The general solution of the field equations in LRS Bianchi-I space-time with perfect fluid equation-of-state (EoS) is presented. The models filled with dust, vacuum energy, Zel’dovich matter and disordered radiation are studied in detail. A unified and systematic treatment of the solutions is presented, and some new solutions are found. The dust, stiff matter and disordered radiation models describe only a decelerated universe, whereas the vacuum energy model exhibits a transition from a decelerated to an accelerated phase.

Keywords: LRS Bianchi I anisotropic model; perfect fluid equation of state.

1. Introduction

Several studies have been carried out on the Bianchi type I cosmological models, which represent the simplest generalisation of the flat Friedmann-Lemaitre-Robertwson-Walker (FRW) models. Various methods have been employed to solve the field equations. In the 1960’s, Thorne presented some solutions of spatially homogeneous axisymmetric anisotropic open, semi-closed and Euclidean models with perfect fluid and magnetic fields and studied anisotropy and elements formation. Jacobs extended the work to the most general Euclidean Bianchi I models following an approach developed by Misner. Solutions of Einstein’s equations for a fluid which exhibit local-rotational-symmetry (LRS) were presented by Stewart and Ellis.

In the 1980’s, Hajj-Boutros introduced a technique to reduce the Einstein field equations to first-order Riccati equations in spherical symmetry. The authors applied this technique for generating exact solutions of LRS Bianchi type I models
filled with a perfect fluid for which the classical barotropic equation-of-state (EoS) 
\( p = (\gamma - 1) \rho \) does not hold. Further, the authors generated two new classes of LRS Bianchi type II models with stiff matter. Hajj-Boutros and Sfeila elaborated this new generation technique in the case of a static spherically-symmetric distribution of charged fluid satisfying a barotropic equation of state, i.e., \( p = (\gamma - 1) \rho \).

In continuation of the series of their work, via a suitable scale transformation, they showed that the condition of isotropy of pressure in a Bianchi I space-time filled with perfect fluid reduces to a linear second-order differential equation which can be used for generating many new LRS Bianchi I solutions. Following their approach, Ram and Singh and Ram also added some new classes of LRS Bianchi type I and type VI\(_0\) perfect fluid models. In 1994, Mazumder showed that the field equations of the LRS Bianchi I space-time filled with a perfect fluid are solvable for arbitrary cosmic scale functions. He tried to generalise the solutions found in Refs. However, the main issue with generating schemes is that the matter energy density \( \rho \) and pressure \( p \) do not satisfy a barotropic equation of state, in general. Another weak point of these models is the consideration of known solutions to obtain exact solution of the field equations. In this paper, we discuss the general solution of the field equations in LRS Bianchi I space-time with a perfect fluid equation of state.

The paper is organized as follows. In Sec. 2, we present the field equations and general solution with a perfect fluid equation of state in the framework of the LRS Bianchi I space-time. In subsections 2.1-2.5, we present the solutions for the dust, vacuum energy, Zel’dovich stiff matter and radiation models, respectively, and discuss their physical and cosmological significance. The summary of the results is accumulated in Sec. 3.

2. The model and solution

The spatially homogenous and anisotropic LRS Bianchi I line-element is given by

\[
\text{d}s^2 = -\text{d}t^2 + A^2 \text{d}x^2 + B^2 (\text{d}y^2 + \text{d}z^2),
\]

where \( A \) and \( B \) are the scale factors, and are functions of cosmic time \( t \).

The energy-momentum tensor for the perfect fluid is given as

\[
T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu},
\]

where \( \rho \) is the energy density and \( p \) is the thermodynamical pressure of the fluid, \( u_\mu \) is the four velocity of the fluid such that \( u_\mu u^\mu = -1 \) and in comoving coordinates \( u^\mu = \delta^\mu_0 \).

The Einstein field equations are

\[
R_{ij} - \frac{1}{2}g_{ij}R = T_{ij},
\]
where we have taken $8\pi G = 1 = c$. The above field equations for the metric (1) and the energy-momentum tensor (2), yield the following independent equations

\[
\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{A}\dot{B}}{AB} = \rho, \tag{4}
\]

\[
\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{B}}{B} = -p, \tag{5}
\]

\[
\frac{\dddot{A}}{B} + \frac{\dddot{B}}{B} + 2\frac{\dot{A}\dot{B}}{AB} = -p. \tag{6}
\]

From (5) and (6), Mazumder\,\cite{16} found the condition for isotropy of pressure as

\[
(\dot{B}A - B\dot{A})B = l, \tag{7}
\]

where $l$ is a constant of integration.

Equations (4)–(6) are three independent equations with four unknowns namely $A$, $B$, $\rho$ and $p$. Therefore, to find the exact solution to the field equations we require a supplementary constraint for the consistency of the system. One may assume any relation between any two arbitrary physical quantities or variables. We consider the perfect fluid equation of state (EoS) which is defined as

\[
p = \omega \rho, \tag{8}
\]

where $\omega$ is the EoS parameter. If the system is to be consistent with causality and mechanically stable, then $-1 \leq \omega \leq 1$.

