BCS-BEC model of high-$T_c$ superconductivity in layered cuprates with unconventional pairing

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High-$T_c$ superconductivity in layered cuprates is described in a BCS-BEC formalism with linearly-dispersive s- and d-wave Cooper pairs moving in quasi-2D finite-width layers about the CuO$_2$ planes. This yields a closed formula for $T_c$, determined by the layer width, the Debye frequency, the pairing energy, and the in-plane penetration depth. The new formula reasonably reproduces empirical values of superconducting $T_s$ for seven different compounds among the LSCO, YBCO, BSCCO and TBCCO layered cuprates.

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I. INTRODUCTION

It seems to be well-established that central to high-$T_c$ superconductivity (HTSC) is the layered two-dimensional (2D) structure of copper oxides and that superconducting pairing occurs mainly on the CuO$_2$ planes. However, the precise nature of the pairing is still the subject of intense research. Recent experiments based on angle-resolved photon emission spectroscopy (ARPES) of underdoped cuprates suggest that fermion Cooper pairs (CPs) form already at and below temperatures higher than the critical transition temperature $T_c$. Furthermore, ARPES studies of the electron spectral function of optimally doped Bi2212 samples show that the magnitude of the isotope effect correlates with the superconducting gap, thus suggesting a role of lattice phonons in the superconducting pairing. ARPES data also suggest that the energy gap $\Delta$ (a measure of the energy needed to break a CP) displays an unconventional $d_{x^2-y^2}$ orbital pairing symmetry, with a functional dependence $\Delta = \Delta_0 \cos 2\theta$ where $\theta = \tan^{-1} (K_y/K_x)$ is the angle between the total or center-of-mass momentum (CMM) $h \mathbf{K} = (hK_x, hK_y)$ of paired electrons in the CuO$_2$ plane and the $a$- or $c$-axis while $\Delta_0$ is the value of the superconducting gap at the antinode ($\theta = 0, \pi/2$). This behavior is also apparent in studies based on electronic Raman scattering and in determinations of the in-plane magnetic penetration depth $\lambda_{ab}$. The so-called “Uemura plot” of data from $\mu$SR, neutron and Raman scattering, and ARPES measurements exhibit $T_c$ vs Fermi temperatures $T_F \equiv E_F/k_B$ where $E_F$ the Fermi energy and $k_B$ the Boltzmann constant. Empirical $T_s$ of many cuprates straddle a straight line parallel to the Uemura-plot line associated with the simple BEC formula $T_B \simeq 3.31 \frac{\hbar^2}{k_B} n_{\text{B}}^{2/3} / m_B k_B \simeq 0.218 T_F$ corresponding to an ideal gas of bosons of mass $m_B = 2m^*$ and number density $n_B = n_s/2$ where $m^*$ is the effective mass and $n_s$ the number density of individual charge carriers. The parallel line is shifted down from $T_B$ by a factor 4-5. This has been judged as “fundamental importance of the BEC concept in cuprates.”

Previous theoretical papers on the possible origin of HTSC proposed that the phenomenon might be rooted in a 2D Bose-Einstein condensate (BEC) of CPs pre-existing above $T_c$, and coupled through a BCS-like phonon mechanism, originally taken as s-wave. As apparently first reported by Schrieffer, the Cooper...
model interaction \([27]\) leads to an approximate linear energy-vs-CMM relation \(\xi_{\text{CP}}=\xi_{0}+c_{1}\hbar K+\left[1-\frac{1}{2}(1-16\pi^{2})E_{F}/B_{2}\right]\hbar^{2}K^{2}/4m^{*}+O(K^{4})\), where \(c_{1}=(2\sqrt{2}/\pi)\) precisely as with the Cooper model interaction [31]. Hence, the leading-order linearity is not induced by the particular interaction binding the CPs but is a consequence of the Fermi sea with \(E_{F} \neq 0\) and in which a CP by definition propagates.

