Gravitational Radiation from Travelling Waves on D-Strings

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Abstract

Boundary states that preserve supersymmetry are constructed for fractional D-strings with travelling waves on a $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. The gravitational radiation emitted between two D-strings with antiparallel travelling waves is calculated.
1. Introduction

Topological defects such as cosmic strings are not thought to be the primary source of large scale structure in the universe while evidence for some sort of inflationary scenario is compelling. Nevertheless, they may play a lesser role in structure formation and other high energy phenomena like baryogenesis, very energetic cosmic rays, and gamma ray bursts. These defects are predicted by string theory and possible observable effects are worth considering.

Recent work has studied unidirectional travelling waves on Dirichlet(D)-strings as an exactly solvable two-dimensional conformal field theory \[1\]. Purely left or right moving waves of arbitrary profile on D-strings preserve one-quarter of the supersymmetry in supersymmetric string theory. These D-strings are dual under the action of type IIB \( SL(2,\mathbb{Z}) \) S-duality to fundamental string states with only left movers or right movers excited. Here, we will calculate the gravitational radiation emitted from the interaction of two fractional D-strings with pulses moving in opposite directions on an \( N = 2 \) supersymmetric \( C^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold of type IIB string theory. The boundary states in this calculation are supersymmetric although the interaction breaks supersymmetry and allows for nontrivial radiation. The interaction of two D-strings with pulses moving in the same direction preserves supersymmetry and does not radiate. In this model the D-string does not produce a conical deficit in the geometry due to the noncompact extra dimensions. Treating more “realistic” four-dimensional models seems difficult, at least in type IIB, because the supergravity with travelling wave requires that the Minkowski dimensions be asymptotically flat far from the D-strings.

In the next section we show that the supergravity travelling wave solution can be applied to the case of a regular (unwrapped) D-string on the conifold. A supergravity solution of purely fractional branes on the conifold that is asymptotically flat in the Minkowski dimensions has yet to be constructed but should exist. Our calculation considers fractional branes in order to restrict movement in the extra dimensions. In section three we construct the relevant boundary states and calculate amplitudes with no insertions, in section four we calculate the gravitational radiation (dilatonic and axionic radiation are included) from the nonsupersymmetric interaction, and in section five we conclude. While this work was progressing there was a paper \[2\] discussing various T-dual phenomena related to the travelling wave. As my work was nearing completion a second related paper \[3\] appeared which has some overlap with section three. This paper constructs boundary states for D-strings with travelling waves in the bosonic string and calculates the amplitude between strings with oppositely directed pulses.
2. Supergravity Solution of a Wiggly D-string on the Conifold

Exact solutions in nonsupersymmetric gravitational theories of travelling waves on cosmic strings were constructed in [4]. These solutions and generalizations could also be constructed in string theory or supergravity [5]. Recent work has analyzed interesting configurations related by various dualities to the travelling wave [6][7].

In this paper we will calculate the amplitude of interactions of fractional branes on the \( \mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold. The resolution of this orbifold contains three \( \mathbb{P}^1 \)'s that each look locally like the conifold. By tuning two of the Kahler parameters to be large, we can approximate the geometry of the conifold. In the resolved orbifold the fractional branes are equivalent to bound states of D3-branes wrapped on a conifold and a D1-brane. Of course, the string theory calculation can only be done in the orbifold limit. In this limit there are three complex plane singularities. To avoid singularities of the metric away from the branes in the orbifold dimensions, we assume that a small resolution of all three \( \mathbb{P}^1 \)'s does not dramatically affect the string theory result. Far from the branes the geometry should be little affected by whether the branes are fractional or regular. Thus, asymptotically the geometry is the conifold (or any noncompact Calabi-Yau threefold) and Minkowski space.

We consider a Dirichlet onebrane parallel to the lightcone directions \( x^\pm = \frac{t \pm x_1}{\sqrt{2}} \). The two transverse directions of Minkowski space we will denote by \( \vec{y} \). The string is localized at \( \vec{y} = 0 \) and at the origin of the conifold \( (r_{\text{con}} = 0) \) where \( r_{\text{con}} \) is a radial coordinate on the resolved conifold. Adding the travelling wave amounts to the \( x^- \) dependent translation \( \vec{y} = \vec{Y}(x^-) \) and requiring the metric to be asymptotically flat in the Minkowski directions. The solution takes the form

\[
\begin{align*}
\text{ds}^2 &= e^{-\phi} g_{\mu\nu} dx^\mu dx^\nu \\
&= -2e^{-\phi} dx^+ dx^- - (e^{-2\phi} - 1) \|\vec{Y}\|^2 dx^- + 2(e^{-2\phi} - 1) \vec{Y} \cdot d\vec{y} dx^- + \|d\vec{y}\|^2 + ds_{\text{con}}^2 \\
B_{++} &= 1 - e^{-2\phi} \\
B_{--} &= \sqrt{2} \dot{Y}_1 (e^{-2\phi} - 1).
\end{align*}
\]

(2.1)

Here, \( \vec{Y} = (Y^2, Y^3) \) depends only on \( x^- \), the dot denotes differentiation with respect to \( x^- \), \( \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu e^{2\phi}) = -Q \delta^{(8)}(\vec{x} - \vec{Y}) \) where \( \phi(\vec{y}, r_{\text{con}}) \) is the dilaton, \( Q = 64\pi^6 \alpha'^3 g_s \), and \( g_s \) is constant. The coordinate \( \vec{y}' = \vec{y} - \vec{Y}(x^-) \). For large distances in the variable \( R = \sqrt{\|\vec{y}'\|^2 + r_{\text{con}}^2} \) and no angular momentum, \( e^{2\phi} \sim 1 + \frac{Q}{64\pi R} \) where \( \Omega_7 = \frac{\pi^4}{3} \). In
the above formulas what we have called $e^\phi$ for simplicity of notation includes a factor $\frac{1}{g_s}$. Supersymmetry is broken to $\frac{1}{16}$ by the conditions

$$\gamma^+ \epsilon = \epsilon^*$$
$$\gamma^- \epsilon = 0$$
$$D_{\mu}^{(conifold)} \epsilon = 0. \quad (2.2)$$

Here, $\epsilon$ is a supersymmetric spinor of type IIB supergravity. The effective theory has two supercharges of negative chirality. There is a T-dual configuration with three fivebranes intersecting on a string (two fivebranes intersect in a threebrane while the third fivebrane intersects one of the two in a threebrane and the other in a string) with momentum in eleven dimensional supergravity. This triple intersection is different from the one analyzed in [8] as it is nonchiral.

3. Boundary States and Interactions for Travelling Waves

In this section we construct boundary states for the fractional D-string with travelling wave on the orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. Most of the ingredients needed for our calculation in this section and the next can be found in [9]. The boundary state formalism was introduced in [10] as an expression of the duality between one loop open string and tree closed string amplitudes and developed in [11] as a solution of a boundary condition on left and right movers.

The four fractional branes satisfying open-closed duality constraints on this orbifold take the form

$$|B_\alpha > = |F_0 > + \sum_j \epsilon_{\alpha j} |F_j > \quad (3.1)$$

where $\prod_j \epsilon_{\alpha j} = 1$, $\epsilon_{\alpha j} = \pm 1$, $\alpha \in \{0, 1, 2, 3\}$, $j \in \{1, 2, 3\}$, $F_0$ corresponds to a onebrane in flat ten dimensions, and $F_j$ corresponds to a fractional onebrane at one of the three possible $\mathbb{C}^2/\mathbb{Z}_2$ orbifolds. The three $|B_\alpha >$ with $\alpha \neq 0$ are equivalent in the resolution of the orbifold to a bound state of two threebranes wrapped on one of the three conifold $\mathbb{P}^1$’s, a threebrane and antithreebrane on the other two conifold $\mathbb{P}^1$’s, plus a onebrane. The state $|B_0 >$ has two threebranes wrapped on each conifold plus a onebrane.

Note that the conjectured theories in [8] with (0,2) supersymmetry obtained by wrapping fivebranes on fourcycles in Calabi-Yau fourfolds have anomalies that cannot be cancelled and are therefore inconsistent.
The interaction of two boundary states is given by a cylinder amplitude in which a closed string is exchanged between the two boundaries. The length of the cylinder is \( l \), and the amplitude takes the form

\[
\mathcal{A}_{ab}^{\alpha\beta} = \int_0^\infty dl < B_\alpha, \vec{Y}_{1a} | e^{-lH} | B_\beta, \vec{Y}_{2b} >
\] (3.2)

where \( H \) is the closed string Hamiltonian and \( \vec{Y}_{ia} \) are the amplitudes for the travelling waves on the D-strings. \( \vec{Y}_{ia} = \vec{Y}_i(x^a) \) where \( a = \pm \). To construct the boundary states, let us write the expansions for closed string bosonic and fermionic fields as a function of \( z = \sigma + i\tau \) where \( \sigma \) is periodic with \( 0 \leq \sigma \leq 1 \), \( 0 \leq \tau \leq l \), and \( \alpha' = \frac{1}{2\pi} \).

\[
X^\mu(z) = \frac{\hat{x}^\mu}{2} - \frac{z}{2} \hat{\beta}^\mu + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} a^\mu_n e^{2\pi inz}
\]

\[
\tilde{X}^\mu(\bar{z}) = \frac{\hat{x}^\mu}{2} + \frac{\bar{z}}{2} \hat{\beta}^\mu + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} \tilde{a}^\mu_n e^{-2\pi in\bar{z}}
\]

\[
Z^i(z) = \frac{i}{\sqrt{4\pi}} \sum_r \frac{1}{r} b^i_r e^{2\pi irz}
\]

\[
\tilde{Z}^i(\bar{z}) = \frac{i}{\sqrt{4\pi}} \sum_r \frac{1}{r} \tilde{b}^i_r e^{-2\pi ir\bar{z}}
\] (3.3)

\[
\Psi^\mu(z) = \sum_p \psi^\mu_p e^{2\pi ipz}
\]

\[
\tilde{\Psi}^\mu(\bar{z}) = \sum_p \tilde{\psi}^\mu_p e^{-2\pi ip\bar{z}}
\]

\[
\Lambda^i(z) = \sum_p \lambda^i_p e^{2\pi i(p+\frac{1}{2})z}
\]

\[
\tilde{\Lambda}^i(\bar{z}) = \sum_p \tilde{\lambda}^i_p e^{-2\pi i(p+\frac{1}{2})\bar{z}}
\]

Here, \( X^\mu(\Psi^\mu) \) are bosonic (fermionic) coordinates in the untwisted directions while \( Z^i(\Lambda^i) \) are coordinates in the directions twisted by a \( \mathbb{Z}_2 \) action, and \( n \in \mathbb{Z}, \quad r \in \mathbb{Z} + \frac{1}{2}, \) while \( p \in \mathbb{Z} \) or \( \mathbb{Z} + \frac{1}{2} \) depending on whether the fermion sector is Ramond (R) or Neveu-Schwarz (NS). The commutation relations are

\[
[a^\mu_n, a^\nu_m] = n\eta^{\mu\nu} \delta_{n+m}
\]

\[
\{\psi^\mu_p, \psi'^\nu_q\} = \eta^{\mu\nu} \delta_{p+q}
\]

\[
[b^i_r, \tilde{b}^j_s] = r\delta^i_j \delta_{r+s},
\] (3.4)
etc. where \( \eta^{00} = -1 \) and \( \eta^{ij} = \delta^{ij} \) for \( i, j \neq 0 \).

