FAMILIES OF SUPERELLIPTIC CURVES, COMPLEX BRAID GROUPS AND GENERALIZED DEHN TWISTS

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Abstract. We consider the universal family $E^d_n$ of superelliptic curves: each curve $\Sigma^d_n$ in the family is a $d$-fold covering of the unit disk, totally ramified over a set $P$ of $n$ distinct points; $\Sigma^d_n \hookrightarrow E^d_n \rightarrow C_n$ is a fibre bundle, where $C_n$ is the configuration space of $n$ distinct points.

We find that $E^d_n$ is the classifying space for the complex braid group of type $B_{p,d,d,n}$.q and we compute a big part of the integral homology of $E^d_n$, including a complete calculation of the stable groups over finite fields by means of Poincaré series. The computation of the main part of the above homology reduces to the computation of the homology of the classical braid group with coefficients in the first homology group of $\Sigma^d_n$, endowed with the monodromy action. While giving a geometric description of such monodromy of the above bundle, we introduce generalized $\frac{1}{d}$-twists, associated to each standard generator of the braid group, which reduce to standard Dehn twists for $d = 2$.

1. Introduction

Let

$$E^d_n := \{(P, z, y) \in C_n \times D \times \mathbb{C} | y^d = (z - x_1) \cdots (z - x_n)\}$$

be the family of superelliptic curves where $D$ is the open unit disk in $\mathbb{C}$, $C_n$ is the configuration space of $n$ distinct unordered points in $D$ and $P = \{x_1, \ldots, x_n\} \in C_n$.

For a fixed $P \in C_n$, the curve

$$\Sigma^d_n := \{(z, y) \in D \times \mathbb{C} | y^d = (z - x_1) \cdots (z - x_n)\}$$

in the family $E^d_n$ is a $d$-fold covering of the disk $D$, totally ramified over $P$, and there is a fibration $\pi : E^d_n \rightarrow C_n$ which takes $\Sigma^d_n$ onto its set of ramification points. One can see $E^d_n$ as a universal family over the Hurwitz space $H^{n,d}$ (for precise definitions see [Ful69, EVW16]).

In addition to the obvious interest for such families, we find a remarkable fact which seems not having been noticed before (even if the proof is not difficult):

Theorem 1.1 (see Theorem [4.1]). The space $E^d_n$ is a classifying space for the complex braid group of type $B(d,d,n)$.

Here we are interested in computing the integral homology of the space $E^d_n$. The rational homology of $E^d_n$ is known, having been computed in [Che17] by using [CMS08]. The bundle $\pi : E^d_n \rightarrow C_n$ has a global section, so $H_*(E^d_n)$ splits into a direct sum $H_*(C_n) \oplus H_*(E^d_n, C_n)$ and by the Serre spectral sequence $H_*(E^d_n, C_n) = H_{*-1}(Br_n; H_1(\Sigma^d_n))$. We use here that $C_n$ is a classifying space for the braid group $Br_n$. We study the geometric action of the braid group over the surface. Each standard generator of the braid group lifts to a particular homeomorphism of the surface that we call a $\frac{1}{d}$-twist. Such twist is associated to an embedding of a regular polygon with an even number of edges, with opposite edges identified, and having
one or two interior holes according to $d$ odd or even (see Figure 5). Suitable rotations of such polygons induce an homeomorphism of the surface that corresponds to the monodromy action (see Figures 6 and 7). Surprisingly, these particular homeomorphisms of a surface have not been considered before. Very recently [KS18] for $d = 3$ and [GM18] for general $d$ independently found a similar description at the same time as ours. The $\frac{1}{2}$-twist reduces to a standard Dehn twist around a simple curve for $d = 2$ (see [PY92], [Wa99]). For $d$ even, the $\frac{d}{2}$-th power of a $\frac{1}{2}$-twist is a standard Dehn twist: so, we obtain explicit roots of Dehn twists which appear to have an easier description than those introduced in [MS09]. In Theorem 3.1 we describe the induced action on the first homology group of the surface.

In this paper we actually compute the integral homology of the braid group with coefficients in the above representation. Since the homology of the braid groups with $d$ one or two interior holes according to the above description of the homology of $E_n$. It would be natural to extend the computation to the homology of the braid group $Br_n$ with coefficients in the symmetric powers of $H_1(\Sigma_n^d)$. In the case of $n = 3$ and $d = 2$ a complete computation (in cohomology) can be found in [CCS13].

Our main results are the following. For reader convenience, we write again the theorem 1.2.

**Theorem 1.2** (see Theorems 8.2, 9.2, 10.6).

1. For odd $n$ or odd $d$ the homology $H_i(\langle Br_n; H_1(\Sigma_n^d); \mathbb{Z} \rangle)$ has no torsion of order $p^k$ if $p^k \nmid d$.

2. For odd $n$ and for $p$ prime such that $p \nmid d$ the rank of $H_i(\langle Br_n; H_1(\Sigma_n^d); \mathbb{Z} \rangle) \otimes \mathbb{Z}_p$ as a $\mathbb{Z}_p$-module is the coefficient of $q^it^n$ in the expansion of the series

$$\tilde{P}_p(q,t) = \frac{qt^3}{(1-t^2q^2)(1-t^2)} \prod_{j>0} \frac{1 + q^{2p^j-1}2p^j}{1-q^{2p^j+1}-2q^{2p^j+1}}.$$

When $d$ is square-free and $d$ or $n$ is odd Theorem 1.2 completely determines the homology groups $H_*(E_n^d, C_n)$ with integer coefficients.

**Theorem 1.3** (see Theorems 10.3, 10.7). Consider homology with integer coefficients.

1. The homomorphism

$$H_1(\langle Br_n; H_1(\Sigma_n^d); \mathbb{Z} \rangle) \rightarrow H_i(\langle Br_{n+1}; H_1(\Sigma_n^d); \mathbb{Z} \rangle)$$

is an epimorphism for $i \leq \frac{n}{2} - 1$ and an isomorphism for $i < \frac{n}{2} - 1$.

2. Let $p$ be a prime that does not divide $d$. For $n$ even the group $H_i(\langle Br_n; H_1(\Sigma_n^d); \mathbb{Z} \rangle)$ has no $p$ torsion when $\frac{mp}{p-1} + 3 \leq n$ and no free part for $i + 3 \leq n$. In particular for $n$ even when $\frac{m}{2} + 3 \leq n$ the group $H_i(\langle Br_n; H_1(\Sigma_n^d); \mathbb{Z} \rangle)$ has only torsion that divides $d$.

3. Let $p$ be a prime that divides $d$. The Poincaré polynomial of the stable homology $H_1(\langle Br_n; H_1(\Sigma_n^d); \mathbb{Z} \rangle) \otimes \mathbb{Z}_p$ as a $\mathbb{Z}_p$-module is the following:

$$P_p(Br; H_1(\Sigma^d))(q) = \frac{q}{1-q^2} \prod_{j>0} \frac{1 + q^{2p^j-1}}{1-q^{2p^j+1}}.$$

When $d$ is square-free Theorem 1.3 completely determines the stable homology groups $H_*(E_n^d, C_n)$ with integer coefficients.

We also find unstable free components in the top and top–1 dimension for $n$ and $d$ both even (see Theorem 10.4), coherently with the computations in [Che17].
Since we have \( H_i(B(d, d, n)) \cong H_i(\Br_n) \oplus H_{i-1}(\Br_n; H_1(\Sigma^d_n; \Z)) \) and the homology of the braid group with trivial coefficients is classically known ([Arn70]) we get in particular:

**Theorem 1.4** (see Theorem 10.5). The homomorphism

\[
H_i(B(d, d, n)) \rightarrow H_i(B(d, d, n + 1))
\]

induced by the natural inclusion \( B(d, d, n) \hookrightarrow B(d, d, n + 1) \) is an epimorphism for \( i \leq \frac{n}{2} \) and an isomorphism for \( i < \frac{n}{2} \).

Notice that there were very few cohomological computations about the homology of complex braid groups of type \( B(d, d, n) \); in fact, the only known computations (see [CM14], not more than the second homology groups) used methods based on a resolution given in [DL03] for a Garside monoid introduced in [CP11]. This method seems too complicated to be used for higher homology groups.

This paper is a natural continuation of [CS17], where we considered the case \( d = 2 \). In particular, the main ingredients which here we generalize are: a Mayer-Vietoris geometrical decomposition of the space \( E_2^n \), which allows to reduce the computation to the local homology of some "pieces" which are identified with regular coverings of the configuration space \( C_{1,n} \) for the Artin group of type \( B \); the adaptation and the use of some of the homology computations given in [CM14] to our case. For reader convenience, we collect most of the results we need in Section 5. Sections 5–8 and 10 parallel Sections 3–7 of [CS17], generalizing the results already obtained for \( E_2^n \). We will directly refer to [CS17] when using the results presented there with no variations.

Some tables with explicit computations for \( d = 3, 4, 5, 6 \) are provided in the final section.

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**2. General setting**

We recall that the fundamental group of the configuration space \( C_n \) introduced before is the classical braid group \( \Art(A_{n-1}) = \Br_n \), and that \( C_n \) is a \( K(\Br_n, 1) \) (see [EN62]). We will make use also of the configuration spaces \( C_{1,n} \) of \( n \) unordered distinct points in \( D \) with one additional distinct marked point. The fundamental
group of \(C_{1,n}\) is the Artin groups \(\text{Art}(B_n)\) of type B and (see for example [Bri73]) the space \(C_{1,n}\) is a \(K(\text{Art}(B_n),1)\).

Fixed \(P = \{x_1, \ldots, x_n\} \in C_n\), the set
\[
\Sigma^d_n := \{(z,y) \in D \times C | y^d = (z - x_1) \cdots (z - x_n)\}
\]
is a connected oriented surface with \(\gcd(n,d)\) boundary components. The genus of \(\Sigma^d_n\) is
\[
g = 1 + \frac{(d - 1)n - d - \gcd(n,d)}{2},
\]
in particular for \(d = 2\) we have \(g = \frac{n - 1}{2}\) for odd \(n\) and \(g = \frac{n}{2}\) for \(n\) even.

Besides the classical braid group, in what follows we will make large use also of the Artin group \(\text{Art}(B_n)\) of type B.

We consider the family
\[
E^d_n := \{(P, z, y) \in C_n \times D \times C | y^d = (z - x_1) \cdots (z - x_n)\}
\]
which is a fibre bundle with natural projection \(\pi : E^d_n \to C_n\), mapping \((P, y, z) \mapsto P\).

The fiber of \(\pi\) is the surface \(\Sigma^d_n\), which is a \(d\)-fold covering of \(D\), totally ramified along \(P\); we will identify \(P\) with a subset of \(\Sigma\).

We define \(\widetilde{D\setminus P}^d := \Sigma^d_n \setminus P\) as the \(d\)-fold covering of \(D \setminus P\) induced by \(\Sigma^d_n \to D\).

Since \(H_1(D\setminus P)\) has rank \(n\) we have that \(H_1(\widetilde{D\setminus P}^d)\) has rank \((n - 1) + 1\).

The projection \(E^d_n \to C_n \times D\) given by \(p : (P, z, y) \mapsto (P, z)\) makes \(E^d_n\) a \(d\)-fold covering of \(C_n \times D\), which ramifies over \(C_{1,n-1} \equiv \{(P, z) \in C_n \times D | z \in P\}\). The complement of \(C_{1,n-1} \subset C_n \times D\) identifies with \(C_{1,n}\), so the complement \(\widetilde{C_{1,n}} := E^d_n \setminus (p^{-1}(C_{1,n-1}))\) is a \(d\)-fold covering of \(C_{1,n}\).

**Remark 1.** The fibre bundle \(\Sigma^d_n \to E^d_n \to C_n\) admits a global section (see Definition 3) so \(H_u(E^d_n) = H_u(E^d_n; C_n) \oplus H_u(C_n)\) and
\[
H_1(E^d_n; C_n) = H_{-1}(C_n; H_1(\Sigma^d_n)).
\]

Recall also that the \(H_1(\Sigma^d_n)\) is endowed with an anti-symmetric form given by the cap product. There is a monodromy action of \(\pi_1(C_n)\) on \(\text{Homeo}(\Sigma^d_n)\) associated to the fiber \(\pi\); the induced action onto \(H_1(\Sigma^d_n)\) preserves this form. We describe this monodromy in more details in the following section. For \(d = 2\) the monodromy representation maps the standard generators of the braid groups to Dehn twists and is called geometric monodromy (see [PV92], [Wa99]). Hence we can consider \(H_1(\Sigma^d_n)\) as a \(\pi_1(C_n) = \text{Br}_n\)-representation; we write also \(V_{n,d} := H_1(\Sigma^d_n)\).

We will need to consider the natural map of the braid group \(\text{Br}_n = \text{Art}(A_{n-1})\) onto the permutation group \(\mathfrak{S}_n\) on \(n\) letters. This map induces a representation of the group \(\text{Br}_n\) onto \(\mathbb{Z}^n\) by permuting codinates. We write \(\Gamma_n\) for this representation of \(\text{Br}_n\).

Along all this paper, when not specified, the homology is understood to be computed with constant coefficients over a ring \(R\).

### 3. Generalized twists

We give a picture of the surface \(\Sigma^d_n\) as a ramified covering of the disk \(D\) by choosing a system of cuts in \(D\) connecting the points of \(P\) with the the boundary, as in Figure 1. The arc \(s_k\) connecting \(x_k\) with \(x_{k+1}\) lifts to the \(i\)th-sheet to a path \(\gamma_i^k\) which connects \(x_k\) to \(x_{k+1}\) \((i = 1, \ldots, d)\); as said in the previous section, we are
identifying the \(x_k\)'s with points in \(E_d^{(n)}\). Locally in \(\Sigma_d^n\), around each ramification point \(x_k\), the sheets follow each other in the anticlockwise ordering (see Figures 2 to 4).

Clearly, \(\Sigma_d^n\) deformation retracts onto the graph with set of vertices \(P = \{x_1, \ldots, x_n\}\) and edges \(\gamma_i^k, \; k = 1, \ldots, n - 1, \; i = 1, \ldots, d\), therefore \(b_1(\Sigma_d^n) = (n - 1)(d - 1)\).

Let us give to \(\gamma_i^k\) the orientation going from \(x_k\) to \(x_{k+1}\). We can consider the circuits \(a_i^k = \gamma_i^k(\gamma_{i+1}^k)^{-1}, \; i = 1, \ldots, d\) (taking indices mod \(d\)); notice the relations \(\sum_{i=1}^d [a_i^k] = 0\) among their classes in \(H_1\). A basis for \(H_1(\Sigma_d^n)\) is given by the classes \([a_i^k], \; k = 1, \ldots, n - 1, \; i = 1, \ldots, d - 1\).
Next, we give a precise description of the monodromy action for the bundle $\Sigma^d_n \to \mathbb{E}^d_n \to \mathbb{C}_n$.

Let $\sigma_k$ be the standard generator for the braid group $\mathbb{B}_k$, given by an anti-clockwise half-twist around the arc $s_k$, exchanging $x_k$ and $x_{k+1}$ and leaving everything outside a neighborhood of $s_k$ pointwise fixed. We find that $\sigma_k$ lifts to an homeomorphism $t_{k,d}$ of $\Sigma^d_n$ which generalizes standard Dehn twist for $d = 2$.

Let $\Gamma_k = \cup_{i=1}^d \gamma^k_i$; first, $t_{k,d}$ is the identity outside a small neighborhood of $\Gamma_k$. Inside a small neighborhood of $\Gamma_k$, $t_{k,d}$ acts as “rotation around $\Gamma_k$ of a $\frac{2\pi}{d}$-angle.” To be precise, we describe the neighborhood of $\Gamma_k$ as a $2$–complex.

Let $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$, with coordinates $(x, y)$ and the standard CW-structure. We orient the horizontal edges according to increasing $x$-coordinate.

