Spin calculations in arbitrary and diagonal spin bases

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Abstract
We consider the spin calculations in arbitrary and diagonal spin bases

1 Introduction
At the present time, the conventional method of direct calculation of the squared modulus of amplitudes is practically not used in calculating of reactions. This is due to the fact that even an insignificant increase in the number of particles participating in a reaction as well as the account of their polarizations sharply complicate calculations. The idea of direct calculation of amplitudes followed by their quadrating is fairly obvious. However, such an approach entails a number of specific problems. Therefore, in the last few decades publications devoted to the development of particular aspect of this method have been appearing in the press (see references in reviews [1], [2]).

These works can conventionally be classified according to the following directions: a covariant or an incovariant approach is used; calculations are carried out in an arbitrary or a particular spin basis. Among the first works devoted to this topic are the works of F.I. Fedorov and his scholars. For details see [3]. They are devoted primarily to covariant calculation of amplitudes in an arbitrary spin basis. For instance, in [4] a method that permits using in calculating amplitudes the main “trick” of the quadrating procedure was proposed. Namely, reduction of calculations to the computation of traces from $\gamma$-matrix operators.

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Let us illustrate this approach by an example of a fermion “sandwich” which can be given in the following form ($Q$ – arbitrary operator):

$$\bar{u}^{\sigma'}(p', s')Qu^\sigma(p, s) = \text{Tr} Qu^\sigma(p, s) \bar{u}^{\sigma'}(p', s').$$ (1)

The main difficulties of this approach are the obtaining of an explicit form of the transition operator

$$u^\sigma(p, s)\bar{u}^{\sigma'}(p', s'),$$ (2)

as well as the solution of such problems as the calculation of the phase factors, the removal of singularities, the calculation of the exchange diagrams, and the account of the specificity of the spin configuration $\sigma' = -\sigma$, and so on. Some of these difficulties can be overcome by fixing a certain manner the spin basis. Transition to another basis is realized with the use of the Wigner $D$-functions. However, this apparatus is exceedingly cumbersome and incovariant.

The present paper shows how, using amplitudes (1) in the basis in which they have the simplest form, to obtain expressions for the squared modulus of an amplitude in an arbitrary spin basis. The paper considers the following questions. In Section 2, one-particle states are considered. It is shown that consistent use of tetrads formalism permits operating with a complete set of bispinors as well as reducing the operations of matrix operators to tensor ones.

In Section 3, matrices that make it possible, using the amplitudes calculated in one spin basis, to calculate the squared modulus of amplitudes in another spin basis.

Section 4 considers the introduced into [5] the diagonal spin basis (DSB). In the DSB, the bispinors of the initial and final states have a common set of spin operators and interrelated in the simplest way. As a result, the central problem of this approach – the obtaining of a simple and compact solution. The diagonal amplitudes have a clear physical interpretation and are the best “building bricks” for calculation of squared modulus of amplitudes in arbitrary spin basis.

In Section 5, some examples of calculation in DSB are considered.

2 One-particle states. Tetrads

The one-particle state is defined by 4-vectors, momentum $p$ and the axis of spin projections $s$. And $ps = 0$. This condition is fulfilled if $s$ is determined in terms of the arbitrary vector $q$

$$s = \frac{(pq)p - m^2q}{m\sqrt{(pq)^2 - m^2q^2}}.$$ (3)

Define

$$s_0 = v = \frac{p}{m}, \quad s_3 = s,$$ (4)
then \( s_0^2 = -s_3^2 = 1, s_0s_3 = 0 \). Any vector having the form \( \alpha s_0 + \beta s_3 \) belongs to the 2-plane \((s_0, s_3)\) in Minkowski space. The orthogonal 2-plane can be determined with the aid of two tensors

\[
g^{\mu\nu} = v^\mu v^\nu - s^\mu s^\nu, \quad \tilde{\varepsilon}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} v_\rho s_\sigma,
\]

where \( \varepsilon_{0123} = 1 \). If the arbitrary vector \( r \) lies incompletely in the 2-plane \((s_0, s_3)\), then the vectors \((g_{\parallel} - g_{\perp})r^\perp, -\tilde{\varepsilon}^{\mu\nu}r_\perp\) satisfy the conditions \( s_1^2 = s_2^2 = -1, s_1s_2 = 0 \) and both vectors are orthogonal to \( s_0 \) and \( s_3 \). Thus, the tetrad \( s_0, s_3, s_1, s_2 \) form orthonormalised basis in the Minkowski space. In all spin reactions, vectors \( s_1 \) and \( s_2 \) only occur in combination \( s_1 \pm is_2 \). From (6) follows

\[
s_1' + i\sigma s_2' = e^{i\sigma \varphi} (s_1 + i\sigma s_2), \quad e^{i\sigma \varphi} = \frac{r_{\perp} T_\sigma r'}{r_{\perp} r'}.
\]

