The $K$-theory of the Compact Quantum Group $SU_q(2)$ for $q = -1$

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Abstract

We determine the $K$-theory of the $C^*$-algebra $C(SU_{-1}(2))$ and describe its spectrum. A calculation of the Haar state allows us to exhibit a continuous $C^*$-bundle over $[-1, 0)$ whose fiber at $q$ is isomorphic to $C(SU_q(2))$.

Introduction

In the Woronowicz’ theory of compact quantum groups [19, 22], $q$-deformations of compact Lie groups serve as fundamental examples [20, 21]. In the algebraic setting, Drinfel’d and Jimbo introduced $q$-deformed semisimple Lie groups [9, 10]. In the case of compact Lie groups Rosso showed in [14] that the approaches of Woronowicz and Drinfel’d and Jimbo are essentially equivalent. Since the quantum group $SU_q(2)$ is a fundamental example of a $q$-deformation, it attracts a great deal of attention. It has been studied intensively from various perspectives, with most research focusing on the case of a positive deformation parameter $q$, see for example [7, 8]. In [23], Zakrzewski shows that $C(SU_{-1}(2))$ can be considered as a sub-$C^*$-algebra of $M_2(C(SU(2)))$. This indicates that the $C^*$-algebra $C(SU_{-1}(2))$ differs significantly from $C(SU_q(2))$ for $|q| < 1$. Still, $SU_{-1}(2)$ and the quantum groups $SU_q(2)$ for $q < 0$ behave as nicely as their companions for positive deformation parameter. Recently, the quantum groups $SU_q(2)$ for $q < 0$ have attracted some attention as well, most notably because of the close relation to free orthogonal quantum groups $A_o(F)$ defined for $F \in GL(n, \mathbb{C})$ with $FF^\dagger \in \mathbb{R} \cdot 1$. [1, 2, 17]. More specifically, we have

$$SU_{-1}(2) \cong A_o(2)$$

by [1, Proposition 7]. In the context of recent work on the Baum-Connes conjecture for free orthogonal groups [18], it is natural to ask for the $K$-theory of $C(SU_{-1}(2))$.

Let us now explain how this work is organized. We start with a preliminary section where we recall some facts about compact quantum groups.
Moreover, we give a proof for Zakrzewski’s result stating that $C(SU_{-1}(2))$ can be seen as a subalgebra of $M_2(C(SU(2)))$. More specifically, this subalgebra turns out to be the intersection of the fixed point algebras of two commuting order two-automorphisms of $M_2(C(SU(2)))$. In Section 2 we use this representation of $C(SU_{-1}(2))$ to determine its spectrum. Section 3 is the main part of this work. We see that both $K$-groups of $C(SU_{-1}(2))$ are isomorphic to $\mathbb{Z}$. Just as in the situation of $C(SU_q(2))$ for $q \neq -1$, the class of the unit and the canonical unitary in $M_2(C(SU_{-1}(2)))$ serve as generators. We finally characterize the Haar state on $C(SU_{-1}(2))$ in Section 4 and use this to describe a continuous bundle of $C^*$-algebras over $[-1,0)$ whose fiber at $q$ is isomorphic to $C(SU_q(2))$.

The following work is based on the main part of the authors diploma thesis and we remark that at least parts of the above results are certainly known to the experts. However, to the best of our knowledge they are not documented in the literature.

1 Preliminaries

We recall the definition and some properties of compact quantum groups. As our main example we consider the compact quantum group $C(SU_q(2))$ for $q \in [-1,1] \setminus \{0\}$. When dealing with tensor products, we always use the minimal tensor product of $C^*$-algebras. For more information on quantum groups we refer to the literature [11, 15].

**Definition 1.1.** Let $A$ be a unital $C^*$-algebra and $\Delta : A \to A \otimes A$ a unital $*$-homomorphism. The pair $(A, \Delta)$ is called a compact quantum group if the following conditions hold:

i) $\Delta$ is coassociative, i.e., $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$,

ii) $(A, \Delta)$ is bisimplifiable, i.e. $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are linearly dense in $A \otimes A$.

If $G$ is a compact group, $C(G)$ is a compact quantum group with comultiplication

$$\Delta : C(G) \to C(G) \otimes C(G) \cong C(G \times G), \quad \Delta(f)(s,t) := f(s \cdot t).$$

In the special case where $G$ is a closed subgroup of the unitary group $\mathcal{U}(n)$, the comultiplication satisfies

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},$$

where $u_{ij}$ denotes the canonical projection onto the $(i,j)$-th coordinate.
We now consider the family of quantum groups $C(SU_q(2))$. In the case of $q = 1$, we recover the commutative compact quantum group $C(SU(2))$. Throughout this work we identify the topological spaces

$$SU(2) \cong S^3 = \{ (a, c) \in \mathbb{C}^2 : |a|^2 + |c|^2 = 1 \}$$

via the homeomorphism sending $(a, c)$ to $\left(\frac{a - \bar{c}c}{a}, \frac{c}{a} \right)$.

**Definition 1.2.** For $q \in [-1, 1]$ let $C(SU_q(2))$ be the universal unital $C^*$-algebra with generators $\alpha$ and $\gamma$ satisfying the relations

$$\alpha^* \alpha + \gamma^* \gamma = 1, \quad \alpha \alpha^* + q^2 \gamma \gamma^* = 1, \quad \alpha \gamma = q \gamma \alpha, \quad \alpha \gamma^* = q \gamma^* \alpha, \quad \gamma \gamma^* = \gamma^* \gamma. \quad (1.1)$$

Note that the above relations imply, that $u_q := \left(\begin{array}{cc} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{array}\right) \in M_2(C(SU_q(2)))$ is unitary. The comultiplication $\Delta_q : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$ given by

$$\Delta_q(\alpha) = \alpha \otimes \alpha - q \gamma^* \otimes \gamma, \quad \Delta_q(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$$

turns $(C(SU_q(2)), \Delta_q)$ into a compact quantum group for $q \neq 0$.

**Definition 1.3.** Let $(A, \Delta)$ be a compact quantum group. A state $h$ on $A$ is called left invariant, if

$$(id_A \otimes h) \circ \Delta(a) = h(a) \cdot 1_A$$

holds for all $a \in A$. It is called right invariant, if

$$(h \otimes id_A) \circ \Delta(a) = h(a) \cdot 1_A$$

holds for all $a \in A$.

It is well-known that every compact quantum group has a unique right and left invariant state [16], called the Haar state. We close this section with a description of the Haar state $h_q$ in the case of $C(SU_q(2))$.

