Electrostatic patch effect in cylindrical geometry: II. Forces

Valerio Ferroni$^{1,2,3}$ and Alexander S Silbergleit$^2$

$^1$ ICRANet, Department of Physics, University of ‘Sapienza’, Rome, Italy
$^2$ Gravity Probe B, W W Hansen Experimental Physics Laboratory, Stanford University, Stanford, CA 94305-4085, USA

E-mail: vferroni@stanford.edu, ferroni@science.unitn.it and gleit@stanford.edu

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Abstract

We continue our study of patch effect (PE) for two close cylindrical conductors with parallel axes, slightly shifted against each other in the radial and by any length in the axial direction. It was started in Ferroni and Silbergleit (2011 Class. Quantum Grav. 28 145001), where the potential and energy in the gap were calculated to the second order in the small transverse shift, and to lowest order in the gap to cylinder radius ratio. Based on these results, here we derive and analyze the PE force. It consists of three parts: the usual capacitor force due to the uniform potential difference, the one from the interaction between the voltage patches and the uniform voltage difference, and the force due to patch interaction, entirely independent of the uniform voltage. General formulas for these forces are found, and their general properties are described. A convenient model of a localized patch is then suggested that allows us to calculate all the forces in a closed elementary form. Using this, a detailed analysis of the patch interaction for one pair of patches is carried out, and the dependence of forces on the patch parameters (width and strength) and their mutual position is examined. The general PE analysis based on our patch model is described.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Electrostatic patch effect (PE) [2–4] (see also its earlier discussion in [5–7]) is a nonuniform distribution of an electrical potential on the surface of a metal. Its implications were first examined theoretically in [8], where the PE force in a plane capacitor with varying boundary

$^3$ Present address: Science Department, University of Trento, Povo, Italy.
voltages has been calculated; the analysis carried out there was particularly motivated by the LISA space experiment to detect gravitational waves (cf [9]). PE can similarly affect the accuracy of any other precision measurement if its set-up includes conducting surfaces in a close proximity to each other. PE torques turned out one of the two major difficulties in the analysis of data from Gravity Probe B (GP-B) Relativity Science Mission [10, 11] that measured the relativistic drift of a gyroscope predicted by Einstein’s general relativity [12]. This required theoretical calculation of PE torques [13] for the case of two concentric spherical conductors.

Here we continue our study of PE in cylindrical geometry that we started in paper [1], henceforth referred to as CPEI, for ‘Cylindrical Patch Effect’. It is strongly motivated by the experimental configuration of the Satellite Test of the Equivalence Principle (STEP) [14–17], where each test mass and its superconducting magnetic bearing is a pair of approximately coaxial conducting cylinders. The goal of STEP is the precise (1 part in 10^{18}) measurement of the relative axial acceleration of a pair of coaxial test masses, so the importance of properly accounting for PE forces is evident. Notably, the axial and transverse forces we find are inversely proportional to the gap and its square, respectively, exactly as in the plane capacitor [8].

Along with the theoretical analysis, experimental measures, such as proper coating of conducting surfaces, should be taken to reduce the PE in every particular experiment. However, it seems that so far no such measures guarantee a complete PE elimination, so its theoretical estimates remain important.

We determine the PE forces between the two slightly shifted cylinders with parallel axes by the energy conservation argument: a small shift, \( \vec{r}_0 \), of one of the conductors relative to the other causes an electrostatic force given by

\[
\vec{F}(\vec{r}_0) = -\frac{\partial W(\vec{r}_0)}{\partial \vec{r}_0}, \quad \vec{F}_0(0) = -\frac{\partial W(\vec{r}_0)}{\partial \vec{r}_0} \bigg|_{\vec{r}_0 = 0},
\]

where \( W(\vec{r}_0) \) is the electrostatic energy as a function of the shift. The latter was found in CPEI to the second order in the small transverse shift, \( \rho_0 \ll \Delta \), where \( \Delta \) is the gap between the cylinders in the coaxial position, so the force to the first order in \( \rho_0 \) is found below. For typical experimental conditions, such as the STEP configuration [15, 16], the gap is much smaller than either of the cylinders’ radii, \( a < b \); thus, two small parameters are actually involved in the problem, \( \rho_0/\Delta \ll 1, \; \Delta/a \sim \Delta/b \ll 1 \). While justifying the model of infinite cylinders, this also allows for a significant simplification of the results to the lowest order (l.o.) in \( \Delta/a \).

In the next section, we summarize the results from CPEI needed for the force calculation. Based on this, we calculate PE forces in section 3 and elucidate their general properties. In section 4, we introduce a convenient model of a localized patch potential allowing one to find simple closed-form expressions for the forces (and torques, in the last, third paper constituting our work). Section 5 contains a detailed analysis of PE forces when a single localized patch described by our model is sitting at each of the cylinder boundaries. Remarks on using our results for an efficient patch modeling in particular experiments, such as STEP, are found in section 6. The details of calculations, in places rather complicated and cumbersome, are found in the appendix.

### 2. Summary of results from CPEI

We use Cartesian and cylindrical coordinates in two frames of the inner and outer cylinders as shown in figure 1. In the inner, or ‘primed’, frame the position of a point is given by the
Figure 1. Geometry of the problem and coordinate systems.

vector radius \( \vec{r}' \), and Cartesian coordinates \( \{x', y', z'\} \) or cylindrical coordinates \( \{\rho', \varphi', z'\} \). In the outer, or ‘unprimed’, frame the corresponding quantities are \( \vec{r} \), \( \{x, y, z\} \), \( \{\rho, \varphi, z\} \). Frame origins are separated by \( \vec{r}^0 \); hence, the coordinates are related by

\[
\vec{r}' = \vec{r} + \vec{r}^0; \quad x' = x + x^0, \quad y' = y + y^0, \quad z' = z + z^0.
\]

We also use an alternative writing \( x^0 = x^0_1 \), \( y^0 = x^0_2 \), \( x = x_1 \), \( y' = x_2' \), etc.

The surfaces of the inner and outer cylinders are \( \rho' = a \) and \( \rho = b \), respectively, with \( d = b - a \). They carry arbitrary voltage distributions given by the conditions

\[
\Phi \bigg|_{\rho' = a} = G(\varphi', z'), \quad \Phi \bigg|_{\rho = b} = V^- + H(\varphi, z),
\]

where \( \Phi \) is the electrostatic potential, and \( V^- \) is the uniform potential difference: all the voltages in the problem are counted from the uniform voltage of the inner cylinder taken as zero. The non-uniform potentials (patch patterns) are described by arbitrary smooth-enough functions \( G(\varphi', z') \) and \( H(\varphi, z) \), whose local nature is emphasized by requiring

\[
||G||^2 = \int_0^{2\pi} \int_{-\infty}^{\infty} d\varphi' dz' |G(\varphi', z')|^2 < \infty, \quad ||H||^2 = \int_0^{2\pi} \int_{-\infty}^{\infty} d\varphi dz |H(\varphi, z)|^2 < \infty.
\]
For any function $u(\varphi, z)$ satisfying the square integrability condition, we denote its Fourier coefficient $u_n(k)$:

$$u(\varphi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_n(k) e^{i(k\varphi + nz)} \, dk \, dz,$$

$$u_n(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi u(\varphi, z) e^{-i(k\varphi + nz)} - |c_n|^2;$$

(5)

in particular, the Fourier coefficients of the patch voltages $G(\varphi', z')$ and $H(\varphi, z)$ are $G_n(k)$ and $H_n(k)$, respectively. Since these functions are real, their Fourier coefficients satisfy

$$G_n(k) = G_{n}^{*}(-k), \quad H_n(k) = H_{n}^{*}(-k);$$

(6)

here and elsewhere the star denotes the complex conjugation. For any two squarely integrable functions $u(\varphi, z)$ and $v(\varphi, z)$, the useful Parseval identity holds:

$$(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi u(\varphi, z) v^\ast(\varphi, z) = \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} u_n(k)v_n^\ast(k).$$

(7)

In the case $u = v$, identity (7) shows that the squared norm, $||u||^2$, of a function $u$ is equal to the squared norm of its Fourier coefficient $u_n(k)$.

