CONVOLUTION ESTIMATES FOR MEASURES ON SOME COMPLEX CURVES

HYUNUK CHUNG AND SEHEON HAM

Abstract. We consider the convolution operator for a measure supported on complex curves. The measure which we consider here is an analogue of the affine arclength measure for real curves. By modifying a combinatorial argument called the band structure argument, we prove the (nearly) optimal Lorentz space estimates. This includes the optimal strong type estimates as special cases. The complex curves we consider here are the ones considered for the Fourier restriction estimates for complex curves in [1].

1. Introduction

Let \( h(z) = (z, z^2, \ldots, z^{d-1}, \phi(z)) \), \( d \geq 2 \), be a complex curve of simple type in \( \mathbb{C}^d \), where \( \phi(z) \) is an analytic function defined on a region \( \Omega \subset \mathbb{C} \). This is regarded as a 2-dimensional surface in \( \mathbb{R}^{2d} \) defined by the real mapping
\[ z = (x, y) \mapsto h(x, y) = (x, y, x^2 - y^2, 2xy, \ldots, \text{Re} (\phi(z)), \text{Im} (\phi(z))). \]

We consider a convolution operator for a measure supported on the range of \( h \), defined by
\[ A f(\xi) = \int_D f(\xi - h(z)) |\phi^{(d)}(z)|^{\frac{4}{4d+1}} d\mu(z). \]

Here \( D \) is the unit ball in \( \mathbb{R}^2 \), \( f \) is a Borel function on \( \mathbb{R}^{2d} \), and \( d\mu(z) \) is the surface measure \( d\mu(z) = d\mu(h(z)) = dx dy \), where \( z = x + iy \). We have \(|\phi^{(d)}(z)|^{\frac{4}{4d+1}} d\mu(z) = c_d |\text{det}(h'(z), h''(z), \ldots, h^{(d)}(z))|^{\frac{4}{4d+1}} d\mu(z)\), which is an analogue of the affine arclength measure in the case of real curves.

It is obvious that \( A : L^\infty(\mathbb{R}^{2d}) \mapsto L^\infty(\mathbb{R}^{2d}) \), and by duality \( A : L^1(\mathbb{R}^{2d}) \mapsto L^1(\mathbb{R}^{2d}) \).

In the nondegenerate case, i.e. when \( \phi(z) = z^d \), one can see that \( A \) is not of restricted weak type \((p, q)\) outside of the (closed) trapezoid \( R \) with vertices at \((0, 0), (1, 1), (1/p_d, 1/q_d), (1/q_d', 1/p_d')\) in the plane. Here \( p_d = \frac{d+1}{2} \) and \( q_d = \frac{d(d+1)}{2(d-1)} \). (We will discuss the necessary conditions in Appendix B.) Hence, if we show the restricted...
weak type for $\mathcal{A}$ at $(p_d, q_d)$ (then by duality $\mathcal{A}$ is also of restricted weak type of $(q_d', p_d')$), then $\mathcal{A}$ is of strong type $(p, q)$ on $\mathcal{R}$ except for the points $(1/p_d, 1/q_d)$ and $(1/q_d', 1/p_d')$. In addition to the restricted weak type at $(p_d, q_d)$, we will show the optimal boundedness of $\mathcal{A}$ on the scale of Lorentz spaces.

The following is our first result.

**Theorem 1.1.** Let $d \geq 2$ and $\phi(z) = z^N$ for a nonnegative integer $N$. Then there exists a constant $C = C(d, N)$ such that

$$
\|Af\|_{L^{q_d,v}({\mathbb{R}}^d)} \leq C\|f\|_{L^{p_d,u}({\mathbb{R}}^d)} \quad \text{for } u < q_d, \ p_d < v, \text{ and } u < v.
$$

Here $C = C(d, N)$ depends only on $d, N$.

This implies that $\mathcal{A}$ is of strong type in $\mathcal{R}$. This result relies on an estimate for a lower bound of some Jacobian (see Section 2) arised from the Fourier restriction theorem for complex curves (see [1]), where a uniform Fourier restriction estimate for polynomial curves of simple type for $d = 3$ was also obtained. Using this, we thus get the following.

**Theorem 1.2.** When $d = 3$ and $\phi(z)$ is a polynomial of degree at most $N$, (1) holds for $h(z) = (z, z^2, \phi(z))$ and a constant $C = C(N)$, which depends only on $N$.

For a two-dimensional surface $h(z) = (z, \phi(z))$ in $\mathbb{R}^4$, where $\phi(z)$ is not necessarily holomorphic, Drury and Guo [16] showed the $L^{3/2}({\mathbb{R}}^4) \rightarrow L^3({\mathbb{R}}^4)$ boundedness for the convolution operator defined by an induced measure on the surface $h(z)$ under some nondegeneracy conditions on $\phi(z)$.

Our approach basically follows the case of real curves. Let $\gamma(t) : I = [0, 1] \mapsto \mathbb{R}^d$ be a smooth space curve in $\mathbb{R}^d$. Let us denote the affine arclength measure along $\gamma(t)$ by

$$
w(t)dt = [\det (\gamma'(t), \gamma''(t), \ldots, \gamma^{(d)}(t))]^{1/(d+1)}dt.
$$

There has been much work about estimates for the convolution operator given by

$$
Bf(x) = \int_I f(x - \gamma(t))w(t)dt.
$$

When $\gamma(t) = (t, t^2, \ldots, t^d)$ i.e. $w(t)dt \sim dt$, Littman [21] ($d = 2$) and Oberlin [22] established the strong type $(p_d, q_d)$ for $d = 3$ and the restricted weak type $(p_d, q_d)$ for $d = 4$. The endpoint case in $d = 2$, i.e. $L^{3/2,r}({\mathbb{R}}^2) \rightarrow L^{3,r}({\mathbb{R}}^2)$ estimate for all $p_2 = 3/2 \leq r \leq 3 = q_2$ was shown by Bak, Oberlin, and Seeger [2]. The estimate $L^2({\mathbb{R}}^3) \rightarrow L^{3/2,2}({\mathbb{R}}^3)$ (which is the case $u = v = p_3 = 2$) was established by Bennett and Seeger in [6], where the result was proved by analyzing the singularities of the phase function of a certain oscillatory integral operator. When $d \geq 2$, Christ [9] proved the restricted weak type $(p_d, q_d)$ for $\mathcal{B}$ by using the band structure argument. This argument was extended by Stovall [30] to establish the $L^{p_d,u} \rightarrow L^{q_d,v}$ estimates for $u < q_d$, $v > p_d$, and $u < v$.

Gressman [20] proved the restricted weak type $(p_a, q_a)$ for a class of monomial curves $\gamma(t) = (t^{a_1}, \ldots, t^{a_d})$ for positive integers $a_1 < \cdots < a_d$, and for $w(t) = |t|^b$, $-1 < b \leq 0$. Here, $(p_a, q_a)$ depends on the exponents $a_1, \ldots, a_d$, where $a = (a_1, \ldots, a_d)$. See also [29, 24] for the $d = 3$ case.
For more details on the history related to general classes of curves, we refer to [31, 11] and the references therein. Here we will focus on some results related to our work. Drury [15] introduced the affine arclength measure to obtain optimal $L^p \to L^q$ estimates for $(t, p(t))$, where $p(t)$ satisfies some technical assumptions. (See also [7].) Let $\gamma(t) = (P_1(t), \ldots, P_d(t))$ for arbitrary polynomials $P_i(t), 1 \leq i \leq d$. When $d = 2$, Oberlin [26] showed optimal $L^p \to L^q$ boundedness of $A$, where the constant depends only on the maximum degree of the polynomials. Dendrinos, Laghi and Wright [11] obtained the uniform boundedness of $B$ in some Lorentz spaces when $d = 3$, which was an extension of the case $\gamma(t) = (t, P_1(t), P_2(t))$ established by Oberlin [26]. For the general dimension, Stovall [31] proved the $L^{p_u} \to L^{q_v}$ boundedness whenever $u < q_d$, $p_d < v$, and $u < v$.

For curves with less regularity, more conditions on the torsion were needed. Oberlin [28] proved the sharp strong type boundedness for $B$ in $d = 2, 3, 4$ for certain flat curves of simple type, where the weight function is monotone and log-concave. For the higher dimensional cases it seems to be difficult to construct the band structure. Recently, Dendrinos and Stovall [13] proved the restricted weak type estimates for certain curves (not necessarily simple) with low regularity under monotonicity and concavity assumptions on the affine arclength measure. They also obtained strong type $(p_d, q_d)$ for the monomial-like curves. By suitably ordering certain parameters, they efficiently avoid the band structure argument.

We basically follow the argument in [31], which exploited geometric inequalities arisen from Fourier restriction estimates for polynomial curves (see [14]) and the band structure argument in [9, 30].

As in the study of the Fourier restriction estimates for space curves, properties of the mapping $(t_1, \ldots, t_d) \mapsto \sum_{i=1}^d \gamma(t_i)$, such as finite generic multiplicity and a lower bound for the Jacobian, are required in this paper to perform a change of variables. As in [31], which relied on the uniform estimates for the Jacobian in [14], we rely on an analogous (but weaker) result in [1], related to the Fourier restriction estimates for complex curves. In fact, we can decompose $C$ into finitely many disjoint regions such that the holomorphic Jacobian of the mapping $(z_1, \ldots, z_d) \mapsto \sum_{i=1}^d h(z_i)$ is bounded below on each region by the complex Vandermonde determinant and the arithmetic mean of $|\phi^{(d)}(z_1)|, \ldots, |\phi^{(d)}(z_d)|$. See Section 2 for more details.

We wish to point out here that the affine arclength measure was first used in the study of the Fourier restriction estimates for various classes of degenerate curves to allow the possibility of uniform estimates by mitigating the degeneracy of the torsion of the curve. (See [3, 4, 5, 12, 14, 15, 17, 18, 32] for the case of degenerate real curves, and see [27, 1] for complex curves.)

The usual treatment of the affine arclength measure involved a lower bound of a certain Jacobian in terms of the geometric mean of the translates of the torsion. Here, following [5, 11], we will exploit a stronger estimate involving the arithmetic mean (or, equivalently, the maximum) of the translates of the torsion (in Lemma 2.1 and 2.2) rather than their geometric mean. Using $\max_{1 \leq i \leq d} |\phi^{(d)}(z_i)|$ instead of $\prod_{1 \leq i \leq d} |\phi^{(d)}(z_i)|^\frac{1}{d}$, our computation can be simplified quite a bit, since it allows us to
manipulate the exponents easily by choosing $c_1, \ldots, c_d$ appropriately such that
\[ \max_{1 \leq i \leq d} |\phi^{(d)}(z_i)| \geq \prod_{1 \leq i \leq d} |\phi^{(d)}(z_i)|^{c_i}, \] where $\sum_{i=1}^d c_i = 1$.

If one can obtain Lemma 2.1 for complex polynomial curves in general dimensions, Theorem 1.1 may be extended to cover those curves by using the modified band structure argument for complex variables. The idea is to consider balls in $\mathbb{C}$ in place of the distance between real variables, which we will explain in more detail in Section 3 (which was motivated by the proof of Lemma 5.3). By this we can obtain an estimate on Lorentz spaces.

For the optimal estimate we need further observations, which will be given in Section 4. For the case $d = 3$, the argument is simpler, since it does not need the band structure argument. This case will be discussed in Section 5 for the sake of completeness. The standard method to obtain Theorem 1.1 from the main lemmas in Sections 3 and 4 was established by Stovall [30]. For the sake of completeness a detailed proof will be given in Appendix A. The necessary conditions on the indexes $p_d, q_d, u, v$ will be discussed in Appendix B.

2. Lower bounds for the Jacobian

In this section, we recall the lower bounds for the Jacobian for $h(z) = (z, z^2, \phi(z))$, where $\phi(z)$ is an arbitrary polynomial. Let $J_C(z_1, z_2, z_3)$ be the determinant of the holomorphic Jacobian of the mapping $(z_1, z_2, z_3) \mapsto \Phi_h(z_1, z_2, z_3) = -h(z_1) + h(z_2) - h(z_3)$. Then $C$ is decomposed into a bounded number of regions, on which a lower bound of $J_C(z_1, z_2, z_3)$ may be given as follows.

**Lemma 2.1** (Lemma 4.2 in [1]). There exists a positive integer $M = M(N)$ and a collection of convex open sets $B_1, \ldots, B_M$, which are pairwise disjoint, such that $C = \bigcup_{\ell=1}^M B_\ell$ ignoring null sets. Moreover, there exists a constant $c(N) > 0$ such that for $1 \leq \ell \leq M$,

\[ |J_C(z_1, z_2, z_3)| \geq c(N)V(z_1, z_2, z_3) \max\{|\phi'''(z_1)|, |\phi'''(z_2)|, |\phi'''(z_3)|\}, \]

whenever $z_1, z_2, z_3 \in B_\ell$. Here $V(z_1, z_2, z_3) = |z_2 - z_1||z_3 - z_1||z_3 - z_2|$ is the Vandermonde determinant.

Let us describe the set $B_\ell$ in brief. Fix a zero $u_1$ of $\phi'''(z)$ and denote the other $N - 4$ zeros by $u_2, \ldots, u_{N-3}$ such that $|u_2 - u_1| \leq \cdots \leq |u_{N-3} - u_1|$. Set $S(u_1) = \{z \in \mathbb{C} : |z - u_1| < |z - u_k| \text{ for } k = 2, \ldots, N - 3\}$. By translation, we may assume that $u_1 = 0$. Then we define the gap annuli $G_k$ and the dyadic annuli $D_k$ by

\[ G_k = \{z_1 \in \mathbb{C} : A|u_k| \leq |z_1| \leq A^{-1}|u_{k+1}|\} \text{ for } 1 \leq k \leq N - 4, \]
\[ D_k = \{z_1 \in \mathbb{C} : A^{-1}|u_k| \leq |z_1| \leq |A_1|u_k|\} \text{ for } 2 \leq k \leq N - 3, \]

and $G_{N-3} = \{z_1 \in \mathbb{C} : A|u_{N-3}| \leq |z_1|\}$. Here $A$ and $A_1$ are appropriate constants. (See the proof of Lemma 4.2 in [1].) In addition, let us consider the collection of narrow sectors $\{\Delta\}$ centered at the origin with angle $\varepsilon$, which cover $C$. Since the operator $\mathcal{A}$ is invariant under an affine transformation, it suffices to consider one...
sector $\Delta = \{ z = re^{i\theta} : 0 < r, \theta \in (0, \varepsilon) \}$. Then $B_\ell$ is a convex subset of $S(u_1) \cap \Delta \cap E_k$ for some $E_k = G_k$ or $D_k$. By the proof of Lemma 4.2 in [1], we have that

$$\phi''(z) = \prod_{n=k+1}^{N-3} |u_n| |z|^k =: H_k |z|^k$$

whenever $z \in B_\ell \subset S \cap \Delta \cap E_k$ with $E_k = G_k$ or $D_k$.

In fact, $|J_\ell(z_1, z_2, z_3)|$ can be reduced to the determinant of the holomorphic Jacobian of the mapping $(z_1, z_2, z_3) \mapsto \Phi_k(z_1, z_2, z_3)$, where $\Phi(z) = (z, z^2, z^{k+3})$ on $B_\ell$. More precisely, it is known that

$$|J_\ell(z_1, z_2, z_3)| \geq H_k \cdot V(z_1, z_2, z_3) \cdot \max\{|z_1|^k, |z_2|^k, |z_3|^k\}$$

if $z_1, z_2, z_3 \in B_\ell$ for some $B_\ell \subset S \cap \Delta \cap E_k$ with $E_k = G_k$ or $D_k$.

When $h(z)$ is a monomial curve of simple type, we have the following.

**Lemma 2.2** (Lemma 3.3 in [1]). Let $h(z) = (z, z^2, \ldots, z^{d-1}, z^N)$ for an integer $N \geq d$ with $d \geq 2$. Set

$$J_d(z_1, \ldots, z_d) = J_\ell(z_1, \ldots, z_d) = \det(h(z_1), \ldots, h(z_d))$$

where $z_j \in \mathbb{C}$, $1 \leq j \leq d$. Then $\mathbb{C}$ may be written as the union (ignoring a null-set) of $C(d, N)$ sectors $\Delta_\ell$ with vertex at the origin such that for each $1 \leq \ell \leq C(d, N)$, we have

$$|J(z_1, \ldots, z_d)| \geq c(d, N) \max_{1 \leq j \leq d} |z_j|^N \prod_{1 \leq i < j \leq d} |z_j - z_i|$$

where $z_j \in \Delta_\ell$. Here, $C(d, N)$ and $c(d, N)$ are positive constants depending only on $d$ and $N$.

In this case, it suffices to consider the case when $\Delta_\ell = \Delta = \{ re^{i\theta} : r > 0 \text{ and } \theta \in (0, \varepsilon) \}$ for some small $\varepsilon$.

### 3. The band structure for complex variables

In this section, we consider a monomial curve $h(z) = (z, z^2, \ldots, z^{d-1}, z^N)$ for a nonnegative integer $N$. We may assume $N \geq d$, because $Af(x) = 0$ for $N < d$. To handle the general dimensional case, we basically follow the ‘band structure’ argument in [3].

By Lemma 2.2 it suffices to consider

$$Tf(x) = \int_{\Delta} f(x - h(z)) d\sigma(z),$$

where $d\sigma(z) \sim |z|^{\frac{d(N-d)}{d(d-n)}} d\mu(z) =: |z|^{\frac{dK}{d(d+n)}} d\mu(z)$ and $\Delta = \{ re^{i\theta} : 0 < r \leq 1 \text{ and } \theta \in (0, \varepsilon) \}$ for a small constant $\varepsilon$ as in Lemma 2.2.

The estimate [1] for $T$ follows from the propositions in the Appendix. Those propositions assume the restricted weak type $(p_d, q_d)$, which we will now prove.
Let us define quantities \( \alpha \) and \( \beta \) by
\[
\alpha = \frac{\langle T \chi_E, \chi_F \rangle}{|F|} \quad \text{and} \quad \beta = \frac{\langle T^* \chi_F, \chi_E \rangle}{|E|},
\]
for measurable sets \( E, F \). Here \( T^* \) is the dual operator of \( T \) given by \( T^*f(x) = \int_{\Delta} f(x + h(z))d\sigma(z) \). Note that \( \alpha|F| = \beta|E| \). Then
\[
\langle T \chi_E, \chi_F \rangle \leq |E|^\frac{1}{p_d}|F|^\frac{1}{q_d}
\]
is equivalent to
\[
|E| \gtrsim \alpha^\frac{d(d+1)}{2} \left( \frac{\beta}{\alpha} \right)^{d-1}
\]
since \( \alpha|F| = \beta|E| = \langle T \chi_E, \chi_F \rangle \).

The following lemma is a refinement of \((5)\).

**Lemma 3.1.** Let \( E_1, E_2, G \subset \mathbb{R}^{2d} \) be measurable sets with finite measure. Suppose that
\[
T \chi_{E_1}(x) \geq \alpha_1 \quad \text{and} \quad T \chi_{E_2}(x) \geq \alpha_2
\]
for all \( x \in G \) and \( \alpha_1 \leq \alpha_2 \). Then
\[
|E_2| \gtrsim \alpha_1^{\frac{d(d+1)}{2}} \left( \frac{\beta}{\alpha_1} \right)^{d-1} \left( \frac{\alpha_2}{\alpha_1} \right)^d,
\]
where \( \beta = \alpha_1 \frac{|G|}{|E_1|} \).

**Remark 3.2.** If we set \( E_1 = E_2 \) and \( \alpha_1 = \alpha_2 \), we obtain \((5)\) i.e. the restricted weak type \((p_d, q_d)\) for \( T \).

First we will find a sequence of subsets of \( \Delta \) and their properties, which is essential to construct a band structure of Lemma 3.5.

