Schrödinger Operators with Singular Rank-Two Perturbations and Point Interactions

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Abstract. Norm resolvent approximation for a wide class of point interactions in one dimension is constructed. To analyse the limit behaviour of Schrödinger operators with localized singular rank-two perturbations coupled with δ-like potentials as the support of perturbation shrinks to a point, we show that the set of limit operators is quite rich. Depending on parameters of the perturbation, the limit operators are described by both the connected and separated boundary conditions. In particular an approximation for a four-parametric subfamily of all the connected point interactions is built. We give examples of the singular perturbed Schrödinger operators without localized gauge fields, which converge to point interactions with the non-trivial phase parameter. We also construct an approximation for the point interactions that are described by different types of the separated boundary conditions such as the Robin–Dirichlet, the Neumann–Neumann or the Robin–Robin types.

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1. Introduction

Solvable Schrödinger type operators have attracted considerable attention both in the physical and mathematical literature in recent years. Such the operators are of interest in applications of mathematics in different fields of science and engineering. The so-called solvable models that are based upon the concept of point interactions also often appear in quantum theory and allow us to calculate explicitly spectral characteristics of systems such as eigenvalues, eigenfunctions or scattering data. The Schrödinger operators with singular distributional potentials supported on discrete sets reveal an
unquestioned effectiveness whenever the exact solvability together with non trivial description of the actual process is required. It is an extensive subject with a large literature, see the books by Albeverio et al. [1] and Albeverio and Kurasov [2] discussing point interactions and more general singular perturbations of the Schrödinger operators and the extensive bibliography lists therein.

In spite of all advantages of the solvable models, they give rise to many mathematical difficulties. One of the main difficulty deals with the problem of defining a multiplication of distributions. It entails that many Schrödinger operators with singular potentials are often only formal differential expressions without a precise mathematical meaning. We cite two linear differential equations with distributions contained in the coefficients as an example.

First let us consider the Schrödinger equation $-y'' + \delta(x)y = k^2y$ with the $\delta$ potential. Here $\delta$ is the Dirac delta-function and $\delta(x)y(x) = y(0)\delta(x)$. It can be also written in the form $-y'' + \langle \delta(x), y \rangle \delta(x) = k^2y$. It is well-known that both the equations have the same 2-dimensional space of solutions in the sense of distributions. All the solutions are continuous at the origin and therefore the product $\delta(x)y(x)$ and the value $\langle \delta(x), y \rangle$ are well-defined. Both the differential expressions $-\frac{d^2}{dx^2} + \delta(x)$ and $-\frac{d^2}{dx^2} + \langle \delta(x), \cdot \rangle \delta(x)$ could be interpreted as the same self-adjoint operator in $L^2(\mathbb{R})$, defined by $Sy = -y''$ on functions in $W^2_2(\mathbb{R} \setminus \{0\})$ obeying the interface conditions $y(-0) = y(+0), y'(+0) - y'(-0) = y(0)$.

At the same time, the equation $-y'' + \delta'(x)y = k^2y$ with the first derivative of the Dirac delta-function as a potential has no mathematical sense, because for it no solution exists in the space of distributions, except the trivial one. Indeed, the product $\delta'(x)y = y(0)\delta'(x) - y'(0)\delta(x)$ is well defined for $y$ that is continuously differentiable at the origin. But this is impossible for a non-trivial solution, because its second derivative is the singular distribution $y(0)\delta'(x) - y'(0)\delta(x) + k^2y$. The equation $-y'' + \langle \delta(x), y \rangle \delta'(x) + \langle \delta'(x), y \rangle \delta(x) = k^2y$, in which potential $\delta'(x)$ is treated as the rank-two perturbation, is also meaningless.

Hence the situation is more obscure with definition of the Schrödinger operators with potential $\delta'$, and one must be careful in using the formal differential expressions

$$ -\frac{d^2}{dx^2} + \alpha \delta'(x) + \beta \delta(x), \quad (1.1) $$

$$ -\frac{d^2}{dx^2} + \alpha(\langle \delta'(x), \cdot \rangle \delta(x) + \langle \delta(x), \cdot \rangle \delta'(x)) + \beta \delta(x). \quad (1.2) $$

However, such the pseudo-Hamiltonians often appear in the models of quantum devices with barrier-well junctions. To get round the problem of multiplication of distributions, we can regularize $\delta'$ by smooth enough localized potentials and then investigate convergence of the Schrödinger operators with the regular potentials. The main goal is to find the limit operator and assign for the quantum system a solvable model (i.e., a point interaction) that governs the quantum process of the true interaction with adequate accuracy. Note that such the results depend on shapes of the approximation sequences.
From a physical point of view, this means that there are many different \(\delta'\) potentials”, namely, the quantum devices with \(\delta'\)-like potentials of various shapes exhibit the different properties.

The Schrödinger operators with \((\alpha\delta' + \beta\delta)\)-like potentials, i.e., the regularization of the pseudo-Hamiltonian (1.1), was studied in [16–20]. The norm resolvent convergence of the corresponding families of operators was established and a class of solvable models that approximate the quantum systems with barrier-well junctions was obtained. The result of [29] about the regularization of \(\delta'\)-potential was revised and adjusted in [17]. Different families of Schrödinger operators with potentials of the dipole type using a regularization by rectangles in the form of a barrier and a well were treated by Zolotaryuk (partly with coauthors) in [15,30–33].

In this paper we study families of Schrödinger operators with localized singular rank-two perturbations coupled with \(\delta\)-like potentials. These operators can be regarded as the regularization of the pseudo-Hamiltonian (1.2), but only in a special case. A careful analysis actually shows that the families describe a variety of quantum interactions and the set of all limit operators, which can be obtained in the norm resolvent topology as the support of perturbation shrinks to the origin, contains a wide class of point interactions. The limit operators are described by both the connected and separated boundary conditions. In the first case, we obtained the approximation for a four-parametric subfamily of all the connected point interactions with a complete matrix in the boundary conditions. Moreover an unexpected fact is that the point interactions with non-trivial phase parameter appear in the limit, although the perturbed Schrödinger operators contain no localized magnetic field. We also constructed an approximation for point interactions that are described by different types of the separated boundary conditions such as the Robin–Dirichlet, the Neumann–Neumann or the Robin–Robin types. A partial case of the problem has been recently published in [21].

Problems of this nature have a long history and the literature on approximation for point interactions as well as finite rank perturbations of the Schrödinger operators is extensive. Among all zero range interactions, the \(\delta'\)-interactions, along with \(\delta\) and \(\delta'\) potentials, are most studied in this kind of research. We want to especially note the paper [4,9–12,28,29] and the references therein. This special case has attracted much attention recently [7,8,24,33]. Many authors have dealt with finite rank perturbations and their relationship with the point interactions. In particular we mention papers on singular finite rank perturbations and nonlocal potentials [3–6,22,23,25–27].

2. Statement of Problem, Main Results and Discussion

From now on, the scalar product and norm in \(L_2(\mathbb{R})\) will be denoted by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\) respectively. Let us consider the Schrödinger operator

\[
H_0 = -\frac{d^2}{dx^2} + V(x)
\]
in $L_2(\mathbb{R})$, where potential $V$ is real-valued, measurable and locally bounded. We also assume that $V$ is bounded from below in $\mathbb{R}$. Let $f$ and $g$ be complex-valued and compactly supported functions in $L_2(\mathbb{R})$ that are linearly independent. We denote by $Q_\varepsilon$ the rank-two operators
\[
(Q_\varepsilon v)(x) = \langle g(\varepsilon^{-1} \cdot), v \rangle f(\varepsilon^{-1} x) + \langle f(\varepsilon^{-1} \cdot), v \rangle g(\varepsilon^{-1} x)
\]
acting in $L_2(\mathbb{R})$. Let us consider the family of self-adjoint operators
\[
H_\varepsilon = H_0 + \varepsilon^{-3} Q_\varepsilon + \varepsilon^{-1} q(\varepsilon^{-1} x),
\]
where $q$ is an integrable real-valued bounded function of compact support. Since the perturbation of $H_0$ has a compact support, we have $\text{dom } H_\varepsilon = \text{dom } H_0$.

One of the questions of our primary interest in this paper is to understand the limiting behavior of the operators $H_\varepsilon$ as the small positive parameter $\varepsilon$ goes to zero, i.e., as the support of perturbation shrinks to the origin. An asymptotic analysis of $H_\varepsilon$ leads us to a few cases of norm resolvent limits. This limiting behaviour is governed primarily by $f$ and $g$ as well as their interaction with the potential $q$.

We introduce notation
\[
f_0 = \int f \, dx, \quad g_0 = \int g \, dx, \quad f_1 = \int x f \, dx, \quad g_1 = \int x g \, dx.
\]
Let us denote by $h^{(-1)}(x) = \int_{-\infty}^{x} h(s) \, ds$ and $h^{(-2)}(x) = \int_{-\infty}^{x} (x - s) h(s) \, ds$ the first and second antiderivatives of a function $h$. The antiderivatives are well-defined for measurable functions of compact support, for instance. In addition, if $h$ has zero mean, then $h^{(-1)}$ is also a function of compact support. We will henceforth use notation
\[
\pi = \|f^{(-1)}\| \cdot \|g^{(-1)}\| - \|\langle f^{(-1)}, g^{(-1)} \rangle + 1],
\]
\[
\omega = e^{i \text{arg}(\langle f^{(-1)}, g^{(-1)} \rangle + 1)} \frac{\|g^{(-1)}\| f^{(-2)} - \|f^{(-1)}\| g^{(-2)}}{\|f^{(-1)}\| g^{(-2)}} ,
\]
\[
a_0 = \int q \, dx, \quad a_1 = \int q \, \omega \, dx, \quad a_2 = \int q \, |\omega|^2 \, dx.
\]
that will be correct only if $f$ and $g$ have zero means, i.e., $f_0 = 0$ and $g_0 = 0$. In this case, $\omega$ is constant outside some interval that contains the supports of $f$ and $g$. We write
\[
\nu = \lim_{x \to +\infty} \omega(x).
\]
Of course, $\lim_{x \to -\infty} \omega(x) = 0$. We also set
\[
\lambda = \|g_0 f^{(-1)} - f_0 g^{(-1)}\|^2 - 2 \text{Re} (f_0 \bar{g}_0),
\]
\[
\sigma = |g_0|^2 \left( f_0 f^{(-2)} - \langle f, f^{(-2)} \rangle \right) - |f_0|^2 \left( \bar{g}_0 g^{(-2)} - \langle g, g^{(-2)} \rangle \right),
\]
\[
\sigma_- = \lim_{x \to -\infty} \sigma(x), \quad \sigma_+ = \lim_{x \to +\infty} \sigma(x), \quad \sigma_* = \int q |\sigma|^2 \, dx.
\]
Remark that \( g_0 f^{(-1)} - f_0 g^{(-1)} \) belongs to \( L_2(\mathbb{R}) \), because \( g_0 f - f_0 g \) is a function of zero mean. The limits \( \sigma_- \) and \( \sigma_+ \) also exist (to be proved later in Lemma 3.2).

