Exponential Growth Constants for Spanning Forests on Archimedean Lattices: Values and Comparisons of Upper Bounds

Shu-Chiuan Chang and Robert Shrock

(a) Department of Physics, National Cheng Kung University, Tainan 70101, Taiwan and (b) C. N. Yang Institute for Theoretical Physics and Department of Physics and Astronomy, Stony Brook University, Stony Brook, NY 11794, USA

We compare our upper bounds on the exponential growth constant \( \phi(\Lambda) \) characterizing the asymptotic behavior of spanning forests on Archimedean lattices \( \Lambda \) with recently derived upper bounds. Our upper bounds on \( \phi(\Lambda) \), which are very close to the respective values of \( \phi(\Lambda) \) that we have calculated, are shown to be significantly better for these lattices than the new upper bounds.

I. INTRODUCTION

Let \( G = (V, E) \) be a graph, defined by its vertex and edge sets \( V \) and \( E \), and denote \( n = n(G) = |V| \) and \( e(G) = |E| \) as the numbers of vertices (= sites) and edges (= bonds) in \( G \). An important problem in mathematics is the determination of the number of subgraphs of \( G \) that satisfy a specified property, and, in particular, the asymptotic behavior of this number as \( n(G) \to \infty \). A spanning subgraph of \( G \), denoted \( G' \), is a graph with the same vertex set, \( V' = V \) and a subset of the edge set of \( G \), \( E' \subseteq E \). A forest is a spanning subgraph that does not contain any cycles. Given a graph \( G \), let us denote the number of spanning forests in \( G \) as \( N_{SF}(G) \) and the number of connected spanning subgraphs in \( G \) as \( N_{CSSG}(G) \). For many families of graphs \( G \), \( N_{SF}(G) \) and \( N_{CSSG}(G) \) grow exponentially rapidly as functions of \( n(G) \) for large \( n(G) \), thereby motivating the definitions of corresponding exponential growth constants \( \phi(\{G\}) \) and \( \sigma(\{G\}) \),

\[
\phi(\{G\}) = \lim_{n(G) \to \infty} [N_{SF}(G)]^{1/n(G)} \quad (1.1)
\]

and

\[
\sigma(\{G\}) = \lim_{n(G) \to \infty} [N_{CSSG}(G)]^{1/n(G)} , \quad (1.2)
\]

where \( \{G\} \) denotes the \( n(G) \to \infty \) limit of the graphs in a given family. Recall that the degree \( \Delta_{v_i} \) of a vertex \( v_i \) in a graph \( G \) is the number of edges connecting to \( v_i \). A graph with the property that all of its vertices have the same degree \( \Delta \) is termed a \( \Delta \)-regular graph. To avoid unimportant complications, we restrict here to loopless graphs.

In Ref. \cite{1} we calculated upper bounds on \( \phi(\Lambda) \) and \( \sigma(\Lambda) \), where \( \Lambda \) denotes the \( n(G) \to \infty \) limit of an Archimedean lattice graph. Here an Archimedean lattice is defined as a uniform tiling of the plane with one or more types of regular polygons, such that all vertices are equivalent, and hence is \( \Delta \)-regular. In general, an Archimedean lattice \( \Lambda \) is identified by the ordered sequence of regular polygons traversed in a circuit around any vertex \( \Lambda \) :

\[
\Lambda = (\prod p_i^{a_i}) , \quad (1.3)
\]

