Chaos in the triple-well $\Phi^6$-van der Pol oscillator driven by periodically external and nonlinear parametric excitations

Liang-qiang Zhou $^{1,2}$*, Fang-qi Chen $^1$*

1. Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, PR China
2. Department of Mechanics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, PR China

Email: zlgrex@tom.com, cfqyf@eyou.com

Abstract. The chaotic dynamics of a nonlinear damped triple-well $\Phi^6$-van der Pol oscillator under periodically external and nonlinear parametric excitations is studied. Chaos arising from homoclinic and heteroclinic crossings is analyzed using Melnikov method. The chaotic behaviors are compared with a periodically external excitation, a linear parametric excitation and a nonlinear parametric excitation. The critical curves which separate the chaotic regions from the non-chaotic regions are plotted. Especially, there is “a controllable frequency” leading to no chaos for all excitation amplitudes.

1. Introduction
Chaotic behavior in nonlinear dynamical systems has attracted much attention over the last 20 years. Many nonlinear damping systems exist chaotic behavior. Yagasaki studied a two-frequency quasi-periodically forced DVP oscillator with single well potential in certain special cases [1]. Zhang et al studied the chaotic dynamics in nonlinear non-planar oscillations of a nonlinear parametrically excited cantilever beam [2]. Yang et al investigated the chaotic behavior of an extended Duffing Vander Pol oscillator in a $\Phi^6$ potential under additive harmonic and bonded noise excitations for a special parametric choice [3]. Sun et al studied the chaotic behaviors of a particle in a triple well $\Phi^6$ potential possessing both homoclinic and heteroclinic orbits under harmonic and Gaussian white noise excitations [4]. Yu et al studied the complex dynamics in an extended Duffing Vander Pol Oscillator [5]. Using Melnikov method, Siewe et al studied the chaotic behavior in the triple-well $\Phi^6$-Van der Pol oscillator driven by external and linear parametric excitations [6].

In this paper, the chaotic behavior in the triple-well $\Phi^6$-van der Pol oscillator driven by periodically external and nonlinear parametric excitations is studied, and some new dynamical phenomena are obtained.

* To whom any correspondence should be addressed.
2. Formulation of the problem

Consider the externally and parametrically forced $\Phi^6$-Van der Pol oscillator described by the following equation:

$$\ddot{x} - \mu(1-x^2)\dot{x} + g(x,t)x + \lambda x^3 + \beta x^5 = f \cos \omega t$$

where $g(x,t) = \omega_0^2 [1 + (\eta + \delta n^2) \cos \omega t]$. 

Scaling $f = \tilde{f}, \mu = \tilde{\mu}$ and $\eta = \tilde{\eta}$, then dropping the tildes, Eq. (1) can be rewritten as a first order system in the form

$$\dot{x} = y$$

$$\dot{y} = -\omega_0^2 x - \lambda x^3 - \beta x^5 + \varepsilon [f \cos \omega t + \mu (1-x^2) y - \omega_0^2 \eta (x + \delta x^3) \cos \omega t]$$

where $\varepsilon$ is a small parameter ($\varepsilon << 1$) characterizing the smallness of the dissipative and forced terms. The unperturbed system ($\varepsilon = 0$) is given by

$$\dot{x} = y$$

$$\dot{y} = -\omega_0^2 x - \lambda x^3 - \beta x^5$$

System (3) is a completely integral Hamiltonian system with Hamiltonian function given by

$$H = y^2 / 2 + \omega_0^2 x^2 / 2 + \lambda x^4 / 4 + \beta x^6 / 6 .$$

We notice that in the case of triple-well potential, $(\lambda < 0, \beta > 0)$, the system has five equilibrium points. Among these five equilibrium points, we have three stable fixed points at $(0,0)$ and $(\pm x_1,0)$, and two unstable saddle points at $(\pm x_2,0)$, where

$$x_1 = \frac{-1}{\sqrt{2} \beta} (\lambda + \sqrt{\lambda^2 - 4\beta \omega_0^2})$$

and

$$x_2 = \frac{-1}{\sqrt{2} \beta} (\lambda - \sqrt{\lambda^2 - 4\beta \omega_0^2})$$