Let us introduce the subspace \( \mathcal{V} \subset L_2(\mathbb{R}) \) as follows. We say that \( h \) belongs to \( \mathcal{V} \) if there exist two functions \( h_- \) and \( h_+ \) belonging to \( \text{dom} \, H_0 \) such that \( h(x) = h_-(x) \) if \( x < 0 \) and \( h(x) = h_+(x) \) if \( x > 0 \). Let \( (c_{ij}) \) be a square matrix of order 2 with real elements and \( \varphi \in \mathbb{R} \). We denote by \( \mathcal{H} \) the operator defined by

\[
\mathcal{H} v = -\frac{d^2 v}{dx^2} + V(x)v
\]

on functions \( v \in \mathcal{V} \) obeying the coupling conditions

\[
\begin{pmatrix}
v_+ \\
v'_+
\end{pmatrix}
= e^{i\varphi} \begin{pmatrix} c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix}
\begin{pmatrix}
v_- \\
v'_-
\end{pmatrix}, \quad (2.4)
\]

at the origin. Here \( v_- = v(-0) \), \( v_+ = v(+0) \) and \( i^2 = -1 \). Remark that \( \mathcal{H} \) is self-adjoint if and only if \( c_{11}c_{22} - c_{12}c_{21} = 1 \).

The following theorem collects the cases of limiting behaviour of \( H_\varepsilon \) in which the limit operators describe non-trivial point interactions; these cases are of special interest in the scattering theory. Under a non-trivial point interaction we understand the point interaction that describes by boundary conditions (2.4).

**Theorem 1.** Let \( f, g : \mathbb{R} \to \mathbb{C} \) and \( q : \mathbb{R} \to \mathbb{R} \) be integrable functions of compact support, and \( f \) and \( g \) are linearly independent in \( L_2(\mathbb{R}) \).

**A1.** If \( f_0 = 0 \), \( g_0 = 0 \), \( \pi = 0 \) and \( a_2 \neq \overline{a}_1 \), then operators \( H_\varepsilon \) converge in the norm resolvent sense as \( \varepsilon \to 0 \) to the operator \( \mathcal{H} \) with coupling conditions

\[
\begin{pmatrix}
v_+ \\
v'_+
\end{pmatrix}
= e^{i \arg(a_2 - \overline{a}_1)} \begin{pmatrix}
|x|^2 a_0 - 2 \text{Re}(\overline{a}_1 a_2) + a_2 \\
\frac{|a_2 - \overline{a}_1|}{a_0 a_2 - |a_1|^2} \\
\frac{|a_2 - \overline{a}_1|}{|a_2 - \overline{a}_1|}
\end{pmatrix}
\begin{pmatrix}
v_- \\
v'_-
\end{pmatrix}, \quad (2.5)
\]

**A2.** Suppose \( \lambda = 0 \), \( f_0g_0 \neq 0 \) and \( \sigma_- \sigma_+ \neq 0 \), then \( \sigma_+ \) is a real number and \( H_\varepsilon \) converge to operator \( \mathcal{H} \) in the norm resolvent sense, where \( \text{dom} \, \mathcal{H} \) consists of functions \( v \in \mathcal{V} \) such that

\[
\begin{pmatrix}
v_+ \\
v'_+
\end{pmatrix}
= e^{-i \arg \sigma_-} \begin{pmatrix}
\sigma_+ \\
\frac{\sigma_+}{\sigma_-} \\
\frac{\sigma_+}{\sigma_-}
\end{pmatrix}
\begin{pmatrix}
v_- \\
v'_-
\end{pmatrix}.
\]

**A3.** Assume \( f_0 = 0 \), \( g_0 = 0 \) and one of the following conditions holds

\( \circ \pi \neq 0 \);

\( \circ \pi = 0 \), \( \kappa = 0 \), \( a_1 = 0 \) and \( a_2 = 0 \).

Then resolvents of \( H_\varepsilon \) converge in norm as \( \varepsilon \to 0 \) to the resolvent of operator \( \mathcal{H} \) with coupling conditions

\[
\begin{pmatrix}
v_+ \\
v'_+
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\
a_0 & 1 \end{pmatrix}
\begin{pmatrix}
v_- \\
v'_-
\end{pmatrix}, \quad (2.7)
\]
Remark that function $g_0f^{(-1)} - f_0g^{(-1)}$ belongs to $L_2(\mathbb{R})$, because $g_0f - f_0g$ has the zero mean.

Let $\mathcal{V}_-$ and $\mathcal{V}_+$ be the spaces obtained by the restriction of all elements of $\mathcal{V}$ to $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively. We introduce the operators

$$D_\pm = -\frac{d^2}{dx^2} + V, \quad \text{dom } D_\pm = \{ v \in \mathcal{V}_\pm : v(0) = 0 \};$$

$$R_\pm(\theta) = -\frac{d^2}{dx^2} + V, \quad \text{dom } R_\pm = \{ v \in \mathcal{V}_\pm : v'(0) = \theta v(0) \},$$

where $\theta \in \mathbb{R}$. In the next theorem we will assemble together all cases, when the limit operator is a direct sum of two operators acting independently on the negative and positive semiaxes. The corresponding point interactions are described by the boundary conditions

$$\alpha_1 v_- + \beta_1 v'_- = 0, \quad \alpha_2 v_+ + \beta_2 v'_+ = 0 \quad (2.8)$$

with real coefficients $\alpha_k$ and $\beta_k$. These conditions are called separated in contrast to conditions (2.4), which are called connected.

**Theorem 2.** Let $f, g : \mathbb{R} \to \mathbb{C}$ and $q : \mathbb{R} \to \mathbb{R}$ be integrable functions of compact support, and $f$ and $g$ are linearly independent in $L_2(\mathbb{R})$.

**B1.** Suppose that $f_0 = 0, g_0 = 0, \pi = 0, \chi \neq 0$, and $a_2 = \mp a_1$. Then operators $H_\varepsilon$ converge to the direct sum $R_-(\theta_1) \oplus R_+(\theta_2)$ as $\varepsilon \to 0$ in the norm resolvent sense, where $\theta_1 = |\chi|^{-2} a_2 - a_0$ and $\theta_2 = |\chi|^{-2} a_2$.

**B2.** If $\lambda = 0$, $f_0g_0 \neq 0$, $f_1g_0 \neq f_1g_0$ and $\sigma_- \sigma_+ = 0$, then

$$H_\varepsilon \to \begin{cases} D_- \oplus R_+(\theta_+), & \text{if } \sigma_- = 0, \\ R_-(\theta_-) \oplus D_+, & \text{if } \sigma_+ = 0, \end{cases} \quad \text{as } \varepsilon \to 0$$

in the norm resolvent sense, where $\theta_+ = \sigma_+ |\sigma_+|^{-2}$ and $\theta_- = -\sigma_- |\sigma_-|^{-2}$.

**B3.** Suppose that one of the following conditions holds

- $\lambda \neq 0$;
- $\lambda = 0$, $f_0g_0 \neq 0$, $f_0g_1 = f_1g_0$, $\sigma_- = 0$ and $\sigma_+ = 0$;
- $f_0 = 0$, $g_0 = 0$, $\pi = 0$, $\chi = 0$, $a_2 = 0$ and $a_1 \neq 0$.

Then the operator family $H_\varepsilon$ converges to the direct sum $D_- \oplus D_+$ in the norm resolvent sense.

As depicted in the graph (Fig. 1), Theorems 1 and 2 cover all limit cases as $\varepsilon \to 0$. We need only note explicitly that if $f_0g_1 = f_1g_0$, then $\sigma_- = \sigma_+$ (we will prove this fact below). We also remark that the case when $\|g_0f^{(-1)} - f_0g^{(-1)}\|^2 = 2 \text{Re}(f_0g_0)$, $f_0g_0 = 0$, but only one of mean values $f_0$ and $g_0$ equals zero (the node “x” of the graph), is impossible under our assumption about linear independence of $f$ and $g$. For instance, if $f_0 = 0$ and $g_0 \neq 0$, then condition $\|g_0f^{(-1)}\|^2 = 0$ yields $f = 0$.

Theorems 1 and 2 can be summarized by saying that the family of operators $H_\varepsilon$ always converges in the norm resolvent sense.

**Theorem 3.** Let $f, g : \mathbb{R} \to \mathbb{C}$ and $q : \mathbb{R} \to \mathbb{R}$ be integrable functions of compact support. Assume $f$ and $g$ are linearly independent in $L_2(\mathbb{R})$ and define the sequences of scaled functions $f^{\varepsilon} = f(\varepsilon^{-1} \cdot)$, $g^{\varepsilon} = g(\varepsilon^{-1} \cdot)$ and $q^{\varepsilon} = q(\varepsilon^{-1} \cdot)$. 


Figure 1. Bifurcation graph of limiting behaviour of $H_\varepsilon$

Then operator family $H_\varepsilon = H_0 + \varepsilon^{-3}\langle g^\varepsilon, \cdot \rangle f^\varepsilon + \varepsilon^{-3}\langle f^\varepsilon, \cdot \rangle g^\varepsilon + \varepsilon^{-1}q^\varepsilon$ converges in the norm resolvent sense as $\varepsilon \to 0$ to some operator $\mathcal{H} = \mathcal{H}(f,g,q)$ and we have estimate

$$\| (H_\varepsilon - \zeta)^{-1} - (\mathcal{H} - \zeta)^{-1} \| \leq C\varepsilon^{1/2}$$

for all $\zeta \in \mathbb{C} \setminus \mathbb{R}$, where the constant $C$ does not depend on $\varepsilon$. The limit operator $\mathcal{H}$ is described in Theorems 1 and 2.

Remark that the $\delta$-like sequence $\varepsilon^{-1}q^\varepsilon$ is obviously subordinated to the rank-two perturbation as $\varepsilon \to 0$, nevertheless it has a considerable influence on the limit behaviour of $H_\varepsilon$. We should note that the most interesting case A1 is possible only if the potential $q$ is different from zero.

