Regularity of almost minimizers with free boundary

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Abstract

In this paper we study the local regularity of almost minimizers of the functional

$$J(u) = \int_{\Omega} |\nabla u(x)|^2 + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u<0\}}(x)$$

where $q_\pm \in L^\infty(\Omega)$. Almost minimizers do not satisfy a PDE or a monotonicity formula like minimizers do (see [AC], [ACF], [CJK], [W]). Nevertheless we succeed in proving that they are locally Lipschitz, which is the optimal regularity for minimizers.

1 Introduction

In this paper we consider a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and study the local regularity of almost minimizers of the functional

$$J(u) = \int_{\Omega} |\nabla u(x)|^2 + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u<0\}}(x)$$

where $q_\pm \in L^\infty(\Omega)$ are bounded real valued functions.

In [AC], Alt and Caffarelli proved existence and regularity results for minimizers in the following context. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $q_+ \in L^\infty(\Omega)$ be given, set

$$K_+(\Omega) = \{ u \in L^1_{\text{loc}}(\Omega) ; u(x) \geq 0 \text{ almost everywhere on } \Omega \text{ and } \nabla u \in L^2(\Omega) \}$$

and

$$J^+(u) = \int_{\Omega} |\nabla u|^2 + q_+^2(x)\chi_{\{u>0\}}$$

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for $u \in K_+(\Omega)$, and let $u_0 \in K_+(\Omega)$ be given, with $J^+(u_0) < \infty$. They prove the existence of a function $u \in K_+(\Omega)$ that minimizes $J^+$ among functions of $K_+(\Omega)$ such that

\begin{equation}
(1.4) \quad u = u_0 \text{ on } \partial \Omega.
\end{equation}

Notice that when $\Omega$ is Lipschitz and $u \in L^1_{\text{loc}}(\Omega)$ is such that $\nabla u \in L^2(\Omega)$, we can define the trace of $u$ almost everywhere on $\partial \Omega$, which lies in a slightly better space than $L^2(\partial \Omega)$, so (1.4) makes sense ([D]).

They also showed that the minimizers are Lipschitz-continuous up to the free boundary $\partial \{u > 0\}$, and that if $q_+$ is Hölder-continuous and bounded away from zero, then

\begin{equation}
(1.5) \quad \partial \{u > 0\} = \partial_* \{u > 0\} \cup E,
\end{equation}

where $H^{n-1}(E) = 0$ and $\partial_* \{u > 0\}$ is the reduced boundary of $\{x \in \Omega; u(x) > 0\}$ in $\Omega$; in addition, $\partial_* \{u > 0\}$ locally coincides with a $C^{1,\alpha}$ submanifold of dimension $n-1$.

Alt, Caffarelli and Friedman [ACF] later showed that if $\Omega$ is a bounded Lipschitz domain, $q_+ \in L^\infty(\Omega)$,

\begin{equation}
(1.6) \quad K(\Omega) = \{u \in L^1_{\text{loc}}(\Omega); \nabla u \in L^2(\Omega)\}
\end{equation}

and $u_0 \in K(\Omega)$, then there exists $u \in K(\Omega)$ that minimizes $J(u)$ under the constraint (1.4). See the proof of Theorem 1.1 in [ACF]. In fact in [ACF] they considered a slightly different functional, for which they prove that the minimizers are Lipschitz. They also prove optimal regularity of the free boundary when $n = 2$ and make important strides toward understanding the regularity of the free boundary in higher dimensions. Later papers by [CJK], [DeJ] and [W] present a more complete picture of the structure of the free boundary in higher dimensions.

In this paper we study the regularity properties of the almost minimizers for $J^+$ and $J$. We consider a domain $\Omega \subset \mathbb{R}^n$ and two functions $q_{\pm} \in L^\infty(\Omega)$. We shall restrict to $n \geq 2$ to simplify the discussion, but $n = 1$ would be simpler. We do not need any boundedness or regularity assumption on $\Omega$, because our results will be local and so we do not need to define a trace on $\partial \Omega$. Also, $q_-$ is not needed when we consider $J^+$, and then we may assume that it is identically zero. Set

\begin{equation}
(1.7) \quad K_{\text{loc}}(\Omega) = \{u \in L^1_{\text{loc}}(\Omega); \nabla u \in L^2(B(x,r)) \text{ for every open ball } B(x,r) \subset \Omega\},
\end{equation}

\begin{equation}
(1.8) \quad K^+_{\text{loc}}(\Omega) = \{u \in K_{\text{loc}}(\Omega); u(x) \geq 0 \text{ almost everywhere on } \Omega\},
\end{equation}

and let constants $\kappa \in (0, +\infty)$ and $\alpha \in (0, 1]$ be given.

We say that $u$ is an almost minimizer for $J^+$ in $\Omega$ (with constant $\kappa$ and exponent $\alpha$) if $u \in K^+_{\text{loc}}(\Omega)$ and

\begin{equation}
(1.9) \quad J_{x,r}^+(u) \leq (1 + \kappa r^\alpha)J_{x,r}^+(v)
\end{equation}
for every ball $B(x, r)$ such that $\overline{B}(x, r) \subset \Omega$ and every $v \in L^1(B(x, r))$ such that $\nabla v \in L^2(B(x, r))$ and $v = u$ on $\partial B(x, r)$, where

\begin{equation}
J_{x,r}(v) = \int_{B(x,r)} |\nabla v|^2 + q_+^2 \chi_{\{v>0\}}.
\end{equation}

Here, when we say that $v = u$ on $\partial B(x, r)$, we mean that they have the same trace on $\partial B(x, r)$, or equivalently that their radial limits on $\partial B(x, r)$, which happen to exist almost everywhere on $\partial B(x, r)$, are equal almost everywhere on $\partial B(x, r)$. See Section 13 of [D]. Note that we could easily restrict to nonnegative competitors $v$ (just because $v_+ = \max(0, v)$ is at least as good as $v$). Thus we can assume that $v \in K^{+}_{\text{loc}}(\Omega)$. Thus we would get an equivalent definition by considering competitors $v \in K^{+}_{\text{loc}}(\Omega)$ such that $v = u$ on $\Omega \setminus B(x, r)$. Similarly, we say that $u$ is an almost minimizer for $J$ in $\Omega$ if $u \in K^{+}_{\text{loc}}(\Omega)$ and

\begin{equation}
J_{x,r}(u) \leq (1 + \kappa r^\alpha)J_{x,r}(v)
\end{equation}

for every ball $B(x, r) \subset \Omega$ and every $v \in L^1(B(x, r))$ such that $\nabla v \in L^2(B(x, r))$ and $v = u$ on $\partial B(x, r)$, where

\begin{equation}
J_{x,r}(v) = \int_{B(x,r)} |\nabla v|^2 + q_+^2 \chi_{\{v>0\}} + q_-^2 \chi_{\{v<0\}}.
\end{equation}

In effect, we would obtain essentially the same results if we only required (1.9) or (1.11) when $r \leq r_0$. Potentially more interestingly, we could try to replace $\kappa r^\alpha$ with larger gauge functions, for instance satisfying some Dini condition of some order. We will not address this question here.

We try to follow the lead of [AC] and [ACF], and study the local regularity in $\Omega$ of almost minimizers $u$ for $J$ and $J^+$. We shall show that they are locally Lipschitz in $\Omega$. Furthermore under the assumption that $q_+$ is bounded below way from zero, we shall prove a non-degeneracy condition for $u$ which translates into some regularity properties for the free boundary $\Gamma = \partial\{x \in \Omega ; u(x) > 0\}$. Although our results are similar to those in [AC] and [ACF], we lack one of their major ingredients, that is almost minimizers to not satisfy a differential equation. Our proofs are mostly based on choosing appropriate competitors for the almost minimizers.

In Section 2 we show that almost minimizers are locally continuous. The main tool in this section and in several other through the paper is to compare, in balls, the almost minimizer with its harmonic extension. In Section 3 we prove higher regularity for almost minimizers inside the open sets $\{u > 0\}$ and $\{u < 0\}$. Once again this amounts to making careful local comparisons with the harmonic extension of the almost minimizer. In Section 4 we start the proof that almost minimizers are locally Lipschitz. In Sections 2, 3 and 4 we do not need to distinguish between almost minimizers for $J$ and $J^+$. In Section 5 we finish the proof of the fact that almost minimizers for $J^+$ are locally Lipschitz. In this section the fact that the almost minimizer does not change signs plays a key role. The proof of the fact that almost minimizers for $J$ are also locally Lipschitz requires, as in the minimizing
case, an additional tool. In our case we need to prove an almost monotonicity formula. The proof of the almost monotonicity formula appears in Sections 6 and 7. In Section 8 we use this almost monotonicity to prove that almost minimizers for $f$ are locally Lipschitz. In Section 9, we study limits of sequences of almost minimizers, paying special attention to blow-up sequences. These sequences play an important role when trying to understand the fine properties and the structure of the free boundary. In Section 10 we prove non-degeneracy results for almost minimizers under mild assumptions for the functions $q_{\pm}$. These results, in particular, ensure that the blow-up limits are non-trivial and that therefore they yield information on the free boundary. In Section 11 we briefly summarize what we know thus far about the free boundary. Understanding the structure and the regularity of the free boundary for almost minimizers is the subject of our current research.

2 Almost minimizers are continuous in $\Omega$.

In this section we do not distinguish between $f^+$ and $f$. We will write $f$ in the proofs with the understanding that $q_-$ might be identically zero and that we may be working with nonnegative functions.

**Theorem 2.1** Almost minimizers of $f$ are continuous in $\Omega$. Moreover if $u$ is an almost minimizer for $f$ there exists a constant $C > 0$ such that if $B(x, r_0) \subset \Omega$ then for $x, y \in B(x, r_0)$

$$|u(x) - u(y)| \leq C|x - y| \left(1 + \log \frac{2r_0}{|x - y|}\right).$$

**Proof.** Let $u$ be an almost minimizer for $f$, and let $x \in \Omega$ and $r > 0$ be such that $B(x, r) \subset \Omega$. For the moment, and up to (2.13) (included), we need no other assumption on $(x, r)$.

For $s \leq r$ let $u_s^*$ denote the function in $L^1(B(x, s))$ with $\nabla u_s^* \in L^2(B(x, s))$ and trace $u$ on $\partial B(x, s)$ which minimizes the Dirichlet energy on $B(x, s)$. The existence and uniqueness of $u_s^*$ are easy to obtain, by convexity, and we shall often refer to $u_s^*$ as the harmonic extension of the restriction of $u$ to $\partial B(x, s)$. Note that this is the case if $u$ is smooth enough on $\partial B(x, s)$.

In the present context, the minimality property is just easier to work with. Many of the estimates in this paper will come from a comparison with functions like $u_s^*$. By definition, for any $t \in \mathbb{R}$

$$\int_{B(x, s)} |\nabla u_s^*|^2 \leq \int_{B(x, s)} |\nabla (u_s^* + t(u - u_s^*))|^2. \tag{2.2}$$

Expanding near $t = 0$ yields $\int_{B(x, s)} \langle \nabla u - \nabla u_s^*, \nabla u_s^* \rangle = 0$ and hence

$$\int_{B(x, s)} |\nabla u_s^*|^2 = \int_{B(x, s)} \langle \nabla u, \nabla u_s^* \rangle. \tag{2.3}$$
Since $u$ is an almost minimizer and $q_{\pm} \in L^\infty$, (2.3) yields
\[
\int_{B(x,s)} |\nabla u - \nabla u_s^*|^2 = \int_{B(x,s)} |\nabla u|^2 - \int_{B(x,s)} |\nabla u_s^*|^2
\leq (1 + \kappa s^\alpha) \int_{B(x,s)} |\nabla u_s^*|^2 - \int_{B(x,s)} |\nabla u_s^*|^2 + C s^n
\leq \kappa s^\alpha \int_{B(x,s)} |\nabla u_s^*|^2 + C s^n \leq \kappa s^\alpha \int_{B(x,s)} |\nabla u|^2 + C s^n,
\]
where in the last inequality we used again the fact that $u_s^*$ is an energy minimizer. Here $C \geq 0$ depends on $\|q_{\pm}\|_{L^\infty}$. Thus (2.4) applied to $B(x,r)$ yields
\[
\int_{B(x,r)} |\nabla u - \nabla u_r^*|^2 \leq \kappa r^\alpha \int_{B(x,r)} |\nabla u_r^*|^2 + C r^n \leq \kappa r^\alpha \int_{B(x,r)} |\nabla u|^2 + C r^n.
\]
For $s > 0$ such that $B(x, s) \subset \Omega$, we set
\[
\omega(x, s) = \left( \frac{1}{|B(x, s)|} \int_{B(x, s)} |\nabla u|^2 \right)^{\frac{1}{2}} = \left( \frac{1}{|B(x, s)|} \int_{B(x, s)} |\nabla u|^2 \right)^{\frac{1}{2}}.
\]
Since $u_r^*$ is an energy minimizer, it is harmonic in $B(x, r)$. This is easy to see. Then $|\nabla u_r^*|^2$ is subharmonic, and therefore for $s \leq r$
\[
\left( \frac{1}{|B(x, s)|} \int_{B(x, s)} |\nabla u_r^*|^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} |\nabla u_r^*|^2 \right)^{\frac{1}{2}}.
\]
Combining the triangle inequality in $L^2$, (2.5), (2.6), and (2.7) we obtain
\[
\omega(x, s) \leq \left( \frac{1}{|B(x, s)|} \int_{B(x, s)} |\nabla u - \nabla u_r^*|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{|B(x, s)|} \int_{B(x, s)} |\nabla u_r^*|^2 \right)^{\frac{1}{2}}
\leq C s^{-n/2} \left( \frac{1}{s} \right)^{n/2} \omega(x, r) + C \left( \frac{r}{s} \right)^{n/2} + \left( \frac{1}{|B(x, s)|} \int_{B(x, s)} |\nabla u_r^*|^2 \right)^{\frac{1}{2}}
\leq C \left( \frac{r}{s} \right)^{n/2} \omega(x, r) + C \left( \frac{r}{s} \right)^{n/2} + \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} |\nabla u_s^*|^2 \right)^{\frac{1}{2}}
\leq \left( 1 + C \left( \frac{r}{s} \right)^{n/2} \right) \omega(x, r) + C \left( \frac{r}{s} \right)^{n/2}.
\]
Here $C$ denotes a constant that only depends on $\kappa$, $\|q_\pm\|_{L^\infty}$, and $n$. Set $r_j = 2^{-j}r$ for $j \geq 0$. By (2.8),

$$\omega(x, r_{j+1}) \leq \left(1 + C 2^{n/2} r_j^{\alpha/2}\right) \omega(x, r_j) + C 2^{n/2}$$

and an iteration yields

$$\omega(x, r_{j+1}) \leq \omega(x, r) \prod_{l=0}^j \left(1 + C 2^{n/2} r_l^{\alpha/2}\right) + C \sum_{l=1}^{j+1} \left(\prod_{k=l}^j \left(1 + C 2^{n/2} r_k^{\alpha/2}\right)\right) 2^{n/2} \leq \omega(x, r) P + CP 2^{n/2} j \leq C \omega(x, r) + C j,$$

where we set $P = \prod_{j=0}^{\infty} \left(1 + C 2^{n/2} r_j^{\alpha/2}\right) = \prod_{j=0}^{\infty} \left(1 + C 2^{n/2} (2^{-j}r)^{\alpha/2}\right)$, and use the fact that $P$ can be bounded, depending on an upper bound for $r$.

At this point we have proved that if $B(x, r) \subset \Omega$, then for $0 < s \leq r$,

$$\omega(x, s) \leq C \omega(x, r) + C \log(r/s),$$

with a constant $C$ that depends only on $\kappa$, $\|q_+\|_{L^\infty}$, $\|q_-\|_{L^\infty}$, $\alpha$, $n$, and an upper bound for $r$. Indeed, if $s \geq r/4$, just observe that $\omega(x, s) \leq 2^n \omega(x, r)$ by (2.6), and otherwise choose $j$ such that $r_{j+2} \leq s \leq r_{j+1}$, observe that $\omega(x, s) \leq 2^{n/2} \omega(x, r_{j+1})$, and use (2.10).

We now return to (2.1). Set $u_j = \int_{B(x, r_j)} u$; Poincaré’s inequality and (2.10) yield

$$\left(\frac{1}{\int_{B(x, r_j)} |u - u_j|^2}\right)^{\frac{1}{2}} \leq C r_j \omega(x, r_j) \leq C r_j \omega(x, r) + C j r_j.$$

Suppose in addition that $x$ is a Lebesgue point for $u$; then $u(x) = \lim_{t \to \infty} u_t$, and

$$|u(x) - u_j| \leq \sum_{l=j}^{\infty} |u_{l+1} - u_l| \leq \sum_{l=j}^{\infty} \int_{B(x, r_{l+1})} |u - u_l| \leq 2^n \sum_{l=j}^{\infty} \int_{B(x, r_l)} |u - u_l| \leq C \sum_{l=j}^{\infty} r_l \left(\omega(x, r) + l\right) \leq C r_j (\omega(x, r) + j + 1)$$

because $\sum_{l=j}^{\infty} 2^{j-l} \frac{l}{j+1} \leq C$.

We are now ready to prove Theorem 2.1. Let $x_0$, $r_0$, $x$, and $y$ be as in the statement. It is enough to prove (2.1) when $x$ and $y$ are Lebesgue points for $u$ (for the other points, we
would use the estimates that we have on the Lebesgue set to modify $u$ on a set of measure zero and get an equivalent continuous function.

We may even assume that $|x - y| \leq r_0/8$; otherwise, use a few intermediate points. From (2.13) with $r = |x - y|$ and $j = 0$ we deduce that

$$
\left| u(x) - \int_{B(x,r)} u \right| \leq Cr(\omega(x, r) + 1).
$$

Similarly,

$$
\left| u(y) - \int_{B(y,r)} u \right| \leq Cr(\omega(y, r) + 1).
$$

Then set $z = \frac{x + y}{2}$ and $m = \int_{B(z,2r)} u$; by Poincaré, Cauchy-Schwarz, and (2.6),

$$
\left| m - \int_{B(z,2r)} u \right| \leq \int_{B(z,2r)} |u - m| \leq 2^n \int_{B(z,2r)} |u - m| \leq Cr \int_{B(z,2r)} |\nabla u| \leq Cr \left( \int_{B(z,2r)} |\nabla u|^2 \right)^{\frac{1}{2}} = Cr\omega(z, 2r).
$$

Similarly,

$$
\left| m - \int_{B(y,r)} u \right| \leq Cr \left( \int_{B(z,2r)} |\nabla u|^2 \right)^{\frac{1}{2}} = Cr\omega(z, 2r).
$$

Altogether,

$$
|u(x) - u(y)| \leq Cr(\omega(x, r) + \omega(y, r) + 1 + \omega(z, 2r)) \leq Cr(1 + \omega(z, 2r))
$$

by (2.14)-(2.17) and because $B(x, r) \cup B(y, r) \subset B(z, 2r)$.

Let $j$ be such that $2^{-j-3}r_0 \leq r \leq 2^{-j-2}r_0$, and apply (2.10) to the pair $(z, r_0/2)$. Notice that $B(z, r_0/2) \subset B(x_0, 3r_0/2) \subset \Omega$ by assumption, so (2.10) holds. We get that

$$
\omega(z, 2r) \leq 2^{n/2}\omega(z, 2^{-j-1}r_0) \leq C\omega(z, r_0/2) + Cj \leq C\omega(x_0, 3r_0/2) + C \log \frac{r_0}{r}
$$

(recall that $r = |x - y| \leq r_0/8$). Now (2.18) shows that

$$
|u(x) - u(y)| \leq C|x - y| \left( \omega(x_0, 3r_0/2) + \log \frac{r_0}{|x - y|} \right).
$$

This holds under the assumptions of Theorem 2.1, for $x, y \in B(x_0, r_0)$ such that $|x - y| \leq r_0/8$, with a constant $C$ that depends only on $\kappa, \|q_+\|_{L^\infty}, \|q_-\|_{L^\infty}, \alpha, n$, and an upper bound for $r$. Obviously (2.1) and Theorem 2.1 follow.

Theorem 2.1 has the following immediate consequence.

**Corollary 2.1** If $u$ is an almost minimizer for $J$, then for each compact set $K \subset \Omega$ there is a constant $C_K > 0$ such that for $x, y \in K$

$$
|u(x) - u(y)| \leq C_K |x - y| \left( 1 + \log \frac{1}{|x - y|} \right).
$$
3 Almost minimizers are $C^{1,\beta}$ in $\{u > 0\}$ and in $\{u < 0\}$

In this section again, we do not distinguish between $J^+$ and $J$; we write $J$ in the proofs with the understanding that $q_-$ might be identically zero and the functions nonnegative. We address the regularity of an almost minimizer $u$, far from the zero set $\{u = 0\}$. Our estimates will depend on the distance to the zero set. We start with Lipschitz bounds; see Theorem 3.2 for $C^{1,\beta}$ bounds.

**Theorem 3.1** Let $u$ be an almost minimizer for $J$ in $\Omega$. Then $u$ is locally Lipschitz in $\{u > 0\}$ and in $\{u < 0\}$.

**Proof.** We start with $\{u > 0\}$, and assume that $B(x_0, 3r_0) \subset \{u > 0\} \subset \Omega$. For $x \in B(x_0, r_0)$ and $r \leq r_0$, denote by $u^*_r$ the function with the same trace (or radial limit almost everywhere) as $u$ on $\partial B(x, r)$ and which minimizes the Dirichlet energy under this constraint. First we address a minor technical issue.

**Remark 3.1** Let us check that in the present case, since $u$ is continuous on $\partial B(x, r)$ because of Theorem 2.1, $u^*_r$ is also the harmonic extension of the restriction of $u$ to $\partial B(x, r)$, obtained by convolution with the Poisson kernel. Indeed, first observe that $u$ is bounded on $\partial B(x, r)$, and hence $u^*_r$ is bounded in $B(x, r)$ (otherwise truncate and get a better competitor). It is also harmonic in $B(x, r)$, by a standard variational argument. Next let $y \in B(x, r)$ be given, and for $|y - x| < t < r$, write $u^*_r(y)$ as the Poisson integral of $u^*_r$ on $\partial B(x, t)$. That is, $u^*_r(y) = \int_{\partial B(x, t)} P_t(y, z)u^*_r(z)dz$. Then let $t$ tend to $r$; with $y$ fixed, the Poisson kernel stays bounded, and so does $u^*_r(z)$; then we can use the fact that $u^*_r$ has radial limits equal to those of $u$ almost everywhere, the dominated convergence theorem, and the fact that $P_t(y, \cdot)$ tends to $P_r(y, \cdot)$ radially, to conclude that $u^*_r(y) = \int_{\partial B(x, r)} P_r(y, z)u^*_r(z)dz$, as needed.

We return to the proof of Theorem 3.1. Since $u$ is an almost minimizer we have

$$J_{x,r}(u) \leq (1 + \kappa r^\alpha) J_{x,r}(u^*_r)$$

as in (1.9) or (1.11). Here $u > 0$ on $\overline{B}(x, r)$, and by the maximum principle $u^*_r > 0$ on $\overline{B}(x, r)$ also; by (1.10) or (1.12), (3.1) becomes

$$\int_{B(x,r)} (|\nabla u|^2 + q_+(x)) \, dx \leq (1 + \kappa r^\alpha) \int_{B(x,r)} (|\nabla u^*_r|^2 + q_+(x)) \, dx.$$

Thus

$$\int_{B(x,r)} |\nabla u|^2 \leq (1 + \kappa r^\alpha) \int_{B(x,r)} |\nabla u^*_r|^2 + \kappa r^\alpha \int_{B(x,r)} q_+(x) \, dx$$

$$\leq (1 + \kappa r^\alpha) \int_{B(x,r)} |\nabla u^*_r|^2 + Cr^{\alpha+n},$$

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where $C = \kappa \|q_+\|^2_{L^\infty} \omega_n$ and $\omega_n$ denotes the measure of the unit ball in $\mathbb{R}^n$. By (2.3),

$$\int_{B(x,r)} |\nabla u^*_r|^2 = \int_{B(x,r)} \langle \nabla u, \nabla u^*_r \rangle,$$

so (3.3) yields

$$\int_{B(x,r)} |\nabla u - \nabla u^*_r|^2 = \int_{B(x,r)} |\nabla u|^2 + \int_{B(x,r)} |\nabla u^*_r|^2 - 2 \int_{B(x,r)} \langle \nabla u, \nabla u^*_r \rangle$$

(3.4)

$$= \int_{B(x,r)} |\nabla u|^2 - \int_{B(x,r)} |\nabla u^*_r|^2$$

$$\leq \kappa r^\alpha \int_{B(x,r)} |\nabla u^*_r|^2 + C r^{\alpha+n} \leq \kappa r^\alpha \int_{B(x,r)} |\nabla u|^2 + C r^{\alpha+n},$$

by the minimizing property of $u^*_r$.

Define $\omega(x,s)$, for $0 < s \leq r$, as in (2.6). The triangle inequality, (2.7), and (3.4) yield, as for (2.8),

$$\omega(x,s) \leq \left( \frac{r}{s} \right)^{\frac{n}{2}} \left( \int_{B(x,\frac{r}{s})} |\nabla u - \nabla u^*_r|^2 \right)^{\frac{1}{2}} + \left( \int_{B(x,\frac{r}{s})} |\nabla u^*_r|^2 \right)^{\frac{1}{2}}$$

(3.5)

$$\leq \left( \frac{r}{s} \right)^{\frac{n}{2}} \kappa^{1/2} r^{\alpha/2} \left( \int_{B(x,\frac{r}{s})} |\nabla u|^2 \right)^{\frac{1}{2}} + C \left( \frac{r}{s} \right)^{\frac{n}{2}} r^{\alpha/2} + \left( \int_{B(x,\frac{r}{s})} |\nabla u|^2 \right)^{\frac{1}{2}}$$

$$\leq \left( 1 + C \left( \frac{r}{s} \right)^{\frac{n}{2}} r^{\alpha/2} \right) \omega(x,r) + C \left( \frac{r}{s} \right)^{\frac{n}{2}} r^{\alpha/2},$$

with $C = C(\kappa, \|q_+\|_{L^\infty})$. Then set $r_j = 2^{-j} r$ for $j \geq 0$, and apply (3.5) repeatedly; we obtain (as in as in (2.9) and as in (2.10)) that

$$\omega(x,r_{j+1}) \leq \left( 1 + C 2^{n/2} r_{j+1}^{\alpha/2} \right) \omega(x,r_j) + C 2^{n/2} r_{j+1}^{\alpha/2}$$

(3.6)

$$\leq \omega(x,r_j) \prod_{l=0}^{j} \left( 1 + C 2^{n/2} r_l^{\alpha/2} \right) + C \sum_{l=1}^{j+1} \left( \prod_{k=l}^{j} \left( 1 + C 2^{n/2} r_k^{\alpha/2} \right) \right) 2^{n/2} r_{j+1}^{\alpha/2}.$$
Let us continue as in (2.12) and (2.13), but with (2.10) replaced by the better estimate (3.7). Set \( u_j = \int_{B(x,r_j)} u \); by Poincaré and (3.7),

\[
\left( \int_{B(x,r_j)} |u - u_j|^2 \right)^{\frac{1}{2}} \leq Cr_j \omega(x, r_j) \leq Cr_j \left( \omega(x, r) + r^{\alpha/2} \right).
\]

Now we know that \( x \) is a Lebesgue point for \( u \), because Theorem 2.1 says that \( u \) is continuous, and

\[
|u(x) - u_j| \leq 2^n \sum_{l=j}^{\infty} \left( \int_{B(x,r_l)} |u - u_l|^2 \right)^{\frac{1}{2}} \leq C \sum_{l=j}^{\infty} r_l \left( \omega(x, r) + r^{\alpha/2} \right)
\]

\[
\leq Cr_j \left( \omega(x, r) + r^{\alpha/2} \right)
\]

as in (2.13) and by (3.7).

All this holds for \( x \in B(x_0, r_0) \) and \( 0 < r \leq r_0 \). Now let \( y \in B(x_0, r_0) \) be such that \( |x - y| \leq r_0 \). Set \( r = |x - y| \); by (3.10) with \( j = 0 \),

\[
\left| u(x) - \int_{B(x,r)} u \right| \leq Cr(\omega(x, r) + r^{\alpha/2})
\]

and similarly

\[
\left| u(y) - \int_{B(y,r)} u \right| \leq Cr(\omega(y, r) + r^{\alpha/2}).
\]

Again set \( z = \frac{x+y}{2} \) and \( m = \int_{B(z,2r)} u \), and notice that \( B(z,2r) \subset B(x_0, 3r_0) \). By Poincaré, Cauchy-Schwarz, and (2.6), and exactly as in (2.16), we have

\[
\left| m - \int_{B(x,r)} u \right| \leq \int_{B(x,r)} |u - m| \leq 2^n \int_{B(z,2r)} |u - m| \leq Cr \int_{B(z,2r)} |\nabla u|
\]

\[
\leq Cr \left( \int_{B(z,2r)} |\nabla u|^2 \right)^{\frac{1}{2}} = Cr \omega(z, 2r).
\]

Similarly,

\[
\left| m - \int_{B(y,r)} u \right| \leq Cr \omega(z, 2r).
\]

Thus (3.11) to (3.14) yield

\[
|u(x) - u(y)| \leq Cr(\omega(x, r) + \omega(y, r) + r^{\alpha/2}) \leq Cr(\omega(z, 2r) + r^{\alpha/2})
\]

because \( B(x, r) \cup B(y, r) \subset B(z, 2r) \). Let us check that

\[
\omega(z, 2r) \leq C(\omega(x_0, 3r_0) + r_0^{\alpha/2}).
\]
If \( r \leq r_0/2 \), this follows from (3.8), applied to the pair \((z,r_0)\) and \( s = 2r \) (recall that \( z \in B(x_0, r_0) \)); otherwise, for \( r \geq r_0/2 \), \( \omega(z, 2r) \leq C\omega(x_0, 3r_0) \) since \( B(z, 2r) \subset B(x_0, 3r_0) \) because \( r = |x - y| \leq r_0 \). Thus we proved that

\[
|u(x) - u(y)| \leq C|x - y|\left(\omega(x_0, 3r_0) + r_0^{\alpha/2}\right) \tag{3.17}
\]

whenever \( B(x_0, 3r_0) \subset \{u > 0\} \) and \( x, y \in B(x_0, r_0) \) are such that \( |x - y| \leq r_0 \), and where the constant \( C \) depends on \( \kappa, \|q_+\|_{\infty} \) and an upper bound for \( r_0 \) (because of the transition from (3.6) to (3.7)). For the record, notice that this implies that when \( B(x_0, 3r_0) \subset \{u > 0\} \),

\[
|\nabla u(x)| \leq C\left(\omega(x_0, 3r_0) + r_0^{\alpha/2}\right) \quad \text{for almost every } x \in B(x_0, r_0). \tag{3.18}
\]

All this gives local Lipschitz bounds for \( u \) in \( \{u > 0\} \). The local Lipschitz bounds in \( \{u < 0\} \) are handled exactly the same way; we prove that (3.17) holds when \( B(x_0, 3r_0) \subset \{u < 0\} \subset \Omega \) and \( x, y \in B(x_0, r_0) \) are such that \( |x - y| \leq r_0 \), where now the constant \( C \) depends on \( \kappa \), an upper bound for \( r_0 \), \( \kappa \), and \( \|q_-\|_{\infty} \) rather than \( \|q_+\|_{\infty} \). This second case is only relevant when we work with \( J \) rather than \( J^+ \). Theorem 3.1 follows.

Let us now improve Theorem 3.1 slightly, and prove local \( C^{1,\beta} \) estimates for \( u \) far from its zero set.

**Theorem 3.2** Let \( u \) be an almost minimizer for \( J \) in \( \Omega \), and set \( \beta = \frac{\alpha}{n+2+\alpha} \). Then \( u \) is \( C^{1,\beta} \) locally in \( \{u > 0\} \) and in \( \{u < 0\} \).

**Proof.** Of course we do not claim that this value of \( \beta \) is optimal. First assume that \( B(x_0, 4r_0) \subset \{u > 0\} \subset \Omega \), and for \( x \in B(x_0, r_0) \) and \( 0 < r \leq r_0 \), let us compare again with \( u^* \), the harmonic extension of the restriction of \( u \) to \( \partial B(x, r) \).

Let \( \tau \in \left(0, \frac{1}{2}\right) \) be small, to be chosen later. Also set \( v(x, r) = \int_{B(x, r)} \nabla u^*_r \); then by (3.4)

\[
\int_{B(x, \tau r)} |\nabla u - v(x, r)|^2 \leq \int_{B(x, \tau r)} |\nabla u - \nabla u^*_r|^2 + 2\int_{B(x, \tau r)} |\nabla u^*_r - v(x, r)|^2 \leq Cr^\alpha \int_{B(x, r)} |\nabla u|^2 + Cr^{\alpha+n} + 2\int_{B(x, \tau r)} |\nabla u^*_r - v(x, r)|^2. \tag{3.19}
\]

By the mean value theorem and because \( u^*_r \) is harmonic in \( B(x, r) \), \( v(x, r) = \nabla u^*_r(x) \). In addition, standard estimates for harmonic functions (use Remark 3.1 and write \( \nabla u^*_r \) as the integral of its Poisson kernel) yield for \( y \in B(x, \tau r) \)

\[
|\nabla u^*_r(y) - v(x, r)| = |\nabla u^*_r(y) - \nabla u^*_r(x)| \leq \tau r \sup_{B(x, \tau r)} |\nabla^2 u^*_r| \leq C\tau r^{\alpha+n} \tag{3.20}
\]

Thus

\[
\int_{B(x, \tau r)} |\nabla u - v(x, r)|^2 \leq C\tau \left(\int_{B(x, r)} |\nabla u^*_r|^2\right)^{\frac{1}{2}} \leq C\tau \left(\int_{B(x, r)} |\nabla u|^2\right)^{\frac{1}{2}} = C\tau \omega(x, r).
\]
because \( u_r^* \) is harmonic and energy minimizing. Combining (3.19) and (3.20) we obtain

\[
\left( \int_{B(x,\tau r)} |\nabla u - v(x,r)|^2 \right)^{\frac{1}{2}} \leq C \tau^{-n/2} r^{\alpha/2} \omega(x,r) + C \tau^{-n/2} r^{\alpha/2} + \sqrt{2} \left( \int_{B(x,\tau r)} |u_r^* - v(x,r)|^2 \right)^{\frac{1}{2}} \\
(3.21) \leq C \tau^{-n/2} r^{\alpha/2} \omega(x,r) + C \tau^{-n/2} r^{\alpha/2} + C \tau \omega(x,r).
\]

By Theorem 3.1, \( u \) is locally Lipschitz. More precisely, notice that for \( x \in B(x_0,r_0) \), \( B(x,3r_0) \subset \{ u > 0 \} \subset \Omega \), so (3.18) holds for \( B(x,3r_0) \) and yields

\[
|\nabla u(y)| \leq C(\omega(x,3r_0) + r_0^{\alpha/2}) \leq C(\omega(x_0,4r_0) + r_0^{\alpha/2}) \tag{3.22}
\]

for almost every \( y \in B(x,r_0) \). Hence \( \omega(x,r) \leq C(\omega(x_0,4r_0) + r_0^{\alpha/2}) \) for \( 0 < r \leq r_0 \), and (3.21) yields

\[
\int_{B(x,\tau r)} |\nabla u - v(x,r)|^2 \leq C_0^2 (r^\alpha \tau^{-n} + \tau^2) \tag{3.23}
\]

where \( C_0 = C(\omega(x_0,4r_0) + r_0^{\alpha/2} + 1) \) depends on \( \kappa, \|q_+\|_\infty \omega(x_0,4r_0) \), and an upper bound for \( r_0 \).