where the \( i \)th polygon has \( p_i \) sides and appears \( a_i \) times contiguously in the sequence (it can also occur non-contiguously). There are three Archimedean lattices which each involve only a single type of polygon, namely honeycomb = \( (6^3) \), square = \( (4^4) \), and triangular = \( (3^6) \), abbreviated as (hc), (sq), and (tri), respectively. The other Archimedean lattices are heteropolygonal, i.e., they involve more than a single type of polygon. Examples are \( (4 \cdot 8) \cdot 8 \), \( (3 \cdot 6 \cdot 3 \cdot 6) \) (often called “kagomé” (kag)), \( (3^3 \cdot 4^2) \), and \( (3 \cdot 3 \cdot 4 \cdot 3 \cdot 4) \). With appropriate boundary conditions, a finite section of an Archimedean lattice is a \( \Delta \)-regular graph. The \( \Delta \) values for the Archimedean lattices range from 3 to 6. Our upper bounds were denoted \( \phi_u(\Lambda) \) and \( \sigma_u(\Lambda) \), where the subscript \( u \) stands for “upper”. Our bounds are, to our knowledge, the best upper bounds on \( \phi(\Lambda) \) and \( \sigma(\Lambda) \) for these lattices. In \cite{1} we also calculated lower bounds on these exponential growth constants, which are very close to the respective upper bounds, with fractional differences ranging from \( 10^{-4} \) to \( 10^{-2} \). This property enabled us to calculate quite accurate approximate values of \( \phi(\Lambda) \) and \( \sigma(\Lambda) \) for these lattices, which we denote here simply as \( \phi(\Lambda) \) and \( \sigma(\Lambda) \).

For each lattice \( \Lambda \), our method made use of calculations of lower and upper bounds on \( \phi \) and \( \sigma \) for a sequence of infinite-width lattice strips of increasing widths. Our approximate values were determined conservatively as the average of our lower and upper bounds on the widest strips with periodic transverse boundary conditions (to minimize finite-width effects). As we noted in \cite{1}, our upper bounds and approximate values for \( \phi(\Lambda) \) and \( \sigma(\Lambda) \) are monotonically increasing functions of the vertex degree \( \Delta \) for the Archimedean lattices that we studied. In the following, we focus on our results for \( \phi(\Lambda) \) in view of new general bounds in \cite{1}. For reference, we list our upper bounds and values for \( \phi(\Lambda) \) from \cite{1} in Table I together with the ratio of our (central) value of \( \phi(\Lambda) \) divided by our upper bound for each \( \Lambda \), namely

\[
R_\phi(\Lambda) = \frac{\phi(\Lambda)}{\phi_u(\Lambda)} . \quad (1.4)
\]

The fact that the ratios \( R_\phi(\Lambda) \) for these lattices are very close to unity shows how close our upper bounds are to being sharp. We found that our upper bounds on \( \phi(\Lambda) \) and \( \sigma(\Lambda) \) approach limiting values more rapidly than our lower bounds, so that the true values of \( R_\phi(\Lambda) \) are ex-
expected to be even closer to unity than the values listed in Table I, i.e., the respective upper bounds are even closer to being sharp for these lattices.

Let $G_\Delta$ denote the set of all $\Delta$-regular $n$-vertex graphs. Recently, in Ref. [4], Bobényi, Csikvári, and Luo (BCL) presented upper bounds on the supremum over all $G \in G_\Delta$ of the quantity $[N_{SF}(G)]^{1/n(G)}$,

$$f_\Delta = \sup_{G \in G_\Delta} [N_{SF}(G)]^{1/n(G)} \ . \tag{1.5}$$

The upper bounds reported in [4] do not depend on $n(G)$, so they also apply in the $n(G) \to \infty$ limit, yielding upper bounds on $\phi$, which we denote as

$$\phi_{u,BCL,i}(\Delta) = \lim_{n(G) \to \infty} \sup_{G \in G_\Delta} [N_{SF}(G)]^{1/n(G)} \ , \tag{1.6}$$

where the subscript $i$ will label the specific BCL bounds. It is of considerable interest to compare the BCL upper bounds with our upper bounds and values for the Archimedean lattices that we considered in [1]. We perform this comparison in the present paper.