All the cases A1–A3, B1–B3 can be realized by a proper choice of triple $(f,g,q)$. For instance, we explain how to choose the triple in case A1. Let us consider two functions $F$ and $G$ of compact support, belonging to $W^1_2(\mathbb{R})$, such that $\|F\| = \|G\| = 1$ and $\langle F, G \rangle = 0$. Then $f = F'$ and $g = G'$ have zero means and satisfy the condition $\pi = 0$. Next, $\omega = F^{(-1)} - G^{(-1)}$, we can calculate $\varkappa = \omega(+\infty)$. The linear independence of $F$ and $G$ implies the linear independence of $f$ and $g$ and also the linear independence of functions 1, $\omega$ and $|\omega|^2$ on each interval $[-r, r]$. Therefore for all $(a_0, a_2) \in \mathbb{R}^2$ and $a_1 \in \mathbb{C}$ there exists a potential $q$ of compact support satisfying (2.2). In particular, the potential can be chosen in such a way that $a_2 \neq \Re a_1$. 
In this connection, the next question arises as to whether any real matrix \((c_{kl})\) with the unit determinant can be realized in coupling conditions (2.5) for some \(f, g\) and \(q\). The answer is negative, the matrix
\[
\begin{pmatrix}
\alpha & 0 \\
\beta & \alpha^{-1}
\end{pmatrix}
\] (2.10)
with \(\alpha \neq 1\) is a counterexample. In fact, if we put \(\kappa = 0\) in (2.5), we immediately obtain the matrix with the unit diagonal. However, such matrices appear in case A2. It is worth mentioning that the point interactions given by (2.10) also arise in analysis of the Schrödinger operators with \((a\delta' + b\delta)\)-like potentials [13,14,18,20,32].

Now we will discuss a few special subcases.

**Regularization of pseudo-Hamiltonian (1.2).** If we suppose that \(f_0 = \alpha, g_0 = 0\) and \(g_1 = -1\), then \(\varepsilon^{-1}f(\varepsilon^{-1}x) \rightarrow \alpha\delta(x)\) and \(\varepsilon^{-2}g(\varepsilon^{-1}x) \rightarrow \delta'(x)\), as \(\varepsilon \rightarrow 0\), in the sense of distributions. Then the family \(H_{\varepsilon}\) can be treated as the regularization of the formal operator (1.2) with \(\beta = a_0\). Suppose that \(\alpha\) is different from zero. Since \(\lambda = \alpha^2\|q(-1)\|^2 \neq 0\), we fall into the conditions of case B3, and so \(H_{\varepsilon}\) converge to the direct sum \(D_- \oplus D_+\) in the norm resolvent sense.

**Generalized \(\delta'\)-interactions.** Suppose that \(f\) and \(g\) are real-valued functions. Under the assumptions of case A1, we assume that \(a_0a_2 = a_1^2\). Then operators \(H_{\varepsilon}\) give us the approximation to the point interactions with matrix
\[
\begin{pmatrix}
\alpha & \beta \\
0 & \alpha^{-1}
\end{pmatrix},
\]
where \(\alpha = a_2^{-1}(a_2 - \chi a_1)\) and \(\beta = (a_2 - \chi a_1)^{-1}\chi^2\). This case has been treated recently in [21]. In particular, if \(a_1 = 0\), then \(\alpha = 1\) and \(\beta = \chi^2 a_2^{-1}\). So we obtain the new approximation to the classic \(\delta'\)-interactions of strength \(\beta\).

**Exotic point interactions.** The case A1 also contains a few types of specific limit point interactions. For instance, if we choose potential \(q\) such that \(a_2 = 0\), but \(a_1 \neq 0\), then the limit operator \(H\) is associated with the point interactions
\[
e^{i\phi} \begin{pmatrix}
\beta & -\alpha \\
\alpha^{-1} & 0
\end{pmatrix},
\]
where \(\alpha = -|\chi||a_1|^{-1}\), \(\beta = |\chi|^{-1}a_1^{-1}(|\chi|^2a_0 - 2\text{ Re}(\overline{a}_1)), \varphi = \text{ arg}(a_2 - \overline{a}_1)\).

In the case when \(a_2 = 2\text{ Re}(\overline{a}_1) - |\chi|^2a_0\), operators \(H_{\varepsilon}\) converge to the operator which describes the point interactions
\[
e^{i\phi} \begin{pmatrix}
0 & -\alpha \\
\alpha^{-1} & \beta
\end{pmatrix},
\]
where \(\alpha = -|\chi|^2|a_2 - \overline{a}_1|^{-1}\) and \(\beta = a_2|a_2 - \overline{a}_1|^{-1}, \varphi = \text{ arg}(a_1 \chi - a_0 |\chi|^2)\).

It is worth noting that Theorem 2 provides the approximation to almost all point interactions given by separated boundary conditions (2.8)—the Robin–Robin type (case B1), the Neumann–Neumann (case B1 with \(q = 0\)), the Robin-Dirichlet and Dirichlet-Robin types (case B2) and the Dirichlet-Dirichlet type (case B3).
3. Half-Bound States

Let us consider the operator

\[ B = -\frac{d^2}{dx^2} + \langle g, \cdot \rangle f(x) + \langle f, \cdot \rangle g(x), \quad \text{dom } B = W_2^2(\mathbb{R}), \]

in \( L_2(\mathbb{R}) \). We will introduce the notion of half-bound state, which plays a crucial role in our considerations.

**Definition 3.1.** We say that the operator \( B \) possesses a half-bound state provided there exists a nontrivial solution of the equation \(-u'' + (g, u)f + \langle f, u \rangle g = 0 \) that is bounded on the whole line.

In this case we also say that \( B \) has a zero-energy resonance. All half-bound states of \( B \) form a linear space.

**Lemma 3.2.** The operator \( B \) possesses a half-bound state if and only if

\[ \|g_0 f^{(-1)} - f_0 g^{(-1)}\|^2 = 2 \text{Re } (f_0 g_0). \]  

(3.1)

The operator can possess one or two linearly independent half-bound states.

(i) If (3.1) holds and \( f_0 g_0 \neq 0 \), then \( B \) has the half-bound state

\[ \sigma = |g_0|^2 \left( f_0 f^{(-2)} - \langle f, f^{(-2)} \rangle \right) - |f_0|^2 \left( g_0 g^{(-2)} - \langle g, g^{(-2)} \rangle \right). \]  

(3.2)

(ii) If \( f_0 = 0, g_0 = 0 \) and \( \pi \neq 0 \), then the only constant function is a half-bound state of \( B \).

(iii) (Double zero-energy resonance) If \( f_0 = 0, g_0 = 0 \) and \( \pi = 0 \), then there exist two linearly independent half-bound states of \( B \), namely the constant function and

\[ \omega = e^{i \arg(\langle f^{(-1)}, g^{(-1)} \rangle + 1)} \|g^{(-1)}\| f^{(-2)} - \|f^{(-1)}\| g^{(-2)}. \]

**Proof.** Equation \( Bu = 0 \) has the general solution

\[ u = c_1 f^{(-2)} + c_2 g^{(-2)} + c_3 + c_4 x, \]

where the constants \( c_k \) satisfy conditions

\[ \langle f, f^{(-2)} \rangle c_1 + (\langle f, g^{(-2)} \rangle - 1) c_2 + \bar{f}_0 c_3 + \bar{f}_1 c_4 = 0, \]

\[ (\langle g, f^{(-2)} \rangle - 1) c_1 + \langle g, g^{(-2)} \rangle c_2 + \bar{g}_0 c_3 + \bar{g}_1 c_4 = 0. \]

These conditions are derived from the equation in view of the linear independence of \( f \) and \( g \). Next, \( u(x) = c_3 + c_4 x \) for large \( x \) with large absolute value, since \( f^{(-2)} \) and \( g^{(-2)} \) vanish in a neighbourhood of the negative infinity. Therefore \( c_4 = 0 \), because we look for bounded solutions. On the other hand,

\[ f^{(-2)}(x) = f_0 x - f_1, \quad g^{(-2)}(x) = g_0 x - g_1 \]  

(3.3)

for large positive \( x \). Indeed, if \( x \) lies on the right of \( \text{supp } f \), then

\[ f^{(-2)}(x) = \int_{-\infty}^{x} (x - s) f(s) \, ds = x \int_{\mathbb{R}} f(s) \, ds - \int_{\mathbb{R}} s f(s) \, ds = f_0 x - f_1. \]

We conclude from (3.3) that \( u(x) = (c_1 f_0 + c_2 g_0)x - c_1 f_1 - c_2 g_1 + c_3 \) for large positive \( x \), and hence that \( f_0 c_1 + g_0 c_2 = 0 \), since \( u \) is bounded. Therefore
vector \( \vec{c} = (c_1, c_2, c_3) \) must be a non-trivial solution of the homogeneous linear system \( A\vec{c} = 0 \) with matrix

\[
A = \begin{pmatrix}
    \langle f, f^{(-2)} \rangle & \langle f, g^{(-2)} \rangle - 1 & \bar{f}_0 \\
    \langle g, f^{(-2)} \rangle - 1 & \langle g, g^{(-2)} \rangle & \bar{g}_0 \\
    f_0 & g_0 & 0
\end{pmatrix}.
\]

For all zero mean functions \( v, w \) of compact support, integrating by parts yields

\[
\langle v, w^{(-2)} \rangle = -\langle v^{(-1)}, w^{(-1)} \rangle, \quad \langle v, v^{(-2)} \rangle = -\|v^{(-1)}\|^2. \tag{3.4}
\]

Then a direct calculation verifies

\[
\det A = -\langle g_0 f - f_0 g, g_0 f^{(-2)} - f_0 g^{(-2)} \rangle - 2 \text{Re} (f_0 \bar{g}_0) = \|g_0 f^{(-1)} - f_0 g^{(-1)}\|^2 - 2 \text{Re} (f_0 \bar{g}_0),
\]

since \( g_0 f - f_0 g \) is a function of zero mean. Hence operator \( B \) possesses a half-bound state if and only if (3.1) holds, i.e., \( \lambda = 0 \).

Suppose that \( f_0 g_0 \neq 0 \) and matrix \( A \) is degenerate. It can only happen if the first and second rows of \( A \) are linearly dependent. In particular, from this we deduce

\[
\bar{g}_0 \left( \langle f, g^{(-2)} \rangle - 1 \right) = \bar{f}_0 \langle g, g^{(-2)} \rangle. \tag{3.5}
\]

Next, vector \( (g_0, -f_0, c_3) \) satisfies the third equation of system \( A\vec{c} = 0 \) for all \( c_3 \). Substituting the vector to the first equation yields

\[
g_0 \langle f, f^{(-2)} \rangle - f_0 (\langle f, g^{(-2)} \rangle - 1) + f_0 c_3 = 0.
\]

From (3.5) we have \( \bar{f}_0 \bar{g}_0 c_3 = |f_0|^2 \langle g, g^{(-2)} \rangle - |g_0|^2 \langle f, f^{(-2)} \rangle \). Hence the vector

\[
\left( |g_0|^2 \bar{f}_0, -|f_0|^2 \bar{g}_0, |f_0|^2 \langle g, g^{(-2)} \rangle - |g_0|^2 \langle f, f^{(-2)} \rangle \right)
\]

solves \( A\vec{c} = 0 \) and therefore function

\[
\sigma(x) = |g_0|^2 \bar{f}_0 f^{(-2)}(x) - |f_0|^2 \bar{g}_0 g^{(-2)}(x) + |f_0|^2 \langle g, g^{(-2)} \rangle - |g_0|^2 \langle f, f^{(-2)} \rangle
\]

is a half-bound state of \( B \).

