Explicit Formulas for Grassmannian Polylogarithms

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Abstract. We give new explicit formulas for Grassmannian and Aomoto polylogarithms in terms of iterated integrals, for arbitrary weight. We also explicitly reduce the Grassmannian polylogarithm in weight 4 and in weight 5 each to depth 2. Furthermore, using this reduction in weight 4 we obtain an explicit, albeit complicated, form of the so-called 4-ratio, which gives an expression for the Borel class in continuous cohomology of $GL_4(\mathbb{C})$ in terms of $Li_4$.

1. Introduction

The classical polylogarithm $Li_m$ is an analytic function defined by the power series

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}, \quad |z| < 1.$$ 

For $m \geq 1$ it extends to a multivalued analytic function on $\mathbb{C} \setminus \{0, 1\}$ as can be seen from the differential equation $\frac{d}{dz} Li_m(z) = \frac{1}{z} Li_{m-1}(z)$ together with $Li_0(z) = \frac{z}{\log z}$. Polylogarithms appear in many diverse areas of mathematics, from hyperbolic geometry to number theory, algebraic geometry and algebraic $K$-theory.

An important open problem in the area, and one of our principal motivations for this paper, is Zagier’s Polylogarithm Conjecture about the connection between classical polylogarithms and special values of Dedekind zeta functions at positive integers. Let us briefly recall one of its formulations. Let $F$ be a number field of discriminant $D_F$ with $r_1$ real embeddings and $r_2$ conjugate pairs of complex embeddings. Recall that the Dedekind zeta function of $F$ is defined by $\zeta_F(s) = \sum_a N(a)^{-s}$, for $Re(s) > 1$, where the sum is taken over all non-zero ideals $a$ in the ring of integers $\mathcal{O}_F$, and $N(a)$ denotes the norm of the ideal $a$. The sum is absolutely convergent for $Re(s) > 1$ and extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. For $m \geq 2$ we define an integer $d_m = d_m(F)$ by the formula

$$d_m = \begin{cases} r_2, & \text{if } m \text{ is even}, \\ r_1 + r_2, & \text{if } m \text{ is odd}. \end{cases}$$

(More conceptually, $d_m(F)$ is the order of vanishing of $\zeta_F(s)$ at $s = 1 - m$.) Let us also define a single-valued version of $Li_m$ due to Zagier [34]:

$$L_m(z) = Re_m \left( \sum_{j=0}^{m-1} 2^j B_j \frac{1}{j!} Li_{m-j}(z) \log^j |z| \right), \quad m \geq 2,$$

where $Re_m(z)$ denotes the real part of $z$ if $m$ is odd and the imaginary part of $z$ if $m$ is even, and $B_j$ denotes the $j$-th Bernoulli number. For $m = 2$ the function $L_2$ is better known as the Bloch-Wigner dilogarithm [2]. The function $L_m$ is real-analytic on $\mathbb{C} \setminus \{0, 1\}$ and continuous on $\mathbb{P}^1(\mathbb{C})$. For convenience we extend $L_m$ to a function on $\mathbb{Z}[\mathbb{C}]$ (formal linear combinations of elements in $\mathbb{C}$) by linearity.

Conjecture 1 (Zagier). Let $\{\sigma_j\}_{j=1,\ldots,n}$ be the set of all complex embeddings of a number field $F$, where $n = [F:\mathbb{Q}] = r_1 + 2r_2$, labeled in such a way that $\sigma_j = \sigma_{r_1 + r_2 + j}$, $j = 1, \ldots, r_2$. Then there exist elements $y_1, \ldots, y_{d_m} \in \mathbb{Z}[F^\times]$ such that

$$\zeta_F(m) \sim_{\mathbb{Q}} |D_F|^{1/2} \cdot \pi^{d_m} \cdot \det \left( L_m(\sigma_j(y_{ij}))_{1 \leq i, j \leq d_m} \right).$$

In fact, the full statement of Zagier’s Conjecture also gives a precise recipe for the choice of the elements $y_1, \ldots, y_{d_m}$: one has to take $y_i$ to be elements of the so-called $m$-th Bloch group $B_m(F)$, a certain subquotient of $\mathbb{Z}[F^\times]$. For a precise definition we refer to [34]. Conjecturally $B_m(F) \otimes \mathbb{Q}$ has

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dimension $d_m$ and is (canonically) isomorphic to $K_{2m-1}(F) \otimes \mathbb{Q}$, see [11, 28]. Here $K_n(F)$ is the $n$-th algebraic $K$-group of $F$.

Zagier’s Conjecture generalizes the regulator part of the analytic class number formula

$$\text{Res}_{s=1}\zeta_F(s) = \frac{2^{n_1}(2\pi i)^{n_2} \cdot h_F \cdot \text{Reg}_F}{w_F \cdot \sqrt{|D_F|}},$$

where $h_F$ is the class number, $\text{Reg}_F$ is the classical regulator, and $w_F$ is the number of roots of unity of $F$. For $m = 2$ Conjecture [11] follows from the results of Bloch [3] and Suslin [32] as well as Beilinson (as laid out in [8]). In a slightly weaker form it was also proved by Zagier in [33]. For $m = 3$ it was proved by Goncharov in [19], where, in particular, he also outlined a general approach towards Zagier’s Conjecture for $m > 3$. Recently the conjecture was also settled in the case $m = 4$ by Goncharov and Rudenko [24]. The conjecture remains open for $m \geq 5$.

Goncharov’s strategy for proving Conjecture [11] relies on a theorem of A. Borel, which we briefly recall. In [4] Borel has defined a regulator map $\kappa_{m, n}^B : K_{2m-1}(\mathbb{C}) \rightarrow \mathbb{R}(m-1)$, where $\mathbb{R}(k) := (2\pi i)^k \mathbb{R}$, and proved that, if $\Sigma_F = \text{Hom}(F, \mathbb{C})$ and $\psi$ is defined by the composition

$$K_{2m-1}(F) \rightarrow \bigoplus_{\sigma \in \Sigma_F} K_{2m-1}(\mathbb{C}) \overset{\kappa_{m, n}^B}{\rightarrow} \mathbb{Z}^{\Sigma_F} \otimes \mathbb{R}(m-1),$$

then $\psi$ is injective modulo torsion, the image of $\psi$ defines a lattice $\Lambda_F^m$ in $(\mathbb{Z}^{\Sigma_F} \otimes \mathbb{R}(m-1))^+$ (the superscript $+$ denoting invariants under complex conjugation), and its covolume $\text{covol}(\Lambda_F^m)$ is related to $\zeta_F(m)$ via

$$\zeta_F(m) \sim_{\mathbb{Q}^*} \sqrt{|D_F|} \pi^{md_{m+1}} \text{covol}(\Lambda_F^m).$$

(The stronger version of Zagier’s conjecture predicts that the image of $\mathcal{B}_m(F)$ under $\mathcal{L}_n$, evaluated on the suitable complex embeddings, is also a lattice in $\mathbb{Z}^{\Sigma_F} \otimes \mathbb{R}(m-1)$, and that the two lattices should be commensurable.)

The Borel regulator can be represented by the so-called Borel class $\tilde{b}_m^{(N)} \in H^{2m-1}_{\text{cts}}(\text{GL}_N(\mathbb{C}), \mathbb{R}(m-1))$, for $N \geq m$. An argument in Goncharov’s paper [19, §2.2] (see also [3]) establishes that to prove Zagier’s conjecture for $\zeta_F(m)$, it is enough to give a formula for this Borel class as a linear combination of $\mathcal{L}_n$’s. For $m = 2$ such a formula was given by Bloch [3] using the Bloch-Wigner dilogarithm $L_2$, and for $m = 3$ Goncharov gave an ingenious formula for the Borel class using $L_4$.

For $m \geq 4$ Goncharov has shown in [22] that the Borel class $b_m^{(N)}$ can be expressed in terms of a certain function $\mathcal{L}_n^{(N)}$, the single-valued Grassmannian polylogarithm, defined on the space of $m$-planes in $\mathbb{C}^{2m}$. However, the function $\mathcal{L}_n^{(N)}$ cannot be expressed in terms of only $\mathcal{L}_n$ for $m \geq 4$.

In their proof of Conjecture [11] for $m = 4$ Goncharov and Rudenko have overcome this difficulty by giving a formula for the Borel regulator using the (multi-valued) Grassmannian polylogarithm $G_4$ from [23] (see Section 3 below), and showing the existence of an $\mathcal{L}_4$-expression for a small modification of $G_4$ that represents the same cohomology class. More precisely, to prove the existence of the $\mathcal{L}_4$ expression they established part of the conjectural structure of the motivic Lie coalgebra in weight 4.

Their proof does not seem to give any practical way of producing an explicit $\mathcal{L}_4$-formula for $b_4^{(N)}$, though.

Motivated by Goncharov’s original work [19] in conjunction with [24], with a view towards Zagier’s Conjecture in weights 5 and higher, we investigate the Grassmannian polylogarithm $G_m$ via explicit formulas in terms of classical iterated integrals. This gives rise to numerous explicit formulas which we now state. In order to introduce our results it will be convenient to introduce some more notation.

**Notation.** First, we introduce new coordinates $\rho_i = (\rho_i^{(m-1, 2m)})$ on the moduli space of configurations of $2m$ points $v_1, \ldots, v_{2m}$ in an $m$-dimensional vector space, modulo the action of $\text{GL}_m$. Explicitly, $\rho_i$ is the ratio $\Delta_{(i, i+1, \ldots, i+m-2, m-1)} / \Delta_{(i, i+1, \ldots, i+m-2, 2m)}$, where $\Delta_{(i_1, \ldots, i_m)}$ denotes the determinant of the $m \times m$-matrix with columns $v_{i_1}, \ldots, v_{i_m}$.

Second, following [24], given $2n$ points $x_1, \ldots, x_{2n} \in \mathbb{P}^1$ we will denote by $(i_1 i_2 \ldots i_{2n})_x$ the cyclic ratio

$$(i_1 i_2 \ldots i_{2n})_x := (-1)^n \frac{x_{i_1} - x_{i_2}}{x_{i_2} - x_{i_3}} \frac{x_{i_3} - x_{i_4}}{x_{i_4} - x_{i_5}} \ldots \frac{x_{i_{2n-1}} - x_{i_{2n}}}{x_{i_{2n-2}} - x_{i_{2n-1}}}. $$

Finally, it will also be convenient to introduce various symmetrization operators: by $\text{Alt}^n f(x_1, \ldots, x_n)$ we denote the skew-symmetrization of $f$ over all permutations of the $x_i$; by $\text{Alt}^n_m f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{2m})$ the skew-symmetrization of $f$ taken over all permutations of indices in $\mathfrak{S}_{(1, \ldots, m)} \times \mathfrak{S}_{(m+1, \ldots, 2m)}$; and by $\text{Cyc}^{(N)}_m f(x_1, \ldots, x_{2m})$ we denote the cyclic linear combination $\sum_{i \pmod{2m}} e^i f(x_{i+1}, \ldots, x_{i+2m})$ with indices written modulo $2m$. 

**Main results.**

1) We give a new and concise formula for Grassmannian polylogarithms $\text{Gr}_m = \text{Gr}_m(v_1, \ldots, v_{2m})$ in arbitrary weight $m$ in terms of iterated integrals up to depth $m$ (Theorem 4) as

$$-\binom{2m}{2} \text{Gr}_m = m!^2 \text{Alt}_{2m} I(0; 0, \rho_1, \rho_2, \ldots, \rho_{m-1}; \rho_m),$$

where $I$ is defined in Section 2 below, and equality is to be understood on the level of symbols (see Section 2,3).

2) In weight 4 we give an expression for $\text{Gr}_4$ using only depth $\leq 2$ iterated integrals (Theorem 9).

Specifically, modulo products the identity can be written as

$$\frac{7}{12} \text{Gr}_4 = \text{Alt}_8 C_4(\rho_1, \rho_2, \rho_3, \rho_4, \infty, 0),$$

where, for general weight $n$, $C_n$ is the depth 2 function of 6 points in $\mathbb{P}^1$, defined by

$$C_n(x_1, \ldots, x_n) = \text{Cy}_6^{(1)}(\{1234\}x, \{4561\}x) - (-1)^n \left\{ \frac{2(n-1)}{3} \right\} \text{Li}_n((123456)x).$$

(For the definition of $I_{n,1}$ see [1] below.)

3) Building on 2) we obtain an explicit, albeit complicated, formula for the elusive quadruple ratio $\text{Gr}_4^4$ given by a slight modification of $\text{Gr}_4$, in terms of $\mathcal{L}_4$ (Theorem 15 and Corollary 17).

4) In weight 5 we find a similar expression to the one in 2), also in terms of depth 2 iterated integrals (Theorem 19), namely, modulo products we have

$$\frac{1}{320} \text{Gr}_5 = \text{Alt}_{10} \left[C_5(\infty, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5) - C_5(0, 0, \rho_2, \rho_3, \rho_4, \rho_5)\right].$$

5) Building on 4) we express in Conjecture 22 a non-zero rational multiple of the Borel class $b_5^{(4)}$, given by a slight modification of $\text{Gr}_5$, as a linear combination of expressions of the form $I_{4,1}^k(\text{FE}_{2}, \cdot)$ and $I_{4,1}^k(\cdot, \text{FE}_{3})$, where $\text{FE}_k$ can be any functional equation for $\mathcal{L}_k$ ($k = 2, 3$) and $I_{4,1}^k$ is a symmetrized version of $I_{4,1}$, defined in [19] below.

This constitutes a substantial stepping stone towards Zagier’s Polylogarithm Conjecture in weight 5 as, assuming the conjectural structure of the motivic Lie coalgebra in weight 5, each such expression can be written in terms of $\mathcal{L}_5$ only.

6) Finally, following a suggestion of Daniil Rudenko, we obtain via 1) a formula for the Aomoto polylogarithm, defined on pairs of simplices and subject to scissors congruence relations and further symmetries (see [23] and Section 3 below), in terms of iterated integrals as well, again for arbitrary weight. Moreover, we find a concise form for it in Theorem 47 yet again using the $\rho$-coordinates introduced above, as

$$-m^2 \mathcal{A}_{m-1}(v_1, \ldots, v_m; v_{m+1}, \ldots, v_{2m}) = \text{Alt}_{m,m} I(0; \rho_{m+1}^{(1,2m)}, \rho_{m}^{(1,2m)}, \ldots, \rho_{3}^{(1,2m)}, \rho_{2}^{(1,2m)}).$$

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## 2. Preliminaries

We briefly recall some of the motivic framework of multiple polylogarithms and iterated integrals from Goncharov’s paper [21], in particular their Hopf algebra structure and $\otimes$-symbols of iterated integrals.

### 2.1. Iterated Integrals.

Recall the definition of the iterated integral function $I(x_0; x_1; \ldots, x_N; x_{N+1})$ as

$$I(x_0; x_1; \ldots, x_N; x_{N+1}) = \int_{x_0 < t_1 < \ldots < t_N < x_{N+1}} \frac{dt_1}{t_1 - x_1} \wedge \frac{dt_2}{t_2 - x_2} \wedge \cdots \wedge \frac{dt_N}{t_N - x_N}.$$

As per standard notation, we put

$$I_{n_1, \ldots, n_d}(x_1, \ldots, x_d) = I(0; x_1; 0)^{n_1-1}, \ldots, x_d; 0)^{n_d-1}; 1),$$

where $I$ is defined in Section 2 below, and equality is to be understood on the level of symbols (see Section 2,3).
and \( \{a\}^n \) is a repeated \( n \) times. These functions are related to the multiple polylogarithms

\[
\text{Li}_{n_1, \ldots, n_d}(z_1, \ldots, z_d) = \sum_{0 < k_1 < \cdots < k_d} \frac{z_1^{k_1} \cdots z_d^{k_d}}{k_1^{n_1} \cdots k_d^{n_d}}
\]

by the formula

\[
I_{n_1, \ldots, n_d}(0; (a_1 \ldots a_d)^{-1}, (a_2 \ldots a_d)^{-1}, \ldots, a_d^{-1}) = (-1)^d \text{Li}_{n_1, \ldots, n_d}(a_1, a_2, \ldots, a_d).
\]

2.2. Motivic iterated integrals. In [21], the iterated integrals \( I(x_0; \ldots; x_{N+1}) \), \( x_i \in \mathbb{Q} \), are upgraded to framed mixed Tate motives, to define motivic iterated integrals \( I^\mathfrak{a}(x_0; \ldots; x_{N+1}) \) living in a graded (by the weight \( N \)) connected Hopf algebra \( \mathcal{A}_\mathfrak{a} = \mathcal{A}(\mathbb{Q}) \). The Hopf algebra \( \mathcal{A}_\mathfrak{a} \) is the ring of regular functions on the unipotent part of the motivic Galois group. (Note that in Brown’s motivic framework, [7], the motivic iterated integrals \( I^\mathfrak{a} \) are denoted by \( I^\mathfrak{a} \).) The coproduct \( \Delta \) on this Hopf algebra is computed via Theorem 1.2 in [21] as

\[
\Delta I^\mathfrak{a}(x_0; x_1, \ldots, x_N; x_{N+1}) = \sum_{0 \leq i_0 < i_1 < \cdots < i_k < k+1 = N+1} I^\mathfrak{a}(x_0; x_{i_1}, \ldots, x_{i_k}; x_{N+1}) \otimes \prod_{p=0}^k I^\mathfrak{a}(x_{i_p}; x_{i_p+1}, \ldots, x_{i_p+1}; x_{i_p+1}).
\]

Here \( I^\mathfrak{a}(a; b; c) \) is regularized as

\[
I^\mathfrak{a}(a; b; c) = \begin{cases} 
\text{log}^\mathfrak{a}(1) = 0 & \text{if } a = b \text{ and } b = c, \\
\text{log}^\mathfrak{a}(\frac{1}{a-b}) & \text{if } a \neq b \text{ and } b = c, \\
\text{log}^\mathfrak{a}(b-c) & \text{if } a = b \text{ and } b \neq c, \\
\text{log}^\mathfrak{a}(\frac{b}{x-z}) & \text{otherwise}.
\end{cases}
\]

Similarly, if \( x_i \in \mathbb{C} \) we follow Goncharov and consider \( I^\mathbb{C}(x_0; \ldots; x_{m+1}) \) as framed Hodge-Tate structures (see [21] §3.2 (ii), p. 232]), where the coproduct for the corresponding Hopf algebra of framed objects is given by the same formula [2], with \( \mathfrak{a} \) replaced by \( \mathbb{C} \) (see [21] Thm. 3.4). Since our results ultimately only use [2], they apply in any of these two setups. We therefore adopt the following convention.

Convention. We will omit the superscripts from the notation, and simply write \( I(x_0; x_1, \ldots, x_m; x_{m+1}) \).

2.3. The \( \otimes \)-symbol modulo products and the Lie coalgebra. Recall from [21] §4.4] the “\( \otimes \)\^m-invariant”, or symbol, of an iterated integral. The symbol \( S(I^\mathfrak{a}) \in \mathcal{A}^\mathfrak{a} \) is an algebraic invariant of \( I^\mathfrak{a} \) that respects functional equations. More precisely, \( S : (\mathcal{A}_\mathfrak{a}, \mathbb{W}, \Delta) \to (T(\mathcal{A}_1), \mathbb{W}, \Delta_{\text{dec}}) \) is a map of graded Hopf algebras, where \( T(\mathcal{A}_1) \) is the tensor algebra of \( \mathcal{A}_1 \), and \( \Delta_{\text{dec}} \) is the deconcatenation coproduct on tensors. It is obtained by iterating the reduced coproduct \( \Delta^\prime \) exactly \( N - 1 \) times. A corollary of the definition is the following recursive formula for the symbol that we will use repeatedly

\[
SI(x_0; \ldots; x_{N+1}) = \sum_{j=1}^{N} SI(x_0; x_1, \ldots, \hat{x}_j, \ldots, x_N; x_{N+1}) \otimes I(x_{j-1}; x_j; x_{j+1}).
\]

Recall also the projectors \( \Pi_n \) from [13] §5.5] which annihilate the symbols of products. The projector \( D_N = \Pi_N \) acts on length \( N \) tensors as follows.

\[
D_N(a_1 \otimes \cdots \otimes a_N) = D_{N-1}(a_1 \otimes \cdots \otimes a_{N-1}) \otimes a_N - D_{N-1}(a_2 \otimes \cdots \otimes a_N) \otimes a_1.
\]

We prefer \( D_N \) to \( \Pi_N \) in order to avoid unnecessary scaling factors, at the expense of that operator no longer being idempotent. Note that the map \( D_N \) is the classical Dynkin operator in the theory of Hopf algebras (see, e.g., [14] Section 4).

We call the composition \( D_N \circ S = S^\mathfrak{m} : \mathcal{A}_\mathfrak{a} \to T(\mathcal{A}_1) \) the mod-products symbol. As usual with symbols, we will drop the \( \otimes \) from the notation and write tensors multiplicatively. The notation might suggest that the symbol entries are elements \( x \in \mathbb{Q} \), when they are really elements of \( \mathbb{Q}^n \otimes \mathbb{Q} \cong \mathcal{A}_n \). In particular, on the level of the symbol 2-torsion vanishes, because in the Hopf algebra \( \mathcal{A}_n \) (as \((2\pi i)^{\otimes \mathfrak{a}} \) is zero) one has the exact equality of motivic logarithms \( \text{log}^\mathfrak{a}(x) = \text{log}^\mathfrak{a}(-x) \). We can therefore ignore signs in the tensor entries, and freely interchange between \( \otimes(-x) \) and \( \otimes x \). To emphasize that certain identities hold only on the level of symbol or on the level of mod-products we shall write \( f \overset{\mathfrak{m}}{=} g \) and \( f \overset{\mathfrak{m}}{=} g \) to denote \( S(f) = S(g) \) and \( S^\mathfrak{m}(f) = S^\mathfrak{m}(g) \), respectively.

To give an example, \( S^\mathfrak{m} \) for classical polylogarithms is given by \( S^\mathfrak{m} \text{Li}_n(x) = (x \wedge (1-x)) \otimes x^{\otimes (m-2)} \), where we write \( a \wedge b = a \otimes b - b \otimes a \). An important property of the single-valued polylogarithms \( \mathcal{L}_m \)
is that if the \( f_j \) are rational functions and \( S^\nu \sum \nu_j \text{Li}_m(f_j(x)) = 0 \), then \( \sum \nu_j \mathcal{L}_m(f_j(x)) \) is constant (see \cite{34} Prop. 1, p. 411]

Finally, recall that the coproduct in a Hopf algebra induces a cobracket \( \delta = \Delta - \Delta^{op} \) (with \( \Delta^{op} \) the opposite coproduct) on the Lie coalgebra of irreducibles \( \mathcal{L}_* := \mathcal{A}_{>0}/\mathcal{A}_{>0}^2 \). The 2-part of this cobracket in weight \( m \), i.e. the composition of \( \delta \) with projection to \( \bigoplus_{k=2}^m \mathcal{L}_k \wedge \mathcal{L}_{m-k} \), can be seen to annihilate all classical polylogarithms, and conjecturally this is the only obstruction, see Conjecture 1.20 and Section 1.6 in \cite{18}. We use the vanishing of the 2-part of \( \delta \) as a guiding principle for possible depth reduction of the weight 5 Grassmannian polylogarithm in Section 3.

