SKEW HOWE DUALITY FOR $U_q(\mathfrak{gl}_n)$ VIA QUANTIZED CLIFFORD ALGEBRAS

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Abstract. We develop an operator commutant version of the First Fundamental Theorem of invariant theory for the general linear quantum group $U_q(\mathfrak{gl}_n)$ by using a double centralizer property inside a quantized Clifford algebra. In particular, we show that $U_q(\mathfrak{gl}_m)$ generates the centralizer of the $U_q(\mathfrak{gl}_n)$-action on the tensor product of braided exterior algebras $\Lambda_q(C^n)^{\otimes m}$. We obtain a multiplicity-free decomposition of the $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-module $\Lambda_q(C^n)^{\otimes m} \cong \Lambda_q(C^{nm})$ by computing explicit joint highest weight vectors. We find that the irreducible modules in this decomposition are parametrized by the same dominant weights as in the classical case of the well-known skew $GL_n \times GL_m$-duality. Clifford algebras are an essential feature of our work: they provide a unifying framework for classical and quantized skew Howe duality results that can be extended to include orthogonal algebras of types $BD$.

1. Introduction

In this article we develop a classical and a quantum skew Howe duality result for Type $A$ algebras. The classical result lays the foundations for our quantized theorem, and the quantized result suggests the generalization to orthogonal types obtained in [Abo22]. Clifford algebras are an essential feature of our work, as they provide a unifying framework for discussing duality results for skew-symmetric variables in both the classical and quantum cases, for Types $ABD$.

Our duality results extend a long and rich tradition in invariant theory tracing back to at least Weyl’s prominent book, The Classical Groups [Wey39]. The fundamental theorems of classical invariant theory may be formulated as a solution to the following problem: given a module $V$ for a reductive group $G$ over a field $k$, provide a complete description of $\text{End}_{k[G]}(V^{\otimes m})$. The First Fundamental Theorem (FFT) gives generators for the centralizer algebra and the Second Fundamental Theorem (SFT) describes all relations amongst them [GW09, Wey39]. A typical case addressed by the fundamental theorems is when $G$ is one of the classical complex Lie groups and $V$ is the natural $G$-module. In these cases, the fundamental theorems are known as Schur-Weyl-Brauer duality. For details, see Section 5.6 in [GW09].

Various results are known in the quantized setting. For instance, when $\mathfrak{g}$ is a classical complex Lie algebra and $V$ is the natural module of the quantized enveloping algebra $U_q(\mathfrak{g})$, $\text{End}_{U_q(\mathfrak{g})}(V^{\otimes m})$ is a quotient of a Hecke algebra if $\mathfrak{g} = \mathfrak{gl}_n$ and it is a quotient of a Birman-Murakami-Wenzl (BMW) algebra in the remaining cases [Jim86, LR97, LZ06]. Lehrer and Zhang describe $\text{End}_{U_q(\mathfrak{g})}(V^{\otimes m})$ for any irreducible $U_q(\mathfrak{gl}_2)$-module $V$ and when $V$ is the 7-dimensional irreducible $U_q(\mathfrak{gl}_2)$-module in [LZ06]. Wenzl covered some cases where $\mathfrak{g}$ is of Lie types $D$ and $E$ in [Wen12, Wen03].

There are various equivalent formulations of the fundamental theorems; in this article and the next we consider operator commutant versions [How95, Theorems 4.2.5, 4.3.4]. We adopt Howe’s

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perspective, which uses multiplicity-free actions satisfying a double centralizer property as an organizing principle, and apply it to the setting of $q$-skew-symmetric variables.

In this article we focus Type $A$ algebras. Our main result, Theorem 4.17, obtains an FFT for $U_q(\mathfrak{gl}_n)$ by computing the multiplicity-free decomposition of a certain quantized exterior algebra under the $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-action induced by embeddings of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ into a quantized Clifford algebra. Diagram (2) summarizes the construction of Section 4.

We begin by re-proving the classical skew $GL_n \times GL_m$-duality Theorem 3.10. This theorem is well-known and may be found in various references, e.g., see Theorem 4.1.1 in [How95] or Theorem 38.2 in [Bum04]. We provide an alternative proof using a Clifford algebra. In particular, Section 3 constructs commuting embeddings of $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$ into the Clifford algebra $Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*)$, as illustrated in the following diagram, and then computes a multiplicity-free decomposition of the exterior algebra $\Lambda(\mathbb{C}^{nm})$ as a module of the tensor product of enveloping algebras $U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_m)$. We describe this novel method because it generalizes to the quantum case, allowing us to prove our skew $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-duality Theorem 4.17. Howe does not provide these Clifford embeddings explicitly in [How95], but they are crucial to the generalization to Types BD developed in [Abo22].

\[
\Lambda(\mathbb{C}^{m})^n \cong \Lambda(\mathbb{C}^{nm}) \cong \mathbb{C}^{n+m} \\
\cap \\
Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*) \\
\lambda \\
\rho \\
\mathfrak{gl}_n \quad \quad \quad \quad \quad \mathfrak{gl}_m
\]

Here $Cl(V \oplus V^*)$ denotes the Clifford algebra generated by the complex space $V \oplus V^*$ equipped with the canonical symmetric bilinear form arising from the duality pairing between $V$ and $V^*$. When necessary, we view the Clifford algebra as a Lie algebra with bracket given by the usual algebra commutator.

With Theorem 3.10 in hand, we turn to our main result: the skew $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-duality Theorem 4.17. The first step is to obtain commuting actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ on the braided exterior algebra $\Lambda_q(V^{(nm)})$, with $V^{(\rho)}$ denoting the natural $U_q(\mathfrak{gl}_p)$-module. Braided exterior algebras were introduced in [BZ05] as $U_q(\mathfrak{g})$-module analogues of the exterior algebra $\Lambda(V)$. Both the classical and the braided exterior algebras can be understood as quotients of a tensor algebra modulo an ideal generated by “symmetric” 2-tensors. In the classical case, these “symmetric” 2-tensors are always the +1 eigenvectors of the trivial flip maps that transpose tensor factors; in the quantum case, they are the eigenvectors with eigenvalue of the form $+q^r$ for some $r \in \mathbb{Q}$ of the braiding operators induced by the $R$-matrix of $U_q(\mathfrak{g})$. Section 4.1 recalls the construction of $\Lambda_q(V^{(n)})$ in detail.

Section 4.2 obtains quantum group actions on $\Lambda_q(V^{(nm)})$ by emulating the classical Clifford algebra construction we develop in Section 3. In particular, we define quantum analogues $\lambda_q$ and...
\( \rho_q \) of \( \lambda \) and \( \rho \) as illustrated in the following diagram.

\[
\begin{array}{ccc}
\Lambda_q(V^{(nm)}) & \cong & \Lambda_q(V^{(n)})^{\otimes m} \\
\otimes & \Downarrow & \otimes \\
Cl_q(nm) & \leftarrow & \rho_q \\
\alpha & \leftarrow & \alpha \\
U_q(\mathfrak{gl}_n) & \longrightarrow & U_q(\mathfrak{gl}_m)
\end{array}
\]

In the top row we have an isomorphism of \( U_q(\mathfrak{gl}_n) \)-modules. We use \( Cl_q(nm) \) to denote the quantum Clifford algebra. The quantized Clifford algebra was first defined by Hayashi in [Hay90]. In this article we use a similar version due to Kwon [Kwo14]. In [AS22] we recall basic properties of the classical Clifford algebras and study their quantized counterparts in depth: we obtain new results on their algebraic structure and representation theory, including a center calculation, a factorization as a tensor product, and a complete list of irreducible representations. Proposition 4.10 shows that \( \lambda_q \) factors through \( U_q(\mathfrak{sl}_{nm}) \), much like \( \lambda : \mathfrak{gl}_n \rightarrow Cl(\mathbb{C}^m \oplus (\mathbb{C}^m)^*) \) factors through \( \mathfrak{gl}_{nm} \) in the classical case.

In contrast to the classical picture of Diagram (1), the map \( \rho_q \) does not factor through \( U_q(\mathfrak{gl}_{nm}) \). In the quantum case, a single choice of weight basis cannot simultaneously describe both the isomorphism of \( U_q(\mathfrak{gl}_n) \)-modules \( \Lambda_q(V^{(nm)}) \cong \Lambda_q(V^{(n)})^{\otimes m} \) and the isomorphism of \( U_q(\mathfrak{gl}_m) \)-modules \( \Lambda_q(V^{(nm)}) \cong \Lambda_q(V^{(m)})^{\otimes n} \): the very construction of \( \Lambda_q(V^{(p)}) \) depends on a choice of weight basis of \( V^{(p)} \) through the \( R \)-matrix of \( U_q(\mathfrak{gl}_p) \), which is defined in terms of its action on a given basis. Remark 3.8 explains that, as illustrated in Figure 1, the action of simple root vectors in \( U_q(\mathfrak{gl}_{nm}) \) on a chosen \( U_q(\mathfrak{gl}_{nm}) \)-weight basis of \( \Lambda_q(V^{(nm)}) \) does not simultaneously “align” with the action simple root vectors in both \( U_q(\mathfrak{gl}_n) \) and \( U_q(\mathfrak{gl}_m) \).

Regardless, \( \lambda_q \) and \( \rho_q \) still define commuting actions on the \( Cl_q(nm) \)-module \( \Lambda_q(V^{(nm)}) \). In fact, Proposition 4.16 shows that \( U_q(\mathfrak{sl}_n) \) and \( U_q(\mathfrak{gl}_m) \) generate mutual commutants in \( \text{End}(\Lambda_q(V^{(nm)})) \). Although \( U_q(\mathfrak{sl}_n) \) and \( U_q(\mathfrak{gl}_m) \) induce commuting subalgebras of module endomorphisms, the images of \( \lambda_q \) and \( \rho_q \) do not define commuting subalgebras in \( Cl_q(nm) \). This discrepancy is explained by the fact that the \( Cl_q(nm) \)-module \( \Lambda_q(V^{(nm)}) \) is not faithful [AS22, Theorem 2.19]. This is contrary to the classical case, where the representation \( \Lambda(\mathbb{C}^m) \) of the simple algebra \( Cl(\mathbb{C}^m \oplus (\mathbb{C}^m)^*) \) is faithful and the images of \( \lambda \) and \( \rho \) indeed define commuting subalgebras in \( Cl(\mathbb{C}^m \oplus (\mathbb{C}^m)^*) \).

We conclude our proof of Theorem 4.17 by computing joint highest weight vectors in \( \Lambda_q(V^{(nm)}) \) with respect to the \( U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{gl}_m) \)-action induced by \( \lambda_q \otimes \rho_q \) and practicing the double-commutant yoga introduced by Howe in [How95].

We note that Theorem 6.16 in [LZZ10] proves a similar related result. It computes the multiplicity-free decomposition of \( \Lambda_q(V^{(n)} \otimes V^{(m)}) \) as a \( U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{gl}_m) \)-module. The braided exterior algebra \( \Lambda_q(V^{(n)} \otimes V^{(m)}) \) is constructed using the braiding induced by the tensor product of the \( R \)-matrices of \( U_q(\mathfrak{sl}_n) \) and \( U_q(\mathfrak{gl}_m) \). While \( \Lambda_q(V^{(n)} \otimes V^{(m)}) \) is isomorphic as a vector space to the braided exterior algebra \( \Lambda_q(V^{(nm)}) \) considered here, which is constructed using the braiding of \( U_q(\mathfrak{gl}_{nm}) \) instead, they have different algebra structures. The structure considered here yields an isomorphism \( \Lambda_q(V^{(nm)}) \cong \Lambda_q(V^{(n)})^{\otimes m} \) of \( U_q(\mathfrak{sl}_n) \)-module algebras that motivates an extension to orthogonal types through the see-saw developed in [Abo22].
Although [LZZ10] also defines a multiplicity-free action of \( U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m) \) on a quantized exterior algebra, it does not consider a double centralizer property inside a quantized Clifford algebra. In our work, Clifford algebras are essential. They provide a unifying framework for skew Howe duality results: in each case we consider certain subalgebras of a Clifford algebra defining commuting actions on a skew-symmetric space. In addition, the Clifford algebras framework elucidates a correspondence between the classical and quantized cases, in the sense that the operators induced by generators of quantized enveloping algebras de-quantize to their classical counterparts in the limit \( q \to 1 \).

The extension to types BD described in [Abo22] is a particularly remarkable feature of our method. In that setting, the Clifford algebras framework facilitates a computation of explicit joint duality results: in each case we consider certain subalgebras of a Clifford algebra defining commuting actions on a tensor product of exterior algebras, and finally the third achieves a multiplicity-free decomposition using the actions just defined.

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### 2. Notation and conventions

We will use \( e_i \) throughout to denote the standard basis of \( \mathbb{R}^n \).

In this section we fix our notation for Lie algebras and recall some definitions.

**Lie algebra presentation.** Following [CP94], we use the Chevalley presentation throughout. If \( \mathfrak{g} \) is a complex semisimple Lie algebra, we let \( A = [a_{ij}] \) denote its generalized Cartan matrix. We let \( D = \text{diag}(d_1, \ldots, d_r) \) denote the diagonal matrix of root lengths such that \( DA \) is symmetric positive definite. Recall \( a_{ii} = 2 \) and \( a_{ij} \leq 0 \) whenever \( i \neq j \).

The semisimple algebra \( \mathfrak{g} \) is generated by \( H_i, E_i, \) and \( F_i \), for \( i = 1, \ldots, r \), subject to the relations

\[
[H_i, H_j] = 0, \quad [H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j, \quad [E_i, F_j] = \delta_{ij} H_i,
\]

along with the Serre relations for \( i \neq j \):

\[
(ad_{E_i})^{1-a_{ij}}(E_j) = 0, \quad \text{and} \quad (ad_{F_i})^{1-a_{ij}}(F_j) = 0
\]

**Remark.** Authors like Jantzen [Jan96], Lusztig [Lus88], and Sawin [Saw96] prefer the notation \( X_i^+ = E_i \) and \( X_i^- = F_i \), suggestive of positive and negative roots in general, and of “raising” and “lowering” operators in this context.
We will denote by $U(\mathfrak{g})$ the universal enveloping algebra associated to $\mathfrak{g}$.

Now fix the Cartan subalgebra $\mathfrak{h}$ generated by the $H_i$. The simple roots of $\mathfrak{g}$ are the linear functionals $\alpha_i : \mathfrak{h} \rightarrow \mathbb{C}$ satisfying
\[ \alpha_i(H_j) = a_{ji}. \]

We let
\[ \mathcal{P} = \{ \lambda \in \mathfrak{h}^* \mid \lambda(H_i) \in \mathbb{Z} \} \quad \text{and} \quad \mathcal{P}^+ = \{ \lambda \in \mathfrak{h}^* \mid \lambda(H_i) \in \mathbb{Z}_+ \} \]
denote the lattice of weights and dominant weights, respectively.

Bilinear form. When $\mathfrak{g}$ is semisimple, there exists a unique non-degenerate symmetric invariant bilinear form $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ such that
\[ \langle H_i, H_j \rangle = d_i^{-1} a_{ij}, \quad \langle H_i, E_j \rangle = \langle H_i, F_j \rangle = 0, \]
\[ \langle E_i, E_j \rangle = \langle F_i, F_j \rangle = 0, \quad \text{and} \quad \langle E_i, F_j \rangle = d_i^{-1} \delta_{ij}, \]
for all $i, j$ [Kac83, Theorem 2.2]. When $\mathfrak{g} = \mathfrak{gl}_n$ we take $\langle , \rangle$ to be the non-degenerate trace bilinear form of the natural representation.

The bilinear form $\langle , \rangle$ induces an isomorphism of vector spaces $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$ under which $H_i$ corresponds to the coroot $\alpha_i^\vee = d_i^{-1} \alpha_i$. The induced form on $\mathfrak{h}^*$ satisfies
\[ \langle \alpha_i, \alpha_j \rangle = d_i a_{ij}. \]

Since $a_{ii} = 2$, we must have $d_i = \langle \alpha_i, \alpha_i \rangle / 2$. Thus we obtain a formula for the Cartan integers:
\[ a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle = \frac{2 \langle \alpha_i^\vee, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}. \]

Notice the bilinear form induced by $\nu$ is normalized so that
\[ \langle \alpha_i, \alpha_i \rangle = 2d_i. \]

When $\mathfrak{g}$ is simple $\langle \alpha_i, \alpha_i \rangle = 2$ for short roots.

For concreteness and convenience, we record some relevant Cartan matrices:
\[ A_n = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ \ddots & \ddots & \ddots \\ -1 & 2 & -1 \\ -1 & -1 & \ddots \\ -1 & -1 & \ddots & \ddots \\ -1 & -1 & \ddots & \ddots & 2 \end{bmatrix}, \]
\[ B_n = \begin{bmatrix} A_{n-1} & 0 & \vdots & 0 \\ 0 & 0 & \vdots & -2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \text{and} \quad D_n = \begin{bmatrix} A_{n-1} & 0 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}. \]

We label the matrices by the Lie type of the root system to which they are associated. The corresponding diagonal root lengths matrices are described by $d = (1, \ldots, 1)$ for the root systems of types $A_n, D_n$, and by $d = (2, \ldots, 2, 1)$ for type $B_n$.

We denote the half sum of the positive roots in $\mathfrak{h}^*$ by $\rho$. The Weyl vector $\rho$ is uniquely characterized by $\langle \alpha_i^\vee, \rho \rangle = 1$. 
Casimir element. Let $X_\alpha$ denote any basis of $\mathfrak{g}$, and let $X^\alpha$ denote the corresponding dual basis with respect to $\langle \cdot, \cdot \rangle$. The next formula uniquely characterizes the Casimir element

$$C = \sum_\alpha X_\alpha X^\alpha = \sum_\alpha X^\alpha X_\alpha \in U(\mathfrak{g}).$$

(7)

The Casimir element is in fact canonical and central [Bum04, Theorem 10.2]. In addition, $C$ acts as the scalar

$$\chi_\lambda(C) = \langle \lambda, \lambda + 2\rho \rangle$$

(8)
on any irreducible $U(\mathfrak{g})$-module with highest weight $\lambda$ [Kac83, Corollary 2.6].