System (3) has heteroclinic orbits connecting the two saddle points defined as

$$(x_{het}, y_{het}) = (\pm \frac{\sqrt{2} x_1 \sinh(T_t/2)}{\sqrt{\xi + \cosh(T_t)}} \pm \frac{\sqrt{2} T_t x_1 (1 - \xi) \cosh(T_t/2)}{2(\xi + \cosh(T_t))^{1/2}}$$

and a symmetric pair of homoclinic trajectories connected each unstable point to itself given by

$$(x_{hom}, y_{hom}) = (\pm \frac{\sqrt{2} x_1 \cosh(T_t/2)}{\sqrt{\xi + \cosh(T_t)}} \pm \frac{\sqrt{2} T_t x_1 (1 - \xi) \sinh(T_t/2)}{2(\xi + \cosh(T_t))^{1/2}}$$

where

$$T_t = x_1^2 \sqrt{2} \beta (\rho^2 - 1) \ , \ A^2 = x_1^2 (\rho^2 + 3) \ , \ \xi = \frac{5 - 3 \rho^2}{3 \rho^2 - 1} \ , \ \rho^2 = \frac{\lambda - \sqrt{\lambda^2 - 4 \beta \omega_0^2}}{\lambda + \sqrt{\lambda^2 - 4 \beta \omega_0^2}}$$

3. Melnikov’s analysis of chaotic behaviors

Here we compute the Melnikov’s function $M(t_o)$, letting the variable $t \to t + t_o$, after some arrangements, the Melnikov function of system (2) is as follows:

For the homoclinic orbits:
\[ M(t_0) = -f \sin \omega_0 \int_{-\infty}^{+\infty} y_{\text{hom}}(t) \sin \omega t dt + \mu \int_{-\infty}^{+\infty} y^2_{\text{hom}}(t)(1-x^2_{\text{hom}}(t))dt \]

\[ -\omega^2_0 \eta_0 \int_{-\infty}^{+\infty} x_{\text{hom}}(t)y_{\text{hom}}(t) \sin \omega t dt - \omega^2_0 \eta_0 \int_{-\infty}^{+\infty} x^3_{\text{hom}}(t)y_{\text{hom}}(t) \sin \omega t dt \]

\[ \equiv -fI_1 \sin \omega_0 + \mu I_0 - \omega^2_0 \eta_0 (J_2 + \delta \overline{J}_3) \sin \omega_0 \]  

(7)

For the heteroclinic orbits:

\[ M(t_0) = f \cos \omega_0 \int_{-\infty}^{+\infty} y_{\text{het}}(t) \cos \omega t dt + \mu \int_{-\infty}^{+\infty} y^2_{\text{het}}(t)(1-x^2_{\text{het}}(t))dt \]

\[ -\omega^2_0 \eta_0 \int_{-\infty}^{+\infty} x_{\text{het}}(t)y_{\text{het}}(t) \sin \omega t dt - \omega^2_0 \eta_0 \int_{-\infty}^{+\infty} x^3_{\text{het}}(t)y_{\text{het}}(t) \sin \omega t dt \]

\[ \equiv fJ_1 \cos \omega_0 + \mu J_0 - \omega^2_0 \eta_0 (J_2 + \delta J_3) \sin \omega_0 \]  

(8)

where \[ I_0 = \mu \int_{-\infty}^{+\infty} y^2_{\text{hom}}(t)(1-x^2_{\text{hom}}(t))dt \]

\[ I_1 = \int_{-\infty}^{+\infty} y_{\text{hom}}(t) \sin \omega t dt = 2 \chi \sin \frac{2\omega}{T_1}, \]

\[ I_2 = \int_{-\infty}^{+\infty} x_{\text{hom}}(t)y_{\text{hom}}(t) \sin \omega t dt \]

\[ I_3 = \int_{-\infty}^{+\infty} x^3_{\text{hom}}(t)y_{\text{hom}}(t) \sin \omega t dt \]

\[ J_0 = \mu \int_{-\infty}^{+\infty} y^2_{\text{het}}(t)(1-x^2_{\text{het}}(t))dt \]

\[ J_1 = \int_{-\infty}^{+\infty} y_{\text{het}}(t) \sin \omega t dt \]

\[ J_2 = \int_{-\infty}^{+\infty} x_{\text{het}}(t)y_{\text{het}}(t) \sin \omega t dt \]

\[ J_3 = \int_{-\infty}^{+\infty} x^3_{\text{het}}(t)y_{\text{het}}(t) \sin \omega t dt \]  

(9)

\[ I_0, J_0 \] are computed by quadrature, while \[ I_1 \sim I_3, J_1 \sim J_3 \] are calculated by the residues’ theorem, and the results are omitted here own to too complex.

