Taylor-Proudman theorems in $\mathbb{E}^3, 4$ and $5$

J.-Z. Zhu

Su-Cheng Centre for Fundamental and Interdisciplinary Sciences, Gaochun, Nanjing, China

Taylor’s circulation-invariance proof of the theorem after Proudman is reinstated in the new derivation in $\mathbb{E}^3$ and in the Taylor-Proudman-type theorems in $\mathbb{E}^5$ and $\mathbb{E}^4$. The generalized Kelvin theorem for the higher-order differential form brings no new constraint. Fast rotations in $\mathbb{E}^4$ do not lead to the complete cylinder condition as in $\mathbb{E}^3$, particularly resulting in a passive but nonlinear scalar problem with simple rotation. Dynamical resonant wave theory is accordingly remarked. For compressible flows, only the velocity component lying in the rotating plane is required to be incompressible in the fast rotation limit, with nontrivial implications, say, for aeroacoustics.

1. Introduction

Taylor’s proof of the theorem after Proudman [1] strictly follows the “geometrical condition” of invariant projection area [2], which is neglected (e.g., [3,4]) in almost all literatures. Chandrasekhar’s [5] recognition is, to our best knowledge, the only but half-way exception. It is curious why Taylor took pains to re-prove it in such a much more ‘indirect’ way than that Proudman had shown. Nevertheless, the tranquil beauty is demonstrated with the Kelvin-Helmholtz theorems carried over to the rotating frame. The flow geometry is however also ‘magnificent’, with unsteady and higher-dimensional invariant integral surface [6] generalizing the vortex line, say.

The Taylor-Proudman [1,2] theorem plays a central role in understanding the rotating flows in $\mathbb{E}^3$. Although uniformity along the axis (cylinder condition) is stated for a steady flow, there are (resonant wave) theories, experiments and simulations supporting the dynamical evolution (at least partially) towards such a state (c.f., Chap. 4 of [4]) with fast rotation. The cylinder condition reduces the (‘vertical’) velocity component along the axis to a linear passive scalar advected by the (‘horizontal’) velocity components in the rotating plane. Obviously, reducing velocity component(s) to passive scalar(s) by taking cylinder condition(s) applies also to the Navier-Stokes equations in $\mathbb{E}^d$ with $d > 3$, but it is to be clarified whether (actually ‘no’!) cylinder condition or the linearly advected passive scalar follows.
Rotation is of interest in the formulation and mathematical analysis of ideal hydrodynamics in 3-space \([7,8]\), as well as in the physical interplay with helicity for the cascade dynamics and structures of turbulence \([9]\), (nonpremixed) passive scalar mixing in homogeneous turbulence (e.g., most recently \([10]\) and references therein), and, appearing of closer engineering connection, Rayleigh-Bénard convection subjected to competitions of various ingredients \([11]\) and transport in channel flows (e.g., \([12–17]\)). While high-dimensional flows have long been studied \([18]\) even in curved spaces, with up to date developments (e.g., very recently, Fecko \([6]\) on generalizing the vortex lines, Besse & Frisch \([19]\) on Cauchy invariants equation and Anco, Dar & Tufail \([20]\) on compressible invariants), the study of the dynamical effects of rotation in either the ideal \([21]\) or the statistical \([22,23]\) hydrodynamics is still wanted.

Note that the classical Taylor-Proudman theorem and the resonant wave theory have been founded on and clarified by somewhat rigorous mathematical calculation and proofs (e.g., \([24,25]\)), particularly useful for understanding laminar flows. Rotating turbulence is however far from clear, with lots of work (e.g., \([26]\), \([23]\]) on generalizing the vortex lines, Besse & Frisch \([19]\) or the statistical \([22,23]\), the study of the dynamical effects of rotation in either the ideal \([21]\) or the statistical \([22,23]\) hydrodynamics is still wanted.

The theme of this note is carrying the flow geometry from inertial to rotating frames. The geometrical analyses and new models of passive scalar with appropriate analytical tractability are called for, which is also a motivation for going to higher dimensions and looking for appropriate reductions.

2. Some geometrical remarks

The theme of this note is carrying the flow geometry from inertial to rotating frames. The geometrical results should reveal themselves in the dynamics, though only the lowest-order one, the invariant velocity circulation, is explicitly used in the calculations. Thus, we also offer brief remarks on other higher-order geometrical objects. If not particularly pointed out, we consider incompressible flows with constant density, which however is not necessary as we will come back in the conclusion with important further remarks.

(a) Flows in the inertial frame

Let \(U := \sum_{i=1}^{d} u_i dx_i\) be an element of 1-forms in the space \(g^\ast\) dual to the Lie algebra \(g\) of the incompressible velocity vector \(u = \sum_i u_i \partial_i\) (or \(u\) for notation of a field), then its exterior derivative \(dU\) is the 2-form vorticity. With the introduction of the Bernoulli function \(B = P + \frac{u^2}{2}\) and the Lie derivative \(L_u\) along \(u\), the ideal Euler equation for the dynamics of such objects in \(\mathbb{R}^d\) reads

\[
(\partial_t + u \cdot \nabla) U = (\partial_t + L_u) U - d\omega_u U = (\partial_t + L_u) U - du^2 = -dB, \tag{2.1}
\]

\[
(\partial_t + L_u) dU = 0. \tag{2.2}
\]

With \(\xi = \partial_t + u\) and \(\Psi = U - B dt\) as proposed in Fecko \([6]\) and with \(\phi\) adding to \(d\) the differential with respect to \(t\), the Euler equation is the interior product between \(\xi\) and \(\phi\Psi\)

\[
\iota_\xi (\phi \Psi) = 0. \tag{2.3}
\]

The above geometrical formulation as the starting point of our derivation of the Taylor-Proudman-type theorems in general \(\mathbb{R}^d\) implicitly reinstates Taylor’s \([2]\) idea, and Fecko’s \([6]\) recent generalization of the Helmholtz vortex lines is of fundamental interests. So, we offer relevant remarks below.

