On the automorphism groups of $q$-enveloping algebras of nilpotent Lie algebras.

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Abstract

We investigate the automorphism group of the quantised enveloping algebra $U_q^+(\mathfrak{g})$ of the positive nilpotent part of certain simple complex Lie algebras $\mathfrak{g}$ in the case where the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity. Studying its action on the set of minimal primitive ideals of $U_q^+(\mathfrak{g})$ we compute this group in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$ confirming a Conjecture of Andruskiewitsch and Dumas regarding the automorphism group of $U_q^+(\mathfrak{g})$. In the case where $\mathfrak{g} = \mathfrak{sl}_3$, we retrieve the description of the automorphism group of the quantum Heisenberg algebra that was obtained independently by Alev and Dumas, and Caldero. In the case where $\mathfrak{g} = \mathfrak{so}_5$, the automorphism group of $U_q^+(\mathfrak{g})$ was computed in [16] by using previous results of Andruskiewitsch and Dumas. In this paper, we give a new (simpler) proof of the Conjecture of Andruskiewitsch and Dumas in the case where $\mathfrak{g} = \mathfrak{so}_5$ based both on the original proof and on graded arguments developed in [17] and [18].

Introduction

In the classical situation, there are few results about the automorphism group of the enveloping algebra $U(\mathcal{L})$ of a Lie algebra $\mathcal{L}$ over $\mathbb{C}$; except when $\dim \mathcal{L} \leq 2$, these groups are known to possess “wild” automorphisms and are far from being understood. For instance, this is the case when $\mathcal{L}$ is the three-dimensional abelian Lie algebra $[22]$, when $\mathcal{L} = \mathfrak{sl}_2$ [14] and when $\mathcal{L}$ is the three-dimensional Heisenberg Lie algebra [1].

In this paper we study the quantum situation. More precisely, we study the automorphism group of the quantised enveloping algebra $U_q^+(\mathfrak{g})$ of the positive nilpotent part of a finite dimensional simple complex Lie algebra $\mathfrak{g}$ in the case where the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity. Although it is a common belief that quantum algebras are ”rigid” and so should possess few symmetries, little is known about the automorphism group of $U_q^+(\mathfrak{g})$. Indeed, until recently, this group was known only in the case where $\mathfrak{g} = \mathfrak{sl}_3$

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whereas the structure of the automorphism group of the augmented form $\tilde{U}_q(b^+)$, where $b^+$ is the positive Borel subalgebra of $\mathfrak{g}$, has been described in [9] in the general case.

The automorphism group of $U_q^+(\mathfrak{sl}_3)$ was computed independently by Alev-Dumas, [2], and Caldero, [8], who showed that

$$\text{Aut}(U_q^+(\mathfrak{sl}_3)) \simeq (\mathbb{C}^*)^2 \rtimes S_2.$$  

Recently, Andruskiewitsch and Dumas, [4] have obtained partial results on the automorphism group of $U_q^+(\mathfrak{so}_5)$. In view of their results and the description of Aut($U_q^+(\mathfrak{sl}_3)$), they have proposed the following conjecture.

**Conjecture (Andruskiewitsch-Dumas, [4, Problem 1]):**

$$\text{Aut}(U_q^+(\mathfrak{g})) \simeq (\mathbb{C}^*)^{\text{rk}(\mathfrak{g})} \rtimes \text{autdiagr}(\mathfrak{g}),$$

where autdiagr($\mathfrak{g}$) denotes the group of automorphisms of the Dynkin diagram of $\mathfrak{g}$.

Recently we proved this conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$, [16], and, in collaboration with Samuel Lopes, in the case where $\mathfrak{g} = \mathfrak{sl}_4$, [18]. The techniques in these two cases are very different. Our aim in this paper is to show how one can prove the Andruskiewitsch-Dumas Conjecture in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$ by first studying the action of Aut($U_q^+(\mathfrak{g})$) on the set of minimal primitive ideals of $U_q^+(\mathfrak{g})$ - this was the main idea in [16] -, and then using graded arguments as developed in [17] and [18]. This strategy leads us to a new (simpler) proof of the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$.

Throughout this paper, $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $q$ is a nonzero complex number that is not a root of unity.

## 1 Preliminaries.

In this section, we present the $\mathcal{H}$-stratification theory of Goodearl and Letzter for the positive part $U_q^+(\mathfrak{g})$ of the quantised enveloping algebra of a simple finite-dimensional complex Lie algebra $\mathfrak{g}$. In particular, we present a criterion (due to Goodearl and Letzter) that characterises the primitive ideals of $U_q^+(\mathfrak{g})$ among its prime ideals. In the next section, we will use this criterion in order to describe the primitive spectrum of $U_q^+(\mathfrak{g})$ in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$.

### 1.1 Quantised enveloping algebras and their positive parts.

Let $\mathfrak{g}$ be a simple Lie $\mathbb{C}$-algebra of rank $n$. We denote by $\pi = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Recall that $\pi$ is a basis of an euclidean vector space $E$ over $\mathbb{R}$, whose inner product is denoted by $(\ ,\ )$ ($E$ is usually
denoted by $\mathfrak{b}_\alpha^\circ$ in Bourbaki). We denote by $W$ the Weyl group of $\mathfrak{g}$, that is, the subgroup of the orthogonal group of $E$ generated by the reflections $s_i := s_{\alpha_i}$, for $i \in \{1, \ldots, n\}$, with reflecting hyperplanes $H_i := \{ \beta \in E \mid (\beta, \alpha_i) = 0 \}$, $i \in \{1, \ldots, n\}$. The length of $w \in W$ is denoted by $l(w)$. Further, we denote by $w_0$ the longest element of $W$. We denote by $R^+$ the set of positive roots and by $R$ the set of roots. Set $Q^+ := \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n$ and $Q := \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$. Finally, we denote by $A = (a_{ij}) \in M_n(\mathbb{Z})$ the Cartan matrix associated to these data. As $\mathfrak{g}$ is simple, $a_{ij} \in \{0, -1, -2, -3\}$ for all $i \neq j$.

Recall that the scalar product of two roots $(\alpha, \beta)$ is always an integer. As in [5], we assume that the short roots have length $\sqrt{2}$.

