Path integral solution for a Klein-Gordon particle in vector and scalar deformed radial Rosen-Morse-type potentials

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Abstract

The problem of a Klein-Gordon particle moving in equal vector and scalar Rosen-Morse-type potentials is solved in the framework of Feynman’s path integral approach. Explicit path integration leads to a closed form for the radial Green’s function associated with different shapes of the potentials. For $q \leq -1$, and $\frac{1}{2}\alpha \ln |q| < r < +\infty$, the energy equation and the corresponding wave functions are deduced for the $l$ states using an appropriate approximation to the centrifugal potential term. When $-1 < q < 0$ or $q > 0$, it is shown that the quantization conditions for the bound state energy levels $E_n$ are transcendental equations which can be solved numerically. Three special cases such as the standard radial Manning-Rosen potential ($|q| = 1$), the standard radial Rosen-Morse potential ($V_2 \rightarrow -V_2, q = 1$) and the radial Eckart potential ($V_1 \rightarrow -V_1, q = 1$) are also briefly discussed.

Keywords: Rosen-Morse potential; Manning-Rosen potential; Green’s function; Path integral; Bound states.

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1 Introduction

The purpose of the present paper is to further study the problem of a spinless relativistic particle of mass $M$ and charge ($-e$) in interaction with a mixed central potential consisting of a vector potential $V_q(r)$ and a scalar potential $S_q(r)$. The scalar potential is added to the rest mass and the whole can be interpreted as an effective position dependent mass. The potentials $V_q(r)$ and
\( S_q (r) \) are equal and of the form (see, for example Ref. \[1\] and references therein)

\[
V_q (r) = S_q (r) = - \frac{V_1}{\cosh^2_q (\alpha r)} - V_2 \tanh_q (\alpha r),
\]

where \( V_1, V_2, \alpha \) and \( q \) are four adjustable real parameters, \( V_1 \) and \( V_2 \) describe the depth of the potential well, \( \alpha \) is the screening parameter related to the range of the potential characterized by the length \( \alpha^{-1} \). The expression (1) is defined in terms of \( q \)-deformed hyperbolic functions

\[
\sinh_q x = \frac{e^x - q e^{-x}}{2}, \quad \cosh_q x = \frac{e^x + q e^{-x}}{2}, \quad \tanh_q x = \frac{\sinh_q x}{\cosh_q x}
\]

which have been introduced for the first time by Arai \[2\]. The deformation parameter \( q \) being a non-zero real number. The expression (1) represents a variant of the deformed Schrödinger potential \[3\] or the Tietz potential \[4\] which serve as modeling potentials for diatomic molecules. In the case where \( q = 1 \), the potentials (1) reduce to the usual radial Rosen-Morse potential that has been investigated from different points of view in the last decade \[3, 6, 7, 8, 9, 10, 11, 12, 13\]. Also, for \(-1 \leq q < 0 \) or \( q > 0 \), various methods have been used to solve the Klein-Gordon and Dirac equations \[14, 15, 16, 17, 18\] with these same potentials. In particular, for \( |q| = 1 \) and for a light modification of expression (1), the relativistic rotational-vibrational energies and the radial wave functions have been approximately calculated with the help of the supersymmetric WKB approach and through the resolution of the Klein-Gordon equation \[19\]. More recently, the usefulness of the Manning-Rosen potential and the Rosen-Morse potential for calculations of the vibrational partition function to study the thermodynamic properties of diatomic molecules has been emphasized by Jia and co-workers \[20, 21\].

In this study, we shall present a complete treatment of the bound state problem for the potentials (1) via the path integral approach for any real value of parameter \( q \neq 0 \).

The organization of the paper is as follows: in section 2, we formulate the radial Green’s function associated with any \( l \)-wave in the framework of Feynman’s path integral. In section 3, we construct the radial Green’s function associated with the deformed Manning-Rosen potential \((q < 0)\). When \( q \leq -1 \) and \( \frac{1}{\alpha} \ln |q| < r < +\infty \), we use an appropriate approximation to the centrifugal potential term to calculate the expression of the radial Green’s function for a state of orbital quantum number \( l \). From this, we shall then obtain the equation for the energy spectrum and the normalized eigenfunctions. For \(-1 < q < 0 \), the \( q \)-deformed Manning-Rosen potential is converted into the standard Manning-Rosen potential which is defined on a half-line. In this case, the radial Green’s function, for \( l = 0 \), is evaluated in a closed form by using the perturbation method which consists in adding a Dirac \( \delta \)-function perturbation to the standard Manning-Rosen potential and making the strength of this perturbation infinitely repulsive to create a totally reflecting boundary. In this fashion, we obtain the radial Green’s function for a particle moving on a half-line. From
the poles of the Green’s function, we derive a transcendental equation for the energy levels. In section 4, the \( q \)-deformed Rosen-Morse potential \( (q > 0) \) is worked out in a similar way. We first transform the path integral associated with this potential into the one of the standard Rosen-Morse potential on a half-line and by means of the same Dirac \( \delta \)-function perturbation trick, we calculate the radial Green’s function and also obtain a transcendental equation for the \( s \)-state energy levels. In section 5, the standard radial Manning-Rosen potential \( (|q| = 1) \), the standard radial Rosen-Morse potential \( (V_2 \rightarrow -V_2, q = 1) \) and the radial Eckart potential \( (V_1 \rightarrow -V_1, q = 1) \) are considered as special cases. Section 6 will be a conclusion.

