A short proof of a symmetry identity for the \( q \)-Hahn distribution

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Abstract

We give a short and elementary proof of a symmetry identity for the \( q \)-moments of the \( q \)-Hahn distribution arising in the study of the \( q \)-Hahn Boson process and the \( q \)-Hahn TASEP. This identity discovered by Corwin in "The \( q \)-Hahn Boson Process and \( q \)-Hahn TASEP", Int. Math. Res. Not., 2014, was a key technical step to prove an intertwining relation between the Markov transition matrices of these two classes of discrete-time Markov chains. This was used in turn to derive exact formulas for a large class of observables of both these processes.

Keywords: Markov duality; \( q \)-Hahn process.

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Introduction

Zero-range and exclusion processes are generic stochastic models for transport phenomena on a lattice. Integrability of these models is an important question. In a short letter [5], Evans, Majumdar and Zia considered spatially homogeneous discrete time zero-range processes on periodic domains. They addressed and solved the question of characterizing the jump distributions for which invariant measures are product measures. Povolotsky [7] further examined the most general form of jump distributions allowing solvability by Bethe ansatz, and found a family depending on three real parameters \( q \), \( \mu \) and \( \nu \), later called the \( q \)-Hahn distribution. In the same article [7], he also studied the corresponding \( q \)-Hahn Boson process and \( q \)-Hahn TASEP, and conjectured exact formulas for the models on the infinite lattice.

Using a Markov duality between the \( q \)-Hahn Boson process and the \( q \)-Hahn TASEP, Corwin [4] showed a variant of these formulas and provided a method to compute a large class of observables. This can be seen as a generalization of a similar work on \( q \)-TASEP and \( q \)-Boson process performed in [3, 2]. In his proof, the intertwining relation between the two Markov transition matrices essentially boils down to a symmetry identity verified by the \( q \)-moments of the \( q \)-Hahn distribution [4, Proposition 1.2]. The proof was adapted from [2, Lemma 3.7] which is the \( \nu = 0 \) case, and required the use of Heine’s summation formula for the basic hypergeometric series \( _2\phi_1 \). In the following, we give a new proof of this identity.

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First, we define the three parameter deformation of the Binomial distribution introduced in [7].

**Definition 0.1.** For $|q| < 1$, $0 \leq \nu \leq \mu < 1$ and integers $0 \leq j \leq m$, define the function

$$
\varphi_{q,\mu,\nu}(j|m) = \mu^j \binom{\nu/\mu; q, \mu; q}_{m-j} \binom{\mu; q}_{m-j}
$$

where

$$
\binom{m}{j}_q = \frac{(q; q)_m}{(q; q)_j (q; q)_{m-j}}
$$

are $q$-Binomial coefficients with, as usual,

$$(z; q)_m = \prod_{i=0}^{m-1} (1 - q^i z).$$

It happens that for each $m \in \mathbb{N} \cup \{\infty\}$, this defines a probability distribution on the set $\{0, \ldots, m\}$. The weights $\varphi_{q,\mu,\nu}(j|m)$ are very closely related to the weights associated with the $q$-Hahn orthogonal polynomials (see (7.2.22) in [6]), hence the use of the name $q$-Hahn.

**Lemma 0.2** (Lemma 1.1, [4]). For any $|q| < 1$ and $0 \leq \nu \leq \mu < 1$,

$$
\sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) = 1.
$$

**Proof.** As shown in [4], this equation is equivalent to a specialization of some known summation formula for basic hypergeometric series $\phi_1$ (Heine’s $q$-generalizations of Gauss’ summation formula).

We now state and prove the main identity.

**Proposition 0.3** (Proposition 1.2, [4]). Fix $|q| < 1$ and $0 \leq \nu \leq \mu < 1$. Let $X$ (resp. $Y$) be a random variable following the $q$-Hahn distribution on $\{0, \ldots, x\}$ (resp. $\{\nu q^j, \nu q^j + 1, \ldots, \nu q^x\}$).

