow immediately provided that different cardinality restrictions on sequents and hence on rule-applications do not count as modifying a rule. As Restall puts it:

these rules also provide us with rules for negation if we are to reason using the strictures of intuitionist logic. If we take the same rules but be careful to use only proofs in which there is at most one formula on the right-hand side of a sequent, the instances of the rules we will use are [...] the intuitionist instances of the rules for negation. (Restall 2014, 283)

All in all, the situation is such that, in the case of classical, intuitionist and dual-intuitionist logic:

we have one language, and three logical consequence relations on that language, by having one family of derivations and various different criteria which may be applied to those derivations. (Restall 2014, 284)

This is not an unproblematic contention. Still, at least the view of rules that underscores it has some prima facie credibility. As already noticed by Haack, since the extant differences between these rules are purely structural and ‘involve no essential reference to any connectives, it is hard to see how [derivability variations] could be explicable as arising from divergence of meaning of connectives’ (Haack 1996, 10).

Fortunately, we need not concern ourselves with the plausibility of this thesis here—on this see Hjortland (2013) and Dicher (2016). This is so mainly because the view just sketched could be seen as a precursor of the intra-theoretical pluralism of Hjortland (2013). Thus it is better to look directly at, as it were, the successor view.

Hjortland’s argument for intra-theoretical pluralism centres around the relation between the logic of paradox (LP) championed by Priest (1979) and Kleene’s strong logic (K3) (Kleene 1952). These are different logics, i.e., different consequence relations. For instance, explosion \((A, \neg A \vdash B)\) is valid in K3 but not in LP; K3
validates disjunctive syllogism \((A \lor B, \neg A \vdash B)\) whereas LP does not; the law of the excluded middle (\(\vdash A \lor \neg A\)) is LP-valid but not K3-valid, etc. However, the connectives of these two logics have identical meanings in the following sense.

Model-theoretically, they are uniformly characterised by the same trivalent matrices:

\[
\begin{array}{ccc|ccc|c}
\wedge & 1 & \frac{1}{2} & 0 & \lor & 1 & \frac{1}{2} & 0 & \neg \\
1 & 1 & \frac{1}{2} & 0 & 1 & 1 & 1 & 1 & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \\
\end{array}
\]

If the meaning of the connectives is determined by their associated truth functions, then one would say that conjunction, negation, etc. have the same meaning in K3 and LP. That they are different crs is due to the choice of designated values: 1 in K3, both 1 and \(\frac{1}{2}\) in LP. Thus, in these two logics consequence is defined, respectively, as follows:

\((\vdash_{K3}) \quad X \vdash_{K3} A \iff \text{the n-sequent } X \upharpoonright X \upharpoonright Y \text{ is derivable;}\)

\((\vdash_{LP}) \quad X \vdash_{LP} A \iff \text{the n-sequent } X \upharpoonright Y \upharpoonright Y \text{ is derivable.}\)

where \(v\) is any valuation extending the matrices above. So, while we have the same truth functions, we get different crs, by, so to speak, valorising differently their output.

Proof-theoretically, both LP and K3 can be presented by the same n-sided sequent calculus. Which is to say that all their rules, operational or structural, are formulated identically, up to and including the contexts. An n-sided sequent (n-sequent for short) is an n-tuple \(X_1|\ldots|X_n\) of sets of formulae where the \(i\)th position (\(1 \leq i \leq n\)) corresponds to the \(i\)th element in the set \(V\) of truth values of the logic in question. (It is assumed here that these truth values are given under some ordering. For \(V\) the set \(\{0, \frac{1}{2}, 1\} \subseteq \mathbb{Q}\), we can take them to be naturally ordered by \(\leq\).) A valuation \(v\) satisfies an n-sequent iff there exists an \(X_i\) such that, for some \(A \in X_i\), \(v(A) = i\). Intuitively n-sequents are read as disjunctions of disjunctions.

The details matter little for what follows so I leave the full presentation of the system for the “Appendix”. To give the reader a taste of these rules, here are the rules for negation:

\[
\frac{A, X|Y|Z}{X|Y|\neg A, Z} \quad \text{R} \quad \frac{X|A, Y|Z}{X|\neg A, Y|Z} \quad \text{M} \quad \frac{X|Y|A, Z}{\neg A, X|Y|Z} \quad \text{L}
\]

The two logics are distinguished by how one reads consequence out of the derivable sequents of the calculus; thus:

\((\vdash_{K3}) \quad X \vdash_{K3} Y \iff \text{the n-sequent } X\upharpoonright X|Y \text{ is derivable;}\)

\((\vdash_{LP}) \quad X \vdash_{LP} Y \iff \text{the n-sequent } X\upharpoonright X|Y \text{ is derivable.}\)

So much for technical details; now let us go back to pluralism. The thought here is that logics that stand in the sort of relation instantiated by LP and K3 come as
close as possible to exemplifying meaning-invariant pluralism, both model- and proof-theoretically.

Model-theoretically, this claim is grounded in the fact that the connectives are characterised by the same truth functions. I will, however, largely bracket this perspective here, except for the following comment. It is unclear whether these alleged model-theoretic benefits really obtain. As Hjortland admits, the model-theoretic picture is open to the accusation that the change of designated values actually testifies to a change of connective meaning. The charge is rather feebly defused by Hjortland, who simply observes (correctly) that the alternative view, according to which changing the set of designated values is merely a change of ‘what is preserved in valid arguments’, not of the meaning of the connectives, is ‘equally cogent’ (Hjortland 2013, 371). This is hardly a knock-out blow in the match. It would seem that we have, at best, a draw between two competing interpretations of the change of designated value. Nevertheless, even model-theoretically, intra-theoretical pluralism is not less plausible than Beall and Restall’s pluralism.