Substituting (8) in (5) and eliminating $\rho$ from (4) and (5), we get

\[
\frac{\dot{B}}{B} + \frac{\dot{A}}{A} = -\left(1 + \frac{\omega}{2}\right) \frac{\dot{B}}{B}, \tag{9}
\]

which is a second order differential equation in $B$ and a first order differential equation in $A$. The variables are thus separated and the equation becomes integrable. We solve it for $A$ to obtain

\[
A^\omega = m B^{-\frac{1+\omega}{\omega}}, \tag{10}
\]

where $m$ is an integration constant which must be positive for an expanding universe. Equation (10) is the most general solution of the LRS Bianchi I model filled with a perfect fluid. However, this solution contains the derivative of $B$, therefore, is itself a first order linear differential equation in $B$. Equation (10) cannot be solved due to the presence of $A(t)$. Therefore, one may explicitly solve (7) and (10) by supplying values to the EoS parameter $\omega$ for various forms of matter. The most common sources of the matter in the universe are non-relativistic matter (dust), ultra-relativistic matter (radiation), Zel’dovich matter (stiff matter) and vacuum energy (the cosmological constant). We shall determine the solution for $\omega = 0$ (dust), $\omega = -1$ (vacuum energy), $\omega = 1$ (stiff matter) in the forthcoming subsections and also study the influence of each of these matter sources in cosmological
evolution. Let us define some cosmological parameters to study the cosmological evolution in LRS Bianchi I space-time.

The average scale factor is defined as
\[ a = (AB^2)^{\frac{1}{3}}. \] (11)
The rates of the expansion along the x, y, and z-axes are defined by
\[ H_x = \frac{\dot{A}}{A}, \quad H_y = H_z = \frac{\dot{B}}{B}, \] (12)
where a dot denotes the ordinary derivative with respect to cosmic time \( t \). The average Hubble parameter (average expansion rate) \( H \), which is the generalization of the Hubble parameter in the isotropic case, is given by
\[ H = \frac{1}{3} \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right). \] (13)
The streamlines of the motion of a cosmic fluid are characterized kinematically by their expansion, \( \theta \), shear, \( \sigma \), and rotation, \( w \). Consider now a time-like congruence with a tangent vector \( u^\mu \). Since any four dimensional quantity can be resolved into its space and time components by projecting it into the three dimensional space orthogonal to the time-like worldlines by means of the operator \( h_{\mu\nu} \), then \( u_{\mu\nu} \) may be decomposed as follows\[^{17}\]
\[ u_{\mu\nu} = w_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3} \theta h_{\mu\nu} - \dot{u}_{\mu} u_{\nu}, \] (14)
where \( w_{\mu\nu} \) is the tensor of rotation (vorticity), \( \sigma_{\mu\nu} \) is the shear tensor, \( h_{\mu\nu} = g_{\mu\nu} - u_{\mu} u_{\nu} \) is the projection tensor and \( \dot{u}^{\mu} = u_{\nu}^{\mu} u^{\nu} \) is the acceleration vector.

The expansion scalar, \( \theta \), and the shear scalar, \( \sigma \), are, respectively, defined by
\[ \theta = u^{\mu}_{\;\mu} = u_{\mu}^{\mu} + \Gamma_{\mu\nu}^{\mu} u^{\nu} = \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} = 3H, \] (15)
\[ \sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} = \frac{1}{3} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right)^2, \] (16)
where the shear tensor, \( \sigma_{\mu\nu} \), is defined as\[^{18}\]
\[ \sigma_{\mu\nu} = u_{(\mu;\nu)} - \ddot{u}_{(\mu} u_{\nu)} - \frac{1}{3} \theta h_{\mu\nu}, \] (17)
where round brackets denote symmetrisation, e.g., \( u_{(\mu;\nu)} = \frac{1}{2}(u_{\mu\nu} + u_{\nu\mu}) \). For the metric (1), the acceleration \( \dot{u}^{\mu} \) and vorticity \( w_{\mu\nu} \) turn out to be zero.

The shear tensor, \( \sigma_{\mu\nu} \), determines the distortion arising in the fluid flow leaving the volume invariant\[^{19}\]. The expansion rates can be different in the different directions, unlike the Robertson-Walker models where the expansion rates are the same in all directions. The directional components of the shear tensor are
\[ \sigma_1 = \frac{2}{3} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right), \quad \sigma_2 = \sigma_3 = -\frac{1}{3} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right), \quad \sigma_4 = 0 \quad \text{and} \quad \sigma_{\mu}^{\nu} = 0, \mu \neq \nu. \] (18)
The quantities we have defined here (expansion and shear) are called the kinematic quantities because they characterize the kinematic features of the fluid flow. For an expanding model $\theta > 0$ and the shear decreases with time. The rate of work done by anisotropic stresses augments the shear dissipation. In the case of expanding models, it is found that the dynamical importance of matter increases while that of shear decreases in the course of evolution. In the spatially homogenous cosmological models in which the matter content of space-time is a perfect fluid and in which the fluid flow vector is not normal to the surfaces of homogeneity, the matter may move with non-zero expansion, shear and rotation. The only shear-free spatially homogenous perfect fluid universes models are Robertson-Walker model\textsuperscript{20}.

