Einstein field equations for Bose-Einstein condensates in cosmology

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Abstract. In this work we consider the Gross-Pitaevskii equation for Bose-Einstein condensates (BECs) in a general Riemannian metric. Given initial conditions dictated by an external potential, we consider the free expansion of the condensate when the external potential is turned off. Focusing on the forces associated with the geometry of the initial configuration, we show how these are related to the Ricci curvature tensor and the Ricci scalar and we find an Einstein field equation governing the steady flow. Some important correlations between the study of defects in BECs and the appearance of cosmological singularities will be addressed, in particular the emergence of an effective Lorentzian spacetime geometry, which is what is needed to obtain Hawking radiation effects.

1. Introduction
In this work we consider the Gross-Pitaevskii equation (GPE) in the context of a generic Riemannian manifold $M$ with metric $g$ in order to derive an Einstein’s field equation governing the kinematics and dynamics of Bose-Einstein condensates (BECs) for possible applications in cosmology. This idea comes from the analogy that exists between the study BECs and cosmological black-hole theory. In recent years many authors advanced the idea [1, 2] that gravity could be a large scale phenomenon represented by a condensed state of matter. With this in mind it has been proven [3, 4] that via linearisation both BEC systems and gravity could be governed by a curved spacetime effective Lorentzian geometry. This theory has led to analog models of gravity [5, 6] where condensates have been used as toy models to provide a powerful tool [7] for testing kinematic aspects of both classical and quantum field theories in curved spacetime, including the possibility to test experimentally phenomena such as the Hawking radiation associated with a black hole formation.

The structure of the paper is as follows. A brief review on the cosmological context where analog models for gravity emerge is given in section 2. In section 3 we introduce the physical system considering the GPE on a generic manifold where we derive the hydrodynamic formulation in the extended form. Here we focus on the extra term coming from the geometric configuration of the manifold and we provide a physical interpretation for the emergence of an Einstein’s field equation governing the dynamics of the condensate. In section 4 we carry out the linearisation around the mean field in order to find the hyperbolic wave equation governing the kinematics of the perturbations (phonons) on an effective Lorentzian geometry.
2. Analog models in cosmological context

The basic idea is that any alteration to the propagation of a field due to the presence of a curved spacetime can be reproduced by an analog field propagating in some material background with space and time dependent properties.

The notion of effective metric was first introduced by Gordon [8] in the description of the effects of a refractive index on the propagation of light, and this idea was further developed by Landau and Lifshitz [9] in optics, where they represent gravitational fields in terms of an equivalent refractive index. It was only in the eighties that Unruh [10, 11] introduced and used the hydrodynamic analog, where sound waves in fluids are interpreted as a scalar field in an effective curved spacetime described by acoustic geometry. The first observation on simulation of gravitational configurations in BECs was made by Garay et al. [12, 13] who investigated the analog of a gravitational black hole in a dilute Bose-Einstein condensate. They showed that, in the hydrodynamic limit, configurations that exhibit behaviours analog to that of a gravitational black hole do exist, simulating a sonic black hole by solving numerically the GPE subject to some particular trapping potential. Visser, Barceló and Liberati [14, 15] proved that the propagation of phononic perturbations in a fluid is described by a wave equation that can be thought to propagate in an effective relativistic curved spacetime with metric entirely determined by the physical properties of the fluid, given by its density $\rho$ and velocity $u$. In particular they showed that BECs may provide a useful model of an approximate Lorentz invariance in the low-momentum limit. This because perturbations about the stationary state are shown to obey the Bogoliubov equations, that within the regime of validity of the hydrodynamic approximation can be reduced to a single second order differential equation for the perturbation of the condensate’s phase $\theta$ in the form of a relativistic wave equation in a curved spacetime described by acoustic metric. This Lorentz invariance is then proven to break down in the context of higher energy physics regimes.

