The Strength of the Grätzer-Schmidt Theorem

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Abstract
The Grätzer-Schmidt theorem of lattice theory states that each algebraic lattice is isomorphic to the congruence lattice of an algebra. A lattice is algebraic if it is complete and generated by its compact elements. We show that the set of indices of computable lattices that are complete is \( \Pi^1_1 \)-complete; the set of indices of computable lattices that are algebraic is \( \Pi^1_1 \)-complete; and that there is a computable lattice \( L \) such that the set of compact elements of \( L \) is \( \Pi^1_1 \)-complete. As a corollary, there is a computable algebraic lattice that is not computably isomorphic to any computable congruence lattice.

Keywords: lattice theory, computability theory.

Introduction
The Grätzer-Schmidt theorem \cite{gratzer1956}, also known as the congruence lattice representation theorem, states that each algebraic lattice is isomorphic to the congruence lattice of an algebra. It established a strong link between lattice theory and universal algebra. In this article we show that this theorem as stated fails to hold effectively in a very strong way.

We use notation associated with partial computable functions, \( \varphi_e, \varphi_{e,s}, \varphi^{\sigma}_{e,s}; \) as in Odifreddi \cite{odifreddi1989}. Following Sacks \cite{sacks1990} page 5, a \( \Pi^1_1 \) subset of \( \omega \) may be written in the form

\[
C_e = \{ n \in \omega \mid \forall f \in \omega^\omega \varphi_e(n) \downarrow \}.
\]

A subset \( A \subseteq \omega \) is \( \Pi^1_1 \)-hard if each \( \Pi^1_1 \) set is \( m \)-reducible to \( A \); that is, for each \( e \), there is a computable function \( f \) such that for all \( n, n \in C_e \) iff \( f(n) \in A \). \( A \) is \( \Pi^1_1 \)-complete if it is both \( \Pi^1_1 \) and \( \Pi^1_1 \)-hard. It is well known that such sets

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exist. Fix for the rest of the paper a number $c_0$ so that $C_{c_0}$ is $\Pi^1_1$-complete. With each $n$, the set $C_{c_0}$ associates a tree $T_n'$ defined by

$$T_n' = \{ \sigma \in \omega^\omega \mid \varphi_{c_0,\sigma}(n) \uparrow \}.$$ 

Note that $T_n'$ has no infinite path iff $n \in C_{c_0}$.

A computable lattice $(L, \preceq)$ has underlying set $L = \omega$ and an lattice ordering $\preceq$ that is formally a subset of $\omega^2$.

We will use the symbol $\preceq$ for lattice orderings, and reserve the symbol $\leq$ for the natural ordering of the ordinals and in particular of $\omega$. Meets and joins corresponding to the order $\preceq$ are denoted by $\land$ and $\lor$. Below we will seek to build computable lattices from the trees $T_n'$; since for many $n$, $T_n'$ will be finite, and a computable lattice must be infinite according to our definition, we will work with the following modification of $T_n'$:

$$T_n = T_n' \cup \{ \langle i \rangle : i \in \omega \} \cup \{ \emptyset \}$$

where $\emptyset$ denotes the empty string and $\langle i \rangle$ is the string of length 1 whose only entry is $i$. This ensures that $T_n$ has the same infinite paths as $T_n'$, and each $T_n$ is infinite. Moreover the sequence $\{T_n\}_{n \in \omega}$ is still uniformly computable.

1 Computational strength of lattice-theoretic concepts

1.1 Completeness

Definition 1. A lattice $(L, \preceq)$ is complete if for each subset $S \subseteq L$, both sup $S$ and inf $S$ exist.

Lemma 2. The set of indices of computable lattices that are complete is $\Pi^1_1$.

Proof. The statement that sup $S$ exists is equivalent to a first order statement in the language of arithmetic with set variable $S$:

$$\exists a[\forall b(b \in S \rightarrow b \preceq a) \& \forall c((\forall b(b \in S \rightarrow b \preceq c) \rightarrow a \preceq c)].$$

The statement that inf $S$ exists is similar, in fact dual. Thus the statement that $L$ is complete consists of a universal set quantifier over $S$, followed by an arithmetical matrix. \hfill $\Box$

Example 3. In set-theoretic notation, $(\omega + 1, \leq)$ is complete. Its sublattice $(\omega, \leq)$ is not, since $\omega = \text{sup } \omega \not\in \omega$.