We want to apply this with \( \tau = \frac{\alpha}{n+2} \); since we want \( \tau < 1/2 \), we only do this for \( r \) small (precisely, so small that \( r^{\frac{n+2}{n+2}} < \frac{1}{2} \)). Then set \( \rho = \tau r = r^{1+\frac{n+2}{n+2}} = r^{\frac{n+2}{n+2} - \frac{n+2}{n+2}} \), and observe that \( r^\alpha \tau^{-n} = \tau^2 = r^{\frac{2n}{n+2}} = \rho^{\frac{2n}{n+2}} \). Also set \( \beta = \frac{\alpha}{n+2+\alpha} \) and \( m(x,\rho) = \int_{B(x,\rho)} \nabla u \); then (3.23) yields

\[
\int_{B(x,\rho)} |\nabla u - m(x,\rho)|^2 \leq \int_{B(x,\rho)} |\nabla u - v(x,r)|^2 \leq 2C_0 \rho^{2\beta} \tag{3.24}
\]

Note that (3.24) holds for all \( x \in B(x_0,r_0) \) (as above) and \( 0 < \rho \leq \rho_0 \), where \( \rho_0 \) is chosen so that if \( \rho \leq \rho_0 \) and if we set \( r = \rho^{\frac{n+2}{n+2} - \frac{n+2}{n+2}} \), then \( r \leq r_0 \) and \( r^{\frac{n+2}{n+2}} < \frac{1}{2} \). In other words, we pick \( \rho_0 \) such that \( \rho_0^{\frac{n+2}{n+2} - \frac{n+2}{n+2}} \leq r_0 \) and \( \rho_0^{\frac{n+2}{n+2} - \frac{n+2}{n+2}} < \frac{1}{2} \).

It follows from the triangle inequality, Cauchy-Schwarz, and (3.24), that for \( 0 < \rho \leq \rho_0 \),

\[
|m(x,\rho/2) - m(x,\rho)| = \left| \int_{B(x,\rho/2)} \nabla u - m(x,\rho) \right| \leq 2^n \int_{B(x,\rho)} |\nabla u - m(x,\rho)| \\
(3.25) \leq 2^n \left( \int_{B(x,\rho)} |\nabla u - m(x,\rho)|^2 \right)^{\frac{1}{2}} \leq CC_0 \rho^{\beta};
\]

then using a standard argument we get that if \( x \) is a Lebesgue point for \( \nabla u \),

\[
|\nabla u(x) - m(x,\rho)| \leq CC_0 \rho^{\beta}. \tag{3.26}
\]
Now let $x, y$ be Lebesgue points for $\nabla u$ such that $x, y \in B(x_0, r_0)$ and $|x - y| \leq \rho_0/2$. Set $\rho = 2|x - y|$, and observe that by (3.24),

$$|m(y, \rho/2) - m(x, \rho)| \leq \int_{B(y, \rho/2)} |\nabla u - m(x, \rho)|$$

$$\leq 2^n \int_{B(x, \rho)} |\nabla u - m(x, \rho)| \leq CC_0^{1/2} \rho^\beta$$

because $B(y, \rho/2) \subset B(x, \rho)$. Since $|\nabla u(x) - m(x, \rho)| + |\nabla u(y) - m(y, \rho/2)| \leq CC_0 \rho^\beta$ by (3.26), we get that

$$|\nabla u(y) - \nabla u(x)| \leq CC_0 \rho^\beta \leq CC_0 |x - y|^\beta.$$ 

Thus (3.28) holds for all Lebesgue points $x \in B(x_0, r_0)$ and $y \in B(x_0, r_0)$ such that $|x - y| > \rho_0/2$, we can connect $x$ to $y$ through less than $4n \rho_0$ intermediate points, and we get that

$$|\nabla u(y) - \nabla u(x)| \leq CC_0 \frac{r_0}{\rho_0} |x - y|^\beta.$$ 

This gives the local $C^{1,\beta}$-regularity of $u$ on $\{u > 0\}$. The argument for $\{u < 0\}$ is the same, and Theorem 3.2 follows.

4 First estimates for the local Lipschitz regularity.

We now focus on the local Lipschitz estimates for $u$ when $u$ is an almost minimizer for $J$ or for $J^+$. The difference with the previous section is that now we also consider balls that meet $\{u > 0\}, \{u < 0\}$, and the zero set of $u$. Our general strategy is to show that the quantity $\omega(x, r)$ (defined in (2.6)) is bounded on compact sets, and does not become too large when $r$ gets small.

In this section we shall prove a few lemmas that work equally well for almost minimizers for $J$ or $J^+$, so we continue with our convention that we do not distinguish between $J$ and $J^+$. For the moment we shall only prove local Lipschitz bounds under an additional condition (namely, that $(x, r) \in G(\tau, C_0, C_1, r_0)$; see below), which we get rid of in later sections.

In addition to $\omega(x, r) = \left(\int_{B(x, r)} |\nabla u|^2\right)^{1/2}$ (see (2.6)), we shall often use the quantities

$$b(x, r) = \int_{\partial B(x, r)} u \quad \text{and} \quad b^+(x, r) = \int_{\partial B(x, r)} |u|$$

which are well defined when $\overline{B}(x, r) \subset \Omega$, because Theorem 2.1 says that $u$ is continuous on $\partial B(x, r)$. 

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For each choice of constants $\tau \in (0, 10^{-2}), C_0 \geq 1$, and $C_1 \geq 3$, and $r_0 > 0$ we introduce the class of pairs $(x, r) \in \Omega \times (0, r_0]$ such that $B(x, 2r) \subset \Omega$,

\begin{align}
(4.2) \quad r^{-1}|b(x, r)| & \geq C_0 \tau^{-n} \left(1 + r^\alpha \omega(x, r)^2\right)^{\frac{1}{2}}, \nonumber \\
\text{and} \nonumber \\
(4.3) \quad b^+(x, r) & \leq C_1 |b(x, r)|. \nonumber
\end{align}

We denote by $G(\tau, C_0, C_1, r_0)$ this class of pairs, i.e., set

\begin{align}
(4.4) \quad G(\tau, C_0, C_1, r_0) = \{(x, r) \in \Omega \times (0, r_0] ; B(x, 2r) \subset \Omega \text{ and } (4.2) \text{ and } (4.3) \text{ hold}\}. \nonumber
\end{align}

Notice that $G(\tau, C_0, C_1, r_0)$ really depends on $C_0 \tau^{-n}$, rather than $C_0$ and $\tau$, but it will be more convenient to use the definition this way. We see (4.2) and as (4.3) nice properties, because they will help us control places where $u$ and the harmonic extension of $u|_{\partial B(x,r)}$ have a given sign. In the mean time we start with a self-improvement lemma.

**Lemma 4.1** Assume that $u$ is an almost minimizer for $J$ in $\Omega$. For each choice of constants $C_1 \geq 3$ and $r_0$, there is a constant $\tau_1 \in (0, 10^{-2})$, that depends only on $n$, $\kappa$, $\alpha$, $r_0 > 0$, and $C_1$, such that if $(x, r) \in G(\tau, C_0, C_1, r_0)$ for some choice of $\tau \in (0, \tau_1)$ and $C_0 \geq 1$, then for each $z \in B\left(x, \frac{r}{2}\right)$ we can find $\rho_z \in (\frac{r}{2}, \frac{r}{2})$ such that $(z, \rho_z) \in G(\tau, 10C_0, 3, r_0)$.

**Proof.** Let $(x, r) \in G(\tau, C_0, C_1, r_0)$ be as in the statement, and as usual denote by $u^*_r$ the harmonic extension of (or initially the smallest energy $W^{1,2}$-function with the same trace as) the restriction of $u$ to $\partial B(x, r)$. Since $0 < \tau \leq 10^{-2}$, standard estimates on harmonic functions ensure that

\begin{align}
(4.5) \quad \sup_{B(x, \tau r)} |\nabla u^*_r| & \leq \frac{C}{r} \sup_{\partial B(x, r/2)} |u^*_r| \leq \frac{C}{r} \int_{\partial B(x, r)} |u^*_r| = \frac{C}{r} b^+(x, r) \nonumber
\end{align}

because $u^*_r = u$ on $\partial B(x, r)$ and by (4.1). Hence, for $y \in B(x, \tau r)$, we deduce from (4.1) and our assumption (4.3) that

\begin{align}
(4.6) \quad |u^*_r(y) - b(x, r)| = |u^*_r(y) - \int_{\partial B(x, r)} u^*_r| = |u^*_r(y) - u^*_r(x)| \nonumber \\
\leq \tau r \sup_{B(x, \tau r)} |\nabla u^*_r| \leq C \tau |b^+(x, r)| \leq CC_1 \tau |b(x, r)|. \nonumber
\end{align}

We choose $\tau_1$ so small that $CC_1 \tau_1 \leq \frac{1}{4}$; since here $\tau \leq \tau_1$, (4.6) yields

\begin{align}
(4.7) \quad |u^*_r(y) - b(x, r)| \leq CC_1 \tau |b(x, r)| \leq \frac{1}{4} |b(x, r)| \quad \text{for } y \in B(x, \tau r). \nonumber
\end{align}

Recall that (2.4) holds as soon as $B(x, r) \subset \Omega$ and $0 < s \leq r$; we take $s = r$ and get that

\begin{align}
(4.8) \quad \int_{B(x, r)} |\nabla u - \nabla u^*_r| \leq \kappa r^\alpha \int_{B(x, r)} |\nabla u|^2 + C = \kappa r^\alpha \omega(x, r)^2 + C. \nonumber
\end{align}
Then, by Poincaré’s inequality,

\( \int_{B(x,r)} |u - u_r^*|^2 \leq C\tau^2 \int_{B(x,r)} |\nabla u - \nabla u_r^*|^2 \leq C\tau^2 \left(r^\alpha \omega(x,r)^2 + 1\right) \)

and by Cauchy-Schwarz

\( \int_{B(x,\tau r)} |u - u_r^*| \leq \left(\int_{B(x,\tau r)} |u - u_r^*|^2\right)^{\frac{1}{2}} \leq C\tau^{-n/2}r \left(r^\alpha \omega(x,r)^2 + 1\right)^{\frac{1}{2}}. \)

Let \( z \in B \left(x, \frac{\tau r}{2}\right) \) be given; since

\( \int_{B(z, \frac{\tau r}{2})} |u - u_r^*| = \int_{0}^{\frac{\tau r}{2}} \int_{\partial B(z,s)} |u - u_r^*| \)

there exists \( \rho_z \in \left(\frac{\tau r}{4}, \frac{\tau r}{2}\right) \) such that

\( \int_{\partial B(z, \rho_z)} |u - u_r^*| \leq \frac{4}{\tau r} \int_{B(z, \frac{\tau r}{2})} |u - u_r^*| \leq \frac{4}{\tau r} \int_{B(x, \tau r)} |u - u_r^*| \)

\( = C(\tau r)^{n-1} \int_{B(x, \tau r)} |u - u_r^*| \leq C(\tau r)^{n-1} r^{-n/2} \left(r^\alpha \omega(x,r)^2 + 1\right)^{\frac{1}{2}}. \)

because \( B \left(z, \frac{\tau r}{2}\right) \subset B(x, \tau r) \) and by (4.10).

Set \( b^* = \int_{\partial B(z, \rho_z)} u_r^* \). Then by (4.1), (4.12), and our assumption (4.2)

\( |b(z, \rho_z) - b^*| = \left| \int_{\partial B(z, \rho_z)} u - \int_{\partial B(z, \rho_z)} u_r^* \right| \leq \int_{\partial B(z, \rho_z)} |u - u_r^*| \)

\( \leq C\tau^{-\frac{\alpha + 1}{2}} \left(1 + r^\alpha \omega(x,r)^2\right)^{\frac{1}{2}} \leq \frac{C\tau^{-\frac{n}{2}}}{C_0 \tau^{-n}} |b(x, r)|. \)

Recall that \( \tau \leq \tau_1 \) and \( C_0 \geq 1 \); if \( \tau_1 \) is chosen small enough, then \( \frac{C\tau^{-\frac{n}{2}}}{C_0 \tau^{-n}} \leq C\tau^\frac{n}{2} \leq \frac{1}{4} \), and (4.13) says that

\( |b(z, \rho_z) - b^*| \leq \frac{1}{4} |b(x, r)|. \)

Since \( \partial B(z, \rho_z) \subset B(x, \tau r) \) by (4.7) we have

\( |b^* - b(x, r)| = \left| \int_{\partial B(z, \rho_z)} (u_r^* - b(x, r)) \right| \leq \int_{\partial B(z, \rho_z)} |u_r^* - b(x, r)| \leq \frac{1}{4} |b(x, r)|. \)

Combining (4.14) and (4.15) we get

\( |b(z, \rho_z) - b(x, r)| \leq |b(z, \rho_z) - b^*| + |b^* - b(x, r)| \leq \frac{1}{2} |b(x, r)|. \)
and hence
\begin{equation}
|b(z, \rho_z)| \geq |b(x, r)| - |b(z, \rho_z) - b(x, r)| \geq \frac{1}{2}|b(x, r)|.
\end{equation}

We can also control $b^*(z, \rho_z)$. Recall that $b(x, r) \neq 0$ by (4.2); by (4.7), $u_r^*$ keeps the same sign as $b(x, r)$ on $\partial B(z, \rho_z) \subset B(x, \tau r)$. Then $\int_{\partial B(z, \rho_z)} |u_r^*| = |\int_{\partial B(z, \rho_z)} u_r^*| = |b^*|$. Hence by (4.1), (4.14), and (4.15)
\begin{equation}
b^*(z, \rho_z) = \int_{\partial B(z, \rho_z)} |u| \leq \int_{\partial B(z, \rho_z)} |u_r^*| + \int_{\partial B(z, \rho_z)} |u - u_r^*| \\
\leq \int_{\partial B(z, \rho_z)} |u_r^*| + \frac{1}{4} |b(x, r)| = |b^*| + \frac{1}{4}|b(x, r)| \leq \frac{3}{2}|b(x, r)|.
\end{equation}

This and (4.17) imply that $b^*(z, \rho_z) \leq 3|b(z, \rho_z)|$ which is (4.3) with $C_1 = 3$.

We still need to check that $(z, \rho_z)$ satisfies (4.2). By (4.17) and (4.2),
\begin{equation}
|b(z, \rho_z)| \geq \frac{1}{2}|b(x, r)| \geq \frac{C_0 \tau^{-n}}{2} r \left(1 + r^\alpha \omega(x, r)^2\right)^{\frac{1}{2}}.
\end{equation}

We want a similar estimate, but in terms of $\omega(z, \rho_z)$, so we need a reasonable upper bound for $\omega(z, \rho_z)$. Let $j \geq 0$ be such that $2^{-j-1} \leq \tau \leq 2^{-j}$, and set $r_j = 2^{-j} \tau$ as before. Observe that (2.10) applies, just because it is valid as soon as $B(x, \tau) \subset \Omega$. It yields
\begin{equation}
\omega(x, r_j) \leq C \omega(x, r) + C j \leq C \omega(x, r) + C(1 + |\log \tau|).
\end{equation}

Here $C$ depends on $n$, $\kappa$, $\alpha$, and our upper bound $r_0$ for $r$. In addition, $B(z, \rho_z) \subset B(x, \tau r) \subset B(x, r_j)$ by definitions, so
\begin{equation}
\omega(z, \rho_z) \leq \left(\frac{r_j}{\rho_z}\right)^{\frac{n}{2}} \omega(x, r_j) \leq 8^{\frac{n}{2}} \omega(x, r_j) \leq C \omega(x, r) + C(1 + |\log \tau|).
\end{equation}

Now
\begin{eqnarray*}
1 + \rho_z^2 \omega(z, \rho_z)^2 & \leq & 1 + C \rho_z^2 \omega(x, \tau)^2 + C \rho_z^2 (1 + |\log \tau|)^2 \\
& \leq & 1 + C r^\alpha \omega(x, r)^2 + C r^\alpha (1 + |\log \tau|)^2 \\
& \leq & 1 + C r^\alpha \omega(x, r)^2 + C r^\alpha [\tau^\alpha (1 + |\log \tau|)^2] \\
& \leq & C(1 + r^\alpha \omega(x, r)^2).
\end{eqnarray*}

By (4.19), (4.22), and because $r \geq 2\tau^{-1} \rho_z$,
\begin{equation}
|b(z, \rho_z)| \geq \frac{C_0 \tau^{-n}}{2} r \left(1 + r^\alpha \omega(x, r)^2\right)^{\frac{1}{2}} \\
\geq C^{-1} C_0 \tau^{-n} \left(2 \tau^{-1} \rho_z\right) \left(1 + \rho_z^2 \omega(z, \rho_z)^2\right)^{\frac{1}{2}} \\
= (C \tau)^{-1} C_0 \tau^{-n} \rho_z \left(1 + \rho_z^2 \omega(z, \rho_z)^2\right)^{\frac{1}{2}}.
\end{equation}

We choose $\tau_1$ so large that $(C \tau)^{-1} \geq 10$ in (4.23), and this gives (4.2) with the constant $10C_0$. This completes our verification that $(z, \rho_z) \in G(\tau, 10C_0, 3, r_0)$. Lemma 4.1 follows. \qed
Lemma 4.2 Let $u$, $x$, and $r$ satisfy the assumptions of Lemma 4.1, so that in particular $(x, r) \in G(\tau, C_0, C_1, r_0)$ for some $C_0 \geq 1$ and $\tau \leq \tau_1$. Recall that $b(x, r) \neq 0$ by (4.2). If $b(x, r) > 0$, then

(4.24) \hspace{1cm} u \geq 0 \text{ on } B(x, \tau r/2) \text{ and } u > 0 \text{ almost everywhere on } B(x, \tau r/2).

If instead $b(x, r) < 0$, then

(4.25) \hspace{1cm} u \leq 0 \text{ on } B(x, \tau r/2) \text{ and } u < 0 \text{ almost everywhere on } B(x, \tau r/2).

Proof. In this proof we apply Lemma 4.1 repeatedly. Let $u$, $x$, and $r$ be as in the statement, and let $z \in B(x, \tau r/2)$ be given. Apply Lemma 4.1 a first time. This gives a radius $\rho_z \in (\tau r/4, \tau t/2)$ such that $(z, \rho_z) \in G(\tau, 10C_0, 3, r_0)$. Set $\rho_0 = \rho_z$ to unify the notation below.

Then iterate, but this time systematically apply Lemma 4.1 with $x$ and $z$ both equal to the recently given $z$. This gives a sequence of radii $\rho_j$, $j \geq 0$, such that for $j \geq 0$

(4.26) \hspace{1cm} (z, \rho_j) \in G(\tau, 10^jC_0, 3, r_0)

which implies by (4.2) that

(4.27) \hspace{1cm} \rho_j^{-1}|b(z, \rho_j)| \geq 10^jC_0\tau^{-n} (1 + \rho_j^\omega(z, \rho_j^2))^{3/2}.

Moreover by construction, we also get that for $j \geq 0$

(4.28) \hspace{1cm} \frac{\tau \rho_j}{4} \leq \rho_{j+1} \leq \frac{\tau \rho_j}{2}.

Let $u^*_j$ denote the energy-minimizing extension of the restriction of $u$ to $\partial B(z, \rho_j)$. This is the function that we call $u^*_j$ when we prove Lemma 4.1 with $x = z$ and $r = \rho_j$. By (4.7) in this context,

(4.29) \hspace{1cm} |u^*_j(y) - b(z, \rho_j)| \leq \frac{1}{4}|b(z, \rho_j)| \text{ for } y \in B(z, \tau \rho_j).

Note that (4.16) ensures that $b(z, \rho_z)$ is not zero, and that it has the same sign as $b(x, r)$. In addition, $b(z, \rho_j) \neq 0$ by (4.27). Thus by (4.16) and an easy induction argument we conclude that $b(z, \rho_j)$ has the same sign for all $j$. Set

(4.30) \hspace{1cm} Z_j = \{y \in B(z, \tau \rho_j); u(y)b(x, r) \leq 0\} = \{y \in B(z, \tau \rho_j); u(y)b(z, \rho_j) \leq 0\};

this is just the subset of $B(z, \tau \rho_j)$ where $u$ does not have the right sign. Notice that for $y \in Z_j$,

(4.31) \hspace{1cm} |u(y) - u^*_j(y)| \geq |u(y) - b(z, \rho_j)| - |b(z, \rho_j) - u^*_j(y)| \geq \frac{3}{4}|b(z, \rho_j)|

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because \(|u(y) - b(z, \rho_j)| \geq |b(z, \rho_j)|\) and by (4.29). Then by Chebyshev, the Lebesgue measure of \(Z_j\) is
\[
m(Z_j) \leq \frac{4}{3|b(z, \rho_j)|} \int_{B(z, \tau \rho_j)} |u - u^*_j|.
\]
But by (4.10)
\[
\int_{B(z, \tau \rho_j)} |u - u^*_j| \leq C(\tau \rho_j)^n \tau^{-n/2} \rho_j (1 + \rho_j^2 \omega(z, \rho_j)^2)^{1/2}.
\]
We now combine (4.32), (4.33), and (4.27) to get that
\[
m(Z_j) \leq C[10^j C_0 \tau^{-n} \rho_j]^{-1} (\tau \rho_j)^n \tau^{-n/2} \rho_j = C[10^j C_0]^{-1} \tau^{3n/2} \rho_j^n.
\]
Let us assume that \(b(x, r) > 0\), and thus \(b(z, \rho_j) > 0\) for \(j \geq 0\), to simplify the discussion. We just proved that for every \(z \in B(x, \tau r/2)\),
\[
\lim_{j \to \infty} \frac{m(\{u \leq 0\} \cap B(z, \tau \rho_j))}{m(B(z, \tau \rho_j))} = \lim_{j \to \infty} \frac{m(Z_j)}{m(B(z, \tau \rho_j))} = 0.
\]
Now (4.35) fails when \(z\) is a Lebesgue point of \(\{u \leq 0\}\), which is the case for almost every \(z \in \{u \leq 0\}\). Hence \(m(\{u \leq 0\} \cap B(x, \tau r/2)) = 0\), that is \(u(x) > 0\) almost everywhere on \(B(x, \tau r/2)\), and (4.24) holds (recall that \(u\) is continuous).
The case when \(b(x, r) < 0\) is dealt with in a similar fashion; Lemma 4.2 follows.

**Lemma 4.3** Let \(u\) be an almost minimizer for \(J\) in \(\Omega\), and let \(x\) and \(r\) satisfy the assumptions of Lemma 4.1, so that in particular \((x, r) \in G(\tau, C_0, C_1, r_0)\) for some \(C_0 \geq 1\) and \(\tau \leq \tau_1\). Then for \(z \in B(x, \tau r/4)\) and \(\rho \in (0, \tau r/8)\)
\[
\omega(z, \rho) \leq C \left( \tau^{-\frac{n}{2}} \omega(x, r) + r^\frac{\alpha}{2} \right).
\]
Moreover for \(y, z \in B(x, \tau r/4)\)
\[
|u(y) - u(z)| \leq C \left( \tau^{-\frac{n}{2}} \omega(x, r) + r^\frac{\alpha}{2} \right) |y - z|.
\]
Here \(C\) depends on \(n, \kappa, \alpha,\) and \(r_0\). Finally, there is a constant \(C(\tau, r)\), that depends on \(n, \kappa, \alpha, r_0, \tau,\) and \(r\), such that
\[
|\nabla u(y) - \nabla u(z)| \leq C(\tau, r) (\omega(x, r) + 1) |y - z|^\beta,
\]
for \(y, z \in B(x, \tau r/8)\) and where \(\beta = \frac{\alpha}{n+2+\alpha}\) as in Theorem 3.2.
Proof. Let \( u, x, \) and \( r \) be as in the statement. Also let \( z \in B \left( x, \frac{\tau r}{4} \right) \) be given. By the proof of Lemma 4.1, we can find \( \rho \in \left( \frac{\tau r}{4}, \frac{\tau r}{4} \right) \) such that \((z, \rho) \in G(\tau, 10C_0, 3, r_0)\). Notice that we made \( \rho \) smaller here, but the proof of Lemma 4.1 still gives this, maybe at the expense of making \( \tau_1 \) a little smaller; we could also get a pair \((z, \rho)\) with a smaller \( \rho \), by applying Lemma 4.1 a second time to the pair \((z, \rho_2)\) given by Lemma 4.1, but this would be a little clumsier and would give slightly worse estimates.

Let \( u^*_\rho \) denote the usual energy-minimizing extension of the restriction of \( u \) to \( \partial B(z, \rho) \). Since \( u \) is an almost minimizer,

\[
J_{z, \rho}(u) \leq (1 + \kappa \rho^\alpha) J_{z, \rho}(u^*_\rho)
\]

as in (1.11), and where \( J_{z, \rho} \) is as in (1.12). Recall that here we shall just take \( q_- = 0 \) if we work with \( J^+; \) see (1.9) and (1.10).

Let us assume that \( b(x, r) > 0 \); the other case would be similar. By Lemma 4.2, \( u \geq 0 \) everywhere and \( u > 0 \) almost everywhere on \( B(x, \tau r/2) \), as in (4.24). We just made sure to chose \( \rho \) so that \( B(z, \rho) \subset B(x, \tau r/2) \), so this happens on \( B(z, \rho) \) too. Then (1.12) yields

\[
J_{z, \rho}(u) = \int_{B(z, \rho)} |\nabla u|^2 + q_+^2 \chi_{\{u > 0\}} + q_-^2 \chi_{\{u < 0\}} = \int_{B(z, \rho)} |\nabla u|^2 + q_+^2
\]

(the term with \( \chi_{\{u < 0\}} \) disappears as \( u \geq 0 \) in \( B(z, \rho) \)). Since \( u^*_\rho \) is harmonic in \( B(z, \rho) \) and \( u^*_\rho = u \) on \( \partial B(z, \rho) \) the maximum principle ensures that \( u^*_\rho \geq 0 \) in \( B(z, \rho) \). Moreover

\[
J_{z, \rho}(u^*_\rho) = \int_{B(z, \rho)} |\nabla u^*_\rho|^2 + q_+^2 \chi_{\{u^*_\rho > 0\}} \leq \int_{B(z, \rho)} |\nabla u^*_\rho|^2 + q_+^2.
\]

So \( \int_{B(z, \rho)} q_+^2 \) in (4.39) cancels partially, and we get that

\[
\int_{B(z, \rho)} |\nabla u|^2 \leq (1 + \kappa \rho^\alpha) \int_{B(z, \rho)} |\nabla u^*_\rho|^2 + \kappa \rho^\alpha \int_{B(z, \rho)} q_+^2 \leq (1 + \kappa \rho^\alpha) \int_{B(z, \rho)} |\nabla u^*_\rho|^2 + C \rho^{n+\alpha}.
\]

This is the same as (3.3), with \((x, r)\) replaced with \((z, \rho)\), even though we obtained the cancellation for different reasons. We continue the argument as in (3.4)-(3.8). Notice that we still have that \( u > 0 \) almost everywhere in smaller balls, so we can iterate the argument as we did in (3.6) We finally obtain, as in (3.8), that

\[
\omega(z, s) \leq C \omega(z, \rho) + C \rho^{n/2} \text{ for } 0 < s \leq \rho.
\]

Since in addition \( \omega(z, \rho) \leq C \tau^{-\frac{\alpha}{2}} \omega(x, r) \) by the definition (2.6), we deduce from (4.43) that for \( 0 < s \leq \tau r/8 \),

\[
\omega(z, s) \leq C \tau^{-\frac{\alpha}{2}} \omega(x, r) + \rho^{n/2},
\]
which is slightly better than (4.36).
We continue the argument as in the proof of Theorem 3.1. Set \( \rho_j = 2^{-j}\rho \) (the analogue of \( r_j = 2^{-j}r \)) and \( u_j = \mathcal{f}_{B(z,\rho_j)} u \), we still have the analogue of (3.10), i.e., that

\[
|u(z) - u_j| \leq C\rho_j^p(\omega(z, \rho) + \rho^{\alpha/2}) \leq C2^{-j\tau}r(\tau^{-\frac{n}{2}}\omega(x, r) + \rho^{\alpha/2}).
\]

Next let \( z' \) be another point of \( B(x, \tau r/4) \), denote by \( \rho' \in (\frac{\tau}{8}, \frac{\tau}{4}) \) the analogue of \( \rho \) for \( z' \), and also define \( \rho'_j = 2^{-j}\rho' \) and \( u'_j = \mathcal{f}_{B(z,\rho'_j)} u \).

\[
|u(z') - u'_j| \leq C\rho'_j(\omega(z', \rho') + (\rho')^{\alpha/2}) \leq C2^{-j\tau}r(\tau^{-\frac{n}{2}}\omega(x, r) + \rho^{\alpha/2}).
\]

Set \( \delta = \frac{|z' - z|}{4} \). Choose \( j \) such that \( \rho_{j+1} \leq \delta < \rho_j \) and \( j' \) such that \( \rho'_{j+1} \leq \delta < \rho'_j \), and set \( B = B(z, \rho_j), B' = B(z', \rho'_j), \) and \( B'' = B(z, (2^{-j}\rho + 2^{-j'}\rho')) \). Notice that \( B \cup B' \subset B'' \) because \( |z' - z| = \delta \leq 2^{-j}\rho \), and that \( u_j = \mathcal{f}_B u \) and \( u'_j = \mathcal{f}_{B'} u \). Also set \( m = \mathcal{f}_{B''} u \). Then by Poincaré

\[
|u'_j - u_j| \leq |m - \mathcal{f}_{B'} u| + |m - \mathcal{f}_B u| \leq C\delta \int_{B''} |u - m| + C\delta \int_{B''} |u - m|.
\]

because the radii \( 2^{-j}\rho, 2^{-j'}\rho', \) and \( 2^{-j}\rho + 2^{-j'}\rho' \) are all comparable to \( \delta \).

Suppose in addition that \( \delta \leq \tau r/32 \). Then \( 2^{-j}\rho + 2^{-j'}\rho' \leq 4\delta \leq \tau r/8 \leq \rho \). By Cauchy-Schwarz and (4.44),

\[
\int_{B''} |\nabla u| \leq \omega(z, 2^{-j}\rho + 2^{-j'}\rho') \leq C(\tau^{-\frac{n}{2}}\omega(x, r) + \rho^{\alpha/2}).
\]

Altogether, by (4.45), (4.46), (4.47), and (4.48),

\[
|u(z') - u(z)| \leq |u(z') - u'_j| + |u'_j - u_j| + |u_j - u(z)|
\]

\[
\leq C(2^{-j}\rho + 2^{-j'}\rho') \left( \tau^{-\frac{n}{2}}\omega(x, r) + \rho^{\alpha/2} \right) + C\delta \int_{B''} |\nabla u|
\leq C\delta \left( \tau^{-\frac{n}{2}}\omega(x, r) + \rho^{\alpha/2} \right) = C|z' - z| \left( \tau^{-\frac{n}{2}}\omega(x, r) + \rho^{\alpha/2} \right).
\]

When \( z, z' \in B(x, \tau r/4) \) are such that \( \delta = |z' - z| > \tau r/32 \), we still get (4.49), by going through a few intermediate points; (4.37) follows.

For (4.38) we follow the proof of Theorem 3.2. Let us nonetheless sketch the argument, since there are a few small differences. First we fix \( z \in B(x, \tau r/8) \), define \( \rho \) and \( u^* \) as above, and proceed as in Theorem 3.2, but with \( (x, r) \) replaced by \( (z, \rho) \). Up to (3.21), we change nothing; then we observe that since (4.49) holds for all \( z \in B(x, \tau r/4) \),

\[
|\nabla u(y)| \leq C \left( \tau^{-\frac{n}{2}}\omega(x, r) + \rho^{\alpha/2} \right)
\]
almost everywhere on \( B(x, \tau r/4) \), and hence
\[
\omega(z, \rho) \leq C \left( \tau^{-\frac{3}{2}} \omega(x, r) + \rho^{\alpha/2} \right) \tag{4.51}
\]
(because \( B(z, \rho) \subset B(x, \tau r/4) \) since \( z \in B(x, \tau r/8) \)). We compare with the analogue of (3.21) and get that
\[
\int_{B(x, \tau \rho)} \left| \nabla u - \nabla u^* \right|^2 \leq C \left( \tau^{-n} \rho^\alpha + \tau^2 \right) \omega(z, \rho) + C \tau^{-n} \rho^\alpha \tag{4.52}
\]
with \( C_0 = C \left( \tau^{-\frac{3}{2}} \omega(x, r) + \rho^{\alpha/2} + 1 \right) \), and as in (3.23). We continue the argument, with (3.23) replaced by (4.52). We find a small \( \rho_0 \), that depends on \( \beta = \frac{\alpha}{n+2+\alpha} \) (see below (3.24)) \( r \), and \( \tau \), such that, as in (3.25),
\[
\left| \nabla u(z) - \nabla u \right| \leq CC_0 \rho^\beta \text{ for } 0 < t \leq \rho_0. \tag{4.53}
\]
This also yields that \( \left| \nabla u(z) - \int B(z,t) \nabla u \right| \leq CC_0 \rho^\beta \) if \( z \) is a Lebesgue point for \( \nabla u \), by summing a geometric series.

Then, if \( y \in B(x, \tau r/8) \) is another Lebesgue point for \( \nabla u \), and if \( |y - z| \leq \rho_0/2 \), we deduce from (4.53), a similar estimate for \( y \), and a comparison as in (3.27) that
\[
|\nabla u(y) - \nabla u(z)| \leq CC_0 |x - y|^{\beta}; \tag{4.54}
\]
our last estimate (4.38) follow easily from this. This completes our proof of Lemma 4.3. \( \square \)

We end this section with a way to obtain pairs \( (x, \rho) \) that satisfy the conditions of Lemmas 4.1-4.3, under somewhat weaker assumptions. We shall now assume that \( b(x, r) = \int_{\partial B(x,r)} u \) is not too small, and use homogeneity to find a smaller radius \( \rho \) where (accounting for scale invariance) it looks big.