It is clear that the tensor \( T_\sigma \) plays the special role in spin calculations. Besides the apparent properties \( T_\sigma = T_{-\sigma}, T_{\mu} = T_\nu, T_\sigma^2 = -2T_\sigma \), the relation

\[
T^\mu_{\sigma} T^{\alpha\beta}_{\sigma} = T^{\mu\beta}_{\sigma} T^{\alpha\nu}_{\sigma}
\]

holds for it. From this relation follows

\[
T^\mu_{\sigma} r_{\alpha} T^{\alpha\beta}_{\sigma} = r_{\perp}^2 T^{\mu\beta}_{\sigma}.
\]

If in (7) we replace the vector \( r \) by \( r' \), then from (6) and (8) we obtain

\[
s_1' + i\sigma s_2' = e^{i\sigma \varphi} (s_1 + i\sigma s_2), \quad e^{i\sigma \varphi} = \frac{r_{\perp} T_\sigma r'}{r_{\perp} r'}.
\]

Thus, variations of \( r \) lead to rotation by angle \( \varphi \) in the 2-plane \((s_1, s_2)\), and in the spin relations an additional phase factor appears. Having fixed \( r \), we thus fix the tetrad in the Minkowski space and thus can write in explicit form the whole set of spin operators:

\[
\Sigma_3 = \gamma_5 \hat{s}_3, \quad \Sigma_\sigma = \frac{1}{2}(\hat{s}_1 + i\sigma \hat{s}_2),
\]

which act on the bispinor \( u^\sigma(p, s, r) \) in following way:

\[
\Sigma_3 u^\sigma(p, s, r) = \sigma u^\sigma(p, s, r); \quad \Sigma_{-\sigma} u^\sigma(p, s, r) = u^{-\sigma}(p, s, r); \quad \Sigma_\sigma u^\sigma(p, s, r) = 0.
\]

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1. \( \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \) can be given in form \( \gamma_5 = i\hat{s}_1\hat{s}_2\hat{s}_3\hat{s}_0 \).
2. We will keep symbols \( s \) and especially \( r \) only where necessary.
The second relation represents the known phase agreement: the action of the spin flip operator does not lead to the appearance of an additional phase factor. Note the following important fact. Having fixed the tetrad, we obtain the possibility of writing for the 4-component bispinor four equations, namely, two equations for the spin flip operator are added to the Dirac equation and the equation for the spin projection.

Let us consider the consequences of this situation. We give the tensor $g$ in the form

$$g^{\mu\nu} = \bar{v}^{\mu}v^{\nu} - s^{\mu}s^{\nu} - s_1^{\mu}s_1^{\nu} - s_2^{\mu}s_2^{\nu}. \quad (13)$$

Then from (11), (13) follows

$$\gamma^{\mu} = g^{\mu\nu}\gamma^{\nu} = \bar{v}^{\mu}\hat{v}^{\nu} - s^{\mu}\hat{s}^{\nu} - s_1^{\mu}\hat{s}_1^{\nu} - s_2^{\mu}\hat{s}_2^{\nu}$$

$$= \bar{v}^{\mu}\hat{v} - \gamma_5 \left\{ s^{\mu}\Sigma_3 + (s_1^{\mu} - i\sigma s_2^{\nu})\Sigma_{\sigma} + (s_1^{\mu} + i\sigma s_2^{\nu})\Sigma_{-\sigma} \right\}. \quad (14)$$

Using the Dirac equation and (12), we have

$$\gamma^{\mu}u^{\sigma}(p, s) = (\bar{v}^{\mu} - \sigma s^{\mu}\gamma_5)u^{\sigma}(p, s) - (s_1^{\mu} + i\sigma s_2^{\mu})\gamma_5u^{\sigma}(p, s). \quad (15)$$

Repeated action of the $\gamma$-matrix on (15) gives

$$\sigma^{\mu\nu}u^{\sigma}(p, s) = \sigma \left( [v \cdot s]^{\mu\nu}\gamma_5 + i[v \cdot s]^{\mu\nu} \right) u^{\sigma}(p, s)$$

$$- [(s_1 + i\sigma s_2) \cdot (\sigma s + \gamma_5 v)]^{\mu\nu} u^{\sigma}(p, s), \quad (16)$$

where the notation for tensor is introduced

$$[a \cdot b]^{\mu\nu} = a^{\mu}b^{\nu} - b^{\mu}a^{\nu}. \quad (17)$$

Any operator $Q$ can be expanded by complete set of Dirac matrices

$$\Gamma = \{1, \gamma_5, \gamma^{\mu}, \gamma_5\gamma^{\mu}, \sigma^{\mu\nu}\}. \quad (18)$$

Thus, formulas (15), (16) provide the possibility of reducing the action of an arbitrary matrix operator on the bispinor: $Qu^{\sigma}(p, s)$ to expansion by a complete set of bispinors $u^{\pm\sigma}(p, s)$, $\gamma_5u^{\pm\sigma}(p, s)$. The expansion coefficients will be tensors whose rank is equal to the number of free Lorentz-indices in the operator $Q$.

