Smooth multisoliton solutions and their peakon limit of Novikov’s Camassa–Holm type equation with cubic nonlinearity

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Abstract

We consider Novikov’s Camassa–Holm type equation with cubic nonlinearity. In particular, we present a compact parametric representation of the smooth bright multisolution solutions on a constant background and investigate their structure. We find that the tau-functions associated with the solutions are closely related to those of a model equation for shallow-water waves (SWW) introduced by Hirota and Satsuma. This novel feature is established by applying the reciprocal transformation to the Novikov equation. We also show by specifying a complex phase parameter that the smooth soliton is converted to a novel singular soliton with single cusp and double peaks. We demonstrate that both the smooth and singular solitons converge to a peakon as the background field tends to zero, whereby we employ a method that has been developed for performing a similar limiting procedure for the multisoliton solutions of the Camassa–Holm equation. In the subsequent asymptotic analysis of the two- and N-soliton solutions, we confirm their solitonic behavior. Remarkably, the formulas for the phase shifts of the solitons as well as their peakon limits coincide formally with those of the Degasperis–Procesi equation. Last, we derive an infinite number of conservation laws of the Novikov equation by using a relation between solutions of the Novikov equation and those of the SWW equation. In appendix, we prove various bilinear identities associated with the tau-functions of the multisoliton solutions of the SWW equation.

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1. Introduction

Recently, Novikov introduced an integrable Camassa–Holm (CH) type equation (the Novikov equation hereafter) in an attempt to classify the nonlocal partial differential equations (PDEs) with quadratic or cubic nonlinearity [1]. It may be written in the form

\[ m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_xx, \]  

(1.1)
where \( u = u(x, t) \) is a function of time \( t \) and a spatial variable \( x \), and the subscripts \( x \) and \( t \) appended to \( m \) and \( u \) denote partial differentiation. Using the perturbative symmetry approach, he found a few symmetries and then derived a scalar Lax representation for it. Subsequently, Hone and Wang gave a matrix Lax representation and showed that the Novikov equation is related by a reciprocal transformation to a negative flow in an integrable hierarchy of the Sawada–Kotera equation [2]. A bi-Hamiltonian structure of the hierarchy was also provided as well as an infinite number of conservation laws. A remarkable feature of the Novikov equation is that it admits peaked waves (or peakons) whose dynamics are shown to obey a finite dimensional integrable Hamiltonian system [2, 3]. As for exact solutions, a few works have been concerned with peakons [2–4]. We emphasize that smooth soliton solutions have not been available as yet.

The purpose of this paper is to construct the smooth \( N \)-soliton solution (\( N \): arbitrary positive integer) of the Novikov equation which satisfies the boundary condition \( u \to u_0 \) (\( u_0 \): real constant) as \(|x| \to \infty\) and analyze its structure. Note that equation (1.1) is invariant under the transformation \( u \to -u \). Hence, we may consider positive solutions only. The constant \( u_0 \) is therefore taken to be positive without loss of generality. While the peakon solutions of the Novikov equation have been constructed under the vanishing boundary condition, we demonstrate for the first time that the smooth solitons converge to the peakons in the limit of zero background field \( (u_0 \to 0) \).

This paper is organized as follows. In section 2, we transform equation (1.1) to a system of equations by a reciprocal transformation and reveal that it can be solved in terms of the tau-function associated with the \( N \)-soliton solution of a model equation for shallow-water waves (SWW equation for short) introduced by Hirota and Satsuma [5]. The various novel bilinear identities are presented among the tau-functions which are related simply to the tau-function for the \( N \)-soliton solution of the SWW equation. In section 3, we provide a compact parametric representation for the \( N \)-soliton solution of the Novikov equation. The proof of the solution is performed by means of a purely algebraic procedure within the framework of the bilinear formalism. We find that the structure of the tau-functions for the \( N \)-soliton solution is relevant to that of the Degasperis–Procesi (DP) equation [6, 7]. In section 4, we investigate the properties of the one-, two- and \( N \)-soliton solutions in detail. The smooth one-soliton solution takes the form of a bright soliton on a constant background. In addition, we show that a novel singular soliton with W-shaped profile is produced from the smooth soliton which exhibits both cusp and peak. This can be established simply by specifying a complex phase parameter for the smooth soliton solution. Last, we demonstrate that the smooth soliton converges to the peakon in the limit of \( u_0 \to 0 \) with the velocity of the soliton being fixed, which we call the peakon limit. This limiting procedure is performed by employing a novel method [8, 9] that has been developed for solving a similar problem for the multisoliton solutions of the CH equation. We also provide numerical evidence for the passage to the peakon. Furthermore, we briefly discuss the passage of the singular soliton to a peakon in the same context. In the subsequent asymptotic analysis of the two- and \( N \)-soliton solutions, we obtain the formulas for the phase shift of solitons and confirm their solitonic behavior. Remarkably, we notice that the formulas coincide formally with those of the \( N \)-soliton solution of the DP equation [6, 7]. These formulas reduce, in the peakon limit, to the corresponding ones for the two- and \( N \)-peakon solutions of the Novikov equation obtained by means of the inverse scattering transform (IST) method [3]. In section 5, we derive an infinite number of conservation laws starting from those of the SWW equation. While the conservation laws have been constructed by using the Lax pair for the Novikov equation [2], our method is based on a purely algebraic procedure without recourse to the IST. Section 6 is devoted to concluding remarks. In the appendix, we prove the various bilinear identities presented in section 2.
2. Reciprocal transformation and SWW equation

2.1. Reciprocal transformation

In accordance with [2], we introduce the coordinate transformation \( (x, t) \rightarrow (y, \tau) \) by
\[
dy = m^{2/3} \, dx - m^{2/3} \, u^2 \, dt, \quad d\tau = dt,
\]
subjected to the restriction \( m > 0 \). Consequently, the \( x \) and \( t \) derivatives can be rewritten as
\[
\frac{\partial}{\partial x} = m^{2/3} \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - m^{2/3} u^2 \frac{\partial}{\partial y}.
\]

It now follows from (2.1b) that the variable \( x = x(y, \tau) \) satisfies a system of linear PDEs
\[
x_y = m^{-2/3}, \quad x_\tau = u^2.
\]

We apply the transformation (2.1b) to the Novikov equation and find that it can be recast into the form
\[
m_\tau + 3m^{4/3} uu_\tau = 0.
\]

On the other hand, \( u \) from (1.1) can be rewritten in terms of \( m \) as
\[
u = m + m^{4/3} u_y + \frac{2}{3} m^{1/3} m_t u_y.
\]

If we define the new variables \( V \) and \( W \) by \( V = m^{2/3} \) and \( W = um^{1/3} \), respectively, then equations (2.3) and (2.4) can be put into the form [2]
\[
\left( \frac{1}{V} \right)_\tau = \left( \frac{W^2}{V} \right)_y,
\]
\[
W_{yy} + U W + 1 = 0, \tag{2.6a}
\]
where
\[
U = -V_x + \frac{V^2}{4V^2} - \frac{1}{V^2}.
\]

The following proposition comes from the compatibility condition \( \psi_{yyy} = \psi_{yyyy} \) of the Lax pair for the Novikov equation written in terms of the variables \( y \) and \( \tau \) [2]
\[
\psi_{yyy} + U \psi_y = \lambda^2 \psi, \quad \psi_\tau = \frac{1}{\lambda^2} (W \psi_{yy} - W_y \psi_y) - \frac{2}{3\lambda^2} \psi. \tag{2.7}
\]

Here, we provide an alternative proof based on (2.5) and (2.6).

**Proposition 2.1.** The variables \( U \) and \( W \) satisfies a linear PDE
\[
U_\tau + 3W_\tau = 0. \tag{2.8}
\]

**Proof.** First, we put \( p = 1/V, Z = W^2 \) and rewrite (2.5) and (2.6b) in terms of \( p \) and \( Z \) as
\[
p_\tau = (pZ)_y, \quad U = \frac{p_{yy}}{2p} - \frac{3 p_y}{4 p^2} - p^2.
\]

A direct computation using the above two equations, as well as equation (2.6a) to replace the term \( W_{yy} \) yields
\[
U_\tau = \frac{1}{2} Z_{yy} + 2 U Z_y + U_\tau Z, \quad Z_{yyy} = -4 U Z_y - 2 U_\tau Z - 6 W_\tau.
\]

Both expressions of (2.10) immediately lead to equation (2.8). \( \square \)

If we eliminate the variable \( W \) from (2.6a) and (2.8), we obtain a single equation for \( U_\tau \):
\[
UU_{yy} - U_\tau U_y + U^2 U_\tau + 3 U_\tau = 0. \tag{2.11}
\]

Note that the evolution equation for \( p \) in the independent variables \( \tau \) and \( y \) is the reciprocal transformation of the Novikov equation which can be derived by substituting \( U \) from (2.9) into equation (2.11). The resultant expression is, however, too formidable to write down explicitly.
2.2. SWW equation

We show that the system of equations (2.6a) and (2.8) for $U$ and $W$ admits the $N$-soliton solution by reducing it to the SWW equation. To this end, we first seek the $N$-soliton solution of equation (2.11) of the form

$$U = U_0 + 6(\ln f)_{yy}, \quad f = f(y, \tau).$$

(2.12)

The above dependent variable transformation enables us to recast (2.11) to the bilinear equation for $f$

$$\left( D_x D_y^2 - 3W_0 D_y^2 + U_0 D_x D_y \right) f \cdot f = 0.$$  

(2.13)

Here, the bilinear operators $D_x$ and $D_y$ are defined by

$$D_x^n D_y^m g = \left. \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial \tau'} \right)^m \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'} \right)^n f(y, \tau) g(y', \tau') \right|_{y'=y, \tau'=-\tau},$$

(2.14)

where $m$ and $n$ are non-negative integers. The constants $U_0$ and $W_0$ are boundary values of $U$ and $W$, respectively as $|y| \to \infty$. Specifically, since $u \to u_0$ and $m \to u_0$ in this limit, it follows from the definition of $V(=m^{2/3})$, $W(=um^{1/3})$ and $U$ from (2.6b) that $U_0 = -u_0^{2/3}$ and $W_0 = u_0^{4/3}$. The tau-function $f$ introduced in (2.12) will be shown to be the most important ingredient in constructing the $N$-soliton solution of the Novikov equation.

A remarkable feature of equation (2.13) is that it coincides with the bilinear form of the SWW equation introduced by Hirota and Satsuma [5]. Actually, by means of the dependent variable transformation $q = 2(\ln f)_y$, as well as the formulas

$$\frac{D_y^2 f}{f^2} = 2(\ln f)_{yy}, \quad \frac{D_x^2 f}{f^2} = 2(\ln f)_y,$$

(2.15a)

$$\frac{D_x D_y^2 f}{f^2} = 2(\ln f)_{y_{yy} + 12(\ln f)_{y_{yy}}},$$

(2.15b)

the bilinear equation (2.13) can be transformed, after rewriting it in terms of the variable $q$, to the SWW equation

$$q_y + 3\kappa^2 q_y - 3\kappa^2 qq_x + 3\kappa^2 q_x \int_q^\infty q_t \, dq - \kappa^2 q_{yy} = 0, \quad q = q(y, \tau),$$

(2.16)

where the positive parameter $\kappa$ has been introduced for later convenience by the relation $\kappa = u_0^{1/3}$ so that $U_0 = -\kappa^{-2}$ and $W_0 = \kappa^2$. Substituting (2.12) into equation (2.8) and integrating once with respect to $y$ under the boundary condition $W \to \kappa^2, |y| \to \infty$, we obtain the expression of $W$ in terms of the tau-function $f$

$$W = \kappa^2 - 2(\ln f)_y.$$  

(2.17)

Lastly, it follows from (2.2) and the definition of $W$ that the variable $x = x(y, \tau)$ obeys the linear PDE

$$x_t = W^2 x_y.$$  

(2.18)

If one can solve equation (2.18) for given $W$, then the expression of $u$ follows immediately from the second equation of (2.2). This provides a parametric representation for the $N$-soliton solution of the Novikov equation. Recall that the integrability of equation (2.18) (or equivalently, that of the system of equations (2.2)) is assured by (2.3). The method of solution for it is the core in the present analysis.
Remark 2.1. It is important that $U$ and $q$ are connected by
\[ U = -\kappa^{-2} + 3q, \]  
which follows from (2.12) and the transformation $q = 2(\ln f)_{yy}$. The relation (2.19) can be interpreted as a Bäcklund transformation between solutions of the equation for $p(t = 1/V)$ and those of the SWW equation via $U$ from (2.9). It can be used to derive an infinite number of conservation laws of the Novikov equation, as will be demonstrated in section 5.