**Lemma 4.4** Let \( u \) be an almost minimizer for \( J \) in \( \Omega \). For each choice of \( \gamma \in (0, 1) \), \( \tau > 0 \), and \( C_0 \geq 1 \), we can find \( r_0, \eta < 10^{-1} \), and \( K \geq 1 \) with the following property. Let \( x \in \Omega \) and \( r > 0 \) be such that \( 0 < r \leq r_0 \), \( B(x, 2r) \subset \Omega \),
\[
|b(x, r)| \geq \gamma r (1 + \omega(x, r)) \tag{4.55}
\]
and
\[
\omega(x, r) \geq K. \tag{4.56}
\]
Then there exists \( \rho \in \left( \frac{\eta r}{2}, \eta r \right) \) such that \( (x, \rho) \in G(\tau, C_0, 3, r_0) \).
In particular, (4.60)\[ \eta \]

We shall choose \( y \)

Then, for \( y \in B(x, \eta r) \) and because \( u^*_r(x) = \int_{\partial B(x, r)} u = b(x, r) \) by harmonicity (and either Remark 3.1 or a small limiting argument to go to the traces on the boundary \( \partial B(x, r) \), which is very easy because our functions are bounded and we could use the dominated convergence theorem),

\[
|u^*_r(y) - b(x, r)| = |u^*_r(y) - u^*_r(x)| \leq \eta r \sup_{B(x, \eta r)} \nabla u^*_r \leq 2^{n/2} \eta r \omega(x, r).
\]

We shall choose \( \eta \) so small that \( \eta^{2n/2} < \gamma/4 \); then (4.55) yields

\[
|u^*_r(y) - b(x, r)| \leq 2^{n/2} \eta r \omega(x, r) \leq \frac{1}{4} \gamma r \omega(x, r) \leq \frac{1}{4} |b(x, r)|.
\]

In particular, \( u^*_r(y) \) keeps the same sign as \( b(x, r) \) on \( B(x, \eta r) \), and

\[
\frac{5}{4} |b(x, r)| \geq |u^*_r(y)| \geq \frac{3}{4} |b(x, r)| \text{ for } y \in B(x, \eta r).
\]

Choose \( \rho \in \left( \frac{\eta r}{2}, \eta r \right) \), as we did near (4.11), such that by Poincaré’s inequality and Cauchy-Schwarz,

\[
\int_{\partial B(x, \rho)} |u - u^*_r| \leq \frac{2}{\eta r} \int_{\eta r/2}^{\eta r} \int_{\partial B(x, \rho)} |u - u^*_r| d\rho \leq \frac{2}{\eta r} \int_{B(x, \eta r)} |u - u^*_r| \\
\leq C \int_{B(x, \eta r)} |\nabla u - \nabla u^*_r| \leq C(\eta r)^{\frac{n}{2}} \left( \int_{B(x, \eta r)} |\nabla u - \nabla u^*_r|^2 \right)^\frac{1}{2} \\
\leq C(\eta r)^{\frac{n}{2}} \left( \int_{B(x, r)} |\nabla u - \nabla u^*_r|^2 \right)^\frac{1}{2}.
\]

By (2.4) and as in (4.8)

\[
\int_{B(x, r)} |\nabla u - \nabla u^*_r|^2 \leq \kappa r^\alpha \int_{B(x, r)} |\nabla u|^2 + C = \kappa r^\alpha \omega(x, r)^2 + C,
\]

so

\[
\int_{\partial B(x, \rho)} |u - u^*_r| \leq C \eta^{\frac{n}{2}} r^\alpha (1 + r^\alpha \omega(x, r)^2)^\frac{1}{2}.
\]
Recall that $r \leq r_0$; then $r^\alpha \leq r_0^\alpha$, and by (4.63) and (4.56)

\[(4.64) \quad \int_{\partial B(x,\rho)} |u - u_r^*| \leq C(\eta r)^{1-n} \int_{\partial B(x,\rho)} |u - u_r^*| \leq C\eta^{1-\frac{n}{2}} r \left(1 + r_0^\alpha \omega(x, r)^2\right)^{\frac{1}{2}} \]
\[\leq C\eta^{1-\frac{n}{2}} r \omega(x, r) \left(K^{-2} + r_0^\alpha\right)^{\frac{1}{2}}.\]

We shall choose $K$ large enough, and $r_0$ small enough, both depending on $\gamma$ and $\eta$, so that in (4.64)

\[(4.65) \quad C\eta^{1-\frac{n}{2}} \left(K^{-2} + r_0^\alpha\right)^{\frac{1}{2}} \leq \frac{\gamma}{4}.\]

Then by (4.64) and (4.55)

\[(4.66) \quad \int_{\partial B(x,\rho)} |u - u_r^*| \leq \frac{\gamma}{4} r \omega(x, r) \leq \frac{|b(x, r)|}{4}.\]

Recall that $u^*_r$ does not change signs on $\partial B(x, \rho)$; then by (4.60) and (4.66)

\[(4.67) \quad |b(x, \rho)| = \left| \int_{\partial B(x,\rho)} u \right| \geq \left| \int_{\partial B(x,\rho)} u_r^* \right| - \int_{\partial B(x,\rho)} |u - u_r^*| \]
\[\quad = \int_{\partial B(x,\rho)} |u_r^*| - \int_{\partial B(x,\rho)} |u - u_r^*| \]
\[\quad \geq \frac{3}{4}|b(x, r)| - \frac{1}{4}|b(x, r)| = \frac{1}{2}|b(x, r)|.\]

The same computations yield

\[(4.68) \quad |b^+(x, \rho)| = \int_{\partial B(x,\rho)} |u| \leq \int_{\partial B(x,\rho)} |u_r^*| + \int_{\partial B(x,\rho)} |u - u_r^*| \]
\[\quad \leq \frac{5}{4}|b(x, r)| + \frac{1}{4}|b(x, r)| \leq \frac{3}{2}|b(x, r)|\]

by the definition (4.1), (4.60), and (4.66). It follows from (4.67) and (4.68) that $(x, \rho)$ satisfies (4.3) with $C_1 = 3$.

We still need to check (4.2). But by (4.67) and (4.55),

\[(4.69) \quad \frac{|b(x, \rho)|}{\rho} \geq \frac{1}{2\rho} |b(x, r)| \geq \frac{\gamma^r}{2\rho} (1 + \omega(x, r)) \geq \frac{\gamma}{2\eta} (1 + \omega(x, r))\]

and now we want a lower bound for $\omega(x, r)$ in terms of $\omega(x, \rho)$.

Recall from (2.10) that whenever $B(x, r) \subset \Omega$, we have that for $j \geq 0$,

\[(4.70) \quad \omega(x, 2^{-j-1}r) \leq C\omega(x, r) + Cj.\]
We apply this to the integer $j$ such that $2^{-j-2}r \leq \rho < 2^{-j-1}r$ and get that
\begin{equation}
\omega(x, \rho) \leq 2^{n/2}\omega(x, 2^{-j-1}r) \leq C\omega(x, r) + Cj \leq C\omega(x, r) + C|\log \eta|
\end{equation}
and now (4.69) yields
\begin{equation}
(1 + \rho^\alpha \omega(x, \rho)^2)^{1/2} \leq 1 + \rho^{\alpha/2}\omega(x, \rho) \leq 1 + Cr_0^{\alpha/2}\omega(x, r) + Cr_0^{\alpha/2}|\log \eta|
\end{equation}
\begin{equation}
\leq \left(1 + Cr_0^{\alpha/2} + Cr_0^{\alpha/2}|\log \eta|\right)(1 + \omega(x, r))
\end{equation}
\begin{equation}
\leq \left(1 + Cr_0^{\alpha/2} + Cr_0^{\alpha/2}|\log \eta|\right)\frac{2\eta |b(x, \rho)|}{\gamma} \rho
\end{equation}
and (multiplying with $C_0\tau^{-n}$)
\begin{equation}
C_0\tau^{-n} (1 + \rho^\alpha \omega(x, \rho)^2)^{1/2} \leq A \frac{|b(x, \rho)|}{\rho},
\end{equation}
with
\begin{equation}
A = C_0\tau^{-n} \left(1 + Cr_0^{\alpha/2} + Cr_0^{\alpha/2}|\log \eta|\right)\frac{2\eta}{\gamma}
\end{equation}
Thus we see that (4.2) holds for the pair $(x, \rho)$ as soon as $A \leq 1$. We choose $\eta$ so small, depending on $C_0$, $\tau$, and $\gamma$, that $C_0\tau^{-n} \frac{2\eta}{\gamma} \leq \frac{1}{2}$, and then $r_0$ so small, depending on $\eta$, that $1 + Cr_0^{\alpha/2} + Cr_0^{\alpha/2}|\log \eta| \leq 2$. This is compatible with our previous constraints (see below (4.58) and below (4.64)). It is true that $C$ in (4.74) depends on $r_0$, but only through an upper bound on $r_0$, so there is no vicious circle here.
We just finished our verification that $(x, \rho) \in G(\tau, C_0, 3, r_0)$; this completes our proof of Lemma 4.4.

\section{Almost minimizers for $J^+$ are locally Lipschitz}

We now distinguish between $J^+$ and $J$. We start with the somewhat easier case of $J^+$, and prove in this section that the almost minimizers of $J^+$ are locally Lipschitz in $\Omega$ (see Theorem 5.1 below).

\begin{lemma}
Let $u$ be an almost minimizer for $J^+$ in $\Omega$. Pick $\theta \in (0, 1/2)$. There exist $\gamma > 0$, $K_1 > 1$, $\beta \in (0, 1)$, and $r_1 > 0$ such that if $x \in \Omega$ and $0 < r \leq r_1$ are such that $B(x, r) \subset \Omega$,
\begin{equation}
b(x, r) \leq \gamma r (1 + \omega(x, r)),
\end{equation}
and
\begin{equation}
\omega(x, r) \geq K_1,
\end{equation}
then
\begin{equation}
\omega(x, \theta r) \leq \beta \omega(x, r).
\end{equation}
\end{lemma}
Proof. Recall from the definition (1.8) that almost minimizers for \( J^+ \) are non-negative almost everywhere, hence everywhere on \( \Omega \) because Theorem 2.1 says that they are continuous (after modification on a set of measure zero).

Let \( x \in \Omega \) and \( r \leq r_1 \) be such that \( B(x, r) \subset \Omega \). Let \( u_r^* \) denote the energy-minimizing extension of the restriction of \( u \) to \( \partial B(x, r) \). Note that \( u_r^* \geq 0 \) in \( B(x, r) \). For \( y \in B(x, r) \), set \( a(y) = u_r^*(x) + \langle \nabla u_r^*(x), y - x \rangle \), and

\[
(5.4) \quad v_r^*(y) = u_r^*(y) - a(y) = u_r^*(y) - u_r^*(x) - \langle \nabla u_r^*(x), y - x \rangle;
\]

then \( v^* \) is harmonic in \( B(x, r) \), \( v_r^*(x) = 0 \), and \( \nabla v_r^*(x) = 0 \).

Recall from the fourth line of (5.4) that for \( 0 < s \leq r \),

\[
(5.5) \quad \omega(x, s) \leq C \left( \frac{r}{s} \right)^{n/2} r^{\alpha/2} \omega(x, r) + C \left( \frac{r}{s} \right)^{n/2} + \left( \int_{B(x, s)} |\nabla u_r^*|^2 \right)^{1/2}.
\]

Next we evaluate \( \int_{B(x, r)} |\nabla u_r^*|^2 \). By (5.4) and because \( \nabla a = \nabla u_r^*(x) \),

\[
(5.6) \quad \int_{B(x, r)} |\nabla u_r^*|^2 = \int_{B(x, r)} |\nabla (a + v_r^*)|^2 = \int_{B(x, s)} |\nabla v_r^*|^2 + \int_{B(x, s)} |\nabla a|^2 + 2 \int_{B(x, s)} \langle \nabla a, \nabla v_r^* \rangle.
\]

Since \( v_r^* \) is harmonic on \( B(x, r) \), so is \( \nabla v_r^* \), and \( \int_{B(x, s)} \nabla v_r^* = \nabla v_r^*(x) = 0 \) by definition of \( v_r^* \). So

\[
(5.7) \quad \int_{B(x, s)} |\nabla u_r^*|^2 = |\nabla u_r^*(x)|^2 + \int_{B(x, s)} |\nabla v_r^*|^2.
\]

The same proof, with \( B(x, s) \) replaced by \( B(x, r) \) shows that

\[
(5.8) \quad \int_{B(x, r)} |\nabla u_r^*|^2 = |\nabla u_r^*(x)|^2 + \int_{B(x, r)} |\nabla v_r^*|^2.
\]

We return to \( \int_{B(x, s)} |\nabla u_r^*|^2 \). By (5.7), because \( \int_{B(x, s)} \nabla v_r^* = \nabla v_r^*(x) = 0 \), by Poincaré’s inequality, and because \( \nabla^2 a = 0 \),

\[
(5.9) \quad \int_{B(x, s)} |\nabla u_r^*|^2 = |\nabla u_r^*(x)|^2 + \int_{B(x, s)} |\nabla v_r^*|^2
\]

\[
\leq |\nabla u_r^*(x)|^2 + C s^2 \int_{B(x, s)} |\nabla^2 u_r^*|^2
\]

\[
\leq |\nabla u_r^*(x)|^2 + C s^2 \int_{B(x, s)} |\nabla^2 u_r^*|^2.
\]

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Now suppose that $s < r/2$; by standard estimates on harmonic functions,

\[(5.10) \quad \int_{B(x,s)} |\nabla^2 u_r^s|^2 \leq \sup_{B(x,s)} |\nabla^2 u_r^s|^2 \leq C \left( r^{-2} \int_{\partial B(x,r)} |u_r^s|^2 \right)^2 = C \left( r^{-2} \int_{\partial B(x,r)} u \right)^2 = C r^{-4} b(x,r)^2 \]

because $u_r^s = u \geq 0$ on $\partial B(x,r)$ and by the definition (4.1). Now (5.9) and (5.10) yield

\[(5.11) \quad \int_{B(x,s)} |\nabla u_r^s|^2 \leq |\nabla u_r^s(x)|^2 + C s^2 \int_{B(x,s)} |\nabla^2 u_r^s|^2 \leq |\nabla u_r^s(x)|^2 + C r^{-4} s^2 b(x,r)^2. \]

By (5.5), (5.11) and since $b(x,r) \geq 0$ (and because $\sqrt{a^2 + b^2} \leq a + b$ for $a, b \geq 0$)

\[(5.12) \quad \omega(x,s) \leq C \left( \frac{r}{s} \right)^{n/2} r^{\alpha/2} \omega(x,r) + C \left( \frac{r}{s} \right)^{n/2} \left( \int_{B(x,s)} |\nabla u_r^s|^2 \right)^{1/2} \leq C \left( \frac{r}{s} \right)^{n/2} r^{\alpha/2} \omega(x,r) + C \left( \frac{r}{s} \right)^{n/2} + |\nabla u_r^s(x)| + C r^{-2} s b(x,r). \]

Let $\theta \in (0,1/2)$ be as in the statement, and take $s = \theta r < r/2$. With this notation, (5.12) yields

\[(5.13) \quad \omega(x, \theta r) \leq |\nabla u_r^s(x)| + C \theta^{-n/2} r^{\alpha/2} \omega(x,r) + C \theta^{-n/2} + C\theta r^{-1} b(x,r) \leq |\nabla u_r^s(x)| + C \theta^{-n/2} \left( r^{\alpha/2} + K^{-1} \right) \omega(x,r) + C \theta \gamma (1 + \omega(x,r)) \leq |\nabla u_r^s(x)| + C \left[ \theta^{-n/2} \left( r^{\alpha/2} + K^{-1} \right) + \theta \gamma \left( K^{-1} + 1 \right) \right] \omega(x,r) \]

by (5.2) and (5.1).

We shall now control $|\nabla u_r^s(x)|$ in terms of $\omega(x,r)$. We consider two cases. Let $\eta > 0$ be small, to be chosen soon. If

\[(5.14) \quad \int_{B(x,r)} |\nabla u_r^s|^2 \geq \eta^2 \int_{B(x,r)} |\nabla u|^2 = \eta^2 \omega(x,r)^2 \]

then we use (5.8) to prove that

\[(5.15) \quad \omega(x,r)^2 = \int_{B(x,r)} |\nabla u|^2 \geq \int_{B(x,r)} |\nabla u_r^s|^2 = |\nabla u_r^s(x)|^2 + \int_{B(x,r)} |\nabla u_r^s|^2 \geq |\nabla u_r^s(x)|^2 + \eta^2 \omega(x,r)^2 \]

and hence by (5.13)

\[(5.16) \quad \omega(x, \theta r) \leq |\nabla u_r^s(x)| + C \left[ \theta^{-n/2} \left( r^{\alpha/2} + K^{-1} \right) + \theta \gamma \left( K^{-1} + 1 \right) \right] \omega(x,r) \leq \sqrt{1 - \eta^2} \omega(x,r) + C \left[ \theta^{-n/2} \left( r^{\alpha/2} + K^{-1} \right) + \theta \gamma \left( K^{-1} + 1 \right) \right] \omega(x,r). \]
We shall deal with the quantifiers soon, but let us get rid of the case when (5.14) fails. Then by (5.8)

\[(5.17) \quad \int_{B(x,r)} |\nabla u^*_r|^2 = |\nabla u^*_r(x)|^2 + \int_{B(x,r)} |\nabla u^*_r|^2 \leq |\nabla u^*_r(x)|^2 + \eta^2 \omega(x,r)^2.\]

Since standard estimates on harmonic functions yield

\[(5.18) \quad |\nabla u^*_r(x)| \leq Cr^{-1} \int_{\partial B(x,r)} |u^*_r| = Cr^{-1} \int_{\partial B(x,r)} |u| = Cr^{-1} \int_{\partial B(x,r)} u = Cr^{-1}b(x,r)\]
as for (5.10), and because \(u^*_r = u \geq 0\) on \(\partial B(x,r)\). Returning to (5.17),

\[(5.19) \quad \int_{B(x,r)} |\nabla u^*_r|^2 \leq |\nabla u^*_r(x)|^2 + \eta^2 \omega(x,r)^2 \leq C r^{-2}b(x,r)^2 + \eta^2 \omega(x,r)^2\]

by (5.1). At the same time, (5.5) with \(s = r\) and then (5.19) and (5.2) yield

\[(5.20) \quad \omega(x,r) \leq C r^{\alpha/2} \omega(x,r) + C + \left( \int_{B(x,r)} |\nabla u^*_r|^2 \right)^{1/2} \leq C r^{\alpha/2} \omega(x,r) + C + C \gamma (1 + \omega(x,r)) + \eta \omega(x,r) \leq C [r_1^{\alpha/2} + K_1^{-1} + \gamma K_1^{-1} + \gamma + \eta] \omega(x,r)\]

(recall our assumption that \(r < r_1\)). If \(K_1\) is large enough, and \(r_1, \gamma,\) and \(\eta\) are small enough, we get a contradiction because \(\omega(x,r) \geq K_1 > 0\). That is, we choose \(\eta\) so that \(C \eta < 1/4\) and only consider \(K_1\) large enough and \(r_1\) small enough so

\[(5.21) \quad C \left( r_1^{\alpha/2} + K_1^{-1} + \gamma K_1^{-1} + \gamma \right) < \frac{1}{4}\]

Under these conditions the second case is impossible and (5.16) holds. To deduce (5.3) choose \(K_1, r_1\) and \(\gamma\) satisfying both (5.21) and

\[(5.22) \quad C \left[ \theta^{-n/2} (r_1^{\alpha/2} + K_1^{-1}) + \theta \gamma (K_1^{-1} + 1) \right] \leq \frac{1 - \sqrt{1 - \eta^2}}{2},\]

where \(\eta\) is as above. Then letting \(\beta \in (\frac{1 + \sqrt{1 - \eta^2}}{2}, 1)\) we have

\[(5.23) \quad \sqrt{1 - \eta^2} + C \left[ \theta^{-n/2} (r_1^{\alpha/2} + K_1^{-1}) + \theta \gamma (K_1^{-1} + 1) \right] \leq \beta,
\]

which ensures that (5.3) holds ; Lemma 5.1 follows.

\[\square\]

**Theorem 5.1** Let \(u\) be an almost minimizer for \(J^+\) in \(\Omega\). Then \(u\) is locally Lipschitz in \(\Omega\).
Let us make two observations before we start the proof. The theorem comes with uniform estimates. That is, there exists $r_2 > 0$ and $C_2 \geq 1$ (that depend on $n$, $\kappa$, and $\alpha$) such that for each choice of $x_0 \in \Omega$ and $r_0 > 0$ such that $r_0 \leq r_2$ and $B(x_0, 2r_0) \subset \Omega$,

\begin{equation}
|u(x) - u(y)| \leq C_2(\omega(x_0, 2r_0) + 1)|x - y| \text{ for } x, y \in B(x_0, r_0).
\end{equation}

Theorem 3.2 ensures that $u$ is more regular away from the free boundary $\partial(\{u > 0\})$. In the good cases, we expect $u$ to behave, near a point of $\partial(\{u > 0\})$, like $a(x)_+ = \max(0, a(x))$ for some non constant affine function that vanishes at the given point. Precise results in this direction are beyond the scope of this paper.

**Proof.** Let $(x, r)$ be such that $B(x, 2r) \subset \Omega$; we want to see whether our different lemmas can bring us to a pair $(x, \rho)$ where we control $u$, and for this we will distinguish between a few cases.

Pick $\theta = \frac{1}{3}$ for definiteness (but smaller values would work as well), and let $\beta, \gamma, K_1$, and $r_1$ be as in Lemma 5.1. Then pick $\tau = \tau_1/2$, where $\tau_1 \in (0, 10^{-2})$ is the constant that we get in Lemma 4.1, applied with $C_1 = 3$ and $r_0 = r_1$.

Next let $r_0, \eta$, and $K$ be as in Lemma 4.4, applied with $C_0 = 10$ and the small $\gamma$ that we just found. Set $r_\gamma = r_0$ to avoid confusion. Set

\begin{equation}
K_2 \geq \max(K_1, K) \text{ and } r_2 \leq \min(r_1, r_\gamma),
\end{equation}

and assume that $r \leq r_2$. We consider three cases.

Case 1:

\begin{equation}
\begin{cases}
\omega(x, r) & \geq K_2 \\
b(x, r) & \geq \gamma r (1 + \omega(x, r))
\end{cases}
\end{equation}

Case 2:

\begin{equation}
\begin{cases}
\omega(x, r) & \geq K_2 \\
b(x, r) & < \gamma r (1 + \omega(x, r))
\end{cases}
\end{equation}

Case 3:

\begin{equation}
\omega(x, r) < K_2.
\end{equation}

We start with Case 1. By (5.26) we can apply Lemma 4.4 (recall that $r \leq r_2$, which by (5.25) is not more than the $r_\gamma$ of Lemma 4.4) and we find $\rho \in \left(\frac{\eta r}{2}, \eta r\right)$ such that $(x, \rho) \in G(\tau, 10, 3, r_\gamma)$. Notice that the pair $(x, \rho)$ satisfies the assumptions of Lemmas 4.1-4.3, which we applied with $r_0 = r_1$. This is the way we defined $\tau_1$ and $\tau$. Lemmas 4.1-4.3 still apply even if $r << r_1$. Now Lemma 4.3 says that $u$ is $C_x$-Lipschitz in $B(x, \frac{\tau \rho}{4})$, and hence also on $B(x, \frac{\tau \rho}{8})$. By (4.37) we can take

\begin{equation}
C_x = C \left(\tau^{-\frac{2}{n}} \omega(x, \rho) + \rho \frac{\omega}{\kappa}\right) \leq C \left(\tau^{-\frac{2}{n}} \eta \frac{\omega}{\kappa} + \omega(x, r) + \frac{\omega}{\kappa}\right).
\end{equation}
By Lemma 4.3, we even know that $u$ is $C^{1,\beta}$ in a neighborhood of $x$, thus Case 1 yields additional regularity.

In the two remaining cases, we set $r_k = \theta^k r = 3^{-k}r$ for $k \geq 0$, and our main task will be to control $\omega(x, r_k)$. If the pair $(x, r_k)$ ever satisfies (5.26) (the definition of Case 1), we denote by $k_{\text{stop}}$ the smallest integer $k$ such that $(x, r_k)$ satisfies (5.26) (notice that $k \geq 1$ because we are not in Case 1); otherwise set $k_{\text{stop}} = +\infty$.

Let $k < k_{\text{stop}}$ be given. If $(x, r_k)$ satisfies (5.27), we can apply Lemma 5.1 to it (this is how we chose $\gamma$, $K_1$, and $r_1$ above), and we get that

$$\omega(x, r_{k+1}) \leq \beta \omega(x, r_k),$$

as in (5.3). Otherwise, $(x, r_k)$ satisfies (5.28) (because (5.27) is false when $k < k_{\text{stop}}$), and we observe that by definitions

$$\omega(x, r_{k+1}) = \left( \int_{B(x, r_{k+1})} |\nabla u|^2 \right)^{\frac{1}{2}} \leq 3^\frac{n}{2} \omega(x, r_k) \leq 3^\frac{n}{2} K_2.$$

By (5.30), (5.31), and an easy induction, we get that for $0 \leq k \leq k_{\text{stop}},$

$$\omega(x, r_k) \leq \max \left( \beta^k \omega(x, r), 3^\frac{n}{2} K_2 \right).$$

If $k_{\text{stop}} = +\infty$, this implies that

$$\limsup_{k \to \infty} \omega(x, r_k) \leq 3^\frac{n}{2} K_2.$$

In particular, if $x$ is a Lebesgue point for $\nabla u$

$$|\nabla u(x)| \leq 3^\frac{n}{2} K_2.$$

If $k_{\text{stop}} < \infty$, we apply our argument for Case 1 to the pair $(x, r_{k_{\text{stop}}})$, and get that $u$ is $C^{1,\beta}$ in a neighborhood of $x$, and by (5.29) and (5.32)

$$|\nabla u(x)| \leq C \left( \tau^{-\frac{n}{2}} \eta^{-\frac{n}{2}} \omega(x, r_{k_{\text{stop}}}) + r_{k_{\text{stop}}}^{\frac{n}{2}} \right) \leq C \tau^{-\frac{n}{2}} \eta^{-\frac{n}{2}} \max \left( \beta^{k_{\text{stop}}} \omega(x, r), 3^\frac{n}{2} K_2 \right) + Cr^\frac{n}{2} \leq C' \omega(x, r) + C',$$

where $C'$ depends on $n$, $\kappa$, $\alpha$ through the various constants above. We still have (5.35) in Case 1 (directly by (5.29)), and since (5.34) is better than (5.35), we proved that if $r \leq r_2,$ (5.35) holds for almost every $x \in \Omega$ such that $B(x, 2r) \subset \Omega$.

Now let $x_0 \in \Omega$ and $r_0 < r_2$ be such that $B(x_0, 2r_0) \subset \Omega$. Then for almost every $x \in B(x_0, r_0)$, (5.35) holds for with $r = r_0/2$ (so that $B(x, 2r) \subset B(x_0, 2r_0) \subset \Omega$), and so

$$|\nabla u(x)| \leq C' \omega(x, r) + C' \leq 2^{n/2} C' \omega(x_0, 2r_0) + C'.$$

We already know that $u$ is in the Sobolev space $W^{1,2}_{\text{loc}}(B(x_0, r_0))$, so we deduce from (5.36) that $u$ is Lipschitz in $B(x_0, r_0)$, with the estimate (5.24). Theorem 5.1 follows. $\blacksquare$
6 Almost monotonicity; statement and first estimates

To prove that almost minimizers for $J$ are locally Lipschitz, we need a variant of the monotonicity result of Alt, Caffarelli, and Friedman \[\text{ACF}\]. We use the functional $\Phi$ they introduced, but we shall only be able to prove that it is almost nondecreasing and has a limit at the origin; see Theorem 6.1 and (6.5).

First we need some notation. As usual, $u$ is an almost minimizer in the domain $\Omega$, we fix $x \in \Omega$, and for $r > 0$ such that $B(x, r) \subset \Omega$, set

\begin{equation}
A_+(r) = \int_{B(x, r)} \frac{|\nabla u^+(y)|^2}{|x - y|^{n-2}} dy \quad \text{and} \quad A_-(r) = \int_{B(x, r)} \frac{|\nabla u^-(y)|^2}{|x - y|^{n-2}} dy
\end{equation}

and

\begin{equation}
\Phi(r) = \frac{1}{r^4} A_+(r) A_-(r) = \frac{1}{r^4} \left( \int_{B(x, r)} \frac{|\nabla u^+(y)|^2}{|x - y|^{n-2}} dy \right) \left( \int_{B(x, r)} \frac{|\nabla u^-(y)|^2}{|x - y|^{n-2}} dy \right).
\end{equation}

We keep the same formula when $n = 2$ (and some proofs will be simpler), and recall that we do not consider $n = 1$ here. We want to study the monotonicity properties of $\Phi$ as a function of $r$. To some extent we follow the argument of \[\text{ACF}\], but since we cannot expect $u$ to be as smooth, we need to avoid integration by parts. We can only expect estimates that hold in average and weaker monotonicity properties. Here is the main result that we prepare for in this section and prove in the next one.

**Theorem 6.1** Let $u$ be an almost minimizer for $J$ in $\Omega$, and let $\delta$ be such that $0 < \delta < \alpha/4(n + 1)$. Then there is a constant $C > 0$, which depends only on $n$, $\alpha$, $\delta$, $\kappa$, and $L^\infty$ bounds for $q_+$ and $q_-$, such that the following holds. Let $x \in \Omega$ and $r_0 > 0$ be such that $B(x, 2r_0) \subset \Omega$. Suppose that $u(x) = 0$. Then for $0 < s < r < \frac{1}{2} \min(1, r_0)$

\begin{equation}
\Phi(s) \leq \Phi(r) + C(x, r_0) r^\delta,
\end{equation}

where

\begin{equation}
C(x, r_0) = C + C \left( \int_{B(x, 3r_0/2)} |\nabla u(y)|^2 \right)^2 + C((\log r_0)_+)^4.
\end{equation}

Of course none of the exponents above are expected to be sharp. Notice that Theorem 6.1 implies that if $u$ is an almost minimizer and $x \in \Omega$ is such that $u(x) = 0$, then

\begin{equation}
\lim_{r \to 0} \Phi(r) \in [0, +\infty) \text{ exists.}
\end{equation}

Indeed, first observe that by (6.3), $\limsup_{s \to 0} \Phi(s) \leq \Phi(r) + C(x, r_0) r^\delta$ for $r$ small. So $0 \leq \liminf_{s \to 0} \Phi(s) \leq \limsup_{s \to 0} \Phi(s) < \infty$. Set $l = \liminf_{s \to 0} \Phi(s)$. For each $\varepsilon > 0$, we can find $r > 0$, arbitrarily small, such that $\Phi(r) \leq l + \varepsilon$. Then by (6.3) again $\limsup_{s \to 0} \Phi(s) \leq$
\[ \sup_{0 < s < r} \Phi(s) \leq l + \varepsilon + C(x, r_0)r^\delta, \] which is arbitrarily close to \( l \) as it holds for all \( r < r_0 \); (6.5) follows.

We start the proof with a few computations using competitors. In the case of minimizers, these calculations would often be obtained using integrations by parts. The next lemma, which will be obtained by replacing \( u^\pm \) with something like \( u^\pm + \lambda \varphi u^\pm \), will be used a few times later, with different choices for \( \varphi \).

**Lemma 6.1** Let \( u \) be an almost minimizer for \( J \) in \( \Omega \), and assume that \( B(x, 2r) \subset \Omega \). Let \( \varphi \in W^{1,2}(\Omega) \cap C(\Omega) \) be such that \( \varphi(y) \geq 0 \) everywhere, \( \varphi(y) = 0 \) on \( \Omega \setminus B(x, r) \), and let \( \lambda \in \mathbb{R} \) be such that

\[ |\lambda \varphi(y)| < 1 \quad \text{on} \ \Omega. \]

Then for each choice of sign \( \pm \),

\[ 0 \leq \kappa r^\alpha J_{x,r}(u) + 2\lambda \left[ \int_{B(x,r)} \varphi |\nabla u^\pm|^2 + \int_{B(x,r)} u^\pm \langle \nabla u^\pm, \nabla \varphi \rangle \right] + \lambda^2 \left[ \int_{B(x,r)} \varphi^2 |\nabla u^\pm|^2 + (u^\pm)^2 |\nabla \varphi|^2 + 2\varphi u^\pm \langle \nabla u^\pm, \nabla \varphi \rangle \right], \]

where \( J_{x,r}(u) = \int_{B(x,r)} |\nabla u|^2 + q^2_+ \chi_{\{u > 0\}} + q^2_- \chi_{\{u < 0\}} \) is as in (1.12), and \( \nabla u^\pm \) denotes the gradient of \( u^\pm \).

