ON AN EXTREMIZATION PROBLEM
CONCERNING FOURIER COEFFICIENTS

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Abstract. Among subsets of Euclidean space with prescribed measure, for which sets is the $L^q$ norm of the Fourier transform of the indicator function maximized? Various partial results concerning this question are established, including the existence of maximizers and the identification of maximizers as ellipsoids for certain exponents sufficiently close to even integers.

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1. Introduction

1.1. An extremization problem. Consider the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) \, dx$$

of a function $f : \mathbb{R}^d \to \mathbb{C}$. Let $q \in (2, \infty)$ and let $p = q' = q/(q - 1) \in (1, 2)$ be the exponent conjugate to $q$. The sharp Hausdorff-Young inequality of Beckner [2] states that

$$\|\hat{f}\|_q \leq C_q^d \|f\|_{q'}$$

for all $f \in L^q(\mathbb{R}^d)$, where $C_q = p^{1/2p} q^{-1/2q}$. The constant $C_q^d$ is optimal. Lieb [15] has shown that $f$ maximizes the ratio $\|\hat{f}\|_q/\|f\|_{q'}$ if and only if $f$ is a Gaussian function $c \exp(-Q(x,x) + v \cdot x)$, where $Q$ is a positive definite real quadratic form, $v \in \mathbb{C}^d$, and $c \in \mathbb{C}$.

In this paper we pose a variant extremization problem by introducing a constraint. Denote by $1_E$ the indicator function of a set $E$. Which sets $E \subset \mathbb{R}^d$, if any, maximize
the ratio $\frac{\|\hat{1}_E\|_q}{\|1_E\|_q}$? That the supremum of this ratio is strictly less than $C^d_q$ is a consequence of \[12\].

By a subset of $\mathbb{R}^d$ or of $S^{d-1}$ we will always mean a Lebesgue measurable subset. Two Lebesgue measurable sets are considered to be equivalent if their symmetric difference is a Lebesgue null set. The ratio $\frac{\|\hat{1}_E\|_q}{\|1_E\|_q}$ respects this equivalence relation. Throughout the discussion, we do not distinguish sets from equivalence classes of sets under this relation.

The exponents $q = 2$ and $q = \infty$ are exceptional, in that all sets of given measure yield a common value for the ratio. For $q = 2$ this is Plancherel’s Theorem, while for $q = 1$ there is the relation $\|\hat{1}_E\|_\infty = \hat{1}_E(0) = |E|$.

Exponents $q \in \{4,6,8,\ldots\}$ are known to be exceptional in other respects. For instance, the optimal constant in the Hausdorff-Young inequality was found first by Babenko \[1\] for these exponents. More directly relevant to our topic is the primordial observation that for $q \in \{4,6,8,\ldots\}$, the Fourier transform can be eliminated from the discussion via the identity

\begin{equation}
(1.2) \quad \|\hat{1}_E\|_q^q = \|1_E * 1_E * \cdots * 1_E\|_2^2,
\end{equation}

where $*$ denotes convolution of functions and there are $q/2$ factors $1_E$ in the convolution. No similar elimination of the Fourier transform is available for any other exponent in $(2, \infty)$.

The Riesz-Sobolev inequality implies that among all sets $E$ of specified measure, the quantity $\|1_E * 1_E * \cdots * 1_E\|_2^2$ is maximized by balls. Therefore balls are among the maximizers $\|1_E\|_q/E^{1/q'}$ for these particular exponents. This reduction leads to a complete answer to our question for these same exponents, through work of Burchard \[6\], who has shown that $\|1_E * 1_E * \cdots * 1_E\|_2^2$ is maximized, among all sets $E$ of specified Lebesgue measure, only by ellipsoids.

It is natural to ask: Might ellipsoids be extremizers for other exponents, perhaps for all exponents in $(2, \infty)$? There are grounds for caution. Indeed, for typical exponents the mapping $f \mapsto \|\hat{f}\|_q$ lacks at least two important monotonicity properties that hold for $q \in \{4,6,8,\ldots\}$. Let $q \geq 2$, and for the sake of simplicity of the statements consider only functions in $L^1 \cap L^2$. Then:

(i) The inequality $\|\hat{f}\|_q \leq \|\hat{f}\|_q$ holds for all functions $f$, if and only if $q \in \{2,4,6,\ldots\}$. The failure of this inequality for $q = 3$ was shown by Hardy and Littlewood, and subsequently was proved by Boas \[3\] for all other exponents not in $\{2,4,6,\ldots\}$. Green and Ruzsa \[14\], and independently Mockenhaupt and Schlag \[15\], have established more quantitative results in this direction.

(ii) Let $f \geq 0$, let $p = q'$, and let $u(t,x)^p$ satisfy the heat equation $\partial_t u^p = \Delta_x u^p$ with $u(0,\cdot) = f$. Then $\|u(t,\cdot)^p\|_p \equiv \|\hat{f}\|_p$ for all $t \geq 0$. For $q \in \{2,4,6,\ldots\}$, the norm $\|u(t,\cdot)^p\|_q$ is a nondecreasing function of $t$ under this flow for arbitrary initial data $f$. Babenko’s inequality is a simple corollary of this monotonicity. Bennett, Bez, and Carbery \[3\] have shown that for every exponent $q \notin \{2,4,6,\ldots\}$ there exist initial data for which this monotonicity fails.

In this paper we explore this extremization problem. We show that for many exponents, ellipsoids are indeed (global) maximizers, and moreover are the only maximizers. We establish some related results along the way. However, all of our global results rely in part on perturbative considerations. We do not bring to light any general principle which would explain such a result for arbitrary exponents, and the question remains open in general.

If ellipsoids are indeed extremizers, then the optimal constant in the inequality satisfies $A^q_{q,d} = |\mathbb{B}|^{(q-1)/q} \int_{\mathbb{R}^d} |1_\mathbb{B}|^q$ where $\mathbb{B}$ is the unit ball in $\mathbb{R}^d$. This author is not aware of an
expression for the right-hand side in more elementary terms for general dimensions and exponents. It is the identity of the extremizing sets, rather than an elementary expression for the values of these constants, that is of interest.

This paper is one of a series in which inverse theorems of additive combinatorics are applied to affine-invariant inequalities. All of our global results rely on a compactness theorem, whose proof relies in turn on such an inverse theorem.

2. Results

2.1. Some notation. Two Lebesgue measurable sets are considered to be equivalent if the Lebesgue measure of their symmetric difference vanishes. Likewise, two functions are equivalent if they agree almost everywhere. By equality of sets or functions we will always mean equivalence in this sense. Thus two Lebesgue measurable functions are said to be disjointly supported if their product vanishes almost everywhere.

For $E \subset \mathbb{R}^d$, we denote by $E^*$ the closed ball in $\mathbb{R}^d$ that is centered at 0 and satisfies $|E^*| = |E|$. $(x) = (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^d$. Denote by $B$ the unit ball $B = \{x \in \mathbb{R}^d : |x| \leq 1\}$ in $\mathbb{R}^d$, and by $\omega_d = |B|$ its Lebesgue measure. $Q^d$ denotes the unit cube $Q^d = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_j \leq 1 \text{ for every } 1 \leq j \leq d\}$. $|E|$ denotes the Lebesgue measure of a set $E$. $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of two sets.

Reflections of functions are denoted by

$$g(x) = g(-x)$$

for any function $g : \mathbb{R}^d \to \mathbb{C}$. We write $\langle f, g \rangle = \int_{\mathbb{R}^d} f \overline{g}$; in the analysis below, often both functions are real-valued. $q' = \frac{q}{q - 1}$ denotes the exponent conjugate to $q$, throughout the paper.

Definition 2.1. For $d \geq 1$ and $q \in (2, \infty)$,

$$\Phi_q(E) = |E|^{-1/q'} \|\hat{1}_E\|_q$$

and

$$A_{q,d} = \sup_E \Phi_q(E).$$

Division by $|E|^{1/q'}$ normalizes $\Phi_q$, making $\Phi_q(\hat{E}) = \Phi_q(E)$ whenever $\hat{E}$ is a dilate of $E$. The supremum defining $A_{q,d}$ is taken over all Lebesgue measurable sets $E \subset \mathbb{R}^d$ with positive, finite Lebesgue measures. The classical Hausdorff-Young inequality guarantees that $A_{q,d} \leq 1 < \infty$; the sharp Hausdorff-Young inequality of Beckner guarantees that $A_{q,d} \leq C_q^d = (p^{1/2p}/q^{1/2q})^d < 1$; the compactness theorem of [12] for the Hausdorff-Young inequality ensures strict inequality $A_{q,d} < C_q^d$.

$\text{Aff}(d)$ denotes the group of all affine automorphisms of $\mathbb{R}^d$. The functional $\Phi_q$ is affine-invariant in the sense that

$$\Phi_q(T(E)) = \Phi_q(E) \text{ for all } T \in \text{Aff}(d)$$

for all Lebesgue measurable sets $E \subset \mathbb{R}^d$ with $|E| \in \mathbb{R}^+$. Consequently

$$A_{q,d} = |B|^{-1/q'} \sup_{|E| = \overline{B}} \|\hat{1}_E\|_q.$$
Throughout the paper, \( q \) denotes a strictly positive quantity which depends on the dimension \( d \) and on the exponent \( q \), and which can be taken to be independent of \( q \) so long as \( q \) is restricted to a compact subset of the indicated domain.

Let \( \mathcal{E} \) denote the set of all ellipsoids \( \mathcal{E} \subset \mathbb{R}^d \). For any Lebesgue measurable subset \( E \subset \mathbb{R}^d \) with \( |E| \in (0, \infty) \) define

\[
\text{dist}(E, \mathcal{E}) = \inf_{E \in \mathcal{E}} \frac{|E \Delta E|}{|E|}
\]

where the infimum is taken over all ellipsoids satisfying \( |E| = |E| \). Again, the normalizing factor makes \( \text{dist}(E, \mathcal{E}) \) invariant under the action of \( \text{Aff}(d) \). We emphasize that this notion of distance is not closely related to the Minkowski distance between compact sets.

The elementary proof of the following lemma is omitted.

**Lemma 2.1.** Let \( d \geq 1 \). For any Lebesgue measurable set \( E \subset \mathbb{R}^d \), \( \text{dist}(E, \mathcal{E}) = 0 \) if and only if there exists an ellipsoid \( E \) satisfying \( |E \Delta E| = 0 \).

The following two functions occur naturally in analysis of \( \hat{1}_E \) for sets \( E \) having small symmetric difference with the unit ball \( B \).

**Definition 2.2.** For \( q \in (2, \infty) \) and \( d \geq 1 \), functions \( K_q \) and \( L_q \) with domain \( \mathbb{R}^d \) are defined by

\[
\begin{align*}
\hat{K}_q &= \frac{\hat{1}_B}{|\hat{1}_B|^q} - 2 \\
\hat{L}_q &= \frac{\hat{1}_B}{|\hat{1}_B|^q} - 2.
\end{align*}
\]

For any \( d \geq 1 \) and \( q \in (2, \infty) \), \( K_q \) and \( L_q \) are both locally integrable, real-valued, radially symmetric functions. This is justified below.

Define

\[
q_d = 4 - 2(d + 1)^{-1}.
\]

This exponent satisfies \( 3 \leq q_d < 4 \) for all dimensions \( d \). A restriction \( q > q_d \) arises in parts of our analysis. It might perhaps be relaxed at some points, but certain aspects of the local analysis do break down in an essential way for all \( q \in (2, 3) \) when \( d = 1 \).

**2.2. Existence and precompactness.** The following precompactness statement, valid for arbitrary exponents \( q \in (2, \infty) \), is a foundational result. \( A_{q,d} \) continues to denote the supremum over all sets \( E \) of \( ||\hat{1}_E||_q / |E|^{1/q'} \).

**Theorem 2.2.** Let \( d \geq 1 \) and \( q \in (2, \infty) \). Let \( (E_\nu) \) be a sequence of Lebesgue measurable subsets of \( \mathbb{R}^d \) with \( |E_\nu| \in \mathbb{R}^+ \). Suppose that \( \lim_{\nu \to \infty} \Phi_q(E_\nu) = A_{q,d} \). There exist a sequence \( \nu_k \) of indices tending to infinity, a Lebesgue measurable set \( E \subset \mathbb{R}^d \) satisfying \( 0 < |E| < \infty \), and a sequence \( (T_k) \) of affine automorphisms of \( \mathbb{R}^d \) such that the sets \( E_k^\pm = T_k(E_{\nu_k}) \) satisfy

\[
\lim_{j \to \infty} |E_j^\pm \Delta E| = 0.
\]

The existence of maximizers is a direct consequence.

**Corollary 2.3.** For any \( d \geq 1 \) and any \( q \in (2, \infty) \), there exists a Lebesgue measurable set \( E \subset \mathbb{R}^d \) with \( |E| \in \mathbb{R}^+ \) such that \( \Phi_q(E) = A_{q,d} \).

An additive combinatorial inverse theorem is central to the proof of Theorem 2.2. Its presence is somewhat obscured by the organization of the proof; the central Lemmas 10.6 and 10.7 of the present paper are direct consequences of Proposition 6.4 of [12], whose proof in turn relies on the inverse theorem.
2.3. First variation: a necessary condition. We say that ellipsoids are local maximizers of \(\Phi_q\) if there exists \(\eta > 0\) such that \(\Phi_q(\mathbb{B}) \geq \Phi_q(E)\) for all Lebesgue measurable sets \(E \subset \mathbb{R}^d\) satisfying \(\text{dist}(E, \mathbb{B}) < \eta\). ess inf and ess sup are the essential infimum and supremum, respectively, in the usual measure-theoretic sense. Equivalently, \(\Phi_q(\mathcal{E}) \geq \Phi_q(E)\) whenever \(\mathcal{E}\) is an ellipsoid and \(|\mathcal{E}| = |E|\).

**Proposition 2.4.** Let \(d \geq 1\) and \(q \in (2, \infty)\). If ellipsoids are local maximizers of the functional \(\Phi_q\) then

\[
\text{ess inf}_{|x| \leq 1} K_q(x) \geq \text{ess sup}_{|x| \geq 1} K_q(x).
\]

For \(d = 1\) there is a partial converse; see Proposition 2.8 below. For \(d > 1\) a second variation comes into play, and the analysis is more intricate.

We are not in possession of any formula for \(K_q\), for general exponents, which would make it possible to determine whether \((2.10)\) holds. Various partial results concerning its validity are established below and in the companion paper \([13]\). Condition \((2.10)\) holds for all sufficiently large \(q\); it holds for all exponents sufficiently close to \(\{4, 6, 8, \ldots\}\); in dimension \(d = 1\) it holds for all exponents sufficiently close to 2 and also for \(q = 3\) \([13]\); numerical evidence \([13]\) suggests that it holds for all \(q \in (2, \infty)\) for \(d = 1\); we are not aware of any case in which it fails.

The condition \((2.10)\) would hold if the restriction of the radially symmetric function \(K_q\) to rays emanating from 0 were strictly decreasing. This monotonicity holds for \(q \in \{4, 6, 8, \ldots\}\), but numerical calculations \([13]\) indicate that it is false for general exponents, and in particular, for \(q = 3\) when \(d = 1\).

2.4. A form of stability. The next result asserts that if ellipsoids maximize the functional \(\Phi_q\) in a strong sense for some exponent \(q\), then they also maximize \(\Phi_r\) for all exponents \(r\) sufficiently close to \(q\). This principle will be used to analyze exponents close to \(\{4, 6, 8, \ldots\}\).

Recall that for \(E \subset \mathbb{R}^d\), \(E^* \subset \mathbb{R}^d\) denotes the closed ball centered at the origin satisfying \(|E^*| = |E|\).

**Proposition 2.5.** Let \(d \geq 1\) and \(q > q_d\). Suppose that there exist \(c, \delta > 0\) such that

\[
\|1_E\|_q^q \leq \|1_{E^*}\|_q^q - c \text{dist}(E, \mathbb{E})^2 |E|^{q-1}
\]

for all Lebesgue measurable sets \(E \subset \mathbb{R}^d\) satisfying \(|E| \in (0, \infty)\) and \(\text{dist}(E, \mathbb{E}) \leq \delta\). Then for any exponent sufficiently close to \(q\), the same inequality holds for some constants \(c, \delta > 0\).

Scaling considerations dictate the exponent \(q - 1\) attached to \(|E|\) in a straightforward manner. More worthy of note is the power 2 attached to \(\text{dist}(E, \mathbb{E})\).

2.5. Large exponents. In arbitrary dimensions, we are able to determine the extremizers for exponents that are simultaneously large and close to some even positive integer:

**Theorem 2.6.** Let \(d \geq 1\). There exists \(M < \infty\) such that for every even integer \(m \in [M, \infty)\) there exists \(\delta(m) > 0\) such that the following hold for all exponents satisfying \(|q - m| \leq \delta(m)\). Ellipsoids are extremizers of \(\Phi_q\), that is,

\[
(2.11) \quad \mathbf{A}_{q,d} = \Phi_q(\mathbb{B}).
\]

Moreover, ellipsoids are the only extremizers; for any Lebesgue measurable set \(E \subset \mathbb{R}^d\) with \(0 < |E| < \infty\), \(\Phi_q(E) = \mathbf{A}_{q,d}\) if and only if \(E\) is an ellipsoid.
We hope to remove the hypothesis that \( q \) is large in a forthcoming work. This is done for \( d = 1 \) in Theorem 2.9 using a feature of the one-dimensional case that does not extend to higher dimensions. Theorems 2.10 and 2.11 accomplish this for the specific case \( d = 2 \) and \( q \approx 4 \), which is representative of the general situation. See the comment following Theorem 2.11 below.

Ellipsoids are local maximizers of \( \Phi_q \), in a strong sense, for all sufficiently large \( q \).

**Theorem 2.7.** Let \( d \geq 1 \). There exists \( Q < \infty \) such that for every exponent \( q \in [Q, \infty) \), there exist \( c, \eta > 0 \) such that

\[
\Phi_q(E) \leq \Phi_q(B) - c \text{dist}(E, \mathcal{E})^2
\]

for every set \( E \subset \mathbb{R}^d \) satisfying \( 0 \leq \text{dist}(E, \mathcal{E}) < \eta \).

2.6. Dimension one. Our partial results are most comprehensive for dimension \( d = 1 \).

**Proposition 2.8.** Let \( d = 1 \). If \( q > 3 \) and if \( K_q \) satisfies

\[
\min_{|x| \leq 1-\delta} K_q(x) > \max_{|x| \geq 1+\delta} K_q(x) \quad \text{for all } \delta > 0
\]

then intervals are local extremizers of \( \Phi_q \), in the strong sense that there exists \( c_q > 0 \) such that

\[
\Phi(E) \leq \Phi(B) - c_q |\text{dist}(E, \mathcal{E})|^2
\]

whenever \( \text{dist}(E, \mathcal{E}) \) is sufficiently small.

This result will be extended to \( q = 3 \), by an argument relying in part on numerical calculations, in [12].

The restriction to large exponents in Theorem 2.6 is no longer needed:

**Theorem 2.9.** Let \( d = 1 \). For every \( m \in \{4, 6, 8, \ldots\} \) there exists \( \delta(m) > 0 \) such that for all exponents satisfying \( |q - m| \leq \delta(m) \), the following hold. Firstly,

\[
A_{q,1} = \Phi_q(B) = 2^{-1/q'} \pi^{-1} \left( \int_{-\infty}^{\infty} |\xi|^{-1} \sin(2\pi \xi)|q| \, d\xi \right)^{1/q}
\]

Secondly, there exists \( c_q > 0 \) such that for every such set \( E \),

\[
\Phi_q(E) \leq A_{q,1} - c_q \text{dist}(E, \mathcal{E})^2.
\]

In particular, for any Lebesgue measurable set \( E \subset \mathbb{R}^d \) with \( 0 < |E| < \infty \), \( \Phi_q(E) = A_{q,1} \) if and only if \( E \) is an interval.

2.7. The special case \( q = 4 \) and \( d = 2 \). It is natural to subject exponents \( q \) close to 4 to extra scrutiny. For \( d = 1 \), Theorem 2.9 applies. For \( d = 2 \), we have carried out a detailed analysis. Recall that \( A_{m,d} = \Phi_m(B) \) for all dimensions \( d \) and all \( m \in \{4, 6, 8, \ldots\} \), and in particular for \( m = 4 \), a direct consequence of the Riesz-Sobolev inequality.

The next two results concern the cases \( q = 4 \), and \( q \approx 4 \), respectively.

**Theorem 2.10.** Let \( d = 2 \). There exist \( c, C \in (0, \infty) \) such that for any Lebesgue measurable subset \( E \subset \mathbb{R}^2 \) with \( |E| \in \mathbb{R}^+ \),

\[
\|\widehat{1}_E\|_{L^4_{\frac{q}{2}}(\mathbb{R}^2)}^4 \leq A_{4,2}^4 |E|^3 - c \text{dist}(E, \mathcal{E})^2 |E|^3
\]

If \( \text{dist}(E, \mathcal{E}) \) is sufficiently small then

\[
\|\widehat{1}_E\|_{L^4_{\frac{q}{2}}(\mathbb{R}^2)}^4 \leq A_{4,2}^4 |E|^3 - \gamma_{4,2} \text{dist}(E, \mathcal{E})^2 |E|^3 + C \text{dist}(E, \mathcal{E})^{2+\varrho} |E|^3
\]

where \( \gamma_{4,2} = \frac{8}{5} \pi^{-1} \) and \( \varrho > 0 \).
Theorem 2.11. Let $d = 2$. For all exponents sufficiently close to 4,
\begin{equation}
A_{q,2} = \Phi_q(B).
\end{equation}
For all Lebesgue measurable sets $E \subset \mathbb{R}^2$ with $0 < |E| < \infty$, $\Phi_q(E) = A_{q,2}$ if and only if $E$ is an ellipsoid. Moreover,
\begin{equation}
\Phi_q(E) \leq A_{q,2} - c_0 \text{dist}(E, \mathcal{E})^2
\end{equation}
where $c_0$ is an absolute constant. Finally, if $\text{dist}(E, \mathcal{E})$ is sufficiently small and if $q$ is sufficiently close to 4 then
\begin{equation}
\|1_E\|_{L_q(B)} \leq A^{q,2}_{\mathcal{E}} |E|^{q-1} - \gamma_{q,2} \text{dist}(E, \mathcal{E})^2 |E|^{q-1} + C \text{dist}(E, \mathcal{E})^2 |E|^{q-1}
\end{equation}
where $\gamma_{q,2} \to \frac{8}{\pi} \pi^{-1}$ as $q \to 4$, and $q > 0$.

The proofs of Theorems 2.10 and 2.11 rely on an explicit calculation of the eigenvalues of a certain rotation-invariant operator on the unit sphere $S^1 \subset \mathbb{R}^2$. Each exponent and dimension give rise in the same way to an operator of this type, on $S^{d-1}$, depending on both parameters. The machinery developed in this paper makes it possible to extend Theorem 2.6 to exponents $q$ sufficiently close to an arbitrary positive integer $\geq 4$ provided that the eigenvalues of the associated operators satisfy suitable inequalities. Calculation of these eigenvalues for $d = 2$ and $q = 4$ underlies Theorems 2.10 and 2.11 but this author has not been able to calculate these eigenvalues in general. Instead, we hope to remove the restriction that $q$ be large in a subsequent work by an argument which will — if all goes as planned — demonstrate the required inequality for the eigenvalues without yielding formulas for them.

2.8. Some variants. (i) It would be natural to generalize this analysis to all exponents sufficiently close to $\{4, 6, 8, \ldots\}$, for all dimensions rather than merely for $d = 1$. Results established in this paper reduce this matter to calculations of the eigenvalues of certain operators on $L^2(S^{d-1})$ that are diagonalized in terms of spherical harmonics; if all eigenvalues except the first two are strictly less than those first two then ellipsoids are local maximizers; if some eigenvalue exceeds these first two then ellipsoids are not. This author has not carried out the calculation of these eigenvalues.

(ii) For $q \in (1, 2)$, the relation between norms in the Hausdorff-Young inequality is reversed: $\|\hat{f}\|_q \geq c_q \|f\|_{q'}$. This reversed inequality is equivalent, by duality, to the Hausdorff-Young inequality in the regime $q \geq 2$. We assert that all of the above theorems have reversed versions for exponents $q$ in the range $(1, 2)$. However, these reversed versions are equivalent by duality not to the theorems stated above, but to their analogues concerning the functional
\begin{equation}
\Phi^*_q(f) = \|\hat{f}\|_{L^{q,\infty}}
\end{equation}
where $L^{q,\infty}$ is the usual weak $L^q$ space with the norm
\begin{equation}
\|g\|_{L^{q,\infty}} = \sup_E |E|^{-1/q'} \int_E |g|.
\end{equation}
This norm is equivalent, though not identical, to other norms commonly used for $L^{q,\infty}$; duality considerations impose the particular norm (2.23) on us.

These analogues follow from the methods developed here. Certain steps are carried out in this paper as parts of the proofs of the main theorems, but these analogues are not fully proved here.
(iii) Another quite natural variant is obtained by replacing indicator functions of sets by bounded multiples, as follows.

Definition 2.3. Let \( f : \mathbb{R}^d \to \mathbb{C} \) and \( E \subset \mathbb{R}^d \) be a Lebesgue measurable function and a Lebesgue measurable set, respectively. Then \( f < E \) if \( f = 0 \) almost everywhere on \( \mathbb{R}^d \setminus E \), and \( |f| \leq 1 \) almost everywhere on \( E \).

Definition 2.4. For \( d \geq 1 \) and \( q \in (2, \infty) \),

\[
(2.24) \quad \Psi_q(E) = \sup_{f < E} \frac{\|f\|_q}{|E|^{1/q}},
\]

\[
(2.25) \quad B_{q,d} = \sup_E \Psi_q(E)
\]

where the supremum is taken over all Lebesgue measurable sets \( E \subset \mathbb{R}^d \) with positive, finite Lebesgue measures.

