Solvable Confining Gauge Theories at Large N

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Abstract

In this letter we consider models with $N U(1)$ gauge fields $A_\mu^n$ together with $N$ Kalb-Ramond fields $B_{\mu\nu}^n$ in the large $N$ limit. These models can be solved explicitly and exhibit confinement for a large class of bare actions. The confining phase is characterized by an approximate "low energy" vector gauge symmetry under which the Kalb-Ramond fields $B_{\mu\nu}^n$ transform. A duality transformation shows that confinement is associated with magnetic monopoles condensation.

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1 Introduction

In various approaches towards a description of the confining phase of gauge theories – on the lattice, through duality with the Higgs phase, ANO-strings or the "confining string" – the introduction of a Kalb-Ramond fields $B_{\mu\nu}$ as effective variables has proved to be very useful [2 – 10]. In the case of pure Yang Mills theories they can be introduced as auxiliary fields for the abelian components of the field strength [8,9] in the maximal abelian gauge [11] and, after performing the path integral over the "non-diagonal" gauge fields (associated with non-diagonal generators), one is left with an effective action involving abelian gauge fields and Kalb-Ramond fields only.

Common to all these approaches is the idea that Kalb-Ramond fields are effective variables only at low energy, i.e. in the infra-red regime. It is thus natural to provide models with Kalb-Ramond fields as effective variables with an ultra-violet cutoff, but to allow for irrelevant operators in the corresponding bare action. On the other hand it is sufficient to restrict oneself to abelian gauge theories (or the abelian subsector of non-abelian theories): Monopole condensation, which is believed to be the mechanism behind confinement [12, 13], is a purely abelian phenomenon. In the context of non-abelian gauge theories the relevance of the abelian sub-sector is conventionally referred to as "abelian dominance".

The purpose of the present paper is the study of models with $N$ abelian gauge fields $A^a_\mu$ and $N$ Kalb-Ramond fields $B^n_{\mu\nu}$ in the large $N$ limit. As we will see, they can be solved using standard functional methods employed for large $N$ field theories. The emergence of a confining phase can be seen explicitly, and the appearance of a "low energy" vector gauge symmetry allows for a duality transformation showing that magnetic monopoles have condensed. In this letter we present the essential results of this approach, leaving many details (as the precise relation to Yang-Mills theories) to a subsequent publication [14].

2 The $A_\mu$ - $B_{\mu\nu}$ - model

The starting point is the partition function for a model with the above field content. Adding sources for $A^a_\mu$ and $B^n_{\mu\nu}$, $n = 1 \ldots N$, and a covariant gauge fixing term the partition function reads
\[ e^{-W(J)} = \frac{1}{N} \int DA \, DB \, e^{-S_{\text{bare}}(A,B)} + \int d^4x \left\{ \frac{1}{2N} \left( \partial_\mu A_\mu^n \right)^2 + J^A_{\lambda, \mu} A_\mu^n + J^B_{\beta, \mu} B_\mu^n \right\}. \] (2.1)

Due to the \( N \) U(1) gauge symmetries \( S_{\text{bare}}(A, B) \) can only depend on \( F_{\mu\nu} \), hence we can write \( S_{\text{bare}}(F, B) \). Next, in order to allow for a large \( N \) expansion, we assume that \( S_{\text{bare}}(F, B) \) depends on \( O(N) \) invariants (singlets) only. The aim would be to allow for a dependence of \( S_{\text{bare}}(F, B) \) on \( O(N) \) singlets as general as possible. Here we confine ourselves to the following ansatz: First we introduce three Lorentz scalar \( O(N) \) singlet operators

\[ \mathcal{O}_1(x) = \sum_{n=1}^{N} F^n_{\mu\nu}(x) F^n_{\mu\nu}(x), \]
\[ \mathcal{O}_2(x) = \sum_{n=1}^{N} F^n_{\mu\nu}(x) B^n_{\mu\nu}(x), \]
\[ \mathcal{O}_3(x) = \sum_{n=1}^{N} B^n_{\mu\nu}(x) B^n_{\mu\nu}(x). \] (2.2)

Then we take \( S_{\text{bare}}(F, B) \) of the form

\[ S_{\text{bare}}(F, B) = \int d^4x \left\{ L_{\text{bare}}(O_i) + \frac{h}{2} \left( \partial_\mu \tilde{B}^n_{\mu\nu} \right)^2 + \frac{\sigma}{2} \left( \partial_\mu B^n_{\mu\nu} \right)^2 \right\} \] (2.3)

where

\[ \tilde{B}^n_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\rho\sigma} B^n_{\rho\sigma}. \] (2.4)

We allow \( L_{\text{bare}} \) in (2.3) to contain arbitrary derivatives acting on the operators \( O_i \). This is still not the most general form of \( S_{\text{bare}}(F, B) \); one can certainly construct infinitely many more \( O(N) \) singlet operators which contain open Lorentz indices and/or "internal" derivatives as the second and third terms in (2.3). It can be argued \[14\], however, that these do not modify the essential features of the results obtained below.

