The error assessment in case of using two models for the
displacement of a linear spring

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Abstract. The displacement of the free end of a spring is not always an axial displacement. In
some circumstances, the free end of the spring displaces along a circular trajectory, and the
spring elongation is considered approximately equal to its axial displacement. Such an
approximation is used for modelling the suspension of railway wagons. The present work
purpose is to exactly determine the elastic force in case of spring free end circular
displacement, in order to compute the error of considering this displacement equal to the spring
axial displacement. Then, the real dynamic behavior of the model is compared with the
approximate one, by numerical simulation.

1. Introduction
It is well-known that the elastic force in a linear spring is computed as the product between the elastic
constant of the spring (stiffness) and the spring elongation. In some mechanisms, the free end of the
spring is attached to a lever. In such circumstances, the free end of the spring displaces on a circular
trajectory, the circle radius being equal to the lever arm. In case of small rotations (±5°), it is
permitted to compute the elastic force like in case of a linear axial displacement of the spring. Such an
approximation is used for modelling the suspension of railway wagons [1], [2], [3], [4], [5]. The
present work purpose is to exactly compute the elastic force in the spring in case of its free end
circular displacement and to assess the error when the linear axial displacement approximation is used.
Based on the required accuracy, the approximate calculus could be or not convenient. At the same
time, a comparative analysis by numerical simulation is performed for the dynamic behavior of the
spring in the two considered cases, in order to assess the validity of approximate calculus in these
circumstances, as well.

2. The exact calculus of spring elongation in case of its free end circular displacement
The spring deformation when its free end displaces along a circular trajectory is detailed pictured in
figure 1. The stiffness of the spring is \( k_e \) and the original length is \( l_0 \). The end of the spring P is
attached to a lever, whose arm is \( b \) in length. It is presumed that the lever rotates an angle \( \theta \) and point
\( P_0 \) reaches the location of point P.

In order to compute the exact elongation of the spring, its final length, after lever rotation at an
angle \( \theta \) should be determined. By applying the cosine theorem in triangle OPQ, it results:
\[ PQ = (OP^2 + OQ^2 - 2 \cdot OP \cdot OQ \cdot \cos(\theta + \varphi))^{1/2} \]  

(1)

**Figure 1.** The model of the circular displacement.

On the other side, from the right triangle \( \text{OP}_0\text{Q} \), it is obtained:

\[ OQ = (\text{OP}_0^2 + \text{P}_0\text{Q}^2)^{1/2} \]

(2)

Because \( \text{OP}_0 = b, \text{OP} = b \) and \( \text{P}_0\text{Q} = l_0 \) equation (1) is has the form:

\[ PQ = (2b^2 + l_0^2 - 2b(b^2 + l_0^2 \cos(\theta + \varphi)))^{1/2} \]

(3)

From the right triangle \( \text{OPQ} \) it also results:

\[ \sin \varphi = l_0 / \sqrt{b^2 + l_0^2} \quad \text{and} \quad \cos \varphi = b / \sqrt{b^2 + l_0^2} \]

(4)

By substituting equations (4) in \( \cos(\theta + \varphi) \) expression, it results:

\[ \cos(\theta + \varphi) = \cos \theta \cos \varphi - \sin \theta \sin \varphi = \cos \theta \cdot b / \sqrt{b^2 + l_0^2} - \sin \theta \cdot l_0 / \sqrt{b^2 + l_0^2} \]

(5)

The length \( PQ \) of the spring in terms of angle \( \theta \) is determined by introducing equation (5) in equation (3):

\[ PQ = (2b^2 + l_0^2 - 2b(b \cos \theta - l_0 \sin \theta))^{1/2} \]

(6)

The elastic force in the spring will be accurately computed according to the following relation:

\[ F_e = k_e(\text{PQ} - l_0) = k_e[(2b^2 + l_0^2 - 2b(b \cos \theta - l_0 \sin \theta))^{1/2} - l_0] \]

(7)

**3. Comparison between the case of circular and linear displacements**

In the case of linear displacement model, it is considered that the elongation is \( \text{P}_0\text{A} = \text{PB} \) and the elastic force becomes:

\[ F_0 = k_e \cdot \text{P}_0\text{A} = k_e b \sin \theta \]

(8)

This is the approximate expression of the elastic force. The elastic forces in the two considered cases, for \( \theta \) lying between \( \pm \pi / 6 \), obtained by using MATHCAD [6], are presented in figure 2. Their variation is a linear one. The relative error related to the exact value of the elastic force is rendered evident by plotting the graph of the function \( |F_e(\theta) - F_{e1}(\theta)| / |F_e(\theta)| \), which is represented in figure 3.

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Figure 2. The diagram of $F_d(\theta)$ and $F_{e1}(\theta)$ for $\theta$ lying between $\pm \pi / 6$.

Figure 3. The relative error of the elastic force with respect to the exact calculus.

The graph shows that between the two variants of computing the elastic force the maximum difference is of 0.7%. In case of using only the elastic force value, the approximate calculus is a very satisfactory one for the considered interval of angle $\theta$.

Because the lever motion is caused by the elastic force moment, it is required to compute it in the two considered circumstances.

The moment with respect to the pole $O$ produced by the elastic force acts upon the lever, as a reaction to the moment given by force $F$ which applies at the opposite end. In case of small oscillations, it is considered that the arm of the elastic force given by equation (8) is $b$, that is:

$$M_1(\theta) = F_{o}b = k_o b^2 \sin \theta$$

(9)

This is the approximate expression of the moment produced by the elastic force with respect to pole $O$.