Proposition 4. The set of indices of computable lattices that are complete is $\Pi^1_1$-hard.
Proof. Let \( L_n \) consist of two disjoint copies of \( T_n \), called \( T_n \) and \( T_n^* \). For each \( \sigma \in T_n \), its copy in \( T_n^* \) is called \( \sigma^* \). Order \( L_n \) so that \( T_n \) has the prefix ordering

\[
\sigma \preceq \sigma^\tau, 
\]

\( T_n^* \) has the reverse prefix ordering, and \( \sigma \prec \sigma^* \) for each \( \sigma \in T_n \). We take the transitive closure of these axioms to obtain the order of \( L_n \); see Figure 1.

Next, we verify that \( L_n \) is a lattice. For any \( \sigma, \tau \in T_n \) we must show the existence of \((1)\ \sigma \lor \tau, \ (2)\ \sigma \land \tau, \ (3)\ \sigma \lor \tau^*, \) and \((4)\ \sigma \land \tau^*; \) the existence of \( \sigma^* \lor \tau^* \) and \( \sigma^* \land \tau^* \) then follows by duality.

We claim that for any strings \( \alpha, \sigma \in T_n \), we have \( \alpha^* \preceq \sigma \) iff \( \alpha \) is comparable with \( \sigma \); see Figure 1. In one direction, if \( \alpha \succeq \sigma \) then \( \alpha^* \succeq \sigma \), and if \( \sigma \succeq \alpha \) then \( \alpha^* \preceq \alpha \). In the other direction, if \( \alpha^* \preceq \sigma \) then by the definition of \( \preceq \) as a transitive closure there must exist \( \rho \) with \( \alpha^* \succeq \rho \succeq \rho \succeq \sigma \). Then \( \alpha \preceq \rho \) and \( \sigma \preceq \rho \), which implies that \( \alpha \) and \( \rho \) are comparable.

Using the claim we get that \((1)\ \sigma \lor \tau\) is \((\sigma \land \tau)^*\), where \((2)\ \sigma \land \tau\) is simply the maximal common prefix of \( \sigma \) and \( \tau\); \((3)\ \sigma \lor \tau^*\) is \( \sigma^* \lor \tau^*\) which is \((\sigma \land \tau)^*\); and \((4)\ \sigma \land \tau^*\) is \( \sigma \land \tau\).

It remains to show that \((L_n, \preceq)\) is complete iff \( T_n \) has no infinite path. So suppose \( T_n \) has an infinite path \( S \). Then \( \sup S \) does not exist, because \( S \) has no greatest element, \( S^* \) has no least element, each element of \( S^* \) is an upper bound of \( S \), and there is no element above all of \( S \) and below all of \( S^* \).

Conversely, suppose \( T_n \) has no infinite path and let \( S \subseteq L_n \). If \( S \) is finite then \( \sup S \) exists. If \( S \) is infinite then since \( T_n \) has no infinite path, there is no infinite linearly ordered subset of \( L_n \), and so \( S \) contains two incomparable elements \( \sigma \) and \( \tau \). Because \( T_n \) is a tree, \( \sigma \land \tau \) is in \( T_n^* \). Now the set of all elements of \( L_n \) that are above \( \sigma \lor \tau \) is finite and linearly ordered, and contains all upper bounds of \( S \). Thus there is a least upper bound for \( S \). Since \( L_n \) is self-dual, i.e. \((L_n, \preceq)\) is isomorphic to \((L_n, \succeq)\) via \( \sigma \mapsto \sigma^* \), infs also always exist. So \( L_n \) is complete.

\( \Box \)

1.2 Compactness

Definition 5. An element \( a \in L \) is compact if for each subset \( S \subseteq L \), if \( a \preceq \sup S \) then there is a finite subset \( S' \subseteq S \) such that \( a \preceq \sup S' \).

Lemma 6. In each computable lattice \( L \), the set of compact elements of \( L \) is \( \Pi_1^1 \).

Proof. Similarly to the situation in Lemma 2, the statement that \( a \) is compact consist of a universal set quantifier over \( S \) followed by an arithmetical matrix.

\( \Box \)

Example 7. Let \( L[a] = \omega + 1 \cup \{a\} \) be ordered by \( 0 < a < \omega \), and let the element \( a \) be incomparable with the positive numbers. Then \( a \) is not compact, because \( a \preceq \sup \omega \) but \( a \not\preceq \sup S' \) for any finite \( S' \subseteq \omega \).
The following result will be useful for our study of the Grätzer-Schmidt theorem.

**Proposition 8.** There is a computable algebraic lattice $L$ such that the set of
compact elements of $L$ is $\Pi^1_1$-hard.