In order to solve the model in the large \( N \) limit we have to make assumptions on the \( N \) dependence of the parameters in \( S_{\text{bare}} \). These assumptions can be summarized by rewriting (2.3) as

\[ S_{\text{bare}}(F, B) = \int d^4x \left\{ N L_{\text{bare}} \left( \frac{O_i}{N} \right) + \frac{h}{2} \left( \partial_\mu \tilde{B}^n_{\mu\nu} \right)^2 + \frac{\sigma}{2} \left( \partial_\mu B^n_{\mu\nu} \right)^2 \right\} \] (2.5)

where now the coefficients of \( L_{\text{bare}} \) are independent of \( N \).
3 The large N solution

The most convenient formalism for the treatment of field theories in the large
$N$ limit is the introduction of auxiliary fields for composite $O(N)$ singlet operators
[15]. In the present case we introduce one auxiliary field $\phi_i$ for each of the bilinear
$O(N)$ singlet operators $\mathcal{O}_i$ in eq. (2.2). This amounts to re-write the term involving
$N\mathcal{L}_{\text{bare}}(\mathcal{O}_i/N)$ in the exponent of (2.1) as

$$e^{-N \int d^4x \mathcal{L}_{\text{bare}}(\frac{\mathcal{O}_i}{N})} = \frac{1}{N} \int D\phi_i \ e^{-N\mathcal{G}_{\text{bare}}(\phi_i)} - \int d^4x \phi_i \mathcal{O}_i \ .$$

In the large $N$ limit the path integral on the right-hand side of (3.1) can be replaced
by its stationary point, and the relation between $\mathcal{G}_{\text{bare}}$ and $\mathcal{L}_{\text{bare}}$ becomes

$$N \int d^4x \mathcal{L}_{\text{bare}}\left(\frac{\mathcal{O}_i}{N}\right) = N\mathcal{G}_{\text{bare}}(\phi_i) + \int d^4x \phi_i \mathcal{O}_i \ .$$

Equation (3.2) allows, in principle, to construct $\mathcal{G}_{\text{bare}}(\phi_i)$ from $\mathcal{L}_{\text{bare}}$, although here
we allow $\mathcal{L}_{\text{bare}}$ to be an arbitrary functional (including derivatives) of $\mathcal{O}_i$. Next we
insert eq. (3.1) into (2.1), which becomes

$$e^{-W(J)} = \frac{1}{N} \int D\phi_i \int DA \ DB \ 
\times e^{-N\mathcal{G}_{\text{bare}}(\phi_i)} - \int d^4x \left\{ \phi_i \mathcal{O}_i + \frac{1}{2} (\partial_\mu \mathcal{B}_\mu^n)^2 + \frac{1}{2} (\partial_\mu \mathcal{B}_{\mu\nu}^{n})^2 + \frac{1}{2} (\partial_\mu \mathcal{A}_\mu^n)^2 - J_{A,\mu}^n A_{\mu}^n - J_{B,\mu\nu}^n B_{\mu\nu}^n \right\}$$

The $DA \ DB$ path integrals have become Gaussian in (3.3). In order to express
the result in compact form we introduce the notation $\varphi_r^n = \{ A_{\mu}^n, B_{\mu\nu}^{n}\}$, i.e. the indices
$r$ attached to the fields $\varphi^n$ denote both the different fields $A^n, B^n$ and the different
Lorentz indices. Correspondingly we introduce notation $J_r^n$ for $\{ J_{A,\mu}^n, J_{B,\mu\nu}^n \}$. The
result of the Gaussian integration over $DA \ DB$ can now be written as

$$e^{-W(J)} = \frac{1}{N} \int D\phi_i \ e^{-N\mathcal{G}_{\text{bare}}(\phi_i)} - N\Delta G(\phi_i) + \frac{1}{2} \int d^4x_1 d^4x_2 \{ J_{A,\mu}^n(x_1) P^{rs}(x_1,x_2,\phi_i) J_{A,\mu}^n(x_2) \}$$

with

$$\Delta G(\phi_i) = -\frac{1}{2} \operatorname{Tr} \log (P^{rs}(x_1, x_2, \phi_i)) \ .$$
The propagators $P^{rs}$ of the $A^n_{\mu}$, $B^n_{\mu\nu}$ system are proportional to $\delta_{n,m}$ with $n, m = 1 \ldots N$ and we took care of the resulting contribution from the trace in (3.3) by the explicit factor $N$ multiplying $\Delta G$ in (3.4). The propagators $P^{AA}_{\mu\nu}$, $P^{AB}_{\mu\rho\sigma}$ and $P^{BB}_{\mu\nu\rho\sigma}$ depend on the terms $\phi_i O_i$, $\frac{i}{2}(\partial\bar{B})^2$, $\frac{e}{2}(\partial B)^2$ and $\frac{1}{2\alpha}(\partial A)^2$ in the exponent of (3.3). Simple explicit expressions can be obtained only for constant fields $\phi_i$; in this case one finds for $\Delta G$ (in the Landau gauge $\alpha \to 0$)

$$\Delta G(\phi_i) = \frac{3}{2} \int d^4x \int \frac{d^4p}{(2\pi)^4} \left[ \log(\phi_1 \sigma p^2 + 4\phi_1 \phi_3 - \phi_2^2) + \log \left( h p^2 + 4\phi_3 \right) \right].$$

(3.6)