**Definition 1.** For $i = 1, \ldots, d$, $j = 1, -1$, let $Q_{i,j}$ be a copy of $Q$, with coordinates $(x, y)_{i,j}$. We define the 2-complex $\tilde{T}_d$ as

$$\tilde{T}_d := \left( \bigsqcup_{i=1}^d Q_{i,j} \right) / \left( \begin{array}{c} (1, y)_{i,1} \sim (1, y)_{i+1, -1} \\ (0, y)_{i,1} \sim (0, y)_{i-1, -1} \end{array} \right)$$

where the indices are considered mod $d$. 
Now define
\[ T_d := \tilde{T}_d / (x, 0)_{i,1} \sim (x, 0)_{i,-1} \quad (i = 1, \ldots, d). \]

Notice that \( \tilde{T}_d \) is connected for odd \( d \), homeomorphic to a cylinder, while it has two connected components for even \( d \), both homeomorphic to a cylinder. Therefore \( T_d \) is an orientable surface with one boundary component for \( d \) odd and two boundary components for \( d \) even. In case \( d \) even, make a preliminary step by attaching in \( \tilde{T}_d \) the two edges corresponding to the index \( i \) : 
\[ \gamma_{x,0}^i \cup \gamma_{x,0}^{i,-1}. \]
Then \( T_d \) is obtained from a regular polygon \( P_d \) having \( 2d \) edges and one hole for odd \( d \), having \( 2d - 1 \) edges and two holes for even \( d \), and attaching the opposite edges, as in Figure 5. One verifies that the genus of \( T_d \) is \( \frac{d-1}{2} \).

**Figure 5.** Examples of \( P_d \) for \( d = 3, 4 \). \( T_d \) is obtained by attaching the opposite edges \( \gamma_i \) according to the arrows.

The image in \( T_d \) of the set \( \Gamma := \bigsqcup_{i,j} \{(0,1) \times \{0\}\}_{i,j} \) is homeomorphic to the graph \( \Gamma_k \) (where the vertices correspond to the images in \( T_d \) of \( \bigsqcup_{i,j} \{(0,1) \times \{0\}\}_{i,j} \)).

Actually, there is an embedding \( j_k : T_d \to \Sigma_k \) taking \( \Gamma \) to \( \Gamma_k \) and \( T_d \) to a small neighborhood of \( \Gamma_k \) in \( \Sigma_k \) (see Figures 3 and 4).

**Definition 2.** For \( h \in \mathbb{R}, \ 0 \leq h \leq 1 \), we define the \( h \)-rotation \( \rho_h \) in \( T_d \) as the map defined by:
\[
\rho_h((x,y)_{i,1}) = \begin{cases} 
(x + h, y)_{i,1} & \text{if } x + h \leq 1 \\
(2 - x - h, y)_{i+1,-1} & \text{if } x + h \geq 1
\end{cases};
\]
\[
\rho_h((x,y)_{i,-1}) = \begin{cases} 
(x - h, y)_{i,-1} & \text{if } x - h \geq 0 \\
(h - x, y)_{i+1,1} & \text{if } x - h \leq 0
\end{cases}.
\]

(in one formula: \( \rho_h((x,y)_{i,j}) = \begin{cases} 
(x + jh, y)_{i,j} & \text{if } 0 \leq x + h \leq 1 \\
(1 - x + j(1 - h), y)_{i+1,-j} & \text{otherwise}
\end{cases} \))

Let \( \varphi : [0,1] \to [0,1] \) be a \( C^\infty \) function such that:

i) \( \varphi(t) = 1 \) for \( t < \epsilon; \ \varphi(t) = 0 \) for \( t > 1 - \epsilon \ (\epsilon << 1); \)

ii) \( \varphi \) is decreasing in \( \epsilon < t < 1 - \epsilon. \)
We define the $\frac{1}{d}$-twist on $T_d$ as the homeomorphism

$$\tau_d((x, y)_{i,j}) = \rho_{\nu(y)}((x, y)_{i,j}).$$

In Figures 6 and 7 we represent the action of $\tau_d$ by using the polygon $P_d$. For odd $d$, $\tau_d$ is the identity in the boundary of $T_d$ and is induced by a rotation of $\frac{\pi}{d}$ in the exterior boundary of $P_d$, in the sense taking $\gamma_i$ into $\gamma_{i+1}$, $i = 1, \ldots, d$. For even $d$, $\tau_d$ is the identity in the boundary of $T_d$ (corresponding to the boundary of the two holes of $P_d$) and is induced by a rotation of $\frac{2\pi}{d}$, one for each of the two halves of $P_d$ determined by $\gamma_1$, in the sense which takes $\gamma_i$ into $\gamma_{i+1}$, $i = 1, \ldots, d$.

**Definition 3 (generalized Dehn twist).** The corresponding $\frac{1}{d}$-twist in the surface $\Sigma_n^d$ is the homeomorphism $t_{k,d}$ induced by $\tau_d$ through the embedding $j_k$.

In particular, $t_{k,d}$ takes $\gamma_i^k$ into $(\gamma_{i+1}^k)^{-1}$ (indices mod $d$).

**Remark 2.** Notice that for $d$ even $(t_{k,d})^2$ is a standard Dehn twist around a simple curve following the path $\prod_{i=1}^{d} (\gamma_i^k)^{(-1)^i-1}$. So, we obtain an explicit root of a Dehn twist which appears to be simpler than the one described in [MS09].
We denote by \((,\)\) the intersection product
\[
(,): H_1(\Sigma_n^d; \mathbb{Z}) \times H_1(\Sigma_n^d; \mathbb{P}; \mathbb{Z}) \rightarrow \mathbb{Z}
\]
By the exact sequence
\[
0 \rightarrow H_1(\Sigma_n^d; \mathbb{Z}) \rightarrow H_1(\Sigma_n^d; \mathbb{P}; \mathbb{Z}) \rightarrow H_0(\mathbb{P}; \mathbb{Z}) = \mathbb{Z}^n \rightarrow H_0(\Sigma_n^d; \mathbb{Z}) = \mathbb{Z} \rightarrow 0
\]
it follows that \(H_1(\Sigma_n^d; \mathbb{P}; \mathbb{Z})\) is free of rank \((n - 1)d\), generated by the classes \([\gamma_i^k]\), \(k = 1, \ldots, n - 1\), \(i = 1, \ldots, d\).

It is easy to verify the following intersection products (we consider indices \(i, j\) mod \(d\)):

\[
([a_i^k], [a_j^k]) = \begin{cases} 
1 & \text{if } j = i + 1 \\
-1 & \text{if } j = i - 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
([a_i^k], [a_j^{k+1}]) = \begin{cases} 
-1 & \text{if } j = i \\
1 & \text{if } j = i + 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
([a_i^k], [\gamma_j^k]) = \begin{cases} 
1 & \text{if } j = i \\
-1 & \text{if } j = i + 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
([a_i^k], [\gamma_j^{k+1}]) = \begin{cases} 
-1 & \text{if } j = i \\
0 & \text{otherwise}
\end{cases}
\]

\[
([a_i^{k+1}], [\gamma_j^k]) = \begin{cases} 
1 & \text{if } j = i \\
0 & \text{otherwise}
\end{cases}
\]

**Theorem 3.1.**

(a) The \(\frac{1}{d}\) – twist \(t_{k,d}\) is the homeomorphism induced by the monodromy of the bundle \(E_n^d \rightarrow \mathbb{C}_n\) applied to the half-twist \(\sigma_k \in \text{Br}_n\).

(b) \(t_{k,d}\) induces on \(H_1(\Sigma_n^d; \mathbb{Z})\) the automorphism

\[
(t_{k,d})_*: a \rightarrow a - \displaystyle\sum_{i=1}^{d} (a, \gamma_i^k)[a_i^k]
\]

(c) Let \(t_{a_i^k}\) be the Dehn twist associated to the simple curve \(a_i^k\), \(i = 1, \ldots, d - 1\); then

\[
(t_{k,d})_* = (t_{a_1^k} \cdots t_{a_{d-1}^k})_*
\]

**Proof.** Statement (a) follows by direct verification that \(t_{k,d}\) projects through \(\pi\) into the half-twist \(\sigma_k\).

Let the class of \(a\) be represented by a cycle \(\tilde{a}\) which intersects transversely every \(\gamma_i^k\). For every point \(p \in \tilde{a} \cap \gamma_i^k\), \(t_{k,d}\) modifies \(\tilde{a}\) by adding a cycle in the class of \([a_i^k]\), according to the sign of the intersection (see Figures 6 and 7). Therefore (b) follows.

We prove (c) by induction on \(d\). First, recall that a Dehn twist around a simple curve \(c\) acts on the first homology group by \(a \rightarrow a - (a, [c])[c]\).

We have

\[
(t_{k,d-1})_*(a) = a - \displaystyle\sum_{i=1}^{d-2} (a, \gamma_i^k)[a_i^k] - (a, \gamma_{d-1}^k)[\tilde{a}_{d-1}^k]
\]
where \( \tilde{a}_{d-1}^k = [\tau_{d-1}^k(\gamma_{d-1}^k)^{-1}] = [a_{d-1}^k] + [a_d^k] \). Therefore one has
\[
t_{k,d}(a) = t_{k,d-1}(a) + (a, [a_{d-1}^k][a_d^k]) = (t_{a_1} \cdots t_{a_{d-2}})u(a) + (a, [a_{d-1}^k][a_d^k])
\]
where the last equality comes from induction. But from equations (1) and \( \sum_{i=1}^d a_i^k = 0 \) it easily follows
\[
(t_{a_1} \cdots t_{a_{d-2}})u([a_{d-1}^k]) = -[a_{d}^k]
\]
therefore
\[
t_{k,d}(a) = (t_{a_1} \cdots t_{a_{d-2}})u(a - (a, [a_{d-1}^k][a_d^k])) = (t_{a_1} \cdots t_{a_{d-1}})u(a).
\]

Part (a) and (c) of the preceding theorem has been proved independently for \( d = 3 \) in [KST18, Thm. 4.1] and for any \( d \) in [GM18, Prop. 5.3], where the \( \frac{1}{d} - \text{twist} \) are called “chain twists”.

4. Motivations and homology of complex braid groups

We recall that the classification of irreducible complex reflection groups is given in [ST54]. The group \( G(d, e, d) \) is the group of monomial matrices such that all the non-zero entries are \((de)\)-th roots of unity and the product of all the non-zero entries is a \( e \)-th root of unity. In particular in this paper we are interested in the complex reflection groups of type \( G(d, d, n) \), that act naturally as complex reflection groups on the space \( \mathbb{C}^n \). The complement of the reflection arrangement associated to the group \( G(d, d, n) \) is
\[
\mathcal{M}(d, n) = \{(z_1, \ldots, z_n) \mid \forall i < j, a \in \mathbb{Z}, z_i \neq \zeta_d^a z_j\}.
\]
and the group \( G(d, d, n) \) acts freely on this space (see for example [Nak83, BMR98]). For \( d = 2 \) the group \( G(d, d, n) \) is the Coxeter reflection group of type \( D_n \).

**Theorem 4.1.** The space \( E_{d,n}^d \) is homotopy equivalent to the regular orbit space of the complex reflection group of type \( G(d, d, n) \). Hence it is a classifying space for the complex braid group of type \( B(d, d, n) \).

Recall that for \( d = 2 \) we have that the group \( B(2, 2, n) \) is the Artin group \( \text{Art}(D_n) \). In this case the monodromy action of the braid group \( B_n \) on the fundamental group of the fiber \( \Sigma_{d,n}^d \) is described in [CP95] (see also [PV96]).

**Proof.** For this proof it will be convenient to replace the space \( E_{d,n}^d \) with the homotopy equivalent space
\[
\tilde{E}_{d,n}^d := \{(P, y) \in \text{Conf}_n(\mathbb{C}) \times \mathbb{C} \times \mathbb{C} \mid y^d = (z - x_1) \cdots (z - x_n)\}.
\]
Then we define the space \( \tilde{E}_{d,n}^d \) that can be identified with the subset of \( E_{d,n}^d \) given by the points \((P, z, y)\) with \( z = 0 \). We set
\[
\tilde{E}_{d,n}^d := \{(P, y) \in \text{Conf}_n(\mathbb{C}) \times \mathbb{C} \mid y^d = (1)^n x_1 \cdots x_n\}
\]
where, as usual, we write \( P \) for the unordered configuration of points \( \{x_1, \ldots, x_n\} \subset \mathbb{C} \). Next we can define a retraction which is an homotopy equivalence: let \( \rho : \tilde{E}_{d,n}^d \to \tilde{E}_{d,n}^d \) be the map
\[
(P, z, y) \mapsto (P - z, y)
\]
where, given the set \( P = \{x_1, \ldots, x_n\} \subset \mathbb{C} \), we write \( P - z \) for the set \( \{x_1 - z, \ldots, x_n - z\} \). Clearly \( \rho \) is an homotopy equivalence since its fiber is contractible.
Finally we consider the regular orbit space of the complex reflection group G(d, d, n). We can define a continuous map
\[ \Xi : \mathcal{M}(d, n) \to \tilde{E}^d_n \]
by
\[ \Xi : (z_1, \ldots, z_n) \mapsto \left( \{ z_1^d, \ldots, z_n^d \}, \zeta_{2d} z_1 \cdots z_n \right) \]
where \( \zeta_{2d} \) is a 2d-th primitive root of unity. It is easy to verify that the map \( \Xi \) is G(d, d, n)-invariant, hence it induces a map on the quotient space
\[ \tilde{\Xi} : \mathcal{M}(d, n)/G(d, d, n) \to \tilde{E}^d_n. \]

It is straightforward to check that the \( \tilde{\Xi} \)-fiber of a point \( X \in \tilde{E}^d_n \) is the G(d, d, n)-orbit of a point \( \tilde{X} \) in the counter-image of \( X \). This implies that the map \( \tilde{\Xi} \) is a bijection.

Moreover, since the map \( \tilde{\Xi} \) is bijective, proper and closed, it induces an homeomorphism between the two spaces (see for example [Loo84, Prop. 1.11] for a detailed argument).

From the fibration
\[ \Sigma^d_n \hookrightarrow E^d_n \to C_n \]
where the left and the right term are \( k(\pi, 1) \) spaces, we obtain that \( E^d_n \) is a \( k(\pi, 1) \) space for \( \pi = G(d, d, n) \). \( \square \)

Since the surface \( \Sigma^d_n \) is a \( k(\pi, 1) \) space, our construction gives a short proof of the result in [Nak83] for the complex arrangements of type G(d, d, n):

**Corollary 4.2.** The space \( \mathcal{M}(d, n) \) is a \( k(\pi, 1) \).

**Remark 3.** Let us consider the subset of \( \mathbb{C}^n \)
\[ \mathcal{M}_0(d, n) := \{ (z_1, \ldots, z_n) \mid \forall i, z_i \neq 0, \forall i < j, a \in \mathbb{Z}, z_i \neq \zeta_d^a z_j \}. \]
with the natural free action of the complex reflection group G(d, 1, n). Since the map \( \tilde{\Xi}_0 : \mathcal{M}_0(d, n)/G(d, 1, n) \to \text{Conf}_n(\mathbb{C}^*) \) is a covering, we obtain that \( \mathcal{M}_0(d, n) \) is a \( k(\pi, 1) \) space as already proved in [Nak83].

5. **Homology of some Artin groups**

We use the notations and the technical results collected in [CSI7] § 3 for the case \( d = 2 \). We indicate here all the generalizations that we need for general \( d \).

From the description given in [CMI4] Thm. 4.5, Thm. 4.12, Rmk. 4.13] we have the following results.

**Proposition 5.1.** Let \( \mathbb{F} \) be a field of characteristic \( p \). For \( p \) an odd prime and \( n \) odd the \( \mathbb{F}[t] \)-module \( \oplus_{i,n} H_i(\text{Art}(B_n); \mathbb{F}[t^{\pm 1}]) \) has a basis given by
\[
\bar{e}(z_{2m+1}t^{-1}y_{j_1} \cdots y_{j_t}x_{i_1} \cdots x_{i_k})
\]
where \( r > 0, j_1 \leq \cdots \leq j_t, i_1 < \cdots < i_k, \) these generators have torsion of order \( (1 + t) \).

We can compute the homology groups \( H_*(\text{Art}(B_n); \mathbb{F}_p[t]/(1 - (-t)^d)) \) by using the explicit description of [CMI4] § 4.3, 4.4, 4.5 and 4.6]. As a special case of [CMI4] Prop. 4.7, 4.14] we have the isomorphism
\[
H_i(\text{Art}(B_n); \mathbb{F}_p[t]/(1 - (-t)^d)) = h_i(n, p) \oplus h'_i(n, p)
\]
where the two summands are determined by the following exact sequence:

\[ 0 \to h'_i(n, p) \to H_i(\text{Art}(B_n); \mathbb{F}_p[t^\pm]) \xrightarrow{(1 - (-t)^d)} H_i(\text{Art}(B_n); \mathbb{F}_p[t^\pm]) \to h_i(n, p) \to 0. \]

For odd \( n \) all the elements of \( H_i(\text{Art}(B_n); \mathbb{F}_p[t^\pm]) \) are multiple of \( x_0 \) for \( p = 2 \) and multiple of \( h \) for odd \( p \); hence they have \((1 + t)\)-torsion (see Proposition 5.1). This implies that the multiplication by \((1 - (-t)^d)\) is the zero map and the generators of \( h'_i(n, p) \) and \( h_i(n, p) \) are in bijection with a set of generators of \( H_i(\text{Art}(B_n); \mathbb{F}_p[t^\pm]) \).

