On Symmetric Operators in Noncommutative Geometry

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Abstract

In Noncommutative Geometry, as in quantum theory, classically real variables are assumed to correspond to self-adjoint operators. We consider the relaxation of the requirement of self-adjointness to mere symmetry for operators $X_i$ which encode space-time information.

1 Introduction

Part of the ‘dictionary’ of Noncommutative Geometry [1] is the assumption that classically real variables correspond to self-adjoint operators in the full theory, in line with the usual quantum mechanical description of observables through self-adjoint operators. The use of self-adjoint operators is, of course, in many ways natural since, for example, they often generate unitary groups of physical significance.

On the other hand, there are indications that the structure of space-time at distances close to the Planck scale is highly nontrivial, see e.g. [3], and [5]. In particular, as we will describe below in more detail, there is evidence which points towards a short-distance structure of space-time which is beyond what can be described by self-adjoint operators, namely beyond continua and lattices.

Let us, therefore, reconsider the basic assumption that operators $X_i$ which encode space-time information within a fundamental theory of quantum gravity are self-adjoint.

Of course, if the fundamental theory of quantum gravity is linear as a quantum theory, i.e. if it obeys a linear superposition principle, then it is natural to assume that it encodes space-time information through operators $X_i$ which are linear. And further, it is natural to assume that all formal expectation values of these operators

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\(X_i\) are real - in order to allow some form of physical interpretation in terms of space-time coordinates. However, these assumptions imply only that these operators \(X_i\) are symmetric on their domain \(D\). Self-adjointness, as is well known, is a stronger condition than symmetry. We are therefore led to consider the possibility that the fundamental theory of quantum gravity may encode space-time information using operators \(X_i\) which are symmetric and not self-adjoint.

Clearly, if we wish to consider merely symmetric rather than self-adjoint operators we will have to address two crucial questions:

1. Which are the implications of the fact that generic symmetric operators, a priori, do not generate unitary groups?

2. Physically, which types of spatial structure are described by merely symmetric linear operators \(X_i\)?

## 2 Preliminaries

Let us begin by briefly recalling von Neumann’s theory of self-adjoint extensions of symmetric operators.

Consider the Möbius transform \(x \rightarrow (x - i)(x + i)^{-1}\), which maps the real line onto the unit circle. An analogous transformation, the so-called Cayley transform,

\[
X \rightarrow S := (X - i1)(X + i1)^{-1}
\]

maps the set of self-adjoint operators \(X\) onto the set of unitaries. The Cayley transform does this bijectively. Because of the bijectivity the Cayley transform is technically to be preferred over the exponentiation as a method to relate self-adjoints to unitaries.

For clarity of the argument, let us now consider the clear-cut case of an operator \(X\) which is simple symmetric, i.e. which is symmetric but not self-adjoint and which is also not self-adjoint on any invariant sub space. Recall that without restricting generality we can assume \(X\) to be closed.

The Cayley transform maps \(X\) into an operator \(S\) which is merely isometric. This is because either the domain or the range, or both, of \(S\) are not the full Hilbert space \(H\) - otherwise \(X\) would be self-adjoint.

The orthogonal complements of the domain and the range

\[
L_+ := ((X + i1).D)^\perp
\]

and

\[
L_- := ((X - i1).D)^\perp
\]

are the so-called deficiency spaces. Their dimensions \((r_+, r_-)\) are the so-called deficiency indices. Let us assume them to be finite. If both are equal, \(r_+ = r_-\), then it
is possible to supplement $S$ by a finite rank isometry $S' : L_+ \to L_-$. In this way, $S$ is extended by $S'$ to a unitary operator $U : H \to H$. The inverse Cayley transform of $U$ yields a self-adjoint extension $X_u$ of the simple symmetric $X$. Of course, the choice of $S'$ is arbitrary and nonunique up to elements of the group $U(r)$ which maps, say, $L_+$ into itself, where $r = r_+ = r_-$ is the deficiency index. Therefore, the set of self-adjoint extensions forms itself a representation of a $U(r)$. We will find this group reappearing in our discussion later.

