A Classical Solution in Six-dimensional Gauge Theory
with Higher Derivative Coupling

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Abstract

We show that the spin connection of the standard metric on a six-dimensional sphere gives an
exact solution to the generalized self-duality equations suggested by Tchrakian some years ago.
We work on an SO(6) gauge theory with a higher-derivative coupling term. The model consists
of vector fields only. The pseudo-energy is bounded from below by a topological charge which
is proportional to the winding number of spatial $S^5$ around the internal space SO(6). The fifth
homotopy group of SO(6) is, indeed, $\mathbb{Z}$. The coupling constant of higher derivative term is quadratic
in the radius of the underlying space $S^6$.

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Classical solutions to field equations often play an important role in study of non-perturbative effects in field theory or string theory. Instantons are such objects. Yang-Mills instantons on $\mathbb{R}^4$ or on a four sphere $S^4$ found by Belavin et al. (BPST) are topological solitons classified by a map from the spatial boundary (three-dimensional sphere $S^3$) to a gauge group $SU(2)$; the third homotopy group is $\Pi_3[SU(2)] \cong \mathbb{Z}$. Since its discovery the Yang-Mills instantons have been studied by many people in both physics and mathematics. One important achievement is the determination of low energy effective action of $\mathcal{N} = 2$ supersymmetric gauge theories [2]. String theory or M-theory is considered to unify space-time, matter and all forces including gravity, but it requires ten or eleven dimensional space-time contrary to our four dimensional space-time. Two different mechanisms to compactify extra space in higher dimensions have been suggested and studied extensively; the Kaluza-Klein compactification [3, 4] and brane world scenarios [5]. Classical solutions (solitons) to field equations play important roles in both scenarios: in the former non-trivial field configurations on a compact space are used for spontaneous compactification [6], and in the latter solitons themselves as extended objects may realize four-dimensional world on their world-volume.

Some years ago Tchrakian suggested a broad class of higher-dimensional generalization of instantons and monopoles [7]. Since then several people have been studying these generalized self-duality equations. Tchrakian constructed the BPST analogues in all $4p$ dimension on $\mathbb{R}^{4p}$ [8]. Due to the scale invariance of these systems, it is also possible to map the systems on $\mathbb{R}^{4p}$ onto the spheres $S^{4p}$ [9]. The solitons on the complex projective space $\mathbb{C}P^n$ were also considered [10]. The Grossman-Kephart-Stasheff solution on an eight-dimensional sphere is an example of an exact solution to the generalized self-duality equations [11]. In [12] non-Abelian Berry’s phases suggest the localized monopoles with codimensions greater than five in ten-dimensional space-time. Finite energy solution in six-dimensional Minkowski space has been recently studied numerically [13, 14].

In this article we construct a new exact instanton solution to the generalized self-duality equations with gauge group $SO(6)$ on a six-dimensional sphere $S^6$. Our solution saturates the Bogomol’nyi type bound for the pseudo-energy and so is stable. Since the sphere $S^6$ is covered by two patches each of which is diffeomorphic to $\mathbb{R}^6$, the topological charge is given by a map from a spatial boundary of one patch (five-dimensional sphere $S^5$) to the gauge group $SO(6)$, as the generalization of BPST instantons on $S^4$. This map is
classified by the fifth homotopy group, $\Pi_5[SO(6)] \simeq \mathbb{Z}$. A six-dimensional compact space is particularly important because a soliton on it may give spontaneous compactification from ten dimensions, suggested by string theory, to four dimensions. In fact we see that the radius of $S^6$ is related to the pseudo energy of our solution, suggesting the stabilization of the modulus of the compactified space. We consider an SO(6) gauge theory with a higher derivative coupling. Several people study higher derivative coupling terms in the context of D-branes as higher derivative corrections to a non-Abelian Dirac-Born-Infeld soliton \cite{15}. We expect that our higher derivative coupling has such a origin.