3. Grassmannian and Aomoto polylogarithms

There are several different constructions of “Grassmannian polylogarithms” in the literature: there is a real-valued Grassmannian logarithm of Gelfand and MacPherson \cite{22}, Grassmannian \( m \)-logarithms constructed by Hanamura and MacPherson \cite{20}, \cite{21}, and Goncharov’s construction of real-analytic single-valued Grassmannian polylogarithms \cite{23}. The subject of our investigations is Goncharov’s complex analytic multi-valued Grassmannian polylogarithm, defined in \cite{23}. We refer to it as the Grassmannian polylogarithm throughout this paper.

For \( m, n \geq 1 \), let \( \text{Conf}_n(m) \) be the space of all \( n \)-tuples of vectors \( (v_1, \ldots, v_n) \) in general position in \( \mathbb{C}^m \) modulo the diagonal action of \( \text{GL}_m(\mathbb{C}) \). Let us denote by \( \Delta(i_1, \ldots, i_m) \) the determinant of the \( m \times m \) matrix with columns \( v_{i_1}, \ldots, v_{i_m} \) (for better readability we will often omit the commas and simply write \( \Delta(i_1 \ldots i_m) \)). The functions \( \Delta(i_1, \ldots, i_m) \) are invariant under the action of \( \text{SL}_m(\mathbb{C}) \) and the ring of regular functions \( \mathcal{O}(\text{Conf}_n(m)) \) is generated by all possible ratios of determinants \( \Delta(i_1 \ldots i_m)/\Delta(j_1 \ldots j_m) \).

First, following \cite{23}, we recall the definition and basic properties of Aomoto polylogarithms. The Aomoto \( n \)-logarithm is a complex analytic function defined on admissible pairs of \( n \)-simplices in \( \mathbb{P}^n(\mathbb{C}) \). Here a simplex in \( \mathbb{P}^n(\mathbb{C}) \) is simply a collection \( (\ell_0, \ldots, \ell_n) \) of \( n+1 \) hyperplanes. We call a simplex \( L = (\ell_0, \ldots, \ell_n) \) non-degenerate if its hyperplanes are in general position. A pair of simplices \( L = (\ell_0, \ldots, \ell_n) \) and \( M = (\ell'_0, \ldots, \ell'_n) \) is called admissible, if \( L \) and \( M \) are non-degenerate and if \( L \) and \( M \) share no faces of the same dimension. To any non-degenerate simplex \( L \) one associates an \( n \)-form \( \omega_L \) in \( \mathbb{P}^n(\mathbb{C}) \) having simple poles at the hyperplanes \( \ell_i \), namely, \( \omega_L = d\log(z_1/z_0) \wedge \cdots \wedge d\log(z_n/z_0) \), where \( z_i = 0 \) denotes the homogeneous equation of the hyperplane \( \ell_i \). Next, to any non-degenerate simplex \( M = (\ell'_0, \ldots, \ell'_n) \) one associates a topological \( n \)-cycle \( \Delta_M \) representing the generator of the rank one (relative) homology group \( H_n(\mathbb{P}^n(\mathbb{C}), M_0 \cup \cdots \cup M_n) \). The Aomoto \( n \)-logarithm is then defined as

\[
\mathcal{A}_n(L; M) = \int_{\Delta_M} \omega_L.
\]

For our purposes it is convenient to dualize and instead view \( \mathcal{A}_n \) as a function of pairs of \( (n+1) \)-tuples of points in \( \mathbb{P}^n(\mathbb{C}) \), which we will also denote by \( \ell_i \) and \( m_j \). To give the simplest example, for \( n = 1 \)

\[
\mathcal{A}_1((\ell_0, \ell_1; m_0, m_1)) = \log \text{cr}(\ell_0, \ell_1; m_0, m_1),
\]

where \( \text{cr}(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)} \) is the classical cross-ratio. The function \( \mathcal{A}_n \) is skew-symmetric in both sets of points, projectively invariant, and most importantly satisfies the additivity relation

\[
\sum_{i=0}^{n+1} (-1)^i \mathcal{A}_n((\ell_0, \ldots, \hat{\ell}_i, \ldots, \ell_{n+1}); (m_0, \ldots, m_n)) = 0
\]

for any configuration \( (\ell_0, \ldots, \ell_{n+1}) \) in \( \mathbb{P}^n(\mathbb{C}) \). It also satisfies analogous additivity relation in the second variable \( M \) as well as dualized versions of additivity \cite{23} §2.1. Abstracting away these properties one considers scissors congruence groups \( \mathcal{A}_n(F) \), abelian groups generated by the elements \( (\ell_0, \ldots, \ell_n; m_0, \ldots, m_n)_{\mathcal{A}_n} \) with \( \ell_i, m_j \in \mathbb{P}^n(F) \), subject to the above-mentioned skew-symmetry, projective invariance, and additivity properties. The graded sum \( \bigoplus_{n \geq 0} \mathcal{A}_n(F) \) also has a graded coassociative coalgebra structure, of which we will only need the component \( \mathcal{A}_{n-1,1} \) of the coproduct (see \cite{23} eq. (7))) that for generic simplices can be expressed in the following form

\[
\mathcal{A}_{n-1,1}((L; M)_{\mathcal{A}_n}) = - \sum_{i,j=0}^{n} (-1)^{i+j} (\ell_i|0, \ldots, \hat{\ell}_i, \ldots, \ell_n; m_0, \ldots, m_j, \ldots, m_n)_{\mathcal{A}_n} \otimes \Delta(l_i, m_0, \ldots, m_j, \ldots, m_n).
\]

As per general setup explained in \cite{23} iterating the \((n-1, 1)\) part of the coproduct using the above formula we obtain the tensor symbol of the Aomoto polylogarithms.
Proposition 2. The Aomoto polylogarithm $A_n(v_1, \ldots, v_{n+1}; v_{n+2}, \ldots, v_{2n+2})$ has $\otimes$-symbol 
\[ (-1)^{k} Alt_{n+1,n+1} \Delta(2, \ldots, n+2) \otimes \Delta(3, \ldots, n+3) \otimes \cdots \otimes \Delta(n+1, \ldots, 2n+1). \]

In [23] Goncharov has defined the Grassmannian $m$-logarithm $Gr_m(v_1, \ldots, v_{2m})$ as a multivalued analytic function on $Conf_{2m}(m)$ by requiring that $Gr_m(v_1, \ldots, v_{2m}) = Alt_{2m} F(v_1, \ldots, v_{2m})$, where, as in the introduction, $Alt_n$ denotes the skew-symmetrization operator
\[ Alt_n f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \]
and the function $F$ is a primitive of the following 1-form
\[ Alt_{2m} \left( A_{m-1}(v_1, \ldots, v_m; v_{m+1}, \ldots, v_{2m}) d \log \Delta(m+1, \ldots, 2m) \right), \]
where $A_{m-1}$ is the Aomoto polylogarithm (see [23] §1.1). Goncharov has proved (loc. cit.) that $Gr_m$ is well-defined, i.e., that the 1-form on the right-hand side of (3) is indeed closed, and that it is projectively invariant, i.e., $Gr_m(\lambda_1 v_1, \ldots, \lambda_2 v_{2m}) = Gr_m(v_1, \ldots, v_{2m})$ for all $\lambda_1, \lambda_2 \in \mathbb{C}^{\times}$. In particular, $Gr_m$ is a well-defined function on the space of configurations of $2m$ points in $\mathbb{P}^{m-1}(\mathbb{C})$. Note that $Gr_m(v_1, \ldots, v_{2m})$ is manifestly skew-symmetric under the permutations of $v_1, \ldots, v_{2m}$. The key property of the Grassmannian polylogarithm is that it satisfies the following functional equations.

Proposition 3 (23). (i) For any generic configuration of $(2m+1)$ vectors $v_0, \ldots, v_{2m}$ in $\mathbb{C}^m$ we have
\[ \sum_{i=0}^{2m} (-1)^i Gr_m(v_0, \ldots, \hat{v}_i, \ldots, v_{2m}) = \text{const}. \]

(ii) For any generic configuration of $(2m+1)$ vectors $w_0, \ldots, w_{2m}$ in $\mathbb{C}^{m+1}$ we have
\[ \sum_{i=0}^{2m} (-1)^i Gr_m(\pi_i(w_0), \ldots, \hat{w}_i, \ldots, \pi_i(w_{2m})) = \text{const}, \]
where $\pi_i$ denotes the canonical projection from $\mathbb{C}^{m+1}$ to $\mathbb{C}^{m+1}/(w_i)$.

These identities follow from the following expression for the symbol of $Gr_m$ (23 Thm. 4.2)
\[ S(Gr_m) = 2(-1)^m(m!)^2 Alt_{2m} \Delta(1, \ldots, m) \otimes \Delta(2, \ldots, m+1) \otimes \cdots \otimes \Delta(m, \ldots, 2m-1). \]

Our first result is an explicit formula that expresses $Gr_m(v_1, \ldots, v_{2m})$ in terms of the classical iterated integrals. To state this and other formulas, for a set $I = \{i_1, \ldots, i_{m-1} \} \subset \{1, \ldots, 2m\}$ we define
\[ \rho^I_{(k,l)} := \frac{\Delta(i_1, i_2, \ldots, l_{m-1}, k)}{\Delta(i_1, i_2, \ldots, l_{m-1}, l)}, \quad I \cap \{k, l\} = \emptyset. \]
Note that $\rho^I_{(k,l)}$ is a rational function that is symmetric and projectively invariant in $v_{i_1}, \ldots, v_{i_{m-1}}$. Furthermore, let us set $\rho^I_{(k,l)} := \rho^I_{(i, \ldots, i+m-2)}$ and $\rho_i := \rho^I_{(2m-1, 2m)}$.

Theorem 4. For $m \geq 2$ we have the following identity on the level of symbols
\[ \frac{2m-1}{m!} Gr_m \overset{\leq}{=} Alt_{2m} I(0; 0, \rho_1, \rho_2, \ldots, \rho_{m-1}; \rho_m). \]

The proof will be given in Section 6. As a corollary from this Theorem and Proposition 3 we obtain explicit geometric functional equations for the iterated integral $I(0; 0, x_1, \ldots, x_{m-1}; x_m)$.

Remark 5. The proof can also be adapted to show the same formula with only the change of the lower bound of the integrals from $0$ to $\infty$
\[ \frac{2m-1}{m!} Gr_m \overset{\leq}{=} Alt_{2m} I(\infty; 0, \rho_1, \rho_2, \ldots, \rho_{m-1}; \rho_m). \]

Here the iterated integrals $I(\infty; a_1, \ldots, a_m; a_{m+1})$ can be shuffle-regularized and written as an explicit combination of iterated integrals evaluated at finite points. These integrals have better symmetry properties, for instance, they are invariant (up to sign and modulo products) under dihedral permutations of the variables $a_1, \ldots, a_{m+1}$. These iterated integrals starting at $\infty$ are also related to the “motivic correlators” $\text{Cor}_\infty(a_1, \ldots, a_{m+1})$ from [23] and [24] (the main difference is that $I(\infty; a_1, \ldots, a_m; a_{m+1})$ lives in the motivic Hopf algebra, while $\text{Cor}_\infty(a_1, \ldots, a_{m+1})$ lives in the motivic Lie coalgebra, a quotient of the latter).
Remark 6. The geometric meaning of \( \rho^{(k,l)} \) is as follows. If we pick projective coordinates on the line \( \ell \) passing through \( P_k \) and \( P_l \) (where \( P_j \in \mathbb{P}^{m-1} \) corresponds to \( v_j \in \mathbb{C}^m \)) in such a way that \( P_k \) has coordinate 0, and \( P_l \) has coordinate \( \infty \), then \( \rho_l \) is the coordinate for the intersection of \( \ell \) with the hyperplane passing through the points \( P_1, \ldots, P_{m-1} \). Note that \( \rho_l \)'s depends on the choice of the vectors \( v_1, v_2 \) that represent \( P_k, P_l \in \mathbb{P}^{m-1} \), but any ratio of two \( \rho_l \)'s is well-defined and projectively invariant.

Theorem 7. For \( m \geq 2 \) we have the following identity

\[
\mathcal{A}_{m-1}(v_1, \ldots, v_m; v_{m+1}, \ldots, v_{2m}) \leq \frac{(-1)^{m-1}}{m^2} \mathrm{Alt}_{m,m} I(0; \rho^{(1,2m)}_{m+1}, \rho^{(1,2m)}_m, \ldots, \rho^{(1,2m)}_2),
\]

where \( \mathrm{Alt}_{m,m} \) denotes the skew-symmetrization over all index permutations in \( \mathfrak{S}_{(1, \ldots, m)} \times \mathfrak{S}_{(m+1, \ldots, 2m)} \).

Let us also remark that the identities (7) and (9) also make sense as equalities between multivalued analytic functions, if one chooses their branches appropriately (compare with the remark after Theorem 3.9 in [20]).

Remark 8. (i) There is a slightly better formula for the Aomoto polylogarithm in which we replace \( \mathrm{Alt}_{m,m} \) by \( \mathrm{Alt}_{m-1,m-1} \) (fixing the two labels 1 and 2—note the different superscripts as opposed to the formula for \( \mathrm{Gr}_m \)). This has the further benefit of cancelling the factor \( m^2 \).

(ii) Furthermore, generalizing (7) and (9) one can also consider the higher weight functions

\[
\mathrm{Alt}_{m-1,m-1} I(0; 0, \ldots, 0, \rho^{(1,2m)}_{m+1}, \rho^{(1,2m)}_m, \ldots, \rho^{(1,2m)}_2)
\]

obtained by adding a string of zeros on the left (highlighted in boldface above) to the iterated integral (this construction is inspired by the construction of the \( \mathrm{QLi} \) functions in [31]).

Then following the initial steps in the proof of Theorem 7 gives a recursive formula for the \( \mathrm{Gr}_{m,m} \) by \( \mathrm{Alt}_{m,m} \) and matching cobracket. This ‘coboundary correction’ is just a version of Goncharov’s \( \delta_2 \) in [20]. Finally we give an explicit expression for the difference, i.e. of \( \mathrm{Gr}_4 \) minus this coboundary correction, in terms of \( \mathrm{Li}_4 \) only (Theorem 13). The resulting \( \mathrm{Li}_4 \) expression is our version of the elusive \textit{quadruple ratio}. (Here by a ‘coboundary’ we mean a linear combination of functions on configurations of 8 points in \( \mathbb{P}^3 \) where each individual term depends on at most 7 of these points. The reason for this terminology is that such a ‘coboundary’ lies in the image of the coboundary operator \( d \) of a suitable cochain complex. Note that any such ‘coboundary’ will trivially vanish when we alternate over 9 points.) While the existence of such formulas follows from the results of Goncharov and Rudenko in [24], their proof does not seem to give any practical approach to obtaining them.

4. The Grassmannian polylogarithm in weight 4

In this section we formulate our main results for the weight 4 Grassmannian polylogarithm. First we give an explicit formula for \( \mathrm{Gr}_4 \) in terms of \( I_{3,1} \) and \( \mathrm{Li}_4 \) (Theorem 9). It is known that \( \mathrm{Gr}_4 \) is not expressible in terms of \( \mathrm{Li}_4 \) alone, the only obstacle to doing so being the non-vanishing of the 2-part of its cobracket (see Section 2). We reproduce a ‘coboundary correction’ for it—the \( \Delta_8 \) term on the left hand side of Theorem 9—with matching cobracket. This ‘coboundary correction’ is just a version of Goncharov’s \( \delta_2 \) in [20]. Finally we give an explicit expression for the difference, i.e. of \( \mathrm{Gr}_4 \) minus this coboundary correction, in terms of \( \mathrm{Li}_4 \) only (Theorem 13). The resulting \( \mathrm{Li}_4 \) expression is our version of the elusive \textit{quadruple ratio}. (Here by a ‘coboundary’ we mean a linear combination of functions on configurations of 8 points in \( \mathbb{P}^3 \) where each individual term depends on at most 7 of these points. The reason for this terminology is that such a ‘coboundary’ lies in the image of the coboundary operator \( d \) of a suitable cochain complex. Note that any such ‘coboundary’ will trivially vanish when we alternate over 9 points.) While the existence of such formulas follows from the results of Goncharov and Rudenko in [24], their proof does not seem to give any practical approach to obtaining them.

4.1. Explicit formula for \( \mathrm{Gr}_4 \) in terms of \( I_{3,1} \) and \( \mathrm{Li}_4 \). Theorem 9 already gives us an explicit formula for \( \mathrm{Gr}_4 \) in terms of iterated integrals, to which one could apply the known reduction formulas in weight 4 (see [11, 12, 15, 31]) to obtain an explicit expression in terms of \( I_{3,1} \) and \( \mathrm{Li}_4 \) (recall that \( I_{3,1}(x, y) = I(0; x, 0, y; 1) \)). This reduction, however, produces a somewhat complicated expression. Instead we will give a direct formula for \( \mathrm{Gr}_4 \) in terms of \( I_{3,1} \) and \( \mathrm{Li}_4 \) that is much shorter.

We are working with the configuration space \( \text{Conf}_4(4) \) and, as in the more general situation in Theorem 4 we set

\[
\rho_i := \rho^{(7,8)}_{i+1, i+2} = \frac{\Delta(i, i + 1, i + 2, 7)}{\Delta(i, i + 1, i + 2, 8)}, \quad 1 \leq i \leq 4.
\]

We will also use the following notation for projected cross-ratios

\[
\text{cr}(ab|cdef) := \frac{\Delta(abce)\Delta(abdf)}{\Delta(abef)\Delta(abde)}.
\]

Geometrically \( \text{cr}(ab|cdef) \) is simply the cross-ratio of the images of \( v_e, v_d, v_c, v_f \) in \( \mathbb{P}(\mathbb{C}^4/(v_a, v_b)) \).
Theorem 9. We have the following identity modulo products
\[ \frac{7}{144} \text{Gr}_4 \equiv \text{Alt}_8 \left[ I_{3,1} \left( \frac{\rho_1 \rho_2 \rho_3}{\rho_4 \rho_1} \right) + 2I_{3,1} \left( \frac{\rho_1 \rho_2 \rho_3}{\rho_1 \rho_2 \rho_3} \right) + 6\text{Li}_4 \left( \frac{\rho_1 \rho_2 \rho_3}{\rho_1 \rho_2 \rho_3} \right) \right], \]
where we denote \( \rho_{i,j} = \rho_i - \rho_j \).

The proof will be given in Section 8.

Remark 10. Note that the combination inside the square brackets on the right-hand side is \( \text{Alt}_8 \)-equivalent to the function \( C_4(\rho_1, \rho_2, \rho_3, \rho_4, \infty, 0) \) defined in the introduction. The function \( C_4 \) essentially coincides with the map \( L^1_k \), [24, eq. (168)], used by Goncharov and Rudenko to construct a map from the chain complex of the Stasheff polytope to the polylogarithmic complex. This suggests a connection between Grassmannian polylogarithms \( \text{Gr}_{2k} \) and the cluster polylogarithm map in weight \( 2k \) from [24] (the latter being a conjectural object for \( k \geq 3 \)). The formula for \( \text{Gr}_5 \) from Theorem 19 and Remark 20 below suggests that something analogous might also be the case for Grassmannian polylogarithms in odd weights.

4.2. Explicit formula for \( \text{Gr}_4 \) minus coboundary in terms of \( \text{Li}_4 \). It is known that \( \text{Gr}_4 \) cannot be written purely in terms of \( \text{Li}_4 \) (this is explained in Section 1 in [19]), so one cannot hope to completely remove \( I_{3,1} \) from \([\text{III}]\). Nevertheless, for the application to Zagier’s Conjecture it is important to have an expression in terms of \( \text{Li}_4 \) for some function that represents the same cohomology class (as before, we interpret this naively: it has to be a function of the form \( \text{Gr}_4(v_1, \ldots, v_8) - \text{Alt}_8 f(v_1, \ldots, v_7) \) for some \( f \)). In Theorem 17 we exhibit an explicit expression of exactly this type. First we recall some terms, denoted by
\[ I_{3,1}(x,y) = -I_{3,1}(-x,-y), \]
\[ I_{3,1}(x,y) + I_{3,1}(1-x, y) \equiv 0. \]

Combining these identities we obtain the following.

Definition 12. The function \( \tilde{I}_{3,1} (x,y) \) is defined by
\[ \tilde{I}_{3,1} (x,y) := \frac{1}{36} \sum_{\sigma, \pi \in S_3} \text{sgn}(\sigma) \text{sgn}(\pi) I_{3,1}(x^\sigma, y^\pi), \]
where \( x^\sigma \) denotes elements of \( \{ x, \frac{1}{x}, 1 - x, 1 - \frac{1}{x}, \frac{1}{1 - x}, \frac{1}{1 - \frac{1}{x}} \} \) (the anharmonic group) corresponding to \( \sigma \in S_3 \) (under some isomorphism).

Proposition 13. There is an explicit combination of \( \text{Li}_4 \) terms, denoted by \( \text{Sym}_{36}(x,y) \), such that
\[ I_{3,1}(x,y) - \tilde{I}_{3,1}(x,y) \equiv \text{Sym}_{36}(x,y). \]

Moreover, the function \( \tilde{I}_{3,1}(x,y) \) satisfies
\[ \tilde{I}_{3,1}(x^\sigma, y^\pi) \equiv \text{sgn}(\sigma) \text{sgn}(\pi) \tilde{I}_{3,1}(x,y), \]
\[ \tilde{I}_{3,1}(y,x) \equiv -\tilde{I}_{3,1}(x,y). \]

For the sake of completeness we give an expression for \( \text{Sym}_{36}(x,y) \) in terms of \( \text{Li}_4 \) in Appendix A. The key property of \( I_{3,1} \) that we need is the following theorem that is the main result of [15].
**Theorem 14** (Gangl, [15]). There is an explicit collection of rational functions \( f_j \in \mathbb{Q}(x,y,z) \) and numbers \( c_j \in \mathbb{Q} \) such that

\[
\tilde{I}_{3,1}(z, [x] + [y] + \left[ \frac{1-x}{1-xy} \right] + [1-xy] + \left[ \frac{1-y}{1-xy} \right]) \equiv \sum_{j=1}^{N} c_j \text{Li}_4(f_j(x, y, z)).
\]

We denote the left-hand side of the above expression by \( V(z; x, y) \); by this theorem it is equal to an explicit combination of \( \text{Li}_4 \) terms. For the sake of completeness we also reproduce the expression for \( V(z; x, y) \) in terms of \( \text{Li}_4 \) in Appendix A.