**Lemma 3.3.** Let \( \gamma = \max\{\alpha_1, \beta\} \), \( \nu = \frac{d(d+1)}{4K + 2d(d+1)} \) under the assumptions in Lemma 3.1. There exist a point \( y_0 \) in \( E_1 \), a constant \( C > 0 \), and a sequence of subsets \( P_1, \ldots, P_{2d} \) of \( \Delta \) such that

1. \( \sigma(P_j) \geq C \beta \) for odd \( j \),
2. \( \sigma(P_j) \geq C \alpha_1 \) for even \( j < 2d \),
3. \( \sigma(P_{2d}) \geq C \alpha_2 \),
4. \( |z_j| \geq (4\pi \nu)^{-\nu} \gamma^\nu \) for \( z_j \in P_j \), \( 1 \leq j \leq 2d - 1 \), and \( |z_{2d}| \geq (4\pi \nu)^{-\nu} \gamma^\nu \),
5. Also there exists a small constant \( c \geq 0 \) such that
6. \( \sigma(P_j) \geq c \beta^{\frac{1}{2}} |z_i|^{\frac{2K}{d(d+1)^2}} \), where \( i < j \leq 2d \),
7. \( \sigma(P_j) \geq c \alpha_1^{\frac{1}{2}} |z_i|^{\frac{2K}{d(d+1)^2}} \), where \( i < j \),
8. \( |z_{2d} - z_j| \geq c \alpha_2^{\frac{1}{2}} |z_{2d}|^{\frac{2K}{d(d+1)^2}} \) if \( |z_j| < \frac{1}{2}(4\pi \nu)^{-\nu} \gamma^\nu \), and
9. \( |z_{2d} - z_j| \geq c \alpha_2^{\frac{1}{2}} |z_j|^{\frac{2K}{d(d+1)^2}} \) if \( |z_j| \geq \frac{1}{2}(4\pi \nu)^{-\nu} \gamma^\nu \).
Proof. We begin by showing that we may consider a truncated operator instead of \( T \) by following the proof of Lemma 1 in \([20]\). Let us define \( B_\gamma = \{ z \in \Delta : |z| < (4\pi \nu)^{-\nu \gamma'} \} \) and a truncated operator \( \tilde{T}f(x) = \int_{\Delta \setminus B_\gamma} f(x - h(z)) d\sigma(z) \).

Since
\[
\sigma(B_\gamma) = \int_{B_\gamma} |z|^{4K} d\mu(z) \leq 2\pi \int_0^{(4\pi \nu)^{-\nu \gamma'}} s^{4K} \frac{4K+1}{4\pi} ds = 2\pi \nu (4\pi \nu)^{-1} \gamma = \frac{\gamma}{2},
\]
we have that
\[
\langle \tilde{T} \chi_{E_1}, \chi_G \rangle \geq \langle T \chi_{E_1}, \chi_G \rangle - \sigma(B_\gamma) |G| \geq (\alpha_1 - \gamma/2) |G|.
\]
If \( \gamma = \alpha_1 \), then \( \langle \tilde{T} \chi_{E_1}, \chi_G \rangle \geq \frac{1}{2} \alpha_1 |G| = \frac{1}{2} \beta |E_1| \).

If \( \gamma = \beta \), we see that
\[
\langle \tilde{T} \chi_{E_1}, \chi_G \rangle = \langle \chi_{E_1}, \tilde{T}^* \chi_G \rangle \geq \langle \chi_{E_1}, T^* \chi_G \rangle - \sigma(B_\gamma) |E_1| \geq \frac{1}{2} \beta |E_1|
\]
since \( \langle \chi_{E_1}, T^* \chi_G \rangle \geq \alpha_1 |G| = \beta |E_1| \).

Therefore we get that
\[
\langle \tilde{T} \chi_{E_1}, \chi_G \rangle \geq \frac{1}{2} \alpha_1 |G| = \frac{1}{2} \beta |E_1|.
\]

Now we show \((i) - (vi)\). By abuse of notation, we will write \( T \), instead of \( \tilde{T} \). Since
\[
\langle T \chi_{E_1}, \chi_G \rangle = \langle \chi_{E_1}, T^* \chi_G \rangle \geq \frac{1}{2} \beta |E_1|,
\]
we can define a set
\[
E_1^1 = \{ y \in E_1 : T^* \chi_G(y) \geq \beta/4 \}.
\]
Let us set
\[
G^1 = \{ x \in G : T \chi_{E_1}(x) \geq \alpha_1/8 \},
\]
considering that
\[
\langle T \chi_{E_1^1}, \chi_G \rangle = \langle T \chi_{E_1}, \chi_G \rangle - \langle \chi_{E_1 \setminus E_1^1}, T^* \chi_G \rangle \geq \frac{1}{2} \beta |E_1| - \frac{1}{4} \beta |E_1| = \frac{1}{4} \beta |E_1| = \frac{1}{4} \alpha_1 |G|.
\]

Continuing this procedure, we can find sequences of sets \( E_i^j \) and \( G^j \) defined by
\[
E_1^0 := E_1, \quad G^0 := G
\]
\[
E_1^j = \{ y \in E_1^{j-1} : T^* \chi_{G^{j-1}}(y) \geq \beta/2^{2j} \}, \quad j = 1, \ldots, d
\]
\[
G^j = \{ x \in G^{j-1} : T \chi_{E_1^j}(x) \geq \alpha_1/2^{2j+1} \}, \quad j = 1, \ldots, d - 1.
\]

We should show that \( E_i^j \) and \( G^j \) are nonempty. It suffices to show that
\[
\langle T \chi_{E_1^j}, \chi_G^j \rangle \geq \beta |E_1|/2^{2j+1} \text{ for } j \geq 0.
\]
The case \( j = 0 \) is clear by (8). Let us assume the claim (8) for some \( j \). Then it follows that
\[
(T \chi_{E_1^{j+1}}, \chi_{G^{j+1}}) = (T \chi_{E_1^{j+1}}, \chi_{G^{j}}) - (T \chi_{E_1^{j+1}}, \chi_{G^{j} \setminus G^{j+1}})
\geq \langle \chi_{E_1^{j+1}} \rangle, T^* \chi_{G^{j+1}} \rangle - 2^{-2j-3} \alpha_1 |G|
\geq \langle \chi_{E_1^{j+1}} \rangle, T^* \chi_{G^{j+1}} \rangle - 2^{-2j-2} \alpha_1 |G| - 2^{-2j-3} \alpha_1 |G|
\geq 2^{-2j-1} \alpha_1 |G| - 2^{-2j-2} \alpha_1 |G| - 2^{-2j-3} \alpha_1 |G| = 2^{-2(j+1)-1} \beta |E_1|.
\]
This gives the claim (8) by induction.

Now we define
\[
H_k(z_1, \ldots, z_k) := \sum_{j=1}^k (-1)^{j+1} h(z_j), \quad k \geq 1.
\]

Fix \( y_0 \in E_1^d \subset E_1 \) and set
\[
P_1 = \{ z_1 \in \Delta \setminus B_{\gamma} : y_0 + H_1(z_1) \in G^{d-1} \}, \quad \sigma(P_1) = T^* \chi_{G^{d-1}}(y_0) \geq \beta / 2^d.
\]
For all \( z_1 \in P_1 \), we also set
\[
P_2 = \{ z_2 \in \Delta \setminus B_{\gamma} : y_0 + H_2(z_1, z_2) \in E_1^{d-1} \}, \quad \sigma(P_2) = T \chi_{E_1^{d-1}}(y_0 + H_1(z_1)) \geq \alpha_1 / 2^{d-1}.
\]

Through the iterative process, we obtain that for \( k = 2, 3, \ldots, d - 1 \)
\[
P_{2k-1} = \{ z_{2k-1} \in \Delta \setminus B_{\gamma} : y_0 + H_{2k-1}(z_1, \ldots, z_{2k-1}) \in G^{d-k} \},
\]
\[
P_{2k} = \{ z_{2k} \in \Delta \setminus B_{\gamma} : y_0 + H_{2k}(z_1, \ldots, z_{2k}) \in E_1^{d-k} \}.
\]

Finally we set
\[
P_{2d-1} = \{ z_{2d-1} \in \Delta \setminus B_{\gamma} : y_0 + H_{2d-1}(z_1, \ldots, z_{2d-1}) \in G \},
\]
\[
P_{2d} = \{ z_{2d} \in \Delta \setminus B_{\alpha_2} : y_0 + H_{2d}(z_1, \ldots, z_{2d}) \in E_2 \}
\]
where \( B_{\alpha_2} = \{ z \in \Delta : |z| \leq (4\pi \nu)^{-\frac{1}{\nu}} \alpha_2^\nu \} \). Then it follows that for \( k = 2, 3, \ldots, d - 1 \)
\[
\sigma(P_{2k-1}) = T^* \chi_{G^{d-k}}(y_0 + H_{2k-2}(z_1, \ldots, z_{2k-2})) \geq \beta / 2^{2(d-k+1)}
\]
\[
\sigma(P_{2k}) = T \chi_{E_1^{d-k}}(y_0 + H_{2k-1}(z_1, \ldots, z_{2k-1})) \geq \alpha_1 / 2^{2(d-k)+1}
\]
provided \( z_j \in P_j \). We also have
\[
\sigma(P_{2d-1}) = T^* \chi_{G}(y_0 + H_{2d-2}(z_1, \ldots, z_{2d-2})) \geq \beta / 2^2,
\]
and
\[
\sigma(P_{2d}) = \int_{\Delta \setminus B_{\alpha_2}} \chi_{E_2}(y_0 + H_{2d}(z_1, \ldots, z_{2d})) d\sigma(z_{2d})
\geq \int_{\Delta} \chi_{E_2}(y_0 + H_{2d}(z_1, \ldots, z_{2d})) d\sigma(z_{2d}) - \sigma(B_{\alpha_2}) \geq \frac{1}{2} \alpha_2
\]
since \( y_0 + H_{2d-2} \in E_1^d \) and \( y_0 + H_{2d-1} \in G \). Hence we get a sequence of sets \( P_j, 1 \leq j \leq 2d \), in which \( |z_i| \geq \gamma^\nu, 1 \leq i < 2d \) and \( |z_{2d}| \geq \alpha_2^\nu \). Thus we obtain (i)-(iii).
To prove (iv)–(vi), we will consider subsets of \( P_i \)'s which maintain the properties (i)–(iii). Let \( B_\beta(z_j) = \{ z \in \Delta : |z - z_j| \leq c_0 \beta^{1/2} |z_j|^{\frac{2K}{d(d+1)}} \} \) for a sufficiently small constant \( c_0 > 0 \). Suppose \( i \) is odd and \( j < i < 2d \). If \( |z| \leq c |z_j| \) for all \( z \in P_i \) and a constant \( c > 0 \), then

\[
\sigma(P_i \cap B_\beta(z_j)) = \int_{P_i \cap B_\beta(z_j)} |z|^{\frac{4K}{d(d+1)}} d\mu(z) \leq (c |z_j|)^{\frac{4K}{d(d+1)}} \mu(B_\beta(z_j)) \leq c_1 \beta
\]

where \( c_1 = 2\pi c^{\frac{4K}{d(d+1)}} c_0^{\frac{2K}{d(d+1)}} \). Since \( \sigma(P_i) \geq \beta/2^{2d-i+1} \) when \( i \) is odd, we have \( \sigma(P_i \setminus B_\beta(z_j)) \geq \beta/2^{2d-i+1} - c_1 \beta \geq \beta/2^{2d-i+2} \) if we choose sufficiently small \( c_0 > 0 \). (Thus (i) still holds for \( P_i \setminus B_\beta(z_j) \).) If \( |z| \leq c |z_j| \) is not valid for all \( z \in P_i \), we use the fact that \( |z_j| \geq (4\pi \nu)^{-\nu} |z_j|^{\frac{2K}{d(d+1)}} \) for \( z_j \in P_j \). It follows that \( (4\pi \nu)^{\frac{1}{2}} |z_j| \geq \gamma \frac{1}{2} |z_j|^{\frac{2K}{d(d+1)}} \geq \beta^{\frac{1}{2}} |z_j|^{\frac{2K}{d(d+1)}} \), and then \( |z - z_j| < c_0 \beta^{\frac{1}{2}} |z_j|^{\frac{2K}{d(d+1)}} < c_0 (4\pi \nu)^{\frac{1}{2}} |z_j| \) on \( B_\beta(z_j) \). In this case, we also obtain \( \sigma(P_i \setminus B_\beta(z_j)) \geq \beta/2^{2d-i+2} \), since \( |z| \leq (1 + c_0 (4\pi \nu)^{1/2}) |z_j| \). Thus we conclude that (iv) holds in any case.

When \( 1 < i < 2d \) is even, we can also obtain (v) by replacing \( \beta \) with \( \alpha_1 \).

To show (vi), we consider two cases for each \( j < 2d \). If \( |z_j| < \frac{1}{2} (4\pi \nu)^{-\nu} \alpha_2^\nu \), then \( |z_{2d} - z_j| > \frac{1}{2} |z_{2d}| \geq \frac{1}{2} (4\pi \nu)^{-1/2} \alpha_2^{1/2} |z_{2d}|^{-2K/[d(d+1)]} \), since \( |z_{2d}| \geq (4\pi \nu)^{-\nu} \alpha_2^\nu \) and \( \nu = d(d + 1)/(4K + 2d(d + 1)) \). If \( |z_j| \geq \frac{1}{2} (4\pi \nu)^{-\nu} \alpha_2^\nu \), i.e. \( \frac{1}{2} (4\pi \nu)^{-1/2} \alpha_2^{1/2} |z_j|^{-2K/[d(d+1)]} \leq |z_j| \), it follows that for \( z \in B_{\alpha_2}(z_j) \), the inequalities \( |z - z_j| < c_0 \alpha_2^{1/2} |z_j|^{-2K/[d(d+1)]} < 2c_0 (4\pi \nu)^{1/2} |z_j| \) imply that \( |z| < (1 + 2c_0 (4\pi \nu)^{1/2}) |z_j| \) for \( z \in B_{\alpha_2}(z_j) \). By this we have that

\[
\sigma(B_{\alpha_2}(z_j)) = \int_{B_{\alpha_2}(z_j)} |z|^{\frac{4K}{d(d+1)}} d\mu(z)
\]

\[
\leq (1 + 2c_0 (4\pi \nu)^{\frac{1}{2}})|z_j|^{\frac{4K}{d(d+1)}} \times \mu(B_{\alpha_2}(z_j))
\]

\[
\leq (1 + 2c_0 (4\pi \nu)^{\frac{1}{2}})|z_j|^{\frac{4K}{d(d+1)}} \times \pi c_0^2 \alpha_2 |z_j|^{-\frac{4K}{d(d+1)}}
\]

\[
= c_1 \alpha_2^2
\]

where \( c_1 = \pi c_0^2 (1 + 2c_0 (4\pi \nu)^{\frac{1}{2}})|z_j|^{-\frac{4K}{d(d+1)}} \). Choosing sufficiently small \( c_0 \), we can get \( \sigma(P_{2d} \setminus B_{\alpha_2}(z_j)) > \alpha_2^2/4 \). Once again, (iii) holds for \( P_{2d} \setminus B_{\alpha_2}(z_j) \) in place of \( P_{2d} \).

Hence \( |z_{2d} - z_j| \geq c_0 \alpha_2^{1/2} |z_j|^{-\frac{2K}{d(d+1)}} \) when \( |z_j| \geq \alpha_2^\nu \). This completes the proof. \( \square \)

To prove Lemma 3.1, we consider two cases \( \beta \geq \alpha_1 \) and \( \beta \ll \alpha_1 \). First let us assume \( \beta \geq \alpha_1 \). Let \( \Phi_{2d} = \{ (z_1, z_2, ..., z_{2d}) : z_i \in P_i \text{ where } 1 \leq i \leq 2d \} \). We define \( z_0 = (z_1, z_2, ..., z_{2d}) \in \Phi_d \) and \( \Phi := \{ z \in \mathbb{C}^d : (z_0, z) \in \Phi_{2d} \} \). Note that \( \sigma(\Phi) \sim \alpha_1^{d/2-1} \beta^{d/2} \alpha_2 \) when \( d \) is even, and \( \sigma(\Phi) \sim \alpha_1^{(d-1)/2} \beta^{(d-1)/2} \alpha_2 \) when \( d \) is odd. For \( d \geq 2 \), let

\[
a_d = \begin{cases} 1 + 3 + \cdots + (d - 1) & \text{if } d \text{ is even,} \\ 2 + 4 + \cdots + (d - 1) & \text{if } d \text{ is odd.} \end{cases}
\]

The following lemma gives Lemma 3.1 whenever \( \beta \gg \alpha_1 \) and \( a_d \geq (d - 1) \).
Lemma 3.4. Let $d \geq 2$. Assume the hypotheses in Lemma 3.1. Then

$$|E_2| \gtrsim \alpha_1^{d(d+1)/2} \left( \frac{\beta}{\alpha_1} \right)^{ad} \left( \frac{\alpha_2}{\alpha_1} \right)^d.$$  

Proof. We assume that $d$ is even. The proof is similar when $d$ is odd. Let $H(z) := y_0 + H_{2d}(z_0, z)$. By Lemma 2.2, we have

$$|E_2| \gtrsim \int |J RH(z)| d\mu(z) = \int |J RH(z)|^2 d\mu(z)$$

$$\gtrsim \int \max_{d+1 \leq i \leq 2d} |z_i|^{2K} \prod_{d+1 \leq i < j \leq 2d} |z_j - z_i|^2 d\mu(z),$$

where $J RH$ is the real Jacobian of $H$ in $\mathbb{R}^{2d}$.

Using $(iv)$ – $(vi)$ in Lemma 3.3, we obtain that

$$|E_2| \gtrsim \alpha_1^{1+3+\ldots+(d-3)/2+2+\ldots+(d-2)/2} \alpha_2^{d-1} \times$$

$$\times \int \max_{d+1 \leq i \leq 2d} |z_i|^{2K} \prod_{i=d+1}^{2d-2} |z_i|^{\frac{4K}{d(d+1)}} (2d-i-1) \prod_{j=d+1}^{2d-1} |z_j|^{\frac{4K}{d(d+1)}} (1-\epsilon_j) d\mu(z),$$

$$= \alpha_1^{\frac{d(d-1)}{2}} \left( \frac{\beta}{\alpha_1} \right)^{2+\ldots+(d-2)/2} \left( \frac{\alpha_2}{\alpha_1} \right)^{d-1} \times$$

$$\times \int \max_{d+1 \leq i \leq 2d} |z_i|^{2K} \prod_{i=d+1}^{2d-1} |z_i|^{\frac{4K}{d(d+1)}} (2d-i-1+\epsilon_i) |z_{2d}|^{\frac{4K}{d(d+1)}} \sum_{i=d+1}^{2d-1} (1-\epsilon_j) d\mu(z),$$

where $\epsilon_j = 0$ if $|z_j| < \frac{1}{2} (4\pi \nu)^{-\nu} \alpha_2^2$, and $\epsilon_j = 1$ if $|z_j| \geq \frac{1}{2} (4\pi \nu)^{-\nu} \alpha_2^2$, for $d + 1 \leq j \leq 2d - 1$. It is clear that $\max_{d+1 \leq i \leq 2d} |z_i|^{2K} \geq \prod_{i=d+1}^{2d} |z_i|^{2K}$ when $\sum_{i=d+1}^{2d} K_i = K$. We can choose $K_i \geq 0$ such that

$$K_i = \frac{2K}{d(d+1)} (1 + (2d - i - 1) + \epsilon_i)$$

where $d + 1 \leq i \leq 2d - 1$, and

$$K_{2d} = \frac{2K}{d(d+1)} (1 + \sum_{i=d+1}^{2d-1} (1 - \epsilon_i)),$$

where $\epsilon_i = 0$ or 1 for $d + 1 \leq i \leq 2d - 1$. 

Then we get that
\[ |E_2| \gtrsim \alpha_1^{d(d-1)/2} \left( \frac{\beta}{\alpha_1} \right)^{2+4+\cdots+(d-2)} \left( \frac{\alpha_2}{\alpha_1} \right)^{d-1} \int_{\Phi} \prod_{i=d+1}^{2d} |z_i|^{-\frac{4K}{\alpha(d+1)}} d\mu(z) \]
\[ = \alpha_1^{d(d-1)/2} \left( \frac{\beta}{\alpha_1} \right)^{2+4+\cdots+(d-2)} \left( \frac{\alpha_2}{\alpha_1} \right)^{d-1} \sigma(\Phi) \]
\[ \sim \alpha_1^{d(d-1)/2} \left( \frac{\beta}{\alpha_1} \right)^{2+4+\cdots+(d-2)} \left( \frac{\alpha_2}{\alpha_1} \right)^{d-1} \frac{\alpha_1^{d-1}}{\alpha_1^{d-1}} \beta^2 \alpha_2 \]
\[ \sim \alpha_1^{d(d+1)/2} \left( \frac{\beta}{\alpha_1} \right)^{a_d} \left( \frac{\alpha_2}{\alpha_1} \right)^d, \]
when \( d \) is even. \( \square \)

3.1. The band structure argument. To handle the case \( \beta \ll \alpha_1 \), we modify the original band structure argument due to Christ \[9\]. We also refer to the works by Stovall \[31\] and Gressman \[20\], which treat the degenerate cases.

We begin with recalling some definitions to describe the band structure argument. We shall decompose an index set into subsets called bands. In each band, the even index or 1 is a ‘free index’. If there is no even index or 1, the smallest index is the free index. If a band has only two indices, the other is a ‘quasi-free’ index, and we say that the quasi-free index is quasi-bound to the free index. If a band has more than three indices, the other indices other than the free index are bound to the free index.

The following lemma is a variant of the real case considered in \[31\].