Quantum groups and their representations. Throughout this work, the term “quantum group” refers to an associative Hopf algebra $U_q(\mathfrak{g})$ as presented in [CP94, Definition 9.1.1]. We note the following identity is often useful for verifying $q$-Serre relations when $a_{ij} = -1$. Writing $[A, B]_q = AB - qBA$, we have

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right]_q X_i^k X_j X_i^{1-a_{ij} - k} = X_i (X_i X_j - q_i X_j X_i) - q_i^{-1} (X_i X_j - q_i X_j X_i) X_i$$

$$= [X_i, [X_i, X_j]_q]_q^{-1}. \tag{9}$$

Unless otherwise stated, we assume that $U_q(\mathfrak{g})$ is equipped with the co-algebra structure defined by the comultiplication map $\Delta: U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ satisfying

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i$$

$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

and

$$\Delta(K_i) = K_i \otimes K_i, \tag{10}$$

as in [LZZ10]. The choice of comultiplication convention is important in defining commuting module actions, where different conventions are needed in order to ensure that relevant quantum groups induce commuting endomorphism subalgebras.

In this work, we only consider finite-dimensional type $(1, \ldots, 1)$ modules for the quantum groups $U_q(\mathfrak{g})$. Refer to Chapters 9 and 10 in [CP94] for definitions. Every type $(1, \ldots, 1)$ finite-dimensional $U_q(\mathfrak{g})$-module is semisimple [Jan96, Theorem 5.17] and each simple $U_q(\mathfrak{g})$-module is in one-to-one correspondence with a $\mathfrak{g}$-module parametrized by a dominant weight of $\mathfrak{g}$ [Jan96, Theorem 5.10]. Moreover, corresponding modules have the same weight multiplicities [Jan96, Lemma 5.14].

### 3. Classical skew duality via Clifford algebras

In this section we (re)-prove the classical skew $GL_n \times GL_m$-duality Theorem 3.10 using a double centralizer property inside a Clifford algebra. This theorem is well-known and there are various proofs in the literature. We develop a Clifford algebra argument here because it generalizes to the quantum case, as illustrated by Diagrams (1) and (2), allowing us to prove our skew $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-duality Theorem 4.17 in Section 4. The enveloping algebra $U(\mathfrak{gl}_n)$ more closely resembles the quantum group $U_q(\mathfrak{gl}_n)$, so we work at the Lie algebra level throughout.

We prove Theorem 3.10 in three steps. First, in Section 3.1 we show that for any complex vector space $V$ the $\mathfrak{gl}(V)$-action by derivations on the exterior algebra $\bigwedge(V)$ factors through the spin action of the Clifford algebra $\text{Cl}(V \oplus V^*)$. Then we construct commuting embeddings of $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$ into $\text{Cl}(\mathbb{C}^n + (\mathbb{C}^m)^*)$ in Section 3.2. In fact, first we embed $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$ into $\mathfrak{gl}_{nm}$ and then use the map $\mathfrak{gl}_{nm} \rightarrow \text{Cl}(\mathbb{C}^n + (\mathbb{C}^m)^*)$ of Section 3.1. Finally, we compute a multiplicity-free decomposition of $\bigwedge(\mathbb{C}^n)$ as a $\mathfrak{gl}_n \otimes \mathfrak{gl}_m$-module in Section 3.3.
3.1. A $\mathfrak{gl}_n$-action on the Clifford spin module. In [AS22, Section 1] we define the Clifford algebra $Cl(V \oplus V^*)$ and its spin action on the exterior algebra $\bigwedge(V)$ via inner and exterior multiplication operators $\iota_f$ and $\varepsilon_v$. For convenience, recall

$$\iota_f(\eta) = \sum_{j=1}^{k} (-1)^{j-1} f(v_j) w_1 \wedge \cdots \wedge \widehat{w_j} \wedge \cdots \wedge w_k \quad \text{and} \quad \varepsilon_v(\eta) = v \wedge \eta$$  \hspace{1cm} (1)

for any $v \in V$, $f \in V^*$, and $\eta = w_1 \wedge \cdots \wedge w_k \in \bigwedge(V)$. Our treatment is fairly standard and may be found in various sources. For instance, see [Bum04, Chapter 31], [GW09, Chapter 6], or [ML90, Chapters 1-2]. The inner multiplication operators are sometimes known as contractions [GW09, Chapter 6]. When necessary or convenient, we view the Clifford algebra as a Lie algebra with bracket given by the usual algebra commutator: $[A, B] = AB - BA$.

In this section we show that the natural $GL(V)$-action on $V$ induces a $\mathfrak{gl}(V)$-action on $\bigwedge(V)$ by derivations that factors through $Cl(V \oplus V^*)$. In particular, we construct a Lie algebra map $\Phi_n$ in Proposition 3.2 making the following diagram commute.

$$\begin{array}{ccc}
Cl(V \oplus V^*) & \xrightarrow{\cong} & \text{End}(\bigwedge(V)) \\
\phi_v & \downarrow & \downarrow \phi_v \\
\mathfrak{gl}(V) & & \\
\end{array}$$

(2)

The constructions in this section generalize to the quantized setting. In particular, Section 4.1 defines maps making Diagram (1) commute, which are key ingredients in the proof our quantized skew $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-duality Theorem 4.17.

It all starts with the natural $GL(V)$-action. The general linear group $GL(V)$ acts on $V$ by matrix multiplication and the diagonal maps $\delta^{(j)}(g) = g \otimes \cdots \otimes g$ extend this action to the tensor algebra $T(V) = \bigoplus_{j=0}^{\infty} V^\otimes j$. Differentiating the $GL(V)$-action on $T(V)$ by automorphisms yields a $\mathfrak{gl}(V)$-action by derivations. In fact, it yields an action of the universal enveloping algebra $U(\mathfrak{gl}(V))$ making the following diagram commute for each $j = 0, 1, 2, \ldots$. In the diagram, $\Delta : U(\mathfrak{gl}(V)) \rightarrow U(\mathfrak{gl}(V))^{\otimes 2}$ denotes the comultiplication of $U(\mathfrak{gl}(V))$.

$$\begin{array}{ccc}
GL(V) & \xrightarrow{\delta^{(j-1)}} & \text{End}(V^{\otimes j}) \\
\exp & \downarrow & \downarrow \exp \\
\mathfrak{gl}(V) & & \\
\end{array}$$

(3)

We define the exterior algebra $\bigwedge(V)$ and the induced $\mathfrak{gl}(V)$-action it carries by considering $U(\mathfrak{gl}(V))$ as a cocommutative Hopf algebra. We take the Hopf algebra point of view here in order to motivate the construction of the braided exterior algebra in Section 4.1: in both cases we consider a tensor algebra modulo an ideal generated by certain eigenvectors of braiding operators induced by an $R$-matrix. The enveloping algebra $U(\mathfrak{gl}(V))$ is cocommutative, so it is (trivially) a braided bialgebra [Kas95, Definition VIII.2.2]. That is, $1 \otimes 1$ is an $R$-matrix of $U(\mathfrak{gl}(V))$. The braiding operator on $V \otimes V$ induced by $R = 1 \otimes 1$ is the flip map $\tau$ taking $v \otimes w \rightarrow w \otimes v$. Braiding operators are module maps, so the $\mathfrak{gl}(V)$-action on $T(V)$ preserves the set

$$\text{Sym}^2(V) = \{v \otimes w + w \otimes v \mid v, w \in V\}$$

(4)

of +1 eigenvectors of $\tau$. Therefore, the $\mathfrak{gl}(V)$-action descends to the exterior algebra

$$\bigwedge(V) := T(V)/\text{Sym}^2(V).$$

(5)
Note that $\wedge(V)$ inherits the $\mathbb{Z}$-grading on $T(V)$ because $\langle\text{Sym}^2(V)\rangle$ is homogeneous.

Section 4.1 defines the braided exterior algebra $\Lambda_q(V^{(n)})$, a quantum analogue of $\wedge(V)$, in a similar fashion, using braiding operators induced by the (non-trivial) $R$-matrix of the quantum group $U_q(\mathfrak{g}l_n)$; compare Relation (5) with Relation (3).

Identifying elements of $\mathfrak{gl}(V)$ with their counterparts in $V \otimes V^*$ allows us to factor the $\mathfrak{gl}(V)$-action on $\wedge(V)$ through $\text{Cl}(V \oplus V^*)$. Recall that there is a canonical isomorphism of vector spaces $\mathfrak{gl}(V) \cong \text{End}(V) \cong V \otimes V^*$ satisfying

$$(v \otimes f)(w) = f(w)v, \quad v, w \in V, \; f \in V^*. $$

At the level of elementary linear algebra, this isomorphism says that square matrices are linear combinations of rank 1 outer products. It becomes an isomorphism of Lie algebras when we define a multiplication on $V \otimes V^*$ using the composition inside $\text{End}(V)$ and take the Lie bracket defined by the usual algebra commutator. Concretely, the product $(v \otimes f)(w \otimes h)$ in $V \otimes V^*$ satisfies

$$((v \otimes f)(w \otimes h))(u) = h(u)f(w)v = (f(w)v \otimes h)(u), \quad u, v, w \in V, \; f, h \in V^*. $$

Since $\mathfrak{gl}(V)$ acts on $\wedge(V)$ by derivations, the action of any $X$ in $\mathfrak{gl}(V)$, identified with $u \otimes f$ in $V \otimes V^*$, factors into an inner and an exterior multiplication:

$$X \triangleright v_1 \wedge \cdots \wedge v_k = (Xv_1) \wedge v_2 \wedge \cdots \wedge v_k + \cdots + v_1 \wedge \cdots \wedge (Xv_k)$$

$$= u \wedge \left( \sum_{j=1}^{k} (-1)^{k-1} f(v_j) v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k \right). \tag{6}$$

Recall the action of inner and exterior multiplication operators $\iota_f$ and $\varepsilon_v$ as in Relation (1). In Relation (6), $\hat{v}_j$ means that $v_j$ is omitted from the wedge product.

Relation (6) is key. It shows that the element $X = v \otimes f$ in $\mathfrak{gl}(V) \cong V \otimes V^*$ acts on $\wedge(V)$ as the Clifford product

$$X = \varepsilon_v \iota_f. \tag{7}$$

Relation (7) defines a Lie algebra map $\mathfrak{gl}(V) \to \text{Cl}(V \oplus V^*)$ making Diagram (2) commute, thereby achieving this subsection’s goal. The map is injective, so $\{\varepsilon_v \iota_f \mid v \in V, \; f \in V^*\}$ generates a subalgebra in $\text{Cl}(V \oplus V^*) \cong \text{End}(\wedge(V))$ identical to $\mathfrak{gl}(V)$.

However, Relation (7) defines the map $\mathfrak{gl}(V) \to \text{Cl}(V \oplus V^*)$ canonically. We are ultimately interested in a quantum version of this embedding, and operators in the quantum setting are described by their action on a weight basis. Thus, in anticipation of the quantum case, we now describe the homomorphism defined by Relation (7) explicitly in terms of a $\mathfrak{gl}(V)$-weight basis.

To this end, let $v_i$, for $i = 1, \ldots, n$, denote a $\mathfrak{gl}(V)$-weight basis of $V$ and let $v_i^*$ denote the corresponding dual basis of $V^*$, defined by $v_j^*(v_i) = \delta_{ij}$. Then the

$$\bar{v}(\ell) = \begin{cases} v_{i_1}^{\ell_1} \wedge v_{i_2}^{\ell_2} \wedge \cdots \wedge v_{i_n}^{\ell_n}, & \text{if } \ell_i \in \{0,1\} \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

for $\ell \in \{0,1\}^n$, form a basis of $\wedge(V)$.

Our choice of weight basis defines an isomorphism $V \cong \mathbb{C}^n$; from now on, $\mathfrak{sl}_n$ denotes $\mathfrak{gl}(V) \cong \mathfrak{gl}(\mathbb{C}^n)$. We work with the presentation of $\mathfrak{sl}_n$ obtained from the Chevalley presentation of $\mathfrak{sl}_n$ described by Relations (1) and (2) by adjoining elements $\bar{L}_i$, for $i = 1, \ldots, n$, subject to the relations

$$H_i = \bar{L}_i - \bar{L}_{i+1}, \quad \text{and} \quad [\bar{L}_i, \bar{L}_j] = 0.$$
The $\bar{L}_i$ correspond to a basis of $\mathfrak{h}^*$ that is orthonormal with respect to the trace bilinear form on $V$.

**Remark 3.1.** We use an overbar in this section to distinguish $\bar{v}(\ell)$ in $\Lambda(V)$ from $v(\ell)$ in the braided exterior algebra $\Lambda_q(V(n))$ defined in Section 4. Similarly, we use an overbar to distinguish the $\bar{L}_i$ in $\mathfrak{gl}_n$ from the $U_q(\mathfrak{gl}_n)$ generators introduced by Definition 4.1.

As in Relation (7), we obtain the map $\mathfrak{gl}_n \rightarrow \text{Cl}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ by identifying elements of $\mathfrak{gl}_n$ with their counterparts in $\mathbb{C}^n \oplus (\mathbb{C}^n)^*$. This time we identify generators *only*. Let $M_{ij}v_k = \delta_{jk}v_i$ denote the matrix units with respect to the $v_i$ basis and recall that there is an isomorphism $\mathfrak{gl}_n \rightarrow \text{Mat}_n(\mathbb{C})$ satisfying

$$E_i \rightarrow M_{i,i+1}, \quad F_i \rightarrow M_{i+1,i}, \quad \text{and} \quad \bar{L}_i \rightarrow M_{i,i}.$$  \hspace{1cm} (9)

Under this isomorphism, the $\bar{L}_i$ become matrices of unit trace. Composing with $\text{Mat}_n(\mathbb{C}) \cong \mathbb{C}^n \otimes (\mathbb{C}^n)^*$ identifies the $\mathfrak{gl}_n$ generators as

$$E_i \rightarrow v_i v_{i+1}^*, \quad F_i \rightarrow v_{i+1} v_i^*, \quad \text{and} \quad \bar{L}_i \rightarrow v_i v_i^*.$$ \hspace{1cm} (10)

On the Clifford algebra side, we consider the generators $\psi_i = e_{v_i}$ and $\psi_i^\dagger = e_{v_i}$, for $i = 1, \ldots, n$, as in Relation (1). The $\psi_i$ and $\psi_i^\dagger$ satisfy the canonical anticommutation relations

$$\psi_i \psi_j + \psi_j \psi_i = \psi_i^\dagger \psi_j^\dagger + \psi_j^\dagger \psi_i^\dagger = 0,$$

$$\psi_i \psi_j^\dagger + \psi_j^\dagger \psi_i = \delta_{ij},$$ \hspace{1cm} (11)

and they act on $\Lambda(V)$ by lowering and raising operators: for any $\bar{v}(\ell)$,

$$\psi_i \bar{v}(\ell) = (-1)^{\ell_1 + \cdots + \ell_{i-1}} \bar{v}(\ell - e_i), \quad \text{and} \quad \psi_i^\dagger \bar{v}(\ell) = (-1)^{\ell_1 + \cdots + \ell_{i-1}} \bar{v}(\ell + e_i).$$ \hspace{1cm} (12)

The next proposition describes the homomorphism $\mathfrak{gl}_n \rightarrow \text{Cl}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ explicitly by composing the map defined by Relation (10) with the natural inclusion $\gamma : \mathbb{C}^n \otimes (\mathbb{C}^n)^* \hookrightarrow \text{Cl}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$.

**Proposition 3.2.** Suppose $n = \dim V$. There is a homomorphism of Lie algebras $\Phi_n : \mathfrak{gl}_n \rightarrow \text{Cl}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ satisfying

$$E_i \rightarrow \psi_i^\dagger \psi_{i+1}, \quad F_i \rightarrow \psi_{i+1}^\dagger \psi_i, \quad \text{and} \quad \bar{L}_i \rightarrow \psi_i^\dagger \psi_i,$$

for each $E_i, F_i$, and $\bar{L}_i$ in $\mathfrak{gl}_n$.

**Proof.** As explained above, this map is a composition of known Lie algebra maps: $\mathfrak{gl}_n \cong \text{Mat}_n(\mathbb{C}) \cong \mathbb{C}^n \otimes (\mathbb{C}^n)^* \hookrightarrow \text{Cl}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$. \hspace{1cm} $\square$

This proposition is the classical precursor of Proposition 4.5.

### 3.2. Commuting $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$ embeddings into the Clifford algebra.

Now suppose that $V = U \otimes W$ with $\dim U = n$ and $\dim W = m$. In this subsection, we construct commuting embeddings of $\mathfrak{gl}(U)$ and $\mathfrak{gl}(W)$ into the Clifford algebra

$$\text{Cl}((U \otimes W) \oplus (U \otimes W)^*) \cong \text{End}(\Lambda(U \otimes W))$$

in Propositions 3.6 and 3.7. These embeddings are depicted in Diagram (1) and they serve as the foundation for our proof the classical skew duality Theorem 3.10.
As in Section 3.1, the story begins with an action by matrix multiplication. The group $GL(U)$ acts on the tensor product $U \otimes W$ via its action on the first factor:

$$g \triangleright (u \otimes w) = (gu) \otimes w, \quad g \in GL(U), \quad v \in U, \ w \in W.$$  

Similarly, $GL(W)$ acts on $U \otimes W$ through its natural action on the second factor. These actions commute, so they induce a group homomorphism

$$GL(U) \times GL(W) \rightarrow GL(U \otimes W) = GL(V)$$

that defines commuting actions on the $GL(V)$-module $\Lambda(V) = \Lambda(U \otimes W)$. We compose the differentials of these actions with the map $gl(V) \rightarrow Cl(V \oplus V^*)$ defined by Relation (7) to obtain commuting actions of $gl(U)$ and $gl(W)$ on $\Lambda(V)$ that factor through $Cl(V \oplus V^*)$.

