ON P-BIHAMRONIC CURVES

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ABSTRACT. In this article we study p-biharmonic curves as a natural generalization of biharmonic curves. In contrast to biharmonic curves p-biharmonic curves do not need to have constant geodesic curvature if \( p = \frac{1}{2} \) in which case their equation reduces to the one of \( \frac{1}{2} \)-elastic curves. We will classify \( \frac{1}{2} \)-biharmonic curves on closed surfaces and three-dimensional space forms making use of the results obtained for \( \frac{1}{2} \)-elastic curves from the literature. By making a connection to magnetic geodesic we are able to prove the existence of \( \frac{1}{2} \)-biharmonic curves on closed surfaces. In addition, we will discuss the stability of p-biharmonic curves with respect to normal variations. Our analysis highlights some interesting relations between p-biharmonic and p-elastic curves.

1. Introduction and Results
Finding interesting curves on a Riemannian manifold is one of the central topics in modern differential geometry. The most prominent examples of such curves are without doubt geodesics as they are the curves with minimal distance between two given points in a Riemannian manifold. Moreover, the existence of geodesics is guaranteed by a number of famous results in differential geometry, such as the Theorem of Hopf-Rinow.

In order to obtain the equation for geodesics it is useful to employ a variational approach as this gives a lot of additional mathematical structure. To this end, we consider a curve \( \gamma : I \rightarrow M \), where \( I \subset \mathbb{R} \) represents an interval, \( (M, g) \) a Riemannian manifold and by \( s \) we denote the parameter of the curve \( \gamma \). Moreover, we use the notation \( \gamma' = \frac{d\gamma}{ds} \). Then, we define the energy of the curve \( \gamma \) by

\[
E_1(\gamma) = E(\gamma) := \frac{1}{2} \int_I |\gamma'|^2 ds. \tag{1.1}
\]

The critical points of (1.1) are precisely geodesics and are characterized by the equation

\[
\tau(\gamma) := \nabla_\gamma \gamma' = 0,
\]

which is a second order non-linear ordinary differential equation for \( \gamma \) and the quantity \( \tau(\gamma) \) is usually referred to as tension field. In the case of a higher-dimensional domain (1.1) is replaced by the energy of a map whose critical points are harmonic maps.

Another interesting class of curves can be obtained by extremizing the bienergy of a curve \( \gamma \) which is given by

\[
E_2(\gamma) := \frac{1}{2} \int_I |\tau(\gamma)|^2 ds.
\]

The critical points of this functional are called biharmonic curves and are characterized by a non-linear ordinary differential equation of fourth order. However, as biharmonic curves necessarily have constant geodesic curvature they are often too rigid in order to allow for applications in the natural sciences. In order to circumvent this drawback there exist several notions similar to biharmonic curves which are not that restricted.

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In this article we will introduce one such further generalization of biharmonic curves. The starting point of our analysis is the energy for \( p \)-biharmonic curves which is defined as follows

\[
E_{2,p}(\gamma) = \frac{1}{p} \int |\tau(\gamma)|^p ds, \quad p > 0.
\]  

(1.2)

This definition is motivated from applications of such curves in elasticity where it is often favorable to choose an exponent different from 2. Of course, for \( p = 2 \) the energy functional (1.2) reduces to the well-studied energy for biharmonic curves.

Another possible variant are the so-called \( p \)-elastic curves which are critical points of the \( p \)-elastic energy given by

\[
E_{p,\text{elastic}}(\gamma) = \frac{1}{p} \int |k|^p ds, \quad p > 0.
\]  

(1.3)

Here, \( k \) represents the geodesic curvature of the curve \( \gamma \).

Although both (1.2) and (1.3) look very similar at first glance we can expect them to be different in general: If we could consider (1.2) and (1.3) in the case of a two-dimensional domain with \( p = 2 \) then (1.2) would become the bienergy while (1.3) would turn into the Willmore energy. As these two variational problems are very different in nature one should also expect substantial differences between (1.2) and (1.3).

From an analytic point of view both (1.2) and (1.3) are favorable to investigate if \( p \geq 2 \) as the resulting Euler-Lagrange equations might become degenerate if \( p < 2 \). One could also allow for the possibility of \( p \) being negative. However, as we always would like to draw a comparison between critical points of (1.2), (1.3) and geodesics we restrict to positive values of \( p \) as for \( p < 0 \) we need to be careful with (1.2), (1.3) should the curve be a geodesic.

Let us give an (non-exhaustive) overview on the mathematical results on \( p \)-elastic curves. One of the most influential works on this subject for the case \( p = 2 \) goes back to Langer and Singer [23] who classified elastic curves on two-dimensional manifolds. Their work was later extended by Watanabe [37] who was able to determine the geodesic curvature of \( p \)-elastic curves for \( p \geq 2 \) making use of generalized elliptic integrals. Recently, Shioji and Watanabe showed some interesting phenomena for \( p \)-elastic curves on the sphere in [35]. Concerning the gradient flow of \( p \)-elastic curves for \( p \geq 2 \) we refer to the recent articles [7, 28, 32]. The elastic flow (\( p = 2 \)) of curves on the sphere was investigated in [16]. For the current status of research regarding the flow of elastic curves we refer to the recent survey [24].

One of the main observations of this article is that in the case of \( p = 1/2 \) the first variations of (1.2) and (1.3) lead to the same equation. While for \( p \neq 1/2 \) the equations for \( p \)-biharmonic curves always require that we have a curve of constant curvature we will see that for \( p = 1/2 \) we do not encounter this restriction.

The idea of studying \( p \)-elastic curves goes back to Bernoulli, and later Blaschke initiated the study of \( 1/2 \)-elastic curves in \( \mathbb{R}^3 \). Recently, \( 1/2 \)-elastic curves have been investigated in the case that the ambient space is a sphere or hyperbolic space in a series of articles by Arroyo, Garay and Menéndez [2, 3] and by Arroyo, Garay and Barros [1]. Another variant of (1.3) that received growing attention is

\[
E_{\mu}(\gamma) = \int |\sqrt{k + \mu}| ds, \quad \mu \in \mathbb{R},
\]  

(1.4)

Recently the functional (1.4) has been investigated in great generality by Arroyo, Garay and Pámano [4, 5]. At various places our analysis is close to the results obtained in [1, 2, 3, 4, 5] in the context of critical points of (1.4). A general discussion on curvature energies of curves can be found in [20].

In the case of a higher-dimensional domain (1.2) becomes the energy for \( p \)-biharmonic maps. A number of geometric classification results for the latter could be achieved in [14, 18, 19]. Some stability results on F-biharmonic maps, which are a further generalization of \( p \)-biharmonic maps, have been obtained in [24]. Recently, \( p \)-biharmonic submanifolds were investigated in [25]. For the stability of biharmonic maps we recommend the reader to consult the recent articles [26, 29, 30] and the references therein. A stability analysis for harmonic self-maps of
COHOMOGENEITY ONE MANIFOLDS, WHICH EMPLOYS SIMILAR IDEAS AS THE ONES USED IN THIS ARTICLE, WAS CARRIED OUT IN [11]. CONCERNING THE CURRENT STATUS OF RESEARCH ON BIHARMONIC CURVES WE REFER TO [13, SECTION 4], A GENERAL INTRODUCTION TO THE FIELD OF BIHARMONIC MAPS IN RIEMANNIAN GEOMETRY IS PROVIDED BY THE RECENT BOOK [31].

LET US ALSO MENTION THE FOLLOWING RESULTS CLOSELY CONNECTED TO THIS ARTICLE. THE CURRENT STATUS OF RESEARCH ON HIGHER ORDER VARIATIONAL PROBLEMS CAN BE FOUND IN [8]. A FAMILY OF CURVES THAT INTERPOLATES BETWEEN GEODESICS AND BIHARMONIC CURVES HAS BEEN INVESTIGATED IN [9, SECTION 3].

Throughout this article we will use the following notation: We use $\eta$ to represent the parameter of the curve $\gamma$ and denote differentiation of the curve $\gamma$ with respect to the curve parameter by $\gamma'$. By $I \subset \mathbb{R}$ we denote an interval and $(M, g)$ represents a Riemannian manifold of dimension $\dim M = n$. For the Riemannian curvature tensor we use the sign convention $\text{Riem}(\cdot) = \text{Riem}(\cdot)^{\ast}$. By $\gamma_1$ we denote an interval and $(\gamma^i_1)_{i=1}^d$.