Other than the above kinematical parameters one of the most important parameter for the present study is the deceleration parameter $q = -\frac{\ddot{a}}{a\dot{a}^2}$, which in terms of hubble parameter reads as

$$q = -1 - \frac{\dot{H}}{H^2}. \quad (19)$$

A positive deceleration parameter corresponds to a decelerated universe, whereas a negative one represents an accelerating universe. In what follows we shall study the dust, vacuum energy and Zel’довich stiff matter models.

### 2.1. The dust model

The dust model corresponds to $p = 0$, i.e., $\omega = 0$. Consequently, (10) reduces to

$$\dot{B} = \frac{m}{\sqrt{B}} \quad (20)$$

which on integration yields

$$B(t) = \left( \frac{3mt}{2} \right)^\frac{2}{3}. \quad (21)$$

We have dropped out the integration constant so that the big-bang singularity occurs at $t = 0$. We have already been considered $m > 0$ for an expanding universe. In (21), the positivity of $m$ ensures the reality of the solution.

Substituting (20) and (21) in (7), we get

$$A(t) = c_1 t^\frac{2}{3} + \left( \frac{2}{3m} \right)^\frac{2}{3} \frac{l}{t^\frac{1}{3}}, \quad (22)$$

where $c_1$ is an integration constant. We must have $l \geq 0$ for an expanding universe. Since $A \to \infty$ as $t \to 0$, therefore, the universe explodes with infinite rate of expansion in the direction of $A$. This is an example of what is called a cigar-type singularity, i.e., the expansion parameter tends to zero in 2 directions, but diverges in the last direction as $t \to 0^{+}$. This is due in this case to the presence of the inverse term $t^{-\frac{1}{3}}$ in (22). However, the expansion slows down as time passes and it speeds up once again as the first term in (19) starts dominating. From (20) and
(22), we note that $A$ and $B$ are related by $A = (\frac{2}{3m})^\frac{2}{3} B + \frac{l}{t^2}$.

The solution in metric form can be written as

$$ds^2 = -dt^2 + \left(c_1 t^\frac{2}{3} + \frac{c_2}{t^\frac{2}{3}}\right)^2 dx^2 + c_3 t^\frac{2}{3} (dy^2 + dz^2),$$

(23)

where $c_2 = l \left(\frac{2}{3m}\right)^\frac{2}{3}$ and $c_3 = \left(\frac{3m}{2}\right)^\frac{2}{3}$.

The average scale factor is

$$a = \left[lt + \left(\frac{3m}{2}\right)^\frac{2}{3} t^2\right]^\frac{1}{3},$$

(24)

where we have taken the integration constant $c_1$ to be unity without any loss of generality. If $l = 0$ then $a(t) \propto t^\frac{2}{3}$, i.e., the solution reduces to the Einstein-de Sitter solution for dust which has homogeneous and isotropic spatial sections.

The average Hubble parameter gives

$$H = \frac{2^{\frac{1}{2}} 3^\frac{2}{3} l + 18m^\frac{2}{3} t}{3 \left(2^{\frac{1}{2}} l + 9m^\frac{2}{3} t\right) t},$$

(25)

The energy density of dust matter becomes

$$\rho = \left[lt + \left(\frac{3m}{2}\right)^\frac{2}{3} t^2\right]^{-1},$$

(26)

which remains always positive. It is to be noted that the energy density is infinite at $t = 0$ which decreases with evolution of the universe and vanishes as $t \to \infty$.

The deceleration parameter gives

$$q = \frac{3 \left[2^{\frac{1}{2}} \sqrt{3} l^2 + 3^\frac{2}{3} l (2m)^{\frac{2}{3}} t + 27m^\frac{2}{3} t^2\right]}{2 \left(\sqrt{2} 3^\frac{2}{3} l + 9m^\frac{2}{3} t\right)^2}.$$  

(27)

Figure 1 plots $q(t)$ versus $t$, which shows that the deceleration parameter starts from $q = 2$ at $t = 0$ and approaches to $q \to 0.5$ as $t \to \infty$. Since the deceleration parameter always remains positive, the dust model describes a decelerated phase of the universe.

The shear scalar ($\sigma$) and the expansion scalar ($\theta$) have the expressions

$$\theta = \frac{2^{\frac{1}{2}} 3^\frac{2}{3} l + 18m^\frac{2}{3} t}{\left(2^{\frac{1}{2}} 3^\frac{2}{3} l + 9m^\frac{2}{3} t\right) t},$$

(28)

$$\sigma = \frac{2^{\frac{4}{3}} 3^\frac{2}{3} l}{\left(2^{\frac{1}{2}} 3^\frac{2}{3} l + 9m^\frac{2}{3} t\right) t},$$

(29)

Consequently, the ratio of the expansion scalar to the shear scalar can be expressed as

$$\frac{\sigma}{\theta} = \left(\frac{3^{5/6} m^{4/3} t}{\sqrt{2} t} + \sqrt{3}\right)^{-1}.$$