As shown in CPEI, section 3, the electrostatic energy in the gap between the two cylinders, as a function of their mutual shift $\vec{r}$, consists of three parts:

$$W(\vec{r}) = W^u(\vec{r}) + W^{int}(\vec{r}) + W^p(\vec{r}),$$

(8)

where the first one is due to the uniform potential difference, the second results from the interaction between the uniform and patch voltages, and the third one is the energy of the patch interaction. Each of these contributions was found in CPEI in the form of an expansion in the small transverse shift $\rho_0$ to the quadratic order in $\rho_0/d \ll 1$, with the coefficients depending generally on the axial shift $z_0$:

$$W^A(\vec{r}) = W^u_0(z_0) + W^{int}_\mu(z_0^\ast/(d) + \cdots + W^p(z_0^\ast)/(d) + O((\rho_0/d)^3);$$

$$A = u, \text{ int, } p; \quad \mu, v = 1, 2.$$}

(9)

Here and everywhere else we adopt the summation rule over repeated Greek indices $\mu, \nu$, etc: the summation over them runs from 1 to 2, corresponding to the transverse coordinates in the cylinder cross-section. The explicit coefficients for each kind of energy follow immediately.

The uniform potential energy is, naturally, proportional to the length of the capacitor, same as its expansion coefficients which, for $|z| \leq L$, are

$$W^u_0(L) = 2\pi \epsilon_0 \frac{a}{d} (V^-)^2, \quad W^u_p(L) = 0, \quad W^{int}_{\mu\nu}(L) = \pi \epsilon_0 \frac{a}{d} (V^-)^2 \delta_{\mu\nu}.$$}

(10)

The first-order term vanishes as it should be due to symmetry (otherwise there would be a non-zero transverse force in a perfectly symmetric configuration of coaxial conductors under uniform potentials).

The expansion coefficients for the interaction energy are

$$W^u_0 = -2\pi \epsilon_0 \frac{a}{d} V^- \left[(G_0(0) - H_0(0))\right],$$

$$W^{\mu} = 4\pi \epsilon_0 \frac{a}{d} V^- \Re[c^{\ast}_\mu (G_1(0) - H_1(0))],$$

$$W^{\mu\nu} = -4\pi \epsilon_0 \frac{a}{d} V^- \left[\Re[c^{\ast}_\mu c^{\ast}_\nu (G_2(0) - H_2(0)) + (\delta_{\mu\nu}/4) (G_0(0) - H_0(0))\right],$$

(11)

where $c^\ast = 0.5$, $c^\ast = 0.5i$, $\Re(\cdot)$ denotes the real part of $\cdot$; recall also that $G_n(k)$ and $H_n(k)$ are the Fourier coefficients of the patch voltages (3). Same as the uniform part of energy,
the interaction one does not depend on the axial shift, because the longitudinal shifting of an electrode with the uniform potential does not change the actual charge configuration.

Finally, the patch energy coefficients are found to be

\[
W^p_p = \frac{\epsilon_0 d}{2} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} |G_n(k) e^{i k z^0} - H_n(k)|^2;
\]

\[
W^p_\mu = -\frac{\epsilon_0 d}{d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \Re\{e_n^* (G_n^*(k) e^{-i k z^0} - H_n^*(k))(G_{n+1}(k) e^{i k z^0} - H_{n+1}(k))\};
\]

\[
W^p_{\mu \nu} = \frac{\epsilon_0 d}{2d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} [\delta_{\mu \nu}/2 |G_n(k) e^{i k z^0} - H_n(k)|^2
+ 2\Re\{e_n^* e_{n+1}^* (G_n^*(k) e^{-i k z^0} - H_n^*(k))(G_{n+2}(k) e^{i k z^0} - H_{n+2}(k))\}];.
\]

These quantities do depend on the axial shift, \( z^0 \). Combining expressions (8) and (9), we can also write for the total energy:

\[
W(\vec{r}^0) = W_0(z^0) + W_\mu(z^0)(x^0_\mu/d) + W_{\mu \nu}(z^0) (x^0_\mu/d)(x^0_\nu/d) + O((\rho_0/d)^2);
\]

\[
W_z(z^0) = W_0^z + W_{\mu}^z(z^0) + W_{\nu}^z(z^0), \quad \xi = 0, \mu, \mu \nu.
\]

Note that all parts of energy are given to l.o. in \( d/a \).

## 3. Patch effect forces

As explained in the introduction, the force is found by formulas (1) and (13):

\[
F_\mu = -\frac{\partial W(\vec{r}^0)}{\partial x^0_\mu} = -\left[ W_\mu(z^0) + 2W_{\mu \nu}(z^0)(x^0_\mu/d) + O((\rho_0/d)^2) \right]
= -\left[ W_\mu(0) + W'_\mu(0)z^0 + 2W_{\mu \nu}(0)(x^0_\mu/d) + O((\rho_0/d)^2) \right], \quad \mu = 1, 2;
\]

\[
r_0 = |\vec{r}^0| = \sqrt{(x^0)^2 + (y^0)^2 + (z^0)^2},
\]

(15)

for the transverse force components, and

\[
F_z = -\frac{\partial W(\vec{r}^0)}{\partial z^0} = -[W_0'(z^0) + W_\mu'(z^0)(x^0_\mu/d) + O((\rho_0/d)^2)]
= -[W_0'(0) + W_{\mu}'(0)z^0 + W_{\mu}'(0)(x^0_\mu/d) + O((\rho_0/d)^2)],
\]

(16)

for the axial force (here and elsewhere the primes denote the derivatives in \( z^0 \)). Same as energy, the force consists, of course, of three parts, which we study below one by one.

### 3.1. The force due to the uniform potential difference

Using formula (15) for the transverse force and expansion (9) with the coefficients (10), we obtain (as usual, \( F_1^u = F_1^u, \ F_2^z = F_2^z \))

\[
F_1^u = -2\pi L \epsilon_0 \frac{a}{d^2} (V^-)^2 (x^0/d), \quad F_2^u = -2\pi L \epsilon_0 \frac{a}{d^2} (V^-)^2 (y^0/d), \quad F_2^u = 0.
\]

(17)

The axial force is equal to zero by translational symmetry resulting in the independence of the energy on the axial shift, \( z^0 \). Recall that \( 2L \) is the cylinder length, so the force per unit length of the cylinders is finite.
3.2. The patch and uniform potential interaction force

To obtain it, we combine formulas (15), (9) and (11). After simplifying algebraic transformations that are rather straightforward, the result becomes

\[ F^\text{int}_x = -2\pi \epsilon_0 \frac{a}{d^2} V^* \left[ 9 \left[ G_1(0) - H_1(0) \right] - 9 \left[ (G_2(0) - H_2(0)) \right] + \Im (G_0(0) - H_0(0)) (x^0/d) + \Im (G_2(0) - H_2(0)) (y^0/d) \right); \]

\[ F^\text{int}_y = -2\pi \epsilon_0 \frac{a}{d^2} V^* \left[ -\Im (G_1(0) - H_1(0)) + \Im (G_2(0) - H_2(0)) (x^0/d) + \Re [(G_2(0) - H_2(0)) (y^0/d)] \right]; \]

\[ F^\text{int}_z = 0; \]

\(\Im(\cdot)\) is the imaginary part of (\(\cdot\)). The axial force vanishes again, by symmetry.

