Universal dynamics of ranking

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Abstract

Virtually anything can be and is ranked; people and animals, universities and countries, words and genes. Rankings reduce the components of highly complex systems into ordered lists, aiming to capture the fitness or ability of each element to perform relevant functions, and are being used from socioeconomic policy to knowledge extraction. A century of research has found regularities in ranking lists across nature and society when data is aggregated over time. Far less is known, however, about ranking dynamics, when the elements change their rank in time. To bridge this gap, here we explore the dynamics of 30 ranking lists in natural, social, economic, and infrastructural systems, comprising millions of elements, whose temporal scales span from minutes to centuries. We find that the flux governing the arrival of new elements into a ranking list reveals systems with identifiable patterns of stability: in high-flux systems only the top of the list is stable, while in low-flux systems the top and bottom are equally stable. We show that two basic mechanisms — displacement and replacement of elements — are sufficient to understand and quantify ranking dynamics. The model uncovers two regimes in the dynamics of ranking lists: a fast regime dominated by long-range rank changes, and a slow regime driven by diffusion. Our results indicate that the balance between robustness and adaptability characterizing the dynamics of complex systems might be governed by random processes irrespective of the details of each system.

Introduction

Rankings are everywhere. From country development indices, academic indicators and candidate poll numbers to music charts and sports scoreboards, rankings are key to how humans measure and make
sense of the world [1,2]. The ubiquity of rankings stems from the generality of their definition: Reducing
the (often high-dimensional) complexity of a system to a few or even a single measurable quantity of
interest [3, 4], dubbed score, leads to an ordered list where elements are ranked, typically from highest
to lowest score. Rankings are, in this sense, a proxy of relevance or fitness to perform a function in the
system. Rankings are being used to identify the most accomplished individuals or institutions, and to
determine the essential pieces of knowledge or infrastructure in society [1]. Since rankings mediate access
to resources (education, jobs, funds) via public and economic policy, they play a role in social mobility;
when misused, they contribute to the disenfranchisement of minorities.

The statistical properties of ranking lists have caught the attention of natural and social scientists for
more than a century. A heavy-tailed decay of score with rank, commonly known as Zipf’s law [5,6], has
been systematically observed in the ranking of cities by population [7,8], words and phrases by frequency
of use [9–14], companies by size [15–17], and many features of the Internet [18]. Zipf’s law appears even
in the score-rank distributions of natural systems, such as earthquakes [19,20], DNA sequences [21], and
metabolic networks [22]. Rankings have also proven useful when analyzing productivity and impact in
science and the arts [23–27], in human urban mobility [28–30], epidemic spreading by influentials [31],
and the development pathways of entire countries [32]. Recently, studies of language use [14,33], sports
performance [34] and many biological and socioeconomic rankings [35] have strengthened the notion of
universality suggested by Zipf’s law: Despite microscopic differences in elements, scores, and types of
interaction, the aggregate, macroscopic properties of ranking lists are remarkably similar throughout
nature and society.

The similarity of score-rank distributions across systems raises the question of the existence of simple
generative mechanisms behind them. While mechanisms of proportional growth [36], cumulative advan-
tage [37], and preferential attachment [38] are often used to explain the heavy-tailed distributions of
ranking lists at single points of time [39,40], they fail to reproduce the way elements actually move in
rank [41], such as the sudden changes in city size throughout history [42]. Here we report on the exis-
tence of generic features of rank dynamics over a wide array of systems, from individuals to countries,
and spatio-temporal scales, from minutes to centuries. By measuring the flux of elements across rank-
ing lists [43,44], we identify a continuum ranging from systems where highly ranked elements are more
stable than the rest, to systems where the least relevant elements are also stable. We show that simple
mechanisms relying on fluxes generated by displacement and replacement of elements can quantitatively
explain all observed patterns of rank stability. A model based on these ingredients uncovers two regimes
in rank dynamics, a fast regime driven by long jumps in rank space, and a slow one driven by diffusion.

Results

We gather 30 ranking lists in natural, social, economic, and infrastructural systems. Data includes human
and animal groups, languages, countries and cities, universities, companies, transportation systems,
online platforms, and sports, with no selection criteria other than having enough information for analysis
(for data details see Supplementary Information [SI] Section S2 and Table S1). Elements in each list
are ranked by a measurable score that changes in time: scientists by citations, businesses by revenue,
regions by number of earthquakes, players by points, etc. Size and temporal scales in the data vary
widely, from the number of people in 636 station entrances of the London Underground every 15 minutes
during a week in 2012 [45], to the written frequency of 124k English words every year since the 17th
Figure 1. Ranking lists in nature and society show generic patterns in their dynamics. (a) Yearly top ranking of universities worldwide according to ARWU score [47]. Elements in the system change rank as their scores evolve in time. (b) (Left) Rank openness $o_t$ at time $t$ for studied systems, defined as the number $N_t$ of elements ever seen in the ranking list up to $t$ relative to list size $N_0$ (see SI Fig. S5). (Right) Correlation between mean openness rate $\dot{o}$ and mean flux $F$ (average probability that an element enters or leaves the list). Ranking lists form a continuum from the most open systems ($F,\dot{o} \sim 1$) to the less open ($F,\dot{o} \sim 0$; for values see SI Table S2). The area between dashed lines has linear scales to show closed systems with $F = \dot{o} = 0$. (c) Time series of rank $R_t/N_0$ occupied by elements across the ranking list in selected systems (all datasets included in SI Fig. S2). In the least open systems available, the top and bottom of the ranking list are stable. In open systems, only the top is stable. (d) Rank change $C$ (average probability that element at rank $R$ changes between $t-1$ and $t$) across ranking lists (see SI Fig. S6), for $F \geq 0.01$ (top) and $F < 0.01$ (bottom). The stable top and bottom ranks of less open systems mean $C$ is roughly symmetric. In open systems, $C$ increases with rank $R$. century [46]. Following an element’s rank through time reveals systematic patterns (Fig. 1). For example, in the Academic Ranking of World Universities (ARWU) [47], published yearly since 2003, institutions like Harvard and Stanford maintain a high score, while institutions down the list change rank frequently (Fig. 1a).

Ranking lists typically have a fixed size $N_0$ (e.g., the Top 100 universities [47], the Fortune 500 companies [48]), so elements may enter or leave the list at any of the $T$ observations $t = 0, \ldots, T - 1$, allowing us to measure the flux of elements across rank boundaries [40, 43, 44] (for the observed values of $N_0$, $T$ see SI Table S1). We introduce two time-dependent measures of flux: the rank openness $o_t = N_t/N_0$, representing the number $N_t$ of elements ever seen in the ranking list until time $t$ relative to the list size $N_0$, and the rank flux $F_t$, representing the probability that an element enters or leaves the ranking list at time $t$. Rank openness is a monotonic increasing function indicating how fast new elements reach the list (Fig. 1b left; all datasets in SI Fig. S5). In turn, flux shows a striking stationarity in time despite differences in temporal scales and potential shocks to the system (SI Fig. S3). By averaging over time, the mean openness rate $\dot{o} = (o_T - o_0)/(T - 1)$ and the mean flux $F = \langle F_t \rangle$ turn out to be highly correlated quantities that uncover a continuum of ranking lists (Fig. 1b right; values in SI Table S2). In one extreme, the most open systems ($F,\dot{o} \sim 1$) have elements that constantly enter and leave the list. Less open systems ($F,\dot{o} \sim 0$) have progressively lower turnout of constituents. Five out of 30 ranking lists are completely closed ($F = \dot{o} = 0$), meaning no single new element is recorded during the observation window.
The measures of rank openness and flux reveal regularities in the stability of ranking processes [41,49]. We follow the time series of the rank $R_t$ occupied by a given element at time $t$ [42] (Fig. 1c; all datasets in SI Fig. S2). In most systems, highly ranked elements like Harvard University and the English word 'the' never change position, showcasing the correspondence between rank stability and notions of relevance like academic prestige [24, 25], grammatical function [14, 46], and underlying network structure [49]. As we go down the ranking list of open systems, rank trajectories increasingly fluctuate in time. In the least open systems where openness and flux are low, however, low ranked elements are also stable. In the ranking of British cities by population, for example, both Birmingham and Nairn remain the most and least populated local authority areas throughout the 20th century [50]. These findings uncover a more fine-grained sense of rank stability: most systems have a stable top ranking, but only the least open systems feature stable bottom ranks as well. The rank change $C$, measured as the average probability that element at rank $R$ changes between times $t - 1$ and $t$, varies between an approximately monotonic increasing function of $R$ for open systems to a symmetric shape as systems become less open (Fig. 1d; all datasets in SI Fig. S6).

Since the stability patterns of an empirical ranking list (as measured by rank change $C$) can be systematically connected to the amount of elements flowing into and out of the list, we build a model of rank dynamics based solely on simple generative mechanisms of flux (Fig. 2). Without assuming system-specific features of elements or their interactions, there are at least two ways to implement flux in rank space. Smooth (but arbitrarily large) changes in the score of an element might make it larger or smaller than other scores, causing elements to move across ranks (the way some scientists gather more citations than others [25], or how population size fluctuates due to historical events [42]). Regardless of score, elements might also disappear from the list and be replaced by new elements: young athletes enter competitions while old ones retire [34]; new words replace anachronisms due to cultural shifts [46]. We implement random mechanisms of displacement and replacement in a simple model by considering a synthetic ranking list of length $N_0$ embedded within a larger system of size $N \geq N_0$. At each time step of length $\Delta t = 1/N$, a randomly chosen element moves to a randomly selected rank with probability $\tau$, displacing others. At the same time, a randomly chosen element gets replaced by a new one with probability $\nu$, leaving other ranks untouched. The dynamics involves all $N$ elements, but to mimic real ranking lists, we only consider the top $N_0$ ranks when comparing with empirical data (Fig. 2a; model details in SI Section S4).

We solve the model analytically by introducing the displacement probability $P_{x,t}$ that an element with rank $r = R/N$ gets displaced to rank $x = X/N$ after a time $t$ (Fig. 2b; uppercase/lowercase symbols denote integer/normalized ranks). Since for small $\Delta x = 1/N$ the probability that at time $t$ an element has not yet been replaced is $e^{-\nu t}$, we have

$$P_{x,t} = e^{-\nu t}(L_t + D_{x,t}).$$

(1)

Here, $L_t = (1 - e^{-\tau t})/N$ is the (rank-independent) probability that up until time $t$ an element gets selected and jumps to any other rank. Jumps have arbitrary length, so they form a Lévi random walk with step length exponent 0 [51] (full derivation in SI Section S4). The probability $D_{x,t} = e^{-\tau t}D(x,t)\Delta x$ that the element in rank $r$ gets displaced to rank $x$ after a time $t$ (due to Lévi walks of other elements) follows approximately the diffusion-like equation

$$\frac{\partial D}{\partial t} = \alpha x(1-x)\frac{\partial^2 D}{\partial x^2},$$

(2)
model of rank dynamics
random displacement random replacement

Figure 2. Model of rank dynamics reproduces features of real-world ranking lists. (a) Model of rank dynamics in system of $N$ elements and ranking list size $N_0$. At time $t$, a random element is moved to a random rank with probability $\tau$. A random element is also replaced by a new element with probability $\nu$. (b) Probability $P_{x,t}$ that element in rank $r = R/N$ moves to $x = X/N$ after time $t$ (uppercase/lowercase symbols are integer/normalized ranks). Elements not replaced diffuse around $x = r$ (with probability $D_{x,t}$) or perform Lévy walks [51] (with probability $L_{t}$). Eq. (2) agrees well with simulations, shown here for $\tau = 0.1$, $\nu = 0.2$, $N = 100$, and $N_0 = 80$ at times $t = 1, 5$ (left/right plots), averaged over $10^5$ realizations. (c) Time series of rank flux $F_t$ over observation period $T$ for data (lines), and mean flux $F$ from fitted model (dashes) (all datasets in SI Fig. S3; for fitting see SI Section S5). (d) Probability $P_{x,t}$ for $t = 1$ and varying $r$ (left) and rank change $C$ (right), shown for selected datasets (lines) and fitted model (dashes; $\tau$ and $\nu$ in plot) (empirical $P_{x,t}$ is passed through a Savitzky–Golay filter; see SI Figs. S6-S9 and SI Table S2). As systems become more open, we lose symmetry in the rank dependence of both $C$ and the height of the diffusion peaks of $P_{x,t}$ (signaled by curved arrows). Data and model agree considerably well in all rank measures.

where $\alpha = \tau/N$. Since $\sum_x D_{x,t} = e^{-\tau t}$, both $D_{x,t}$ and $D(x,t)$ are not conserved in time. Instead of a standard diffusion equation, Eq. (2) is equivalent to the Wright-Fisher equation of random genetic drift in allele populations [52,53]. The solution $D(x,t)$ of Eq. (2) is well approximated by a decaying Gaussian distribution with mean $r$ and standard deviation $\sqrt{2\alpha r(1-r)t}$, i.e. a diffusion kernel (Fig. 2b). Overall, local displacement makes elements slowly diffuse around their initial rank, while Lévy walks and the replacement dynamics reduces exponentially the probability that old elements remain in the ranking list.

An explicit expression for the displacement probability $P_{x,t}$ allows us to derive the mean flux

$$ F = 1 - e^{-\nu[p + (1-p)e^{-\tau}]}, $$

and the mean openness rate

$$ \dot{o} = \nu + \tau \frac{\nu + \tau}{\nu + pt}, $$

where $p = N_0/N$ is the length of the ranking list relative to system size (see SI Section S4). In order to fit the model to each empirical ranking list, we obtain $N_0$ from the data and approximate $N = N_T-1$ as the number of distinct elements ever seen in the list during the observation period $T$, thus fixing $p$ (values for all datasets in SI Tables S1-S2). The remaining free parameters $\tau$ and $\nu$ (regulating the mechanisms of displacement and replacement) come from numerically solving Eqs. (3)-(4) with $F$ and $\dot{o}$ fixed by the
Figure 3. Model uncovers regimes of dynamical behavior in open ranking lists. (a) Rescaled model parameters \( \tau_r \) and \( \nu_r \) in open ranking lists, obtained from fitted parameters \( \tau \) and \( \nu \), relative ranking list size \( p \), and mean openness rate \( \dot{o} \) [see Eq. (5) and SI Section S5; only systems with \( \dot{o} > 10^{-3} \) are shown]. Values collapse onto the universal curve \( \tau_r \nu_r = 1 \), so an inverse relationship between displacement and replacement is enough to emulate empirical rank dynamics. (b)–(d) Average probability that an element changes rank by Lévi walk (\( W_{\text{levi}} \)), diffusion (\( W_{\text{diff}} \), or is replaced (\( W_{\text{repl}} \)) between consecutive observations in the data. Probabilities shown both for selected datasets (dots), and for the model moving along the curve \( \tau_r \nu_r = 1 \) with the same \( p \) and \( \dot{o} \) as the data (lines) (for rest of systems see SI Fig. S17). The simulated system in (d) is the model itself for given values of \( \tau \), \( \nu \), and \( p \) (shown in plot). The model reveals a crossover in real-world ranking lists between a regime dominated by Lévi walks (b) to one driven by diffusion (c). Although not seen in data, the model also predicts a third regime driven by replacement (d).

data (Fig. 2c and Fig. 1b; for model fitting see SI Section S5). These simple generative mechanisms of flux are enough to recover the behavior of ranking lists as quantified by \( P_{x,t} \) and \( C \) (Fig. 2d): When rank flux is low, both the top and bottom of the list are similarly stable and rank dynamics is mostly driven by an interplay between Lévi walks and diffusion. As systems become more open, however, this symmetry gets broken due to a growing flux of elements at the bottom of the ranking list (see SI Fig. S4). Regardless of whether we rank people or animals, words or countries, the pattern of stability across a ranking list is accurately emulated by random mechanisms of flux that disregard the microscopic details of the individual system.

