Probes of universality in AdS/CFT

This content has been downloaded from IOPscience. Please scroll down to see the full text.
2013 J. Phys.: Conf. Ser. 462 012044
(http://iopscience.iop.org/1742-6596/462/1/012044)

View the table of contents for this issue, or go to the journal homepage for more

Download details:

IP Address: 188.184.3.52
This content was downloaded on 13/11/2014 at 16:25

Please note that terms and conditions apply.
Probes of universality in AdS/CFT

Adam Ritz
Department of Physics and Astronomy, University of Victoria
Victoria BC, V8P 5C2, Canada

Abstract. We discuss some universal relations that hold for conformal field theories which admit a dual classical gravitational description in anti-de Sitter space. These relations therefore apply in a suitable large $N$ limit, and imply that various hydrodynamic transport coefficients for conserved currents and thermodynamic state functions are entirely fixed by central charges.

1. Introduction
From the Wilsonian viewpoint [1], the primary building blocks of the space of quantum field theories are those special points which exhibit conformal invariance. Indeed, the quantum field theories which play a role in nature are all understood to be either conformal field theories (CFT) themselves, or relevant deformations of a CFT. Among other things, such field theories describe the interactions of all known elementary particles, and the universal scaling behaviour near critical points in statistical mechanics.

In attempting to study CFTs at the quantitative level, there are several avenues available beyond $d = 1 + 1$ where symmetry alone is sufficiently powerful to constrain the allowed space of theories. One approach to arrive at an interacting CFT is by taking a critical limit of a statistical-mechanical model (or, more generally, by following the renormalization group flow to the endpoint) [1]. Another is to start with a Lagrangian formulation of a classically conformal theory, and use extra symmetry (such as supersymmetry) to argue that the quantum theory must be conformally invariant as well [2]. A prime example of such a CFT is the $N = 4$ supersymmetric Yang-Mills theory in 3+1 dimensions. However, the last decade has exhibited another remarkable approach to arrive at a CFT, namely through a higher-dimensional theory of (quantum) gravity in Anti-de Sitter (AdS) space, and the AdS/CFT correspondence [3, 4, 5].

In the limit where classical gravity is a reliable approximation, which involves taking both the number of CFT degrees of freedom and their interactions to be parametrically large, this latter approach is particularly tractable and provides a new class of tools. It is therefore natural to ask which class of CFTs, and thus which universality classes, can be described in this tractable classical gravity regime. In exploring this question, particularly in the condensed matter context, it is important to distinguish two notions of universality:

- Static universality: This reflects the universal scaling behaviour of thermodynamic functions, such as the specific heat, as one approaches the critical point. Static universality classes are generally determined by the spatial dimension and the symmetry of the order parameter distinguishing the phases. A classic example is the fact that the liquid-gas critical point and the ferromagnetic critical point are both in the same static universality class, that of the $d = 3$ Ising CFT.
Figure 1. A schematic illustration of the phase diagram near a quantum critical point at $T = 0$ and $g = g_c$, where $g$ is some coupling or control parameter for the system. The shaded region is controlled by the CFT associated with the critical point.

- **Dynamic universality**: This goes beyond static universality and describes classes of systems in which dynamical transport coefficients also exhibit universal scaling laws [6]. Two fixed points in the same static universality class may exhibit quite distinct dynamical behaviour. Indeed, the liquid-gas and ferromagnetic critical points are in different dynamical universality classes [6]. The additional characteristic which plays a crucial role in defining these classes is the presence or otherwise of conserved currents, which provide additional hydrodynamic degrees of freedom.

Given this dichotomy, it is natural to ask whether there are special fixed points in which these two notions of universality are the same, so that all static and dynamical behaviour is described near the fixed point by a Lorentzian CFT, formulated in spacetime (rather than just space). Physical examples are indeed provided by quantum critical phenomena [7, 8], where the phase transition occurs at zero temperature, driven by quantum fluctuations (see Fig. 1). Such CFTs are relativistic, although their speed of “light” $v$ need not equal $3\times10^8\text{m/s}$. The CFT then dictates static and dynamical quantities in any nearby regime dominated by a single dynamical scale, i.e. temperature, which breaks conformal invariance.