The $d^4p$ integral in (3.6) has to be performed with an UV cutoff $\Lambda^2$. The result simplifies considerably if one introduces

$$\Sigma = \frac{4\phi_1 \phi_3 - \phi_2^2}{\sigma \phi_1}. \quad (3.7)$$

Up to field independent terms one then obtains

$$\Delta G(\phi_i) = \frac{3}{32\pi^2} \int d^4x \left[ (\Lambda^4 - \Sigma^2) \log \left( \sigma \phi_1 (\Lambda^2 + \Sigma) \right) + \Sigma^2 \log (\sigma \phi_1 \Sigma) + \Lambda^2 \Sigma \right.$$

$$+ \left. \left( \Lambda^4 - \frac{16\phi_3^2}{h^2} \right) \log \left( \Lambda^2 h + 4\phi_3 \right) + \frac{16\phi_3^2}{h^2} \log (4\phi_3) + 4\Lambda^2 \phi_3 \right].$$

(3.8)

This expression for $\Delta G(\phi_i)$ has to be inserted into (3.4) and, in the large $N$ limit, the $D\phi_i$ path integral is again dominated by its stationary point(s). Hence $W(J)$ becomes

$$W(J) = NG(\hat{\phi}_i) - \frac{1}{2} \int d^4x_1 d^4x_2 \left\{ J^n_r(x_1) P^{rs}(x_1, x_2, \hat{\phi}_i) J^n_s(x_2) \right\}$$

(3.9)

where

$$G(\phi_i) = G_{bare}(\phi_i) + \Delta G(\phi_i) \quad (3.10)$$

and $\hat{\phi}_i \equiv \hat{\phi}_i(J)$ satisfy the three equations (recall $i = 1, 2, 3$)

$$\left[ \frac{\delta}{\delta \phi_i} \left( NG(\phi_i) - \frac{1}{2} \int d^4x_1 d^4x_2 J^n_r(x_1) P^{rs}(x_1, x_2, \phi_i) J^n_s(x_2) \right) \right]_{\hat{\phi}_i(J)} = 0. \quad (3.11)$$

The model is thus solved, for given $G_{bare}(\phi_i)$, up to the technical problem of finding the stationary points $\hat{\phi}_i(J)$.
Next we wish to show that the particular configuration where
\[
4 \hat{\phi}_1 \hat{\phi}_3 - \hat{\phi}_2^2 = 0 \quad .
\] (3.12)
(or \(\hat{\Sigma} = 0\)) is a “natural” solution of the three stationary point equations (3.11), i.e. a solution which requires no fine tuning of the parameters in \(G_{\text{bare}}(\phi_i)\). Below we will see that the phase where (3.12) holds is the confining phase of the model.

In order to “see” the solution (3.12) of the eqs. (3.11) it is necessary to regularize the singularity of the derivatives of \(\Delta G_{\phi_i}(\phi_i)\) w.r.t. the fields at \(\Sigma = 0\). The origin of the non-analytic behaviour of \(\Delta G_{\phi_i}(\phi_i)\) at \(\Sigma = 0\) is the infrared behaviour of the propagators \(P^{rs}(\phi_i)\) which, in momentum space, behave like \(P^{rs}(q^2, \phi_i) \sim q^{-4}\) for \(\Sigma = 0\). In order to regularize these infrared singularities we perform the \(d^4p\) integral in (3.6) also with an infrared cutoff \(k^2\). For \(J^n_r = 0\) (hence we write \(\hat{\phi}_0^i\) instead of \(\hat{\phi}_i\)) the three stationary point equations (3.11) can then be brought into the form
\[
\begin{align*}
\left[ \frac{\delta G_{\text{bare}}}{\delta \phi_1} + \frac{3}{32\pi^2\phi_1} \left( \Lambda^4 - k^4 \right) \right] \hat{\phi}_0^i &= 0 , \\
\left[ \frac{\delta G_{\text{bare}}}{\delta \phi_3} + \frac{3}{4\pi^2 h^2} \left\{ 4\phi_3 \log \left( \frac{4\phi_3 + h k^2}{4\phi_3 + h \Lambda^2} \right) + h \left( \Lambda^2 - k^2 \right) \right\} \right] \hat{\phi}_0^i &= 0 , \\
\left[ \frac{\delta G_{\text{bare}}}{\delta \Sigma} + \frac{3}{16\pi^2} \left\{ \Sigma \log \left( 1 + \frac{\Sigma}{\Lambda^2} \right) + \Lambda^2 - k^2 \right\} \right] \hat{\Sigma}_0^0 &= 0 .
\end{align*}
\] (3.13)

The solutions \(\hat{\phi}_0^i\) of (3.13), and hence \(\hat{\Sigma}_0^0\), depend on the infrared cutoff \(k^2\). Let us now assume that
\[
\left. - \frac{\delta G_{\text{bare}}}{\delta \Sigma} \right|_{\Sigma_0^0} = \frac{3}{16\pi^2} \Lambda^2 > 0 .
\] (3.14)

Then the last of the stationary point equations (3.13) implies \(\hat{\Sigma}_0^0(k^2) < 0\) for \(\Lambda^2 > k^2 > 0\), and \(\hat{\Sigma}_0^0(k^2)\) behaves as follows for \(k^2 \to 0\):
\[
\hat{\Sigma}_0^0(k^2) \to 0_{-\epsilon} , \quad \frac{3}{16\pi^2} \Sigma_0^0 \log \left( \frac{\Sigma_0^0 + k^2}{\Sigma_0^0 + \Lambda^2} \right) \to - \left. \frac{\delta G_{\text{bare}}}{\delta \Sigma} \right|_{\Sigma_0^0} = - \frac{3}{16\pi^2} \Lambda^2 .
\] (3.15)

Hence, under the condition (3.14), we obtain \(\hat{\Sigma}(0) = 0\) naturally. Note that this stationary point would not have been observed if one puts \(k^2 = 0\) from the start.