The characterization of flux in ranking lists with mechanisms of displacement and replacement of elements reveals regimes of dynamical behavior that are not apparent from the data alone (Fig. 3). By rescaling the fitted parameter values of the model as

\[
\tau_r = \frac{\tau}{p(1-p)\dot{o}} \quad \text{and} \quad \nu_r = \frac{\nu - p\dot{o}}{\dot{o}},
\]

all open ranking lists \((F, \dot{o} > 0)\) are predicted to follow the universal curve

\[
\tau_r \nu_r = 1,
\]

which suggests that ranking dynamics are regulated by a single effective parameter (Fig. 3a; derivation in SI Section S5). Even if, potentially, displacement and replacement could appear in any relative quantity,
adjusting the model to observations of rank flux and openness (Fig. 1b) leads to an inverse relationship between parameters regulating their generative mechanisms. Real-world ranking lists lie in a spectrum where their dynamics is either mainly driven by score changes that displace elements in rank (high $\tau_r$ and low $\nu_r$, like for GitHub software repositories [54] ranked by daily popularity), or by birth-death processes triggering element replacement (low $\tau_r$ and high $\nu_r$, like for Fortune 500 companies [48] ranked by yearly revenue). While the symmetry (or lack thereof) in rank change $C$ may seemingly imply two distinct classes of systems (see Fig. 1d and Fig. 2d), Eq. (6) reveals the existence of a continuum of open ranking lists, which can be captured by a single model with a single effective parameter.

The universal curve in Eq. (6) displays three regimes in the dynamics of open ranking lists, as measured by the average probabilities that, between consecutive observations in the data, an element performs either a Lévi walk \[ W_{\text{levi}} = e^{-\nu}(1 - e^{-\tau}) \], changes rank by diffusion \[ W_{\text{diff}} = e^{-\nu}e^{-\tau} \], or is replaced \[ W_{\text{repl}} = 1 - e^{-\nu} \], with \[ W_{\text{levi}} + W_{\text{diff}} + W_{\text{repl}} = 1 \]. In systems with the largest rank flux and openness (the ranking lists by daily popularity of both GitHub software repositories and of online readers of the British newspaper The Guardian [55]), elements tend to change rank via long jumps, following a Lévi walk, where \[ W_{\text{levi}} > W_{\text{diff}}, W_{\text{repl}} \] (Fig. 3b). Here, long-range rank changes take elements in and out of a short ranking list within a big system (low $p$), thus generating large mean flux $F$ (see SI Table S2). Most datasets, like the yearly rankings of scientists by citations in American Physical Society journals [25, 56] and of countries by economic complexity [32, 57], belong instead to a diffusion regime with \[ W_{\text{diff}} > W_{\text{levi}}, W_{\text{repl}} \] (Fig. 3c). In this regime, a local, diffusive rank dynamics is the result of elements smoothly changing their scores and overcoming their neighbors in rank space. The model also predicts a third regime dominated by replacement \( W_{\text{repl}} > W_{\text{levi}}, W_{\text{diff}} \); Fig. 3d), where elements are more likely to disappear than change rank. Such ranking lists replace most constituents from one observation to the next, forming a highly fluctuating regime that we do not observe in empirical data. To showcase the crossover between regimes, we simulate the model along the universal curve of Eq. (6) while keeping $p$ and $\dot{o}$ fixed in Eq. (5) (lines in Fig. 3b–d). These curves show how close systems are to a change of regime, i.e. from one dominated by Lévi walks to one driven by diffusion. When a ranking list is close to a regime boundary, external shocks (amounting to variations in parameters $\tau$ and $\nu$) may change the main mechanism behind rank dynamics, thus affecting the overall stability of the system.

**Discussion**

Ranking lists reduce the elements of high-dimensional complex systems into ordered values of a summary statistic, allowing us to compare seemingly disparate phenomena in nature and society [2, 40]. The diversity of their components (people, animals, words, institutions, countries) stands in contrast with the statistical regularity of score-rank distributions when aggregated over time [6, 35]. By exploring the flux of elements of 30 ranking lists in natural, social, economic, and infrastructural systems, we present evidence of universal temporal patterns of rank dynamics. While open systems (large flux) keep the same elements only in top ranks, less open systems (lower flux) also have stable bottom ranks, forming a continuum of ranking lists explained by a single class of models. We show that simple random processes of displacement and replacement of elements are sufficient to recover this balance between robustness (fixed elements in top ranks) and variability (more changes in intermediate and bottom ranks). The model reveals two regimes of dynamical behavior for systems with nonzero flux. Real-world ranking lists are driven either by Lévi random walks [51] that change the rank of elements abruptly, or by a more local,
diffusive movement similar to genetic drift [52, 53], both alongside relatively low levels of new elements entering the system.

When considering open systems in close interaction with their surroundings, there is evolutionary advantage in preserving the functionality of the most essential elements (to maintain robustness [58]), while allowing for fast variability of less crucial components (conferring adaptivity [59]). An understanding of the balance between stability and variability has so far been mostly constrained to critical phase transitions in homogeneous systems [60–62]. If, however, we assume varying timescales of dynamical behavior according to rank (with elements at the top of a ranking list changing more slowly than the rest), systems benefit by having robustness and adaptability at the same time, independently of critical parameters: “slower” elements provide robustness, while “faster” elements provide adaptivity. The ability of a simple model to reproduce rank dynamics in a wide variety of phenomena, regardless of their domain, suggests that such balance can be achieved with random rank change. Thus, complex systems may need much less to be evolvable than previously thought (on the basis of homogeneous timescales only), implying that random variation of natural selection, if heterogeneous, is enough to produce the complex adaptations seen in evolutionary biology and computer science [63].

A more nuanced understanding of the universal features of ranking dynamics might help us limit resource exhaustion in competitive environments, such as information overload in online social platforms and prestige biases in scientific publishing [64], via better algorithmic rating tools. The observation of a generic interplay between ranking robustness and variability can be refined by exploring the relationship between ranking lists associated to the same system, or by incorporating networked interactions that lead to macroscopic ordering [42, 65]. Given that rankings often mediate access to resources via policy, similar mechanisms to those explored here may play a role in finding better ways to avoid social and economic disparity. In general, a better understanding of rank dynamics is promising for regulating systems by adjusting their temporal heterogeneity.

Data and code availability

Code to reproduce the results of the paper is publicly available at https://github.com/iniguezg/social. For data availability see SI Section S2. Non-public data is available from the authors upon reasonable request.

Acknowledgments

We acknowledge José A. Morales and Sergio Sánchez for data handling at the start of the project. We are grateful for data provision to Gustavo Carreón, Syed Haque, Kay Holekamp, Amiyaal Ilany, Márton Karsai, Raj Kumar Pan, Roberto Murcio, and Roberta Sinatra. G.I. thanks Tiina Nasi for valuable suggestions. G.I. acknowledges partial support from the Air Force Office of Scientific Research under award number FA8655-20-1-7020, and by the EU H2020 ICT48 project Humane AI Net under contract 952026. C.P. and C.G. acknowledge support by projects CONACyT 285754 and UNAM-PAPIIT IG100518, IG101421, IN107919, and IV100120. A.-L.B. was supported by the European Union’s Horizon 2020 SYNERGY grant under grant agreement No 810115-DYNASNET, the John Templeton Foundation grant #61066 and AFOSR award number FA9550-19-1-0354.

Author contributions

G.I., C.P., C.G., and A.-L.B. designed the study. G.I. performed data analysis and model fitting. G.I. and C.P. derived analytical results and performed numerical simulations. G.I., C.P., C.G., and A.-L.B. wrote the paper.
Competing interest statement

A.-L.B. is founder of Foodome, ScipherMedicine, and Datapolis, companies that explore the role of networks in health and urban environments. G.I. is founder of Predify, a data science consulting startup in Mexico. All other authors declare no competing interest.

References

[1] P. ´Erdi, *Ranking: The unwritten rules of the social game we all play*. Oxford University Press, 2019.
[2] A. N. Langville and C. D. Meyer, *Who’s #1?: The science of rating and ranking*. Princeton University Press, 2012.
[3] J. Diamond, *Guns, germs, and steel*. W. W. Norton & Company, 1997.
[4] P. Turchin et al., “Quantitative historical analysis uncovers a single dimension of complexity that structures global variation in human social organization,” *Proc. Nat. Acad. Sci. USA*, vol. 115, no. 2, pp. E144–E151, 2018.
[5] G. K. Zipf, *Human behavior and the principle of least effort: An introduction to human ecology*. Cambridge, MA, USA: Addison-Wesley Press, 1949.
[6] M. E. J. Newman, “Power laws, Pareto distributions and Zipf’s law,” *Contemp Phys.*, vol. 46, no. 5, pp. 323–351, 2005.
[7] F. Auerbach, “Das gesetz der bevölkerungskonzentration,” *Petermanns Geographische Mitteilungen*, vol. 59, pp. 74–76, 1913.
[8] K. T. Rosen and M. Resnick, “The size distribution of cities: An examination of the Pareto law and primacy,” *J. Urban Econ.*, vol. 8, no. 2, pp. 165–186, 1980.
[9] A. D. Booth, “A “law” of occurrences for words of low frequency,” *Inform. Control*, vol. 10, no. 4, pp. 386–393, 1967.
[10] L. Q. Ha, E. I. Sicilia-Garcia, J. Ming, and F. J. Smith, “Extension of Zipf’s law to words and phrases,” in *Proceedings of the 19th International Conference on Computational Linguistics*, pp. 1–6, 2002.
[11] R. Ferrer i Cancho and R. V. Solé, “Least effort and the origins of scaling in human language,” *Proc. Nat. Acad. Sci. USA*, vol. 100, no. 3, pp. 788–791, 2003.
[12] B. Corominas-Murtra, J. Fortuny, and R. V. Solé, “Emergence of Zipf’s law in the evolution of communication,” *Phys. Rev. E*, vol. 83, no. 3, p. 036115, 2011.
[13] P. S. Dodds, K. D. Harris, I. M. Kloumann, C. A. Bliss, and C. M. Danforth, “Temporal patterns of happiness and information in a global social network: Hedonometrics and twitter,” *PloS ONE*, vol. 6, no. 12, p. e0121898, 2015.
[14] R. E. Lucas Jr, “On the size distribution of business firms,” *Bell J. Econ.*, pp. 508–523, 1978.
[15] M. H. Stanley, L. A. Amaral, S. V. Buldyrev, S. Havlin, H. Leschhorn, P. Maass, M. A. Salinger, and H. E. Stanley, “Scaling behaviour in the growth of companies,” *Nature*, vol. 379, no. 6568, pp. 804–806, 1996.
[16] R. L. Axtell, “Zipf distribution of US firm sizes,” *Science*, vol. 293, no. 5536, pp. 1818–1820, 2001.
[17] L. A. Adamic and B. A. Huberman, “Zipf’s law and the internet,” *Glottometrics*, vol. 3, no. 1, pp. 143–150, 2002.
[18] Y. Ogata and K. Katsura, “Analysis of temporal and spatial heterogeneity of magnitude frequency distribution inferred from earthquake catalogues,” *Geophys. J. Int.*, vol. 113, no. 3, pp. 727–738, 1993.
A. Clauset, S. Arbesman, and D. B. Larremore, “Systematic inequality and hierarchy in faculty hiring networks,” Sci. Adv., vol. 1, no. 1, p. e1400005, 2015.

R. Sinatra, D. Wang, P. Deville, C. Song, and A.-L. Barabási, “Quantifying the evolution of individual scientific impact,” Science, vol. 354, no. 6312, 2016.

S. P. Fraiberger, R. Sinatra, M. Resch, C. Riedl, and A.-L. Barabási, “Quantifying reputation and success in art,” Science, vol. 362, no. 6416, pp. 825–829, 2018.

M. Janosov, F. Musciotto, F. Battiston, and G. Iñiguez, “Elites, communities and the limited benefits of mentorship in electronic music,” Sci. Rep., vol. 10, no. 1, pp. 1–8, 2020.

M. C. González, C. A. Hidalgo, and A.-L. Barabási, “Understanding individual human mobility patterns,” Nature, vol. 453, no. 7196, pp. 779–782, 2008.

A. Nonius, S. Scellato, R. Lambiotte, M. Pontil, and C. Mascolo, “A tale of many cities: Universal patterns in human urban mobility,” PloS one, vol. 7, no. 5, p. e37027, 2012.

L. Alessandretti, P. Sapiezynski, V. Sekara, S. Lehmann, and A. Baronchelli, “Evidence for a conserved quantity in human mobility,” Nat. Hum. Behav., vol. 2, no. 7, pp. 485–491, 2018.

J. Gu, S. Lee, J. Saramäki, and P. Holme, “Ranking influential spreaders is an ill-defined problem,” Europhys. Lett., vol. 118, no. 6, p. 68002, 2017.

C. A. Hidalgo and R. Hausmann, “The building blocks of economic complexity,” Proc. Nat. Acad. Sci. USA, vol. 106, no. 26, pp. 10570–10575, 2009.

J. A. Morales, E. Colman, S. Sánchez, F. Sánchez-Puig, C. Pineda, G. Iñiguez, G. Cocho, J. Flores, and C. Gershenson, “Rank dynamics of word usage at multiple scales,” Front. Phys., vol. 6, p. 45, 2018.

J. A. Morales, S. Sánchez, J. Flores, C. Pineda, C. Gershenson, G. Cocho, J. Zizumbo, R. F. Rodríguez, and G. Iñiguez, “Generic temporal features of performance rankings in sports and games,” EPJ Data Science, vol. 5, no. 1, p. 33, 2016.

G. Martínez-Mekler, R. A. Martínez, M. B. del Río, R. Mansilla, P. Miramontes, and G. Cocho, “Universality of rank-ordering distributions in the arts and sciences,” PLoS One, vol. 4, no. 3, p. e4791, 2009.

H. A. Simon, “On a class of skew distribution functions,” Biometrika, vol. 42, no. 3/4, pp. 425–440, 1955.

D. Price, “A general theory of bibliometric and other cumulative advantage processes,” J. Am. Soc. Inform. Sci., vol. 27, no. 5, pp. 292–306, 1976.

A.-L. Barabási and R. Albert, “Emergence of scaling in random networks,” Science, vol. 286, no. 5439, pp. 509–512, 1999.

T. Maillart, D. Sornette, S. Spaeth, and G. von Krogh, “Empirical tests of Zipf’s law mechanism in open source Linux distribution,” Phys. Rev. Lett., vol. 101, no. 21, p. 218701, 2008.

P. S. Dodds, J. R. Minot, M. V. Arnold, T. Alshaabi, J. L. Adams, D. R. Dewhurst, T. J. Gray, M. R. Frank, A. J. Reagan, and C. M. Danforth, “Allotaxonometry and rank-turbulence divergence: A universal instrument for comparing complex systems,” Eprint arXiv:2002.09770, 2020.

N. Blumm, G. Ghoshal, Z. Forró, M. Schich, G. Bianconi, J.-P. Bouchaud, and A.-L. Barabási, “Dynamics of ranking processes in complex systems,” Phys. Rev. Lett., vol. 109, no. 12, p. 128701, 2012.

M. Batt, “Rank clocks,” Nature, vol. 444, no. 7119, pp. 592–596, 2006.

M. Gerlach, F. Font-Clos, and E. G. Altmann, “Similarity of symbol frequency distributions with heavy tails,” Phys. Rev. X, vol. 6, no. 2, p. 021009, 2016.

E. A. Pechenick, C. M. Danforth, and P. S. Dodds, “Is language evolution grinding to a halt? the scaling of lexical turbulence in english fiction suggests it is not,” J. Comput. Sci., vol. 21, pp. 24–37, 2017.

R. Murcio, C. Zhong, E. Manley, and M. Batty, “Identifying risk profiles in the London’s public transport system,” in 14th International Conference on Computers in Urban Planning and Urban Management, 2015.

J.-B. Michel et al., “Quantitative analysis of culture using millions of digitized books,” Science, vol. 331, no. 6014, pp. 176–182, 2011.