2.3. Bilinear identities for the tau-functions

The tau-function $f$ for the $N$-soliton solution of the SWW equation is given compactly by [5]
\[ f = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i \xi_i + \sum_{1 \leq i < j \leq N} \mu_i \mu_j \gamma_{ij} \right], \]  
(2.20a)
with
\[ \xi_i = k_i \left( y - \frac{3 \kappa^4}{1 - (\kappa k_i)^2} \tau - y_0 \right), \quad (i = 1, 2, \ldots, N). \]  
(2.20b)
\[ e^{\phi_i} = \frac{(k_i - k_j)^2 [\left( k_i^2 + k_j^2 \right) \kappa^2 - 3]}{(k_i + k_j)^2 [\left( k_i^2 + k_j^2 \right) \kappa^2 - 3]}, \quad (i, j = 1, 2, \ldots, N; i \neq j). \]  
(2.20c)

Here, $k_i$ and $y_0$ are the amplitude and phase parameters of the $i$th soliton, respectively, and the notation $\sum_{\mu=0,1}$ implies the summation over all possible combination of $\mu_1 = 0, 1$, $\mu_2 = 0, 1, \ldots, \mu_N = 0, 1$. We shall prove in the appendix that the tau-function $f$ solves the bilinear equation (2.13).

To proceed, let us introduce some notations. The $N$-soliton solution from (2.20) is parametrized by the $N$ phase variables $\xi_i$ ($i = 1, 2, \ldots, N$) and hence we use a vector notation $f = f(\xi)$ with an $N$-component row vector $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$. Let $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$ be an $N$-component row vector with the elements
\[ e^{-\phi_i} = \sqrt{\frac{(1 - \kappa k_i)(1 - \kappa k_i)}{(1 + \kappa k_i)(1 + \kappa k_i)}}, \quad (i = 1, 2, \ldots, N). \]  
(2.21)
Define the tau-functions $f_1, f'_1, f_2$ and $f'_2$ by making use of the above notation
\[ f_1 = f(\xi - \phi), \quad f'_1 = f(\xi - 2\phi), \quad f_2 = f(\xi + \phi), \quad f'_2 = f(\xi + 2\phi). \]  
(2.22)

Then, the following proposition provides various identities among the tau-functions (2.22).

**Proposition 2.2.** The tau-functions $f, f'_1$ and $f'_2$ satisfy the bilinear identities
\[ D_x f'_1 \cdot f'_2 + \frac{2}{k} f'_1 f'_2 = \frac{2}{k^3} (k^2 f^2 - D_x D_y f \cdot f), \]  
(2.23)
\[ D_x f_1 \cdot f'_2 + 2k^3 f'_1 f'_2 = \frac{2}{k^3} (k^6 f^2 + D_x^2 f \cdot f), \]  
(2.24)
\[ D_x^3 f_1 \cdot f'_2 + \frac{6}{k} D_x^2 f'_1 \cdot f'_2 + \frac{11}{k^2} D_x f'_1 \cdot f'_2 + \frac{6}{k^3} (f'_1 f'_2 - f^2) = 0, \]  
(2.25)
\[ D_x f_1 \cdot f'_2 + \kappa D_x D_y f_1 \cdot f'_2 + \frac{k^2}{4} D_x D_y^2 f'_1 \cdot f'_2 + k^3 (f'_1 f'_2 - f^2) + \frac{k^4}{2} D_x f'_1 \cdot f'_2 + \frac{k^5}{2} D_x^2 f'_1 \cdot f'_2 \]
\[ = \frac{1}{2k} (D_y^2 D_x^2 f \cdot f + k^6 D_x^2 f \cdot f). \]  
(2.26)

The proof of the above proposition will be presented in the appendix. These bilinear identities as well as (2.13) play the central role in constructing the $N$-soliton solution of the Novikov equation.
3. The $N$-soliton solution

Let us introduce the tau-function $g = g(\xi)$

$$
g = \sum_{\mu, \nu = 0, 1} \exp \left[ \sum_{i=1}^{N} (\mu_i + \nu_i) \xi_i + \sum_{i=1}^{N} (2\mu_i \nu_i - \mu_i - \nu_i) \ln a_i \right. 
+ \left. \frac{1}{2} \sum_{i,j=1}^{N} (\mu_i \mu_j + \nu_i \nu_j) A_{2i-1,2j-1} + \frac{1}{2} \sum_{i,j=1}^{N} (\mu_i \nu_j + \mu_j \nu_i) A_{2i-1,2j} \right].
$$

(3.1a)

Here

$$
a_i = \sqrt{\frac{1 - \kappa^2 k_i^2}{4}}, \quad (i = 1, 2, \ldots, N),
$$

(3.1b)

$$
\exp [A_{2i-1,2j-1}] = \frac{(p_i - p_j)(q_i - q_j)}{(p_i + q_j)(q_i + p_j)}, \quad (i, j = 1, 2, \ldots, N; i \neq j),
$$

(3.1c)

$$
\exp [A_{2i-1,2j}] = \frac{(p_i - q_j)(q_i - p_j)}{(p_i + q_j)(q_i + q_j)}, \quad (i, j = 1, 2, \ldots, N; i \neq j),
$$

(3.1d)

$$
p_i = \frac{k_i}{2} \left[ 1 + \frac{2}{\kappa k_i} \sqrt{\frac{1}{3} \left( 1 - \frac{1}{4} \kappa^2 k_i^2 \right)} \right], \quad (i = 1, 2, \ldots, N),
$$

(3.1e)

$$
q_i = \frac{k_i}{2} \left[ 1 - \frac{2}{\kappa k_i} \sqrt{\frac{1}{3} \left( 1 - \frac{1}{4} \kappa^2 k_i^2 \right)} \right], \quad (i = 1, 2, \ldots, N),
$$

(3.1f)

and $\xi_i$ ($i = 1, 2, \ldots, N$) are already given by (2.20b).

The tau-functions $g_1$ and $g_2$ are defined by

$$
g_1 = g(\xi - \phi), \quad g_2 = g(\xi + \phi),
$$

(3.2)

where $\phi$ is the $N$-component row vector introduced by (2.21).

**Remark 3.1.** The tau-function $g$ has already appeared in constructing the $N$-soliton solution of the DP equation. See the expression (2.11) of [7], where the notation $f$ is used in place of $g$.

Now, the main result in our paper is given by the following theorem.

**Theorem 3.1.** The Novikov equation (1.1) admits the parametric representation for the $N$-soliton solution

$$
u^2 = \nu^2(y, \tau) = \kappa^3 + \frac{1}{2} \partial \nu \ln \frac{g_1}{g_2},
$$

(3.3a)

$$
x = x(y, \tau) = \frac{\nu}{\kappa} + \kappa^3 \tau + \frac{1}{2} \ln \frac{g_1}{g_2} + d,
$$

(3.3b)

where the tau-functions $g_1$ and $g_2$ are given by (3.1) and (3.2) and $d$ is an arbitrary constant.

The proof of theorem 3.1 will be carried out by a sequence of steps. We shall start our discussion with the proposition 3.1.
Proposition 3.1. The following relation holds among the tau-functions $g$, $f_1$ and $f_2$
\[ g = f_1 f_2 + \kappa D_\tau f_1 \cdot f_2, \]  
where $f_1$ and $f_2$ are defined by (2.22).

**Proof.** This relation stems from (3.13) of [7] if one replaces $f$, $g_1$ $g_2$ by $g$, $f_1$ and $f_2$, respectively. \qed 

If we use (3.4) and take into account the notation (2.22), we can express $g_1 = g(\xi - \phi)$ and $g_2 = g(\xi + \phi)$ in terms of $f$, $f_1$ and $f_2$. Explicitly,
\[ g_1 = f_1 f + \kappa D_\tau f_1 \cdot f, \]  
\[ g_2 = f_2 f - \kappa D_\tau f_2 \cdot f. \]

The proposition below connects the tau-functions $g_1$ and $g_2$ with the tau-function $f$.

Proposition 3.2. The tau-functions $f$, $g_1$ and $g_2$ satisfy the relations
\[ \left( D_\tau + \frac{2}{\kappa} \right) g_1 \cdot g_2 = \frac{2}{\kappa} f^4, \]  
\[ (D_\tau + 2\kappa^3)g_1 \cdot g_2 = \frac{2}{\kappa} (\kappa^2 f^2 - D_\tau D_\tau f \cdot f)^2. \]

**Proof.** First, we prove (3.6a). Substituting (3.5) into (3.6a), we obtain, after some straightforward calculations,
\[ \left( D_\tau + \frac{2}{\kappa} \right) g_1 \cdot g_2 - \frac{2}{\kappa} f^4 = \frac{\kappa^2}{4} \left[ D_\tau^2 f_1 \cdot f_2 + \frac{6}{\kappa} D_\tau^2 f_1 \cdot f_2 + \frac{11}{\kappa^2} D_\tau f_1 \cdot f_2 + \frac{6}{\kappa^3} (f_1 f_2 - f^3) \right] f^2 \]
\[ + \kappa^3 \left( -\frac{1}{4} f F_\tau + f_1 f_2 + f_3 f + \frac{1}{4\kappa^2} f F - \frac{1}{2\kappa^3} f^3 \right). \]

where we have put $F = D_\tau f_1 \cdot f_2 + (2/\kappa) f_1^2 f_2^2$ for simplicity. Since $F$ is equal to $(2/\kappa^3)(\kappa^2 f^2 - D_\tau D_\tau f \cdot f)$ by (2.23), the second term on the right-hand side of (3.7) divided by $\kappa^2 f$ simplifies to
\[ -\frac{1}{4} f F_\tau + f_1 f_2 + f_3 f + \frac{1}{4\kappa^2} f F - \frac{1}{2\kappa^3} f^3 = -\frac{1}{2\kappa^3} (W_\tau + U W + 1) f^3, \]

after using (2.12) and (2.17). This expression becomes zero by (2.6a). The first term of (3.7) also turns out to be zero by virtue of (2.25), completing the proof of (3.6a). 

To prove (3.6b), let $P = (D_\tau + 2\kappa^3) g_1 \cdot g_2 - (2/\kappa) (\kappa^2 f^2 - D_\tau D_\tau f \cdot f)^2$. This expression can be modified, after introducing (3.5) into it, to $P = P_1 + P_2$ with
\[ P_1 = \kappa^3 (-f_2^2 G + f_1 f_2 G_\tau + \frac{1}{4} F D_\tau D_\tau f \cdot f), \]
\[ P_2 = [D_\tau f_1 \cdot f_2 + \kappa D_\tau D_\tau f_1 \cdot f_2 + \kappa^2 (-f_1 f_2) f_3 f_2 + f_3 f_2] + \kappa^4 D_\tau f_1 \cdot f_2 - 2\kappa^5 f_1 f_2 f_3 f_2], \]

where $G = D_\tau f_1 \cdot f_2 + 2\kappa^3 f_1^2 f_2^2$. Using the right-hand side of (2.23) for $F$ and that of (2.24) for $G$, respectively, $P_1$ becomes
\[ P_1 = \frac{2}{\kappa} \left[ 2 f_3 f_1 f_2 - 2 f_3 f_2 + \frac{\kappa^4}{2} \left(f_2^2 - f_1 f_2 - \frac{\kappa}{2} D_\tau f_1 \cdot f_2 \right) \right] f^2. \]

Let $\tilde{P} = P/f^2$. Then,
\[ \tilde{P} = \frac{2}{\kappa} (2 f_3 f_1 f_2 - 2 f_3 f_2 + \kappa^6 f_2^2) + D_\tau f_1 \cdot f_2 + \kappa D_\tau D_\tau f_1 \cdot f_2 + \kappa^2 (-f_1 f_2) f_3 f_2 + \kappa^3 (f_2 f_1 f_2 - 2\kappa^5 f_1 f_2 f_3), \]

+ $\kappa^3 (f_2^2 - f_1 f_2) + \frac{\kappa^4}{2} D_\tau f_1 \cdot f_2 - 2\kappa^5 f_1 f_2 f_3.$
Differentiating (2.24) twice with respect to $y$, we deduce
\[ \kappa^2 (-f^1_{1,y} f^2_y + f^1_{1,y} f^2_y) = \frac{\kappa^2}{4} D_1 D_2 f^1_1 f^2_2 + \frac{\kappa^5}{2} (f^1_1 f^2_y + f^2_1 f^1_y), \]
which, substituted into the corresponding term in $\tilde{P}$, gives
\[ \tilde{P} = -\frac{1}{2\kappa} (D_1^2 f^2 f + \kappa^6 D^2_2 f f) + D_2 f^1_1 f^2_2 + \kappa D_1 D_2 f^1_1 f^2_2 + \frac{\kappa^2}{4} D_1 D_2 f^1_1 f^2_2 \]
\[ + \kappa^3 (f^2 - f^1_1 f^2_2) + \frac{\kappa^4}{2} D_2 f^1_1 f^2_2 + \frac{\kappa^5}{2} D_1 f^1_1 f^2_2. \]
This expression vanishes by virtue of (2.26), completing the proof of (3.6b).