In the cases (ii) and (iii), \( f \) and \( g \) have zero means. The matrix \( A \) becomes

\[
A = -\begin{pmatrix}
    \|f^{(-1)}\|^2 & \langle f^{(-1)}, g^{(-1)} \rangle + 1 & 0 \\
    \langle g^{(-1)}, f^{(-1)} \rangle + 1 & \|g^{(-1)}\|^2 & 0 \\
    0 & 0 & 0
\end{pmatrix}, \tag{3.6}
\]

by (3.4). The rank of \( A \) equals 1 if and only if \( \|f^{(-1)}\| \|g^{(-1)}\| = |\langle f^{(-1)}, g^{(-1)} \rangle| + 1 \), i.e., \( \pi = 0 \), where \( \pi \) is given by (2.1). Then the kernel of \( A \) is the span of two vectors \( \vec{c}_1 = (0, 0, 1) \) and \( \vec{c}_2 = (e^{i\vartheta} \|g^{(-1)}\|, -\|f^{(-1)}\|, 0) \), where \( \vartheta = \arg (\langle f^{(-1)}, g^{(-1)} \rangle + 1) \). In fact, substituting \( \vec{c}_2 \) into the first equation gives us

\[
e^{i\vartheta} \|f^{(-1)}\| \|g^{(-1)}\| - \|f^{(-1)}\| (\langle f^{(-1)}, g^{(-1)} \rangle + 1)
\]

\[= \|f^{(-1)}\| \left( e^{i\vartheta} |\langle f^{(-1)}, g^{(-1)} \rangle + 1| - \langle f^{(-1)}, g^{(-1)} \rangle - 1 \right) = 0.
\]
Figure 2. Half-bound state of $B$

Hence operator $B$ possesses half-bound states $1$ and $\omega$. Of course, the only constant function is a half-bound state of $B$ if $\pi \neq 0$. □

The last lemma partially explains the origination of some conditions in Theorems 1 and 2.

4. Auxiliary Statements

From now on, we assume that the supports of $f$, $g$ and $q$ lie in interval $I = [-1, 1]$. This involves no loss of generality. Then a half-bound state of $B$ is constant outside the interval $I$ as a solution of equation $u'' = 0$, which is bounded at infinity (see Fig. 2). Therefore the restriction of $u$ to $I$ is a nonzero solution of the Neumann boundary value problem

$$-u'' + (g, u)f + (f, u)g = 0 \quad \text{in } I, \quad u'(-1) = 0, \quad u'(1) = 0, \quad (4.1)$$

where $(\cdot, \cdot)$ is the scalar product in $L_2(I)$.

Given $r \in L_2(I)$ and $a, b \in \mathbb{C}$, we consider the nonhomogeneous problem

$$-v'' + (g, v)f + (f, v)g = r \quad \text{in } I, \quad v'(-1) = a, \quad v'(1) = b. \quad (4.2)$$

If operator $B$ has a half-bound state, i.e., (4.1) admits a non-trivial solution, then problem (4.2) is generally unsolvable. In this case, even if the problem has a solution for some $r$, $a$ and $b$, this solution is ambiguously determined, according to Fredholm’s alternative. But we can always choose a solution of (4.2) for which the estimate

$$\|v\|_{W^2_2(I)} \leq c(|a| + |b| + \|r\|_{L_2(I)}) \quad (4.3)$$

holds with some constant $c$ depending only on $f$ and $g$.

**Proposition 4.1.**

(i) Suppose that operator $B$ possesses the half-bound state $\sigma$ given by (3.2). Then problem (4.2) admits a solution if and only if

$$a\sigma_- - b\sigma_+ = (\sigma, r). \quad (4.4)$$

(ii) If the only constant function is a half-bound state of $B$, then problem (4.2) is solvable iff $a - b = (1, r)$.

(iii) If $B$ has the double zero-energy resonance, then (4.2) is solvable iff

$$a - b = (1, r), \quad b\omega = - (\omega, r). \quad (4.5)$$

(iv) Suppose that (4.2) is solvable for given $a$, $b$ and $r$. Then it always admits a solution that satisfies (4.3).
Proof. Parts (i)–(iii) are a simple consequence of Fredholm’s alternative for the self-adjoint operator

$$B_0 = -\frac{d^2}{dx^2} + (g, \cdot) f + (f, \cdot) g,$$

$$\text{dom } B_0 = \{ h \in W^2_2(\mathcal{I}) : h'(-1) = 0, h'(1) = 0 \}$$

in space $L_2(\mathcal{I})$. For instance, two solvability conditions (4.5) can be easily obtained by multiplying equation (4.2) by $y_1$ and $\overline{y}$ in turn and then integrating by parts twice in view of the boundary conditions. According to Fredholm’s alternative, these conditions are also sufficient.

The problem (4.1) has a trivial solution only, if $\lambda \neq 0$. Then (4.2) is uniquely solvable for all $a, b, r,$ and the solution satisfies (4.3). Otherwise, there are infinitely many solutions of (4.1), and therefore (4.2) is solvable under the conditions stated above. The proof of part (iv) is similar for all the cases (i)–(iii) of non-uniqueness. We will focus our attention on more difficult case (iii). Now since $f_0 = 0$ and $g_0 = 0$, antiderivatives $f^{(-1)}$ and $g^{(-1)}$ have compact supports lying in $\mathcal{I}$. Hence

$$f^{(-1)}(-1) = 0, \quad g^{(-1)}(-1) = 0, \quad f^{(-1)}(1) = 0, \quad g^{(-1)}(1) = 0. \quad (4.6)$$

Let us find a partial solution of (4.2) of the form

$$v_* = c_1 f^{(-1)} + c_2 g^{(-1)} - r^{(-1)} + a(x + 1),$$

where $r^{(-1)}(x) = \int_{-1}^{x}(x - s) r(s) \, ds$. For all $c_1$ and $c_2$ function $v_*$ satisfies boundary conditions in (4.2). Indeed, from (4.6) and the first solvability condition in (4.5) we see that

$$v'_*(-1) = c_1 f^{(-1)}(-1) + c_2 g^{(-1)}(-1) - r^{(-1)}(-1) + a = a,$n$$

$$v'_*(1) = c_1 f^{(-1)}(1) + c_2 g^{(-1)}(1) - r^{(-1)}(1) + a = a - (1, r) = b,$n$$

since $r^{(-1)}(1) = (1, r)$. Let us introduce the temporary notation $n_f, n_g$ and $p$ for $\|f^{(-1)}\|$, $\|g^{(-1)}\|$ and $(f^{(-1)}, g^{(-1)})$ respectively. Next, from (3.3), which now holds for $x \geq 1$, we have $f^{(-2)}(1) = -f_1, g^{(-2)}(1) = -g_1$. Then

$$\varpi = \omega(1) = e^{i\vartheta} n_g f^{(-2)}(1) - n_f g^{(-2)}(1) = n_f g_1 - e^{i\vartheta} n_g f_1, \quad (4.7)$$

where $\vartheta = \arg(p + 1)$. Also

$$(\omega'', r^{(-2)}) = -\overline{\varpi} r^{(-1)}(1) + (\omega, r) = -\overline{\varpi}(1, r) + (\omega, r) = - (\varpi - \omega, r). \quad (4.8)$$

Direct substitution $v_*$ into equation (4.2) yields the linear system

$$\begin{pmatrix} n_f^2 & p + 1 \\ p + 1 & n_g^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad (4.9)$$

with $z_1 = a f_1 - (f, r^{(-2)})$ and $z_2 = a g_1 - (g, r^{(-2)})$ (cf. (3.6) in the proof of Lemma 3.2). Since $\pi = 0$ in this case, we have $|p + 1| = n_f n_g$, and the system is consistent, if and only if $(p + 1)z_2 = n_g z_1$. This condition can be written in the form $e^{i\vartheta} n_f z_2 = n_g z_1$. Recalling (4.7) and (4.8) gives us

$$e^{i\vartheta} n_f z_2 - n_g z_1 = e^{i\vartheta} n_f (a g_1 - (g, r^{(-2)})) - n_g (a f_1 - (f, r^{(-2)})) = e^{i\vartheta} a (n_f g_1 - e^{-i\vartheta} n_g f_1) + e^{i\vartheta} (e^{i\vartheta} n_g f - n_f g, r^{(-2)}) = e^{i\vartheta} \left( a \varpi + (\omega'', r^{(-2)}) \right) = e^{i\vartheta} (a \varpi - (\varpi - \omega, r)) = 0,$$
because $a\varpi = (\varpi, r) + b\varpi = (\varpi - \omega, r)$ by solvability conditions (4.5). Hence system (4.9) is solvable and admits solution $\tilde{c} = (n_f^{-2}z_1, 0)$. Therefore

$$v_* = n_f^{-2}(a\tilde{f}_1 - (f, r^{(-2)})) f^{(-2)} - r^{(-2)} + a(x + 1).$$

(4.10)

is a solution of (4.2). In the case $\varpi \neq 0$, we can modify $v_*$ to obtain the solution

$$v = v_* + \varpi^{-1}\left(f_1 n_f^{-2}(a\tilde{f}_1 - (f, r^{(-2)})) + r^{(-2)}(1 - 2a)\right) \omega$$

(4.11)

of (4.2) satisfying $v(-1) = 0$ and $v(1) = 0$, as is easy to check. If $\varpi = 0$, then we write

$$v = v_* - (\omega, \omega)^{-1}(\omega, v_*) \omega;$$

(4.12)

this solution fulfills the following conditions $v(-1) = 0$ and $(\omega, v) = 0$.

