BPS surface operators and calibrations

Nadav Drukker\(^1\) and Maxime Trépanier\(^{1,2,\ast}\)

\(^1\) Department of Mathematics, King’s College London, London WC2R 2LS, United Kingdom
\(^2\) Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada

E-mail: trepanier.maxime@gmail.com

Received 3 November 2022; revised 9 March 2023
Accepted for publication 24 March 2023
Published 6 April 2023

Abstract

We present here a careful study of the holographic duals of BPS surface operators in the 6d \(\mathcal{N} = (2,0)\) theory. Several different classes of surface operators have been recently identified and each class has a specific calibration form—a 3-form in \(AdS_7 \times S^4\) whose pullback to the M2-brane world-volume is equal to the volume form. In all but one class, the appropriate forms are exact, so the action of the M2-brane is easily expressed in terms of boundary data, which is the geometry of the surface. Specifically, for surfaces of vanishing anomaly, it is proportional to the integral of the square of the extrinsic curvature. This can be extended to the case of surfaces with anomalies, by taking the ratio of two surfaces with the same anomaly. This gives a slew of new expectation values at large \(N\) in this theory. For one specific class of surface operators, which are Lagrangian submanifolds of \(\mathbb{R}^4 \subset \mathbb{R}^6\), the structure is far richer and we find that the M2-branes are special Lagrangian submanifold of an appropriate six-dimensional almost Calabi-Yau submanifold of \(AdS_7 \times S^4\). This allows for an elegant treatment of many such examples.

Keywords: supersymmetry, M-theory, calibration, 6d \(\mathcal{N} = (2,0)\), Lagrangian submanifold, surface operators

1. Introduction and conclusions

In a recent paper [1] we presented four classes of BPS surface operators in the 6d \(\mathcal{N} = (2,0)\) theory [2–5]. In each case the shape of the surfaces and their scalar coupling (or local
R-symmetry breaking) have a geometric relation that guarantees that the projector Equations for supersymmetry breaking, which have 16 independent solutions at each point, also share some solutions along the entire surface. These are surface operators in 6d analogues of BPS Wilson loops of the types studied in [6–9].

Such surfaces preserve some (conformal) Killing spinor in $\mathbb{R}^6$. The holographic description of surface operators is in terms of an M2-brane in $AdS_7 \times S^4$, which is also BPS and preserves the Killing spinor $\epsilon$ related to the one on the boundary. The supergravity projector equation is a consequence of the $\kappa$-invariance of the M2-brane action [10] and reads

$$-\frac{i}{6} \epsilon_{MNP} \partial_m X^M \partial_n X^N \partial_P X^p \varepsilon^{mnp} = \epsilon,$$

(1.1)

where $X^M$ are $AdS_7 \times S^4$ coordinates and $\varepsilon^{mnp}$ is the Levi-Civita tensor density and includes $1/\sqrt{g}$ where $g_{mn}$ is the induced metric.

Given a preserved Killing spinor $\epsilon$, we can construct a 3-form [11]

$$\phi = -i \frac{\epsilon_{MNP}}{\epsilon \varepsilon} \epsilon \epsilon \varepsilon^\dagger dX^M \wedge dX^N \wedge dX^P.$$

(1.2)

By construction, the pullback of this 3-form to an M2-brane satisfying (1.1) is its volume form. If $\phi$ is closed $d\phi = 0$, then the form is a calibration [12–14]; its integral is the same on all 3-volumes of the same homology class and is equal to the minimal volume, so the action of the classical M2-brane.

If $\phi$ is exact then the action comes only from the boundary and at leading order is simply a divergence proportional to the area of the surface operator. This divergent term is removed by the Legendre transform of [15–17] or equivalently by renormalization [18]. The expectation value of the surfaces is then given by subleading terms. If there are logarithmic divergences, the surface is anomalous and the coefficient of the divergence, or the anomaly is given by a universal formula unraveled in [16, 19–31]. See [17] for a discussion of the state of the art. If there are no anomalies, the surface operators may have finite expectation values, which is what we focus on below. In cases with anomalies we can find finite ratios between expectation values of two different surface operators, again giving a finite computable quantity, see [32].

Explicit expressions for the 3-forms for all families of BPS surface operators identified in [1] were presented there and are repeated below. In this paper we note that for three of the classes: Type-$R$, Type-$H$ and Type-$C$ the calibration forms are exact and use that to evaluate the finite part of their expectation values.

The calculation does not require to find the minimal surfaces, which is generically a hard problem, even in the BPS case. But it does provide explicit expressions for the classical action which we can compare to expressions found for all known examples. This leaves only one class of BPS surfaces, Type-S in the parlance of [1], where the form is not exact and thus we cannot determine the expectation value directly from the boundary data without solving the equations of motion. This may require generalized calibrations, as in [33–35].

In the second part of this paper we look at a subclass of surfaces within Type-$H$ of Lagrangian surfaces in $\mathbb{R}^4$. We find that the M2-branes are special Lagrangian surfaces within a 6d subspace of $AdS_7 \times S^4$. We apply some of the methods of constructing special Lagrangian surfaces to this setting, recovering previously found solutions of tori, cylinder and crease [32, 36] and some generalizations thereof.

The rich spectrum of BPS operators of this theory [1] continue to provide an opening to performing explicit calculations in the $\mathcal{N} = (2,0)$ theory in increasingly wider settings. We expect them to continue to be an ideal laboratory for the study of the 6d theory.
Throughout this paper we use the metric
\[ ds^2 = \frac{y}{L} dx_m dx_n + \frac{L^2}{y^2} dy dy, \tag{1.3} \]
for \( AdS_7 \times S^4 \). To avoid confusion with exponents, we use subscripts for flat indices \( m = 1, \ldots, 6 \) and \( l = 1, \ldots, 5 \) and denote \( y = \sqrt{y_l y_l} \).

