Gal($\overline{\mathbb{Q}_p}/\mathbb{Q}_p$) as a geometric fundamental group

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1 Introduction

Let $p$ be a prime number. In this article we present a theorem, suggested by Peter Scholze, which states that $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is the étale fundamental group of certain object $Z$ which is defined over an algebraically closed field. As a consequence, $p$-adic representations of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ correspond to $\mathbb{Q}_p$-local systems on $Z$.

The precise theorem involves perfectoid spaces, [Sch12]. Let $C/\mathbb{Q}_p$ be complete and algebraically closed. Let $D$ be the open unit disk centered at 1, considered as a rigid space over $C$, and given the structure of a $\mathbb{Z}_p$-module where the composition law is multiplication, and $a \in \mathbb{Z}_p$ acts by $x \mapsto x^a$. Let

$$\tilde{D} = \lim_{x \to x^p} D.$$ 

Then $\tilde{D}$ is no longer a classical rigid space, but it does exist in Huber’s category of adic spaces, and is in fact a perfectoid space. Note that $\tilde{D}$ has the structure of a $\mathbb{Q}_p$-vector space. Let $\tilde{D}^\times = \tilde{D}\setminus \{0\}$; this admits an action of $\mathbb{Q}_p^\times$.

**Theorem 1.0.1.** The category of $\mathbb{Q}_p^\times$-equivariant finite étale covers of $\tilde{D}^\times$ is equivalent to the category of finite étale $\mathbb{Q}_p$-algebras.

The object $Z$ of the first paragraph is then the quotient $Z = \tilde{D}^\times/\mathbb{Q}_p^\times$. This quotient doesn’t belong to the category of adic spaces. Rather, one has a Yoneda-style construction.

The category $\text{Perf}_C$ of perfectoid spaces over $C$ has a pro-étale topology, [Sch12], and one has a sheaf of sets $Z$ on $\text{Perf}_C$, namely the sheafification of $X \mapsto \text{Hom}(X, \tilde{D}^\times)/\mathbb{Q}_p^\times$. Thus $Z$ belongs to the category of sheaves of sets on $\text{Perf}_C$ which admit a surjective map from a representable sheaf. In this category, a morphism $\mathcal{F}' \to \mathcal{F}$ is called finite...
étale if the pull-back to any representable sheaf is finite étale in the usual sense. Then one may define the fundamental group for such objects, and Thm. 1.0.1 asserts that \( \pi^{\text{ét}}_1(\mathbb{Z}) = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \).

Theorem 1.0.1 can be generalized to a finite extension \( E/\mathbb{Q}_p \). Let \( \pi \in E \) be a uniformizer, and let \( H \) be the corresponding Lubin-Tate formal \( \mathcal{O}_E \)-module. Then \( D \) gets replaced by the generic fiber \( H^{\text{ad}}_C \); this is the unit disc centered at 0, considered as an adic space over \( C \). \( H^{\text{ad}}_C \) is endowed with the \( \mathcal{O}_E \)-module structure coming from \( H \). Form the universal cover \( \tilde{H}^{\text{ad}}_C = \varprojlim H^{\text{ad}}_C \), where the inverse limit is taken with respect to multiplication by \( \pi \) in \( H \). Then \( \tilde{H}^{\text{ad}}_C \) is an \( E \)-vector space object in the category of perfectoid spaces over \( C \). Form the quotient \( Z_E = (\tilde{H}^{\text{ad}}_C \setminus \{0\})/\mathbb{E}^\times \).

Then the main theorem (Thm. 4.0.10) states that the categories of finite étale covers of \( Z_E \) and \( \text{Spec} \, E \) are equivalent, so that \( \pi^{\text{ét}}_1(Z_E) = \text{Gal}(\mathbb{E}/E) \).

The proof hinges on a combination of two themes: the fundamental curve \( X \) of p-adic Hodge theory, due to Fargues-Fontaine, and the tilting equivalence, due to Scholze. Let us sketch the proof in the case \( E = \mathbb{Q}_p \).

Let \( C^{\phi} \) be the tilt of \( C \), a perfectoid field in characteristic \( p \). Consider the punctured open disc \( D^{\ast}_C \) (with parameter \( t \)) and its universal cover \( \tilde{D}^{\ast}_C \).

Then \( \tilde{D}^{\ast}_C \) is simultaneously a perfectoid space over two fields:

\[
\text{Spa} \, \mathbb{F}_p((t^{1/p^\infty})) \quad \xrightarrow{\alpha} \quad \text{Spa} C^{\phi} \quad \xrightarrow{\beta} \quad \text{Spa} \, \mathbb{Q}_p((t^{1/p^\infty}))
\]

Considered as a perfectoid space over \( C^{\phi} \), \( \tilde{D}^{\ast}_C \) has an obvious “un-tilt”, namely \( \tilde{D}^*_C \). The other field \( \mathbb{F}_p((t^{1/p^\infty})) \) is the tilt of \( \mathbb{Q}_p(\mu_{p^\infty}) \). It turns out that there is a perfectoid space over \( \mathbb{Q}_p(\mu_{p^\infty}) \) whose tilt is also \( \tilde{D}^*_C \), and here is where the Fargues-Fontaine curve comes in.

The construction of the Fargues-Fontaine curve \( X \) is reviewed in §2. \( X \) is an integral noetherian scheme of dimension 1 over \( \mathbb{Q}_p \), whose closed points parametrize un-tilts of \( C^{\phi} \) modulo Frobenius. For our purposes we need the adic version \( X^{\text{ad}} \), which is the quotient of another adic space \( Y^{\text{ad}} \) by
a Frobenius automorphism \( \phi \). The extension of scalars \( Y^{\text{ad}} \otimes \hat{\mathbb{Q}}_{p}(\mu_{p^\infty}) \) is a perfectoid space; by a direct calculation (Prop. 4.0.8) we show that its tilt is isomorphic to \( \tilde{D}_{C^b}^{*} \). In this isomorphism, the action of \( \text{Gal}(\mathbb{Q}_{p}(\mu_{p^\infty})/\mathbb{Q}_{p}) \cong \mathbb{Z}_{p}^{\times} \) on the field of scalars \( \hat{\mathbb{Q}}_{p}(\mu_{p^\infty}) \) corresponds to the geometric action of \( \mathbb{Z}_{p}^{\times} \) on \( \tilde{D}_{C^b}^{*} \), and the automorphism \( \phi \) corresponds (up to absolute Frobenius) to the action of \( p \) on \( \tilde{D}_{C^b}^{*} \).

Therefore under the tilting equivalence, finite étale covers of \( \tilde{D}_{C^b}^{*} \) and \( Y^{\text{ad}} \otimes \hat{\mathbb{Q}}_{p}(\mu_{p^\infty}) \) are identified. The same statement is proved in [FF11] for the algebraic curve \( X^{\text{ad}} \); we have adapted the proof for \( X^{\text{ad}} \) in Prop. 3.8.4. Thus finite étale covers of \( \tilde{D}_{C^b}^{*}/p\mathbb{Z} \) are equivalent to finite étale \( \mathbb{Q}_{p}(\mu_{p^\infty}) \)-algebras. Now we can descend to \( \mathbb{Q}_{p} \): \( \mathbb{Z}_{p}^{\times} \)-equivariant finite étale covers of \( \tilde{D}_{C^b}^{*}/p\mathbb{Z} \) are equivalent to finite étale \( \mathbb{Q}_{p} \)-algebras, which is Thm. 1.0.1.

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2 The Fargues-Fontaine curve

2.1 The rings \( B \) and \( B^{+} \)

Here we review the construction of the Fargues-Fontaine curve. The construction requires the following two inputs:

- A finite extension \( E/\mathbb{Q}_{p} \), with uniformizer \( \pi \) and residue field \( \mathbb{F}_{q}/\mathbb{F}_{p} \),
- A perfectoid field \( F \) of characteristic \( p \) containing \( \mathbb{F}_{q} \).

Note that a perfectoid field \( F \) of characteristic \( p \) is the same as a perfect field of characteristic \( p \) which is complete with respect to a nontrivial (rank 1) valuation. Let \( x \mapsto |x| \) denote one such valuation.

Let \( W(F) \) be the ring of Witt vectors of the perfect field \( F \), and let \( W_{\mathcal{O}_{E}}(F) = W(F) \otimes_{W(\mathbb{F}_{q})} \mathcal{O}_{E} \). A typical element of \( W_{\mathcal{O}_{E}}(F) \) is a series

\[
x = \sum_{n \geq -\infty} [x_{n}] \pi^{n},
\]

where \( x_{n} \in F \). Let \( B^{b} \subset W_{\mathcal{O}_{E}}(F)[1/\pi] \) denote the subalgebra defined by the condition that \( |x_{n}| \) is bounded as \( n \to \infty \). Equivalently, \( B^{b} = \)
$W_{\mathcal{O}_E}(\mathcal{O}_F)[1/\pi, 1/|\varpi|]$, where $\varpi \in F$ is any element with $0 < |\varpi| < 1$. Then $B^b$ clearly contains $B^{b,+} = W(\mathcal{O}_F) \otimes_{W(k)} \mathcal{O}_E$ as a subring.

