Resolution of type IV singularities in quantum cosmology

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We discuss the fate of classical type IV singularities in quantum cosmology. The framework is Wheeler–DeWitt quantization applied to homogeneous and isotropic universes with a perfect fluid described by a generalized Chaplygin gas. Such a fluid can be dynamically realized by a scalar field. We treat the cases of a standard scalar field with positive kinetic energy and of a scalar field with negative energy (phantom field). We first present the classical solutions. We then discuss in detail the Wheeler–DeWitt equation for these models. We are able to give analytic solutions for a special case and to draw conclusions for the general case. Adopting the criterion that singularities are avoided if the wave function vanishes in the region of the classical singularity, we find that type IV singularities are avoided only for particular solutions of the Wheeler–DeWitt equation. We compare this result with earlier results found for other types of singularities.

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I. INTRODUCTION

It is well known that Einstein’s theory of general relativity predicts the occurrence of spacetime singularities. A sufficient condition for this is the validity of certain classical energy conditions, which leads to the classic singularity theorems [1]. In many interesting situations, these conditions are, however, violated. It has thus been suggested that the classical energy conditions be replaced by semiclassical conditions, which are often fulfilled in cases of interest [2]. Independent of this generalization, it is a fact that singularities occur even in situations in which the classical energy conditions, notably the dominant energy condition, are violated.

Such violations occur quite frequently in situations where Dark Energy plays a role. Since observations indicate that our Universe is currently accelerating, the occurrence of singularities may be relevant for its future evolution. Such singularities have been classified in [3], see also [4, 5, 6] and the references therein. Depending on the variables that become divergent, they are called type I (big rip) [7, 8], type II (big brake, sudden or big démarrage) [9, 10], type III (big freeze) [3, 11–13] and type IV [3, 14]. The mildest among these singularities is the type IV singularity, which is the subject of this paper. It is characterized by a divergence of higher derivatives of the Hubble rate $H$, with $H$ and $H$ itself being finite at the singularity; it is a singularity only in derivatives of curvature invariants, not in the invariants themselves. Such a singularity takes place at a finite scale factor and at a finite cosmic time. Since geodesics can be extended through the type IV singularity, it is not a singularity in the sense of the standard definition used in general relativity.

It is generally believed that a theory of quantum gravity should avoid such singularities. Unfortunately, there does not yet exist a full theory in complete form, but only a couple of approaches, such as quantum geometrodynamics, path-integral quantization, loop quantum gravity, and string theory [15]. The question of singularity avoidance can thus only be addressed in a concrete approach and only for concrete simplified situations. These situations are typically either gravitational collapse with spherically symmetric metrics or homogeneous cosmologies. Most investigations in cosmology deal with Friedmann–Lemaître–Robertson–Walker (FLRW) models, because our observed Universe can be described approximately by such models. We shall also do this here in our investigation of the fate of type IV singularities in quantum cosmology.

Our analysis is based on the most conservative approach to quantum gravity — quantum geometrodynamics with the Wheeler–DeWitt equation as its central equation. For the situation here, this is a partial differential equation for a wave function that depends on the scale factor $a$, of the FLRW model, as well as on matter degrees of freedom (below, a homogeneous scalar field $\phi$).

It has already been shown that singularity avoidance can happen in the quantum versions of models with a big rip [16, 17], a big brake or big démarrage [18, 19], and big freeze [20], cf. [21, 22] for reviews. As sufficient (though by no means necessary) conditions for singularity avoidance, the vanishing of the wave function at the region of the classical singularity [23] or the breakdown of the semiclassical approximation (dispersion of wave packets) [24] were postulated.

Type IV singularities are essentially different from the singularities discussed in these earlier papers, and this is why they deserve a separate treatment. It will become clear in the course of this paper that the rather mild na-
ture of these singularities leads to a much more restrictive degree of avoidance than the other types of singularities. For singularity avoidance, we adopt here the criterion that the wave function vanishes in the corresponding region in configuration space. This is a sufficient (but not necessary) criterion that goes back to DeWitt’s pioneering paper on canonical quantum gravity [24].

We must emphasize that it does not make sense to talk of quantum avoidance of singularities without specifying the concrete model, in particular, the form of the potential in the Wheeler–DeWitt equation. The situation is well known from quantum mechanics, which also by itself does not cure the classical singularities. In the case of the Coulomb potential, singularity avoidance is obvious. But for many other singular potentials, this does not happen [25]. It is an amazing aspect of Nature that potentials which are physically relevant are singularity free. The same may happen in quantum cosmology.

Our paper is organized as follows. In Sec. II, we discuss classical models with a standard and a phantom scalar field which lead to a type IV singularity. The classical equation of state is given by a generalized Chaplygin gas. We address, in particular, the trajectories in configuration space and the exact form of the potential. Sec. III is devoted to the quantum analysis of these models and constitutes the central part of our paper. We show that singularity avoiding solutions to the Wheeler–DeWitt equation exist, but that they form only a subset of all normalizable solutions. In Sec. IV we present our conclusions and an outlook on further investigations.