It is clear that a simple symmetric operator with unequal deficiency indices cannot have self-adjoint extensions, since its isometric Cayley transform does not have unitary extensions.

3 Symmetric Operators and Unitary Groups

Let us now prepare the ground for answering the first of the questions which we raised in the introduction:

A merely symmetric operator cannot be exponentiated to yield a unitary. Its exponentiation only yields isometries, and similarly, as we recalled in the previous section, also the Cayley transform yields mere isometries.

Does this mean that simple symmetric operators cannot play a significant role? Perhaps unexpectedly, the answer is that simple symmetric operators, at least those with equal deficiency indices, are nevertheless very much at the heart of the algebra $B(H)$!

Let us first consider the simple but instructive example of the differential operator $X = i \partial_\lambda$:

We consider this operator $X$ on a dense domain $D \subseteq L^2([0, 1])$, where all functions $\phi(\lambda) \in D$ are infinitely differentiable and obey the boundary condition $\phi(0) = \phi(1) = 0$. On $D$, the operator $X$ is simple symmetric with deficiency indices $(1, 1)$.

The exponentiation $S(a) := \exp(-iaX)$, being the exponentiation of a differentiation operator, $S(a) = \exp(a\partial_\lambda)$, is of course a translation operator $S(a)\phi(\lambda) = \phi(\lambda + a)$. However, since $S(a)$ is translating functions confined to an interval, $S(a)$ is not unitary, but is instead a mere isometry: $S(a)$ can only act on functions who’s support is such that $S(a)$ is not translating their support beyond the ends of the interval.

We can, however, extend the operator $X$ to a self-adjoint operator $X_u$, by extending its domain to include functions with the boundary condition $\phi(0) = u \phi(1)$. Here, the choice of the phase $u \in U(1)$ labels the self-adjoint extension. In this way, one arrives at operators $S_u(a) := \exp(-iaX_u)$ which are self-adjoint. These translation operators do in fact translate arbitrary functions. Whatever part of the function is translated beyond the interval boundary, reappears from the other boundary into the interval - phase rotated by the phase $u$. 
Let us now consider the set of all unitaries which can be generated by the self-adjoint extensions $X_u$. Clearly, we can compose unitaries $S_u(a), S_{u'}(a')$ from different self-adjoint extensions, i.e. different boundary conditions, to obtain more unitaries. For example, we can translate in one direction with one phase relation followed by a translation back by the same amount - crucially, however, the forward and the backward translations need not have the same boundary conditions, i.e. they need not be using the same self-adjoint extension. For example,

$$T := S_{u'}(-a)S_u(a)$$

(4)

for some arbitrary fixed $0 < a < 1$ is a unitary which does not translate at all. Instead, it phase rotates functions $\phi(\lambda)$ by the phase $(u')^{-1}u \in U(1)$ on the interval $(1-a, 1)$ and acts as the identity for that part of functions which is defined on the interval $(0, 1-a)$. It becomes clear that the self-adjoint extensions of $X$ can be used not only to translate functions. In addition, by suitably combining unitaries $S_u(a)$ for different $u \in U(1)$ and $a \in (-1, 1)$, we can arrive at unitaries which describe arbitrary local phase rotations in $\lambda$-space!

The same argument goes through if we define a differential operator $X$ which acts on $r$ copies of the interval. Then, its deficiency indices are $(r, r)$, and the only change to the above discussion is that the boundary conditions of the self-adjoint extensions are now determined by unitary matrices $u \in U(r)$, rather than simply by a phase:

$$\phi_i(0) = \sum_{j=1}^{r} u_{ij} \phi_j(1)$$

(5)

We conclude that the exponentiated self-adjoint extensions of $X$ are able to generate arbitrary local $U(r)$-transformations - in addition to the translations which they of course also generate.

This means that the set of self-adjoint extensions of this simple symmetric $X$ with equal deficiency indices $r$ generate translations, accelerations, and $r$-dimensional local iso-rotations. In fact, as will be shown in detail in [4], the self-adjoint extensions always generate the whole of $B(H)$! In order to be precise, let us now use the unitary extensions of the Cayley transform, as discussed in section 2: The statement is that the weak closure of the *-algebra generated by the coset of all unitary extensions of the Cayley transform of a simple symmetric $X$ with finite equal deficiency indices, is $B(H)$. Therefore, if simple symmetric $X$ emerge in a physical theory, they may be expected to play a central rôle.