Our model consists of gauge fields $A_{i}^{[ab]} (i, a, b = 1, \cdots, 6)$. The group SO(6) is locally isomorphic to SU(4). We use the Clifford algebra of SO(6) for the representation of gauge group. \cite{18}

It is generated by the gamma matrices $\Gamma_a$, where $a = 1, \cdots, 6$. These generators $\{\Gamma_a\}$ are realized as Hermitian $8 \times 8$ matrices with complex coefficients. Thus the Lie algebra is embedded in the $\text{su}(8)$ algebra. They satisfy the anti-commutation relation $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$ and their commutators $\Gamma_{ab} = \frac{1}{2} [\Gamma_a, \Gamma_b]$ generate the Lie algebra $\text{so}(6)$. The chirality operator $\gamma_7 = -i\Gamma_1 \cdots \Gamma_6$ plays an important role. The operator $\gamma_7$ anti-commutes with $\Gamma_a$ and commutes with $\Gamma_{ab}$. $\Gamma_a$'s and $\gamma_7$ are Hermitian ($\Gamma_a^\dagger = \Gamma_a$, $\gamma_7^\dagger = \gamma_7$) whereas $\Gamma_{ab}$'s are anti-Hermitian. Antisymmetric products of $\Gamma$'s are defined as

$$\Gamma_{a_1 \cdots a_s} = \frac{1}{s!} \sum_{\sigma \in \mathfrak{S}_s} \text{sign}(\sigma) \Gamma_{a_{\sigma(1)}} \cdots \Gamma_{a_{\sigma(s)}} ,$$

where the sum is taken over all permutations of $\{1, \cdots, s\}$. A set of the matrices $\{1, \Gamma_a, \Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}, \Gamma_a \gamma_7, \Gamma_a \gamma_7, \gamma_7\}$ generates the total Clifford algebra. Thus the dimension of the Clifford algebra is equal to $2^6 = 64$. The Clifford algebra can be graded by the number of $\Gamma_a$'s where $a$ runs from 1 to 6. For instance the grade of the generator $\Gamma_{ab} \gamma_7$ is fourth. The (anti-)commutation relations of $\Gamma_{ab}$'s are given as follows,

$$\{\Gamma_{ab}, \Gamma_{cd}\} = 2(\delta_{bc}\delta_{ad} - \delta_{bd}\delta_{ac} + \Gamma_{abcd}) ,$$

$$[\Gamma_{ab}, \Gamma_{cd}] = 2(\delta_{bc}\Gamma_{ad} - \delta_{bd}\Gamma_{ac} - \delta_{ac}\Gamma_{bd} + \delta_{ad}\Gamma_{bc}) .$$

The anti-commutation relations imply that $\text{Tr} \Gamma_{ab} \Gamma_{cd} = -8(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) = -8\delta_{[cd]}^{ab}$. The matrices $\Gamma_{abcd}$ is related to $\Gamma_{ef}$ by the chiral operator $\gamma_7$ and totally antisymmetric tensor $\epsilon_{abcdef}$:

$$\Gamma_{abcd} = -\frac{i}{2!} \gamma_7 \epsilon_{abcdef} \Gamma_{ef} ,$$

where $\epsilon_{abcdef}$ is the totally antisymmetric tensor.
where $\epsilon$ is normalized as $\epsilon_{123456} = 1$.

The base space which we consider is a six-dimensional sphere $S^6$. The sphere $S^6$ is covered by the two patches which are diffeomorphic to $\mathbb{R}^6$. We work on one of them. We denote the space coordinates by $x^i$. We consider the standard metric on the sphere: the metric and curvature tensors are

\[
 ds^2 = \frac{\delta_{ij}}{(1 + x^2/4R_0^2)^2}dx^i dx^j = g_{ij}dx^i dx^j, \quad R^i_{jkl} = \frac{1}{R_0^2} \left( \delta^i_k g_{jl} - \delta^i_l g_{jk} \right), \quad \mathcal{R} = \frac{30}{R_0^2},
\]

