In a recent paper [Stud. Hist. Phil. Mod. Phys. 36, 355 (2005)] it is argued that to properly understand the thermodynamics of Landauer’s principle it is necessary extend the concept of logical operations to include indeterministic operations. Here we examine the thermodynamics of such operations in more detail, extending the work of Landauer to include indeterministic operations and to include logical states with variable entropies, temperatures and mean energies. We derive the most general statement of Landauer’s principle and prove its universality, extending considerably the validity of previous proofs. This confirms conjectures made that all logical operations may, in principle, be performed in a thermodynamically reversible fashion, although logically irreversible operations would require special, practically rather difficult, conditions to do so. We demonstrate a physical process that can perform any computation without work requirements or heat exchange with the environment. Many widespread statements of Landauer’s principle are shown to be only special cases of our generalised principle.

I. INTRODUCTION

Landauer’s principle holds a special place in the thermodynamics of computation. It has been described as “the basic principle of the thermodynamics of information processing” [1]. Yet the literature on Landauer’s principle is focused almost exclusively on a single, logically irreversible operation and a particular physical procedure by which this operation is performed [1].

In this paper we seek to analyse the form of Landauer’s principle in a more general context, building upon the consideration of the thermodynamics of indeterministic logical operations [2, 3]. We will explicitly be considering situations where logical states do not necessarily have uniform mean energies, entropies or even temperatures and we will work in a framework in which logically reversible and irreversible and logically deterministic and indeterministic operations can be treated on an equal footing. Once we have done this we will have a single framework in which the different aspects of Landauer’s principle can be united. Doing so will help to address criticisms [4, 5] of the limited validity of previous proofs of Landauer’s principle, and criticisms [6] of the conclusions of [3].

This will lead us to the following generalisation of Landauer’s principle:

Generalised Landauer’s principle

A physical implementation of a logical transformation of information has minimal expectation value of the work requirement given by:

$$\langle \Delta W \rangle \geq \langle \Delta E \rangle - T \Delta S$$  \hspace{1cm} (1)

where $\langle \Delta E \rangle$ is the change in the mean internal energy of the information processing system, $\Delta S$ the change in the Gibbs-von Neumann entropy of that system and $T$ is the temperature of the heat bath into which any heat is absorbed.

The equality is reachable, in principle, by any logical transformation of information, and if the equality is reached the physical implementation is thermodynamically reversible.

We start by considering what we mean by a logical state, a logical operation and the requirements for a physical system to be an embodiment of such an operation. We will be considering only the processing of discrete, classical information here, although we will be assuming the fundamental physics is quantum [2].

We then construct an explicit physical process, based upon the familiar “atom in a box” model, that implements a generic logical operation. Thermodynamically optimising this model will, in general, require consideration of the probability distribution over the input logical states. As this probability distribution is also required to quantify the Shannon information stored in the system, we will refer to the combination of the logical operation and the probability distribution as a logical transformation of information. The optimal implementation, using

1 Although there little consensus on the naming of these, we will refer to the logical operation as RESET TO ZERO (or RTZ) and the widely used physical process which embodies this operation will be referred to as Landauer Erasure (or LE).

2 The analysis would proceed largely unchanged for classical physics, but it would be unnecessarily cumbersome to attempt both. See [6] for a classical treatment.
the “atom in a box” physical process, shows that the above limit is reachable in principle. We then demonstrate that this limit cannot be exceeded by any system evolving according to a Hamiltonian evolution.

We then consider in more detail the implications of this limit, including several special cases that correspond to more familiar expressions of Landauer’s principle, when the physical implementation conforms to a set of conditions which we refer to as “uniform computing”. We will show a less familiar set of conditions, which are nevertheless physically possible, which we call “adiabatic equilibrium computing”, and which can embody any logical operation without either exchanging heat with the environment or requiring work to be performed. We conclude that any logical transformation of information can be performed in a thermodynamically reversible manner. As this conclusion may seem surprising, we discuss some of the practical barriers to achieving this and the particular problems presented by logically irreversible operations.

II. LOGICAL STATES AND OPERATIONS

Although Landauer’s principle is about the thermodynamics of information processing, very little of the literature surrounding it attempts to define what is meant by a logical operation and what are then the minimal requirements of a physical system for it to be regarded as the embodiment of a logical operation. Without first answering this question, it cannot be certain that the most general relationship between information and thermodynamics has been discovered. In this Section the abstract properties of logical states and operations will be considered. This leads to constraints upon a physical system which is required to embody the logical states and operations.

A. Logical States

A logical state simply consists of a variable \( \alpha \), which takes a value from a set \( \{ 1, \ldots, n \} \). If the variable \( \alpha \) takes the value \( x \) then this means that the logical proposition represented by the statement \( \alpha = x \) is true. This paper will consider only classical information processing on finite machines. This produces additional properties, whose assumption is usually implicit:

1. The set of values is a finite set (and by implication, discrete).

2. The values are distinct. In any given instance the variable takes one, and only one, of the possible values.

3. The values are distinguishable. In any given instance the value taken by the variable can be ascertained.

4. The values are stable. The value taken by the variable cannot change except as a result of a logical operation.

B. Logical Operations

A logical operation \( LOp \) maps input logical states from the set \( \{ \alpha \} \) to output logical states from the set \( \{ \beta \} \):

\[
LOp : \alpha \rightarrow \beta
\]

The number of input and output states need not be the same. The output states from one logical operation may be used as input states to another logical operation. Tables I and II show the maps for two of the most commonly encountered logical operations that act upon two input states 0 and 1, the \textit{NOT} operation and the \textit{Reset To Zero} (\textit{RTZ}) operation. These rules can be represented by:

\[
\begin{array}{c|c}
\text{NOT} & \\
\hline
\text{IN} & \text{OUT} \\
0 & 1 \\
1 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{RTZ} & \\
\hline
\text{IN} & \text{OUT} \\
0 & 0 \\
1 & 0 \\
\end{array}
\]

TABLE I: Logical NOT

TABLE II: Reset To Zero

where use has been made of the fact that the \textit{RTZ} operation transforms both input states into the 0 output state. The \textit{RTZ} operation is logically \textit{irreversible}:

\[
\begin{align*}
\text{NOT} : 0 & \rightarrow 1 \\
\text{NOT} : 1 & \rightarrow 0 \\
\text{RTZ} : \{0, 1\} & \rightarrow 0
\end{align*}
\]

4 There is an even more trivial logical operation: logical Do Nothing \textit{IDN}. Including this as a logical operation is not a trivial step, as this is the identity operator! It must also be included as a time delay operator when one considers a sequence of logical operations.
We shall call a device logically irreversible if the output of a device does not uniquely define the inputs. 

If multivalued maps (see [8] for example) are to be considered, it is necessary to also define logically indeterministic computation:

We shall call a device logically indeterministic if the input of a device does not uniquely define the outputs.

| \( UFZ \) | \( \text{IN} \) | \( \text{OUT} \) |
|--------|--------|--------|
| 0      | 0      | 1      |

TABLE III: Unset From Zero

| \( RND \) | \( \text{IN} \) | \( \text{OUT} \) |
|--------|--------|--------|
| 0      | 0      | 1      |
| 0      | 1      | 1      |

TABLE IV: Randomisation

Logically indeterministic operations such as Unset From Zero \( (UFZ) \) and Randomise \( (RND) \) are given in Tables III and IV which follow the rules

\[
UFZ : 0 \rightarrow \{0, 1\} \\
RND : \{0, 1\} \rightarrow \{0, 1\}
\]  

(4)

\( UFZ \) is logically reversible, while \( RND \) is logically irreversible.

Logically indeterministic operations are perhaps less commonly encountered than logically deterministic operations, and it has been questioned whether these are really logical operations [8], for example, take it as part of the definition of a logical operation that it be a single valued map). We include them for a number of reasons:

1. Most importantly, such operations play a significant role in the theory of computational complexity classes for actual computers. The complexity class BPP, (Bounded-error Probabilistic Polynomial-time), represents a class of computational problems for which the inclusion of logically indeterministic operations can produce an accurate answer exponentially faster than any known algorithm consisting only of logically deterministic operations (see [12] for example). Excluding them excludes a genuine class of computational procedures;

2. By including them we are able to derive a more coherent general framework for the thermodynamics of computation. Excluding them creates an artificial asymmetry and physical properties ascribed to logically irreversible operations in the literature may be artefacts of the asymmetry caused by this exclusion;

3. Logically indeterministic transformations of information involve the use of probabilistic inferences. There is a point of view, [10, 11], that regards probabilistic inferences as a natural generalisation of deductive logical inferences;

4. Finally, there seems no special reason not to include them as they form a natural counterpart to the concept of logically irreversible operations. Any conclusion we can draw that applies to the set of all such logical operations must necessarily apply to all logically deterministic operations. Including logically indeterministic operations in our analysis will not invalidate its applicability to logically deterministic operations.

C. Logical Transformation of Information

To quantify the information being processed by the logical operation, the Shannon information measure will be used. This requires the specification of a probability distribution over the input and output states. If the logical states input to a computation occur with probabilities \( P(\alpha) \), then the Shannon information represented by the input states is

\[
H_\alpha = - \sum_\alpha P(\alpha) \log_2 P(\alpha)
\]  

(5)

During the logical operation these input states are transformed into output states \( \beta \). When an input state may be transformed into more than one output state, one must specify the probability \( P(\beta|\alpha) \) for each possible output state. For logically deterministic operations, specifying \( P(\beta|\alpha) \) is trivial as \( \forall \alpha \exists \beta P(\beta|\alpha) = 1 \) (or equivalently \( \forall \alpha \exists \beta \forall \beta' \neq \beta P(\beta'|\alpha) = 0 \)). Specifying all the non-zero \( P(\beta|\alpha) \) completely specifies the rules of the logical operation. We will therefore take the set \( \{P(\beta|\alpha)\} \) as the definition of a general logical operation. For logically deterministic operations, this is the list of all combinations of input and output states that have conditional probability one, which is simply the truth table for the operation.
After the logical operation, the output states $\beta$ will occur with probability

$$P(\beta) = \sum_\alpha P(\beta|\alpha)P(\alpha) \quad (6)$$

so the Shannon information represented by the output states is

$$H_\beta = -\sum_\beta P(\beta) \log_2 P(\beta) \quad (7)$$

When we refer to a logical transformation of information, we will mean a logical operation, acting upon input states $\alpha$, which occur with probabilities $P(\alpha)$, which transforms the input states to output states $\beta$ with conditional probabilities $P(\beta|\alpha)$.

The conditional probability that a given output state $\beta$ was generated by the input state $\alpha$ is:

$$P(\alpha|\beta) = \frac{P(\alpha)P(\beta|\alpha)}{P(\beta)} \quad (8)$$

and the joint probability that there was an input state $\alpha$ and output state $\beta$ is

$$P(\alpha, \beta) = P(\alpha)P(\beta|\alpha) = P(\beta)P(\alpha|\beta) \quad (9)$$

This gives an equivalent formulation of logical determinism and logical reversibility:

A logically deterministic computation is one for which

$$\forall \alpha, \beta \ P(\beta|\alpha) \in \{0, 1\} \quad (10)$$

A logically reversible computation is one for which

$$\forall \alpha, \beta \ P(\alpha|\beta) \in \{0, 1\} \quad (11)$$

This is defined in terms of the set $\{P(\alpha|\beta)\}$. A logical operation has been defined only by the set $\{P(\beta|\alpha)\}$, with the $P(\alpha|\beta)$ dependant upon the input logical state probabilities $P(\alpha)$.

From

$$P(\alpha|\beta) = 0 \Rightarrow P(\alpha, \beta) = P(\beta|\alpha) = 0$$

$$P(\alpha|\beta) = 1 \Rightarrow P(\alpha' \neq \alpha|\beta) = 0$$

$$\Rightarrow P(\alpha' \neq \alpha, \beta) = P(\beta|\alpha' \neq \alpha) = 0 \quad (12)$$

there is an equivalent definition of logically reversible computations. An operation is logically reversible, if and only if,

$$\forall \beta \ [P(\beta|\alpha) \neq 0 \Rightarrow [\forall \alpha' \neq \alpha \ P(\beta|\alpha') = 0]] \quad (13)$$

This definition is now independant of the input probability distribution.

We summarise these properties and some consequences.

1. Logically deterministic operations

$$\forall \alpha, \beta \ P(\beta|\alpha) \in \{0, 1\}$$

$$\forall \alpha \ [P(\alpha|\beta) \neq 0 \Rightarrow [\forall \alpha' \neq \alpha \ P(\beta|\alpha') = 0]]$$

$$\forall \alpha, \beta \ P(\alpha|\beta) \in \left\{0, \frac{P(\alpha)}{P(\beta)}\right\} \quad (14)$$

In the case where a particular $\alpha \rightarrow \beta$ transition has $P(\beta|\alpha) = 1$, we may refer to this as a logically deterministic transition, even if the overall operation is not logically deterministic.

2. Logically reversible operations

$$\forall \alpha, \beta \ P(\alpha|\beta) \in \{0, 1\}$$

$$\forall \beta \ [P(\beta|\alpha) \neq 0 \Rightarrow [\forall \alpha' \neq \alpha \ P(\beta|\alpha') = 0]]$$

$$\forall \alpha, \beta \ P(\alpha|\beta) \in \left\{0, \frac{P(\beta)}{P(\alpha)}\right\} \quad (15)$$

In the case where a particular $\alpha \rightarrow \beta$ transition has $P(\alpha|\beta) = 1$, we may refer to this as a logically reversible transition, even if the overall operation is not logically reversible.

D. Physical Representation of Logical States

We will now consider what the properties above imply for the physical embodiment of logical states and operations upon them. The physical system will have a state space of possible microstates $\{\mu\}$. How can these be used to embody the logical states?

1. A particular logical state $\alpha$ will be identified with a set of microstates $\{\mu_\alpha\}$ in the state space, in the sense that when the physical state of the system is one of the microstate $\mu \in \{\mu_\alpha\}$, then the logical state takes the value $\alpha$.

2. As logical states are distinct, a given microstate can be identified with one, and only one, input state. Each set of microstates $\{\mu_\alpha\}$ is therefore non-intersecting with any other such set of microstates:

$$\{\mu_\alpha\} \cap \{\mu_{\alpha'} \neq \alpha\} = \emptyset \quad (16)$$

3. For the logical states to be distinguishable, it is necessary that it is possible to ascertain the set to which the microstate belongs. We are not considering analogue information processing, so the physical interactions must not need to be sensitive to arbitrarily close (using a natural distance measure) states in state space. We replace the point in state space $\mu$ with the neighbourhood of that point $R(\mu)$. The logical state $\alpha$ is now identified with the region of state space corresponding to the union of all the neighbourhoods $\{R(\mu_\alpha)\}$. The neighbourhoods corresponding to different logical states must be non-overlapping.

---

6 For simplicity, input states for which $P(\alpha) = 0$, i.e. which are certain to not occur, will not be included in this set.
4. We can now identify the proposition for the logical state $\alpha$ with the projector $K_\alpha$ onto the region of state space \{R(\mu_\alpha)\}

$$K_\alpha K_{\alpha'} = \delta_{\alpha\alpha'} K_\alpha$$

$$\sum_{\alpha} K_\alpha = I$$

$$K_\alpha [R(\mu_\alpha)] = R(\mu_\alpha)$$

$$K_\alpha [R(\mu_{\alpha'} \neq \alpha)] = 0 \quad (17)$$

The proposition $\alpha$ is true if the state $\mu$ is in the region of state space \{R(\mu_\alpha)\} projected out by $K_\alpha$.

5. For the physical representation of the logical states to be complete, then it must also be the case that if the state $\mu$ is in the region of state space \{R(\mu_\alpha)\} projected out by $K_\alpha$, then the logical proposition corresponding to the logical state $\alpha$ is true.

6. For the logical states to be stable, then under the normal evolution of the system, a microstate within the region of state space corresponding to a given logical state must stay within that region of state space$^7$. The normal evolution of the system is, trivially, a physical embodiment of the 'logical Do Nothing IDN' operation.

E. Physical Representation of Logical Operations

During the normal evolution of a system, logical states do not change. To perform non-trivial logical operations new interactions must alter the evolution of the state space. All the essential characteristics of a logical operation are included in the set \{P(\beta|\alpha)\}. It follows that a physical process is an embodiment of a logical operation if, and only if, the evolution of the microstates in the physical process are such that, over an ensemble of microstates in the region \{R(\mu_\alpha)\}, the probability that the microstate ends up in the region \{R(\mu_\beta)\} is just $P(\beta|\alpha)$.