Then the conditions for homoclinic or heteroclinic orbits crossing are as follows, respectively.

For the homoclinic orbits:

If the external excitation amplitude \( f \) is fixed, letting

\[ R_{\eta_{\text{hom}}}^{\text{min}} = \min_{t_0 \in (0,2\pi/\omega )} \max \left\{ \omega_0^2 (I_2 + \delta \overline{J}_3) \sin \omega_0, 0 \right\} \]

\[ \mu J_0 - fI_1 \sin \omega_0 \]

\[ R_{\eta_{\text{hom}}}^{\text{max}} = \max_{t_0 \in (0,2\pi/\omega )} \max \left\{ \omega_0^2 (I_2 + \delta \overline{J}_3) \sin \omega_0, 0 \right\} \]

the condition is
If the parametric excitation amplitude $\eta$ is fixed, letting
\[
Rf^\text{min}_\text{hom} = \min_{t_0 \in (0, 2\pi / \omega)} \max\left\{ \frac{I_1 \sin \omega t_0}{\mu t_0 - \omega_0^2 \eta (I_2 + \delta I_3) \sin \omega t_0}, 0 \right\}
\]
\[
Rf^\text{max}_\text{hom} = \max_{t_0 \in (0, 2\pi / \omega)} \max\left\{ \frac{I_1 \sin \omega t_0}{\mu t_0 - \omega_0^2 \eta (I_2 + \delta I_3) \sin \omega t_0}, 0 \right\}
\]
the condition is
\[
Rf^\text{min}_\text{hom} < 1 / f < Rf^\text{max}_\text{hom}.
\] (11)

If the external excitation amplitude is equal to the parametric excitation amplitude, i.e. $f = \eta$, letting
\[
R^\text{min}_\text{hom} = \min_{t_0 \in (0, 2\pi / \omega)} \max\left\{ \frac{I_1 \sin \omega t_0 + (I_2 + \delta I_3) \sin \omega t_0}{\mu t_0}, 0 \right\}
\]
\[
R^\text{max}_\text{hom} = \max_{t_0 \in (0, 2\pi / \omega)} \max\left\{ \frac{I_1 \sin \omega t_0 + (I_2 + \delta I_3) \sin \omega t_0}{\mu t_0}, 0 \right\}
\]
the condition is
\[
R^\text{min}_\text{hom} < 1 / f = 1 / \eta < R^\text{max}_\text{hom}.
\] (12)

Analogously, for the heteroclinic orbits:

When the external excitation amplitude $f$ is fixed, letting
\[
R^\text{min} = \min_{J_0 \in (0, 2\pi / \omega)} \max\left\{ \frac{\omega_0^2 (J_2 + \delta J_3) \sin \omega t_0}{f J_1 \cos \omega t_0 + \mu J_0}, 0 \right\}
\]
\[
R^\text{max} = \max_{J_0 \in (0, 2\pi / \omega)} \max\left\{ \frac{\omega_0^2 (J_2 + \delta J_3) \sin \omega t_0}{f J_1 \cos \omega t_0 + \mu J_0}, 0 \right\}
\]
the condition is
\[
R^\text{min}_\text{het} < 1 / \eta < R^\text{max}_\text{het}.
\] (13)

When the parametric excitation amplitude $\eta$ is fixed, letting
\[
Rf^\text{min}_\text{het} = \min_{t_0 \in (0, 2\pi / \omega)} \max\left\{ \frac{J_1 \cos \omega t_0}{\omega_0^2 \eta (J_2 + \delta J_3) \sin \omega t_0 - \mu J_0}, 0 \right\}
\]
\[
Rf^\text{max}_\text{het} = \max_{t_0 \in (0, 2\pi / \omega)} \max\left\{ \frac{J_1 \cos \omega t_0}{\omega_0^2 \eta (J_2 + \delta J_3) \sin \omega t_0 - \mu J_0}, 0 \right\}
\]
the condition is
When the external excitation amplitude is equal to the parametric excitation amplitude, i.e. \( f = \eta \), letting

\[
R^{\text{min}}_{\text{het}} = \min_{t_0, \omega(0,2\pi/\omega)} \max \left\{ \frac{\omega_0^2 (J_2 + \delta J_1) \sin \omega_0 - J_1 \cos \omega_0}{\mu J_0}, 0 \right\}
\]

\[
R^{\text{max}}_{\text{het}} = \max_{t_0, \omega(0,2\pi/\omega)} \min \left\{ \frac{\omega_0^2 (J_2 + \delta J_1) \sin \omega_0 - J_1 \cos \omega_0}{\mu J_0}, 0 \right\}
\]

the condition is

\[
R^{\text{min}}_{\text{het}} < 1/ f < R^{\text{max}}_{\text{het}}.
\] (14)