In [20] it is shown that for $q \neq 0$

$$\mathcal{B} := \left\{ \alpha^k \gamma^l \gamma^* m : k, l, m \in \mathbb{N}_0, \ p \in \mathbb{N} \right\}$$

is a linearly independent set spanning the canonical dense $*$-subalgebra of $C(SU_q(2))$. For convenience, we set

$$\eta^{k,m} := \begin{cases} 
\alpha^k \gamma^l \gamma^* m, & \text{if } k \geq 0, \ m, n \geq 0, \\
\alpha^* k^l \gamma^* m, & \text{if } k < 0, \ m, n \geq 0.
\end{cases}$$
Proposition 1.4. For \( q \in (-1,1) \setminus \{0\} \) the Haar state \( h_q \) is the unique state on \( C(SU_q(2)) \) satisfying
\[
h_q(\eta^{klm}) = \begin{cases} \frac{1-q^2}{1-q^{k+m+2}}, & \text{if } k = 0, \ l = m, \\ 0, & \text{else}. \end{cases}
\]
In the case \( q = 1 \), the Haar state \( h_1 \) is uniquely defined by
\[
h_1(\eta^{klm}) = \begin{cases} \frac{1}{m+1}, & \text{if } k = 0, \ l = m, \\ 0, & \text{else}. \end{cases}
\]
Furthermore all the Haar states are faithful.

Proof. For \( |q| < 1 \) see [11, 4.3, Theorem 14] for the formula and [12] for the faithfulness of \( h_q \). In the commutative case consider the Haar integral on \( SU(2) \) and use the spherical coordinate system to verify the above equalities directly.

We now represent \( C(SU_{-1}(2)) \) as a sub-\( C^* \)-algebra of \( M_2(C(SU(2))) \). Shortly after this work was finished, Adam Skalski informed the author of Zakrzewskis work [23]. He shows a more general result, namely that a \( 2 \times 2 \) quantum group can be considered as a sub-\( C^* \)-algebra of the \( C^* \)-algebra of continuous functions on the classical group with values in \( M_2(\mathbb{C}) \). For the convenience of the reader we shall give our proof here.

For \((a, c) \in SU(2)\) we define a representation \( \pi_{(a,c)} \) as follows. If \( a = 0 \), let
\[
\pi_{(a,0)}: C(SU_{-1}(2)) \to \mathbb{C}, \quad \pi_{(a,0)}(\alpha) := a, \quad \pi_{(a,0)}(\gamma) := 0,
\]
and similarly, we set
\[
\pi_{(0,c)}: C(SU_{-1}(2)) \to \mathbb{C}, \quad \pi_{(0,c)}(\alpha) := 0, \quad \pi_{(0,c)}(\gamma) := c,
\]
if \( a = 0 \). In all other cases, we define
\[
\pi_{(a,c)}: C(SU_{-1}(2)) \to M_2(\mathbb{C}), \quad \pi_{(a,c)}(\alpha) := \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad \pi_{(a,c)}(\gamma) := \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}.
\]

Proposition 1.5. For every \((a, c) \in SU(2)\) the representation \( \pi_{(a,c)} \) is irreducible. Furthermore every irreducible representation is equivalent to some \( \pi_{(a,c)} \). In particular, the \( C^* \)-algebra \( C(SU_{-1}(2)) \) is 2-subhomogeneous and hence nuclear.
Proof. Every \( \pi_{(a,c)} \) is surjective and thus irreducible. Obviously, there are no further characters. Let \( \pi : C(SU_{-1}(2)) \to \mathcal{L}(H) \) be an irreducible representation on a Hilbert space \( H \) not isomorphic to \( \mathbb{C} \). The relations (1.1) imply that \( \alpha^2 \) and \( \gamma^2 \) are central. Hence \( \pi(\alpha^2) \) and \( \pi(\gamma^2) \) are scalar, whereas irreducibility prevents \( \pi(\alpha) \) and \( \pi(\gamma) \) from being scalar. Thus the spectrum of \( \pi(\alpha) \) consists of two distinct points, some \( a \in \mathbb{C} \) with \( |a| \leq 1 \) and its negative. The same holds for the spectrum of \( \pi(\gamma) \) and some \( c \in \mathbb{C} \) with \( |c| \leq 1 \). By functional calculus there are projections \( P, Q \) in the image of \( \pi \) such that

\[
\pi(\alpha) = aP - a(1 - P) = 2aP - a1, \quad \pi(\gamma) = 2cQ - c1.
\]

In fact, the image of \( \pi \) coincides with \( C^*(P,Q,1) \), hence is equal to the image of a (necessarily irreducible) representation of the unital universal \( C^* \)-algebra generated by two projections. This is a 2-subhomogeneous \( C^* \)-algebra [4, Example IV.1.4.2] and we can restrict our attention on irreducible representations \( \pi : C(SU_{-1}(2)) \to M_2(\mathbb{C}) \). Since \( \alpha \) is normal, we can assume \( \pi(\alpha) \) to be of the form

\[
\pi(\alpha) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}.
\]

The fact that \( \alpha \) and \( \gamma \) anticommute implies that \( \pi(\gamma) \) is an off-diagonal matrix. As \( \pi(\gamma^* \gamma) \) is scalar, we also get that the off-diagonal entries have the same modulus. Adjoining \( \pi(\gamma) \) with the symmetry \( v \in M_2(\mathbb{C}) \) given by

\[
v := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

we conclude

\[
\pi(\gamma) = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}.
\]

Finally, \( \alpha^* + \gamma^* = 1 \) implies that \( (a, c) \in SU(2) \), thus \( \pi = \pi_{(a,c)} \). For nuclearity of \( C(SU_{-1}(2)) \) see [5].

We can now describe \( C(SU_{-1}(2)) \) as a sub-\( C^* \)-algebra of \( M_2(C(SU(2))) \). By the universal property of \( C(SU_{-1}(2)) \) there is a \( * \)-homomorphism

\[
\phi : C(SU_{-1}(2)) \to M_2(C(SU(2))), \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad \gamma \mapsto \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}.
\]

Define \( G := \mathbb{Z}/2 \oplus \mathbb{Z}/2 \), \( r := (1,0) \), and \( s := (0,1) \), and consider the canonical \( G \)-action on \( C(SU(2)) \) given by
\[(f \cdot r)(a,c) := f(-a,c), \quad (f \cdot s)(a,c) := f(a,-c).\]

This leads to a \(G\)-action \(\beta\) on \(M_2(C(SU(2)))\) given by the commuting order two automorphisms

\[
\beta_1 \left( \begin{pmatrix} f & g \\ h & k \end{pmatrix} \right) := \begin{pmatrix} f \cdot s & -g \cdot s \\ -h \cdot s & k \cdot s \end{pmatrix}, \quad \beta_2 \left( \begin{pmatrix} f & g \\ h & k \end{pmatrix} \right) := \begin{pmatrix} k \cdot r & h \cdot r \\ g \cdot r & f \cdot r \end{pmatrix}.
\]

**Theorem 1.6.** The \(\ast\)-homomorphism \(\phi\) is injective and its image coincides with the fixed point algebra of the \(G\)-action \(\beta\).