1. We will assume that the laws of physics are Hamiltonian. The evolution of microstates over the state space of the combined system of the logical processing apparatus and the environment must be described by a Hamiltonian evolution operator.

2. If the interaction of the microstates of the system and the environment are such that any individual microstate $\mu_\alpha$ starting in state $\alpha$ is randomised so that it ends up in the output state $\beta$ with probability $P(\beta|\alpha)$ then we do not need to be sensitive to the initial probability distribution of the ensemble of microstates within the logical state $\alpha$. In general, however, we may need to be sensitive to the initial probability distribution $\rho_\alpha$ over microstates corresponding to the logical state $\alpha$.

3. The complete statistical state of the logical processing system input to the logical operation is

$$\sum_{\alpha} P(\alpha)\rho_\alpha \quad (18)$$

where

$$\forall \alpha, K_\alpha \rho_\alpha K_\alpha = \rho_\alpha \quad (19)$$

4. The complete statistical state of the logical processing system output from the logical operation is required to be

$$\sum_{\beta} P(\beta)\rho_\beta \quad (20)$$

where

$$\forall \beta, K_\beta \rho_\beta K_\beta = \rho_\beta \quad (21)$$

We have not considered separate systems for the logical input states, the logical processing apparatus or the output states. At first, this seems to assume that the system embodying the logical input states must be the same as the system embodying the logical output states and that the logical processing apparatus cannot have internal states - which would seem to be quite a strong restriction. This is not the case. Let us consider the case where there are three distinct systems: the input state system, with states \{\rho_\alpha\}; an output state system, with states \{\rho_\beta\}; and an auxiliary system corresponding to all internal and external components of the process, with states \{\rho_{App}\}.

The statistical state is described at the start of the operation by

$$\sum_{\alpha} P(\alpha)\rho_\alpha \otimes \sum_{\beta} f_\beta \rho_{\beta_{App}} \otimes \rho_\beta \quad (22)$$

where we have assumed that the input state system is initially uncorrelated to the apparatus but have not assumed the output state system is initially uncorrelated to the internal states \{\rho_{\beta_{App}}\} of the apparatus. The effect of the operation would be to evolve the combined system into some new correlated state, combining the three systems:

$$\sum_{\alpha,\beta} P(\alpha,\beta)\rho_{\alpha,\beta} \otimes \rho_{\alpha,\beta}^{App} \otimes \rho_\beta \quad (23)$$

Our approach here is then to consider the state space of the combined system of input, output and apparatus as a single state space, with input states for $\alpha$ of:

$$\rho_\alpha \otimes \left(\sum w_\beta \rho_{\beta_{App}} \otimes \rho_\beta\right) \quad (24)$$

$^7$ A weaker condition, acceptable for most practical needs, is that the probability of the microstate leaving the region of a given logical state, during the time scale of the information processing, must be very low.
and output states for $\beta$ of:

$$\left( \sum_{\alpha} P(\alpha, \beta) \rho_{\alpha, \beta} \otimes \rho_{\alpha, \beta}^{\text{App}} \right) \otimes \rho_\beta$$ (25)

We then consider a Hamiltonian evolution on the combined state space to be the operation. This cleanly separates the logical states, embodied by the physical state of the combined state space, from the logical operation, embodied by the Hamiltonian evolution on that state space. We have not restricted ourselves by the assumption of a Hamiltonian evolution on a single state space, as we have the full generality of all possible Hamiltonian interactions allowed between the input state system, the output state system and the logical processing apparatus. We have avoided, on the other hand, any need to consider the restrictions and complications that would arise if we constructed models based upon specific assumptions as to how the input state, output state and logical processing apparatus systems are allowed to interact. This completes the physical characterisation of logical states and operations.

F. Logical vs Microscopic Determinism and Reversibility

There is one final issue that needs to be stated, for the sake of clarity, regarding the (absence of a) relationship between logical and microscopic indeterminism and irreversibility.

**Logical indeterminism** does not imply or require the existence of any fundamental indeterminism in the microscopic dynamics of the physical states. Neither is **logical determinism** incompatible with the existence of fundamental indeterministic dynamics.

A specific microstate from the input logical state may evolve deterministically into a specific microstate of an output logical state, while the operation remains logically indeterministic, provided the set of the microstates corresponding to the same input logical state do not all evolve, with certainty, into microstates of the same output logical state.

A specific microstate from an input logical state may evolve indeterministically into a number of possible microstates, while the operation remains logically deterministic, provided that the set of the input microstates corresponding to the same input logical state can only evolve into microstates from the set corresponding to the same output logical state.

**Logical irreversibility** does not imply or require the existence of any fundamental irreversibility in the microscopic dynamics of the physical states. Neither is **logical reversibility** incompatible with the existence of fundamental irreversible dynamics.

A specific microstate from the input logical state may evolve reversibly into a specific microstate of an output logical state, while the operation remains logically irreversible, provided the set of the microstates corresponding to the same output logical state have not all evolved, with certainty, from microstates of the same input logical state.

A specific microstate from an input logical state may evolve irreversibly into a specific microstate, while the operation remains logically reversible, provided that the set of microstates corresponding to the same output logical state can only have evolved from microstates in the set corresponding to the same input logical state.

III. THERMODYNAMICS OF LOGICAL OPERATIONS

We now undertake the main task of this paper: to determine the limiting thermodynamic cost to a logical operation. We will do this in two steps.

Firstly, we will construct a physical process, capable of implementing any logical operation, as we have defined them, and we will consider the optimum thermodynamic cost to the process. This optimum will be considered in two ways: for individual transitions between specific logical states; and as an expectation value over an ensemble of operations. Both work required to perform the process and heat generated by the process will be calculated, where it is assumed that all heat generated is absorbed by a heat bath at some reference temperature $T_R$.

To calculate the expectation values, we must consider the probability distribution over the input logical states. For this we will use the probability distribution used to calculate the Shannon information being processed. The optimum process will, of necessity, involve various idealisations (such as frictionless motion and quasi-static processes) that cannot be achieved in practice. The purpose is to demonstrate not that it is possible to build such optimal operations, but rather that there is no physical limitation, in principle, on how close one can get to them.

Then we will prove that there cannot exist any physical process that can implement the same logical transformation of information, but with a lower expectation value for either the work requirement or the heat generation. The optimum process, for our particular implementation of a logical transformation of information, is also the optimum for any possible implementation of that transformation.

A. Statistical mechanical assumptions

We will now clearly state the statistical mechanical assumptions that are being made. There are a number of
different approaches to the foundations of statistical mechanics and, as the models discussed here involve such idealisations as the treatment of individual atoms, it is important to be clear which approach is being taken. In this article we will assume the standard structure of Gibbs canonical statistical mechanics: we will be dealing with Hamiltonian flows with probability distributions over a state space, we will assume that a system that has been thermalised can be represented by a canonical distribution over its accessible state space, and will be initially statistically independent of any other system. While these assumptions are clearly open to debate, a full discussion or justification of them lies outside the scope of this article (although see [13]).

1. The system consists of the logical processing apparatus (including auxiliary systems as discussed in Section [IIE]) and a number of heat baths. A heat bath is simply a system that has been allowed to thermalise at some temperature and is sufficiently large that any energy transfer with the logical processing apparatus will have negligible effect upon the heat baths internal energy. The Hamiltonian for the combined system is:

\[ H = H_L + \sum_i (H_i + V_i) \]  

where \( H_L \) is the internal Hamiltonian for the logical processing apparatus, \( H_i \) the internal Hamiltonian of the heat bath \( i \) and \( V_i \) is the interaction Hamiltonian between the logical processing apparatus and the heat bath \( i \). We assume there is no interaction between heat baths. The density matrix of the combined system is \( \rho_{CL} \), and \( \rho_L \) is the marginal density matrix after tracing over the heat bath subsystems.

2. Work is performed upon the apparatus through the variation of some externally controlled parameter \( X \), which affects the energy eigenvalues and eigenstates\(^8\).

\[ H_L(X) = \sum_n E_n(X) |E_n(X)\rangle \langle E_n(X)| \]  

The mean work performed, as the parameter is varied from \( X_0 \) to \( X_1 \) is given by:

\[ \Delta W = \int_{X_0}^{X_1} \text{Tr} \left[ \frac{\partial H_L(X)}{\partial X} \rho_L(X) \right] dX \]  

Note that the density matrix \( \rho_L \) may be varying as \( X \) varies. We will assume that neither the internal Hamiltonians of the heat baths nor the interaction Hamiltonians have controllable parameters: work is only performed upon the logical processing apparatus itself.

3. The mean change in internal energy of the logical processing apparatus is

\[ \Delta E = \text{Tr} [H_L(X_1)\rho_L(X_1)] - \text{Tr} [H_L(X_0)\rho_L(X_0)] \]  

4. If we now assume negligible changes in interaction energies:

\[ \forall i, \text{Tr} [V_i\rho_C(X_1)] \approx \text{Tr} [V_i\rho_C(X_0)] \]  

then

\[ \Delta W - \Delta E = \sum_i \Delta Q_i \]  

where

\[ \Delta Q_i = \text{Tr} [H_i\rho_i(X_1)] - \text{Tr} [H_i\rho_i(X_0)] \]  

is the increase in internal energy of the heat bath \( i \), and \( \rho_i \) is the marginal density matrix of the heat bath, after tracing over the logical processing apparatus and all other heat baths.

5. If the evolution of density matrix is such that it always remains diagonalised by the energy eigenstate basis, so that

\[ \rho_L(X) = \sum_n p_n(X) |E_n(X)\rangle \langle E_n(X)| \]  

then:

\[ \Delta W = \int_{X_0}^{X_1} \sum_n p_n(X) \frac{\partial E_n(X)}{\partial X} dX \]  

\[ \sum_i \Delta Q_i = \int_{X_0}^{X_1} \sum_n \frac{\partial p_n(X)}{\partial X} E_n(X) dX \]  

It is important to note that these 5 points make no assumption regarding the identification of either thermodynamic entropy or thermal distributions. Neither the canonical distribution nor the Gibbs-von Neumann entropy has been used.

The following results depend upon the assumption that a heat bath is represented by a canonical distribution and that a limiting ideal case exists of thermalisation through a succession of brief interactions with small subsystems of a heat bath. The calculations are well known (see [13, 14, 15], for example) and the results are stated here purely for clarity. No formal identification of the Gibbs-von Neumann entropy with thermodynamic entropy is required to derive these results.

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\(^8\) More generally, one should consider a number of controllable parameters, which are each varying in time.
6. A system that is brought into contact with an ideal heat bath will, over time periods long with respect to its thermal relaxation time, be well represented by a canonical probability distribution

$$\rho_\alpha = \frac{e^{-H/kT}}{\text{Tr}[e^{-H/kT}]}$$

over accessible states of the system, with the $T$ being the temperature of the heat bath, and $H$ the Hamiltonian of the system over the accessible subspace.

7. In the limit of isothermal quasistatic processes, the system is in contact with an ideal heat bath at some temperature, and system stays in thermal equilibrium with the heat bath at all times.

8. In the limit of adiabatic quasistatic processes (or essentially isolated\(^{14}\) processes) the system always remains in a (canonically distributed) thermal state but there is zero mean energy flow out of the system ($\Delta W = \Delta E$). The temperature of this state may vary.

9. We will assume that the only systems with which the information processing system interacts are ideal heat baths at temperatures $\{T_\alpha\}, \{T_\beta\}$ and $T_R$, and a work reservoir, and that there are no initial correlations between the system and the heat baths.

While these assumptions involve significant idealisations, they are the kind of idealisations that are standard in thermodynamics and statistical mechanics. Rather than representing a physically achievable process, they represent the limit of what can be physically achieved. There is no physical reason why one cannot, in principle, get arbitrarily close to these results.

Although the value of the Gibbs-von Neumann entropy, $-k\text{Tr}[\rho \ln \rho]$ will be calculated for the input and output logical states, all results in this Section, in terms of work required and heat generated, are derivable, from the assumptions stated, without needing to identify this property with thermodynamic entropy\(^9\).

### B. Generic logical operation

#### 1. Input logical states

We start the operation with the logical states represented by physical states with the properties:

1. An input logical state, $\alpha$, to the logical computation is physically embodied by a system confined to some region of state space. The distribution over the microstates of that region gives the density matrix $\rho_\alpha$.

2. $\rho_\alpha$ has mean energy $E_\alpha = \text{Tr}[H_L\rho_\alpha] = \text{Tr}[H_\alpha\rho_\alpha]$, where $H_\alpha = K_\alpha H_L K_\alpha$.

3. For simplicity, in the main section, we will assume the input logical state $\alpha$ is canonically distributed, as if it has been thermalised with a heat bath at temperature $T_\alpha$.

$$\rho_\alpha = \frac{e^{-H_\alpha/kT_\alpha}}{\text{Tr}[e^{-H_\alpha/kT_\alpha}]}$$

This assumption is not essential, and can easily be relaxed without affecting any result. If the initial density matrix is not a canonical distribution, then it is possible to construct a unitary operator that acts upon the system in isolation and rotates it into a canonical state, with neither heat nor work requirement.

To give an explicit construction, suppose the initial Hamiltonian and density matrix are $H^{(i)}_\alpha$ and $\rho^{(i)}_\alpha$, such that

$$\rho^{(i)}_\alpha \neq \frac{e^{-H^{(i)}_\alpha/kT_\alpha}}{\text{Tr}[e^{-H^{(i)}_\alpha/kT_\alpha}]}$$

for any $T_\alpha$. Given the diagonal representation

$$\rho^{(i)}_\alpha = \sum_n p_n |\lambda_n\rangle \langle \lambda_n|$$

then the Hamiltonian, acting between $0 < t < \tau$,

$$H_A = \left[ \cos^2 \left( \frac{\pi t}{2\tau} \right) - \sin^2 \left( \frac{\pi t}{2\tau} \right) \right] H^{(i)}_\alpha$$

$$+ \left[ \sin^2 \left( \frac{\pi t}{2\tau} \right) - \sin^2 \left( \frac{\pi t}{2\tau} \right) \right] H_\alpha$$

$$- \frac{2\hbar}{\tau} \sin^2 \left( \frac{\pi t}{\tau} \right) \ln \left[ \sum_n |\gamma_n\rangle \langle \lambda_n| \right]$$

with

$$H_\alpha = \sum_n E_n |\gamma_n\rangle \langle \gamma_n|$$

$$E_n = E_\alpha - kT_\alpha \left( \ln(p_n) - \sum_m p_m \ln(p_m) \right)$$

varies continuously from $H^{(i)}_\alpha$ to $H_\alpha$, and has the effect of leaving the system, after time $\tau$, in the stationary canonical state

$$\rho_\alpha = \frac{e^{-H_\alpha/kT_\alpha}}{\text{Tr}[e^{-H_\alpha/kT_\alpha}]} = \sum_n p_n |\gamma_n\rangle \langle \gamma_n|$$

\(^9\) See \[13\] and \[2\] [Chapter 6] where this kind of calculation is carried out in detail for same the kinds of systems considered here.
\(\rho_\alpha\) is unitarily equivalent to \(\rho_\alpha^{(i)}\) and \(\text{Tr} [H_\alpha \rho_\alpha] = \text{Tr} \left[ H_\alpha^{(i)} \rho_\alpha^{(i)} \right]\). The mean work requirement is zero and no heat is exchanged with the environment. It should be noted that this construction holds even if \(\rho_\alpha^{(i)}\) is not diagonalised in the eigenstates of \(H_\alpha^{(i)}\).

4. The Gibbs-von Neumann entropy of the input logical state is:

\[ S_\alpha = -k \text{Tr} [\rho_\alpha \ln(\rho_\alpha)] \]

5. There are \(M\) possible input logical states.

2. \textit{Output logical states}

The output logical states may be similarly characterised by:

1. An output logical state, \(\beta\), from the logical computation is physically embodied by a system confined to some region of state space. The distribution over the microstates of that region gives the density matrix \(\rho_\beta\).

2. \(\rho_\beta\) has mean energy \(E_\beta = \text{Tr} [H_\beta \rho_\beta] = \text{Tr} \left[ H_\beta' \rho_\beta \right]\), where \(H_\beta' = K_\beta H_\beta K_\beta\).