This article is organized as follows: In Section 2 we study the first and second variation of the energy functional for $p$-biharmonic curves and $p$-elastic curves. Afterwards, in Section 3, we turn to the analysis of $p$-biharmonic curves on surfaces. In particular, we study their stability with respect to normal variations. Moreover, employing a number of results on magnetic geodesics from the literature we are able to obtain some existence results for $p$-biharmonic curves. Section 4 is devoted to the analysis of $p$-biharmonic curves on three-dimensional space forms also including an analysis of their stability.

## 2. Variational Formulas

In this section we derive the first and second variation of (1.2) and discuss the relation between $p$-biharmonic and $p$-elastic curves.

### 2.1. The first variation formula.

We start by deriving the Euler-Lagrange equation of (1.2). To this end, we fix a small number $\varepsilon > 0$ and let $\gamma_\varepsilon: (-\varepsilon, \varepsilon) \times I \to M$ be a variation of the curve $\gamma$ satisfying $\frac{\partial \gamma_\varepsilon}{\partial t}|_{t=0} = \eta$. For the moment we assume that $\eta \in \Gamma(\gamma^s TM)$. Here, $I \subset \mathbb{R}$ is a closed interval.

**Lemma 2.1.** Let $\gamma_\varepsilon$ be a variation of $\gamma$ as described above. Then the following formula holds

$$
\frac{d}{dt}\bigg|_{t=0} \frac{1}{p} \int_I |\nabla \gamma_\varepsilon^t|^{p} ds = \int_I \left( \langle \eta, \nabla \gamma' \nabla \gamma' |(\nabla \gamma^2 \gamma')^{p-2} \nabla \gamma' \gamma' \rangle - |\nabla \gamma' \gamma'|^{p-2} R^M(\gamma', \nabla \gamma' \gamma') \gamma' \right) ds.
$$

(2.1)

**Proof.** It is straightforward to show that

$$
\frac{\partial}{\partial t}\bigg|_{t=0} \nabla \gamma_\varepsilon^t = \nabla \gamma' \nabla \gamma' \eta + R^M(\eta, \gamma') \gamma',
$$

where we first interchanged covariant derivatives and then employed the torsion-freeness of the Levi-Civita connection.

Then, we get

$$
\frac{d}{dt}\bigg|_{t=0} \frac{1}{p} \int_I |\nabla \gamma_\varepsilon^t|^{p} ds = \int_I |\nabla \gamma' \gamma'|^{p-2} \left( \langle \nabla \gamma' \nabla \gamma' \eta, \nabla \gamma' \gamma' \rangle + \langle R^M(\eta, \gamma') \gamma', \nabla \gamma' \gamma' \rangle \right) ds.
$$

Finally, using the symmetries of the Riemannian curvature tensor and integration by parts completes the proof. □

A direct consequence of the previous Lemma is the following

**Proposition 2.2.** Let $\gamma_\varepsilon$ be a variation of $\gamma$ as described above.

1. If $\eta \in \Gamma(\gamma^s TM)$ then (2.1) yields the equation for $p$-biharmonic curves

$$
\nabla \gamma' \nabla \gamma' |(\nabla \gamma^2 \gamma')^{p-2} \nabla \gamma' \gamma' \rangle - |\nabla \gamma' \gamma'|^{p-2} R^M(\gamma', \nabla \gamma' \gamma') \gamma' = 0.
$$

(2.2)
(2) If $\eta \in \Gamma((\gamma^*TM)^\perp)$ then (2.1) yields the equation for $p$-elastic curves
\[ (\nabla_\gamma \nabla_\gamma (|\nabla_\gamma \gamma'|^{p-2} \nabla_\gamma \gamma') - |\nabla_\gamma \gamma'|^{p-2} R^M(\gamma', \nabla_\gamma \gamma') \gamma')^\perp = 0. \] (2.3)

2.2. The Frenet-equations. For the further analysis it turns out to be useful to rewrite the equations for both $p$-biharmonic (2.2) as well as $p$-elastic curves (2.3) in terms of its Frenet-frames. To this end, we recall the following

Definition 2.3 (Frenet-frame). Let $\gamma : I \rightarrow M$ be a curve which is parametrized with respect to arclength. Then its Frenet-frame is defined by the following set of equations
\[ F_1 = \gamma', \quad \nabla_\gamma F_1 = k_1 F_2, \]
\[ \nabla_\gamma F_i = - k_{i-1} F_{i-1} + k_i F_{i+1}, \quad i = 2, \ldots, n - 1, \]
\[ \nabla_\gamma F_n = - k_{n-1} F_{n-1}. \]

For more details on Frenet-frames and their application in geometry we refer to the book [38].

Lemma 2.4. Let $\gamma : I \rightarrow M$ be a smooth curve parametrized by arclength with its associated Frenet-frame. Then the following equations hold
\[ \nabla_\gamma \nabla_\gamma (|\nabla_\gamma \gamma'|^{p-2} \nabla_\gamma \gamma') = ((1 - 2p)k_1^{p-1} k_1' F_1 + ((p - 1)(p - 2)k_1^{p-3} k_1'^2 + (p - 1)k_1^{p-2} k_1'' - k_1^{p-1} k_2 - k_1^{p-1} k_2^2) F_2 + (2(p - 1)k_1^{p-2} k_1' k_2 + k_1^{p-1} k_2') F_3 + (k_1^{p-1} k_2 k_3) F_4, \]
\[ |\nabla_\gamma \gamma'|^{p-2} R^M(\gamma', \nabla_\gamma \gamma') \gamma' = k_1^{p-1} R^M(F_1, F_2) F_1. \]

Proof. This follows by a direct calculation using the Frenet-frame (2.4).

Proposition 2.5. Let $\gamma : I \rightarrow M$ be a smooth curve parametrized by arclength with its associated Frenet-frame. A curve is $p$-biharmonic, that is a critical point of (1.2), if the following system holds
\[ 0 = (1 - 2p)k_1^{p-1} k_1', \]
\[ 0 = (p - 1)(p - 2)k_1^{p-3} k_1'^2 + (p - 1)k_1^{p-2} k_1'' - k_1^{p-1} k_2 + k_1^{p-1} \langle R^M(F_1, F_2) F_2, F_1 \rangle, \]
\[ 0 = 2(p - 1)k_1^{p-2} k_1' k_2 + k_1^{p-1} k_2' - k_1^{p-1} \langle R^M(F_1, F_2) F_1, F_3 \rangle, \]
\[ 0 = k_1^{p-1} k_2 k_3 - k_1^{p-1} \langle R^M(F_1, F_2) F_1, F_4 \rangle, \]
\[ 0 = k_1^{p-1} \langle R^M(F_1, F_2) F_1, F_j \rangle, \quad j = 5, \ldots, n. \]

A curve is $p$-elastic if it solves the system (2.5) neglecting the first equation.

Proof. This is a direct consequence of the previous Lemma.

At this point we have to make the following case distinction:

Corollary 2.6. (1) If $p \neq \frac{1}{2}$ the system (2.1) reduces to the well-known system of equations that characterizes a proper biharmonic curve (see for example [12]), i.e.
\[
\begin{cases}
    k_1 = \text{const} \neq 0, \\
    k_2^2 + k_3^2 = \langle R^M(F_1, F_2) F_2, F_1 \rangle, \\
    k_2^2 = \langle R^M(F_1, F_2) F_1, F_3 \rangle, \\
    k_2 k_3 = \langle R^M(F_1, F_2) F_1, F_4 \rangle, \\
    \langle R^M(F_1, F_2) F_1, F_j \rangle = 0, \quad j = 5, \ldots, n.
\end{cases}
\]
If \( p = \frac{1}{2} \) the first equation of \((2.5)\) is always satisfied and we obtain the system:

\[
\begin{aligned}
\frac{3}{4}k_1^{-1}k_1^2 - \frac{1}{4}k_2^2 - k_3^2 - k_1k_2^2 + k_1\langle R^M(F_1,F_2)F_2,F_1 \rangle = 0, \\
k_2k_3 - \langle R^M(F_1,F_2)F_1,F_4 \rangle = 0, \\
\langle R^M(F_1,F_2)F_1,F_j \rangle = 0, \quad j = 5, \ldots, n.
\end{aligned}
\]

Let us make the following remarks:

**Remark 2.7.** (1) Let us give some further explanations on the common features and differences of the energy functionals \([12]\) and \([13]\). If we start with the energy for \( p \)-biharmonic curves \((1.2)\) and use the Frenet-equations \((2.4)\) then formally the energy functionals \((1.2)\) and \((1.3)\) coincide. However, by employing the Frenet-equations \((2.4)\) in \((1.2)\) we tacitly assume that we are switching to an arclength parametrization and thus, in general, both variational problems will be different.