II. CLASSICAL MODEL

The generalized Chaplygin gas (GCG) is a perfect fluid with a relatively simple equation of state and a surprisingly wide range of applications [26, 31]. It can, for example, describe and unify different matter contents in the universe; moreover, it can model a universe with almost all kinds of singularities [15]. It can, in particular, also induce a type IV singularity. This is the case of interest here.

The GCG fulfills the equation of state [26, 28]
\[ P = -\frac{A}{\rho^\beta}, \]
where \( A \) and \( \beta \) are constants with arbitrary sign. (The usual Chaplygin gas corresponds to the choices \( A > 0 \) and \( \beta = 1 \)) Imposing the conservation of the energy–momentum tensor of such a fluid, one obtains the equation \( \dot{\rho} + 3H(\rho + p) = 0 \), which can readily be solved to yield
\[ \rho = \left( A + \frac{B}{a^{3(1+\beta)}} \right)^{\frac{1}{1+\beta}}, \]
where \( B \) as an arbitrary (real) constant. We shall now discuss the case for which this behavior corresponds to a type IV singularity [15]. We restrict ourselves to the case of a spatially flat FLRW universe.

A. Standard GCG and type IV singularity

A GCG fulfilling the null, strong, and weak energy conditions can induce a type IV singularity in the future if \( A < 0, B > 0 \) and \( -\frac{1}{2} < \beta < 0 \) being \( \beta \neq 1/(2p) - 1/2 \), where \( p \) is a positive integer [15]. Then the energy density \( \rho \) and pressure can be expressed as
\begin{align*}
\rho &= |A|^{\frac{1}{1+\beta}} \left( \frac{a_{\text{max}}}{a} \right)^{3(1+\beta)} - 1 \right)^{\frac{1}{1+\beta}}, \quad (3) \\
P &= |A|^{\frac{1}{1+\beta}} \left( \frac{a_{\text{max}}}{a} \right)^{3(1+\beta)} - 1 \right)^{-\frac{\beta}{1+\beta}}, \quad (4)
\end{align*}
where \( a_{\text{max}} \) is defined by
\[ a_{\text{max}} := \left| \frac{B}{A} \right|^{\frac{1}{1+\beta}}, \quad (5) \]
which will play the role of the maximum scale factor. We recognize from (3) and (4) that the energy density and pressure go to zero as the scale factor approaches \( a_{\text{max}} \). Nevertheless, this FLRW universe will face at \( a = a_{\text{max}} \) a type IV singularity [15], see the remarks below.

The Friedmann equation for flat spatial sections with this matter content can be integrated analytically, resulting in
\begin{align*}
\frac{B}{2(1+\beta)} &\left[ \frac{1}{2(1+\beta)} + \frac{2\beta + 1}{2(1+\beta)} \right] \\
&- \frac{B}{\left( \frac{a}{a_{\text{max}}} \right)^{3(1+\beta)}} \left( \frac{1}{2(1+\beta)} + \frac{2\beta + 1}{2(1+\beta)} \right) \\
&= \sqrt{3k|A|^2(1+\beta)t}.
\end{align*}
where \( B[\gamma, \delta] \) and \( B[x, \gamma, \delta] \) denote the beta function and the incomplete beta function, respectively (cf. Sec. 6.2. in [32]); \( \kappa \) is defined by \( \kappa^2 = 8\pi G \), where \( G \) is the gravitational constant. Finally, \( t \) stands for the time that elapses from a given finite value of the scale factor to its maximum value \( a_{\text{max}} \). For \(-\frac{1}{2} < \beta < 0 \), it assumes a finite value, but it becomes infinite in the limiting case \( \beta \to -\frac{1}{2} \). We can rewrite the previous expression as (cf. Eq. 15.1.20 in [32])
\begin{align*}
2(1+\beta) &\left\{ \frac{1}{1+\beta} + \frac{1}{1+\beta} - \frac{1}{1+\beta} \right\} \\
&- \left( \frac{a}{a_{\text{max}}} \right)^{\frac{1}{1+\beta}} \left[ \frac{1}{1+\beta} + \frac{1}{1+\beta} + \frac{a}{a_{\text{max}}}^{3(1+\beta)} \right] \\
&= \sqrt{3k|A|^2(1+\beta)t}.
\end{align*}
where \( F[\gamma, \delta; c; x] \) denotes a hypergeometric function (cf. Chap. 15. in [32]). One can show directly from this
expression that \( t \) is finite until \( \beta \to -\frac{1}{2} \) where it becomes infinite. This exact result coincides, as it should, with the approximation that is presented in \[15\].

The \( n \)-th derivative of the Hubble parameter blows up at \( a = a_{\text{max}} \) if \( \beta \neq 1/(2\rho) - 1/2 \), where \( \rho \) is a positive integer. It can be expressed as \( n = 1 + E(1/(1+2\beta)) \), where \( E \) denotes the integer value function \[15\]. Therefore, the \((n-1)\)-th derivative of the scalar curvature diverges at \( a = a_{\text{max}} \), resulting in a type IV singularity at \( a = a_{\text{max}} \).