**Proof.** Let \(ev_{(a,c)} \in C(SU(2))\) be the evaluation at \((a,c) \in SU(2)\). For non-zero \(a\) and \(c\), we have \(ev_{(a,c)} \circ \phi = \pi_{(a,c)}\). On the other hand, if \(c = 0\), \(ev_{(a,c)} \circ \phi = \pi_{(a,0)} \oplus \pi_{(-a,0)}\) holds. If \(a = 0\), we have \(ev_{(0,c)} \circ \phi = Ad(v) \circ \pi_{(0,c)} \oplus \pi_{(0,-c)}\). So for \(0 \neq x \in C(SU_{-1}(2))\), we find \((a,c) \in SU(2)\) such that \(ev_{(a,c)} \circ \phi(x) \neq 0\) and injectivity of \(\phi\) is shown.

It is obvious that \(\beta_1\) and \(\beta_2\) fix \(\phi(\alpha)\) and \(\phi(\gamma)\), thus the image of \(\phi\) is contained in the fixed point algebra of \(\beta\). For the other inclusion note that for every element in \(x \in M_2(SU(2))\) there is a unique decomposition of the form

\[x = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} + \begin{pmatrix} g & 0 \\ 0 & -g \end{pmatrix} + \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix} + \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}\]

for some \(f, g, h, k \in C(SU(2))\). An easy calculation shows that such a linear combination is fixed by \(\beta_1\) and \(\beta_2\) if and only if this holds for each summand. In this case the following symmetry relations hold

\[f = f \cdot r = f \cdot s, \quad g = -g \cdot r = g \cdot s, \quad h = h \cdot r = -h \cdot s, \quad k = -k \cdot r = -k \cdot s.\]

Now, each of these functions can be approximated by linear combinations of elements in \(B\) satisfying the same symmetry relations. Hence \(C(SU_{-1}(2))\) is mapped onto the fixed point algebra of \(\beta\).

### 2 The spectrum of \(C(SU_{-1}(2))\)

In this section we determine the spectrum of \(C(SU_{-1}(2))\). Since a 2-subhomogeneous \(C^\ast\)-algebra is GCR, the canonical surjection

\[C(SU_{-1}(2)) \twoheadrightarrow \text{Prim}(C(SU_{-1}(2)))\]

is injective \[11\]. Hence, it induces a homeomorphism and we only have to understand the Jacobson topology on \(\text{Prim}(C(SU_{-1}(2)))\). Denote by

\[X := SU(2) / \sim\]

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the quotient space, where the equivalence relation is defined as follows. If 
\((a, c) \in SU(2)\) satisfies \(ac \neq 0\), we have
\[
(a, c) \sim (b, d) \iff (b, d) \in G \cdot (a, c).
\]
Otherwise, the equivalence class of \((a, c)\) only consists of one element. As a
set, the canonical map
\[
\Pi : X \rightarrow C(SU_{-1}(2))^\sim,
\]
\([[(a, c)] \mapsto [\pi_{(a,c)}]]
\]
is a bijection. To see that this map is well-defined and surjective, consider
the symmetries \(v_r\) and \(v_s\) given by
\[
v_r := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad v_s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
(2.1)
Every 2-dimensional representation \(\pi_{(a,c)}\) gets intertwined with \(\pi_{(a,-c)}\) by
\(v_s\), and similarly \(v_r\) intertwines \(\pi_{(a,c)}\) and \(\pi_{(a,-c)}\). Thus, we also get the
equivalence of \(\pi_{(a,c)}\) and \(\pi_{(a,-c)}\). Injectivity of \(\Pi\) can be seen by comparing
the determinant of the images of \(\alpha\) and \(\gamma\) for given representations \(\pi_{(a,c)}\) and
\(\pi_{(b,d)}\).

**Theorem 2.1.** The bijection \(\Pi\) is a homeomorphism of \(X\) onto the spectrum
of \(C(SU_{-1}(2))\).

**Proof.** Let \(N \subseteq \text{Prim}(C(SU_{-1}(2)))\) and \(M := \Pi^{-1}(N) \subseteq X\). We have to
show that \(\overline{M} = \Pi^{-1}(\overline{N})\) holds. Let \(x \in C(SU_{-1}(2)) \subseteq C(SU(2), M_2(\mathbb{C}))\)
satisfy \(\pi_{(a,c)}(x) = 0\) for all \((a, c) \in SU(2)\) with \([(a, c)] \in M\). Since the
coefficient functions of \(x\) and the canonical projection \(P : SU(2) \rightarrow X\) are
continuous, we have
\[
\{ \ker(\pi_{(a,c)}) : [(a, c)] \in \overline{M} \} \subseteq \overline{N}.
\]
For the other implication, assume \(\emptyset \neq \overline{M} \neq X\) and fix \((b, d) \in X \setminus \overline{M}\). We
first treat the case, where \(b\) and \(d\) are non-zero. Denote by \(Y \subseteq SU(2)\) the
\(G\)-Orbit of \(P^{-1}(\overline{M}) \cup \{(b, d)\}\). Let \(y \in C(Y, M_2(\mathbb{C}))\) be given by
\[
y_{|G \cdot P^{-1}(\overline{M})} = 0, \quad y_{|(b,d)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Obviously, \(y\) lies in the fixed point algebra of the \(G\)-action on \(C(Y, M_2(\mathbb{C}))\)
induced by \(\beta\). If \(\tilde{y} \in C(SU(2), M_2(\mathbb{C}))\) is an extension of \(y\), the element
\[
x := \frac{1}{4} (\tilde{y} + \beta_1(\tilde{y})\beta_2(\tilde{y}) + \beta_1\beta_2(\tilde{y})) \in C(SU(2), M_2(\mathbb{C}))
\]
is a $\beta$-invariant lift of $y$, and therefore lies in $C(SU_{-1}(2))$. By construction, $x \in \ker(\pi(a,c))$ for every $(a,c) \in P^{-1}(M)$, but $x \notin \ker(\pi(b,d))$.

If $d = 0$, we also assume that $(-b,0) \in P^{-1}(M)$ (otherwise we can proceed as above). Define the closed subset

\[ Y := \{ (a, \pm c) \in SU(2) : (a,c) \in P^{-1}(M) \} , \]

and let $f \in C(SU(2))$ be a function with $f|_Y = 0$ and $f(b,0) = 1$. Now, 

\[ x := \begin{pmatrix} \frac{1}{2} (f + f \cdot s) & 0 \\ 0 & \frac{1}{2} (f + f \cdot s) \cdot r \end{pmatrix} \in C(SU_{-1}(2)) \]

satisfies $\pi(b,0)(x) = 1$, $\pi(a,c)(x) = 0$ for all $(a,c) \in P^{-1}(M) \subseteq Y$. If $b = 0$, we can proceed analogously by considering the automorphism on $C(SU_{-1}(2))$ exchanging $\alpha$ and $\gamma$. \hfill \blacksquare

3 \hspace{1em} \textbf{K-Theory of } C(SU_{-1}(2))

To determine the $K$-theory, we make use of a suitable filtration of $SU(2)$, which induces a cofiltration of $C(SU_{-1}(2))$. The main difficulty in describing the quotients is to keep track of the symmetry relations holding for the coefficient functions of an element in $C(SU_{-1}(2)) \subseteq M_2(C(SU(2)))$, when it is restricted to the respective closed subspace. The reader is referred to [3] and [13] for information on $K$-theory.