N. C. Liu and Y. Cheng, “The academic ranking of world universities,” High. Educ. Eur., vol. 30, no. 2, pp. 127–136, 2005.
[48] J. Zhu, “Multi-factor performance measure model with an application to Fortune 500 companies,” *Eur. J. Oper. Res.*, vol. 123, no. 1, pp. 105–124, 2000.

[49] G. Ghoshal and A.-L. Barabási, “Ranking stability and super-stable nodes in complex networks,” *Nat. Commun.*, vol. 2, no. 1, pp. 1–7, 2011.

[50] R. Edwards and M. Batty, “City size: Spatial dynamics as temporal flows,” *Environ. Plann. A*, vol. 48, no. 6, pp. 1001–1003, 2016.

[51] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, “Strange kinetics,” *Nature*, vol. 363, no. 6424, pp. 31–37, 1993.

[52] L. Chen and D. W. Stroock, “The fundamental solution to the Wright–Fisher equation,” *SIAM J. Math. Anal.*, vol. 42, no. 2, pp. 539–567, 2010.

[53] C. L. Epstein and R. Mazaero, “Wright–Fisher diffusion in one dimension,” *SIAM J. Math. Anal.*, vol. 42, no. 2, pp. 568–608, 2010.

[54] B. Vedres and O. Vasarhelyi, “Gendered behavior as a disadvantage in open source software development,” *EPJ Data Science*, vol. 8, no. 1, p. 25, 2019.

[55] N. Thurman, “Forums for citizen journalists? Adoption of user generated content initiatives by online news media,” *New Media & Society*, vol. 10, no. 1, pp. 139–157, 2008.

[56] B. Vedres and O. Vasarhelyi, “Gendered behavior as a disadvantage in open source software development,” *EPJ Data Science*, vol. 8, no. 1, p. 25, 2019.

[57] C. A. Hidalgo, B. Klinger, A.-L. Barabási, and R. Hausmann, “The product space conditions the development of nations,” *Science*, vol. 317, no. 5837, pp. 482–487, 2007.

[58] A. Wagner, *Robustness and Evolvability in Living Systems*. Princeton, NJ, USA: Princeton University Press, 2005.

[59] F. Heylighen, “The science of self-organization and adaptivity,” in *The Encyclopedia of Life Support Systems* (L. D. Kiel, ed.), Oxford, UK: EOLSS Publishers, 2003.

[60] T.-D. Lee and C.-N. Yang, “Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model,” *Phys. Rev.*, vol. 87, no. 3, p. 410, 1952.

[61] S. A. Kauffman, *The Origins of Order*. Oxford, UK: Oxford University Press, 1993.

[62] M. Aldana, S. Coppersmith, and L. P. Kadanoff, “Boolean dynamics with random couplings,” in *Perspectives and Problems in Nonlinear Science* (E. Kaplan, J. E. Marsden, and K. R. Sreenivasan, eds.), New York, NY, USA: Springer, 2003.

[63] G. P. Wagner and L. Altenberg, “Perspective: Complex adaptations and the evolution of evolvability,” *Evolution*, vol. 50, no. 3, pp. 967–976, 1996.

[64] V. Sekara, P. Deville, S. E. Ahnert, A.-L. Barabási, R. Sinatra, and S. Lehmann, “The chaperone effect in scientific publishing,” *Proc. Nat. Acad. Sci. USA*, vol. 115, no. 50, pp. 12603–12607, 2018.

[65] P. L. Krapivsky and S. Redner, “Statistics of changes in lead node in connectivity-driven networks,” *Phys. Rev. Lett.*, vol. 89, no. 25, p. 258703, 2002.
Supplementary Information for

Universal dynamics of ranking

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S1 Summary of notation

We consider a ranking list at times $t = 0, \ldots, T - 1$, that is, an ordered set of $N_0$ elements (with $N_0$ constant in time) where each element $i$ in the list at time $t$ has a score $s_i(t)$, and elements are ordered across the list with decreasing score. Thus, elements and their order may change throughout time. The most important element (with the largest score) at time $t$ has rank $R_t = 1$; the least important element (with the lowest score) has rank $R_t = N_0$, and the ranking is the particular order of elements and scores across ranks $R = 1, \ldots, N_0$, which changes in time as elements vary their scores. We consider the case where we do not have access to the scores at all times, but only in a discrete set of $T$ observations, separated in average by a real time interval $\langle \tau_{\text{days}} \rangle$. We also consider the case where we may not have empirical data on the scores of all elements at all times, either because elements enter/leave the ranking at some point in time, or because score data is unavailable. Thus, there are $N_t \geq N_0$ distinct elements that have ever been in the ranking up to (and including) time $t$. If $N_t = N_0$ for all $t$ the ranking list is closed, since elements do not leave or enter the ranking list, and if $N_t > N_0$ the ranking list is open. In what follows we characterize the temporal variability of ranking lists across many systems by analysing the flow of elements in and out of the ranking list as time goes by.

S2 Data description

We analyse 30 datasets in a wide range of structural and temporal scales, comprising several definitions of elements and scores. Table 1 lists the observed system size $N_{T-1}$ (number of distinct elements ever seen in the ranking), ranking list size $N_0$, number of observations $T$, and real time interval $\langle \tau_{\text{days}} \rangle$ for all datasets considered. Table 1 also includes a system classification based on the nature of the elements in the system, and the corresponding definitions of elements and scores for each system. Social systems reflect human interactions at the individual and organizational levels. Language datasets show how word usage has changed across centuries. Economic rankings illustrate value at different scales. Systems in the infrastructure category are specific to public transport and city populations. Nature datasets gather information from biological and geological phenomena. Finally, rankings in sports capture the relative performance of players and teams according to sets of rules. We now describe the datasets in more detail.

In their original state some datasets are not homogeneous in time and size, since they do not have the same real time interval between any times $t$ and $t + 1$ or the same number of elements per observation (i.e. they have a variable ranking list size). To consistently analyse all datasets, we crop the data to obtain roughly homogeneous time intervals and have a constant ranking list size across observations, while trying to retain as large $T$ and $N_0$ as possible.

S2.1 Society

GitHub repositories. GitHub [1] is perhaps the most popular web-based version control repository, mostly used for source code. This dataset contains daily rankings of repositories, based on the number of users watching each project, from April 1, 2012 to December 30, 2014.

\[^{1}\text{The time unit is arbitrary and may be anything other than days.}\]
| Dataset | Element | Score | Measure |
|---------|---------|-------|---------|
|         |         |       | N<sub>T−1</sub> | N<sub>0</sub> | T | ⟨τ<sub>days</sub⟩ |
| **Society** | | | | | | |
| GitHub repositories [1] | repository # watchers | 450655 | 4773 | 727 | 1.00 |
| The Guardian readers (recc) [2] | person avg # recommends | 29165 | 753 | 182 | 1.00 |
| The Guardian readers (comm) [2] | person # comments | 18244 | 753 | 182 | 1.00 |
| Enron emails [3] | person # emails | 4720 | 209 | 101 | 7.00 |
| Scientists [4,5] | person # citations | 2614 | 1041 | 45 | 365.25 |
| Universities [6] | university ARWU score [7] | 140 | 100 | 14 | 365.23 |
| **Languages** | | | | | | |
| Russian [8–11] | word frequency | 281346 | 35494 | 210 | 365.24 |
| Spanish [8–11] | word frequency | 233323 | 31750 | 260 | 365.24 |
| German [8–11] | word frequency | 195455 | 22661 | 262 | 365.25 |
| French [8–11] | word frequency | 182507 | 17645 | 367 | 365.24 |
| Italian [8–11] | word frequency | 139645 | 20879 | 244 | 365.24 |
| English [8–11] | word frequency | 124464 | 17750 | 334 | 365.24 |
| **Economics** | | | | | | |
| Companies [12] | company revenue | 1895 | 500 | 51 | 365.26 |
| Countries [13–15] | country complexity [13–15] | 139 | 99 | 49 | 365.25 |
| **Infrastructure** | | | | | | |
| Cities (RU) [16] | city population | 1639 | 936 | 8 | 4383.00 |
| Metro stations (London) [17] | station # passengers | 636 | 636 | 69 | 0.0104 |
| Cities (GB) [18] | city population | 458 | 458 | 11 | 3652.50 |
| Metro stations (Mexico) | station # passengers | 175 | 175 | 365 | 1.00 |
| **Nature** | | | | | | |
| Hyenas [19] | animal association index | 303 | 43 | 23 | 365.23 |
| Regions JP (quake mag) [20,21] | region avg quake magnitude | 264 | 264 | 176 | 28.00 |
| Regions JP (quakes) [20,21] | region # quakes | 264 | 264 | 176 | 28.00 |
| **Sports** | | | | | | |
| Chess players (male) [22] | person Elo rating [23] | 16568 | 13500 | 46 | 30.44 |
| Chess players (female) [22] | person Elo rating [23] | 16539 | 12681 | 46 | 30.44 |
| Poker players [24] | person GPI score [25] | 9799 | 1795 | 221 | 7.04 |
| Tennis players [26] | person ATP points [27] | 4793 | 1600 | 400 | 7.00 |
| Golf players [28] | person OWGR points [29] | 3632 | 1150 | 768 | 7.00 |
| Football scorers [30] | person # goals [31] | 2397 | 400 | 53 | 7.04 |
| NASCAR drivers (Busch) [32] | person NASCAR points | 676 | 76 | 34 | 365.24 |
| NASCAR drivers (Winston Cup) [32] | person NASCAR points | 272 | 50 | 35 | 365.26 |
| National football teams [33] | team FIFA points [34] | 210 | 200 | 71 | 30.44 |

Table S1. Datasets used in this study. Characteristics of the available datasets, including the observed system size N<sub>T−1</sub>, ranking list size N<sub>0</sub>, number of observations T, and real time interval between observations ⟨τ<sub>days</sub⟩⟩. The table includes a system type based on the nature of the elements in the ranking list, as well as the corresponding definitions of elements and scores for each system, and references regarding the dataset and score system.

The Guardian readers. The Guardian [2] is a British national daily newspaper, publishing online articles (news and opinion pieces) in diverse subjects. Users registered to the website can post comments to some of the articles, which readers may ‘recommend’ (by clicking a ‘like’ button akin to those of social network sites). Focusing on the period from November 1, 2011 till May 1, 2012, we crawl articles appearing in three sections of The Guardian (‘politics’, ‘sport’, and ‘comment is free’), and rank users daily by the number of comments they write (denoted ‘comm’), as well as by the average number of...
recommends their comments receive (denoted ‘recc’).

**Enron emails.** The Federal Energy Regulatory Commission, during its investigation of the company Enron, made public about half a million emails from roughly 150 users, mostly senior managers. A cleaned, current version of the dataset is available online [3]. From this dataset we rank users (email accounts) by the number of emails sent, on a weekly basis.

**Scientists.** We use yearly citation data from journals of the American Physical Society to rank the most cited scientists by number of citations [4, 5].

**Universities.** ShangaiRanking Consultancy is an independent organization dedicated to higher education research. During the period 2003–2016 [6], the organization ranked top universities according to the Academic Ranking of World Universities (ARWU) score, which considers several criteria such as number of students, publications, Nobel Prizes, Fields Medals, etc. [7].

**S2.2 Languages**

We use a subset of the publicly available Google Books Ngram dataset [8, 9], the result of the digitalization and conversion of millions of books in several languages (about 4% of all published books until 2009). From this data we extract the frequency of words by year for Russian, Spanish, German, French, Italian, and English. The original dataset is case sensitive, so we merge words with different cases. We have already analyzed the temporal variability of ranks at several scales for these languages in previous studies [10, 11].

**S2.3 Economics**

**Companies.** Since 1955, *Fortune* magazine has compiled a yearly dataset of the top 500 corporations in the world based on yearly revenue. Here we use the freely accessible archives of 1955–2005 [12].

**Countries.** MIT’s Observatory of Economic Complexity [13] has compiled a yearly ranking dataset of the economic complexity of countries between 1962 and 2015 [14, 15], where a complexity score reflects the diversity of exports of countries around the globe.

**S2.4 Infrastructure**

**Cities.** To obtain ranking datasets of cities by population, we use data from previous studies of Russia (RU) [16] and Great Britain (GB) [18]. In Russia, urban population has decreased, while in most other countries it has increased. For Great Britain, population data corresponds to 63 primary urban areas from England, Scotland, and Wales between 1901 and 2011.

**Metro stations.** The London Underground has 270 stations, serving about 5 million passengers per day. For London, the ranking dataset corresponds to aggregated Oyster smart-card data of 636 station
entrances in a week of 2012, considering 15-minute intervals [17]. For Mexico City (Mexico), we requested and obtained data directly from Sistema de Transporte Colectivo Metro, the part of Mexican government directing metro services, and used daily entrance data for 175 stations during 2014.

S2.5 Nature

Hyenas. A 23-year field study has monitored the social relationships of a spotted hyena population in Kenya [19]. Using this dataset, we rank individual hyenas according to the sum of an association index describing the strength of the relationships with the rest of the hyenas over a particular year.

Regions JP. The Japan University Network Earthquake Catalog openly records earthquake events happening in Japan (JP) [20]. Following a previous study [21], we only consider earthquakes with Richter magnitude larger than 2.0, happening between July 1, 1985 and December 31, 1998, taking main-shocks and after-shocks as separate events. We rank administrative regions in Japan by both the monthly number of earthquakes happening in a region (denoted ‘quakes’), and the average earthquake magnitude in a month (denoted ‘quake mag’).

S2.6 Sports

Chess players. Every month, the Fédération Internationale des Échecs (FIDE, International Chess Federation) ranks top male and female players [22] based on the Elo rating system [23].

Poker players. Weekly rankings of Poker players are listed online [24]. Rankings are based on the Global Poker Index (GPI) score [25], which takes into account players who participated in tournaments in the last 18 months, including an aging factor and the scores obtained in different tournaments.

Tennis players. The Association of Tennis Professionals (ATP) [26] has an intricate ranking system based on ATP points [27], which is updated every week to rank male players in Singles.

Golf players. The Official World Golf Ranking (OWGR) publishes weekly lists of player rankings, accumulating points achieved in tournaments in the previous two years [28]. To calculate OWGR points, the points achieved by each player on all tournaments during these two years are averaged over the number of tournaments involved [29].

Football scorers. The Football World Rankings publishes a weekly list of top scorers [30], considering the number of goals scored by each player in the previous year [31].

NASCAR drivers. We use an open dataset containing the ranking of drivers of the National Association for Stock Car Auto Racing (NASCAR), both for the Busch Series and the Winston Cup [32].

National football teams. The Fédération Internationale de Football Association (FIFA, International Federation of Association Football) [33] publishes a monthly list of national teams ranked according to
Figure S1. Fitness in open to closed ranking lists. Probability $P_f$ that a randomly selected element $i$ has fitness $f_i$ at time $t$, normalized by the maximum value of $P_f$ at the time (denoted $P_m$). We plot $P_f$ for various times $t$ (colored lines). To compare across observations $t = 0, \ldots, T-1$, we calculate fitness as $f_i(t) = s_i(t)/s_m(t)$, with $s_i(t)$ the score of element $i$ and $s_m(t)$ the maximum score in the ranking at time $t$ [Eq. (S1)]. Datasets are ordered from most open (upper row) to closed (lower row) according to mean rank flux $F$ [Eq. (S2)]. Fitness distributions are either broad (spanning orders of magnitude) or constrained to a small interval, and show wild variations across datasets.

the points obtained in the previous four years, using several criteria to determine FIFA points [34]. Data for female teams has a lower temporal scale, so we restrict our analysis to male teams.