(2) In the case of \( p \)-biharmonic curves with \( p \neq \frac{1}{2} \) we always have to look for curves of constant geodesic curvature \( k_1 \). This condition comes from the Frenet-equation that describes the tangential direction of the curve \( \gamma \). This is in sharp contrast to \( p \)-elastic curves which allow for solutions with non-constant curvature for all values of \( p \). However, for \( \frac{1}{2} \)-biharmonic curves the tangential equation from the Frenet-frame is automatically satisfied due to our special choice of \( p \) allowing for solutions of non-constant curvature.

(3) Note that also for \( p = \frac{1}{2} \) we always have the constant geodesic curvature solutions. However, there may be additional solutions and understanding the latter is the content of this article.

(4) We would like to point out that, in particular for \( p < 2 \), one has to be very careful if a geodesic also is a solution of the equation for \( p \)-biharmonic curves as there might be a negative power of the geodesic curvature involved. One way to get around this problem is to remove geodesics from the spaces of curves we are working with.

Motivated from the previous remark we make the following definition:

**Definition 2.8.** A curve \( \gamma : I \to M \) is called a proper \( \frac{1}{2} \)-biharmonic curve if it has non-constant geodesic curvature.

**Remark 2.9.** Note that the notion of proper \( \frac{1}{2} \)-biharmonic curve also excludes the case of geodesics as these have constant geodesic curvature zero.

**Remark 2.10.** We can also show that \( p \)-biharmonic curves can have non-constant geodesic curvature without using any Frenet-frame. Testing \((2.2)\) with \( \gamma' \) we get

\[
0 = \langle \nabla_{\gamma'}\nabla_{\gamma'}(|\nabla_{\gamma'}\gamma'|^{p-2}\nabla_{\gamma'}\gamma'),\gamma' \rangle,
\]

from which we immediately deduce

\[
0 = -2\frac{d}{ds}|\nabla_{\gamma'}\gamma'|^p + |\nabla_{\gamma'}\gamma'|^{p-2}\langle \nabla_{\gamma'}\nabla_{\gamma'}\gamma',\nabla_{\gamma'}\gamma' \rangle.
\]

Here, we used that \( \gamma \) is parametrized with respect to arclength. The above equation directly implies that

\[
0 = ( -2 + \frac{1}{p} ) \frac{d}{ds}|\nabla_{\gamma'}\gamma'|^p
\]

and holds trivially for \( p = \frac{1}{2} \).

Let us also make a comment on the Euler-Lagrange method for \( p \)-biharmonic curves which is a straightforward generalization of the corresponding result for biharmonic curves from \([27]\). We define the Lagrangian for \( p \)-biharmonic curves by

\[
\mathcal{L}_{2,p} := |\tau(\gamma)|^p = g(\tau(\gamma),\tau(\gamma))^\frac{p}{p-2}.
\]

(2.6)
Then, either by a direct calculation, or by the general theory of one-dimensional variational problems, we obtain the following

**Theorem 2.11.** The equation for p-biharmonic curves

\[
\nabla_{\gamma'} \nabla_{\gamma'} \left( |\nabla_{\gamma'} \gamma' p-2^2 | \nabla_{\gamma'} \gamma' \right) - |\nabla_{\gamma'} \gamma p-2^2 \mathcal{R}^M (\gamma', \nabla_{\gamma'} \gamma') \gamma' = 0
\]

is equivalent to the system of Euler-Lagrange equations

\[
\frac{d^2}{ds^2} \left( \frac{\partial \mathcal{L}_{2,p}}{\partial \gamma''} \right) - \frac{d}{ds} \left( \frac{\partial \mathcal{L}_{2,p}}{\partial \gamma'} \right) + \frac{\partial \mathcal{L}_{2,p}}{\partial \gamma} = 0, \quad k = 1, \ldots, n,
\]

where the Lagrangian \( \mathcal{L}_{2,p} \) is defined in (2.6).

2.3. The second variation formula. In this subsection we calculate the second variation of the energy of p-biharmonic curves (1.2).

Again, we fix a small number \( \varepsilon > 0 \) and let \( \gamma_t : (-\varepsilon, \varepsilon) \times I \to M \) be a variation of the curve \( \gamma \), satisfying \( \frac{\partial}{\partial t} \gamma(t, 0) = \eta \), where \( \eta \in \Gamma(\gamma^* TM) \) and \( I \subset \mathbb{R} \) is a closed interval.

**Proposition 2.12.** The second variation of (1.2), evaluated at a critical point, that is a solution of (2.2), is given by the following expression

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \frac{1}{p} \int_I |\nabla_{\gamma_t} \gamma_t|^p ds
\]

\[
= \int_I \langle \eta, \mathcal{R}^M(\eta, \gamma') \nabla_{\gamma'} \left( |\nabla_{\gamma'} \gamma' p-2^2 | \nabla_{\gamma'} \gamma' \right) \rangle ds - 2 \int_I |\nabla_{\gamma'} \gamma' p-2^2 \mathcal{R}^M(\nabla_{\gamma'} \eta, \nabla_{\gamma'} \gamma') \gamma', \eta \rangle ds
\]

\[
+ (p - 2) \int_I |\nabla_{\gamma'} \gamma' p-4^2 | \langle \nabla_{\gamma'} \nabla_{\gamma'} \eta, \nabla_{\gamma'} \gamma' \rangle |^2 ds
\]

\[
+ 2(p - 2) \int_I |\nabla_{\gamma'} \gamma' p-4^2 (\nabla_{\gamma'} \nabla_{\gamma'} \eta, \nabla_{\gamma'} \gamma') \mathcal{R}^M(\eta, \gamma'), \nabla_{\gamma'} \gamma' \rangle ds
\]

\[
+ \int_I |\nabla_{\gamma'} \gamma' p-2^2 | \langle \nabla_{\gamma'} \nabla_{\gamma'} \eta \rangle |^2 ds + 2 \int_I |\nabla_{\gamma'} \gamma' p-2^2 \mathcal{R}^M(\eta, \gamma'), \nabla_{\gamma'} \nabla_{\gamma'} \eta \rangle ds
\]

\[
+ (p - 2) \int_I |\nabla_{\gamma'} \gamma' p-4^2 | \langle \mathcal{R}^M(\gamma, \nabla_{\gamma'} \gamma'), \eta \rangle |^2 ds
\]

\[
- \int_I |\nabla_{\gamma'} \gamma' p-2^2 (\nabla_{\eta} \mathcal{R}^M(\gamma', \nabla_{\gamma'} \gamma' \gamma), \eta) ds + \int_I |\nabla_{\gamma'} \gamma' p-2^2 | \mathcal{R}^M(\gamma', \eta) \gamma' |^2 ds
\]

\[
- \int_I |\nabla_{\gamma'} \gamma' p-2^2 \mathcal{R}^M(\gamma', \nabla_{\gamma'} \gamma') \nabla_{\gamma'} \eta, \eta \rangle ds.
\]

**Proof.** Let us recall the first variation formula derived in Lemma 2.1 that is

\[
\frac{d}{dt} \bigg|_{t=0} \frac{1}{p} \int_I |\nabla_{\gamma_t} \gamma_t|^p ds = \int_I \langle \eta, \nabla_{\gamma'} \gamma' \left( |\nabla_{\gamma'} \gamma' p-2^2 | \nabla_{\gamma'} \gamma' \right) - |\nabla_{\gamma'} \gamma' p-2^2 \mathcal{R}^M(\gamma', \nabla_{\gamma'} \gamma') \gamma' \rangle ds
\]

and the identity

\[
\frac{\partial}{\partial t} \bigg|_{t=0} \nabla_{\gamma_t} \gamma_t' = \nabla_{\gamma'} \nabla_{\gamma'} \eta + \mathcal{R}^M(\eta, \gamma') \gamma'.
\]