3. Again, for convenience we will assume the output logical state \(\beta\) is canonically distributed as if it has been thermalised at a temperature \(T_\beta\). This will make the density matrix:

\[ \rho_\beta = \frac{e^{-H_\beta'/kT_\beta}}{\text{Tr} \left[ e^{-H_\beta'/kT_\beta} \right]} \]  

Again, this assumption is easily dropped. If the final state is required to be a non-canonical density matrix \(\rho_\beta^{(f)}\), with Hamiltonian \(H_\beta^{(f)}\), then \(\rho_\beta^{(f)}\) can be obtained from \(\rho_\beta\) by constructing \(H_\beta\) and \(H_B\) in the same manner as \(H_\alpha\) and \(H_A\) above.

4. The Gibbs-von Neumann entropy of the output logical state is:

\[ S_\beta = -k \text{Tr} [\rho_\beta \ln(\rho_\beta)] \]

5. There are \(N\) possible output logical states.

6. The output logical state \(\beta\) must occur with probability \(P(\beta|\alpha)\) given input logical state \(\alpha\).

We will also note here that probabilities enter the calculation at two levels: as a probability distribution over the microstates within a given logical state, and as a probability distribution over the different logical states. We will attempt to keep these formally separate. From this point onwards, the microstate probability will be represented only by the density matrix. Explicitly appearing probabilities and averages will always refer to the probability distribution over the logical states.

**C. The transformation of information.**

If we consider the actual microstate of the system, within the region corresponding to a given logical state, as being some free parameter, then the physical representation of each logical state may initially be regarded as a potential well with some arbitrary shape, such that the free parameter is confined within the well. The potential wells associated with different logical states are in different regions of physical space, separated by high potential barriers, such that there is a very low possibility of transitions between different logical states.

This can be represented as an atom in one of a number of boxes, where the free parameter is the location of the atom in the box. The logical state is represented by the particular box, or potential well, within which the atom is confined. The physical transformation of the information will take place in nine steps. Steps 1 through to 3 will bring the input logical states into standardised physical states at a shared reference temperature. Steps 4 is the logically indeterministic implementation of the \(P(\beta|\alpha)\) transition. Step 5 and Step 6 implement the joining together of the \(\beta\) output states from the different \(\alpha\) input states, giving the logically irreversible stage. Steps 6 through to 9 then alter each output logical state to the required final physical state.

Calculations for work requirements, heat generation and so forth, follow the statistical mechanical calculations above. Particularly detailed calculations for 'atom in a box' type systems are considered in references such as [2, 16, 17, 18]. The key results can be summarised. The Hamiltonian for an infinite square well potential, of width \(l\), holding an atom of mass \(m\) is

\[ H(l) = \sum_n \frac{\hbar^2 l}{8m l^2} n^2 |E_n| \langle E_n | \]  

Work is performed upon the system by varying the \(l\) parameter (width of the box).

In a canonical thermal state at temperature \(T\), the mean energy is

\[ E = \frac{\sum_n \frac{\hbar^2 l^2}{8m l^2} n^2 e^{-\frac{\hbar^2 l^2}{8m l^2} n^2/kT}}{\sum_n e^{-\frac{\hbar^2 l^2}{8m l^2} n^2/kT}} \approx \frac{1}{2}kT \]

and the Gibbs-von Neumann entropy is

\[ S = \frac{\sum_n \frac{\hbar^2 l^2}{8m l^2} n^2 e^{-\frac{\hbar^2 l^2}{8m l^2} n^2/kT}}{\sum_n e^{-\frac{\hbar^2 l^2}{8m l^2} n^2/kT}} + k \ln \left[ \sum_n e^{-\frac{\hbar^2 l^2}{8m l^2} n^2/kT} \right] \]

\[ \approx \frac{k}{2} \ln \left[ \frac{2e\hbar^2}{\pi \hbar^2} T \right] \]  

\[ \text{Not externally controlled.} \]
The approximations hold when the temperature is high with respect to the ground state:

$$kT \gg \frac{\pi \hbar^2}{2em}$$

(48)

1. The first step is to continuously and slowly deform the potential well of each separate logical state into a square well potential.

The square well should be deformed to width $d^{(1)}_{\alpha}$

$$d^{(1)}_{\alpha} = \left( \sqrt{\frac{\pi \hbar^2}{2emkT}} \right) e^{S_{\alpha}/k}$$

(49)

where $m$ is the mass of atom.

This state has mean energy and entropy

$$E^{(1)}_{\alpha} = \frac{1}{2} kT_{\alpha}$$

$$S^{(1)}_{\alpha} = S_{\alpha}$$

(50)

If this deformation is carried out sufficiently slowly, the mean heat generation is zero and the work requirement is

$$W^{(1)}_{\alpha} = \frac{1}{2} kT_{\alpha} - E_{\alpha}$$

(51)

This is a mean work requirement for the operation. Fluctuations may occur around this value.

The system is then pictured as a box, divided with $M-1$ partitions. When the atom is located between the $\alpha-1$ and $\alpha$ partitions, the system is in logical state $\alpha$. This can be seen in Figure 1(a).

2. Remove the system from all contact with heat baths and then, slowly, adiabatically vary the width of each square well to $d^{(2)}_{\alpha}$:

$$d^{(2)}_{\alpha} = d^{(1)}_{\alpha} \sqrt{\frac{T_{\alpha}}{T_R}}$$

(52)

At the limit of a slow, quasistatic process, this will leave each logical state with a density matrix equal to a canonical thermal system with temperature $T_R$. Mean energy, entropy and mean work requirements are:

$$E^{(2)}_{\alpha} = \frac{1}{2} kT_R$$

$$S^{(2)}_{\alpha} = k \ln \left[ d^{(2)}_{\alpha} \left( \sqrt{\frac{2emkT_R}{\pi \hbar^2}} \right) \right] = S_{\alpha}$$

$$W^{(2)}_{\alpha} = \frac{1}{2} kT_R - \frac{1}{2} kT_{\alpha}$$

(53)

As the total width of the box is now

$$L = \sum_{\alpha'} d^{(2)}_{\alpha'} = \left( \sqrt{\frac{\pi \hbar^2}{2emkT_R}} \right) \sum_{\alpha'} e^{S_{\alpha'}/k}$$

(54)

FIG. 1: Arranging input states

then

$$d^{(2)}_{\alpha} = \frac{L e^{S_{\alpha}/k}}{\sum_{\alpha'} e^{S_{\alpha'}/k}}$$

(55)

3. Now bring the entire system into contact with heat baths at the reference temperature $T_R$. Slowly and isothermally move the positions of the potential barriers separating the square wells (see Figure 1(a-b)). Move the $i^{th}$ barrier to the position $x_i$:

$$x_i = L \sum_{\alpha=i}^n w_{\alpha}$$

(56)

where $\sum_{\alpha} w_{\alpha} = 1$. The values of $w_{\alpha}$ have not been specified. Varying these will be used to optimise the operation.

Each logical state now has a width $d^{(3)}_{\alpha} = w_{\alpha} L$. If $w_{\alpha} = 0$ for one of the input states $\alpha$ this stage will compress the volume of that state to zero. Clearly this can only be allowed to take place if there is no possibility that the partition is occupied by the atom!
\[ E^{(3)}_\alpha = \frac{1}{2} kT_R \]
\[ S^{(3)}_\alpha = k \ln \left[ \frac{d^{(3)}_\alpha}{d^{(2)}_\alpha} \left( \frac{2emkT_R}{\pi \hbar^2} \right) \right] \]
\[ W^{(3)}_\alpha = kT_R \ln \left[ \frac{d^{(3)}_\alpha}{d^{(2)}_\alpha} \right] \]
\[ Q^{(3)}_\alpha = kT_R \ln \left[ \frac{d^{(3)}_\alpha}{d^{(2)}_\alpha} \right] \]  

where \( Q^{(3)}_\alpha \) is the heat generated in the heat bath if the atom is in the \( \alpha \) partition.

4. Insert \( N - 1 \) new potential barriers slowly into each partition. Within a given partition \( \alpha \), the barriers should be spaced according to the probabilities \( P(\beta|\alpha) \) of the logical operation (see Figure 2). They have a width
\[ d^{(4)}_{\alpha,\beta} = P(\beta|\alpha)w_\alpha L \]

This is the logically indeterministic step of the computation, in Figure 1 (b-c). There are \( M \) partitions, each with \( N \) subpartitions.

If the atom was located in partition \( \alpha \) beforehand, and the system has been allowed to thermalise at temperature \( T_R \) for a period of time greater than the thermal relaxation time, then the probability of the atom now being located in the \((\alpha, \beta)\) subpartition is \( P(\beta|\alpha) \). For logically deterministic operations, then all non-zero \( P(\beta|\alpha) \) are equal to one and no partitions need be inserted.

The mean energy, and the entropy, associated with the atom being located within a particular \((\alpha, \beta)\) subpartition is:
\[ E^{(4)}_{\alpha,\beta} = \frac{1}{2} kT_R \]
\[ S^{(4)}_{\alpha,\beta} = k \ln \left[ d^{(4)}_{\alpha,\beta} \left( \frac{2emkT_R}{\pi \hbar^2} \right) \right] \]

5. Now rearrange the subpartitions so that, for each \( \beta \), all the \( \beta \) output partitions are adjacent. From each \( \alpha \) partition, we gather first subpartition, corresponding to output logical state \( \beta = 1 \), and collect them together. Repeat this for each set of \( \beta \) subpartitions, from all the \( \alpha \) partitions. Finally this produces a sequence of \( N \) \( \beta \) partitions, each with \( M \) \( \alpha \) subpartitions. This is illustrated in Figure 3.
6. Remove the potential barriers within each $\beta$ output partition and leave the box for a time that is long in comparison to the atoms thermal relaxation time.

The $\beta$ partition has a width

$$d_{\beta}^{(0)} = w_{\beta} L$$  \hspace{1cm} (60)

where

$$w_{\beta} = \sum_{\alpha} w_{\alpha} P(\beta|\alpha)$$  \hspace{1cm} (61)

and if the atom is located in the $\beta$ partition, then

$$E_{\alpha}^{(6)} = \frac{1}{2} kT_{R}$$

$$S_{\alpha}^{(6)} = k \ln \left[ d_{\beta}^{(6)} \left( \frac{2 \hbar m k T_{R}}{\pi \hbar^2} \right) \right]$$  \hspace{1cm} (62)

This is the logically irreversible stage and is illustrated in Figure 1(d-e). This stage is trivial for logically reversible computations, for which each $\beta$ output partition is composed of only one $\alpha$ subpartition, and so has no internal barriers. Note also that if $\forall \alpha, P(\beta|\alpha) = 0$, then the atom can never be located in the $\beta$ partition.

7. Now slowly and isothermally resize the output partitions. The barriers should be moved until the $\beta$ partition has width

$$d_{\beta}^{(7)} = \left( \frac{\pi \hbar^2}{2 \hbar m k T_{R}} \right) e^{S_{\beta}/k}$$  \hspace{1cm} (63)

See Figure 1(e) to (f).

The overall width of the box may change by this operation, and is now:

$$L' = \sum_{\beta'} d_{\beta'}^{(7)} = L \sum_{\beta'} \frac{e^{S_{\beta}/k}}{\sum_{\alpha'} e^{S_{\alpha'}/k}}$$  \hspace{1cm} (64)

so that

$$d_{\beta}^{(7)} = \frac{L'}{\sum_{\beta'} e^{S_{\beta'}/k}}$$  \hspace{1cm} (65)

For the atom located in the $\beta$ partition, we have:

$$E_{\beta}^{(7)} = \frac{1}{2} kT_{R}$$

$$S_{\beta}^{(7)} = k \ln \left[ d_{\beta}^{(7)} \left( \frac{2 \hbar m k T_{R}}{\pi \hbar^2} \right) \right] = S_{\beta}$$

$$W_{\beta}^{(7)} = kT_{R} \ln \left[ \frac{d_{\beta}^{(7)}}{d_{\beta}^{(6)}} \right]$$

$$Q_{\beta}^{(7)} = kT_{R} \ln \left[ \frac{d_{\beta}^{(7)}}{d_{\beta}^{(6)}} \right]$$  \hspace{1cm} (66)

where $Q_{\beta}^{(7)}$ is the heat generated in the heat bath.

8. Now remove all contact from the $T_{R}$ heat baths. With the system thermally isolated, slowly and adiabatic resize the output partitions to the widths:

$$d_{\beta}^{(8)} = d_{\beta}^{(7)} \sqrt{\frac{T_{R}}{T_{\beta}}}$$  \hspace{1cm} (67)

If the atom is in the $\beta$ partition, the effect of this quasistatic, adiabatic evolution is to leave the atom in a canonical thermal state with temperature $T_{\beta}$.

$$E_{\beta}^{(8)} = \frac{1}{2} kT_{\beta}$$

$$S_{\beta}^{(8)} = S_{\beta}$$

$$W_{\beta}^{(8)} = \frac{1}{2} kT_{\beta} - \frac{1}{2} kT_{R}$$  \hspace{1cm} (68)

9. The output logical states $\beta$ are now all at the required temperature, and entropy. For completeness, bring each separate $\beta$ partition into thermal contact with a heat bath at the appropriate temperature $T_{\beta}$ and slowly, continuously and isothermally deform the shape of each square well potential into the final potential for the output logical state.

$$E_{\beta}^{(9)} = E_{\beta}$$

$$S_{\beta}^{(9)} = S_{\beta}$$

$$W_{\beta}^{(9)} = E_{\beta} - \frac{1}{2} kT_{\beta}$$  \hspace{1cm} (69)

This completes the physical implementation of the logical operation.

**D. Thermodynamic costs**

The procedure detailed in the previous Section fulfils the requirements of a generic logical operation. The input logical states are represented by the appropriate physical input states, the output logical states are represented by the appropriate physical output states, and the transitions between them occur with probabilities $P(\beta|\alpha)$.

1. **Individual transitions**

Adding up the work and heat values, across all steps, for a system which starts in logical state $\alpha$ and ends in logical state $\beta$ gives:

$$\Delta W_{\alpha,\beta} = (E_{\beta} - T_{R} S_{\beta}) - (E_{\alpha} - T_{R} S_{\alpha}) + kT_{R} \ln \left[ \frac{w_{\beta}}{w_{\alpha}} \right]$$

$$\Delta Q_{\alpha,\beta} = T_{R} \left( S_{\alpha} - S_{\beta} + k \ln \left[ \frac{w_{\beta}}{w_{\alpha}} \right] \right)$$  \hspace{1cm} (70)
1. For a logically reversible transition,
\[
\frac{w_\beta}{w_\alpha} = P(\beta|\alpha)
\]  
(71)

and so is independent of the choice of \(w_\alpha\). If the transition is also logically deterministic, \(P(\beta|\alpha) = 1\) and the logarithmic term is zero. The work requirements are
\[
\Delta W_{\alpha,\beta} = (E_\beta - T_R S_\beta) - (E_\alpha - T_R S_\alpha)
\]  
(72)

2. If the logically reversible transition is indeterministic, the work requirement is reduced by the quantity \(-kT_R \ln [P(\beta|\alpha)]\). If \(P(\beta|\alpha)\) is small, this term can be large, even to the extent of making the work requirement negative (i.e. implying work may be extracted from the process).

3. Now consider logically irreversible transitions. When the transition is logically deterministic, \(w_\beta\) is the sum of all the \(w_\alpha\) values where the transition is permitted. It is therefore always the case that \(\frac{w_\beta}{w_\alpha} \geq 1\). This implies an increased work requirement compared to a logically reversible, deterministic transition between equivalent \((\alpha, \beta)\) states.

4. Finally logically irreversible, indeterministic transitions may, in principle, take values for \(\frac{w_\beta}{w_\alpha}\) both above and below 1.

