PATHWISE SOLUTION TO ROUGH STOCHASTIC LATTICE DYNAMICAL SYSTEM DRIVEN BY FRACTIONAL NOISE

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Abstract. The fBm-driving rough stochastic lattice dynamical system with a general diffusion term is investigated. First, an area element in space of tensor is desired to define the rough path integral using the Chen-equality and fractional calculus. Under certain conditions, the considered equation is proved to possess a unique local mild path-area solution.

1. Introduction. Due to the spatially discrete structure, the deterministic lattice dynamical systems occur in a wide variety of applications and various properties of solutions to lattice dynamical systems have been studied, see e.g. [11, 28, 2, 33, 34, 30, 35] and references therein. As we know, dynamical systems are often under random perturbation, and the noise can drastically modify the deterministic dynamics, and even induce new types of dynamical behavior. Therefore, stochastic lattice dynamical systems (SLDS) arise naturally when taking into account these random influences or uncertainties. Since Bates et al. firstly studied the existence of a compact global random attractor within the set of tempered random bounded sets of the one-dimensional SLDS [1], many works in the literature have been extensively done regarding the existence of global random attractors for first (or second)-order SLDS with additive white noises [27, 26, 25, 31, 7], or a multiplicative white noise [6, 22, 23, 8, 3, 5].

In physics, the uncertainties are used to describe the interaction between the (small) system and its (large) environment. The driving noises could be white or
colored, Markovian or non-Markovian, Gaussian or non-Gaussian, and semimartingale or non-semimartingale. There is no a-priori reason to assume that the stochastic forces are independent of disjoint time intervals. This fact motivates us to choose the so-called fractional Brownian motion (fBm) as driving process for the SLDS. In this respect, the random attractor of the SLDS driven by an fBm with Hurst parameter bigger than 1/2 was shown to be a singleton sets under usual dissipativeness conditions [19, 20]. The synchronous phenomena of the solutions to system driven by fractional environmental noises on finite lattice was studied in [21] when $H > 1/2$. Later on, the existence of a global forward attracting set of a stochastic lattice system with a Caputo fractional time derivative was established in the weak mean-square topology [32]. Recently, stochastic lattice dynamical systems driven by fractional Brownian motion with $H \in (1/2, 1)$ was proved to have a unique pathwise mild solution, which is exponentially stable under suitable conditions [4]. There has, however, been little mention about the SLDS in the rough paths case, say, $H \in (0, 1/2]$. To define the stochastic integral of rough functions, Hu and Nualart formulated a second equation for the so-called area in the space of tensors and proved an existence and uniqueness result for finite-dimensional stochastic differential equations [24], in which the fBm is such that $H \in (1/3, 1/2]$. Later on, Garrido-Atienza et al. have extended this idea to infinite-dimensional setting and proved the existence of a unique local mild solution consisting in a path and an area component [13]. Under some more restrictive conditions, they further proved the existence and uniqueness of a global mild pathwise solution [14]. To the best of our knowledge, the existence and uniqueness of a mild pathwise solution to the SLDS driven by an fBm with $H \in (0, 1/2]$ is a challenging and rather open problem up to now.

In this paper, we consider the stochastic lattice equation driven by fractional Brownian motion with $H \in (1/3, 1/2]$: 

$$
\frac{du_i(t)}{dt} = (\nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + f_i(u_i))dt + g_i(u_i)dB^H_i(t), \quad i \in Z
$$

(1)

with initial condition $u_i(0) = u^0_i$, where $Z$ denotes the integers set, $u_i \in \mathbb{R}$, $\nu, \lambda > 0$, each $B^H_i(t)$ is a one-dimensional two-sided fBm with Hurst parameter $H \in (1/3, 1/2]$, and $f_i, g_i$ are smooth functions satisfying proper conditions, which will be made precise below.

This paper is organized as follows. In Section 2 we present a brief introduction to some spaces, definitions, and preliminary results. In Section 3 we consider the local existence and uniqueness of mild solution to (1). Specifically, in Subsection 3.1 we give assumptions of $f_i$ and $g_i$, and then rewrite (1) into a stochastic integral equation at first. In Subsection 3.2 we define the mild path-area solution in a limit points space. At last, in Subsection 3.3 we present the local existence and uniqueness theorem and prove it with some key technique estimates.

2. Preliminaries. We now present a brief introduction to some spaces, definitions, and preliminary facts, which are used throughout this paper.

Let $V = (V, \langle \cdot, \cdot \rangle, | \cdot |)$ be a separable Hilbert space, and $V \times V$, $V \otimes V$ be the cartesian product and the tensor product of $V$ respectively. For convenience, we denote the norm of $V \otimes V$ by $| \cdot |_{V \otimes V}$ and the rank-one tensor by $x \otimes y$ for $x, y \in V$. Then $(e_i \otimes V e_j)_{i,j \in \mathbb{N}}$ is a complete orthogonal system of $V \otimes V$ where $(e_i)_{i \in \mathbb{N}}$ can be any complete orthogonal system for $V$. For $0 \leq T_1 < T_2 < \infty$ and $0 < \beta < 1$, we use $C_\beta(T_1, T_2; V)$ to denote the Banach space of $V$-valued Hölder continuous
function with the norm
\[ ||u||_{\beta,T_1,T_2} = \sup_{t \in [T_1,T_2]} |u(t)| + ||u||_{\beta,T_1,T_2}, \]
where
\[ ||u||_{\beta,T_1,T_2} = \sup_{T_1 \leq s < t \leq T_2} \frac{|u(t) - u(s)|}{(t-s)^{\beta}}. \]

Let \( \Delta_{T_1,T_2} \) be given by
\[ \Delta_{T_1,T_2} = \{(s,t) : T_1 \leq s < t \leq T_2 \}. \]

\[ ||v||_{\beta+\beta',\Delta_{T_1,T_2}} = \sup_{T_1 \leq s < t \leq T_2} \frac{|v(s,t)|_{V \otimes V}}{(t-s)^{\beta+\beta'}} < \infty. \]

As shown in [13, Lemma 4], \( C_{\beta+\beta'}(\Delta_{T_1,T_2}; V \otimes V) \) is Banach space.

In order to treat the stochastic integral of rough functions, we introduce several main features of fractional calculus [29]. To do this, assume that for some \( 0 \leq T_1 < T_2 < \infty, u \in C_\beta([T_1,T_2]; V), \omega \in C_\beta([T_1,T_2]; V) \) and \( v \in C_{\beta+\beta'}(\Delta_{T_1,T_2}; V \otimes V) \) for \( 1/3 < \beta < \beta' < 1/2 \), and that these three elements satisfy the Chen-equality [9], given by
\[ v(s,r) + v(r,t) + (u(r) - u(s)) \otimes V (\omega(t) - \omega(r)) = v(s,t) \quad (2) \]
for \( T_1 < s < r < t \leq T_2 \). Let \( V_1, V_2 \) be two separable Hilbert spaces and \( L_2(V_1; V_2) \) be the space of Hilbert-Schmidt operators from \( V_1 \) to \( V_2 \). Throughout this paper, we assume that \( 0 < \alpha < 1 \).

**Definition 2.1.** The Weyl fractional derivative of general measurable functions \( u : [s,t] \to V_1 \) and \( \omega : [s,t] \to V_2 \) of order \( \alpha \) and \( 1 - \alpha \), respectively, are defined for \( s < r < t \) by
\[ D_{s+}^\alpha u[r] = \frac{1}{\Gamma(1-\alpha)} \left( \frac{u(r)}{(r-s)^{\alpha}} + \alpha \int_s^r \frac{u(r) - u(q)}{(r-q)^{1+\alpha}} dq \right) \in V_1, \]
\[ D_{t-}^{1-\alpha} \omega_{t-}[r] = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left( \frac{\omega(r) - \omega(t-)}{(t-r)^{1-\alpha}} + (1-\alpha) \int_r^t \frac{\omega(r) - \omega(q)}{(q-r)^2^{2-\alpha}} dq \right) \in V_2, \]
where \( \omega_{t-} = \omega(t) - \omega(t-) \) and \( \Gamma(\cdot) \) denotes the Euler Gamma function.

**Definition 2.2.** The generalized fractional derivative of the tensor valued element \( v \) is defined for \( T_1 \leq r < t \leq T_2 \) by
\[ D_{t-}^{1-\alpha} v[r] = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left( \frac{v(r,t)}{(t-r)^{1-\alpha}} + (1-\alpha) \int_r^t \frac{v(r,q)}{(q-r)^2^{2-\alpha}} dq \right). \]

**Definition 2.3.** For \( g : V_1 \to L_2(V_1; V_2) \), the compensated fractional derivative of order \( \alpha \) of \( g(u(\cdot)) \) is given by
\[ \dot{D}_{s+}^\alpha g(u(\cdot))[r] = \frac{1}{\Gamma(1-\alpha)} \left( \frac{g(u(r))}{(r-s)^{\alpha}} + \alpha \int_s^r \frac{g(u(r)) - g(u(q)) - Dg(u(q))(u(r) - u(q))}{(r-q)^{1+\alpha}} dq \right). \]

It is easy to check that \( D_{s+}^\alpha g(u(\cdot))[r] \) is not well-defined provided that \( \beta < \alpha \), \( g(u(\cdot)) \in C_\beta([T_1,T_2]; V) \) and \( Dg \) is bounded. Hence the classical Young integral
the separable Hilbert space of square summable sequences, equipped with the norm provided that $u, v, \omega$ area

Next, we turn to introduce the driving process. Denote by $\ell^2$ the separable Hilbert space of square summable sequences, equipped with the norm

$$\|u\| = \left(\sum_{i \in \mathbb{Z}} |u_i|^2\right)^{1/2}, \text{ } u = (u_i)_{i \in \mathbb{Z}} \in \ell^2,$$

and the inner product

$$\langle u, v \rangle = \sum_{i \in \mathbb{Z}} u_i v_i, \text{ } u = (u_i)_{i \in \mathbb{Z}}, v = (v_i)_{i \in \mathbb{Z}} \in \ell^2.$$

Additionally, we denote by $\ell^2_\kappa$ the subspace of $\ell^2$ as

$$\ell^2_\kappa := \left\{ u \in \ell^2 : \sum_{i \in \mathbb{Z}} (1 + |i|)^{2\kappa} |u_i|^2 < \infty, \kappa > \frac{1}{2}\right\}.$$