The proof-theoretic perspective is rather more interesting. LP and K3 have exactly the same derivations. The rules used to build them, structural or operational, are all shared between LP and K3. Moreover, there are no context differences between the rules of these two logics. So they are indisputably the same and meaning-invariance seems to be assured (cf. Hjortland 2013, 370).

Intra-theoretical pluralism is likely the strongest form of meaning-invariant pluralism discussed in the literature so far. This needs to be qualified by a glossa on the meaning of ‘strongest’. Hjortland’s position is more restrictive than Restall’s: fewer logics are amenable to a treatment that uniformises in the required fashion the (model- and) proof-theoretic characterisation of their connectives. So while intra-theoretical pluralism brings with it a smaller plurality of logics, it is stronger in the sense that it appears to better sustain the case for meaning-invariance. Just how easy life could be for the intra-theoretical pluralist is a matter of debate. What is clear is that, at least proof-theoretically, the position does away with significant complications arising from the existence of differences in contexts. That, in turn, might just be a crucial advantage over its less fortunate relatives.

3 The 1st variation: Calculi and consequence relations

So far we have learned that one and the same calculus can express distinct consequence relations. The flat truth of intra-theoretical pluralism can be squeezed in the fairly simple observation that:

**Proposition 2** (The truism) Given a logical calculus, the question ‘Which logic is a calculus for?’ does not have an unambiguous answer.

This Truism is hardly informative as long as the very concept of a cr remains unexplained. Absent such an explanation, there is no way to know whether a calculus expresses or not a cr. The obvious thing to do then would be to define the

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concept. But the difficulty with doing the obvious thing is that there is little consensus as to what are the incontrovertible properties of a cr.

A long time ago, as a preamble to his famed model-theoretic definition of classical first-order consequence, Tarski noticed that:

[w]ith respect to the clarity of its content the common concept of consequence is in no way superior to other concepts of everyday language. Its extension is not sharply bounded and its usage fluctuates. Any attempt to bring into harmony all possible vague, sometimes contradictory, tendencies which are connected with the use of this concept, is certainly doomed to failure. We must reconcile ourselves from the start to the fact that every precise definition of this concept will show arbitrary features to a greater or lesser degree. (Tarski 1956, 409)

Today, when there is a remarkable proliferation of logics, this observation is even more pertinent. Nevertheless, even while acknowledging the point of Tarski’s remark, it is still possible to pick some suitable notion of cr to be used as a conventional yardstick, perhaps without really committing to it. Indeed, we will eventually do this—though not just yet. For the moment it is more instructive to pursue a different route. Instead of starting with a controversial standard, we will start by looking at how Gentzen systems are usually taken to express crs.

The usual way of associating a cr to a sequent calculus proceeds by reading sequents as statements to the effect that the succedent follows from the antecedent. In other words, one reads sequents as entailments. This means that, ultimately, sequent calculi are interpreted as a kind of meta-calculi for natural deduction derivations. This position is clearly articulated in, e.g., Hacking (1979, 292), where a sequent calculus is characterised as ‘a metatheory [i]n [a] metalanguage [in which] we make statements about deducibility relations in the object language’; see also Negri and von Plato (2001, 14) or Dummett (1991, 186). On this view, the target phenomenon of sequent calculi is some natural deduction derivability (= ‘deducibility’ in Hacking’s words) relation.

The expectation is that this natural deduction derivability relation will itself be a cr. Reading sequents as entailments gives us what Avron (1988, 1991) calls the internal consequence relation of a calculus, defined as follows:

**Definition 3** (Internal cr) $X \vdash Y$ iff the sequent $X : Y$ is derivable.

But if no constraints are pre-imposed on what a cr is, this really is a no-standards strategy of associating crs to sequent calculi. The properties of ‘$\vdash$’ depend essentially on the properties of ‘:’ and on the kind of collections $X$ and $Y$ are.

A very simple example of how the no-standards strategy might work is provided by Gentzen’s sequent calculus for classical logic, LK. Its sequents are pairs of sequences; so the internal cr of classical logic appears to hold between sequences of

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3 To keep things simple, we will look at sequent calculi simpliciter; the observations to follow generalise to other types of Gentzen systems.
formulae. *Pari passu*, it seems to be multiple-conclusion: the length of the sequence \( Y \) can be greater than 1.

Of course, one need not apply the ‘sequents = entailments’ policy mindlessly. As far as \( \vdash_{\text{LK}} \) being multiple-conclusion is concerned, this can be avoided in several ways, by far the easiest being to turn a blind eye to sequents whose succedents contain more than one formula occurrence. One can think of extracting consequence claims out of sequents as proceeding via some function that works as a ‘translation’ from sequents to entailments. Then this could be a partial function, defined only over a subset of its domain. Specifically, the function would be defined only for sequents whose succedents contain at most one formula occurrence. Thus \( A : A \) but not \( A : A, B \) expresses a consequence claim, despite both these sequents being \( \text{LK} \)-derivable.

This already shows the existence of two distinct *crs* that could be naturally associated with \( \text{LK} \), one of which would be single-conclusion and the other multiple-conclusion. Since otherwise these two *crs* share many of their formal properties, this makes for a rather unspectacular illustration of the truism stated at the beginning of this section. It is also a cheaply obtained form of intra-theoretical pluralism that is definitely stronger than Hjortland’s original version: no suspicion of meaning variance can arise in this case. However, such examples would not help best Hjortland’s position. While the multiple-conclusion *cr* associated with classical logic is definitely different from its single-conclusion counterpart, this difference is, after all, trivial. Taking these two *crs* to be different logics may be an accurate move given our setup, but it is, nonetheless, a very ‘formalistic’ take on the matter.\(^4\) The letter of pluralism receives its due; its spirit, however, is smothered in formal compliance.