Remark 18. Borel regulator for \( V \) (extend \( F \))

\[
\text{Borel regulator for } V \text{ is the number of terms in } V \text{ Borel class } b \text{ (recall that cr(34|2567, cr(67|1345))}
\]

\[
\equiv \text{Alt}_8 \left[ -V(\frac{\rho_1}{\rho_2}; \frac{\rho_3 \rho_4}{\rho_1 \rho_2 \rho_3}) - [43|2685; 48|7653] + \frac{1}{3} [43|1256; 42|1365] \right] + V(\frac{\rho_1}{\rho_2}; -[43|2685; 48|7653] + [48|7235; 42|7623] + \frac{1}{2} [46|5238; 43|2568])
\]

\[
+ \text{Sym}_{36}(\frac{\rho_1 \rho_2 \rho_4}{\rho_3 \rho_5 \rho_6}) + 2 \text{Sym}_{36}(\frac{\rho_1 \rho_2 \rho_4}{\rho_3 \rho_5} + 6 \text{Li}_4(\frac{\rho_1 \rho_2 \rho_4}{\rho_3 \rho_5 \rho_6})).
\]

The proof is based on the identity from Theorem 9 and will be given in Section 9.

**4.3. Corollaries.** Let us denote the formal linear combination of the arguments of \( \text{Li}_4 \) in (11) by \( \mathcal{Q}(v_1, \ldots, v_8) \) (one can think of \( \mathcal{Q} \) as a function from \( \text{Conf}_8(4) \) to \( \mathbb{Z}[P^1(C)] \)). This is an explicit form of the map \( f_8(4) \) from [16], it is also related to the map \( r_8^*(4) \) from [24] (more precisely, it is \( r_8^*(4) \) together with the correction term used in the proof of Theorem 1.17 in [24]).

Recall that the symbol of the left-hand side of (11) is equal, in view of (6), to \( \text{Alt}_8 f(v_1, \ldots, v_7) \) for some \( f \). So by symmetrizing over 9 points we immediately obtain the following 'cohomological' functional equation for \( \mathcal{L}_4 \).

**Corollary 16.** The tetralogarithm function satisfies the following 9-term relation

\[
\sum_{j=0}^8 (-1)^j \mathcal{L}_4(\mathcal{Q}(v_0, \ldots, \hat{v}_j, \ldots, v_8)) = 0,
\]

where \( v_0, \ldots, v_8 \in C^4 \) are vectors in general position.

This functional equation is an analogue of the 5-term relation for the dilogarithm and of Goncharov’s 840-term relation for the trilogarithm [18].

Moreover, if \( v \in C^4 \) is any non-zero vector, then the function \( \varphi: \text{GL}_4(C)^8 \rightarrow \mathbb{R} \) defined by

\[
\varphi(g_1, \ldots, g_8) = \mathcal{L}_4(\mathcal{Q}(g_1 v, \ldots, g_8 v))
\]

defines a measurable 7-cocycle for \( \text{GL}_4(C) \), and, as explained for example in [19] §1.4 and in [3] §9.7, also defines a continuous cohomology class in \( H_{cts}^7(\text{GL}_4(C), \mathbb{R}) \) (that is independent of \( v \)). The following result follows from Theorem 15 together with the proofs of Th. 1.17 and Th. 1.2 in the work of Goncharov and Rudenko [24] p. 72-73.

**Corollary 17.** The cohomology class \([i \varphi]\) in \( H_{cts}^7(\text{GL}_4(C), \mathbb{R}(3)) \) is a non-zero rational multiple of the Borel class \( b_4(4) \in H_{cts}^7(\text{GL}_4(C), \mathbb{R}(3)) \) (here, as before, \( \mathbb{R}(m) := (2\pi i)^m \mathbb{R} \)).

Due to the validity of the Rank Conjecture for number fields [3], this result suffices to compute the Borel regulator for \( K_7(F) \) in terms of \( \mathcal{L}_4 \), where \( F \) is a number field (for details see [3] §9.4).

**Remark 18.** The number of different \( \mathcal{S}_8 \)-orbits of arguments in \( \mathcal{Q} \) is at most \( 7N + 2M + 1 \), where \( N \) is the number of terms in \( \mathbf{V} \) and \( M \) is the number of terms in \( \text{Sym}_{36} \). If we use the expressions for \( \mathbf{V} \) and \( \text{Sym}_{36} \) given in Appendix A below, then we get that \( \mathcal{Q} \) is a sum of \( 7 \cdot 2340 + 2 \cdot 26 + 1 = 16433 \) orbits. A slightly better version of \( \mathbf{V} \) has 246 terms, so we obtain \( 7 \cdot 246 + 2 \cdot 26 + 1 = 1775 \) orbits. However,
some of the $S_8$-orbits in the resulting expression turn out to coincide, and some of them cancel out after skew-symmetrization over $S_8$. A more careful (computer assisted) analysis of $Q$ using techniques similar to those used in [10] gives an expression with 368 orbits of $Li_4$ arguments.

5. The Grassmannian Polylogarithm in Weight 5

Here we are now working with the configuration space $Conf_{10}(5)$ and, as before, we define

$$p_i := p_{(9,0)}(i, i+1, i+2, i+3) = \frac{\Delta(i, i+1, i+2, i+3, 3, 9)}{\Delta(i, i+1, i+2, i+3, 0)}, \quad 1 \leq i \leq 5,$$

where for notational reasons we use the digit 0 for the 10-th point. Again we also use the following notation for projected cross-ratios

$$cr(abc|def) := \frac{\Delta(abcdf)\Delta(abceg)}{\Delta(abcdf)\Delta(abceg)}.$$

Finally, we also use the following notation for the projected triple-ratio (note that Goncharov’s triple-ratio is the $Alt_5$-skew-symmetrization of this, and the single term is denoted by $r'_3$ in [18])

$$r_3(ab|cdf, fgh) := \frac{\Delta(abcf)\Delta(abch)\Delta(abdg)}{\Delta(abcf)\Delta(abch)\Delta(abch)}.$$

By analogy with Theorem 19 we have the following expression for $Gr_5$ in terms of $I_{4,1}$ and $Li_5$.

**Theorem 19.** We have the following identity modulo products

$$\frac{1}{640} Gr_5 \equiv Alt_{10} \left[ I_{4,1} \left( \frac{\rho_{2,3}}{\rho_{2,1}}, \frac{\rho_{4,3}}{\rho_{4,5}} \right) + 2I_{4,1} \left( \frac{\rho_{1,3}}{\rho_{2,1}}, \frac{\rho_{1,4}}{\rho_{3,1}}, \frac{\rho_{3,4}}{\rho_{4,4}} \right) + 8Li_5 \left( \frac{\rho_{1,2}}{\rho_{2,3}}, \frac{\rho_{1,3}}{\rho_{2,4}}, \frac{\rho_{1,4}}{\rho_{3,4}} \right) \right],$$

where we denote $p_{i,j} = p_i - p_j$.

**Remark 20.** If we define $C_5(x_1, \ldots, x_6)$ as in the introduction, then the first row of (12) is $Alt_{10}$ equivalent to $C_5(\infty, \rho_1, \ldots, \rho_5)$ and the second row is $Alt_{10}$ equivalent to $-C_5(\infty, 0, \rho_2, \ldots, \rho_5)$. In particular, we see that the second line is a specialization of the first.

This identity can be proved similarly to the proof of Theorem 8 as it is done in Section 8. We have checked it symbolically using a computer. Naively one could write out all $10! \cdot 6$ terms on the right, and compute the symbol, but since both sides of the identity are given by $Alt_{10}$ of a small number of terms, it is much more manageable to expand the symbol of a single term in the square brackets in terms of irreducible $SL_4$-invariant polynomials, and then compute its canonical form (say, lexicographically minimal representatives) modulo the action of $S_8$ for each of the tensors.

As for $Gr_4$, the non-vanishing of the 2-part of the cobracket of $Gr_5$ shows that one cannot express it in terms of depth 1 functions alone. In analogy to the weight 4 case we first look for a ‘coboundary correction’ with matching cobracket, which in this case amounts to a linear combination of functions on configurations of 10 points in $P^4$, where each individual term depends on at most 9 of these points.

In order to be able to write a concise formula, we introduce the following shorthand for the invariant and the anti-invariant under the swap of the left and right triples inside the triple ratio

$$r^+_3(ab, cd) := [r_3(ab, cd)] \pm [r_3(cd, ab)].$$

(Note that $1 - r_3(ab, cd)$ and $1 - r_3(cd, ab)$ share a nontrivial irreducible factor.) We also adopt the notation $r^+_3(ab|cd, ef, gh)$ in line with the standard notation for the projected version of the triple ratio.

**Proposition 21.** The following combination has a vanishing 2-part of the cobracket

$$\frac{1}{640} Gr_5 + \frac{5}{3} Alt_{10} \left[ 2I_{4,1}(cr(136|2459), r^+_3(12345, 678)) + I_{4,1}(cr(346|1279), r^-_3(12345, 678)) \right].$$

According to the conjectured structure of the motivic Lie coalgebra (of a field) in weight 5 as predicted by Goncharov (see e.g. [19], [18]), this combination should actually lie in depth 1, and we expect the following relationship with the Borel class $b_5^{(3)}$ in weight 5.
Conjecture 22. \(\text{(i)}\) There exists a formal linear combination \(Q_5(v_1,\ldots,v_{10})\) of rational functions on \(\text{Conf}_{10}(5)\) such that 
\[\text{Li}_5(Q_5(v_1,\ldots,v_{10}))\]
equals \([13]\), modulo products.
\(\text{(ii)}\) Assuming (i) holds, the function \(\varphi : \text{GL}_5(\mathbb{C})^{10} \to \mathbb{R}\) defined by
\[\varphi(g_1,\ldots,g_{10}) = \mathcal{L}_5(Q_5(g_1v,\ldots,g_{10}v))\]
is a bounded measurable 5 cocycle for \(\text{GL}_5(\mathbb{C})\), whose corresponding continuous cohomology class is a non-zero rational multiple of the Borel class \(b_5^{(5)} \in H^5_{\text{con}}(\text{GL}_5(\mathbb{C}),\mathbb{R})\).

Explicitly, in view of (12) the combination
\[
\begin{align*}
\text{Alt}_{10} & \left[ I_{4,1} \left( \frac{p_2 p_3 p_4}{p_2 p_1 p_4}, x, y \right) + 2 I_{4,1} \left( \frac{p_1 p_4}{p_3 p_4}, x, y \right) + I_{4,1} \left( \frac{p_1 p_2 p_3}{p_3 p_1 p_4}, x, y \right) \right] \\
& \quad + \frac{10}{3} \left[ I_{4,1}(cr(136|2459), r_3^{+}(12|345, 678)) + \frac{5}{3} \left[ I_{4,1}(cr(346|1279), r_3^{-}(12|345, 678)) \right] \right]
\end{align*}
\]
should be reducible to purely \(\text{Li}_5\) terms.

In order to state this more precisely, we will decompose the above combination into pieces expressed in terms of \(I_{4,1}^+(x, y)\), where
\[
I_{4,1}^+(x, y) := \frac{1}{2} \left( I_{4,1}(x, y) + I_{4,1}(x, y^{-1}) \right).
\]

The conjectural behaviour of \(I_{4,1}^+\) under \(\text{Li}_2\) functional equations in \(x, y\) should play a key part in any explicit reduction of \(\text{Gr}_5\) to \(\text{Li}_5\). By a conjecture of Goncharov about the structure of the motivic Lie coalgebra in weight 5 ([13], §4.5, §4.8), we expect \(I_{4,1}^+(x, y)\) to reduce to \(\text{Li}_5\) under dilogarithm functional equations in \(x, y\), and also under trilogarithm functional equations in \(y\) (that is, \(I_{4,1}^+(x, y)\) modulo \(\text{Li}_5\) should satisfy the same relations as \(\text{Li}_2(x) \circ \text{Li}_3(y)\)).

We can express \(I_{4,1}^+(x, y)\) in terms of \(I_{4,1}^+\) as follows
\[
I_{4,1}(x, y) \equiv I_{4,1}^+(x, y) + I_{4,1}^+(y, x) + \frac{1}{2} \text{Li}_5(x) + \frac{1}{2} \text{Li}_5(y) + 2 \text{Li}_5(xy).
\]
Let us give some partial results supporting the conjectural behavior of \(I_{4,1}^+\). By analogy to Theorem 11 for \(I_{3,1}\) we have the following.

Theorem 23. Modulo products and explicit \(\text{Li}_5\) terms, the function \(I_{4,1}^+(x, y)\) satisfies the dilogarithm 6-fold symmetries in \(x, y\), and the \(\text{Li}_3\) inversion and three-term relation in \(y\):

- \(\text{(i)}\) \(I_{4,1}^+(x, y) + I_{4,1}^+(x^{-1}, y) = 0 \mod \text{Li}_5, \text{products}\),
- \(\text{(ii)}\) \(I_{4,1}^+(x, y) + I_{4,1}^+(1 - x, y) = 0 \mod \text{Li}_5, \text{products}\),
- \(\text{(iii)}\) \(I_{4,1}^+(x, y) - I_{4,1}^+(x, y^{-1}) = 0 \mod \text{Li}_5, \text{products}\), and
- \(\text{(iv)}\) \(I_{4,1}^+(x, y) + I_{4,1}^+(x, 1 - y) + I_{4,1}^+(x, 1 - y^{-1}) - I_{4,1}^+(x, 1) = 0 \mod \text{Li}_5, \text{products}\).

Explicit expressions for these identities are given in Appendix [13]. Additionally, in [10] we have established the 5-term relation for \(I_{4,1}^+(x, 1) = I_{4,1}(x, 1)\) in \(x\) (modulo \(\text{Li}_5\) and products), or rather the Nielsen polylogarithm \(S_{3,2}(x) \equiv I_{4,1}(x, 1) + 4 \text{Li}_5(x)\), although we will not use this. Analogously to the weight 4 case it is convenient to introduce the following symmetrization of \(I_{4,1}^+\).

Definition 24. Let us define
\[
\tilde{I}_{4,1}^+(x, y) := \frac{1}{6} \sum_{\sigma \in S_3} \text{sgn}(\sigma) I_{4,1}(x^\sigma, y),
\]
where \(x^\sigma\) denotes elements of \(\{x, 1/x, x - 1, 1 - x, 1/(1 - x), x/(x - 1)\}\) (the anharmonic group) corresponding to \(\sigma \in S_3\) (under some isomorphism).

Proposition 25. The combination
\[
I_{4,1}^+(x, y) - \tilde{I}_{4,1}^+(x, y)
\]
is equal modulo products to an explicit sum of \(\text{Li}_5\) terms. Moreover, the function \(\tilde{I}_{4,1}^+(x, y)\) satisfies
\[
\tilde{I}_{4,1}^+ (x^\sigma, y^{\pm 1}) \equiv \text{sgn}(\sigma) \tilde{I}_{4,1}^+(x, y).
\]

We now claim that, after converting \(I_{4,1}\) to \(I_{4,1}^+\) via (16), the combination in (13) breaks up into 3 nontrivial subsums which are combinations of \(\text{Li}_2\) and \(\text{Li}_3\) functional equations, and a piece that constitutes a trivial coboundary.
Proposition 26. The expression (13) breaks up as follows

\[ A = I^+_{4,1}(\rho_{1,2,3}, \rho_{2,3,4}) - I^+_{4,1}(\rho_{3,2,4}, \rho_{1,2,3}), \]

\[ B = 2I^+_{4,1}(\rho_{1,2,3,4}, \rho_{2,3,4}, \rho_{1,4}, \rho_{2,4}), \]

\[ C = 2I^+_{4,1}(\rho_{1,2,3,4}, \rho_{2,3,4}, \rho_{1,4}, \rho_{2,4}) + 2I^+_{4,1}(\rho_{3,2,4}, \rho_{1,2,3,4}, \rho_{1,4}, \rho_{2,4}) + 2I^+_{4,1}(\rho_{1,2,3,4}, \rho_{1,4}, \rho_{2,4}) \]

\[ - \frac{4}{3} I^+_{4,1}(\rho_{1,2,3,4}, \rho_{1,4}, \rho_{2,4}) + \frac{5}{3} I^+_{4,1}(\rho_{1,2,3,4}, \rho_{1,4}, \rho_{2,4}) \]

\[ D = I^+_{4,1}(\rho_{1,2,3,4}, \rho_{1,4}, \rho_{2,4}) + \frac{10}{3} I^+_{4,1}(\rho_{1,2,3,4}, \rho_{1,4}, \rho_{2,4}) + \frac{10}{3} I^+_{4,1}(\rho_{1,2,3,4}, \rho_{1,4}, \rho_{2,4}) \]

\[ + \frac{5}{3} I^+_{4,1}(\rho_{1,2,3,4}, \rho_{1,4}, \rho_{2,4}) \]

have the following properties. The expressions \( A, B, \) and \( C \) can be explicitly written as \( I^+_{4,1} \) of \( \text{Li}_2 \) functional equations in the first argument, and \( \text{Li}_3 \) functional equations in the second argument. The expression \( D \) is a coboundary term.

We prove this Proposition in Section 10. We expect the expression \( D \) to also have a decomposition in terms of \( \text{Li}_2 \) and \( \text{Li}_3 \) functional equations, but since it is a coboundary, we can simply add it to the last 4 terms (which are also coboundaries) in (14).

6. Proof of Theorem 4

To prove the result we will compute the symbol of the right-hand side of (7) and show that it is equal to a constant multiple of (4). To compute the symbol we repeatedly apply the \((n - 1, 1)\)-part of the coproduct to the expression on the right of (7). After the first application we obtain

\[ \text{Alt}_{2m}[I(0; \rho_1, \ldots, \rho_m) \otimes \rho_1 + I(0; \rho_2, \ldots, \rho_m) \otimes \left(1 - \frac{\rho_2}{\rho_1}\right) + \sum_{j=2}^{m-1} I(0; \rho_1, \ldots, \rho_j, \ldots, \rho_m) \otimes \left(\frac{\rho_j - \rho_{j+1}}{\rho_j - \rho_{j-1}}\right)]. \]

Let us denote

\[ A_j = \Delta(j, \ldots, j + m - 2, j + m - 1), \]

\[ B_j = \Delta(j, \ldots, j + m - 2, 2m - 1), \]

\[ C_j = \Delta(j + 1, \ldots, j + m - 2, j + m - 1, 2m), \]

\[ D_j = \Delta(j + 1, \ldots, j + m - 2, 2m - 1, 2m) \]

(note the offset in the definition of \( C_j \) and \( D_j \)). Then for \( 2 \leq j \leq m - 1 \) we have the following factorization (that follows from two applications of projected Plücker relations)

\[ \frac{\rho_j - \rho_{j+1}}{\rho_j - \rho_{j-1}} = \frac{C_{j-2}D_j}{C_jA_{j-1}} \cdot \frac{A_j}{D_{j-1}}, \]

where the first factor is fixed by the transposition \((j, j - 1)\), and the second one is fixed (up to sign) by \((j + m - 2, j + m - 1)\). Since \( I(0; 0, \rho_1, \ldots, \rho_j, \ldots, \rho_m) \) is trivially invariant under both of these transpositions, we obtain that

\[ \text{Alt}_{2m}\left[I(0; 0, \rho_1, \ldots, \rho_j, \ldots, \rho_m) \otimes \left(\frac{\rho_j - \rho_{j+1}}{\rho_j - \rho_{j-1}}\right)\right] = 0, \quad 2 \leq j \leq m - 1. \]

This leaves us with

\[ \text{Alt}_{2m}\left[I(0; 0, \rho_1, \ldots, \rho_{m-1}; \rho_m) \otimes \rho_1 + I(0; 0, \rho_2, \ldots, \rho_{m-1}; \rho_m) \otimes \left(1 - \frac{\rho_2}{\rho_1}\right)\right]. \]

Next, after the second application of the coproduct we get

\[ \text{Alt}_{2m}\left[I(0; \rho_2, \ldots, \rho_{m-1}; \rho_m) \otimes \left(1 - \frac{\rho_2}{\rho_1}\right) \otimes \rho_1 + \sum_{j=2}^{m-1} I(0; \rho_1, \ldots, \rho_j, \ldots, \rho_m) \otimes \left(\frac{\rho_j - \rho_{j+1}}{\rho_j - \rho_{j-1}}\right) \otimes \rho_1 \right. \]

\[ + I(0; \rho_2, \ldots, \rho_{m-1}; \rho_m) \otimes \rho_2 \otimes \left(1 - \frac{\rho_2}{\rho_1}\right) + I(0; 0, \rho_3, \ldots, \rho_{m-1}; \rho_m) \otimes \left(1 - \frac{\rho_2}{\rho_1}\right) \otimes \left(1 - \frac{\rho_2}{\rho_1}\right) \otimes \left(1 - \frac{\rho_2}{\rho_1}\right) \]
By the same reasoning as before, we see that after skew-symmetrization each of the two sums above cancels out termwise, so we are left with

$$\text{Alt}_{2m} \left[ I(0; \rho_2, \ldots, \rho_{m-1}; \rho_m) \otimes (1 - \frac{\rho_j}{\rho_j}) \otimes \rho_1 + I(0; \rho_2, \ldots, \rho_{m-1}; \rho_m) \otimes (1 - \frac{\rho_j}{\rho_j}) \right].$$