3.3. Forces due to the patch interaction

By formula (15) and the energy expansion (9) with the coefficients (12), we find

\[ F^P_x = \frac{\epsilon_0 d}{2d^2} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \left[ 9 \left[ G^*_n(k) e^{-ikz^0} - H^*_n(k) \right] (G_{n+1}(k) e^{ikz^0} - H_{n+1}(k)) \right] \]

\[ - \left[ \left[ G^*_n(k) e^{ikz^0} - H^*_n(k) \right] (G_{n+1}(k) e^{ikz^0} - H_{n+1}(k)) \right] (x^0/d) + \Im \left[ \left[ G^*_n(k) e^{-ikz^0} - H^*_n(k) \right] (G_{n+1}(k) e^{ikz^0} - H_{n+1}(k)) \right] (y^0/d); \]

\[ F^P_y = \frac{\epsilon_0 d}{2d^2} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \left[ \Im \left[ \left[ G^*_n(k) e^{-ikz^0} - H^*_n(k) \right] (G_{n+1}(k) e^{ikz^0} - H_{n+1}(k)) \right] \right] \]

\[ - \left[ \left[ G^*_n(k) e^{ikz^0} - H^*_n(k) \right] (G_{n+1}(k) e^{ikz^0} - H_{n+1}(k)) \right] (x^0/d) + \Re \left[ \left[ G^*_n(k) e^{-ikz^0} - H^*_n(k) \right] (G_{n+1}(k) e^{ikz^0} - H_{n+1}(k)) \right] (y^0/d); \]

\[ F^P_z = \frac{\epsilon_0 d}{2d^2} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \left[ \Re \left[ \left[ G^*_n(k) e^{-ikz^0} - H^*_n(k) \right] (G_{n+1}(k) e^{ikz^0} - H_{n+1}(k)) \right] \right] \]

\[ - \left[ \left[ G^*_n(k) e^{ikz^0} - H^*_n(k) \right] (G_{n+1}(k) e^{ikz^0} - H_{n+1}(k)) \right] (x^0/d) + \Im \left[ \left[ G^*_n(k) e^{-ikz^0} - H^*_n(k) \right] (G_{n+1}(k) e^{ikz^0} - H_{n+1}(k)) \right] (y^0/d). \]

(19)

To obtain the axial force, we use formulas (9), (12), and (16):

\[ F^P_z = -\frac{\epsilon_0 d}{d} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \left[ \Im \left[ k G^*_n(k) H^*_n(k) e^{ikz^0} \right] \right] \]

\[ + \Im \left[ \frac{\pi}{2} \left[ G^*_n(k) H_{n+1}(k) e^{-ikz^0} - H^*_n(k) G_{n+1}(k) e^{ikz^0} \right] k(x^0/d) \right] \]

\[ + \Re \left[ \frac{\pi}{2} \left[ G^*_n(k) H_{n+1}(k) e^{-ikz^0} - H^*_n(k) G_{n+1}(k) e^{ikz^0} \right] k(y^0/d) \right]. \]

(20)

Formulas (19) and (20) provide the force due to patches to linear order in a small transverse shift for an arbitrary axial displacement. In many cases, such as the STEP experimental set-up, the axial shift is also small, and only PE forces to linear order in all the shifts are needed. We obtain these expressions by replacing \( \exp(\pm ikz^0) \) with the two terms of its Maclaurin expansion, assuming that all the arising integrals in \( k \) converge.

\[ F^P = \frac{\epsilon_0 d}{2d^2} \int_{-\infty}^{\infty} dk \sum_{n=-\infty}^{\infty} \left[ 9 \left[ G^*_n(k) - H^*_n(k) \right] (G_{n+1}(k) - H_{n+1}(k)) \right] \]

\[ - \left[ \left[ G^*_n(k) - H^*_n(k) \right] (G_{n+1}(k) - H_{n+1}(k)) \right] (x^0/d) + \Im \left[ \left[ G^*_n(k) - H^*_n(k) \right] (G_{n+1}(k) - H_{n+1}(k)) \right] (y^0/d). \]
Unlike the transverse shifts $x_0$ and $y_0$, the axial shift enters here not in the ratio to the gap, $d$, but in the product with $k$, which is the inverse characteristic length of the voltage change in the axial direction. Note that all the forces are derived as acting on the inner cylinder, the forces on the outer one have the opposite sign.

### 3.4. General properties of electrostatic PE forces

The above results allow for some general conclusions regarding the patch interaction.

1. To lowest order, the axial patch force is inversely proportional to the gap width; the transverse force components go as its inverse square.

2. Forces and shifts are directionally coupled: a transverse shift causes generally some axial force, and vice versa a transverse force appears due to an axial shift.

3. The axial force vanishes when the patches are present on one of the cylinders only, i.e. when either $G_n(k) = 0$ or $H_n(k) = 0$.

4. The transverse force does not vanish when the patches are on one of the cylinders only. Moreover, expressions (19) for its components can be given in terms of the non-zero patch potential (and not its Fourier coefficient) using the Parseval identity (7). For example, in the case when there are no patches on the inner cylinder [$V_a(\varphi', z') \equiv 0, G_n(k) \equiv 0$] the transverse components are

$$F_p^x = -\frac{\varepsilon_0 a}{2d^2} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[ \mathcal{Z}[(G_n(k) - H_n^*(k))(G_{n+1}(k) - H_{n+1}(k))] 
+ \mathcal{Z}[(G_n(k)H_{n+1}(k) - H_n(k)G_{n+1}(k))(k^2)] \right];$$

$$F_p^y = -\frac{\varepsilon_0 a}{2d^2} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[ \mathcal{Z}[kG_n(k)H_{n+1}(k)] + \frac{\mathcal{Z}}{2}[G_n(k)H_{n+1}(k)] 
- H_n^*(k) G_{n+1}(k) [k(x_0/d) + \frac{\mathcal{Z}}{2}[G_n(k)H_{n+1}(k)] 
- H_n^*(k) G_{n+1}(k) [k(y_0/d) + \frac{\mathcal{Z}}{2}[G_n(k)H_{n+1}(k)]k(z_0)] \right].$$

(21)

5. Uniformly charged cylinders give rise to a mutual restoring (elastic) force, so that the coaxial condenser is at a neutrally stable equilibrium.

6. The interaction between patch and uniform potentials involves only the first and second polar angle harmonics of the patch distribution if the force is taken, as above, to linear order in the transverse shift.
Some of the above conclusions might be rather obvious or intuitively clear: for instance, no 3 comes from the translational invariance, while no 1 seems to be a dimensional effect (the system is bounded in both transverse dimensions, but infinite along its symmetry axis). However, all of them are now accurately established by our analysis.

4. The patch model

To better understand patch interaction, it is natural to examine the case when just a couple of patches are present, and the PE forces are described by as simple expressions as possible. One thus needs some convenient model of a patch as a localized deviation from the uniform potential described by some particular functions with just a few parameters involved. One of them should control the patch potential, two more have to govern the spot width in the axial and azimuthal directions, and two more parameters specify the patch position.

Such a patch model is desired for another reason as well. In the experiments, there is usually no way to directly measure the patch distribution at the electrode surfaces. Instead, one should infer it from some other signals, like the patch forces. However, our force formulas are not fit for immediate modeling, since the unknowns in them are the Fourier coefficients $G_n(k)$ and $H_n(k)$, with no means to estimate these functions unless properly parameterized. The existing experience of such parameterizations, along with the common sense, demonstrates clearly that only the models based on the underlying physics, rather than ad hoc ones, turn out efficient and work successfully.

So, the goal of an effective patch model is to find such functions that (a) both Fourier coefficients $G_n(k)$ and $H_n(k)$ are found in a closed form, and (b) all the series and integrals in formulas (19) and (20) for the forces are computed analytically in a closed form. Separation of variables is the simplest representation leading to these goals, so we set

$$V(\phi - \phi_0, z - z_0) = V_0 f(z - z_0)u(\phi - \phi_0).$$  \hspace{1cm} (22)

Here the normalizing constant $V_0$ has the dimension of a potential, and the dimensionless functions $f(z)$ and $u(\phi)$ are chosen so that $|f(z)| \leq 1$, $|u(\phi)| \leq 1$; $f(0) = 1$, $u(0) = 1$.