This brings us to the collections of which \( \vdash_{\text{LK}} \) holds.\(^5\) Despite \( \text{LK} \)-sequents being pairs of sequences, it is easy to get set-based *crs* out of them. (Anticipating slightly, it is important to remind the reader that this is the orthodox take on *crs.*) This is so because \( \text{LK} \) has the full complement of structural rules (see the “Appendix”), which in effect collapses sequences into sets. Therefore, the sequent \( \langle A \rightarrow B, A : B \rangle \) (\( = \langle A \rightarrow B, A \rangle, \langle B \rangle \)) corresponds to the consequence claim \( \{ A \rightarrow B, A \} \vdash_{\text{LK}} B \). The same structural rules ensure that \( \vdash_{\text{LK}} \) is:

- reflexive, because of the zero-premise rule *Identity* (*Id*);
- (left-) monotonic, on account of *Weakening* on the left (*Wl*);\(^6\)
- transitive, in virtue of *Cut*.

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\(^4\) For more on the relation between the various internal *crs* that can be associated with a calculus, see Avron (1991).

\(^5\) The observations to follow are formulated without disambiguating the symbol \( \vdash_{\text{LK}} \) according to whether it is the single- or the multiple-conclusion relation that is under discussion because, by and large, the differences do not matter. Besides, such disambiguation is trivial.

\(^6\) Weakening on the right is restricted in the single-conclusion case, so it is only the multiple-conclusion relation that is fully monotonic on the right.
Many logics lack the full complement of structural rules and therefore generate derivability relations—and hence (putative) crs—with different properties. Such is the case of, say, relevant logics, which, lacking Weakening, have non-monotonic internal crs. So, for a sequent calculus like that for, e.g., distributionless R (q.v. Paoli 2002), since its sequents do not collapse into sets, its internal cr cannot relate sets. Its relata are multisets of formulae, that is, sets with repetitions or sequences without order. The same holds for the internal cr of linear logic, which lacks both Weakening and Contraction. Indeed, there are many examples in the same vein and the phenomenon is pervasive among substructural logics—logics obtained from classical logic by dropping or restricting some of its structural rules (Paoli 2002; Restall 2000).

I will mention two more examples, very eloquent for the far-reaching consequences of the no-standards strategy. Nevertheless, I will do so while reversing the dialectical punchline and also considering the matter of why one might want crs that stray from the properties exhibited by the classical cr. First, we have reflexivity, as codified by $Id$. Once thought to be ‘the archetypal form of inference, the trivial foundation of all reasoning’ (Anderson and Belnap 1975), reflexivity is, in principle, open to challenges. For instance, going non-reflexive is one possible way of dealing with the paradoxes (French 2016). Second, consider $Cut$, which can be seen as a transitivity principle for crs. It too has been recently called into question, again in connection with the paradoxes (Cobreros et al. 2013). Dropping it allows one to reason classically in the presence of paradoxical sentences—or so the story goes; more on this follows in Sect. 5.

So much by way of examples. The picture that emerges is as follows. Under the previously sketched interpretation of sequent calculi, internal crs inherit the structural properties of the sequent calculi which determine them. So if two calculi have different structural rules and properties, then they codify different crs—unless remedial measures are taken. The no-standards strategy ensues in a very liberal—one would say anarchic—understanding of consequence.

Not everyone is comfortable with this situation. Some derivability relations are (rightly) seen as so strange that they are unworthy of being considered logics. Indeed such relations would count as relations of logical consequence by ‘courtesy only’, to use an apt phrase from Asmus and Restall (2012, p. 53). This is not so much an indictment of the ‘sequents = entailments’ principle, as of the ‘no-standards’ policy that yielded too variegated a multitude of (putative and indeed fake) crs out of otherwise respectable but not really logical derivability relations.

But is there some way of avoiding this strategy, rather than mitigate its effects on an ad hoc basis? The only way is to go back to setting a determinate prior standard for something being a consequence relation. Our best candidate is the traditional conception of a cr embodied in the single-conclusion consequence relation of classical logic.7

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7 This is not to say that classical logic itself has some special dignity, for the same kind of cr can be encountered in the case of other logics, such as, for instance, intuitionist logic. It’s just that it is the most familiar cr that fits the bill.
This conception was articulated in a general fashion by Tarski (1956, chs. 3, 5, 12). His previously quoted sceptical remarks notwithstanding, Tarski proposed an account of consequence relations that has become the standard in the literature. On it, a consequence relation is a reflexive, monotonic, and transitive relation between a set of formulae (the premises) and one formula (the conclusion). To put it more formally:

**Definition 4** Let \( \mathcal{L} \) be a language, \( \text{Form}(\mathcal{L}) \) the set of its formulae, and \( \mathcal{P}(\text{Form}(\mathcal{L})) \) its powerset. A (logical) cr over \( \mathcal{L} \) is a (substitution-invariant) relation \( \vdash \subseteq \mathcal{P}(\text{Form}(\mathcal{L})) \times \text{Form}(\mathcal{L}) \) satisfying, for every \( X, Y \subseteq \text{Form}(\mathcal{L}) \) and every \( A, B \in \text{Form}(\mathcal{L}) \):

- (Reflexivity) if \( A \in X \), then \( X \vdash A \);
- (Monotonicity) if \( X \vdash A \) and \( X \supseteq Y \), then \( Y \vdash A \);
- (Transitivity) if \( X \vdash A \) and, for every \( B \in X \), \( Y \vdash B \), then \( Y \vdash A \).\(^8\)

These are precisely the properties of the single-conclusion classical consequence relation discussed above. Moreover, these properties mark the nowadays dominant understanding of a cr. Alas, as we have seen already, literally every property mentioned in Definition 4 has been contested and its hegemony is waning dramatically. So Tarski’s axiomatic characterisation of consequence is at best the conventional yardstick mentioned above. But that is quite enough right now; indeed, it is a lot!