Only in the vacuum limit \(E_{F} \rightarrow 0\) or \(\xi_{\text{CP}}=\xi_{0}+c_{1}\hbar K\) does that general dispersion relation reduce by inspection to the expected quadratic form \(\xi_{\text{CP}}=\xi_{0}+\hbar^{2}K^{2}/4m^{*}\) for a composite object of mass \(2m^{*}\).

For either inter-electron interaction model, the linear term is a consequence of the presence of the Fermi sea. The formation of a BEC of CPs in 2D does not violate Hohenberg’s theorem [32] as this holds only for quadratically-dispersive particles.

The predicted 2D BEC temperature is \(T_{c} \propto (n^{2D})^{1/2} \propto (\Theta_{D}T_{F})^{1/2}\) where \(n^{2D}\) is the CP number per unit area. This leads to values of \(T_{c}\) that are too high compared with empirical values. However, these schemes provide a correct description of other relevant physical properties of HTSCs such as a short coherence length, a type II magnetic behavior, and the temperature dependence of the electronic heat capacity [14]. They also lead to excellent fits of the condensate fraction curves for quasi-2D cuprates just below \(T_{c}\) [32], as well as for 3D and even quasi-1D SCs. To go beyond the simple \(s\)-wave interaction, an \(l\)-wave formulation of BCS theory was discussed by Schrieffer [29] himself and studied in considerable detail by Anderson and Morel [34] in the weak-coupling limit. This has been successfully employed [4, 33] to describe thermodynamic and transport properties of high-\(T_{c}\) cuprates. The \(d\)-wave extension in strong-coupling Eliashberg theory is reported in Refs. [36].

Here we develop a general \(l\)-wave BCS-type theory which is then applied in a quasi-2D BEC picture with either \(l=0\) or \(l=2\) pairing symmetry. In § II the \(l\)-wave BCS theory within the framework of the present model is discussed. In § III we study a quasi-2D BEC of linearly-dispersive, massless-like CPs and we evaluate the number density. In § IV the areal density \(n^{2D}\) of charge carriers is estimated by calculating the magnetic penetration depth arising from the CPs. In § V an analytical expression for the critical BEC temperature is derived, which is then applied in § VI for various superconducting materials including YBCO under different doping levels. Discussion and conclusions are given in § VII.
where \( a_k = a_k Y_l^m(\Omega_k) \). Eq. 5 can be now rewritten as

\[
a^{(l)}_k = C^{(l)} \frac{V^{(l)}_0 f^{(l)}_k}{\mathcal{E}_K^{(l)} - \epsilon_k + K/2 - \epsilon_k - K/2}
\]

where \( \sum_k a_k f^{(l)}_k \equiv C^{(l)} \) is a constant. One thus obtains a BCS-type integral relation for a CP in the eigenstate characterized by \((l, m)\)

\[
1 = V^{(l)}_0 \sum_k \frac{|f^{(l)}_k|^2}{\epsilon_k^{(l)} - \epsilon_k + K/2 - \epsilon_k - K/2}.
\]

Following Schrieffer 26 we assume that the angular-independent \( l \) component of the generalized BCS model interaction \( \omega \) is given by

\[
V^{(l)}_0 f^{(l)}_k f^{* (l)}_k = -V_0
\]

with \( V_0 > 0 \) for CPs with relative momenta \((\mathbf{k}, \mathbf{k}')\) lying in the neighborhood of the Fermi surface

\[
k_F < |\mathbf{k} + \mathbf{K}/2|, |\mathbf{k} - \mathbf{K}/2| < K_{\text{max}}
\]

and \( V^{(l)}_0 f^{(l)}_k f^{* (l)}_k = 0 \) otherwise. Here \( K_{\text{max}} = \sqrt{k_F^2 + k_D^2} \) with \( k_D \) defined in terms of the Debye energy \( \hbar \omega_D \equiv \hbar^2 k_D^2 / 2m^* \). A straightforward analysis \( 26 \) reveals that \( 7 \) yields a bound state with energy \( \mathcal{E}^{(l)}_0 < 0 \) for arbitrarily weak coupling so long as the potential is attractive in the region \( 9 \) in \( k \)-space. Then, a bosonic CP can form only if the tip of vector \( \mathbf{k} \) lies within the intersection of the two spherical shells defined by \( 9 \) whose center-to-center separation is \( K \); fermions with wave vectors lying outside this overlap are unpairable \( 24 \).

