Further de-Entangling Entanglement

Elemér E Rosinger

Department of Mathematics
and Applied Mathematics
University of Pretoria
Pretoria
0002 South Africa
eerosinger@hotmail.com

Abstract

A recent general model of entanglement, [6], that goes much beyond the usual one based on tensor products of vector spaces is further developed here. It is shown that the usual Cartesian product can be seen as two extreme particular instances of non-entanglement. Also the recent approach to entanglement in [9] is incorporated in the general model in [6]. The idea pursued is that entanglement is by now far too important a phenomenon in Quantum Mechanics, in order to be left confined to its present exclusive modelling by tensor products. Once this is realized, it turns out that one can quite easily de-entangle entanglement from tensor products, and in fact, one can do so in a large variety of ways. Within such general settings the issue of entanglement becomes connected with the issue of "System versus Subsystem" in General Systems Theory, where synthesizing subsystems into a system may often be less difficult than identifying subsystems in a system.

0. Preliminaries

It is by now well and widely understood that entanglement is an essential physical phenomenon, and in fact, also a major resource, in Quantum Mechanics. As such, entanglement appeared for the first
time in the celebrated 1935 EPR paper of Einstein, Podolsky and Rosen, [1]. The term *entanglement* itself, however, was not used in that paper. Instead, it was introduced, in German, as *Verschränkung*, in the subsequent papers of Schrödinger, [3,4], in which he commented on the state of Quantum Mechanics at the time, and among others, brought to attention the problematic situation which ever since would be called *Schrödinger’s cat*. 

In this regard, it may be instructive to note how often in science fundamental ideas, concepts or phenomena may take quite some time to reach their better appreciation and understanding. Indeed, until recently, hardly any of the major texts on Quantum Mechanics found it necessary to deal with entanglement as a subject on its own, let alone, an important one.

As it happened, independently, and prior to the EPR paper, in Multilinear Algebra, the concept of *tensor product* was introduced by mathematicians in view of its *universal* property of establishing a natural connection between multilinear and linear mappings, see Appendix in [6]. And it took some time, both for physicists and mathematicians, to become aware of the fact that a natural mathematical formulation of quantum entanglement can be obtained with tensor products of Hilbert spaces.

It may be of a certain historical interest to find out the first publication in Quantum Mechanics where entanglement was treated using tensor products.

The aim of [6] was as follows. So far, entanglement has been modelled mathematically only by tensor products of vector, and in particular, Hilbert spaces. However, in view of the significant importance of entanglement, one could ask the question:

- Can entanglement be modelled in other more *general* ways, than by tensor products of vector spaces?

In [6] an affirmative answer was given to that question, by presenting general, and yet simple ways of entanglement, which contain as a particular case tensor products. In fact, in that general approach to entanglement, and unlike in the particular case of tensor products of vector spaces, the spaces involved can be rather arbitrary sets, just as
in the case of Cartesian products, thus in particular, they need \textit{not} be vector spaces, and not even groups or semigroups. This paper further develops the respective results.

Here however, it is important to note the following. The issue of entanglement - when de-entangled from its present tensor product based exclusive representation - becomes clearly connected with the well known and rather deep issue of the relationship "system versus subsystem" in General Systems Theory. And this system-subsystem relationship has two \textit{dual} aspects. One is to \textit{synthesize} given subsystems into a system, while the other is to \textit{identify} subsystems in a given system. And as is well known, typically neither of the two are easy issues, with the latter being often considerably more difficult, than the former, [9-11].

An easy to grasp illustration of that \textit{asymmetry} in difficulty can be given by the so called "Universal Law of Unintended Effects" in human affairs, a law which operates, among others, precisely due to the usual lack of appropriate insight into the structure of subsystems in a given system.

By the way, in medicine, the so called side-effects of treatments are such typical unintended effects.

As it happens, entanglement has so far only been considered in the context of \textit{synthesis}, that is, when independent quantum systems $S_1, \ldots, S_n$, where $n \geq 2$, with the corresponding Hilbert spaces $H_1, \ldots, H_n$, are constituted into an aggregate quantum system $S$, with the resulting Hilbert space $H = H_1 \otimes \cdots \otimes H_n$.