Let us consider the infinite sequence $(e_i)_{i \in \mathbb{Z}}$ in $\ell^2$, which has 1 in position $i$ and 0 in other positions. Then $(e_i)_{i \in \mathbb{Z}}$ is a complete orthonormal basis of $\ell^2$, and $(\frac{e_i}{(1 + |i|)^\kappa})_{i \in \mathbb{Z}}$ is the complete orthonormal basis of $\ell^2_\kappa$. One can check that $\ell^2_\kappa$ is a separable Hilbert space with inner product

$$\langle u, v \rangle_{\ell^2_\kappa} = \sum_{i \in \mathbb{Z}} (1 + |i|)^{2\kappa} u_i v_i,$$

and norm

$$\|u\|_{\ell^2_\kappa} = \left(\sum_{i \in \mathbb{Z}} (1 + |i|)^{2\kappa} |u_i|^2\right)^{1/2}.$$

Throughout this paper, we will consider a fractional Brownian motion (fBm) with values in $\ell^2$ and Hurst parameter $H \in (1/3, 1/2]$. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let us denote $(B^H_t)_{t \in \mathbb{R}}$ by an independent and identically distributed sequence of fBm with same Hurst parameter $H$. This means that each $B^H_t$ is a centered Gaussian process on $\mathbb{R}$ with the covariance

$$\text{cov} \left( B^H_t(s), B^H_t(t) \right) = \frac{1}{2} \left( |s|^{2H} + |t|^{2H} - |t - s|^{2H} \right) \text{ for } s, t \in \mathbb{R}.$$ 

Let $Q$ be a linear operator on $\ell^2$ such that $Qe_i = \sigma_i^2 e_i$, $\sigma = (\sigma_i)_{i \in \mathbb{Z}}$. Thus $Q$ is a non-negative, symmetric, and trace class operator. Then a continuous $\ell^2$-valued fBm $B^H$ with covariance operator $Q$ and Hurst parameter $H$ is defined by

$$B^H(t) = \sum_{i \in \mathbb{Z}} \sigma_i B^H_i(t) e_i, \text{ } t \in \mathbb{R}.$$ 

(3)
In the sequel, we will work with the canonical version of the fBm. Let \( \Omega = C_{0}(\mathbb{R}; \ell^{2}) \) be the space of continuous paths in \( \ell^{2} \) with values zero at zero equipped with the compact open topology. \( \mathcal{F} \) is defined as the Borel-\( \sigma \)-algebra and \( \mathbb{P}_{H} \) is the distribution of \( B^{H}(t) \). Let us consider the Wiener shift given by \( \theta_{t}\omega(\cdot) = \omega(\cdot + t) - \omega(t) \) for \( t \in \mathbb{R} \) and \( \omega \in C_{0}(\mathbb{R}; \ell^{2}) \). It follows from [18, Theorem 1] that the quadruple \( (\Omega, \mathcal{F}, \mathbb{P}_{H}, \theta) \) is an ergodic metric dynamical system. Also, we limit this metric dynamical system to the set \( \Omega \) of \( \beta \)-Hölder continuous paths on \([-m, m]\) for any \( m \in \mathbb{N} \) and \( \beta' \in (1/3, H) \) and denote the restricted metric dynamical system again by \( (\Omega, \mathcal{F}, \mathbb{P}_{H}, \theta) \) with a slight abuse of notation. In particular, let us also identify \( B^{H}(\cdot, \omega) \) and \( \omega(\cdot) \) in the following discussion.

3. Existence and uniqueness of local mild solution.

3.1. Reformulation of the equation (1). Let \( A \) be a linear bounded operator from \( \ell^{2} \) to \( \ell^{2} \) defined by \( Au = ((Au)_{i})_{i \in \mathbb{Z}} \), where

\[
(Au)_{i} = -\nu(u_{i-1} - 2u_{i} + u_{i+1}), \quad i \in \mathbb{Z}.
\]

It is easy to check that \( A = BB^{*} = B^{*}B \), where

\[
(Bu)_{i} = \sqrt{\nu}(u_{i+1} - u_{i}), \quad (B^{*}u)_{i} = \sqrt{\nu}(u_{i-1} - u_{i})
\]

and thus \( \langle Au, u \rangle \geq 0 \) for any \( u \in \ell^{2} \). Let us further consider the linear bounded operator

\[
A_{\lambda}u = Au + \lambda u,
\]

where \( \lambda \) is defined in (1). Then \( \langle A_{\lambda}u, u \rangle \geq 0 \), which implies that \( -A_{\lambda} \) is a negative defined and bounded operator, and thus it generates a uniformly continuous semigroup \( S := e^{-A_{\lambda}t} \) on \( \ell^{2} \). Then we have the following estimates for the semigroup \( S \):

\[
\begin{align*}
\|S(t)\|_{L(\ell^{2}, \ell^{2})} & \leq e^{-\lambda t}, & (4a) \\
\|S(t) - id\|_{L(\ell^{2}, \ell^{2})} & \leq \|A_{\lambda}\|t, & (4b) \\
\|S(t) - S(s)\|_{L(\ell^{2}, \ell^{2})} & \leq \|A_{\lambda}\|\|(t-s)e^{-\lambda s}\|, & (4c)
\end{align*}
\]

for all \( 0 \leq s \leq t \). The first one can be obtained by the energy inequality and the last two follow by the mean value theorem, see also [4]. Furthermore, using these inequalities we can deduce the following properties of \( S(t) \). In fact, for any \( 0 \leq q \leq r \leq s \leq t \) these inequalities hold

\[
\begin{align*}
\|S(t-r) - S(t-q)\|_{L(\ell^{2}, \ell^{2})} & \leq \|A_{\lambda}\|(r-q)e^{-\lambda(t-r)} & (5a) \\
\|S(t-r) - S(s-r) - S(t-q) + S(s-q)\|_{L(\ell^{2}, \ell^{2})} & \leq c\|A_{\lambda}\|^{2}(t-s)(r-q)e^{-\lambda(s-r)} & (5b)
\end{align*}
\]

We are now in position to formulate the needed assumptions for the nonlinear functions \( f_{i} \) and \( g_{i} \):

(A1): The process \( \omega \) is a canonical \( \ell^{2} \)-valued continuous fBm with covariance \( Q_{i} \) defined by (3). In particular, the parameters are chosen to satisfy that \( 1/3 < H \leq 1/2, \ 1/3 < \beta < \beta' < H, \ 1 - \beta < \alpha < 2\beta, \ 2\alpha < \beta + 1 \).

(A2): \( f_{i} \in C^{1}(\mathbb{R}; \mathbb{R}), \sum_{i \in \mathbb{Z}} f_{i}^{2}(0) < \infty \), and there exists a constant \( D_{f} \geq 0 \) such that

\[
|f'_{i}(\zeta)| \leq D_{f}, \quad \zeta \in \mathbb{R}, \ i \in \mathbb{Z}.
\]
(A3): $g_i \in C^4(\mathbb{R}; \mathbb{R})$, and there exist constants $D_g, M_g, F_g \geq 0$ such that for any $\zeta \in \mathbb{R}$, $i \in \mathbb{Z}$, the following inequalities hold:

$$\sqrt{\sum_{i \in \mathbb{Z}} (1 + |i|)^{2\kappa} |g_i(\zeta)|^2} < \infty, \quad \sqrt{\sum_{i \in \mathbb{Z}} (1 + |i|)^{2\kappa} |g_i'(\zeta)|^2} \leq D_g,$$

$$\sqrt{\sum_{i \in \mathbb{Z}} (1 + |i|)^{2\kappa} |g_i''(\zeta)|^2} \leq M_g, \quad \sqrt{\sum_{i \in \mathbb{Z}} (1 + |i|)^{2\kappa} |g_i'''(\zeta)|^2} \leq F_g.$$

Let $u = (u_i)_{i \in \mathbb{Z}}$ be an element of $\ell^2$. Then by (A2) and (A3) one can define the operators $f : \ell^2 \to \ell^2$, $f(u) := (f_i(u_i))_{i \in \mathbb{Z}}$ and $g : \ell^2 \times \ell^2 \to \ell^2_2$, $g(u)v := (g_i(u_i)v_i)_{i \in \mathbb{Z}}$ with $\sum_{i \in \mathbb{Z}} (1 + |i|)^{2\kappa} |g_i(u_i)v_i|^2 < \infty$. Then, similar to [4, Lemma 3.2], $f$ and $g$ are proved to be well-posed in the next result, as well as their main regularity properties are given.

**Lemma 3.1.** (i) The operator $f : \ell^2 \to \ell^2$ is well-defined and is Lipschitz continuous with the Lipschitz constant $D_f$.

(ii) The operator $g : \ell^2 \times \ell^2 \to \ell^2_2$ is well-defined and continuously differentiable. Moreover, its first derivative $Dg$, second derivative $D^2g$ and the third derivative $D^3g$ are bounded with the bounds $D_g, M_g$ and $F_g$, respectively. Furthermore, for $u, v, w, z \in \ell^2$ we have the following regularity properties:

$$\|g(u)\|_{L_2(\ell^2; \ell^2_2)} \leq D_g\|u\| + \|g(0)\|_{L_2(\ell^2; \ell^2_2)}; \tag{6a}$$

$$\|g(u) - g(v)\|_{L_2(\ell^2; \ell^2_2)} \leq D_g\|u - v\|; \tag{6b}$$

$$\|Dg(u) - Dg(v)\|_{L_2(\ell^2; \ell^2_2)} \leq M_g\|u - v\|; \tag{6c}$$

$$\|g(u) - g(v) - Dg(v)(u - v)\|_{L_2(\ell^2; \ell^2_2)} \leq \sqrt{2}D_g\|u - v - (w - z)\| + 2M_g\|u - w\| + \|w - z\|; \tag{6d}$$

$$\|Dg(u) - Dg(v) - (Dg(w) - Dg(z))\|_{L_2(\ell^2; \ell^2_2)} \leq \sqrt{2}M_g\|u - v - (w - z)\| + 2F_g\|u - w\|\|u - v\| + \|w - z\|; \tag{6e}$$

$$\|g(u) - g(v) - Dg(v)(u - v) - (g(w) - g(z) - Dg(z)(w - z))\|_{L_2(\ell^2; \ell^2_2)} \leq \sqrt{6}M_g\|u - v - (w - z)\| + 2\sqrt{2}F_g\|u - v\| + \|w - z\|\|u - v\| + \|u - w - (v - z)\|. \tag{6g}$$

