A NOTE ON THE KOSZUL COMPLEX IN DEFORMATION QUANTIZATION

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ABSTRACT. The aim of this short note is to present a proof of the existence of an $A_\infty$-quasi-isomorphism between the $A_\infty$-$S(V^*)\wedge(V)$-bimodule $K$, introduced in [1], and the Koszul complex $K(V)$ of $S(V^*)$, viewed as an $A_\infty$-$S(V^*)\wedge(V)$-bimodule, for $V$ a finite-dimensional (complex or real) vector space.

1. INTRODUCTION

The main result of [1] is a Formality Theorem in presence of two subspaces $U_1$, $U_2$ of a real or complex finite-dimensional vector space $V$, constructed using the graphical techniques of Kontsevich [7]. Such a Formality Theorem has interesting by-products even in the simplest case, when $U_1 = V$ and $U_2 = \{0\}$:

1. It implies that the deformation quantization of $A = \mathcal{O}_V = S(V^*)$ [2] and of $B = \mathcal{O}_{V^*[1]} = \wedge(V)$ [1] w.r.t. a given polynomial Poisson structure on $V$ preserves Koszul duality.

2. It yields a proof of a conjecture of B. Shoikhet [2, 9] about the possibility of realizing Kontsevich’s deformed algebra $A_\hbar = A[h]$ [7] w.r.t. a polynomial Poisson structure via generators and relations.

In this particular context, the main object appearing in the Formality Theorem is $K = \mathbb{K}$ (here, $\mathbb{K}$ is any field of characteristic 0 containing $\mathbb{R}$), endowed with a nontrivial $A_\infty$-$A$-$B$-bimodule structure. This bimodule structure is constructed explicitly in [1, Subsection 6.2], using Kontsevich’s graphical techniques, specializing earlier results of Cattaneo–Felder [3, 4]. The main result of this note is the following

Theorem 1.1. The $A_\infty$-bimodule $K$ is $A_\infty$-quasi-isomorphic to the Koszul complex $K(V)$ of $A$, endowed with the usual $A$-$B$-bimodule structure.

Theorem 1.1 has been previously stated as a conjecture, see [1, Conjecture 1.3], where the $A_\infty$-$A$-$B$-bimodule has been introduced.

In fact, the Koszul complex, which is a dg $A$-$B$-bimodule, has been considered in the framework of Deformation Quantization, in [10], where the author used Tamarkin’s Formality in order to quantize both algebras $A$ and $B$ and the Koszul complex itself, in order to prove that Koszul duality is preserved by Deformation Quantization.

The approach of [1] makes use of Kontsevich’s Formality: the $A_\infty$-$A$-$B$-bimodule structure originates from perturbative expansion of the Poisson Sigma model, and it turns out that it realizes, in the $A_\infty$-framework, Koszul duality between $A$ and $B$: it turns out that it behaves well w.r.t. Deformation Quantization, hence it is the right candidate for showing that Kontsevich’s Deformation Quantization techniques preserve also Koszul duality.

For the proof we check that both morphisms in the sequence

$$K(V) \to A_\infty \otimes K \to K$$

are quasi-isomorphisms of $A_\infty$-bimodules. Here $(- \otimes -)$ denotes a tensor product of $A_\infty$-bimodules, defined in Section 3. As for ordinary bimodules, one verifies that the tensor product with the algebra itself, i.e., $(A \otimes A)$, is (quasi-)isomorphic to the identity, and hence the right arrow is a quasi-isomorphism. The fact that the left arrow is a quasi-isomorphism of $A_\infty$-bimodules is due to a peculiarity of the bimodule structure on $K$, and is proven in Section 3.

Remark 1.2. It has been pointed out to us by B. Keller and A. Khoroshkin independently that Theorem 1.1 should follow from the results of [1] and general arguments of homological algebra: however, we do not know, for the time being, a rigorous proof in this framework.

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2. NOTATION AND CONVENTIONS

Throughout the paper, $\mathbb{K}$ is a field of characteristic 0, which contains $\mathbb{R}$. $V$ is a finite-dimensional vector space over $\mathbb{K}$, $V^*$ its dual. Further, we denote by $\{x_i\}$, $i = 1, \ldots, d = \dim V$, a basis of $V^*$, which yields automatically global linear coordinates on $V$, which we denote by $y_i$.

Let $\mathcal{Mdg}_k$ be the monoidal category of graded vector spaces, with graded tensor product, and with inner spaces of morphisms (i.e., morphisms, which are finite sums of morphisms of any degree); $[\bullet]$ denotes the degree-shifting functor on $\mathcal{Mdg}_k$. In particular, the identity morphism of an object $M$ of $\mathcal{Mdg}_k$ induces a canonical isomorphism $s : M \to M[1]$ of degree $-1$ with inverse $s^{-1} : M[1] \to M$ (suspension and de-suspension isomorphisms); for the sake of simplicity, we will use the following short-hand notation

$$(v_1| \cdots |v_n) = s(v_1) \otimes \cdots \otimes s(v_n).$$

The degree of an element $m$ of a homogeneous component of an object $M$ of $\mathcal{Mdg}_k$ is denoted by $|m|$. Unadorned tensor products are meant to be over $\mathbb{K}$.