### 3 The transit matrices between the different spin bases

Let us assume that we know the set of amplitudes in a certain spin basis

$$\mathcal{M}_{\delta\delta} = \bar{u}^{\delta'}(p', s')Qu^{\delta}(p, s), \quad (19)$$
but the initial state is represented so that the axis of the spin projections is the vector \( l \).

Thus, we are interested in the amplitudes

\[
\mathcal{M}^\dagger \delta \sigma = \bar{u}^\dagger(p', s') Qu^\sigma(p, l),
\]

or to be more exact, in the final analysis, the squares of their modules.

Using the equation

\[
\sum_\delta \frac{1}{4}(1 + \hat{v})(1 + \delta \gamma_5 \hat{s}) u^\sigma(p, l) = u^\sigma(p, l)
\]

we obtain the expression for the squared module of the amplitude (20) (arguments in bispinors are omitted):

\[
|\mathcal{M}^\dagger \delta \sigma|^2 = \bar{u}^\dagger Qu^\sigma \bar{u}^\sigma \bar{Q} u^\delta
\]

(22)

where

\[
\mathcal{K}_{\delta_1 \delta_2} = \bar{u}^\delta_1 u^\sigma u^\delta_2 = \text{Tr} u^\sigma \bar{u}^\delta_1 \bar{u}^\delta_2 .
\]

Thus, if we know the explicit form of matrix (23), then using (19), we calculate the squared modules of amplitudes (20).

If \( \delta_1 = \delta_2 = \delta \), then

\[
\mathcal{K}_{\delta \delta} = \frac{1}{16} \text{Tr}(1 + \hat{v})(1 + \sigma \gamma_5 \hat{l})(1 + \hat{v})(1 + \delta \gamma_5 \hat{s}) = \frac{1}{2} (1 - \sigma \delta(s \hat{l})) .
\]

(24)

Using the operators of spin flipping \( \frac{1}{2} \gamma_5 (s_1 - i \delta s_2) \), we obtain the expression for the configuration \( \delta_1 = -\delta_2 = \delta \):

\[
\mathcal{K}_{-\delta \delta} = \frac{1}{32} \text{Tr}(1 + \hat{v})(1 + \sigma \gamma_5 \hat{l}) \gamma_5 (s_1 - i \delta s_2)(1 + \hat{v})(1 + \delta \gamma_5 \hat{s})
\]

\[
= \frac{1}{16} \text{Tr} \gamma_5 (s_1 - i \delta s_2)(\sigma \gamma_5 \hat{l} + \sigma \delta \hat{s} \hat{l})
\]

(25)

In deriving (25), we used the representation \( s_3 = s, s_0 = v \)

\[
\gamma_5 = i s_1 \hat{s}_2 \hat{s}_3 s_0 .
\]

(26)

From the explicit form (24), (23) of the elements of the matrix \( \mathcal{K}_{\delta_1 \delta_2} \) it follows that it can be written with the aid of Pauli matrices \( \sigma_i \):

\[
\mathcal{K}_{\delta_1 \delta_2} = (1 - \sigma_i(s \hat{l})) \delta_1 \delta_2 .
\]

(27)
can be given in a covariant 4-dimensional form. To do this, we introduce the isotropic 4-vector

\[ L = (1; \sigma(ls_i)) \]    

and take into account that \( \sigma_0 = 1 \). Then

\[ \mathcal{K} = \sigma_\mu L^\mu . \]  

It should be noted that 1 is nothing but the square of the 4-velocity of the particle, and \( l \) and \( s_3 \) are the axes of spin projections in different spin bases.

Note that into (27), (28) enter only the vectors of the tetrad in which we calculated amplitudes (19). It can be shown that the finite expression (22) is independent of not only the phase vector \( r \), but also the vectors \( s_1 \) and \( s_2 \), since they enter into (22) in combinations \( s_1^\mu s_1^\nu + s_2^\mu s_2^\nu = s_3^\mu s_3^\nu - g^{\mu\nu} \) and \( \varepsilon^{\mu\nu\rho\sigma} s_1^\rho s_2^\sigma = [v_0 \cdot s_3]^\mu\nu \).

Using formulas (24), (25), (27), (29) we can construct a matrix \( \mathcal{K} \) for any fermion participating in the reaction. And the representation of (27), (29) provides the possibility of using, if necessary, the Fierz transformation.

Note the following important fact. Formula (22) can also be used in the case where the beam is partially polarized and the states are not pure. To do this, it is enough to carry out substitution \( \sigma l \rightarrow a, |a^2| < 1 \) in (24), (25), (28). \( a \) is the vector of partial polarization (see [6]).

To investigate the polarization phenomena, besides the squared modules of amplitudes (22), it is necessary to know the values of \( \mathcal{M}_{\sigma_1} \mathcal{M}^{\ast}_{\sigma_2} \) since various polarization characteristics are expressed in their real and imaginary parts.