**Proof.** Recall that \( u^+ = \max\{u, 0\} \) and \( u^- = \max\{-u, 0\} \). Let us do the verification for \( u^+ \), the proof for \( u^- \) would be similar. Define \( v \) on \( \Omega \) by

\[ v(y) = u(y) + \lambda \varphi(y) u(y) = (1 + \lambda \varphi(y))u^+(y) \quad \text{if} \ y \in B(x, r) \quad \text{and} \ u(y) > 0 \]

and \( v(x) = u(x) \) otherwise. Recall that \( u \) is continuous, and then \( v \) is also continuous, because \( \varphi(y) = 0 \) on \( \Omega \setminus B(x, r) \), and the expression in (6.8) yields \( v(x) = 0 \) when \( u(x) = 0 \). Next, \( v(y) \) is obtained from \( u(y) \) by multiplying it by either 1 or \( 1 + \lambda \varphi(y) \), which is also positive by (6.6); so

\[ \{ y \in \Omega; v(y) > 0 \} = \{ y \in \Omega; u(y) > 0 \} \quad \text{and} \quad \{ y \in \Omega; v(y) < 0 \} = \{ y \in \Omega; u(y) < 0 \}. \]

In addition, \( v^- = u^- \) and \( v^+ = (1 + \lambda \varphi)u^+ \) everywhere on \( \Omega \) (recall that \( \varphi(y) = 0 \) on \( \Omega \setminus B(x, r) \)). We know, for instance from Corollary 2.1.8 in [Z], that \( u^- \) and \( u^+ \) lie in \( W^{1,2}(\Omega) \), and that \( \nabla u^\pm = \chi_{\{u^\pm > 0\}} \nabla u \). Then \( v^\pm \in W^{1,2}(\Omega) \), with

\[ \nabla v^+ = (1 + \lambda \varphi) \nabla u^+ + \lambda u^+ \nabla \varphi. \]

Thus \( v \in W^{1,2}(\Omega) \), and we can apply the definition (1.11) of almost minimality. This yields

\[ J_{x,r}(u) \leq (1 + \kappa r^\alpha)J_{x,r}(v) = (1 + \kappa r^\alpha) \int_{B(x,r)} |\nabla v|^2 + q^2_+ \chi_{\{v > 0\}} + q^2_- \chi_{\{v < 0\}} \]

\[ + \lambda^2 \int_{B(x,r)} \varphi^2 |\nabla u^\pm|^2 + (u^\pm)^2 |\nabla \varphi|^2 + 2\varphi u^\pm \langle \nabla u^\pm, \nabla \varphi \rangle \]
by (1.12). Now \( q^2 \chi_{\{v>0\}} + q^2 \chi_{\{v<0\}} = q^2 \chi_{\{u>0\}} + q^2 \chi_{\{u<0\}} \) by (6.9). Also,

\[
\int_{B(x,r)} |\nabla v|^2 = \int_{B(x,r)} |\nabla v^+|^2 + \int_{B(x,r)} |\nabla v^-|^2 = \int_{B(x,r)} |\nabla v^+|^2 + \int_{B(x,r)} |\nabla u^-|^2
\]

(6.12)

\[
= \int_{B(x,r)} |\nabla u|^2 - \int_{B(x,r)} |\nabla u^+|^2 + \int_{B(x,r)} |\nabla v^+|^2
\]

because \( \nabla v^\pm = \chi_{\{v>0\}} \nabla v \), similarly for \( u \), and by (6.8) again. So

\[
J_{x,r}(v) = J_{x,r}(u) + \int_{B(x,r)} |\nabla v^+|^2 - \int_{B(x,r)} |\nabla u^+|^2
\]

and (6.11) yields

\[
0 \leq (1 + \kappa r^\alpha) J_{x,r}(v) - J_{x,r}(u) = \kappa r^\alpha J_{x,r}(v) + J_{x,r}(v) - J_{x,r}(u)
\]

(6.14)

\[
= \kappa r^\alpha J_{x,r}(v) + \int_{B(x,r)} |\nabla v^+|^2 - \int_{B(x,r)} |\nabla u^+|^2
\]

\[
= \kappa r^\alpha J_{x,r}(u) + (1 + \kappa r^\alpha) \int_{B(x,r)} [ |\nabla v^+|^2 - |\nabla u^+|^2 ].
\]

Next by (6.10)

\[
|\nabla v^+|^2 = (1 + \lambda \phi - |\nabla u^+|^2 + 2\lambda (1 + \lambda \phi) u^+ \langle \nabla u^+, \nabla \phi \rangle + \lambda^2 (u^+)^2 |\nabla \phi|^2
\]

(6.15)

\[
= |\nabla u^+|^2 + 2\lambda [\phi |\nabla u^+|^2 + u^+ \langle \nabla u^+, \nabla \phi \rangle]
\]

+ \lambda^2 \phi^2 |\nabla u^+|^2 + 2\phi (u^+)^2 |\nabla \phi|^2
\]

by (6.10). We integrate this, replace in (6.14), and get that

\[
0 \leq \kappa r^\alpha J_{x,r}(u) + 2\lambda (1 + \kappa r^\alpha) \left[ \int_{B(x,r)} \phi |\nabla u^+|^2 + \int_{B(x,r)} u^+ \langle \nabla u^+, \nabla \phi \rangle \right]
\]

(6.16)

\[
+ \lambda^2 (1 + \kappa r^\alpha) \left[ \int_{B(x,r)} \phi^2 |\nabla u^+|^2 + (u^+)^2 |\nabla \phi|^2 + 2\phi u^+ \langle \nabla u^+, \nabla \phi \rangle \right].
\]

We divide by \( 1 + \kappa r^\alpha \), add \( \kappa r^\alpha J_{x,r}(u) - \frac{\kappa r^\alpha}{1 + \kappa r^\alpha} J_{x,r}(u) \geq 0 \), and get (6.7) for \( u^+ \). The proof for \( u^- \) is the same.

The next lemma will only be needed in ambient dimensions \( n \geq 3 \); when \( n = 2 \), the algebra leading to the monotonicity formula will just be simpler.

**Lemma 6.2** Suppose \( n \geq 3 \). Let \( u \) be an almost minimizer for \( J \) in \( \Omega \), assume that \( B(x, 2r_0) \subset \Omega \) and that \( u(x) = 0 \). Then for \( 0 < r < \min(1, r_0) \),

\[
\frac{c_n}{r^2} A_+(r) - \frac{1}{n(n-2) \int_{B(x,r)} |\nabla u^+|^2 - \frac{1}{2} \int_{\partial B(x,r)} \left( \frac{u^+}{r} \right)^2
\]

(6.17)

\[
geq -Cr^{\frac{n}{n+1}} \left( 1 + \int_{B(x,3r_0/2)} |\nabla u|^2 + \log^2(3r_0/r) + \log^2(1/r) \right).
\]
where we set
\begin{equation}
(6.18) \quad c_n = \frac{1}{n(n-2)\omega_n} \quad \text{and} \quad \omega_n = |B(0,1)|,
\end{equation}
and \(C\) depends only on \(\kappa\), the \(|q_\pm|\infty\), \(\alpha\), and \(n\).

This will be generalized later, with a similar proof; see Lemma 6.3.

**Proof.** We use the Green function for \(B(x,r)\) with pole \(x\) defined by
\begin{equation}
(6.19) \quad G_r(y) = \frac{c_n}{|y-x|^{n-2}} - \frac{c_n}{r^{n-2}}.
\end{equation}

Fix \(s < r\), and set
\begin{equation}
(6.20) \quad \varphi(y) = \varphi_{r,s}(y) = \begin{cases} 0 & \text{for } y \in \Omega \setminus B(x,r) \\ G_r(y) & \text{for } y \in B(x,r) \setminus B(x,s) \\ c_ns^{2-n} - c_nr^{2-n} & \text{for } y \in B(x,s). \end{cases}
\end{equation}

We want to apply Lemma 6.1 to \(\varphi\), with a \(\lambda > 0\) to be chosen soon. First observe that \(\varphi \in W^{1,2}(\Omega) \cap C(\Omega)\) and \(\varphi(y) = 0\) on \(\Omega \setminus B(x,r)\), as needed. Also,
\begin{equation}
(6.21) \quad ||\varphi||\infty \leq \frac{c_n}{s^{n-2}} \quad \text{and} \quad ||\nabla \varphi||\infty \leq \frac{c_n(n-2)}{s^{n-1}}
\end{equation}
so that we just need to take
\begin{equation}
(6.22) \quad \lambda < \frac{s^{n-2}}{c_n}
\end{equation}
for (6.6) to hold. Then the lemma applies and (6.7) yields
\begin{equation}
(6.23) \quad 0 \leq \kappa r^\alpha J_{x,r}(u) + 2\lambda \left[ \int_{B(x,r)} \varphi |\nabla u^+|^2 + \int_{B(x,r)} u^+ \langle \nabla u^+, \nabla \varphi \rangle \right] + 2\lambda^2 \left[ \int_{B(x,r)} \varphi^2 |\nabla u^+|^2 + (u^+)^2 |\nabla \varphi|^2 \right],
\end{equation}
where we just applied Cauchy-Schwarz to take care of \(2\varphi u^+ \langle \nabla u^+, \nabla \varphi \rangle\) in the last bracket.

We now estimate the various terms. First
\begin{equation}
(6.24) \quad \int_{B(x,r)} \varphi |\nabla u^+|^2 = c_n \int_{B(x,s)} \left( \frac{1}{s^{n-2}} - \frac{1}{r^{n-2}} \right) |\nabla u^+|^2 + c_n \int_{B(x,r) \setminus B(x,s)} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{r^{n-2}} \right) |\nabla u^+|^2
\end{equation}
\begin{equation}
\leq c_n A_+(r) - \frac{c_n}{r^{n-2}} \int_{B(x,r)} |\nabla u^+|^2
\end{equation}
by (6.1). Let us check now that

\begin{equation}
2 \int_{B(x,r)} u^+ \langle \nabla u^+, \nabla \varphi \rangle = 2 \int_{B(x,r) \setminus \overline{B(x,s)}} u^+ \langle \nabla u^+, \nabla G_r \rangle = \int_{\partial B(x,s)} (u^+)^2 - \int_{\partial B(x,r)} (u^+)^2
\end{equation}

The first part follows from (6.20). The second one will come from the fact that \( G_r \) is a Green function. In fact for \( 0 < \rho < 2r \), Set

\begin{equation}
F(\rho) = \int_{\partial B(x,\rho)} (u^+)^2 = \int_{\partial B(0,1)} u^+(x + \rho \theta)^2.
\end{equation}

We know that \( u \in W^{1,2}(B(x,2r)) \cap C(B(x,2r)) \), and hence \((u^+)^2 \in W^{1,1}(B(x,3r/2))\), which implies that \( F \in W^{1,1}((0,3r/2)) \), with a derivative

\begin{equation}
F'(\rho) = 2 \int_{\partial B(0,1)} u^+(x + \rho \theta) \frac{\partial u^+}{\partial \rho}(x + \rho \theta) = \frac{2}{|\partial B(0,1)|} \int_{\partial B(x,\rho)} u^+(y) \frac{\partial u^+}{\partial \rho}(y).
\end{equation}

At the same time,

\begin{equation}
2 \int_{\partial B(x,\rho)} u^+ \langle \nabla u^+, \nabla G_r \rangle = -2 \int_{\partial B(x,\rho)} u^+(y) \frac{\partial u^+}{\partial \rho}(y) c_n(n-2) \frac{y - x}{|y - x|^{n-1}} = -F'(\rho),
\end{equation}

because \( \frac{1}{|\partial B(0,1)|} = c_n(n-2) \) (see definition (6.18)). Now (6.25) follows from (6.26) and (6.28) because \( F \in W^{1,1}((0,3r/2)) \).

We estimate the \( \lambda^2 \) term in (6.23) using (6.21):

\begin{equation}
\int_{B(x,r)} \varphi^2 |\nabla u^+|^2 + (u^+)^2 |\nabla \varphi|^2 \leq \frac{c^2_n}{s^{2n-4}} \int_{B(x,r)} |\nabla u^+|^2 + \frac{c^2_n(n-2)^2}{s^{2n-2}} \int_{B(x,r)} (u^+)^2.
\end{equation}

Combining (6.23), (6.24), (6.25) and (6.29) we get that

\begin{equation}
0 \leq \kappa r^\alpha J_{x,r}(u) + 2 \lambda \left[ c_n A_+(r) - \frac{c_n}{n-2} \int_{B(x,r)} |\nabla u^+|^2 \right] + \lambda \left[ \int_{\partial B(x,s)} (u^+)^2 - \int_{\partial B(x,r)} (u^+)^2 \right]
\end{equation}

\begin{equation}
+ 2 \lambda^2 \left[ \frac{c_n^2}{s^{2n-4}} \int_{B(x,r)} |\nabla u^+|^2 + \frac{c^2_n(n-2)^2}{s^{2n-2}} \int_{B(x,r) \setminus \overline{B(x,s)}} (u^+)^2 \right].
\end{equation}

Set

\begin{equation}
A = \frac{c_n}{r^2} A_+(r) - \frac{1}{n(n-2)} \int_{B(x,r)} |\nabla u^+|^2 - \frac{1}{2} \int_{\partial B(x,r)} \left( \frac{u^+}{r} \right)^2;
\end{equation}

this is the quantity that we want to estimate in (6.17). Recall from (6.18) that \( c_n = \frac{1}{n(n-2) \omega_n} \); thus

\begin{equation}
A = \frac{c_n}{r^2} A_+(r) - c_n r^{-n} \int_{B(x,r)} |\nabla u^+|^2 - \frac{1}{2r^2} \int_{\partial B(x,r)} (u^+)^2;
\end{equation}

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we move the corresponding pieces of (6.30) to the right, divide by $2\lambda r^2$, and get that
\[ -A \leq \frac{\kappa r^\alpha J_{x,r}(u)}{2\lambda r^2} + \frac{1}{2\lambda r^2} \int_{B(x,s)} (u^+)^2 \]
\[ + \lambda r^{-2} \left[ \frac{c_n^2}{s^{2n-4}} \int_{B(x,r)} |\nabla u^+|^2 + \frac{c_n^2(n-2)^2}{s^{2n-2}} \int_{B(x,r) \setminus B(x,s)} (u^+)^2 \right]. \]  
(6.33)

Recall from (2.20) that
\[ |u(y) - u(z)| \leq C |y - z| \left( \omega(x, 3r_0/2) + \log \frac{r_0}{|y - z|} \right) \]
when $y, z \in B(x, r_0)$ are such that $|y - z| \leq r_0/8$. We apply this with $z = x$ (and possibly a few intermediate points if $|y - x| > r_0/8$) and get that
\[ |u(y)| = |u(y) - u(x)| \leq C |y - x| \left( \omega(x, 3r_0/2) + \log \frac{2r_0}{|y - x|} \right) \]
for $y \in B(x, r)$, because we assumed that $u(x) = 0$. Hence
\[ \int_{\partial B(x,s)} (u^+)^2 \leq C s^2 \left( \omega(x, 3r_0/2) + \log \frac{r_0}{s} \right)^2 \]
(6.36)
and
\[ \int_{B(x,r) \setminus B(x,s)} (u^+)^2 \leq C r^{n+2} \left( \omega(x, 3r_0/2) + \log \frac{r_0}{s} \right)^2. \]
(6.37)

By (2.6) and (2.11)
\[ \int_{B(x,r)} |\nabla u^+|^2 \leq \int_{B(x,r)} |\nabla u|^2 = |B(x, r)| \omega(x, r)^2 \leq C r^n \left( \omega(x, r_0) + \log(r_0/r) \right)^2. \]
(6.38)

In all these estimates (starting with (2.11) and (2.20)), $C$ depends only on $\kappa$, $\|q_+\|_{L^\infty}$, $\|q_-\|_{L^\infty}$, $\alpha$, and $n$.

It is now time to choose $\lambda$ and $s$. We take
\[ \lambda = c_n^{-1} r^{n-2+\beta}, \quad \beta = \frac{n\alpha}{n+1}, \quad \text{and} \quad s = r^{1+\frac{\beta}{n}}. \]
(6.39)

Since $r < 1$ and $n - 2 + \beta > (n - 2)(1 + \frac{\beta}{2n})$ then (6.22) holds. Thus Lemma 6.1 applies, and we get (6.33). Let us now estimate each term in (6.33). First,
\[ \frac{\kappa r^\alpha J_{x,r}(u)}{2\lambda r^2} = \frac{\kappa r^\alpha}{2\lambda r^2} \int_{B(x,r)} |\nabla u|^2 + q_+^2 \chi_{\{u > 0\}} + q_-^2 \chi_{\{u < 0\}} \]
(6.40)
by (6.38). Given the choice of \( \lambda \) in (6.39), the exponent of \( r \) is \( n + \alpha - (n - 2 - \beta) - 2 = \alpha - \beta = \frac{\alpha}{n+1} \), so this first term is in accordance with (6.17). Next
\[
 r^{-2} \int_{\partial B(x,s)} (u^+)^2 \leq Cr^{-2}s^2 \left( \omega(x,3r_0/2) + \log \frac{r_0}{s} \right)^2
\]
(6.41)
\[
 \leq Cr^{-2}r^{2+\frac{\beta}{n}} \left( \omega(x,3r_0/2) + \log \frac{r_0}{r} + \log \frac{r}{s} \right)^2
\]

by (6.36); the exponent is again \( \frac{\beta}{n} = \frac{\alpha}{n+1} \), and \( \log \frac{\xi}{s} = \log(r^{-\frac{\beta}{n}}) = \frac{\beta}{2n} \log(1/r) \), so this term fits with (6.17) too. By (6.38) the third term in (6.33) is
\[
\lambda r^{-2} \frac{c_n^2}{s^{2n-4}} \int_{B(x,r)} |\nabla u^+|^2 \leq Cr^{n-2+\beta}r^{-2}r^{-(2n-4)(1+\frac{\beta}{2n})}r^n \left( \omega(x, r_0) + \log(r_0/r) \right)^2.
\]
(6.42)
The power is \( 2n - 4 + \beta - (2n - 4) \left( 1 + \frac{\beta}{2n} \right) = \beta - (2n - 4) \frac{\beta}{2n} = \frac{2\beta}{2n} = \frac{2\alpha}{n+2} > \frac{\alpha}{n+1} \), which is again all right. By (6.37) the last term in (6.33) is
\[
\lambda r^{-2} \frac{c_n^2(n-2)^2}{s^{2n-2}} \int_{(B(x,r) \setminus B(x,s))} (u^+)^2 \leq Cr^{n-2+\beta}r^{-2}r^{-2n-2(1+\frac{\beta}{2n})}r^n + 2 \left( \omega(x,3r_0/2) + \log \frac{r_0}{s} \right)^2.
\]
(6.43)
The power is \( 2n - 2 + \beta - (2n - 2) \left( 1 + \frac{\beta}{2n} \right) = \beta(1 - \frac{2n-2}{2n}) = \frac{2\beta}{n} \), like the previous one, and this last term fits with (6.17). The terms \( \log r_0/s \) is handled as in (6.41). Thus this proves (6.17) and Lemma 6.2.

The following generalization of Lemma 6.2 contains three other cases, that will be treated similarly.

Lemma 6.3 Still assume that \( n \geq 3 \). Let \( u \) be an almost minimizer for \( J \) in \( \Omega \), and assume that \( B(x,2r_0) \subset \Omega \) and that \( u(x) = 0 \). Then for \( 0 < r < \min(1,r_0) \) and each choice of sign \( \pm \),
\[
\frac{|c_n|}{r^2} A_\pm(r) = \frac{1}{n(n-2)} \int_{B(x,r)} |\nabla u_\pm|^2 - \frac{1}{2} \int_{\partial B(x,r)} \left( \frac{u_\pm}{r} \right)^2
\]
(6.44)
\[
\leq C r^{n+\frac{\alpha}{n+1}} \left( 1 + \int_{B(x,3r_0/2)} |\nabla u|^2 + \log^2(r_0/r) + \log^2(1/r) \right).
\]

Here again \( c_n = [n(n-2)\omega_n]^{-1} \) as in (6.18) and \( C \) depends only on \( \kappa \), the \( \|q_\pm\|_\infty \), \( \alpha \), and \( n \).

Proof. We continue with the case of \( A_+ \) and \( u^+ \), and prove the upper bound. We still apply Lemma 6.1, but this time with \( \lambda' = -\lambda = -c_n^{-1}r^{n-2+\beta} \) in (6.39). The requirement
that $|\lambda' \varphi| < 1$ is still satisfied, so we get (6.23) with $-\lambda$. That is,

$$0 \leq \kappa r^\alpha J_{x,r}(u) - 2\lambda \left[ \int_{B(x,r)} \varphi |\nabla u^+|^2 + \int_{B(x,r)} u^+ (\nabla u^+, \nabla \varphi) \right] + 2\lambda^2 \left[ \int_{B(x,r)} \varphi^2 |\nabla u^+|^2 + (u^+ \nabla \varphi)^2 \right].$$

(6.45)

Because of (6.25), this is the same as

$$0 \leq \kappa r^\alpha J_{x,r}(u) - \lambda \left[ 2 \int_{B(x,r)} \varphi |\nabla u^+|^2 + \int_{\partial B(x,s)} (u^+)^2 - \int_{\partial B(x,r)} (u^+)^2 \right] + 2\lambda^2 \left[ \int_{B(x,r)} \varphi^2 |\nabla u^+|^2 + (u^+ \nabla \varphi)^2 \right].$$

(6.46)

or equivalently (since $\lambda > 0$)

$$2 \int_{B(x,r)} \varphi |\nabla u^+|^2 - \int_{\partial B(x,r)} (u^+)^2 \leq - \int_{\partial B(x,s)} (u^+)^2 + \frac{\kappa r^\alpha J_{x,r}(u)}{\lambda} + 2\lambda \left[ \int_{B(x,r)} \varphi^2 |\nabla u^+|^2 + (u^+ \nabla \varphi)^2 \right].$$

(6.47)

This time we can even drop $\int_{\partial B(x,s)} (u^+)^2$ (which was small anyway), and get that

$$2 \int_{B(x,r)} \varphi |\nabla u^+|^2 - \int_{\partial B(x,r)} (u^+)^2 \leq \frac{\kappa r^\alpha J_{x,r}(u)}{\lambda} + 2\lambda \left[ \int_{B(x,r)} \varphi^2 |\nabla u^+|^2 + (u^+ \nabla \varphi)^2 \right].$$

(6.48)

By the first part of (6.24),

$$\int_{B(x,r)} \varphi |\nabla u^+|^2 = c_n \int_{B(x,s)} \left( \frac{1}{s^{n-2}} - \frac{1}{r^{n-2}} \right) |\nabla u^+|^2 + c_n \int_{B(x,r) \setminus B(x,s)} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{r^{n-2}} \right) |\nabla u^+|^2.$$

(6.49)

Set

$$\Delta = c_n \int_{B(x,s)} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{s^{n-2}} \right) |\nabla u^+|^2,$$

then

$$\int_{B(x,r)} \varphi |\nabla u^+|^2 = c_n \int_{B(x,r)} \left( \frac{1}{|x-y|^{n-2}} - \frac{1}{r^{n-2}} \right) |\nabla u^+|^2 - \Delta = c_n A^+(r) - \frac{c_n}{r^{n-2}} \int_{B(x,r)} |\nabla u^+|^2 - \Delta.$$
and (6.48) yields

\[
2c_n A_+(r) = \frac{2c_n}{r^{n-2}} \int_{B(x,r)} \nabla u^+|^2 - \int_{\partial B(x,r)} (u^+)^2 \leq \frac{k r^\alpha J_{x,r}(u)}{\lambda} + \Delta + 2\lambda \left[ \int_{B(x,r)} \varphi^2 |\nabla u^+|^2 + (u^+)^2 |\nabla \varphi|^2 \right].
\]

(6.52)

The left-hand side of (6.52) is \(2r^2 A\) (see (6.32)), so

\[
A \leq \frac{k r^\alpha J_{x,r}(u)}{2r^2 \lambda} + \frac{\Delta}{2r^2} + \lambda r^{-2} \left[ \int_{B(x,r)} \varphi^2 |\nabla u^+|^2 + (u^+)^2 |\nabla \varphi|^2 \right].
\]

(6.53)

We just need to estimate \(\Delta\), since the other terms are the same as before, and were already estimated for Lemma 6.2. But, with the notation of (2.6) and by (2.11) (used twice),

\[
\Delta \leq \sum_{j \geq 0} \frac{2^{(j+1)(n-2)} s^{2-j}}{2^{n-1} n^2} \int_{B(\tilde{x}, 2^{-j} s)} |\nabla u|^2 \leq \sum_{j \geq 0} \frac{2^{(j+1)(n-2)} s^{2-j} n^2} {2^{n-1} n^2} \int_{B(x,2^{-j} s)} |\nabla u|^2
\]

(6.54) = \(C s^2 \sum_{j \geq 0} 2^{-2j} \omega(x, 2^{-j} s)^2 \leq C s^2 \sum_{j \geq 0} 2^{-2j} (\omega(x, s) + j)^2 \leq C s^2 (\omega(x, s) + 1)^2 \leq C s^2 (\omega(x, 3r_0/2) + 1 + \log \frac{3r_0}{2s})^2 .
\]

Thus

\[
\frac{\Delta}{2r^2} \leq C r^{-2} s^2 (\omega(x, 3r_0/2) + 1 + \log \frac{3r_0}{2s})^2 \leq C r^{\frac{\alpha}{n+1}} (\omega(x, 3r_0/2) + 1 + \log \frac{r_0}{r} + \log \frac{r}{s})^2
\]

(6.55)

because \(s = r^{1+\frac{\alpha}{n+1}}\) and \(\beta = \frac{n\alpha}{n+1}\). Again this is dominated by the right-hand side of (6.44), and this completes our proof of (6.44) for \(u^+\). The proof for \(u^-\) is the same: just replace + by - in all the proof, or apply the result for \(u^+\) to \(-u\), with \(q_+\) and \(q_-\) exchanged. This proves Lemma 6.3.

In the next lemma we return to the general case of \(n \geq 2\). We would have preferred to obtain an estimate on the difference of integrals (as in the case of minimizers), but unfortunately we only get an estimate on the average of the integrals on a small interval near \(r\). Notice that in (6.56) we shall not even get the absolute values to be inside the \(s\)-integral.
Lemma 6.4 Let $u$ be an almost minimizer for $J$ in $\Omega$, and assume that $B(x,2r_0) \subset \Omega$ and that $u(x) = 0$. For $0 < r \leq \frac{1}{2} \min(1,r_0)$, set $t = t(r) = (1 - \frac{r^{1/4}}{10})r$. Then for $0 < r < \min(1/2, r_0)$ and each choice of sign $\pm$,

$$\left| \int_{t(r)}^{r} \left( \int_{B(x,s)} |\nabla u^\pm(y)|^2 dy \right) ds - \int_{t(r)}^{r} \left( \int_{\partial B(x,s)} u^\pm \frac{\partial u^\pm}{\partial n} \right) ds \right| \leq Cr_n^{\alpha/4} \left( 1 + \int_{B(x,3r_0/2)} |\nabla u|^2 + \log^2 \frac{r_0}{r} \right),$$

(6.56)

were $\frac{\partial u^\pm}{\partial n}$ denotes the radial derivative of $u^\pm$, and $C$ depends only on $\kappa$, the $||q^\pm||_\infty$, $\alpha$, and $n$.

Proof. We want to apply Lemma 6.1 with yet another choice of function $\varphi$. We choose a radial cut-off function such that

$$\varphi(y) = 0 \text{ for } y \in \Omega \setminus B(x,r) \text{ and } \varphi(y) = 1 \text{ for } y \in B(x,t)$$

In the remaining annulus, we interpolate linearly, i.e. set

$$\varphi(y) = \frac{r - |y - x|}{r - t} \text{ for } y \in B(x,r) \setminus B(x,t).$$

(6.57)

The assumptions of Lemma 6.1 are satisfied as long as $|\lambda| < 1$. In this case (6.7) yields

$$-2\lambda \left[ \int_{B(x,r)} \varphi |\nabla u^\pm|^2 + \int_{B(x,r)} u^\pm \langle \nabla u^\pm, \nabla \varphi \rangle \right] \leq \kappa r^{\alpha} J_{x,r}(u)$$

$$+ \lambda^2 \left[ \int_{B(x,r)} \varphi^2 |\nabla u^\pm|^2 + (u^\pm)^2 |\nabla \varphi|^2 + 2\varphi u^\pm \langle \nabla u^\pm, \nabla \varphi \rangle \right].$$

(6.59)

First observe that by our choice of $\varphi$,

$$\int_{t(r)}^{r} \left( \int_{B(x,s)} |\nabla u^\pm(y)|^2 dy \right) ds = \int_{B(x,r)} |\nabla u^\pm(y)|^2 \left( \frac{1}{r - t} \int_t^r \chi_{|y-x| \leq s} ds \right) dy$$

$$= \int_{B(x,r)} \varphi(y) |\nabla u^\pm(y)|^2 dy.$$

(6.60)

Next notice that $\nabla \varphi = 0$, except on $B(x,r) \setminus B(x,t)$ where its only component is $\frac{\partial \varphi}{\partial n} = -\frac{1}{r-t}$. Then

$$\int_{t(r)}^{r} \left( \int_{\partial B(x,s)} u^\pm \frac{\partial u^\pm}{\partial n} \right) ds = \frac{1}{r - t} \int_t^r \left( \int_{\partial B(x,s)} u^\pm \frac{\partial u^\pm}{\partial n} \right) ds$$

$$= \frac{1}{r - t} \int_{B(x,r) \setminus B(x,t)} u^\pm \frac{\partial u^\pm}{\partial n}$$

$$= - \int_{B(x,r) \setminus B(x,t)} u^\pm \langle \nabla u^\pm, \nabla \varphi \rangle.$$

(6.61)
Let $A$ denote the quantity that we need to estimate for (6.56); that is, set
\begin{equation}
A = \oint_{(r)} \left( \int_{B(x,s)} |\nabla u^\pm(y)|^2 dy \right) ds - \oint_{(r)} \left( \int_{\partial B(x,s)} u^\pm \frac{\partial u^\pm}{\partial n} \right) ds
\end{equation}
Then by (6.60) and (6.61), $A$ is equal to the content of the first brackets in (6.59), and so (6.59) says that
\begin{equation}
-2\lambda A \leq \kappa r^\alpha J_{x,r}(u) + \lambda^2 \left[ \int_{B(x,r)} \varphi^2 |\nabla u^\pm|^2 + (u^\pm)^2 |\nabla \varphi|^2 + 2\varphi u^\pm \langle \nabla u^\pm, \nabla \varphi \rangle \right]
\end{equation}
\begin{equation}
\leq \kappa r^\alpha J_{x,r}(u) + 2\lambda^2 \left[ \int_{B(x,r)} \varphi^2 |\nabla u|^2 + u^2 |\nabla \varphi|^2 \right]
\end{equation}
by Cauchy-Schwarz, straightforward estimates on $\varphi$, and because $u^\pm \leq |u|$ and $|\nabla u^\pm| \leq |\nabla u|$. We now take $\lambda = r^\alpha/2$, and then $\lambda = -r^\alpha/2$ (both authorized because $r < 1$), and (6.63) yields
\begin{equation}
|A| \leq \frac{\kappa r^\alpha J_{x,r}(u)}{2r^\alpha/2} + r^\alpha/2 \left[ \int_{B(x,r)} |\nabla u|^2 + (r-t)^{-2} \int_{B(x,r) \setminus B(x,t)} u^2 \right]
\end{equation}
\begin{equation}
\leq C r^\alpha/2 \left[ r^n + \int_{B(x,r)} |\nabla u|^2 + (r-t)^{-2} \int_{B(x,r) \setminus B(x,t)} u^2 \right]
\end{equation}
because $J_{x,r}(u) \leq \int_{B(x,r)} |\nabla u|^2 + Cr^\alpha$ by definition.
We can still use (6.35) and (6.38) (our assumptions are the same as in the previous lemmas); the second one yields
\begin{equation}
\int_{B(x,r)} |\nabla u^\pm|^2 \leq Cr^n (\omega(x, r_0) + \log(r_0/r))^2
\end{equation}
and by the first one,
\begin{equation}
|u(y)| \leq C |y - x| \left( \omega(x, 3r_0/2) + \log \frac{2r_0}{|y - x|} \right) \text{ for } y \in B(x,r).
\end{equation}
We apply this for $y \in B(x,r) \setminus B(x,t)$ (and thus $|y - x| \geq r/2$) and get that
\begin{equation}
(r-t)^{-2} \int_{B(x,r) \setminus B(x,t)} u^2 \leq C (r-t)^{-2} |B(x,r) \setminus B(x,t)| r^2 \left( \omega(x, 3r_0/2) + \log \frac{2r_0}{r} \right)^2
\end{equation}
\begin{equation}
\leq C (r-t)^{-2} r^{n-1} (r-t) r^2 \left( \omega(x, 3r_0/2) + \log \frac{2r_0}{r} \right)^2
\end{equation}
\begin{equation}
= C r^{n-\alpha/4} \left( \omega(x, 3r_0/2) + \log \frac{2r_0}{r} \right)^2
\end{equation}
(recall that \( r - t = r^{1+\alpha/4}/10 \)). Combining (6.64), (6.65), and (6.67) we obtain
\[
|A| \leq Cr^n r^{\alpha/4} \left[ (\omega(x, 3r_0/2) + \log \frac{2r_0}{r})^2 + 1 \right],
\]
which implies (6.56); Lemma 6.4 follows. \[\blacksquare\]

7 Almost monotonicity 2: we put things together

In this section we use the inequalities proved in the last section to complete the proof of the near monotonicity result stated in Theorem 6.1. We first compute the derivatives of \( A_\pm \).

It follows from definition (6.1) that \( A_\pm \) are absolutely continuous and differentiable almost everywhere, with
\[
A_\pm'(r) = r^{2-n} \int_{\partial B(x,r)} |\nabla u_\pm|^2.
\]
Then \( \Phi(r) = r^{-4}A_+(r)A_-(r) \) is also differentiable almost everywhere, with
\[
\Phi'(r) = -4r^{-5}A_+(r)A_-(r) + r^{-4}A'_+(r)A_-(r) + r^{-4}A_+(r)A'_-(r).
\]
Moreover, a fairly simple manipulation of multiple integrals shows that \( \Phi \) is the integral of its derivative.

We shall denote by \( S \) or \( S^{n-1} \) the unit sphere of \( \mathbb{R}^n \). For \( 0 < s < r \), we shall use Poincaré-type estimates on the domains \( \Gamma_\pm(s) \) of \( S \) defined by
\[
\Gamma_\pm(s) = \{ \theta \in S^{n-1}; \pm u(s\theta + x) > 0 \},
\]
which are open because \( u \) is continuous. Define \( \alpha_\pm(s) \) by
\[
\alpha_\pm(s) = \sup \left\{ \frac{\int_{\Gamma_\pm(s)} |v|^2}{\int_{\Gamma_\pm(s)} |\nabla_\theta v|^2}; v \in W_0^{1,2}(\Gamma_\pm(s)), v \neq 0 \right\} \in [0, +\infty],
\]
where \( W_0^{1,2}(\Gamma_\pm(s)) \) is the closure, in \( W^{1,2}(\Gamma_\pm(s)) \), of the set of smooth functions compactly supported on \( \Gamma_\pm(s) \), and \( \nabla_\theta v \) is our notation for the gradient on the sphere. Note that \( 1/\alpha_\pm(s) \) corresponds to the first eigenvalue of the Laplacian on \( \Gamma_\pm(s) \). Moreover \( \alpha_\pm(s) > 0 \) as soon as \( \Gamma_\pm(s) \neq \emptyset \) (because we can easily find nontrivial smooth functions \( v \) with compact support in \( \Gamma_\pm(s) \)), but it is reasonable to take \( \alpha_\pm(s) = 0 \) when \( \Gamma_\pm(s) = \emptyset \), because \( \alpha_\pm(s) \) is a nondecreasing function of the domain. Finally, \( \alpha_\pm(s) < +\infty \) if the complement of \( \Gamma_\pm(s) \) contains a ball, because then there is a Poincaré inequality for compactly supported functions in the complement of that ball.

