ON THE QUANTUM POTENTIAL

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Abstract. We survey various origins and expressions for the quantum potential, expanding and extending the treatment given in a previous paper [35].

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1. INTRODUCTION

The quantum potential arises in various forms, some of which were summarized in [35], and we want to return to this in a more systematic manner, with some new embellishments. Historically this arises by putting \( \psi = R \exp(iS/\hbar) \) into the Schrödinger equation (SE)

\[
i\hbar \frac{\partial \psi}{\partial t} = -\left(\frac{\hbar^2}{2m}\right) \psi_{xx} + V \psi \quad \text{(1-D for simplicity)},
\]

yielding

\[
S_t + \frac{(S')^2}{2m} - \left(\frac{\hbar^2}{2m}\right) \left(\frac{R''}{R}\right) + V = 0; \quad \partial_t R^2 + \partial \left(\frac{R^2S'}{m}\right) = 0
\]

The quantity \( Q = -(h^2/2m)(R''/R) \) (or more generally \( Q = -(h^2/2m)(\Delta R/R) \)) is the quantum potential and one takes \( p = S' \) with \( v = p/m \) (or \( p = \nabla S \) with \( v = p/m \)) for the momentum and velocity. We mention here in passing the refinements in [16, 29, 31, 66, 67, 68, 69] relative to the stationary situation \( S_t = -E \), which precludes the use of \( (A1) \) as such and leads to \( (A5) \) \( v = p/m_Q \), where \( m_Q = m(1 - \partial E Q) \); this will be discussed in more detail below. In any event the quantum potential does enter

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into any trajectory theory of deBroglie-Bohm (dBB) type. The history is discussed for example in [91] (cf. also [10, 18, 19, ?, 24]) and we will show how this quantum potential idea can be formulated in various ways in terms of statistical mechanics, hydrodynamics, information and entropy, etc. when dealing with different versions and origins of the SE. Given the existence of particles we finds the pilot wave of thinking very attractive, with the wave function serving to choreograph the particle motion (or perhaps to “create” particles and/or spacetime paths). However the existence of particles itself is not such an assured matter and in field theory approaches for example one will deal with particle currents (cf. [129] and see also e.g. [20, 21, 64, 75, 173]). The whole idea of quantum particle path seems in any case to be either fractal (cf. [1, 2, 3, 35, 42, 44, 48, 122, 136, 139, 140, 145, 146, 147], stochastic (see e.g. [10, 35, 71, 72, 82, 83, 84, 99, 100, 120, 121, 127, 154], or field theoretic (cf. [20, 21, 64, 75, 128, 129, 130, 131, 132, 133, 134, 173]. The fractal approach sometimes imagines an underlying micro-spacetime where paths are perhaps fractals with jumps, etc. and one possible advantage of a field theoretic approach would be to let the fields sense the ripples, which as e.g. operator valued Schwartz type distributions, they could well accomplish. In fact what comes into question here is the structure of the vacuum and/or of spacetime itself. One can envision microstructures as in [35, 79, 122, 123] for example, textures (topological defects) as in [12, 27, 179], Planck scale structure and QFT, along with space-time uncertainty relations as in [11, 61, 62, 111, 181], vacuum structures and conformal invariance as in [118, 119, 160, 161, 162, 163, pilot wave cosmology as in [167], ether theories as in [163, 170], etc. Generally there seems to be a sense in which particles cannot be measured as such and hence the idea of particle currents (perhaps corresponding to fuzzy particles or ergodic clumps) should prevail perhaps along with the idea of probability packets. A number of arguments work with a (representative) trajectory as if it were a single particle but there is no reason to take this too seriously; it could be thought of perhaps as a “typical” particle in a cloud but conclusions should perhaps always be constructed from an ensemble point of view. We will try to develop some of this below. The sticky point as we see it now goes as follows. Even though one can write stochastic equations for (typical) particle motion as in the Nelson theory for example one runs into the problem of ever actually being able to localize a particle. Indeed as indicated in [61, 62] (working in a relativistic context but this should hold in general) one expects space time uncertainty relations even at a semiclassical level since any localization experiment will generate a gravitational field and deform spacetime. Thus there are relations \[ q_\mu q_\nu = i\lambda_P^2 Q_{\mu\nu} \] where \( \lambda_P \) is the Planck length and the picture of spacetime as a local Minkowski manifold should break down at distances of order \( \lambda_P \). One wants the localization experiment to avoid creating a black hole (putting the object out of “reach”) for example and this suggests \[ \Delta x_{0}(\sum_{i}^{3} \Delta x_{i}) \geq \lambda_P^2 \] with \( \Delta x_{1}\Delta x_{2} + \Delta x_{2}\Delta x_{3} + \Delta x_{3}\Delta x_{1} \geq \lambda_P^2 \) (cf. [61, 62]). On the other hand in [129] it is shown that in a relativistic bosonic field theory for example one can speak of currents and n-particle wave functions can have particles attributed to them with well defined trajectories, even though the probability of their experimental detection is zero. Thus one enters an arena of perfectly respectable but undetectable particle trajectories. The discussion in [13, 64, 174, 175, 174] is also relevant here; some recourse to the idea of beables and reality and observables as beables is also involved (cf. [20, 21, 45, 175]). We will have something to say about all these matters.
2. POINTS OF VIEW

We collect here some different ways in which the quantum potential arises with a sketch of the derivation (cf. [35] for more details, derivations and references); additional origins will be given subsequently.

2.1. SCHRÖDINGER EQUATIONS.

EXAMPLE 2.1. Take (A1), (A2), and (1.1) with \( P = R^2 (\sim |\psi|^2) \) and \( Q \) as in (A3). This gives (A6) \( S_t + (S')^2 + Q + V = 0; \) \( P_t + \frac{1}{m}(PS')' = 0 \) which has some hydrodynamical interpretations in the spirit of Madelung. Indeed going to [53] for example we take \( p = S' \) with \( p = m\dot{q} \) for \( \dot{q} \) a velocity (or “collective” velocity - unspecified). Then (A6) leads to

\[
\rho_t + (\rho v)_t = 0 \quad \text{(2.1)}
\]

\[
\rho_t + \frac{\rho v}{m} = 0 \quad \text{(2.2)}
\]

This is similar to an “Euler” type equation (cf. [53]) and it definitely has a hydrodynamic flavor (cf. also [80]). Now go to [148] and write from (2.1)

\[
\partial_v + (v \cdot \nabla) v = - \frac{1}{m} \nabla (V + Q); \quad v_t + vv' = -(1/m)\partial(v + Q)
\]

The higher dimensional form is not considered here but matters are similar there. This equation (and (2.2)) is incomplete as a hydrodynamical equation as a consequence of a missing term \(-\rho^{-1}\nabla p\) where \( p \) is the pressure (cf. [112]). Hence one “completes” the equation in the form

\[
m \left( \frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) = -\nabla (V + Q) + \nabla F; \quad m v_t + m v v' = -\partial (V + Q) - F'
\]

where (A7) \( \nabla F = (1/R^2)\nabla p \) (or \( F' = (1/R^2)p' \)). By the derivations above this would then correspond to an extended SE of the form (A8) \( i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta \psi + V \psi + F \psi \). Provided one can determine \( F \) in terms of the wave function \( \psi \). This suggests a role for \( Q \) in the form \( (F = 0) \) (A9) \( \partial Q = (1/\rho)p' \Rightarrow p' = -(\hbar^2/2m)\rho(\partial^2 \sqrt{\rho}/\sqrt{\rho}) = -(\hbar^2/2m)R^2 \partial R'/R \).

Alternatively one can form a nonlinear SE (NLSE) with \( p \) a suitable function of \( \psi \).

EXAMPLE 2.2. We turn next to [100] for a statistical origin for QM (cf. also [29, 50, 99, 100, 127, 144, 155]). The idea is to build a program in which the microscopic motion, underlying QM, is described by a rigorous dynamics different from Brownian motion (thus avoiding unnecessary assumptions about the Brownian nature of the underlying dynamics). The Madelung approach gives rise to fluid dynamical type equations with a quantum potential, the latter being capable of interpretation in terms of a stress tensor of a quantum fluid. Thus one shows in [100] that the quantum state corresponds to a subquantum statistical
ensemble whose time evolution is governed by classical kinetics in the phase space. The equations take the form

\[(2.5) \quad \rho_t + \partial_x (\rho u) = 0; \quad \partial_t (\rho u) + \partial_j (\rho \phi_j) + \rho \partial_x V = 0; \quad \partial_t (\rho E) + \partial_x (\rho S) - \rho \partial_t V = 0\]

with (A10) \( \frac{\partial S}{\partial t} + \frac{1}{2\mu} \left( \frac{\partial S}{\partial x} \right)^2 + W + V = 0 \) for two scalar fields \( \rho, S \) determining a quantum fluid. These can be rewritten as

\[(2.6) \quad \frac{\partial \xi}{\partial t} \left( 1 + \frac{1}{2\mu} \frac{\partial^2 S}{\partial x^2} \right) + \frac{\partial S}{\partial x} \frac{\partial \xi}{\partial x} = 0; \quad \frac{\partial S}{\partial t} - \frac{\eta^2}{4\mu} \frac{\partial^2 \xi}{\partial x^2} - \frac{\eta^2}{8\mu} \left( \frac{\partial \xi}{\partial x} \right)^2 + \frac{1}{2\mu} \left( \frac{\partial S}{\partial x} \right)^2 + V = 0\]

where \( \xi = \log(\rho) \) and for \( Q = (\xi/2) + (\eta^2/4\mu) \log(S) \) with \( m = N \mu, \ V = NV \), and \( h = N \eta \) one arrives at a SE (A11) \( ih \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \hbar \Psi \). Further one can write \( \Psi = \rho^{1/2} \exp(iS/\hbar) \) with \( \Phi = NS \) and here \( N = \int |\Psi|^2 dx \). Thus from a statistical origin in classical kinetics there emerges a SE with quantum potential \( W \sim Q \) (before scaling with \( N \)) related to the stress tensor of a quantum fluid.

**Example 2.3.** Now in [63] one is obliged to use the form \( \psi = R \exp(iS/\hbar) \) to make sense out of the constructions (this is no problem with suitable provisos, e.g. that \( S \) is not constant - cf. [29, 66, 67] and comments later). This leads to (1.1) and (2.1) with \( Q = -\hbar^2 R''/2mR \) as in (A3). In [63] one emphasizes configurations based on coordinates whose motion is choreographed by the SE according to the rule (1-D only here)

\[(2.7) \quad \dot{q} = v = \frac{\hbar}{m} \frac{\psi^* \psi'}{|\psi|^2} = \frac{\hbar}{m} \Im \left( \frac{\psi'}{\psi} \right)\]

where (A12) \( ih \psi_t = -(h^2/2m)\psi'' + V \psi \). The argument for (2.7) is based on obtaining the simplest Galilean and time reversal invariant form for velocity, transforming correctly under velocity boosts. This leads directly to (2.7) so that Bohmian mechanics (BM) is governed by (2.7) and (A12). It’s a fairly convincing argument and no recourse to Floydian time seems possible (cf. [29, 66, 67]. Note however that if \( S = \epsilon \) then \( \dot{q} = v = (\hbar/m) \Im(R/R') = 0 \) while \( p = S' = 0 \) so perhaps this formulation avoids the \( S = 0 \) problems indicated in [29, 66, 67, 68, 69]. Note however that if \( S = \epsilon \) then \( \dot{q} = v = (\hbar/m) \Im(R/R') = 0 \) while \( p = S' = 0 \) so perhaps this formulation avoids the \( S = 0 \) problems indicated in [29, 66, 67, 68, 69].