Somehow surprisingly, there is a way of associating a Tarskian cr with pretty much any sequent calculus. Again, the formal details are due to Avron (1988), where the notion of an external cr of a sequent calculus is defined as follows:

**Definition 5** (External cr) \( X \vdash^e A \) iff the sequent \( (\vdash A) \) is derivable when the sequents \( (\vdash B_j) \), for each \( B_j \in X \) are taken as axioms, with Cut taken as primitive.\(^9\)

As defined here, an external cr is single-conclusion. Nothing could have prevented us from generalising the concept alongside the exact same lines as in the case of the internal cr. However, there are heuristic reasons for not doing so. Internal crs have been definitionally allowed to be multiple-conclusion in order to underscore the continuity between their features and the structural properties of the sequent calculi from which they were generated. In the case of external crs, the dialectical punch is different: this is about getting Tarskian crs from a sequent calculus irrespective of the calculus’s structural features.

It is best to illustrate this notion of cr in the context of a wider discussion. To begin with, it is often the case that the internal and the external crs of a calculus coincide extensionally (modulo the tweaks required to put the two definitions in

\(^8\) Logical crs are usually required to be substitution invariant. I have bracketed this feature in the definition because it will play no role subsequently.

\(^9\) The parentheses around the sequents with empty antecedents are added in order to facilitate readability.
accord with respect to the cardinality of their second relatum. For instance, in a logic that has the standard rules for the conditional:

\[
\frac{X : Y, A}{A \to B, X, Z : Y, W} (\to \text{L}) \quad \frac{A, X : Y, B}{X : Y, A \to B} (\to \text{R})
\]

as well as the standard structural rules, \( B \) follows from \( A \) and \( A \to B \) both internally and externally. Internally, this is so because the sequent \( A, A \to B : B \) is derivable by one application of \( \to \text{L} \):

\[
\frac{A : A}{A, A \to B : B}
\]

\( B \) is also an external consequence of \( A \) and \( A \to B \), which can be shown by extending the derivation above with two cuts on \( A \) and \( A \to B \) respectively:

\[
\frac{\frac{A : A}{A, A \to B : B} \to \text{R}}{A \to B} \quad \frac{\frac{\frac{\frac{A : A}{A, A \to B : B} \to \text{R}}{A \to B} \text{Cut}}{A \to B \text{ Cut}} \text{Cut}}{B}
\]

It is easy to check that the same holds in the case of intuitionist logic; in fact, the derivations above are both intuitionistically correct.

But the internal and external crs of a calculus need not coincide. Indeed, whenever one of the structural rules codifying the properties stipulated in Definition 4 is missing from the calculus, they will not coincide. A simple example is provided by the entailment from \( A, B \) to \( A \land B \) in the context of a calculus lacking \( \text{Wl} \). If conjunction is governed by its usual Gentzen rules and thus the \( \land \text{R} \) rule is

\[
\frac{X : Y, A \quad X : Y, B}{X : Y, A \land B}
\]

then the sequent \( A, B : A \land B \) cannot be derived: Starting with the axiom sequents \( A : A \) and \( B : B \), there is no way to apply \( \land \text{R} \) without first uniformising the antecedents of these sequents by weakening in an instance of \( B \) and \( A \), respectively. Thus \( A, B \not\models A \land B \). Nevertheless, \( A, B \models^e A \land B \), as the following derivation proves:

\[
\frac{\frac{A \quad B}{A \land B} \land \text{R}}{A \land B}
\]

A similar example shows more clearly the advantages—or problems, depending on one’s taste—of taking external crs to be thoroughly Tarskian. Staying with the same sequent calculus lacking \( \text{Wl} \), observe that in it the sequent \( A, B : B \) cannot be obtained from the sequent \( B : B \). Therefore, in such a calculus \( A, B \not\models B \). Yet if \( (A) \) and \( (B) \) are added as extra axioms to the calculus, then there is a derivation of \( (B) \), i.e., \( (B) \) itself. Thus \( B \models^e B \) from which \( A, B \models^e B \) follows by monotonicity. That this is so has nothing to do with \( (A) \). That, however, is perfectly fine—external crs
are Tarskian by fiat, so they are, trivially, insensitive to the available stock of structural rules.

One may believe that, apart from their technical possibility, external crs are somehow less than appealing and this may well threaten the subsequent train of thought. What benefits could there be in, e.g., associating a monotonic cr to a relevant logic? Doing so seems to defeat the purpose of having a relevant logic. Yet whether the external cr of a non-monotonic logic is appealing, relative to some determinate purpose that may have prompted the development of that logic, is neither here nor there for my present purposes. It is precisely the technical possibility of exhibiting external crs that is of interest here, as a means of backing what I have deemed the ‘plain truth’ of intra-theoretical pluralism. And it is important to observe that there’s nothing technically wrong with tracking external crs that are Tarskian by fiat.