A universe filled with this kind of matter content is dust-dominated at small scale factors, that is, \( p/\rho \ll 1 \), facing a big bang singularity where the energy density and pressure diverge. When the universe approaches \( a_{\text{max}} \), this universe encounters a type IV singularity. For \( \beta = -1/2 \), even though it takes an infinite time for the universe to reach its maximum size, the Hubble parameter and all its cosmic time derivatives are finite (in fact, they vanish).

A perfect fluid of this type can be dynamically implemented by a scalar field that is either minimally coupled or kinetically driven (a kind of Born-Infeld scalar field or K-essence field; see, for example, \[29\]). For simplicity, we shall stick here to a minimally coupled scalar field with standard energy density and pressure, that is,

\[
\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi),
\]

where the dot stands for the derivative with respect to cosmic time \( t \). In terms of the scale factor, the kinetic energy and the potential of the scalar field can be expressed as

\[
\dot{\phi}^2 = |A|^{-1+\beta} \left[ \left( \frac{a_{\text{max}}}{a} \right)^{3(1+\beta)} - 1 \right]^{-1+\beta}, \quad V(\phi) = \frac{1}{2} |A|^{-1+\beta} \left[ \left( \frac{a_{\text{max}}}{a} \right)^{3(1+\beta)} - 2 \right]^{-1+\beta}.
\]

Consequently, the scalar field scales with the scale factor as

\[
|\phi - \phi_{\text{max}}|(a) = \frac{2\sqrt{3}}{3\kappa|1+\beta|} \ln \left[ \left( \frac{a_{\text{max}}}{a} \right)^{1/2(1+\beta)} + \sqrt{\left( \frac{a_{\text{max}}}{a} \right)^{3(1+\beta)} - 1} \right],
\]

where \( \phi_{\text{max}} \) stands for the value acquired by the scalar field at \( a = a_{\text{max}} \), where the singularity is situated. For simplicity, we will set \( \phi_{\text{max}} \) to zero. In Fig. 1 we show the kinetic energy of the scalar field and the dependence of the field on the scale factor, that is, the classical trajectory in configuration space.

\[\text{FIG. 1. The kinetic energy of the scalar field (top) and the dependence of the field on the logarithmic scale factor } \alpha = \ln(a/a_{\text{max}}) \text{ (bottom). In the upper Figure, the value } \beta = -\sqrt{3}/3 \text{ is chosen. The singularity is at } \phi = 0, \text{ where } a = a_{\text{max}}.\]

The scalar field potential can be written as

\[V(\phi) = V_1 \left[ \sinh^{-\frac{\kappa}{2\sqrt{3}}}(\sqrt{\frac{3}{2}}\kappa|1+\beta||\phi|) - \sinh^{-\frac{\kappa}{2\sqrt{3}}}(\sqrt{\frac{3}{2}}\kappa|1+\beta||\phi|) \right],\]

where \( V_1 = |A|^{1+\beta}/2 \), cf. \[21\]. The potential is displayed in Fig. 2 for a typical value of \( \beta \).

Notice that near \( a_{\text{max}} (\phi = 0) \) the potential is negative and finite. This is not surprising, since in a type IV singularity both the energy density and the pressure are finite. We emphasize that the potential \[12\] is of the form of a double-well potential and is regular everywhere. This is in stark contrast to the cases discussed in \[18, 20, 21\] and is connected with the soft nature of the type IV singularity. It will have direct consequences for the study of the quantum theory below.

Close to the type IV singularity, the potential can be
approximated as
\[ V(\phi) \simeq -V_1 \left( \frac{\sqrt{3}}{2} \kappa |1 + \beta||\phi| \right)^{-\frac{2\beta}{1+\beta}}, \tag{13} \]
cf. Eq. (16) in [21]. In the limiting case \( \beta = -1/2 \), this corresponds to an inverted harmonic oscillator.

At small scale factor (or large value of the scalar field), the potential can be approximated by the exponential form
\[ V(\phi) \simeq 2^{-\frac{1+\beta}{\beta}} V_1 \exp \left( \sqrt{3} \kappa |\phi| \right). \tag{14} \]

Such a potential occurs also in the cases of the big rip with a phantom field [18] and the big bang with an anti-Chaplygin gas [21]. In the latter case, it was shown that the big-bang singularity is avoided in the quantum theory simultaneously with the big-brake singularity, which is present in the classical version of this model. Indeed, a similar expression to (14) can be found as well for a big freeze model induced by a standard GCG, but where the dust-like behavior \((p/\rho \sim 0)\) is reached at large scale factors rather than at small scale factors [21].

