FIDELITY PRESERVING MAPS ON DENSITY OPERATORS

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Abstract. We prove that any bijective fidelity preserving transformation on the set of all density operators on a Hilbert space is implemented by an either unitary or antiunitary operator on the underlying Hilbert space.

Let $H$ be a Hilbert space. The set of all density operators on $H$, that is, the set of all positive self-adjoint operators on $H$ with finite trace is denoted by $C_1^+(H)$. (We note that one may prefer normalized density operators; see the first remark at the end of the paper.)

According to Uhlmann [6, 7], for any $A, B \in C_1^+(H)$ we define the fidelity of $A$ and $B$ by

$$F(A, B) = \text{tr}(A^{1/2}BA^{1/2})^{1/2}.$$ 

This is in fact the square-root of the transition probability introduced by Uhlmann in [3] for density operators which later Jozsa called fidelity and showed its use in quantum information theory [3]. The reason that Uhlmann defined the fidelity in the way above is that after taking square-root the function $F$ behaves significantly better.

Clearly, the fidelity is in intimate connection with the transition probability between pure states. Wigner’s theorem describing the form of all bijective transformations on the set of all pure states which preserve the transition probability plays fundamental role in the theory of quantum systems. By analogy, it seems to be of some importance to describe all bijective transformations on the density operators which preserve the fidelity. This is exactly what we are performing in the present paper. We show that the fidelity preserving transformations on $C_1^+(H)$ are implemented by an either unitary or antiunitary operator of the underlying Hilbert space.

The main result of the paper reads as follows.

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Theorem 1. Let $\phi : C_1^+ (H) \to C_1^+ (H)$ be a bijective transformation with the property that

$$F(\phi(A), \phi(B)) = F(A, B) \quad (A, B \in C_1^+ (H)).$$

Then there is an either unitary or antiunitary operator $U : H \to H$ such that

$$\phi(A) = UAU^* \quad (A \in C_1^+ (H)).$$

Proof. The main point of the proof is to reduce the problem to Wigner's classical result. In order to do so, we first prove that $\phi$ preserves the order $\leq$ (which comes from the usual order between bounded self-adjoint operators on $H$) on $C_1^+ (H)$. If $A, B \in C_1^+ (H)$, $A \leq B$, then for any $C \in C_1^+ (H)$ we have

$$C^{1/2} AC^{1/2} \leq C^{1/2} BC^{1/2}.$$ 

Since the square-root function is operator monotone we have

$$(C^{1/2} AC^{1/2})^{1/2} \leq (C^{1/2} BC^{1/2})^{1/2}.$$ 

Taking trace we obtain

$$F(A, C) \leq F(B, C)$$

which implies

$$F(\phi(A), \phi(C)) \leq F(\phi(B), \phi(C))$$

for every $C \in C_1^+ (H)$. Let $\phi(C)$ run through the set of all rank-one projections. If $P$ is the rank-one projection projecting onto the subspace generated by the unit vector $x \in H$, then we have

$$\langle \phi(A)x, x \rangle^{1/2} = F(\phi(A), P) \leq F(\phi(B), P) = \langle \phi(B)x, x \rangle^{1/2}.$$ 

Since this holds for every unit vector $x \in H$, we obtain $\phi(A) \leq \phi(B)$. Since $\phi^{-1}$ has the same properties as $\phi$, it follows that $\phi$ preserves the order in both directions.

We next show that $\phi$ preserves the rank-one operators. In fact, one can easily see that an element $A \in C_1^+ (H)$ is of rank one if and only if the set $\{ T \in C_1^+ (H) : T \leq A \}$ is infinite and total in the sense that any two elements in it are comparable with respect to the order $\leq$. By the order preserving property of $\phi$ it now follows that $\phi$ preserves the rank-one elements of $C_1^+ (H)$ in both directions.

Clearly, a rank-one operator $A \in C_1^+ (H)$ is a rank-one projection if and only if its trace is 1, that is, if $F(A, A) = 1$. It follows that $\phi$ preserves the rank-one projections.