For every $0 < \rho < 1$ we define a norm $|\cdot|_\rho$ on $B^b$ by

$$|x|_\rho = \sup_{n \in \mathbb{Z}} |x_n| \rho^n.$$ 

**Definition 2.1.1.** Let $B$ denote the Fréchet completion of $B^b$ with respect to the family of norms $|\cdot|_\rho$, $\rho \in (0, 1)$.

$B$ can be expressed as the inverse limit $\lim_{\leftarrow} I_B$ of Banach $E$-algebras, where $I$ ranges over closed subintervals of $(0, 1)$, and $B_I$ is the completion of $B^b$ with respect to the norm

$$|x|_I = \sup_{\rho \in I} |x|_\rho.$$  

(Note that $|x|_\rho$ is continuous in $\rho$ and therefore bounded on $I$.)

It will be useful to give an explicit description of $B_I$.

**Lemma 2.1.2.** Let $I \subset (0, 1)$ be a closed subinterval whose endpoints lie in the value group of $F$: $I = [\varpi_1, \varpi_2]$. Let $B^{b,0}_I$ be subring of elements $x \in B^b$ with $|x|_I \leq 1$. Then $B^{b,0}_I := B^{b,+} + \left[\frac{\varpi_1}{\varpi}, \frac{\varpi}{\varpi_2}\right]$.

**Proof.** It is easy to see that

$$\left|\frac{\varpi_1}{\varpi}\right|_I = \left|\frac{\pi}{\varpi_2}\right| = 1,$$

which proves that $B^{b,+} + \left[\frac{\varpi_1}{\varpi}, \frac{\varpi}{\varpi_2}\right] \subset B^{b,0}_I$.

Conversely, suppose $x = \sum_{n \gg -\infty} [x_n] \pi^n \in B^{b,0}_I$. Then $|x_n| \leq |\varpi_1|^{-n}$ for $n < 0$ and $|x_n| \leq |\varpi_2|^{-n}$ for $n \geq 0$. This shows that the “tail term” $\sum_{n < 0} [x_n] \pi^n$ lies in $B^{b,+}[\varpi_1/\pi]$. It also shows that for $n \geq 0$, each term $[x_n] \pi^n$ lies in $B^{b,+}[\pi/\varpi_2]$. Since $x \in B^b$, there exists $C > 0$ with $|x_n| \leq C$ for all $n$. Let $N$ be large enough so that $|\varpi_2|^N C \leq 1$; then for $n \geq N$ we have $|\varpi_2^N x_n| \leq 1$. Then the sum of the terms of $x$ of index $\geq N$ are

$$\pi^N (\lfloor x_N \rfloor + [x_{N+1}] \pi + \ldots) = \frac{\pi^N}{\varpi_2^N} ((\varpi_2^N x_N) + [\varpi_2^N x_{N+1}] \pi + \ldots),$$

which lies in $B^{b,+} + \left[\frac{\varpi_1}{\varpi}, \frac{\varpi}{\varpi_2}\right]$.

\qed
Lemma 2.1.3. If $B^I_1$ is the closed unit ball in $B_1$, then $B^I_1$ is the $\pi$-adic completion of $B^{I,0}_1$.

Proof. $B_I$ is the completion of $B^b$ with respect to the norm $||_I$. This norm induces the $\pi$-adic topology on $B^{I,0}_1$, so that the $\pi$-adic completion of $B^{I,0}_1$ is the same as the closed unit ball in $B_I$. □

The $q$th power Frobenius automorphism of $F$ induces a map $\phi: B^{b,+} \to B^{b,+}$. For $\rho \in (0,1)$ we have $|x^\phi|_{\rho} = |x|_{\rho^{1/q}}^q$. As a result, $\phi$ induces an endomorphism of $B$. We have $B^{\phi=1} = E$. (See Prop. 7.1 of [FF11].) We put

$$P = \bigoplus_{d \geq 0} B^{\phi=p^d},$$

a graded $E$-algebra.

Definition 2.1.4. Let $X_{E,F} = \text{Proj } P$.

Theorem 2.1.5, [FF11], Théorème 10.2

1. $X_{F,E}$ is an integral noetherian scheme which is regular of dimension 1.

2. $H^0(X_{F,E}, \mathcal{O}_{X_{F,E}}) = E$. ($E$ is the field of definition of $X_{F,E}$)

3. For a finite extension $E'/E$ there is a canonical isomorphism $X_{F,E'} \cong X_{F,E} \otimes_E E'$.

3 The adic curve $X_{F,E}^{ad}$

3.1 Generalities on formal schemes and adic spaces

The category of adic spaces is introduced in [Hub94]. In brief, an adic space is a topological space $X$ equipped with a sheaf of topological rings $\mathcal{O}_X$, which is locally isomorphic to Spa($R, R^+$). Here $(R, R^+)$ is an “affinoid ring”, $R^+ \subset R$ is an open and integrally closed subring, and Spa($R, R^+$) is the space of continuous valuations $||$ on $R$ with $|R^+| \leq 1$.

Definition 3.1.1. A topological ring $R$ is $f$-adic if it contains an open subring $R_0$ whose topology is generated by a finitely generated ideal $I \subset R_0$. An affinoid ring is a pair $(R, R^+)$, where $R$ is $f$-adic and $R^+ \subset R$ is an open and integrally closed subring consisting of power-bounded elements. A morphism of affinoid algebras $(R, R^+) \to (S, S^+)$ is a continuous homomorphism $R \to S$ which sends $R^+ \to S^+$.

$R$ is a Tate ring if it contains a topologically nilpotent unit.
Note that if \((R, R^+)\) and \(R\) is Tate, say with topologically nilpotent \(\pi \in R\), then \(R = R^+[1/\pi]\) and \(\pi R \subset R\) is open. Note also that if \((R, R^+)\) is an affinoid algebra over \((E, \mathcal{O}_E)\), then \(R\) is Tate.

It is important to note that a given affinoid ring \((R, R^+)\) does not necessarily give rise to an adic space \(\text{Spa}(R, R^+)\), because the structure sheaf on \(\text{Spa}(R, R^+)\) is not necessarily a sheaf. Let us say that \((R, R^+)\) is sheafy if the structure presheaf on \(\text{Spa}(R, R^+)\) is a sheaf. Huber shows \((R, R^+)\) is a sheaf when \(R\) is "strongly noetherian", meaning that \(R\langle X_1, \ldots, X_n \rangle\) is noetherian for all \(n\). In §2 of [SW13] we constructed a larger category of "general" adic spaces, whose objects are sheaves on the category of complete affinoid rings (this category can be given the structure of a site in an obvious way). If \((R, R^+)\) is a (not necessarily sheafy) affinoid ring, then \(\text{Spa}(R, R^+)\) belongs to this larger category. If \(X\) is an adic space in the general sense, let us call \(X\) an honest adic space if it belongs to the category of adic spaces in the sense of Huber; i.e. if it is locally \(\text{Spa}(R, R^+)\) for a sheafy \((R, R^+)\).

Now suppose \(R\) is an \(\mathcal{O}_E\)-algebra which is complete with respect to the topology induced by a finitely generated ideal \(I \subset R\) which contains \(\pi\). Then \(\text{Spf} R\) is a formal scheme over \(\text{Spf} \mathcal{O}_E\), and \((R, R)\) is an affinoid ring. One can form the (general) adic space \(\text{Spa}(R, R)\), which is fibered over the two-point space \(\text{Spa}(\mathcal{O}_E, \mathcal{O}_E)\).

By [SW13], Prop. 2.2.1, \(\text{Spf} R \mapsto \text{Spa}(R, R)\) extends to a functor \(M \mapsto M^\text{ad}\) from the category of formal schemes over \(\text{Spf} \mathcal{O}_E\) locally admitting a finitely generated ideal of definition, to the category of (general) adic spaces over \(\text{Spa}(\mathcal{O}_E, \mathcal{O}_E)\).

**Definition 3.1.2.** Let \(M\) be a formal scheme over \(\text{Spf} \mathcal{O}_E\) which locally admits a finitely generated ideal of definition. The adic generic fiber \(M^\text{ad}_\eta\) is the fiber of \(M^\text{ad}\) over the generic point \(\eta = \text{Spa}(E, \mathcal{O}_E)\) of \(\text{Spa}(\mathcal{O}_E, \mathcal{O}_E)\). We will also notate this as \(M^\text{ad}_E\).