The center of the patch is at $\phi = \phi_0$, $z = z_0$, where the potential achieves the maximum magnitude $V_0$ (positive or negative). The Fourier coefficient of the function (22) is

$$V_n(k) = V_0 \tilde{f}(k) e^{-ikz_0} u_n e^{-in\phi_0} = V_0 \tilde{\mathcal{f}}(k) u_n e^{-ikz_0} e^{-in\phi_0};$$

$$\tilde{\mathcal{f}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz f(z) e^{-ikz}, \quad u_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\phi u(\phi) e^{-in\phi}.$$  \hspace{1cm} (23)

Successful implementation of the above requirements (a) and (b) hinges now on the choice of functions $f(z)$ and $u(\phi)$. The first of them is natural to choose as the Gaussian exponent in $z$,

$$f(z) = \exp \left[ - \left( \frac{z}{\sqrt{2}\Delta z} \right)^2 \right], \quad \tilde{\mathcal{f}}(k) = \Delta z \exp \left[ - \left( \frac{k\Delta z}{\sqrt{2}} \right)^2 \right],$$  \hspace{1cm} (24)

with the Fourier coefficient a Gaussian exponent, too, and the parameter $\Delta z$ giving the axial half-width of the spot. The force expressions (19) and (20) contain the product of two Gaussians, which is again a Gaussian; hence, the integrals in $k$ there are combinations of the elementary functions, as desired.

A good choice of the second function, $u(\phi)$, which is $2\pi$-periodic, turns out much more difficult. Nevertheless, eventually one can come up with the following:

$$u(\phi) = u(\phi, \lambda) = \frac{(1 - \lambda)^2}{2} \frac{1 + \cos \phi}{1 - 2\lambda \cos \phi + \lambda^2}.$$  \hspace{1cm} (25)
where $-1 \leq \lambda < 1$ is controlling the width of the peak at $\phi = 0$, see figure 2. Indeed, when $\lambda = -1$, $u(\phi, -1) \equiv 1$, for $\lambda = 0$, $u(\phi, 0) = 0.5(1 + \cos \phi)$, and finally, when $\lambda \rightarrow 1 - 0$, the function demonstrates a ‘bounded delta-like’ behavior by going to zero everywhere except $\phi = 0$, where the limit is unity (zero width peak). We introduce the azimuthal patch half-width, $\Delta \phi$, in an accurate way as the abscissa at the point where $u$ coincides with its mean value over the whole interval, $u(\Delta \phi) = u_{av}$, which gives

$$\cos \Delta \phi = \lambda, \quad \Delta \phi = \arccos \lambda. \quad (26)$$

In complete agreement with the above, when $\lambda \rightarrow 1 - 0$, the width shrinks according to $\Delta \phi \approx \sqrt{2(1 - \lambda)} \rightarrow 0$; in the opposite case $\lambda = -1$, one naturally has $2\Delta \phi = 2\pi$.

What makes the choice (25) really invaluable for calculations is its Fourier coefficients obtained by simply expanding the function in powers of $\lambda \exp(-i\phi)$:

$$u_n = u_n(\lambda) = \sqrt{\frac{2\pi}{\Delta \phi}} \frac{1 - \lambda^2}{4\lambda} \lambda^{|n|}, \quad n \neq 0; \quad u_0 = u_0(\lambda) = \sqrt{\frac{2\pi}{\Delta \phi}} \frac{1 - \lambda}{2}, \quad (27)$$

they are essentially just exponents of $|n|$, as in the geometric progression. Apparently, expressions (25) and (27) satisfy our requirements (a) and (b), perhaps even in the simplest possible way. The profiles of $u(\phi)$ are plotted in figure 2 for various width values.

With $f(z)$ and $u(\phi)$ defined by formulas (24) and (25), our patch model (22) becomes

$$V(\phi - \phi_*, z - z_*) = V(\phi - \phi_*, z - z_*, \Delta z, \Delta \phi) = V_* V(\phi - \phi_*, z - z_*, \Delta z, \Delta \phi)$$

$$= V_* \frac{(1 - \lambda_*)^2}{2} \frac{1 + \cos(\phi - \phi_*)}{1 - 2\lambda_* \cos(\phi - \phi_*) + \lambda_*) \exp\left[-\left(\frac{z - z_*}{\sqrt{2\lambda}}\right)^2\right]. \quad (28)$$

The corresponding Fourier coefficients are found by expressions (23), (24), and (27) as

$$V_n(k) = \sqrt{2\pi \Delta z_*} V_* \frac{1 - \lambda_*^2}{4\lambda_*} \lambda_*^{|n|} \exp\left[-\left(\frac{k \Delta z_*}{\sqrt{2}}\right)^2\right] e^{-i(\phi_*, k \Delta z_*)}, \quad n = \pm 1, \pm 2, \ldots, \quad (29)$$

$$V_0(k) = \sqrt{2\pi \Delta z_*} V_* \frac{1 - \lambda_*}{2} \exp\left[-\left(\frac{k \Delta z_*}{\sqrt{2}}\right)^2\right] e^{i\Delta z_*}.$$
Certainly, these functions are smooth enough to satisfy conditions (4); in fact, the function (28) and all its derivatives in $\phi$ and $z$ are squarely integrable. The image of equipotentials of the patch (28) normalized by the maximum voltage is shown in figure 3.

Finally, it is worth mentioning that separated variables in our patch model (28) do not limit its applicability to complicated voltage distributions where the dependences on $\phi$ and $z$ are strongly entangled. Such distributions can be modeled by linear combinations of our patches with different parameters and locations (see section 6): the set of patch functions (28) with different magnitudes $(V^*_i)$, widths $(\Delta_\phi, \Delta_z)$, and centers $(\phi^*_i, z^*_i)$ is clearly complete.

5. Single patch at each of the electrodes: a picture of patch interaction

We consider now two patches of the form (28), one at the inner boundary and the other at the outer boundary. Patch voltages in the boundary conditions (3) become ($i = 1, 2$)

\[
G(\phi', z') = V(\phi' - \phi_1, z' - z_1), \quad H(\phi, z) = V(\phi - \phi_2, z - z_2);
\]

\[
V(\phi - \phi_i, z - z_i) = V_i \left( \frac{1 - \lambda_i}{2} \right)^2 \frac{1 + \cos(\phi - \phi_i)}{1 - 2\lambda_i \cos(\phi - \phi_i) + \lambda_i^2} \exp \left[ - \left( \frac{z - z_i}{\Delta z_i} \right)^2 \right]. \tag{30}
\]

Here, according to relation (26) between $\lambda$ and the angular width, $\Delta \phi$, $0 < \Delta \phi_i \leq \pi$, $0 \leq \Delta z_i < \infty$, $-\pi < \phi_i \leq \pi$, $-\infty < z_i < \infty$, $i = 1, 2$. The forces corresponding to these distributions are derived in the appendix. We study here the interaction of two identical patches, $\Delta z_1 = \Delta z_2 = \Delta z$, $\Delta \phi_1 = \Delta \phi_2 = \Delta \phi$, and $V_1 = \pm V_2 = V_0$. 