A direct consequence of their definitions is that \( A_{q,d} \leq B_{q,d} \). Adapted from the periodic setting to \( \mathbb{R} \), the example of Hardy and Littlewood demonstrates that \( \Psi_3(E) \) is strictly greater than \( \Phi_3(E) \) for some sets \( E \). According to the results of subsequent authors, the same holds for all exponents \( q \in (2, \infty) \setminus \{4, 6, 8, \ldots \} \).

For the functionals \( \Psi_q \) there are partial results parallel to those for \( \Phi_q \). For many exponents, \( A_{q,d} = B_{q,d} \), and all extremizers take the form \( e^{ix} \xi \mathbf{1}_E \) where \( E \) is an ellipsoid and \( \xi \in \mathbb{R}^d \) is arbitrary. We plan to present details in a subsequent paper.

(iv) For \( q = 2 \), all sets are extremizers. Because the derivative with respect to \( q \) of \( \|\widehat{1}_E\|_q^q \), evaluated at \( q = 2 \), is \( \Psi(E) = 2 \int |\widehat{1}_E|^2 \ln|\widehat{1}_E| \), it is natural to pose the same questions concerning \( \Psi \) as for \( \int |\widehat{1}_E|^q \).

3. Preliminaries

3.1. Continuous dependence of \( A_{q,d} \) on \( q \). Part of our analysis is perturbative with respect to the exponent \( q \). It will be important that \( A_{q,d} \) and \( \|\widehat{1}_E\|_q \) depend continuously on \( q \).

Lemma 3.1. Let \( d \geq 1 \) and \( r \in (2, \infty) \). As \( E \) varies over all subsets of \( \mathbb{R}^d \) satisfying \( |E| = 1 \), the functions \( q \mapsto \|\widehat{1}_E\|_q \) form an equicontinuous family of functions of \( q \) on any compact subset of \( (2, \infty) \).

Proof. For all \( y \in (0, 1] \) and all \( \theta \in (0, 1] \),

\[
y^\theta \leq y + \ln(1/y)(1 - \theta)
\]

because equality holds for \( \theta = 1 \), and the derivative of \( y^\theta \) with respect to \( \theta \) has absolute value \( \ln(1/y)y^\theta \leq \ln(1/y) \). If \( \theta \in [\frac{1}{2}, 1] \) then also \( y^\theta \leq y^{1/2} \leq y + y^{1/2} \), so

\[
y^\theta \leq y + \min \left( \ln(1/y)(1 - \theta), y^{1/2} \right)
\]

\[
\leq y + (1 - \theta)^{1/2} \left( \ln(1/y)y^{1/2} \right)^{1/2}
\]

\[
\leq y + C(1 - \theta)^{1/2}.
\]

Let \( r \in (2, \infty) \) be given and assume \( |E| = 1 \). Consider any \( q \in [r, 2r] \) and set \( \theta = r/q \in [\frac{1}{2}, 1] \). Then since \( \|\widehat{1}_E\|_s \leq 1 \) for all \( s \in [2, \infty] \),

\[
\|\widehat{1}_E\|_q \leq \|\widehat{1}_E\|_r^{1-\theta} \|\widehat{1}_E\|_\infty^{\theta} \leq \|\widehat{1}_E\|_r^{1-\theta} \|\widehat{1}_E\|_r + C(1 - \theta)^{1/2} \leq \|\widehat{1}_E\|_r + C_r|q - r|^{1/2}.
\]
This same inequality holds for $q \in (2, r)$ sufficiently close to $r$. Indeed, define $\theta$ by $q^{-1} = \theta r^{-1} + (1 - \theta)2^{-1}$. Provided that $q$ is sufficiently close to $r$ to ensure that $\theta \geq \frac{1}{2}$,

$$\| I_E \|_q \leq \| I_E \|_{r}^{\theta} \| I_E \|_{2}^{1-\theta} = \| \hat{I}_E \|_{r}^{\theta} |E|^{(1-\theta)/2}$$

$$= \| \hat{I}_E \|_{r}^{\theta} \leq \| \hat{I}_E \|_{r} + C|1 - \theta|^{1/2} \leq \| \hat{I}_E \|_r + C_r |q - r|^{1/2}.$$  

The constant $C_r$ is uniformly bounded for $r$ in any compact subinterval of $(2, \infty)$, so these upper bounds can be reversed to yield lower bounds. We have shown that

$$\| \hat{I}_E \|_q - \| \hat{I}_E \|_r = O(|q - r|^{1/2})$$

uniformly for all sets satisfying $|E| = 1$, so long as $q,r$ vary over some compact subset of $(2, \infty)$. □

This equicontinuity has the following immediate consequence.

**Corollary 3.2.** For each dimension $d \geq 1$, the mapping $(2, \infty) \ni q \mapsto A_{q,d} \in \mathbb{R}^+$ is continuous.

The following variant will make possible a perturbation analysis with respect to the exponent $q$.

**Corollary 3.3.** Let $d \geq 1$ and $q_0 \in (2, \infty)$. Let $S$ be a collection of Lebesgue measurable subsets $E \subset \mathbb{R}^d$, satisfying $|E| \in \mathbb{R}^+$. Let $\eta > 0$, and suppose that $\Phi_{q_0}(E) \leq A_{q_0,d} - \eta$ for all $E \in S$. Then there exists $\delta > 0$ such that

$$\Phi_q(E) \leq A_{q,d} - \frac{1}{2}\eta$$

for all $E \in S$ and all exponents $q$ satisfying $|q - q_0| < \delta$.

**Proof.** Because $\Phi_q$ is invariant under dilations, it suffices to prove this under the hypothesis that $|E| = 1$ for all $E \in S$. Then the conclusion follows from the preceding corollary and lemma. □

If $q_0$ is an exponent for which one knows that $\Phi_{q_0}$ attains its maximum value on ellipsoids and on no other sets, then for any $\tau > 0$, one can apply this corollary with $S$ equal to the collection of all sets satisfying $\text{dist}(E, \mathcal{E}) \geq \tau$ to conclude that only sets satisfying $\text{dist}(E, \mathcal{E}) \leq \tau$ are candidates to be extremizers of $\Phi_q$ for $q$ close to $q_0$.

### 3.2. Taylor expansion with respect to $E$

Let $d \geq 1$ and $q \in (2, \infty)$. Let $K_q, L_q$ be the functions defined in (2.7), whose Fourier transforms are essentially powers of $|1_\mathcal{B}|$.

Much of our analysis will involve comparison of $1_E$ with $1_B$ for sets $E \subset \mathbb{R}^d$ that are close to $\mathcal{B}$. Two senses of closeness arise naturally in this analysis; first, the measure $|E \Delta \mathcal{B}|$ of their symmetric difference may be small, and second, $E \Delta \mathcal{B}$ may be contained in a small neighborhood of the boundary of $\mathcal{B}$. To facilitate the comparison we use the function

$$f = 1_E - 1_B.$$

This notation $f$ will be employed throughout the discussion. Recall the notations $\widehat{g}(x) = g(-x)$ and $\langle f, g \rangle = \int_{\mathbb{R}^d} f \hat{g}$. Convolution in $\mathbb{R}^d$ will be denoted by $\ast$.

**Lemma 3.4.** Let $d \geq 1$ and $q \in (3, \infty)$. There exist $\varrho > 0$ and $C < \infty$ with the following property. Let $E \subset \mathbb{R}^d$ and set $f = 1_E - 1_B$. If $|E \Delta \mathcal{B}|$ is sufficiently small then

$$\| \hat{E} \|_q^2 = \| \hat{B} \|_q^2 + q \langle K_q, f \rangle$$

$$+ \frac{1}{2} q^2 \langle f \ast L_q, f \rangle + \frac{1}{2} q(q - 2) \langle f \ast L_q, \hat{f} \rangle + O(|E \Delta \mathcal{B}|^2 + \varrho).$$
So long as $q$ belongs to any compact subset of $(q_d, \infty)$, the exponent $q$ and the constant implicit in the notation $O(\cdot)$ in the term $O(|E \Delta B|^{2+\varrho})$ may be taken to be independent of $q$.

Because we are dealing with a constrained optimization problem, it will not the case that the first perturbation term $q(K_q, f)$ vanishes when $B$ is an extremizer. Nor is this term strictly greater than 2. Therefore, the first perturbation term $q(K_q, f)$ is not of quadratic order. Thus in order to determine whether the favorable term $q(K_q, f)$ is sufficiently negative to outweigh the unfavorable terms involving $L_q$.

**Proof of Lemma 3.4** Let $q \in (3, \infty)$. For any $t \in \mathbb{C}$,

$$|1 + t|^q = 1 + q \Re(t) + \frac{1}{2}q(q - 1) \Re(t)^2 + \frac{1}{2}q \Im(t)^2 + O(|t|^3 + |t|^q).$$

Indeed, if $|t| \leq \frac{1}{2}$ this holds by the binomial theorem. If $|t| \geq \frac{1}{2}$ then

$$|1 + t|^q = O(|t|^q) = 1 + q \Re(t) + \frac{1}{2}q(q - 1) \Re(t)^2 + \frac{1}{2}q \Im(t)^2 + O(|t|^q).$$

For $q \geq 2$ and $|t| \geq \frac{1}{2}$, the quantities $1$, $|t|$, $|t|^2$ are all $O(|t|^q)$.

Since $\hat{1}_B$ is real-valued,

$$|\hat{1}_B + \hat{f}|^q = |\hat{1}_B|^q + q \Re(\hat{f})|\hat{1}_B|^q + \frac{1}{2}q(q - 1)(\Re(\hat{f}))^2|\hat{1}_B|^q - \frac{1}{2}q \Im(\hat{f})^2|\hat{1}_B|^q + O(|\hat{f}|^3|\hat{1}_B|^{q-3}) + O(|\hat{f}|^q).$$

(3.4)

Since $f$, $K_q$, and $L_q$ are real-valued, integrating the pointwise inequality (3.4) over $\mathbb{R}^d$ and invoking Plancherel’s theorem gives

$$\|\hat{1}_E\|_q^q = \|\hat{1}_B\|_q^q + q\langle K_q, f \rangle + \frac{1}{2}q(q - 1) \int (\Re(\hat{f}))^2|\hat{1}_B|^q - \frac{1}{2}q \int (\Im(\hat{f}))^2|\hat{1}_B|^q + O(||f||_q^3||\hat{1}_B||_q^{q-3}) + O(||f||_q^q).$$

Now $\|f\|_{q'} = |E \Delta B|^{1/q'}$. For $q > 3$, both $3/q' = 3(q - 1)/q = 3 - 3q^{-1}$ and $q/q' = q - 1$ are strictly greater than 2. Therefore

$$\|\hat{1}_E\|_q^q = \|\hat{1}_B\|_q^q + q\langle K_q, f \rangle + \frac{1}{2}q(q - 1) \int (\Re(\hat{f}))^2|\hat{1}_B|^q - \frac{1}{2}q \int (\Im(\hat{f}))^2|\hat{1}_B|^q + O(|E \Delta B|^{2+\varrho}).$$

where $\varrho = \rho(q) > 0$.

Since $f$ is real-valued, $\hat{f}$ is the complex conjugate of $\hat{f}$ and therefore

$$\frac{1}{2}q(q - 1)(\Re(\hat{f}))^2 + \frac{1}{2}q(\Im(\hat{f}))^2 = \frac{1}{8}q(q - 1)(\hat{f} + \bar{\hat{f}})^2 - \frac{1}{8}q(q - 2)(\hat{f} - \bar{\hat{f}})^2$$

$$= \frac{1}{8}q(q - 2)\hat{f}^2 + \frac{1}{8}q(q - 2)(\bar{\hat{f}})^2 + \frac{1}{4}q^2 \hat{f} \cdot \bar{\hat{f}}$$

$$= \frac{1}{2}q(q - 2)\hat{f} \cdot \bar{\hat{f}} + \frac{1}{8}q(q - 2)\hat{f} \cdot \bar{\hat{f}} + \frac{1}{4}q^2 \hat{f} \cdot \bar{\hat{f}}.$$
Using this identity together with Plancherel’s Theorem gives
\[
\frac{1}{2}q(q-1) \int (\Re \hat{f})^2 |\hat{1}_B|^q - 2 + \frac{1}{2}q \int (\Im \hat{f})^2 |\hat{1}_B|^q - 2 = \frac{1}{4}q^2 (f \ast L_q, f) + \frac{1}{4}q(q-2) (f \ast L_q, \hat{f})
\]
since $L_q$ is real-valued and even. Likewise,
\[
\Re (\int \hat{f} \hat{1}_B |\hat{1}_B|^q - 2) = \Re (\int \hat{f} \hat{K}_q) = \Re \langle f, K_q \rangle = \langle f, K_q \rangle
\]
since $f, K_q$ are real-valued. □

A variant of Lemma 3.4 holds for $q = 3$, and follows from the same proof.

**Lemma 3.5.** Let $q = 3$. For each $d \geq 1$ there exists $C < \infty$ with the following property. Let $E \subset \mathbb{R}^d$. If $|E \Delta B|$ is sufficiently small then
\[
(3.5) \quad ||\hat{1}_E||_3^2 = ||\hat{1}_B||_3^2 + 3 \langle K_3, f \rangle + \frac{q}{4} (f \ast L_3, f) + \frac{3}{4} (f \ast L_3, \hat{f}) + O(|E \Delta B|^2)
\]
where $f = 1_E - 1_B$.

The distinction between the two lemmas is that the remainder term becomes merely $O(|E \Delta B|^2)$ for $q = 3$, rather than $O(|E \Delta B|^{2+\phi})$. This lemma will be exploited in [13].

For $q \in (2, 3)$ the situation changes. For simplicity we consider only the case $d = 1$.

**Lemma 3.6.** Let $d = 1$ and $q \in (2, 3)$. For any set $E \subset \mathbb{R}^1$ satisfying $|E| = |B|$ and $|E \Delta B| \ll 1$,
\[
(3.6) \quad ||\hat{1}_E||_q^2 = ||\hat{1}_B||_q^2 + q \langle K_q, f \rangle + O(|E \Delta B|^{q-1}).
\]

What is different in this regime is that the leading term $q \langle K_q, f \rangle$ can have the same order of magnitude as occurs when $E$ is a translate of $B$.

**Proof.** Consider any dimension $d$. As in the proof of Lemma 3.4
\[
||\hat{1}_E||_q^2 \leq ||\hat{1}_B||_q^2 + q \langle K_q, f \rangle + O(\int ||\hat{1}_B(\xi)||_2^q |\hat{f}^{(\xi)}|^2 d\xi) + O(||\hat{f}||_q^q)
\]
\[
= ||\hat{1}_B||_q^2 + q \langle K_q, f \rangle + O(||\hat{1}_B||_q^{q-2} ||\hat{f}||_q^2) + O(||\hat{f}||_q^q)
\]
\[
= ||\hat{1}_B||_q^2 + q \langle K_q, f \rangle + O(|E \Delta B|^{2(q-1)/q}) + O(|E \Delta B|^{q-1}).
\]
\[
= ||\hat{1}_B||_q^2 + q \langle K_q, f \rangle + O(|E \Delta B|^{1+\phi})
\]
where $1 + \phi = 2(q-1)q^{-1} \in (1, q-1)$ since $q > 2$.

To do better for $d = 1$, use the majorization $\hat{1}_B(\xi) = O(|\xi|^{-(d+1)/2})$ and specialize to $d = 1$ to obtain
\[
\int ||\hat{1}_B(\xi)||_q^{q-2} |\hat{f}(\xi)|^2 d\xi \leq C \int |\xi|^{-(q-2)} |\hat{f}(\xi)|^2 d\xi
\]
\[
= C \iint f(x)f(y) |x - y|^{q-3} dx dy
\]
\[
\leq C \iint 1_E \Delta B(x) 1_E \Delta B(y) |x - y|^{q-3} dx dy
\]
\[
\leq C |E \Delta B|^{q-1}.
\]

□
3.3. A necessary condition. Here we prove Proposition 3.4. Recall that $K_q$ is real-valued and even, and is continuous for all $q > 3 - \frac{2}{d+1}$. In particular, for $d = 1$ it is continuous for all $q > 2$. $L_q$ is continuous for all $q > 4 - \frac{2}{d+1} = q_d$.

Because $\hat{K}_q(\xi) = O((\xi)^{-(q-1)(d+1)/2})$, $\hat{K}_q \in L^s$ for all $s > 2d(d+1)^{-1}(q-1)^{-1}$. If $2d(d+1)^{-1}(q-1)^{-1} < 1$ it follows that $\hat{K}_q \in L^1$ and hence $K_q$ is a continuous function. For any $q > 2$, $2d(d+1)^{-1}(q-1)^{-1} \leq 2d(d+1)^{-1} < 2$ and therefore $\hat{K}_q \in L^s$ for some $s < 2$. By the Hausdorff-Young inequality, $K_q \in L^{s'}$. In particular, $K_q$ is well-defined as a locally integrable function.

Proof of Proposition 3.4. Let $x', x'' \in \mathbb{R}^d$ be any Lebesgue points of $K_q$ that satisfy $|x'| > 1$ and $|x''| < 1$. Let $B', B''$ be the balls of measures $\delta$ with centers $x', x''$ respectively. Consider any $\delta > 0$ that is sufficiently small to ensure that these balls are contained in $\mathbb{R}^d \setminus B$ and $B$, respectively. Let $E = (B \setminus B'') \cup B'$. Then $|E| = |B|$ and $f = 1_{B'} - 1_{B''}$.

Apply Lemma 3.6. The leading term is

$$\langle K_q, f \rangle = \left( K_q(x') - K_q(x'') \right) \delta + o_{x', x''}(\delta).$$

Therefore

$$\|\mathbf{1}_E\|_q = \|\mathbf{1}_B\|_q + q(\langle K_q(x') - K_q(x'') \rangle \delta + o_{x', x''}(\delta)).$$

By letting $\delta \to 0$ while $x', x''$ remain fixed we conclude that if $K_q(x') - K_q(x'') > 0$ then $B$ is not a local maximizer. \hfill \Box

3.4. The kernels $K_q$, $L_q$, $\hat{1}_B$ is a radially symmetric real-valued real analytic function which satisfies

$$(3.7) \quad |\hat{1}_B(\xi)| + |\nabla \hat{1}_B(\xi)| \leq C_d(1 + |\xi|)^{-(d+1)/2}.$$ 

The following lemmas are direct consequences of these properties. Recall that $q_d = 4 - 2(d+1)^{-1}$.

Lemma 3.7. Let $d \geq 1$ and $q \in (q_d, \infty)$. The functions $K_q$, $L_q$ are real-valued, radially symmetric, bounded, and Hölder continuous of some positive order. Moreover, $K_q(x) \to 0$ as $|x| \to \infty$ and likewise for $L_q(x)$. The function $K_q$ is continuously differentiable, and $x \cdot \nabla K_q$ is likewise real-valued, radially symmetric, and Hölder continuous of some positive order. These conclusions hold uniformly for $q$ in any compact subset of $(q_d, \infty)$.

The Hölder continuity of $L_q$ is a direct consequence of the fact that $(1 + |\xi|)^{\rho}|\hat{1}_B| \in L^1$ for some $\rho > 0$, which holds by virtue of $(3.7)$ since $(q_d - 2)(d+1)/2 = d$ and $q > q_d$.

Lemma 3.8. For each $d \geq 1$, $K_q$, $L_q$, and $x \cdot \nabla K_q$ depend continuously on $q \in (q_d, \infty)$. This holds in the sense that for each compact subset $\Lambda \subset (q_d, \infty)$, the mappings $q \mapsto K_q$ and $q \mapsto L_q$ are continuous from $\Lambda$ to the space of continuous functions on $\mathbb{R}^d$ that tend to zero at infinity. Moreover, there exists $\rho > 0$ such that this mapping from $\Lambda$ to the space of bounded Hölder continuous functions of order $\rho$ on any bounded subset of $\mathbb{R}^d$ is continuous. The two conclusions also hold for $q \mapsto x \cdot \nabla K_q$.

Lemma 3.9. Let $d \geq 1$ and $q \in (q_d, \infty)$. Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set with finite measure, and let $f = \mathbf{1}_E - \mathbf{1}_B$. Then

$$(3.8) \quad \langle f * L_q, f \rangle = O(|E \Delta B|^2) \quad \text{and} \quad \langle f * L_q, \hat{f} \rangle = O(|E \Delta B|^2).$$

This is an immediate consequence of the boundedness of $L_q$ together with the relation $|f| = \mathbf{1}_E \Delta B$. \hfill \Box
Lemma 3.10. For each $d \geq 1$ and each even integer $m \geq 4$ there exists $\eta = \eta(d, m) > 0$ such that whenever $|q - m| < \eta$, there exists $c > 0$ such that whenever $|y| \leq 1 \leq |x| \leq 2$,

$$K_q(y) \geq K_q(x) + c|x - |y|).\quad (3.9)$$

Proof. Recall that $4 > q_d$, so exponents satisfying the hypothesis satisfy $q > q_d$ provided that $\eta$ is chosen to be sufficiently small.

The conclusion holds when $q$ is an even integer $m \geq 4$. Indeed, $K_m$ is the convolution product of $m - 1$ factors of $1_B$. This is a nonnegative radially symmetric function, which is supported in the closed ball of radius $m - 1$ centered at 0, and is strictly positive in the corresponding open ball. The convolution product of two factors of $1_B$ is a radial function $f_2$ whose support equals the ball of radius 2 centered at 0. For $|y| = 1$, the function $[0, 2] \ni t \mapsto f_2(ty)$ is nonincreasing, with strictly negative derivative for all $t \in (0, 2)$. By induction on the number of factors in such a convolution product we conclude that $t \mapsto K_m(ty)$ has strictly negative derivative for all $t \in (0, m - 1)$, a stronger result than (3.9).

For $q$ close to some even integer $m$, (3.9) for $q$ follows from (3.9) for $m$, because $x \cdot \nabla_x K_q(x)$ depends continuously on $q$. \qed

4. Localization of perturbation near the boundary

The next lemma asserts that if $|E \Delta B|$ is small, then in the expansion (3.3) of $\|1_E\|_q$ about $\|1_B\|_q$, the first-order perturbation term $q(K_q, 1_E - 1_B)$ is rather negative unless most of the symmetric difference $E \Delta B$ lies close to the boundary of $B$. To facilitate the discussion, for $E \subset \mathbb{R}^d$ and $\eta \in (0, 1]$ define

$$E_\eta = \{x \in E \Delta B : |x| - 1 \geq \eta\}.\quad (4.1)$$

This notation is employed at several points of the analysis below. The reader should beware that although $E_\eta$ is always constructed from $E$ as in (4.1), $A_\eta, B_\eta$ are not constructed from $A, B$ respectively in exactly this same way in the proof of Lemma 4.1.

Lemma 4.1. Let $d \geq 1$. Let $q \in (q_d, \infty)$. Suppose that $q > q_d$ and that $K_q$ satisfies both (2.13) and (3.9). Then there exist $c, C, \eta \in \mathbb{R}^+$ with the following property. Let $\eta \in (0, 1]$ and let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set satisfying $|E| = |B|$. Then

$$\|1_E\|_q^q \leq \|1_{E \setminus E_\eta}\|_q^q - c\eta|E_\eta| + C|E \Delta B| \cdot |E_\eta| + C|E \Delta B|^{2+\eta}.\quad (4.2)$$

If $q$ is sufficiently large then this holds with $q = 1$.

The key point is that the negative term $-c\eta|E_\eta|$ is proportional to the first power of $|E_\eta|$, albeit with a factor that tends to zero with $\eta$.

Proof. Continue to express $E = (B \cup A) \setminus B$ where $B \subset B$ and $A \subset \mathbb{R}^d \setminus B$ satisfy $|A| = |B| = \frac{1}{2}|E \Delta B|$. Set $A_\eta = A \cap E_\eta$ and (although this is inconsistent with the general notation (3.4)) $B_\eta = B \cap E_\eta$. Consider first the case in which $|B_\eta| \geq |A_\eta|$. Choose a measurable subset $A'_\eta$ satisfying $A_\eta \subset A'_\eta \subset A$ that satisfies $|A'_\eta| = |B_\eta|$. Such a set exists since $|A| = |B| \geq |B_\eta|$. Set $f = 1_E - 1_B = 1_A - 1_B$, $f_\eta = 1_{A'_\eta} - 1_{B_\eta}$, and

$$f^\dagger = f - f_\eta = 1_{A \setminus A'_\eta} - 1_{B \setminus B_\eta}.\quad (4.3)$$

Because $L_q$ is bounded and $|f_\eta| = |A'_\eta \cup B_\eta|$ satisfies

$$\|f_\eta\|_{L_1} \leq |A'_\eta| + |B_\eta| = 2|B_\eta| \leq 2|E_\eta|,$$
one has
\[ \langle L_q f, f \rangle = \langle L_q f^\dagger, f^\dagger \rangle + O(|E \Delta \mathbb{B}| \cdot |E_\eta|). \]
 Likewise \( \langle L_q f, f^\dagger \rangle = \langle L_q f^\dagger, f^\dagger \rangle + O(|E \Delta \mathbb{B}| \cdot |E_\eta|). \)

By the hypothesis \((3.9)\),
\[ K_q(x) \geq K_q(y) + c\eta \text{ whenever } |x| \leq 1 - \eta \text{ and } |y| \leq 1 + \eta. \]
 Consequently
\[ \langle K_q, f_\eta \rangle \leq -c\eta\|f_\eta\|_1 = -c\eta|B_\eta| - c\eta|A'_\eta| \leq -c\eta|E_\eta| \]
and in total,
\[ (4.4) \quad \langle K_q, f \rangle \leq -2c\eta|E_\eta| + \langle K_q, f^\dagger \rangle. \]

Therefore by Lemma \([3.4]\)
\[ \| \mathbf{1}_E \|_q = \| \mathbf{1}_E \|_q + q(K_q, f) + \frac{1}{4}q^2(L_q f, f) + \frac{1}{4}q(q - 2)\langle L_q f, f \rangle + O(|E \Delta \mathbb{B}|^{2+\epsilon}) \]
\[ \leq \| \mathbf{1}_E \|_q - c\eta|E_\eta| + q(K_q, f^\dagger) + \frac{1}{4}q^2(L_q f^\dagger, f^\dagger) + \frac{1}{4}q(q - 2)\langle L_q f^\dagger, f^\dagger \rangle + O(|E \Delta \mathbb{B}| \cdot |E_\eta|) + O(|E \Delta \mathbb{B}|^{2+\epsilon}) \]
\[ = \| \mathbf{1}_{E \setminus E_\eta} \|_q - c\eta|E_\eta| + O(|E \Delta \mathbb{B}| \cdot |E_\eta|) + O(|E \Delta \mathbb{B}|^{2+\epsilon}) \]

since Lemma \([3.4]\) also gives
\[ \| \mathbf{1}_{E \setminus E_\eta} \|_q = q(K_q, f^\dagger) + \frac{1}{4}q^2(L_q f^\dagger, f^\dagger) + \frac{1}{4}q(q - 2)\langle L_q f^\dagger, f^\dagger \rangle + O((|E \setminus E_\eta| \Delta \mathbb{B}|)^{2+\epsilon}) \]
and \( (E \setminus E_\eta) \Delta \mathbb{B} \subset E \Delta \mathbb{B} \). This completes the analysis of the case in which \( |B_\eta| \geq |A'_\eta| \).