**Lemma 3.5.** Let \( \varepsilon > 0 \). There exist parameters \( \delta, \delta' \) satisfying \( 0 < c_{d,\varepsilon} < \delta' < \varepsilon \delta \), a constant \( \tilde{c} \), an integer \( d \leq k < 2d \), a set \( \omega \subset C^k \) with \( \sigma(\omega) \gtrsim \alpha_1^{[k/2]} \beta^{[k/2]} (\alpha_2/\alpha_1) \), and a band structure on \( \{2d-k+1, \ldots, 2d\} \), such that the following properties hold:

(i) There are exactly \( d \) free or quasi-free indices. In particular, each even index is free.

(ii) \( |z_i - z_j| > \delta \alpha_1^d |z_i z_j|^{-\frac{K}{\alpha(d+1)}} \), unless \( i \) and \( j \) lie in the same band.

(iii) \( \tilde{c} \beta^d |z_i z_j|^{-\frac{K}{\alpha(d+1)}} < |z_i - z_j| \leq \delta \alpha_1^d |z_i z_j|^{-\frac{K}{\alpha(d+1)}} \) whenever \( i \) is quasi-bound to \( j \).

(iv) \( \delta' \alpha_1^d |z_i z_j|^{-\frac{K}{\alpha(d+1)}} > |z_i - z_j| \) whenever \( i \) is bound to \( j \).

**Proof.** It is not enough to arrange the absolute values of \( z_1, \ldots, z_{2d} \) in order, since two variables with the same size can be separated. Thus our approach is slightly different at the beginning. We start by observing that \( z_j \)'s for even \( j \) are separated from each other. In other words, all the balls centered at \( z_j \) for even \( j \) can be made mutually disjoint by choosing the radii appropriately.

**Step 1.** By Lemma 3.3, there exist constants \( c \) such that
\[ |z_i - z_j| > c \alpha_1^d |z_i z_j|^{-\frac{K}{\alpha(d+1)}} \]
for all even indices \( j \leq 2d \) and for all \( i < j \). In fact, we have that \( |z_i - z_j| \gtrsim \alpha_1^d |z_i|^{-\frac{2K}{\alpha(d+1)}} \) for even \( j \leq 2d - 2 \) and \( i < j \) by (v) in Lemma 3.3. If \( |z_i| \lesssim |z_j| \), it
follows that $|z_i - z_j| \gtrsim \alpha^1, \frac{K}{d(d+1)} |z_i z_j|^{-\frac{1}{d(d+1)}}$. If $|z_i| \gg |z_j|$, then we get $|z_i - z_j| \gtrsim |z_i| \gg |z_j|$. Since $|z_j| \geq (4\pi \nu)^{-\nu} \gamma''$ by Lemma 3.3 and $\nu = \frac{d(d+1)}{4K+2d(d+1)}$, one can see that $|z_i - z_j| \gtrsim |z_j| \gtrsim \alpha^2, \frac{K}{d(d+1)} |z_i z_j|^{-\frac{1}{d(d+1)}}$.

Next, let us consider the case of $j = 2d$. If $|z_{2d}| \sim |z_i|$ for any $i < 2d$, then (vi) in Lemma 3.3 gives $|z_{2d} - z_i| \gtrsim \alpha^1, \frac{K}{d(d+1)} |z_i z_{2d}|^{-\frac{1}{d(d+1)}}$. (Note that we are assuming here that $\alpha_1 \leq \alpha_2$.) For the case $|z_{2d}| \ll |z_i|$ or $|z_{2d}| \gg |z_i|$, we see that $|z_{2d} - z_i| \gtrsim |z_{2d}| = |z_i| \frac{1}{d(d+1)} |z_i z_{2d}|^{-\frac{1}{d(d+1)}}$. Since $|z_{2d}| \gtrsim \alpha^2 \geq \alpha^1$ and $|z_i| \gtrsim \gamma'' \geq \alpha^1$ for $i < 2d$, it follows that

$$|z_{2d} - z_i| \gtrsim \alpha^1, \frac{K}{d(d+1)} |z_i z_{2d}|^{-\frac{1}{d(d+1)}} = \alpha^2, \frac{K}{d(d+1)} |z_i z_{2d}|^{-\frac{1}{d(d+1)}}.$$

Therefore, (10) is valid for any even $j$ and $i < j$.

Similarly, there exists a constant $\tilde{c} > 0$ such that

$$\left| z_i - z_j \right| > \tilde{c} \beta_1, \frac{1}{d(d+1)} |z_i z_j|^{-\frac{K}{d(d+1)}}$$

for any odd $j$ and $i < j$.

Let us define a ball centered at $z_j$ with small $\delta < c$, to be chosen later, by setting

$$B_{\delta, \alpha_1}(z_j) = \{ z : |z - z_j| \leq \delta \alpha_1, \frac{1}{d} |z z_j|^{-\frac{K}{d(d+1)}} \}.$$

For each even index $j$, let $b(j)$ be a subset of $\{1, 2, \ldots, 2d\}$ such that

- $j \in b(j)$,
- $j_1 \in b(j)$ if $z_{j_1} \in B_{\delta, \alpha_1}(z_j)$,
- $j_{k+1} \in b(j)$ if $z_{j_{k+1}} \in B_{\delta, \alpha_1}(z_j)$ for some $j_k \in b(j)$.

If there is no $j_1$ such that $z_{j_1} \in B_{\delta, \alpha_1}(z_j)$, then we set $b(j) = \{j\}$.

First we show how to construct $b(j)$ for each even $j$. By (10), $b(2d) = \{2d\}$ holds. In fact, we have that

$$\left| z_i - z_{2d} \right| \gtrsim c \alpha^1, \frac{1}{d} |z_i z_{2d}|^{-\frac{K}{d(d+1)}}$$

for all $i < 2d$.

For $z_{2d-2}$, (10) holds for all odd and even $i < j = 2d - 2$. On the other hand, $z_{2d-1}$ may be contained in $B_{\delta, \alpha_1}(z_{2d-2})$. In general, each odd $i$ may be contained in either exactly one $B_{\delta, \alpha_1}(z_j)$ satisfying $i > j$ or none of them. Then each $B_{\delta, \alpha_1}(z_j)$ may contain odd indices greater than $j$. Those odd indices belong to $b(j)$. If $B_{\delta, \alpha_1}(z_j)$ has no odd index, then let $b(j) = \{j\}$.

Let $j_1$ be one of the odd indices contained in $b(j)$, and consider $B_{\delta, \alpha_1}(z_{j_1})$. If there exists an odd index $\ell$ such that $z_{\ell} \in B_{\delta, \alpha_1}(z_{j_1})$, then we denote it by $j_2$ (of course, $j_2 \in b(j)$). For $z_{j_2}$ we consider $B_{\delta, \alpha_1}(z_{j_2})$ and repeat the process as above. Hence, each $b(j)$ consists of a unique even index $j$ and some odd indices greater than $j$.

Now consider indices which belong to none of the $b(j)$ for even $j$. In this case, we choose the smallest index among the remaining indices. By (10) this index must be 1. In fact, $B_{\ell/3, \alpha_1}(z_1)$ is disjoint from the other $B_{\ell/3, \alpha_1}(z_{j})$ for even $j$. By choosing sufficiently small $\delta$, it is valid that $B_{\delta, \alpha_1}(z_1) \subset B_{\ell/3, \alpha_1}(z_{j})$ for all $i \in b(j)$. Then we can construct $b(1)$ in the same manner as above.
If there are still remaining indices, we choose the smallest one and repeat the procedure. By this we can construct bands for odd indices.

Hence the index set \( \{1, \ldots, 2d\} \) can be decomposed into the bands \( b(j) \)'s. Here \( j \) represents the free index in \( b(j) \). Each \( b(j) \) has less than \( d \) elements because of (10). (The case where \( b(j) \) has \( d \) elements can only occur when \( b(1) \) contains all odd indices.) Also the free index is the smallest in the band.

Now we can check that the properties (ii) and (iii) hold. Let us assume that \( j' \in b(j) \) and \( k' \in b(k) \) for \( j \neq k \), i.e. \( j' \) and \( k' \) are in different bands. From the construction of \( b(j) \) and \( b(k) \), it follows that \( z_{j'} \notin B_{\bar{\delta}, \alpha_1}(z_{k'}) \) and \( z_{k'} \notin B_{\bar{\delta}, \alpha_1}(z_{j'}) \). This immediately implies (ii) for \( j' \) and \( k' \).

If \( j' \) is quasi-bound to \( j \), i.e. \( b(j) = \{j, j'\} \), then \( j' \) must be an odd number. Hence (iii) follows from the construction of \( b(j) \) and (11).

Step 2. First, we need to verify that \( |z_{jk}| \sim |z_{jk+1}| \) whenever \( j_k, j_{k+1} \in b(j) \). Since \( z_{jk+1} \in B_{\bar{\delta}, \alpha_1}(z_{jk}) \) and \( |z_j| \geq (4\pi\nu)^{\nu} \alpha_1^\nu \) for all \( j \), it follows that

\[
|z_{jk}| \geq |z_{jk+1} - z_{j_k} - z_{j_{k+1}}| \geq |z_{jk+1}| - \delta \alpha_1^\frac{1}{4}|z_{j_k}z_{j_{k+1}}|^{\frac{K}{d(d+1)}} \\
\geq |z_{jk+1}| - \alpha_1^\frac{1}{4}(\alpha_1^{2\nu})^{-\frac{K}{d(d+1)}} = |z_{jk+1}| - \alpha_1^\nu,
\]

where \( c = \delta(4\pi\nu)^{2\nu}/(d+1) \). Thus we have that \( |z_{jk+1}| \leq \alpha_1^\nu + |z_{jk}| \leq (\delta(4\pi\nu)^{1/2} + 1)|z_{jk}| \). If we exchange \( z_{jk} \) and \( z_{jk+1} \), then it also holds that \( |z_{jk}| \leq (\delta(4\pi\nu)^{1/2} + 1)|z_{jk+1}| \). Hence we have that for any \( j_k, j_{k+1} \in b(j) \)

\[
|z_{jk}| \leq (1 + \bar{c})|z_{jk+1}| \leq (1 + \bar{c})^2|z_{jk+2}| \leq \cdots \leq (1 + \bar{c})^d|z_{ji}|,
\]

where \( \bar{c} = (4\pi\nu)^{1/2} \). (Note that each \( b(j) \) has at most \( d \) elements.) By exchanging \( z_{jk} \) for \( z_{ji} \), we finally get \( (1 + \bar{c})^{-d}|z_{ji}| \leq |z_{jk}| \leq (1 + \bar{c})^{d}|z_{ji}| \). Therefore we obtain that

\[
|z_{jk}| \sim |z_{ji}| \quad \text{whenever } j_k \text{ and } j_i \text{ are in the same band.}
\]

(The implicit constant can be adjusted by choosing sufficiently small \( \delta \) when we use [13] in the proof of Lemma 3.0.)

If \( j_k \) is bound to \( j \), we see that

\[
|z_{jk} - z_j| \leq |z_{jk} - z_{jk-1}| + \cdots + |z_{j_1} - z_j| \\
\leq \delta \alpha_1^\frac{1}{4}|z_{jk}z_{jk-1}|^{-\frac{K}{d(d+1)}} + \cdots + \delta \alpha_1^\frac{1}{4}|z_{j_1}z_j|^{-\frac{K}{d(d+1)}} \\
\leq \delta \alpha_1^\frac{1}{2}d(1 + \bar{c})^{-\frac{K}{d(d+1)}},
\]

by (12). Then (iv) is not guaranteed. If (iv) holds on a subset \( \mathcal{Z}' \subset \mathcal{Z} := P_1 \times P_2 \times \cdots \times P_{2d} \) with \( \sigma(\mathcal{Z}') \geq \sigma(\mathcal{Z})/2 \), then we proceed to the next step.

Otherwise, there exist a subset \( \mathcal{Z}'' \subset \mathcal{Z} \) satisfying \( \sigma(\mathcal{Z}'') \geq \sigma(\mathcal{Z})/2 \), a band \( b(j) \), and an index \( j_0 \) such that \( |z_{j_0} - z_j| \geq \delta \alpha_1^\frac{1}{4}|z_{j_0}z_j|^{-\frac{K}{d(d+1)}} \). Then we replace \( \delta \) by \( \delta'/d \), with which we repeat Step 1 until we get

\[
|z_{jk} - z_j| < \delta' \alpha_1^\frac{1}{4}|z_{jk}z_j|^{-\frac{K}{d(d+1)}}
\]

for each bound index \( j_k \in b(j) \) on a subset \( \tilde{\mathcal{Z}} \subset \mathcal{Z} \) satisfying \( \sigma(\tilde{\mathcal{Z}}) \geq \sigma(\mathcal{Z})/2 \).
Step 3. Adopting the notations in [9], we denote by $\mathcal{M}, \mathcal{N}$ the number of free and quasi-free indices, respectively. We have at least $d + 1$ free indices, which are even indices and 1, from the previous steps. Using a projection repeatedly, we will reduce the value $\mathcal{M} + \mathcal{N}$ to $d$, which yields (i). First, we discard the index 1 by fixing $z_1 \in P_1$ and classify $\{2, \ldots, 2d\}$ as free, quasi-free and bound indices. Then the number $\mathcal{M} + \mathcal{N}$ can decrease by 1. Of course, it can be unchanged or increased by 1 when 1 was the free index of a band with two or three elements. After discarding indices $\{1, 2, \ldots, 2d - k\}$ appropriately, we obtain a band structure on $\{2d - k + 1, \ldots, 2d\}$ with $\mathcal{M} + \mathcal{N} = d$. Note that discarding an index does not affect properties (ii) – (iv).

We denote by $\omega \in \mathbb{C}^k$ a set of $(z_{2d-k+1}, \ldots, z_{2d}) \in P_{2d-k+1} \times \cdots \times P_{2d}$ such that $\sigma(\omega) \geq c\sigma(P_{2d-k+1} \times \cdots \times P_{2d})$ is valid for some constant $c < 1$. Thus we obtain $\sigma(\omega) \sim \sigma(P_{2d-k+1} \times \cdots \times P_{2d}) \gtrsim \alpha_1^{[k/2]} \beta^{[k/2]}(\alpha_2/\alpha_1)$.

3.2. The slice argument. Let $\Lambda = \{\lambda_1, \ldots, \lambda_d\} \subset \{2d-k+1, \ldots, 2d\}$ be a set of all indices which are free or quasi-free. Also we set $\tau = (\tau_1, \ldots, \tau_d) \in \mathbb{C}^d$ with $\tau_i = z_{\lambda_i}$, $\lambda_i \in \Lambda$. For $\lambda_i' \in \Lambda' := \{2d-k+1, \ldots, 2d\} \setminus \Lambda$, 1 \leq j \leq k - d, there is a free index $\lambda_i' \in \Lambda$ to which $\lambda_j$ is bound. Let $s_j = (z_{\lambda_j'} - z_{\lambda_j})z_{\lambda_j}^{-2K/\delta^2}$ and $s = (s_1, \ldots, s_{k-d}) \in \mathbb{C}^{k-d}$. It is clear that the map $z := (z_{2d-k+1}, \ldots, z_{2d}) \mapsto (\tau, s)$ is a diffeomorphism, so its inverse $z(\tau, s)$ exists and is differentiable. Set $x_0 + H_{2d}(z_1, \ldots, z_{2d-k}, z) := H(z(\tau, s))$, then $H(\omega) \subset E_2$ for $\omega$ as in Lemma 3.5. (Recall that $H_{2d}$ is defined in (9).)

For any $s \in \mathbb{C}^{k-d}$, we consider a slice $\omega_s = \{\tau : z(\tau, s) \in \omega\} \subset \mathbb{C}^d$. By the construction of $P_{2d}$ in the proof of Lemma 3.3, $z_{2d}$ is clearly one exponent of $\tau$, $H(\omega_s)$ is contained in $E_2$ for each $s$. By Bézout’s theorem, for each $s \in \mathbb{C}^{k-d}$ there are finitely many preimages under the map $\tau \mapsto H(z(\tau, s))$. Thus we have

$$|E_2| \gtrsim \int_{\omega_s} \left| \det \left( \frac{\partial H(z(\tau, s))}{\partial \tau} \right) \right|^2 d\mu(\tau).$$

The following lemma gives a lower bound for the integrand in (15).

Lemma 3.6. For a given $\varepsilon > 0$, there exist a band structure on $\{2d-k+1, \ldots, 2d\}$ and a set $\omega$ which satisfy Lemma 3.5. Then for all $(\tau, s) \in \omega$, and for some $C > 0$,

$$\left| \det \left( \frac{\partial H(z(\tau, s))}{\partial \tau} \right) \right|^2 \geq C\alpha_1^{d(d-1)/2} \left( \frac{\beta}{\alpha_1} \right)^{N} \left( \frac{\alpha_2}{\alpha_1} \right)^{-d - 1} \prod_{i=1}^{d} |\tau_i|^{-\frac{4K}{\delta(d+1)}}$$

holds. Here $N$ is the number of quasi-free indices associated to $\omega$, and $\varepsilon$ and $C$ depend only on $d, \delta$ by Lemma 3.3.

We postpone its proof for a moment and prove Lemma 3.1.

Proof of Lemma 3.1. By integrating both sides of (15), and by (16), we see that

$$\int_{\{s : z(\tau, s) \in \omega\}} |E_2| d\mu(s) \gtrsim \alpha_1^{d(d-1)/2} \left( \frac{\beta}{\alpha_1} \right)^N \left( \frac{\alpha_2}{\alpha_1} \right)^{-d - 1} \int_{\{s : z(\tau, s) \in \omega\}} d\sigma(\tau) d\mu(s).$$

For any $s = (s_1, \ldots, s_{k-d}) \in \mathbb{C}^{k-d}$ satisfying $z(\tau, s) \in \omega$,

$$|s_j| = |z_{\lambda_j'} - z_{\lambda_j}|z_{\lambda_j}^{-2K/\delta^2 \alpha_1} \leq \delta' \alpha_1^{1/2} |z_{\lambda_j'} z_{\lambda_j}|^{-\frac{K}{\delta(d+1)}} \leq \delta' \alpha_1^{1/2}.$$
Recall that

By reversing the change of variables \((\tau, s) \mapsto z\), we obtain that

\[
\int_{\{x: z(\tau, s) \in \omega\}} d\sigma(\tau) d\mu(s) = \int_{\{x: z(\tau, s) \in \omega\}} \prod_{i=1}^{d} |\tau_i|^{\frac{2K}{d(d+1)}} d\mu(\tau) d\mu(s)
\]

\[
\sim \int_{\omega} \prod_{j=1}^{d-k} |z_{\lambda_j'}|^{\frac{2K}{d(d+1)}} \prod_{i=1}^{d} |z_{\lambda_i}|^{\frac{2K}{d(d+1)}} d\mu(z) = \sigma(\omega)
\]

since \(|\det\left(\frac{\partial (x, z)}{\partial x}\right)| \sim \prod_{j=1}^{k-d} |z_{\lambda_j'}|^{\frac{2K}{d(d+1)}}\) by \([13]\). Thus we conclude that

\[
|E_2| \gtrsim \alpha_1 \binom{d(d-1)}{2} \left(\frac{\beta}{\alpha_1}\right)^N \alpha_1^{d-1} \alpha_1^{-d_k} \alpha_1^{[k/2]} \left(\frac{\alpha_2}{\alpha_1}\right) = \alpha_1^{\frac{d(d-1)}{2}} \left(\frac{\beta_1}{\alpha_1}\right)^{N+[k/2]} \left(\frac{\alpha_2}{\alpha_1}\right)^d.
\]

Since \(N + [k/2] \leq d - 1\) and \(\beta_1 \ll \alpha_1\), this gives Lemma 3.1. \(\Box\)

**Proof of Lemma 3.6** Recall that \(\tau_i = z_{\lambda_i}\) for \(\lambda_i \in \Lambda\) and \(s_j = (z_{\lambda_j'} - z_{\lambda_i})z_{\lambda_i}^{-\frac{2K}{d(d+1)}}\). Then

\[
H(z(\tau, s)) = x_0 + H_{2d}(z_1, \ldots, z_{2d-k}, z(\tau, s))
\]

\[
= x_0 + H_{2d-k}(z_1, \ldots, z_{2d-k}) + \sum_{1 \leq i \leq d} \left((-1)^{\lambda_i + 1} h(\tau_i) + \sum_{j \neq i} (-1)^{\lambda_j + 1} h(s_j \tau_i^{-\frac{2K}{d(d+1)}} + \tau_i)\right)
\]

\[
= x_0 + H_{2d-k}(z_1, \ldots, z_{2d-k}) + \sum_{1 \leq i \leq d} \left(\theta_i h(\tau_i) + \sum_{j \neq i} (-1)^{\lambda_j + 1} h(s_j \tau_i^{-\frac{2K}{d(d+1)}} + \tau_i) - h(\tau_i)\right),
\]

where \(j \Rightarrow i\) means \(\lambda_j' \in \Lambda'\) is bound to \(\lambda_i \in \Lambda\), and \(\theta_i = (-1)^{\lambda_i + 1} + \sum_{j \Rightarrow i} (-1)^{j + 1}\). Note that \(\theta_i\) cannot be 0. In fact, for each \(\lambda_i\), the number of indices which is bound to \(\lambda_i\) is 0 or at least 2. Also, all the indices bound to each \(\lambda_i\) are odd. Thus, \(\sum_{j \Rightarrow i} (-1)^{j + 1}\) should be at least 2 (0 or 0 if there is no index bound to \(i\)). Hence \(\theta_i\) cannot be 0.