Thus, in the general case, it is necessary to know the matrix

\[ \mathcal{K}_{\delta_1 \delta_2}^{\sigma_1 \sigma_2} = \text{Tr} \ u^{\sigma_1 \sigma_2} \bar{u}^{\delta_1 \delta_2} , \]  

and at \( \sigma_2 = \sigma_1 \), we obtain (24), (25).

It can be shown that the matrix \( \mathcal{K} \) is of the form (\( \delta_{\alpha\beta} \) are Kroneker symbols)

\[ \mathcal{K}_{\delta_1 \delta_2}^{\sigma_1 \sigma_2} = \frac{1}{2} \left\{ \delta_{\sigma_1 \sigma_2} \delta_{\delta_1 \delta_2} - \sigma_1^{\sigma_2 \sigma_1} \sigma_2^{\delta_1 \delta_2} (l_i s_i) \right\} . \]  

It should be recalled that with respect to \( \delta_1 \) and \( \delta_2 \) summation with the amplitudes calculated in the spin basis with the tetrad \( \{v, s_3, s_1, s_2\} \) is made.

In conclusion, note that the formula

\[ \mathcal{M}_{\sigma_1} \mathcal{M}^{\ast}_{\sigma_2} = \sum_{\delta_1 \delta_2} \mathcal{M}_{\delta_1} \mathcal{K}_{\delta_1 \delta_2}^{\sigma_1 \sigma_2} \mathcal{M}^{\ast}_{\delta_2} \]  

represent a new way of describing the transition from one spin basis to another without using the extremely cumbersome and noncovariant apparatus of D-Wigner functions.

The projection of only that particle whose polarization we are investigating is given in the amplitude designations.
4 Diagonal spin basis (DSB)

In any process, only an even number of fermions can participate. If the number of free fermions is $2n$, each diagram contains $n$ open fermion lines and is described by a structure consisting of contractings between $n$ fermion “sandwiches” of the type

$$\bar{\Phi}^{\sigma}(p', s')Q\Phi^{\sigma}(p, s).$$

(33)

For the fermion line $\Phi$ are bispinors $u$ (see (1)) and for the antifermion line – bispinors $v$; for annigilation and for creation of an pair $\Phi$ are bispinors $u$ and $v$.

For certainty, we shall consider the structures (1), (2).

We shall not consider the group aspects of introduction of the DSB. We only note that in DSB the vectors $s$ and $s'$ are chosen such that they belong to the 2-plane $(p, p')$ or $(v, v')$; $v = \frac{p}{m}$, $v' = \frac{p'}{m'}$. Satisfying this requirement, we obtain

$$s = \frac{(vv')v - v'}{\sqrt{(vv')^2 - 1}}, \quad s' = -\frac{(vv')v' - v}{\sqrt{(vv')^2 - 1}}.$$  

(34)

Thus, the support vectors (see (3)) for the on- and off- states of the fermion line are vectors $q = v'$, $q' = -v$. With such a choice of signs in special reference systems vectors $\vec{s}$ and $\vec{s}'$ coincide with the direction of the 3-momentum of the initial particle and are opposite to the 3-momentum of the final particle. By special reference systems is meant the Breit system for fermion or antifermion lines ($t$-lines) and s.c.m. for the pair being annihilated or created ($s$-lines).

¿From (34) it can easily be seen that in the derived reference systems diagonality gets the meaning of helicity, with $\delta = \lambda$, $\delta' = -\lambda'$. Thus, the DSB is covariant description of the helicity in the special reference systems. Exactly in these systems helicity the pairs of particles as well as such notions as non-flip and flip amplitudes have a clear physical meaning. Indeed, in helicity basis in arbitrary reference system the spin of the initial particle is projected on the 3-momentum $\vec{p}$ and that of the final particle on $\vec{p}'$, than what can be said about the non-flip or flip process? These notions in helicity basis have a only marking meaning.

In our opinion, neglect of this fact is the main reason why the process of constructing operators (2) convenient for calculation has been extended to decades. Attempts to construct covariant operators (2) in helicity basis look unreasonable. The helicity of massive particle is a “bad” quantum number, since it is not invariant at a Lorentz transformation. Any declaration of covariancy of operators (2) in the helicity basis usually is a disguised transition to the special reference system. It should be noted that other fermion pair “droop”, since each pair has its own special reference system. The only exception is $e^+e^- \rightarrow \mu^+\mu^-$ type reactions. No wonder that just these reactions are chosen, as a rule, for examples of calculations of processes.
Before to go to the construction of tetrads for the initial and final states, we introduce into the 2-plane \((v, v')\) two \(v\)- and \(v'\)-symmetrized, orthonormalized vectors

\[
\begin{align*}
n_0 &= \frac{v + v'}{2V_+}, \quad n_3 = \frac{v - v'}{2V_-}; \\
V_\pm &= \sqrt{\frac{vv' \pm 1}{2}}. 
\end{align*}
\]