Consider the following closed subspaces of $SU(2)$

\begin{align*}
X_1 & := \{ (1,0), (0,1) \} , \\
X_2 & := \{ (a,c) \in SU(2) : \text{Im}(a) = \text{Im}(c) = 0 \} , \\
X_3 & := \{ (a,c) \in SU(2) : \text{Im}(a), \text{Im}(c) \geq 0 \text{ and } \text{Im}(a) \cdot \text{Im}(c) = 0 \} , \\
X_4 & := \{ (a,c) \in SU(2) : \text{Im}(a), \text{Im}(c) \geq 0 \} .
\end{align*}

These subspaces obviously define a filtration of $SU(2)$ and the restriction homomorphisms give rise to a cofiltration

\[ C(SU_{-1}(2)) \xrightarrow{\pi_4} A_4 \xrightarrow{\pi_3} A_3 \xrightarrow{\pi_2} A_2 \xrightarrow{\pi_1} A_1 . \]

Since every coefficient function of an element in the fixed point algebra of $\beta$ is already uniquely determined by its values on $X_4$, we conclude that $\pi_4$ is an isomorphism. Via the above cofiltration, we compute $K_*(A_{k+1})$ using $K_*(A_k)$ and the six-term sequence induced by $K_*(\pi_k)$.

Since we have 

\[ A_1 \cong C^*(v_r) \oplus C^*(v_s) \cong \mathbb{C}^2 \oplus \mathbb{C}^2 , \]
our first task is to compute $K_*(A_2)$. For any subset $M \subset SU(2)$ we extend the notation of the last section and write $f \cdot r$ and $f \cdot s$ for $f \in C(M)$, whenever it makes sense. We have

$$A_2 := \left\{ \begin{pmatrix} f & g \\ g \cdot r & f \cdot r \end{pmatrix} \in M_2(C(X_2)) : f = f \cdot s, \ g = -g \cdot s \right\}.$$ 

Again, the symmetry relations imply that an element in $A_2$ is already uniquely determined by its values on the compact subset of $X_2$ consisting of all $(a, c)$ where $\text{Re}(a)$ and $\text{Re}(c)$ are non-negative. Using a homeomorphism between this subset and $[0, 1]$ sending $(1, 0)$ to 0 and $(0, 1)$ to 1, we get an isomorphism between $A_2$ and

$$C := \{ f \in C([0, 1], M_2(C)) : f(0) \in C^*(v_r), \ f(1) \in C^*(v_s) \}.$$

**Lemma 3.1.** We have $K_0(C) = \mathbb{Z}^3$ and $K_1(C) = 0$. Moreover, $K_0(C)$ is generated by $[p_0], [q_0], \text{and } [1]$, where the projections $p_0(t), q_0(t) \in M_2(\mathbb{C})$ are given for $t \in [0, 1]$ by

$$p_0(t) := \begin{pmatrix} 1 - \frac{t}{2} & \sqrt{\frac{t}{2} - \frac{t^2}{4}} \\ \sqrt{\frac{t}{2} - \frac{t^2}{4}} & \frac{t}{2} \end{pmatrix}, \quad q_0(t) := \begin{pmatrix} \frac{t}{2} & \sqrt{\frac{t^2}{4} - \frac{t^2}{4}} \\ \sqrt{\frac{t^2}{4} - \frac{t^2}{4}} & 1 - \frac{t}{2} \end{pmatrix}.$$ 

**Proof.** The kernel of $ev_0 \oplus ev_1 : C \to C^*(v_r) \oplus C^*(v_s)$ is $C_0((0, 1), M_2(\mathbb{C}))$, and we get the following six-term exact sequence

$$0 \to K_0(C) \to K_0(C^*(v_s)) \oplus K_0(C^*(v_r)) \to K_1(C_0([0, 1])) \to 0 \quad \rho$$

As generators for $K_0(C^*(v_s)) \oplus K_0(C^*(v_r))$ we can choose

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad e_4 := \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$ 

To compute the images of these elements under the exponential map $\rho$, set

$$f_1(t) := \begin{pmatrix} 1 - t & 0 \\ 0 & 0 \end{pmatrix}, \quad f_2(t) := \begin{pmatrix} 0 & 0 \\ 0 & 1 - t \end{pmatrix},$$

$$f_3(t) := \begin{pmatrix} \frac{t}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{t}{2} \end{pmatrix} \quad \text{and} \quad f_4(t) := \begin{pmatrix} \frac{t}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{t}{2} \end{pmatrix}.$$ 

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for \(t \in [0, 1]\). Then we have \(\rho(e_j) = \exp(2\pi if_j)\) for \(1 \leq j \leq 4\) and
\[
\rho(e_1) = \rho(e_2) = -\rho(e_3) = -\rho(e_4) = -[z],
\]
where \(z \in C(T)\) is the identity map. Thus \(\rho\) is surjective and we have
\[
\ker(\rho) = \langle e_1 + e_3, e_1 - e_2, e_3 - e_4 \rangle.
\]
If \(r_0 \in C\) denotes the projection given by \(r_0(t) := 1 - q_0(t)\), we get
\[
K_0(ev_0 \oplus ev_1)(\langle p_0 \rangle) = e_1 + e_3, \quad K_0(ev_0 \oplus ev_1)(\langle p_0 - [r_0] \rangle) = e_1 - e_2, \quad K_0(ev_0 \oplus ev_1)(\langle p_0 - [r_0] \rangle) = e_3 - e_4.
\]
The claim now follows by exactness of the six-term sequence above.

Let us denote by \(\tilde{p}, \tilde{q} \in A_2\) the unique extensions of \(p_0, q_0 \in C\). Note that these projections satisfy the relations
\[
\tilde{p}(1, 0) = \tilde{q}(-1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{p}(-1, 0) = \tilde{q}(1, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
\tilde{p}(0, 1) = \tilde{q}(0, 1) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \quad \tilde{p}(0, -1) = \tilde{q}(0, -1) = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}.
\]
For the \(K\)-theory of \(A_3\), we write \(X_3\) as the disjoint union
\[
X_3 = X_2 \cup U_1 \cup U_2,
\]
with
\[
U_1 := \{ (a, c) \in SU(2) : \text{Im}(a) = 0, \text{Im}(c) > 0 \},
\]
\[
U_2 := \{ (a, c) \in SU(2) : \text{Im}(a) > 0, \text{Im}(c) = 0 \}.
\]
Since the surjection \(\pi_2\) is induced by the inclusion of \(X_2\) into \(X_3\), the kernel of \(\pi_2\) lies in \(M_2(C_0(U_1 \cup U_2))\). Considering the symmetry relations, we actually get an exact sequence
\[
\begin{array}{cccccc}
0 & \longrightarrow & I_1 \oplus I_2 & \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow & 0, \\
\end{array}
\]
where
\[
I_1 := \left\{ \begin{pmatrix} f & g \\ h & k \end{pmatrix} \in M_2(C_0(U_1)) : f = k \cdot s, \ g = h \cdot s \right\},
\]
\[
I_2 := \left\{ \begin{pmatrix} f & g \\ h & k \end{pmatrix} \in M_2(C_0(U_2)) : f = f \cdot r, \ g = -g \cdot r, \ h = -h \cdot r, \ k = k \cdot r \right\}.
\]
Note that $I_1$ and $I_2$ are isomorphic via the automorphism of $C(SU_{-1}(2))$ which exchanges $\alpha$ and $\gamma$. If $\mathbb{D}$ denotes the closed unit disk, we obtain a homeomorphism