We work under the following assumptions for the next computations. Let \( u \) be an almost minimizer for \( J \). Fix \( x \in \Omega \) and radii \( 0 < r < r_0 \), with \( B(x, 2r_0) \subset \Omega \) and \( r < \min(1/2, r_0) \).
We assume that \( u(x) = 0 \), and to simplify the notation we take \( x = 0 \). Notice that these assumptions correspond to those needed in the hypothesis of the lemmas in the previous section. As in Lemma 6.4 set

\[
(7.5) \quad t = t(r) = (1 - \frac{r^{\alpha/4}}{10}) r.
\]

Our next long term goal is to estimate \( \Phi(r) - \Phi(t) \) (see (7.58)). Choose \( s_0 \in (t, r) \), such that \( u^\pm(s_0) \in W^{1,2}_0(\Gamma^\pm(s_0)) \). We want to avoid problems coming from the variations of the \( \Gamma^\pm(s) \) as a function of \( s \), so we shall try to reduce to the single \( \Gamma^\pm(s_0) \). Observe that for \( \theta \in S^{n-1} \) and \( s \in [t, r] \),

\[
|u(s\theta) - u(s_0\theta)| \leq C|s - s_0|(\omega(x, 3r_0/2) + \log \frac{2r_0}{|s - s_0|}) \leq C|r - t|(\omega(x, 3r_0/2) + \log \frac{2r_0}{|r - t|}) =: a(r)
\]

by (6.34) (or directly by (2.20)), and using the fact that for \( 0 < \tau < 1/e \) the function \(-\tau \log \tau\) is increasing. The last identity gives the definition of \( a(r) \). Set

\[
(7.7) \quad w_s(\theta) = (u(s\theta) - 2a(r))_+.
\]

We claim that \( w_s \in W^{1,2}_0(\Gamma^+(s_0)) \) for almost every \( s \in [t, r] \). First, the fact that \( w_s \in W^{1,2}(S) \) for almost every \( s \in [t, r] \) is classical (locally, after a change of variables, it comes from the fact that the restriction to almost every hyperplane lies in \( W^{1,2} \)). Next, \( u \) is continuous and \( u(s_0\theta) \leq 0 \) on \( S \setminus \Gamma^+(s_0) \); then by (7.6) \( u(s\theta) \leq a(r) \) and \( w_s(\theta) = 0 \) on that set. So \( w_s \) is continuous and compactly supported in \( \Gamma^+(s_0) \). It is easily approximated by smooth compactly supported functions, and our claim follows. The definition (7.4) yields for almost every \( s \in [t, r] \) that

\[
(7.8) \quad \int_S |w_s|^2 = \int_{\Gamma^+(s_0)} |w_s|^2 \leq \alpha^+(s_0) \int_{\Gamma^+(s_0)} |\nabla \theta w_s|^2.
\]

This is still true when \( \Gamma^+(s_0) \) is empty (and we set \( \alpha^+(s_0) = 0 \)), because then \( w_s = 0 \). Return to \( u(s\theta) \), and cut \( S \) into \( E = \{ \theta \in S; u(s\theta) \leq 2a(r) \} \) and \( F = \{ \theta \in S; u(s\theta) > 2a(r) \} \); then

\[
(7.9) \quad \int_S |u(s\theta)|^2 \leq \int_E |u(s\theta)|^2 + \int_F |u(s\theta)|^2 \leq 4a(r)^2|E| + \int_F |w_s(\theta) + 2a(r)|^2
\]

\[
\leq 4a(r)^2|S| + 4a(r) \int_F w_s(\theta) + \alpha^+(s_0) \int_{\Gamma^+(s_0)} |\nabla \theta w_s|^2
\]
Observe that for $\theta \in F$, 
\[
0 \leq w_s(\theta) \leq u(s\theta) \leq C|s\theta - x| \left( \omega(x, 3r_0)/2 + \log \frac{2r_0}{|s\theta - x|} \right)
\]  
(7.10) 
\[\leq Cr(\omega(x, 3r_0)/2 + \log \frac{2r_0}{r})\]
by (6.35) and the fact that $-\tau \log \tau$ is increasing for $\tau \in (0, e^{-1})$ (we did not need $n \geq 3$ there). Also, (7.7) says that $\nabla_{\theta} w_s = \nabla_{\theta} u(s \cdot) = \nabla_{\theta} u^+(s \cdot)$ on the open set $F$, while $\nabla_{\theta} w_s = 0$ almost everywhere on $S \setminus F$. (By Calderón’s extension of Rademacher’s theorem, almost everywhere on that set, $\nabla_{\theta} w_s(\theta)$ comes from a true differential, which has to vanish if $\theta$ is a point of density of $S \setminus F$.) So $\int_{\mathbb{S}^n} |\nabla_{\theta} w_s|^2 \, d\theta = \int_F |\nabla_{\theta} u^+(s \cdot)|^2 \, d\theta$. Thus (7.9) and (7.10) yield 
\[
\int_S |u(s\theta)|^2 \leq \mathcal{E}_1 + \alpha^+(s_0) \int_F |\nabla_{\theta} u^+(s \cdot)|^2,
\]  
(7.11) 
with 
\[
\mathcal{E}_1 = Ca(r)^2 + Ca(r)r(\omega(x, 3r_0)/2 + \log \frac{2r_0}{r}).
\]  
(7.12)
It turns out that if $\alpha^+(s_0)$ is too small, (7.11) is too good to be used like this (due to the way we shall write the error terms), so we choose $\alpha^+ > 0$, with 
\[
\alpha^+ \geq \alpha^+(s_0),
\]  
(7.13) 
and then of course 
\[
\int_S |u(s\theta)|^2 \leq \mathcal{E}_1 + \alpha^+ \int_F |\nabla_{\theta} u^+(s \cdot)|^2.
\]  
(7.14)
Return to the computation of $A'_{+}$ in (7.1); for almost every $s \in [t, r]$, 
\[
A'_{+}(s) = s^{2-n} \int \! \! \int_{\partial B(x,s)} |\nabla_{\theta} u^+|^2 = s^{2-n} \int \! \! \int_{\partial B(x,s)} \left( \frac{\partial u^+}{\partial n} \right)^2 + |\nabla_{\theta} u^+|^2,
\]  
(7.15) 
where $\frac{\partial u^+}{\partial n}$ is the radial derivative. By (7.14), 
\[
s^{2-n} \int \! \! \int_{\partial B(x,s)} \left| \nabla_{\theta} u^+ \right|^2 = s \int \! \! \int |(\nabla_{\theta} u^+)(s \cdot)|^2 = s^{-1} \int \! \! \int |\nabla_{\theta} u^+(s \cdot)|^2 \geq s^{-1} \int \! \! \int |\nabla_{\theta} u^+(s \cdot)|^2 \geq \frac{1}{s\alpha^+} \left( \int \! \! \int |u(s\theta)|^2 - \mathcal{E}_1 \right) = \frac{s^{-n}}{\alpha^+} \int \! \! \int_{\partial B(x,s)} |u|^2 - \frac{\mathcal{E}_1}{s\alpha^+}.
\]  
(7.16)
Now introduce $\beta_+ \in [0, 1]$, to be chosen later, and split 
\[
A'_+(s) = s^{2-n} \int \! \! \int_{\partial B(x,s)} \left( \frac{\partial u^+}{\partial n} \right)^2 + |\nabla_{\theta} u^+|^2 \geq s^{2-n} \int \! \! \int_{\partial B(x,s)} \left( \frac{\partial u^+}{\partial n} \right)^2 + \frac{s^{-n}}{\alpha^+} \int \! \! \int_{\partial B(x,s)} |u|^2 - \frac{\mathcal{E}_1}{s\alpha^+}
\]  
(7.17) 
\[= s^{2-n} \int \! \! \int_{\partial B(x,s)} \left( \frac{\partial u^+}{\partial n} \right)^2 + \frac{s^{-n}(1 - \beta_+^2)}{\alpha^+} \int \! \! \int_{\partial B(x,s)} |u|^2 + \frac{s^{-n}(1 - \beta_+^2)}{\alpha^+} \int \! \! \int_{\partial B(x,s)} |u|^2 - \frac{\mathcal{E}_1}{s\alpha^+}.\]
By Cauchy-Schwarz, since \(|u| \geq u^+\), the first two terms in (7.17) combine as
\[
s^{-n} \int_{\partial B(x,s)} \left( \frac{\partial u^+}{\partial n} \right)^2 + s^{-n} \int_{\partial B(x,s)} |u|^2 \geq \frac{2s^{1-n} \beta_+}{\alpha^+} \int_{\partial B(x,s)} |u^+ \frac{\partial u^+}{\partial n}| \geq \frac{2s^{1-n} \beta_+}{\sqrt{\alpha^+}} \int_{\partial B(x,s)} (u^+ \frac{\partial u^+}{\partial n}).
\]
(7.18)

Hence
\[
A'_+(s) \geq \frac{2s^{1-n} \beta_+}{\sqrt{\alpha^+}} \int_{\partial B(x,s)} (u^+ \frac{\partial u^+}{\partial n}) + \frac{s^{-n}(1 - \beta_+^2)}{\alpha^+} \int_{\partial B(x,s)} |u^+|^2 - \frac{\mathcal{E}_1}{s \alpha^+}.
\]
(7.19)

When \(n \geq 3\), pick \(\beta_+\) so that
\[
\beta_+ = \frac{1 - \beta_+^2}{(n - 2) \alpha^+};
\]
this is possible, and \(\beta_+\) is unique, because the difference between the two sides of (7.20) is a strictly monotone function of \(\beta_+\) on \([0, 1]\), whose endpoint values have different signs. When \(n = 2\), take \(\beta_+ = 1\). Then for \(n \geq 3\) set
\[
\gamma_+ = \frac{2\beta_+}{\sqrt{\alpha^+}} = \frac{1 - \beta_+^2}{(n - 2) \alpha^+}
\]
and \(\gamma_+ = \frac{2}{\sqrt{\alpha^+}}\) when \(n = 2\). Return to (7.19), which now can be written as
\[
A'_+(s) \geq \gamma_+ s^{1-n} \int_{\partial B(x,s)} (u^+ \frac{\partial u^+}{\partial n}) + \frac{(n - 2) \gamma_+}{2} s^{-n} \int_{\partial B(x,s)} |u^+|^2 - \frac{\mathcal{E}_1}{s \alpha^+}.
\]
(7.22)

When \(n \geq 3\), use Lemma 6.3 to write for \(s \in [t, r]\)
\[
\frac{1}{2} \int_{\partial B(x,s)} |u^+|^2 = c_n A_+(s) - \frac{s^2}{n(n - 2)} \int_{B(x,s)} |\nabla u^+|^2 + \mathcal{E}_2,
\]
with
\[
|\mathcal{E}_2| \leq C r^{n-1} \left( 1 + \int_{B(x,3r^2/2)} |\nabla u|^2 + \log^2(r_0/r) + \log^2(1/r) \right).
\]
(7.23)

We shall not need to do this when \(n = 2\), because the middle term in (7.22) vanishes. Continue with \(n \geq 3\) for the moment, multiply (7.23) by \(H^{n-1}(\partial B(x,s)) = ns^{n-1} \omega_n\) and get that
\[
\frac{1}{2} \int_{\partial B(x,s)} |u^+|^2 = ns^{n-1} \omega_n c_n A_+(s) - \frac{s}{n - 2} \int_{B(x,s)} |\nabla u^+|^2 + C s^{n-1} \mathcal{E}_2
\]
\[
= \frac{s^{n-1} A_+(s)}{n - 2} - \frac{s}{n - 2} \int_{B(x,s)} |\nabla u^+|^2 + C s^{n-1} \mathcal{E}_2
\]
(7.25)
because \( c_n = \left( n(n - 2)\omega_n \right)^{-1} \). We multiply by \((n - 2)\gamma s^{-n}\), replace in (7.22), and get that
\[
A'_+(s) \geq \gamma s^{1-n} \int_{\partial B(x,s)} \left( u^+ \frac{\partial u^+}{\partial n} \right) + \frac{\gamma + A'_+(s)}{s} s^{1-n} \gamma + \int_{B(x,s)} |\nabla u^+|^2 + C_2 \frac{\gamma + \mathcal{E}_2}{s} - \mathcal{E}_1 \frac{s}{s\alpha^+},
\]
(7.26)
where we set
\[
Z^+(s) = \int_{\partial B(x,s)} \left( u^+ \frac{\partial u^+}{\partial n} \right) - \int_{B(x,s)} |\nabla u^+|^2.
\]
When \( n = 2 \), we also get this directly from (7.22) and (6.1), and with one less error term. We know from Lemma 6.4 that \( Z^+(s) \) is rather small in average, i.e., that
\[
\left| \int_{\partial B(x,s)} Z^+(s) ds \right| \leq \mathcal{E}_3
\]
with
\[
\mathcal{E}_3 = Cr^\nu + \frac{\alpha + 2}{4} \left( 1 + \int_{B(x,3r_0/2)} |\nabla u|^2 + \log^2 \frac{r_0}{r} \right),
\]
so let us treat \( Z^+(s) \) as another error term and continue the computation.
We now consider \( A_- \). We keep the same radius \( s_0 \), so \( \alpha^-(s_0) \) is defined; then we shall pick some \( \alpha^- \geq \alpha^-(s_0) \) (as in (7.13)), and also choose \( \beta_- \) and \( \gamma_- \), so that
\[
\gamma_- = \frac{2\beta_-}{\sqrt{\alpha^-}} = \frac{1 - \beta_-^2}{(n-2)\alpha^-},
\]
(7.30)
(as in (7.21), and where we choose \( \beta_- = 1 \) and forget the second part when \( n = 2 \)). Then we proceed as we did for \( A_+ \), and find that
\[
A'_-(s) \geq \frac{\gamma_- A'_-(s)}{s} + \gamma_- s^{1-n} Z^-(s) + C \frac{\gamma_- \mathcal{E}'_2}{s} - \mathcal{E}_1 \frac{s}{s\alpha^-},
\]
(7.31)
(7.32)
(as in (7.26)), where \( \mathcal{E}'_2 \) also satisfies (7.24) and \( Z^- \) also satisfies (7.28).
Write (7.26) and (7.31) as \( A'_\pm \geq s^{-1} \gamma \pm A_\pm(s) + R_\pm \), and plug this back in (7.2). This yields
\[
s^5 \Phi'(s) = -4A_+(r)A_-(s) + sA'_+(s)A_-(s) + sA_+(s)A'_-(s) \geq \left[ \gamma_+ - 4\gamma_+ A_+(r)A_-(s) + sR_+(s)A_-(s) + sA_+(s)R_-(s) \right).
\]
The whole point of the computation is that we can now choose \( \alpha^+ \) and \( \alpha^- \) so that \( \gamma_+ + \gamma_- \geq 4 \).
To see this, let us compute the numbers \( \beta_\pm \) and \( \gamma_\pm \) in terms of \( \alpha^\pm \).
Let us start with the case when \( n \geq 3 \). First, \( \beta_\pm = \beta(\alpha^\pm) \), where \( \beta(\alpha) \) is the unique solution in \([0,1]\) of \( \frac{\beta}{\sqrt{\alpha}} = \frac{1 - \beta^2}{(n-2)\alpha} \) (as in (7.20) or (7.30)). That equation is the same as
(n - 2)\sqrt{\alpha} \beta = 1 - \beta^2 \text{ or as } \beta^2 + (n - 2)\sqrt{\alpha} \beta - 1 = 0; \text{ the discriminant is } \Delta = (n - 2)^2 \alpha + 4 \text{ and solutions are } \beta = -\frac{1}{2}[(n - 2)\sqrt{\alpha} \pm \sqrt{\Delta}]. \text{ We keep the only positive solution, hence } 
\beta(\alpha) = -\frac{1}{2}[(n - 2)\sqrt{\alpha} - \sqrt{\Delta}]. \text{ Then } \gamma_\pm = \gamma(\alpha), \text{ where }
\gamma(\alpha) = \frac{2\beta}{\sqrt{\alpha}} + \frac{\Delta - ((n - 2)\sqrt{\alpha})^2}{\sqrt{\Delta} + (n - 2)\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}} \cdot \frac{4}{\sqrt{(n - 2)^2\alpha + 4} + (n - 2)\sqrt{\alpha}}.
(7.33)

The function \( \gamma \) is continuous and (strictly) decreasing on \((0, +\infty)\). It goes from \(+\infty\) to 0. Let \( \alpha_0 \) be the solution of \( \gamma(\alpha_0) = 4 \), and pick

\( \alpha^+ = \max(\alpha^+(s_0), \alpha_0) \) and \( \alpha^- = \max(\alpha^-(s_0), \alpha_0) \).

Then (7.13) and its analogue for \( \alpha^- \) hold. Also recall that

\( \gamma(\alpha^+(s_0)) + \gamma(\alpha^-(s_0)) \geq 4. \)

This is the same nontrivial result about first eigenvalues on disjoint domains of the sphere that was already used in \([\text{ACF}]\). See (5.7) in \([\text{ACF}]\), which is itself derived from results in \([\text{FH}]\) and \([\text{Sp}]\). Now it is easy to see that

\( \gamma_+ + \gamma_- = \gamma(\alpha^+) + \gamma(\alpha^-) \geq 4, \)

because either \( \alpha^+ = \alpha^+(s_0) \) and \( \alpha^- = \alpha^-(s_0) \), and (7.36) follows from (7.35), or else one of the \( \alpha^\pm \) is equel to \( \alpha_0 \), and then \( \gamma_\pm = \gamma(\alpha_0) = 4 \).

Let us also check (7.36) when \( n = 2 \). It is well know, and easy to check, that when \( I \) is an interval and \( v \in W^{1,2}_0(I) \), we have that \( \int_I |v|^2 \leq (|I|/\pi)^2 \int_I |v'|^2 \). Moreover, for \( I = [0, l] \) the optimal functions are multiples of \( \sin(\pi x/l) \). Thus \( \alpha^\pm(s_0) = (l^\pm /\pi)^2 \), where \( l^\pm \) is the length of the longest component of \( \Gamma^\pm(s_0) \) (compare with the definition (7.4)). The analogue of (7.35) is then

\( \gamma(\alpha^+(s_0)) + \gamma(\alpha^-(s_0)) = \frac{2}{\sqrt{\alpha^+(s_0)}} + \frac{2}{\sqrt{\alpha^-(s_0)}} = \frac{2\pi}{l^+} + \frac{2\pi}{l^-} \geq 4. \)

We choose \( \alpha^\pm = \max(\alpha^\pm(s_0), \alpha_0) \), where \( \alpha_0 = 1/4 \) is again chosen so that \( \gamma(\alpha_0) = \frac{2}{\sqrt{\alpha_0}} = 4 \), and then we get (7.36) as before.

We may now return to the general case. Notice that (7.32) yields

\( s^5\Phi'(s) \geq sR_+(s)A_-(s) + sA_+(s)R_-(s) \)

and (using the fact that \( \Phi \) is the integral of its derivative),

\( \Phi(r) - \Phi(t) \geq \int_t^r [R_+(s)A_-(s) + A_+(s)R_-(s)] \frac{ds}{s^4}. \)

(7.38)
We are now left with the task of giving lower bounds for the various terms in the integral. We shall concentrate on \( R_+(s)A_+(s) \); the other terms would be treated the same way, by exchanging the roles of \( A_+ \) and \( A_- \). Recall that

\[
R_+(s) = \gamma_+ s^{1-n} Z^+(s) + C \frac{\gamma_+ E_2}{s} - \frac{E_1}{s^{\alpha^+}}
\]

(see (7.26)). We start with

\[
E_1 = \int_t^r \frac{E_1}{s^{\alpha^+}} A_-(s) \frac{ds}{s^4}.
\]

(7.40)

Observe that by (7.1) and (2.11) (used twice),

\[
A_- (s) = \int_{B(x,s)} \frac{1}{|x-y|^{n-2}} |\nabla u^-|^2 \leq \int_{B(x,r)} \frac{1}{|x-y|^{n-2}} |\nabla u|^2
\]

\[
\leq \sum_{j \geq 0} \int_{B(x,2^{-j}r) \setminus B(x,2^{-j-1}r)} \frac{1}{|x-y|^{n-2}} |\nabla u|^2
\]

(7.41)

\[
\leq \sum_{j \geq 0} 2^{(j+1)(n-2)-2n} \int_{B(x,2^{-j}r)} |\nabla u|^2 \leq C \sum_{j \geq 0} 2^{j(n-2)} r^{2-n} (2^{-j}r)^n \int_{B(x,2^{-j}r)} |\nabla u|^2
\]

\[
= C r^2 \sum_{j \geq 0} 2^{-2j} \omega(x, 2^{-j}r)^2 \leq C r^2 \sum_{j \geq 0} 2^{-2j} (\omega(x, r) + j)^2
\]

\[
\leq C r^2 (\omega(x, r) + 1)^2 \leq C r^2 (\omega(x, 3r_0/2) + 1 + \log \frac{3r_0}{2r})^2;
\]

then

\[
E_1 \leq C \frac{r-t}{r^3} [C(a) r^2 + C(a) r (\omega(x, 3r_0/2) + \log \frac{2r_0}{r})] r^2 (\omega(x, 3r_0/2) + 1 + \log \frac{3r_0}{2r})^2
\]

(7.42)

by (7.12). We may drop \( \alpha^+ \), because we made sure that it is never less than the constant \( \alpha_0 \). Also

\[
a(r) = C |r-t| (\omega(x, 3r_0/2) + \log \frac{2r_0}{|r-t|})
\]

(7.43)

by (7.6). Recall from (7.5) that \(|r-t| = 10^{-1} r^{1+\frac{\alpha}{2}}\); hence, for \( r < 1 \)

\[
|r-t| \log \frac{2r_0}{|r-t|} \leq r^{1+\frac{\alpha}{2}} \log \frac{2r_0}{r} + r^{1+\frac{\alpha}{2}} \log \frac{r}{|r-t|} \leq r \log \frac{2r_0}{r} + Cr.
\]

(7.44)

Then (7.42) yields

\[
E_1 \leq C \frac{r-t}{r^2} a(r) (1 + \omega(x, 3r_0/2) + \log \frac{2r_0}{r})^3
\]

(7.45)

\[
\leq C \frac{(r-t)^2}{r^2} (1 + \omega(x, 3r_0/2) + \log \frac{2r_0}{r})^4
\]

\[
\leq C r^{\frac{\alpha}{2}} (1 + \omega(x, 3r_0/2) + \log \frac{2r_0}{r})^4.
\]
Next we deal with

\[ E_2 = \int_t^r \left| C \frac{\gamma_+ E_2}{s} \right| A_-(s) \frac{ds}{s^4}. \]

Here again we may drop \( \gamma_+ \), since \( \gamma_+ = \gamma(\alpha^+) \leq \gamma(\alpha_0) \); we use (7.24) and (7.41) and get that

\[
E_2 \leq C \frac{r-t}{r^5} r^2 \left( \omega(x, 3r_0/2) + 1 + \log \frac{3r_0}{2r} \right)^2.
\]

\[
r^{2+\frac{n}{n+1}} \left( 1 + \int_{B(x, 3r_0/2)} |\nabla u|^2 + \log^2 \left( \frac{r_0}{r} \right) + \log^2 \left( \frac{1}{r} \right) \right)
\]

\[
\leq C \frac{r-t}{r} r^{\frac{n}{n+1}} \left( 1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r} + \log \frac{1}{r} \right)^4
\]

by (7.5). Finally set

\[
E_3 = \int_t^r \gamma_+ s^{1-n} Z^+(s) A_-(s) \frac{ds}{s^4}.
\]

Since we could only estimate \( Z^+ \) in average, we first estimate

\[
E_{31} = \int_t^r \gamma_+ r^{1-n} Z^+(s) A_-(r) \frac{ds}{r^4} = \gamma_+ r^{3-n} A_-(r) \int_t^r Z^+(s) ds,
\]

for which we use (7.28), (7.29), and (7.41) and get

\[
|E_{31}| \leq C r^{3-n} A_-(r) (r-t) E_3 \leq C r^{3-n} r^2 \left( \omega(x, 3r_0/2) + 1 + \log \frac{3r_0}{2r} \right)^2
\]

\[
(r-t) r^{n+\frac{n}{n+1}} \left( 1 + \int_{B(x, 3r_0/2)} |\nabla u|^2 + \log^2 \left( \frac{r_0}{r} \right) \right)
\]

\[
\leq C r^{t} \frac{r-t}{r} r^{\frac{n}{n+1}} \left( 1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r} \right)^4 \leq C r^t \left( 1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r} \right)^4.
\]

The other piece is

\[
E_{32} = \gamma_+ \int_t^r Z^+(s) \left[ s^{-3-n} A_-(s) - r^{-3-n} A_-(r) \right] ds.
\]

which we shall only estimate under the assumption that

\[
\left| s^{-3-n} A_-(s) - r^{-3-n} A_-(r) \right| \leq K \frac{(r-t)}{r^{n+4}} A_-(r),
\]

where the numerical constant \( K \geq 1 \) is to be chosen soon. Recall from (7.27) that

\[
|Z^+(s)| \leq \int_{\partial B(x,s)} |u^+ \frac{\partial u^+}{\partial n}| + \int_{B(x,s)} |\nabla u^+|^2.
\]
By (6.38) (or directly (2.11)),

\[ \int_{B(x,s)} |\nabla u^+|^2 \leq \int_{B(x,r)} |\nabla u^+|^2 \leq Cr^n (\omega(x, r_0) + \log \frac{r_0}{r})^2. \]  

(7.54)

It is more convenient to integrate the other term:

\[ \int_t^r \int_{\partial B(x,s)} |u^+ \frac{\partial u^+}{\partial n}| = \int_{B(x,r)\setminus B(x,t)} |u^+ \frac{\partial u^+}{\partial n}| \]

\[ \leq Cr \left( \omega(x, 3r_0/2) + \log \frac{2r_0}{r} \right) \int_{B(x,r)\setminus B(x,t)} |\nabla u^+| \]

\[ \leq Cr \left( \omega(x, 3r_0/2) + \log \frac{2r_0}{r} \right) \]

(7.55)

because by (6.35) \(|u(y)| \leq r \left( \omega(x, 3r_0/2) + \log \frac{2r_0}{r} \right) \) in \( B(x, r) \), and then by (7.54). Notice that this term gives a bigger contribution than the one in (7.54) (after it is integrated on \([t, r]\)). Thus, under our additional assumption (7.52),

\[ |E_{32}| \leq C r \left( \frac{r - t}{r^{n+1}} A_-(r)r^{n+1} \left( \frac{r - t}{r} \right)^{1/2} (\omega(x, 3r_0/2) + \log \frac{2r_0}{r})^2 \right) \]

\[ \leq Cr^{-2} \left( \frac{r - t}{r} \right)^{3/2} A_-(r)(\omega(x, 3r_0/2) + \log \frac{2r_0}{r})^2 \]

(7.56)

by (7.41). We now sum the pieces from (7.45), (7.47), (7.50), and (7.56) and get that

\[ \int_t^r R_+(s) A_-(s) \frac{ds}{s^4} \geq -Cr \frac{\omega}{r^{4(n+1)}} \left( 1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r} + \log \frac{1}{r} \right)^4. \]

(7.57)

Under the same assumption (7.52), but for \( A_+ \), we get the same estimate for \( \int_t^r R_-(s) A_+(s) \frac{ds}{s^4} \), and the we sum, compare with (7.38), and get that

\[ \Phi(r) - \Phi(t) \geq -Cr \frac{\omega}{r^{4(n+1)}} \left( 1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r} + \log \frac{1}{r} \right)^4. \]

(7.58)

This will be good enough for us, but we still need to study the case when (7.52), or its analogue for \( A_+ \), fails. In this case, we shall prove that \( \Phi(r) \geq \Phi(t) \) by a more direct argument, just because the large increase in \( A_- \) or \( A_+ \) is enough.
Suppose for instance that (7.52) fails, i.e., that

\[(7.59) \quad \left| s^{-3-n}A_-(s) - r^{-3-n}A_-(r) \right| > \frac{K(r-t)}{r^{n+4}}A_-(r) \]

for some \( s \in [t, r] \). Observe that since \( A_- \) is nondecreasing by (6.1)) and

\[(7.60) \quad s^{-3-n}A_- (s) - r^{-3-n}A_- (r) \leq (s^{-3-n} - r^{-3-n})A_- (r) \leq (3 + n)s^{-4-n}(r-s)A_- (r) \]

by the fundamental theorem of calculus and if we choose \( r < 1 \) and \( K \geq (3 + n)(10/9)^{n+4} \) (see the definition (7.5)). So (7.59) actually says that

\[(7.61) \quad r^{-3-n}A_- (r) - s^{-3-n}A_- (s) > \frac{K(r-t)}{r^{n+4}}A_- (r) \]

because the other sign is impossible. Then

\[(7.62) \quad A_- (s) \leq \frac{s^{3+n}}{r^{3+n}} \left[ 1 - \frac{K(r-t)}{r} \right] A_- (r) \leq \left[ 1 - \frac{K(r-t)}{r} \right] A_- (r) \]

and

\[(7.63) \quad \Phi(t) = t^{-4}A_+(t)A_- (t) \leq t^{-4}A_+(r)A_- (s) \leq t^{-4}A_+(r)A_- (r) \left[ 1 - \frac{K(r-t)}{r} \right] \]

if \( K \) is large enough. The case when the analogue of (7.52) for \( A_+ \) fails is treated the same way. Thus (7.58) is established in full generality.

Next we want to use (7.58) to estimate \( \Phi(s) \) for all \( s < r \), and not just \( s = t(r) \). We start with estimates along the slowly decreasing sequence \( \{r_j\} \), where \( r_0 = r \) and \( r_{j+1} = t(r_j) = (1 - \frac{r_j^{1/4}}{10})r_j \) (see (7.5)).

First observe that since \( \{r_j\} \) is decreasing and nonnegative, it has a limit \( \ell \geq 0 \). In addition, since \( t(\ell) = \ell \), we get that \( \ell = 0 \). For the moment, fix \( r \) and set

\[(7.64) \quad \Psi(s) = \Phi(s) + C(r)s^\delta, \]

where the exponent \( \delta \) is chosen so that \( 0 < \delta < \frac{\alpha}{4(\alpha+1)} \), and \( C(r) \) is to be chosen soon. We want to show that for \( j \geq 0 \),

\[(7.65) \quad \Psi(r_{j+1}) \leq \Psi(r_j) \]

By (7.58)

\[(7.66) \quad \Phi(r_j) - \Phi(r_{j+1}) \geq -C_0r_j^{\alpha}r_j^{(n+1)/4} \left( 1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j} + \log \frac{1}{r_j} \right)^4, \]

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where we call the constant $C_0$ to avoid confusion. We just need to check that

$$\tag{7.67} C_0 r_j^{\alpha} r_j^{\alpha/(n+1)} \left(1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j} + \log \frac{1}{r_j}\right)^4 \leq C(r)[r_j^{\delta} - r_j^{\delta+1}].$$

Since $r_{j+1} = (1 - \frac{r_j^{\alpha/4}}{10}) r_j$,

$$\tag{7.68} r_j^{\delta} - r_{j+1}^{\delta} = r_j^{\delta} - (1 - \frac{r_j^{\alpha/4}}{10})^{\delta} r_j^{\delta} = r_j^{\delta} \left[1 - (1 - \frac{r_j^{\alpha/4}}{10})^{\delta}\right].$$

Set $u = \frac{r_j^{\alpha/4}}{10}$; then by Taylor’s formula (or just concavity) $(1 - u)^\delta \leq 1 - \delta u$ (the second derivative is negative), so $1 - (1 - u)^\delta \geq \delta u$ and

$$\tag{7.69} r_j^{\delta} - r_{j+1}^{\delta} \geq \delta r_j^{\delta} \frac{r_j^{\alpha/4}}{10}.$$  

Thus, returning to (7.67), it is enough to prove that

$$\tag{7.70} C_0 r_j^{\alpha} r_j^{\alpha/(n+1)} \left(1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j} + \log \frac{1}{r_j}\right)^4 \leq C(r)\delta r_j^{\delta} \frac{r_j^{\alpha/4}}{10}.$$  

Let $R = 1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j} + \log \frac{1}{r_j}$; then

$$\left(1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j} + \log \frac{1}{r_j}\right)^4 = \left(R + 2 \log \frac{r_j}{r_j}\right)^4 \leq R^4 \left(1 + 2 \log \frac{1}{r_j}\right)^4$$

$$\tag{7.71} \leq R^4 \left(1 + 2 \log \frac{1}{r_j}\right)^4 \leq C(\varepsilon) R^4 r_j^{-\varepsilon}$$

because $R \geq 1$, $r \leq 1$, and where we choose $\varepsilon = \frac{\alpha}{4(n+1)} - \delta > 0$ so that the powers will match. Then we choose $C(r)$ such that

$$\tag{7.72} \delta C(r) = 10 C_0 C(\varepsilon) R^4 = 10 C_0 C(\varepsilon) \left(1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j} + \log \frac{1}{r_j}\right)^4 ,$$

and we get (7.70), (7.67), and (7.65). In particular,

$$\tag{7.73} \Phi(r_j) \leq \Psi(r_j) \leq \Psi(r) = \Phi(r) + C(r)r_j^{\delta} \text{ for } j \geq 0 .$$

We now turn to any $s \in (0, r)$. Let $j$ be such that $r_{j+1} \leq s \leq r_j$; then

$$\tag{7.74} \Phi(s) = s^{-4} A_+(s) A_-(s) \leq s^{-4} A_+(r_j) A_-(r_j) = \frac{r_j^4}{s^4} \Phi(r_j) .$$

Recall that $r_{j+1} = (1 - \frac{r_j^{\alpha/4}}{10}) r_j$, and set $u = \frac{r_j - s}{r_j}$. Obviously $u \leq \frac{r_j - r_{j+1}}{r_j} = \frac{r_j^{\alpha/4}}{10}$. Also,

$$1 - u = \frac{s}{r_j} \text{ and hence}$$

$$\tag{7.75} \frac{r_j^4}{s^4} = (1 - u)^{-4} \leq 1 + 10u \leq 1 + r_j^{\alpha/4} .$$

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Next, we know from the proof of (7.41) that \( A_{\pm}(r_j) \leq C r_j^2 (\omega(x, r_j) + 1)^2 \), so

\[
\Phi(r_j) = r_j^{-4} A_+(r_j) A_-(r_j) \leq C (\omega(x, r_j) + 1)^4 \leq C (1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j})^4
\]

(7.76)

\[
\leq C (1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j})^4 (1 + \log \frac{r}{r_j})^4
\]

by (2.11). Thus

\[
r_j^{\alpha/4} \Phi(r_j) \leq C (1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j})^4 r_j^{\alpha/4} \left[ \left( \frac{r_j}{r} \right)^{\alpha/4} (1 + \log \frac{r}{r_j})^4 \right]
\]

(7.77)

\[
\leq C (1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j})^4 r^{\alpha/4}.
\]

Finally by (7.74), (7.75), (7.77), and (7.73),

\[
\Phi(s) \leq \frac{r^4}{s^4} \Phi(r) \leq \Phi(r_j) + r_j^{\alpha/4} \Phi(r_j)
\]

(7.78)

\[
\leq \Phi(r_j) + C (1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j})^4 r_j^{\alpha/4}
\]

\[
\leq \Phi(r) + C'(r)r^\delta
\]

with a formula for \( C'(r) \) that is just like (7.72). Hence

\[
C'(r) = C (1 + \omega(x, 3r_0/2) + \log \frac{3r_0}{2r_j})^4
\]

(7.79)

\[
\leq C (1 + \omega(x, 3r_0/2) + (\log r_0)_+)^4 (1 + \log \frac{1}{r})^4.
\]

Now this works with any \( \delta < \alpha/4(n + 1) \). Thus for \( 0 < \delta < \delta' < \alpha/4(n + 1) \) we have

\[
\Phi(s) \leq \Phi(r) + \tilde{C}(x, r_0) \left( 1 + \log \frac{1}{r} \right)^4 r^{\delta'}
\]

(7.80)

with

\[
\tilde{C}(x, r_0) = C (1 + \omega(x, 3r_0/2) + (\log r_0)_+)^4.
\]

Hence modulo making \( \tilde{C}(x, r_0) \) a little larger depending on \( \delta \) we obtain

\[
\Phi(s) \leq \Phi(r) + C(x, r_0)r^{\delta}
\]

(7.81)

with \( C(x, r_0) \) as in (6.4). This proves (6.3) and completes the proof of Theorem 6.1.
8 Almost minimizers for $J$ are locally Lipschitz

We are now ready to prove the following.