We are now ready to prove theorem 3.1. Actually, it is a consequence of proposition 3.2.

**Proof of theorem 3.1.** We establish the theorem by showing that expression (3.3b) for $x$ satisfies equation (2.18). To this end, we rewrite the latter equation as $u^2 = W u$, by referring to the second equation of (2.2). If we substitute (2.17), (3.3a) and (3.3b) into this equation, then the equation to be proved becomes
\[ (D_x + 2\kappa^3) g_1 g_2 = \frac{1}{f^2} (\kappa^2 f^2 - D_x f f)^2 \left(D_x + \frac{2}{\kappa}\right) g_1 g_2. \]
In view of (3.6a) and (3.6b), the above equation holds identically. This completes the proof of theorem 3.1.

**Remark 3.2.** The determinant of the transformation matrix corresponding to the coordinate transformation (2.1a) is $m^{3/3}(= 1/\kappa_0)$. With use of (3.3b) and (3.6a), this can be evaluated as $\kappa g_1 g_2 / f^4$, which would turn out to be a positive quantity provided that the conditions $0 < \kappa k_i < 1$ ($i = 1, 2, \ldots, N$) are satisfied for the amplitude parameters of solitons. In this setting, the mapping (2.1a) becomes one-to-one. Then, the expression of $u$ from (3.3) gives rise to a single-valued function of $x$. Explicit examples will be presented in the next section for the one- and two-soliton solutions.

### 4. Properties of soliton solutions

#### 4.1. One-soliton solution

**4.1.1. Smooth soliton.** The tau-functions corresponding to the one-soliton solution are given from (3.1) and (3.2) (or from (2.20), (2.22) and (3.5)) with $N = 1$. They read
\[ g_1 = 1 + \frac{4(1 - \alpha)}{2 + \alpha} e^{\xi} + \frac{2}{2 + \alpha} \frac{1}{1 + \alpha} e^{2\xi}, \quad \text{for } \xi = k(y - \tilde{c} \tau - y_0), \]
\[ g_2 = 1 + \frac{4(1 + \alpha)}{2 - \alpha} e^{\xi} + \frac{2 + \alpha}{2 - \alpha} \frac{1}{1 - \alpha} e^{2\xi}, \quad \text{for } \tilde{c} = \frac{3\kappa^4}{1 - \alpha^2}, \]
where we have put $\xi = \xi_1$, $k = k_1$, $\alpha = \kappa k_1$ and $y_0 = y_{10}$ for simplicity. We assume $k > 0$ hereafter and the condition $0 < \alpha < 1$ is imposed to assure the smoothness of the solution.
Figure 1. The profile of smooth solitons with \( \kappa = 1 \), \( \alpha = 0.7 \) (dashed curve), \( \alpha = 0.85 \) (dotted curve), \( \alpha = 0.95 \) (solid curve).

The parametric representation of the smooth one-soliton solution follows from (3.3) and (4.1). It can be written in the form

\[
\begin{align*}
\rho^2 & = \kappa^3 + \frac{12k\alpha^2}{4 - \alpha^2} \frac{\cosh \xi + \frac{1}{2} \frac{2 + \alpha^2}{1 - \alpha^2}}{\cosh 2\xi + \frac{81(2 + \alpha^2)}{4 - \alpha^2} \cosh \xi + \frac{8(4 - \alpha^2)^2 + 3\alpha^4}{(1 - \alpha^2)(4 - \alpha^2)}} \\
& = \frac{2\kappa^3}{\cosh 2\xi + \frac{81(2 + \alpha^2)}{4 - \alpha^2} \cosh \xi + \frac{8(4 - \alpha^2)^2 + 3\alpha^4}{(1 - \alpha^2)(4 - \alpha^2)}} \left( \cosh \xi + \frac{1 + 2\rho^2}{1 - \rho^2} \right)^2, \\
X & \equiv x - ct - x_0 = \frac{\xi}{\alpha} + \frac{1}{2} \ln \left( \frac{\tanh^2 \frac{\xi}{2} - \frac{2}{\alpha} \tanh \frac{\xi}{2} + \frac{4 - \alpha^2}{8\alpha^2}}{\tanh^2 \frac{\xi}{2} + \frac{2}{\alpha} \tanh \frac{\xi}{2} + \frac{4 - \alpha^2}{8\alpha^2}} \right),
\end{align*}
\]

(4.2a)

(4.2b)

(4.2c)

where

\[ c = \frac{\kappa^3}{\kappa + \rho^2} = \frac{\kappa^3}{1 - \alpha^2}, \]

(4.2c)

is the velocity of the soliton in the \((x, t)\) coordinate system and \(x_0 = y_0/\kappa\). The constant \(d\) has been chosen such that \(\xi = 0\) corresponds to \(X = 0\). Notice that the form of \(\rho^2\) in the second line of (4.2a) can be anticipated from (3.3a) and (3.6b). Actually, the numerator of \(\rho^2\) is represented by a square of exponential functions.

Figure 1 depicts the profile of smooth solitons against the stationary coordinate \(X\) for three distinct values of \(\alpha\) with \(\kappa = 1\). The one-soliton solution represents a bright soliton on a constant background \(u = \kappa^{3/2}\) whose center position \(x_c\) is located at \(x_c = ct + x_0\). The amplitude of the soliton with respect to the background field, which we denote by \(A\), is found to be

\[ A = \kappa^{3/2} \left( \frac{2 + \alpha^2}{\sqrt{(1 - \alpha^2)(4 - \alpha^2)}} - 1 \right). \]

(4.3)

Eliminating the parameter \(\alpha\) from (4.2c) and (4.3), we obtain the amplitude–velocity relation

\[ c = \frac{1}{\kappa} \left[ (A + \kappa^{3/2})^2 + 4\kappa^3 + (A + \kappa^{3/2}) \sqrt{(A + \kappa^{3/2})^2 + 8\kappa^3} \right]. \]

(4.4)

We see from this expression that the velocity becomes a monotonically increasing function of the amplitude. Only in the small amplitude limit does (4.4) recover a linear relation between
c and A. Of particular interest is the limit \( \kappa \to 0 \) for which \( c \) is equal to \( A^2 \). This limiting value of the velocity coincides with the velocity of the peakon, as will be revealed in section 4.1.3.

To investigate the feature of the solution in more detail, we particularly focus on the profile of \( u \) near the crest. This can be accomplished if one expands \( u \) and \( X \) with respect to \( \xi \) as

\[
    u = e^{c\xi/2} \left[ \frac{2 + \alpha^2}{4 - \alpha^2} - \frac{9\alpha^2(1 - \alpha^2)}{4\alpha(4 - \alpha^2)^3} \xi^2 + O(\xi^4) \right],
\]

(4.5a)

\[
    X = \frac{4(1 - \alpha^2)}{\alpha(4 - \alpha^2)} \xi + \frac{2\alpha(2 - \alpha^2 - \alpha^4)}{(4 - \alpha^2)^3} \xi^3 + O(\xi^5).
\]

(4.5b)

By eliminating the variable \( \xi \) from (4.5a) and (4.5b), we can see that near the crest \( X \sim 0 \), \( u \) and its \( X \) derivative behave like

\[
    u = e^{c\xi/2} \left[ \frac{2 + \alpha^2}{4 - \alpha^2} - \frac{9\alpha^2}{16(1 - \alpha^2)(4 - \alpha^2)} \xi^2 + O(\xi^3) \right],
\]

(4.6a)

\[
    \frac{du}{dX} = \frac{9}{8} \frac{\alpha^4 e^{c\xi/2}}{(1 - \alpha^2)(4 - \alpha^2)} X + O(X^3).
\]

(4.6b)

The expression (4.6b) indicates that as \( \alpha \) increases, the crest of the smooth soliton becomes sharp. Note, however that the expansion breaks down in the vicinity of \( \alpha = 1 \) for which we need a separate treatment. The asymptotic behavior of \( u \) as \( \alpha \) tends to 1 will be explored in detail in section 4.1.3, showing that it forms a peak.

4.1.2. Singular soliton. The singular soliton is obtained from the smooth soliton (4.2) if one replaces the phase variable \( x_0 \) and \( y_0 \) by \( x_0 + \pi i/\alpha \) and \( y_0 + \pi i/\kappa \), respectively. In this setting, \( \cosh \xi \to -\cosh \xi \) and \( \tanh(\xi/2) \to \coth(\xi/2) \), giving rise to the parametric representation of \( u \)

\[
    u^2 = \kappa^3 + \frac{12\kappa\alpha\tilde{\kappa}}{4 - \alpha^2} \frac{\cosh 2\xi - \frac{1(2 + \alpha^2)}{2(1 - \alpha^2)} \cosh \xi + \frac{3(4 - \alpha^2 + 3\alpha^4)}{(1 - \alpha^2)(4 - \alpha^2)}}{\cosh 2\xi - \frac{8(2 + \alpha^2)}{4 - \alpha^2} \cosh \xi + \frac{3(4 - \alpha^2 + 3\alpha^4)}{(1 - \alpha^2)(4 - \alpha^2)}}.
\]

(4.7a)

\[
    X \equiv x - ct \to x_0 = \frac{\xi}{\alpha} + \frac{1}{2} \ln \left( \frac{\cosh^2 \frac{\xi}{2} - \frac{2}{3} \coth \frac{\xi}{2} + \frac{4 - \alpha^2}{3\alpha^2}}{\cosh^2 \frac{\xi}{2} + \frac{2}{3} \coth \frac{\xi}{2} + \frac{4 - \alpha^2}{3\alpha^2}} \right).
\]

(4.7b)

Figure 2 shows the typical profile of singular solitons for three distinct values of \( \alpha \) with \( \kappa = 1 \). We can observe that the singularities appear both at the crest \( X = 0 \) and at \( X = \pm X_0 \), where \( X_0 \) is a positive constant specified later.

We first analyze the structure of the soliton near the crest. Expanding \( u \) and \( X \) near \( \xi = 0 \), we obtain

\[
    u = e^{c\xi/2} \left[ 1 - \frac{1 - \alpha^2}{24\alpha^2} \xi^4 + O(\xi^6) \right],
\]

(4.8a)

\[
    X = \frac{(1 - \alpha^2)(4 - \alpha^2)}{180\kappa^2} \xi^5 + \frac{(1 - \alpha^2)(2 - \alpha^2)(4 - \alpha^2)}{1512\alpha^2} \xi^7 + O(\xi^9).
\]

(4.8b)

Elimination of the variable \( \xi \) from the above expressions yields the approximate expressions of \( u \) and \( du/dX \) near \( X = 0 \). They read

\[
    u = e^{c\xi/2} \left[ 1 - \frac{180^4/5}{24} \frac{(1 - \alpha^2)^{1/5}}{(4 - \alpha^2)^{4/5}} X^{4/5} + O(X^{6/5}) \right],
\]

(4.9b)
We expand (4.11) and (4.12) if one replaces $\alpha$ by $\frac{1}{\alpha}$.