$$U_2 \rightarrow \mathbb{D} \setminus T,$$

$$(a + ib, c) \mapsto a + ic.$$ 

An element in $I_2$ is uniquely determined by its values on $(a, c) \in U_2$ with $c \geq 0$. Hence, the above homeomorphism induces an isomorphism between $I_2$ and

$$\{ f \in C_0(\mathbb{D} \setminus T \geq 0, M_2(\mathbb{C}) ) : f(x) \in C^*(v_r), x \in T \setminus T \geq 0 \} ,$$

where $T \geq 0$ consists of those elements in $T$ with non-negative real part. As a consequence, $I_2$ is homotopy equivalent to

$$I := \{ f \in C_0(\mathbb{D} \setminus \{1\} , M_2(\mathbb{C}) ) : f(x) \in C^*(v_r), x \in T \setminus \{1\} \} .$$

The two $C^*$-algebras have isomorphic $K$-theory, which can be obtained from the exact sequence

$$0 \rightarrow C_0(\mathbb{D} \setminus T, M_2(\mathbb{C})) \rightarrow I \rightarrow C_0((0,1),C^*(v_r)) \rightarrow 0$$

induced by restriction on $T \setminus \{1\}$.

**Lemma 3.2.** We have $K_0(I) = 0$ and

$$\ker(\rho : K_1(C_0((0,1), C^*(v_r)))) \rightarrow K_0(C_0(\mathbb{D} \setminus T))) = \mathbb{Z} \cdot [\text{diag}(z, \bar{z})] ,$$

induces an isomorphism $K_1(I) \cong \mathbb{Z}$, where $\rho$ defines the index map associated to the above exact sequence.

**Proof.** Consider the exact six-term sequence

$$\begin{array}{c}
\mathbb{Z} \cong K_0(C_0(\mathbb{D} \setminus T)) \\
\rho \downarrow \\
\mathbb{Z}^2 \cong K_1(C_0((0,1), C^*(v_r))) \\
\uparrow \\
K_1(I) \cong K_0(I) \cong 0
\end{array}$$

As generators for $K_1(C_0((0,1), C^*(v_r)))$ we choose

$$e_1 := \left[ \begin{array}{c} z \\
0 \\
1 \end{array} \right], \quad e_2 := \left[ \begin{array}{c} 1 \\
0 \\
z \end{array} \right].$$

It suffices to show that $\rho$ sends $e_1$ and $e_2$ to the Bott element. We will only show this for $e_1$, the other case is similar. Let $I^\sim$ be the unitarization of $I$ and $u \in M_2(I^\sim)$ the unitary lift for $\text{diag}(z,1,\bar{z},1)$ given by

$$u(t) := \left( \begin{array}{cccc}
t & 0 & -\sqrt{1-|t|^2} & 0 \\
0 & 1 & 0 & 0 \\
\sqrt{1-|t|^2} & 0 & \bar{t} & 0 \\
0 & 0 & 0 & 1 \end{array} \right), \quad t \in \mathbb{D}. $$
If $1_2 \in M_2(I)$ denotes the unit, we get

$$\rho(e_1) = [u \cdot \text{diag}(1_2, 0) \cdot u^*] - [1_2]$$

$$= \left[ \begin{pmatrix} |z|^2 & z \sqrt{1 - |z|^2} \\ z \sqrt{1 - |z|^2} & 1 - |z|^2 \end{pmatrix} \right] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right],$$

and this completes the proof.

Actually, Lemma 3.2 shows that the natural homomorphisms

$$K_1(I_1) \to K_1(C_0(V_1, C^*(v_s))), \quad K_1(I_2) \to K_1(C_0(V_2, C^*(v_r)))$$

are injective, where

$$V_1 := \{ (a, c) \in U_1 : a = 0 \} \quad \text{and} \quad V_2 := \{ (a, c) \in U_2 : c = 0 \}.$$

We are now able to determine $K_\ast(A_3)$, which turns out to be the most difficult part of the computation of $K_\ast(C(SU_{-1}(2))).$

**Lemma 3.3.** It holds that $K_0(A_3) = \mathbb{Z} \cdot [1]$ and $K_1(A_3) = \mathbb{Z}_2$.

**Proof.** The exact sequence (3.1) and Lemma 3.2 induce the following 6-term-sequence

$$0 \to K_0(A_3) \xrightarrow{K_0(\pi_3)} K_0(A_2) \cong \mathbb{Z}^3 \to 0$$

$$0 \to K_1(A_3) \xrightarrow{K_1(\pi_3)} K_1(I_1) \oplus K_1(I_2) \cong \mathbb{Z} \oplus \mathbb{Z} \to 0.$$

It is clear from Lemma 3.1 that $[1] \in K_0(A_3)$ generates a copy of $\mathbb{Z}$. So it suffices to consider the restriction of $\rho$ to the subgroup of $K_0(A_2)$ generated by $[\tilde{p}]$ and $[\tilde{q}]$. As we will see, $\rho$ can be considered as the injective endomorphism on $\mathbb{Z}^2$ sending the canonical generators to $(1, 1)$ and $(1, -1)$, respectively. Hence, $K_1(A_1) \cong \mathbb{Z}_2$ follows.

If we set

$$t(a, c) := \left( \frac{\text{Re}(a)}{\sqrt{\text{Re}(a)^2 + \text{Re}(c)^2}}, \frac{\text{Re}(c)}{\sqrt{\text{Re}(a)^2 + \text{Re}(c)^2}} \right),$$

we can define positive lifts $p, q \in A_3$ for $\tilde{p}$ and $\tilde{q}$ by

$$p(a, c) := \begin{cases} (1 - \text{Im}(a)^2) \cdot \tilde{p}(t(a, c)) & \text{if } \text{Im}(a) \neq 1, \quad \text{Im}(c) = 0, \\ (1 - \text{Im}(c)^2) \cdot \tilde{p}(t(a, c)) & \text{if } \text{Im}(a) = 0, \quad \text{Im}(c) \neq 1, \\ 0 & \text{if } \text{Im}(a) = 1 \text{ or } \text{Im}(c) = 1, \end{cases}$$

and similarly for $q$. The exact sequence (3.1) and Lemma 3.2 induce the following 6-term-sequence

$$0 \to K_0(A_3) \xrightarrow{K_0(\pi_3)} K_0(A_2) \cong \mathbb{Z}^3 \to 0$$

$$0 \to K_1(A_3) \xrightarrow{K_1(\pi_3)} K_1(I_1) \oplus K_1(I_2) \cong \mathbb{Z} \oplus \mathbb{Z} \to 0,$$
\[
q(a,c) := \begin{cases} 
(1 - \text{Im}(a)^2) \cdot \tilde{q}(t(a,c)), & \text{if } \text{Im}(a) \neq 1, \text{ Im}(c) = 0, \\
(1 - \text{Im}(c)^2) \cdot \tilde{q}(t(a,c)), & \text{if } \text{Im}(a) = 0, \text{ Im}(c) \neq 1, \\
0, & \text{if } \text{Im}(a) = 1 \text{ or } \text{Im}(c) = 1.
\end{cases}
\]