In the quasi-2D limit the fundamental expression \( 17 \) can be evaluated by substituting the summation over \( \mathbf{k} \) by a 2D integration. In addition, for small \( \delta \) the only term in \( K_{\text{max}} \) that yields a finite contribution is \( n = 0 \). By assuming a 2D cylindrical Fermi surface we obtain

\[
1 = \frac{V_0}{2(2\pi)^2} \int_0^{2\pi} \int_0^{k_2} \frac{k dk}{\left| \mathcal{E}^{(l)}_K + \epsilon_k + K/2 + \epsilon_k - K/2 \right|}
\]

\[
\simeq \frac{m^* V_0}{4\pi} \left\langle \ln \left| \left| \mathcal{E}^{(l)}_K + 2\hbar \omega_D - v_F h K \cos \theta \right| \right. \right|_F
\]

where \( k_1 = k_F + (K/2) \cos \theta, k_2 = k_F + k_D - (K/2) \cos \theta \), and the approximate equality in the second row holds up to terms of order \( (k_D/k_F)^2 \equiv \Theta_D/T_F \). The angular brackets denote an average over a 2D cylindrical Fermi surface \( < ... > = (1/2\pi) \int_0^{2\pi} d\theta \). The Fermi average then gives the energy spectrum of excited CPs \( 26 \):

\[
\mathcal{E}^{(l)}_K \simeq \mathcal{E}^{(0)}_0 + c_1 \hbar K + O(K^2)
\]

where \( c_1 \equiv 2v_F/\pi \) in 2D and \( \mathcal{E}^{(0)}_0 \) is the binding energy of the CP ground state \( (h \mathbf{K}, 0) \) \( 27 \):

\[
\mathcal{E}^{(l)}_0 = -2\hbar \omega_D / [\exp(2/N_0 V^{(l)}_0) - 1].
\]

The dispersion relation \( 11 \) is linear in leading order rather than quadratic as would be expected in vacuo. As a consequence, all excited CPs behave like a gas of free massless-like bosons with a common group velocity \( c_1 = \hbar^{-1} d\mathcal{E}^{(l)}_K / dK \), but a variable energy determined by their CMM \( \hbar K \). The dispersion relation \( 11 \) implies that in order for a CP to remain bound (i.e., \( \mathcal{E}^{(l)}_K < 0 \)) its maximum CMM wavenumber must not exceed the value \( |\mathcal{E}^{(0)}_0| / c_1 < K_0 \) since CPs with \( K > K_0 \) have \( \mathcal{E}^{(l)}_K > 0 \) and thus break up \( 26 \).

Explicit expressions of relevant thermodynamic variables and transport coefficients evaluated within the weak-coupling limit of the \( l \)-wave BCS theory have been derived in Refs. 6, 34, 32. In these papers it is shown that the average behavior of most of these quantities over the cylindrical Fermi surface exhibits small variation due to the explicit realization of an \( l = 0 \) or \( l = 2 \) symmetry \( 6, 35 \). In particular, the temperature-dependent gap equation is given by \( 6, 22, 34 \)

\[
1 = N_0 V_0 \int_0^{\hbar \omega_D} d\epsilon_k \left\langle \frac{\text{tanh} \left( \frac{\beta}{2} \sqrt{\epsilon_k^2 + \Delta^{(l)}(k)^2} \right)}{\sqrt{\epsilon_k^2 + \Delta^{(l)}(k)^2}} \right\rangle_F
\]