For all $i \in \{1, \ldots, n\}$, set $q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}$ and

$$
\begin{bmatrix}
  m \\
  k
\end{bmatrix}_i := \frac{(q_i - q_{i}^{-1}) \cdots (q_{i}^{m-1} - q_{i}^{1-m})(q_{i}^{m} - q_{i}^{-m})}{(q_i - q_{i}^{-1}) \cdots (q_{i}^{k-1} - q_{i}^{-k})(q_i - q_{i}^{-1}) \cdots (q_{i}^{m-k} - q_{i}^{-m})}
$$

for all integers $0 \leq k \leq m$. By convention,

$$
\begin{bmatrix}
  m \\
  0
\end{bmatrix}_i := 1.
$$

The quantised enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{C}$ associated to the previous data is the $\mathbb{C}$-algebra generated by the indeterminates $E_1, \ldots, E_n, F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$ subject to the following relations:

$$
K_i K_j = K_j K_i
$$

$$
K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j \quad \text{and} \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j
$$

$$
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}
$$

and the quantum Serre relations:

$$
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix}
  1 \\
  k
\end{bmatrix}_i E_i^{1-a_{ij} - k} E_j E_i^k = 0 \quad (i \neq j)
$$

(1)

and

$$
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix}
  1 \\
  k
\end{bmatrix}_i F_i^{1-a_{ij} - k} F_j F_i^k = 0 \quad (i \neq j).
$$

We refer the reader to [5, 13, 15] for more details on this (Hopf) algebra. Further, as usual, we denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \ldots, E_n$ (resp. $F_1, \ldots, F_n$) and by $U^0$ the subalgebra of $U_q(\mathfrak{g})$ generated by $K_1^{\pm 1}, \ldots, K_n^{\pm 1}$. Moreover, for all $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n \in Q$, we set

$$
K_\alpha := K_1^{a_1} \cdots K_n^{a_n}.
$$
As in the classical case, there is a triangular decomposition as vector spaces:

\[ U_q^{-}(\mathfrak{g}) \otimes U^0 \otimes U_q^{+}(\mathfrak{g}) \simeq U_q(\mathfrak{g}). \]

In this paper we are concerned with the algebra \( U_q^{+}(\mathfrak{g}) \) that admits the following presentation, see [13, Theorem 4.21]. The algebra \( U_q^{+}(\mathfrak{g}) \) is (isomorphic to) the \( \mathbb{C} \)-algebra generated by \( n \) indeterminates \( E_1, \ldots, E_n \) subject to the quantum Serre relations \((1)\).

### 1.2 PBW-basis of \( U_q^{+}(\mathfrak{g}) \).

To each reduced decomposition of the longest element \( w_0 \) of the Weyl group \( W \) of \( \mathfrak{g} \), Lusztig has associated a PBW basis of \( U_q^{+}(\mathfrak{g}) \), see for instance [19, Chapter 37], [13, Chapter 8] or [5, I.6.7]. The construction relates to a braid group action by automorphisms on \( U_q^{+}(\mathfrak{g}) \).

Let us first recall this action. For all \( s \in \mathbb{N} \) and \( i \in \{1, \ldots, n\} \), we set

\[ [s]_i := \frac{q_i^s - q_i^{-s}}{q_i - q_i^{-1}} \quad \text{and} \quad [s]_i! := [1]_i \cdots [s - 1]_i [s]_i. \]

As in [5, I.6.7], we denote by \( T_i \), for \( 1 \leq i \leq n \), the automorphism of \( U_q^{+}(\mathfrak{g}) \) defined by:

\[
T_i(E_i) = -F_i K_i, \\
T_i(E_j) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)}, \quad i \neq j \\
T_i(F_i) = -K_i^{-1} E_i, \\
T_i(F_j) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^s F_i^{(s)} F_j F_i^{(-a_{ij}-s)}, \quad i \neq j \\
T_i(K_\alpha) = K_{s_\alpha(\alpha)}, \quad \alpha \in Q,
\]

where \( E_i^{(s)} := \frac{E_i^s}{[s]_i!} \) and \( F_i^{(s)} := \frac{F_i^s}{[s]_i!} \) for all \( s \in \mathbb{N} \). It was proved by Lusztig that the automorphisms \( T_i \) satisfy the braid relations, that is, if \( s_i s_j \) has order \( m \) in \( W \), then

\[ T_i T_j T_i \cdots = T_j T_i T_j \cdots, \]

where there are exactly \( m \) factors on each side of this equality.

The automorphisms \( T_i \) can be used in order to describe PBW bases of \( U_q^{+}(\mathfrak{g}) \) as follows. It is well-known that the length of \( w_0 \) is equal to the number \( N \) of positive roots of \( \mathfrak{g} \). Let \( s_{i_1} \cdots s_{i_N} \) be a reduced decomposition of \( w_0 \). For \( k \in \{1, \ldots, N\} \), we set \( \beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \). Then \( \{\beta_1, \ldots, \beta_N\} \) is exactly the set of positive roots of \( \mathfrak{g} \). Similarly, we define elements \( E_{\beta_k} \) of \( U_q(\mathfrak{g}) \) by

\[ E_{\beta_k} := T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}). \]

Note that the elements \( E_{\beta_k} \) depend on the reduced decomposition of \( w_0 \). The following well-known results were proved by Lusztig and Levendorskii-Soibelman.
We denote by \( \text{Spec}(U) \) where each situation, every prime ideal of unity, it was proved by Ringel \cite{Ringel21} (see also \cite{Goodearl10}*{Theorem 2.3}) that, as in the classical subalgebra. In particular, Theorem 1.1 (Lusztig and Levendorskii-Soibelman)

1. For all \( k \in \{1, \ldots, N\} \), the element \( E_{\beta_k} \) belongs to \( U_q^+(g) \).
2. If \( \beta_k = \alpha_i \), then \( E_{\beta_k} = E_i \).
3. The monomials \( E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N} \), with \( k_1, \ldots, k_N \in \mathbb{N} \), form a linear basis of \( U_q^+(g) \).
4. For all \( 1 \leq i < j \leq N \), we have
   \[
   E_{\beta_j} E_{\beta_i} - q^{-(\beta_i, \beta_j)} E_{\beta_i} E_{\beta_j} = \sum a_{k_i+1, \ldots, k_j-1} E_{\beta_{i+1}}^{k_{i+1}} \cdots E_{\beta_{j-1}}^{k_{j-1}},
   \]
   where each \( a_{k_i+1, \ldots, k_j-1} \) belongs to \( \mathbb{C} \).

As a consequence of this result, \( U_q^+(g) \) can be presented as a skew-polynomial algebra:
\[
U_q^+(g) = \mathbb{C}[E_{\beta_1}, E_{\beta_2}, \ldots, E_{\beta_N}; \sigma_1, \sigma_2, \ldots, \sigma_N, \delta_N],
\]
where each \( \sigma_i \) is a linear automorphism and each \( \delta_i \) is a \( \sigma_i \)-derivation of the appropriate subalgebra. In particular, \( U_q^+(g) \) is a noetherian domain and its group of invertible elements is reduced to nonzero complex numbers.

### 1.3 Prime and primitive spectra of \( U_q^+(g) \).

We denote by \( \text{Spec}(U_q^+(g)) \) the set of prime ideals of \( U_q^+(g) \). First, as \( q \) is not a root of unity, it was proved by Ringel \cite{Ringel21} (see also \cite{Goodearl10}*{Theorem 2.3}) that, as in the classical situation, every prime ideal of \( U_q^+(g) \) is completely prime.