Since \(dU\) is Lie invariant, the exact 4-form

\[
dU \wedge dU = d(U \wedge dU) \tag{2.4}
\]

is also invariant, and its Hodge dual \(\ast (dU \wedge dU)\) is a 1-form in \(\mathbb{R}^5\), the latter corresponding to a 5-vector \(\gamma\) generating the flow \(\gamma(t)\). The vanishing interior product \(\iota_\gamma d(U \wedge dU) = 0\) defines a distribution \(\mathcal{D}\) \([27,28]\) whose integral manifold is tracked by the flow \(\gamma(t)\). The integrability of \(\mathcal{D}\) is assured by the Frobenius
theorem (involutivity) which can be validated directly: Suppose \( \gamma_1, \gamma_2 \in \mathcal{D} \), we have

\[ t_{[\gamma_1, \gamma_2]} = [L_{\gamma_1}, t_{\gamma_2}] = [\gamma_1, d\gamma_2 + d\gamma_1, t_{\gamma_2}], \]

thus \( t_{[\gamma_1, \gamma_2]}d(U \wedge dU) = 0 \). So, for \( d = 5 \), the integral manifold is precisely a line (the ‘bi-vorticity line’ if a terminology is needed), a special case of the extension of Fecko’s [6] generalization of the Helmholtz theorem. We then have the Helmholtz-type theorem for the frozen-in bi-vorticity line. This result is obviously extendable to any odd dimension \( d = 2m + 1 \), by working with \((dU)^m\) in a similar way.

Note that the interior product with the volume form \( \iota_\omega \mu = dU \wedge dU \) defines the vorticity vector \( \omega \) [18] in \( \mathbb{E}^3 \), which also generates a ‘vorticity line’ which however should not be confused with the above \( \gamma(t) \leftrightarrow r(t) \) as the bi-vorticity line of the integrated submanifold.

The lines can of course be generalized to more general integral surfaces of the distributions of dimension \( \Delta < d - (2\delta) \) (because the object is not decomposable in general: c.f., Appendix C of Fecko [6]) with \( \delta \) replacing \( m \) in \((dU)^m\), to have \((dU)^{2\delta}\), in the above for defining the distribution \( \mathcal{D} \).

What is of closer relevance to our work when carried over to rotating frames is the integral surface, with dimension \( \Delta \leq d - 2 \), of the distribution in \( \mathbb{E}^d \)

\[ \mathcal{D} := \{ \text{vectors } \gamma \text{ such that } t_\gamma dU = 0 \text{ holds } \}. \] (2.5)

When \( d = 3 \), the integral surface is the vorticity line which is of classic importance [3]. Fecko [6] also has used the form as Eq. (2.3) to introduce over the flowing closed chain \( c(t) \) the relative invariant \( \int c \cdot U \) which includes the well-known Kelvin theorem of circulation invariance (with respect to the flow \( u \)). Of course there are also other higher-order dynamical integral invariants [6] corresponding to the above mentioned higher-order Lie-invariant forms.

(b) Flows in the rotating frame

For flows in a rotating frame, an inertial force should thus be included in Eqs. (2.1,2.2). This amounts to calculate the acceleration, i.e., the change rate of the velocity of a point of the rotating frame, which, together with the transformation into differential forms when necessary, is direct in whatever \( d \). Alternatively, one can adopt the functional action (Lagrangian [7,8] or Hamiltonian [21]) and variational approach. The latter has been clearly presented by Shashikanth [21], and we won’t repeat the details here but just summarize, explain and develop the result for our purpose:

\( \Omega_R \) denoting the 2-form corresponding to the rotation of the frame and \( X \) the 1-form corresponding to the position vector \( x \), the ‘inertial force’ entering the left hand side of Eq. (2.1) reads

\[ L_u[*(\ast \Omega_R \wedge X)] = (\partial_\mu \hat{u}_t \ast_d \partial_\mu \ast \Omega_R), \] (2.6)

with the frame rotating velocity 1-form

\[ U_R := *(\ast \Omega_R \wedge X). \] (2.7)

The exact \( \partial_\mu \hat{u}_t \) part on the right hand side of Eq. (2.6) contributes to the centrifugal force, and the \( \hat{u}_t \) part to the ‘Coriolis force’, following the terminology in \( \mathbb{E}^3 \). Later, we will further apply the spectral theorem to the skew-symmetric matrix corresponding to \( \Omega_R \), for simplifying the explicit calculations and analyses in \( \mathbb{E}^3 \).

Combining Eqs. (2.1,2.2,2.6,2.7), we have the Lie-invariance law in the rotating \( \mathbb{E}^d \)

\[ (\partial_\mu + L_u) \partial(U + U_R) = 0. \] (2.8)

Thus, with the replacement of \( dU \rightarrow \partial(U + U_R) \), many results in the inertial frame carry over, mutatis mutandis, in particular, Fecko’s integral surface generalizing the Helmholtz vortex line and tube, and, the relative integral invariant generalizing the circulation in Kelvin’s theorem.

