Thermal radiation effect on unsteady mixed convection boundary layer flow and heat transfer of nanofluid over permeable stretching surface through porous medium in the presence of heat generation

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Abstract
Here, we study the effect of mixed convection and thermal radiation on unsteady boundary layer of heat transfer and nanofluid flow over permeable moving surface through a porous medium. The effect of heat generation is also discussed. The equations governing the system are the continuity equation, momentum equation and the heat transfer equation. These governing equations transformed into a system of nondimensional equations contain many physical parameters that describe the study. The transformed equations are solved numerically using an implicit finite difference technique with Newton’s linearization method. The thermo-physical parameters describe the study are the mixed convection parameter \( \alpha \), \( 0 \ll \alpha \ll 10 \), the Radiation parameter \( R_d \), \( 0 \ll R_d \ll 1.0 \), porous medium parameter \( k \), \( 0 \ll k \ll 1.0 \), the nanoparticles volume \( \phi \), \( 0 \ll \phi \ll 0.2 \), the suction or injection parameter \( f_m \), \( -1 \ll f_m \ll 1 \), the unsteadiness parameter \( A_t \), \( 1 \ll A_t \ll 2 \) and the heat source parameter \( \lambda = 0.5 \). The influence of the thermo-physical parameters is obtained analytically.
and displayed graphically. Comparisons of some special cases of the present study are performed with previously published studies and a good agreement is obtained.

**Keywords**
Thermal radiation, mixed convection, heat transfer, nanofluid flow, porous medium, heat generation

**Introduction**

The fluids have relatively low thermal conductivities when compared to the thermal conductivity of metals. The thermal conductivity of the fluid can be increased by adding small metal particles to that fluid. A fluid containing suspensions of metallic or non-metallic solid particles with sizes on the order of nanometers or micrometers is termed as nanofluid. Nanofluids can be used to cool automobile engines and welding equipment and high heat-flux devices such as high-power microwave tubes and high-power laser diode arrays. A nanofluid coolant could flow through tiny passages in micro-electro-mechanical systems to improve its efficiency. The measurement of nanofluids critical heat flux in a forced convection loop is useful for nuclear applications. Nanofluids are crucial applications in science and technology, industrial applications such as plastic, marine engineering, polymer industries, home cancer therapy, and building sciences. Flows through moving vertical flat plates also have enormous applications in aerosols engineering, aerodynamics and civil engineering, therefore researchers in these areas are interested in this field.

The wide range of applications of nanofluids has carried out significant researches in recent years to study the heat transfer characteristics of nanofluids. Xuan and Li\(^1\) investigated the enhancement of the heat transfer for nanofluids, while Eastman et al.\(^2\) studied the anomalously increased effective thermal conductivities of ethylene glycol-based nanofluids containing copper nanoparticles. Khan and Pop\(^3\) studied the boundary-layer flow of a nanofluid past a Stretching sheet. Syakila et al.\(^4\) investigated the Blasius and Sakiadis problems in nanofluids. Makinde and Aziz\(^5\) considered the boundary layer flow of a nanofluid past a stretching sheet with a convective boundary condition. Bachok et al.\(^6\) studied the boundary layer flow of nanofluids over a moving surface in a flowing fluid. Hamad\(^7\) found the analytical solutions of convective flow and heat transfer of an incompressible viscous nanofluid past a semi-infinite vertical stretching sheet in the presence of a magnetic field. Hamad and Ferdows\(^8\) investigated heat and mass transfer analysis for boundary layer stagnation-point flow over a stretching sheet in a porous medium saturated by a nanofluid with internal heat generation/absorption.

An unsteady flows, such as a start-up process and periodic fluid motion, are very much important in engineering practices. In many engineering problems, such as helicopter rotor, the ship propeller, the cascades of blades of the turbo-machinery unsteady environment occur. The investigation of the simultaneous effects of magnetic field and unsteadiness of boundary layer flow and heat transfer due to stretching sheet in the presence of heat source is investigated by Ibrahim and Shanker.\(^9\) Elsayed et al.\(^10\) investigated the effect of magnetic on flow and heat transfer of a nanofluid over an unsteady continuous moving surface in the presence of suction. Mukhopadhyay\(^11\) investigated the thermal radiation effects on unsteady mixed convection flow and heat transfer over a porous
stretching surface in a porous medium. Ishak et al.\textsuperscript{12} investigated the flow of an unsteady boundary layer in a micropolar fluid over a stretching sheet. Rohnia et al.\textsuperscript{13} discussed the heat transfer and flow over an unsteady shrinking sheet with suction in nanofluids. Mahdy\textsuperscript{14} investigated numerically unsteady mixed convection heat transfer of nanofluid over a stretching vertical surface. Bakier\textsuperscript{15} investigated the effect of thermal radiation on mixed convection from vertical surfaces in saturated porous media.

