Noether Symmetries of Bianchi Type II Spacetime Metrics

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ABSTRACT

We classify a class of Bianchi type II spacetimes according to their Noether symmetries. We briefly discuss the conservation laws admitted by these Noether symmetries and classify their possible algebras and determine their structures also.

Introduction

The Noether symmetry, also known as variational symmetry, is associated with mechanical systems possessing a Lagrangian $L$. In general Relativity, this Lagrangian is given by $L = g_{ab} \dot{x}^a \dot{x}^b$ and is obtained from the spacetime metric $g_{ab}$ given by $ds^2 = g_{ab} dx^a dx^b$. The Euler-Lagrange equations corresponding to the above Lagrangian determine the trajectories of the particles and are solutions of $\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0$, also known as geodesic equations. In Relativity, the spacetime metric, $g_{ab}$ is of particular significance as it determines exact solutions of the Einstein field Equations, $R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab}$, where $R_{ab}, T_{ab}, R$ and $\kappa$ respectively represent Ricci & Stress tensors, Ricci scalar and the gravitational constant.

As for Einstein, they are highly non-linear coupled system of partial differential equations. This
non-linearity makes Einstein field equations very hard to solve for their exact solutions. On the other hand if one chooses energy momentum tensor to define Einstein field equation, then any arbitrarily chosen metric will form a solution of these equations, which may be totally un-physical. To date reasonably small number of solutions of the Einstein field equations are known and their full account can be found in [1, 2]. Generally, exact solutions are categorized as vacuum and non-vacuum solutions. One of the most interesting solutions of the vacuum Einstein field equations is given by the famous Schwarzschild solution, whose predictions (planetary orbits and black holes etc) are responsible for the fundamental importance attached to general theory of Relativity. In the non-vacuum case, there are several classes of solutions that have been widely discussed. Brans and Dicke [3] theory of gravitation is a well known modified version of Einstein’s theory of Relativity. It is a scalar tensor theory in which the gravitational interaction is mediated by a scalar field $\phi$ as well as the tensor field $g_{ab}$ [4]. In a recent work Singh and Rai [5] list a detailed discussion of Brans-Dicke cosmological models. In particular, spatially homogeneous Bianchi models in Brans-Dicke theory in the presence of perfect fluid are quite important to discuss the early stages of evolution of the universe. Other important aspects of the Brans and Dicke theory are discussed in Belinskii and Khalatnikov [6], and Reddy et al. [7]. Chakraborty and Bali et.al discussed Bianchi type IV strings models in general Relativity [9][10][11]. A detailed discussion of Bianchi type II, VIII and IX models in scale covariant theory of gravity is given in the work published by Reddy et.al [8]. Shanthi et.al have studied Bianchi type-VIII & IX models in Lyttleton-Bondi Universe [12]. Also, Rao et.al have studied Bianchi type-VIII & IX models in Zero mass scalar fields and self creation cosmology [13]. More recently Valagapudi studied Bianchi type-II, VIII
and IX perfect fluid cosmological models in a scalar tensor theory [14].

Bianchi spacetimes have also been discussed in literature from the point of view of symmetry approach. Following this course, Capozziello et. al have investigated homogeneous theories of gravity, in which a scalar field is minimally coupled to gravity, searching for point symmetries in the cosmological Lagrangian density which allow to solve exactly the dynamical problem used. In particular, they have used Noether symmetry approach to study the Einstein equations minimally coupled with a scalar field, in the case of Bianchi universes of certain classes [15]. In an interesting recent work Camci et. al have investigated matter collineations (MC) of Bianchi type II spacetime according to the degenerate and non-degenerate energy momentum tensor [16]. It is shown that when the energy-momentum tensor is degenerate, most of the cases yield infinite dimensional MCs whereas some cases give finite dimensional Lie algebras in which there are three, four or five MCs. For the non-degenerate matter tensor cases they found that the Lie algebra of MCs is finite dimensional, in which the number of MCs is three, four or five. Furthermore, they discussed the physical implications of the obtained MCs in the case of perfect fluid as source.

Our main focus in this paper is to classify Bianchi type II spacetimes according to their metrics and Noether symmetries. These models are worth studying because they present a middle way between Friedmann Robertson Walker models and completely inhomogeneous and anisotropic universes and thus play an important role in modern cosmological studies. Further, the Bianchi type-II models have been widely studied for the simplification and description of the large scale behavior of the actual universe. With this in mind, it is argued that it might be worth investigating the variational conservation laws admitted by the Lagrangian of the Bianchi spacetimes.
The plan of this investigations is as follows: In the next section we give a brief description of
the equations giving Noether symmetries. In section 3, we give a complete classification of the
Noether symmetries of Bianchi type II spacetime metric and then classify Lie algebras of Noether
symmetries. A brief summary and discussion of the work is given in the last section.