### II. COMPARISON OF UPPER BOUNDS ON $\phi(\Delta)$

We first recall a general upper bound for any set of spanning subgraphs, including spanning trees, spanning forests, and connected spanning subgraphs. In the construction of a spanning subgraph, there is choice for each edge of $G$, namely whether it is present or absent. Since this is a two-fold choice for each edge, it follows that the number of spanning subgraphs of $G$ is

$$N_{SSC}(G) = 2^{v(G)} \ . \tag{2.1}$$

This is an upper bound for any specific subclass of spanning subgraphs. Hence, in particular, for spanning forests,

$$N_{SF}(G) \leq N_{SSC}(G) \ . \tag{2.2}$$

Since for $\Delta$-regular graphs $G \in G_\Delta$,

$$e(G) = \frac{n(G)\Delta}{2} \ , \tag{2.3}$$

it follows that for these $\Delta$-regular graphs,

$$f_\Delta \leq 2^{\Delta/2} \ , \tag{2.4}$$

and hence, in the limit $n(G) \to \infty$,

$$\phi(G_\Delta) \leq 2^{\Delta/2} \ . \tag{2.5}$$

Before comparing our upper bounds on $\phi(\Delta)$ to the $n(G) \to \infty$ limits of upper bounds recently derived in [4], we mention some previous bounds. After early work [3], Ref. [6] obtained the upper limit

$$\phi(sq) \leq 3.7410018 \ . \tag{2.6}$$

Before our work in [1], the best upper bound on $\phi(sq)$ was from Mani, in Ref. [7], namely

$$\phi(sq) \leq 3.705603 \ . \tag{2.7}$$

In [1] we derived the upper bound

$$\phi(sq) \leq 3.699659 \ . \tag{2.8}$$

As we noted, our upper bounds on $\phi(\Lambda)$ for this and the other Archimedean lattices that we studied are, to our knowledge, the best upper bounds on $\phi(\Lambda)$ for these lattices. Our results in [1] were part of a general program of calculating bound on, and values of, exponential growth constants for various classes of subgraphs on Archimedean lattices [8–11].

A first upper bound proved in [4] is

$$N_{SF}(G) \leq \prod_{v_i \in V} (\Delta_{v_i} + 1) \ . \tag{2.9}$$

The special case of this bound for a $\Delta$-regular graph $G$ is

$$f_\Delta \leq \Delta + 1 \ , \tag{2.10}$$

which also applies in the limit as $n(G) \to \infty$ as

$$\phi(G_\Delta) \leq \Delta + 1 \ . \tag{2.11}$$

We denote the right-hand side of (2.11) as $\phi_{u,BCL1}(\Delta) = \Delta + 1$. Generalizing $\Delta$ from positive integral values to positive real values, we find that the upper bound (2.10) is more stringent than (2.11) if $\Delta < 5.3197$.

A second upper bound for $\Delta$-regular graphs discussed in [4] is

$$f_\Delta \leq f_{u,BCL2}(\Delta) \ , \tag{2.12}$$

where

$$f_{u,BCL2}(\Delta) = \left(\frac{\Delta + 1}{\eta(\Delta)}\right)\left(\frac{\Delta - 1}{\Delta - \eta(\Delta)}\right)^{\frac{\Delta - 2}{2}} \ , \tag{2.13}$$

with

$$\eta(\Delta) = \frac{(\Delta + 1)(\Delta + 1 - \sqrt{\Delta^2 - 2\Delta + 5})}{2(\Delta - 1)} \ . \tag{2.14}$$

(See also [11] for related work.) Since this bound applies uniformly for any $n(G)$, it also applies to the limit as $n(G) \to \infty$:

$$\phi(\Lambda) \leq \phi_{u,BCL2}(\Delta) \ , \tag{2.15}$$

where $\phi_{u,BCL2}(\Delta) = f_{u,BCL2}(\Delta)$. We list below the analytic expressions of the upper bound $\phi_{u,BCL2}(\Delta)$ and the corresponding numerical values (given to the indicated number of significant figures) for the values of $\Delta$ that are relevant for comparison with our bounds:

$$\phi_{u,BCL2}(2) = \frac{3 + \sqrt{5}}{2} = 2.618034 \tag{2.16}$$

...
limit, but is less than the upper bounds for the case of $\Delta = 2$, we recall the elementary values on Archimedean lattices, namely $\Delta = 3$.