### 2. Surface expectation value from exact calibrations

In this section we study surfaces of Type-\( \mathbb{R} \), Type-\( \mathbb{H} \) and Type-\( \mathbb{C} \) that have finite expectation values and rely on the calibration equations to find their expectation values at leading order at large \( N \). Surfaces of Type-\( \mathbb{R} \) do not have anomalies, while for surfaces of Type-\( \mathbb{H} \) and Type-\( \mathbb{C} \) there are constraints on their topology that ensure that the anomaly vanishes. This was pointed out in [1] and is reviewed below.

For now, let us assume that there is a surface calibrated by an exact form \( \phi = d\Phi \). Then the expectation value of the surface operator is
\[ \langle V_{\Sigma} \rangle = \exp \left[ - T_{M2} \int_{\Sigma} \phi \right] = \exp \left[ - \frac{N}{4\pi L^3} \int_{\Sigma = \partial V} \Phi \right], \tag{2.1} \]
where the tension of the M2-brane is
\[ T_{M2} = \frac{1}{4\pi^2 L_p^2} = \frac{N}{4\pi L^3}. \tag{2.2} \]
The expression for \( \Phi \) depends on the type of operator, but in all cases it needs to be evaluated near the boundary of \( AdS_7 \), where we can rely on the near boundary solution found by Graham and Witten [21] relying on the Fefferman-Graham expansion [37].

Using the coordinates \( u \) and \( v \) on the surface \( \Sigma \), the surface is given by \( x^{(0)}_\mu(u,v) \in \mathbb{R}^6 \) and \( n^{(0)}_l(u,v) \in S^4 \). For the M2-brane solution we employ the same coordinates and in addition \( y \) from (1.3). The coordinates \( n_l \) become the angular components of \( y_l = y n_l \). The solution is represented perturbatively as
\[ x^{(1)} = x^{(0)} + \frac{L^3}{y} x^{(1)} + \ldots, \quad n = n^{(0)} + \frac{L^3}{y} n^{(1)} + \ldots. \tag{2.3} \]

\( x^{(1)} \) is determined [21] to be the mean curvature vector
\[ x^{(1)}_\mu(u,v) = H_{\mu l}(u,v) = h^{ab} \partial_\mu x^{(0)}_a \partial_\nu x^{(0)}_b \left( \delta^\lambda_\mu - h^{cd} \partial_\mu x^{(0)}_c \partial_\nu x^{(0)}_d \right). \tag{2.4} \]
Here \( h_{ab} \) is the induced metric on \( \Sigma \). \( n^{(1)} \) can be calculated following [17], but is not required for any of our calculations below. The reason is that it always appears in the combination \( n^{(0)}_l n^{(1)}_l \) which vanishes, since the normalization \( n_l n_l = 1 \) is always assumed.

We now apply this to the various examples of BPS surfaces.

#### 2.1. Type-\( \mathbb{R} \)

These surfaces are a product of a line in the \( x_b = v \) direction and an arbitrary curve in the transverse direction \( x^{(0)}_I(u) \in \mathbb{R}^5 (I = 1, \ldots, 5) \). The scalar coupling then is chosen to be along the tangent vector to the curve.
\[ n_j^{(0)}(u,v) = \left. \frac{\partial \alpha^{(1)}(u)}{\partial u} \right|_{x_0} \]  \hspace{1cm} (2.5)

Examples of surfaces in this class were previously studied in [32, 36, 38].

The M2-branes in AdS\(_5\) \times S\(_4\) are naturally also homogeneous along the \(x_6\) direction and are calibrated by the 3-form

\[ \phi^\mathbb{R} = -dx_6 \wedge \omega^\mathbb{R}, \quad \omega^\mathbb{R} = \sum_{i=1}^{5} (dx_i \wedge dy_i). \]  \hspace{1cm} (2.6)

This form is clearly exact. In fact we can write it in two inequivalent ways

\[ \phi^\mathbb{R} = -d(x_6 \omega^\mathbb{R}) = d \left( \sum_{i=1}^{5} y_i dx_i \wedge dx_6 \right). \]  \hspace{1cm} (2.7)

The second expression is more relevant for us, and using \(y_i = y n_j\) we find that the action of the M2-brane is equal to

\[ S = T_{M2} \int dx_6 du \sum_{i=1}^{5} y n_i \partial_u x_i \bigg|_{y=\infty}. \]  \hspace{1cm} (2.8)

Plugging in the large \(y\) expansion (2.3) and the asymptotic expression for \(n_j\) in (2.5), we find

\[ S = \frac{N}{4\pi} L^3 \int dx_6 du \left( y n_j^{(0)} \partial_u x_j^{(0)} + L^3 n_j^{(0)} \partial_u x_j^{(1)} + L^3 n_j^{(1)} \partial_u x_j^{(0)} \right) + O(y^{-1}). \]  \hspace{1cm} (2.9)

The term proportional to \(y\) is the usual divergent term, proportional to the area of the surface, which is removed by an appropriate counterterm.

The last term in (2.9) is proportional to \(n_j^{(0)} n_j^{(1)}\) and as mentioned above vanishes. We are left with the \(n_j^{(0)} \partial_u x_j^{(1)}\) terms which using (2.3), (2.5) and integration by parts is

\[ S_{\text{ren}} = \frac{N}{4\pi} \int dx_6 du \frac{\partial u x_j^{(0)}}{\partial u (x_0^{(0)})} \partial_u H^6 + O(y^{-1}) = -\frac{N}{4\pi} \int dx_6 du |\partial_u x^{(0)}| H^6 + O(y^{-1}). \]  \hspace{1cm} (2.10)

Note that for a curve of Type-\(\mathbb{R}\) the mean curvature vector is simply the curvature of the curve \(\partial_u x^{(0)}\) (in the unit speed parametrization).