Consequently, we have \(\rho([\tilde{p}]) = [\exp(2\pi ip)]\) and \(\rho([\tilde{q}]) = [\exp(2\pi i q)]\). For simplicity, write
\[
e = (e_1, e_2) := [\exp(2\pi i p)], \\
f = (f_1, f_2) := [\exp(2\pi i q)].
\]

Let \((a, 0) \in V_2\) be an arbitrary element, i.e., \(-1 < \text{Re}(a) < 1\) and \(\text{Im}(a) > 0\).
If we set \(\lambda(a) := \exp(-2\pi i \text{Im}(a)^2)\),
\[
\exp(2\pi i p)(a, 0) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
0 & \lambda(a) \end{pmatrix}, & \text{if } \text{Re}(a) < 0, \\
\begin{pmatrix} \lambda(a) & 0 \\
0 & 1 \end{pmatrix}, & \text{if } \text{Re}(a) > 0, \\
\begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}, & \text{else}
\end{cases}
\]
and
\[
\exp(2\pi i q)(a, 0) = \begin{cases} 
\begin{pmatrix} \lambda(a) & 0 \\
0 & 1 \end{pmatrix}, & \text{if } \text{Re}(a) < 0, \\
\begin{pmatrix} 1 & 0 \\
0 & \lambda(a) \end{pmatrix}, & \text{if } \text{Re}(a) > 0, \\
\begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}, & \text{else}
\end{cases}
\]
holds. Because of \(\overline{V_2} = V_2 \cup \{(i, 0), (-i, 0)\}\) and \(\lambda(i) = \lambda(-i) = 1\),
we can consider the restrictions of \(\exp(2\pi ip)\) and \(\exp(2\pi i q)\) as elements in \(C_0(V_2, C^*(v_r))^\sim \cong C(T, C^*(v_r))\). This identification can be chosen, such that
\[
\exp(2\pi ip) \sim_h \text{diag}(z, \bar{z}), \quad \exp(2\pi i q) \sim_h \text{diag}(\bar{z}, z) \text{ in } U(C(T, C^*(v_r))).
\]
Lemma 3.2 shows that \(e_2 = -f_2 \in K_1(I_2)\) is a generator. The first coordinate behaves similarly. The two functions \(\exp(2\pi ip)\) and \(\exp(2\pi i q)\) agree.

on $V_1$, since the relations
\[ \tilde{p}(0, \text{dir}0\alpha) = \tilde{q}(0, 1) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \]
\[ \tilde{p}(0, -1) = \tilde{q}(0, -1) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]
hold, hence $e_1 = f_1 \in K_1(I_1)$. Considering the order two automorphism of $C(SU_{-1}(2))$ which exchanges $\alpha$ and $\gamma$, the element $e_1$ is a generator for $K_1(I_1)$, since this holds for $e_2 \in K_1(I_2)$. This completes the proof. \[ \square \]

Finally, to deduce the $K$-theory of $C(SU_{-1}(2))$, consider the exact sequence
\[ 0 \longrightarrow M_2(C_0(X_4 \setminus X_3)) \longrightarrow A_4 \overset{\pi}{\longrightarrow} A_3 \longrightarrow 0. \]
In fact, $X_4 \setminus X_3$ is homeomorphic to $\mathbb{R}^3$ and we conclude with the help of Lemma 3.3 that $K_0(C(SU_{-1}(2))) = \mathbb{Z} \cdot [1]$. Moreover there is a short exact sequence
\[ 0 \longrightarrow \mathbb{Z} \longrightarrow K_1(C(SU_{-1}(2))) \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \]
This implies that $K_1(C(SU_{-1}(2)))$ can only be $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$.

**Theorem 3.4.** We have $K_0(C(SU_{-1}(2))) = \mathbb{Z} \cdot [1]$ and $K_1(C(SU_{-1}(2))) = \mathbb{Z} \cdot [u_{-1}]$.

**Proof.** The only thing left to show is that $K_1(C(SU_{-1}(2)))$ is isomorphic to $\mathbb{Z}$ and generated by the class of $u_{-1}$. We have a commutative diagramm

\[
\begin{array}{ccc}
M_2(C_0(X_4 \setminus X_3)) & \overset{\iota}{\longrightarrow} & C(SU_{-1}(2)) \\
\phi \downarrow & & \phi \downarrow \\
M_2(C(SU(2))) & \overset{\phi}{\longrightarrow} & C(SU_{-1}(2)) \\
\end{array}
\]
where $\phi$ is the injective $*$-homomorphism from Theorem 1.6 and $\iota$ comes from the canonical embedding into $A_4 \cong C(SU_{-1}(2))$. An easy calculation shows $K_1(\phi)([u_{-1}]) = [x] + [y]$, with $x, y \in \mathcal{U}(C(SU(2), M_2(\mathbb{C})))$ given by
\[ x(a, c) := \begin{pmatrix} a & \bar{c} \\ c & -\bar{a} \end{pmatrix}, \quad y(a, c) := \begin{pmatrix} -a & \bar{c} \\ c & \bar{a} \end{pmatrix}. \]
Actually, $[u_1] = [x] = [y] \in K_1(C(SU(2)))$. For $f \in C_0(X_4 \setminus X_3, M_2(\mathbb{C}))$, let $\tilde{f} \in C(SU(2), M_2(\mathbb{C}))$ denote the canonical extension. We get
\[ \varphi(f) = \tilde{f} + \tilde{f} \cdot r + \tilde{f} \cdot s + \tilde{f} \cdot rs. \]

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Thus, the image of $K_1(\varphi)$ is contained in $4 \cdot K_1(C(SU(2))$.

Now assume $K_1(C(SU_{-1}(2))) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. In this case, $K_1(\iota)$ can be viewed as the canonical embedding of the direct summand isomorphic to $\mathbb{Z}$. Consequently, the image of $K_1(\varphi)$ coincides with the image of $K_1(\phi)$. But this implies

$$2 [u_1] \in 4 \cdot K_1(C(SU(2)),$$

which contradicts the fact, that $[u_1]$ is a generator of $K_1(C(SU(2)))$. Thus, $K_1(C(SU_{-1}(2))) \cong \mathbb{Z}$ must be true. It also implies that $K_1(\iota)$ is multiplication with 2, and so $K_1(\phi)$ has to be multiplication with 2, as well. Hence, $[u_{-1}]$ is a generator for $K_1(C(SU_{-1})(2))$, which completes the proof.