**Example 2.4.** The Fisher information connection (cf. [82, 83, 84]) involves a classical ensemble with particle mass \( m \) moving under a potential \( V \) (A13) \( S_t + \frac{1}{2m}(S')^2 + V = 0; \ P_t + \frac{1}{m} \partial_t (PS')' = 0 \) where \( S \) is a momentum potential; note that no quantum potential is present but this will be added on in the form of a term \( (1/2m) \int dt (\Delta N)^2 \) in the Lagrangian which measures the strength of fluctuations. This can then be specified in terms of the
probability density $P$ leading to a SE. A “neater” approach is given in following [151] leading in 1-D to

$$S_t + \frac{1}{2m} (S')^2 + V + \frac{\lambda}{m} \left( \frac{(P')^2}{P^2} - \frac{2P''}{P} \right) = 0$$

(2.8)

Note that $Q = -(\hbar^2/2m)(R''/R)$ becomes for $R = P^{1/2}$ (A14) $Q = -(\hbar^2/2m)[(2P''/P) - (P'/P)^2]$. Thus the addition of the Fisher information serves to quantize the classical system. One also defines an information entropy (IE) via (A15) $\mathcal{S} = - \int \rho \log(\rho) d^3x$ ($\rho = |\psi|^2$) leading to

$$\frac{\partial \mathcal{S}}{\partial t} = \int (1 + \log(\rho)) \partial(\rho \rho) \sim \int \frac{(\rho')^2}{\rho}$$

(2.9)

modulo constants involving $D \sim \hbar/2m$. $\mathcal{S}$ is typically not conserved and $\partial_t \rho = -\nabla \cdot (\rho \rho \mathbf{u})$ \textit{(u = D\nabla\log(\rho)} with $\mathbf{v} = -\mathbf{u}$ corresponds to standard Brownian motion with $d\mathcal{S}/dt \geq 0$. Then high IE production corresponds to rapid flattening of the probability density. Note here also that $\mathfrak{S} \sim -(2/D^2) \int \rho Q dx = \int dx [(\rho')^2/\rho]$ is a functional form of Fisher information. Entropy balance is discussed in [74]. \[■\]

The Nagasawa theory (based in part on Nelson’s approach) is very revealing and fascinating (cf. [120] [121] and for the nonrelativistic theory one has

**THEOREM 2.1.** Let $\psi(t,x) = \exp[R(t,x) + iS(t,x)]$ be a solution of the SE (A16) $i\partial_t \psi + (1/2)\Delta \psi - \mathbf{a}(t,x) \cdot \nabla \psi - V(t,x) \psi = 0$ ($\hbar$ and $m$ omitted) then (A17) $\phi(t,x) = \exp[R(t,x) + S(t,x)]$ and $\hat{\phi} = \exp[R(t,x) - S(t,x)]$ are solutions of

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \Delta \phi + \mathbf{a}(t,x) \cdot \nabla \phi + c(t,x,\phi) \phi = 0;$$

$$-\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2} \Delta \hat{\phi} - \mathbf{a}(t,x) \cdot \nabla \hat{\phi} + c(t,x,\hat{\phi}) \hat{\phi} = 0$$

(2.10)

where the creation and annihilation term $c(t,x,\phi)$ is given via

$$c(t,x,\phi) = -V(t,x) - 2 \frac{\partial S}{\partial t}(t,x) - (\nabla S)^2(t,x) - 2a \cdot \nabla S(t,x)$$

(2.11)

Conversely given $\phi, \hat{\phi}$ as in (A17) satisfying (2.10) it follows that $\psi$ satisfies the SE (A16) with $V$ as in (2.11) (note $R = (1/2)\log(\hat{\phi})$, $S = (1/2)\log(\phi/\hat{\phi})$, and $\exp(R) = (\hat{\phi}/\phi)^{1/2}$). \[■\]

Thus in short: $\psi = \exp(R + iS)$ satisfies the SE (A18) $i\psi_t + (1/2)\psi'' + i\mathbf{a}\psi' - V\psi = 0$ if and only if

$$V = -S_t + \frac{1}{2} R'' + \frac{1}{2} (R')^2 - \frac{1}{2} (S')^2 - aS; 0 = R_t + \frac{1}{2} S'' + S'R' + aR'$$

(2.12)

Changing variables via $X = (\hbar/\sqrt{m})x$ and $T = \hbar t$ one gets (A19) $i\hbar \psi_T = -(\hbar^2/2m)\psi_{XX} - iA\psi_X + V\psi$ where $A = ah/\sqrt{m}$ and

$$i\hbar R_T + (\hbar^2/m^2)R_X S_X + (\hbar^2/2m^2)S_{XX} + AR_X = 0;$$

$$V = -i\hbar S_T + (\hbar^2/2m)R_{XX} + (\hbar^2/2m^2)R_X^2 - (\hbar^2/2m^2)S_X^2 - AS_X$$

(2.13)
The quantum potential is \( Q \) and suitable "applied" forces (e.g. radiative reactive for ces). 

Derivations of Schrödinger equations via diffusion ideas `a la Nelson, Markov wave equations, (cf. also Example 3.4). We remark also that the papers in [52] contain very interesting

Equation (2.16) becomes then

\[
(2.21) \quad V = 2D\partial S; \quad S = \log(\rho^{1/2}) + iS; \quad \psi = \sqrt{\rho}e^{iS} = e^{iS}; \quad V = -2i\partial\log(\psi)
\]
For Lagrangian $\mathcal{L} = (1/2)m\ddot{y}^2 - mV$ one gets a SE (A21) $i\hbar \dot{\psi} = -\hbar^2/(2m) \partial^2 \psi + \Omega \psi$ coming from Newton’s law (A22) $-\partial \mathcal{U} = -2i\mathcal{D}(d'/dt)\partial \log(\psi) = m(d'/dt)\dot{V}$. If $\mathcal{U} = 0$ we see that free motion $m(d'/dt)V = 0$ yields the SE as a geodesic equation in fractal spacetime. To clarify this we write out $(d'/dt)\partial \log(\psi) = 0$ as $(\partial_t - i\partial^2 + V\partial)\partial \log(\psi) = 0$ with $V = -2i\mathcal{D}\partial \log(\psi)$. Using identities as in [137] this is

$$\partial_t \partial \log(\psi) - i\partial^3 \log(\psi) - 2i\partial \log(\psi)\partial \log(\psi) = 0 \Rightarrow i\partial_t \psi + D\partial^2 \psi = 0$$

with $D = \hbar/2m$ which means $i\hbar \partial_t \psi = -(\hbar^2/2m)\partial^2 \psi$ as desired (the multidimensional case is in [137]). Next since the quantum potential in this context arises via $Q = -((\hbar^2/2m)(\Delta\sqrt{p}/\sqrt{p})$ (as usual) and since $V = -(i\hbar/m)\partial \log(\psi) = V - iU = -(i\hbar/m)\partial[\log\sqrt{p} + iS]$ one can write (★) $V = (i\hbar/m)\partial S$ and $U = (i\hbar/m)(\partial\sqrt{p}/\sqrt{p})$. Consequently

$$\partial U = -\frac{\hbar}{m}\left(\frac{2m}{\hbar^2}Q + \frac{m^2}{\hbar^2}U^2\right)$$

and (★★) $\partial U = -(2/\hbar)Q - (mh/U^2) \Rightarrow Q = -(m/2)U^2 - (h/2)\partial U$ (cf. (A23) below). Hence the quantum potential arises directly from the geodesic equation in fractal spacetime based on continuous nonsmooth paths. We refer here to Remark 5.2 for more discussion on the derivation of the SE.

**Remark 2.1.** It is perhaps a little unsettling to see operators $\partial$ and $\partial^2$ appearing in the equations above since the paths $y(t)$ are not smooth. However the functions $f(x(t), t)$ can well be smooth functions of $x$ so there is less of a problem (see e.g. [2, 35, 47, 48, 137] and Section 5.1 - the development of Cresson et al is rigorous and polished). For a sketch of explanation we follow [18] and consider $x \rightarrow f(x(t), t) \in C^{n+1}$ with $X(t) \in H^{1/n}$ (i.e. $c\epsilon^{1/n} \leq |X(t') - X(t)| \leq C\epsilon^{1/n}$). Define (f real valued)

$$\nabla_{\pm} f(t) = \frac{f(t + \epsilon) \mp f(t)}{\epsilon}; \quad \square_{t} f(t) = \frac{1}{2}(\nabla_{+}^2 + \nabla_{-}^2) f - \frac{i}{2}(\nabla_{+} - \nabla_{-}) f;$$

$$a_{\epsilon,j}(t) = \frac{1}{2}[(\Delta_{+}^{\epsilon} y)^j - (-1)^j(\Delta_{-}^{\epsilon} y)^j] - \frac{i}{2}[(\Delta_{+}^{\epsilon} y)^j + (-1)^j(\Delta_{-}^{\epsilon} y)^j]$$

Assume some minimal control over the lack of differentiability (cf. [18]) and then for $f$ now complex valued with $\square_{t} f/\square t = (\square_{e} f_R/\square t) + i(\square_{e} f_I/\square t)$ (note the mixing of i terms is not trivial) one has

$$\square_{t} f = \frac{\partial f}{\partial t} + \frac{\partial x}{\partial t} \frac{\partial f}{\partial x} + \frac{n}{2} \sum j! a_{\epsilon,j}(t) \frac{\partial^{j} f}{\partial x^{j}} e^{-j} + o(\epsilon^{1/n})$$

We refer to [2, 47, 18] for a full exposition.

**Example 2.6.** The development in [42] involves thinking of nonlinear QM as a fractal Brownian motion with complex diffusion coefficient. In particular one uses (A22) and (A24) and arrives at

$$-\nabla U = -2im[D\partial_t \nabla \log(\psi)] - 2D\nabla \left( D\frac{\nabla^2 \psi}{\psi} \right)$$
Thus putting in a complex diffusion coefficient leads to the NLSE

\[
(2.27) \quad i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \alpha \nabla^2 \psi + U \psi - \frac{i\hbar^2}{2m} (\nabla \log(\psi))^2 \psi
\]

with \( h = \alpha + i\beta = 2mD \) complex.

In [3] one writes again \( \psi = R \exp(iS/\hbar) \) with field equations in the hydrodynamical picture

\[
(2.28) \quad d_t(m_0 \rho v) = \partial_t(m_0 \rho v) + \nabla(m_0 \rho v) = -\rho \nabla(u + Q); \quad \partial_t \rho + \nabla \cdot (\rho v) = 0
\]

where \( Q = -(h^2/2m_0)(\Delta \sqrt{\rho}/\sqrt{\rho}) \). One works with the Nottale approach as above with \( d_v \sim d_t \) and \( d_u \sim d_t \) (cf. \([24]\)). One assumes that the velocity field from the hydrodynamical model agrees with the real part of the complex velocity \( V = v - iu \) so (cf. \([24]\)) \( v = (1/m_0)\nabla s \sim 2Dd\sigma \) and \( u = -(1/m_0)\nabla \sigma \sim D\partial \log(\rho) \) where \( D = h/2m_0 \). In this context the quantum potential \( Q = -(h^2/2m_0)\Delta \sqrt{\rho}/\sqrt{\rho} \) becomes \([A23]\) \( Q = -\rho \sqrt{\rho} V \) or \( -(h^2/2m_0)\nabla u - (1/2)m_0 u^2 \sim -(h^2/2m_0)u - (1/2)m_0 u^2 \). Consequently \( Q \) arises from the fractal derivative and the nondifferentiability of spacetime. Further one can relate \( u \) (and hence \( Q \)) to an internal stress tensor \([A24]\) \( \sigma_{ik} = \eta[(\partial u_i/\partial x_k) + (\partial u_k/\partial x_i)] \) whereas the \( v \) equations correspond to systems of Navier-Stokes type.

**EXAMPLE 2.7.** The equivalence principle (EP) of Faraggi-Matone (cf. \([16, 29, 30, 37, 67]\)) is based on the idea that all physical systems can be connected by a coordinate transformation to the free situation with vanishing energy (i.e. all potentials are equivalent under coordinate transformations). This automatically leads to the quantum stationary Hamilton-Jacobi equation (QSHJE) which is a third order nonlinear differential equation providing a trajectory representation of quantum mechanics (QM). The theory transcends in several respects the Bohm theory and in particular utilizes a Floydian time (cf. \([68, 69]\)) leading to \([A25]\) \( \dot{q} = p/m_Q \neq p/m \) where \([A26]\) \( m_Q = m(1 - \partial_E Q) \) is the “quantum mass” and \( Q \) the “quantum potential”. Thus the EP is reminscient of the Einstein equivalence of relativity theory. This latter served as a midwife to the birth of relativity but was somewhat inaccurate in its original form. It is better put as saying that all laws of physics should be invariant under general coordinate transformations (cf. \([143]\)). This demands that not only the form but also the content of the equations be unchanged. More precisely the equations should be covariant and all absolute constants in the equations are to be left unchanged (e.g. \( c, h, e, m \) and \( \eta_{\mu\nu} = \text{Minkowski tensor} \). Now for the EP, the classical picture with \( S^d(q, Q, t) \) the Hamilton principal function (\( p = \partial S^d/\partial q \)) and \( P^d, Q^d \) playing the role of initial conditions involves the classical HJ equation (CHJE) \([A27]\) \( H(q, p, (\partial S^d/\partial q), t) + (\partial S^d/\partial t) = 0 \). For time independent \( V \) one writes \( S^d = S_0^d(q, Q^0) - Et \) and arrives at the classical stationary HJ equation (CSHJE) \([A28]\) \((1/2m) (\partial S^d_0/\partial q)^2 + W = 0 \) where \( W = V(q) - E \). In the Bohm theory one looked at Schrödinger equations \([A29]\) \( i\hbar \dot{\psi} = -(h^2/2m)\psi'' + V\psi \) with \( \psi = \psi(q) \exp(-iEt/\hbar) \) and \([A30]\) \( \psi(q) = R(q) \exp(iW/\hbar) \) leading to

\[
(2.29) \quad \left( \frac{1}{2m} \right) (\dot{W}'')^2 + V - E - \frac{\hbar^2 R''}{2mR} = 0; \quad (R^2 \dot{W}')' = 0
\]

where \([A31]\) \( \dot{Q} = -h^2 R''/2mR \) was called the quantum potential; this can be written in the Schwartzian form \([A32]\) \( \dot{Q} = (h^2/4m)\{\dot{W}; q \} \) (via \( R^2 \dot{W}' = c \)). Here \([A33]\) \( \{f; q\} = \)
concerning the Klein-Gordon (KG) equation and the equivalence principle (EP) of V. One starts with the relativistic classical Hamilton-Jacobi equation (RCHJE) with a potential example. A number of interesting philosophical points arise (e.g., the emergence of space from the wave function) and we connected this to various features of space coordinate and the wave function via a prepotential (free energy) in the form

\begin{equation}
(\frac{f''}{f'}) - (3/2)(f''/f')^2. \text{ Writing } \mathfrak{W} = V(q) - E \text{ as in (A28) we have the quantum stationary HJ equation (QSHJE) (A34) } (1/2m)(\partial W'/\partial q)^2 + \mathfrak{W}(q) + \hat{Q}(q) = 0 \text{ (} \equiv \mathfrak{W} = -(\hbar^2/4m) \exp(2iS_0/h; q)). \text{ This was worked out in the Bohm school (without the Schwarzian connections) but (A30) is not appropriate for all situations; the Bohm theory is incomplete and can lead to incorrect predictions (} \hat{W} = \text{ constant must be excluded). The technique of Faraghi-Matone (FM) is completely general and with only the EP as guide one exploits the relations between Schwarzians, Legendre duality, and the geometry of a second order differential operator } D_x^2 + V(x) \text{ (Möbius transformations play an important role here) to arrive at the QSHJE in the form}

\begin{equation}
(2.30) \quad \frac{1}{2m} \left( \frac{\partial S_0^v(q^v)}{\partial q^v} \right)^2 + \mathfrak{W}(q^v) + \Omega^v(q^v) = 0
\end{equation}

where } v : q \rightarrow q^v \text{ represents an arbitrary locally invertible coordinate transformation. Note in this direction for example that the Schwarzian derivative of the the ratio of two linearly independent elements in } \ker(D_x^2 + V(x)) \text{ is twice } V(x). \text{ In particular given an arbitrary system with coordinate } q \text{ and reduced action } S_0(q) \text{ the system with coordinate } q^0 \text{ corresponding to } V - E = 0 \text{ involves (A35) } \mathfrak{W}(q) = (q^0; q) \text{ where } (q^0, q) \text{ is a cocycle term which has the form (A36) } (q^a; q^b) = -(\hbar^2/4m)\{a; q^b\}. \text{ In fact it can be said that the essence of the EP is the cocycle condition (A37) } (q^a; q^b) = (\partial_{q^c}q^b)^2 [(q^a; q^b) - (q^c; q^b)].