Proof. Let $L$ consist of disjoint copies of the trees $T_n$, $n \in \omega$, each having the prefix ordering; least and greatest elements 0 and 1; and elements $a_n$, $n \in \omega$, such that $\sigma \prec a_n$ for each $\sigma \in T_n$, and $a_n$ is incomparable with any element not in $T_n \cup \{0, 1\}$ (see Figure 3).

Suppose $T_n$ has an infinite path $S$. Then $a_n = \sup S$ but $a_n \nless \sup S'$ for any finite $S' \subseteq S$, since $\sup S'$ is rather an element of $S$. Thus $a_n$ is not compact.

Conversely, suppose $T_n$ has no infinite path, and $a_n \preceq \sup S$ for some set $S \subseteq L$. If $S$ contains elements from $T_m \cup \{a_m\}$ for at least two distinct values of $m$, say $m_1 \neq m_2$, then $\sup S = 1 = \sigma_1 \lor \sigma_2$ for some $\sigma_i \in S \cap (T_{m_i} \cup \{a_{m_i}\})$, $i = 1, 2$. So $a_n \preceq \sup S'$ for some $S' \subseteq S$ of size two. If $S$ contains 1, there is nothing to prove. The remaining case is where $S$ is contained in $T_m \cup \{a_m, 0\}$ for some $m$. Since $a_n \preceq \sup S$, it must be that $m = n$. If $S$ is finite or contains $a_n$, there is nothing to prove. So suppose $S$ is infinite. Since $T_n$ has no infinite path, there must be two incomparable elements of $T_n$ in $S$. Their join is then $a_n$, since $T_n$ is a tree, and so $a_n \preceq \sup S'$ for some $S' \subseteq S$ of size two.

Thus we have shown that $a_n$ is compact if and only if $T_n$ has no infinite path. There is a computable presentation of $L$ where $a_n$ is a computable function of $n$, for instance we could let $a_n = 2^n$. Thus letting $f(n) = 2n$, we have that $T_n$ has no infinite path iff $f(n)$ is compact, i.e. $\{a \in L : a$ is compact$\}$ is $\Pi^1_1$-hard. □

![Figure 3: The lattice $L$ from Proposition 8](image)

1.3 Algebraicity

Definition 9. A lattice $(L, \preceq)$ is compactly generated if $C = \{a \in L : a$ is compact$\}$ generates $L$ under sup, i.e., each element is the supremum of
its compact predecessors. A lattice is algebraic if it is complete and compactly generated.

**Lemma 10.** The set of indices of computable lattices that are algebraic is $\Pi^1_1$.

**Proof.** $L$ is algebraic if it is complete (this property is $\Pi^1_1$ by Lemma 2) and each element is the least upper bound of its compact predecessors, i.e., any element that is above all the compact elements below $a$ is above $a$:

$$\forall b (\forall c \in C \& c \preceq a \rightarrow c \preceq b) \rightarrow a \preceq b)$$

Equivalently,

$$\forall b (\exists c (c \in C \& c \preceq a \& c \not\preceq b) \text{ or } a \preceq b)$$

This is equivalent to a $\Pi^1_1$ statement since, by the Axiom of Choice, any statement of the form $\exists c \forall S A(c, S)$ is equivalent to $\forall (S_c)_{c \in \omega} \exists c A(c, S_c)$. □

**Example 11.** The lattice $(\omega + 1, \leq)$ is compactly generated, since the only non-compact element $\omega$ satisfies $\omega = \sup \omega$. The lattice $L[a]$ from Example 7 and Figure 2 is not compactly generated, as the non-compact element $a$ is not the supremum of $\{0\}$.

**Proposition 12.** The set of indices of computable lattices that are algebraic is $\Pi^1_1$-hard.

**Proof.** Let the lattice $T_n[a]$ consist of $T_n$ with the prefix ordering, and additional elements $0 \prec a \prec 1$ such that $a$ is incomparable with each $\sigma \in T_n$, and $0$ and $1$ are the least and greatest elements of the lattice. Note that $T_n[a]$ is always complete, since any infinite set has supremum equal to 1. We claim that $T_n[a]$ is algebraic iff $T_n$ has no infinite path.

Suppose $T_n$ has an infinite path $S$. Then $a \preceq \sup S$, but $a \not\preceq \sup S'$ for any finite $S' \subseteq S$. Thus $a$ is not compact, and so $a$ is not the sup of its compact predecessors (0 being its only compact predecessor), which means that $T_n[a]$ is not an algebraic lattice.