It has become clear that, for these reasons, the quantum critical regime is a primary target to be modeled using the AdS/CFT correspondence. Understanding CFTs in the language of classical gravity has already proved useful in studies of quantum critical transport [9], and this motivates further exploration of the relation of the AdS/CFT correspondence to quantum criticality. In this contribution, we will discuss a number of constraints that naturally apply to any CFT dual to classical gravity in AdS (see [10, 11, 12] for more details). In broad terms, these constraints relate the thermodynamic and hydrodynamic peroperties of the CFT at finite temperature to each other, and furthermore to the central charges which determine the zero-temperature correlators of conserved currents.

Recall that at very short distances, the effects of temperature are irrelevant, and the natural physical questions involve the leading short-distance singularities of the correlation functions. On the other hand, at long distances the effects of temperature become important, and the natural questions are related to thermodynamics and hydrodynamic transport phenomena. In CFTs, however, short and long distances are related by a scaling symmetry, and one may anticipate a universal relation between the long-distance transport coefficients and the parameters which describe the short-distance singularities. Unfortunately, in general this expectation seems to be quantitatively true only in 1+1 dimensions. Nonetheless, this expectation does indeed hold true in CFTs dual to classical gravity in any dimension.

At zero temperature, the (Euclidean) vacuum correlation functions of the energy-momentum tensor $T_{\mu\nu}$ and a $U(1)$ conserved current $J_\mu$ in a CFT are fixed to be

$$\langle J_\mu(x)J_\nu(0) \rangle = \frac{k}{x^{2(d-1)}} \frac{1}{\omega_{d-1}^2} I_{\mu\nu} ,$$

(1.1)
\[ (T_{\mu\nu}(x)T_{\alpha\beta}(0)) = \frac{c}{x^d} \frac{1}{\omega_{d-1}^2} \left( I_{\mu\alpha}I_{\nu\beta} + I_{\mu\beta}I_{\nu\alpha} - \frac{2}{d} \delta_{\mu\nu} \delta_{\alpha\beta} \right), \quad (1.2) \]

where \( I_{\mu\nu} = \delta_{\mu\nu} - 2x_{\mu}x_{\nu}/x^2 \), and \( k \) and \( c \) are central charges, which are dimensionless constants. [We use units in which \( h = v = 1 \) where \( v \) is the speed of \textit{light} in the CFT.] The factors of \( \omega_{d-1} \equiv 2\pi^{d/2}/\Gamma(d/2) \) are inserted for notational convenience. At non-zero temperature \( T \), the equilibrium state is characterized by pressure \( P \), as well as by the charge susceptibility \( \chi = \langle Q^2 \rangle/(VT) \), where \( Q \) is the conserved charge associated with the current \( J_{\mu} \), and we take the thermodynamic limit \( V \to \infty \). The susceptibility can be evaluated by introducing a small chemical potential \( \mu \), so that \( \chi(T) = \partial \rho/\partial \mu |_{\mu=0} \), where \( \rho(T, \mu) = \langle Q \rangle/V \) is the charge density. In a CFT, temperature remains the only scale which dictates,

\[ P(T) = c'T^d, \quad \chi(T) = k'T^{d-2}, \quad (1.3) \]

where \( c' \) and \( k' \) are dimensionless constants. Physically, \( c \) and \( c' \) provide a measure of the total number of degrees of freedom in the system, while \( k \) and \( k' \) measure the number of charged degrees of freedom. In two dimensions, \( c \) is uniquely related to \( c' \) [13, 14], while \( k \) is uniquely related to \( k' \):

\[ c' = \frac{\pi}{6} c, \quad k' = \frac{1}{2\pi} k, \quad (1.4) \]

which means that thermodynamics is uniquely fixed by the central charges. The reason is that in two dimensional CFTs, the vacuum state is related to the thermal state by a symmetry transformation. In \( d > 2 \), the conformal symmetry group is not large enough to enforce a relation similar to Eq. (1.4), and therefore thermodynamics is not determined by the central charges. However, there does exist a large class of CFTs in \( d > 2 \), whose pressure is determined by the central charge \( c \), and whose susceptibility is determined by the central charge \( k \), resembling the two-dimensional case [10, 11]. The crucial property of these models is that they admit a dual description in terms of classical gravity on a \((d+1)\) dimensional anti-de Sitter space (AdS). Namely, we have the following relations [10, 11]:

\[ \frac{c'}{c} = \frac{1}{4\pi^{d/2}} \left( \frac{4\pi}{d} \right)^d \frac{\Gamma(d/2)^3}{\Gamma(d)} \frac{(d-1)}{d(d+1)}, \quad \frac{k'}{k} = \frac{1}{2\pi^{d/2}} \left( \frac{4\pi}{d} \right)^{d-2} \frac{\Gamma(d/2)^3}{\Gamma(d)}. \quad (1.5) \]

In \( d = 2 \) they reproduce the universal relations (1.4).