2 Green’s function

The Green’s function associated with a particle moving in a vector potential and a scalar potential with spherical symmetry can develop into partial waves \(^{22}\) in spherical coordinates:

\[
G\left(r'', t'', r', t'\right) = \frac{1}{r'' r'} \sum_{l=0}^{\infty} \frac{2l + 1}{4\pi} G_l(r'', t'', r', t') P_l(\cos \Theta), \tag{3}
\]

where the radial Green’s function is given by

\[
G_l(r'', t'', r', t') = \frac{1}{2i} \int_0^\infty d\Lambda \, \langle r'', t'' | \exp \left[ \frac{i}{2} \left( -P_r^2 + (P_0 - V_q)^2 \right) - \frac{l(l+1)}{r^2} - (M + S_q)^2 \right] \Lambda | r', t' \rangle, \tag{4}
\]

and \( P_l(\cos \Theta) \) is the Legendre polynomial of degree \( l \) in \( \cos \Theta = \frac{\vec{r}'' \cdot \vec{r}'}{r'' r'} = \cos \theta'' \cos \theta' + \sin \theta'' \sin \theta' \cos (\phi'' - \phi') \).

In Feynman’s formulation \(^{23, 24}\), the radial Green’s function \( G_l(r'', t'', r', t') \) is explicitly expressed in the form of a path integral,

\[
G_l(r'', t'', r', t') = \frac{1}{2i} \int_0^\infty d\Lambda P_l(r'', t'', r', t'; \Lambda), \tag{5}
\]

where

\[
P_l(r'', t'', r', t'; \Lambda) = \lim_{N \to \infty} \prod_{n=1}^N \left[ \int dr_n dt_n \right] \prod_{n=1}^{N+1} \left[ \int \frac{d(P_r)_n \ d(P_0)_n}{(2\pi)^2} \right] \times \exp \left[ i \sum_{n=1}^{N+1} A_1^n \right], \tag{6}
\]
with the short time action $A_1^n$ given by

$$A_1^n = -(P_r)_n \Delta r_n + (P_0)_n \Delta t_n + \frac{\varepsilon_\Lambda}{2} \left[ -(P_r)_n^2 + ((P_0)_n - V_q(r_n))^2 - \frac{l(l+1)}{r_n^2} - (M + S_q(r_n))^2 \right],$$

(7)
in which the notation used is $\Delta r_n = r_n - r_{n-1}, \Delta t_n = t_n - t_{n-1}, r_n = r(t_n)$, and $\varepsilon_\Lambda = \frac{\Lambda}{N+1}$.

Let us notice first that the integrations on the variables $t_n$ in the expression (6) give $N$ Dirac distributions $\delta((P_0)_n - (P_0)_{n+1})$. Thereafter, performing the integrations on the variables $(P_0)_n$, one finds that

$$(P_0)_1 = (P_0)_2 = ... = (P_0)_{N+1} = E.$$

(8)

Consequently, the propagator (6) becomes

$$P_l(r'', t'', r', t'; \Lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dE \exp \{ iE(t'' - t') \} P_l(r'', r'; \Lambda),$$

(9)

with the kernel $P_l(r'', r'; \Lambda)$ given by

$$P_l(r'', r'; \Lambda) = \lim_{N \to \infty} \prod_{n=1}^{N} \left[ \int dr_n \right] \prod_{n=1}^{N+1} \left[ \int \frac{d(P_r)_n}{2\pi} \right] \exp \left[ i \sum_{n=1}^{N+1} A_2^n \right],$$

(10)

where the new short time action is then

$$A_2^n = -(P_r)_n \Delta r_n + \frac{\varepsilon_\Lambda}{2} \left[ -(P_r)_n^2 + (E - V_q(r_n))^2 - \frac{l(l+1)}{r_n^2} - (M + S_q(r_n))^2 \right].$$

(11)

Then, performing the integrations over the variables $(P_r)_n$, we find

$$P_l(r'', r'; \Lambda) = \frac{1}{\sqrt{2\pi i \varepsilon_\Lambda}} \lim_{N \to \infty} \prod_{n=1}^{N} \left[ \int \frac{dr_n}{\sqrt{2\pi i \varepsilon_\Lambda}} \right] \exp \left\{ i \sum_{n=1}^{N+1} \left[ \frac{(\Delta r_n)^2}{2\varepsilon_\Lambda} \right] - \frac{\varepsilon_\Lambda}{2} \left[ (M + S_q(r_n))^2 - (E - V_q(r_n))^2 + \frac{l(l+1)}{r_n^2} \right] \right\},$$

(12)

and substituting (9) into (5), we notice that the term depending on time $t$ does not contain the pseudo-time variable $\Lambda$. Thus, we can rewrite the Green’s function (5) in the form:

$$G_l(r'', t'', r', t') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dE \exp \{ iE(t'' - t') \} G_l(r'', r'),$$

(13)
where
\[ G_l(r'', r') = \frac{1}{2i} \int_0^\infty d\Lambda P_l(r'', r'; \Lambda). \] (14)