Since we have an eye on the quantum case, we will define the embeddings of $gl(U)$ and $gl(W)$ into $Cl((U \otimes W) \oplus (U \otimes W)^*)$ explicitly with respect to a $gl(V)$-weight basis of $V = U \otimes W$. We start $U$ and $W$: let $u_i$ and $w_j$ denote $gl(U)$- and $gl(W)$-weight bases of $U$ and $W$, and let $u_i^*$ and $w_j^*$ denote the corresponding dual bases of $U^*$ and $W^*$. Then the products $u_i \otimes w_j$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, define a $gl(V)$-weight basis of $V$ because

$$gl(V) \cong \text{End}(U \otimes W) \cong (U \otimes U^*) \otimes (W \otimes W^*) \cong gl(U) \otimes gl(W). \quad (13)$$

Setting

$$v_{i+(j-1)n} = u_i \otimes w_j \quad (14)$$

arranges this basis in the column-major order $u_1 \otimes w_{1}, \ldots, u_n \otimes w_{1}, u_1 \otimes w_{2}, \ldots, u_n \otimes w_{2}, \ldots, u_1 \otimes w_{m}, \ldots, u_n \otimes w_{m}$ and defines isomorphism $\mathbb{C}^{nm} \cong U \otimes W$. We then obtain a $gl(V)$-weight basis of $\Lambda(V) \cong \Lambda(\mathbb{C}^{nm})$ by considering the $\bar{\epsilon}(\ell)$ as in Relation (8) for $\ell \in \{0, 1\}^{nm}$.

In the quantized setting of Section 4.2, the isomorphism defined by Relation (14) allows us to consider the braided exterior algebra $\bigwedge_q(V^{(nm)})$ defined using the $R$-matrix of the single quantum group $U_q(gl_{nm})$, instead of the braided exterior algebra $\bigwedge(V^{(n)} \otimes V^{(m)})$ used in [LZZ10], which requires the $R$-matrices of both $U_q(gl_{n})$ and $U_q(gl_{m})$.

Having chosen explicit weight bases of $U$, $W$, and $V = U \otimes W$, we refer to $gl(U) \cong gl(\mathbb{C}^{n})$, $gl(W) \cong gl(\mathbb{C}^{m})$, and $gl(U \otimes W) \cong gl(\mathbb{C}^{nm})$ as $gl_n$, $gl_m$, and $gl_{nm}$.

The Clifford algebra $Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*)$ is generated by the inner and exterior multiplication operators $\psi_k = \epsilon_{v_k}^1$ and $\psi^k = \epsilon_{v_k}$, for $k = 1, \ldots, nm$, because the functionals $v_{i+(j-1)n} = u_i^* \otimes w_j^*$ define a basis of $(\mathbb{C}^{nm})^* \cong (U \otimes W)^*$ dual to $v_1, \ldots, v_{nm}$. As in Section 3.1, the $\psi_k$ and $\psi^k$ satisfy the canonical anticommutation relations (11) and they act on $\Lambda(U \otimes W)$ as in Relation (12). In particular, the operator $\psi_{i+(j-1)n}$ vacates the $(i, j)$ position in the $n \times m$ grid defined by $\epsilon(\ell)$, while $\psi^k_{i+(j-1)n}$ occupies it.

Propositions 3.6 and 3.7 define the desired embeddings of $gl_n$ and $gl_m$ into $Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*) \cong \text{End}(\Lambda(U \otimes W))$. These maps rely crucially on the factorizations $gl(U \otimes W) \cong gl(U) \otimes gl(W)$ and $X = \epsilon_{u_t f}$ of Relations (13) and (7); the former takes an element in $gl_n$ or $gl_m$ into $gl_{nm}$ by tensoring with the identity $I$ and then the latter identifies the resulting element in $gl_{nm} \cong \mathbb{C}^{nm} \otimes (\mathbb{C}^{nm})^*$ with a Clifford product in $Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*)$.

Therefore the first step is to describe the maps $gl_n \rightarrow gl_{nm}$ and $gl_m \rightarrow gl_{nm}$ explicitly. The isomorphism $gl(V) \cong gl(U) \otimes gl(W)$ of Relation (13) implies that $gl(U) \cong gl(U) \otimes id_W \subset gl(V)$. Identifying the elements $X \otimes I$, with $X$ a generator in $gl_n \cong \text{Mat}_n(\mathbb{C}) \cong \mathbb{C}^n \otimes (\mathbb{C}^n)^*$, with their counterparts in $\text{Mat}_{nm}(\mathbb{C}) \cong gl_{nm}$ results in the map $gl_n \rightarrow gl_{nm}$. For instance, the simple positive
root vector \( E^{(n)}_{i} \) in \( \mathfrak{gl}_{n} \) is identified with \( u_{i} \otimes u_{i+1}^{*} \) in \( \mathbb{C}^{n} \otimes (\mathbb{C}^{n})^{*} \), and the matrix \( I \) is identified with the canonical vector \( \sum_{j=1}^{m} w_{j} \otimes w_{j}^{*} \) in \( \mathbb{C}^{m} \otimes (\mathbb{C}^{m})^{*} \), so the map \( \mathfrak{gl}_{n} \to \mathfrak{gl}_{nm} \) takes

\[
E^{(n)}_{i} \mapsto \sum_{j=1}^{m} u_{i} \otimes u_{i+1}^{*} \otimes w_{j} \otimes w_{j}^{*}
\]

\[
\leftrightarrow \sum_{j=1}^{m} M_{i+(j-1)n, i+1+(j-1)n}
\]

\[
\leftrightarrow \sum_{j=1}^{m} E^{(nm)}_{i+(j-1)n}.
\]

As usual, the \( M_{ab} \) denote matrix units defined by \( M_{ab}v_{c} = \delta_{bc}v_{a} \). The third identity is a consequence of the isomorphism \( \mathfrak{gl}_{nm} \to \text{Mat}_{nm}(\mathbb{C}) \), as in Relation (9).

**Remark 3.3.** The superscript on a Chevalley generator indicates the algebra to which it belongs. For instance, \( E^{(n)}_{i} \) is an element of \( \mathfrak{gl}_{n} \) while \( E^{(nm)}_{i+(j-1)n} \) lives in \( \mathfrak{gl}_{nm} \).

The matrix units in Relation (15) shift certain occupied positions upwards, to a previous adjacent position in the column-major order defined by Relation (14). This action coincides with the effect of the simple root vectors in \( \mathfrak{gl}_{nm} \), as illustrated in Figure 1, so we obtain the following proposition.

**Proposition 3.4.** There is a Lie algebra homomorphism \( \mathfrak{gl}_{n} \to \mathfrak{gl}_{nm} \) mapping

\[
X^{(n)}_{i} \mapsto \sum_{j=1}^{m} X^{(nm)}_{i+(j-1)n},
\]

where \( X_{a} \) denotes one of \( E_{a}, F_{a}, \) or \( L_{a} \).

**Proof.** As explained by Relation (15), this map is the composition \( \mathfrak{gl}_{n} \cong (U \otimes U^{*}) \otimes \text{id}_{W} \subset \text{End}(U) \otimes \text{End}(W) \cong \mathfrak{gl}_{nm} \) of known Lie algebra maps. \( \square \)

Dually, the factorization \( \mathfrak{gl}(V) \cong \text{End}(U) \otimes \text{End}(W) \cong \mathfrak{gl}(U) \otimes \mathfrak{gl}(W) \) of Relation (13) implies that \( \mathfrak{gl}(W) \cong \text{id}_{U} \otimes \mathfrak{gl}(W) \subset \mathfrak{gl}(V) \). In this case \( \text{id}_{U} \) corresponds to \( \sum_{i=1}^{n} u_{i} \otimes u_{i}^{*} \) in \( \mathbb{C}^{n} \otimes (\mathbb{C}^{n})^{*} \), so the map \( \mathfrak{gl}_{m} \to \mathfrak{gl}_{nm} \cong \text{Mat}_{nm}(\mathbb{C}) \) satisfies

\[
E^{(m)}_{j} \mapsto \sum_{i=1}^{n} u_{i} \otimes u_{i}^{*} \otimes w_{j} \otimes w_{j+1}^{*}
\]

\[
\leftrightarrow \sum_{i=1}^{n} M_{i+(j-1)n, i+jn}.
\]

In this case, the matrix units in Relation (16) shift certain occupied positions leftwards, to non-adjacent positions in the column-major order defined by Relation (14). These matrix units no longer correspond to simple root vectors in \( \mathfrak{gl}_{nm} \); however, they are identified to non-simple root vectors given by nested commutators.

**Proposition 3.5.** There is a Lie algebra homomorphism \( \mathfrak{gl}_{m} \to \mathfrak{gl}_{nm} \) satisfying

\[
E^{(m)}_{j} \mapsto \sum_{i=1}^{n} \left[ [E^{(nm)}_{i+(j-1)n}, E^{(nm)}_{i+1+(j-1)n}], E^{(nm)}_{i+2+(j-1)n}, \ldots, E^{(nm)}_{i+jn} \right], \quad \text{and}
\]
Proof. Relation (16) explains that this map is the composition \( gl_m \cong \text{id}_U \odot (W \otimes W^*) \subset \text{End}(U) \otimes \text{End}(W) \cong gl_{nm} \) of known Lie algebra maps.

The next two propositions define the embeddings \( \lambda \) and \( \rho \) of \( gl_n \) and \( gl_m \) into \( Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*) \) by composing the maps defined in Propositions 3.4 and 3.5 with the map \( \Phi_{nm} : gl_n \rightarrow Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*) \) of Proposition 3.2. These maps make the following diagram commute and they motivate the quantized \( \lambda_q : U_q(gl_n) \rightarrow Cl_q(nm) \) and \( \rho_q : U_q(gl_m) \rightarrow Cl_q(nm) \) defined in Propositions 4.10 and 4.15.

\[
\begin{align*}
\text{gl}_n & \xrightarrow{\Delta^{(m-1)}} \text{gl}_n^{\otimes m} & \xrightarrow{\Phi_n^{\otimes m}} & Cl(\mathbb{C}^{n} \oplus (\mathbb{C}^{n})^*)^{\otimes m} \xrightarrow{=} & \text{End}(\Lambda(\mathbb{C}^n)^{\otimes m}) \\
\text{gl}_n \otimes \text{gl}_m & \xrightarrow{\Phi_{nm}} Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*) \xrightarrow{=} & \text{End}(\Lambda(\mathbb{C}^{nm})^*) \\
\text{gl}_m & \xrightarrow{\Delta^{(n-1)}} \text{gl}_m^{\otimes n} & \xrightarrow{\Phi_m^{\otimes n}} & Cl(\mathbb{C}^{m} \oplus (\mathbb{C}^{m})^*)^{\otimes n} \xrightarrow{=} & \text{End}(\Lambda(\mathbb{C}^{m})^n) \\
\end{align*}
\]

Recall that [AS22, Proposition 1.7] defines \( \Gamma_U \) and \( \Gamma_W \).

**Proposition 3.6.** Recall the superscript notation explained in Remark 3.3. There is a Lie algebra map \( \lambda : gl_n \rightarrow Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*) \) satisfying

\[
E_i^{(n)} \rightarrow \sum_{j=1}^{m} \psi_i^{j(j-1)n} \psi_i^{1+(j-1)n},
\]

\[
F_i^{(n)} \rightarrow \sum_{j=1}^{m} \psi_i^{1+(j-1)n} \psi_i^{1+(j-1)n}, \quad \text{and}
\]

\[
L_i^{(n)} \rightarrow \sum_{j=1}^{m} \psi_i^{1+(j-1)n} \psi_i^{1+(j-1)n},
\]

for every \( E_j^{(n)}, F_j^{(n)}, \) and \( L_j^{(n)} \) in \( gl_n \).

*Proof. This map is the composition of the homomorphism \( gl_n \rightarrow gl_{nm} \) defined in Proposition 3.4 with the map \( \Phi_{nm} : gl_{nm} \rightarrow Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*) \) defined in Proposition 3.2.\]

**Proposition 3.7.** There is a Lie algebra map \( \rho : gl_m \rightarrow Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*) \) satisfying

\[
E_j^{(m)} \rightarrow \sum_{i=1}^{n} \psi_i^{j(j-1)n} \psi_i^{j+1},
\]
and 

\[ F_j^{(m)} \rightarrow \sum_{i=1}^{n} \psi_{i+jn}^i \psi_{i+(j-1)n}, \quad \text{and} \]

\[ L_j^{(m)} \rightarrow \sum_{i=1}^{n} \psi_{i+(j-1)n}^i \psi_{i+(j-1)n}, \]

for every \( E_j^{(m)}, F_j^{(m)}, \) and \( L_j^{(m)} \) in \( \mathfrak{gl}_m \).

**Proof.** Much like \( \lambda \) of Proposition 3.6, this map is the composition of the homomorphism \( \mathfrak{gl}_m \rightarrow \mathfrak{gl}_{nm} \) defined in Proposition 3.5 with \( \Phi_{nm} \). \( \square \)

**Remark 3.8.** The embeddings defined in Propositions 3.6 and 3.7 both factor through \( \mathfrak{gl}_{nm} \), but the column-major ordering defined by Relation (14) makes it so that only the \( \mathfrak{gl}_n \) root vectors “align” with those of \( \mathfrak{gl}_{nm} \), as illustrated in Figure 1. In contrast, simple root vectors in \( \mathfrak{gl}_m \) correspond to non-simple root vectors in \( \mathfrak{gl}_{nm} \), which are implemented by highly nested commutators. In the quantized setting, Propositions 4.10 and 4.15 describe quantum analogues of the \( \mathfrak{gl}_n \) and \( \mathfrak{gl}_m \) embeddings, but only the \( U_q(\mathfrak{gl}_n) \) embedding factors through \( U_q(\mathfrak{gl}_{nm}) \). In the quantum case, a single choice of \( U_q(\mathfrak{gl}_{nm}) \)-weight basis cannot simultaneously describe both the isomorphism \( \bigwedge_q(V^{(nm)}) \cong \bigwedge_q(V^{(n)})^{\otimes m} \) of \( U_q(\mathfrak{gl}_n) \)-modules and the isomorphism \( \bigwedge_q(V^{(nm)}) \cong \bigwedge_q(V^{(m)})^{\otimes n} \) of \( U_q(\mathfrak{gl}_m) \)-modules.

Now we take a moment to offer some explanatory remarks. To help interpret our formulas, we introduce the shorthand \( v_{ij} = v_{i+(j-1)n} \) and arrange the \( v_{ij} \) in a rectangular array as follows:

\[
\begin{array}{ccccccc}
 v_{11} & v_{12} & v_{13} & \cdots & v_{1m} \\
 v_{21} & v_{22} & v_{23} & \cdots & v_{2m} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 v_{n1} & v_{n2} & v_{n3} & \cdots & v_{nm}
\end{array}
\]  

(18)

Then if \( \ell \in \{0,1\}^{nm} \), the wedge product \( \bar{v}(\ell) \) in \( \bigwedge(C^{nm}) \cong \bigwedge(U \otimes W) \) describes a state of occupied and vacant positions in the \( n \times m \) array (18).

Figure 1 illustrates the action of simple positive root vectors in \( \mathfrak{gl}_n \), \( \mathfrak{gl}_m \), and \( \mathfrak{gl}_{nm} \) with respect to the \( \mathfrak{gl}_{nm} \)-weight basis defined by Relation (14). Under \( \lambda \) and \( \rho \), each simple root vector \( \mathfrak{gl}_n \) and \( \mathfrak{gl}_m \) becomes the sum of a product of a creation and an annihilation operator, so the \( E_i^{(n)}, F_i^{(n)}, E_j^{(m)}, \) and \( F_j^{(m)} \) preserve spaces of homogeneous degree. More specifically, if we let \( \alpha_{i,j}^v = e_{i,j} - e_{i+1,j} \) and \( \alpha_{i,j}^h = e_{i,j} - e_{i,j+1} \), we see that Propositions 3.6 and 3.7 imply that

\[
\lambda(E_i^{(n)}) \bar{v}(\ell) = \sum_{j=1}^{m} (-1)^{p_j} \bar{v}(\ell + \alpha_{i,j}^v) \quad \text{and}
\]

\[
\rho(E_j^{(m)}) \bar{v}(\ell) = \sum_{i=1}^{n} (-1)^{p'_i} \bar{v}(\ell + \alpha_{i,j}^h)
\]

(19)

for some integers \( p_j, p'_i \). Notice that \( \lambda(E_i^{(n)}) \) attempts to shift each occupied box in the \((i+1)\)st row **upwards**, as illustrated by Figure 2. Dually, \( \rho(E_j^{(m)}) \) attempts to shift each occupied position in the \((j+1)\)st column **leftwards**, as in Figure 3. Attempting to fill an occupied position, or vacate an empty one, annihilates the state.
Figure 1. Some $\mathfrak{gl}_n$, $\mathfrak{gl}_m$, and $\mathfrak{gl}_{nm}$ root vectors acting on the $v_i$ weight basis of $V \cong U \otimes W$. The green arrows illustrate the action of $E^{(nm)}_{i+(j-1)n}$ generators.

Now observe that each $L_i^{(n)}$ acts on $\wedge(U \otimes W)$ as a diagonal operator measuring degree of homogenous elements in the $i$th row: for any $\bar{v}(\ell)$,

$$L_i^{(n)} \triangleright \bar{v}(\ell) = \sum_{j=1}^{m} \psi^{i+(j-1)n}_j \psi^{i+(j-1)n}_1 \bar{v}(\ell) = \sum_{j=1}^{m} \ell_{i+(j-1)n} \bar{v}(\ell). \tag{21}$$
Similarly, $L_j^{(m)}$ acts as the diagonal operator measuring degree of homogenous elements in the $j$th column. Figure 4 illustrates the action $L_j^{(n)}$ and $L_j^{(m)}$ on the same given $\bar{v}(\ell)$. Since root vectors preserve components of homogeneous degree, their action on any state vector $v(\ell)$ must commute with that of every row and column degree operator $L_i^{(n)}$ and $L_i^{(m)}$. The representation $Cl((U \otimes W) \oplus (U \otimes W)^*) \cong \text{End}(\Lambda(V \otimes W))$ is faithful, so commutation relations amongst module endomorphisms imply commutation relations in the Clifford algebra.

\[
\begin{array}{ccc}
\begin{array}{ccc}
\v_1 & \v_2 & \v_3 \\
\v_2 & \v_3 & \v_4 \\
\v_3 & \v_4 & \v_1 \\
\end{array} & \lambda(L_3^{(n)}) & \begin{array}{ccc}
\v_1 & \v_2 & \v_3 \\
\v_2 & \v_3 & \v_4 \\
\v_3 & \v_4 & \v_1 \\
\end{array} \\
\begin{array}{ccc}
\v_1 & \v_2 & \v_3 \\
\v_2 & \v_3 & \v_4 \\
\v_3 & \v_4 & \v_1 \\
\end{array} & \rho(L_4^{(m)}) & \begin{array}{ccc}
\v_1 & \v_2 & \v_3 \\
\v_2 & \v_3 & \v_4 \\
\v_3 & \v_4 & \v_1 \\
\end{array} \\
\end{array}
\]

**Figure 4.** The $gl_n$ generator $L_3^{(n)}$ measures the degree of a given state vector along the 3rd row. Dually, the $gl_m$ generator $L_4^{(m)}$ measures degree along the 4th column. As in Figure 2, we take $n = 3$ and $m = 4$.