S2.7 Data acknowledgments and accessibility

We acknowledge José A. Morales and Sergio Sánchez for data handling in the initial stages of the project. We are grateful for data provision to Syed Haque (GitHub repositories and Enron emails), Raj Kumar Pan and author Gerardo Iñiguez (The Guardian), Roberta Sinatra (Scientists), Roberto Murcio (Cities [RU and GB], Metro stations [London]), Gustavo Carreón (Metro stations [Mexico]), Amiyaal Ilany and Kay Holekamp (Hyenas), and Márton Karsai (Regions JP). All data is either openly accessible, can be crawled directly from the web, or can be requested directly from the previous researchers under reasonable conditions.
Figure S2. Rank dynamics in open to closed ranking lists. Normalized rank $R_t/N_0$ for selected elements in ranking as a function of normalized time $t/T$. Rank time series are colored according to the last observed rank of the element, $R_{T-1}/N_0$. The lower part of plots ($R > N_0$, below dashed lines) represents the unobserved part of an open system, containing elements without known score/rank at the time. Datasets are ordered from most open (upper row) to closed (lower row) according to mean rank flux $F$ [Eq. (S2)]. Relatively open ranking lists are more stable at the top (low $R_t$) than at the bottom (high $R_t$). Closed ranking lists are more stable both at the top (low $R_t$) and bottom (high $R_t$), and less stable in the middle.

S3  Rank measures

S3.1  Fitness

Scores $s_i(t)$ vary across elements and observations of the ranking list. To compare their distribution between times $t$, we normalize the score of element $i$ as the fitness

$$f_i(t) = \frac{s_i(t)}{s_m(t)},$$

with $s_m(t)$ the maximum score in the ranking at time $t$. We define $P_f$ as the probability that a randomly selected element $i$ has fitness $f_i$ at time $t$, and $P_m$ as the maximum value of $P_f$ at time $t$. The normalized fitness distribution $P_{f}/P_{m}$ for each time $t = 0, \ldots, T - 1$ varies greatly between systems, having either a broad or narrow functional form (Fig. S1). In most ranking lists, $P_f$ is relatively constant in time.

S3.2  Rank dynamics

We consider the dynamics of rank $R_t = 1, \ldots, N_0$ of a given element as a function of time $t$. In most open ranking lists ($N_t > N_0$), the time series $R_t$ at the top of the ranking ($R_t \ll N_0$) tend to be stable, while $R_t$ fluctuates more as we go down in ranking (middle rows in Fig. S2). If at some time $t$ an element
**Figure S3. Rank flux in open to closed ranking lists.** Flux $F_t$ as a function of normalized time $t/T$, calculated as the probability that an element in (out of) the ranking list at time $t-1$ leaves (enters) the ranking at time $t$, averaged over all elements in the ranking at the time. Flux is shown for data (continuous lines) and the model of Section S4 with Eq. (S27) for $s = N$ (dashed lines) (for parameter fitting see Table 2 and Section S5). Datasets are ordered from most open (upper row) to closed (lower row) according to mean rank flux $F$ [Eq. (S2)]. Flux is roughly constant over time, with occasional large deviations from the mean. In languages, flux has a decreasing trend over time. Closed ranking lists have zero flux, consistent with $N_t = N_0$.

is not in the ranking, its rank $R_t$ is not observable from empirical data (because we do not know the corresponding score). For some open ranking lists even the top of the ranking is not stable, since the time scale $T$ is larger than the typical lifetime of an element in the system (see, e.g., upper rows in Fig. S2, or datasets where people or companies have not been active during the whole observation period). In closed ranking lists ($N_t = N_0$), both the top and the bottom of the ranking are stable, while the middle part of the ranking shows more variation in $R_t$ (lower rows in Fig. S2).

### S3.3 Rank flux

In open ranking lists, elements flow into and out of the ranking (due to scores not being measured all the time), meaning that an element may not have a well-defined rank $R_t = 1, \ldots, N_0$ for all times $t = 0, \ldots, T - 1$. We define rank flux $F_t$ as the probability that an element in (out of) the ranking list at time $t - 1$ leaves (enters) the ranking at time $t$, averaged over all elements in the ranking list $^2$. For most open ranking lists, flux is roughly constant throughout time (upper rows in Fig. S3), i.e. $F_t$ does not considerably deviate from its mean over time,

$$F = \frac{1}{T-1} \sum_{t=1}^{T-1} F_t.$$  

---

$^2$Since the ranking size $N_0$ is constant in time, the number of elements that enter and leave the ranking between two consecutive observations are equal.
Figure S4. Rank out-flux in open systems. Out-flux $F_R^-$ as a function of normalized rank $R/N_0$, defined as the probability that an element in rank $R$ at time $t-1$ leaves the ranking at time $t$, averaged over time. Out-flux is shown for data (continuous lines) and the model of Section S4 with Eq. (S29) for $s = N$ (dashed lines) (for parameter fitting see Table 2 and Section S5). Datasets are ordered from most open (upper row) to closed (lower row) according to mean rank flux $F$ [Eq. (S2)]. Out-flux has very similar qualitative behavior to in-flux in all considered datasets.

The constant trend in rank flux is sometimes disrupted by large deviations when a larger or smaller number of elements enter/leave the ranking list. In language datasets, $F_t$ has a decreasing trend over time (Fig. S3), arguably due to the exceptionally long observation period (centuries instead of days/years, see Table 1), or to a very heterogeneous sampling of words throughout observations. Closed ranking lists have zero flux, since $N_t = N_0$ for all times $t$. Values of the constant flux $F$ in all datasets are very well reproduced by the model of Section S4 [Eq. (S27)], except for the most open ranking list explored [The Guardian readers (recc)], where the fitting process is less precise (see Section S5).

We define out-flux $F_R^-$ as the probability that an element in rank $R$ at time $t - 1$ leaves the ranking list at time $t$, averaged over all times $t = 1, \ldots, T - 1$ (Fig. S4). Similarly, we define in-flux $F_R^+$ as the probability that an element out of the ranking list at time $t - 1$ gets rank $R$ at time $t$, averaged over time. Out-/in-fluxes are time averages of flux that determine what part of the ranking contributes more to the flow of elements out of (and into) it. Even though out-/in-fluxes do not need to be equal in general, they are very similar in all considered datasets. In most open systems the top of the ranking (low $R$) is more stable than the bottom (high $R$), since high out-flux tends to appear only at the bottom. This functional form is recovered by the model of Section S4 [Eq. (S29)], except for very open systems [upper rows in Fig. S4]. On the other hand, the change in concavity of out-flux (as a function of $R$) for some of the most open systems is not captured by the model.
Figure S5. Rank openness in open to closed ranking lists. Rank openness $\alpha_t = N_t / N_0$ as a function of normalized time $t/T$, defined as the number $N_t$ of distinct elements that have been in the ranking at any time up to $t$, in units of ranking list size $N_0$ (Eq. (S3)). Openness is shown for data (continuous lines) and the model of Section S4 with Eq. (S38) for $s = N$ (dashed lines) (for parameter fitting see Table 2 and Section S5). Datasets are ordered from most open (upper row) to closed (lower row) according to mean rank flux $F$ [Eq. (S2)]. Most datasets have a concave (or roughly linear) openness in time, while closed ranking lists have constant openness $\alpha_t = 1$ for all $t$.

### S3.4 Rank openness

As mentioned in Section S1, $N_t$ is the number of distinct elements that have ever been in the ranking list up to time $t$ (i.e. at any $t' = 0, \ldots, t$). Closed systems have $N_t = N_0$ for any $t$, and open systems have $N_t > N_0$ for some $t > 0$. The ranking list size $N_0$ is the initial condition of the time series $N_t$, since in our initial observation we can only measure $N_0$ scores. Since $N_t$ counts the elements that have visited the ranking up until time $t$, it is a monotonically increasing function ($N_t \leq N_t \leq N_{T-1}$ for $t' \leq t$). The value $N_{T-1}$ is an observable proxy for the (unknown) size of the system that may increase with larger $T$, since more observations potentially mean access to scores of new elements (see related discussion in the model definition of Section S4.1). To compare ranking lists of different size $N_0$, we define the rank openness of a ranking list at time $t$ as the number of elements seen up until that time in units of ranking list size,

$$\alpha_t = \frac{N_t}{N_0}$$

(S3)

In open ranking lists, the time series $\alpha_t$ increases from $\alpha_0 = 1$ to $\alpha_{T-1} = N_{T-1}/N_0$ (see Table 1), with average slope $\bar{\alpha}_t = (\alpha_t - \alpha_0)/t$ (Fig. S5). The mean openness rate after $T$ observations, $\dot{\alpha} \equiv \dot{\alpha}_{T-1} \in [0, 1]$, is a single number that characterises the openness of a ranking list [just like the mean flux $F$ in Eq. (S2)], from $\dot{\alpha} = 0$ for closed systems, to $\dot{\alpha} = 1$ for the most open system possible (where any element in the ranking list is only seen once across observations). In most open datasets considered, openness $\alpha_t$ has a concave (or roughly linear) functional form in $t$. Our minimal model of rank dynamics introduced
Figure S6. Rank change in open to closed ranking lists. Rank change $C$ as a function of normalized rank $R/N_0$, calculated as the probability that elements in rank $R$ at times $t$ and $t-1$ are not the same, averaged over time. Rank change is shown for data (continuous lines) and the model of Section S4 with Eq. (S41) for $s = N$ (dashed lines) (for parameter fitting see Table 2 and Section S5). Datasets are ordered from most open (upper row) to closed (lower row) according to mean rank flux $F$ [Eq. (S2)]. In relatively open ranking lists, rank change is asymmetric: $C$ is lower at the top of the ranking (low $R$) than at the bottom (high $R$). In progressively more closed ranking lists, rank change turns roughly symmetric in $R$, with low $C$ also at the bottom of the ranking.

in Section S4 captures both the openness and its derivative very well [see Section S4.5, particularly Eqs. (S38)-(S40)].

S3.5 Rank change

In order to measure the stability of rank dynamics, we introduce rank change $C$ as the probability that elements in rank $R$ at times $t-1$ and $t$ are not the same, averaged over all times $t=1,...,T-1$ (Fig. S6). In most open ranking lists, rank change is an asymmetric function of $R$, with low values at the top of the ranking (low $R$) and high values elsewhere. Closed ranking lists, in turn, have symmetric $C$, with low rank change also at the bottom of the ranking (high $R$). In other words, most systems (except the most open datasets studied) have a stable ranking at the top where elements keep their rank most of the time. Yet closed ranking lists are also stable at the bottom, while open ranking lists have an unstable ranking at the bottom due to a constant flow of elements in and out of the system (Figs. S3-S4).

The qualitative behavior of $C$ is similar to that of rank diversity $d_R$, defined as the number of distinct elements occupying rank $R$ over all times $t=1,...,T-1$, normalized by $T$. Rank diversity has been previously introduced and studied by us in detail, particularly in the case of ranking lists in language and sports [10,11,35,36]. While both rank change and diversity capture the qualitative difference in ranking behavior between open and closed systems, rank change is more analytically tractable in our model (see Section S4). Following Eq. (S41), the model captures the functional shape and symmetry of
Figure S7. Success in open to closed ranking lists. Success $S_{t}^{++}$ as a function of normalized time $t/T$, calculated as the probability that an element in the top of the ranking at time $t'$ will stay there after $t$ observations, averaged over $t'$. The top of the ranking is defined as all ranks lower than an arbitrary threshold ($R = 1, \ldots, cN_0$), set here to $c = 1/2$. Success is shown for data (continuous lines) and the model of Section S4 with Eq. (S43) for $s = N$ (dashed lines) (for parameter fitting see Table 2 and Section S5). Datasets are ordered from most open (upper row) to closed (lower row) according to mean rank flux $F$ [Eq. (S2)]. In all ranking lists, the probability of staying in the top longer times decreases with $t$.

$C$ extremely well, for both open and closed ranking lists.

S3.6 Success

We further analyze the flux of elements within the ranking list by dividing it into two regions via an arbitrary threshold $c \in (0,1)$: we define the top (+) of the ranking as all ranks lower than a threshold ($R \leq cN_0$), and the bottom (−) as the rest of the elements in the ranking. We define the matrix element $S_{i}^{ij}$ as the probability that an element in region $i$ will move to region $j$ after $t$ observations (averaged over all compatible times $t' = 0, \ldots, T - t - 1$), with $i, j \in \{+, -\}$. The matrix element $S_{i}^{ij}$ characterizes the flux of elements within the ranking list over time periods of size $t$. We focus on success $S_{t}^{++}$, the probability that an element stays in the top of the ranking list over a time $t$. Success $S_{t}^{++}$ is a decreasing function of $t$ for all datasets: while in some ranking lists the probability of staying in the top is roughly constant, for most systems it decays linearly, exponentially, or with a long tail (Fig. S7). In all cases, rank success show the same qualitative behavior for intermediate values of the arbitrary threshold $c$ (defining the relative sizes of the top and bottom of the ranking), although for some ranking lists, $c$ has an effect on the functional form of $S_{t}^{++}$. The model of Section S4 captures the decay of $S_{t}^{++}$ with lag [Eq. (S43)], although there are deviations from the empirical values for most datasets.
S4  Model of rank dynamics

Here we introduce and explore in detail a minimal model for the dynamics of ranking lists. We derive approximate master equations and their solutions for both the probability of an element changing rank and some of the rank measures introduced in Section S3. Our aim is to show how closely numerical simulations of the model (and their analytical approximations) emulate microscopic and macroscopic properties of the ranking dynamics of the datasets described in Section S2.

S4.1 Model definition

Our model is simple and corresponds to the null hypothesis that rank dynamics is driven by two processes: i) random displacements of elements across the ranking list, leading to Lévi flight- and diffusion-like movement in rank (tuned by a model parameter $\tau \in [0,1]$); and ii) random replacement of elements by new ones, leading to an increase in rank flux and openness (tuned by a model parameter $\nu \in [0,1]$).

We assume that the model system is a list of $N \geq N_0$ elements (with $N$ a parameter of the model), and define the normalized rank $r = R\Delta r = \Delta r, 2\Delta r, \ldots, p, \ldots, 1$, with $\Delta r = 1/N$ and $p = N_0/N$. If $N > N_0$ the system is larger than its ranking list, in which case we choose to ‘ignore’ the elements in ranks $R = N_0 + 1, \ldots, N$ (despite their continuing dynamics), emulating lack of access to their scores in empirical data. To compare time scales between model and data, we assume that the real time interval between observations at times $t$ and $t+1$ corresponds to $N$ time steps in the model, defining the time step variable $s = 0, 1, \ldots, N, \ldots, T - 1$, and thus $t = s\Delta r = 0, \Delta r, \ldots, 1, \ldots, T - 1$. In other words, $t$ is a macroscopic time variable that describes the (finite number of) observations of the ranking list in the data, while $s$ is a microscopic time variable that describes the dynamics in the model in the time between empirical observations.

At each time step $s$ we perform two independent rank updates. The first update, happening with probability $\tau$, will cause Lévi flight- and diffusion-like dynamics. The second update, happening with probability $\nu$, will lead to the replacement of elements in the ranking.

For the first update, we select an element uniformly at random, remove it from the list and place it in one of the $N$ spaces to the right or left of the remaining $N-1$ elements. More precisely, we choose rank $R_{\text{old}} = 1, \ldots, N$ uniformly at random and take its element temporarily out of the system (thus making the rank change $R \mapsto R - 1$ for elements previously in ranks $R = R_{\text{old}} + 1, \ldots, N$). Then we re-introduce this element to the rank $R_{\text{new}} = 1, \ldots, N$ chosen uniformly at random (making the rank change $R \mapsto R + 1$ for elements previously in ranks $R = R_{\text{new}}, \ldots, N - 1$). In this way, elements initially between (and including) ranks $R_{\text{old}}$ and $R_{\text{new}}$ will change rank (if $R_{\text{old}} \neq R_{\text{new}}$), while elements outside this range will keep their rank $^3$ (Fig. S8).

In the second update, we choose a rank $R_{\text{rep}} = 1, \ldots, N$ uniformly at random and replace its element with a new element from outside the system, leaving the rest of the ranks untouched. The old element is removed from the system and ceases to have any dynamics, while the new element participates in the rank updates of time step $s + 1$ and beyond (Fig. S8).