The CFTs which admit a dual gravitational description have many more universal properties beyond the above relation between thermodynamics and the central charges. A surprising feature of these CFTs (and of their relevant deformations) is that momentum transport in these models is completely determined by thermodynamics. In particular, their viscosity is given by \( \eta = s/4\pi \) in any dimension [15, 16], where \( s = \partial P/\partial T \) is the entropy density. This is surprising because transport coefficients are typically determined by the mean-free path even in CFTs [17], and are not fixed by thermodynamics. Moreover, charge transport in these models is also completely determined by thermodynamics. In particular, the dc electrical conductivity \( \sigma \) obeys a similar relation, so we have

\[ \frac{\eta}{s} = \frac{1}{4\pi}, \quad \frac{\sigma}{\chi} = \frac{1}{4\pi T} \frac{d}{d-2}. \quad (1.6) \]

Even though the ratio \( \sigma/\chi \) was derived for integer \( d \geq 3 \), it can be analytically continued to any real positive \( d > 2 \). The singularity at \( d = 2 \) is precisely what one expects in 1+1 dimensional CFTs.

In what follows, we describe the derivation of these relations between central charges and thermodynamics, and between thermodynamics and hydrodynamics. In the final section, we will discuss some comparisons to known systems, and deviations from these universal relations beyond the classical gravity limit.
2. Thermodynamics from central charges

We will focus on quantum field theories which admit a dual description in terms of classical gravity in Anti-de Sitter (AdS) space within the AdS/CFT correspondence [3, 4, 5]. For such field theories, a large-volume thermal state in a \(d\)-dimensional CFT is described by a \((d+1)\)-dimensional black hole in AdS. The black hole solution follows from the Einstein-Maxwell action,

\[
S = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-\mathcal{g}} \left[ R + \frac{d(d-1)}{L^2} \right] - \frac{1}{4g_{d+1}^2} \int d^{d+1}x \sqrt{-\mathcal{g}} F^2,
\]

where \(L^2\) sets the value of the cosmological constant, and \(g_{d+1}^2\) is the \((d+1)\)-dimensional gauge coupling constant, which has dimension of \((\text{length})^{d-3}\). An equilibrium state at finite temperature and density is described by the Reissner-Nordstrom black hole in AdS. The thermodynamics of these black holes has been studied extensively, see for example [18]. The metric in the thermodynamic limit is given by

\[
ds^2 = \frac{r^2}{L^2} \left( -V(r)dt^2 + dx^2 \right) + \frac{L^2}{r^2} \frac{dr^2}{V(r)},
\]

where \(V(r) = 1 - m/r^d + m_q^2/r^{2d-2}\), and the boundary is at \(r \to \infty\). The parameter \(m\) determines the mass of the black hole, and \(m_q\) determines its charge. The background gauge field is,

\[A_t = \mu - \frac{g_{d+1}^2 L^{d-1} \rho}{(d-2)r^{d-2}},\]

where the charge density \(\rho\) is defined by the variation of the action with respect to the boundary value of the gauge field \(A^{(b)}_t = A_t(r \to \infty)\), and is given by \(\rho = \mu(d-2)/g_{d+1}^2 L^{d-1}\) in terms of the chemical potential \(\mu\). This relation is fixed by requiring that \(A_t\) vanishes at the horizon \(r = r_0\).

We will actually work in the limit \(\mu \to 0\), so that temperature is the dominant scale breaking conformal invariance, given by \(T = r_0d/(4\pi L^2)\). The charge density then vanishes, but the results above allow us to compute the susceptibility.