(35)

From (6), (37) it follows that the choice of common phase vector \(r' = r\) leads to the coincidence for both vectors of the tetrads lying in the orthogonal 2-plane, i.e.

\[
\begin{align*}
n_1^\mu &= s_1^\mu = s'^{\mu}_1 = (g_{\mu\nu} - g^\mu_{\nu'}) \frac{r_\nu}{r_\perp}, \\
n_2^\mu &= s_2^\mu = s'^{\mu}_2 = -\varepsilon_{\mu\nu\rho\sigma} \frac{r_\nu}{r_\perp}. 
\end{align*}
\]

(38)

Next, let us coincidence the plane Lorentz transformation that transforms \(v\) to \(v'\). In the representation of the group \(SL(2, C)\), it is of the form

\[
\Lambda(v \rightarrow v') = \frac{1 + \hat{v}' \hat{v}}{2V_+},
\]

(39)

and \(\Lambda \hat{v} \Lambda^+ = \hat{v}'\). From (34) it also follows that \(\Lambda \hat{s} \Lambda^+ = \hat{s}'\). Transformation (39) does not change vectors lying in the orthogonal 2-plane. Thus, the Lorentz transformation (39) converts the tetrad of the initial particle into the tetrad of final particle:

\[
\Lambda(v \rightarrow v') \{\hat{v}, \hat{s}, \hat{n}, \hat{n}_2\} \Lambda^+(v \rightarrow v') = \{\hat{v}', \hat{s}', \hat{n}', \hat{n}'_2\},
\]

(40)

and this in turn means that the relation between the bispinors of the initial and final states in DSB is of the form

\[
u^\delta(p', s') = \Lambda(v \rightarrow v') u^\delta(p, s).
\]

(41)

In the DSB we choose the normalization

\[
\bar{u}^\delta(p) u^\delta(p) = \bar{u}^\delta(p') u^\delta(p') = 1.
\]

(42)

Then the relation (41) describes the cases \(m' \neq m\) too.
To restore the generally accepted normalization, it is necessary to multiply the amplitudes calculated in the DSB by factor
\[
\prod_{i=1}^{n} \sqrt{2m_{i}2m_{i}'} ,
\] (43)
where \(n\) is the number of open fermion lines.

Using the Dirac equation and \((14)\), we can write relation \((11)\) in different representations
\[
u^\delta(p', s') = \frac{\hat{v}'+1}{2V_+}u^\delta(p, s) = \hat{n}_0u^\delta(p, s) = (V_+ - \delta\gamma_5V_-)u^\delta(p, s).
\] (44)

Formula \((44)\) makes it possible to express in the DSB the explicit form of the transition operator \((2)\) in terms of the projective operators of the initial (or final) state
\[
\bar{u}^\delta(p', s')\{1; \gamma_5; \gamma_{5}^\mu; \sigma_{\mu\nu}\}u^\delta(p, s) = \frac{1}{4}(\hat{v} + 1)(1 + \delta\gamma_5\delta).
\] (45)

As relation \((14)\), the transition operators \((2)\) can be given in different form. Some of them are shown below.

\[
4u^\delta(p, s)\bar{u}^\delta(p', s') = (\hat{v} + 1)\left(\frac{1}{2V_+} - \frac{\delta\gamma_5}{2V_-}\right)(\hat{v}' + 1) = (V_+ + \delta\gamma_5V_-)(\hat{n}_0 + \delta\gamma_5\hat{n}_3) = \left(1 + \frac{1}{2}(V_+ + \delta\gamma_5V_-)(\hat{n}_0 + \delta\gamma_5\hat{n}_3)\right)(\hat{n}_0 + \delta\gamma_5\hat{n}_3),
\] (46)

\[
\frac{\delta}{r_\perp}(\hat{v} + 1)\left(\frac{1}{2V_-}(\hat{r} - \frac{r(v + v')}{vv' + 1}) - \frac{\delta\gamma_5}{2V_+}(\hat{r} - \frac{r(v - v')}{vv' - 1})\right)(\hat{v}' + 1) = \gamma_5(V_+ + \delta\gamma_5V_- - \hat{n}_0)\hat{n}_2 = \gamma_5\left(V_+ + \delta\gamma_5V_- - \frac{1}{2}(\hat{n}_0 + \delta\gamma_5\hat{n}_3)\right)(\hat{n}_1 + i\delta\hat{n}_2).
\] (47)

In deriving formula \((17)\) from \((16)\), the spin flip operator \((1)\) is used and the relation valid for an arbitrary orthonormalized tetrad also hold:
\[
\hat{n}_0(\hat{n}_1 + i\delta\hat{n}_2) = \delta\gamma_5\hat{n}_3(\hat{n}_1 + i\delta\hat{n}_2) , \ (\hat{n}_0 + \delta\gamma_5\hat{n}_3)^2 = 2(1 - \delta\gamma_5\hat{n}_0\hat{n}_3).
\] (48)