Such a synthesis approach, in a significantly generalized manner, was also pursued in [6].

In [9], the dual systems approach, that is, of \textit{identifying} subsystems in a system, is pursued to a good extent, even if the authors may insists, as their very title indicates, to have gone beyond any systemic considerations, by introducing observers in the process of instituting and identifying entanglement. Indeed, the introduction of observers $"O"$ does not do more than simply \textit{extend} the initial quantum system $"S"$
assumed to be without observers, to a new system ”S and O” which this time contains both the quantum system and the observers. Of course, one may miss that point, or simply refuse to have mixed together into a whole the quantum system and the observers. However, the fact remains that, as for instance in [9], the observers enter into a highly relevant interaction with the quantum system, not least in the ways entanglement is instituted and then identified by them. Therefore, the merit - and novelty - in [9] is precisely in the fact that their approach to the issue of entanglement is no longer confined to the usual systems synthesis leading to a given tensor product, thus to one and only way to have entanglement. Instead, in [9], even if still tensor products are used exclusively in modelling entanglement, the way entanglement is instituted and identified is due to the observers who have a certain latitude in identifying subsystems in the given quantum system.

In the sequel when going significantly beyond the usual tensor products, we shall mainly pursue the approach in [6] which is along the usual systems synthesis. What is somewhat unexpected with such an approach is that it can help in a similarly general way beyond tensor products, this time along the dual approach of subsystems identification as well which, quite likely, is to be considerably more difficult in its fuller study. In this regard, the subsystems identification approach related to entanglement suggested in [9], namely, by the introduction of observers, can be seen as a particular case of the approach in this paper. The advantage of such a particularization is in the fact mentioned above, namely that, the subsystems identification approach is typically far more difficult than that of systems synthesis.

As for a general enough approach to entanglement along subsystems identification lines, this may quite likely be a considerably difficult task.

Two further observations are important, before proceeding with the paper.

The usual, tensor product based concept of entanglement is in fact
given by the *negation* of a certain kind of rather simple representation. Consequently, any extension and/or deepening of that concept is bound to open up a large variety of meaningful possibilities. This is precisely one of the features - often overlooked - of the concept of entanglement which makes it nontrivial. Indeed, the role in Physics of definitions by negation is an issue which can touch upon fundamental aspects, [5].

Quantum physics arguments expressing quite some concern related to the usual tensor product based concept of entanglement were recently presented in [8]. And they indicate what may be seen as a lack of *ontological robustness* of that concept. As an effect, one may expect that what appears to be entanglement in terms of tensor products may in fact correspond to considerably deeper and more general aspects. In this regard, the old saying that "the whole is *more* than the sum of its parts" may in fact mean that what is involved in that "more" in the case of entanglement can correspond to very different things, depending on the specific situations.

A likely consequence of these two facts is that, when seen in its depth and generality, the concept of entanglement may naturally and necessarily branch into a larger variety of rather different concepts which are only somewhat loosely related to one another.

The main definition, [6,7], is presented in section 3, and it is further extended in section 5. The main new results can be found in sections 4 and 5.

1. Generators and Bases

For convenience, here and in the next two sections we recall a few concepts and results in [6,7]. These are within the usual, that is, *systems synthesis* approach to entanglement.

**Definition 1.1.**

Given any set $X$, a mapping $\psi : \mathcal{P}(X) \to \mathcal{P}(X)$ will be called a
generator, if and only if
\[(1.1) \quad \forall \ A \subseteq X \ : \ A \subseteq \psi(A)\]

and
\[(1.2) \quad \forall \ A \subseteq A' \subseteq X \ : \ \psi(A) \subseteq \psi(A')\]

Let us denote by
\[(1.3) \quad \mathcal{G}en(X)\]

the set of generators on \(X\).

**Examples 1.1.**

1) A trivial, yet as we shall see important, example of generator is given by \(\psi = id_{2^X}\), that is, \(\psi(A) = A\), for \(A \subseteq X\).

