A Dynamical Thermostat Approach To Financial Asset Price Dynamics

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Abstract. A dynamical price formation model for financial assets is presented. It aims to capture the essence of speculative trading where mispricings of assets are used to make profits. It is shown that together with the incorporation of the concept of risk aversion of agents the model is able to reproduce several key characteristics of financial price series. The approach is contrasted to the conventional view of price formation in financial economics.

INTRODUCTION

A remarkable feature of (financial) economics is that an immensely complex system - consisting of a relatively large number of human beings, interacting through the exchange of goods, information, services, and money - reduces to one-dimensional functions in time, the prices of assets. These prices potentially carry substantial information about the underlying system, and are therefore an intriguing subject to study.

Technically the asset price, be it of a stock, option, or any other liquid financial derivative can change whenever a trade (interaction of market participants) takes place. Prices are formed through the actions of three types of agents: A seller offers an asset at a certain price, usually through a sell order to the market maker. Potential buyers offer to buy the asset at a certain price, or at any price. The market maker sets the actual unique price at which trades then occur. For convenience one frequently works with the log-return process rather than with the price process itself. The log-return over a specified time interval (aggregation time $\tau$) is simply given by the logarithm of the quotient of prices separated by time $\tau$:

$$r_\tau(t) = \log(p(t + \tau)) - \log(p(t)) \quad .$$

Due to the addition property of logarithms it is easy to change from a given aggregation time $\tau$ to higher ones by taking the prices which form the boundary of the time interval of interest. These return series possess a highly non-trivial structure,
and over the last decades considerable effort has been devoted to quantitatively characterize such processes.

Maybe the best known fact about financial price time-series is that the distribution of returns is leptocurtic. On short time-scales ($\tau$ ranging from minutes to several days), for small price fluctuations the return distribution can be well approximated by a Lévy stable distribution [1], while its tails - emerging from violent price changes - generally follow a power law, see e.g. [2]. It is well established that for longer time-scales, i.e., aggregation times $\tau$ of three weeks and longer, the distribution functions of returns are almost Gaussian. Historically there has been some confusion on this subject: In the 1960s Mandelbrot and Fama found empirical evidence that the cumulative distribution functions of returns of time-scale $\tau$, $f(r_\tau)$, behave like stable Lévy distributions, which are characterized by a parameter $1 \leq \mu \leq 2$, where $\mu = 2$ corresponds to the special case of a normal distribution. For $\mu < 2$ the distributions are fat tailed and higher momenta - including the second - are infinite. They estimated an empirical value of $\mu = 1.7$, indicating that higher momenta will not exist and therefore putting in doubt most statistical quantities. In the 1980s power laws for the tails of the distributions have been found, i.e. $f(r_\tau) \sim |r_\tau|^{-\alpha}$, with $\alpha > 2$. This result has been confirmed throughout the 1990s and the exponent is nowadays believed to be close to $\alpha = 3$ [3].

For liquid assets, $O(100 - 1000)$ trades per day, the stock returns are basically uncorrelated, i.e., the autocorrelation functions with time lags larger than one are usually compatible with zero. However, one notices the fact that there are periods in time when price fluctuations are stronger than at other times. This is called volatility clustering and can be captured quantitatively by the autocorrelation function of the absolute values (or squares) of the returns. These volatility autocorrelations do not vanish exponentially but again show an empirical power-like decay, $\text{lag}^{-\beta}$, with $\beta \sim 0.1 - 0.5$, see e.g. [4–6].

There have been observations that higher momenta of volatility $\langle |r_\tau|^q \rangle$ are scaling processes with respect to aggregation time $\tau$ and that the corresponding scaling exponents are not linear functions in $q$, which is an indication for a multiscaling process [4,7].

Further empirical features of price processes are that prices are non-stationary, the second moment of returns $\langle r^2 \rangle$ exists (but converges slowly), whereas the fourth moment $\langle r^4 \rangle$ does not. Prices are usually quantized (1/16 dollar at the NYSE), and can not jump arbitrarily much from one trade to the other. The closing price is the price of the asset at the end of a trading day.