For fixed \(s\), each column of \(\frac{\partial H(z(\tau, s))}{\partial \tau}\) is given by \(\theta_i h'(\tau_i)\) if there is no \(j\) such that \(j \Rightarrow i\), or

\[
\theta_i h'(\tau_i) + \sum_{j \Rightarrow i} (-1)^{\lambda_j + 1} \times
\]

\[
\times \left(h'(s_j \tau_i^{-\frac{2K}{d(d+1)}} + \tau_i) - h'(\tau_i) - \frac{2K}{d(d+1)} \left(s_j \tau_i^{-\frac{2K}{d(d+1)}} \frac{2K}{d(d+1)} + \tau_i\right)\right).
\]

Then by multilinearity, we have

\[
\det\left(\frac{\partial H(z(\tau, s))}{\partial \tau}\right) = C J_d(\tau) + \text{error terms},
\]
where \( C = \prod_{1 \leq i \leq d} \theta_i \), and \( J_d(\tau) = \det(h(\tau_1), \ldots, h(\tau_d)) \) is the determinant of the complex Jacobian of the map \((\tau_1, \ldots, \tau_d) \mapsto \sum_{1 \leq i \leq d} h(\tau_i)\).

Our claim is that the error terms can be bounded by \( O(\varepsilon) \times |J_d(\tau)| \). If it is proven, we can see that \(|\det(\frac{\partial h(z(\tau)))}{\partial \tau})| \geq |J_d(\tau)|\) by choosing sufficiently small \( \varepsilon \). Recall that

\[
|J_d(\tau)| \geq \max_{1 \leq i \leq d} |\tau_i|^K \prod_{1 \leq i < j \leq d} |\tau_i - \tau_j|
\]

from Lemma 2.2. We may assume that \( \tau_d = z_{2d} \). One can see that, by Lemma 3.2 and Lemma 3.3,

\[
\prod_{1 \leq i < j \leq d} |\tau_i - \tau_j| \geq \alpha_1 \frac{d(d-1)}{4} \left( \frac{\beta}{\alpha_1} \right)^{\frac{d-1}{2}} \prod_{i=1}^{d-2} |\tau_i| - \frac{2K}{a(d+1)}(d-1-i) \prod_{j=1}^{d-1} |\tau_j| - \frac{2K}{a(d+1)}\varepsilon_j |\tau_d| - \frac{2K}{a(d+1)}(1-\varepsilon_j)
\]

where \( \varepsilon_j = 0 \) if \( |\tau_j| < \frac{1}{2}(4\pi \nu)^{-\nu} \alpha_2^2 \), and \( \varepsilon_j = 1 \) otherwise. (See (vi) in Lemma 3.3)

Also it is obvious that

\[
\max_{1 \leq i \leq d} |\tau_i|^K \geq \prod_{1 \leq i \leq d} |\tau_i|^K_i,
\]

for some \( K_i > 0 \) satisfying \( \sum_{i=1}^{d} K_i = K \). By choosing appropriate \( K_i \)’s to cancel out the exponents \( \varepsilon_j \)’s, we can obtain (16). In fact, one can choose \( K_i = \frac{2K}{a(d+1)}(d-i+\varepsilon_i) \) for \( 1 \leq i \leq d-2 \), \( K_{d-1} = \frac{2K}{a(d+1)}(1+\varepsilon_{d-1}) \), and \( K_d = \frac{2K}{a(d+1)}(d-\sum_{j=1}^{d-1} \varepsilon_j) \) satisfying \( \sum_{i=1}^{d} K_i = K \). Hence this gives the desired inequality (16).

Now we turn to the error terms. It suffices to consider two types of error terms, which are

\[
det(h'(\tau_1), \ldots, h'(\tau_{i-1}), h'(u_j(i) + \tau_i) - h'(\tau_i), \ldots, h'(u_j(d) + \tau_d) - h'(\tau_d)),
\]

and

\[
det(h'(\tau_1), \ldots, h'(\tau_{i-1}), -\frac{u_j(i)}{\tau_i} h'(u_j(i) + \tau_i), \ldots, -\frac{u_j(d)}{\tau_d} h'(u_j(d) + \tau_d)),
\]

where \( u_j(i) = s_j \tau_i \frac{2K}{a(d+1)} \) for some \( j \Rightarrow i \).

**An estimate for the second type** (18). First, we shall find an upper bound of (18). Note that \( u_j(i) + \tau_i = u_j(i) + z_{\lambda_i} = z_{\lambda'_j} \) such that \( \lambda'_j \) is bound to \( \lambda_i \). (See the definition of \( s_j \) at the beginning of the proof.) Using (iv) in Lemma 3.3 or (14), we observe that

\[
|u_j(i)| = |z_{\lambda'_j} - z_{\lambda_i}| < \varepsilon \delta \alpha_1 \frac{1}{2} |z_{\lambda'_j} z_{\lambda_i}|^{-\frac{K}{a(d+1)}} \leq \varepsilon \delta \alpha_1 \frac{1}{2} \alpha_1 \frac{2K}{a(d+1)} = \varepsilon \delta \alpha_1 \frac{1}{2} \alpha_1 \frac{2K}{a(d+1)}
\]
since we assume that $|z_i| \geq \gamma^r = \max\{\alpha_1, \beta_1\}^r$ for $1 \leq i \leq 2d-1$, and $|z_{2d}| \geq \alpha_2^r \geq \alpha_1^r$ in Lemma 3.3. (Recall that $\nu = d(d + 1)/(4K + 2d(d + 1))$.) It follows that

$$|u_{j(i)}| \lesssim \varepsilon \delta |z_{\lambda_i}| = \varepsilon \delta |\tau_i|$$

whenever $j \Rightarrow i$.

Thus we see that

$$|\nu i \tau_i| \lesssim (\varepsilon \delta)^{d-i+1} |\det(h'(\tau_1), \ldots, h'(\tau_{i-1}), h'(u_{j(i)} + \tau_i), \ldots, h'(u_{j(d)} + \tau_d))|.$$  

By (13), we see that

$$|u_{j(i)} + \tau_i| = |z_{\lambda_k}| \sim |z_{\lambda_i}| = |\tau_i|$$

for $i \leq l \leq d$ and $j \Rightarrow i$.

Recall that if $h(z) = (z, z^2, \ldots, z^{d-1}, z^N)$, then

$$\det(h'(z_1), \ldots, h'(z_d)) = N (d-1)! V(z_1, \ldots, z_d) Q_{N-d}(z_1, \ldots, z_d),$$

where $V(z_1, \ldots, z_d) = \prod_{1 \leq i < j \leq d} (z_j - z_i)$ is the complex Vandermonde determinant and $Q_m$ is a homogeneous monic polynomial of degree $m$ defined by

$$Q_m(z_1, \ldots, z_d) = \sum_{a_1 + \cdots + a_d = m} z_1^{a_1} \cdots z_d^{a_d}.$$

We refer to Section 3 in [1] for further details of (22).

If we set $z_\nu := z_{\lambda_k}' = u_{j(i)} + \tau_i$, we obtain that

$$|\nu i \tau_i| \lesssim (\varepsilon \delta)^{d-i+1} |V(\tau_1, \ldots, \tau_{i-1}, z_\nu', \ldots, z_{d'})| \times |Q_{N-d}(\tau_1, \ldots, \tau_{i-1}, z_\nu', \ldots, z_{d'})|.$$  

We first show that

$$|V(\tau_1, \ldots, \tau_{i-1}, z_\nu', \ldots, z_{d'})| \lesssim |V(\tau_1, \ldots, \tau_d)|.$$  

To see this, we will show that $|\tau_i - z_\nu| \lesssim |\tau_i - \tau_l|$ and $|z_\nu - z_\nu| \lesssim |\tau_i - \tau_l|$ for $i \neq l$.

By the triangle inequality, it suffices to show that $|\tau_i - z_\nu| \lesssim \varepsilon |\tau_i - \tau_l|$ for $i \neq l$.

If $|\tau_i| \ll |\tau_l|$ or $|\tau_l| \ll |\tau_i|$, we have $|\tau_i - \tau_l| \gtrsim |\tau_l|$. By (21) we also have that $|\tau_i - z_\nu| \lesssim \varepsilon |\tau_i - \tau_l|$ as desired.

If $|\tau_i| \sim |\tau_l|$, then (14) and (21) gives

$$|\tau_i - z_\nu| \leq \varepsilon \delta |\tau_i| \frac{1}{\alpha_1^{d+1}} \sim \varepsilon \delta |\tau_i| \frac{1}{\alpha_1^{d+1}} \sim \varepsilon \delta |\tau_i| \sim \varepsilon |\tau_i - \tau_l|.$$  

Here the last inequality holds by (ii) in Lemma 3.5 since $i$ and $l$ are in different bands. Hence we obtain (23).

Also, we obtain from (21) that

$$|Q_{N-d}(\tau_1, \ldots, \tau_{i-1}, z_\nu', \ldots, z_{d'})| \leq \sum_{a_1 + \cdots + a_d = N-d} |\tau_1|^{a_1} \cdots |\tau_{i-1}|^{a_{i-1}} |z_\nu'|^{a_i} \cdots |z_{d'}|^{a_d} \sim \sum_{a_1 + \cdots + a_d = N-d} |\tau_1|^{a_1} \cdots |\tau_d|^{a_d} \lesssim \max_{1 \leq i \leq d} |\tau_i|^{N-d}.$$
Therefore it follows that
\[
|I_{13}| \lesssim (\varepsilon \delta)^{d-i+1} \max_{1 \leq l \leq d} |\tau_i|^{N-d} |V(\tau_1, \ldots, \tau_d)| \lesssim (\varepsilon \delta)^{d-i+1} |J_d(\tau)|.
\]
This finishes the error estimate for the second type.

An estimate for the first type (17). We can write
\[
\begin{align*}
(17) &= \int_{\tau_i}^{u_{j(i)}+\tau_i} \cdots \int_{\tau_d}^{u_{j(d)}+\tau_d} \prod_{l=1}^d \frac{\partial}{\partial \tau_l} \bigg|_{\tau_k = \zeta_k} \det(h'(\tau_1), \ldots, h'(\tau_d)) d\zeta_1 \cdots d\zeta_d. \\
&\quad \text{By (22), we get} \\
&\quad \prod_{l=1}^d \frac{\partial}{\partial \tau_l} \bigg|_{\tau_k = \zeta_k} \det(h'(\tau_1), \ldots, h'(\tau_d)) \\
&\quad = N (d-1)! Q_{N-d}(\tau_1, \ldots, \tau_{i-1}, \zeta_i, \ldots, \zeta_d) \prod_{l=1}^d \frac{\partial}{\partial \tau_l} \bigg|_{\tau_i = \zeta_i} V(\tau_1, \ldots, \tau_d) + \\
&\quad + N (d-1)! V(\tau_1, \ldots, \tau_{i-1}, \zeta_i, \ldots, \zeta_d) \prod_{l=1}^d \frac{\partial}{\partial \tau_l} \bigg|_{\tau_i = \zeta_i} Q_{N-d}(\tau_1, \ldots, \tau_d) \\
&\quad =: I + II.
\end{align*}
\]

First, we consider the following product of derivatives of the Vandermonde determinant:
\[
\prod_{l=1}^d \frac{\partial}{\partial \tau_l} V(\tau_1, \ldots, \tau_d),
\]
which is given by a finite sum of terms \(V(\tau_1, \ldots, \tau_d)/\prod_{l=i}^d (\tau_l - \zeta_{m(l)})\). Here, \(m(l)\) is an index strictly less than \(l\). Then \(|I|\) is bounded by terms such as
\[
\frac{N (d-1)! |Q_{N-d}(\tau_1, \ldots, \tau_{i-1}, \zeta_i, \ldots, \zeta_d)| V(\tau_1, \ldots, \tau_d)}{\prod_{l=i}^d |\tau_l - \zeta_{m(l)}|}.
\]
Note that \(\zeta_{m(l)} = \tau_{m(l)}\) if \(m(l) \leq i - 1\). To handle the denominator, we need to make some observations. Since \(\zeta_l\) is on the line segment between \(\tau_l\) and \(u_{j(l)} + \tau_l\), we see that \(|\tau_l - \zeta_l| \leq |u_{j(l)}|\). If \(|\tau_l| \ll |\tau_{m(l)}|\) or \(|\tau_{m(l)}| \ll |\tau_l|\), it follows from (20) that
\[
|\tau_l - \zeta_l| \leq |u_{j(l)}| \leq \varepsilon \delta |\tau_l| \lesssim \varepsilon \delta |\tau_l - \tau_{m(l)}|. 
\]
If \(|\tau_l| \sim |\tau_{m(l)}|\), then (19) (with (13)) and (ii) in Lemma 3.5 gives that
\[
|\tau_l - \zeta_l| \leq |u_{j(l)}| \leq \varepsilon \delta \alpha_1^{1/2} |\tau_l|^{-(2K/d+1)} \sim \varepsilon \delta \alpha_1^{1/2} |\tau_l \tau_{m(l)}|^{-(K/d+1)} < \varepsilon |\tau_l - \tau_{m(l)}|.
\]
Hence we obtain that \(|\tau_l - \zeta_l| < \varepsilon |\tau_l - \tau_{m(l)}|\). In the same way, one can see that \(|\tau_{m(l)} - \zeta_{m(l)}| < \varepsilon |\tau_l - \tau_{m(l)}|\) for \(m(l) \geq i\). By this and the triangle inequality, it follows that
\[
|\zeta_l - \zeta_{m(l)}| \gtrsim |\tau_l - \tau_{m(l)}|
\]
for \(1 \leq m(l) < l \leq d\).
Once again we consider two cases. If $|\tau| \ll |\tau_m(t)|$ or $|\tau_m(t)| \ll |\tau|$, 

$$|\zeta_l - \zeta_m(t)| \gtrsim |\tau_l - \tau_m(t)| \gtrsim |\tau_l| \gtrsim \alpha^{\frac{1}{d}} |\tau_l|^{-\frac{2K}{d(d+1)}},$$

where the last inequality is from the fact that $|\tau_l| \gtrsim \alpha^{\nu} = \alpha \frac{d(d+1)}{d+2(d+1)}$. If $|\tau| \sim |\tau_m(t)|$, it follows from $(ii)$ in Lemma $3.3$ that 

$$|\zeta_l - \zeta_m(t)| \gtrsim |\tau_l - \tau_m(t)| \gtrsim \delta \alpha^{\frac{1}{d}} |\tau_l|^{-\frac{2K}{d(d+1)}} \sim \delta \alpha^{\frac{1}{d}} |\tau_l|^{-\frac{2K}{d(d+1)}}.$$

Thus we obtain that 

$$\prod_{l=i}^{d} |\zeta_l - \tau_m(t)| \gtrsim \prod_{l=i}^{d} \delta \alpha^{\frac{1}{d}} |\tau_l|^{-\frac{2K}{d(d+1)}}.$$

Also, the numerator can be bounded by $|J_d(\tau)|$ by following the same argument as the second type estimate.

As a result, we obtain that 

$$\left| \int_{\tau_1}^{u_j(\tau_1) + \tau_1} \cdots \int_{\tau_d}^{u_j(\tau_d) + \tau_d} I d\zeta_l \cdots d\zeta_l \right| \lesssim |J_d(\tau)| \frac{\prod_{l=i}^{d} |u_j(l)|}{\prod_{l=i}^{d} \delta \alpha^{\frac{1}{d}} |\tau_l|^{-\frac{2K}{d(d+1)}}} \lesssim |J_d(\tau)| \frac{\prod_{l=i}^{d} \varepsilon \delta \alpha^{\frac{1}{d}} |\tau_l|^{-\frac{2K}{d(d+1)}}}{\prod_{l=i}^{d} \delta \alpha^{\frac{1}{d}} |\tau_l|^{-\frac{2K}{d(d+1)}}} = \varepsilon^{d-i+1} |J_d(\tau)|.$$

The last inequality is valid by (19) and (21).

Now we consider an estimate for $II$. Using the definition of the monic polynomial, we see that 

$$\left| \prod_{l=i}^{d} \frac{\partial}{\partial \tau_l} Q_{N-d}(\tau_1, \ldots, \tau_d) \right| = \left| \sum_{a_1, \ldots, a_d = N-d; \ a_1, \ldots, a_d \geq 1} \left( \prod_{l=i}^{d} a_l \right) \tau_1^{a_1} \cdots \tau_i^{a_i-1} \tau_{i+1}^{a_{i+1}} \cdots \tau_d^{a_d-1} \right| \lesssim \frac{\max_{1 \leq l \leq d} |\tau_l|^{N-d}}{\prod_{l=i}^{d} |\tau_l|}.$$

Also, the fact that $|V(\tau_1, \ldots, \tau_i, \zeta, \ldots, \zeta_d)| \lesssim |V(\tau_1, \ldots, \tau_d)|$ is already obtained.

It follows that 

$$|II| \lesssim \frac{N (d-1)! |V(\tau_1, \ldots, \tau_d)| \max_{1 \leq l \leq d} |\tau_l|^{N-d}}{\prod_{l=i}^{d} |\tau_l|} \lesssim \frac{|J_d(\tau)|}{\prod_{l=i}^{d} |\tau_l|},$$

and then 

$$\left| \int_{\tau_1}^{u_j(\tau_1) + \tau_1} \cdots \int_{\tau_d}^{u_j(\tau_d) + \tau_d} II d\zeta_l \cdots d\zeta_l \right| \lesssim |J_d(\tau)| \prod_{l=i}^{d} \frac{|u_j(l)|}{|\tau_l|} \lesssim (\varepsilon \delta)^{d-i+1} |J_d(\tau)|$$

by (21). Hence the error terms of both types can be bounded by $O(\varepsilon) \times |J_d(\tau)|$. This completes the proof. \qed
4. Lemmas for optimal Lorentz space inequalities

In this section, we prove Lemma 4.1 which is crucial to show the (nearly) optimal Lorentz boundedness of $T$. (See Lemma A.2.)

\textbf{Lemma 4.1.} Let $F_1, F_2, E \subset \mathbb{R}^{2d}$ be measurable sets with finite measure. Suppose that

\[ T^* \chi_{F_1}(y) \geq \beta_1 \text{ and } T^* \chi_{F_2}(y) \geq \beta_2 \]

for all $y \in E$ and $\beta_1 \leq \beta_2$. Suppose that $\alpha_i$ for $i = 1, 2$ such that $(T^* \chi_{F_1}, \chi_E)/|F_1| \geq \alpha_i$ and $\alpha_2 \leq \alpha_1$. Then there exists a constant $C > 0$, depending on $N$ and $d$, such that

\[ |F_2| \geq C \alpha_1 \alpha_2 r_s \beta_1^s \beta_2^s, \]

where $r_1 + r_2 = \frac{d(d-1)}{2}$, $s_1 + s_2 = d$, and $\frac{s_2}{q_2} - \frac{s_1}{q_1} - 1 > 0$.

\textbf{Proof of Lemma 4.1} Similarly to Lemma 3.3, we need the following to prove Lemma 4.1.

\textbf{Lemma 4.2.} Let $\gamma_1 = \max\{\alpha_1, \beta_1\}$, $\nu = \frac{d(d-1)}{4K+2d(d+1)}$. There exist a point $y_0$ in $E$, a constant $C > 0$, and a sequence of sets $P_1, \ldots, P_{2d-1}$ in $\Delta$ such that

(i) $\sigma(P_j) \geq C \beta_1$ for odd $j < 2d - 1$,

(ii) $\sigma(P_j) \geq C \alpha_1$ for even $j < 2d - 1$,

(iii) $\sigma(P_{2d-1}) \geq C \beta_2$,

and $|z_j| \geq (4\pi \nu)^{-\nu} \gamma_1'$ for $z_j \in P_j$ for $1 \leq j \leq 2d - 2$, $|z_{2d-1}| \geq (4\pi \nu)^{-\nu} \beta_2''$.