The boundary interaction action of the travelling wave at \( \tau = 0 \) takes the form

\[
S_{+, \eta} = \oint d\sigma \{ \bar{Y}(X^+) \cdot \partial_\tau \tilde{X} + \frac{1}{2} \partial_+ \bar{Y}(X^+) \cdot \bar{\Psi} + \Psi^+ \} 
\]

(3.5)

where \( X^\mu = X^\mu(z) + \tilde{X}^\mu(\bar{z}) \), \( \Psi^\mu_\pm \equiv \Psi^\mu(z) \pm i \eta \bar{\Psi}^\mu(\bar{z}) \), \( \Lambda^i_\pm \equiv \Lambda^i(z) \pm i \eta \bar{\Lambda}(\bar{z}) \), and \( \eta = \pm 1 \).

The action is invariant under the supersymmetries \( \epsilon \) with

\[
\begin{align*}
\delta_\epsilon X^+ &= \epsilon \Psi^+ \\
\delta_\epsilon \tilde{X} &= \epsilon \bar{\Psi}_- \\
\delta_\epsilon \Psi^+ &= -2i \epsilon \partial_\sigma X^+ \\
\delta_\epsilon \bar{\Psi}_+ &= -2 \epsilon \partial_\tau \tilde{X}.
\end{align*}
\]

(3.6)

The boundary equations of motion yield

\[
\begin{align*}
\partial_\tau X^+ |_{boundary} &= 0 \\
\partial_\tau X^- - \partial_+ \tilde{Y} \cdot \partial_\tau \tilde{X} - \frac{1}{2} \partial_+^2 \tilde{Y} \cdot \bar{\Psi} + \Psi^+ |_{boundary} &= 0 \\
\partial_\sigma \tilde{X} - \partial_+ \bar{Y} \partial_\sigma X^+ |_{boundary} &= 0 \\
\partial_\sigma Z^i |_{boundary} &= 0 \\
\Psi^+ |_{boundary} &= 0 \\
\bar{\Psi}_- - \partial_+ \bar{Y} \Psi^+ |_{boundary} &= 0 \\
\bar{\Psi}_- - \partial_+ \bar{Y} \Psi^+ |_{boundary} &= 0 \\
\Lambda^i_\pm |_{boundary} &= 0.
\end{align*}
\]

(3.7)

To expand these equations in oscillator modes, let us assume that we can expand \( \bar{Y} \) and \( \partial_+ \bar{Y} \) in the form

\[
\begin{align*}
Y^i(X^+(\sigma)) &= \sum_{n \in \mathbb{Z}} f^i_{-n} e^{2\pi i n \sigma} \\
\partial_+ Y^i(X^+(\sigma)) &= \sum_{n \in \mathbb{Z}} g^i_{-n} e^{2\pi i n \sigma}
\end{align*}
\]

(3.8)

where \( f^i_{-n} \) and \( g^i_{-n} \) are functions of the commuting modes \( a^+_n, \tilde{a}^+_n, \) and \( \hat{x}^+ \) of the form

\[
f^i_n = \sum_{i_1, \ldots, i_l} c^i_{\mu_1 \mu_2 \ldots \mu_l}(\hat{x}^+) a^+_{i_1} \cdots a^+_{i_l} \tilde{a}^+_{l_1} \cdots \tilde{a}^+_{l_m},
\]

(3.9)
$p_i, q_j$ are negative integers, and $n \neq 0$. These operators are well defined quantum operators without ordering ambiguities, and $p^+ = 0$ at the boundary. Note that $\vec{f}_0 = \vec{Y}(\hat{x}^+) + \cdots$ and $\vec{g}_0 = \partial_+ \vec{Y}(\hat{x}^+) + \cdots$. We also use the relations of oscillators below to make the $p_i$ and $q_j$ negative integers.

The boundary conditions for oscillator modes at $\tau = 0$ are the following:

\[
(a_n^+ + \bar{a}_{-n}^-) F_1, \vec{Y}_+, \eta > = 0
\]
\[
(a_n^- + \bar{a}_{-n}^- - \sum_m \vec{g}_m \cdot (\bar{a}_{n+m} + \bar{a}_{-n-m}) - \frac{1}{\sqrt{\pi}} \bar{g}_n \cdot \hat{\vec{p}} + O(\partial^2_+ \vec{Y})) |F_1, \vec{Y}_+, \eta > = 0
\]
\[
(a^\mu_n - \bar{a}^\mu_{-n}^- - \sum_m \bar{g}_m^\mu \delta^\mu_1 (a^+_n + \bar{a}^+_{-n}^-) ) |F_1, \vec{Y}_+, \eta > = 0
\]
\[
(b^i_r - \bar{b}^i_{-r}^-) |F_1, \vec{Y}_+, \eta > = 0
\]
\[
(\psi^+_p + i \eta \bar{\psi}^+_p - \sum_m \bar{g}_m \cdot (\bar{\psi}_{p+m} + i \eta \bar{\psi}_{-p-m})) |F_1, \vec{Y}_+, \eta > = 0
\]
\[
(\psi^\mu_p - i \eta \bar{\psi}^\mu_p - \sum_m \bar{g}_m (\psi^+_{p+m} - i \eta \bar{\psi}^+_{-p-m})) |F_1, \vec{Y}_+, \eta > = 0
\]
\[
(\lambda^i_{p+\frac{1}{2}} - i \eta \bar{\lambda}^i_{-p-\frac{1}{2}})|F_1, \vec{Y}_+, \eta > = 0
\]

(3.10)

for $\mu \in \{2,3,4,5\}, i \in \{2,3\}$, and $n \neq 0$. Terms with $g_n$ and $n \neq 0$ are also of order $\partial^2_+ \vec{Y}$.

For bosonic zero modes we have the conditions:

\[
\hat{p}^+ |F_1, \vec{Y}_+, \eta > = 0
\]
\[
(\hat{p}^- - \bar{g}_0 \cdot \hat{\vec{p}} - \sqrt{\pi} \sum_{m \neq 0} \bar{g}_m \cdot (\bar{a}_m - \bar{a}_{-m}) + O(\partial^2_+ \vec{Y})) |F_1, \vec{Y}_+, \eta > = 0
\]
\[
(\hat{x}^\mu - f^i_0 \delta^\mu_i ) |F_1, \vec{Y}_+, \eta > = 0
\]

(3.11)

We have verified to order $\partial_+ \vec{Y}$ that the boundary state satisfying the above equations is invariant under a linear combination of the left and right moving local supersymmetries as well as the Virasoro constraints which imply conformal invariance. Note that translational invariance in one of the light cone directions is broken, and global supersymmetry, accordingly, is broken in half. The boundary state satisfying the above constraints can be written as

\[
|F_1, \vec{Y}_+, \eta > = \int dx^+ dx^- \int \frac{d^4q}{(2\pi)^4} O(\vec{Y}_{+,\eta}) |F_1, \eta >osc \otimes |F_1, \eta >0 \otimes |ghost >
\]

(3.12)
\[ |F_1, \eta >_{osc} = \exp \left( \sum_{n>0} \frac{1}{n} \left( a_{-n} a^\dagger_{-n} + a^\dagger_{-n} a_{-n} + a^\mu_{-n} \cdot \tilde{a}^\mu_{-n} \right) \right. \]
\[ + \sum_{r>0} \frac{1}{r} \left( b^i_{-r} \cdot \tilde{b}^i_{-r} + b^i_{-r} \cdot \tilde{b}^i_{-r} \right) \]
\[ + i\eta \sum_{p>0} (\psi_p \tilde{\psi}_p^+ + \psi_p^+ \tilde{\psi}_p^- + \psi^\mu_p \cdot \tilde{\psi}^\mu_p \]
\[ + \lambda^i_{p+\frac{1}{2}} \cdot \tilde{\lambda}^i_{p+\frac{1}{2}} + \tilde{\lambda}^i_{p+\frac{1}{2}} \cdot \tilde{\lambda}^i_{p+\frac{1}{2}} \right) |0 > \]
\[ |F_1, \eta >_{RR} = \exp \left( i\eta (\psi_0 \tilde{\psi}_0^+ + \tilde{\psi}^2 \psi^2 + \tilde{\psi}^4 \psi^4) \right) \]
\[ (|0 > \otimes |0 >_{\bar{q}, \bar{x}^\pm}) (3.13) \]
\[ O(\bar{Y}^+, \eta) = P e^{\int d\sigma \left( \bar{Y} (x^+) \cdot \partial_x \bar{Y} + \bar{Y} (x^+) \cdot \tilde{\partial}_x \psi^+ \psi_0 \right)} (3.14) \]

The oscillator vacuum is annihilated by modes with \( n, p, r > 0 \). In the NS-NS sector there are additional fermion zero modes for the \( \lambda^i \), but the \( \psi^\mu \) zero modes are not present. Here, \( \psi^\mu = \frac{-i \psi^\mu + \psi^\mu_0}{\sqrt{2}} \) for \( \mu \in \{2, 4\} \).

The zero mode vacuum in the RR sector is defined by the conditions:
\[ \hat{x}^\pm (|0 > \otimes |0 >_{\bar{q}, \bar{x}^\pm}) = x^\pm \]
\[ \hat{p} (|0 > \otimes |0 >_{\bar{q}, \bar{x}^\pm}) = \bar{q} \]
\[ \psi_0^+ |0 > = \tilde{\psi}_0^- |0 > = 0 \]
\[ \psi^\mu |0 > = \tilde{\psi}^\mu |0 > = 0. \] (3.16)

Interchanging \( O(\bar{Y}^+, \eta) \) and \( O(\bar{Y}^-, \eta) \) gives the boundary state for the \( \bar{Y} (x^-) \) pulse. We will not write the ghost contribution explicitly. The ghosts will cancel the contribution to the partition function from two untwisted directions.

The full boundary state \( |F_\alpha, \bar{Y}^+ > \) satisfying a GSO projection can be written as
\[ |F_1, \bar{Y}^+ > = c_1 (|F_1, \bar{Y}^+, + >_{NS-NS} + |F_1, \bar{Y}^+, - >_{NS-NS} + |F_1, \bar{Y}^+, + >_{RR} + |F_1, \bar{Y}^+, - >_{RR}) \]
\[ |F_0, \bar{Y}^+ > = c_2 (|F_0, \bar{Y}^+, + >_{NS-NS} - |F_0, \bar{Y}^+, - >_{NS-NS} + |F_0, \bar{Y}^+, + >_{RR} + |F_0, \bar{Y}^+, - >_{RR}) \] (3.17)

up to overall normalizations \( c_1, c_2 \). The closed string Hamiltonian is
\[ H^s = \frac{p^2}{2} + 2\pi \sum_{n>0} (a_{-n} a^\dagger_{-n} - a^\dagger_{-n} a_{-n} + a^\mu_{-n} \cdot a^\mu_{-n} - \tilde{a}^\mu_{-n} \tilde{a}^\mu_{-n} - \tilde{a}^\mu_{-n} a^\mu_{-n} + a^\mu_{-n} \cdot \tilde{a}^\mu_{-n}) \]
\[ + \sum_{r>0} (b^i_{-r} \cdot \tilde{b}^i_{-r} + b^i_{-r} \cdot \tilde{b}^i_{-r}) \]
\[ + \sum_{p>0} p (-\psi^+_p \psi^-_p + \psi^+_p \tilde{\psi}^+_p - \psi^-_p \tilde{\psi}^-_p - \tilde{\psi}^+_p \tilde{\psi}^-_p + \psi^+_p \tilde{\psi}^-_p + \tilde{\psi}^+_p \psi^-_p) \]
\[ + \sum_{p+\frac{1}{2}>0} (p + \frac{1}{2}) (\lambda^i_{p+\frac{1}{2}} \cdot \tilde{\lambda}^i_{p+\frac{1}{2}} + \tilde{\lambda}^i_{p+\frac{1}{2}} \cdot \tilde{\lambda}^i_{p+\frac{1}{2}} + a^s_{\alpha}) + H_{ghost} \] (3.18)
where the index \( s = 1 \) denotes the NS-NS sector while \( s = 2 \) indicates the RR sector, \( a_0^1 = -1 \) and the other \( a_i^s = 0 \).