The emergence of Lorentzian-signature effective metrics is an almost generic aspect of low momentum physics. Such effective metrics are consequence of performing a normal modes analysis on an arbitrary field theory, namely a linearisation of a classical scalar field theory around some background configuration. This technique is of great interest since it provides analog models with the ability to generate analog horizons and analog black holes that simulate aspects of general relativity. Indeed, the linearisation of any Lagrangian dynamics, or of any hyperbolic second order PDE, leads to an effective geometry that governs the propagation of perturbations. The existence of a low momentum regime, where phase perturbations of the wave function behave as if they were coupled to an effective Lorentzian metric, is a general property of the non-linear Schrödinger types of equations. An effective Lorentzian metric is also what is needed to obtain simulations of the Hawking radiation effect. In this context BECs, considered in their hydrodynamic formulation given by the Gross-Pitaevskii equation, provide an approximation of the Hawking temperature which is only few orders of magnitude less than the BEC temperature. For these reasons Bose-Einstein condensates appear to be extremely promising systems for cosmological applications.

3. Hydrodynamic form of the Gross-Pitaevskii equation in Riemannian metric

A Bose-Einstein condensate at zero temperature is the ground-state of a second-quantized many-body Hamiltonian for $N$ interacting bosons of mass $m$, trapped by an external potential $V$, where only binary collisions are relevant and characterized by a single parameter that is the scattering length $a_s$ [16]. In the mean field approximation all the particles are thought to be in the same single particle quantum state, hence we can assume the condensate to be described completely by a single wave function $\Psi$. This wave function evolves under the Gross-Pitaevskii equation
[17, 18], a non-linear Schrödinger equation given by

\[ i\hbar \partial_t \Psi = \left( -\frac{\hbar^2}{2m} \Delta + g |\Psi|^2 + V(x, t) \right) \Psi, \tag{1} \]

where \( g = 4\pi\hbar^2a_s/m \) takes account of the inter-particle interaction.

Let us consider the GPE (1) on a generic Riemannian manifold \( M \) with metric \( g \) and recall that the Laplacian operator can be written in terms of the metric as

\[ \Delta \Psi = \frac{1}{|g|} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j \Psi \right), \tag{2} \]

where \( g_{ij} = g(\partial_i, \partial_j) \) is the metric matrix, \( g^{ij} \) is the inverse matrix, and \( |g| = |\det g_{ij}| \). By writing the wave function \( \Psi \) as a complex function in polar coordinates in terms of density \( \rho \) and phase \( \theta \), we apply the Madelung transform [19]

\[ \Psi = \sqrt{\rho} e^{i(\frac{m}{\hbar} \theta)}, \tag{3} \]

and by substituting (3) into (1) we obtain the hydrodynamic formulation of the GPE (see [20]). By equating the imaginary parts we obtain an equation for the density which can be easily recast in the form of a continuity equation

\[ \partial_t \rho = -\nabla \rho \cdot \nabla \theta - \rho \text{div} \nabla \theta = - \text{div} (\rho \nabla \theta); \tag{4} \]

by equating the real parts we obtain the evolution equation for the phase \( \theta \) in the form of a Hamilton-Jacobi equation

\[ \partial_t \theta = \frac{\hbar^2}{2m \sqrt{\rho}} \Delta \sqrt{\rho} - \frac{1}{2} |\nabla \theta|^2 - \frac{g}{m} \rho - \frac{V}{m}. \tag{5} \]

The term \( Q = \hbar^2 \Delta \sqrt{\rho} / (2m^2 \sqrt{\rho}) \) bears the name of quantum potential and describes quantum forces due to spatial variation of \( \rho \), while \( U = -V/m - g\rho/m \) represents the classical potential. If spatial variations of the density occur on length scales \( l \) smaller than, or of the order of the healing length \( \xi = \hbar / \sqrt{2m\rho g} \), the quantum potential dominates over the classical potential, while \( Q \) can be considered negligible on larger length scales.