**Proof.** In fact, it follows that for $h, y \in \ell^2$

$$\|Dg(u) - Dg(v) - (Dg(w) - Dg(z))\|^2_{L_2(\ell^2; \ell^2_2)}$$

$$= \sup_{\|h\|=1} \|Dg(u)h - Dg(v)h - (Dg(w)h - Dg(z)h)\|^2_{L_2(\ell^2_2)}$$

$$= \sup_{\|h\|=1, \|y\|=1} \sum_{i \in \mathbb{Z}} (1 + |i|)^{2\kappa} |g'_i(u_i)h_iy_i - g'_i(v_i)h_iy_i - g'_i(w_i)h_iy_i + g'_i(z_i)h_iy_i|^2$$

$$\leq \sum_{i \in \mathbb{Z}} (1 + |i|)^{2\kappa} |g'_i(u_i) - g'_i(v_i) - g'_i(w_i) + g'_i(z_i)|^2$$

$$\leq 2M_g^2 \sum_{i \in \mathbb{Z}} |u_i - v_i - (w_i - z_i)|^2 + 4F_g^2 \sum_{i \in \mathbb{Z}} |u_i - w_i|^2 + |w_i - z_i|^2$$

$$= 2M_g^2\|u - v - (w - z)\|^2 + 4F_g^2\|u - w\|^2 + \|w - z\|^2,$$
which implies the required estimate (6f). Following the same procedure, one can obtain (6g). In particular, other desired estimates were proved in [4, Lemma 3.2]. Therefore, the proof of this Lemma is completed. □

With the above discussion at hand, we are going to reformulate the equation (1) as the following evolution equation with values in $\ell^2$:

$$\begin{cases}
    du(t) = (-A_1u(t) + f(u(t)))dt + g(u(t))d\omega(t), \\
    u(0) = u_0.
\end{cases} \tag{7}$$

For our purpose, we look for a mild solution to equation (7), namely, for the existence of $u(t) = (u_i(t))_{i \in \mathbb{Z}} \in \ell^2$ satisfying the integral operator equation

$$\begin{cases}
    u(t) = S(t)u_0 + \int_0^t S(t-r)f(u(r))dr + \int_0^t S(t-r)g(u(r))d\omega(r), \\
    u(0) = u_0.
\end{cases} \tag{8}$$

### 3.2. Path-area solution to (8)

In order to understand the notion of path-area solution to (8), we first consider the case where the driving noise is regular, to later on consider the Hölder case which we are interested in. In particular, the fractional integration techniques allow us to shed light on this issue. Inspired by [13], we give the second equation for the so-called *area* as follows. To this, assume $\omega \in C^1([0, T]; \ell^2)$ be a smooth enough $\ell^2$-valued function, $u \in C_\beta([0, T]; \ell^2)$. Define

$$(u \otimes \omega)(s,t) = \int_s^t D^\alpha_+ \int_s^t S(\cdot - r)f(u(r))dr [\xi] \otimes \ell^2 D^\alpha_1 \omega_1[\xi]d\xi$$

$$+ (-1)\alpha \int_s^t D^\alpha_+((S(s) - id)(u(s))[\xi] \otimes \ell^2 D^\alpha_1 \omega_1[\xi]d\xi$$

$$- (-1)^\alpha \int_s^t D^\alpha_+ g(u(\cdot))dr [\xi] \otimes \ell^2 D^\alpha_1 \omega_1[\xi]d\xi$$

$$+ (-1)^{2\alpha - 1} \int_s^t D^2 \omega(u(\cdot))dr [\xi] \otimes \ell^2 D^\alpha_1 \omega_1w(t, \cdot, \cdot)dr,$$

where the element $w = (u \otimes (\omega \otimes S \omega))$ is defined by

$$ZW(t, s, q) := - \int_s^q D^2 \omega(S(s), t)Z(u(\cdot) - u(s), \cdot)[\xi] \otimes \ell^2 D^\alpha_1 \omega_1[\xi]d\xi$$

$$+ (+1)^{\alpha - 1} \int_s^q D^2 \omega(u(\cdot) - u(s), \cdot)[\xi] \otimes \ell^2 D^\alpha_1 \omega_1[\xi]d\xi$$

$$+ (+1)^{\alpha - 1} \int_s^q D^2 \omega(u(\cdot) - u(s), \cdot)[\xi] \otimes \ell^2 D^\alpha_1 \omega_1[\xi]d\xi,$$

for $0 < s < q \leq t \leq T$ and $Z \in L_2(\ell^2 \otimes \ell^2; \ell^2)$. Moreover, for $e \in L_2(\ell^2; \ell^2)$$$
eq \int_s^t S(\xi - \tau)\omega'(\tau)d\tau \otimes \ell^2 \omega'(\xi)d\xi$$

$$\omega_S(t, e) := (-1)^{-\alpha} \int_s^t S(\xi - \tau)e \otimes \ell^2 d\omega_1(\xi),$$

and

$$e(\omega_S(t) \otimes \omega)(s, \tau) = \int_s^t (\omega_S(r, t) - \omega_S(s, t))ed\omega(r),$$
for $s \leq \tau \leq t$. Herein, the driving noise $\omega$ was assumed to be a smooth enough function, which does not include the fBm with $H \in (1/3, 1/2]$. To avoid this embarrassing situation, it is desirable to import a linear approximation of a given fixing $\omega$. Precisely, we introduce one more assumption condition:

**Remark 1.** In Appendix A, we have proven that when $n \to \infty$, and $\omega^n \to \omega$ in $C_{\beta'}([0, T]; \ell^2)$, the sequence $(\omega^n \otimes \omega^n)_{n \in \mathbb{N}}$ converges to $(\omega \otimes \omega)$ in $C_{\beta'}([0, T]; \ell^2) \times C_{2\beta'}(\Delta_{0,T}; L_2(L_2(\ell^2; \ell^2)))$ for any $\beta' < H$ on a set of full measure.

**Remark 1.** Fixing $\omega \in C_{\beta'}([0, T]; \ell^2)$, denote by $(W_{0,T}, \| \cdot \|_{W_{0,T}})$ the subspace of elements $U = (u, v)$ of the Banach space $C_{\beta'}([0, T]; \ell^2) \times C_{\beta'}(\Delta_{0,T}; \ell^2 \otimes \ell^2)$ such that the Chen-equality (2) holds with respect to $\omega$. We are ready to consider $\bar{W}_{0,T}$ be the subspace given by the limit points in the space of the set

$$\{(u^n, (\omega^n \otimes \omega^n))_{n \in \mathbb{N}}: u^n \in C_{\beta'}([0, T]; \ell^2), u^n(0) \in \ell^2, (\omega^n, (\omega^n \otimes \omega^n)) \text{ satisfies (A4)}\}.$$ 

In fact, $\bar{W}_{0,T}$ is a complete metric space depending on $\omega$ with the norm

$$d_{\bar{W}_{0,T}}(U_1, U_2) = d_{W_{0,T}}(U_1, U_2) := \|u_1 - u_2\|_{\beta,0,T} + \|v_1 - v_2\|_{\beta+\beta',\Delta_{0,T}}.$$

In addition, each element from $U = (u, v) \in \bar{W}_{0,T}$ satisfies the Chen-equality (2) with respect to $\omega$. Letting $U = (u, v) \in \bar{W}_{0,T}$, we consider the operator

$$\mathcal{T}(U, \omega, u_0) = (\mathcal{T}_1(U, \omega, u_0), \mathcal{T}_2(U, \omega, \omega \otimes \omega, u_0))$$

defined on $\bar{W}_{0,T}$ by the expressions

$$\mathcal{T}_1(U, \omega, u_0)(t) = S(t)u_0 + \int_0^t S(t-r)f(u(r))dr$$

$$+ (-1)^{\alpha} \int_0^t \hat{D}_{\alpha+}^{-1}(S(t-\cdot)g(u(\cdot)))(r)D_{1-\alpha}^{-1}\omega_{\alpha-}[r]dr$$

$$+ (-1)^{\alpha} \int_0^t \hat{D}_{\alpha+}^{-1}(S(t-\cdot)g(u(\cdot)))(r)D_{1-\alpha}^{-1}\omega_{\alpha-}[r]dr,$$

and

$$\mathcal{T}_2(U, \omega, (\omega \otimes \omega), u_0)(s,t)$$

$$+ (-1)^{\alpha} \int_0^t \hat{D}_{\alpha+}^{-1}(S(t-\cdot)g(u(\cdot)))(r)D_{1-\alpha}^{-1}\omega_{\alpha-}[r]dr,$$

and

$$\mathcal{T}_2(U, \omega, (\omega \otimes \omega), u_0)(s,t)$$

$$= (-1)^{\alpha} \int_s^t \hat{D}_{\alpha+}^{-1}(S(\cdot-\cdot)g(u(\cdot)))(r)D_{1-\alpha}^{-1}\omega_{\alpha-}[\xi]d\xi$$

$$+ (-1)^{\alpha} \int_s^t \hat{D}_{\alpha+}^{-1}(S(\cdot-\cdot)g(u(\cdot)))(r)D_{1-\alpha}^{-1}\omega_{\alpha-}[\xi]d\xi$$

$$- (-1)^{\alpha} \int_0^t \hat{D}_{\alpha+}^{-1}(S(t-\cdot)g(u(\cdot)))(r)D_{1-\alpha}^{-1}\omega_{\alpha-}[\cdot,t-\cdot]rdr$$

$$+ (-1)^{\alpha} \int_s^t \hat{D}_{\alpha+}^{-1}(S(\cdot-\cdot)g(u(\cdot)))(r)D_{1-\alpha}^{-1}\omega_{\alpha-}[\cdot,t-\cdot]rdr.$$

Having the above discussion in mind, we are ready to define a path-area solution to (8) as follows.
Definition 3.2. For any $T > 0$, the element $U \in \hat{W}_{0,T}$ such that $U = T(U, \omega, (\omega \otimes S \omega), u_0)$ is called a path-area solution to equation (8).

3.3. Main results.

Theorem 3.3. Suppose that the assumptions (A1), (A2), (A3) and (A4) hold. For any $T > 0$, $t \in [0, T]$, equation (7) possesses a unique local mild path-area solution $U \in \hat{W}_{0,T}$.

Before proceeding with the existence of the solution to the considered equation, we pause to provide some key estimates, which are very useful for proving our main results. For simplicity and without causing ambiguity, we denote $\| \cdot \|$, $\| \cdot \|_{L(\ell^2, \ell^2)}$, and $\| \cdot \|_{L(\ell^2, \ell^2)}$ and all norms of operator by $\| \cdot \|$. Throughout the whole paper we will write very often a positive constant $c$, which can change from line to line but is always chosen independent of time parameter $T$.