As in [CS17] § 3, prop. 3.5] for \( d = 2 \), we have:

**Proposition 5.2.** For odd \( n \) the homology \( H_n(\text{Art}(B_n); \mathbb{F}_2[t]/(1 - (-t)^d)) \) is generated, as an \( \mathbb{F}_2[t] \)-module, by the classes of the form

\[
\tilde{\gamma}(z_c, x_0x_{i_1} \cdots x_{i_k} := \frac{1 - (-t)^d}{1 + t} z_{c+1}x_{i_1} \cdots x_{i_k}.
\]

that correspond to the generators of \( h'_i(n, 2) \) and

\[
\gamma(z_c, x_0x_{i_1} \cdots x_{i_k}) := \frac{1 - (-t)^d}{1 + t} z_{c+1} h^{r-1} y_{i_1} \cdots y_{i_k} x_{i_1} \cdots x_{i_k}.
\]

that correspond to generators of \( h_i(n, 2) \). Here \( 0 \leq i_1 \leq \cdots i_k \), \( c \) is even and both kind of generators have torsion of order \((1 + t)\).

For odd \( n \) and \( p \) an odd prime the homology \( H_n(\text{Art}(B_n); \mathbb{F}_p[t]/(1 - (-t)^d)) \) is generated, as an \( \mathbb{F}_p[t] \)-module, by the classes of the form

\[
\tilde{\gamma}(z_c, h^r y_{j_1} \cdots y_{j_t} x_{i_1} \cdots x_{i_k} := \frac{1 - (-t)^d}{1 + t} z_{c+1} h^{r-1} y_{j_1} \cdots y_{j_t} x_{i_1} \cdots x_{i_k}.
\]

that correspond to the generators of \( h'_i(n, p) \) and

\[
\gamma(z_c, h^r y_{j_1} \cdots y_{j_t} x_{i_1} \cdots x_{i_k}) := \frac{1 - (-t)^d}{1 + t} z_{c+1} h^{r-1} y_{j_1} \cdots y_{j_t} x_{i_1} \cdots x_{i_k}.
\]

that correspond to generators of \( h_i(n, p) \). Here \( 0 < r, j_1 \leq \cdots \leq j_t, i_1 < \cdots < i_k \), \( c \) is even and both kind of generators have torsion of order \((1 + t)\).

Generalizing the sets introduced in [CS17] §3, we provide sets of elements \( B' \), \( B'' \) of the \( \mathbb{Z} \)-modules \( H_i(C_{1,n}; \mathbb{Z}) \cong H_i(\text{Art}(B_n); \mathbb{Z}[t]/(1 + t)) \) and \( H_i(C_{1,n}; \mathbb{Z}) \) for \( n \) odd, such that the following two condition are satisfied:

(i) \( B' \) (resp. \( B'' \)) induces a base of the homology of \( H_i(C_{1,n}; \mathbb{Q}) \) (resp. \( H_i(C_{1,n}; \mathbb{Q}) \));
(ii) the images of the elements of \( B' \) (resp. \( B'' \)) in \( H_i(C_{1,n}; \mathbb{Z}_p) \) (resp. \( H_i(C_{1,n}; \mathbb{Z}_p) \)) are linearly independent for any prime \( p \).

**Definition 4.** Let \( n \) be an odd integer. We define the sets \( B' \subset H_i(C_{1,n}; \mathbb{Z}) \) (for \( d = 1 \)) and \( B'' \subset H_i(C_{1,n}; \mathbb{Z}) \) (for \( d > 1 \)) given by the following elements:

\[
\omega^{(d)}_{2i,j,0} := \frac{\tilde{\gamma}(z_{2i+1} x_{j-1}^0)}{(1 + t)} \text{ and } \tilde{\omega}^{(d)}_{2i,j,0} := \frac{(1 - (-t)^d) z_{2i+1} x_{j-1}^0}{(1 + t)} \text{ for } j > 0;
\]

and

\[
\omega^{(d)}_{2i,j,1} := \frac{\tilde{\gamma}(z_{2i+1} x_{j-1}^1 x_1)}{(1 + t)} \text{ and } \tilde{\omega}^{(d)}_{2i,j,1} := \frac{(1 - (-t)^d) z_{2i+1} x_{j-1}^1 x_1}{(1 + t)} \text{ for } j > 0.
\]
The elements above give a basis for \( H_\bullet(C_{1,n};\mathbb{Q}) \) (resp. \( H_\bullet(C_{1,n};\mathbb{Q}) \)) for \( n \) odd (condition (i)) (see \cite{CM14} §4.2).

For condition (ii), one verifies that the elements in \( \mathcal{B}' \) and \( \mathcal{B}'' \) define, mod 2, a subset of the bases of \( H_i(\text{Art}(B_n);\mathbb{Z}[t]/(1 + t)) \) and \( H_i(\text{Art}(B_n);\mathbb{Z}[t]/(1 - (-t)^d)) \) given in \cite{CM14} §4.4 and, mod \( p \) for an odd prime, a subset of the bases of \( H_i(\text{Art}(B_n);\mathbb{Z}_p[t]/(1 + t)) \) and \( H_i(\text{Art}(B_n);\mathbb{Z}_p[t]/(1 - (-t)^d)) \) given in \cite{CM14} §4.6.

So we have:

**Proposition 5.3.** For \( n \) odd the elements of \( \mathcal{B}' \) (resp. \( \mathcal{B}'' \)) are a free set of generators of a maximal free \( \mathbb{Z} \)-submodule of \( H_i(C_{1,n};\mathbb{Z}) \) (resp. \( H_i(C_{1,n};\mathbb{Z}) \)).

**Proposition 5.4.** Let \( n \) be an even integer. Let \( \mathbb{F} \) be a field of characteristic 0. The Poincaré polynomial of \( H_\bullet(\text{Art}(B_n);\mathbb{F}[t]/([d]_{(-t)})) \) is trivial for \( d \) odd \( (d > 1) \) and is \( (1 + q)q^{n-1} \) for \( d \) even. For \( d \) even a basis of the homology is given by the following generators

\[
\overline{\mathcal{E}}(z_n) = \frac{1}{1 - t}, z_n.
\]

**Proof.** This follows from \cite{CS17} Prop. 3.2 and by studying the long exact sequence associated to

\[
0 \to \mathbb{F}[t^\pm 1][d]_{(-t)} \mathbb{F}[t^\pm 1] \to \mathbb{F}[t]/([d]_{(-t)}) \to 0
\]

as in \cite{CM14} §4.2. \( \square \)

6. Exact sequences

This section generalizes the results given in \cite{CS17} §4.

The fibration \( \pi : \widetilde{C}_{1,n}^d \to C_n \) has a continuous section \( s \) that can be defined as follows.

**Definition 5.** If \( p \) is a monic polynomial with \( n \) distinct roots \( x_1, \ldots, x_n \) such that \( |x_i| < 1, i = 1, \ldots, n \), we define

\[
s : p \mapsto \left( p, \ z := \frac{\max_i |x_i| + 1}{2}, \sqrt[p]{p(z)} \right).
\]

Here, if \( p(z) = \prod (z - x_i) \) we have \( \Re(z - x_i) > 0 \), and we choose \( \sqrt[p]{z - x_i} \) as the unique \( d \)-th root with maximum real part; this defines \( \sqrt[p]{p(z)} := \prod_i \sqrt[p]{z - x_i} \) as a continuous function.

The section \( s : C_n \to \widetilde{C}_{1,n}^d \) lifts the section \( \overline{s} : C_n \to C_{1,n} \) taking \( p \mapsto (p, z) \). It follows from the exact sequence of the pair \( (\widetilde{C}_{1,n}^d, C_n) \) the splitting \( H_i(\widetilde{C}_{1,n}^d) \cong H_i(C_n) \oplus H_i(C_{1,n}^d, C_n) \). In a similar way we have \( H_i(C_{1,n}^d) \cong H_i(C_n) \oplus H_i(C_{1,n}, C_n) \).

**Remark 4.** The group \( \mathbb{Z}/d \) acts on \( \widetilde{C}_{1,n}^d \) as a group of automorphisms of the covering \( \widetilde{C}_{1,n}^d \to C_{1,n} \). Hence we can define \( d \) sections \( s^{(j)} : C_n \to \widetilde{C}_{1,n}^d \), for \( j \in \mathbb{Z}/d \) as follows:

\[
s^{(j)} : p \mapsto \left( p, \ z := \frac{\max_i |x_i| + 1}{2}, \ e^{\frac{2\pi ij}{d}} \sqrt[p]{p(z)} \right).
\]
where we choose $\sqrt[p(z)]{q}$ as above (and $s = s^{(0)}$).

We can include $S : C_n \times \mathbb{Z}/d \to \widetilde{C}_{1,n}^d$ and the action of $\mathbb{Z}/d$ exchanges the components $C_n \times \{i\}$. Hence we can understand the inclusion $s : C_n \to \widetilde{C}_{1,n}^d$ in homology via the following diagram:

$$H_i(C_n) \xrightarrow{\text{long}} H_i(C_n) \otimes R[t]/(1 - (-t)^d) \xrightarrow{S} H_i(\widetilde{C}_{1,n}^d)$$

and the composition is injective.

**Remark 5.** If $\gcd(n, d) = 1$ any two sections $s^{(k)}, s^{(j)}$ are homotopic. In fact, given $a, b$ integers such that $an + bd = 1$ we can define in a unique way a continuous family of maps $s_t : C_n \to \widetilde{C}_{1,n}^d$ such that $s_0 = s^{(k)}$ and

$$s_t := \left( p, z_t := e^{2\pi i (k-j)a} \frac{\max |x_i| + 1}{2}, \sqrt[p(z_t)]{q} \right)$$

where we choose the $d$-th root $\sqrt[p(z_t)]{q}$ as above for $t = 0$ and we extend it continuously for $t > 0$. Since $p(z_t)$ is a product of $n$ factors we have $s_1 = s^{(j)}$. As a consequence $s^{(i)} = s^{(j)}$.

Generalizing the construction given in [CS17, § 4] (see also [Bia16]) for the case $d = 2$, we consider the following decomposition. We have $E_n^d = \widetilde{C}_{1,n}^d \cap N$, where $N$ is a small tubular neighborhood of the subset $C_{1,n-1}$ corresponding to the ramification locus. The intersection $\widetilde{C}_{1,n}^d \cap N$ contracts onto $C_{1,n-1} \times S^1$. Moreover, a copy of the subspace $C_n = s(C_n)$ is contained into $\widetilde{C}_{1,n}^d$.

Therefore the relative Mayer-Vietoris long exact sequence gives the following long exact sequence:

$$\cdots \to H_i(C_{1,n-1}) \otimes H_i(S^1) \to H_i(\widetilde{C}_{1,n}^d, C_n) \to H_i(E_n^d, C_n) \to \cdots$$

From Kunneth decomposition the map $\iota$ factorizes taking the first (second) factor into the first (second) factor; so the exact sequence reduces to:

$$(10) \quad \cdots \to H_{i-1}(C_{1,n-1}) \otimes H_i(S^1) \to H_i(\widetilde{C}_{1,n}^d, C_n) \to H_i(E_n^d, C_n) \to \cdots$$

The argument given in [CS17, § 4] gives

$$H_i(\text{Br}_n; \Gamma_n) = H_i(C_n; \Gamma_n) \simeq H_i(C_{1,n-1}) = H_i(\text{Art}(B_{n-1}))$$

where $\Gamma_n$ is the permutation representation (see section 2) and $H_i(S^1 \times \mathbb{P}) \simeq \Gamma_n$. Moreover, from Remark 4

$$H_i(E_n^d, C_n) \simeq H_{i-1}(\text{Br}_n; H_i(\Sigma_n^d))$$

and finally the term $H_i(\widetilde{C}_{1,n}^d, C_n)$ is isomorphic to $H_{i-1}(\text{Br}_n; H_i(\mathbb{D} \setminus \mathbb{P}^d))$.

Therefore we can rewrite (10) as an exact sequence involving non local homology of the braid groups:

$$(11) \quad \cdots \to H_{i-1}(\text{Br}_n; H_i(S^1 \times \mathbb{P})) \xrightarrow{\iota} H_{i-1}(\text{Br}_n; H_i(\mathbb{D} \setminus \mathbb{P}^d)) \to H_{i-1}(\text{Br}_n; H_i(\Sigma_n^d)) \to \cdots$$

Sequence (11) is the long exact sequence for the homology of the group $\text{Br}_n$ associated to the short exact sequence of coefficients

$$0 \to H_1(S^1 \times \mathbb{P}) \to H_1(\mathbb{D} \times \mathbb{P}) \oplus H_1(\mathbb{D} \setminus \mathbb{P}^d) \to H_1(\Sigma_n^d) \to 0$$
Proposition 6.1. Let \( \tau : H_1(C_{1,n}) \to H_1(\widetilde{C}_{1,n}^d) \) be the transfer map induced by the \( d \) fold covering \( \widetilde{C}_{1,n}^d \to C_{1,n} \). The following diagram commutes:

\[
\begin{array}{ccc}
H_{i-1}(\text{Br}_n; H_1(S^1 \times P)) & \overset{\iota}{\longrightarrow} & H_{i-1}(\text{Br}_n; H_1(\widetilde{D}^d)) \\
\Downarrow \cong & & \Downarrow \cong \\
H_{i-1}(C_{1,n-1}) \otimes H_1(S^1) & \overset{\mu_*}{\longrightarrow} & H_i(\widetilde{C}_{1,n}^d, C_n) \\
\Downarrow \mu_* & & \Downarrow J \\
H_i(C_{1,n}) & \overset{\tau}{\longrightarrow} & H_i(\widetilde{C}_{1,n}^d) 
\end{array}
\]

where \( \mu_* \) is induced by the map \( \mu \) above and \( J \) is induced by the inclusion \( \widetilde{C}_{1,n}^d \to (\widetilde{C}_{1,n}^d, C_n) \).

Some properties of \( \mu_* \) are investigated in [CS17, Prop. 4.1, Prop. 4.2, Rmk. 5]. In particular we have that \( \mu_* \) is injective and when we consider homology with coefficients in a field \( F \) of characteristic 0 the map

\[
\mu_* : H_{i-1}(C_{1,n-1}) \otimes H_1(S^1) \to H_i(C_{1,n})
\]

is an isomorphism.

Recall the notation \([d]_q\) for the \( q \)-analog \( 1 + q + \cdots + q^{d-1} \).

Proposition 6.2. The following diagram commutes:

\[
\begin{array}{ccc}
H_i(C_{1,n}) & \overset{\tau}{\longrightarrow} & H_i(\widetilde{C}_{1,n}^d) \\
\Downarrow \cong & & \Downarrow \cong \\
H_i(\text{Art}(B_n); R[t]/(1 + t)) & \overset{[d]_{-1}}{\longrightarrow} & H_i(\text{Art}(B_n); R[t]/(1 - (-t)^d)).
\end{array}
\]
where in the bottom row we are considering the map induced by the \([d]_{(-t)}\)-multiplication map \(C_*(\text{Art}(B_n), R[t]/(1 + t)) \to C_*(\text{Art}(B_n), R[t]/(1 - (-t)^d))\).

The proofs of Propositions 6.1, 6.2 are analogous to the proofs of [CS17, Prop. 4.3, Prop. 4.4].

We also have the following analogue of [CS17, Rmk. 6]:

**Remark 6.** Consider the isomorphism \(H_*(\widetilde{C}_{1,n}^{-d}) \cong H_*(\text{Art}(B_n); R[t]/(1 - (-t)^d))\). Let \(R\) be a field of characteristic \(p\). For \(p \nmid d\) the second term decomposes as

\[
H_*(\text{Art}(B_n); R[t]/(1 + t)) \cong H_*(\text{Art}(B_n); R[t]/[d]_{(-t)})
\]

and moreover for \(n\) odd and \(p \nmid d\) the term \(H_*(\text{Art}(B_n); R[t]/[d]_{(-t)})\) is trivial, which follows from the fact that for \(n\) odd the module \(H_*(\text{Art}(B_n); R[t^{\pm 1}])\) has \((1 + t)\)-torsion and from the homology long exact sequence associated to

\[
0 \to R[t^{\pm 1}]/[d]_{(-t)} \to R[t]/[d]_{(-t)} \to 0
\]
since \([d]_{(-t)} \equiv d \pmod{(1 + t)}\). Since for \(n\) even the module \(H_*(\text{Art}(B_n); R[t^{\pm 1}])\) has \((1 + t^2)^k\)-torsion for suitable \(k\) (see [CML14, Thm. 4.12]), the same argument applies for \(n\) even, when \(d\) is odd and \(p \nmid d\). In fact for \(p \neq 2\) we have that \([d]_{(-t)}\) is co-prime both with \((1 + t)\) and with \((1 - t)\) and, for \(p\) odd, \((1 + t)\) and \((1 - t)\) are co-primes;

for \(p = 2\) and \(d\) odd we have that \([d]_{(-t)}\) and \((1 - t^2)\) are co-primes.