A (possibly curved) $A_\infty$-algebra $A$ over $\mathbb{K}$ is equivalent to the structure of a codifferential cofree coalgebra with counit on $T(A[1]) = \bigoplus_{n \geq 0} A[1]^{\otimes n}$, for an object $A$ of $\mathcal{Mdg}_k$. The codifferential $d_A$ is uniquely determined by its Taylor components

$$d^n_A : A[1]^{\otimes n} \to A[1], \quad n \geq 0,$$

all of degree 1, and the condition that $d_A$ squares to 0 translates into an infinite family of quadratic relations between its Taylor components. We further set $m^n_A = s^{-1} \circ d^n_A \circ s^{\otimes n}$. By construction, $m^n_A$ are $\mathbb{K}$-linear maps from $A^{\otimes n}$ to $A$ of degree $2 - n$. We refer to $m^n_A$ as to the curvature of $A$. It is an element of $A$ of degree 2, which measures the failure of $(A, m^n_A)$ to be a differential graded (shortly, from now on, dg) vector space over $\mathbb{K}$. If $m^n_A = 0$, then $A$ is said to be flat.

Finally, given two (possibly curved) $A_\infty$-algebras $A, B$, an $A_\infty$-$A$-$B$-bimodule structure on an object $K$ of $\mathcal{Mdg}_k$ is equivalent to the structure of a codifferential cofree bicomodule on $T(A[1]) \otimes K[1] \otimes T(B[1])$. As for $A_\infty$-algebras, such a codifferential $d_K$ is uniquely determined by its Taylor components

$$d^{m,n}_K : A[1]^{\otimes m} \otimes K[1] \otimes B[1]^{\otimes n} \to K[1], \quad m, n \geq 0,$$

all of degree 1. As before, we introduce the maps $m^{m,n}_K = s^{-1} \circ d^{m,n}_K \circ s^{\otimes m+1+n}$, of degree $1 - m - n$. The condition that $d_K$ squares to 0 is equivalent to an infinite family of quadratic relations between the Taylor components of $d_A$, $d_B$ and $d_K$. For more details on $A_\infty$-bimodules over (possibly curved) $A_\infty$-algebras, we refer to [1, Sections 3 and 4].

Remark 2.1. We observe that, if $A$ and $B$ are both flat, then an $A_\infty$-$A$-$B$-bimodule structure on $K$ yields both a left $A_\infty$-$A$- and right $A_\infty$-$B$-module structure on $K$ in the sense of [3, Section 3.3]; but, if either $A$ or $B$ or both are curved, then the $A_\infty$-bimodule structure does not restrict to (left or right) $A_\infty$-module structures, see e.g. [11 and 1, Subsection 4.1].

3. THE TENSOR PRODUCT OF $A_\infty$-BIMODULES

We consider now three (possibly curved) $A_\infty$-algebras $A$, $B$ and $C$. Furthermore, we consider an $A_\infty$-$A$-$B$-bimodule $K_1$ and an $A_\infty$-$B$-$C$-bimodule $K_2$. The tensor product of $K_1$ and $K_2$ over $B$, as an element of $\mathcal{Mdg}_k$, is defined as

$$K_1 \otimes_B K_2 = K_1 \otimes T(B[1]) \otimes K_2.$$

The $A_\infty$-structures on $A$, $B$, $C$, $K_1$ and $K_2$ determine a unique structure of $A_\infty$-$A$-$C$-bimodule over $K_1 \otimes_B K_2$, which we now describe explicitly. By the arguments in Section 2, it suffices to construct a codifferential on the cofree bicomodule

$$T(A[1]) \otimes (K_1 \otimes_B K_2) \otimes T(C[1]) \cong T(A[1]) \otimes K_1[1] \otimes T(B[1]) \otimes K_2[1] \otimes T(C[1]),$$

where the isomorphism is induced by suspension and de-suspension, and has degree $-1$; on the latter dg vector space, we define a bicodereivation via

$$(1) \quad d_{K_1} \otimes 1 \otimes 1 + 1 \otimes d_{K_2} - 1 \otimes d_B \otimes 1 \otimes 1,$$

where 1 denotes here the identity operator on the corresponding factor.