Since any interaction operator can be expanded by the complete set of Dirac matrices \((18)\), it is tempting to calculate the matrix elements of this set in DSB. From \((2)\), \((16)\), \((17)\) it follows that
\[
\bar{u}^\delta(p', s')\{1; \gamma_5; \gamma_5^\mu; \gamma_5\gamma_5^\mu; \sigma_{\mu\nu}\}u^\delta(p, s) = \left\{V_+ ; \delta V_- ; n_0^b ; -\delta n_3^b ; V_- [n_0 \cdot n_3]_{\mu\nu} - i\delta V_+ [n_0 \cdot n_3]_{\mu\nu} \right\},
\] (49)
\[ \bar{u}^{-\delta}(p', s') \{1; \gamma_5; \gamma_5 \gamma^\mu; \gamma^\mu; \sigma^{\mu\nu}\} u^\delta(p, s) = \{0; 0; \delta V_-(n_1 + i\delta n_2)^\mu; -V_+(n_1 + i\delta n_2)^\mu; \delta [n_3 \cdot (\hat{n}_1 + i\delta \hat{n}_2)]^{\mu\nu}\}. \] (50)

The matrix elements (49), (50) can be interpreted as spin characteristics of exchange particles under the scalar, pseudoscalar, axial and tensor interaction, respectively.

From (50) it follows that the exchange (emission, absorption) of a scalar or pseudoscalar particle does not change the fermion spin projection in DSB. Thus, the choice of DSB not only leads to simple calculation formulas, but also adequately reflects the physical meaning of the processes being described. To support this thesis, we consider in DSB the matrix elements of nucleon current

\[ J^{\delta,\delta}_\mu = \bar{u}^{\delta'}(p', s') \left(F_1(q^2)\gamma_\mu - \frac{F_2(q^2)}{2m} \sigma_{\mu\nu}q^\nu\right) u^\delta(p, s); \quad q = p' - p. \] (51)

From (49), (50) we have

\[ J^{\delta,\delta}_\mu = G_E n_0^\mu, \quad J^{\delta,\delta}_\mu = \delta V_+ G_M (n_1 + i\delta n_2)_\mu. \] (52)

It is noteworthy that in DSB formfactors \( F_1 \) and \( F_2 \) independently go in Saks formfactors having a clear physical meaning:

\[ G_E = F_1 + \frac{q^2}{4m^2} F_2, \quad G_M = F_1 + F_2, \] (53)

\( G_E \) and \( G_M \) describe the distribution of the electric charge and the magnetic moment of nucleon, respectively. Thus, from (52) it follows that the situation where the non-flip processes are responsible for the electric interaction and the flip spin processes – for the magnetic interaction is covariantly described in DSB. It should be noted that the space-like \( (q^2 < 0) \) virtual photon has a scalar or a circular polarization, respectively.

In helicity basis, grouping of (53) occurs only in Breit’s system in which the Fourier transformations becomes three-dimensional since

\[ \exp(-iqr) = \exp i(qr - q_0 t) = \exp(iqr). \]

Formulas (46), (47) described the fermion \( t \)-line. To describe the antifermion line as well as the \( s \)-line for the annihilating and creating pairs, one should make use by the relation

\[ u^\delta(p, s) = -\delta \gamma_5 u^{-\delta}(p, s), \] (54)

which relates the particle and antiparticle bispinors.

If in (46), (47) we restore by the recipe (43) the normalization \( \bar{u}^\delta(p) u^\delta(p) = 2m \) and perform the limiting transition \( m \to 0 \) or/and \( m' \to 0 \), we will obtain the transition operators for processes involving massless fermions found in [7].

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For calculating concrete processes, it may be convenient to utilize a formalism, in which the basis spinor $u^\delta(n_0, n_3; n_1, n_2)$ common for initial and final bispinors and satisfying the conditions

$$\hat{n}_0 u^\delta(n_0, n_3) = u^\delta(n_0, n_3), \quad \gamma_5 \hat{n}_3 u^\delta(n_0, n_3) = \delta u^\delta(n_0, n_3),$$

is used.

It is easy to see\footnote{For this the relation $\frac{V_-}{V_+ + 1} = \frac{V_+ - 1}{V_-}$ is used.} that plane Lorentz transformations $\Lambda(n_0 \to v) = \frac{1 + \hat{v} \hat{n}_0}{\sqrt{2(V_+ + 1)}}$ and

$$\Lambda(n_0 \to v') = \frac{1 + \hat{v}' \hat{n}_0}{\sqrt{2(V_+ + 1)}}$$

change the basis spinor tetrad to tetrads of the initial and final states respectively. Therefore, the transition operators (46), (47) can be given in the form

$$4u^\delta(p, s)\bar{u}^{\delta'}(p', s') = \frac{2}{V_+ + 1}(\hat{\nu} + 1)u^\delta(n_0, n_3)\bar{u}^{\delta'}(n_0, n_3)(\hat{\nu}' + 1) = \frac{1}{2(V_+ + 1)}(\hat{\nu} + 1)(\hat{n}_0 + 1)(\delta_{\delta\delta'} + \gamma_5 \hat{n}_i \sigma_{\delta\delta'})(\hat{\nu}' + 1).$$

In this equality the Bouchiat and Michel relation [8] is used.