By assuming that \( V_q(r) = S_q(r) \), the radial Green’s function (14) reduces to
\[ G_l(r'', r') = \frac{1}{2i} \int_0^\infty d\Lambda \exp \left( i\tilde{E}_0^2 \Lambda \right) K_l(r'', r'; \Lambda), \] (15)
where
\[ K_l(r'', r'; \Lambda) = \frac{1}{\sqrt{2\pi\varepsilon\Lambda}} \lim_{N \to \infty} \prod_{n=1}^N \left[ \int \frac{dr_n}{\sqrt{2\pi\varepsilon\Lambda}} \exp \left\{ i \sum_{n=1}^{N+1} \left[ \frac{(\Delta r_n)^2}{2\varepsilon\Lambda} - \frac{\varepsilon\Lambda}{2} \left( 2(E + M)V_q(r_n) + \frac{l(l+1)}{r_n^2} \right) \right] \right\}, \] (16)
and
\[ \tilde{E}_0^2 = \frac{E^2 - M^2}{2}. \] (17)

The radial propagator (16) and the radial Green’s function (15) depend on the arbitrary real deformation parameter \( q \). When \( -1 < q < 0 \) or \( q > 0 \), the radial Green’s function (15) can be only evaluated exactly for the \( s \) states, but, when \( q \leq -1 \), the \( l \)–state problem can be solved by using a proper approximation to the centrifugal potential term. So to undertake this study, three interesting cases must be distinguished according to the values of the deformation parameter \( q \).

3 Deformed radial Manning-Rosen Potentials

3.1 First case: \( q \leq -1 \).

When \( q \leq -1 \), the potentials (1) are written in the form:
\[ V_q(r) = S_q(r) = -\frac{V_1}{\sinh^2_q(\alpha r)} - V_2 \coth_q(\alpha r). \] (18)

The motion takes place in the half-space \( r > r_0 = \frac{1}{2\alpha} \ln(-q) \). The figure 1 represents the variations with \( (\alpha r) \) of the deformed Manning-Rosen potential (18) for three different \( |q| \) values. In order to construct the path integral for a state of orbital quantum number \( l \), we first use the expression
\[ \frac{1}{r^2} \approx \alpha^2 \left( \frac{1}{3} + \frac{|q|}{\sinh^2_q(\alpha r)} \right) \] (19)
as a good approximation for \( r \) in the centrifugal potential term when \( |q| \geq 1 \) as it can be seen in Fig. 2 which contains a plot of \( \frac{1}{(\alpha r)^2} \) and of \( \left( \frac{1}{3} + \frac{|q|}{\sinh^2(\alpha r)} \right) \) for some characteristic values of \( |q| \). Note that the approximation (19) is equivalent to those of the literature \([25, 26]\) for \( |q| = 1 \). We next transform the variable \( r \in [r_0, +\infty[ \) into a new variable \( x \in [0, +\infty[ \) by

\[
 r = \frac{1}{\alpha} \left( x + \frac{1}{2} \ln |q| \right). 
\]

(20)

In addition to the change of variable, we rescale the local time interval \( \varepsilon_\Lambda \) to a new time interval \( \alpha^{-2} \varepsilon_\Lambda \). Putting these considerations together, we can rewrite the Green’s function (15) as:

\[
 G_l(r'', r') = -\frac{1}{2\alpha} G^l_{MR}(x'', x'; \tilde{E}_l^2),
\]

(21)

with

\[
 G^l_{MR}(x'', x'; \tilde{E}_l^2) = i \int_0^\infty dS \exp \left( \frac{i}{\alpha^2} \tilde{E}_l^2 S \right) K^l_{MR}(x'', x'; S),
\]

(22)

where

\[
 \tilde{E}_l^2 = E_0^2 - \frac{l(l + 1)\alpha^2}{6},
\]

(23)

and

\[
 K^l_{MR}(x'', x'; S) = \int Dx(s) \exp \left\{ i \int_0^S \left[ \frac{x^2}{2} - V^l_{MR}(x) \right] ds \right\}
\]

(24)

is the propagator associated with the standard Manning-Rosen potential \([27]\)

\[
 V^l_{MR}(x) = -\tilde{V}_2 \coth x + \frac{\tilde{V}_1}{\sinh^2 x}; \quad x > 0,
\]

(25)

in which we have put

\[
 \tilde{V}_1 = -(E + M) \frac{V_1}{|q| \alpha^2} + \frac{l(l + 1)}{2}; \quad \tilde{V}_2 = (E + M) \frac{V_2}{\alpha^2}.
\]

(26)

The Green’s function can be stated in a closed form as is known from the literature \([28, 29, 30]\).
The quantities $L_E$, $M_1$, and $M_2$ are given by
\begin{align*}
L_E &= -\frac{1}{2} + \frac{1}{2\alpha} \sqrt{(M + E)(M - E + 2V_2) + \frac{\alpha^2}{4}(l + 1)}; \\
M_{1,2} &= \sqrt{\left(l + \frac{1}{2}\right)^2 - 2(M + E)\frac{V_2}{\alpha|q|} \pm \frac{1}{2\alpha} \sqrt{(M + E)(M - E - 2V_2) + \frac{\alpha^2}{4}(l + 1)}}.
\end{align*}
(28)