4 Interlude

So what is the upshot of all this? The picture sketched above is indicative of a kind of logical pluralism—let’s call it internal/external pluralism. Moreover, this seems to be related to that defended by Hjortland. But is internal/external pluralism deserving the qualification intra-theoretical?

It is difficult to say what is the relation between these two species of intra-theoretical pluralism, apart from them both being children of the same Truism (Proposition 2). On the face of it, internal/external pluralism cannot best Hjortland’s intra-theoretical pluralism. The point is straightforward from the model-theoretic perspective: there can be no assurance that, relative to the two crs, the truth functions corresponding to the connectives could be presented via the same matrices. In the next section we will encounter an example where this worry does not occur; right now it is the general perspective that matters. Still, the model-theoretic perspective is, in any case, the Achille’s heel of Hjortland’s intra-theoretical pluralism. So giving up the requirements that may originate from it does not seem to be too big a sacrifice.

The proof-theoretic perspective presents a slightly more complicated image. Hjortland’s proposal boasted perfect uniformity with respect to the general properties of the two crs obtained out of the same calculus. Internal/external pluralism cannot match this. Indeed, it thrives on there being a mismatch between the structural properties of the two crs. So this may be another aspect on which internal/external pluralism loses some brownie points, perhaps to the point of not qualifying as intra-theoretical. Whether this is so or not may depend on how we disambiguate—for disambiguating we must—the notion of a ‘theory’ evoked by the locution ‘intra-theoretical’. Hjortland’s explains it as follows:

by a single logical theory we might mean a proof system $S$. Normally there is a corresponding notion of $S$-provability, but it is perfectly possible to devise systems where two or more definitions of provability cohabit [...]. (Hjortland 2013, 365)

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10 The reader can find a considered defence of them in Mares and Paoli (2014).
There is no indication here that the notions of provability at stake must fit with but one set of general properties of a cr. Hjortland’s glossa does not explicitly rule out, for instance, the situation whereby one would distinguish, relative to a given proof system, between, say what is relevantly provable and what is constructively provable in it.¹¹ So it is difficult to fault internal/external pluralism on this count.

Notice, however, that despite the un-demandingness of this official explanation, Hjortland’s example is intra-theoretical in a stronger sense. As we are dealing with just one set of structural rules, we are dealing with a single (implicit) theory of what a cr is. Moreover, both K3 and LP have Tarskian internal crs.

One may, in fact, take these two senses of ‘intra-theoretical’ to express independent conditions of adequacy for intra-theoretical pluralism. Then, one should observe the following: Complying with the latter condition is not sufficient for intra-theoretical pluralism. There are logics that obey it, but nevertheless cannot be characterised by a single calculus as LP and K3 can be.

It is unclear whether compliance with it is necessary. Elsewhere, Hjortland worries about what he calls the meta-Quinean objection to meaning-invariant pluralism, namely that changing the properties of the cr may itself bring about meaning variance, albeit with respect to consequence and validity. This would seem to suggest that the condition is necessary and hence that, for intra-theoretical pluralism to occur, it must be that the two (or more) logics concerned exhibit the same kind of cr.¹²

However, the harder one presses for this, the more embarrassing the fact that intra-theoretical pluralism flags different sequents as expressing consequence claims becomes. For surely picking different kinds of sequents, even if constructed with the same tools, to express distinct consequence relations is a willful equivocation of the meaning of ‘follows from’. Internal/external pluralism does not do this. Instead, it takes exactly the same derivable sequents and extracts different crs from them. In a sense, this seems to be an advantage over intra-theoretical pluralism, simply because the different crs are generated by the same derivable sequents.

5 The 2nd variation: the strange case of ST

To variate further, we will look at these matters from a different and richer perspective, that will allow us to explore another plausible case of intra-theoretical pluralism. The discussion will revolve, again, around the difference between the internal and external crs of some substructural logics, though in a slightly more general fashion.

¹¹ Of course, this distinction would have to be generated by something other than context-differences in the formulation of the rules. Otherwise, there would no progress over Restall’s version of pluralism.

¹² It is unclear to me whether Hjortland actually made this connection. But this is the easiest way to reconcile the views he expressed in Hjortland (2013, 2014). Incidentally, it’s worth mentioning that the initial impetus for considering external crs was precisely an attempt to defuse the ‘meta-Quinean objection’, cf. Paoli (2014) and also Dicher and Paoli (forthcoming a).
The starting point is the logic ST, in whose case the internal/external divide has been extensively studied lately. ST promises to attain a seemingly impossible desideratum: deploying classical reasoning in the presence of transparent truth or of vague predicates (Cobreros et al. 2013; Ripley 2012, 2013). With it one is able to reason classically—that is, one obtains a classical $cr$—even when paradoxes like the Liar or the Sorites lurk around.  

Ripley (2013) puts forward a simple way of understanding ST, that builds on Restall’s (2005) bilateral interpretation of sequent calculi. On this interpretation, a sequent is a structure that says, roughly, that given the assertion of the antecedent, the denial of the succedent is out of bounds. ST-theorists distinguish between two kinds of assertions and denials: strict and tolerant. A statement is tolerantly asserted iff its assertion does not rule out the statement also being denied. The same holds, mutatis mutandis, of denial. Such is the case with, e.g., the (strengthened) Liar: ‘This sentence is not true’. Asserting it does not rule out its denial and vice-versa. This behaviour of assertions and denials is readily captured by the same three-valued matrices that characterise K3 and LP. Strict assertability is identified with receiving value 1 and tolerant assertability with receiving a value greater than or equal to $\frac{1}{2}$. To be strictly deniable is to receive the value 0, whereas tolerant deniability is represented by receiving a value from $\{0, \frac{1}{2}\}$.