Also there exists a positive small constant $c$ such that

(iv) if $z_j \in P_j$ for odd $j < 2d - 1$, then $|z_j - z_i| \geq c \beta_1^\frac{j}{2K} |z_i|^{\frac{1}{2K(d+1)}}$, where $i < j$,

(v) if $z_j \in P_j$ for even $j < 2d - 1$, then $|z_j - z_i| \geq c \alpha_1^\frac{j}{2K} |z_i|^{\frac{1}{2K(d+1)}}$, where $i < j$,

(vi) for $z_{2d-1} \in P_{2d-1}$ and $j < 2d - 1$, $|z_{2d-1} - z_j| \geq c \beta_2^\frac{j}{2} |z_{2d-1}|^{\frac{1}{2K(d+1)}}$ if $|z_j| < \frac{1}{2} (4\pi \nu)^{-\nu} \beta_2''$, and $|z_{2d-1} - z_j| \geq c \beta_2^\frac{j}{2} |z_j|^{\frac{1}{2K(d+1)}}$ if $|z_j| \geq \frac{1}{2} (4\pi \nu)^{-\nu} \beta_2''$.

We begin with the easy case $\beta_1 \geq \alpha_1$. Let $\Phi_{2d-1} = \{(z_1, z_2, \ldots, z_{2d-1}) : z_i \in P_i \text{ where } 1 \leq i \leq 2d - 1\}$. We define $z_0 = (z_1, z_2, \ldots, z_{d-1})$ and $\Phi := \{z \in \mathbb{C}^d : (z_0, z) \in \Phi_{2d-1}\}$. Note that $\sigma(\Phi) \sim \alpha_1^{d/2} \beta_1^{d/2} / (\beta_2 / \beta_1)$. For $d \geq 2$, let us define

\[ a_d' = \begin{cases} 1 + 3 + \cdots + d & \text{if } d \text{ is odd}, \\ 2 + 4 + \cdots + d & \text{if } d \text{ is even}. \end{cases} \]

Then the following lemma is obtained by the same argument as in the proof of Lemma 3.4.

\textbf{Lemma 4.3.} Let $d \geq 2$. Assume the hypotheses in Lemma 4.1. Then

\[ |F_2| \geq \alpha_1 \frac{d(d+1)}{2} \left( \frac{\beta_1}{\alpha_1} \right)^{a_d'} \left( \frac{\beta_2}{\beta_1} \right)^d. \]
Since $\beta_1 \gtrsim \alpha_1$, this implies Lemma 4.1. In fact, $(\beta_1/\alpha_1)^d \geq (\beta_1/\alpha_1)\rho$, and $(\beta_2/\beta_1)^d \geq (\beta_2/\beta_1)^2$ by the assumption $\beta_2 \geq \beta_1$ of Lemma 4.1.

Now suppose that $\beta_1 \ll \alpha_1$. In this case we obtain the following lemma similar to Lemma 3.5.

**Lemma 4.4.** Let $\varepsilon > 0$. Then there exist parameters $\delta, \delta'$ satisfying $c_{d, \varepsilon} < \delta' < \varepsilon \delta < \varepsilon c$, a positive constant $c_0$, an integer $d \leq k < 2d$, an element $z_0$, a set $\omega \subset \mathbb{C}^k$ with

\[
\sigma(\omega) \sim \alpha_1^{\lfloor \frac{k}{2} \rfloor} \beta_1^{\lceil \frac{k}{2} \rceil} \left( \frac{\beta_2}{\beta_1} \right),
\]

and a band structure on $\{2d - k, \ldots, 2d - 1\}$, such that the following properties hold:

(i) There are exactly $d$ free or quasi-free indices. In particular, each even index is free.

(ii) $|z_i - z_j| > \delta\alpha_1^{\frac{1}{2}}|z_i z_j|^{-K/d(d+1)}$, unless $i$ and $j$ lie in the same band.

(iii) $c_0\beta_1^\frac{1}{d} |z_i z_j|^{-K/d(d+1)} < |z_i - z_j| \leq \delta\alpha_1^{\frac{1}{2}}|z_i z_j|^{-K/d(d+1)}$ whenever $i$ is quasi-bound to $j$.

(iv) $\delta'\alpha_1^{\frac{1}{d}} |z_i z_j|^{-K/d(d+1)} > |z_i - z_j|$ whenever $i$ is bound to $j$.

Note that (24) may be deduced from $(i) - (iii)$ in Lemma 4.2.

Now suppose that $\beta_2 \gtrsim \alpha_1$. It follows that $2d - 1$ must be a free index without quasi-bound and bound indices after carrying out Lemma 4.4. Thus (16) will be modified as follows:

\[
\left| \det \left( \frac{\partial H(z(\tau, s))}{\partial \tau} \right) \right|^2 \gtrsim \alpha_1^{\frac{d(d-1)}{2}} \left( \frac{\beta_1}{\alpha_1} \right)^N \left( \frac{\beta_2}{\alpha_1} \right)^{d-1} \prod_{i=1}^d |\tau_i|^{\frac{4K}{d(d+1)}}
\]

where $N$ is the number of quasi-free indices. Similarly to the proof of Lemma 3.1, it follows by (24) that

\[
|F_2| \gtrsim \alpha_1^{\frac{d(d-1)}{2}} \left( \frac{\beta_1}{\alpha_1} \right)^N \left( \frac{\beta_2}{\alpha_1} \right)^{d-1} \alpha_1^{d-k} \alpha_1^{\lfloor \frac{k}{2} \rfloor} \beta_1^{\lceil \frac{k}{2} \rceil} \left( \frac{\beta_2}{\beta_1} \right)
\]

\[
= \alpha_1^{\frac{d(d+1)}{2}} \left( \frac{\beta_1}{\alpha_1} \right)^{N + \lfloor \frac{k}{2} \rfloor} \left( \frac{\beta_2}{\alpha_1} \right)^{d-1} \left( \frac{\beta_2}{\beta_1} \right).
\]

Since $2d - k$, $2d - 1$, and all even indices between $2d - k$ and $2d - 1$ are free indices, the number of free indices is at least $\lfloor \frac{k}{2} \rfloor + 2$. Hence $N + \lfloor \frac{k}{2} \rfloor \leq d - 1$. Since we have $\beta_2 \gtrsim \alpha_1$, we conclude that $|F_2| \gtrsim \alpha_1^{\frac{d(d-1)}{2}} \beta_1^{d-2} \beta_2^2$. This satisfies the relations in Lemma 4.1.

We assume that $\beta_2 \ll \alpha_1$. Then the index $2d - 1$ may not be free. The number of free indices is at least $\lfloor \frac{k}{2} \rfloor + 1$, which means $N + \lfloor \frac{k}{2} \rfloor \leq d$. One can see that this is not enough for the desired bound. So we follow the argument using a two-stage band structure due to Stovall [30].

Let $\mathcal{B}(2d - 1)$ be the band containing $2d - 1$ after carrying out Lemma 4.4. Now, we decompose $\mathcal{B}(2d - 1)$ into sub-bands as follows. For $\varepsilon > 0$ and $c_{d, \varepsilon}$ in Lemma 4.4, there exist $\rho$ and $\rho'$ such that $c_{d, \varepsilon} < \rho' < \varepsilon \rho < \delta'$, and a subset $\omega'$ of $\omega$ satisfying that $\sigma(\omega') \sim \sigma(\omega)$. Then the following properties hold:
(i) \(|z_i - z_j| > \rho |z_i z_j|^{-K/d(d+1)}\) unless \(i\) and \(j\) lie in the same band.

(ii) \(a_0 \rho^{1/2} |z_i z_j|^{-K/d(d+1)} < |z_i - z_j| \leq \rho |z_i z_j|^{-K/d(d+1)}\) whenever \(i\) is quasi-bound to \(j\).

(iii) \(\rho^{1/2} |z_i z_j|^{-K/d(d+1)} > |z_i - z_j|\) whenever \(i\) is bound to \(j\), for \(i, j \in \mathcal{B}(2d - 1)\), \(\gamma_2 = \max\{\alpha_2, \beta_2\}\) and some constant \(c_0 > 0\).

After this step, the number of free and quasi-free indices in \(\{2d - k, \ldots, 2d - 1\}\) may increase. Then we repeat \textit{Step 3} in the proof of Lemma 3.5 (the step of eliminating some indices) until we get exactly \(d\) free and quasi-free indices in \(\{2d - k', \ldots, 2d - 1\}\) for some integer \(k'\). By abuse of notation we will write \(k\) instead of \(k'\).

Let \(F_1\) and \(Q_1\) be the number of free and quasi-free indices which are contained in \(\{2d - k, \ldots, 2d - 1\}\) \(\mathcal{B}(2d - 1)\). Also, let \(F_2\) and \(Q_2\) be the number of free and quasi-free indices which are contained in \(\mathcal{B}(2d - 1)\). Note that \(F_1 + Q_1 + F_2 + Q_2 = d\).

We set \(M = F_2 + Q_2\), and the number of elements of \(\mathcal{B}(2d - 1)\) is denoted by \(N\). Then \(N - M\) denotes the number of bound indices in \(\mathcal{B}(2d - 1)\).

The case when \(\mathcal{B}(2d - 1) = 2d - 1\) is the same as the case \(\beta_2 \geq \alpha_1\) above. Hence we consider the following three cases:

1. \(2d - 1\) is free and there is at least one free index other than \(2d - 1\) in \(\mathcal{B}(2d - 1)\).
2. \(2d - 1\) is quasi-free.
3. \(2d - 1\) is bound to some \(j\) in \(\mathcal{B}(2d - 1)\).

Case (1). Since we are in the case that \(2d - 1\) is free, we have \(F_1 + F_2 \geq \left\lceil \frac{k}{2} \right\rceil + 2\).

In this case, we get the lower bound of Jacobian (16) as follows:

\[
\left| \det \left( \frac{\partial H(z(\tau, s))}{\partial \tau} \right) \right|^2 > C \alpha_1^{d(d-1)/2} \left( \frac{\gamma_2}{\alpha_1} \right)^{M(M-1)/2} \left( \frac{\beta_1}{\gamma_2} \right)^{Q_1} \left( \frac{\beta_1}{\gamma_2} \right)^{Q_2} \prod_{i=1}^{d} |\alpha_i|^{2K/(d+1)}.
\]

In addition, note that \(\sigma(\{s : z(\tau, s) \in \omega\})\) is bounded above by \(\alpha_1^{k-d} \left( \frac{\gamma_2}{\alpha_1} \right)^{N-M}\) in this case. Thus, by combining these, we obtain that

\[
|F_2| \geq \alpha_1^{d(d-1)/2} \left( \frac{\gamma_2}{\alpha_1} \right)^{M(M-1)/2} \left( \frac{\beta_1}{\gamma_2} \right)^{Q_1} \left( \frac{\beta_1}{\gamma_2} \right)^{Q_2} \left( \frac{\gamma_2}{\alpha_1} \right)^{M-N} \times \alpha_1^{d-k} \left( \frac{\gamma_2}{\alpha_1} \right)^{M-N} \times \alpha_1^{l_2} \left( \frac{\beta_2}{\alpha_1} \right)^{M-N}.
\]

The last inequality holds, because \(N - M \geq 0\) and \(\gamma_2 \ll \alpha_1\). Since \(2d - 1\) is free, we have that \(Q_1 + Q_2 + \left\lceil \frac{k}{2} \right\rceil - 1 \leq Q_1 + Q_2 + F_1 + F_2 - 2 = d - 2\).

If \(M(M-1)/2 \leq d - Q_1 - \left\lceil \frac{k}{2} \right\rceil\), then we may write \((\gamma_2/\alpha_1)^{M(M-1)/2} \geq (\beta_2/\alpha_1)^{d-Q_1-Q_2-\left\lceil \frac{k}{2} \right\rceil}\).

Then we have that \(|F_2| \geq \alpha_1^{r_1} \alpha_2^{r_2} \beta_1^{s_1} \beta_2^{s_2}\) with \(r_1 = d(d-1)/2\), \(r_2 = 0\) and \(s_1 = Q_1 + Q_2 + \left\lceil \frac{k}{2} \right\rceil - 1\), \(s_2 = d - Q_1 - Q_2 - \left\lceil \frac{k}{2} \right\rceil + 1\).
It is easy to verify that the relations of $r_1, r_2, s_1, s_2$ in Lemma 4.4 are valid by the fact that $s_2 \geq 2$ implies $s_2/q_d > 1$.

If $\frac{M(M-1)}{2} > d - Q_1 - \lceil \frac{k}{2} \rceil$, we may write $\gamma_2^\frac{M(M-1)}{2} - Q_2 \geq \beta_2^{d - Q_1 - Q_2 - \lceil \frac{k}{2} \rceil} \alpha_2^M \frac{M(M-1)}{2} + Q_1 + \lceil \frac{k}{2} \rceil - d$. Then $|F_2| \geq \alpha_1^r \alpha_2^s \beta_1^s \beta_2^s$ holds with

$$r_1 = \frac{d(d+1)}{2} - \frac{M(M-1)}{2} - Q_1 - \lceil \frac{k}{2} \rceil,$$
$$r_2 = \frac{M(M-1)}{2} + Q_1 + \lceil \frac{k}{2} \rceil - d,$$
$$s_1 = Q_1 + Q_2 + \lceil \frac{k}{2} \rceil - 1,$$
$$s_2 = d - Q_1 - Q_2 - \lceil \frac{k}{2} \rceil + 1.$$ 

It is easy to check that $r_1 + r_2 = d(d - 1)/2$ and $s_1 + s_2 = d$. Now let us verify $s_2/q_d - r_2/q_d - 1 > 0$. Since $2d - 1$ is free and is the largest index, we see that $2d - 1$ has no quasi-bound index, hence $Q_2 \leq (M - 1)/2$. Also it is valid that $Q_1 + \lceil k/2 \rceil \leq d - M + 1$, since $F_1$ is at least $\lceil k/2 \rceil - 1$ and $d = F_1 + Q_1 + F_2 + Q_2 = F_1 + Q_1 + M$. Hence we see that $s_2/q_d - r_2/q_d - 1 \geq (M - 1)/2 - 1/q_d - (M - 1)^2/2q_d$, which is positive for all $2 \leq M \leq d/2 + 1$. Here, we may assume that $M \geq 2$, because $B(2d - 1)$ contains at least two elements. Then the least element and $2d - 1$ are free. Also, we get $M \leq d/2 + 1$ from the fact that $Q_1 + \lceil k/2 \rceil \leq d - M + 1$ implies $M \leq d - \lceil k/2 \rceil + 1 \leq d/2 + 1$. By the concavity, $(M - 1)/2 - 1/q_d - (M - 1)^2/2q_d > 0$ if it is positive when $M = 2$ and $M = d/2 + 1$. For the case $M = 2$, the condition $d \geq 4$ is necessary. Thus we conclude that $s_2/q_d - r_2/q_d - 1 > 0$ holds for all possible $M$ whenever $d \geq 4$.

When $d = 3$, one can see that the only possible cases are $k = 3$ and all indices 3, 4, 5 are free. (Note that 4, 5 should be free by the assumptions.) Thus $Q_1 = 0$ and then $M(M - 1)/2 = d - Q_1 - \lceil \frac{k}{2} \rceil$ since $M = 2$. We will consider more general $C^3$ curves in Section 5.

Case (2). When $2d - 1$ is quasi-free, we have that $F_1 + F_2 \geq \lfloor \frac{k}{2} \rfloor + 1$. In this case, the lower bound of Jacobian (16) can be modified as

$$|\text{det} \left( \frac{\partial H(z, \tau, s)}{\partial \tau} \right) |^2 \geq C \alpha_1^{\frac{d(d-1)}{2}} \left( \frac{\gamma_2}{\alpha_1} \right)^{\frac{M(M-1)}{2}} \left( \frac{\beta_1}{\alpha_1} \right)^{Q_1} \left( \frac{\beta_1}{\gamma_2} \right)^{Q_2-1} \left( \frac{\gamma_2}{\gamma_2} \right)^{Q_2} \prod_{i=1}^{d} |\tau_i|^{\frac{4K}{M+d+1}}.$$ 

Then it follows that

$$|F_2| \geq \alpha_1^{\frac{d(d-1)}{2}} \left( \frac{\beta_1}{\alpha_1} \right)^{Q_1+Q_2+\lceil \frac{k}{2} \rceil - 2} \left( \frac{\gamma_2}{\alpha_1} \right)^{\frac{M(M-1)}{2} - (N-M) - Q_2} \left( \frac{\beta_2}{\alpha_1} \right)^2 \left( \frac{\gamma_2}{\alpha_1} \right)^{\frac{M(M-1)}{2} - Q_2} \left( \frac{\beta_2}{\alpha_1} \right)^2.$$


The last inequality holds, because \( N - M \geq 0 \) and \( \gamma_2 \ll \alpha_1 \).

If \( \frac{M(M-1)}{2} \leq d - Q_1 - \left\lceil \frac{k}{2} \right\rceil \), then we may write \( (\gamma_2/\alpha_1)^{\frac{M(M-1)}{2}} Q_2 \geq (\beta_2/\alpha_1)^{d - Q_1 - Q_2 - \left\lceil \frac{k}{2} \right\rceil} \).

Also we see that \( (\beta_2/\alpha_1)^{Q_1 + Q_2 + \left\lceil \frac{k}{2} \right\rceil - 2} \geq (\beta_2/\alpha_1)^{d - 2} \), since \( Q_1 + Q_2 + \left\lceil \frac{k}{2} \right\rceil \leq Q_1 + Q_2 + F_1 + F_2 \leq d \). Then we have that \( r_1 = (d - 1)/2, r_2 = 0, s_1 = d - 2, \) and \( s_2 = 2 \). It is easy to check the relations of \( r_1, r_2, s_1, s_2 \) in Lemma [4, 1].

If \( \frac{M(M-1)}{2} > d - Q_1 - \left\lceil \frac{k}{2} \right\rceil \), we may write \( r_2 \geq \beta_2^{d - Q_1 - Q_2 - \left\lceil \frac{k}{2} \right\rceil} \frac{M(M-1)}{2} + Q_1 + \left\lceil \frac{k}{2} \right\rceil - d \).

Then we have that
\[
\begin{align*}
r_1 &= \frac{d(d+1)}{2} - \frac{M(M-1)}{2} - Q_1 - \left\lceil \frac{k}{2} \right\rceil, \\
r_2 &= \frac{M(M-1)}{2} + Q_1 + \left\lceil \frac{k}{2} \right\rceil - d, \\
s_1 &= Q_1 + Q_2 + \left\lceil \frac{k}{2} \right\rceil - 2, \\
s_2 &= d - Q_1 - Q_2 - \left\lceil \frac{k}{2} \right\rceil + 2.
\end{align*}
\]

It is easy to check that \( r_1 + r_2 = d(d - 1)/2 \) and \( r_1 + r_2 = d \). Now let us verify \( s_2/q_d - r_2/q_d - 1 > 0 \). Since \( Q_2 \leq M/2 \) and \( Q_1 + \left\lceil k/2 \right\rceil \leq d - M + 1 \), we see that \( s_2/q_d - r_2/q_d - 1 \geq M/2 - 2/q_d - M(M - 2)/2q_d \), which is positive for all \( 2 \leq M \leq d/2 + 1 \). Since \( 2d - 1 \) is quasi-free, \( M \) is at least 2. Also, \( Q_1 + \left\lceil k/2 \right\rceil \leq d - M + 1 \) implies \( M \leq d - \left\lceil k/2 \right\rceil + 1 \leq d/2 + 1 \). Thus we conclude that \( s_2/q_d - r_2/q_d - 1 > 0 \) holds for all possible \( M \) whenever \( d \geq 4 \).

When \( d = 3 \), we can check that \( \frac{M(M-1)}{2} = d - Q_1 - \left\lceil \frac{k}{2} \right\rceil \) holds. In fact, the only possible cases are that 5 is quasi-bound to 3 or 4 with \( k = 3 \). Then \( Q_1 = 0 \) and \( M = F_2 + Q_2 = 2 \).