In (56), vectors $v$ and $v'$ can be expanded in vectors $n_0$ and $n_3$ with the aid of relations (35).

In conclusion of this section, we give the recipe for calculating in DSB exchange diagrams. As an example, we consider the electron-electron scattering. For certainty, let particles 1, 3 and 2, 4 are paired. Then the exchange diagram has the structure

$$\bar{u}^\delta_1(p_4)\gamma_\mu u^\delta_1(p_1)\bar{u}^{\delta_3}(p_3)\gamma_\mu u^{\delta_2}(p_2) = \text{Tr} \gamma_\mu u^\delta_1(p_1)\bar{u}^{\delta_3}(p_3)\gamma_\mu u^{\delta_2}(p_2)\bar{u}^\delta_4(p_4),$$

i.e. it is expressed in terms of the transition operators entering into direct diagram.

### 5 Examples of calculations in DSB

In Section 4, it is shown how the main task of the approach based on the use of formula (1) is solved in DSB in compact, covariant form. Namely, the problem of finding in explicit form the transition operators (2). In each particular reaction, a further simplification of calculations can be achieved by simplifying the form of the interaction operators $Q$. Let us demonstrate this by an example of calculating the processes of radiation of bremsstrahlung photons. Let the initial fermion in the state $u^\delta(p, s, r)$ emits a photon with a momentum $k$ and helicity $\lambda$. We do not concentrate the spin projection vector of the fermion so far.

Choose the photon polarization vector in the form [9]

$$e^\lambda(k) = \frac{1}{\sqrt{2k_\perp}}(\varepsilon_\parallel + i \lambda \tilde{\varepsilon}_\parallel - \lambda \sigma)k,$$

\begin{equation}
\end{equation}
\[ \varepsilon_{\parallel}^{\mu
u} = v^\mu s^\nu - s^\mu v^\nu, \quad \tilde{\varepsilon}_{\parallel}^{\mu
u} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\parallel}^{\rho\sigma}. \] (59)

It is easy to see that vector (58) meets the necessary requirement for the polarization vector: \( k e^\lambda = (e^\lambda)^2 = 0 \), \( e^\lambda e^{-\lambda} = -1 \). From the last equation in (12) and condition (14) it follows

\[ i\gamma \tilde{\varepsilon}_{\parallel} k u^\sigma(p, s) = -\sigma \gamma (g_{\parallel} - g) k u^\sigma(p, s), \] (60)

therefore, taking into account that \( e^\lambda v = (s - \lambda \sigma v) k \sqrt{2/k_\perp} \), we obtain

\[ \hat{e}^\lambda u^\sigma(p, s) = e^\lambda v(1 + \lambda \gamma_5) u^\sigma(p, s). \] (61)

In amplitude, the operator \( \hat{e}^\lambda \) is followed by operator \( \hat{p} - \hat{k} + m \). Using (61) and the commutation relation \( \hat{p} \hat{e} = 2 \hat{p} e - \hat{e} \hat{p} \), it can easily be seen that

\[ (\hat{p} - \hat{k} + m) \hat{e}^\lambda u^\sigma(p, s) = 2 e^\lambda p \left(1 - \frac{1}{2m}(1 - \lambda \gamma_5) \hat{k}\right) u^\sigma(p, s). \] (62)

The fact is rather surprising that if \( n \) photons with equal helicity \( \lambda \) are emitted from the fermion line, then the structure avalanche-like increasing in complexity acquires in this approach a very trivial form

\[ (\hat{f}_n + m) \hat{e}_n \cdots (\hat{f}_1 + m) \hat{e}_1 u^\sigma(p, s) = 2^n f_n e_n \cdots f_1 e_1 \left(1 - \frac{1}{2m}(1 - \lambda \gamma_5) \hat{k}\right) u^\sigma(p, s), \] (63)

here \( f_i = p - k_i - \ldots - k_1 \) is the momentum of fermion between the photons \( k_i \) and \( k_{i+1} \), \( k = k_1 + \ldots + k_n \).

We now consider the following problem. In those diagrams where a photon is emitted from the final fermion line, we shall still use vector (58). However, we shall not obtain a structure similar to (62) because tensors (59) were constructed for the initial fermion line. This problem disappears in DSB where these tensors coincide (37).