ST-consequence is (model-theoretically) obtained by stipulating that a formula $A$ follows from a set of premises $X$ iff, whenever all the formulae in $X$ are strictly assertable, $A$ is at least tolerantly assertable. This single-conclusion notion of consequence is readily transposable into a multiple-conclusion one—as indeed it is customary to do.

On the proof-theoretic side of things, everything is doubly-fun. ST can be presented by the classical sequent calculus, without Cut.  

While the discussion to follow may safely be framed relative to LK, in light of some of the results mentioned subsequently, it is technically convenient to start from a variant of it, christened PK (Dicher and Paoli forthcoming a). The rules are presented in detail in the “Appendix”; right now it is enough to know the following: PK contains Cut. PK$^-$ is PK without Cut. PK$^-$ together with the inverses of the operational rules is PK$^-_{INV}$.

The calculi in the PK family derive precisely the same sequents. For PK and PK$^-_{}$ this follows from the Cut-elimination theorem; when PK$^-_{INV}$ is in the picture, it follows from the Cut-elimination theorem plus the admissibility of the inverses of the operational rules. Thus, the internal $cr$ of each of these calculi is the classical relation of consequence. As for Cut itself, it is no longer admissible in ST-theories that contain, e.g., a transparent truth predicate. More generally, Cut is not admissible if there really are tolerantly assertible/deniable statements around. If it

---

13 Whether this really is the case is a contentious matter. But I have no reasons to go into this matter here. An argument against ST’s alleged classicality, recently proposed in Dicher and Paoli (forthcoming b), will be briefly mentioned below.

14 This is not the only option; in particular, there are n-sided presentations of it, of the kind discussed in the previous section. See Ripley (2012) and Barrio et al. (2015).

15 Notice, however, that as long as the truth rules are not in the picture, Cut is perfectly safe, cf. Ripley (2012).
were, it would be possible to cut all the way down to the empty sequent, \( \emptyset : \emptyset \), from which, by Weakening, it would follow that everything follows from everything.

It is easy to see that ST lacks a great deal many other structures similar to Cut: \textit{largo sensu} inferential structures that output sequents from sequents. For instance, it loses the following form of \textit{disjunctive syllogism}:

\[
\begin{array}{c}
X : A \lor B \\
X : \neg A
\end{array} \quad \frac{\text{\( X : B \)}}{}
\]

A ST-counterexample to it is provided by the valuation that assigns 1 (to \( X, \frac{1}{2} \)) to \( A \) and 0 to \( B \). In this case, the consequents of both premise-sequents are valued at \( \frac{1}{2} \) which verifies both premise-sequents. Nevertheless, the sequent \( X : B \) is not satisfied since \( B \) had value 0. The same valuation \( (v(X) = 1; v(A) = v(\neg A) = \frac{1}{2}; v(B) = 0) \) shows that ST loses the sequent version of explosion:

\[
\begin{array}{c}
X : A \\
X : \neg A
\end{array} \quad \frac{\text{\( X : B \)}}{}
\]

But what exactly are these structures? As long as sequents are construed as entailments, losing Cut and similar entities that prescribe sequent(s)-to-sequent inferential passages does not amount to losing \textit{inferences}. Rather, it amounts to losing a kind of inferential structures relating sequents to sequents, usually called \textit{metainferences}.\(^{16}\) These were characterised as ‘closure properties’ (Cobreros et al. 2013), or ‘schemata’ (Barrio et al. 2015), or even as objects in their own right, i.e., set-theoretic objects manipulable within a calculus (Dicher and Paoli, forthcoming b; Barrio et al. 2018). The last view is better suited here, as it renders metainferences ‘concrete’ syntactic objects. Accordingly, a metainference is an ordered pair of sets of sequents, the first element of which is possibly empty and at most finite, while the second is a singleton. Just like inferences, metainferences too can be assessed as valid or not. Generically, a metainference is \textit{valid} iff, for every valuation that satisfies the premise-sequents, the conclusion-sequent is likewise satisfied.\(^{17}\)

The valuations that invalidate the metainferences discussed above also invalidate the inferences from \( \neg A \) and \( A \lor B \) and, respectively, from \( A \) and \( \neg A \), to \( B \) in LP. This is a recurring pattern and, to cut a long story short, it turns out that the valid metainferences of ST correspond precisely to the valid inferences of LP (Barrio et al. 2015; Pynko 1995). At the same time, it has been proved that PK\(_{\text{INV}}\) is sound and complete for the locally valid ST-metainferences (Dicher and Paoli forthcoming b).

\(^{16}\) A more precise name would be \textit{metainferences of level one}, cf. Barrio et al. (2018).

\(^{17}\) This is in contradistinction to the definition of metainferential validity in Cobreros et al. (2013), on which a metainference is valid iff every valuation satisfies the premises, the conclusion is likewise satisfied. This ‘global’ notion of metainferential validity is too weak to be of much interest: any metainference whose premises are not logically valid will be valid.
It is not difficult to spot that the structures that engender the external cr of a calculus are, in fact, a species of metainferences. Specifically, they are metainferences such that their premises, as well as their conclusion, consist of sequents with empty antecedents. So the last result mentioned above simply generalises the observation that while the internal cr of ST is classical logic, its external cr is LP.

