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Yamabe Spectra

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0. Introduction

The Uniformization Program of William Thurston prescribes, for a large class of compact closed three-manifolds, the existence of a hyperbolic metric, i.e. a metric of constant negative curvature $-1$. One approach to find such a metric is a two-step variational problem, which we now describe. Consider a compact Riemannian manifold $(M^n, g)$, and for any metric $\tilde{g}$ in the conformal class $C = C(g)$, define

$$Y(\tilde{g}) = \int_M R_{\tilde{g}} d\text{Vol}_{\tilde{g}} \left[ \text{Vol}(\tilde{g}) \right]^{\frac{n-2}{n}}$$

Then one easily shows that the critical points $\tilde{g}$ of $Y(\tilde{g})$ in $C$ are metrics of constant scalar curvature [16]. The minimum $\inf Y(\tilde{g})$, $\tilde{g} \in C$ is called the Yamabe invariant of $(M, g)$ and denoted $\lambda(g)$. The solution to the Yamabe problem, obtained in the last thirty years due to the efforts of Yamabe [16], Trudinger [14], Aubin [1], and Schoen [11], [12] establishes the existence of the minimizing metric $\tilde{g}$ in $C$. For $M$ three-dimensional one knows, under certain topological restrictions, that the constant scalar curvature of a metric on $M$ may not be positive. In fact, hyperbolizable manifolds never carry a metric of nonnegative scalar curvature, as shown by Schoen-Yau [13] and Gromov-Lawson [3]. So for such manifolds, $\lambda(g)$ is always negative. The importance of the functional $C \rightarrow \lambda(g)$ stems from the hope to find a conformal class $C$, maximizing this functional. Then an easy computation show that the corresponding constant scalar curvature metric $\tilde{g}$ in $C$ is actually Einstein, hence hyperbolic, since $\dim M = 3$. It is more convenient to introduce a modified function $\text{Vol} : C \rightarrow \text{Vol}(C)$, as follows: take the appropriately scaled metric $\tilde{g}$ in $C$ of constant scalar curvature $-1$, and denote $\text{Vol}(C) = \text{Vol}(\tilde{g})$. We will call the set $\{\text{Vol}(C)\}$ the Yamabe spectrum of $M$. 
The expectation for the global minimum of $Vol$ at the hyperbolic metric is justified by the following known facts: firstly, if $g_0$ is a (necessarily unique up to a diffeomorphism) hyperbolic metric on $M$, then $(D^2 Vol)_{g_0} > 0$, see [4], theorem 8.2. Secondly, if $\dim M = 4$ and $g_0$ is hyperbolic, then the Euler characteristic and signature computations of Johnson and Millson [4], theorem 8.3 show that $Vol$ attains its global minimum at $g_0$.

In the three-dimensional case we do not have Gauss-Bonnet type formulas, and one should look for other ways for estimation $Vol$. In the present paper, we deal with Haken three-manifolds with infinite first homology group. For every conformal class, $C$, we introduce an invariant of $C$, coming from the $L^3$-geometry of the Jacobian variety $J^1(M) = H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$. We prove that this invariant estimates $Vol(C)$ from below and use it to show our main result:

**Theorem 2.** Let $M$ be a compact oriented homologically atoroidal three-manifold with $\pi_2(M) = 0$ and $H_1(M, \mathbb{R}) \neq 0$. The $\sup_C Vol(C) = \infty$.

We use our estimates for demonstrating new global obstructions for the Nash isometrical immersions $M^3 \to N$ of arbitrary codimension, which induce nontrivial map in the first homology. Finally, we establish, for all three-manifolds with the pinched negative curvature $-K \leq K(M) \leq -k < 0$, the following principle: the Thurston genus norm in $H_2(M)$ is uniformly equivalent to the area norm.