In addition FM developed a theory of } (x, \psi) \text{ duality (cf. ref. [1]) which related the space coordinate and the wave function via a prepotential (free energy) in the form } \mathfrak{F} = (1/2)i\psi \bar{\psi} + iX/\epsilon \text{ for example. A number of interesting philosophical points arise (e.g., the emergence of space from the wave function) and we connected this to various features of dispersionless KdV in [29,37] in a sort of extended WKB spirit. One should note here that although a form (A30) is not generally appropriate it is correct when one is dealing with two independent solutions of the Schrödinger equation } \psi \text{ and } \bar{\psi} \text{ which are not proportional. In this context we utilized some interplay between various geometric properties of KdV which involve the Lax operator } L^2 = D_x^2 + V(x) \text{ and of course this is all related to Schwartzians, Virasoro algebras, and vector fields on } S^1 \text{ (see e.g. [29,38,39,10,11]). Thus the simple presence of the Schrödinger equation (SE) in QM automatically incorporates a host of geometrical properties of } D_x = d/dx \text{ and the circle } S^1. \text{ In fact since the FM theory exhibits the fundamental nature of the SE via its geometrical properties connected to the QSHJE one could speculate about trivializing QM (for 1-D) to a study of } S^1 \text{ and } \partial_x. \quad \blacksquare

2.2. **THE KLEIN-GORDON EQUATION.** We import here some comments based on [16] concerning the Klein-Gordon (KG) equation and the equivalence principle (EP) of Example 2.7 (details are in [16] and cf. also [56,57,61,116,117] for the KG equation). One starts with the relativistic classical Hamilton-Jacobi equation (RCHJE) with a potential } V(q, t) \text{ given as}

\begin{equation}
(2.31) \quad \frac{1}{2m} \sum_{q=1}^D (\partial_k S_{cl}(q, t))^2 + \mathfrak{W}_{rel}(q, t) = 0; \quad \mathfrak{W}_{rel}(q, t) = \frac{1}{2mc^2}[m^2c^4 - (V(q, t) + \partial_t S_{cl}(q, t))^2]
\end{equation}
In the time-independent case one has $S^{c\ell}(q,t) = S^{c\ell}_0(q) - Et$ and (2.30) becomes

$$\frac{1}{2m} \sum_{i=1}^{D} (\partial_k S^{c\ell}_0)^2 + \mathcal{W}_{\text{rel}} = 0; \quad \mathcal{W}_{\text{rel}}(q) = \frac{1}{2mc^2} [m^2 c^4 - (V(q) - E)^2]$$

In the latter case one can go through the same steps as in the nonrelativistic case and the relativistic quantum HJ equation (RQHJE) becomes (A38) $(1/2m)(\nabla S^0)^2 + \mathcal{W}_{\text{rel}} - (h^2/2m)(\Delta R/R) = 0$ with $\nabla \cdot (R^2 \nabla S^0) = 0$; these equations imply the stationary KG equation (A39) $-\hbar^2 c^2 \Delta \psi + (m^2 c^4 - V^2 + 2E V - E^2) \psi = 0$ where $\psi = R \exp(iS^0/h)$. Now in the time dependent case the (D+1)-dimensional RCHJE is (A40) $(1/2m) \eta^{\mu\nu} \partial_\mu S^{c\ell} \partial_\nu S^{c\ell} + \mathcal{W}'_{\text{rel}} = 0$ where $\eta^{\mu\nu} = \text{diag}(-1,1,\cdots,1)$ and (A41) $\mathcal{W}'_{\text{rel}} = (1/2mc^2)[m^2 c^4 - V^2(q) - 2Ev(q)\partial_0 S^{c\ell}(q)]$ with $q = (ct,q_1,\cdots,q_D)$. Thus (A40) has the same structure as (2.32) with Euclidean metric replaced by the Minkowskian one. We know how to implement the EP by adding $Q$ via (A42) $(1/2m)(\partial S)^2 + \mathcal{W}_{\text{rel}} + Q = 0$ (cf. [67] and Example 2.7). Note now that $\mathcal{W}'_{\text{rel}}$ depends on $S^{c\ell}$ one requires an identification (A43) $\mathcal{W}_{\text{rel}} = (1/2mc^2)[m^2 c^4 - V^2(q) - 2Ev(q)\partial_0 S(q)]$. (S replacing $S^{c\ell}$) and implementation of the EP requires that for an arbitrary $\mathcal{W}^{a}$ state ($q \sim \alpha$) one must have (A44) $\mathcal{W}^{b}_{\text{rel}}(q^b) = (p^b|p^a)\mathcal{W}^{a}_{\text{rel}}(q^a) + (q^b;q^a)$ (cf. (A36)) and (A45) $Q^b(q^b) = (p^b|p^a)Q(q^a) - (q^b;q^a)$. (p^b;p^a) = [\eta^{\mu\nu}p^b_\mu p^a_\nu/\eta^{\mu\nu}p^a_\mu p^b_\nu] = p^T J_\eta J^T p/p^T n p$ and (A47) $J^\mu_\nu = \partial q^\mu/\partial q^\nu$ (J is a Jacobian and these formulas are the natural multidimensional generalization - see [16] for details). Furthermore there is a cocycle condition (A48) $(q^a;q^c) = (p^a|p^b)[(q^a,q^b) - (q^c;q^b)]$ (cf. (A37)).

Next one shows that (A49) $\mathcal{W}_{\text{rel}} = (h^2/2m)[\Box(\text{Reexp}(iS/h))/\text{Reexp}(iS/h)]$ and hence the corresponding quantum potential is (A50) $Q_{\text{rel}} = -(h^2/2m)[\Box R/R]$. Then the RQHJE becomes (A51) $(1/2m)(\partial S)^2 + \mathcal{W}_{\text{rel}} + Q = 0$ with $\partial \cdot (R^2 \partial S) = 0$ (here $\Box R = \partial_\mu \partial^\mu R$) and this reduces to the standard SE in the classical limit $c \to \infty$. To see how the EP is simply implemented one considers the so called minimal coupling prescription for an interaction with an electromagnetic four vector $A_\mu$. Thus set $P^{c\ell}_\mu = p^{c\ell}_\mu + eA_\mu$ where $p^{c\ell}_\mu$ is a particle momentum and $P^{c\ell}_\mu = \partial_\mu S^{c\ell}$ is the generalized momentum. Then the RCHJE reads as (A52) $(1/2m)(\partial S^{c\ell} - eA)^2 + (1/2mc^2)^2 = 0$ where $A_0 = -V/e$. Then (A53) $\mathcal{W} = (1/2mc^2)$ and the critical case $\mathcal{W} = 0$ corresponds to the limit situation where $m = 0$. One adds the standard Q correction for implementation of the EP to get (A54) $(1/2m)(\partial S - eA)^2 + (1/2mc^2)^2 + Q = 0$ and there are transformation properties

$$\mathcal{W}(q^b) = (p^b|p^a)\mathcal{W}^{a}(q^a) + (q^b;q^a); \quad Q^b(q^b) = (p^b|p^a)Q^a(q^a) - (q^b;q^a)$$

$$\frac{(p^b;p^a)}{(p - eA)^2} = \frac{(p - eA)^T J_\eta J^T (p - eA)}{(p - eA)^T \eta (p - eA)}$$

Here J is a Jacobian (A55) $J^\mu_\nu = \partial q^\mu/\partial q^\nu$ and this all implies the cocycle condition (A37) again. One finds now that

$$\text{(2.34) } (\partial S - eA)^2 = \hbar^2 \left( \frac{\Box R}{R} - \frac{D^2(Re^{iS/h})}{Re^{iS/h}} \right); \quad D_\mu = \partial_\mu - i/eA_\mu$$
and it follows that
\begin{equation}
W = \frac{\hbar^2}{2m} \frac{D^2(Re^{iS/h})}{Re^{iS/h}}; \quad Q = - \frac{\hbar^2}{2m} \frac{\Box R}{R}; \quad D^2 = \Box - \frac{2ieA\partial}{\hbar} - \frac{e^2A^2}{\hbar^2} - \frac{ie\partial A}{\hbar},
\end{equation}

\begin{equation}
(\partial S - eA)^2 + m^2c^2 - \hbar^2 \frac{\Box R}{R} = 0; \quad \partial \cdot (R^2(\partial S - eA)) = 0.
\end{equation}

Note also that (A40) coincides with (A52) after setting \(W_{\text{rel}} = \frac{mc^2}{2}\) and replacing \(\partial_{\mu}S_{\text{cl}}\) by \(\partial_{\mu}S_{\text{cl}} - eA_{\mu}\). One can check that (2.36) implies the KG equation (A56) \((ih\partial + eA)^2\psi + \frac{mc^2}{2}\psi = 0\) with \(\psi = Rexp(iS/h)\).

**REMARK 2.2.** We extract now a remark about mass generation and the EP from [16]. Thus a special property of the EP is that it cannot be implemented in classical mechanics (CM) because of the fixed point corresponding to \(W = 0\). One is forced to introduce a uniquely determined piece to the classical HJ equation (namely a quantum potential Q). In the case of the RCHJE (A52) the fixed point \(W(q^0) = 0\) corresponds to \(m = 0\) and the EP then implies that all the other masses can be generated by a coordinate transformation. Consequently one concludes that masses correspond to the inhomogeneous term in the transformation properties of the \(W^0\) state, i.e. (A57) \((1/2)mc^2 = (q^0; q)\). Furthermore by (2.35) masses are expressed in terms of the quantum potential (A58) \((1/2)mc^2 = (p|p^0)Q^0(q^0) - Q(q)\). In particular in [67] the role of the quantum potential was seen as a sort of intrinsic self energy which is reminiscent of the relativistic self energy and (A58) provides a more explicit evidence of such an interpretation.

### 3. QUANTUM FIELD THEORY

In trying to imagine particle trajectories of a fractal nature or in a fractal medium we are tempted to abandon (or rather relax) the particle idea and switch to quantum fields (QF). Let the fields sense the bumps and fractality; if one can think of fields as operator valued distributions for example then fractal supports for example are quite reasonable. There are other reasons of course since the notion of particle in quantum field theory (QFT) has a rather fuzzy nature anyway. Then of course there are problems with QFT itself (cf. [178]) as well as arguments that there is no first quantization (except perhaps in the Bohm theory - cf. [129, 184]). We review here some aspects of particles arising from QF and QFT methods, especially in a Bohmian spirit (cf. [13, 45, 64, 130, 131, 180]).

#### 3.1. EMERGENCE OF PARTICLES

We refer to [87, 178] for interesting philosophical discussion about particles and localized objects in a QFT and will extract here from [13, 45, 64, 130, 131]; for QFT we refer to [86, 168]. We omit many details and assume standard QFT techniques are known. First [131] is impressive in producing a local operator describing the particle density current for scalar and spinor fields in an arbitrary gravitational and electromagnetic background. This enables one to describe particles in a local, general covariant, and gauge invariant manner. The current depends on the choice of a 2-point Wightman function and a most natural choice based on the Green’s function à la Schwinger-deWitt leads to local conservation of the current provided that interaction with quantum
fields is absent. Interactions lead to local nonconservation of current which describes local
particle production consistent with the usual global description based on the interaction
picture. Thus for suitable choice of a 2-point Wightman function \( W(x,x') \) the formula
for scalar fields is given by (B1) \( j_\mu(x) = (1/2) \int_\Sigma d\Sigma' \{ W(x,x') \overset{\leftrightarrow}{\partial}_\mu \phi(x)\phi(x') + h.c. \}. \) Upon
extracting formulas for \( \phi^\pm \) in \( \phi = \phi^+ + \phi^- \) in

\[
\phi^+(x) = i \int_\Sigma d\Sigma' W^+(x,x') \overset{\leftrightarrow}{\partial}_\mu \phi(x'); \quad \phi^-(x) = -i \int_\Sigma d\Sigma' W^-(x,x') \overset{\leftrightarrow}{\partial}_\mu \phi(x')
\]

one arrives at

\[
j_\mu(x) = \frac{1}{2} \int_\Sigma d\Sigma' \left[ W^+(x,x') \overset{\leftrightarrow}{\partial}_\mu x \overset{\leftrightarrow}{\partial}_\nu x \phi(x') \right]
\]

and thence to (B2) \( j_\mu = i\phi^- \overset{\leftrightarrow}{\partial}_\mu \phi^+ \) or (B3) \( j_\mu = (i/2) N_- \phi \overset{\leftrightarrow}{\partial}_\nu \phi \) where \( N_- \phi^+ \phi^- = -\phi^- \phi^+ \).