The same reasoning can be applied in the alternative situation when \( |A_\eta| \geq |B_\eta| \), by replacing \( B_\eta \) by a superset \( B'_\eta \) satisfying \( B_\eta \subset B'_\eta \subset B \) and \( |B'_\eta| = |A'_\eta| \).

A natural choice of the parameter \( \eta \) in the preceding lemma leads to a bound which will be combined with a more detailed analysis of \( \| \mathbf{1}_{E \setminus E_\eta} \|_q \) below.

**Corollary 4.2.** Under the hypotheses of Lemma \([4.1]\) if \( \lambda \) is a sufficiently large constant depending only on \( d, q \) then for any set \( E \subset \mathbb{R}^d \) that satisfies \( |E| = |\mathbb{B}| \) and \( \lambda|E \Delta \mathbb{B}| \leq 1 \), if \( \eta \) is chosen to equal \( \lambda|E \Delta \mathbb{B}| \) then
\[ (4.5) \quad \| \mathbf{1}_E \|_q^2 \leq \| \mathbf{1}_{E \setminus E_\eta} \|_q^2 - c\lambda|E_\eta| \cdot |E \Delta \mathbb{B}| + C_\lambda|E \Delta \mathbb{B}|^{2+\epsilon} \]
where \( c, q \in \mathbb{R}^+ \) depend only on \( d, q \) while \( C_\lambda \) depends only on \( d, q, \lambda \). If \( q \) is sufficiently large then the conclusion holds with \( q = 1 \).

These statements hold uniformly for all \( q \) in any fixed compact subset of \((q_d, \infty)\).

**5. Connection with a spectral problem on \( S^{d-1} \)**

Let \( d \geq 1 \). Let \( E \subset \mathbb{R}^d \) be a Lebesgue measurable set with \( |E| = |\mathbb{B}| \). For any set whose symmetric difference \( E \Delta \mathbb{B} \) is contained in a small neighborhood of the unit sphere, we next show how \( \| \mathbf{1}_E \|_q^2 \) can be expressed in terms of associated functions on the unit sphere and rotation-invariant quadratic forms involving these functions, modulo a small error.

Let \( \sigma \) denote surface measure on the unit sphere \( S^{d-1} \subset \mathbb{R}^d \). Continue to express \( E = (\mathbb{B} \cup A) \setminus B \) where \( B \subset \mathbb{B} \) and \( A \subset \mathbb{R}^d \setminus \mathbb{B} \). Let \( K_q, L_q \) be as defined above.
Definition 5.1.
\[ \gamma_{q,d} = -x \cdot \nabla K_q(x) \big|_{x=1}. \]

Definition 5.2. Let \( E \subset \mathbb{R}^d \) be a bounded Lebesgue measurable set. The functions \( a, b, F : S^{d-1} \to [0, \infty) \) are defined by
\[
\begin{align*}
(5.2) & \quad a(\alpha) = \int_0^\infty 1_A(r\alpha)r^{d-1} \, dr \\
(5.3) & \quad b(\alpha) = \int_0^\infty 1_B(r\alpha)r^{d-1} \, dr \\
(5.4) & \quad F(\alpha) = b(\alpha) - a(\alpha)
\end{align*}
\]
for \( \alpha \in S^{d-1} \).

These satisfy
\[
\int_{S^{d-1}} a \, d\sigma = |A|, \quad \int_{S^{d-1}} b \, d\sigma = |B|, \quad \int_{S^{d-1}} F \, d\sigma = 0.
\]
Consequently
\[
(5.5) \quad \int_{S^{d-1}} (a^2 + b^2) \, d\sigma \geq \sigma(S^{d-1})^{-1}(|A|^2 + |B|^2) = \frac{|E \Delta \mathbb{B}|^2}{2\sigma(S^{d-1})}
\]
by Cauchy-Schwarz since \( |A| = |B| = \frac{1}{2} |E \Delta \mathbb{B}| \). Moreover,
\[
(5.6) \quad \int_{S^{d-1}} F^2 \, d\sigma \leq \int_{S^{d-1}} (a^2 + b^2) \, d\sigma
\]
since \( F^2 = (a^2 + b^2 - 2ab) \leq a^2 + b^2 \).

Lemma 5.1. Let \( d \geq 1 \). For each compact subset \( \Lambda \subset (q_d, \infty) \) there exists \( q > 0 \) such that the following holds for all \( q \in \Lambda \). Let \( \lambda \in [1, \infty) \). Suppose that \( \lambda|E \Delta \mathbb{B}| \leq 1 \) and that the set \( E \) satisfies
\[
(5.7) \quad E \Delta \mathbb{B} \subset \{ x : |1 - |x|| \leq \lambda|E \Delta \mathbb{B}| \}.
\]
Then
\[
(5.8) \quad \langle L_q \ast f, f \rangle = \iint_{S^{d-1} \times S^{d-1}} F(\alpha)F(\beta)L_q(\alpha - \beta) \, d\sigma(\alpha) \, d\sigma(\beta) + O_\lambda(|E \Delta \mathbb{B}|^{2+\epsilon}).
\]
Likewise
\[
(5.9) \quad \langle L_q \ast f, \tilde{f} \rangle = \iint_{S^{d-1} \times S^{d-1}} F(\alpha)F(-\beta)L_q(\alpha - \beta) \, d\sigma(\alpha) \, d\sigma(\beta) + O_\lambda(|E \Delta \mathbb{B}|^{2+\epsilon}).
\]
Finally,
\[
(5.10) \quad \langle K_q, f \rangle \leq -\frac{1}{2} \gamma_{q,d} \int_{S^{d-1}} (a^2 + b^2) \, d\sigma + O_\lambda(|E \Delta \mathbb{B}|^{2+\epsilon}).
\]
The constants implicit in the expressions \( O_\lambda(|E \Delta \mathbb{B}|^{2+\epsilon}) \) depend only on \( \lambda, \Lambda, d \).

The conclusion \( (5.10) \) is an inequality, rather than an identity, even up to the remainder \( O(|E \Delta \mathbb{B}|^{2+\epsilon}) \). It is important to note the form of \( (5.10) \), in which the right-hand side is quadratic in \( a, b \), even though the left-hand side is formally linear in \( f \). Indeed, the left-hand side is roughly of second order in \( |E \Delta \mathbb{B}| \) when the set \( E \Delta \mathbb{B} \) satisfies \( (5.7) \), but is roughly of first order when \( E \Delta \mathbb{B} \) lies a fixed distance from this boundary. Corollary 4.2 addresses the contribution of the latter situation, while Lemma 5.1 deals with the former.
Proof.

\[
\langle L_q * f, f \rangle = \int\int_{S^{d-1} \times S^{d-1}} \int_0^\infty \int_0^\infty f(r^\alpha f(\rho^\beta) L_q(r^\alpha - \rho^\beta) r^{d-1} dr \rho^{d-1} d\rho d\sigma(\alpha) d\sigma(\beta)
\]

\[
= \int\int_{S^{d-1} \times S^{d-1}} \int_0^\infty \int_0^\infty f(r^\alpha f(\rho^\beta) L_q(r^\alpha - \rho^\beta) d\sigma(\alpha) d\sigma(\beta) r^{d-1} dr \rho^{d-1} d\rho
\]

\[
+ O_\lambda(|E \Delta \mathbb{B}|^{2+\varepsilon})
\]

since \(L_q\) is Hölder continuous and the hypothesis (5.7) ensures that \(|r - 1| + |\rho - 1| = O_\lambda(|E \Delta \mathbb{B}|)\) whenever \(f(r^\alpha f(\rho^\beta) \neq 0\). Each factor of \(f\) accounts for one factor \(O_\lambda(|E \Delta \mathbb{B}|)\), and the Hölder continuity of \(L_q\) together with the support restriction result in another factor \(O_\lambda(|E \Delta \mathbb{B}|^{\varepsilon})\).

By performing the integrals with respect to \(r, \rho\) and invoking the definition of \(F\), one obtains

\[
\langle L_q * f, f \rangle = \int\int_{S^{d-1} \times S^{d-1}} F(\alpha) F(\beta) L_q(\alpha - \beta) d\sigma(\alpha) d\sigma(\beta) + O_\lambda(|E \Delta \mathbb{B}|^{2+\varepsilon}).
\]

The same reasoning applies to \(\langle L_q * f, \tilde{f} \rangle\).

To prove (5.10), define \(K^*_q(s) = K_q(x)\) where \(|x| = s\). Since \(\nabla K_q\) is Hölder continuous,

\[
K^*_q(r) = K^*_q(1) - \gamma_q, d(r - 1) + O_\lambda(|r - 1|^{1+\varepsilon})
\]

where \(\rho > 0\). Consequently

\[
\langle K_q, f \rangle = \int_{S^{d-1}} \int_{R^+} K^*_q(r) f(r^\alpha) r^{d-1} dr d\sigma(\alpha)
\]

\[
= K^*_q(1) \int_{S^{d-1}} F(\alpha) d\sigma(\alpha) - \gamma_q, d \int_{S^{d-1}} \int_{R^+} f(r^\alpha) (r - 1) r^{d-1} dr d\sigma(\alpha)
\]

\[
+ O_\lambda(|E \Delta \mathbb{B}|^{1+\varepsilon}||f||_1)
\]

since \(E \Delta \mathbb{B} \subset \{x : |x| - 1 \leq \lambda|E \Delta \mathbb{B}|\}\). Since \(\int_{S^{d-1}} F(\alpha) d\sigma(\alpha) = \int_{R^d} f(x) dx = 0\) and \(||f||_1 = |E \Delta \mathbb{B}|\),

\[
\langle K_q, f \rangle = -\gamma_q, d \int_{S^{d-1}} \int_{R^+} f(r^\alpha) r^{d-1} (r - 1) dr d\sigma(\alpha) + O_\lambda(|E \Delta \mathbb{B}|^{2+\varepsilon})
\]

\[
= -\gamma_q, d \int_{S^{d-1}} \int_{R^+} (1_A - 1_B)(r^\alpha) r^{d-1} (r - 1) dr d\sigma(\alpha) + O_\lambda(|E \Delta \mathbb{B}|^{2+\varepsilon}).
\]

Consider the contribution of \(1_A\) to the last integral. The set \(A \subset R^d\) is by hypothesis contained in \(\{x : 1 \leq |x| \leq \lambda|E \Delta \mathbb{B}|\}\), so the factor \(r - 1\) is nonnegative whenever \(a(\alpha) = \int_{R^+} 1_A(r^\alpha) r^{d-1} dr\) is nonzero. Define \(\tilde{a}(\alpha)\) by the relation

\[
\int_1^{1+\tilde{a}(\alpha)} r^{d-1} dr = a(\alpha).
\]

Then

\[
\tilde{a}(\alpha) = a(\alpha) + O_\lambda(\alpha^2) = a(\alpha) + O_\lambda(|E \Delta \mathbb{B}|^2)
\]

since the hypotheses \(A \subset \{x : |x| \leq 1 + \lambda|E \Delta \mathbb{B}|\}\) and \(\lambda|E \Delta \mathbb{B}| \leq 1\) imply that

\[
a(\alpha) \leq \int_1^{1+\lambda|E \Delta \mathbb{B}|} r^{d-1} dr = O_\lambda(|E \Delta \mathbb{B}|).
Fix $\alpha$ momentarily. Among all sets $A \subset \mathbb{R}^d \setminus \mathbb{B}$ that satisfy $\int_1^\infty 1_A(r\alpha)r^{d-1} \, dr \equiv a(\alpha)$, plainly the integral $\int_1^\infty 1_A(r\alpha)r^{d-1}(r - 1) \, dr$ is minimized when $\{r : r\alpha \in A\}$ is equal to the interval $[1, 1 + \tilde{a}(\alpha)]$. Therefore

$$\int_{S^{d-1}} \int_{\mathbb{R}^+} 1_A(r\alpha) r^{d-1}(r - 1) \, dr \, d\sigma(\alpha) \geq \int_{S^{d-1}} \int_1^{1+\tilde{a}(\alpha)} r^{d-1}(r - 1) \, dr \, d\sigma(\alpha)$$

$$= \int_{S^{d-1}} ( (d+1)^{-1} r^{d+1} - d^{-1} r^{d} ) [1+\tilde{a}(\alpha)] \, d\sigma(\alpha)$$

$$= \int_{S^{d-1}} \left( \frac{1}{2} \tilde{a}(\alpha)^2 + O_\lambda(\tilde{a}(\alpha)^3) \right) \, d\sigma(\alpha)$$

$$= \int_{S^{d-1}} \frac{1}{2} a(\alpha)^2 \, d\sigma(\alpha) + O_\lambda(|E\Delta\mathbb{B}|^3).$$

The same analysis applies to $\int_{S^{d-1}} \int_{\mathbb{R}^+} 1_B(r\alpha) r^{d-1}(1 - r) \, dr$ with appropriate reversals of signs and inequalities, establishing (5.10). \qed

**Definition 5.3.** The quadratic form $Q_{q,d}$ acts on a pair of real-valued functions $\varphi, \psi \in L^2(S^{d-1})$ by

$$(5.11) \quad Q_{q,d}(\varphi, \psi) = \int_{S^{d-1} \times S^{d-1}} \varphi(\alpha)\psi(\beta)L_q(\alpha - \beta) \, d\sigma(\alpha) \, d\sigma(\beta).$$

This form is invariant with respect to the diagonal action of the rotation group $O(d)$ on $(\varphi, \psi)$, hence is potentially amenable to analysis in terms of spherical harmonic decomposition.

We have shown

**Lemma 5.2.** Let $d \geq 1$, and let $\Lambda \subset (q_d, \infty)$ be a compact set. There exists $q > 0$ such that for any sufficiently large $\lambda \in \mathbb{R}^+$ and any measurable set $E \subset \mathbb{R}^d$ satisfying $|E| = |\mathbb{B}|$ and $\lambda|E\Delta\mathbb{B}| \leq 1$, if $E\Delta\mathbb{B} \subset \{x : |x| - 1 \leq \lambda|E\Delta\mathbb{B}|\}$ then $f = 1_E - 1_{\mathbb{B}}$ satisfies

$$q(K_1, f) + \frac{1}{q} q^2 \langle K_2 * f, f \rangle + \frac{1}{q} q(q - 2) \langle K_2 * f, \tilde{f} \rangle$$

$$(5.12) \quad \leq -\frac{1}{2} q \gamma_{q,d} \int_{S^{d-1}} (a^2 + b^2) \, d\sigma + \frac{1}{q} q^2 Q_{q,d}(F, F) + \frac{1}{q} q(q - 2) Q_{q,d}(F, \tilde{F})$$

$$+ O_\lambda(|E\Delta\mathbb{B}|^{2+q})$$

where $F$ is as defined in (5.4). The notation $O_\lambda$ indicates an implicit constant that depends only on $d, \lambda, \Lambda$.

6. Balancing

Recall that $\text{Aff}(d)$ denotes the group of all affine automorphisms of $\mathbb{R}^d$. For any $\phi \in \text{Aff}(d)$ and measurable set $E \subset \mathbb{R}^d$, $\Phi_q(\phi(E)) = \Phi_q(E)$ for all $q \in [2, \infty)$. Therefore in analyzing $\|\hat{1}_E\|_q$ for measurable sets $E$ that have small symmetric difference with $\mathbb{B}$, rather than writing $1_E = 1_{\mathbb{B}} + 1_{E\Delta\mathbb{B}}$ and expanding $\|\hat{1}_E\|_q$ about $1_{\mathbb{B}}$, we wish to exploit an expansion based on a representation $1_E = 1_{\mathbb{E}} + 1_{E\Delta\mathbb{E}}$ for an optimally chosen element $\mathbb{E}$ of the orbit of $\mathbb{B}$ under $\text{Aff}(d)$. Equivalently, we seek to replace $E$ by $\phi(E)$ for some $\phi \in \text{Aff}(d)$, chosen so that $\phi(E)$ is best approximated by $\mathbb{E}$. In this section we specify what approximation is to be considered to be best — see Lemma 6.1 — and prove that $\phi$ exists.
Definition 6.1. A bounded Lebesgue measurable set \( E \subset \mathbb{R}^d \) satisfying \( |E| = |B| \) is balanced if the function \( F : S^{d-1} \to \mathbb{R} \) associated to \( E \) as in Definition 5.2 satisfies
\[
\int_{S^{d-1}} F(y) P(y) \, d\sigma(y) = 0
\]
for every polynomial \( P : \mathbb{R}^d \to \mathbb{R} \) of degree less than or equal to 2.

This equation for constant functions \( P \) is the redundant assertion that \( \int (1_E - 1_B) = 0 \), a restatement of the hypothesis \( |E| = |B| \).

Denote by \( \mathcal{M}_d \) the vector space of all \( d \times d \) square matrices with real entries, and by \( \mathcal{M}_d \oplus \mathbb{R}^d \) the set of all ordered pairs \((S, v)\) where \( S \in \mathcal{M}_d \) and \( v \in \mathbb{R}^d \), with the natural vector space structure. Identify elements of \( \mathcal{M}_d \) with linear endomorphisms of \( \mathbb{R}^d \) in the usual way. Fix any norm \( \| \cdot \|_{\mathcal{M}_d} \) on \( \mathcal{M}_d \).

Elements \( \phi \in \text{Aff}(d) \) take the form \( \phi(x) = T(x) + v \) where \( (T, v) \in \mathcal{M}_d \oplus \mathbb{R}^d \) is uniquely determined by \( \phi \), and \( T : \mathbb{R}^d \to \mathbb{R}^d \) is an invertible linear transformation. Define \( \| \phi \|_{\text{Aff}(d)} = \|T\|_{\mathcal{M}_d} + \|v\|_{\mathbb{R}^d} \). We abuse notation by writing \( \det(\phi) \) for the determinant of the unique \( T \in \mathcal{M}_d \) thus associated to \( \phi \), and likewise \( \text{trace}(\phi) = \text{trace}(T) \).

In this section we prove:

**Lemma 6.1.** Let \( d \geq 1 \). There exists \( c > 0 \) with the following property. For every \( \lambda \geq 1 \) and every Lebesgue measurable set \( E \subset \mathbb{R}^d \) satisfying \( |E| = |B| \), \( \lambda|E \Delta B| \leq c \), and \( E \Delta B \subset \{x : |x - 1| \leq \lambda|E \Delta B|\} \), there exists a measure-preserving transformation \( \phi \in \text{Aff}(d) \) such that
\[
\phi(E) \text{ is balanced,}
\]
\[
\|\phi - I\|_{\text{Aff}(d)} = O_\lambda(|E \Delta B|),
\]
\[
\phi(E) \Delta B \subset \{x : |x - 1| \leq C_\lambda|E \Delta B|\}.
\]

The constant \( C_\lambda \) depends on \( \lambda, d \) but not on \( E \). It is the case of large \( \lambda \) that is of interest. A point worthy of notice is that the inequality in the final conclusion takes the form \( |1 - |x|| \leq C_\lambda|E \Delta B| \), whereas the variant \( |1 - |x|| \leq C_\lambda \phi(E) \Delta B| \) will be required as a hypothesis in the subsequent analysis. Because the functional \( \Phi_q \) is affine-invariant, at the outset we may replace \( E \) by \( \psi \) where \( \psi \in \text{Aff}(d) \) is measure-preserving and \( |\psi(E) \Delta B| \leq 2 \inf_{E \in E} |E \Delta \mathcal{E}| \), taking the infimum over all ellipsoids satisfying \( |\mathcal{E}| = |E| \). For this modified set \( E \), if \( \phi \in \text{Aff}(d) \) satisfies the conclusions of Lemma 6.1 then \( |\phi(E) \Delta B| \geq \inf_{E \in E} |E \Delta \mathcal{E}| \geq \frac{1}{2} |E \Delta B| \) and therefore
\[
|1 - |x|| \leq C_\lambda|E \Delta B| \leq |1 - |x|| \leq 2C_\lambda|\phi(E) \Delta B|.
\]

To prepare for the proof of Lemma 6.1 denote by \( W_2 \) the real vector space of all polynomials \( P : \mathbb{R}^d \to \mathbb{R} \) that are finite linear combinations of homogeneous harmonic polynomials of degrees \( \leq 2 \). Denote by \( V_2 \) the real vector space of all restrictions to \( S^{d-1} \) of real-valued polynomials of degrees \( \leq 2 \). The natural linear mapping from \( W_2 \) to \( V_2 \) induced by restriction from \( \mathbb{R}^d \) to \( S^{d-1} \) is a bijection.

Regard \( V_2 \) as a real inner product space, with the \( L^2(S^{d-1}, \sigma) \) inner product. Denote by \( \Pi \) the orthogonal projection of \( L^2(S^{d-1}) \) onto its subspace \( V_2 \). Define \( \mathcal{T} : \mathcal{M}_d \to V_2 \) by
\[
\mathcal{T}(S)(\alpha) = \Pi(\alpha \cdot S(\alpha)),
\]
that is, the right-hand side equals the restriction to \( S^{d-1} \) of the quadratic polynomial \( \mathbb{R}^d \ni x \mapsto x \cdot S(x) \).

**Lemma 6.2.** \( \mathcal{T} : \mathcal{M}_d \to V_2 \) is surjective.
Proof. The range of $\mathcal{T}$ is the collection of all functions $S^{d-1} \ni \alpha \mapsto S(\alpha) \cdot \alpha$, as the function $S$ varies over all affine mappings from $\mathbb{R}^d$ to $\mathbb{R}^d$. Because $S \mapsto \mathcal{T}(S)$ is linear, this range is a subspace of $V_2$.

Firstly, the constant function $\alpha \mapsto 1$ equals $\mathcal{T}(S)$ when $S(x) \equiv x$, since $S(\alpha) \cdot \alpha = \alpha \cdot \alpha \equiv 1$ for $\alpha \in S^{d-1}$. Secondly, a linear monomial $\alpha = (\alpha_1, \ldots, \alpha_d) \mapsto \alpha_k$ is expressed by choosing $S(x) \equiv e_k$, the $k$-th coordinate vector. Thirdly, to express a monomial $\alpha \mapsto \alpha_j \alpha_k$ in the form $S(\alpha) \cdot \alpha$, define $S(x) = (S_1(x), \ldots, S_d(x))$ by $S_i(x) \equiv 0$ for all $i \neq j$, and $S_j(x) = x_k$. Then $\alpha_j \alpha_k = S(\alpha) \cdot \alpha$. Functions of these three types span $V_2$, so $\mathcal{T}$ is indeed surjective. \hfill $\square$

Proof of Lemma 6.1. If $c \leq \frac{1}{2}$ then $E$ contains the ball of radius $\frac{1}{2}$ centered at 0, so if $f \in \text{Aff}(d)$ is sufficiently close to the identity then $\phi(E)$ contains the ball of radius $\frac{1}{4}$ centered at 0.

Let $k \in \{0, 1, 2\}$. Let $P : \mathbb{R}^d \to \mathbb{R}$ be a homogeneous harmonic polynomial of degree $k$. Set $g(x)$ be a smooth function that agrees with $|x|^{-k}P(x)$ in $\{x : |x| - 1 \leq \frac{1}{4}\}$.

For $f \in \text{Aff}(d)$ let $f_{\phi(E)} = 1_{\phi(E)} - 1_B$ and $F_{\phi(E)}$ be the functions associated to $\phi(E)$ in the same way that $f = 1_E - 1_B$ and $F$ are associated to $E$. Then

$$
\int_{S^{d-1}} F_{\phi(E)}(y)P(y) \, d\sigma(y) = \int_{S^{d-1}} \int_0^\infty (1_{\phi(E)} - 1_B)(ry) r^{d-1} \, dr \, P(y) \, d\sigma(y)
= \int_{\mathbb{R}^d} (1_{\phi(E)} - 1_B)(x)|x|^{-k}P(x) \, dx
= \int_{\mathbb{R}^d} (1_E \circ \phi^{-1} - 1_B) g
= \int_{\mathbb{R}^d} (f \circ \phi^{-1}) g + \int_{\mathbb{R}^d} (1_B \circ \phi^{-1} - 1_B) g.
$$

(6.3)

The second to last equation holds because both $B$ and $\phi(E)$ contain the ball of radius $\frac{1}{4}$ centered at 0, and $g(x) \equiv |x|^{-k}P(x)$ for all $x$ in the complement of this ball. All of these quantities depend linearly on $P$.

We seek the desired $\phi \in \text{Aff}(d)$ in the form

$$
\phi = I + S,
$$

where $\|S\|_{\text{Aff}(d)}$ is small and $I$ is the identity matrix; that is, $\phi(x) = x + S(x)$ where $S$ is an affine mapping. The second term on the right-hand side of (6.3) is independent of $E$. Moreover,

$$
\int_{\mathbb{R}^d} (1_B \circ \phi^{-1}) g = |\det(\phi)| \int_B g \circ \phi
= (1 + \text{trace } (S)) \int_B g \circ \phi + O_P(\|S\|^2_{\text{Aff}(d)})
= (1 + \text{trace } (S)) \int_B g(x + S(x)) \, dx + O_P(\|S\|^2_{\text{Aff}(d)}).
$$

Here and below, $O_P(\|S\|^2_{\text{Aff}(d)})$ denotes a quantity which depends linearly on $P$, whose norm or absolute value, as appropriate, is bounded above by a constant multiple of the norm of $P$ times the $\text{Aff}(d)$ norm squared of $S$. 