For a given $\Delta$, these BCL upper bounds are all less stringent than the upper bounds that we derived on $\phi(\Lambda)$ and also the values of $\phi(\Lambda)$ for lattices with the same value of $\Delta$. As before, their upper bound for $\phi(\Lambda) = 2 = 2$. This saturates the upper bound (2.5), and hence, in the infinite-length limit, $\phi$ is a monotonically increasing function of the strip width. For the smallest widths, one obtains simple algebraic expressions for these exponential growth constants.

In the two cases with $\Delta = 5$ and $\Delta = 6$ relevant for Archimedean lattices, these BCL upper bounds are all less stringent than the upper bounds that we derived on $\phi(\Lambda)$ for the Archimedean lattices $\Lambda$ in [4].

This is supported in part by the Taiwan Ministry of Science and Technology grant MOST 109-2112-M-006-008 (S.-C.C.) and by the U.S. National Science Foundation grant No. NSF-PHY-1915093 (R.S.).

In Appendix A: Some Background from Graph Theory, in this appendix we briefly review some background from graph theory, in particular, a connection of $N_{SF}(G)$ and $N_{CSSG}(G)$ with evaluations of the Tutte polynomial. As in the text, let $G = (V, E)$ be a graph defined by its vertex and edge sets $V$ and $E$. Further, let $n = n(G) = |V|$, $e(G) = |E|$, $k(G)$, and $c(G)$ denote the numbers of vertices, edges, connected components, and linearly independent cycles in $G$, respectively. The Tutte polynomial of a graph $G$, denoted $T(G, x, y)$, is defined as

$$T(G, x, y) = \sum_{G' \subseteq G} (x-1)^{k(G')-k(G)} (y-1)^{c(G')}, \quad (A1)$$
where $G'$ is a spanning subgraph of $G$ (see, e.g., [12, 13]). The numbers of spanning forests and connected spanning subgraphs are evaluations of the Tutte polynomial:

$$N_{SF}(G) = T(G, 2, 1) \quad (A2)$$

and

$$N_{CSSG}(G) = T(G, 1, 2). \quad (A3)$$

For a general graph $G$, the calculation of $N_{SF}(G)$ and $N_{CSSG}(G)$ are # P hard [20]. This is why it is useful to have bounds on these quantities, and also on the corresponding exponential growth constants.

A remark is in order here concerning graphs with loops. Recall that a loop is an edge that connects a vertex back to itself. The reason that we restrict to loopless graphs in our work is that if one allows loops, then one loses a connection between the vertex degree of a $\Delta$-regular graph $G$ and $\phi(\{G\})$. This can be illustrated in the simple case of the circuit graph $C_n$, which is $\Delta$-regular with $\Delta = 2$. One has $T(C_n, x, y) = y + \sum_{j=1}^{n-1} x^j$, so that, in the limit $n \to \infty$, $\phi(\{C\}) = 2$. Now let us attach $m$ loops ($\ell$) to each vertex of $C_n$. We denote the resultant graph as $C_{n,m\ell}$. This is again a $\Delta$-regular graph with vertex degree $\Delta = 2(1 + m)$. The Tutte polynomial is

$$T(C_{n,m\ell}, x, y) = y^{mn} T(C_n, x, y) = y^{mn} \left( y + \sum_{j=1}^{n-1} x^j \right). \quad (A4)$$

Hence,

$$N_{SF}(C_{n,m\ell}) = T(C_{n,m\ell}, 2, 1) = T(C_n, 2, 1) = N_{SF}(C_n) \quad (A5)$$

and, in the limit $n \to \infty$, the corresponding values of $\phi$ are the same for the $C_n$ and $C_{n,m\ell}$ families of graphs, although the vertex degrees are different for these families. Thus, if one were to allow modifications of Archimedean lattices with loops, one would lose the informative connection between the vertex degree and the value of $\phi(\Lambda)$.