Noether Symmetries in Bianchi Type II Spacetime

The Bianchi type models represent a middle way between Friedmann Robertson Walker models
and the completely inhomogeneous and anisotropic universes and thus play an important role in
modern cosmological studies. In particular, the Bianchi type-II models have been widely studied
for the simplification and description of the large scale behavior of the universe. Mathematically,
these models are represented by a metric of the form [16],

\[ ds^2 = -dt^2 + A(t)^2 dx^2 + B(t)^2 (dy^2 + x^2 dz^2 - 2x dy dz) + C(t)^2 dz^2. \]  (1)

The Lagrangian associated with the above metric is,

\[ L = -t^2 + A(t)^2 x^2 + B(t)^2 (y^2 + x^2 \dot{z}^2 - 2x y \dot{z}) + C(t)^2 \dot{z}^2, \]  (2)

where the dot (.) represents derivative with respect to the affine parameter \( s \). According to
Noether’s theorem, to each differentiable symmetry of the action of a physical system there cor-
responds a conservation law. For the time translation symmetry, Noether’s theorem states that the
energy of the system is conserved. Similarly, a translation in space gives rise to conservation of
linear momentum, while existence of a rotational symmetry provides conservation of angular mo-
mentum. Mathematically, the Noether Symmetry arises from the invariance properties admitted
by the lagrangian \( \mathcal{L} = \int_{\Omega} L(t(s), x(s), y(s), z(s); s) \, ds \) defined over \( \Omega \). Including point dependent
gauge functions $f$, the Noether symmetries associated with the given lagrangian are given by \[17\]

$$X^1 L + L D_s \xi = D_s f,$$  \hspace{1cm} (3)

where

$$X^1 = X + \tau^1 \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial y} + \phi^1 \frac{\partial}{\partial z},$$  \hspace{1cm} (4)

is the 1st prolongation of the symmetry generator $X = \mu \frac{\partial}{\partial s} + \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial z}$ associated with 4-spacetime, in which $\mu, \tau, \xi, \eta, \phi$ are functions of $s, t, x, y, z$. Also, in (4) the components $\tau^1, \xi^1, \eta^1, \phi^1$ of the 1st prolongation are defined by

$$
\begin{align*}
\tau^1 &= D_s \tau - iD_s \mu, \\
\xi^1 &= D_s \xi - iD_s \mu, \\
\eta^1 &= D_s \eta - iD_s \mu, \\
\phi^1 &= D_s \phi - iD_s \mu,
\end{align*}
$$

with

$$D_s = \frac{\partial}{\partial s} + i \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z};$$

(6)

defining total derivative operator. Using equation (2), gives rise to a polynomial equation in derivatives of $(t, x, y, z)$:

$$
\begin{align*}
&\left[ \mu \frac{\partial^2}{\partial s^2} + \tau \frac{\partial^2}{\partial t^2} + \xi \frac{\partial^2}{\partial x^2} + \eta \frac{\partial^2}{\partial y^2} + \phi \frac{\partial^2}{\partial z^2} + \tau^1 \frac{\partial^2}{\partial t \partial s} + \xi^1 \frac{\partial^2}{\partial x \partial s} + \eta^1 \frac{\partial^2}{\partial y \partial s} + \phi^1 \frac{\partial^2}{\partial z \partial s} \right] - i t^2 + A(t)^2 x^2 + \\
&B(t)^2 (y^2 + x^2 z^2 - 2xyz) + C(t)^2 z^2] + [-i t^2 + A(t)^2 x^2 + B(t)^2 (y^2 + x^2 z^2 - 2xyz) + \\
&C(t)^2 z^2] \left[ \frac{\partial^2}{\partial s^2} + i \frac{\partial^2}{\partial t^2} + x \frac{\partial^2}{\partial x^2} + y \frac{\partial^2}{\partial y^2} + z \frac{\partial^2}{\partial z^2} \right] \mu - \left[ \frac{\partial^2}{\partial s^2} + i \frac{\partial^2}{\partial t^2} + x \frac{\partial^2}{\partial x^2} + y \frac{\partial^2}{\partial y^2} + z \frac{\partial^2}{\partial z^2} \right] f = 0
\end{align*}
$$