For example, a circle of radius \(R\) is given in the unit speed parametrization by \(x^{(0)}(u) = (R \cos(u/R), R \sin(u/R))\) and \(H^2 = R^{-2}\). The action is then

\[ S_{\text{ren}} = \frac{N}{4\pi} \int_{-D/2}^{D/2} du \int_{0}^{2\pi} \frac{dx}{R^2} \frac{1}{R^2} = -\frac{N D}{2 R}, \]  \hspace{1cm} (2.11)

where \(D\) is a cutoff on the \(x_6\) coordinate. This agrees precisely with the result of [36] derived there by solving the minimal surface equations of motion.

If we dimensionally reduce the theory along \(x_5\) we find a line operator in 5d along the curve \(x_7\). If that curve is at a fixed value of \(x_5\), we can further reduce to 4d where we find a BPS Wilson loop in \(\mathcal{N} = 4\) SYM. These Wilson loops with the exact relation (2.5) were found by Zarembo in [6]. They are known to have vanishing expectation value [13, 39, 40]. This is consistent with the calculation above, since in this limit \(D\), the length of the \(x_6\) direction vanishes. Thus the 6d observables have non-trivial expectation value per unit length, while the 4d ones are trivial.
2.2. Type-\(H\)

Type-\(H\) surfaces are formed of any oriented surface in \(\mathbb{R}^4\) accompanied by the scalar coupling \([1]\)

\[
n_I^{(0)} = \frac{1}{2} \eta_{\mu
u} \partial_\mu \chi^{(0)}_I \partial_\nu \chi^{(0)}_I \epsilon^{ab} = \frac{1}{2} \epsilon_{IJK} \partial_\mu \chi^{(0)}_I \partial_\nu \chi^{(0)}_J \epsilon^{ab} + \partial_\mu \chi^{(0)}_I \partial_\nu \chi^{(0)}_I \epsilon^{ab},
\]

(2.12)

where \(\eta_{\mu\nu}\) is the 't Hooft chiral symbol expressed on the right in terms of the 3d epsilon symbol (with \(I,J,K \in \{1,2,3\}\)) and \(\epsilon^{ab}\) is the 2d epsilon tensor (including a factor \(h^{-1/2}\)). This has a nice interpretation in terms of the Gauss map from the surface to its tangent space in \(S^2 \times S^2\). The vector \(n_I^{(0)}\) is a coordinate on one of those spheres, so this is the projection of the Gauss map to a single \(S^2\).

The case when the surface is a cone over a curve on \(S^3\), or more precisely its conformal transform to a surface in \(\mathbb{R} \times S^5\) has been described and studied in great detail in \([11]\).

The holographic dual of surface operators of Type-\(H\) are M2-branes restricted to an \(AdS_5 \times S^5\) subspace and calibrated with respect to the form

\[
\phi^H = \frac{1}{2} \eta_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu - \left(\frac{L}{\epsilon}\right)^3 \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3.
\]

(2.13)

Using spherical coordinates \(y, \vartheta\) and \(\varphi\) instead of \(y_I\), we can see that this form is exact

\[
\phi^H = \mathcal{d}\left[\frac{1}{2} \eta_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu - \mathcal{L}^3 \ln \mathcal{V} \mathcal{M}\right], \quad \mathcal{V}^2 = \sin \vartheta \mathrm{d}\vartheta \wedge \mathrm{d}\varphi.
\]

(2.14)

The classical action again reduces to a boundary term

\[
S = T_{M2} \int_{\Sigma} \phi^H = T_{M2} \int_{\Sigma} \frac{1}{2} n_I^{(0)} \eta_{\mu\nu} \partial_\mu \chi^{(0)}_I \partial_\nu \chi^{(0)}_I \epsilon^{ab} \sqrt{h} \mathrm{d}^2 \sigma - T_{M2} L^3 \ln \mathcal{V} \mathcal{M} + O(y^0).
\]

(2.15)

The first term is the usual area divergence that can be cancelled by a counter-term. The second one is logarithmically divergent, signifying a conformal anomaly. The integral is over the 2-sphere at \(y_I \to \infty\) and is the same as that appearing in the Gauss map. The integral is equal to \(4\pi\) times the degree \(\nu\) of the map, giving a total anomaly

\[
S = 8\pi T_{M2} L^3 \nu \log \epsilon + \text{finite} = 2N\nu \log \epsilon + \text{finite}.
\]

(2.16)

Here we identified \(y \sim \epsilon^{-2}\), where \(\epsilon\) is a short distance cutoff in the CFT. This expression was already found in \([1]\) relying on the original surface anomaly calculation of Graham and Witten \([21]\).

In cases without an anomaly we can evaluate the finite term by again looking at \(1/y\) corrections to the first term in (2.15). Using (2.4), we find

\[
S_{\text{ren}} = \frac{N}{4\pi} \int_{\Sigma} \sqrt{h} \epsilon^{ab} \eta_{\mu\nu} \left( n_I^{(0)} \partial_\mu \chi^{(0)}_J \partial_\nu \chi^{(0)}_J + \frac{1}{2} n_J^{(1)} \partial_\mu \chi^{(0)}_J \partial_\nu \chi^{(0)}_J \right) d^2 \sigma + O(y^{-1}).
\]

(2.17)

The second term is proportional to \(n_I^{(1)} n_J^{(0)}\) and therefore vanishes. Plugging the expression for \(n_I^{(0)}\) (2.12), using the orthogonality relation for \(\eta_{\mu\nu}\) and integration by parts we find

\[
S_{\text{ren}} = \frac{N}{4\pi} \int_{\Sigma} \sqrt{h} \epsilon^{ab} \eta_{\mu\nu} \left( n_I^{(0)} \partial_\mu \chi^{(0)}_J \partial_\nu \chi^{(0)}_J + \frac{1}{2} n_J^{(1)} \partial_\mu \chi^{(0)}_J \partial_\nu \chi^{(0)}_J \right) d^2 \sigma + O(y^{-1})
\]

\[
= \frac{N}{4\pi} \int_{\Sigma} \sqrt{h} \epsilon^{ab} \eta_{\mu\nu} \partial_\mu \chi^{(0)}_J \partial_\nu \chi^{(0)}_J d^2 \sigma + O(y^{-1}) = -\frac{N}{4\pi} \int_{\Sigma} \sqrt{h} \chi^{(0)}_J \partial_\mu \chi^{(0)}_J d^2 \sigma + O(y^{-1}).
\]

(2.18)

Interestingly, this is exactly the same expression as in the case of Type-\(H\) surfaces (2.10).
In cases with an anomaly, the expectation value depends on the choice of $\varepsilon$ so is not well-defined. However, since the anomaly is topological, the ratio of expectation values of two surface operators with the same topology is a well-defined quantity. It is given by the difference in action, which as above is simply the difference in integrated mean curvature.

