THE SPINOR BUNDLE OF RIEMANNIAN PRODUCTS

FRANK KLINKER

Abstract. In this note we compare the spinor bundle of a Riemannian manifold \((M = M_1 \times \cdots \times M_N, g)\) with the spinor bundles of the Riemannian factors \((M_i, g_i)\). We show that - without any holonomy conditions - the spinor bundle of \((M, g)\) for a special class of metrics is isomorphic to a bundle obtained by tensoring the spinor bundles of \((M_i, g_i)\) in an appropriate way. For \(N = 2\) and an one dimensional factor this construction was developed in [Baum 1989a]. Although the fact for general factors is frequently used in (at least physics) literature, a proof was missing.

I would like to thank Shahram Biglari, Mario Listing, Marc Nardmann and Hans-Bert Rademacher for helpful comments. Special thanks go to Helga Baum, who pointed out some difficulties arising in the pseudo-Riemannian case.

We consider a Riemannian manifold \((M = M_1 \times \cdots \times M_N, g)\), which is a product of Riemannian spin manifolds \((M_i, g_i)\) and denote the projections on the respective factors by \(p_i\). Furthermore the dimension of \(M_i\) is \(D_i\) such that the dimension of \(M\) is given by \(D = \sum_{i=1}^{N} D_i\).

The tangent bundle of \(M\) is decomposed as
\[
T_{(x_0, \ldots, x_N)} M = p_1^*T_{x_1}M_1 \oplus \cdots \oplus p_N^*T_{x_N}M_N.
\]
We omit the projections and write \(TM = \bigoplus_{i=1}^{N} TM_i\).

The metric \(g\) of \(M\) need not be the product metric of the metrics \(g_i\) on \(M_i\), but is assumed to be of the form
\[
g_{ab}(x)\big|_{TM_i} = A_{ic}(x)g_{cd}(x_i)A_{db}(x),
\]
for \(D_1 + \cdots + D_{i-1} + 1 \leq a, b \leq D_1 + \cdots + D_i, 1 \leq i \leq N\)

In particular, for those metrics the splitting 1 is orthogonal, i.e. the frame bundle of \(M\) can be reduced to a \(SO(D_1) \times \cdots \times SO(D_N)\)-principal bundle, and this is isomorphic to the product of the frame bundles over \(M_i\). The explicit form of the isomorphism is
\[
P_{SO}(M_1) \times P_{SO}(M_N) \ni (E_1(x_1), \ldots, E_N(x_N)) \mapsto (A_1(x)E_1(x_1), \ldots, A_N(x)E_N(x_N)) \in P_{SO}(M)
\]

It is clear that such a manifold need not have a splitting of the holonomy group into subgroups of \(SO(D_i)\), which would lead to an, at least local, Riemannian product (cf. [Joyce 2000 sect. 3.2]). Examples for such spaces are the Eguchi
Hanson space – where we have $M \subset \mathbb{R} \times S^3$ with metric $\alpha(r)dr^2 + h_r(S^3)$ – or warped products of metrics.

We consider spinor bundles $S_i$ over $M_i$ and we are going to construct a bundle $S$ over $M$ from these spinor bundles. We will discuss what conditions are necessary for the bundle $S$ to be the spinor bundle over $M$ and how we have to modify the given Clifford multiplication on $S_i$ to extend it to $S$.

As is well known, the Clifford algebra of the sum of two vector spaces is the $(\mathbb{Z}_2,\mathbb{Z}_2)$-graded) tensor product of the Clifford algebras of the two summands (This is denoted by $\text{Cl}(V \oplus W) = \text{Cl}(V) \hat{\otimes} \text{Cl}(W)$ in [Lawson and Michelsohn 1989]). Of course this is not restricted to two factors, but can be iterated. The case of two factors, i.e. $N = 2$, will always be emphasized.