where \( \beta = 1/k_B T \), \( g^{(0)}(\mathbf{k}) = 1 \) for \( l = 0 \), and \( g^{(2)}(\mathbf{k}) = \cos(2\theta) \) for \( l = 2 \). The critical temperature is determined from \( 13 \) by the condition \( \Delta^{(l)}(T_c) = 0 \). In the weak-coupling limit \( \hbar \omega_D / k_BT_c \gg 1 \) it can be calculated analytically and it follows that \( T_c \) is independent of the \( l \)-state \( 34 \):

\[
k_BT_c = \frac{2e^\gamma}{\pi} \hbar \omega_D \exp(-1/N_0 V_0) \approx 1.13 \hbar \omega_D \exp(-1/N_0 V_0)
\]

with \( \gamma \approx 0.577 \cdots \) Euler’s constant. In the zero-temperature limit \( \Delta^{(l)}_0 = \Delta^{(l)}(T = 0) \) the energy integration in \( 13 \) leads to the gap relation \( 34 \):

\[
\Delta^{(l)}_0 = 2\Gamma^{(l)} \hbar \omega_D \exp(-1/N_0 V^{(l)}_0)
\]

where \( \Gamma^{(l)} = \exp[-(g^{(l)}(k)^2) \ln|g^{(l)}(k)|]/F \). For \( l = 0 \), \( \Gamma^{(0)} = 1 \), while for \( l = 2 \), \( \Gamma^{(2)} = 2 \exp(-1/2) \approx 1.213 \), so that combining \( 14 \) and \( 15 \) we are led to the gap-to-\( T_c \) ratios

\[
\frac{2\Delta^{(0)}_0}{k_BT_c} \approx 3.53 \quad \frac{2\Delta^{(2)}_0}{k_BT_c} \approx 4.28.
\]

For \( l = 0 \) one recovers the standard BCS result \( 27 \) and the somewhat higher value for the \( l = 2 \)-wave case. We note that the quantity \( \Delta^{(l)}_0 / \Gamma^{(l)} \) has the same functional dependence as the zero-temperature gap of the BCS theory \( 27 \). Considering that measurements of the energy gap for any given cuprate show some scatter about a central value \( \Delta^{(l)}_0 \) in the following we shall assume that \( \Delta^{(0)}_0 \neq \Delta^{(2)}_0 / \Gamma^{(2)} \approx \Delta^{(0)}_0 \neq \Delta_0 \).

On the other hand, the average superfluid density \( \rho_s(T) = \lambda_{ab}^2(0)/\lambda_{ab}^2(T) \) exhibits a more pronounced an-
gular momentum dependence. This is given by \[ \rho_s^{(l)} = 1 - \beta \int_0^\infty dk \left( \frac{|g^{(l)}(k)|^2}{\cosh^2 \left( \frac{1}{2} \beta \sqrt{\frac{\Delta^{(l)}_0}{k_B T}} \right) + \mu} \right). \] (17)

In the low-temperature limit \[ \beta \mu \ll 1 \] yields for \( l = 0 \) an exponential \( T \)-dependence

\[ \rho_s^{(0)}(T) \simeq 1 - \left( \frac{2\pi \Delta^{(0)}_0}{k_B T} \right)^{1/2} \exp(-\Delta^{(0)}_0/k_B T) \] (18)

while for \( l = 2 \) it gives the linear \( T \)-dependence

\[ \rho_s^{(2)}(T) \simeq 1 - \left( \frac{2 \ln 2}{\Delta^{(0)}_0} k_B T \right). \] (19)

Experiments \[ 8, 9 \] on the temperature variation of the magnetic penetration depth \( \lambda_{ab}(T) \) are consistent with the quasi-linear behavior \[ 19 \] which is a signature of \( d \)-wave symmetry. However, its asymptotic value \( \lambda_{ab}(T \to 0) \) is independent of \( l \), a result that we apply below.