It needs elementary computation to show that for any rank-one projections $P, Q$ we have

$$F(P, Q) = (\text{tr } PQ)^{1/2}.$$
Hence, we have proved that if we restrict $\phi$ onto the set of all rank-one projections, we have a bijective function on this set which satisfies
\[ \text{tr} \phi(P)\phi(Q) = \text{tr} PQ \]
for every $P, Q$. Now, we can apply Wigner’s theorem and get that there exists an either unitary or antiunitary operator $U : H \to H$ such that
\[ \phi(P) = UPU^* \]
holds for every rank-one projection $P$. Replacing $\phi$ by the transformation $A \mapsto U^*\phi(A)U$ if necessary, we can obviously assume that our original transformation $\phi$ satisfies $\phi(P) = P$ for every rank-one projection $P$. It remains to show that $\phi(A) = A$ holds for every density operator $A \in C_1^+(H)$. If $x \in H$ is a unit vector and $P$ is the corresponding rank-one projection, then we compute
\[ \langle \phi(A)x, x \rangle^{1/2} = \text{tr}(P\phi(A)P)^{1/2} = F(\phi(A), P) = F(\phi(A), \phi(P)) = F(A, P) = \langle Ax, x \rangle^{1/2}. \]
Since this holds for every unit vector $x \in H$, we conclude that $\phi(A) = A$ ($A \in C_1^+(H)$). This completes the proof.

If the underlying Hilbert space is finite dimensional, then we can get rid of the assumption on bijectivity and obtain our second result which follows.

**Theorem 2.** Let $H$ be a finite dimensional Hilbert space and let $\phi : C_1^+(H) \to C_1^+(H)$ be a transformation such that
\[ F(\phi(A), \phi(B)) = F(A, B) \quad (A, B \in C_1^+(H)). \]
Then there is an either unitary or antiunitary operator $U : H \to H$ such that
\[ \phi(A) = UAU^* \quad (A \in C_1^+(H)). \]

**Proof.** If $A, B$ are self-adjoint operators, then we say that $A, B$ are mutually orthogonal if $AB = 0$. Clearly, $A, B$ are mutually orthogonal if and only if they have mutually orthogonal ranges.

Let us assume that the dimension of $H$ is $d \geq 2$. It is easy to see that one can characterize the positive rank-one operators in the following way: a positive operator $A$ is of rank one if and only if $A \neq 0$ and there exists a system $A_1, \ldots, A_{d-1}$ of nonzero positive operators such that the elements of $A, A_1, \ldots, A_{d-1}$ are mutually orthogonal.

It is clear that $\phi$ preserves the nonzero operators. Indeed, this follows form the equality $F(A, A) = \text{tr} A$. Let $A, B$ be positive operators with $AB = 0$. Then $A, B$ are commuting and by the properties of the positive square-root of positive operators, we have the same for $A, B^{1/2}$. Therefore, we infer
\[ B^{1/2}AB^{1/2} = AB = 0. \]
So, we have $\text{tr}(\phi(B)^{1/2}\phi(A)\phi(B)^{1/2})^{1/2} = 0$. But this implies that

$$(\phi(B)^{1/2}\phi(A)\phi(B)^{1/2})^{1/2} = 0.$$ 

Hence, we have

$$\phi(B)^{1/2}\phi(A)\phi(B)^{1/2} = 0$$

from which we get

$$(\phi(A)^{1/2}\phi(B))^{*}(\phi(A)^{1/2}\phi(B)) = \phi(B)\phi(A)\phi(B) = 0.$$ 

Consequently, we have $\phi(A)^{1/2}\phi(B) = 0$ which implies $\phi(A)\phi(B) = 0$. This shows that $\phi$ preserves the orthogonality in one direction. By the characterization of rank-one operators given in the beginning of the proof, we infer that $\phi$ sends rank-one operators to rank-one operators. Now, similarly to the corresponding part of the proof of our previous theorem one can check that $\phi$ sends rank-one projections to rank-one projections. Just in that proof one can readily verify that

$$\text{tr} \phi(P)\phi(Q) = \text{tr} PQ$$

holds for every rank-one projection $P, Q$. Now, by the nonsurjective version of Wigner’s theorem [1] (also see [4, Theorem 3]), we have an isometry or antiisometry $U : H \rightarrow H$ such that

$$\phi(P) = UPU^{*}$$

holds for every rank-one projection $P$. Since $H$ is finite dimensional, $U$ is in fact a unitary or antiunitary operator. The proof can now be completed very similarly to the proof of our first theorem.