It turns out from (4.11) that $u_0$ from (4.7a) has two zeros $\xi = \pm \xi_0$ with $\xi_0 = \cosh^{-1}[1 + 2\alpha^2/(1 - \alpha^2)]$. The corresponding value of $X_0$ is given from (4.7b) by

$$X_0 = \frac{1}{\alpha} \ln \left[ \frac{1 + 2\alpha^2 + \alpha \sqrt{3(2 + \alpha^2)}}{1 - \alpha^2} \right] + \frac{1}{2} \ln \left[ \frac{3 - \sqrt{3(2 + \alpha^2)}}{3 + \sqrt{3(2 + \alpha^2)}} \right].$$

We expand $u$ and $X$ near $\xi = \xi_0$ to obtain

$$u = e^{1/2} \left[ \frac{\sqrt{(1 - \alpha^2)(2 + \alpha^2)}}{3\alpha^2} |\xi - \xi_0| + \frac{1}{6\sqrt{3}} (4 + \alpha^2 - 2\alpha^4) \sqrt{1 - \alpha^2} (\xi - \xi_0)^2 + O((\xi - \xi_0)^3) \right],$$

$$X = X_0 + \frac{1 + 3\alpha - \alpha^2}{3\alpha^2} (\xi - \xi_0) + O((\xi - \xi_0)^2).$$

It turns out from (4.11) that

$$u = e^{1/2} \left[ \frac{\sqrt{(1 - \alpha^2)(2 + \alpha^2)}}{1 + 3\alpha - \alpha^2} |X - X_0| + O((X - X_0)^2) \right],$$

$$\frac{du}{dX} = e^{1/2} \left[ \frac{\sqrt{(1 - \alpha^2)(2 + \alpha^2)}}{1 + 3\alpha - \alpha^2} \text{sgn}(X - X_0) + O(X - X_0) \right].$$

where $\text{sgn} X$ denotes the sign function, i.e., $\text{sgn} X = 1$ for $X > 0$ and $\text{sgn} X = -1$ for $X < 0$. The corresponding expressions of $u$, $X$ and $du/dX$ near $\xi = -\xi_0$ or $X = -X_0$ follows from (4.11) and (4.12) if one replaces $\xi_0$ and $X_0$ by $-\xi_0$ and $-X_0$, respectively. The expression (4.12a) indicates that $u$ exhibits a peak at $X = \pm X_0$. Notice from (4.10) that $X_0$ is a monotonically increasing function of $\alpha$ and has a limiting value $1.303$ when $\alpha \to 0$. The profile of $u$ with $\alpha = 0.1$ drawn in figure 2 is closest to the limiting form as $\alpha \to 0$. 

Figure 2. The profile of singular solitons with $\kappa = 1$. $\alpha = 0.1$ (dashed curve), $\alpha = 0.85$ (dotted curve), $\alpha = 0.95$ (solid curve).
4.1.3. Peakon. It has been shown that the Novikov equation admits no smooth solutions which vanish at infinity. Under the same boundary condition, however, it exhibits a peaked wave (or peakon) solution of the form [2–4]

\[ u = \sqrt{c} e^{-|x - ct - x_0|}. \]  

(4.13)

More generally, the Novikov equation has the multipeakon solutions whose dynamics are governed by an integrable finite dimensional dynamical system for the positions and amplitudes of peakons [2, 3]. Here, we demonstrate that the single peakon solution can be reduced from the smooth one-soliton solution by taking an appropriate limit. The similar limiting procedure has been performed for the smooth multisoliton solutions of the CH equation [8, 9]. Here, we apply the method developed in [9] to obtain the peakon solution (4.13).

First, we observe that the tau-function \( g_2 \) from (4.1b) exhibits a singularity at \( \alpha = 1 \). This becomes an obstacle in the limiting process. Hence, we remove it by replacing the phase variable as \( \xi \to \xi - \phi \), where \( e^{i\phi} = \sqrt[4]{[(1 - \alpha/2)(1 - \alpha)/(1 + \alpha/2)(1 + \alpha)]} \) (see (2.21)). Consequently, the tau-functions \( g_1 \) and \( g_2 \) can be put into the form

\[ g_1 = 1 + 2 \left( \frac{1 - \alpha}{1 + \frac{\alpha}{2}} \right)^{3/2} \left( \frac{1 - \frac{\alpha}{2}}{1 + \alpha} \right)^{1/2} e^{\xi} + \left[ \frac{(1 - \frac{\alpha}{2})(1 - \alpha)}{(1 + \frac{\alpha}{2})(1 + \alpha)} \right] e^{2\xi}, \]  

(4.14a)

\[ g_2 = 1 + 2 \left( \frac{1 - \alpha^2}{1 - \frac{\alpha^2}{4}} \right)^{1/2} e^{\xi} + e^{2\xi}. \]  

(4.14b)

We now take the limit \( \kappa \to 0 \) with \( c \) being fixed, where \( c \) is the velocity of the smooth soliton in the \((x, t)\) coordinate system which is given by (4.2c). The constancy of \( c \) demands that one must takes the limit \( \alpha \to 1 \) simultaneously. By eliminating the variable \( y \) from (3.3b) and (4.1c) and substituting the result for \( \xi \) into \( e^{\xi} \), we obtain

\[ e^{\xi} = \left( \frac{g_2}{g_1} \right)^{u/2} e^{\alpha}, \quad z = e^{-ct - x_0}, \quad \xi = x - ct - \frac{1}{2} \ln \frac{g_1}{g_2} - x_0. \]  

(4.15)

Note from (4.14) that both \( g_1 \) and \( g_2 \) are positive and hence \( g_2/g_1 \) is a positive quantity. In the limit \( \kappa \to 0 \), \( \alpha \) from (4.2c) is expanded in powers of \( \kappa \) as \( \alpha = 1 - (3/2c)\kappa^3 + O(\kappa^6) \). Substituting this expansion and (4.15) into (4.14), the tau-functions \( g_1 \) and \( g_2 \) are approximated by

\[ g_1 \sim 1 + \frac{\kappa^{9/2}}{c^{3/2}} \sqrt{\frac{g_2}{g_1}} z + \frac{\kappa^6}{16c^2 g_1} \frac{g_2}{g_1} z^2, \]  

(4.16a)

\[ g_2 \sim 1 + \frac{\kappa^{3/2}}{c^{3/2}} \sqrt{\frac{g_2}{g_1}} z + \frac{g_2}{g_1} z^2. \]  

(4.16b)

If we put \( r = \sqrt{g_1/g_2} \) and introduce a small parameter \( \epsilon = (\kappa^3/c)^{1/2} \), then we deduce from (4.16) that

\[ r^2 \sim \frac{r^2 + \epsilon^3 r z + \epsilon^4 z^2}{r^2 + 4\epsilon z r + \epsilon^2}. \]  

(4.17)

It turns out that \( r \) satisfies the quartic equation

\[ r^2 + 4\epsilon z r^3 + (z^2 - 1) r^2 - \epsilon^3 z r - \frac{\epsilon^4}{16} z^2 + O(\epsilon^5) = 0. \]  

(4.18)

The expression of \( u^2 \) from (3.3a) can be expressed in terms of \( r \) as \( u^2 = \kappa^3 + (\ln r)_r \). We rewrite the \( \tau \)-derivative in accordance with the relation \( \partial/\partial \tau = \partial/\partial t + u^2 \partial/\partial x \) which follows
from (2.1b) and then solve the resultant equation with respect to \( u^2 \), giving rise to an important relation

\[
u^2 = \frac{r + \epsilon^3 r}{r - r_s} \tag{4.19}\]

Differentiating (4.18) by \( t \) and \( x \), respectively and using the relations \( \zeta = -\epsilon z, \zeta_i = z \), which stem simply from the second expression of (4.15) for \( z \), we obtain

\[
\begin{align*}
    r_s & \sim \frac{2\epsilon z^2 r^2 + 4\epsilon czr^3 - \epsilon^3 czr - \epsilon^4 cz^2}{4r^3 + 2(z^2 - 1)r + 12czr^2 - \epsilon^3 z}. \\
    r_n & \sim \frac{-2\epsilon z^2 r^2 - 4\epsilon czr^3 + \epsilon^3 czr + \epsilon^4 cz^2}{4r^3 + 2(z^2 - 1)r + 12czr^2 - \epsilon^3 z}. \tag{4.20a}
\end{align*}
\]

Next, if we introduce (4.20) into (4.19) and use (4.18) to eliminate a term \( r^2 \), we arrive at an approximate expression of \( u^2 \) in terms of \( r \):

\[
u^2 \sim \frac{cz(2zr^2 + 4\epsilon r^3 - \epsilon^3 r - \epsilon^4 z)}{2r^2 + 2\epsilon zr + \frac{1}{8}\epsilon^4 z^2}. \tag{4.21}
\]

The last step is to solve equation (4.18) for \( r \) and then substitute the solution into (4.21). The key feature of our method is that we need not solve the equation exactly and instead require only the approximate solution. To perform this procedure, we expand \( r \) in powers of \( \epsilon \) as \( r = \sum_{n=0}^{\infty} r_n \epsilon^n \) \((r_n \neq 0)\) and substitute this into (4.18). Comparing the coefficients of \( \epsilon^n \) \((n = 0, 1, \ldots)\), we obtain a system of algebraic equations for \( r_n \). Thus, for \( r_0 \neq 0 \), the first three equations arising from the system are found to be

\[
\begin{align*}
r_0^4 + (z^2 - 1)r_0^2 & = 0, \tag{4.22a} \\
\{4r_0^3 + 2(z^2 - 1)r_0\}r_1 + 4r_0^3 z & = 0, \tag{4.22b} \\
\{4r_0^3 + 2(z^2 - 1)r_0\}r_2 + (6r_0^3 + z^2 - 1)r_1^2 + 12zr_0^2 r_1 & = 0. \tag{4.22c}
\end{align*}
\]

For \( r_0 = 0 \), on the other hand, equations (4.22a) and (4.22b) are satisfied automatically and the first three nontrivial equations read

\[
\begin{align*}
(z^2 - 1)r_1^2 & = 0, \tag{4.23a} \\
2(z^2 - 1)r_1 r_2 & = 0, \tag{4.23b} \\
2(z^2 - 1) r_1 r_3 + (z^2 - 1) r_2^2 + r_1^4 + 4zr_1^3 - zr_1 - \frac{z^2}{16} & = 0. \tag{4.23c}
\end{align*}
\]

Thus, if \( 0 < z \leq 1 \) \((x - ct - x_0 \leq 0)\), then \( r_0 = \sqrt{1 - z^2} \) from (4.22a) which, substituted into (4.21), yields the limiting form of \( u \) when \( \epsilon \to 0 \)

\[
u \sim \sqrt{cz} = \sqrt{\epsilon} e^{-(x - ct - x_0)}. \tag{4.24a}
\]

If, on the other hand, \( z > 1 \) \((x - ct - x_0 > 0)\), then from (4.23), \( r_0 = r_1 = 0 \) and \( r_2 = (1/4)z/\sqrt{z^2 - 1} \). Since then \( r \sim O(\epsilon^2) \), we can see that both the numerator and denominator of (4.21) have a leading term of order \( \epsilon^4 \), which gives a limiting form of \( u \) as \( \epsilon \to 0 \)

\[
u \sim \sqrt{cz} \left[ \frac{r_2 - 1}{r_2 + \frac{z^2}{16}} \right] = \sqrt{cz}^{-1} = \sqrt{\epsilon} e^{-(x - ct - x_0)}. \tag{4.24b}
\]

If we combine (4.24a) with (4.24b), we see that the resulting expression of \( u \) coincides perfectly with the peckon solution (4.13).
The peakon limit of the smooth soliton with $c = 1$, $\kappa = 0.3$ (dashed curve), $\kappa = 0.2$ (dotted curve), $\kappa = 0.1$ (bold solid curve), $\kappa = 0.01$ (thin solid curve).

Figure 4. The peakon limit of the singular soliton with $c = 1$, $\kappa = 0.3$ (dashed curve), $\kappa = 0.2$ (dotted curve), $\kappa = 0.1$ (bold solid curve), $\kappa = 0.01$ (thin solid curve).

The passage to the peakon solution described above is illustrated in figure 3 for four distinct values of $\kappa$. We can observe that the profile drawn by the thin solid curve fits very well with the peakon solution (4.13) with $c = 1$. This provides a numerical evidence for the validity of the limiting procedure developed here.

Remark 4.1. The asymptotic method employed here for the smooth one-soliton solution can be applied as well to the general $N$-soliton solution. The resulting limiting form of the solution should coincide with the $N$-peakon solution which has been obtained by means of the IST [3]. The detailed analysis will be reported elsewhere.