III. BOSE-EINSTEIN CONDENSATION

We assume that charge carriers are an ideal binary mixture of non-interacting unpaired fermions plus breakable bosonic linearly-dispersive CPs \[ 14, 22, 23, 26 \]. Let the fermion number per unit area be \( n_f = n_{f1} + n_{f2} \) where \( n_{f1} \) and \( n_{f2} \) are the number densities of unpairable and pairable fermions, respectively. Unpairable fermions lie outside the interaction region of \[ 8 \] unlike the pairable fermions whose \( T \)-dependent density \( n_{f2}(T) \) is

\[ n_{f2}(T) = 2 \left[ n_{0}^{2D}(T) + n_{0}^{2D}_{0<K<K_0}(T) \right] + n_{f2}^{(1)}(T). \] (20)

Here \( n_{0}^{2D} \) is the bosonic number density of CPs with CMM wavenumber \( K = 0 \), \( n_{0}^{2D}_{0<K<K_0} \) that with \( 0 < K < K_0 \), and \( n_{f2}^{(1)} \) the number density of pairable but unpaired fermions. By asserting that in thermal equilibrium these kinds of fermions arise precisely from broken CPs \[ 22 \] we identify \( n_{f2}^{(1)}(T) = 2n_{0}^{2D}(T) \left( 2\pi n_{f2}^{(1)} + 2n_{f2}^{(2)} \right) \) while \( n_{0}^{2D} \) is the total boson number per unit area. The number equation for pairable fermions may thus be reexpressed in terms of boson quantities alone, namely \( n_{0}^{2D} = n_{0}^{2D}(T) + n_{0}^{2D}_{0<K<K_0}(T) + n_{0}^{2D}_{0<K<K_{max}}(T) \equiv n_{0}^{2D}(T) + n_{0}^{2D}_{0<K<K_{max}}(T) \). Thus

\[ n_{0}^{2D} = n_{0}^{2D}(T) + \int_{0}^{K_{max}} \frac{d^2K}{(2\pi)^2} \frac{1}{z-1} \exp(\beta E^{(0)}_K) - 1 \] (21)

where \( \beta \equiv 1/k_BT \), \( \mu \) the boson chemical potential and \( z \equiv \exp(\beta\mu) \) is the fugacity \((0 < z < 1)\). On introducing \[ 21 \] the energy-shifted boson dispersion relation \( E^{(0)}_K = \hbar c_1 K \) for \( K > 0 \) the integral can evaluated by changing to the variable \( x = \beta \hbar c_1 K \). Since

\[ c_1 = 2v_F/\pi \] and \( K_{max} = k_F(1+k_F^2/k_B^2) \), the upper integration limit \( x_{max} \) in \[ 21 \] is then be very large, namely \( x_{max} = \beta \hbar v_F k_F = 2E_F/k_B T \gg 1 \). The last inequality is consistent with the maximum empirical value for the ratio \( k_B T_c/E_F \leq 0.05 \) reported \[ 21 \] for all SCs including cuprate SCs. Given the rapid convergence of Bose integrals the upper integration limit \( x_{max} \) may safely be taken as infinite in \[ 21 \] so that the integrals can then be evaluated exactly by expanding the integrand in powers of \( z \exp(-x) \) and integrating term by term. The number density \[ 21 \] becomes

\[ n_{0}^{2D} = n_{0}^{2D}(T) + \frac{(k_B T)^2}{2\pi \hbar^2 c_1} \sum_{n=1}^{\infty} \frac{z^n}{n^2} \] (22)

The critical BEC temperature \( T_c \) is now determined by solving \[ 22 \] for \( n_{0}^{2D}(T_c) = 0 \) and \( z(T_c) = 1 \). One obtains

\[ T_c = \frac{\hbar c_1}{k_B} \left( \frac{2\pi n_{0}^{2D}}{\zeta(2)} \right)^{1/2} \] (23)

where \( \zeta(2) = \pi^2/6 \).