Moreover, \( T \) represents the term \( O \) simply because 
\[
\int_{\nabla S \cdot T} + O_P(\|S\|_{\text{Aff}(d)}^2) = \int_B \left( g \nabla S \cdot T \right) + O_P(\|S\|_{\text{Aff}(d)}^2)
\]

in (6.4). \( P, Q \) are polynomials, the sum of whose degrees equals 4. Therefore at least one of \( P, Q \) must have degree less than or equal to 2. The corresponding integral vanishes.

The second to last equality is justified by the divergence theorem, and the last by the identity \( g \equiv P \) on \( S^{d-1} \). Thus

\[
\int_{\mathbb{R}^d} (1_B \circ \phi^{-1}) g = \int_{S^{d-1}} P(\alpha) S(\alpha) \cdot \alpha d\sigma(\alpha) + O_P(\|S\|_{\text{Aff}(d)}^2).
\]

Since \( \int_{\mathbb{R}^d} (f \circ \phi^{-1}) g = \int_{\mathbb{R}^d} (f \circ \phi^{-1}) P(x) |x|^{-k} \), by returning to (6.3) we find that the equation \( \int_{S^{d-1}} F_{\phi(E)} P d\sigma = 0 \) for a still unknown \( S \in \text{Aff}(d) \) takes the form

\[
(6.4) \quad \int_{S^{d-1}} P(\alpha) S(\alpha) \cdot \alpha d\sigma(\alpha) = - \int_{\mathbb{R}^d} (f \circ \phi^{-1}) P(x) |x|^{-k} dx + O_P(\|S\|_{\text{Aff}(d)}^2).
\]

All three terms in this equation depend linearly on \( P \), so we may regard this as an equation in \( V_2^* \), or equivalently as an equation in \( V_2 \) since this is a Hilbert space.

Write this equation as

\[
(6.5) \quad \mathcal{T}(S) = \mathcal{N}_f(S) + \mathcal{R}(S)
\]

where \( \mathcal{T} \) is defined above, \( \mathcal{N}_f(S) \) is the mapping \( P \mapsto - \int_{\mathbb{R}^d} (f \circ \phi^{-1}) P(x) |x|^{-k} dx \), and \( \mathcal{R} \) represents the term \( O_P(\|S\|_{\text{Aff}(d)}^2) \). Both \( \mathcal{N}_f \) and \( \mathcal{R} \) are twice continuously differentiable. Moreover,

\[
(6.6) \quad \|\mathcal{N}_f(S)\|_{V_2^*} \leq C|E \Delta B|
\]

simply because \( |f| \leq 1_{|E \Delta B|} \) and \( f \) is supported where \( \frac{1}{2} \leq |x| \leq \frac{3}{2} \). Since \( \mathcal{T} : \mathcal{M}_d \rightarrow V_2 \) is surjective, the Implicit Function Theorem guarantees that the equation \( \mathcal{T}(S) = \mathcal{N}_f(S) + \mathcal{R}(S) \), for an unknown \( S \in \mathcal{M}_d \), admits a solution satisfying \( \|S\|_{\mathcal{M}_d} \leq C_N|E \Delta B| \). \( \Box \)

**Lemma 6.3.** Let \( E \subset \mathbb{R}^d \) be balanced with respect to \( B \). Then

\[
(6.7) \quad \int_{S^{d-1} \times S^{d-1}} F(\alpha) F(\beta) |\alpha - \beta|^{2k} d\sigma(\alpha) d\sigma(\beta) = 0 \quad \text{for} \quad k \in \{0, 1, 2\}.
\]

**Proof.** Expand \( |\alpha - \beta|^{2k} \) as a linear combination of monomials in \( (\alpha, \beta) \), and expand the double integral accordingly. Each monomial gives rise to a double integral that factors as a product of two single integrals of the form

\[
\int_{S^{d-1}} F P d\sigma \cdot \int_{S^{d-1}} F Q d\sigma
\]

where \( P, Q \) are polynomials, the sum of whose degrees equals 4. Therefore at least one of \( P, Q \) must have degree less than or equal to 2. The corresponding integral vanishes,
by the balancing hypothesis, which asserts that the function \( F \) associated to \( E \) satisfies
\[
\int_{\mathbb{R}^d} F P \, d\sigma = 0 \text{ for all polynomials } P \text{ of degrees less than or equal to } 2.
\]

\[\qed\]

7. **Large exponents**

In this section we examine asymptotic behavior as \( q \to \infty \) while the dimension \( d \) remains fixed. The notation \( O(\psi(q)) \) indicates a quantity whose absolute value is bounded above by \( C\psi(q) \) where \( C \) may depend on \( d \) and on other parameters but is independent of \( q \) provided only that \( q \) is sufficiently large.

**Proposition 7.1.** Let \( d \geq 1 \). There exists \( q_d^* < \infty \) such that for any \( q \in [q_d^*, \infty) \) there exist \( \delta(q,d), c(q,d), C(q,d) \in \mathbb{R}^+ \) with the following property. Let \( E \subset \mathbb{R}^d \) be a Lebesgue measurable set satisfying \( |E| = |B| \). If \( \text{dist}(E, \mathcal{E}) \leq \delta \) then
\[
(7.1) \quad \|\hat{1}_E\|_q^q \leq \|\hat{1}_\mathbb{R}\|_q^q - c(q,d) \text{dist}(E, \mathcal{E})^2.
\]

More quantitative information will be implicit in the proof. In particular, the constant \( c(q,d) \) has order of magnitude \( \omega_d^q q^{-(d+2)/2} \) as \( q \to \infty \).

7.1. **The functions \( K_q \) and \( L_q \) for large \( q \).** \( L_q \) is by definition the inverse Fourier transform of \( |\hat{1}_B|^{q-2} \). The function \( |\hat{1}_B| \) has maximum value \( \omega_d = |B| \), and achieves this value only at the origin. In studying \( L_q \) for large \( q \), it is natural to examine the normalized function \( \omega_d^{-q} L_q \); likewise \( \omega_1^{-q} K_q \). Recall that \( K_q \) is real-valued, and that \( K_q, L_q \) are three times continuously differentiable globally bounded functions for all sufficiently large \( q \), which tend to zero as \( |x| \to \infty \).

**Lemma 7.2.** Let \( d \geq 1 \). The following hold for all sufficiently large \( q < \infty \). Firstly, \( L_q(x) \) is a three times continuously differentiable bounded function, which tends to zero as \( |x| \to \infty \). Secondly, there exist \( \kappa_j = \kappa_j(q,d) \) such that as functions of \( x \) in any compact subset of \( \mathbb{R}^d \),
\[
(7.2) \quad L_q(x) = \kappa_0 + \kappa_1 |x|^2 + \kappa_2 |x|^4 + O(\omega_d^q q^{-(d+6)/2}).
\]

This holds in the \( C^3 \) norm as a function of \( x \) on any fixed compact subset of \( \mathbb{R}^d \).

The proof gives explicit asymptotic values for \( \kappa_j \), and expansions modulo lower order terms, but the contributions of these three leading terms in the expansion for \( L_q \) will vanish in the analysis below, so that only the order of magnitude of the remainder term will subsequently be relevant.

**Proof.** Express
\[
\hat{1}_B(\xi) = \int_{|x| \leq 1} e^{-2\pi i x \cdot \xi} \, dx = \omega_d^{-1} \int_{-1}^1 e^{-2\pi i s |\xi|} (1 - s^2)^{(d-1)/2} \, ds.
\]

In particular, \( \hat{1}_B(0) = \omega_d \). It is immediate that
\[
(7.4) \quad \nabla(\hat{1}_B)(0) = 0 \quad \text{and} \quad |\hat{1}_B(\xi)| < \omega_d \text{ for all } \xi \neq 0.
\]
Expanding the exponential factor $e^{-2\pi is|\xi|}$ in the last integral in Maclaurin series, terms of odd degree with respect to $s$ contribute zero. Therefore for $\xi$ in any bounded set,

$$\hat{1}_B(\xi) = \omega_{d-1} \int_{-1}^{1} (1 + \frac{1}{2}(-2\pi i|\xi|s) + O(|\xi|^4))(1 - s^2)^{(d-1)/2} \, ds$$

$$= \omega_d - 2\pi^2\omega_{d-1}|\xi|^2 \int_{-1}^{1} s^2(1 - s^2)^{(d-1)/2} \, ds + O(|\xi|^4)$$

$$= \omega_d(1 - \pi \rho_d|\xi|^2) + O(|\xi|^4)$$

where $\rho_d$ is defined by

$$(7.5) \quad \rho_d = 2\pi\omega_{d-1}\omega_d^{-1} \int_{-1}^{1} s^2(1 - s^2)^{(d-1)/2} \, ds.$$ 

Provided that $|\xi|$ is sufficiently small, this can be rewritten as

$$(7.6) \quad \omega_d^{-1}\hat{1}_B(\xi) = \exp\left(-\pi \rho_d|\xi|^2 + O(|\xi|^4)\right).$$

Therefore for $|\xi|$ in some neighborhood of 0 that depends only on $d$, 

$$(7.7) \quad \omega_d^{-2-q}\hat{1}_B(\xi)|^q - 2 = e^{-\pi \rho_d(q-2)|\xi|^2} + O(q|\xi|^4).$$

Moreover for $\xi$ in any fixed bounded region,

$$(7.8) \quad |\omega_d^{-1}\hat{1}_B(\xi)| \leq e^{-c|\xi|^2}$$

where $c > 0$ depends only on the dimension $d$, while $|\hat{1}_B(\xi)| = O(|\xi|^{-(d+1)/2})$ as $|\xi| \to \infty$.

Therefore for arbitrary $k \geq 0$,

$$\int_{|\xi| \geq \xi^{-1/4}} |\xi|^k \cdot |\omega_d^{-1}\hat{1}_B(\xi)|^q - 2 \, d\xi = O(q^{-N}) \quad \text{for all $N$}$$

as $q \to \infty$ while $k$ remains fixed. Therefore uniformly for all $x \in \mathbb{R}^d$, for any $N < \infty$,

$$(7.9) \quad \omega_d^{-2-q} L_q(x) = \int_{\mathbb{R}^d} e^{2\pi ix} \omega_d^{-2-q} \hat{1}_B(\xi)|^q - 2 \, d\xi$$

$$= \int_{\mathbb{R}^d} e^{2\pi ix} h(\xi) \, d\xi$$

where $h = h_{q,d}$ is a radially symmetric function that satisfies

$$(7.10) \quad \int_{\mathbb{R}^d} |\xi|^k h(\xi) \, d\xi = O(q^{-(d+k)/2})$$

for any nonnegative real number $k$.

If $k$ is a nonnegative integer then by virtue of the radial symmetry of $h$, 

$$(7.11) \quad \int_{|\xi| \leq \xi^{-1/4}} (x \cdot \xi)^k h(\xi) \, d\xi = c_{k,q}|x|^k$$

where $c_{k,q} = 0$ for all odd $k$ and $c_{k,q} = O_k(q^{-(d+2+k)/2})$. Expand

$$e^{2\pi ix \cdot \xi} = \sum_{k=0}^{5} (2\pi ix \cdot \xi)^k / k! + O(|x \cdot \xi|^6)$$

and invoke (7.11) to obtain

$$\int_{|\xi| \leq \xi^{-1/4}} e^{2\pi ix} h(\xi) \, d\xi = c_0(q) + c_1(q)|x|^2 + c_2(q)|x|^4 + O(q^{-(d+6)/2})$$
where $c_j(q)$ are independent of $x$ while the remainder term $O(q^{-(d+6)/2})$ satisfies this upper bound uniformly for all $x$ in an arbitrary compact set.

Recall the notation $\gamma_{q,d} = -x \cdot \nabla K_q(x)|_{|x|=1}$.  

**Lemma 7.3.** Let $d \geq 1$. There exists a real number $\kappa > 0$ that depends only on the dimension $d$ such that as $q \to \infty$,  
\begin{equation}
\gamma_{q,d} = \kappa q^{-(d+2)/2} \omega_d^q + O(q^{-(d+4)/2} \omega_d^q) .
\end{equation}

For any bounded set $S \subset \mathbb{R}^d$ there exists $r < \infty$ such that for all $q \geq r$ and all $x, y \in S$,  
\begin{equation}
K_q(x) > K_q(y) \text{ whenever } |x| < 1 < |y| .
\end{equation}

**Proof.** Let $S \subset \mathbb{R}^d$ be a bounded set, and consider any point $x \in S$. In the expression
\begin{equation}
x \cdot \nabla K_q(x) = \int e^{2\pi i x \cdot \xi} (2\pi i x \cdot \xi) \widehat{1_B}(\xi)|1_B(\xi)|^{q-2} d\xi ,
\end{equation}

expand $e^{2\pi i x \cdot \xi} = \sum_{k=0}^{\infty} (2\pi i x \cdot \xi)^k / k! + O(|x| \cdot \xi^3)$ and argue as in the proof of Lemma 7.2 to obtain
\begin{align*}
x \cdot \nabla K_q(x) &= -4\pi^2 \int (x \cdot \xi)^2 \widehat{1_B}(\xi)|\widehat{1_B}(\xi)|^{q-2} d\xi + O(|x|^{q-4} q^{-4}/2) \\
&= -4\pi^2 |x|^2 \int |\xi|^2 \widehat{1_B}(\xi)|\widehat{1_B}(\xi)|^{q-2} d\xi + O(|x|^{q-4} q^{-4}/2).
\end{align*}

Comparing to the proof of Lemma 7.2, there is one extra factor of $\widehat{1_B}$ in the integral. This makes no essential difference in the inequalities.

The last line is obtained by expanding $(x \cdot \xi)^2 = \sum_{j,k} x_j x_k \xi_j \xi_k$ and noting that for any $j \neq k$,
\begin{equation}
\int_{\mathbb{R}^d} \xi_j \xi_k \widehat{1_B}(\xi)|\widehat{1_B}(\xi)|^{q-2} d\xi = 0
\end{equation}
since $\widehat{1_B}$ is radial. By using (7.7) one obtains
\begin{equation}
\int_{\mathbb{R}^d} |\xi|^2 \widehat{1_B}(\xi)|\widehat{1_B}(\xi)|^{q-2} d\xi = \kappa \omega_d^q q^{-(d+2)/2} + O(\omega_d^q q^{-(d+4)/2})
\end{equation}
where $\kappa = \kappa(d) > 0$.  

We require some global control over $K_q$.

**Lemma 7.4.** Let $d \geq 1$. For all sufficiently large exponents $q < \infty$, $K_q$ satisfies
\begin{equation}
\min_{|x| \leq 1-\delta} K_q(x) > \max_{|x| \geq 1+\delta} K_q(x) \text{ for all } \delta > 0
\end{equation}
\begin{equation}
K(x) - K(y) \geq c_q |x| - |y| \text{ whenever } |x| \leq 1 \leq |y| \leq 2
\end{equation}
where $c_q > 0$.

These are the hypotheses (2.13) and (3.9) of Lemma 4.1.

**Proof.** It suffices to establish the first conclusion, for the second follows from the first together with Lemma 7.3.

As in the proof of Lemma 7.2, $\omega_d^{1-q} K_q(x)$ is the inverse Fourier transform of the function $e^{-\pi \rho_d(x-1)} |\xi|^2 + O(|\xi|^4) 1_{|\xi| \leq q^{-1/4}}$ plus $O(q^{N})$ for all $N$. The main term equals
\begin{equation}
(q-1)^{-d/2} \int_{|\xi| \leq q^{1/4}} e^{-\pi \rho_d |\xi|^2 + O(q^{-1} |\xi|^4)} e^{2\pi i y \cdot \xi} d\xi
\end{equation}
plus $O(q^{-N})$ for all $N$, where $y = (q-1)^{-1/2}x$. Now the two functions $e^{-\pi \rho_d |\xi|^2} + O(q^{-1}|\xi|^4)1_{|\xi| \leq q^{1/4}}$ and $e^{-\pi \rho_d |\xi|^2}$ differ by $O(q^{-1})$ in $L^1$ norm. Therefore uniformly for all $x \in \mathbb{R}^d$ and all sufficiently large $q$,

$$ \omega_d^{1-q}K_q(x) = (q - 1)^{-d/2} \left( e^{-\pi \rho_d |\xi|^2} e^{2\pi i y \xi} d\xi \right) + O\left(q^{-d-2/2}\right) $$

where $y = (q - 1)^{-1/2}x$. The leading term is a positive Gaussian function of $y$, and has order of magnitude $q^{-d/2}$ for $y$ in any compact set. (7.16) follows from the monotonicity of Gaussians together with these asymptotics. □

7.2. The coronal case. By the coronal case, we mean that in which $E$ is close to the unit ball in the strong sense that $E \Delta \mathbb{B} \subset \{x : |x| - 1 \leq \lambda |E \Delta \mathbb{B}|\}$ for a suitable constant $\lambda$, which is at our disposal and is to depend only on the dimension $d$ and on the exponent $q$.

**Lemma 7.5.** Let $d \geq 1$ and let $q$ be sufficiently large. There exists $\lambda_0 = \lambda_0(q, d) < \infty$ with the following property. Let $\lambda \geq \lambda_0$. Suppose that $E \subset \mathbb{R}^d$ satisfies $|E| = |\mathbb{B}|$, $|E \Delta \mathbb{B}| \leq 2|E| \cdot \text{dist}(E, \mathcal{E})$, $\lambda |E \Delta \mathbb{B}| \leq 1$, and

$$ E \subset \{x : |x| - 1 \leq \lambda |E \Delta \mathbb{B}|\}. $$

Then

$$ \left\| \hat{1}_E \right\|_q^q \leq \left\| \hat{1}_\mathbb{B} \right\|_q^q - c \text{dist}(E, \mathcal{E})^2 + O_{\lambda}(\text{dist}(E, \mathcal{E})^{2+\theta}) $$

where $c, q$ are positive constants that depend on $d, q$ but are independent of $\lambda$.

**Proof.** If $\lambda_0$ is sufficiently small then by Lemma 6.1 there exists a measure-preserving affine automorphism $\phi \in \text{Aff}(d)$ such that $\phi(E)$ is balanced with respect to $\mathbb{B}$, and $\phi(E) \subset \{x : |x| \leq 1 + O_{\lambda}(1) \text{dist}(E, \mathcal{E})\}$. By its definition, $\text{dist}(E, \mathcal{E}) = \text{dist}(\phi(E), \mathcal{E})$ satisfies $\text{dist}(E, \mathcal{E}) \leq |E| \cdot |\phi(E) \Delta \mathbb{B}|$. Therefore $\phi(E) \subset \{x : |x| \leq 1 + O_{\lambda}(1) |\phi(E) \Delta \mathbb{B}|\}$. Replace $E$ by $\phi(E)$ for the remainder of this proof, recalling that $\left\| \hat{1}_{\phi(E)} \right\|_q = \left\| \hat{1}_E \right\|_q$.

Let $f = \mathbf{1}_E - \mathbf{1}_\mathbb{B}$ and let $a, b, F$ be the functions associated to $E, f$ as in Definition 5.2. We seek upper bounds for $|\langle L_q * f, f \rangle|$ and $|\langle L_q * f, \hat{f} \rangle|$, and a negative upper bound for $\langle K_q, f \rangle$ for all sufficiently large exponents $q$, strong enough to guarantee that in magnitude, $\langle K_q, f \rangle$ dominates the other two terms. Lemma 7.3 guarantees that the hypotheses of Lemma 4.1 are satisfied for all sufficiently large $q$. According to the latter lemma,

$$ \langle K_q, f \rangle \leq -\frac{1}{2} \gamma(q, d) \int (a^2 + b^2) \, d\sigma + O_{\lambda}(|E \Delta \mathbb{B}|^{2+\theta}) $$

$$ \leq -c q^{-(d+2)/2} \omega_d^q \int (a^2 + b^2) \, d\sigma + O(\omega_d^q q^{-(d+4)/2} \|F\|_{L^2}^2) + O_{\lambda}(|E \Delta \mathbb{B}|^{2+\theta}) $$

where $c > 0$ is independent of $q$ so long as $q$ is sufficiently large. Since $\|F\|_{L^2}^2 = \int (a - b)^2 \, d\sigma \leq \int (a^2 + b^2) \, d\sigma$ (recall that $a, b$ are nonnegative by their definitions), by replacing $c$ by $c/2$ we obtain

$$ \langle K_q, f \rangle \leq -c q^{-(d+2)/2} \omega_d^q \int (a^2 + b^2) \, d\sigma + O_{\lambda}(|E \Delta \mathbb{B}|^{2+\theta}). $$

We have shown in Lemma 5.1 that

$$ \langle L_q * f, f \rangle = \int_{S^{d=1} \times S^{d=1}} F(\alpha) F(\beta) L_q(\alpha - \beta) \, d\sigma(\alpha) \, d\sigma(\beta) + O_{\lambda}(|E \Delta \mathbb{B}|^{2+\theta}). $$
According to Lemma 7.22, \( L_q(\alpha - \beta) = \kappa_0 + \kappa_1 |\alpha - \beta|^2 + \kappa_2 |\alpha - \beta|^4 + O(\omega^q q^{-(d+6)/2}) \) where \( \kappa_k \) depend on \( q, d \) but are independent of \( \alpha, \beta \). The contributions of \( \kappa_k |\alpha - \beta|^{2k} \) to the double integral vanish for \( k = 0, 1, 2 \), by Lemma 6.3. Therefore

\[
\langle L_q * f, f \rangle = O(\omega^q q^{-(d+6)/2}\|F\|_{L^2}^2).
\]

By the same reasoning, \( \langle L_q * f, \tilde{f} \rangle = O(\omega^q q^{-(d+6)/2}\|F\|_{L^2}^2) \).

Recall that \( \|f\|_{L^2} = \frac{1}{2} q \langle L_q * f, f \rangle + \frac{1}{2} q(q - 2) \langle L_q * f, \tilde{f} \rangle \)

\[
\leq -cq \cdot q^{-(d+2)/2} \omega_q \int (a^2 + b^2) d\sigma + O(q^2 q^{-(d+6)/2} \omega_q^q \|F\|_{L^2}^2) + O_\lambda(\|E \Delta B\|^{2+\var})
\]

\[
= -c \cdot q^{-d/2} \omega_q \int (a^2 + b^2) d\sigma + O(q^{-d+2}/2 \omega_q^q \|F\|_{L^2}^2) + O_\lambda(\|E \Delta B\|^{2+\var})
\]

\[
\leq -c' \cdot q^{-d/2} \omega_q \int (a^2 + b^2) d\sigma + O_\lambda(\|E \Delta B\|^{2+\var})
\]

where \( c', \var > 0 \) for all sufficiently large exponents \( q \). The crucial point is that for large \( q \), the factor \( \omega_q^q q^{-d/2} \) in the leading term dominates the factor \( \omega_q^q q^{-(d+2)/2} \) in the remainder term \( O(q^{-(d+2)/2} \omega_q^q \|F\|_{L^2}^2) \).

Since \( \|F\|_{L^2} = O(\|E \Delta B\|) \) and \( \int (a^2 + b^2) d\sigma \geq \frac{1}{2} \sigma(S^d)^{-1} \|E \Delta B\|^2 \), this yields

\[
\|1_E\|_q^q \leq \|1_B\|_q^q - c \omega_q^q q^{-d/2} \|E \Delta B\|^2 + O_\lambda(\|E \Delta B\|^{2+\var})
\]

\[
\leq \|1_B\|_q^q - c \omega_q^q q^{-d/2} \text{dist}(E, \mathcal{C})^2 + O_\lambda(\text{dist}(E, \mathcal{C})^{2+\var})
\]

since \( |E \Delta B| \geq |E| \cdot \text{dist}(E, \mathcal{C}) \).

\( \square \)

7.3. Reduction to the coronal case. The next step is to show how the case of sets satisfying the weaker condition \( |E \Delta B| \ll |E| \) can be reduced to the coronal case, completing the proof of Proposition 7.1. Recall the notation \( E_\eta \) introduced in 4.11 and employed in Corollary 2.2. \( E_\eta = \{x \in E : |x| - 1 > \eta\} \).

Proof of Proposition 7.1. Let \( \lambda \) be a large constant to be chosen below. Suppose that \( |E| = |B| \), and that \( B \) is a quasi-optimal approximation to \( E \) among all ellipses of measure \( |E| \) in the sense that

\[
|E \Delta B| \leq 2 \inf_{|E| = |E|} |E \Delta \mathcal{E}| = 2|E| \cdot \text{dist}(E, \mathcal{C}) \ll 1.
\]

Express \( E \) as \( E = (B \cup A) \setminus B \) where \( A \subset B \), \( B \cap B = \emptyset \), and \( |A| = |B| \). Set \( \eta = \lambda |E \Delta B| \) and consider \( E_\eta, A_\eta, B_\eta \). Define sets \( B^\dagger \subset B \setminus B_\eta \) and \( A^\dagger \subset A \setminus A_\eta \) as follows. If \( |B_\eta| \geq |A_\eta| \) choose a measurable set \( A' \subset A_\eta \) satisfying \( |A'| = |B_\eta| \) and define \( B^\dagger = B \setminus B_\eta \) and \( A^\dagger = A \setminus A' \). If \( |A_\eta| \geq |B_\eta| \) choose a measurable set \( B' \subset B_\eta \) satisfying \( |B'| = |A_\eta| \), and define \( B^\dagger = B \setminus B' \) and \( A^\dagger = A \setminus A_\eta \). Define

\[
E^\dagger = (B \cup A^\dagger) \setminus B^\dagger
\]

\[
f^\dagger = 1_{A^\dagger} - 1_{B^\dagger}
\]

and \( f_\eta = f - f^\dagger = 1_{A \setminus A^\dagger} - 1_{B \setminus B^\dagger} \). Then \( |E^\dagger| = |B| \) and

\[
|E \Delta B| = |E \Delta E^\dagger| + |E^\dagger \Delta B|.
\]

Let \( \tilde{f}^\dagger(x) = f^\dagger(-x) \).
By Corollary 4.2 for all sufficiently large \( q \),

\[
(7.24) \quad \| \hat{1}_E \|_q^q \leq \| \hat{1}_{E^\dagger} \|_q^q - c\lambda |E \Delta B| \cdot |E \Delta E^\dagger| + O_\lambda(|E \Delta B|^{2+\varepsilon}).
\]

Since \( |E^\dagger| = |E| \),

\[
\inf_{|E'|=|E^\dagger|} |E^\dagger \Delta E'| = \inf_{|E'|=|E|} |E^\dagger \Delta E'| \leq |E^\dagger \Delta B| \leq |E \Delta B|.
\]

Moreover, a simplification of the analysis above establishes the simple bound

\[
(7.25) \quad \| \hat{1}_{E^\dagger} \|_q^q \leq \| \hat{1}_B \|_q^q + O(|E^\dagger \Delta B|^2)
\]

since \( \langle K_q, f^\dagger \rangle \) is nonpositive for \( q \) sufficiently large. Together these give

\[
\| \hat{1}_E \|_q^q \leq \| \hat{1}_B \|_q^q - c\lambda |E \Delta B| \cdot |E \Delta E^\dagger| + O(|E \Delta B|^2).
\]

From this we deduce the desired conclusion \( \| \hat{1}_E \|_q^q \leq \| \hat{1}_B \|_q^q - c|E \Delta B|^2 \) unless

\[
(7.26) \quad |E^\dagger \Delta B| \leq C\lambda^{-1}|E \Delta B|.
\]

Here the constant \( C \) depends only on \( q, d \).