What is the meaning of the condition (3.14) or, better, \(\hat{\Sigma}_0^0(k^2) < 0\)? To this end we push the infrared cutoff \(k^2\) upwards until it reaches the UV cutoff \(\Lambda^2\), and
investigate the consequence of $\Sigma_0(\Lambda^2) < 0$. Given the definition (3.7) for $\Sigma$, and for $\phi_1^0(\Lambda^2) > 0$, this latter condition reads

$$4\phi_1^0(\Lambda^2)\phi_2^0(\Lambda^2) - (\phi_2^0(\Lambda^2))^2 < 0 .$$

(3.16)

For $k^2 = \Lambda^2$ the contribution $\Delta G(\phi_i)$ to $G(\phi_i)$ in (3.10) vanishes, and the configurations $\phi_i^0(\Lambda^2)$ are the stationary points of $G_{bare}(\phi_i)$. From the Legendre transformation (3.2) and the definitions (2.2) of the operators $O_i$ it is now straightforward to see that the inequality (3.16) implies (with $\varphi^a_r$ as below (3.3))

$$\text{Det} \left( \frac{\delta^2 L_{bare}}{\delta \varphi^a_r \delta \varphi^a_s} \right) \bigg|_{F=B=0} < 0 .$$

(3.17)

(3.17) corresponds to a $L_{bare}(F, B)$ which is non-convex at the origin of field space.

All this is very similar to the case of a non-convex bare scalar potential: The effective scalar potential has to be semi-convex, and if the bare scalar potential is sufficiently non-convex the effective potential becomes flat in its "inner" region. As in the present case the observation of this phenomenon requires the introduction of an "artificial" infrared cutoff $k^2$, and a careful discussion of the limit $k^2 \to 0$ [16]. We emphasize that consequently the emergence of the "confining" phase (3.12) (see below) is not a particular feature of the large N limit; the advantage of the large N limit is only to allow for an explicit study of this phenomenon (for given parametrizations of $L_{bare}(F, B)$ or $G_{bare}(\phi_i)$).

4 Properties of the confining phase

In the following we assume that the necessary inequality on the parameters of $L_{bare}(F, B)$ (a sufficiently negative curvature at the origin of field space) for the reach of the confining phase (3.12) is satisfied. In order to discuss its properties it is more convenient to switch from $W(J)$ in (3.9) to the effective action $\Gamma(A, B)$ via a Legendre transform with respect to the sources $J$:

$$\Gamma(A, B) = W(J) + \int d^4x \left( J^n_{A,\mu} A^n_\mu + J^n_{B,\mu\nu} B^n_{\mu\nu} \right) .$$

(4.1)

As discussed in detail in [14] one obtains

$$\Gamma(A, B) = NG(\phi_i) + \int d^4x \left( \phi_i O_i + \frac{\hbar}{2} \left( \partial_\mu \bar{B}^n_{\mu\nu} \right)^2 + \frac{\sigma}{2} \left( \partial_\mu B^n_{\mu\nu} \right)^2 \right) .$$

(4.2)
with $G(\hat{\phi}_i)$ as in (3.10), and the stationary point equations (3.11) for $\hat{\phi}_i$ can be written as

\[
\left[ \frac{\delta \Gamma}{\delta \hat{\phi}_i} \right]_{\hat{\phi}_i(A,B)} = 0 .
\] (4.3)

First we note that because of the relation (3.12) the expression $\hat{\phi}_i O_i$ in (4.2) becomes

\[
\hat{\phi}_i O_i = \sum_n \left( \sqrt{\hat{\phi}_1} F^m_{\mu\nu} + \sqrt{\hat{\phi}_3} B^n_{\mu\nu} \right)^2 .
\] (4.4)

Consequently it is invariant under the following gauge symmetry involving vector-like gauge parameters $\Lambda^n_\mu$ [1]:

\[
\delta A^\mu_n(x) = \Lambda^n_\mu(x) , \quad \delta F^m_{\mu\nu}(x) = \partial_\mu \Lambda^n_\nu(x) - \partial_\nu \Lambda^n_\mu(x) \equiv \Lambda^n_{\mu\nu}(x) ,
\]

\[
\delta B^n_\mu(x) = \sqrt{\frac{\phi_1}{\phi_3}} \Lambda^n_\mu(x) .
\] (4.5)

In addition one finds that the term $\sim (\partial_\mu \tilde{B}^n_{\mu\nu})^2$ in (1.2) is also invariant under (4.5) thanks to a Bianchi identity, provided the configurations $\hat{\phi}_i$ are constant in $x$. (Note that, from eq. (4.3) with $\Gamma$ as in (1.2), constant configurations $\hat{\phi}_i$ result from constant configurations $O_i$; however, from their definition (2.2), constant $O_i$ do not necessarily imply constant configurations $F^n_{\mu\nu}$ and $B^n_{\mu\nu}$.) The last term $\sim (\partial_\mu B^n_{\mu\nu})^2$ in (1.2) behaves as a gauge fixing term of the symmetry (4.5), and its presence insures the existence of the propagators.