2) Another example which is important in the sequel is obtained as follows. Given any binary operation \(\alpha : X \times X \rightarrow X\), we call a subset \(A \subseteq X\) to be \(\alpha\)-stable, if and only if
\[(1.4) \quad x, y \in A \implies \alpha(x, y) \in A\]

Obviously, \(X\) itself is \(\alpha\)-stable, and the intersection of any family of \(\alpha\)-stable subsets is also \(\alpha\)-stable. Consequently, for every subset \(A \subseteq X\), we can define the smallest \(\alpha\)-stable subset which contains it, namely
\[(1.5) \quad \left[A\right]_{\alpha} = \bigcap_{A \subseteq B, \ B \ \text{\(\alpha\)-stable}} B\]

Therefore, we can associate with \(\alpha\) the mapping \(\psi_{\alpha} : 2^X \rightarrow 2^X\) defined by
\[(1.6) \quad \psi_{\alpha}(A) = \left[A\right]_{\alpha}, \quad A \subseteq X\]

which is obviously a generator. Furthermore, we have in view of (1.5)
(1.7) \( \forall \ A \subseteq X : \psi_\alpha(\psi_\alpha(A)) = \psi_\alpha(A) \)

since as mentioned, \([A]_\alpha \) is \( \alpha \)-stable, and obviously \([A]_\alpha \subseteq [A]_\alpha \).

We note that, in general, the relation \( \psi(\psi(A)) = \psi(A) \), with \( A \subseteq X \), need not hold for an arbitrary generator \( \psi \).

3) A particular case of 2) above is the following. Let \((S, \ast)\) be a semigroup with the neutral element \( e \). Then \([\{e\}]_\ast = \{e\} \), while for \( a \in S, \ a \neq e \), we have \([\{a\}]_\ast = \{a, a \ast a, a \ast a \ast a, \ldots\} \).

For instance, if \((S, \ast) = (\mathbb{N}, +)\), then \([\{1\}]_+ = \mathbb{N} \setminus \{0\} = \mathbb{N}_+ \).

4) A further case, which is of relevance in tensor products, is when we are given a vector space \( E \) over some field of scalars \( \mathbb{K} \). If now we have any subset \( A \subseteq E \), then we can define \( \psi(A) \) as the vector subspace in \( E \) generated by \( A \). Clearly, we obtain a generator in the sense of Definition 1.1.

Here one should, however, note that this generator is no longer of the simple form in 2) above. Indeed, the generation of a vector subspace does involve two algebraic operations, and not only one, as in 2) above, namely, addition of vectors, and multiplication of vectors with scalars.

5) The general pattern corresponding to 4), and which contains 2) as a particular case is as follows. Given on a set \( X \) the mappings \( \alpha_1, \ldots, \alpha_n : X \rightarrow X \) and \( \beta_1 : K_1 \times X \rightarrow X, \ldots, \beta_m : K_m \times X \rightarrow X \), where \( K_1, \ldots, K_m \) are certain sets. Then we call a subset \( A \subseteq X \) to be \( \alpha, \beta \)-stable, if and only if, see (3.1)

\[
(1.8) \quad x, y \in A \implies \alpha_i(x, y) \in A, \quad 1 \leq i \leq n
\]

and

\[
(1.9) \quad x \in A \implies \beta_j(c_j, x) \in A, \quad c_j \in K_j, \quad 1 \leq j \leq m
\]

Obviously, \( X \) itself is \( \alpha, \beta \)-stable, and the intersection of any family of \( \alpha, \beta \)-stable subsets is again \( \alpha, \beta \)-stable. Thus, for every subset \( A \subseteq X \),
we can define the smallest $\alpha, \beta$-stable subset which contains it, namely

\[(1.9) \quad [A]_{\alpha, \beta} = \cap_{A \subseteq B, \ B \ \alpha, \beta-stable} B\]

Therefore, we can define the mapping $\psi_{\alpha, \beta} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ as given by

\[(1.11) \quad \psi_{\alpha, \beta}(A) = [A]_{\alpha, \beta}, \quad A \subseteq X\]

which is obviously a generator in the sense of Definition 1.1. Furthermore, we have in view of (1.10)

\[(1.12) \quad \forall \ A \subseteq X : \psi_{\alpha, \beta}(\psi_{\alpha, \beta}(A)) = \psi_{\alpha, \beta}(A)\]

since as mentioned, $[A]_{\alpha, \beta}$ is $\alpha, \beta$-stable, and obviously $[A]_{\alpha, \beta} \subseteq [A]_{\alpha, \beta}$.