Continuing in this fashion we arrive at the following expression for the symbol of the RHS of (7)

$$\text{Alt}_{2m} \left[ \sum_{j=1}^{m} \left( 1 - \frac{\rho_m}{\rho_m-j} \right) \otimes \left( 1 - \frac{\rho_{m-j}}{\rho_{m-j}} \right) \otimes \cdots \otimes \left( 1 - \frac{\rho_{j+1}}{\rho_{j+1}} \right) \otimes \rho_j \otimes \left( 1 - \frac{\rho_{j-1}}{\rho_{j-1}} \right) \cdots \otimes \left( 1 - \frac{\rho_2}{\rho_2} \right) \right],$$

where $\rho_j$ is inserted in the $j$-th position from the right. For $m = 2$ we directly compute

$$\text{Alt}_4 \left[ \left( 1 - \frac{\rho_2}{\rho_2} \right) \otimes \rho_1 + \rho_2 \otimes \left( 1 - \frac{\rho_2}{\rho_2} \right) \right] = -12 \text{Alt}_4 \left[ \Delta(12) \otimes \Delta(23) \right],$$

which proves (7) in this case. From now on we assume that $m \geq 3$. Noting the following factorizations

$$\rho_j = \frac{B_j}{C_j - 1}, \quad 1 - \frac{\rho_{j+1}}{\rho_j} = \frac{A_j D_j}{B_j C_j},$$

let us look at the $k$-th term of the sum in (18):

$$\frac{A_{m-1} D_{m-1}}{B_{m-1} C_{m-1}} \otimes \cdots \otimes \frac{A_k D_k}{B_k C_k} \otimes \frac{B_k}{C_k - 1} \otimes \frac{A_{k-1} D_{k-1}}{B_{k-1} C_{k-1}} \otimes \cdots \otimes \frac{A_1 D_1}{B_1 C_1},$$

and hence these terms cancel out. Therefore, we need to compute the sum

$$\sum_{k=1}^{m} (T_k^B - T_k^C),$$

where $T_k$ are given by

$$T_k^B = \frac{A_{m-1} D_{m-1}}{B_{m-1} C_{m-1}} \otimes \cdots \otimes \frac{A_k D_k}{B_k C_k} \otimes \frac{B_k}{C_k - 1} \otimes \frac{A_{k-1} D_{k-1}}{B_{k-1} C_{k-1}} \otimes \cdots \otimes \frac{A_1 D_1}{B_1 C_1},$$

$$T_k^C = \frac{A_{m-1} D_{m-1}}{B_{m-1} C_{m-1}} \otimes \cdots \otimes \frac{A_k D_k}{B_k C_k} \otimes \frac{C_k - 1}{C_k - 1} \otimes \frac{A_{k-1} D_{k-1}}{B_{k-1} C_{k-1}} \otimes \cdots \otimes \frac{A_1 D_1}{B_1 C_1}.$$

For $\pi \in \mathfrak{S}_{2m}$ we denote by $\sigma_{\pi}$ the automorphism of the ring of regular functions on $\text{Conf}_{2m}(m)$ induced by $\pi$. It is easy to check that the involution

$$a_1 \otimes \cdots \otimes a_m \mapsto \sigma_{\pi}(a_1) \otimes \cdots \otimes \sigma_{\pi}(a_m),$$

where $\pi = (2m - 2)(1,2m - 2)(2,2m - 3) \cdots (m - 1, m)$ interchanges $T_k^B$ and $T_k^C$ (for now we ignore the sign). Indeed, the permutation $(2m - 1, 2m)$ interchanges $B_j$ and $C_j$, the permutation $(1, 2m - 2)(2,2m - 3) \cdots (m - 1, m)$ maps $X_j \leftrightarrow X_{m-j}$ for $X = A, D, B_j \leftrightarrow B_{m+1-j}$, and $C_j \leftrightarrow C_{m-j}$, so their composition maps $B_j \leftrightarrow C_{m-j}$ and $A_j \leftrightarrow A_{m-j}$, and after reversing the order of the tensors we get the involution $T_k^B \leftrightarrow T_k^C$. Thus, it is enough to calculate the terms $T_k^B$ modulo $\text{Alt}_{2m}$ for $k = 1, \ldots, m$. We will expand $T_k^B$ into a sum of “elementary tensors” and describe all such tensors that are not fixed by any transposition in $\mathfrak{S}_{2m}$ (since any such term will vanish after skew-symmetrization). We have the following lemma.

**Lemma 27.** Let $w = X_{m-1} \otimes \cdots \otimes X_k \otimes B_k \otimes X_{k-1} \otimes \cdots \otimes X_1$, where $X_i \in \{A_i, B_i, C_i, D_i\}$, be any term in the expansion of $T_k^B$, $1 \leq k \leq m$, that is not fixed by any transposition, and let $2 \leq j \leq m - 1$ be such that $j \neq k$. Then

(i) if $X_{j-1} \in \{A_{j-1}, B_{j-1}\}$, then $X_j \in \{A_j, B_j\}$;

(ii) if $X_{j-1} \in \{B_{j-1}, D_{j-1}\}$, then $X_j \in \{B_j, D_j\}$;

(iii) if $X_j \in \{C_j, D_j\}$, then $X_{j-1} \in \{C_{j-1}, D_{j-1}\}$;

(iv) if $X_j \in \{A_j, C_j\}$, then $X_{j-1} \in \{A_{j-1}, C_{j-1}\}$;

(v) if $k < m$ and $X_1 \in \{A_1, C_1\}$, then $X_{m-1} \in \{C_{m-1}, D_{m-1}\}$;
Since the transformation \((20)\) maps each term on the right hand side of the above equation to its negation to \(v\) \(v\) \(v\) \(v\) 

(21)

\[ v_{j,k}^D = (-1)^{k-j} D_{m-1} \otimes \cdots \otimes D_k \otimes B_k \otimes B_{k-1} \otimes \cdots \otimes B_1, \]

for \(1 \leq j \leq k\). Let \(w = X_{m-1} \otimes \cdots X_k \otimes B_k \otimes B_{k-1} \otimes \cdots \otimes X_1\) be any term in the expansion of \(T_k^B\) that is not fixed by any transposition (we ignore the coefficient with which \(w\) appears in \(T_k^B\)). First, note that if any \(X_i = C_i\), then so must be \(X_k = C_k\) by repeatedly using parts (iii)-(vi) of the lemma, but looking at \((19)\) we see that \(X_k \neq C_k\). Thus there are no \(C_i\) among \(X_i\)’s. Next, if \(X_1 = D_1\), then \(X_{m-1} = A_{m-1}\) by (vi), and by parts (i) and (ii) we must have \(X_i = A_i\), \(i \geq k\), and \(X_i = D_i\) for \(i < j\) for some \(k \geq j > 1\), i.e., \(v_{j,k}^A\) for some \(j\). Similarly, if \(X_1 = A_1\), then we get \(v_{i,k}^D\) for some \(j\). Finally, let \(X_1 = B_1\). Then by (i) and (ii) we have \(w = X_{m-1} \otimes \cdots X_k \otimes B_k \otimes B_{k-1} \otimes \cdots \otimes B_j\). If \(X_k = B_k\), then \(X_i = B_i\) for all \(i\), but then \(w\) would be fixed by the transposition \((m-1, 2m-1)\). If \(X_k = A_k\), then by (i) and (ii) we have \(X_i = A_i\) for \(i = k, \ldots, l\), and \(X_i = B_i\) for \(l < i \leq m-1\). In this case if \(l < m-1\), then \((2m-2, 2m)\) fixes \(w\). So the only term that is not fixed by transpositions in this case is \(v_{1,k}^D\). Similarly, in the case \(X_k = D_k\) we get \(v_{1,k}^D\).

Case \(k = m\). In this case parts (i) and (ii) of the above lemma immediately imply that in the expansion of \(T_m^B\) only the following \(m^2\) terms are not fixed by any transposition:

\[ (22) \]

\[ w_{i,j}^A = (-1)^{m-j+i-1} B_m \otimes \cdots \otimes B_j \otimes A_{j+1} \otimes A_{j-2} \otimes \cdots \otimes A_1 \otimes C_{j+1} \otimes \cdots \otimes C_1, \]

for \(1 \leq i \leq j \leq m\) (here \(w_{i,j}^A = w_{i,j}^D\), so we only count them once).

What is left to show is that all of the terms in \((21)\) and \((22)\) are \(\text{Alt}_{2m}\)-equivalent to

\[ R = (-1)^{m-1} \Delta(1, \ldots, m) \otimes \Delta(2, \ldots, m+1) \otimes \cdots \otimes \Delta(m, \ldots, 2m-1). \]

First, we show that all terms are pairwise \(\text{Alt}_{2m}\)-equivalent. The permutation \((j+1, m-2, 2m)\) sends \(v_{j,k}^A\) to \(v_{j-1,k}^A\), and the permutation \((j-1, 2m-1)\) sends \(v_{j,k}^D\) to \(v_{j-1,k}^D\), thus each \(v_{j,k}^\tau\) is equivalent to \(v_{i,k}^A\).

Similarly, the permutation \((i-1, 2m)\) sends \(w_{i,j}^A\) to \(w_{i-1,j}^A\) and the permutation \((i+1, m-2, 2m-1)\) sends \(w_{i,j}^D\) to \(w_{i-1,j}^D\), thus each \(w_{i,j}^\tau\) is equivalent to \(w_{i,j}^A\). Next, applying \((k, 2m-1)\) to \(v_{i,k}^A\) we get \(v_{i,k+1}^A\), and applying \((k+1, m-1, 2m)\) to \(v_{i,k}^D\) we get \(v_{i,k+1}^D\), where we set both \(v_{1,m}\) and \(v_{m,m}\) to be equal to \(v_{1}^A\). Similarly, applying \((j+1, m-2, 2m-1)\) to \(w_{i,j}^A\) we get \(w_{i,j+1}^A\), and applying \((j-1, 2m)\) to \(w_{i,j}^D\) we get \(w_{i,j+1}^D\). Thus all terms are equivalent to \(w_{1,1}^A\), from which we get \(R\) after first applying \((2m-2, 2m-1)\) and \((1, m)\) (both permutations have sign \((-1)^{m-1}\)).

This shows that \(\text{Alt}_{2m}(\sum_{k=1}^{m} T_k^B)\) is equal to

\[ m(2m-1)(-1)^{m-1} \text{Alt}_{2m} \left[ \Delta(1, \ldots, m) \otimes \Delta(2, \ldots, m+1) \otimes \cdots \otimes \Delta(m, \ldots, 2m-1) \right]. \]

Since the transformation \((20)\) maps each term on the right hand side of the above equation to its negation (modulo \(\text{Alt}_{2m}\)), and it maps \(T_k^B\) to \(T_{m-k}^C\), we get that \(\text{Alt}_{2m}(\sum_{k=1}^{m} T_k^C)\) equals

\[ m(2m-1)(-1)^{m} \text{Alt}_{2m} \left[ \Delta(1, \ldots, m) \otimes \Delta(2, \ldots, m+1) \otimes \cdots \otimes \Delta(m, \ldots, 2m-1) \right]. \]

Combining these two identities together with \((6)\) we obtain the claim of the theorem.
Remark 28. The proof of (8) is analogous, except that (15) becomes
\[ Alt_{2m} \left[ \sum_{j=1}^{m} (\rho_m - \rho_{m-1}) \otimes \cdots \otimes (\rho_{j+1} - \rho_j) \otimes \rho_j \otimes \left( 1 - \frac{\rho_j}{\rho_{j+1}} \right) \otimes \cdots \otimes \left( 1 - \frac{\rho_2}{\rho_1} \right) \right], \]
and the rest of the combinatorial analysis needs to be changed accordingly. We then again use
\[
\frac{A_{m-1}D_{m-1}}{C_mC_{m-2}} \otimes \cdots \otimes \frac{A_kD_k}{C_kC_{k-1}} \otimes \frac{B_k}{C_{k-1}} \otimes \frac{A_{k-1}D_{k-1}}{B_kC_{k-1}} \otimes \cdots \otimes \frac{A_1D_1}{B_1C_1}.
\]
where again the terms of the form \( \cdots \otimes C_k \otimes B_k \otimes \cdots \) cancel between consecutive values of \( k \). All other terms in the expansion are either invariant under a transposition and thus are \( Alt_{2m} \)-equivalent to 0 or are \( Alt_{2m} \)-equivalent to \( R \), where \( R \) is defined in (24).

7. Proof of Theorem 7

To simplify notation, in this section we will write \( \rho_i \) for \( \rho_i^{(1,2m)} \), contrary to the convention throughout the rest of the paper. The proof follows the same strategy as the proof of Theorem 4 above: to prove (27) we calculate the symbol of
\[ Alt_{m,m} \left[ I(0; \rho_{m+1}, \rho_m, \ldots, \rho_2) \right] \]
by repeatedly applying the \((n-1,1)\)-part of the coproduct to it. After the first application we obtain
\[ Alt_{m,m} \left[ I(0; \rho_m, \ldots, \rho_2) \otimes \frac{\rho_{m+1} - \rho_m}{\rho_m} + \sum_{j=3}^{m} I(0; \rho_{m+1}, \ldots, \rho_j, \ldots, \rho_2) \otimes \frac{\rho_j - \rho_{j-1}}{\rho_j - \rho_{j+1}} \right]. \]
As in the proof of Theorem 4 we note that the \( \frac{\rho_j - \rho_{j-1}}{\rho_j - \rho_{j+1}} \) decomposes into a product of two terms, one fixed by the involution \((j-1,j)\), and the other by \((j+m-2,j+m-1)\). Since both of these involutions fix \( I(0; \rho_{m+1}, \ldots, \rho_j, \ldots, \rho_2) \) each term in the sum \( \sum_{j=3}^{m} I(0; \rho_{m+1}, \ldots, \rho_j, \ldots, \rho_2) \) vanishes under \( Alt_{m,m} \). Applying the \((n-1,1)\)-part of the coproduct again to the remaining term we get
\[ Alt_{m,m} \left[ \left( I(0; \rho_{m-1}, \ldots, \rho_2) \otimes \frac{\rho_m - \rho_{m-1}}{\rho_m} + \sum_{j=3}^{m-1} I(0; \rho_m, \ldots, \rho_j, \ldots, \rho_2) \otimes \frac{\rho_j - \rho_{j-1}}{\rho_j - \rho_{j+1}} \right) \otimes \frac{\rho_{m+1} - \rho_m}{\rho_{m+1}} \right]. \]
By the same argument as above, all terms in the sum \( \sum_{j=3}^{m-1} \) vanish under \( Alt_{m,m} \) (since for \( 3 \leq j \leq m-1 \) both \((j-1,j)\) and \((j+m-2,j+m-1)\) leave \( \frac{\rho_{m+1} - \rho_m}{\rho_{m+1}} \) invariant). After repeating this argument \((m-2)\) times we get that the symbol of (24) is equal to
\[ Alt_{m,m} \left[ \frac{\rho_m - \rho_2}{\rho_3} \otimes \cdots \otimes \frac{\rho_m - \rho_{m-1}}{\rho_m} \otimes \frac{\rho_{m+1} - \rho_m}{\rho_{m+1}} \right]. \]
Let us denote
\[ A_j = \Delta(j+1, \ldots, j+m-2, j+m-1, 1), \quad C_j = \Delta(j+1, \ldots, j+m-2, 2m, 1), \quad B_j = \Delta(j+1, \ldots, j+m-2, j+m-1, j), \quad D_j = \Delta(j+1, \ldots, j+m-2, 2m, j). \]
Then for \( 3 \leq j \leq m \) we have the following corollary of the (projected) Plücker relations
\[ \frac{\rho_{j+1} - \rho_j}{\rho_{j+1}} = \frac{B_jC_j}{A_jD_j}. \]
We expand the expression in (25), and let \( T = X_2 \otimes X_3 \otimes \cdots \otimes X_m \) be any tensor in it with \( X_j \in \{ A_j, B_j, C_j, D_j \} \). By analogy with the proof of Lemma 27, if for some \( j \) we have \( (X_j, X_{j+1}) \in \{(B_j, A_{j+1}), (B_j, C_{j+1}), (C_j, A_{j+1}), (C_j, B_{j+1}), (D_j, A_{j+1}), (D_j, B_{j+1}, (D_j, C_{j+1}) \} \), then \( Alt_{m,m} T = 0 \), since \( T \) will be fixed by the transposition \((j, j+1)\) or by \((j+m-1, j+m)\). Thus any non-vanishing \( T \) must follow one of the patterns \( A^r B^s D^t \) \((m+1) \) such with \( r, s, t \geq 0 \) or \( A^r C^s D^t \), and a simple counting argument shows that there are \( m^2 (\frac{m+1}{2}) \) such terms. Let
\[ w_{k,t} = (-1)^{m-t+k} A_2 \otimes \cdots \otimes A_{k-1} \otimes B_k \otimes \cdots \otimes B_t \otimes D_{t+1} \otimes \cdots \otimes D_m. \]
As in the proof of Theorem 8, applying the transpositions \((1, k - 1)\) (resp. \((l + m, 2m)\)) to \(w_{k,l}\) gives \(w_{k-1,l}\) (resp. \(w_{k,l+1}\)). Thus all terms \(w_{k,l}\) are \(\text{Alt}_{m,m}\)-equivalent to \(w_{2,m} = B_2 \otimes \cdots \otimes B_m\). Similarly, all terms of the form
\[
(-1)^{m-l+k} A_2 \otimes \cdots \otimes A_{k-1} \otimes C_k \otimes \cdots \otimes C_l \otimes D_{l+1} \otimes \cdots \otimes D_m
\]
are also \(\text{Alt}_{m,m}\)-equivalent to \(B_2 \otimes \cdots \otimes B_m\). Thus, \((28)\) equals
\[
m^2 \text{Alt}_{m,m}[B_2 \otimes \cdots \otimes B_m],
\]
which by Proposition 2 equals \((-1)^{m-1}/m^2\) times the symbol of the Aomoto polylogarithm, as claimed.

8. Proof of Theorem 9

Proving Theorem 9 amounts to computing \(S^u\) of both sides and checking that they are equal. The easiest way to do this verification is to use a computer algebra system, and in fact this is the way in which the identity of Theorem 9 was discovered. In this section we will give a direct proof that does not require any computer calculations.

First, note the following identities
\[
S^u I_{3,1}(x, y) = S^u I_{2,1}(x, y) \otimes \frac{y}{x} - S^u (\text{Li}_3(x) - \text{Li}_3(x/y)) \otimes (1 - y^{-1})
\]
\[
+ S^u (\text{Li}_3(y) - \text{Li}_3(y/x)) \otimes (1 - x^{-1}),
\]
\[
S^u I_{2,1}(x, y) = S^u I_{1,1}(x, y) \otimes \frac{y}{x} + S^u (\text{Li}_2(x) - \text{Li}_2(x/y)) \otimes (1 - y^{-1})
\]
\[
+ S^u (\text{Li}_2(y) - \text{Li}_2(y/x)) \otimes (1 - x^{-1}).
\]

Recall also that \(S^u \text{Li}_2(x) = x \wedge (1 - x)\) and that for \(k \geq 3\) we have \(S^u \text{Li}_k(x) = S^u \text{Li}_{k-1}(x) \otimes x\). In this section we will simply write \(I_{k,1}(x, y), \text{Li}_m(x)\) instead of \(S^u I_{k,1}(x, y), S^u \text{Li}_m(x)\).

Throughout the proof we work modulo the kernel of the skew-symmetrization operator \(\text{Alt}_8\); in particular, any term that is invariant under an odd permutation is annihilated by \(\text{Alt}_8\). We denote such identities by \(\llbracket \text{Alt}_8 \rrbracket\). In the proof we will repeatedly use the fact that the permutation \((16)(25)(34)\) permutes \(\rho_1, \ldots, \rho_4\) as \(\rho_1 \leftrightarrow \rho_1, \rho_3 \leftrightarrow \rho_2\). Note also that the transposition \((78)\) maps each \(\rho_i\) to its inverse \(\rho_i^{-1}\). All tensor products are written with respect to multiplication and we work modulo torsion, so that \(a \otimes b \otimes c = a \otimes b + a \otimes c\) and \(a \otimes (-b) = a \otimes b\). In particular, recalling the notation \(\rho_{i,j} = \rho_i - \rho_j\), the above implies that \(\rho_{i,j} \otimes x = \rho_{i,j} \otimes x\) which we will use freely. Finally, we denote by \(a \land b\) the difference \(a \otimes b - a \otimes a\) and by \(a \land b\) the sum \(a \otimes b + b \otimes a\).

First, we give a different expression for the mod-products symbol of \(\text{Gr}_4\).

**Lemma 29.** We have the following identity
\[
\frac{7}{144} S^u(\text{Gr}_4) = 8 \text{Alt}_8 \left[ S^u \text{Li}_2 \left( \frac{\rho_{2,4}}{\rho_{3,4}} \right) \otimes \rho_{1,2} \otimes \frac{\rho_1}{\rho_4} \right].
\]

**Proof.** Note that we have
\[
\text{Alt}_8 \left[ \frac{\rho_{3,4}}{\rho_{3,4}} \otimes f(\rho_1, \rho_2, \rho_4) \right] = 0,
\]
where \(f(\rho_1, \rho_2, \rho_4)\) is any expression that depends only on \(\rho_1, \rho_2, \rho_4\). This follows from the fact that \(\rho_1, \rho_2, \rho_4\) are fixed by both \((23)\) and \((56)\), while \(\frac{\rho_{2,4}}{\rho_{3,4}} = \frac{\Delta(2345)\Delta(4568)}{2\Delta(2345)\Delta(4568)} = \frac{\Delta(3478)}{2\Delta(4567)}\) is a product of two terms, one invariant under \((23)\) and the other under \((56)\). Using the fact that \(S^u \text{Li}_2(x) = x \wedge (1 - x)\), we calculate right-hand side of \((28)\) to be equal to
\[
8 \text{Alt}_8 \left[ \frac{\rho_{2,4}}{\rho_{3,4}} \wedge \frac{\rho_{1,2}}{\rho_{3,4}} \otimes \rho_{1,2} \otimes \frac{\rho_1}{\rho_4} \right] = 8 \text{Alt}_8 \left[ - \rho_{4,3} \otimes \rho_{3,2} \otimes \rho_{2,1} \otimes \frac{\rho_1}{\rho_4} \right]
\]
\[
= 8 \text{Alt}_8 \left[ - \rho_{4,3} \otimes \rho_{3,2} \otimes \rho_{2,1} \otimes \rho_1 - \rho_{1,2} \otimes \rho_{3,2} \otimes \rho_{3,4} \otimes \frac{\rho_1}{\rho_4} \right]
\]
\[
= 8 \text{Alt}_8 \left[ - D_3(\rho_{4,3} \otimes \rho_{3,2} \otimes \rho_{2,1}) \otimes \rho_1 \right],
\]
where on the second line we have applied \((16)(25)(34)\) to one of the terms, and \(D_3\) is the operator that annihilates shuffle products (see Section 2.3). Since
\[
D_4(a \otimes b \otimes c \otimes d) = D_3(a \otimes b \otimes c) \otimes d - D_3(d \otimes c \otimes b) \otimes a,
\]
we have
\[
8 \text{Alt}_8 \left[ - \rho_{4,3} \otimes \rho_{3,2} \otimes \rho_{2,1} \otimes \rho_1 - \rho_{1,2} \otimes \rho_{3,2} \otimes \rho_{3,4} \otimes \rho_1 \right] = \frac{7}{144} S^u(\text{Gr}_4),
\]
a contradiction. Therefore, \(S^u(\text{Gr}_4) = 0\) as required.
and the odd permutation (1 7)(2 6)(3 5) maps $\Delta(1234) \otimes \Delta(2345) \otimes \Delta(3456) \otimes \Delta(4567)$ to its reversal, it is enough to prove

\[(30) \quad \text{Alt}_8 \left[ \rho_{4,3} \otimes \rho_{3,2} \otimes \rho_{2,1} \otimes \rho_1 \right] = -14 \text{Alt}_8 \left[ \Delta(1234) \otimes \Delta(2345) \otimes \Delta(3456) \otimes \Delta(4567) \right].\]

This identity can be seen by either doing the same combinatorial analysis as in the proof of Theorem 4 or by directly expanding the 128 terms on the left and noting that only 14 of them are not fixed by any transposition and that the rest is $\text{Alt}_8$-equivalent to $-\Delta(1234) \otimes \Delta(2345) \otimes \Delta(3456) \otimes \Delta(4567)$ (also compare with (30) below).