Estimate (4.3) for $v$ immediately follows from explicit formulae (4.10), (4.11) and the continuity of operator $L_2(\mathcal{I}) \ni r \mapsto r^{(-2)} \in W_2^2(\mathcal{I})$. Indeed, all terms in (4.10) and (4.11) contain either $a$ or $r^{(-2)}$. For instance, we can estimate

$$\|a\tilde{f}_1 n_f^{-2} f^{(-2)}\|_{W_2^2(\mathcal{I})} \leq c_1|a|\|f^{(-2)}\|_{W_2^2(\mathcal{I})} \leq c_2|a|\|f\|_{L_2(\mathcal{I})} \leq c_3|a|,$$

$$\|n_f^{-2}(f, r^{(-2)}) f^{(-2)}\|_{W_2^2(\mathcal{I})} \leq c_4\|f^{(-2)}\|_{W_2^2(\mathcal{I})}\|f, r^{(-2)}\|_{L_2(\mathcal{I})} \leq c_5\|f\|_{L_2(\mathcal{I})}\|r^{(-2)}\|_{L_2(\mathcal{I})} \leq c_6\|r\|_{L_2(\mathcal{I})}. \quad \square$$

Remark 4.2. In case A1, $\varpi \neq 0$, a slight variant of the proof above provides the estimate

$$\|v\|_{W_2^2(\mathcal{I})} \leq c\|r\|_{L_2(\mathcal{I})}$$

(4.13)

for the solution given by (4.11). In fact, owing to (4.5) we see that for any $r \in L_2(\mathcal{I})$ there exists a unique pair of numbers

$$a(r) = (1 - \varpi^{-1}\omega, r), \quad b(r) = -\varpi^{-1}(\omega, r)$$

(4.14)

for which (4.2) is solvable. These numbers can be regarded as linear functionals in $L_2(\mathcal{I})$. Hence we have the bounds $|a(r)| + |b(r)| \leq c\|r\|_{L_2(\mathcal{I})}$, from which (4.13) follows.

On the other hand, if $\varpi = 0$ in (4.5), then (4.2) is solvable only for $r$ such that $(\omega, r) = 0$. It is interesting to note that even single equation (4.2), without any boundary conditions, is unsolvable if this condition does not hold.

Proposition 4.3. The half-bound state $\sigma$ given by (3.2) possesses the following properties:

(i) the limit $\sigma_+ = \lim_{x \to +\infty} \sigma(x)$ is always a real number;

(ii) $\sigma_+ - \sigma_- = \tilde{f}_0\tilde{g}_0 (f_0 g_1 - f_1 g_0)$.

Proof. As above, we assume that $f$ and $g$ vanish outside of interval $\mathcal{I}$. Then $f^{(-1)}(1) = f_0$. We also note that $f^{(-2)}(1) = f_0 - f_1$, by (3.3), and also that

$$(f, f^{(-2)}) = \tilde{f}_0(f_0 - f_1) - \|f^{(-1)}\|^2_{L_2(\mathcal{I})},$$
where $\| \cdot \|_I$ is a norm in $L_2(I)$. In fact,

$$\int_{-1}^{1} \tilde{f} f^{(-2)} \, dx = \tilde{f}^{(-1)}(1) f^{(-2)}(1) - \int_{-1}^{1} |f^{(-1)}|^2 \, dx = \tilde{f}_0(f_0 - f_1) - \|f^{(-1)}\|_I^2.$$ 

The same formulae are valid for $g$ as well. We have

$$\sigma_+ = \sigma(1) = |g_0|^2 \left( \tilde{f}_0(f_0 - f_1) - \langle f, f^{(-2)} \rangle \right)$$

$$-|f_0|^2 \left( \tilde{g}_0(g_0 - g_1) - \langle g, g^{(-2)} \rangle \right) = |g_0|^2 \|f^{(-1)}\|_I^2 - |f_0|^2 \|g^{(-1)}\|_I^2,$$

and therefore $\sigma_+ \in \mathbb{R}$. Next write

$$\sigma_- = \sigma(-1) = -|g_0|^2 \langle f, f^{(-2)} \rangle + |f_0|^2 \langle g, g^{(-2)} \rangle$$

$$= -|g_0|^2 \left( \tilde{f}_0(f_0 - f_1) - \|f^{(-1)}\|_{L_2(I)}^2 \right)$$

$$+ |f_0|^2 \left( \tilde{g}_0(g_0 - g_1) - \|g^{(-1)}\|_{L_2(I)}^2 \right) = |g_0|^2 \|f^{(-1)}\|_I^2 - |f_0|^2 \|g^{(-1)}\|_I^2 + |g_0|^2 \tilde{f}_0 f_1 - |f_0|^2 \tilde{g}_0 g_1$$

$$\sigma_- = \tilde{f}_0 \tilde{g}_0 (f_0 g_1 - f_1 g_0),$$

which establishes (ii). □

At the end of the section, we record some technical assertion. Let $[w]_\xi$ denote the jump $w(\xi + 0) - w(\xi - 0)$ of function $w$ at a point $\xi$.

**Proposition 4.4.** Let $U$ be the real line with two removed points $x = -\varepsilon$ and $x = \varepsilon$, i.e., $U = \mathbb{R} \setminus \{-\varepsilon, \varepsilon\}$. Assume that function $w \in W^2_{2, \text{loc}}(U)$ along with its first derivative has jump discontinuities at points $x = -\varepsilon$ and $x = \varepsilon$. There exists a function $\rho \in C^\infty(U)$ such that $w + \rho$ belongs to $W^2_{2, \text{loc}}(\mathbb{R})$. Moreover, $\rho$ is a function of compact support, $\rho$ vanishes in $(-\varepsilon, \varepsilon)$ and

$$|\rho^{(k)}(x)| \leq C \left( |[w]_{-\varepsilon}| + |[w]_{\varepsilon}| + |[w']_{-\varepsilon}| + |[w']_{\varepsilon}| \right) \quad (4.15)$$

for $|x| \geq \varepsilon$, $k = 0, 1, 2$, where the constant $C$ does not depend on $w$ and $\varepsilon$.

**Proof.** Let us introduce functions $\varphi$ and $\psi$ that are smooth outside the origin, have compact supports contained in $[0, \infty)$, and such that $\varphi'(0) = 0$, $\psi'(0) = 0$ and $\psi'(0) = 1$. We set

$$\rho(x) = [w]_{-\varepsilon} \varphi(x - \varepsilon) - [w']_{-\varepsilon} \psi(x - \varepsilon) - [w]_{\varepsilon} \varphi(x + \varepsilon) - [w']_{\varepsilon} \psi(x + \varepsilon).$$

By construction, $\rho$ has a compact support and vanishes in $(-\varepsilon, \varepsilon)$. An easy computation also shows that

$$[\rho]_{-\varepsilon} = -[w]_{-\varepsilon}, \quad [\rho]_{\varepsilon} = -[w]_{\varepsilon}, \quad [\rho']_{-\varepsilon} = -[w']_{-\varepsilon}, \quad [\rho']_{\varepsilon} = -[w']_{\varepsilon}.$$ 

Therefore $w + \rho$ is continuous on $\mathbb{R}$ along with the first derivative and consequently belongs to $W^2_{2, \text{loc}}(\mathbb{R})$. Finally, the explicit formula for $\rho$ makes it obvious that inequality (4.15) holds. □
5. Proof of Theorem 1

5.1. How to Guess the Limit Operator

Given $h \in L_2(\mathbb{R})$ and $\zeta \in \mathbb{C}$ with $\text{Im} \zeta \neq 0$, we set $y_\varepsilon = (H_\varepsilon - \zeta)^{-1} h$. Let us find a formal asymptotics of $y_\varepsilon$, as $\varepsilon \to 0$, in the form

$$y_\varepsilon(x) \sim \begin{cases} y(x) + \cdots & \text{if } |x| > \varepsilon, \\ u \left( \frac{a}{\varepsilon} \right) + \varepsilon v \left( \frac{a}{\varepsilon} \right) + \cdots & \text{if } |x| < \varepsilon, \end{cases} \quad (5.1)$$

provided the coupling conditions $[y_\varepsilon]_{\pm \varepsilon} = 0$, $[y'_\varepsilon]_{\pm \varepsilon} = 0$ hold. Function $y_\varepsilon$ is a $L_2(\mathbb{R})$-solution of the equation

$$-y''_\varepsilon + V(x)y_\varepsilon + \varepsilon^{-3} Q_\varepsilon y_\varepsilon + \varepsilon^{-1} q(\varepsilon^{-1} x)y_\varepsilon = \zeta y_\varepsilon + h \quad \text{in } \mathbb{R}.$$ 

Since the interval on which the perturbation is localized shrinks to a point, $y$ must solve the equation

$$-y'' + V(x)y = \zeta y + h \quad \text{in } \mathbb{R} \setminus \{0\} \quad (5.2)$$

and, of course, it must belong to $L_2(\mathbb{R})$. This solution can not be uniquely determined without additional conditions at the origin. One naturally expects that these conditions depend on the perturbation.

Set $t = \varepsilon^{-1} x$ and $z_\varepsilon(t) = y_\varepsilon(\varepsilon t)$. Then, for $|t| < 1$, we have

$$-\frac{d^2 z_\varepsilon}{dt^2} + (g, z_\varepsilon) f(t) + (f, z_\varepsilon) g(t) + \varepsilon q(t) z_\varepsilon = \varepsilon^2 (\zeta z_\varepsilon + V(\varepsilon t) + h(\varepsilon t))$$

Since $z_\varepsilon \sim u + \varepsilon v + \cdots$, we see that $-u'' + Qu = 0$ and $-v'' + Qv = -qu$ for $t \in \mathcal{I}$, where $Q = (g, \cdot) f + (f, \cdot) g$ is a rank-two operator in $L_2(\mathcal{I})$.

Next, the asymptotic equalities $y(\pm \varepsilon) \sim u(\pm 1) + \varepsilon v(\pm 1) + \cdots$ and $y'(\pm \varepsilon) \sim \varepsilon^{-1} u'(\pm 1) + v'(\pm 1) + \cdots$ imply in particular that

$$y_- = u(-1), \quad y_+ = u(1), \quad (5.3)$$

and also that $u'(-1) = 0$, $u'(1) = 0$, $v'(-1) = y'_- \text{ and } v'(1) = y'_+$. Here and subsequently, $y_\pm = y(\pm 0)$ and $y'_\pm = y'(\pm 0)$.