Partition function amplitudes where both pulses are a function of \( x^+ \) or \( x^- \) are supersymmetric, and one obtains the usual result up to the zero mode contribution—these amplitudes vanish. When we take pulses \( \tilde{Y}_1(x^-) \) and \( \tilde{Y}_2(x^+) \), supersymmetry is broken, and the oscillator mode contributions of \( \mathcal{O}(\tilde{Y}_{2+}, \eta) \) and \( \mathcal{O}(\tilde{Y}_{1-}, \eta) \) are relevant. An exact calculation is perhaps possible, but since we will be considering low energies compared to the string scale, we will solve the equations (3.10) to linear order in oscillators and to first derivatives of \( \tilde{Y} \). One obtains

\[
\mathcal{O}(\tilde{Y}_{+}, \eta) = \mathcal{O}_{osc} \cdot O_0
\]

\[
O_{osc} = \exp\left\{-2 \sum_{n > 0} \frac{1}{n} \partial_+ \tilde{Y}_+ \cdot (a_+^n \tilde{a}_-^n + \tilde{a}_-^n \tilde{a}_+^n - \partial_+ \tilde{Y}_+ a_+^n \tilde{a}_-^n)\right\}
\]

\[
-2i\eta \sum_{p > 0} \partial_+ \tilde{Y}_+ \cdot \left( \psi_+^p \tilde{\psi}_-^p + \tilde{\psi}_-^p \psi_+^p - \partial_+ \tilde{Y}_+ \psi_+^p \psi_+^p\right)
\]

\[
O_0 = \exp\left\{-i\tilde{Y}(x^+) \cdot \tilde{q} + 2i\eta \tilde{\psi}_0^+ \tilde{\psi}_0^2 \partial_+ Y_+^{(2)} - 2\tilde{\psi}_0^+ \tilde{\psi}_0^2 \partial_+ \tilde{Y}_+^{(2)}\right\}
\]

where \( Y_+^{(2)} = \frac{-iY_2^2 + Y_3^2}{\sqrt{2}} \).

To calculate the nonzero mode oscillator contribution to the nonsupersymmetric amplitude, one can go to a T-dual configuration of threebranes at the orbifold point and then Wick rotate in two directions to obtain threebrane instantons and a time direction corresponding to one of the untwisted transverse directions. (One can alternatively do a direct calculation on the fermions and use supersymmetry.) The Wick rotation makes the electric field imaginary. The matrix rotating the oscillators belongs to \( SO(4, C) \) and takes the form

\[
M_+ = \begin{pmatrix}
1 + \partial_+ \tilde{Y}_+ \cdot \partial_+ \tilde{Y}_+ & i \partial_+ \tilde{Y}_+ \cdot \partial_+ \tilde{Y}_+ & \sqrt{2i} \partial_+ Y_+^2 & \sqrt{2i} \partial_+ Y_+^2 \\
i \partial_+ \tilde{Y}_+ \cdot \partial_+ \tilde{Y}_+ & 1 - \partial_+ \tilde{Y}_+ \cdot \partial_+ \tilde{Y}_+ & -\sqrt{2} \partial_+ Y_+^2 & -\sqrt{2} \partial_+ Y_+^2 \\
-\sqrt{2} \partial_+ Y_+^2 & \sqrt{2} \partial_+ Y_+^2 & 1 & 0 \\
-\sqrt{2} \partial_+ Y_+^3 & \sqrt{2} \partial_+ Y_+^3 & 0 & 1
\end{pmatrix}.
\]

The eigenvalues of this matrix are all equal to one, and there are two independent eigenvectors. The action of \( M_- \) alone commutes with the Hamiltonian so that the relevant rotation for the nonsupersymmetric case is \( R = M_+ M_2+ \). Let \( y_+ = |\partial_+ \tilde{Y}_2+|, y_- = |\partial_- \tilde{Y}_1-| \), and \( \cos \theta = \frac{\partial_+ \tilde{Y}_2+ \partial_- \tilde{Y}_1-}{y_+ y_-} \). We find that the eigenvalues of \( R \) take the form \( \lambda_1 = e^{i\pi \alpha}, \)
\[ \lambda_2 = e^{-i\pi\alpha}, \lambda_3 = e^{i\pi\beta}, \text{ and } \lambda_4 = e^{-i\pi\beta} \] where we trust that the context of using \( \alpha \) and \( \beta \) will preclude confusion with the subscripts of boundary states and

\[
\cos \pi\alpha = 1 - 2y_+y_- \cos \theta + y_+^2y_-^2 \\
+ \sqrt{(-2y_+y_- \cos \theta + y_+^2y_-^2)^2 + 4y_+^2y_-^2\sin^2 \theta} \\
\cos \pi\beta = 1 - 2y_+y_- \cos \theta + y_+^2y_-^2 \\
- \sqrt{(-2y_+y_- \cos \theta + y_+^2y_-^2)^2 + 4y_+^2y_-^2\sin^2 \theta}. \tag{3.21}
\]

One can see that \( \alpha \) is imaginary for \( \theta \neq 0 \) while \( \beta \) is imaginary for large \( y_+y_- \). At least one of the two is always imaginary for both \( y_+ \) and \( y_- \) not zero if \( \theta \neq 0 \).

The amplitudes are as follows:

\[
\int_0^\infty dl < F_1, \bar{Y}_1_-|e^{-iH}l F_1, \bar{Y}_2_+ > = \frac{c_1^2}{4\pi^2} \int dx^+ dx^- \int_0^\infty \frac{dl}{l^2} e^{\frac{-l^2(y_+^2 + y_-^2)}{2\pi}} Z_{F_1} \\
\int_0^\infty dl < F_0, \bar{Y}_0_-|e^{-iH}l F_0, \bar{Y}_2_+ > = \frac{c_2^2}{16\pi^4} \int dx^+ dx^- \int_0^\infty \frac{dl}{l^3} e^{\frac{-l^2(y_+^2 + y_-^2)}{2\pi}} Z_{F_0} \tag{3.22}
\]

where \( q = e^{-2\pi l} \) and

\[
Z_{F_1} = \prod_i \frac{f_2^4(q)f_3^4(q)}{f_4^4(q)f_1^4(q)} - \prod_i \frac{f_3^4(q)f_2^4(q)}{f_1^4(q)f_4^4(q)} Z_0 \\
Z_{F_0} = \prod_i \frac{f_3^4(q)f_3^4(q)}{f_4^4(q)f_1^4(q)} - \prod_i \frac{f_4^4(q)f_4^4(q)}{f_1^4(q)f_4^4(q)} - \prod_i \frac{f_2^4(q)f_2^4(q)}{f_1^4(q)f_3^4(q)} Z_0 \tag{3.23}
\]

with

\[
b(x^+, x^-) = |\bar{Y}_2_+ - \bar{Y}_1_-| \\
f_1^\lambda(q) = q^{\frac{1}{2\lambda}} \prod_{n=1}^\infty (1 - \lambda q^{2n}) \\
f_2^\lambda(q) = \sqrt{2} q^{\frac{1}{2\lambda}} \prod_{n=1}^\infty (1 + \lambda q^{2n}) \tag{3.24} \\
f_3^\lambda(q) = q^{\frac{1}{2\lambda}} \prod_{n=1}^\infty (1 + \lambda q^{2n-1}) \\
f_4^\lambda(q) = q^{\frac{1}{2\lambda}} \prod_{n=1}^\infty (1 - \lambda q^{2n-1}).
\]

The constants \( c_1 \) and \( c_2 \) can be determined from the supersymmetric case by comparing open and closed cylinders. We find that \( c_2 = \frac{\sqrt{2}}{2} c_1 = \pi^2 \alpha' c_1 \) where \( c_1^2 = \frac{\alpha'}{2} = \alpha^2 \). The potentials vanish when either \( y_+ \) or \( y_- \) is zero.

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The total amplitude in the field theory \((l \to \infty)\) limit is
\[
\lim_{l \to \infty} A_{\alpha\beta}^- = \frac{1}{8\pi^2\alpha'}(-1 + 4\delta_{\alpha\beta}) \int dx^+ dx^- \left( \int_1^\infty \frac{dl}{l^2} e^{-\frac{b^2(x^+, x^-)}{4\pi^2\alpha'}} y_+ y_- \cos \theta \right)
+ \frac{1}{64\pi^2\alpha'} \int dx^+ dx^- \left( \int_1^\infty \frac{dl}{l^4} e^{-\frac{b^2(x^+, x^-)}{4\pi^2\alpha'}} \frac{y_+^2 y_-^2}{2} \right).
\] (3.25)

The absolute value of the potential decreases with separation of the D-strings, but the sign of the potential can change as a function of the two-dimensional position since \(\theta\) depends on \(x_+\) and \(x_-\). The maximum nonunitary eigenvalue \(\lambda\) generates a logarithmic divergence in the amplitude as \(\lambda q^2 \to 1\) and should be indicative of a decay process. For small \(l\) a naive modular transformation continued to nonunitary \(\lambda, l \to \frac{1}{2\pi}\), reveals a tachyon in the open string spectrum for \(b(x^+, x^-) \sim \sqrt{2\pi^2\alpha'}\), and the amplitude gains an imaginary piece.

Although the potential vanishes when the wave is unidirectional, supersymmetry is not generally restored continuously. To analyze this issue, we multiply \(\vec{Y}_1^-\) by \(0 < \epsilon << 1\) and examine the correlation function
\[
A_{F_1^+, \epsilon} = \epsilon \int_0^\infty dl < F_1, \eta_1 | S_{\eta_2, \eta_1}^{\dagger} e^{-lH} | F_1, \eta_2, \vec{Y}_{2+} > .
\] (3.26)

The purely bosonic part of this correlation function vanishes by supersymmetry. The fermion zero mode piece in the even spin structure of the RR sector (the odd spin structure vanishes due to zero modes) takes the form
\[
\frac{\epsilon}{\alpha'} \int dx^+ dx^- \int_0^\infty \frac{dl}{l^2} f_2^A(q) f_3^A(q) e^{-\frac{b^2(x^+, x^-)}{4\pi^2\alpha'}} (y_+ y_- \cos \theta + O(\alpha'^2)).
\] (3.27)

A modular transformation shows that there is a tachyon for \(b(x^+, x^-) < b_{\text{crit}} = \sqrt{2\pi^2\alpha'}\). The NS-NS terms do not cancel the tachyon. There is also a divergence in the \(\alpha'\) expansion for \(l \to 0\). Without a detailed calculation, the expansion for small \(l\) is a series with terms of the form \((2\pi^2\alpha')^{2n}\) multiplied by \((2n)^{\text{th}}\) derivatives of \(y_+\) and \(y_-\). For \(b >> b_{\text{crit}}\), the exponential damping factor should cancel this divergence. The divergence for small \(b\) is localized in a region of the D-strings if the pulses have finite energy. Because the divergence exists for infinitesimal \(\epsilon\), supersymmetry is not restored continuously unless \(b >> b_{\text{crit}}\). The target space supersymmetry being broken by the presence of the D-string with an antiparallel travelling wave is chiral, and one might expect a discontinuity when the mass of the string communicating this breaking is sufficiently small.
The above potentials imply that the vacuum is not stable unless either \( y_+ \) or \( y_- \) is zero or, alternatively, \( b(x^+, x^-) \to \infty \) for \( x^+ \) and \( x^- \) such that \( y_+ y_- \neq 0 \). The interaction is localized in space and time on the D-strings. Depending on the details of the pulses, the interacting pieces are pushed apart or they attract. In the case of attraction, a tachyon develops at a critical separation, and one expects the system to decay to a supersymmetric set of D-strings with a unidirectional travelling wave. Both possibilities can occur in different regions of the D-string worldvolume and the two possibilities are exchanged by interchanging the \( \alpha = \beta \) and \( \alpha \neq \beta \) boundary states. For the radiation calculation of the next section, we will take \( b >> b_{\text{crit}} \) and \( g_s = \lim_{R \to \infty} e^\phi << 1 \) so that the perturbative calculation is approximately valid. Also, we will take only the lowest order in the \( \alpha' \) derivative expansion as we will be studying radiation in the field theory limit. Note that \( y_+ \) and \( y_- \) are dimensionless since \( \vec{Y} \) has the dimension of length, and without additional input we would need to include arbitrary powers of \( y_+ \) and \( y_- \) obtained from the boundary state of equation (3.19). To make the radiation calculation less than horrendous as well as to minimize back reaction effects, we will take \( y_+, y_- << 1 \).