For the remainder of the proof we may assume that (7.26) holds. If \( \lambda \) is sufficiently large, then (7.26) together with the relation \( |E \Delta B| = |E \Delta E^\dagger| + |E^\dagger \Delta B| \) imply that

\[
|E^\dagger \Delta B| \geq \frac{1}{2}|E \Delta B|.
\]

This condition ensures that

\[
(7.27) \quad E^\dagger \Delta B \subset E \Delta B \subset \{ x : |x| - 1 \leq \lambda|E \Delta B| \} \subset \{ x : |x| - 1 \leq 2\lambda|E^\dagger \Delta B| \}.
\]

Moreover, for any ellipsoid \( E \) satisfying \( |E| = |E^\dagger| = |E| \),

\[
|E^\dagger \Delta E| \geq |E \Delta E| - |E \Delta E^\dagger| \geq \frac{1}{2}|E \Delta B| - |E \Delta E^\dagger| \geq (\frac{1}{2} - C\lambda^{-1})|E \Delta B| \geq \frac{1}{4}|E \Delta B|
\]

provided that \( \lambda \) is sufficiently large. Thus \( \text{dist}(E^\dagger, E) \geq \frac{1}{4} \text{dist}(E, E) \).

By invoking first (7.24), then Lemma 7.5 we obtain

\[
\| \hat{1}_E \|_q^q \leq \| \hat{1}_{E^\dagger} \|_q^q - c\lambda |E \Delta B| \cdot |E \Delta E^\dagger| + O_\lambda(|E \Delta B|^{2+\varepsilon})
\]

\[
\leq \| \hat{1}_B \|_q^q - \zeta \text{dist}(E^\dagger, E)^2 + O_\lambda(\text{dist}(E^\dagger, E)^{2+\varepsilon})
\]

\[
- c\lambda |E \Delta B| \cdot |E \Delta E^\dagger| + O_\lambda(|E \Delta B|^{2+\varepsilon})
\]

\[
\leq \| \hat{1}_B \|_q^q - \frac{1}{2}\zeta \text{dist}(E, E)^2 + O_\lambda(\text{dist}(E, E)^{2+\varepsilon})
\]

where \( \zeta = \zeta(d, q) \) is a positive constant. \( \square \)

### 7.4. Large exponents near even integers.

**Proposition 7.6.** Let \( d \geq 1 \). Let \( q \geq 4 \) be any even integer. Let \( E \subset \mathbb{R}^d \) be a Lebesgue measurable set satisfying \( 0 < |E| < \infty \). Let \( E^* \subset \mathbb{R}^d \) be the closed ball centered at 0 satisfying \( |E^*| = |E| \). Then

\[
(7.28) \quad \| \hat{1}_E \|_q^q \leq \| \hat{1}_{E^*} \|_q^q.
\]

Moreover, \( \| \hat{1}_E \|_q^q = \| \hat{1}_{E^*} \|_q^q \) if and only if \( E \) is an ellipsoid.
Proof. Let \( q = 2m \). By Plancherel’s Theorem,
\[
(7.29) \quad \| \widehat{1_E} \|_{2m}^2 = \int_{\Lambda(d,m)} \prod_{j=1}^{m} 1_E(x_j) \prod_{k=1}^{m} 1_E(y_k) \, d\lambda(x,y)
\]
where
\[
\Lambda(d,m) = \left\{ (x_1, \ldots, x_m, y_1, \ldots, y_m) \in (\mathbb{R}^d)^{2m} : \sum_{j=1}^{m} x_j = \sum_{k=1}^{m} y_k \right\}
\]
and \( \lambda \) is the natural measure on \( \Lambda(d,m) \) defined by \( d\lambda(x,y) = \prod_{j=1}^{m} dx_j \prod_{k=1}^{m-1} dy_k \), or by various equivalent expressions.

Define
\[
T(f_1, \ldots, f_m, g_1, \ldots, g_m) = \int_{\Lambda(d,m)} \prod_{j=1}^{m} f_j(x_j) \prod_{k=1}^{m} g_k(y_k) \, d\lambda(x,y)
\]
for any nonnegative measurable functions \( f_j, g_k : \mathbb{R}^d \to [0, \infty] \). By the Riesz-Sobolev inequality iterated, or as a special case of an inequality of Brascamp-Lieb [5], if \( |\{ x : f_j(x) > t \}| < \infty \) for all \( t > 0 \) and all indices \( j \), and likewise for all \( g_k \),
\[
T(f_1, \ldots, g_m) \leq T(f_1^*, \ldots, g_m^*)
\]
where \( h^* \) denotes the radial nonincreasing majorant of \( h \). Specializing to \( f_1 = \cdots = g_m = 1_E \) gives \( \| \widehat{1_E} \|_q^2 \leq \| \widehat{1_{E^*}} \|_q^2 \).

By the theorem of Burchard [6], if \( E_j, F_k \) are Lebesgue measurable sets with finite, positive measures such that the vector \( (|E_1|, \ldots, |E_m|, |F_1|, \ldots, |F_m|) \) satisfies a certain admissibility hypothesis, then
\[
T(1_{E_1}, \ldots, 1_{E_m}, 1_{F_1}, \ldots, 1_{F_m}) = T(1_{E_1^*}, \ldots, 1_{E_m^*}, 1_{F_1^*}, \ldots, 1_{F_m^*})
\]
only if each of the sets \( E_j, F_k \) differs from some ellipsoid by a Lebesgue null set. Moreover, this admissibility hypothesis is satisfied if \( |E_j| = |F_k| \in (0, \infty) \) for all indices \( j, k \). Specializing to \( E_j = F_k = E \) for all indices \( j, k \) completes the proof. \( \square \)

Lemma 7.7. Let \( d \geq 1 \). Let \( n \) be a sufficiently large even positive integer. For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any Lebesgue measurable set \( E \subset \mathbb{R}^d \) that satisfies \( |E| = |B| \) and
\[
|\widehat{1_E}|_n^p \geq (1 - \delta)|\widehat{1_B}|_n^p,
\]
there exists an ellipsoid \( E \subset \mathbb{R}^d \) satisfying \( |E| = |E| \) such that
\[
(7.30) \quad |E \Delta E| < \varepsilon.
\]

Proof. If this lemma were not true then there would exist \( \eta > 0 \) and a sequence of sets \( E_k \) satisfying \( |E_k| = |B| \) and \( |\widehat{1_E_k}|_n^p \to |\widehat{1_B}|_n^p \) as \( k \to \infty \), such that for any measure-preserving affine automorphism \( \psi \) of \( \mathbb{R}^d \) and any ellipse \( E \subset \mathbb{R}^d \), \( |\psi(E_k) \Delta E| \geq \eta \) for all \( k \).

By Proposition 7.6 \( |\widehat{1_E}|_n^p \leq |\widehat{1_B}|_n^p \) whenever \( |E| = |B| \). Thus the hypothesis is that \( \Phi_q(E_k) \to \mathcal{A}_{q,d} \). According to Theorem 2.2 the compactness principle to be proved below, there exists a sequence of measure-preserving affine automorphisms \( \phi_k \) of \( \mathbb{R}^d \) such that some subsequence \( (\widehat{E}_{k_j} = \phi_{k_j}(E_{k_j}) : j \in \mathbb{N}) \) converges to a measurable set \( S \subset \mathbb{R}^d \) in the sense that \( |\widehat{E}_{k_j} \Delta S| \to 0 \). This convergence forces
\[
|\widehat{1_S}|_n^p = \lim_{j \to \infty} |\widehat{1_{\phi_{k_j}(E_{k_j})}}|_n^p = \lim_{j \to \infty} |\widehat{1(E_{k_j})}|_n^p = |\widehat{1_B}|_n^p.
\]
By Proposition 7.6 \( S \) must be an ellipse. We have \( |E_{k_j} \Delta S| \to 0 \), contradicting the hypothesis (7.30). \( \square \)
In the next proof, \( r' = r/(r - 1) \) denotes the exponent conjugate to \( r \).

**Proof of Theorem 2.6** Let \( d \geq 1 \). Recall that \( \text{dist}(E, \mathcal{E}) = \inf_E |E|^{-1}|E \Delta \mathcal{E}| \), where \( E \subset \mathbb{R}^d \) and the infimum is taken over all ellipsoids \( E \subset \mathbb{R}^d \) satisfying \( |E| = |E| \). We have shown that for any sufficiently large integer \( n \), there exist \( \delta(n), c > 0 \) such that for all exponents \( q \) sufficiently close to \( n \), for every set \( E \subset \mathbb{R}^d \) satisfying \( |E| = |\mathbb{B}| \) with \( \text{dist}(E, \mathcal{E}) \leq \delta(n) \),

\[
\|1_E\|_q^q \leq \|\mathbf{1}_\mathbb{B}\|_q^q - c \text{dist}(E, \mathcal{E})^2.
\]

By Lemma 7.1 there exists \( \varepsilon > 0 \) such that for any Lebesgue measurable set \( E \subset \mathbb{R}^d \) satisfying \( |E| = |\mathbb{B}| \), if \( \text{dist}(E, \mathcal{E}) \geq \delta(n) \) then

\[
\|1_E\|_q^q \leq \|1_E\|_n^q - \varepsilon = A_{n,d}^n|E|^{n-1} - \varepsilon.
\]

From this together with Corollary 3.3 we conclude that there exists \( \tau > 0 \) such that for any exponent \( q \) satisfying \( |q - n| < \tau \), if \( |E| = |\mathbb{B}| \) and \( \text{dist}(E, \mathcal{E}) \geq \delta(n) \) then

\[
\|1_E\|_q^q \leq A_{q,d}^q|E|^{q-1} - \frac{1}{2}\varepsilon.
\]

This conclusion can be rewritten in the form

\[
\|1_E\|_q^q \leq A_{q,d}^q|E|^{q-1} - c' \text{dist}(E, \mathcal{E})^2,
\]

still assuming that \( \text{dist}(E, \mathcal{E}) \geq \delta(n) \).

Combining this inequality with (7.3) one concludes that whenever \( |E| = |\mathbb{B}| \), \( \|1_E\|_q^q \leq \|1_E\|_q^q = c' \text{dist}(E, \mathcal{E})^2 \) where \( c' = \min(c, c') \). Therefore

\[
\|1_E\|_q^q \leq \|1_E\|_q^q |\mathbb{B}|^{-q(|q-1)-1} - c \text{dist}(E, \mathcal{E})^2 |E|^{q-1}
\]

whenever \( 0 < |E| < \infty \). In particular, \( \|1_E\|_q^q \leq \|1_E\|_q^q |\mathbb{B}|^{-q(|q-1)-1} |E|^{q-1} \), with equality if and only if \( E \) differs from some ellipse by a set of Lebesgue measure zero. \( \square \)

### 8. \( q = 4 \) Analysis for Dimension \( d = 2 \)

Here we prove Theorem 2.10. Let \( d = 2 \) and \( q = 4 \). Define

\[
(8.1) \quad K = K_4 = 1_\mathbb{B} \ast 1_\mathbb{B} \ast 1_\mathbb{B}
\]

\[
(8.2) \quad L = L_4 = 1_\mathbb{B} \ast 1_\mathbb{B}.
\]

These are radially symmetric functions on \( \mathbb{R}^2 \). The same reasoning as employed above for the case of large exponents \( q \) reduces Theorem 2.10 to the following special case.

**Proposition 8.1.** Let \( E \subset \mathbb{R}^2 \) be a Lebesgue measurable set satisfying \( |E| = |\mathbb{B}| \). Suppose that \( E \) is balanced and lies close to \( \mathbb{B} \) in the sense (5.7). Then

\[
\|1_E\|_{L^4}^4 \leq \|1_\mathbb{B}\|_{L^4}^4 - \frac{8}{q^2} |E \Delta \mathbb{B}|^2 + O(|E \Delta \mathbb{B}|^{2+\varepsilon})
\]

where \( \varepsilon > 0 \).

As in the analysis of the case of large exponents \( q \) developed above, the proof of Proposition 8.1 reduces to the estimation of

\[
-\frac{1}{2}q_4 \gamma_{4,2} \int_{S^1} (a^2 + b^2) d\sigma + \frac{1}{4}q_2 \mathcal{Q}(F, F) + \frac{1}{4}q(q - 2) \mathcal{Q}(F, \tilde{F})
\]

\[
= -2\gamma_{4,2} \int_{S^1} (a^2 + b^2) d\sigma + 4\mathcal{Q}(F, F) + 2\mathcal{Q}(F, \tilde{F})
\]

where \( \mathcal{Q} \) is the quadratic form defined in (5.11) and \( \gamma_{4,2} = -x \cdot \nabla K |_{|x|=1} \).
8.1. **Fourier series analysis.** Identify $S^1$ with $[0, 2\pi]$ via $S^1 \ni \alpha = (\cos(\theta), \sin(\theta))$. The quadratic form $Q = Q_{4,2}$ defined in (5.11) becomes

\begin{equation}
Q(F, G) = \int_{S^1 \times S^1} F(\alpha)G(\beta)L(\alpha - \beta) d\sigma(\alpha) d\sigma(\beta) = \langle F \ast \mathcal{L}, G \rangle
\end{equation}

where $\ast$ denotes convolution on $S^1$ and

\begin{equation}
\mathcal{L}(\theta) = L(x) \text{ where } |x| = \sqrt{2 - 2\cos(\theta)}.
\end{equation}

We utilize the Fourier transform, mapping $L^2(S^1)$ to $\ell^2(\mathbb{Z})$, normalized by

\begin{equation}
\hat{h}(n) = (2\pi)^{-1} \int_0^{2\pi} h(\theta)e^{-in\theta} d\theta.
\end{equation}

Thus $\int_0^{2\pi} g \hat{h} = 2\pi \sum_{n=-\infty}^{\infty} \hat{h}(n)\overline{g(n)}$, and $g \ast h = 2\pi \hat{g} \hat{h}$. Thus

\begin{equation}
Q(F, G) = 2\pi \sum_{n \in \mathbb{Z}} \overline{F} \ast \mathcal{L}(n) \overline{G(n)} = 4\pi^2 \sum_{n \in \mathbb{Z}} \hat{F}(n) \overline{G(n)} \hat{\mathcal{L}}(n).
\end{equation}

For any function $F \in L^2(S^1)$,

\begin{equation}
4Q(F, F) + 2Q(F, \tilde{F}) = 4\pi^2 \sum_n (4 + 2(-1)^n)\hat{\mathcal{L}}(n)|\hat{F}(n)|^2.
\end{equation}

Indeed, specializing to $G = F$ in (8.7) gives $Q(F, F) = 4\pi^2 \sum_{n \in \mathbb{Z}} |\hat{F}(n)|^2 \hat{\mathcal{L}}(n)$. The reflection $\tilde{F}(\theta) = F(-\theta)$ of any real-valued function $F \in L^2(S^1)$ has Fourier coefficients $\hat{\tilde{F}}(n) = e^{-in\pi}\hat{F}(n) = (-1)^n \hat{F}(n)$ for all $n$. Setting $G = \tilde{F}$ instead gives $Q(F, \tilde{F}) = 4\pi^2 \sum_n (-1)^n |\hat{F}(n)|^2 \hat{\mathcal{L}}(n)$.

The next two lemmas will be proved in §8.2

**Lemma 8.2.** For any nonzero $n \in \mathbb{Z}$,

\begin{equation}
\hat{\mathcal{L}}(n) = \begin{cases} 
2n^{-2}\pi^{-1} & \text{if } n \text{ is odd} \\
2(n^2 - 1)^{-1}\pi^{-1} & \text{if } n \text{ is even}.
\end{cases}
\end{equation}

**Lemma 8.3.** $\gamma_{4,2} = 4$.

By Lemma 8.2, the numbers $\{4 + 2(-1)^n\} \hat{\mathcal{L}}(n)$ are real and positive, and satisfy

\begin{equation}
\{4 + 2(-1)^n\} \hat{\mathcal{L}}(n) \begin{cases} 
= 4\pi^{-1} & \text{for } n \in \{\pm 1, \pm 2\} \\
\leq \frac{8}{3}\pi^{-1} < 4\pi^{-1} & \text{for } |n| \geq 3.
\end{cases}
\end{equation}

**Lemma 8.4.** Let $d = 2$. Suppose that $E \subset \mathbb{R}^2$ is well approximated by $B$ in the sense (5.1), and that $E$ is balanced. Then

\begin{equation}
4\langle K, f \rangle + 4\langle L \ast f, f \rangle + 2\langle L \ast f, \tilde{f} \rangle \leq -\frac{8}{9}|E \Delta B|^2 + O(|E \Delta B|^{2+\delta}).
\end{equation}
Proof. The hypothesis that $E$ is balanced is equivalent to $\widehat{F}(n) = 0$ for all $n \in \{0, \pm 1, \pm 2\}$. Therefore using the relation $F = a - b$,

\[
4(K, f) + 4(L*, f, f) + 2(L*, f, \widehat{f}) = 4 \cdot (-\frac{1}{2^{4/2}}) \int_{S^1} (a^2 + b^2)(\alpha) \, d\sigma(\alpha) + 4\pi^2 \sum_{n \in \mathbb{Z}} |\widehat{F}(n)|^2 (4 + 2(-1)^n)\widehat{L}(n) + O(|E \Delta \mathbb{B}|)^{2+\varepsilon}
\]

\[
= -8 \int_{S^1} (a^2 + b^2)(\alpha) \, d\sigma(\alpha) + 4\pi^2 \sum_{|n| \geq 3} |\widehat{F}(n)|^2 (4 + 2(-1)^n)\widehat{L}(n) + O(|E \Delta \mathbb{B}|)^{2+\varepsilon}
\]

\[
\leq -8 \int_{S^1} (a^2 + b^2)(\alpha) \, d\sigma(\alpha) + 4\pi^2 \sup_{|n| \geq 3} ((4 + 2(-1)^n)\widehat{L}(n)) \cdot \sum_{|n| \geq 3} |\widehat{F}(n)|^2 + O(|E \Delta \mathbb{B}|)^{2+\varepsilon}
\]

\[
= -8 \int_{S^1} (a^2 + b^2)(\alpha) \, d\sigma(\alpha) + 4\pi^2 \cdot \frac{4}{5}\pi^{-1} \sum_{n \in \mathbb{Z}} |\widehat{F}(n)|^2 + O(|E \Delta \mathbb{B}|)^{2+\varepsilon}
\]

\[
= -8 \int_{S^1} (a^2 + b^2)(\alpha) \, d\sigma(\alpha) + \frac{8}{5} \int_{S^1} (a - b)^2 \, d\sigma(\alpha) + O(|E \Delta \mathbb{B}|)^{2+\varepsilon}
\]

\[
\leq -8 \int_{S^1} (a^2 + b^2)(\alpha) \, d\sigma(\alpha) + \frac{8}{5} \int_{S^1} (a^2 + b^2) \, d\sigma(\alpha) + O(|E \Delta \mathbb{B}|)^{2+\varepsilon}
\]

\[
= -\frac{32}{5} \int_{S^1} (a^2 + b^2)(\alpha) \, d\sigma(\alpha) + O(|E \Delta \mathbb{B}|)^{2+\varepsilon}
\]

\[
= -\frac{8}{5}\pi^{-1}|E \Delta \mathbb{B}|^2 + O(|E \Delta \mathbb{B}|)^{2+\varepsilon}
\]

for a certain constant $\varepsilon > 0$. We have used the trivial inequality $(a - b)^2 = a^2 + b^2 - 2ab \leq a^2 + b^2$, which holds since $a, b$ are by their definitions nonnegative real-valued functions. The final line is obtained from (5.5):

\[
\int_{S^1} (a^2 + b^2) \, d\sigma \geq \frac{1}{2}\sigma(S^1)^{-1}|E \Delta \mathbb{B}|^2 = (4\pi)^{-1}|E \Delta \mathbb{B}|^2.
\]

\[\square\]

One may observe that if $|E| = |\mathbb{B}|$ but $E$ is not assumed to be balanced, then $\sup_{|n| \geq 3} (4 + 2(-1)^n)\widehat{L}(n)$ must be replaced in this calculation by $\sup_{n \neq 0} (4 + 2(-1)^n)\widehat{L}(n)$. With this change, the terms involving $\int_{\mathcal{S}^1} (a^2 + b^2) \, d\sigma$ in the above calculation cancel exactly, and consequently no useful conclusion is reached.

8.2. Two calculations.

Proof of Lemma 8.3. For $d = 2$,

\[
(1_{\mathbb{B}} * 1_{\mathbb{B}})(x) = 4 \int_{|x|/2}^{1} (1 - s^2)^{1/2} \, ds
\]

so

\[
\frac{\partial}{\partial x_1}(1_{\mathbb{B}} * 1_{\mathbb{B}})(x_1, x_2) = -4(1 - |x|^2/4)^{1/2} \cdot \frac{1}{2} \frac{x_1}{|x|} = -2x_1|x|^{-1}(1 - |x|^2/4)^{1/2}
\]
for $|x| < 2$, and this function vanishes for $|x| > 2$. Therefore

$$\frac{\partial}{\partial x_1}(1_B * 1_B * 1_B)(1, 0) = \left(\frac{\partial}{\partial x_1}(1_B * 1_B) * 1_B\right)(1, 0) = -2 \int_{|y-(1,0)| \leq 1} y_1 |y|^{-1}(1 - |y|^2/4)^{1/2} dy.$$ 

In polar coordinates $y = (r \cos(\theta), r \sin(\theta))$, the domain of integration is

$$\{y : (y_1 - 1)^2 + y_2^2 \leq 1\} = \{(r, \theta) : 0 \leq r \leq 2 \cos(\theta) \text{ and } |\theta| \leq \pi/2\}$$

and the integral becomes

$$-2 \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos(\theta)} \cos(\theta)(1 - \frac{1}{4} r^2)^{1/2} r dr d\theta = 2 \int_{-\pi/2}^{\pi/2} \cos(\theta) \cdot \frac{1}{3} (1 - \frac{1}{4} r^2)^{3/2} |\sin(\theta)| d\theta$$

$$= \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos(\theta) (|\sin(\theta)|^3 - 1) d\theta = \frac{16}{3} (\frac{1}{2} \sin^4(\theta) - 1) d\theta = \frac{16}{3} \cdot (\frac{1}{4} - 1) = -4.$$

\[\square\]

**Proof of Lemma 8.2.** The Fourier coefficients of $\mathcal{L}$ are

\[(8.12) \quad \hat{\mathcal{L}}(n) = (2\pi)^{-1} \int_0^{2\pi} L^*(\sqrt{2 - 2 \cos(\theta)}) e^{-i n \theta} d\theta = (2\pi)^{-1} \int_0^{2\pi} L^*(\sqrt{2 - 2 \cos(\theta)}) \cos(n \theta) d\theta\]

since the $2\pi$–periodic function $\theta \mapsto L^*(\sqrt{2 - 2 \cos(\theta)})$ is even. Assume that $n \neq 0$. Integrate by parts to obtain

$$2\pi \hat{\mathcal{L}}(n) = -n^{-1} \int_{-\pi}^{\pi} \sin(n \theta) \frac{d}{d\theta}(L^*(\sqrt{2 - 2 \cos(\theta))) d\theta$$

$$= -n^{-1} \int_{-\pi}^{\pi} \sin(n \theta) \left( \frac{d}{d\theta} L^*(\sqrt{2 - 2 \cos(\theta))) \cdot \sin(\theta) \cdot (2 - 2 \cos(\theta))^{-1/2} d\theta$$

$$= -n^{-1} \int_{-\pi}^{\pi} \sin(n \theta) \left( - \omega_1 (1 - \frac{1}{4} (2 - 2 \cos(\theta))^{2/2})^{1/2} \cdot \sin(\theta) \cdot (2 - 2 \cos(\theta))^{-1/2} d\theta$$

$$= n^{-1} \int_{-\pi}^{\pi} \sin(n \theta) (1 + \cos(\theta))^{1/2} \cdot \sin(\theta) \cdot (1 - \cos(\theta))^{-1/2} d\theta$$

$$= n^{-1} \int_{-\pi}^{\pi} \sin(n \theta) (1 + \cos(\theta)) \cdot \sin(\theta) \cdot (1 - \cos(\theta))^{-1/2} d\theta$$

$$= 2n^{-1} \int_{0}^{\pi} \sin(n \theta) (1 + \cos(\theta)) \cdot \sin(\theta) \cdot (1 - \cos(\theta))^{-1/2} d\theta$$

$$= 2n^{-1} \int_{0}^{\pi} \sin(n \theta) (1 + \cos(\theta)) d\theta.$$
The integral is an odd function of \( n \) and splits naturally as a sum of two terms. For the first of these,

\[
\int_{0}^{\pi} \sin(n\theta) \, d\theta = \begin{cases} 
0 & \text{if } n \text{ is even} \\
2n^{-1} & \text{if } n \text{ is odd}. 
\end{cases}
\]

The second, for \( n > 0 \), is

\[
\int_{0}^{\pi} \sin(n\theta) \cos(\theta) \, d\theta = (4i)^{-1} \int_{0}^{\pi} (e^{in\theta} - e^{-in\theta})(e^{i\theta} + e^{-i\theta}) \, d\theta
\]

\[
= \begin{cases} 
0 & \text{if } n \text{ is odd} \\
2n(n^2 - 1)^{-1} & \text{if } n \text{ is even}. 
\end{cases}
\]

\[
\square
\]

9. ON DIMENSION \( d = 1 \)

Now \( S^{d-1} = S^0 = \{ \pm 1 \} \), and \( \sigma(S^0) = 2 \). If \( E \) is balanced then \( F = b - a \) on \( S^0 \) is orthogonal in \( L^2(S^0) \) to \( x \) and to constant functions. Thus if \( E \) is balanced then

\[
F \equiv 0.
\]

Equivalently, \( b \equiv a \).

