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Nonexistence of spectral gaps in Hölder spaces for continuous time dynamical systems

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Abstract

We show that there is a natural restriction on the smoothness of spaces where the transfer operator for a continuous dynamical system has a spectral gap. Such a space cannot be embedded in a Hölder space with Hölder exponent greater than $\frac{1}{2}$ unless it consists entirely of coboundaries.

1 Introduction

Decay of correlations (rates of mixing) and strong statistical properties are well-understood for Axiom A diffeomorphisms since the work of [2, 9, 10]. Mixing rates are computed with respect to any equilibrium measure with Hölder potential. Up to a finite cycle, such diffeomorphisms have exponential decay of correlations for Hölder observables. In the one-sided (uniformly expanding) setting, this is typically proved by establishing quasicompactness and a spectral gap for the associated transfer operator $L$. Such a spectral gap yields a decay rate $\|L^n v - \int v\| \leq C_v e^{-an}$ for $v$ Hölder, where $\| \|$ is a suitable Hölder norm and $a, C_v$ are positive constants. Decay of correlations for Hölder observables is an immediate consequence of the decay for $L^n$. This philosophy has been extended to large classes of nonuniformly expanding dynamical systems with exponential [13] and subexponential decay of correlations [14].

For continuous time dynamical systems, the usual techniques [5, 7, 8] bypass spectral gaps; the only exceptions that we know of being Tsujii [11, 12]. However, the result in [11] is for suspension semiflows over the doubling map with a $C^3$ roof function, where the smoothness of the roof function is crucial and very restrictive. A spectral gap for contact Anosov flows is obtained in [12]; unfortunately it seems nontrivial to extend this to nonuniformly hyperbolic contact flows (or uniformly hyperbolic

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contact flows with unbounded distortion), see [1] which proves exponential decay of correlations for billiard flows with a contact structure but does not establish a spectral gap. Indeed, apart from [11, 12], there are no results on spectral gaps of transfer operators for semiflows and flows.

The results of Tsujii [11, 12] provide a spectral gap in an anisotropic Banach space. In this paper we obtain a restriction on the Banach spaces that can yield a spectral gap. We work in the following general setting:

Let $(\Lambda, d)$ be a bounded metric space with Borel probability measure $\mu$, and let $F_t : \Lambda \to \Lambda$ be a measure-preserving semiflow. We suppose that $t \to F_t$ is Lipschitz a.e. on $\Lambda$. Let $L_t : L^1(\Lambda) \to L^1(\Lambda)$ denote the transfer operator corresponding to $F_t$ (so $\int_\Lambda L_t v w \, d\mu = \int_\Lambda v w \circ F_t \, d\mu$ for all $v \in L^1(\Lambda)$, $w \in L^\infty(\Lambda)$, $t > 0$). Let $v \in L^\infty(\Lambda)$ and define $v_t = \int_0^t v \circ F_r \, dr$ for $t \geq 0$.

**Theorem 1.1** Let $\eta \in (\frac{1}{2}, 1)$. Suppose that $L_t v \in C^\eta(\Lambda)$ for all $t > 0$ and that $\int_0^\infty \|L_t v\|_\eta \, dt < \infty$. Then $v_t$ is a coboundary:

$$v_t = \chi \circ F_t - \chi \text{ for all } t \geq 0, \text{ a.e. on } \Lambda$$

where $\chi = \int_0^\infty L_t v \, dt \in C^\eta(\Lambda)$. In particular, $\sup_{t \geq 0} |v_t|_\infty < \infty$.

Here, $|g|_\infty = \text{ess sup}_\Lambda |g|$ and $\|g\|_\eta = |g|_\infty + \sup_{x \neq y} |g(x) - g(y)|/d(x, y)^\eta$.

Theorem 1.1 implies that any Banach space admitting a spectral gap and embedded in $C^\eta(\Lambda)$ for some $\eta > \frac{1}{2}$ is cohomologically trivial. However, for (non)uniformly expanding semiflows and (non)uniformly hyperbolic flows of the type in the aforementioned references, coboundaries are known to be exceedingly rare, see for example [3, Section 2.3.3]. Hence, Theorem 1.1 can be viewed as an “anti-spectral gap” result for such continuous time dynamical systems.

2 Proof of Theorem 1.1

Let $v \in L^\infty(\Lambda)$, with $L_t v \in C^\eta(\Lambda)$ for all $t > 0$ and $\int_0^\infty \|L_t v\|_\eta \, dt < \infty$ where $\eta \in (\frac{1}{2}, 1)$. Following Gordin [6] we consider a martingale-coboundary decomposition. Define $\chi = \int_0^\infty L_t v \, dt \in C^\eta(\Lambda)$, and

$$v_t = \int_0^t v \circ F_r \, dr, \quad m_t = v_t - \chi \circ F_t + \chi,$$

for $t \geq 0$. Let $\mathcal{B}$ denote the Borel $\sigma$-algebra on $\Lambda$.