It pays to look more carefully into this matter, by dwelling a bit longer on Cut, sequents, and sequent-to-sequent transitions. The first pertinent observation is that all the calculi in the PK family have Cut on sequents:

\[
\begin{align*}
X_1 : Y_1, \ldots, X_n : Y_n \vdash^s X : Y & \quad X : Y, Z_1 : W_1, \ldots, Z_m : W_m \vdash^s Z : W & \text{Cut}_S
\end{align*}
\]

although some lack Cut on formulae. To put it slightly differently, the sequent-to-sequent derivability relations of the PK-calculi are transitive. They are also reflexive, monotonic and single-conclusion. So with each of these calculi one can associate a sequent-to-sequent analogue of a Tarskian cr. Needless to say, this corresponds to their natural derivability relation, because in the PK calculi and, for that matter, in any sequent calculus, one is deriving sequents from sequents, not formulae from formulae.

This relation is a so-called Blok-Jónsson consequence relation: essentially, a Tarskian cr that is not restricted to being carried by sets of formulae (Blok and Jónsson 2006). Instead, any set whatsoever can be used to define a Blok-Jónsson cr. A fortiori, any set of sequents can serve this purpose.

From this perspective, PK, PK−, and PKINV, having different rules, generate different sequent-to-sequent crs. In turn, these can be used to generate (different) formula-to-formula crs via various sequent-to-formula translations. So now suppose that one has a cr on sequents, as well as a cr on formulae. These are equivalent iff they satisfy the conditions stated in full generality in the definition below:

**Definition 6** (Equivalence of crs) Let \( U_1 \) and \( U_2 \) be two sets and \( \vdash_1 \subseteq \mathcal{P}(U_1) \times U_1 \) and, respectively, \( \vdash_2 \subseteq \mathcal{P}(U_2) \times U_2 \) be their associated consequence relations. \( \vdash_1 \) and \( \vdash_2 \) are Blok-Jónsson equivalent iff there exist mappings \( \tau : U_1 \rightarrow \mathcal{P}(U_2) \) and \( \rho : U_2 \rightarrow \mathcal{P}(U_1) \) such that:

1. \( X \vdash_1 a \text{ iff } \tau(X) \vdash_2 \tau(a) \)
2. \( b \not\vdash_2 \tau(\rho(b)) \)

for every \( X \cup \{a\} \subseteq U_1 \) and every \( b \in U_2 \).

For instance, the sequent-to-sequent cr of PK generates the classical cr via the transformers \( \tau \) and \( \rho \) defined below only for sequents with nonempty antecedents and succedents:

\[\text{Definition 6 (Equivalence of crs)}\]

---

18 In Blok and Jónsson (2006), this is called ‘similarity’. Here I follow the terminology from Dicher and Paoli (forthcoming a, 2017).
\[
\tau(A) = \{ : A \} \]
\[
\rho(A_1, \ldots, A_n : B_1, \ldots, B_2) = \{(A_1 \land \ldots \land A_n) \to (B_1 \lor \ldots \lor B_2)\}.
\]

It is easy to check that, via the same transformers, the sequent-to-sequent \(cr\) of \(\text{PK}^{\text{INV}}\) is equivalent to the Tarskian formula-to-formula \(cr\) of LP.

In the present context, this is important because it exhibits another sense in which one can associate two different \(crs\) on formulae to one and the same theory (=logical calculus). The usual ‘sequents as entailments’ interpretation of \(\text{PK}^{\text{INV}}\) yields the classical \(cr\), although, at least in the presence of, e.g., transparent truth, Cut on formulae will not be available. If, on the other hand, we opt to associate to \(\text{PK}^{\text{INV}}\) a \(cr\) on formulae via its sequent-to-sequent \(cr\), then this calculus expresses LP. Obviously, the \(cr\) thus associated is our old friend, the external \(cr\) of ST. The significant difference lies in the manner in which we have come across this \(cr\).

Might this be, in some sense, another case of intra-theoretical logical pluralism? At first blush, the answer must be affirmative. Clearly, the ‘syntactic’ condition for intra-theoretical pluralism is satisfied: a single calculus generates two distinct \(crs\).\(^{19}\)

Is this enough to settle the matter?

One may think that it is not. Perhaps intra-theoretical pluralism is not just about having two or more \(crs\) in the same calculus, but also about these being obtained by the same process or method of associating \(crs\) to logical calculi.\(^{20}\)

Such an objector would be unfazed by generating different \(crs\) by taking different sequents to express consequence claims, but would not abide getting distinct consequence relations by tracking derivability relations between different types of entities. For that, they would have to argue that while alterations of the reading of the formal elements that constitute a logical calculus do not amount to a change of theory, modifying the route whereby one extracts \(crs\) from a calculus does. This is a rather strange, if not disingenuous, position. It is difficult to see why a sceptical stance would be more warranted in the latter case than in the former. Perhaps a case for this claim could be mounted by ascribing some kind of—for lack of another word—superiority to internal \(crs\) over their external counterparts. But, while it is true that internal \(crs\) have been traditionally the object of attention in logical inquiries, there are no principled reasons why they should be preferred to the detriment of external \(crs\). Or, at the very least, I am unaware of there being such reasons.

All in all, one can conclude that while the two forms of intra-theoretical pluralism are different in many ways, they are, nevertheless, rightfully classified as species of the same genus.

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\(^{19}\) Even model-theoretically, we have it that the connectives of these two logics are given via the same truth functions.