B. Phantom GCG and type IV singularity

A phantom GCG violating the null energy condition can induce a type IV singularity in the past if \( A > 0 \), \( B < 0 \) and, as above, \(-1 < \beta \leq 0\), being \( \beta \neq 1/(2p) - 1/2 \), where \( p \) is a positive integer [17]. Then, the energy density \((2)\) and the pressure can be expressed as
\[ \rho = |A|^{-\frac{1}{1+\beta}} \left[ 1 - \left( \frac{a_{\text{min}}}{a} \right)^{3(1+\beta)} \right]^{\frac{1}{1+\beta}}, \tag{15} \]
\[ P = -|A|^{-\frac{1}{1+\beta}} \left[ 1 - \left( \frac{a_{\text{min}}}{a} \right)^{3(1+\beta)} \right]^{\frac{1}{1+\beta}}, \tag{16} \]
where here
\[ a_{\text{min}} := \left| \frac{B}{A} \right|^{\frac{1}{3(1+\beta)}}, \tag{17} \]
thus leading to a minimal value for the scale factor instead of a maximum value as in the corresponding case for the standard field. The type IV singularity is now located at \( a = a_{\text{min}} \). Notice that the cosmic time derivatives of the Hubble rate, in this case, are similar to those presented in the previous subsection and therefore the proof of the existence of a type IV singularity follows directly.

The Friedmann equations can again be integrated analytically, resulting in (cf. section 6.2, in [32])
\[ B \left[ \left( \frac{a}{a_{\text{min}}} \right)^{3(1+\beta)}, 0, 2 \beta + 1 \right] = \sqrt{3} \kappa A^{-\frac{1}{2(1+\beta)}} (1 + \beta) t, \tag{18} \]
where \( t \) stands for the time that has elapsed from the beginning of the expansion at \( a_{\text{min}} \), that is, at a type IV singularity, until it has reached a given finite size \( a \). Notice that even though the incomplete beta function in the previous expression assumes the value zero in its second argument, it is well defined.

At very large values of the scale factor, the universe becomes asymptotically de Sitter. For the limiting case \( \beta = -1/2 \), even though it takes an infinite time for the universe to reach a given size, the Hubble rate and all its cosmic derivatives are finite (in fact, they vanish), similar to the case in Sec. IIA.

Again, the fluid can be mapped to a scalar field that is either minimally coupled (the case considered here) or kinetically driven. Here, energy density and pressure read
\[ \rho_\phi = -\frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_\phi = -\frac{1}{2} \dot{\phi}^2 - V(\phi). \tag{19} \]

Note the change of sign in the kinetic terms compared to (3). In terms of the scale factor, the kinetic energy and the potential of the scalar field can be expressed as
\[ \dot{\phi}^2 = A^{-\frac{1}{1+\beta}} \left[ 1 - \left( \frac{a_{\text{min}}}{a} \right)^{3(1+\beta)} \right]^{\frac{1}{1+\beta}}, \tag{20} \]
\[ V(\phi) = \frac{1}{2} A^{-\frac{1}{1+\beta}} \left[ 1 - \left( \frac{a_{\text{min}}}{a} \right)^{3(1+\beta)} \right]^{\frac{1}{1+\beta}}. \tag{21} \]

Consequently, the equation in configuration space is given by
\[ |\phi - \phi_{\text{min}}(a)| = 2 \kappa \sqrt{3} \frac{1}{1+\beta} \arccos \left( \frac{a_{\text{min}}}{a} \right)^{\frac{3(1+\beta)}{1+\beta}}, \tag{22} \]
where \( \phi_{\text{min}} \) stands for the value acquired by the scalar field at \( a_{\text{min}} \). In Fig. [3] we have displayed the absolute
The potential can be written as

\[ V(\phi) = V_{-1} \left[ \sin \frac{2\beta}{1+\beta} \sqrt{\frac{3}{2}} |1+\beta||\phi| \right] + \sin \frac{2\beta}{3} \sqrt{\frac{3}{2}} |1+\beta||\phi| \], \quad (23) \]

where \( V_{-1} = A_{1+\beta}/2 \) and \( 0 < (\sqrt{3}/2)|1+\beta||\phi| \leq \pi/2 \), cf. Eq. (21) in [21]. Notice that near \( a_{\text{min}} (\phi = 0) \), the potential is positive and finite, in contrast to the cases discussed in [21]. This is, again, not surprising, as in a type IV singularity both the energy density and the pressure are finite. The potential (23) is, in contrast to the case of the standard field, periodic in \( \phi \). The shape of the potential in terms of the scalar field is shown in Fig. 4.

In the expanding branch, the evolution starts from the singularity located at \( \phi = 0 \), then the scalar field rolls up the potential and asymptotically reaches the top of the potential, which is located at \( \sqrt{3(1+\beta)}\phi/2 = \pi/2 \), while \( a \to \infty \). Classically, the various parts (extensions of the part shown in Fig. 4; i.e., for example, outside the maxima of the potential marked with two vertical lines) correspond to different classical solutions. This may have consequences in the quantum theory.

Close to the singularity, the potential can be approximated by

\[ V(\phi) \simeq V_{-1} \left( \frac{\sqrt{3}}{2} |1+\beta||\phi| \right)^{-2\beta/(1+\beta)}. \quad (24) \]

We now turn to the quantum versions of these models.