4. Gravitational Radiation

Calculation of gravitational radiation from a cosmic string with travelling waves was performed in [12]. In this section we calculate the massless NS-NS closed string emission from the interaction of two D-strings with antiparallel travelling waves. A similar calculation of radiation from moving D-particle interactions is extensively discussed in [9], and many of the results there will be useful in our calculation. We want to determine the following amplitude:

\[
A_{\alpha\beta}^\pm(p) = \int_0^\infty dl \int_0^l d\tau <B_\alpha, \vec{Y}_{1-}|e^{-lH}V_p(\sigma, \tau)|B_\beta, \vec{Y}_{2+}> \tag{4.1}
\]

where we approximate the boundary states as in equation (3.19) with \( y_+, y_- << 1 \), the closed string vertex operator

\[
V_p(z, \bar{z}) = e_{\mu\nu}(\partial X^\mu - \frac{1}{2} p \cdot \tilde{\Psi}^\mu)(\partial X^\nu + \frac{1}{2} p \cdot \tilde{\Psi} \tilde{\Psi}^\nu)e^{ip \cdot X}, \tag{4.2}
\]

and \( 2p^+ p^- = ||\vec{p}||^2 \equiv p^2 \). We restrict the index \( \mu \in \{+, -, 2, 3\} \). Here, \( X^\mu = X^\mu(z) + \vec{X}^\mu(\bar{z}) \). Because there are extra fermion zero modes for the \( Z_2 \times Z_2 \) orbifold, the odd spin structure vanishes unless the \( V_p \) insertion has these zero modes. If we consider polarizations
in the extra dimensions, then there will be a term with two zero modes. We will not discuss this case further here.

Our gauge choice is the following. The axion has antisymmetric components \( e_{+i} = \frac{-a^i p_1 - b p^j \epsilon_{ij} p^l}{p^2} \), \( e_{-i} = \frac{a^i p_1 - b p^j \epsilon_{ij} p^l}{p^2} \), \( e_{++} = a^p \), and \( e_{ij} = b^p \epsilon_{ij} \). The dilaton has a symmetric polarization tensor with nonvanishing components \( e_{++} = \frac{\phi^p(p^-)^2}{p^2} \), \( e_{--} = \frac{\phi^p(p^-)^2}{2p^2} \), and \( e_{ij} = \phi^p(\delta_{ij} - \frac{p_1 p_2}{p^2}) \) for \( i, j \in \{2, 3\} \). The graviton has symmetric components \( e_{++} = -h_2^p (p^-)^2 \), \( e_{--} = h_2^p (p^+)^2 \), \( e_{i+} = h_1^p p^- \epsilon_{ij} p^l \), and \( e_{i-} = h_1^p p^+ \epsilon_{ij} p^l - h_2^p \frac{p^+ p^-}{2p^2} \) with the others vanishing and satisfies \( \eta^{\mu \nu} e_{\mu \nu} = 0 \). All polarization tensors satisfy \( p^\mu e_{\mu \nu} = 0 \), and the dilaton has \( \eta^{\mu \nu} e_{\mu \nu} = 2 \phi^p \). The index \( p \) indicates that these components are a Fourier transform that will be defined more precisely near the finish of this calculation. This gauge is consistent with working in the rescaled metric \((2.1)\). We will use the notation \( \vec{v}_1 \times \vec{v}_2 = v_1^i v_2^j \epsilon_{ij} \). Note that \( h_1^p \) is gauge invariant while \( h_2^p \) mixes with the axion so that \( h_2^p - 2a^p \) is gauge invariant. As we realized after doing the calculation, \( b^p \) is the fourth physical degree of freedom that can either be packaged as the \( ij \) component of the axion or as part of the graviton.

The amplitude can be split into a zero mode and oscillator part. Defining \( l' = l - \tau \) so that \( l' = 0 \) at the boundary with pulse \( \vec{Y}_1(x^-) \) and \( l' = l \) at the boundary with \( \vec{Y}_2(x^+) \), the amplitude between two \( F_1 \) D-strings can be written as (see [9] for details)

\[
\mathcal{A}_{F_1}(p) = c_1^2 \int dx^+ dx^- (e^{i\vec{Y}_2(x^+) \cdot \vec{p} - ip^+ x^- - ip^- x^+}). \tag{4.3}
\]

Note that \( k^+ = p^+ \), \( q^- = -p^- \), and \( \vec{k} = \vec{q} + \vec{p} \) with \( k^- = q^+ = 0 \). Also, \( \vec{b}(x^+, x^-) = \vec{Y}_1(x^-) - \vec{Y}_2(x^+) \). To lowest order in derivatives, the integrals over \( x^+ \) and \( x^- \) enforce the further conditions that \( p^+ = \partial_- \vec{Y}_1 \cdot \vec{k} \) and \( p^- = -\partial_+ \vec{Y}_2 \cdot \vec{q} \). The factor

\[
\mathcal{N}_{F_1} \equiv \sum_{s, a} Z_{F_1, s, a} \mathcal{M}_{s, a} \epsilon_{F_1, s, a} \tag{4.4}
\]

where

\[
Z_{F_1, s, a} = < F_1, \eta_1, \vec{Y}_{1-}, e^{-tH} | F_1, \eta_2, \vec{Y}_{2+} >_{osc} \tag{4.5}
\]

\( a = \pm 1 \) with \( \eta_1 \eta_2 = a \), and \( \epsilon_{F_1, s, a} \) is determined by \((3.17)\). The \( Z_{F_1, s, a} \) can be read from equation \((3.23)\) where we only take linear terms in \( y_+ \) and \( y_- \).

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In the even spin structure correlation functions are defined as
\[
< \mathcal{O}(\sigma, \tau) >_{osc}^s \equiv \frac{< F_i, \eta_1, \bar{Y}_- | e^{-iH} \mathcal{O}(\sigma, \tau) | F_i, \eta_2, \bar{Y}_+ >_{osc}^s}{Z_{F_i}}. \tag{4.6}
\]

Bosonic correlators can be calculated with \( y_+ = y_- = 0 \) since they get multiplied by a fermion zero mode term proportional to \( y_+ y_- \). The correlation function
\[
< e^{ip \cdot X} >_{osc} = \prod_{n=0}^{\infty} (1 - q^{2n} e^{-4\pi \tau})^{-\frac{\pi}{2}} (1 - q^{2n} e^{-4\pi \tau'})^{-\frac{\pi}{2}}.
\tag{4.7}
\]

The odd spin structure will only contribute to the \( F_0 \) amplitudes in the NS-NS sector and is defined as above.

The factor \( \mathcal{M}^{s_a} \) takes the form
\[
\mathcal{M}^{s_a} = e_{\mu\nu} \{ < \partial X^\mu \bar{\partial} X^\nu >_{osc} - < \partial X^\mu p \cdot X >_{osc} < \bar{\partial} X^\nu p \cdot X >_{osc}
+ \frac{1}{4} ( < p \cdot \Psi \bar{\Psi} >_{osc}^s < \Psi \bar{\Psi}^\nu >_{osc}^s
- < p \cdot \Psi \Psi^\mu >_{osc}^s < p \cdot \bar{\Psi} \bar{\Psi}^\nu >_{osc}^s + < p \cdot \bar{\Psi} \Psi^\mu >_{osc}^s < p \cdot \bar{\Psi} \bar{\Psi}^\nu >_{osc}^s)
+ \frac{i}{2} ( < \partial X^\mu p \cdot X >_{osc} - < \bar{\partial} X^\nu p \cdot X >_{osc} < p \cdot \Psi \Psi^\mu >_{osc}^s)
- \frac{1}{2} k^\mu ( i < \bar{\partial} X^\nu p \cdot X >_{osc} + \frac{1}{2} < p \cdot \bar{\Psi} \bar{\Psi}^\nu >_{osc}^s)
+ \frac{1}{2} k^\nu ( i < \partial X^\mu p \cdot X >_{osc} - \frac{1}{2} < p \cdot \Psi \Psi^\mu >_{osc}^s) - \frac{1}{4} k^\mu k^\nu \}. \tag{4.8}
\]