Actually, according to the Lie theory [28], rotation \( R \in SO(d) \) can be represented as the exponential of an anti-symmetric matrix \( A \) and we have the spectral theory [29] (see also, e.g., Ref. [30] and references therein): \( A \) is a normal matrix, subjecting to the spectral theorem, thus (block) diagonalizable by a special
orthogonal transformation. More definitely, for \( d = 2m \) we have \( A = Q \Lambda Q^T \) where \( Q \) is orthogonal and

\[
A = \begin{bmatrix}
0 & \lambda_1 & 0 & \cdots & 0 \\
-\lambda_1 & 0 & \lambda_2 & 0 & \cdots \\
0 & 0 & 0 & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \lambda_n \\
& & & & -\lambda_n & 0 \\
& & & & & \ddots \\
& & & & & & 0 \\
\end{bmatrix}
\] (2.9)

with real \( \lambda_i \)'s, in the \( 2 \times 2 \) blocks along the diagonal, being the coefficients of the purely imaginary eigenvalues. When \( d = 2m + 1 \), \( \Lambda \) presents at least one extra row and column of 0s.

The spectral theorem indicates that the rotation can be decomposed into sub-rotations in mutually orthogonal 2-planes. Thus, \( \Omega_R \) can be decomposed into 'orthogonal' 2-forms (the wedge product of one with another never vanishes), explicitly in terms of the coordinates of \( E^d \), say,

\[
\Omega_R = \sum_{i=1}^{m} \lambda_i \, d\mathbf{x}_{2i-1} \wedge d\mathbf{x}_{2i} \quad \text{for} \quad 2m = d \quad \text{or} \quad 2m + 1 = d,
\] (2.10)

with which we can directly verify

\[
\int [\star (\star \Omega_R \wedge \mathbf{x})] = dU_R = 2\Theta_R
\] (2.11)

and turn Eq. (2.8) into the nice form

\[
(\partial_t + L_u) (dU + 2\Omega_R) = 0.
\] (2.12)

3. Taylor-Proudman-type hydrodynamics

We know that (e.g., [6]), to first order accuracy for small \( t \), the integral over a circuit (1-cycle)

\[
\int_{c(t)} (U + U_R) = \int_{c(0)} (U + U_R) + t \int_{c(0)} L_u(U + U_R),
\] (3.1)

where \( c(t) = \Phi_t[c(0)] \) and \( U + U_R \) in the integrands on the right hand side is the pull back (\( \Phi_T^* \)) of that on the left hand side. And, the following integral invariant, as the generalization of the circulation in Kelvin’s theorem [6], is implied by Eqs. (2.7, 2.8) in \( E^d \)

\[
\int_{c(t)} (U + U_R) = \int_{c(0)} (U + U_R).
\] (3.2)

For small motion or fast rotation leaving only the terms denoted with \( R \), Eq. (2.10) and the Stokes theorem lead to

\[
\int_{c(t)} U_R = \int_{S(0)} dU_R = 2 \sum_{i=1}^{m} \lambda_i \int_{S(0)} d\mathbf{x}_{2i-1} \wedge d\mathbf{x}_{2i} =: 2\lambda_i A_i,
\] (3.3)

That is, the invariant is actually in general \( E^d \) the sum of each area of the projection of the circuit \( A_i \) in the respective rotating plane multiplied by \( 2\lambda_i \), a beautiful generalization of Taylor’s [2] observation.
Taking the time derivative of Eq. (3.1), we have the change rate of the projection area (weighted by $\lambda_i$), as Taylor calculated in $\mathbb{E}^3$ (except for the weight/signature $\lambda$),

$$\int_{c(0)} L_u \mathcal{U}_R = \int_{S(0)} \mathcal{D}L_u \mathcal{U}_R = 2 \int_{S(0)} L_u \Omega_R = 0. \quad (3.4)$$

Because for every 1-boundary $c(0)$ the second term of the right hand side should vanish for all possible $S(0)$ satisfying $\partial S(0) = c(0)$, we have the differential version

$$2L_u \Omega_R = L_u d\mathcal{U}_R = 0. \quad (3.5)$$

**a) Reinstating Taylor’s geometry in $\mathbb{E}^3$**

For $d = 3$, $n = 1$ in Eq. (2.9) and that the rotation rate/vorticity 2-form reads

$$\Omega_R = \lambda dx_1 \wedge dx_2, \quad (3.6)$$

i.e., rotation in the 1-2 plane or around the $x_3$ axis. Eq. (3.5) turns into

$$L_u d\mathcal{U}_R = 2\lambda \left[ (u_{1,1} + u_{2,2}) dx_1 \wedge dx_2 + u_{1,3} dx_3 \wedge dx_2 + u_{2,3} dx_1 \wedge dx_3 \right] = 0. \quad (3.7)$$

Thus, together with the incompressibility condition, we have the cylinder condition $\partial_3 u \equiv 0$ which is the *Taylor-Proudman theorem* in $\mathbb{E}^3$.

The content in the square bracket of Eq. (3.7) is, in Taylor’s notation with slight adjustments, $(u_{1,3}\bar{x}_1 + u_{2,3}\bar{x}_2 + u_{3,3}\bar{x}_3) \cdot d\mathcal{S} = (u_{1,3}\bar{x}_1 + u_{2,3}\bar{x}_2 + u_{3,3}\bar{x}_3) \cdot n dS$ where $\bar{x}_*$ and $n$ are unite vectors normal to each coordinate plane and the directed surface element. Since the result should hold for all possible circuits, he obtained the theorem, which is now reinstated in the above in terms of differential forms: For exact consistence, be aware that $dx_1 \wedge dx_3$ is understood to be anti-parallel to the $x_1$ axis in the right-handed coordinate system. We actually originally just started with Eq. (3.5), but once seeing the very illustrative results we found Taylor’s insight and reformulated the derivations.

