A Remark on Stress of a Spatially Uniform Dislocation Density Field

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1 Introduction

1.1 Main objective

In an interesting recent paper [1], Acharya proved that the stress produced by a spatially uniform dislocation density field in a body comprising a nonlinear elastic material may fail to vanish under no loads. The class of counterexamples constructed in [1] is essentially 2-dimensional: it works with the subgroup $SO(2) \oplus \langle \text{Id} \rangle \subset O(3)$. The objective of this note is to extend Acharya’s result in [1] to the whole $O(3)$, subject to an additional structural condition and less regularity assumptions.

1.2 Nomenclature

Throughout $\Omega \subset \mathbb{R}^3$ is a simply-connected bounded domain with outward unit normal vectorfield $\mathbf{n}$. The group of $3 \times 3$ orthogonal matrices is denoted by $O(3)$; i.e., $M \in O(3)$ if and only if $M^\top = M^{-1}$. The special orthogonal group $SO(2)$ consists of the matrices in $O(2)$ with determinant 1. The matrix field $F : \Omega \to \mathfrak{gl}(3; \mathbb{R})$ designates the elastic distortion, and $W := F^{-1}$ whenever $F$ is invertible. $T : \mathfrak{gl}(3; \mathbb{R}) \to O(3)$ denotes a generally nonlinear, frame-indifferent stress response function, where $\mathfrak{gl}(3; \mathbb{R})$ is the space of $3 \times 3$ matrices. The composition $T(F)$ is the symmetric Cauchy stress field applied to the configuration of body $\Omega$. The constant matrix $\alpha \in \mathfrak{gl}(3; \mathbb{R})$ denotes the dislocation density distribution specified on $\Omega$.

For a matrix field $M = \{M_{ij}\}_{1 \leq i,j \leq m} : \Omega \to \mathfrak{gl}(3; \mathbb{R})$, its curl and divergence are understood in the row-wise sense. In local co-ordinates it means the following: for each $i, j, k, \ell \in \{1, 2, 3\}$, curl $M$ is the matrix field

$$[\text{curl } M]_j := \nabla_k M_{ij}^\ell - \nabla_\ell M_{ik}^j$$

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where \((k, \ell, j)\) is an even permutation of \((1, 2, 3)\), and \(\text{div } M\) is the vectorfield

\[
[\text{div } M]^j = \sum_j \nabla_j M^j.
\]

Moreover, recall that the \textit{Leray projector} is the \(L^2\)-orthogonal projection \(P : L^2(\mathbb{R}^3; \mathbb{R}^3) \to L^2(\mathbb{R}^3; \mathbb{R}^3)\) sending a vectorfield in \(\mathbb{R}^3\) to its divergence-free part. On \(\mathbb{R}^3\) it can be defined via Fourier transform:

\[
\hat{P}v(\xi) := \left(\text{Id} - \frac{\xi \otimes \xi}{|\xi|^2}\right)\hat{v}(\xi).
\]

The Leray projector plays an important role in the mathematical analysis of incompressible Navier–Stokes equations. See, e.g., Constantin–Foias [4] and Temam [11]. For a matrix field \(M\), \(P(M)\) is again understood in the row-wise sense. We denote by

\[
\mathcal{Q} := \text{Id} - P
\]

the complementary projection of \(P\).

### 1.3 Differential Equations

In the above setting, the governing equations for the internal stress field in the body subject to the Cauchy stress field \(T(F)\) was derived by Willis [12]. See also Eq. (3) in [1]:

\[
\begin{cases}
\text{curl } W = -\alpha & \text{in } \Omega, \\
\text{div } (T(F)) = 0 & \text{in } \Omega, \\
T(F) \cdot n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Here \(\alpha\) is a prescribed constant matrix. This PDE system is considered under the following assumption:

\textbf{Assumption 1.1} \(T(F) = 0\) if and only if \(F\) takes values in \(O(3)\).

Acharya proved in [1] the following result:

\textbf{Theorem 1.2} Let \(\Omega, W, T, F, \text{ and } n\) be as in Sect. 1.2 above. Let \(\alpha\) be any nonzero constant matrix. Then, under Assumption 1.1, there does not exist \(\theta \in C^2(\Omega; \mathbb{R})\) such that \(W = R_\theta\) is a solution for Eq. (1); here

\[
R_\theta := \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The proof of Theorem 1.2 in [1] follows from concrete computations: with ansatz (2), Eq. (1) reduces to a system of algebraic equations for \(\sin \theta\) and \(\cos \theta\) that is not soluble unless \(\alpha \equiv 0\).
1.4 Mechanics

In terminologies of continuum mechanics, Theorem 1.2 means that in the \textit{nonlinear} regime and under Assumption 1.1, there is no $C^2$-stress-free spatially uniform (with respect to Cartesian co-ordinates) dislocation density field, unless such uniform dislocation density is everywhere vanishing.