where the parameter $R_0$ is the radius. The determinant of the metric is $g = \det g_{ij} = (1 + x^2/4R_0^2)^{-12}$. We often abbreviate the basis of forms; $dx^{i_1} \wedge \cdots \wedge dx^{i_k} = dx^{i_1 \cdots i_k}$. The six-form $dv = dx^{1 \cdots 6} \sqrt{g}$ is an invariant volume form. Indeed the form is invariant under general coordinate transformations. The integration $\int_{S^6} dv$ gives the volume of $S^6$. The Hodge dual of a base of differential forms, $dx^{i_1 \cdots i_s}$, is given as

\[
 *dx^{i_1 \cdots i_s} = \frac{1}{(6 - s)!} \sqrt{g} \epsilon_{i_1 \cdots i_s j_1 \cdots j_{6-s}} g_{j_{1} j'_{1}} \cdots g_{j_{6-s} j'_{6-s}} dx^{j'_{1} \cdots j'_{6-s}},
\]

where $\epsilon_{ijklmn}$ is again a totally antisymmetric tensor and is normalized as $\epsilon_{123456} = 1$. One must take care of the position of the indices which should be raised and lowered by the metric tensor. The use of sechsbein $e^I = dx^I/(1 + x^2/4R_0^2)$, $(I = 1, 2, \cdots, 6)$ makes notation simple. They form an orthonormal frame, $ds^2 = \delta_{IJ} e^I e^J$. Thus we can relax the index position of $I, J$. The spin connection is defined by the relation; $de^I = -\omega^{IJ} e^J$. One can easily see that $\omega^{IJ} = (x^I e^J - x^J e^I)/2R_0^2$. The Riemann curvature is expressed as $R^{IJ} = d\omega^{IJ} + \omega^{IK} \wedge \omega^{KJ} = e^I \wedge e^J/R_0^2$. We often mix up these indices with the internal indices $a, b, \cdots$. The Hodge dual of their products becomes

\[
 *e^{I_1} \wedge \cdots \wedge e^{I_s} = \frac{1}{(6 - s)!} e^{I_1 \cdots I_{6-a}} e^{I_{a+1}} \cdots e^{I_6}.
\]

The dynamical degree of freedom of our model is a Lie-algebra valued one form $A = \frac{1}{2} A^{ab}_i dx^i \Gamma_{ab}$. The corresponding field strength $F$ is defined as usual, $F = dA + eA \wedge A$. Here $e$ is a gauge coupling constant. Having finished with the description of our notation, we move to defining the pseudo-energy, $E$, of the system,

\[
 E = \frac{1}{16} \int \text{Tr} \left\{ -F \wedge *F + \alpha^2 (F \wedge F) \wedge *(F \wedge F) \right\}.
\]
Here the word “pseudo” means that this system is not an ordinary dynamical system because the space is a Riemannian space which has a positive definite metric $g_{ij}$ without a time direction. If we consider the space-time $\mathbb{R} \times S^6$ by adding a time coordinate $\mathbb{R}$, then $E$ can be regarded as a usual energy for static configurations. In the case of the theory defined on a ten-dimensional space-time, the Riemannian space $S^6$ can be considered as the compactified space in $M^4 \times S^6$ with $M_4$ a pseudo Riemannian manifold \[16\]. The (mass) dimension of the gauge field is equal to 2. The couplings $e$ and $\alpha$ have (mass) dimension $-1$ and $-3$, respectively. Thus this system is, of course, non-renormalizable. We leave the discussion of the physical meaning of our solution. We show the pseudo-energy in components expression:

$$F = \frac{1}{4} F_{ij}^{[ab]} dx^{ij} \Gamma_{ab}, \quad \text{Tr} F \wedge *F = -2 F_{ij}^{[ab]} F^{ij,[ab]} dv$$  \hspace{1cm} (9)$$

The square of the Hodge star acts identically on these even forms, $**F = F, **(F \wedge F) = F \wedge F$.