It is also demonstrated that energy production corresponds exactly to particle production. ■

3.2. FIELD THEORY MODELS.

**EXAMPLE 3.1.** In the bosonic theory of [B30] for a relativistic KG equation (B4) \( (\partial^2_0 - \nabla^2 + m^2) \phi = 0 \), one has a corresponding particle current (B5) \( j_\mu = i\psi^* \overset{\leftrightarrow}{\partial}_\mu \psi \) and particles
have a trajectory velocity (B6) \( (d\vec{x}/dt) = \vec{j}(t,\vec{x})/j_0(t,\vec{x}) \). One obtains a HJ equation
(1/2m)\( \Box S - (c^2m/2) + Q = 0 \) with (B7) \( Q = -(1/2m)(\partial^\mu \partial_\mu R)/R = -(1/2m)(\Box R/R) \) (quantum potential). In fact this leads to (B8) \( m d^2 x_\mu = \partial^\mu Q \) and the physical number of particles \( N_{phys} = \int d^3 x |j_0| \) is not conserved (although \( N = \int d^3 x j_0 \) is). In an interaction
picture with (B9) \( (\partial^2_0 - \nabla^2 + m^2) \phi = J(\phi) \) (and \( c = \hbar = 1 \)) one has a dBB interpretation via
(\( \Psi \sim \Psi[\phi(x0,0)] \))

\[
(\partial^2_0 - \nabla^2) \phi(x) = J(\phi(x)) - \left( \frac{\delta Q[\phi,t]}{\delta \phi(x)} \right)_{\phi(x)=\phi(x)};
\]

\( Q = -\frac{1}{2} |\Psi| \int d^3 x \frac{\delta^2 |\Psi|}{\delta \phi^2(x)} \)

with quantum potential (B10) \( Q = -(1/2|\Psi|) \int d^3 x |\overset{\leftrightarrow}{\partial}^2 \phi^2(x)| \) in functional form and
standard physics abuse of notation (cf. Example 3.2 to conclude that Q generates mass).

The n particles attributed to the wave function \( \psi_n \) also have trajectories given via

\[
\frac{d\mathbf{x}_{n,j}}{dt} = \left( \frac{\psi_n^*(x^{(n)}) \overset{\leftrightarrow}{\partial}_\mu \psi_n(x^{(n)})}{\psi_n(x^{(n)}) \overset{\leftrightarrow}{\partial}_\mu \psi_n(x^{(n)})} \right)_{t_1=\ldots=t_n=t}
\]

An effectivity parameter (B11) \( \epsilon_n[\phi,t] = |\Psi_n[\phi,t]|^2 / \sum_{n'}^\infty |\Psi_{n'}[\phi,t]|^2 \) is defined to be a non-local hidden variable attributed to the particle introduced to provide a deterministic
description of the creation and annihilation of particles. Here \( \Psi[\phi,t] = \sum_0^\infty \Psi_n[\phi,t] \) where \( \Psi_n \)
are unnormalized n-particle wave functionals and \( \psi_n(x,t) = <0|\hat{\phi}(t,x)\ldots\hat{\phi}(t,x)|\Psi> \). ■

**EXAMPLE 3.2.** Quantum fields are also discussed briefly in [B1] and we extract here from
this source. The approach follows [?] and one takes \( \mathcal{L} = (1/2) \partial^\mu \psi \partial^\mu \psi = (1/2)[\pi^2 - (\nabla \psi)^2] \) as Lagrangian where \( \dot{\psi} = \partial_t \psi \) and variational technique yields the wave equation \( \Box \psi = 0 \)
\( (\hbar = c = 1) \). Define conjugate momentum as \( \pi = \partial \mathcal{L}/\partial \dot{\psi} \), the Hamiltonian via \( \mathcal{H} = \pi \dot{\psi} - \mathcal{L} = (1/2)[\pi^2 + (\nabla \psi)^2] \), and the field Hamiltonian by \( \hat{\mathcal{H}} = \int \mathcal{H} d^3x \). Replacing \( \pi \) by
\( \frac{\partial S}{\partial t} + \frac{1}{2} \int d^3x \left[ \left( \frac{\delta S}{\delta \psi} \right)^2 + (\nabla \psi)^2 \right] = 0 \)

The term \( \frac{1}{2} \int d^3x (\nabla \psi)^2 \) plays the role of an external potential. To quantize the system one treats \( \psi(x) \) and \( \pi(x) \) as Schrödinger operators with \([\psi(x), \psi(x')] = [\pi(x), \pi(x')] = 0 \) and \([\psi(x), \pi(x')] = i\delta(x - x') \). Then one works in a representation \( |\psi(x)\rangle \) in which the Hermitian operator \( \psi(x) \) is diagonal. The Hamiltonian becomes an operator \( \hat{H} \) acting on a wavefunction \( \Psi[\psi(x), t] = \langle \psi(x) | \Psi(t) \rangle \) which is a functional of the real field \( \psi \) and a function of \( t \). This is not a point function of \( x \) since \( \Psi \) depends on the variable \( \psi \) for all \( x \).

Now the SE for the field is

\[ \frac{\partial \Psi}{\partial t} + \frac{1}{2} \int d^3x \left[ -\frac{\delta^2}{\delta \psi^2} + (\nabla \psi)^2 \right] \Psi = 0; \]

Thus \( \psi \) is playing the role of the space variable \( x \) in the particle SE and the continuous index \( x \) here is analogous to the discrete index \( i \) in the many particle theory. To arrive at a causal interpretation now one writes \( \Psi = R \exp(iS) \) for \( R, S[\psi, t] \) real functionals and decomposes (3.6) as

\[ \frac{\partial S}{\partial t} + 1 \int d^3x \left[ \left( \frac{\delta S}{\delta \psi} \right)^2 + (\nabla \psi)^2 \right] + Q = 0; \]

where the quantum potential is now \( B_{12} = \left(-\frac{1}{2}R\right) \int d^3x (\delta^2 R/\delta \psi^2) \). (3.7) now gives a conservation law wherein at time \( t \) \( R^2 D\psi \) is the probability for the field to lie in an element of volume \( D\psi \) around \( \psi \), where \( D\psi \) means roughly \( \prod_x d\psi \) and there is a normalization \( \int |\Psi|^2 D\psi = 1 \). Now introduce the assumption that at each instant \( t \) the field \( \psi \) has a well defined value for all \( x \) as in classical field theory, whatever the state \( \Psi \). Then the time evolution is obtained from the solution of the “guidance” formula

\[ \frac{\partial \psi(x, t)}{\partial t} = \frac{\delta S[\psi(x, t)]}{\delta \psi(x)} \bigg|_{\psi(x) = \psi(x, t)} \]

(analogous to \( m\ddot{x} = \nabla S \)) once one has specified the initial function \( \psi_0(x) \) in the HJ formalism. To find the equation of motion for the field coordinates apply \( \delta/\delta \psi \) to the HJ equation (3.7) to get

\[ \frac{d}{dt} \dot{\psi} = -\frac{\delta}{\delta \psi} \left[ Q + \frac{1}{2} \int d^3x (\nabla \psi)^2 \right]; \]

This is analogous to \( m\ddot{x} = -\nabla (V + Q) \) and, noting that \( d\dot{\psi}/dt = \partial \dot{\psi}/dt \) and taking the classical external force term to the right one arrives at

\[ \square \psi(x, t) = -\frac{\delta Q[\psi(x, t)]}{\delta \psi(x)} \bigg|_{\psi(x) = \psi(x, t)} \]

The quantum force term on the right side is responsible for all the characteristic effects of QFT. In particular comparing to a classical massive KG equation \( \square \psi + m^2 \psi = 0 \) with
suitable initial conditions one can argue that the quantum force generates mass in the sense that the massless quantum field acts as if it were a classical field with mass given via the quantum potential (cf. Remark 2.1).

EXAMPLE 3.3. In the fermionic theory of [130] one defines trajectory velocities

\[
\frac{dx^P}{dt} = \frac{j^P(t, x^P)}{j_0^P(t, x^P)}, \quad \frac{dx^A}{dt} = \frac{j^A(t, x^A)}{j_0^A(t, x^A)}
\]

for a causal interpretation of the Dirac equation. For the field theory the Grassman fields are bosonized in terms of \( \phi \) fields and it is shown how to create sources and velocities leading to an equation \((B13)\) \( d\bar{\phi}/dt = \bar{v}(\phi, t) \). This is made explicit and effectivity parameters are defined again (cf. [130] for details and philosophy).

EXAMPLE 3.4. Going now to [63, 64, 65] the philosophy revolves around Bell type QFT containing the idea of stochastic jumps via Markov processes but expressed field theoretically. (We refer to [121] for the diffusion approach to relativistic QM via Markov processes with jumps; the approach is however very different.) We mention also again that the KG equation is not covered (yet) in this theory (cf. however [16, 91]). The central formula is

\[
\sigma(dq|q') = \left\{ \frac{(2/\hbar)^3}{\langle \Psi|P(dq)|H^\dagger(dq')|\Psi > \rangle} \right\}^+ - \frac{\langle \Psi|P(dq')|\Psi > \rangle}{\langle \Psi|P(dq)|\Psi > \rangle} + \frac{\langle \Psi|P(dq)|\Psi > \rangle}{\langle \Psi|P(dq)|\Psi > \rangle}
\]

which describes the jump probability in a space \( \Omega = \cup \Omega^\alpha \) of world lines. Transition probabilities for the Markov process \( Q_t \) are described by forward and backward generators \( L_t \) and \( L_t^\dagger \) which are dual via \((B14)\) \( \int f(q)\mathcal{L}_t\rho(dq) = \int L_t f(q)\rho(dq) \). One looks for equivariant transition probabilities so that \( |\Psi_t|^2 = \rho_t \) for all \( t \) corresponds to \( L_t\rho_t = \partial_t \rho_t = \partial_t |\Psi|^2 \). Jump processes will correspond to integral operator type Hamiltonians and one will have jump relations \((B15)\) \( L^\dagger\rho(dq) = \int_{q' \in \Omega} \left( \sigma(dq'|q')\rho(dq') - \sigma(dq)d(q')\rho(dq) \right) \) subsequent to which one goes through various constructions involving positive operator valued measures. We refer to [63, 64, 65] for details and many examples (cf. also [9, 10, 25, 51, 75, 175] for information on Bohmian theory).

REMARK 3.1. For a discussion of a possible quantum origin of the gravitational interaction via the quantum potential see [114].

REMARK 3.2. There are a number of papers by T. Arimitsu et al dealing with quantum stochastic diffusion equations for boson and fermion systems in the context of nonequilibrium thermo field dynamics (NETFD). The materia also goes into the thermodynamics of multifractal systems and turbulence and we refer here to [5, 6, 95, 103]. This material seems very interesting but we postpone discussion for now.

REMARK 3.3. We mention also [105, 106, 107] where some interesting thermal equations arise related to Schrödinger equations, Klein-Gordon equations, etc. One deals with systems (micro or macro) having a thermal history described via \((B16)\) \( q(t) = \int_{t=\infty}^t K(t - t')\nabla T(t')dt' \). Here \( q(t) \) is the density of the energy flux and \( K \) describes the thermal memory, often of the form \((B17)\) \( K(t-t') = (K/\tau) \exp[-(t-t')/\tau] \) where \( K \) is constant and \( \tau \)
denotes the relaxation time. There are three principal situations

\[
K(t-t') = \begin{cases} 
K \lim_{t_0 \to 0} \delta(t-t' - t_0) & \text{diffusion} \\
K = \text{constant} & \text{wave} \\
(K/\tau) \exp \left[ -\frac{(t-t')}{\tau} \right] & \text{damped wave or hyperbolic diffusion}
\end{cases}
\]

The damped wave or hyperbolic diffusion equation is \((B18)\) \(\partial_t^2 T + (1/\tau)\partial_t T = (D_T/\tau)\nabla^2 T\) which for \(\tau \to 0\) becomes \((B19)\) \(\partial_t T = D_T\nabla^2 T\) where \(D_T\) is the thermal diffusion coefficient. The systems with very short relaxation times have very short memory. For \(\tau \to \infty\) \((B18)\) has the form of an undamped thermal wave equation or ballistic thermal equation. In solid state physics the ballistic phonons or electrons are those for which \(\tau \to \infty\). Experiments with ballistic phonons or electrons demonstrate the existence of wave motion on the lattice scale or on the electron gas scale \((B20)\) \(\partial_t^2 T = (D_T/\tau)\nabla^2 T\). Now define in \((B18)\) the quantity \((B21)\) \(v = (D_T/\tau)^{1/2}\) (velocity of thermal wave propagation) and \((B22)\) \(\lambda = v\tau\) (\(\lambda\) is the mean free path of the heat carriers). Then \((B18)\) can be written as \((B23)\) \((1/v^2)\partial_t^2 T + (1/\tau v^2)\partial_t T = \nabla^2 T\). Formally with substitutions \(t \leftrightarrow it\) and \(T \leftrightarrow \psi\) this is \((B24)\) \(i\hbar\psi_t = -(\hbar^2/2m)\nabla^2 \psi - \hbar\tau \psi_T\) where one uses \(D_T = \hbar^2/2m\) and \(\tau = h/2mv^2\). One could also embellish this with a potential term \(V\psi \sim VT\) to get

\[
(i\hbar)\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi - \hbar\tau \psi_T
\]

The term \(\tau \hbar \psi_T\) could be envisioned as Zitterbewegung describing the interaction of “thermal particles” (say electrons) with “spacetime” \(\sim\) a vacuum full of virtual particle pairs (say electron-positron pairs). One can argue that in certain realistic situations \((B25)\) \(\tau \sim \tau_P = \tau_{Planck} = (1/2)(\hbar G/c^5)^{1/2} = \hbar/2Mc^2\) where \(M_P \sim\) Planck mass; then \((3.14)\) can be written as

\[
(i\hbar)\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi - \frac{\hbar^2}{2M_P} \nabla^2 \psi + \frac{\hbar^2}{2M_P} \left( \nabla^2 \psi - \frac{1}{c^2} \psi_T \right)
\]

where the last two terms with \((B26)\) \(\nabla^2 \psi - (1/c^2)\psi_T = 0\) could perhaps be considered as describing a Bohmian type pilot wave (cf. \([21, 19, ?]\) for general philosophy). Note that the pilot wave does not depend on the particle mass \(m\) (nor on \(M_P\)) and for \(m \ll M_P\) Schrödinger mechanics prevails; however the pilot wave still exists.