(7)

Setting the coefficients of $i^3, x^3, y^3, z^3, i^2, x^2, y^2, z^2, ix, iy, iz, xy, yz, t, x, y, z$ and the constant term, respectively, to zero, gives rise to an over determined system of differential
equations given by:

\[ \mu_t = 0, \quad (8) \]

\[ A^2 \mu_x = 0, \quad (9) \]

\[ B^2 \mu_y = 0, \quad (10) \]

\[ (C^2 + B^2 x^2) \mu_z = 0, \quad (11) \]

\[ \mu_s - 2 \tau_t = 0, \quad (12) \]

\[ 2AA' \tau + 2A^2 \xi_x - A^2 \mu_s = 0, \quad (13) \]

\[ 2BB' \tau + 2B^2 \eta_y - B^2 \mu_s - 2B^2 x \phi_y = 0, \quad (14) \]

\[ 2B^2 x \xi + 2BB' x^2 \tau + 2CC' \tau - 2B^2 x \eta_z - C^2 \mu_s - B^2 x^2 \mu_y + 2C^2 \phi_z + 2B^2 x^2 \phi_z = 0, \quad (15) \]

\[ A^2 \xi_t - \tau_x = 0, \quad (16) \]

\[ B^2 \eta_t - \tau_y - B^2 x \phi_x = 0, \quad (17) \]

\[ -B^2 x \eta_z + C^2 \phi_t + B^2 x^2 \phi_x = 0, \quad (18) \]

\[ B^2 \eta_x + A^2 \xi_y - B^2 x \phi_x = 0, \quad (19) \]

\[ -B^2 x \eta_x + A^2 \xi_z + C^2 \phi_x + B^2 x^2 \phi_x = 0, \quad (20) \]

\[ -B^2 \xi - 2BB' x \tau - B^2 x \eta_y + B^2 \eta_z + B^2 x \mu_s + C^2 \phi_y + B^2 x^2 \phi_y - B^2 x \phi_z = 0, \quad (21) \]
In order to classify Noether symmetries, the aim is to determine coefficient of the Noether’s operator \( \mathcal{X}' \) as functions of \( t, x, y, z \) and \( s \). The method adopted to classify Noether symmetries admitted by (1) is to solve the system (8) - (26) by following an exhaustive procedure followed in [18]. This method of solving above system gives rise to Noether symmetries admitted by (1) corresponding to allowable conditions satisfied by conditions of the metric components. This process of classification gives a total of ten cases. Each case satisfies certain differential constraints given in equation (1) as detailed below.

**Case I.** \( A'' = 0, \ A = B = C \).

The component form of the Noether symmetry \( X = \mu \frac{\partial}{\partial s} + \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \varphi \frac{\partial}{\partial z} \) are:

\[
\mu = \frac{\alpha}{2} s^2 + \alpha_3 s + \alpha_5; \ \tau = \frac{\alpha}{2} s t + \alpha_4 s + \alpha_6; \ \xi = \alpha_7; \ \eta = \alpha_7 z + \alpha_8; \ \varphi = -\alpha_9,
\]

along with conformal factor \( f = -\frac{\alpha}{2} t^2 - 2\alpha_4 t \).

The Noether symmetry generators in this case are:

\[
X_1 = \frac{s^2}{2} \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t}; \ X_2 = \frac{s}{2} \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t}; \ X_3 = \frac{\partial}{\partial t}; \ X_4 = \frac{\partial}{\partial s}; X_5 = s \frac{\partial}{\partial t}, f = -2t; \ X_6 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}; \ X_7 = \frac{\partial}{\partial y}; \ X_8 = -\frac{\partial}{\partial z}.
\]

The non-commuting generators satisfy the algebra given by:
In this case the components of the Noether symmetry generator

\[ [X_1, X_2] = -X_1, \quad [X_1, X_3] = \frac{1}{2} X_5, \quad [X_1, X_4] = -X_2, \quad [X_2, X_3] = -\frac{1}{2} X_1, \quad [X_2, X_4] = -X_4, \quad [X_2, X_5] = X_5, \]
\[ [X_4, X_5] = X_3, \quad [X_6, X_8] = X_7. \]

All the remaining commutation relations are zero.