In order to study the prime and primitive spectra of \( U_q^+(g) \), we will use the stratification theory developed by Goodearl and Letzter. This theory allows the construction of a partition of these two sets by using the action of a suitable torus on \( U_q^+(g) \). More precisely, the torus \( \mathcal{H} := (\mathbb{C}^*)^n \) acts naturally by automorphisms on \( U_q^+(g) \) via:
\[
(h_1, \ldots, h_n).E_i = h_i E_i \text{ for all } i \in \{1, \ldots, n\}.
\]

(It is easy to check that the quantum Serre relations are preserved by the group \( \mathcal{H} \).) Recall (see \cite{Goodearl11}*{3.4.1}) that this action is rational. (We refer the reader to \cite{Goodearl10}*{II.2.} for the definition of a rational action.) A non-zero element \( x \) of \( U_q^+(g) \) is an \( \mathcal{H} \)-eigenvector of \( U_q^+(g) \) if \( h.x = C^h.x \) for all \( h \in \mathcal{H} \). An ideal \( I \) of \( U_q^+(g) \) is \( \mathcal{H} \)-invariant if \( h.I = I \) for all \( h \in \mathcal{H} \). We denote by \( \mathcal{H} \text{-Spec}(U_q^+(g)) \) the set of all \( \mathcal{H} \)-invariant prime ideals of \( U_q^+(g) \). It turns out that this is a finite set by a theorem of Goodearl and Letzter about iterated Ore extensions, see \cite{Goodearl11}*{Proposition 4.2}. In fact, one can be even more precise in our situation. Indeed, in \cite{Gorelik12}, Gorelik has also constructed a stratification of the prime spectrum of \( U_q^+(g) \) using tools coming from representation theory. It turns out that her stratification coincides with the \( \mathcal{H} \)-stratification, so that we deduce from \cite{Gorelik12}*{Corollary 7.1.2} that

**Proposition 1.2 (Gorelik)** \( U_q^+(g) \) has exactly \( |W| \) \( \mathcal{H} \)-invariant prime ideals.
The action of $\mathcal{H}$ on $U_q^+(g)$ allows via the $\mathcal{H}$-stratification theory of Goodearl and Letzter (see [5 II.2]) the construction of a partition of $\text{Spec}(U_q^+(g))$ as follows. If $J$ is an $\mathcal{H}$-invariant prime ideal of $U_q^+(g)$, we denote by $\text{Spec}_J(U_q^+(g))$ the $\mathcal{H}$-stratum of $\text{Spec}(U_q^+(g))$ associated to $J$. Recall that $\text{Spec}_J(U_q^+(g)) := \{P \in \text{Spec}(U_q^+(g)) \mid \bigcap_{h \in \mathcal{H}} h.P = J\}$. Then the $\mathcal{H}$-strata $\text{Spec}_J(U_q^+(g))$ $(J \in \mathcal{H}-\text{Spec}(U_q^+(g)))$ form a partition of $\text{Spec}(U_q^+(g))$ (see [5 II.2]):

$$\text{Spec}(U_q^+(g)) = \bigsqcup_{J \in \mathcal{H}-\text{Spec}(U_q^+(g))} \text{Spec}_J(U_q^+(g)).$$

Naturally, this partition induces a partition of the set $\text{Prim}(U_q^+(g))$ of all (left) primitive ideals of $U_q^+(g)$ as follows. For all $J \in \mathcal{H}-\text{Spec}(U_q^+(g))$, we set $\text{Prim}_J(U_q^+(g)) := \text{Spec}_J(U_q^+(g)) \cap \text{Prim}(U_q^+(g))$. Then it is obvious that the $\mathcal{H}$-strata $\text{Prim}_J(U_q^+(g))$ $(J \in \mathcal{H}-\text{Spec}(U_q^+(g)))$ form a partition of $\text{Prim}(U_q^+(g))$:

$$\text{Prim}(U_q^+(g)) = \bigsqcup_{J \in \mathcal{H}-\text{Spec}(U_q^+(g))} \text{Prim}_J(U_q^+(g)).$$

More interestingly, because of the finiteness of the set of $\mathcal{H}$-invariant prime ideals of $U_q^+(g)$, the $\mathcal{H}$-stratification theory provides a useful tool to recognise primitive ideals without having to find all its irreductible representations! Indeed, following previous works of Hodges-Levasseur, Joseph, and Brown-Goodearl, Goodearl and Letzter have characterised the primitive ideals of $U_q^+(g)$ as follows, see [11, Corollary 2.7] or [5, Theorem II.8.4].

**Theorem 1.3 (Goodearl-Letzter)** $\text{Prim}_J(U_q^+(g))$ $(J \in \mathcal{H}-\text{Spec}(U_q^+(g)))$ coincides with those primes in $\text{Spec}_J(U_q^+(g))$ that are maximal in $\text{Spec}_J(U_q^+(g))$.

2 **Automorphism group of $U_q^+(g)$.**

In this section, we investigate the automorphism group of $U_q^+(g)$ viewed as the algebra generated by $n$ indeterminates $E_1, \ldots, E_n$ subject to the quantum Serre relations. This algebra has some well-identified automorphisms. First, there are the so-called torus automorphisms; let $\mathcal{H} = (\mathbb{C}^\times)^n$, where $n$ still denotes the rank of $g$. As $U_q^+(g)$ is the $\mathbb{C}$-algebra generated by $n$ indeterminates subject to the quantum Serre relations, it is easy to check that each $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathcal{H}$ determines an algebra automorphism $\phi_\lambda$ of $U_q^+(g)$ with $\phi_\lambda(E_i) = \lambda_i E_i$ for $i \in \{1, \ldots, n\}$, with inverse $\phi_\lambda^{-1} = \phi_{\lambda^{-1}}$. Next, there are the so-called diagram automorphisms coming from the symmetries of the Dynkin diagram of $g$. Namely, let $w$ be an automorphism of the Dynkin diagram of $g$, that is, $w$ is an element of the symmetric group $S_n$ such that $(\alpha_i, \alpha_j) = (\alpha_{w(i)}, \alpha_{w(j)})$ for all $i, j \in \{1, \ldots, n\}$. Then one defines an automorphism, also denoted $w$, of $U_q^+(g)$ by: $w(E_i) = E_{w(i)}$. Observe that

$$\phi_\lambda \circ w = w \circ \phi(\lambda_{w(1)}, \ldots, \lambda_{w(n)}).$$
We denote by $G$ the subgroup of $\text{Aut}(U^+_q(\mathfrak{g}))$ generated by the torus automorphisms and the diagram automorphisms. Observe that

$$G \simeq \mathcal{H} \rtimes \text{autdiagr}(\mathfrak{g}),$$

where $\text{autdiagr}(\mathfrak{g})$ denotes the set of diagram automorphisms of $\mathfrak{g}$.