It is to be expected that the symmetry (4.5) is broken by higher derivative terms (beyond the gauge fixing term): The bare action $S_{\text{bare}}$ (2.3) of the model does certainly not exhibit the symmetry (4.3), and the Green functions at large non-exceptional Euclidean momenta with $p^2 \to \Lambda^2$ are generated by $S_{\text{bare}}$. This fact is realized by the dependence of the effective action on higher derivative terms. The symmetry (4.5) is thus a pure “low energy” phenomenon. The implication of the gauge symmetry (4.3) on modes of the $U(1)$ gauge fields $A^\mu_n$ which correspond to constant configurations $O_i$ is that they can be “gauged away” and “eaten” by the (massive or even infinitely massive) Kalb-Ramond fields $B^n_{\mu\nu}$, in some analogy to the ordinary Higgs effect [1].

In addition the symmetry (4.5) allows for a duality transformation: A priori the dual of a $U(1)$ gauge field $A^\mu_n$ (in $d = 4$) is again a $U(1)$ gauge field $C^m_\mu$ (whose field
strength tensor will be denoted by $F^{c,n}_{\mu\nu}$, and the dual of a Kalb-Ramond field $B^n_{\mu\nu}$ is a (pseudo-) scalar $\varphi^n$. In the present case the duality transformations mix the fields and read

\[
\frac{1}{2} F^{c,n}_{\mu\nu} = \hat{\phi}_1 \tilde{F}^n_{\mu\nu} + \sqrt{\hat{\phi}_1 \hat{\phi}_3} \tilde{B}^n_{\mu\nu},
\]

\[
\partial_{\mu} \varphi^n + C^n_{\mu} = \frac{h}{2} \sqrt{\hat{\phi}_1 \hat{\phi}_3} \partial_{\mu} \tilde{B}^n_{\nu\mu}
\]

(4.6)

where the tildes on $\tilde{B}^n_{\mu\nu}$ and $\tilde{F}^n_{\mu\nu}$ have been defined in (2.4). The corresponding dual action reads (as obtained from (4.2) without $NG(\hat{\phi}_i)$ and without the ”gauge fixing” term)

\[
\Gamma_{Dual}(C, \varphi) = \int d^4x \left\{ \frac{1}{4\hat{\phi}_1} F^{c,n}_{\mu\nu} F^{c,n}_{\mu\nu} + 2 \frac{\hat{\phi}_3}{h \hat{\phi}_1} \left( \partial_{\mu} \varphi^n + C^n_{\mu} \right)^2 \right\}.
\]

(4.7)

Note that, due to the implicit dependence of $\hat{\phi}_i$ on $F$ and $B$, these duality transformation are non-linear. Actually one finds [14] that only half of the equations of motion and Bianchi identities are exactly interchanged through (4.6) and (4.7), whereas the other half holds again only for constant configurations $\hat{\phi}_i$ and hence $O_i$. Thus duality is realized at the non-linear level again only in the corresponding ”low energy” regime.

The physical interpretation of the dual action (4.7) is obviously the one of an abelian $U(1)^N$ Higgs model in the spontaneously broken phase where $\varphi^n$ represent the Goldstone bosons, and where the gauge fields $C^n_{\mu}$ have acquired a mass $2(\hat{\phi}_3/h)^{1/2}$. Since this represents the “low energy effective action” of a theory in which the ”dual” electric charge has condensed in the vacuum, the original action (1.2) with (4.4) corresponds to the situation where the “magnetic” charge has condensed in the vacuum.

Let us turn to the response of the model in the confining phase with respect to external sources. The expression for $W(J)$ has been given in eq. (3.9) in the preceding section, and first we concentrate ourselves on the term quadratic in the sources $J^n_{r}$. Let us start with a source $J^n_{A,\mu}(x)$ for the fields $A^n_{\mu}$ only, which is of the form of a Wilson loop:

\[
J^n_{A,\mu}(x) = ig_A \int_C dx' \delta^4(x - x')
\]

(4.8)
The term quadratic in $J$ in (3.9) then becomes

$$\frac{Ng^2}{2} \int_C dx_{1,\mu} \int_C dx_{2,\nu} P_{\mu,\nu}^{AA}(x_1 - x_2)$$

where $P_{\mu,\nu}^{AA}$ has to be obtained from the action (4.2) with (4.4):

$$P_{\mu,\nu}^{AA}(z) = \frac{1}{16\pi^2\phi_1} \left\{ \delta_{\mu\nu} \left( \frac{1}{|z|^2} - \frac{2\phi_3}{\sigma} \log |z| + \text{const.} \right) - \frac{1}{2} \partial_\mu \partial_\nu \left( \log |z| - \frac{\phi_3 |z|^2}{2\sigma} (\log |z| + \text{const.}) \right) \right\}.$$  (4.10)

The (actually divergent) constants in (4.10) disappear in the expression (4.9). In the limit where the (minimal) surface $S$ enclosed by the loop $C$ in (1.8) becomes very large one finds that the expression (4.9) is proportional to $S$, thus one obtains the area law for the expectation value of the Wilson loop.