**Case (3).** Finally we consider the case that \( 2d - 1 \) is bound to some \( j \in \mathcal{B} \). In this case, we have that
\[
\left| \det \left( \frac{\partial H(z(\tau, s))}{\partial \tau} \right) \right|^2 \geq C \alpha_1^{d(d-1)/2} \frac{\gamma_2}{\alpha_1} \left( \frac{\beta_1}{\beta_2} \right)^{Q_1} \left( \frac{\alpha_1}{\alpha_1} \right)^{Q_2} \prod_{i=1}^{d} |\tau_i|^{4k}. \]

Thus one can see that
\[
|F_2| \geq \alpha_1^{d(d+1)/2} \frac{\beta_1}{\alpha_1} \frac{\gamma_2}{\alpha_1} \left( \frac{\beta_1}{\beta_2} \right)^{Q_1 + Q_2 + \left\lceil \frac{k}{2} \right\rceil - 1} \left( \frac{\gamma_2}{\alpha_1} \right)^{M(N-1)} \frac{\beta_1^{\left\lfloor \frac{k}{2} \right\rfloor} \beta_2^{\left\lceil \frac{k}{2} \right\rceil}}{\alpha_1^k}.
\]

Since \( 2d - 1 \) is bound to some \( j \), there are at least two indices which are bound to \( j \). Then we have \( N - M \geq 2 \), which implies \( (\gamma_2/\alpha_1)^{-(N-M)} \geq (\gamma_2/\alpha_1)^{-2} \). Thus we obtain that
\[
|F_2| \geq \alpha_1^{d(d+1)/2} \left( \frac{\beta_1}{\alpha_1} \right)^{Q_1 + Q_2 + \left\lceil \frac{k}{2} \right\rceil - 1} \left( \frac{\gamma_2}{\alpha_1} \right)^{M(N-1) - Q_2 - 2} \left( \frac{\beta_2}{\alpha_1} \right). \]
Note that \( d - Q_1 - Q_2 - \lceil \frac{k}{2} \rceil \geq 0 \) and we may assume \( \gamma_2 = \alpha_2 > \beta_2 \). (If \( \alpha_2 \leq \beta_2 \), then \( 2d - 1 \) should be free.) Hence we may write
\[
\gamma_2 \frac{M(M-1)}{2} - Q_2 - 2 \geq \beta_2 d - Q_1 - Q_2 - \lceil \frac{k}{2} \rceil \frac{M(M-1)}{2} + Q_1 + \lceil \frac{k}{2} \rceil - d - 2.
\]
It follows that \( |F_2| \gtrsim \alpha_1^{r_1} \alpha_2^{r_2} \beta_1^s \beta_2^s \) holds with
\[
\begin{align*}
r_1 &= \frac{d(d+1)}{2} - Q_1 - \left\lceil \frac{k}{2} \right\rceil - \frac{M(M-1)}{2} + 2, \\
r_2 &= \frac{M(M-1)}{2} + Q_1 + \left\lceil \frac{k}{2} \right\rceil - d - 2, \\
s_1 &= Q_1 + Q_2 + \left\lceil \frac{k}{2} \right\rceil - 1, \\
s_2 &= d - Q_1 - Q_2 - \left\lceil \frac{k}{2} \right\rceil + 1.
\end{align*}
\]
It is easy to check that \( r_1 + r_2 = d(d-1)/2 \) and \( s_1 + s_2 = d \). Now let us verify \( s_2/q_4' - r_2/q_d - 1 > 0 \) for all \( d \geq 3 \). Since \( Q_2 \leq (M-1)/2 \) and \( Q_1 + \lfloor k/2 \rfloor \leq d - M + 1 \), it follows that \( s_2/q_4' - r_2/q_d - 1 \geq (M-1)/2 + 1/q_d - (M+1)^2/2q_d \), which is positive for all \( 1 \leq M \leq d/2 + 1 \). Since the assumption that \( 2d - 1 \) is bound implies \( M \geq 1 \), and the condition \( Q_1 + \lfloor k/2 \rfloor \leq d - M + 1 \) implies \( M \leq d/2 + 1 \) for all \( d \geq 3 \). This completes the proof.

\( \square \)

5. The polynomial curves in \( \mathbb{C}^3 \)

In this section, we consider polynomial curves of simple type in \( \mathbb{C}^3 \). Let \( h(z) = (z, z^2, \phi(z)) \) with a complex polynomial \( \phi(z) \) of degree at most \( N \). Then \( d \sigma(z) = |\phi'''(z)|^{1/3} d\mu(z) \). By Lemma \( \ref{lemma:2.1} \), it is enough to consider a restricted domain \( B := D \cap B_\ell \) for some \( 1 \leq \ell \leq M \). By \( \ref{lemma:2} \), we also see that \( |\phi'''(z)| \sim |z|^k \) whenever \( z \in B_\ell \subseteq S \cap \Delta \cap E_k \) with \( E_k = G_k \) or \( D_k \).

Thus we can set
\[
T f(x) = \int_B f(x - h(z)) \, d\sigma(z) = \int_B f(x - h(z)) |z|^k d\mu(z).
\]

This section is actually redundant, because \( \ref{lemma:25} \) is the same as \( \ref{lemma:1} \). However, we will show that the \( d = 3 \) case can be treated directly without the band structure. Also, the complex version of the band structure in Section 3 was motivated by the proof of Lemma \( \ref{lemma:5.3} \).

Again, we will show that two refinements of the restricted weak type estimates hold:
\[
|E| \gtrsim \alpha^4 \beta^2 \quad \text{or} \quad |F| \gtrsim \alpha^3 \beta^3.
\]

First, we prove the following, by which one can obtain the strong type \((2,3)\) estimate. (See Lemma \( \ref{lemma:A.1} \) and the beginning of Appendix A.)
Lemma 5.1. Let $E_1, E_2, G \subset \mathbb{R}^6$ be measurable sets with finite measure. Suppose that

$$T \chi_{E_1}(x) \geq \alpha_1 \text{ and } T \chi_{E_2}(x) \geq \alpha_2$$

for all $x \in G$ and $\alpha_1 \leq \alpha_2$. Then

$$|E_2| \geq \alpha_1 \alpha_2^3 \beta^2,$$

where $\beta = \alpha_1^{|G|} |E_1|^{|G|}$.

The proof is similar to the real case, since we make a comparison between the absolute values of complex variables. (See [11].)

**Proof.** Since $\langle T^* \chi_G, \chi_{E_1} \rangle = \langle \chi_G, T \chi_{E_1} \rangle \geq \alpha_1 |G| = \beta |E_1|$, we can define a set

$$E_1^1 = \{ y \in E_1 : T^* \chi_G(y) \geq \beta/2 \}.$$

It follows that

$$\langle T \chi_{E_1^1}, \chi_G \rangle = \langle \chi_{E_1^1}, T^* \chi_G \rangle = \langle \chi_{E_1^1}, T^* \chi_G \rangle - \langle \chi_{E_1 \setminus E_1^1}, T^* \chi_G \rangle \geq \alpha_1 |G| - \frac{\beta}{2} |E_1| = \frac{\alpha_1}{2} |G|.$$

Thus we can define

$$G^1 = \{ x \in G : T \chi_{E_1^1}(x) \geq \alpha_1/4 \}.$$

One can see that $G^1$ is not empty.

For $x_0 \in G^1$, we set

$$P = \{ z_1 \in B : x_0 - h(z_1) \in E_1^1 \} \text{ and then } \sigma(P) = T \chi_{E_1^1}(x_0) \geq \alpha_1/4.$$

For all $z_1 \in P$, we also set

$$Q_{z_1} = \{ z_2 \in B : x_0 - h(z_1) + h(z_2) \in G \} \text{ and then } \sigma(Q_{z_1}) = T^* \chi_G(x_0 - h(z_1)) \geq \beta/2.$$

For $z_1 \in P$, and $z_2 \in Q_{z_1}$, we set

$$R_{z_1, z_2} = \{ z_3 \in B : x_0 - h(z_1) + h(z_2) - h(z_3) \in E_2 \}.$$

Then we see that $\sigma(R_{z_1, z_2}) = T \chi_{E_2}(x_0 - h(z_1) + h(z_2)) \geq \alpha_2$.

Let $S = \{ (z_1, z_2, z_3) : z_1 \in P, z_2 \in Q_{z_1}, z_3 \in R_{z_1, z_2} \}$. And if we set

$$\Phi_h(z_1, z_2, z_3) = -h(z_1) + h(z_2) - h(z_3),$$

then $x_0 + \Phi_h(S) \subset E_2$. From Lemma 2.1 and Bézout’s theorem, we have that

$$|E_2| \geq C \iiint_S |J_R \Phi_h(z_1, z_2, z_3)| \, d\mu(z_1) \, d\mu(z_2) \, d\mu(z_3)$$

$$= \iiint_S |J_c \Phi_h(z_1, z_2, z_3)|^2 \, d\mu(z_1) \, d\mu(z_2) \, d\mu(z_3)$$

$$\geq C \iiint_S \max\{|z_1|, |z_2|, |z_3|\}^{2k} |z_2 - z_1|^2 |z_3 - z_1|^2 |z_3 - z_2|^2 \, d\mu(z_1) \, d\mu(z_2) \, d\mu(z_3)$$

whenever $z_1, z_2, z_3 \in S$. The last integrand is obtained from (23).

To obtain a lower bound of the last integral, we follow the argument in [11]. Let us define a set

$$B_\alpha = \{ z \in B : |z| \leq (16\pi \nu)^{-\nu} \alpha^\nu \},$$
for \( \nu = \frac{3}{k+6} \) and a fixed \( k \) as in the definition of \( T \). Then we see that

\[
\sigma(B_\alpha) = \int_{B_\alpha} |z|^\frac{k}{4} d\mu(z) \leq 2\pi \int_0^{(16\pi\nu)^{-\nu}} r^{\frac{k}{4}+1} dr = 2\pi \nu (16\pi\nu)^{-1} \alpha = \frac{\alpha}{8}.
\]

Thus we may assume that \( |z_1| \geq (16\pi\nu)^{-\nu}\alpha_2^2 \) on \( P \). In fact, we can replace \( P \) by \( P \setminus B_{\alpha_1} \) since \( \sigma(P \setminus B_{\alpha_1}) \geq \sigma(P) - \sigma(B_{\alpha_1}) \geq \frac{\alpha}{8} \). We also assume that \( |z_2| \geq (16\pi\nu)^{-\nu}\beta \nu \) on \( Q_{z_1} \) and \( |z_3| \geq (16\pi\nu)^{-\nu}\alpha_2^2 \) on \( R_{z_1,z_2} \) in the same way.

The following lemma implies that we may assume \( z_1, z_2 \) and \( z_3 \) are separated from each other.

**Lemma 5.2.** There exists a small constant \( c > 0 \) such that for \((z_1, z_2, z_3) \in \mathcal{S}\)

- (i) \(|z_2 - z_1| \geq c\beta^\frac{1}{2}|z_2|^{-\frac{\beta}{8}|z_1|^{-\frac{1}{8}(1-\varepsilon_1)}}, \) where \( \varepsilon_1 = 0 \) if \( z_1 \in B_{\beta/2} \), or \( \varepsilon_1 = 1 \) if \( z_1 \notin B_{\beta/2} \).

- (ii) \(|z_3 - z_1| \geq c\alpha_2^\frac{1}{2}|z_3|^{-\frac{\beta}{8}|z_1|^{-\frac{1}{8}(1-\varepsilon_2)}}, \) where \( \varepsilon_2 = 0 \) if \( z_1 \in B_{\alpha_2/2} \), and \( \varepsilon_2 = 1 \) if \( z_1 \notin B_{\alpha_2/2} \).

- (iii) \(|z_3 - z_2| \geq c\alpha_2^\frac{1}{2}|z_3|^{-\frac{\beta}{8}|z_2|^{-\frac{1}{8}(1-\varepsilon_3)}}, \) where \( \varepsilon_3 = 0 \) if \( z_2 \in B_{\alpha_2/2} \), and \( \varepsilon_3 = 1 \) if \( z_2 \notin B_{\alpha_2/2} \).

**Proof.** We will show (i). The remaining cases can be shown in a similar way. First, we consider the case \( z_1 \in B_{\beta/2} \), where \(|z_1| \leq (16\pi\nu)^{-\nu}\beta/2 \nu < |z_2|/2 \nu \). Then we have that

\[ |z_2 - z_1| \geq (16\pi\nu)^{-\frac{k}{2}}|z_2|^{-\frac{k}{8}} |z_1|^{-\frac{k}{8}(1-\varepsilon_1)}, \]

since \(|z_2| > (16\pi\nu)^{-\nu}\beta \nu = (16\pi\nu)^{-3/(k+6)} \beta^{3/(k+6)} \).

If \( z_1 \notin B_{\beta/2} \), then \(|z_1| > (16\pi\nu)^{-\nu}\beta/2 \nu \), which also implies \(|z_1| > (32\pi\nu)^{-\frac{k}{2}}|z_1|^{-\frac{k}{8}} \).

Let us define a set \( B_\beta(w) \) for \( w \in B \) and \( c_0 > 0 \) by

\[ B_\beta(w) = \{ z \in B : |z - w| \leq c_0 \beta^\frac{1}{2}|w|^{-\frac{k}{8}} \}. \]

When \( z \in B_\beta(z_1) \) for \( z_1 \notin B_{\beta/2} \), we see that

\[ |z - z_1| \leq c_0 \beta^\frac{1}{2}|z_1|^{-\frac{k}{8}} \leq c_0 (32\pi\nu)^{\frac{1}{2}} |z_1|, \]

and then

\[ |z| \leq |z - z_1| + |z_1| \leq (1 + c_0 (32\pi\nu)^{\frac{1}{2}}) |z_1|. \]

It follows that

\[ \sigma(B_\beta(z_1)) = \int_{B_\beta(z_1)} |z|^\frac{k}{4} d\mu(z) \leq \left(1 + c_0 (32\pi\nu)^{\frac{1}{2}} |z_1|\right)^{\frac{k}{8}} \times \pi c_0^2 \beta |z_1|^{-\frac{k}{8}}. \]

By choosing sufficiently small \( c_0 \), we have that \( \sigma(Q_{z_1} \setminus B_\beta(z_1)) \geq \beta \). Thus we can regard \( Q_{z_1} \setminus B_\beta(z_1) \) as \( Q_{z_1} \), and we can say (i) holds for \( z_1 \in P \) and \( z_2 \in Q_{z_1} \). \( \square \)

Now we turn to obtaining a lower bound of \( E' \). We may assume that \( \mathcal{S} \) can be replaced by a suitable subset of \( \mathcal{S} \), where \( z_1, z_2, z_3 \) satisfy the observations above. By the above lemma, the Vandermonde determinant \(|z_2 - z_1||z_3 - z_1||z_3 - z_2|\) can be treated in each case. In fact, we have

\[ |z_2 - z_1|^2 |z_3 - z_1|^2 |z_3 - z_2|^2 \geq \beta \alpha_2^2 |z_1|^{-\frac{k}{8}(2 - \varepsilon_1)} |z_2|^{-\frac{k}{8}(1-\varepsilon_3)} |z_3|^{-\frac{k}{8}(2 + \varepsilon_3)} \].
Also it is obvious that
\[ \max\{ |z_1|, |z_2|, |z_3| \}^{2k} \geq |z_1|^a |z_2|^b |z_3|^c \]
for positive constants \(a,b,c\) satisfying \(a + b + c = 2k\).

Therefore, if we set
\[ a = \frac{k}{3} + \frac{k}{3}(2 - \varepsilon_1 - \varepsilon_2), \quad b = \frac{k}{3} + \frac{k}{3}(\varepsilon_1 + 1 - \varepsilon_3), \quad \text{and} \quad c = \frac{k}{3} + \frac{k}{3}(\varepsilon_2 + \varepsilon_3), \]
we have
\[
|E_2| \geq C \int_{\mathbb{R}^3} \max\{ |z_1|, |z_2|, |z_3| \}^{2k} |z_2 - z_1|^2 |z_3 - z_1|^2 |z_3 - z_2|^2 d\mu(z_1) d\mu(z_2) d\mu(z_3)
\]
\[
\geq \beta \alpha^2 \int_{\mathbb{R}^3} |z_1|^3 \int_{\mathbb{R}^3} |z_2|^3 \int_{\mathbb{R}^3} |z_3|^3 d\mu(z_3) d\mu(z_2) d\mu(z_1)
\]
\[
\geq \alpha \beta \alpha^3.
\]
This completes the proof. \(\square\)

To obtain sharper Lorentz space estimates, we need another refinement of (26) as follows.

**Lemma 5.3.** Let \(F_1, F_2, G \subset \mathbb{R}^6\) be measurable sets with finite measure. Suppose that
\[ T^*\chi_{F_1}(y) \geq \beta_1 \quad \text{and} \quad T^*\chi_{F_2}(x) \geq \beta_2 \]
for all \(y \in G\) and \(\beta_1 \leq \beta_2\). Then
\[ |F_2| \geq \alpha^3 \beta_1^2 \beta_2, \]
where \(\alpha = \beta_1 \frac{|G|}{|F_1|}\).

**Proof.** Following the proof of Lemma 5.1, we have that
\[
F_1^1 = \{ x \in F_1 : T^*\chi_G(x) \geq \alpha/2 \}
\]
\[
G^1 = \{ y \in G : T^*\chi_{F_1^1}(y) \geq \beta_1/4 \}.
\]
Then we can construct the sets contained in \(B\) as follows. For \(y_0 \in G^1\), we define
\[
P' = \{ z_1 \in B : y_0 + h(z_1) \in F_1^1 \}, \quad (\sigma(P') \geq \beta_1/4),
\]
\[
Q'_{z_1} = \{ z_2 \in B : y_0 + h(z_1) - h(z_2) \in G \}, \quad (\sigma(Q'_{z_1}) \geq \alpha/2),
\]
\[
R'_{z_1,z_2} = \{ z_3 \in B : y_0 + h(z_1) - h(z_2) + h(z_3) \in F_2 \}, \quad (\sigma(R'_{z_1,z_2}) \geq \beta_2).
\]
If we set \(S' = \{ (z_1, z_2, z_3) : z_1 \in P', z_2 \in Q'_{z_1}, z_3 \in R'_{z_1,z_2} \}\) and \(\Phi'_h(z_1, z_2, z_3) = h(z_1) - h(z_2) + h(z_3), \) then \(y_0 + \Phi'_h(S') \subset F_2\). Again we have that
\[
|F_2| \geq C \int_{\mathbb{R}^3} \max\{ |z_1|, |z_2|, |z_3| \}^{2k} |z_2 - z_1|^2 |z_3 - z_1|^2 |z_3 - z_2|^2 d\mu(z_1) d\mu(z_2) d\mu(z_3)
\]
We also have an analogue of Lemma 5.2
(i) \(|z_1| \geq (16\pi\nu)^{-\nu} \beta_1^\gamma\), \(|z_2| \geq (16\pi\nu)^{-\nu} \alpha^\gamma\), and \(|z_3| \geq (16\pi\nu)^{-\nu} \beta_2^\gamma\).
We closely follow the argument due to Stovall [30]. (See also [10].) We begin with assuming Lemmas 3.1 and 4.1. (Theorem 1.2 is implied by Lemmas 5.1 and 5.3.)

We consider two cases: $\beta_2 \gtrsim \alpha$ and $\beta_2 \ll \alpha$.

**Case 1 :** $\beta_2 \gtrsim \alpha$. In this case, we follow the proof of Lemma 5.1. Then we obtain

$$|F_2| \gtrsim \alpha^2 \beta_1 \beta_2^3 \gtrsim \alpha^3 \beta_1 \beta_2^2$$

by (i)–(iv) above.

**Case 2 :** $\beta_2 \ll \alpha$. In this case, we assume that $z_1 \in B_{\alpha/2}$. Then $|z_2 - z_1| \geq c \alpha^{1/2} |z_1|^{-b}$. (When $z_1 \notin B_{\alpha/2}$, we start with $|z_2 - z_1| \geq c \alpha^{1/2} |z_2|^{-b}$.)

We consider two balls given by

$$B(z_1) = \{ z : |z - z_1| < \frac{1}{3} c \alpha^{1/2} |z_1|^{-b} \}$$

$$B(z_2) = \{ z : |z - z_2| < \frac{1}{3} c \alpha^{1/2} |z_1|^{-b} \}.$$ 

Then $z_3$ can be located in $B(z_1)$, $B(z_2)$, or $(B(z_1) \cup B(z_2))^c$. If $z_3 \in B(z_1)$, then (iv) can be replaced by

$$|z_3 - z_2| \geq \frac{1}{3} c \alpha^{1/2} |z_1|^{-b}.$$ 

By \((28)\) with $a = k + \frac{b}{3}(1 - \varepsilon_2)$, $b = \frac{b}{3}$, $c = \frac{k}{3} + \frac{b}{3} \varepsilon_2$, and by (i)–(iii) above, it follows that

$$|F_2| \geq C \iint_{S'} \max\{|z_1|, |z_2|, |z_3|\}^{2k} |z_2 - z_1|^2 |z_3 - z_1|^2 |z_3 - z_2|^2 d\mu(z_1)d\mu(z_2)d\mu(z_3)$$

$$\gtrsim \iint_{S'} |z_1|^a |z_2|^b |z_3|^c \alpha^{1/2} \beta_2^2 |z_1|^{-b/2 - a/4 - 1/2 \varepsilon_2} |z_3|^{-2/4 \varepsilon_2} d\mu(z_1)d\mu(z_2)d\mu(z_3)$$

$$\gtrsim \alpha^{1/2} \beta_2^2 \iint_{S'} |z_1|^a |z_2|^b |z_3|^c \alpha^{1/2} \beta_2^2 |z_1|^{-b/2 - a/4 - 1/2 \varepsilon_2} d\mu(z_1)d\mu(z_2)d\mu(z_3)$$

$$\gtrsim \alpha^3 \beta_1 \beta_2^2.$$ 

If $z_3 \in B(z_2)$, then we replace (iii) by $|z_3 - z_1| \geq \frac{1}{3} c \alpha^{1/2} |z_1|^{-b}$. By choosing appropriate $a, b, c$, we obtain $|F_2| \gtrsim \alpha^3 \beta_1 \beta_2^2$ again.