Let us consider optimising the thermodynamic cost of an individual \(\alpha \rightarrow \beta\) transition. The only free variables are the \(w_\alpha\). For logically reversible transitions, these have no effect and the cost is always:
\[
\begin{align*}
\Delta W_{\alpha,\beta} &= (E_\beta - T_R S_\beta) - (E_\alpha - T_R S_\alpha) + kT_R \ln [P(\beta|\alpha)] \\
\Delta Q_{\alpha,\beta} &= T_R (S_\alpha - S_\beta + k \ln [P(\beta|\alpha)])
\end{align*}
\]  
(73)

For logically irreversible transitions, the quantity \(\frac{w_\beta}{w_\alpha}\) should be made as small as possible, subject to the constraint that \(\sum \alpha w_\alpha = 1\). From \(w_\beta = \sum_\alpha w_\alpha P(\beta|\alpha')\) it must be the case that
\[
w_\beta \geq w_\alpha P(\beta|\alpha)
\]  
(74)

Equality is reached by setting \(w_{\alpha'} = 0\), for all the input logical states \(\alpha' \neq \alpha\) where \(P(\beta|\alpha') \neq 0\). This gives \(w_\beta = w_\alpha P(\beta|\alpha)\). If the transition is a logically deterministic one, \(\frac{w_\beta}{w_\alpha} = 1\), otherwise \(\frac{w_\beta}{w_\alpha} < 1\) and the work requirement is reduced (as for a logically reversible, indeterministic transition). The result is similar to logically reversible transitions:
\[
\begin{align*}
\Delta W_{\alpha,\beta} &\geq (E_\beta - T_R S_\beta) - (E_\alpha - T_R S_\alpha) + kT_R \ln [P(\beta|\alpha)] \\
\Delta Q_{\alpha,\beta} &\geq T_R (S_\alpha - S_\beta + k \ln [P(\beta|\alpha)])
\end{align*}
\]  
(75)

2. Expectation values

The problem with optimising for an individual transition is that this can go catastrophically wrong if the operation is performed upon any of the other \(\alpha'\) input logical states. For logically irreversible processes, as \(w_{\alpha'} \to 0\), then \(\Delta W_{\alpha',\beta} \to \infty\).

We need to consider an optimisation over the full set of input logical states, rather than with respect to a single input logical state. For the set of all possible transitions, we will seek to minimise the expectation value, or mean cost, of performing the operation.

This is not the only criteria that could be used. One may seek instead, for example, to optimise by a minimax criteria: minimising the maximum cost that might be incurred. This would lead to a different set of \(w_\alpha\) to those we will calculate here. The maximum cost that might be incurred with such a set would, for certainty, be no higher than the maximum cost we will arrive at here. However, the expectation value for the cost, with the different set, would be at least as high as the expectation value we will find.

To be able to calculate an expectation value, a probability distribution over the input logical states is needed. For this we will use the probabilities that go into the calculation of the Shannon information of the input state: \(P(\alpha)\). The probability of the transition \(\alpha \rightarrow \beta\) occurring is then \(P(\beta|\alpha)P(\alpha)\) and the expectation values for the work requirement is:
\[
\langle \Delta W \rangle = \sum_\beta P(\beta) (E_\beta - T_R S_\beta) - \sum_\alpha P(\alpha) (E_\alpha - T_R S_\alpha)
\]  
(76)

\[
+ kT_R \sum_{\alpha,\beta} P(\alpha, \beta) \ln \left[ \frac{w_\beta}{w_\alpha} \right]
\]

where \(P(\alpha, \beta) = P(\beta|\alpha)P(\alpha)\) and \(P(\beta) = \sum_\alpha P(\alpha, \beta)\).

For logically reversible transformations, this is fixed:
\[
\frac{w_\beta}{w_\alpha} = P(\beta|\alpha) = \frac{P(\beta)}{P(\alpha)}
\]  
(77)

For logically irreversible transformations, we must vary the \(w_\alpha\) to minimise the function
\[
X = \sum_{\alpha,\beta} P(\alpha, \beta) \ln \left[ \frac{w_\beta}{w_\alpha} \right]
\]  
(78)

Consider the similar function
\[
Y = \sum_\beta P(\beta) \ln P(\beta) - \sum_\alpha P(\alpha) \ln P(\alpha)
\]  
(79)

\[
X - Y = \sum_{\alpha,\beta} P(\alpha, \beta) \ln \left[ \frac{w_\beta P(\alpha)}{w_\alpha P(\beta)} \right]
\]
\[
= \sum_{\alpha,\beta} P(\alpha, \beta) \ln \left[ \frac{P(\alpha, \beta)}{P(\alpha)w(\alpha)\beta} \right]
\]
\[
\geq 0
\]  
(80)
and the equality occurs iff \( P(\alpha, \beta) = P(\beta)w(\alpha|\beta) \). As \( Y \) is independent of the values of the \( w_\alpha \), the minimum value of \( X \) is precisely the value of \( Y \). This minimum value of \( X \) is reached when \( w_\alpha = P(\alpha) \), which leads to \( w_\beta = P(\beta) \).

The result can easily be re-expressed as:

\[
\langle \Delta W \rangle \geq \sum_\beta P(\beta) \left( E_\beta - T_R (S_\beta - k \ln P(\beta)) \right) - \sum_\alpha P(\alpha) \left( E_\alpha - T_R (S_\alpha - k \ln P(\alpha)) \right)
\]

This is the minimum expectation value of the work requirement for the logical operation, using the physical procedure we have described. The same expression holds for logically reversible, irreversible, deterministic and indeterministic operations. It is not hard to see that this also minimises the expectation value of the heat generated:

\[
\langle \Delta Q \rangle \geq -T_R \left( \sum_\beta P(\beta) (S_\beta - k \ln P(\beta)) \right) - \sum_\alpha P(\alpha) (S_\alpha - k \ln P(\alpha))
\]

As was noted for the case of LE in \( \text{[8]} \), to achieve the optimal physical implementation of a logically irreversible operation requires the physical process to be designed for the particular probability distribution \( P(\alpha) \) over the input logical states\(^{11}\). A physical implementation optimised for one input probability distribution will not, in general, be optimised for a different input probability distribution. For logically irreversible operations it is only possible to thermodynamically optimise the logical transformation of information (where the input probability is specified). Without a probability distribution (even a default assumption of equiprobable input states) it does not even make sense to talk about optimising the expectation value for the work or heat requirements, or about the Shannon information of the input and output states.

\(^{11}\) It is worth noting that this is not the same as having a prior knowledge of the input logical states. Having prior knowledge of which input state occurs allows one, trivially, to do rather better than this, by choosing \( w'_\alpha = 0 \) for all other input states. This optimises for all individual transitions that come from the known \( \alpha \) input state, but requires a different physical implementation each time a different input logical state occurs. That different physical implementation is, in each case, equivalent to a logically reversible operation.

3. Multiple Heat Baths

For completeness, we note that if there are several heat baths available, at different temperatures, the equations may be easily generalised. Defining:

\[
\langle \Delta Q_i \rangle = \sum_i \langle \Delta Q_i \rangle \quad \text{and} \quad \langle \Delta Q \rangle = \sum_i \langle \Delta Q_i \rangle \quad \text{and} \quad \langle \Delta Q \rangle = \sum_i \langle \Delta Q_i \rangle
\]

where \( \langle \Delta Q_i \rangle \) is the mean heat generated in a heat bath at temperature \( T_i \), we may simply replace \( T_R \) with \( T \) and \( \langle \Delta Q \rangle \) with \( \langle \Delta Q \rangle \), in Equation \([82]\) and all subsequent equations. In effect, this is equivalent to the possibility of using reversible Carnot cycles to rearrange heat between any heat baths available, in addition to performing the logical operation with a single heat bath.

The introduction of multiple heat baths has little practical significance though. If

\[
\sum_\beta P(\beta) (S_\beta - k \ln P(\beta)) - \sum_\alpha P(\alpha) (S_\alpha - k \ln P(\alpha)) < 0
\]

then the least work is required by generating all the heat in the coolest heat bath available. If

\[
\sum_\beta P(\beta) (S_\beta - k \ln P(\beta)) - \sum_\alpha P(\alpha) (S_\alpha - k \ln P(\alpha)) > 0
\]

the opposite is true. The least work involves only generating heat in the hottest heat bath.

E. Optimum physical process

We have shown that a particular physical process can implement a logical operation, with a minimum expectation value for the work required or heat generated. Perhaps other physical processes might exist which can perform the same logical operation at a lower cost? We will now prove that no physical process can implement the same logical transformation of information at a lower cost.

The initial statistical state of the logical processing apparatus is

\[
\rho_I = \sum_\alpha P(\alpha) \rho_\alpha
\]

The final statistical state is

\[
\rho_F = \sum_\beta P(\beta) \rho_\beta
\]

We assume that the environment is initially well described by a canonical thermal state \( \rho_E(T_R) \), at temperature \( T_R \), and that it is uncorrelated with the initial state of the logical processing system.
Now consider the initial density matrix of the joint system of the logical processing system and the apparatus

\[ \rho = \rho_I \otimes \rho_E(T_R) \]  

so

\[ \text{Tr} [\rho \ln [\rho]] = \text{Tr} [\rho_I \ln [\rho_I]] + \text{Tr} [\rho_E(T_R) \ln [\rho_E(T_R)]] \]  

For any unitary evolution upon the combined system to be a physical representation of the logical state, it must evolve the system to some state \( \rho' \) such that the marginal distribution of the information processing apparatus is:

\[ \rho_F = \text{Tr}_E [\rho'] \]  

The marginal distribution of the environment is then:

\[ \rho'_E = \text{Tr}_F [\rho'] \]  

From the well known\[14,19,20\] properties of unitary evolutions and density matrices:

\[ \text{Tr} [\rho \ln [\rho]] = \text{Tr} [\rho' \ln [\rho']] \]  

\[ \text{Tr} [\rho' \ln [\rho']] \geq \text{Tr} [\rho_F \ln [\rho_F]] + \text{Tr} [\rho'_E \ln [\rho'_E]] \]  

As \( \rho_E(T_R) \) is a canonical distribution

\[ \frac{H_E}{kT_R} = - \ln [\rho_E(T_R)] - \ln Z \]  

so

\[ \text{Tr} \left[ \rho'_E \left( \ln [\rho'_E] + \frac{H_E}{kT_R} \right) \right] - \text{Tr} \left[ \rho_E(T_R) \left( \ln [\rho_E(T_R)] + \frac{H_E}{kT_R} \right) \right] = \text{Tr} [\rho'_E \ln [\rho'_E] - \ln [\rho_E(T_R)] ] \geq 0 \]  

where \( H_E \) is the internal Hamiltonian of the environment. A simple rearrangement gives

\[ \text{Tr} [\rho_I \ln [\rho_I]] - \text{Tr} [\rho_F \ln [\rho_F]] \geq \frac{\text{Tr} [H_E \rho_E(T_R)]}{kT_R} - \frac{\text{Tr} [H_E \rho'_E]}{kT_R} \]  

As the physical representations of the logical states are non-overlapping:

\[ -k \text{Tr} [\rho_I \ln [\rho_I]] = \sum_\alpha P(\alpha) (S_\alpha - k \ln P(\alpha)) \]  

\[ -k \text{Tr} [\rho_F \ln [\rho_F]] = \sum_\beta P(\beta) (S_\beta - k \ln P(\beta)) \]  

The expectation value for the work performed upon the system must equal\[12\] the expectation value for the change in the internal energy of the system plus the expectation value for the change in the internal energy of the environment:

\[ \langle \Delta W \rangle = \sum_\beta P(\beta) E_\beta + \text{Tr} [H_E \rho'_E] \]  

\[ - \sum_\alpha P(\alpha) E_\alpha - \text{Tr} [H_E \rho_E(T_R)] \]  

12 We assume that the interaction energy between system and environment is negligible at the start and end of the operation. Both this assumption, and the assumption that the environment is initially an uncorrelated Gibbs state, do not appear to hold in \[21,22\].
there is no physical representation of the logical operation that has a lower expectation value for the work requirements or heat generation.

IV. GENERALISED LANDAUER’S PRINCIPLE

There are several different, but formally equivalent, ways of expressing the Generalised Landauer’s principle (GLP). It will be convenient to use the notation:

\[
\langle \Delta E \rangle = \sum_{\beta} P(\beta)E_{\beta} - \sum_{\alpha} P(\alpha)E_{\alpha}
\]

\[
\Delta S = \sum_{\beta} P(\beta)(S_{\beta} - k \ln P(\beta)) - \sum_{\alpha} P(\alpha)(S_{\alpha} - k \ln P(\alpha))
\]

\[
\Delta H = -\sum_{\beta} P(\beta) \log P(\beta) + \sum_{\alpha} P(\alpha) \log P(\alpha)
\]

for: the change in the expectation value for of the internal energy of the information processing apparatus; the change in the Gibbs-von Neumann entropy of the statistical ensemble describing the information processing system; and the change in the Shannon information of the logical states over the course of the operation.

A. Work requirements

GLP1: Work

A logical transformation of information has a minimal expectation value for the work requirement given by:

\[
\langle \Delta W \rangle \geq \langle \Delta E \rangle - T_R \Delta S
\]

B. Heat generation

Noting that

\[
\langle \Delta Q \rangle = \langle \Delta W \rangle - \langle \Delta E \rangle
\]

is equal to the expectation value of the heat generated in the heat bath:

GLP2: Heat

A logical transformation of information has a minimal expectation value for the heat generated in the environment of:

\[
\langle \Delta Q \rangle \geq -T_R \Delta S
\]

It is important to remember that the term \(\Delta S\) appearing in GLP1 and GLP2 is not the change in Shannon information \(\Delta H\) between the input and output states. It is the change in the Gibbs-von Neumann entropy of the logical system, taking into account any changes in the entropies of the subensembles that represent the input and output logical states. It can be related to the change in the Shannon information by

\[
\Delta S = \sum_{\beta} P(\beta)S_{\beta} - \sum_{\alpha} P(\alpha)S_{\alpha} + k \Delta H \ln 2
\]

C. Entropic cost

The change in the Gibbs-von Neumann entropy of the environmental heat bath is given by:

\[
\Delta S_{HB} = -k T_R [\rho'_E \ln [\rho'_E]] + k T_R [\rho_E(T_R) \ln [\rho_E(T_R)]]
\]

which gives the entropic form of the Generalised Landauer’s principle:

GLP3: Entropy

A logical transformation of information requires a minimal change in the Gibbs-von Neumann entropies of the marginal statistical states of an information processing apparatus \(\Delta S\) and its environment \(\Delta S_{HB}\) of:

\[
\Delta S_{HB} + \Delta S \geq 0
\]

This is a trivial consequence of the requirements that the evolution be unitary and that the statistical states of the logical processing system and the environment be initially uncorrelated. The expectation value of the heat generated in the environment is at least equal to the increase in the Gibbs-von Neumann entropy of the marginal state of the heat bath:

\[
\langle \Delta Q \rangle \geq T_R \Delta S_{HB}
\]

This allows us to deduce GLP1 or GLP2 from GLP3, but not reverse\(^{13}\).

D. Information

If we define the term:

\[
\Delta S_L = \sum_{\beta} P(\beta)S_{\beta} - \sum_{\alpha} P(\alpha)S_{\alpha}
\]

we get

\[
\Delta S_{HB} + \Delta S_L \geq -k \Delta H \ln 2
\]

\(^{13}\) In the limiting case of an ideal heat bath and quasistatic processes, the equality is reached and the deduction can then go in both directions.
This expression seems suggestive. If we regard the terms $\Delta S_H$ and $\Delta S_L$ as changes in the entropies of the ‘non-information bearing degrees of freedom’ of the environment and the apparatus, respectively, then we appear to have provided a quantitative version of Bennett’s statement that any logically irreversible manipulation of information $[\Delta H]$ ... must be accompanied by a corresponding $[k \ln 2]$ entropy increase in the non-information bearing degrees of freedom of the information processing apparatus $[\Delta S_L]$ or its environment $[\Delta S_H]$.

Although unlike Bennett, we do not restrict this to irreversible transformations of data.

This produces what may be taken as the information form of the GLP:

GLP4: Information

A logical transformation of information requires an increase of entropy of the non-information bearing degrees of freedom of the information processing apparatus and its environment of at least $-k \ln 2$ times the change in the total quantity of Shannon information over the course of the operation:

$$\Delta S_{NIBDF} \geq -k \Delta H \ln 2 \quad (113)$$

where $\Delta S_{NIBDF} = \Delta S_H + \Delta S_L$. This is quite generally true and follows directly from GLP3 and the definition of $\Delta S_L$.

V. MODELS OF COMPUTING

We will now discuss some of the consequences that can be drawn from the Generalised Landauer’s principle by varying the thermodynamic properties of the input and output states. This allows us to consider the effects of having different energies and entropies for the physical states that embody the logical states and has some surprising consequences.

A. Uniform Computing

When we make the assumption that the computation takes place at the same temperature throughout, such that

Assumption 1: Isothermal

$$\forall \alpha, \beta \ T_R = T_\alpha = T_\beta \quad (114)$$

then we shall call this isothermal computing.