However, at first sight an inconsistency arises due to the long-range behaviour of the propagator $P_{\mu,\nu}^{AA}(z)$ in (4.10): Let us imagine that space-time is filled with "virtual" Wilson loops (originating, e.g., from vacuum bubbles of virtual quark-antiquark pairs), and let us compute the corresponding contribution to the action due to the interactions among different "virtual" Wilson loops. Even if one assumes that these "virtual" Wilson loops are arbitrarily tiny in size, localized in space-time and if one averages over their orientation in space-time, the contribution to the action induced by the long-range behaviour of the propagator $P_{\mu,\nu}^{AA}(z)$ in (4.10) diverges (logarithmically) in the infinite volume limit.

This infinity can be avoided, however, once one realizes that all components $\{r, s\} = \{A, B\}$ of the propagators $P_{rs}(z)$ in (3.9) have a "bad" long range behaviour. The precise expressions for all propagators, as obtained from the action (1.2), will be given in (14). One finds that all terms in the propagators which decrease not sufficiently fast at infinity in order to avoid the above infrared divergence (which originate from $q^{-4}$-terms in momentum space) cancel in the sum over $r$ and $s$ in $J^a_r P_{rs} J^a_s$ in (3.9) if the sources $J^a_{A,\mu}$ and $J^a_{B,\mu\nu}$ satisfy

$$\sqrt{\phi_3} \ J^a_{A,\mu}(x) = 2 \sqrt{\phi_1} \partial_\nu J^a_{B,\nu\mu}(x).$$  (4.11)

This result can also be phrased as follows: In the presence of an arbitrary background of "virtual" Wilson loops it costs an infinite amount of action (or energy) to "switch on" sources $J^a_{A,\mu}$ and/or $J^a_{B,\mu\nu}$ which are not related as in (4.11), due to the
interactions induced between the sources and the background of "virtual" Wilson loops induced by the long-range terms in the propagators.

If the source \( J_{A, \mu}^n \) is of the form of a Wilson loop \((4.8)\) one finds that \((4.11)\) implies that the source \( J_{B, \mu \nu}^n \) is of the form of a "Wilson surface",

\[ J_{B, \mu \nu}^n(x) = i g_B \int_S d^2 \sigma_{\mu \nu}(z) \, \delta^4(x - z) \quad , \tag{4.12} \]

where the surface \( S \) is bounded by the loop \( C \) in \((4.8)\) (but otherwise arbitrary) and where \( g_B \) satisfies \( \sqrt{\hat{\phi}_1} g_B = \sqrt{\hat{\phi}_3} \, g_A / 2 \).

It is straightforward to see that the condition \((4.11)\) on the sources is equivalent to the condition that the couplings \( J_{A, \mu}^n A_{\mu}^n + J_{B, \mu \nu}^n B_{\mu \nu}^n \) of the fields to the sources respect the "low energy" gauge symmetries \((4.5)\). In the case of conventional gauge symmetries these conditions can (and have to) be imposed by hand in order to ensure renormalizability and unitarity of the theory. In the present model, on the one hand, they cannot be imposed from the beginning, since the associated (low energy) gauge symmetries appear only at the level of the effective action once the equations of motion of the fields \( \phi_i \) are satisfied. Although renormalizability is not an issue here, since we consider an effective low energy theory with a fixed UV cutoff, it is interesting to see that the corresponding condition on the sources is generated dynamically in the sense that its violation costs infinite action.

Clearly we now have to reconsider the expectation value of the Wilson loop, which consists now of a source \((4.8)\) for \( A_{\mu}^n \) and a source \((4.12)\) for \( B_{\mu \nu}^n \) inserted into the term quadratic in \( J \) in \((3.9)\) (higher orders in \( J \) will be discussed below). Using all propagators \( P^{rs} \) from \([14]\) this term becomes

\[ \int_{S} d^2 \sigma_{\mu \nu}(z_1) \int_{S} d^2 \sigma_{\rho \sigma}(z_2) \left( -g_F^2 / 16 \pi^2 \phi_1 \right) \frac{\sqrt{\hat{\phi}_3 h}}{\phi_3} \times \left( T_{1, \mu \nu, \rho \sigma}^{\nu \rho} (\partial) - \frac{4 \phi_3}{h} T_{2, \mu \nu, \rho \sigma} \right) \frac{1}{|z_1 - z_2|} K_1 \left( 2 |z_1 - z_2| \sqrt{\frac{\phi_3}{h}} \right) \tag{4.13} \]

with

\[ T_{1, \mu \nu, \rho \sigma}(\partial) = \delta_{\mu \rho} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \rho} - \delta_{\nu \rho} \partial_\nu \partial_\rho - \delta_{\nu \rho} \partial_\rho \partial_\sigma + \delta_{\nu \sigma} \partial_\mu \partial_\rho , \]

\[ T_{2, \mu \nu, \rho \sigma} = \delta_{\mu \rho} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \rho} , \tag{4.14} \]
and where $K_1$ is a Bessel function. In the limit where the surface $S$ becomes large
the expression (4.13) behaves as

$$S \cdot \frac{2g^2_F}{\pi \hat{\phi}_1} \left( \frac{\hat{\phi}_3}{h} \right)^{3/2} \int_0^\infty dz \ K_1 \left( 2z \sqrt{\frac{\hat{\phi}_3}{h}} \right).$$

(4.15)

Hence it implies the area law in spite of the cancellations of the long range contributions of the propagators. (Since we had omitted the UV cutoff in the space-time propagators the expression (4.15) is seemingly UV divergent).