Hence, we have

\[
\frac{d}{dt} \bigg|_{t=0} |\nabla_{\gamma_t} \gamma_t|^{p-2} = (p - 2) |\nabla_{\gamma'} \gamma' p-4^2 \left( \langle \nabla_{\gamma'} \nabla_{\gamma'} \eta, \nabla_{\gamma'} \gamma' \rangle + \mathcal{R}^M(\eta, \gamma') \gamma' \right) \rangle.
\]
By a direct calculation we find

\[
\frac{\nabla}{\partial t}|_{t=0} \left( \nabla_{\gamma'} \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \nabla_{\gamma'} \gamma' \right) \right)
\]

\[= R^M(\eta, \gamma') \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \nabla_{\gamma'} \gamma' \right) + \nabla_{\gamma'} \left( \nabla_{\gamma'} \left| \nabla_{\gamma'} \gamma' \right|^{p-2} R^M(\eta, \gamma') \nabla_{\gamma'} \gamma' \right)
\]

\[+ \nabla_{\gamma'} \nabla_{\gamma'} \left( \frac{\nabla}{\partial t}|_{t=0} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \nabla_{\gamma'} \gamma' \right) \right)
\]

\[= R^M(\eta, \gamma') \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \nabla_{\gamma'} \gamma' \right) + \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} R^M(\eta, \gamma') \nabla_{\gamma'} \gamma' \right)
\]

\[+ (p-2) \nabla_{\gamma'} \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-4} \left( \nabla_{\gamma'} \nabla_{\gamma'} \eta, \nabla_{\gamma'} \gamma' \right) \nabla_{\gamma'} \gamma' + (R^M(\eta, \gamma') \nabla_{\gamma'} \gamma') \nabla_{\gamma'} \gamma' \right)
\]

\[+ \nabla_{\gamma'} \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} (\nabla_{\gamma'} \nabla_{\gamma'} \eta + R^M(\eta, \gamma') \nabla_{\gamma'} \gamma') \right).
\]

In addition, we find

\[
\frac{\nabla}{\partial t}|_{t=0} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} R^M(\gamma', \nabla_{\gamma'} \gamma') \right)
\]

\[= (p-2) \left| \nabla_{\gamma'} \gamma' \right|^{p-4} \left( \langle \nabla_{\gamma'} \nabla_{\gamma'} \eta, \nabla_{\gamma'} \gamma' \rangle + R^M(\eta, \gamma') \nabla_{\gamma'} \gamma' \right) R^M(\gamma', \nabla_{\gamma'} \gamma') \gamma'
\]

\[+ \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \left( \langle \nabla_{\gamma} R^M(\gamma', \nabla_{\gamma'} \gamma') \rangle \gamma' + R^M(\nabla_{\gamma} \eta, \nabla_{\gamma'} \gamma') \gamma' \right.
\]

\[+ \left. R^M(\gamma', \nabla_{\gamma'} \eta) \gamma' + R^M(\gamma', \nabla_{\gamma'} \gamma') \nabla_{\gamma'} \eta \right).
\]

Combining the previous equations we get

\[
\frac{d^2}{dt^2}|_{t=0} \int_I \left| \nabla_{\gamma'} \gamma' \right|^p ds
\]

\[= \int_I \langle \eta, R^M(\eta, \gamma') \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \nabla_{\gamma'} \gamma' \right) \rangle ds + \int_I \langle \eta, \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} R^M(\eta, \gamma') \nabla_{\gamma'} \gamma' \right) \rangle ds
\]

\[+ (p-2) \int_I \langle \eta, \nabla_{\gamma'} \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-4} \left( \nabla_{\gamma'} \nabla_{\gamma'} \eta, \nabla_{\gamma'} \gamma' \right) \nabla_{\gamma'} \gamma' \rangle ds
\]

\[+ (p-2) \int_I \langle \eta, \nabla_{\gamma'} \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-4} R^M(\eta, \gamma') \nabla_{\gamma'} \gamma' \rangle \nabla_{\gamma'} \gamma' \rangle ds
\]

\[+ \int_I \langle \eta, \nabla_{\gamma'} \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \nabla_{\gamma'} \eta \right) \rangle ds + \int_I \langle \eta, \nabla_{\gamma'} \nabla_{\gamma'} \left( \left| \nabla_{\gamma'} \gamma' \right|^{p-2} R^M(\eta, \gamma') \right) \rangle ds
\]

\[- (p-2) \int_I \left| \nabla_{\gamma'} \gamma' \right|^{p-4} \langle \nabla_{\gamma'} \nabla_{\gamma'} \eta, \nabla_{\gamma'} \gamma' \rangle \langle R^M(\gamma', \nabla_{\gamma'} \gamma') \gamma', \eta \rangle ds
\]

\[- (p-2) \int_I \left| \nabla_{\gamma'} \gamma' \right|^{p-4} \langle R^M(\eta, \gamma') \gamma', \nabla_{\gamma'} \gamma' \rangle \langle R^M(\gamma', \nabla_{\gamma'} \gamma') \gamma', \eta \rangle ds
\]

\[- \int_I \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \langle \nabla_{\gamma} R^M(\gamma', \nabla_{\gamma'} \gamma') \rangle \gamma', \eta \rangle ds - \int_I \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \langle R^M(\nabla_{\gamma} \eta, \nabla_{\gamma'} \gamma') \rangle \gamma', \eta \rangle ds
\]

\[- \int_I \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \langle R^M(\gamma', R^M(\eta, \gamma') \gamma', \eta \rangle ds - \int_I \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \langle R^M(\gamma', \nabla_{\gamma} \nabla_{\gamma} \eta) \gamma', \eta \rangle ds
\]

\[- \int_I \left| \nabla_{\gamma'} \gamma' \right|^{p-2} \langle R^M(\gamma', \nabla_{\gamma'} \gamma') \rangle \nabla_{\gamma'} \eta \rangle ds
\]
\[ \sum_{i=1}^{13} J_i. \]

Now, we will manipulate the terms on the right-hand side of (2.9). First, we note that
\[ J_2 + J_{10} = -2 \int_I |\nabla_{\gamma'}^p|^p - 2 \langle R^M(\nabla_{\gamma'}^p, \nabla_{\gamma'}^p) \gamma', \eta \rangle ds. \]

Concerning the \( J_3 \)-term integration by parts yields
\[ J_3 = (p - 2) \int_I |\nabla_{\gamma'}^0|^p - 4 |\langle \nabla_{\gamma'}\nabla_{\gamma'}^0, \nabla_{\gamma'}^0 \rangle |^2 ds. \]

Moreover, it is easy to see, using integration by parts again, that
\[ J_4 = (p - 2) \int_I |\nabla_{\gamma'}^0|^p - 4 |\langle \nabla_{\gamma'}\nabla_{\gamma'}^0, \nabla_{\gamma'}^0 \rangle |^2 ds. \]

Hence, we can conclude that
\[ J_4 + J_7 = 2(p - 2) \int_I |\nabla_{\gamma'}^0|^p - 4 |\langle \nabla_{\gamma'}\nabla_{\gamma'}^0, \nabla_{\gamma'}^0 \rangle |^2 ds. \]

For the \( J_5 \)-term we get
\[ J_5 = \int_I |\nabla_{\gamma'}^0|^p - 2 |\nabla_{\gamma'}| ds. \]

In addition, similar arguments as before show that
\[ J_6 + J_{12} = 2 \int_I |\nabla_{\gamma'}^0|^p - 2 \langle R^M(\eta, \gamma') \gamma', \nabla_{\gamma'}^0 \eta \rangle ds. \]

Regarding the \( J_8 \)-term, we use the symmetries of the Riemannian curvature tensor and find
\[ \langle R^M(\eta, \gamma') \gamma', \nabla_{\gamma'}^0 \rangle \langle R^M(\gamma', \nabla_{\gamma'}^0) \gamma', \eta \rangle = -|\langle R^M(\gamma, \nabla_{\gamma'}^0) \gamma', \eta \rangle|^2. \]

Concerning the \( J_11 \)-term, we use
\[ \langle R^M(\gamma', R^M(\eta, \gamma') \gamma', \eta \rangle = -|R^M(\gamma', \eta \gamma')|^2. \]

Inserting these identities into (2.10) completes the proof. \( \square \)

**Remark 2.13.** Note that for \( p = 2 \) equation (2.10) coincides with the calculation of Jiang [21] for the second variation of the bienergy for maps between arbitrary Riemannian manifolds.

In the following we will assume that the manifold \( M \) is a space form, that is it has a metric of constant curvature \( K \). In this case the Riemannian curvature tensor can be written in the following form
\[ R^M(X, Y)Z = K(\langle Z, Y \rangle X - \langle Z, X \rangle Y) \quad (2.10) \]

for vector fields \( X, Y, Z \).