Theorem 8.1 Let $u$ be an almost minimizer for $J$ in $\Omega$. Then $u$ is locally Lipschitz in $\Omega$.

We want to follow the same general scheme as for Theorem 5.1 (in the case of $J^+$), and here is the analogue of Lemma 5.1. Recall that we set $b(x, r) = \int_{\partial B(x, r)} u$ in (4.1).

Lemma 8.1 Let $u$ be an almost minimizer for $J$ in $\Omega$, and let $B_0 = B(x_0, 2r_0) \subset \Omega$ be given. Then there exist $\gamma > 0$, $K_1 > 1$, and $r_1 > 0$ such that if $x \in B(x_0, r_0)$ and $0 < r \leq r_1$ are such

\[ u(y) = 0 \text{ for some } y \in B(x, 2r/3), \]

\[ |b(x, r)| \leq \gamma r (1 + \omega(x, r)), \]

and

\[ \omega(x, r) \geq K_1, \]

then

\[ \omega(x, r/3) \leq \omega(x, r)/2. \]

So the main difference with Lemma 5.1 is that we now add the constraint (8.1), but we shall see later that things are easier if (8.1) does not hold. We also used $B_0$ to localize a little more and get a uniform control on the function $\Phi$ as defined in (6.2).

Proof. Let $x$ and $r$ be as in the statement, and let $\Phi$ be as in (6.2). Apply Theorem 6.1, but with everything centered at some $\tilde{x} \in B(x, r)$ such that $u(\tilde{x}) = 0$, with $\tilde{s} = 2r, \tilde{r} = r_0/4$, and $\tilde{r}_0 = r_0/2$. Thus $B(\tilde{x}, 2\tilde{r}_0) \subset B_0$ and $\tilde{s} < \tilde{r}$ if $r_1$ is small enough. We get that

\[ \Phi(2r) = \Phi(\tilde{s}) \leq \Phi(\tilde{r}) + C(\tilde{x}, \tilde{r}_0)^{-\delta} \leq \Phi(\tilde{r}) + C(\tilde{x}, \tilde{r}_0)\tilde{r}_0^{-\delta} \]

where

\[ C(\tilde{x}, \tilde{r}_0) = C + C \left( \int_{B(\tilde{x}, 3\tilde{r}_0/2)} |\nabla u|^2 \right)^2 + C(\log(\tilde{r}_0) + 1)^4 \]

\[ \leq C + C \left( \int_{B_0} |\nabla u|^2 \right)^2 + C(\log(r_0) + 1)^4. \]

Since

\[ \Phi(\tilde{r}) = \tilde{r}^{-4} A_+(\tilde{r}) A_-(\tilde{r}) \leq C(\omega(\tilde{x}, \tilde{r}) + 1)^4 \leq C \left( \int_{B(x, 3r_0/2)} |\nabla u|^2 \right)^2 \]
as in (7.76) and (7.41), we see that
\[ \Phi(2r) \leq C(B_0), \quad \text{with} \quad C(B_0) = C + C \left( \int_{B(x,3r_0/2)} |\nabla u|^2 \right)^2. \] (8.8)

Set
\[ \omega_\pm(x,r) = \left( \int_{B(x,r)} |\nabla u^\pm|^2 \right)^{1/2}; \]
then
\[ \omega_+(x,r)^2 + \omega_-(x,r)^2 = \omega(x,r)^2 \geq K_1^2 \] (8.10)
by (2.6) and (8.3). At the same time, \(B(x,r) \subset B(\bar{x}, 2r)\), so by (6.1), (6.2), and (8.8)
\[ \omega_+(x,r)^2 \omega_-(x,r)^2 \leq C \omega_+(\bar{x}, 2r)^2 \omega_-(\bar{x}, 2r)^2 \leq C r^{-4} A_+(2r) A_+(2r) = C \Phi(2r) \leq CC(B_0). \] (8.11)

Let us assume that \(\omega_+(x,r) \leq \omega_-(x,r)\); the other case would be treated the same way. Then by (8.11)
\[ \omega_+(x,r)^2 \leq \sqrt{CC(B_0)} \]
and by (8.10)
\[ \omega_-(x,r)^2 \geq K_1^2 - \omega_+(x,r)^2 \geq K_1^2 - \sqrt{CC(B_0)} \geq K_1^2/2 \] (8.13)
if we choose \(K_1^2 \geq 2\sqrt{CC(B_0)}\). By (8.11) and (8.13) we have
\[ \omega_+(x,r)^2 \leq 2K_1^{-2} CC(B_0), \]
which is still very small for \(K_1\) large.

Our next task is to estimate \(\int_{\partial B(x,r)} u^+\). By (8.1), we can find \(z \in B(x,2r/3)\) such that \(u(z) = 0\). Let \(\eta < 1/6\) be small, to be chosen soon, and let us apply (2.20) with \(B(z, \eta r)\) replacing \(B(x_0, r_0)\). We get that for \(y \in B(z, \eta r/8)\),
\[ u^+(y) \leq |u(y)| = |u(y) - u(z)| \leq C |y - z| \left( \omega(z, 2\eta r) + \log \frac{\eta r}{|y - z|} \right) \]
(8.15)
\[ \leq C \eta r (1 + \omega(z, 2\eta r)) \leq C \eta r (1 + \omega(z, r/3) + \log(1/\eta)) \]
\[ \leq C \eta r (1 + \omega(x, r) + \log(1/\eta)) \]
by (2.11) and because \(B(z, r/3) \subset B(x, r)\). Next applying the fundamental theorem of calculus along rays from \(z\) and between \(\partial B(z, \eta r)\) and \(\partial B(x, r)\) then averaging we have
\[ \int_{\partial B(x,r)} u^+ - \int_{\partial B(z,\eta r)} u^+ \leq C(\eta)r \int_{B(x,r)} |\nabla u^+|, \]
(8.16)
where of course \(C(\eta)\) depends on \(\eta\). In turn

\[
(8.17) \quad \int_{B(x,r)} |\nabla u^+| \leq C \left( \int_{B(x,r)} |\nabla u^+|^2 \right)^{1/2} = C\omega_+(x, r) \leq C[K_1^{-2}C(B_0)]^{1/2}
\]

by (8.14). So by (8.15), (8.16), and (8.17)

\[
\int_{\partial B(x,r)} u^+ \leq \sup_{\partial B(z,\eta r)} u^+ + \left| \int_{\partial B(x,r)} u^+ - \int_{\partial B(z,\eta r)} u^+ \right| \leq C\eta (1 + \omega(x, r) + \log(1/\eta)) + C(\eta)r [K_1^{-2}C(B_0)]^{1/2}.
\]

Since \(u = u^+ - u^-\),

\[
(8.19) \quad \int_{\partial B(x,r)} u^- = \int_{\partial B(x,r)} u^+ - \int_{\partial B(x,r)} u
\]

and hence by (8.2)

\[
(8.20) \quad \int_{\partial B(x,r)} |u| = \int_{\partial B(x,r)} u^+ + \int_{\partial B(x,r)} u^- \leq 2 \int_{\partial B(x,r)} u^+ - \int_{\partial B(x,r)} u \leq \xi r,
\]

where

\[
(8.21) \quad \xi = \gamma(1 + \omega(x, r)) + C\eta (1 + \omega(x, r) + \log(1/\eta)) + C(\eta)[K_1^{-2}C(B_0)]^{1/2}
\]

is still small compared to \(\omega(x, r)\).

We again want to compare \(u\) to \(u^*_r\), the harmonic energy minimizing extension defined near (2.2). Recall from Remark 3.1 that \(u^*_r\) can also be computed from the values of \(u\) on \(\partial B(x, r)\) by convolving with the Poisson kernel. Then using the Poisson kernel we have

\[
(8.22) \quad u^*_r(y) \leq C \int_{\partial B(x,r)} |u| \leq C\xi r \quad \text{for } y \in B(x, 3r/4) \quad \text{and } |\nabla u^*_r(y)| \leq C\xi \quad \text{for } y \in B(x, r/2).
\]

Recall from (2.5) that

\[
(8.23) \quad \int_{B(x,r)} |\nabla u - \nabla u^*_r|^2 \leq \kappa r^\alpha \int_{B(x,r)} |\nabla u|^2 + C r^n;
\]

then

\[
\omega(x, r/3)^2 = \int_{B(x,r/3)} |\nabla u|^2 \leq 2 \int_{B(x,r/3)} |\nabla u^*_r|^2 + 2 \int_{B(x,r/3)} |\nabla u - \nabla u^*_r|^2
\]

\[
(8.24) \quad \leq \tilde{C}\xi^2 + C \kappa r^\alpha \int_{B(x,r)} |\nabla u|^2 + C = \tilde{C}\xi^2 + C \kappa r^\alpha \omega(x, r)^2 + C.
\]

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If $K_1$ is large enough, the last term is $C \leq C K_1^{-4} \omega(x, r)^2 \leq \omega(x, r)^2/20$, by (8.3). If $r_1$ is small enough, then $C K_1 r^4 \omega(x, r)^2 \leq \omega(x, r)^2/20$. Concerning $\xi$, if $\gamma$ is small enough, then $\tilde{C} \gamma^2 (1 + \omega(x, r))^2 \leq \omega^2(x, r)/100$ in (8.21); if $\eta$ is small enough, then $\tilde{C} C^2 \eta^2 (1 + \omega(x, r) + \log(1/\eta))^2 \leq \omega^2(x, r)/100$. Finally, if $K_1$ is large enough, depending also on $\eta$, then $\tilde{C} \sigma^2 \leq \omega^2(x, r)/10$ and altogether $\omega(x, r/3)^2 \leq \omega(x, r)^2/5$, which implies (8.4). Lemma 8.1 follows.

We are now ready to prove Theorem 8.1. We shall proceed as for Theorem 5.1, with a slight modification to take care of the extra assumption (8.1). Notice that except for (8.1) and the minor fact that we now demand that $B(x, 2r) \subset \Omega$ instead of $B(x, r) \subset \Omega$, Lemma 8.1 is the same as Lemma 5.1, with $\theta = 1/3$ and $\beta = 1/2$.

So we start again with a pair $(x, r)$ such that $B(x, 2r) \subset \Omega$, and make a construction to find a small ball $B(x, \rho) \subset B(x, r)$, on which $u$ is Lipschitz with estimates that depend only on $B(x_0, r_0)$.

We choose the constants the same way as in the proof of Theorem 5.1 and now split Case 2 into two subcases, depending on whether we can apply Lemma 8.1 or not. That is,

**Case 2a:**

$$(8.25) \quad \begin{cases} \omega(x, r) \geq K_2 \\ |b(x, r)| < \gamma r (1 + \omega(x, r)) \end{cases} \quad (8.1) \text{ holds}$$

and

**Case 2b:**

$$(8.26) \quad \begin{cases} \omega(x, r) \geq K_2 \\ |b(x, r)| < \gamma r (1 + \omega(x, r)) \end{cases} \quad (8.1) \text{ fails}.$$

The other cases stay the same. We treat Case 1 and Case 3 just as we did before. We treat Case 2a as Case 2 before, except that we apply Lemma 8.1 instead of Lemma 5.1. In Case 2b, and since (8.1) fails, we know that $u$ does not vanish anywhere on $B(x, 2r/3)$, so there is a sign $\pm$ such that $\pm u > 0$ on $B(x, 2r/3)$. We may then apply (3.18) to $B(x, 2r/3)$ (and to $-u$ if $\pm = -$), and get that $u$ is Lipschitz on $B(x, 2r/9)$, with

$$(8.27) \quad |\nabla u(y)| \leq C(\omega(x, 2r/3) + r^{\alpha/2}) \text{ for almost every } y \in B(x, 2r/9).$$

Then we just stop, with an even better estimate as in Case 1. The rest of the argument is the same as for Theorem 5.1. This completes our proof of Theorem 8.1.

**9 Limits of almost minimizers**

The main result of this section says under suitable uniformity assumptions, limits of sequences of almost minimizers for $J$, or for $J^+$, are also almost minimizers.
Let $\Omega \subset \mathbb{R}^n$ be a given open set; there will be no need here to let $\Omega$ vary along the sequence. For the sake of the discussion, let us generalize slightly our notion of almost minimizers, and replace the function $\kappa r^\alpha$ by more general functions $h$. We shall only consider continuous nondecreasing functions $h : (0, +\infty) \to [0, +\infty]$, with $\lim_{r \to 0} h(r) = 0$; we shall call such function a gauge function, but as before our main example is $h(r) = \kappa r^\alpha$.

We say that $u \in K_{\text{loc}}(\Omega)$ (see the definition (1.7)) is an almost minimizer for $J$ in $\Omega$ and with the gauge function $h$ if

\begin{equation}
J_{x,r}(u) \leq (1 + h(r))J_{x,r}(v)
\end{equation}

for each ball $B(x, r) \subset \Omega$ such that $\overline{B}(x, r) \subset \Omega$ and every $v \in L^1(B(x, r))$ such that $\nabla v \in L^2(B(x, r))$ and $v = u$ on $\partial B(x, r)$. Here $J$ is still as in (1.12) and our definition is a very mild generalization of (1.11).

Similarly, we say that $u$ is an almost minimizer for $J^+$ in $\Omega$ and with the gauge function $h$ if $u \in K^+_{\text{loc}}(\Omega)$ (see (1.8)) and

\begin{equation}
J^+_{x,r}(u) \leq (1 + h(r))J^+_{x,r}(v)
\end{equation}

for each ball $B(x, r) \subset \Omega$ such that $\overline{B}(x, r) \subset \Omega$ and every $v \in L^1(B(x, r))$ such that $\nabla v \in L^2(B(x, r))$ and $v = u$ on $\partial B(x, r)$. See (1.9) and compare with (1.10).

For our main statement, we consider a sequence $\{u_k\}$ of almost minimizers in $\Omega$, and we even allow the functions $q_{\pm}$ that define the functional $J$ or $J^+$ to depend on $k$. That is, for each $k$, we are given functions $q_{k,\pm}$ and $q_{k,-}$ (here and below, just forget about $q_{k,-}$ if we deal with $J^+$).

We nonetheless assume that for each ball $B_0$ with $\overline{B}_0 \subset \Omega$, there is a constant $M(B_0) \geq 0$ such that

\begin{equation}
|q_{k,\pm}(x)| + |q_{k,-}(x)| \leq M(B_0) \quad \text{for all} \ x \in B_0 \ \text{and} \ k \geq 0.
\end{equation}

We also assume that the functions $q_{k,\pm}$ converge, in $L^1_{\text{loc}}(\Omega)$, to a limit $q_{\infty,\pm}$. That is, for each ball $B_0$ with $\overline{B}_0 \subset \Omega$ and each sign $\pm$,

\begin{equation}
\lim_{k \to \infty} \int_{B_0} |q_{\infty,\pm} - q_{k,\pm}| = 0.
\end{equation}

We denote by $J^k$ (or $J^{k,+}$) the functional defined by the $q_{k,\pm}$, and similarly for $J^\infty$ (or $J^{\infty,+}$). We also give ourselves functions $u_k$ on $\Omega$, and assume that for some fixed gauge function $h$ and every $k \geq 0$,

\begin{equation}
\text{9.5} \quad u_k \text{ is an almost minimizer for } J^k \text{ in } \Omega, \text{ with gauge function } h,
\end{equation}

or, if we work with $J^+$,

\begin{equation}
\text{9.6} \quad u_k \text{ is an almost minimizer for } J^{k,+} \text{ in } \Omega, \text{ with gauge function } h.
\end{equation}
Let us assume that we can find $r_0 > 0$, $\alpha > 0$, and $\kappa \geq 0$ such that
\begin{equation}
\tag{9.7}
h(r) \leq \kappa r^\alpha \text{ for } 0 < r \leq r_0.
\end{equation}
Or even, a little more generally, that we can cover $\Omega$ with open balls $B_j$, such that $2B_j \subset \Omega$, so that for each $j$ we can find $\alpha = \alpha_j > 0$ and $\kappa = \kappa_j \geq 0$ such that
\begin{equation}
\tag{9.8}
u_k \text{ is an almost minimizer for } J^k \text{ in } 2B_j, \text{ with the function } \kappa r^\alpha,
\end{equation}
or, if we work with $J^+$,
\begin{equation}
\tag{9.9}
u_k \text{ is an almost minimizer for } J^{k,+} \text{ in the interior of } 2B_j, \text{ with the function } \kappa r^\alpha.
\end{equation}
We add this assumption in order to be able to apply the results of the previous sections. The fact that we can localize here is not really important.

Our last uniformity assumption is that for each ball $B_0$ with $B_0 \subset \Omega$, there is a constant $C(B_0) \geq 0$ such that
\begin{equation}
\tag{9.10}
\int_{B_0} |\nabla u_k|^2 \leq C(B_0) \text{ for } k \text{ large,}
\end{equation}
where of course it is important that $C(B_0)$ does not depend on $k$.

We claim that under these assumptions, for each ball $B$ with $B \subset \Omega$ there is a constant $L(B)$ such that for $k$ large,
\begin{equation}
\tag{9.11}
each u_k \text{ is Lipschitz in } B, \text{ with } |\nabla u_k| \leq L(B) \text{ almost everywhere in } B.
\end{equation}
Indeed, cover $\overline{B}$ with the $B_j$ above; by compactness we only need a finite collection of $B_j$. By (9.10), we get a uniform bound for $\int_{\frac{1}{2}B_j} |\nabla u_k|^2$. Then we can apply Theorem 5.1 or Theorem 8.1, and we get that for $k$ large, $u_k$ is $L_j$-Lipschitz on $B_j$. This implies that for $k$ large, $u_k$ is locally $L$-Lipschitz in $B$, with $L = \max_j L_j$; (9.11) follows.

Our final assumption is that there is a function $u_\infty$ defined on $\Omega$ such that
\begin{equation}
\tag{9.12}
\lim_{k \to \infty} u_k(x) = u_\infty(x) \text{ for } x \in \Omega.
\end{equation}

**Remark 9.1.** We only assume pointwise convergence, but (9.11), we know that it implies uniform convergence on compact subsets of $\Omega$. It also implies that $u_\infty$ is locally Lipschitz, with the same bounds as in (9.11). Indeed, each compact subset of $\Omega$ can be covered by a finite collection of balls $B$ such that $2B \subset \Omega$, so it is enough to prove the uniform convergence on each such ball $B$, which easily follows from (9.11), (9.12) and Arzela-Ascoli. Note also that $u_k \rightharpoonup u_\infty$ in $W^{1,2}_{\text{loc}}(\Omega)$.

**Remark 9.2.** Similarly, if we have a sequence $\{u_k\}$ that satisfies the assumptions above, except (9.12), and if we also know that for each connected component of $\Omega$ there is a point $x$ such that the family $\{u_k(x)\}$ is bounded, then there is a subsequence of $\{u_k\}$ that converges pointwise on $\Omega$. Indeed each ball $B$ with $\overline{B} \subset \Omega$ can be connected to one of these points $x$ by a finite chain of compact balls in $\Omega$, so (9.11) shows that the $u_k$, $k$ large, are uniformly bounded on $B$, and once again Arzela-Ascoli guarantees the claim.
Theorem 9.1  Let $\Omega$ and the functions $q_{k,\pm}$ and $u_k$ satisfy the conditions above. Then $u_\infty$ is an almost minimizer for $J^\infty$ (for $J^{\infty,+}$ if we assumed (9.6) and (9.9)) in $\Omega$, with the same gauge function $h$ as the $u_k$’s. In addition, for each ball $B(x,r)$ such that $\overline{B}(x,r) \subset \Omega$ and for each choice of sign $\pm$, we have that

$$\lim_{k \to \infty} \nabla u_k^{\pm} = \nabla u_\infty^{\pm} \text{ in } L^2(B(x,r)),$$

$$\int_{B(x,r)} \chi_{\{\pm u_\infty > 0\}} \ q_{\infty,\pm} = \lim_{k \to \infty} \int_{B(x,r)} \chi_{\{\pm u_k > 0\}} \ q_{k,\pm}. $$

Hence

$$\int_{B(x,r)} |\nabla u_\infty^{\pm}|^2 = \lim_{k \to \infty} \int_{B(x,r)} |\nabla u_k^{\pm}|^2,$$

$$\int_{B(x,r)} |\nabla u_\infty|^2 = \lim_{k \to \infty} \int_{B(x,r)} |\nabla u_k|^2,$$

$$J^\infty_{x,r}(u_\infty) = \lim_{k \to \infty} J^k_{x,r}(u_k),$$

and similarly for $J^{\infty,+}$.

Proof. We prove all this in the special case of almost minimizers for $J$; the reader will easily see that the proof carries through to minimizers for $J^{\infty,+}$ with very minor modifications.

We start with lower semicontinuity estimates, that is the upper bounds in (9.14) and (9.15). Fix $B(x,r)$ such that $\overline{B}(x,r) \subset \Omega$ and set, for $\varepsilon > 0$,

$$W^{\pm}_\varepsilon = \{y \in B(x,r) ; \pm u_\infty(y) > \varepsilon\}.$$

By Remark 9.1, the $u_k$’s converge to $u_\infty$ uniformly on $\overline{B}(x,r)$, so $\pm u_k(y) \geq \varepsilon/2$ on $W^{\pm}_\varepsilon$ for $k$ large enough. For such $k$,

$$\int_{W^{\pm}_\varepsilon} \chi_{\{\pm u_\infty > 0\}} \ q_{\infty,\pm} = \int_{W^{\pm}_\varepsilon} q_{\infty,\pm} = \int_{W^{\pm}_\varepsilon} \chi_{\{\pm u_k > 0\}} \ q_{k,\pm} \leq \int_{W^{\pm}_\varepsilon} \chi_{\{\pm u_k > 0\}} \ q_{k,\pm} + \int_{W^{\pm}_\varepsilon} |q_{\infty,\pm} - q_{k,\pm}|$$

$$\leq \int_{B(x,r)} \chi_{\{\pm u_k > 0\}} \ q_{k,\pm} + \int_{B(x,r)} |q_{\infty,\pm} - q_{k,\pm}|. $$

By (9.4) the second term tends to 0 when $k$ tends to $\infty$, so

$$\int_{W^{\pm}_\varepsilon} \chi_{\{\pm u_\infty > 0\}} \ q_{\infty,\pm} \leq \liminf_{k \to \infty} \int_{B(x,r)} \chi_{\{\pm u_k > 0\}} \ q_{k,\pm}$$

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and, since this holds for all \( \varepsilon > 0 \) and \( \{ \chi_{W^\varepsilon} \} \) is nondecreasing in \( \varepsilon \) we have

\[
\int_{B(x,r)} \chi_{\{(\pm u_{\infty})>0\}} q_{\infty,\pm} \leq \liminf_{k \to \infty} \int_{B(x,r)} \chi_{\{(\pm u_k)>0\}} q_{k,\pm}.
\]

Let us also check that

\[
\int_{B(x,r)} |\nabla u_{\infty,\pm}^\pm|^2 \leq \liminf_{k \to \infty} \int_{B(x,r)} |\nabla u_k^\pm|^2.
\]

Observe that the \( u_k^\pm \) are uniformly Lipschitz on \( \overline{B}(x,r) \), just because the \( u_k \)'s are (by (9.11)). Moreover they converge uniformly to \( u_\infty^\pm \) (multiply by \( \pm 1 \) and compose with \( \max(0, \cdot) \)). Then \( \{u_k^\pm\} \) also converges weakly in \( W^{1,2}_{\text{loc}}(\Omega) \) to \( u_\infty^\pm \) and (9.22) follows from the lower semicontinuity of the norm in \( W^{1,2}(B(x,r)) \). Notice that by (9.21), (9.22) and the definition of \( J_{x,r} \) we have

\[
J_{x,r}^\infty(u_\infty) \leq \liminf_{k \to \infty} J_{x,r}^k(u_k).
\]

Next we show that \( u_\infty \) is an almost minimizer for \( J^\infty \). Let \( \overline{B}(x,r) \subset \Omega \), and let \( v \in L^1(B(x,r)) \) be such that \( \nabla v \in L^2(B(x,r)) \) and \( v = u_\infty \) on \( \partial B(x,r) \). We want to use \( v \) to construct a good competitor for \( u_k \), \( k \) large, in a slightly larger ball; then we will use the fact that \( u_k \) is an almost minimizer and get valuable information. Let \( \varepsilon > 0 \) be small, and define \( v_{k,\varepsilon} \) by

\[
v_{k,\varepsilon}(y) = \begin{cases} v(y) & \text{for } y \in B(x,r) \\ u_k(y) & \text{for } y \in \Omega \setminus B(x,(1+\varepsilon)r) \\ (1-a(y))u_\infty(y) + a(y)u_k(y) & \text{for } y \in B(x,(1+\varepsilon)r) \setminus B(x,r), \end{cases}
\]

where we set \( a(y) = \frac{|y-x|}{\varepsilon r} \).

We want to use \( v_{k,\varepsilon} \) as a competitor, so let us check that

\[
v_{k,\varepsilon} \in W^{1,2}_{\text{loc}}(\Omega).
\]

It is enough to show that

\[
v_{k,\varepsilon} \in W^{1,2}(B(x,(1+\varepsilon/2)r)),
\]

because it is clear that it is locally Lipschitz in \( \Omega \setminus B(x,r) \).

Let us start the proof of (9.26). Consider the function \( w \) such that \( w(y) = v(y) - u_\infty(y) \) for \( y \in B(x,r) \), and \( w(y) = 0 \) elsewhere. We know that \( w \in W^{1,2}(B(x,r)) \), and that the trace of \( w \) on \( \partial B(x,r) \) is zero. Then \( w \in W^{1,2}(\Omega) \), i.e., the gluing along \( \partial B(x,r) \) does not create any additional part of the distribution derivative of \( w \). See for instance Lemma 14.4 in [D]. Let \( w_1 \) be such that \( w_1(y) = 0 \) for \( y \in B(x,r) \), and \( w_1(y) = a(y)(u_k(y) - u_\infty(y)) \) on \( B(x,(1+\varepsilon)r) \setminus B(x,r) \). The function \( w_1 \) is Lipschitz, and \( w_1 \in W^{1,2}(B(x,(1+\varepsilon/2)r)) \) as well. Since \( v_{k,\varepsilon} = u_\infty + w + w_1 \) in \( B(x,(1+\varepsilon)r) \), we get (9.26) and (9.25). Since \( v_{k,\varepsilon} = u_k \)
on $\partial B(x, (1+\varepsilon)r)$, we use it as a competitor for $u_k$, in the ball of radius $\tilde{r} = (1+\varepsilon)r$ and center $x$. By (9.5) and (9.1),

\begin{equation}
J_{x,\tilde{r}}^k(u_k) \leq (1 + h(\tilde{r})) J_{x,\tilde{r}}^k(v_{k,\varepsilon}).
\end{equation}

Next

\begin{align*}
J_{x,\tilde{r}}^k(v_{k,\varepsilon}) &= J_{x,\tilde{r}}^k(v_{k,\varepsilon}) + \int_{B(x,\tilde{r}) \setminus B(x, r)} |\nabla v_{k,\varepsilon}|^2 + \int_{B(x,\tilde{r}) \setminus B(x, r)} [\chi_{\{v_{k,\varepsilon} > 0\}} q_{\varepsilon,+,k}^k + \chi_{\{v_{k,\varepsilon} < 0\}} q_{\varepsilon,-,k}^k]
\leq J_{x,r}^k(v_{k,\varepsilon}) + \int_{B(x,\tilde{r}) \setminus B(x, r)} |\nabla v_{k,\varepsilon}|^2 + C|B(x, \tilde{r}) \setminus B(x, r)|
\leq J_{x,r}^k(v_{k,\varepsilon}) + \int_{B(x,\tilde{r}) \setminus B(x, r)} |\nabla v_{k,\varepsilon}|^2 + C\varepsilon r^n,
\end{align*}

where $C$ is a constant that may depend on $B$, our sequence, and even $v$, but not on $\varepsilon$ or $k$. Notice that $J_{x,r}^k(v_{k,\varepsilon}) = J_{x,r}^k(v)$ (by (9.24)), and that on $B(x, \tilde{r}) \setminus B(x, r)$,

\begin{equation}
|\nabla v_{k,\varepsilon}| \leq |\nabla u_\infty| + |\nabla u_k| + |\nabla v| |u_\infty - u_k| \leq C + C\varepsilon^{-1}||u_\infty - u_k||_{L^\infty(B(x,\tilde{r}))},
\end{equation}

where $C$ depends on $n$ and $L(B)$ in (9.11). Thus by (9.28)

\begin{equation}
J_{x,r}^k(v_{k,\varepsilon}) \leq J_{x,r}^k(v) + C\varepsilon^{-2}r^n ||u_\infty - u_k||_{L^\infty(B(x,\tilde{r}))}^2 + C\varepsilon r^n \leq J_{x,r}^k(v) + C\varepsilon r^n
\end{equation}

if $k$ is large enough (recall that $\{u_k\}$ converges to $u_\infty$ uniformly on compact subsets, and we restrict to $\varepsilon$ so small that $\overline{B(x, \tilde{r})} \subset \Omega$).

By (9.23), $J_{x,r}^\infty(u_\infty) \leq J_{x,r}^k(u_k) + \varepsilon$ if $k$ is large enough, and so

\begin{equation}
J_{x,r}^\infty(u_\infty) \leq J_{x,r}^k(u_k) + \varepsilon \leq J_{x,r}^k(v_{k,\varepsilon}) + \varepsilon \leq (1 + h(\tilde{r})) J_{x,\tilde{r}}^k(v_{k,\varepsilon}) + \varepsilon
\leq (1 + h(\tilde{r})) J_{x,r}^k(v) + C(1 + h(\tilde{r})) \varepsilon r^n + C\varepsilon
\end{equation}

by (9.27) and (9.30). Since in addition

\begin{equation}
|J_{x,r}^k(v) - J_{x,r}^\infty(v)| = \int_{B(x,r)} \chi_{\{v > 0\}} [q_{k,+} - q_{\infty,+}] + \int_{B(x,r)} \chi_{\{v < 0\}} [q_{k,-} - q_{\infty,-}] 
\leq \int_{B(x,r)} |q_{k,+} - q_{\infty,+}| + |q_{k,-} - q_{\infty,-}| \leq \varepsilon
\end{equation}

for $k$ large (by (9.4)), we get that

\begin{equation}
J_{x,r}^\infty(u_\infty) \leq (1 + h(\tilde{r})) J_{x,r}^\infty(v) + C(1 + h(\tilde{r}))(1 + r^n)\varepsilon.
\end{equation}

Letting $\varepsilon$ tend to 0, and using the continuity of $h$, we get that

\begin{equation}
J_{x,r}^\infty(u_\infty) \leq (1 + h(r)) J_{x,r}^\infty(v).
\end{equation}
So $u_\infty$ is an almost minimizer.

Next we want to take care of the lower bounds in (9.14) and (9.15). For this the main point is to control what happens when $u_\infty = 0$. Again fix $\overline{B}(x, r) \subset \Omega$. Set $Z = \{ y \in \overline{B}(x, r) : u_\infty(y) = 0 \}$. Then let $Z_0$ be the set of Lebesgue density point of $Z$, i.e., points $y \in Z$ such that $\lim_{t \to 0} t^{-d} |B(y, t) \setminus Z| = 0$, and recall that $|Z \setminus Z_0| = 0$. Let $\varepsilon_0 > 0$ be so small that

$$ (9.35) \quad \overline{B}(x, r + 10\varepsilon_0) \subset \Omega $$

Then let $\varepsilon \in (0, \varepsilon_0)$ be small. For each $y \in Z_0$, pick a ball $B_y = B(y, r_y)$ such that

$$ (9.36) \quad r_y < \varepsilon \quad \text{and} \quad |B(y, 10r_y) \setminus Z| < \varepsilon^n |B(y, r_y)|. $$

Then use Vitali’s covering lemma (see the first pages of [S]) to find a covering of $Z_0$ by a countable collection of balls $B(y, 5r_y)$, $y \in Y \subset Z_0$, such that the $B_y$’s are disjoint. To complete the covering of $Z$, we cover $Z \setminus Z_0$ with a collection of balls $D_j = B(z_j, t_j)$, so that $t_j \leq \varepsilon$ for all $j$, and $\sum_j t_j^n \leq \varepsilon$. Finally, we use the fact that $Z$ is compact to cover it with only a finite subcollection of the $B_y$ and the $D_j$.

Notice that all the $B(y, 10r_y)$ and the $D_j$ are contained in $\overline{B}(x, r + 10\varepsilon_0)$. By (9.11) there is a constant $L \geq 0$, independent of $\varepsilon$, such that each of the $u_k$’s and $u_\infty$ are $L$-Lipschitz on each $B(y, 10r_y)$ and each $D_j$. This will be used to estimate the contribution of these balls to $J^k(u_k)$.

For the $D_j$’s a rough estimate yields

$$ (9.37) \quad J_{z_j, t_j}^k(u_k) \leq \int_{B(z_j, t_j)} |\nabla u_k|^2 + q_k^+ + q_k^- \leq Ct_j^n. $$

Note that (9.36) guarantees that for $z \in B(y, 5r_y)$, $B(z, \varepsilon r_y)$ meets $Z$. Since $u_\infty$ vanishes on $Z$ and is $L$-Lipschitz we have that

$$ (9.38) \quad |u_\infty(z)| \leq L \varepsilon r_y \quad \text{for} \quad z \in B(y, 5r_y). $$

From (9.38) and the uniform convergence of the $u_k$’s on $B(y, 5r_y) \subset B(x, r + 10\varepsilon_0)$ we have that for $k$ large,

$$ (9.39) \quad |u_k(z)| \leq 2L \varepsilon r_y \quad \text{for} \quad z \in B(y, 5r_y). $$

Since we only have a finite collections of balls $B(y, 5r_y)$, $k$ large enough works for all of them at once.