[1] S.-C. Chang and R. Shrock, Asymptotic behavior of spanning forests and connected spanning subgraphs on two-dimensional lattices, Int. J. Mod. Phys. B 34, 205029 (2020) arXiv:2002.07150.
[2] B. Grünbaum and G. C. Shephard, Tilings and Patterns: An Introduction (Freeman, New York, 1989).
[3] R. Shrock and S.-H. Tsai, Lower bounds and series for the ground state entropy of the Potts antiferromagnet on Archimedean lattices and their duals, Phys. Rev. E 56, 4111-4124 (1997).
[4] M. Borbély, P. Csikvári, and H. Luo, On the number of forests and connected spanning subgraphs, arXiv:2005.12752.
[5] C. Merino and D. J. A. Welsh, Forest, colorings, and acyclic orientations of the square lattice, Ann. Comb. 3, 417-429 (1999).
[6] N. Calkin, C. Merino, S. Noble and M. Noy, Improved bounds for the number of forests and acyclic orientations in the square lattice, Electron. J. Combin. 10, 1-18 (2003).
[7] A. P. Mani, On some Tutte polynomial sequences in the square lattice, J. Combin. Theory B 102, 436-453 (2012).
[8] S.-C. Chang and R. Shrock, Tutte polynomials and related asymptotic limiting functions for recursive families of graphs (talk given by R. Shrock at Workshop on Tutte polynomials, Centre de Recerca Matemática (CRM), Sept. 2001, Univ. Autonoma de Barcelona), Adv. Appl. Math. 32, 44-87 (2004).
[9] S.-C. Chang and R. Shrock, Study of exponential growth constants of directed heteropolYGONAL Archimedean lattices, J. Stat. Phys. 174, 1288-1315 (2019).
[10] S.-C. Chang and R. Shrock, Asymptotic behavior of acyclic and cyclic orientations of directed lattice graphs, Physica A 540, 123059 (2020).
[11] N. Kahale and L. J. Schulman, Bounds on the chromatic polynomial and the number of acyclic orientations of a graph, Combinatorica 16, 383-397 (1996).
[12] W. T. Tutte, On dichromatic polynomials, J. Combin. Theory 2, 301-320 (1967).
[13] For relevant graph theory background, see, e.g., N. Biggs, Algebraic Graph Theory (Cambridge Univ. Press, Cambridge, UK, 1993) and B. Bollobás, Modern Graph Theory (Springer, New York, 1998).
[14] R. Shrock, Exact Potts Model Partition Functions for Ladder Graphs, Physica A 283, 388-446 (2000).
[15] S.-C. Chang and R. Shrock, Exact Potts Model Partition Functions on Strips of the Triangular Lattice, Physica A 286, 189-238 (2000).
[16] S.-C. Chang and R. Shrock, Exact Potts Model Partition Functions on Strips of the Honeycomb Lattice, Physica A 296, 183-233 (2001).
[17] S.-C. Chang and R. Shrock, Exact Partition Function for the Potts Model with Next-Nearest Neighbor Couplings on Strips of the Square Lattice, Int. J. Mod. Phys. B 15, 443-478 (2001).
[18] S.-C. Chang and R. Shrock, Exact Potts Model Partition Functions on Wider Arbitrary-Length Strips of the Square Lattice, Physica A 296, 234-288 (2001).
[19] S.-C. Chang and R. Shrock, Complex-Temperature Phase Diagrams for the $q$-State Potts Model on Self-Dual Families of Graphs and the Nature of the $q \to \infty$ Limit, Phys. Rev. E 64, 066116 (2001).
[20] F. Jaeger, D. L. Vertigan, and D. J. A. Welsh, On the computational complexity of the Jones and Tutte polynomials, Math. Proc. Camb. Phil. Soc. 108, 35-53 (1990).
TABLE I: For each Archimedean lattice Λ, this table lists the value of φ(Λ) and the upper bound, \( \phi_u(\Lambda) \), both from Ref. [1], together with the ratio \( R_\phi(\Lambda) = \frac{\phi(\Lambda)}{\phi_u(\Lambda)} \). The lattices are listed in order of increasing vertex degree \( \Delta(\Lambda) \).