The preceding results, based on a treatment of the term quadratic in $J$ in (3.9),
have obvious interpretations in the context of the stochastic vacuum model [17] for Yang-Mills theories: There, in the Gaussian approximation, the expectation value of the Wilson loop is given by the expectation value of the field strength correlator, which plays the same role as the term quadratic in $J$ in (3.9). At first sight an ambiguity appears: A priori it is not clear, whether the Yang-Mills Wilson loop reappears in our "effective low energy model" (after integrating out the off-diagonal gluons in the MAG, see [14]) in the form of a source $J_A$, $J_B$ or, most likely, as a combination of both (or even in the form of additional terms in $S_{\text{bare}}$). The condition (4.11) fixes this ambiguity.

If we identify naively the term quadratic in $J$ in (3.9) with the expectation value of the field strength correlator, the area law obtained from (4.13) above corresponds to a function $D_1(x^2)$ in the standard decomposition of the field strength correlator [17] which decreases only as $|x|^{-2}$ for large $|x|$. Such a behaviour is strongly disfavoured by lattice measurements [18] of the Yang-Mills field strength correlator. On the other hand the results (4.13) and (4.15), which are the consequence of the condition (4.11), agree well with with the lattice measurements of the Yang-Mills field strength correlator [18] and are in fact identical to the results for this correlator obtained in various models [10, 13, 20].

We recall, however, that up to now we have only discussed the term quadratic in $J$ in (3.9). Let us note that this term coincides (up to an irrelevant constant) with $W(J)$ to $O(J^2)$, provided that we replace $\hat{\phi}_i$ by $\hat{\phi}_i^0$ in $P^{rs}(\hat{\phi}_i)$: From the stationary point equations (3.11) for $\hat{\phi}_i$ we have $\hat{\phi}_i(J) = \hat{\phi}_i^0 + O(J^2)$, and – since the $\hat{\phi}_i^0$ are stationary points of $NG(\phi_i)$ – we thus have $NG(\hat{\phi}_i(J)) = NG(\hat{\phi}_i^0) + O(J^4)$. Beyond an expansion in powers of $J$ the stationary point equations (3.11) are, however, cumbersome to solve since they involve the full propagators $P^{rs}(\phi_i)$.

To this end it is wiser to start with the effective action $\Gamma(A,B)$ as given in eq.
One should solve the combined equations of motion for $A_{\mu}^n$, $B_{\mu\nu}^n$ in the presence of sources $J_{A,\mu}^n$ and $J_{B,\mu\nu}^n$ together with the equations (4.3) for $\phi_i$ (in some analogy with the approach in [19] based on the dual abelian Higgs model). The corresponding solutions have to be inserted into $\Gamma(A, B)$ in (4.2), and then one has to “undo” the Legendre transformation (4.1) in order to obtain $W(J)$. This last step is actually trivial since $\Gamma(A, B)$ is quadratic in $A, B$: It suffices to change the sign of the second term in the expression (4.2). Then one can study the dependence of $W(J)$ on $J$ in its full beauty.

As discussed in somewhat more detail in [14] the effect of such a more complete calculation can be estimated in the simple case where the sources $J$ are non-vanishing only inside a finite volume $V$ (to be identified with a ”Wilson surface” of finite width $\Delta L$), and where derivative terms in $G(\phi_i)$ are neglected: Then the solutions $\hat{\phi}_i(x)$ of the stationary point equations differ from $\hat{\phi}_i^0$ inside $V$, but coincide with $\hat{\phi}_i^0$ outside $V$. Hence one obtains an additional contribution to the action proportional to $V$, or a contribution to the energy of the configuration proportional to the diameter of the Wilson surface (at fixed time $t$). This picture supports ”flux-tube” models for the origin of the string tension. Its details (such as the shape of the fields $\hat{\phi}_i$ perpendicular to the string axis) depend, however, on the precise form of $G_{\text{bare}}(\phi_i)$ and hence of $L_{\text{bare}}(O_i)$. (At this point it is useful to note that the auxiliary fields $\phi_i$, as introduced in (3.1), parametrize vevs of the bilinear operators $O_i$ in (2.2)).