**Case II.** $A' = 0, B' = 0, C' = 0$;

In this case the components of the Noether symmetry generator $X = \mu \frac{\partial}{\partial s} + \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial z}$, along with gauge term, are given by

\[ \mu = \alpha_1; \tau = \alpha_2 s + \alpha_3; \xi = \alpha_4 z + \alpha_5; \eta = \alpha_4 \left( \frac{z^2 - x^2}{2} \right) + \alpha_5 z + \alpha_6; \phi = -\alpha_4 x - \alpha_7 \text{ with } f = -2\alpha_5 t \]

In generator form, the above give rise to following 7 symmetry generators:

\[ X_1 = \frac{\partial}{\partial s}, \quad X_2 = \mathbf{s} \frac{\partial}{\partial t} \text{ with } f = -2t; \quad X_3 = \frac{\partial}{\partial x} + \left( \frac{z^2 - x^2}{2} \right) \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial t}, \quad X_5 = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}, \quad X_6 = \frac{\partial}{\partial x}, \quad X_7 = -\frac{\partial}{\partial z} \]

The above symmetries form a closed form Lie-algebra of which all generators commute except

\[ [X_1, X_2] = X_4; \quad [X_5, X_3] = X_7; \quad [X_3, X_7] = X_5; \quad [X_5, X_7] = X_6. \]

**Case III.** $A' = 0, B' = 0, C'' = 0$;

The components of the Noether symmetry generator $X = \mu \frac{\partial}{\partial s} + \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial z}$ are:

\[ \mu = \alpha_1 s + \alpha_2; \tau = \frac{\alpha_5}{2} t + \alpha_3; \xi = \alpha_4 x + \alpha_4; \eta = \alpha_4 \frac{y}{2} + \alpha_5 z + \alpha_5; \phi = -\alpha_6 \]

Generator form of above Noether symmetries is:

\[ X_1 = s \frac{\partial}{\partial s} + \frac{\tau}{2} \frac{\partial}{\partial t} + \frac{\xi}{2} \frac{\partial}{\partial x} + \frac{\eta}{2} \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = \frac{\partial}{\partial t} + z \frac{\partial}{\partial y}, \quad X_5 = \frac{\partial}{\partial x}, \quad X_6 = -\frac{\partial}{\partial z} \]

**Case IV.** $A' = 0, B' \neq 0, C' = 0$.

In this case the components of Noether symmetry generator $X = \mu \frac{\partial}{\partial s} + \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial z}$ take the form,

\[ \mu = \alpha_1; \tau = 0; \xi = \alpha_2 z + \alpha_3; \eta = \alpha_2 \left( \frac{z^2 - x^2}{2} \right) + \alpha_3 z + \alpha_4; \phi = -\alpha_3 x - \alpha_5. \]
Noether Symmetries generators associated with above components are:

\[ X_1 = \frac{\partial}{\partial s}, \quad X_2 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial y}, \quad X_5 = -\frac{\partial}{\partial z}. \]

Except \([X_2, X_3] = X_5, [X_3, X_2] = X_3, [X_3, X_5] = X_4\), all the remaining commutation relations vanish.

**Case V.** \(A'' \neq 0, B'' = 0 = C''\).

Non-zero components of the Noether symmetry generator, \(X = \mu \frac{\partial}{\partial s} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial z}\) become,

\[ \mu = \alpha_1 s + \alpha_2; \quad \tau = \frac{\alpha_1}{2} t + \alpha_3; \quad \eta = \alpha_4; \quad \phi = -\alpha_5. \]

Generator form of above symmetries is given by:

\[ X_1 = s \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t}; \quad X_2 = \frac{\partial}{\partial s}; \quad X_3 = y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}; \quad X_4 = \frac{\partial}{\partial y}; \quad X_5 = -\frac{\partial}{\partial z}. \]

Two non-commuting relations are \([X_2, X_1] = X_2, [X_3, X_1] = \frac{1}{2} X_3\).

**Case VI.** \(A'' = 0, B' = 0, C = C(t)\).

In this case \(C\) is an arbitrary function of \(t\) and the non-zero components of the Noether symmetry generator take the form,

\[ \mu, \tau, \eta, \phi \text{ with } \xi = 0. \]

The five Noether symmetry generators associated with above generator are;

\[ \mu = \alpha_1 s + \alpha_2; \quad \tau = \alpha_1 (\frac{t}{2} + 1); \quad \eta = \alpha_3 y + \alpha_4; \quad \phi = \alpha_3 z - \alpha_5. \]

In the light of above, the symmetry generators take the form,

\[ X_1 = s \frac{\partial}{\partial s} + (\frac{t}{2} + 1) \frac{\partial}{\partial t}; \quad X_2 = \frac{\partial}{\partial s}; \quad X_3 = y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}; \quad X_4 = \frac{\partial}{\partial y}; \quad X_5 = -\frac{\partial}{\partial z}. \]

The non-commuting commutation relation satisfied by these symmetries are: \([X_2, X_1] = X_2, [X_4, X_3] = X_4, [X_5, X_3] = X_5\).