Remark 3.1. We now need a caveat regarding the fact that we use suspension and de-suspension in order to construct an $A_\infty$-structure on the tensor product of two $A_\infty$-bimodules, namely, while suspension or de-suspension of an $A_\infty$-(bi)module is again an $A_\infty$-(bi)module in a natural way, this is not true anymore for an $A_\infty$-algebra. In fact, an $A_\infty$-algebra structure on $A$ cannot be transported on $A[-1]$, because of the fact that twisting w.r.t. suspension and de-suspension of the Taylor components of the $A_\infty$-structure on $A$ would produce maps of the wrong degree.
It is more instructive to write down explicit formulæ for the Taylor components of the previous bicodervation
\[
d^2_{\Delta_{K_1 \otimes \mathbb{R}^n K_2}}(a_1 \cdots a_m | k_1 \otimes (b_1 \cdots | b_q) \otimes k_2 | c_1 \cdots | c_n) = 0, \quad m, n > 0
\]
\[
d^{m,n}_{\Delta_{K_1 \otimes 2 m K_2}}(a_1 \cdots | a_m | k_1 \otimes (b_1 \cdots | b_q) \otimes k_2) = \sum_{l=0}^{q} s \left( s^{-1} (d_{\Delta_{K_2}}^{m,n}(a_1 \cdots | a_m | k_1 | b_1 \cdots | b_l) \otimes (b_{l+1} \cdots | b_q) \otimes k_2) \right), \quad m > 0
\]
\[
d^{0,n}_{\Delta_{K_1 \otimes 2 K_2}}(k_1 \otimes (b_1 \cdots | b_q) \otimes k_2 | c_1 \cdots | c_n) = (-1)^{|k_1|} \sum_{l=0}^{q} s (k_1 \otimes (b_1 \cdots | b_l) \otimes \sum_{\sigma \in S_{q-l}} \sum_{i=1}^{q-l} \left( d_{\Delta_{K_2}}^{n-1,0}(b_{l+1} \cdots | b_q) \otimes k_2 \right) + \left( -1 \right)^{|k_1|} \sum_{\sigma \in S_{q-l}} \sum_{i=1}^{q-l} s (k_1 \otimes (b_1 \cdots | b_l) \otimes s^{-1} \left( d_{\Delta_{K_2}}^{n-1,0}(b_{l+1} \cdots | b_q) \otimes k_2 \right) )
\]

Remark 3.2. We observe that the signs in (2) are dictated by Koszul’s sign rule, together with the signs arising from the previous isomorphism \(T(A[1]) \otimes (K_1 \otimes 2 m K_2) \otimes T(C[1]) \cong T(A[1]) \otimes K_1 \otimes T(B[1]) \otimes K_2 \otimes T(C[1])\) of said degree \(-1\).

Proposition 3.3. For \(A_\infty\)-algebras \(A, B, C\) and \(A_\infty\)-bimodules \(K_1, K_2\) as above, \((K_1 \otimes 2, d_{K_1 \otimes 2} K_2)\), where the bicodervation \(d_{K_1 \otimes 2} K_2\) is defined in (2), is an \(A_\infty\)-A-C-bimodule.

The proof of Proposition 3.3 follows from the following argument: since the bicoderviall \(d_{K_1 \otimes 2} K_2\) is a twist of \(1\), it suffices to prove that \(1\) squares to 0. Using the fact that \(d_B, d_{K_1}\) and \(d_{K_2}\) all square to 0, we are reduced to prove that
\[
(1 \otimes 1) (d_{K_1} \otimes 1) (1 \otimes 1) - (d_{K_1} \otimes 1) (1 \otimes 1) (d_B \otimes 1) + (1 \otimes 1) (d_{K_2}) (d_{K_1} \otimes 1) (1 \otimes 1) - (1 \otimes 1) (d_{K_1}) (d_{K_2}) (1 \otimes 1) + (1 \otimes 1) (d_{K_2}) (1 \otimes 1) + 1 \otimes 1 (d_{K_2})
\]
vanishes.

Since \(d_{K_1}\) and \(d_{K_2}\) are bicoderviations, and \(d_B\) is a coderivation, the sum of the first and second line in (3) can be rewritten as
\[
1 \otimes 1 \otimes d_B \otimes 1 \otimes d_C + (1 \otimes 1 \otimes d_B \otimes (pr_{K_1} \circ d_{K_1})) (1 \otimes 1 \otimes \Delta_B \otimes 1 \otimes 1) - (1 \otimes 1 \otimes d_B \otimes (pr_{K_1} \circ d_{K_1})) (1 \otimes 1 \otimes d_B \otimes 1 \otimes 1) - (1 \otimes 1 \otimes d_B \otimes 1 \otimes d_A) - ((pr_{K_1} \circ d_{K_1}) \otimes d_B \otimes 1 \otimes 1) (1 \otimes 1 \otimes \Delta_B \otimes 1 \otimes 1),
\]
where we have used the short-hand notation
\[
pr_{K_1} \circ d_{K_1} = (1 \otimes (pr_{K_1} \circ d_{K_1}) \otimes 1) (\Delta_B \otimes 1 \otimes 1),
\]
\(pr_{K_1}\), denoting the projection from \(T(A[1]) \otimes K_1 \otimes T(B[1])\) onto \(K_1[1]\) and \(\Delta_A\) is the comultiplication of \(T(A[1])\); similar notation holds for \(pr_{K_1} \circ d_{K_2}\). The notation \(pr_{K_1} \circ d_{K_2}\) means therefore simply that we select only that part of \(d_{K_2}\) given by the Taylor components \(d_{K_1}^{m,n}\), thus forgetting about the action of \(d_B\) and \(d_C\) on the left and on the right, and similarly for \((pr_{K_1} \circ d_{K_2})\).

Using the same arguments, together with the nilpotence of \(d_B\), yields the same expression for the last line of (3), up to global minus sign, coming from Koszul’s sign rule.