The exponent \( q_d = 4 - \frac{2}{d+1} \) equals 3 for \( d = 1 \), so the range \((q_d, \infty)\) in which results for general dimensions were established specializes to \((3, \infty)\).

Together with results shown for general dimensions above, for balanced sets \( E \subset \mathbb{R}^1 \) this establishes the following simplified comparison of \( \| \hat{1}_E \|_q^q \) to \( \| \hat{1}_B \|_q^q \), in which no quadratic terms in \( f = 1_E - 1_B \) appear.

**Lemma 9.1.** Let \( d = 1 \). For each \( q \in (3, \infty) \) there exists \( \varrho > 0 \) such that for any bounded, balanced set \( E \subset \mathbb{R}^1 \) that satisfies \( |E| = |B| \),

\[
\| \hat{1}_E \|_q^q = \| \hat{1}_B \|_q^q + q(K_q, 1_E - 1_B) + O(|E \Delta B|^{2+\varrho}).
\]

Another simplification is the availability of a simple expression for \( \hat{1}_B \). Since

\[
\hat{1}_B(\xi) = \int_{-1}^{1} e^{-2\pi ix} \, dx = (-2\pi i)^{-1} \xi^{-1} (e^{-2\pi i \xi} - e^{2\pi i \xi}) = \frac{\sin(2\pi \xi)}{\pi \xi},
\]

one has

\[
K_q(x) = \pi^{-(q-1)} \int_{-\infty}^{\infty} e^{2\pi i x \xi} \sin(2\pi \xi) \xi^{-1} \sin(2\pi \xi) |\xi|^{-2-q} \, d\xi
\]

\[
= \pi^{-(q-1)} \int_{-\infty}^{\infty} \cos(2\pi x \xi) \xi^{-1} \sin(2\pi \xi) \xi^{-1} \sin(2\pi \xi) |\xi|^{-2-q} \, d\xi
\]

and

\[
\gamma_{1,q} = -\frac{dK_q}{dx} \bigg|_{x=1} = 2\pi^{2-q} \int_{-\infty}^{\infty} \xi^{2-q} \sin(2\pi \xi) |\xi|^{-q} \, d\xi > 0
\]

for all \( q \in (2, \infty) \). Moreover, if \( q \geq 4 \) is an even integer then \( K_q \), the convolution product of \( q-1 \) factors of \( 1_B \), is an even function that is strictly decreasing on the interval \([0, q-1] \), and vanishes identically for \( |x| > q - 1 \). In particular, \((2.13)\) holds for any even integer \( q \geq 4 \).
Since $F \equiv 0$, $\langle L_q \ast f, f \rangle = O(|E \Delta B|)^{2+\epsilon}$ and likewise $\langle L_q \ast f, \tilde{f} \rangle = O(|E \Delta B|)^{2+\epsilon}$ where $\epsilon > 0$. Therefore for any $q \in \{4, 6, 8, \ldots \}$,

$$\|1_E\|^q_q = \|1_B\|^q_q - \frac{1}{q} q \gamma_1 q \int_{S^n} (a^2 + b^2) d\sigma(a) + O(|E \Delta B|)^{2+\epsilon} \leq \|1_B\|^q_q - \frac{1}{q} q \gamma_1 q |E \Delta B|^2 + O(|E \Delta B|)^{2+\epsilon}.$$

Theorem 2.9 follows from Lemma 9.1 by results and reasoning shown in earlier sections. $\square$

Proposition 2.8 is a consequence of Lemma 9.1 and these formulas. $\square$

10. Precompactness of extremizing sequences

We turn to the proof of the compactness principle, Theorem 2.2. The method used is that of [12].

10.1. Notation. By a measurable partition of a Lebesgue measurable set $E$ we mean a pair of Lebesgue measurable subsets $A, B \subset E$ satisfying $A \cup B = E$ and $A \cap B = \emptyset$ up to Lebesgue null sets. We say that two Lebesgue measurable functions $g, h$ are disjointly supported if their product vanishes almost everywhere.

$\|y\|_{\mathbb{R} / \mathbb{Z}}$ will denote the distance from $y \in \mathbb{R}$ to $\mathbb{Z}$. For $d > 1$, if $x = (x_1, \ldots, x_d)$,

$$\|x\|_{\mathbb{R}^d / \mathbb{Z}^d} = \max_{1 \leq j \leq d} \|x_j\|_{\mathbb{R} / \mathbb{Z}}.$$ There is a triangle inequality $\|x + y\|_{\mathbb{R}^d / \mathbb{Z}^d} \leq \|x\|_{\mathbb{R}^d / \mathbb{Z}^d} + \|y\|_{\mathbb{R}^d / \mathbb{Z}^d}$.

$O_\delta(1)$ denotes a quantity whose absolute value is less than or equal to a finite constant that depends only on $\delta$ and on the dimension $d$, or sometimes on the exponent $q$ under discussion, as well. Similarly, $o_\delta(1)$ denotes a quantity that tends to zero as $\delta \to 0$, at a rate that may depend on $d$ or on both $d$ and $q$. All bounds are implicitly asserted to be uniform for $q$ in any compact subset of $(2, \infty)$.

$\text{Gl}(d)$ denotes the group of all invertible linear transformations $T : \mathbb{R}^d \to \mathbb{R}^d$. $\text{Aff}(d)$ denotes the group of all affine automorphisms of $\mathbb{R}^d$, that is, all maps $x \mapsto \mathcal{T}(x) = T(x) + v$ where $v \in \mathbb{R}^d$ and $T \in \text{Gl}(d)$. $\det(T)$ denotes the determinant of $T$, and $|J(\mathcal{T})|$ denotes the Jacobian determinant of $\mathcal{T}$, which is equal to $|\det(T)|$.

$|S|$ denotes the Lebesgue measure of a subset $S$ of a Euclidean space of arbitrary dimension. Most frequently $S$ will be a subset of $\mathbb{R}^d$, but other spaces, such as $\mathbb{R}^{d^2}$ and $\mathbb{R}^{d(d-1)}$, will also arise. $\#(S)$ denotes the cardinality of a finite set $S$, and equals $\infty$ if $S$ is an infinite set.

$\mathbb{Q}^d$ denotes the unit cube in $\mathbb{R}^d$, the set of all $x = (x_1, \ldots, x_d)$ satisfying $0 \leq x_j \leq 1$ for every $1 \leq j \leq d$. $\mathbb{T}$ is the quotient group $\mathbb{T} = \mathbb{R} / \mathbb{Z}$.

Definition 10.1. A discrete multiprogression $\mathbf{P}$ in $\mathbb{R}^d$ of rank $r$ is a function

$$\mathbf{P} : \prod_{i=1}^r \{0, 1, \ldots, N_i - 1\} \to \mathbb{R}^d$$

of the form

$$\mathbf{P}(n_1, \ldots, n_r) = \{a + \sum_{i=1}^r n_i v_i : 0 \leq n_i < N_i\},$$

for some $a \in \mathbb{R}^d$, some $v_j \in \mathbb{R}^d$, and some positive integers $N_1, \ldots, N_r$. The size of $\mathbf{P}$ is $\sigma(\mathbf{P}) = \prod_{i=1}^r N_i$. The multiprogression $\mathbf{P}$ is said to be proper if this mapping is injective.
A continuum multiprogression $P$ in $\mathbb{R}^d$ of rank $r$ is a function

$$P : \prod_{i=1}^{r} \{0, 1, \ldots, N_i - 1\} \times \mathbb{Q}^d \to \mathbb{R}^d$$

of the form

$$(n_1, \ldots, n_d; y) \mapsto a + \sum_{i} n_i v_i + sy$$

where $a, v_i \in \mathbb{R}^d$ and $s \in \mathbb{R}^+$. The size of $P$ is defined to be

$$\sigma(P) = s^d \prod_{i} N_i.$$

$P$ is said to be proper if this mapping is injective.

We will loosely identify a multiprogression with its range, and will thus refer to multiprogressions as if they were sets rather than functions. If $P$ is proper then the Lebesgue measure of its range equals its size. A more invariant definition would replace $\mathbb{Q}^d$ by an arbitrary convex set of positive and finite Lebesgue measure. The above definition is equivalent for our purposes, according to a theorem of John, and is more convenient.

10.2. Generalities concerning norm inequalities.

**Lemma 10.1.** For each $d \geq 1$ and $q \in (2, \infty)$ there exist $\delta_0, c, C' \in \mathbb{R}^+$ with the following property. Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set satisfying $0 < |E| < \infty$. Let $\delta \in (0, \delta_0]$. Suppose that $\|\hat{1}_E\|_q \geq (1 - \delta)A_{q,d}|E|^{1/q'}$. For any measurable subset $A \subset E$ satisfying $|A| \geq C'\delta|E|$,  

$$\|\hat{1}_A\|_q \geq c\delta^{1/q}A_{q,d}|A|^{1/q'}.$$  

**Proof.** Let $p = q'$. We may assume without loss of generality that $|E| = 1$, by the scaling symmetry of the inequality. Set $B = E \setminus A$.

There exists $C < \infty$ such that for arbitrary functions $G, H \in L^q$,  

$$\|G + H\|_q^q \leq \|G\|_q^q + C\|G\|_q^{q-1}\|H\|_q + C\|H\|_q^q.$$

Consequently

$$\|\hat{1}_A + \hat{1}_B\|_q^q \leq \|\hat{1}_B\|_q^q + C\|\hat{1}_B\|_q^{q-1}\|\hat{1}_A\|_q + C\|\hat{1}_A\|_q^q \leq A_{q,d}|B|^{q/p} + CA_{q,d}^{q-1}|B|^{(q-1)/p}\|\hat{1}_A\|_q + C\|\hat{1}_A\|_q^q.$$

Set $y = |A|^{1/p} \in (0, 1]$ and $x = \frac{\|\hat{1}_A\|_q}{A_{q,d}|A|^{1/p}} \in [0, 1]$. Then $|B|^{1/p} = (1 - y^p)^{1/p} \leq 1 - c_p y^p$ for a certain constant $c_p > 0$.

Since $|E| = 1$, the hypothesis $\|\hat{1}_E\|_q \geq (1 - \delta)A_{q,d}|E|^{1/p}$ can be rewritten as $\|\hat{1}_A + \hat{1}_B\|_q \geq (1 - \delta)A_{q,d}$. Together with the upper bound derived above, this gives  

$$(1 - \delta)^q \leq (1 - y^p)^{q/p} + C(1 - y^p)^{(q-1)/p}xy + Cx^q y^q \leq 1 - c_p y^p + Cxy$$

since $(1 - y^p) \leq 1$ and $x^q y^q \leq xy$. Therefore $x \geq cy^{p-1} - C\delta y^{-1} \geq c\delta^{(p-1)/p}$ provided that $|A| \geq C'\delta$ for a sufficiently large constant $C'$. Inserting the definition of $x$, this inequality becomes

$$\|\hat{1}_A\|_q \geq c\delta^{(p-1)/p}A_{q,d}|A|^{1/p} = c\delta^{1/q}A_{q,d}|A|^{1/q'}.$$  

$\square$
We also require the corresponding statement concerning the dual inequality. This is the inequality
\begin{equation}
\|\hat{f}\|_{L^{q',\infty}} \leq A_{q,d} \|f\|_{L^{q'}}
\end{equation}
where $A_{q,d}$ is the same constant as above. Recall that we use for the Lorentz space $L^{q,\infty}$ the norm
\begin{equation}
\|g\|_{L^{q,\infty}} = \|g\|_{q,\infty} = \sup_E \left| \int_E g \right| \cdot |E|^{-1/q'},
\end{equation}
with the supremum taken over all Lebesgue measurable sets satisfying $0 < |E| < \infty$. Since $L^q$ is embedded continuously in $L^{q,\infty}$, the Fourier transform is bounded from $L^{q'}$ to $L^{q,\infty}$ for each $q \in (2, \infty)$. The norm that we have chosen for $L^{q,\infty}$ is the one naturally associated with the duality between $L^{q,\infty}$ and the Lorentz space $L^{q,1}$, so that the optimal constant in the inequality $\|\hat{f}\|_{L^{q,\infty}(\mathbb{R}^d)} \leq C \|f\|_{L^{q'}(\mathbb{R}^d)}$ is $C = A_{q,d}$.

**Lemma 10.2.** For each $d \geq 1$ and $q \in (2, \infty)$ there exist $\delta_0, c, C' \in \mathbb{R}^+$ with the following property. Let $f = g + h$ where $f, g, h \in L^{q'}(\mathbb{R}^d)$ and $g, h$ are disjointly supported. Let $\delta \in (0, \delta_0]$. Suppose that $\|\hat{f}\|_{q,\infty} \geq (1 - \delta)A_{q,d}\|f\|_{q'}$. If $\|h\|_{q'} \geq C'\delta \|f\|_{q'}$ then
\begin{equation}
\|\hat{h}\|_{q,\infty} \geq c\delta^{q'/q}A_{q,d}\|h\|_{q'}.
\end{equation}

**Proof.** We may assume without loss of generality that $\|f\|_{q'} = 1$. Set $p = q'$. Let $A, B$ be disjoint measurable sets such that $g = 1_A g$ and $h = 1_B h$ almost everywhere. By definition of the $L^{q,\infty}$ norm, there exists a Lebesgue measurable set $E \subset \mathbb{R}^d$ with $0 < |E| < \infty$ satisfying $\int_E \hat{f} \geq (1 - 2\delta)A_{q,d}|E|^{1/p}$. Then
\begin{align*}
\int_E \hat{h} &\geq \int_E \hat{f} - \int_E \hat{g} \\
&\geq (1 - 2\delta)A_{q,d}|E|^{1/p} - A_{q,d}|E|^{1/p}\|g\|_p \\
&= A_{q,d}|E|^{1/p}((1 - 2\delta) - (1 - \|h\|_p^{1/p})) \\
&= A_{q,d}|E|^{1/p}(p^{-1}\|h\|_p^p - 2\delta + O(\|h\|_p^{2p})).
\end{align*}
If $\|h\|_p^p \geq 4p\delta$, and if $\delta$ is sufficiently small, we conclude that
\begin{equation}
\int_E \hat{h} \geq cA_{q,d}|E|^{1/p}\|h\|_p^p \geq cA_{q,d}|E|^{1/p}\delta^{p/q}\|h\|_p.
\end{equation}
Since $\|\hat{h}\|_{q,\infty} \geq |E|^{-1/p}\int_E \hat{h}$ by definition of this norm, this completes the proof. \hfill \Box

**Lemma 10.3.** For each $d \geq 1$ and $q \in (2, \infty)$ there exist $\delta_0, c', C' \in \mathbb{R}^+$ with the following property. Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set satisfying $0 < |E| < \infty$. Let $\delta \in (0, \delta_0]$. Suppose that $\|\hat{1}_E\|_{q} \geq (1 - \delta)A_{q,d}|E|^{1/q'}$. Suppose that $E = A \cup B$ where $A \cap B = \emptyset$, and $A, B$ are Lebesgue measurable. If $\min(|A|, |B|) \geq C'\delta |E|$ then
\begin{equation}
\|\hat{1}_A \hat{1}_B\|_{q/2} \geq cA_{q,d}^2|E|^{2/q'}.
\end{equation}

**Proof.** Set $p = q'$.\[
\|\hat{1}_E\|_{q}^2 = \|\hat{1}_A + \hat{1}_B\|_{q}^2 \\
= \|\hat{1}_A + \hat{1}_B\|_{q}^2 \\
\leq \|\hat{1}_A\|_{q}^2 + \|\hat{1}_B\|_{q}^2 + 2\|\hat{1}_A \hat{1}_B\|_{q/2} \\
\leq A_{q,d}^2|A|^{2/p} + A_{q,d}^2|B|^{2/p} + 2\|\hat{1}_A \hat{1}_B\|_{q/2}.
\]
Now setting $\rho = 2p^{-1} - 1 > 0$,
\begin{align*}
|A|^{2/p} + |B|^{2/p} &\leq \max(|A|, |B|)^\rho(|A| + |B|) \\
&= \max(|A|, |B|)^\rho|E| \\
&= \max(|A||E|, |B||E|)^\rho|E|^{2/p} \\
&\leq (1 - C'\delta)^\rho|E|^{2/p} \\
&\leq (1 - C'c\delta)|E|^{2/p}.
\end{align*}
Therefore
\[
\|\widehat{1_A 1_B}\|_{q/2} \geq \frac{1}{2} \mathcal{A}_{q,d}^2 \left( (1 - \delta)^2 - (1 - C'c\delta) \right)|E|^{2/p}.
\]
If $C'$ is sufficiently large then $(1 - \delta)^2 - (1 - C'c\delta) \geq 2\delta$ for all sufficiently small $\delta > 0$. □

**Lemma 10.4.** For each $d \geq 1$ and $q \in (2, \infty)$ there exist $\delta_0, c, C_0 \in \mathbb{R}^+$ with the following property. Let $f = g + h$ where $f, g, h \in L^q(\mathbb{R}^d)$ and $g, h$ are disjointly supported. Let $\delta \in (0, \delta_0]$. Suppose that $\|\hat{f}\|_{q, \infty} \geq (1 - \delta)\mathcal{A}_{q,d}\|f\|_{q'}$. If $\min(\|g\|_{q'}, \|h\|_{q'}) \geq C_0\delta^{1/q'}\|f\|_{q'}$ then
\begin{equation}
\|\hat{g}\hat{h}\|_{q, \infty} \geq c\delta^{3/2}\mathcal{A}_{q,d}\|f\|_{q'}.
\end{equation}

**Proof.** Choose some Lebesgue measurable set $E \subset \mathbb{R}^d$ with $0 < |E| < \infty$ satisfying $\int_E \hat{f} \geq (1 - 2\delta)\mathcal{A}_{q,d}\|f\|_{q'}|E|^{1/q'}$. Define
\begin{align*}
E_g &= \{ \xi \in E : |\hat{g}(\xi)| \geq \delta^{-1}|\hat{h}(\xi)| \}, \\
E_h &= \{ \xi \in E : |\hat{h}(\xi)| \geq \delta^{-1}|\hat{g}(\xi)| \}, \\
E_0 &= E \setminus (E_g \cup E_h).
\end{align*}
Then
\[
\left| \int_E \hat{f} \right| \leq (1 + \delta)\mathcal{A}_{q,d}\left| \int_{E_g} \hat{g} \right| + (1 + \delta)\mathcal{A}_{q,d}\left| \int_{E_h} \hat{h} \right| + \delta^{-1/2} \int_{E_0} |\hat{g}\hat{h}|^{1/2}
\leq (1 + \delta)\mathcal{A}_{q,d}|E_g|^{1/q'}\|g\|_{q'} + (1 + \delta)\mathcal{A}_{q,d}|E_h|^{1/q'}\|h\|_{q'} + \delta^{-1/2} \int_E |\hat{g}\hat{h}|^{1/2}
\]
so that
\[
\int_E |\hat{g}\hat{h}|^{1/2} \geq \delta^{1/2} \mathcal{A}_{q,d} \left( (1 - 2\delta)\|f\|_{q'}|E|^{1/q'} - (1 + \delta)|E_g|^{1/q'}\|g\|_{q'} - (1 + \delta)|E_h|^{1/q'}\|h\|_{q'} \right).
\]
By Hölder’s inequality,
\[
|E_g|^{1/q'}\|g\|_{q'} + |E_h|^{1/q'}\|h\|_{q'} \leq |E_g| + |E_h|^{1/q'} \left( \|g\|_{q'} + \|h\|_{q'}^{q'} \right)^{1/q}
\leq |E|^{1/q'}\|f\|_{q'}^{q'/q} \max \left( \|g\|_{q'}, \|h\|_{q'} \right)^{(q-q')/q}
\leq |E|^{1/q'}\|f\|_{q}(1 - C''\delta)
\]
where $C'' = cC_0^{q'/q}$, since
\[
\max \left( \|g\|_{q'}, \|h\|_{q'} \right)^{q'} = \|f\|_{q'}^{q'} - \min(\|g\|_{q'}, \|h\|_{q'}) \leq \|f\|_{q'}^{q'} - C_0^{q'}\delta\|f\|_{q'}^{q'}.
\]
by the hypothesis that \( \min(\|g\|_{q'}, \|h\|_{q'}) \geq C_0 \delta \|f\|_{q'}. \) Therefore

\[
\int_E |\hat{g}\hat{h}|^{1/2} \geq \delta^{1/2} A_{q,d} \|f\|_{q'} |E|^{1/q'} \left( (1 - 2\delta) - (1 + \delta)(1 - C'\delta) \right)
\]

\[
\geq \delta^{1/2} A_{q,d} \|f\|_{q'} |E|^{1/q'} (C'\delta - 2\delta - \delta)
\]

\[
\geq \delta^{3/2} A_{q,d} \|f\|_{q'} |E|^{1/q'}
\]

provided that \( C_0 \) is sufficiently large to ensure that \( C' \geq 4. \) \( \square \)

### 10.3. Compatibility of multiprogressions

The next result is Lemma 5.5 of \[12\], to which we refer for a proof.

**Lemma 10.5.** Let \( d \geq 1 \) and let \( \Lambda \subset (2, \infty) \) be compact. Let \( \lambda > 0 \) and \( R < \infty. \) There exists \( C < \infty \), depending only on \( R, d, \Lambda, \lambda \), with the following property for all \( q \in \Lambda. \) Let \( P, Q \subset \mathbb{R}^d \) be nonempty proper continuum multiprogressions of ranks \( \leq R. \) Let \( \varphi, \psi \) be functions supported on \( P, Q \) respectively that satisfy \( \|\varphi\|_\infty |P|^{1/q'} \leq 1 \) and \( \|\psi\|_\infty |Q|^{1/q'} \leq 1. \) If

\[
(10.8) \quad \| \hat{\varphi} \hat{\psi} \|_{1/2} \geq \lambda
\]

then

\[
(10.9) \quad |P + Q| \leq C \min(|P|, |Q|).
\]

If \( \| \hat{\varphi} \hat{\psi} \|_{1/2} \geq \lambda \) then the hypothesis of Lemma \[10.5\] is satisfied, since a constant multiple of the \( L^q \) norm majorizes the \( L^{q, \infty} \) norm. Both situations arise below in the analysis.

### 10.4. Quasi-extremizers

**Lemma 10.6.** Let \( d \geq 1 \), let \( \Lambda \subset (2, \infty) \) be compact, and let \( \eta \in (0, 1] \). There exist \( C_\eta, c_\eta \in \mathbb{R}^+ \) with the following property for all \( q \in \Lambda. \) Suppose that \( E \subset \mathbb{R}^d \) is a Lebesgue measurable set with \( |E| \in \mathbb{R}^+ \) satisfying \( \|1_E\|_q \geq \eta |E|^{1/q'}. \) Then there exists a proper continuum multiprogression \( P \) satisfying

\[
|P \cap E| \geq c_\eta |E|,
\]

\[
|P| \leq C_\eta |E|,
\]

\[
\text{rank}(P) \leq C_\eta.
\]

**Lemma 10.7.** Let \( d \geq 1 \), let \( \Lambda \subset (2, \infty) \) be compact, and let \( \eta \in (0, 1] \). There exist \( C_\eta, c_\eta \in \mathbb{R}^+ \) with the following property for all \( q \in \Lambda. \) Suppose that \( 0 \neq f \in L^q(\mathbb{R}^d) \) satisfies \( \|f\|_{q, \infty} \geq \eta \|f\|_{q'}. \) Then there exist a proper continuum multiprogression \( P \) and a disjointly supported Lebesgue measurable decomposition \( f = g + h \) such that

\[
g \text{ is supported on } P,
\]

\[
\|g\|_{q'} \geq c_\eta \|f\|_{q'},
\]

\[
\|g\|_{\infty} |P|^{1/q'} \leq C_\eta \|f\|_{q'},
\]

\[
\text{rank}(P) \leq C_\eta.
\]

**Proofs of Lemmas.** These two lemmas are simple consequences of Proposition 6.4 of \[12\], in which a function \( f \in L^d \) is assumed to satisfy \( \|\hat{f}\|_q \geq \eta \|f\|_{q'}. \) and the conclusions are the
same as those of Lemma 10.7. Lemma 10.7 follows directly from that result; its hypotheses are stronger than required since \( \| \hat{f} \|_{q, \infty} \leq C_q \| \hat{f} \|_q \).

Lemma 10.6 is obtained by specializing Proposition 6.4 of [12] to the case \( f = 1_E \). In the resulting disjointly supported decomposition \( f = g + h \), the summands \( g, h \) are necessarily indicator functions of disjoint sets. Thus the conclusions of the Proposition are those of Lemma 10.6.