Conversely, suppose $T_n[a]$ is not algebraic. Then some element of $T_n[a]$ is not the join of its compact predecessors. In particular, some element of $T_n[a]$ is not compact. So there exists a set $S \subseteq T_n[a]$ such that for all finite subsets $S' \subseteq S$, $\sup S' < \sup S$. In particular $S$ is infinite. Since each element except 1 has only finitely many predecessors, we have $\sup S = 1$. Notice that $T_n[a]\{1\}$ is actually a tree, so if $S$ contains two incomparable elements then their join is already 1, contradicting the defining property of $S$. Thus $S$ is linearly ordered, and infinite, which implies that $T_n$ has an infinite path. □

2 Lattices of equivalence relations

Let $\operatorname{Eq}(A)$ denote the set of all equivalence relations on $A$. Ordered by inclusion, $\operatorname{Eq}(A)$ is a complete lattice. In a sublattice $L \subseteq \operatorname{Eq}(A)$, we write $\sup_L$ for the
supremum in $L$ when it exists, and sup for the supremum in $\text{Eq}(A)$, and note that $\text{sup} \leq \text{sup}_L$.

A complete sublattice of $\text{Eq}(A)$ is a sublattice $L$ of $\text{Eq}(A)$ such that $\text{sup}_L = \text{sup}$ and $\text{inf}_L = \text{inf}$. A sublattice of $\text{Eq}(A)$ that is a complete lattice is not necessarily a complete sublattice in this sense. The following lemma is well known. A good reference for lattice theory is the monograph of Grätzer [1].

**Lemma 13.** Suppose $A$ is a set and $(L, \subseteq)$ is a complete sublattice of $\text{Eq}(A)$. Then an equivalence relation $E$ in $L$ is a compact member of $L$ if and only if $E$ is finitely generated in $L$.

**Proof.** One direction only uses that $L$ is a sublattice of $\text{Eq}(A)$ and $L$ is complete as a lattice. Suppose $E$ is not finitely generated in $L$. Let $C_{(a,b)}$ denote the infimum of all equivalence relations in $L$ that contain $(a,b)$. Then $E \subseteq \text{sup}_L \{C_{(a,b)} : aEb\}$, but $E$ is not below any finite join of the relations $C_{(a,b)}$. So $E$ is not compact.

Suppose $E$ is finitely generated in $L$. So there exists an $n$ and pairs $(a_1, b_1), \ldots, (a_n, b_n)$ such that $a_iEb_i$ for all $1 \leq i \leq n$, and for all equivalence relations $F$ in $L$, if $a_iFb_i$ for all $1 \leq i \leq n$ then $E \subseteq F$. Suppose $E \subseteq \text{sup}_L \{E_i : 1 \leq i < \infty\}$ for some $E_1, E_2, \ldots \in L$. Since $L$ is a complete sublattice of $\text{Eq}(A)$, $\text{sup}_L = \text{sup}$, so $E \subseteq \text{sup}\{E_i : 1 \leq i < \infty\}$. Note that $\text{sup}\{E_i : 1 \leq i < \infty\}$ is the equivalence relation generated by the relations $E_i$ under transitive closure. So there is some $j = j_n < \infty$ such that $\{(a_i, b_i) : 1 \leq i \leq n\} \subseteq \bigcup_{i=1}^j E_i$ and hence $E \subseteq \bigcup_{i=1}^j E_i$. Thus $E$ is compact.

A computable complete sublattice of $\text{Eq}(\omega)$ is a uniformly computable col-
lection $\mathcal{E} = \{E_i\}_{i \in \omega}$ of distinct equivalence relations on $\omega$ such that $(\mathcal{E}, \subseteq)$ is a complete sublattice of $\text{Eq}(\omega)$. We say that the lattice $L = (\omega, \preceq)$ is computably isomorphic to $(\mathcal{E}, \subseteq)$ if there is a computable function $\varphi : \omega \to \omega$ such that for all $i, j$, we have $i \preceq j \iff E_{\varphi(i)} \subseteq E_{\varphi(j)}$.

**Lemma 14.** The indices of compact congruences in a computable complete sublattice of $\text{Eq}(\omega)$ form a $\Sigma^0_2$ set.