2.3. Type-C

Surfaces of Type-C are holomorphic with respect to the usual complex structure on $\mathbb{R}^6 = \mathbb{C}^3$. They have a constant $n_3^{(0)} = \delta_{ij}$. Their holographic duals are calibrated with respect to $\phi_H$.

\[ \phi_C = (dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6) \wedge dy_1. \]  

Clearly this integrates to

\[ \Phi_C = y_1(dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6). \]  

In this case holomorphicity ensures that $x^{(1)}_i = H_{\mu} = 0$, so the action written in terms of complex 2d and 6d coordinates is

\[ S = T_{M2}y \int_\Sigma d\sigma d\bar{\sigma} \partial_\sigma x_i \partial_{\bar{\sigma}} x_i. \]  

This is simply the area divergence, so the renormalized action vanishes. Note that at subleading orders in $N$, surfaces of Type-C may have an anomaly [1].

3. Lagrangian surfaces

We shift now to focus on a subclass of Type-H surfaces, referred to as Type-L which are Lagrangian in $\mathbb{R}^4$ with the symplectic form $\omega_0 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$, meaning that $\omega_0|_\Sigma = 0$.

One can see that in this case $n_3^{(0)} = 0$ (2.12), so the image of the projection of the Gauss map is a circle inside $S^2$.

Such surfaces preserve two supercharges, which is double the number of a generic Type-H surface. An easy way to see this, is that these surfaces are also BPS under an alternative Type-H construction with $n_3^{(0)} \rightarrow -n_3^{(0)}$. As such, the M2-branes describing them in $AdS_7 \times S^4$ are calibrated with respect to two different forms, $\phi^H$ in (2.13) and $\phi^{\overline{H}}$ with $dy_3 \rightarrow -dy_3$. Being calibrated with respect to both, we can define

\[ \phi^L = \frac{1}{2} (\phi^H + \phi^{\overline{H}}), \]  

and then

\[ \phi^L|_V = vol_V, \quad (\phi^H - \phi^{\overline{H}})|_V = 0. \]  

From (2.13) we can get the explicit expressions

\[ \phi^L = (dx_1 \wedge dx_4 + dx_2 \wedge dx_3) \wedge dy_1 + (dx_3 \wedge dx_1 + dx_2 \wedge dx_4) \wedge dy_2, \]  

\[ \phi^H - \phi^{\overline{H}} = 2 \left( dx_1 \wedge dx_2 + dx_3 \wedge dx_4 - \left( \frac{L}{5} \right)^3 dy_1 \wedge dy_2 \right) \wedge dy_3. \]  

Since the pullback of $\phi^L$ is the volume form, we conclude that the M2-brane is in the subspace of $AdS_7 \times S^4$ at constant $x_5$, $x_6$, $y_{3,4,5}$. 

\[ 6 \]
This is an even dimensional manifold and it is natural to associate a complex structure. Taking the usual complex structure $J$ for $z_1 = x_1 + i x_2$, $z_2 = x_3 + i x_4$, $z_3 = -(y_2 + i y_1)$ and the symplectic form $\omega$ to be the extension of $\omega_0$ appearing in (3.5)

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 - \left( \frac{L}{y} \right)^3 dy_1 \wedge dy_2$$

$$= \frac{i}{2} \left[ dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + \left( \frac{L}{y} \right)^3 dz_3 \wedge d\bar{z}_3 \right]. \quad (3.6)$$

this defines a Kähler manifold with metric

$$\tilde{G} = dx_\mu dx_\nu + \left( \frac{L}{y} \right)^3 dy_\mu dy_\nu = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + \left( \frac{L}{y} \right)^3 dz_3 d\bar{z}_3. \quad (3.7)$$

The metric $\tilde{G}$ is not the same as $G$ in (1.3). It is related to it (on an appropriate subspace) by the conformal transformation

$$G = \frac{y}{L} \tilde{G}. \quad (3.8)$$

In addition, an (almost) Kähler manifold has a holomorphic 3-form

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3, \quad (3.9)$$

related to $\omega$ as

$$\frac{1}{3!} \left( \frac{y}{L} \right)^3 \omega^3 = -i \left( \frac{i}{2} \right)^3 \bar{\Omega} \wedge \Omega = -dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2. \quad (3.10)$$

This makes this space an almost Calabi–Yau/special Kähler manifold according to [41].

We now prove that the M2-brane world-volume $V$ is a special Lagrangian submanifold, which requires

$$\text{Im} \omega|_V = 0, \quad \omega|_V = 0. \quad (3.11)$$

By expanding $\Omega$ we find

$$\Omega = \frac{1}{2} \sum_{i=1}^{2} \eta_{\mu \nu} (\delta_{ij} - i \varepsilon_{ij}) dx_\mu \wedge dx_\nu \wedge dy_j. \quad (3.12)$$

So its real part is $\text{Re} \Omega = \phi^{\parallel}$ (3.4). Since the M2-brane are calibrated by $\phi^{\parallel}$, the pullback of $\text{Re} \omega$ is the volume form, so the M2-branes are parallel to it. From (3.6) it is clear that $\text{Im} \Omega$ is orthogonal to $\text{Re} \omega$, so the pullback $\text{Im} \Omega|_V$ vanishes. Finally note that $\omega = \partial_{z_3} \cdot \phi^{\parallel} = \partial_{y_1}$, $\phi^{\parallel}$, so $\omega|_V = 0$ by virtue of the second equation in (3.3). This proves (3.11) and therefore $V$ is a special Lagrangian submanifold.