We consider the pullback bundles $p^*_iS_i \to M$ of $S_i$ over $M$ and once more we omit the projections in our further notation. From $S_i$ we construct a bundle $W$ on $M$

$$W = (S_i^+ \oplus S_i^-) \otimes \cdots \otimes (S_N^+ \oplus S_N^-)$$

(4)

For $N = 2$ this is

$$W = (S_1^+ \oplus S_1^-) \otimes (S_2^+ \oplus S_2^-) = (S_1^+ \otimes S_2^+ \oplus S_1^- \otimes S_2^-) \oplus (S_1^+ \otimes S_2^- \oplus S_1^- \otimes S_2^+)$$

The bundles $S_i^\pm$ are subbundles of $S_i$ with the $\mathbb{Z}_2$-degree defined by the label

$$\epsilon_i = \mathbb{Z}_2.$$ 

This induces on $W$ the natural $\mathbb{Z}_2$-grading given by the decomposition (4). An element $\Xi \in W$ is called totally homogenous, if $\Xi = \xi_1^1 \otimes \cdots \otimes \xi_N^N$, i.e. it has only contributions from one of the summands in the decomposition (4). We will use the multi index notation $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$.

The subbundles of $S_i$ are chosen in such a way that the Clifford multiplication acts via $TM_i \otimes S^\pm_i \to S^\pm_i$, i.e.

$$X_i\xi_i = X_i\begin{pmatrix} \xi_i^+ \\ \xi_i^- \end{pmatrix} = \begin{pmatrix} 0 & X_i \\ X_i & 0 \end{pmatrix} \xi_i.$$ 

(5)

Clifford multiplication is an odd operation on $S_i^+ \oplus S_i^-$, i.e. $|X_i\phi_i| = |\phi_i| + 1$.

We will consider the continuation of this to the bundle $W$. Therefore we introduce the linear map $\delta_k : W \to W$, which is defined by its action on totally homogenous elements $\Xi = \xi_1^1 \otimes \cdots \otimes \xi_N^N$

$$\delta_k(\Xi) = (-)^{\epsilon_1 + \cdots + \epsilon_k - 1} \Xi,$$

(6)

$\delta_k$ can be seen as the sign, we get by “putting an odd operator acting on $S_k$ at the right place in the tensor product”. We define $\delta_1 = \text{id}.$

1We identify $\mathbb{Z}_2 = \{0 = +, 1 = -\}.$
For $X = X_1 + \cdots + X_N \in TM$ and totally homogenous $\Xi^e = \xi^{i_1}_1 \otimes \cdots \otimes \xi^{i_N}_N \in W$ we define
\begin{equation}
X \Xi^e := \sum_{i=1}^{N} (1 \otimes \cdots \otimes A_i X_i \otimes \cdots \otimes 1) \Xi^e
= \sum_{i=1}^{N} \delta_i(\Xi^e) \xi^{i_1}_1 \otimes \cdots \otimes A_i X_i \xi^{i_i}_i \otimes \cdots \otimes \xi^{i_N}_N.
\end{equation}

In particular for $N = 2$
\begin{equation}
(X_1 + X_2)\Xi^{(e_1,e_2)} = (1 \otimes A_1 X_1 + A_2 X_2 \otimes 1)\Xi^{(e_1,e_2)}
= A_1 X_1 \xi^{i_1}_1 \otimes \xi^{i_2}_2 + (-)^{e_1} \xi^{i_1}_1 \otimes A_2 X_2 \xi^{i_2}_2
\end{equation}

**Proposition 1.** Let $(M, g), (M_i, g_i), S_i$ and $W$ be as before. The Clifford relation
\begin{equation}
(XY + YX)\Xi = -2g(X, Y)\Xi
\end{equation}
holds for all $X, Y \in TM$ and $\Xi \in W$.

**Proof.** Because of linearity it is sufficient to prove the statement for $X_i \in TM_i$ and $Y_j \in TM_j$ and totally homogenous $\Xi^e$. We have to distinguish the cases $i = j$ and $i < j$. We recall the property $\delta_k(X, \Xi^e) = \begin{cases} -\delta_k(\Xi^e) & i < k \\ \delta_k(\Xi^e) & i \geq k \end{cases}$.