IV. CHARGE CARRIER DENSITY

The areal density of charge carriers was formerly estimated from measurements of the London penetration
depth $\lambda_L$ which is the distance over which an external magnetic field decays within the superconductor. For superelectrons with a 3D density $n_s$, charge $e$, and effective mass $m^*$, one has the well-known relation $1/\lambda_L^2 = 4\pi e^2 n_s/m^*c^2$. By introducing $\lambda$ the average interlayer spacing $c_{int}$ between CuO$_2$ planes in HTSCs it follows that $n^{2D} \simeq c_{int} n_s$. Penetration-depth data spanning a wide range of critical temperatures are consistent with the phenomenological Uemura relation $T_c \propto 1/\lambda_L^2 \propto n^{2D}/m^*$.\[20\\]

Within the present model we evaluate the magnetic penetration depth due to linearly-dispersive CPs with charge $2e$, and constrained to move within a thin layer of width $\delta$ with a uniform CM speed $c_1$. Thus, we first consider the expression for the 3D supercurrent of excited CPs $\mathbf{J}_s = n^{3D}(2e)c_1 \mathbf{K}$ with $\mathbf{K} \equiv \mathbf{K}/K$. We now introduce the contour integral of the CP wavefunction phase within a homogeneous medium and in the presence of an external magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. The following integral along any closed path vanishes

$$ \oint \mathbf{A} \cdot d\mathbf{r} = 0 \quad (24) $$

where $c$ is the speed of light in vacuum. By expressing $\mathbf{K}$ in terms of $\mathbf{J}_s$ and using Stoke’s theorem to evaluate $\oint \mathbf{A} \cdot d\mathbf{r}$ we get a modified version of London’s equation $\mathbf{J}_s = -\mathbf{\Lambda}_p \mathbf{A}$ where $\mathbf{\Lambda}_p \equiv 4\pi e^2 n^{3D}/\hbar K$. Taking now the curl of this modified London equation and introducing Ampere’s law $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}_s$, it follows that the magnetic induction $\mathbf{B}$ satisfies the Helmholtz equation $\nabla^2 \mathbf{B} = \lambda^{-2} \mathbf{B}$, where

$$ 1/\lambda^2 \equiv \frac{(2e)^2 4\pi e^2 n^{3D}}{\hbar K}. \quad (25) $$

Note that London’s result is recovered for quasi-particles with density $n^{3D} \to n_s/2$, momentum $\hbar K \to 2m^*c_1$, and charge $2e \to e$. This expression for $\lambda$ varies between its minimum value $\lambda = 0$ when $K = 0$ (perfect diamagnetism), and its maximum, say $\lambda_0$, for $K = K_0$ (CP breakup). It seems natural to identify $\lambda_0$ with the experimentally observed value of the in-plane penetration depth at $T = 0$, namely $\lambda_0 = \lambda_{ab}(T = 0)$. Here, $\lambda_{ab}^{-1}(0) = \lambda_a^{-1}(0) + \lambda_b^{-1}(0)$ is the geometric mean of this parameter measured along crystallographic in-plane directions $a$ and $b$. As shown in § II, this parameter is independent of the explicit value of the angular momentum $l$. By substituting the dispersion relation (11) to eliminate $K_0$ from $\lambda_0$ and imposing the relation $n^{2D} = \delta \eta^{3D}$ the 2D charge carrier density becomes

$$ n^{2D} = \frac{e^2 \delta \eta^{(l)}(0)}{c^2 16\pi^2 c^2 \lambda_{ab}^2}. \quad (26) $$

This latter expression can be reformulated by considering the relation (15) and the weak-coupling limit of (12). It follows that $\delta \eta^{(l)}(0) = (\delta \eta^{(l)}(0))/2\hbar \omega_D$ so that

$$ n^{2D} = \frac{e^2 \delta \eta^{(l)}(0)}{32\pi^2 c^2 \hbar \omega_D \lambda_{ab}} \quad (27) $$

where the approximate relation $\Delta^{(l)}(0)/\Gamma^{(l)} \simeq \Delta_0$ as justified in § II was used.