**Lemma 3.1.3.** Let \(R\) be a flat \(\mathcal{O}_E\)-algebra which is complete with respect to the topology induced by a finitely generated ideal \(I\) containing \(\pi\). Let \(f_1, \ldots, f_r, \pi\) be generators for \(I\). For \(n \geq 1\), let \(S_n = R[\overline{f_1^n/\pi}, \ldots, \overline{f_r^n/\pi}]^\wedge\) (\(\pi\)-adic completion). Let \(R_n = S_n[1/\pi]\), and let \(R_{n+1}^+\) be the integral closure of \(S_n\) in \(R_n\). There are obvious inclusions \(R_{n+1} \hookrightarrow R_n\) and \(R_{n+1}^+ \hookrightarrow R_n^+\). Then

\[
M^\text{ad}_E = \varprojlim_n \text{Spa}(R_n, R_n^+).
\]

**Proof.** This amounts to showing that whenever \((T, T^+)\) is a complete affinoid
algebra over \((E, \mathcal{O}_E)\), then

\[
M_E^{ad}(T, T^+) = \lim_{\longrightarrow n} \text{Hom}((R_n, R_n^+/\pi_n),(T, T^+)).
\]

An element of the left hand side is a continuous \(\mathcal{O}_E\)-linear homomorphism \(g: R \to T^+\); we need to produce a corresponding homomorphism \(R_n^+/\pi_n \to T^+\). We have that \(g(f_i) \in T^+\) is topologically nilpotent for \(i = 1, \ldots, r\), since \(f_i\) is. Since \(\pi T^+\) is open in \(T\), there exists \(n \geq 0\) so that \(g(f_i)^n \in \pi T^+, i = 1, \ldots, r\). Thus \(g\) extends to a map \(R^+[f_1^n/\pi, \ldots, f_r^n/\pi] \to T^+\). Passing to the integral closure of the completion gives a map \(R_n \to T^+\) as required.

**Example 3.1.4.** The adic generic fiber of \(R = \text{Spf} \mathcal{O}_E[[t]]\) is the rigid open disc over \(E\).

### 3.2 Formal schemes with perfectoid generic fiber

In this section \(K\) is a perfectoid field of characteristic 0, and \(\varpi \in K\) satisfies 0 < \(|\varpi| \leq |p|\). Assume that \(\varpi = \varpi^\flat\) for some \(\varpi^\flat \in \mathcal{O}_K^\flat\), so that \(\varpi^{1/p^n} \in \mathcal{O}_K\) for all \(n \geq 1\).

**Definition 3.2.1.** Let \(S\) be a ring in characteristic \(p\). \(S\) is semiperfect if the Frobenius map \(S \to S\) is surjective. If \(S\) is semiperfect, let \(S^\flat = \lim_{\leftarrow x \to x^p} S\), a perfect topological ring.

If \(R\) is a topological \(\mathcal{O}_K\)-algebra with \(R/\varpi\) semiperfect, then write \(R^\flat = (R/\varpi)^\flat\), a perfect topological \(\mathcal{O}_K^\flat\)-algebra.

If \(R\) is complete with respect to a finitely generated ideal of definition, then so is \(R^\flat\).

**Proposition 3.2.2.** Let \(R\) be an \(\mathcal{O}_K\)-algebra which is complete with respect to a finitely generated ideal of definition. Assume that \(R/\pi\) is a semiperfect ring. Then \((\text{Spf} R)^{ad\flat}_K\) and \((\text{Spf} R^\flat)^{ad\flat}_K\) are a perfectoid spaces over \(K\) and \(K^\flat\) respectively, so that in particular they are honest adic spaces. There is a natural isomorphism of perfectoid spaces over \(K^\flat\):

\[
(\text{Spf} R)^{ad\flat}_K \cong (\text{Spf} R^\flat)^{ad\flat}_K.
\]

**Proof.** First we show that \((\text{Spf} R^\flat)^{ad\flat}_K\) is a perfectoid space over \(K^\flat\). Let \(f_1, \ldots, f_r\) be generators for an ideal of definition of \(R^\flat\). Then \((\text{Spf} R^\flat)^{ad\flat}_{\eta^\flat}\)
is the direct limit of subspaces $\text{Spa}(R^+_n, R^+_n)$ as in Lemma 3.1.3. Here $R^0_n = R^0 \langle f_i^n / \varpi^b \rangle [1/\varpi^b]$. Let 

$$S^{\hat{o},o}_n = R^0 \left\langle \left( \frac{f_1^n}{\varpi^b} \right)^{1/p^\infty}, \cdots, \left( \frac{f_r^n}{\varpi^b} \right)^{1/p^\infty} \right\rangle \subset R^0_n$$

Then $S^{\hat{o},o}_n$ is a $\varpi^b$-adically complete flat $\mathcal{O}_{K^s}$-algebra. We claim that $S^{\hat{a},o}_n$ is a $\varpi^b$-adically complete flat $\mathcal{O}_{K^s}$-algebra, which is to say that the Frobenius map $S^{\hat{a},o}_n / \varpi^1/p \to S^{\hat{a},o}_n / \varpi$ is an almost isomorphism. For this we refer to [Sch12], proof of Lemma 6.4(i) (case of characteristic $p$). This shows that $R^0_n = S^{\hat{a},o}_n [1/\varpi^b]$ is a perfectoid $K^s$-algebra, and thus $\text{Spf}(R^0_n, R^+_n)$ is a perfectoid affinoid, and in particular it is an honest adic space.

To conclude that $(\text{Spf} R^0_n)^{\text{ad}}$ is a perfectoid space, one needs to show that its structure presheaf is a sheaf. (I thank Kevin Buzzard for pointing out this subtlety.) But since $(\text{Spf} R^0_n)^{\text{ad}}$ is the direct limit of the $\text{Spa}(R_n^0, R^+_n)$, its structure presheaf is the inverse limit of the pushforward of the structure presheaves of the $\text{Spa}(R_n^0, R^+_n)$, which are all sheaves. Now we can use the fact that an arbitrary inverse limit of sheaves is again a sheaf.

We now turn to characteristic 0. Recall the sharp map $f \mapsto f^2$, which is a map of multiplicative monoids $R^0 \to R$. The elements $f^2_1, \ldots, f^2_r$ generate an ideal of definition of $R$. Then $(\text{Spf} R^0)^{\text{ad}}$ is the direct limit of subspaces $\text{Spa}(R_n, R^+_n)$, where $R_n = R \langle f^n / \varpi \rangle [1/\varpi]$. The proof of [Sch12], Lemma 6.4(i) (case of characteristic 0) shows that $\text{Spa}(R_n^0, R^+_n)$ is a perfectoid affinoid, and Lemma 6.4(iii) shows that the tilt of $\text{Spa}(R_n, R^+_n)$ is $\text{Spa}(R_n^0, R^+_n)$.

### 3.3 The adic spaces $Y_{F,E}^{\text{ad}}$ and $X_{F,E}^{\text{ad}}$

Once again, $E/\mathbb{Q}_p$ is a finite extension.

A scheme $X/E$ of finite type admits a canonical analytification $X^{\text{ad}}$: this is a rigid space together with a morphism $X \to X^{\text{ad}}$ of locally ringed spaces satisfying the appropriate universal property. GAGA holds as well: If $X$ is a projective variety, then the categories of coherent sheaves on $X$ and $X^{\text{ad}}$ are equivalent.

The Fargues-Fontaine curve $X = X_{F,E}$, however, is not of finite type, and it is not obvious that such an analytification exists. However, in [Far13] there appears an adic space $X^{\text{ad}}$ defined over $E$, such that $X$ and $X^{\text{ad}}$ satisfy a suitable formulation of GAGA.
**Definition 3.3.1.** Give $B^{b,+} = W_{O_F}(O_F)$ the $I$-adic topology, where $I = ([\varpi], \pi)$ and $\varpi \in O_F$ is an element with $0 < |\varpi| < 1$. Note that $B^{b,+}$ is $I$-adically complete. Let

$$Y^{\text{ad}} = (\text{Spf } B^{b,+})_E^{\text{ad}} \setminus \{0\},$$

where “0” refers to the valuation on $B^{b,+} = W_{O_F}(O_F)$ pulled back from the $\pi$-adic valuation on $W_{O_F}(k_E)$.

**Proposition 3.3.2.** As (general) adic spaces we have

$$Y^{\text{ad}} = \lim_{\rightarrow} I \text{Spa}(B_I, B_I^\circ).$$

This shows that our definition of $Y^{\text{ad}}$ agrees with the definition in [Far13], Définition 2.5.

**Proof.** By Prop. 3.1.3 we have

$$(\text{Spf } B^{b,+})_E^{\text{ad}} = \lim_{\rightarrow} \varpi_1 \text{Spa}(R_{\varpi_1}, R_{\varpi_1}^+),$$

where $\varpi_1$ runs over elements of $O_F$ with $0 < |\varpi_1| < 1$, $R_{\varpi_1}^+$ is the $\pi$-adic completion of $B^{b,+}\left[\frac{[\varpi_1]}{\pi}\right]$, and $R_{\varpi_1} = R_{\varpi_1}^+[1/\pi]$.