Remark 4.2. The peakon limit of the singular soliton can be performed in the same way whereby we use the tau-functions $g_1$ and $g_2$ from (4.14) with $\xi$ replaced by $\xi + \pi i$. Note that this recipe is equivalent to changing the sign of the parameter $\epsilon$ in (4.17). The subsequent calculation parallels the smooth soliton case. As a result, the limiting form of the singular soliton as $\epsilon \to 0$ (or equivalently $\kappa \to 0$) is precisely given by (4.24), reproducing the peakon solution (4.13). Specifically, the cusp at the origin turns out to be a peak, whereas the two peak positions $X = \pm X_0$ move to infinity as evidenced from (4.10) by taking the limit $\alpha \to 1$. The passage to the peakon limit is illustrated in figure 4 for four distinct values of $\kappa$. 

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4.2. Two-soliton solution

The asymptotic form of the solution follows from (3.1) and (3.2) (or from (2.20), (2.22) and (3.5)) with \( N = 2 \). They read

\[
g_1 = 1 + 2b_1 e^{\xi_1} + 2b_2 e^{\xi_2} + (a_1 b_1)^2 e^{2\xi_1} + (a_2 b_2)^2 e^{2\xi_2} + 2\nu b_1 b_2 e^{\xi_1 + \xi_2} + 2\delta b_1 (a_2 b_2)^2 e^{\xi_1 + 2\xi_2} + \delta^2 (a_1 a_2 b_1 b_2)^2 e^{2\xi_1 + 2\xi_2},
\]

\[
g_2 = 1 + \frac{2}{a_1^2 b_1} e^{\xi_1} + \frac{2}{a_2^2 b_2} e^{\xi_2} + \frac{1}{(a_1 b_1)^2} e^{2\xi_1} + \frac{1}{(a_2 b_2)^2} e^{2\xi_2} + \frac{2\nu}{(a_1 a_2)^2 b_1 b_2} e^{\xi_1 + \xi_2} + \frac{2\delta}{(a_1 a_2 b_1 b_2)^2} e^{\xi_1 + 2\xi_2} + \frac{\delta^2}{(a_1 a_2)^2 (b_1 b_2)^2} e^{2\xi_1 + 2\xi_2},
\]

(4.25a)

where

\[
\xi_i = k_i (y - c_i t - x_0), \quad \bar{c}_i = \frac{3k_i^4}{1 - (k_i k_2)^2}, \quad (i = 1, 2),
\]

(4.25c)

\[
a_i = \sqrt{1 - \frac{4k_i^2}{(1 - (k_i k_2)^2)^2}}, \quad b_i = \frac{1 - k_i}{1 + k_i^2}, \quad (i = 1, 2),
\]

(4.25d)

\[
\delta = \frac{(k_1 - k_2)^2 [(k_1^2 - k_1 k_2 + k_2^2) k x^2 - 3]}{(k_1 + k_2)^2 [(k_1^2 + k_1 k_2 + k_2^2) k x^2 - 3]}, \quad \nu = \frac{(2k_2^4 - k_1^2 k_2^2 + 2k_2^4) k^2 - 6(k_1^2 + k_2^2)}{(k_1 + k_2)^2 [(k_1^2 + k_1 k_2 + k_2^2) k x^2 - 3]},
\]

(4.25e)

Note that the relation \( e^{-\phi} = a_1 b_1 \) holds by (2.21) and (4.25d).

Here, we restrict our consideration to the interaction process of two smooth solitons. The other types of interaction such as soliton–peakon interaction and smooth soliton–singular soliton interaction will not be considered.

Let \( c_1 \) \((i = 1, 2)\) be the velocity of the \( i \)th soliton in the \((x, t)\) coordinate system and assume that \( 0 < c_2 < c_1 \) which, by (4.2c), is equivalent to imposing the conditions \( 0 < \kappa k_2 < \kappa k_1 < 1 \).

First, we take the limit \( t \to -\infty \) with \( \xi_1 \) being fixed. In this limit, \( \xi_2 \to -\infty \). Then, the leading-order asymptotics of the tau-functions are given by

\[
g_1 \sim 1 + 2b_1 e^{\xi_1} + (a_1 b_1)^2 e^{2\xi_1},
\]

(4.26a)

\[
g_2 \sim 1 + \frac{2}{a_1^2 b_1} e^{\xi_1} + \frac{1}{(a_1 b_1)^2} e^{2\xi_1},
\]

(4.26b)

The asymptotic form of the solution follows from (3.3) and (4.26). It can be written as

\[
u = u_1(\xi_1),
\]

(4.27a)

\[
x - c_1 t - x_10 \sim \frac{\xi_1}{\alpha_1} + \frac{1}{2} \ln \left( \frac{\tan^2 \frac{\xi_1}{2} - \frac{2}{\alpha_1} \tanh \frac{\xi_1}{2} + \frac{4 - \alpha_1^2}{3\alpha_1^2}}{\tan^2 \frac{\xi_1}{2} + \frac{2}{\alpha_1} \tanh \frac{\xi_1}{2} + \frac{4 - \alpha_1^2}{3\alpha_1^2}} \right),
\]

(4.27b)

where \( u_1 = u(\xi) \) is the one-soliton solution given by (4.7a) and

\[
c_1 = \frac{\bar{c}_1}{\kappa} + \kappa^3 = \frac{\kappa^3 (4 - \alpha_1^2)}{1 - \alpha_1^2}, \quad \alpha_1 = \kappa k_1.
\]

(4.27c)

In the limit \( t \to +\infty \), on the other hand, \( \xi_2 \to +\infty \). The expressions corresponding to (4.26) and (4.27) read
solution of the DP equation. In the latter case, the parameter $u - \delta(\frac{\ln a}{\Delta_1})$

\[ g_1 \sim (a_2b_2)^2e^{2\xi_1}(1 + 2\delta b_1e^{\xi_1} + \delta^2(a_1b_1)^2e^{2\xi_1}), \]  

(4.28a)

\[ g_2 \sim \frac{1}{(a_2b_2)^2}e^{2\xi_1}\left(1 + \frac{2\delta}{a_1^2b_1}e^{\xi_1} + \frac{\delta^2}{(a_1b_1)^2}e^{2\xi_1}\right). \]  

(4.28b)

\[ u \sim u_1(\xi_1 + \delta_1^{(+)}). \]  

(4.29a)

\[ x - c_1t - x_{10} \sim \frac{\xi_1}{\alpha_1} + \frac{1}{2\alpha_1}\ln\left(\frac{\tanh^2\frac{1}{2}(\xi_1 + \delta_1^{(+)}) - \frac{2}{a_1}\tanh\frac{1}{2}(\xi_1 + \delta_1^{(+)}) + \frac{4 - \alpha_1^2}{\alpha_1^2}}{\tanh^2\frac{1}{2}(\xi_1 + \delta_1^{(+)}) + \frac{2}{a_1}\tanh\frac{1}{2}(\xi_1 + \delta_1^{(+)}) + \frac{4 - \alpha_1^2}{\alpha_1^2}}\right) + 2\ln(a_2b_2), \]  

(4.29b)

where $\delta_1^{(+)} = \ln \delta$.

Let $x_{ic}$ be the center position of the $i$th soliton. Then, as $t \to -\infty$, we find

\[ x_{ic} \sim c_1t + x_{10}, \quad (\xi_1 = 0). \]  

(4.30)

As $t \to +\infty$, on the other hand, $x_{ic}$ reads

\[ x_{ic} \sim c_1t + x_{10} - \frac{\ln \delta}{\alpha_1} + 2\ln(a_2b_2), \quad (\xi_1 = -\delta_1^{(+)}). \]  

(4.31)

The above analysis shows that the asymptotic state of the solution for large time is represented by a superposition of two single solitons in the rest frame of reference. The net effect of the interaction between solitons is the phase shift, which we shall now evaluate. To this end, we define the phase shift of the $i$th soliton by

\[ \Delta_i = x_{ic}(t \to +\infty) - x_{ic}(t \to -\infty), \quad (i = 1, 2). \]  

(4.32)

Then, we see from (4.25), (4.30) and (4.31) that the large soliton suffers a phase shift

\[ \Delta_1 = -\frac{1}{\kappa k_1}\ln\left(\frac{(k_1 - k_2)^2\left((k_1^2 - k_1k_2 + k_2^2)\kappa^2 - 3\right)}{(k_1 + k_2)^2\left((k_1^2 + k_1k_2 + k_2^2)\kappa^2 - 3\right)}\right) - \ln\left(\frac{1 + \frac{\kappa k_1}{2}}{1 - \frac{\kappa k_1}{2}}\right)\left(1 + \kappa k_2\right)\left(1 - \kappa k_2\right). \]  

(4.33)

By a similar asymptotic analysis, the phase shift of the small soliton is found to be

\[ \Delta_2 = \frac{1}{\kappa k_2}\ln\left(\frac{(k_1 - k_2)^2\left((k_1^2 - k_1k_2 + k_2^2)\kappa^2 - 3\right)}{(k_1 + k_2)^2\left((k_1^2 + k_1k_2 + k_2^2)\kappa^2 - 3\right)}\right) + \ln\left(\frac{1 + \frac{\kappa k_2}{2}}{1 - \frac{\kappa k_2}{2}}\right)\left(1 + \kappa k_1\right)\left(1 - \kappa k_1\right). \]  

(4.34)

It is interesting that the above formulas coincide formally with those of the two-soliton solution of the DP equation. In the latter case, the parameter $\kappa^3$ is the coefficient of the linear dispersive term $u$. See formula (4.37) of [6]. We can see that there exists a critical curve along which $\Delta_1 = \Delta_2$ and beyond which $\Delta_1 < \Delta_2$, implying that the phase shift of the small soliton is greater than that of the large soliton. Such a phenomenon has never been observed in the interaction process of solitons for the Korteweg–de Vries and SWW equations.

Last, we address the peakon limit of the formula for the phase shift. To this end, we take the limit $\kappa \to 0$ with $c_1$ and $c_2$ being fixed. Substituting the expansion of $\alpha_i(= \kappa k_i)$ for small $\kappa$, $\alpha_i = 1 - (3/2c_i)\kappa^3 + O(\kappa^5), \quad (i = 1, 2)$ into (4.33) and (4.34), we obtain the following limiting form of the phase shift:

\[ \Delta_1 = \ln\left(\frac{c_1(c_1 + c_2)}{(c_1 - c_2)^2}\right), \quad \Delta_2 = \ln\left(\frac{(c_1 - c_2)^2}{c_2(c_1 + c_2)}\right). \]  

(4.35)

This result reproduces the formulas for the phase shift of the two-peakon solution of the Novikov equation [3]. We recall that they coincide formally with the corresponding formulas for the two-peakon solution of the DP equation [6].
The profile of the smooth two-soliton solution and its limiting form for small $\kappa$ are depicted in figure 5 for four distinct values of $t$. This figure obviously shows the solitonic behavior of the solution. We can observe that the amplitudes of the large and small peakons are 1.41 and 1.0, respectively which are in accordance with the those evaluated from the one-peakon solution (4.13).

4.3. N-soliton solution

Here, we describe the asymptotic behavior of the general $N$-soliton solution. Since the analysis almost parallels that of the two-soliton case, we outline the result. To this end, we order the magnitude of the velocity of each soliton as $0 < c_N < c_{N-1} < \cdots < c_1$ by imposing the conditions $0 < \kappa k_N < \kappa k_{N-1} < \cdots < \kappa k_1 < 1$.