In particular, under the condition stated above the homology groups \(H_*(\widetilde{C}_{1,n}^{-d}) \cong H_*(\text{Art}(B_n); R[t]/(1 - (-t)^d))\) and \(H_*(\text{C}_{1,n}) \cong H_*(\text{Art}(B_n); R[t]/(1 + t))\) are isomorphic and the isomorphism is induced by the quotient map \(R[t]/(1 - (-t)^d) \to R[t]/(1 + t)\) (again, here we are using that \(p \nmid d\) and \(n\) is odd). So we can consider the commuting diagram

\[
\begin{array}{ccc}
0 & \to & H_0(\text{C}_{1,n}) \\
\downarrow \cong & & \downarrow \cong \\
0 & \to & H_1(\text{C}_{1,n})
\end{array}
\]

where the last vertical map is an isomorphism from the five lemma. From the right square we have that the map \(J\) corresponds to the homomorphism

\[
J: H_1(\text{Art}(B_n); R[t]/(1 + t)) \to H_1((\text{Art}(B_n), \text{Art}(A_{n-1})); R[t]/(1 + t))
\]

associated to the inclusion \(\text{Art}(A_{n-1}) \hookrightarrow \text{Art}(B_n)\) induced by \(C_{1,n} \hookrightarrow C_{1,n}\). From the short exact sequence in the second row of the diagram above we have that the homomorphism

\[
\overline{J}: H_1(\text{C}_{1,n}) = H_1(\text{C}_{1,n}, C_n) \oplus H_1(C_n) \to H_1(\text{C}_{1,n}, C_n)
\]
is the projection on the first term of the direct sum.

**Lemma 6.3.** If \(p\) is a prime such that \(p \nmid d\) or \(p = 0\), \(R\) is a field of characteristic \(p\) then the homomorphism

\[
\overline{\tau}: H_1(\text{Art}(B_n); R[t]/(1 + t)) [d]_{(-t)} H_1(\text{Art}(B_n); R[t]/(1 - (-t)^d))
\]
is invertible if at least one between \(n\) and \(d\) is odd.
Proof. This follows since, for odd $n$, the homology group $H_i(\text{Art}(B_n); R[t^{\pm 1}])$ has $(1 + t)$-torsion, while for $n$ even the group has torsion of order $(1 - t^2)^k$ for suitable $k$.

For $p \nmid d$ (and with $d$ odd when $n$ is even) we have that

$$H_i(\text{Art}(B_n); R[t]/[d]_{(-1)}) = 0$$

(see Remark 6) and hence $H_q$ is injective and its cokernel is trivial for $d$ and has Poincaré polynomial $k$. 

Proposition 6.4. 

Hence the following consequences. Since $H_i(\text{Art}(B_n); R[t]/([d]_{(-1)}))$ is trivial for $d$ odd and has Poincaré polynomial $(1 + q)q^{n-1}$ for $d$ even (see Proposition 5.4) and since $H_u(C_n)$ has Poincaré polynomial $(1 + q)$ (see [CS17] § 3.1), we have that the map

$$J : H_i(C_{1,n} \rightarrow H_i(C_{1,n})$$

is an isomorphism for $i > 1$. Moreover the argument of Lemma 6.3 implies that the map

$$\tau : H_u(C_{1,n}) \rightarrow H_u(C_{1,n})$$

is injective and its cokernel is trivial for $d$ odd and has Poincaré polynomial $(1 + q)q^{n-1}$ for $d$ even.

Proposition 6.4. Let $n$ be even and $F$ a field of characteristic 0. Then for $i > 1$ the map

$$i : H_{i-1}(Br_n; H_1(S^1 \times P)) \rightarrow H_{i-1}(Br_n; H_1(D(P^d))$$

is injective; its cokernel has rank 1 if $d$ is odd and $i = n - 1, n - 2$; otherwise its rank is 0.

Proof. The proposition follows from [CS17] Rmk. 5 and from Remark 7. 

Theorem 6.5. Consider the inclusions $H_i(C_n) \subset H_i(C_{1,n}, C_n)$ and $H_i(C_n) \subset H_i(C_{1,n})$ associated to the sections $s : C_n \hookrightarrow C_{1,n}$ and $s : C_n \hookrightarrow C_{1,n}$. If $n$ is odd

$$\tau(H_u(C_n)) \subset H_u(C_n)$$

and for $x \in H_u(C_n)$ we have $\tau(x) = dx$. Moreover if $n$ is even, $d$ is odd and $R$ is a ring of characteristic $p$ with $p \nmid d$, then for $x \in H_u(C_n; R)$ we have $\tau(x) = dx$. 


Proof: The complex $C^\bullet_\ast(A_{n-1}), R)$ described in Section 5 that computes the homology $H^\ast_\ast(C_n; R)$, can be seen as a subcomplex of

$$C^\bullet_\ast(B_n), R) \cong C^\bullet_\ast(B_n), R[t]/(1 + t))$$

mapping $A \mapsto \overline{A}$. This inclusion of complexes induces the homology homomorphism associated to the section $\overline{s} : C_n \hookrightarrow C_1$,n. Moreover, this map of complexes lift to a map $\overline{s} : C^\bullet_\ast(A_{n-1}), R) \to C^\bullet_\ast(B_n), R[t]/(1 + t))$ and hence to the map $\overline{s} : C^\bullet_\ast(A_{n-1}), R) \to C^\bullet_\ast(B_n), R[t]/(1 - (-t)^d))$ that induces the homology homomorphism associated to the section $s : C_n \hookrightarrow C_1$,n. $\overline{d}_n$.

This implies that we have the following commutative diagram with exact columns:

\[
\begin{array}{cccc}
H^\ast_\ast(B_n), R[t^\pm]) & 1 & H^\ast_\ast(B_n), R[t^\pm]) \\
& 1 & & 1 - (-t)^d \\
& 1 & & 1 - (-t)^d \\
& & & \\
& & & \\
\end{array}
\]

Notice that the restriction of the map $\tau$ to the image of $\overline{s}$ lifts, via $\overline{s}$, to the multiplication by $[d]_{-t}$.

We claim that for $n$ odd the image of $s$ in $H^\ast_\ast(B_n), R[t]/(1 - (-t)^d))$ has only $(1 + t)$-torsion. This follows since any the inclusion

$$C^\bullet_\ast(A_{n-2}), R) \to C^\bullet_\ast(A_{n-1}), R)$$

defined on generators by

$$A \mapsto 0A$$

induces an isomorphism in homology and hence any class $x \in s(A^\bullet_\ast(C_n; R))$ is represented by a cycle of the form $00x'$ and we have the relation $(1 + t)00x' = \overline{710}x'$.

For $n$ even and $d$ odd, when $R$ is a field of characteristic $p$ such that $p \nmid d$ we have the $R[t]$-module $H^\ast_\ast(B_n), R[t]/(1 - (-t)^d))$ has only $(1 + t)$-torsion and is isomorphic to $H^\ast_\ast(B_n), R[t]/(1 + t))$, as seen in Remark 4 where the isomorphism is induced by the quotient map $R[t]/(1 - (-t)^d) \to R[t]/(1 + t)$.

In both cases this imply that we have the inclusion $\tau(H^\ast_\ast(C_n)) \subset H^\ast_\ast(C_n)$ and, since $[d]_{-t} \equiv d \mod (1 + t)$, $\tau(x) = dx$ for $x \in H^\ast_\ast(C_n)$.

From theorems 6.3 and Lemma 6.3 we obtain:

**Corollary 6.6.** If $R$ is a field of characteristic $p$, $p \nmid d$, and at least one between $d$ and $n$ is odd, then the projection on $H^\ast_\ast(C_1, C_n)$ of the restriction of the homomorphism $\tau$ to $H^\ast_\ast(C_1, n, n)$

$$\tau : H^\ast_\ast(C_1, n, n) \to H^\ast_\ast(C_1, n, n)$$

is an isomorphism.
Proof. Under the given assumptions $\tau$ is an isomorphism (Lemma 6.3) and its restriction to $H_i(C_n)$ maps to $H_i(C_n)$. So the corollary follows. □

We now describe the odd torsion of the homology $H_i(\text{Br}_n; H_1(\Sigma_n^d))$, by using the map $J$ in the diagram (13).

Let $\mathcal{F}$ and $E^2_{ij}$ be the filtration and the associated spectral sequence defined in [CS17] § 4. The proof of the following lemma is completely analogous to the one of [CS17] Lem. 4.9.

**Lemma 6.7.** Let $E^2_{ij} = H_j(\text{Br}(n-i); R[t]/([d]_{(-t)}))$ be the spectral sequence induced by the filtration $\mathcal{F}$. The inclusion $\text{Art}(A_{n-1}) \hookrightarrow \text{Art}(B_n)$ induces the isomorphism

$$H_j(\text{Art}(A_{n-1}); R[t]/([d]_{(-t)})) \cong E^2_{0j}$$

for all $j$.

**Theorem 6.8.** Let at least one between $n$ and $d$ be odd. Consider the integral monodromy representation of the braid group $\text{Br}_n$ on the group $H_1(\Sigma_n^d; \mathbb{Z})$. Then the homology $H_i(\text{Br}_n; H_1(\Sigma_n^d; \mathbb{Z}))$ is a torsion $\mathbb{Z}$-module with only $p^j$ torsion for $p | d$.

**Proof.** Let $n$ and $d$ be as in the theorem, and let $p \nmid d$. It derives from the description of the map $\mu_p$ (see [CS17] Prop. 4.1 and Cor. 4.10), from the results about the map $\tau$ (Proposition 6.1, Lemma 6.3, Corollary 6.6) and from Remark 6 concerning the map $J$ that the map $\iota$ in diagram (13) is an isomorphism in characteristic $p$. Then the result follows from sequence (11). □

7. A FIRST BOUND FOR TORSION ORDER

In this section we prove that if $p^j \mid d$ and $p^{j+1} \nmid d$, then torsion in $H_i(\text{Br}_n; H_1(\Sigma_n^d))$ appears with order at most $p^{j+1}$.

Following lemma generalizes [CS17] Lem. 5.2.

**Lemma 7.1.** Let $R = \mathbb{Z}$. For $n$ odd, the homology

$$H_i(\widetilde{C}_{1,n}^d) \cong H_i(\text{Art}(B_n); R[t]/(1 - (-t)^d))$$

has no $p^2$-torsion for any prime $p$.

**Proof.** It will suffice to show that the dimension over $\mathbb{F}_p$ of the homology of the complex $(H_*(\text{Art}(B_n); \mathbb{F}_p[t]/(1 - (-t)^d)), \beta_p)$, where $\beta_p$ is the Bockstein homomorphism, is the same as the dimension over $\mathbb{Q}$ of $H_*(\text{Art}(B_n); \mathbb{Q}[t]/(1 - (-t)^d))$ (see [Hat02] Thm. 3E.4). We split our proof in several steps

**Step a)** According to [Leh04] Thm. 6.1, case $n$ odd], the Poincaré polynomial of the homology groups $H_*(\text{Art}(B_n); \mathbb{Q}[t]/(1 - (-t)^d))$ is

$$P(\text{Art}(B_n), t) = (1 + t)(1 + t + t^2 + \cdots + t^{n-1}).$$

**Step b)** The explicit computation of the Bockstein homomorphism $\beta_p$ of the homology group $H_*(\text{Art}(B_n); \mathbb{F}_p[t]/(1 - (-t)^d))$ is the following. Let

$$0 \to \mathbb{Z}_p[t]/(1 - (-t)^d) \xrightarrow{\pi_p} \mathbb{Z}_p[t]/(1 - (-t)^d) \xrightarrow{\pi_p} \mathbb{Z}_p[t]/(1 - (-t)^d) \to 0$$

be the short exact sequence of coefficients. Then for $p = 2$ (see Prop. 5.2)

$$\frac{(1 - (-t)^d)}{1 + t} z_{c+1} x_{i_1} \cdots x_{i_k} = \pi_2 \left( \frac{(1 - (-t)^d)}{1 + t} z_{c+1} x_{i_1} \cdots x_{i_k} \right)$$
and
\[
\frac{1}{1+t} (1 - (-t)^d) \frac{2}{1+t} z_{c+1} x_{i_1} \cdots x_{i_k} = \sum_{j=1}^{i_1 \cdots k} \frac{2}{1+t} (1 - (-t)^d) z_{c+1} x_{i_1} \cdots x_{i_{j-1}} \cdots x_{i_k} = \\
= i_2 \left( \sum_{j=1}^{i_1 \cdots k} \frac{1}{1+t} (1 - (-t)^d) z_{c+1} x_{i_1} \cdots x_{i_{j-1}} \cdots x_{i_k} \right)
\]
and hence, using the notation introduced in Proposition 5.2 we have
\[
\beta_2 \gamma (z_c, x_0 x_{i_1} \cdots x_{i_k}) = \sum_{j=1}^{i_1 \cdots k} \beta_2 \gamma (z_c, x_0 x_{i_1} \cdots x_{i_{j-1}} \cdots x_{i_k}) 
\]
for all generators of the form given in (6). Moreover
\[
\frac{\varepsilon (z_{c+1} x_{i_1} \cdots x_{i_k})}{(1+t)} = \\
= \pi_2 \left( \frac{1}{1+t} \left( \varepsilon (z_{c+1} x_{i_1} \cdots x_{i_k}) - 2 \sum_{j=1}^{i_1 \cdots k} z_{c+1} x_{i_1} \cdots x_{i_{j-1}} \cdots x_{i_k} \right) \right)
\]
and
\[
\frac{\varepsilon}{(1+t)} \left( \frac{1}{1+t} \left( \varepsilon (z_{c+1} x_{i_1} \cdots x_{i_k}) - 2 \sum_{j=1}^{i_1 \cdots k} z_{c+1} x_{i_1} \cdots x_{i_{j-1}} \cdots x_{i_k} \right) \right) = \\
= \varepsilon \left( \frac{1}{1+t} \left( -2 \sum_{j=1}^{i_1 \cdots k} z_{c+1} x_{i_1} \cdots x_{i_{j-1}} \cdots x_{i_k} \right) \right) = \\
= -2 \sum_{j=1}^{i_1 \cdots k} \frac{\varepsilon (z_{c+1} x_{i_1} \cdots x_{i_{j-1}} \cdots x_{i_k})}{1+t} = i_2 \left( \sum_{j=1}^{i_1 \cdots k} \frac{\varepsilon (z_{c+1} x_{i_1} \cdots x_{i_{j-1}} \cdots x_{i_k})}{1+t} \right)
\]
hence we have
\[
\beta_2 \gamma (z_c, x_0 x_{i_1} \cdots x_{i_k}) = \sum_{j=1}^{i_1 \cdots k} \gamma (z_c, x_0 x_{i_1} \cdots x_{i_{j-1}} \cdots x_{i_k}) 
\]
for all generators of the form given in (7).

For \( p \) odd we have the following analogous computation (again, see Prop. 5.2)
\[
\frac{1}{1+t} z_{c+1} h^{r-1} y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k} = \pi_p \left( \frac{1}{1+t} z_{c+1} h^{r-1} y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k} \right)
\]
and
\[
\frac{\varepsilon}{1+t} \left( \frac{1}{1+t} z_{c+1} h^{r-1} y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k} \right) = \\
= \sum_{j=1}^{i_1 \cdots k} p \left( \frac{1}{1+t} z_{c+1} h^{r-1} y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_{j-1}} y_{j_i} x_{i_{j+1}} \cdots x_{i_k} \right)
\]
and hence, using the notation introduced in Proposition 5.2 we have

\[ (16) \quad \beta_p \gamma(z_c, h^r y_{j_1} \cdots y_{j_l} x_{i_1} \cdots x_{i_k}) = \sum_{j=1}^{\infty} \gamma(z_c, h^r y_{j_1} \cdots y_{j_l} x_{i_1} \cdots x_{i_j} x_{i_{j+1}} \cdots x_{i_k}) \]

for all generators of the form given in (8). Moreover

\[ \beta_p \gamma(z_c, h^r y_{j_1} \cdots y_{j_l} x_{i_1} \cdots x_{i_k}) = \frac{1}{1 + t} \left( \sum_{j=1}^{\infty} \gamma(z_c, h^r y_{j_1} \cdots y_{j_l} x_{i_1} \cdots x_{i_j} x_{i_{j+1}} \cdots x_{i_k}) \right) + \]

and computing \( \gamma \) on the term above we get

\[ (17) \quad \beta_p \gamma(z_c, h^r y_{j_1} \cdots y_{j_l} x_{i_1} \cdots x_{i_k}) = \sum_{j=1}^{\infty} \gamma(z_c, h^r y_{j_1} \cdots y_{j_l} x_{i_1} \cdots x_{i_j} x_{i_{j+1}} \cdots x_{i_k}) \]

for all generators of the form given in (9).