In the most commonly encountered set of assumptions for the thermodynamics of computation, we have, in addition to the assumption of isothermal computing, the physical states, that represent the logical states, all have the same entropy and mean energy, so that

Assumption 2: Uniform states

$$\forall \alpha, \beta \ E_R = E_\alpha = E_\beta$$
$$\forall \alpha, \beta \ S_R = S_\alpha = S_\beta \quad (115)$$

This reduces the Generalised Landauer’s principle to the form of:

$$\langle \Delta W \rangle \geq -k T_R \Delta H \ln 2$$
$$\langle \Delta Q \rangle \geq -k T_R \Delta H \ln 2 \quad (116)$$

where $\Delta H$ is the change in Shannon information over the course of the transformation. This is the usual form in which Landauer’s principle is encountered.

The necessary and sufficient conditions for these to hold is the weaker condition:

Assumption 3: Uniform computing

$$\sum_\alpha P(\alpha) E_\alpha = \sum_\beta P(\beta) E_\beta$$
$$\sum_\alpha P(\alpha) S_\alpha = \sum_\beta P(\beta) S_\beta \quad (117)$$

B. Equilibrium Computing

The simplifying assumption of uniform computing is made so universally, that it might be questioned whether there is any value to considering non-uniform computing. To answer this, consider what happens if the input and output states are constructed to be canonical thermal systems, at temperature $T_R$, with the properties:

Assumption 4: Equilibrium Computing

$$E_\alpha - T_R S_\alpha + k T_R \ln P(\alpha) = C_A$$
$$E_\beta - T_R S_\beta + k T_R \ln P(\beta) = C_A \quad (118)$$

where $C_A$ is a constant, related to the overall size of the logical processing apparatus. This yields the relationships

$$\langle \Delta E \rangle - T_R \Delta S = 0 \quad (119)$$

and reduces the Generalised Landauer’s principle to

$$\langle \Delta W \rangle \geq 0 \quad (120)$$

although

$$\langle \Delta Q \rangle \geq -T_R \Delta S \quad (121)$$

still. The equality can, of course, only be reached in the limit of slow processes.

The necessary and sufficient assumption for Equation [120] to hold is
Assumption 5: Zero mean work

\[
\sum_\alpha P(\alpha) (E_\alpha - T_R (S_\alpha + k \ln P(\alpha))) = \sum_\beta P(\beta) (E_\beta - T_R (S_\beta + k \ln P(\beta)))
\] (122)

Assumption 5 only implies the average work requirement can approach zero, over all the possible transitions between logical states. Assumption 4 ensures that there is a zero mean work requirement \(\Delta W_{\alpha,\beta}\) for all individual \((\alpha, \beta)\) transitions.

C. Adiabatic Computing

To eliminate mean heat generation in the ideal limit, the necessary and sufficient condition is:

Assumption 6: Zero mean heat generation

\[
\sum_\alpha P(\alpha) (S_\alpha + k \ln P(\alpha)) = \sum_\beta P(\beta) (S_\beta + k \ln P(\beta))
\] (123)

leading to

\[
\langle \Delta Q \rangle \geq 0
\] (124)

although this does not eliminate mean work requirements

\[
\langle \Delta W \rangle = \langle \Delta E \rangle
\] (125)

Again, this is only the expectation value over all transitions. To ensure that the mean heat generated \(\Delta Q_{\alpha,\beta}\) is zero for each individual \((\alpha, \beta)\) transition, requires:

Assumption 7: Adiabatic computing

\[
S_\alpha + k \ln P(\alpha) = C_B \\
S_\beta + k \ln P(\beta) = C_B
\] (126)

where \(C_B\) is an apparatus related constant.

D. Adiabatic Equilibrium Computing

Combining the assumptions of adiabatic and equilibrium computing gives the requirement

Assumption 8: Adiabatic equilibrium computing

\[
E_\alpha = E_\beta = E_R \\
S_\alpha + k \ln P(\alpha) = C_C \\
S_\beta + k \ln P(\beta) = C_C
\] (127)

which yields, \(\forall \alpha, \beta\)

\[
\Delta W_{\alpha,\beta} \geq 0 \\
\Delta Q_{\alpha,\beta} \geq 0
\] (128)

with equality being reachable as a limiting case, and \(C_C\) again a machine dependant constant.

This result may seem surprising. It suggests that it is possible to design a computer to perform any combination of logical operations, with no exchange of heat with the environment and requires no work to be performed upon it. This must be as true for logically irreversible operations as for logically reversible operations, and as true for logically indeterministic operations as for logically deterministic operations.

To understand this better, let us consider what happens in adiabatic equilibrium computing. We can use the square well potential as the physical model of the logical states, as the internal energy of these states is \(\frac{1}{2} kT\). Varying the width of the square well potential for each input and output logical state satisfies the remaining conditions.

Implementing the model of adiabatic equilibrium computing on the processes of Section III simplifies the procedure significantly:

1. There is no need to resize the input states, as these will already be canonically distributed. Steps 1 through to 3 are redundant.

2. Potential barriers are inserted into the \(\alpha\) states, corresponding to the conditional probabilities \(P(\beta|\alpha)\), as in Step 4.

3. The separate portions of the \(\beta\) output states are brought into adjacent positions as in Step 5.

4. The potential barriers within each \(\beta\) output states are removed, as in Step 6.

5. These output states are already canonically distributed. There is therefore no need for a resizing of the output states and Step 7 through to 9 are unnecessary.

None of these stages require any work to be performed upon the system or exchange of heat with the environment. The computation is reduced to a process of rearranging a canonical ensemble from one set of canonically distributed orthogonal subensembles into a different set of canonically distributed orthogonal subensembles, in accordance with the computational probabilities \(P(\beta|\alpha)\).

As the probabilities of the different output states cannot change between logical operations, then the canonically distributed output states can be used as canonically distributed input states to any new logical operation. This thermodynamic model may therefore proceed indefinitely without generating any heat or requiring any work.

\[14\] By definition anything that changes the probabilities of a state must be a logical transformation of the data.
Before leaving this subject, let us just note one feature of equilibrium computing. Logically deterministic, irreversible computations are able to avoid generating heat, in this model, by increasing the size of the physical states representing the logical states. This does not mean that the logical processing apparatus itself needs to be increasing in size. Although the size of the individual states has increased, the number of logical states has decreased (by the definition of a logically deterministic, irreversible computation!). Whenever the equality in Assumption [6] holds, the two effects cancel out and the overall size of the logical processing apparatus can remain constant.

VI. THERMODYNAMIC REVERSIBILITY

We have not yet examined the question of whether these operations are thermodynamically reversible. This is a subtle question and depends upon what one takes to be the statistical mechanical generalisation of thermodynamic entropy and thermodynamic reversibility. We will first discuss how this appears from the perspective of three different approaches to entropy, and then from a definition based on thermodynamic cycles that is not directly based upon any definition of entropy.

It is worth remembering that a net increase in entropy is considered is taken as a sign of irreversibility because net decreases in entropy cannot occur (or are unlikely). An ‘entropy’ that can be systematically decreased may be a useful indicator of some properties, but its increase cannot automatically be regarded as an indicator of irreversibility, whether thermodynamic or of some other kind.

We will consider three possible conditions for thermodynamic reversibility and irreversibility:

1. The thermodynamic entropy is the entropy of the individual state. If the system is in logical state $\alpha$, then the thermodynamic entropy is $S_\alpha$. The net entropy change for a particular logical transition, from logical state $\alpha$ to logical state $\beta$ is:

$$S_\beta - S_\alpha + \frac{\Delta Q_{\alpha,\beta}}{T_R}$$  \hspace{1cm} (129)

A transition is thermodynamically reversible if the decrease in individual state entropy from the input to output logical states is equal to the heat generated in the heat bath, divided by the temperature of the heat bath. A transition is thermodynamically irreversible if the decrease in individual state entropy is less than this. Decreases in individual state entropy greater than this cannot occur.

2. The thermodynamic entropy is the entropy of the individual state, but is only non-decreasing on average. If the system is in logical state $\alpha$, then the thermodynamic entropy is $S_\alpha$, but this may decrease provided it does not decrease on average. The average change is:

$$\sum_\beta P(\beta) S_\beta - \sum_\alpha P(\alpha) S_\alpha + \frac{\langle \Delta Q_{\alpha,\beta} \rangle}{T_R}$$  \hspace{1cm} (130)

A logical transformation of information is thermodynamically reversible if the average decrease in individual state entropy over all the transitions from input to output logical states is equal to the average heat generated in the heat bath, divided by the temperature of the heat bath. The transformation is thermodynamically irreversible if the average decrease in individual state entropy is less than this. Average decreases in individual state entropy greater than this cannot occur.

3. The thermodynamic entropy is the Gibbs-von Neumann entropy of the marginal statistical states. If the statistical state of the system is $\rho = \sum_\alpha P(\alpha) \rho_\alpha$, the thermodynamic entropy is $-k \text{Tr} \rho \ln |\rho|$. A logical transformation of information is thermodynamically reversible if the decrease in Gibbs-von Neumann entropy from the input to output statistical states is equal to the average heat generated in the heat bath, divided by the temperature of the heat bath. The transformation is thermodynamically irreversible if the decrease in Gibbs-von Neumann entropy is less than this. Decreases in Gibbs-von Neumann entropy greater than this cannot occur.

The first two conditions imply thermodynamic irreversibility for logically deterministic, irreversible operations. Unfortunately, it will be shown neither condition can consistently account for logically indeterministic operations, which can systematically decrease the relevant entropy measure by quantities greater than should be permitted.

The third condition gives an entropy that is consistently non-decreasing (provided there are no spontaneous or pre-existing correlations with heat baths). Logically indeterministic operations do not decrease this entropy. On the other hand, logically irreversible operations no longer necessarily increase this entropy measure either. According to the Gibbs-von Neumann measure, all logical operations may be implemented in a thermodynamically reversible manner.

We will be making the standard assumptions that all processes can take place with ideal heat baths and sufficiently slowly that equalities are reached as the limiting cases. Without these assumptions no process can be thermodynamically reversible. We will therefore replace the appropriate inequalities with equalities.
A. Individual logical state entropy

The net individual state entropy change, for a particular logical transition, gives:

\[ S_\beta - S_\alpha + \frac{\Delta Q_{\alpha,\beta}}{T_R} \geq k \ln |P(\beta|\alpha)| \]  

(131)

Allowed logically deterministic transitions require \( P(\beta|\alpha) = 1 \). The equality is automatically reached for logically deterministic, reversible transitions, and which are therefore thermodynamically reversible. For logically deterministic, irreversible transitions, the equality requires \( w_\alpha = 1 \). This is only possible if no other input logical states are allowed. Such an operation would be trivially logically reversible as there is only one permissible input logical state. So according to this entropy measure, logically deterministic irreversible transitions must be thermodynamically irreversible.

\[ \text{As } \ln |P(\beta|\alpha)| \leq 0 \text{ it is possible that } S_\beta - S_\alpha + \frac{\Delta Q_{\alpha,\beta}}{T_R} < 0 \]  

(132)

This gives a net decrease in individual state entropy. For this to happen, the transition must be logically indeterministic. Optimally implemented, logically indeterministic, reversible transitions will always decrease individual state entropy.

If an entropy increase is indicative of thermodynamic irreversibility because entropy decreases are impossible, this measure of entropy cannot be seen as a good indicator of thermodynamic irreversibility. Any apparent irreversibility can actually be reversed.

B. Average state entropy

In statistical mechanics fluctuations occur. Perhaps the demand for a strictly non-decreasing entropy might be the problem. What of the average change in entropy? Does this give a good indicator of thermodynamic irreversibility?

This gives:

\[ \sum_\beta P(\beta)S_\beta - \sum_\alpha P(\alpha)S_\alpha + \frac{\sum_\beta \Delta Q_{\alpha,\beta}}{T_R} = -k \Delta H \ln 2 \]  

(133)

What we have here is the ideal limit case of GLP4, with \( \Delta S_{\text{NIBDF}} \) now representing the average change in entropy:

\[ \Delta S_{\text{NIBDF}} = \sum_\beta P(\beta)S_\beta - \sum_\alpha P(\alpha)S_\alpha + \frac{\sum_\beta \Delta Q_{\alpha,\beta}}{T_R} \]  

(134)

Logically deterministic, irreversible operations have \( \Delta H < 0 \) then

\[ \Delta S_{\text{NIBDF}} > 0 \]  

(135)

and the net mean change in individual state entropy of the system and environment is strictly increasing. Again logically deterministic, irreversible operations must be, on average, individual state entropy increasing.

The problem with the argument should be immediately apparent: for logically reversible, indeterministic operations \( \Delta H > 0 \) and by the same reasoning and arguments it is possible that

\[ \Delta S_{\text{NIBDF}} = -k \Delta H \ln 2 < 0 \]  

(136)

Not only can logically indeterministic operations reduce individual state entropy on individual transitions, they can even reduce this entropy on average.

C. Gibbs-von Neumann entropy

The entropy measure which includes the effects of the statistical mixture over the states, the Gibbs-von Neumann entropy over the ensemble, gives the initial entropy of the logical processing system:

\[ S_I = -k \text{Tr} \left[ \rho_I \ln \rho_I \right] \]  

(137)

where

\[ \rho_I = \sum_\alpha P(\alpha)\rho_\alpha \]  

(138)

and the final entropy:

\[ S_F = -k \text{Tr} \left[ \rho_F \ln \rho_F \right] \]  

(139)

where

\[ \rho_F = \sum_\beta P(\beta)\rho_\beta \]  

(140)

In the paper it is argued that the Gibbs-von Neumann entropy is indeed the correct statistical mechanical generalisation of thermodynamic entropy, although this identification has not been assumed anywhere within this paper.

When we consider the Gibbs-von Neumann entropy, the most appropriate form of the GLP is \( \text{GLP}^3 \). In this case, the limiting behaviour gives

\[ \Delta S_{\text{HB}} + \Delta S = 0 \]  

(141)

As any logical operation may reach this limit, the Gibbs-von Neumann entropy regards all logical operations as being possible in a thermodynamically reversible manner.

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15 We have calculated the Gibbs-von Neumann entropies for individual states, in canonical distributions, but even here the calculation of the mean work requirements and mean heat generated did not depend upon any identification of this as a thermodynamic entropy.
D. Discussion

As some of these results may seem surprising or counter-intuitive, and appear to contradict widely stated expressions of the implications of the thermodynamics of logically irreversible operations, let us examine them in more detail.

1. The RLE-LE cycle

First, let us take the examples of the logically deterministic, irreversible Reset to Zero (RTZ) operation and the logically reversible, indeterministic Unset from Zero (UFZ) operation (Appendix A). If the argument is accepted that the optimal procedure to implement RTZ is entropy increasing, then it must also be accepted that the optimal procedure for UFZ can be entropy decreasing.

That this must be the case can be seen by considering the Reverse Landauer Erasure (RLE) operation immediately followed by the Landauer Erasure (LE) operation. If these two procedures are matched in terms of the probabilities, input and output states, then the result is to leave both the logical system and the environment in their initial states. The total entropy must be the same at the end of such a procedure, as at the start, and it follows if it increases during LE, then it must decrease during RLE.

As a simple example, using the assumptions of uniform computing, and an initial input state of 0, the process of RLE extracts $kT \ln 2$ heat from the environment, and converts it into work. The output state of RLE is an equiprobable distribution of logical states 0 and 1, each of which has the same entropy as the initial 0 state.

This is input to the LE procedure, which requires $kT \ln 2$ heat to be generated in the environment and leaves the output state as 0. The system and environment are left in the same logical and thermodynamic states as at the beginning of the process. There is a zero net work requirement and a zero net heat generation. The combination of RLE followed by LE is clearly a thermodynamically reversible cycle.

It follows that the net change in entropy over the course of the two operations must be zero for both system and environment. To argue that the net change in entropy for the LE procedure is $k \ln 2$, requires, for the overall change in entropy to be zero, the change in entropy during the RLE operation to be $-k \ln 2$.

Both the individual state entropies, and the average state entropy, do indeed decrease by $k \ln 2$ during the RLE operation. The Gibbs-von Neumann entropy remains constant, as the mixing entropy increases by $k \ln 2$ to compensate. During the course of the LE operation, the individual state, and average state, entropies increase by $k \ln 2$. In the conventional operation of the LE process, this is associated with heat generated in the environment, and is often considered to be the source of an irreversible entropy increase. However, we can clearly see that from the point of view of the Gibbs-von Neumann entropy, there is a compensating reduction of $k \ln 2$ associated with the reduction in the mixing entropy.