### 4. SCALE RELATIVITY

In \([35]\) and Example 2.6 here we sketched a few developments in the theory of scale relativity. This is by no means the whole story and we want to give a taste of the main ideas along with deriving KG and Dirac equations in this context (cf. \([2, 14, 17, 45, 136, 137, 138, 139, 140, 141]\)). A main idea here is that the Schrödinger, Klein-Gordon, and Dirac equations are all geodesic equations in the fractal framework. They have the form \(D^2/ds^2 = 0\) where \(D/ds\) represents the appropriate covariant derivative. The complex nature of the SE and KGE arises from a discrete time symmetry breaking based on nondifferentiability. For the Dirac equation further discrete symmetry breakings are needed on the spacetime variables in a biquaternionic context (cf. here \([14]\)). First we go back to \([137, 140, 141]\) and sketch some of the fundamentals of scale relativity. This is a very rich and beautiful theory extending in both spirit and generality the relativity theory of Einstein but it does not yet appear...
to have been sanctified by the establishment (cf. also [13] for variations involving Clifford theory). The basic idea here is that (following Einstein) the laws of nature apply whatever the state of the system and hence the relevant variables can only be defined relative to other states. Standard scale laws of power-law type correspond to Galilean scale laws and from them one actually recovers quantum mechanics (QM) in a nondifferentiable space. The quantum behavior is a manifestation of the fractal geometry of spacetime. In particular (as indicated in Example 3.6) the quantum potential is a manifestation of fractality in the same way as the Newton potential is a manifestation of spacetime curvature. In this spirit one can also conjecture (cf. [14]) that this quantum potential may explain various dynamical effects presently attributed to dark matter (cf. also [4]). Now for basics one deals with a continuous but nondifferentiable physics. It is known for example that the length of a continuous nondifferentiable curve is dependent on the resolution $\epsilon$.

Here only $V$ differences or scale differences have a physical meaning. Thus $V$ is a "state of scale" just as velocity is a state of motion. In this spirit laws of linear transformation of

$$\log \frac{\phi(x',\epsilon)}{\phi_0} = \log \frac{\phi(x,\epsilon)}{\phi_0} + V\delta(\epsilon); \quad V = \log \frac{\epsilon}{\epsilon'}$$

In the same way that only velocity differences have a physical meaning in Galilean relativity here only $V$ differences or scale differences have a physical meaning. Thus $V$ is a "state of scale" just as velocity is a state of motion. In this spirit laws of linear transformation of
fields in a scale transformation \( \epsilon \rightarrow \epsilon' \) amount to finding \( A, B, C, D(V) \) such that

\[
\log \frac{\phi(\epsilon')}{\phi_0} = A(V) \log \frac{\phi(\epsilon)}{\phi_0} + B(V) \delta(\epsilon); \quad \delta(\epsilon') = C(V) \log \frac{\phi(\epsilon)}{\phi_0} + D(V) \delta(\epsilon)
\]

Here \( A = 1, B = V, C = 0, D = 1 \) corresponds to the Galilean dilation law for scales larger than the quantum transitional transition. Note also \((C8) \epsilon \rightarrow \epsilon' \rightarrow \epsilon'' \Rightarrow V'' = V + V'\). For the analogue of Lorentz transformations there is a need to preserve the Galilean dilation law for scales that follow that \((C9) \epsilon \rightarrow \epsilon', \rho' : \epsilon' \rightarrow \epsilon''\) and \(\rho'' : \epsilon \rightarrow \epsilon''\) with compositions (the notation is meant to correspond to \(C2\)). We recall the Einstein-Lorentz law \((C10) w = (u + v)/[1 + (uv/c^2)]\) but one now has several regimes to consider. Following \([11]\) small scale symmetry is broken by mass via the emergence of \((C11) \lambda = \lambda_c = \hbar/mc \) (Compton length) and \(\lambda_{dB} = \hbar/mv \) (deBroglie length), while for extended objects \(\lambda_{dB} = \hbar/m \nu^2 > 1/2 \) (thermal deBroglie length) determines the transition scale. The transition scale in \([11]\) is the Einstein-deBroglie scale (in rest frame \(\lambda = \tau = \hbar/mc^2\) and in the cosmological realm the scale symmetry is broken by the emergence of static structure of typical size \(\lambda_g = GM/3 < \nu^2\)). The scale space consists of three domains (quantum, classical - scale independent, and cosmological). Another small scale transition factor appears in the Planck length scale \(\lambda_P = (\hbar G/c^3)^{1/2}\) and at large scales the cosmological constant \(\Lambda\) comes into play. With this background the composition of dilatations is taken to be

\[
\log \frac{\epsilon'}{\lambda} = \frac{\log \rho + \log \frac{\epsilon'}{\lambda}}{1 + \log \frac{\log(\epsilon/\lambda)}{C^2}} = \frac{\log \rho + \log \frac{\epsilon'}{\lambda}}{1 + \log \frac{\log(\epsilon/\lambda)}{C^2}}
\]

where \(L \sim \lambda_P\) near small scales and \(L \sim \Lambda\) near large scales (note \(\epsilon = L \Rightarrow \epsilon' = L\) in \([11]\)). Comparing with \((C10)\) one thinks of \(\log(L/\lambda) = C \sim c\) (note here \(\log^2(a/b) = \log^2(b/a)\)) in comparing formulas in \([10, 11]\). Lengths now change via

\[
\log \frac{\epsilon'}{\tau_0} = \frac{\log(\ell/\ell_0) + \delta \log \rho}{\sqrt{1 - \frac{\log^2 \rho}{C^2}}}
\]

and the scale variable \(\delta\) (or djinn) is no longer constant but changes via

\[
\delta(\epsilon') = \frac{\delta(\epsilon) + \log \frac{\log(\ell/\ell_0)}{C^2}}{\sqrt{1 - \frac{\log^2 \rho}{C^2}}}
\]

where \(\lambda \sim \text{fractal-nonfractal transition scale}\).
\( D = 2 \) via \((C12)\) \( dX^\mu_\pm = d_\pm x^\mu + d_\pm \xi^\mu = v^\mu_\pm ds + u^\mu_\pm \sqrt{2D} ds^{1/2} \). Here \( u^\mu_\pm \) is a dimensionless fluctuation and the length scale \( 2D \) is introduced for dimensional purposes. The large scale forward and backward derivatives \( d/ds_+ \) and \( d/ds_- \) are defined via

\[
\frac{d}{ds_-} f(s) = \lim_{s \to 0} \frac{\mathcal{LS} \left( f(s + \delta s) - f(s) \right)}{\delta s}
\]

Applied to \( x^\mu \) one obtains the forward and backward large scale four velocities of the form \((C13)\) \((d/dx_+) x^\mu(s) = v^\mu_+ \) and \((d/dx_-) x^\mu = v^\mu_- \). Combining yields

\[
\frac{d'}{ds} x^\mu = V^\mu - iU^\mu = \frac{v^\mu_+ + v^\mu_-}{2} - i \frac{v^\mu_+ - v^\mu_-}{2}
\]

For the fluctuations one has \((C14)\) \( \mathcal{LS} < d\xi^\mu_+ d\xi^\nu_- > = +2D \eta^{\mu\nu} ds \). One chooses here \((+, -, -, -)\) for the Minkowski signature for \( \eta^{\mu\nu} \) and there is a mild problem because the diffusion (Wiener) process makes sense only for positive definite metrics. Various solutions were given in \([60, 165, 183]\) and they are all basically equivalent, amounting to the transformation of a Laplacian into a D'Alembertian. Thus the two forward and backward differentials of \( f(x, s) \) should be written as \((C15)\) \((df/ds_\pm) = (\partial_s + v^\mu_\pm \partial_\mu + D \partial^\mu \partial_\mu) f \). One considers now only stationary functions \( f \), not depending explicitly on the proper time \( s \), so that the complex covariant derivative operator reduces to \((C16)\) \((df/ds) = (V^\mu + iD \partial^\mu) \partial_\mu f \).

Now assume that the large scale part of any mechanical system can be characterized by a complex action \( \mathcal{S} \) leading one to write \((C17)\) \( \delta \mathcal{S} = -mc^2 \int_0^b ds = 0 \) where \( ds = \mathcal{LS} < \sqrt{dX^\nu dX_\nu} > \). This leads to \((C18)\) \( \delta \mathcal{S} = -mc^2 \int_0^b V_\nu d(\delta x^\nu) \) with \( \delta x^\nu = \mathcal{LS} < dX^\nu > \). Integrating by parts yields \((C19)\) \( \delta \mathcal{S} = -[mc \delta x^\nu]_a^b + mc \int_0^b \delta x^\nu (dV_\mu/ds) ds \). To get the equations of motion one has to determine \( \delta \mathcal{S} = 0 \) between the same two points, i.e. at the limits \( (\delta x^\nu)_a = (\delta x^\nu)_b = 0 \). From \((C19)\) one obtains then a differential geodesic equation \((C20)\) \( dV/ds = 0 \). One can also write the elementary variation of the action as a functional of the coordinates. So consider the point \( a \) as fixed so \( (\delta x^\nu)_a = 0 \) and consider \( b \) as variable. The only admissible solutions are those satisfying the equations of motion so the integral in \((C19)\) vanishes and writing \( (\delta x^\nu)_b \) as \( \delta x^\nu \) gives \((C21)\) \( \delta \mathcal{S} = -mc \nu_\nu \delta x^\nu \) (the minus sign comes from the choice of signature). The complex momentum is now \((C22)\) \( \mathcal{P}_\nu = mc \nu_\nu = -\partial_\nu \mathcal{S} \) and the complex action completely characterizes the dynamical state of the particle. Hence introduce a wave function \( \psi = \exp(i \mathcal{S}/mc) \) and via \((C22)\) one gets \((C23)\) \( \mathcal{V}_\nu = (i\mathcal{S}/mc) \partial_\nu \log(\psi) \). Now for the scale relativistic prescription replace the derivative in \((C20)\) by its covariant expression \((C16)\). Using \((C23)\) one transforms \((C20)\) into

\[
- \frac{\mathcal{S}_0^2}{m^2 c^2} \partial^\mu \log(\psi) \partial_\mu \partial_\nu \log(\psi) - \frac{\mathcal{S}_0 D}{mc} \partial^\mu \partial_\mu \partial_\nu \log(\psi) = 0
\]

The choice \( \mathcal{S}_0 = \hbar = 2mcD \) allows a simplification of \((4.9)\) when one uses the identity

\[
\frac{1}{2} \left( \frac{\partial_\mu \partial_\nu \psi}{\psi} \right) = \left( \partial_\mu \log(\psi) + \frac{1}{2} \partial_\mu \right) \partial^\mu \partial^\nu \log(\psi)
\]
Dividing by \( D^2 \) one obtains the equation of motion for the free particle \((C24) \) \( \partial' \left[ \partial^\mu \partial_\mu \psi / \psi \right] = 0. \) Therefore the KG equation (no electromagnetic field) is \((C25) \) \( \partial^\mu \partial_\mu \psi + (m^2 c^2 / \hbar^2) \psi = 0 \) and this becomes an integral of motion of the free particle provided the integration constant is chosen in terms of a squared mass term \( m^2 c^2 / \hbar^2. \) Thus the quantum behavior described by this equation and the probabilistic interpretation given to \( \psi \) is reduced here to the description of a free fall in a fractal spacetime, in analogy with Einstein’s general relativity. Moreover these equations are covariant since the relativistic quantum equation written in terms of \( d'/ds \) has the same form as the equation of a relativistic macroscopic and free particle using \( d/\)s. One notes that the metric form of relativity, namely \( V^\mu V_\mu = 1 \) is not conserved in QM and it is shown in \([151]\) that the free particle KG equation expressed in terms of \( V \) leads to a new equality \((C26) \) \( V^\mu V_\mu + 2i D \partial^\mu V_\mu = 1. \) In the scale relativistic framework this expression defines the metric that is induced by the internal scale structures of the fractal spacetime. In the absence of an electromagnetic field \( \mathcal{F} \) and \( \mathcal{G} \) are related by \((C22) \) which can be written as \((C27) \) \( V_\mu = - (1/mc) \partial_\mu \mathcal{G} \) so \((C26) \) becomes \((C28) \) \( \partial^\mu \mathcal{G} \partial_\mu \mathcal{G} - 2imcD \partial^\mu \partial_\mu \mathcal{G} = m^2 c^2 \) which is the new form taken by the Hamilton-Jacobi equation.