**Case VII.** \(A' = 0, B' = 0, C'' \neq 0\);
Here the components of Noether symmetry generator $X = \mu \frac{\partial}{\partial s} + \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial z}$ are,

$\mu = \alpha_1; \tau = 0; \xi = \alpha_2; \eta = \alpha_3 + \alpha_3; \phi = -\alpha_4$. In generator form, the above give rise to following 4 symmetry generators:

$X_1 = \frac{\partial}{\partial s}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = -\frac{\partial}{\partial z}.$

All of the symmetry generators commute except $[X_2, X_4] = x_3$. 

**Case VIII.** $A'' \neq 0, B' = 0 = C''$. 

The non zero components of the Noether symmetry generator, $X = \mu \frac{\partial}{\partial s} + \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial z}$ are,

$\mu = \alpha_1; \tau = \alpha_2; \eta = \alpha_3; \phi = -\alpha_4$. 

Noether symmetry generator associated with above components are:

$X_1 = \frac{\partial}{\partial s}, \quad X_2 = \frac{\partial}{\partial t}; \quad X_3 = \frac{\partial}{\partial y}; \quad X_4 = -\frac{\partial}{\partial z}.$

All Noether symmetry generators in this case commute, i.e., $[X_i, X_j] = 0$ for all $i$ and $j$.

**Case IX.** $A'' \neq 0, B = B(t), C'' = 0$. 

In this case we get minimal set of 3 Noether symmetries, which in component form are given by:

$\mu = \alpha_1; \eta = \alpha_2; \phi = -\alpha_3$. 

The three Noether symmetry generators corresponding to above components are;

$X_1 = \frac{\partial}{\partial s}; \quad X_2 = \frac{\partial}{\partial y}; \quad X_3 = \frac{\partial}{\partial z}$, all of which commute.

We do not include the remaining cases in this discussion because they either belong to one of the above cases or are their proper subalgebras.

**Structure of Lie Algebras of Noether Symmetries**

We give the detailed structure of the Lie algebra of Noether symmetries in two cases, (I) and (II).
The main reason for this is that the Lie algebra in case I is non-soluble, while it is soluble in all the remaining cases, which can be dealt with as case (case II).

**Structure of the Lie Algebra in Case I**

The Lie algebra of Noether symmetries in this case is $L_8 = \langle X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8 \rangle$. The Killing form is defined by $\kappa(X_i, X_j) = Tr(AdX_i \circ AdX_j)$. It is straightforward to verify that $\kappa(X_1, X_4)$ and $\kappa(X_2, X_2)$ are not equal to zero, while all the remaining $\kappa(X_i, X_j) = 0$. The Levi decomposition is $\langle X_1, X_2, X_4 \rangle \oplus \text{Rad}(L_1)$, where the first summand is isomorphic to $sl(2, R)$.

**Structure of the Lie Algebra in Case II**

The Lie algebra of Noether symmetries in this case is $L_1 = \langle X_1, X_2, X_3, X_4, X_5, X_6, X_7 \rangle$. A straightforward computation shows that the derived series of $L$ is $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq L^3 = \{0\}$. This can be used to give an ascending sequence of ideals each of codimension 1, which is useful in obtaining conjugacy classes of 1–dimensional subalgebra as in [20].

**Summary and Discussions**

A classification of Noether symmetries of a Lagrangian associated with the Bianchi type II space-time metric is obtained. This classification arises from a variety of differential constraints obtained on the coefficients of the metric representing the Bianchi models. It is shown that for a total of 9 nontrivial cases arise giving Noether symmetries admitting maximal group $G < 8 >$ and a minimal group $G < 3 >$ of Noether symmetries. The maximal group $G < 8 >$ contains all the remaining seven to three parameter subalgebras as proper subalgebra.

All these cases yield physically interesting conservation laws giving linear momentum conservation along $y$ and $z$ directions, whereas cases I – III, IV and VIII additionally admit time transla-
tional invariance giving conservation of energy.

The detailed structure of the algebras given above can be used to construct conserved quantities for each conjugacy classes of 1–dimensional subalgebras as in ([20],[21]).

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