In this paper we model the one-dimensional price process, by casting some fundamental characteristics of economic systems and human behavior into a set of differential equations, and compare the emerging price time-series statistics with known facts of real price series. The paper is organized in the following way: The next section is a mini-overview on the standard views in (micro) economics of price formation. We then motivate and propose a dynamical system, show some of its features, and discuss the results.
STANDARD TREATMENT IN ECONOMICS

Financial economics literature, in particular the field of micro-economics, is dominated by several concepts which are powerful from a theoretical standpoint but which are known to have their limits when it comes to the analysis of real time-series. These concepts are:

**Dogma 1:** Prices are the equilibrium result of trades, i.e., all sellers and buyers come to an optimum state (Pareto optimum), where everybody is better off than without doing the trades. It can be shown quite generally with the help of the Brower fixed point theorem, that such an equilibrium state exists which also provides an optimum [8]. For this to be true it is important that the influence of trading barriers is kept small. In this approach all the agents involved in trading maximize their von Neumann-Morgenstern utility functions. These are monotonically rising convex functions, which describe the “happiness” of the individual agent as a function of his wealth. The fact that utility functions rise monotonically reflects that “more is preferred to less”, the convexity takes into account the risk-aversion of agents. The main problem associated with this view is that it is not able to account for speculative trading at all. On the other hand one knows that speculative trading is dominating entire markets. For example in 1995 the daily trading volume at the foreign exchange markets exceeded $10^{12}$ USD which was about 20 times the daily world gross national product.

**Dogma 2:** Markets are efficient. The basic requirement for efficiency is that all participating agents are fully rational, that there is rapid information flow, there are no transaction costs or other market friction, and that everybody has identical access to all the relevant information. As a consequence, prices simply reflect the expectations on future earnings from investments made today. Thus trading would only occur at the appearance of news which would influence those expectations. Other than that there would be no trading activity and speculative trading would be entirely useless. Another consequence of the efficiency hypothesis is that there exist no patterns in financial time-series which could be exploited in one or the other fashion. Also there should not exist arbitrage (risk-less profit) opportunities. In fact the non-existence of arbitrage forms the basis of many celebrated results in financial mathematics, especially in the field of pricing financial derivatives.

Obviously the efficient market hypothesis is rather unrealistic and there are strong problems related with it: Agents are not fully rational, but often tend to have non-rational strategies, such as heard behavior, believe in experts’ opinions, etc. Moreover it has been clearly demonstrated that most trading is not influenced by news [9]. Finally, it is needless to mention that there exist financial arbitrage companies which are specialized in exploiting patterns in financial time-series.

**Dogma 3:** Mathematical tools: In financial literature the most commonly used mathematical tool is game theory. Most of the time, non-iterative one or two period games are considered, which severely limits any study of the dynamics of a system. If dynamics is studied, usually the corresponding models are based on Wiener processes (Brownian motion). For example as a starting point for many
models a price process is assumed to have the form

$$dp(t) = \mu p(t) \, dt + \sigma p(t) \, dW(t)$$

(2)

with $\mu$ the so-called drift term, $\sigma$ the volatility, and $W(t)$ the Wiener process. The first term on the right side determines the overall exponential growth of an asset price with growth-rate $\mu$. Superimposed on that, the second term introduces a source of stochasticity, whose relative importance is controlled by the volatility $\sigma$.

Wiener processes are nowadays mathematically well understood and are relatively easy to handle, since they are Gaussian processes. These kind of processes have led to a number of celebrated results in option pricing [10], interest rate models, etc. For an overview see [11]. However, there are severe problems associated with Wiener processes if one tries to explain real price series. Clearly, in Gaussian price processes no power laws are present in the return distributions, there is no volatility clustering and there exist no power laws in the volatility autocorrelations.

In order to make things more realistic, so-called ARCH-GARCH processes have been introduced [12]. Such models are nowadays frequently used in everyday econometrics and risk management. A GARCH(1,1) process has one source of randomness and obeys the following evolution equations:

$$\epsilon_t = \sigma_t z_t \quad ; \quad z_t \sim \mathcal{N}(0, 1)$$
$$\sigma_t^2 = a\sigma_{t-1}^2 + b\epsilon_{t-1}^2 + c$$

(3)

where $z_t$ are uncorrelated normal random variables, and $a, b$ and $c$ are real numbers. Such models are able to explain clustered volatility, but fail to give power laws in the volatility autocorrelations. The main problem with GARCH models is that they are ad hoc models and do not relate the variables to an economic context.

Obviously there is a need for a better understanding of the origin of the basic features of price processes. Recently there have been put forward some new ideas from the mathematics and physics community, some of which are certainly more successful in describing reality than the standard methods used in economics. Just to mention a few directions of new developments, there are contributions from evolutionary game theory [13], agent based models and minority games [14], and spinglass-inspired models [15]. Stochastic models of market maker behavior have been discussed in [16] and in market impact models [17] several realistic price formation scenarios are studied. In the following we present a dynamical systems approach, which tries to model fundamental behavior of speculating market participants. The aim is to capture this behavior in a set of coupled differential equations. The solution to this system yields the price process.