Combining the equalities above, we obtain two boundary value problems

$$-u'' + Qu = 0, \quad t \in \mathcal{I}, \quad u'(-1) = 0, \quad u'(1) = 0; \quad (5.4)$$

$$-v'' + Qv = -qu, \quad t \in \mathcal{I}, \quad v'(-1) = y'_-, \quad v'(1) = y'_+. \quad (5.5)$$

**Case A1.** In view of Lemma 3.2 (iii) problem (5.4) has the two-dimensional space of solutions generated by $1$ and $\omega$. We set

$$u(t) = y_- + \kappa^{-1}(y_+ - y_-) \omega(t), \quad t \in \mathcal{I}, \quad (5.6)$$

provided $\kappa \neq 0$. Recall the $\omega(1) = \kappa$. Hence, $u$ is a restriction of half-bound state to $\mathcal{I}$ such that $u(-1) = y_-$ and $u(1) = y_+$. Problem (5.5) with the introduced $u$ in the right hand side of the equation is solvable if conditions (4.5) hold, namely $y'_- - y'_+ = -1, qu$ and $\kappa y'_+ = (\omega, qu)$. We now substitute (5.6) into the last equalities and recall notation (2.2). After some calculations we thus write the solvability conditions in the matrix form

$$\begin{pmatrix} \frac{a_1}{\kappa} - 1 & -1 \\ \frac{a_2}{\kappa^2} - 1 \end{pmatrix} \begin{pmatrix} y_+ \\ y'_+ \end{pmatrix} = \begin{pmatrix} \frac{a_1 - \kappa a_0}{\kappa^2} - 1 \\ \frac{a_2 - \kappa a_1}{\kappa^2} - 0 \end{pmatrix} \begin{pmatrix} y_- \\ y'_- \end{pmatrix}. \quad (5.7)$$
Since \( a_2 \neq \Re a_1 \) in case A1, the matrix on the left is invertible. From this we deduce
\[
\begin{pmatrix}
y_+ \\
y'_+
\end{pmatrix} = \frac{1}{a_2 - \Re a_1} \begin{pmatrix} |\epsilon|^2 a_0 - 2\Re(\Re a_1) + a_2 |\epsilon|^2 \, a_0 a_2 - |a_1|^2 \end{pmatrix} \begin{pmatrix} y_- \\
y'_-
\end{pmatrix}.
\tag{5.8}
\]
As \( a_2 - \Re a_1 = e^{i \arg(a_2 - \Re a_1)} |a_2 - \Re a_1| \) and \( e^{-i \arg(a_2 - \Re a_1)} = e^{i \arg(a_2 - \Re a_1)} \), we see that function \( y \) in asymptotics (5.1) must be a solution of (5.2) belonging to \( \mathcal{V} \) and satisfying conditions (2.5). Since coupling conditions (5.8) are simultaneously the solvability conditions for (5.5), there exists a solution \( v \) of this problem defined up to terms \( c_1 + c_2 \omega \). We can fix \( v \) such that \( v(-1) = 0 \) and \( v(1) = 0 \) [see (4.11)].

As we see in Fig. 1, there is also another path going to node A1, which is described by conditions \( \epsilon = 0 \) and \( a_2 \neq 0 \). Since \( \epsilon = 0 \), half-bound state \( \omega \) now vanishes not only at \( t = -1 \), but also at \( t = 1 \). Hence \( \omega \) is a bound state of operator \( B \). Then for any solution \( u = c_1 + c_2 \omega \) of (5.4) we have \( u(-1) = u(1) = c_1 \). Therefore
\[
y_+ = y_-
\tag{5.9}
\]
by (5.3), and \( u = y(0) + c_2 \omega \). As above, applying solvability conditions (4.5) to problem (5.5) yields
\[
y'_+ - y'_- = a_0 y(0) + a_1 c_2, \quad a_1 y(0) + a_2 c_2 = 0,
\tag{5.10}
\]
from which we have
\[
y'_+ = y'_- + a_2^{-1}(a_0 a_2 - |a_1|^2) y(0).
\tag{5.11}
\]
Hence, in the case \( \epsilon = 0 \), the leading term \( y \) in asymptotics for \( y_\epsilon \) is a solution of (5.2) obeying conditions (5.9) and (5.11) (cf. coupling conditions (2.5) for \( \epsilon = 0 \)). We also have \( u = y(0) \left( 1 - \bar{a}_1 a_2^{-1} \omega \right) \). Our choice of \( y \) ensures the solvability of (5.5); we also fix \( v \) as in (4.12).

**Case A2.** Problem (5.4) admits a one-parametric family of solutions \( u = c_0 \sigma \), where \( \sigma \) is given by (3.2). Applying (5.3) and solvability condition (4.4) for problem (5.5), we deduce \( y_- = c_0 \sigma_- \), \( y_+ = c_0 \sigma_+ \) and \( \bar{\sigma}_+ y'_+ - \bar{\sigma}_- y'_- = c_0 \sigma_* \), because the limits \( \sigma_- \) and \( \sigma_+ \) in (2.3) coincide with values \( \sigma(-1) \) and \( \sigma(1) \) respectively. Since both the limits \( \sigma_- \) and \( \sigma_+ \) are different from zero, we have
\[
c_0 = \frac{y_+}{\sigma_+} = \frac{y_-}{\sigma_-}, \quad \bar{\sigma}_+ y'_+ - \bar{\sigma}_- y'_- = \frac{\sigma_*}{\sigma_-} y_-.
\]
From this we readily deduce the conditions
\[
y_+ = \frac{\sigma_+}{\sigma_-} y_- \quad \text{and} \quad y'_+ = \frac{\sigma_+}{\sigma_-} y'_- + \frac{\sigma_*}{\sigma_+} y_-,
\]
since \( \sigma_+ \) is real by Proposition 4.3 (i). The last equalities can be written in matrix form (2.6) using identities \( \sigma_- = e^{i \arg \sigma_- |\sigma_-|} \) and \( \sigma_- = e^{-i \arg \sigma_- |\sigma_-|} \). Then problem (5.5) admits a solution \( v \) such that \( (\sigma, v) = 0 \).
**Case A3.** There two paths going to node A3 in the graph (see Fig. 1). If $\pi = 0$, $\kappa = 0$, $a_1 = 0$ and $a_2 = 0$, then (5.9), (5.10) reduce to the coupling conditions

$$y_+ = y_-, \quad y'_+ = y'_- + a_0 y(0)$$

(5.12)

that correspond to point interactions (2.7). In this case, $u = y(0)$ and $v$ solves (5.5) and satisfies additional conditions $v(-1) = 0$ and $(v, \omega) = 0$.

Going the other way, for which $\pi \neq 0$, we have that $u$ is a constant function, in view of Lemma 3.2 (ii). Then (5.3) imply $y_+ = y_-$ and $u = y(0)$. Next, $v$ must be a solution of the problem

$$-v'' + Qv = -y(0)q, \quad t \in I, \quad v'(-1) = y'_-, \quad v'(1) = y'_+,$$

which is solvable iff the second condition in (5.12) holds. Hence, in this case $y$ also satisfies the conditions (5.12). We fix a solution of (5.5) by condition $v(-1) = 0$.

### 5.2. Uniform Approximation

The basic idea of the proof is to construct a good approximation to $y_\varepsilon = (H_\varepsilon - \zeta)^{-1} h$, uniformly for $h$ in bounded subsets of $L_2(\mathbb{R})$. In addition, this approximation must belong to the domain of $H_\varepsilon$. The function $y = (\mathcal{H} - \zeta)^{-1} h$ is a satisfactory approximation to $y_\varepsilon$ for $|x| > \varepsilon$, whereas the problem of finding a close approximation on the support $(\varepsilon, \epsilon)$ of perturbation is rather subtle. Recall that $\mathcal{H}$ stands for the limit operator in the case under study.

**Case A1.** Let $m$ be a $L_2(\mathcal{I})$-function of zero mean such that $(\omega, m) = 1$. We consider the problem involving three parameters $\alpha_\varepsilon$, $\beta_\varepsilon$ and $\gamma_\varepsilon$

$$-\vartheta''_\varepsilon + Q \vartheta_\varepsilon = h(\varepsilon \cdot) + \gamma_\varepsilon m \quad \text{in } \mathcal{I}, \quad \vartheta'_\varepsilon(-1) = \alpha_\varepsilon, \quad \vartheta'_\varepsilon(1) = \beta_\varepsilon.$$  

(5.13)

The problem is solvable if and only if

$$\mathcal{R} \alpha_\varepsilon + \gamma_\varepsilon = (\kappa - \omega, h(\varepsilon \cdot)), \quad \alpha_\varepsilon - \beta_\varepsilon = (1, h(\varepsilon \cdot)),$$

provided operator $B$ has the double zero-energy resonance.

Assume that $\kappa \neq 0$. Let us introduce function $\vartheta_\varepsilon$ as a solutions of (5.13) with $\gamma_\varepsilon = 0$. Then $\alpha_\varepsilon$ and $\beta_\varepsilon$ can be uniquely defined $\alpha_\varepsilon(h) = (1 - \kappa^{-1} \omega, h(\varepsilon \cdot)), \beta_\varepsilon(h) = -(\mathcal{R})^{-1} (\omega, h(\varepsilon \cdot))$ for given $h$ [see (4.14)]. By (4.11), there exists a unique solution $\vartheta_\varepsilon$ satisfying the additional conditions $\vartheta_\varepsilon(-1) = 0$ and $\vartheta_\varepsilon(1) = 0$.

Set $w_\varepsilon(x) = y(x)$ if $|x| > \varepsilon$ and $w_\varepsilon(x) = u \left( \frac{x}{\varepsilon} \right) + \varepsilon v \left( \frac{x}{\varepsilon} \right) + \varepsilon^2 \vartheta_\varepsilon \left( \frac{x}{\varepsilon} \right)$ if $|x| < \varepsilon$. By construction, $w_\varepsilon$ belongs to $W^2_{2,\text{loc}}(\mathbb{R} \setminus \{-\epsilon, \epsilon\})$, but this function is in general discontinuous at the points $x = \pm \varepsilon$; its jumps and the jumps of its first derivative are small enough as we will show below. This observation allows us to correct $w_\varepsilon$ to a $W^2_{2,\text{loc}}(\mathbb{R})$-function by a small perturbation. We can find $\rho_\varepsilon$ such that

$$Y_\varepsilon(x) = w_\varepsilon(x) + \rho_\varepsilon(x) = \begin{cases} y(x) + \rho_\varepsilon(x) & \text{if } |x| > \varepsilon, \\ u \left( \frac{x}{\varepsilon} \right) + \varepsilon v \left( \frac{x}{\varepsilon} \right) + \varepsilon^2 \vartheta_\varepsilon \left( \frac{x}{\varepsilon} \right) & \text{if } |x| < \varepsilon \end{cases}$$

(5.14)

belongs to $W^2_{2,\text{loc}}(\mathbb{R})$, by Proposition 4.4. Recall that $\rho_\varepsilon$ is zero in $(-\varepsilon, \varepsilon)$. Obviously $Y_\varepsilon$ also belongs to the domain of $H_\varepsilon$, since $y \in \mathcal{V}$ and $\rho_\varepsilon$ has a compact support.
Now we suppose that \( \varkappa = 0 \). According to the second part of Remark 4.2, the solvability of (5.13) cannot be ensured by parameters \( \alpha \) and \( \beta \) only. We can find \( \vartheta_\varepsilon \) by setting \( \alpha_\varepsilon(h) = 0, \beta_\varepsilon(h) = -(1,h(\varepsilon \cdot)) \) and \( \gamma_\varepsilon = -(\omega,h(\varepsilon \cdot)) \). Then the problem admits a unique solution such that \( \vartheta_\varepsilon(-1) = 0 \) and \( (\omega,\vartheta_\varepsilon) = 0 \), by (4.12). Finally we define \( Y_\varepsilon \in \text{dom} H_\varepsilon \) by (5.14), as for \( \varkappa \neq 0 \).