Next, we compute the right-hand side of (30). After applying (26) to compute the $S^w$ of the right hand side of (10), we obtain the skew-symmetrization of

\[ I_{2,1} \left( \frac{\rho_{1,2,3,4}}{\rho_{3,2,1,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) + 2I_{2,1} \left( \frac{\rho_{1,2,3,4}}{\rho_{1,2,3,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) + 6 \text{Li}_3(U) \otimes U \]

\[ + \left[ - \text{Li}_3 \left( \frac{\rho_{1,2,3,4}}{\rho_{1,2,3,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) \otimes \frac{\rho_2}{\rho_{1,2,3,4}} + \text{Li}_3 \left( \frac{\rho_1}{\rho_{1,2,3,4}} \right) \otimes \frac{\rho_2}{\rho_{1,2,3,4}} \right] \cong -2 \text{Li}_3(U) \otimes U, \]

where we have used the fact that (16)(25)(34)(78) leaves $U$ fixed and sends the other parenthesized term to its inverse (we have added extra parentheses to emphasize that we work multiplicatively). This leaves us with $\text{Alt}_8$ of

\[ I_{2,1} \left( \frac{\rho_{1,2,3,4}}{\rho_{3,2,1,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) + 2I_{2,1} \left( \frac{\rho_{1,2,3,4}}{\rho_{1,2,3,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) + 4 \text{Li}_3(U) \otimes U \]

\[ - \text{Li}_3 \left( \frac{\rho_{1,2,3,4}}{\rho_{1,2,3,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) \otimes \frac{\rho_2}{\rho_{1,2,3,4}} + \text{Li}_3 \left( \frac{\rho_1}{\rho_{1,2,3,4}} \right) \otimes \frac{\rho_2}{\rho_{1,2,3,4}} \right] \cong -2 \text{Li}_3(U) \otimes U, \]

where the crossed-out terms cancel since they expand out to a combination of terms that are fixed either by (12) or by (23) and hence vanish. Next, we apply (27) to the expression in the square brackets. Rearranging the terms with $\text{Li}_3(U)$ as above we get (skew-symmetrization of

\[ I_{1,1} \left( \frac{\rho_{1,2,3,4}}{\rho_{3,2,1,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) + 2I_{1,1} \left( \frac{\rho_{1,2,3,4}}{\rho_{1,2,3,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) + 2 \text{Li}_2(U) \otimes U \]

\[ - \text{Li}_3 \left( \frac{\rho_{1,2,3,4}}{\rho_{1,2,3,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) \otimes \frac{\rho_2}{\rho_{1,2,3,4}} + 2 \text{Li}_3 \left( \frac{\rho_1}{\rho_{1,2,3,4}} \right) \otimes \frac{\rho_2}{\rho_{1,2,3,4}} \right] \cong -2 \text{Li}_3(U) \otimes U, \]

We use the following identity that is easy to verify directly (it holds without any symmetrization)

\[ I_{1,1} \left( \frac{\rho_{1,2,3,4}}{\rho_{3,2,1,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) + I_{1,1} \left( \frac{\rho_{1,2,3,4}}{\rho_{1,2,3,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) + 2 \text{Li}_2(U) \]

\[ = \text{Li}_2 \left( \frac{\rho_1}{\rho_{1,2,3,4}} \right) + \text{Li}_2 \left( \frac{\rho_2}{\rho_{1,2,3,4}} \right) + \text{Li}_2 \left( \frac{\rho_3}{\rho_{1,2,3,4}} \right) + \text{Li}_2 \left( \frac{\rho_4}{\rho_{1,2,3,4}} \right). \]

Since the second and third $I_{1,1}$ terms above get interchanged by (16)(25)(34)(78), applying this identity to the expression in the first square brackets of (31) we get that (31) is equal to

\[ I_{2,1} \left( \frac{\rho_{1,2,3,4}}{\rho_{3,2,1,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) + I_{2,1} \left( \frac{\rho_{1,2,3,4}}{\rho_{1,2,3,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) + 2 \text{Li}_2(U) \]

\[ - \text{Li}_3 \left( \frac{\rho_{1,2,3,4}}{\rho_{1,2,3,4}} \otimes \frac{\rho_1}{\rho_{1,2,3,4}} \right) \otimes \frac{\rho_2}{\rho_{1,2,3,4}} + 2 \text{Li}_3 \left( \frac{\rho_1}{\rho_{1,2,3,4}} \right) \otimes \frac{\rho_2}{\rho_{1,2,3,4}} \right] \cong -2 \text{Li}_3(U) \otimes U, \]
We claim that the three crossed-out terms vanish under \( Alt_8 \). Indeed, the first term is
\[
\text{Li}_2 \left( \frac{p_2}{p_4} \right) \otimes U \otimes U = \text{Li}_2 \left( \frac{p_2}{p_4} \right) \otimes \left[ \frac{p_1}{p_{1,2}} \otimes \frac{p_1}{p_{1,2}} + \frac{p_1}{p_{1,2}} \otimes \frac{p_3,3}{p_{3,4}} + \frac{p_3,3}{p_{3,4}} \otimes \frac{p_1}{p_{1,2}} + \frac{p_1}{p_{1,2}} \otimes \frac{p_3,3}{p_{3,4}} \right].
\]
Here the term \( \text{Li}_2 \left( \frac{p_2}{p_4} \right) \otimes \frac{p_1}{p_{1,2}} \otimes \frac{p_3,3}{p_{3,4}} \) vanishes after \( Alt_8 \) since the even permutation \((2\,6)(3\,5)\) changes its sign, and the other three summands vanish by \((29)\). More generally, \((29)\) shows that \( Alt_8 \langle f(p_1, p_2, p_4) \rangle \otimes U = 0 \), and by applying the symmetry \((1\,6)(2\,5)(3\,4)(7\,8)\) we see also that \( Alt_8 \langle f(p_1, p_3, p_4) \rangle \otimes U = 0 \). This immediately gives us the vanishing of the third crossed-out term \( \text{Li}_2 \left( \frac{p_2}{p_{1,2}} \right) \otimes p_{3,4} \otimes U \), and for the second crossed-out term we have
\[
\text{Li}_2 \left( \frac{p_1}{p_{1,4}} \right) \otimes \frac{p_1,3,3}{p_{1,2,3,4}} \otimes U \overset{\text{Alt}_8}{=} \text{Li}_2 \left( \frac{p_1}{p_{1,4}} \right) \otimes \frac{p_2}{p_{1,2}} \otimes U + \text{Li}_2 \left( \frac{p_1}{p_{1,4}} \right) \otimes \frac{p_1,3}{p_{3,4}} \otimes U \overset{\text{Alt}_8}{=} 0.
\]
Next, we apply the five-term relation
\[
\text{Li}_2 \left( \frac{p_2,3,3}{p_{2,3,4}} \right) = \text{Li}_2 \left( \frac{p_2,3,3}{p_{1,3,4}} \right) + \text{Li}_2 \left( \frac{p_2,4,3}{p_{3,4}} \right) + \text{Li}_2 \left( \frac{p_1,4,3}{p_{1,3,4}} \right)
\]
and order the resulting expression by the \( \text{Li}_2 \) terms:
\[
\text{Li}_2 \left( \frac{p_1}{p_{1,2}} \right) \otimes \left[ U \otimes U - 2p_{3,2} \otimes U + 2 \frac{p_1,2}{p_1} \otimes p_{3,2} \right] + \text{Li}_2 \left( \frac{p_2,4,3}{p_{3,4}} \right) \otimes \left[ U \otimes U + \frac{p_1,4,3}{p_{1,3,4}} \right]
\]
\[
- 2 \frac{p_2}{p_{2,1}} \otimes U - \frac{p_1,2,3,4}{p_{3,2,1,4}} \otimes \frac{p_4}{p_1} - 2 \frac{p_3,2}{p_{3,4}} \otimes \frac{p_2}{p_{2,1}} - \frac{p_4}{p_1} U \otimes \frac{p_1,4,3}{p_{1,3,4}} + \frac{p_1,4,4}{p_{1,4,4}} U - 2 \frac{p_3,2}{p_{3,4}} \otimes \frac{p_4}{p_1} \overset{\text{Alt}_8}{=} 0,
\]
where we again have used \((29)\). Here \( \text{Li}_2 \left( \frac{p_2,3,3}{p_{1,3,4}} \right) \otimes p_{1,4} \otimes \frac{p_4}{p_1} \overset{\text{Alt}_8}{=} 0 \), since after expanding out \( \text{Li}_2 \left( \frac{p_2,3,3}{p_{1,3,4}} \right) \) all the terms cancel by transpositions except for the term
\[
\Delta(2348) \wedge \Delta(3458) \otimes p_{1,4} \otimes \frac{p_4}{p_1},
\]
which vanishes under \( Alt_8 \) since it is fixed by the odd permutation \((1\,6)(2\,5)(3\,4)\). Thus, in view of Lemma \((29)\) it is enough to prove the following identity
\[
\text{Li}_2 \left( \frac{p_2,4,3}{p_{3,4}} \right) \otimes \frac{p_4}{p_{1,2}} \otimes \frac{p_1,2,3,4}{p_{3,2,1,4}} - \frac{p_1,2,3,4}{p_{3,2,1,4}} \otimes \frac{p_4}{p_1} - 2 \frac{p_2}{p_{2,1}} \otimes \frac{p_1,3,3}{p_{1,2,3,4}} - 2 \frac{p_3,2}{p_{3,4}} \otimes \frac{p_2}{p_{2,1}} \overset{\text{Alt}_8}{=} 0.
\]
To prove \((34)\) we decompose it into parts that are symmetric and skew-symmetric in the last two tensor positions.

**Skew-symmetric part.** For the skew-symmetric part of \((34)\) we need to show that
\[
\text{Li}_2 \left( \frac{p_2,4,3}{p_{3,4}} \right) \otimes \left[ - \frac{p_2}{p_{2,1}} \wedge \frac{p_1}{p_{1,2}} + 4 \frac{p_1,2}{p_1} \wedge \frac{p_4}{p_1} - \rho_1 \wedge \rho_2 \right] + \text{Li}_2 \left( \frac{p_1}{p_{1,2}} \right) \otimes (p_{3,2} \wedge p_{3,4}) \overset{\text{Alt}_8}{=} 0.
\]
Here the term with \( p_1 \wedge p_4 \) cancels for the same reason as the term \( \text{Li}_2 \left( \frac{p_2,4,3}{p_{3,4}} \right) \otimes p_{1,4} \otimes \frac{p_4}{p_1} \) above. Using \((29)\)
and the mod-products symbol for \( \text{Li}_2 \) we get
\[
\text{Alt}_8 \left[ (p_{3,2} \wedge p_{3,4}) \otimes - \frac{p_2}{p_{2,1}} + 4 \frac{p_1,2}{p_1} \wedge \frac{p_4}{p_1} + \text{Li}_2 \left( \frac{p_1}{p_{1,2}} \right) \otimes (p_{3,2} \wedge p_{3,4}) \right].
\]
Since the original expression that we are computing (i.e., $S^\omega$ applied to the RHS of (30) lies in the image of the projector $D_4$, and since symmetrizing $D_4(a \otimes b \otimes c \otimes d)$ in the last two tensor positions results in $(a \wedge b) \wedge (c \wedge d)$, we see that the above expression is skew-symmetric under interchanging the first two and the last two tensors. Thus we need to show

$$(p_{3,2} \wedge p_{3,4}) \otimes \left[ -2 \text{Li}_2 \left( \frac{p_1}{p_2} \right) + 4 p_{1,2} \wedge \frac{\partial}{\partial p_1} \right] \overset{\text{Alt}_8}{=} 0.$$ 

The crossed-out term vanishes by

$$(p_{3,2} \wedge p_{3,4}) \otimes (p_{1,2} \wedge p_1) \overset{\text{Alt}_8}{=} -(p_{3,2} \wedge p_{2,1}) \otimes (p_{4,3} \wedge p_1)$$

$$\overset{\text{Alt}_8}{=} 14(\Delta(2345) \wedge \Delta(3456)) \otimes (\Delta(1234) \wedge \Delta(4567)) \overset{\text{Alt}_8}{=} 0,$$

where in the first equality we have applied the permutation $(16)(25)(34)$, in the second we have used (30), and in the third we have used the fact that the term is fixed by an odd permutation $(17)(26)(35)$. Thus we only need to prove

$$(p_{3,2} \wedge p_{3,4}) \otimes \left[ p_{1,2} \wedge p_1 + p_{1,2} \wedge p_2 - p_1 \wedge p_2 \right] \overset{\text{Alt}_8}{=} 0.$$ 

We claim that

$$\text{Alt}_8 \left[ p_{4,3} \otimes p_{3,2} \otimes p_{1,2} \otimes p_2 \right] = \text{Alt}_8 \left[ \Delta(1234) \otimes \Delta(2345) \otimes \Delta(3456) \otimes \Delta(4567) \otimes \Delta(4567) \right] - 13\Delta(1234) \otimes \Delta(2345) \otimes \Delta(4567) \otimes \Delta(3456).$$

Taking the skew-symmetrization of (30) and (36) in the last two tensor positions we obtain (35). We can prove (30) by direct expansion using the notation from the proof of Theorem 4

$$p_{4,3} \otimes p_{3,2} \otimes p_{1,2} \otimes p_2 = A_1 D_3 \otimes A_2 D_2 \otimes B_1 D_1 C_2 C_1 \otimes B_1 C_1 \otimes B_2 C_1.$$ 

Omitting any terms that are invariant under odd permutations we get

$$A_3 \otimes A_2 \otimes \frac{\partial}{\partial p_1} \otimes B_2 + D_3 \otimes D_2 \otimes \frac{\partial}{\partial p_1} \otimes B_2 + C_2 A_2 D_2 \otimes B_1 \otimes B_2 - C_2 \otimes C_1 \otimes B_1 \otimes B_2$$
$$+ A_3 \otimes C_2 \otimes D_1 \otimes C_1 + D_3 \otimes C_2 \otimes A_1 \otimes C_1 - C_3 \otimes C_2 \otimes A_1 D_1 \otimes C_1 - C_3 \otimes C_2 \otimes C_1 \otimes B_2,$$

where $C_3 \otimes C_2 \otimes C_1 \otimes B_2$ is equivalent to $\Delta(1234) \otimes \Delta(2345) \otimes \Delta(3456) \otimes \Delta(4567)$, while the other 13 terms are equivalent to $\Delta(1234) \otimes \Delta(2345) \otimes \Delta(4567) \otimes \Delta(3456)$.

**Symmetric part.** For the symmetric part we need to prove

$$\text{Li}_2 \left( \frac{p_{2,4}}{p_{3,4}} \right) \otimes \left[ \frac{p_1}{p_1} \otimes \frac{p_{1,2}}{p_{1,2}} - 2 \frac{p_{3,2}}{p_{3,2}} \otimes \frac{p_2}{p_{2,1}} \right] \overset{\text{Alt}_8}{=} 2 \text{Li}_2 \left( \frac{p_1}{p_2} \right) \otimes \left[ \frac{p_1}{p_1 \otimes p_{1,2}} \right].$$

where we denote $a \otimes b = a \otimes (b + b \otimes a)$.

**Lemma 30.** The following identities hold

$$\text{Li}_2 \left( \frac{p_{2,4}}{p_{3,4}} \right) \otimes \left[ \frac{p_1}{p_1} \otimes \frac{p_{1,2}}{p_{1,2}} - 2 \frac{p_{3,2}}{p_{3,2}} \otimes \frac{p_2}{p_{2,1}} \right] \overset{\text{Alt}_8}{=} 2 \text{Li}_2 \left( \frac{p_1}{p_2} \right) \otimes \left[ \frac{p_1}{p_1 \otimes p_{1,2}} \right],$$

and

$$\text{Li}_2 \left( \frac{p_{2,4}}{p_{3,4}} \right) \otimes \left[ - \frac{p_2}{p_{2,1}} \otimes \frac{p_1}{p_{1,2}} \right] + \text{Li}_2 \left( \frac{p_1}{p_2} \right) \otimes \left[ \frac{1}{2} p_{3,4} \otimes p_{3,4} - \frac{1}{2} p_{3,2} \otimes p_{3,2} \right] \overset{\text{Alt}_8}{=} 0.$$

**Proof.** First, we prove (35). By rearranging the terms in the first set of square brackets we get an equivalent identity

$$\text{Li}_2 \left( \frac{p_{2,4}}{p_{3,4}} \right) \otimes \left[ \frac{p_4}{p_2} \otimes \frac{p_{2,4}}{p_{3,4}} - \frac{p_{2,1}}{p_{2,1}} \otimes \frac{p_{4}}{p_{1}} \right] \overset{\text{Alt}_8}{=} 2 \text{Li}_2 \left( \frac{p_1}{p_2} \right) \otimes \left[ \frac{p_1}{p_1 \otimes p_{1,2}} \right],$$

where the crossed-out term vanishes since it is anti-invariant under $(26)(35)$. We rewrite

$$- \frac{p_2 p_1}{p_{2,1}} \otimes \frac{p_3}{p_{3,2}} = - \frac{1}{2} \frac{p_2 p_1}{p_{2,1}} \otimes \frac{p_{2,1}^2 p_4}{p_{3,2} p_{3,4}} + \frac{1}{2} \frac{p_2 p_1}{p_{2,1}} \otimes \frac{p_4}{p_{2,1}},$$

where the first term on the right is fixed by (78). Since (78) maps $\text{Li}_2 \left( \frac{p_{2,4}}{p_{3,4}} \right)$ to $\text{Li}_2 \left( \frac{p_{2,2} p_{4}}{p_{3,4} p_{2}} \right)$, we have

$$\text{Li}_2 \left( \frac{p_{2,4}}{p_{3,4}} \right) \otimes \left[ \frac{p_2 p_1}{p_{2,1}} \otimes \frac{p_{2,1}^2 p_4}{p_{3,2} p_{3,4}} \right] \overset{\text{Alt}_8}{=} \frac{1}{2} \left[ \text{Li}_2 \left( \frac{p_{2,4}}{p_{3,4}} \right) - \text{Li}_2 \left( \frac{p_{3,2} p_{4}}{p_{2,1}} \right) \right] \otimes \left[ \frac{p_2 p_1}{p_{2,1}} \otimes \frac{p_{2,1}^2 p_4}{p_{3,2} p_{3,4} p_{2}} \right].$$
and hence we can use the five-term relation
\begin{equation}
\text{Li}_2 \left( \frac{\rho_4}{\rho_3} \right) + \text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) = \text{Li}_2 \left( \frac{\rho_3 \rho_2}{\rho_2 \rho_3} \right) - \text{Li}_2 \left( \frac{\rho_2 \rho_4}{\rho_3} \right) + \text{Li}_2 \left( \frac{\rho_4}{\rho_2} \right)
\end{equation}
to rewrite the LHS of (40) as
\begin{equation}
\frac{1}{4} \left[ \text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) + \text{Li}_2 \left( \frac{\rho_4}{\rho_3} \right) - \text{Li}_2 \left( \frac{\rho_4}{\rho_2} \right) \right] \otimes \frac{\rho_3 \rho_2}{\rho_2^2} \otimes \frac{\rho_2^2 \rho_4}{\rho_3^2 \rho_2} \\
+ \text{Li}_2 \left( \frac{\rho_3 \rho_2}{\rho_2^2} \otimes \frac{1}{2} \rho_2 \rho_1 \otimes \frac{\rho_3 \rho_2}{\rho_2} + 3 \rho_1 \rho_2 \otimes \frac{\rho_4}{\rho_3} \right) - 2 \text{Li}_2 \left( \frac{\rho_1}{\rho_2} \right) \otimes \frac{\rho_1}{\rho_1} \otimes \rho_{3,2}.
\end{equation}

Here the first crossed-out term cancels by (29) and the second crossed-out term cancels since it is invariant under (78). We claim that (49) vanishes as a corollary of the following identities:
\begin{align}
(44) & \quad \text{Li}_2 \left( \frac{\rho_2 \rho_4}{\rho_3,4} \right) \otimes \left[ \rho_2 \otimes \rho_1 \right] \overset{\text{Alt}_s}{=} 0, \\
(45) & \quad \text{Li}_2 \left( \frac{\rho_3}{\rho_1} \right) \otimes \left[ \rho_1 \rho_2,3 \otimes \rho_3,4 \right] - \text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \left[ \rho_3,2 \otimes \rho_2,1 \right] + 2 \text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \left[ \rho_3,2 \otimes \rho_2,1 \right] \overset{\text{Alt}_s}{=} 0, \\
(46) & \quad \text{Li}_2 \left( \frac{\rho_4}{\rho_1} \right) \otimes \left[ \rho_1 \rho_2,2 \otimes \rho_3,4 \right] + \text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \left[ \rho_3,1 \otimes \rho_4 \right] + \text{Li}_2 \left( \frac{\rho_2 \rho_4}{\rho_3,4} \right) \otimes \left[ \rho_1 \rho_1 \rho_2 \right] \overset{\text{Alt}_s}{=} 0, \\
(47) & \quad \text{Li}_2 \left( \frac{\rho_4}{\rho_3} \right) \otimes \left[ \rho_1,2 \otimes \rho_2,3 \right] + 2 \text{Li}_2 \left( \frac{\rho_2 \rho_4}{\rho_3,4} \right) \otimes \left[ \rho_1,2 \otimes \rho_1 \right] \overset{\text{Alt}_s}{=} 0, \\
(48) & \quad \text{Li}_2 \left( \frac{\rho_2 \rho_4}{\rho_3,4} \right) \otimes \left[ \rho_2 \otimes \rho_1 \rho_2,1 \right. \\
(49) & \quad \left. - \rho_1,2 \otimes \rho_1 \right] \overset{\text{Alt}_s}{=} 0, \\
(50) & \quad \text{Li}_2 \left( \frac{\rho_1}{\rho_2} \right) \otimes \left[ \rho_1,2 \otimes \rho_3,4 \right] - 2 \text{Li}_2 \left( \frac{\rho_1}{\rho_2} \right) \otimes \left[ \rho_3,2 \otimes \rho_1 \right] + \text{Li}_2 \left( \frac{\rho_2 \rho_4}{\rho_3,4} \right) \otimes \left[ \rho_1 \rho_1,2 \otimes \rho_4 \right] \overset{\text{Alt}_s}{=} 0.
\end{align}