Remark 3.4. We point out that, in general, the right-hand side of (1) should be
\[
(1 \otimes (pr_{K_1} \circ d_{K_1}) \otimes 1) (\Delta_B \otimes 1 \otimes \Delta_B),
\]
but, in this framework, the second coproduct \(\Delta_B\) would be redundant and would produce too many terms, due to the fact that there is already a coproduct \(\Delta_B\), whence the reason, why we suppressed it in (1).

Alternatively, straightforward computations involving quadratic relations w.r.t. the Taylor components of \(d_{A}, d_{C}\) and \(d_{K_1 \otimes 2} K_2\), which in turn can be re-written (after a very tedious book-keeping of all signs involved) in terms of the quadratic relations between the Taylor components of the codifferentials on \(A, B, C, K_1\) and \(K_2\). Such computations are similar to the computations in [3] Chapter 4 in the case of right \(A_\infty\)-modules.
It is easy to verify that, if both $A$, $C$ are flat, then the Taylor component \( m_{K, A, B}^{0, 0} \) yields a structure of dg vector space on \( K \otimes_{B} K' \) (while \( B \) may be curved).

### 3.1. The \( A_\infty \)-bar construction of an \( A_\infty \)-bimodule

We consider two (possibly curved) \( A_\infty \)-algebras, and an \( A_\infty \)-\( A \)-\( B \)-bimodule \( K \). It is easy to verify that e.g. \( A \) can be endowed with the structure of an \( A_\infty \)-\( A \)-bimodule, whose Taylor components are specified via the assignment \( a^{m+1} = a^{m} + a_{A}^{m+1} \).

In particular, we may form the tensor product \( A \otimes_{A} K \), which has the structure of an \( A_\infty \)-\( A \)-\( B \)-bimodule, according to Proposition 4.1. We may consider the \( A_\infty \)-\( A \)-\( B \)-bimodule \( K \otimes_{B} B \).

An \( A_\infty \)-algebra \( A \) is said to be \textbf{unitial}, if it possesses an element 1 of degree 0, such that

\[
m_{1}(1 \otimes a) = m_{1}(a \otimes 1) = a, \quad m_{n}(a_{1} \otimes \cdots \otimes a_{n}) = 0, \quad n \neq 2,
\]

if \( a_{i} = 1 \), for some \( i = 1, \ldots, n \). If \( A \) is unital, and \( K \) is an \( A_\infty \)-\( A \)-\( B \)-bimodule, then \( K \) is (left-)unital w.r.t. \( A \), if the identities hold true

\[
m_{K}^{1}(1 \otimes k) = k, \quad m_{K}^{m,n}(a_{1} \otimes \cdots \otimes a_{m} \otimes k \otimes b_{1} \otimes \cdots \otimes b_{n}) = 0, \quad m \neq 1, \quad n \geq 0,
\]

if \( a_{1} = 1 \), for some \( i = 1, \ldots, m \).

Now, there is a natural morphism \( \mu \) of \( A_\infty \)-\( A \)-\( B \)-bimodules from \( A \otimes_{A} K \) to \( K \); the cofreeness of \( A_\infty \)-\( A \)-\( B \)-bimodules implies that such a morphism is uniquely specified by its Taylor components

\[
\mu^{m,n}(a_{1} \cdots a_{m} \otimes b_{1} \cdots b_{n}) = (-1)^{\sum_{i=1}^{m}|a_{i}|} |b_{1}| + \sum_{i=1}^{q-1} (|b_{i}| - 1) d_{K}^{m+1, n+1}(a_{1} \cdots a_{m} a_{\overline{a}_{1}} \cdots \overline{a}_{q} | b_{1} \cdots b_{n}), \quad m, n \geq 0.
\]

Similar formulæ hold true for the case of the \( A_\infty \)-\( A \)-\( B \)-bimodule \( K \otimes_{B} B \).

**Proposition 3.5.** For two (possibly curved) \( A_\infty \)-algebras \( A, B \) and an \( A_\infty \)-\( A \)-\( B \)-bimodule \( K \), there is a natural morphism \( \mu \), of \( A_\infty \)-\( A \)-\( B \)-bimodules from \( A \otimes_{A} K \) to \( K \).

If \( A, B \) are both flat, and \( A, K \) are (left-)unital, then the \( A_\infty \)-morphism \( \mu \) is an \( A_\infty \)-quasi-isomorphism.

We first observe that the Taylor components in (4) have the right degree. Then, the proof of Proposition 4.1 consists in checking the aforementioned quadratic identities: in view of (2), and after a straightforward, but tedious, book-keeping of all signs involved, such quadratic identities can be re-written as the quadratic identities for the Taylor components of the \( A_\infty \)-structures on \( A \otimes_{A} K \) and \( K \). We notice that the isomorphism in the middle is of degree \(-1\).