Define the velocity field by

\[ u = \frac{\hbar}{m} \frac{\nabla \Psi}{|\Psi|^2} = i\hbar \frac{\Psi \nabla \Psi^* - \Psi^* \nabla \Psi}{2m |\Psi|^2}; \tag{6} \]

substituting (3) into (6) we find \( u = \nabla \theta \), so we can write the continuity equation (4) in the usual form

\[ \partial_t \rho = -\text{div} (\rho u). \tag{7} \]

We see that the condensate corresponds to a potential flow and, if \( \theta \) is non-singular, we can also conclude that the motion of the condensate is irrotational. All the vorticity is therefore confined in defect lines where \( \rho = 0 \) and \( \theta \) is ill-defined. In particular it is around these defect lines, inside a tubular neighbourhood of radius the healing length \( \xi \), that the quantum potential dominates the dynamics.
Taking the gradient of (5) in Riemannian metric we end up with an equation for the velocity field in the form of an Euler equation (see [21] for details):

\[ \partial_t u + \nabla u \cdot u = \nabla \left( \frac{\hbar^2}{2m\sqrt{\rho}} \Delta \sqrt{\rho} \right) - \frac{g}{m} \nabla \rho \frac{\nabla V}{m} = \nabla Q - \nabla U, \]

(8)

where \( \nabla u \cdot u \) is the covariant derivative of the velocity field along itself, that by applying Koszul’s formula is equal to the gradient \( \nabla (|u|^2/2) \).

Since we want to study the free expansion of the condensed cloud after being trapped [22] let’s consider the GPE in absence of a trapping potential \( (V = 0) \). Multiplying equation (8) by the density \( \rho \) by some straightforward algebra it can take the form [21] similar to a Navier-Stokes equation

\[ \rho (\partial_t u + \nabla u \cdot u) = -\frac{g}{m} \nabla \left( \frac{\rho^2}{2} \right) + \frac{\hbar^2}{4m^2} \text{div} (\rho \text{Hess}(\ln \rho)) + \frac{\hbar^2}{4m^2} E = -\nabla p + \text{div} \tau + \tilde{E} \]

(9)

where

- \( p = \frac{g\rho^2}{2m} \) denotes a pressure-like term,
- \( \tau = \frac{\hbar^2}{4m^2} \rho \text{Hess}(\ln \rho) \) denotes a stress tensor,
- \( \tilde{E} = \frac{\hbar^2}{4m^2} E \) denotes a new term.

\( E \) depends on the geometry of the manifold and its corresponding 1–form is given by

\[ E^b = E_j \, dx^j = -g^{ik} R_{ij} \partial_k \rho \, dx^j, \]

(10)

where \( R_{ij} \) is Ricci curvature tensor, given by \( R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij} \Gamma^a_{ka} - \Gamma^k_{ik} \Gamma^a_{ja} \).

This additional term \( \tilde{E} \) determines a sort of external force acting on the BEC that depends only on the geometry of the initial configuration and the slope of the density profile.

Following the same procedure we can also write a continuity equation for the momentum \( \rho u \) in the form:

\[ \partial_t (\rho u) = -\text{div}(\Pi) + \varepsilon = -\text{div} \left( D + pg - \tau + \frac{\hbar^2}{4m^2} G \rho \right) - \frac{\hbar^2}{8m^2} R d\rho, \]

(11)

where \( \Pi = \Pi_{ij} \, dx^i \wedge dx^j \) denotes the momentum flux tensor, given by

\[ \Pi_{ij} = \rho u_i u_j + pg_{ij} - \tau_{ij} - \rho \tilde{G}_{ij}, \]

(12)

where \( u^b = u_j \, dx^j \) is the 1–form associated to the velocity field, \( \tilde{G}_{ij} = \hbar^2 G_{ij}/(4m^2) \) is proportional to Einstein’s tensor \( G_{ij} \), and \( \varepsilon = -\hbar^2 R d\rho/(8m^2) \) is a geometric correction term proportional to the Ricci scalar.