The energy spectrum for the bound states can be obtained from the poles of the Green's function [27] or the poles of the Euler function $\Gamma(M_1 - L_E)$. These poles are given by
\begin{equation}
M_1 - L_E = -n_r; \quad n_r = 0, 1, 2, \ldots
\end{equation}
(29)

By inserting the values of $L_E$ and $M_1$ in (28), we obtain
\begin{align*}
M^2 - E_{n_r,l}^2 &= \frac{(M + E_{n_r,l})^2 V_2^2}{\alpha^2 \left(n_r + \frac{1}{2} \mp \sqrt{\left(l + \frac{1}{2}\right)^2 - 2(M + E_{n_r,l}) \frac{V_2}{\alpha|q|}} \right)^2} \\
&+ \alpha^2 \left(n_r + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - 2(M + E_{n_r,l}) \frac{V_1}{\alpha|q|}} \right)^2 \\
&- \frac{\alpha^2}{3} l(l + 1).
\end{align*}
(30)

The corresponding wave functions are of the form:
\begin{equation}
u_{n_r,l}^{q \leq -1}(r) = r \Phi_{n_r,l}^{q \leq -1}(r) = N_{n_r,l} (1 - |q| e^{-2\sigma})^{\delta_l} \left(|q| e^{-2\sigma}\right)^{w_l} P_{n_r}^{(2\sigma l, 2\sigma l - 1)} \left(1 - 2|q| e^{-2\sigma}\right),
\end{equation}
(31)
where
\begin{align*}
w_l &= \frac{1}{2\alpha} \sqrt{M^2 - E_{n_r,l}^2 - 2(M + E_{n_r,l}) V_2 + \frac{\alpha^2}{3} l(l + 1)}; \\
\delta_l &= \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - 2(M + E_{n_r,l}) \frac{V_1}{\alpha|q|}}.
\end{align*}
(32)
The normalization constant \( N_{n_r} \) follows from the condition 
\[
\int_{\ln|q|}^{+\infty} \left| u_{n_r,l}^{-1}(r) \right|^2 dr = 1. \tag{33}
\]

The calculation leads to
\[
N_{n_r,l} = \left[ \frac{4\omega l(n_r + w_l + \delta_l)}{n_r + \delta_l} \frac{n_r \Gamma(n_r + 2w_l + 2\delta_l)}{\Gamma(n_r + 2w_l + 1)\Gamma(n_r + 2\delta_l)} \right]^{\frac{1}{2}}. \tag{34}
\]

### 3.2 Second case: \(-1 < q < 0\)

In this case, the potential \( (18) \) is defined in the interval \( \mathbb{R}^+ \). A plot of this potential for three different \(|q|\) values is shown in Fig. 3. The transformation \( r = \frac{1}{\alpha}(x + \frac{1}{2} \ln |q|) \) converts \( r \in [0, +\infty[ \) into \( x \in ]-\frac{1}{2} \ln |q|, +\infty[ \). This means that the kernel \( (24) \) is the propagator describing the evolution of a particle in the presence of a Manning-Rosen-type potential on the half-line \( x > x_0 = -\frac{1}{2} \ln |q| \).

As we cannot perform a direct path integration to evaluate the propagator \( (24) \), the problem can be solved by a trick that consists in incorporating an auxiliary term potential defined by a Dirac \( \delta \) function in equation \((24)\) to form an impenetrable barrier \( [31] \) at \( x = x_0 \). Since the approximation \((19)\) is not suitable for \( 0 < |q| < 1 \), we limit ourselves to the evaluation of the Green’s function associated with the \( s \)-waves. So the Green’s function \( (22) \), for \( l = 0 \), becomes
\[
G^{\delta}_{MR}(x'', x'; E_0^2) = i \int_0^\infty dS \exp \left( \frac{iE_0^2}{\alpha^2} S \right) K^{\delta}_{MR}(x'', x'; S), \tag{35}
\]
where
\[
K^{\delta}_{MR}(x'', x'; S) = \int Dx(s) \exp \left\{ i \int_0^S \left[ \frac{x'^2}{2} - V^{\delta}_{MR}(x) \right] ds \right\}. \tag{36}
\]

The path integral \((36)\) is the propagator of a particle which moves in a potential of the form:
\[
V^{\delta}_{MR}(x) = V^{0}_{MR}(x) - \eta \delta(x - x_0), \tag{37}
\]
where \( V^{0}_{MR}(x) \) is the expression of the potential \((26)\) for \( l = 0 \). As is quite clear, given the complicated form of the potential \((37)\), the calculation of the Green’s function \((35)\) cannot be performed directly. We propose to apply the perturbation approach in order to express \( \exp \left( i\eta \int_{x''}^{x'} \delta(x - x_0) ds \right) \) in power series. Then, the propagator \((36)\) can be written as:
\[ K^\delta (x'', x'; S) = K^0_{MR} (x'', x'; S) + \sum_{n=1}^{\infty} \frac{(i\eta)^n}{n!} \Pi_{j=1}^{n} \left[ \int_{s_{j-1}}^{s_{j+1}} \int_{-\infty}^{s_{j+1}} ds_j dx_j \right] \times K^0_{MR} (x_1, x_1'; s_1 - s_1) \delta (x_1 - x_0) K^0_{MR} (x_2, x_1; s_2 - s_1) \times \ldots \times \delta (x_{n-1} - x_0) K^0_{MR} (x_n, x_{n-1}; s_n - s_{n-1}) \times \delta (x_n - x_0) K^0_{MR} (x'', x_n; S - s_n) \]