With these informations we get
\[
(X_i Y_j + Y_j X_i)\Xi^e = X_i \delta_j(\Xi^e) \xi^{i_1}_1 \otimes \cdots \otimes A_j Y_j \xi^{i_j}_j \otimes \cdots \otimes \xi^{i_N}_N \\
+ Y_j \delta_i(\Xi^e) \xi^{i_1}_1 \otimes \cdots \otimes A_i X_i \xi^{i_i}_i \otimes \cdots \otimes \xi^{i_N}_N
\]
\[
= \begin{cases} \delta_i(A_i Y_j \Xi^e) \delta_j(\Xi^e) \xi^{i_1}_1 \otimes \cdots \otimes A_i X_i \xi^{i_i}_i \otimes \cdots \otimes A_j Y_j \xi^{i_j}_j \otimes \cdots \otimes \xi^{i_N}_N \\
+ \delta_j(A_i X_i \Xi^e) \delta_i(\Xi^e) \xi^{i_1}_1 \otimes \cdots \otimes A_i X_i \xi^{i_i}_i \otimes \cdots \otimes A_j Y_j \xi^{i_j}_j \otimes \cdots \otimes \xi^{i_N}_N & i < j \\
+ \delta_i(\Xi^e) \delta_i(\Xi^e) \xi^{i_1}_1 \otimes \cdots \otimes (A_i Y_j)(A_i X_i) \xi^{i_i}_i \otimes \cdots \otimes \xi^{i_N}_N & i = j \\
0 & i < j \end{cases}
\]
\[
= \begin{cases} \xi^{i_1}_1 \otimes \cdots \otimes (A_i X_i)(A_j Y_j) \xi^{i_i}_i \otimes \cdots \otimes \xi^{i_N}_N = g(X_i, Y_j)\Xi^e & i = j \\
-2g_i(A_i X_i, A_i Y_i)\Xi^e = -2g(X_i, Y_j)\Xi^e & \text{other cases} \end{cases}
\]
\[\square\]

Up to now we did not specify the subbundles of $S_i \subset S_i$. We have to distinguish two different situations:
\begin{itemize}
\item If $\dim M_i = 2n_i$ is even, the bundle $S_i$ (of rank $2^{n_i}$) itself admits a natural $\mathbb{Z}_2$-grading induced by the volume element. And we take exactly this one.
\item If $\dim M_i = 2n_i + 1$ is odd, the spinor bundle $S_i$ (of rank $2^{n_i}$) does not admit a natural $\mathbb{Z}_2$-grading. In this case we double the bundle and define $S_i^+ := S_i$ and $S_i^- := \Pi_i S_i$.
\end{itemize}
In this definition of the subbundles, \( \Pi_i : S^+_i \oplus S^-_i \to S^-_i \oplus S^+_i \) denotes the parity operator with \( (\Pi_i(S^+_i \oplus S^-_i))^\pm = S^\pm_i \) and \( \Pi^2_i = \text{id} \). Explicitly we have

\[
\Pi_i \begin{pmatrix} \xi_i^+ \\ \xi_i^- \end{pmatrix} = \begin{pmatrix} \xi_i^- \\ \xi_i^+ \end{pmatrix}.
\]

The parity operator is naturally extended to \( W \) by its action on totally homogenous elements

\[
\Pi(\Xi) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (-1)^{\epsilon_1^{i} + \cdots + \epsilon_{i-1}^{i}} \xi_1^{\epsilon_1} \otimes \cdots \otimes \Pi_i \xi_i^{\epsilon_i} \otimes \cdots \otimes \xi_N^{\epsilon_N}
\]

where we use that the parity operator is formally odd. The proof for \( \Pi^2 = \text{id} \) is similar to that for the Clifford relation.

Let \( N_e \) and \( N_o \) be the number of even and odd dimensional manifolds in the product \( M = M_1 \times \cdots \times M_N \), respectively (i.e. \( N_e + N_o = N \)). The dimension of the product manifold is \( \dim M = 2n + N_o \) with \( n := \sum_{i=1}^{N} n_i \) and the rank of the bundle \( W \) is \( 2n + N_o \).