![Figure 3. Equipotentials of the patch model for $\Delta \phi = \pi/8$.](image)
5.1. Transverse force

5.1.1. Transverse force due to patch and uniform potential interaction. By formula (A.1) of the appendix, the transverse force due to the interaction between the uniform potential and patches reduces to the following expressions:

\[
\begin{align*}
\frac{F_{\text{int}}^x}{F_0} &= -\pi \sqrt{\frac{\pi}{2}} \frac{\Delta z}{a} \sin^2 \Delta \phi \left\{ (\cos \phi_1 \mp \cos \phi_2) \\
&\quad - \frac{x^0}{d} \left[ (\cos 2\phi_1 \mp \cos 2\phi_2) \cos \Delta \phi + \frac{2(1 \mp 1)}{1 + \cos \Delta \phi} \right] \\
&\quad - \frac{y^0}{d} \left[ (\cos 2\phi_1 \mp \cos 2\phi_2) \cos \Delta \phi \right] \right\}; \\
\frac{F_{\text{int}}^y}{F_0} &= -\pi \sqrt{\frac{\pi}{2}} \frac{\Delta z}{a} \sin^2 \Delta \phi \left\{ (\sin \phi_1 \mp \sin \phi_2) \\
&\quad - \frac{x^0}{d} \left[ (\sin 2\phi_1 \mp \sin 2\phi_2) \cos \Delta \phi \right] \\
&\quad - \frac{y^0}{d} \left[ \frac{2(1 \mp 1)}{1 + \cos \Delta \phi} - (\cos 2\phi_1 \mp \cos 2\phi_2) \cos \Delta \phi \right] \right\}. \quad (31)
\end{align*}
\]

Here \( F_0 \) is the characteristic force defined as

\[
F_0 = \epsilon_0 V_0 V^- (a/d)^2, \quad (32)
\]

and \( V^- \) is the uniform voltage difference \((3)\). The minus or plus sign is taken for the patch voltages of the same or opposite sign, respectively; the signs of the charges induced by patches on the cylinders are opposite in the first case, and same in the second. As expected, the magnitude of the transverse force is proportional to \((a/d)^2\). It is also proportional to the relative axial width, \(\Delta z/a\), and entirely independent of the axial positions \(z_{1,2}\) of the patches. The dependence on the angular width is more complicated: the maximum force is at \(\Delta \phi = \pi/2\), as prompted by geometry; for a small width, the force goes to zero as \((\Delta \phi)^2\). When \(\Delta \phi = \pi\), i.e. the patches are the ‘belts’ of voltage uniform in \(\phi\), the zeroth-order force vanishes, and the total becomes proportional to the shift and directed along it:

\[
\vec{F}_{\text{int}} = (2\pi)^{3/2} (1 \mp 1) F_0 \left( \frac{\Delta z}{a} \right) \left( \vec{\rho}_0 \right) \cdot d. \quad (33)
\]

This is zero when the patch voltages are equal: the forces from each of them have the same magnitude and opposite directions (see more on this below).

The main contribution, i.e. the force in the centered position, is best characterized by its polar components, namely

\[
\begin{align*}
\frac{F_{\text{int}}^\rho}{F_0} &= -\pi \sqrt{\frac{\pi}{2}} \frac{\Delta z}{a} \sin^2 \Delta \phi \left[ \cos(\phi - \phi_1) \mp \cos(\phi - \phi_2) \right]; \\
\frac{F_{\text{int}}^\phi}{F_0} &= \pi \sqrt{\frac{\pi}{2}} \frac{\Delta z}{a} \sin^2 \Delta \phi \left[ \sin(\phi - \phi_1) \mp \sin(\phi - \phi_2) \right].
\end{align*}
\]

The total force is a superposition of the two forces from each of the patches. They act along the radial direction to the corresponding patch center, and the total is a vector sum of these two radial vectors, see figure 4. That is why, the interaction force vanishes when the patches are one opposite the other \((\phi_1 = \phi_2)\): their contributions, aligned and of the opposite signs,
Figure 4. Interaction force due to two patches with (a) the same sign of the voltages and (b) the opposite sign of the voltages.

exactly cancel each other. So here a patch behaves as an effective point charge, $q_{\text{eff}}$, in the uniform radial field $E^r = -V^-/d$: the force due to it is just $q_{\text{eff}}E^r$. The effective charges are readily found by comparison with the above expressions of the force.

The forces of the first order are directionally coupled to the shifts, meaning an $x$-force depends on the $y$-shift, and vice versa. The forces consist of a constant term and the second harmonics of the patch angular position.

5.1.2. Transverse force due to patch interaction. The general expressions (A.8) and (A.10) simplify for the same size patches to

\[
\frac{F_{p_x}^p}{F_r} = 2(\pi)^{3/2} \frac{\Delta z}{a} \sin^2 \left( \frac{\Delta \psi}{2} \right) \left\{ \mathcal{N}_1 \mp 2\mathcal{M}_1 \exp \left[ -\left( \frac{z_1 - z_2}{2\Delta z} \right)^2 \right] \right\} (\cos \varphi_1 + \cos \varphi_2)
\]

\[
- \frac{x^0}{a} \left[ 2\mathcal{N}_0 + \mathcal{N}_2 (\cos 2\varphi_1 + \cos 2\varphi_2) \right]
\]

\[
\mp 2 \left( \mathcal{M}_0 + \mathcal{M}_2 \cos (\varphi_1 + \varphi_2) \right) \exp \left[ -\left( \frac{z_1 - z_2}{2\Delta z} \right)^2 \right] \]

\[
- \frac{y^0}{a} \left[ \mathcal{N}_2 (\sin 2\varphi_1 + \sin 2\varphi_2) \mp 2\mathcal{M}_2 \sin (\varphi_1 + \varphi_2) \exp \left[ -\left( \frac{z_1 - z_2}{2\Delta z} \right)^2 \right] \right]
\]

\[
\mp \frac{z^0}{\Delta z} \left[ \mathcal{M}_1 (\cos \varphi_1 + \cos \varphi_2) \frac{z_1 - z_2}{\Delta z} \exp \left[ -\left( \frac{z_1 - z_2}{2\Delta z} \right)^2 \right] \right]
\]

\[
\frac{F_{p_y}^p}{F_r} = 2(\pi)^{3/2} \frac{\Delta z}{a} \sin^2 \left( \frac{\Delta \psi}{2} \right) \left\{ \mathcal{N}_1 \mp 2\mathcal{M}_1 \exp \left[ -\left( \frac{z_1 - z_2}{2\Delta z} \right)^2 \right] \right\} (\sin \varphi_1 + \sin \varphi_2) \quad (34)
\]

\[
- \frac{x^0}{a} \left[ \mathcal{N}_2 (\sin 2\varphi_1 + \sin 2\varphi_2) \mp 2\mathcal{M}_2 \sin (\varphi_1 + \varphi_2) \exp \left[ -\left( \frac{z_1 - z_2}{2\Delta z} \right)^2 \right] \right]
\]
with just a natural replacement of $V^-$ with $V_0$. The coefficients involved are found by combining formulas (A.5), (A.6) and (A.7) of the appendix with the representation (A.12):

$$N_0 = \frac{3 - \lambda}{8}; \quad N_1 = \frac{1 + \lambda}{8} (2 - \lambda); \quad N_2 = \frac{1 + \lambda}{16} (1 + 4\lambda - 3\lambda^2);$$

$$M_0 = \frac{1}{4} \left[ 1 + (1 + \lambda)^2 \frac{\cos(\phi_1 - \phi_2) - \lambda^2}{2D} \right]; \quad D = 1 - 2\lambda^2 \cos(\phi_1 - \phi_2) + \lambda^4;$$

$$M_1 = \frac{1 - \lambda^2}{8} \left[ 1 - \lambda (1 + \lambda) \frac{1 + \lambda^2 - 2 \cos(\phi_1 - \phi_2)}{2D} \right];$$

$$M_2 = \frac{1 - \lambda^2}{8} \left[ \frac{1 + \lambda}{2} + \lambda \left( 2 \cos(\phi_1 - \phi_2) + (1 + \lambda) \frac{\cos 2(\phi_1 - \phi_2) - \lambda^2 \cos(\phi_1 - \phi_2)}{D} \right) \right].$$

As before, the signs $\mp$ correspond to the case of the same or opposite signs of the patch voltages. The force (34) is again proportional to $(a/d)^2$ and $\Delta z/a$. The dependence on the angular width here is even more complicated than in the previous case; still, the force is $\propto (\Delta \phi)^2$ in the narrow patch limit. For $\Delta \phi = \pi$, an analog of formula (33) holds:

$$\tilde{F}_p = 2(\pi)^{3/2} F_u \left( \frac{\Delta z}{a} \right) \left\{ 1 \mp \exp \left[ -\left( \frac{\Delta \phi}{2} \right)^2 \left( \frac{\rho_0}{a} \right) \right] \right\}.$$  

This force, however, does not vanish for identical voltages, unless the patches are one right against each other.