**Proposition 2.1** (i) $t \to m_t$ is $C^\eta$ a.e. on $\Lambda$.

(ii) $\mathbb{E}(m_t|F_t^{-1}\mathcal{B}) = 0$ for all $t \geq 0$. 

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Proof (i) For $0 \leq s \leq t \leq 1$ and $x \in \Lambda$,
\[
|m_s(x) - m_t(x)| \leq |v_s(x) - v_t(x)| + |\chi(F_s x) - \chi(F_t x)| \\
\leq |s - t||v|_\infty + |\chi|_{\eta} d(F_s x, F_t x)^{\eta}.
\]
Since $t \mapsto F_t$ is a.e. Lipschitz, it follows that $t \mapsto m_t$ is a.e. $C^{\eta}$.

(ii) Let $U_t v = v \circ F_t$, and recall that $L_t U_t = I$ and $\mathbb{E}(\cdot|F_t^{-1} \mathcal{B}) = U_t L_t$. Then
\[
L_t m_t = L_t(v_t - U_t \chi + \chi) = \int_0^t L_t U_t v dr - \chi + \int_0^\infty L_t L_r v dr \\
= \int_0^t L_{t-r} v dr - \chi + \int_0^\infty L_{t+r} v dr = \int_0^t L_r v dr - \chi + \int_t^\infty L_r v dr = 0.
\]
Hence $\mathbb{E}(m_t|F_t^{-1} \mathcal{B}) = U_t L_t m_t = 0$.

Proof of Theorem 1.1 Fix $T > 0$, and define
\[
M_T(t) = m_t - m_{t-s} = m_t \circ F_{t-s}, \quad t \in [0, T].
\]
Also, define the filtration $\mathcal{G}_{T,t} = F_{T-t}^{-1} \mathcal{B}$. It is immediate that $M_T(t) = m_t \circ F_{t-s}$ is $\mathcal{G}_{T,t}$-measurable. Also, for $s < t$ we have $M_T(t) - M_T(s) = m_{t-s} - m_{t-t} = m_{t-s} \circ F_{t-s}$, so
\[
\mathbb{E}(M_T(t) - M_T(s)|\mathcal{G}_{T,s}) = \mathbb{E}(m_{t-s} \circ F_{t-s}|F_{T-s}^{-1} \mathcal{B}) = \mathbb{E}(m_{t-s}|F_{T-s}^{-1} \mathcal{B}) \circ F_{t-s} = 0
\]
by Proposition 2.1(ii). Hence $M_T$ is a martingale for each $T > 0$. Next,
\[
|M_T(t)|_\infty = |m_t \circ F_{T-t}|_\infty \leq |m_t|_\infty \leq |v_t|_\infty + 2|\chi|_\infty \leq T|v|_\infty + 2|\chi|_\infty.
\]
Hence $M_T(t)$, $t \in [0, T]$, is a bounded martingale.

By Proposition 2.1(i), $M_T$ has $C^{\eta}$ sample paths. Since $\eta > \frac{1}{2}$, it follows from general martingale theory that $M_T \equiv 0$ a.e. Taking $t = T$, we obtain $m_T = 0$ a.e. Hence $v_T = \chi \circ F_T - \chi$ a.e. for all $T > 0$ as required.

For completeness, we include the argument that $M_T \equiv 0$ a.e. We require two standard properties of the quadratic variation process $t \mapsto [M_T](t)$; a reference for these is [2] Theorem 4.1. First, $[M_T](t)$ is the limit in probability as $n \to \infty$ of
\[
S_n(t) = \sum_{j=1}^n (M_T(jt/n) - M_T((j-1)t/n))^2.
\]
Second (noting that $M_T(0) = 0$),
\[
[M_T](t) = M_T(t)^2 - 2 \int_0^t M_T dM_T,
\]
where the stochastic integral has expectation zero. In particular, $\mathbb{E}([M_T]) = \mathbb{E}(M_T^2)$.

Since $M_T$ has Hölder sample paths with exponent $\eta > \frac{1}{2}$, we have a.e. that
\[
|S_n(t)| = O(t^{2\eta}n^{-2\eta}) \to 0 \quad \text{as } n \to \infty.
\]
Hence $[M_T] \equiv 0$ a.e. It follows that $\mathbb{E}(M_T^2) \equiv 0$ and so $M_T \equiv 0$ a.e.
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