**Definition 1.2.**

Given a generator $\psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, a subset $B \subseteq X$ is called a $\psi$-basis for $X$, if and only if

\[(1.13) \quad \psi(B) = X\]

and $B$ is minimal with that property, namely

\[(1.14) \quad \forall \ B' \subseteq B : \psi(B') \nsubseteq X\]

Let us denote by

\[(1.15) \quad \mathcal{B}_{\psi}(X)\]

the set of all $B \subseteq X$ which are a $\psi$-basis for $X$.

**Note 1.1.**

1) In view of 3) in Examples 1.1., it follows that neither $\{0\}$, nor $\{1\}$ are $\psi_{+}$-bases in $(\mathbb{N}, +)$, while on the other hand, $\{0, 1\}$ is.
2. Covered Generators

The usual systems synthesis approach to entanglement is extended now considerably beyond tensor products, based on the previous section 1.

Definition 2.1.

Given the sets $X$ and $Y$, with the corresponding generators $\psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, $\varphi : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$, and $\chi : \mathcal{P}(X \times Y) \rightarrow \mathcal{P}(X \times Y)$. We say that $\chi$ is covered by $\psi, \varphi$, if and only if

\[
\forall \ A \subseteq X, \ B \subseteq Y : \ \chi(A \times B) \subseteq \psi(A) \times \varphi(B)
\]

Examples 2.1.

1) Obviously, if $\psi = id_{\mathcal{P}(X)}$, $\varphi = id_{\mathcal{P}(Y)}$ and $\chi = id_{\mathcal{P}(X \times Y)}$, then $\chi$ is a covering for $\psi, \varphi$.

2) Let now $\alpha : X \times X \rightarrow X$ and $\beta : Y \times Y \rightarrow Y$ be two binary operations and, as usual, let us associate with them the binary operation $\alpha \times \beta : (X \times Y) \times (X \times Y) \rightarrow (X \times Y)$ given by

\[
(\alpha \times \beta)((x, y), (u, v)) = (\alpha(x, u), \beta(y, v)), \quad x, u \in X, \ y, v \in Y
\]

Then $\psi_{\alpha \times \beta}$ is covered by $\psi_{\alpha}, \ \psi_{\beta}$, see [6], Lemma 2.1.

3) The case of interest for tensor products is the following. Given two vector spaces $E$ and $F$ on a scalar field $\mathbb{K}$, let $\psi$ and $\varphi$ be the corresponding generators as defined in 4) in Examples 1.1. Further, on the vector space $E \times F$, let $\chi$ be the generator defined in the same manner.

Then it follows easily that $\chi$ is a covering for $\psi$ and $\varphi$. 

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3. A General Concept of Entanglement

Within the usual systems synthesis approach, and based on the above, we can now give a very general definition of entanglement, [6, section 4, 7, section 3], much beyond that of the usual tensor tensor products, or in fact, of any algebraic nature as such.

Definition 3.1.

Given the sets $X$ and $Y$, with the corresponding generators $\psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, $\varphi : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$.

An element $w \subseteq X \otimes_{\psi, \varphi} Y$ is called entangled, if and only if it is not of the form

$$w = x \otimes_{\psi, \varphi} y$$

for some $x \in X$ and $y \in Y$.

Note 3.1.

1) In view of [7, Theorem 3.3.], the above Definition 3.1. contains as a particular case the customary concept of entanglement as formulated in terms of usual tensor products.

2) The interest in the general concept of entanglement in Definition 3.1. is, among others, in the fact that it is no longer confined within any kind of algebraic context. In this way, this paper, following [6,7], shows that entanglement can, so to say, be de-entangled not only from tensor products, but also more generally, from all algebra as well.

Consequently, the "quantum way of composition of system", namely, by the tensor product of their state spaces, is no longer limited to quantum systems alone. Indeed, in order to be composable by tensor products, such state spaces need no longer be limited to vector spaces, and instead, can now be given by arbitrary sets.
3) The generality of the concept of entanglement given in Definition 3.1., which at first appears confined to the systems synthesis approach, has nevertheless the further advantage to allow a certain incursion into the dual subsystems identification approach.

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