Transfer of heat in a liquid film on an unsteady stretching sheet explained by Liu.\textsuperscript{16} Sheikholeslami and Ganji\textsuperscript{17} investigated the heat transfer and unsteady nanofluid flow in the presence of thermal radiation considering the magnetic field. Keller\textsuperscript{18} used numerical methods in boundary layer theory. Cebeci and Bradshaw\textsuperscript{19} discussed the computational and physical aspects of convective heat transfer. Elbashbeshy et al.\textsuperscript{20} investigated the thermal radiation effect on natural convection heat transfer around thermal spheres embedded in porous media by using the Keller box method.\textsuperscript{18,19} Maleki et al.\textsuperscript{21} investigated the heat transfer and nanofluid flow over a porous plate with radiation and slip boundary conditions.

Much work has been done concerning the stagnation-point flow due to its essential practical applications to the industrial environment. Therefore, several research works are available in the literature regarding the heat transfer enhancement using nanofluids in addition to the study of the free and forced stagnation point of the viscous fluid flows.\textsuperscript{22,23} In recent years, the phenomenon of mixed convection flow at the inaction point of the two-dimensional geometric for mutilated physical characteristics has been discussed by many researchers, for example, see references.\textsuperscript{24,25} Some new contributions in heat transfer in magnetohydrodynamics have been investigated in Refs.\textsuperscript{26–29}

The current study aims to investigate numerically the combined effects of the mixed convection and thermal radiation on unsteady boundary layer of heat transfer and nanofluid flow over permeable moving surface through a porous medium in the presence of heat source or sink. To the best of our knowledge, this problem has not been considered before.

### Analysis and formulation of the problem

We consider the two-dimensional mixed convection boundary layer of an incompressible viscous liquid through porous medium along a permeable stretching vertical wall in the presence of thermal radiation. The vertical sheet is taken along $x$–axis and normal to $y$–axis. The velocity $U_w(x,t)$ and the temperature of the sheet $T_w(x,t)$ are supposed to have linear variation with $x$ and an inverse for its decrease with time (Mahdy.\textsuperscript{14} The fluid is a water-based nanofluid containing Copper Cu nanoparticles. The base fluid and the nanoparticles are assumed to be in equilibrium and no slip occurs between them as shown in Figure 1. The Thermophysical properties for the nanofluid are given in Table 1.

### Table 1. Thermophysical properties of the fluid (water) and the nanoparticles of Cu.

| Properties     | $C_p$ (J/kgK) | $\rho$ (kg/m$^3$) | $K$ (W/mK) | $\alpha \times 10^7$ |
|----------------|---------------|-------------------|------------|---------------------|
| Fluid (water)  | 4179          | 997.1             | 0.613      | 1.47                |
| Cu             | 385           | 8933              | 400        | 1163.1              |
The Hamilton-Crosser’s model of thermal conductivity of nanofluid is given by

$$k_{nf} = \left[ \frac{(k_s + (n - 1)k_f) - (n - 1)\phi(k_f - k_s)}{(k_s + (n - 1)k_f) + \phi(k_f - k_s)} \right]k_f$$  \hspace{1cm} (1)

where $k_{nf}$ denotes the effective thermal conductivity of the nanofluid, $k_f$ the thermal conductivity of the fluid and $n$ is the empirical shape factor for the nanoparticles. For spherical nanoparticles $n = 3$ and for cylindrical nanoparticles $n = 3/2$. Throughout this work we analyze the impact of these two models, cylindrical and spherical models, on heat transfer rate with Cu nanoparticles.

The governing equations of the model are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \hspace{1cm} (2)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left( \frac{\mu_{nf}}{\rho_{nf}} \right) \frac{\partial^2 u}{\partial y^2} + g\beta_{nf}(T - T_{\infty}) - \left( \frac{\mu_{nf}}{\rho_{nf}} \right) \frac{u}{k_0} \hspace{1cm} (3)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \left( \frac{k_{nf}}{(\rho C_P)_{nf}} \right) \frac{\partial^2 T}{\partial y^2} + \left( \frac{Q}{(\rho C_P)_{nf}} \right) (T - T_{\infty}) - \frac{1}{(\rho C_P)_{nf}} \frac{\partial q_r}{\partial y} \hspace{1cm} (4)$$