The nonzero correlators, where we need to keep track of \( \eta \), are the following:
\[
< \partial X^+(z) \bar{X}^-(\bar{z}) >_{osc} = - < \bar{\partial} \bar{X}^+(\bar{z}) X^-(z) >_{osc} = \frac{i}{2} K(\tau, l) \tag{4.9}
\]
\[
< \partial X^i(z) \bar{X}^j(\bar{z}) >_{osc} = \delta^{ij} \frac{i}{2} K(\tau, l) \tag{4.10}
\]
\[
< \Psi^+(z) \bar{\Psi}^-(\bar{z}) >_{osc}^s = < \Psi^-(z) \bar{\Psi}^+(\bar{z}) >_{osc}^s = \frac{-i \eta_1}{2Z_0} \delta^s_2 \delta^s_\pm + i \eta_1 F^{s_a} - 4i \eta_1 \bar{Y}^s_+ \cdot \bar{Y}^s_- (H^{s_a}(l', l) + \eta_1 \eta_2 H^{s_a}(\tau, l)) \tag{4.11}
\]
\[
< \Psi^i(z) \bar{\Psi}^j(\bar{z}) >_{osc}^s = (\frac{-i \eta_1}{2Z_0} \delta^i_j - \frac{i \eta_1}{2Z_0} (y^i_+ y^j_- + y^j_+ y^i_-)) \delta^s_2 \delta^s_\pm
+ i \eta_1 \delta^i_j F^{s_a} - 4i \eta_1 (y^i_+ y^j_- + y^j_+ y^i_-) (H^{s_a}(l', l) + \eta_1 \eta_2 H^{s_a}(\tau, l)) \tag{4.12}
\]
\[
< \Psi^i(z) \bar{\Psi}^+(\bar{z}) >_{osc}^s = \frac{-i \eta_1}{2Z_0} y^i_- \delta^s_2 \delta^s_\pm - 2i \eta_2 y^i_- G^{s_a}(l', l) \tag{4.13}
\]
\begin{align}
\langle \Psi^i(z) \bar{\Psi}^{-}(\bar{z}) \rangle^{sa} &= \frac{-i\eta_1 y_i^l \delta^a_\pm}{\varepsilon Z_0} \delta^a_\pm - 2i\eta_1 y_i^l + G^{sa}(\tau, l) \tag{4.14} \\
\langle \Psi^+(z) \bar{\Psi}^+(\bar{z}) \rangle^{sa} &= -2i\eta_2 \bar{y}_- \cdot \bar{y}_- G^{sa}(l', l) \tag{4.15} \\
\langle \Psi^-(z) \bar{\Psi}^-(\bar{z}) \rangle^{sa} &= -2i\eta_1 \bar{y}_+ \cdot \bar{y}_+ G^{sa}(\tau, l) \tag{4.16} \\
\langle \Psi^+(z) \Psi^-(z) \rangle^{sa} &= -N_{\infty}^{osc} \tag{4.17} \\
\langle \bar{\Psi}^+(z) \bar{\Psi}^-(\bar{z}) \rangle^{sa} &= \frac{1 - 2Z_0}{Z_0} \delta^a_\pm - N_{\infty}^{osc} \\
\langle \Psi^i(z) \Psi^j(z) \rangle^{sa} &= \delta^{ij} \left( \frac{1}{2} \delta^a_\pm + N_{\infty}^{osc} \right) \tag{4.18} \\
\langle \bar{\Psi}^i(z) \bar{\Psi}^j(\bar{z}) \rangle^{sa} &= \left( \frac{1}{2} \delta^a_\pm + \frac{i\epsilon^{ij} Z_0 - 1}{Z_0} \right) \delta^a_\pm + \delta^{ij} N_{\infty}^{osc} \\
\langle \Psi^i(z) \Psi^+(z) \rangle^{sa} &= \langle \bar{\Psi}^i(\bar{z}) \bar{\Psi}^+(\bar{z}) \rangle^{sa} = \frac{-y_i^l}{2Z_0} \delta^a_\pm \tag{4.19} \\
\langle \Psi^i(z) \Psi^-(z) \rangle^{sa} &= \langle \bar{\Psi}^i(\bar{z}) \bar{\Psi}^-(\bar{z}) \rangle^{sa} = \frac{y_i^l}{2Z_0} \delta^a_\pm \tag{4.20} \\
K(\tau, l) &= -\sum_{n=0}^{\infty} \left( \frac{q^{2n-4\pi \tau}}{1 - q^{2n-4\pi \tau}} - \frac{q^{2n-4\pi l'}}{1 - q^{2n-4\pi l'}} \right) \tag{4.21} \\
F^{2+} &= -\sum_{n=0}^{\infty} (-1)^n \left( \frac{q^{2n-4\pi \tau}}{1 - q^{2n-4\pi \tau}} + \frac{q^{2n-4\pi l'}}{1 - q^{2n-4\pi l'}} \right) \\
F^{1\pm} &= -\sum_{n=0}^{\infty} (\mp)^n \left( \frac{q^{n-2\pi \tau}}{1 - q^{2n-4\pi \tau}} \pm \frac{q^{n-2\pi l'}}{1 - q^{2n-4\pi l'}} \right) \tag{4.22} \\
G^{2+}(x, l) &= \sum_{n=0}^{\infty} (-1)^n(n + 1) \frac{q^{2n-4\pi x}}{1 - q^{2n-4\pi x}} \tag{4.23} \\
G^{1\pm}(x, l) &= \sum_{n=0}^{\infty} (\mp)^n(n + 1) \frac{q^{n-2\pi x}}{1 - q^{2n-4\pi x}} \\
H^{2+}(x, l) &= \sum_{n=0}^{\infty} (-1)^n \frac{(n + 2)!}{n!^2} \frac{q^{2n+2-4\pi x}}{1 - q^{2n+2-4\pi x}} \tag{4.24} \\
H^{1\pm}(x, l) &= \sum_{n=0}^{\infty} (\mp)^n \frac{(n + 2)!}{n!^2} \frac{q^{n+1-2\pi x}}{1 - q^{2n+2-4\pi x}}.
\end{align}

To simplify notation \(\bar{y}_+ = \partial_+ \bar{Y}_{2+}\) and \(\bar{y}_- = \partial_- \bar{Y}_{1-}\). The infinite constant \(N_{\infty}^{osc}\) does not contribute to \(\mathcal{M}^{sa}\) because \(p^\mu e_{\mu\nu} = 0\). Note that \(H^{2+}\) will not contribute in the field theory limit.
The axion emission comes from the asymmetry of left and right zero mode fermions in the even spin RR sector. We obtain the term

$$\mathcal{M}_{ax}^{sa} = \frac{a_p}{4} \left( 1 - \frac{2Z_0}{Z_0} \right) (p^2 - \tilde{k} \cdot \tilde{p}) \delta_2^s \delta_+^a. \quad (4.25)$$

There is no dependence on nonzero mode fermion correlations. For the dilaton and the graviton the term $\mathcal{M}^{sa}$ simplifies to

$$\mathcal{M}^{sa}_{\phi} = \phi^p \left\{ \frac{p^2}{16} + \frac{1}{8} (\tilde{p} \cdot \tilde{y}_+)(\tilde{p} \cdot \tilde{y}_-) - \frac{1}{8} p^2 \tilde{y}_+ \cdot \tilde{y}_- ight. \\
+ \frac{1}{8} (\tilde{p} \cdot \tilde{y}_+)(\tilde{k} \cdot \tilde{y}_-) - \frac{1}{8} (\tilde{y}_+ \cdot \tilde{k})(\tilde{p} \cdot \tilde{y}_-) - \frac{1}{4} (\tilde{k} \cdot \tilde{p})(\tilde{k} \cdot \tilde{y}_-) - \frac{1}{4} (\tilde{k} \cdot \tilde{y}_+)(\tilde{k} \cdot \tilde{y}_-) \\
+ \frac{1}{4} \tilde{k} \cdot \tilde{p} (\tilde{y}_+ \cdot \tilde{y}_+)(\tilde{p} \cdot \tilde{y}_-) + \frac{1}{4} \tilde{k} \cdot \tilde{p} (\tilde{p} \cdot \tilde{y}_+)(\tilde{k} \cdot \tilde{y}_-) - \frac{i}{8} (\tilde{k} \cdot \tilde{y}_+)(\tilde{y}_+ \cdot \tilde{y}_-) \delta_2^s \delta_+^a \\
+ \left. \frac{1}{2} \left( p^2 (\tilde{y}_+ \cdot \tilde{y}_-) - (\tilde{p} \cdot \tilde{y}_+)(\tilde{p} \cdot \tilde{y}_-) \right) F^{sa} \right\} \quad (4.26)$$

$$\mathcal{M}^{sa}_{h_1} = \hbar^p \left\{ \frac{1}{8} (\tilde{y}_+ \times \tilde{p})(\tilde{y}_- \times \tilde{p}) - (\tilde{y}_+ \cdot \tilde{p})(\tilde{y}_- \times \tilde{p}) - \frac{i}{8} p^2 (\tilde{y}_+ \cdot \tilde{y}_-) - \frac{1}{4} (\tilde{k} \times \tilde{p})(\tilde{y}_+ \cdot \tilde{y}_-) \\
+ \frac{1}{8} (\tilde{p} \cdot \tilde{y}_+)(\tilde{y}_- \times \tilde{p}) + \frac{1}{4} (\tilde{y}_+ \times \tilde{p})(\tilde{y}_- \times \tilde{k}) \\
- \frac{1}{4} (\tilde{k} \times \tilde{p})(\tilde{y}_+ \times \tilde{k}) + \frac{1}{4} (\tilde{k} \times \tilde{p})(\tilde{y}_+ \times \tilde{k}) \delta_2^s \delta_+^a \\
+ \left. \left[ 1 (\tilde{y}_+ \cdot \tilde{p})(\tilde{y}_- \times \tilde{p})(G^{sa}(\tau, l) - G^{sa}(l', l)) + \frac{1}{4} (\tilde{y}_+ \times \tilde{p})(\tilde{y}_- \cdot \tilde{k})(G^{sa}(\tau, l) - G^{sa}(l', l)) \\
- \frac{1}{2} (\tilde{y}_+ \cdot \tilde{p})(\tilde{y}_- \times \tilde{p})(F^{sa} - 2G^{sa}(l', l)) - \frac{1}{2} (\tilde{y}_+ \times \tilde{p})(\tilde{y}_- \cdot \tilde{k})(F^{sa} - 2G^{sa}(\tau, l)) \delta_2^s \delta_+^a \\
- a((\tilde{y}_+ \times \tilde{p})(\tilde{y}_- \cdot \tilde{p}) - (\tilde{y}_+ \cdot \tilde{p})(\tilde{y}_- \times \tilde{p})) G^{sa}(\tau, l) G^{sa}(l', l) \\
- 2a(\tilde{y}_+ \cdot \tilde{q})(\tilde{y}_- \times \tilde{q}) G^{sa}(l', l) F^{sa} - 2(\tilde{y}_+ \times \tilde{p})(\tilde{k} \cdot \tilde{y}_-) G^{sa}(\tau, l) F^{sa} - \frac{1}{4} (\tilde{k} \times \tilde{p}) K - \frac{1}{4} (\tilde{k} \times \tilde{p}) \right\} \quad (4.27)$$
\[ \mathcal{M}_{h2}^{sa} = \frac{\hbar^2}{16} \left( \frac{p^2}{16} (\vec{y}^2 \cdot \vec{y}^2) - \frac{1}{8} (\vec{k} \cdot \vec{p})(\vec{y}^2 \cdot \vec{y}^2) + \frac{3}{16} (\vec{q} \cdot \vec{y}^2)(\vec{p} \cdot \vec{y}^2) + \frac{1}{8} (\vec{p} \cdot \vec{y}^2)(\vec{k} \cdot \vec{y}^2) \right. \\
\left. - \frac{1}{8} \frac{(\vec{k} \cdot \vec{p})}{p^2} (\vec{q} \cdot \vec{y}^2)(\vec{p} \cdot \vec{y}^2) + \frac{1}{8} \frac{(\vec{k} \cdot \vec{p})}{p^2} (\vec{q} \cdot \vec{y}^2)(\vec{k} \cdot \vec{y}^2) \right) \delta_s^a \\
+ \frac{1}{4} (\vec{p} \cdot \vec{y}^2)(\vec{q} \cdot \vec{y}^2)(G^{sa}(\tau, l) - G^{sa}(l', l)) \\
+ \frac{1}{2} (\vec{q} \cdot \vec{y}^2)(\vec{p} \cdot \vec{y}^2)(G^{sa}(l', l) - F^{sa}) + \frac{1}{2} (\vec{p} \cdot \vec{y}^2)(\vec{k} \cdot \vec{y}^2)(G^{sa}(\tau, l) - F^{sa}) \right) \delta_s^a \\
- \frac{1}{8} p^2 K + \frac{1}{16} p^2 - \frac{1}{8} (\vec{k} \cdot \vec{p}) \right). \] (4.28)