Various dislocation distributions producing no stress have been observed in the limit of continuum elastic descriptions; \textit{cf.} Mura [10], Head–Howison–Ockendon–Tighe [7], Yavari–Goriely [13], etc. On the other hand, we refer to Acharya–Knops–Sivaloganathan [2] for the linear theory of nonsingular dislocations, which is in sharp contrast with the nonlinear theory. This is the background for our work.

This note aligns with the themes of the classical works by Kröner–Seeger [8], Willis [12], and many others: the key question to be addressed is that \textit{“given the dislocation density, determine the state of stress of a body”}. In this setting, the tensor $\alpha$ should be understood as the \textit{“true dislocation density tensor”} in [12]. More precisely, in terms of notations therein, let $\{X_I\}$ be coordinates on the initial/reference configuration, let $\{x_j\}$ be coordinates on the deformed/current configuration, and define $A$ ($=\hat{W}$ in our paper) by $dX^I = \sum_j A^I_j \, dx^j$. Then, as stated in Eq. (3.3) in Willis and the ensuing remarks, $\alpha$ has components

$$
\alpha^{rI} = \sum_k \sum_s \epsilon^{rks} \frac{\partial A^I_k}{\partial x^s},
$$

where, “if the index $r$ is strangled, $\alpha^{rI}$ is a vector in the (initial) $X$-space”. Thus, that $\alpha$ equals to a constant matrix field corresponds to the uniformity of the dislocation density \textit{in the reference configuration}. In addition, as proposed by Willis in [12, p.177, §4], the true dislocation density $\alpha$ is an easily measurable physical quantity, “since it would involve (in principle) only counting the number of dislocation lines, and measuring their Burgers vectors approximately”. Meanwhile, the author frankly admits that the physical significance of the constancy of $\alpha$ is yet to be exploited. It appears to be a natural assumption near the dislocation core.

2 Main Result

The goal of this note is to extend Acharya’s Theorem 1.2 in order to include more general form of $\hat{W}$ and assuming lower regularity requirements. At the moment we are not able to generalise to all of $\mathcal{O}(3)$-valued $\hat{W}$; an additional structural condition is needed —

\textbf{Assumption 2.1} $\Omega(\hat{W})$ is a $\mathcal{O}(3)$-valued matrix field, where $\Omega = \text{Id} - \mathcal{P}$ is the complement of Leray projector as in Sect. 1.2.

Our main result is the following:

\textbf{Theorem 2.2} Let $\Omega$, $\hat{W}$, $\alpha$, $T$, $F$, and $n$ be as in Sect. 1.2. Under Assumptions 1.1 and 2.1, Eq. (1) has no solution $\hat{W}$ in $C^1(\Omega; \mathcal{O}(3))$ unless the constant matrix field $\alpha \equiv 0$.

\textbf{Remark 2.3} $\hat{W} = \mathcal{R}_\alpha$ in Theorem 1.2 (Eq. (2)) satisfies Assumption 2.1. Direct computation in polar co-ordinates shows that $\text{div} \mathcal{R}_\alpha \equiv 0$; hence $\Omega \hat{W} \equiv \mathcal{R}_\alpha$, which is indeed $\mathcal{O}(3)$-valued.
3 Proof

Proof of Theorem 2.2 Throughout the proof we denote by \( W^1, W^2, W^3 \) the row-vector fields of the matrix field \( W \). Also, let \( \widetilde{\alpha} \) be the field of differential 2-forms dual to \( \alpha \), namely that
\[
\widetilde{\alpha}^i = \alpha_1^i \, dx^2 \wedge dx^3 + \alpha_2^i \, dx^3 \wedge dx^1 + \alpha_3^i \, dx^1 \wedge dx^2.
\]
Thus, by Hodge duality, the first equation in Eq. (1) becomes
\[
dW^i = -\widetilde{\alpha}^i \quad \text{for each } i \in \{1, 2, 3\},
\]
which is an identity of 2-forms. Here and hereafter, we identify \( W^i \) with a 1-form (not relabelled).

Under Assumption 1.1 the second and the third equations in Eq. (1) are satisfied automatically. So it remains to solve for Eq. (3) in the space of \( O(3) \)-valued matrix fields.