Where $u$ and $v$ are velocity components in the $x$ and $y$ directions respectively, $t$ is the time, $g$ is the gravity field, $\mu_{nf}$ is the nanofluid dynamic viscosity, $\rho_{nf}$ is the nanofluid density,
\( \alpha_{nf} \) is the nanofluid thermal diffusion and \( \beta_{nf} \) is the nanofluid volumetric coefficient of thermal expansion. \( T \) is the nanofluid temperature, \( k_0 \) is the permeability of porous medium where \( k_0 = k_1 (1 - \gamma t) \) and \( k_1 \) is the initial permeability. \( k_{nf} \) is the nanofluid coefficient of thermal conductivity, \((C_p)_{nf}\) is the nanofluid specific heat at constant pressure, \( Q \) is the heat source \( Q(x) = Q_0 / x \), \( Q_0 \) is a constant and \( q_r \) is the radiative heat flux. The appropriate boundary conditions are

\[ u = U_w, \quad v = V_w, \quad T = T_w \text{ at } y = 0 \]
\[ u = 0, \quad T \rightarrow T_\infty \text{ as } y \rightarrow y_\infty \]  

(5)

Where

\[ U_w = \frac{ax}{1 - \gamma t}, \quad \text{and } T_w = T_\infty + \frac{bx}{(1 - \gamma t)^2} \]  

(6)

Here \( V_w \) is the suction or injection velocity of the fluid (suction for \( V_w > 0 \) and injection for \( V_w < 0 \)), \( \nu_f \) is the fluid kinematic viscosity and \( T_\infty \) is ambient temperature. \( a \) and \( \gamma \) are constants with dimension time\(^{-1}\) where \( a > 0 \), \( \gamma \geq 0 \) and \( \gamma t < 1 \), \( b \) is a constant with dimension temperature(length\(^{-1}\)), \( b > 0 \) for assisting flows, \( b < 0 \) for opposing flows while \( b = 0 \) for vanishing forced convection\(^{14}\).

The physical properties of the fluid are assumed to be constants and the viscous dissipation is neglected. The Rosseland approximation for radiation is considered and the radiative heat flux is given as

\[ q_r = -\frac{4\sigma}{3\sigma^*} \frac{\partial T^4}{\partial y} \]  

(7)

Here \( \sigma \) is the Stefan-Boltzmann constant and \( \sigma^* \) is the mean absorption coefficient. \( T^4 \) can be expanded by Taylor series about \( T_\infty \) and neglecting higher-order terms, we have \( T^4 = 4T_\infty^3 T - 3T_\infty^4 \). Substituting in radiative heat flux \( q_r \), in equation (7), it becomes

\[ q_r = -\frac{16\sigma T_\infty^3}{3\sigma^*} \frac{\partial T}{\partial y} \]  

(8)

The nanofluid properties are

\[ \mu_{nf} = \frac{\mu_f}{(1 - \phi)^{2.5}}, \quad \alpha_{nf} = \frac{k_{nf}}{(\rho C_p)_{nf}}, \quad \rho_{nf} = (1 - \phi)\rho_f + \phi \rho_s, \]
\[ (\rho \beta)_{nf} = (1 - \phi)(\rho \beta)_f + \phi (\rho \beta)_s, \quad (\rho C_P)_{nf} = (1 - \phi)(\rho C_P)_f + \phi (\rho C_P)_s. \]  

(9)

Where \( \mu_f \) is the fluid dynamic viscosity, \( \rho_f \) is the base fluid density, \( \rho_s \) is the nanoparticles solid density, \( \alpha_{nf} \) represents the nanofluid thermal diffusion and \( \alpha_f \) is the fluid thermal diffusion. \( \beta_{nf} \) represents the volumetric coefficient of nanofluid thermal expansion, \( \beta_f \) is the fluid volumetric coefficient of thermal expansion, \( k_{nf} \) is the nanofluid coefficient of thermal conductivity and \( k_f \) is the fluid coefficient of thermal conductivity. \((C_p)_{nf}\) is the nanofluid specific heat at constant pressure, \((C_p)_f\) is the fluid specific heat at constant pressure, \((C_p)_s\) is the effect of the heat capacity of the nanoparticles solid and \( \phi \) is
nanoparticle volume fraction. It is worth mentioning that when \( \phi = 0 \), the study reduces to those of a viscous regular fluid. For similarity solution, the following nondimensional variables are considered as

\[
\eta = \frac{y}{\sqrt{\nu_f (1 - \gamma t)}}, \quad \psi = \frac{a \nu_f}{(1 - \gamma t)^{1/2}} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}
\]

(10)