4 The Haar state and $C(SU_q(2))$ as a continuous bundle

In this section we show, that $C(SU_q(2))$ can be considered as a continuous bundle of $C^*$-algebras in a natural way, where $q$ runs through $[-1,0)$. This result is inspired by the analogous result of Blanchard [5] for positive $q$. The crucial point is that the Haar states satisfy a suitable continuity condition, which is easily verified once we know the Haar state for $C(SU_{-1}(2))$.

**Proposition 4.1.** The Haar state $h_{-1}$ is uniquely determined by

$$h_{-1}(\eta^{klm}) = \begin{cases} \frac{1}{m+1}, & \text{if } k = 0, \ l = m, \\ 0, & \text{else}. \end{cases}$$

Moreover, $h_{-1}$ is faithful.

**Proof.** If $[\cdot, \cdot]$ denotes the commutator on $C(SU_{-1}(2)) \otimes C(SU_{-1}(2))$,

$$[\alpha \otimes \alpha, \gamma^* \otimes \gamma] = [\gamma \otimes \alpha, \alpha^* \otimes \gamma] = 0$$

holds. For $k$, $l$, and $m \in \mathbb{N}_0$ we have

$$\Delta_{-1}(\eta^{klm}) = \sum_{i=0}^{k} \binom{k}{i} \alpha^i \gamma^{*k-i} \otimes \alpha^i \gamma^{k-i} \sum_{j=0}^{l} \binom{l}{j} \gamma^j \alpha^{l-j} \otimes \alpha^j \gamma^{l-j} \sum_{p=0}^{m} \binom{m}{p} \gamma^{*p} \alpha^{m-p} \otimes \alpha^p \gamma^{*m-p}. $$

Right invariance of the Haar state implies

$$h_{-1}(\eta^{klm}) \cdot 1 = \sum_{i=0}^{k} \sum_{j=0}^{l} \sum_{p=0}^{m} \binom{k}{i} \binom{l}{j} \binom{m}{p} h_{-1}(\alpha^i \gamma^{*k-i} \gamma^j \alpha^{l-j} \gamma^{*p} \alpha^{m-p}) \cdot \alpha^i \gamma^{k-i} \alpha^j \gamma^{l-j} \alpha^p \gamma^{*m-p},$$

$$4.1$$
and similarly left invariance shows
\[ h_{-1}(\eta^{klm}) \cdot 1 = \]
\[ \sum_{i=0}^{k} \sum_{j=0}^{l} \sum_{p=0}^{m} (k^i l^j m^p) h_{-1}(a^i \gamma^k \alpha^j \gamma^l \alpha^p \gamma^m) \cdot \alpha^{k-i} \gamma^j \alpha^{l-j} \gamma^p \alpha^{m-p}. \]
\[ (4.2) \]

Let \( \pi_{(a,0)} \) be the character associated to \( a \in T \), as described in Section 1. Since \( \pi_{(a,0)}(\gamma) = 0 \), the only summand on the right hand side of equality (4.1) which does not vanish is the one associated to \( i = k, j = l \) and \( p = m \). In fact,
\[ h_{-1}(\eta^{klm}) = h_{-1}(\eta^{klm}) \cdot a^{k+l-m}. \]
The analogous observation for equality (4.2) gives
\[ h_{-1}(\eta^{klm}) = h_{-1}(\eta^{klm}) \cdot a^{k-l+m}. \]
Since these equations hold for every \( a \in T \), we conclude that \( h_{-1}(\eta^{klm}) \neq 0 \) can only hold for
\[ k + l - m = k - l + m = 0. \]
This is equivalent to \( k = 0 \) and \( l = m \). Using the anticommutativity relations, (4.1) simplifies to
\[ h_{-1}(\eta^{0mm}) \cdot 1 = \sum_{p=0}^{m} (m^p)^2 h_{-1}(\gamma^p \gamma^p \alpha^{m-p} \alpha^{m-p}) \cdot \alpha^p \alpha^p \gamma^{m-p} \gamma^{m-p}. \]
Thus, by applying the binomial theorem twice, we end up with
\[ h_{-1}(\eta^{0mm}) \cdot 1 = \sum_{p=0}^{m} \sum_{i=0}^{m-p} \sum_{j=0}^{p} (m^p)^2 (\gamma^p)^i (\gamma^p)^j (-1)^{i+j} h_{q}(\eta^{0(p+i)(p+j)}) \cdot \eta^{0(m-p+j)(m-p+j)}. \]
Since the family \( \{ \eta^{pp} : p \in N_0 \} \) is linearly independent, we can consider the coefficient of \( \eta^{011} \) in the equation above. This amounts to
\[ -m \cdot h_{-1}(\eta^{0mm}) + m^2 \cdot h_{-1}(\eta^{0(m-1)(m-1)}) - m^2 \cdot h_{-1}(\eta^{0mm}) = 0. \]
Since \( h_{-1} \) is a state, \( h_{-1}(1) = 1 \) holds. Moreover, assume that we already have
\[ h_{-1}(\eta^{0(m-1)(m-1)}) = \frac{1}{m}. \]
In this case
\[ h_{-1}(\eta_{mm}^0) = \frac{m^2}{m^2 + m} \cdot h_{-1}(\eta_{(m-1)(m-1)}^0) = \frac{1}{m + 1} \]
follows. It remains to check faithfulness of \( h_{-1} \). An easy calculation shows that \( h_{-1} \) coincides with the restriction of \( h_1 \otimes \text{tr} \) on \( C(SU(2)) \otimes M_2(\mathbb{C}) \), when we consider \( C(SU_{-1}(2)) \) as a subalgebra of \( C(SU(2)) \otimes M_2(\mathbb{C}) \). Finally, \( h_1 \otimes \text{tr} \) is faithful, since it is a vector state consisting of faithful states \([6]\). □

At this point, we should mention that Zakrzewski already noted in \([23]\) that the Haar state on \( C(SU_{-1}(2)) \) is given by restriction of \( h_1 \otimes \text{tr} \). We recall some basic facts about \( C_0(X) \)-algebras and continuous bundles of \( C^* \)-algebras. For further details on this subject, we refer to \([5]\).

**Definition 4.2.** Let \( X \) be a locally compact Hausdorff space. A \( C_0(X) \)-algebra is a \( C^* \)-algebra \( A \) together with a non-degenerate \( * \)-homomorphism from \( C_0(X) \) into the center of \( M(A) \), the multiplier algebra of \( A \).