Indeed, as we will discuss at the end, there automatically arises an $r$-dimensional ‘isospinor’ structure if the deficiency indices are $(r, r)$.
4 Symmetry and self-adjointness

Let us now prepare the ground for answering the second of the questions raised in the introduction.

In order to clarify in which ways space-times described by simple symmetric operators would differ from space-times described by self-adjoint operators, let us recall from section 2 that simple symmetric operators can be categorized by their deficiency indices into two classes. The \( X \) with finite, equal deficiency indices \((r, r)\) have a \( U(r) \)-family of self-adjoint extensions, and those with unequal finite deficiency indices do not have self-adjoint extensions.

The spatial short-distance structures described by simple symmetric operators were named ‘fuzzy’ in \([\text{5}]\). The particular subclasses of operators of equal and unequal deficiency indices were denoted fuzzy-A and fuzzy-B, respectively, in \([\text{6}]\). There, it was also first described that there exists a physically more intuitive, equivalent definition of the fuzzy-A case:

\[ X \text{ is simple symmetric with equal deficiency indices exactly iff there exists a positive function } \Delta X_{\text{min}}(\xi) > 0, \]

so that for each \( \xi \in \mathbb{R} \), all normalized \(|\phi\rangle \in D\) with expectation \( \langle \phi | X | \phi \rangle = \xi \) obey

\[ (\Delta X)_{|\phi\rangle} \geq \Delta X_{\text{min}}(\xi). \]

Using this characterization, we can now describe operators of the type fuzzy-A as yielding spaces in which the standard deviation in positions - we are here formally using quantum mechanical terminology - could not be made smaller than some finite lower bound \( \Delta X_{\text{min}}(\xi) \). Since the lower bound is in general some function of the expectation value \( \xi \) the amount of ‘fuzzyness’ can in general vary from place to place.

The fuzzy-B case, i.e. the case simple symmetric operators \( X \) with unequal deficiency indices can be shown to describe short distance structures which are such that there exist sequences of vectors in the physical domain such that \( \Delta x \) converges to zero. The fuzzy-B type short-distance structures are ‘fuzzy’ in the sense that vectors of increasing localization around different expectation values then in general do not become orthogonal.

Among studies into the structure of space-time at the Planck scale there appeared evidence for effective correction terms to the uncertainty relation, in the simplest case

\[ \Delta x \Delta p \geq \frac{\hbar}{2} \left( 1 + \beta (\Delta p)^2 + \ldots \right) \]

for some \( \beta > 0 \), related either to the Planck scale or to a string scale. For a sufficiently small constant \( \beta \), the correction term is negligible at present-day experimentally accessible scales. The correction term nevertheless implies a crucial new feature, namely
that $\Delta x$ is now finitely bounded from below by

$$\Delta x_{\text{min}} = h \sqrt{\beta} \quad (9)$$

For reviews on the origins of this type of uncertainty relation, see e.g. [7, 8].

It is clear from the physical characterization of the fuzzy-A case above, that any operator, within any theory, which obeys this type of uncertainty relation, is simple symmetric with equal deficiency indices.

5 Conclusions

Let us now try to tie the discussed issues together, in order to answer the questions raised in the beginning. We asked about the mathematical and physical implications of relaxing the ‘dictionary entry’ for real variables from self-adjointness to mere symmetry. We have discussed some aspects already, but it remains to address the question which rôle unitaries and isometries may play in a theory with simple symmetric $X_i$. To this end we will also have to address the question how, in practice, a physical theory may determine that its operators $X_i$ are merely symmetric rather than self-adjoint.

Let us begin by noting that a generic symmetric operator may be self-adjoint and simple symmetric on different subspaces. This means that it can describe arbitrary mixtures of the basic cases of short-distance structures, namely continua, lattices and the two fuzzy cases.