Note that Eq. (3.7) only implies the incompressibility of the flow component lying in the rotating plane, and $u_{3,3} = 0$ is deduced from incompressibility of the total flow. Incompressibility condition however can be relaxed for our analysis of rotating flows with non-trivial implications, as we will address in the concluding discussions.

**b) Taylor-Proudman theorem in $\mathbb{E}^4$**

Now $\Omega_R = \lambda_1 dx_1 \wedge dx_2 + \lambda_2 dx_3 \wedge dx_4$ for $m = 2$ in the mutually orthogonal planes, which is the decomposition of the rotation in $\mathbb{E}^4$ [30,31]. Eq. (3.5) reads

$$L_u \left[ (u_1 \mathcal{U} + \lambda \Omega_R \wedge \mathcal{X}) \right] = 2\lambda \left( \lambda_1 dx_1 \wedge dx_2 + \lambda_2 dx_3 \wedge dx_4 \right)$$

$$= 2\lambda_1 \left[ (u_{1,1} + u_{2,2}) dx_1 \wedge dx_2 + u_{1,3} dx_3 \wedge dx_2 + u_{1,4} dx_4 \wedge dx_2 - u_{2,3} dx_3 \wedge dx_1 - u_{2,4} dx_4 \wedge dx_1 \right] +$$

$$+ 2\lambda_2 \left[ (u_{3,1} dx_1 \wedge dx_4 + u_{3,2} dx_2 \wedge dx_4 + (u_{3,3} + u_{4,4}) dx_3 \wedge dx_4 - u_{4,1} dx_1 \wedge dx_3 - u_{4,2} dx_2 \wedge dx_3 \right]$$

$$= 0, \quad (3.8)$$

following which is the 4-space *Taylor-Proudman theorem*:

- in the case of constant double-rotation for $\lambda_1 = r \lambda_2$ and $r > 0$ without loss of generality, coordinate transformations $x'_1 = \sqrt{r} x_1$ and $x'_2 = \sqrt{r} x_2$ can be applied to have $\Omega'_1 \wedge x' = \lambda_2$, or, in other words, $r = 1$ can always be used, thus assumed here, with appropriate re-scaling

$$u_{1,1} + u_{2,2} = u_{3,3} + u_{4,4} = 0 = u_{1,3} + u_{4,4} = u_{2,3} - u_{4,4} = u_{4,4} - u_{3,2} = u_{2,4} + u_{3,1}; \quad (3.9)$$
\[ u_{1,1} + u_{2,2} = u_{3,3} + u_{4,4} = 0 = u_{1,3} = u_{1,4} = u_{2,3} = u_{2,4}. \]  

Eq. (3.9) shows that in the fast double-rotation limit, although the incompressibility is reached respectively in each rotating plane, we have the mixed constraints instead of the complete cylinder condition. In terms of Taylor’s [2] geometrical interpretation, this is the vanishing variation of the weighted sum of the arbitrary invariant circuit’s projection area in each rotating plane: Or, in turn, this explains why we do not get the pure cylinder condition.

For the fast simple rotation limit, cylinder condition of the ‘horizontal flow’ along the axises perpendicular to the rotating ‘horizontal’ plane is indeed reached, as shown by the four equalities on the right of Eqs. (3.10). However, the complete cylinder condition with extra \( u_{3,3} = u_{4,4} = 0 \) can not be derived.

Note that for general \( r \) the latter parts of Eq. (3.9) read \( 0 = ru_{1,3} + u_{4,2} = ru_{2,3} - u_{4,1} = ru_{1,4} - u_{3,2} = ru_{2,4} + u_{4,1} \), so taking \( r \to \infty \) before \( |\lambda_1| \to \infty \) leads to the corresponding latter parts of Eq. (3.10).

The fast simple rotation limit deserves more remarks. Denote the incompressible horizontal velocity in the \( h \)-plane \( x_1 \cdot x_2 \) by \( u_h = [u_1(x_1, x_2), u_2(x_1, x_2)] \) and the incompressible vertical velocity ‘along’ the \( v \)-plane \( x_3 \cdot x_4 \) by \( u_v = [u_3(x_1, x_2, x_3, x_4), u_4(x_1, x_2, x_3, x_4)] \), and, the pressure \( P = P_h(x_1, x_2) + P_v(x_3, x_4) \). Eq. (3.10) leads to the decomposition of the 4D Navier-Stokes into a two-fluid system

\[
\partial_t u_v + (u_v \cdot \nabla) u_v + u_h \cdot \nabla h - \nu_h \Delta h - \nu_v \Delta h) u_v = -\nabla v P_v \quad \text{with} \quad \nabla v \cdot u_v = 0, \ \nabla v P_v = 0, \hspace{1cm} (3.11)
\]

and

\[
\partial_t u_h + (u_h \cdot \nabla h - \nu_h \Delta h) u_h = -\nabla h P_h \quad \text{with} \quad \nabla h \cdot u_h = 0, \ \nabla h P_h = 0, \hspace{1cm} (3.12)
\]