By performing a suitable change of variable to Beta function, the following integral formula is straightforward. Precisely, for any $0 < s < t \leq T$, there exists a constant $c = c(\mu, \nu) > 0$ such that

$$\int_s^t (r - s)\mu(t - r)^\nu dr = c(t - s)^{\mu+\nu+1}. \tag{9}$$

We now in position to state and prove several key estimations.

Lemma 3.4. Assume that (A1), (A3) and (A4) hold. For $0 \leq s \leq t \leq T$, it fulfills that

$$\| D_{s+}^\alpha S(t - \cdot)g(u(\cdot))[r] \| \leq cD_g(1 + \| u \|) \left[ \|(r - s)^{-\alpha} + (r - s)^{1-\alpha} \right] + cM_g \| u \|_{\beta,0,T}^2(r - s)^{2\beta-\alpha}, \tag{10}$$

and

$$\| D_{s+}^{2\alpha-1} S(t - \cdot)Dg(u(\cdot))[r] \| \leq cM_g(1 + \| u \|) \left[ \|(r - s)^{-2\alpha} + (r - s)^{2-2\alpha} \right] + cM_g \| u \|_{\beta,0,T}^2(r - s)^{\beta + 1 - 2\alpha}. \tag{11}$$

Proof. By (5b), (6e), (9) and Definition 2.3, we have

$$\| D_{s+}^\alpha S(t - \cdot)g(u(\cdot))[r] \| = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{S(t - r)g(u(r))}{(r - s)^{\alpha}} \right)$$

$+ \alpha \int_s^r \frac{S(t - r)g(u(r)) - S(t - q)g(u(q))}{(r - q)^{1+\alpha}} dq$

$+ \alpha \int_s^r \frac{S(t - r)Dg(u(q))(u(r) - u(q))}{(r - q)^{1+\alpha}} dq$

$\leq c  \left( \frac{\| S(t - r)g(u(r)) \|}{(r - s)^{\alpha}} + \alpha \int_s^r \frac{\| S(t - r) - S(t - q)g(u(r)) \|}{(r - q)^{1+\alpha}} dq \right)$

$+ \alpha \int_s^r \frac{\| S(t - q)[g(u(r)) - g(u(q)) - Dg(u(q))(u(r) - u(q))] \|}{(r - q)^{1+\alpha}} dq$

$\leq c \left( \frac{\| g(u) \|}{(r - s)^{\alpha}} + \frac{\| f(u) \|}{(r - q)^{1+\alpha}} dq + \int_s^r \frac{M_g \| u(r) - u(q) \|^2}{(r - q)^{1+\alpha}} dq \right)$

$\leq cD_g(1 + \| u \|) \left[ \|(r - s)^{-\alpha} + (r - s)^{1-\alpha} \right] + cM_g \| u \|_{\beta,0,T}^2(r - s)^{2\beta-\alpha}.$
Similarly, direct computation gives that
\[
\left\| \frac{1}{\Gamma(2\alpha)} \int_s^{r} \frac{S(t-r)Dg(u(r)) - S(t-q)Dg(u(q))}{(r-q)^{2\alpha-1}} dq \right\| \\
\leq c \left( \left\| S(t-r)Dg(u(r)) \right\|_{r-s}^{2\alpha-1} + \int_s^{r} \left\| S(t-r) - S(t-q) \right\| Dg(u(r)) dq \\
+ \int_s^{r} \left\| S(t-q) \right\| Dg(u(r)) - Dg(u(q)) dq \right) \\
\leq cM_g(1 + \| u \|) \left[ (r-s)^{1-2\alpha} + (r-s)^{2-2\alpha} \right] + cM_g \| u \|_{\beta,0,T}(r-s)^{\beta-2\alpha+1}.
\]
Therefore, the proof is completed.

Lemma 3.5. Assume that (A1), (A3) and (A4) hold. Then the following statements hold:
\[
\left\| \int_s^{t} S(t-r)Dg(u(r)) d\omega(r) \right\| \\
\leq c \| \omega \|_{\beta',0,T} \left[ D_g(1 + \| u \|)(1 + t - s) + M_g(t-s)^{2\beta} \| u \|^2_{\beta,0,T} \right] (t-s)^{\beta'} \\
\quad + c \left( \| u \|_{\beta+\beta',0,T} + \| u \|_{\beta,0,T} \| \omega \|_{\beta',0,T} \right) \left[ M_g(1 + \| u \|)(1 + t - s) \\
\quad + M_g(t-s)^{\beta} \| u \|_{\beta,0,T} \right] (t-s)^{\beta+\beta'},
\]
and
\[
\left\| \int_0^{s} \int_s^{t} S(t-r) - S(s-r)S(t-s)Dg(u(r)) d\omega(r) \right\| \\
\leq c \| \omega \|_{\beta',0,T} \left[ D_g(1 + \| u \|)(1 + s) + M_g s^{2\beta} \| u \|^2_{\beta,0,T} \right] (t-s)s^{\beta'} \\
\quad + c \left( \| u \|_{\beta+\beta',0,T} + \| u \|_{\beta,0,T} \| \omega \|_{\beta',0,T} \right) \left[ D_g + sD_g \\
\quad + M_g s^{\beta} \| u \|_{\beta,0,T} \right] (t-s)s^{\beta+\beta'}.
\]

Proof. By (10) and the following inequality from [13, Lemma 5]
\[
\left\| D_{\cdots}^{\alpha} \omega_{t-} \right\| \leq c \| \omega \|_{\beta',0,T}(t-r)^{\alpha+\beta'-1},
\]

we obtain
\[
\left\| \int_s^t \hat{D}_{s+}^\alpha S(t - \cdot) g(u(\cdot)) [r] D_{t-}^{1-\alpha} \omega_{t-} [r] dr \right\| \leq \int_s^t \| \hat{D}_{s+}^\alpha S(t - \cdot) g(u(\cdot)) [r] \| \| D_{t-}^{1-\alpha} \omega_{t-} [r] \| dr \leq c \| \omega \|_{\beta', 0, T} \left( \int_s^t c \| g(u(\cdot)) \| (r - s)^{-\alpha} (t - r)^{\alpha + \beta' - 1} (t - s)^{\beta'} \right) + c M_g \| u \|_{\beta', 0, T}^2 (r - s)^{2\beta - \alpha} (t - r)^{\alpha + \beta' - 1} dr \leq c \| \omega \|_{\beta', 0, T} \left[ D_g (1 + \| u \|)(1 + t - s) + M_g (t - s)^{2\beta} \| u \|_{\beta', 0, T}^2 \right] (t - s)^{\beta'}.
\]
On the other hand, by (11) and the inequality given by (14)
\[
\| D_{t-}^{1-\alpha} D_{l-}^{1-\alpha} v[r] \| \leq c (\| v \|_{\beta + \beta', \Delta_0, T} + \| u \|_{\beta, 0, T} \| \omega \|_{\beta', 0, T}) (t - r)^{\beta + \beta' + 2\alpha - 2},
\]
we further have
\[
\left\| \int_s^t D_{s+}^{2\alpha - 1} S(t - \cdot) D_g(u(\cdot)) [r] D_{l-}^{1-\alpha} D_{t-}^{1-\alpha} v[r] dr \right\| \leq c \left( \| v \|_{\beta + \beta', \Delta_0, T} + \| u \|_{\beta, 0, T} \| \omega \|_{\beta', 0, T} \right) \times \int_s^t \left[ D_g (r - s)^{-2\alpha} (t - r)^{\beta + \beta' + 2\alpha - 2} + D_g (t - r)^{\beta + \beta' + 2\alpha - 2} \times (r - s)^{2\alpha} + M_g \| u \|_{\beta, 0, T} (t - r)^{\beta + \beta' + 2\alpha - 2} (r - s)^{-2\alpha + 1} \right] dr \leq c \left( \| v \|_{\beta + \beta', \Delta_0, T} + \| u \|_{\beta, 0, T} \| \omega \|_{\beta', 0, T} \right) \left[ D_g + (t - s) D_g \right. + M_g (t - s)^{\beta} \| u \|_{\beta, 0, T}^2 (t - s)^{\beta + \beta'}.
\]
Then
\[
\left\| \int_s^t S(t - r) g(u(r)) dr \right\| \leq \left\| \int_s^t \hat{D}_{s+}^\alpha S(t - \cdot) g(u(\cdot)) [r] D_{t-}^{1-\alpha} \omega_{t-} [r] dr \right\| + \left\| \int_s^t D_{s+}^{2\alpha - 1} S(t - \cdot) D_g(u(\cdot)) [r] D_{l-}^{1-\alpha} D_{t-}^{1-\alpha} v[r] dr \right\| \leq c \| \omega \|_{\beta', 0, T} \left[ D_g (1 + \| u \|)(1 + t - s) + M_g (t - s)^{2\beta} \| u \|_{\beta, 0, T}^2 \right] (t - s)^{\beta'} + c \left( \| v \|_{\beta + \beta', \Delta_0, T} + \| u \|_{\beta, 0, T} \| \omega \|_{\beta', 0, T} \right) \left[ D_g + (t - s) D_g \right. + M_g (t - s)^{\beta} \| u \|_{\beta, 0, T}^2 (t - s)^{\beta + \beta'}.
\]
Thus the former required inequality is achieved. Now we treat the later applying similar procedure. What is worth being noted is that $S(t - r)$ in the former is
replaced by $S(t - r) - S(s - r)$ in later and the range of integration becomes $[0, s]$. Moreover, similar to Lemma 3.4, we have

$$
\left\| \hat{D}_0^\alpha [S(t - \cdot) - S(s - \cdot)]g(u(\cdot))[r] \right\| \\
\leq c(t - s)r^{-\alpha} \left[ D_g(1 + \|u\|) + rD_g(1 + \|u\|) + r^{2\beta}\|u\|_{r,0,T}^2 \right],
$$

and

$$
\left\| \hat{D}_0^{2\alpha - 1} [S(t - \cdot) - S(s - \cdot)] Dg(u(\cdot))[r] \right\| \leq c(t - s)r^{1-2\alpha} \left[ D_g + rD_g + r^{\beta}M_g \|u\|_{r,0,T} \right].
$$
Hence

$$
\left\| \int_0^s [S(t - r) - S(s - r)] Dg(u(r))d\omega(r) \right\| \\
\leq c\|\omega\|_{r,0,T} \left[ D_g(1 + \|u\|(1 + s)) + M_g s^{2\beta}\|u\|_{r,0,T}^2 \right] (t - s)s^{\beta'} \\
+ c \left( \|v\|_{r,0,T} + \|u\|_{r,0,T}\|\omega\|_{r,0,T} \right) \left[ D_g + sD_g \right] \\
+ M_g s^{\beta}\|u\|_{r,0,T} (t - s)s^{\beta + \beta'}.
$$

\[\square\]

Next we state a useful estimate about $\mathcal{T}_1$ based on Lemmas 3.4 and 3.5.