The field theory \((l \to \infty)\) limits of \(K, F^{sa}\), and \(G^{sa}\) are

\[
K \to -(f(\tau) - f(l')) \\
F^{2+} \to -(f(\tau) + f(l')) \\
G^{2+}(x, l) \to f(x)
\] (4.29)

where \(f(x) = \frac{e^{-4\pi x}}{1 + e^{-4\pi x}}\). Integration by parts has been used to calculate \(\mathcal{M}^{sa}\) with

\[
\int_0^\infty d\tau \int_0^\infty dl' (\partial_\tau - \partial_\tau') \left\{ e^{-\frac{\tau^2 + (m^2)^2}{2}} e^{-\frac{k^2 + m^2 l'}{2}} < e^{ipX} >_{osc} \right\} = 0 \quad (4.30)
\]

where the infinite surface terms have been set to zero by an analytic continuation from \(p^2 < 0\). This integration by parts in the \(l \to \infty\) limit allows one to replace \(f(\tau)\) by \(-\frac{1}{4} \frac{q^2 + m^2}{p^2}\) and \(f(l')\) by \(-\frac{1}{4} \frac{k^2 + m^2}{p^2}\) where \(m^2\) is the momentum in the zero mode directions orthogonal to the four Minkowski dimensions. There are cancellations between different sectors so that only terms at the most linear in the \(f(x)\) appear. We also make use of the relation

\[
\int_0^\infty dx e^{-\frac{x}{2}(1 - e^{-4\pi x})} = \frac{1}{4\pi} B\left(\frac{a^2}{8\pi}, 1 - \frac{b^2}{2\pi}\right) = \frac{1}{4\pi} \frac{\Gamma\left(\frac{a^2}{8\pi}\right) \Gamma\left(1 - \frac{b^2}{2\pi}\right)}{\Gamma\left(\frac{a^2}{8\pi} + 1 - \frac{b^2}{2\pi}\right)}. \] (4.31)

Equation (4.3) yields

\[
\mathcal{N}_{F_1} = \sum_{\eta_1} (Z_{F_1}^{1+} \mathcal{M}_{F_1}^{1+} + Z_{F_1}^{2+} \mathcal{M}_{F_1}^{2+}) \\
\mathcal{N}_{F_0} = \sum_{\eta_0} (Z_{F_0}^{1+} \mathcal{M}_{F_0}^{1+} - Z_{F_0}^{1-} \mathcal{M}_{F_0}^{1-} + Z_{F_0}^{2+} \mathcal{M}_{F_0}^{2+}). \] (4.32)

Calculating these terms in the \(l \to \infty\) limit yields

\[
\lim_{l \to \infty} \mathcal{N}_{F_1}^{sa} = a^p (1 - 2\vec{y}_+ \cdot \vec{y}_-)(p^2 - \vec{k} \cdot \vec{p}) \] (4.33)
\[ \lim_{t \to \infty} N_{\phi}^{\alpha\xi} = 4N_{\phi}^{\alpha\xi} \]  \hspace{1cm} (4.34)

\[ \lim_{t \to \infty} N_{\phi}^{\alpha_f} = \phi \{ -\frac{1}{4}p^2 + \left( \frac{1}{4}p^2 - \frac{3}{4}k^2 + \frac{1}{2}(\vec{k} \cdot \vec{p})^2 \right) + \frac{i}{2}(\vec{k} \times \vec{p} - \frac{1}{2}m_1^2)\vec{y}_+ \cdot \vec{y}_- + \frac{1}{2}(\vec{k} - \vec{p})^2 \} + \frac{1}{2}(\vec{k} \cdot \vec{p})(\vec{p} \cdot \vec{y}_+)(\vec{p} \cdot \vec{y}_-) + 2(\vec{k} \cdot \vec{y}_+)(\vec{k} \cdot \vec{y}_-) \]  \hspace{1cm} (4.35)

\[ \lim_{t \to \infty} N_{\phi}^{\rho_f} = \phi \{ -p^2 + (2p^2 + 2i\vec{p} \times \vec{k})\vec{y}_+ \cdot \vec{y}_- \]  \hspace{1cm} (4.36)

\[ \lim_{t \to \infty} N_{h_1}^{\rho_f} = h_1 \{ -\frac{1}{4}p^2 + \frac{2\vec{k} \cdot \vec{p}}{2p^2}(\vec{y}_+ \cdot \vec{p})(\vec{y}_- \cdot \vec{p}) + \frac{3p^2 + 2\vec{k} \cdot \vec{p}}{4p^2}(\vec{y}_+ \cdot \vec{p})(\vec{y}_- \cdot \vec{p}) \]  \hspace{1cm} (4.37)

\[ \lim_{t \to \infty} N_{h_2}^{\rho_f} = h_2 \{ -\frac{1}{8}(p^2 - 2\vec{k} \cdot \vec{p})\vec{y}_+ \cdot \vec{y}_- - \frac{3(2 - \frac{1}{2}m_1^2)}{p^2}(\vec{p} \cdot \vec{y}_+)(\vec{p} \cdot \vec{y}_-) \]  \hspace{1cm} (4.38)

\[ \lim_{t \to \infty} N_{h_2}^{\rho_f} = h_2 \{ -\frac{1}{8}(p^2 - 2\vec{k} \cdot \vec{p})\vec{y}_+ \cdot \vec{y}_- - \frac{3(2 - \frac{1}{2}m_1^2)}{p^2}(\vec{p} \cdot \vec{y}_+)(\vec{p} \cdot \vec{y}_-) \]  \hspace{1cm} (4.39)
\[ \lim_{l \to \infty} N_{F_0}^{h_2} = 4N_{F_1}^{h_2} + \left( \frac{1}{2} p^2 - \vec{k} \cdot \vec{p} \right) \vec{y}_+ \cdot \vec{y}_-. \]  

(4.40)

We are considering radiated energies well below the string scale so we only take the lowest order terms in \( \alpha' p^2 \) and utilize the relation

\[ \lim_{\alpha' p^2 \to 0} B\left( \frac{x^2}{8\pi}, 1 - \alpha' p^2 \right) = \frac{8\pi}{x^2}. \]  

(4.41)

Finally, the amplitudes reduce to

\[ \lim_{\alpha' \rightarrow 0} A_{\alpha\beta}(p) = \left( \int dx^+ dx^- (e^{i S_2(x^+)} \vec{p} - ip^+ x^- - ip^- x^+ \right). \]  

(4.42)

\[ \lim_{\alpha' \rightarrow 0} A_0(p) = \left( \int dx^+ dx^- (e^{i S_2(x^+)} \vec{p} - ip^+ x^- - ip^- x^+ \right). \]  

(4.43)

where \( e_{\alpha\beta} = \pm 1 \) and can be determined from (3.1). We have introduced a cutoff on momentum, \( \Lambda \sim \frac{1}{\sqrt{\alpha'}} \) because the large momentum region picks out the small \( l \) region where the above calculation is no longer valid, but for \( b >> b_{\text{crit}} \) taking \( \Lambda \to \infty \) should be okay. Again, we can do a modular transformation \( l \to \frac{l}{2\pi} \) to show that there is no divergence so long as the average separation of the D-strings is sufficiently large. We can calculate the above integrals using the formula

\[ \frac{1}{4\pi^2} \int d^2 k \frac{e^{i \vec{k} \cdot \vec{b}}}{k^2 + m^2} = - \frac{1}{2\pi} K_0(mb) \]  

(4.44)

where \( K_0 \) is the modified Bessel function of order zero. We also need the following integrals which can be found in [13].

\[ \int_0^\infty xK_0(ax)dx = \frac{\pi}{a^2} \]  

\[ \int_0^\infty xK_0(ax)K_0(bx)dx = \frac{\ln \frac{b}{a}}{a^2 - b^2} \]  

(4.45)

Defining

\[ g(\vec{b}, \vec{p}) = \frac{1}{4\pi^2} \int d^2 x \frac{\ln \frac{|\vec{b} - \vec{x}|}{|\vec{b}|} e^{i \vec{p} \cdot \vec{x}}}{b^2 - 2\vec{b} \cdot \vec{x}} \]  

(4.46)
where \( g(\vec{b}, \vec{p}) \) is finite for \( pb > 0 \), we obtain

\[
\lim \limits_{\alpha' \to \infty} A_{\alpha \beta}^{\alpha x}(p) = 4c_1^2(-1 + 4\delta_{\alpha \beta}) \int dx^+ dx^- \left( e^{i\vec{Y}_2(x^+)} \vec{p}^- \cdot \vec{p}^+ \vec{p}^- \cdot \vec{p}^+ x^- \right).
\] (4.47)

\[
[a^p(1 - 2\vec{y}_+ \cdot \vec{y}_-)(p^2 g + i\vec{g}_b \cdot \vec{p})] + A_0^{\alpha x}(p)
\]

\[
\lim \limits_{\alpha' \to \infty} A_{\alpha \beta}^{\phi}(p) = 4c_1^2(-1 + 4\delta_{\alpha \beta}) \int dx^+ dx^- \left( e^{i\vec{Y}_2(x^+)} \vec{p}^- \cdot \vec{p}^+ x^- \right).
\] (4.48)

\[
h_1^{\alpha \beta}(p) = 4c_1^2(-1 + 4\delta_{\alpha \beta}) \int dx^+ dx^- \left( e^{i\vec{Y}_2(x^+)} \vec{p}^- \cdot \vec{p}^+ x^- \right).
\] (4.49)

\[
\lim \limits_{\alpha' \to \infty} A_{\alpha \beta}^{h_2}(p) = 4c_1^2(-1 + 4\delta_{\alpha \beta}) \int dx^+ dx^- \left( e^{i\vec{Y}_2(x^+)} \vec{p}^- \cdot \vec{p}^+ x^- \right).
\] (4.50)

\[
A_0^{\alpha x}(p) + \cdots
\]
where $\Delta_b$ is the two-dimensional Laplacian with respect to $\vec{b}$ while $(g_b)_i = \partial_{b_i} g(\vec{b}, \vec{p})$ and $(g_{bb})_{ij} = \partial_{b_i} \partial_{b_j} g(\vec{b}, \vec{p})$. We leave as an exercise the calculation of $A_0(p)$ which should vanish more quickly for large $b$.

The function $g$ can be calculated with formulas from [13], and we obtain

$$g(\vec{b}, \vec{p}) = -\frac{e^{ib\vec{p}(2)}(2)}{4\pi\vec{p} \times \vec{b}} \{ Ei\left(\frac{ib\vec{p}(2)}{2}\right) \sinh \frac{\vec{p} \times \vec{b}}{2} + Ei\left(-\frac{ib\vec{p}(2)}{2}\right) e^{\frac{\vec{p} \times \vec{b}}{2}} \sinh \frac{\vec{p} \times \vec{b}}{2} \}$$

where $p^{(2)} = \frac{\vec{p} \cdot \vec{b} + i\vec{p} \times \vec{b}}{b}$, and the exponential integral takes the form

$$Ei(x) = -\int_{-\infty}^{\infty} \frac{e^{-t}}{t} dt$$

for $x < 0$ with the general case defined by analytic continuation. Asymptotically, if $\vec{p} \cdot \vec{b} \to \infty$, the function $g \to e^{i\vec{p} \cdot \vec{b} - \frac{\vec{p} \times \vec{b}}{2}}$. If $\vec{p} \cdot \vec{b} = 0$ and $\vec{p} \times \vec{b} \to \infty$, $g \sim (\vec{p} \times \vec{b})^{-2}$.

For $p \to 0$, $g$ diverges as $\ln pb$, but there is always a factor of $p^2$ so that this part of the amplitude vanishes.