Central charges: To find the central charge \(c\) in dimension \(d \geq 3\), we consider the vacuum geometry \((T = 0)\) and a convenient momentum space representation of the correlator (1.2),

\[G_{\mu\nu,\alpha\beta}(k) = \left( P_{\mu\alpha}P_{\nu\beta} + P_{\mu\beta}P_{\nu\alpha} - \frac{2}{d-1} P_{\mu\nu}P_{\alpha\beta} \right) G(k^2),\]

where \(P_{\mu\nu} = \delta_{\mu\nu} - k_\mu k_\nu/k^2\). The central charge is related to \(G(k^2)\) by

\[
c = \frac{d+1}{d-1}\frac{\omega_{d-1}^2}{(2\pi)^d} \int \frac{d^dk}{(2\pi)^d} e^{ikx} G(k^2).
\]

Choosing \(k\) along \(x_i\) with \(i \neq 1\) or \(2\), one has \(G_{12,12}(k) = G(k^2)\), and therefore it suffices to evaluate \(G_{12,12}\) to find the central charge. According to the AdS/CFT prescription, \(G_{12,12}\) is given by the second variation of the gravitational action with respect to the boundary value of the metric perturbation \(h_{12}\). The \(h_{12}\) perturbation decouples from all other perturbations, and obeys the equation of motion coming from the action of a massless scalar in the \(AdS_{d+1}\) background. The two-point correlation function for the massless scalar can be evaluated using the standard AdS/CFT prescription, and one finds for the central charge,

\[c = \frac{d+1}{d-1}\frac{L^{d-1}}{4\pi G_{N}^{d+1}} \frac{\Gamma(d+1)\pi^{d/2}}{\Gamma(d/2)^3}.
\]
The value of the central charge \( k \) can be found in a similar manner using the Maxwell action for the gauge field dual to the current \( J^\mu \). However, we will skip the details here, as it was obtained previously in the work of Freedman et al. [19]:

\[
k = \frac{L^{d-3}}{g_{d+1}^{2}} \frac{\Gamma(d)(d-2)}{2\pi^{d/2}\Gamma(d/2)} \omega_{d-1}^2.
\]  

(2.13)

**Thermodynamics:** The entropy is proportional to the \( d-1 \) dimensional area of the horizon, \( S = A_d/4G_N \) where \( G_N \) is the \( d+1 \)-dimensional Newton’s constant. Dividing by the (infinite) \( (d-1) \)-volume \( V \), and working with \( T \gg \mu \) as described above, one finds for the entropy density,

\[
s = \frac{1}{4G_N} \left( \frac{4\pi L}{d} \right)^{d-1} T^{d-1}.
\]  

(2.14)

To find the susceptibility, we need the relation between \( \rho \) and \( \mu \) to linear order in \( \mu \). Expressing \( r_0 \) in terms of temperature when \( \mu \to 0 \),

\[
T = \frac{r_0}{d/(4\pi L^2)},
\]

we find \( \rho(T, \mu) = \chi(T) \mu \), where the susceptibility is

\[
\chi = \frac{(d-2)L^{d-3}}{g_{d+1}^2} \left( \frac{4\pi}{d} \right)^{d-2} T^{d-2}.
\]  

(2.15)

Comparing the results above for \( s \) and \( \chi \) to those for \( c \) and \( k \), we deduce the universal relations for \( c' \) and \( k' \) given in Eq. (1.5).

3. **Hydrodynamics from central charges**

**Momentum transport:** The well-known universal result for the shear viscosity to entropy ratio, \( \eta/s = 1/(4\pi) \) can now be translated one step further to a universal relation of \( \eta \) to the central charge \( c \). Given [20],

\[
\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \text{Im} \langle T_{xy}T_{xy} \rangle(\omega, q = 0)_{\text{ret}},
\]

(3.16)

one obtains

\[
\eta = \frac{s}{4\pi} \left[ \frac{1}{16\pi^{d/2+1}} \frac{d-1}{d+1} \left( \frac{4\pi}{d} \right)^{d} \frac{\Gamma(d/2)^2}{\Gamma(d)} \right] eT^{d-1}.
\]  

(3.17)

This exhibits the universal relation between hydrodynamics, thermodynamics, and the vacuum in the shear channel associated with the conserved energy momentum tensor.