**Lemma 3.6.** Assume that (A1), (A2), (A3) and (A4) hold, $u_0 \in \ell^2$. For $U = (u, v) \in \hat{W}_{0,T}$, the following statements are true:

$$
\|T_1(U, \omega, u_0)\|_{\beta,0,T} \leq c(T^{1-\beta} + 1)\|u_0\| + cT'(1 + \|\omega\|_{r,0,T})(1 + \|U\|_{W_0,T} + \|U\|_{\hat{W}_0,T}^2),
$$

and

$$
\|T_1(U, \omega, u_0)(T)\| \leq c(T^{1-\beta} + 1)\|u_0\| + cT'(1 + \|\omega\|_{r,0,T})(1 + \|U\|_{W_0,T} + \|U\|_{\hat{W}_0,T}^2).
$$

Furthermore, for $U^1 = (u^1, v^1), U^2 = (u^2, v^2) \in \hat{W}_{0,T}$

$$
\|T_1(U^1) - T_1(U^2)\|_{\beta,0,T} \leq c(T^{1-\beta} + 1) \left( \|u_0^1 - u_0^2\| + cT'\|U^1 - U^2\|_{W_0,T} \right) (1 + \|\omega\|_{r,0,T}) \\
\times (1 + \|U^1\|_{W_0,T} + \|U^2\|_{W_0,T}^2),
$$

where

$$
T' = \begin{cases} 
T^{\beta'-\beta}, & T < 1, \\
T^{2\beta'+2\beta+3}, & T \geq 1.
\end{cases}
$$

\[\text{Proof.}\] The definition $T_1$ gives that

$$
\|T_1(U, \omega, u_0)\|_{\beta,0,T} \leq \|S(\cdot)u_0\|_{\beta,0,T} + \left\| \int_0^s S(\cdot - r)f(u(r))dr \right\|_{\beta,0,T} \\
+ \left\| \int_0^s S(\cdot - r)g(u(r))d\omega(r) \right\|_{\beta,0,T}.
$$
It is easy to check that \( \| S(\cdot)u_0 \|_{\beta,0,T} \leq c(T^{1-\beta} + 1)\| u_0 \| \). We now calculate the second term by expending each part as follows

\[
\left\| \int_0^t S(\cdot - r)f(u(r))dr \right\|_{\beta,0,T} \leq \sup_{0 \leq t \leq T} \int_0^t \| S(t - r)f(u(r)) \| dr + \sup_{0 < s < t \leq T} \frac{\| \int_0^s S(t - r)f(u(r))dr - \int_0^s S(s - r)f(u(r))dr \|}{(t - s)^{\beta}}.
\]

Regarding the last integral term of the above expression, we have

\[
\left\| \int_0^t S(\cdot - r)f(u(r))dr - \int_0^s S(s - r)f(u(r))dr \right\|
\leq \left\| \int_0^s [S(t - r) - S(s - r)]f(u(r))dr \right\| + \left\| \int_s^t S(t - r)f(u(r))dr \right\|
\leq c(t - s)\| f(u) \| \int_0^s e^{-\lambda(s - r)}dr + c(t - s)\| f(u) \| 
\leq cD_f(t - s)(1 + \| u \|).
\]

Hence

\[
\left\| \int_0^t S(\cdot - r)f(u(r))dr \right\|_{\beta,0,T} \leq cD_fT'(1 + \| u \|) \leq cT'(1 + \| U \|_{W_{0,T}}).
\]

Herein, we refer to the property of \( f \), which is in Lemma 3.1. Next we turn to calculate \( \| \int_0^t S(\cdot - r)g(u(r))d\omega(r) \|_{\beta,0,T} \). Based on Lemma 3.5, we have

\[
\left\| \int_0^t S(\cdot - r)g(u(r))d\omega(r) \right\|_{\beta,0,T} \leq \sup_{t \in [0,T]} \left\| \int_0^t S(t - r)g(u(r))d\omega(r) \right\|
\leq \sup_{0 < s < t \leq T} \frac{\left\| \int_0^s [S(t - r) - S(s - r)]g(u(r))d\omega(r) \right\|}{(t - s)^{\beta}}
\leq cT'(1 + \| \omega \|_{\beta',0,T}) \left(1 + \| U \|_{W_{0,T}} + \| U \|_{W_{0,T}}^2 \right),
\]

together with the assumption \( \| g(u) \| \leq D_g\| u \| + \| g(0) \| \) here. we further get

\[
\| T_1(U,\omega,u_0) \|_{\beta,0,T} \leq c(T^{1-\beta} + 1)\| u_0 \| + cT'(1 + \| \omega \|_{\beta',0,T}) \left(1 + \| U \|_{W_{0,T}} + \| U \|_{W_{0,T}}^2 \right).
\]

This is the first required result in Lemma 3.6. As for the second part, we denote \( \Delta u = u^1 - u^2 \) and \( \Delta U = U^1 - U^2 \), then

\[
\| g(u^1(r)) - g(u^2(r)) \| \leq D_g\| \Delta u \|_{\beta,0,T},
\]
and
\[\|g(u^1(r)) - g(u^1(q)) - Dg(u^1(q))[u^1(r) - u^1(q)] - g(u^2(r)) + g(u^2(q)) + Dg(u^2(u(q)))\| \]
\[\leq cM_g\|(u^1(r) - u^1(q))\| + \|u^2(r) - u^2(q)\|\|u^1(r) - u^2(r) - u^1(q) + u^2(q)\| \]
\[+ cF_g\|u^2(r) - u^2(q)\|\|u^1(q) - u^2(q)\|\|(u^1(r) - u^1(q))\| \]
\[+ \|u^1(r) - u^2(r) - u^1(q) + u^2(q)\|\]
\[\leq cM_g\|(u^1)\|_{\beta,0,T} + \|u^2\|\|\Delta u\|_{\beta,0,T}(r - q)^{2\beta} \]
\[+ F_g\|u^2\|\|\Delta u\|_{\beta,0,T}(2\|u^1\|_{\beta,0,T} + \|u^2\|_{\beta,0,T})(r - q)^{2\beta} \]
\[\leq c\|u^1\|_{W_{0,T}} + \|u^2\|_{W_{0,T}}\|\Delta u\|_{W_{0,T}}(1 + \|u^2\|_{W_{0,T}})(r - q)^{2\beta}.\]
Thus, it follows from the proof procedure of Lemma 3.4 that
\[\|D^s_{x^*}S(t - \cdot)[g(u^1(\cdot)) - g(u^2(\cdot))]\| \]
\[\leq c(r - s)^{-\alpha \left[\|\Delta u\|_{W_{0,T}} + (r - s)^{2\beta/2}\right]} \]
\[+ c\|\|u^1\|_{W_{0,T}} + \|u^2\|_{W_{0,T}}\|\Delta u\|_{W_{0,T}}(1 + \|u^2\|_{W_{0,T}})(r - s)^{2\beta - \alpha},\]
and
\[\|D^{2\alpha-1}s_{x_{s+1}}[Dg(u^1(\cdot)) - Dg(u^2(\cdot))]\| \]
\[\leq c\|\Delta u\|_{W_{0,T}}[(r - s)^{-2\alpha} + (r - s)^{2 - 2\alpha}] \]
\[+ c\|\|u^1\|_{W_{0,T}} + \|u^2\|_{W_{0,T}}\|\Delta u\|_{W_{0,T}}(r - s)^{\beta + 1 - 2\alpha}.\]
Similar to Lemma 3.5, we have
\[\left\|\int_s^t S(t - r)[g(u^1(r)) - g(u^2(r))]d\omega(r)\right\| \]
\[\leq c\|\Delta u\|_{W_{0,T}}(1 + \|\omega\|_{\beta,r,0,T})\left[(t - s)^{\beta + \beta'} \right. \]
\[+ (t - s)^{\beta' + 1} + (1 + \|U^1\|_{W_{0,T}})(t - s)^{2\beta + \beta'} + (t - s)^{\beta'} \]
\[\left. + (t - s)^{\beta' + 1} \right) \]
\[+ (\|U^1\|_{W_{0,T}} + \|U^2\|_{W_{0,T}})(1 + \|U^2\|_{W_{0,T}})(t - s)^{2\beta + \beta'},\]
and
\[\left\|\int_0^s [S(t - r) - S(s - r)][g(u^1(r)) - g(u^2(r))]d\omega(r)\right\| \]
\[\leq c(t - s)^{\|\Delta u\|_{W_{0,T}}(1 + \|\omega\|_{\beta,r,0,T})\left[s^{\beta + \beta'} + s^{\beta' + 1} + (1 + \|U^1\|_{W_{0,T}})s^{2\beta + \beta'} \right. \]
\[+ s^{\beta' + 1} + (\|U^1\|_{W_{0,T}} + \|U^2\|_{W_{0,T}})(1 + \|U^2\|_{W_{0,T}})(s^{2\beta + \beta'} \left.\right].\]
Hence
\[\left\|\int_0^s S(\cdot - r)[g(u^1(\cdot)) - g(u^2(\cdot))]d\omega(r)\right\|_{\beta,0,T} \]
\[\leq cT^{\|\Delta u\|_{W_{0,T}}(1 + \|\omega\|_{\beta,r,0,T})(1 + \|U^1\|_{W_{0,T}} + \|U^2\|_{W_{0,T}})^2}.\]
We further obtain $\|S(\cdot)(u_1 - u_2^0)\|_{\beta,0,T} \leq cT'\|u_1^0 - u_0^0\|$, and
\[
\left\| \int_0^T S(\cdot - r)[f(u^1(r)) - f(u^2(r))]dr \right\|_{\beta,0,T} \leq cD_f T' \|\Delta U\|_{W_0,T}.
\]
In conclusion, we have
\[
\|\mathcal{T}_1(U^1) - \mathcal{T}_2(U^2)\|_{\beta,0,T} \leq cT'\|\Delta U\|_{W_0,T}(1 + \|\omega\|_{\beta',0,T})
\]
\[
\times (1 + \|U^1\|_{W_0,T} + \|U^2\|_{W_0,T}^2) + c(T^{1-\beta} + 1)\|\Delta u_0\|.
\]
Hence we complete this proof. \(\square\)

We have finished the calculation of $\mathcal{T}_1$ up to now. In the following, we are going to estimate the another component $\mathcal{T}_2$ of $\mathcal{T}$. In order to achieve this goal, we must give the following Lemma treating the operator $w$.