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1. The Homological Plateau Problem

The following version of the existence theorem for minimal surfaces of Sacks-Uhlenbeck-Schoen-Yau (c.f. [10], [13]) has been established recently by Marina Ville [16]:

**Theorem A.** Let $M$ be a compact Riemannian manifold and let $0 \neq z \in H_2(M, \mathbb{Z})$. Then there exists a (branched) minimal immersion $f$ of a compact oriented, possibly disconnected Riemannian surface $\Sigma^g$ to $M$, representing $z$. Moreover, one can take $\Sigma^g$ of the least possible genus among all surfaces, representing $z$ and the map $f$ to be globally area minimizing in its homology class.

For the reader’s convenience we sketch the proof here. Start with the following lemma.

**Lemma 1.** (comp. [8], [17]). Let $X$ be a $CW$-complex and let $z \in H_2(X, \mathbb{Z})/\pi_2(X)$. There exists a collection of oriented Riemannian surfaces $\Sigma^{g_1}, \ldots, \Sigma^{g_k}$, $g_i \geq 1$ and maps $f_i : \Sigma^{g_i} \to X$ such that

(i) $\sum (2g_i - 2) = ||z||_g$ (see the section 2).

(ii) $\sum [f_i] = z$

(iii) For any essential simple loop $\gamma$ in $\Sigma^{g_i}$, $f_i([\gamma]) \neq 0$ in $\pi_1(x)$.

**Proof:** Start with any collection of $\Sigma^{g_i}$ satisfying (i) and (ii). Suppose (iii) is not valid, that is, for some $\gamma$ in $\Sigma^{g_i}$, there exists a map $\varphi : D^2 \to X$ with $\varphi|\partial D^2 = f|\gamma$ (we identify $\gamma$ with $\partial D^2$). Cut $\Sigma^{g_i}$ along $\gamma$ and paste two copies of $D^2$ along the new boundaries. Denote $\tilde{\Sigma}_i$ the resulting surface and let $\tilde{f}_i : \tilde{\Sigma}_i \to X$ be the map, obtained by patching $\varphi$ and $f_i$.

Consider the two following cases.

1) $\gamma$ is not separating. Then genus $(\tilde{\Sigma}_i) = g_i - 1$

2) $\gamma$ is separating. Then if $\tilde{\Sigma}_i^{(1)}$ and $\tilde{\Sigma}_i^{(2)}$ are the two components of $\Sigma^{g_i} - \gamma$, then

$$\tilde{\Sigma}_i = \tilde{\Sigma}_i^{(1)} \sqcup \tilde{\Sigma}_i^{(2)}$$

and genus $(\tilde{\Sigma}_i^{(1)}) + \text{genus } (\tilde{\Sigma}_i^{(2)}) = g_i$.

In any case, we have, first, that the new collection of surfaces and maps still represents $z$, and, second, that $\sum |\chi(\Sigma^{g_i})|$ strictly decreases. This contradicts (i) and thus (iii) is valid.

Returning to the proof of Theorem A, we first note, that by the theorem of Sacks and Ulenbeck, there exists a set of minimal spheres in $M$, generating $\pi_2(M)$ as a $\pi_1(M)$ module, so their images in $\pi_2(M)$ generate $\pi_2(M)$. Hence we may work modulo $\pi_2(M)$.
Fix $z \in H_2(M)/\pi_2(M)$ and consider a collection of surfaces and maps $\Sigma^{g_i}, f_i$ as in the lemma 1, so that $\sum (2g_i - 2)$ is the least possible. By the Hopf exact sequence, we have $H_2(M)/\pi_2(M) = H_2(\pi_1(M))$. Now, the condition (iii) implies, by the theorem of Schoen-Yau [13] and Sacks-Ulenbeck [10], that there exist, for any $i$, a branched minimal immersion $\psi_i : \Sigma^{g_i} \to M$, inducing the same action on $\pi_1(\Sigma^{g_i})$ as $f_i$, up to a conjugation. In particular, $\Sigma[\psi_i] = z$ in $H_2(\pi_1(M))$, as desired.

We refer to [8] for the further refinement of this result and numerous algebraic applications.

If the dimension of $M$ is three, we may (and will) consider $f$ to be unbranched, by the same argument as in [13].