We recall that there are numerous physical systems in which the self-adjoint extensions of various differential operators correspond physically to choices of boundary conditions - which are imposed externally onto the physical system. The question arises, therefore, whether a theory can intrinsically specify that an operator $X_i$ is simple symmetric - even if self-adjoint extensions exist in the Hilbert space. To see that theories can easily yield such intrinsic domain specifications, we note that any dynamical or kinematical operator equation within a theory involves a domain condition, in particular if unbounded operators are involved. If a theory requires equations among its operators, then a physical domain can only be a domain on which all of the operators are well-defined.

For example, the stringy uncertainty relation above, can be induced by the commutation relation $[x,p] = i\hbar (1 + \beta p^2)$. It is clear that in any domain $D$ in a Hilbert space $H$, on which this equation holds, the operator $X$ can only be simple symmetric, as is easily concluded from the implied uncertainty relation. This is in spite of $x$ having self-adjoint extensions in the Hilbert space. We remark that the functional analysis of generalized uncertainty relations of this type was first considered in [9].

In this way, any dynamical or kinematical equation between the $X_i$ and arbitrary other operators within a fundamental theory of quantum gravity can induce domain
specification for the operators $X_i$. Any such theory-intrinsic domain specification could determine that certain $X_i$ are simple symmetric and cannot be extended within the physical domain, i.e. within the domain on which the theory’s equation hold. Those dynamical or kinematical equations could of course also specify unequal deficiency indices. Domain specifications and possibly a noncommutativity of the $X_i$ may also arise with path integrals. In this context, see [10, 11].

Of course, the structure of space-time may in general vary arbitrarily from place to place in particular, in an in general $n$ dimensional noncommutative space. This is here reflected by the fact that the functional analysis of any one $X_i$ is a function of the analysis of the other $X_i$.

We can now finally address the question of isometries and unitaries: Simple symmetric operators, though densely defined cannot be defined on the entire Hilbert space, since they are discontinuous operators, i.e. they are blind to part of the Hilbert space. As we have seen in the case of equal deficiency indices, all vectors which would describe structure smaller than a finite smallest uncertainty are cutoff from the domain. Unitaries and isometries, however, are bounded operators which by continuity can see all of the Hilbert space. No part of the Hilbert space can be hidden from them. This indeed yields a mechanism by which the cutoff degrees of freedom reappear, producing an isospinor structure of internal degrees of freedom:

Let us consider the case of a single $X$ with equal deficiency indices in more detail: For each real $\xi$ there can be shown [4] to exist a self-adjoint extension in which $\xi$ is an $r$-fold degenerate eigenvalue. Thus, each real $\xi$ is an $r$-fold eigenvalue of the adjoint $X^*$ with eigenvector say $|\xi, i\rangle$, where $i = 1, 2, ..., r$. Using these, any vector can be expanded in an ‘isospinor function’ $\phi_i(\xi) := \langle \xi, i|\phi\rangle$. At large distances the eigenvectors become orthogonal, yielding an ordinary isospinor function. From a cutoff with deficiency indices $(r, r)$ an isospinor structure and a corresponding local $U(r)$ structure have emerged automatically!

A precise definition of ‘gauge transformations’, can be given which indeed yields ordinary local gauge transformations at large distances: the idea is to define gauge transformations as isometries which commute with the $X$ and which map from a physical domain into a physical domain. The emergence of gauge transformations in this way was first discussed in [12]. A more detailed study, including also the case of unequal deficiency indices, will given in [4].

To summarize, we conclude that the mathematics of symmetric versus self-adjoint and isometric versus unitary operators offers new possibilities for the description of space-time short-distance structures within theories which are linear as quantum theories. In particular, there emerges a new mechanism by which ultraviolet cutoff degrees of freedom can reappear by inducing an isospinor structure of internal degrees of freedom.

It should be most interesting to study potential applications of the tools of non-commutative geometry in this context, in particular, as cohomological methods tend
to focus on larger scale structures.

Finally, we remark that in [13] the fuzzy-A type short-distance structure has recently been applied to the resolution of the transplanckian energy problem of black hole radiation.

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