### V. CRITICAL TEMPERATURE

The final explicit expression for the critical BEC temperature $T_c$ is now obtained by substituting (27) in (28). This leaves

$$ T_c = \frac{\hbar c}{2\pi k_B e} \left( \frac{3\delta}{2\hbar \omega_D} \right)^{1/2} \frac{\Delta_0}{\lambda_{ab}} \quad (28) $$

which is independent of the CP speed $c_1$. We observe that for fixed values of $\omega_D$, $\Delta_0$, and $\delta$, the critical temperature increases linearly with $\lambda_{ab}^{-1}$. This dependence has been observed by Zuev et al.\[57\] in experiments in underdoped YBCO films with $T_c$s ranging from 6 to 50K. They conclude that, within some noise, their data fall on the same curve $\rho_\alpha \propto \lambda_{ab}^{-2} \propto T^{2.3\pm0.4}$, irrespective of annealing procedure, oxygen content, etc. Thus, by assuming that except for $\lambda_{ab}$ the other YBCO parameters are approximately independent of the doping level, we introduced in (28) the values: $\Theta_D = 410K$\[38\], $\Delta_0 = 14.5$...
meV \textsuperscript{38}, and \( \delta = 2.15 \, \text{Å} \) \textsuperscript{39, 40} to get the relation

\[
T_c = \frac{16.79[(\text{pm})^{-1}K]}{\lambda_{ab}}. \tag{29}
\]

Figure 1 is adapted from Ref.\textsuperscript{37} and compares theoretical predictions \textsuperscript{29} with experimental data, as well as with data pertaining to higher doping regimes. We see that \textsuperscript{29} gives an excellent fit to the experimental data. The same functional dependence has been observed in single YBCO crystals near the optimally-doped regime \textsuperscript{41}. More recently, Broun et al. \textsuperscript{9} found that their samples of high-purity single-crystal YBCO followed the rule \( T_c \propto \lambda_{ab}^{-1} \propto n_{s}^{1/2} \propto (p - p_c)^{1/2} \) where the doping \( p \) is the number of holes per copper atom in the \( CuO_2 \) planes and \( p_c \) the minimal doping for superconductivity onset. The measured value of the penetration length in YBCO crystals is an order of magnitude bigger than in thin films \textsuperscript{9, 41}, so that the specific values of \( T_c \)s derived from \textsuperscript{28} are not in such good agreement as in the YBCO films. However, one should expect variations of parameters such as the energy gap associated to crystals and film systems. It has been pointed out \textsuperscript{37} that YBCO films seem to behave more like other cuprates such as BiSrCaCuO or LaSrCuO than do YBa\textsubscript{2}CuO\textsubscript{4} crystals. Furthermore, a different approach \textsuperscript{42} based on measurements of the lower critical magnetic field \( H_{c1}(T) \) for highly underdoped YBCO indicates that experimental data may be consistently described only by assuming \( T_c \propto n_{s}^{0.61} \), in close agreement with studies mentioned above.