For large $b$ the dominant contributions to the amplitudes are

$$A_{\alpha \beta}^{ax}(p) \sim (-1 + 4\delta_{\alpha \beta}) \alpha' \pi \int dx^+ dx^- (e^{i\frac{\vec{p} \cdot (x^+ - x^-)}{2} + i\frac{\vec{y}_+(x^+ - x^-)}{2}} - ip^+ x^- - ip^- x^+) \left[ ia^p (1 - 2\vec{y}_+ \cdot \vec{y}_-) \frac{p^2}{2} e^{-\frac{i\vec{p} \cdot \vec{y}_-}{2}} \right]$$

$$A_{\alpha \beta}^{\phi}(p) \sim (-1 + 4\delta_{\alpha \beta}) \alpha' \pi \int dx^+ dx^- (e^{i\frac{\vec{p} \cdot (x^+ - x^-)}{2} + i\frac{\vec{y}_+(x^+ - x^-)}{2}} - ip^+ x^- - ip^- x^+) \left[ \phi^p \{ -\frac{p^2 \sin \frac{\vec{p} \cdot \vec{b}}{2}}{4} \frac{\vec{p} \cdot \vec{b}}{\vec{p} \cdot \vec{b}} + \frac{p^2 \sin \frac{\vec{p} \cdot \vec{b}}{2}}{4} \frac{\vec{p} \cdot \vec{b}}{\vec{p} \cdot \vec{b}} - i\frac{p^2 e^{i\vec{p} \cdot \vec{b}}}{4} \frac{\vec{p} \cdot \vec{b}}{\vec{p} \cdot \vec{b}} \} (\vec{p} \cdot \vec{y}_+) (\vec{p} \cdot \vec{y}_-) \right]$$

$$- \left( \frac{1}{2} \frac{\vec{p} \cdot \vec{b}}{\vec{p} \cdot \vec{b}} + i\frac{e^{\frac{i\vec{p} \cdot \vec{b}}{2}}}{4} \frac{\vec{p} \cdot \vec{b}}{\vec{p} \cdot \vec{b}} \right) (\vec{p} \cdot \vec{y}_+) (\vec{p} \cdot \vec{y}_-) \right]$$

For large $b$, the dominant contributions to the amplitudes are

$$A_{ax}^{ax}(p) \sim (-1 + 4\delta_{\alpha \beta}) \alpha' \pi \int dx^+ dx^- (e^{i\frac{\vec{p} \cdot (x^+ - x^-)}{2} + i\frac{\vec{y}_+(x^+ - x^-)}{2}} - ip^+ x^- - ip^- x^+) \left[ ia^p (1 - 2\vec{y}_+ \cdot \vec{y}_-) \frac{p^2}{2} e^{-\frac{i\vec{p} \cdot \vec{y}_-}{2}} \right]$$

$$A_{ax}^{\phi}(p) \sim (-1 + 4\delta_{\alpha \beta}) \alpha' \pi \int dx^+ dx^- (e^{i\frac{\vec{p} \cdot (x^+ - x^-)}{2} + i\frac{\vec{y}_+(x^+ - x^-)}{2}} - ip^+ x^- - ip^- x^+) \left[ \phi^p \{ -\frac{p^2 \sin \frac{\vec{p} \cdot \vec{b}}{2}}{4} \frac{\vec{p} \cdot \vec{b}}{\vec{p} \cdot \vec{b}} + \frac{p^2 \sin \frac{\vec{p} \cdot \vec{b}}{2}}{4} \frac{\vec{p} \cdot \vec{b}}{\vec{p} \cdot \vec{b}} - i\frac{p^2 e^{i\vec{p} \cdot \vec{b}}}{4} \frac{\vec{p} \cdot \vec{b}}{\vec{p} \cdot \vec{b}} \} (\vec{p} \cdot \vec{y}_+) (\vec{p} \cdot \vec{y}_-) \right]$$

$$- \left( \frac{1}{2} \frac{\vec{p} \cdot \vec{b}}{\vec{p} \cdot \vec{b}} + i\frac{e^{\frac{i\vec{p} \cdot \vec{b}}{2}}}{4} \frac{\vec{p} \cdot \vec{b}}{\vec{p} \cdot \vec{b}} \right) (\vec{p} \cdot \vec{y}_+) (\vec{p} \cdot \vec{y}_-) \right]$$

20
\[ A^h_{\alpha\beta}(p) \sim (-1 + 4\delta_{\alpha\beta})\alpha' \pi \int dx^+ dx^- (e^{i\frac{E_1(x^-)}{p}} + e^{i\frac{E_2(x^+)}{p}} - ip^+ x^- - ip^- x^+). \]

\[
\left[ h_1^p \left\{ \frac{ip^2}{2} \sin \frac{p^+ \vec{y}}{b} (\vec{y}^+ \cdot \vec{y}^-) \right. \right.
- \left( \frac{1}{4} \frac{\sin \frac{p^+ \vec{b}}{b}}{\vec{p} \cdot \vec{b}} + \frac{i e^{i\frac{p^+ \vec{b}}{2}}}{2 \vec{p} \cdot \vec{b}} \right)(\vec{y}^+ \cdot \vec{p})(\vec{y}^- \cdot \vec{p}) \right.
+ \left( \frac{3}{4} \frac{\sin \frac{p^+ \vec{b}}{b}}{\vec{p} \cdot \vec{b}} + \frac{i e^{i\frac{p^+ \vec{b}}{2}}}{4 \vec{p} \cdot \vec{b}} \right)(\vec{y}^+ \cdot \vec{p})(\vec{y}^- \cdot \vec{p}) \}
+ h_2^p \left\{ - \frac{ip^2}{8} \cos \frac{p^+ \vec{b}}{b} (\vec{y}^+ \cdot \vec{y}^-) - \frac{i e^{i\frac{p^+ \vec{b}}{2}}}{8 \vec{p} \cdot \vec{b}} (\vec{p} \cdot \vec{y}^+)(\vec{p} \cdot \vec{y}^-) \right\} \right].
\]

Notice that the amplitudes vanish for unidirectional travelling waves. The dominant radiation propagates parallel or antiparallel to \( \vec{b} \). The amplitudes for radiation transverse to \( \vec{b} \) (including the string direction) are suppressed by \( \frac{1}{b} \). The axion amplitude is proportional to the electric field on the D-strings. This effect of causing axion radiation by turning on an electric field along the string can be compared to the effect of causing a bulk anomaly inflow current for a chiral string by an electric field on the string. If the fluctuations of the axion, dilaton, and graviton do not depend on both \( x^+ \) and \( x^- \), their Fourier transforms will contain delta functions of \( p^+ \) or \( p^- \), and the radiation will be confined to the string.

Assume that the strings are far enough apart that there is an approximate solution in supergravity that is the sum of the right and left moving fluctuations or that there is a linear approximation even though supersymmetry is broken. We assume that the above result is not greatly affected by a small resolution of the orbifold. The metric for the resolved orbifold is more complicated than for the conifold but for large distances from the D-strings, the warping factor should depend on this distance in the same way. The radiation is observed at \( r_{con} = 0 \) (see (2.1)) and \( R >> b >> \sqrt{\alpha'} \) so we will ignore the angular dependence due to nonzero separation. The fluctuation of the graviton and dilaton can be written in position space as

\[ e_{\mu\nu} \sim T'_{\mu\nu}(1 - e^{-2\phi}) \sim \frac{T'_{\mu\nu}}{R^6} \]

(4.56)

where the prime denotes that we drop the delta function of the transverse space for the energy-momentum tensor from the D-string source. The energy-momentum tensor for the
travelling wave on the D-string was calculated from the action for the D-string in [1] with the result to lowest order in $\alpha'$

$$T_{+-} = T_D (\delta^{(8)}(\vec{x} - \vec{Y}_1) + \delta^{(8)}(\vec{x} - \vec{Y}_2))$$

$$T_{++} = T_D ||\partial_+ \vec{Y}_2||^2 \delta^{(8)}(\vec{x} - \vec{Y}_2)$$

$$T_{--} = T_D ||\partial_- \vec{Y}_1||^2 \delta^{(8)}(\vec{x} - \vec{Y}_1)$$

$$T_{+i} = -T_D \partial_+ Y^i_2 \delta^{(8)}(\vec{x} - \vec{Y}_2)$$

$$T_{-i} = -T_D \partial_- Y^i_1 \delta^{(8)}(\vec{x} - \vec{Y}_1)$$

(4.57)

where $T_D = \frac{1}{2\pi \alpha' g_s}$ is the D-string tension. In this linear approximation there is only radiation along the strings because there are no terms in the energy-momentum tensor containing an interaction of the two pulses, and the Fourier transform of the fields will contain delta functions of $p^+$ or $p^-$. There will be higher order terms with these interactions leading to gravitational radiation transverse to the strings, but the calculation will break down for $b \rightarrow b_{\text{crit}}$.

To obtain the second order of the graviton fluctuation is difficult. There are $\alpha'$ corrections to the D-string action and to the supergravity fields which couple to this action. In section three we have shown that there is a potential for the nonsupersymmetric interaction proportional to $\vec{y}_+ \cdot \vec{y}_-$. For $b >> b_{\text{crit}}$, this interaction is of order $\frac{1}{b^2}$. To second order the energy-momentum tensor calculated from the D-string action by itself will not be covariantly conserved because one has to include the energy-momentum of the radiation in the bulk. Rather than trying to determine precisely the second order fluctuation we will assume that a conserved energy-momentum tensor of the full theory incorporating the nonlocal interaction can be defined where we still take $b >> b_{\text{crit}}$ so that we are in a weak gravity limit and can use a linear approximation. This energy-momentum tensor takes the form

$$T_{\mu\nu} = T_{(4)}^{(\mu\nu)} \delta^{(6)}(\vec{x})$$

(4.58)

where $\mu \in \{0,1,2,3\}$, and we are ignoring in this formula a small amount of smearing on the resolved orbifold. We choose a gauge such that the three degrees of freedom of the graviton and dilaton satisfy $(\triangle^{(9)} - \partial_t^2) e_{\mu\nu} = (2\pi \alpha')^{\frac{7}{2}} g_s T_{\mu\nu}$ where $\triangle^{(9)}$ is the Laplace operator in nine dimensions. The factor of $(2\pi \alpha')^{\frac{7}{2}}$ is introduced so that the vertex operator is dimensionless in target space units. This calculation should be taken with a grain of salt. We only need to know $e_{\mu\nu}$ far from the source in the Minkowski dimensions. Assuming as we are that there is no angular dependence in the extra dimensions, we should be able to
approximate the result by using the flat space Laplacian. The retarded Green function for the ten-dimensional wave equation takes the form

\[
G(x, x') = \frac{-1}{i\Omega_8 r^7} \left( \delta(\Delta_{x,x'}) - r \delta'(\Delta_{x,x'}) + \frac{2}{5} r^2 \delta''(\Delta_{x,x'}) - \frac{1}{15} r^3 \delta'''(\Delta_{x,x'}) \right)
\]  

(4.59)

where \( \Delta_{x,x'} = t' - t + |x_9 - x'_9|, r = |x_9 - x'_9|, \) and \( \Omega_8 = \frac{32\pi^3}{105} \).