There is a natural candidate for the morphism \( \mu \), namely,

\[
\mu^{m,n}(a_{1} \cdots a_{m} \otimes b_{1} \cdots b_{n}) = (-1)^{\sum_{i=1}^{m}|a_{i}|} |b_{1}| + \sum_{i=1}^{q-1} (|b_{i}| - 1) d_{K}^{m+1, n+1}(a_{1} \cdots a_{m} a_{\overline{a}_{1}} \cdots \overline{a}_{q} | b_{1} \cdots b_{n}), \quad m, n \geq 0.
\]

If \( A, K \) are unital, we set

\[
\nu^{0,0}(k) = 1 \otimes k, \quad \sigma(a \otimes b_{1} \cdots b_{n}) = 1 \otimes a b_{1} \cdots b_{n}, \quad q \geq 0.
\]

It is not difficult to check that the degree of (4) is 0, while the degree of (5) is \(-1\). Moreover, direct computations involving the fourth identity in (2) and the fact that \( A, K \) are unital imply that (6) is homotopically inverse to \( \mu^{0,0} \), with explicit homotopy (7), whence the second claim follows.

**Remark 3.6.** We observe that Proposition 3.5 has been stated and proved in the framework of right \( A_\infty \)-modules in [8].

### 3.2. The \( A_\infty \)-bimodule structure on the bar resolution of the augmentation module

For \( V \) as in Section 2 we consider the symmetric algebra \( A = S(V) \) and the exterior algebra \( B = \wedge(V) \); both are unital dg algebras with trivial differential, \( A \) is concentrated in degree 0, while \( B \) is non-negatively graded. In particular, \( A \) and \( B \) can be viewed as flat, unital \( A_\infty \)-algebras, whose only non-trivial Taylor components are \( d_{A}^{0,1} \) and \( d_{B}^{0,1} \) respectively.

According to [1], \( K = \mathbb{K} \) can be endowed with a non-trivial \( A_\infty \)-\( A \)-\( B \)-bimodule structure, which restricts to the natural augmentation left- and right-modules; non-triviality, here, means that there are non-trivial Taylor components \( d_{K}^{m,n} \), for both \( m \) and \( n \) non-zero, e.g.

\[
m_{K}^{1,1}(a \otimes 1 \otimes b) = \langle b, a \rangle, \quad a \in V, \quad b \in V^*.
\]
and \((\bullet, \bullet)\) denotes the duality pairing between \(V^*\) and \(V\). For a complete description of the \(A_\infty\)-\(A\)-\(B\)-bimodule structure on \(K\), we refer to [1] Subsection 6.2.

Since \(A, B\) are flat, and \(A, K\) are unital, Proposition [5,5] implies that there is an \(A_\infty\)-quasi-isomorphism of \(A_\infty\)-\(A\)-\(B\)-bimodules from \(A_\infty^{\mathbb{A}}K\) to \(K\). A direct computation implies that \(A_\infty^{\mathbb{A}}K\) is a dg vector space concentrated in non-positive degrees; recalling [2], its \(A_\infty\)-\(A\)-\(B\)-bimodule structure is given by

\[
\begin{align*}
d^0_{A_\infty^{\mathbb{A}}K}(a \otimes (\bar{a}_1) \cdots (\bar{a}_q) \otimes 1) &= s \left( (a\bar{a}_1) \otimes (\bar{a}_2) \cdots (\bar{a}_q) \otimes 1 + \sum_{i=1}^{q-1} (-1)^i a \otimes (\bar{a}_1) \cdots (\bar{a}_i\bar{a}_{i+1}) \cdots (\bar{a}_q) \otimes 1 + \\
&\quad + (-1)^q a \otimes (\bar{a}_1) \cdots (\bar{a}_{q-1}) \otimes \bar{a}_q(0) \right), \\
d^0_{A_\infty^{\mathbb{A}}K}(a_1 \otimes (\bar{a}_1) \cdots (\bar{a}_q) \otimes 1) &= s(aa_1 \otimes (\bar{a}_1) \cdots (\bar{a}_q) \otimes 1), \\
d^0_{A_\infty^{\mathbb{A}}K}(a \otimes (\bar{a}_1) \cdots (\bar{a}_q) \otimes \{b_1|\cdots|b_n\}) &= (-1)^q \sum_{i=0}^q s(i_1 \otimes (\bar{a}_1) \cdots (\bar{a}_i) \otimes s^{-1}(d_{A_\infty^{\mathbb{A}}K}^{-i}(\bar{a}_{i+1}) \cdots (\bar{a}_q)|1|b_1|\cdots|b_n)),
\end{align*}
\]

and in all other cases, the Taylor components are trivial.

The first identity in (5) yields the identification between the dg vector space \(\left( A_\infty^{\mathbb{A}}K, d^0_{A_\infty^{\mathbb{A}}K} \right)\) identifies with the bar complex of the left augmentation module \(K = \mathbb{K}\) over \(A\).

Further, the identities (5) imply that the left \(A_\infty\)-\(A\)-module structure on the bar complex \(A_\infty^{\mathbb{A}}K\) of \(K\) is the standard one, while the non-triviality of the \(A_\infty\)-\(A\)-\(B\)-bimodule structure on \(K\) yields non-triviality of the right \(A_\infty\)-\(B\)-module structure on \(A_\infty^{\mathbb{A}}K\).