The group $\text{Aut}(U^+_q(\mathfrak{sl}_3))$ was computed independently by Alev and Dumas, see [2, Proposition 2.3], and Caldero, see [3, Proposition 4.4]; their results show that, in the case where $\mathfrak{g} = \mathfrak{sl}_3$, we have

$$\text{Aut}(U^+_q(\mathfrak{sl}_3)) = G.$$ 

About ten years later, Andruskiewitsch and Dumas investigated the case where $\mathfrak{g} = \mathfrak{so}_5$, see [4]. In this case, they obtained partial results that lead them to the following conjecture.

**Conjecture (Andruskiewitsch-Dumas, [4, Problem 1]):**

$$\text{Aut}(U^+_q(\mathfrak{g})) = G.$$ 

This conjecture was recently confirmed in two new cases: $\mathfrak{g} = \mathfrak{so}_5$, [16], and $\mathfrak{g} = \mathfrak{sl}_4$, [18]. Our aim in this section is to show how one can use the action of the automorphism group of $U^+_q(\mathfrak{g})$ on the primitive spectrum of this algebra in order to prove the Andruskiewitsch-Dumas Conjecture in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$.

### 2.1 Normal elements of $U^+_q(\mathfrak{g})$.

Recall that an element $a$ of $U^+_q(\mathfrak{g})$ is normal provided the left and right ideals generated by $a$ in $U^+_q(\mathfrak{g})$ coincide, that is, if

$$aU^+_q(\mathfrak{g}) = U^+_q(\mathfrak{g})a.$$

In the sequel, we will use several times the following well-known result concerning normal elements of $U^+_q(\mathfrak{g})$.

**Lemma 2.1** Let $u$ and $v$ be two nonzero normal elements of $U^+_q(\mathfrak{g})$ such that $\langle u \rangle = \langle v \rangle$. Then there exist $\lambda, \mu \in \mathbb{C}^*$ such that $u = \lambda v$ and $v = \mu u$.

**Proof.** It is obvious that units $\lambda, \mu$ exist with these properties. However, the set of units of $U^+_q(\mathfrak{g})$ is precisely $\mathbb{C}^*$.

\[\square\]
2.2 N-grading on $U_q^+(\mathfrak{g})$ and automorphisms.

As the quantum Serre relations are homogeneous in the given generators, there is an $\mathbb{N}$-grading on $U_q^+(\mathfrak{g})$ obtained by assigning to $E_i$ degree 1. Let

$$U_q^+(\mathfrak{g}) = \bigoplus_{i \in \mathbb{N}} U_q^+(\mathfrak{g})_i$$

be the corresponding decomposition, with $U_q^+(\mathfrak{g})_i$ the subspace of homogeneous elements of degree $i$. In particular, $U_q^+(\mathfrak{g})_0 = \mathbb{C}$ and $U_q^+(\mathfrak{g})_1$ is the $n$-dimensional space spanned by the generators $E_1, \ldots, E_n$. For $t \in \mathbb{N}$ set $U_q^+(\mathfrak{g})_{\geq t} = \bigoplus_{i \geq t} U_q^+(\mathfrak{g})_i$, and define $U_q^+(\mathfrak{g})_{\leq t}$ similarly.

We say that the nonzero element $u \in U_q^+(\mathfrak{g})$ has degree $t$, and write $\deg(u) = t$, if $u \in U_q^+(\mathfrak{g})_{\leq t} \setminus U_q^+(\mathfrak{g})_{\leq t-1}$ (using the convention that $U_q^+(\mathfrak{g})_{\leq -1} = \{0\}$). As $U_q^+(\mathfrak{g})$ is a domain, $\deg(uv) = \deg(u) + \deg(v)$ for $u, v \neq 0$.

**Definition 2.2** Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be an $\mathbb{N}$-graded $\mathbb{C}$-algebra with $A_0 = \mathbb{C}$ which is generated as an algebra by $A_1 = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. If for each $i \in \{1, \ldots, n\}$ there exist $0 \neq a \in A$ and a scalar $q_{i,a} \neq 1$ such that $x_i a = q_{i,a} a x_i$, then we say that $A$ is an $\mathbb{N}$-graded algebra with enough $q$-commutation relations.

The algebra $U_q^+(\mathfrak{g})$, endowed with the grading just defined, is a connected $\mathbb{N}$-graded algebra with enough $q$-commutation relations. Indeed, if $i \in \{1, \ldots, n\}$, then there exists $u \in U_q^+(\mathfrak{g})$ such that $E_i u = q^\bullet u E_i$ where $\bullet$ is a nonzero integer. This can be proved as follows. As $\mathfrak{g}$ is simple, there exists an index $j \in \{1, \ldots, n\}$ such that $j \neq i$ and $a_{ij} \neq 0$, that is, $a_{ij} \in \{-1, -2, -3\}$. Then $s_i s_j$ is a reduced expression in $W$, so that one can find a reduced expression of $w_0$ starting with $s_i s_j$, that is, one can write

$$w_0 = s_i s_j s_{i_3} \ldots s_{i_N}.$$

With respect to this reduced expression of $w_0$, we have with the notation of Section 1.2

$$\beta_1 = \alpha_i \quad \text{and} \quad \beta_2 = s_i(\alpha_j) = \alpha_j - a_{ij} \alpha_i$$

Then it follows from Theorem 1.1 that $E_{\beta_1} = E_i$, $E_{\beta_2} = E_{\alpha_j - a_{ij} \alpha_i}$ and

$$E_i E_{\beta_2} = q^{(\alpha_i, \alpha_j - a_{ij} \alpha_i)} E_{\beta_2} E_i,$$

that is,

$$E_i E_{\beta_2} = q^{-(\alpha_i, \alpha_j)} E_{\beta_2} E_i.$$

As $a_{ij} \neq 0$, we have $(\alpha_i, \alpha_j) \neq 0$ and so $q^{-(\alpha_i, \alpha_j)} \neq 1$ since $q$ is not a root of unity. So we have just proved:

**Proposition 2.3** $U_q^+(\mathfrak{g})$ is a connected $\mathbb{N}$-graded algebra with enough $q$-commutation relations.
One of the advantages of \( \mathbb{N} \)-graded algebras with enough \( q \)-commutation relations is that any automorphism of such an algebra must conserve the valuation associated to the \( \mathbb{N} \)-gradation. More precisely, as \( U^+_q(\mathfrak{g}) \) is a connected \( \mathbb{N} \)-graded algebra with enough \( q \)-commutation relations, we deduce from [18] (see also [17, Proposition 3.2]) the following result.

**Corollary 2.4** Let \( \sigma \in \text{Aut}(U^+_q(\mathfrak{g})) \) and \( x \in U^+_q(\mathfrak{g}) \setminus \{0\} \). Then \( \sigma(x) = y_d + y_{>d} \), for some \( y_d \in U^+_q(\mathfrak{g})_d \setminus \{0\} \) and \( y_{>d} \in U^+_q(\mathfrak{g})_{>d+1} \).