### 3.1. Lagrangian surfaces with $U(1)$ symmetry

If the surface has a symmetry under $(z_1, z_2) \to (e^{i \alpha} z_1, e^{-i \alpha} z_2)$, we expect the M2 world volume to have the same symmetry, which then matches the conditions on Lagrangian surfaces studied in [42, 43].

Following [43], we define $w = \text{Im}(z_1 z_2) = x_1 x_4 + x_2 x_3$ and $v = \text{Re}(z_1 z_2) = x_1 x_3 - x_2 x_4$, then based on the symmetry we can parametrize the surface in terms of $y_2$, $w$ and an angular direction such that the nontrivial dependence is
The conserved quantity associated with the $U(1)$ symmetry is
\[ |z_1|^2 - |z_2|^2 = 2a, \tag{3.14} \]
with a constant $a$.

In these coordinates it is easy to check that $\omega$ (3.6) takes the form
\[ \omega = \frac{dv \wedge dw}{|z_1|^2 + |z_2|^2} - \frac{L^3}{y^3} \frac{dy_1}{|z_1|^2 + |z_2|^2}. \tag{3.15} \]
In this form the constraint $\omega|_{v=0}$ is straightforward to read, and with a bit more work one can obtain the constraint $\text{Im}\Omega|_{v=0}$ as well. They respectively give
\[ \frac{\partial v}{\partial y_2} = -2\sqrt{w^2 + v^2 + a^2} \frac{L^3}{y^3} \frac{\partial y_1}{\partial w}, \quad \frac{\partial v}{\partial y_2} = \frac{\partial y_1}{\partial y_2}. \tag{3.16} \]
Note that $2\sqrt{w^2 + v^2 + a^2} = |z_1|^2 + |z_2|^2$. These can be thought of as modified Cauchy-Riemann equations and are related to the equations in [43] by the factor of $L^3/y^3$.

Alternatively, the $U(1)$ symmetry can act on $(z_1, z_3) \rightarrow (e^{i\chi}z_1, e^{-i\chi}z_3)$. Adapting the case above, we define $w' = -\text{Im}(z_1 z_3) = x_1 y_1 + x_2 y_2$ and $v' = -\text{Re}(z_1 z_3) = x_1 y_2 - x_2 y_1$, then we parametrize the surface in terms of $x_3, w$ and an angular direction such that the nontrivial dependence is
\[ x_3(x_3, w'), \quad v'(x_3, w'). \tag{3.17} \]

Now the analog of the constraint (3.14) is
\[ |z_1|^2 + \frac{2L^3\sqrt{|z_1|^2 + y_3^2}}{|z_3|^2 + y_3^2} = |z_1|^2 + \frac{2L^3}{y} = 2a, \tag{3.18} \]
and $\omega|_{v=0} = \text{Im}\Omega|_{v=0}$ imply
\[ \frac{\partial v'}{\partial x_3} = -\left(2\frac{y_3}{L^3} - y_3^2 - y_3^2\right) \frac{\partial x_4}{\partial w'}, \quad \frac{\partial v'}{\partial w} = \frac{\partial x_4}{\partial x_3}. \tag{3.19} \]
The parenthesis in the first equation is simply $y_3|z_1|^2/L^3 + |z_3|^2$. It can be expressed in terms of $v, w, a$ only by recasting the expression for $a$ as a cubic equation for $y$
\[ 2a = \frac{v^2 + w^2}{y^2 - y_3^2} + \frac{2L^3}{y}. \tag{3.20} \]

### 3.1.1. Simplest solutions.

The simplest way to solve (3.16) is if the left equation vanishes identically, so
\[ v = \alpha w + \beta, \quad y_1 = \alpha y_2 + \gamma. \tag{3.21} \]
In this family of solutions, the coordinates $z_1$ and $z_2$ do not depend on the $y_i$ coordinates, as is also the case for holomorphic surfaces in section 2.3. This is not a coincidence, as for large $y$, the surface approaches a single point with $y_1/y_2 = \alpha$, exactly as in the case of surfaces of Type-C. To see the relation, define the complex coordinates
\[ \zeta = x_1 + \frac{i}{\sqrt{1 + \alpha^2}}(x_3 - \alpha x_4), \quad \eta = x_2 - \frac{i}{\sqrt{1 + \alpha^2}}(x_4 + \alpha x_3). \tag{3.22} \]
Then the equations
\[ x_1^2 + x_2^2 - x_3^2 - x_4^2 = 2a, \quad x_1 x_3 - x_2 x_4 = \alpha (x_1 x_4 + x_2 x_3) + \beta, \tag{3.23} \]
combine into the holomorphic quadratic equation \( \zeta^2 + \eta^2 = 2a + 2i\beta/\sqrt{1 + \alpha^2} \).

This solution is more general than that in section 2.3 because of the constant \( \gamma \). When it is different from zero, the surface does not reach \( y_1 = y_2 = 0 \) and instead continues to negative values of \( y_1 \) and \( y_2 \) and another asymptotic region, representing the correlation function of two coincident surfaces with opposite orientation and scalar couplings. This is a rather singular configuration, where the same boundary conditions have degenerate solutions with arbitrary \( \gamma \). These solutions are similar in spirit to those studied in [44], and are also related to zig-zag symmetry, see for example [15, 45, 46]. We find more such degeneracies below in section 3.1.2.

The same can be done with (3.19), where now
\[ v' = \alpha w' + \beta, \quad x_4 = \alpha x_3 + \gamma. \tag{3.24} \]
This simple dependence of \( x_3 \) and \( x_4 \) is a version of Type-\( R \), see section 2.1.