10.5. Structure of near-extremizers.

**Lemma 10.8** (Multiprogression structure of near-extremizers). Let \( d \geq 1 \), and let \( \Lambda \subset (2, \infty) \) be compact. For any \( \varepsilon > 0 \) there exist \( \delta > 0 \) and \( C_\varepsilon < \infty \) with the following property for all \( q \in \Lambda \). Let \( E \subset \mathbb{R}^d \) be a Lebesgue measurable set with \( |E| \in \mathbb{R}^+ \) that satisfies \( \| \hat{1}_E \|_q \geq (1 - \delta) A_{q,d} |E|^{1/q'} \). There exist a measurable partition \( E = A \cup B \) and a continuum multiprogression \( P \) satisfying

\[
|B| \leq \varepsilon |A| \\
A \subset P \\
|P| \leq C_\varepsilon |E| \\
\text{rank}(P) \leq C_\varepsilon.
\]

**Lemma 10.9** (Multiprogression structure of near-extremizers). Let \( d \geq 1 \), and let \( \Lambda \subset (2, \infty) \) be compact. For any \( \varepsilon > 0 \) there exist \( \delta > 0 \) and \( C_\varepsilon < \infty \) with the following property for all \( q \in \Lambda \). Let \( 0 \neq f \in L^{q'}(\mathbb{R}^d) \) satisfy \( \| \hat{f} \|_{q, \infty} \geq (1 - \delta) A_{q,d} \| f \|_q \). There exist a disjointly supported Lebesgue measurable decomposition \( f = g + h \) and a continuum multiprogression \( P \) satisfying

\[
\| h \|_{q'} < \varepsilon \| f \|_{q'} \\
\| g \|_{\infty} |P|^{1/q'} \leq C_\varepsilon \| f \|_{q'} \\
g \text{ is supported on } P \\
\text{rank}(P) \leq C_\varepsilon.
\]

**Proofs.** These two lemmas are analogues of Lemma 7.3 of [12]. They are consequences of the lemmas of the preceding three subsections, by the same reasoning used in [12] to deduce that Lemma 7.3 from corresponding ingredients. Therefore the details are omitted.

10.6. Discrete and hybrid groups. In [10.7] we will gain information about near-extremizers by lifting sets and operators from \( \mathbb{R}^d \) to \( \mathbb{Z}^d \times \mathbb{R}^d \). In the present subsection we establish basic facts about near-extremizers of the corresponding Fourier transform inequality in this product setting.

We use the same notation \( \hat{\cdot} \) to denote the Fourier transform for each of the groups \( G = \mathbb{Z}^\kappa \times \mathbb{R}^d \), mapping functions on \( G \) to functions on the dual group \( \widehat{G} = \mathbb{T}^\kappa \times \mathbb{R}^d \) by

\[
\hat{f}(\theta, \xi) = \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}^\kappa} e^{-2\pi i n \cdot \xi} e^{-2\pi i n \cdot \theta} f(n, x) \, dx
\]

where \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \). Integration on \( \mathbb{Z}^\kappa \times \mathbb{R}^d \) will always be with respect to the product of Lebesgue measure with counting measure; likewise integration on \( \mathbb{T}^\kappa \times \mathbb{R}^d \) is with respect to the natural Lebesgue measure.
Lemma 10.10. Let \( d, \kappa \geq 1 \), and \( q \in (2, \infty) \). The optimal constant \( A(q, d, \kappa) \) in the inequality

\[
\| \hat{1}_E \|_q \leq A(q, d, \kappa) |E|^{1/q'}
\]

for \( \mathbb{Z}^\kappa \times \mathbb{R}^d \) satisfies

\[
A(q, d, \kappa) = A_{q,d}.
\]

The optimal constant \( A'(q, d, \kappa) \) in the inequality

\[
\| \hat{f} \|_{q,\infty} \leq A'(q, d, \kappa) \| f \|_q
\]

for \( \mathbb{Z}^\kappa \times \mathbb{R}^d \) is likewise equal to \( A_{q,d} \).

The proof requires some additional notation. Let \( q \in [2, \infty] \), and let \( p \in [1, 2] \) be the conjugate exponent. The product Fourier transform can be expressed as a composition \( \mathcal{F} \circ \hat{F} \) of commuting operators, with

\[
\mathcal{F} g(\theta, \xi) = \sum_{n \in \mathbb{Z}^\kappa} g(n, \xi) e^{-2\pi i n \cdot \theta}
\]

\[
\hat{F} f(n, \xi) = \int_{\mathbb{R}^d} f(n, x) e^{-2\pi i x \cdot \xi} \, dx.
\]

Here \( f, g : \mathbb{Z}^\kappa \times \mathbb{R}^d \to \mathbb{C} \). Thus \( \mathcal{F} \) is a partial Fourier transform in the first variable, while \( \hat{F} \) is a partial Fourier transform in the second.

There are corresponding partial Fourier transforms which act in the reverse order; one maps a function of \((n, x)\) to a function of \((\theta, x)\) while another maps a function of \((\theta, x)\) to a function of \((\theta, \xi)\). We will always work in \( \mathbb{Z}^\kappa \times \mathbb{R}^d \) or \( \mathbb{T}^\kappa \times \mathbb{R}^d \), with the coordinate for \( \mathbb{Z}^\kappa \) or \( \mathbb{T}^\kappa \) regarded as the first coordinate and that for \( \mathbb{R}^d \) as the second coordinate. We will denote these corresponding partial Fourier transforms by \( \mathcal{F} \) (partial Fourier transform with respect to the first variable) and \( \hat{F} \) (partial Fourier transform with respect to the second variable) respectively, despite the ambiguity in notation. These operators commute in the sense that

\[
\mathcal{F} \circ \hat{F} = \hat{F} \circ \mathcal{F},
\]

although the operators on the left-hand side of this equation are not exactly identical to the corresponding ones on the right with the same symbols. It will be clear from context which operators are meant in the discussion below if one bears in mind that \( \mathcal{F} \) acts with respect to the first coordinate, and \( \hat{F} \) with respect to the second.

The norms for \( L^p_L L^q_\xi (\mathbb{Z}^\kappa \times \mathbb{R}^d_\xi) \) and \( L^q_\xi L^p_\kappa (\mathbb{Z}^\kappa \times \mathbb{R}^d_\xi) \) for a function \( g : \mathbb{Z}^\kappa \times \mathbb{R}^d \to \mathbb{C} \) are defined respectively by

\[
\| g \|_{L^p_L L^q_\xi} = \left( \sum_n \left( \int |g(n, \xi)|^q \, d\xi \right)^{p/q} \right)^{1/q}
\]

\[
\| g \|_{L^q_\xi L^p_\kappa} = \left( \int \left( \sum_n |g(n, \xi)|^p \right)^{q/p} \, d\xi \right)^{1/p}.
\]

There are corresponding norms for \( L^p_L L^q_\theta (\mathbb{T}^\kappa \times \mathbb{R}^d_\theta) \) and for \( L^q_\theta L^p_x (\mathbb{T}^\kappa \times \mathbb{R}^d_x) \).

Since \( q \geq p \), \( L^p_L L^q_\theta (\mathbb{T}^\kappa \times \mathbb{R}^d_\theta) \) is contained in \( L^q_\theta L^p_x (\mathbb{T}^\kappa \times \mathbb{R}^d_x) \). Moreover, this inclusion is a contraction;

\[
\| g \|_{L^q_\theta L^p_x (\mathbb{T}^\kappa \times \mathbb{R}^d_x)} \leq \| g \|_{L^q_\theta L^p_x (\mathbb{T}^\kappa \times \mathbb{R}^d_x)}.
\]
by Minkowski’s integral inequality. Furthermore, $\hat{F}$ is a contraction from $L^q_\xi L^p_\theta(\mathbb{Z}_n^\kappa \times \mathbb{R}^d_\xi)$ to $L^q_\xi L^p_\theta(T_\theta^\kappa \times \mathbb{R}^d_\xi)$.

For $n \in \mathbb{Z}^\kappa$ define $E_n \subset \mathbb{R}^d$ by

$$E_n = \{ x \in \mathbb{R}^d : (n, x) \in E \}.$$  

(10.16)

For each $n \in \mathbb{Z}^\kappa$, $\| \hat{F}1_{E_n} \|_{L^q(\mathbb{R}^d)} \leq A_{q,d}|E_n|^{1/p}$. Therefore

$$\| \hat{F}1_E \|_{L^p_q L^p_\theta} \leq A_{q,d}(\sum_{n \in \mathbb{Z}^\kappa} |E_n|^{p/p})^{1/p} = |E|^{1/p}.$$  

**Proof of Lemma 10.10** First consider the optimal constant $A(q, d, \kappa)$. One has for any measurable $E \subset \mathbb{R}^d$ with finite Lebesgue measure

$$\|1_E\|_{L^q_\theta(T_\theta^\kappa \times \mathbb{R}^d)} = \|F \hat{F}1_E\|_{L^q_\xi L^p_\theta} \leq \| \hat{F}1_E \|_{L^q_\theta L^p_\theta} \leq \| \hat{F}1_E \|_{L^p_q L^p_\theta} \leq A_{q,d}|E|^{1/p}.$$  

Thus $A(q, d, \kappa) \leq A_{q,d}$. Consideration of product sets $E = A \times \{0\}$ immediately yields the reverse inequality.

Although these partial Fourier transforms commute in the sense (10.15), this reasoning breaks down if they are applied in reversed order, because the partial Fourier transform of $1_E$ with respect to the $\mathbb{Z}^\kappa$ coordinate, a function of $(\theta, x) \in T_\theta^\kappa \times \mathbb{R}^d$, is no longer a scalar multiple of the indicator function of a subset of $\mathbb{R}^d$ for fixed values of $\theta$. For the same reason, in the analysis of $A'(q, d, \kappa)$ these will have to be composed in the reverse order.

Next consider $A'(q, d, \kappa)$. For any $f \in L^p(T_\theta^\kappa \times \mathbb{R}^d)$ and any Lebesgue measurable set $E \subset T_\theta^\kappa \times \mathbb{R}^d$ with $|E| \in \mathbb{R}^+$, writing $E_\theta = \{ \xi \in \mathbb{R}^d : (\theta, \xi) \in E \}$,

$$\left| \int_E \hat{f} \right| = \left| \int \int \hat{F} \check{F}(f)(\theta, \xi)1_E(\theta, \xi) \, d\xi \, d\theta \right| \leq \int \| \hat{F} \check{F}(f)(\theta, \cdot) \|_{L^q_\theta L^p_\theta} |E_\theta|^{1/p} \, d\theta \leq \int A_{q,d} \| \check{F}(f)(\theta, \cdot) \|_{L^q_\theta} |E_\theta|^{1/p} \, d\theta,$$

since $A_{q,d}$ is the optimal constant in the dual inequality $\| \check{h} \|_{L^q_\theta(\mathbb{R}^d)} \leq C\| h \|_{L^p(\mathbb{R}^d)}$. Therefore

$$\left| \int_E \hat{f} \right| \leq A_{q,d}|E|^{1/p} (\int \| \check{F}(f)(\theta, \cdot) \|_{L^q_\theta}^q \, d\theta)^{1/q} \leq A_{q,d}|E|^{1/p} \| \check{F}(f) \|_{L^q_\theta L^p_\theta} \leq A_{q,d}|E|^{1/p} \| f \|_{L^p_q L^p_\theta} \leq A_{q,d}|E|^{1/p} \| f \|_{L^p_q}.$$  

Thus $A'(q, d, \kappa) \leq A_{q,d}$. Once again, the reverse inequality follows by consideration of functions $f(n, x)$ that are supported on a single value of $n$.  

For a function $f : \mathbb{Z}^\kappa \times \mathbb{R}^d \rightarrow \mathbb{C}$ and $m \in \mathbb{Z}^\kappa$ define $f_m : \mathbb{R}^d \rightarrow \mathbb{C}$ by

$$f_m(x) = f(m, x).$$  

(10.17)

Continue to write $E_m = \{ x \in \mathbb{R}^d : (m, x) \in E \}$ for $E \subset \mathbb{Z}^\kappa \times \mathbb{R}^d$.

The main results of this subsection are the two parallel Propositions 10.11 and 10.12. They will be proved at the end of this subsection after discussion of requisite lemmas.
**Proposition 10.11.** Let \( d, \kappa \geq 1 \) and \( q \in (2, \infty) \). Let \( \delta > 0 \) be small. Let \( E \subset \mathbb{Z}^\kappa \times \mathbb{R}^d \) be a Lebesgue measurable set with \( |E| \in \mathbb{R}^+ \). If \( \| \mathbf{1}_E \|_q \geq (1 - \delta) A_{q,d} |E|^{1/p} \) then there exists \( m \in \mathbb{Z}^\kappa \) such that

\[
|E_m| \geq (1 - o_\delta(1)) |E|.
\]

In fact the proof will give \( |E_m| \geq (1 - C \delta^p)|E| \) for some constants \( C, \rho \in \mathbb{R}^+ \), but the \( o_\delta(1) \) bound suffices in the sequel.

The proof will use the representation \( \mathcal{F} = \mathcal{F} \circ \mathcal{F} \) together with specific properties of the factor \( \mathcal{F} \), but the only properties of \( \mathcal{F} \) used are that it is linear, acts with respect to the \( \mathbb{R}^d \) variable alone, and satisfies a restricted type \((p, q)\) inequality with optimal constant \( A_{q,d} \).

**Proposition 10.12.** Let \( d, \kappa \geq 1 \) and \( q \in (2, \infty) \). Let \( \delta > 0 \) be small. Let \( 0 \neq f \in L^q(D) \). If \( \| \mathbf{1}_E \|_q \geq (1 - \delta) A_{q,d} |E|^{1/p} \) then there exists \( m \in \mathbb{Z}^\kappa \) such that

\[
\| f_m \|_L^q(\mathbb{R}^d) \geq (1 - o_\delta(1)) \| f \|_L^q(\mathbb{Z}^\kappa \times \mathbb{R}^d).
\]

The following lemma will be used in the proof of Proposition 10.11. It is a direct consequence of the chain of inequalities in the analysis of \( \mathcal{A}'(q, d, \kappa) \) in the proof of Lemma 10.10 so requires no further proof.

**Lemma 10.13.** Let \( d, \kappa \geq 1 \) and \( q \in (2, \infty) \). Set \( p = q' \). Let \( \delta > 0 \) be small. Let \( E \subset \mathbb{Z}^\kappa \times \mathbb{R}^d \) be a Lebesgue measurable set with \( |E| \in \mathbb{R}^+ \). If \( \| \mathbf{1}_E \|_q \geq (1 - \delta) A_{q,d} |E|^{1/p} \) then all of the following hold:

\[
\| \mathbf{F} \mathbf{1}_E \|_{L^q_\xi L^p_\eta} \geq (1 - \delta) \| \tilde{\mathbf{F}}(\mathbf{1}_E) \|_{L^q_\xi L^p_\eta},
\]

\[
\| \tilde{\mathbf{F}}(\mathbf{1}_E) \|_{L^q_\xi L^p_\eta} \geq (1 - \delta) \| \tilde{\mathbf{F}}(\mathbf{1}_E) \|_{L^p_\xi L^q_\eta},
\]

\[
\| \tilde{\mathbf{F}}(\mathbf{1}_E) \|_{L^p_\eta L^q_\xi} \geq (1 - \delta) A_{q,d} |E|^{1/p}.
\]

To simplify notation, we continue to write \( p = q' \) below.

**Lemma 10.14.** Let \( E \subset \mathbb{Z}^\kappa \times \mathbb{R}^d \) satisfy \( \| \mathbf{1}_E \|_q \geq (1 - \delta) A_{q,d} |E|^{1/p} \) where \( p = q' \). There exists a disjointly supported decomposition

\[
\tilde{\mathbf{F}} \mathbf{1}_E(n, \xi) = g(n, \xi) + h(n, \xi)
\]

where

\[
\| h \|_{L^q_\xi L^p_\eta} \leq o_\delta(1) |E|^{1/p}
\]

and for each \( \xi \in \mathbb{R}^d \) there exists \( n(\xi) \in \mathbb{Z}^\kappa \) such that

\[
g(n, \xi) = 0 \text{ for all } n \neq n(\xi).
\]

The conclusion is weaker than the one we seek eventually to establish in Proposition 10.12 in that \( n(\xi) \) is permitted here to depend on \( \xi \).

**Proof of Lemma 10.14.** Let \( \eta = \delta^{1/2} \). For \( \xi \in \mathbb{R}^d \), define \( \varphi_\xi : \mathbb{Z}^\kappa \rightarrow \mathbb{C} \) by

\[
\varphi_\xi(n) = \tilde{\mathbf{F}} \mathbf{1}_E(n, \xi).
\]

This is well-defined for almost every \( \xi \).

Define

\[
\mathcal{G} = \left\{ \xi \in \mathbb{R}^d : \| \varphi_\xi \|_{L^p(\mathbb{Z}^\kappa)} \neq 0 \text{ and } \| \hat{\varphi}_\xi \|_{L^q(\mathbb{Z}^\kappa)} \geq (1 - \eta) \| \varphi_\xi \|_{L^p(\mathbb{Z}^\kappa)} \right\}.
\]
The Fourier transform here is that for the group $\mathbb{Z}^k$. With this definition, 
\[
\|\mathcal{F}\hat{F}(1_E)\|_{L^q_{\xi}L^p_n} = \int_{\mathbb{R}^d\setminus G} \|\hat{\varphi}\|_{L^q(\mathbb{T}^k)}^q d\xi + \int_{G} \|\hat{\varphi}\|_{L^q(\mathbb{T}^k)}^q d\xi \\
\leq \int_{\mathbb{R}^d\setminus G} (1-\eta)^{-1}\|\varphi\|_p^q d\xi + \int_{G} \|\varphi\|_p^q d\xi \\
\leq \int_{\mathbb{R}^d} \|\varphi\|_p^q d\xi - c\eta \int_{\mathbb{R}^d\setminus G} \|\varphi\|_p^q d\xi.
\]

Therefore by $\text{(10.20)}$, 
\[
(10.24) \quad (1-\delta)^q \int_{\mathbb{R}^d} \|\varphi\|_p^q d\xi = (1-\delta)^q \|\mathcal{F}\hat{F}(1_E)\|_{L^q_{\xi}L^p_n} \\
\leq \int_{\mathbb{R}^d} \|\varphi\|_p^q d\xi - c\eta \int_{\mathbb{R}^d\setminus G} \|\varphi\|_p^q d\xi.
\]

Consequently 
\[
(10.25) \quad \int_{\mathbb{R}^d\setminus G} \|\varphi\|_p^q d\xi \leq C\eta^{-1}\delta \int_{\mathbb{R}^d} \|\varphi\|_p^q d\xi \leq C\delta^{1/2} \|\mathcal{F}F(1_E)\|_{L^q_{\xi}L^p_n}
\]

by the choice $\eta = \delta^{1/2}$.

According to [2], if $\xi \in G$, that is, if $\|\hat{\varphi}\|_{L^q(\mathbb{T}^k)} \geq (1-\eta)$, then there exists $n(\xi) \in \mathbb{Z}^k$ such that 
\[
\|\varphi\|_{L^p(\mathbb{Z}^k \setminus \{n(\xi)\})} \leq o(1)\|\varphi\|_{L^p(\mathbb{Z}^k)}.
\]

In this case define 
\[
g(n, \xi) = \begin{cases} 
\varphi(n) = \mathcal{F}1_E(n, \xi) & \text{for the single point } n = n(\xi), \\
0 & \text{for all other } n \in \mathbb{Z}^k.
\end{cases}
\]

For $\xi \in G$ define $h(n, \xi) = \mathcal{F}1_E(n, \xi)$ for all $n \neq n(\xi)$.

On the other hand, for $\xi \notin G$ define $h(n, \xi) = \varphi(n) = \mathcal{F}1_E(n, \xi)$ for all $n \in \mathbb{Z}^k$, and $g(n, \xi) \equiv 0$. Thus 
\[
\mathcal{F}1_E(n, \xi) = g(n, \xi) + h(n, \xi)
\]

for almost all $(n, \xi) \in \mathbb{Z}^k \times \mathbb{R}^d$, and this is a disjointly supported decomposition of $\mathcal{F}1_E$.

The function $g$ satisfies the stated conclusion of the lemma by its definition. On the other hand, the summand $h$ has small $L^q_{\xi}L^p_n$ norm. Indeed, 
\[
\int_{\xi \in \mathbb{G}} \left( \sum_{n \in \mathbb{Z}^k} |h(n, \xi)|^p \right)^{q/p} d\xi \leq \int_{\xi \in \mathbb{G}} o(1) \left( \sum_{n \in \mathbb{Z}^k} |\varphi(n)|^p \right)^{q/p} d\xi \\
= o(1) \|\mathcal{F}1_E\|_{L^q_{\xi}L^p_n}
\]

while 
\[
\int_{\xi \in \mathbb{G}} \left( \sum_{n \in \mathbb{Z}^k} |h(n, \xi)|^p \right)^{q/p} d\xi = \int_{\xi \in \mathbb{G}} \|\varphi\|_{L^p_n}^q d\xi \leq C\delta^{1/2} \|\mathcal{F}1_E\|_{L^q_{\xi}L^p_n}
\]

by $\text{(10.25)}$. 

The conclusion of Lemma $\text{[10.14]}$ with $1_E(n, \xi)$ supported at a single $\xi$–dependent point $n(\xi)$ for nearly all $\xi$, is manifestly weaker than the type of conclusion asserted by Proposition $\text{[10.11]}$ that $1_E(n, \xi)$ is supported at a single value of $n$, independent of $\xi$, up to a
small norm remainder. The balance of the proof of Proposition \textbf{10.11} is a simple argument to bridge this gap.

\textit{Proof of Proposition \textbf{10.11}} Let \( p = q' \). Let \( E \subset \mathbb{Z}^n \times \mathbb{R}^d \) satisfy \( \| \mathbf{1}_E \|_q \geq (1 - \delta)A_{q,d}|E|^{1/p} \). Decompose \( \tilde{F}(1_E)(n, \xi) = g(n, \xi) + h(n, \xi) \) as in Lemma \textbf{10.14}.

The upper bound \( \| h \|_{L^q_\xi L^p_n} \leq o_\delta(1)|E|^{1/p} \) and the lower bounds \textbf{(10.21)}, \textbf{(10.22)} together imply

\[
(10.26) \quad \| g \|_{L^q_\xi L^p_n} \geq (1 - o_\delta(1))\| g \|_{L^q_\xi L^p_n}.
\]

Now because \( n \mapsto g(n, \xi) \) is supported on a set of cardinality at most one for each \( \xi \),

\[
(10.27) \quad \| g \|_{L^q_\xi L^p_n} = \int (\sum_n |g(n, \xi)|^p)^{1/p} d\xi = \int |g(n(\xi), \xi)|^q d\xi = \| g \|_{L^q_\xi L^p_n} = \| g \|_{L^q_\xi L^p_n},
\]

so

\[
(10.28) \quad \| g \|_{L^q_\xi L^p_n} \geq (1 - o_\delta(1))\| g \|_{L^q_\xi L^p_n}.
\]

Furthermore

\[
(10.29) \quad \| g \|_{L^q_\xi L^p_n} \leq \sup_{m \in \mathbb{Z}^n} \| g(m, \cdot) \|_{L^q_\xi} \| g \|_{L^p_n L^q_\xi}.
\]

Since \( q > p \), \textbf{(10.28)} and \textbf{(10.29)} together give

\[
(10.30) \quad \sup_{m \in \mathbb{Z}^n} \| g(m, \cdot) \|_{L^q_\xi} \geq (1 - o_\delta(1))\| g \|_{L^q_\xi L^p_n} \geq (1 - o_\delta(1))A_{q,d}|E|^{1/p}.
\]

Because \( \tilde{F}(1_E) = g + h \) and \( g, h \) have disjoint supports, \( \| \tilde{F}(1_E)(m, \cdot) \|_{L^q_\xi} \geq \| g(m, \cdot) \|_{L^q_\xi} \) for any \( m \) and therefore there exists \( m \) satisfying

\[
(10.31) \quad \| \tilde{F}(1_E)(m, \cdot) \|_{L^q_\xi} \geq (1 - o_\delta(1))A_{q,d}|E|^{1/p}.
\]

On the other hand, by definition of \( A_{q,d} \) as the optimal constant in the inequality,

\[
(10.32) \quad \| \tilde{F}(1_E)(m, \cdot) \|_{L^q_\xi} \leq A_{q,d}|E_m|^{1/p}.
\]

Together with the lower bound \textbf{(10.31)} for \( |E_m| \), this implies that

\[
(10.33) \quad |E_m| \geq (1 - o_\delta(1))|E|,
\]

concluding the proof of Proposition \textbf{10.11}. \( \Box \)

Proposition \textbf{10.12} is proved by combining the reasoning in the proof of Proposition \textbf{10.11} with the proof of the second conclusion of Lemma \textbf{10.10}. \( \Box \)

10.7. \textbf{Lifting to} \( \mathbb{Z}^d \times \mathbb{R}^d \). Let \( Q_d = [-\frac{1}{2}, \frac{1}{2}]^d \) be the unit cube of sidelength 1 centered at 0 \( \in \mathbb{R}^d \). Identify \( \mathbb{R}^d \) with \( \mathbb{Z}^d + Q_d \) via the map \( \mathbb{Z}^d \times Q_d \ni (n, y) \mapsto n + y \in \mathbb{R}^d \).

\textbf{Definition 10.2}. To any function \( f : \mathbb{R}^d \to \mathbb{C} \) associate the function \( f^\dagger : \mathbb{Z}^d \times \mathbb{R}^d \to \mathbb{C} \) defined by

\[
(10.34) \quad f^\dagger(n, x) = \begin{cases} f(n + x) & \text{if } x \in Q_d \\ 0 & \text{if } x \notin Q_d. \end{cases}
\]

For any set \( E \subset \mathbb{R}^d \), the associated set \( E^\dagger \subset \mathbb{Z}^d \times \mathbb{R}^d \) is

\[
(10.35) \quad E^\dagger = \{(n, x) : n + x \in E \text{ and } x \in Q_d\}.
\]

Thus \( |E^\dagger| = |E| \).
Lemma 10.15. Let $d \geq 1$ and $q \in (2, \infty)$. Let $\delta, \eta > 0$ be small. Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set with $|E| \in \mathbb{R}^+$. Suppose that

$$
(10.36) \quad \text{distance } (x, \mathbb{Z}^d) \leq \eta \text{ for all } x \in E
$$

and that

$$
(10.37) \quad \|1_E\|_{L^q(\mathbb{R}^d)} \geq (1 - \delta) A_{q,d}|E|^{1/q'}.
$$

Then

$$
(10.38) \quad \|1_{E^\dagger}\|_{L^q(\mathbb{T}^d \times \mathbb{R}^d)} \geq (1 - \delta - o_\eta(1)) A_{q,d}|E^\dagger|^{1/q'}.
$$

The notation $\widehat{\cdot}$ is used for the Fourier transform for any of the groups $\mathbb{R}^d$, $\mathbb{Z}^\kappa$, and $\mathbb{Z}^\kappa \times \mathbb{R}^d$.