We first take the limit $t \to -\infty$ with the coordinate $\xi_i$ of the $i$th soliton being fixed. Then, $\xi_1, \xi_2, \ldots, \xi_{i-1} \to +\infty$, and $\xi_{i+1}, \xi_{i+2}, \ldots, \xi_N \to -\infty$. We employ (2.20), (2.22) and (3.5) to derive the following asymptotic forms of $g_1$ and $g_2$

\[ g_1 \sim \beta_1^2 \exp \left[ 2 \sum_{j=1}^{i-1} (\xi_j - \phi_j) \right] \left[ 1 + \frac{4(1 - \alpha_i)}{2 + \alpha_i} e^{\xi_j + \phi_j} + \frac{(1 - \frac{\alpha_i}{2}) (1 - \alpha_j)}{(1 + \frac{\alpha_i}{2}) (1 + \alpha_j)} e^{2(\xi_j + \phi_j)} \right] \],

(4.36a)

\[ g_2 \sim \beta_1^2 \exp \left[ 2 \sum_{j=1}^{i-1} (\xi_j + \phi_j) \right] \left[ 1 + \frac{4(1 - \alpha_i)}{2 + \alpha_i} e^{\xi_j + \phi_j} + \frac{(1 + \frac{\alpha_i}{2}) (1 + \alpha_j)}{(1 - \frac{\alpha_i}{2}) (1 - \alpha_j)} e^{2(\xi_j + \phi_j)} \right] ,

(4.36b)
where
\[
\delta_i^{-1} = \sum_{j=1}^{i-1} Y_{ij}, \quad \beta_i = \exp \left\{ \sum_{1 \leq j < k \leq i-1} Y_{jk} \right\}, \quad \alpha_i = \kappa k_i. \quad (i = 1, 2, \ldots, N).
\]

The asymptotic form of the solution is computed by using (3.3) and (4.36) to give
\[
u \sim u_i (\xi_i + \delta_i^{(-)}),
\]
\[
x - c_d t - x_0 \sim \frac{\xi_i}{\alpha_i} + \frac{1}{2} \ln \left( \frac{\tanh^2 \frac{1}{2} (\xi_i + \delta_i^{(+)} - \frac{2}{\alpha_i} \tanh \frac{1}{2} (\xi_i + \delta_i^{(-)}) + \frac{4 - \alpha_i^2}{\alpha_i^2})}{\tanh^2 \frac{1}{2} (\xi_i + \delta_i^{(+)} - \frac{2}{\alpha_i} \tanh \frac{1}{2} (\xi_i + \delta_i^{(-)}) + \frac{4 - \alpha_i^2}{\alpha_i^2})} \right) - 2 \sum_{j=1}^{i} \phi_j.
\]

In the limit \( t \to +\infty \), on the other hand, \( \xi_1, \xi_2, \ldots, \xi_{i-1} \to -\infty \), and \( \xi_{i+1}, \xi_{i+2}, \ldots, \xi_N \to +\infty \) and the expressions corresponding to (4.36) and (4.37) take the form
\[
g_1 \sim \beta_i^{(2)} \exp \left\{ \sum_{j=i+1}^{N} (\xi_j - \phi_j) \right\} \left[ 1 + \frac{4(1 - \alpha_i)}{2 + \alpha_i} e^{\xi_i + \delta_i^{(+)}} + \frac{(1 - \frac{\alpha_i}{2}) (1 - \alpha_i)}{(1 + \frac{\alpha_i}{2}) (1 + \alpha_i)} e^{2(\xi_i + \delta_i^{(+)})} \right],
\]
\[
g_2 \sim \beta_i^{(2)} \exp \left\{ \sum_{j=i+1}^{N} (\xi_j + \phi_j) \right\} \left[ 1 + \frac{4(1 + \alpha_i)}{2 - \alpha_i} e^{\xi_i + \delta_i^{(+)}} + \frac{(1 + \frac{\alpha_i}{2}) (1 + \alpha_i)}{(1 - \frac{\alpha_i}{2}) (1 - \alpha_i)} e^{2(\xi_i + \delta_i^{(+)})} \right],
\]
where
\[
\delta_i^{(+)} = \sum_{j=i+1}^{N} Y_{ij}, \quad \beta_i' = \exp \left\{ \sum_{i+1 \leq j < k \leq N} Y_{jk} \right\},
\]
and
\[
u \sim u_i (\xi_i + \delta_i^{(+)})
\]
\[
x - c_d t - x_0 \sim \frac{\xi_i}{\alpha_i} + \frac{1}{2} \ln \left( \frac{\tanh^2 \frac{1}{2} (\xi_i + \delta_i^{(+)}) - \frac{2}{\alpha_i} \tanh \frac{1}{2} (\xi_i + \delta_i^{(+)}) + \frac{4 - \alpha_i^2}{\alpha_i^2}}{\tanh^2 \frac{1}{2} (\xi_i + \delta_i^{(+)}) + \frac{2}{\alpha_i} \tanh \frac{1}{2} (\xi_i + \delta_i^{(+)}) + \frac{4 - \alpha_i^2}{\alpha_i^2}} \right) - 2 \sum_{j=1}^{N} \phi_j.
\]

The phase shift of the \( i \)th soliton is evaluated from (4.37) and (4.39). It reads
\[
\Delta_i = \frac{1}{\kappa k_i} \sum_{j=1}^{i-1} \ln \left[ \frac{(k_i - k_j)^2 \{(k_i^2 - k_j k_i + k_j^2)k_i^2 - 3\}}{(k_i + k_j)^2 \{(k_i^2 + k_j k_i + k_j^2)k_i^2 - 3\}} \right]
\]
\[
- \frac{1}{\kappa k_i} \sum_{j=i+1}^{N} \ln \left[ \frac{(k_i - k_j)^2 \{(k_i^2 - k_j k_i + k_j^2)k_i^2 - 3\}}{(k_i + k_j)^2 \{(k_i^2 + k_j k_i + k_j^2)k_i^2 - 3\}} \right]
\]
\[
+ \sum_{j=1}^{i-1} \ln \left[ \frac{(1 + \frac{k_i}{2}) (1 + \kappa k_j)}{(1 - \frac{k_i}{2}) (1 - \kappa k_j)} \right]
\]
\[
- \sum_{j=i+1}^{N} \ln \left[ \frac{(1 + \frac{k_i}{2}) (1 + \kappa k_j)}{(1 - \frac{k_i}{2}) (1 - \kappa k_j)} \right], \quad (i = 1, 2, \ldots, N).
\]
The above formulas for the phase shift clearly show that each soliton has a pairwise interaction with other solitons. For the special case of $N = 2$, they reduce to the corresponding formulas for the two-soliton solution (4.33) and (4.34). The first two terms on the right-hand side of (4.40) coincide with the phase shift of the $i$th soliton of the SWW equation whereas the last two additional terms stem from the coordinate transformation (2.1). A novel feature of the phase shift would appear due to the latter terms. Note that, remarkably, the formulas (4.40) are formally the same as those of the DP equation. See formulas (4.52) of [6] as well as formulas (4.11) of [7].

The peakon limit of (4.40) can be carried out straightforwardly to give the formulas

$$
\Delta_i = \sum_{j=1}^{i-1} \ln \left[ \frac{(c_i - c_j)^2}{c_i(c_i + c_j)} \right] - \sum_{j=i+1}^{N} \ln \left[ \frac{(c_i - c_j)^2}{c_i(c_i + c_j)} \right], \quad (i = 1, 2, \ldots, N),
$$

which reproduce the corresponding formulas for the $N$-peakon solution of the Novikov equation [3] and coincide formally with those of the DP equation [6].

5. Conservation laws

The existence of an infinite number of conservation laws is a common feature of integrable nonlinear PDEs. The conservation laws of the Novikov equation have been derived by means of the IST [2]. Actually, this has been established by making use of the Lax pair (2.7) rewritten in terms of the variables $x$ and $t$. Here, we present an alternative method on the basis of the relation (2.19) which connects solutions of the Novikov equation and those of the SWW equation (2.16). To this end, we substitute $U$ from (2.9) into (2.19) to obtain

$$
q = \frac{1}{3} \left( p_{yx} - \frac{3}{4} \frac{p_x^2}{p^2} - p^2 + \frac{1}{\kappa^2} \right),
$$

(5.1a)

Upon substituting the definition of $p$, i.e., $p = m^{2/3}$ and rewriting the result by the variables $x$ and $t$ in accordance with (2.1b), relation (5.1a) can be put into the form

$$
q = -\frac{1}{3} \left( m^{-4/3} \left( 1 - \frac{4}{9} m^{-2} m_x^2 + \frac{1}{3} m^{-1} m_{xx} \right) - \frac{1}{\kappa^2} \right).
$$

(5.1b)

The above relation enables us to construct conservation laws of the Novikov equation from those of the SWW equation, as we shall now demonstrate. The latter equation has local conservation laws of the form

$$
w_{n,t} = j_{n,y}, \quad (n = 0, 1, 2, \ldots),
$$

(5.2)

where the conserved density $w_n$ and associated flux $j_n$ are polynomials of $q$ and its $y$-derivatives. We rewrite (5.2) in terms of the variables $x$ and $t$ whereby we employ equation (1.1) modified in the form $(m^{2/3})_t + (m^{2/3} u^2)_x = 0$. The resulting expression reads

$$
(m^{2/3} w_{n,t})_t = (j_n - m^{2/3} u^2 w_n)_x.
$$

(5.3)

It turns out that the quantities

$$
I_n = \int_{-\infty}^{\infty} (m^{2/3} w_n - \kappa w_{n0}) \, dx, \quad (n = 0, 1, 2, \ldots),
$$

(5.4)

are conserved, where $w_{n0}$ is the boundary value of $w_n$ as $|x| \to \infty$. The conserved densities $w_n$ for the SWW equation have been obtained by means of the Bäcklund transformation. According to [10], for instance, the first three of them can be written as

$$
w_0 = 1, \quad w_1 = q, \quad w_2 = q^3 - q_y^2 - \frac{1}{\kappa^2} q_x^2, \quad w_3 = q^4 - 3qq_x^2 + \frac{1}{3} q_{yy}^2 - \frac{2}{3\kappa^2} \left( q_x^2 - \frac{1}{2} q_y^2 \right).
$$

(5.5)
Note that the terms expressed by a perfect derivative \((W_n)\) have been dropped in (5.5) since they become zero after integrating with respect to \(x\). Actually, we can show by using (2.1b) that 
\[
\int_{-\infty}^{\infty} m^{2/3}(W_n)_x \, dx = \int_{-\infty}^{\infty} (W_n)_x \, dx = 0.
\]
The conservation laws constructed in this way have very lengthy expressions and hence we quote the first two of them here:
\[
I_0 = \int_{-\infty}^{\infty} \left( m^{2/3} - \kappa \right) \, dx, \quad I_1 = \int_{-\infty}^{\infty} \left[ m^{-2/3} \left( 1 - \frac{4}{9} m^{-2} m_x^2 + \frac{1}{3} m^{-1} m_{xx} \right) - \frac{1}{\kappa} \right] \, dx.
\]
(5.6)

Recall that \(I_0\) follows directly from (1.1) and \(I_1\) coincides with the corresponding law obtained by the IST [2].

6. Concluding remarks

In this paper, we have developed a systematic method for solving the Novikov equation. In particular, we have obtained the parametric representation of the \(N\)-soliton solution in terms of the tau-functions associated with those of the SWW equation. A detailed analysis of the structure of the solutions reveals various new features which have never been seen in existing soliton solutions. Although there are many more things to be explored for the Novikov equation, the present paper will shed some light on the subject.

In conclusion, it will be worthwhile to compare the results obtained here with those of another version of the modified CH equation, which is given by
\[
\frac{\partial m}{\partial t} + \left[ m \left( u^2 - u_x^2 \right) \right]_x = 0, \quad m = u - u_{xx}.
\]
(6.1)
The above equation has been derived by several researchers as an integrable generalization of the CH equation and attracted a lot of interest in recent years [11–18]. It exhibits smooth solitons [17, 18] on a constant background as well as singular peaked solitons with W-shaped profile and cusp solitons [14, 15]. Although both equations have cubic nonlinearities, their structure differs substantially as observed by comparing their Lax representations. Actually, the spatial part of the Lax pair for the Novikov equation is the third-order ODE [1, 2] whereas the corresponding one for the modified CH equation is the second-order ODE [14]. Generally speaking, the former problem is more difficult to analyze than the latter one. This would be a reason why the IST has not still succeeded in obtaining smooth multisoliton solutions of the Novikov equation. In this respect, however, we recall that the construction of peakon solutions has gone ahead as in the cases of the CH and DP equations [2, 3].