where \( \nu_h \) and \( \nu_v \) may be different to take into account the possibility of anisotropic damping for whatever reason (including the re-scaling with a non-unit \( r \)). Thus, for \( u_h \), we see that \( u_v \) becomes passive, though the latter is not of linear-advection but of the nonlinear-Euler-dynamics nature. We may call such \( u_v \) “non-linear passive scalar (vector)”. As a check, if we further let \( \partial_4 \equiv 0 \), or, in other words, let \( u_4 \) be a passive scalar, the results automatically reduce to the Taylor-Proudman theorem in \( \mathbb{E}^3 \) with \( \partial_4 \equiv 0 \). Note that in reducing to the 3-space Taylor-Proudman theorem, it has not been deduced that \( u_{4,3} \) be zero. This might appear somewhat counter intuitive on the first thought, but note that Taylor’s [2] experimental observation does not indicate the uniformity of the color along the axis. On the other hand, in reducing to the common 3-space Taylor-Proudman theorem, \( u_{3,4} \) actually does not need to be zero, which means that allowing \( u_4 \) to be non-passive, but advecting \( u_3 \), through non-vanishing \( u_{3,4} \) does not prevent the reduction.

(c) Taylor-Proudman theorem in \( \mathbb{E}^d \) with \( d \geq 5 \)

The block-diagonal form (2.9) for \( d = 5 \) is the same as that for \( d = 4 \) except for an additional row and column of zeros. Thus, the decomposition of the rotation is formally the same and one can similarly find the Taylor-Proudman theorem in \( \mathbb{E}^5 \), and generally for higher \( d \).

Indeed, extending the calculation, we find the following extra terms in addition to those in \( \mathbb{E}^4 \)

\[
\lambda_1 (u_{1,5} dx_5 \wedge dx_2 - u_{2,5} dx_5 \wedge dx_1) + \lambda_2 (u_{3,5} dx_5 \wedge dx_4 - u_{4,5} dx_5 \wedge dx_3),
\]

and that the Taylor-Proudman theorem in \( \mathbb{E}^5 \) is just adding to that in \( \mathbb{E}^4 \) the following cylinder condition along the rotation axis (taken to be \( x_5 \) here, and again \( r = 1 \)), without loss of generality:

\[
u_{1,5} = u_{2,5} = u_{3,5} = u_{4,5} = 0 = u_{5,5} \]

for the double-rotation case. [The last equality is also due to the incompressibility condition.] This cylinder condition imposed on the Navier-Stokes equation, \( u_5 \) becomes a passive scalar. For the simple rotation case with \( \lambda_2 = 0 \), similar to that in \( \mathbb{E}^4 \), \( u_{3,5} \) and \( u_{4,5} \) are not required to vanish.

The circuit is a 1-boundary and may be chosen as the submanifold of the integral surface (itself being a submanifold, of dimension less than 3 because of the indecomposibility of \( \partial U_R \) except for the simple rotation case) of the distribution (2.5), again with \( U \) now replaced by \( U_R \) as the first-order approximation.

Since the circuit, as a horizontal curve, can be chosen arbitrary in the integral surface, we may say Taylor’s projection area of the circuit is now generalized to be of the whole integral surface: Such remarks in this
paragraph have assumed that the integral surface be the boundary of some chain or at least contain a 1-boundary, which may not always be true and thus the invalid assumption could not be used to derive the Taylor-Proudman theorem. As discussed in Sec. 2, there are rich classes of distributions and the corresponding integral surfaces, each of which contains interesting geometry, systematic discussion of which is highly nontrivial and far beyond the scope of this note.

(d) Inertial waves

In the derivation of the Taylor-Proudman theorem, we have assumed that the time variation term is also much smaller than the Coriolis term or have simply let it be zero (steady state) for a more rigorous treatment. Intuitively, if we keep increasing the rotation rate, the period of time for developing fluid structures varying as fast as the rotation should also increase. Thus, at least for some finite time two-dimensionalization of the flow towards the cylinder condition defined by the 3-space Taylor-Proudman theorem should happen, and similarly for other cases in \( \mathbb{E}^d \) such as \( d = 5, 4 \) as derived in the above. Indeed, there is a resonant wave theory for such a scenario.

Here, we offer the inertial waves as the basis for the resonant wave theory in \( \mathbb{E}^{5, 4} \) but leave the more rigorous treatments, including the development of the rigorous mathematical analyses comparable to those in \( \mathbb{E}^3 \) \cite{24, 25} and the numerical simulations of 4-space rotating turbulence, for future study.

We start with Eqs. (2.1, 2.6) and, in the appropriately chosen coordinates according to the spectral theorem mentioned earlier, (2.11), transformed to the following form, with \( L_u = \iota_4 u_0 + \iota_1 u_1 \).

\[
\partial_t \tilde{u} + \iota_4 (\partial_t \Omega + 2\Omega_R) = -\partial \{ P - \iota_4 \{ \iota_4 (k \cdot \nabla + \iota_4) \} \} =: -\partial \Pi. \tag{3.14}
\]

The standard Fourier normal-mode analysis for the above system with \( u_i = \tilde{u}_i \exp\left\{ i (k \cdot x + \iota_4 t) \right\} (+c.c.) \) and \( i^2 = -1 \), and similarly for \( \Pi \), leads to the dispersion relation between the wave frequency \( \iota_4 \) and vector \( k \).