III. QUANTUM ANALYSIS

In this section, we investigate the question whether the classical type IV singularity can be avoided in the quantum theory or not. In treating the Wheeler–DeWitt equation, we apply the methods used in the earlier papers [18, 21, 21]. We also want to emphasize that the quantum cosmology of a GCG was first discussed in [23]. In our case, we have for the wave function \( \Psi (\alpha, \phi) \) a Wheeler–DeWitt equation of the form

\[ \frac{\hbar^2}{2} \left( \frac{\kappa^2}{6} \frac{\partial^2}{\partial \alpha^2} - \ell^2 \frac{\partial^2}{\partial \phi^2} \right) \Psi (\alpha, \phi) + a_0^6 e^{6\phi} V(\phi) \Psi (\alpha, \phi) = 0, \quad (25) \]

where \( a_0 \) corresponds to the location of the singularity, which is \( a_0 = a_{\text{max}} \) for the model of Sec. II.A and \( a_0 = a_{\text{min}} \) for the model in Sec. II.B. Here, we have used...
the Laplace–Beltrami factor ordering, but our main results should be insensitive to the particular choice of ordering. The potential $V(\phi)$ is given by $\text{(12)}$ for the standard scalar field and by $\text{(28)}$ for the phantom scalar field model. We shall use the rescaled scale factor $\tilde{a} := a/a_0$ instead of $a$ in the following, which implies $\tilde{a}_0 = 1$; but for simplicity, we shall drop the tilde. We have introduced in $\text{(25)}$ as well $\alpha := \ln(a/a_0)$. In order to treat the phantom and non-phantom cases in one equation, we have introduced the parameter $\ell$, which assumes the value $\ell = -1$ for the phantom scalar field and $\ell = 1$ for the ordinary scalar field.

In order to solve this equation, we use the Born–Oppenheimer (BO) type of ansatz first used in $\text{(34)}$ and write

$$
\Psi(\alpha, \phi) = \varphi_k(\alpha, \phi)C_k(\alpha),
$$

where $k$ is a general (complex) parameter. In the BO limit, we require that the functions $\varphi_k$ satisfy the equation

$$
-\ell^2 \frac{h^2}{2} \frac{\partial^2 \varphi_k}{\partial \phi^2} + a_0^6 e^{6\alpha} V(\phi) \varphi_k = E_k(\alpha) \varphi_k.
$$

In general, the study of the singularity structure (in the mathematical sense) of this eigenvalue equation should give us some insight into the quantum avoidance or non-avoidance of the cosmological singularities $\text{(33)}$.

A. Standard field

Let us first treat the standard (non-phantom) case $\ell = 1$; we set $\hbar = 1$ for simplicity. For a general value of $\beta$, it is difficult to treat this equation analytically. For this reason, we shall choose the particular value $\beta = -1/2$. Strictly speaking, this value lies outside the range $-1/2 < \beta < 0$ that we imposed earlier, but we nevertheless expect that the qualitative features for this limiting case reflect the generic situation; after all, the appearance of the potential in Fig. 2 remains unchanged in this limit. We shall draw conclusions for the general case below.

For the value $\beta = -1/2$, $\text{(27)}$ takes the following form:

$$
-\frac{1}{2} \frac{\partial^2 \varphi_k}{\partial \phi^2} + a_0^6 V_1 e^{6\alpha} \left[ \sin^4 \left( \frac{\sqrt{3}}{4} \kappa \phi \right) - \sin^2 \left( \frac{\sqrt{3}}{4} \kappa \phi \right) \right] \varphi_k = E_k(\alpha) \varphi_k.
$$

We introduce now the variable

$$
x := \sinh \left( \frac{\sqrt{3}}{4} \kappa \phi \right),
$$

where $x > 0$ corresponds to the upper branch of the trajectory displayed in Fig. 1 (right) and $x < 0$ to the lower branch. We skip the index $k$ for simplicity. Equation $\text{(28)}$ now assumes the form

$$
(1 + x^2) \frac{\partial^2 \varphi}{\partial x^2} + x \frac{\partial \varphi}{\partial x} - \xi x^2(x^2 - 1) \varphi = -\epsilon \varphi,
$$

where

$$
\xi := \frac{32V_1 a_0^6 e^{6\alpha}}{3\kappa^2} > 0, \quad \epsilon := \frac{32E_k(\alpha)}{3\kappa^2}.
$$

Both $\xi$ and $\epsilon$ depend on $\alpha$, although we suppress this dependence for notational simplification. Since $\text{(30)}$ is symmetric under $x \mapsto -x$, both branches can be treated on an equal footing.

With the separation ansatz $\varphi = \exp \left( -\frac{\sqrt{x^2}}{2} \right) H(x)$, we find that $H(x)$ obeys the following differential equation $\text{(1)}$

$$
(1 + x^2) \frac{d^2 H}{dx^2} + \left( x - 2x(1 + x^2) \sqrt{\xi} \right) \frac{dH}{dx} - \left( (1 + 2x^2) \sqrt{\xi} - 2x^2 \xi \right) H = -\epsilon H.
$$

We can transform this equation into a standard form for the confluent Heun differential equation (see e.g. $\text{[36]}$ for details on these functions) by performing the transformation $z := -x^2$. Equation $\text{(30)}$ then takes the following form:

$$
\frac{d^2 H}{dz^2} + \frac{\sqrt{\xi} z^2 - (\sqrt{\xi} - 1) z - \frac{1}{4}}{z(z - 1)} \frac{dH}{dz} - \frac{(2\xi - 2\sqrt{\xi}) z + \sqrt{\xi} - \epsilon}{4z(z - 1)} H = 0.
$$