Notice that, if the scalar curvature \( R \) is constant on the manifold, then the correction term can be written as

\[ -\frac{\hbar^2}{8m^2} R d\rho = -\text{div} \left( \frac{\hbar^2}{8m^2} R g \right) \]

(13)
so to obtain the equation for the momentum in the standard form

$$\partial_t (\rho u) = - \text{div} \left( D + pg - \tau + \frac{\hbar^2}{4m^2} G \rho + \frac{\hbar^2}{8m^2} R \rho g \right)$$

$$= - g^{ik} \nabla_k \left( \rho u_i u_j + \frac{g \rho^2}{2m} g_{ij} - \frac{\hbar^2}{4m^2} \rho \text{Hess}(\ln \rho) + \frac{\hbar^2}{4m^2} G_{ij} \rho + \frac{\hbar^2}{8m^2} R \rho g_{ij} \right).$$

(14)

In this case we can consider the steady state given by requesting Ricci’s tensor to satisfy

$$R_{ij} = \text{Hess}_{ij}(\ln \rho) - \frac{4m^2}{\hbar^2} u_i u_j - 8\pi a_s \rho g_{ij}.$$  

(15)

Going back to the general case, we can re-writing equation (11) in the form

$$\partial_t (\rho u) = - g^{ik} \nabla_k \left( \rho u_i u_j - \frac{\hbar^2}{4m^2} \rho \left( \text{Hess}_{ij}(\ln \rho) - G_{ij} \right) \right) \text{d}x^j - \left( \frac{g}{m} \rho + \frac{\hbar^2}{8m^2} R \right) \partial_j \rho \text{d}x^j,$$  

(16)

and we can impose a steady flow through a geometric configuration of the space by letting

$$G_{ij} = \text{Hess}_{ij}(\ln \rho) - \frac{4m^2}{\hbar^2} u_i u_j$$

(17)

and

$$R = - \frac{8mg}{\hbar^2} \rho = - 32\pi a_s \rho,$$  

(18)

(remembering that \( g = 4\pi \hbar^2 a_s / m \)) so to find a form of Einstein’s field equations for the GPE.

4. GPE linearisation on a manifold

Following Visser et al. [23, 24], we proceed with the linearisation of the GPE and its hydrodynamic counterparts in order to find the effective Lorentzian geometry on the manifold \( M \). The aim of a linearisation process is to obtain a Poisson equation for the perturbation of the velocity potential, i.e. \( \Delta \theta = 0 \), namely

$$\frac{1}{\sqrt{|h|}} \partial_{\mu} \left( \sqrt{|h|} h^{\mu\nu} \partial_{\nu} \theta \right) = 0,$$  

(19)

with \( h = h_{\mu\nu}(x, t) \) the effective metric.

Let us write each quantity as the sum of a mean field and a perturbation, i.e.

$$\rho = \rho_0 + \epsilon \rho_1,$$

$$p = p_0 + \epsilon p_1 = \frac{g(\rho_0 + \epsilon \rho_1)^2}{2m} = \frac{g\rho_0^2}{2m} + \epsilon \frac{g \rho_0 \rho_1}{m},$$

$$\theta = \theta_0 + \epsilon \theta_1,$$

$$u = u_0 + \epsilon u_1 = \nabla(\theta_0 + \epsilon \theta_1) = \nabla \theta_0 + \epsilon \nabla \theta_1.$$  

(20)

Substituting the above expressions into the continuity equation (4) we get the equations for the density mean field and perturbation, given by

$$\partial_t \rho_0 = - \text{div}(\rho_0 u_0)$$  

(21)

and

$$\partial_t \rho_1 = - \text{div}(\rho_1 u_0 + \rho_0 u_1).$$  

(22)
Substituting into the Hamilton-Jacobi equation (5), we get
\begin{equation}
\partial_t \theta_0 + \epsilon \partial_t \theta_1 = -\frac{V}{m} - \frac{g \rho_0}{m} - \epsilon \frac{g \rho_1}{m} - \frac{1}{2} |\nabla \theta_0|^2 - \epsilon \nabla \theta_0 \cdot \nabla \theta_1 + \frac{\hbar^2}{m^2} Q, \tag{23}
\end{equation}
where $Q$ is given by
\begin{equation}
Q = \left( \frac{1}{2\sqrt{\rho_0}} - \epsilon \frac{\rho_1}{4\sqrt{\rho_0}} \right) \left( \Delta \sqrt{\rho_0} + \epsilon \Delta \left( \frac{\rho_1}{2\sqrt{\rho_0}} \right) \right) \tag{24}
\end{equation}
where $\Delta$ denotes a second order differential operator acting on $\rho_1$, given by
\begin{equation}
\mathcal{D}(\rho_1) = \frac{1}{4\sqrt{\rho_0}} \Delta \sqrt{\rho_0} \cdot \rho_1 - \frac{1}{4\sqrt{\rho_0}} \Delta \left( \frac{\rho_1}{\sqrt{\rho_0}} \right), \tag{25}
\end{equation}
that depends explicitly on the metric of the manifold through the Laplacian operator.