**Lemma 3.7.** Assume that (A1), (A2), (A3) and (A4) hold. Then the two mappings
\[
e \in V \rightarrow \omega_S(r,t)e \in \ell^2
\]
and
\[
Z \in L_2(\ell^2 \otimes \ell^2; \ell^2) \rightarrow Zw(t,s,q) \in \ell^2 \otimes \ell^2
\]
have the following properties: for $0 \leq s \leq r \leq t \leq T$, $U \in \hat{W}_{0,T}$,
\[
\|\omega_S(r,t)e - \omega_S(s,t)e\| \leq \hat{c}\|\omega\|_{\beta',0,T}\{(r-s)(t-r)^{\beta'} + (t-r)^{\beta'+1}\}
\]
\[
+ (r-s)^{\beta'} + (r-s)^{\beta'+1}\|e\|;
\]
\[
\|\omega_S(s,t)e\| \leq \hat{c}(t-s)^{\beta'} + (t-s)^{\beta'+1}\|\omega\|_{\beta',0,T}\|e\|;
\]
\[
\|w(t,s,q)Z\| \leq \hat{c}(1 + \|\omega\|_{\beta',0,T})^2\|Z\|\|U\|_{W_0,T}[(t-s)^{\beta'} + (t-s)^{\beta'+1}]
\]
\[
\times [(q-s)^{\beta'+1} + (q-s)^{\beta'+1}],
\]
where the constant $\hat{c}$ depends on $\|\omega\|_{\beta',0,T}$ and $\|\omega \otimes S\omega\|_{2\beta',0,T}$.

**Proof.** According to the expression of $\omega_S$ given before, it is straightforward that the first two inequalities hold sure. For the last one, we first separate $w$ into three parts $I_1, I_2$ and $I_3$,
\[
Zw(t,s,q) = -\int_s^q \hat{D}^\alpha_{s+} \omega_S(\cdot,t)Z(u(\cdot) - u(s),\cdot)[r]D^{1-\alpha}_q\omega[r]dr
\]
\[
+ (-1)^{\alpha-1}\int_s^q D^{2\alpha-1}_s Z(u(\cdot) - u(s),\cdot)[r]D^{1-\alpha}_q D^{1-\alpha}(\omega_S(t) \otimes \omega)[r]dr
\]
\[
+ (-1)^{\alpha-1}\int_s^q D^{2\alpha-1}_s \omega_S(\cdot,t)[r]ZD^{1-\alpha}_q D^{1-\alpha}(u \otimes \omega)(\cdot,q)[r]dr
\]
\[
:= I_1 + I_2 + I_3.
\]
Herein we adopt the similar idea in proving \cite[Lemma 20]{13} but just point out the main differences. As for $I_2$, there are two substantial distinctions
\[
\left\| \omega_S(\tau,t) \int_s^\tau (S(\tau - r) - id) Zd\omega(r) \right\| \leq \hat{c}\|Z\|\|\omega\|^2_{\beta',0,T}(\tau-s)^{1+\beta'}[\tau - \tau)^{\beta'} + (t-\tau)^{\beta'+1}],
\]
and applying [13, Lemma 21] step by step, we can obtain
\[ \| (1 - t)\{ \omega \otimes_S \omega \}(s, t) + (\omega_S(t, s) - \omega_S(t, s)) \| \leq \tilde{c}(t - s)^{\beta + 1}[(t - r)^{\beta'} + (t - r)^{\beta' + 1}] + \tilde{c}(t - s)^{2\beta'} + \tilde{c}(t - s)^{2\beta' + 1}. \]

Thus the required claim is sure.

But for \( J_3 \), there are two different estimations
\[
\| D_{\alpha} \cdot \alpha \| \leq c(\| u \|_{\beta} \cdot \Delta_0, T) \| (q - r)^{\beta + \beta'} + (q - s)^{\beta + \beta'} - 2. \]

and
\[
\| D_{\alpha} \cdot \alpha \| \leq c(\| u \|_{\beta} \cdot \Delta_0, T) \| (q - r)^{\beta + \beta'} + (q - s)^{\beta + \beta'} - 2. \]

Substituting their counterpart with these different estimations above into the proof of [13, Lemma 20] step by step, we obtain
\[
\| I_1 \| \leq c(\| u \|_{\beta} \cdot \Delta_0, T) \| (q - r)^{\beta + \beta'} + (q - s)^{\beta + \beta'} - 2. \]

and
\[
\| I_2 \| \leq c(\| u \|_{\beta} \cdot \Delta_0, T) \| (q - r)^{\beta + \beta'} + (q - s)^{\beta + \beta'} - 2. \]

and
\[
\| I_3 \| \leq c(\| u \|_{\beta} \cdot \Delta_0, T) \| (q - r)^{\beta + \beta'} + (q - s)^{\beta + \beta'} - 2. \]

Thus the required claim is sure.

Combining the estimations of Lemma 3.7 and the following Chen-equality
\[ Zw(t, s, r) + Zw(t, r, t) - Z(u(r, s), \cdot) \omega_S \otimes \omega \cdot (r, t) = Zw(t, s, t), \]

and applying [13, Lemma 21] step by step, we can obtain
\[ \| D_{\alpha} \cdot \alpha \| \leq c(\| u \|_{\beta} \cdot \Delta_0, T) \| (q - r)^{\beta + \beta'} + (q - s)^{\beta + \beta'} - 2. \]

for \( U \in W_{0, T} \) and \( 0 \leq r \leq t \leq T \).

**Lemma 3.8.** Assume that (A1), (A2), (A3) and (A4) hold and \( u_0 \in \ell^2 \). Then for \( U \in W_{0, T} \), we have
\[ \| T_2(U, \omega_S \omega, u_0) \|_{\beta + \beta', \Delta_0, T} \leq \tilde{c} T^{1 - \beta} \| u_0 \| + \tilde{c} \left( 1 + \| U \|_{W_{0, T}}^2 \| U \|_{W_{0, T}}^2 \right). \]

In addition, for two elements \( U^1, U^2 \in W_{0, T} : \)
\[ \| T_2(U^1, \omega_S \omega, u^1_0) - T_2(U^2, \omega_S \omega, u^2_0) \|_{\beta + \beta', \Delta_0, T} \leq \tilde{c} T^{1 - \beta} \| u^1_0 - u^2_0 \| + \tilde{c} T \left( 1 + \| U^1 \|_{W_{0, T}}^2 \| U^2 \|_{W_{0, T}}^2 \right)^2 \| U^1 - U^2 \|_{W_{0, T}}. \]

Herein, \( \tilde{c} \) depends on either \( \| \omega_S \omega \|_{2\beta', \Delta_0, T}, \| \omega \|_{\beta', \Delta_0, T}, \) or both. \( T' \) is defined in (14).

**Proof.** Let us denote
\[ T_2(U)(s, t) = B_0(s, t) + B_1(s, t) + B_2(s, t) + B_3(s, t), \]
corresponding to the four different addends of $T_2$. In fact, it follows that

\[
\left\| D^{\alpha}_s \left[ \int_s^T S(\cdot - r) f(u(r)) dr \right] \xi \right\| \\
\leq c \| f(u) \| ((\xi - s)^{1-\alpha} + \left\| \int_s^T \left[ \int_s^q S(\xi - r) - S(q - r) \right] f(u(r)) dr \frac{dq}{(\xi - q)^{1+\alpha}} \right\| \\
+ \int_s^T S(\xi - r) f(u(r)) dr \frac{dq}{(\xi - q)^{1+\alpha}} \right\| \\
\leq c \| f(u) \| ((\xi - s)^{1-\alpha}.
\]

Hence

\[
\| B_0 \|_{\beta + \beta', \Delta_0, T} \\
= \left\| (-1)^\alpha \int_s^T D^{\alpha}_s \left[ \int_s^T S(\cdot - r) f(u(r)) dr \right] \xi \otimes_{\mathcal{F}^t} D^{1-\alpha}_r \omega [\xi] dr \right\|_{\beta + \beta', 0, T} \\
\leq c \| \omega \|_{\beta', 0, T} \| f(u) \| \sup_{0 < s < t < T} \frac{\int_t^s (t - \xi)^{\alpha + \beta' - 1}(\xi - s)^{1-\alpha} d\xi}{(t - s)^{\beta + \beta'}} \\
\leq c T^{1-\beta} \| \omega \|_{\beta', 0, T} \| f(u) \|.
\]

On the other hand, dividing $B_1(s, t)$ into three parts

\[
B_1(s, t) = \int_s^t (S(\xi - s) - id) u(s) \otimes_{\mathcal{F}^t} d\omega(\xi) \\
= \int_s^t (S(\xi) - S(s)) u_0 \otimes_{\mathcal{F}^t} d\omega(\xi) \\
+ \int_s^t (S(\xi) - s) - id \left[ \int_0^s (S(s - r)) f(u(r)) dr \right] \otimes_{\mathcal{F}^t} d\omega(\xi) \\
+ \int_s^t (S(\xi) - s) - id \left[ \int_0^s S(s - r) g(u(r)) d\omega(r) \right] \otimes_{\mathcal{F}^t} d\omega(\xi) \\
= B_{11}(s, t) + B_{12}(s, t) + B_{13}(s, t).
\]

It is easy to check that

\[
\| B_{11} \|_{\beta + \beta', \Delta_0, T} \leq c T^{1-\beta} \| \omega \|_{\beta', 0, T} \| u_0 \|.
\]