**Proposition 2.14.** Let \( \gamma : I \to M \) be a \( p \)-biharmonic curve and suppose that \( M \) is a space form of constant curvature \( K \). Then, the second variation of (1.2) evaluated at a critical point, that
is a solution of (2.2), simplifies as follows
\begin{align}
\frac{d^2}{dt^2} |\nabla_\gamma \gamma'|^p |\Pi| ds = & \quad (p - 2) \int |\nabla_\gamma \gamma'|^{p-2} \left( \langle \nabla_\gamma \gamma' \nabla_\gamma \gamma \nabla_\gamma \gamma' \rangle \right) \frac{ds}{\Pi} \\
& + (p - 2) K^2 \int |\nabla_\gamma \gamma'|^{p-2} \left( \langle \eta, \gamma' \rangle \langle \gamma', \nabla_\gamma \gamma' \rangle \right) \frac{ds}{\Pi} \\
& + K^2 \int |\nabla_\gamma \gamma'|^{p-2} \left( \langle \eta, \gamma' \rangle \langle \nabla_\gamma \gamma', \eta \rangle \right) \frac{ds}{\Pi} \\
& + 2K \int |\nabla_\gamma \gamma'|^{p-2} \left( \langle \gamma', \nabla_\gamma \gamma \rangle \langle \gamma', \nabla_\gamma \gamma \rangle \right) \frac{ds}{\Pi} \\
& + 2(p - 2) K \int |\nabla_\gamma \gamma'|^{p-2} \left( \langle \gamma', \nabla_\gamma \gamma \rangle \langle \gamma', \nabla_\gamma \gamma \rangle \right) \frac{ds}{\Pi} \\
& - K \int |\nabla_\gamma \gamma'|^{p-2} \left( \langle \gamma', \nabla_\gamma \gamma \rangle \langle \gamma', \nabla_\gamma \gamma \rangle \right) \frac{ds}{\Pi}.
\end{align}

**Proof.** The claim follows by inserting the definition of the curvature tensor of a space form (2.10) into (2.8) and a direct calculation. Note that the term proportional to $\nabla R$ drops out due to the assumption of constant curvature.

\[\square\]

### 2.4. Qualitative aspects of $p$-biharmonic curves

It is well-known that every biharmonic curve on a manifold with negative curvature must be a geodesic. In the case of $p$-biharmonic maps we have a similar result, see for example [19]. Here, we prove the following

**Theorem 2.15.** Let $\gamma: I \to M$ be a $p$-biharmonic curve parametrized by arclength. If $M$ has non-positive curvature then $\gamma$ must be a geodesic.

**Proof.** From the second equation of the system (2.5) we get
\[k_1^2 + k_2^2 = \langle R^M(F_1, F_2)F_2, F_1 \rangle\]
which immediately completes the proof. \[\square\]

**Remark 2.16.** We can also show the above claim by a different method in the case that $I \subset \mathbb{R}$ is a closed interval. We test the equation for $p$-biharmonic curves (2.2) with $|\nabla_\gamma \gamma'|^{p-2} \nabla_\gamma \gamma'$ (assuming that $\gamma$ is non-geodesic) and obtain
\[\int |\nabla_\gamma \gamma'|^{p-2} \langle \nabla_\gamma \gamma', \nabla_\gamma \gamma \rangle \, ds = 0.\]
Using the assumption of $M$ having negative sectional curvature together with integration by parts we can deduce that
\[\int |\nabla_\gamma \gamma'|^{2p-4} \langle R^M(\gamma', \nabla_\gamma \gamma'), \gamma' \rangle \, ds = 0.\]
which implies that $\int |\nabla_\gamma \gamma'|^{p-2} \nabla_\gamma \gamma' \, ds \leq 0$. Testing this equation with $\gamma'$ and using integration by parts one more time we find
\[0 = \int |\nabla_\gamma \gamma'|^p \, ds\]
from which we deduce $\nabla_\gamma \gamma' = 0$ completing the proof.

**Remark 2.17.** We would like to point out once more that one has to be very careful with results such as Theorem 2.15 since there might be negative powers of the geodesic curvature involved.
3. \( p \)-BIHARMONIC CURVES IN TWO DIMENSIONS

In this section we study \( p \)-biharmonic curves on two-dimensional domains. We will pay special attention to the Euclidean plane \( \mathbb{R}^2 \) modelling the zero curvature case \( K = 0 \), the round sphere \( S^2 \) with constant curvature \( K = 1 \) and hyperbolic surfaces with constant curvature \( K = -1 \).

We have already seen that for \( p \neq \frac{1}{2} \) the solutions of the equation for \( p \)-biharmonic curves are precisely biharmonic curves, see [13], such that we do not have to further investigate this case. On the other hand, the case of \( p = \frac{1}{2} \) has already been studied in the context of \( p \)-elastic curves on surfaces and critical points of (1.4) in [1, 2, 5, 6]. Our results are very close to the ones obtained in that references.

On a two-dimensional manifold we have the Frenet-frame \( \{ T, N \} \) satisfying the following equations

\[
\nabla_T T = kN, \quad \nabla_T N = -kT
\]

(3.1)
corresponding to (2.4) where we relabelled \( k_1 = k, F_1 = T, F_2 = N \) in order to be consistent with the notation usually employed in the case of a curve on a surface.

Writing the equation for \( p \)-biharmonic curves in terms of the Frenet-frame (3.1) we obtain the system

\[
(p-1)(p-2)k^{-1}k'^2 N + (p-1)k''N - (2p-1)kk'T - k^3 N - kRM(T, N)T = 0.
\]

Taking the scalar product with \( T \) and \( N \) we get the two equations

\[
(2p-1)kk' = 0,
\]

\[
(p-1)(p-2)k^{-1}k'^2 + (p-1)k'' - k^3 + kK = 0.
\]

In the case that \( p \neq \frac{1}{2} \) we get the well-known system

\[
k = \text{const}, \quad k^2 = K
\]

(3.2)
which characterizes biharmonic curves on surfaces [13].

However, in the case that \( p = \frac{1}{2} \) the first equation is satisfied trivially and we get the following

**Proposition 3.1.** Let \( \gamma: I \to M \) be a curve parametrized by arclength and \( M \) a surface with Gauss curvature \( K \). Then the curve \( \gamma \) is proper \( \frac{1}{2} \)-biharmonic if its geodesic curvature \( k \) is a non-constant solution of the following ordinary differential equation

\[
3\frac{k^2}{4} - \frac{1}{2}k''k - k^4 + k^2K = 0.
\]

(3.3)

It can be directly seen that \( k^2 = K = \text{const} \) gives a solution of (3.3). Hence, in the following we will assume that \( k \neq \text{const} \), then the following identity holds

\[
k^\frac{5}{2}(k^{-\frac{1}{2}})^{'''} = \frac{3}{4}k^2 - \frac{1}{2}k''k
\]

(3.4)
and (3.3) simplifies to

\[
(k^{-\frac{1}{2}})'' - k^\frac{3}{2} + k^{-\frac{1}{2}}K = 0.
\]

Setting \( f := k^{-\frac{1}{2}} \), which we assume to be non-constant, and multiplying by \( f' \) we find

\[
0 = (f'^2)' + K(f'^2)' + \left(\frac{1}{f^2}\right)'.
\]

This allows us to obtain the simpler equation

\[
c_1 = f'^2 + Kf'^2 + \frac{1}{f^2}
\]

for some integration constant \( c_1 \in \mathbb{R} \).

To further simplify this equation we define \( h(s) := \frac{1}{2}f'^2(s) \) such that we are left with the differential equation

\[
0 = h'^2 + 4K\frac{h'}{2} - 2c_1h + 1.
\]

(3.5)
The solutions of \((3.5)\) then give the geodesic curvature via the assignment \(k(s) = \frac{1}{2h(s)}\). The next theorem discusses the solutions of \((3.5)\) in the three cases \(K = 1, 0, -1\) which is very similar to the mathematical analysis carried out in \([1, 2, 5, 6]\) in the context of generalized elastic curves and is included in \([4, \text{Proposition 3.1}]\).

**Theorem 3.2.** Let \(\gamma: I \to M\) be a smooth curve parametrized by arclength and \(M\) a surface with constant curvature \(K\).