The spinor bundle of \( M \) should have the rank \( 2n + [N_o/2] \). We will construct a subbundle \( S \) of \( W \) of this rank.

The trick is to diagonalize some of the bundles which come from the odd dimensional manifolds. The diagonalization is denoted by \( \Delta \) and the bundle \( S \) is constructed as follows:

Choose \( N_o - [N_o/2] \) of the odd dimensional bundles\(^2\) and consider the subbundle \( S \subset W \) given by

\[
S := \bigotimes_{i=1}^{N_e} S_i \otimes \bigotimes_{j=N_e+1}^{N-N_o+[N_o/2]} (S_j \oplus \Pi_j S_j) \otimes \bigotimes_{k=N-N_o+[N_o/2]+1}^{N} \Delta(S_k \oplus \Pi_k S_k)
\]

which has the rank \( 2^K \) with

\[
K = \sum_{i=1}^{N_e} n_i + \sum_{j=N_e+1}^{N-N_o+[N_o/2]} n_j + (N - N_o + [N_o/2] - N_e) + \sum_{k=N-N_o+[N_o/2]+1}^{N} n_k
\]

\[
= \sum_{i=1}^{N} n_i + [N_o/2] = n + [N_o/2]
\]

We specialize this to the case \( N = 2 \):

- \( \dim M_1 = 2n \), \( \dim M_2 = 2m \): \( S \) over \( M = M_1 \times M_2 \) is given by

\[
S = S_1 \otimes S_2
\]

- \( \dim M_1 = 2n \), \( \dim M_2 = 2m + 1 \): The subbundle \( S \) over \( M \) is given by

\[
S = S_1 \otimes \Delta(S_2 \oplus \Pi_2 S_2) \simeq S_1 \otimes S_2.
\]

\(^2\)This does not depend on the choice, because the resulting (non graded) bundles are isomorphic.
THE SPINOR BUNDLE OF RIEMANNIAN PRODUCTS

• dim \( M_1 = 2n + 1 \), \( dim \ M_2 = 2m + 1 \): In this case \( S \) is defined by

\[
S = (S_1 \oplus \Pi_1 S_1) \otimes \Delta(S_2 \oplus \Pi_2 S_2) \cong (S_1 \oplus S_1) \otimes S_2 \\
\cong S_1 \otimes S_2 \oplus S_1 \otimes S_2 \cong (S_2 \oplus S_2) \otimes S_1,
\]

where we have to emphasize that the equivalences may not respect the \( \mathbb{Z}_2 \)-grading.

That this bundle is indeed a spinor bundle will be clear from the following remark. Although we have established the Clifford multiplication, the bundle is not a priori a spinor bundle. For that it should be constructed as an associated vector bundle to the \( Spin \)-principal bundle \( P_{Spin(D)}(M) \) or of one reduction of this. If such a reduction does not exist in an appropriate way, then \( W \) is not of this kind. The reason is that we do not have an action of \( Spin(D) \) on the standard fibre of \( W \).

The maximal subgroup of \( Spin(D) \) which is compatible with the structure of the standard fibre is

\[
S(Pin(D_1) \times \cdots \times Pin(D_N)) := Pin(D_1) \times \cdots \times Pin(D_N) \cap \text{Cl}^+(D) \\
\subset \otimes_{i=1}^N \text{Cl}(D_i).
\]

This contains the subgroup \( Spin(D_1) \times \cdots \times Spin(D_N) \), which will be of interest soon.

In the next proposition we show that we are indeed able to write \( W \) as an associated bundle, if we demand the reduction of the structure group of \( M \) to \( SO(D_1) \times \cdots \times SO(D_N) \) compatible with the natural splitting \( \mathbb{H} \). This is a weaker condition than \( \mathbb{Q} \), which was at least necessary to get the Clifford multiplication.