We have seen that if the source contains a noninteracting piece for large separations, the radiation is confined to the string. In what follows we assume that the source is interacting. If we consider finite energy pulses as physically sensible, then the source for the radiation is not the entire worldvolume of the D-strings but only the localized regions where both pulses are nonvanishing. We assume that the spatial regions can be enclosed in a compact region of radius \( r_s << r \) where \( r \) is the observer's radius and \( r_s \approx b \). Note that the observer can be along the string far from the source. The dilaton and graviton satisfy

\[
e_{\mu\nu}(\vec{x}, t) \sim \frac{(2\pi\alpha')^{\frac{5}{2}} g_s}{7\Omega_8} \int d^3 x' \frac{1}{|\vec{x} - \vec{x}'|^7} (T^{(4)}_{\mu\nu}(\vec{x}', t - |\vec{x} - \vec{x}'|) - |\vec{x} - \vec{x}'| \partial_t T^{(4)}_{\mu\nu}(\vec{x}', t - |\vec{x} - \vec{x}'|) + \frac{2}{5} |\vec{x} - \vec{x}'|^2 \partial^2_t T^{(4)}_{\mu\nu}(\vec{x}', t - |\vec{x} - \vec{x}'|) - \frac{1}{15} |\vec{x} - \vec{x}'|^3 \partial^3_t T^{(4)}_{\mu\nu}(\vec{x}', t - |\vec{x} - \vec{x}'|))
\]

(4.60)

Taking the Fourier transform of \( T^{(4)}_{\mu\nu} \) in the time direction yields

\[
T^{(4)}_{\mu\nu}(\vec{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega T^{(4)}_{\mu\nu}(\vec{x}, \omega) e^{-i\omega t}
\]

(4.61)

where the origin of the coordinate \( \vec{x} \) lies at the center of the compact region and at the (resolved) orbifold point, \( \omega = \frac{p^+ - p^-}{\sqrt{2}} \), and \( T^{(4)}_{\mu\nu}(\vec{x}, -\omega) = T^{(4)}_{\mu\nu}(\vec{x}, \omega) \). Note that the coordinate \( \vec{x} \) includes the extra dimensions (these are zero for \( T^{(4)}_{\mu\nu} \)), but our observations will always be at the (resolved) orbifold point. Then

\[
e_{\mu\nu}(\vec{x}, t) \sim \frac{4\pi^2 \alpha'^2 g_s}{7\Omega_8} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \int \frac{d\vec{x}'}{|\vec{x} - \vec{x}'|^7} e^{i(\omega|\vec{x} - \vec{x}'|)} (1 + i\omega|\vec{x} - \vec{x}'| - \frac{2}{5} \omega^2 |\vec{x} - \vec{x}'|^2 + \frac{1}{15} i\omega^3 |\vec{x} - \vec{x}'|^3). \]

(4.62)

In the limit that the observation distance \( r \) is much larger than the extent of the source, we can approximate \( |\vec{x} - \vec{x}'| \) by \( r - \frac{\vec{x} \cdot \vec{x}'}{r} \). Thus, we obtain

\[
e_{\mu\nu}(\vec{x}, t) \sim \frac{i\alpha'^2 g_s}{8\pi^2 r^4} \int_{-\infty}^{\infty} d\omega r^3 e^{-i\omega t + i\vec{k} \cdot \vec{x}} \int d\vec{x}' e^{-i\vec{k} \cdot \vec{x}'} T^{(4)}_{\mu\nu}(\vec{x}', \omega)
\]

(4.63)
where $\vec{k} = \frac{\omega \vec{x}}{r}$. Obviously, there is a divergence for travelling waves that are highly localized in the lightcone directions so for this model, we must consider waves with compact support in $\omega$ or, alternatively, temporally periodic waves with discrete frequencies. For periodic waves we would calculate the power rather than the energy.

The polarization tensor of the graviton and dilaton are obtained from the above equation as

$$e_{\mu \nu}(\vec{x}, \omega) = \frac{i \alpha' \omega^3 g_s}{2 r^4} T^{(4)}_{\mu \nu}(\vec{k}, \omega).$$

From this formula we can extract the forms of $\phi^p$, $h_1^p$, and $h_2^p$ at large distances $r$.

$$\phi^p \sim \frac{i \alpha' \omega^3 g_s}{2 r^4} \phi(\vec{k}, \omega)$$

$$h_1^p \sim \frac{i \alpha' \omega^3 g_s}{2 r^4} h_1(\vec{k}, \omega)$$

$$h_2^p \sim \frac{i \alpha' \omega^3 g_s}{2 r^4} h_2(\vec{k}, \omega)$$

where

$$\phi(\vec{k}, \omega) = \frac{1}{2} \eta^{\mu \nu} T^{(4)}_{\mu \nu}(\vec{k}, \omega)$$

$$h_1(\vec{k}, \omega) = \frac{p^+ T^{(4)}_+ \times \vec{p} - p^- T^{(4)}_- \times \vec{p}}{p^2}$$

$$h_2(\vec{k}, \omega) = \frac{2 p^+ T^{(4)}_+ \cdot \vec{p} - p^- T^{(4)}_- \cdot \vec{p}}{p^2}$$

with $\vec{k} = (\frac{p^+ - p^-}{\sqrt{2}}, p^2, p^3)$ and $T^{(4)}_+ \times \vec{p} = T^{(4)}_{+i} p_j \epsilon^{ij}$. The condition $2p^+ p^- = p^2$ is imposed by hand. Note again that $h_2$ and $\phi$ are not gauge invariant, but we replace them in the gauge we have chosen by the right hand side of the above formulas which is gauge invariant. The boundary states break Lorentz invariance.

We now calculate the energy radiated through a large sphere of radius $r$ surrounding the interaction region of the D-strings. Not all of the radiation is in four dimensions–there is also radiation in the extra dimensions. To not violate energy conservation we need to consider an eight-sphere of radius $r$ surrounding the source. Then we set all but two of the angles on the eight-sphere to $\pi/2$ so that we observe the radiation through a two-sphere at the (resolved) orbifold point. Again, we are assuming no angular dependence in the extra dimensions. The energy flux per unit solid angle observed at the (resolved) orbifold point is given by

$$\frac{dE}{d\Omega_8} = r^8 \int_0^\infty d\omega \omega^8 |A|^2$$

(4.67)
where $A$ is the amplitude from the string calculation. At large distances energy is conserved because there is no $r$ dependence, and we obtain

$$\frac{dE_\phi}{d\Omega_8} \sim \frac{g_s^2 \alpha'^7}{4} \int_0^\infty d\omega \omega^{14} |f^\phi \phi|^2$$

(4.68)

$$\frac{dE_h}{d\Omega_8} \sim \frac{g_s^2 \alpha'^7}{4} \int_0^\infty d\omega \omega^{14} (|f^{h_1} h_1|^2 + |f^{h_2} h_2|^2)$$

(4.69)

where $f^\phi = \frac{A^{\phi}(p)}{\phi \alpha'}$, $f^{h_1} = \frac{A^{h_1}(p)}{h_1 \alpha'}$, and $f^{h_2} = \frac{A^{h_2}(p)}{h_2 \alpha'}$.

The factors $f^\phi$, $f^{h_1}$, and $f^{h_2}$ greatly restrict the range of $\omega$ for slowly varying pulses so that these integrals should converge despite the enormous powers of frequency. In the field theory limit we might expect a suppression of short wavelengths relative to the separation of the strings. We find a suppression of order $|g|^2 |\phi|^2 \sim \sin^2 (kk')$ (since the amplitude in section 3 is $\sim 1/k^2$ and $\phi$ is a dimensionless function of $\vec{k}$, $\phi \sim 1/k^2$) so the suppression is a power rather than exponential. Radiation with the wave vector transverse to the separation direction $\vec{b}$ is suppressed by an additional factor of $1/b_k^2$ relative to radiation parallel to this direction. For the pulses to have finite energy $y_+$ and $y_-$ must decrease to zero for large $|x^+|$ and $|x^-|$, and the higher derivatives are responsible for the amplitudes $A(p)$ not vanishing. For more intense pulses our approximations break down, and these integrals will diverge. Reducing the number of dimensions reduces the possible frequency divergence. For instance, for six noncompact dimensions, ignoring Kaluza-Klein effects, one would obtain

$$\frac{dE_{\phi \text{six}}}{d\Omega_4} \sim \frac{g_s^2 \alpha'^3}{4} \int_0^\infty d\omega \omega^6 f^\phi_6 |\phi|^2$$

(4.70)

Note that had we taken spatially periodic waves with finite energy per unit length, the source of radiation would be the entire string not just localized pieces and the calculation would be modified as in Reference [12].

5. Conclusions

Let us summarize our results. We have constructed supersymmetric boundary states for fractional D-strings with travelling waves on a $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. The underlying orbifold with D-strings is supersymmetric. The interaction between two of these D-strings with waves travelling in opposite directions breaks supersymmetry and was calculated. Whether the interaction was attractive or repulsive depended on the sign of $\vec{y}_+ \cdot \vec{y}_-$ as
well as whether or not the fractional branes were of the same type. This interaction was localized on the D-strings to the regions where \( \vec{y}_+ \cdot \vec{y}_- \) was nonvanishing and could change sign along the D-string worldvolume. In all cases a tachyon developed at small separation of the two strings as an open string mode became massless in the region where the interaction was nonvanishing. One might expect that there would be a tachyonic condensation with antiparallel momentum modes annihilating. The end result would be a BPS state with momentum in one direction. The tachyon is related to massless modes in the RR sector. The relevant term in the supergravity would be of the form \( \int H^{RR}_+ \cdot H^{RR}_- \) where \( H^{RR}_\pm \) is the three-form RR field strength due to the D-string with a right or left-moving pulse. We found that for small separations of the D-strings such that the tachyon was present, supersymmetry could not be restored continuously. This result was physically reasonable since the target space supersymmetry was chiral.

Under the assumption that for large separations of the D-strings compared to the critical separation, perturbative string theory was not entirely invalid, we calculated the axion, dilaton, and graviton radiation emitted from the interaction of antiparallel pulses. By a gauge transformation one can consider the axion and dilaton as part of the gravitational field. At large enough separations the effective energy-momentum tensor does not incorporate the nonlocal interaction between the two pulses, and the radiation is restricted to the D-strings. The gravitational radiation is the result of left and right moving momenta annihilating into gravitons. This interaction deforms the waves over time so that one must somehow obtain an energy-momentum tensor that incorporates the D-strings as well as the bulk. We were not able at this time to obtain the appropriate interacting energy-momentum tensor but assuming its existence showed that there was gravitational radiation. There are two variables in the model governing the intensity of the interaction. Increasing the distance between the D-strings decreases the interaction. The amount of radiation can be tuned to an arbitrarily small value by studying boundary states in which the frequency distribution has \( |p^+| \) or \( |p^-| \) very small so that, for example, the pulse on the first D-string is a very slowly varying function of \( x^- \), while the pulse on the second D-string is an arbitrary function of \( x^+ \). One could perhaps have enough control over the time dependent interaction to calculate the variation of the waves leading to a supersymmetric configuration. This interaction is significantly less intense than brane-antibrane annihilation so it might be more easily understood. At the next order of \( \alpha' \), this calculation probably requires a fast computer. The discontinuity between the supersymmetric and nonsupersymmetric configurations necessitates a nonperturbative formulation.
There are differences between this string theory calculation and the classical gravity calculation of cosmic string radiation as in [12]. We are required by supersymmetry to separate the right and left moving pulses so that the interaction is nonlocal. The situation of having both type of pulses on the same string would not give a controllable interaction in string theory. How to obtain the energy-momentum tensor for this kind of nonlocal interaction is challenging. We are also required to include noncompact extra dimensions so that the metric is asymptotically flat at large distances, and the string theory calculation makes sense. As a result the energy spectrum is unrealistic. Perhaps an analogous calculation could be done in four dimensions in an orientifolded theory. The time dependence of these backgrounds is somewhat trivial because the wave only travels in one spatial dimension. Having constructed supersymmetric boundary states, studying waves in perpendicular directions could be a means for obtaining backgrounds with nontrivial three-dimensional dependence. The calculations in this paper were done in the closed string boundary state framework. One could possibly study these questions from the dual open string viewpoint via the S-matrix formalism developed in [I]. One still needs to understand the nonperturbative dynamics of supersymmetry breaking.

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