2. The Thurston Norm and the Area Norm

Let $M$ be a smooth Riemannian manifold and let $z \in H_2(M, \mathbb{Z})/\pi_2(M)$ where $\pi_2(M)$ is the image of the Hurewitz map. In [15], Thurston introduced a seminorm $||z||_g$, which we will take in a form

$$||z||_g = \inf_{[\Sigma] = z} |\chi(\Sigma)|$$

taken over all singular surfaces $f : \Sigma \to M$ of genus $\geq 1$, representing $z$. One makes $||z||_g$ to a norm by the standard normalization procedure (see [15]) and shows that this norm is essentially equivalent to the Gromov’s simplicial norm.

On the other hand, given a metric on $M$, one has the usual area norm, or the mass, of $z$:

$$||z||_a = \inf_{[\Sigma] = z} \text{area}(\Sigma).$$

The celebrated Thurston’s inequality relates $||z||_g$ to $||z||_a$ in the case when $M$ is of negative curvature. The following sharp version was established in [7]: if $-K \leq K(M) \leq -k < 0$ and $M$ is compact, then

$$(2) \quad ||z||_a \leq \frac{2\pi}{k} ||z||_g.$$
Theorem 1. Let $M$ be compact three-dimensional homologically atoroidal Riemannian manifold, and let $R(M) = \sup_{M}(-R(x))$. Then for any $z \in H_{2}(M, \mathbb{Z})/\pi_{2}(M)$, one has

$$||(z)||_{a} \geq \frac{2\pi}{R(M)}||z||_{g}$$

which is sharp.

Proof: We recall that a three-manifold $M$ is called homologically atoroidal if any map of a torus $T^{2}$ to $M$ induces zero homomorphism in the second homology. All hyperbolizable manifolds are atoroidal.

We begin with finding a minimal map $f : \Sigma^{g} \to M$ representing $z$ which exists by the Theorem A. We assume $|\chi(\Sigma)|$ is minimal possible and $f$ is an immersion, $(\Sigma^{g}$ may be disconnected$)$. Let $\Sigma^{g_{i}}, i = 1, \ldots, q$, be the components of $\Sigma$. Then $g_{i} > 1$ since $\chi(\Sigma)$ is minimal and $M$ is atoroidal. Thus the second variation formula gives (comp. [13], (5.3))

$$\sum_{i} \int_{\Sigma^{g_{i}}} R(x)darea \leq \sum_{i} \int_{\Sigma^{g_{i}}} Kdarea$$

which implies by Gauss-Bonnet

$$R(M)Area(f) \geq 2\pi \sum -\chi(\Sigma^{g_{i}}) = 2\pi ||z||_{g}.$$ 

Since $f$ is the minimizing map, we get $Area(f) = ||z||_{a}$, so

$$R(M)||z||_{a} \geq 2\pi ||z||_{g},$$

as prescribed by (3).

Corollary 2. Let $T(k, K)$ denotes a class of compact three-manifolds with negative curvature satisfying $-K \leq K(x) \leq -k < 0$. Then the Thurston norm and the area norm are uniformly equivalent in $T(k, K)$. More precisely, for any $M \in T(k, K)$ and $z \in H_{2}(M, \mathbb{Z})$, one has

$$\frac{1}{k} ||z||_{g} \geq \frac{1}{2\pi} ||z||_{a} \geq \frac{1}{3K} ||z||_{g}$$

and the left hand side is sharp.
3. Geometry of Jacobians and Conformal Invariants

For a compact Riemannian manifold $N$ we denote $J^1(N) = H^1(N, \mathbb{R})/H^1(N, \mathbb{Z})$. Let $\Omega^*(N)$ stand for the de Rham complex of $N$. For $p \geq 1$ and $\omega \in \Omega^k(N)$ we denote as usual $||\omega||_{L^p} = (\int_N |\omega|^p d\text{Vol})^{1/p}$. This induces a norm on $H^k(N, \mathbb{R})$ by the formula $||w||_{L^p} = \inf_{\omega \in w} ||\omega||_{L^p}$. We claim:

**Proposition 3.** Let $M$ be a homologically atoroidal three-manifold with $\pi_2(M) = 0$ and let $w \neq 0 \in H_1(M, \mathbb{Z})$. Then

$$||w||_{L^1} \geq \frac{4\pi}{R(M)}.$$  \hspace{1cm} (5)

**Corollary 4.** The volume of the Jacobian $J^1(M)$ in the $L^2$-metric is at least

$$\left(\frac{4\pi}{R(M) \text{Vol}^{1/2}(M)}\right)^m \text{Vol}B_m,$$

where $m = b_1(M)$, and $B_m$ stands for the Euclidean ball.