**Step c)** Next we need to consider the following definition.

**Definition 6.** Let \( a, b \) be two non-negative integers, with \( a \in \mathbb{N}_{\geq 0}, b \in \{0, 1\} \). Moreover let \( I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_h) \), where we assume that:

(i) \( j_1 < \cdots < j_h \),

(ii) min \( J \geq 2 \),

(iii) for all \( s \in 1, \ldots, k \) there exists an integer \( t \in 1, \ldots, h \) such that \( i_s + 1 = j_t \).

We define the following sub-modules of \( H_*(\text{Art}(B_n); \mathbb{F}_2[t]/(1 - (-t)^d)) \):

\[ M(c, a, b, I, J)_2 := \langle \gamma(z_c, x_0^a x_1^b x_2^2 \cdots x_i^2 \epsilon(x_{j_1}) \cdots \epsilon(x_{j_h})) \rangle \quad \text{where} \quad \epsilon(x_{j_t}) = x_{j_t} \quad \text{or} \quad x_{j_{t-1}} \]

and

\[ \tilde{M}(c, a, b, I, J)_2 := \langle \gamma(z_c, x_0^a x_1^b x_2^2 \cdots x_i^2 \epsilon(x_{j_1}) \cdots \epsilon(x_{j_h})) \rangle \quad \text{where} \quad \epsilon(x_{j_t}) = x_{j_t} \quad \text{or} \quad x_{j_{t-1}} \]

Moreover let \( p \) be an odd prime and let \( I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_h) \), where we assume that:

(i') \( j_1 < \cdots < j_h \),

(ii') min \( J \geq 1 \),

(iii') for all \( s \in 1, \ldots, k \) there exists an integer \( t \in 1, \ldots, h \) such that \( i_s = j_t \)
We define the following sub-modules of $H^*(\text{Art}(B_n); \mathbb{F}_p[t]/(1 - (-t)^d))$:

$M(c, a, b, I, J)_p := \langle (z_c, h^a x_0^b y_{i_1} \cdots y_{i_k} \epsilon(x_{j_1}) \cdots \epsilon(x_{j_n})) \mid \epsilon(x_{j_1}) = \epsilon(y_{j_2}) \rangle.$

and

$\tilde{M}(c, a, b, I, J)_p := \langle \tilde{\gamma}(z_c, h^a x_0^b y_{i_1} \cdots y_{i_k} \epsilon(x_{j_1}) \cdots \epsilon(x_{j_n})) \mid \epsilon(x_{j_n}) = \epsilon(y_{j_1}) \rangle.$

**Remark 8.** Notice that conditions (iii) and (iii') above imply that if $J = \emptyset$ then also $I = \emptyset$ and hence for any prime $p$ the modules $M(c, a, b, \emptyset, \emptyset)_p$ and $\tilde{M}(c, a, b, \emptyset, \emptyset)_p$ have rank 1 concentrated in degree $c + b$ and $c + b + 1$ respectively.

The modules $M(c, a, b, I, J)_p$ and $\tilde{M}(c, a, b, I, J)_p$ are free $\mathbb{Z}_p[t]/(1 + t)$-modules, closed for $\beta_p$, as follows from formulas [14], [15], [16], [17].

**Step d)** If $J \neq \emptyset$, the complexes $(M(c, a, b, I, J)_p, \beta_p)$ and $(\tilde{M}(c, a, b, I, J)_p, \beta_p)$ are acyclic. This fact can be proven by the same argument used in [Cal06] Lem. 4.4. The argument can be expressed with the following statement:

**Lemma 7.2.** Let $P(J)$ be the chain complex with $\mathbb{F}_p$ coefficients associated to the boolean lattice of the subsets of $J$. The complexes $(M(c, a, b, I, J)_p, \beta_p)$ and $(\tilde{M}(c, a, b, I, J)_p, \beta_p)$ are isomorphic to $P(J)$. In particular if $J \neq \emptyset$ the complexes $(M(c, a, b, I, J)_p, \beta_p)$ and $(\tilde{M}(c, a, b, I, J)_p, \beta_p)$ are acyclic.

**Proof of Lemma 7.2.** Let first construct an isomorphism $\theta$ between the boolean complex $P(J)$ and the complex $M(c, a, b, I, J)_p$. For $p = 2$ we can map the generator $e_K$ of $P(J)$ associated to a subset $K$ of $J$ to the element

$$\theta(e_K) := \gamma(z_c, h^a x_0^b y_{i_1} \cdots y_{i_k} \epsilon(x_{j_1}) \cdots \epsilon(x_{j_n})),$$

where $\epsilon(x_{j_1}) = \epsilon(y_{j_2})$. For $p > 2$ we can map the generator $e_K$ of $P(J)$ associated to a subset $K$ of $J$ to the element

$$\theta(e_K) := \gamma(z_c, h^a x_1^b y_{i_1} \cdots y_{i_k} \epsilon(x_{j_1}) \cdots \epsilon(x_{j_n})),$$

where $\epsilon(x_{j_1}) = \epsilon(y_{j_2})$. It is easy to check that $\theta \circ d = \beta_p \circ \theta$. Similarly we can construct an isomorphism $\theta' : P(J) \rightarrow \tilde{M}(c, a, b, I, J)_p$ with

$$\theta'(e_J) := \tilde{\gamma}(z_c, h^a x_0^b y_{i_1} \cdots y_{i_k} \epsilon(x_{j_1}) \cdots \epsilon(x_{j_n}))$$

for $p = 2$ and

$$\theta'(e_J) := \tilde{\gamma}(z_c, h^a x_1^b y_{i_1} \cdots y_{i_k} \epsilon(x_{j_1}) \cdots \epsilon(x_{j_n}))$$

for $p > 2$ and check that $\theta' \circ d = \beta_2 \circ \theta'$.

**Step e)** Given multi-indices $I = (i_1, \ldots, i_k)$ (resp. $J = (j_1, \ldots, j_l)$) we write $x_I$ for $x_{i_1} \cdots x_{i_k}$ (resp. $y_J$ for $y_{j_1} \cdots y_{j_l}$).

We recall from Proposition 5.2 that the elements of the form $\tilde{\gamma}(z_c, x_0 x_1)$ and $\gamma(z_c, x_0 x_1)$ (resp. $\gamma(z_c, s y_{j_2})$ and $\gamma(z_c, s y_{j_2} y_{j_3})$) for $p$ an odd prime are basis of $H_*(\text{Art}(B_n); \mathbb{F}_2[t]/(1 - (-t)^d))$ (resp. $H_*(\text{Art}(B_n); \mathbb{F}_p[t]/(1 - (-t)^d))$ for $p$ an odd prime. We assume this basis fixed.

**Step f)** For a given prime $p$ we claim that any two distinct modules $M(c, a, b, I, J)_p$ (or $\tilde{M}(c, a, b, I, J)_p$) are contained in the span of disjoint subsets of the given basis and in particular they are in direct sum. The case of modules of the form $M(c, a, b, I, J)_2$ can be proved as follows (the cases of modules of the form $\tilde{M}(c, a, b, I, J)_2$, $\tilde{M}(c, a, b, I, J)_p$, $M(c, a, b, I, J)_p$, and $\tilde{M}(c, a, b, I, J)_p$ are analogous).

...
and of the form $M(p, a, b, I, J)_p$ and $\tilde{M}(c, a, b, I, J)_p$ when $p$ is odd, are analogous). Let

$$\gamma_0 = \gamma(z, x_0 x_1^p x_2^2 \cdots x_i^2 \gamma(x_{j_1}) \cdots \gamma(x_{j_n}))$$

with $\gamma(x_{j_1}) = x_{j_1}$ or $x_{j_1}^2$ be a generator of $M(c, a, b, I, J)$. We can choose an element $\gamma_1$ of the form

$$\gamma_1 = \gamma(z, x_0 x_1^p x_2^2 \cdots x_i^2 \gamma(x_{j_1}) \cdots \gamma(x_{j_n}))$$

such that $\gamma_0$ appears as a summand in $\beta_2(\gamma_1)$. The constrains on the multi-indexes $I$ and $J$ and formula (15) imply that such an element $\gamma_1$ exists if and only if for at least one index $l$ we have that $\gamma(x_{j_l}) = x_{j_l}^2$. If such an element $\gamma_1$ exists we have that $\gamma_1 \in M(c, a, b, I, J)$ and we say that $\gamma_0$ lifts to $\gamma_1$. Hence in a finite number of steps we have that $\gamma_0$ lifts to

$$\gamma(z, x_0 x_1^p x_2^2 \cdots x_i^2 \gamma(x_{j_1}) \cdots \gamma(x_{j_n})).$$

This implies that an element $\gamma_0$ does not belong at the same time to two different modules $M(c, a, b, I, J)_p$ and $M(c', a', b', I', J')_p$.

Therefore if we fix a prime $p$ the modules $M(c, a, b, I, J)_p$ for all admissible $c, a, b, I, J$ are in direct sum and the modules $\tilde{M}(c, a, b, I, J)_p$ for all admissible $c, a, b, I, J$ are in direct sum.

Step g) Next we claim that every element of the form $\tilde{\gamma}(z, x_0 x_1 \cdots x_{l_k})$ or of the form $\gamma(z, x_0 x_1 \cdots x_{l_k})$ (resp. $\tilde{\gamma}(z, h^n y, x_I)$ and $\gamma(z, h^n y, x_I)$ for $p$ odd) appears in at least one complex $M(c, a, b, I, J)_p$ or $\tilde{M}(c, a, b, I, J)_p$. Let us prove this when $p = 2$, in the case of a generator of the form $\gamma(z, x_0 x_1 \cdots x_{l_k})$, the other cases being analogous. We can write the monomial $x_0 x_1 \cdots x_{l_k}$ as

$$x_0^{q_0} \cdots x_{l_k}^{q_r}$$

with $q_1 < q_2 < \cdots < q_r$. Then we define the strictly ordered multi-index $J$ as follows: $j \in J$ if and only if $j > 1$ and one of the following conditions is satisfied:

(a) $p_j$ is odd;

(b) $p_j = 0$ and $p_{j-1}$ is even and non-zero.

Moreover if $p_j$ is odd we set $\gamma(x_j) = x_j$, otherwise we set $\gamma(x_j) = x_{j-1}^2$. Next we define the multi-index $I$ suitably in the unique way such that

$$x_0^{p_0} x_1^{p_1} \cdots x_{l_k}^{p_{l_k}} = x_0^{p_0} x_1^{p_1} \cdots x_{l_k}^{p_{l_k}} \gamma(x_{j_1}) \cdots \gamma(x_{j_n}).$$

It is straightforward to check that

$$\gamma(z, x_0 x_1 x_2 x_3 \cdots x_i \gamma(x_{j_1}) \cdots \gamma(x_{j_n})) = \gamma(z, x_0 x_1 x_2 x_3 \cdots x_i \gamma(x_{j_1}) \cdots \gamma(x_{j_n})) =$$

and that the multi-indexes $I$ and $J$ satisfies the condition of Definition 6.

Step h) Hence $h_i(n, p)$ is the direct sum of all admissible modules $M(c, a, b, I, J)_p$ and $h'_i(n, p)$ is the direct sum of all admissible modules $\tilde{M}(c, a, b, I, J)_p$ and we recall (equation 5) that

$$H^*_\beta(\text{Art}(B_n); F_p[t]/(1 - (-t)^d)) = h_i(n, 2) \oplus h'_i(n, 2).$$

This direct sum decomposition implies that the homology $H_{\beta_p}$ of the complex $(H^*_\beta(\text{Art}(B_n); F_p[t]/(1 - (-t)^d)), \beta_p)$ is given as follows:

$$H_{\beta_p} = \bigoplus M(c, a, b, \emptyset, \emptyset)_p \oplus \bigoplus \tilde{M}(c, a, b, \emptyset, \emptyset)_p$$
for \( c \) even, \( a \in \mathbb{N}_{>0}, b \in \{0,1\} \). In fact if \( J = \emptyset \) then also \( J = \emptyset \) and for all non-empty \( J \) we have from Lemma 7.2 that the complexes \( (M(c,a,b,I,J)_p,\beta_p) \) and \( (\hat{M}(c,a,b,I,J)_p,\beta_p) \) are acyclic.

Step i) Using Remark 8 it is easy to check that the complex \( H_{\beta_p} \) has Poincaré polynomial \( (1+t)(1+t^2 + \cdots + t^{n-1}) \), hence Lemma 7.1 follows.

\[ \square \]

In [CMI14] it was proved that \( H_*(B(2d,d,n);R) = H_*(\text{Art}(B_n);R[t]/(1-(-t)^d)) \) : the homology of the complex braid group \( B(2d,d,n) \) with coefficients in any field was computed, but no computation about the Bockstein homomorphism was given in this case. As a consequence of Lemma 7.1 we can add:

**Corollary 7.3.** For \( n \) odd the integer homology of \( B(2d,d,n) \) has no \( p^2 \)-torsion.

**Lemma 7.4.** Let \( R = \mathbb{Z} \). Let \( p \) be a prime and assume that \( p^k \mid d \), but \( p^{k+1} \nmid d \). For \( n \) odd the cokernel of the homomorphism

\[
\tau : H_i(C_{1,n}) \to H_i(\tilde{C}_{1,n})
\]

has no \( p^{k+1} \)-torsion.

**Proof.** The proof is practically identical to [CS17, Lem. 5.4]: we use here the sets \( B' \subset H_i(C_{1,n} ; \mathbb{Z}) \) and \( B'' \subset H_i(\tilde{C}_{1,n} ; \mathbb{Z}) \) introduced in Definition 4. These are free sets of generators of a maximal free \( \mathbb{Z} \)-submodule of \( H_i(C_{1,n} ; \mathbb{Z}) \) and \( H_i(\tilde{C}_{1,n} ; \mathbb{Z}) \) respectively (Proposition 5.3).

The map \( \tau \) acts on the elements of \( B' \) by:

\[
\begin{align*}
\tau : \omega_{2i,j,0}^{(1)} &\mapsto \frac{1+(-t)^d}{1+t} \omega_{2i,j,0}^{(d)} = d\omega_{2i,j,0}^{(d)}, \\
\tau : \tilde{\omega}_{2i,j,0}^{(1)} &\mapsto \tilde{\omega}_{2i,j,0}^{(d)}, \\
\tau : \omega_{2i,j,1}^{(1)} &\mapsto \frac{1+(-t)^d}{1+t} \omega_{2i,j,1}^{(d)} = d\omega_{2i,j,1}^{(d)}, \\
\tau : \tilde{\omega}_{2i,j,1}^{(1)} &\mapsto \tilde{\omega}_{2i,j,1}^{(d)}.
\end{align*}
\]

Therefore \( \tau \) diagonally maps each element of \( B' \) to 1 or \( d \) times the corresponding element of \( B'' \).

Then the result follows since the \( \mathbb{Z} \)-modules \( H_i(C_{1,n} ; \mathbb{Z}) \) and \( H_i(\tilde{C}_{1,n} ; \mathbb{Z}) \) have no \( p^2 \)-torsion and \( \tau \) is an isomorphism mod \( p \) if \( p \nmid d \) (see Lemma 6.3).

Now we consider homology with coefficient in \( R = \mathbb{Z} \). As stated in [CS17, Cor. 4.10], the image of \( \mu_* \) is the submodule \( H_i(C_{1,n},C_n) \) in \( H_i(C_{1,n}) \) and we consider the composition \( \iota = J \circ \tau \circ \mu_* : H_{i-1}(C_{1,n},1) \otimes H_1(S^1) \to H_i(\tilde{C}_{1,n},C_n) \).

**Lemma 7.5.** Let \( p \) be a prime and assume that \( p^k \mid d \), but \( p^{k+1} \nmid d \). For \( n \) odd the cokernel of the composition \( J \circ \tau \circ \mu_* \) has no \( p^{k+1} \)-torsion.