We conclude this subsection by noting that $\lambda$ and $\rho$ indeed define commuting subalgebras of $Cl((U \otimes W) \oplus (U \otimes W)^*) \cong \text{End}(\Lambda(\mathbb{C}^{nm}))$. This is guaranteed by the form of our embeddings, since they were obtained by tensoring each linear factor with an identity operator. In fact, even more is true: the images of $\lambda$ and $\rho$ generate each others full commutants.

**Proposition 3.9.** The maps $\lambda$ and $\rho$ given in Propositions 3.6 and 3.7 define commuting subalgebras of $Cl((U \otimes W) \oplus (U \otimes W)^*)$. Moreover, $gl(U)$ and $gl(W)$ generate mutual commutants in $\text{End}(\Lambda(U \otimes W))$.

**Proof.** The first statement holds because the image of $gl_n$ under $\lambda$ factors through $gl_n \otimes \{I\} \subset gl_{nm}$ while the image of $gl_m$ under $\rho$ factors through $\{I\} \otimes gl_m \subset gl_{nm}$.

Howe proves the second statement in [How95, Section 4.2.5].

3.3. **Multiplicity-free decomposition of $\Lambda(\mathbb{C}^{nm})$.** We now recall the classical duality result for Type A. Recall that every irreducible finite dimensional representation of a semisimple Lie group $G$ contains a highest weight vector that is essentially unique (all highest weight vectors are scalar multiples of each other), and the highest weight vector identifies the module [Bum04, Theorem 22.3].

The highest weight vector in an irreducible $G$-module is fixed by the unipotent subgroup of any Borel subgroup of $G$ [Bum04, Theorem 26.5]. At the Lie algebra level, this means that the highest weight vector of an irreducible module is the unique weight vector that is annihilated by all the simple positive root vectors. In particular, a vector $v$ in some $g$-module is a highest weight vector if and only if $v$ is a joint eigenvector of the $H_i$ and $E_i v = 0$ for each $i = 1, \ldots, r$. There is a one-to-one correspondence between highest weight vectors and dominant weights, as specified by Relation (3).
When \( \mathfrak{g} = \mathfrak{gl}_n \), the dominant weights \( \mu \in \mathbb{Z}^n \) satisfy
\[
\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.
\]
We refer the reader to Proposition 3.1.20 in [GW09] for more details. In what follows, we let \( \pi^G_\mu \) denote the irreducible \( G \)-module with highest weight \( \mu \).

**Theorem 3.10** (Skew \( GL_n \times GL_m \)-duality). Consider the natural action of \( GL(U) \times GL(W) \) on \( V = U \otimes W \) and its extension to \( \bigwedge (U \otimes W) \), as described by the commutative diagram (3). As a \( GL(U) \times GL(W) \)-module, the exterior algebra \( \bigwedge (U \otimes W) \) decomposes as
\[
\bigwedge (U \otimes W) \cong \sum_\mu \pi^{GL(U)}_\mu \otimes \pi^{GL(W)}_\mu.
\]
Here, the sum ranges over all partitions \( \mu \) with at most \( n = \dim U \) rows of length at most \( m = \dim W \): all partitions fitting inside an \( n \times m \) rectangle.

**Remark 3.11.** At the character level, skew \( GL_n \times GL_m \)-duality is the well-known dual Cauchy identity, which states that if \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_m \) are complex numbers of absolute value \( < 1 \) then
\[
\prod_{i=1}^n \prod_{j=1}^m (1 + \alpha_i \beta_j) = \sum_\lambda s_\lambda(\alpha_1, \ldots, \alpha_n)s_\lambda^*(\beta_1, \ldots, \beta_m).
\]
The sum ranges over all partitions \( \lambda \) fitting in an \( n \times m \) box and each \( s_\lambda \) is a Schur polynomial. For details, see e.g. [Bum04, Theorem 38.2].

There is a short proof of Theorem 3.10 based on Schur Duality, due to Howe [How95, Theorem 4.1.1]. We provide an alternative proof at the end of this subsection, whose structure serves as a blueprint for establishing quantized duality results in Section 4.3 and in [Abo22, Section 3.3]. Our alternative proof, outlined below, is independent of Schur Duality. Instead, it relies on a double commutant theorem of Weyl, which we reproduce below for the reader’s convenience.

**Theorem 3.12** (Double commutant). [GH18, Theorem B.1.1]

(a) Let \( \mathcal{H} \) be a finite dimensional vector space. Let \( \mathcal{A}, \mathcal{A}' \) be two subalgebras of \( \text{End}(\mathcal{H}) \) such that
1. The algebra \( \mathcal{A} \) acts semi-simply on \( \mathcal{H} \).
2. Each of \( \mathcal{A} \) and \( \mathcal{A}' \) is the full commutant of the other in \( \text{End}(\mathcal{H}) \).
Then \( \mathcal{A}' \) acts semi-simply on \( \mathcal{H} \), and as a representation of \( \mathcal{A} \otimes \mathcal{A}' \), we have
\[
\mathcal{H} \cong \bigoplus_{i \in I} \mathcal{H}_i \otimes \mathcal{H}'_i,
\]
where \( \mathcal{H}_i \) are all the irreducible representations of \( \mathcal{A} \), and \( \mathcal{H}'_i \) are all the irreducible representations of \( \mathcal{A}' \). In particular, we have a bijection between irreducible representations of \( \mathcal{A} \) and \( \mathcal{A}' \), and moreover, every isotypic component for \( \mathcal{A} \) is an irreducible representation of \( \mathcal{A} \otimes \mathcal{A}' \).

(b) On the other hand, if \( \mathcal{A} \) and \( \mathcal{A}' \) commute and the decomposition (22) holds as a representation of \( \mathcal{A} \otimes \mathcal{A}' \), then each of \( \mathcal{A} \) and \( \mathcal{A}' \) is the full commutant of the other in \( \text{End}(\mathcal{H}) \).

To establish Theorem 3.10, first we use Proposition 3.9 to prove that \( GL(U) \) and \( GL(W) \) generate mutual commutants in \( \text{End}(\bigwedge (U \otimes W)) \), thereby guaranteeing that the decomposition of the \( GL(U) \otimes GL(W) \)-module \( \bigwedge (U \otimes W) \) is multiplicity-free by the so-called double commutant yoga of Theorem 3.12(a) [How95]. Next, we use Lemma 3.13 to characterize the set of joint highest weights that can possibly appear in \( \bigwedge (U \otimes W) \). Finally, we explicitly construct joint highest weight vectors...
in Lemma 3.14, thereby exhibiting a \textit{Howe correspondence}, that is, a bijection \( \mu \rightarrow f(\mu) \) such that the irreducible \( GL(U) \times GL(W) \)-module \( \pi^U_\mu \otimes \pi^W_\mu \) appears in the decomposition \( \Lambda(U \otimes W) \).

We are interested in developing a quantum analogue of Theorem 3.10, so we work at the Lie algebra level throughout. Showing that \( GL(U) \) and \( GL(W) \) generate full mutual commutants is equivalent to showing that \( \mathfrak{gl}(U) \) and \( \mathfrak{gl}(W) \) generate full mutual commutants: Proposition 3.14 in [LZ06] proves that if \((\pi, Z)\) is any representation of the connected complex Lie group \( G \) and \((d\pi, Z)\) is the corresponding representation of the Lie algebra \( \mathfrak{g} = \text{Lie}(G) \), then

\[
\text{End}_G(Z) = \text{End}_\mathfrak{g}(Z).
\]

Proposition 3.9 shows that \( \mathfrak{gl}(U) \) and \( \mathfrak{gl}(W) \) generate mutual commutants in \( \mathcal{C} := (U \otimes W) \oplus (U \otimes W)^* \). Combining it with the Double Commutant Theorem 3.12 guarantees that the decomposition of the \( GL(U) \times GL(W) \)-module \( \Lambda(U \otimes W) \) is multiplicity-free.

The next proposition characterizes the set of highest weights that can possibly appear in the decomposition of \( \Lambda(U \otimes W) \).

**Lemma 3.13.** If the irreducible \( GL(U) \times GL(W) \)-module \( \pi^U_\mu \otimes \pi^W_\mu \) appears in the decomposition of \( \Lambda(U \otimes W) \) into irreducibles, then \( \mu_1 \leq m \) and \( \nu_1 \leq n \), with \( n = \dim V \) and \( m = \dim W \).

**Proof.** Every weight of \( GL(U) \times GL(W) \) is a weight of the induced \( \mathfrak{gl}(U) \otimes \mathfrak{gl}(W) \)-action. We show that the upper bounds in fact hold for any weights of the \( \mathfrak{gl}(U) \otimes \mathfrak{gl}(W) \)-module \( \Lambda(U \otimes W) \).

For every \( \ell \in \{0, 1\}^{nm} \), the vector \( \bar{v}(\ell) \) is a simultaneous eigenvector of each \( \bar{L}_i^{(n)} \) and each \( \bar{I}_j^{(m)} \). Relation (21) shows that the \( \bar{L}_i^{(n)} \)-eigenvector of \( \bar{v}(\ell) \) is the degree of \( \bar{v}(\ell) \) in the \( i \)th row, which is at most \( m \) by construction. Similarly, the \( \bar{L}_j^{(m)} \)-eigenvector of \( \bar{v}(\ell) \) is its degree in the \( j \)th column, which is at most \( n \). The \( \bar{L}_i^{(n)} \) and \( \bar{L}_j^{(m)} \) generate a Cartan subalgebra of \( \mathfrak{gl}(U) \otimes \mathfrak{gl}(W) \), and there are \( |\{0, 1\}^{nm}| = 2^{nm} = \dim (\Lambda(U \otimes W)) \) eigenvectors \( \bar{v}(\ell) \), so it follows that every \( \mathfrak{gl}(U) \)-weight \( \mu \) in \( \Lambda(U \otimes W) \) satisfies \( \mu_1 \leq m \). Dually, every \( \mathfrak{gl}(W) \)-weight \( \nu \) satisfies \( \nu_1 \leq n \).

The next proposition provides a supply of joint \( GL_n \times GL_m \)-highest weight vectors in \( \Lambda(U \otimes W) \), one for each partition \( \mu \) fitting in an \( n \times m \) rectangle. The highest weight vectors have a neat diagrammatical representation: the highest weight vector corresponding to \( \mu \) is the wedge product of the \( v_{ij} \), arranged as in Diagram 18, filling the boxes occupied by the Young diagram corresponding to \( \mu \). For example, the highest weight vector corresponding to \( \mu = (3, 2, 1) \) is

\[
(v_{11} \wedge v_{12} \wedge v_{13}) \wedge (v_{21} \wedge v_{22}) \wedge (v_{31}) = (v_{11} \wedge v_{21} \wedge v_{31}) \wedge (v_{12} \wedge v_{22}) \wedge (v_{13}),
\]

the wedge product of the basis vectors illustrated in Figure 5.

\[
v_\mu = \begin{array}{c}
v_{11} \\
v_{12} \\
v_{13} \\
v_{21} \\
v_{22} \\
v_{31}
\end{array}
\]

**Figure 5.** The joint \( GL_n \times GL_m \)-highest weight vector corresponding to the partition \( \mu = (3, 2, 1) \) is the wedge product of the basis vectors filling the boxes of the Young diagram defined by \( \mu \).
Up to sign, this procedure determines a unique vector in $\bigwedge(U \otimes W)$ corresponding to a partition $\mu$ fitting in an $n \times m$ box. We note that Lemma 4.19 constructs joint highest weight vectors of the general linear quantum groups $U_q(\mathfrak{g}_n)$ and $U_q(\mathfrak{g}_m)$ using essentially the same procedure.

**Lemma 3.14.** For $1 \leq \ell \leq n$, let $r_\ell = v_{11} \wedge \cdots \wedge v_{\ell t}$. Given a partition $\mu = (\mu_1, \ldots, \mu_n)$, define $v_\mu = r_1(\mu_1) \wedge \cdots \wedge r_n(\mu_n)$. The vector $v_\mu$ is a joint $GL_n \times GL_m$-highest weight vector in $\bigwedge(U \otimes W)$. Its weight as an eigenvector of the direct product of diagonal maximal tori $(T \times T') \subset GL_n \times GL_m$ is given by $\psi_\mu(t, t') = t^\mu(t')^{\mu'}$. Here $t \in T$ is given by

$$
t = \begin{bmatrix} t_1 & \cdots & t_n \end{bmatrix},
$$

$t^\mu = \prod_i t_i^{\mu_i}$, and $(t')^{\mu'}$ is defined similarly.

**Proof.** As usual, we work at the Lie algebra level. Given Lemma 3.13, it suffices to show that each $v_\mu$ is annihilated by each $E_i^{(n)}$ and each $E_j^{(m)}$. But this is immediate from the shape of $v_\mu$.

Recall Relations (15) and (16) and Figures 1–3, which explain that the operators induced by the simple positive root vectors $E_i^{(n)}$ and $E_j^{(m)}$ in $\mathfrak{g}_n$ and $\mathfrak{g}_m$ shift certain occupied positions in $v(\ell)$ above and to the left. Since $\mu$ is a left-justified partition, every position above and to the left of an occupied position in $v_\mu$ is occupied already, so

$$
\lambda(E_i^{(n)} v_\mu = \rho(E_j^{(m)} v_\mu = 0.
$$

Hence, each $v_\mu$ is a joint highest weight vector for the $\mathfrak{gl}(U) \otimes \mathfrak{gl}(W)$-action. \hfill \Box

**Proof of Theorem 3.10.** Together, Proposition 3.9 and Lemmata 3.13 and 3.14 imply Theorem 3.10. Proposition 3.9 proves that $GL(U)$ and $GL(W)$ generate mutual commutants in $End(\bigwedge(U \otimes W))$, so the decomposition of $\bigwedge(U \otimes W)$ into $GL(U) \times GL(W)$ irreducibles is multiplicity-free by the Double Commutant Theorem 3.12(a). Lemma 3.13 characterizes the set of weights that can possibly appear in the decomposition of $\bigwedge(U \otimes W)$, and Lemma 3.14 explicitly constructs a joint $GL(U) \times GL(W)$ highest weight vector for each dominant weight allowed by Lemma 3.13. \hfill \Box

4. QUANTUM SKEW DUALITY VIA QUANTUM CLIFFORD ALGEBRAS

In this section we prove our skew $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-duality Theorem 4.17. Definition 4.1 introduces general linear quantum groups by adjoining certain pairwise commuting generators to $U_q(\mathfrak{gl}_n)$. Our proof of Theorem 4.17 largely mirrors that of its classical counterpart, Theorem 3.10. Recall that to prove the classical skew duality Theorem 3.10, we first constructed the exterior algebra $\bigwedge(U \otimes W)$ as a module of the Clifford algebra $Cl((U \otimes W) \oplus (U \otimes W)^*)$ in Section 3.1, and then we found commuting embeddings of $\mathfrak{gl}(U)$ and $\mathfrak{gl}(W)$ into $Cl((U \otimes W) \oplus (U \otimes W)^*)$ in Section 3.2. Finally, we proved that $\mathfrak{gl}(U)$ and $\mathfrak{gl}(W)$ generate mutual commutants in $Cl((U \otimes W) \oplus (U \otimes W)^*)$ and used the Double Commutant Theorem 3.12(a) and a highest weight vector computation to conclude.

In the quantum case, we prove Theorem 4.17 by first introducing the braided exterior algebra $\bigwedge_q(V^{(n)})$, a quantum analogue of $\bigwedge(U \otimes W)$ as a $Cl_q(nm)$-module in Section 4.1. The quantum Clifford algebra $Cl_q(nm)$ is the quantum analogue of $Cl(\mathbb{C}^n \oplus (\mathbb{C}^m)^*)$ studied in [AS22]. As mentioned in the introduction, $Cl_q(nm)$ is shorthand for $Cl_q(nm, 1)$: in this article we use the quantized Clifford algebra introduced in [Kwo14]. Then we embed $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ into
Our homomorphisms define commuting subalgebras of module endomorphisms, so we compute highest weight vectors in Section 4.3 and appeal to the Double Commutant Theorem 3.12(b) to conclude.

4.1. The braided exterior algebra as a quantum Clifford algebra module. In this subsection we define the braided exterior algebra \( \Lambda_q(V^{(n)}) \) as a module for \( Cl_q(n) \), a quantum analogue of the Clifford algebra \( Cl(\mathbb{C}^n \oplus (\mathbb{C}^n)^*) \) studied in [AS22]. See also [Kwo14] for a definition. Braided exterior algebras were first introduced in [BZ05] as \( U_q(\mathfrak{g}) \)-module analogues of the classical \( \mathfrak{g} \)-module \( \Lambda(V) \). Much like in the classical case, we then define an action of \( U_q(\mathfrak{gl}_n) \) on \( \Lambda_q(V^{(n)}) \) that factors through \( Cl_q(n) \) and makes the following diagram commute:

\[
\begin{array}{ccc}
Cl_q(n) & \longrightarrow & \text{End}(\Lambda_q(V^{(n)})) \\
\phi_{q,n} & & \\
U_q(\mathfrak{gl}_n) & \downarrow & \\
\end{array}
\tag{1}
\]

Definition 4.1 introduces the general linear quantum group \( U_q(\mathfrak{gl}_n) \). Diagram (1) is the quantum analogue of Diagram (2). In this case, the top arrow \( Cl_q(n) \rightarrow \text{End}(\Lambda_q(V^{(n)})) \) is not an isomorphism: in the quantum case, the \( Cl_q(n) \)-module \( \Lambda_q(V^{(n)}) \) is not faithful [AS22, Theorem 2.19].