(30)
We have $\sigma = \frac{1}{\sqrt{3}}$ at $t = 0$ and $\lim_{t \to \infty} \frac{\sigma}{\theta} = 0$, therefore, the model is anisotropic at early times but tends to isotropy for late times which is consistent with observations.\cite{23,24,25,26,27}

The solution presented in here is a special case of the general Bianchi type I solution found by Robinson\cite{28} for a universe containing only dust. Kompaneets and Chernov\cite{20} first obtained the axially symmetric solution in a different notation. Later on, the solutions were also obtained by Vajk and Eltgroth\cite{29} by transforming independent variables and rediscovered by Iyer and Vishveshwara\cite{30}. The general solution has also been given in a different form by Jacobs\cite{2}. Similar solutions have also been obtained by Hajj-Boutros and Sfeila\cite{11} applying a reverse way of generating technique to the Einstein-de Sitter metric. Ram\cite{13} obtained the above solution from the Kasner vacuum metric\cite{4} by implementing the generating technique. In another paper, Ram\cite{14} obtained a similar solution starting from the dust-filled solution of Hajj-Boutros and Sfeila\cite{11}.

2.2. The vacuum energy model

The expansion of the universe is accelerating in the present epoch\cite{32,33,34}. But the Einstein field equations lead to a decelerated expansion of the universe if the matter content is only ordinary baryonic matter. The accelerated expansion can be described by supplying some exotic component of the matter to the field equations. An unknown matter called dark energy (DE) is supposed to be responsible for the present accelerating universe. The past two decades have produced a flood of candidates for DE. However, a cosmological constant, $\Lambda$ is not only the simplest

The Kasner vacuum solution is given by

$$ds^2 = -dt^2 + t^{2a}dx^2 + t^{2b}(dy^2 + dz^2)$$ where $a = -\frac{1}{3}$ and $b = \frac{2}{3}$.
Vacuum energy corresponds to \( \omega = -1 \), for which (10) gives

\[
A = \frac{\dot{B}}{m},
\]

(31)

Consequently, (7) can be written as

\[
\ddot{B}B^2 - \dot{B}^2 B = k,
\]

(32)

where \( k = lm \). The above equation possesses two real solutions

\[
B_1(t) = e^{-\sqrt{\beta}t} \left( \frac{e^{3\sqrt{\beta}t}}{6\beta} - k \right)^{\frac{1}{2}},
\]

(33)

\[
B_2(t) = e^{-\sqrt{\beta}t} \left( \frac{ke^{3\sqrt{\beta}t} - \frac{1}{6\beta}}{2} \right)^{\frac{1}{2}},
\]

(34)

where \( \beta \) is a positive integration constant. Substituting the above expressions in (31), we get

\[
A_1(t) = \frac{e^{-\sqrt{\beta}t} \left( e^{3\sqrt{\beta}t} + 6\beta k \right)}{\frac{1}{2} \beta \frac{1}{2} m \left( e^{3\sqrt{\beta}t} - 6\beta k \right)^{\frac{1}{2}}},
\]

(35)

\[
A_2(t) = \frac{e^{-\sqrt{\beta}t} \left( 6\beta ke^{3\sqrt{\beta}t} + 1 \right)}{\frac{1}{2} \beta \frac{1}{2} m \left( 6\beta ke^{3\sqrt{\beta}t} - 1 \right)^{\frac{1}{2}}}.
\]

(36)

The average scale factors for both solutions become, respectively,

\[
a_1(t) = \left[ e^{-3\sqrt{\beta}t} \left( \frac{e^{6\sqrt{\beta}t} - 36\beta^2 k^2}{36\beta^2 m} \right) \right]^{\frac{1}{2}},
\]

(37)

\[
a_2(t) = \left[ e^{-3\sqrt{\beta}t} \left( \frac{36\beta^2 k^2 e^{6\sqrt{\beta}t} - 1}{36\beta^2 m} \right) \right]^{\frac{1}{2}}.
\]

(38)

The average Hubble parameters give

\[
H_1 = \left( \frac{1}{\sqrt{\beta}} - \frac{72\beta^{3/2} k^2}{36\beta^2 k^2 + e^{6\sqrt{\beta}t}} \right)^{-1},
\]

(39)

\[
H_2 = \sqrt{\beta} \left( 1 - \frac{2}{36\beta^2 k^2 e^{6\sqrt{\beta}t} + 1} \right)^{-1}.
\]

(40)

The energy density and pressure of vacuum energy are constants, i.e., \( \rho_1 = 3\beta = -p \). Thus the integration constant \( \beta \) stands for a cosmological constant which represents the vacuum energy in the present model.
Let us first discuss the model for the solution of $A_1$ and $B_1$. The deceleration parameter takes the form

$$q_1 = -1 + \frac{432 \beta^2 k^2 e^{6\sqrt{\beta}t}}{(36 \beta^2 k^2 + e^{6\sqrt{\beta}t})^2}. \tag{41}$$

Figure 2 plots $q_1(t)$ versus $t$ for different values of $k$ and $\beta$.