By definition, $Y^{\text{ad}}$ is the complement in $(\text{Spf } B^{b,+})_E^{\text{ad}}$ of the single valuation pulled back from the $\pi$-adic valuation on $W_{O_F}(k)$. Thus if $x \in Y^{\text{ad}}$ we have $|\varpi(x)| \neq 0$ for all $\varpi \in O_F \setminus \{0\}$. Thus there exists $\varpi_2 \in O_F$ with $|\varpi_2(x)| \geq |\pi(x)|$. On the other hand we have $x \in \text{Spa}(R_{\varpi_1}, R_{\varpi_1}^+)$ for some $\varpi_1$. Thus $x$ belongs to the rational subset $\text{Spa}(B_I, B_I^\circ) \subset \text{Spa}(R_{\varpi_1}, R_{\varpi_1}^+)$, where $I = ([\varpi_1], |\varpi_2|)$. This shows that the $\text{Spa}(B_I, B_I^\circ)$ cover $Y^{\text{ad}}$. 

The following conjecture is due to Fargues.

**Conjecture 3.3.3.** Let $I \subset (0,1)$ be a closed interval. The Banach algebra $B_I$ is strongly noetherian. That is, for every $n \geq 1$, $B_I \langle T_1, \ldots, T_n \rangle$ is noetherian.

Despite not being able to prove Conj. 3.3.3, we have the following proposition (Théorème 2.1 of [Far13]), which is proved by extending scalars to a perfectoid field and appealing to results of [Sch12].

**Proposition 3.3.4.** $Y^{\text{ad}}$ is an honest adic space.

**Definition 3.3.5.** The adic space $X^{\text{ad}}$ is the quotient of $Y^{\text{ad}}$ by the automorphism $\phi$. 

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3.4 The adic Fargues-Fontaine curve in characteristic $p$

The entire story of the Fargues-Fontaine curve can be retold when $E$ is replaced by a local field of residue field $\mathbb{F}_q$. The construction is very similar, except that $B^{h,+} = W(O_F) \otimes W(\mathbb{F}_q) O_E$ must be replaced by $O_F \otimes_{\mathbb{F}_q} O_E$. One arrives at a curve $X_{F,E}$ defined over $E$ satisfying the same properties as Thm. 2.1.5.

One also gets spaces $X_{F,E}^{\text{ad}}$ and $Y_{F,E}^{\text{ad}}$. Namely,

$$Y_{F,E}^{\text{ad}} = \text{Spf} \left( O_F \otimes_{\mathbb{F}_q} O_E \right)^{\text{ad}} \setminus \{0\},$$

where "0" refers to the pullback of the valuation on $O_F$ through the quotient map $O_F \otimes_{\mathbb{F}_q} O_E \to O_F$. As before, $X_{F,E}^{\text{ad}}$ is defined as the quotient of $Y_{F,E}^{\text{ad}}$ by the automorphism $\phi$ coming from the Frobenius on $F$.

Since $E \cong \mathbb{F}_q((t))$, we have

$$Y_{F,E}^{\text{ad}} = \text{Spf} \left( O_F[[t]]^{\text{ad}} \setminus \{0\} \right).$$

This is nothing but the punctured rigid open disc $D^*_F$. We have the quotient $X_{F,E}^{\text{ad}} = Y_{F,E}^{\text{ad}} / \phi \mathbb{Z}$. Note that since $\phi$ does not act $F$-linearly, $X_{F,E}^{\text{ad}}$ does not make sense as a rigid space over $F$.

3.5 Tilts

Suppose once again that $E$ has characteristic 0. Let $K$ be a perfectoid field containing $E$. Let $Y_{F,E}^{\text{ad}} \otimes K$ be the strong completion of the base change to $K$ of $Y^{\text{ad}}$. Then

$$Y_{F,E}^{\text{ad}} \otimes K = \text{Spf} \left( W_{O_E}(O_F) \otimes_{W(\mathbb{F}_q)} O_K \right)^{\text{ad}} \setminus \{0\}. $$

Note that $(W_{O_E}(O_F) \otimes_{W(\mathbb{F}_q)} O_K)/\pi = O_F \otimes_{\mathbb{F}_q} O_K / \pi$ is semiperfect, and $(W_{O_E}(O_F) \otimes_{W(\mathbb{F}_q)} O_K)^{\flat} = O_F \otimes_{\mathbb{F}_q} O_K^{\flat}$. By Prop. 3.2.2, $Y_{F,E}^{\text{ad}} \otimes K$ is a perfectoid space, and

$$(Y_{F,E}^{\text{ad}} \otimes K)^{\flat} \cong \text{Spf} \left( O_F \otimes_{\mathbb{F}_q} O_K^{\flat} \right)^{\text{ad}} \setminus \{0\}. \quad (3.5.1)$$

As a special case, let $E_n$ be the field obtained by adjoining the $\pi^n$-torsion in a Lubin-Tate formal group over $E$, and let $E_\infty = \bigcup_{n \geq 1} E_n$. Then $E_\infty$ is a perfectoid field. Let $L(E)$ be the imperfect field of norms for the extension $E_\infty / E$. As a multiplicative monoid we have

$$L(E) = \varprojlim_{\pi} E_n.$$
where the inverse limit is taken with respect to the norm maps $E_{n+1} \to E_n$. $L(E) \cong \mathbb{F}_q((t))$ is a local field, and $\tilde{E}_\infty \cong \mathbb{F}_q((t^{1/q^\infty}))$ is the completed perfection of $L(E)$.

The following proposition follows immediately from Eq. (3.5.1).

**Proposition 3.5.1.**

$$(Y_{F,E}^{\text{ad}} \hat{\otimes} \hat{E}_\infty)^b \cong Y_{F,L(E)}^{\text{ad}} \hat{\otimes} \hat{E}_\infty^b.$$ 

This is Thm. 2.7(2) of [Far13].

### 3.6 Classification of vector bundles on $X_{F,E}$

In this section we review the results of [FF11] concerning the classification of vector bundles on $X_{F,E}$.

Recall that $X_{F,E} = \text{Proj } P$, where $P$ is the graded ring $\bigoplus_{d \geq 0} (B^+)^{\phi=d}$. For $d \in \mathbb{Z}$, let $P[d]$ be the graded $P$-module obtained from $P$ by shifting degrees by $d$, and let $O_{X_{E}}(d)$ be the corresponding line bundle on $X_{E}$. For $h \geq 1$, let $E_h/E$ be the unramified extension of degree $h$, let $\pi_h : X_{F,E_h} = X_{F,E} \otimes E_h \to X_{F,E}$ be the projection. If $(d,h) = 1$, define $O_{X_{F,E_h}}(d/h) = \pi_h^* O_{X_{F,E_h}}(d)$, a vector bundle on $O_{X_{F,E}}$ of rank $h$. One thus obtains a vector bundle $O_{X_{F,E}}(\lambda)$ for any $\lambda \in \mathbb{Q}$, which satisfy $O_{X_{F,E}}(\lambda) \otimes O_{X_{F,E}}(\lambda') \cong O_{X_{F,E}}(\lambda + \lambda')$.

**Proposition 3.6.1.** Let $\lambda \in \mathbb{Q}$. Then $H^0(\lambda) \neq 0$ if and only if $\lambda \geq 0$.

**Theorem 3.6.2.** Every vector bundle $\mathcal{F}$ on $X_{F,E}$ is isomorphic to one of the form $\bigoplus_{i=1}^s O_{X_{F,E}}(\lambda_i)$, with $\lambda_i \in \mathbb{Q}$. Furthermore, $\mathcal{F}$ determines the $\lambda_i$ up to permutation.

### 3.7 Classification of vector bundles on $X_{F,E}^{\text{ad}}$

In the absence of the noetherian condition of Conj. 3.3.3, one does not have a good theory of coherent sheaves on $X_{F,E}^{\text{ad}}$ when $E$ has characteristic 0. This frustrates attempts to prove an analogue of Thm. 3.6.2 for $X_{F,E}^{\text{ad}}$. Nonetheless, in [Far13], Fargues gives an ad hoc notion of vector bundle on $X_{F,E}^{\text{ad}}$, and proves a GAGA theorem relating vector bundles on $X_{F,E}$ to those on $X_{F,E}^{\text{ad}}$.

When $E$ has characteristic $p$, however, $Y_{F,E}^{\text{ad}}$ is isomorphic to the punctured disc $D^* = (\text{Spf } O_F[t])^{\mathbb{P}^1 \setminus \{0\}}$ over $F$. In particular it is a rigid space, which does have a well-behaved theory of coherent sheaves. In this case we
are in the setting of [HP04], which classifies $\phi$-equivariant vector bundles $\mathcal{F}$ on the punctured disc $D^* = Y^\text{ad}_{F,E}$. In [HP04], these are called $\sigma$-bundles, and they admit a Dieudonné-Manin classification along the lines of Thm. 3.6.2. A $\sigma$-bundle is one and the same thing as a vector bundle on the quotient $Y^\text{ad}_{F,E}/\phi\mathbb{Z} = X^\text{ad}_{F,E}$. One gets a vector bundle $\mathcal{O}(\lambda)$ on $X^\text{ad}_{F,E}$ for every $\lambda \in \mathbb{Q}$.