Where \( \eta \) is the similarity variable, \( \nu_f \) denotes the kinematic viscosity of the liquid, \( \theta \) is the dimensionless temperature and \( \psi \) is the stream function, which is related to the velocity components by

\[
u = -\frac{\partial \psi}{\partial x}, \quad u = \frac{ax}{(1 - \gamma t)} f'(\eta)
\]

(11)

Substituting in the governing equations, we get

\[
f'''' + \left( \frac{\rho_f}{\rho_f} \right) \left( f f'' - f'' - A \left( f' + \frac{\eta}{2} f'' \right) \right) - k f' + \left( \frac{\rho \beta_f}{\rho_f} \right) \left( \frac{\eta}{\frac{1}{2}} f' \right) \alpha \theta = 0
\]

(13)

\[
\left( \left( \frac{k_{nf}}{k_f} + Rd \right) Pr \right) \theta'' + \left( \frac{\rho C_p f}{\rho C_p f_f} \right) \left( f \theta' - f' \theta - A \left( 2 \theta + \frac{\eta}{2} \theta' \right) \right) + \lambda \theta = 0
\]

(14)

The boundary conditions are

\[
f(0) = f_w, \quad f'(0) = 1, \quad \theta(0) = 1, \quad f'(\eta_\infty) = 0 \quad \text{and} \quad \theta(\eta_\infty) = 0
\]

(15)

Where prime is differentiation with respect to \( \eta \) and \( \alpha \) is the mixed convection parameter, \( Rd \) is the Radiation parameter and \( k \) is the porous medium parameter are computed respectively as

\[
a = \frac{Gr_x}{Re_x^2}, \quad Rd = \left( \frac{16 \sigma T_\infty^3}{3 \sigma^* k_f} \right), \quad k = \frac{1}{Re_x Da_x} = \left( \frac{\nu_f}{a k_f} \right)
\]

(16)

\( f_w \) is the suction or injection parameter (suction for \( f_w > 0 \) and injection for \( f_w < 0 \)), \( A \) is the unsteadiness parameter and \( \lambda \) is the heat source parameter, are defined by

\[
f_w = -V_w \sqrt{\left( \frac{x}{\nu_f U_w} \right)} = -\frac{V_w}{U_w} Re_x^{1/2}, \quad A = \left( \frac{Y}{a} \right), \quad \lambda = \frac{Q_0}{U_w (\rho C_p) f_f}
\]

(17)

Here \( Pr \) is the Prandtl number, \( Gr_x \) is the local Grashof number, \( Re_x \) is the local Reynolds number and \( Da_x \) is the local Darcy number which are given by

\[
Pr = \left( \frac{\nu C_p}{k} \right)_f, \quad Gr = \frac{g \beta_f (T_w - T_\infty) x^3}{\nu_f^2}, \quad Re_x = \frac{U_w x}{\nu_f}, \quad Da_x = \frac{x^2}{k_0}
\]

(18)
The important quantities for this study are the skin friction coefficient $C_f$ and Nusselt number $N_u$, which indicate physically to the surface shear stress and the rate of heat transfer respectively, are defined as

$$C_f = \frac{2\tau_w}{\rho U_w^2}, \quad N_u = \frac{x q_w}{k(T_w - T_\infty)}$$

(19)

Where $\tau_w$ is the surface shear stress and $q_w$ is the rate of heat transfer which are given by

$$\tau_w = \mu_f \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k_f \left( \frac{\partial T}{\partial y} \right)_{y=0} + q_r \bigg|_{y=0}$$

(20)

Substituting from equations (10)–(12), we get

$$\tau_w = \mu_f \left( \frac{\partial u}{\partial y} \right)_{y=0} = \frac{\mu_f U_w}{(1 - \phi)^{1.5}} \sqrt{\frac{U_w}{2\pi f}(0)}$$

(21)

Table 2. Comparing with the results of Mahdy$^{14}$ (table 4, model I) values of the skin friction coefficient and rate of heat transfer for various values of nanoparticles volumes $\phi$ with $Pr = 6.5$, $A = 0.5$, $\alpha = 0.5$.

| $\phi$ | $-f''$ | $-\theta'$ | $-f''$ | $-\theta'$ |
|-------|-------|-------|-------|-------|
| 0.0   | 1.07432 | 3.78318 | 1.074228 | 3.782332 |
| 0.02  | 1.13585 | 3.65296 | 1.13574 | 3.652169 |
| 0.04  | 1.18736 | 3.53000 | 1.187232 | 3.529295 |
| 0.06  | 1.23031 | 3.41352 | 1.230166 | 3.41328 |
| 0.08  | 1.26579 | 3.30291 | 1.265637 | 3.302253 |
| 0.1   | 1.29466 | 3.19760 | 1.294503 | 3.196963 |