If $z_3 \in (B(z_1) \cup B(z_2))^c$, we have $|z_3 - z_1| \geq \frac{1}{3} c \alpha^{1/2} |z_1|^{-b}$ and $|z_3 - z_2| \geq \frac{1}{3} c \alpha^{1/2} |z_1|^{-b}$. Thus we see that $|F_2| \gtrsim \alpha^3 \beta_1 \beta_2^2$. 

\[\square\]

**Appendix A. Proof of Theorems 1.1 and 1.2**

In this section, for the sake of completeness we present a detailed proof of (1) assuming Lemmas 3.1 and 4.1. (Theorem 1.2 is implied by Lemmas 5.1 and 5.3.) We closely follow the argument due to Stovall [30]. (See also [10].) We begin with
establishing a weaker version \( T : L^{p,u}(\mathbb{R}^d) \to L^{q,\infty}(\mathbb{R}^d) \), which implies the weak type \((p, q)\). By the argument in \([10]\), the weak type \((p, q)\) gives the Lorentz space boundedness of \( T : L^p(\mathbb{R}^d) \to L^{q,p+\epsilon}(\mathbb{R}^d) \) for any \( \epsilon > 0 \).

**Lemma A.1.** Let \( 1 \leq u < q = \frac{d(d+1)}{2(d-1)} \) and \( p = \frac{d+1}{2} \). Also let \( F \) be a (Borel) measurable set and \( f \in L^{p,u} \). Then there exists a constant \( C > 0 \), depending only on \( p, q, u \), such that

\[
\langle Tf, \chi_F \rangle \leq C \|f\|_{L^{p,u}} |F|^\frac{1}{q'}.
\]

Here, \( |F| \) is the Lebesgue measure of \( F \) on \( \mathbb{R}^d \) and \( 1/q' = 1 - 1/q \).

**Proof.** Let us set \( f = \sum_{k=\infty}^{\infty} 2^k \chi_{E_k} \) where the \( E_k \)'s are pairwise disjoint measurable sets in \( \mathbb{R}^d \). Let us assume that

\[
|f|_{L^{p,u}} \simeq \left( \sum_{k=\infty}^{\infty} (2^k |E_k|^\frac{1}{p})^u \right)^\frac{1}{u} = 1.
\]

Then it suffices to show that

\[
\sum_{k \in \mathbb{Z}} 2^k \langle T \chi_{E_k}, \chi_F \rangle \lesssim |F|^\frac{1}{q'}.
\]

Let \( \mathcal{I}(E_k, F) \) denote \( \langle T \chi_{E_k}, \chi_F \rangle \). We classify \( E_k \) depending on the restricted weak type estimate obtained in Section 3 and the normalization of \( f \) above. For nonnegative integers \( m \) and \( n \), we define

\[
\mathcal{K}_0^F = \{ k \in \mathbb{Z} : \mathcal{I}(E_k, F) = 0 \},
\]

\[
\mathcal{K}_m^F = \{ k \in \mathbb{Z} : C 2^{-m-1} |E_k|^\frac{1}{p} |F|^\frac{1}{q'} \leq \mathcal{I}(E_k, F) \leq C 2^{-m} |E_k|^\frac{1}{p} |F|^\frac{1}{q'} \},
\]

\[
\mathcal{K}_{m,n}^F = \{ k \in \mathcal{K}_m^F : 2^{-n-1} < (2^k |E_k|^\frac{1}{p})^u \leq 2^{-n} \}.
\]

Here the constant \( C \) arose from the restricted weak type estimates for \( T \).

We split \( \mathcal{K}_{m,n}^F \) into \( Am \)-separated sets so we define \( \{ \mathcal{K}_{m,n,i}^F \}_{i=1}^{[Am]} \) to be a partition of \( \mathcal{K}_{m,n}^F \). Here \( A \) is some constant that will be determined later. Note that \( |k - k'| \geq Am \) for any \( k, k' \in \mathcal{K}_{m,n,i}^F \). Fix \( i \) and set \( \mathcal{K} = \mathcal{K}_{m,n,i}^F \) for convenience.

By the normalization of \( f \), we have \( \sum_{k \in \mathcal{K}} 2^k |E_k|^{1/p} u \leq 1 \). We may assume that \( \# \mathcal{K} > 0 \). By (32), we see that \( \sum_{k \in \mathcal{K}} 2^{-n} \lesssim 1 \), so we obtain \( \# \mathcal{K} \lesssim 2^n \). Now we will find two upper bounds for \( \sum 2^k \mathcal{I}(E_k, F) \).

Firstly, we obtain that

\[
\sum_{k \in \mathcal{K}} 2^k \mathcal{I}(E_k, F) \lesssim \sum_{k \in \mathcal{K}} 2^k 2^{-m} |E_k|^\frac{1}{p} |F|^\frac{1}{q'}
\]

\[
\lesssim \sum_{k \in \mathcal{K}} 2^{-m} 2^{-\frac{n}{u}} |F|^\frac{1}{q'}
\]

\[
\lesssim (\# \mathcal{K}) 2^{-\frac{n}{u}} 2^{-m} |F|^\frac{1}{q'}
\]

\[
\lesssim 2^{n(1-\frac{1}{p')}-m} |F|^\frac{1}{q'}
\]

since \( \mathcal{K} \) is a subset of (31) and (32).
Secondly, we consider a subset of $F$ related to an average of $T \chi_{E_k}$ on $F$. For each $k \in \mathcal{K}$, let
\[
G_k = \{ x \in F : T \chi_{E_k}(x) \geq \frac{\mathcal{T}(E_k, F)}{|F|} \}.
\]
It follows that
\[
\mathcal{T}(E_k, F \setminus G_k) < \frac{\mathcal{T}(E_k, F)}{2|F|}|F \setminus G_k| \leq \frac{1}{2}\mathcal{T}(E_k, F),
\]
and then $\mathcal{T}(E_k, F) \sim \mathcal{T}(E_k, G_k)$.

Since
\[
C2^{-(m-1)}|E_k|^\frac{1}{p}|F|^\frac{1}{q} \leq \mathcal{T}(E_k, F) \sim \mathcal{T}(E_k, G_k) \leq C|E_k|^\frac{1}{p}|G_k|^\frac{1}{q},
\]
we have
\[
(35) \quad 2^{-mq}|F| \lesssim |G_k|.
\]
Also we can observe that
\[
\sum_{k \in \mathcal{K}} |G_k| = \sum_{k \in \mathcal{K}} \int_F \chi_{G_k} \leq \left( \int_F \left( \sum_{k \in \mathcal{K}} \chi_{G_k} \right)^2 \right)^{\frac{1}{2}} |F|^{\frac{1}{2}}.
\]
It follows that
\[
\left( \frac{1}{|F|} \sum_{k \in \mathcal{K}} |G_k| \right)^2 \leq \frac{1}{|F|} \int_F \left( \sum_{k \in \mathcal{K}} \chi_{G_k} \right)^2 \leq \frac{1}{|F|} \sum_{k \in \mathcal{K}} |G_k| + \frac{1}{|F|} \sum_{k \neq \ell} |G_k \cap G_\ell|.
\]
Therefore we have two cases, i.e.
\[
\frac{1}{|F|} \sum_{k \in \mathcal{K}} |G_k| \lesssim 1 \quad \text{or} \quad \left( \frac{1}{|F|} \sum_{k \in \mathcal{K}} |G_k| \right)^2 \lesssim \frac{1}{|F|} \sum_{k \neq \ell} |G_k \cap G_\ell|.
\]
From the latter inequality and (35), it holds that
\[
(36) \quad |G_k \cap G_\ell| \gtrsim 2^{-2mq}|F|
\]
for some $k \neq \ell$. In fact, one can see that
\[
|F|^{-1} \sum_{k \neq \ell} |G_k \cap G_\ell| \leq |F|^{-1} (\# \mathcal{K})^2 \max_{k \neq \ell} |G_k \cap G_\ell|
\]
and
\[
(|F|^{-1} \sum_{k \in \mathcal{K}} |G_k|)^2 \gtrsim (|F|^{-1} 2^{-mq}|F| (\# \mathcal{K}))^2
\]
by (35). So we obtain $\max_{k \neq \ell} |G_k \cap G_\ell| \gtrsim 2^{-2mq}|F|$.

Our claim is that (36) yields a contradiction by Lemma 4.1 or 5.1. We postpone the proof for a moment.
Then we may assume that $\sum_{k \in \mathcal{X}} |G_k| \lesssim |F|$. By this inequality, the second bound for $\sum 2^k \mathcal{J}(E_k, F)$ can be derived from

$$\sum_{k \in \mathcal{X}} 2^k \mathcal{J}(E_k, G_k) \lesssim \sum_{k \in \mathcal{X}} 2^k |E_k|^{\frac{1}{\eta}} |G_k|^{\frac{1}{\gamma}}$$

(37)

$$\lesssim \left( \sum_{k \in \mathcal{X}} (2^k |E_k|^{\frac{1}{\eta}})^q \right)^{\frac{1}{q}} \left( \sum_{k \in \mathcal{X}} |G_k|^{\frac{1}{\gamma}} \right)^{\frac{1}{\gamma}}$$

$$\lesssim (\# \mathcal{X} 2^{\frac{m}{u}})^{\frac{1}{q}} |F|^{\frac{1}{\gamma}}$$

$$\lesssim 2^{n(\frac{1}{q} - \frac{1}{u})} |F|^{\frac{1}{\gamma}}.$$ 

By (33) and (37), we obtain that

$$\sum_{k \in \mathcal{X}} 2^k \mathcal{J}(E_k, F) \lesssim \min\left(2^{n(1 - \frac{1}{u}) - m}, 2^{n(\frac{1}{q} - \frac{1}{u})}\right) |F|^{\frac{1}{\gamma}}.$$ 

(38)

Since $1 \leq u < q$, we have

$$\sum_{k \in \mathcal{X}} 2^k \mathcal{J}(E_k, F) = \sum_{n=0}^{\infty} \sum_{k \in \mathcal{X}} 2^k \mathcal{J}(E_k, F)$$

(39)

$$\lesssim Am \min_{n=0}^{\infty} \left(2^{n(1 - \frac{1}{u}) - m}, 2^{n(\frac{1}{q} - \frac{1}{u})}\right) |F|^{\frac{1}{\gamma}}$$

$$\lesssim Am \left( \sum_{n > [mq']} 2^{n(\frac{1}{q} - \frac{1}{u})} + \sum_{n \leq [mq']} 2^{n(1 - \frac{1}{u}) - m} \right) |F|^{\frac{1}{\gamma}}$$

$$\lesssim Am 2^{-mq'(\frac{1}{q} - \frac{1}{u})} |F|^{\frac{1}{\gamma}}.$$ 

Let us set $0 \leq \varepsilon < q'(\frac{1}{u} - \frac{1}{q'})$. Since there exists a constant $c$ such that $m \leq 2^em$ for $m \geq c$, we sum (39) over $0 \leq m < \infty$ as follows:

$$\sum_{m=0}^{\infty} \sum_{k \in \mathcal{X}_m} 2^k \mathcal{J}(E_k, F) = \sum_{0 \leq m < c} Am 2^{-mq'(\frac{1}{q} - \frac{1}{u})} |F|^{\frac{1}{\gamma}} + \sum_{m \geq c} A2^{-m(q'(\frac{1}{q} - \frac{1}{u}) - \varepsilon)} |F|^{\frac{1}{\gamma}} \lesssim |F|^{\frac{1}{\gamma}}$$

This gives the desired inequality (29).

Now we turn to proving that a contradiction occurs if we assume (30). Let us denote $G = G_k \cap G_{\ell}$ where $k > l$, $E_1 = E_k$, $E_2 = E_{\ell}$, $\alpha_1 = 2^{-m}|E_1|^{\frac{1}{\gamma}} |F|^{-\frac{1}{\gamma}}$, $\beta = \alpha_1 |G||E_1|^{-1}$, and $\alpha_2 = 2^{-m}|E_2|^{\frac{1}{\gamma}} |F|^{-\frac{1}{\gamma}}$. Then, by Lemma 3.11 or Lemma 5.1, and (36), we obtain that

$$|E_2| \gtrsim \alpha_1^{\frac{d(d+1)}{2}} \left(\frac{\beta}{\alpha_1}\right)^{d-1} \left(\frac{\alpha_2}{\alpha_1}\right)^d$$

$$\gtrsim (2^{-m}|E_1|^{\frac{1}{\gamma}} |F|^{-\frac{1}{\gamma}})^{\frac{d+1}{2}} (2^{-2m q' |F||E_1|^{-1}})^{d-1} (|E_2|^{\frac{1}{\gamma}} |E_1|^{-\frac{1}{\gamma}})^d$$

$$\gtrsim 2^{-m(\frac{d(d+1)}{2}+2(d-1)q')} |E_1|^{\frac{1}{2}} |E_2|^{\frac{d}{2}}.$$
It follows that \(|E_1| \gtrsim 2^{-m(d+1)(q+2q')}\)|\(E_2|\). Together with the fact \(|E_1| \sim 2^{-np/u-kp}\) and \(|E_2| \sim 2^{-np/u-\ell p}\) by (32), we have that \(2^{m(d+1)(q+2q')} \gtrsim 2^{(k-\ell)p}\). Since \(k > \ell\), and the case \(k < \ell\) can be obtained in a similar way, we finally obtain that \(|k-\ell| \lesssim m(d+1)(q+2q')/p|\). Since we can take the constant \(A\) to be sufficiently large, this contradicts our construction of \(K\). This completes the proof. \(\square\)

Recall that \(p = \frac{d+1}{2}\), \(q = \frac{d(d+1)}{2(d-1)}\). The following lemma implies Theorem 1.1 and Theorem 1.2. In fact, the other three cases \((u \leq p < q, p \leq u < q \leq v,\) and \(u \leq p < q \leq v)\) follow from the next lemma and the fact that \(\|f\|_{L^p,u} \leq \|f\|_{L^{p,u'}}\) whenever \(u' \leq u\).

**Lemma A.2.** Let \(1 \leq p \leq u < v \leq q \leq \infty\). For \(f \in L^{p,u}\) there exists a constant \(C > 0\) such that

\[
\|Tf\|_{L^{p,u}(\mathbb{R}^{2d})} \leq C\|f\|_{L^{p,v}(\mathbb{R}^{2d})}.
\]

**Proof.** Let \(f = \sum_k 2^k \chi_{E_k}\) and \(g = \sum_j 2^j \chi_{F_j}\) where \(E_k\) and \(F_j\) are pairwise disjoint measurable sets. We also assume that

\[
\|f\|_{L^{p,u}} \sim \left( \sum_{k = -\infty}^{\infty} (2^k |E_k|^{1/p})^u \right)^{1/u} = 1
\]

and

\[
\|g\|_{L^{q,v'}} \sim \left( \sum_{j = -\infty}^{\infty} (2^j |F_j|^{1/q'})^{v'} \right)^{1/v'} = 1.
\]

Then it suffices to show that

\[
\sum_{j \in \mathbb{Z}} 2^j \sum_{k \in \mathbb{Z}} 2^k T(E_k, F_j) \lesssim 1,
\]

where \(T(E_k, F_j) = \langle T \chi_{E_k}, \chi_{F_j}\rangle\).

Fix a nonnegative integer \(m\), and we set \(K^{F_j}_m\) as above and

\[
K^{F_j}_m = \{k \in \mathbb{Z} : c2^{-m-1}|E_k|^{1/p}|F_j|^{1/q'} < \mathcal{I}(E_k, F_j) \leq c2^{-m}|E_k|^{1/p}|F_j|^{1/q'}\}
\]

for some constant \(c > 0\). Instead of \(K^{F_j}_{m,n}\), we define

\[
\mathcal{I}_n = \{j \in \mathbb{Z} : 2^{-n-1} < (2^j |F_j|^{1/q'})^{v'} \leq 2^{-n}\}
\]

from the normalization of \(g\). We will divide \(\mathcal{I}_n\) into \([A'm]\) subsets, where \(A'\) will be specified later. For \(1 \leq i \leq [A'm]\), let us denote by \(\mathcal{I}^{m,i}_{n}\) an \(A'm\)-separated subset of \(\mathcal{I}_n\). Let \(\mathcal{J} = \mathcal{I}^{m,i}_{n}\) for convenience. Using the method used to get \#\(K\), one can see that \#\(\mathcal{J}\) \(\leq 2^n\) for each \(\mathcal{J}\).

By (39), we have that

\[
\sum_{j \in \mathcal{J}} 2^j \sum_{k \in K^{F_j}_m} 2^k \mathcal{I}(E_k, F_j) \lesssim Am 2^{-mq'(1-\frac{1}{q})} \sum_{j \in \mathcal{J}} 2^j |F_j|^{1/q'}
\]

\[
\lesssim Am 2^{-mq'(1-\frac{1}{q})} 2^{-n/v'} \#\mathcal{J}
\]

\[
\lesssim Am 2^{-mq'(1-\frac{1}{q})} 2^n(1-\frac{1}{v'}).
\]
For the second bound, we consider

\[(41)\quad E_{k,j} = \{ x \in E_k : T^* \chi_{F_j}(x) \geq \frac{\mathcal{T}(E_k, F_j)}{2|E_k|} \},\]

which implies that

\[\mathcal{T}(E_{k \setminus E_{k,j}}, F_j) \leq \langle \chi_{E_{k \setminus E_{k,j}}}, T^* \chi_{F_j} \rangle < \frac{1}{2} \mathcal{T}(E_{k,j}, F_j).\]

Therefore, \(\mathcal{T}(E_k, F_j) \sim \mathcal{T}(E_{k,j}, F_j)\). It follows that \(2^{-mp}|E_k| \lesssim |E_{k,j}|\), which is similar to (35). By the same argument, we get two cases:

\[\frac{1}{|E_k|} \sum_{j \in J} |E_{k,j}| \lesssim 1 \quad \text{or} \quad \left( \frac{1}{|E_k|} \sum_{j \in J} |E_{k,j}| \right)^2 \lesssim \frac{1}{|E_k|} \sum_{j \neq l} |E_{k,j} \cap E_{k,l}|.\]

Again, we may assume that \(\sum_{j \in J} |E_{k,j}| \leq |E_k|\) for each \(k \in \mathcal{K}_m\). In fact, the second inequality and \(2^{-mp}|E_k| \lesssim |E_{k,j}|\) lead to \(|E_{k,j} \cap E_{k,l}| \gtrsim 2^{-mp}|E_k|\), which implies a contradiction to the definition of \(J\). We will check this at the end.

By the assumption \(u/p \geq 1\), it follows that

\[\sum_{j \in J} \left( \frac{1}{|E_k|} \sum_{j \in \delta} |E_{k,j}| \right)^{\frac{u}{p}} \leq \left( \sum_{j \in \delta} |E_{k,j}| \right)^{\frac{u}{p}} \leq |E_k|^{\frac{u}{p}},\]

and then

\[(42)\quad \sum_{j \in \delta} \sum_{k \in \mathcal{K}_m F_j} 2^{nu} |E_{k,j}|^{\frac{u}{p}} \leq \sum_{k \in Z} 2^{nu} |E_k|^{\frac{u}{p}}.\]

By Lemma (A.1), H"older’s inequality, and (42), we thus obtain that

\[\sum_{j \in \delta} 2^{n j} \sum_{k \in \mathcal{K}_m F_j} 2^{k \mathcal{T}(E_k, F_j)} \sim \sum_{j \in \delta} 2^{n j} \sum_{k \in \mathcal{K}_m F_j} 2^{k \mathcal{T}(E_{k,j}, F_j)} \]

\[\lesssim \sum_{j \in \delta} 2^{n j} \left( \sum_{k \in \mathcal{K}_m F_j} \left( 2^k |E_{k,j}|^{\frac{1}{p}} \right)^u \right)^{\frac{1}{u}} |F_j|^{\frac{1}{q'}}.\]

\[\lesssim \left( \sum_{j \in \delta} \sum_{k \in \mathcal{K}_m F_j} 2^{ku} |E_{k,j}|^{\frac{u}{p}} \right)^{\frac{1}{u}} \left( \sum_{j \in \delta} (2^j |F_j|^{\frac{1}{q'}})^u ' \right)^{\frac{1}{u'}} \]

\[\lesssim \left( \sum_{k \in Z} 2^{ku} |E_k|^{\frac{u}{p}} \right)^{\frac{1}{u}} (\# \mathcal{J} 2^{-\frac{n u'}{u'}})^{\frac{1}{u'}} \]

\[\lesssim 2^{n(\frac{1}{u} - \frac{1}{u'})}.\]
Now, let $a = q'(1/u - 1/q)$. Then we have

$$
\langle T f, g \rangle = \sum_{j,k} 2^j 2^k \mathcal{T}(E_k, F_j)
$$

$$
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A' \sum_{j=0}^{\infty} \sum_{k \in \mathcal{K}'_m} 2^j 2^k \mathcal{T}(E_k, F_j)
$$

$$
\lesssim \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=1}^{A' m} \min\{ A \mu 2^{-am} 2^{n(1-\frac{1}{\gamma'})}, 2^n (\frac{1}{\gamma' - \frac{1}{\gamma}}) \}
$$

$$
\lesssim \sum_{m=0}^{\infty} \left( \sum_{n > [am]} A' m^2 2^{-am + n(1-\frac{1}{\gamma'})} + \sum_{n \leq [am]} A' m 2^n (\frac{1}{\gamma' - \frac{1}{\gamma}}) \right)
$$

$$
\lesssim \sum_{m=0}^{\infty} A' m^2 2^{am (\frac{1}{\gamma'} - \frac{1}{\gamma}) m}
$$

$$
\lesssim \sum_{m \leq [c]} A' m^2 2^{am (\frac{1}{\gamma'} - \frac{1}{\gamma}) m} + \sum_{m \leq [c]} A' 2^{am (\frac{1}{\gamma'} - \frac{1}{\gamma}) m + \epsilon m}
$$

$$
\lesssim 1
$$

where we can take a constant $c > 0$ such that $m^2 \leq 2^c m$ when $m \geq c$ and $0 \leq \epsilon < a \mu (1/\gamma' - 1/u')$.