By substituting (25) into (23) we get the equation for the phase mean field
\begin{equation}
\partial_t \theta_0 = -\frac{V}{m} - \frac{g \rho_0}{m} - \frac{1}{2} |\nabla \theta_0|^2 + \frac{\hbar^2}{2m^2\sqrt{\rho_0}} \Delta \sqrt{\rho_0} \tag{26}
\end{equation}
and the equation for the perturbation
\begin{equation}
\partial_t \theta_1 = -\frac{g \rho_1}{m} - \nabla \theta_0 \cdot \nabla \theta_1 - \frac{\hbar^2}{m^2} \mathcal{D}(\rho_1). \tag{27}
\end{equation}

If the operator $\mathcal{D}$ is invertible, from (27) we can find $\rho_1$:
\begin{equation}
\rho_1 = -\left( \frac{g}{m} + \frac{\hbar^2}{m^2} \mathcal{D} \right)^{-1} \left( \partial_t \theta_1 + \nabla \theta_0 \cdot \nabla \theta_1 \right) \tag{28}
\end{equation}
and by substituting the above into (22) we obtain
\begin{equation}
- \partial_t \left[ \left( \frac{g}{m} + \frac{\hbar^2}{m^2} \mathcal{D} \right)^{-1} \left( \partial_t \theta_1 + \nabla \theta_0 \cdot \nabla \theta_1 \right) \right] \\
+ \text{div} \left( \rho_0 \nabla \theta_1 - \nabla \theta_0 \left[ \left( \frac{g}{m} + \frac{\hbar^2}{m^2} \mathcal{D} \right)^{-1} \left( \partial_t \theta_1 + \nabla \theta_0 \cdot \nabla \theta_1 \right) \right] \right) = 0, \tag{29}
\end{equation}
that is a wave-like equation that describes the propagation of the linearised phase $\theta_1$: this equation can be written in the contracted form
\begin{equation}
\partial_\mu (f^{\mu\nu} \partial_\nu \theta_1) = 0 \tag{30}
\end{equation}
by putting
\begin{equation}
\begin{align*}
f^{00} &= -\left( \frac{g}{m} + \frac{\hbar^2}{m^2} \mathcal{D} \right)^{-1}, \\
f^{0j} &= -\left( \frac{g}{m} + \frac{\hbar^2}{m^2} \mathcal{D} \right)^{-1} g^{ij} \partial_i \theta_0, \\
f^{ij} &= -g^{ij} \partial_j \theta_0 \left( \frac{g}{m} + \frac{\hbar^2}{m^2} \mathcal{D} \right)^{-1}, \\
f^{ij} &= \rho_0 g^{ij} - g^{ik} \partial_k \theta_0 \left( \frac{g}{m} + \frac{\hbar^2}{m^2} \mathcal{D} \right)^{-1} g^{lj} \partial_l \theta_0. \tag{31}
\end{align*}
\end{equation}
If we finally set
\[ \sqrt{|h|} h^{\mu \nu} = f^{\mu \nu}, \]
we have the required Poisson equation
\[ \Delta \theta_1 = \frac{1}{\sqrt{|h|}} \partial_\mu (\sqrt{|h|} h^{\mu \nu} \partial_\nu \theta_1) = 0. \]

This represents the emergence of the effective Lorentzian geometry in the generic case of a Riemannian manifold. Detailed analysis of the invertibility condition of \( D \) and conditions for the emergence of this effective geometry will be investigated in a forthcoming paper [25].

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