**REMARK 4.1.** We go back to \([141, 151]\) now and repeat some of their steps in a perhaps more primitive but revealing form. Thus one omits the \( LS \) notation and uses \( \lambda \sim 2D; \) equations \((C12)-(C16) \) and \((4.5) \) are the same and one writes now \( \partial / ds \) for \( d'/ds. \) Then \( \partial / ds = V^\mu \partial_\mu + (i\lambda / 2) \partial^\mu \partial_\mu \) plays the role of a scale covariant derivative and one simply takes the equation of motion of a free relativistic quantum particle to be given as \((C29) \) \( (\partial / ds) V^\mu = 0, \) which can be interpreted as the equations of free motion in a fractal spacetime or as geodesic equations. In fact now \((C29) \) leads directly to the KG equation upon writing \( \psi = \exp (i \mathcal{G} / mc \lambda) \) and \( \mathfrak{F}^\mu = - \partial^\mu \mathcal{G} = mcV^\mu \) so that \( i \mathcal{G} = mc \lambda \log (\psi) \) and \( V^\mu = i \lambda \partial^\mu \log (\psi) \). Then

\[
(4.11) \quad \left( V^\mu \partial_\mu + \frac{i\lambda}{2} \partial^\mu \partial_\mu \right) \partial^\nu \log (\psi) = 0 = i\lambda \left( \frac{\partial^\mu \psi}{\psi} \partial_\mu + \frac{1}{2} \partial^\mu \partial_\mu \right) \partial^\nu \log (\psi)
\]

Now some identities are given in \([151]\) for aid in calculation here, namely

\[
(4.12) \quad \partial^\nu \left( \frac{\partial^\mu \psi}{\psi} \partial_\mu \psi \right) = \partial^\mu \left( \frac{\partial^\nu \psi}{\psi} \partial_\mu \psi \right) = \partial^\nu \left( \frac{\partial^\mu \psi}{\psi} \right) \partial_\mu \left( \frac{\partial^\nu \psi}{\psi} \right) + \frac{\partial^\mu \partial^\nu \partial_\mu \psi}{\psi} = \frac{\partial^\mu \partial_\mu \psi}{\psi}
\]

The first term in the last equation of \((4.11) \) is then \((1/2)[(\partial^\mu \psi / \psi)(\partial_\mu \psi / \psi)] \) and the second is

\[
(4.13) \quad (1/2) \partial^\mu \partial_\mu \partial^\nu \log (\psi) = (1/2) \partial^\nu \partial^\mu \partial_\mu \log (\psi) = (1/2) \partial^\nu \left( \frac{\partial^\mu \partial_\mu \psi}{\psi} - \frac{\partial^\mu \psi \partial_\mu \psi}{\psi^2} \right)
\]

Combining we get \((C30) \) \((1/2) \partial^\nu (\partial^\mu \partial_\mu \psi / \psi) = 0 \) which integrates then to a KG equation \((C31) \) \(- (\hbar^2 / m^2 c^2) \partial^\mu \partial_\mu \psi / \psi = \psi \) for suitable choice of integration constant (note \( \hbar / mc \) is the Compton wave length).
Now in this context or above we refer back to Section 2.2 and write \( Q = -(1/2m)(\square R/R) \) (cf. (A50) and take \( h = c = 1 \) for convenience here). Then recall (C32) \( \psi = \exp(iS/m\lambda) \) and \( \mathfrak{V}_\mu = mV_\mu = -\partial_\mu \mathfrak{S} \) with \( i\mathfrak{S} = m\log(\psi) \). Also (C33) \( V_\mu = -(1/m)\partial_\mu \mathfrak{S} = i\lambda\partial_\mu \log(\psi) \) with \( \psi = R\exp(iS/m\lambda) \) so \( \log(\psi) = i\mathfrak{S}/m\lambda = \log(R) + iS/m\lambda \), leading to

\[
V_\mu = i\lambda[\partial_\mu \log(R) + (i/m\lambda)\partial_\mu S] = -\frac{1}{m}\partial_\mu S + i\lambda\partial_\mu \log(R) = V_\mu + iU_\mu
\]

Then \( \square = \partial^\mu \partial_\mu \) and \( U_\mu = \lambda\partial_\mu \log(R) \) leads to (C34) \( \partial^\mu U_\mu = \lambda\partial^\mu \partial_\mu \log(R) = \lambda\square \log(R) \).

Further \( \partial^\mu \partial_\mu \log(R) = (\partial^\mu \partial_\mu R/R) - (R/R_\mu/R^2) \) so (C35) \( \square \log(R) = \partial^\mu \partial_\mu \log(R) = (\square R/R) - (\sum R_i^2/R^2) = (\square R/R) - \sum (\partial_\mu R/R)^2 = (\square R/R) - |U|^2 \) for \(|U|^2 = \sum U_\mu^2 \). Hence via \( \lambda = 1/2m \) for example one has

\[
Q = -(1/2m)(\square R/R) = -\frac{1}{2m} \left[ |U|^2 + \frac{1}{\lambda} \square \log(R) \right] = -\frac{1}{2m} |U|^2 - \frac{1}{2} \text{div}(U)
\]

(cf. Example 2.6 and Section 2.2).

**REMARK 4.2.** The words fractal spacetime as used in the scale relativity methods of Nottale et al for producing geodesic equations (SE or KG equation) are somewhat misleading in that essentially one is only looking at continuous nondifferentiable paths for example. Scaling as such is of course considered extensively at other times as partially indicated in Section 4. It would be nice to create a fractal derivative based on scaling properties and H-dimension alone for example which would permit the powerful techniques of calculus to be used in a fractal context. There has been of course some work in this direction already in e.g. [46, 76, 81, 88, 104, 101, 142, 149, 156] and we will sketch some of this below.

## 5. FRACTAL CALCULUS

We sketch first (in summary form) from [149] where a calculus based on fractal subsets of the real line is formulated. A local calculus based on renormalizing fractional derivatives à la [104] is subsumed and embellished. Consider first the concept of content or \( \alpha \)-mass for a (generally fractal) subset \( F \subset [a,b] \) (in what follows \( 0 < \alpha \leq 1 \)). Then define the flag function for a set \( F \) and a closed interval \( I \) as (D1) \( \theta(F, I) = 1 \) if \( F \cap I \neq \emptyset \) and otherwise \( \theta = 0 \). Then a subdivision \( P_{[a,b]} \sim P \) of \([a,b]\) \((a < b)\) is a finite set of points \( \{a = x_0, x_1, \ldots, x_n = b\} \) with \( x_i < x_{i+1} \). If \( Q \) is any subdivision with \( P \subset Q \) it is called a refinement and if \( a = b \) the set \( \{a\} \) is the only subdivision. Define then

\[
\sigma^\alpha[F,p] = \sum_{0}^{n-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)} \theta(F, [x_i, x_{i+1}])
\]

For \( a = b \) one defines \( \sigma^\alpha[F,p] = 0 \). Next given \( \delta > 0 \) and \( a \leq b \) the coarse grained mass \( \gamma^\alpha_\delta(F,a,b) \) of \( F \cap [a,b] \) is given via

\[
\gamma^\alpha_\delta(F,a,b) = \inf_{|P| \leq \delta} \sigma^\alpha[F,P] \quad (|P| = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i))
\]
where the infimum is over \( P \) such that \( |P| \leq \delta \). Various more or less straightforward properties are:

- For \( a \leq b \) and \( \delta_1 < \delta_2 \) one has (D2) \( \gamma_{\alpha}^\delta(F,a,b) \geq \gamma_{\alpha}^{\delta_1}(F,a,b) \).
- For \( \delta > 0 \) and \( a < b < c \) one has (D3) \( \gamma_{\alpha}^\delta(F,a,b) \leq \gamma_{\alpha}^\delta(F,a,c) \) and \( \gamma_{\alpha}^\delta(F,b,c) \leq \gamma_{\alpha}^\delta(F,a,c) \).
- (D4) \( \gamma_{\alpha}^\delta \) is continuous in \( b \) and \( a \).

Now define the mass function \( \gamma_{\alpha}(F,a,b) \) via (D5) \( \gamma_{\alpha}(F,a,b) = \lim_{\delta \to 0} \gamma_{\alpha}^\delta(F,a,b) \). The following results are proved

1. (D6) If \( F \cap (a,b) = \emptyset \) then \( \gamma_{\alpha}(F,a,b) = 0 \).
2. Let \( a < b < c \) and \( \gamma_{\alpha}(F,a,c) < \infty \). Then (D7) \( \gamma_{\alpha}(F,a,c) = \gamma_{\alpha}(F,a,b) + \gamma_{\alpha}(F,b,c) \).
   Hence \( \gamma_{\alpha}(F,a,b) \) is increasing in \( b \) and decreasing in \( a \).
3. (D8) Let \( a < b \) and \( \gamma_{\alpha}(F,a,b) \neq 0 \) be finite. If \( 0 < y < \gamma_{\alpha}(F,a,b) \) then there exists \( c, a < c < b \) such that \( \gamma_{\alpha}(F,a,c) = y \). Further if \( \gamma_{\alpha}(F,a,b) \) is finite then (D9) \( \gamma_{\alpha}(F,a,x) \) is continuous for \( x \in (a,b) \).
4. For \( F \subseteq \mathbb{R} \) and \( \lambda \in \mathbb{R} \) let (D9) \( F + \lambda = \{ x + \lambda; x \in F \} \). Then (D10) \( \gamma_{\alpha}(F + \lambda,a + \lambda,b + \lambda) = \gamma_{\alpha}(F,a,b) \) and \( \gamma_{\alpha}(x F, \lambda a, \lambda b) = \lambda^\alpha \gamma_{\alpha}(F,a,b) \).

Now for \( a_0 \) an arbitrary fixed real number one defines the integral staircase function of order \( \alpha \) for \( F \) as

\[
S^\alpha_F(x) = \begin{cases} 
\gamma_{\alpha}(F,a_0,x) & x \geq a_0 \\
-\gamma_{\alpha}(F,x,a_0) & \text{otherwise}
\end{cases}
\]  

(5.3)

The following properties of \( S^\alpha_F \) are restatements of properties for \( \gamma_{\alpha} \). Thus

- \( S^\alpha_F(x) \) is increasing in \( x \).
- If \( F \cap (x,y) = \emptyset \) then \( S^\alpha_F \) is constant in \([x,y]\).
- \( S^\alpha_F(y) - S^\alpha_F(x) = \gamma_{\alpha}(F,x,y) \).
- \( S^\alpha_F \) is continuous on \((a,b)\).

Now one considers the sets \( F \) for which the mass function \( \gamma_{\alpha}(F,a,b) \) gives the most useful information. Indeed one can use the mass function to define a fractal dimension. If \( 0 < \alpha \leq 1 \) one writes

\[
\sigma^\beta[F,P] \leq |P|^\beta - \alpha \sigma^\alpha[F,P] \frac{\Gamma(\alpha + 1)}{\Gamma(\beta + 1)}; \quad \gamma_{\alpha}^\beta(F,a,b) \leq \delta^{\beta - \alpha} \gamma_{\alpha}^\delta(F,a,b) \frac{\Gamma(\alpha + 1)}{\Gamma(\beta + 1)}
\]  

(5.4)

Thus in the limit \( \delta \to 0 \) one gets \( \gamma_{\alpha}^\beta(F,a,b) = 0 \) provided \( \gamma_{\alpha}^\beta(F,a,b) < \infty \) and \( \alpha < \beta \). It follows that \( \gamma_{\alpha}(F,a,b) \) is infinite up to a certain value \( a_0 \) and then jumps down to zero for \( \alpha > a_0 \) (if \( a_0 < 1 \)). This number is called the \( \gamma \)-dimension of \( F \); \( \gamma_{\alpha}(F,a,b) \) may itself be zero, finite, or infinite. To make the definition precise one says that the \( \gamma \)-dimension of \( F \cap [a,b] \), denoted by \( \dim_{\gamma}(F \cap [a,b]) \), is

\[
\dim_{\gamma}(F \cap [a,b]) = \begin{cases} 
\inf\{\alpha; \ \gamma_{\alpha}(F,a,b) = 0\} \\
\sup\{\alpha; \ \gamma_{\alpha}(F,a,b) = \infty\}
\end{cases}
\]  

(5.5)

One shows that (D11) \( \dim_H(F \cap [a,b]) \leq \dim_{\gamma}(F \cap [a,b]) \) where \( \dim_H \) denotes Hausdorff dimension. Further (D12) \( \dim_{\gamma}(F \cap [a,b]) \leq \dim_B(F \cap [a,b]) \) where \( \dim_B \) is the box dimension. Some further analysis shows that (D14) For \( F \subseteq \mathbb{R} \) compact \( \dim_{\gamma}F = \dim_H F \).
Next one notes that the correspondence $F \rightarrow S_F^\alpha$ is many to one (examples from Cantor sets) and one calls the sets giving rise to the same staircase function “staircasewise congruent”. The equivalence class of congruent sets containing $F$ is denoted by $E_F$; thus if $G \in E_F$ it follows that $S_G^\alpha = S_F^\alpha$ and $E_G^\alpha = E_F^\alpha$. One says that a point $x$ is a point of change of $f$ if $f$ is not constant over any open interval $(c,d)$ containing $x$. The set of all points of change of $f$ is denoted by $Sch(f)$. In particular if $G \in E_F^\alpha$ then $S_G^\alpha(x) = S_F^\alpha(x)$ so $Sch(S_G^\alpha) = Sch(S_F^\alpha)$.