More precisely, one gets (49) by summing up the above 7 identities with coefficients given by (1,1,1,2,−1/2,−1,1,1). We prove these identities as follows. Relation (44) follows from anti-invariance under (26)(35). Equation (45) after rearranging terms (using cyclic shifts and (16)(25)(34)) becomes
\[\text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \frac{\rho_3 \rho_2}{\rho_2^2} \otimes \frac{\rho_2 \rho_4}{\rho_3 \rho_2} \overset{\text{Alt}_s}{=} 0,\]
which is true by (29). Equation (46), after applying (78) and splitting \(\text{Li}_2 \left( \frac{\rho_2 \rho_4}{\rho_3,4} \right)\) analogously to (41) becomes
\[\frac{1}{2} \left[ \text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) + \text{Li}_2 \left( \frac{\rho_4}{\rho_3} \right) - \text{Li}_2 \left( \frac{\rho_4}{\rho_2} \right) \right] \otimes \frac{\rho_3 \rho_2}{\rho_2^2} \otimes \frac{\rho_2^2 \rho_4}{\rho_3^2 \rho_2} \overset{\text{Alt}_s}{=} 0,
\]
which clearly follows from (42) and (29). Identity (47) is proved completely analogously to (46). Equation (48) follows from (43) and (40) after using the symmetry (16)(25)(34). Identity (49) also follows from (40), (46) and the following identity
\begin{align}
\text{Alt}_s \left[ \rho_3,2 \otimes \rho_3,4 \right] &= \text{Alt}_s \left[ \Delta(1234) \otimes \Delta(43) \Delta(1245) \otimes \Delta(34) \right] \\
&\quad + \Delta(1245) \otimes \Delta(2345) \otimes \Delta(1245) \otimes \Delta(34).
\end{align}

that is easily proved by expanding in \(\Delta(ijk)\). Finally, for equation (50) we use cyclic shift in the second term to rewrite the first two terms of the LHS as
\begin{align}
\text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \left[ \rho_1,2 \otimes \rho_3,4 \right] &= -2 \text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \left[ \rho_1,2 \otimes \rho_1 \right] \overset{\text{Alt}_s}{=} \text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \left[ \rho_1,2 \otimes \rho_3,4 \right], \\
\text{Alt}_s \text{Li}_2 \left( \frac{\rho_4}{\rho_3,4} \right) &\overset{\text{Alt}_s}{=} \text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \left[ \rho_1,2 \otimes \rho_3,4 \right],
\end{align}
where we used the permutation (16)(25)(34) and (29) in the second equality, and (78) in the third. Using the five-term relation (42) we rewrite
\[\text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \left[ \rho_1 \otimes \rho_4 \right] \overset{\text{Alt}_s}{=} -\frac{1}{2} \text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \left[ \rho_1 \otimes \rho_4 \right],\]
and thus, in view of (50), equation (50) follows from the following identity
\begin{equation}
\text{Li}_2 \left( \frac{\rho_3}{\rho_2} \right) \otimes \left[ \rho_1 \otimes \rho_4 \right] \overset{\text{Alt}_s}{=} 28 \left( \Delta(34) \otimes \Delta(34) \right) \otimes \Delta(1245) \otimes \Delta(34).
\end{equation}
that we verify by direct expansion with the following simplifications. Since \( \frac{\rho_2}{\rho_2} = \text{cr}(34|2578)^{-1} \), we can expand \( \text{Li}_2\left(\frac{\rho_2}{\rho_2}\right) \) as

\[
\text{Li}_2\left(\frac{\rho_2}{\rho_2}\right) = \frac{1}{2} \text{Alt}_{(2,5,7,8)} \Delta(3425) \wedge \Delta(3427)
\]

Since the above expression is fixed by (34) and (1 6), we expand \( \frac{\rho_2}{\rho_1} \otimes \frac{\rho_2}{\rho_1} - \frac{1}{2} \rho_1 \otimes \rho_4 \) as

\[
\Delta(2348) \otimes \Delta(4567) - \Delta(1237) \otimes \Delta(3456) + \Delta(1237) \otimes \Delta(3458) - \Delta(1237) \otimes \Delta(4578)
\]

\[
+ \Delta(1234) \otimes \Delta(4578) - \Delta(1234) \otimes \Delta(4567) - \Delta(2378) \otimes \Delta(4567) + \Delta(2378) \otimes \Delta(4565)
\]

\[
+ \frac{1}{2} \Delta(1237) \otimes \Delta(4567) - \Delta(1238) \otimes \Delta(4568) + \frac{1}{2} \Delta(1238) \otimes \Delta(4567)
\].

Here we did not include the terms that are invariant under (34) or (1 6). Moreover, the three terms on the last line vanish under skew-symmetrization after multiplying by \( \text{Li}_2\left(\frac{\rho_2}{\rho_2}\right) \). After this we simply expand the remaining expression and collect the terms modulo \( \text{Alt}_8 \). This proves (52) and thus (53) and (54).

Next, we prove (30), the second claim of the lemma. By applying the permutation (1 6)(2 5)(3 4) and cyclic shifts to the last two terms in (30) we see that (30) equals (mod \( \text{Alt}_8 \))

\[
\text{Li}_2\left(\frac{\rho_2}{\rho_3,4} \otimes \left[\frac{- \rho_2}{\rho_2,1} \otimes \frac{\rho_1}{\rho_1,2} - \frac{1}{2} \rho_1 \otimes \rho_2 \right]\right) = \text{Li}_2\left(\frac{\rho_2}{\rho_3,4} \otimes \left[\frac{1}{2} \rho_1,2 \otimes \rho_1,2\right]\right) - \text{Li}_2\left(\frac{\rho_2}{\rho_3,4} \otimes \left[\frac{1}{2} \rho_1,2 \otimes \rho_1,2\right]\right).
\]

Using the five-term relation (42) and noting that \( f(\rho_1, \rho_2, \rho_4) \equiv 0 \) and that \( \text{Li}_2(\frac{\rho_2}{\rho_3,4}) \) is equivalent to \( \text{Li}_2(\frac{\rho_2}{\rho_3,4}) \otimes \frac{1}{2} \rho_1 \otimes \rho_2 \), we get that (53) is equal to

\[
\text{Li}_2\left(\frac{\rho_2}{\rho_3,4} \otimes \left[\rho_1 \otimes \rho_1 + \rho_2 \otimes \rho_2\right]\right) \equiv 0 \mod \text{Alt}_8.
\]

This in turn, follows from

\[
(\rho_1 \otimes \rho_1 + \rho_2 \otimes \rho_2) \equiv 0 \mod \text{Alt}_8,
\]

which we now check by direct expansion. In what follows we cancel terms by transpositions:

\[
\frac{\Delta(3456) \Delta(4578)}{\Delta(3458) \Delta(4568)} \equiv \frac{\Delta(2345) \Delta(34278)}{\Delta(2348) \Delta(3458)} \equiv \frac{\Delta(1234) \Delta(2378)}{\Delta(1238) \Delta(2348)} \equiv \frac{1}{3} \Delta(1237).
\]

by (12) and (23), then

\[
\frac{\Delta(3456) \Delta(4578)}{\Delta(3458) \Delta(4568)} \equiv \frac{\Delta(2345) \Delta(34278)}{\Delta(2348) \Delta(3458)} \equiv \frac{\Delta(1234) \Delta(2378)}{\Delta(1238) \Delta(2348)} \equiv \frac{1}{3} \Delta(1237).
\]

by (23) and (12), then

\[
\frac{\Delta(3456) \Delta(4578)}{\Delta(3458) \Delta(4568)} \equiv \frac{\Delta(2345) \Delta(34278)}{\Delta(2348) \Delta(3458)} \equiv \frac{\Delta(1234) \Delta(2378)}{\Delta(1238) \Delta(2348)} \equiv \frac{1}{3} \Delta(1237).
\]

by (45) and (5 6), and finally (note that the two terms \( \Delta(3458) \otimes \Delta(2348) \) cancel out)

\[
\frac{\Delta(3456) \Delta(2348) + \Delta(3458) \Delta(2378)}{\Delta(3458) \Delta(2378)} \equiv \frac{1}{3} \Delta(1237).
\]

by (5 6) and (45).

Combining (38) and (39) gives (37), and hence concludes the proof of Theorem 9.

9. PROOF OF THEOREM 15

We will use the identity from Theorem 9

\[
7 \over 144 \text{Gr}_4 \equiv \text{Alt}_8 \left[ I_{3,1} \left(\frac{\rho_1,2 \rho_3,4}{\rho_3,2 \rho_1,4}, \frac{\rho_1}{\rho_1,2}\right) + 2I_{3,1} \left(\frac{\rho_1,2 \rho_3,2}{\rho_1,2 \rho_3,4}, \frac{\rho_1}{\rho_1,2}\right) + 6 \text{Li}_4 \left(\frac{\rho_1 \rho_3,2,3}{\rho_1,2 \rho_3,4}\right)\right].
\]

Observe that the \( \text{Li}_4 \) term in the above expression corresponds exactly to the single \( \text{Li}_4 \) term in (10), and also that the two \( \text{Sym}_{16} \) terms correspond to passing from \( \text{I}_{3,1} \) to \( \text{I}_{3,1} \) using Proposition (18). Thus to prove the theorem it is enough to establish the following two lemmas.
Lemma 31. We have
\[
\text{Alt}_8 \left[ \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right) \right] = \text{Alt}_8 \left[ \begin{array}{c} \frac{\rho_1}{\rho_4} \\ \frac{\rho_3}{\rho_4} \\ \frac{\rho_3}{\rho_1} \end{array} \right] - \left[ \begin{array}{c} \frac{\rho_1}{\rho_4} \\ \frac{\rho_3}{\rho_4} \\ \frac{\rho_3}{\rho_1} \end{array} \right] + [34|2685; 48|7653] - \frac{1}{4}[43|1256; 43|1268] + \frac{1}{2}[43|1256; 42|1365].
\]

Lemma 32. We have
\[
\text{Alt}_8 \left[ \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right) + \tilde{I}_{3,1}(\text{cr}(34|2567), \text{cr}(67|1345)) \right] = \text{Alt}_8 \left[ \begin{array}{c} \frac{\rho_1}{\rho_4} \\ \frac{\rho_3}{\rho_4} \\ \frac{\rho_3}{\rho_1} \end{array} \right] - \frac{1}{8}[34|2685; 48|7653] + \frac{1}{4}[48|7235; 48|7263] + \frac{1}{8}[46|5238; 43|2568].
\]

Proof of Lemma 31. First, using a five-term relation equivalent to (53) we get
\[
\tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right) = \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right)
\]
\[
= \text{V} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right) - \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right) - \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right) - \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right)
\]
\[
= \text{Alt}_8 \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right) - 2 \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right).
\]
Here the last two terms on the second line vanish under Alt$_8$, since they are fixed by (23) and (12) respectively, and using the involution (16)(25)(34) and the 6-fold symmetry of $\tilde{I}_{3,1}$ we see that the term $\tilde{I}_{3,1}(\frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1})$ is Alt$_8$-equivalent to $\tilde{I}_{3,1}(\frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1})$. Note that
\[
\frac{\rho_3}{\rho_4} = \text{cr}(34|2685) / \text{cr}(48|7653).
\]
Thus, if we denote $r_1 = \text{cr}(34|2685)$, $r_2 = \text{cr}(48|7653)$, then
\[
\text{V} \left( \frac{\rho_1}{\rho_4}, r_1, r_2 \right) = \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_1}{\rho_4} - [r_1] + [r_2] - [(1 - \sigma_{56}(r_0))^{-1}] - [1 - r_0] - [(1 - r_3)^{-1}] \right)
\]
\[
= \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, [r_0] - [\sigma_{56}(r_0)] - [r_1] + [r_2] - [r_3] \right),
\]
where we denote $r_0 = \frac{\rho_3}{\rho_4}$, $r_3 = \frac{\rho_3}{\rho_1}$. as and before $\sigma_{56}$ denotes the action of $\pi \in S_8$. Since $\sigma_{56}(r_3) = r_3$ and $\sigma_{56}(r_2) = r_2$ (and these two involutions fix $\rho_1/\rho_4$) the last two terms in (56) cancel out after skew-symmetrization, and hence we obtain
\[
-2 \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1} \right) \overset{\text{Alt}_8}{=} \text{V} \left( \frac{\rho_1}{\rho_4}, r_1, r_2 \right).
\]
For $\tilde{I}_{3,1}(\frac{\rho_1}{\rho_4}, \text{cr}(34|2685))$ we use the following five-term identities:
\[
\text{V} \left( \frac{\rho_1}{\rho_4}, \text{cr}(43|1256), \text{cr}(42|1365) \right)
\]
\[
= \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, [43|1256] + [42|1365] + [45|1326] + [41|3526] + [46|1235] \right)
\]
\[
= \text{Alt}_8 \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, 3[43|1256] \right),
\]
\[
\text{V} \left( \frac{\rho_1}{\rho_4}, \text{cr}(43|1256), \text{cr}(43|1268) \right)
\]
\[
= \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, [43|1256] + [43|1268] + [43|1586] + [43|1528] + [43|2856] \right)
\]
\[
= \text{Alt}_8 \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, [43|1256] + 2[43|1268] + 2[43|2856] \right) \overset{\text{Alt}_8}{=} \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, 4[43|2856] \right),
\]
where we have used the fact that $\frac{\rho_1}{\rho_4}$ is fixed by arbitrary permutations of $\{1, 2, 3\}$ and $\{4, 5, 6\}$, and on the last line we have used the fact that it is mapped to its inverse under the involution (16)(25)(34). From this we obtain
\[
\tilde{I}_{3,1}(\frac{\rho_1}{\rho_4}, r_1) \overset{\text{Alt}_8}{=} \text{V} \left( \frac{\rho_1}{\rho_4}, \frac{1}{4}[43|1256; 43|1268] - \frac{1}{16}[43|1256; 42|1365] \right),
\]
which together with (53) and (57) proves the claim. 

Proof of Lemma 32. We start by rewriting $\tilde{I}_{3,1}(\frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1}) = \tilde{I}_{3,1}(\frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_4}, \frac{\rho_3}{\rho_1})$. Using (53) and the same five-term relation as in (56) we get
\[
\text{V} \left( \frac{\rho_1}{\rho_2}, r_1, r_2 \right) = \tilde{I}_{3,1} \left( \frac{\rho_1}{\rho_4}, [r_0] - [\sigma_{56}(r_0)] - [r_1] + [r_2] - [r_3] \right),
\]
where as before $r_0 = \frac{\rho_3}{\rho_2}$, $r_1 = \text{cr}(34|2685)$, $r_2 = \text{cr}(48|7635)$, and $r_3 = \frac{\Delta(2346)\Delta(4578)}{\Delta(2345)\Delta(3467)}$. Since $\sigma_{(23)}(r_3) = r_2$ and $\sigma_{(23)}(\rho_i) = \rho_i$ for $i = 1, 2$, the term with $r_3$ vanishes after skew-symmetrization, and we obtain

$$\tilde{I}_{3,1}(\frac{\rho_2}{\rho_1},\frac{\rho_3}{\rho_1}) = \frac{1}{2} V\left(\frac{\rho_1}{\rho_2} : r_1, r_2^{-1}\right) + \frac{1}{2} \tilde{I}_{3,1}(\frac{\rho_3}{\rho_1}, \text{cr}(34|2685)) - \frac{1}{2} \tilde{I}_{3,1}(\frac{\rho_2}{\rho_1}, \text{cr}(48|7635)) .$$

Since $\frac{\rho_i}{\rho_1} = \text{cr}(23|1487)$, we can rewrite the remaining combination of $\tilde{I}_{3,1}$’s (modulo $\text{Alt}_8$) together with the term $\tilde{I}_{3,1}(\text{cr}(34|2567), \text{cr}(67|1345))$ as

$$\tilde{I}_{3,1}(\frac{\rho_2}{\rho_1}, \frac{1}{2}[34|2685] - \frac{1}{2}[48|7635] + [48|7235]).$$

To express it in terms of $V$ we will use two five-term relations. The first is

$$V\left(\frac{\rho_2}{\rho_1}; \text{cr}(48|7235), \text{cr}(48|7263)\right) = \tilde{I}_{3,1}(\frac{\rho_2}{\rho_1}, [48|7235] + [48|7263] + [48|2563] + [48|2576] + [48|3567])$$

$$\text{Alt}_8 \tilde{I}_{3,1}(\frac{\rho_2}{\rho_1}, 2[48|7235] - 2[48|7635] + [48|2356]) = \text{Alt}_8 \tilde{I}_{3,1}(\frac{\rho_2}{\rho_1}, 2[48|7235] - 2[48|7635]),$$

where we have used the fact that the permutations $(23)$ and $(56)$ both leave $\frac{\rho_2}{\rho_1}$ fixed, and the term $\tilde{I}_{3,1}(23|1487), [48|2356]$ vanishes after skew-symmetrization, since the even permutation $(15)(24)(38)(67)$ maps it to $\tilde{I}_{3,1}(48|5236, 23|4817)$ that is equal to $-\tilde{I}_{3,1}(23|1487), [48|2356]$ by Proposition 23. For the second five-term relation we use the symmetry $\tilde{I}_{3,1}(x, y) = -\tilde{I}_{3,1}(y, x)$ together with 36-fold symmetry for $\tilde{I}_{3,1}$ to rewrite

$$\tilde{I}_{3,1}(23|1487), [48|7635]) = -\tilde{I}_{3,1}([48|7635], [23|1487]) \text{ Alt}_8 \tilde{I}_{3,1}(23|1487), [46|5238]),$$

where in the second equality we have used the cyclic permutation $(15783426)$. Then

$$V\left(\frac{\rho_2}{\rho_1}; \text{cr}(46|5238), \text{cr}(43|2568)\right) = \tilde{I}_{3,1}(\frac{\rho_2}{\rho_1}, [46|5238] + [43|2568] + [42|3568] + [48|5326] + [45|2638])$$

$$\text{Alt}_8 \tilde{I}_{3,1}(\frac{\rho_2}{\rho_1}, 2[46|5238] + 2[34|2685] - [48|2356]) = \text{Alt}_8 \tilde{I}_{3,1}(\frac{\rho_2}{\rho_1}, 2[46|5238] + 2[34|2685]),$$

where we have used again that $\tilde{I}_{3,1}(23|1487), [48|2356]$ vanishes under skew-symmetrization. Combining the two five-term relations together with (59) we get

$$\tilde{I}_{3,1}(\frac{\rho_2}{\rho_1}, \frac{1}{2}[34|2685] - \frac{1}{2}[48|7635] + [48|7235]) \text{ Alt}_8 V\left(\frac{\rho_2}{\rho_1}; \frac{1}{2}[48|7235] + [48|7635] + \frac{1}{2}[46|5238] + [43|2568]\right),$$

which together with (58) proves the claim. \qed

10. Proof of Proposition 26

The decomposition of (13) in Equation (17) into the $I_{4,1}^+$ subsums is a direct rewriting of the $I_{4,1}$ expression. It is clear that the last subsum is a trivial coboundary piece: every summand depends only on 9 of the 10 points. The claim will follow from the following lemmas which express the first three subsums as combinations of $\text{Li}_2$ and $\text{Li}_3$ functional equations.

We will write

$$V(x, y) = [x] + [y] + \left[\frac{1 - x}{1 - xy}\right] + [1 - xy] + \left[\frac{1 - y}{1 - xy}\right]$$

for the dilogarithm five-term relation in its 5-cyclic form.

Without loss of generality, we can replace $I_{4,1}$ with $\tilde{I}_{4,1}$, since they are equal up to explicit $\text{Li}_5$ terms.

**Lemma 33.** The combination

$$\text{Alt}_{10} \left[ I_{4,1}^+ \left( \frac{\rho_3}{\rho_2}, \frac{\rho_4}{\rho_1} \right) - I_{4,1}^+ \left( \frac{\rho_4}{\rho_3}, \frac{\rho_3}{\rho_2} \right) \right]$$

vanishes identically.

**Lemma 34.** We have

$$\text{Alt}_{10} \left[ 2 \tilde{I}_{4,1}^+ \left( \frac{\rho_1}{\rho_4}, \frac{\rho_3}{\rho_2}, \frac{\rho_4}{\rho_1} \right) \right]$$

$$= \text{Alt}_{10} \left[ \tilde{I}_{4,1}^+ \left( -2V \left( \frac{\rho_3}{\rho_2}, \frac{\rho_4}{\rho_3} \right) - V([340|9625], [234|1650]) + V([450|9736], [345|2760]) \right) \right.$$

$$+ \frac{1}{3} V([340|9625], [340|9657]) + \frac{1}{3} V([345|2706], [347|2560], \frac{\rho_4}{\rho_1}) \right].$$
We first note a useful \( L_{i3} \) functional equation which will enter as part of the reduction.

**Lemma 35.** Let

\[
\mathcal{T} := 3 \left[ \frac{\rho_{1,2}\rho_{3,4}}{\rho_{1,4}\rho_{3,2}} \right] - \left[ 3[r_3(14|259,370)] - 3[r_3(54|179,260)] + [r_3(14|259,369)] + [r_3(14|259,376)] - [r_3(94|150,276)] - [r_3(94|157,260)] - [cr(154|2769)] + [cr(194|2570)] \right].
\]

Then following is a \( L_{i3} \) functional equation

\[
\text{Alt}_{\xi(1,2,3) \times \xi(5,6,7)} L_{i3}(\mathcal{T}) \equiv 0.
\]

Currently, we do not reduce the third orbit directly to the 22-term, or the 840-term \( L_{i3} \) functional equation. Instead we invoke more general functional equations to simplify the reduction for the moment.