The normal-mode analysis of Eq. (3.14) in \( \mathbb{E}^4 \) results in

\[
\begin{bmatrix}
   i\iota_4 & -2\lambda_1 & 0 & 0 & ik_1 \\
   2\lambda_1 & i\iota_4 & 0 & 0 & ik_2 \\
   0 & 0 & i\iota_4 & -2\lambda_2 & ik_3 \\
   0 & 0 & 2\lambda_2 & i\iota_4 & ik_4 \\
   k_1 & k_2 & k_3 & k_4 & 0
\end{bmatrix}
\begin{bmatrix}
   \tilde{u}_1 \\
   \tilde{u}_2 \\
   \tilde{u}_3 \\
   \tilde{u}_4 \\
   \Pi
\end{bmatrix} = 0,
\tag{3.15}
\]

the existence of whose nontrivial solutions leads to the equation for the vanishing of the determinant of the coefficient matrix. The solutions are

\[
\omega_{\pm} = \pm 2\sqrt{\lambda_2^2 k_2^2 + \lambda_1^2 k_1^2 + \lambda_4^2 k_4^2 + \lambda_3^2 k_3^2}/|k|, \tag{3.16}
\]

\[
\omega_0 = 0. \tag{3.17}
\]

Letting \( \lambda_2 = 0 = k_4 \), Eq. (3.16) reduces to the well-known dispersion relations for the circularly polarized Rossby waves in \( \mathbb{E}^3 \), which also explains the wave properties of the components lying in the two respective rotating planes in \( \mathbb{E}^4 \).

Taking Eq. (3.17) for the ‘natural’ vortical mode into Eq. (3.15), we of course obtain Eq. (3.9) for the Taylor-Proudman theorem in \( \mathbb{E}^4 \). In the simple rotation case, with \( \lambda_2 = 0 \), say, we see that Eq. (3.17) for the ‘natural’ vortical/slow mode (3.16) and the ‘imposed’ vortical mode with vanishing

\[
k_3 = k_4 = 0
\]

in Eq. (3.16) leads to the Taylor-Proudman theorem (3.10). Note that taking the above into Eq. (3.15), we see that the third and fourth equations are actually \( 0 = 0 \) without restriction on \( \tilde{u}_3 \) and \( \tilde{u}_4 \), precisely Eq. (3.10) instead of stronger condition as one might think on the first sight.

Such results appear to be the indication of a resonant wave theory as in \( \mathbb{E}^3 \) \cite{4} and references therein) for predicting the approach to those Taylor-Proudman states: A simple consistent intuition is that inertial
waves are anisotropic, while resonant interactions due to nonlinearity tend to isotropize the system, thus reducing the former. We also believe that other higher-order geometrical laws discussed in Sec. 2 (see, also [6,19]) should works in constraining the higher-order interactions to favor the dynamical evolution towards the Taylor-Proudman-type states, because in the Lie-transport equations for higher-order differential forms, say, \(d\mathbf{u} \wedge d\mathbf{u}\), higher-order nonlinearities are involved, which of course involves the higher-order resonant interactions. Although there are rigorous mathematical proofs/estimations (e.g., [24,25]) in \(E^4\) supporting the resonant wave theory under appropriate conditions, there are also unsettled subtleties (e.g., [4] and references therein) in the turbulent regime involving the large Reynolds number, near resonant interactions and higher order effects in the long time. In \(E^4\), the number and distribution of the resonant and near-resonant modes are at least quantitatively different to \(E^3\), thus may offer a model with changing effects of the output to validate relevant ideas developed to attack the subtle issues in the latter. We intuitively speculate that with the change of dimensionality and that the geometry and topology of the slow manifold, or the change of the population and distribution of the (near) resonant modes, there could be some phase transition point (number of dimensionality) crossing which the asymptotic limit fast rotation are different. This is however beyond the scope of this note and we hope to address with other tools the relevant statistical hydrodynamics issues.

The situation in \(E^5\) can be similarly discussed, but the results are much more lengthy. There are four (two pairs of) dispersion relations, and, as in \(E^4\) or odd dimension \(d = 2m + 1\) in general, there is no ‘natural’ vortical mode like Eq. (3.17) in \(E^4\). We just put down in the Appendix A one of the frequency for the interested readers to see (it is easiest to start with the simple rotation with \(\lambda_2 = 0\), as an example, how the mode reduce to the vortical one and recover the Taylor-Proudman theorem with additional (3.13).

4. Conclusions and further discussions

It is reasonable to infer that Taylor believed that the geometry in the Kelvin-Helmholtz theorems, carried over to the rotating frame, should be more fundamental and illustrative, and/or he was fully led by his keen observation of the geometry in the experiment.

Eq. (2.12) has been obtained and used to cleanly derive the Taylor-Proudman-type theorems, with Eqs. (3.1,3,2,3,3) reinstating and generalizing Taylor’s geometrical idea, and the re-derived formula in \(E^4\) in terms of differential forms demonstrate in more explicit geometrical notations his details. Inertial waves in \(E^5,4\) are derived, followed by the consistency between the vortical modes and the steady-state Taylor-Proudman-type theorems, which promotes our remarks on the connection with the resonant wave theory in \(E^4\).

As a by-product, we obtain a nonlinear passive scalar problem from the Taylor-Proudman-type theorem in \(E^4\) with simple fast rotation, with the nonlinearity coming from the absence of complete cylinder condition in the vertical velocity \(u_v\). Note that, unlike in \(E^3\), where the absence of cylinder condition in the vertical velocity makes the latter ‘active’, with back reaction to the horizontal velocity through the incompressibility condition, the incompressible \(u_v\) is now still passive. This is, to our best knowledge, a novel feature for the passive scalar problem, and we hope it can be used to bring new lights on the latter in the future.