The nature of the force due to the patch interaction is most clearly seen in the main term corresponding to coaxial cylinders with no axial shift:

1. $V_1 = V_2 = V_0$

$$\frac{F_p}{F_u} = 4(\pi)^{3/2} \frac{\Delta z}{a} \sin^2 \left( \frac{\Delta \phi}{2} \right) \frac{\cos \frac{\phi_1 - \phi_2}{2}}{2} \left\{ N_1 - 2M_1 \exp \left[ -\left( \frac{\Delta \phi}{2} \right)^2 \left( \frac{\rho_0}{a} \right) \right] \right\};$$

2. $V_1 = -V_2 = V_0$

$$\frac{F_p}{F_u} = 4(\pi)^{3/2} \frac{\Delta z}{a} \sin^2 \left( \frac{\Delta \phi}{2} \right) \frac{\cos \frac{\phi_1 - \phi_2}{2}}{2} \left\{ N_1 - 2M_1 \exp \left[ -\left( \frac{\Delta \phi}{2} \right)^2 \left( \frac{\rho_0}{a} \right) \right] \right\}.$$
proportional to $N_1$ depending only on the angular patch width and non-negative, by formulas (36). The other is the negative exponent of $(z_1 - z_2)^2$ with the coefficient $M_1$ depending on $|\phi_1 - \phi_2|$. The coefficient $M_1$, along with $M_0$ and $M_2$, is shown in figure 5 versus the angular distance. They all have a sharp maximum at $\phi_1 = \phi_2$ and drop quickly away from it; the sharper the maximum, and the faster the drop, the smaller the angular width of the patch is. This is a strong manifestation of screening of patch charges: the patches interact strongest of all when their centers are right opposite each other; the interaction drops when one charge stops ‘seeing’ the other due to the obstruction by the inner cylinder.

In figure 6, $F_{\perp}^p$ is plotted as a function of the angular distance $|\phi_1 - \phi_2|$ for both cases (1) and (2). The two curves differ significantly for the moderate angular separations and tend to zero when $\phi_1 - \phi_2 \to \pi$. The dependence of $F_{\perp}^p$ on the axial distance $|z_1 - z_2|$ is shown in figure 7. The force does not vanish at infinity but goes instead to the asymptotic value common for the two cases.

As prompted by the similarity between the coefficients $M_0$–$M_2$ and $N_0$–$N_2$, being just some bounded functions of $\Delta \phi$, the terms in the force (34) proportional to the transverse shifts have the structure, and thus the behavior, similar to that of the zeroth-order expressions. The term with the axial shift is rather different, first of all because of an additional factor, $(d/\Delta z)$. Thus $z'^0$ no longer compares to the gap, $d$: quite naturally, it is the ratio $(z'^0/\Delta z)$ that stands as a small parameter at this part of the force. The other peculiarity is the additional factor proportional to $z_1 - z_2$. This part of the force vanishes in both limits $(z_1 - z_2)/\Delta z \to \infty$ and $(z_1 - z_2)/\Delta z = 0$, with the maximum magnitude at $|z_1 - z_2| = \sqrt{2} \Delta z$ as seen in figure 8. Finally, it is proportional to the first harmonics of the angular patch positions and the coefficient $M_1$. The coefficient, and thus the force, is maximum when $\phi_1 = \phi_2$, then it decreases monotonically as $|\phi_1 - \phi_2|$ increases, figure 5.
Figure 6. Normalized transverse force versus the angular separation of the patches for $\Delta \varphi = \pi / 8$.

Figure 7. Normalized transverse force versus the axial separation of the patches.
5.2. Axial force

As demonstrated in section 3, the axial PE force is only due to the patch interaction. Its general expression (A.11) reduces in our case to

$$\frac{F_p^z}{F_{ax}} = \pm 2\pi^{3/2} \sin^2 \left( \frac{\Delta \varphi}{2} \right) \frac{z_1 - z_2}{\Delta z} \exp \left[ - \left( \frac{z_1 - z_2}{2\Delta z} \right)^2 \right] \left\{ M_0 - \frac{x^0}{d} M_1 (\cos \varphi_1 + \cos \varphi_2) \right. \left. \frac{y^0}{d} M_1 (\sin \varphi_1 + \sin \varphi_2) - \frac{z^0}{z_1 - z_2} M_0 \left[ 1 - \left( \frac{z_1 - z_2}{\sqrt{2} \Delta z} \right)^2 \right] \right\}. \quad (37)$$

The coefficients $M_0$, $M_1$ are found in formulas (36), and the characteristic force is inversely proportional to the relative gap, $d/a$, and not to its square, as before:

$$F_{ax} = \varepsilon_0 V_0^2 a/d. \quad (38)$$

The sign of the force (37) switches from plus to minus for patch voltages of the same or opposite signs, respectively. The most striking feature of the axial force is its overall dependence on the ratio $(z_1 - z_2)/\Delta z$. As one expects, the axial force tends to zero in both limits $\Delta z \to 0$ and $\Delta z \to \infty$, due to the Gaussian exponent of the above argument multiplied by this same argument. It has the maximum $\pi^{3/2} e^{-2} \sin^2 (\Delta \varphi/2) F_{ax}$ at $|z_1 - z_2| = \sqrt{2} \Delta z$, same as the axial shift part of the transverse force from figure 8.

The overall dependence of the axial force on $\Delta \varphi$ is just like that of the transverse force; particularly, for belt-like patches, $\Delta \varphi = \pi$, the force is

$$\frac{F_p^z}{F_{ax}} = \pm \pi^{3/2} \frac{z_1 - z_2}{\Delta z} \exp \left[ - \left( \frac{z_1 - z_2}{2\Delta z} \right)^2 \right] \left\{ 1 - \frac{z^0}{z_1 - z_2} \left[ 1 - \left( \frac{z_1 - z_2}{\sqrt{2} \Delta z} \right)^2 \right] \right\}. \quad \left[ \Delta \varphi = \frac{\pi}{2}, z_1 = z_2 \right.$$

Here the main term vanishes when the two patches are in the same cross-section, $z_1 = z_2$, but the ‘correction’ proportional to $z^0$ does not: $F_p^z / F_{ax} = \mp \pi^{3/2} (z^0 / \Delta z)$, $\Delta \varphi = \pi$, $z_1 = z_2$. 

---

Figure 8. Normalized $x$-component of the force due to the axial shift as a function of the axial distance between the patches ($\varphi_2 = 0$ is taken).
Figure 9. Normalized $z$-component of the force due to the axial shift as a function of the axial distance between the patches.

The first-order force proportional to the axial shift has a factor with $(z_1 - z_2)^2$, instead of $(z_1 - z_2)$, in front of the Gaussian. This contribution changes sign at $z_1 - z_2 = \sqrt{2} \Delta z$, and is maximum when the patches are in the same $z$ plane, as illustrated by figure 9.

The axial force decreases with the angular distance between the patches. The zeroth-order component of the axial force together with the first-order term proportional to the axial shift depends on $|\phi_1 - \phi_2|$ through the coefficient $M_0$ only. The two other components due to the transverse shifts are proportional to the harmonics of the patch angular positions and the coefficient $M_1$; they both vanish in just one case, when $\phi_1 - \phi_2 = \pi$. Despite these minor differences, each of these terms follows basically the characteristic behavior of the coefficients $M_0$ or $M_1$ versus the distance $|\phi_1 - \phi_2|$, as presented in figure 5.