**Proposition 2.** Let \( M \) be a Riemannian spin manifold. If the structure group of \( M \) can be reduced to \( SO(D_1) \times \cdots \times SO(D_N) \), we have a reduction of the spin principal bundle to \( Spin(D_1) \times \cdots \times Spin(D_N) \).

**Proof.** We use the following notations: \( G := Spin(D) \), \( \tilde{G} := Spin(D_1) \times \cdots \times Spin(D_N) \), \( H := SO(D) \) and \( \tilde{H} := SO(D_1) \times \cdots \times SO(D_N) \).

Let \( P_H \) be the principal bundle of orthonormal frames of \((M, g)\) and \( P_{\tilde{H}} \) be the reduction to \( \tilde{H} \), which we denote by \( \iota \). Furthermore let \( P_G \) the \( Spin \)-principal bundle over \( M \) with the two fold covering \( \lambda : P_G \rightarrow P_H \), compatible with the right action of \( G \) and \( H \) and the cover \( \lambda : G \rightarrow H \) for which we take the same symbol.

We collect this by writing \( P_G \xrightarrow{\lambda} P_H \xleftarrow{\iota} P_{\tilde{H}} \). We use the pullback construction for fibre bundles cf. ([Kolár et al. 1993]) to complete this edge to a commutative diagram and we denote the pullback by \((P_G \times P_{\tilde{H}})/P_H \).

\[
(P_G \times P_{\tilde{H}})/P_H \xrightarrow{\lambda} P_H \xleftarrow{\iota} P_{\tilde{H}}
\]

Its total space is given by

\[
\{(p_G, p_{\tilde{H}}) \in P_G \times P_{\tilde{H}} \mid \lambda(p_G) = \iota(p_{\tilde{H}})\}
\]
The bundle \((PG \times PH)/PH\) is a principal bundle with the right action of its standard fibre \(G \times \tilde{H}/H = \{(g, \tilde{h}) | \lambda(g) = \iota(\tilde{h}) = \tilde{h}\} \simeq \tilde{\lambda}^{-1}(\tilde{H}) \simeq \tilde{G}\) on the total space defined in the obvious way. We take the action on the Cartesian product \((PG, PH)\alpha = (PG\alpha, PH\lambda(\alpha))\) which is compatible with the quotient:

\[
\lambda(pG\alpha) = \lambda(pG)\lambda\alpha = \iota(pH\lambda(\alpha))
\]

This shows that the Spin-principal bundle \(PG\) is reduced to a Spin\((D_1) \times \cdots \times Spin(D_N)\)-principal bundle. The reduction is just the left vertical arrow in the diagram.

From this we get

**Theorem 3.** Let \((M = M_1 \times \cdots \times M_N, g)\) be a Riemannian manifold, given as a (not necessarily Riemannian) product of the simply connected Riemannian spin manifolds \((M_i, g_i)\). The metric \(g\) is connected to the metrics \(g_i\) via (2).

Then \(M\) is spin and the spinor bundle is isomorphic to the subbundle \(S\) of \(W\) constructed in this section.

**Proof.** That \(M\) is spin, follows immediately from the fact that the second Stiefel-Whitney class behaves additively under products of manifolds. The ON-frame bundles w.r.t. \(g\) and \(g_1 + \cdots + g_N\) are isomorphic cf. [3]. On \(M\) there exists exactly one spin structure, because \(M\) is simply connected. So the spin principal bundles obtained by \(g\) and \(g_1 + \cdots + g_N\) are isomorphic. From the last proposition we get that the Spin-principal bundle \(PG\) over \((M, g)\) is reducible to Spin\((D_1) \times \cdots \times Spin(D_N)\).

With the notations from the previous proposition and the construction for \(S \subset W\) we have established the following isomorphism

\[
(18) \quad S \simeq ((PG \times PH)/PH) \times \tilde{\hat{S}}
\]

where \(\tilde{\hat{S}}\) denotes the standard fibre of \(S\).

**Remark 4.** An important example of this construction is the case \(N = 2\) with one of the factors being one dimensional and the metric is a warped product. This has been discussed in detail in [Baum 1989a] and [Baum 1989b].