2. Uniform Computing

We can easily generalise this to situations where the quantity of information erased is less than 1 bit\(^{16}\), and in doing so will see more clearly the need to optimise the operation to the probability distribution. We simply need to implement an UFZ(p) operation, followed by an RTZ(p) operation.

We start with a standard “atom in a box”, and the partition divides the box exactly in half. The atom is on the left hand side (which represents logical state 0) with certainty.

1. RLE(p).

The RLE(p) operation consists of the following steps:

(a) Isothermally move the partition to the right hand side, extracting $W_1 = kT \ln 2$ heat as work.

(b) Insert the partition at location $x = pL$ in the box, where the width of the box is $L$. The atom is, with probability $p$, on the left hand side of the partition.

(c) Isothermally move the partition to the centre of the box ($x = \frac{1}{2}L$). If the atom is on the left hand side, the work requirement is $kT \ln(2p)$ while if it is on the right, the work requirement is $kT \ln(2(1-p))$. The mean work required in this stage is

$$W_2 = kT (p \ln p + (1 - p) \ln(1 - p) + \ln 2) \quad (142)$$

so the net work for the operation is

$$W_1 + W_2 = kT (p \ln p + (1 - p) \ln(1 - p)) \quad (143)$$

which is negative, representing a net extraction of work.

Now, we find that the individual state entropy, and average state entropy, remains the same as at the start of the operation, despite the fact that $kT (p \ln p + (1 - p) \ln(1 - p))$ work has been extracted from the heat bath. From the point of view of the Gibbs-von Neumann entropy, this is compensated by the increase in mixing entropy between the two logical states.

2. LE(p).

If we follow this with an LE(p), we have the steps:

\(^{16}\) This cycle was detailed in A.
(a) Isothermally move the partition to the position $x = pL$. If the atom is on the left hand side, the work requirement is $- kT \ln(2p)$ while if it is on the right, the work requirement is $- kT \ln(2(1 - p))$. The mean work required in this stage is

$$W_3 = - kT (p \ln p + (1 - p) \ln(1 - p) + 2) \quad (144)$$

(b) Remove the partition from the box.

(c) Insert the partition in the right hand side of the box and isothermally move it to the centre. This requires $W_4 = kT \ln 2$ work, so the net work is

$$W_3 + W_4 = - kT (p \ln p + (1 - p) \ln(1 - p)) \quad (145)$$

Again, both the individual and average state entropy are unchanged, while work is converted to heat in the environment. The Gibbs-von Neumann entropy, however, shows a compensating decrease in mixing entropy.

The net work and net heat generated, over the course of the cycle, is zero:

$$W_1 + W_2 + W_3 + W_4 = 0 \quad (146)$$

If the heat generated in the environment during the $LE(p)$ operation is an indicator of an irreversible entropy increase, we have to explain a corresponding systematic reduction in entropy during the $RLE(p)$ operation. As we noted, entropy increases are associated with irreversibility precisely because corresponding systematic entropy decreases are supposed to be impossible.

3. $LE(p')$.

Let us now consider following the $RLE(p)$ operation with $LE(p')$, where the erasure operation has been optimised for a different probability distribution.

(a) Isothermally move the partition to the position $x = p'L$. If the atom is on the left hand side, the work requirement is $- kT \ln(2p')$ while if it is on the right, the work requirement is $- kT \ln(2(1 - p'))$. The mean work required in this stage is

$$W_5 = - kT (p \ln p' + (1 - p) \ln(1 - p') + 2) \quad (147)$$

(b) Remove the partition from the box.

(c) Insert the partition in the right hand side of the box and isothermally move it to the centre. This requires $W_6 = kT \ln 2$ work, so the net work is

$$W_5 + W_6 = - kT (p \ln p' + (1 - p) \ln(1 - p')) \quad (148)$$

The net work required over the $RLE(p)$-$LE(p')$ cycle is

$$W_1 + W_2 + W_3 + W_4 + W_5 + W_6 = kT \left[ p \ln \left( \frac{p}{p'} \right) + (1 - p) \ln \left( \frac{1 - p}{1 - p'} \right) \right] \geq 0$$

with equality occurring if, and only if, $p = p'$. Once again, both the individual and average state entropy are unchanged. In this case, however, the cycle generates a net heat in the environment, unless $p = p'$. This cycle is, in general, thermodynamically irreversible.

From the point of view of the Gibbs-von Neumann entropy, it is the removal of the partition from the location $x = p'L$, when the probability is $p$, that associated with an uncompensated entropy increase. We can see this by noting that if we reinsert the partition at $x = p'L$, we do not recover the previous statistical state, as the probability of the atom being on the left hand side would then be $p'$. To recover the statistical state we need to reinsert the partition at $x = pL$, then move it isothermally to $x = p'L$. This isothermal movement of the partition requires, on average:

$$kT \left[ p \ln \left( \frac{p}{p'} \right) + (1 - p) \ln \left( \frac{1 - p}{1 - p'} \right) \right] \geq 0$$

work to be performed.

Even so, let us note that had $LE(p')$ in fact followed the $RLE(p')$ operation, it would have been thermodynamically reversible. The physical process involved in performing the $LE(p)$ (or $LE(p')$) operation cannot be said to be intrinsically thermodynamically reversible (or irreversible) in itself. Whether it is thermodynamically reversible, or not, depends upon the statistical state upon which it acts.

3. Adiabatic Equilibrium Computing

Let us look at the same logical cycle, but with a different computing model: adiabatic equilibrium. Again we start with a standard “atom in a box”. As the atom is in logical state $0$ with certainty, the conditions of Equation 127 require that logical state $0$ occupies the entire box.

1. $RLE(p)$.

The $RLE(p)$ operation now consists of the single step:

(a) Insert the partition at location $x = pL$ in the box, where the width of the box is $L$. The atom is, with probability $p$, on the left hand side of the partition.

No work is required or heat generated. The individual and average state entropies have decreased, with the average state entropy decreasing by $k (p \ln p + (1 - p) \ln(1 - p))$. The Gibbs-von
Neumann entropy remains the same, as the mixing entropy compensates for this.

2. \( LE(p) \).

If we follow this with an \( LE(p) \), we have the step:

(a) Remove the partition from the box.

Both the individual and average state entropy are increased, with the average state entropy increasing by \( k \left( p \ln p + (1 - p) \ln (1 - p) \right) \). The Gibbs-von Neumann entropy, however, shows a compensating decrease in mixing entropy.

We see how, in the case of adiabatic equilibrium computing, the generation of heat in the environment is replaced by changes in the entropies of the individual states (or, as [1] refers to it, the non-information bearing degrees of freedom of the apparatus). Although there is an increase in such entropies during the \( LE(p) \) process, there is an exactly equivalent decrease during the \( RLE(p) \) process. Again, if we take the increase during \( LE(p) \) to be indicative of a thermodynamic irreversibility, we are left with the challenge of accounting for the systematic decrease during the \( RLE(p) \) operation.

3. \( LE(p') \)

Following \( RLE(p) \) with an \( LE(p') \) operation under the assumptions of adiabatic equilibrium does not entirely make sense, as adiabatic equilibrium requires the physical representation of the logical states to be tailored to the probability of the state occurring. However, we may consider the optimum implementation of \( RTZ(p') \), on the assumption that the probability of the logical state 0 is \( p' \), with the partition initially located at \( x = p'L \) and the process leaving the system in a state compatible with adiabatic equilibrium computation.

(a) Isothermally move the partition to the position \( x = p'L \). If the atom is on the left hand side, the work requirement is \( -kT \ln \left( \frac{p'}{p} \right) \) while if it is on the right, the work requirement is \( -kT \ln \left( \frac{1 - p'}{1 - p} \right) \). The mean work required\(^{17} \) in this stage is

\[
kT \left( p \ln \left( \frac{p}{p'} \right) + (1 - p) \ln \left( \frac{1 - p}{1 - p'} \right) \right) \geq 0 \quad (151)
\]

with equality occurring if, and only if, \( p = p' \).

(b) Remove the partition from the box.

The net work required over the \( RLE(p) - LE(p') \) cycle is again

\[
kT \left( p \ln \left( \frac{p}{p'} \right) + (1 - p) \ln \left( \frac{1 - p}{1 - p'} \right) \right) \geq 0 \quad (152)
\]

Once again, from the point of view of the Gibbs-von Neumann entropy, it is the removal of the partition from the location \( x = p'L \), when the probability is \( p \), that is associated with an uncompensated entropy increase.

4. Generic logical operations

Now let us consider a generic logical transformation of information. Start with input logical states \( \alpha \), physically represented by states with energies and entropies \( E_\alpha \) and \( S_\alpha \), and define a logical operation by the transition probabilities \( P(\beta|\alpha) \) to the output logical states \( \beta \) with physical state energies and entropies \( E_\beta \) and \( S_\beta \).

To thermodynamically optimise the physical process, we need a probability distribution \( P(\alpha) \). The \( \beta \) output states will then occur with probabilities

\[
P(\beta) = \sum_\alpha P(\beta|\alpha)P(\alpha) \quad (153)
\]

Writing

\[
\rho_I = \sum_\alpha P(\alpha)\rho_\alpha
\]
\[
\rho_F = \sum_\beta P(\beta)\rho_\beta
\]

then the optimal thermodynamic cost of this is:

\[
\langle \Delta W \rangle = \langle \text{Tr} [H\rho_I] - T_R S[\rho_I] \rangle - \langle \text{Tr} [H\rho_F] - T_R S[\rho_F] \rangle
= \sum_\beta P(\beta) (E_\beta - T_R S_\beta) - \sum_\alpha P(\alpha) (E_\alpha - T_R S_\alpha)
+ kT_R \sum_{\alpha,\beta} P(\alpha,\beta) \ln \left( \frac{P(\beta)}{P(\alpha)} \right)
\]

\[
\langle \Delta Q \rangle = -T_R S[\rho_I] + T_R S[\rho_F] \quad (154)
\]

We can now define a physical process, that acts upon the physical states \{ \beta \}, and evolves them into the physical states \{ \alpha \}, with probabilities\(^{18} \) given by

\[
\Pi(\alpha|\beta) = \frac{P(\beta|\alpha)P(\alpha)}{\sum_\alpha P(\beta|\alpha)P(\alpha)} \quad (155)
\]

\(^{17} \) Note that, had the probability of logical state 0 \( \text{actually} \) been \( p' \), the work required would have been:

\[
kT \left( p' \ln \left( \frac{p'}{p'} \right) + (1 - p') \ln \left( \frac{1 - p'}{1 - p'} \right) \right) \leq 0 \quad (150)
\]

so work would have been extracted in the process.

\(^{18} \) For logical operations taking as input states \{ \beta \} and producing output states \{ \alpha \}, we will use the notation \( \Pi \) for the corresponding probabilities.
It is straightforward to see that if this acts upon states \( \{ \beta \} \), occurring with probabilities \( P(\beta) \), then it produces the states \( \{ \alpha \} \) with probabilities \( P(\alpha) \). If the physical process is optimised for these probabilities, then the thermodynamic cost is

\[
\langle \Delta W_{\Pi} \rangle = -\langle \Delta W \rangle \\
\langle \Delta Q_{\Pi} \rangle = -\langle \Delta Q \rangle
\]

(156)

So for any logical transformation of information, optimally implemented, there exist a second operation, which when optimally implemented restores the original statistical state, and for which the total expectation value of the work requirement, and the total expectation value of the energy generated in the environment, is zero. This is true regardless of whether the original operation is logical reversible, irreversible, deterministic or indeterministic.

As we have noted before, however, to achieve this optimum for logically irreversible operations, the physical process must take into account the probability distribution \( P(\alpha) \) over the input logical states. One cannot create a physical process, that implements a logically irreversible operation, which will be thermodynamically optimal for every probability distribution over the input logical states. This differs from logically reversible operations, which may be represented by a physical process which is thermodynamically optimal for any probability distribution over the input logical states.

We will now look at the effect of an operation that is optimised for the input probability distribution \( \Pi(\alpha) \), above, but the physical process has been optimised for the input probability distribution \( \Pi(\beta) \). The output states are expected to occur with probabilities

\[
\Pi(\alpha) = \sum_{\beta} \Pi(\alpha|\beta) \Pi(\beta)
\]

(157)

and the expected thermodynamic cost, from Equation [70] is:

\[
\langle \Delta W_{\Pi} \rangle = \sum_{\alpha} \Pi(\alpha) (E_\alpha - T_R S_\alpha) - \sum_{\beta} \Pi(\beta) (E_\beta - T_R S_\beta)
\]

\[
+ kT_R \sum_{\alpha,\beta} \Pi(\alpha,\beta) \ln \left[ \frac{w_\alpha}{w_\beta} \right]
\]

(158)

with \( w_\alpha = \Pi(\alpha) w_\beta = \Pi(\beta) \) and \( \Pi(\alpha,\beta) = \Pi(\alpha|\beta) \Pi(\beta) \).

The input states do not occur with \( \Pi(\alpha) \) but with \( P(\beta) \). The actual thermodynamic cost incurred is:

\[
\langle \Delta W'_{\Pi} \rangle = \sum_{\alpha} P(\alpha) (E_\alpha - T_R S_\alpha) - \sum_{\beta} P(\beta) (E_\beta - T_R S_\beta)
\]

\[
+ kT_R \sum_{\alpha,\beta} P(\alpha,\beta) \ln \left[ \frac{w_\alpha}{w_\beta} \right]
\]

(159)

The combined cycle now has a cost

\[
\langle \Delta W'_{\Pi} \rangle + \langle \Delta W \rangle = kT_R \sum_{\alpha,\beta} P(\alpha,\beta) \ln \left[ \frac{w_\alpha P(\beta)}{w_\beta P(\alpha)} \right]
\]

(160)

which can be rearranged to give

\[
\langle \Delta W'_{\Pi} \rangle + \langle \Delta W \rangle = kT_R \sum_{\alpha,\beta} P(\alpha,\beta) \ln \left[ \frac{P(\alpha,\beta)}{\Pi(\alpha|\beta)P(\alpha)} \right] \geq 0
\]

(161)

where \( \Pi(\beta|\alpha) \Pi(\alpha) = \Pi(\alpha|\beta) \Pi(\beta) \).

Equality can occur in two ways. Firstly, and most simply, if \( \Pi(\beta) = P(\beta) \). The input states to the \( \Pi(\alpha|\beta) \) operation occur with the optimal probabilities.

Secondly, if the second operation is a logically reversible operation, then

\[
\forall \beta \ [\Pi(\alpha|\beta) \neq 0 \Rightarrow \forall \alpha' \neq \alpha \ \Pi(\alpha'|\beta) = 0] \quad (162)
\]

As \( \Pi(\alpha|\beta) = P(\alpha|\beta) \), it follows the first operation must have been logically deterministic:

\[
\forall \beta \ [P(\alpha|\beta) \neq 0 \Rightarrow \forall \alpha' \neq \alpha \ P(\alpha'|\beta) = 0] \quad (163)
\]

Together this means

\[
P(\alpha|\beta) = \frac{P(\alpha)}{P(\beta)} = \Pi(\alpha|\beta) = \frac{w_\alpha}{w_\beta} \quad (164)
\]

and \( \langle \Delta W'_{\Pi} \rangle + \langle \Delta W \rangle = 0 \), regardless of the values of \( w_\beta \). This shows, once more, that logically reversible operations may be thermodynamically optimised without reference to the probability distribution over their input states.

A corollary to this is worth noting. While the second logical operation, if logically reversible, may be implemented and optimised without reference to the probability distribution over the input states, its very definition depends upon the probability distribution over the input states of the first operation. The first operation is defined by the set of transition probabilities \( \{ P(\beta|\alpha) \} \), while the second is defined by

\[
\Pi(\alpha|\beta) = P(\alpha|\beta) = \frac{P(\beta|\alpha)P(\alpha)}{\sum_{\alpha} P(\beta|\alpha)P(\alpha)} \quad (165)
\]

There is, in general, only one way to make this independent of \( \{ P(\alpha) \} \): if the first operation is logically reversible, then \( P(\alpha|\beta) \in \{0,1\} \). The second operation is now logically deterministic and \( \Pi(\alpha|\beta) \in \{0,1\} \) does not require the \( \{ P(\alpha) \} \).