Table 3. The effect of mixed convection.

| $\alpha$ | $-f''$ | $-\theta'$ | $-f''$ | $-\theta'$ |
|---------|-------|-------|-------|-------|
| 0       | 2.430156 | 4.38328 | 2.430156 | 4.66556 |
| 0.3     | 2.389166 | 4.385921 | 2.391285 | 4.668019 |
| 0.7     | 2.334684 | 4.389415 | 2.33960 | 4.671724 |
| 1.0     | 2.293948 | 4.392015 | 2.300941 | 4.673699 |
| 2.0     | 2.15893 | 4.40056 | 2.172717 | 4.681582 |
| 3.0     | 2.0215038 | 4.408926 | 2.04544 | 4.689516 |
| 5.0     | 1.760433 | 4.42516 | 1.793567 | 4.704769 |
| 10      | 1.115367 | 4.463222 | 1.177962 | 4.740764 |
Now we get

\[ C_f = \left( \frac{2}{(1 - \phi)^2 \pi} \right) f''(0) \text{ and } Nu = \left( \frac{k_r}{k_f} + Rd \right) \sqrt{Re_x \theta'(0)} \] (23)

Grashof number, \( Gr \), is a nondimensional parameter used in the correlation of heat and mass transfer due to thermally induced natural convection at a solid surface immersed in a fluid. Since Reynolds number, \( Re \), represents the ratio of momentum to viscous forces. The relative magnitudes of \( Gr \) and \( Re \) are an indication of the relative importance of natural and forced convection in determining heat transfer. Forced convection effects are usually insignificant when \( Gr/Re^2 \gg 1 \) and conversely, natural convection effects

Table 4. The thermal radiation effect.

| Rd | \(-f''\) | \(-\theta'\) | \(-f''\) | \(-\theta'\) |
|----|-----------|-----------|-----------|-----------|
| 0  | 2.380589  | 6.348166  | 2.385076  | 7.065226  |
| 0.2| 2.37386   | 5.490245  | 2.377965  | 5.986901  |
| 0.4| 2.367658  | 4.865216  | 2.371448  | 5.231242  |
| 0.6| 2.361901  | 4.387672  | 2.365432  | 4.66965   |
| 0.8| 2.356528  | 4.009719  | 2.359818  | 4.234282  |
| 1.0| 2.351492  | 3.702346  | 2.354578  | 3.885859  |

Table 5. The effect of nanoparticles volume \( \phi \), unsteadies parameter \( A \) and \( f_w \).

| \( \phi \) | A | \( f_w \) | \(-f''\) | \(-\theta'\) | \(-f''\) | \(-\theta'\) |
|-----------|---|---------|-----------|-----------|-----------|-----------|
| 0.0       | 1 | -1      | 0.910478  | 1.665125  | 0.910478  | 1.665125  |
|           |   |         | 1.902480  | 5.113165  | 1.902480  | 5.113165  |
| 2         | 1 | -1      | 1.185988  | 2.506720  | 1.185988  | 2.506720  |
|           |   |         | 2.132633  | 5.864460  | 2.132633  | 5.864460  |
| 0.05      | 1 | -1      | 0.991499  | 1.616635  | 0.991986  | 1.634490  |
|           |   |         | 2.181402  | 4.735548  | 2.183911  | 4.887712  |
| 2         | 1 | -1      | 1.289040  | 2.428618  | 1.289533  | 2.460282  |
|           |   |         | 2.433662  | 5.472567  | 2.435198  | 5.634311  |
| 0.1       | 1 | -1      | 1.042069  | 1.569162  | 1.042843  | 1.604714  |
|           |   |         | 2.361901  | 4.387672  | 2.365423  | 4.669650  |
| 2         | 1 | -1      | 1.351898  | 2.351359  | 1.352678  | 2.413634  |
|           |   |         | 2.626586  | 5.107962  | 2.629243  | 5.408644  |
may be neglected when $Gr/Re^2 << 1$. When the ratio is of the order of one, combined effects of natural and forced convection have to be taken into account.