Now we show that $|E_{k,j} \cap E_{k,l}| \gtrsim 2^{-2mp}|E_k|$ leads to a contradiction in view of Lemma 4.1. Let us assume $j > l$ and $m \geq 1$. (The case $m = 0$ is excluded in the construction of the set $\mathcal{J}$.) To apply Lemma 4.1, let us set $E := E_{k,j} \cap E_{k,l}$, $F_1 := F_j$, $F_2 := F_l$. By (41) and by the definition of $\mathcal{K}'_m$, we have that

$$
T^x \chi_{F_1}(x) \geq \mathcal{T}(E_k, F_1) \geq 2^{-m-2} |E_k| \frac{1}{\gamma'} |F_1|^{-\frac{1}{\gamma'}} \text{ for } x \in E,
$$

where $i = 1, 2$. Let us set $\beta_i = 2^{-3m} |E_k|^{-1/p'} |F_i|^{1/q'}$ and $\alpha_i := \beta_i |E||F_i|^{-1}$ for $i = 1, 2$, which satisfy the assumption of Lemma 4.1. (Note that $2^{-m-2} \geq 2^{-3m}$ for $m \geq 1$.)

Then, by Lemma 4.1 and by the assumption $|E| = |E_{k,j} \cap E_{k,l}| \gtrsim 2^{-2mp}|E_k|$, we obtain that

$$
|F_2| \gtrsim \alpha_1^2 \beta_1 \beta_1^{s_1} \beta_2^{s_2}
$$

$$
\gtrsim (2^{-3m} |E_k|^{-\frac{1}{\gamma'}} |F_1|^{-\frac{1}{\gamma'}})^{r_1 + s_1} (2^{-3m} |E_k|^{-\frac{1}{\gamma'}} |F_2|^{-\frac{1}{\gamma'}})^{r_2 + s_2} |E|^{r_1 + r_2} |F_1|^{-r_1} |F_2|^{-r_2}
$$

$$
\gtrsim 2^{-\frac{md(d+1)}{2} - mpd(d-1)} |E_k|^{-\frac{d(d+1)}{2p'q} + \frac{d(d-1)}{2}} |F_1|^{\frac{r_1 + r_1 s_1}{q}} |F_2|^{\frac{r_2 + r_2 s_2}{q} - r_2}
$$

$$
= 2^{-\frac{md(d+1)(d+2)}{2}} |F_1|^{\frac{r_2}{q} - \frac{r_2 s_2}{q}} |F_2|^{\frac{r_2}{q} - \frac{r_2}{q} - 1},
$$

using the conditions $r_1 + r_2 = d(d-1)/2$, $s_1 + s_2 = d$ in Lemma 4.1 and $p = (d+1)/2$, $q = d(d+1)/2(d-1)$.

It follows that

$$
|F_1|^{\frac{r_2}{q} - \frac{r_2 s_2}{q} - 1} \gtrsim 2^{-\frac{md(d+1)(d+2)}{2}} |F_2|^{\frac{r_2}{q} - \frac{r_2}{q} - 1}.
$$
Since we have $|F_1| \sim 2^{-nq'/v'-jd'}$ and $|F_2| \sim 2^{-nq'/v'-jd'}$ by (10), we have that $2^{Cm} \gtrsim 2^{-l}$. Here, $C = d(d+1)(d+2)/(2q'(s_2/q' - r_2/q + 1))$. Since the case $l < j$ can be obtained in a similar way, we finally obtain that $|j - l| \lesssim Cm$. If we take the constant $A'$ to be sufficiently large, this contradicts our construction of $J$. □

**Appendix B. The necessary conditions**

We use the notation and terminology in [9, 30]. We will only treat the nondegenerate case when $h(z) = (z, z^2, \ldots, z^{d-1}, z^d)$, which is an analogue of the moment curve in the real case. Then $ds(z) \sim d\mu(z) = dudv$ and we may assume that $A f(x) = \int f(x - h(z)) d\mu(z)$.

Let $D_r$ be an anisotropic scaling in $\mathbb{R}^d$ given by

$$D_r(x_1, y_1, x_2, y_2, \ldots, x_d, y_d) = (rx_1, ry_1, r^2 x_2, r^2 y_2, \ldots, r^d x_d, r^d y_d).$$

We also define a ball $B(x, \varepsilon)$ of radius $\varepsilon$ centered at $x$ in $\mathbb{R}^d$. Then $B(x, \varepsilon) - h(z)$ for $z \in D$ is an $\varepsilon$-neighborhood of $-h(z)$, translated by $x$. Hence let us set $N(x, \varepsilon) := B(x, \varepsilon) - h(z)$ for $z \in D$ so that $y - h(z) \in N(x, \varepsilon)$ whenever $y \in B(x, \varepsilon)$.

First we show that the restricted weak type $(p, q)$ of $A$ may hold only for $(p, q) \in \mathcal{R}$, where $\mathcal{R}$ is a trapezoid with vertices $(0, 0), (1, 1), (1/p_d, 1/q_d)$, and $(1-1/q_d, 1-1/p_d)$.

Let $0 < r < 1$ be a small constant. We consider $D_r N(0, 1)$ in $\mathbb{R}^d$. For each $y \in \mathbb{R}^d$, one can see that $y - h(rz) = y - D_r h(z) \in D_r N(0, 1)$ whenever $y \in D_r B(0, 1)$ and $|z| < r^{-1}$. Hence it follows from the dilation $z \mapsto rz$ that

$$A \chi_{D_r N(0, 1)}(y) = \int_{|z| < 1} \chi_{D_r N(0, 1)}(y - h(z)) d\mu(z)$$

$$= r^2 \int_{|z| < r^{-1}} \chi_{D_r N(0, 1)}(y - h(rz)) d\mu(z)$$

$$\gtrsim r^2 \chi_{D_r B(0, 1)}(y),$$

which implies $\|A \chi_{D_r N(0, 1)}\|_{L^{q, \infty}} \gtrsim r^{2d(d+1)/q + 1}$. Since $|D_r N(0, 1)|^{1/p} \sim r^{d(d+1)/p}$, we obtain $r^{2d(d+1)/q + 1} \lesssim r^{d(d+1)/p}$ from the restricted weak type $(p, q)$ for $A$. Since $0 < r < 1$ we get $1 + d(d+1)/(2q) \geq d(d+1)/(2p)$.

Now for $0 < \varepsilon < 1$ we consider $B(0, \varepsilon)$ and $N(0, \varepsilon)$ so that $\|A \chi_{N(0, \varepsilon)}\|_{L^{q, \infty}} \gtrsim |B(0, \varepsilon)|^{1/q} = \varepsilon^{2d/q}$. Also we have $|N(0, \varepsilon)| \sim \varepsilon^{2(d-1)}$. By the restricted weak type $(p, q)$ for $\mathcal{A}$, we get $\varepsilon^{2d/q} \lesssim \varepsilon^{2(d-1)/p}$. Hence, $d/q \geq (d-1)/p$, and duality gives $1 + (d-1)/q \geq d/p$.

Finally, the condition $q \geq p$ follows by the fact that $\mathcal{A}$ is translation invariant. In fact, if any nonzero linear operator which is translation invariant is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, then $q \geq p$ is necessary. (See Section 2.5.3 in [19].)

As a result, we can see that $\mathcal{A}$ is of restricted weak type $(p, q)$ only if $(p, q) \in \mathcal{R}$.

**Lorentz space estimates.** Now we show that if (11) holds, then $u \leq q_d, p_d \leq v$, and $u \leq v$, where $(p_d, q_d) = \left( \frac{d+1}{2}, \frac{d(d+1)}{2(d-1)} \right)$.

- $u \leq v$: For a positive integer $M$, we choose $0 < \varepsilon < 1$ and $x_j$ for $j = 1, \ldots, M$ such that $B(x_j, \varepsilon^j)$ are pairwise disjoint and $N(x_j, \varepsilon^j)$ are also pairwise disjoint.
Let \( f(y) = \sum_{j=1}^{M} e^{-2(d-1)j/pd} \chi_{N(x_j, \varepsilon^j)}(y) \). Then one can see that
\[
\|f\|_{L^{p,d,u}} \sim \left( \sum_{j=1}^{M} \varepsilon^{-2(d-1)j/pd} |N(x_j, \varepsilon^j)|^{u/pd} \right)^{1/u} \sim M^{\frac{1}{u}}.
\]
Since \( \mathcal{A}_{\chi_{N(x_j, \varepsilon^j)}}(y) \gtrsim \chi_{B(x_j, \varepsilon^j)}(y) \) and \( B(x_j, \varepsilon^j) \) are pairwise disjoint, we obtain that
\[
\|\mathcal{A}f\|_{L^{q,d,u}} \gtrsim \left( \sum_{j=1}^{M} \varepsilon^{-2(d-1)j/pd} \chi_{B(x_j, \varepsilon^j)}\| \right)_{L^{q,d,u}} \sim \left( \sum_{j=1}^{M} \varepsilon^{-2(d-1)j/pd} |B(x_j, \varepsilon^j)|^{u/qd} \right)^{1/u} = M^{\frac{1}{u'}}.
\]
For the last equality we use the fact that \( (pd, qd) \) satisfies \( d/qd = (d-1)/pd \). Therefore \( u \leq v \) gives \( M^{1/u} \leq M^{1/u'} \) for any positive integer \( M \geq 2 \). Hence we obtain \( u \leq v \).

- \( u \leq qd \): Let \( N_{\varepsilon,r}(x) = D_r N(x, \varepsilon) \) and \( B_{\varepsilon,r}(x) = D_r B(x, \varepsilon) \). We begin with observing \( \mathcal{A}_{\chi_{N_{\varepsilon,r}}}(y) \gtrsim r^2 \chi_{B_{\varepsilon,r}}(y) \) for some \( x \) and \( 0 < \varepsilon, r < 1 \). Since \( h(rz) = D_r h(z) \), we see that
\[
\mathcal{A}_{\chi_{N_{\varepsilon,r}}}(y) = \int_{|z|<1} \chi_{N_{\varepsilon,r}}(y - h(z))d\mu(z) = r^2 \int_{|z|<r^{-1}} \chi_{N_{\varepsilon,r}}(y - h(rz))d\mu(z) = r^2 \int_{|z|<r^{-1}} \chi_{N_{\varepsilon,r}}(D_r^{-1}y - h(z))d\mu(z) \gtrsim r^2 \chi_{B_{\varepsilon,r}}(y).
\]

Let us set \( \varepsilon_j = 2^{-(M+j)p_d}, r_j = 2^{-j} \), and choose \( x_j \) so that \( B_{\varepsilon_j,r_j}(x_j) \) are pairwise disjoint and also \( N_{\varepsilon_j,r_j}(x_j) \) are pairwise disjoint. (This choice of \( \varepsilon_j \) and \( r_j \) is borrowed from Section 3 in [30].) If we set \( f = \sum_{j=1}^{M} 2^{2j} \chi_{N_{\varepsilon_j,r_j}} \), it follows that \( \|f\|_{L^{p,d,u}} \sim M^{1/u} 2^{-2M(d-1)} \). Also, we have that
\[
\|\mathcal{A}f\|_{L^{q,d,u}} \gtrsim \left( \sum_{j=1}^{M} 2^{2j} 2^{-2j} \chi_{B_{\varepsilon_j,r_j}}(x_j) \right)_{L^{q,d,u}} = \left( \bigcup_{j=1}^{M} B_{\varepsilon_j,r_j}(x_j) \right)^{\frac{1}{qd}} = M^{\frac{1}{u'}} 2^{-\frac{2dMpd}{qd}}.
\]
Thus (1) implies that \( M^{1/u} 2^{-2dMpd/qd} \leq M^{1/u'} 2^{-2M(d-1)} \). Since \( 2^{dMpd}(d/qd-(d-1)/pd) = 1 \), we get \( u \leq qd \) whenever \( M > 1 \). Note that if \( M = 1 \), one can obtain \( 1 + d(d + 1)/(2q) \geq d(d + 1)/(2p) \).

- \( p_d \leq v \): In this case, we make use of \( \mathcal{A}^* \chi_{B_{\varepsilon,r}} \gtrsim \varepsilon^2 \gamma^2 \) on \( N_{\varepsilon,r} \), which can be obtained by the same calculation as above. For a positive integer \( M > 1 \) and \(-M \leq j \leq -1 \), we set \( \varepsilon_j = 2^{-(j+M+1)p_d}, r_j = 2^{j/qd}/qd \), and choose \( x_j \) so that \( B_{\varepsilon_j,r_j}(x_j) \) are pairwise disjoint and \( N_{\varepsilon_j,r_j}(x_j) \) are also pairwise disjoint.

Since \( 2^{2j} |B_{\varepsilon_j,r_j}(x_j)|^{1/qd} = 2^{2j} 2^{-2d(j+M)+d(d+1)/qd} = 2^{-2dM} \) for \( qd = \frac{d(d+1)}{2(d-1)} \). If we set \( f = \sum_{j=-1}^{M} 2^{2j} \chi_{E_j} \), it follows that \( \|f\|_{L^{q',v'}} \sim M^{1/u'} 2^{-2dM} \).
Also, we have that

\[ \|A^* f\|_{L_p^d,u'} \gtrsim \sum_{j=-1}^{M} 2^{2j} 2^{-2(j+M)q_d(1+2jq_d/q_d)} \chi_{N_{s_j},r_j(x_j)} \|_{L_p^d,u'} \]

\[ = 2^{-2Mq_d} \sum_{j=1}^{M} N_{s_j,r_j(x_j)} \|_{L_p^d,u'} = \frac{1}{p_d} 2^{-2Mq_d} \frac{2M(d-1)q_d}{p_d} \chi_{N_j,r_j(x_j)} \]

If (1) holds, then we get

\[ M \frac{1}{p_d} 2^{-2Mq_d} \frac{2M(d-1)q_d}{p_d} \chi_{N_j,r_j(x_j)} \|_{L_p^d,u'} \lesssim M^{1/v'} 2^{-2dM} \]

Since \((p_d,q_d) = (\frac{d+1}{2}, \frac{d(d+1)}{2(d-1)})\) satisfies \(d = \frac{d-1}{p_d}\) or \(1 + \frac{d-1}{p_d} = \frac{d}{q_d}\), it follows that \(v' \leq p_d\) for any positive integer \(M > 1\).

**References**

[1] J.-G. Bak and S. Ham, *Restriction of the Fourier transform to some complex curves*, J. Math. Anal. Appl. **409** (2014), 1107–1127.
[2] J.-G. Bak, D. Oberlin, and A. Seeger, *Two endpoint bounds for generalized Radon transforms in the plane*, Rev. Mat. Iberoamericana. **18** (2002), 231–247.
[3] J.-G. Bak, D. Oberlin, and A. Seeger, *Restriction of Fourier transforms to curves and related oscillatory integrals*, Amer. J. Math. **131** (2009), no. 2, 277–311.
[4] J.-G. Bak, D. Oberlin, and A. Seeger, *Restriction of Fourier transforms to curves, II: Some classes with vanishing torsion*, J. Austral. Math. Soc. **85** (2008), 1–28.
[5] J.-G. Bak, D. Oberlin, and A. Seeger, *Restriction of Fourier transforms to curves: An endpoint estimate with affine arclength measure*, J. Reine Angew. Math. **682** (2013), 167–205.
[6] J. Bennett and A. Seeger, *The Fourier extension operator on large spheres and related oscillatory integrals*, Proc. London Math. Soc. **98** (3) (2009), 45–82.
[7] Y. Choi, *Convolution operators with the affine arclength measure on plane curves*, J. Korean Math. Soc. **36** (1) (1999), 193–207.
[8] Y. Choi, *The \(L^p-L^q\) mapping properties of convolution operators with the affine arclength measure on space curves*, J. Aust. Math. Soc. **75** (2) (2003), 247–261.
[9] M. Christ, *Convolution, curvature, and combinatorics: a case study*, Int. Math. Res. Not. **19** (1998), 1033–1048.
[10] M. Christ, *Quasi-extremals for a Radon-like transform*, preprint.
[11] S. Dendrinos, N. Laghi, and J. Wright, *Universal \(L^p\) improving for averages along polynomial curves in low dimensions*, J. Funct. Anal. **257** (2009), 1355–1378.
[12] S. Dendrinos, D. Müller, *Uniform estimates for the local restriction of the Fourier transform to curves*, Trans. Amer. Math. Soc. **365** (2013) 3477–3492.
[13] S. Dendrinos and B. Stovall, *Uniform bounds for convolution and restricted X-ray transforms along degenerate curves*, J. Funct. Anal. **268** (2015), 585–633.
[14] S. Dendrinos and J. Wright, *Fourier restriction to polynomial curves I: a geometric inequality*, Amer. J. Math. **132** (4) (2010) 1031-1076.
[15] S. Drury, *Degenerate curves and harmonic analysis*, Math. Proc. Camb. Phil. Soc. **108** (1990), 89–96.
[16] S. Drury and K. Guo, *Convolution estimates related to surfaces of half the ambient dimension*, Math. Proc. Camb. Phil. Soc. **110** (1991), 151–159.
[17] S. Drury, B. Marshall, *Fourier restriction theorems for curves with affine and Euclidean arclengths*, Math. Proc. Cambridge Philos. Soc. **97** (1985), 111–125.
[18] S. Drury, B. Marshall, *Fourier restriction theorems for degenerate curves*, Math. Proc. Cambridge Philos. Soc. **101** (1987), 541–553.
[19] L. Grafakos, Classical Fourier analysis, Springer-Verlag, Graduate Texts in Mathematics 249, (2008).
[20] P. Gressman, Convolution and fractional integration with measures on homogeneous curves in \( \mathbb{R}^n \), Math. Res. Lett. 11 (2004), 869–881.
[21] W. Littman, \( L^p-L^q \) estimates for singular integral operators arising from hyperbolic equations, Partial differential equations (Berkeley, Calif., 1971), Proc. Sympos. Pure Math., 23, Amer. Math. Soc., Providence, (1973), 479–481.
[22] D. Oberlin, Convolution estimates for some measures on curves, Proc. Amer. Math. Soc. 99 (1987), 56–60.
[23] D. Oberlin, A convolution estimate for a measure on a curve in \( \mathbb{R}^4 \), Proc. Amer. Math. Soc. 125 (5), (1997), 1355-1361.
[24] D. Oberlin, Convolution with measures on curves in \( \mathbb{R}^3 \), Canad. Math. Bull. 41 (4), (1998), 478-480.
[25] D. Oberlin, A convolution estimate for a measure on a curve in \( \mathbb{R}^4 \) II, Proc. Amer. Math. Soc. 127 (1), (1999), 217-221.
[26] D. Oberlin, Convolution with measures on polynomial curves, Math. Scand. 90 (1) (2002), 126–138.
[27] D. Oberlin, Some convolution inequalities and their applications, Trans. Amer. Math. Soc. 354 (2002), no. 6, 2541–2556.
[28] D. Oberlin, Convolution with measures on flat curves in low dimensions, J. Funct. Anal. 259 (7) (2010), 1799–1815.
[29] Y. Pan, \( L^p \)-improving properties for some measures supported on curves, Math. Scand. 78 (1) (1996), 121-132.
[30] B. Stovall, Endpoint bound for a generalized Radon transform, J. London Math. Soc. 80 (2) (2009), 357–374.
[31] B. Stovall, Endpoint \( L^p \rightarrow L^q \) bounds for integration along certain polynomial curves, J. Funct. Anal. 259 (12) (2010) 3205-3229.
[32] B. Stovall, Uniform estimates for Fourier restriction to polynomial curves in \( \mathbb{R}^d \), to appear in Amer. J. Math.

Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Korea
E-mail address: nuki@postech.ac.kr

School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Korea
E-mail address: hamsh@kias.re.kr