We conclude the paper with some remarks.

(1) In the introduction we have mentioned that one may prefer to restrict the investigation to normalized density operators, that is, to positive self-adjoint operators with trace 1. Although following Uhlmann, in our treatment we did not insist on normalization, we point out to the fact that one can get the same result also in that case. The only thing that should be done is the following. If $\phi$ is a bijective transformation on the set of all normalized density operators, then we define $\tilde{\phi} : C_{1}^{+}(H) \rightarrow C_{1}^{+}(H)$ in the following way: $\tilde{\phi}(0) = 0$ and for any $0 \neq A \in C_{1}^{+}(H)$ we set

$$\tilde{\phi}(A) = \text{tr} A\phi \left( \frac{A}{\text{tr} A} \right).$$

It is apparent that $\tilde{\phi} : C_{1}^{+}(H) \rightarrow C_{1}^{+}(H)$ is a bijective transformation extending $\phi$ and it preserves the fidelity. Now, Theorem 3 applies.

(2) One can easily generalize our results to obtain the same description of transformations on density operators preserving not the "full" fidelity but a certain part of it. We mean the quantity $F_{m}^{+}(A, B)$ denoting the sum of the $m$ largest eigenvalues of the operator $(A^{1/2}BA^{1/2})^{1/2}$ $(A, B \in C_{1}^{+}(H))$ [7, Definition]. Here $m$ is fixed and when we speak about eigenvalues we always take into account the multiplicities. Now, one can formulate the same
assertions as in Theorem 1 and 2 with $F_m^+$ in the place of $F$. As for the proofs, one can follow quite the same argument. In fact, the only additional thing that should be observed concerns the order preserving property. Namely, one should verify that $A \leq B$ if and only if $F_m^+(A, C) \leq F_m^+(B, C)$ holds for every density operator $C$. The sufficiency is clear if $C$ runs through the set of all rank-one projections. The necessity follows from Weyl’s monotonicity theorem stating that if $A \leq B$, then the the $k$th largest eigenvalue of $A$ is less than or equal to the $k$th largest eigenvalue of $B$ (cf. [2, Lemma 1.1, p. 26]).

It would certainly be of interest to obtain similar results concerning the "partial" fidelities introduced by Uhlmann in [5].

(3) It is easy to see that just as in Wigner’s classical theorem, the implementing unitary or antiunitary operator $U$ is unique up to a phase factor (a scalar of modulus one).

(4) Following the lines in the proof of our first result one can easily see that there is no need to assume the injectivity of the transformation $\phi$. We set this condition only for the sake of "symmetry" in the formulation.

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References

1. V. Bargmann, Note on Wigner’s theorem on symmetry operations, J. Math. Phys. 5 (1964), 862–868.
2. I.C. Gohberg and M.G. Krein, Introduction to The Theory of Linear Nonselfadjoint Operators, American Mathematical Society, 1969.
3. R. Jozsa, Fidelity for mixed quantum states, J. Modern Opt. 41 (1994), 2315–2323.
4. L. Molnár, Transformations on the set of all $n$-dimensional subspaces of a Hilbert space preserving principal angles, Commun. Math. Phys. (to appear)
5. A. Uhlmann, The "transition probability" in the state space of a $*$-algebra, Rep. Math. Phys. 9 (1976), 273–279.
6. A. Uhlmann, On "partial" fidelities, Rep. Math. Phys. 45 (2000), 407–418.
7. A. Uhlmann, Simultaneous decomposition of two states, arXiv:quant-ph/00070118.