4. The Koszul complex of \(A = S(V^*)\): A brief memento

For \(V\) as in Section [2] we consider the symmetric algebra \(A = S(V^*)\) and the exterior algebra \(B = \wedge(V)\).

We further consider the Koszul complex \(K(V)\) of \(A\). As a graded vector space,

\[
K^q(V) = \wedge^q_{\mathbb{C}[V]} \Omega^1_{\mathbb{C}[V]/\mathbb{K}}, \quad q \geq 0,
\]

where \(\Omega^1_{\mathbb{C}[V]/\mathbb{K}}\) denotes the module of Kähler differentials on \(\mathbb{C}[V] = S(V^*) = A\) as a \(\mathbb{K}\)-algebra; in particular, \(K(V)\) is non-positively graded. The differential \(\partial\) on \(K(V)\) is induced by the (left) contraction w.r.t. the Euler vector field on \(V\); further, \(K(V)\) admits an obvious left \(A\)-action, and contraction w.r.t. polyvector fields on \(V\) induces by restriction (and keeping track of Koszul’s sign rule) a right \(B\)-action on \(K(V)\). In particular, \(K(V)\) admits the structure of an \(A_\infty\)-\(A\)-\(B\)-bimodule, whose only non-trivial Taylor components are labeled by pairs of indices \((m, n)\), such that \(m + n \leq 1\). For later computations, we choose a \(\mathbb{K}\)-basis \(\{x_i\}\) on \(V^*\) as in Section [2] in particular, we may write

\[
K(V) \cong \mathbb{K}[x_i, \theta_i],
\]

where \(\{\theta_i\}\) denotes a set of odd coordinates of degree \(-1\), which anticommute with each other and commute with \(x_i\); w.r.t. the previous algebra isomorphism, \(x_i \mapsto x_i, dx_i \mapsto \theta_i, \partial\) is uniquely determined by the graded Leibniz rule (from the left) and by \(\partial(x_i) = 0, \partial(\theta_i) = x_i\).

Furthermore, we may also write \(B = \mathbb{K}[[\theta]]\), where the partial derivative \(\partial_{\theta_j}\) has degree 1, and acts in an obvious way from the left on \(K(V)\); thus, the right \(B\)-action on \(K(V)\) takes the explicit form

\[
K(V) \otimes B \ni \eta \otimes b_1 \mapsto (-1)^{\eta(b_1)}b_1(\eta).
\]

We finally observe that \((K(V), \partial)\) is a free resolution of the left \(A\)-module \(K\) via augmentation; it is also a free resolution of the right \(B\)-module \(K\) via augmentation, because of the isomorphism \(B \cong (\mathbb{K}[[\theta]])[[-d]]\) induced by contraction w.r.t. polyvector fields.

5. An explicit \(A_\infty\)-quasi-isomorphism between the bar and the Koszul complex of \(A\)

We use the same notation as in the preceding sections; we only observe that in the whole Section, \(A = S(V^*)\), \(B = \wedge(V)\) and \(K = \mathbb{K}\) with the \(A_\infty\)-\(A\)-\(B\)-bimodule structure from [1].

Since \(A_\infty^{\mathbb{A}}K\) and \(K(V)\) are both resolutions of the left augmentation module \(K = \mathbb{K}\) over \(A\), arguments from abstract (co)homological algebra imply that they are quasi-isomorphic to each other as complexes of free left \(A\)-modules. More precisely, the quasi-isomorphism from \(K(V)\) to \(A_\infty^{\mathbb{A}}K\) as complexes of left \(A\)-modules has an explicit form, namely

\[
\Phi(\theta_{i_1} \cdots \theta_{i_q}) = \sum_{\sigma \in \mathfrak{S}_q} (-1)^\sigma (x_{\sigma(i_1)}|\cdots|x_{\sigma(i_q)}) \otimes 1, \quad 1 \leq i_1 < \cdots < i_q \leq d.
\]
It suffices to define the morphism $\Phi$ on monomials of the form $\theta_{i_1} \cdots \theta_{i_k}$, and then extend it $A$-linearly on the left. It follows immediately that $\Phi$ is of degree 0 and commutes with left $A$-action. An easy computation shows that $\Phi$ commutes with differentials. It is a quasi-isomorphism since $\Phi(1) = 1$.

**Theorem 5.1.** For a finite-dimensional $\mathbb{K}$-vector space $V$, the morphism $\Phi$ extends to a quasi-isomorphism of $A_\infty$-$A$-$B$-bimodules from $K(V)$ to $A_\infty \otimes_\mathbb{K} K$, where the $A_\infty$-$A$-$B$-bimodule structures on $K(V)$ and $A_\infty \otimes_\mathbb{K} K$ are described in Sections 4 and 7, respectively.

**Proof.** We know that $\Phi$ is a morphism of degree 0 from $K(V)$ to $A_\infty \otimes_\mathbb{K} K$: we declare (the conjugation w.r.t. $s$ of) $\Phi$ to be the $(0,0)$-th Taylor component of the desired $A_\infty$-quasi-isomorphism, while for $(m,n)$ such that $m+n \geq 1$, we set simply 0.