Solutions to this equation are the Heun functions denoted by $\mathcal{H}_c(u, v, w, \delta, \eta; z)$, which depend on five parameters. The canonical form of this differential equation is given by (see e.g. p. 59, Eq. (13) in $\text{[37]}$)

$$
\frac{d^2 \mathcal{H}_c}{dz^2} + \frac{u z^2 - (u - v - w - 2) z - v - 1}{z(z - 1)} \frac{d\mathcal{H}_c}{dz} + \frac{\delta + \frac{1}{2} u(v + w + 2)}{z(z - 1)} \frac{\mathcal{H}_c}{z + \frac{1}{2}(w - u)(v + 1) + \frac{1}{2} + \eta} = 0.
$$

Comparing this general form with our special case $\text{(30)}$, we find that $\text{(33)}$ is solved by

$$
\mathcal{H}(z) = \mathcal{H}_c \left( \sqrt{\xi}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \xi, \frac{3}{8} + \frac{1}{4} \epsilon; z \right).
$$

Consequently, $\text{(32)}$ is solved by

$$
\mathcal{H}(x) = \mathcal{H}_c \left( \sqrt{\xi}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \xi, \frac{3}{8} + \frac{1}{4} \epsilon; -x^2 \right).
$$

\[3\] From now on, we only indicate the dependence on the variable $x$ and thus write ordinary differentials.
A linearly independent solution to (32) is given by (see e.g. p. 61, proposition 2-1, in [37])

\[ H(x) = x H_c \left( \sqrt{\xi}, \frac{1}{2} \right) \exp \left( \frac{1}{2} - \frac{1}{2} \xi, \frac{1}{4} \xi; -x^2 \right). \]

Therefore, \( \varphi \) can be written as the following linear combination:

\[ \varphi(x) = c_1 e^{-\frac{x^2}{2}} H_c \left( \sqrt{\xi}, \frac{1}{2} \right) \exp \left( \frac{1}{2} - \frac{1}{2} \xi, \frac{1}{4} \xi; -x^2 \right) + c_2 x e^{-\frac{x^2}{2}} H_c \left( \sqrt{\xi}, \frac{1}{2} \right) \exp \left( \frac{1}{2} - \frac{1}{2} \xi, \frac{1}{4} \xi; -x^2 \right), \quad (35) \]

with constants \( c_1 \) and \( c_2 \). It is, in fact, well known from quantum mechanics that the stationary Schrödinger equation for a potential of the form appearing in (28) is analytically solvable in terms of Heun functions; see, for example, p. 265, Table II, in [33].

Although the general choice of Hilbert space in quantum gravity is an open issue [17], it is reasonable to proceed here as in ordinary quantum mechanics and to demand that the physically allowed wave functions \( \varphi(x) \) approach zero for large argument. The Heun function \( H_c \) appearing above has the property that it is regular at the origin \( x = 0 \) ([32], p. 98),

\[ H_c(\cdot, \cdot, \cdot, \cdot; 0) = 1, \quad (36) \]

and that it increases as a power for large \( x \) ([32], p. 101). This increase as a power is compensated by the decrease induced by the Gaussian factor in (35); the wave function \( \varphi(x) \) in (35) thus satisfies the physical requirement that it approaches zero at infinity.

We note that in the first term of (35) the variable \( x \) appears only quadratically, so that this part of the wave function is symmetric, whereas the second part of (35) is antisymmetric and takes the value zero at the origin due to the presence of the additional term \( x \). Since \( x = 0 \) corresponds to the location of the singularity at \( \phi = 0 \), it is this second part that fulfills the condition of singularity avoidance.

For general \( \beta \), the equations become much more complicated, but one can nevertheless draw general conclusions without making explicit calculations. Let us present the general arguments.

A sufficient criterium for singularity avoidance is the vanishing of the wave function at the point of the classical singularity. This corresponds in our case to the value \( \phi = 0 \). Can we implement here \( \varphi(\alpha, 0) = 0 \)? As one knows from quantum mechanics, for a potential of the form shown in Fig. 2 one has a spectrum that consists of infinitely many discrete bound states. The ground state \( \varphi_0 \) is symmetric, and the excited states \( \varphi_n \) are alternately antisymmetric and symmetric and have \( n \) nodes; between two consecutive nodes of \( \varphi_n \), there is a node of \( \varphi_n+1 \). From this, it is clear that the antisymmetric solutions vanish at \( \phi = 0 \), while the symmetric solutions do not. The difference to the cases discussed in [28] and [21] is thus the following: whereas in these earlier papers the vanishing of the wave function at the point of the classical singularity is (at least in some of the cases considered) enforced by its normalizability with respect to the \( L^2 \) inner product, the type IV case discussed here allows such solutions but does not enforce them. Singularity avoiding solutions can here be constructed as superpositions of states of the form (28), in which \( \varphi_k \) is an antisymmetric eigenstate of (34). This argument holds for general \( \beta \) in the allowed range, while the above solutions for \( \beta = -1/2 \) in terms of Heun functions is a special case that can be written in explicit form; for the allowed eigenstates, the ‘energy’ \( \epsilon \) is quantized.