6. Perspectives of PE force modeling

The results of this paper allow, in fact, for an essentially more realistic patch force modeling than the analysis above, even though the physical picture is so clear in this case and the pertinent formulas are rather simple. This can be done in the following way.

One represents both patch potentials $V_a(\phi', z')$ and $V_b(\phi, z)$ as a superposition of some number, $N_{a, b}$, of model patches (28):

$$V_\mu(\phi, z) = \sum_{n=0}^{N_\mu} V(\phi - \phi_\mu^n, z - z_\mu^n; \Delta z_\mu^n, \Delta \phi_\mu^n)$$

$$= \sum_{n=0}^{N_\mu} V_\mu^n(\phi - \phi_\mu^n, z - z_\mu^n; \Delta z_\mu^n, \Delta \phi_\mu^n), \quad \mu = a, b, \quad (39)$$

with different voltages, sizes, and positions. By the formulas of section 3 the PE forces corresponding to the distributions (39) can be explicitly calculated, as it is done here for a
single patch at each of the boundaries. Being cumbersome, this general calculation is otherwise straightforward, without any new technical difficulties. It leads to the force expression as a quadratic form of the patch voltages $V^\mu_n$, with the coefficients depending on all other parameters in a known way.

Having these formulas at hand, one then carries out simulations by specifying parameter sets in various ways and computing the patch forces. One can pick the parameters randomly and eventually come up with the patch force statistics. One can also use any lab information on the patch distributions, arranging for a semi-random patch sets, as was done, for instance, when simulating magnetic trapped flux distribution on GP-B rotors [18].

The motion of the cylinders can then be examined based on the motion equations with the obtained PE forces (longitudinal motion and small transverse oscillations). The full motion analysis, however, should include the results of the third part of this work, namely the PE torques: rotation of the cylinders makes the patch distributions, and hence the PE forces, time dependent.

All this applies, of course, to any experimental set-up with cylindrical geometry. In particular, such exhaustive analysis is strongly recommended before the STEP flight. On the other hand, the same general formulas for the transverse forces can be used for fitting control effort data obtained during the experiment, to restore the voltage patch patterns on the proof masses and bearings. If the latter are known, the axial forces can be computed, and the systematic experimental error due to them can thus be bounded.

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Appendix A. Calculation of the transverse and axial forces for a single patch at each of the cylinders

Here we consider one patch described by our patch model (28) at each of the two boundaries, i.e. for the boundary voltage distributions (30). The force (18) due to the interaction between the uniform potential and patches is a linear function of the Fourier coefficients of the boundary patch voltages computed for $k = 0$ and $n = 0, 1, 2$. Thus formulas (29) allow one to obtain ($l_i = \sqrt{2} \Delta z_i$, $i = 1, 2$):

$$F^\text{int}_x = \pi \sqrt{\frac{\epsilon_0 a^2}{d^2}} V \left\{- \frac{V_1 l_1}{2} \frac{1 - \lambda_1^2}{2} \cos \varphi_1 + \frac{V_2 l_2}{2} \frac{1 - \lambda_2^2}{2} \cos \varphi_2 \right. $$

$$+ \frac{x^0}{d} \left[ V_1 l_1 (1 - \lambda_1) \left(1 + \frac{1 + \lambda_1}{2} \lambda_1 \cos 2\varphi_1 \right) \right. $$

$$- V_2 l_2 (1 - \lambda_2) \left(1 + \frac{1 + \lambda_2}{2} \lambda_2 \cos 2\varphi_2 \right) \left. \right] $$

$$+ \frac{y^0}{d} \left[ V_1 l_1 \frac{1 - \lambda_1^2}{2} \lambda_1 \sin 2\varphi_1 - V_2 l_2 \frac{1 - \lambda_2^2}{2} \lambda_2 \sin 2\varphi_2 \right] \right\} ;$$
\[ F_p^{\text{int}} = \pi \sqrt{\pi} \frac{e^{ilp}}{d^2} V \left\{ -V_1 \left( 1 - \frac{\lambda_2^2}{2} \right) \sin \varphi_1 + V_2 \left( 1 - \frac{\lambda_2^2}{2} \right) \sin \varphi_2 \right. \\
+ \frac{x^0}{d} \left[ V_1 \left( 1 - \frac{\lambda_1^2}{2} \lambda_1 \sin 2\varphi_1 - V_2 \left( 1 - \frac{\lambda_2^2}{2} \lambda_2 \sin 2\varphi_2 \right) \right. \\
+ \frac{y^0}{d} \left[ V_1 l_1 (1 - \lambda_1) \left( 1 + \frac{\lambda_1}{2} \lambda_1 \cos 2\varphi_1 \right) \\
\left. - V_2 l_2 (1 - \lambda_2) \left( 1 + \frac{\lambda_2}{2} \lambda_2 \cos 2\varphi_2 \right) \right] \right\}. \quad (A.1) \]

The transverse force due to the patch interaction is essentially more cumbersome to derive, with the additional difficulty of computing some integrals in \( k \) and sums over \( n \) containing products of Fourier coefficients. We start with \( F_p^p \) from the first of formulas (21). Using expressions (23) and (30), combined with equality (24), we find

\[ F_p^p = -\frac{\pi}{2} \frac{e^{ilp}}{d^2} \int_{-\infty}^{\infty} dk \left\{ -N_1 (\varphi_1, \lambda_1) e^{i\varphi_1} + V_1 \frac{e^{ilp}}{d^2} (\lambda_1) e^{i\varphi_1} \right. \\
\frac{y^0}{d} \left[ N_1 (\varphi_1, \lambda_1) + N_2 (\varphi_1) e^{i\varphi_1} \right. \\
\left. + V_2 \frac{e^{ilp}}{d^2} (\lambda_2) e^{i\varphi_1} \right. \\
\left. + V_1 \frac{e^{ilp}}{d^2} (\lambda_2) e^{i\varphi_1} \right. \\
\left. + V_2 \frac{e^{ilp}}{d^2} (\lambda_2) e^{i\varphi_1} \right. \\
\left. - V_1 \frac{e^{ilp}}{d^2} (\lambda_2) e^{i\varphi_1} \right. \\
\left. - k^2 \left\{ V_1 \frac{e^{ilp}}{d^2} (\lambda_2) e^{i\varphi_1} \right. \\
\left. - \left( \frac{e^{ilp}}{d^2} \right) (\lambda_2) e^{i\varphi_1} \right) \right\}. \quad (A.2) \]

Here we have introduced the following notations for the coefficients:

\[ M_q = M_q (\varphi_1, \lambda_1, \varphi_2, \lambda_2) \equiv \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} u_n (\lambda_1) e^{-im\varphi_1} u_n^* (\lambda_2) e^{i(n+q)\varphi_2}; \quad (A.3) \]

\[ N_q (\lambda) \equiv \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} u_n (\lambda) u_n^* (\lambda) = M(0, \lambda; 0, \lambda); \quad q = 0, 1, 2; \quad (A.4) \]