For the convenience of the reader we state the main definitions and constructions of [HP04]. A vector bundle on $D^*$ is by definition a locally free coherent sheaf of $\mathcal{O}_{D^*}$-modules. The global sections functor is an equivalence between the category of vector bundles on $D^*$ and the category of finitely generated projective modules over $\mathcal{O}_{D^*}$. The Frobenius automorphism $x \mapsto x^q$ on $F$ induces an automorphism $\sigma : D^* \to D^*$ (the arithmetic Frobenius). Note that the $F$-points of $D^*$ are $\{x \in F | 0 < |x| < 1\}$, and on this set $\sigma$ acts as $x \mapsto x^{q-1}$.

**Definition 3.7.1.** A $\sigma$-bundle on $D^*$ is a pair $(\mathcal{F}, \tau_{\mathcal{F}})$, where $\mathcal{F}$ is a vector bundle on $D^*$ and $\tau_{\mathcal{F}}$ is an isomorphism $\sigma^* \mathcal{F} \cong \mathcal{F}$. If $(\mathcal{F}, \tau_{\mathcal{F}})$ is a $\sigma$-bundle, then a global section of $\mathcal{F}$ is a global section of $\mathcal{F}$ which is invariant under $\tau_{\mathcal{F}}$.

We will refer to the pair $(\mathcal{F}, \tau_{\mathcal{F}})$ simply as $\mathcal{F}$. Then the space of global sections of $\mathcal{F}$ will be denoted $H^0(\mathcal{F})$.

The trivial vector bundle $\mathcal{O} = \mathcal{O}_{D^*}$ is a $\sigma$-bundle whose global sections $H^0(\mathcal{O})$ are the ring of power series $\sum_{n \in \mathbb{Z}} a_n t^n$ which converge on $D^*$ and which satisfy $a_n^q = a_n$ (thus $a_n \in F_q$). This ring is easily seen to be the field $\mathbb{F}_q((t))$ (see [HP04], Prop. 2.3).

Let $n \in \mathbb{Z}$. The twisting sheaf $\mathcal{O}(n)$ is a $\sigma$-bundle with underlying sheaf $\mathcal{O}$. The isomorphism $\tau_{\mathcal{O}(n)}$ is defined as the composition of $\tau_{\mathcal{O}} : \sigma^* \mathcal{O}_{D^*} \to \mathcal{O}_{D^*}$ followed by multiplication by $t^{-n}$.

For $r \geq 1$, let $[r]$ denote the morphism $[r] : D^* \to D^*, t \mapsto t^r$. Then for an integer $d$ relatively prime to $r$ we set $\mathcal{F}_{d,r} = [r]\mathcal{O}(d)$, together with the induced isomorphism $\tau_{\mathcal{F}_{d,r}} = [r]\tau_{\mathcal{O}(d)}$. Then $\mathcal{F}_{d,r}$ is a $\sigma$-bundle of rank $r$.

In keeping with the notation of [FP11], let us write $\mathcal{O}(d/r) = \mathcal{F}_{d,r}$ whenever $d$ and $r$ are relatively prime integers with $r \geq 1$. Then $\mathcal{O}(\lambda)$ makes sense for any $\lambda \in \mathbb{Q}$. We have $\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda') \cong \mathcal{O}(\lambda + \lambda')$, and $\mathcal{O}(\lambda)^* \cong \mathcal{O}(-\lambda)$.
Proposition 3.7.2. [HP04], Prop. 8.4 Let $\lambda \in \mathbb{Q}$. Then $H^0(\lambda) \neq 0$ if and only if $\lambda > 0$.

Hartl and Pink give a complete description of the category of $\sigma$-bundles. In particular we have the following classification theorem of Dieudonné-Manin type.

Theorem 3.7.3. [HP04], Thm. 11.1 and Cor. 11.8 Every $\sigma$-bundle $\mathcal{F}$ is isomorphic to one of the form $\bigoplus_{i=1}^{s} O(\lambda_i)$, with $\lambda_i \in \mathbb{Q}$. Furthermore, $\mathcal{F}$ determines the $\lambda_i$ up to permutation.

Since $Y_{F,E}^{ad} \cong D^*$, with $\phi$ corresponding to $\sigma$, vector bundles on $X_{F,E}^{ad} = Y_{F,E}^{ad}/\phi^*$ correspond to $\sigma$-bundles, and we have a similar classification of them.

3.8 $X_{F,E}^{ad}$ is geometrically simply connected

In [FT11], Thm. 18.1, it is shown that the scheme $X = X_{F,E}$ is geometrically connected, which is to say that every finite étale cover of $X_{F,E}$ is split. We need a similar result for the adic curve $X_{F,E}^{ad}$:

Theorem 3.8.1. $E' \mapsto X_{F,E}^{ad} \otimes E'$ is an equivalence between the category of finite étale $E$-algebras and the category of finite étale covers of $X_{F,E}^{ad}$.

Unfortunately, Thm. 3.8.1 cannot be deduced directly from the corresponding theorem about the scheme $X$. Following a suggestion of Fargues, we can still mimic the proof of the simple-connectedness of $X$ in this adic context, by first translating the problem into characteristic $p$, and applying the Hartl-Pink classification of $\sigma$-bundles (Thm. 3.7.3).

Lemma 3.8.2. Let $f : Y \to X$ be a finite étale morphism of adic spaces of degree $d$. Then $\mathcal{F} = f_* \mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module, and

$$
\left( \bigwedge^d \mathcal{F} \right)^{\otimes 2} \cong \mathcal{O}_X.
$$

Proof. If $A$ is a ring, and if $B$ is a finite étale $A$-algebra, then $B$ is flat and of finite presentation, hence locally free. Furthermore, the trace map

$$
\text{tr}_{B/A} : B \otimes B \to A
$$

is perfect, so that $B$ is self-dual as an $A$-module. Globalizing, we get that $\mathcal{F}$ is locally free and $\mathcal{F} \cong \mathcal{F}^* = \text{Hom}(\mathcal{F}, \mathcal{O}_X)$. Taking top exterior powers shows that $\bigwedge^d \mathcal{F} \cong \bigwedge^d \mathcal{F}^* = \left( \bigwedge^d \mathcal{F} \right)^*$, so that the tensor square of $\bigwedge^d \mathcal{F}$ is trivial. \qed
The same lemma appears in [FF11], Prop. 4.7.

Let $H_E$ be the Lubin-Tate formal $\mathcal{O}_E$-module corresponding to the uniformizer $\pi$, so that $[\pi]H_E(T) \equiv T^q \pmod{\pi}$. Let $K$ be the completion of the field obtained by adjoining the torsion points of $H_E$ to $E$. Then $K$ is a perfectoid field and $K^b = \mathbb{F}_q((t^{1/q^\infty}))$.

**Proposition 3.8.3.** Suppose $E$ has characteristic $p$. Then $X_{F,E}^{ad}$ is geometrically simply connected. In other words, $E' \mapsto X_{F,E}^{ad} \otimes E'$ is an equivalence between the category of finite étale $E$-algebras and finite étale covers of $X_{F,E}^{ad}$.

**Proof.** It suffices to show that if $f: Y \to X_{F,E}^{ad}$ is a finite étale cover of degree $n$ with $Y$ geometrically irreducible, then $n = 1$. Given such a cover, let $\mathcal{F} = f_* \mathcal{O}_Y$. This is a sheaf of $\mathcal{O}_{X_{F,E}^{ad}}$-algebras, so we have a multiplication morphism $\mu: \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$.

Consider $\mathcal{F}$ as a $\sigma$-bundle. By Thm. 3.7.3, $\mathcal{F} \cong \bigoplus_{i=1}^n \mathcal{O}(\lambda_i)$, for a collection of slopes $\lambda_i \in \mathbb{Q}$ (possibly with multiplicity). Assume that $\lambda_1 \geq \cdots \geq \lambda_n$.