The proof of the lemma above is analogous to the proof of [CS17, Lem. 5.5], with suitable modification. For the reader’s convenience we give here the adapted proof.

**Proof.** First we can consider the homomorphism \( \pi_* : H_*(C_n) \to H_*(C_{1,n}) \) induced by the inclusion \( \pi : C_n \hookrightarrow C_{1,n} \); given the decomposition \( H_*(C_{1,n}) = H_*(C_n) \oplus H_*(\tilde{C}_{1,n},C_n) \), we call \( \pi_1 \) and \( \pi_2 = J \) respectively the projections of \( H_*(\tilde{C}_{1,n}) \) onto...
We can consider the following diagram:

\[ \xrightarrow{\tau} H_*(C_n) \xrightarrow{\tau_1} H_*(C_{1,n}) \]

From Theorem 6.5 we have that \( \pi \) the first and the second summand, and hence the map \( \tau_1 : H_*(C_n) \to H_*(C_{1,n}) \) defined by the composition

\[ H_*(C_n) \xrightarrow{\tau_1} H_*(C_{1,n}) \xrightarrow{\tau} H_*(C_{1,n}) \]

We can consider the following diagram:

\[ \xrightarrow{\tau_{11}} H_*(C_n) \xrightarrow{\tau} H_*(C_{1,n}) \]

From Theorem 6.5 we have that \( \tau_{11} = d \text{Id}_{H_*(C_n)} \) and \( \pi_2 \tau_1 \pi_2(H_*(C_n)) = 0 \).

Now, let \( x_2 \in H_*(C_{1,n}) \) and let \( \tau_{12} : H_*(C_{1,n}, C_n) \to H_*(C_{1,n}, C_n) \) be the map induced by \( \tau \) by restricting to \( H_*(C_{1,n}, C_n) \) and projecting to \( H_*(C_{1,n}, C_n) \).

Recall from [CS17, Prop. 4.2] that \( \mu_\pi \) is injective. If there exists \( y \in H_*(C_{1,n}) \) such that \( \pi_2 \tau(y) = p^{k+1} x_2 \), then let \( x_1 := \pi_1(\tau(y)) \). We can consider \( -x_1 + dy \in H_*(C_{1,n}) \) and we have that \( \tau(-x_1 + dy) = -dx_1 + d(x_1 + p^{k+1} x_2) = dp^{k+1} x_2 \). Since the cokernel of \( \tau \) has at most \( p^k \)-torsion (see Lemma 7.4) it follows that \( dx_2 = 0 \) in coker \( \tau \) and finally, since \( \pi_2 \tau_1 \pi_2(H_*(C_n)) = 0 \), \( p^k x_2 = 0 \) in coker \( \tau_{12} \).

From [CS17, Lem. 5.1] and 7.5 we have that, with integer coefficients, the kernel of the map

\[ \iota : H_{i-1}(\text{Br}_n; H_1(S^1 \times P)) \to H_{i-1}(\text{Br}_n; H_1(D \setminus P)) \]

in diagram (11) has no \( p^2 \)-torsion and the cokernel of \( \iota \) has no \( p^{k+1} \)-torsion. Hence we have:

**Theorem 7.6.** Let \( p \) be a prime and assume that \( p^k \mid d \), but \( p^{k+1} \nmid d \). For \( n \) odd the homology \( H_i(\text{Br}_n; H_1(\Sigma_d)) \) computed with coefficients in the ring \( R = \mathbb{Z} \) has \( p \)-torsion of order at most \( p^{k+1} \).

8. No \( p^{k+1} \)-torsion

In this section we will show that if \( p \) is a prime such that \( p^k \mid d \) \((k > 0)\), but \( p^{k+1} \nmid d \) and \( n \) is odd then \( H_i(\text{Br}_n; H_1(\Sigma_d)) \) has \( p \)-torsion of order at most \( p^k \). In particular for all this section we will assume that \( p \) is a prime that divides \( d \).

We will consider for \( n \) odd the following short exact sequence associated to (11), with coefficients in \( \mathbb{Z}_p \) and in \( \mathbb{Z} \)

\[ 0 \to \text{coker } \iota \to H_i(\text{Br}_n; H_1(\Sigma_d)) \to \ker \iota \to 0. \]

**(22)**

**Proposition 8.1.** The short exact sequence (22) with \( \mathbb{Z} \) coefficients splits.

**Proof.** Let us fix an odd integer \( n \). We write

\[ H_i(\text{Br}_n; H_1(\Sigma_d; \mathbb{Z})) \otimes \mathbb{Z}_p = \mathbb{Z}_p^{a_i}, \]

\[ \text{coker}(\iota_i : H_i(\text{Br}_n; H_1(S^1 \times P; \mathbb{Z})) \to H_i(\text{Br}_n; H_1(D \setminus P; \mathbb{Z})) \otimes \mathbb{Z}_p = \mathbb{Z}_p^{b_i}, \]

and

\[ \text{ker}(\iota_i : H_i(\text{Br}_n; H_1(S^1 \times P; \mathbb{Z})) \to H_i(\text{Br}_n; H_1(D \setminus P; \mathbb{Z})) \otimes \mathbb{Z}_p = \mathbb{Z}_p^{c_i}. \]
Since for \( n \) odd we have seen that \( H_1(Br_n; H_1(\Sigma^d_n; \mathbb{Z})) \) is all torsion (see Theorem 6.8), we have that the \( p \)-torsion part of (22) splits if and only if
\[
 u_i + v_{i-1} = a_i.
\]
Moreover, with coefficients in \( \mathbb{Z}_p \), we have
\[
 H_1(Br_n; H_1(\Sigma^d_n; \mathbb{Z}_p)) = \mathbb{Z}_p^{a_0 + a_1}. 
\]

Let
\[
 \overline{u}_i := \text{rk ker}(t_i : H_1(Br_n; H_1(S^1 \times P; \mathbb{Z}_p)) \to H_1(Br_n; H_1(D^d \setminus P; \mathbb{Z}_p)))
\]
and
\[
 \overline{v}_i := \text{rk ker}(t_i : H_1(Br_n; H_1(S^1 \times P; \mathbb{Z}_p)) \to H_1(Br_n; H_1(D^d \setminus P; \mathbb{Z}_p))).
\]

It follows that the \( p \)-torsion part of (22) splits if and only if
\[
 2 \sum_i (u_i + v_i) = \sum_i (\overline{u}_i + \overline{v}_i).
\]

Hence we can compute the rank of the modules above.

A basis of the homology \( H_i(Br_n; H_1(S^1 \times P; \mathbb{Z}_p)) \) is given as follows. Following [CM14], the homology \( H_n(\text{Art}(B_n); F_p[t]/(1 + t)) \) for \( n \) odd is generated, as an \( F_p[t] \)-module, by the classes of the following form (see Proposition 5.2):

- for \( p = 2 \),
  \[
    \gamma_1(z_c, x_0 x_{i_1} \cdots x_{i_k}) := \frac{\gamma(z_{c+1} x_{i_1} \cdots x_{i_k})}{(1 + t)},
  \]
  where we assume \( 0 \leq i_1 \leq \cdots i_k \) and \( c \) even;

- for \( p \) an odd prime,
  \[
    \gamma_1(z_c, h^r y_{j_1} \cdots y_{j_l} x_{i_1} \cdots x_{i_k}) := \frac{\gamma(z_c, y_{j_1} \cdots y_{j_l} x_{i_1} \cdots x_{i_k})}{(1 + t)},
  \]

In particular for \( p = 2 \) the image of \( \mu_a \) if generated by all elements of the form \( \gamma_1(z_c, x_0 x_{i_1} \cdots x_{i_k}) \) and all elements of the form \( \gamma_1(z_c, x_0 x_{i_1} \cdots x_{i_k}) \) with \( c > 0 \), while for \( p \) odd the image of \( \mu_a \) is generated by all elements of the form \( \gamma_1(z_c, h^r y_{j_1} \cdots y_{j_l} x_{i_1} \cdots x_{i_k}) \) and all elements of the form \( \gamma_1(z_c, h^r y_{j_1} \cdots y_{j_l} x_{i_1} \cdots x_{i_k}) \) with \( c > 0 \).

As seen in Section 5 the homology \( H_n(\text{Art}(B_n); F_p[t]/(1 - (-t)^d)) \) for \( n \) odd is generated, as an \( F_p[t] \)-module, by the following classes

- for \( p = 2 \), by the classes (already introduced in [6] and [7], Proposition 5.2):
  \[
    \gamma(z_c, x_0 x_{i_1} \cdots x_{i_k}) := \frac{1 - (-t)^d}{1 + t} z_{c+1} x_{i_1} \cdots x_{i_k}
  \]
  and
  \[
    \gamma(z_c, x_0 x_{i_1} \cdots x_{i_k}) := \frac{\gamma(z_{c+1} x_{i_1} \cdots x_{i_k})}{(1 + t)}
  \]
  where we assume \( 0 \leq i_1 \leq \cdots i_k \) and \( c \) even;
b) for \( p \) odd, by the classes (already introduced in [8], [9], Proposition 5.2)

\[
\gamma(z_c, h^r y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k}) := \frac{1 - (-t)^d}{1 + t} z_c + 1 h^{r-1} y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k}.
\]

and

\[
\gamma(z_c, h^r y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k}) := \frac{\gamma(z_c + 1 h^{r-1} y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k})}{1 + t}.
\]

The following classes generate the image of \( s_* \) and hence these are the generators of the kernel of \( J \):

a) for \( p = 2 \), by the classes \( \gamma(z_c, x_0 x_{i_1} \cdots x_{i_k}) \) for \( c = 0 \);

b) for \( p \) odd, by the classes \( \gamma(z_c, h^r y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k}) \) with \( c = 0 \)

Hence for any \( p \) the map \( \tau \) acts as follows:

a) for \( p = 2 \):

\[
\tau : \gamma_1(z_c, x_0 x_{i_1} \cdots x_{i_k}) \mapsto \gamma(z_c, x_0 x_{i_1} \cdots x_{i_k});
\]

\[
\tau : \gamma_1(z_c, x_0 x_{i_1} \cdots x_{i_k}) \mapsto d \gamma(z_c, x_0 x_{i_1} \cdots x_{i_k});
\]

b) for \( p \) odd:

\[
\tau : \gamma_1(z_c, h^r y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k}) \mapsto \tilde{\gamma}(z_c, h^r y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k});
\]

\[
\tau : \gamma_1(z_c, h^r y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k}) \mapsto d \gamma(z_c, h^r y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k}).
\]

Hence if we assume that \( p \nmid d \) we have that \( \tau \) is an isomorphism with coefficients in \( \mathbb{Z}_p \). As a consequence using the Universal Coefficients Theorem we can state a stronger version of Theorem 6.8.

Let now focus on a prime \( p \) that divides \( d \).

**Remark 9.** Let \( p \) a prime such that \( p \nmid d \) and let \( n \) be an odd integer. A basis of

\[
\ker(\iota_i : H_i(\text{Br}_n ; H_1(S^1 \times P ; \mathbb{Z}_p)) \to H_i(\text{Br}_n ; H_1(D^d \mathbb{P} ; \mathbb{Z}_p)))
\]

is given:

a) for \( p = 2 \), by the elements \( \gamma(z_c, x_0 x_{i_1} \cdots x_{i_k}) \) with \( c \) even, \( c > 0 \), of degree \( n \) and homological dimension \( i + 1 \);

b) for \( p \) odd, by the elements \( \gamma(z_c, h^r y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k}) \) with \( c \) even, \( c > 0 \), \( r > 0 \), of degree \( n \) and homological dimension \( i + 1 \);

A basis of

\[
\ker(\iota_i : H_i(\text{Br}_n ; H_1(S^1 \times P ; \mathbb{Z}_p)) \to H_i(\text{Br}_n ; H_1(D^d \mathbb{P} ; \mathbb{Z}_p)))
\]

is given:

a) for \( p = 2 \), by the elements \( \gamma_1(z_c, x_0 x_{i_1} \cdots x_{i_k}) \) with \( c \) even, \( c > 0 \), of degree \( n \) and homological dimension \( i + 1 \);

b) for \( p \) odd by the elements \( \gamma_1(z_c, h^r y_{j_1} \cdots y_{j_i} x_{i_1} \cdots x_{i_k}) \) with \( c \) even, \( c > 0 \), \( r > 0 \), of degree \( n \) and homological dimension \( i + 1 \).

Clearly in both cases a) and b) the base of the kernel is in bijection with the corresponding base of the cokernel and we have \( \pi_i = \pi_i \).

Now recall (see Definition 4 and Proposition 5.3) the bases \( B' \), \( B'' \) generating the homology of \( H_i(C_{1,n} ; \mathbb{Q}) \) and \( H_i(\tilde{C}_{1,n} ; \mathbb{Q}) \) and spanning a maximal free \( \mathbb{Z} \) submodule of \( H_i(C_{1,n} ; \mathbb{Z}) \) and \( H_i(\tilde{C}_{1,n} ; \mathbb{Z}) \). Recall also the action of \( \tau \) over these
bases given in equations (18–21). The elements of $\mathcal{B}'$ (resp. $\mathcal{B}''$) map, modulo $p$, to elements of the form $\gamma_1$ (resp. $\gamma$) and in particular the elements the form $\omega_{2i,j,e}^{(1)}$ (resp. $\omega_{2i,j,e}^{(d)}$) with $i = 0$ map to elements of the form $\gamma_1(z_0, \ldots)$ (resp. $\gamma(z_0, \ldots)$) with $e = 0$. The elements of $\mathcal{B}'$ (resp. $\mathcal{B}''$) of the form $\omega_{2i,j,e}^{(1)}$ (resp. $\omega_{2i,j,e}^{(d)}$) map, modulo $p$, to elements of the form $\gamma_1$ (resp. $\gamma$). Let

$$w_i = |\{\omega_{0,j,\varepsilon}^{(1)} \in H_i(C_{1,n} ; \mathbb{Q})\}| = |\{\omega_{0,j,\varepsilon}^{(d)} \in H_i(C_{1,n} \widetilde{\Sigma}^d)\}|.$$

From the Universal Coefficients Theorem and from the description of $\tau$ given in equations (18–21) we have that

$$\sum_i u_i = \sum_i \frac{\overline{u}_i - w_i}{2} + \sum_i w_i$$

and

$$\sum_i v_i = \sum_i \frac{\overline{v}_i - w_i}{2}.$$

Then it is straightforward to see that

$$2 \sum_i (u_i + v_i) = \sum_i (\overline{u}_i + \overline{v}_i)$$

and this imply that the exact sequence (22) splits and Proposition 8.1 follows. □

As a corollary of Proposition 8.1, under the hypothesis of Theorem 7.6 we have the following result.

**Theorem 8.2.** For odd $n$ the homology $H_i(\Br_n ; H_1(\Sigma_n^d ; \mathbb{Z}))$ has no torsion of order $p^k$ if $p^k \nmid d$. □

**Remark 10.** For $d = 2$ and $n$ odd Theorem 8.2 allows to compute the homology $H_i(E_n^2 ; \mathbb{Z}) = H_i(\Br(n) ; \mathbb{Z}) \oplus H_{i-1}(\Br_n ; H_1(\Sigma_n^d ; \mathbb{Z}))$ that corresponds (see Theorem 4.1) to the homology of the Artin group of type $D_n$ (see also [Gor78]).

9. **Natural maps between families**

Let $d, d'$ two positive integers such that $d \mid d'$ and $d' = md$. There is a natural ramified covering

$$\Sigma_n^d \rightarrow \Sigma_n^{d'}$$

given by mapping $(z, y) \mapsto (z, y^m)$ and hence a corresponding map between the spaces

$$\rho = \rho^{(d',d)}_n : E_n^{d'} \rightarrow E_n^d$$

given by $(P, z, y) \mapsto (P, z, y^m)$. The map $\rho = \rho^{(d',d)}_n$ commutes with the projections $\pi : E_n^{d'} \rightarrow C_n$ and $\pi : E_n^d \rightarrow C_n$ and with the sections $s : C_n \rightarrow E_n^{d'}$ and $s : C_n \rightarrow E_n^d$.