\( u_n (\lambda) \) is defined by formula (27). Coefficients (A.3) are clearly symmetric in the pairs of arguments \( \varphi, \lambda \) corresponding to each of the patches, \( M_q (\varphi_1, \lambda_1, \varphi_2, \lambda_2) = M_q (\varphi_2, \lambda_2, \varphi_1, \lambda_1) \), which we have already used in the above expressions. Now, formulas (27) lead to the explicit sums of the series (A.3), since they reduce to geometric progressions:

\[ M_0 (\varphi_1, \lambda_1, \varphi_2, \lambda_2) = \frac{1 - \lambda_1 (1 - \lambda_2)}{4} \left[ 1 + (1 + \lambda_1) (1 + \lambda_2) \cos (\varphi_1 - \varphi_2) - \lambda_1 \lambda_2 \right]; \]

\[ M_1 (\varphi_1, \lambda_1, \varphi_2, \lambda_2) = \frac{1 - \lambda_1 (1 - \lambda_2)}{8} \left\{ e^{i\varphi_1} (1 + \lambda_1) \left[ 1 + \frac{\lambda_1}{2} \frac{e^{i(\varphi_1 - \varphi_2)} - \lambda_1 \lambda_2}{D} \right] + e^{i\varphi_2} \left[ 1 + \frac{\lambda_2}{2} \frac{e^{i(\varphi_2 - \varphi_1)} - \lambda_1 \lambda_2}{D} \right] \right\}; \quad (A.5) \]
\[ M_2(\psi_1, \lambda_1, \varphi_2, \lambda_2) = \frac{(1 - \lambda_1)(1 - \lambda_2)}{8} \left\{ e^{2\psi_1(1 + \lambda_1)}\lambda_1 \left[ \frac{1 + \lambda_1}{2} (1 + \lambda_2) \frac{e^{(-\psi_1 - \psi_2)}}{D} - \lambda_1 \lambda_2 \right] + e^{2\psi_1(1 + \lambda_2)}\lambda_2 \left[ \frac{1 + \lambda_2}{2} (1 + \lambda_1) \frac{e^{(-\psi_1 - \psi_2)}}{D} - \lambda_1 \lambda_2 \right] + \frac{(1 + \lambda_1)(1 + \lambda_2)}{2} e^{(-\psi_1 + \psi_2)} \right\}; \]

\[ D = 1 - 2\lambda_1 \lambda_2 \cos(\psi_1 - \psi_2) + (\lambda_1 \lambda_2)^2. \]  

By definition (A.4), the coefficients \( N_0(\lambda) \) are found as a particular case of expressions (A.5) when \( \lambda_1 = \lambda_2 = \lambda \) and \( \psi_1 = \psi_2 = 0 \):

\[ N_0(\lambda) = \frac{1 - \lambda}{8}(3 - \lambda); \quad N_1(\lambda) = \frac{1 - \lambda^2}{8}(2 - \lambda); \quad N_2(\lambda) = \frac{1 - \lambda^2}{16}(1 + 4\lambda - 3\lambda^2). \]  

For the closed form of \( F_p^\varphi \), two well-known integrals in \( k \) are used:

\[ \int_{-\infty}^{\infty} dk \exp \left[ -\frac{k^2(l_1^2 + l_2^2)}{4} \right] e^{ik(z_1 - z_2)} = \frac{2\sqrt{\pi}}{\sqrt{l_1^2 + l_2^2}} \exp \left[ -\frac{(z_1 - z_2)^2}{l_1^2 + l_2^2} \right]; \]

\[ \int_{-\infty}^{\infty} dk \exp \left[ -\frac{k^2(l_1^2 + l_2^2)}{4} \right] k e^{ik(z_1 - z_2)} = \pm i 4\sqrt{\pi} \frac{z_1 - z_2}{(l_1^2 + l_2^2)^{3/2}} \exp \left[ -\frac{(z_1 - z_2)^2}{l_1^2 + l_2^2} \right]. \]

In the case when \( z_1 = z_2 \) and \( l_1 = l_2 = l \), the first integral becomes \( \int_{-\infty}^{\infty} dk \exp \left[ -\frac{k^2l_1}{2} \right] = \sqrt{2\pi}/l \). Thanks to all the above results for the series and integrals, we obtain the expression for \( F_p^\varphi \), to linear order in the parameter \( r^0/d \ll 1 \):

\[ F_p^\varphi = \frac{\pi^{3/2} \epsilon_0 a}{\sqrt{2}} \left\{ [V_1^2 l_1 N_1(\lambda_1) \cos \varphi_1 + V_2^2 l_2 N_1(\lambda_2) \cos \varphi_2 - V_1 V_2 \tilde{I}\tilde{I}(M_1) e^{-iz}] - \frac{\gamma^{0}}{d} [V_1^2 l_1 (N_0(\lambda_1) + N_2(\lambda_1) \cos 2\varphi_1) + V_2^2 l_2 (N_0(\lambda_2) + N_2(\lambda_2) \cos 2\varphi_2) - V_1 V_2 \tilde{I}\tilde{I}(M_2) + M_0] e^{-iz}] - \frac{\gamma^0}{d} [V_1^2 l_1 N_2(\lambda_1) \sin 2\varphi_1 + V_2^2 l_2 N_2(\lambda_2) \sin 2\varphi_2 - V_1 V_2 \tilde{I}\tilde{I}(M_2)] e^{-iz}] - \frac{\gamma^0 t}{23/2 l_1 l_2} [2V_1 V_2 \tilde{I}\tilde{M}(M_1) e^{-iz}] \right\}; \]

\[ \tilde{z} = \frac{z_1 - z_2}{l_1^2 + l_2^2}; \quad \tilde{t} = \frac{2^{3/2} l_1 l_2}{l_1^2 + l_2^2}. \]  

Calculation of \( F_p^\varphi \) is pretty similar, and its result, to linear order in \( r^0/d \ll 1 \), is

\[ F_p = \frac{\pi^{3/2} \epsilon_0 a}{\sqrt{2}} \left\{ [V_1^2 l_1 N_1(\lambda_1) \sin \varphi_1 + V_2^2 l_2 N_1(\lambda_2) \sin \varphi_2 - V_1 V_2 \tilde{I}\tilde{M}(M_1) e^{-iz}] - \frac{\gamma^{0}}{d} [V_1^2 l_1 N_2(\lambda_1) \sin 2\varphi_1 + V_2^2 l_2 N_2(\lambda_2) \sin 2\varphi_2 - V_1 V_2 \tilde{I}\tilde{M}(M_2)] e^{-iz}] \right\}.
Finally, we find a closed-form expression for the axial force. Starting from the last of formulas (21), we just need to repeat the same steps as with $F_p^\lambda$. The only significant difference is that here we need a slightly different integral,

$$
\int_{-\infty}^{\infty} dk \exp \left[ -\frac{k^2 (l_1^2 + l_2^2)}{4} \right] k^2 e^{i k (z_1 - z_2)}
$$

In this way, using definitions (A.9) and (A.5), we arrive at the final expression for $F_c^\lambda$:

$$
F_c^\lambda = \left( \frac{\pi^3}{4} \right)^{1/2} \frac{\epsilon_0 a d}{V_1 V_2 l_1 l_2} e^{-\frac{z_0}{d}} \left\{ \frac{\Re (M_0) - \frac{z_0}{d} \Re (M_1)}{\Re (M_1)} \right\} - \frac{z_0}{d} \left[ \frac{\Im (M_1)}{\Re (M_1)} \right].
$$

The calculation of the PE forces for the case when a single patch is placed on each of the cylinders is now completed. Expressions (A.8), (A.10) and (A.11) are for the patches with different sizes and magnitudes. When these are identical, the results simplify essentially due to $\lambda_1 = \lambda_2 = \lambda, l_1 = l_2 = l, |V_1| = |V_2|$, in particular

$N_0(\lambda) = (1 - \lambda) N_0; \quad N_1(\lambda) = (1 - \lambda) N_1; \quad N_2(\lambda) = (1 - \lambda) N_2; \quad M_0(\lambda, \phi_1, \phi_2) = (1 - \lambda) M_0; \quad M_1(\lambda, \phi_1 - \phi_2) [\cos (\phi_1 + \cos \phi_2) + t (\sin \phi_1 + \sin \phi_2)]; \quad M_2(\lambda, \phi_1 - \phi_2) [\cos (\phi_1 + \phi_2) + t \sin (\phi_1 + \phi_2)].$

The coefficients $N_1$ and $M_1$ here are given explicitly by formulas (36).

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