The critical curves of three cases, that is, the external excitation amplitude equals zero \(( f = 0 \)\), the parametric excitation amplitude equals zero \(( \eta = 0 \)\), or the external excitation amplitude equals to the external excitation amplitude \(( f = \eta \)\), are shown as follows:

We know that the critical curves separate the non-chaotic regions (above) from the possibly chaotic regions (below). From Fig.1(a)-(e), we can obtain the following conclusions:

1. No chaos take place for the system when \((1/f, \omega)\) or \((1/\eta, \omega)\) are in region A. The homoclinic orbits are chaotically excited, but the heteroclinic orbits aren’t while \((1/f, \omega)\) or \((1/\eta, \omega)\) belong to region B. The cases are opposite while \((1/f, \omega)\) or \((1/\eta, \omega)\) belong to region C. Both the homoclinic and heteroclinic orbits are chaotically excited when \((1/f, \omega)\) or \((1/\eta, \omega)\) are in region D.

2. For the system with a periodically linear parametric excitation \((\delta = 0, f = 0)\), the critical curve has the classical bell shape, this means that, with the excitations possessing sufficiently small or very large frequency, the system is not chaotically excited via the homoclinic orbits. The critical curve of the heteroclinic orbits is decreased monotonously with \(\omega\). This means for large frequency, the system is not chaotically excited via the heteroclinic orbits. The critical curve of the homoclinic orbits is always below that of the heteroclinic orbits, which means the homoclinic orbits are more easily chaotically excited than the heteroclinic orbits.

3. For the system with a periodically external excitation \((\delta = 0, \eta = 0)\), the critical curve of the homoclinic orbits approaches infinite when \(\omega \to 0\), so that the system is always chaotically excited via homoclinic orbits when the frequency of excitation is small. The critical curve of the homoclinic orbits is above that of the heteroclinic orbits when \(\omega \to 0\), but below that of the heteroclinic orbits when \(\omega\) is large. This means, when \(\omega\) is near 0, as the increasing of excitation amplitude, first the homoclinic orbits are chaotically excited, then both the homoclinic and heteroclinic orbits are chaotically excited. The critical curves of both the homoclinic and heteroclinic orbits are deceased monotonously with \(\omega\). The case is similar for the system with combined external and linear parametric excitation \((\delta = 0, f = \eta)\).

4. For the system with periodically nonlinear excitations \((\delta = 1, \eta \neq 0)\), the critical curve of the homoclinic orbits first decreases quickly to zero from infinity and then increases, at last it decreases to zero as \(\omega\) increases from zero. There exists a “controllable frequency \(\omega^*\)” with \(\omega\) near 0. This means for this class of system, excited at the “controllable frequency \(\omega^*\)” chaotic motions don’t take place via the homoclinic orbits no matter how large the excitation amplitude is. The critical curve of the heteroclinic orbits decreases to zero from infinity with some sway as \(\omega\) increases from zero. So chaos
always takes place via both the homoclinic orbits and the heteroclinic orbits when the frequency $\omega$ is near zero.

![Graphs showing critical curves for different cases](image)

Fig.1 The critical curves for the case of (a) $\delta = 0$, $f = 0$, (b) $\delta = 0$, $\eta = 0$, (c) $\delta = 0$, $f = \eta$, (d) $\delta = 1$, $f = 0$, (e) $\delta = 1$, $f = \eta$

4. Conclusions

With Melnikov method, the chaotic motion of a nonlinear damped triple-well $\Phi^s$-Van der Pol oscillator under periodically external and nonlinear parametric excitations is studied. For the system with a nonlinear periodically parametric excitation, when the frequency $\omega$ is near zero, chaos always take place via both the homoclinic orbits and the heteroclinic orbits. There exists “a controllable frequency $\omega$” for the homoclinic orbits. This means, for this class of system excited at the “controllable frequency $\omega$”, chaos don’t occur via the homoclinic orbits no matter how large the
excitation amplitude is. The results provide some inspiration and guidance for the analysis and
dynamic design for this class of structures.

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