Figure 2 plots $q_1(t)$ versus $t$ which shows the transition from a decelerated to an accelerated universe which is consistent with many recent observations \cite{40,41,42,43,44,45}.

The expansion scalar and the shear scalar have expressions

$$\theta_1 = 3 \left( \frac{1}{\sqrt{\beta}} - \frac{72 \beta^2 k^2}{36 \beta^2 k^2 + e^{6\sqrt{\beta}t}} \right)^{-1}, \tag{42}$$

$$\sigma_1 = \frac{12 \sqrt{3} \beta^2 k^2 e^{3\sqrt{\beta}t}}{e^{6\sqrt{\beta}t} - 36 \beta^2 k^2}. \tag{43}$$

The ratio of shear scalar to expansion scalar can be written as

$$\frac{\sigma_1}{\theta_1} = \frac{4 \sqrt{3} \beta k^2 e^{3\sqrt{\beta}t}}{36 \beta^2 k^2 + e^{6\sqrt{\beta}t}}. \tag{44}$$

Figure 3 plots $\frac{\sigma_1}{\theta_1}$ versus $t$ which shows that the universe was anisotropic at early times but becomes isotropic at late times.

Now for the second solution corresponding to $H_2$, the deceleration parameter takes the form

$$q_2 = -1 + \frac{432 \beta^2 k^2 e^{6\sqrt{\beta}t}}{(1 + 36 \beta^2 k^2 e^{6\sqrt{\beta}t})^2}. \tag{45}$$

Figure 4 plots $q_2$ versus $t$, which also describes the transition from a decelerated to an accelerated phase of the universe.
The expressions for expansion scalar and shear scalar are

\[ \theta_2 = 3\sqrt{3} \left( 1 - \frac{2}{1 + 36\beta^2 k^2 e^{6\sqrt{3}t}} \right)^{-1}, \]  

(46)

\[ \sigma_2 = \frac{12\sqrt{3}\beta k^2 e^{3\sqrt{3}t}}{36\beta^2 k^2 e^{6\sqrt{3}t} - 1}, \]  

(47)

respectively. Consequently

\[ \frac{\sigma_2}{\theta_2} = \frac{4\sqrt{3}\beta k e^{3\sqrt{3}t}}{1 + 36\beta^2 k^2 e^{6\sqrt{3}t}} \]  

(48)

One may observe that the behavior of \( \frac{\sigma_2}{\theta_2} \) is similar to that shown in Fig. 3. Therefore, this case also shows the anisotropic behavior of the model at early times which becomes isotropic at late times. Thus, the characteristics of the models with both solutions of \( a_1 \) and \( a_2 \) are similar.

It is to be noted that Iyer and Vishveshwara also found solution for constant
vacuum energy density. The present solutions are different from those obtained in Ref. [31] since we have not assumed a constant vacuum energy density. Rather, it is the natural outcome of our procedure. To the best of our knowledge the solutions obtained here are new.

2.3. Zel’ dovich stiff matter model

The Zel’ dovich stiff matter corresponds to \( \omega = \frac{4}{3} \). In this case (10) reduces to

\[
A = \frac{m}{BB}. 
\]

(49)

Consequently, (7) can be written as

\[
m \left( \frac{\dot{BB}}{B^2} + 2 \right) - n = 0,
\]

(50)

where we have taken \( l = -n \) \((n > 0)\) for reality of the solution. The above equation yields

\[
B(t) = \begin{cases} 
\beta [(n + 3m)t]^\frac{n+m}{n+3m}, & n \neq 3m; \\
\beta e^{\alpha t}, & n = 3m,
\end{cases}
\]

(51)

Here \( \alpha \) and \( \beta \) are constants of integration and one integration constant in case of \( n \neq 3m \), has been taken zero so that the big-bang singularity occurs at \( t = 0 \). We must have \( \beta > 0 \) for an expanding universe.

Substituting the values of \( B(t) \) in (49), we get

\[
A(t) = \begin{cases} 
\frac{1}{\beta^2} [(n + 3m)t]^\frac{n+m}{n+3m}, & n \neq 3m; \\
\frac{m}{\alpha^2} e^{-2\alpha t}, & n = 3m.
\end{cases}
\]

(52)

The directional scale factors are related by \( A = \frac{4}{\beta^2} [(n + 3m)t]^\frac{n+m}{n+3m} \) for \( n \neq 3m \) and \( A = \frac{mBe^{-3\alpha t}}{\alpha^2} \) for \( n = 3m \). For \( n \neq 3m \), \( A = 0 = B \) at \( t = 0 \), which shows a point type singularity, whereas for \( n = 3m \), \( B = \beta \) and \( A = \frac{m}{\alpha^2} \) at \( t = 0 \), which is a singularity-free model. Let us express both solutions in metric form:

\[
ds^2 = -dt^2 + c_4 e^{\frac{2(n+m)}{n+3m} \alpha t} dx^2 + c_5 t^{\frac{2m}{n+3m}} (dy^2 + dz^2); \quad n \neq 3m,
\]