**Results and discussion**

The system of nonlinear PDEs (2–6) governing the problem is converted by the similarity transformations (10) into a system of non-dimensional ODEs (13–15) containing a set of physical factors controlling the study. The problem is solved by a computational method consisting of the finite difference technique with Newton’s linearization scheme. It was first introduced by Keller$^{18}$ and laboratory described by Cebeci and Bradshaw.$^{19}$ The computational results gained in terms of velocity profiles and temperature distributions are illustrated in graphical forms while the results obtained in terms of velocity gradient and the rate of heat transfer are performed in tabular forms. The computational results analyze the influence of the physical parameters governing the problem, namely the thermal radiation $Rd$, mixed convection $\alpha$, porous medium parameter $k$, heat generation $\lambda$, and nanoparticles volume fraction $\phi$. To validate the computational results, a comparison of special cases of the current investigation is prepared with the previous reported study by Mahdy.$^{14}$ The comparisons are performed in Table 2 and an excellent agreement is obtained. The considered values used in the computational results are $Pr = 6.2$, $\alpha = 0.5$, $Rd = 0.6$, $\lambda = 0.2$, $Rc = 0.1$, $Sc = 1.0$, $M = 0.2$, $m = 0.5$, $\phi = 0.1$, $f_w = 1.0$, and $Nt = 1.0$. The positive sign in computational results of the skin friction and Nusselt number

**Table 6.** The effect of porous medium parameter $k$.

|   | Model I | Model II |
|---|---------|----------|
| $k$ | $-f''$  | $-\theta'$ | $-f''$  | $-\theta'$ |
| 0  | 2.269392 | 4.395437 | 2.29697 | 4.397264 |
| 0.1 | 2.30087 | 4.392777 | 2.304453 | 4.674759 |
| 0.2 | 2.331691 | 4.39019 | 2.335244 | 4.672171 |
| 0.3 | 2.361901 | 4.387672 | 2.365423 | 4.66965 |
| 0.4 | 2.391536 | 4.385218 | 2.395028 | 4.667192 |
| 0.5 | 2.420634 | 4.382823 | 2.424097 | 4.664793 |

**Table 7.** The effect of suction or injection parameters.

| $f_w$ | Model I | Model II |
|-------|---------|----------|
|       | $-f''$  | $-\theta'$ | $-f''$  | $-\theta'$ |
| 0.1   | 1.624359 | 2.756103 | 1.626859 | 2.882663 |
| 0.3   | 1.76744 | 3.068072 | 1.770281 | 3.223352 |
| 0.5   | 1.922699 | 3.410655 | 1.925828 | 3.598393 |
| 0.7   | 2.08989 | 3.782088 | 2.093236 | 4.005557 |
| 1.0   | 2.361901 | 4.387672 | 2.365423 | 4.66965 |
indicates that the fluid affects a drag force on the surface while the negative sign points out that the force works in a reverse direction.

The effect of mixed convection parameter $\alpha$ on skin friction coefficient and the rate of heat transfer is presented in Table 3 with other physical parameters values $k = 0.3$, $f_w = 1.0$, $A = 1.0$, $Pr = 6.2$, $Rd = 0.60$, $\lambda = 0.5$, $\phi = 0.1$, $\alpha = 0.5$. It is clear from Table 3 that the increase of mixed convection parameter leads to a decrease of skin friction coefficient but a slight increase in the rate of heat transfer for both two models (see, Mukhopadhyay.11)

**Figure 2.** The velocity $f'$ for various values of mixed convection parameter.

**Figure 3.** The temperature $\theta$ for various values of mixed convection parameters.
The thermal radiation parameter $Rd$ effect on skin fraction coefficient and the rate of heat transfer with other physical parameters values $\alpha = 0.5, k = 0.3, f_w = 1.0, A = 1.0, Pr = 6.2, \lambda = 0.5$ is reported in Table 4. Table 4 shows that an increase of thermal radiation parameter leads to a slight decrease of skin fraction coefficient and also decreases in the rate of heat transfer for both two models.

The effect of the nanoparticles volume $\phi$, the unsteadies parameter $A_t$ and the suction parameter $f_w$ on the skin friction coefficient and the rate of heat transfer is presented in Table 5 with other physical parameters values $k = 0.3, Pr = 6.2, Rd = 0.60, \lambda = 0.5, \alpha = 0.5$. It is seen in Table 5 that the increase of nanoparticles volume $\phi$ leads to increases of skin fraction coefficient but decreases in the rate of heat transfer for both two models. Also, it is seen in Table 5 that the increase of an unsteady parameter $A$ leads to an increase of both skin fraction coefficient and the rate of heat transfer.