If \( A \) is a \( C_0(X) \)-algebra, then for every \( x \in X \), \( C_0(X \setminus \{x\})A \) forms a closed two-sided ideal. The respective quotient \( A_x \) is called the fiber at \( x \). If \( a \in A \), let \( a_x \) be its image under the canonical surjection onto \( A_x \). One fundamental property of \( C_0(X) \)-algebras is that the map
\[ X \rightarrow \mathbb{R} , \quad x \mapsto \|a_x\| \]
is upper-semicontinuous for every \( a \in A \). We call \( A \) a continuous \( C^* \)-bundle over \( X \) if all these maps are continuous.

**Theorem 4.3.** The \( C^* \)-algebras \( C(SU_q(2)) \), \( q \in [-1, 0) \), form a continuous \( C^* \)-bundle over \([-1, 0)\).

**Proof.** Let \( B \) be the universal unital \( C^* \)-algebra with generators \( \alpha, \gamma \) and \( f \), satisfying the following relations: \( f \) is normal, has spectrum \([-1, 0] \), commutes with \( \alpha \) and \( \gamma \) and
\[ \left( \begin{array}{cc} \alpha & -f\gamma^* \\ \gamma & \alpha^* \end{array} \right) \in M_2(B) \]
is unitary. Since \( f \) is central and normal, \( B \) can be considered as a \( C([-1, 0]) \)-algebra. Using the respective universal properties of \( C(SU_q(2)) \) and \( B_q \), we see that the natural homomorphism \( C(SU_q(2)) \rightarrow B_q \) is an isomorphism. Thus \( A := C_0([-1, 0])B \) is a \( C_0([-1, 0]) \)-algebra with fibres \( C(SU_q(2)) \).

Since all Haar states are faithful, \([5]\) Theorem 3.3] says that \( A \) is a continuous bundle if there is a \( C_0([-1, 0]) \)-linear, positive map
\[ h: A \rightarrow C_0([-1, 0])) , \]
with $h_q = ev_q \circ h$. Analogous to the case of $C(SU_q(2))$, the $C_0([-1,0))$-linear span of

$$\eta^{klm} := \begin{cases} \alpha^{k,l,m} & \text{if } k \geq 0, \ m, n \geq 0, \\ \alpha^{s-k,l,m} & \text{if } k < 0, \ m, n \geq 0 \end{cases}$$

is dense in $A$. For $a \in A$ we define a map

$$h(a): [-1,0) \to \mathbb{C}, \quad h(a)(q) := h_q(a_q),$$

and see that it satisfies

$$h(\eta^{klm})(q) = \delta_{k=0} \frac{1 - q^2}{1 - q^{2m+1}} \frac{q^{-1}}{m+1} = h(\eta^{klm})(-1).$$

Hence, $h(a) \in C_0([-1,0))$ for each element $a$ of a dense subalgebra of $A$. Using an $\varepsilon^3$-argument, we conclude that $h$ is well-defined. Moreover, it is $C_0([-1,0))$-linear and positive, since each $h_q$ is a state. The proof is complete.

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**References**

[1] T. Banica, *Le Groupe Quantique Compact Libre $U(n)$*, Comm. Math. Phys., 190 (1997), pp. 143–172.

[2] J. Bichon, A. De Rijdt, and S. Vaes, *Ergodic Coactions with large Multiplicity and Monoidal Equivalence of Quantum Groups*, Comm. Math. Phys., 262 (2006), pp. 703–728.

[3] B. Blackadar, *K-Theory for Operator Algebras*, vol. 5 of Mathematical Sciences Research Institute Publications, Springer-Verlag, New York, 1986.

[4] B. Blackadar, *Operator Algebras: Theory of $C^*$-algebras and von Neumann Algebras*, vol. 122 of Encyclopaedia of Mathematical Sciences, Springer, 2005.

[5] É. Blanchard, *Déformations de C*-algèbres de Hopf*, Bull. Soc. Math. France, 124 (1996), pp. 141–215.
[6] N. P. Brown and N. Ozawa, $C^*$-algebras and finite-dimensional Approximations, vol. 88 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2008.

[7] P. S. Chakraborty and A. Pal, Equivariant Spectral Triples on the Quantum $SU(2)$ Group, K-Theory, 28 (2003), pp. 107–126.

[8] L. Dąbrowski, L. Giovanni, A. Sitarz, W. Suijlekom, and J. C. Várilly, The Dirac Operator on $SU_q(2)$, Comm. Math. Phys., 259 (2005), pp. 729–759.

[9] V. G. Drinfel’d, Quantum Groups., in Proceedings of the international congress of mathematicians (ICM), Berkley, USA, August 3–11, 1986, vol. II, Providence, R.I.: American Mathematical Society, 1987, pp. 798 –820.

[10] M. Jimbo, A q-Difference Analogue of $U(g)$ and the Yang-Baxter Equation., Lett. Math. Phys., 10 (1985), pp. 63–69.

[11] A. Klimyk and K. Schmüdgen, Quantum Groups and their Representations, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997.

[12] G. Nagy, On the Haar Measure of the Quantum $SU(N)$ Group, Communications in Mathematical Physics, 153 (1993), pp. 217–228.

[13] M. Rørdam, F. Larsen, and N. Laustsen, An Introduction to K-Theory for $C^*$-Algebras, London Mathematical Society Student Texts, Cambridge University Press, 2000.

[14] M. Rosso, Algèbres Enveloppantes Quantifiées, Groupes Quantiques Compacts de Matrices et Calcul Différentiel Non Commutatif, Duke Math. J., 61 (1990), pp. 11–40.

[15] T. Timmermann, An Invitation to Quantum Groups and Duality: From Hopf Algebras to Multiplicative Unitaries and Beyond, EMS Textbooks in Mathematics, European Mathematical Society, 2008.

[16] A. Van Daele, The Haar Measure on a Compact Quantum Group, Proceedings of The American Mathematical Society, 123 (1995).

[17] A. Van Daele and S. Wang, Universal Quantum Groups, Internat. J. Math., 7 (1996), pp. 255–263.

[18] C. Voigt, The Baum-Connes Conjecture for Free Orthogonal Quantum Groups., Adv. Math., 227 (2011), pp. 1873–1913.

[19] S. L. Woronowicz, Compact Matrix Pseudogroups., Commun. Math. Phys., 111 (1987), pp. 613–665.
[20] S. L. Woronowicz, *Twisted SU(2) Group. An Example of a Noncommutative Differential Calculus*, Publ. Res. Inst. Math. Sci., 23 (1987), pp. 117–181.

[21] S. L. Woronowicz, *Tannaka-Krein Duality for Compact Matrix Pseudogroups. Twisted SU(N) Groups.*, Invent. Math., 93 (1988), pp. 35–76.

[22] S. L. Woronowicz, *Compact Quantum Groups*, in Quantum symmetries/ Symétries quantiques. Proceedings of the Les Houches summer school, Session LXIV, Les Houches, France, August 1 – September 8, 1995, A. Connes et al., eds., Amsterdam: North-Holland, 1998, pp. 845–884.

[23] S. Zakrzewski, *Matrix Pseudogroups Associated with Anti-Commutative Plane*, Letters in Mathematical Physics, 21 (1991), pp. 309–321.

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