**Proof of the Proposition 3:** Let $\omega \in \Omega^1(M)$ with $[\omega] = w$. Since all periods of $\omega$ are integers, there exists a smooth map $\varphi : M \to S^1 = \mathbb{R}/\mathbb{Z}$ with $\varphi^*(dt) = \omega$. Let $S(t) = \varphi^{-1}(t)$ and write

$$\int_M ||\nabla \varphi|| d\text{Vol} = \int_0^1 \text{Area}(S(t)) dt$$

by the coarea formula. For almost all $t$, $S(t)$ is smooth and $[S(t)] \in H_2(M, \mathbb{Z})$ is Poincaré dual to $w$. Applying Theorem 1, we get

$$||\omega||_{L^1} \geq 2\pi R^{-1}(M) ||PD(w)||_g \geq \frac{4\pi}{R(M)},$$

so $||w||_{L^1} = \inf_{[\omega] = w} ||\omega||_{L^1} \geq \frac{4\pi}{R(M)}$. \hspace{1cm} Q.E.D.

**Proof of the Corollary 4:** This follows readily from (5) and the Hölder inequality. Let $M, w$ be as in the proposition 3, and let $\omega \in \Omega^1(M)$ with $[\omega] = w$. Let $g$ be the metric of $M$ and let $h$ be its conformal perturbation. Say $h = \varphi \cdot g$ for some positive $\varphi \in C^\infty(M)$. Using the proposition 3, we get

$$R_h(M) \cdot \int_M ||\omega||_h d\text{Vol}_h \geq 4\pi,$$
or
\[ R_h(M) \cdot \int_M \lVert \omega \rVert_g \cdot \varphi^2 d\text{Vol}_g \geq 4\pi. \]

This gives
\[ \int_M \varphi^2 d\text{Vol}_g \geq \frac{4\pi}{R_h(M) \cdot \lVert \omega \rVert_{L^\infty}_g}, \]

and, by Hölder,
\[ \text{Vol}_h(M) = \int_M \varphi^3 d\text{Vol}_g \geq \left( \frac{4\pi}{R_h(M) \cdot \lVert \omega \rVert_{L^\infty}_g} \right)^{3/2} \cdot \text{Vol}_g^{-1/2}(M), \]

so
\[ R_h^{3/2}(M) \cdot \text{Vol}_h(M) \geq \left( \frac{4\pi}{\lVert \omega \rVert_{L^\infty}_g} \right)^{3/2} \cdot \text{Vol}_g^{-1/2}(M). \]

We wish to improve this, letting the original metric \( g \) to change within its conformal class. Put \( \hat{g} = \psi \cdot g \) and write (5) for \( \hat{g} \) instead of \( g \) to get
\[ (4\pi)^{3/2} R_h^{-3/2}(M) \text{Vol}_h^{-1}(M) \leq (\text{sup} \lVert \omega(x) \rVert \cdot \psi^{-1}(x))^{3/2} \left( \int \psi^3 d\text{Vol}_g \right)^{1/2}. \]

The infimum of the right hand side taken over all \( \psi > 0 \) is easily seen to be \( \lVert \omega \rVert_{L^3}^{3/2} \), so we get finally
\[ 4\pi R_h^{-1}(M) \text{Vol}_h^{-2/3}(M) \leq \lVert \omega \rVert_{L^3}^3. \]

Letting \( h \) be the Yamabe metric in \( C_g \), we arrive to the following result

**Proposition 5.** Let \( M \) be compact three-dimensional homologically atoroidal manifold with \( \pi_2(M) = 0 \) and infinite \( H_1(M, \mathbb{Z}) \). For any conformal class \( C \) we have
\[ \text{Vol}(C) \geq \frac{4\pi}{\lVert w \rVert_{L^3}}, \]

where \( w \) is any class in \( H^1(M, \mathbb{Z}) \) and the \( L^3 \)-norm is taken according to any metric \( g \in C \).