The proof now follows that of [FF11], Théorème 18.1. We claim that $\lambda_1 \leq 0$. Assume otherwise, so that $\lambda_1 > 0$. After pulling back $\mathcal{F}$ through some $[d]: D^* \to D^*$, we may assume that $\lambda_i \in \mathbb{Z}$ for $i = 1, \ldots, s$ (see [HP04], Prop. 7.1(a)). The restriction of $\mu$ to $\mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_1)$ is the direct sum of morphisms

$$
\mu_{1,1,k}: \mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_1) \to \mathcal{O}(\lambda_k)
$$

for $k = 1, \ldots, n$. The morphism $\mu_{1,1,k}$ is tantamount to a global section of

$$
\text{Hom}(\mathcal{O}(\lambda_1)^{\otimes 2}, \mathcal{O}(\lambda_k)) = \mathcal{O}(\lambda_k - 2\lambda_1).
$$

But since $\lambda_k - 2\lambda_1 < 0$, Prop. 3.7.2 shows that $H^0(\mathcal{O}(\lambda_k - 2\lambda_1)) = 0$. Thus the multiplication map $\mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_1) \to 0$ is 0. This means that $H^0(Y, \mathcal{O}_Y)$ contains zero divisors, which is a contradiction because $Y$ is irreducible.

Thus $\lambda_1 \leq 0$, and thus $\lambda_i \leq 0$ for all $i$. By Lemma 3.8.2, $(\bigwedge^n \mathcal{F})^{\otimes 2} \cong O_{D^*}$, from which we deduce $\sum_{i=1}^n \lambda_i = 0$. This shows that $\lambda_i = 0$ for all $i$, and therefore $\mathcal{F} \cong O_X$. We find that $E' = H^0(Y, \mathcal{O}_Y)$ is an étale $E$-algebra of degree $n$. Since $Y$ is geometrically irreducible, $E'$ must be a field for every separable field extension $E''/E$, which implies that $E' = E$ and $n = 1$.

**Proposition 3.8.4.** Suppose $E$ has characteristic 0. Then $X_{F,E}^{ad}$ is geometrically simply connected.
Proof. Let $C$ be a complete algebraically closed field containing $E$. We want to show that $X_{F,E}^{\text{ad}} \otimes C$ admits no nontrivial finite étale covers. Such covers are equivalent to covers of its tilt, which is

$$(X_{F,E}^{\text{ad}} \otimes C)^\flat \cong X_{F,L(E)}^{\text{ad}} \otimes C^\flat.$$ 

The latter is simply connected, by Prop. 3.8.3 (note that $C^\flat$ is algebraically closed), which shows there are no nontrivial covers.

4 Proof of the main theorem

As before, $H_E$ is the Lubin-Tate formal $\mathcal{O}_E$-module attached to the uniformizer $\pi$. Let us recall the construction of $H_E$: choose a power series

$$f(T) \in T \mathcal{O}_E[T]$$

with $f(T) \equiv T^q \pmod{\pi}$. Then $H_E$ is the unique formal $\mathcal{O}_E$-module satisfying $[\pi]_{H_E}(T) = f(T)$. Let $t = t_1, t_2, \ldots$ be a compatible family of roots of $f(T), f(f(T)), \ldots$, and let $E_n = E(t_n)$. For each $n \geq 1$, $H_E[\pi^n]$ is a free $(\mathcal{O}_E/\pi^n)$-module of rank 1, and the action of Galois induces an isomorphism

$$\text{Gal}(E_n/E) \cong (\mathcal{O}_E/\pi^n)^\times.$$

Let $H_{E,0} = H_E \otimes_{\mathcal{O}_E} F_q$.

Lemma 4.0.5. For all $n \geq 2$ we have an isomorphism $H_{E,0}[\pi^{n-1}] \cong \text{Spec} \mathcal{O}_{E_n}/t$. The inclusion $H_{E,0}[\pi^{n-1}] \rightarrow H_{E,0}[\pi^n]$ corresponds to the $q$th power Frobenius map $\text{Frob}_q: \mathcal{O}_{E_{n+1}}/t \rightarrow \mathcal{O}_{E_n}/t$.

Proof. We have $H_{E,0}[\pi^{n-1}] = \text{Spec} F_q[T]/f^{(n-1)}(T) = \text{Spec} F_q[T]/T^{q^{n-1}}$. Meanwhile $\mathcal{O}_{E_n} = \mathcal{O}_{E_1}[T]/(f^{(n-1)}(T) - t)$, so that $\mathcal{O}_{E_n}/t = F_q[T]/T^{q^{n-1}}$. Thus $H_{E,0}[\pi^{n-1}] \cong \text{Spec} \mathcal{O}_{E_n}/t$. The second claim in the lemma follows from $[\pi]_{H_{E,0}}(T) = T^q$.

Let $\tilde{H}_E$ be the universal cover:

$$\tilde{H}_E = \varprojlim_{\pi} H_E.$$ 

Then $\tilde{H}_E$ is an $E$-vector space object in the category of formal schemes over $\mathcal{O}_E$. We will call such an object a formal $E$-vector spaces.

Let $\tilde{H}_{E,0} = \tilde{H}_E \otimes_{\mathcal{O}_E} F_q$, a formal $E$-vector space over $F_q$. Since $\tilde{H}_{E,0} =$
Spf \(F_q[T]\) and \([\pi]_{H_{E,0}}(T) = T^q\), we have
\[
\tilde{H}_{E,0} = \lim_{\pi} \text{Spf } F_q[T] \\
= \text{Spf} \left( \lim_{T \to T^q} F_q[T] \right)^{\wedge} \\
= \text{Spf} F_q[T^{1/q^\infty}].
\]
In fact we also have \(\tilde{H}_E = \text{Spf } \mathcal{O}_E[T^{1/q^\infty}]\), see [Wei14], Prop. 2.4.2(2).

**Lemma 4.0.6.** We have an isomorphism of formal \(E\)-vector spaces over \(F_q\):
\[
\tilde{H}_{E,0} = \lim_{\pi} \lim_{n} H_{E,0}[\pi^n].
\]

**Proof.** For each \(n \geq 1\) we have the closed immersion \(H_{E,0}[\pi^n] \to H_{E,0}\). Taking inverse limits gives a map \(\lim_{\pi} H_{E,0}[\pi^n] \to \tilde{H}_{E,0}\), and taking injective limits gives a map \(\lim_{\pi} H_{E,0}[\pi^n] \to \tilde{H}_{E,0}\). The corresponding homomorphism of topological rings is
\[
F_q[T^{1/q^\infty}] \to \lim_{\pi} \lim_{n} F_q[T^{1/q^n}] / T = \lim_{T \to T^q} F_q[T^{1/q^\infty}] / T,
\]
which is an isomorphism. \(\square\)

**Proposition 4.0.7.** There exists an isomorphism of formal schemes over \(F_q\):
\[
\tilde{H}_{E,0} \cong \text{Spf } \mathcal{O}_{\hat{E}_\infty}.
\]
This isomorphism is \(E^\times\)-equivariant, where the action of \(E^\times\) on \(\mathcal{O}_{\hat{E}_\infty}\) is defined as follows: \(\mathcal{O}_{\hat{E}_\infty}^\times\) acts through the isomorphism \(\mathcal{O}_{\hat{E}_\infty}^\times \cong \text{Gal}(E_\infty/E)\) of class field theory, and \(\pi \in E^\times\) acts as the \(q\)th power Frobenius map.

**Proof.** Combining Lemmas 4.0.5 and 4.0.6 we get
\[
\tilde{H}_{E,0} \cong \lim_{\pi} \lim_{n} H_{E,0}[\pi^n] \\
\cong \lim_{\text{Frob}_q} \text{Spec } \mathcal{O}_{E_\infty} / t \\
\cong \text{Spf } \mathcal{O}_{\hat{E}_\infty}.
\]
The compatibility of this isomorphism with the \(\mathcal{O}_{\hat{E}}^\times\) follows from the definition of the isomorphism \(\mathcal{O}_{\hat{E}}^\times \cong \text{Gal}(E_\infty/E)\) of local class field theory. The compatibility of the action of \(\pi\) follows from the second statement in Lemma 4.0.5 \(\square\)
Let $C$ be a complete algebraically closed field containing $E$. Write $\hat{H}^{\text{ad}}_{E,C}$ for the generic fiber of $\hat{H} \otimes \mathcal{O}_C$. By Prop. 3.2.2, $\hat{H}^{\text{ad}}_{E,C}$ is a perfectoid space.

**Proposition 4.0.8.** There exists an isomorphism of adic spaces

\[
(\hat{H}^{\text{ad}}_{E,C} \setminus \{0\})^b \cong \left(Y^{\text{ad}}_{C^\circ, E} \hat{\otimes} \hat{E}_{\infty}\right)^b
\]

equivariant for the action of $\mathcal{O}_E^\times$ (which acts on $E_\infty$ by local class field theory). The action of $\pi \in E^\times$ on the left corresponds to the action of $\phi^{-1} \otimes 1$ on the right, up to composition with the absolute Frobenius morphism on $\left(Y_{C^\circ, E} \hat{\otimes} \hat{E}_{\infty}\right)^b$.