If a photon is emitted from another fermion line, then by virtue of gauge invariance we can make the following substitution in these diagrams:

\[ e^\lambda \rightarrow e^{-i\lambda \varphi} e^{i\lambda}, \quad e^{i\lambda \varphi} = e^{*\lambda} e^{i\lambda}, \] (64)

(see [9]) where the vector \( e' \) constructed by the recipe of (58) on basis vectors of another fermion line.

If we pass to limit of massless fermions, we will obtain the known results CALCUL-group [11], including the factorization of the sum of the contributions of all the diagrams obtained on permutation of photons in (63).

If considering the radiation of gluons, it is necessary to take into account details associated with the gauge invariance in non-Abel theory [12].
In the processes involving two bremsstrahlung photons, the case \( \lambda_1 = \lambda_2 = \lambda \) is described by vector (58). For the polarization \( \lambda_1 = -\lambda_2 = \lambda \), it is possible to construct a polarization vector common for two photons by formulas analogous to (38). In this case, the amplitude structure takes on the form of a one-photon process and fact of the participation of two photons is only reflected by the kinematics factor \([13]\).

In \([13]\) and review \([1]\), a number of processes going to field of an intense laser wave are considered. One of these processes, \( e^\pm + n\gamma^* = e^\pm + \gamma \) (\( n \) is number of coherently absorbing laser photons) is basic in the operation of \( e\gamma \) and \( \gamma\gamma \) colliders \([14]\), the others are calibration and test ones. Calculation at level of amplitudes makes it possible to investigate the nonlinear and spin effects. And the spin effects are especially significant exactly in the nonlinear processes.

If reaction involves \( W^\pm, Z^0 \) bosons, the structure of the transition operators (46), (47) is considerably simplified. This is due to the relations of the type

\[
(1 \pm \gamma_5)(\hat{n}_0 + \delta\gamma_5\hat{n}_3) = (1 \pm \gamma_5)(\hat{n}_0 + \delta\hat{n}_3),
\]

i.e. these operators are expressed in terms isotropic tetrads, and the relation \((1 \pm \gamma_5)(1 \mp \gamma_5) = 0\) reduces to zero some of the terms.

If two bosons participate in the process, their tetrads are plotted in the same manner as for a pair of fermions. In so doing, the circular polarization vectors of bosons coincide. With the participation of tree bosons, as, for example, in the reaction \( e^+e^- \rightarrow W^+W^-Z^0 \), it is convenient to choose for them a common support vector \( q = k_1 + k_2 + k_3 = p_1 + p_2 \) and a common vector \( n_2 \). And \( n_2^\mu \sim \varepsilon^{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma \). This will considerably simplify the interaction operator \( Q \) whose structure comprises up to 5 \( \gamma \)-matrices.

References

[1] M.V. Galynskii, S.M. Sikach, Physics of Particles and Nuclei 29 (1998) 496, hep-ph/9910284
[2] A.L. Bondarev, hep-ph/9710398
[3] F.I. Fedorov, Lorentz Group. Nauka, Moscow (1979) (in Russian)
[4] A.A. Bogush, F.I. Fedorov, Vesti AN BSSR. Ser. fiz.-mat. navuk 2 (1962) 26 (in Russian)
[5] S.M. Sikach, in Covariant Methods in Theoretical Physics, Institute of Physics, Academy of Sciences of Belarus, Minsk (1981), p.91 (in Russian)
[6] L.D. Landau, E.M. Lifshitz, Quantum Electrodynamics. Pergamon, New York (1980)
[7] S.M. Sikach, IP ASB preprints no. 658, 659 (1992)
[8] C. Bouchiat, L. Michel, Nucl.Phys. 5 (1958) 416
[9] S.M. Sikach, in Covariant Methods in Theoretical Physics, Institute of Physics, Academy of Sciences of Belarus, Minsk (1997), p.151 (in Russian)
[10] S.M. Sikach, in *Covariant Methods in Theoretical Physics*, Institute of Physics, Academy of Sciences of Belarus, Minsk (2001), p.134 (in Russian)

[11] CALCUL collab.:
  P. de Gausmaehecker at al., Nucl. Phys. B206 (1982) 53
  F.A. Berends at al., Nucl. Phys. B206 (1982) 61
  F.A. Berends at al., Nucl. Phys. B239 (1984) 382

[12] Z. Xu, D.H. Zhang, L. Chang, Nucl. Phys. B291 (1987) 392

[13] S.M. Sikach, in Proceedings of 10 Annual Seminar *Nonlinear Phenomena in Complex Systems*, Minsk (2001), p.297, [hep-ph/0103323](http://arxiv.org/abs/hep-ph/0103323)

[14] I.F. Ginzburg, G.L. Kotkin, V.G. Serbo, V.I. Telnov, Nucl. Instr. Meth. 205 (1983) 47
     I.F. Ginzburg, G.L. Kotkin, S.L. Panfil, V.G. Serbo, V.I. Telnov, Nucl. Instr. Meth. 219 (1984) 5