THE DYNAMICAL THERMOSTAT MODEL

The model is supposed to capture three fairly general features of price dynamics and investor behavior:
**Fact 1:** If money is invested in a bank or in government bonds, which is often referred to as a risk-less investment, wealth will increase exponentially with a fixed interest rate (for short-term investments). If one decides to invest in risky assets like stock, this exponential tendency (drift) should also be present with a somewhat larger expected rate of return, which compensates for taking the risk (risk premium). The price dynamics of a risky asset will be more complicated since it is coupled to a complex dynamical system - the market. This can be written as

\[ \dot{p}(t) = \mu p(t) - c \xi_1(t)p(t) \]  

(4)

with \( \mu \) the drift coefficient and \( c \) a coupling constant. \( \xi_1 \) is a coupling or friction variable, whose dynamics is the main subject of this work. Note that this equation is formally similar to the standard price dynamics using Wiener processes, Eq. (2).

**Fact 2:** The essence of speculative trading is that agents try to make profits by using mispricings of assets. Suppose there exists the “true” value of an asset at a given time \( t \), \( T_0(t) \). The actual price \( p(t) \) can, and in general will, differ from this value. As an example of how such a mispricing can be used to make profits imagine that the asset is “under-priced”, i.e., the actual price is lower than the true value, \( p < T_0(t) \). Alert agents who notice this difference will buy the asset and hold it until the market as a whole (not that alert) values the asset correctly. In this course the price of the asset will rise by an increased demand, and the mispricing gets reduced. Obviously the rise in price can be used by the agents to realize profits. If an asset is “over-priced” there are also ways to make profits by so-called “short-selling”. In this context it seems reasonable to view the price changes \( \xi_1 \) in Eq. (4) proportional to the mispricing:

\[ \dot{\xi}_1 = \frac{1}{\tau_1} [p - T_0(t)] \]  

(5)

with \( \tau_1 \) being a constant, which can be interpreted as the on-set time of the correctional movements. As long as the asset is under-priced, \( \xi_1 \) is negative, \( \xi_1 \) will decline and eventually fall below zero, which leads to a positive second term on the right side of Eq. (4), which makes the price rise.

**TABLE 1.** Meaning of model constants and variables and its relation to physics

| symbol | meaning in model | relates to in physics |
|--------|------------------|-----------------------|
| \( p \) | price function | momentum |
| \( \mu \) | risk-free rate (bank rate) | “force” term |
| \( c \) | primary coupling of the asset to the market | |
| \( \xi_1 \) | relative price-change variable | thermostat variable |
| \( \xi_2 \) | control variable for \( \xi_1 \) | thermostat variable |
| \( T_0(t) \) | fundamental or “true” value of asset | temperature |
| \( T_1(t) \) | collective (inverse) risk-aversion factor | temperature |
| \( \tau_i \) | response on-set times | response on-set times |
Fact 3: Price changes are not un-restricted, which can be explained by the fact that agents are risk averse. Agents are aware that they can misjudge the situation about mispricings and as a consequence make losses. Since they are not entirely sure if their investment will be profitable or not, they will invest only limited amounts. The model should thus have a handle to restrict the price changes $\xi_1$ to a region around zero with controllable variance. Technically this can be done by introducing a new dynamical variable $\xi_2$ and by adding a term $-\xi_1 \xi_2$ to the left hand side of Eq. (5). The dynamics of $\xi_2$, which keeps the variance of $\xi_1$ at a value of $T_1$, is given by

$$
\dot{\xi}_2 = \frac{1}{\tau_2} \left[ \xi_2^2 - T_1(t) \right],
$$

where $\tau_2$ is another time constant. $T_1(t)$ is a factor (maybe time dependent) that can be interpreted as a measure of collective inverse risk aversion of the involved agents. If $T_1$ is small, agents are not willing to take risk and only small price changes will occur. By gathering the above arguments we propose the following model:

$$
\begin{align*}
\dot{p} &= \mu p - cp \xi_1 \\
\dot{\xi}_1 &= \frac{1}{\tau_1^2} \left[ p - T_0(t) \right] - \xi_1 \xi_2 \\
\dot{\xi}_2 &= \frac{1}{\tau_2^2} \left[ \xi_1^2 - T_1(t) \right].
\end{align*}
$$

The variables are collected in Table 1. The first two equations have the form of dynamical thermostat equations which have been introduced some time ago [18]. To build the bridge to statistical physics, if one considers $p^2$ instead of $p$ in Eqs.
TABLE 2. Parameter dependence of log-return distribution functions. \( \alpha \) corresponds to the tail exponent at a given aggregation time \( \tau \). For all simulations we took 1000 time steps, and fixed \( T_1 = T_0 \).