We also have to look for the gravitational part of the wave function (26). Inserting the ansatz (28) into the Wheeler–DeWitt equation (25), we get an equation for \( C_k(\alpha) \),

\[ \frac{k^2}{6} \left( 2 \dot{C}_k \varphi_k + C_k \ddot{\varphi}_k \right) + \left( \frac{k^2}{6} C_k^2 + 2 E_k(\alpha) C_k \right) \varphi_k = 0, \quad (37) \]

where a dot indicates a derivative with respect to \( \alpha \). In the BO approximation, one assumes that \( C_k \) varies much more rapidly with \( \alpha \) than with \( \varphi_k \) and neglects the backreaction of the matter part on the gravitational part; it then follows that we can neglect the terms \( \dot{C}_k \varphi_k \) and \( C_k \ddot{\varphi}_k \) [35]. This means that the matter part only contributes its energy to the gravitational part via the term \( E_k(\alpha) \). With this approximation, we then have

\[ \left( \frac{k^2}{6} C_k + 2 E_k(\alpha) C_k \right) \varphi_k = 0. \quad (38) \]

Note that the parameter \( \ell \) does not appear here, so that this equation holds for both the standard scalar field and the phantom field.

We do not know the exact expression for \( E_k(\alpha) \), because these are the eigenvalues of (28), which cannot be given in explicit form. But we can solve (35) in a WKB approximation to obtain

\[ C_k(\alpha) \sim \left( \frac{12 E_k(\alpha)}{\kappa^2} \right)^{-\frac{1}{4}} \left( b_1 \exp \left[ i \int \sqrt{ \frac{12 E_k(\alpha)}{\kappa^2} } d\alpha \right] + b_2 \exp \left[ -i \int \sqrt{ \frac{12 E_k(\alpha)}{\kappa^2} } d\alpha \right] \right), \quad (39) \]

with constants \( b_1 \) and \( b_2 \). We note that \( E_k(\alpha) \) is an (\( \alpha \)-dependent) eigenvalue of the Hermitian operator appearing in (25) and is thus real. They are positive in the classically allowed region \( (a \leq a_{\text{max}}) \) and negative in the classically forbidden region \( (a > a_{\text{max}}) \). In order to respect the correspondence to the classical limit, the wave functions \( C_k(\alpha) \) should exponentially decrease for large \( \alpha \) [39]. This is an important consistency condition. By the standard WKB connection formulae, this then introduces a relation between \( b_1 \) and \( b_2 \). Since all these solutions are regular, they do not spoil our conclusions on singularity avoidance.
One would, of course, also get a solution that vanishes at the classical singularity if one demanded that the $C_k$ vanish there. This would entail a certain condition between the constants $b_1$ and $b_2$. Since the ensuing functions $C_k$ would then not decrease in the classically forbidden region, we shall, however, disregard this possibility.

An interesting aspect of this model is the possibility of tunnelling from one well to the other, as can be seen from the form of the potential displayed in Fig. 4. In this way, the universe could avoid the singular region present at the origin. A detailed study of tunnelling is, however, beyond the scope of this paper.

In summary, singularity avoidance for type IV singularities occurs only in special cases. In general, the singularity is not avoided.

B. Phantom field

The case of the phantom case can be treated analogously to the case of the standard field, so we only report the main steps.

Choosing $\ell = -1$, $\beta = -1/2$, and using the phantom potential (29), we arrive instead of (28) at the equation

$$-\frac{1}{2} \frac{\partial^2 \varphi_k}{\partial \phi^2} - V_{-1} e^{6\alpha} \left[ \sin^4 \left( \frac{\sqrt{3}}{4} k \phi \right) + \sin^2 \left( \frac{\sqrt{3}}{4} k \phi \right) \right] \varphi_k = -E_k(\alpha) \varphi_k.$$  

We introduce here the variable

$$y := \sin \left( \frac{\sqrt{3}}{4} k \phi \right),$$  

which then leads to the following equation for $\varphi_k$ (dropping, as before, the index $k$ from now on):

$$(1 - y^2) \frac{\partial \varphi}{\partial y^2} - y \frac{\partial \varphi}{\partial y} + \xi y^2 (1 + y^2) \varphi = \epsilon \varphi,$$

with

$$\xi := \frac{32 V_{-1} e^{6\alpha}}{3k^2}, \quad \epsilon := \frac{32 E_k(\alpha)}{3k^2}.$$  

Equation (42) replaces the equation (30) for the standard case.