Thus if $F \subseteq \mathbb{R}$ is such that $S_F^\alpha(x)$ is finite for all $x$ ($\alpha = dim_F F$) then $H = Sch(S_F^\alpha) \in E_F^\alpha$. This takes some proving which we omit (cf.\ [149]). As a consequence let $F \subseteq \mathbb{R}$ be such that $S_F^\alpha(x)$ is finite for all $x \in \mathbb{R}$ ($\alpha = dim_F F$). Then the set $H = Sch(S_F^\alpha)$ is perfect (i.e. $H$ is closed and every point is a limit point). Hence given $S_F^\alpha(x)$ finite for all $x$ ($\alpha = dim_F F$) one calls $Sch(S_F^\alpha)$ the $\alpha$-perfect representative of $E_F^\alpha$ and one proves that it is the minimal closed set in $E_F^\alpha$. Indeed an $\alpha$-perfect set in $E_F^\alpha$ is the intersection of all closed sets $G$ in $E_F^\alpha$. One can also say that if $F \subseteq \mathbb{R}$ is $\alpha$-perfect and $x \in F$ then for $y < x < z$ either $S_F^\alpha(y) < S_F^\alpha(x)$ or $S_F^\alpha(x) < S_F^\alpha(z)$ (or both). Thus for an $\alpha$-perfect set it is assured that the values of $S_F^\alpha(y)$ must be different from $S_F^\alpha(x)$ at all points $y$ on at least one side of $x$. As an example one shows that the middle third Cantor set $C = E_{1/3}$ is $\alpha$-perfect for $\alpha = \log(2)/\log(3) = d_H(C)$ so $C = Sch(S_C^\alpha)$.

Now look at $F$ with the induced topology from $\mathbb{R}$ and consider the idea of F-continuity.

**DEFINITION 5.1.** Let $F \subseteq \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ with $x \in F$. A number $\ell$ is said to be the limit of $f$ through the points of $F$, or simply $F$-limit, as $y \rightarrow x$ if given $\epsilon > 0$ there exists $\delta > 0$ such that $y \in F$ and $|y - x| < \delta \Rightarrow |f(y) - \ell| < \epsilon$. In such a case one writes $\ell = F$-limit$_{y \rightarrow x} f(y)$. A function $f$ is $F$-continuous at $x \in F$ if $f(x) = F$-limit$_{y \rightarrow x} f(y)$ and uniformly $F$-continuous on $E \subseteq F$ if for $\epsilon > 0$ there exists $\delta > 0$ such that $x \in F$, $y \in E$ and $|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon$. One sees that if $f$ is $F$-continuous on a compact set $E \subseteq \mathbb{R}$ then it is uniformly $F$-continuous on $E$.

**DEFINITION 5.2.** The class of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which are bounded on $F$ is denoted by $B(F)$. Define for $f \in B(F)$ and $I$ a closed interval

$$M[f,F,I] = \begin{cases} \sup_{x \in F \cap I} f(x) & F \cap I \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$m[f,F,I] = \begin{cases} \inf_{x \in F \cap I} f(x) & F \cap I \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

**DEFINITION 5.3.** Let $S_F^\alpha(x)$ be finite for $x \in [a,b]$ and $P$ be a subdivision with points $x_0, \ldots, x_n$. The upper $F^\alpha$ and lower $F^\alpha$ sums over $P$ are given respectively by

$$U^\alpha[f,F,P] = \sum_{i=0}^{n-1} M[f,F,[x_i,x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i));$$

$$L^\alpha[f,F,P] = \sum_{i=0}^{n-1} m[f,F,[x_i,x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i))$$

This is sort of like Riemann-Stieltjes integration and in fact one shows that if $Q$ is a refinement of $P$ then $U^\alpha[f,F,Q] \leq U^\alpha[f,F,P]$ and $L^\alpha[f,F,Q] \geq L^\alpha[f,F,P]$. Further
One specifies also \( U^\alpha[f,F,P] \geq L^\alpha[f,F,Q] \) for any subdivisions of \([a,b]\) and this leads to the idea of \( F \)-integrability. Thus assume \( S^\alpha_F \) is finite on \([a,b]\) and for \( f \in B(F) \) one defines lower and upper \( F^\alpha \)-integrals via

\[
(5.8) \quad \int_a^b f(x)\,d_F^\alpha x = \sup \{ L^\alpha[f,F,P] : \int_a^b f(x)\,d_F^\alpha x = \inf f_U^\alpha[f,F,P] \}
\]

One then says that \( f \) is \( F^\alpha \)-integrable if \((D15)\) \( \int_a^b f(x)\,d_F^\alpha x = \int_a^b f(x)\,d_F^\alpha x = \int_a^b f(x)\,d_F^\alpha x \).

One shows then

1. if \( f \in B(F) \) is \( F^\alpha \)-integrable on \([a,b]\) if and only if for any \( \epsilon > 0 \) there is a subdivision \( P \) of \([a,b]\) such that \( U^\alpha[f,F,P] < L^\alpha[f,F,P] + \epsilon \).
2. Let \( F \cap [a,b] \) be compact with \( S^\alpha_F \) finite on \([a,b]\). Let \( f \in B(F) \) and \( a < b \); then if \( f \) is \( F \)-continuous on \( F \cap [a,b] \) it follows that \( f \) is \( F^\alpha \)-integrable on \([a,b]\).
3. Let \( a < b \) and \( f \) be \( F^\alpha \)-integrable on \([a,b]\) with \( c \in (a,b) \). Then \( f \) is \( F^\alpha \)-integrable on \([a,c]\) and \([c,b]\) with \((D16)\) \( \int_x^c f(x)\,d_F^\alpha x = \int_a^c f(x)\,d_F^\alpha x + \int_a^c f(x)\,d_F^\alpha x \).
4. If \( f \) is \( F^\alpha \)-integrable then \((D17)\) \( \int_a^b \chi(x)\,d_F^\alpha x = \int_a^b \chi(x)\,d_F^\alpha x \) and, for \( g \) also \( F^\alpha \)-integrable, \((D18)\) \( \int_a^b (f(x) + g(x))\,d_F^\alpha x = \int_a^b f(x)\,d_F^\alpha x + \int_a^b g(x)\,d_F^\alpha x \).
5. If \( f, g \) are \( F^\alpha \)-integrable and \( f(x) \geq g(x) \) for \( x \in F \cap [a,b] \) then \((D19)\) \( \int_a^b f(x)\,d_F^\alpha x \geq \int_a^b g(x)\,d_F^\alpha x \).

One specifies also \((D20)\) \( \int_a^b f(x)\,d_F^\alpha x = -\int_a^b f(x)\,d_F^\alpha x \) and it is easily shown that if \( \chi_F(x) \) is the characteristic function of \( F \) then \((D21)\) \( \int_a^b \chi_F(x)\,d_F^\alpha x = S^\alpha_F(b) - S^\alpha_F(a) \). Now for differentiation one writes

\[
(5.9) \quad D^\alpha_F f(x) = \begin{cases} F - \lim_{y \to x} \frac{f(y) - f(x)}{S^\alpha_F(y) - S^\alpha_F(x)} & x \in F \\ 0 & \text{otherwise} \end{cases}
\]

if the limit exists. One shows then

1. If \( D^\alpha_F f(x) \) exists for all \( x \in (a,b) \) then \( f(x) \) is \( F \)-continuous in \((a,b)\).
2. With obvious hypotheses \( D^\alpha_F(\lambda f(x)) = \lambda D^\alpha_F f(x) \) and \( D^\alpha_F(f + g)(x) = D^\alpha_F f(x) + D^\alpha_F g(x) \). Further if \( f \) is constant then \( D^\alpha_F f = 0 \).
3. \( D^\alpha_F(S^\alpha_F(x)) = \chi_F(x) \).
4. (Rolle’s theorem) Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous with \( \text{Sch}(f) \subset F \) where \( F \) is \( \alpha \)-perfect and assume \( D^\alpha_F f(x) \) is defined for all \( x \in [a,b] \) with \( f(a) = f(b) = 0 \). Then there is a point \( c \in F \cap [a,b] \) such that \( D^\alpha_F f(c) \geq 0 \) and a point \( d \in F \cap [a,b] \) where \( D^\alpha_F f(d) \leq 0 \).

**Example 5.1.** This is the best that can be done with Rolle’s theorem since for \( C \) the Cantor set \( E_{1/3} \) take \( f(x) = S^\alpha_F(x) \) for \( 0 \leq x \leq 1/2 \) and \( f(x) = 1 - S^\alpha_F(x) \) for \( 1/2 < x \leq 1 \). This function is continuous with \( f(0) = f(1) = 0 \) and the set of change \((\text{Sch}(f))\) is \( C \). The \( C^\alpha \)-derivative is given by \( D^\alpha_F f(x) = \chi_C(x) \) for \( 0 \leq x \leq 1/2 \) and by \(-\chi_C(x) \) for \( 1/2 < x \leq 1 \). Thus \( x \in C \Rightarrow D^\alpha_F f(x) = \pm 1 \neq 0 \).

As a corollary one has the following result: Let \( f \) be continuous with \( \text{Sch}(f) \subset F \) where \( F \) is \( \alpha \)-perfect; assume \( D^\alpha_F f(s) \) exists at all points of \([a,b]\) and that \( S^\alpha_F(b) \neq S^\alpha_F(a) \). Then
there are points \(c, d \in F\) such that

\[
(5.10) \quad D^\alpha_F f(c) \geq \frac{f(b) - f(a)}{S^\alpha_F(b) - S^\alpha_F(a)}; \quad D^\alpha_F f(d) \leq \frac{f(b) - f(a)}{S^\alpha_F(b) - S^\alpha_F(a)}
\]

Similarly if \(f\) is continuous with \(Sch(f) \subset F\) and \(D^\alpha_F f(x) = 0 \forall x \in [a, b]\) then \(f(x)\) is constant on \([a, b]\).

There are also other fundamental theorems as follows

1. (Leibniz rule) If \(u, v : R \to R\) are \(F^\alpha\)-differentiable then \((D22)\) \(D^\alpha_F (uv)(x) = (D^\alpha_F u(x))v(x) + u(x)(D^\alpha_F v)(x)\).

2. Let \(F \subset R\) be \(\alpha\)-perfect. If \(f \in B(F)\) is \(F\)-continuous on \(F \cap [a, b]\) with \(g(x) = \int_x^a f(y)d^\alpha_F y\) for all \(x \in [a, b]\) then \(D^\alpha_F g(x) = f(x)\chi_F(x)\).

3. Let \(f : R \to R\) be continuous and \(F^\alpha\)-differentiable with \(Sch(f)\) contained in an \(\alpha\)-perfect set \(F\); let also \(h : R \to R\) be \(F\)-continuous such that \(h(x)\chi_F(x) = D^\alpha_F f(x)\). Then \(\int_a^b h(x)d^\alpha_F x = f(b) - f(a)\).

4. (Integration by parts) Assume: (i) \(u\) is continuous on \([a, b]\) and \(Sch(u) \subset F\). (ii) \(D^\alpha_F u(x)\) exists and is \(F\)-continuous on \([a, b]\). (iii) \(v\) is \(F\)-continuous on \([a, b]\). Then

\[
(5.11) \quad \int_a^b uv d^\alpha_F x = \left[ u(x) \int_a^x v(x') d^\alpha_F x' \right]_a^b - \int_a^b D^\alpha_F u(x) \int_a^x v(x') d^\alpha_F x'
\]

Some examples are given relative to applications and we mention e.g.

**EXAMPLE 5.2.** Following [104] one has a local fractal diffusion equation

\[
(5.12) \quad D^\alpha_F W(x, t) = \frac{\partial^2}{\partial x^2} W(x, t)
\]

with solution

\[
(5.13) \quad W(x, t) = \frac{1}{(2\pi S^\alpha_F(t))^{1/2}} \exp \left(-\frac{x^2}{2S^\alpha_F(t)}\right)
\]

The appendix to [149] also gives some formulas for repeated integration and differentiation. For example it is shown that

\[
(5.14) \quad (D^\alpha_F)^2 (S^\alpha_F(x))^2 = 2\chi_F(x); \quad \int_a^{x'} (S^\alpha_F(x))^n d^\alpha_F x = \frac{1}{n+1} (S^\alpha_F(x'))^{n+1}
\]

We refer to [104] [149] for much other interesting stuff.

### 5.1. POSSIBLE APPLICATIONS.

Example 5.2 gives a model of a local fractal diffusion equation where the transition times \(t \in F\) are rare (or infrequent) and this was developed also in [104] for local fractional derivatives in a broader context. It is not clear whether or not the fractal calculus above will be useful in describing quantum paths of Hausdorff dimension two for example or other fractal curves such as Peano or von Koch curves (for which [104] looks more promising - see also [2] [17] [43]; such curves are not of the form \(y(t)\) with \(t \in F\) for sets \(F\) of zero topological dimension (such as Cantor sets).