We also note that the fast rotation in \(E^3\) does not require cylinder condition on the passive scalar advected by the flow subjecting to the Taylor-Proudman theorem. This means that the steady fast rotation limit itself does not have restriction on the passive scalar, which applies of course to general \(E^d\). Whether or not the passive scalar reveals anything relevant to the two-dimensionalization through resonant interactions is another problem. Since there is neither pressure nor other linear term (except for the diffusion) in the linear passive scalar advection equation, the passive scalar does not admit harmonic waves in the normal-mode analysis, unless the wave is specifically pumped in. However, in our nonlinear passive scalar equation (3.11), the pressure gradient appears and that \(u_v\) permits inertial wave if the \(u_v\) equation is modified to be governed by the Navier-Stokes equation, say, in the frame rotating in \(E^3\).

As noticed in Sec. 2, the geometrical formulation in general carries over to the rotating frame. For instance we have \((\partial_t + \mathbf{L}_\mathbf{u})(\mathbf{u} + \mathbf{u}_R) = -\alpha(P - \mathbf{u}^2/2)\) and Eq. (2.8), and thus

\[
(\partial_t + \mathbf{L}_\mathbf{u})[(\mathbf{u} + \mathbf{u}_R) \wedge d(\mathbf{u} + \mathbf{u}_R)] = -d[(P - \mathbf{u}^2/2) \wedge d\mathbf{u}],
\]  
(4.1)
So, $H_t := (U + \mathbb{U}_R) \land d(U + \mathbb{U}_R)$ is precisely in the same form of situation of $U$, and that an integral invariant as the ‘circulation’ of $H_t$ over a ‘circuit’ follows: The ‘circuit’ is now any 3-boundary. Then following the same reasoning and calculation procedure in Sec. 3, we derive from $dL_{ni} H_t = 0$, with the rotation dominance approximation $H_t$ replaced by $H := (\mathbb{U}_R) \land d(\mathbb{U}_R)$, for the fast rotation limit $\sum_{i=1}^{4} u_{i,i} = 0$ in $E^4$ with double rotation (null constraint from fast simple rotation), and additionally $\sum_{i=1}^{4} u_{i,i} = 0$ in $E^5$ with double rotation, nothing new. And, there is no next-order contribution to $dL_{ni} H_t = 0$ from $H_t$. This actually should not be too surprising, because the classical Kelvin theorem already uses the information of both $U$ and $dU$.

An interesting and probably previously unnoticed point is that the Taylor-Proudman-type theorem itself actually only requires the incompressibility of the flow components lying in the rotating plane. Kelvin circulation theorem applies also to compressible barotropic flows [6] for which Eqs. (2.1,2.2,2.3) describe (see also [19,20]). The Taylor-Proudman-type theorem in $E^4$ however requires total incompressibility by the individual incompressibility in each rotating plane. In $E^5$, previous $u_{5,5} = 0$ was derived from the total incompressibility, and now can be relaxed. Similarly, we can only conclude incompressibility and cylindricity for the horizontal flow from Eq. (3.7) in $E^5$, and $u_3$ can vary in three dimensions, a two-component-two-dimensional couple with one-component-three-dimensional (2C2Dcw1C3D) system. In time-varying, and even turbulent, flows, it is then indicated, by the discussions in Sec. (d) concerning the relevance of the Taylor-Proudman theorem, that the horizontal compressibility would be reduced by rotation and the aeroacoustic noise may be more relevant to the variations along the rotating axis of the velocity component along it. It is a shame that this author [32] did not recognize such a handy example when proposing and analyzing 2C2Dcw1C3D flows, although the barotropic case did not escape from his attention.

Ethics. N/A

Data Accessibility. This article has no additional data.

Authors’ Contributions. The author complete this work independently.

Competing Interests. We have no competing interests.

Funding. NSFC (No. 11672102) and Tián-Yuán-Xué-Pài (No. 31415926) fundings.

Acknowledgements. The author thanks Wei X, Xu K, Zhang P and Zhong JQ for interactions, especially the latter for explaining Ref. [11], which also motivates his knowledge of Refs. [12–17].

APPENDIX

A. Dispersion relation of rotating flows in $E^5$

Since the derivation is standard and the computed dispersion relations are lengthy and similar, we only present one of the four results for reference:

$$
\omega = \sqrt{2} \{ (k_5^2 + k_4^2 + k_3^2 + k_2^2 + k_1^2) \\
\lambda_2^2 k_2^2 + \lambda_2^2 k_5^2 + \lambda_1^2 k_4^2 + \lambda_1^2 k_3^2 + k_5^2 \lambda_1^2 + \lambda_2^2 k_1^2 + \\
(2 \lambda_2^2 k_2^2 k_5^2 + 2 \lambda_2^4 k_2^2 k_1^2 - 2 \lambda_2^2 k_5^4 \lambda_1^2 + \\
2 \lambda_2^2 k_5^2 k_1^2 + 2 \lambda_1^4 k_4^2 k_3^2 + 2 \lambda_1^4 k_4^2 k_5^2 + 2 \lambda_1^4 k_4^2 k_3^2 + 2 \lambda_1^4 k_3^2 k_5^2 + \\
2 \lambda_2^2 k_2^2 \lambda_1^2 k_2^2 + 2 \lambda_2^2 k_2^2 \lambda_1^2 k_3^2 - \\
2 \lambda_2^2 k_2^2 k_5^2 \lambda_1^2 - 2 \lambda_2^2 \lambda_5^2 \lambda_1^2 k_4^2 - 2 \lambda_2^2 k_5^2 \lambda_1^2 k_3^2 + \\
2 \lambda_1^2 k_4^2 \lambda_2^2 k_1^2 + 2 \lambda_1^2 k_3^2 \lambda_2^2 k_1^2 - 2 \lambda_2^2 \lambda_1^2 \lambda_2^2 k_1^2 + \\
\lambda_2^4 k_4^2 + \lambda_2^4 k_5^4 + \lambda_1^4 k_4^4 + \lambda_1^4 k_3^4 + k_5^4 \lambda_1^4 + \lambda_2^4 k_1^4) \}^{1/2} \\
/(k_5^2 + k_4^2 + k_3^2 + k_2^2 + k_1^2).$$