We have

$$
E[q^{xY}] = E[q^{yX}].
$$

**Proof.** Let $S_{x,y} := \sum_{j=0}^{x} \varphi_{q,\mu,\nu}(j|x) q^{\nu y}$. We have to show that $S_{x,y} = S_{y,x}$ for all integers $x, y \geq 0$. Our proof is based on the fact that $S_{x,y}$ satisfies a recurrence relation which is invariant when exchanging the roles of $x$ and $y$. First notice that by Lemma 0.2, $S_{x,0} = 1$ for all $x \geq 0$, and by definition $S_{0,y} = 1$ for all $y \geq 0$.

The Pascal identity for $q$-Binomial coefficients, (see 10.0.3 in [1]),

$$
\binom{x+1}{j}_q = \binom{x}{j}_q q^j + \binom{x}{j-1}_q,
$$

yields

$$
S_{x+1,y} = \sum_{j=0}^{x+1} \mu^j \binom{\nu/\mu; q, \mu; q}_{x+1-j} \binom{\mu; q}_{x+1-j} \binom{x}{j}_q q^{\nu y} + \sum_{j=0}^{x+1} \mu^j \binom{\nu/\mu; q, \mu; q}_{x+1-j} \binom{\mu; q}_{x+1-j} \binom{x}{j}_q q^{y - 1/j},
$$

$$
S_{x+1,y} = \sum_{j=0}^{x} \varphi_{q,\mu,\nu}(j|x) \frac{1 - \mu q^{x-j}}{1 - \nu q^y} q^j q^{\nu y} + \sum_{j=0}^{x} \varphi_{q,\mu,\nu}(j|x) \mu \frac{1 - \nu q^j}{1 - \nu q^y} q^{\nu y} q^j q^{y}. \tag{16}
$$
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The last equation can be rewritten

\[
(1 - \nu q^x)S_{x+1, y} = (S_{x, y+1} - \mu q^x S_{x, y}) + (\mu q^y (S_{x, y} - \nu / \mu S_{x, y+1})),
\]

\[
= (1 - \nu q^y)S_{x, y+1} + \mu (q^y - q^x)S_{x, y}.
\]

Thus, the sequence \((S_{x, y})_{(x, y) \in \mathbb{N}^2}\) is completely determined by

\[
\begin{cases}
(1 - \nu q^x)S_{x+1, y} = (1 - \nu q^y)S_{x, y+1} + \mu (q^y - q^x)S_{x, y}, \\
S_{x, 0} = S_{0, y} = 1.
\end{cases}
\]

Setting \( T_{x, y} = S_{y, x} \), one notices that the sequence \((T_{x, y})_{(x, y) \in \mathbb{N}^2}\) enjoys the same recurrence, which concludes the proof.

Remark 0.4. To completely avoid the use of basic hypergeometric series, one would also need a similar proof of the Lemma above. One can prove the result by recurrence on \( m \) (as in the proof of [2, Lemma 1.3]), but the calculations are less elegant when \( \nu \neq 0 \).

More precisely, fix some \( m \) and suppose that for any \( 0 \leq \nu \leq \mu < 1 \), \( S_{m, 0}(q, \mu, \nu) := \sum_{j=0}^{m} \varphi_{q, \mu, \nu}(j|m) = 1 \). Pascal’s identity yields

\[
S_{m+1, 0}(q, \mu, \nu) = \frac{1 - \mu}{1 - \nu} S_{m, 0}(q, \mu, q^\nu) + \sum_{j=0}^{m} \varphi_{q, \mu, \nu}(j|m) \frac{1 - \nu / \mu q^j}{1 - \nu q^m},
\]

\[
= \frac{1 - \mu}{1 - \nu} S_{m, 0}(q, \mu, q^\nu) + \frac{\mu}{1 - \nu q^m} (S_{m, 0}(q, \mu, \nu) - \nu / \mu S_{m, 1}(q, \mu, \nu)).
\]

Then, using the recurrence formula (0.1) for \( S_{m, 1}(q, \mu, \nu) \), and applying the recurrence hypothesis, one obtains \( S_{m+1, 0}(q, \mu, \nu) = 1 \).

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