The remaining coordinates satisfy
\[ x_1 y_2 - x_2 y_1 = \alpha (x_1 y_1 + x_2 y_2) - \beta, \quad x_1^2 + x_2^2 + \frac{1}{2y} = 2a. \tag{3.25} \]
For \( \beta = 0 \), this is up to rotations the cylinder solution of [36] reviewed in appendix ‘Cylinder’. The case \( \beta \neq 0 \) has a similar interpretation as the degenerate solutions for two coincident surfaces discussed above.

3.1.2. Torus. Example 5.1 in [43], which was first presented in section III.3.A of [12], has \( U(1)^2 \) symmetry. It is the intersection of the two construction above, so
\[ |z_1|^2 - |z_2|^2 = 2a, \quad |z_1|^2 + \frac{2L^3}{y} = 2b, \quad \text{Im}(z_1 z_2 z_3) = 0. \tag{3.26} \]
The third condition (up to a constant) is a consequence of the second of the equations in (3.16) and (3.19), which can be seen by writing \( z_1, z_2 \) and \( z_3 \) in polar coordinates. From the limit \( y \to \infty \) we find that both \( |z_1| \) and \( |z_2| \) are constants, making a torus with radii squared \( R_1^2 = 2b \) and \( R_2^2 = R_1^2 - 2a \). The third equation gives the relation between the three phases. This completely specifies the surface and one can verify that the differential equations above are indeed satisfied.

One still needs to check whether the solution is regular. An obvious condition is \( R_1^2 > 0, R_2^2 > 0 \). Without loss of generality we can assume \( R_1 \leq R_2 \). Then if we impose \( b = L^3/y_3 \), then at \( z_3 = 0 \) the surface closes off smoothly at \( z_1 = 0 \). This is the solution for the toroidal surface operator found in [36] and reviewed in appendix ‘Torus’.

For \( b < L^3/y_3 \) the surface cannot extend to \( z_3 = 0 \) and for \( b > L^3/y_3 \) it cannot reach \( |z_1| = 0 \). In the former case we can extend the surface to cover the \( |z_1|^2 \leq 2b \) disc twice, while reaching a minimal value of \( |z_3|^2 = L^6/b^2 - y_3^2 \) at \( z_1 = 0 \). This surface has two asymptotic regions with \( y \to \infty \), so is the correlation function of two coincident surface operators with opposite orientation but opposite scalar couplings that allow them to be mutually BPS. For the second case, with \( b > L^3/y_3 \), the surface is a double cover of the \( z_3 \) plane, but covers only annuli in the \( z_1 \) and \( z_2 \) planes (also twice).

In these cases we find the same tori at large \( y \) irrespective of the value of \( y_3 \), so the solutions are degenerate. This resembles the situation studied in [44].
3.1.3. Generalized cylinder. Consider the limit of \( a \to -\infty \) in (3.14). This means that \( |z_1| \ll |z_2| \) and if we look at the \( U(1) \) action for small \( \chi \), it is essentially a translation along the tangent in the \( z_2 \) direction and does not act on \( z_1 \). Assume \( x_4 \sim \sqrt{2|a|} \) and \( x_3 \sim 0 \), then the surface is locally a curve in the \( z_1 \) plane times the \( x_3 \) direction, so a special class that is both Lagrangian and of Type-R (one can easily verify that the ansätze for \( n^{(1)}_I \) agree).

Now we have \( w \sim \sqrt{|a|} x_1 \) and \( v \sim -\sqrt{|a|} x_2 \). Equation (3.16) then becomes

\[
\frac{\partial x_2}{\partial y_2} = \frac{L^3}{y^3} \frac{\partial y_1}{\partial x_1}, \quad \frac{\partial x_1}{\partial y_2} = -\frac{\partial y_1}{\partial y_2}. \tag{3.27}
\]

These equations can also be derived directly from \( \text{Im} \Omega |V = \omega V = 0 \) when we impose the extra constraint \( x_4 = 0 \). It is not too hard to check that they are satisfied by the crease solution of \([32]\) reviewed in appendix 'Crease'.

One can try to take a similar limit on the other \( U(1) \) symmetric configuration (3.18), such that there is translation invariance in the \( y_I \) directions. While the limit is a bit more subtle, the equation (3.19) should be the regular Cauchy-Riemann equations in \( \mathbb{R}^4 \). This reproduces the results from section 2.3, where the surfaces were homogeneous in the \( y_I \) directions.

3.2. Perturbed cylinder

The cylinder solution in appendix 'Cylinder' solves the generalized Cauchy-Riemann equation (3.27). Following Joyce [47], we can study solutions close to it by taking a function \( f(y_2, x_1) \) and taking

\[
y_1 \to y_1 - \frac{\partial f}{\partial x_1}, \quad x_2 \to x_2 + \frac{\partial f}{\partial y_2}. \tag{3.28}
\]

Equation (3.27) now gives

\[
\frac{\partial x_2}{\partial y_2} - \frac{L^3}{y^3} \frac{\partial y_1}{\partial x_1} = \frac{\partial^2 f}{\partial y_2^2} + \frac{L^3}{y^3} \frac{\partial^2 f}{\partial x_1^2} = 0. \tag{3.29}
\]

Alternatively, this can be written as

\[
L^3 \frac{\partial^2 f}{\partial x_1^2} + \left( \frac{y_1 - \frac{\partial f}{\partial x_1}}{y^2 + y_2^2 + y_3^2} \right)^{3/2} \frac{\partial f}{\partial y_2} = 0. \tag{3.30}
\]

Here \( y_1 \) is as in the cylinder solution (A.4), which is an explicit function of \( x_1 \) and \( y_2 \). For small \( f \) we can linearize the equation to

\[
L^3 \frac{\partial^2 f}{\partial x_1^2} + (y_1(x_1, y_2)^2 + y_2^2 + y_3^2)^{3/2} \frac{\partial f}{\partial y_2} = 0. \tag{3.31}
\]

In this equation \( y_1 \) is the original solution we are perturbing about, so generally it is not a separable equation. It would still be nice to find examples where it can be solved explicitly. Other results on special Lagrangian manifolds could also lead to new M2-brane solutions.