| Λ             | \( \Delta(\Lambda) \) | \( g(\Lambda) \) | \( \phi(\Lambda) \)      | \( \phi_u(\Lambda) \) | \( R_\phi(\Lambda) \) |
|---------------|-------------------------|-------------------|---------------------------|------------------------|------------------------|
| \( (4 \cdot 8^2) \) | 3                       | 4                 | 2.77931 ± 0.00018         | 2.779486               | 0.99994                |
| \( (6^3) = hc \) | 3                       | 6                 | 2.80428 ± 0.00050         | 2.804781               | 0.99982                |
| \( (3 \cdot 6 \cdot 3 \cdot 6) \) | 4                       | 3                 | 3.602 ± 0.012             | 3.614045               | 0.99667                |
| \( (4^4) = sq \) | 4                       | 4                 | 3.687 ± 0.012             | 3.699659               | 0.99658                |
| \( (3^1 \cdot 4^2) \) | 5                       | 3                 | 4.530 ± 0.024             | 4.553665               | 0.99480                |
| \( (3^2 \cdot 4 \cdot 3 \cdot 4) \) | 5                       | 3                 | 4.503 ± 0.065             | 4.568231               | 0.98572                |
| \( (3^6) = tri \) | 6                       | 3                 | 5.444 ± 0.051             | 5.494840               | 0.99075                |

TABLE II: Comparison of upper bounds on φ(Λ) for Archimedean lattices Λ. The most stringent upper bounds on φ(Λ) are those from Ref. [1], denoted \( \phi_u(\Lambda) \). The table also lists the general upper bound \( 2^{\Delta/2} \) and, where applicable, the upper bounds \( \phi_{u,BCLi}(\Delta) \), \( i = 1, 2, 3, 4 \) from [4]. The BCL3 bound applies for \( \Delta = 4 \), while the BCL4 bound applies for \( \Delta = 5, 6 \). For lattices where a given BCLi bound is not applicable, we denote this by a dash.

| Λ             | \( \Delta(\Lambda) \) | \( g(\Lambda) \) | \( \phi_u(\Lambda) \) | \( 2^{\Delta/2} \) | \( \phi_{u,BCL1}(\Delta) \) | \( \phi_{u,BCL2}(\Delta) \) | \( \phi_{u,BCL3,4}(\Delta) \) |
|---------------|-------------------------|-------------------|------------------------|---------------------|-----------------------------|-----------------------------|-----------------------------|
| \( (4 \cdot 8^2) \) | 3                       | 4                 | 2.779486               | 2.82843             | 4                           | 3.57081                     | –                           |
| \( (6^3) = hc \) | 3                       | 6                 | 2.804781               | 2.82843             | 4                           | 3.57081                     | –                           |
| \( (3 \cdot 6 \cdot 3 \cdot 6) \) | 4                       | 3                 | 3.614045               | 4                   | 5                           | 4.54845                     | 3.994                       |
| \( (4^4) = sq \) | 4                       | 4                 | 3.699659               | 4                   | 5                           | 4.54845                     | 3.994                       |
| \( (3^1 \cdot 4^2) \) | 5                       | 3                 | 4.553665               | 5.65685             | 6                           | 5.53618                     | 5.1965                      |
| \( (3^2 \cdot 4 \cdot 3 \cdot 4) \) | 5                       | 3                 | 4.568231               | 5.65685             | 6                           | 5.53618                     | 5.1965                      |
| \( (3^6) = tri \) | 6                       | 3                 | 5.494840               | 8                   | 7                           | 6.52864                     | 6.3367                      |