**Proof.** Suppose the complete sublattice is $E = \{E_i\}_{i \in \omega}$. By Lemma 13, $E_k$ is compact if and only if it is finitely generated, i.e.,

$$\exists n \exists a_1, \ldots, a_n \exists b_1, \ldots, b_n \left[ \bigwedge_{i=1}^n a_i E_k b_i \land \forall j \left( \bigwedge_{i=1}^n a_i E_j b_i \to E_k \subseteq E_j \right) \right].$$

Here $E_k \subseteq E_j$ is $\Pi^0_1$: $\forall x \forall y \left( x E_k y \to x E_j y \right)$, so the formula is $\Sigma^0_2$.

**Theorem 15.** There is a computable algebraic lattice that is not computably isomorphic to any computable complete sublattice of $\text{Eq}(\omega)$.

**Proof.** Let $L$ be the lattice of Proposition 8 and let $f$ be the $m$-reduction of Proposition 8. Suppose $\varphi$ is a computable isomorphism between $L$ and a computable complete sublattice of $\text{Eq}(\omega)$, $(\mathcal{E}, \subseteq)$. Since being compact is a lattice-theoretic property, it is a property preserved under isomorphisms. Thus an element $a \in L$ is compact if and only if $E_{\varphi(a)}$ is a compact congruence relation. This implies that $T_n$ has no infinite path if and only if $E_{\varphi(f(n))}$ is a compact congruence relation. By Lemma 13 this implies that $C_{\text{co}} = \{n : T_n \text{ has no infinite path}\}$ is a $\Sigma^0_2$ set, contradicting the fact that this set is $\Pi^0_1$-complete.

3 Congruence lattices

An algebra $\mathfrak{A}$ consists of a set $A$ and functions $f_i : A^{n_i} \to A$. Here $i$ is taken from an index set $J$ which may be finite or infinite, and $n_i$ is the arity of $f_i$. Thus, an algebra is a purely functional model-theoretic structure. A congruence relation of $\mathfrak{A}$ is an equivalence relation on $A$ such that for each unary $f_i$ and all $x, y \in A$, if $x E y$ then $f_i(x) E f_i(y)$, and the natural similar property holds for $f_i$ of arity greater than one.

The congruence relations of $\mathfrak{A}$ form a lattice under the inclusion (refinement) ordering. This lattice $\text{Con}(\mathfrak{A})$ is called the congruence lattice of $\mathfrak{A}$.

The following lemma is well-known and straight-forward.

**Lemma 16.** If $\mathfrak{A}$ is an algebra on $A$, then $\text{Con}(\mathfrak{A})$ is a complete sublattice of $\text{Eq}(A)$.

Thus, may define a computable congruence lattice to be a computable complete sublattice of $\text{Eq}(\omega)$ which is also $\text{Con}(\mathfrak{A})$ for some algebra $\mathfrak{A}$ on $\omega$. 
Theorem 17. There is a computable algebraic lattice that is not computably isomorphic to any computable congruence lattice.

Proof. By Theorem 15, there is a computable algebraic lattice that is not even computably isomorphic to any computable complete sublattice of $\text{Eq}(\omega)$. 

Thus, we have a failure of a certain effective version of the following theorem.

Theorem 18 (Grätzer-Schmidt [2]). Each algebraic lattice is isomorphic to the congruence lattice of an algebra.

Remark 19. Let $A$ be a set, and let $L$ be a complete sublattice of $\text{Eq}(A)$. Then $L$ is algebraic [4], and so by Theorem 18 $L$ is isomorphic to $\text{Con}(\mathfrak{A})$ for some algebra $\mathfrak{A}$ on some set, but it is not in general possible to find $\mathfrak{A}$ such that $L$ is equal to $\text{Con}(\mathfrak{A})$. Thus, Theorem 15 is not a consequence of Theorem 17.

Remark 20. The proof of Theorem 17 shows that not only does the Grätzer-Schmidt theorem not hold effectively, it does not hold arithmetically. We conjecture that within the framework of reverse mathematics, a suitable form of Grätzer-Schmidt may be shown to be equivalent to the system $\Pi^1_1$-$\text{CA}_0$ (arithmetic comprehension). On the other hand, W. Lampe has pointed out that the Grätzer-Schmidt theorem is normally proved as a corollary of a result which may very well hold effectively: each upper semilattice with least element is isomorphic to the collection of compact congruences of an algebra.

Conjecture 21. An upper semilattice with least element has a computably enumerable presentation if and only if it is isomorphic to the collection of compact congruences of some computable algebra.

The idea for the only if direction of Conjecture 21 is to use a restate and slightly modify Jónsson and Pudlák’s construction [4]. The if direction appears to be straightforward.

Remark 22. The lattices used in this paper are not modular, and we do not know if our results can be extended to modular, or even distributive, lattices.

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