By the previous arguments, the only non-trivial identities to check are

$$
\begin{align*}
\eta_{l,n}^{0,1}(\Phi(\eta)|b_1|) &= \Phi(d^{0,1}_{K(V)}(\eta|b_1)), \\
\eta_{l,n}^{0,n}(\Phi(\eta)|b_1| \cdots |b_n|) &= 0, \quad n \geq 2,
\end{align*}
$$

for $b_i$, $i=1,\ldots,n$, resp. $\eta$, a general element of $B$, resp. $K(V)$.

We begin by proving Identity (11): $A$-linearity implies that we may take $\eta$ of the form $\theta_{i_1} \cdots \theta_{i_k}$, $1 \leq i_1 < \cdots < i_q \leq d$. Recalling now Identity (9), we rewrite the left-hand side in (11) as

$$
\eta_{l,n}^{0,n}(\Phi(\eta)|b_1| \cdots |b_n|) = (-1)^n \sum_{l=0}^{n} \sum_{\sigma \in S_n} (-1)^{s} \left( 1 \otimes (x_{\sigma(i_1)}| \cdots |x_{\sigma(i_q)}) \otimes s^{-1}(d^{l-n}_{K}(x_{\sigma(i_{q+1})})| \cdots |x_{\sigma(i_1)}|b_1| \cdots |b_n) \right).
$$

We now analyze the last factor on the right-hand side: degree reasons imply that, for $0 \leq l \leq q$,

$$
-(q-l) - 1 + \sum_{j=1}^{n} |b_j| - 1 \leq 0 \iff \sum_{j=1}^{n} |b_j| = n + q - l - 1 > q - l,
$$

because $n \geq 2$ by assumption.

We recall now from [13] Subsection 6.2] that the Taylor components of the $A_\infty$-$A$-$B$-bimodule structure on $K$ are constructed explicitly via admissible graphs, in a way reminiscent of Kontsevich’s graphical technique of [7]. Using the same notation of [7], admissible graphs of type $(m,n)$ (elements of $\mathcal{G}_{m,n}$) are graphs embedded in $\mathbb{R} \sqcup \mathbb{H}$ with $m$ vertices of the first type (i.e. lying in the complex upper half-plane $\mathbb{H}$), $n$ vertices of the second type (i.e. $n$ ordered vertices on the real axis $\mathbb{R}$), and with a certain number of oriented edges between them. In the present framework, we consider typically elements of $\mathcal{G}_{0,m+1+n}$, where to the first $m$ vertices of the second type we associate elements of $A$, to the $m+1$-st vertex of the second type 1 as an element of $K$, and to the last $n$ vertices of the second type we associate elements of $B$: accordingly, oriented edges are associated to elements of $B$, which we view as translation-invariant poly-derivations acting on $A = \mathcal{O}_V$. To such admissible graphs are associated polydifferential operators on $A$, $B$ and $K$ with values in $K$ (by the obvious rule that oriented edges correspond to derivatives) and integral weights, for whose precise treatment we refer to [13] Subsection 6.2] again: suffice it to recall here that the integral weight of a given admissible graph $\Gamma$ in $\mathcal{G}_{0,m+1+n}$ is the integral over the compactified configuration space $\mathcal{C}_{0,m+1+n}$ of $m+1+n$ ordered points on the real axis (modulo rescalings and real translations) of a differential form depending explicitly on $\Gamma$, roughly defined via the rule that a closed 1-form is associated to an edge connecting two vertices.

The previous strict inequality, which is a consequence of the non-vanishing of the integral weight of any admissible graph appearing in the formula for $d^{l-n}_{K}$, implies the claim: in fact, any admissible graph $\Gamma$ in $\mathcal{G}_{0,n+q-1-1}$ in $d^{l-n}_{K}$ must have exactly $n + q - l - 1$ arrows departing from the vertices on the right-hand side of the $q - l + 1$-st vertex and incoming on the vertices on the left-hand side; no other arrows or loops are allowed. Since $n \geq 2$ and $A$, $B$ and $K$ are unital, then all vertices (except the $q - l + 1$-st vertex) must be at least univalent: more precisely, the vertices on the left-hand side of the $q - l + 1$-st vertex must have exactly one incoming arrow (because of degree reasons), while the vertices on the right-hand side must have at least one outgoing arrow, and the previous strict inequality proves that no such graphs exist.

It remains to prove Identity (10). We first evaluate the right-hand side: using the isomorphism at the end of Section 4 we may write $b_1 = \partial_{\theta_{j_1}} \cdots \partial_{\theta_{j_p}}$, whence the right-hand side takes the form $(-1)^{(p+1)q} \Phi(b_1(\eta))$. 