The parametric representation for the bright \(N\)-soliton solution of equation (6.1) has been obtained quite recently by means of a reciprocal transformation [18]. Specifically, it has been demonstrated that the transformed equation is closely related to the SWW equation introduced by Ablowitz et al [19]
\[
q_t + 2\kappa^3 q_y - 4\kappa^2 q q_x + 2\kappa^2 q_x \int_y^{\infty} q_t \, dy - \kappa^2 q_{yy} = 0, \quad q = q(y, \tau).
\]
(6.2)
It is well-known that equation (6.2) has a different mathematical structure from that of the SWW equation of Hirota and Satsuma [5, 20]. The \(N\)-soliton solution of equation (6.1) is represented in terms of the tau-functions associated with the \(N\)-soliton solution of the equation (6.2) [18]. This intriguing feature is similar to that between the CH and DP equations. Actually, the structure of the soliton solutions of the Novikov (modified CH) equation bears a close resemblance to that of the DP (CH) equation. This fact can be seen if one compares the formulas for the phase shift of solitons, for instance. See also a few papers related to a similar approach to that developed here [21–25].
As for the singular solutions, equation (6.1) exhibits both W-shaped solitons with three peaks and cusp solitons, whereas the Novikov equation admits the solitons with a similar W-shaped profile but with single cusp and double peaks, as shown in figure 2. Another distinct feature is that the peakon solutions of the Novikov equation can be reduced from the smooth soliton solutions as the background field tends to zero. However, a similar limiting procedure has not been undertaken for the bright soliton solutions of equation (6.1) [26]. This is indeed a very interesting issue to be clarified by further study.

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Appendix. Proof of the bilinear identities

In this appendix, we perform the proof of the various identities presented in proposition 2.2 as well as (2.13) for the N-soliton solution of the SWW equation. The proof will be done by mathematical induction, similar to that used for the proof of the bilinear identities associated with the N-soliton solution of the DP equation. See section 3 of [7]. We first prove (2.13) and then proceed to (2.23)–(2.26).

A.1. Proof of (2.13)

Proof. Substituting the tau-function $f$ from (2.20) into (2.13) and using the formula

$$D_i^n D_j^p \exp \left[ \sum_{i=1}^{N} \mu_i \xi_i \right] \cdot \exp \left[ \sum_{i=1}^{N} v_i \xi_i \right] = \left\{ - \sum_{i=1}^{N} (\mu_i - v_i) k_i c_i \right\}^m \left\{ \sum_{i=1}^{N} (\mu_i - v_i) k_i \right\}^n \times \exp \left[ \sum_{i=1}^{N} (\mu_i + v_i) \xi_i \right] \quad (m, n = 0, 1, 2, \ldots),$$

the identity to be proved becomes

$$\sum_{\mu, v=0, 1} \left[ - \left\{ \sum_{i=1}^{N} (\mu_i - v_i) k_i c_i \right\} \left\{ \sum_{i=1}^{N} (\mu_i - v_i) k_i \right\}^3 - 3 \kappa^2 \left\{ \sum_{i=1}^{N} (\mu_i - v_i) k_i \right\}^2 \right]$$

$$+ \frac{1}{\kappa^4} \left\{ \sum_{i=1}^{N} (\mu_i - v_i) k_i c_i \right\} \left\{ \sum_{i=1}^{N} (\mu_i - v_i) k_i \right\}$$

$$\times \exp \left[ \sum_{i=1}^{N} (\mu_i + v_i) \xi_i + \sum_{1 \leq i < j \leq N} (\mu_i + v_i) y_{ij} \right] = 0. \quad (A.1)$$

Let $P_{n,m}$ be the coefficient of the factor $\exp \left[ \sum_{i=1}^{n} \xi_i + \sum_{i=m+1}^{m} 2 \xi_i \right]$ $(1 \leq n < m \leq N)$ on the left-hand side of (A.1). Correspondingly, the summation with respect to $\mu$ and $v_i$ must be performed under the conditions

$$\mu_i + v_i = 1 \quad (i = 1, 2, \ldots, n), \quad \mu_i = v_i = 1 \quad (i = n + 1, n + 2, \ldots, m),$$

$$\mu_i = v_i = 0 \quad (i = m + 1, m + 2, \ldots, N). \quad (A.2)$$

To simplify the notation, we introduce the new summation indices $\sigma_i$ by the relations $\mu_i = (1 + \sigma_i)/2$, $v_i = (1 - \sigma_i)/2$ for $i = 1, 2, \ldots, n$, where $\sigma_i$ either takes the value +1 or −1. It turns out that $\mu_i \mu_j + v_i v_j = (1 + \sigma_i \sigma_j)/2$. 

\[\text{J. Phys. A: Math. Theor. 46 (2013) 365203} \quad \text{Y Matsumo}\]
Now, under the conditions (A.2), we deduce that
\[
\sum_{1 \leq i < j \leq N} (\mu_i \mu_j + v_i v_j) \gamma_{ij} = \frac{1}{2} \sum_{1 \leq i < j \leq n} (1 + \sigma_i \sigma_j) \gamma_{ij} + \sum_{m j=n+1}^{m} \sum_{(j \neq i)} \gamma_{ij}.
\]  
(A.3)

Using (A.3), \( P_{m,n} \) can be written in the form
\[
P_{m,n} = \sum_{\sigma = \pm 1} \left[ -\left( \sum_{i=1}^{n} \sigma_i k_i \right)^2 \left( \sum_{i=1}^{n} \sigma_i k_i \right) - 3 \kappa^2 \left( \sum_{i=1}^{n} \sigma_i k_i \right)^2 + \frac{1}{\kappa^2} \left( \sum_{i=1}^{n} \sigma_i k_i \right) \left( \sum_{i=1}^{n} \sigma_i k_i \right) \right]
\times \exp \left[ \frac{1}{2} \sum_{1 \leq i < j \leq n} (1 + \sigma_i \sigma_j) \gamma_{ij} + \sum_{m j=n+1}^{m} \sum_{(j \neq i)} \gamma_{ij} \right].
\]  
(A.4)

The following relation follows from (2.20c) and the definition of \( \sigma_i \):
\[
\exp \left[ \frac{1}{2} \sum_{1 \leq i < j \leq n} (1 + \sigma_i \sigma_j) \gamma_{ij} \right] = \frac{(\sigma_i k_i - \sigma_j k_j)^2 [(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) \kappa^2 - 3]}{(k_i + k_j)^2 [(k_i^2 + k_i k_j + k_j^2) \kappa^2 - 3]}.
\]  
(A.5)

Introducing (A.5) into (A.4), the identity to be proved reduces, after multiplying \( P_{m,n} \) by a factor \( \prod_{i=1}^{n} (1 - \kappa^2 k_i^2) \) and using the definition of \( \tilde{c}_i \) from (4.25c), to
\[
P_n(k_1, k_2, \ldots, k_n) = \sum_{\sigma = \pm 1} \left[ -3 \kappa^2 \left( \sum_{i=1}^{n} \sigma_i k_i \right)^2 \left( \sum_{i=1}^{n} \sigma_i k_i \right) - \sum_{j=1}^{n} \left( 1 - \kappa^2 k_j^2 \right) \right] \left( \sum_{i=1}^{n} \sigma_i k_i \right)^3
- \sum_{i=1}^{n} \left( \sigma_i k_i - \sigma_j k_j \right)^2 [(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) \kappa^2 - 3] = 0, \quad (n = 1, 2, \ldots, N),
\]  
(A.6)

where a multiplicative factor independent of the indices \( \sigma \) has been dropped.

Before proving (A.6), we first establish the following identity:
\[
\tilde{P}_n(k_1, k_2, \ldots, k_n) = \sum_{\sigma = \pm 1} \left( \kappa \sum_{i=1}^{n} \sigma_i k_i \right) \left( 1 + \kappa \sum_{i=1}^{n} \sigma_i k_i \right) \left( \frac{1}{2} \sum_{i=1}^{n} \sigma_i k_i \right)
\times \prod_{i=1}^{n} \left( 1 - \kappa \sigma_i k_i \right) \left( 1 - \kappa \sigma_i k_i \right) \prod_{1 \leq i < j \leq n} \left( \sigma_i k_i - \sigma_j k_j \right) [(k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) \kappa^2 - 3]
= 0, \quad (n = 1, 2, \ldots, N).
\]  
(A.7)

The proof proceeds by mathematical induction. A direct calculation shows that \( \tilde{P}_1 = \tilde{P}_2 = 0 \).

Assume that \( \tilde{P}_n = \tilde{P}_{n-1} = 0 \). Then,
\[
\tilde{P}_n|_{k_1=0} = \prod_{i=2}^{n} k_i^2 (k_i^2 - 3) \tilde{P}_{n-1}(k_2, k_3, \ldots, k_n) = 0.
\]  
(A.8)

\[
\tilde{P}_n|_{k_1=1/\kappa} = 0.
\]  
(A.9)

\[
\tilde{P}_n|_{k_1=k_2} = -12 \kappa^2 (1 - \kappa^2 k_1^2)^2 \left( 1 - \frac{1}{4} \kappa^2 k_1^2 \right)
\times \prod_{i=3}^{n} (k_i^2 - k_j^2) [(k_1^2 + k_i k_j^2 + k_j^2) \kappa^4 - 6(k_1^2 + k_j^2) \kappa^2 + 9] \tilde{P}_{n-2}(k_3, k_4, \ldots, k_n) = 0.
\]  
(A.10)
Note that (A.9) is proved without use of the assumption of induction. The polynomial $\bar{P}_n$ is symmetric and even function of $k_i$ ($i = 1, 2, \ldots, n$). With this fact and (A.8)–(A.10) in mind, we see that $\bar{P}_n$ can be factored by a polynomial

$$\prod_{i=1}^{n} k_i^2 \left( k_i^2 - \frac{1}{\kappa^2} \right) \prod_{1 \leq i < j \leq n} (k_i^2 - k_j^2)^2,$$

of $k_i$ ($i = 1, 2, \ldots, n$) of degree $2n^2 + 2n$. On the other hand, the degree of $\bar{P}_n$ from (A.7) is $2n^2 + 3$ at most, which is impossible except for $\bar{P}_n \equiv 0$. This completes the proof of (A.7).

We now prove (A.6) following the same procedure as that used for (A.7). It is easy to check that $P_1 = P_2 = 0$. Assume that $P_{n-2} \equiv 0$. Then,

$$P_n|_{k_1 = 0} = \prod_{i=2}^{n} k_i^2 (\kappa^2 k_i^2 - 3) P_{n-1} (k_2, k_3, \ldots, k_n) = 0, \quad (A.11)$$

$$P_n|_{k_1 = \kappa} = -12 \prod_{i=2}^{n} \left( -\frac{2}{\kappa^2} \right) (1 - \kappa^2 k_i^2)^2 \bar{P}_{n-1} (k_2, k_3, \ldots, k_n) = 0, \quad (A.12)$$

$$P_n|_{k_1 = k_2} = 4k_1^2 (1 - \kappa^2 k_1^2)^2 \prod_{i=3}^{n} (k_i^2 - k_1^2)^2 \left[ (k_1^4 + k_1^2 k_i^2 + k_i^4) \kappa^4 - 6(k_1^2 + k_i^2) \kappa^4 + 9 \right] \times P_{n-2} (k_3, k_4, \ldots, k_n) = 0, \quad (A.13)$$

where we have used (A.7). The relations (A.11)–(A.13) together with the symmetry and evenness of $P_n$ in $k_i$ ($i = 1, 2, \ldots, n$) imply that $P_n$ has a factor

$$\prod_{i=1}^{n} k_i^2 \left( k_i^2 - \frac{1}{\kappa^2} \right) \prod_{1 \leq i < j \leq n} (k_i^2 - k_j^2)^2,$$

whose degree in $k_i$ ($i = 1, 2, \ldots, n$) is $2n^2 + 2n$ whereas the degree of $P_n$ from (A.6) is $2n^2 + 2$ at most. This is impossible except for $P_n \equiv 0$, completing the proof of (A.6) and hence (2.13).