**Lemma 36.** The combination

\[
\text{Alt}_{10} \left[ 2\tilde{I}^+_{4,1} \left( \frac{\rho_{1,3}}{\rho_{4,1}}, \frac{\rho_{2,3}}{\rho_{4,1}} \right) + 2\tilde{I}^+_{4,1} \left( \frac{\rho_{4,3}}{\rho_{1,4}}, \frac{\rho_{1,2}\rho_{3,4}}{\rho_{1,4}\rho_{3,2}} \right) + \tilde{I}^+_{4,1} \left( \frac{\rho_{1,2}}{\rho_{1}}, \frac{\rho_{1,2}}{\rho_{1}} \right) + 2\tilde{I}^+_{4,1} \left( \frac{\rho_{1,2}}{\rho_{1}}, \frac{\rho_{3,2}}{\rho_{3,2}} \right) \right]
\]

\[
- \frac{4}{3} \tilde{I}^+_{4,1} \left( \frac{\rho_{1,2}}{\rho_{1,4}}, \frac{\rho_{1,2}}{\rho_{1}} \right) + \frac{5}{3} \tilde{I}^+_{4,1} \left( \text{cr}(346[1279], v_3[12|345,678]) \right)
\]

can be decomposed into a sum of the form

\[
\text{Alt}_{10} \left[ \sum_i \xi_i \tilde{I}^+_{4,1}(\xi_i, y_i) + \sum_j \lambda_j \tilde{I}^+_{4,1}(x_j, A_j) \right]
\]

where \( \xi_i \) are \( L_{i2} \) functional equations, and \( A_j \) are \( L_{i3} \) functional equations.

**Proof of Lemma 36.** The involution \((18)(27)(36)(45)\) of signature +1, induces the map \( \rho_i \mapsto \rho_{6-i}, \ i = 1, \ldots, 5 \). Under this, the second summand maps exactly to the first, and the combination equals

\[
\text{Alt}_{10} \left[ \tilde{I}^+_{4,1} \left( \frac{\rho_{3,4}}{\rho_{2,3}}, \frac{\rho_{3,4}}{\rho_{2,3}} \right) - \tilde{I}^+_{4,1} \left( \frac{\rho_{3,4}}{\rho_{1,4}}, \frac{\rho_{3,4}}{\rho_{1,4}} \right) \right]
\]

\[
= \text{Alt}_{10} \left[ (1 - \sigma(18)(27)(36)(45))I^+_{4,1} \left( \frac{\rho_{2,3}}{\rho_{2,1}}, \frac{\rho_{4,3}}{\rho_{4,1}} \right) \right]
\]

\[
= 0.
\]

**Proof of Lemma 37.** Choosing \( x = \frac{\rho_{2,1}}{\rho_{2,3}}, \ y = \frac{\rho_{3,4}}{\rho_{4,1}} \), we obtain

\[
\tilde{I}^+_{4,1} \left( V \left( \frac{\rho_{2,1}}{\rho_{2,3}}, \frac{\rho_{3,4}}{\rho_{4,1}}, \frac{\rho_{3,4}}{\rho_{4,1}} \right) \right) = \tilde{I}^+_{4,1} \left( \frac{\rho_{2,1}\rho_{2,3}}{\rho_{2,1}\rho_{2,3}}, \frac{\rho_{4,5}}{\rho_{4,1}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right) + \tilde{I}^+_{4,1} \left( \frac{\rho_{2,1}}{\rho_{2,3}}, \frac{\rho_{4,5}}{\rho_{4,1}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right) + \tilde{I}^+_{4,1} \left( \frac{\rho_{3,4}}{\rho_{3,4}}, \frac{\rho_{4,5}}{\rho_{4,1}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right)
\]

\[
+ \tilde{I}^+_{4,1} \left( \frac{\rho_{3,4}}{\rho_{3,4}}, \frac{\rho_{4,5}}{\rho_{4,1}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right) + \tilde{I}^+_{4,1} \left( \frac{\rho_{3,4}}{\rho_{3,4}}, \frac{\rho_{4,5}}{\rho_{4,1}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right)
\]

Observe that, under the six-fold symmetry in the first argument, we have

\[
\tilde{I}^+_{4,1} \left( \frac{\rho_{1,3}\rho_{2,4}}{\rho_{1,4}\rho_{2,3}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right) = -\tilde{I}^+_{4,1} \left( \frac{\rho_{1,2}\rho_{3,4}}{\rho_{1,4}\rho_{3,2}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right).
\]

Moreover, note that the two \( \tilde{I}^+_{4,1} \) terms on the second line are invariant under the transposition \((2,3)\) and \((1,2)\), respectively. Hence under \( \text{Alt}_{10} \) they vanish identically. Overall

\[
2\tilde{I}^+_{4,1} \left( \frac{\rho_{1,2}\rho_{3,4}}{\rho_{1,4}\rho_{3,2}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right) \text{Alt}_{10} = -2\tilde{I}^+_{4,1} \left( V \left( \frac{\rho_{2,1}}{\rho_{2,3}}, \frac{\rho_{3,4}}{\rho_{4,1}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right) \right) + 2\tilde{I}^+_{4,1} \left( \frac{\rho_{2,1}}{\rho_{2,3}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right) + 2\tilde{I}^+_{4,1} \left( \frac{\rho_{2,3}}{\rho_{2,4}}, \frac{\rho_{4,5}}{\rho_{4,1}} \right)
\]

Note that

\[
\frac{\rho_{2,1}}{\rho_{2,3}} = \frac{\text{cr}(340|9625)}{\text{cr}(342|1605)}
\]

If we write \( r_1 = \text{cr}(340|9625) \), \( r_2 = \text{cr}(342|1605) \), then

\[
\tilde{I}^+_{4,1} \left( V(r_1, r_2^{-1}), \frac{\rho_{4,5}}{\rho_{4,1}} \right) = \tilde{I}^+_{4,1} \left( [r_1] - [r_2] + [1 - r_0] + [(1 - \sigma(5,6)r_0)^{-1}] - [(1 - r_3)^{-1}] \frac{\rho_{4,5}}{\rho_{4,1}} \right)
\]

\[
= \tilde{I}^+_{4,1} \left( -[r_0] + [\sigma(5,6)r_0] + [r_1] - [r_2] + [r_3], \frac{\rho_{4,5}}{\rho_{4,1}} \right)
\]
where \( r_0 = \frac{\rho_{2,3}}{\rho_{3,2}} \) and \( r_3 = \frac{\Delta(12345)\Delta(34690)}{\Delta(1246)\Delta(34590)} \). Notice that \( \sigma_{(23)} r_2 = r_2 \) and \( \sigma_{(12)} r_3 = r_3 \) and both of these involutions fix \( \frac{\rho_{2,3}}{\rho_{3,2}} \). Hence the last two terms in (61) vanish after skew-symmetrization. Since (56) also fixes the second argument, we obtain

\[
-2 I_{4,1}^+ (r_0, \frac{\rho_{4,5}}{\rho_{4,1}})^{\text{Alt}_{10}} = I_{4,1}^+ (V(r_1, r_2^{-1}), \frac{\rho_{4,5}}{\rho_{4,1}}) - \tilde{I}_{4,1}^+ (r_1, \frac{\rho_{4,5}}{\rho_{4,1}})
\]

Next, notice that up to six-fold symmetries

\[
\tilde{I}_{4,1}^+ (\frac{\rho_{2,3}}{\rho_{2,4}}, w) = -\tilde{I}_{4,1}^+ (\frac{\rho_{3,2}}{\rho_{3,4}}, w)
\]

and

\[
\frac{\rho_{3,2}}{\rho_{3,4}} = \sigma_{(1234567)} \frac{\rho_{2,1}}{\rho_{2,3}}.
\]

Unfortunately this permutation does not fix the second argument of \( \tilde{I}_{4,1}^+ \). Nevertheless by applying it the generators of the five-term relation, we immediately obtain

\[
\tilde{I}_{4,1}^+ (V(r_1', (r_2')^{-1}), \frac{\rho_{4,5}}{\rho_{4,1}}) = \tilde{I}_{4,1}^+ (\frac{\rho_{4,5}}{\rho_{4,1}}) + \tilde{I}_{4,1}^+ (r_0', \frac{\rho_{4,5}}{\rho_{4,1}}),
\]

where

\[
\begin{align*}
 r_0' &= \frac{\rho_{3,2}}{\rho_{3,4}}, \\
 r_1' &= \frac{\rho_{3,2}}{\rho_{3,4}}, \\
 r_2' &= \frac{\rho_{3,2}}{\rho_{3,4}}, \\
 r_3' &= \frac{\rho_{3,2}}{\rho_{3,4}}.
\end{align*}
\]

This time \( \sigma_{(12)} r_1' = r_1', \sigma_{(23)} r_2' = r_2' \) and each of the involutions \( (12), (23) \) and \( (67) \) fixes \( \frac{\rho_{3,2}}{\rho_{3,4}} \). Hence after skew-symmetrization we find

\[
-2 I_{4,1}^+ (r_0', \frac{\rho_{4,5}}{\rho_{4,1}})^{\text{Alt}_{10}} = I_{4,1}^+ (V(r_1', (r_2')^{-1}), \frac{\rho_{4,5}}{\rho_{4,1}}) + \tilde{I}_{4,1}^+ (r_2', \frac{\rho_{4,5}}{\rho_{4,1}}).
\]

From (60), (61) and (62) we obtain

\[
2 I_{4,1}^+ (\frac{\rho_{1,2} \rho_{3,4}}{\rho_{1,4} \rho_{3,2}}, \frac{\rho_{4,5}}{\rho_{4,1}})^{\text{Alt}_{10}} = \tilde{I}_{4,1}^+ \left[ 340|9625], \frac{\rho_{4,5}}{\rho_{4,1}} \right] + \tilde{I}_{4,1}^+ \left[ 345|2706], \frac{\rho_{4,5}}{\rho_{4,1}} \right] + \tilde{I}_{4,1}^+ \left[ 450|9736], \frac{\rho_{4,5}}{\rho_{4,1}} \right].
\]

To reduce \( \tilde{I}_{4,1}^+ ([340|9625] + [345|2706], \frac{\rho_{4,5}}{\rho_{4,1}}) \), consider the following five-term relations. Firstly

\[
\tilde{I}_{4,1}^+ (V([340|9625], [340|9657]), \frac{\rho_{4,5}}{\rho_{4,1}}) = \tilde{I}_{4,1}^+ \left[ 340|9625] + [340|9657] + [340|7652] + [340|9267] + [340|9275], \frac{\rho_{4,5}}{\rho_{4,1}} \right] + \tilde{I}_{4,1}^+ \left[ 3|340|9625] + [340|2567], \frac{\rho_{4,5}}{\rho_{4,1}} \right],
\]

using the six-fold anharmonic symmetry, and the invariance of the second argument under arbitrary permutations of \( \{1, 2, 3\} \) and \( \{5, 6, 7\} \). The term containing cross-ratio \([340|9657]\) vanishes because it is invariant under \( (12) \). Then

\[
\tilde{I}_{4,1}^+ (V([345|2706], [347|2560]), \frac{\rho_{4,5}}{\rho_{4,1}})
\]

\[
= \tilde{I}_{4,1}^+ \left[ 345|2706] + [347|2560] + [342|5076] + [340|2576] + [346|2750], \frac{\rho_{4,5}}{\rho_{4,1}} \right]
\]

again by the six-fold symmetry, and by the invariance of the second argument under permutations of \( \{5, 6, 7\} \). The third term vanishes because it is invariant under \( (23) \).

From (63) and (65) we conclude

\[
\tilde{I}_{4,1}^+ ([340|9625] + [345|2706], \frac{\rho_{4,5}}{\rho_{4,1}})
\]
\[ Alt_{10} \frac{1}{3} \tilde{T}^+_4 \left( V([340][9625], [340][9657]) + V([345][2706], [347][2560]), \frac{\rho_{4,5}}{\rho_{4,1}} \right). \]

Together with \( \text{(83)} \) this establishes the claim. \( \square \)

**Proof of Lemma 26.** Let \( \Omega \) denote the combination

\[ \Omega = 2 \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) + 2 \tilde{T}^+_4 \left( \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) + \tilde{T}^+_4 \left( \frac{\rho_{1,4}}{p_{1,4,1}}, \frac{\rho_{1,2,3,4}}{p_{1,2,3,2}} \right) + 2 \tilde{T}^+_4 \left( \frac{\rho_{1,2}}{p_1}, \frac{\rho_{3,4}}{p_{3,4}} \right) \]

\[ - \frac{4}{3} \tilde{T}^+_4 \left( \frac{p_{1,2}}{p_{1,4}}, \frac{p_{2,3,4}}{p_{1,4}} \right) + 5 \tilde{T}^+_4 \left( \frac{\rho_{1,2}}{p_1}, \frac{\rho_{3,4}}{p_{3,4}} \right) \]

First note the following five-term relation (here 0 is not a vector index, but the number)

\[ \tilde{T}^+_4 \left( \frac{\sigma (\rho_{4,0} \rho_0 p_0), \sigma (\rho_{4,0} \rho_0 p_0)}{\rho_{1,4,1} \rho_{1,4,3,2}} \right) = \tilde{T}^+_4 \left( \frac{\sigma (\rho_{4,0} \rho_0 p_0), \sigma (\rho_{4,0} \rho_0 p_0) \sigma (\rho_{4,0} \rho_0 p_0), \sigma (\rho_{4,0} \rho_0 p_0)}{\rho_{2,3,4} \rho_{2,3,2}} \right). \]

Up to six-fold symmetries this is equal to

\[ \tilde{T}^+_4 \left( \frac{\rho_{1,4,1}}{\rho_{1,4,1}}, \frac{\rho_{2,3,4}}{\rho_{2,3,2}} \right) \]

\[ \text{the equality after skew-symmetrization following since } \frac{\rho_{1,2}}{\rho_{1,2}} \text{ is invariant under the transposition } (9,10). \]

We claim now that modulo \( Alt_{10} \) the orbit \( \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \) is a combination of \( L_3 \) functional equations in the second argument. Indeed, since the first argument is invariant under \( \sigma (1,2,3,4) \times \sigma (5,6,7,8) \), we get

\[ \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) = \frac{1}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ + \frac{2}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ + \frac{1}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ \text{the equality after skew-symmetrization following since } \frac{\rho_{1,2}}{\rho_{1,2}} \text{ is invariant under the transposition } (9,10). \]

We claim now that modulo \( Alt_{10} \) the orbit \( \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \) is a combination of \( L_3 \) functional equations in the second argument. Indeed, since the first argument is invariant under \( \sigma (1,2,3,4) \times \sigma (5,6,7,8) \), we get

\[ \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) = \frac{1}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ + \frac{2}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ + \frac{1}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ \text{the equality after skew-symmetrization following since } \frac{\rho_{1,2}}{\rho_{1,2}} \text{ is invariant under the transposition } (9,10). \]

We claim now that modulo \( Alt_{10} \) the orbit \( \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \) is a combination of \( L_3 \) functional equations in the second argument. Indeed, since the first argument is invariant under \( \sigma (1,2,3,4) \times \sigma (5,6,7,8) \), we get

\[ \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) = \frac{1}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ + \frac{2}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ + \frac{1}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ \text{the equality after skew-symmetrization following since } \frac{\rho_{1,2}}{\rho_{1,2}} \text{ is invariant under the transposition } (9,10). \]

We claim now that modulo \( Alt_{10} \) the orbit \( \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \) is a combination of \( L_3 \) functional equations in the second argument. Indeed, since the first argument is invariant under \( \sigma (1,2,3,4) \times \sigma (5,6,7,8) \), we get

\[ \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) = \frac{1}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ + \frac{2}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ + \frac{1}{3} \tilde{T}^+_4 \left( \frac{p_{1,4,5}}{p_{1,4,1}}, \frac{p_{1,2,3,4}}{p_{1,2,3,2}} \right) \]

\[ \text{the equality after skew-symmetrization following since } \frac{\rho_{1,2}}{\rho_{1,2}} \text{ is invariant under the transposition } (9,10). \]
So we find that $\Omega$ reduces to

$$[1 - \sigma_{(67)(90)} \frac{\rho_{1,2}}{\rho_{3,4}}] + \left(1 - \frac{\rho_{1,2}}{\rho_{3,4}} \right)^{-1} + \left(1 - \frac{\Delta(23457)\Delta(45690)}{\Delta(23456)\Delta(45790)} \right)^{-1} \frac{\rho_{1,2}}{\rho_{1}}.$$  

In the former the last term is invariant under (3 4) so vanishes after skew-symmetrization, in the latter it is invariant under (2 3) and so vanishes also. In the latter, the term $[cr(453|0276)]$ also vanishes due to invariance under (3 4). From this we obtain

$$\text{Alt}_{10} \left[ - \frac{4}{3} \tilde{\tau}_{4,1} \left( \frac{\rho_{1,2}}{\rho_{3,4}}, \frac{\rho_{1,2}}{\rho_{1}} \right) + 2 \tilde{\tau}_{4,1} \left( \frac{\rho_{1,2}}{\rho_{1}}, \frac{\rho_{1,2}}{\rho_{3,4}} \right) \right]$$

$$= \text{Alt}_{10} \left[ \tilde{\tau}_{4,1} \left( V(cr(564|0387), cr(560|4987)^{-1}), \frac{\rho_{1,2}}{\rho_{1}} \right) - \frac{2}{3} \tilde{\tau}_{4,1} \left( V(cr(453|0276), cr(450|3976)^{-1}), \frac{\rho_{1,2}}{\rho_{1}} \right) - \left[ cr(564|0387) + cr(560|4987) - \frac{2}{\rho_{1}}[cr(450|3976)] \right] \right].$$

From $\mathcal{T}$, we again obtain

$$\text{Alt}_{10} \left[ \tilde{\tau}_{4,1} \left( \frac{\rho_{1}}{\rho_{5}}, \frac{\rho_{1,2}}{\rho_{1,4}}, \frac{\rho_{3,4}}{\rho_{2,1}} \right) \right]$$

$$= \text{Alt}_{10} \left[ \frac{1}{3} \tilde{\tau}_{4,1} \left( \frac{\rho_{1}}{\rho_{5}}, \frac{\rho_{1,2}}{\rho_{1,4}}, \frac{\rho_{3,4}}{\rho_{2,1}} \right) + \frac{1}{3} \tilde{\tau}_{4,1} \left( \frac{\rho_{1}}{\rho_{5}}, \frac{\rho_{1,2}}{\rho_{1,4}}, \frac{\rho_{3,4}}{\rho_{2,1}} \right) - \frac{1}{3} \tilde{\tau}_{4,1} \left( \frac{\rho_{1}}{\rho_{5}}, \frac{\rho_{1,2}}{\rho_{1,4}}, \frac{\rho_{3,4}}{\rho_{2,1}} \right) \right].$$

We can substitute (66), (67) and (68) into the original combination $\Omega$, and rewrite the remaining arguments in terms of cross-ratios and triple-ratios using

$$\frac{\rho_{1}}{\rho_{2}} = cr(234|1590), \quad \frac{\rho_{1}}{\rho_{5}} = cr(567|4890),$$

$$\frac{\rho_{2,3}}{\rho_{2,1}} = r_{3}([254|1290], 3[254|1290], 3[254|1290]), \quad \frac{\rho_{3,2}}{\rho_{3,1}} = r_{3}(345|2619), \quad \frac{\rho_{2,3}}{\rho_{2,1}} = r_{3}(345|2619).$$

Moreover, we can put the second argument into a canonical form, namely $cr(123|4567)$ or $r_{3}(123|4567)$ respectively, by choosing the inverse of the permutation which maps $\{1, \ldots, 10\}$ to the points which appear in the second argument, and then the complementary points in order of index. Drop, for simplicity, the functional equations appearing in [66], [67] and [68]. Note also that

$$\tilde{\tau}_{4,1} \left( cr(346|1279), 3 \left( \frac{12}{345|678} \right) \right)$$

$$= \tilde{\tau}_{4,1} \left( cr(346|1279), r_{3}(123|4567), 3 \left( \frac{12}{345|678} \right) \right) - \tilde{\tau}_{4,1} \left( cr(346|1279), r_{3}(123|4567), 3 \left( \frac{12}{345|678} \right) \right)$$

$$= \tilde{\tau}_{4,1} \left( cr(346|1279), r_{3}(123|4567), 3 \left( \frac{12}{345|678} \right) \right) - \sigma_{(34)(57)(18)} \tilde{\tau}_{4,1} \left( cr(368|1259), 3 \left( \frac{12}{345|678} \right) \right).$$

So we find that $\Omega$ reduces to

$$\frac{1}{3} \tilde{\tau}_{4,1} \left( cr(256|3970), 3 \left( \frac{12}{345|678} \right) \right) + 2 \left( cr(356|2798), 3 \left( \frac{1}{3} \right) \right) + cr(123|4567)$$

$$+ \tilde{\tau}_{4,1} \left( -2 \left[ cr(136|2958) \right] - cr(147|2058) \right) + cr(123|590] - cr(356|2809)$$

$$+ cr(479|2058) + \frac{5}{3} \left[ cr(346|1279) \right] + \frac{2}{3} \left[ cr(368|1259) \right] - \frac{1}{3} \left[ cr(457|1820) \right]$$

$$- \frac{1}{3} \left[ cr(457|2980) \right] - \frac{1}{3} \left[ cr(478|1520) \right] - \frac{1}{3} \left[ cr(478|2950) \right].$$

Under the automorphisms of $g = r_{3}(123|4567$, including inverting, and the six-fold symmetries, we note the following equalities

$$\sigma_{(34)(67)(90)} \tilde{\tau}_{4,1} \left( cr(136|2958), g \right) = \tilde{\tau}_{4,1} \left( cr(147|2058), g \right),$$

$$\sigma_{(34)(67)(90)} \tilde{\tau}_{4,1} \left( cr(346|1279), g \right) = \tilde{\tau}_{4,1} \left( cr(457|1280), g \right) = - \tilde{\tau}_{4,1} \left( cr(457|1280), g \right),$$

$$\sigma_{(34)(67)(90)} \tilde{\tau}_{4,1} \left( cr(356|2809), g \right) = \tilde{\tau}_{4,1} \left( cr(457|2809), g \right) = \tilde{\tau}_{4,1} \left( cr(457|2809), g \right).$$

We also note, under the automorphisms of $p = cr(123|4567)$, that

$$\sigma_{(23)(80)} \tilde{\tau}_{4,1} \left( cr(256|3970), p \right) = \tilde{\tau}_{4,1} \left( cr(356|2798), p \right) = - \tilde{\tau}_{4,1} \left( cr(356|2798), p \right).$$
So the above combination (69) is $\text{Alt}_{10}$-equivalent to
\begin{align}
\frac{1}{2} \tilde{I}_{4,1}^+ & (\{- \text{cr}(256|3970)\} + \{\text{cr}(569|2730), \text{cr}(123|4567)\})
\end{align}
\begin{align}
&+ \frac{1}{4} \tilde{I}_{4,1}^+ (\{\frac{1}{2}\text{cr}(346|1279)\} + \{\text{cr}(368|1259)\} - \frac{1}{2}\{\text{cr}(356|2809)\} - \{\frac{1}{4}\text{cr}(478|2950)\})
\end{align}
\begin{align}
&\quad - \{\text{cr}(136|2958)\} + \{\text{cr}(236|1590)\} + \{\text{cr}(479|2058)\}, r_3(12|345, 678)) .
\end{align}