\[ = K^0_{MR} (x'', x'; S) + \sum_{n=1}^{\infty} (i\eta)^n \int_{s_i}^{s_{i+1}} ds_i \int_{s_{i+1}}^{s_{i+2}} ds_{i+1} \ldots \int_{s_2}^{s_1} ds_1 \times K^0_{MR} (x_0, x'; s_1 - s_1) K^0_{MR} (x_0, x; s_2 - s_1) \times \ldots \times K^0_{MR} (x_0, x; s_n - s_{n-1}) K^0_{MR} (x'', x_0; S - s_n), \tag{38} \]

where we ordered the time as follows: \( s' = s_0 < s_1 < s_2 < \ldots < s_n < s_{n+1} = s'' \).

To perform the successive integrations on the variables \( s_j \) in equation (38), we insert (38) into (35) and use the convolution theorem of Fourier transformation, we then obtain

\[
G^\delta_{MR} (x'', x'; \tilde{E}_0) = G^0_{MR} (x'', x'; \tilde{E}_0) - \frac{1}{\eta} \frac{G^0_{MR} (x'', x_0; \tilde{E}_0) \frac{\partial}{\partial \tilde{E}_0} G^0_{MR} (x_0, x'; \tilde{E}_0)}{G^0_{MR} (x_0, x_0; \tilde{E}_0)}, \tag{39} \]

where \( G^0_{MR} (x'', x'; \tilde{E}_0) \) is the Green’s function \( 27 \) associated with the standard Manning-Rosen potential \( 25 \), for \( l = 0 \).

If we now take the limit \( \eta \to -\infty \), the physical system will be forced to move in the potential \( V^0_{MR} (x) \) bounded by an infinitely repulsive barrier \( 28, 31 \) located at \( x = x_0 \). In this case, the Green’s function is then given by:

\[
\tilde{G}^0_{MR} (x'', x'; \tilde{E}_0) = \lim_{\eta \to -\infty} G^4_{MR} (x'', x'; \tilde{E}_0) \]

\[
= G^0_{MR} (x'', x'; \tilde{E}_0) - \frac{G^0_{MR} (x'', x_0; \tilde{E}_0) \frac{\partial}{\partial \tilde{E}_0} G^0_{MR} (x_0, x'; \tilde{E}_0)}{G^0_{MR} (x_0, x_0; \tilde{E}_0)}. \tag{40} \]

Finally, when \(-1 < q < 0\), the radial Green’s function of our problem is expressed as:

\[
G_0 (r'', r') = -\frac{1}{2\alpha} \tilde{G}^0_{MR} (x'', x'; \tilde{E}_0^2). \tag{41} \]

The energy spectrum is determined from the poles of the expression (40), i.e., by the equation \( G^0_{MR} (x_0, x_0; \tilde{E}_0^2) = 0 \), or as well by the transcendental
equation
\[ 2F_1(\delta + w - p, \delta + p + w, 2w + 1; |q|) = 0, \quad (42) \]
where the quantities \( \delta, p \) and \( w \) are defined as
\[
\begin{align*}
\delta &= \frac{1}{2} \left( 1 + \sqrt{1 - \frac{8(E_{nr} + M)V_1}{\alpha^2 |q|}} \right); \\
p &= \frac{\alpha}{2} \sqrt{(M + E_{nr})(M - E_{nr} + 2V_2)}; \\
w &= \frac{\alpha}{2} \sqrt{(M + E_{nr})(M - E_{nr} - 2V_2)}. 
\end{align*}
\]

The equation (42) can be solved numerically to determine the discrete energy levels of the particle. The corresponding wave functions are of the form:
\[
u_{n_r}^{-1 < q < 0}(r) = r\Psi_{n_r}^{-1 < q < 0}(r) = C \left( 1 - |q| e^{2\alpha r} \right)^\delta \left( |q| e^{2\alpha r} \right)^w 
\times 2F_1(\delta + w - p, \delta + p + w, 2w + 1; |q| e^{2\alpha r}), \quad (44)\]
where \( C \) is a constant factor. Note that these wave functions well satisfy the boundary conditions
\[
u_{n_r}^{-1 < q < 0}(r) \to 0, \quad r \to 0, \quad (45)\]
and
\[
u_{n_r}^{-1 < q < 0}(r) \to 0, \quad r \to \infty. \quad (46)\]

### 4 Deformed radial Rosen-Morse potentials

For \( q > 0 \), the potentials (11) have the form of the deformed Rosen-Morse potential which is defined in the interval \( \mathbb{R}^+ \). Figure 4 contains a plot of the deformed radial Rosen-Morse potential for six different \( q \) values. In order to bring back the integral (15), for \( l = 0 \), to a solvable form, we proceed as in the previous case. We perform the following coordinate transformation:
\[
\begin{align*}
r \in \mathbb{R}^+ \to x \in \left[-\frac{1}{2} \ln q, +\infty\right]
\end{align*}
\]
defined by
\[
r = \frac{1}{\alpha} \left( x + \frac{1}{2} \ln q \right). \quad (48)\]