A NOTE ON THE KOSZUL COMPLEX IN DEFORMATION QUANTIZATION

The left-hand side of Identity (10) has the explicit form

\[ d_{A, K}^p(\Phi(\eta)b_1) = (-1)^q \sum_{l=0}^q (-1)^{pl} s \left( 1 \otimes [x_{\sigma(i_1)}] \cdots [x_{\sigma(i_l)}] \otimes s^{-1} (d_{K, x}^{p-1}(x_{\sigma(i_{l+1})}) \cdots [x_{\sigma(i_k)}] 1 b_1) \right) = \]

\[ = (-1)^q \sum_{\sigma \in \mathcal{G}_q} (-1)^p s \left( 1 \otimes [x_{\sigma(i_1)}] \cdots [x_{\sigma(i_{p+1})}] \otimes s^{-1} (d_{K, x}^{p-1}(x_{\sigma(i_{p+2})}) \cdots [x_{\sigma(i_k)}] 1 b_1) \right), \]

where the second equality follows because of degree reasons.

First, if \( q \leq p - 1 \), both sides of Identity (10) vanish: this follows immediately from the previous formulae. It remains therefore to prove the claim in the case \( q \leq p \).

We now take a closer look at the left-hand side of Identity (10): we need to understand, in particular, the Taylor component \( d_{K, x}^{p-1} \). We assume \( p \geq 1 \), because the case \( p = 0 \) follows immediately by direct computations and previous considerations. In view of [1, Subsection 6.2], \( d_{K, x}^{p-1} \) is a sum over admissible graphs \( \Gamma \) in \( \mathcal{G}_{0,p+2} \): in fact, there is only one such admissible graph contributing non-trivially, pictorially

![Figure 1 - The only admissible graph contributing non-trivially to \( d_{K, x}^{p-1}(x_{\sigma(i_{p+1})} \cdots [x_{\sigma(i_k)}] 1 b_1) \)]

The differential operator \( \mathcal{O}_\Gamma \) associated to the admissible graph \( \Gamma \) as in Figure 1 can be explicitly evaluated, following the prescriptions in [1, Subsection 6.2], namely

\[ s^{-1}(d_{K, x}^{p-1}(x_{\sigma(i_{p+1})}) \cdots [x_{\sigma(i_k)}] 1 b_1) = \frac{(-1)^p}{p!} b_1(\theta_{\sigma(i_{p+1})} \cdots \theta_{\sigma(i_k)}). \]

The claim now follows from Identity (12).

5.1. A final remark. In this final Subsection, we want to point out that the techniques portrayed in Sections [8, 9] and in the present one apply as well to the more general situation examined in [1]: namely, for any two subspaces \( U_i, i = 1, 2 \), of a finite-dimensional \( K \)-vector space \( V \), we may associate an \( A_\infty \)-category with two objects, i.e. \( U_1 \) and \( U_2 \), two \( A_\infty \)-algebras \( A \) and \( B \) and an \( A_\infty \)-\( A-B \)-bimodule \( K \).

More explicitly, \( A \), resp. \( B \), is the \( A_\infty \)-algebra associated to the graded vector space of global (regular) sections of the exterior algebra of the normal bundle of \( U_1 \), resp. of \( U_2 \), in \( V \) (with obvious product and trivial differential); \( K \) is the graded vector space of global (regular) sections of the exterior algebra of the quotient bundle \( TV/(TU_1 + TU_2) \) over \( U_1 \cap U_2 \), and the \( A_\infty \)-\( A-B \)-bimodule structure extends the natural left \( A \)- and right \( B \)-action: of course, \( A \), resp. \( B \), resp. \( K \), must be understood as the endomorphism space of \( U_1 \), resp. of \( U_2 \), resp. the space of morphisms from \( U_1 \) to \( U_2 \), declaring trivial the remaining one. We do not indulge in its explicit construction: suffice it to say that it involves (once again) Kontsevich’s diagrammatic techniques, and we observe that it reduces, when \( U_1 = V \) and \( U_2 = \{0\} \), to the one we have made explicit in the previous computations.

As already remarked in [1, Subsection 7.3], there is also a Koszul complex \( K(V, U_1, U_2) \) e.g. for \( A \), whose construction is similar to the one of \( K(V) \) in Section 4 with obvious due changes; again, \( K(V, U_1, U_2) \) can be given the structure of a DG \( A-B \)-bimodule (hence, of an \( A_\infty \)-\( A-B \)-bimodule).

Furthermore, there is a natural (graded) version of the quasi-isomorphism [9] from the Koszul complex \( K(V, U_1, U_2) \) to the bar resolution of \( K \) as a left \( A \)-module: then, a more involved vanishing lemma, proved in the same spirit of Theorem [5, 1] Identity (10), and a more refined version of Identity (11), imply that such a quasi-isomorphism extends to an \( A_\infty \)-quasi-isomorphism of \( A-B \)-bimodules. In particular, Theorem [1] holds true in the more general situation of [1].

Since Keller’s condition in [8, 9] holds true in the more general situation already examined, i.e. for a DG category with two objects \( U_i, i = 1, 2 \), and with spaces of morphisms given by \( A \), \( B \) (the endomorphisms of \( U_i \) and \( U_2 \) respectively) and \( K(V, U_1, U_2) \) (the morphisms from \( U_1 \) to \( U_2 \)), a natural question is, if it is possible to formulate and prove the main result of [10] in this more general setting, using Tamarkin’s results instead of Kontsevich’s.
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