Observe that \( \lVert w \rVert_{L^3} \) does not depend on the choice of the metric. In fact, \( L^3 \) geometry of the Jacobain \( J^1(M) \) depends only on \( C \). The number \( \inf_{w \in H^1(M, \mathbb{Z})} \lVert w \rVert_{L^3} \) denoted \( j(C) \), is therefore a conformal invariant. We can write (6) in the form
\[ \text{Vol}(C) \geq 4\pi j^{-1}(C). \]

One can view (8) as a three-dimensional version of the Li-Yau estimates, c.f. [5].
Theorem 2. Let $M$ be as in the Proposition 5. Then

$$\sup_{C} Vol(C) = \infty.$$ 

Proof: Fix $w \in H^1(M, \mathbb{Z})$ and $\omega \in w$. Fix $\varepsilon > 0$. In view of the proposition 5, it is enough to find a metric $g$ on $M$ such that $||\omega||_{L^3_g} < \varepsilon$. For that, fix a smooth measure $\mu$ on $M$. We will always assume that the density $\mu|_g = 1$. Set $||\omega||_g < \varepsilon^{1/3}$ everywhere and correct $g_x$ if necessary in the kernel of $\omega_x$ to achieve $\mu|_g = 1$, keeping $||\omega||_g$ unchanged. This would be a desired metric.

Remark.

We show in [9] that the “most” of homology three-spheres are Haken. It would be very interesting to know if the Theorem 2 is still valid for such manifolds.

4. Obstructions to Nash Embeddings

Let $M^m$ and $Q^q$ be Riemannian manifolds with $q \geq \frac{m(m+1)}{2}$. Then any distance-decreasing map $\varphi : M \to Q$ can be $C^0$-approximated by an isometrical embedding (c.f. Gromov [2] for the contemporary survey of related results). In particular, there always exists such an embedding, homotopic to a constant map. The situation changes if we wish to prescribe the topological properties of the embedding, e.x. its action in homology/homotopy groups. For example, if we demand that $\varphi_* : H_1(M) \to H_1(Q)$ is nonzero, then, evidently, there is a necessary condition that the length of the shortest geodesic in the homology class $C \in H_1(M, \mathbb{Z})$ is not less than that of $\varphi_*C$. Using the machinery developed above, we arrive to more obstructions of global character.

Theorem 3. Let $M^3$ and $Q^q$ be compact Riemannian manifolds. Suppose $M$ is homologically atoroidal with $\pi_2(M) = 0$ and $b_1(M), b_1(Q) \neq 0$. Then there is a constant $C(Q)$, such that if there exists an isometrical immersion $f : M \to Q$ inducing a nontirivial map in the first real homology, then

$$R(M) \cdot Vol(M) \geq C(Q).$$
Proof: Let \( \omega_1, \ldots, \omega_r \) be a basis of harmonic 1-forms representing integer classes in \( H^1(Q, \mathbb{R}) \). Put \( C^{-1}(Q) = \max_i ||\omega_i||_{L^\infty} \). If the action of \( f_* : H_1(M, \mathbb{R}) \to H_1(Q, \mathbb{R}) \) is nontrivial, then \( [f^*\omega_i] \neq 0 \) for some \( i \). Since \( [\omega_i] \in H_1(Q, \mathbb{Z}) \), also \( [f^*\omega_i] \in H_1(M, \mathbb{Z}) \).

Applying the proposition 3, we get

\[
||f^*\omega_i||_{L^1} \geq \frac{4\pi}{R(M)}.
\]

But \( ||f^*\omega_i||_{L^1} = \int_M ||f^*\omega_i|| dVol \leq ||\omega_i||_{L^\infty} \cdot Vol(M) \), so \( R(M) \cdot Vol(M) \geq C(Q) \). Q.E.D.

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