Data availability statement

No new data were created or analyzed in this study.
Acknowledgments

We are grateful to J Maldacena, M Martone, D Panov and especially M Probst for helpful discussions. N D would like to thank l’École Polytechnique Fédérale de Lausanne, the Simons Center for Geometry and Physics, Stony Brook and the KITP, Santa Barbara for their hospitality in the course of this work. N D’s research is supported by the Science Technology & Facilities council under the Grants ST/T000759/1 and ST/P000258/1 and by the National Science Foundation under Grant No. NSF PHY-1748958. M T gratefully acknowledges the support from the Institute for Theoretical and Mathematical Physics (ITMP, Moscow) where this project began, and Université Laval, the Simons Center for Geometry and Physics, Stony Brook and New York University where parts of this project were realised. M T’s research is funded by the Engineering & Physical Sciences Research Council under the Grant EP/W522429/1.

Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Research and Innovation.

Appendix. Review of known BPS solutions

**Torus**

The torus solution was presented in [36]. The world volume coordinates are $\rho$, $\varphi_1$ and $\varphi_2$ and in the metric (1.3) it takes the form

$$
\begin{align*}
    x_1 &= r_1(\rho) \cos \varphi_1, & x_2 &= r_1(\rho) \sin \varphi_1, \\
    x_3 &= r_2(\rho) \cos \varphi_2, & x_4 &= r_2(\rho) \sin \varphi_2, \\
    y_1 &= -\rho \sin(\varphi_1 + \varphi_2), & y_2 &= \rho \cos(\varphi_1 + \varphi_2),
\end{align*}
$$

(A.1)

with

$$
\begin{align*}
    r^2_1(\rho) &= R^2_1 - \frac{2L^3}{y}, & r^2_2(\rho) &= R^2_2 - \frac{2L^3}{y}, & y_3 &= \frac{2L^3}{R^2_1}.
\end{align*}
$$

(A.2)

Here $y^2 = y_1^2 + y_2^2 + y_3^2 = \rho^2 + y_3^2$ and $R_1$ and $R_2$ are constants, the radii of the torus at the boundary $y \to \infty$ and we assume $R_1 \leq R_2$.

**Cylinder**

The cylinder is a limit of the torus, where effectively $R_2 \to \infty$. We take $\rho$, $\varphi_1$ and $v$ as world-volume coordinates and

$$
\begin{align*}
    x_1 &= r_1(\rho) \cos \varphi_1, & x_2 &= r_1(\rho) \sin \varphi_1, \\
    x_3 &= 0, & x_4 &= v, \\
    y_1 &= -\rho \sin \varphi_1, & y_2 &= \rho \cos \varphi_1,
\end{align*}
$$

(A.3)

with

$$
\begin{align*}
    r^2_1(\rho) &= R^2_1 - \frac{2L^3}{y}, & y_3 &= \frac{2L^3}{R^2_1}.
\end{align*}
$$

(A.4)
The crease solution was found in [32]. In the BPS case, if we take $r$, $u$ and $v$ as worldvolume coordinates, it can be expressed as

$$
x_1 = r \cos \varphi(u), \quad x_2 = r \sin \varphi(u), \quad x_3 = \sqrt{1 + u^2 v}, \quad x_4 = 0,
$$

$$
y_1 = \frac{4L^3}{u^2 r^2} \sin \vartheta(u), \quad y_2 = -\frac{4L^3}{u^2 r^2} \cos \vartheta(u), \quad y_3 = 0,
$$

with

$$
\tan \vartheta = \frac{2J \sqrt{1 + u^2} - J^2 u^4}{1 + (1 + 2J^2) u^2}, \quad \sin(\varphi - \vartheta) = \frac{Ju^2}{\sqrt{1 + u^2}}.
$$

$J$ is an arbitrary constant, leading to a one-parameter family of solutions.

It is simple to check that this also implies

$$
\frac{\sin \vartheta}{\cos \varphi} = \frac{2J}{\sqrt{1 + u^2}},
$$

where in our notation $u = 2L^{3/2}/r \sqrt{y}$. This can also be written as

$$
r \cos \varphi = \frac{1}{J} \sqrt{\frac{r^2}{4} + \frac{L^3}{y}} \sin \vartheta = 0.
$$

**ORCID iDs**

Nadav Drukker [https://orcid.org/0000-0003-4984-5736](https://orcid.org/0000-0003-4984-5736)
Maxime Trépanier [https://orcid.org/0000-0003-3117-5703](https://orcid.org/0000-0003-3117-5703)