We can summarise this, as follows: if an operation, \( \{ P(\beta|\alpha) \} \), is logically reversible, then it is possible to calculate a (logically deterministic) reverse operation, \( \{ \Pi(\alpha|\beta) \} \), independantely of the first input probability distribution, \( \{ P(\alpha) \} \). However, if \( \{ P(\beta|\alpha) \} \) is logically indeterministic, then optimising the reverse operation requires the output probability distribution \( \{ P(\beta) \} \).

Conversely, if an operation \( \{ P(\beta|\alpha) \} \) is logically deterministic, then it is possible to thermodynamically optimise a (logically reversible) reverse operation \( \{ \Pi(\alpha|\beta) \} \), independantely of the first output probability distribution \( \{ P(\beta) \} \). However, if \( \{ P(\beta|\alpha) \} \) is logically irreversible, then the very calculation of the probabilities \( \{ \Pi(\alpha|\beta) \} \) require the first input probability distribution \( \{ P(\alpha) \} \).
In general, it is only for logically deterministic, reversible operations (which are permutations) that one can construct optimal reverse operations independently of the probability distributions.

**E. Thermodynamic irreversibility**

The reverse operations considered in the preceding discussion have the property of restoring the original statistical state of the logical system. They do not, in general, restore the original logical state. The question of what is the ‘correct’ thermodynamic entropy to use in such situations is not uncontroversial and can depend upon differing physical interpretations of the probabilities of the initial and final logical states. It will therefore be helpful to consider an approach to thermodynamic reversibility which does not depend upon such definitions.

We will use this to discuss further that the thermodynamic optimisation of logically irreversible operations is not possible without specifying the probability distribution over the input states. Then we consider two additional sources of thermodynamic irreversibility that occur in the practical construction of information processing systems.

1. **Thermodynamic cycles**

   In phenomenological thermodynamics, in any closed cycle, where a system returns to its initial state, the total heat generated in heat baths in the process, must satisfy

   \[
   \sum_i \frac{Q_i}{T_i} \geq 0 \quad (166)
   \]

   As is well known, in statistical mechanics this can no longer be relied upon. There is some probability for the equality being violated. However, provided the system does return to its initial macroscopic state with certainty, then

   \[
   \sum_i \frac{(Q_i)}{T_i} \geq 0 \quad (167)
   \]

   still holds. We will regard such a cycle for which the equality holds, to be a thermodynamically reversible cycle, and use the following definition\(^{19}\) of a thermodynamically reversible process:

   If a given physical process can, in principle, be included in at least one thermodynamically reversible cycle, then it is a thermodynamically reversible process.

   To say otherwise would require one either to say that the overall cycle is thermodynamically reversible, although one of the steps in the cycle is not (which challenges what it could possibly mean to refer to that step as thermodynamically irreversible) or to say that the overall cycle is thermodynamically irreversible, despite the fact that it restores the original state with certainty and generates no net heat in any heat bath (and which means that the entropy of the universe must be the same at the end as the start of the cycle).

   Conversely

   If a given physical process cannot, even in principle, be included in any thermodynamically reversible cycles, then it is a thermodynamically irreversible process.

   To avoid interpretational problems over probability, we will require that the thermodynamically reversible cycle starts, and ends, with the system in a physical state that represents a fixed logical state \(a\), with certainty.

2. **Optimal Implementations**

   Take any logical operation, defined by the set \(\{P(\beta|\alpha)\}\), and construct a physical implementation of that operation, optimised for the values \(w_\alpha\) and \(w_\beta = \sum_{\alpha} P(\beta|\alpha) w_\alpha\). This physical process will implement the \(\{P(\beta|\alpha)\}\) operation regardless of the input state probabilities.

   We now also construct two further operations: a logically reversible, indeterministic operation, generalising the \(UFZ\) operation, that acts on \(a\) as the sole possible logical input state, and outputs state \(\alpha\) with probability \(P(\alpha) = w_\alpha\); and a logically irreversible, deterministic operation, generalising \(RTZ\), that acts on the logical states \(\{\beta\}\), and always outputs logical state \(a\). The physical implementation of this second operation is optimised for probabilities \(P(\beta) = w_\beta\). Both these are well defined physical processes.

   It is clear that the sequence of these three operations forms a closed cycle, starting and ending in logical state \(a\), with certainty. It is trivial to show that the optimal implementation of these operations produces a net thermodynamic cost of zero, over the course of the cycle. The cycle is, unquestionably, a thermodynamically reversible cycle. The given physical process that implements the logical operation must, then, be regarded as a thermodynamically reversible process.

3. **Suboptimal Implementations**

   If we had used a different initial operation, generating the logical state \(a\) with probability \(P'(\alpha)\), and a final operation optimised for probabilities \(P'(\beta) = \sum_{\alpha} P(\beta|\alpha) P'(\alpha)\), then it is straightforward to show the

---

\(^{19}\) We define the condition in this way to take into account the fact that for any physical process, it is always trivially possible to find some closed cycle incorporating that process for which inequality is strictly positive.
of the partition becomes thermodynamically reversible. Now the off-centre removal and reinsertion of the partition in the centre, but now with gas initially prepared to the off-centre position of partition. Start with the partition in the centre, and in the container is a macroscopic gas. The pressure on both sides of the partition is initially equal, and the gas is always kept in isothermal contact with a single heat bath. Removing and reinserting the partition is clearly thermodynamically reversible.

If we slowly, isothermally, slide the partition to the left, compressing half the gas and expanding the other half, until the compressed gas occupies only one-third the container, the pressure on the left side is double the pressure on the right side (net work is required). Removing and reinserting the partition at this off-centre position is not thermodynamically reversible.

This thermodynamic irreversibility is not simply due to the off-centre position of partition. Start with the partition in the centre, but now with gas initially prepared to be at twice the pressure on the right hand side of the container as on the left hand side. Simply removing the partition from the centre of the box is now thermodynamically irreversible. Isothermally moving the partition to the left until the left hand side holds only one third of the container’s volume equalises the pressure (and extracts work). Now the off-centre removal and reinsertion of the partition becomes thermodynamically reversible.

The parallel to the model used for logical operations should be clear20. A given sequence of actions cannot, in general, be regarded as thermodynamically reversible independently of the state on which they act. To describe a phenomenological thermodynamic process as thermodynamically reversible it is necessary to specify both the sequence of actions and the state on which they act in the definition of the physical process. This carries over into statistical mechanics and, as we have seen above, into the thermodynamics of computation.

The situation also bears some similarity to data compression from a signal source. A given coding scheme will only be optimal for a particular distribution of probabilities of signals from the source. Should the signals, in fact, be generated with a different probability distribution, then the mean length of the encoded signals will be greater than the Shannon information of the source. That Shannon’s coding theorem is of practical utility indicates that it is not inconceivable that there may be information processing problems where the probability distribution over the logical states may be available when designing optimal physical implementations.

4. Uncertain Operations

If the logical operation acts upon a set of statistical states, but it is uncertain which operations have acted upon the system in the past, an additional source of thermodynamic irreversibility may occur. As an example of this, let us consider a bit that has been deterministically set to either zero or one, from a standard state \( a \), and now needs to be reset to the standard state.

If the first operation set the bit to zero, the operation is \( UFZ(1) \), and the work required was

\[
\Delta W_0 = (E_0 - T_R S_0) - (E_a - T_R S_a) \tag{169}
\]

and if set to one, \( UFZ(0) \) gives

\[
\Delta W_1 = (E_1 - T_R S_1) - (E_a - T_R S_a) \tag{170}
\]

If the reset operation is optimised with values \( w_0 + w_1 = 1 \), then it is \( RTZ(w_0) \),

\[
\Delta W_{R_0} = (E_a - T_R S_a) - (E_0 - T_R S_0) - kT_R \ln w_0
\]
\[
\Delta W_{R_1} = (E_a - T_R S_a) - (E_1 - T_R S_1) - kT_R \ln w_1 \tag{171}
\]

giving total costs

\[
\Delta W_{T_0} = \Delta W_{R_0} + \Delta W_0 = -kT_R \ln w_0 \geq 0
\]
\[
\Delta W_{T_1} = \Delta W_{R_1} + \Delta W_1 = -kT_R \ln w_1 \geq 0 \tag{172}
\]

\[20\] Indeed, if we are considering a statistical mechanical N-atom gas, with \( N=1 \), it is exactly the same model.
The equalities can be reached by setting $w_0 = 1$ or $w_1 = 1$, respectively, but this is only possible if the other is zero - which would require an infinite amount of work if the wrong operation had taken place!

If we assign non-zero probabilities to the set operations of $p_0$ and $p_1$, then the expected cost for the cycle is

$$
\Delta W_T = -kT \sum_{i=0,1} p_i \ln w_i \geq -kT \sum_i p_i \ln p_i > 0
$$

with the equality occurring if $w_i = p_i$. Clearly this is a thermodynamically irreversible cycle, despite the fact that each of the three logical operations ($UFZ(1)$, $UFZ(0)$, $RTZ(w_0)$) can be individually incorporated in a thermodynamically reversible cycle. What is the source of the irreversibility?

There are a number of ways one can regard this. Both the deterministic set operations are, in themselves, thermodynamically reversible. It could be argued that the irreversibility in whichever of the $\Delta W_T$ cycles that actually took place, is then through the reset operation, which was designed for the possibility of either deterministic set operation.

A different way to perceive the situation is to regard the situation as either being $\Delta W_{T0}$, which may be thermodynamically optimised by setting $w_0 = 1$, or $\Delta W_{T1}$ which may be optimised by $w_1 = 1$. In either case the cycle becomes thermodynamically reversible. The source of thermodynamic irreversibility would then be that the reset operation was not optimised for the correct probabilities (which must now be regarded as either $p_0 = 1$ or $p_1 = 1$, corresponding to which operation actually did take place).

Yet another way would be to consider a new class of operation: an ‘uncertain’ operation, where there is an uncertainty as to which actual operation took place. In this case we have an ‘Uncertain Set’ operation, which could be defined as $p_0 UFZ(1) + p_1 UFZ(0)$. This operation has a work requirement:

$$
\Delta W_U = \sum_{i=0,1} p_i(E_i - TR S_i) - (E_a - TR S_a)
$$

(174)

Viewed as a logical transformation, this would take as input logical state 0 with probability one, and output states 0 and 1 with probabilities $p_0$ and $p_1$. The optimal implementation of such a logical transformation of information would be $UFZ(p_0)$, which has cost

$$
\Delta W = \sum_{i=0,1} p_i(E_i - TR S_i) - (E_a - TR S_a) + kT \sum_i p_i \ln p_i
$$

(175)

As a logical transformation of information, the ‘Uncertain Set’ operation is clearly sub-optimal. It is thermodynamically irreversible, as it cannot be included in any thermodynamically reversible cycle.

What is the ‘correct’ way to view this? We are not sure this is a well-posed question. However, what all three explanations have in common is that the thermodynamic irreversibility is a consequence of the uncertainty over which logical operation took place. It is this that prevents the construction of a thermodynamically reversible cycle.

Suppose we have a number of different process, labeled with $\gamma$, and each implements a logical operation $\{P(\beta|\alpha, \gamma)\}$, optimised for input state probabilities $P(\alpha)$. The optimal cost for operation $\gamma$ is

$$
\Delta W_\gamma = \sum_{\beta} P(\beta|\gamma) (E_\beta - TR S_\beta + kT \ln P(\beta|\gamma)) - \sum_{\alpha} P(\alpha) (E_\alpha - TR S_\alpha + kT \ln P(\alpha))
$$

(176)

where $P(\beta|\gamma) = \sum_{\alpha} P(\beta|\alpha, \gamma) P(\alpha)$.

Now we assign a probability $P(\gamma)$ to each logical operation occurring (and take for granted $P(\alpha, \gamma) = P(\alpha) P(\gamma)$). The cost of this ‘Generic Uncertain Operation’ is

$$
\langle \Delta W_\gamma \rangle = \sum_{\alpha, \beta, \gamma} P(\alpha, \beta, \gamma) \left( E_\beta - E_\alpha - TR \left( S_\beta - S_\alpha - k \ln \left( \frac{P(\beta|\gamma)}{P(\alpha)} \right) \right) \right)
$$

(177)

This produces the output states $\{\beta\}$ with probabilities $P(\beta) = \sum_{\alpha, \gamma} P(\beta|\alpha, \gamma) P(\alpha) P(\gamma)$.

Now to complete the cycle, we consider an optimised Reset operation, the acts upon states $\{\beta\}$ to produce the standard state $a$, and an optimal operation that acts upon $a$ and produces the logical states $\{\alpha\}$ with probability $P(\alpha)$. Combining these two has the cost

$$
\Delta W_R = \sum_{\alpha} P(\alpha) (E_\alpha - TR S_\alpha + kT \ln P(\alpha)) - \sum_{\beta} P(\beta) (E_\beta - TR S_\beta + kT \ln P(\beta))
$$

(178)

giving a total cost for the cycle of

$$
\langle \Delta W_\gamma \rangle + \Delta W_R = kTR \sum_{\beta, \gamma} P(\beta, \gamma) \ln \left( \frac{P(\beta, \gamma)}{P(\beta)P(\gamma)} \right) \geq 0
$$

(179)

Equality is reached only if $P(\beta, \gamma) = P(\beta)P(\gamma)$, i.e. there is no correlation between the occurrence of the $\beta$ output states and which $\gamma$ operation actually took place. The thermodynamic irreversibility that occurs if $P(\beta|\gamma) \neq P(\beta)$ does not depend upon whether the operation required to restore the original statistical state is logically reversible or logically irreversible.

In the familiar case of the ‘Uncertain Set’-Reset cycle there is a compression of the logical state space during the reset operation and the compensating increase in the non-information bearing degrees of freedom of system or environment may give the impression that the source of the thermodynamic irreversibility is the logical irreversibility of the Reset operation. The ‘Generic Uncertain Operation’ shows this is not the case. In fact
an optimal operation that restores the \( P(\alpha) \) distribution from the \( P(\beta) \) distribution could be logically reversible and the cycle still be thermodynamically irreversible provided \( P(\beta|\gamma) \neq P(\beta) \). It is the uncertainty over which \( \gamma \) operation took place that is the source of the thermodynamic irreversibility.

As before, this situation has well known parallels in standard statistical mechanics. The spread of gas molecules into a box, shielded from any outside interference, can in principle be reversed. (Spin-echo experiments have even demonstrated similar reversals to this in the laboratory.) However, this reversal is very sensitive to uncertainty in the outside forces that act upon the gas. In a famous calculation, Borel showed that the gravitational influence of remote stars could change the microscopic state of an expanding macroscopic gas within seconds. Reversing that expansion would be possible, in principle, if there was highly detailed knowledge of the gravitational influence of the remote bodies on the gas (or if microscopic state of the expanded gas molecules turned out to be independent of that influence) but becomes impossible when the gravitational influence is uncertain.

5. Partial Operations

A third reason for the occurrence of thermodynamic irreversibility is that the physical implementation of the logical operation is not able to take into account the existence of correlations between systems, and can only act upon part of the total logical state\(^{21}\). We will show that, in this case, logically reversible operations are able to avoid the thermodynamic irreversibility, although logically irreversible operations are still not always thermodynamically irreversible.