1. If \(M = \mathbb{R}^2\) with the flat metric the curve \(\gamma\) is proper \(\frac{1}{2}\)-biharmonic if its geodesic curvature is given by
   \[k(s) = \frac{c_1}{c_1^2(c_2 + s)^2 + 1},\]  
   where \(c_1, c_2 \in \mathbb{R}\).

2. If \(M = S^2\) with the round metric with curvature \(K = 1\) the curve \(\gamma\) is proper \(\frac{1}{2}\)-biharmonic if its geodesic curvature is given by
   \[k(s) = \frac{2}{c_1 + \sqrt{c_1^2 - 4\sin 2(c_3 + s)}},\]  
   where \(c_1, c_3 \in \mathbb{R}\).

3. If \(M\) is a hyperbolic surface with a metric of constant curvature \(K = -1\) the curve \(\gamma\) is proper \(\frac{1}{2}\)-biharmonic if its geodesic curvature is given by
   \[k(s) = \frac{4e^{2(c_4 + s)}}{(c_1 - e^{2(c_4 + s)})^2 + 4},\]  
   where \(c_1, c_4 \in \mathbb{R}\).

**Remark 3.3.** Note that by shifting the curve parameter \(s\) appropriately the constants \(c_2, c_3, c_4\) can assumed to be zero.

**Proof.** In the case of \(M = \mathbb{R}^2\) with the flat Euclidean metric we have \(K = 0\) and \((3.5)\) simplifies to
\[0 = h'^2 - 2c_1 h + 1.\]
This equation can be integrated as
\[h(s) = \frac{1 + c_1^2(c_2 + s)^2}{2c_1},\]
where the integration constant \(c_2 \in \mathbb{R}\) is determined via the initial data.

In the case of \(M = S^2\) with the round metric with \(K = 1\) equation \((3.5)\) acquires the form
\[0 = h'^2 + 4h^2 - 2c_1 h + 1.\]
This equation can again be integrated directly as
\[h(s) = \frac{c_1}{4} + \frac{1}{4} \sqrt{c_1^2 - 1\sin 2(c_3 + s)},\]
where, again, the integration constant \(c_3 \in \mathbb{R}\) is determined by the initial data.

On a hyperbolic surface we can always choose a metric of constant Gaussian curvature \(K = -1\) such that \((3.5)\) reduces to
\[0 = h'^2 - 4h^2 - 2c_1 h + 1.\]
A solution of this equation is given by
\[h(s) = \frac{1}{8} e^{-2(c_4 + s)}((c_1 - e^{2(c_4 + s)})^2 + 4),\]
with the integration constant \(c_4 \in \mathbb{R}\). Now, the proof is complete. \(\square\)
3.1. Existence results via magnetic geodesics. In the case of $M = S^2$, or $M$ being a hyperbolic surface we can make use of the results obtained for (closed) magnetic geodesics by Schneider [33, 34] to obtain an existence result for proper $\frac{1}{2}$-biharmonic curves.

Magnetic geodesics describe the trajectory of a particle evolving in a Riemannian manifold $M$ under the influence of an external magnetic field. However, there also is a more geometric point of view on such curves by asking: Given a function $k: M \to \mathbb{R}$, does there exist a (closed) curve $\gamma$ on $M$ with geodesic curvature $k$? In order to answer this question, or to find a magnetic geodesic, one has to solve the equation

$$\nabla_\gamma \gamma' = k J_g(\gamma'),$$

(3.9)

where $J_g$ represents the rotation by $\pi/2$ in $TM$ measured with respect to the Riemannian metric $g$, which corresponds to the Frenet-equations (3.1). Note that in the mathematics literature on magnetic geodesics one usually thinks of magnetic geodesics as closed curves without explicitly mentioning this.

Now, we recall the following result (see [33, Theorem 1.3])

**Theorem 3.4** (Schneider, 2011). Consider $S^2$ with the round metric of constant curvature $K = 1$. For any positive constant $k_0 > 0$ there exists a positive smooth function $k: S^2 \to \mathbb{R}$, which can be chosen arbitrarily close to $k_0$ such that there are exactly two simple solutions of (3.9).

In other words, this means that on the sphere with the round metric we can solve equation (3.9) for any prescribed curvature $k$.

Recall that due to Theorem 3.2 case (2), the geodesic curvature of a proper $\frac{1}{2}$-biharmonic curve on $S^2$ with the round metric is given by

$$k(s) = \frac{2}{c_1 + \sqrt{c_1^2 - 4 \sin 2s}},$$

where $c_1 \in \mathbb{R}$.

Hence, if the integration constant $c_1$ satisfies $c_1 > 2$ we have $k(s) > 0$ such that we can apply Theorem 3.4 to obtain an existence result for proper $\frac{1}{2}$-biharmonic curves on $S^2$.

In addition, we recall the following result (see [34, Theorem 1.2]):

**Theorem 3.5** (Schneider, 2012). Let $(M, g)$ be a closed oriented surface with negative Euler characteristic $\chi(M)$ and $k: M \to \mathbb{R}$ be a positive function. Assume that there exists $K_0 > 0$ such that $k$ and the Gaussian curvature $K_g$ of $(M, g)$ satisfy

$$k > (K_0)^{\frac{1}{2}} \text{ and } K_g \geq -K_0.$$  

(3.10)

Then, there exists an Alexandrov embedded curve $\gamma \in C^2(S^1, M)$ that solves (3.9) and the number of solutions is at least $-\chi(M)$ provided they are all non-degenerate.

Recall that due to Theorem 3.2 case (3), the geodesic curvature of a proper $\frac{1}{2}$-biharmonic curve on a hyperbolic surface with constant curvature $K = -1$ is given by

$$k(s) = \frac{4e^{2s}}{(c_1 - e^{2s})^2 + 4},$$

where $c_1 \in \mathbb{R}$. In order to satisfy the conditions (3.10) we could assume that $s \geq 0$ or otherwise we would need to require that $c_1^2$ is sufficiently large. Then, we can directly apply Theorem 3.5 to get an existence result for proper $\frac{1}{2}$-biharmonic curves on hyperbolic surfaces.

**Remark 3.6.** In [33, 34] there are also existence results for magnetic geodesics on $S^2$ and hyperbolic surfaces in case that these do not carry the standard metric of constant curvature. In the case of constant Gauss curvature $K$ there exist many results on $\frac{1}{2}$-elastic curves we could refer to [1, 2, 3, 5, 17] in order to obtain the curvature for proper $\frac{1}{2}$-biharmonic curves. Note that these results rely on explicitly solving the equations for the curvature of the curve (3.3). However, in the case of a surface with non-constant Gauss curvature $K$ it might be very
Note that by choosing \( N \) the unit normal \( \nabla \), By the uniformization theorem every closed surface admits a metric of constant curvature\( \). Proof. Assume that \( \gamma \) is parametrized by arclength such that we can use the Frenet-equations (3.1). Inserting this choice into (2.11) we find

\[
\text{Hess}(E_{2,p}(\gamma))(\eta,\eta) := \frac{d^2}{dt^2} \big|_{t=0} E_{2,p}(\gamma(t)),
\]

where \( \eta := \frac{\gamma''}{\|\gamma''\|} \big|_{t=0} \in \Gamma(\gamma^*TM) \) denotes the variational vector field.

Proposition 3.8. Let \( \gamma : I \to M \) be a \( p \)-biharmonic curve parametrized by arclength and suppose that \( M \) is a closed surface of constant curvature \( K \). Then the Hessian of (1.2) evaluated at the unit normal \( N \) of the surface is given by

\[
\text{Hess}(E_{2,p}(\gamma))(N,N) = \int_I \left( (p-1)k^{p+2} + k^2k^{p-2} + (p-1)K^2k^{p-2} - (2p+2)Kk^p \right) ds,
\]

where \( k \) is the geodesic curvature of the curve \( \gamma \).

Proof. By the uniformization theorem every closed surface admits a metric of constant curvature \( K \). Hence, we can apply the second variation formula derived in Proposition 2.14 and choose the unit normal \( N \) as variational vector field.