### 2.3 The case where \( \mathfrak{g} = \mathfrak{sl}_3 \).

In this section, we investigate the automorphism group of \( U^+_q(\mathfrak{g}) \) in the case where \( \mathfrak{g} = \mathfrak{sl}_3 \). In this case the Cartan matrix is

\[
A = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix}
\]

so that \( U^+_q(\mathfrak{sl}_3) \) is the \( \mathbb{C} \)-algebra generated by two indeterminates \( E_1 \) and \( E_2 \) subject to the following relations:

\[
\begin{align*}
E_1^2E_2 - (q + q^{-1})E_1E_2E_1 + E_2E_1^2 &= 0 \quad (3) \\
E_2^2E_1 - (q + q^{-1})E_2E_1E_2 + E_1E_2^2 &= 0 \quad (4)
\end{align*}
\]

We often refer to this algebra as the quantum Heisenberg algebra, and sometimes we denote it by \( \mathbb{H} \), as in the classical situation the enveloping algebra of \( \mathfrak{sl}_3^+ \) is the so-called Heisenberg algebra.

We now make explicit a PBW basis of \( \mathbb{H} \). The Weyl group of \( \mathfrak{sl}_3 \) is isomorphic to the symmetric group \( S_3 \), where \( \sigma_1 \) is identified with the transposition \( (1\,2) \) and \( \sigma_2 \) is identified with \( (2\,3) \). Its longest element is then \( w_0 = (1\,3) \); it has two reduced decompositions:

\[
w_0 = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.
\]

Let us choose the reduced decomposition \( \sigma_1\sigma_2\sigma_1 \) of \( w_0 \) in order to construct a PBW basis of \( U^+_q(\mathfrak{sl}_3) \). According to Section 1.2, this reduced decomposition leads to the following root vectors:

\[
E_{\alpha_1} = E_1, \quad E_{\alpha_1+\alpha_2} = T_1(E_2) = -E_1E_2 + q^{-1}E_2E_1 \quad \text{and} \quad E_{\alpha_2} = T_1T_2(E_1) = E_2.
\]

In order to simplify the notation, we set \( E_3 := -E_1E_2 + q^{-1}E_2E_1 \). Then, it follows from Theorem 1.2 that

- The monomials \( E_{\alpha_1}^{k_1}E_{\alpha_2}^{k_2}E_{\alpha_3}^{k_3} \), with \( k_1, k_2, k_3 \) nonnegative integers, form a PBW-basis of \( U^+_q(\mathfrak{sl}_3) \).

- \( \mathbb{H} \) is the iterated Ore extension over \( \mathbb{C} \) generated by the indeterminates \( E_1, E_2, E_3 \) subject to the following relations:

\[
E_3E_1 = q^{-1}E_1E_3, \quad E_2E_3 = q^{-1}E_3E_2, \quad E_2E_1 = qE_1E_2 + qE_3.
\]

In particular, \( \mathbb{H} \) is a Noetherian domain, and its group of invertible elements is reduced to \( \mathbb{C}^* \).
It follows from the previous commutation relations between the root vectors that $E_3$ is a normal element in $\mathbb{H}$, that is, $E_3 \mathbb{H} = \mathbb{H}E_3$.

In order to describe the prime and primitive spectra of $\mathbb{H}$, we need to introduce two other elements. The first one is the root vector $E'_3 := T_2(E_1) = -E_2 E_1 + q^{-1} E_1 E_2$. This root vector would have appeared if we have chosen the reduced decomposition $s_2 s_1 s_2$ of $w_0$ in order to construct a PBW basis of $\mathbb{H}$. It follows from Theorem 1.1 that $E'_3$ $q$-commutes with $E_1$ and $E_2$, so that $E'_3$ is also a normal element of $\mathbb{H}$. Moreover, one can describe the centre of $\mathbb{H}$ using the two normal elements $E_3$ and $E'_3$. Indeed, in [3, Corollaire 2.16], Alev and Dumas have described the centre of $U^+_q(s\ell_n)$; independently Caldero has described the centre of $U^+_q(g)$ for arbitrary $g$, see [7]. In our particular situation, their results show that the centre $Z(\mathbb{H})$ of $\mathbb{H}$ is a polynomial ring in one variable $Z(\mathbb{H}) = \mathbb{C}[\Omega]$, where $\Omega = E_3 E'_3$.

We are now in position to describe the prime and primitive spectra of $\mathbb{H} = U^+_q(s\ell(3))$; this was first achieved by Malliavin who obtained the following picture for the poset of prime ideals of $\mathbb{H}$, see [20, Théorème 2.4]:

$\langle \langle E_1, E_2 - \beta \rangle \rangle$  $\langle \langle E_1, E_2 \rangle \rangle$  $\langle \langle E_1 - \alpha, E_2 \rangle \rangle$

$\langle E_1 \rangle$  $\langle E_2 \rangle$

$\langle \langle E_3 \rangle \rangle$  $\langle \langle \Omega - \gamma \rangle \rangle$  $\langle \langle E'_3 \rangle \rangle$

$\langle 0 \rangle$

where $\alpha, \beta, \gamma \in \mathbb{C}^\ast$.

Recall from Section 1.3 that the torus $\mathcal{H} = (\mathbb{C}^\ast)^2$ acts on $U^+_q(s\ell_3)$ by automorphisms and that the $\mathcal{H}$-stratification theory of Goodearl and Letzter constructs a partition of the prime spectrum of $U^+_q(s\ell_3)$ into so-called $\mathcal{H}$-strata, this partition being indexed by the $\mathcal{H}$-invariant prime ideals of $U^+_q(s\ell_3)$. Using this description of $\text{Spec}(U^+_q(s\ell_3))$, it is easy to identify the $6 = |W|$ $\mathcal{H}$-invariant prime ideals of $\mathbb{H}$ and their corresponding $\mathcal{H}$-strata. As $E_1, E_2, E_3$ and $E'_3$ are $\mathcal{H}$-eigenvectors, the 6 $\mathcal{H}$-invariant primes are:

$\langle 0 \rangle, \langle E_3 \rangle, \langle E'_3 \rangle, \langle E_1 \rangle, \langle E_2 \rangle$ and $\langle E_1, E_2 \rangle$.