For now, let \( \mathfrak{g} \) be a simple complex Lie algebra and let \( V \) be any \( U_q(\mathfrak{g}) \)-module. Later we will specialize to the case where \( V \) is the natural \( U_q(\mathfrak{gl}_n) \)-module. The quantum group \( U_q(\mathfrak{g}) \) acts on the tensor algebra \( T(V) := \bigoplus_{j=0}^{\infty} V^\otimes j \) through its comultiplication. Unlike the classical enveloping algebra \( U(\mathfrak{g}) \), the Hopf algebra \( U_q(\mathfrak{g}) \) is not cocommutative, so the \( U_q(\mathfrak{g}) \)-action on \( T(V) \) does not preserve the ideal generated by the set of symmetric tensors \( \text{Sym}^2(V) \) defined by Relation (4).

Recall Relation (5), which explains that \( \Lambda(V) := T(V)/\langle \text{Sym}^2(V) \rangle \).

Regardless, the category of \( U_q(\mathfrak{g}) \)-modules is braided [CP94, Corollary 10.1.20]. For any pair of \( U_q(\mathfrak{g}) \)-modules \( V,W \), there is a natural isomorphism \( \tilde{R}^{V,W} : V \otimes W \rightarrow W \otimes V \) satisfying

\[
\tilde{R}^{V,W}(v \otimes w) = \tau(\mathcal{R} \triangleright v \otimes w),
\]

with \( \mathcal{R} \) denoting the universal \( R \)-matrix of \( U_q(\mathfrak{g}) \). We view \( \mathcal{R} \) as an invertible element of a suitable completion of \( U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \) with a well-defined action on any tensor product of finite-dimensional \( U_q(\mathfrak{g}) \)-modules. For details, see [CP94, Section 10.1.D]. In the related topological Hopf algebra \( U_{\tilde{q}}(\mathfrak{g}) \), the \( R \)-matrix has a unit constant term, so we may view the braiding operator as a deformation of the flip map \( \tau \) [Kas95, Section XX.4]. The deformed map \( \tilde{R} := \tilde{R}^{V,V} \) is no longer involutive, but it satisfies the braid relations

\[
\tilde{R}_i \tilde{R}_{i+1} \tilde{R}_i = \tilde{R}_{i+1} \tilde{R}_i \tilde{R}_{i+1}.
\tag{2}
\]

When \( T \) is an operator on a tensor product \( V \otimes V \), we write \( T_i \) to denote the element of \( \text{End}(V^\otimes n) \) acting by \( T \) on the \( i \)th and \( (i+1) \)st factors:

\[
T_i = \text{id}_V^\otimes (i-1) \otimes T \otimes \text{id}_V^\otimes (n-1-i).
\]

Relation (2) is ultimately a consequence of the quasitriangularity axioms defining \( U_q(\mathfrak{g}) \) [Maj02, Definition 5.1], which guarantee \( \mathcal{R} \) satisfies the celebrated Yang-Baxter equation \( \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} \) [Maj02, Lemma 5.2].

We use the braiding operator \( \tilde{R} \) to define a quantum analogue \( \text{Sym}_{\tilde{q}}^2(V) \) of \( \text{Sym}^2(V) \) that is preserved by the \( U_q(\mathfrak{g}) \)-action. Since the braiding operator \( \tilde{R} \) belongs to the centralizer \( \text{End}U_q(\mathfrak{g})(V \otimes V) \), the space of \( \mu \)-eigenvectors of \( \tilde{R} \) is a \( U_q(\mathfrak{g}) \)-module for each eigenvalue \( \mu \). It is a well-known fact,
see e.g. Corollary 2.22 in [LR97] or Proposition XVII.3.2 in [Kas95], that if \( V_\mu \) is the irreducible \( U_q(\mathfrak{g}) \)-module with highest weight \( \mu \) and the decomposition \( V_\mu \otimes V_\mu \) is multiplicity-free, then the braiding operator \( \hat{R}^{V_\mu \otimes V_\mu} \) acts on \( V_\mu \otimes V_\mu \) by the scalar

\[
\pm q^{\frac{1}{2}\chi_\mu(C) - \chi_\mu(C)}.
\]

Relation (8) defines the eigenvalue \( \chi_\mu(C) = \langle \gamma, \gamma + 2\rho \rangle \) of the Casimir operator of \( U(\mathfrak{g}) \) acting on the irreducible \( V_\mu \).

We obtain the \textit{braided exterior algebra} \( \wedge_q(V) \) as the quotient of the tensor algebra \( \mathcal{T}(V) \) modulo the ideal generated by the \textit{positive} eigenvectors of \( \hat{R} \) [BZ05, Remark 2.3]. That is, we set

\[
\wedge_q(V) := \mathcal{T}(V)/\langle \text{Sym}_q^2(V) \rangle,
\]

with \( \text{Sym}_q^2(V) \) denoting the set of eigenvectors of \( \hat{R} \) with eigenvalue of the form \( +q^r \) for some \( r \in \mathbb{Q} \). Relation (3) is the quantum analogue of Relation (5), which constructs the classical exterior algebra using the \( +1 \) eigenvectors of \( \tau \), the classical enveloping algebra’s (symmetric) braiding operator. In fact, a continuity argument shows that the positive eigenvectors of \( \hat{R} \) become elements of \( \text{Sym}_q^2(V) = \ker (\tau + i\text{d}V) \) in the limit as \( q \to 1 \) [LR97, Corollary 2.22(3)].

We now specialize \( \mathfrak{g} = \mathfrak{gl}_n \) and focus on the braided exterior algebra \( \wedge_q(V^{(n)}) \), with \( V^{(n)} \) denoting the natural \( U_q(\mathfrak{gl}_n) \)-module. We begin by defining \( U_q(\mathfrak{gl}_n) \) and computing its action on \( \wedge_q(V^{(n)}) \).

**Definition 4.1.** The \textit{general linear quantum group} \( U_q(\mathfrak{gl}_n) \) is the unital associative algebra generated by \( U_q(\mathfrak{sl}_n) \) along with the pairwise commuting group-like generators \( L_i^{\pm 1} \), for \( i = 1, \ldots, n \) satisfying the following relations:

\[
K_i = L_i L_{i+1}^{-1}, \quad L_i E_j L_i^{-1} = q^{\delta_{ij}} E_j, \quad \text{and} \quad L_i F_j L_i^{-1} = q^{-\delta_{ij}} F_j.
\]

**Remark 4.2.** The algebra \( U_q(\mathfrak{gl}_n) \) is known as the \textit{semisimple} form of the quantum group associated to the Cartan type \( A_{n-1} \), since for each element \( \mu = \sum \mu_i \omega_i \) in the \( \mathfrak{sl}_n \) weight lattice there is \( \hat{K}_\mu = \prod \hat{L}_i^{\mu_i} \) in \( U_q(\mathfrak{gl}_n) \) [CP94, Remark 9.1.1]. This is consistent with the following observation.

Note that if \( \epsilon_i \) denotes the standard basis \( \mathfrak{h}^*(\mathfrak{sl}_n) \), which is orthonormal with respect to inner product \( \langle \cdot, \cdot \rangle \) defined by Relation (4), then \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) is the \( i \)th simple positive root of \( \mathfrak{sl}_n \) and the relations \( L_i E_j L_i^{-1} = q^{\delta_{ij}} \) and \( L_i F_j L_i^{-1} = q^{-\delta_{ij}} F_j \) are equivalent to

\[
L_i E_j L_i^{-1} = q^{\langle \epsilon_i, \alpha_j \rangle} E_j \quad \text{and} \quad L_i F_j L_i^{-1} = q^{-\langle \epsilon_i, \alpha_j \rangle} F_j.
\]

In this chapter we consider \( U_q(\mathfrak{gl}_n) \) as a co-algebra with the comultiplication described in Relation (10). Although usually merely a convention, the choice of comultiplication map is important in Section 4.2, where we must use different conventions for each quantum group to ensure that \( U_q(\mathfrak{gl}_n) \) and \( U_q(\mathfrak{sl}_n) \) induce commuting subalgebras of module endomorphisms.

The natural \( U_q(\mathfrak{gl}_n) \)-module \( V^{(n)} \) has a weight basis \( v_i \), for \( i = 1, \ldots, n \), so that

\[
K_i \triangleright v_j = q^{\langle \alpha_i, \epsilon_j \rangle} v_j.
\]

Since the natural \( \mathfrak{sl}_n \)-module is minuscule, it lifts to the quantum group \( U_q(\mathfrak{sl}_n) \) in such a way that the matrices of Relation (9) describing the Chevalley generators remain the same [Jan96, Section 5A.1]. The following explicit formulae define the natural \( U_q(\mathfrak{gl}_n) \)-module:

\[
E_i \triangleright v_j = \delta_{i+1,j} v_i, \quad F_i \triangleright v_j = \delta_{ij} v_{i+1}, \quad \text{and} \quad L_i \triangleright v_j = q^{\delta_{ij}} v_j.
\]

Now we compute \( \text{Sym}_q^2(V^{(n)}) \) by studying the decomposition of \( V^{(n)} \otimes V^{(n)} \) as a \( U_q(\mathfrak{gl}_n) \)-module. We decompose \( V^{(n)} \otimes V^{(n)} \) by finding \textit{highest weight vectors}, that is, simultaneous eigenvectors of the \( K_i \) that are annihilated by every \( E_i \).
Lemma 4.3. Let $V^{(n)}$ denote the natural $U_q(\mathfrak{gl}_n)$-module, as in Relation (4). As a $U_q(\mathfrak{gl}_n)$-module, $V^{(n)} \otimes V^{(n)}$ decomposes as

$$V^{(n)} \otimes V^{(n)} \cong V_{\epsilon_1+\epsilon_2} \oplus V_{2\epsilon_1}.$$  

The irreducible module $V_{\epsilon_1+\epsilon_2}$ is spanned by

$$v_i \otimes v_j - q^{-1}v_j \otimes v_i, \quad i < j,$$

and $V_{2\epsilon_1}$ is spanned by

$$v_i \otimes v_i, \quad v_i \otimes v_j + qv_j \otimes v_i, \quad i < j.$$

Proof. Using the formulae above, is easy to check that $v_1 \otimes v_2 - q^{-1}v_2 \otimes v_1$ and $v_1 \otimes v_1$ are weight vectors of weights $\epsilon_1 + \epsilon_2$ and $2\epsilon_1$. In addition, a straightforward computation shows they are annihilated by all the $E_i$. The irreducible modules $V_{\epsilon_1+\epsilon_2}$ and $V_{2\epsilon_1}$ are cyclic, so the spanning property follows from an inductive argument using the formulas of Relation (4). The $U(\mathfrak{gl}_n)$- and $U_q(\mathfrak{gl}_n)$-modules parametrized by the same dominant weight have the same dimension, so the decomposition follows. □

Notice that the vectors spanning $V_{2\epsilon_1}$ become elements of $\text{Sym}^2(V)$ in the limit $q \to 1$. This means that $V_{2\epsilon_1} \subset V^{(n)} \otimes V^{(n)}$ contains all positive eigenvectors of $\bar{R}$ and therefore $\bigwedge_q(V^{(n)})$ is the quotient

$$\mathcal{T}(V^{(n)})/\langle \text{Sym}_q^2(V^{(n)}) \rangle = \mathcal{T}(V^{(n)})/\langle v_i \otimes v_i, v_i \otimes v_j + qv_j \otimes v_i \mid i < j \rangle$$

when $V^{(n)}$ is the natural $U_q(\mathfrak{gl}_n)$-module. Equivalently, $\bigwedge_q(V^{(n)})$ the unital associative algebra generated by $v_1, \ldots, v_n$ subject to the following relations:

$$v_i^2 = 0, \quad i = 1, \ldots, n$$

$$v_iv_j = -qv_jv_i, \quad i < j.$$  \hspace{1cm} (5)

In particular, the set of

$$v(\ell) = \begin{cases} v_1^{\ell_1}v_2^{\ell_2} \cdots v_n^{\ell_n}, & \text{if } \ell_i \in \{0,1\} \\ 0, & \text{otherwise}, \end{cases} \hspace{1cm} (6)$$

for $\ell \in \{0,1\}^n$, forms a basis of $\bigwedge_q(V^{(n)})$ [BZ05, Lemma 2.32]. This basis closely resembles the basis of $\bigwedge(\mathbb{C}^n)$ described by Relation (8). In fact, $\bigwedge_q(V^{(n)}) \cong \bigwedge(\mathbb{C}^n)$ as vector spaces but not as algebras.

Since $U_q(\mathfrak{gl}_n)$ acts on the tensor algebra $\mathcal{T}(V^{(n)})$ via the comultiplication map, it preserves the underlying algebraic structure of $\bigwedge_q(V^{(n)})$. In other words, $\bigwedge_q(V^{(n)})$ is a $U_q(\mathfrak{gl}_n)$-module algebra.

Definition 4.4. [Mon93, Definition 4.1.1] An associative algebra $(A, \mu)$ with multiplication $\mu$ is a $U_q(\mathfrak{g})$-module algebra if $A$ is a $U_q(\mathfrak{g})$-module such that the $U_q(\mathfrak{g})$-action preserves the algebraic structure of $A$, meaning that

$$X \triangleright \mu(a \otimes b) = \mu(\Delta(X)(a \otimes b)), \quad \text{and} \quad X \triangleright 1 = \epsilon(X)1,$$

for all $a, b \in A$ and $X \in U_q(\mathfrak{g})$.

We are now ready to show that the $U_q(\mathfrak{gl}_n)$-action on $\bigwedge_q(V^{(n)})$ factors through the quantum Clifford algebra $C_{\ell_q}(n)$, so that Diagram (1) commutes. Recall Definition 2.1. In this section, we
fix \( k = 1 \), use the shorthand \( \text{Cl}_q(n) \) in place of \( \text{Cl}_q(n, 1) \), and only consider the module \( \Omega^0 \) defined in Proposition 2.16. For convenience, we review the \( \text{Cl}_q(n) \)-action on \( \Lambda_q(V^{(n)}) \):

\[
\begin{align*}
\psi_i \triangleright v(\ell) &= (-1)^{\ell_1 + \cdots + \ell_{j-1}} v(\ell - e_j), \\
\psi_j \triangleright v(\ell) &= (-1)^{\ell_1 + \cdots + \ell_{j-1}} v(\ell + e_j), & \text{and} \\
\omega_j \triangleright v(\ell) &= q^{-\ell_j} v(\ell).
\end{align*}
\]

The next proposition constructs the map \( \Phi_{q,n} : U_q(\mathfrak{gl}_n) \to \text{Cl}_q(n) \) illustrated in Diagram (1) by factoring the \( U_q(\mathfrak{gl}_n) \)-action on \( \Lambda_q(V^{(n)}) \) into products of \( \text{Cl}_q(n) \) operators, much like Relation (7) factors the \( gl(V) \)-action in the classical case into products of the inner and exterior multiplication operators of Relation (1). This result is a quantum analogue of Proposition 3.2. In this case we consider the quantized inner and exterior multiplication operators \( \iota_i^q \) and \( \varepsilon_i^q \) satisfying

\[
\begin{align*}
i_i^q(v(\ell)) &= (-q)^{\ell_1 + \cdots + \ell_{i-1}} v(\ell - e_i) = \pi_0 \left( \prod_{k < i} \omega_k^{-1} \psi_i \right) (v(\ell)) \quad \text{and} \\
\varepsilon_i^q(v(\ell)) &= (-q^{-1})^{\ell_1 + \cdots + \ell_{i-1}} v(\ell + e_i) = \pi_0 \left( \prod_{k < i} \omega_k \psi_i \right) (v(\ell))
\end{align*}
\]

for any \( v(\ell) \in \Lambda_q(V^{(n)}) \). These operators consider the underlying algebra structure of \( \Lambda_q(V^{(n)}) \) and include correction terms to correctly account for powers of \( (-q) \) that result from the exchange of \( v_i \)’s needed to express the result of an inner or exterior multiplication in terms of the \( v(\ell) \) basis defined by Relation (6).

**Proposition 4.5.** Let \( \iota_i^q \) and \( \varepsilon_i^q \) denote the quantized inner and exterior multiplication operators defined by Relation (8). There is an algebra map \( \Phi_{q,n} : U_q(\mathfrak{gl}_n) \to \text{Cl}_q(n) \) satisfying

\[
E_i \to \varepsilon_i^q \iota_{i+1}^q \quad \text{and} \quad F_i \to \varepsilon_i^q \iota_i^q
\]

for \( i = 1, \ldots, n - 1 \). In terms of \( \text{Cl}_q(n) \) generators, \( \Phi_{q,n} \) maps

\[
E_i \to q^{-1} \omega_i^{-1} \psi_i \psi_{i+1}, \quad F_i \to \omega_i \psi_i \psi_{i+1}, \quad \text{and} \quad L_i \to \omega_i^{-1}
\]

**Remark 4.6.** Recall Relation (12) and Proposition 3.2: in the classical case the root vectors \( E_i \) and \( F_i \) of \( \mathfrak{gl}_n \) map to \( \varepsilon_i \iota_{i+1} \) and \( \varepsilon_i \iota_i \) in \( Cl(\mathbb{C}^n + (\mathbb{C}^n)^* ) \).

**Proof.** The claims follow from direct calculation. It suffices to show that the images \( \tilde{E}_i, \tilde{F}_i, \) and \( \tilde{K}_i \) of \( E_i, F_i, \) and \( K_i \) under \( \Phi_{q,n} \) satisfy the relations defining \( U_q(\mathfrak{gl}_n) \). Most relations are easy consequences of the relations defining \( \text{Cl}_q(n) \) and [AS22, Lemma 2.4], which is a generalized analogue of [Bay90, Lemma 3.1]. For instance, we may easily obtain

\[
[\tilde{E}_i, \tilde{F}_i] = [\psi_{i+1} \psi_i, \psi_i \psi_{i+1}] = \frac{(\omega_i \omega_{i+1}) - (\omega_i \omega_{i+1}^{-1})^{-1}}{q - q^{-1}} = \tilde{K}_i - \tilde{K}_i^{-1}
\]

When \( |i - j| > 1 \), the quantum Serre relations follow immediately from the anticommutation relations \( \psi_i \psi_j = q^{\delta_{i,j}} \psi_j \psi_i \). We use Relation (9) to verify the remaining quantum Serre relations. Applying [AS22, Relation (7)], we find that

\[
[\tilde{E}_i, \tilde{E}_{i+1}] = (\psi_{i+1} \psi_i)(\psi_{i+1} \psi_i) - q(\psi_{i+1} \psi_i)(\psi_{i+1} \psi_i) = \omega_i^{-1} \psi_i \psi_{i+2},
\]

so

\[
[\tilde{E}_i, [\tilde{E}_i, \tilde{E}_{i+1}]] q^{-1} = -\omega_i^{-1} (q \psi_{i+1} \psi_i)^2 \psi_{i+2} - q^{-1} \psi_{i+2} (\psi_i)^2 \psi_{i+1} = 0.
\]
Since $\Phi_{q,n}(F_i) = \psi_{i+1}^i \psi_i (q\omega_i^{-1}) = \Phi_{q,n}(E_i)^*$, it follows that the $\Phi_{q,n}(F_i)$ also satisfy the $q$-Serre relations. \hfill \Box

The next proposition proves that the $U_q(\mathfrak{gl}_n)$-action on $\Lambda_q(V^{(n)})$ induced by $\Phi_{q,n}$ coincides with the $U_q(\mathfrak{gl}_n)$-module algebra action.