With the separation ansatz $\varphi = \exp \left( -\frac{\sqrt{3}}{2} y^2 \right) H(y)$, we find that $H(y)$ obeys

$$(1 - y^2) \frac{d^2 H}{dy^2} - \left( y + 2y^2 (1 - y^2) \frac{\sqrt{\xi}}{\epsilon} \right) \frac{dH}{dy} + \left( (2y^2 - 1) \sqrt{\xi} + 2y^2 \xi \right) H = \epsilon H.$$

We can again transform this equation into a standard form for the confluent Heun differential equation by making now the transformation $z := y^2$. Equation (44) then takes the form

$$\frac{d^2 H}{dz^2} \frac{\sqrt{\xi} z^2 - (\sqrt{\xi} + 1) z + \frac{1}{2}}{z(z-1)} \frac{dH}{dz} - \frac{(2\xi + 2\sqrt{\xi}) z - \sqrt{\xi} - \epsilon}{4z(z-1)} H = 0.$$  

Comparing this with the above canonical form (31), we find now that our equation (45) is solved by

$$H(z) = H_c \left( -\sqrt{\xi}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{3}{8} + \frac{1}{4}; z \right),$$

and (44) is solved by

$$H(y) = H_c \left( -\sqrt{\xi}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{3}{8} + \frac{1}{4}; -y^2 \right).$$

The main difference to the standard case is thus the sign of the first entry in the Heun functions; another difference is that $y$ is here a periodic variable and normalizability is thus not required (although the wave functions are still not allowed to increase exponentially for large $x$). Except for this change of sign, the solution for $\varphi(y)$ is the same as the earlier solution $\varphi(x)$ in (29). The arguments presented for the standard case still apply, and we arrive at the same conclusions for singularity avoidance as before. As already remarked at the end of the last subsection, the solution for the gravitational part is independent of the parameter $\ell$ and is thus given also here by (30).

In spite of the close similarities between the standard and the phantom cases, there exist nevertheless important differences. In contrast to the potential for the standard case given by (12), the phantom potential (29) is periodic in $\phi$; more precisely, it is symmetric under $\sqrt{3} \kappa \phi/4 \to \sqrt{3} \kappa \phi/4 + \pi$. Fig. 4 shows mainly that part of the potential which by itself describes the entire classical solutions (bounded by the two vertical lines at the maximum of the potential). Other parts of the potential correspond to different, though equivalent, classical solutions. While these various branches correspond to entirely independent classical solutions, the quantum theory allows the occurrence of tunnelling. In quantum mechanics, this leads to the well studied concepts of Bloch states, Brouilin zones, and energy bands for e.g. electrons in a crystal. In quantum cosmology, this would be relevant for the concept of a multiverse. A detailed study of this issue is beyond the scope of this paper and will be discussed elsewhere.

We can thus conclude that the type IV singularity is in general also not avoided in the phantom case.

IV. CONCLUSIONS AND OUTLOOK

In this paper, we have investigated the fate of the type IV cosmological singularity in quantum geometrodynamics. The mild nature of this singularity at the classical level (geodesics can be extended through it and tidal
forces remain finite) has left its imprint at the quantum level: generic solutions to the Wheeler–DeWitt equation do not vanish in the region of the classical singularity. This is different from the situations encountered in earlier papers. Guided from this example, one may formulate the conjecture that weak singularities are not generically avoided in quantum cosmology. A proof of this conjecture may involve a general discussion of the singularity structure of Equations 26.

As for the type IV singularity, singularity non-avoidance is also prevalent in loop quantum cosmology [41]. Except for some special cases (a certain parameter choice for closed universes), the type IV singularity remains there, too.

It has been remarked earlier that our condition of a vanishing wave function signals singularity avoidance is not sufficient; see, for example, [41]. The reason given is the lacking knowledge about the physical inner product for the Wheeler–DeWitt equation. While this is certainly true for the full equation, we can impose consistently the standard $L^2$ inner product in the case of the homogeneous models considered here. Taking account the standard measure (the square root of the determinant of the DeWitt metric), the integrand appearing in this inner product vanishes if the wave function vanishes. At the heuristic level considered here, our model is self-contained. We emphasize in this context that we do not introduce a massless scalar field as an effective time variable, in contrast to [41].

The conclusions drawn in this paper may, of course, change if a different formalism or a different interpretation is used. It has been argued, for example, that one should use the method of ‘time-depending gauge fixing’ and ‘reduction to physical degrees of freedom’ instead of the Wheeler–DeWitt equation 22, 12. For the big brake discussed in 20, it was found that this reduction method leads to a quantum non-avoidance instead of an avoidance, and it has been claimed that the same is true for all weak singularities; one can thus expect that it predicts quantum non-avoidance for the type IV singularities, too. Type IV singularities are also not resolved by invoking quantum effects due to the conformal anomaly in a certain class of $f(R, T)$ gravity models [13].

An example for a different interpretation is the use of the Bohm approach. Here, it has been concluded that singularities for the case of a flat universe and a massless scalar field are avoided in the sense that the Bohmian trajectories are non-singular [44]. One may expect that this will be the case also for the cases discussed in our paper. Other recent discussions of singularity avoidance (although not for the type IV singularity) include the maximal acceleration found from spinfoam theory [45] and the avoidance of the big bang singularity in a multiverse picture [46]. Future discussions of singularity avoidance should attempt to obtain statements that can be proven within a wide class of cosmological models.

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