Moreover the corresponding $\mathcal{H}$-strata are:

$\text{Spec}_{\langle 0 \rangle}(\mathbb{H}) = \{ \langle 0 \rangle \} \cup \{ \langle \Omega - \gamma \rangle \mid \gamma \in \mathbb{C}^\ast \}$,
\[ \text{Spec}(E_3)(\mathbb{H}) = \{ \langle E_3 \rangle \}, \]
\[ \text{Spec}(E_3')(\mathbb{H}) = \{ \langle E_3' \rangle \}, \]
\[ \text{Spec}(E_3)(\mathbb{H}) = \{ \langle E_3 \rangle \} \cup \{ \langle E_1, E_2 - \beta \mid \beta \in \mathbb{C}^* \} , \]
\[ \text{Spec}(E_3)(\mathbb{H}) = \{ \langle E_3 \rangle \} \cup \{ \langle E_1 - \alpha, E_2 \mid \alpha \in \mathbb{C}^* \} \]
and \[ \text{Spec}(E_3,E_3')(\mathbb{H}) = \{ \langle E_1, E_2 \rangle \} . \]

We deduce from this description of the \( \mathcal{H} \)-strata and the the fact that primitive ideals are exactly those primes that are maximal within their \( \mathcal{H} \)-strata, see Theorem 1.3, that the primitive ideals of \( U_q^+(\mathfrak{sl}_3) \) are exactly those primes that appear in double brackets in the previous picture.

We now investigate the group of automorphisms of \( \mathbb{H} = U_q^+(\mathfrak{sl}_3) \). In that case, the torus acting naturally on \( U_q^+(\mathfrak{sl}_3) \) is \( \mathcal{H} = (\mathbb{C}^*)^2 \), there is only one non-trivial diagram automorphism \( w \) that exchanges \( E_1 \) and \( E_2 \), and so the subgroup \( G \) of \( \text{Aut}(U_q^+(\mathfrak{sl}_3)) \) generated by the torus and diagram automorphisms is isomorphic to the semi-direct product \( (\mathbb{C}^*)^2 \rtimes S_2 \). We want to prove that \( \text{Aut}(U_q^+(\mathfrak{sl}_3)) = G \).

In order to do this, we study the action of \( \text{Aut}(U_q^+(\mathfrak{sl}_3)) \) on the set of primitive ideals that are not maximal. As there are only two of them, \( \langle E_3 \rangle \) and \( \langle E_3' \rangle \), an automorphism of \( \mathbb{H} \) will either fix them or permute them.

Let \( \sigma \) be an automorphism of \( U_q^+(\mathfrak{sl}_3) \). It follows from the previous observation that

\[ \text{either } \sigma(\langle E_3 \rangle) = \langle E_3 \rangle \text{ and } \sigma(\langle E_3' \rangle) = \langle E_3' \rangle , \]

or \[ \sigma(\langle E_3 \rangle) = \langle E_3' \rangle \text{ and } \sigma(\langle E_3' \rangle) = \langle E_3 \rangle . \]

As it is clear that the diagram automorphism \( w \) permutes the ideals \( \langle E_3 \rangle \) and \( \langle E_3' \rangle \), we get that there exists an automorphism \( g \in G \) such that

\[ g \circ \sigma(\langle E_3 \rangle) = \langle E_3 \rangle \text{ and } g \circ \sigma(\langle E_3' \rangle) = \langle E_3' \rangle . \]

Then, as \( E_3 \) and \( E_3' \) are normal, we deduce from Lemma 2.1 that there exist \( \lambda, \lambda' \in \mathbb{C}^* \) such that

\[ g \circ \sigma(E_3) = \lambda E_3 \text{ and } g \circ \sigma(E_3') = \lambda' E_3'. \]

In order to prove that \( g \circ \sigma \) is an element of \( G \), we now use the \( \mathbb{N} \)-graduation of \( U_q^+(\mathfrak{sl}_3) \) introduced in Section 2.2. With respect to this graduation, \( E_1 \) and \( E_2 \) are homogeneous of degree 1, and so \( E_3 \) and \( E_3' \) are homogeneous of degree 2. Moreover, as \( (q^{-2} - 1)E_1 E_2 = E_3 + q^{-1} E_3' \), we deduce from the above discussion that

\[ g \circ \sigma(E_1 E_2) = \frac{1}{q^{-2} - 1} (\lambda E_3 + q^{-1} \lambda' E_3') \]

has degree two. On the other hand, as \( U_q^+(\mathfrak{sl}_3) \) is a connected \( \mathbb{N} \)-graded algebra with enough \( q \)-commutation relations by Proposition 2.3, it follows from Corollary 2.4 that \( \sigma(E_1) = a_1 E_1 + a_2 E_2 + u \) and \( \sigma(E_2) = b_1 E_1 + b_2 E_2 + v \), where \( (a_1, a_2), (b_1, b_2) \in \mathbb{C}^2 \setminus \{(0, 0)\} \), and \( u, v \in U_q^+(\mathfrak{sl}_3) \) are linear combinations of homogeneous elements of degree greater than one. As \( g \circ \sigma(E_1), g \circ \sigma(E_2) \) has degree two, it is clear that \( u = v = 0 \). To conclude that
Proposition 2.5 \( \text{Aut}(U_q^+(\mathfrak{sl}_3)) \simeq (\mathbb{C}^*)^2 \rtimes \text{autdiagr}(\mathfrak{sl}_3) \)

This result was first obtained independently by Alev and Dumas, [2 Proposition 2.3], and Caldero, [8 Proposition 4.4], but using somehow different methods; they studied this automorphism group by looking at its action on the set of normal elements of \( U_q^+(\mathfrak{sl}_3) \).

2.4 The case where \( g = \mathfrak{so}_5 \).

In this section we investigate the automorphism group of \( U_q^+(\mathfrak{g}) \) in the case where \( g = \mathfrak{so}_5 \). In this case there are no diagram automorphisms, so that the Andruskiewitsch-Dumas Conjecture asks whether every automorphism of \( U_q^+(\mathfrak{so}_5) \) is a torus automorphism. In [16] we have proved their conjecture when \( g = \mathfrak{so}_5 \). The aim of this section is to present a slightly different proof based both on the original proof and on the recent proof by S.A. Lopes and the author of the Andruskiewitsch-Dumas Conjecture in the case where \( g \) is of type \( A_3 \).