Each empirical dataset allow us to fix $N_0$ and $T$ in the model (see Table 1). In what follows we will derive analytical approximations of the model dynamics for all values of the free parameters $\tau$, $\nu$ and $N$ (or, alternatively, $\tau$, $\nu$ and $p$). Still, it’s good to highlight the two parameter ranges that are sufficient to

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$^3$For a mathematical treatment of similar shuffling models, see [37].
Figure S8. Model of rank dynamics. At each time step of the model dynamics we perform two independent rank updates. (a) In the first update (random displacements of elements), an element (selected uniformly at random) is reallocated to a random position, also chosen uniformly at random. This element has a (potentially long) displacement in rank space, while the elements between the previous and new position of the selected element (here underlined) move only one rank. Other elements (i.e. in the boundaries of rank space) do not change rank. (b) In the second update (random replacement of elements), we replace an element selected uniformly at random by a new element not previously in the system.

emulate the dynamics of empirical open and closed ranking lists as described in Section S3. For closed ranking lists we take $N = N_0$ (i.e. $p = 1$) and $\nu = 0$, so the only free parameter is $\tau$. In this parameter range the ranking list covers the whole model system, the ranks of all elements are known at all times, no new elements enter the system, and the dynamics of rank is purely driven by Lévi flights and diffusion (as we see in Section S4.2 and Section S4.3). For open ranking lists we take $N = N_{T-1}$ (i.e. the model system size is the observable size of the empirical system, $p = 1/\nu_{T-1}$), so the free parameters are $\tau$ and $\nu$. In this parameter range the ranking list does not cover the whole model system ($N_{T-1} > N_0$), so elements may enter/leave the ranking list from/to the unobserved ranks $R = N_0 + 1, \ldots, N$ due to Lévi flights and diffusion, and new elements may replace old ones. Our model assumes that the flow of elements into and out of an open ranking list has two sources: one coming from a lack of score data outside the ranking list, and another due to birth/death of elements.

S4.2 Displacement probability

At each time step of the dynamics, with probability $\nu \Delta r$ the element in rank $R$ is replaced by a new element, thus leaving the system. Alternatively (with probability $1 - \nu \Delta r$), the element in rank $R$ is displaced to a new rank $X$ in the system with probability $\tau$, either by being randomly selected with the displacement update rule of the model (Lévi flight), or because of the rank change of another element (diffusion). Like with $R$, we can define the normalized rank $x = X \Delta r = \Delta r, 2 \Delta r, \ldots, p, \ldots, 1$. Picturing the rank space $R = 1, \ldots, N$ as a discrete line, negative/positive displacement means movement to the left/right of $R$ in the line (i.e. to more/less important ranks in the ranking list) \(^4\). We explore the temporal evolution of the model by formulating a master equation for the displacement probability $P_{x,t}$, the probability that an element in the normalized rank $r = R \Delta r$ moves to the normalized rank $x = X \Delta r$ after a time $t = s \Delta r$.

We first consider the displacement probability after a single time step of the dynamics takes place ($s = 1$), $P^s_x \equiv P_{x, \Delta r}$. An element in rank $r$ (that is not replaced) moves to rank $x$ with probability $\tau$, either by performing a Lévi flight (with probability $L$) or due to the rank change of another element (with probability $D^x_r$), or stays in place with probability $1 - \tau$. Then we have

$$P^s_x = (1 - \nu \Delta r) \left[ \tau (L + D^x_r) + (1 - \tau) \delta^x_r \right], \quad (S4)$$

with $\delta^x_r$ a Kronecker delta. Note that for $\tau = 1$ and $\nu = 0$ (in a closed ranking list with maximum

\(^4\)Note that the discrete rank space $r, x = \Delta r, \ldots, 1$ becomes the continuous interval $[0, 1]$ in the limit $N \to \infty$. 

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number of displacements in a given time interval), the single-step displacement probability reduces to $P_{x,t}^r = L + D_{x,r}$. The Lévi probability $L$ is straightforward to calculate: the element in rank $r$ gets picked up by the dynamics with probability $\Delta r$, and it is placed back in rank $x$ with the same probability $\Delta r$, since both processes are uniformly random in rank space. Thus,

$$L = \begin{cases} \Delta^2 & x \in \{\Delta r, 2\Delta r, \ldots, 1\} \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (S5)$$

Eq. (S5) explains our motivation to use the term Lévi flight: a Lévi flight is a random walk where displacement length follows a heavy-tailed probability distribution; since Eq. (S5) does not depend on the normalized displacement $d = x - r$, the probability $L$ can be thought of as a discrete power law with exponent $0$ (and cut-off due to finite system size) [38, 39]. In other words, our model implements maximally heavy-tailed Lévi flights where all displacements are equally likely, beyond diffusion.

The diffusion probability $D_{x,r}^r$ is slightly more involved. The element in rank $r$ moves one step to the right ($x = r + \Delta r$) if we choose an element to its right (with probability $1 - x + \Delta r$) and place it back to its left (with probability $x - \Delta r$). Conversely, the element in rank $r$ moves one step to the left ($x = r - \Delta r$) if we choose an element to its left (with probability $x$) and place it back to its right (with probability $1 - x$). The element at rank $r$ can stay in place ($x = r$) only if another element is picked and placed back to its left [with probability $(x - \Delta r)^2$] or to its right [with probability $(1 - x)^2$]. Joining these expressions we obtain

$$D_{x,r}^r = \begin{cases} x(1 - x) & x = r - \Delta r \\ (x - \Delta r)^2 + (1 - x)^2 & x = r \\ (x - \Delta r)(1 - x + \Delta r) & x = r + \Delta r \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (S6)$$

Note that $\sum_x D_{x,r}^r = 1 - \Delta r$. In other words, the probability loss from the diffusive process in Eq. (S6) goes to the uniform contribution made by the Lévi term in Eq. (S5), meaning that the continuous limit of Eq. (S6) does not exactly correspond to a diffusion equation. Moreover, $\sum_x P_{x,t}^r = 1 - \nu \Delta r < 1$ (for $\nu > 0$), which implies a global loss of probability reflecting the fact that we replace elements in the dynamics.

Moving forward in the dynamics ($t = s\Delta t \geq 0$), we write a recurrence relation for the displacement probability $P_{x,t}$. We observe that, for an element in rank $r$ to move to rank $x$ after $s$ time steps, it needs to have moved to an arbitrary rank $x' = \Delta r, \ldots, 1$ over the first $s - 1$ time steps, and then from $x'$ to $x$ in the last time step, i.e.,

$$P_{x,t} = \sum_{x'} P_{x',t-\Delta r} P_{x'}^x$$  \hspace{1cm} (S7)$$

with a Kronecker delta as initial condition ($P_{x,t} = \delta_x^r$), since at the start of the dynamics, an element in rank $r$ can only be in that rank ($x = r$). Note that Eq. (S7) for $s = 1$ recovers $P_{x,\Delta r} = P_{x}^r$ in Eq. (S4), as expected.

Let’s focus for a moment on the cumulative $P_t = \sum_x P_{x,t}$, the probability that, after a time $t$, the element in rank $r$ moves to any other rank in the system. Summing up over $x$ in Eq. (S7) gives the recurrence relation $P_t = (1 - \nu \Delta r)P_{t-\Delta r}$, with initial condition $P_0 = 1$ (due to $P_{x,0} = \delta_x^r$) and
\[ P_{\Delta r} = 1 - \nu \Delta r. \] Solving the master equation gives

\[ P_t = (1 - \nu \Delta r)^s \simeq e^{-\nu s \Delta r} = e^{-\nu t}. \] (S8)

The approximation in Eq. (S8) comes from using the time scale relation \( t = s \Delta r \), the binomial theorem, and the power series of the exponential, and becomes exact in the limit \( N \to \infty \). In other words, the probability of an element staying in the system under a dynamics of replacement (\( \nu > 0 \)) decays exponentially with time. When there is only element displacement (\( \nu = 0 \)), \( P_t = 1 \) and the ranking list is indeed closed.

Inserting Eqs. (S4)-(S6) into Eq. (S7) we obtain a master equation for the displacement probability after \( s \) time steps,

\[ P_{x,t} = (1 - \nu \Delta r)^s \{ P_{x,t} - \Delta r + \tau \left[ \Delta r^2 (1 - \nu \Delta r)^{s-1} + x(1-x)P_{x+\Delta r,t-\Delta r} \right. \right. \\
- 2 \left[ x(1 - x + \Delta r) - \frac{1}{2} \Delta r^2 \right] P_{x,t-\Delta r} + (x-\Delta r)(1 - x + \Delta r)P_{x-\Delta r,t-\Delta r} \}. \] (S9)

Eq. (S9) is reminiscent of a discretized diffusion equation in one dimension (the rank space \( x \)). The multiplicative factor \( 1 - \nu \Delta r \) represents the dynamics of elements being replaced, thus decreasing their probability of staying in the system as \( s \) increases. The second factor on the right hand side accounts for elements being selected by the displacement dynamics of the model and moving (possibly long) distances in rank space as Lévy flights. The last three terms on the right hand side describe a (rank-dependent) local, diffusive movement of elements in the system due to other elements performing Lévy flights. The time scale of both Lévy flights and diffusion-like movement is regulated by \( \tau \).

We can actually measure the displacement probability \( P_{x,t} \) for \( t = 1 \) directly from empirical data as the probability that the element at rank \( R \) at time \( t' - 1 \) moves to rank \( X \) at time \( t' \), averaged over all times \( t' = 1, \ldots, T - 1 \) (Fig. S9). The use of an average over time is supported by the stationarity of the empirical rank dynamics as seen in the roughly constant flux time series of most datasets (see Fig. S3). As we will see in Section S4.3 below, the empirical displacement probability resembles either a flat Lévy sea or a diffusion peak, respectively corresponding to late and early times of the rank dynamics in the model. By calculating \( P_{x,t} \) in the model for \( s = N \) (i.e. between times \( t = 0 \) and \( t = 1 \)), we see that the model reproduces the empirical displacement probability very well for most datasets and any rank [see Eq. (S22)].

As seen in Eq. (S8), the probability \( P_t = \sum_x P_{x,t} \) of an element staying in the system for \( \nu > 0 \) decays exponentially with time, i.e. \( P_t \) is not conserved. In order to approximately solve Eq. (S9) with a diffusion ansatz (see Section S4.3), we can renormalize \( P_{x,t} \) by introducing the probability distribution

\[ Q_{x,t} = \frac{P_{x,t}}{(1 - \nu \Delta r)^s} \simeq P_{x,t} e^{\nu t}, \] (S10)

which is indeed conserved, since \( \sum_x Q_{x,t} = 1 \) for all \( t \). As before, the approximation holds for \( t = s \Delta r \).
Figure S9. Displacement probability in open to closed ranking lists. Displacement probability $P_{x,t}$ for $t = 1$ as a function of normalized rank $X/N_0$ for various $R$ values (colored lines), calculated as the probability that the element at rank $R$ at time $t' - 1$ moves to rank $X$ at time $t'$, averaged over all times $t'$. $P_{x,t}$ is shown for data (continuous lines) and the model of Section S4 with Eq. (S22) for $s = N$ (dashed lines) (for parameter fitting see Table 2 and Section S5). Datasets are ordered from most open (upper row) to closed (lower row) according to mean rank flux $F$ [Eq. (S2)]. In some datasets $P_{x,t}$ is roughly flat, reminiscent of a uniform Lévy sea, while in others the displacement probability resembles a diffusion peak (see Section S4.3). To smooth the curves, empirical data is passed through a Savitzky–Golay filter (with polynomial order 2 and filter window length ranging from $N_0/200$ to $N_0/10$).

and is exact for $N \to \infty$. Substituting Eq. (S10) into Eq. (S9) gives

$$Q_{x,t} = Q_{x,t-\Delta t} + \tau \left\{ \Delta r^2 + x(1-x)Q_{x+\Delta r,t-\Delta t} - 2 \left[ r(1-x+\Delta r) - \frac{1}{2} \Delta r^2 \right] Q_{x,t-\Delta t} + (x-\Delta r)(1-x+\Delta r)Q_{x-\Delta r,t-\Delta t} \right\}.$$ (S11)

Eq. (S11) is a simplified master equation for the displacement probability that effectively decouples the dynamics of displacement ($\tau$) and replacement ($\nu$) in the model. We can first look for an approximate solution $Q_{x,t}$ of the displacement dynamics governed by Eq. (S11), and then consider the replacement dynamics explicitly with $P_{x,t} = Q_{x,t} e^{-\nu t}$.

S4.3 Approximation for displacement probability

Rather than solving Eq. (S11) explicitly, we derive an approximate expression for the renormalized displacement probability $Q_{x,t}$ that becomes more accurate as $N \to \infty$. Considering the shape of Eq. (S4) (i.e. for $s = 1$), we propose an ansatz of $Q_{x,t}$ for any $t$ separated into Lévy flight and diffusive components,
\[ Q_{x,t} = L_t + D_{x,t}. \]  

As we will see below, the probability \( L_t \) of moving across rank space in Lévy flights is rank- and displacement-independent and grows slowly in time (a uniform Lévy sea), while the probability \( D_{x,t} \) of diffusing in rank space due to the movement of other elements is approximately Gaussian-distributed in \( x \) (a diffusion peak that widens and decreases in height as time goes by) (Fig. S10).

First, we write a recurrence relation for \( L_t \) by noting that after a time \( t = s\Delta r \), an element initially in rank \( r \) can only move due to diffusion within the \( 2s + 1 \) ranks around \( r \), so \( Q_{x,t} > L_t \) for \( |x - r| \leq t \). Conversely, the only way for an element to move more than \( r \) in rank is by Lévy flight, meaning \( Q_{x,t} = L_t \) for \( |x - r| > t \). Rewriting Eq. (S11) for \( |x - r| > t \) we obtain

\[ L_t = \tau \Delta r^2 + (1 - \tau \Delta r)\Delta r, \]  

a recurrence relation for \( L_t \) with initial condition \( L_0 = 0 \) (since at the start of the dynamics no Lévy flights have yet occurred), which consistently gives \( L_{\Delta r} = \tau L \) as in Eqs. (S4)-(S5). Eq. (S13) can be directly solved and gives an exact expression for the probability of moving across rank space in Lévy flights,

\[ L_t = \Delta r[1 - (1 - \tau \Delta r)^t] \simeq \Delta r(1 - e^{-\tau t}), \]  

with \( t = s\Delta r \) and the approximation improving as \( N \to \infty \). In other words, the Lévy sea in Eq. (S14) is a uniform probability (both in ranks \( r \), \( x \) and displacement \( d = x - r \) that increases asymptotically from \( \tau \Delta r^2 \) to \( \Delta r \) as \( t \to \infty \).

By inserting Eqs. (S12)-(S13) into Eq. (S11), we find a master equation for the probability \( D_{x,t} \) of diffusing in rank space due to the movement of other elements,

\[ D_{x,t} = D_{x,t-\Delta r} + \tau \left\{ x(1-x)D_{x+\Delta r,t-\Delta r} - 2\left[ x(1-x+\Delta r) - \frac{1}{2}\Delta r^2 \right] D_{x,t-\Delta r} + (x-\Delta r)(1-x+\Delta r)D_{x-\Delta r,t-\Delta r} \right\}, \]  

with initial condition \( D_{x,0} = \delta_r^x \). Just like Eq. (S11), Eq. (S15) is difficult to solve exactly for any \( t \). We can, however, write an expression for \( A_t = \sum_x D_{x,t} \), the area under the diffusion peak [disregarding the Lévy sea; see Eq. (S12)]. First, we know that \( \sum_x Q_{x,t} = 1 \) due to its definition in Eq. (S10). Then, by plugging Eq. (S14) into this normalization condition we obtain

\[ A_t = \sum_x D_{x,t} = (1 - \tau \Delta r)^t \simeq e^{-\tau t}, \]  

with \( t = s\Delta r \) and the approximation getting better as \( N \to \infty \). In other words, \( L_t = \Delta r(1 - A_t) \). Intuitively, the area \( A_t \) of the diffusion peak decreases exponentially as time goes by, ‘leaking probability’ into a Lévy sea that increases in height with \( t \).