We focus first on the (cross-ratio, cross-ratio) terms in (70). Consider the five-term
\begin{align}
\tilde{I}_{4,1}^+ (V(256|7390), [567|3209]), [123|4567])
&= \tilde{I}_{4,1}^+ ([256|3790] + [567|3209] + [356|2970] + [569|3270] + [560|3729], [123|4567]) .
\end{align}
The second term is invariant under (67) since it maps the second argument to its inverse; this term vanishes after skew-symmetrization. Note that
\begin{align}
\sigma_{(23)} \tilde{I}_{4,1}^+ ([562|3790], [123|4567]) &= \tilde{I}_{4,1}^+ ([563|2970], [123|4567]) = -\tilde{I}_{4,1}^+ ([563|2970], [123|4567]) ,
\end{align}
\begin{align}
\sigma_{(09)} \tilde{I}_{4,1}^+ ([569|3270], [123|4567]) &= \tilde{I}_{4,1}^+ ([560|3729], [123|4567]) = -\tilde{I}_{4,1}^+ ([560|3729], [123|4567]) ,
\end{align}
so after skew-symmetrization the first and third, and fourth and fifth terms combine to give
\begin{align}
\frac{1}{6} \tilde{I}_{4,1}^+ (V(256|7390), [567|3209]), [123|4567])
&= \frac{1}{3} \tilde{I}_{4,1}^+ ([256|3790] + [569|3270], [123|4567])
\end{align}
\begin{align}
&= \frac{1}{3} \tilde{I}_{4,1}^+ ([256|3790] + [569|2730], [123|4567]) .
\end{align}

This leaves only the following (cross-ratio, triple-ratio) terms in (70) to reduce. Unfortunately, the reduction here relies on finding a suitable decomposition purely with computer assistance. Introduce the following combination
\begin{align}
\Psi = & \quad 2\{134|2569\} + \{134|2569\} - 2\{134|2689\} + \{134|2689\} + \{134|5690\} - \{134|6890\}
\end{align}
\begin{align}
&+ 24\{136|2479\} + 20\{136|2490\} - 4\{136|2790\} - 2\{137|2459\} - 2\{137|2489\} + 2\{137|2569\}
\end{align}
\begin{align}
&+ 2\{137|2590\} - 2\{137|2689\} + 2\{137|2890\} - \{137|4590\} - \{137|4890\} + \{137|5690\}
\end{align}
\begin{align}
&- \{137|6890\} - 6\{139|2460\} + 6\{139|2670\} - 2\{167|2359\} - 2\{167|2389\} + \{167|2590\}
\end{align}
\begin{align}
&+ \{167|2890\} - \{167|3590\} - \{167|3890\} + 6\{169|2340\} + 6\{169|2370\} + 5\{346|1259\}
\end{align}
\begin{align}
&- 18\{346|1279\} + 5\{346|1289\} - 8\{346|1290\} + 4\{346|1579\} - 3\{346|1590\} - 4\{346|1789\}
\end{align}
\begin{align}
&+ 38\{346|1790\} - 3\{346|1890\} - 5\{349|1260\} - 18\{356|1249\} + 5\{367|1298\} - 5\{367|1299\} + 5\{368|1299\}
\end{align}
\begin{align}
&- 8\{367|1290\} - 4\{367|1459\} - 4\{367|1489\} + 14\{367|1490\} - 3\{367|1590\} - 3\{367|1890\}
\end{align}
\begin{align}
&- 14\{369|1240\} - 14\{369|1270\} + 24\{369|1470\} - 5\{379|1240\} - 5\{379|1260\} + 5\{780|1259\} .
\end{align}

Denote the (cross-ratio, triple-ratio) terms in (70) by
\begin{align}
\Omega' = \tilde{I}_{4,1}^+ (\frac{1}{2}\{\text{cr}(346|1279)\} + \frac{1}{4}\{\text{cr}(368|1259)\} - \frac{1}{2}\{\text{cr}(356|2809)\} - \{\frac{1}{4}\text{cr}(478|2950)\})
\end{align}
\begin{align}
&\quad - \{\text{cr}(136|2958)\} + \{\text{cr}(236|1590)\} + \{\text{cr}(479|2058)\}, r_3(12|345, 678)) .
\end{align}

Then one can check that
\begin{align}
\Omega' - \frac{1}{24} \tilde{I}_{4,1}^+ (\Psi, r_3(12|345, 678))
\end{align}
is a Li2-functional equation in the first arguments, under automorphisms and inversion of the triple-ratio $r_3(12|345, 678)$. In particular it will be expressible as a combination of five-term relations. One can also check after permuting the points so that the first argument is $\text{cr}(123|4567)$, that
\begin{align}
\Omega' + \frac{1}{24} \tilde{I}_{4,1}^+ (\Psi, r_3(12|345, 678))
\end{align}
is a Li3-functional equation in the second argument, under automorphisms of the cross-ratio $\text{cr}(123|4567)$ and the 6-fold symmetries. From the sum of (72) and (73), we conclude that $\Omega'$ decomposes into $I_{4,1}^+$ combinations of purely Li2 functional equations, and purely Li3 functional equations, in the first and second argument, respectively. This completes the decomposition of $\Omega$ into such functional equations, and hence establishes the claim. \qed
APPENDIX A. AN EXPLICIT EXPRESSION FOR $\text{Sym}_{36}(x, y)$ AND $V(z; x, y)$ IN TERMS OF $I_4$

For the sake of completeness we give explicitly the combination of $I_4$ terms appearing on the right-hand side of

$$ I_{3,1}(x, y) - \tilde{I}_{3,1}(x, y) \equiv \sum_j \lambda_j I_4(f_j(x, y)) , $$

which we denoted by $\text{Sym}_{36}(x, y)$. The combination can be obtained by applying Theorem [11] to relate every $I_{3,1}(x^2, y^2)$ in $I_{3,1}(x, y)$ back to $\text{sgn}(\sigma) \text{sgn}(\pi) I_{3,1}(x, y)$. The resulting expression is as follows.

$$
I_{3,1}(x, y) - \tilde{I}_{3,1}(x, y) \equiv \\
- \frac{1}{12} I_4\left(\frac{(1-x)y^2}{(1-x)(1-y)^2}\right) + \frac{1}{12} I_4\left(\frac{x^2 y}{(1-x)(1-y)^2}\right) + \frac{1}{12} I_4\left(\frac{xy^2}{(1-x)(1-y)^2}\right) + \frac{1}{6} I_4\left(\frac{(1-x)xy}{(1-x)(y-1)}\right) - \frac{1}{4} I_4\left(\frac{-(1-x)y}{(1-x)(1-y)}\right) + 4 I_4\left(\frac{1-y}{1-x}\right) + 3 I_4\left(\frac{x(1-y)}{x-1}\right) - \frac{1}{4} I_4\left(\frac{(1-x)xy}{(1-x)(y-1)}\right) - \frac{1}{4} I_4\left(\frac{(1-x)^2 y}{(1-x)(1-y)^2}\right) - \frac{1}{4} I_4\left(\frac{-y}{(1-x)(1-y)}\right) - \frac{1}{4} I_4\left(\frac{(1-x)y}{1-x}\right) - \frac{1}{2} I_4\left(\frac{xy}{x(1-y)}\right) - \frac{1}{4} I_4\left(\frac{y}{x(1-y)}\right) - \frac{1}{4} I_4\left(\frac{1-y}{1-x}\right) + \frac{1}{12} I_4\left(\frac{(1-x)y}{1-x}\right) - \frac{1}{12} I_4\left(\frac{(1-x)x}{1-y}\right) + \frac{1}{2} I_4\left(\frac{x}{1-y}\right) + \frac{3}{2} I_4\left(\frac{1}{1-y}\right) + \frac{3}{2} I_4\left(\frac{y}{1-y}\right) .
$$

We also give explicitly the combination of $I_4$ terms appearing on the right-hand side of

$$
\tilde{I}_{3,1}(z, [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right]) \equiv \sum_j \nu_j I_4(f_j(x, y, z)),
$$

which we denote by $V(z; x, y)$. The expression we give is only slightly different from the one given in [15] in that we give a relation only for the 36-fold symmetrization of $I_{3,1}$. We write the identity in the following symmetric form. Choose $z_1, \ldots, z_9 \in \mathbb{P}^1(\mathbb{C})$ in such a way that $z = \text{cr}(z_1, z_2, z_3, z_4)$, $x = \text{cr}(z_5, z_6, z_7, z_8)$, $y = \text{cr}(z_9, z_5, z_6, z_9)$, for example, we can take $(z_1, \ldots, z_9) = (\infty, 0, 1, 1-x, 0, 1-\frac{1}{y}, 1, \infty)$. Then the left-hand side of (78) is skew-symmetric under the action of $S_4 \times S_5$ on the 9 points $z_1, \ldots, z_9$. Thus we can decompose the $I_4$ terms into orbits under the action of $S_4 \times S_5$. The resulting expression is as follows. Note that we write $(abcd) = \text{cr}(abcd)$ as shorthand for the individual cross-ratio in the $I_4$ arguments, to differentiate them from the notation for formal linear combinations elsewhere.

$$
I_{3,1}(z, [x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right]) \equiv 4 \frac{\text{Alt}_{\sigma} \times \text{Alt}_{\tau}}{\text{Alt}_{\tau} \times \text{Alt}_{\sigma}} \left[ - I_4\left(\frac{(1234)(5678)((7569) - (1234))}{(5968)(5987)(5986)(5968)}\right) + 2 I_4\left(\frac{(5978)(8659) - (1234)}{(7659) - (1234)}\right) \right.
- 2 I_4\left(\frac{(1234)(7569) - (1234)}{(5876)(5876) - (1234)}\right) + 2 I_4\left(\frac{(5987)(7569) - (1234)}{(1234)(8659) - (1234)}\right) + 3 I_4\left(\frac{(5968)(7659) - (1234)}{(1234)(1234)}\right) + 4 I_4\left(\frac{(5986)(7659) - (1234)}{(1234)(1234)}\right) + 4 I_4\left(\frac{(1234)(1234)}{(1234)(1234)}\right) - 6 I_4\left(\frac{(7569) - (1234)}{(1234)(1234)}\right) + 8 I_4\left(\frac{(7569) - (1234)}{(1324)(7659)(8679)}\right) .
$$

APPENDIX B. EXPLICIT EXPRESSIONS FOR SYMMETRIES OF $I_{4,1}^+(x, y)$ IN TERMS OF $I_5$

Recall the function

$$ I_{4,1}^+(x, y) := \frac{1}{2}(I_{4,1}(x, y) + I_{1,1}(x, y^{-1})) . $$

Modulo products, and explicit $I_5$ terms, it satisfies the $I_2$ anharmonic symmetries in $x$, and the $I_3$ inversion in $y$. It also satisfies the $I_5$ three-term relation (including constant term) $I_3(y) + I_4(1-y) + I_3(1-y^{-1}) \equiv I_9(1)$ in $y$. Explicitly, we have the following identities.

**Theorem 37.** The function $I_{4,1}^+(x, y)$ satisfies the following symmetries and identities.
(i) We have $I_{4,1}^+(x, y) - I_{4,1}^-(x, y^{-1}) = 0$.

(ii) Modulo products the combination $I_{4,1}^+(x, y) + I_{4,1}^+(x^{-1}, y)$ is equal to
\[
-2 \text{Li}_5 \left( \frac{y}{x} \right) - 2 \text{Li}_5(xy) - \text{Li}_5(x) - \text{Li}_5(y). 
\]

(iii) Modulo products the combination $I_{4,1}^+(x, y) + I_{4,1}^+(1 - x, y)$ is equal to
\[
\frac{1}{12} \text{Li}_5 \left( \frac{x^2 y}{(1 - x)(1 - y)^2} \right) + \frac{1}{12} \text{Li}_5 \left( \frac{(1 - x)^2 y}{x(1 - y)^2} \right) + \frac{1}{6} \text{Li}_5 \left( \frac{(1 - x)xy^2}{y - 1} \right) + \frac{1}{6} \text{Li}_5 \left( \frac{(1 - y)y}{(1 - x)x} \right) \\
- \frac{1}{2} \text{Li}_5 \left( \frac{1 - x}{x(y - 1)} \right) - \frac{1}{2} \text{Li}_5 \left( \frac{xy}{(1 - x)(1 - y)} \right) - \frac{1}{2} \text{Li}_5 \left( \frac{(1 - x)(1 - y)}{-x} \right) - \frac{1}{2} \text{Li}_5 \left( \frac{(1 - x)y}{x(1 - y)} \right) \\
- \frac{7}{4} \text{Li}_5 \left( \frac{y}{1 - x} \right) - \frac{7}{4} \text{Li}_5((1 - x)y) - \frac{7}{4} \text{Li}_5 \left( \frac{y}{x} \right) - \frac{7}{4} \text{Li}_5(xy) \\
- \text{Li}_5 \left( \frac{(1 - y)}{x} \right) - \text{Li}_5 \left( \frac{1 - x}{1 - y} \right) - \text{Li}_5 \left( \frac{(1 - x)y}{y - 1} \right) - \text{Li}_5 \left( \frac{xy}{y - 1} \right) \\
+ \frac{1}{2} \text{Li}_5(1 - x) + \frac{1}{2} \text{Li}_5 \left( \frac{1}{x} \right) + \text{Li}_5 \left( \frac{x - 1}{x} \right) + \text{Li}_5 \left( \frac{1}{1 - y} \right) + \text{Li}_5 \left( \frac{y}{y - 1} \right).
\]

(iv) Modulo products the combination $I_{4,1}^+(x, y) + I_{4,1}^+(1, y) + I_{4,1}^+(x, 1 - y^{-1}) - I_{4,1}^+(1, 1)$ is equal to
\[
- \frac{1}{18} \text{Li}_5 \left( \frac{(1 - x)y^2}{x^2(1 - y)} \right) - \frac{1}{18} \text{Li}_5 \left( \frac{x^2y}{(1 - x)(1 - y)^2} \right) - \frac{1}{18} \text{Li}_5 \left( \frac{x^2(1 - y)y}{x - 1} \right) \\
+ \frac{1}{36} \text{Li}_5 \left( \frac{xy}{(1 - x)^2(1 - y)} \right) + \frac{1}{36} \text{Li}_5 \left( \frac{(1 - x)^2y^2}{y - 1} \right) + \frac{1}{36} \text{Li}_5 \left( \frac{(1 - x)^2y^2}{x(1 - y)^2} \right) \\
+ \frac{1}{9} \text{Li}_5 \left( \frac{x - 1}{1 - y} \right) + \frac{1}{9} \text{Li}_5 \left( \frac{(1 - y)y}{x - 1} \right) + \frac{1}{9} \text{Li}_5 \left( \frac{y}{x - 1} \right) \\
- \frac{1}{2} \text{Li}_5 \left( \frac{1}{1 - x} \right) - \frac{1}{2} \text{Li}_5 \left( \frac{(1 - x)}{1 - y} \right) - \frac{1}{2} \text{Li}_5 \left( \frac{1 - y}{x} \right) \\
- \frac{1}{2} \text{Li}_5 \left( \frac{1 - y}{(1 - x)(1 - y)} \right) - \frac{1}{2} \text{Li}_5 \left( \frac{(1 - x)y}{y - 1} \right) - \frac{1}{2} \text{Li}_5((1 - x)y) \\
- \frac{5}{4} \text{Li}_5 \left( \frac{x}{1 - y} \right) - \frac{5}{4} \text{Li}_5(1 - y) - \frac{5}{4} \text{Li}_5 \left( \frac{y}{x} \right) - \frac{5}{4} \text{Li}_5(xy) \\
- \frac{5}{4} \text{Li}_5 \left( \frac{y}{x(y - 1)} \right) - \frac{5}{4} \text{Li}_5 \left( \frac{xy}{y - 1} \right) + \text{Li}_5 \left( \frac{1}{1 - x} \right) + \frac{3}{2} \text{Li}_5(x).
\]

Proof. Each identity is checked directly on the level of the mod-products symbol.

The identity in (i) is immediate from the definition of $I_{4,1}^+$. The identity in (ii) follows from the inversion property of $I_{a,b}(x^{-1}, y^{-1})$ given in Theorem 6.1.2 of [2] (see also [29] for a more general version of the inversion property).

The identity in (iii) can be obtained from the case $a = 1, b = 0$ of the reduction of $I_{4,1}^+$ under the so-called algebraic Li$_2$ functional equation $\sum \text{Li}_2(p_i(t)) = 0$ where $p_i(t)$ are the roots counted with multiplicity of $x^a(1 - x)^b = t$. This is given in Theorem 7.4.6 of [4] for the related function $I_{4,1}^-$ and in Corollary 7.4.9 of [2] for $I_{4,1}^+(x, y)$ itself.

The identity in (iv) can be obtained from Theorem 7.4.17 in [4] where it is stated for the related function $I_{4,1}^-$. Note that the constant term is written using the Nielsen polylogarithm $S_{3,2}$ instead of $I_{4,1}^+$ with one argument specialized to 1, but they are related via $S_{3,2}(x) \equiv I_{4,1}(x, 1) + 4 \text{Li}_5(x)$.

References

[1] A. A. Beilinson, P. Deligne, Motivic polylogarithm and Zagier’s Conjecture, preprint, 1992.
[2] S. Bloch, Higher regulators, algebraic K-theory, and zeta-functions of elliptic curves. Lecture Notes, U.C. Irvine, 1977.
[3] S. Bloch, Applications of the dilogarithm function in algebraic K-theory and algebraic geometry, Proc. of the International Symp. on Alg. Geometry, Kinokuniya, Tokyo (1978).
[4] A. Borel, Cohomologie de $\text{SL}_n$ et valeurs de fonctions zêta aux points entiers, Ann. Sc. Norm. Sup. Pisa 4, pp. 613–636 (1977).
[5] A. Borel, Values of zeta-functions at integers, cohomology and polylogarithms, Current Trends in Mathematics and Physics, a Tribute to Harish-Chandra, pp. 1-44 (1995).
[6] A. Borel, J. Yang, The rank conjecture for number fields, Math. Res. Let., Vol. 1, pp. 689–699 (1994).
[7] F. C. S. Brown Motivic periods and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, Proceedings of the International Congress of Mathematics, Seoul, 2014.
[8] J. I. Burgos Gil. The regulators of Beilinson and Borel. CRM Monograph Series, 15. American Mathematical Soc., 2002.
[9] S. Charlton, Identities arising from coproducts on multiple zeta values and multiple polylogarithms, Thesis, Durham, 2016.
[10] S. Charlton, H. Gangl, D. Radchenko, On functional equations for Nielsen polylogarithms, arXiv:1908.04770
[11] N. Dan, Sur la conjecture de Zagier pour n = 4, arXiv:0809.3984
[12] N. Dan, Sur la conjecture de Zagier pour n = 4, II, arXiv:1101.1557
[13] C. Duhr, H. Gangl, J. R. Rhodes, From polygons and symbols to polylogarithmic functions, Journal of High Energy Physics, vol. 10, p. 75 (2012)
[14] K. Ebrahimi-Fard, J. M. Garcia-Bondiá, F. Patras, A Lie theoretic approach to renormalization, Commun. Math. Phys. 276, pp. 519–549 (2007)
[15] H. Gangl, Multiple polylogarithms in weight 4, arXiv:1609.05557
[16] H. Gangl, The Grassmannian complex and Goncharov’s motivic complex in weight 4, arXiv:1801.07816
[17] I. M. Gelfand, R. MacPherson, Geometry in Grassmannians and a generalization of the dilogarithm, Adv. Math. 44, p. 279–312 (1982)
[18] A. B. Goncharov, Polylogarithms and motivic Galois groups, In Motives (Seattle, WA, 1991), volume 55 of Proc. Sympos. Pure Math., pp. 43–96. Amer. Math. Soc., Providence, RI, 1994.
[19] A. B. Goncharov, Geometry of configurations, polylogarithms, and motivic cohomology, Adv. Math. 114, no. 2, pp. 197–298 (1995)
[20] A. B. Goncharov, Geometry of the trilogarithm and the motivic Lie algebra of a field, Regulators in analysis, geometry and number theory, Progress Math. Vol. 171, pp. 127–165. Birkhäuser Boston, Boston, MA, 2000.
[21] A. B. Goncharov, Galois symmetries of fundamental groupoids and non-commutative geometry, Duke Math. J. 128, no. 2, pp. 209–284 (2005)
[22] A. B. Goncharov, Polylogarithms, regulators and Arakelov motivic complexes. J. Amer. Math. Soc. 18 (1), pp. 1–60 (2005)
[23] A. B. Goncharov, A simple construction of Grassmannian polylogarithms, Adv. Math. 241, pp. 79–102 (2013)
[24] A. B. Goncharov, D. Rudenko, Motivic correlators, cluster varieties, and Zagier’s conjecture on $\zeta_F(4)$, arXiv:1806.00855
[25] A. B. Goncharov, Hodge correlators, J. Reine Angew. Math. 748, pp. 1–138 (2019)
[26] M. Hanamura, R. MacPherson, Geometric construction of polylogarithms, Duke Math. J. 70, pp. 481–516 (1993)
[27] M. Hanamura, R. MacPherson, Geometric construction of polylogarithms, II, Progress in Math. 132, pp. 215–282 (1996)
[28] R. de Jeu, Zagier’s Conjecture and wedge complexes in algebraic K-theory. Compos. Math., 96 (2), pp. 197–247 (1995)
[29] E. Panzer, The parity theorem for multiple polylogarithms, J. Number Theory, Vol. 172, pp. 93-113, (2017)
[30] D. Radchenko, Higher cross-ratios and geometric functional equations for polylogarithms. PhD thesis, Bonn University, 2016.
[31] D. Rudenko, On the Goncharov depth conjecture and a formula for volumes of orthoschemes, arXiv:1802.05599
[32] A. A. Suslin, Algebraic K-theory of fields, Proc. Internat. Congr. Math. (Berkeley, 1986), Amer. Math. Soc., Providence, R.I., pp. 222–244 (1987)
[33] D. Zagier, Hyperbolic manifolds and special values of Dedekind zeta-functions, Invent. Math. 83, pp. 285–301 (1986)
[34] D. Zagier, Polylogarithms, Dedekind zeta functions, and the algebraic K-theory of fields. Progress Math. Vol. 89, pp. 392–430. Birkhäuser, Boston, MA, 1991.

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