**Proposition 4.7.** The $U_q(\mathfrak{gl}_n)$-action on its module algebra $\Lambda_q(V^{(n)})$ factors through the quantum Clifford algebra $\text{Cl}_q(n)$. Concretely, Diagram (1) commutes.

**Proof.** Since $\Lambda_q(V^{(n)})$ is a $U_q(\mathfrak{gl}_n)$-module algebra, it follows that

$$E_i \triangleright v(\ell) = (E_i v_1^{e_1})(K_i v_2^{e_2})(K_i v_3^{e_3}) \cdots (K_i v_n^{e_n}) + v_1^{e_1} (E_i v_2^{e_2})(K_i v_3^{e_3}) \cdots (K_i v_n^{e_n})$$

$$+ \cdots + v_1^{e_1} v_2^{e_2} \cdots v_n^{e_n-1} (E_i v_n^{e_n})$$

$$= \sum_{j=1}^n \delta_{i+1,j} \cdot \ell_j \cdot q^{\sum_{k>j} (\alpha_i, e_k)} (v_i^{e_1} \cdots v_{j-1}^{e_{j-1}} \cdot v_i \cdot v_j^{e_j} \cdots v_n^{e_n})$$

$$= (-q)^{e_i+\cdots+e_{j-1}} v_i \cdot \ell_{i+1} (v_i^{e_1} \cdots v_i^{e_{j-2}} v_j^{e_j} \cdots v_n^{e_n})$$

$$= v_i \cdot e_{i+1} v(\ell)$$

$$= \varepsilon_i e_{i+1} v(\ell)$$

$$= \Phi_{q,n}(E_i) v(\ell).$$

In the second-to-last equality we use the commutation relations (5) defining the braided exterior algebra $\Lambda_q(V^{(n)})$. A similar calculation shows that $F_i \triangleright v(\ell) = \Phi_{q,n}(F_i) v(\ell)$.

Finally, notice that

$$L_i \triangleright v(\ell) = (L_i v_1^{e_1}) \cdots (L_i v_n^{e_n}) = q^{e_i} v(\ell) = \omega_i^{-1} v(\ell) = \Phi_{q,n}(L_i) v(\ell). \hfill \Box$$

The map $\Phi_{q,n}$ immediately yields a multiplicity-free decomposition of the $U_q(\mathfrak{gl}_n)$-module algebra $\Lambda_q(V^{(n)})$. The irreducible $U_q(\mathfrak{gl}_n)$-modules in $\Lambda(V^{(n)})$ are parametrized by the same dominant weights as in the corresponding classical result.

**Proposition 4.8.** As a $U_q(\mathfrak{gl}_n)$-module,

$$\Lambda_q(V^{(n)}) \cong \bigoplus_{j=0}^n V^{(n)}_{\gamma_j}.$$  

Here $V^{(n)}_{\mu}$ denotes the irreducible $U_q(\mathfrak{gl}_n)$-module with highest weight $\mu$, and each $\gamma_j = \sum_{k=1}^j e_k$ denotes a fundamental weight of $\mathfrak{gl}_n$.

**Proof.** Let $\alpha_i = e_i - e_{i+1}$. Notice that $v(\gamma_j)$ is a $U_q(\mathfrak{gl}_n)$-highest weight vector, since it is annihilated by each $E_i$:

$$E_i \triangleright v(\gamma_j) = \psi_{i+1}^i \psi_i v(\gamma_j) = v(\gamma_j + \alpha_i) = 0.$$

The claim now follows by a dimension count, considering the classical decomposition of $\mathfrak{gl}_n$-modules $\Lambda(V) \cong \bigoplus_{j=0}^n \Lambda^j(V)$. \hfill \Box
4.2. Commuting actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ on $\bigwedge_q(V^{(nm)})$. Fix $n$ and $m$. Recall Diagram (17), which illustrates the construction of commuting embeddings $\lambda$ and $\rho$ of $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$ into the Clifford algebra $Cl(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*)$ as a composition of known Lie algebra maps. These embeddings are critical to our proof of the skew $GL_n \times GL_m$-duality Theorem 3.10 in the classical case.

In this subsection, we mimic that construction, or at least part of it, to obtain the quantum analogues $\lambda_q$ and $\rho_q$ mapping $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ into $Cl_q(nm)$. Much like their classical counterparts, the maps $\lambda_q$ and $\rho_q$ play a prominent role in the proof of our skew $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-duality Theorem 4.17: Proposition 4.16 proves that they define commuting subalgebras of $\text{End}(\bigwedge_q(V^{(nm)}))$, which we use to compute the multiplicity-free decomposition of $\bigwedge_q(V^{(nm)})$ in Section 4.3.

For the rest of this section, we focus on the braided exterior algebra $\bigwedge_q(V^{(nm)})$, as defined by Relation (3), with $V^{(nm)}$ denoting the natural $U_q(\mathfrak{gl}_{nm})$-module. Recall that $V^{(nm)}$ has a $U_q(\mathfrak{gl}_{nm})$-weight basis $v_i$, for $i = 1, \ldots, nm$. In addition, recall that $\bigwedge_q(V^{(nm)})$ is generated as an algebra by the same $v_i$, subject to the relations in (5). In particular, $\bigwedge_q(V^{(nm)})$ has a basis $v(\ell)$, for $\ell \in \{0,1\}^{nm}$, as described by Relation (6). In this case, we say that each basis element $v(\ell)$ describes a unique state of occupied and vacant positions in an $n \times m$ grid.

To begin, we obtain $\lambda_q: U_q(\mathfrak{gl}_n) \rightarrow Cl_q(nm)$ as the composition resulting from the following diagram, which is the quantum analogue of the top half of Diagram (17):

$\begin{array}{cccc}
U_q(\mathfrak{gl}_n) & \xrightarrow{\Lambda^{(m-1)}} & U_q(\mathfrak{gl}_n)^{\otimes m} & \xrightarrow{\Phi_{q,nm}^{\otimes m}} & Cl_q(n)^{\otimes m} & \xrightarrow{\Gamma_q} & \text{End} \left( \bigwedge_q(V^{(n)})^{\otimes m} \right) \\
\Phi_q & \downarrow{\lambda_q} & \downarrow{\Phi_{q,nm}} & \downarrow{\Gamma_q} & \downarrow{\cong} \\
U_q(\mathfrak{gl}_{nm}) & \xrightarrow{\Phi_{q,nm}} & Cl_q(nm) & \xrightarrow{\text{End} \left( \bigwedge_q(V^{(nm)}) \right)} \\
\end{array}$

Recall that Theorem 2.11 defines the isomorphism $\Gamma_q$.

On one hand, there is a $U_q(\mathfrak{gl}_n)$-action on $\bigwedge_q(V^{(nm)})$ that factors through $U_q(\mathfrak{gl}_{nm})$. Motivated by the embedding $\mathfrak{gl}_n \rightarrow \mathfrak{gl}_{nm}$ of Proposition 3.4, we construct an algebra map $\Theta: U_q(\mathfrak{gl}_n) \rightarrow U_q(\mathfrak{gl}_{nm})$ in Proposition 4.9. Recall Relation (4), which describes the action of the $U_q(\mathfrak{gl}_{nm})$ generators on $V^{(nm)}$. As in the classical case, the $U_q(\mathfrak{gl}_n)$ root vectors “align” with those of $U_q(\mathfrak{gl}_{nm})$, as depicted in Figure 1. Composing $\Theta$ with the $Cl_q(nm)$-representation $\Phi_{q,nm}$ defined in Proposition 4.5 results in an algebra map $U_q(\mathfrak{gl}_n) \rightarrow \text{End} \left( \bigwedge_q(V^{(nm)}) \right)$.

On the other hand, the braided exterior algebra $\bigwedge_q(V^{(n)})$ is a $U_q(\mathfrak{gl}_n)$-module, so $U_q(\mathfrak{gl}_n)$ acts on $\bigwedge_q(V^{(n)})^{\otimes m}$ via comultiplication.

Proposition 4.12 proves that $\bigwedge_q(V^{(n)})^{\otimes m} \cong \bigwedge_q(V^{(nm)})$ as $U_q(\mathfrak{gl}_n)$-modules, so the two $U_q(\mathfrak{gl}_n)$-actions on $\bigwedge_q(V^{(nm)})$ in fact coincide.

The next proposition defines the map $\Theta: U_q(\mathfrak{gl}_n) \rightarrow U_q(\mathfrak{gl}_{nm})$ illustrated in Diagram (17) as a first step in defining a $U_q(\mathfrak{gl}_n)$-action on $\bigwedge_q(V^{(nm)})$. It is the quantum analogue of Proposition 3.4. To the best of our knowledge, this map is new.

**Proposition 4.9.** Recall the superscript notation of Remark 3.3: $X_j^{(p)}$ denotes a $U_q(\mathfrak{gl}_p)$ generator. Define

$$\Lambda_{i,<j} = \prod_{p<j} K_{i+(p-1)n}^{(nm)}, \text{ and } \Lambda_{i,>j} = \prod_{p>j} K_{i+(p-1)n}^{(nm)},$$
by taking the product of all $K_p^{(nm)}$ generators on the $i$th row to the left, and respectively to the right, of the $j$th column.

There is an algebra homomorphism $\Theta : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_{nm})$ satisfying

\[
E_i^{(n)} \to \sum_{j=1}^m E_{i+(j-1)n}^{(nm)} \Lambda_{i,>j}^{1-n}, \\
F_i^{(n)} \to \sum_{j=1}^m \Lambda_{i,>j}^{1-n} E_{i+(j-1)n}^{(nm)}, \\
L_i^{(n)} \to \prod_{j=1}^m L_{i+(j-1)n}^{(nm)}.
\]

**Proof.** It suffices to show that the images $\tilde{E}_i$, $\tilde{F}_i$ and $\tilde{K}_i$ of $E_i^{(n)}$, $F_i^{(n)}$ and $K_i^{(n)}$ under $\Theta$ satisfy the relations of Definition 4.1. This follows from the definition of $U_q(\mathfrak{gl}_{nm})$ directly from a calculation.

Recall notation of Relation (4). In what follows $a_{ij}$ denotes the $(i, j)$ entry of the Cartan matrix of $\mathfrak{gl}_n$, while each $\alpha_k$, for $k = 1, \ldots, nm$, denotes a simple positive root of $U_q(\mathfrak{gl}_{nm})$. In addition, the form $\langle \cdot, \cdot \rangle$ denotes an inner product on $b_n^{*} \mathfrak{sl}_{nm} \times b_n^{*} \mathfrak{sl}_{nm}$. To begin, note that

\[
\tilde{K}_j \tilde{E}_i \tilde{K}_j^{-1} = \sum_{b=1}^m q^{\langle \alpha_i+(b-1)n, \alpha_j+(-1)n \rangle} E_{i+(b-1)n}^{(nm)} \Lambda_{i,>b} = q^{2\delta_{ij}-\delta_{i,j+1}-\delta_{i,j+1}} \tilde{E}_i = q^{a_{ij}} \tilde{E}_i.
\]

A similar calculation shows that $\tilde{K}_j \tilde{F}_i \tilde{K}_j^{-1} = q^{-a_{ij}} \tilde{F}_i$.

Next observe that

\[
[E_i, \tilde{F}_j] = \sum_{a,b} q^{\langle \alpha_i+(a-1)n, \sum \alpha_j+(c-1)n \rangle} \Lambda_{j,>a}^{1-n} \Lambda_{i,>a} E_{i+(a-1)n}^{(nm)} F_{j+(b-1)n}^{(nm)} - \sum_{a,b} q^{\langle \alpha_i+(b-1)n, \sum \alpha_j+(c-1)n \rangle} \Lambda_{j,>b}^{1-n} \Lambda_{i,>b} E_{i+(a-1)n}^{(nm)} F_{j+(b-1)n}^{(nm)}
\]

\[
= \sum_{a} \Lambda_{j,>a}^{1-n} \Lambda_{i,>a} [E_{i+(a-1)n}^{(nm)}, F_{j+(a-1)n}^{(nm)}]
\]

\[
= \delta_{ij} \left( \sum_{a} \prod_{c < a} (K_{i+(c-1)n}^{(nm)})^{-1} \prod_{c > a} K_{i+(c-1)n}^{(nm)} - \sum_{a} \prod_{c \leq a} (K_{i+(c-1)n}^{(nm)})^{-1} \prod_{c > a} K_{i+(c-1)n}^{(nm)} \right)
\]

\[
= \delta_{ij} \frac{q - q^{-1}}{q - q^{-1}} \left( \prod_{c=1}^m K_{i+(c-1)n}^{(nm)} - \prod_{c=1}^m (K_{i+(c-1)n}^{(nm)})^{-1} \right)
\]

\[
= \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_i^{(-1)}}{q - q^{-1}}.
\]

In the second equality we use the commutators $[E_i^{(nm)}, F_j^{(nm)}] = 0$ for $i \neq j$.

Finally, we verify the Serre relations. Recall our notation and conventions and let $a_{ij}$ denote the entries of the Cartan matrix for $\mathfrak{gl}_n$, as in Relation (6). For any $1 \leq i, j \leq n - 1$, the inner product $\langle \alpha_i+(a-1)n, \alpha_j+(-1)n \rangle$ vanishes whenever $a \neq b$, since then $|(i + (a - 1)n) - (j + (b - 1)n)| \geq 2$. 
Therefore,
\[ [E^{(nm)}_{i+(a-1)n}, E^{(nm)}_{j+(b-1)n}] = 0 \]
for any \( 1 \leq i, j \leq n - 1 \) whenever \( a \neq b \). In other words, simple positive root vectors \( E^{(nm)}_{i+(a-1)n} \) and \( E^{(nm)}_{j+(b-1)n} \) acting on different columns always commute. First suppose \( |i - j| > 1 \). In this case \([E^{(nm)}_{i+(a-1)n}, E^{(nm)}_{j+(a-1)n}] = 0\) for each \( a = 1, \ldots, m \), so
\[
[E_i, E_j] = \sum_{a,b} q^{-(\alpha_{i+(a-1)n} + \sum_{c \geq a} \alpha_{j+(a-1)n})} \Lambda_{j,b} \Lambda_{i,a} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n} \\
- \sum_{a,b} q^{-\alpha_{j+(b-1)n} + \sum_{c \geq a} \alpha_{i+(a-1)n})} \Lambda_{j,b} \Lambda_{i,a} E^{(nm)}_{j+(b-1)n} E^{(nm)}_{i+(a-1)n} \\
= \sum_a \Lambda_{j,a} \Lambda_{i,a} [E^{(nm)}_{i+(a-1)n}, E^{(nm)}_{j+(a-1)n}] \\
= 0.
\]
Now suppose \( j = i + 1 \). With Relation (9) in mind, we first compute
\[
[E_i, E_j] = \sum_{a,b} (1 - q^{1-a_{iji}}) \Lambda_{j,b} \Lambda_{i,a} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n} \\
+ \sum_a \Lambda_{j,a} \Lambda_{i,a} [E^{(nm)}_{i+(a-1)n}, E^{(nm)}_{j+(a-1)n}] q \\
= (1 - q^2) \sum_{a < b} \Lambda_{j,b} \Lambda_{i,a} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n} \\
+ \sum_a \Lambda_{j,a} \Lambda_{i,a} [E^{(nm)}_{i+(a-1)n}, E^{(nm)}_{j+(a-1)n}] q \\
= (1 - q^2) I + II.
\]
Next, we calculate
\[
[E_i, II] = \sum_{a,b} q^{-(\alpha_{i+(a-1)n} + \sum_{c \geq a} \alpha_{j+(a-1)n})} \Lambda_{j,b} \Lambda_{i,a} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n} q^{-1} \\
+ \sum_{a,b} q^{-\alpha_{i+(a-1)n} + \sum_{c \geq a} \alpha_{j+(a-1)n})} \Lambda_{j,b} \Lambda_{i,a} E^{(nm)}_{j+(b-1)n} E^{(nm)}_{i+(a-1)n} q^{-1} \\
- \sum_{a,b} \Lambda_{j,a} \Lambda_{i,a} [E^{(nm)}_{i+(a-1)n}, E^{(nm)}_{j+(b-1)n}] q^{-1} \\
= (1 - q^{-2}) \sum_{a < b} \Lambda_{j,b} \Lambda_{i,a} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n} q^{-1} \\
+ \sum_a \Lambda_{j,a} \Lambda_{i,a} [E^{(nm)}_{i+(a-1)n}, E^{(nm)}_{j+(b-1)n}] q^{-1} \\
= (1 - q^{-2}) III.
\]
In the second equality we used the $q$-Serre relations defining $U_q(\mathfrak{gl}_{nm})$, which guarantee that $[E^{(nm)}_{i+(b-1)n}, E^{(nm)}_{j+(b-1)n}]|_{q^{-1} = 0}$. Continuing, we see that

$$[	ilde{E}_i, I]_{q^{-1}} = \sum_{c, a < b} q^{-(\alpha_i+\alpha_{c-1})n + \sum_{s > a} \alpha_i(s+1)n + \sum_{c > a} \alpha_i(s-1)n} \Lambda_{j, > b} \Lambda_{i, > a} \Lambda_{i, > c+}$$

$$\cdot (E^{(nm)}_{i+(c-1)n} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n})$$

$$- q^{-1} \sum_{c, a < b} q^{-(\alpha_i+\alpha_{c-1})n + \alpha_{j+1}n + \sum_{s > a} \alpha_i(s-1)n} \cdot (E^{(nm)}_{i+(c-1)n} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n})$$

$$= \sum_{c < a < b} (1 - q^{-1-a_{i-1}}) \Lambda_{j, > b} \Lambda_{i, > a} \Lambda_{i, > c+} E^{(nm)}_{i+(c-1)n} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n}$$

$$+ \sum_{a < c < b} (q^{-a_{i-1}} - q^{-a_{i-1}}) \Lambda_{j, > b} \Lambda_{i, > a} \Lambda_{i, > c+} E^{(nm)}_{i+(c-1)n} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n}$$

$$+ \sum_{a < b < c} (q^{-a_{i-1}} - q^{-a_{i-1}}) \Lambda_{j, > b} \Lambda_{i, > a} \Lambda_{i, > c+} E^{(nm)}_{i+(c-1)n} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n}$$

$$+ \sum_{c = a < b} (1 - q^{-1-a_{i-1}}) \Lambda_{j, > b} \Lambda_{i, > a} \Lambda_{i, > b} (q^{-a_{i-1}} E^{(nm)}_{i+(a-1)n}) E^{(nm)}_{j+(b-1)n}$$

$$+ \sum_{a < b = c} \Lambda_{j, > b} \Lambda_{i, > a} \Lambda_{i, > b} (q^{-a_{i-1}} E^{(nm)}_{i+(a-1)n}) E^{(nm)}_{j+(b-1)n}$$

$$- q^{-1} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n} E^{(nm)}_{i+(b-1)n} \Lambda_{j, > b} \Lambda_{i, > a} \Lambda_{i, > c+}.$$