The pseudo energy is bounded from below by a topological charge with higher derivative coupling with coupling constant $\alpha$,

$$E = \frac{1}{16} \int \text{Tr} \left( i F \mp \alpha \gamma_7 * (F \wedge F) \right) \wedge * \left( i F \mp \alpha \gamma_7 * (F \wedge F) \right) \pm \frac{i}{8} \alpha \int \text{Tr} \gamma_7 F \wedge F \wedge F \geq \pm \frac{i}{8} \alpha \int \text{Tr} \gamma_7 F \wedge F \wedge F = \mp \frac{1}{23} \int \epsilon_{abcdef} F^{[ab]} \wedge F^{[cd]} \wedge F^{[ef]} \equiv Q,$$  \hspace{1cm} (10)$$

where the pseudo-energy is bound by the topological number $Q$.

The charge $Q$ is obtained by integrating a total derivative term and it reduces to a surface integral over a five-dimensional sphere as the boundary of one patch. We consider a trivial configuration for the field strength in the other patch. The two configurations of the two patches are transformed to each other by a gauge transformation in the overlap region (isomorphic to $S^5$). Thus the integration is proportional to the winding number of $\Pi_5[S\text{O}(6)]$. The minimal group leading to a non-trivial fifth homotopy group is $\text{SU}(3)$. In general, the fifth homotopy group is closely related to non-abelian anomaly in four-dimensional space-time. The spin connection with respect to the standard metric of six-dimensional sphere provides an exact solution of the Tchrakian’s generalized self-duality equation. If the field strength fulfills the generalized (anti-)self-duality equation

$$F = \pm i \alpha \gamma_7 * (F \wedge F),$$  \hspace{1cm} (11)$$
the pseudo-energy becomes $Q$. Note that the product $F \wedge F$ and the Bogomol’nyi equation written by it depend on the representation matrices of $F$; The pseudo-energy and its Bogomol’nyi equation written by the SO(6) matrices are different from ours.

Now we study an exact nontrivial solution of the self-duality (Bogomol’nyi) equation. We consider a hedge-hog connection $A$ and the field strength,

$$A = \frac{1}{4e R_0^2} x^a e^b \Gamma_{ab}, \quad F = \frac{1}{4e R_0^2} e^a \wedge e^b \Gamma_{ab}. \quad (12)$$

We can easily check that the field strength $F$ fulfills the self-duality equation with $\alpha = e R_0^2 / 3$,

$$F \wedge F = -\frac{3}{e R_0^2} i \gamma_7 \star F, \quad F = \frac{e R_0^2}{3} i \gamma_7 \star (F \wedge F). \quad (13)$$

Then the pseudo-energy is

$$E = \frac{4\pi^3 R_0^2}{e^2}. \quad (14)$$

This is a solution to the generalized self-duality equation (III). The third power of the field strength gives a term proportional to volume form up to a numerical coefficient. The winding number of this solution is equal to 1. We also obtained a solution with charge $Q = -1$. The solution is given as $A^{(-)} := \Gamma_6 A \Gamma_6$. The energy is given by the winding number of spatial $S^5$. Thus the spin connection with respect to the standard metric of six-dimensional sphere gives an exact solution to the Tchrakian’s generalized self-duality equation (III). Our solution is the first one in $(4p + 2)$-dimensional theories with $p$ an integer. It remains as an open question whether the minimal group needed for these solutions is SO(6) or not, since the minimal group leading to a non-trivial fifth homotopy group is SU(3). The realization of this solution in a gauge theory coupled with gravity is a natural extension [16]. It is needed in order to make contact with our universe. In particular the fact that the pseudo energy (14) is related to the radius $R_0$ of $S^6$ suggests that compactification from ten dimensions to four dimensions on $S^6$ with our solution is stable without unwanted massless modes (moduli) [6]. In addition, we can consider the similar system on another geometry. We have checked the case of complex projective space $\mathbb{C}P^3$. To make the Bogomol’nyi completion in the case of $\mathbb{C}P^3$, we must add another higher derivative coupling term which is not a single trace term. Investigating possible relations with supergravity with higher derivative correction terms (see, e.g., [17]) or with non-Abelian Dirac-Born-Infeld solitons is also an interesting subject.
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[18] The reason why we use the Clifford algebra is that the pseudo-energy defined below in Eq. (8) can be expressed in a single trace term. The pseudo-energy expressed in SO(6) indices equivalent to ours cannot be expressed in a single trace and is very complicated.