After changing \( \varepsilon_\Lambda \) into \( \alpha^{-2} \varepsilon_s \) or \( \Lambda \) into \( \alpha^{-2} S \), we can write (15), for the \( s \) states, in the following form:
\[
G_0(r'', r') = -\frac{1}{2\alpha} \tilde{G}^0_{RM} \left( x'', x'; \tilde{E}_0^2 \right), \quad (49)\]
where
\[
\tilde{G}^0_{RM} \left( x'', x'; \tilde{E}_0^2 \right) = i \int_0^\infty dS \exp \left( i \frac{\tilde{E}_0^2 S}{\alpha^2} \right) R_{RM} \left( x'', x'; S \right), \quad (50)\]
and
\[ K_{RM}^0 (x'', x'; S) = \int Dx(s) \exp \left\{ i \int_0^S \left[ \frac{x''^2}{2} - \frac{V_1}{q \cosh^2 x} \right] ds \right\}. \] (51)

The constants \( \widetilde{V}_1 \) and \( \widetilde{V}_2 \) are given by
\[ \widetilde{V}_1 = (E + M) \frac{V_1}{2}; \quad \widetilde{V}_2 = -(E + M) \frac{V_2}{2}. \] (52)

The propagator \((51)\) has the same shape as the path integral relative to the potential originally introduced by Rosen and Morse to discuss the vibrational states of the polyatomic molecules \[32\]. The Rosen-Morse potential is defined for \( x \in \mathbb{R} \), but in this case we have transformed the path integral for the potentials \((1)\) into a path integral for a standard Rosen-Morse-type potential via the transformation \( r \rightarrow r(x) \) which converts \( r \in \mathbb{R}^+ \rightarrow x \in (1/2) \ln q, +\infty \). This means that the motion of the particle takes place in the half-space \( x > x_0 = -(1/2) \ln q \). Then, to calculate the Green’s function relative to the \( s \)-waves, we proceed as in the previous case and we obtain

\[ \tilde{G}_{RM}^0 (x'', x'; \tilde{E}_0^2) = G_{RM}^0 (x'', x'; \tilde{E}_0^2) - \left( \frac{G_{RM}^0 (x', x_0; \tilde{E}_0^2) G_{RM}^0 (x_0, x'; \tilde{E}_0^2)}{G_{RM}^0 (x_0, x_0; \tilde{E}_0^2)} \right), \] (53)

where \( G_{RM}^0 (x'', x'; \tilde{E}_0^2) \) is the Green’s function associated with the standard Rosen-Morse potential \[32\]
\[ V_{RM}(x) = \widetilde{V}_2 \tanh x - \frac{\widetilde{V}_1}{q \cosh^2 x}; \quad x \in \mathbb{R}. \] (54)

It is known that the solution by the path integral for this potential leads to the following expression of the Green’s function \[28\ [29\ [30\]

\[ G_{RM}^0 (x'', x'; \tilde{E}_0^2) = \frac{\Gamma(M_1 - L_E)\Gamma(L_E + M_1 + 1)}{\Gamma(M_1 + M_2 + 1)\Gamma(M_1 - M_2 + 1)} \times \left( \frac{1 - \tanh x' - 1 - \tanh x''}{2} \right)^{(M_1 + M_2)/2} \times \left( \frac{1 + \tanh x' + 1 + \tanh x''}{2} \right)^{(M_1 - M_2)/2} \times \left( \frac{1 + \tanh x_0}{2} \right)^{(M_1 + M_2)/2} \times 2F_1 \left( M_1 - L_E, M_1 + M_1 + 1, M_1 - M_2 + 1; \frac{1 + \tanh x_0}{2} \right) \times 2F_1 \left( M_1 - L_E, M_1 + 1, M_1 + M_2 + 1; \frac{1 - \tanh x_0}{2} \right), \] (55)
with the notation

\[
\begin{align*}
L_E &= -\frac{1}{2} + \frac{1}{2} \sqrt{1 + 8 \frac{(E + M)V_1}{\alpha^2 q}}; \\
M_{1,2} &= \frac{1}{2^n} \left( \sqrt{(M + E)(M - E + 2V_2)} \pm \sqrt{(M + E)(M - E - 2V_2)} \right).
\end{align*}
\]

The bound state energy levels are determined from the poles of the Green’s function (53), i.e., by the equation

\[G_{RM}^{0}(x_0, x_0; \tilde{E}_0^2) = 0,\]

or as well by the following quantization condition which is a transcendental equation involving the hypergeometric function

\[\text{Fig. 12}\]

\[\text{Fig. 13}\]

where the parameters \(\delta, p, w\) are defined by

\[
\begin{align*}
\delta &= \frac{1}{2} \left( 1 + \sqrt{1 + 8 \frac{(E_{nr} + M)V_1}{\alpha^2 q}} \right); \\
p &= \frac{1}{2^n} \sqrt{(M + E_{nr})(M - E_{nr} - 2V_2)}; \\
w &= \frac{1}{2^n} \sqrt{(M + E_{nr})(M - E_{nr} + 2V_2)}.
\end{align*}
\]

The equation (57) can be also solved numerically.