![Figure 4](image1.png)

**Figure 4.** The velocity $f'$ for various values of porous medium parameter $k$.

![Figure 5](image2.png)

**Figure 5.** The temperature $\theta$ for various values of porous medium parameter $k$. 
The porous medium parameter effect on skin friction coefficient and the rate of heat transfer is reported in Table 6 where porous medium parameter values are $k = 0.0$, $0.1$, $0.2$, $0.3$, $0.4$ and $0.5$ while the other parameters values are $\alpha = 0.5$, $f_w = 1.0$, $A = 1.0$, $Pr = 6.2$, $Rd = 0.60$, $\lambda = 0.5$. Table 6 shows that the increase of porous medium leads to increases in values of the skin friction coefficient and decreases in values of the rate of heat transfer for both models. More details about the impact of nanoparticles volume parameter $\phi$ and the unsteadiness parameter $A$ on velocity and temperature of nanofluid are discussed in.\textsuperscript{13,14,17}

The suction parameter $f_w$ ($f_w > 0$) effect on skin friction coefficient and the rate of heat transfer is presented in Table 7 while other parameters values $Rd = 0.6$, $\alpha = 0.5$, $k = 0.3$, $A = 1.0$, $Pr = 6.2$, $\lambda = 0.5$. It is noted from Table 7 that the increase of the values of suction

\textbf{Figure 6.} The velocity $f'$ for various values of thermal radiation parameter $Rd$.

\textbf{Figure 7.} The temperature $\theta$ for various values of thermal radiation parameter $Rd$. 

parameter $f_w$ leads to an increase of both values of the skin friction coefficient and the rate of heat transfer for models I, II.

The effect of the mixed convection effect on the velocity and the temperature profiles is presented in Figures 2 and 3 where the various values of mixed convection parameter $\alpha$ are 0, 0.3, 0.7 and 1.0 while other parameter values are $k = 0.3$, $f_w = 1.0$, $A = 1.0$, $Pr = 6.2$, $Rd = 0.60$, $\lambda = 0.5$, $\phi = 0.1$ and $\alpha = 0.5$. It is found from Figures (2 and 3) that as mixed convection increases, the profiles of velocity $f'$ increase but the profiles of the temperature $\theta$ decrease.

For such behavior, the physical explanation is that the fluid is brought closer to the surface and reduces the thermal boundary layer thickness in this case.

Figure 8. Velocity $f'$ for various values of suction parameter $f_w$.

![Figure 8](image)

Figure 9. The temperature $\theta$ for various values of suction parameter $f_w$.

![Figure 9](image)
The porous medium parameter $k$ effect on the velocity and the temperature profiles is presented in Figures (4 and 5) where the various values of porous parameter $k$ are 0.0, 0.3, 0.7 and 1.0 while other parameter values are $\alpha = 0.5$, $f_w = 1.0$, $A = 1.0$, $Pr = 6.2$, $Rd = 0.60$, $\lambda = 0.5$. It is shown in Figures 4 and 5 that as the porous medium parameter $k$ increases, the values of temperature increase but the profiles of velocity $f^*$ decrease.

The thermal radiation parameter $Rd$ effect on the velocity and temperature profiles are presented in Figures (6 and 7) where $Rd$ values are 0.0, 0.4, 0.6 and 1.0 with $\alpha = 0.5$, $k = 0.3$, $f_w = 1.0$, $A = 1.0$, $Pr = 6.2$ $R = 0.60$ and $\lambda = 0.5$. It is noted from Figures (6 and 7) that as the thermal radiation parameter increase, the profiles of both velocity $f^*$ and temperature $\theta$ increase.

The suction parameter $f_w$ effect on velocity and temperature profiles is presented in Figures (8 and 9) with other parameter values $Rd = 0.6$, $\alpha = 0.5$, $k = 0.3$, $A = 1.0$, $Pr = 6.2$ $R = 0.60$ and $\lambda = 0.5$. It is found from Figures (8 and 9) that as the increase of the suction parameter $f_w$, the profiles of both velocity $f^*$ and temperature $\theta$ decrease.

Physically, it clear that all external parameters have a strong impact on the phenomenon of thermal radiation effect in the presence of heat generation on unsteady mixed convection boundary layer flow and heat transfer of nanofluid over permeable stretching surface through porous medium.