| \( c \) | \( \tau_1 = \tau_2 \) (response) | \( \langle r_{\tau=1} \rangle \) | \( \langle r_{\tau=1}^2 \rangle \) | \( \alpha_{\tau=1} \) |
|------|-------------------------------|-----------------|-----------------|-----------------|
| 0.18 | 0.3                           | 0.0038          | 0.037           | 2.49            |
| 0.18 | 0.4                           | 0.0041          | 0.051           | 3.25            |
| 0.18 | 0.5                           | 0.0034          | 0.131           | 2.27            |
| 0.18 | 0.6                           | 0.0040          | 0.097           | 2.12            |
| 0.30 | 0.3                           | 0.0036          | 0.067           | 2.62            |
| 0.30 | 0.4                           | 0.0039          | 0.064           | 3.07            |
| 0.30 | 0.5                           | 0.0039          | 0.081           | 1.60            |
| 0.30 | 0.6                           | 0.0037          | 0.120           | 1.36            |

(7), the first equation stays the same up to a rescaling factor 2, and the second equation can be seen as a dynamical thermostat, which keeps the kinetic energy \( \langle p^2 \rangle \) at a temperature \( T_0(t) \). \( \tau_1 \) is then the thermostat on-set time.

Model features

The model is written in continuous variables. However, trading is not a continuous process but occurs at specific points in time. To capture this feature we solve the equations numerically with a Runge Kutta 4th order solver with time increments \( \Delta t \) of 1/1000 - 1/10000. Every time increment is supposed to model one single trade (tic-scale). The prices are recorded at integer times 1,2,...,N which represent the “closing prices”, i.e., one time unit is 1 day. \( \Delta t \) can thus be considered an additional (hidden) parameter of the model. In the following we decided to use a dynamical step size, according to predetermined error tolerances. The variable of interest is \( p \), which will be analyzed in the following in the identical manner as realistic stock data. In Fig. 1a we show by numerical simulation that the third equation in Eqs. (7) controls the variance of price changes, such that \( T_1=\text{Var}(\xi_1) \). The mean values of \( \xi_1(t) \) (boxes) are clearly compatible with zero, the variance (circles) rises linearly with the (constant) value of \( T_1 \). The corresponding distributions of the \( \xi_1(t) \) appear to be fat tailed but have the desired variance. The time-series and corresponding histograms of friction variables \( \xi_1 \) in a generic case are seen in Fig. 1b. The solid line in the histograms is \( \exp[-(\xi_1/T_1)^2] \).

For simplicity we set \( \mu = 0 \) for the remainder of the paper. To model a realistic situation, we chose \( T_0(t) = \exp(rt) \) with \( r = 0.0007 \) which corresponds to an average growth of about 19% per year (\( \sim 250 \) trading days). We assume that the risk aversion of agents becomes less as their wealth increases (convexity of utility functions), which we took into account by simply setting \( T_0(t) = T_1(t) \). Certainly one could think of other choices which capture the same feature. The on-set times \( \tau_1 \) and \( \tau_2 \) have been chosen to be less than one, which states that decisions on
mispricings are made on a time-scale below one day, which is nowadays a realistic assumption for liquid assets (intra day trading). With the coupling constant $c$, it is possible to weigh the fluctuations relatively to the drift term. In the following we set it around 0.1 and 0.3. In Fig. 2 we show model results for the price series (a) and the corresponding log-returns (b). Clearly, the price follows the exponential form of $T_0(t)$ (dashed line) and the returns show clusters of enhanced volatility.

To become more quantitative we looked at the first 100 lags of the normalized autocorrelation functions of squared log-returns. In Fig. 2c the situation is shown for the same parameters as before. The first 100 lags of the autocorrelation is seen to be compatible with a power decay (straight line) with an exponent of the order of $\beta \sim 0.1$, which is within the realistic range.

In Table 2 we gather the power exponents $\alpha$ of the return distribution functions for an aggregation time of $\tau = 1$ “day”. These values have been obtained by linearly fitting the tails of the distribution functions in a log-log plot, and by averaging over 3 independent runs. In most cases $\alpha$ lies in the range between 2 and 3, depending
somewhat on the on-set times $\tau_1$ and $\tau_2$. This result is again in good agreement with observed data.