### S4.3.1 Continuum limit for diffusion peak

We may find an explicit, approximate solution for the diffusion peak \( D_{x,t} \) by analyzing the continuum limit of Eq. (S15) as \( N \to \infty \). As mentioned in Section S4.1, in order to match the time scales between model \( (s) \) and data \( (t) \) we take \( t = s\Delta r \) with \( \Delta r = 1/N \). We also use the ansatz \( Q_{x,t} = L_t + D_{x,t} \) with the Lévy sea \( L_t \) given by Eq. (S14). Since the diffusion peak leaks probability into the Lévy sea as time
goes by [see Eq. (S16)], we further propose the ansatz
\[
D_{x,t} = (1 - \tau \Delta r)^s D(x,s \Delta r) \Delta r \simeq e^{-\tau t} D(x,t) dx,
\]
where \( D(x,t) \) is an unknown probability density, \( x \) is taken as a continuous variable, and the approximation improves for larger \( N \). Inserting Eq. (S17) into Eq. (S15) we obtain a master equation for \( D(x,t) \),
\[
\frac{\partial D}{\partial t} = \alpha x(1-x) \frac{\partial^2 D}{\partial x^2}.
\]
Eq. (S18) is a diffusion-like equation with a quadratic, rank-dependent diffusion coefficient \( \alpha x(1-x) \) and \( \alpha \simeq \tau \Delta r \) (for large but finite \( N \)). Note that the probability density \( D(x,t) \) decreases in time as the Lévy sea increases, so Eq. (S18) is not a standard diffusion equation. If we would modify Eq. (S6) to force \( A_t = 1 \) by, say, adding a term \( \Delta r \) in \( D_r \), we would obtain a diffusion equation in which the rank-dependent coefficient is inside the first rank derivative. The initial condition \( D_{x,0} = \delta_x \) in Eq. (S15) leads to the initial condition \( D(x,0) = \delta(x-x_0) \) (with \( x_0 \equiv r \in [0,1] \) the initial rank of the element) and to the Dirichlet boundary conditions \( D(0,t) = D(1,t) = 0 \). Notice that Eq. (S18) is the Wright–Fisher equation [40], a continuous model of genetic drift that has been extensively studied from a mathematical point of view (see [41–43] and references therein).

We can solve Eq. (S18) exactly by separation of variables. Proposing the ansatz \( D(x,t) = u(x)v(t) \) leads to \( v(t) = v(0)e^{-\lambda t} \) with \( \lambda \) a separation constant, while \( u(x) \) needs to satisfy the eigenvalue equation
\[
\alpha x(1-x)u'' + \lambda u = 0.
\]
By using the Frobenius method and the boundary conditions of Eq. (S18), we determine the allowed values of \( \lambda \) implicitly and the associated eigenfunctions \( u(x) \) as infinite series, over which the initial condition \( D(x,0) \) can be expanded to obtain the particular solution of Eq. (S18) we are looking for.

However, it may be more instructive to find a closed-form approximate solution \( D(x,t) \) that lets us qualitatively understand the behaviour of the displacement probability in rank and time. In a small enough interval around the initial rank \( x_0 = r \), we can approximate \( x(1-x) \simeq x_0(1-x_0) \) in Eq. (S18), which leads to the standard diffusion equation with diffusion coefficient \( \alpha x_0(1-x_0) \). Then, the fundamental solution (in the infinite rank domain \( -\infty < x < \infty \)) for \( D(x,0) = \delta(x-x_0) \) is
\[
D(x,t) = \frac{1}{\sigma_t \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x-x_0}{\sigma_t} \right)^2 \right] \equiv G(x_0, \sigma_t^2),
\]
the Gaussian distribution with mean \( x_0 \) and time-dependent standard deviation \( \sigma_t = \sqrt{2\alpha x_0(1-x_0)t} \), which spreads symmetrically around \( x_0 \) as time goes by \(^5\).

Finally, we derive an approximate expression for the displacement probability \( P_{x,t} = Q_{x,t} e^{-\nu t} \) that becomes more accurate as \( N \to \infty \). Inserting Eq. (S20) into Eq. (S17), and using the ansatz of Eq. (S12) alongside Eq. (S14) we obtain
\[
P_{x,t} \simeq e^{-\nu t} \left[ L_t + e^{-\tau t} G(r, \sigma_t^2) \Delta r \right],
\]
\(^5\)Note that the Gaussian approximation fails progressively as we approach the ‘tails’ of the solution \( D(x,t) \) at \( x \simeq 0,1 \), both due to the failing assumption \( x(1-x) \simeq x_0(1-x_0) \) and to the mismatch between the boundary conditions of Eq. (S18) and the infinite rank domain of Eq. (S20). The overall effect is more accuracy for intermediate values of \( x_0 = r \) rather than at the extremes of the interval \( [0,1] \) (Fig. S10).
Figure S10. Displacement probability in model. Displacement probability \( P_{x,t} \) as a function of normalized rank \( x/p \) for \( t = 1 \) \((s = N)\), with \( N = 10^2 \) and \( p = 0.8 \), in numerical simulations of the model (dots) and the analytical approximation of Eq. (S22) (lines). (top row) Dependence of \( P_{x,t} \) on \( \tau \) for fixed \( \nu = 0.5 \), with initial rank \( r = 0.08 \) (left) and 0.4 (right). As \( \tau \) increases, the diffusion peak around \( x = r \) (dashed line) widens and decreases in height, while the uniform Lévy sea grows according to Eq. (S14). (bottom row) Dependence of \( P_{x,t} \) on \( \nu \) for fixed \( \tau = 0.5 \), with initial rank \( r = 0.08 \) (left) and 0.4 (right). As \( \nu \) increases, the displacement probability is exponentially lower due to element replacement [see Eq. (S8)]. Eq. (S22) is accurate even for relatively low \( N = 10^2 \). Note that the probability peak is less diffusive at the extremes of rank space (left) than at the center (right). Simulations are averages over \( 10^5 \) realizations (using \( T = 10 \)).

with \( t = s \Delta r \), or writing everything explicitly,

\[
P_{x,t} \simeq e^{-\nu t} \left[ \Delta r(1 - e^{-\tau t}) + \sqrt{\frac{\Delta r}{4\pi \tau(1-r)t}} \exp \left( -\frac{(x-r)^2}{4\tau r(1-r)t\Delta r - \tau t} \right) \right].
\]  (S22)

Eq. (S21) captures the approximate behavior of the displacement probability \( P_{x,t} \) intuitively: a Gaussian diffusion peak \( G(r, \sigma^2_t) \) widening in time and leaking probability to the uniform Lévy sea \( L_t \), all regulated by an exponential loss in probability due to new elements. Eq. (S22) is a good approximation for \( P_{x,t} \) even for relatively low \( N = 10^2 \), as we can see by comparing with numerical simulations of the model described in Section S4.1 for \( s = N \) (i.e. \( t = 1 \)) (Fig. S10). Eq. (S22) also shows that the model captures the displacement probability of the empirical datasets very well (see Fig. S9).

S4.4 Approximation for rank flux

Beyond the microscopic description of the dynamics given by the displacement probability \( P_{x,t} \), we also explore the temporal evolution of the model by approximating the rank measures introduced in Section S3 with closed expressions. We start with the mean rank flux \( F \), measured for data as the probability that any element in the ranking list at time \( t - 1 \) leaves the ranking at time \( t \), averaged over all recorded (observable) elements in the ranking list and over time [see Section S3.3 and Eq. (S2)]. To find \( F \) in the model, we define the time-dependent flux \( F_t \) as the probability that a given element in rank \( r \leq p \) leaves
**Figure S11. Rank flux in model.** Rank flux $F$ as a function of $\tau$ for fixed $\nu$ (left), and as a function of $\nu$ for fixed $\tau$ (right), in both numerical simulations of the model (dots) and the analytical approximation of Eq. (S27) (lines) for varying ranking list size $p$. Rank flux increases with both $\tau$ and $\nu$, since the displacement/replacement dynamics takes elements out of the ranking list. A relatively longer ranking list (large $p$) makes it less likely that elements leave the ranking list. Simulations are averages over $10^2$ realizations (with $T = 10$) in a system of size $N = 10^4$.

any of these ranks after time $t = s\Delta r$ (either by displacement or by replacement). Following Eq. (S10), flux is given by

$$F_t = 1 - \sum_{x=\Delta r}^p P_{x,t} = 1 - (1 - \nu \Delta r)^a \sum_{x=\Delta r}^p Q_{x,t}. \quad (S23)$$

where the step size in the sum is $\Delta r$, i.e. $x = \Delta r, 2\Delta r, \ldots, p$.

We now find a master equation for the partial cumulative $Q_{[a,b],t} \equiv \sum_{x=a}^b Q_{x,t}$ for arbitrary $a, b = \Delta r, \ldots, 1$ and $b > a$. Using Eq. (S11), summing over $x$, and changing dummy indices, we obtain

$$Q_{[a,b],t} = (1 - \tau \Delta r)Q_{[a,b],t-\Delta r} + \tau [(b - a + \Delta r)\Delta r + (a - \Delta r)(1 - a + \Delta r)(Q_{a-\Delta r,t-\Delta r} - Q_{a,t-\Delta r}) + (1 - b)(Q_{b+\Delta r,t-\Delta r} - Q_{b,t-\Delta r})]. \quad (S24)$$

Eq. (S24) is not closed on the cumulative $Q_{[a,b],t}$, since it depends directly on the renormalized displacement probability (last two terms on the right side of the equation). To close the master equation we make the approximations $Q_{a-\Delta r,t} \simeq Q_{a,t}$ and $Q_{b+\Delta r,t} \simeq Q_{b,t}$ for all $t$ (accurate as long as the diffusion peak of $P_{x,t}$ is far enough from $a$ or $b$), which leads to

$$Q_{[a,b],t} = (1 - \tau \Delta r)Q_{[a,b],t-\Delta r} + \tau (b - a + \Delta r)\Delta r. \quad (S25)$$

We go back to the particular case of the flux $F_t$. With $a = \Delta r$, $b = p$, and the initial condition $Q_{[\Delta r,p],0} = 1$ (since the element starts off within the ranking list), we can solve Eq. (S25) using common

---

This general notation will be useful to determine both the flux $F_t$ via Eq. (S23) by setting $a = \Delta r$ and $b = p$, as well as the success $S^{++}_1$ via Eq. (S42) by setting $a = \Delta r$ and $b = cp$ (see Section S4.7 below).
Figure S12. Rank out-flux in model. Rank out-flux $F_{r,t}$ as a function of normalized rank $r/p$ for $t = 1$ ($s = N$) in both numerical simulations of the model (dots) and the analytical approximation of Eq. (S29) (lines) for fixed ranking list size $p$. Results correspond to varying $\tau$ for fixed $\nu$ (left), and to varying $\nu$ for fixed $\tau$ (right). Rank out-flux increases with both $\tau$ and $\nu$, since the displacement/replacement dynamics takes elements out of the ranking list. $F_{r,t}$ is also relatively constant across the ranking list, except for bottom ranks ($r \sim p$) where out-flux increases. Simulations are averages over $10^5$ realizations (with $T = 10$) in a system of size $N = 10^2$.

Formulas for geometric series to obtain

$$\sum_{x=\Delta r}^{p} Q_{x,t} = p(1-p)(1-\tau \Delta r)^{x} \simeq p + (1-p)e^{-\tau t}$$  \hspace{1cm} (S26)

for $t = s \Delta r$ and sufficiently large $N$. Due to the approximation in Eq. (S25), $Q_{[a,b],t}$ and $F_t$ do no longer depend explicitly on $r$. For $t = 1$ (i.e. $s = N$) we finally obtain

$$F \simeq 1 - e^{-\nu}[p + (1-p)e^{-\tau}].$$  \hspace{1cm} (S27)

Additionally, we define $F = 0$ for $p = 0$ for consistency. Eq. (S27) is an analytical approximation for the probability that any element in the ranking list at time $t - 1$ leaves the ranking at time $t$, according to our model. Intuitively, flux $F$ increases with $\tau$ and $\nu$, since more displacements or replacements make it more likely that elements will leave the ranking list. Conversely, larger $p$ makes the ranking list longer (relative to system size) and thus less likely to lose elements due to out-flux. Eq. (S27) is a good approximation of $F$ for all parameters considered, as we can see by comparing with numerical simulations of the model described in Section S4.1 (Fig. S11). Eq. (S27) also shows that the model quantitatively reproduces the empirical flux of most datasets very well (see Fig. S3).

As described in Section S3.3, in the data we can also measure what part of a ranking list contributes most to the flow of elements out of it by calculating the out-flux $F_{R,t}$: the probability that the element in rank $R$ at time $t - 1$ leaves the ranking list at time $t$, averaged over all observed times. Similarly, in the model we define out-flux $F_{r,t}$ as the probability that the element in the normalized rank $r$ leaves the ranking list after time $t = s \Delta r$. We can compare $F_{R,t}$ in the data with $F_{r,t}$ for $t = 1$ (i.e. $s = N$) in the model. An element contributes to out-flux either because it is replaced with a new element, or because

$$\sum_{x=\Delta r}^{p} Q_{x,t} = p(1-p)(1-\tau \Delta r)^{x} \simeq p + (1-p)e^{-\tau t}$$  \hspace{1cm} (S26)

for $t = s \Delta r$ and sufficiently large $N$. Due to the approximation in Eq. (S25), $Q_{[a,b],t}$ and $F_t$ do no longer depend explicitly on $r$. For $t = 1$ (i.e. $s = N$) we finally obtain

$$F \simeq 1 - e^{-\nu}[p + (1-p)e^{-\tau}].$$  \hspace{1cm} (S27)

Additionally, we define $F = 0$ for $p = 0$ for consistency. Eq. (S27) is an analytical approximation for the probability that any element in the ranking list at time $t - 1$ leaves the ranking at time $t$, according to our model. Intuitively, flux $F$ increases with $\tau$ and $\nu$, since more displacements or replacements make it more likely that elements will leave the ranking list. Conversely, larger $p$ makes the ranking list longer (relative to system size) and thus less likely to lose elements due to out-flux. Eq. (S27) is a good approximation of $F$ for all parameters considered, as we can see by comparing with numerical simulations of the model described in Section S4.1 (Fig. S11). Eq. (S27) also shows that the model quantitatively reproduces the empirical flux of most datasets very well (see Fig. S3).

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it moves out of the ranking list via the displacement dynamics. Thus,

\[ F_{r,t}^- = 1 - (1 - \nu \Delta r)^s + \sum_{x=p+\Delta r}^1 P_{x,t}, \]  

where the displacement probability \( P_{x,t} \) is given explicitly by Eq. (S22) in the limit of large \( N \). Approximating the sum in Eq. (S28) by an integral in the interval \( x \in [p, \infty] \), changing variables and integrating by parts, we obtain

\[ F_{r,t}^- \simeq 1 - e^{-\nu t} \left\{ 1 - (1 - p)(1 - e^{-\tau t}) - \frac{1}{2} e^{-\tau t} \left[ 1 - \text{erf} \left( \frac{p-r}{2\sqrt{\tau r(1-r)\Delta r}} \right) \right] \right\}, \]  

with \( t = s \Delta r \) and \( \text{erf} \) the error function. In the case \( \tau = 0 \), \( P_{x,t}^r = (1 - \nu \Delta r) \delta_x^r \) in Eq. (S4) leads to \( F_{r,t}^- \simeq 1 - e^{-\nu t} \), i.e. a constant out-flux with respect to rank \( r \). Eq. (S29) is a very good approximation for out-flux in the model, showing how \( F_{r,t}^- \) increases with both \( \tau \) and \( \nu \) (due to enhanced displacement and replacement dynamics), see Fig. S12. Out-flux is also relatively constant across the ranking list, except for bottom ranks (\( r \sim p \)) where it increases as we approach the end of the ranking list. Eq. (S29) reproduces the empirical out-flux of several datasets very well (Fig. S4).