Suppose the input logical states factorise into the product of two subsystems, with the logical states of the first system in the set \( \{\alpha\} \) and the second system in \( \{\gamma\} \), so the joint system is described by the logical states \( \{\alpha, \gamma\} \). Now consider a logical operation that acts only on the \( \alpha \) states, with probabilities \( P(\beta|\alpha) \). If the physical implementation of this logical operation has no access to the \( \gamma \) system, then the physical implementation can only be optimised with respect to the marginal probabilities

\[
P(\alpha) = \sum_{\gamma} P(\alpha, \gamma)
\]

The system ends up in output states from the product of the states of the \( \{\gamma\} \) and \( \{\beta\} \) systems, \( \{\beta, \gamma\} \), with probabilities

\[
P(\beta, \gamma) = \sum_{\alpha} P(\beta|\alpha)P(\alpha, \gamma)
\]

The resulting thermodynamic cost of the partially optimised operation is:

\[
\Delta W_P = \sum_{\beta} P(\beta) (E_\beta - T_R (S_\beta - k \ln P(\beta))) - \sum_{\alpha} P(\alpha) (E_\alpha - T_R (S_\alpha - k \ln P(\alpha)))
\]

where \( P(\beta) = \sum_{\gamma} P(\beta, \gamma) \) and we have assumed

\[
E_{\alpha, \gamma} = E_\alpha + E_\gamma
E_{\beta, \gamma} = E_\beta + E_\gamma
S_{\alpha, \gamma} = S_\alpha + S_\gamma
S_{\beta, \gamma} = S_\beta + S_\gamma
\]

An optimal operation for restoring the states \( (\alpha, \gamma) \), with probabilities \( P(\alpha, \gamma) \) has a thermodynamic cost of

\[
\Delta W_R = \sum_{\alpha, \gamma} P(\alpha, \gamma) (E_\alpha - T_R (S_\alpha - k \ln P(\alpha, \gamma))) - \sum_{\beta, \gamma} P(\beta, \gamma) (E_\beta - T_R (S_\beta - k \ln P(\beta, \gamma)))
\]

so the net cost for the cycle is

\[
\frac{\Delta W_R + \Delta W_P}{k T_R} = k T_R \left( \sum_{\alpha, \gamma} P(\alpha, \gamma) \ln \frac{P(\alpha, \gamma)}{P(\alpha)} \right) - \sum_{\beta, \gamma} P(\beta, \gamma) \ln \frac{P(\beta, \gamma)}{P(\beta)}
\]

This can be expressed as changes in conditional or correlation information:

\[
\frac{\Delta W_R + \Delta W_P}{k T_R} = - \sum_{\alpha, \beta, \gamma} P(\alpha, \beta, \gamma) \left( \ln P(\gamma|\beta) - \ln P(\gamma|\alpha) \right) - \sum_{\alpha, \beta, \gamma} P(\alpha, \beta, \gamma) \left( \ln \frac{P(\beta, \gamma)}{P(\beta) P(\gamma)} \right) - \ln \frac{P(\alpha, \gamma)}{P(\alpha) P(\gamma)}
\]

Using the identity

\[
P(\alpha, \gamma|\beta)P(\beta|\alpha) = P(\beta, \gamma|\alpha)P(\alpha|\beta)
\]

gives the form of conditional correlations:

\[
\frac{\Delta W_R + \Delta W_P}{k T_R} = - \sum_{\alpha, \beta, \gamma} P(\alpha, \beta, \gamma) \left( \ln \frac{P(\beta, \gamma|\alpha)}{P(\beta|\alpha) P(\gamma|\alpha)} \right) - \ln \frac{P(\alpha, \gamma|\beta)}{P(\alpha|\beta) P(\gamma|\beta)}
\]

As \( P(\beta|\alpha) = P(\beta|\alpha) \), then

\[
P(\gamma, \beta|\alpha) = P(\beta|\alpha) P(\gamma|\alpha) = P(\beta|\alpha) P(\gamma|\alpha)
\]

\[\text{See also }^{34}\]

\[\text{See also }^{34}\]
The $\alpha$ states screen off any correlation between the $\beta$ and $\gamma$ states and the first term is zero, so:

$$\frac{\Delta W_R + \Delta W_P}{kT_R} = \sum_{\alpha,\beta,\gamma} P(\alpha, \beta, \gamma) \ln \frac{P(\alpha, \gamma | \beta)}{P(\alpha | \beta) P(\gamma | \beta)} \geq 0$$  \hspace{1cm} (190)

Equality occurs if, and only if, $\beta$ screens off any correlations between $\alpha$ and $\gamma$:

$$P(\alpha, \gamma | \beta) = P(\alpha | \beta) P(\gamma | \beta)$$  \hspace{1cm} (191)

This can happen directly if there is no initial correlation between the $\alpha$ and $\gamma$ systems, so that $P(\alpha, \gamma) = P(\alpha) P(\gamma)$. With $P(\beta | \alpha, \gamma) = P(\beta | \alpha)$ it follows $P(\alpha, \beta, \gamma) = P(\alpha, \beta) P(\gamma)$ and from that Equation 191 holds, as might be expected.

To see the effect of logical reversibility, rewrite Equation 190 as

$$\frac{\Delta W_R + \Delta W_P}{kT_R} = \sum_{\alpha, \beta, \gamma} P(\alpha, \beta, \gamma) \left( \ln P(\alpha | \gamma, \beta) - \ln P(\alpha | \beta) \right)$$  \hspace{1cm} (192)

If the operation is logically reversible $P(\alpha | \beta) \in \{0, 1\}$. This gives

$$P(\alpha | \beta) = 0 \Rightarrow P(\alpha, \beta, \gamma) = 0$$

$$P(\alpha | \beta) = 1 \Rightarrow P(\alpha, \gamma, \beta) = 1$$  \hspace{1cm} (193)

and the summation is identically zero. Logically reversible operations avoid the thermodynamically irreversible cost\textsuperscript{22}.

### VII. CONCLUSIONS

The focus on the process of Landauer Erasure can give the impression that Landauer’s principle should be exclusively about the thermodynamics of logically irreversible processes and further that the heat generation of such processes implies thermodynamic irreversibility:

To erase a bit of information in an environment at temperature $T$ requires dissipation of energy $\geq kT \ln 2$. \cite{23,24} in erasing one bit . . . of information one dissipates, on average, at least $k_B T \ln (2)$ of energy into the environment. \cite{25} a logically irreversible operation must be implemented by a physically irreversible device, which dissipates heat into the environment $\Delta W \geq kT \ln 2[H_i - H_f]$ \cite[pg. 362]{3}

any logically irreversible manipulation of data . . . must be accompanied by a corresponding entropy increase in the non-information bearing degrees of freedom of the information processing apparatus or its environment. Conversely, it is generally accepted that any logically reversible transformation of information can in principle be accomplished by an appropriate physical mechanism operating in a thermodynamically reversible fashion. \cite{1}

though it should be noted that not all advocates of Landauer’s principle regard the process of erasure as necessarily thermodynamically irreversible:

a logically irreversible operation . . . may be thermodynamically reversible or not depending on the data to which it is applied. If it is applied to random data . . . it is thermodynamically reversible, because it decreases the entropy of the data while increasing the entropy of the environment by the same amount \cite{1}

In \cite{3} it was argued that there exists a valid thermodynamically reverse process to Landauer Erasure, but which needs to be classified as logically indeterministic, which we called Reverse Landauer Erasure (or RLE). Consideration of the thermodynamic consequences of the existence of this process led us to conclude there was no convincing evidence that logically irreversible operations had special thermodynamic characteristics. Instead, we hypothesised that a generalised form of Landauer’s principle should be possible that made no reference to irreversibility, whether logical or thermodynamic. This was expressed in two conjectures:

\textbf{(E)}: Any logically irreversible transformation of information can in principle be accomplished by an appropriate physical mechanism operating in a thermodynamically reversible fashion.

\textbf{(F)}: A logical operation needs to generate heat equal to at least $-kT \ln 2$ times the change in the total quantity of Shannon information over the operation, or:

$$\Delta W \geq kT \ln 2[H_i - H_f]$$  \hspace{1cm} \cite[pg. 362]{3}

In this paper we have both proved and generalised these conjectures. Our approach has been to take the widest definition of logical operations available and most general procedure for physically implementing these operations that we can. This requires us to consider logically indeterministic operations as well as deterministic

\textsuperscript{22} However, one should note that it is still possible for some logically irreversible operation to satisfy the conditions for thermodynamic reversibility, Equation 191 for particular correlations between the $\alpha$ and $\gamma$ systems.
ones, logically reversible operations as well as irreversible ones.

Other papers have made some consideration of Landauer Erasure in the context of non-uniform temperatures, entropy, and energy, while input probabilities are considered in the proofs of the thermodynamics of logically indeterministic operations does not seem to be considered before [25, 26]. Although [25] (Chapter VI) is close, and it is noticeable that [1] refers throughout to deterministic computation. During the preparation of this paper, a paper has appeared by Turgut deriving similar results using classical phase space arguments.

General proofs of Landauer’s principle seem hard to come by (as pointed out in [3]) although [22] derives similar results to those of Section III E but restricted to the setting for physically implementing classical logical operations, covering and extending these earlier results. We derive the most general statement of Landauer’s principle, prove it cannot be exceeded and give a limiting process which can achieve it.

The general statement of Landauer’s principle we arrived at is:

**Generalised Landauer’s principle**

A physical implementation of a logical transformation of information has minimal expectation value of the work requirement given by:

\[ \langle \Delta W \rangle \geq \langle \Delta E \rangle - T \Delta S \tag{194} \]

where \( \langle \Delta E \rangle \) is the change in the mean internal energy of the information processing system, \( \Delta S \) the change in the Gibbs-von Neumann entropy of that system and \( T \) is the temperature of the heat bath into which any heat is absorbed.

The equality is reachable, in principle, by any logical operation, and if the equality is reached the physical implementation is thermodynamically reversible.

We have then shown how various additional assumptions and simplifications can lead to more familiar versions of Landauer’s principle that can be found in the literature and these are special cases of the GLP. Generalisations about the relationship between information processing and thermodynamic entropy based upon these special cases can be misleading.

In particular, we have argued, counter to a widespread version of Landauer’s principle, that there is nothing in principle, that prevents a logically irreversible operation from being implemented in a thermodynamically reversible manner. What differs between logically irreversible operations and logically reversible operations is that to thermodynamically optimise physical implementations of the former it is necessary to take into account the probability distribution over the complete set of input logical states. A physical implementation of a logically irreversible operation, optimised for a particular input probability distribution, will not be thermodynamically irreversible for a different input probability distribution. If the physical implementation cannot access a correlated system, then logically irreversible operations may incur additional costs.

As the practical business of actually building physical devices to implement logical operations will typically not be able to make such optimisations, it is natural to assume an equiprobable distribution over a subsystem, and expect thermodynamic irreversibility. Nevertheless the point remains: in principle it is always possible to physically implement logically irreversible transformations of information in thermodynamically reversible ways. There are many practical reasons why a logically irreversible operation may not be thermodynamically optimised, and it is clearly important and useful to explore such problems. In this paper, however, we are primarily concerned with the question: what is the *fundamental limit* for thermodynamically optimising the physical implementation of a given logical operation?

We have demonstrated that, under the same conditions of uniform computing that imply logically deterministic, irreversible operations generate heat, logically indeterministic, reversible operations extract heat from the environment which can be converted into work. At the same time we have demonstrated that under other conditions, adiabatic equilibrium computing, information processing is able to progress without any exchange of work or heat, regardless of the type of logical operation.

The thermodynamic reversibility of all logical operations is, of course, based upon the definition of thermodynamic reversibility given in Sections VI C and VI E. Other approaches to thermodynamics (such as [4, 5, 6]) use different concepts of entropy and correspondingly different definitions of thermodynamic irreversibility to this paper. Ultimately the most important question is not what particular quantity one chooses to label as ‘thermodynamic’ entropy. The GLP we have derived here is valid, whether one chooses to regard the Gibbs-von Neumann entropy as the true ‘thermodynamic’ entropy, or not. What is important is the actual work required to drive a system, the actual heat generated by that system. As there is no disagreement over the fundamental microscopic dynamics, it would be surprising if we were unable to be able to agree on these values, regardless of the definition of entropy to which we choose to adhere.

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APPENDIX A: ONE BIT LOGICAL OPERATIONS

1. **Do Nothing: IDN**

The simplest operation is

\[
P(\beta = 0|\alpha = 0) = 1 \\
P(\beta = 0|\alpha = 1) = 0 \\
P(\beta = 1|\alpha = 0) = 0 \\
P(\beta = 1|\alpha = 1) = 1
\]

(A1)

This is logically deterministic and irreversible.

2. **Logical NOT: NOT**

Logical NOT, acting upon an input bit with probability \( p \) of being in state 0, is very simple:

\[
P(\beta = 0|\alpha = 0) = 0 \\
P(\beta = 0|\alpha = 1) = 1 \\
P(\beta = 1|\alpha = 0) = 1 \\
P(\beta = 1|\alpha = 1) = 0
\]

(A2)

and is logically deterministic and reversible.

3. **Reset To Zero: RTZ\((p)\)**

If the input state 0 occurs with probability \( p \), then the \( RTZ(p) \) operation has the properties:

\[
P(\beta = 0|\alpha = 0) = 1 \\
P(\beta = 0|\alpha = 1) = 1
\]

(A5)

giving

\[
P(\alpha = 0|\beta = 0) = p \\
P(\alpha = 1|\beta = 0) = 1 - p
\]

(A6)

This is logically deterministic and irreversible. As \( \forall \alpha P(\beta = 1|\alpha) = 0 \) the state \( \beta = 1 \) is not an output state of the operation and we leave it out of the table.

4. **Unset From Zero: UFZ\((p)\)**

The reverse operation to \( RTZ \), where the state 0 is taken to state 0 with probability \( p \), will be called here the UNSET FROM ZERO operation. In [2] this operation was described in terms of the physical process that reverses \( LE \), so was called ‘Reverse Landauer Erasure’ or \( RLE \). In this paper we will refer to the logical operation as \( UFZ \), and to the specific physical process that can be used to embody it as \( RLE \). This operation may also be characterised as a random number generator.

\[
P(\beta = 0|\alpha = 0) = p \\
P(\beta = 1|\alpha = 0) = 1 - p
\]

(A7)

giving

\[
P(\alpha = 0|\beta = 0) = 1 \\
P(\alpha = 0|\beta = 1) = 1
\]

(A8)

This is indeterministic but reversible. As \( \forall \beta P(\alpha = 1|\beta) = 0 \) the state \( \alpha = 1 \) is not an input state of the operation and we leave it out of the table.

5. **Randomize: RND\((p,p')\)**

The operation which takes an input probability of \( p \) of the state being 0 and produces 0 with an output probability of \( p' \), regardless of input state:

\[
P(\beta = 0|\alpha = 0) = p' \\
P(\beta = 0|\alpha = 1) = p' \\
P(\beta = 1|\alpha = 0) = 1 - p' \\
P(\beta = 1|\alpha = 1) = 1 - p'
\]

(A9)

giving

\[
P(\alpha = 0|\beta = 0) = p \\
P(\alpha = 0|\beta = 1) = p \\
P(\alpha = 1|\beta = 0) = 1 - p \\
P(\alpha = 1|\beta = 1) = 1 - p
\]

(A10)
This is indeterministic and irreversible.

We note that $RTZ(p) \equiv RND(p, 1)$ and $UFZ(p) \equiv RND(1, p)$.

6. General One Bit: $GOB(p, p_{00}, p_{11})$

Finally, we consider the most generic operation possible for 1 input bit and 1 output bit. The operation can be wholly defined by one input probability $p$ and two conditional probabilities $p_{00}$ and $p_{11}$.

$$P(\alpha = 0) = p$$
$$P(\alpha = 1) = 1 - p$$
$$P(\beta = 0|\alpha = 0) = p_{00}$$
$$P(\beta = 0|\alpha = 1) = 1 - p_{11}$$
$$P(\beta = 1|\alpha = 0) = 1 - p_{00}$$
$$P(\beta = 1|\alpha = 1) = p_{11}$$

(A11)

giving

$$P(\alpha = 0, \beta = 0) = pp_{00}$$
$$P(\alpha = 0, \beta = 1) = p(1 - p_{00})$$
$$P(\alpha = 1, \beta = 0) = (1 - p)(1 - p_{11})$$
$$P(\alpha = 1, \beta = 1) = (1 - p)p_{11}$$

(A12)

and

$$P(\beta = 0) = pp_{00} + (1 - p)(1 - p_{11})$$
$$P(\beta = 1) = p(1 - p_{00}) + (1 - p)p_{11}$$

(A13)

so

$$P(\alpha = 0|\beta = 0) = \frac{pp_{00}}{p(1 - p_{00})}$$
$$P(\alpha = 0|\beta = 1) = \frac{p(1 - p_{00})}{p(1 - p_{00}) + (1 - p)p_{11}}$$
$$P(\alpha = 1|\beta = 0) = \frac{(1 - p)(1 - p_{11})}{p_{00} + (1 - p)(1 - p_{11})}$$
$$P(\alpha = 1|\beta = 1) = \frac{(1 - p)p_{11}}{p(1 - p_{00}) + (1 - p)p_{11}}$$

(A14)

In general, this is logically indeterministic and irreversible, but can become logically reversible or deterministic under the right limits:

$$IDN \equiv GOB(p, 1, 1)$$
$$NOT \equiv GOB(p, 0, 0)$$
$$RTZ(p) \equiv GOB(p, 1, 0)$$
$$UFZ(p) \equiv GOB(1, p, -)$$
$$RND(p, p') \equiv GOB(p, p', 1 - p')$$

(A15)
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