**Charge transport:** An analogous procedure allows us to compute the electrical conductivity of CFTs with a dual gravity description. To define conductivity, we imagine gauging a global \( U(1) \) symmetry of the theory with a small coupling \( e \). To leading order in \( e \), the effects of the gauge field can be ignored, and the electromagnetic response can be determined from the original theory. This essentially amounts to sending \( J^\mu \to eJ^\mu \), and a factor of \( e^2 \) will appear in both the conductivity and the susceptibility. The dc conductivity is determined from the real-time retarded correlation function of \( U(1) \) currents in thermal equilibrium,

\[
\sigma = e^2 \lim_{\omega \to 0} \frac{1}{\omega} \text{Im} \langle J_xJ_x \rangle(\omega, q = 0)_{\text{ret}}.
\]  

(3.18)

To evaluate the retarded correlator, we use the standard AdS/CFT recipe of [21, 20], and consider Maxwell fields propagating on the \( (d+1) \)-dimensional background (2.8) at \( m_q = 0 \). Skipping the details [11], the Kubo formula (3.18) then gives the conductivity,

\[
\sigma = \frac{e^2}{g_{d+1}^2} \left( \frac{L}{20} \right)^{d-3}.
\]  

(3.19)
On the other hand, using the susceptibility (2.15), we arrive at the simple result (1.6) for the conductivity to susceptibility ratio. For systems in which charge transport proceeds by diffusion, conductivity is related to the diffusion constant \( D \) by the Einstein relation \( \sigma = \chi D \). Therefore, our result can be interpreted as a remarkably simple diffusion constant in \( d \) spacetime dimensions,

\[
D = \frac{\sigma}{\chi} = \frac{1}{4\pi T} \frac{d}{d-2},
\]

(3.20)

which agrees with the known results in \( d=4 \) [20], and \( d=3,6 \) [22]. The electrical conductivity takes a particularly simple form in 2+1 dimensional CFTs where it is frequency-independent \( \sigma(\omega) = \frac{e^2}{g_4^2} \) [9], and shows no crossover regime at \( \hbar \omega \sim k_B T \).

Putting the pieces together, we have

\[
\sigma = \frac{\chi}{4\pi T} \frac{d}{d-2} = \left[ \frac{1}{8\pi^{d/2+1}} \frac{d}{d-2} \left( \frac{4\pi}{d} \right)^{d-2} \frac{\Gamma(d/2)}{\Gamma(d)} \right] kT^{d-3},
\]

(3.21)

again exhibiting the relation between hydrodynamics, thermodynamics, and the vacuum, this time in relation to the conserved U(1) current.

4. Discussion

In this concluding section, we will comment on some corollaries of these results, some comparisons with known systems, and deviations from universality away from the classical gravity (i.e. large \( N \)) limit.

Transport coefficient ratios: The general relations, \( \eta \sim c T^{d-1} \), and \( \sigma \sim k T^{d-3} \) also imply that the ratio of viscosity to conductivity is proportional to the ratio of the central charges,

\[
\frac{\eta e^2}{\sigma T^2} = \frac{8\pi^2(d-1)(d-2)}{d^3(d+1)} \frac{c}{k} \propto \frac{g_{d+1}^2}{G_N}.
\]

(4.22)

If \( c \) and \( k \) indeed provide a suitable measure of the number of degrees of freedom in the system, it is natural to conjecture a bound of the form \( \sigma T^2 \leq \lambda_d \eta e^2 \), with some order one constant \( \lambda_d \), given that \( k \) is only sensitive to charged degrees of freedom.\(^1\) Another suggestive viewpoint is that the ratio in (4.22) is proportional to the ratio of bulk couplings, \( g_{d+1}^2/G_N \), and from this dual gravitational point of view, such an inequality between \( c \) and \( k \) is related to a version of the “weak gravity conjecture” [24] in AdS space. However, one should keep in mind that the definition of \( \sigma \) (or \( \eta \)) involves an arbitrary choice of normalization for the corresponding current and therefore any bound on conductivity will more naturally involve a quantity which is independent of the normalization, such as \( \sigma/\chi \) [11].

Nonetheless, it is interesting to note that the ratio \( c/k \) also determines when the U(1) symmetry is unstable to condensation forming a superfluid (or superconducting) phase. In \( d = 3 \), the instability occurs when [25]

\[
q^2 \geq \left( 3 + 2\Delta(\Delta - 3) \right) \frac{k}{c},
\]

(4.23)

with \( q \) (\( \Delta \)) the charge (dimension) of the operator breaking U(1).