S4.5 Approximation for rank openness

Here we derive an approximate, closed expression for the rank openness \( o_t = N_t/N_0 \), where \( N_t \) is the number of distinct element that have been in the ranking list up to (and including) time \( t \) [see Eq. (S3)]. Similarly, in the model \( N_t \) is the number of distinct elements that have been in the ranking list (\( r = \Delta r, \ldots, p \)) at any time step \( s' \leq s \) (with \( t = s \Delta r \)). Thus, our task is to find an explicit expression for \( N_t \). We start by introducing the probability \( p_t \) that a randomly chosen element has been in the ranking list at any time \( t' \leq t \),

\[ p_t = N_t/M_t, \]  

where \( M_t \) is the number of distinct elements that have been in the whole system at any time \( t' \leq t \). Since the replacement dynamics adds one new element every time step with probability \( \nu \), on average we have \( M_t \simeq N + \nu s \), which leads to

\[ N_t \simeq (N + \nu s)p_t. \]  

In order to find \( p_t \), we write a master equation for the probability \( p_{x,t} \) that the element in any rank \( x \) of the model system\(^7\) has been in the ranking list at any time \( t' \leq t \), which we average to obtain

\[ p_t = \Delta r \sum_x p_{x,t}. \]

We first note that the initial condition for \( p_{x,t} \) is

\[ p_{x,0} = \begin{cases} 1 & x = \Delta r, \ldots, p \\ 0 & \text{otherwise} \end{cases}, \]

a step function over rank space. Within the ranking list, \( p_{x,t} \) is always 1 and does not change in time, i.e., \( p_{x,t} = p_{x,t-\Delta r} = 1 \) for \( x = \Delta r, \ldots, p \) and \( t \geq \Delta r \). In the rest of the system, \( p_{x,t} \) increases from its initial condition 0. We now write the corresponding equation for elements outside of the ranking list, in

\[ \text{Eq. (S32)} \]

\[^7\text{Eq. (S32)}\]
a similar way to Eq. (S7). For an element in rank \( x \) to have ever been in the ranking list, it must have belonged to the ranking list at some point in its dynamics, and finally moved from some rank \( r \) to rank \( x \) in one time step. We can thus write

\[
p_{x,t} = \sum_{r=\Delta r}^{1} P_{x}^{r} p_{r,t-\Delta r}, \quad \forall x > p.
\]  

(S33)

With Eqs. (S4)-(S6) and some simplification we obtain a master equation for \( p_{x,t} \),

\[
p_{x,t} = (1 - \nu \Delta r) \left\{ p_{x,t-\Delta r} + \tau \left[ p_{t-\Delta r} \Delta r + x(1-x)p_{x+\Delta r,t-\Delta r} + \frac{2}{2} x(1-x) + \Delta r \Delta r \right] \right\}, \quad \forall x > p. \quad \text{(S34)}
\]

Eq. (S34) is a discrete, diffusion-like equation similar to Eq. (S9), except that a term with \( p_{t-\Delta r} \) couples the equations for all \( x \). The temporal evolution of \( p_{x,t} \) starts from a step function in rank space at \( t = 0 \) [Eq. (S32)], after which the front \( p_{p,t} = 1 \) propagates to larger ranks \( x > p \), making \( p_{x,t} \) increase. At the same time, the replacement dynamics (regulated by \( \nu \)) decreases the probability that elements visit the ranking list. The asymptotic final state of the dynamics \( p_{x,t} \) for \( t \to \infty \) depends on the parameters \( \tau \), \( \nu \), and \( p \) [see Eq. (S37) below], as well as on the rank \( x \).

Instead of solving Eq. (S34) exactly, we find an approximate analytical solution for the corresponding master equation of \( p_{t} \). Summing up \( x \) both Eq. (S34) and the trivial solution for \( x = \Delta r, \ldots, p \) [Eq. (S32)], we obtain

\[
p_{t} = p \left[ 1 - (1 - \nu \Delta r)(1 - \tau \Delta r) \right] + (1 - \nu \Delta r)(1 - pr \Delta r)p_{t-\Delta r}, \quad \text{(S35)}
\]

where we have made the approximation \( 1 = p_{p,t} \approx p_{p+\Delta r,t} \) in order to close the equation. We explicitly solve Eq. (S35) by recursion from the initial condition \( p \) via a geometric series,

\[
\frac{p_{t}}{p} = e^{-\nu \Delta r t} + \frac{\nu + \tau}{\nu + pr}(1 - e^{-\nu \Delta r t}), \quad \text{(S36)}
\]

which leads to the asymptotic value

\[
\frac{p_{t}}{p} = \frac{\nu + \tau}{\nu + pr}, \quad t \to \infty. \quad \text{(S37)}
\]

Eq. (S37) clarifies the range of accuracy of the approximation \( p_{p+\Delta r,t} \approx 1 \). Depending only on the parameters of the model and as long as \( t \) is sufficiently large, if the fraction in the right hand side of Eq. (S37) is close to \( 1/p \), then Eq. (S35) is an accurate master equation for \( p_{t} \). This typically happens for small \( \tau \) or \( \nu \) (see Fig. S13), or for \( p \) close to 1.

With Eq. (S31), Eq. (S3), and the explicit expression for \( p_{t} \) in Eq. (S36), we finally obtain

\[
o_{t} \simeq (1 + \nu t) \left[ e^{-(\nu + pr)t} + \frac{\nu + \tau}{\nu + pr}\left(1 - e^{-(\nu + pr)t}\right)\right], \quad \text{(S38)}
\]

with \( t = s\Delta r \) and sufficiently large \( N \), an analytical approximation for the fraction of elements in the model that visit its ranking list as time goes by. Eq. (S38) reproduces openness in numerical simulations of the model reasonably well (Fig. S13). In agreement with empirical measurements of \( o_{t} \) (see Fig. S5),
Eq. (S38) starts off with a concave shape from its initial condition $o_0 = 1$, and has a linear behavior for long times,

$$o_t \simeq \frac{\nu + \tau}{\nu + \nu / \tau} (1 + \nu t), \quad t \gg 0.$$  \hfill (S39)

where the slope of the line is regulated by the same fraction we see in Eq. (S37). Moreover, the mean openness rate after $t$ observations, $\dot{o}_t = (o_t - o_0)/t$, has an asymptotic value $\dot{o} \equiv \dot{o}_{T-1}$ given by

$$\dot{o} \simeq \frac{\nu + \tau}{\nu + \nu / \tau}, \quad T \gg 0.$$  \hfill (S40)

**S4.6  Approximation for rank change**

The approximation for the displacement probability $P_{x,t}$ in Eq. (S22) leads us directly to an explicit expression for the rank change $C$, defined for data as the probability that elements in rank $R$ at times $t - 1$ and $t$ are not the same, averaged over all times (see Section S3.5 and Fig. S6). In the model we define $C_t$ as the probability that an element located in rank $r$ changes place after time $t$ (we will finally compare $C$ with $C_t$ for $t = 1$, i.e. $s = N$). Since $C_t = 1 - P_{r,t}$, we have

$$C_t \simeq 1 - e^{-\nu t} \left[ \Delta r (1 - e^{-\tau t}) + \sqrt{\frac{\Delta r}{4\pi \nu (1 - r) t} e^{-\tau t}} \right]$$  \hfill (S41)

for $t = s\Delta r$ and sufficiently large $N$. Eq. (S41) is a good approximation for rank change in the model that fails slightly at the extremes of the ranking list (Fig. S14). In agreement with empirical data (see Fig. S6), $C_t$ has a symmetric concave shape as a function of $r$, which looks asymmetric if the system is open ($p < 1$). Thus, in closed systems, the probability of changing rank is larger for elements in the middle of the ranking list, and lower at its extremes. In open systems, the bottom of the ranking list
Figure S14. Rank change in model. Rank change $C_t$ as a function of normalized rank $r/p$ for $t = 1$ ($s = N$) in both numerical simulations of the model (dots) and the analytical approximation of Eq. (S41) (lines) for fixed ranking list size $p$. Results correspond to varying $\tau$ for fixed $\nu$ (left), and to varying $\nu$ for fixed $\tau$ (right). Rank change increases with both $\tau$ and $\nu$, since the displacement/replacement dynamics moves elements away from their previous ranks. $C_t$ has a symmetric concave shape across the ranking list, which looks asymmetric if the system is open ($p < 1$). Simulations are averages over $10^5$ realizations (with $T = 10$) in a system of size $N = 10^2$. 

(r $\sim$ p) is also unstable.

S4.7 Approximation for success

Here we derive an approximate, closed expression for the success measure $S_t^{++}$ introduced in Section S3.6. Similarly to the case of empirical data, in the model we define the matrix element $S_{ij}^t$ as the probability that an element in region $i$ will move to region $j$ after a time $t = s\Delta r$. Regions $i, j$ are divided by an arbitrary threshold $c = \Delta r/p, \ldots, 1$ into the top (+) of the ranking ($r = \Delta r, \ldots, cp$) and the bottom (−) of the ranking ($r = cp + \Delta r, \ldots, p$). Success $S_t^{++}$ is then given by

$$S_t^{++} = \sum_{x=\Delta r}^{cp} P_{x,t} = (1 - \nu \Delta r)^s \sum_{x=\Delta r}^{cp} Q_{x,t},$$

with $r = \Delta r, \ldots, cp$. The sum in the right hand side of Eq. (S42) is a particular case of the partial cumulative $Q_{[a,b],t} = \sum_{x=a}^{b} Q_{x,t}$ we introduced in Section S4.4, with $a = \Delta r$ and $b = cp$. We can solve directly the associated recurrence relation [Eq. (S25)] with initial condition $Q_{[\Delta r,cp],0} = 1$ using simple formulas for geometric series. Substituting the solution into Eq. (S42) we finally obtain

$$S_t^{++} \simeq e^{-\nu t} [cp + (1 - cp)e^{-\tau t}],$$

for $t = s\Delta r$ and sufficiently large $N$. Eq. (S43) is an analytical approximation for the probability that elements will stay in the top of the ranking across arbitrarily long times, according to our model. Success $S_t^{++}$ has an exponential decay with time $t$, regulated by both $\tau$ and $\nu$, which matches numerical simulations of the model very well. Denoting $S^{++} \equiv S_t^{++}$ for $t = 1$ (i.e. $s = N$), we see that success decreases as $\tau$ and $\nu$ increase (since there are more displacements and replacements of elements), and
**S5 Fitting data with model**

Here we describe the process of fitting the rank measures of all considered datasets with the minimal model of rank dynamics introduced in Section S4. First, we set $N_0$, $T$ and the time series $N_t$ with the empirical values of each dataset listed in Table 1. As described in Section S1 and Section S4.1, we classify ranking lists as closed if $N_t = N_0$ (for all $t$) and as open otherwise. Then, closed ranking lists have flux $F_t = 0$, out-flux $F^-_R = 0$, and openness $a_t = 1$ for all $t$ and $R$, while open ranking lists have non-zero flux and out-flux, and openness that increases in time. We find five strictly closed systems [Metro stations (London and Mexico), Cities (GB), and Regions JP (quake mag and quakes)], while the rest have varying...
Figure S16. Fitting data with model in open ranking lists. (left) Graphical solution of the transcendental equation for \( \nu \) defined by Eq. (S45). The right hand side (rhs) of Eq. (S45b) has a singularity at \( \nu = \dot{p} \) and crosses the identity in the left hand side (lhs) of Eq. (S45b) in two fixed points. We only consider the solution \( \nu^* \) in the interval that ensures \( \tau > 0 \) in Eq. (S45a) (shaded area). (right) Eq. (S45a) for rescaled \( \tau \) and \( \nu \) [see Eq. (S47)], which allows us to find \( \tau^* \). The solution \((\tau^*, \nu^*)\) collapses to the universal curve \( \tau, \nu = 1 \) when rescaling by the empirical values of ranking list size \( p \), flux \( F \), and mean openness rate \( \dot{o} \). Data corresponds to the Scientists dataset [4] (see Table 1 and Section S2), but the graphical solution is qualitatively the same for all open ranking lists.

We consider open ranking lists first. From Section S4.4 and Section S4.5 we have explicit expressions for \( F \) [Eq. (S27)] and \( \dot{o} \) [Eq. (S40)] in the model, which become more accurate as \( N \) and \( T \) increase. These relations lead to a non-linear system of equations for \( \tau \) and \( \nu \),

\[
\begin{align*}
\tau &= \nu \frac{\dot{o} - \nu}{\nu - \dot{p} \dot{o}}, \\
\nu &= \ln \left( \frac{p + (1 - p)e^{-\tau}}{1 - F} \right),
\end{align*}
\]

\((S45a)\) \((S45b)\)
determined by the specific values of ranking list size, flux, and openness of each dataset. Solving for \( \nu \) (or \( \tau \)) leads to a transcendental equation in \( \nu \) (or \( \tau \)) with no explicit solution. Still, we can gain insight by applying a graphical method and find the solution \((\tau^*, \nu^*)\) numerically (Fig. S16). From Eq. (S45a) we first note that, in order to ensure \( \tau > 0 \), \( \nu \) has to be in the interval \( \dot{p} < \nu < \dot{o} \). Since \( \dot{o} \) is relatively low for all studied datasets (less than \( \sim 0.3 \); see Table 2), we only need to consider small \( \nu \) values in the fitting process. Within this range we find a single solution \( \nu^* \) by graphically equating the two sides of Eq. (S45b) (left panel in Fig. S16). This value can be inserted into Eq. (S45a) to find the remaining solution \( \tau^* \) (right panel in Fig. S16).