Let us manipulate the terms of the above equation which do not have a sign with the help of the Frenet-frame (3.1) as

\[
\int_I \langle \gamma', \nabla_{\gamma'} (|\gamma'|^{p-2} \nabla_{\gamma'} \gamma') \rangle ds = - \int_I k^p ds,
\]

\[
\int_I \langle \nabla_{\gamma'} \gamma' |^{p-2} \langle \nabla_{\gamma'} \gamma', N \rangle \rangle ds = - \int_I k^p ds,
\]

\[
\int_I \langle \nabla_{\gamma'} \gamma', N \rangle \rangle |\gamma'|^{p-4} \langle N, \nabla_{\gamma'} \gamma' \rangle ds = - \int_I k^p ds,
\]

\[
\int_I \langle \nabla_{\gamma'} \gamma', \nabla_{\gamma'} \gamma' \rangle |\gamma'|^{p-2} \langle N, \nabla_{\gamma'} \gamma' \rangle ds = - \int_I k^p ds.
\]
Remark 3.11. From (3.12) we can follow that non-geodesic surfaces are characterized by different from the stability of the curve, calculates the second variation and evaluates the associated quadratic form on.

For $p$-biharmonic curves, see [17] and [1]. While this may seem strange at first glance, there is a straightforward explanation for this phenomena: In the case of $p$-biharmonic curves one calculates the second variation of the $L^p$-norm of $\nabla_\gamma \gamma'$ and inserts the normal $N$ in the quadratic form associated with the second variation, while in the case of $p$-elastic curves one starts with the $L^p$-norm of the curvature of the curve, calculates the second variation and evaluates the associated quadratic form on $N$. As these procedures are both slightly different in nature, one gets different results in the end.

Theorem 3.10. Let $\gamma: I \to M$ be a $p$-biharmonic curve parametrized by arclength and suppose that $M$ is a closed surface of constant curvature $K$. Consider the second variation of (1.2) evaluated at a critical point for a variation in the normal direction. Then the following statements hold:

1. For $p \neq \frac{1}{2}$ we get
   \[
   \text{Hess}(E_{2,p}(\gamma))(N, N) = -4K |I|^p.
   \]  

(3.12)

2. For $p = \frac{1}{2}$ we get
   \[
   \text{Hess}(E_{2,p}(\gamma))(N, N) = \int_I \left( -\frac{1}{2} k_1^2 + k^2 k^{-\frac{3}{2}} - \frac{1}{2} K^2 k^{-\frac{3}{2}} - 3Kk^\frac{1}{2} \right) ds.
   \]  

(3.13)

Proof. First, we consider the case $p \neq \frac{1}{2}$. Under this assumption all $p$-biharmonic curves on surfaces are characterized by $k^2 = K = \text{const}$. Inserting into (3.11) we obtain the claim after a direct calculation.

For $p = \frac{1}{2}$ we get the statement directly from (3.11).

Let us make the following remarks concerning the previous Theorem:

Remark 3.11. (1) From (3.12) we can follow that non-geodesic $p$-biharmonic curves for $p \neq \frac{1}{2}$ on a surface with constant, positive curvature will always be unstable.

(2) In the case of $p = \frac{1}{2}$ we can insert the solutions obtained in Theorem 3.2 into equation (3.13) in order to check the sign of the second variation. Unfortunately, this leads to a number of complicated expressions and it does not seem possible to give a definite answer.

(3) The instability of $p$-elastic curves on $S^2$ was recently shown in [17] and one should thus expect that proper $\frac{1}{2}$-biharmonic curves on $S^2$ will be unstable as well.
We have to distinguish the two cases:

1. Curve which is parametrized with respect to arclength together with its associated Frenet-frame \( \{T, N, B\} \) along the curve \( \gamma \) which satisfies the following Frenet-equations

\[
\nabla T T = kN, \quad \nabla T N = -kT + \tau B, \quad \nabla T B = -\tau N. \tag{4.1}
\]

2. Proposition 4.1. This directly implies the following

\[
\text{Hess}(E_2(\phi))(\tau(\phi), \tau(\phi)) = -4K|\tau(\phi)|^4\text{vol}(M) \leq 0
\]

which clearly shows that such biharmonic maps are unstable.

This result is in perfect agreement with (3.12): If we choose \( p = 2 \), the variational vector field \( \eta = \tau(\gamma) = kN \) and the equation for biharmonic curves on surfaces \( k^2 = K = \text{const} \) then we would get the same result.

### 4. \( p \)-biharmonic curves on three-dimensional manifolds

In this section we extend the previous analysis to the case of a three-dimensional manifold, again assuming that we have a metric of constant curvature.

We choose the standard Frenet-frame \( \{T, N, B\} \) along the curve \( \gamma \) which satisfies the following Frenet-equations

\[
\nabla T T = kN, \quad \nabla T N = -kT + \tau B, \quad \nabla T B = -\tau N. \tag{4.1}
\]

Here, \( k \) represents the geodesic curvature of the curve \( \gamma \) and \( \tau \) denotes the torsion of the curve.

Note that we again relabelled the Frenet-frame \( \{T, N, B\} \) in order to be consistent with the notation usually applied in the three-dimensional setting.

Using the above Frenet-equations the condition for a curve to be \( p \)-biharmonic is given by

\[
0 = ((p - 1)(p - 2)k^{p-3}k^2 + (p - 1)k^{p-2}k'' - k^{p+1} - k^{p-1}k^2)N \tag{4.2}
\]

\[
+ ((1 - 2p)k^{p-1}k')T
\]

\[
+ (2(p - 1)k^{p-2}k\tau + k^{p-1}k')B
\]

\[
- k^{p-1}\text{Hess}(M, T)N, T
\]

This directly implies the following

**Proposition 4.1.** Let \( \gamma: I \to M \), where \( M \) is a three-dimensional Riemannian manifold, be a curve which is parametrized with respect to arclength together with its associated Frenet-frame \( \{T, N, B\} \). Then \( \gamma \) is \( p \)-biharmonic if the following equations hold

\[
0 = ((p - 1)(p - 2)k^{p-3}k^2 + (p - 1)k^{p-2}k'' - k^{p+1} - k^{p-1}k^2)N \tag{4.2}
\]

\[
+ ((1 - 2p)k^{p-1}k')T
\]

\[
+ (2(p - 1)k^{p-2}k\tau + k^{p-1}k')B
\]

\[
- k^{p-1}\text{Hess}(M, T)N, T
\]

We have to distinguish the two cases:

1. If \( p \neq \frac{1}{2} \) the curve \( \gamma \) is \( p \)-biharmonic if

\[
k = \text{const} \neq 0, \quad \tau' = \langle R^M(T, N)T, B \rangle, \quad k^2 + \tau^2 = \langle R^M(T, N)N, T \rangle. \tag{4.3}
\]

2. If \( p = \frac{1}{2} \) the curve \( \gamma \) is \( p \)-biharmonic if the following system holds

\[
0 = \frac{3}{4}k^2 - \frac{1}{2}kk'' - k^4 - \tau^2k^2 + k^2\langle R^M(T, N)N, T \rangle, \tag{4.4}
\]

\[
0 = -k'\tau + k\tau' - k\langle R^M(T, N)T, B \rangle.
\]

Again, we realize that a \( p \)-biharmonic curve in a three-dimensional manifold is biharmonic in general but for \( p = \frac{1}{2} \) it may also admit additional solutions corresponding to generalized elastic curves. Generalized elastic curves in three-dimensional space forms have already been classified in \cite[Section 3]{[4]}, but we will again give a short proof for the corresponding results on proper \( \frac{1}{2} \)-biharmonic curves using our setup, which is a special case of \cite[Proposition 3.1]{[4]}.

**Theorem 4.2.** Let \( \gamma: I \to M \) be a smooth curve parametrized by arclength and \( M \) a three-dimensional Riemannian manifold with constant curvature \( K \).
If \( M = S^3 \) with the round metric of curvature \( K = 1 \) the curve \( \gamma \) is proper \( \frac{1}{2} \)-biharmonic if

\[
k(s) = \frac{2}{b_2 + \sqrt{b_2^2 - 4 - 4b_1^2} \sin 2(s + b_3)}, \quad k(s) = b_1 \tau(s),
\]

where \( b_1, b_2, b_3 \in \mathbb{R} \).

(2) If \( M = \mathbb{R}^3 \) with the flat metric the curve \( \gamma \) is proper \( \frac{1}{2} \)-biharmonic if

\[
k(s) = \frac{b_4}{b_5^2(s - b_3)^2 + 1 + b_1^2}, \quad k(s) = b_1 \tau(s),
\]

where \( b_1, b_4, b_5 \in \mathbb{R} \).