Free-field comparisons: It is natural to ask if there are “conventional” CFTs which obey the AdS/CFT relations for \( c'/c \) and \( k'/k \). This cannot happen for free (or weakly-interacting)

\(^1\) There are, however, known counter-examples to the decrease of \( c \) and \( k \) along renormalization group trajectories in supersymmetric theories [23], but in these examples \( k \) corresponds to the \( R \)-current, and therefore \( k \propto c \) by supersymmetry.
theories when \( d \) is odd, because \( c' \) is proportional to \( \zeta(d) \), while \( c \) contains no such factors. In even dimensions, this is no longer the case and in \( d = 4 \), for a free theory with an arbitrary number of scalar, fermionic and vector degrees of freedom one finds that \( 3/8 \leq \frac{c'/c|_{\text{free}}}{c'/c|_{\text{AdS}}} \leq 9/4 \).

Importantly, we observe that free theories with \( 2n_s + n_f = 8n_v \) (including e.g. \( N = 4 \text{ SYM} \)) do satisfy the AdS relation. Thus we conclude that, at least in \( d = 4 \), the AdS relation \( c'/c \) is necessary but not sufficient for a given CFT to possess a gravity dual.

In \( d = 3 \), as noted above we do not expect free theories to exhibit the AdS relation, but we can consider an interacting example, namely the 3-dimensional \( O(N) \) model at large \( N \) with fields \( \phi^\alpha, \alpha = 1, \ldots, N \) subject to a constraint \( \phi^\alpha \phi^\alpha = 1 \). In this system, the ratio of \( c'/c|_{O(N)} \approx 0.2041 \) was computed in the large-\( N \) limit by Sachdev [26], which differs by only a few percent from the holographic answer, \( c'/c = \frac{\pi^3}{162} \approx 0.1914 \) [10]. With regard to the proposal of Klebanov and Polyakov [27] – that the large-\( N \) dual is a higher-spin gauge theory in AdS – this result amounts to a prediction for the bulk spin-two sector and implies a (small) quantitative difference with pure Einstein gravity. A similar comparison is possible for the ratio \( k'/k \), where a natural variant of the vector current studied above lies in the adjoint, \( J_{\mu}^{\alpha \beta} = \phi^{[\alpha} \partial_{\mu} \phi^{\beta]} \). Computation of the central charge at large \( N \) leads to \( k'/k|_{O(N)} \approx 0.1713 \), which differs by about 24% from the result \( k'/k = \pi/24 \approx 0.1309 \) for models with gravitational duals in AdS. On the other hand, the conductivity in the \( O(N) \) model is large, \( \sigma/\chi \) is \( O(N) \) [8], reflecting the fact that the model becomes weakly coupled at large \( N \). Therefore, the comparison with the \( O(N) \) model in \( d = 3 \) provides an example of a situation where two systems have very similar static thermodynamic properties, but vastly different transport properties.

**Deviation from universality:** We have argued that in a large class of CFTs in \( d > 2 \), there are universal relations between the thermodynamic and transport properties, and the central charges which dictate the short distance behaviour of current-current correlators. One way of defining this class of theories is that they possess dual descriptions within AdS at the level of classical gravity and Maxwell electrodynamics.

It is natural to ask about the regime of validity, or alternatively the constraints on the CFTs which may enter such universality classes. Indeed, the analysis we have performed using the AdS/CFT duality required the validity of a classical gravity approximation, and thus some kind of large-\( N \) limit. On the gravity side of the duality, universality follows from the uniqueness of the lowest dimension operator which determines the dynamics of the metric and/or the gauge field dual to the current in question, i.e. the uniqueness of the Einstein-Hilbert and Maxwell actions respectively. From this point of view, once we move to finite \( N \), it appears that a large number of higher derivative corrections will also be required, thus limiting the possibility for universal behaviour.