Since $D^{\alpha}_s (S(\xi - s) - id)[\int_0^s S(s - r) f(u(r)) dr] [\xi]$ is well-defined on $\mathbb{R}$, one can apply the classical fractional integral on $B_{12}(s, t)$ directly, that is

\[
\| B_{12}(s, t) \| = \left\| \int_s^t (S(\xi) - s) - id \left[ \int_0^s (S(s - r)) f(u(r)) dr \right] \otimes_{\mathcal{F}^t} d\omega(\xi) \right\| \\
= \left\| \int_s^t D^{\alpha}_s (S(\xi) - s) - id \left[ \int_0^s S(s - r) f(u(r)) dr \right] [\xi] \\
\otimes_{\mathcal{F}^t} D^{1-\alpha}_r \omega [\xi] dr \right\|,
\]

which implies that

\[
\| B_{12} \|_{\beta' + \beta, \Delta_0, T} \leq c T^{1-\beta} \| \omega \|_{\beta', 0, T} \| f(u) \|.
\]
It follows similar steps that
\[ \|B_{13}\|_{\beta + \beta', \Delta_0, T} \leq c T' \left( \|\omega\|_{\beta', 0, T} + \|\omega\|^2_{\beta', 0, T} \right) (1 + \|U\|_{W_0, T} + \|U\|^2_{W_0, T}). \]

In conclusion
\[ B_1 \|_{\beta + \beta', \Delta_0, T} \leq c T^{1-\beta} \|\omega\|_{\beta', 0, T} \|u_0\| + c T' \left( \|\omega\|_{\beta', 0, T} + \|\omega\|^2_{\beta', 0, T} \right) \times (1 + \|U\|_{W_0, T} + \|U\|^2_{W_0, T}). \]

Due to \( \omega \otimes_S \omega \) be 2\( \beta' \)-Hölder continuous. Following similar steps, we can obtain
\[ \|B_2\|_{\beta + \beta', \Delta_0, T} \leq c T' \|\omega \otimes_S \omega\|_{2\beta', 0, T} \left( \|g(u)\| + M_g \|u\|^2_{\beta, 0, T} \right) \leq c T' \|\omega \otimes_S \omega\|_{2\beta', 0, T} (1 + \|U\|_{W_0, T} + \|U\|^2_{W_0, T}). \]

Note that the estimates of \( D_1 - \alpha D_1 - \alpha w(t, \cdot, \cdot) \) have been given in (15). \( \|B_3\|_{\beta + \beta', 0, T} \) can be obtained directly that
\[ \|B_3\|_{\beta + \beta', \Delta_0, T} \leq \hat{c} T' \left( 1 + \|U\|_{W_0, T} + \|U\|^2_{W_0, T} \right). \]

In conclusion, we have
\[ \|T_2(U, (\omega \otimes_S \omega), u_0)\|_{\beta + \beta', \Delta_0, T} \leq \hat{c} T^{1-\beta} \|u_0\| + \hat{c} T' \left( 1 + \|U\|_{W_0, T} + \|U\|^2_{W_0, T} \right); \]
where \( \hat{c} \) is depended on either \( \|\omega\|_{\beta', 0, T} \), \( \|\omega \otimes_S \omega\|_{2\beta', 0, T} \), or both. Thus the first part of this Lemma has been completed. As for the second part, it is similar to the procedures of the second part of Lemma 3.6 and method applied in the first part of this Lemma, thus we omit it here. \( \square \)

**Lemma 3.9.** Suppose that \( U^n = (u^n, u^n \otimes \omega^n) \in C_{\beta}([0, T]; \ell^2) \times C_{\beta + \beta'}([0, T]; \ell^2 \otimes \ell^2), n \in \mathbb{Z} \) is an approximation of \( U = (u, v) \in \dot{W}_{0, T} \) satisfying (A4). Then
\[ \lim_{n \to \infty} T(U, (\omega^n \otimes_S \omega^n), u_0) = T(U, \omega, (\omega \otimes_S \omega), u_0) \]
in \( C_{\beta}([0, T]; \ell^2) \times C_{\beta + \beta'}([0, T]; \ell^2 \otimes \ell^2) \) and \( T(U, \omega, (\omega \otimes_S \omega), u_0) \in \dot{W}_{0, T}. \)

**Proof.** According to the definition of \( T \), it is dependent on \( \omega \) and \( \omega \otimes_S \omega \) linearly. Then the first result can be obtained easily. Herein, we just prove the second one. As our knowledge, \( T_3(U^n, \omega^n, u_0^n) \in C_{\beta}([0, T]; \ell^2), T_3(U^n, \omega^n, u_0^n)(0) = u_0^n \in \ell^2 \) and \( T_3(U, \omega^n, (\omega^n \otimes \omega^n), u_0^n) \) can be written as \( T_1(U^n, \omega^n, u_0^n) \otimes \omega^n \). On the other hand, the following inequality holds
\[
\|T(U^n, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n) - T(U, \omega, (\omega \otimes_S \omega), u_0)\|
\leq \|T(U^n, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n) - T(U^n, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n)\|
+ \|T(U^n, \omega^n, (\omega^n \otimes \omega^n), u_0^n) - T(U, \omega^n, (\omega^n \otimes S \omega^n), u_0^n)\|
+ \|T(U, \omega^n, (\omega^n \otimes \omega^n), u_0^n) - T(U, \omega, (\omega \otimes_S \omega), u_0)\|.
\]

By the results of Lemma 3.6, Lemma 3.8 and the first part of this Lemma, we can obtain that last inequality is convergent to 0 as \( n \to \infty \), \( u_0^n \to u_0 \), which means that \( T(U^n, \omega^n, (\omega^n \otimes_S \omega^n), u_0^n) \) is the approximation of \( T(U, \omega, (\omega \otimes_S \omega), u_0) \). Hence \( T(U, \omega, (\omega \otimes_S \omega), u_0) \in \dot{W}_{0, T}. \) \( \square \)

With above estimates at hand, it is enough to prove Theorem 3.3 for us now.
Proof of Theorem 3.3. Inspired by Lemmas 3.6 and 3.8, one can find a small enough $T_1$ such that the operator $T$ on $W_{0,T_1} \subseteq W_{0,T}$ is a self mapping and contractive.

Let $C = (\epsilon + \hat{\epsilon})(T^{1-\beta} + 1)$, where $\hat{\epsilon}$ is defined before. Thus there exists small enough $0 < K < 1$ such that the following inequalities hold:

$$
C < 1, \\
3C^2K^2 + 4C^2K\|u_0\| + 2CK < 1, \\
2CK(1 + 3CK + 4C\|u_0\|)^2 \leq (1 - CK)^2.
$$

(16)

Solving the equation

$$
x = CK x^2 + CK x + CK + C\|u_0\|,
$$

the second inequality of (16) ensures that

$$(CK - 1)^2 - 4CK(CK + C\|u_0\|) > 0.
$$

Denote $r_1$ by the minor root of this equation, then

$$
r_1 = \frac{2(CK + C\|u_0\|)}{1 - CK + \sqrt{(CK - 1)^2 - 4CK(CK + C\|u_0\|)}} \leq \frac{2(CK + C\|u_0\|)}{1 - CK}.
$$

By using the definition of $T'$ in (14), one can choose $T_1 = K^{1/(\beta' - \beta)}$ such that the claim of (16) is valid. Thus $\|T(U)\|_{W_{0,T_1}} \leq r_1$, the mapping $T$ is self-mapping on $B(0, r_1) \subseteq W_{0,T_1}$.

On the other hand, it can be deduced from the third inequality of (16) that

$$
\|T(U^1) - T(U^2)\|_{W_{0,T_1}} \leq CT'(1 + 2r_1)^2 \|U^1 - U^2\|_{W_{0,T_1}} \leq \frac{1}{2} \|U^1 - U^2\|_{W_{0,T_1}}.
$$

Hence $T$ is contractive on $B(0, r_1)$. Moreover, it follows from Lemma 3.9 that $T(U) \in W_{0,T_1}$, which means that $T$ has a unique fixed point $U = (u,v) \in W_{0,T_1} \subseteq W_{0,T}$ by the Banach fixed point theorem. Hence, we have found a solution whose path component $u(t)$ is defined on $[0, T_1]$. Furthermore, Lemma 3.5 gives that

$$
\|u(T_1)\| \leq \|u_0\| + CK \left[ 1 + \|U\|_{W_{0,T_1}} + \|U\|_{W_{0,T_1}}^2 \right]
$$

$$
\leq \|u_0\| + CK \left[ 1 + \frac{2(CK + C\|u_0\|)}{1 - CK} + \frac{4(CK + C\|u_0\|)^2}{(1 - CK)^2} \right]
$$

$$
\leq \|u_0\| + CK \left[ 1 + \frac{2(CK + C\|u_0\|)}{1 - CK} + \frac{CK + C\|u_0\|}{CK} \right]
$$

$$
\leq \|u_0\| + \frac{(1 + CK)(CK + C\|u_0\|)}{1 - CK} < \infty.
$$

Hence we complete this proof.

Appendix A. 

Lemma A.1. Considering $\omega \in C^1([0,T]; \ell^2)$, the operator $(\omega \otimes_S \omega)$ defined in Section 3 is in $C_{2\beta'}(\Delta_{0,T}; L_2(\ell^2; \ell^2; \ell^2 \otimes \ell^2))$.

Proof. Consider a complete orthonormal basis $E_{ij}$ of $L_2(\ell^2; \ell^2)$. It can follows from $(e_i)_{i \in \mathbb{Z}}$ and $(\frac{e_i}{(1+|i|^2)})_{i \in \mathbb{Z}}$ that

$$
E_{ij}e_k = \begin{cases} 
0, & k \neq j; \\
\frac{e_i}{(1+|i|^2)}, & k = j.
\end{cases}
$$
Then for $s, t \in [0, T]$, we have
\[
\|E_{ij}(\omega \otimes \omega)(s, t)\|^2 = \sum_{k,l} \left( \int_s^t \int_r^t S(\xi - r)E_{ij}(r) \otimes e_l, e_k \right)^2 \\
= \sum_{k,l} \left( \int_s^t \int_r^t \left( \frac{S(\xi - r)e_i, e_l}{(1 + |i|)^\kappa} \right)(\omega'(r), e_j)\omega'(\xi) d\xi dr \right)^2 \\
\leq \sum_k \left[ \int_s^t \frac{(\omega'(r), e_j)}{(1 + |i|)^\kappa} \left( \int_r^t (S(\xi - r)e_i, e_l)^2 d\xi \right)^{1/2} \right]^2 \int_s^t (\omega'(r), e_k)^2 dr \\
\leq \int_s^t (\omega'(r), e_j)^2 dr \int_s^t \int_s^t \sum_i (S(\xi - r)e_i, e_l)^2 d\xi dr \int_s^t \|\omega'(r)\|^2 dr \\
\leq c(t - s)^2 (1 + |i|)^{-2\kappa} \int_s^t (\omega'(r), e_j)^2 dr \int_s^t \|\omega'(r)\|^2 dr.
\]
It shows that
\[
\sum_{i,j} \|E_{ij}(\omega \otimes \omega)(s, t)\|^2 \\
\leq c \sum_i \frac{(t - s)^2}{(1 + |i|)^{2\kappa}} \int_s^t (\omega'(r), e_j)^2 dr \sum_j \int_s^t (\omega'(r), e_j)^2 dr \\
\leq c \sum_i \frac{(t - s)^2}{(1 + |i|)^{2\kappa}} \left( \int_s^t \|\omega'(r)\|^2 dr \right)^2 < \infty.
\]
Hence $E_{ij}(\omega \otimes \omega)(s, t) \in L_2(L_2(\ell^2; \ell^2_\kappa); \ell^2 \otimes \ell^2)$ for any $s, t \in [0, T]$, and
\[
\sum_{i,j} \|E_{ij}(\omega \otimes \omega)(s, t)\|^2 \leq c(t - s)^{2 - 4\beta'} < \infty.
\]
Then we have acquired the result that $(\omega \otimes \omega)$ is in $C_{2\beta'}(\Delta_{0,T}; L_2(L_2(\ell^2; \ell^2_\kappa); \ell^2 \otimes \ell^2))$. \hfill \Box