From the construction it is clear that the spin connection obtained from the Levi-Civita connection of \(g\) need not be compatible with the tensor structure of \(S\). Explicit formulas for the connection in the case of an one dimensional factor and warped products can also be found in [Baum 1989b]. By claiming that the holonomy of \(M\) is contained in SO\((D_1) \times \cdots \times SO(D_N)\) – which is the same as to say that the projections \(p_i\) are parallel – we make sure this further compatibility. As we mentioned above, this further assumption forces \(M\) to be a local Riemannian product (in the case of \(M\) being simply connected and complete this decomposition is global).

This constructions yields for \(A = \text{id}\) the following

**Proposition 5.** Let \(M\) be the Riemannian product of \((M_i, g_i)\). Then the spinor bundle of \(M\) with respect to the induced spin structure is given by \(S\) from the construction above.

We add some remarks, which explain our further notation and draw the attention to the pseudo-Riemannian case.
Remark 6.  

(1) Proposition 5 is also true in the case of metrics which are not of Euclidean signature.

(2) For the special case $N = D$, i.e. $D_1 = 1$, our constructions ends up with the Clifford action on $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ as given in [Baum et al. 1991].

(3) For another special case $N = 2$ and $D_2 = 1$ this construction leads to the discussion in Baum 1989a.

(4) For $N = 2$ in the cases $\bullet_1$ and $\bullet_2$, our construction yields the – at least in physics literature – frequently used decomposition of the $\gamma$-matrices $\Gamma^A$, for $1 \leq A \leq D$ in tensor products of $\gamma$-matrices of the respective factors $\gamma^a$, $\gamma^\alpha$ for $1 \leq a \leq D - 2n = D_2$, $1 \leq \alpha \leq 2n = D_1$. The $\mathbb{Z}_2$-grading is ensured by using the volume element $\hat{\gamma} := \gamma^1 \cdots \gamma^{2n}$ which anticommutes with all $\gamma^a$ and define

\begin{equation}
\Gamma^a = \gamma^a \otimes \hat{\gamma}, \quad \Gamma^\alpha = 1 \otimes \gamma^\alpha,
\end{equation}

compare e.g. [Duff et al. 1986].

References

[Baum 1989a] Baum, Helga: Complete Riemannian manifolds with imaginary Killing spinors. In: Ann. Global Anal. Geom. 7 (1989), No. 3, pp. 205–226

[Baum 1989b] Baum, Helga: Odd-dimensional Riemannian manifolds with imaginary Killing spinors. In: Ann. Global Anal. Geom. 7 (1989), No. 2, pp. 141–153

[Baum et al. 1991] Baum, Helga; Friedrich, Thomas; Grunewald, Ralf; Kath, Ines: Twistors and Killing Spinors on Riemannian Manifolds. Teubner-Texte zur Mathematik, 124. Stuttgart etc.: B. G. Teubner Verlagsgesellschaft. 180 p., 1991

[Duff et al. 1986] Duff, M. J.; Nilsson, B. E. W.; Pope, C. N.: Kaluza-Klein supergravity. In: Phys. Rep. 130 (1986), No. 1-2, pp. 1–142

[Joyce 2000] Joyce, Dominic D.: Compact manifolds with special holonomy. Oxford : Oxford University Press, 2000 (Oxford Mathematical Monographs). – xii+436 p

[Klinker 2003] Klinker, Frank: Supersymmetric Killing Structures, University Leipzig, Germany, PhD., 2003

[Kolár et al. 1993] Kolář, Ivan ; Michor, Peter W. ; Slovák, Jan: Natural operations in differential geometry. Berlin : Springer-Verlag, 1993. – vi+434 p

[Lawson and Michelsohn 1989] Lawson, H. B. ; Michelsohn, Marie-Louise: Princeton Mathematical Series. Bd. 38: Spin geometry. Princeton, NJ : Princeton University Press, 1989. – xii+427 p

Department of Mathematics, University of Dortmund, D–44221 Dortmund

frank.klinker@math.uni-dortmund.de