In the second equality, the second summation is the negative of the first so their sum vanishes; each term in the third and fourth summations is trivial so they too vanish. The last summation may be re-written using $q$-commutators:

$$\sum_{a < b} \Lambda_{j, > b} \Lambda_{i, > a} \Lambda_{i, > b} (q^{-a_{i-1}} E^{(nm)}_{i+(a-1)n}) E^{(nm)}_{j+(b-1)n} E^{(nm)}_{i+(b-1)n} - q^{-1} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n} E^{(nm)}_{i+(b-1)n}$$

$$= q^{-2} \sum_{a < b} \Lambda_{j, > b} \Lambda_{i, > a} \Lambda_{i, > b} E^{(nm)}_{i+(a-1)n} E^{(nm)}_{j+(b-1)n} E^{(nm)}_{i+(b-1)n},$$

$$= q^{-2} II.$$

Hence, $[\tilde{E}_i, I]_{q^{-1}} = q^{-2} III$. Combining results, we conclude that $[\tilde{E}_i, [\tilde{E}_i, I]]_{q^{-1}} = (1 - q^2)[\tilde{E}_i, I]_{q^{-1}} + [\tilde{E}_i, II]_{q^{-1}} = 0,$

as desired. We verify that the $\tilde{E}_i$ also satisfy the $q$-Serre relations using a similar calculation, which we omit here. \hfill \Box

We obtain an algebra map $\lambda_q: U_q(\mathfrak{gl}_n) \to Cl_q(nm)$ by composing $\Theta$ with $\Phi_{q,nm}$, as in Diagram (11).

**Proposition 4.10.** Define

$$\kappa_{i, < j} = \prod_{p < j} \omega_{i+(p-1)n} \omega_{i+1+(p-1)n} \quad \text{and} \quad \kappa_{i, > j} = \prod_{p > j} \omega_{i+(p-1)n} \omega_{i+1+(p-1)n},$$

where $\omega_{i+(p-1)n}$ and $\omega_{i+1+(p-1)n}$ are the elements from the Clifford algebra.
by taking an appropriate product of \( \omega \) generators in the \( i \)th and \((i+1)\)st rows to the left, and respectively to the right, of the \( j \)th column.

There is an algebra homomorphism \( \lambda_q : U_q(\mathfrak{gl}_n) \to Cl_q(nm) \) satisfying

\[
E_{i}^{(n)} \to q^{-1} \sum_{j=1}^{m} \omega_{i+(j-1)n}^{-1} \psi_{i+(j-1)n}^{\dagger} \psi_{i+(j-1)n} \kappa_{i,>j},
\]

\[
F_{i}^{(n)} \to \sum_{j=1}^{m} \omega_{i+(j-1)n} \kappa_{i,>j} \psi_{i+(j-1)n}^{\dagger} \psi_{i+(j-1)n}, \quad \text{and}
\]

\[
L_{i}^{(n)} \to \prod_{j=1}^{m} \omega_{i+(j-1)n}^{-1}.
\]

We use superscripts on \( U_q(\mathfrak{gl}_n) \) generators as in Remark 3.3.

Remark 4.11. Notice that when \( m = 1 \), \( \lambda_q \) is exactly the homomorphism \( \Phi_{q,n} \) of Proposition 4.5.

Proof. The map \( \lambda_q \) is simply the composition \( \Phi_{q,n} \circ \Theta \), so the statement follows from Proposition 4.5 and Proposition 4.9. \( \square \)

The map \( \Theta : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_nm) \) of Proposition 4.9 equips the \( U_q(\mathfrak{gl}_nm) \)-module \( \bigwedge_q(V^{(nm)}) \) with a \( U_q(\mathfrak{gl}_n) \)-module structure. Composing the map \( \Delta^{(m-1)} \circ \Phi_{q,n} \) with the isomorphism \( \Gamma_q : Cl_q(n)^{\otimes m} \to Cl_q(nm) \) gives \( \bigwedge_q(V^{(nm)}) \) another \( U_q(\mathfrak{gl}_nm) \)-module structure. The next proposition proves the two actions coincide.

Proposition 4.12. Diagram (11) commutes. In particular, there is an isomorphism of \( U_q(\mathfrak{gl}_n) \)-modules

\[
\bigwedge_q(V^{(nm)}) \cong \bigwedge_q(V^{(n)})^{\otimes m}.
\]

Proof. Using the definitions, it is straightforward to check that \( \lambda_q = \Gamma_q \circ \Delta^{(m-1)} \circ \Phi_{q,n} \). Combining with the commutativity of Diagram (18) proves the claim. \( \square \)

Remark 4.13. The isomorphism of \( U_q(\mathfrak{gl}_n) \)-modules can be promoted to an isomorphism of \( U_q(\mathfrak{gl}_n) \)-module algebras if we deform the multiplication in \( \bigwedge_q(V^{(n)})^{\otimes m} \) as in Theorem 2.3 of [LZZ10].

We now turn to studying the commutant of the \( U_q(\mathfrak{gl}_n) \)-action on \( \bigwedge_q(V^{(nm)}) \). We begin by defining a \( U_q(\mathfrak{gl}_m) \)-module structure on \( \bigwedge_q(V^{(nm)}) \) that factors through \( Cl_q(nm) \) in Proposition 4.15. Our skew Howe duality Theorem 4.17 proves that this action generates the centralizer algebra \( \text{End}_{U_q(\mathfrak{gl}_n)} \left( \bigwedge_q(V^{(nm)}) \right) \).

Unlike its classical counterpart \( \rho \) of Proposition 3.7, the map \( \rho_q \) does not factor through \( U_q(\mathfrak{gl}_nm) \). That is, there is no quantum analogue of Proposition 3.5. This has to do with our choice of weight basis; see Remark 3.8. Two issues arise in attempting to quantize Proposition 3.5. First, the classical \( \mathfrak{gl}_m \) generators \( E_{j}^{(m)} \) map to non-simple \( \mathfrak{gl}_nm \) root vectors, implemented by highly nested commutators of \( E_{a}^{(nm)} \) generators. Although root vectors corresponding to non-simple roots have a quantum analogue in terms of \( q \)-commutators as defined in Section 7.3.1 of [KS97], the resulting expressions are rather complicated. Second, the commutation relations amongst images of \( \mathfrak{gl}_m \) generators rely on the Jacobi identity, which can be quantized in many different ways.
Notwithstanding, our map $\rho_q: U_q(\mathfrak{gl}_m) \to Cl_q(nm)$ makes the following diagram commute:

\[ \begin{array}{cccc}
U_q(\mathfrak{gl}_m) & \xrightarrow{\Delta^{(n-1)}} & U_q(\mathfrak{gl}_m)^{\otimes n} & \xrightarrow{\Phi^{\otimes n}_{q,m}} & Cl_q(m)^{\otimes n} \\
\rho_q & & & & \gamma_q
\end{array} \]  \hspace{1cm} (12)

Note that Diagram (12) quantizes part of Diagram (17).

**Remark 4.14.** In this article the choice of comultiplication is important when discussing commuting quantum group actions. In order to ensure that the $U_q(\mathfrak{gl}_m)$-action on $\bigwedge_q(V^{(nm)})$ commutes with the $U_q(\mathfrak{gl}_n)$-action defined by $\lambda_q$, we must use the comultiplication $\tilde{\Delta}: U_q(\mathfrak{gl}_m) \to U_q(\mathfrak{gl}_m)^{\otimes 2}$ satisfying

\[
\begin{align*}
\tilde{\Delta}(E_i) &= E_i \otimes 1 + K_i \otimes E_i \\
\tilde{\Delta}(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i \\
\tilde{\Delta}(L_i) &= L_i \otimes L_i.
\end{align*}
\hspace{1cm} (13)
\]

This comultiplication does not agree with that defined by $\Delta$ as in Relation (10). In $\tilde{\Delta}(E_i)$, for instance, $K_i$ appears in the first tensor factor. It is also possible to swap the comultiplications for $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$. While choosing the same convention for both factors leads to well-defined embeddings into $Cl_q(nm)$, the resulting actions on $\bigwedge_q(V^{(nm)})$ do not commute. In the setting of orthogonal algebras discussed in [Abo22], the formulas embedding a commuting factor of $U'_q(\mathfrak{so}_m)$ into $U_q(\mathfrak{gl}_m)$ depend on a choice of comultiplication for $U_q(\mathfrak{so}_n)$.

**Proposition 4.15.** Define

\[
k_{<i,j} = \prod_{p<i} \omega_{p+(j-1)n}^{-1} \omega_{p+jn} \quad \text{and} \quad k_{>i,j} = \prod_{p>i} \omega_{p+(j-1)n}^{-1} \omega_{p+jn}
\]

by taking an appropriate product of $\omega_a$ generators in the $j$th and $(j+1)$st columns above, and respectively below, of the $i$th row. There is an algebra homomorphism $\rho_q: U_q(\mathfrak{gl}_m) \to Cl_q(nm)$ mapping

\[
\begin{align*}
E_j^{(m)} &\to \sum_{i=1}^n k_{<i,j} \psi_{i+(j-1)n}^\dagger \psi_{i+jn}, \\
F_j^{(m)} &\to \sum_{i=1}^n \psi_{i+jn}^\dagger \psi_{i+(j-1)n} k_{>i,j}^{-1}, \quad \text{and} \\
L_j^{(m)} &\to \prod_{i=1}^n \omega_{i+(j-1)n}^{-1}.
\end{align*}
\]

As usual, we use superscript indices as in Remark 3.3.

**Proof.** It suffices to show that the images $\bar{E}_j, \bar{F}_j, \text{ and } \bar{K}_j$ of $E_j^{(m)}, F_j^{(m)}$, and $K_j^{(m)}$ under $\rho_q$ satisfy the relations defining $U_q(\mathfrak{gl}_m)$. The following calculations are very similar to those proving
Proposition 4.9. For starters, notice that
\[ \tilde{L}_j \tilde{E}_\ell \tilde{L}_j^{-1} = \sum_a q^{\delta_{j\ell} \kappa_{<a,j} \tilde{\psi}_{a+(\ell-1)n} \tilde{\psi}_{a+\ell n}} = q^{\delta_{j\ell} \tilde{E}_j} \]

Next we verify that
\[ [\tilde{E}_j, \tilde{F}_\ell] = \frac{\delta_{j\ell}}{q-q^{-1}} \]

by summing over different summation index regimes independently. Notice that when \( a < b \), the product \( \kappa_{>b,\ell} \) commutes with \( \tilde{\psi}_{a+(j-1)n} \tilde{\psi}_{a+jn} \). Similarly, \( \kappa_{<a,j} \) commutes with \( \tilde{\psi}_{b+\ell n} \tilde{\psi}_{b+(\ell-1)n} \) when \( a < b \). Therefore, the double sum
\[ [\tilde{E}_j, \tilde{F}_\ell] = \sum_{a,b} \left( q^{a\ell} - q^{a\ell'} \right) \kappa_{<a,j} \kappa_{>b,\ell} \tilde{\psi}_{a+(j-1)n} \tilde{\psi}_{a+jn} \tilde{\psi}_{b+\ell n} \tilde{\psi}_{b+(\ell-1)n} \]

vanishes in the regime \( a < b \). Hence
\[ [\tilde{E}_j, \tilde{F}_\ell] = \frac{\delta_{j\ell}}{q-q^{-1}} \sum_a \left( \kappa_{<a,j} \kappa_{>a,\ell} \right) \left( (\omega_{a+(j-1)n} \omega_{a+jn}) - (\omega_{a+(j-1)n} \omega_{a+jn})^{-1} \right) \]

Note that the sum in the second equality is telescoping.

Finally, we check that \( \tilde{E}_j \) and \( \tilde{F}_\ell \) satisfy the Serre relations. First suppose that \( |j-\ell| > 1 \). In this case,
\[ [\tilde{E}_j, \tilde{F}_\ell] = \sum_{a,b} \left( q^{-a\ell} - 1 \right) \kappa_{<a,j} \kappa_{>b,\ell} \tilde{\psi}_{a+(j-1)n} \tilde{\psi}_{a+jn} \tilde{\psi}_{b+\ell n} \tilde{\psi}_{b+(\ell-1)n} \]

The first and second sums vanish because \( a_{j\ell} = 0 \) when \( |j-\ell| > 1 \). The creation and annihilation operators in the last summation may be arranged in the order specified by the third sum with no overall change in sign. This means the fourth sum is the negative of the third.
Now suppose $\ell = j + 1$, so that $a_{j\ell} = -1$. In this case we compute
\[
\begin{align*}
[\tilde{E}_j, \tilde{E}_j]_q & = \sum_{a > b} (1 - q^{1-a_{ij}}) K_{a, j} K_{b, \ell} \omega_{a+b+1-n} \psi_{a+jn}^\dagger \psi_{b-(\ell-1)n}^\dagger + \sum_a K_{a, j} K_{a, \ell} [\psi_{a+jn}^\dagger \psi_{a+n}^\dagger]_q \\
& = (1 - q^2) \sum_{a > b} K_{a, j} K_{b, \ell} \omega_{a+b+1-n} \psi_{a+jn}^\dagger \psi_{b-(\ell-1)n}^\dagger + \sum_a K_{a, j} K_{a, \ell} \omega_{a+n} \psi_{a+n}^\dagger \psi_{a+n} \\
& = (1 - q^2) I + II.
\end{align*}
\]
In the third equality we used [AS22, Relation (7)] with \( \varphi_1 = \psi_{a+(j-1)n}^\dagger \) and \( \varphi_k = \psi_{a+n}^\dagger \).

Further, observe that
\[
\begin{align*}
[\tilde{E}_j, II]_{q^{-1}} & = \sum_{a > b} \left(1 - q^{-1-(a_{ij}, e_{ij})}\right) K_{a, j} K_{b, \ell} \omega_{a+b+n} \psi_{a+jn}^\dagger \psi_{b-(\ell-1)n}^\dagger + q \sum_a \left(1 - q^{-1} \right) K_{a, j} K_{b, \ell} \omega_{a+n} \psi_{a+n}^\dagger \psi_{a+n} \\
& = (1 - q^{-2}) \sum_{a > b} K_{a, j} K_{b, \ell} \omega_{a+b+n} \psi_{a+jn}^\dagger \psi_{b-(\ell-1)n}^\dagger + q \sum_a \left(1 - q^{-1} \right) K_{a, j} K_{b, \ell} \omega_{a+n} \psi_{a+n}^\dagger \psi_{a+n} \\
& = (1 - q^{-2}) III.
\end{align*}
\]
In addition, we see that
\[
\begin{align*}
[\tilde{E}_j, I]_{q^{-1}} & = \sum_{c < b < a} (q^{-a_{ij}} - q^{-1}) K_{a, j} K_{b, \ell} K_{c, j} \\
& \quad \cdot \psi_{c+(j-1)n}^\dagger \psi_{c+jn} \psi_{a+jn}^\dagger \psi_{b-(\ell-1)n}^\dagger + \sum_{b < c < a} (q^{-a_{ij}} - q^{-1} a_{ij}) K_{a, j} K_{b, \ell} K_{c, j} \\
& \quad \cdot \psi_{c+(j-1)n}^\dagger \psi_{c+jn} \psi_{a+jn}^\dagger \psi_{b-(\ell-1)n}^\dagger \\
& + \sum_{b < a < c} (1 - q^{1-a_{ij}}) K_{c, j} K_{b, \ell} K_{a, j} \\
& \quad \cdot \psi_{a+(j-1)n}^\dagger \psi_{a+jn} \psi_{c+(j-1)n}^\dagger \psi_{c+jn} \psi_{b-(\ell-1)n}^\dagger \psi_{b+(\ell-1)n}^\dagger
\end{align*}
\]
attempts to shift occupied positions in the $\omega^a$.

Attempting to vacate an unoccupied position kills the state, as does attempting to occupy a filled position. Analogous comments apply to the $\omega^b$.

Replacing the signs by suitable powers of $q$ merely scales $v$.

In the third equality we used [AS22, Relation (7)] with $\phi_i = \psi^a_{b+(j-1)n}$ and $\phi_k = \psi_{b+\ell n}$. Combining results, and using identity (9), we conclude that

\[
\lbrack \tilde{E}_j, \tilde{E}_j, \tilde{E}_{\ell} \rbrack_{q^{-1}} = (1-q^2)q^{-2} III + (1-q^{-2})III = 0,
\]

as required. Since $\tilde{E}_j = \rho_q(E_j)^*$, we may apply the map $\star: Cl_q(nm) \rightarrow Cl_q(nm)$ of [AS22, Relation (10)] to conclude that the $\tilde{E}_j$ also satisfy the quantum Serre relations.