Theoretical values of \( T_c \) for superconducting cuprates with different compositions have been also calculated using \textsuperscript{28}. Here we report on these seven layered-cuprate superconducting compounds: \((\text{La}_{0.65}\text{Sr}_{0.35})_2\text{CuO}_4\); YBa\textsubscript{2}Cu\textsubscript{3}O\textsubscript{6.60}; YBa\textsubscript{2}Cu\textsubscript{3}O\textsubscript{6.95}; Tl\textsubscript{2}Ba\textsubscript{2}Ca\textsubscript{2}Cu\textsubscript{2}O\textsubscript{6}; Tl\textsubscript{2}Ba\textsubscript{2}Ca\textsubscript{2}Cu\textsubscript{2}O\textsubscript{6}; Bi\textsubscript{2}Sr\textsubscript{2}Ca\textsubscript{2}Cu\textsubscript{2}O\textsubscript{6}; and Bi\textsubscript{2}Sr\textsubscript{2}Ca\textsubscript{2}Cu\textsubscript{2}O\textsubscript{6}. Characteristic parameters for these materials were taken from tables compiled in Ref.\textsuperscript{38} (see also \textsuperscript{43, 44}). Concerning the layer width \( \delta \) no direct experimental data are available. We have employed results derived from energy band-structure calculations for cuprates. Contour plots \textsuperscript{34, 40} of the charge distribution for La\textsubscript{2}CuO\textsubscript{4}, YBa\textsubscript{2}Cu\textsubscript{3}O\textsubscript{7}, and Bi\textsubscript{2}Sr\textsubscript{2}Ca\textsubscript{2}Cu\textsubscript{2}O\textsubscript{8} suggest that charge carriers in each of these systems are concentrated within slabs of average width \( \delta \approx 2.61\text{Å}, 2.15\text{Å}, \) and 2.28Å, respectively, about their \( CuO_2 \) planes. As \( c_{int} \) denotes the average separation between adjacent \( CuO_2 \) planes, it follows from crystallographic data \textsuperscript{38} that the yttrium and bismuth compounds give \( \delta \approx 0.64 \, c_{int} \) and 0.68 \( c_{int} \), respectively. Taking into account that Bi\textsubscript{2}Sr\textsubscript{2}Ca\textsubscript{n}Cu\textsubscript{n+1}O\textsubscript{6+n} compounds possess the same layering scheme as their TlBa\textsubscript{2}Ca\textsubscript{n}Cu\textsubscript{n+1}O\textsubscript{6+n} counterparts \textsuperscript{38}, we assumed that the condition \( \delta \approx 0.68c_{int} \) holds also for the thallium compounds. The former estimations are congruent with Uemura’s surmise \textsuperscript{20} that SC charge carriers in layered cuprates are concentrated within slabs of width \( \delta = c_{int} \).

Table I shows results obtained using the foregoing assumptions, together with the physical parameters involved in the calculation. In most cases we find rather satisfactory agreement between predicted and measured values of \( T_c \). We also find very good agreement between theoretical and experimental gap-to-\( T_c \) ratios \( \Delta_0/k_B T_c \). Average theoretical and experimental such ratios presented in Table I are \( (2\Delta_0/k_B T_c)^{th} \approx 4.45 \) and \( (2\Delta_0/k_B T_c)^{exp} \approx 4.59 \), respectively. Both are consistent with the ratio \( 2\Delta^2/\kappa_B T_c \approx 4.28 \) predicted by the \( l = 2 \) BCS theory in \textsuperscript{10}. We have not attempted to estimate uncertainties of our theoretical results since the accumulated data of the physical parameters involved in the calculation, particularly \( \Delta_0 \) and \( \lambda_{ab} \), show a wide scatter.

VI. DISCUSSION AND CONCLUSIONS

We have shown that layered-cuprate HTSC can be described by means of an \( l \)-wave BCS for a quasi-2D BEC of Cooper pairs. The theory involves a \textit{linear}, as opposed to quadratic, dispersion relation in their total or CM momenta. The theory yields a simple formula for the critical transition temperature \( T_c \) with a functional relation \( T_c \propto l/\lambda_{ab} \propto n_s^{1/2} \) which applies to a variety of cuprate SCs over a wide range of dopings. Although this behavior apparently disagrees with the phenomenological Uemura relation \( T_c \propto 1/\lambda_{ab}^2 \) \textsuperscript{20}, different experimental studies \textsuperscript{9, 41, 42} show consistency with the inverse linear dependence of \( T_c \). Additional consistency is also seen with the reported dependence \( T_c \propto n_s^{0.61} \) arising from measurements of the lower critical magnetic field \textsuperscript{42}. When averaged over a cylindrical Fermi surface, the physical quantities involved in the theory show small dependence on the angular momentum state \( l \). However, the gap-to-\( T_c \) ratio \( 2\Delta_0/k_B T_c \) is closer to that predicted by the extended BCS theory for \( l = 2 \) than for \( l = 0 \). It is shown elsewhere \textsuperscript{40} that all relevant 2D expressions derived here arise in the limit \( k_B T_\delta/h c_1 \rightarrow 0 \) and higher general 3D BCS-BEC theory for layered materials.

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