Examples of higher curvature corrections to the Einstein-Hilbert action have been shown to correct the universal relation \( \eta/s = 1/(4\pi) \) [30]. Here we will discuss an analogous correction to \( \sigma \), arising from a bulk higher-derivative coupling to the Weyl tensor [12],

\[
\mathcal{L} = \frac{1}{4g_s^2} (F^{\mu \nu} F_{\mu \nu} - 4\gamma C^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} + \cdots),
\]

(4.24)

This setting provides a very clean test of the universality relation (3.21) at higher order, because the geometry dual to the zero temperature state, namely pure AdS, is Weyl flat. It follows immediately that the 2-point function \( \langle J_\mu(x) J_\nu(0) \rangle \) at \( T = 0 \), and thus the central charge \( k \), are uncorrected by turning on the perturbation \( \gamma \),

\[
k(\gamma) = k(\gamma = 0).
\]

(4.25)

However, the Weyl tensor does not vanish for the AdS black hole and a straightforward
calculations shows that \[12\]

\[
\sigma = \frac{\pi LT}{\gamma^2} \left( 1 + \frac{8\gamma}{L^2} \right), \quad D = \frac{\sigma}{\chi} = \frac{1}{2\pi T} \left( 1 + \frac{16\gamma}{L^2} + \cdots \right),
\]

(4.26)

and the universal relation is violated beyond the classical gravity limit. This result is not unexpected, but it is interesting that causality \[30\] provides limits on \(\gamma\) ensuring that deviations cannot be large \[12\].

It would be interesting to see if there are other symmetries which might preserve these universality relations beyond the classical gravity limit, i.e. beyond leading order in \(N\). It would also be interesting to explore other regimes, analogous to quantum critical points in which these techniques may be useful. Recall that the primary link was the presence of just one dimensionful scale breaking conformal invariance. One can easily imagine scenarios where this scale is something other than temperature. For example, it would be interesting to consider cold systems dominated instead by the chemical potential, which should have a dual description in terms of extremal black holes in AdS.

**Acknowledgements:** I’d like to thank the organizers for their invitation to this stimulating meeting, and Pavel Kovtun and John Ward for enjoyable collaboration on the work described here, which was supported in part by NSERC, Canada.

**References**

[1] Wilson K G and Kogut J B 1974 *Phys. Rept.* 12 75
[2] See e.g. Leigh R G and Strassler M J 1995 *Nucl. Phys. B* 447 95
[3] Maldacena J M 1998 *Adv. Theor. Math. Phys.* 2 231
[4] Gubser S S et al. 1998 *Phys. Lett. B* 428 105
[5] Witten E 1998 *Adv. Theor. Math. Phys.* 2 253
[6] Hohenberg P C and Halperin B I 1977 *Rev. Mod. Phys.* 49 435
[7] For a review, see Sachdev S 2008 *Nature Physics* 4 173
[8] Sachdev S 1999 *Quantum phase transitions*, (Cambridge University Press, UK)
[9] Herzog C P et al. 2007 *Phys. Rev. D* 75 085020
[10] Kovtun P and Ritz A 2008 *Phys. Rev. Lett.* 100 171606
[11] Kovtun P and Ritz A 2008 *Phys. Rev. D* 78 066009
[12] Ritz A and Ward J 1009 *Phys. Rev. D* 79 066003
[13] Bloete H W J et al. 1986 *Phys. Rev. Lett.* 56 742
[14] Affleck I 1986 *Phys. Rev. Lett.* 56 746
[15] Kovtun P et al. 2005 *Phys. Rev. Lett.* 94 116101
[16] Buchel A 2005 *Phys. Lett. B* 609 392
[17] Damle K and Sachdev S 1997 *Phys. Rev. B* 56 8714
[18] Chamblin A et al. 1999 *Phys. Rev. D* 60 064018
[19] Freedman D Z et al. 1999 *Nucl. Phys. B* 546 96
[20] Policastro G et al. 2002 *JHEP* 0209 043
[21] Son D T and Starinets A O 2002 *JHEP* 0209 042
[22] Herzog C P 2002 *JHEP* 0212 026
[23] Anselmi D et al. 1998 *Phys. Rev. D* 57 7570
[24] Arkani-Hamed N et al. 2007 *JHEP* 0706 060
[25] Denef F and Hartnoll S A 2009 *Phys. Rev. D* 79 126008
[26] Sachdev S 1993 *Phys. Lett. B* 309 285
[27] Kabanov I R and Polyakov A M 2002 *Phys. Lett. B* 550 213
[28] Petkou A 1996 *Annals Phys.* 249 180
[29] Chubukov A V et al. 1994 *Phys. Rev. B* 49 11919
[30] Buchel A et al. 2005 *Nucl. Phys. B* 707 56; Brigante M et al. 2008 *Phys. Rev. D* 77 126006; Kats Y and Petrov P *JHEP* 0901 044