Recall that the divergence operator acting on differential 1-forms over \( \Omega \subset \mathbb{R}^3 \) is nothing but the codifferential \( d^* := \star d \star \), where \( \star \) is the Hodge star operator. Also, the Laplacian equals
\[
\Delta = dd^* + d^* d.
\]
Let us split \( W \) into
\[
W^i = d^* II^i + d\phi^i + c^i \quad \text{on } \Omega,
\]
where \( II^i \) is a field of differential 2-form, \( \phi^i \) is a scalar field, and \( c^i \) is a constant in \( \mathbb{R}^3 \). This is done by the Hodge decomposition theorem and the simple-connectedness of \( \Omega \); see, e.g., [5]. In local co-ordinates, Eq. (5) can be expressed as follows:
\[
W^i_j = \sum_{k=1}^3 \nabla_k II^i_{kj} + \nabla_j \phi^i + c^i_j \quad \text{for each } i, j \in \{1, 2, 3\},
\]
where \( II^i_{kj} = II^i_{jk} \) for any \( i, j, k \in \{1, 2, 3\} \). Note that by standard elliptic regularity theory ([6]), \( II^i \) and \( \phi^i \) are \( C^{1, \gamma} \) for any \( \gamma \in [0, 1] \). To be precise, the divergence-free condition of \( d^* II^i \) here should be understood in the distributional sense.

Now we claim that
\[
\left\{ \nabla_j \phi^i \right\}_{1 \leq i, j \leq 3} \text{ is a constant } O(3)-\text{matrix.}
\]
Indeed, since the Leray projector maps onto the divergence-free part of \( W \), we have \( \mathcal{Q}W^i = d\phi^i \) for \( \phi^i \in C^{1, \gamma}(\Omega) \). This can be seen directly via the formula
\[
\mathcal{Q}W := \text{grad} \circ \Delta^{-1} \circ \text{div}W.
\]
By Assumption 2.1 we have
\[
\sum_{k=1}^3 \nabla_k \phi^i \nabla_k \phi^j = \delta^{ij},
\]
namely that $\phi$ is an isometric embedding from $\Omega \subset \mathbb{R}^3$ into $\mathbb{R}^3$. The classical rigidity theorem of Liouville [9] yields that $\phi^i$ is an affine map globally on $\Omega$ — In fact, $C^1$-regularity of $\phi^i$ suffices here. Thus the claim (6) follows.

To conclude the proof, taking $d^\ast$ on both sides of Eq. (5) and invoking the claim (6), we obtain that

$$d^\ast W^i = 0.$$ This together with Eqs. (3) and (4) implies that

$$\Delta W^i = 0. \quad (7)$$

That is, $W^i$ is a harmonic 1-form for each $i \in \{1, 2, 3\}$. Equation (7) is understood in the sense of distributions; nevertheless, by Weyl’s lemma (see [6]) $W^i$ is automatically $C^\infty$. In view again of the Hodge theory (see [12, Chap. 6]), it is represented by generators of the first cohomology group of $\Omega$. But $\Omega$ is simply-connected, so there is no such generator except for the trivial one. Thus $W^i$ is constant. Therefore, by Eq. (3), $\alpha^i$ equals zero. The proof is complete. $\square$

### 4 Remarks

It would be interesting to consider the same problem for $\Omega$ being a 3-dimensional manifold, which falls into the framework of incompatible (non-Euclidean) elasticity.

The mechanical problem considered in this paper may have deep underlying geometrical connotations. In particular, it is related to constructions for coframes with prescribed (closed) differential. See Bryant–Clelland [3] for analyses via exterior differential systems.

At the moment, we do not know if Assumption 2.1 can be weakened or even removed. It seems reasonable to conjecture that any $W : \Omega \to \mathcal{O}(3)$ with sufficient regularity such that $\text{curl} \, W = \text{a constant matrix}$ must satisfy $\text{div} \, W \equiv 0$. If this can be proved, then Assumption 2.1 is automatically verified. This will yield the generalisation of Acharya’s Theorem 1.2 to the whole $\mathcal{O}(3)$. On the other hand, one should note that merely being $\mathcal{O}(3)$-valued does not ensure $\text{div} \, W \equiv 0$. For example, the following $\mathcal{O}(3)$-valued matrix field is $C^\infty$ on $B \left( (1, 0, 0)^\top, 1/2 \right)$:

$$W(x, y, z) = \begin{bmatrix} x & \sqrt{1-x^2} & 0 \\ 0 & 0 & 1 \\ -\sqrt{1-x^2} & x & 0 \end{bmatrix}$$

while $\text{div} \,(W^1) = 1$. But $\text{curl} \,(W^1) = \left( 0, 0, -\frac{x}{\sqrt{1-x^2}} \right)^\top$ is non-constant here.

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### Declarations

**Competing Interests** The author declares no competing interests.
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