Using the Green’s function (55) for the Rosen-Morse potential and the link between (49) and (55), we show that the wave functions corresponding to the bound states have the form:

\[u_{q>0}^{>0}(r) = r \Psi_{q>0}^{>0}(r) = C \left( \frac{1}{1 + qe^{-2\alpha r}} \right)^p \left( \frac{q}{q + e^{2\alpha r}} \right)^w \times \text{Fig. 12},\]

\[\text{Fig. 13},\]

where \(C\) is a constant factor. These wave functions are physically acceptable since they satisfy the boundary conditions

\[u_{q>0}^{>0}(r) \to 0, \quad r \to 0,\]

and

\[u_{q>0}^{>0}(r) \to 0, \quad r \to \infty.\]

5 Particular cases

5.1 First case: standard radial Manning-Rosen potentials

If we take \(|q| = 1\), the potentials (18) turn to the standard radial Manning-Rosen potential
\[ V(r) = S(r) = -\frac{V_1}{\sinh^2(\alpha r)} - V_2 \coth(\alpha r). \]  

The energy equation and the normalized wave functions of the bound states can be deduced from expressions (60) and (61).

\[ M^2 - E_{n,r,l}^2 = \frac{(M + E_{n,r,l})^2 V_2^2}{\alpha^2 \left(n_r + \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - 2(M + E_{n,r,l}) V_2} \right)^2} + \alpha^2 \left(n_r + \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - 2(M + E_{n,r,l}) V_1^2} \right)^2 - \frac{\alpha^2}{3} l(l+1), \]  

(63)

where

\[ w_l = \frac{1}{2\alpha} \sqrt{M^2 - E_{n,r,l}^2 - 2(M + E_{n,r,l}) V_2 + \frac{\alpha^2}{3} l(l+1)}; \]

\[ \tilde{\delta}_l = \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 - 2(M + E_{n,r,l}) V_1^2}. \]  

(65)

and

\[ \bar{N}_{n,r,l} = \left[ \frac{4 \alpha w_l(n_r + w_l + \tilde{\delta}_l) n_r! \Gamma(n_r + 2 w_l + 2 \tilde{\delta}_l)}{n_r + \tilde{\delta}_l \Gamma(n_r + 2 w_l + 1) \Gamma(n_r + 2 \tilde{\delta}_l)} \right]^{\frac{1}{2}}. \]  

(66)

### 5.2 Second case: standard radial Rosen-Morse potentials

By setting \( q = 1 \), and by changing \( V_2 \) in \((-V_2)\), the expression (1) becomes the so-called standard Rosen-Morse potential

\[ V(r) = S(r) = -\frac{V_1}{\cosh^2(\alpha r)} - V_2 \tanh(\alpha r). \]  

(67)

The energy levels \( E_{n,r} \) are deduced from (57) by the transcendental equation

\[ _2F_1 \left( p + w - \delta + 1, p + w + \delta, 2p + 1; \frac{1}{2} \right) = 0, \]  

(68)

and the non-normalized wave functions (59) become in this case:
\[ u_{n_r}^{q=1}(r) = r \Psi_{n_r}^{q=1}(r) = C \left( \frac{1}{1 + e^{-2\alpha r}} \right)^p \left( \frac{1}{1 + e^{2\alpha r}} \right)^w \times {_2F_1} \left( p + w - \delta + 1, p + w + \delta, 2p + 1; \frac{1}{1 + e^{-2\alpha r}} \right), \quad (69) \]

where \( p \) is replaced by \( w \) and conversely.

### 5.3 Third case: radial Eckart potentials

By taking \( q = 1 \), and by changing \( V_1 \) in \((- V_1)\), the potentials reduce to the Eckart potential

\[ V(r) = S(r) = \frac{V_1}{\cosh^2(\alpha r)} - V_2 \tanh(\alpha r). \quad (70) \]

The quantization condition of energy levels and the non-normalized wave functions can be derived from equations (57) and (59). They are written respectively,

\[ {_2F_1} \left( p + w - \delta + 1, p + w + \delta, 2p + 1; \frac{1}{2} \right) = 0, \quad (71) \]

and

\[ u_{n_r}^{q=1}(r) = r \Psi_{n_r}^{q=1}(r) = C \left( \frac{1}{1 + e^{-2\alpha r}} \right)^p \left( \frac{1}{1 + e^{2\alpha r}} \right)^w \times {_2F_1} \left( p + w - \overline{\delta} + 1, p + w + \overline{\delta}, 2p + 1; \frac{1}{1 + e^{-2\alpha r}} \right), \quad (72) \]

with \( \overline{\delta} = \frac{1}{2} + \sqrt{1 - 8 \frac{(E_{n_r} + M) V_1}{\alpha^2}}. \)