**Case A3.** Approximation (5.14) constructed above for the case A1 with \( \varkappa = 0 \) is also suitable when \( a_1 = a_2 = 0 \). In the case when \( \pi \neq 0 \), the double zero-energy resonance for \( B \) is absent. In view of Lemma 3.2 (ii), the only constant functions are half-bound states of \( B \). Hence (5.13) admits a solution if \( \alpha_\varepsilon - \beta_\varepsilon = (1,h(\varepsilon \cdot)) \) by Proposition 4.1 (ii). We set \( \alpha_\varepsilon(h) = 0, \gamma_\varepsilon(h) = 0 \) and \( \beta_\varepsilon(h) = -(1,h(\varepsilon \cdot)) \), and fix the solution \( \vartheta_\varepsilon \) by additional condition \( \vartheta_\varepsilon(-1) = 0 \).

**Case A2.** In this operator \( B \) possesses the half-bound state \( \sigma \). Problem (5.13) admits a solution iff \( \alpha_\varepsilon(h)\tilde{\sigma}_- - \beta_\varepsilon(h)\sigma_+ = (\sigma,h(\varepsilon \cdot)) + \gamma_\varepsilon(h)(\sigma,m) \). Set \( \alpha_\varepsilon(h) = 0, \gamma_\varepsilon(h) = 0 \) and \( \beta_\varepsilon(h) = -\sigma_+^{-1}(\sigma,h(\varepsilon \cdot)) \), for instance. Then we choose a solution \( \vartheta_\varepsilon \) in (5.14) such that \( \vartheta_\varepsilon(-1) = 0 \).

Regardless of the case under consideration, the values \( \alpha_\varepsilon, \beta_\varepsilon \) and \( \gamma_\varepsilon \) can be estimated by the norm of \( h \):

\[
|\alpha_\varepsilon(h)| + |\beta_\varepsilon(h)| + |\gamma_\varepsilon(h)| \leq c_1 \|h(\varepsilon \cdot)\|_{L_2(\mathbb{I})} \leq c_2 \varepsilon^{-1/2} \|h\|. \tag{5.15}
\]

Here we used the obvious estimate

\[
\int_{-\varepsilon}^{\varepsilon} |h(\varepsilon s)|^2 \, ds = \varepsilon^{-1} \int_{-\varepsilon}^{\varepsilon} |h(x)|^2 \, dx \leq \varepsilon^{-1} \int_{\mathbb{R}} |h(x)|^2 \, dx.
\]

### 5.3. Remainder Estimates

We will show that \( Y_\varepsilon \) solves the equation

\[
(H_\varepsilon - \zeta)Y_\varepsilon = h + r_\varepsilon,
\]

where the remainder term \( r_\varepsilon \) is small in \( L_2 \)-norm uniformly with respect to \( h \). Let us compute \( r_\varepsilon \). If \( |x| > \varepsilon \), then

\[
r_\varepsilon(x) = -\frac{d^2}{dx^2}(Y_\varepsilon(\frac{x}{\varepsilon})) + (V(x) - \zeta)Y_\varepsilon(\frac{x}{\varepsilon})
\]

\[+
\varepsilon^{-3} \int_{-\varepsilon}^{\varepsilon} \left( \tilde{g}(\frac{x}{\varepsilon}) f(\frac{x}{\varepsilon}) + \tilde{f}(\frac{x}{\varepsilon}) g(\frac{x}{\varepsilon}) \right) Y_\varepsilon(\frac{x}{\varepsilon}) \, d\tau + \varepsilon^{-1} q(\frac{x}{\varepsilon}) Y_\varepsilon(\frac{x}{\varepsilon}) - h(x)
\]

\[=
\varepsilon^{-2}\left(-u''(\frac{x}{\varepsilon}) + (Q u)(\frac{x}{\varepsilon})\right) + \varepsilon^{-1}\left(-v''(\frac{x}{\varepsilon}) + (Q v)(\frac{x}{\varepsilon}) + q(\frac{x}{\varepsilon}) u(\frac{x}{\varepsilon})\right)
\]

\[+
(\theta''(\frac{x}{\varepsilon}) + (Q \theta)(\frac{x}{\varepsilon}) - h(x) - \gamma_\varepsilon m(\frac{x}{\varepsilon}) + q(\frac{x}{\varepsilon}) v(\frac{x}{\varepsilon}) + \gamma_\varepsilon m(\frac{x}{\varepsilon}) + (V(x) - \zeta)Y_\varepsilon(\frac{x}{\varepsilon})
\]

by (5.4), (5.5) and (5.13). Hence

\[
r_\varepsilon(x) = \begin{cases}
-\rho''(x) + (V(x) - \zeta)\rho_\varepsilon(x), & \text{if } |x| > \varepsilon,
q(\frac{x}{\varepsilon}) v(\frac{x}{\varepsilon}) + \gamma_\varepsilon m(\frac{x}{\varepsilon}) + (V(x) - \zeta)Y_\varepsilon(\frac{x}{\varepsilon}), & \text{if } |x| < \varepsilon.
\end{cases}
\]
We will prove that \( r_\varepsilon \) is small in the \( L_2(\mathbb{R}) \)-norm and also show that the non-zero contribution in the norm \( \| Y_\varepsilon \| \), as \( \varepsilon \to 0 \), is produced by the function \( y \) only.

**Proposition 5.1.** For all \( h \in L_2(\mathbb{R}) \) functions \( r_\varepsilon = (H_\varepsilon - \zeta)Y_\varepsilon - h \) and \( s_\varepsilon = Y_\varepsilon - y \) satisfy the estimate

\[
\| r_\varepsilon \| + \| s_\varepsilon \| \leq c\varepsilon^{1/2}\| h \|,
\]

where the constant \( c \) does not depend on \( h \) and \( \varepsilon \).

**Proof.** First we record some estimates on \( y, u, v \) and \( \vartheta_\varepsilon \). We observe that \((\mathcal{H} - \zeta)^{-1}\) is a bounded operator from \( L_2(\mathbb{R}) \) to the domain of \( \mathcal{H} \) equipped with the graph norm, and the domain is a subspace of \( W^2_{2,\text{loc}}(\mathbb{R} \setminus \{0\}) \). Therefore

\[
\| y \|_{W^2(a,0)} + \| y \|_{W^2(0,a)} \leq c_1 \| h \|
\]

for any \( a > 0 \), and thus

\[
\| y \|_{C^1(-a,0)} + \| y \|_{C^1(0,a)} \leq c_2 \| h \|,
\]

(5.16)

by the Sobolev embedding theorem. In particular, we have

\[
|y_j - y_j^*| \leq c_3 \| h \|.
\]

It follows from (5.6) and the last bound that

\[
\| u \|_{L_2(I)} \leq c_4 \| |y_j - y_j^*| \| \leq c_5 \| h \|.
\]

(5.17)

Using the bound (4.3) along with (5.17), we estimate

\[
\| v \|_{W^2(I)} \leq c_6 \| |y_j - y_j^*| + \| qu \|_{L_2(I)} \| \leq c_7 \| h \|,
\]

(5.18)

since \( q \) is bounded. To estimate \( \vartheta_\varepsilon \), we apply (4.3) to problem (5.13)

\[
\| \vartheta_\varepsilon \|_{W^2(I)} \leq c_8 (|\alpha_\varepsilon(h)| + |\beta_\varepsilon(h)| + |\gamma_\varepsilon(h)| + \| h(\varepsilon\cdot) \|_{L_2(I)} \leq c_9 \varepsilon^{-1/2} \| h \|,
\]

(5.19)

where we used (5.15). Hence inequalities (5.17), (5.18) and (5.19) provide the bound

\[
\| Y_\varepsilon(\varepsilon^{-1}\cdot) \|_{L_2(-\varepsilon,\varepsilon)} = \varepsilon^{1/2} \| Y_\varepsilon \|_{L_2(I)} = \varepsilon^{1/2} \| u + \varepsilon v + \varepsilon^2 \vartheta_\varepsilon \|_{L_2(I)} \\
\leq \varepsilon^{1/2} \| u \|_{L_2(I)} + \varepsilon^{3/2} \| u \|_{L_2(I)} + \varepsilon^{5/2} \| \vartheta_\varepsilon \|_{L_2(I)} \leq c_{10} \varepsilon^{1/2} \| h \|.
\]

(5.20)

In order to estimate \( \rho_\varepsilon \) we calculate the jumps of \( w_\varepsilon \). Recalling that \( v(-1) = 0 \) and \( \vartheta_\varepsilon(-1) = 0 \) for all cases, we have

\[
[w_\varepsilon]_{-\varepsilon} = y_j - y_j(-\varepsilon), \quad [w_\varepsilon]_{\varepsilon} = y(\varepsilon) - y_j + \varepsilon v(1) + \varepsilon^2 \vartheta_\varepsilon(1), \quad [w_\varepsilon']_{-\varepsilon} = y_j' - y_j'(-\varepsilon) + \varepsilon \alpha_\varepsilon, \quad [w_\varepsilon']_{\varepsilon} = y_j'(\varepsilon) - y_j' + \varepsilon \beta_\varepsilon.
\]

(5.21)

There exists a constant being independent of \( \varepsilon \) and \( y \) such that

\[
|y^{(k)}(-\varepsilon) - y^{(k)}_j(\varepsilon) - y^{(k)}_j(\varepsilon)| \leq C \varepsilon^{1/2} \| h \|
\]

for \( k = 0, 1 \), since

\[
|y^{(k)}(\pm\varepsilon) - y^{(k)}_{\varepsilon}| \leq \left| \int_{0}^{\pm\varepsilon} y^{(k+1)}(x) \, dx \right| \leq C \varepsilon^{1/2} \| y \|_{W^2_2((-1,1) \setminus \{0\})}.
\]