Now we take a moment to interpret the actions of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ on $\wedge_q(V^{(nm)})$ defined by Proposition 4.10. Recall the explanatory remarks below Proposition 3.7. As in the classical case, it is convenient to arrange the $U_q(\mathfrak{gl}_{nm})$-weight vectors $\psi_{i+(j-1)n}$ of $V^{(nm)}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, in an $n \times m$ rectangular array like (18). The $Cl_q(nm)$ generators $\psi_{i+(j-1)n}$ and $\psi^a_{i+(j-1)n}$ act on a given state vector $v(\ell)$ by attempting to vacate and occupy the $(i, j)$ position. Attempting to vacate an unoccupied position kills the state, as does attempting to occupy a filled position. Each $\omega_i$ merely scales $v(\ell)$ by $q^{-\epsilon_i}$, so the $\kappa$ factors in the definition of $\lambda_q$ act diagonally on state vectors to account for the action of $K^{(n)}$ generators in $\Delta^{(m-1)}(E^{(n)})$ and $\Delta^{(m-1)}(F^{(m)})$. Analogous comments apply to the $\kappa$ in the definition of $\rho_q$. Therefore, if we let $p_j$ and $p'_j$ denote some integers, and we set $\alpha^{v}_{i,j} = e_{i,j} - e_{i+1,j}$ and $\alpha^{h}_{i,j} = e_{i+1,j} - e_{i,j+1}$, we see, for instance, that

\begin{align}
\lambda_q(E^{(n)}_{i}) v(\ell) &= \sum_{j=1}^{m} (-q)^{p_j} v(\ell + \alpha^{v}_{i,j}) \quad \text{and} \\
\rho_q(E^{(m)}_{j}) v(\ell) &= \sum_{i=1}^{n} (-q)^{p'_i} v(\ell + \alpha^{h}_{i,j})
\end{align}

for any state vector $v(\ell)$. That is, the $i$th positive root vector $E^{(n)}_{i}$ in $U_q(\mathfrak{gl}_n)$ attempts to shift each occupied position in the $(i + 1)$st row upwards, while the $j$th positive root vector $E^{(m)}_{j}$ in $U_q(\mathfrak{gl}_m)$ attempts to shift occupied positions in the $(j + 1)$st column to the left. This means that Figures 2 and 3 still illustrate the action of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ root vectors induced by $\lambda_q$ and $\rho_q$, if we replace the signs by suitable powers of $-q$. This means the $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ root vectors act as
their classical counterparts in the limit $q \to 1$: compare Relations (14) and (15) to Relations (19) and (20).

In addition, observe that each $L_i^{(n)}$ and each $L_j^{(m)}$ induces a quantized degree operator that is the exponential of a degree operator acting as in Figure 4. For instance, the action of $\lambda_q(L_i^{(n)})$ counts occupied positions in the $i$th row:

$$L_i^{(n)} \triangleright \psi(\ell) = q \sum_{j=1}^n \ell_{i+j-1} = q^{\lambda(L_i)} \psi(\ell).$$

Recall Proposition 3.6 defines $\lambda: \mathfrak{gl}_n \to Cl(C^n \oplus (C^n)^*)$ in the classical case. Dually, the action of $\rho_q(L_i^{(m)})$ counts occupied positions in the $j$th column. The action of root vectors in $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ preserves spaces of homogeneous column and row degree, so it must commute with the action of the $L_i^{(n)}$ and the $L_i^{(n)}$, respectively.

The next proposition shows that although $\lambda_q: U_q(\mathfrak{gl}_n) \to Cl_q(nm)$ and $\rho_q: U_q(\mathfrak{gl}_m) \to Cl_q(nm)$ do not define commuting subalgebras of $Cl_q(nm)$, they do induce commuting actions on $\Lambda_q(V^{nm})$. Recall that in the classical case the maps $\lambda: \mathfrak{gl}_n \to Cl(C^{nm} \oplus (C^{nm})^*)$ and $\rho: \mathfrak{gl}_m \to Cl(C^{nm} \oplus (C^{nm})^*)$ of Propositions 3.6 and 3.7 define commuting subalgebras of $Cl(C^{nm} \oplus (C^{nm})^*) \cong \text{End}(\Lambda(C^{nm}))$. Theorem 2.19 implies that quantum representation $Cl_q(nm) \to \text{End}(\Lambda_q(V^{nm}))$ is not faithful. This means, for instance, that the actions of $\lambda_q(E_i^{(n)})$ and $\rho_q(E_j^{(m)})$ may commute even if $[\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})]$ is not the zero element in $Cl_q(nm)$.

**Proposition 4.16.** The embeddings $\lambda_q$ and $\rho_q$ of Proposition 4.10 and Proposition 4.15 define commuting subalgebras of $\text{End}(\Lambda_q(V^{nm}))$.

**Proof.** The proof follows from direct calculation. Considering the explanatory remarks preceding the proposition, it remains to show that the action induced by the root vectors of $U_q(\mathfrak{gl}_n)$ commutes with the action induced by the root vectors of $U_q(\mathfrak{gl}_m)$. We begin by computing commutators in the quantum Clifford algebra. As usual, our strategy is to consider different summation index regimes independently. For instance, observe that when $a > j + 1$ or $b < i$, the product $\kappa_{i,a}$ commutes with $\psi_{i+(a-1)n}^\dagger \psi_{i+1+(a-1)n}$ and similarly the product $\kappa_{i,a}$ commutes with $\psi_{b+(j-1)n}^\dagger \psi_{b+jn}$. Therefore, the sum

$$[\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})] = \sum_{a=1}^m \sum_{b=1}^n (\kappa_{i,a} \psi_{i+(a-1)n}^\dagger \psi_{i+1+(a-1)n} \kappa_{b,j} \psi_{b+(j-1)n}^\dagger \psi_{b+jn}$$

$$- \kappa_{b,j} \psi_{b+(j-1)n}^\dagger \psi_{b+jn} \kappa_{i,a} \psi_{i+(a-1)n} \psi_{i+1+(a-1)n})$$

vanishes in the regime where $b < i$ or $a > j + 1$ because we can appropriately rearrange the creation and annihilation operators. Thus we see that the commutator

$$[\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})] = \sum_{a=1}^m \sum_{b=1}^n (\kappa_{i,a} \psi_{i+(a-1)n}^\dagger \psi_{i+1+(a-1)n} \kappa_{b,j} \psi_{b+(j-1)n}^\dagger \psi_{b+jn}$$

$$- \kappa_{b,j} \psi_{b+(j-1)n}^\dagger \psi_{b+jn} \kappa_{i,a} \psi_{i+(a-1)n} \psi_{i+1+(a-1)n})$$

$$= \sum_{a=1}^m \sum_{b=1}^n (\kappa_{i,a} \psi_{i+(a-1)n}^\dagger \psi_{i+1+(a-1)n} \kappa_{b,j} \psi_{b+(j-1)n}^\dagger \psi_{b+jn}$$

$$[\kappa_{i,a} \psi_{i+(a-1)n}^\dagger \psi_{i+1+(a-1)n} \kappa_{b,j} \psi_{b+(j-1)n}^\dagger \psi_{b+jn}].$$

We now take a closer look at the four remaining commutators. When $(a, b) = (j, i)$, each term in the commutator $[\kappa_{i,a} \psi_{i+(a-1)n}^\dagger \psi_{i+1+(a-1)n} \kappa_{b,j} \psi_{b+(j-1)n}^\dagger \psi_{b+jn}]$ contains a factor of $(\psi_{i+(j-1)n})^2$, so
it vanishes. Similarly, when \((a, b) = (j + 1, i + 1)\), each term contains a factor of \((\psi_{i+1+jn})^2\), so it also vanishes. Only two terms remain:

\[
\begin{align*}
[\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})] &= q^{-1} \kappa_{i, > j} \kappa_{< i, j} \left[ \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \right] + \kappa_{i, > j} \kappa_{< i, j} \left[ \psi_{i+1+jn} \psi_{i+1+jn} \psi_{i+1+(j-1)n} \psi_{i+1+jn} \right] \\
&= q^{-1} \kappa_{i, > j} \kappa_{< i, j} \left[ \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \right] + \kappa_{i, > j} \kappa_{< i, j} \left[ \psi_{i+1+jn} \psi_{i+1+jn} \psi_{i+1+(j-1)n} \psi_{i+1+jn} \right] \\
&= \kappa_{i, > j} \kappa_{< i, j} \left[ q^{-1} \omega_1 \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \right] \\
&= \{ \psi_{i+1+jn}, \psi_{i+1+(j-1)n} \} \psi_{i+1+(j-1)n}.
\end{align*}
\]

In the second-to-last equality we used Relation (8) to slide the anticommutators to the left of \(\psi_{i+1+(j-1)n}\).

Using the last equality, we see that the commutator \([\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})]\) annihilates \(\Lambda_q(V^{(nm)})\).

Indeed, Relation (7) implies each anticommutator \(\{\psi_a, \psi_b^\dagger\}\) acts as the identity map, so for any basis vector \(v(\ell)\) of \(\Lambda_q(V^{(nm)})\),

\[
\begin{align*}
[\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})] v(\ell) &= (-1)^{\sum_{k=1}^{j-1} n_k} \kappa_{i, > j} \kappa_{< i, j} \\
&\quad \cdot \left\{ q^{-1} \omega_1 \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \psi_{i+1+(j-1)n} \right\} \\
&= \left\{ (-1)^{\sum_{k=1}^{j-1} n_k} \kappa_{i, > j} \kappa_{< i, j} \right\} \left\{ v(\ell + e_{i+1+(j-1)n} - e_{i+1+jn}) - v(\ell + e_{i+1+jn}) \right\} \\
&= 0.
\end{align*}
\]

Similar calculations show that \([\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})]\) also induces the zero map. Applying the \(*\)-operation defined by Relation (10) shows that the remaining commutators also induce the zero map, since

\[
[\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})]^* = -[\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})], \quad \text{and}
\]

\[
[\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})]^* = -[\lambda_q(E_i^{(n)}), \rho_q(E_j^{(m)})].
\]

4.3. Multiplicity-free decomposition of \(\Lambda_q(V^{(nm)})\). In the last section we found homomorphic images of \(U_q(\mathfrak{gl}_n)\) and \(U_q(\mathfrak{gl}_m)\) in the quantum Clifford algebra \(\mathcal{C}_q(nm)\) that generate commuting actions on the \(\mathcal{C}_q(nm)\)-module \(\Lambda_q(V^{(nm)})\). In this subsection we use those actions to compute a multiplicity-free decomposition of \(\Lambda_q(V^{(nm)})\) as a \(U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)\)-module. We rely on the classical skew duality Theorem 3.10 and on the Double Commutant Theorem 3.12(b) to prove our main quantized duality result, Theorem 4.17.

**Theorem 4.17.** As a \(U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)\)-module,

\[
\Lambda_q(V^{(nm)}) = \bigoplus_{\mu} V_{\mu}^{(n)} \otimes V_{\mu'}^{(m)}.
\]

(16)
The sum ranges over all partitions $\mu$ fitting in an $n \times m$ rectangle, $V^{(\mu)}_\mu$ denotes the irreducible $U_q(\mathfrak{gl}_n)$-module parametrized by $\mu$, and $\mu'$ denotes the conjugate of $\mu$. In particular, $\Lambda_q(V^{(nm)})$ is multiplicity-free as a $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-module.

Consequently, the maps $\lambda_q: U_q(\mathfrak{gl}_n) \rightarrow Cl_q(nm)$ and $\rho_q: U_q(\mathfrak{gl}_m) \rightarrow Cl_q(nm)$ generate mutual commutants in $\text{End} \left( \Lambda_q(V^{(nm)}) \right)$.

Remark 4.18. Recall that in the classical case, we first establish that $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$ generate each others commutants in $\text{End} \left( \Lambda(C^n \otimes C^m) \right)$, and then we use the double commutant yoga of Theorem 3.12(a) to conclude that the decomposition of $\Lambda(C^n \otimes C^m)$ as a $U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_m)$-module is multiplicity-free.

In the quantum case, the argument flows the opposite way. First we construct a multiplicity-free decomposition of $\Lambda_q(V^{(nm)})$ as a $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-module and then we use the Double Commutant Theorem 3.12(b) to conclude that $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{gl}_m)$ indeed generate each others full commutants in $\text{End} \left( \Lambda_q(V^{(nm)}) \right)$. Our proof of Theorem 4.17 requires the classical result and a dimension count to guarantee that the decomposition of $\Lambda_q(V^{(nm)})$ is in fact multiplicity-free.

Theorem 4.17 proves that the irreducibles appearing in the decomposition of $\Lambda_q(V^{(nm)})$ as a $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-module are parametrized by the same weights as the irreducibles in the decomposition of $\Lambda(C^n \otimes C^m) \cong \Lambda(C^n \otimes C^m)$ as a $\mathfrak{gl}_n \otimes \mathfrak{gl}_m$-module. Moreover, our decomposition de-quantizes appropriately, which means that the endomorphisms induced by the quantum group generators tend to those induced by their classical counterparts in the limit $q \to 1$. For instance, compare the action of $E_i^{(n)}$ and $E_j^{(m)}$ on an arbitrary basis vector in the quantum case as computed in Relation (14) and Relation (15) to the action of their classical counterparts in the classical case, as computed in Relation (19) and Relation (20).

Our proof of the first statement in Theorem 4.17 mimics the argument establishing Theorem 6.16 in [LZZ10]. First we construct a joint highest weight vector for each isotypic component in the decomposition of Relation (16) in the next lemma, and then we use a dimension count and the classical skew $GL_n \times GL_m$ duality Theorem 3.10 to conclude.

Recall Lemma 3.14, which computes highest weight vectors in the classical case. In the quantum case, the highest weight vectors are essentially the same: for each partition $\mu$ fitting in an $n \times m$ rectangle we simply take the product of all basis vectors in the boxes occupied by the Young diagram corresponding to $\mu$. Figure 5 illustrates an example.

Lemma 4.19. For each partition $\mu = (\mu_1, \ldots, \mu_n)$ satisfying $\mu_1 \leq m$, let

$$v_\mu = (v_1 \cdots v_1 + (\mu_1 - 1)n) (v_2 \cdots v_2 + (\mu_2 - 1)n) \cdots (v_n \cdots v_2 + (\mu_2 - 1)n) .$$

Then

1. The element $v_\mu$ is a highest weight vector with respect to the actions of $U_q(\mathfrak{gl}_n)$ and of $U_q(\mathfrak{gl}_m)$.
2. The $U_q(\mathfrak{gl}_n)$-weight of $v_\mu$ is $\mu$, and its $U_q(\mathfrak{gl}_m)$-weight is $\mu'$.

Proof. It is understood that if $\mu_i = 0$, then $v_\mu$ contains no $v_{i + (b - 1)n}$ as a factor. Since $\mu$ is a partition, this also implies that there is no $v_{j + (b - 1)n}$ factor for every $j \geq i$.

Much like in the classical case, Relations (14) and (15) show that positive root vectors in $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$ shift occupied positions in any given state vector above and to the left, as illustrated by Figures 2 and 3. Since $\mu$ is a left-justified partition, every position above and to the left of an occupied position in $v_\mu$ is occupied already, so

$$\lambda_q(E_i^{(n)}) v_\mu = \rho_q(E_j^{(m)}) v_\mu = 0 ,$$
for \(i = 1, \ldots, n - 1\) and \(j = 1, \ldots, m - 1\). Hence, each \(v_{\mu}\) is a joint \(U_q(\frak{gl}_n) \otimes U_q(\frak{gl}_m)\)-highest weight vector.

We can readily prove (2) using Propositions 4.10 and 4.15. For instance, we see that
\[
\lambda_q(K_i^{(n)}) v(\ell) = q^{\sum_{j=1}^n \ell_j + (j-1)\alpha_i} v(\ell), \quad \rho_q(K_j^{(m)}) v(\ell) = q^{\sum_{n=1}^m \ell_n - \ell_{n+j} \alpha_j} v(\ell),
\]
for any \(v(\ell)\), so
\[
\lambda_q(K_i^{(n)}) v_{\mu} = q^{\mu_i - \mu_{i+1}} v_{\mu} = q^{(\alpha_i^{(n)}, \mu)} v_{\mu}, \quad \text{and}
\rho_q(K_j^{(m)}) v_{\mu} = q^{\mu_j - \mu_{j+1}} v_{\mu} = q^{(\alpha_j^{(m)}, \mu')} v_{\mu}.
\]

Here we used \(\alpha_i^{(p)} = \epsilon_i - \epsilon_{i+1}\) to denote the \(i\)th simple positive root of \(\frak{gl}_p\).

\[\Box\]

**Proof of Theorem 4.17.** For any \(\mathbb{Z}_{\geq 0}\)-graded module \(M\), let \(M_k\) denote the graded component of degree \(k\). It follows from Lemma 4.19 that \(\bigoplus_\mu V^{(n)}_\mu \otimes V^{(m)}_\mu\) is a \(U_q(\frak{gl}_n) \otimes U_q(\frak{gl}_m)\)-submodule of \(\bigwedge_q(V^{(nm)})\). Let \(|\mu|\) denote the size of the partition \(\mu\), that is, the sum of its parts. Relations (14) and (15) prove that the \(U_q(\frak{gl}_n)\)- and the \(U_q(\frak{gl}_m)\)-actions preserve subspaces of homogeneous degree, so that in fact
\[
\bigoplus_{|\mu| = k} V^{(n)}_\mu \otimes V^{(m)}_\mu \subseteq \left(\bigwedge_q(V^{(nm)})\right)_{=k}
\]
for each \(k \leq nm\). The classical skew duality Theorem 3.10 then implies that
\[
\sum_{|\mu| = k} \dim_{\mathbb{C}(q)}(V^{(n)}_\mu \otimes V^{(m)}_\mu) = \dim_{\mathbb{C}} \bigwedge_q(\mathbb{C}^n \otimes \mathbb{C}^m),
\]
since irreducible modules of \(\frak{gl}_n\) and of \(U_q(\frak{gl}_n)\) parametrized by the same dominant weight have the same dimension. The \(U_q(\frak{gl}_{nm})\)-module \(V^{(nm)}\) is flat in the sense of Lemma 2.32 in [BZ05], so the \(v(\ell)\) form a basis of \(\bigwedge_q(V^{(nm)})\), which implies that \(\dim_{\mathbb{C}(q)} \bigwedge_q(V^{(nm)}) = \dim_{\mathbb{C}}(\mathbb{C}^n \otimes \mathbb{C}^m)\) and concludes our proof. \[\Box\]

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