In this section we investigate the induced homology map

$$\rho_* = (\rho^{(d',d)}_n)_* : H_*(E_n^{d'} , C_n) \rightarrow H_*(E_n^d , C_n).$$
Remark 11. The map $\rho = (d':d)$ commutes with the ramified covering $E_n^d \to C_n \times D$ and $E_n^d \to C_n \times D$ preserving their ramification loci. Hence $\rho$ induces a morphism between the corresponding long exact sequences (11) for $d'$ and $d$, giving a commuting diagram

\[
\begin{array}{ccccccc}
\cdots & \to & H_{i-1}(C_{1,n-1}) \otimes H_1(S^1) & \overset{i}{\to} & H_i(\widetilde{C}_{1,n}^d, C_n) & \to & H_{i-1}(\text{Br}_n; H_1(\Sigma_n^d)) & \to & \cdots \\
\downarrow^m & & \downarrow \varphi_* & & \downarrow \rho_* & & \downarrow \rho_* & & \cdots \\
\cdots & \to & H_{i-1}(C_{1,n-1}) \otimes H_1(S^1) & \overset{i}{\to} & H_i(\widetilde{C}_{1,n}^d, C_n) & \to & H_{i-1}(\text{Br}_n; H_1(\Sigma_n^d)) & \to & \cdots 
\end{array}
\]

where the map $\varphi_*$ is induced by the restriction of $\rho$ to the space $\widetilde{C}_{1,n}^d$:

\[
\varphi : \widetilde{C}_{1,n}^d \to \widetilde{C}_{1,n}.
\]

Notice that the vertical arrow on the left is induced by the restriction of $\rho$ to a tubular neighborhood of the ramification locus. In particular the restriction induces an $m$-fold covering $S^1 \to S^1$ and hence the map induced in homology is the multiplication by $m = \frac{d}{d'}$.

Lemma 9.1. Consider homology with coefficients in a field $\mathbb{F}$ of characteristic $p$. Assume that $2 \mid d$ or that $2 \nmid d'$. Moreover assume that $p \nmid m$. Then the map $\varphi$ induces an isomorphism

\[
\varphi_* : H_*(\widetilde{C}_{1,n}^d, C_n; \mathbb{F}) \to H_*(\widetilde{C}_{1,n}^d, C_n; \mathbb{F}).
\]

Proof. We recall that we can consider the ring $\mathbb{F}[t^{\pm 1}]$ as a module over the group $\text{Art}(B_n)$ as with the usual action (see [CS17, §3]). Since we have the relation

\[
1 - (-t)^{d'} = (1 - (-t)^d)[m](-t)^d
\]

we can consider the following commuting diagram of $\text{Art}(B_n)$-modules:

\[
\begin{array}{cccccc}
0 & \to & \mathbb{F}[t^{\pm 1}]^{1-(-t)^{d'}} & \to & \mathbb{F}[t]/(1-(-t)^d) & \to & 0 \\
\downarrow^{[m](-t)^d} & & \downarrow \text{id} & & \downarrow \eta & & \\
0 & \to & \mathbb{F}[t^{\pm 1}]^{1-(-t)^d} & \to & \mathbb{F}[t]/(1-(-t)^d) & \to & 0
\end{array}
\]

where $\eta$ is the quotient map. The induced diagram of homology groups is

\[
\begin{array}{cccccc}
\cdots & \to & H_i(C_{1,n}; \mathbb{F}[t^{\pm 1}])^{1-(-t)^{d'}} & \to & H_i(C_{1,n}; \mathbb{F}[t^{\pm 1}]) & \to & \cdots \\
\downarrow^{[m](-t)^d} & & \downarrow \text{id} & & \downarrow \varphi_* & & \\
\cdots & \to & H_i(C_{1,n}; \mathbb{F}[t^{\pm 1}])^{1-(-t)^d} & \to & H_i(C_{1,n}; \mathbb{F}[t^{\pm 1}]) & \to & \cdots
\end{array}
\]

We recall from [CM14, Thm. 4.5, 4.12] that the group $H_i(C_{1,n}; \mathbb{F}[t^{\pm 1}])$ is a direct sum of $\mathbb{F}[t^{\pm 1}]$-modules of the form $\mathbb{F}[t]/(1+t)$ and $\mathbb{F}[t]/(1-t)^{(p-1)p^s}$.

Assume that $2 \nmid d$. Then if $I = (1+t)$ or $I = (1-t^2)^k$ for some $k > 0$ then $[m](-t)^d \equiv m \mod I$ and hence, since $p \nmid m$, $[m](-t)^d$ is invertible in $\mathbb{F}[t]/I$. From the five Lemma it follows that the map $\varphi_*$ is an isomorphism.
Now let us assume that $2 \nmid d'$, and hence $2 \nmid m$. If $p = 2$ then $H_*(C_1; \mathbb{F}_2)$ is a direct sum of $\mathbb{F}_2$-modules of the form $\mathbb{F}_2/(1 + t)^k$. As seen before, $[m](-t)^d$ is invertible in $\mathbb{F}_2/(1 + t)^k$ and the same argument gives that $\tilde{p}_*$ is an isomorphism.

If $p \neq 2$ we can split the quotients $\mathbb{F}_2/(1 - t^2)(p-1)p^n$ as a direct sum

$$\mathbb{F}_2/(1 - t^2)(p-1)p^n \simeq \mathbb{F}_2/(1 - t)(p-1)p^n \oplus \mathbb{F}_2/(1 + t)(p-1)p^n.$$  

Since we are assuming that $d$ and $p$ are odd the polynomials $1 - (t)^d$ and $1 - (-t)^d$ are invertible in $\mathbb{F}_2/(1 - t)(p-1)p^n$. Hence if we restrict to the modules of the form $\mathbb{F}_2/(1 - t)(p-1)p^n$ the kernel and the cokernel of the horizontal maps corresponding to multiplications by $1 - (t)^d$ and $1 - (-t)^d$ are trivial. On the other hand if we restrict to the modules of the form $\mathbb{F}_2/(1 + t)(p-1)p^n$ we have that $[m](-t)^d$ is invertible. Again we obtain that $\tilde{p}_*$ is an isomorphism.

In all cases since $\tilde{p}_*$ restricts to the identity on $H_*(C_1, \mathbb{F})$, it is straightforward to see that the restriction of $\tilde{p}_*$ to $H_*(\tilde{C}_1, \mathbb{F})$ is an isomorphism as well. \hfill \qed

**Theorem 9.2.** Consider homology with coefficients in a field $\mathbb{F}$ of characteristic $p$. Let $d' = dm$. Assume that $2 \mid d$ or that $2 \nmid d'$. Moreover assume that $p \nmid m$. Then the map

$$\rho_* : H_*(E_n^d, C_0) \to H_*(E_n^d, C_n)$$

is an isomorphism.

**Proof.** The theorem follows from Lemma 9.1 and from the five lemma applied to the diagram of long exact sequences \cite{29}. \hfill \qed

**Corollary 9.3.** Let $d$ be a positive integer and $p$ a prime.

a) If $p = 2$ and $d = 2^a m$, with $m$ odd, then

$$H_*(E_n^d, C_n; \mathbb{F}_2) = H_*(E_n^{2^a}, C_n; \mathbb{F}_2);$$

in particular if $d$ is odd then $H_*(E_n^d, C_n; \mathbb{F}_2) = H_*(E_n^1, C_n; \mathbb{F}_2) = 0$;

b) If $p > 2$ and $d = p^a m$ with $p \nmid m$ and $m$ odd then

$$H_*(E_n^d, C_n; \mathbb{F}_p) = H_*(E_n^{p^a}, C_n; \mathbb{F}_p);$$

in particular if $p \nmid d$ and $d$ is odd then $H_*(E_n^d, C_n; \mathbb{F}_p) = H_*(E_n^1, C_n; \mathbb{F}_p) = 0$;

c) If $p > 2$ and $d = p^a m$ with $p \nmid m$ and $m$ even then

$$H_*(E_n^d, C_n; \mathbb{F}_p) = H_*(E_n^{2p^a}, C_n; \mathbb{F}_p).$$

**Proof.** The statement follows from Theorem 9.2 applied to $E_n^d$ and $E_n^{2p^a}$ or $E_n^{2p^a}$. For the case when $d$ is odd we also note that the fibration $E_n^1 \to C_n$ has fiber $\Sigma_n^1 = D$, which is contractible. Hence the spaces $E_n^1$ and $C_n$ are homotopy equivalent and $H_*(E_n^1, C_n; \mathbb{F}_p) = 0$ for any prime $p$. \hfill \qed

**Remark 12.** Corollaries 9.3 and Theorem 8.2 imply that when $d$ is odd or $n$ is odd the group $H_*(E_n^d, C_n; \mathbb{Z})$ is finite and has $p$-torsion only if $p \mid d$.

Notice that when $n, d$ are both even the group $H_*(E_n^d, C_n; \mathbb{Z})$ can have $p$-torsion for $p \mid d$ (see Tables 4, 3, 5).

**Remark 13.** Notice that the isomorphisms given in Theorem 9.2 and Corollary 9.3 are induced by the map $p$, hence they are natural and commute with homology operations. It follows from the Bockstein spectral sequence (see McC01 Thm. 10.3, 10.4) that the isomorphisms above induce isomorphisms for homology with integer coefficients localized to the prime $p$. 

10. Stabilization and computations

10.1. Stabilization results. Applying the results of Wahl and Randal-Williams \cite{Wahl2017} for the stability of family of groups with twisted coefficients it is possible to prove that the groups $H_i(B_n; H_1(\Sigma_{n}^d))$ stabilize for all $i$. In particular the map

$$H_i(B_n; H_1(\Sigma_{n}^d)) \rightarrow H_i(B_{n+1}; H_1(\Sigma_{n+1}^d))$$

is an epimorphism for $i \leq \frac{n}{2} - 1$ and an isomorphism for $i \leq \frac{n}{2} - 2$. In this section we will prove a slightly sharper result using the explicit description of the homology.

We recall from \cite{CS17} Def. 7 that the stabilization map $st : C_{1,n} \rightarrow C_{1,n+1}$ is defined by

$$(p_1, \ldots, p_{n+1}) \mapsto (p_1, \ldots, p_{n+1}, \frac{1 + \max_{1 \leq i \leq n+1}(|p_i|)}{2}).$$

We recall that $C_{1,n}$ is a classifying space for $\text{Art}(B_n)$, that is the Artin group of type B. Moreover from Shapiro Lemma (see \cite{Bro94}) we have the isomorphism

$$H_n(B(2d, d, n); R) = H_n(\text{Art}(B_n); R[t]/(1 - (-t)^d))$$

(see \cite{CML14} for more details). Finally we recall from \cite{CML14} Cor. 4.17, 4.18,4.19 the following result:

**Proposition 10.1.** Let $p$ be a prime or 0. Let $\mathbb{F}$ be a field of characteristic $p$. Let us consider the stabilization homomorphisms

$$st_n : H_i(\text{Art}(B_n); \mathbb{F}[t]/(1 + t)) \rightarrow H_i(\text{Art}(B_{n+1}); \mathbb{F}[t]/(1 + t))$$

and

$$st_n : H_i(\text{Art}(B_n); \mathbb{F}[t]/(1 - (-t)^d)) \rightarrow H_i(\text{Art}(B_{n+1}); \mathbb{F}[t]/(1 - (-t)^d)).$$

a) If $p = 2$ $st_n$ is an epimorphism for $2i \leq n$ and an isomorphism for $2i < n$.

b) If $p > 2$ $st_n$ is an epimorphism for $\frac{p(i-1)}{p-1} + 2 \leq n$ and an isomorphism for $\frac{p(i-1)}{p-1} + 2 < n$.

c) If $p = 0$ $st_n$ is an epimorphism for $i + 1 \leq n$ and an isomorphism for $i + 1 < n$.

The map $\mu$ commutes, up to homotopy, with the stabilization map $st : C_{1,n} \rightarrow C_{1,n+1}:

$$\begin{array}{c}
C_{1,n-1} \times S^1 \xrightarrow{\mu} C_{1,n} \\
\downarrow st \times \text{Id} \quad \quad \downarrow st \\
C_{1,n} \times S^1 \xrightarrow{\mu} C_{1,n+1}
\end{array}$$

The map $\tau$ naturally commutes with the stabilization homomorphism $st_n$ in homology, since $\tau$ is given by the multiplication by $(1 - (-t)^d)/(1 + t)$.

We can also define a geometric stabilization map $gst : \hat{C}_{1,n} \rightarrow \hat{C}_{1,n+1}$ as follows:

$$gst : (P, z, y) \mapsto (P \cup \{p_x\}, z, y^\frac{1}{d} - p_x)$$

where we set $p_x := \max\{|p_i|, p_i \in P, |z| = |z|\}$ and since $\Re(z - p_x) < 0$ we choose $z^\frac{1}{d} - p_x$ to be the unique $d$-th root with maximum real part among the roots with strictly positive imaginary part.
The following diagram is homotopy commutative:

\[
\begin{array}{ccc}
C_n & \xrightarrow{s} & \widetilde{C}_{1,n} \\
\downarrow{st} & & \downarrow{gst} \\
C_{n+1} & \xrightarrow{s} & \widetilde{C}_{1,n+1}
\end{array}
\]

and this imply that \( J \) commutes with the stabilization homomorphism \( gst_\ast \).

We also need to prove that the following diagram commutes:

\[
\begin{array}{ccc}
H_\ast(\text{Art}(B_n); R[t]/(1 - (-t)^d)) & \xrightarrow{\cong} & H_\ast(\widetilde{C}_{1,n}) \\
\downarrow{gst_\ast} & & \downarrow{gst_\ast} \\
H_\ast(\text{Art}(B_{n+1}); R[t]/(1 - (-t)^d)) & \xrightarrow{\cong} & H_\ast(\widetilde{C}_{1,n+1})
\end{array}
\]

This is true since the homomorphism

\[
st_\ast : H_\ast(\text{Art}(B_n); R[t]/(1 - (-t)^d)) \rightarrow H_\ast(\text{Art}(B_{n+1}); R[t]/(1 - (-t)^d))
\]

is induced by the map \( st : C_{1,n} \rightarrow C_{1,n+1} \) previously defined and it is obtained applying the Shapiro lemma to \( C_{1,n} = k(\text{Art}(B_n), 1) \), with \( R[t]/(1 - (-t)^d) = R[\mathbb{Z}_d] \). It is straightforward to check that the diagram

\[
\begin{array}{ccc}
\widetilde{C}_{1,n} & \xrightarrow{d} & C_{1,n} \\
\downarrow{gst} & & \downarrow{st} \\
\widetilde{C}_{1,n+1} & \xrightarrow{d} & C_{1,n+1}
\end{array}
\]

commutes, where the horizontal maps are the usual \( d \)-fold coverings. As a consequence we have the following result.

**Lemma 10.2.** The following diagram is commutative

(30)

\[
H_{i-1}(C_{1,n-1}) \otimes H_1(S^1) \xrightarrow{\mu_\ast} H_i(C_{1,n}) \xrightarrow{\tau} H_i(\widetilde{C}_{1,n}) \xrightarrow{J} H_i(\widetilde{C}_{1,n}, C_n) \\
\downarrow{st_\ast \otimes \text{id}} \quad \downarrow{st_\ast} \quad \downarrow{gst_\ast} \quad \downarrow{gst_\ast} \\
H_{i-1}(C_{1,n}) \otimes H_1(S^1) \xrightarrow{\mu_\ast} H_i(C_{1,n+1}) \xrightarrow{\tau} H_i(\widetilde{C}_{1,n+1}) \xrightarrow{J} H_i(\widetilde{C}_{1,n+1}, C_{n+1})
\]

**Theorem 10.3.** Consider homology with integer coefficients. The homomorphism

\[
H_i(\text{Br}_n; H_1(\Sigma_n^d)) \rightarrow H_i(\text{Br}_{n+1}; H_1(\Sigma_{n+1}^d))
\]

is an epimorphism for \( i \leq \frac{n}{2} - 1 \) and an isomorphism for \( i < \frac{n}{2} - 1 \).