(53)

\[
ds^2 = -dt^2 + c_6 e^{-4\alpha t} dx^2 + c_7 e^{2\alpha t} (dy^2 + dz^2); \quad n = 3m,
\]

(54)

where \( c_4 = \frac{(n+3m)2(n+m)}{\beta^2} \), \( c_5 = \beta^2(n + 3m)\frac{2m}{n+3m} \), \( c_6 = \frac{m^2}{\alpha^2} \), and \( c_7 = \beta^2 \). These metrics represent the most general solutions of the stiff matter model in the LRS Bianchi I spacetime model, which are completely different from those obtained by generating methods in previous works [11,13]. If we choose \( \frac{m}{n+3m} = k \) then one of the solutions (53) can be represented by a one-parameter family of solutions to Einstein’s equation with a perfect stiff-matter fluid first obtained by Jacobs [2], i.e.,

\[
ds^2 = -dt^2 + c_4 t^{2(1-2k)} dx^2 + c_5 t^{2k} (dy^2 + dz^2).
\]

(55)
It is to be noted that the above solution is different from the general LRS Kasner stiff-matter metric. Jacobs also discussed the nature of the singularity in detail for this solution. Vajk and Eltgroth found some general solutions for rational values between $-1 < \omega < 1$ and a particular solution for stiff matter with different parameterizations. Later on, Iyer and Vishveshwara rediscovered stiff matter solutions identical to (55) in searching for exact solutions of the Einstein equations in which the Dirac equation separates. The present solutions are different from the solutions of Hajj-Boutros and Sfeila obtained by applying a generating technique to a flat FRW metric with unity expansion rate. The solutions obtained by Ram by implementing generating technique to LRS Kasner stiff-matter metric do not satisfy classical EoS of perfect fluid.

In particular, if $m = -\frac{2n}{3}$ then the solution is given in (53) reduces to

$$ds^2 = -dt^2 + ct^{-\frac{2}{3}} dx^2 + ct^\frac{2}{3} (dy^2 + dz^2).$$

(56)

The above metric is identical to the solution of Singh and Ram, which also does not satisfy a perfect fluid equation of state due to following the solutions generating method.

The average scale factor is given as

$$a(t) = \begin{cases} \left[ (n+3m) \frac{t}{a} \right]^\frac{1}{4}, n \neq 3m; \\ \left( \frac{m}{n} \right)^\frac{1}{4}, n = 3m. \end{cases}$$

(57)

The scale factor for $n \neq 3m$ describes a power-law expansion of the universe, whereas the scale factor for $n = 3m$ corresponds to a static universe. However, only the volume remains constant for the static universe and we can see that the shape of the universe changes exponentially in the spatial directions of A and B.

The average Hubble parameter is given by

$$H = \begin{cases} \frac{1}{3t}, n \neq 3m; \\ 0, n = 3m. \end{cases}$$

(58)

The deceleration parameter also has the constant values, $q = 2$ for $n \neq 3m$ and $q = 0$ for $n = 3m$. Hence, the Zel’dovich stiff matter model describes a decelerating universe for $n \neq 3m$ and a marginal inflationary cosmology for $n = 3m$.

For stiff matter, the energy density and pressure are equal. In the present model they become

$$\rho(=p) = \begin{cases} \frac{m(2n+3m)}{t^2(n+3m)^2}, n \neq 3m; \\ -3a^2, n = 3m. \end{cases}$$

(59)

For $n \neq 3m$, the energy density (or pressure) decreases with the evolution of the universe and vanishes as $t \to \infty$. The energy density is negative for $n = 3m$, which

\[\text{The general LRS Kasner stiff-matter metric is of the form}\]

$$ds^2 = -dt^2 + t^{2a} dx^2 + t^{2b} (dy^2 + dz^2) \text{ where } a + 2b = 1.$$
does not represent a realistic model of the universe.

The expansion and shear scalars are given by, respectively

\[ \theta = \begin{cases} \frac{1}{t}, & n \neq 3m; \\ 0, & n = 3m. \end{cases} \]

\[ \sigma = \begin{cases} \frac{n}{\sqrt{3(n+3m)t}}, & n \neq 3m; \\ \sqrt{3}\alpha, & n = 3m. \end{cases} \]

The ratio of the shear scalar to the expansion scalar for \( n \neq 3m \) has the constant value \( \frac{\sigma}{\theta} = \frac{n}{\sqrt{3(n+3m)}} \), which shows that the stiff matter model remains always anisotropic for all finite values of \( n \neq 3m \). However, the model becomes isotropic in the case of \( n \neq 3m \) when \( n \to 0 \) or \( m \to \infty \). There is no expansion of the universe for \( n = 3m \), but it has finite shear \( \sqrt{3}\alpha \).