(5.22)
Then utilizing estimate (4.15) in Proposition 4.4 (with $\rho_\varepsilon$ and $w_\varepsilon$ in place of $\rho$ and $w$, respectively) we obtain the bound
\[
|\rho_\varepsilon(x)| + |\rho_\varepsilon''(x)| \leq c_{11} \left( |y(-\varepsilon) - y_-| + |y(\varepsilon) - y_+| + |y'(\varepsilon) - y'_-| \right.
\]
\[\left. + |y'(\varepsilon) - y'_+| + \varepsilon(|v(1)| + |\alpha_\varepsilon| + |\beta_\varepsilon|) + \varepsilon^2|\vartheta_\varepsilon(1)| \right) \]
\[\leq c_{12}\varepsilon^{1/2}\|h\| \tag{5.23}
\]
for $|x| \geq \varepsilon$, in view of (5.15), (5.18), (5.19) and (5.22). Using this bounds along with (5.20), we estimate
\[
\|r_\varepsilon\| \leq c_{13}\left( \|\rho_\varepsilon'' + (V - \zeta)\rho_\varepsilon\|
\right.
\[\left. + \|q(\varepsilon^{-1}\cdot)v(\varepsilon^{-1}\cdot) + \gamma_\varepsilon m(\varepsilon^{-1}\cdot) + (V - \zeta)Y_\varepsilon(\varepsilon^{-1}\cdot)\|_{L_2(-\varepsilon, \varepsilon)} \right)
\]
\[\leq c_{14}\max_{|x| > \varepsilon}(\|\rho_\varepsilon\| + |\rho_\varepsilon''|) + c_{15}\varepsilon^{1/2}(\|v\|_{L_2(\mathbb{I})} + \|Y_\varepsilon\|_{L_2(\mathbb{I})}) \leq c_{16}\varepsilon^{1/2}\|h\|
\]
as desired. We still have to estimate
\[
s_\varepsilon(x) = \begin{cases} 
\rho_\varepsilon(x) & \text{if } |x| > \varepsilon, \\
u\left(\frac{x}{\varepsilon}\right) + \varepsilon v\left(\frac{x}{\varepsilon}\right) + \varepsilon^2\vartheta_\varepsilon\left(\frac{x}{\varepsilon}\right) - y(x) & \text{if } |x| \leq \varepsilon.
\end{cases}
\]
We can as before invoke (5.16), (5.20) and (5.23) to derive the bound
\[
\|s_\varepsilon\| \leq c_1\left( \|\rho_\varepsilon\| + \|Y_\varepsilon(\varepsilon^{-1}\cdot)\|_{L_2(-\varepsilon, \varepsilon)} + \|y\|_{L_2(-\varepsilon, \varepsilon)} \right)
\]
\[\leq c_2\varepsilon^{1/2}(\|h\| + \max_{|x| \leq \varepsilon} |y(x)|) \leq c_3\varepsilon^{1/2}\|h\|,
\]
which completes the proof of the proposition. \qed

5.4. End of the Proof
Recall that $y_\varepsilon = (H_\varepsilon - \zeta)^{-1}h$ and $y = (\mathcal{H} - \zeta)^{-1}h$ for given $h \in L_2(\mathbb{R})$ and a complex number $\zeta$ with non-zero imaginary part. By definition of $r_\varepsilon$ and $s_\varepsilon$ we have $(H_\varepsilon - \zeta)Y_\varepsilon = h + r_\varepsilon$ and $Y_\varepsilon = (\mathcal{H} - \zeta)^{-1}h + s_\varepsilon$. We conclude from this that $(H_\varepsilon - \zeta)^{-1}h = Y_\varepsilon - (H_\varepsilon - \zeta)^{-1}r_\varepsilon$ and $(\mathcal{H} - \zeta)^{-1}h = Y_\varepsilon - s_\varepsilon$, hence that
\[
\|(H_\varepsilon - \zeta)^{-1}h - (\mathcal{H} - \zeta)^{-1}h\| = \|s_\varepsilon - (H_\varepsilon - \zeta)^{-1}r_\varepsilon\|
\]
\[\leq \|s_\varepsilon\| + \|(H_\varepsilon - \zeta)^{-1}\|\|r_\varepsilon\| \leq \|s_\varepsilon\| + \|\varIm \zeta|^{-1}\|\|r_\varepsilon\| \leq C\varepsilon^{1/2}\|h\|,
\]
in view of Proposition 5.1. The last bound establishes the norm resolvent convergence of $H_\varepsilon$ to the operator $\mathcal{H}$, which is the desired conclusion.

6. Proof of Theorem 2
6.1. Case B1
We begin from the case in which operator $B$ possesses two linearly independent half-bound states. We assume that $f_0 = g_0 = 0$, $\pi = 0$ and $\chi \neq 0$. But suppose now instead of $a_2 \neq \varIm a_1$, as in the case A1, that the equality $a_2 = \varRe a_1$ holds (see the graph in Fig. 1). Starting the proof as in 5.1, we look for uniform approximation $Y_\varepsilon$ in the form (5.14) to a solution of equation $(H_\varepsilon - \zeta)y_\varepsilon = h$. In this case, we first see the difference in matrix
condition (5.7), because $a_2 = \Re a_1$ and therefore the matrix on the left is now degenerate. Since $\Re a_2 = |\Re|^2 a_1$, (5.7) can be written in the form

$$\begin{pmatrix} \frac{a_2}{|\Re|^2} & -1 \\ \frac{a_2}{|\Re|^2} & -1 \end{pmatrix} \begin{pmatrix} y_+ \\ y'_+ \end{pmatrix} = \begin{pmatrix} \frac{a_2}{|\Re|^2} - a_0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_- \\ y'_- \end{pmatrix}.$$ 

It follows immediately from this that

$$y'_- - (|\Re|^2 a_2 - a_0) y_-' = 0, \quad y'_+ - |\Re|^2 a_2 y_+ = 0. \quad (6.1)$$

Hence we introduce the limit operator $H$ as the Schrödinger operator on the line acting via $\Re \psi'' + V \psi$ on the domain

$$\text{dom } \Re = \{ \psi \in \mathcal{V} : \psi(0) = \theta_1 \psi(-0), \quad \psi'(0) = \theta_2 \psi(+0) \},$$

where $\theta_1 = |\Re|^2 a_2 - a_0$ and $\theta_2 = |\Re|^2 a_2$. Thus $\Re = \Re_-(\theta_1) \oplus \Re_+(\theta_2)$.

Turning to approximation $Y$, we assume that $y = (\Re - \zeta)^{-1} h$ is a $L_2(\mathbb{R})$-function solving the equation $-y'' + (V - \zeta) y = h$, subject to coupling conditions (6.1). Next, $u$ is a half-bound state given by (5.6), $\nu$ and $\sigma$ are solutions to problems (5.5) and (5.13) respectively such that $\nu(\pm 1) = 0$ and $\sigma(\pm 1) = 0$, by Proposition 4.1. The jumps $[w_\varepsilon]_{\pm \varepsilon}$ and $[w'_{\varepsilon}]_{\pm \varepsilon}$ given by (5.21) are small as $\varepsilon \to 0$ uniformly on $h \in L_2(\mathbb{R})$. Hence there exists a small corrector $\rho_\varepsilon$ satisfying estimate (5.23) such that $Y_\varepsilon \in \text{dom } \Re$. In addition, $Y_\varepsilon$ satisfies the equation $(H_\varepsilon - \zeta) Y_\varepsilon = h + r_\varepsilon$ with the remainder $r_\varepsilon$ that can be estimated as in Proposition 5.1. By the argument used at the end of the proof of Theorem 1, we show that $H_\varepsilon$ converge to $\Re_-(\theta_1) \oplus \Re_+(\theta_2)$ as $\varepsilon \to 0$ in the norm resolvent sense.

6.2. Case B2

Suppose that $\lambda = 0$, $f_0 g_0 \neq 0$, $f_0 g_1 \neq f_1 g_0$ and $\sigma_- \sigma_+ = 0$. According to Lemma 3.2 (i), $B$ possesses half-bound state $\sigma$. Looking for approximation $Y_\varepsilon$, we set $u = c_0 \sigma$ with some constant $c_0$. As in the case A2, we have

$$y_- = c_0 \sigma_-, \quad y_+ = c_0 \sigma_+, \quad \sigma_+ y'_+ - \sigma_- y'_- = c_0 \sigma, \quad (6.2)$$

In view of Proposition 4.3 (ii) only one of the values $\sigma_-$ and $\sigma_+$ is equal to zero. Assume for instance $\sigma_- = 0$. Then we deduce $y_- = 0$, $c_0 = \sigma_+^{-1} y_+$ and therefore $y'_+ = \sigma_+ \sigma_-^{-2} y_+$. Hence in this case the limit operator $\Re$ as $\varepsilon \to 0$ is the direct sum $\Re_-(\theta) \oplus \Re_+(\theta)$, where $\theta = \sigma_+ \sigma_-^{-2}$.

Let now $y = (\Re - \zeta)^{-1} h$. Since a solution $\nu$ of (5.5) is defined up to term $c_1 \sigma$ and $\sigma(1) = \sigma_+ \neq 0$, we can find a unique solution $\nu$ such that $\nu(1) = 0$. We also assume that $\nu$ solves the problem

$$-\nu'' + Q \nu = h(\varepsilon \cdot), \quad t \in I, \quad \nu'(-1) = 0, \quad \nu'(1) = \beta_\varepsilon,$$

where $\beta_\varepsilon = -\sigma_+^{-1} (\sigma, h(\varepsilon \cdot))$. The problem has a solution such that $\nu_\varepsilon(1) = 0$. Thus we built approximation $Y_\varepsilon \in \text{dom } H_\varepsilon$ and the rest of the proof is word for word as in the proof of the previous theorem. The subcase $\sigma_+ = 0$ is treated similarly.
6.3. Case B3

This case collects all the subcases, in which the limit operator is the direct sum $D_- \oplus D_+$ of the unperturbed half-line Schrödinger operators with potential $V$, subject to the Dirichlet boundary condition at the origin. In fact, if $\lambda \neq 0$, then operator $B$ has no zero-energy resonance, i.e., problem (5.4) admits a trivial solution $u = 0$ only. In view of coupling conditions (5.3), it immediately follows that $y_- = 0$ and $y_+ = 0$. If $f_0g_0 \neq 0$, $f_0g_1 = f_1g_0$, $\sigma_- = 0$ and $\sigma_+ = 0$, then (6.2) also implies $y_- = 0$ and $y_+ = 0$. Finally, in the case $f_0 = 0$, $g_0 = 0$, $\pi = 0$, $\alpha = 0$, $a_2 = 0$ and $a_1 \neq 0$, it follows from the second condition in (5.10) that $y(0) = 0$. The same proof, as in the previous cases, works in the case B3.

The proof of Theorem 3 is actually contained in the proofs of Theorem 1 and Theorem 2. Estimate (2.9) immediately follows from Proposition 5.1. The order of convergence is optimal, because the estimate $\|\tilde{h}(\varepsilon)\|_{L_2(I)} \leq c\varepsilon^{-1/2}\|h\|$ in the proof of Proposition 5.1 is precise and cannot be improved for $L_2$-functions.

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