The fitting process for closed ranking lists is, in turn, quite straightforward. Closed ranking lists have trivial values of flux and openness, so we cannot use Eq. (S45) for fitting. Instead we consider the explicit expression of success \( S^{++} \) (for time \( t = 1 \)) derived in Section S4.7. \( S^{++} \) is reminiscent of \( F \) in
| Dataset                              | Data measure | Model parameter |
|-------------------------------------|--------------|-----------------|
|                                     | F            | \( \dot{o} \)  | \( S^{++} \) | \( p \) | \( \tau \) | \( \nu \) |
| GitHub repositories [1]             | 0.6461       | 0.1287          | 0.4339       | 0.0106 | 1.0000 | 0.0015 |
| The Guardian readers (rec) [2]      | 0.7996       | 0.2085          | 0.1288       | 0.0258 | 1.0000 | 0.0001 |
| The Guardian readers (comm) [2]     | 0.6118       | 0.1283          | 0.3764       | 0.0413 | 1.0000 | 0.0000 |
| Enron emails [3]                    | 0.5215       | 0.2158          | 0.5394       | 0.0443 | 0.7747 | 0.0129 |
| Scientists [4, 5]                   | 0.1578       | 0.0343          | 0.8083       | 0.3982 | 0.2767 | 0.0147 |
| Universities [6]                    | 0.0477       | 0.0308          | 0.9532       | 0.7143 | 0.9097 | 0.0238 |
| Languages                           |              |                 |              |        |        |        |
| Russian [8–11]                      | 0.1796       | 0.0331          | 0.8436       | 0.1262 | 0.2244 | 0.0048 |
| Spanish [8–11]                      | 0.1603       | 0.0245          | 0.8585       | 0.1361 | 0.2006 | 0.0037 |
| German [8–11]                       | 0.1605       | 0.0292          | 0.8677       | 0.1159 | 0.1957 | 0.0039 |
| French [8–11]                       | 0.1849       | 0.0255          | 0.8338       | 0.0967 | 0.2238 | 0.0027 |
| Italian [8–11]                      | 0.1489       | 0.0234          | 0.8684       | 0.1495 | 0.1875 | 0.0039 |
| English [8–11]                      | 0.1272       | 0.0181          | 0.8928       | 0.1426 | 0.1566 | 0.0029 |
| Economics                           |              |                 |              |        |        |        |
| Companies [12]                      | 0.0734       | 0.0558          | 0.9323       | 0.2639 | 0.0689 | 0.0260 |
| Countries [13–15]                   | 0.0372       | 0.0084          | 0.8967       | 0.7122 | 0.1137 | 0.0061 |
| Infrastructure                      |              |                 |              |        |        |        |
| Cities (RU) [16]                    | 0.1221       | 0.1073          | 0.8824       | 0.5711 | 0.1230 | 0.0793 |
| Metro stations (London) [17]        | 0.0000       | 0.0000          | 0.9207       | 1.0000 | 0.1728 | 0.0000 |
| Cities (GB) [18]                    | 0.0000       | 0.0000          | 0.9517       | 1.0000 | 0.1015 | 0.0000 |
| Metro stations (Mexico)             | 0.0000       | 0.0000          | 0.9100       | 1.0000 | 0.1984 | 0.0000 |
| Nature                              |              |                 |              |        |        |        |
| Hyenas [19]                         | 0.3023       | 0.2748          | 0.5289       | 0.1419 | 0.3213 | 0.0911 |
| Regions JP (quake mag) [20, 21]     | 0.0000       | 0.0000          | 0.7293       | 1.0000 | 0.7797 | 0.0000 |
| Regions JP (quakes) [20, 21]        | 0.0000       | 0.0000          | 0.7625       | 1.0000 | 0.6444 | 0.0000 |
| Sports                              |              |                 |              |        |        |        |
| Chess players (male) [22]           | 0.0106       | 0.0051          | 0.9877       | 0.8148 | 0.0354 | 0.0042 |
| Chess players (female) [22]         | 0.0099       | 0.0068          | 0.9929       | 0.7667 | 0.0193 | 0.0055 |
| Poker players [24]                  | 0.0377       | 0.0203          | 0.9714       | 0.1832 | 0.0402 | 0.0058 |
| Tennis players [26]                 | 0.0112       | 0.0050          | 0.9916       | 0.3338 | 0.0137 | 0.0021 |
| Golf players [28]                   | 0.0058       | 0.0028          | 0.9904       | 0.3166 | 0.0068 | 0.0012 |
| Football scorers [30]               | 0.1340       | 0.0960          | 0.9091       | 0.1669 | 0.1377 | 0.0305 |
| NASCAR drivers (Busch) [32]         | 0.3648       | 0.2392          | 0.6348       | 0.1124 | 0.4744 | 0.0455 |
| NASCAR drivers (Winston Cup) [32]   | 0.1865       | 0.1306          | 0.8145       | 0.1838 | 0.2052 | 0.0422 |
| National football teams [33]        | 0.0031       | 0.0007          | 0.9728       | 0.9524 | 0.0532 | 0.0000 |

Table S2. Data measures and fitted model parameters. Values of empirical measures used in the fitting process (mean flux \( F \), mean openness rate \( \dot{o} \), and success \( S^{++} \)) and of fitted model parameters (relative ranking list size \( p \), displacement probability \( \tau \), and replacement probability \( \nu \)) for all considered datasets (see Table 1 and Section S2). For open ranking lists, we fit the model to empirical data by setting \( N = N_{T−1} \) (i.e. \( p = N_0/N_{T−1} \), see Section S4.1) and by computing \( \tau \) and \( \nu \) numerically from Eq. (S45) in terms of \( F \) and \( \dot{o} \) (Fig. S16). For closed ranking lists, we set \( N = N_0 \) and \( \nu = 0 \), and obtain \( \tau \) explicitly from Eq. (S46) in terms of \( S^{++} \) (for time \( t = 1 \) and threshold \( c = 0.5 \)). Datasets are classified by an (arbitrary) system type based on the nature of the elements in the ranking list.

the sense that it measures the flux of elements out of (into) the top of the ranking list into (from) the bottom, where top and bottom are arbitrarily separated by the threshold \( c \). Solving for \( \tau \) in Eq. (S44) gives

\[
\tau = \ln \left( \frac{1 - cp}{S^{++} - cp} \right), \quad F = 0,
\]
which allows us to find $\tau$ for all closed ranking lists considered.

Table 2 summarizes the results of the fitting process described above. Our minimal model of rank dynamics fits most rank measures of many datasets rather well (see Figs. S3-S7). In the model, rank flux $F_1$ is a constant over time that recovers the empirical mean with high precision, apart from fluctuations (Fig. S3). Out-flux $F_R$ is relatively constant for most $R$ values, and in open systems has a sharp increase for large $R$. This is only a good fit for relatively closed ranking lists, since in very open systems out-flux is a more gradually increasing function of $R$ (Fig. S4). Openness $o_t$ is concave and monotonically increasing in time, which follows empirical data well (Fig. S5). Rank change $C$ recovers the asymmetry of most open systems (low values only in the top of the ranking list) and the rough symmetry in closed systems (low values in both the top and bottom of the ranking list), although empirical closed systems are not as symmetric as the model predicts (Fig. S6). Success $S_{t^+}$ is a decaying function of time $t$ that fits several datasets well. Even if the difference in qualitative behavior between open and closed ranking lists is less pronounced for $S_{t^+}$ than for other rank measures, we still see that success tends to decay faster in open systems (Fig. S7).

### S5.1 Dynamical regimes of open ranking lists

Having explicit expressions in Eq. (S45) lets us further analyze the solution ($\tau^*, \nu^*$). We introduce the rescaled parameters

$$\tau_r = \frac{\tau}{p(1-p)\dot{o}}, \quad \text{and} \quad \nu_r = \frac{\nu - p\dot{o}}{\dot{o}},$$

which allows us to rewrite Eq. (S45a) for sufficiently small $\nu$ (relative to $p$ and $1-p$),

$$\tau_r \nu_r \simeq 1.$$  (S48)

Eq. (S48) shows that, when fitting the model to datasets with given values of $p$, $F$, and $\dot{o}$, the probabilities $\tau$ and $\nu$ can be rescaled to collapse into a universal curve, a power law with exponent 1 (Fig. S16). In order to reproduce the dynamics of empirical ranking lists, is it enough to consider an inverse relationship between the rates of displacement and replacement of elements in the model, meaning that the considered datasets have either an active dynamics of Lévi flights and diffusion plus a few replacements, or less displacements with a larger exchange of elements. This makes the fitting process effectively one-dimensional across open ranking lists, even if the model can be studied for arbitrary $\tau$ and $\nu$.

The universal curve in Eq. (S48) displays three regimes in the dynamics of open ranking lists, as measured by the average probabilities that, between consecutive observations in the data (i.e. between times $t$ and $t+1$), an element performs either a Lévi walk,

$$W_{\text{levi}} = e^{-\nu}(1 - e^{-\tau}),$$  (S49)

changes rank by diffusion

$$W_{\text{diff}} = e^{-\nu}e^{-\tau},$$  (S50)

or is replaced,

$$W_{\text{repl}} = 1 - e^{-\nu},$$  (S51)

where $W_{\text{levi}} + W_{\text{diff}} + W_{\text{repl}} = 1$. In the most open systems we study, elements mostly change rank via
Figure S17. Regimes of dynamical behavior in open ranking lists. Average probability that an element changes rank by Lévi walk ($W_{\text{levi}}$), diffusion ($W_{\text{diff}}$), or is replaced ($W_{\text{repl}}$) between consecutive observations in the data. Probabilities shown both for all datasets with $\dot{\sigma} > 10^{-3}$ (dots), and for the model moving along the curve $\tau_r \nu_r = 1$ [Eq. (S48)] with the same $p$ and $\dot{\sigma}$ as the data (lines). Datasets are ordered from most open (upper row) to least open (lower row) according to mean rank flux $F$ [Eq. (S2)]. The model reveals a crossover in empirical ranking lists between Lévi walk ($W_{\text{levi}} > W_{\text{diff}}, W_{\text{repl}}$) and diffusion ($W_{\text{diff}} > W_{\text{levi}}, W_{\text{repl}}$) regimes. Although not seen in data, there is also a third regime driven by replacement ($W_{\text{repl}} > W_{\text{levi}}, W_{\text{diff}}$).

long jumps, forming a Lévi walk regime where $W_{\text{levi}} > W_{\text{diff}}, W_{\text{repl}}$ (Fig. S17). Here, long-range rank changes take elements in and out of a small ranking list within a big system (low $p$), thus generating large mean flux $F$. The rest of the datasets belong to a diffusion regime with $W_{\text{diff}} > W_{\text{levi}}, W_{\text{repl}}$. A local, diffusive rank dynamics is the result of elements smoothly changing their scores and overcoming their neighbors in rank space. Although not observed in empirical data, the model predicts a third regime dominated by replacement ($W_{\text{repl}} > W_{\text{levi}}, W_{\text{diff}}$). The curves in Fig. S17 show how close a ranking list might be to a change of regime. If a dataset has values of $p$ and $\dot{\sigma}$ that make it fall close to the boundaries of these regimes, small changes in the parameters regulating the displacement ($\tau$) and replacement ($\nu$) can change the dynamical regime of the ranking list.

References

[1] List of repositories, GitHub. https://github.com/explore. Accessed: 2018-05-29.

[2] The Guardian, UK edition. https://www.theguardian.com/uk. Accessed: 2018-07-04.

[3] Enron email dataset, CALO Project. https://www.cs.cmu.edu/~enron/. Accessed: 2018-05-29.

[4] R. Sinatra, P. Deville, M. Szell, D. Wang, and A.-L. Barabási. A century of physics. Nat. Phys. 11, 791–797 (2015).
[5] R. Sinatra, D. Wang, P. Deville, C. Song, and A.-L. Barabási. Quantifying the evolution of individual scientific impact. *Science* **354**, aaf5239 (2016).

[6] Statistics 2016, Academic Ranking of World Universities (ARWU). [http://www.shanghairanking.com/ARWU-Statistics-2016.html](http://www.shanghairanking.com/ARWU-Statistics-2016.html). Accessed: 2018-05-29.

[7] About Academic Ranking of World Universities (ARWU). [http://www.shanghairanking.com/aboutarwu.html](http://www.shanghairanking.com/aboutarwu.html). Accessed: 2018-05-29.

[8] Google Books Ngram Viewer. [http://storage.googleapis.com/books/ngrams/books/datasetsv2.html](http://storage.googleapis.com/books/ngrams/books/datasetsv2.html). Accessed: 2018-05-29.

[9] J.-B. Michel *et al.*, Quantitative analysis of culture using millions of digitized books. *Science* **331**, 176 – 182 (2011).

[10] G. Cocho, J. Flores, C. Gershenson, C. Pineda, and S. Sánchez, Rank diversity of languages: Generic behavior in computational linguistics. *PLoS ONE* **10**, e0121898 (2015).

[11] J. A. Morales *et al.*, Rank dynamics of word usage at multiple scales. *Front. Phys.* **6**, 45 (2018).

[12] Database 1955–2005, Fortune 500. [http://archive.fortune.com/magazines/fortune/fortune500_archive/full/1955/index.html](http://archive.fortune.com/magazines/fortune/fortune500_archive/full/1955/index.html). Accessed: 2018-05-29.

[13] The Observatory of Economic Complexity. [https://atlas.media.mit.edu/en/](https://atlas.media.mit.edu/en/). Accessed: 2018-05-29.

[14] C. A. Hidalgo and R. Hausmann, The building blocks of economic complexity. *Proc. Nat. Acad. Sci USA* **106**, 10570–10575 (2009).

[15] R. Hausmann *et al.*, *The Atlas of Economic Complexity: Mapping Paths to Prosperity* (MIT Press, 2014).

[16] C. Cottineau. A multilevel portrait of shrinking urban Russia. *Espace Populations Sociétés* (online), 2015/3-2016/1 (2016). Accessed: 2018-06-12.

[17] R. Murcio, C. Zhong, E. Manley, and M. Batty, Identifying risk profiles in the London’s public transport system. In: *14th International Conference on Computers in Urban Planning and Urban Management* (Boston, MA, 2015).

[18] R. Edwards and M. Batty, City size: Spatial dynamics as temporal flows. *Environ. Plann. A* **48**, 1001–1003 (2016).

[19] A. Ilany, A. S. Booms, and K. E. Holekamp, Topological effects of network structure on long-term social network dynamics in a wild mammal. *Ecol. Lett.* **18**, 687–695 (2015).

[20] Japan University Network Earthquake Catalog. [https://wwweic.eri.u-tokyo.ac.jp/db/junec/index.html](https://wwweic.eri.u-tokyo.ac.jp/db/junec/index.html). Accessed: 2018-07-04.

[21] M. Karsai, K. Kaski, A.-L. Barabási, and J. Kertész, Universal features of correlated bursty behaviour. *Sci. Rep.* **2**, 397 (2012).

[22] FRL June 2018, World Chess Federation. [https://ratings.fide.com/download.phtml](https://ratings.fide.com/download.phtml). Accessed: 2018-06-12.
[23] Elo rating system, Wikipedia. https://en.wikipedia.org/wiki/Elo_rating_system. Accessed: 2018-06-12.

[24] Rankings, Global Poker Index (GPI). http://www.globalpokerindex.com/rankings/. Accessed: 2018-06-13.

[25] About the Global Poker Index (GPI). http://www.globalpokerindex.com/about/. Accessed: 2018-06-13.

[26] Rankings (singles), Association of Tennis Professionals (ATP) World Tour. http://www.atpworldtour.com/en/rankings/singles. Accessed: 2018-06-13.

[27] Rankings (FAQ), Association of Tennis Professionals (ATP) World Tour. http://www.atpworldtour.com/en/rankings/rankings-faq. Accessed: 2018-06-13.

[28] Official World Golf Ranking (OWGR). http://www.owgr.com/ranking. Accessed: 2018-06-13.

[29] How the ranking system works, Official World Golf Ranking (OWGR). http://www.owgr.com/about. Accessed: 2018-06-13.

[30] Ranking top scorers, Football World Rankings. https://www.clubworldranking.com/ranking-topscorers. Accessed: 2018-06-13.

[31] How are the Football World Rankings calculated?, Football World Rankings. https://www.clubworldranking.com/how-are-the-football-world-rankings-calculated. Accessed: 2018-06-13.

[32] National Association for Stock Car Auto Racing (NASCAR) standings, Entertainment and Sports Programming Network. http://www.espn.com/racing/standings. Accessed: 2018-06-13.

[33] World Ranking, Fédération Internationale de Football Association (FIFA). https://www.fifa.com/fifa-world-ranking/. Accessed: 2018-06-13.

[34] Men’s ranking procedure, Fédération Internationale de Football Association (FIFA). https://www.fifa.com/fifa-world-ranking/procedure/men.html. Accessed: 2018-06-13.

[35] J. A. Morales et al., Generic temporal features of performance rankings in sports and games. EPJ Data Science 5, 33 (2016).

[36] S. Sánchez et al., Trajectory stability in the traveling salesman problem. Complexity 2018, 2826082 (2018).

[37] D. Aldous, P. Diaconis, Shuffling cards and stopping times. Am. Math. Mon. 93, 333–348 (1986).

[38] J. M. Kleinberg, Navigation in a small world. Nature 406, 845 (2000).

[39] G. Li et al., Towards design principles for optimal transport networks. Phys. Rev. Lett. 104, 018701 (2010).

[40] C. A. Muirhead, Genetic drift, models of random. Encyclopedia of Evolutionary Biology, 136–143 (2016).

[41] T. D. Tran, J. Hofrichter, J. Jost, An introduction to the mathematical structure of the Wright-Fisher model of population genetics. Theor. Biosci. 132, 73–82 (2013).
[42] L. Chen, D. W. Stroock, The fundamental solution to the Wright-Fisher equation. *SIAM J. Math. Anal.* **42**, 539–567 (2010).

[43] C. L. Epstein, R. Mazzeo, Wright-Fisher diffusion in one dimension. *SIAM J. Math. Anal.* **42** 568–608 (2010).