A.2. Proof of (2.23)

Proof. We can perform the proof by following the procedure used in the proof of (2.13) and hence we describe the outline. In the present case, the identity to be proved can be written in the form

$$Q_n (k_1, k_2, \ldots, k_n) = \sum_{\sigma = \pm 1} \left[ \left( \sum_{i=1}^{n} \sigma k_i + \frac{2}{\kappa} \right) \prod_{j=1}^{n} \left( 1 + \frac{\kappa \sigma k_j}{1 - \kappa k_j} \right)^2 \right] \times \prod_{1 \leq i < j \leq n} \left( \sigma k_i - \sigma k_j \right)^{2} \left( k_i^2 - k_j^2 \right)^{2} \kappa^4 - 3 = 0, \quad (A.14)$$

($n = 1, 2, \ldots, N$).
Let \( Q'_n \) be the first term of \( Q_n \) multiplied by a factor \( \prod_{i=1}^{n} (1 - \frac{1}{2} \kappa^2 k_i^2) \), i.e.

\[
Q'_n = \sum_{\sigma=\pm 1} \left( \sum_{i=1}^{n} \sigma_i k_i + \frac{2}{\kappa} \right) \prod_{i=1}^{n} \left( 1 - \frac{\kappa \sigma_i k_i}{2} \right)^2 (1 - \kappa \sigma_i k_i)^2 \times \prod_{1 \leq i < j \leq n} (\sigma_i k_i - \sigma_j k_j)^2 \left[ (k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) \kappa^2 - 3 \right].
\]

We can show that \( Q'_n \bigg|_{k_i = 2/\kappa} = -Q'_n \bigg|_{k_i = 2/\kappa} \) and hence \( Q'_n \big|_{k_i = 2/\kappa} = 0 \). The polynomial \( Q'_n \) is symmetric and even function of \( k_i \) \((i = 1, 2, \ldots, n)\). Thus, the polynomial \( Q'_n \) has a factor \( \prod_{i=1}^{n} (1 - \frac{1}{4} \kappa^2 k_i^2) \), implying the first term of \( Q_n \) is indeed a polynomial. Taking into account this fact, we now start the proof of (A.14). A direct calculation shows that \( Q_1 = Q_2 = 0 \). Assume that \( Q_{n-2} = Q_{n-1} = 0 \). Then,

\[
Q_n|_{k_i = 0} = \prod_{i=2}^{n} k_i^2 (k^2 - 3) Q_{n-1}(k_2, k_3, \ldots, k_n) = 0, \tag{A.15}
\]

\[
Q_n|_{k_i = k} = -12k_i^2 (1 - \kappa^2 k_i^2)^3 \prod_{i=3}^{n} (k_i^2 - k_i^2)^2 \left[ (k_i^2 + k_i^2 k_i^2 + k_i^2) \kappa^4 - 6(k_i^2 + k_i^2) \kappa^2 + 9 \right] \times Q_{n-2}(k_3, k_4, \ldots, k_n) = 0. \tag{A.16}
\]

The symmetry and evenness of \( Q_n \) in \( k_i \) \((i = 1, 2, \ldots, n)\) as well as the relations (A.15)–(A.17) show that \( Q_n \) has a factor

\[
\prod_{i=1}^{n} k_i^2 \left( k_i^2 - \frac{1}{\kappa^2} \right) \prod_{1 \leq i < j \leq n} (k_i^2 - k_j^2)^2,
\]

whose degree in \( k_i \) \((i = 1, 2, \ldots, n)\) is \( 2n^2 + 2n \). On the other hand, the degree of \( Q_n \) from (A.14) is \( 2n^2 + 1 \) at most. This is impossible except for \( Q_n \equiv 0 \), completing the proof of (A.14) and hence (2.23).

\[\square\]

### A.3. Proof of (2.24)

**Proof.** The identity to be proved reads

\[
R_0(k_1, k_2, \ldots, k_n) \equiv \sum_{\sigma=\pm 1} \left\{ -3 \kappa^4 \sum_{i=1}^{n} \sigma_i k_i \prod_{j=1}^{n} \left( 1 - \kappa^2 k_i^2 \right) + 2 \kappa^4 \prod_{j=1}^{n} \left( 1 - \kappa^2 k_i^2 \right) \right\} \times \prod_{j=1}^{n} \left( \frac{(1 - \kappa \sigma_i k_i) (1 - \kappa \sigma_j k_j)}{1 + \kappa \sigma_i k_i} \right) \times \prod_{1 \leq i < j \leq n} \left( \sigma_i k_i - \sigma_j k_j \right)^2 \left[ (k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) \kappa^2 - 3 \right] = 0, \quad (n = 1, 2, \ldots, N). \tag{A.18}
\]
The \( R_n \) is a polynomial in \( k_i (i = 1, 2, \ldots, n) \), as shown by an argument similar to that given for \( Q_n' \). A direct calculation shows that \( R_1 = R_2 = 0 \). Assume that \( R_{n-2} = R_{n-1} = 0 \). Then,

\[
R_n|_{k_1 = 0} = \sum_{i=2}^{n} k_i^2 (k^2 k_i^2 - 3) S_{n-1} (k_2, k_3, \ldots, k_n) = 0,
\]

(A.19)

\[
R_n|_{k_1 = 1/\kappa} = 0,
\]

(A.20)

\[
\frac{\partial R_n}{\partial k_1}|_{k_1 = 1/\kappa} = 0,
\]

(A.21)

\[
R_n|_{k_i = 0} = -12 k_i^2 (1 - \kappa^2 k_i^2)^4 \prod_{i=3}^{n} (k_i^2 - k_j^2)^2 \left[ (k_i^2 + k_j^2 + k_i^2)\kappa^4 - 6(k_i^2 + k_j^2)\kappa^2 + 9 \right]
\]

\[
\times R_{n-2} (k_3, k_4, \ldots, k_n) = 0.
\]

(A.22)

Note that (A.20) and (A.21) follow by direct computation without use of the assumption of induction. The symmetry and evenness of \( R_n \) in \( k_i (i = 1, 2, \ldots, n) \) as well as the relations (A.19)–(A.22) show that \( Q_n \) has a factor

\[
\prod_{i=1}^{n} k_i^2 \left( k^2 - \frac{1}{\kappa^2} \right)^2 \prod_{1 \leq i < j \leq n} (k_i^2 - k_j^2)^2,
\]

whose degree in \( k_i (i = 1, 2, \ldots, n) \) is \( 2n^2 + 4n \). On the other hand, the degree of \( R_n \) from (A.18) is \( 2n^2 + 2n \) at most. This is impossible except for \( R_n \equiv 0 \), completing the proof of (A.18) and hence (2.24).

\[\square\]

A.4. Proof of (2.25)

Proof. The identity to be proved reads

\[
S_n (k_1, k_2, \ldots, k_n) = \sum_{\sigma \in S} \left[ \left( \sum_{i=1}^{n} \sigma_i k_i \right)^3 + \frac{6}{\kappa^2} \left( \sum_{i=1}^{n} \sigma_i k_i \right)^2 + \frac{11}{\kappa^4} \sum_{i=1}^{n} \sum_{j \neq i} \sigma_i k_i + \frac{6}{\kappa^3} \sum_{i=1}^{n} \sum_{j \neq i} \sigma_i k_i \right] \prod_{i=1}^{n} \left( 1 - \frac{\sigma_i k_i}{2} \right) \prod_{i=1}^{n} \left( 1 + \frac{\sigma_i k_i}{2} \right)
\]

\[
\times \prod_{1 \leq i < j \leq n} \left( \sigma_i k_i - \sigma_j k_j \right)^2 \prod_{1 \leq i < j \leq n} \left( \sigma_i k_i - \sigma_j k_j \right)^2 \prod_{1 \leq i < j \leq n} \left[ (k_i^2 - \sigma_i k_i k_j + k_j^2)\kappa^2 - 3 \right] = 0, \quad (n = 1, 2, \ldots, N).
\]

(A.23)

The \( S_n \) can be shown to be a polynomial in \( k_i (i = 1, 2, \ldots, n) \). A direct calculation shows that \( S_1 = S_2 = 0 \). Assume that \( S_{n-2} = S_{n-1} = 0 \). Then,

\[
S_n|_{k_1 = 0} = \sum_{i=2}^{n} k_i^2 (k^2 k_i^2 - 3) S_{n-1} (k_2, k_3, \ldots, k_n) = 0,
\]

(A.24)

\[
S_n|_{k_1 = k_2} = -12 k_i^2 (1 - \kappa^2 k_i^2)^4 \prod_{i=3}^{n} (k_i^2 - k_j^2)^2 \left[ (k_i^2 + k_j^2 + k_i^2)\kappa^4 - 6(k_i^2 + k_j^2)\kappa^2 + 9 \right]
\]

\[
\times S_{n-2} (k_3, k_4, \ldots, k_n) = 0.
\]

(A.25)

The symmetry and evenness of \( S_n \) in \( k_i (i = 1, 2, \ldots, n) \) as well as the relations (A.24) and (A.25) show that \( S_n \) has a factor

\[
\prod_{i=1}^{n} k_i^2 \prod_{1 \leq i < j \leq n} (k_i^2 - k_j^2)^2.
\]
whose degree in \( k_i \) (\( i = 1, 2, \ldots, n \)) is \( 2n^2 \). On the other hand, the degree of \( S_n \) from (A.23) is \( 2n^2 - 2n + 3 \) at most. This is impossible except for \( S_n \equiv 0 \), completing the proof of (A.23) and hence (2.25).

\[\square\]

A.5. Proof of (2.26)

**Proof.** The identity to be proved reads

\[
\begin{align*}
T_n(k_1, k_2, \ldots, k_n) &\equiv \sum_{\sigma=\pm 1} \left[ \left( -3 \kappa^4 \sum_{i=1}^{n} \sigma_i k_i \prod_{j=1, j \neq i}^{n} (1 - \kappa^2 k_j^2) \right) \left( 1 + \frac{\kappa}{2} \sum_{i=1}^{n} \sigma_i k_i \right)^2 \\
&\quad + \frac{\kappa^3}{2} \left( \kappa \sum_{i=1}^{n} \sigma_i k_i - 1 \right) \left( \kappa \sum_{i=1}^{n} \sigma_i k_i + 2 \right) \prod_{j=1}^{n} (1 - \kappa^2 k_j^2) \right] \\
&\quad \times \prod_{i=1}^{n} \left( \frac{1 - \kappa \sigma_i k_i}{1 + \kappa \sigma_i k_i} \right) \\
&\quad - 9 \kappa^3 \left( \kappa \sum_{i=1}^{n} \sigma_i k_i \prod_{j=1, j \neq i}^{n} (1 - \kappa^2 k_j^2) \right) \left( \kappa \sum_{i=1}^{n} \sigma_i k_i \right)^2 \\
&\quad + \kappa^3 \left( -\frac{1}{2} \left( \kappa \sum_{i=1}^{n} \sigma_i k_i \right)^2 + 1 \right) \prod_{j=1}^{n} (1 - \kappa^2 k_j^2) \right] \\
&\quad \times \prod_{1 \leq i < j \leq n} (\sigma_i k_i - \sigma_j k_j)^2 \left[ (k_i^2 - \sigma_i \sigma_j k_i k_j + k_j^2) \kappa^2 - 3 \right] \\
&= 0, \quad (n = 1, 2, \ldots, N), \quad (A.26)
\end{align*}
\]

The \( T_n \) is a polynomial in \( k_i \) (\( i = 1, 2, \ldots, n \)). A direct calculation shows that \( T_1 = T_2 = 0 \). Assume that \( T_{n-2} = T_{n-1} = 0 \). Then,

\[
T_n|_{k_2=0} = \prod_{i=2}^{n} k_i^2 (k_i^2 - 3) T_{n-1}(k_2, k_3, \ldots, k_n) = 0, \quad (A.27)
\]

\[
T_n|_{k_2=1/\kappa} = 0, \quad (A.28)
\]

\[
\frac{\partial T_n}{\partial k_1}|_{k_1=1/\kappa} = 0, \quad (A.29)
\]

\[
T_n|_{k_1=k_2} = -12 k_1^2 (1 - \kappa^2 k_1^2) \prod_{i=3}^{n} (k_i^2 - k_i^2) \left[ (k_1^4 + k_2^2 k_i^2 + k_1^4) \kappa^4 - 6 (k_i^4 + k_1^2) \kappa^2 + 9 \right] \\
\quad \times T_{n-2}(k_3, k_4, \ldots, k_n) = 0. \quad (A.30)
\]

The symmetry and evenness of \( T_n \) in \( k_i \) (\( i = 1, 2, \ldots, n \)) as well as the relations (A.27)–(A.30) show that \( T_n \) has a factor

\[
\prod_{i=1}^{n} k_i^2 \left( \frac{k_i^2 - 1}{\kappa^2} \right)^2 \prod_{1 \leq i < j \leq n} (k_i^2 - k_j^2)^2,
\]

whose degree in \( k_i \) (\( i = 1, 2, \ldots, n \)) is \( 2n^2 + 4n \). On the other hand, the degree of \( T_n \) from (A.26) is \( 2n^2 + 2n + 2 \) at most. This is impossible except for \( T_n \equiv 0 \), completing the proof of (A.26) and hence (2.26).

\[\square\]
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