Nevertheless for paths with rare transitions we indicate here heuristically how this fractal calculus from [149] might be applied to the SE in the spirit of Nottale et al as sketched in Examples 2.6 and 2.7 along with Remark 2.1. Thus instead of a continuous nondifferentiable curve think of an \(F\)-continuous path \(y(t)\) based on \(t \in F \subset [a, b]\) where \(F\) is an
\(\alpha\)-perfect fractal set (e.g. a Cantor set) of H-dimension \(\alpha\) (assume here that \(y\) is actually defined on all \([a, b]\)). If \(y(t)\) changes direction (e.g. increasing to decreasing) at \(t_0\) then one expects different left and right side derivatives \(D^+_F y(t_0)\). Thus we define

\[
\begin{align*}
D^+_F y(t_0) &= F - \text{limit}_{t \to t_0^+} \frac{y(t) - y(t_0)}{S^+_F(t) - S^+_F(t_0)}; \\
D^-_F y(t_0) &= F - \text{limit}_{t \to t_0^-} \frac{y(t) - y(t_0)}{S^-_F(t) - S^-_F(t_0)}.
\end{align*}
\]  

Then as in Example 2.6 for example we write (D23) \(b_\pm(t) = \pm D^+_F y(t)\) and set (D24) \(V = (1/2)(b_+ + b_-)\) with \(U = (1/2)(b_+ - b_-)\) with \(V = V - iU\). Now for functions \(f(y(t), t)\) which are suitably smooth in the \(y\) variable one wants to differentiate in \(t\). We recall that \(y\) so defined on all \([D_{23}]\)

Then as in Example 2.6 for example we write (D23) \(b_\pm(t) = \pm D^+_F y(t)\) and set (D24) \(V = (1/2)(b_+ + b_-)\) with \(U = (1/2)(b_+ - b_-)\) with \(V = V - iU\). Now for functions \(f(y(t), t)\) which are suitably smooth in the \(y\) variable one wants to differentiate in \(t\). We recall that for \(F\)-limits as in (5.15) \(t \in F\) along with \(t_0\). The set \(F\) could perhaps be regarded as \(Sch(y)\) so \(y(t)\) is characterized by changes on \(F\) with no specified behavior otherwise.

**PROPOSITION 5.1.** Let \(f\) be smooth and \(y(t)\) continuous on \([a, b] \supset F\) and \(D^+_F y(t) = 0\) for \(y \neq F\) as specified in (5.15). Then

\[
D^+_F f(y(t)) = [D^+_F y(t)] f'(y(t)) \quad (t \in F)
\]  

*Proof:* One has \(f(s) - f(\sigma) = f'(c)(s - \sigma)\) where \(\sigma \leq c \leq s\). Now write

\[
\frac{f(y(t')) - f(y(t))}{S^+_F(t') - S^+_F(t)} = \frac{(y(t') - y(t)) f'(c)}{S^+_F(t') - S^+_F(t)} \to D^+_F y(t) f'(y(t))
\] for \(t \in F\) (with limit zero otherwise). Here \(y(t) \leq c \leq y(t')\) and \(y\) is assumed continuous which squeezes \(c\). QED

Instead of assuming one sided derivatives everywhere on \([a, b]\) as in Section 2 we suppose only one sided \(\alpha\)-derivatives on \(F\) as in (5.15). The undercurrent here is to imagine a fractal space time with some average dimension 3 or 4 say, as in the unstructured \(E^\infty\) of El Naschie. Thus we might be able to “see” a continuous path but in fact the underlying phenomena could involve points \((t, y(t))\) for \(t \in F\) as a fractal graph (whose dimension is basically irrelevant at the moment). One could imagine a geodesic having an underlying fractal structure, no matter what we may fuzzily see in an ambient continuum. Hence one could work with \(b_\pm\) and \((U, V)\) as in (D23)-(D24) and think of some Brownian fluctuations (suitably appended) in addition, as in (2.17)-(2.18). The eventual “geodesics” in the microstructure involve derivatives \(D^+_F\) in the \(t\) variable and consequently would be tied to \(F\) intrinsically. Away from \(F\) there is no microstructure but we may see something in the continuum by visually blending together points.

Thus we introduce a Wiener process \(d\xi_t\) as in Example 2.6 with (D25) \(<d\xi^2_t> = 2Dt = -<d\xi^2_\perp>(where D = h/2m). Although dt makes no sense in \(F\) we know \(\pm D^+_F y(t) = 0\) for \(t \neq F\) so one can envision equations as in (104) involving \(\pm D^+_F y(t)\) provided \(\chi_F(t)\) is inserted at appropriate places (i.e. there will only be evolution for \(t \in F\)). First as in (35), following (104) (127) (137), one writes heuristically for 1-D and \(f = f(t, y(t))\) smooth in the \(y\) variable

\[
\frac{d_+ f}{dt} = \chi_F \partial_t + b_+(t) \partial_y f + \chi_F D \partial_y^2 f; \quad \frac{d_- f}{dt} = \chi_F \partial_t f + b_- \partial_y f - \chi_F D \partial_y^2 f
\]
REMARK 5.1. There are a number of papers about diffusion and fractional diffusion on fractals (see e.g. [26, 76, 156]) and some anomalous features arise. We omit discussion of this for the moment.

REMARK 5.2. Proceeding now from (5.18) we now obtain (using (D23)-(D24) the formula (D26) \( d'/dt = \chi_F \partial_t - iD\chi_F \partial_y^2 + \nabla \partial_y \)) where of course \( V = V - iU = 0 \) for \( t \neq F \). Now the geodesic equation for a free “particle” is from (A22) (D27) \( m(d'/dt)V = 0 \). Here we note that \( V = V(y(t)) \) must be assumed to be of the form \( f(y(t), t) \) smooth in some sense in \( y \) and differentiable in some sense in \( t \). This problem seems to arise also in the derivations we note that in \( \nabla \) and the context of (137) for example since \( \partial_t U \) and \( \partial_t V \) are not a priori well defined and this is one reason we include the present discussion. It would be interesting to pursue it further here and in the context of (137). Further since e.g. \( V \) is (1/2)(b+ + b-) with \( b_+(t) = \pm D^\alpha_F y(t) \) there is even no assurance that \( D^\alpha_y b_+(t) \) will be defined (let alone one sided derivatives \( b_\pm \)) as in (137) and Section 2 or (D23)). We will examine this now in more detail and refer to [2, 47, 18] for variations and refinements. The derivation of Nottale given here and in Example 2.5, although heuristic, is however more revealing in showing just how the formal \( U \) and \( V \) play a role in creating the SE (and a quantum potential). Thus from (137) (cf. also [35, 127]) one defines \( d_\pm \) as in (5.18) (with one sided derivatives \( d_\pm \)) and sets (D28) \( d_v/dt = (1/2)(d_+ + d_-)/dt \) with \( d_a/dt = (1/2)(d_+ - d_-)/dt \) so that (D29) \( d' = a_0 - id_a \) agrees with the formula (D26). Then (D30) \( d'V = (d_v - id_a)(V - iU) = (d_vV - d_aU) - i(d_aV + d_vU) \). Given no driving force both real and imaginary parts of the complex acceleration \( d'V/dt \) vanish and we note that in (127) one defines acceleration via \( \dot{y} = (1/2)(d_+ d_- + d_- d_+) y/dt^2 \) which is exactly the real part of \( d'V/dt \), namely (D31) \( d_vV - d_aU \). Hence in our situation involving \( \pm D^\alpha_F \) one has for the real part of (D27) \( (d'/dt)V = 0 \) the expression

\[
(\pm D^\alpha_F - D^\alpha_F + \mp D^\alpha_F + D^\alpha_F)y(t) = 0 \quad (t \in F)
\]

which requires of course that this combination of derivatives makes sense. For the imaginary part of \( (d'/dt)V \) = 0 we have (D32) (1/2)(2 \( d_+^2 - d_-^2 \)) \( y = 0 \) so in our situation we require (D33) \( (\mp D^\alpha_F)^2 - (-D^\alpha_F)^2 \) \( y(t) = 0 \) \( (t \in F) \) which would be automatic if \( -D^\alpha_F y = \pm D^\alpha_F y = D^\alpha_y y \).

Now using (5.18) one has expressions for \( d_\pm \) acting on suitable functions \( f(y(t), t) \) and the transition to the SE involves using (D34) \( d_v/dt = \partial_t + V \partial \) and \( (d_a/dt) = D \partial^2 + U \partial \) which are then applied to \( U, V \) considered as functions of \( (y(t), t) \). Hence one must assume here that e.g. \( V(y(t), t) \) is smooth in \( y \) and differentiable in the second variable \( t \), while \( y(t) \) remains fractal in nature. Now let \( \rho(y(t), t) \) be the probability density of \( y(t) \) in a Wiener process context. Classically one knows there are then forward and backward Fokker-Planck (FP) equations

\[
\partial_t \rho + \text{div} (\rho b_+) = D \Delta \rho; \quad \partial_t \rho + \text{div} (\rho b_-) = -D \Delta \rho
\]

and by adding and subtracting one gets (D35) \( \partial_t \rho + \text{div} (\rho V) = 0 \) and \( \text{div}(\rho U) - D \Delta \rho = 0 \) (here \( U, V \) are essentially regarded as statistical averages \( < u > \) and \( < v > \)). Hence let us assume a similar situation prevails in our case and stipulate that (D36) \( U = D \partial \log(\rho) \) in the 1-D fractal situation. Now using (D34) the imaginary part of the acceleration satisfies (D37) \( d_vV + d_aU = 0 = \partial_t U + \partial(UV) + D \partial^2 V \) so \( \partial_t U = -D \partial^2 V - \partial(UV) \). One can also
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assume $V = 2D\partial S$ via Lagrangian arguments which leads to $\psi = \sqrt{p}\exp(iS)$ and thence easily to the SE (cf. [35], [137]). In fact we can simply write (D38) $2iD(d'/dt)(\partial \log(\psi)) = 0$ where (D39) $\log(\psi) = (1/2)\log(p) + iS$ so $iD\partial \log(\psi) = iD\partial \log(p) - 2D\partial S = (V - iU) = V$. To produce the SE use then the expression (D40) $(d'/dt) = (\partial_t - iD\partial^2) + V\partial = (d_u - id_a)/dt$ to get (using (D39))

$$\tag{5.21} 0 = 2iDm[\partial_t\partial \log(\psi) - iD\partial^2\partial \log(\psi) - 2iD(\partial \log(\psi)\partial)(\partial \log(\psi))]$$

leading to the SE (D41) $ih\partial_t \psi + (\hbar^2/2m)\partial^2 \psi = 0$ (as a geodesic equation - recall we are taking $D = \hbar/2m$).

In the present situation the meaning of $\rho$ as a probability density can surely be retained (on F). It would be nice if one could rephrase the assumptions of $U, V, \rho$, as functions of $(y(t), t)$, being smooth in $y$ and differentiable in the second variable $t$, and reduce the whole context to behavior on $t \in F$ and $(y(t), t) \in P$ where $P$ is some kind of “quantum” path. To use the calculus of [149] however for $(U, V, \rho)(y(t), t)$ we would need to further develop concepts like the chain rule (cf. Proposition 5.1) and perhaps various equalities would have to be replaced by inequalities. Thus for now we will assume $y(t)$ is continuous for $t \in [a, b]$ and treat $\partial_y \sim \partial$ symbolically while $\partial_t$ could naturally mean $D^t_F$ in the $t$ variable written as $\partial^t_F$. The symbolic $\partial$ could perhaps eventually reduce at times to $D^t_G \sim \partial^t_G$ in $y$ where $(y(t), t) \in P$ implies that $y(t) \in G$ for $t \in F$ where $G$ is some fractal set of Hausdorff dimension $\beta$. Then formally repeating the above calculations one arrives heuristically at a SE

$$\tag{5.22} ih\partial^t_F \psi = -\frac{\hbar^2}{2m} (\chi_F \partial^2 \psi); \ \psi = \sqrt{p}\exp(iS); \ U = \frac{\hbar}{2m} \partial \log(p); \ V = \frac{\hbar}{m} \partial S$$

REMARK 5.3. One notes that (with the assumptions above) the SE in $\psi$ does not explicitly involve the fractal effect described by $U$. However the corresponding HJ equation (cf. Example 2.6)

$$\tag{5.23} \partial^t_F S + \chi_F \frac{\partial S}{2m} + Q = 0; \ Q = -\frac{m}{2} U^2 - \frac{\hbar}{2} \partial U$$

reveals the effect of $U$. Here (D42) $U = (1/2)(b_+ - b_-) = (1/2)(+D^t_Fy(t) - -D^t_Fy(t))$. We remark also that the above procedure illustrates again the strong connection between diffusion and the SE (cf. [120], [121], [127]).

REMARK 5.4. The idea of having a fractal microstructure for the vacuum is of course reminiscent of the ether idea about which there already seems to be a substantial amount of recent material (cf. [55], [94], [161], [170], [172], [176], [177]). We will return to this at another time in noting that explicit modifications to the SE are indicated in [170] (cf. also [161] regarding the Bohmian theory). It appears also that the viewpoints and information arising from [176], [177] should be of critical importance in the future development of quantum theory and cosmology.
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