We now need to compare $u_k$ with an appropriate competitor. Fix $y$ and set

$$ (9.40) \quad \begin{cases} 
\begin{array}{ll}
 v(z) = u_k(z) & \text{for} \quad z \in \Omega \setminus B(y, 5r_y) \\
 v(z) = 0 & \text{for} \quad z \in B(y, (5 - \varepsilon)r_y) \\
 v(z) = u_k(z) - (\varepsilon r_y)^{-1} [5r_y - |y - z|]u_k(z) & \text{for} \quad z \in B(y, r_y) \setminus B(x, (5 - \varepsilon)r_y).
\end{array}
\end{cases} $$
Notice that \( v \) is piecewise Lipschitz and continuous near \( B(y, 5r_y) \), so it is an acceptable competitor for \( u_k \). Thus

\[
J^k_{y, 5r_y}(u_k) \leq (1 + h(5r_y))J^k_{y, 5r_y}(v) \leq (1 + h(5\varepsilon))J^k_{y, 5r_y}(v). \tag{9.41}
\]

Since on \( B(y, 5r_y) \), by (9.40) and (9.39) we have

\[
|\nabla v| \leq |\nabla u_k| + (\varepsilon r_y)^{-1}\|u_k\|_{L^\infty(B(y, 5r_y))} \leq 3L,
\]

then

\[
J^k_{y, 5r_y}(v) \leq \int_{B(x, 5r_y) \setminus B(x, (5 - \varepsilon)r_y)} |\nabla v|^2 + q^+ + q^- \leq C(1 + L)^2\varepsilon r_y^n \tag{9.43}
\]

and hence by (9.41)

\[
J^k_{y, 5r_y}(u_k) \leq (1 + h(5\varepsilon))J^k_{y, 5r_y}(v) \leq C(1 + h(5\varepsilon))(1 + L)^2\varepsilon r^n_y \leq 2C(1 + L)^2\varepsilon r^n_y \tag{9.44}
\]

if \( \varepsilon_0 \) was chosen so small that \( h(5\varepsilon_0) \leq 1 \). Using (9.37), (9.44), the definition of the \( D_j \), and the fact that the \( B(y, r_y) \)'s are disjoint and contained in \( B(x, r + 10\varepsilon_0) \), we get

\[
\sum \limits_j J^k_{z_j, t_j}(u_k) + \sum \limits_y J^k_{y, 5r_y}(u_k) \leq C \sum \limits_j t^n_j + C(1 + L)^2\varepsilon \sum \limits_y r^n_y \leq C\varepsilon + C(1 + L)^2\varepsilon \sum \limits_y |B(y, r_y)| \leq C'. \tag{9.45}
\]

Here \( C' \) depends on the Lipschitz constant, \( r, \varepsilon_0 \) and \( n \), but not on \( \varepsilon \) or \( k \).

Set

\[
V = (\cup_j D_j) \cup (\cup_y B(y, 5r_y)) \quad \text{and} \quad A = \overline{B(x, r)} \setminus V. \tag{9.46}
\]

Notice that \( V \) is an open set that contains the compact set \( Z \), so \( A \) is compact and \( |u_\infty| > 0 \) on \( A \). Let \( \eta > 0 \) be such that \( |u_\infty| \geq \eta \) on \( A \). We can see \( A \) as the union of two compact subsets \( A_+ \), where \( u_\infty \geq \eta \), and \( A_- \), where \( u_\infty \leq -\eta \). Choose an open neighborhood \( U_\pm \) of \( A_\pm \), such that \( \overline{U_-} \) is still contained in \( \{ y \in \Omega; \pm u_\infty \geq \eta/2 \} \). By Theorem 3.2 and the more precise (3.29) we get that the \( \nabla u_k \)'s are Hölder continuous on \( U_\pm \), with a fixed exponent \( \beta \) and uniform bounds on the Hölder constant. Recall that the \( u_k^\pm \) themselves converge to \( u_\infty^\pm \), uniformly on \( U_\pm \).

We claim that the \( \nabla u_k \)'s also converge uniformly to \( \nabla u_\infty \) on \( U_\pm \). Otherwise there exist \( \tau > 0 \) and a subsequence \( \{ \nabla u_{k'} \} \) of \( \{ \nabla u_k \} \) such that \( \| \nabla u_{k'} - \nabla u_\infty \|_\infty \geq \tau \) for all \( k' \). On the other hand by the uniform \( C^{1,\beta} \) bounds on the \( u_k \)'s, we can extract a subsequence that converges uniformly on \( U_\pm \). But this subsequence also converges weakly and its limit is \( u_\infty \) (see Remark 9.1), which contradicts our assumption. So \( \{ \nabla u_k \} \) converges to \( \nabla u_\infty \), uniformly on \( A_\pm \), and

\[
\int_{A_\pm} |\nabla u_\infty^\pm|^2 = \lim_{k \to \infty} \int_{A_\pm} |\nabla u_k^\pm|^2. \tag{9.47}
\]

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Furthermore, by (9.46) and (9.45) we have that for \( k \) large enough,

\[
(9.48) \quad \int_{B(x,r) \setminus A_\pm} |\nabla u_k^\pm|^2 \leq \int_V |\nabla u_k^\pm|^2 \leq C' \varepsilon.
\]

Thus combining (9.47) and (9.48) we obtain

\[
(9.49) \quad \limsup_{k \to \infty} \int_{B(x,r)} |\nabla u_k^\pm|^2 \leq \int_{B(x,r)} |\nabla u_\infty^\pm|^2 + C' \varepsilon.
\]

Since (9.49) holds for \( \varepsilon > 0 \) arbitrarily small, (9.49) and (9.22) prove (9.15).

Next fix a sign \( \pm \) and observe that for \( k \) large, both \( \pm u_\infty \) and \( \pm u_k \) are positive on \( A_\pm \). Then for \( k \) large enough, by (9.4) we have

\[
(9.50) \quad \int_{A_\pm} \chi_{\{ \pm u_k > 0 \}} q_{k,\pm} = \int_{A_\pm} \chi_{\{ \pm u_\infty > 0 \}} q_{k,\pm} \leq \int_{A_\pm} \chi_{\{ \pm u_\infty > 0 \}} q_{\infty,\pm} + \int_{B(x,r)} |q_{k,\pm} - q_{\infty,\pm}| \leq \int_{A_\pm} \chi_{\{ \pm u_\infty > 0 \}} q_{\infty,\pm} + \varepsilon.
\]

Since \( \chi_{\{ \pm u_k > 0 \}} = 0 \) on \( A_\pm \), we also have that

\[
(9.51) \quad \int_{B(x,r) \setminus A_\pm} \chi_{\{ \pm u_k > 0 \}} q_{k,\pm} = \int_{V} \chi_{\{ \pm u_k > 0 \}} q_{k,\pm} \leq \sum_j J_{kj}(u_k) + \sum_j J_{kj}(u_\infty) \leq C' \varepsilon
\]

by (9.45). We add (9.50) and (9.51) to obtain

\[
(9.52) \quad \limsup_{k \to \infty} \int_{B(x,r)} \chi_{\{ \pm u_k > 0 \}} q_{k,\pm} \leq \int_{B(x,r)} \chi_{\{ \pm u_\infty > 0 \}} q_{\infty,\pm} + C' \varepsilon + \varepsilon.
\]

Since \( \varepsilon \) is arbitrarily small, (9.21) and (9.52) yield (9.14).

Now we prove (9.16) and (9.17). Recall that for all \( k \), \( \nabla u_k^+ = \nabla u_k \) and \( \nabla u_k^- = 0 \) on the open set \( \{ x \in \Omega ; u_k(x) > 0 \} \). We have a similar description on \( \{ x \in \Omega ; u_k(x) < 0 \} \), and on the remaining set \( \{ x \in \Omega ; u_k(x) = 0 \} \), we know that \( \nabla u_k^+ \) and \( \nabla u_k^- \) exist almost everywhere (because these functions are locally Lipschitz). Since these derivatives can be computed to be zero on any Lebesgue density point of \( \{ x \in \Omega ; u_k(x) = 0 \} \), we get that \( \nabla u_k^+ = \nabla u_k^- = \nabla u_k = 0 \) almost everywhere on that set. Then \( \int_{B(x,r)} |\nabla u_k|^2 = \int_{B(x,r)} |\nabla u_k^+|^2 + \int_{B(x,r)} |\nabla u_k^-|^2 \).

A similar description holds for \( \nabla u_\infty \), and now (9.16) follows from (9.15). Also, (9.17) follows from (9.14) and (9.16).

To prove (9.13) recall than in \( B(x,r) \), \( u_\infty \) is the uniform limit of the \( u_k \). By composing with the Lipschitz function \( \max(0, \cdot) \), we get that \( u_\infty^\pm \) is the uniform limit of the \( u_k^\pm \)'s. Recall that these functions are Lipschitz in \( B(x,r) \), with uniform estimates. Then (by elementary distribution theory) \( \nabla u_\infty^\pm \) is the weak limit of the \( \nabla u_k^\pm \), in \( L^2(B(x,r)) \). Moreover (9.15) ensures that \( \{ \nabla u_\infty^\pm \} \) converges strongly in \( L^2(B(x,r)) \) to \( \nabla u_\infty^\pm \). This proves (9.13) for \( u_\infty^\pm \).

This completes our proof of Theorem 9.1.\[\square\]
Remark 9.3. Suppose that instead of (9.5) or (9.6), we have that either
\[ (9.53) \quad u_k \text{ is an almost minimizer for } J^k \text{ in } \Omega, \text{ with gauge function } h_k, \]
or, if we work with \( J^+ \), that
\[ (9.54) \quad u_k \text{ is an almost minimizer for } J^{k,+} \text{ in } \Omega, \text{ with gauge function } h_k, \]
where the \( h_k \) are continuous gauge functions such that
\[ (9.55) \quad \lim_{k \to \infty} h_k(r) = 0 \text{ for every } r > 0. \]
Then the function \( u_\infty \) which appears in (9.12) satisfies
\[ (9.56) \quad u_\infty \text{ is a minimizer for } J^\infty \text{ in } \Omega \]
or, if we work with \( J^+ \),
\[ (9.57) \quad u_\infty \text{ is a minimizer for } J^{\infty,+} \text{ in } \Omega, \]
where minimizer means almost minimizer with the gauge function \( h = 0 \).
Indeed, set \( H_l(r) = \sup_{k \geq l} h_k(r) \). It is clear that each \( H_l \) is a gauge function, and it is easy
to see that it is also continuous. Most often we consider the case when the sequence \( \{h_k\} \)
is nondecreasing and \( H_l = h_l \). Apply Theorem 9.1 to the sequence \( \{u_k\}, k \geq l \). We obtain
that \( u_\infty \) is an almost minimizer, with the gauge function \( H_l \). This means that for every ball
\( B(x, r) \), (9.1) or (9.2) holds with \( H_l(r) \). But for each \( r \), \( \lim_{l \to \infty} H_l(r) = 0 \), so \( u_\infty \) is in fact a
minimizer.

Next we apply Theorem 9.1 and Remark 9.3 to the special case of blow-up limits. Let \( \Omega \) be
given, and let \( u \) be an almost minimizer in \( \Omega \) (for \( J \) or \( J^+ \)). Here we just work with one
gauge function \( h \), typically \( h(r) = \kappa r^\alpha \), and one pair of bounded functions \( q_{\pm} \).
Before we discuss blow-up sequences, let us say a few words about dilations. For every \( x \in \Omega \)
and \( r > 0 \), set
\[ (9.58) \quad \Omega_{x,r} = \{ y \in \mathbb{R}^n; x + ry \in \Omega \} = \frac{1}{r}[\Omega - x], \]
\[ (9.59) \quad q^{(x,r)}_{\pm}(y) = q_{\pm}(x + ry) \text{ for } y \in \Omega_{x,r}, \]
and
\[ (9.60) \quad u^{(x,r)}(y) = \frac{1}{r}u(x + ry) \text{ for } y \in \Omega_{x,r}. \]
We use the functions \( q^{(x,r)}_{\pm} \) to define a functional \( J^{x,r} \), or \( J^{x,r,+} \) if we work with \( J^+ \). We claim that
\[ u^{(x,r)} \text{ is an almost minimizer in } \Omega_{x,r}, \text{ for } J^{x,r} \text{ or } J^{x,r,+}, \]
\[ (9.61) \quad \text{with the gauge function } h(r \cdot). \]
This claim is a straightforward exercise on the chain rule. We do the computations for $J$; those for $J^+$ are analogous. Let $v$ be a competitor for $u(x, r)$ in the ball $B = B(y, t)$, which means that $B(y, t) \subset \Omega_{x, r}$, $v \in W^{1,2}(B)$, and its trace on $\partial B$ is the same as the trace of $u$. Keep $v(y) = u(x, r)(y)$ on $\Omega_{x, r} \setminus B$. Set $w(z) = rv(r^{-1}(z - x))$ for $z \in \Omega$. It is easy to see that $w$ is a competitor for $u$ in $B' = x + rB$. Hence $J_{B'}(u) \leq (1 + h(tr))J_{B'}(w)$, where for short we also set $J_B(u) = J_{y, r}(u)$ when $B = B(y, t)$. Now

$$J_{B'}^{x,r}(u) = \int_B |\nabla u(x, r)|^2 + \chi_{\{u(x, r) > 0\}} q^+_r + \chi_{\{u(x, r) < 0\}} q^-_r$$

(9.62)

$$= r^{-n} \int_{B'} |\nabla u|^2 + \chi_{\{u > 0\}} q_+ + \chi_{\{u < 0\}} q_- = r^{-n}J_{B'}(u).$$

The same computation yields $J_{B'}^{x,r}(v) = r^{-n}J_{B'}(w)$, and now

$$J_{B'}^{x,r}(u(x, r)) = r^{-n}J_{B'}(u) \leq r^{-n}(1 + h(tr))J_{B'}(w) = (1 + h(tr))J_{B'}^{x,r}(v),$$

(9.63)

as needed for (9.61).

We now focus on blow-up sequences for almost minimizers. That is let $u$ be an almost minimizer, fix a point $x$, and take a sequence $r_k$ of radii, with

$$\lim_{k \to \infty} r_k = 0.$$

(9.64)

Set $u_k = u(x, r_k)$. That is, define $u_k$ on $\Omega_k = \frac{1}{r_k} \Omega - x$ by

$$u_k(y) = \frac{1}{r_k} u(x + r_k y).$$

(9.65)

We say that the sequence $\{u_k\}$ converges if there is a function $u_\infty$, defined on $\mathbb{R}^n$, such that

$$u_\infty(y) = \lim_{k \to \infty} u_k(y) \text{ for every } y \in \mathbb{R}^n.$$

(9.66)

Notice that for each $y \in \mathbb{R}^n$, $y \in \Omega_k$ for $k$ large (because if $B(x, a) \subset \Omega$, then $\Omega_k$ contains $B(0, a/r_k)$), so (9.66) makes sense. We apparently take a weak definition of convergence, but we shall see soon that when $h(t) \leq kr^{\alpha}$, it implies uniform convergence on compact sets, and even (if the $q_\pm$ are continuous, say) the convergence of the gradients in $L^2_{loc}(B)$ as in Theorem 9.1.

Notice also that if $\{u_k\}$ converges, then

$$u(x) = 0,$$

(9.67)

because otherwise $\{u_k(0)\}$ diverges. This does not disturb us as we are not interested in blow ups at points where $|u| > 0$.

A blow-up limit of $u$ at $x$ is a function $u_\infty : \mathbb{R}^n \to \mathbb{R}$ such that, for some choice of $\{r_k\}$ with $\lim_{k \to \infty} r_k = 0$, the sequence $\{u_k\}$ converges to $u_\infty$ (as in (9.66)). Of course different sequences may give different blow-up limits.
For the following discussion, we shall assume that for some choice of \( \kappa, \alpha > 0, r_0 > 0 \),

\[
(9.68) \quad h(r) \leq \kappa r^\alpha \quad \text{for} \quad 0 < r < r_0
\]

so that we can use the results of the previous sections. Moreover we assume that \( x \) is a Lebesgue point of \( q_+ \) and \( q_- \), in the precise sense that

\[
(9.69) \quad \lim_{r \to 0} \int_{B(x,r)} |q_+(y) - q_+(x)| + |q_-(y) - q_-(x)| = 0.
\]

This is the case if \( q_+ \) and \( q_- \) are continuous at \( x \). The following will be an easy consequence of Theorem 9.1.

**Theorem 9.2** Let \( u \) be an almost minimizer for \( J \) or \( J^+ \) (associated to bounded functions \( q_\pm \)) in \( \Omega \), with a gauge function \( h \) such that (9.68) holds, and let \( x \in \Omega \) be such that \( u(x) = 0 \) and (9.69) holds. Then for each sequence \( \{r_k\} \) in \((0, \infty)\) that tends to 0, we can extract a subsequence \( \{u_{k_j}\} \) such that \( \{u_{k_j}\} \) converges.

Also, if \( \{r_k\} \) is a sequence in \((0, \infty)\) that tends to 0 and for which \( \{u_{k_j}\} \) converges to a limit function \( u_\infty \), then \( u_\infty \) is a minimizer in \( \mathbb{R}^n \), for the functional \( J^\infty \) or \( J^{\infty, +} \) associated to the constant functions \( q_\pm^R = q_\pm(x) \). In addition, for each \( R > 0 \),

\[
(9.70) \quad \{u_{k_j}\} \text{ converges to } u_\infty \text{ uniformly in } B(0, R)
\]

and

\[
(9.71) \quad \{\nabla u_{k_j}\} \text{ converges to } \nabla u_\infty \text{ in } L^2(B(0, R)).
\]

**Proof.** Let \( u \) and \( x \) be as in the statement. Also let \( \{r_k\} \) be any sequence of positive numbers that tends to 0, and \( R > 0 \) be given. Set \( B = B(0, R) \) and \( B_k = B(x, r_k R) \), and observe that \( B_k \subset \Omega \) for \( k \) large. For such \( k \), by (9.61) \( u_k \) is an almost minimizer in \( B \), with the gauge function \( h_k = h(r_k \cdot) \), and with the functional associated to \( q_\pm^R(y) = q_\pm(x + r_k y) \). Notice that for \( k \) large, by (9.69)

\[
(9.72) \quad h_k(t) = h(r_k t) \leq \kappa (r_k t)^\alpha = r_k^\alpha \kappa t^\alpha \quad \text{for} \quad 0 < t < 2R.
\]

To apply Theorem 9.1 to the \( u_k \)'s, in the domain \( B \), we need to check the assumptions. We have (9.3) because \( q_+ \) and \( q_- \) are always assumed to be bounded, and (9.4) (even for the full \( B \)) follows from (9.69) (compute the average on \( B(x, r_k R) \)). The limiting functions are \( q_+(x) \) and \( q_-(x) \), as expected.

We just proved (9.5) or (9.6), with the gauge given by (9.72), so that (9.7) (and its consequence (9.8) or (9.9)) is also immediate). Finally (9.10) holds because we know from Theorem 5.1 or 8.1 that \( u \) is Lipschitz near \( x \) (notice that (9.65) preserves the Lipschitz bounds).

Thus the remarks below (9.9) apply, and we have uniform bounds on the \( \nabla u_k \) in \( L^\infty(B(0, R/2)) \). We also know that \( u_k(0) = r_k^{-1} u(x) = 0 \) for all \( k \).
By Arzela-Ascoli, from the sequence \( \{u_k\} \) we can extract a subsequence that converges uniformly in \( B(0, R/2) \). Since this is true for every \( R > 0 \), by a diagonal argument, we see that there is a subsequence of \( \{u_k\} \) that converges uniformly on every compact subset of \( \mathbb{R}^n \). This takes care of the existence of blow-up limits.

Now suppose that we started with a sequence such that \( \{u_k\} \) converges to some limit \( u_\infty \). Because of our uniform bounds on the \( \|\nabla u_k\|_{L^\infty(B(0,R/2))} \), we see that the convergence is uniform on compact sets, i.e., (9.70) holds. Also applying Theorem 9.1, we get that \( u_\infty \) is an almost minimizer on \( B(0, R) \). In fact, since our gauge functions \( h_k \) tend to 0 uniformly, Remark 9.3 ensures that \( u_\infty \) is a minimizer in \( B(0, R) \) (with constant functions \( q^\pm(x) \)). Since this holds for every \( R \), \( u \) is a minimizer in the whole space \( \mathbb{R}^n \).

Finally, (9.71) for \( B(0, R/2) \) is a consequence of (9.13). This proves Theorem 9.2.

**Remark 9.4.** Under the assumptions of Theorem 9.2, if the blow-up sequence \( \{u_k\} \) converges to a limit \( u_\infty \), we have the following estimates, which follow from the proof above, the estimates (9.13), (9.14), and (9.17), and the change of variables in (9.62):

\[
\lim_{k \to \infty} \nabla u_k^\pm = \nabla u_\infty^\pm \quad \text{in} \quad L^2(B(0, R)) \quad \text{for every} \quad R > 0,
\]

and, for every ball \( B(z,t) \subset \mathbb{R}^n \),

\[
q^\pm(x) \int_{B(z,t)} \chi_{\{u_\infty > 0\}} = \lim_{k \to \infty} \int_{B(z,t)} \chi_{\{u_k > 0\}} q^k = \lim_{k \to \infty} r_k^{-n} \int_{B(x+r_k z,r_k t)} \chi_{\{u > 0\}} q^k,
\]

\[
\int_{B(z,t)} |\nabla u_\infty^\pm|^2 = \lim_{k \to \infty} \int_{B(z,t)} |\nabla u_k^\pm|^2 = \lim_{k \to \infty} \int_{B(x+r_k z,r_k t)} |\nabla u^\pm|^2,
\]

\[
J_{z,t}^\infty(u_\infty) = \lim_{k \to \infty} J_{z,t}^k(u_k) = \lim_{k \to \infty} r_k^{-n} J_{x+r_k z,r_k t}(u),
\]

and similarly for the \( J^+ \).

We conclude this section with a simple consequence of Theorems 9.1 and 9.2, relative to the functional \( \Phi \) from [ACF].

**Corollary 9.1** Let \( u \) be an almost minimizer for \( J \) (associated to bounded functions \( q^\pm \)) in \( \Omega \), with a gauge function \( h \) such that (9.68) holds. Let \( x \in \Omega \) be such that \( u(x) = 0 \) and (9.69) holds. Set

\[
\Phi_x(t) = \frac{1}{t^4} \left( \int_{B(x,t)} \frac{|\nabla u^+(y)|^2}{|x-y|^{n-2}} dy \right) \left( \int_{B(x,t)} \frac{|\nabla u^-(y)|^2}{|x-y|^{n-2}} dy \right)
\]

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for $0 < t < \text{dist}(x, \mathbb{R}^n \setminus \Omega)$. Then let $u_\infty$ be any blow-up limit of $u$ at $x$, and set

$$
\Phi(s) = \frac{1}{s^4} \left( \int_{B(0,s)} \frac{|\nabla u_\infty^+(y)|^2}{|y|^{n-2}} dy \right) \left( \int_{B(0,s)} \frac{|\nabla u_\infty^-(y)|^2}{|y|^{n-2}} dy \right)
$$

for $s > 0$; this is the corresponding function to $u_\infty$ at the origin. Then $\Phi$ is constant, and

$$
\Phi(s) = \lim_{t \to 0} \Phi_x(t) \quad \text{for } s > 0.
$$

The fact that we get a constant function $\Phi$ in the statement above is often quite useful, especially when this limit is nonzero, but we shall not worry about this in this section.

**Proof.** Let $\{r_k\}$ be a sequence in $(0, +\infty)$ that tends to 0, and for which $u_\infty$ is the limit of the $u_k$ defined by (9.65). We want to show that for each $s > 0$,

$$
\Phi(s) = \lim_{k \to 0} \Phi_x(r_k_s);
$$

(9.79) will follow because we already know from (6.5) that the limit exists. Fix $s > 0$, and set

$$
A_\pm = \int_{B(0,s)} \frac{|\nabla u_\infty^\pm(y)|^2}{|y|^{n-2}} dy \quad \text{and} \quad A^k_\pm = \int_{B(0,s)} \frac{|\nabla u_k^\pm(y)|^2}{|y|^{n-2}} dy
$$

(9.81)

Notice that $\Phi(s) = s^{-4} A_\pm A_{-\pm}$. But also,

$$
A^k_\pm = \int_{B(0,s)} \frac{|\nabla u_k^\pm(x + r_k y)|^2}{|y|^{n-2}} dy = r_k^{-n} \int_{B(x, r_k s)} \frac{|\nabla u_k^\pm(z)|^2}{r_k^{2-n}|z - x|^{n-2}} dy
$$

by (9.65) and where we set $z = x + r_k y$. Hence

$$
\Phi_x(r_k s) = (r_k s)^{-4} \left( \int_{B(x, r_k s)} \frac{|\nabla u^\pm(z)|^2}{|z - x|^{n-2}} dy \right) \left( \int_{B(x, r_k s)} \frac{|\nabla u^\mp(z)|^2}{|z - x|^{n-2}} dy \right) = s^{-4} A_\pm A^k_{-\pm}.
$$

(9.83)

It is enough to show that $\lim_{k \to \infty} A^k_{\pm} = A_{\pm}$ and that all these numbers are finite. Let $\eta > 0$ be small, then

$$
|A_\pm - A^k_{\pm}| \leq \int_{B(0,\eta)} \frac{|\nabla u_\infty^\pm(y)|^2}{|y|^{n-2}} - \frac{|\nabla u^\pm_k(y)|^2}{|y|^{n-2}} dy + \int_{B(0,\eta) \setminus B(0,\eta)} \frac{|\nabla u_\infty^\pm(y)|^2}{|y|^{n-2}} - \frac{|\nabla u^\pm_k(y)|^2}{|y|^{n-2}} dy
$$

$$
\leq \int_{B(0,\eta)} \frac{1}{|y|^{n-2}} dy + \eta^{n-2} \int_{B(0,\eta) \setminus B(0,\eta)} \frac{|\nabla u_\infty^\pm(y)|^2}{|y|^{n-2}} - \frac{|\nabla u^\pm_k(y)|^2}{|y|^{n-2}} dy
$$

(9.84)

because $\nabla u_\infty$ and the $\nabla u_k$ are bounded on $B(0,s)$. The first term can be made as small as we want by taking $\eta$ small, and for $\eta > 0$ fixed, the second term tends to 0, because (9.73) ensures that $\nabla u^\pm_k$ converges in $L^2(B(0,s))$ to $\nabla u_\infty^\pm$. Thus $\lim_{k \to \infty} A^k_{\pm} = A_{\pm}$. Note that a similar computation yields that $A^k_\pm$ and $A_{\pm}$ are finite. Corollary 9.1 follows. 



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10 Nondegeneracy near the free boundary

One of the main results of this section states that if $u$ is an almost minimizer for $J$ or $J^+$, and if $q_+$ is bounded below away from zero, if $u^+$ is very small on a small ball $B$, then it actually vanishes on $\frac{1}{4}B$. See Theorem 10.1 below. This statement implies various other nondegeneracy properties of $u$ near the free boundary $\partial\{u > 0\}$, that we also prove in this section. It is reminiscent of some of the estimates that appear in [AC] and [ACF]. Our proofs use a different approach though.

We start with a variant of the result of result in [AC] that states that $u^+$ is subharmonic when $u$ is a minimizer for $J$ or $J^+$. Note that we do not require more than $L^\infty$ bounds on $q_+$ and $q_-$.

**Lemma 10.1** Let $u$ be an almost minimizer for $J$ or $J^+$ in $\Omega$. Let $B(x,r)$ be such that $\overline{B(x,r)} \subset \Omega$, and let $u^*$ denote the harmonic extension to $B(x,r)$ of the restriction of $u^+$ to $\partial B(x,r)$. Then

\[
\int_{B(x,r)} [(u - u^*)^+]^2 \leq C\kappa r^{2+\alpha} (1 + \int_{B(x,r)} |\nabla u|^2),
\]

where $C$ depends only on $n$.

**Proof.** Let us define a function $v$ by

\[
\begin{cases}
  v(y) = u(y) & \text{for } y \in \Omega \setminus B(x,r) \\
  v(z) = \min(u(y), u^*(y)) & \text{for } y \in B(x,r).
\end{cases}
\]

We first check that $v \in W^{1,2}(B(x,r))$. First recall that $u$ and $u^+$ are Lipschitz in $B(x,r)$, and that $W^{1,2}(B(x,r))$ is stable under minima and maxima. So it is enough to check that $u^* \in W^{1,2}(B(x,r))$. But the argument given in Remark 3.1 guarantees that the energy minimizing extension of $u^+$ in $B(x,r)$, which lies in $W^{1,2}(B(x,r))$ by definition, coincides with $u^*$.

Note that $v$ is locally Lipschitz in $\Omega \setminus B(x,r)$ because $u$ is. Finally, $v$ is continuous across $\partial B(x,r)$; then $v \in W^{1,2}_{\text{loc}}(\Omega)$, for instance by the welding Lemma 14.4 in [D]. Since $v = u$ outside of $B(x,r)$, the definition of almost minimizers yields

\[
J_{x,r}(u) \leq (1 + \kappa r^\alpha) J_{x,r}(v)
\]

(or a similar estimate on $J^+$, that would be treated the same way). Notice that $v(y) = u(y)$ when $u(y) \leq 0$, because $u^*(y) \geq 0$ everywhere. Then

\[
\int_{B(x,r)} \chi_{\{v > 0\}} q_+ + \chi_{\{v < 0\}} q_- \leq \int_{B(x,r)} \chi_{\{u > 0\}} q_+ + \chi_{\{u < 0\}} q_-.
\]

Also set $V = \{ y \in B(x,r) ; v(y) \neq u(y) \} = \{ y \in B(x,r) ; u^*(y) < u(y) \}$. Since $\nabla v = \nabla u$ almost everywhere on $B(x,r) \setminus V$ and $\nabla v = \nabla u^*$ everywhere on $V$, by (10.4)

\[
J_{x,r}(v) - J_{x,r}(u) \leq \int_{B(x,r)} |\nabla v|^2 - |\nabla u|^2 = \int_V |\nabla u^*|^2 - |\nabla u|^2.
\]
Next we want to show that

\[(10.6) \quad \int_V |\nabla u|^2 = \int_V |\nabla u^*|^2 + \int_V |\nabla u - \nabla u^*|^2.\]

We observed earlier that, by the proof of Remark 3.1, \(u^*\) is also the function in \(W^{1,2}(B(x, r))\) which minimizes \(\int_{B(x, r)} |\nabla v|^2\) under the condition that the trace of \(v\) on \(\partial B(x, r)\) be equal to \(u^+\).

Now set \(w(y) = \max(u(y), u^*(y))\) for \(y \in B(x, r)\). Then \(w \in W^{1,2}(B(x, r))\) because \(u\) and \(u^*\) both lie in \(W^{1,2}(B(x, r))\), and the trace of \(w\) coincides with \(u^*\) and \(u^+\) on \(\partial B(x, r)\) because \(u\) and \(u^+\) are continuous. So \(w\) is a competitor in the minimizing definition of \(u^*\), and so would be \(u^* + \lambda(w - u^*)\) for any \(\lambda \in \mathbb{R}\). By the usual computation, the scalar product \(\int_{B(x, r)} (\nabla u^*, \nabla (w-u^*)) = 0\). But \(w = u^*\) on \(B(x, r) \setminus V\), so \(\nabla (w-u^*) = 0\) almost everywhere on \(B(x, r) \setminus V\), and we are left with \(0 = \int_V (\nabla u^*, \nabla (u-u^*)) = \int_V (\nabla u^*, \nabla (u-u^*))\); (10.6) follows. An immediate consequence of (10.5) and (10.6) is that \(J_{x,r}(v) \leq J_{x,r}(u)\), and now (10.6), (10.5), and (10.3) yield

\[(10.7) \quad \int_V |\nabla u - \nabla u^*|^2 = \int_V |\nabla u|^2 - \int_V |\nabla u^*|^2 \leq J_{x,r}(u) - J_{x,r}(v) \leq \kappa r^\alpha J_{x,r}(v) \leq C\kappa r^\alpha (r^n + \int_{B(x, r)} |\nabla u|^2).\]

Now we apply Poincaré’s inequality to the function \(h = (u-u^*)_+\). Notice that \(h \in W^{1,2}(B(x, r))\), and \(h\) is continuous on \(\overline{B(x, r)}\), with vanishing boundary values. Then we get that

\[(10.8) \quad \int_{B(x, r)} [(u-u^*)_+]^2 \leq C\alpha \int_{B(x, r)} \nabla [(u-u^*)_+]^2 \leq C\kappa r^{2+\alpha} (1 + \int_{B(x, r)} |\nabla u|^2),\]

because \(u \leq u^*\) on \(B(x, r) \setminus V\), hence \(\nabla [(u-u^*)_+] = \chi_V (\nabla u - \nabla u^*)\). Lemma 10.1 follows. 

In order to simplify the statements below, let us decide that \(n, \alpha, \kappa\), and the \(\|q_+\|_\infty\) will be referred to as the usual constants.

**Theorem 10.1** Let \(u\) be an almost minimizer for \(J\) or \(J^+\) in \(\Omega\). For each choice of \(\rho_0 > 0\) and \(L \geq 1\), there are constants \(\eta_0 > 0\) and \(r_0 > 0\), that depend only on \(\rho_0\), \(L\), and the usual constants, such that if \(x \in \Omega\) and \(0 < r \leq r_0\) are such that \(\overline{B(x, r)} \subset \Omega\),

\[(10.9) \quad \int_{\partial B(x, r)} u^+ \leq r \eta_0,\]

\[(10.10) \quad u \text{ is } L\text{-Lipschitz on } B(x, r),\]

and

\[(10.11) \quad q_+(y) \geq \rho_0 \text{ for } y \in B(x, r),\]

then \(u(y) \leq 0\) for \(y \in B(x, r/4)\).
The same proof will also yield that if \( u \) is an almost minimizer for \( J, \int_{\partial B(x,r)} u^- \leq Cr\eta_0, \) (10.10) holds, and \( q_-(y) \geq \rho_0 \) on \( B(x,r) \), then \( u(y) \geq 0 \) for \( y \in B(x, r/4) \). Notice that for Theorem 10.1, we only need the nondegeneracy assumption (10.11) on \( q_+ \), and that in the case of \( J \) we do not need to assume that \(|\{u = 0\}| = 0 \). The Lipschitz assumption (10.10) is not an issue because we proved earlier that \( u \) is locally Lipschitz. We specify the constant to be used as a normalization factor.