In the case where \( g = \mathfrak{so}_5 \), the Cartan matrix is \( A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \), so that \( U_q^+(\mathfrak{so}_5) \) is the \( \mathbb{C} \)-algebra generated by two indeterminates \( E_1 \) and \( E_2 \) subject to the following relations:

\[
\begin{align*}
E_1^3 E_2 - (q^2 + 1 + q^{-2}) E_1^2 E_2 E_1 + (q^2 + 1 + q^{-2}) E_1 E_2^2 + E_2 E_1^3 &= 0 \\
E_2^3 E_1 - (q^2 + q^{-2}) E_2 E_1 E_2 + E_1 E_2^2 &= 0
\end{align*}
\]

We now make explicit a PBW basis of \( U_q^+(\mathfrak{so}_5) \). The Weyl group of \( \mathfrak{so}_5 \) is isomorphic to the dihedral group \( D(4) \). Its longest element is \( w_0 = -id \); it has two reduced decompositions: \( w_0 = s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 \). Let us choose the reduced decomposition \( s_1 s_2 s_1 s_2 \) of \( w_0 \) in order to construct a PBW basis of \( U_q^+(\mathfrak{so}_5) \). According to Section 1.2, this reduced decomposition leads to the following root vectors:

\[
E_{\alpha_1} = E_1, \quad E_{2\alpha_1+\alpha_2} = T_1(E_2) = \frac{1}{(q + q^{-1})} (E_1^2 E_2 - q^{-1}(q + 1) E_1 E_2 E_1 - q^{-2} E_2 E_1^2),
\]

\[
E_{\alpha_1+\alpha_2} = T_1 T_2(E_1) = -E_1 E_2 + q^{-1} E_2 E_1 \text{ and } E_{\alpha_2} = T_1 T_2 T_1(E_2) = E_2.
\]

In order to simplify the notation, we set \( E_3 := -E_{\alpha_1+\alpha_2} \) and \( E_4 := E_{2\alpha_1+\alpha_2} \). Then, it follows from Theorem 1.1 that

- The monomials \( E_1^{k_1} E_4^{k_4} E_3^{k_3} E_2^{k_2} \), with \( k_1, k_2, k_3, k_4 \) nonnegative integers, form a PBW-basis of \( U_q^+(\mathfrak{so}_5) \).
\( U^+_q(\mathfrak{so}_5) \) is the iterated Ore extension over \( \mathbb{C} \) generated by the indeterminates \( E_1, E_4, E_3, E_2 \) subject to the following relations:

\[
\begin{align*}
E_4E_1 &= q^{-2}E_1E_4 \\
E_3E_1 &= E_1E_3 - (q + q^{-1})E_4, \quad E_3E_4 = q^{-2}E_4E_3, \\
E_2E_1 &= q^2E_1E_2 - q^2E_3, \quad E_2E_4 = E_4E_2 - \frac{q^2-1}{q+q^{-1}}E_3^2, \quad E_2E_3 = q^{-2}E_3E_2.
\end{align*}
\]

In particular, \( U^+_q(\mathfrak{so}_5) \) is a Noetherian domain, and its group of invertible elements is reduced to \( \mathbb{C}^* \).

Before describing the automorphism group of \( U^+_q(\mathfrak{so}_5) \), we first describe the centre and the primitive ideals of \( U^+_q(\mathfrak{so}_5) \). The centre of \( U^+_q(\mathfrak{g}) \) has been described in general by Caldero, \([7]\). In the case where \( \mathfrak{g} = \mathfrak{so}_5 \), his result shows that \( Z(U^+_q(\mathfrak{so}_5)) \) is a polynomial algebra in two indeterminates

\[
Z(U^+_q(\mathfrak{so}_5)) = \mathbb{C}[z, z'],
\]

where

\[
z = (1 - q^2)E_1E_3 + q^2(q + q^{-1})E_4
\]

and

\[
z' = -(q^2 - q^{-2})(q + q^{-1})E_4E_2 + q^2(q^2 - 1)E_3^2.
\]

Recall from Section 1.3 that the torus \( \mathcal{H} = (\mathbb{C}^*)^2 \) acts on \( U^+_q(\mathfrak{so}_5) \) by automorphisms and that the \( \mathcal{H} \)-stratification theory of Goodearl and Letzter constructs a partition of the prime spectrum of \( U^+_q(\mathfrak{so}_5) \) into so-called \( \mathcal{H} \)-strata, this partition being indexed by the \( 8 = |W| \) \( \mathcal{H} \)-invariant prime ideals of \( U^+_q(\mathfrak{so}_5) \). In \([16]\), we have described these eight \( \mathcal{H} \)-strata. More precisely, we have obtained the following picture for the poset \( \text{Spec}(U^+_q(\mathfrak{so}_5)) \),
where $\alpha, \beta, \gamma, \delta \in \mathbb{C}^\ast$, $E'_3 := E_1E_2 - q^2E_2E_1$ and

$$\mathcal{I} = \{\langle P(z, z') \rangle \mid P \text{ is a unitary irreducible polynomial of } \mathbb{C}[z, z'], \ P \neq z, z'\}.$$

As the primitive ideals are those primes that are maximal in their $\mathcal{H}$-strata, see Theorem 1.3, we deduced from this description of the prime spectrum that the primitive ideals of $U^+_q(so_5)$ are the following:

- $\langle z - \alpha, z' - \beta \rangle$ with $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.
- $\langle E_3 \rangle$ and $\langle E'_3 \rangle$.
- $\langle E_1 - \alpha, E_2 - \beta \rangle$ with $(\alpha, \beta) \in \mathbb{C}^2$ such that $\alpha\beta = 0$.

(They correspond to the “double brackets” prime ideals in the above picture.)

Among them, two only are not maximal, $\langle E_3 \rangle$ and $\langle E'_3 \rangle$. Unfortunately, as $E_3$ and $E'_3$ are not normal in $U^+_q(so_5)$, one cannot easily obtain information using the fact that any automorphism of $U^+_q(so_5)$ will either preserve or exchange these two prime ideals. Rather than using this observation, we will use the action of $\text{Aut}(U^+_q(so_5))$ on the set of maximal ideals of height two. Because of the previous description of the primitive spectrum of $U^+_q(so_5)$, the height two maximal ideals in $U^+_q(so_5)$ are those $\langle z - \alpha, z' - \beta \rangle$ with $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. In [16, Proposition 3.6], we have proved that the group of units of the factor algebra $U^+_q(so_5)/\langle z - \alpha, z' - \beta \rangle$ is reduced to $\mathbb{C}^\ast$ if and only if both $\alpha$ and $\beta$ are nonzero. Consequently, if $\sigma$ is an automorphism of $U^+_q(so_5)$ and $\alpha \in \mathbb{C}^\ast$, we get that:

$$\sigma(\langle z - \alpha, z' \rangle) = \langle z - \alpha', z' \rangle \text{ or } \langle z, z' - \beta' \rangle,$$
where $\alpha', \beta' \in \mathbb{C}^*$. Similarly, if $\sigma$ is an automorphism of $U^+_q(\mathfrak{so}_5)$ and $\beta \in \mathbb{C}^*$, we get that:

$$
\sigma(\langle z, z' - \beta \rangle) = \langle z - \alpha', z' \rangle \text{ or } \langle z, z' - \beta' \rangle,
$$

(7)

where $\alpha', \beta' \in \mathbb{C}^*$.

We now use this information to prove that the action of $\text{Aut}(U^+_q(\mathfrak{so}_5))$ on the centre of $U^+_q(\mathfrak{so}_5)$ is trivial. More precisely, we are now in position to prove the following result.