**Proof.** By Prop. 4.0.7 we have an isomorphism $\hat{H}_0 \cong \text{Spf} \mathcal{O}_{E_\infty}$. Extending scalars to $\mathcal{O}_C$, taking the adic generic fiber and applying tilts gives an isomorphism (cf. Prop. 3.2.2)

\[
\left(\hat{H}^{\text{ad}}_{E,C}\right)^b \cong \text{Spf} \left(\mathcal{O}_{C^\circ} \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_{E_\infty}\right)^{b,\text{ad}}_{C^\circ}
\]

The adic space $\text{Spf} \left(\mathcal{O}_{C^\circ} \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_{E_\infty}\right)^{\text{ad}}_{C^\circ}$ has two “Frobenii”: one coming from $\mathcal{O}_{C^\circ}$ and the other coming from $\mathcal{O}_{E_\infty}$. Their composition is the absolute $q$th power Frobenius. The action of $\pi$ on $\hat{H}^{\text{ad}, b}_{C^\circ}$ corresponds to the Frobenius on $\mathcal{O}_{E_\infty}$. Removing the “origin” from both sides of this isomorphism and using Prop. 3.5.1 gives the claimed isomorphism.

**Lemma 4.0.9.** Let $X$ be a perfectoid space which is fibered over $\text{Spec} \mathbb{F}_q$, and suppose $f : X \to X$ is an $\mathbb{F}_q$-linear automorphism. Let $\text{Frob}_q : X \to X$ be the absolute Frobenius automorphism of $X$. Then the category of $f$-equivariant finite étale covers of $X$ is equivalent to the category of $f \circ \text{Frob}_q$-equivariant finite étale covers of $X$.

**Proof.** First observe that a perfectoid algebra in characteristic $p$ is necessarily perfect ([Sch12], Prop. 5.9), which implies that absolute Frobenius is an automorphism of any perfectoid space in characteristic $p$. Then note that if $Y \to X$ is a finite étale cover, then $Y$ is also perfectoid ([Sch12], Thm. 7.9(iii)), so that $\text{Frob}_q$ is an automorphism of $Y$.

The proof of the lemma is now formal: if $g : Y \to X$ is a finite étale cover and $f_Y : Y \to Y$ lies over $f : X \to X$, then $f_Y \circ \text{Frob}_q : Y \to Y$ lies
over $f \circ \text{Frob}_q : X \to X$. Since $\text{Frob}_q$ is invertible on $Y$, the functor is invertible.

We can now prove the main theorem.

**Theorem 4.0.10.** There is an equivalence between the category of $E^\times$-equivariant étale covers of $\tilde{H}_C^\text{ad} \setminus \{0\}$ and the category of finite étale $E$-algebras.

**Proof.** In the following chain of equivalences, “$G$-cover of $X$” is an abbreviation for “$G$-equivariant finite étale cover of $X$”.

\[
\begin{align*}
\{ & E^\times \text{-covers of } \tilde{H}_C^\text{ad} \setminus \{0\} \} \\
\cong & \{ & E^\times \text{-covers of } \tilde{H}_C^\text{ad,} \setminus \{0\} \} & \text{[Sch12], Thm. 7.12} \\
\cong & \{ & O_E^\times \times (\phi^{-1} \circ \text{Frob}_q)^\mathbb{Z}\text{-covers of } Y_C^\text{ad,},L(E) \otimes \hat{E}_\infty \} & \text{Prop. 4.0.8} \\
\cong & \{ & O_E^\times \times \phi^\mathbb{Z}\text{-covers of } Y_{C,0,L(E)}^\text{ad,} \otimes \hat{E}_\infty \} & \text{Lemma. 4.0.9} \\
\cong & \{ & O_E^\times \text{-covers of } X_{C,0,L(E)}^\text{ad,} \otimes \hat{E}_\infty \} & \text{Defn. of } X_{C,0,L(E)}^\text{ad,} \\
\cong & \{ & O_E^\times \text{-equivariant finite étale } \hat{E}_\infty \text{-algebras} \} & \text{Prop. 3.8.3} \\
\cong & \{ & \text{Finite étale } E\text{-algebras} \} & L \mapsto L^{O_E^\times} \\
\end{align*}
\]

**4.1 Functoriality in $E$.**

Let us write

\[ Z_E = (\tilde{H}_E^\text{ad} \setminus \{0\})/E^\times. \]

As in the introduction, $Z_E$ may be considered as a sheaf on Perf$_C$, although the reader may also interpret the definition of $Z_E$ purely formally, keeping in mind that a “finite étale cover of $Z_E$” means the same thing as an $E^\times$-equivariant finite étale cover of $\tilde{H}_E^\text{ad}$.

Let us check that $Z_E$ really only depends on $E$. The construction depends on the choice of Lubin-Tate formal $O_E$-module $H = H_E$, which depends in turn on the choice of uniformizer $\pi$. If $\pi' \in E$ is a different uniformizer, with corresponding $O_E$-module $H'$, then $H$ and $H'$ become isomorphic after base extension to $O_{E_{\pi'}}$, the ring of integers in the completion of the maximal unramified extension of $E$. Such an isomorphism is unique up to multiplication by $E^\times$. Thus the adic spaces $\tilde{H}_C^\text{ad}$ and $(\tilde{H}')_{C}^\text{ad}$ are isomorphic,
and we get a canonical isomorphism \((\tilde{H}^{ad}_C \setminus \{0\})/E^\times \to (\tilde{H}^{ad}_C \setminus \{0\})/E^\times\). Thus \(Z_E\) only depends on \(E\).

It is also natural to ask whether the formation of \(Z_E\) is functorial in \(E\). That is, given a finite extension \(E'/E\) of degree \(d\), there ought to be a "norm" morphism \(N_{E'/E}: Z_{E'} \to Z_E\), which makes the following diagram commute:

\[
\begin{array}{ccc}
\pi\text{\text{\`et}}_1(Z_{E'}) & \xrightarrow{N_{E'/E}} & \pi\text{\text{\`et}}_1(Z_E) \\
\sim & & \sim \\
\Gal(E/E') & \longrightarrow & \Gal(E'/E).
\end{array}
\] (4.1.1)

There is indeed such a norm morphism; it is induced from the determinant morphism on the level of \(\pi\)-divisible \(\mathcal{O}_E\)-modules. The existence of exterior powers of such modules is the subject of [Hed10]. Here is the main result we need\(^1\) (Thm. 4.34 of [Hed10]): let \(G\) be a \(\pi\)-divisible \(\mathcal{O}_E\)-module of height \(h\) (relative to \(E\)) and dimension 1 over a noetherian ring \(R\). Then for all \(r \geq 1\) there exists a \(\pi\)-divisible \(\mathcal{O}_E\)-module \(\bigwedge^r G\) of height \((\binom{h}{r})\) and dimension \((\binom{h-1}{r-1})\), together with a morphism \(\lambda: G^r \to \bigwedge^r G\) which satisfies the appropriate universal property. In particular the determinant \(\bigwedge^h G\) has height 1 and dimension 1.

Let \(\pi'\) be a uniformizer of \(E'\), and let \(H'\) be a Lubin-Tate formal \(\mathcal{O}_{E'}\)-module. Then \(H'[(\pi')^\infty]\) is a \(\pi'\)-divisible \(\mathcal{O}_{E'}\)-module over \(\mathcal{O}_{E'}\) of height 1 and dimension 1. By restriction of scalars, it becomes a \(\pi\)-divisible \(\mathcal{O}_E\)-module \(H'[(\pi)^\infty]\) of height \(d\) and dimension 1. Then \(\bigwedge^d H'[(\pi)^\infty]\) is a \(\pi\)-divisible \(\mathcal{O}_E\)-module of height 1 and dimension 1, so that it is the \(\pi\)-power torsion in a Lubin-Tate formal \(\mathcal{O}_E\)-module \(\bigwedge^d H'\) defined over \(\mathcal{O}_{E'}\). For all \(n \geq 1\) we have an \(\mathcal{O}_E/\pi^n\)-alternating morphism

\[
\lambda: H'[(\pi)^n]^d \to \bigwedge^d H'[(\pi)^n]
\]

of \(\pi\)-divisible \(\mathcal{O}_E\)-modules over \(\mathcal{O}_{E'}\). Let \(H'_0 = H' \otimes \mathcal{O}_{E'}/\pi'.\) Reducing mod \(\pi'\), taking inverse limits with respect to \(n\) and applying Lemma [4.0.6] gives a morphism

\[
\lambda_0: (\tilde{H}'_0)^d \to \bigwedge^d H'_0
\]

\(^1\)This result requires the residue characteristic to be odd, but we strongly suspect this is unnecessary. See [SW13], §6.4 for a construction of the determinant map (on the level of universal covers of formal modules) without any such hypothesis.
of formal vector spaces over $\mathcal{O}_{E'}/\pi'$. By the crystalline property of formal vector spaces ([SW13], Prop. 3.1.3(ii)), this morphism lifts uniquely to a morphism

$$\tilde{\lambda}: (\tilde{H}')^d \to \bigwedge^d H'$$

of formal vector space over $\mathcal{O}_{E'}$.