**References**

[1] Drukker N and Trépanier M 2021 Observations on BPS observables in 6D J. Phys. A **54** 20
[2] Witten E 1995 Some comments on string dynamics *Future Perspectives in String Theory. Proc., Conf. (Strings’95) (Los Angeles, USA, 13–18 March 1995)* pp 501–23
[3] Strominger A 1996 Open $p$-branes Phys. Lett. B **383** 44–47
[4] Ganor O J 1997 Six-dimensional tensionless strings in the large $N$ limit Nucl. Phys. B **489** 95–121
[5] Howe P S, Lambert N D and West P C 1998 The selfdual string soliton Nucl. Phys. B **515** 203–16
[6] Zarembo K 2002 Supersymmetric Wilson loops Nucl. Phys. B **643** 44–47
[7] Drukker N, Giombi S, Ricci R and Trancanelli D 2007 More supersymmetric Wilson loops Phys. Rev. D **76** 107703
[8] Drukker N, Giombi S, Ricci R and Trancanelli D 2008 Wilson loops: from four-dimensional SYM to two-dimensional YM Phys. Rev. D **77** 047901
[9] Drukker N, Giombi S, Ricci R and Trancanelli D 2008 Supersymmetric Wilson loops on $S^3$ J. High Energy Phys. JHEP05(2008)017
[10] Bergshoeff E, Sezgin E and Townsend P K 1987 Supermembranes and eleven-dimensional supergravity Phys. Lett. B **189** 75–78
[11] Mezei M, Pufu S S and Wang Y 2019 Chern-Simons theory from M5-branes and calibrated M2-branes J. High Energy Phys. JHEP08(2019)165
[12] Harvey R and Lawson H B 1982 Calibrated geometries Acta Math. **148** 47–157
[13] Dymarsky A, Gubser S S, Guralnik Z and Maldacena J M 2006 Calibrated surfaces and supersymmetric Wilson loops J. High Energy Phys. JHEP09(2006)057
[14] Joyce D 2007 *Riemannian Holonomy Groups and Calibrated Geometry* vol 12 (Oxford: Oxford University Press)
[15] Drukker N, Gross D J and Ooguri H 1999 Wilson loops and minimal surfaces Phys. Rev. D **60** 125006
[16] Rodgers R 2019 Holographic entanglement entropy from probe M-theory branes J. High Energy Phys. JHEP03(2019)092
[17] Drukker N, Probst M and Trépanier M 2020 Surface operators in the 6d $\mathcal{N} = (2, 0)$ theory J. Phys. A 53 365410
[18] Bianchi M, Freedman D Z and Skenderis K 2002 Holographic renormalization Nucl. Phys. B 631 159–94
[19] Deser S and Schwimmer A 1993 Geometric classification of conformal anomalies in arbitrary dimensions Phys. Lett. B 309 279–84
[20] Berenstein D E, Corrado R, Fischler W and Maldacena J M 1999 The operator product expansion for Wilson loops and surfaces in the large $N$ limit Phys. Rev. D 59 105023
[21] Graham C R and Witten E 1999 Conformal anomaly of submanifold observables in AdS/CFT correspondence Nucl. Phys. B 546 52–64
[22] D’Hoker E, Estes J, Gutperle M and Krym D 2008 Exact half-BPS flux solutions in M-theory I: local solutions J. High Energy Phys. JHEP08(2008)028
[23] D’Hoker E, Estes J, Gutperle M and Krym D 2008 Exact half-BPS flux solutions in M-theory II: global solutions asymptotic to $AdS_7 \times S^4$ J. High Energy Phys. JHEP12(2008)044
[24] Bullimore M and Kim H-C 2015 The superconformal index of the $(2, 0)$ theory with defects J. High Energy Phys. JHEP05(2015)048
[25] Gentle S A, Gutperle M and Marasiniou C 2015 Entanglement entropy of Wilson surfaces from bubbling geometries in M-theory J. High Energy Phys. JHEP08(2015)019
[26] Estes J, Krym D, O’Bannon A, Robinson B and Rodgers R 2019 Wilson surface central charge from holographic entanglement entropy J. High Energy Phys. JHEP05(2019)032
[27] Jensen K, O’Bannon A, Robinson B and Rodgers R 2019 From the Weyl anomaly to entropy of two-dimensional boundaries and defects Phys. Rev. Lett. 122 241602
[28] Chalabi A, O’Bannon A, Robinson B and Sisti J 2020 Central charges of 2d superconformal defects J. High Energy Phys. JHEP05(2020)095
[29] Drukker N, Giombi S, Tseytlin A A and Zhou X 2020 Defect CFT in the 6d $\mathcal{N} = (2,0)$ theory with defects JHEP07(2020)101
[30] Drukker N, Probst M and Trépanier M 2021 Defect CFT techniques in the 6d $\mathcal{N} = (2,0)$ theory J. High Energy Phys. JHEP03(2021)261
[31] Wang Y 2021 Surface defect, anomalies and $b$-extremization J. High Energy Phys. JHEP11(2021)122
[32] Drukker N and Trépanier M 2022 Ironing out the crease J. High Energy Phys. JHEP08(2022)193
[33] Gutowski J and Papadopoulos G 1999 AdS calibrations Phys. Lett. B 462 81–88
[34] Gutowski J, Papadopoulos G and Townsend P K 1999 Supersymmetry and generalized calibrations Phys. Rev. D 60 106006
[35] Gutowski J 2001 Generalized calibrations Progress in String Theory and M-Theory (Nato Science Series C vol 564) (Dordrecht: Springer) pp 343–6
[36] Drukker N and Trépanier M 2022 M2-doughnuts J. High Energy Phys. JHEP02(2022)071
[37] Fefferman C and Graham C R 2011 The ambient metric Ann. Math. Stud. 178 1–128
[38] Lee K-M and Yee H-U 2007 BPS String Webs in the 6-dim (2,0) Theories J. High Energy Phys. JHEP03(2007)057
[39] Guralnik Z and Kulik B 2004 Properties of chiral Wilson loops J. High Energy Phys. JHEP01(2004)065
[40] Guralnik Z, Kovacs S and Kulik B 2005 Less is more: non-renormalization theorems from lower dimensional superspace Int. J. Mod. Phys. A 20 4546–53
[41] Joyce D Lectures on special Lagrangian geometry (arXiv:math/0111111)
[42] Joyce D 2000 Special Lagrangian $m$-folds in $\mathbb{C}^n$ with symmetries Duke Mathematical Journal 115 1–51
[43] Joyce D 2003 $U(1)$-invariant special Lagrangian 3-folds in $\mathbb{C}^3$ and special Lagrangian fibrations Turk. J. Math. 27 99–114
[44] Klebanov I R, Maldacena J M and Thorn III C B 2006 Dynamics of flux tubes in large $N$ gauge theories J. High Energy Phys. JHEP04(2006)024
[45] Polyakov A M 1998 String theory and quark confinement Nucl. Phys. B Proc. Suppl. 68 1–8
[46] Polychronakos A M and Rychkov V S 2000 Gauge field strings duality and the loop equation Nucl. Phys. B 581 116–34
[47] Joyce D 2005 $U(1)$-invariant special Lagrangian 3-folds. I. Nonsingular solutions Adv. Math. 192 35–71