### 2.4. Disordered radiation model

Klein\(^{47}\) and Teixeira et al.\(^{48}\) investigated a source free disordered distribution of electromagnetic radiation. The EoS of disordered radiation is \( p = 3\rho \). In this case (10) reduces to

\[ A^3 = \frac{m}{BB^2}. \]

Consequently, (7) can be written as

\[ B^3 \left( \frac{m}{BB^2} \right)^\frac{2}{3} \left( \dot{BB} + 5\dot{B}^2 \right) = 3lm. \]

The only real solution which above equation possesses is

\[ B(t) = l \left( \frac{2t^2}{3m} \right)^\frac{1}{2}, \]

where both integration constants have been taken zero, one is for the choice of the big-bang singularity at \( t = 0 \) and another for the reality of the solution. For an expanding universe we have must have \( l > 0 \).

Substituting (64) in (62), we get

\[ A(t) = \left( \frac{3m}{2} \right)^\frac{2}{3} \frac{1}{lt^\frac{2}{3}}. \]

Form (64) and (65), the directional scale factors are related by \( B = \frac{2l^2A^2t}{3m} \). The solution in metric form can be written as

\[ ds^2 = -dt^2 + c_8t^\frac{2}{3}dx^2 + c_9t^\frac{2}{3} \left( dy^2 + dz^2 \right), \]

where \( c_8 = l \left( \frac{3m}{2} \right)^\frac{2}{3} \) and \( c_9 = l \left( \frac{2t}{3m} \right)^\frac{2}{3} \). As far as we are aware, the above metric adds a new class of solutions to the LRS Bianchi I model.

The average scale factor is given by

\[ a(t) = (l)^\frac{2}{3}. \]
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The average Hubble parameter, \( H = \frac{1}{t} \), deceleration parameter, \( q = 2 \), and expansion scalar, \( \frac{1}{t} \), are similar to the solution of the Zel’dovich model for \( n \neq 3m \). Therefore, the disordered radiation model describes a decelerating universe. The energy density and pressure vanish, i.e., \( \rho = 0 = p \), which shows that the disordered radiation is source free.

The shear scalar is given by

\[
\sigma = \frac{1}{\sqrt{3} t}.
\]

(68)

The ratio of the shear scalar to the expansion scalar has a constant value \( \frac{\sigma}{\theta} = \frac{1}{\sqrt{3} t} \), which shows that the universe filled with disordered radiation remains anisotropic.

This solution is different from that of Hajj-Boutros and Sfeila\(^{11}\) obtained from the Tolman metric for disordered radiation. The classical EoS does not hold for their solution, and in particular leads to stiff matter.

2.5. The radiation model

Ultra-relativistic radiation corresponds to \( \omega = \frac{1}{3} \), for which (10) gives

\[
A = \frac{m^3}{B^3 B^2}.
\]

(69)

Consequently, (7) can be written as

\[
3m^3 \left( B \ddot{B} + \dot{B}^2 \right) = l \dot{B}^4 B,
\]

(70)

The above equation possesses two real solutions

\[
B_1(t) = \text{InverseFunction}\left[ -\sqrt{3}m^3/2 t \left( 2t^2 + 9m^3 t + 9m^6 t^2 \right) - 2t^2 \sqrt{3} \left( 2t^3 + 3m^3 t + 3m^6 t \right) \log \left( \sqrt{3}m^{3/2} 2t^{3/2} + 3m^{3} t + 3m^6 t \right) \right]
\]

\[
B_2(t) = \text{InverseFunction}\left[ -\sqrt{3}m^3/2 t \left( 1 + 3m^3 t \right) \left( 2t^2 + 9m^3 t + 9m^6 t^2 \right) - 2t^2 \sqrt{3} \left( 2t^3 + 3m^3 t + 3m^6 t \right) \log \left( m^{3/2} 2t^{3/2} + 9m^{3} t + 3m^6 t \right) \right]
\]

(71)

where one integration constant has been taken zero and another unity. Since the above expressions involve complicated inverse functions, it is not possible to give a simple physical interpretation in this case. However, one may also write the scale factors \( A_1 \) and \( A_2 \) for the above expressions which would be more complicated expressions of inverse functions. It is to be noted that Iyer and Vishveshwar\(^{31}\) have found the solution for radiation.

3. Conclusion

In this paper, we have presented the general solution of the field equations in LRS Bianchi-I space-time with perfect fluid equation of state. In different cases of particular interest, we have studied dust, vacuum energy, Zel’dovich stiff matter and disordered radiation models. Though most of these solutions were known earlier, we present a unified and systematic treatment by solving the field equations in a
straight forward manner. However, as far we know, the vacuum energy and disordered radiation solutions are new. It has been found that the dust, Zel’dovich stiff matter and disordered radiation models describe only decelerated universes, whereas the vacuum energy model exhibits a transition from a decelerated to an accelerated universe.

It is well known that the anisotropic models may represent the cosmos during its early stages of evolution. But the investigation of the vacuum energy model shows that the anisotropic models can also successfully describe a sudden change from deceleration to acceleration. The models describe anisotropic behavior at early times and becomes isotropic at late times, except in the disordered radiation model and in a particular case of the Zel’dovich model. The disordered radiation model remains anisotropic throughout the evolution of the universe.

The straight forward procedure used to solve the field equations is much more appealing. We hope that this will make it useful in future applications of anisotropic cosmological models. We shall explore more solutions in other Bianchi space-time models in our future work.