**Proposition 2.6** Let $\sigma \in \text{Aut}(U^+_q(\mathfrak{so}_5))$. There exist $\lambda, \lambda' \in \mathbb{C}^*$ such that

$$
\sigma(z) = \lambda z \quad \text{and} \quad \sigma(z') = \lambda' z'.
$$

**Proof.** We only prove the result for $z$. First, using the fact that $U^+_q(\mathfrak{so}_5)$ is noetherian, it is easy to show that, for any family $\{\beta_i\}_{i \in \mathbb{N}}$ of pairwise distinct nonzero complex numbers, we have:

$$
\langle z \rangle = \bigcap_{i \in \mathbb{N}} P_{0, \beta_i} \quad \text{and} \quad \langle z' \rangle = \bigcap_{i \in \mathbb{N}} P_{\beta_i, 0},
$$

where $P_{\alpha, \beta} := \langle z - \alpha, z' - \beta \rangle$. Indeed, if the inclusion

$$
\langle z \rangle \subseteq I := \bigcap_{i \in \mathbb{N}} P_{0, \beta_i}
$$

is not an equality, then any $P_{0, \beta_i}$ is a minimal prime over $I$ for height reasons. As the $P_{0, \beta_i}$ are pairwise distinct, $I$ is a two-sided ideal of $U^+_q(\mathfrak{so}_5)$ with infinitely many prime ideals minimal over it. This contradicts the noetherianity of $U^+_q(\mathfrak{so}_5)$. Hence

$$
\langle z \rangle = \bigcap_{i \in \mathbb{N}} P_{0, \beta_i} \quad \text{and} \quad \langle z' \rangle = \bigcap_{i \in \mathbb{N}} P_{\beta_i, 0},
$$

and so

$$
\sigma(\langle z \rangle) = \bigcap_{i \in \mathbb{N}} \sigma(P_{0, \beta_i}).
$$

It follows from (7) that, for all $i \in \mathbb{N}$, there exists $(\gamma_i, \delta_i) \neq (0, 0)$ with $\gamma_i = 0$ or $\delta_i = 0$ such that

$$
\sigma(P_{0, \beta_i}) = P_{\gamma_i, \delta_i}.
$$

Naturally, we can choose the family $\{\beta_i\}_{i \in \mathbb{N}}$ such that either $\gamma_i = 0$ for all $i \in \mathbb{N}$, or $\delta_i = 0$ for all $i \in \mathbb{N}$. Moreover, observe that, as the $\beta_i$ are pairwise distinct, so are the $\gamma_i$ or the $\delta_i$.

Hence, either

$$
\sigma(\langle z \rangle) = \bigcap_{i \in \mathbb{N}} P_{\gamma_i, 0},
$$

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or

\[ \sigma((z)) = \bigcap_{i \in \mathbb{N}} P_{0, \delta_i}, \]

that is,

either \( \langle \sigma(z) \rangle = \sigma(\langle z \rangle) = \langle z' \rangle \) or \( \langle \sigma(z) \rangle = \sigma(\langle z \rangle) = \langle z \rangle \).

As \( z, \sigma(z) \) and \( z' \) are all central, it follows from Lemma 2.1 that there exists \( \lambda \in \mathbb{C}^* \)

such that either \( \sigma(z) = \lambda z \) or \( \sigma(z) = \lambda z' \).

To conclude, it just remains to show that the second case cannot happen. In order to do this, we use a graded argument. Observe that, with respect to the \( g \)-graduation of \( U_q^+(\mathfrak{so}_5) \) defined in Section 2.2, \( z \) and \( z' \) are homogeneous of degree 3 and 4 respectively. Thus, if \( \sigma(z) = \lambda z' \), then we would obtain a contradiction with the fact that every automorphism of \( U_q^+(\mathfrak{so}_5) \) preserves the valuation, see Corollary 2.1. Hence \( \sigma(z) = \lambda z \), as desired. The corresponding result for \( z' \) can be proved in a similar way, so we omit it.

Andruskiewitsch and Dumas, \[4, \text{Proposition 3.3}\], have proved that the subgroup of those automorphisms of \( U_q^+(\mathfrak{so}_5) \) that stabilize \( \langle z \rangle \) is isomorphic to \((\mathbb{C}^*)^2\). Thus, as we have just shown that every automorphism of \( U_q^+(\mathfrak{so}_5) \) fixes \( \langle z \rangle \), we get that \( \text{Aut}(U_q^+(\mathfrak{so}_5)) \)

itself is isomorphic to \((\mathbb{C}^*)^2\). This is the route that we have followed in [16] in order to prove the Andruskiewitsch-Dumas Conjecture in the case where \( \mathfrak{g} = \mathfrak{so}_5 \). Recently, with Samuel Lopes, we proved this Conjecture in the case where \( \mathfrak{g} = \mathfrak{sl}_4 \) using different methods and in particular graded arguments. We are now using (similar) graded arguments to prove that every automorphism of \( U_q^+(\mathfrak{so}_5) \) is a torus automorphism (without using results of Andruskiewitsch and Dumas).

In the proof, we will need the following relation that is easily obtained by straightforward computations.

**Lemma 2.7** \((q^2 - 1)E_3E_3' = (q^4 - 1)zE_2 + q^2z'\).

**Proposition 2.8** Let \( \sigma \) be an automorphism of \( U_q^+(\mathfrak{so}_5) \). Then there exist \( a_1, b_2 \in \mathbb{C}^* \) such that

\[ \sigma(E_1) = a_1E_1 \quad \text{and} \quad \sigma(E_2) = b_2E_2. \]

**Proof.** For all \( i \in \{1, \ldots, 4\} \), we set \( d_i := \deg(\sigma(E_i)) \). We also set \( d_3' := \deg(\sigma(E_3')) \). It follows from Corollary 2.4 that \( d_1, d_2 \geq 1, \ d_3, d_3' \geq 2 \) and \( d_4 \geq 3 \). First we prove that \( d_1 = d_2 = 1 \).

Assume first that \( d_1 + d_3 > 3 \). As \( z = (1 - q^2)E_1E_3 + q^2(q + q^{-1})E_4 \) and \( \sigma(z) = \lambda z \)

with \( \lambda \in \mathbb{C}^* \) by Proposition 2.6, we get:

\[ \lambda z = (1 - q^2)\sigma(E_1)\sigma(E_3) + q^2(q + q^{-1})\sigma(E_4). \quad (8) \]

Recall that \( \deg(uv) = \deg(u) + \deg(v) \) for \( u, v \neq 0 \), as \( U_q^+(\mathfrak{g}) \) is a domain. Thus, as \( \deg(z) = 3 < \deg(\sigma(E_1)\sigma(E_3)) = d_1 + d_3 \), we deduce from (8) that \( d_1 + d_3 = d_4 \). As \( z' = -(q^2 - q^{-2})(q + q^{-1})E_4E_2 + q^2(q^2 - 1)E_3' \) and \( \deg(z') = 4 < d_1 + d_3 + d_2 = d_4 + d_2 = \)

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