### 6 Conclusion

In this article, we have solved the problem of a relativistic spinless particle in the presence of equal vector and scalar \( q \)-deformed radial Rosen-Morse-type potentials by path integration. This problem has only partially been discussed through the resolution of the Klein-Gordon equation [14]. However, a complete solution can be given for any deformation parameter \( q \neq 0 \). As we have shown, the path integral for the Green’s function associated with this mixture of equal vector and scalar potentials can not be evaluated in a unified manner whatever the value of the deformation parameter. When \( q \leq -1 \) and \( \frac{1}{2} \ln |q| < r < \infty \), the radial Green’s function for any \( l \) state is directly calculated by using an appropriate approximation to the centrifugal potential term. The energy equation and the corresponding wave functions are then obtained. For
−1 < q < 0 or q > 0, we have limited ourselves to the evaluation of the Green’s functions for the s waves (l = 0). We have shown that the transformation of the Green’s function relative to the starting potential defined on the interval \( \mathbb{R}^+ \) into a path integral associated with a Manning-Rosen or Rosen–Morse-type potential reduces the problem to that of a particle forced to move in a half-space \( x > x_0 \). This problem with the Dirichlet boundary conditions is treated by using the perturbation approach. In both cases, the quantization conditions are transcendental equations involving the hypergeometric function which can be solved numerically to determine the energy levels of the bound states. The radial wave functions, expressed in terms of the hypergeometric functions are also derived.

References

[1] A Kadja, F Benamira and L Guechi *Indian J. Phys.* **91** 259 (2016)
[2] A Arai *J. Phys. A: Math. Gen.* **34** 4281 (2001)
[3] A Amrouche, A Diaf and M Hachama *Can. J. Phys.* **95** 25 (2017)
[4] C S Jia, Y F Diao, X J Liu, P Q Wang, J Y Liu and G D Zhang *J. Chem. Phys.* **137** 014101 (2012)
[5] K J Oyewumi and C O Akoshile *Eur. Phys. A* **45** 311 (2010)
[6] S M Ikhdair *J. Math. Phys.* **51** 023525 (2010)
[7] M R Setare and S Haidari *Phys. Scr.* **81** 015201 (2010)
[8] T T Ibrahim, K J Oyewumi and S M Wyngaardt *Eur. Phys. J. Plus* **127** 100 (2012)
[9] E V Aguda, *Can. J. Phys.* **91** 689 (2013)
[10] T Chen, S R Lin and C S Jia *Eur. Phys. J. Plus* **128** 69 (2013)
[11] S M Ikhdair and M Hamzavi *Chin. Phys. B* **22** 040302 (2013)
[12] M S Tan, S He and C S Jia *Eur. Phys. J. Plus* **129** 264 (2014)
[13] J Y Liu, X T Hu and C S Jia *Can. J. Chem.* **92** 40 (2014)
[14] L Z Yi, Y F Diao, J Y Liu and C S Jia *Phys. Lett. A* **333** 212 (2004)
[15] X Q Zhao, C S Jia and Q B Yang *Phys. Lett. A* **337** 189 (2005)
[16] X C Zhang, Q W Liu, C S Jia and L Z Wang *Phys. Lett. A* **340** 59 (2005)
[17] S M Ikhdair *J. Quantum Inf. Sci.* **1** 73 (2011)
[18] A Ghoumaid, F Benamira and L Guechi *J. Math. Phys.* **57** 024102 (2016)
[19] C S Jia, T Chen and S He *Phys. Lett. A* **377** 682 (2013)
[20] C S Jia, L H Zhang and C W Wang *Chem. Phys. Lett.* **667** 211 (2017)
[21] X Q Song, C W Wang and C S Jia *Chem. Phys. Lett.* **673** 50 (2017)
[22] D Peak and A Inomata *J. Math. Phys.* **10** 1422 (1969)
[23] R P Feynman and A Hibbs *Quantum Mechanics and Path Integrals* (New York : Mc Graw Hill) (1965)
[24] H Kleinert *Path Integrals in Quantum Mechanics, Statistics Polymer Physics and Financial Markets* ( Singapore : World Scientific) (2009)
[25] C S Jia, T Chen and L G Cui *Phys. Lett. A* **373** 1621 (2009)
[26] A Ghoumaid, F Benamira and L. Guechi *Can. J. Phys.* **91** 120 (2013)
[27] M F Manning and N Rosen *Phys. Rev.* **44** 953 (1933)
[28] C Grosche *J. Phys. A: Math. Gen.* **38** 2947 (2005)
[29] F Benamira, L Guechi, S Mameri and M A Sadoun *J. Math. Phys.* **48** 032102 (2007)
[30] F Benamira, L Guechi, S Mameri and M A Sadoun *J. Math. Phys.* **51** 032301 (2010)
[31] T E Clark, R Menikoff and D H Sharp *Phys. Rev. D* **22** 3012 (1980)
[32] N Rosen and P M Morse *Phys. Rev.* **42** 210 (1932)

Figure captions

**Fig. 1.** A plot of the Manning-Rosen potential (18) with $V_2 = V_1/4$, for different values of $|q| \geq 1$.

**Fig. 2.** A plot of the expression $\left(1/3 + |q|/\sinh^2 |q| (\alpha r)\right)$ compared to $\left(1/(\alpha r)^2\right)$, for different values of $|q| > 0$.

**Fig. 3.** A plot of the Manning-Rosen potential (18) with $V_2 = V_1/4$, for different values of $0 < |q| < 1$.

**Fig. 4.** A plot of the Rosen-Morse potential (1) with $V_2 = V_1/4$, for different values of $q > 0$. 

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