(3) If \( M = \mathbb{H}^3 \) with the metric of curvature \( K = -1 \) the curve \( \gamma \) is proper \( \frac{1}{2} \)-biharmonic if

\[
k(s) = \frac{4e^{-2(b_4 + s)}}{(b_4 + e^{-2(b_5 + 2)})^2 + 4 + 4b_1^2}, \quad k(s) = b_1 \tau(s),
\]

where \( b_1, b_6, b_7 \in \mathbb{R} \).

**Remark 4.3.** As in the two-dimensional case the constants \( b_3, b_5, b_7 \) can be set to zero by shifting the curve parameter \( s \).

**Proof.** By assumption the manifold \( M \) is a three-dimensional space form such that the second equation of (4.4) can be written as

\[
(\log k)' = (\log \tau)'
\]

and is solved by

\[
k(s) = b_1 \tau(s),
\]

where \( b_1 \in \mathbb{R} \) is an integration constant. Then, assuming that we are looking for a non-geodesic solution, the first equation of (4.4) simplifies to

\[
\frac{3}{4} k'^2 - \frac{1}{2} kk'' - (1 + b_1^2)k^4 + Kk^2 = 0.
\]

As in the proof of Theorem 3.2 it is straightforward to see that the first equation of (4.4) is equivalent to

\[
(k^{-\frac{1}{2}})'' - (1 + b_1^2)k^\frac{3}{2} + Kk^{-\frac{1}{2}} = 0.
\]

Setting \( f := k^{-\frac{1}{4}} \) and multiplying by \( f' \) we find

\[
0 = (f'^2)' + K(f^2)' + (1 + b_1^2)(\frac{1}{f^2})',
\]

which we integrate to obtain

\[
b_2 = f'^2 + Kf^2 + \frac{1 + b_1^2}{f^2}
\]

for some \( b_2 \in \mathbb{R} \).

To further simplify this equation we set \( h(s) := \frac{1}{2} f^2(s) \) and get

\[
0 = h'^2 + 4Kh^2 - 2b_2h + 1 + b_1^2.
\]

This equation can now easily be integrated distinguishing between the cases \( K = 1, 0, -1 \) and then using that \( k(s) = \frac{1}{2\sqrt{h(s)}} \). \( \square \)
4.1. Stability of $p$-biharmonic curves on three-dimensional space forms. After having determined the structure of $\frac{1}{2}$-biharmonic curves on three-dimensional space forms we now turn to the analysis of their stability where we again focus on the stability with respect to normal variations.

**Proposition 4.4.** Let $\gamma: I \to M$ be a $p$-biharmonic curve parametrized by arclength and suppose that $M$ is a three-dimensional Riemannian space form of constant curvature $K$. Then the Hessian of (1.2) evaluated in the normal direction $N$ is given by

\[
\operatorname{Hess}(E_{2,p}(\gamma))(N, N) = (p-2) \int |\nabla \gamma'|^4 |(\nabla \gamma \nabla \gamma', N, \nabla \gamma')|^2 ds
+ \int |\nabla \gamma'|^{p-2} |\nabla \gamma \nabla \gamma, N| ds + (p-2)K^2 \int |\nabla \gamma'|^{p-4} |(\nabla \gamma', N)|^2 ds
+ K^2 \int |\nabla \gamma'|^{p-2} ds + K \int \langle \gamma', \nabla \gamma'(|\nabla \gamma'|^{p-2} \nabla \gamma') \rangle ds
+ 3K \int |\nabla \gamma'|^{p-2} (\nabla \gamma' N, \gamma') \langle \nabla \gamma', N \rangle ds
+ 2(p-2)K \int \langle \nabla \gamma', \nabla \gamma' N, \nabla \gamma' \rangle |\nabla \gamma'|^{p-4} (N, |\nabla \gamma'| ds
+ 2K \int |\nabla \gamma'|^{p-2} (\nabla \gamma' \nabla \gamma, N, N) ds.
\]

where $k$ is the geodesic curvature and $\tau$ the torsion of the curve $\gamma$.

**Proof.** We apply the second variation formula derived in Proposition 2.14, choose $N$ as variational vector field and employ the Frenet-equations (4.1). Inserting this choice into (2.11) we find as in the two-dimensional case

\[
\operatorname{Hess}(E_{2,p}(\gamma))(N, N) = (p-2) \int |\nabla \gamma'|^4 |(\nabla \gamma \nabla \gamma', N, \nabla \gamma')|^2 ds
+ \int |\nabla \gamma'|^{p-2} |\nabla \gamma \nabla \gamma, N| ds + (p-2)K^2 \int |\nabla \gamma'|^{p-4} |(\nabla \gamma', N)|^2 ds
+ K^2 \int |\nabla \gamma'|^{p-2} ds + K \int \langle \gamma', \nabla \gamma'(|\nabla \gamma'|^{p-2} \nabla \gamma') \rangle ds
+ 3K \int |\nabla \gamma'|^{p-2} (\nabla \gamma' N, \gamma') \langle \nabla \gamma', N \rangle ds
+ 2(p-2)K \int \langle \nabla \gamma', \nabla \gamma' N, \nabla \gamma' \rangle |\nabla \gamma'|^{p-4} (N, |\nabla \gamma'| ds
+ 2K \int |\nabla \gamma'|^{p-2} (\nabla \gamma' \nabla \gamma, N, N) ds.
\]

Using the Frenet-equations (4.1) it is straightforward to calculate that

\[
\langle \nabla \gamma' \nabla \gamma, N, \nabla \gamma' \rangle = -k(k^2 + \tau^2),
|\nabla \gamma' \nabla \gamma, N|^2 = k^2 + \tau^2 + (k^2 + \tau^2)^2.
\]

In addition, we find

\[
\int \langle \gamma', \nabla \gamma'(|\nabla \gamma'|^{p-2} \nabla \gamma') \rangle ds = \int |\nabla \gamma'|^{p-2} (\nabla \gamma' N, \gamma') \langle \nabla \gamma', N \rangle ds = -\int k^p ds,
\]

\[
\int \langle \nabla \gamma' \nabla \gamma' N, \nabla \gamma' \rangle |\nabla \gamma'|^{p-4} (N, \nabla \gamma') ds = -\int (k^2 + \tau^2) k^{p-2} ds,
\]

\[
\int |\nabla \gamma'|^{p-2} (\nabla \gamma' \nabla \gamma, N, N) ds = -\int (k^2 + \tau^2) k^{p-2} ds.
\]

Inserting into the above equation completes the proof. □

**Theorem 4.5.** Let $\gamma: I \to M$ be a $p$-biharmonic curve parametrized by arclength and suppose that $M$ is a three-dimensional Riemannian manifold of constant curvature $K$. Consider the second variation of (1.2) for a variation in the normal direction $N$. Then the following statements hold:

1. For $p \neq \frac{1}{2}$ we get

\[
\operatorname{Hess}(E_{2,p}(\gamma))(N, N) = -4K|I|k^p.
\]

(4.9)
For $p = \frac{1}{2}$ we get
\[
\text{Hess}(E_{2,p}(\gamma))(N,N) = \int_I \left( -\frac{(1+a^2)^2}{2}k_5^2 + (1+a^2)k_2k_{-\frac{3}{2}} - \frac{1}{2}Kk_{5^2} - \frac{1}{2}3Kk_{\frac{1}{2}} + a^2Kk_{\frac{1}{2}} \right)ds,
\]
where $a \in \mathbb{R}$.

**Proof.** First, we consider the case $p \neq \frac{1}{2}$. Then, from (4.3) we know that the condition for a curve to be $p$-biharmonic is $k^2 + \tau^2 = K$. Inserting into (4.8) then completes the proof for this case.

Concerning the case $p = \frac{1}{2}$ we know that the geodesic curvature and the torsion of the curve are related by $\tau(s) = ak(s)$ with $a \in \mathbb{R}$. Inserting into (4.8) we get the result. □

**Remark 4.6.**
(1) Note that for $p \neq \frac{1}{2}$ the torsion $\tau$ of the $p$-biharmonic curve enters into (4.9) via $K = k^2 + \tau^2$. Consequently, we get the same formula as in the two-dimensional case, that is (3.12) and (4.9) coincide.

(2) The same statement as made in Remark 3.12 also holds in the case of a three-dimensional target $M$.

(3) As in the surface case we can insert the solutions obtained in Theorem 4.2 for $p = \frac{1}{2}$ into equation (4.10) in order to check the sign of the second variation. Unfortunately, this again leads to a number of complicated expressions such that no information can be extracted.

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