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References

1. K. S. Thorne, Astrophys. J. 148, 51 (1967).
2. K. C. Jacobs, Astrophys. J. 153, 661–678 (1968).
3. C. W. Misner, Phys. Rev. Lett. 19, 443 (1967).
4. C. W. Misner, Astrophys. J. 151, 431 (1968).
5. J. M. Stewart and G. F. R. Ellis, J. Math. Phys. 9, 1072 (1968).
6. J. Hajj-Boutros, Gravitation, Geometry and Relativity Physics, Lec. Notes Phys., Vol. 212, (Springer, Berlin, 1984), p.51.
7. J. Hajj-Boutros, J. Math. Phys. 26, 2297 (1985).
8. J. Hajj-Boutros, Class. Quantum Grav. 3, 311 (1986).
9. J. Hajj-Boutros, J. Math. Phys. 27, 1592 (1986).
10. J. Hajj-Boutros and J. Sfeila, Gen. Rel. Grav. 18, 395 (1986).
11. J. Hajj-Boutros and J. Sfeila, Int. J. Theor. Phys. 26, 98 (1987).
12. S. Ram, Astrophys. Space Sci. 133, 397 (1987).
13. S. Ram, Gen. Rel. Grav. 21, 697 (1989).
14. S. Ram, Int. J. Theor. Phys. 28, 917 (1989).
15. J. K. Singh and S. Ram, Astrophys. Space Sci. 225, 57 (1995).
16. A. Mazumder, Gen. Rel. Grav. 26, 307 (1994).
17. A. Urankar, Analen der Physik 24, 110–118 (1970).
18. H. M. F. Goenner and F. Kowalewski, Gen. Rel. Grav. 21, 467–488 (1970).
19. A. A. Cooley and R. J. van den Hoogen, J. Math. Phys. 35, 4117–4144 (1994).
20. A. R. King and G. F. R. Ellis, Commun. Math. Phys. 31, 209–242 (1973).
21. H. Bondi, Proc. Roy. Soc. A (London) 282, 303 (1965).
22. M. A. H. MacCullam, Commun. Math. Phys. 20, 57–84 (1971).
23. C. B. Netterfield et al., Astrophys. J. 571, 604–614 (2002), astro-ph/0104460
24. D. N. Spergel et al., Astrophys. J. Suppl. 148, 175–194 (2003), astro-ph/0302209
25. C. L. Bennett et al., Astrophys. J. Suppl. 208, 20 (2013), astro-ph/1212.5225.
26. L. Anderson et al., Mon. Not. Roy. Astron. Soc. 427, 3435 (2013), astro-ph/1203.6594.
27. P. A. R. Ade et al., Astron. Astrophys. 594, A13 (2016), astro-ph/1502.01589
28. B. B. Robinson, Proc. Natl. Acad. Sci. 47, 1852 (1961).
29. A. S. Kompaneets and A. S. Chernov, Zh. Eksp. Teor. Fiz. 47, 1939 (1964), [Sov. Phys. JETP 20, 1303 (1965)].
30. J. P. Vajk and P. G. Eltgroth, J. Math. Phys. 11, 2212 (1970).
31. B. R. Iyer and C. V. Vishveshwara, J. Math. Phys. 28, 1377 (1987).
32. A. G. Riess et al., Astron. J. 116, 1009–1038 (1998), astro-ph/9805201.
33. S. Perlmutter et al., Astrophys. J. 517, 565–586 (1999), astro-ph/9812133.
34. B. P. Schmidt et al., Astrophys. J. 507, 46 (1998), astro-ph/9805200.
35. V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D 9, 373–444 (2000), astro-ph/9904398.
36. T. Padmanabhan, Phys. Rept. 380, 235–320 (2003), hep-th/0212290.
37. P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559–606 (2003), astro-ph/0207347.
38. S. M. Carroll, Living Rev. Rel. 4, 1 (2001), astro-ph/0004075.
39. E. Komatsu et al., Astrophys. J. Suppl. 192, 18 (2011), astro-ph/1001.4538.
40. A. G. Riess et al., Astrophys. J. 560, 49–71 (2001), astro-ph/0010455.
41. L. Amendola, Mon. Not. R. Astron. Soc. 342, 221–226 (2003), astro-ph/0209494.
42. A. G. Riess et al., Astrophys. J. 607, 665–687 (2004), astro-ph/0402512.
43. A. G. Riess et al., Astrophys. J. 659, 98–121 (2007), astro-ph/0611572.
44. Z. Li, P. Wu and H. Yu, Phys. Lett. B 695, 1–8 (2011), gr-qc/1011.1982.
45. R. Giostri et al., J. Cosm. Astropart. Phys. 1203, 027 (2012), [arXiv:astro-ph/1203.3213].
46. Y. B. Zel’dovich, J. Exp. Theor. Phys. 14, 1143 (1962).
47. O. Klein, Arkiv. Mat. Astron. Phys. A 34, 1 (1947).
48. A. F. Da F. Teixeira, I. Wolf and M. M. Som, IL Nuovo Cim. 41B, 387 (1977).