In the case of circular displacement model, the moment produced by the elastic force equals the moment given by its component normal to the lever, $F_n$, whose magnitude is:

$$F_n = F \sin \theta$$

(10)
According to the sinus theorem applied in triangle $OPQ$, it results:

$$\sin \alpha = \sin(\theta + \varphi) \cdot OQ / PQ$$

(11)

By substituting equation (11) in equation (10) and using relations (4), the following final expression for $F_n$ is determined:

$$F_n = k_e [1 - l_0 / (2b^2 + l_0^2 - 2b(b\cos \theta - l_0 \sin \theta)^{1/2})](b\sin \theta + l_0 \cos \theta)$$

(12)

The arm of this force with respect to pole 0 is $b$, therefore its moment with respect to this point is given by the following relation:

$$M_z(\theta) = F_n b = k_e b [1 - l_0 / (2b^2 + l_0^2 - 2b(b\cos \theta - l_0 \sin \theta)^{1/2})](b\sin \theta + l_0 \cos \theta)$$

(13)

This is the accurate expression of the moment produced by elastic force with respect to the pole $O$ because the circular rotations are not considered as small rotations.

Figure 4. The diagram of $M_1(\theta)$ and $M_2(\theta)$ for $\theta$ lying between $\pm \pi / 6$.

In order to assess the difference between the two moments, the following numerical values are considered: $l_0 = 0.6m$, $b = 0.2m$, $k_e = 500N/m$.

The graphs of functions $M_1(\theta)$ and $M_2(\theta)$ for $\theta$ lying between $\pm \pi / 6$ are presented in figure 4. It can be noticed that in absolute value the moment $M_2(\theta)$ is lower than $M_1(\theta)$, which means that the exact calculus leads to lower moments. The graph of function

$$\left| M_1(\theta) - M_2(\theta) \right| / \left| M_1(\theta) \right|$$

is pictured in figure 5 and represents the relative error related to the exact value of the moment.

Because the error between the exact and the approximate values of the elastic force moment is significant when angle for $\theta$ lying between $\pm \pi / 6$, it has been represented only the interval for which the error is lower than 5%, that is for an angle $\theta$ lying between $\pm 16^{\circ}30'$. A relative error lower than 3% is obtained for $\theta$ lying between $\pm 13^{\circ}$. As a conclusion, the approximate calculus can be used when a certain error is assumed.

4. The dynamic behavior modelling

The dynamic behavior is analyzed in free regime, that is, an initial angular displacement of the lever, $\theta_0$ is considered. The motion differential equations are:

$$J\ddot{\theta} = M_1(\theta) \text{ and } J\ddot{\theta} = M_2(\theta)$$

(15)
where $J$ is the moment of inertia of the lever. In the numerical simulation $J = 1Kgm^2$ and $k_e = 500N/m$.

The lever has an oscillatory motion, as shown in figure 6. The continuous line is the representation of exact values, while the dashed line is used for approximate values.

**Figure 5.** The relative error of the moment related to the exact moment.

**Figure 6.** The angle $\theta$ of free oscillation of the lever.

By using the numerical simulation in MATLAB [7] [8], for different angles $\theta_0$, it has been stated that the exact motion period is greater than the approximate period and the differences between them increases when angle $\theta_0$ increases.

For $\theta_0 = 22^o30'$, the real period is greater by 3% than the approximate period, which is considered as acceptable. For $\theta \leq 5^o$ the results coincide.

For the maximum angle of acceptable error, that is $\theta_0 = 22^o30'$, another variable has been considered: the spring stiffness. It has been stated that when this stiffness lies between $50N/m$ and $10000N/m$, the period relative error is around 3%. These results are shown in table 1.
Table 1. The relative error of the time period.

| Spring stiffness [N/m] | T1 [s]   | T2 [s]   | (T1-T2)/T1 |
|-----------------------|---------|---------|------------|
| 10                    | 10.350  | 10.03   | 0.0309     |
| 50                    | 4.625   | 4.485   | 0.0302     |
| 100                   | 3.270   | 3.170   | 0.0305     |
| 500                   | 1.4624  | 1.4185  | 0.0300     |
| 1000                  | 1.0340  | 1.003   | 0.0299     |
| 5000                  | 0.4625  | 0.4485  | 0.0302     |
| 10000                 | 0.3270  | 0.3170  | 0.0305     |

5. Conclusions
The modelling of circular oscillations by approximation with linear oscillations is valid for the case of small oscillations. The work points out that this approximation could be used even for angles greater than $5^\circ$. A mechanical system consisting in a lever with a spring connected to an end is considered. This is the most common situation in engineering when, the circular oscillations are approximated with linear ones. The numerical simulations are performed in order to find out if the error of this approximation can be of convenient accuracy and for this reason, two thresholds are considered: one of 3% and the other of 5%.

The first numerical simulation refers to the elastic force and a good accuracy of 0.7% results when the rotation angle of the lever is in the interval $\pm \pi / 6$.

The second numerical simulation refers to the elastic force moment that acts on the lever. The simulation results show that for rotations up to $\pm 16^\circ 30'$, the approximate assessment of elastic force moment leads to an error of 5%, while for rotations up to $\pm 13^\circ$, the error is around 3%.

The third numerical simulation refers to the dynamical behavior of the mechanism. An initial value $\theta_0$ of the rotation angle is given and a free oscillating motion is generated. In case of the exact assessment, the period of these oscillations increases when the angle $\theta_0$ increases. The error is less the 3% if the angle $\theta_0$ is less than $22^\circ 30'$. For this initial value, the influence of the spring stiffness is then analyzed. The period decreases when the spring stiffness increases but the error between the exact calculus and the approximate calculus remains constant. The exact period is 3% greater than the period given by the approximate calculus, for all spring stiffness.

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