Lemma A.2. Consider $\omega \in C_{2\beta'}([0, T]; \ell^2)$ and $(\omega^n)_{n \in \mathbb{N}}$ be the linear approximation of $\omega$, where $\sigma_i^{1/2} \omega^n_i$, which is the value of $\omega^n$ in position $i$, is the linear approximation of $\sigma_i^{1/2} \omega$. Then $(\omega^n \otimes \omega^n)_{n \in \mathbb{N}}$ is convergent to $(\omega \otimes \omega)$ on $C_{2\beta'}(\Delta_{0,T}; L_2(L_2(\ell^2; \ell^2_\kappa); \ell^2 \otimes \ell^2))$.

Proof. Let $E \in L_2(\ell^2; \ell^2_\kappa)$ and $s, t \in [0, T]$. By the partial integration, one can get
\[
E(\omega^n \otimes \omega^n)(s, t) = \int_s^t E(\omega^n(\xi) - \omega^n(s)) \otimes e_2 d\omega^n(\xi) \\
- \int_s^t A_\lambda \int_s^\xi S(\xi - r)E(\omega^n(r) - \omega^n(s)) dr \otimes e_2 d\omega^n(\xi). \quad (17)
\]
We denote $\int_s^t E(\omega^n(\xi) - \omega^n(s)) \otimes e_2 d\omega^n(\xi)$ by $(\omega^n \otimes \omega^n)(s, t)$. First, we will prove $(\omega^n \otimes \omega^n)(s, t)$ is convergent to a element in $C_{2\beta'}(\Delta_{0,T}; L_2(L_2(\ell^2; \ell^2_\kappa); \ell^2 \otimes \ell^2))$ on a set of full measure. We denote the convergent element by $(\omega \otimes \omega)$. According to
the definition of \((\omega^n \otimes \omega^n)\), we have

\[
\|(\omega^n \otimes \omega^n)(s, t)\|^2 = \sum_{i,j} \sum_{l,k} (E_{ij}(\omega^n \otimes \omega^n)(s, t), e_i \otimes e_j e_k)^2 \\
\leq \sum_i (1 + |i|)^{-2\alpha} \sum_{j,k} \sigma_j \sigma_k \left( \int_s^t (\omega_j^n(\xi) - \omega_j^n(s)) d\omega_j^n(\xi) \right)^2.
\]

Then, similar to the statements given by [16, Theorem 5.1], we can obtain \((\omega^n \otimes \omega^n)(s, t)\) is convergent to an element since \(\omega^n\) is the linear approximation of \(\omega\).

Now, we begin to prove that

\[
\lim_{n \to \infty} \int_s^t A_{ij} \int_s^t S(\xi - r)(\omega_j^n(r) - \omega_j^n(s)) dr \otimes e_j d\omega_i^n(\xi)
= \int_s^t A_{ij} \int_s^t S(\xi - r)(\omega_j(r) - \omega_j(s)) dr \otimes e_j d\omega_i(\xi)
\]

in \(C_{2\beta'}(\Delta_{0,T}; L_2(\ell^2; \ell^2_k); \ell^2 \otimes \ell^2)\) on a set of full measure. In order to do that, we need to prove \(\int_s^t A_{ij} \int_s^t S(\xi - r)(\omega(r) - \omega(s)) dr \otimes e_j d\omega_i(\xi) \in C_{2\beta'}(\Delta_{0,T}; L_2(\ell^2; \ell^2_k); \ell^2 \otimes \ell^2)\) firstly. For \(s \leq \xi \leq \xi \leq t\),

\[
\left\| A_{ij} \int_s^\xi (S(\xi - r)E_{ij}(\omega(r) - \omega(s)), e_i) dr \right\|
= \frac{1}{(1 + |i|) \alpha} \left\| A_{ij} \int_s^\xi (S(\xi - r)e_i, e_i)(\omega_j(r) - \omega_j(s)) dr \right\|
\leq c(\xi - s)^{2\beta'} \frac{1}{(1 + |i|) \alpha} \sigma_j \omega_j \left( \int_s^\xi (S(\xi - r)e_i, e_i)^2 dr \right)^{1/2},
\]

and

\[
\left\| A_{ij} \int_s^\xi (S(\xi - r)E_{ij}(\omega(r) - \omega(s)), e_i) dr - A_{ij} \int_s^{\xi - s} (S(\xi' - r)E_{ij}(\omega(r) - \omega(s)), e_i) dr \right\|
\leq c(\xi' - s)^{2\beta'} \frac{1}{(1 + |i|) \alpha} \sigma_j \omega_j \left( \int_s^{\xi - s} (A_{ij}[S(\xi - r) - S(\xi' - r)]e_i, e_i)^2 dr \right)^{1/2}
+ c(\xi - \xi')(\xi - s)^{2\beta'} \frac{1}{(1 + |i|) \alpha} \sigma_j \omega_j \left( \int_s^\xi (A_{ij}S(\xi - r)e_i, e_i)^2 dr \right)^{1/2}.
\]
Hence the definition of derivative shows that
\[
\left\| D^\alpha_{s+} A_\lambda \int_s^t (S(\cdot - r) e_i, e_l)(\omega_j(r) - \omega_j(s))dr \right\|^2 \\
\leq c \left( \int_s^t (A_\lambda S(\xi - r) e_i, e_l)(\omega_j(r) - \omega_j(s))dr \right)^2 \\
+ c \left( \int_s^t (\xi - \xi')^{\beta' + 2} \| \omega_j \|_{\beta', 0, T} \left( \int_s^t (A_\lambda [S(\xi - r) - S(\xi' - r)] e_i, e_l)^2 dr \right)^{1/2} d\xi' \right)^2 \\
+ c \left( \int_s^t (\xi - \xi')^{\beta'} \| \omega_j \|_{\beta', 0, T} \left( \int_s^t (A_\lambda S(\xi - r) e_i, e_l)^2 dr \right)^{1/2} d\xi' \right)^2.
\]

At this moment, we can obtain that
\[
\left\| \int_s^t \int_s^\xi A_\lambda S(\xi - r)(\omega(r) - \omega(s)) dr \otimes \omega \right\|^2
\leq \sum_{k, l} \sum_{i, j} \left( \int_s^t A_\lambda \int_s^\xi S(\xi - r) E_{ij}(\omega(r) - \omega(s)) dr \otimes \omega \right)^2
\leq \sum_{i, j} \sigma_j \sigma_k (1 + |i|)^{-2\kappa} \left\| \int_s^t \int_s^\xi (A_\lambda S(\xi - r) e_i, e_l)(\omega_j(r) - \omega_j(s)) dr d\omega_k(\xi) \right\|^2
\leq \sum_{i, j} \sigma_j \sigma_k (1 + |i|)^{-2\kappa} \left\| \int_s^t D_\lambda^\alpha S(\xi - r) e_i, e_l)(\omega_j(r) - \omega_j(s)) dr \right\|^2 d\xi
\times \int_s^t \left\| D_\lambda^{1-\alpha} \omega_k \right\|^2 dr
\leq c \left[ (t-s)^{4\beta' + 3} + (t-s)^{4\beta' + 2} \right] \sum_{i} (1 + |i|)^{-2\kappa} \sum_{j, k} \sigma_j \sigma_k \| \omega_j \|^2_{\beta', 0, T} \| \omega_k \|^2_{\beta' 0, T} < \infty,
\]
where we have applied \( \sum_{j, k} \sigma_j \sigma_k \| \omega_j \|^2_{\beta', 0, T} \| \omega_k \|^2_{\beta' 0, T} < \infty, \) see [16, Corollary 5.2].

Hence we complete the proof of \( \int_s^t A_\lambda \int_s^\xi S(\xi - r)(\omega(r) - \omega(s)) dr \otimes \omega \) \( \in C_{2\beta'}(\Delta_{0, T}; L_2(\ell^2; \ell^2); \ell^2 \otimes \ell^2) \). At the reminder part, we will give the convergence of it. Applying the same calculate method as before, we have
\[
\left\| \int_s^t A_\lambda \int_s^\xi S(\xi - r)(\omega^n(r) - \omega^n(s)) dr \otimes \omega^n(\xi) \right\|^2
\leq c \left[ (t-s)^{4\beta' + 3} + (t-s)^{4\beta' + 2} \right] \sum_{j, k} \sigma_j \sigma_k \left\{ \left( \| \omega^n_j \|^2_{\beta', 0, T} - \| \omega_j \|^2_{\beta' 0, T} \right)^2 \right\} \sum_{i} \frac{1}{(1 + |i|)^{2\kappa}}.
\]
where
\[
\sum_{j,k} \sigma_j \sigma_k \left( \left\| \omega_j^n \right\|_{\beta',0,T} \left( \left\| \omega_k^n \right\|_{\beta',0,T} - \left\| \omega_k \right\|_{\beta',0,T} \right) \right)^2 \xrightarrow{n \to \infty} 0,
\]
\[
\sum_{j,k} \sigma_j \sigma_k \left( \left( \left\| \omega_j^n \right\|_{\beta',0,T} - \left\| \omega_j \right\|_{\beta',0,T} \right) \right)^2 \left\| \omega_k \right\|_{\beta',0,T}^2 \xrightarrow{n \to \infty} 0,
\]
on a set of full measure, see [16, Theorem 5.1]. Then we complete this proof.

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