\(^{20}\) Depending on how one construes sameness of process, this putative objection may be used against internal/external pluralism even in the guise discussed in the previous section.
6 Postlude

Much of the preceding discussion revolved around Proposition 2, i.e., the truism that logical calculi do not always determine unequivocally a consequence relation. But should this really matter when we are concerned with pluralism? Or is it rather a kind of technical weakness, perhaps of a contrived nature, which we can shrug off? It may be that the significance of the Truism is that a calculus, not unlike a hammer, does not tell us what to do with it. No surprise then that LP- and K3-theorists, be they pluralists or monists, can do different things with the n-sided calculus used to present these logics. In the particular case of internal/external pluralism, one may even be tempted to put the matter not so much in terms of weakness, as of pathology. Indeed, this is how Barrio et al. (2015) seem to see the matter. As they point out in regards to ST, the ambiguity of a logical calculus may be disturbing, particularly for a monist. It is not inconceivable that even some pluralists may feel ill at ease with it.

These are important matters and I cannot do them justice here. For what is worth, I am optimistic that we have good reasons to admit, if only provisionally, that the internal/external divide is a plausible source of a kind of logical pluralism. At the very least, that means that there is more to intra-theoreticality than meets the eye.

Perhaps even more important is a lesson that one can draw from the last section of this paper. The case of ST shows that the relation between logical calculi and consequence relations is more intricate than it is customarily taken to be. How we identify logics qua consequence relations is an important issue in formulating and ultimately adjudicating debates on, among other things, pluralism. It remains to be seen what novel light this may shed on our thinking about logic.

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Appendix

The n-sided calculus for K3 and LP

The structural rules of this calculus are:

\[
\begin{align*}
\frac{}{A \vdash A} & \text{Id} \\
\frac{X \vdash Y \vdash Z}{A, X \vdash Y} & \text{Wl} \\
\frac{X \vdash Y \vdash Z}{X \vdash A, Y \vdash Z} & \text{Wm} \\
\frac{X \vdash Y \vdash Z}{X \vdash Y, A, Z} & \text{Wr} \\
\frac{A, X \vdash Y \vdash Z}{X \vdash Y, A, Z} & \text{Cut L/R} \\
\frac{A, X \vdash Y \vdash Z}{X \vdash A, Y \vdash Z} & \text{Cut L/M} \\
\frac{X \vdash A, Y \vdash Z}{X \vdash Y, Z} & \text{Cut M/R}
\end{align*}
\]

The official language of this calculus contains only conjunction, disjunction and negation. I present here only the rules for negation and conjunction (the rules for disjunction are dual to these last).

\[
\begin{align*}
\frac{A, X \vdash Y \vdash Z}{X \vdash Y, \neg A, Z} & \neg R \\
\frac{X \vdash A \vdash Y \vdash Z}{X \vdash \neg A, Y \vdash Z} & \neg M \\
\frac{X \vdash Y \vdash Z}{X \vdash A \vdash Y \vdash Z} & \neg L \\
\frac{A, X \vdash Y \vdash Z}{X \vdash Y, A \vdash B, Z} & \wedge R \\
\frac{A, B, X \vdash Y \vdash Z}{A \vdash B, X \vdash Y \vdash Z} & \wedge L \\
\frac{X \vdash A, Y \vdash Z}{X \vdash B, Y \vdash Z} & \wedge M
\end{align*}
\]

The structural rules of LK

\[
\begin{align*}
\frac{}{A \vdash A} & \text{Id} \\
\frac{X \vdash Y}{A \vdash X, Y} & \text{Wl} \\
\frac{X \vdash Y}{A, X \vdash Y} & \text{Wl} \\
\frac{A, X \vdash Y}{X \vdash A \vdash Y} & \text{Wr} \\
\frac{X \vdash Y, A \vdash B}{A, X \vdash Y, B} & \text{Cl} \\
\frac{X \vdash Y, A \vdash B}{A, X \vdash Y, B} & \text{Cr} \\
\frac{X \vdash Y}{A \vdash X, Y} & \text{El} \\
\frac{X \vdash Y}{B \vdash X, Y} & \text{El} \\
\frac{X \vdash Y, A}{X \vdash Y, B, A} & \text{Er}
\end{align*}
\]
The PK family of calculi

These calculi have set-based sequents, rendering explicit Contraction and Exchange rules unnecessary. They all contain Id and W. PK consists these alongside Cut and the rules below that do not have the subscript \( i \) appended to their labels. PK\(-\) is PK minus Cut. PK\(_{\text{INV}}\) is obtained by adding to PK\(-\) the inverses of the operational rules—those rules depicted below that have \( i \) subscripted to their label.\(^{21}\)

\[
\begin{align*}
\frac{A, B, X : Y}{A \land B, X : Y} & \quad \land L & \frac{X : Y, A}{X : Y, A \land B} & \quad \land R \\
\frac{A, X : Y}{A \lor B, X : Y} & \quad \lor L & \frac{X : Y, A, B}{X : Y, A \lor B} & \quad \lor R \\
\frac{\neg A, X : Y}{X : Y, \neg A} & \quad \neg L & \frac{X : Y, A}{X : Y, \neg A} & \quad \neg R \\
\frac{A \land B, X : Y}{A, B, X : Y} & \quad \land L_1 & \frac{X : Y, A \land B}{X : Y, A} & \quad \land R_{11} & \frac{X : Y, A \land B}{X : Y, B} & \quad \land R_{12} \\
\frac{A \lor B, X : Y}{A, X : Y} & \quad \lor L_1 & \frac{A \lor B, X : Y}{A, X : Y} & \quad \lor L_2 & \frac{X : Y, A \lor B}{X : Y, A} & \quad \lor R_1 \\
\frac{\neg A, X : Y}{X : Y, \neg A} & \quad \neg L & \frac{X : Y, A}{X : Y, \neg A} & \quad \neg R_i
\end{align*}
\]

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