We close this section with a comment on the physical spectrum of the model. In momentum space the propagators $P^{rs}(q^2)$ for the $A_{\mu}-B_{\mu\nu}$ system have, in the confining phase, $q^{-4}$ singularities for $q^2 \to 0$ as well as poles at $q^2 = -4\hat{\phi}_3/h$. The $q^{-4}$ singularities for $q^2 \to 0$ do not correspond to asymptotic physical states. The poles at $q^2 = -4\hat{\phi}_3/h$, on the other hand, would disappear if we would replace the constant $h$ in the ansatz (2.3) for $S_{\text{bare}}$ by a function $h(q^2)$ such that $h(q^2)$ vanishes sufficiently rapidly for large $|q^2|$. This would not modify any of our essential results, but would be motivated by the idea that the Kalb-Ramond fields $B_{\mu\nu}^n$ have originally been introduced into a more ”microscopic” theory (as a Yang-Mills theory) as auxiliary fields (for, e.g., the abelian field strengths $F_{\mu\nu}^n$ [8, 9, 14]), and their kinetic terms are thus loop-effects of modes which have been integrated out (as the off-diagonal gluons). Then none of the degrees of freedom of the $A_{\mu}-B_{\mu\nu}$ system would appear as asymptotic states, consistent with the absence of $(N_c - 1)$ -plets in a $SU(N_c)$ Yang-Mills theory.

We have also searched for ”bound states”, i.e. poles in the propagators of the $\phi_i$. 

(12).
fields. Such poles are not present, essentially because bosonic loop contributions to kinetic terms of auxiliary $O(N)$ singlet fields differ in sign with respect to fermionic loop contributions (which do generate propagating bound states [15]). One may be deceived by the absence of “glueballs”, if the present model is interpreted as an effective low energy theory for $SU(N_c)$ Yang-Mills theory. However, we recall that the present model would only describe the abelian subsector of $SU(N_c)$ Yang-Mills theory in the MAG [8, 9, 14], and that we finally have to add the off-diagonal gluons as well. Our theory induces confining interactions among all fields which couple $A_\mu^n$ and $B^{\mu\nu}_n$, hence the off-diagonal gluons will necessarily form bound states which will correspond to the desired glueballs.

Thus, if we use functions $h(q^2)$ with the above-mentioned properties in the ansatz (2.3) for $S_{\text{bare}}$, the model has no asymptotic states at all. Its only ”meaning” is then to react to external sources, and to confine them as discussed before.

5 Discussion and Outlook

We have studied a class of four-dimensional $U(1)$ gauge theories including Kalb-Ramond fields, which exhibit confinement and allow nevertheless – in the large $N$ limit – for controllable computations in the infrared regime. Some features of the confining phase correspond quite to our expectations, notably the possibility to perform a duality transformation of the low energy part of the effective action and thus to interpret confinement as monopole condensation. A technically related phenomenon is the appearance of a low energy vector gauge symmetry, which allows to “gauge away” the low momentum modes of the abelian gauge fields $A^n_\mu$.

An interesting feature is the origin of the relation among the parameters of the effective action which generates the above symmetry: The bare action, as a functional of $A_\mu^n$ and $B_{\mu\nu}$, has to be (sufficiently) non-convex, which renders the effective action ”flat” in some region around the origin in field space. This ”flatness” corresponds to the above symmetry with all its consequences. This phenomenon is evidently independent from the large $N$ limit employed here. The auxiliary scalar fields $\phi_i$ parametrize various bilinear condensates which do not, however, break any internal symmetry.

We have argued – but not shown in detail (to this end see [14]) – that the present class of models can be obtained from $SU(N_c)$ Yang-Mills theories in the maximal abelian gauge after integrating out the off-diagonal gluons. This application clarifies
why it is sensible to consider these models as effective low energy theories equipped with an UV cutoff, but with a bare action including non-renormalizable interactions. Our ansatz (2.3) for the bare action is already quite general and exhibits the most interesting phenomena, but it could easily be generalized by including further bilinear operators with "internal" derivatives and/or external Lorentz indices. It can be argued [14] that further bilinear operators with "internal" derivatives do not affect the low energy limit of the model (and just modify somewhat the relation between the bare and effective actions), whereas operators with external Lorentz indices will have no vevs (but can affect the response of the model with respect to external sources). More detailed investigations in this direction would be quite straightforward.

A grain of salt constitutes the fact that the large N limit in the present class of models does not coincide with the large $N_c$ limit of $SU(N_c)$ Yang-Mills theories [14]. First, the powers of $N_c$ in $S_{bare}$ would not correspond to the powers of N required in (2.3), and second $S_{bare}$ would not necessarily depend only on $O(N)$ singlets since this $O(N)$ is not a sub-group of $SU(N_c)$ (with $N_c = N + 1$). A discrete reflection symmetry, under which $A_{\mu}^a$ and $B_{\mu\nu}^a$ (or the abelian components of $F_{\mu\nu}$) change sign is, however, a symmetry of $SU(N_c)$ Yang-Mills theories, which justifies at least the introduction of bilinear composite operators. We recall again, on the other hand, that the essential features of the confining phase do not rely on the large N limit.

We have seen that the "physics" of the confining phase is by no means unique. On the one hand confinement can always be interpreted as monopole condensation (and hence the vacuum as a dual superconductor), but many features like the most important contributions to the string tension, (non-local) vacuum correlators and the shape of the vacuum energy distribution perpendicular to a flux tube depend on the non-universal properties of the model incoded in $S_{bare}$. Hence, if we wish to learn more about the way confinement is realized in Yang-Mills theories we have to find ways to learn more about the bare action, and eventually to handle the present class of models beyond the large N limit. Nevertheless the solvable version of the present models can certainly play the role of a useful laboratory for the study of the properties of a confining phase in the future.

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