Proof. Corollary 9.2 of [12] asserts a result for general functions $f : \mathbb{Z}^d \to \mathbb{C}$, whose specialization to indicator functions of sets is Lemma 10.15. \hfill $\Box$

Lemma 10.16. Let $d \geq 1$ and $q \in (2, \infty)$. Let $\delta, \eta > 0$ be small. Let $0 \neq f \in L^d(\mathbb{R}^d)$. Suppose that

$$
(10.39) \quad f(x) \neq 0 \Rightarrow \text{distance } (x, \mathbb{Z}^d) \leq \eta
$$

and that

$$
(10.40) \quad \|\hat{f}\|_{q,\infty} \geq (1 - \delta) A_{q,d}\|f\|_{q'}.
$$

Then

$$
(10.41) \quad \|\hat{f}^\dagger\|_{q,\infty} \geq (1 - \delta - o_\eta(1)) A_{q,d}\|f^\dagger\|_{q'}.
$$

In the statement of the lemma and in its proof, $\hat{f}$ and $\hat{f}^\dagger$ denote the Fourier transform of $f$ for the group $\mathbb{R}^d$, and the Fourier transform of $f^\dagger$ for the group $\mathbb{Z}^d \times \mathbb{R}^d$, respectively.

Proof. Let $p = q'$. Choose $E \subset \mathbb{R}^d$ with $|E| \in \mathbb{R}^+$ satisfying $|\int_E \hat{f}| \geq (1 - 2\delta) A_{q,d}\|f\|_p|E|^{1/p}$.

Represent elements of $\mathbb{R}^d$ as $\xi = n(\xi) + \alpha(\xi)$ where $n(\xi) \in \mathbb{Z}^d$ and $\alpha(\xi) \in \mathbb{Q}_d$. It is shown in the proof of Lemma 9.1 of [12] that

$$
(10.42) \quad \|\hat{f}(\theta, \xi) - \hat{f}(n(\xi) + \theta)\|_{L^q(\mathbb{T}^d \times \mathbb{R}^d)} \leq o_\eta(1)\|f\|_p.
$$

The quantity $o_\eta(1)$ depends also on $q, d$ but not on $f$.

Identify $\mathbb{T}^d$ with $\mathbb{Q}_d$ in the natural way, by associating to each element of $\mathbb{Q}_d \subset \mathbb{R}^d$ its equivalence class in $\mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d$. Then

$$
|\int_{E^\dagger} \hat{f}^\dagger| = \left| \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} 1_{E^\dagger}(\theta, \xi) \hat{f}(\theta, \xi) d\theta d\xi \right|
$$

$$
= \left| \int_{\mathbb{R}^d} \int_{\mathbb{Q}_d} 1_{E^\dagger}(\theta, \xi) \hat{f}(n(\xi) + \theta) d\theta d\xi \right| + o_\eta(1)\|f\|_p|E^\dagger|^{1/p}
$$

$$
= \left| \int_{\mathbb{R}^d} \int_{\mathbb{Q}_d} 1_E(n(\xi) + \theta) \hat{f}(n(\xi) + \theta) d\theta d\xi \right| + o_\eta(1)\|f\|_p|E^\dagger|^{1/p}.
$$
Now writing $\xi = n(\xi) + \alpha$ with $\alpha \in Q_d$ gives

$$\int_{\mathbb{R}^d} \int_{Q_d} 1_E(n(\xi) + \alpha) f(n(\xi) + \alpha) d\alpha d\xi = \sum_{n \in \mathbb{Z}^d} \int_{Q_d} 1_E(n + \alpha) f(n + \alpha) d\alpha$$

$$= \sum_{n \in \mathbb{Z}^d} \int_{Q_d} 1_E(n + \alpha) f(n + \alpha) d\alpha$$

$$= \int_{\mathbb{R}^d} 1_E(\xi) f(\xi) d\xi$$

since $|Q_d| = 1$. Thus

$$|\int_{E^1} f^\dagger| = |\int_E \tilde{f} | + o_\eta(1)\|f\|_p|E|^{1/p} \geq (1 - 2\delta)A_{q,d}\|f\|_p|E|^{1/p} + o_\eta(1)\|f\|_p|E|^{1/p},$$

which is equal to $(1 - 2\delta)A_{q,d}\|f^\dagger\|_p|E^1|^{1/p} + o_\eta(1)\|f^\dagger\|_p|E^1|^{1/p}, \quad \Box$

10.8. Spatial localization.

**Proposition 10.17.** Let $d \geq 1$ and $q \in (2, \infty)$. For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property. Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set with $|E| \in \mathbb{R}^+$. If $1_E|q \geq (1 - \delta)A_{q,d}|E|^{1/q}$ then there exists an ellipsoid $E \subset \mathbb{R}^d$ such that

$$|E \setminus E| \leq \varepsilon|E|$$

(10.43)

$$|E| \leq C_\varepsilon|E|.$$  

(10.44)

**Proposition 10.18.** Let $d \geq 1$ and $q \in (2, \infty)$. For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property. Let $0 \neq f \in L^q(\mathbb{R}^d)$ satisfy $\|f\|_{q,\infty} \geq (1 - \delta)A_{q,d}\|f\|_{q'}$. There exist an ellipsoid $E \subset \mathbb{R}^d$ and a decomposition $f = \varphi + \psi$ such that

$$\|\psi\|_{q'} < \varepsilon\|f\|_{q'}$$

(10.45)

$$\varphi \equiv 0 \text{ on } \mathbb{R}^d \setminus E$$

(10.46)

$$\|\varphi\|_{\infty}E^{1/q'} \leq C_\varepsilon\|f\|_{q'}.$$  

(10.47)

**Proof of Proposition 10.17.** Continue to write $p = q'$. Let $\delta$ be small, and assume that $1_E|q \geq (1 - \delta)A_{q,d}|E|^{1/p}$. By Lemma 10.8 there exists a continuum multiprogression $P$ of rank $O_\delta(1)$ such that $|P| \leq O_\delta(1)|E|$ and $|E \setminus P| \leq o_\delta(1)|E|$. By replacing $E$ by its dilate $|P|^{-1/d}E$ we may assume that $|P| = 1$.

Lemma 5.2 of [12] states that for any $r < \infty$ and any continuum multiprogression $P$ of rank at most $r$ and Lebesgue measure equal to 1 there exists an invertible affine automorphism $T$ of $\mathbb{R}^d$ such that distance $(x, Z^d) = o_\delta(1)$ for every $x \in P$, and $|T(P)|$ is bounded below by a positive quantity that depends only on $d, q, \delta, r$; the same lower bound holds for $|T(E \cap P)|$. Choose such an affine automorphism $T$.

Change notation, denoting the set $T(E)$ again by $E$ and $T(P)$ by $P$. Affine transformations of ellipsoids are ellipsoids, so it suffices to establish the stated conclusions for the new set $E$. No upper bound on the Jacobian determinant of $T$ is possible, so we obtain no upper bound for $|T(E)|$.

The ratio $1_E|q / |A|^{1/p}$ is invariant under affine transformations of $A$. Consequently the modified set $E$ still satisfies $1_E|q \geq (1 - \delta)A_{q,d}|E|^{1/p}$. Since $|E \setminus P| \leq o_\delta(1)|E|$,
\[ \|1_{E\cap P}\|_q \leq o_\delta(1)|E|^{1/p}. \] Therefore
\[
\|1_{E\cap P}\|_q \geq \|1_E\|_q - \|1_{E\setminus P}\|_q \\
\geq (1 - \delta(1))A_{q,d}|E|^{1/p} - o_\delta(1)|E|^{1/p} \\
\geq (1 - o_\delta(1))A_{q,d}|P \cap E|^{1/p}.
\]

Consider the lifted set \((E \cap P)^\dagger \subset \mathbb{Z}^d \times \mathbb{R}^d\) introduced in Definition 10.2, that is, the set of all \((n, x)\) such that \(n + x \in E \cap P\) and \(x \in Q_d\). By Lemma 10.15
\[
\|1_{(E \cap P)^\dagger}\|_q \geq (1 - o_\delta(1))|(E \cap P)^\dagger|^{1/p},
\]
where \(\hat{\cdot}\) denotes the Fourier transform for \(\mathbb{Z}^d \times \mathbb{R}^d\).

By Proposition 10.11 there exists \(m \in \mathbb{Z}^d\) such that
\[
(10.48) \quad \|(E \cap P)^\dagger \setminus (\{m\} \times \mathbb{R}^d)\| \leq o_\delta(1)|(E \cap P)^\dagger|.
\]
In terms of the given set \(E\), this means that \(|(E \cap P) \setminus (m + Q_d)| \leq o_\delta(1)|E|\) and hence
\[
(10.49) \quad |E \setminus (m + Q_d)| \leq |(E \cap P) \setminus (m + Q_d)| + |E \setminus P| \leq o_\delta(1)|E|.
\]
The cube \(m + Q_d\) is contained in a ball of comparable measure, which is equivalent by affine invariance to the stated conclusion. \(\square\)

The proof of Proposition 10.18 is nearly identical to that of Proposition 10.17; the necessary preliminary results are established above. Therefore the details are omitted. \(\square\)

10.9. Frequency localization.

**Proposition 10.19.** Let \(d \geq 1\) and \(q \in (2, \infty)\). For every \(\varepsilon > 0\) there exists \(\delta > 0\) with the following property. Let \(E \subset \mathbb{R}^d\) be a Lebesgue measurable set with \(|E| \in \mathbb{R}^+\) satisfying
\[
\|1_E\|_q \geq (1 - \delta)A_{q,d}|E|^{1/q'}. \tag{10.50}
\]
Then there exist an ellipsoid \(E' \subset \mathbb{R}^d\) and a disjointly supported decomposition \(1_E = \Phi + \Psi\) such that
\[
\|\Psi\|_q < \varepsilon|E|^{1/q'} \tag{10.51}
\]
\[\Phi \equiv 0 \text{ on } \mathbb{R}^d \setminus E' \tag{10.52}\]
\[\|\Phi\|_{\infty}E'|^{1/q'} \leq C_\varepsilon|E|^{1/q'}. \tag{10.53}\]

**Proof.** Set \(p = q'\). Set \(f = \hat{1}_E : \hat{1}_E|q^{-2}.\) Then \(|f| = \|1_E|^{q-1},\) so \(|f|^p = \|1_E|^{p(q-1)} = \|\hat{1}_E|^q, f \in L^p.\) This function \(f\) satisfies
\[
\|\hat{f}\|_{q,\infty} \geq (1 - \delta)^qA_{q,d}\|f\|_p. \tag{10.54}
\]
Indeed,
\[
\|f\|_p = \|\hat{1}_E|_{q}^{q-1} \leq A_{q,d}|E|^{(q-1)/p}. \tag{10.55}
\]
Moreover, denoting by \(g^\dagger\) the inverse Fourier transform of a function \(g,\)
\[
\|\hat{1}_E\|_{q'} = \langle 1_E, f \rangle = \langle 1_E, f^\dagger \rangle \leq |E|^{1/p}\|f^\dagger\|_{q,\infty} = |E|^{1/p}\|\hat{f}\|_{q,\infty}; \tag{10.56}
\]
the relation \(|\langle 1_E, f^\dagger \rangle| \leq |E|^{1/p}\|f^\dagger\|_{q,\infty}\) is a tautology because of the definition chosen for the \(L^{q,\infty}\) norm. Consequently
\[
\|\hat{f}\|_{q,\infty} \geq |E|^{-1/p}\|\hat{E}\|_{q} \geq (1 - \delta)^qA_{q,d}|E|^{(q-1)/p} \geq (1 - \delta)^qA_{q,d}\|f\|_p,
\]
using (10.54) to obtain the final inequality.
Apply Proposition \[10.18\] to \(f\) to obtain an ellipsoid \(E'\) and a disjointly supported decomposition \(f = \varphi + \psi\) with the properties listed in that Proposition. Define \(\Phi, \Psi\) by the relations
\[
\Phi = \varphi|\varphi|^{(2-q)/(q-1)} \quad \text{and} \quad \Psi = \psi|\psi|^{(2-q)/(q-1)}
\]
to conclude the proof. 
\[\square\]

Although the dual inequality \(\|\hat{f}\|_{q,\infty} \leq A_{d,q,\delta}\|f\|_{q'}\) is not our main object of study, the various results developed above concerning it are required in this proof.

We have now associated two ellipsoids to any near-extremizing set \(E\). The set \(E\) itself is nearly contained in \(E\), while \(\hat{1}_E\) is nearly supported in \(E'\). It is clear that by the uncertainty principle, broadly construed, the product \(|E| \cdot |\hat{E}|\) is bounded below by some positive constant, provided that \(\varepsilon\) is sufficiently small in Propositions \[10.17\] and \[10.19\]. See \[12\]. We need to show next that \(|E| \cdot |\hat{E}|\) is bounded above.

\textbf{Definition 10.3.} The polar set \(E^*\) of an ellipsoid \(E \subset \mathbb{R}^d\) centered at \(0\) is
\[
E^* = \{y : |\langle x, y \rangle| \leq 1 \text{ for every } x \in E\}
\]
where \(\langle \cdot, \cdot \rangle\) denotes the Euclidean inner product.

\textbf{Definition 10.4.} Let \(\Theta : \mathbb{R}^+ \to \mathbb{R}^+\) be a continuous function satisfying \(\lim_{t \to \infty} \Theta(t) = 0\). Let \(r \in [1, \infty)\) be an exponent. Let \(\eta > 0\). We say that a function \(0 \neq f \in L^r(\mathbb{R}^d)\) is \(\eta\)–normalized with respect \(\Theta, r\) if for all \(R \in [1, \infty)\),
\[
\int_{|x| \geq R} |f|^r \, dx \leq (\Theta(R) + \eta) \|f\|_r^r
\]
\[
\int_{|f(x)| \geq R} |f|^r \, dx \leq (\Theta(R) + \eta) \|f\|_r^r.
\]

Inequality \[10.57\] prevents the mass represented by \(|f|^r\) from being too diffuse, while inequality \[10.58\] prevents excessive concentration.

For any ellipsoid \(E \subset \mathbb{R}^d\) there exists \(T_E \in \text{Aff}(d)\) satisfying \(T_E(\mathbb{B}) = E\). If \(T, T'\) are any two such transformations then \(T' = T \circ S\) for some element \(S\) of the orthogonal group.

\textbf{Definition 10.5.} Let \(E \subset \mathbb{R}^d\) be any ellipsoid. A function \(0 \neq f \in L^r(\mathbb{R}^d)\) is said to be \(\eta\)–normalized with respect \(\Theta, r, E\) if \(f \circ T_E\) is \(\eta\)–normalized with respect to \(\Theta, r\). A measurable set \(E\) is \(\eta\)–normalized with respect to \(\Theta, r, E\) if \(|E| \in \mathbb{R}^+\) and the function \(1_E\) has this property.

Although \(T_E\) is not uniquely defined, all choices lead to identical definitions because of the rotation-invariance of Definition \[10.4\].

The result shown thus far can be reformulated in these terms:

\textbf{Lemma 10.20.} Let \(d \geq 1\) and \(q \in (2, \infty)\). There exists a continuous function \(\Theta : \mathbb{R}^+ \to \mathbb{R}^+\) satisfying \(\lim_{t \to \infty} \Theta(t) = 0\) with the following property. For every sufficiently small \(\delta > 0\) and any Lebesgue measurable set \(E \subset \mathbb{R}^d\) with \(|E| \in \mathbb{R}^+\) that satisfies \(\|1_E\|_q \geq (1 - \delta)A_{d,q,\delta}|E|^{1/p}\), there exist ellipsoids \(E, E' \subset \mathbb{R}^d\) such that \(E\) is \(o_\delta(1)\)–normalized with respect to \(\Theta, q', E\) and \(1_E\) is \(o_\delta(1)\)–normalized with respect to \(\Theta, q, E'\).

The quantities denoted by \(o_\delta(1)\) depend only on \(\delta, q, d\) and on a choice of auxiliary function \(\Theta\), but not on \(E\), and tend to zero as \(\delta \to 0\) while \(d, q, \Theta\) remain fixed.
10.10. Compatibility of approximating ellipsoids. The next step is to show that the ellipsoids \( E, E' \) are dual to one another, up to bounded factors and independent translations. For \( s \in \mathbb{R}^+ \) and \( E \subset \mathbb{R}^d \), we consider the dilated set \( sE = \{ sy : y \in E \} \).

**Lemma 10.21.** Let \( d \geq 1 \) and \( q \in (2, \infty) \). Let \( \rho > 0 \). Let \( \Theta : \mathbb{R}^+ \to \mathbb{R}^+ \) be a continuous function satisfying \( \lim_{t \to \infty} \Theta(t) = 0 \). There exist \( \eta > 0 \) and \( C < \infty \) with the following property. Let \( E, E' \subset \mathbb{R}^d \) be ellipsoids centered at \( 0 \in \mathbb{R}^d \), and let \( u, v \in \mathbb{R}^d \). Let \( E \subset \mathbb{R}^d \) be a Lebesgue measurable set, and suppose that \( \|1_E\|_q \geq \rho |E|^{1/q} \). Suppose moreover that \( E \) is \( \eta \)-normalized with respect to \( \Theta, q, E + u \), and that \( 1_E \) is \( \eta \)-normalized with respect to \( \Theta, q, E' + v \). Then

\[
\mathcal{E} \subset C\mathcal{E}^* \quad \text{and} \quad \mathcal{E}' \subset C\mathcal{E}^*.
\]

**Proof.** A more general result, with \( 1_E \) replaced by an arbitrary function in \( L^q(\mathbb{R}^d) \), is Lemma 11.2 of [12]. \( \square \)

10.11. Precompactness.

**Lemma 10.22.** Let \( d \geq 1 \) and \( q \in (2, \infty) \). Let \( p = q' \). There exists a continuous function \( \Theta : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying \( \lim_{t \to \infty} \Theta(t) = 0 \) with the following property. For any sufficiently small \( \delta > 0 \) and any Lebesgue measurable set \( E \subset \mathbb{R}^d \) with \( |E| \in \mathbb{R}^+ \) that satisfies \( \|1_E\|_q \geq (1 - \delta)A_{q,d}|E|^{1/q} \) there exists \( T \in \text{Aff}(d) \) such that \( T(E) \) is \( o_q(1) \)-normalized with respect to \( \Theta, q', \mathbb{B}, \) and \( 1_{T(E)} \) is \( o_q(1) \)-normalized with respect to \( \Theta, q, \mathbb{B} \).

The corresponding step in [12] is immediate from the analogue of Lemma 10.21 by the affine- and modulation-invariance of the ratio \( \|\hat{f}\|/\|f\|_p \), but here no modulation-invariance is available.

**Proof.** Let the ellipsoids \( \mathcal{E}, \mathcal{E}' \) and auxiliary function \( \Theta \) satisfy the conclusions of Lemmas 10.20 and 10.21. Since the ratio \( \|\hat{f}\|/\|f\|_p \) is invariant under precomposition of \( f \) with affine automorphisms of \( \mathbb{R}^d \), we may assume without loss of generality that \( \mathcal{E} = \mathbb{B} \). Then by Lemma 10.21 we may take \( \mathcal{E}' \) to also be a ball, with \( |\mathcal{E}'| \approx 1 \). If \( \delta \) is sufficiently small then \( |E \setminus C_0\mathbb{B}| \ll |E| \) for a certain constant \( C_0 \) that depends only on \( d, q \).

There exists a constant \( \rho_0 > 0 \) which depends only on \( d, q \) such that \( |\hat{1}_E(\xi)| \geq \rho_0 \) whenever \( |\xi| \leq \rho_0 \). Indeed, split \( E = E' \cup E'' \) where \( E' = E \cap s\mathbb{B} \) where \( s \in \mathbb{R}^+ \) is chosen sufficiently large to ensure that \( |E''| \leq \frac{1}{4}|E| = \frac{1}{2} \). Such a parameter \( s \) may be taken to be independent of \( E \) so long as \( \delta \) is sufficiently small since \( E \) is normalized. Then \( |\hat{1}_{E'}(0)| = |E'| \geq \frac{1}{4}, \hat{1}_{E'} \) is Lipschitz continuous with Lipschitz constant \( O(s) \), and \( \|\hat{1}_{E''}\|_\infty \leq |E''| \leq \frac{1}{4} \).

It is given that the function \( \hat{1}_E \) is normalized with respect to \( \mathcal{E}' \). The center of \( \mathcal{E}' \) must lie within a bounded distance of the origin, or the lower bound \( |\hat{1}_E(\xi)| \geq \frac{1}{2} \) for all \( |\xi| \leq Cs^{-1} \) would contradict the first normalization inequality (10.57). Therefore \( \mathcal{E}' \) can be replaced by a ball centered at the origin, whose radius is bounded above and below by positive constants that depend only on \( q, d \). With respect to this modified \( \mathcal{E}' \), \( \hat{1}_E \) is still normalized, with respect to a modified function \( \hat{\Theta} \) which still satisfies \( \lim_{t \to \infty} \hat{\Theta}(t) = 0 \), and depends only on the function \( \Theta \) given by Lemmas 10.20 and 10.21, not on \( E, \delta \). \( \square \)

**Proposition 10.23.** Let \( d \geq 1 \) and \( q \in (2, \infty) \). Let \( (E_\nu) \) be a sequence of Lebesgue measurable subsets of \( \mathbb{R}^d \) with \( |E_\nu| \in \mathbb{R}^+ \). Suppose that \( \lim_{\nu \to \infty} |E_\nu|^{-1/q'} \|1_{E_\nu}\|_q = A_{q,d} \). Then there exists a sequence of elements \( T_\nu \in \text{Aff}(d) \) such that \( |T_\nu(E_\nu)| = 1 \) for all \( \nu \) and the sequence of indicator functions \( (1_{T_\nu(E_\nu)}) \) is precompact in \( L^q(\mathbb{R}^d) \).
If a sequence $f_\nu$ of indicator functions of sets converges in $L^d$ norm, then the limiting function is necessarily the indicator function of a set, so the conclusion is the type of convergence desired. Nonetheless, we will directly prove convergence in a stronger norm.

**Proof.** According to Lemma 10.22, there exist a positive continuous auxiliary function $\Theta$ that satisfies $\lim_{t \to \infty} \Theta(t) = 0$, and a sequence of affine automorphisms $(T_\nu)$ of $\mathbb{R}^d$ satisfying the conclusions of that lemma. By composing $T_\nu$ with a dilation of $\mathbb{R}^d$ we may arrange that $|T_\nu(E_\nu)| = 1$ for all $\nu$. The other conclusions of Lemma 10.22 are unaffected. Thus both $E_\nu$ and $\hat{E}_\nu$ satisfy uniform upper bounds with respect to $\Theta$ of the type formulated in Definition 10.4, with parameters $\eta = \eta_\nu$ that satisfy $\lim_{\nu \to \infty} \eta_\nu = 0$.

Write $E_\nu$ in place of $T_\nu(E_\nu)$ to simplify notation. Since the ratio $\|\hat{1}_{E_\nu}\|_q/|E_\nu|^{1/p}$ is invariant under replacement of $E$ by any affine image of $E$, we still have an extremizing sequence; $\|\hat{1}_{E_\nu}\|_q/|E_\nu|^{1/p} \to A_{q,d}$ as $\nu \to \infty$.

As in the proof of Lemma 12.2 of [12], it follows immediately that the sequence of functions $\hat{1}_{E_\nu}$ is precompact in $L^q(\mathbb{R}^d)$. Therefore there exists a subsequence, which we continue to denote by $(E_\nu)$, such that $(\hat{E}_\nu)$ converges to some limit $g \in L^q$ in $L^q$ norm.

We claim that this subsequence $(1_{E_\nu})$ converges in $L^p$, or equivalently, $|E_\mu \Delta E_\nu| \to 0$ as $\mu, \nu \to \infty$. This is proved as was done for the corresponding situation in [12]. Define $f_\nu = 1_{E_\nu}$. Then $\|f_\nu\|_q \to A_{q,d}$. Therefore the function $g = \lim_{\nu \to \infty} f_\nu$ satisfies $\|g\|_q = A_{q,d}$. Define

$$h_{\mu,\nu} = \frac{1}{2}(f_\nu + f_\mu) = 1_{E_\mu \cap E_\nu} + \frac{1}{2}1_{E_\mu \Delta E_\nu}.$$  

Then $h_{\mu,\nu} = \frac{1}{2}f_\nu + \frac{1}{2}f_\mu \to g$ in $L^q$ norm as $\mu, \nu \to \infty$. Therefore $\|h_{\mu,\nu}\|_q \to A_{q,d}$, so $\liminf_{\mu,\nu \to \infty} \|h_{\mu,\nu}\|_{p,1} \geq 1$ since $A_{q,d}$ is the optimal constant in the inequality. Since $\|h_{\mu,\nu}\|_p \leq 1$, $\lim_{\mu,\nu \to \infty} \|\frac{1}{2}f_\mu + \frac{1}{2}f_\nu\|_p = 1$.

The exponent $p$ lies in $(1, 2)$. $f_\nu$ lies in the unit ball of $L^p$ for all $\nu$. From the uniform strict convexity of this unit ball it follows that $\lim_{\mu,\nu \to \infty} \|f_\mu - f_\nu\|_p = 0$. That is, $|E_\mu \Delta E_\nu| \to 0$ as $\mu, \nu \to \infty$.

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1 This reasoning relies on the Hausdorff-Young inequality and does not apply directly to $(1_{E_\nu})$ in $L^p$ norm. If $\hat{f} = \hat{g} + \hat{h}$ where $\|\hat{h}\|_q$ is small, one cannot conclude that $\|\hat{h}\|_p$ is small.
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