**Proof.** Let us first prove that if \( B(x, r) \) is as in the statement, and if \( r_0 \) is small enough (depending on \( L \) and \( \eta_0 \) in particular), then

\[
(10.12) \quad u(y) \leq 4^{n+1}\eta_0 r \quad \text{for} \quad y \in B(x, r/2).
\]

Otherwise there is \( y \in B(x, r/2) \) such that \( u(y) \geq 4^{n+1}\eta_0 r \). By (10.10),

\[
(10.13) \quad u(z) \geq u(y) - \eta_0 r \geq (4^{n+1} - 1)\eta_0 r \quad \text{for} \quad z \in B(y, \frac{\eta_0 r}{L}).
\]

Choose \( \eta_0 \in (0, \frac{1}{4}) \); then \( B(y, \frac{\eta_0 r}{4}) \subset B(x, \frac{3}{4}) \), and since \( u^*(x) = \int_{\partial B(x,r)} u^+ \leq r\eta_0 \), for \( z \in B(y, \frac{\eta_0 r}{4}) \) the Poisson formula yields

\[
(10.14) \quad u^*(z) = \frac{r^2 - |x - z|^2}{\sigma_{n-1} r} \int_{\partial B(x,r)} \frac{u(\zeta)}{|\zeta - z|^n} d\zeta \leq 4^n \int_{\partial B(x,r)} u^+ \leq 4^n \eta_0 r
\]

Combining (10.13) and (10.14) we have

\[
(10.15) \quad \int_{B(x,r)} [(u - u^*)_+]^2 \geq \int_{B(y, \frac{\eta_0 r}{L})} [(u - u^*)_+]^2 \\
\geq (4^{n+1} - 1 - 4^n)\eta_0^2 r^2 |B(y, \frac{\eta_0 r}{L})| \geq \eta_0^2 r^2 |B(y, \frac{\eta_0 r}{L})|,
\]

which contradicts (10.8) if \( r_0 \) (and hence \( r \)) is small enough.

Let \( \varphi \) be a smooth function such that \( 0 \leq \varphi \leq 1 \) everywhere, \( \varphi(y) = 1 \) for \( y \in B(x, r/2) \), \( \varphi(y) = 0 \) for \( y \in \Omega \setminus B(x, 3r/4) \), and \(|\nabla \varphi| \leq 5r^{-1} \). Then set

\[
(10.16) \quad \begin{cases} v(y) = [u(y) - 4^{n+1}\eta_0 r\varphi(y)]_+ & \text{when} \ u(y) \geq 0 \\
v(y) = u(y) & \text{when} \ u(y) < 0.
\end{cases}
\]

Notice that \( v \in W^{1,2}_{\text{loc}}(\Omega) \), for instance because we can write \( v = [u - 2\eta_0 r\varphi]_+ - u_- \), and \( u \in W^{1,2}_{\text{loc}}(\Omega) \). Also, \( v = u \) outside \( B(x, 3r/4) \), so \( u \) and \( v \) have the same trace on \( \partial B(x, r) \). Hence, by almost minimality of \( u \), \( J_{x,r}(u) \leq (1 + \kappa r^\alpha)J_{x,r}(v) \).

Observe that \( \{v < 0\} = \{u < 0\} \) and \( \{v > 0\} \subset \{u > 0\} \), but in addition \( v = 0 \) on \( B(x, r/2) \), by (10.12). So in fact \( \{v > 0\} \subset \{u > 0\} \setminus B(x, r/2) \) and by (10.11)

\[
(10.17) \quad \int_{B(x,r)} [\chi_{\{v>0\}}q_+ + \chi_{\{v<0\}}q_-] \leq \int_{B(x,r)} [\chi_{\{u>0\}}q_+ + \chi_{\{u<0\}}q_-] - \int_{B(x,r/2) \cap \{u>0\}} q_+ \\
\leq \int_{B(x,r)} [\chi_{\{u>0\}}q_+ + \chi_{\{u<0\}}q_-] - \rho_0 |B(x, r/2) \cap \{u > 0\}|.
\]
Notice that $\nabla v = \nabla u$ almost everywhere on $\{u < 0\}$, and $|\nabla v| \leq |\nabla (u - 4^{n+1}\eta_0 r \varphi)|$ almost everywhere on $\{u \geq 0\}$, so that

$$
\int_{B(x,r)} |\nabla v|^2 - \int_{B(x,r)} |\nabla u|^2 \leq 2 \cdot 4^{n+1}\eta_0 r \int_{B(x,r)} |\nabla u||\nabla \varphi| + 4^{2(n+1)}\eta_0^2 r^2 \int_{B(x,r)} |\nabla \varphi|^2
$$

(10.18)

$$
\leq \frac{1}{10} \cdot 4^{n+1}\eta_0 \int_{B(x,r)} |\nabla u| + 25 \cdot 4^{2(n+1)}\eta_0^2 |B(x,r)|
\leq C\eta_0(1 + L)r^n
$$

by Cauchy-Schwarz, (10.10), and because $\eta_0$ will be chosen small. Combining (10.17) and (10.18) we get that

$$
J_{x,r}(v) - J_{x,r}(u) \leq -\rho_0 |B(x, r/2) \cap \{u > 0\}| + C\eta_0(1 + L)r^n
$$

(10.19)

and then since $J_{x,r}(u) \leq C(1 + L^2)r^n$

$$
J_{x,r}(u) \leq (1 + \kappa r^\alpha)J_{x,r}(v)
\leq (1 + \kappa r^\alpha)J_{x,r}(u) - (1 + \kappa r^\alpha)\rho_0 |B(x, r/2) \cap \{u > 0\}| + C(1 + \kappa r^\alpha)\eta_0(1 + L)r^n
$$

(10.20)

$$
\leq J_{x,r}(u) + C\kappa r^\alpha(1 + L^2)r^n - \rho_0 |B(x, r/2) \cap \{u > 0\}| + C\eta_0(1 + L)r^n.
$$

We simplify (10.20) to obtain

$$
|B(x, r/2) \cap \{u > 0\}| \leq C\rho_0^{-1}\kappa r^\alpha(1 + L^2)r^n + C\rho_0^{-1}\eta_0(1 + L)r^n.
$$

(10.21)

Then by (10.12)

$$\int_{B(x,r/2)} u^+ \leq \int_{B(x,r/2) \cap \{u > 0\}} 4^{n+1}\eta_0 r \leq 4^{n+1}\eta_0 r |B(x, r/2) \cap \{u > 0\}|
\leq C\eta_0\rho_0^{-1}\kappa r^\alpha(1 + L^2)r^{n+1} + C\rho_0^{-1}\eta_0^2(1 + L)r^{n+1}.
$$

(10.22)

Pick $z \in B(x, r/4)$. We want to find a decreasing sequence of radii $r_j > 0$, such that the balls $B(z, r_j)$ satisfy the hypotheses of Theorem 10.1. First use Chebyshev to find $r_1 \in (r/8, r/4)$ such that

$$
r_1^{-1} \int_{\partial B(z, r_1)} u^+ \leq C r^{-n} \int_{\partial B(z, r_1)} u^+ \leq C r^{-n-1} \int_{r/8}^{r/4} \int_{\partial B(z, r_1)} u^+
\leq C r^{-n-1} \int_{B(x,r/2)} u^+
\leq C\eta_0\rho_0^{-1}\kappa r^\alpha(1 + L^2) + C\rho_0^{-1}\eta_0^2(1 + L) < \eta_0
$$

(10.23)

if $\eta_0$ and $r_0$ are small enough (depending on $\rho_0$ and $L$ in particular).

This means that the ball $B(z, r_1)$ satisfies the assumptions of Theorem 10.1. By the same argument as before, we prove that (see (10.21))

$$
|B(z, r_1/2) \cap \{u > 0\}| \leq C\rho_0^{-1}\kappa r_1^\alpha(1 + L^2)r_1^n + C\rho_0^{-1}\eta_0(1 + L)r_1^n.
$$

(10.24)
and
\[
\int_{B(z,r/2)} u^+ \leq C\eta_0\rho_0^{-1}kr_1^\alpha(1 + L^2)r_1^{n+1} + C\rho_0^{-1}\eta_0^2(1 + L)r^{n+1}
\]
as in (10.22). Thus as before we can choose \( r_2 \in (r_1/8, r_1/4) \) such that the ball \( B(z,r_2) \) also satisfies the assumptions of Theorem 10.1. Notice that we no longer need to change the base point, thus we keep the point \( z \in B(x,r/2) \).

We iterate the argument, and find a sequence of radii \( r_j \), with \( r_j \in (r_{j-1}/8, r_{j-1}/4) \), such that
\[
|B(z,r_j/2) \cap \{ u > 0 \}| \leq C\rho_0^{-1}kr_j^\alpha(1 + L^2)r_j^n + C\rho_0^{-1}\eta_0(1 + L)r^n,
\]
and
\[
\int_{B(z,r_j/2)} u^+ \leq C\eta_0\rho_0^{-1}kr_j^\alpha(1 + L^2)r_j^{n+1} + C\rho_0^{-1}\eta_0^2(1 + L)r_j^{n+1}.
\]

But if \( u(z) > 0 \), then \( u(w) > 0 \) in a neighborhood of \( w \) because \( u \) is continuous, and this contradicts (10.26) for \( j \) large (again, if \( \eta_0 \) and \( r_0 \) are small enough). Since \( z \) was an arbitrary point of \( B(x,r/4) \), we get that \( u(z) \leq 0 \) on \( B(x,r/4) \), as needed for Theorem 10.1. \( \Box \)

Let us now derive a few simple consequences of Proposition 10.1. We shall be interested in the rough behaviour of the almost minimizer \( u \) near the free boundary
\[
\Gamma = \Omega \cap \partial\{ x \in \Omega ; u(x) > 0 \}.
\]
Notice that Theorem 10.1 ensures that \( \frac{1}{2} \int_{\partial B(x,r)} u^+ \geq \eta_0 \) when \( x \in \Gamma \) and \( r \leq r_0 \), where \( \eta_0 \) and \( r_0 \) depend on local bounds for the Lipschitz constant for \( u \) and local lower bounds for \( q_+ \).

**Lemma 10.2** Let \( u \) be an almost minimizer for \( J \) or \( J^+ \) in \( \Omega \). For each choice of \( \rho_0 > 0 \) and \( L \geq 1 \), let \( \eta_0 > 0 \) and \( r_0 > 0 \) be as in Theorem 10.1. Then if \( x \in \Gamma \) and \( 0 < r \leq r_0 \) are such that \( B(x,r) \subset \Omega \), and (10.10) and (10.11) hold, then there exists \( y \in \partial B(x,r/2) \) such that
\[
u(y) \geq \eta_0 r
\]
and hence \( u(z) > \eta_0 r / 4 \) for \( z \in B(y, \eta_0 r / 4) \).

**Proof.** Let \( B(x,r) \) be as in the statement; by Proposition 10.1, applied to \( B(x,r/2) \), we get that \( \frac{1}{2} \int_{\partial B(x,r/2)} u^+ \geq \eta_0 \). By Chebyshev, we can find \( y \in \partial B(x,r/2) \) such that \( u(y) \geq \eta_0 r / 2 \). But by (10.10) \( u \) is \( L \)-Lipschitz on \( B(x,r) \), so \( u(z) > \eta_0 r / 4 \) on \( B(y, \eta_0 r / 4) \), as needed. \( \Box \)

**Lemma 10.3** Let \( u \) be an almost minimizer for \( J \) or \( J^+ \) in \( \Omega \), and let \( B(x,r) \) satisfy the assumptions of Lemma 10.2. Then
\[
\int_{B(x,r)} |\nabla u^+| \geq c_0,
\]
where \( c_0 \) depends only on \( L, \rho_0 \), and the usual constants.
Proof. Indeed, let \( y \in \partial B(x, r/2) \) be as in Lemma 10.2; thus \( u(z) \geq \eta_0 r/4 \) for \( z \in B(y, \eta_0 r/8L) \). On the other hand, \( u(x) = 0 \), so \( u^+(z) \leq |u(z)| \leq \eta_0 r/8 \) for \( z \in B(x, \eta_0 r/8L) \). Set \( m = \int_{B(x,r)} u^+ \), and apply Poincaré’s inequality to \( u^+ \) in \( B(x,r) \); this yields

\[
\eta_0 r/8 \leq \int_{B(y, \eta_0 r/8L)} u^+ - \int_{B(x, \eta_0 r/8L)} u^+ \leq |m - \int_{B(y, \eta_0 r/8L)} u^+| + |m - \int_{B(x, \eta_0 r/8L)} u^+|
\]

\[
\leq \int_{B(y, \eta_0 r/8L)} |u^+ - m| + \int_{B(x, \eta_0 r/8L)} |u^+ - m|
\]

(10.31) \[
\leq C(\eta_0, L) \int_{B(x,r)} |u^+ - m| \leq C'(\eta_0, L) r \int_{B(x,r)} |\nabla u^+|;
\]

(10.30) and the lemma follow. \( \blacksquare \)

Next we prove that locally, if \( q_+ \) is bounded below away from zero then the function \( u^+ \) is equivalent to the distance to the zero set, which we denote by

\[
\delta(y) = \text{dist} \left( y, \{ z \in \Omega : u(z) = 0 \} \right).
\]

Notice that \( \delta(y) = \text{dist} \left( y, \Gamma \right) \) when \( u(y) > 0 \) and \( \delta(y) < \text{dist} \left( y, \partial \Omega \right) \). In fact if \( z \in \{ u = 0 \} \) minimizes the distance to \( y \), then \( z \in \Gamma \) because \( u > 0 \) on \([y, z]\).

**Theorem 10.2** Let \( u \) be an almost minimizer for \( J \) or \( J^+ \) in \( \Omega \). For each choice of \( \rho_0 > 0 \) and \( L \geq 1 \), there are constants \( \eta_0 > 0 \) and \( r_1 > 0 \), that depend only on \( \rho_0, L \), and the usual constants, such that if \( x \in \Gamma \) and \( 0 < r \leq r_1 \) are such that \( \overline{B}(x,r) \subset \Omega \), and if (10.10) and (10.11) hold, then

\[
u^+(y) \geq \eta_0 \delta(y)/4 \quad \text{for} \quad y \in B(x, r/2).
\]

**Proof.** We shall be able to keep the same \( \eta_0 \) as in the previous statements, but \( r_1 \) will possibly need to be smaller than \( r_0 \). Let \( B(x,r) \) be as in the statement, and let \( y \in B(x, r/2) \) be given; since (10.33) is obvious when \( \delta(y) = 0 \), we can assume that \( \delta(y) > 0 \). Also, \( \delta(y) \leq r/2 \) because \( u(x) = 0 \).

Apply Theorem 10.1 to the ball \( B_1 = B(y, \delta(y)/2) \); the assumptions are satisfied (if \( r_1 \leq r_0 \)) because \( B_1 \subset B(x, r) \). Since \( u(y) \neq 0 \), we get that

\[
\int_{\partial B(y, \delta(y)/2)} u^+ \geq \eta_0 \delta(y)/2.
\]

Then denote by \( u^* \) the energy-minimizing extension of \( u|_{\partial B_1} \) to \( B_1 \). As we have seen a few times, this is an acceptable competitor for \( u \) in \( B_1 \), and since both \( u \) and \( u^* \) are positive in \( B_1 \), we get that

\[
J_{y, \delta(y)/2}(u) - J_{y, \delta(y)/2}(v) = \int_{B_1} |\nabla u|^2 - |\nabla u^*|^2 = \int_{B_1} |\nabla (u - u^*)|^2 \geq 0
\]

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where the last equation comes from the minimizing property of $u^*$; see for instance (2.3). The almost minimizing property of $u$ yields $J_{y, \delta(y)/2}(u) \leq (1 + \kappa \delta(y)^\alpha) J_{y, \delta(y)/2}(v)$, hence by (10.35)

$$\int_{B_1} |\nabla (u - u^*)|^2 = J_{y, \delta(y)/2}(u) - J_{y, \delta(y)/2}(v) \leq \kappa \delta(y)^\alpha J_{y, \delta(y)/2}(v)$$

(10.36)

$$\leq \kappa \delta(y)^\alpha J_{y, \delta(y)/2}(u) \leq C \kappa \delta(y)^\alpha (1 + L^2) \delta(y)^n.$$ Moreover since $u^*$ is harmonic, $u > 0$ on $\partial B_1$, and by (10.34) we have

$$u^*(y) = \int_{\partial B_1} u = \int_{\partial B_1} u^+ \geq \eta_0 \delta(y)/2.$$ We want to compare this to $u(y)$. First observe that $|u(z) - u(y)| \leq L \delta(y)/2$ for $z \in \partial B_1$, by (10.10), so $|\nabla u^*| \leq CL$ on $\frac{1}{2} B_1$, by an easy estimate with the Poisson kernel. Let $\tau > 0$ be small, to be chosen soon, and set $B_2 = B(y, \tau \delta(y))$; then $|u^*(z) - u^*(y)| \leq C \kappa \tau \delta(y)$ for $z \in B_2$. Similarly, $|u(z) - u(y)| \leq L \tau \delta(y)$ on $B_2$, just by (10.10), so

$$|u(y) - u^*(y)| \leq |u(z) - u^*(y)| + |u(z) - u^*(z)| + |u^*(z) - u^*(y)|$$

(10.38)

for $z \in B_2$, and now we can apply Poincaré’s inequality to the function $u - u^*$, which vanishes at the boundary of $B_1$, and get that

$$|u(y) - u^*(y)| \leq L \tau \delta(y) + \int_{z \in B_2} |u(z) - u^*(z)|$$

$$\leq L \tau \delta(y) + \tau^{-n} \int_{z \in B_1} |u(z) - u^*(z)|$$

(10.39)

$$\leq L \tau \delta(y) + C \tau^{-n} \delta(y) \int_{z \in B_1} |\nabla (u - u^*)|$$

$$\leq L \tau \delta(y) + C \tau^{-n} \delta(y) [\kappa \delta(y)^\alpha (1 + L^2)^{1/2}]$$

$$\leq L \tau \delta(y) + C \tau^{-n} \delta(y) [\kappa r_1^\alpha (1 + L^2)^{1/2}]$$

by (10.36). We choose $\tau$ so small (depending on $L$ and $\eta_0$), and then $r_1$ so small (depending also on $\tau$) that (10.39) yields $|u(y) - u^*(y)| \leq \eta_0 \delta(y)/4$. Then (10.37) implies that $u(y) \geq \eta_0 \delta(y)/4$, as needed for Theorem 10.2. 

We now prove that $\{u \leq 0\}$ contains non-tangential balls. Our initial lemma is better suited for $J^+$.

**Lemma 10.4** Let $u$ be an almost minimizer for $J^+$ in $\Omega$. For each choice of $\rho_0 > 0$ and $L \geq 1$, there exist $\eta_1 > 0$ and $r_2 > 0$, that depend only on $\rho_0$, $L$, and the usual constants, such that if $x \in \Gamma$ and $0 < r \leq r_2$ are such that $\overline{B}(x, r) \subset \Omega$ and (10.10) and (10.11) hold,

$$|\{y \in B(x, r) \colon u(y) = 0\}| \geq \eta_1 r^n.$$
It is not clear to what extent the lower bound (10.11) on \(q_+\) is necessary for this lemma. It might be possible that Lemma 10.4 holds with (10.11) replaced by a bound on the modulus of continuity for \(q_+\).

**Proof.** Let \(B(x, r)\) be as in the statement. We shall take \(r_2 \leq r_0\), where \(r_0\) is as in Theorem 10.1, thus

\[
\int_{\partial B(x, r)} u \geq \eta_0 r.
\]

Let \(u^\ast\) denote the harmonic extension of the restriction of \(u\) to \(\partial B(x, r)\). We know that \(u^\ast\) is an acceptable competitor for \(u\) in \(B(x, r)\), and by the usual orthogonality argument in (2.3),

\[
\int_{B(x, r)} |\nabla u|^2 - \int_{B(x, r)} |\nabla u^\ast|^2 = \int_{B(x, r)} |(u - u^\ast)|^2
\]

Set \(X = \left| \left\{ y \in B(x, r) : u(y) = 0 \right\} \right|\). Since

\[
\int_{B(x, r)} [\chi_{\{u^\ast > 0\}} - \chi_{\{u > 0\}}] q_+ \leq \int_{B(x, r)} \chi_{\{u = 0\}} q_+ \leq ||q_+||_{\infty} X,
\]

we see that

\[
J_{x, r}(u^\ast) \leq J_{x, r}(u) + ||q_+||_{\infty} X - \int_{B(x, r)} |\nabla(u - u^\ast)|^2
\]

and, by the almost minimality condition,

\[
J_{x, r}(u) - J_{x, r}(u^\ast) \leq (1 + kr^\alpha) J_{x, r}(u^\ast) - J_{x, r}(u^\ast) \leq kr^\alpha J_{x, r}(u^\ast)
\]

\[
\leq kr^\alpha J_{x, r}(u) + kr^\alpha ||q_+||_{\infty} X \leq Ckr^\alpha (1 + L^2)r^n
\]

by (10.44) and our Lipschitz bound (10.10). We compare with (10.44) and get that

\[
\int_{B(x, r)} |\nabla(u - u^\ast)|^2 \leq Ckr^\alpha (1 + L^2)r^n + ||q_+||_{\infty} X.
\]

We now continue almost as in Theorem 10.2. Observe that

\[
u^\ast(x) = \int_{\partial B(x, r)} u^\ast = \int_{\partial B(x, r)} u \geq \eta_0 r,
\]

because \(u^\ast\) is harmonic in \(B(x, r)\), has the same trace as \(u\) on \(\partial B(x, r)\), and by (10.41). Then let \(\tau > 0\) be small, to be chosen soon, and observe that \(u\) is \(L\)-Lipschitz on \(B(x, r)\), then \(|u(y)| \leq Lr\) on \(\partial B(x, r)\) (because \(u(x) = 0\)). As in the proof of Theorem 10.2 then \(u^\ast\) is \(CL\)-Lipschitz on \(B(x, \tau r) \subseteq B(x, r/2)\). Hence, for \(z \in B(x, \tau r)\),

\[
|u(x) - u^\ast(x)| \leq |u(x) - u(z)| + |u(z) - u^\ast(z)| + |u^\ast(z) - u^\ast(x)|
\]

\[
\leq |u(z) - u^\ast(z)| + C L \tau r
\]
and, applying Poincaré’s inequality to the function $u - u^*$ which vanishes at the boundary,

\[(10.49) \quad \eta_0 r \leq |u(x) - u^*(x)| \leq CL\tau r + \int_{z \in B(x,\tau r)} |u(z) - u^*(z)|\]

\[\leq CL\tau r + \tau^{-n} \int_{z \in B(x,\tau r)} |u(z) - u^*(z)|\]

\[\leq CL\tau r + C\tau^{-n} r \int_{z \in B(x,r)} |\nabla (u - u^*)|\]

\[\leq CL\tau r + C\tau^{-n} r \left\{ r^{-n} \int_{z \in B(x,r)} |\nabla (u - u^*)|^2 \right\}^{1/2}\]

\[\leq CL\tau r + C\tau^{-n} r \left[ \kappa r^{\alpha} (1 + L^2) + r^{-n} ||q_+||_{\infty X} \right]^{1/2}\]

by \((10.47)\) and \((10.46)\). We now choose $\tau$ so small that $CL\tau \leq \eta_0/2$ in \((10.49)\), and we are left with

\[(10.50) \quad \left[ \kappa r^{\alpha} (1 + L^2) + r^{-n} ||q_+||_{\infty X} \right]^{1/2} \geq C^{-1} \tau^{n} \eta_0/2.\]

We now choose $r_2$ so small, depending on $\tau$ and $\eta_0$, that for $r \leq r_2$, \((10.50)\) implies that $r^{-n} ||q_+||_{\infty X} \geq \frac{1}{2} [C^{-1} \tau^{n} \eta_0/2]^2$. Then \((10.40)\) holds, with $\eta_1 = C'(\tau^n \eta_0)^2 ||q_+||_{\infty}^{-1}$. The reader should not worry about $||q_+||_{\infty}^{-1}$, which seems to give a very large bound when $q_+$ is small, because we also assumed that $q_+ \geq \rho_0$ on $B(x,r)$. Lemma 10.4 follows. \[\square\]

Observe that if $u$ us a minimizer for $J$, and $u \geq 0$ on $B(x,r)$, the proof of Lemma 10.4 can be implemented exactly as before, because $u^* \geq 0$ on $B(x,r)$ and there is never a contribution of $q_-$. If $u$ takes negative values, then $u^*$ could also take negative values, even on some places where $u > 0$, and and it could happen that $q_-$ is much larger than $q_+$ in those places, and then $u^*$ is not such a great competitor. This case is carefully dealt with in the next statement.

**Lemma 10.5** Let $u$ be an almost minimizer for $J$ in $\Omega$. For each choice of $\rho_0 > 0$ and $L \geq 1$, there exist $\eta_2 > 0$ and $r_3 > 0$, that depend only on $\rho_0$, $L$, and the usual constants, such that if $x \in \Gamma$ and $0 < r \leq r_3$ are such that $\overline{B}(x,r) \subset \Omega$, \((10.10)\) and \((10.11)\) hold, and in addition,

\[(10.51) \quad u(y) \geq 0 \text{ for all } y \in B(x,r),\]

or

\[(10.52) \quad q_-(y) \leq q_+(y) \text{ for all } y \in B(x,r),\]

or

\[(10.53) \quad q_-(y) \geq \rho_0 \text{ for all } y \in B(x,r),\]

then

\[(10.54) \quad \left\{ z \in B(x,r) : u(z) \leq 0 \right\} \geq \eta_2 r^n.\]
Proof. We already explained what happens in the first case when \( u \geq 0 \) on \( B(x, r) \). In the case when (10.52) holds, we still want to use a similar proof, but some estimates need to be replaced. We start with (10.43). We want to check that

\[
\int_{B(x, r)} \chi_{\{u^* > 0\}} q_+ + \chi_{\{u^* < 0\}} q_- - \int_{B(x, r)} \chi_{\{u > 0\}} q_+ + \chi_{\{u < 0\}} q_- \leq \int_{A_0} q_+ \leq ||q_+||_\infty X,
\]

where \( X = \left| \{ z \in B(x, r) : u(z) \leq 0 \} \right| \), so we cut \( B(x, r) \) into the sets

\[
A_0 = \left\{ z \in B(x, r) ; u(z) \leq 0 \right\},
A_1 = \left\{ z \in B(x, r) ; u(z) > 0 \text{ and } u^*(z) > 0 \right\},
A_2 = \left\{ z \in B(x, r) ; u(z) > 0 \text{ and } u^*(z) \leq 0 \right\},
\]

and estimates their contributions one by one. On \( A_0 \), we do not know how large \( u^* \) is, so we just pay the maximum \( \int_{A_0} q_+ \leq ||q_+||_\infty X \). On \( A_1 \), we integrate both functions against \( q_+ \), so the contribution of the difference is zero. On \( A_2 \), the contribution of \( u \) is larger or equal than the contribution of \( u^* \), because we assumed in (10.52) that \( q_+ \geq q_- \) on \( B(x, r) \). This proves (10.55).

The conclusion of (10.55) is the same as in (10.43) (where there was no \( q_- \) to worry about) and we can continue the argument up to (10.47), which we also need to replace.

Let \( z \) be any point of \( \partial B(x, r) \). If \( X > \eta_2 r^n \), then (10.54) holds by definition. Otherwise, if the next constant \( C \) is large enough, \( B(x, r) \cap B(z, C\eta_2^{1/n} r) \) is not contained in \( \{ u \leq 0 \} \), and we can find \( \xi \in B(x, r) \) such that \( |\xi - z| \leq C\eta_2^{1/n} r \) and \( u(\xi) > 0 \). Then \( u(z) \geq -CL\eta_2^{1/n} r \). This proves that

\[
\int_{\partial B(x, r)} u_- \leq CL\eta_2^{1/n} r.
\]

Combining (10.56) and (10.41), we get that

\[
(10.57) \quad u^* (x) = \int_{\partial B(x, r)} u^* = \int_{\partial B(x, r)} u = \int_{\partial B(x, r)} u^+ - \int_{\partial B(x, r)} u^- \geq \eta_0 r - CL\eta_2^{1/n} r \geq \eta_0 r / 2
\]

if \( \eta_2 \) is small enough, depending on \( L \) and \( \eta_0 \). This is a good enough substitute for (10.47).

We may now continue the proof as we did for Lemma 10.4, and our second case follows.

In the case when (10.53) holds, we distinguish between two possibilities. If \( u(y) \geq 0 \) on \( B(x, r/2) \), we just apply our first case to the ball \( B(x, r/2) \), and get (10.54) with a slightly worse constant. Otherwise, pick \( y \in B(x, r/2) \) such that \( u(y) < 0 \), and observe that since \( u(x) = 0 \) we can find \( z \in [y, x] \) such that \( u(z) = 0 \) and \( z \in \partial \{ w \in B(x, r) ; u(w) < 0 \} \). In other words, \( z \) lies on the analogue of the set \( \Gamma \) of (10.28), but for the function \(-u\). Then \( B(z, r/2) \) satisfies the hypothesis of Lemma 10.2, which gives a ball of radius \( \eta_0 r / 8L \) which is contained in \( B(z, r/2) \subset B(x, r) \) and where \( u < 0 \); (10.54) holds in this case also, and Lemma 10.5 follows.

We may now improve the statement of Lemmas 10.4 and 10.5, to obtain a non-tangential ball in \( \{ u \leq 0 \} \cap B(x, r) \) for \( x \in \Gamma \). This will only require a porosity argument.
Proposition 10.3 Let $u$ be an almost minimizer for $J$ or $J^+$ in $\Omega$. For each choice of $\rho_0 > 0$ and $L \geq 1$, there exist $\eta_3 \in (0, 1/3)$ and $r_3 > 0$, that depend only on $\rho_0$, $L$, and the usual constants, such that if $x \in \Gamma$ and $B(x, r)$ satisfies the hypotheses of Lemma 10.4 or Lemma 10.5, then there is $y \in B(x, r/2)$ such that

(10.58) $u(z) \leq 0$ for $z \in B(y, \eta_3 r)$.

We can see Proposition 10.3 as an analogue of Lemma 10.2 for $u^-$. Even when $u$ is a minimizer for $J$, we cannot a priori determine whether there is a large ball in $\{u < 0\}$, or $\{u = 0\}$. The proof yields that there is a large ball in the union of these two sets.

Proof. Our proof will use two large integer parameter $N$ and $M$, to be chosen later. Let $B(x, r)$ satisfy the hypothesis of Proposition 10.3, and denote by $Q_0$ a cube of diameter $r$ centered at $x$. Cut $Q_0$ into $N^d$ almost disjoint cubes of diameter $N^{-1}r$ in the natural way. Call $\Delta_1$ the set of these cubes. For $1 \leq k \leq M + 1$ define collections $\Delta_k$ of cubes, in the following inductive way: for $k \geq 2$ if $Q \in \Delta_{k-1}$, cut $Q$ into $N^n$ almost disjoint cubes of diameter $N^{-k}r$, which we shall call the children of $Q$, and denote by $\Delta_k$ the collection of cubes of diameter $N^{-k}r$ obtained this way. For completeness, set $\Delta_0 = \{Q_0\}$ and call the cubes of $\Delta_1$ the children of $Q_0$.

Set $W = \{z \in B(x, r) ; u(z) > 0\}$, and assume that we cannot find $y$ as in the statement i.e., that

(10.59) $B(y, \eta_3 r)$ meets $W$ for every $y \in B(x, r/2)$.

We show that if $\eta_3$ is small, this assumption contradicts Lemma 10.4 or Lemma 10.5. Let us first check that for each $k \leq M$ and each $Q \in \Delta_k$,

(10.60) at least one of the children of $Q$ is contained in $W$.

Let $k \leq M$ and $Q \in \Delta_k$ be given. Choose a child $R$ of $Q$ that touches the center $x_Q$ of $Q$ (if $N$ is odd, $R$ is unique and the picture is nicer). Let $B_0$ denote the largest ball which is contained in $R$; its radius is

(10.61) $\ell_0 = \frac{N^{-k-1}r}{\sqrt{n}} \geq \frac{N^{-M-1}r}{\sqrt{n}} \geq \eta_3 r$

if $\eta_3$ is small enough (depending on $M$ and $N$). Then $B_0$ meets $W$, by (10.59). If $R \subset W$, (10.60) holds. Otherwise, $R$ meets both $W$ and its complement, so we can find $z \in R \cap \Gamma$ (see (10.28)). We want to apply Lemma 10.2 to the ball $B(z, \ell_1)$, where we set

(10.62) $\ell_1 = \frac{8LN^{-k-1}r}{\eta_0}$.

Notice that $B(z, \ell_1) \subset Q \subset B(x, r)$ for $N$ large enough, because $Q$ contains the ball of radius $N\ell_0$ centered at $x_Q \in R$. In fact it is enough to choose $N$ so that $8L\ell_0^{-1} + 1 \leq N/\sqrt{n}$. Then
Lemma 10.2 applies, and ensure that there is \( y \in \partial B(z, \ell_1/2) \) such that \( B(y, \frac{\eta_0 \ell_1}{4L}) \subset W \). The radius \( \eta_0 \ell_1/4L = 2N^{-k-1}r \) is twice the diameter of any cube of \( \Delta_{k+1} \). This means that the cube of \( \Delta_{k+1} \) that contains \( y \) is contained in \( W \). Thus we conclude that (10.60) holds in all cases.

Now we evaluate the measure of \( A = Q_0 \setminus W \). For each \( k \), denote by \( \Delta'_k \) the set of cubes \( Q \in \Delta_k \) that meet \( A \). Notice that if \( Q \in \Delta_k \) does not meet \( A \), then none of its children meets \( A \). And if \( Q \) meets \( A \), (10.60) guarantees that at least one of its children does not meet \( A \). Thus, if \( n_k \) denotes the cardinal of \( \Delta'_k \), we get that \( n_{k+1} \leq (N^n - 1)n_k \). Equivalently, if \( S_k = \bigcup_{Q \in \Delta'_k} Q \), that \( |S_{k+1}| \leq \frac{N^n - 1}{N^n} |S_k| \). After our \( M \) steps, we obtain that

\[
(10.63) \quad |A| \leq |S_{M+1}| \leq \left(1 - \frac{1}{N^n}\right)^M |Q_0|,
\]

We already chose \( N \) large (below (10.62)), and now we choose \( M \) large enough so \( (1 - N^{-n})^M \leq \min\{\eta_1, \eta_2\} \) where \( \eta_1 \) and \( \eta_2 \) come from Lemma 10.4 and Lemma 10.5. Then (10.63) yields \( |A| \leq \min\{\eta_1, \eta_2\} r^n \), which combined with Lemma 10.4 and Lemma 10.5 gives the desired contradiction. This completes our proof of Proposition 10.3.

This paper settles the question of the regularity for almost minimizers of the functional \( J \), but leaves open problems concerning the structure and the regularity of the corresponding free boundary. Some of our current work focuses on these issues. We have already obtained partial results in this direction, but they are only first steps toward what we expect to be the optimal result.

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