Since $\bigwedge^d H'$ and $H$ are both height 1 and dimension 1, they become isomorphic after passing to $\mathcal{O}_C$. Let $\alpha_1, \ldots, \alpha_d$ be a basis for $E'/E$, and define a morphism of formal schemes

$$\tilde{H}'_C \to \bigwedge^d H'_C \cong \tilde{H}_C$$

$$x \mapsto \tilde{\lambda}(\alpha_1 x, \ldots, \alpha_d x)$$

After passing to the generic fiber we get a well-defined map $N_{E'/E}: Z_{E'} \to Z_E$ which does not depend on the choice of basis for $E'/E$.

The commutativity of the diagram in Eq. (4.1.1) is equivalent to the following proposition.

**Proposition 4.1.1.** The following diagram commutes:

\[
\begin{array}{ccc}
\{ \text{Étale } E\text{-algebras} \} & \xrightarrow{A \mapsto A \otimes E'} & \{ \text{Étale } E'\text{-algebras} \} \\
\downarrow & & \downarrow \\
\{ \text{Étale covers of } Z_E \} & \longrightarrow & \{ \text{Étale covers of } Z'_E \}.
\end{array}
\]

*Here the bottom arrow is pullback via $N_{E'/E}$.***

**Proof.** Ultimately, the proposition will follow from the functoriality of the isomorphism in Lemma 4.0.5. For $n \geq 1$, let $E'_n$ be the field obtained by adjoining the $\pi'^n$-torsion in $H'$ to $E'$. (Note that the $\pi^n$-torsion is the same as the $(\pi')^\text{en}$ torsion, where $e$ is the ramification degree of $E'/E$.) The existence of $\lambda$ shows that $E_n$ contains the field obtained by adjoining the $\pi^n$-torsion in $\bigwedge^d H'$ to $E_n$. Namely, let $x \in H[\pi^n](\mathcal{O}_{E'_n})$ be a primitive element, in the sense that $x$ generates $H'[\pi^n](\mathcal{O}_{E'_n})$ as an $\mathcal{O}_{E'}/\pi^n$-module. If $\alpha_1, \ldots, \alpha_d$ is a basis for $\mathcal{O}_{E'}/\mathcal{O}_E$ then $\lambda(\alpha_1 x, \ldots, \alpha_d x)$ generates $\bigwedge H'[\pi^n](\mathcal{O}_{E'_n})$ as an $\mathcal{O}_E/\pi^n$-module. Since $\bigwedge^d H'$ and $H$ become isomorphic over $\mathcal{E}'_{\text{ur}}$, we get a compatible family of embeddings $E'_n \hookrightarrow E''_{\text{ur}}$. 

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By construction, these embeddings are compatible with the isomorphisms in Lemma 4.0.5 so that the following diagram commutes:

\[
\begin{array}{c}
\text{Spf } \mathcal{O}_{E_n} / t' \\
\downarrow \\
\text{Spf } \mathcal{O}_{E_n} / t
\end{array}
\begin{array}{c}
\sim \\
\sim \\
\sim
\end{array}
\begin{array}{c}
\tilde{H}_0[\pi^{-1}] \otimes \overline{\mathbb{F}}_q \\
N_{E'/E} \\
\tilde{H}_0[\pi^{-1}] \otimes \overline{\mathbb{F}}_q
\end{array}
\]

Here \( t' \in \mathcal{O}_{E_1} \) is a uniformizer. From here we get the commutativity of the following diagram:

\[
\begin{array}{c}
\text{Spf } \left( \mathcal{O}_{E_n} \otimes \mathcal{O}_{C'} \right) / \pi \otimes \mathcal{O}_{C'} \\
\downarrow \\
\text{Spf } \left( \mathcal{O}_{E_n} \otimes \overline{\mathbb{F}}_q \otimes \mathcal{O}_{C'} \right)
\end{array}
\begin{array}{c}
\tilde{H}_0^{l_0} \\
N_{E'/E} \\
\tilde{H}_0^{l_0}
\end{array}
\]

One can now trace this compatibility with the chain of equivalences in the proof of Thm. 4.0.10 to get the proposition. The details are left to the reader. \( \square \)

### 4.2 Descent from \( C \) to \( E \)

The object \( Z_E = (\tilde{H}_{E,C}^{l_0} \setminus \{0\})/E^\times \) is defined over \( C \), but it has an obvious model over \( E \), namely \( Z_{E,0} = (\tilde{H}_E^{l_0} \setminus \{0\})/E^\times \). Since the geometric fundamental group of \( Z_{E,0} \) is \( \text{Gal}(\overline{E}/E) \), we have an exact sequence

\[
1 \to \text{Gal}(\overline{E}/E) \to \pi_1^{\text{ét}}(Z_{E,0}) \to \text{Gal}(\overline{E}/E) \to 1.
\]

This exact sequence splits as a direct product. In other words:

**Proposition 4.2.1.** Let \( E'/E \) be a finite extension, and let \( Z'_E \to Z_E \) be the corresponding finite étale cover under Thm. 4.0.10. Then \( Z'_E \) descends to a finite étale cover \( Z'_{E,0} \to Z_{E,0} \).

**Proof.** Let us write \( R = H^0(\tilde{H}, \mathcal{O}_R) \) for the coordinate ring of \( \tilde{H} \), so that \( R \cong \mathcal{O}_E[[t^{1/q}\pi]] \). Then \( R/\pi \cong \overline{\mathbb{F}}_q[[t^{1/q}\pi]] \) is a perfect ring and \( R = W_{\mathcal{O}_E}(R/\pi) \). By Prop. 4.0.7 we have \( R/\pi = \mathcal{O}_{E_0}^\sim \). Thus \( Z_{E,0} = \text{Spf } W_{\mathcal{O}_{E_0}^\sim}(\mathcal{O}_{E_0}^\sim) \setminus \{0\} \).

Now suppose \( E'/E \) is a finite extension. Let \( \hat{E}_\infty' = E' \otimes _E \hat{E}_\infty \), a finite étale \( \hat{E}_\infty \)-algebra. Then \( \hat{E}_\infty' \) is a perfectoid \( \hat{E}_\infty \)-algebra (in fact it a product of perfectoid fields), so that we can form the tilt \( \hat{E}_\infty'^{l_0} \), a perfect ring
containing $\hat{E}_\infty$. Let $\mathcal{O}_{E,\mathrm{tr}}$ be its subring of power-bounded elements. We put

$$Z'_{E,0} = \text{Spf} W_{\mathcal{O}_{E}}((\mathcal{O}_{E,\mathrm{tr}})^{\mathrm{ad}}_E \setminus \{0\}),$$

a finite étale cover of $Z_{E,0}$.

We claim that $Z'_E = Z_{E,0} \otimes C$. It suffices to show that the tilts are the same. We have

$$(Z_{E,0} \otimes C)^\flat = \text{Spf} \left( W_{\mathcal{O}_{E}}((\mathcal{O}_{E,\mathrm{tr}}^{\flat})^\flat, \otimes_{\mathcal{O}_C}^{\mathrm{ad}} \otimes_{\mathcal{O}_{E,\mathrm{tr}}}^{\mathrm{ad}} \setminus \{0\}) / E^\times \right)$$

$$= \left( \text{Spf} \left( (\mathcal{O}_{E,\mathrm{tr}}^{\flat})^\flat, \otimes_{\mathcal{O}_C}^{\mathrm{ad}} \otimes_{\mathcal{O}_{E,\mathrm{tr}}}^{\mathrm{ad}} \setminus \{0\} \right) / E^\times \right),$$

which by construction is the tilt of $Z'_E$. □

References

[Far13] Laurent Fargues, Quelque résultats et conjectures concernant la courbe, Submitted to Actes de la conférence en l'honneur de Gérard Laumon, 2013.

[FF11] Laurent Fargues and Jean-Marc Fontaine, Courbes et fibrés vectoriels en théorie de Hodge $p$-adique, preprint, 2011.

[Hed10] Hadi Hedayatzadeh, Exterior powers of Barsotti-Tate groups, Ph.D. thesis, ETH Zürich, 2010.

[HP04] U. Hartl and R. Pink, Vector bundles with a frobenius structure on the punctured unit disc, Compositio Mathematica 140 (2004), no. 3, 365–380.

[Hub94] R. Huber, A generalization of formal schemes and rigid analytic varieties, Math. Z. 217 (1994), no. 4, 513–551.

[Sch12] Peter Scholze, Perfectoid spaces, Publ. Math. Inst. Hautes Études Sci. 116 (2012), 245–313.

[Sch13] Peter Scholze, $p$-adic Hodge theory for rigid-analytic varieties, Forum Math. Pi 1 (2013), e1, 77.

[SW13] Peter Scholze and Jared Weinstein, Moduli of $p$-divisible groups, Cambridge Journal of Mathematics 1 (2013), no. 2, 145–237.

[Wei14] Jared Weinstein, Semistable models for modular curves of arbitrary level, Preprint, 2014.