**Conclusions**

In this study, the mixed convection and thermal radiation effect on unsteady boundary layer of heat transfer and nanofluid flow over permeable moving surface through a porous medium is investigated. These governing equations are transformed into a system of nondimensional equations contain many physical parameters. With the help of similarity transformations, the governing time-dependent boundary layer equations for momentum and thermal are reduced to couple ordinary differential equations which are then solved numerically. The transformed equations are solved numerically using an implicit finite difference with Newton’s linearization method. The influence of the thermo-physical parameters is discussed and the following conclusions may be drawn:

- The increase of mixed convection parameter leads to a decrease of skin fraction coefficient but a slight increase in the temperature profiles for both two models. Also as mixed convection increases, the profiles of velocity $f^*$ and the rate of heat transfer $\theta'$ increase.
- The increase of the thermal radiation parameter leads to a slight decrease of skin fraction coefficient and also a decrease in the rate of heat transfer for both two models but as the thermal radiation parameter increase, the profiles of velocity $f^*$ and temperature $\theta$ increase.
- The increase of nanoparticles volume $\phi$ leads to an increase in skin fraction coefficient but a decrease in the rate of heat transfer for both two models while the increase of unsteady parameter leads to an increase of both the skin fraction coefficient and the rate of heat transfer.
- As the porous medium parameter $k$ increases, the values of skin fraction coefficient and the profiles of temperature increase but the profiles of velocity $f^*$ and the rate of heat transfer decrease.
The increase of the suction parameter $f_w$ leads to an increase in the values of skin fraction coefficient $f''$ and the rate of heat transfer $\theta'$ but the profiles of velocity $f'$, the temperature $\theta$ decrease.

Finally, we concluded that the results agree with the practical results and are applicable in related fields as medicine, pharmacology, jets flow, and chemical industries.

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**Data availability**

All data, models, and code generated or used during the study are included within the article.

**Declaration of conflicting interests**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Nomenclature

\( a \)  constant with dimension \( \text{time}^{-1} \) where \( a > 0 \)
\( b \)  constant with dimension temperature. length\(^{-1}\)
\( A \)  the unsteadiness parameter
\( \text{Cf} \)  the skin friction coefficient
\( (C_p)_nf \)  the nanofluid specific heat at constant pressure
\( (C_p)_f \)  the fluid specific heat at constant pressure
\( (C_p)_s \)  the effective of heat capacity of the nanoparticles solid
\( Cu \)  the Copper nanoparticles
\( Da_x \)  the local Darcy number
\( f_w \)  the suction or injection parameter
\( g \)  the gravity field
\( Gr \)  the Grashof number
\( k \)  porous medium parameter
\( k_{nf} \)  the effective thermal conductivity of the nano fluid
\( k_f \)  the thermal conductivity of the fluid
\( k_0 \)  the permeability of porous medium
\( k_1 \)  the initial permeability
\( n \)  the empirical shape factor for the nanoparticles
\( Nu \)  Nusselt number
\( Pr \)  the Prandtl number
\( Rd \)  the Radiation parameter
\( Re_x \)  the local Reynolds number
\( q_r \)  the radiative heat flux
\( q_w \)  the rate of heat transfer at the wall
\( Q \)  the heat source
\( Q_0 \)  the heat source
\( t \)  the time
\( T \)  the nano fluid temperature
\( T_w \)  the temperature of the sheet \( T_w(x,t) \)
\( T_\infty \)  the ambient temperature
\( u \)  velocity component in the \( x \) direction
\( U_w \)  the velocity of the wall \( U_w(x,t) \) along \( x \)-axis
\( v \)  velocity components in the \( y \) direction
\( V_w \)  the suction or injection velocity of the fluid
\( x \)  \( x \)-axis is along the velocity of the vertical wall
\( y \)  \( y \)-axis is normal to the wall

Greek symbols

\( \eta \)  the similarity variable
\( \nu_f \)  the kinematic viscosity of the liquid
\( \theta \)  the dimensionless temperature
\( \psi \)  the stream function
\( \tau_w \) the surface shear stress
\( \mu_f \) the fluid dynamic viscosity
\( \mu_{nf} \) the nanofluid dynamic viscosity
\( \rho_f \) the base fluid density
\( \rho_s \) the nanoparticles solid density
\( \rho_{nf} \) the nanofluid density
\( \alpha_{nf} \) the nanofluid thermal diffusion
\( \alpha_f \) the fluid thermal diffusion
\( \beta_{nf} \) the volumetric coefficient of nanofluid thermal expansion
\( \beta_f \) the fluid volumetric coefficient of thermal expansion
\( \sigma \) the Stefan-Boltzman constant
\( \sigma^* \) the mean absorption coefficient
\( \phi \) the nanoparticles volume
\( \lambda \) the heat source parameter
\( \gamma \) constants with dimension time\(^{-1}\) where \( \gamma \geq 0 \), and \( \gamma t < 1 \)
\( \alpha \) the mixed convection parameter

**Subscripts**

- \( f \) fluid
- \( nf \) nanofluid
- \( w \) conditions at the wall
- \( s \) solid
- \( \square \) conditions in the free stream

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