References

1. Proudman J. 1916 On the motion of solids in a liquid possessing vorticity. P. Roc. R. Soc. Lond. A. 92, 408–424.
2. Taylor GI. 1921 Experiments with Rotating Fluids. P. Roc. R. Soc. Lond. A. 100, 114–21.
3. Wu JZ, Ma HY & Zhou MD. 2006 Vorticity and vortex dynamics. Berlin: Springer.
4. Sagaut P & Cambon C. 2008 Homogeneous turbulence dynamics. Cambridge, UK: Cambridge University Press.
5. Chandrasekhar S. 1961 Hydrodynamic and Hydromagnetic Stability. Dover, New York.
6. Fecko M. 2018 A generalization of vortex lines. Journal of Geometry and Physics 124, 64–73.
7. Lynden-Bell D & Katz J. 1981 Isocirculational Flows and their Lagrangian and Energy Principles. Proc. R. Soc. Lond. A. 378, 179–205.
8. Gjaja I & Holm DD. 1996 Self-consistent Hamiltonian dynamics of wave mean-flow interaction for a rotating stratified incompressible fluid. Phys. D 98, 343–378.
9. Pouquet A & Mininni PD. 2010 The interplay between helicity and rotation turbulence: implications for scaling laws and small-scale dynamics. Phil. Trans. R. Soc. A 368, 1635–1662.
10. Imazio PR & Mininni PD. 2017 Passive scalars: Mixing, diffusion, and intermittency in helical and nonhelical rotating turbulence. Phys. Rev. E 95, 033103.
11. Zhong JQ & Ahler G. 2010 Heat transport and the large-scale circulation in rotating turbulent Rayleigh-Bénard convection. J. Fluid Mech. 665, 300–333.
12. Liu NS, & Lu XY. 2007 Direct numerical simulation of spanwise rotating turbulent channel flow with heat transfer. Intl J. Numer. Meth. Fluids 53, 1689–1706.
13. Yang Z, Cui G, Xu C & Zhang Z. 2011 Study on the analogy between velocity and temperature fluctuations in the turbulent rotating channel flows. J. Phys.: Conf. Ser. 318, 022008.
14. Yang YT, & Wu JZ. 2012 Channel turbulence with spanwise rotation studied using helical wave decomposition. J. Fluid Mech. 692, 137–152.
15. Xia Z, Shi Y & Chen S. 2016 Direct numerical of turbulent channel flow with spanwise rotation. J. Fluid Mech. 788, 42–56.
16. Dai YJ, Huang WX & Xu CX. 2016 Effects of Taylor-Görtler vortices on turbulent flows in a spanwise-rotating channel. Phys. Fluids 28, 115104.
17. Brethouwer G. 2018 Passive scalar transport in rotating turbulent channel flow. J. Fluid Mech. 844, 297–322.
18. Arnold VI & Khesin BA. 1998 Topological methods in hydrodynamics. Springer.
19. Besse N & Frisch U. 2017 Geometric formulation of the Cauchy invariants for incompressible Euler flow in flat and curved spaces. J. Fluid Mech. 825, 412.
20. Anco SC, Dar A and Tufail N. 2015 Proc. R. Soc. A 471, 20150223.
21. Shashikanth BN. 2012 Vortex dynamics in R4. J. Math. Phys., 53, 013103.
22. Nikitin N. 2011 Four-dimensional turbulence in a plane channel. J. Fluid Mech. 680, 67–79.
23. Gotoh T, Watanabe Y, Shiga Y & Suzuki E. 2007 Statistical properties of four-dimensional turbulence. Phys. Rev. E 75, 016310.
24. Embid PF & Majda AJ. 1998 Low Froude number limiting dynamics for stably stratified flow with small or finite Rossby numbers. Geophys. Astrophys. Fluid Dyn. 87, 1–50.
25. Majda AJ & Embid PF. 1998 Averaging over fast gravity waves for geophysics flows with unbalanced initial data. Theor. Comput. Fluid Dyn. 11, 155–169.
26. Frisch U. 2016 The collective birth of multifractals. J. Phys. A: Math. Theor. 49.
27. Lee JM. 2000 Introduction to Smooth Manifolds. Springer-Verlag, New York.
28. Fecko M. 2006 Differential Geometry and Lie Groups for Physicists. Cambridge University Press.
29. Youla DC. 1961 A normal form for a matrix under the unitary congruence group. Can. J. Math. 13, 694–704.
30. Hanson J. Rotations in three, four, and five dimensions. arXiv:1103.5263 [math.MG].
31. Weiner JL & Wilkins GR. 2005 Quaternions and rotations in E4. Am. Math. Mon. 112, 69–76; Coxeter HSM. 1946 Quaternions and reflections. Am. Math. Mon. 53, 136–146.
32. Zhu JZ. 2018 Vorticity and helicity decompositions and dynamics with real Schur form of the velocity gradient. Phys. Fluids 30, 031703.