Finally we computed the higher momenta $\langle |r_\tau|^q \rangle$ of the return processes for various $q$ values, Fig. 3 (top). We observe that $\langle |r_\tau|^q \rangle$ scales with respect to aggregation time $\tau$. The corresponding log-log slopes do not depend on $q$ linearly, which is indicated by a slightly declining $\langle |r_\tau|^q / \langle |r_\tau| \rangle^q \rangle$, Fig. 3 (bottom). This suggests that the underlying process is a multifractal. When comparing to realistic stock data of about the same data length (left), again, nice agreement is found.

The presented model has a rich dynamical structure which will be subject for a more detailed investigation in future work. Here we just mention that formally the stationary solutions are given by $p = 0$, $\xi_1 = \pm \sqrt{T_1}$, and $\xi_2 = \pm \frac{1}{\tau_1^2}(-T_0)/\sqrt{T_1}$. The determinant of the Jacobian matrix is $\text{Det}(M) = -2c \xi_1^3/\tau_2^2$ which for the stationary solutions reduces to $-2c T_1^{3/2}/\tau_2^2$. For the same set of parameters as before the attractor is plotted in Fig. 2d for several time steps. Note that the attractor is on the “tic-scale” and that the price process used for the above analysis takes only every 1000th point (on average) from that trajectory.

**DISCUSSION AND CONCLUSION**

We have presented a non-equilibrium price evolution model, which captures regulatory price movements through the agents’ desire to use mispricings to make profits, and which takes into account risk-aversion of those agents. We contrasted this model to the standard believes of financial economics where prices emerge from equilibrium, by agents maximizing their utility functions. In our approach we do not use the standard mathematical tools like Wiener processes or game theory.
The apparent differences of our dynamical system with the standard view is the explicit use of a non-equilibrium concept, which is realized by the introduction of thermostats which model the behavior of agents. Our model does not contain any sources of randomness, the erratic behavior originates from the non-linear nature of the model. Similarities to the standard approach are that we also use the concept of risk-aversion and that the basic price equation is formally similar to the standard formulation. The presented model - even though very simple conceptually - leads to realistic looking price dynamics over a wide range of parameter settings. Volatility clustering, fat-tailed distributions of returns, correct looking autocorrelation functions, higher momenta and multifractal spectra are well reproduced. Of course, a purely deterministic model can never be the full truth of a complex system like financial markets, but it is intriguing that it is possible to explain most stylized facts of its resulting time series. Whether there are practical sides to the model remains to be seen. To mention an apparent consequence, in the present framework the pricing of financial derivatives would be fundamentally different from current practice where arbitrage free prices are derived from replicating portfolios, which are chosen such that the (Wiener) random sources cancel each other.

REFERENCES

1. Mandelbrot, B.B., *J. Business* **36**, 394 (1963); Fama, E.F., *J. Business* **38**, 34 (1965).
2. Koedijk K.G., Schafgans M.M.A., and De Vries C.G., *J. International Economics* **29**, 93 (1990).
3. Plerou, V., et al., arXiv:cond-mat/9907161.
4. Ghashghaie S., et al., *Nature* **381**, 767 (1996).
5. Arneodo A., Muzy J.-F., and Sornette D., *Europ. Phys. J. B* **2**, 277 (1998).
6. Ding Z., Granger C.W.J., and Engle R.F., *J. Emp. Finance* **1**, 83 (1993).
7. Schmitt F., Schertzer D., and Lovejoy S., *Applied Stochastic Models and Data Analysis* **15**, 29 (1999).
8. Arrow K., and Debreu G., *Econometrica* **22**, 265 (1954).
9. Cutler D.M, Poterba J.M, and Summers L.H., *Journal of Portfolio Management* 4 (Spring 1989).
10. Black F., and Scholes M., *Journal of Political Economy* **3**, 637 (1973).
11. Björk, T., *Arbitrage theory in continuous time*, Oxford: Oxford University Press, (1998).
12. Bollerslev T., Chou R.Y., and Kroner K.F., *Journal of Econometrics* **52**, 5 (1992).
13. Sigmund, K., these proceedings.
14. LeBaron B., Arthur W.B., and Palmer R., *J. Economic Dynamics and Control* **23**, 1487 (1999).
15. Iori, G., arXiv:adap-org/9905005.
16. Maslov, S., arXiv:cond-mat/9910502.
17. Farmer, J.D., SFI preprint 98-12-117.
18. Nosé, S., *J. Chem. Phys.* **81**, 511 (1984); *Molec. Phys.* **52**, 255 (1984); *Prog. Theor. Phys. Suppl.* **103**, 1 (1991); Hoover, W.G., *Phys. Rev. A* **31**, 1695 (1985).