Let \( p \) be a prime that does not divide \( d \). For \( n \) even the group \( H_i(\text{Br}_n; H_1(\Sigma_n^d)) \) has no \( p \) torsion when \( \frac{pi}{p-1} + 3 \leq n \) and no free part for \( i + 3 \leq n \). In particular for \( n \) even when \( \frac{3i}{2} + 3 \leq n \) the group \( H_i(\text{Br}_n; H_1(\Sigma_n^d)) \) has only torsion that divides \( d \).
Proof. The maps in the diagram \[(31)\] with \(\mathbb{Z}_p\) coefficients fits in the map of long exact sequences
\[
\cdots \overset{\partial}{\longrightarrow} H_i(C_{1, n}, C_n; \mathbb{Z}_p) \overset{\text{gst}}{\longrightarrow} H_{i-1}(\text{Br}_n; H_1(S^d; \mathbb{Z}_p)) \overset{\text{st}}{\longrightarrow} H_{i-2}(C_{1, n+1}; \mathbb{Z}_p) \times H_1(S^1) \overset{\partial}{\longrightarrow} \cdots
\]
\[
\cdots \overset{\partial}{\longrightarrow} H_i(C_{1, n+1}, C_{n+1}; \mathbb{Z}_p) \overset{\text{gst}}{\longrightarrow} H_{i-1}(\text{Br}_{n+1}; H_1(S^d; \mathbb{Z}_p)) \overset{\text{st} \otimes \text{Id}}{\longrightarrow} H_{i-2}(C_{1, n+1}; \mathbb{Z}_p) \times H_1(S^1) \overset{\partial}{\longrightarrow} \cdots
\]
For \(p = 2\), from Proposition \[10.1\] and Lemma \[10.2\] we have that the vertical map gst on the left of diagram \[(31)\] is an epimorphism for \(i \leq \frac{n}{2} - 1\) and isomorphisms for \(i < \frac{n}{2} - 1\). The vertical map st \(\otimes\) Id on the right of diagram \[(31)\] is an isomorphism for \(i \leq \frac{n}{2}\).

This implies that
\[
\text{st}_*: H_i(\text{Br}_n; H_1(S^d; \mathbb{Z}_2)) \to H_i(\text{Br}_{n+1}; H_1(S^d; \mathbb{Z}_2))
\]
is epimorphism for \(i \leq \frac{n}{2} - 1\) and an isomorphism for \(i < \frac{n}{2} - 1\).

For \(p > 2\), from Proposition \[10.1\] and Lemma \[10.2\] we have that the vertical map gst on the left of diagram \[(31)\] is an epimorphism for \(\frac{n(i-1)}{p} + 2 < n\) and isomorphisms for \(\frac{n(i-1)}{p} + 2 < n\). The vertical map st \(\otimes\) Id on the right of diagram \[(31)\] is an isomorphism for \(\frac{p(i-1)}{p-1} + 2 \leq n\). We notice that actually these conditions for \(n\) are weaker than the condition that holds when \(p = 2\).

This implies that for \(p > 2\)
\[
\text{st}_*: H_i(\text{Br}_n; H_1(S^d; \mathbb{Z}_p)) \to H_i(\text{Br}_{n+1}; H_1(S^d; \mathbb{Z}_p))
\]
is epimorphism for \(\frac{p(i-1)}{p-1} + 2 \leq n\) and an isomorphism for \(\frac{p(i-1)}{p-1} + 2 < n\).

The same argument for \(p = 0\) shows that
\[
\text{st}_*: H_i(\text{Br}_n; H_1(S^d; \mathbb{Q})) \to H_i(\text{Br}_{n+1}; H_1(S^d; \mathbb{Q}))
\]
is an epimorphism for \(i + 2 \leq n\) and an isomorphism for \(i + 2 < n\).

From the Universal Coefficients Theorem for homology we get that the homomorphism
\[
H_i(\text{Br}_n; H_1(S^d)) \to H_i(\text{Br}_{n+1}; H_1(S^d))
\]
is an epimorphism for \(i \leq \frac{n}{2} - 1\) and an isomorphism for \(i < \frac{n}{2} - 1\).

In order to prove the second part of the theorem recall (Theorem \[8.2\]) that for \(n\) odd the integer homology \(H_i(\text{Br}_n; H_1(S^d))\) has only torsion that divides \(d\).

The stabilization implies that if \(p \nmid d\), for \(n\) even \(H_i(\text{Br}_n; H_1(S^d))\) has no \(p\) torsion for \(\frac{p(n+1)}{p-1} + 3 \leq n\) and no free part for \(i + 3 \leq n\) and for \(n\) odd, \(H_i(\text{Br}_n; H_1(S^d))\) has no \(p\) torsion for \(\frac{p(n+1)}{p-1} + 2 \leq n\) and no free part for \(i + 2 \leq n\).

In particular, for \(n\) even, \(\frac{3n}{2} + 3 \leq n\) we have that \(H_i(\text{Br}_n; H_1(S^d))\) has only torsion that divides \(d\). \(\Box\)

The following statement is the analog of the result in [Che17, Prop. 9] for cohomology.

**Theorem 10.4.** For \(n\) even the groups \(H_i(\text{Br}_n; H_1(S^d; \mathbb{Z}))\) are torsion, except when \(d\) is even and \(i = n - 1, n - 2\) where \(H_i(\text{Br}_n; H_1(S^d; \mathbb{Q})) = \mathbb{Q}\).
Proof. The result follows from Proposition 6.4 and, for \( i < 2 \), from the stabilization Theorem 10.3. For \( n = 4 \) and \( i < 2 \) the result follows from a direct computation (see also \[CM14\] Thm. 6.4). □

Since from Theorem 4.1 we have the isomorphism

\[ H_i(B(d, d, n)) \cong H_i(\operatorname{Br}_n) \oplus H_{i-1}(\operatorname{Br}_n; H_1(\Sigma_n^d; \mathbb{Z})) \]

the result of Theorem 10.3 above and the stability for the homology of the classical braid groups (see \[Arn70\]) imply a stabilization of the homology of the complex braid groups of type \( B(d, d, n) \).

Theorem 10.5. The homomorphism

\[ H_i(B(d, d, n)) \to H_i(B(d, d, n + 1)) \]

induced by the natural inclusion \( B(d, d, n) \hookrightarrow B(d, d, n + 1) \) is an epimorphism for \( i \leq \frac{n}{2} \) and an isomorphism for \( i < \frac{n}{2} \).

In Tables 1 to 5 we present some computations of the groups \( H_i(\operatorname{Br}_n; H_1(\Sigma_n^d)) \) for \( d = 2 \) to 6, with integer coefficients. The computations are obtained using an Axiom implementation of the complex introduced in \[Sal94\]. Notice that the homology groups in the Table 1 coincide with the homology groups \( H_{i+1}(\operatorname{Art}(D_n), \operatorname{Br}_n) \) (see Theorem 4.1 and Remark 10).

|   | 1  | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|---|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 3 | \( \mathbb{Z}_2 \) |     |     |     |     |     |     |     |     |     |     |
| 4 | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |     |     |     |     |     |     |     |     |
| 5 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |     |     |     |     |     |     |     |     |
| 6 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_2 \mathbb{Z}_3 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |     |     |     |     |     |     |
| 7 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_3 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |     |     |     |     |     |
| 8 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^3 \) | \( \mathbb{Z}_2 \mathbb{Z}_3 \) | \( \mathbb{Z}_3 \mathbb{Z}_2 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |     |     |     |     |
| 9 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_2^3 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |     |     |     |     |
| 10 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_3 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_3 \mathbb{Z}_2 \) | \( \mathbb{Z}_3 \mathbb{Z}_2 \mathbb{Z}_5 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |     |     |
| 11 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_2^3 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |     |     |
| 12 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_2^3 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |     |
| 13 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_2^3 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |

Table 1. Computations of \( H_i(\operatorname{Br}_n; H_1(\Sigma_n^d)) \). For each column the first stable group is highlighted.
10.2. Poincaré polynomials. We use the previous results to compute explicitly
the Poincaré polynomials for odd $n$.

**Theorem 10.6.** For odd $n$, for $p$ prime such that $p \mid d$ the rank of $H_i(\text{Br}_n; H_1(\Sigma_n^d; \mathbb{Z}))$ as a $\mathbb{Z}_p$-module is the coefficient of $q^it^n$ in the expansion of the series

$$P_p(q,t) = \frac{qt^3}{(1-tq^2)(1-t^2)} \prod_{j=0}^{d-1} \frac{1 + q^{2^{p^j+1}-1}t^{2^{p^j+1}}}{1 - q^{2^{p^{j+1}}-2}t^{2^{p^{j+1}}}}$$
that specialize in the case $p = 2$ to the series

$$\bar{P}_2(q, t) = \frac{qt^3}{(1 - t^2q^2)} \prod_{i > 0} \frac{1}{1 - q^{2i} - 1t^{2i}}.$$ 

Proof. Let $P_p(\text{Br}_n; H_1(\Sigma^d_n)) (q)$ be the Poincaré polynomial for the homology group $H_*(\text{Br}_n; H_1(\Sigma^d_n; \mathbb{Z})) \otimes \mathbb{Z}_p$ as a $\mathbb{Z}_p$-module. Since we already know that for $n$ odd the homology group $H_i(\text{Br}_n; H_1(\Sigma^d_n; \mathbb{Z}))$ has only torsion of order that divides $d$,.

Table 4. Computations of $H_i(\text{Br}_n; H_1(\Sigma^5_n))$. For each column the first stable group is highlighted.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----|---|---|---|---|---|---|---|---|---|----|----|
| 3   | $\mathbb{Z}_5$ |   |   |   |   |   |   |   |   |    |    |
| 4   | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ |   |   |   |   |   |   |   |    |    |
| 5   | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ |   |   |   |   |   |   |    |    |
| 6   | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ |   |   |   |   |   |    |    |
| 7   | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ |   |   |   |   |    |    |
| 8   | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ |   |   |   |    |    |
| 9   | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ |   |   |    |    |
| 10  | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ | $\mathbb{Z}_5$ |    |    |

Table 5. Computations of $H_i(\text{Br}_n; H_1(\Sigma^6_n))$. For each column the first stable group is highlighted. Notice that $5$-torsion appears for $n$ even.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|
| 3   | $\mathbb{Z}_6$ |   |   |   |   |   |   |   |   |
| 4   | $\mathbb{Z}_2\mathbb{Z}_6$ | $\mathbb{Z}_3\mathbb{Z}_6$ | $\mathbb{Z}_6$ |   |   |   |   |   |   |
| 5   | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ |   |
| 6   | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ |   |
| 7   | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ |   |
| 8   | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ |   |
| 9   | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ |   |
| 10  | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ | $\mathbb{Z}_6$ |
we can compute the polynomial $P_p(\text{Br}_n, H_1(\Sigma^d_n))(q)$ from the Universal Coefficients Theorem as follows. We compute the Poincaré polynomial $P_p(\text{Br}_n, H_1(\Sigma^d_n; \mathbb{Z}_p))(q)$ for the group $H_i(\text{Br}_n; H_1(\Sigma^d_n; \mathbb{Z}_p)) \otimes \mathbb{Z}_p$ and we divide by $1 + q$.

In order to compute $P_p(\text{Br}_n, H_1(\Sigma^d_n; \mathbb{Z}_p))(q)$ we consider the short exact sequence, which splits

$$0 \rightarrow \text{coker} \, \nu_i \rightarrow H_i(\text{Br}_n; H_1(\Sigma^d_n; \mathbb{Z}_p)) \rightarrow \ker \, \nu_{i-1} \rightarrow 0$$

where we recall that $\nu_i$ is the map

$$\nu_i : H_i(\text{Br}_n; H_1(S^1 \times P; \mathbb{Z}_p)) \rightarrow H_i(\text{Br}_n; H_1(\overline{D} \_ P; \mathbb{Z}_p)).$$

It follows from the Remark 9 and from the description in Section 8 that for a fixed odd $n$ the ranks of $\ker \, \nu$ and $\text{coker} \, \nu$ are respectively the coefficients of $q^i \, t^n$ in the following series:

$$P_p(\text{coker} \, \nu) = P_p(\ker \, \nu) = \frac{qt^3}{(1 - t^2)(1 - t^2)} \prod_{j \geq 0} \frac{1 + q^{2p^j - 1}t^{2p^j}}{1 - q^{2p^j + 1}t^{2p^j + 1}}$$

that specialize for $p = 2$ to the series

$$P_2(\text{coker} \, \nu) = P_2(\ker \, \nu) = \frac{qt^3}{1 - t^2} \prod_{j \geq 0} \frac{1}{1 - q^{2^j - 1}t^{2^j}}$$

Clearly the polynomial for $H_i(\text{Br}_n; H_1(\Sigma^d_n; \mathbb{Z}_p)) \otimes \mathbb{Z}_p$ is given by the coefficient of $t^n$ in the sum

$$P_p(\text{coker} \, \nu) + q \, P_p(\ker \, \nu)$$

and hence dividing by $(1 + q)$ we get our result.

\begin{remark}
The same argument of Theorem 10.6 could be used when $d$ is odd to compute a (more complicated) formula of the Poincaré polynomial of the groups $H_*(\text{Br}_n; H_1(\Sigma^d_n; \mathbb{Z})) \otimes \mathbb{Z}_p$ for any $n$.
\end{remark}

\begin{remark}
Notice that the polynomials given in Theorem 10.6 do not depend on the integer $d$. Hence we have that when $n$ is odd and $p \mid d$ there is a (non-natural) isomorphism

$$H_i(\text{Br}_n; H_1(\Sigma^d_n; \mathbb{Z})) \otimes \mathbb{Z}_p = H_i(\text{Br}_n; H_1(\Sigma^n_p; \mathbb{Z})) \otimes \mathbb{Z}_p.$$

\begin{remark}
We know (Theorem 6.8 and 7.6) that when $n$ is odd and $d$ is squarefree the group $H_1(\text{Br}_n; H_1(\Sigma^d_n; \mathbb{Z}))$ is finite (with the exception of $H_0$) and has no $p$-torsion for $p \mid d$ and no $p^2$-torsion for any prime $p$. Then in such cases Theorem 10.6 completely determines the homology groups $H_i(\text{Br}_n; H_1(\Sigma^d_n; \mathbb{Z}))$ and hence (using Theorem 4.1) the groups $H_{i+1}(B(d, d, n), \text{Br}_n; \mathbb{Z})$ for $n$ odd and $d$ squarefree.

The same argument of the previous proof can be applied in stable rank. From the Remark 9 the Stable Poincaré polynomial of both $\text{coker} \, \nu$ and $\ker \, \nu$ with $\mathbb{Z}_p$ coefficients is the following:

$$\frac{q}{1 - q^2} \prod_{j \geq 0} \frac{1 + q^{2p^j - 1}}{1 - q^{2p^j + 1}}$$

and in particular for $p = 2$ we obtain

$$\frac{q}{1 - q^2} \prod_{j \geq 1} \frac{1}{1 - q^{2^j - 1}}.$$
Since for $n$ odd there is no free part in $H_i(B_{n}; H_1(\Sigma_n^d; \mathbb{Z}))$ all these groups have only torsion that divides $d$. In particular for integer coefficients we get the following statement.

**Theorem 10.7.** Let $p$ be a prime and let $d$ be an integer such that $p \parallel d$ the Poincaré polynomial of the stable homology $H_i(B_{n}; H_1(\Sigma_n^d; \mathbb{Z})) \otimes \mathbb{Z}_p$ as a $\mathbb{Z}_p$-module is the following:

$$P_p(B; H_1(\Sigma^d))(q) = \frac{q}{1 - q^2} \prod_{j \geq 0} \frac{1 + q^{2p^j - 1}}{1 - q^{2p^{j+1} - 2}}.$$  

In particular when $d$ is an even integer the Poincaré polynomial of the stable homology $H_i(B_{n}; H_1(\Sigma_n^d; \mathbb{Z})) \otimes \mathbb{Z}_2$ as a $\mathbb{Z}_2$-module is the following:

$$P_2(B; H_1(\Sigma^d))(q) = \frac{q}{1 - q^2} \prod_{j \geq 1} \frac{1}{1 - q^{2^{2j} - 1}}.$$  

**Remark 17.** The stabilization result for the homology of the complex braid groups of type $B(d, d, n)$ (Theorem 10.7) together with Theorem 8.2 implies that the stable homology of $B(d, d, n)$ with integer coefficients has no $p^k$-torsion only if $p^k \parallel d$.  

Hence Theorem 4.4 implies that when $p^2 \parallel d$ the stable Poincaré polynomials given in Theorem 10.7 determines the corresponding $p$-torsion component of the stable homology group $H_{i+1}(B(d, d, n), B_{n}; \mathbb{Z})$ and this component is the same for all $d = pm$ where $p \parallel m$ and is trivial when $p \parallel d$.

An explicit computation of the first terms of the stable series $P_2(B; H_1(\Sigma^d))(q)$ gives

$$q + q^2 + 2q^3 + 3q^4 + 4q^5 + 5q^6 + 7q^7 + 9q^8 + 11q^9 + 14q^{10} + 17q^{11} + \ldots$$

while for $P_2(\Sigma_n^d; H_1(\Sigma_n^d; \mathbb{Z}))$ we have

$$q + q^2 + q^3 + q^4 + 2q^5 + 3q^6 + 3q^7 + 3q^8 + 4q^9 + 5q^{10} + 5q^{11} + 6q^{12} + \ldots$$

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