Study on the Location of Small Reservoir in Hilly Area by Newton Method

Xiaoli Chi¹, Enxian Chi², Chunhua Tang³,*

¹School of Electrical Engineering, University of Jinan, Shandong Jinan, 250022, China
²Shandong University, Shandong Jinan, 250100, China
³Shandong Vocational University of Foreign Affairs, Shandong Jinan, 250100, China

*Corresponding author e-mail: 576422761@qq.com

Abstract. The purpose of this paper is to solve the problem that small reservoirs in hilly areas are located according to the terrain characteristics. According to the terrain height difference characteristics in hilly areas, Newton method is proposed to solve this problem. Newton method has basic Newton method and damping Newton method, both of which search in the given search direction, find the extreme value of the function, and construct the search function through Hessian matrix and negative gradient of the function, Calculate the next iteration point. The following iteration point is the initial point. Continue to calculate the next iteration point until the condition of extreme point is met. The iteration point is the extreme point. This method has fast iteration speed and accurate location. The accuracy of Newton's method is verified by simulation data.

Keywords: Reservoirs, Terrain characteristics, Newton method

1. Introduction

The terrain of China is complex, and the land area of hills accounts for about 10% of the land area of China. The elevation of hills is not high, and the slope surface does not fluctuate much. It is composed of continuous hills. The light, water and heat conditions are suitable, which is conducive to the development of agricultural products such as fruit tree cultivation. Due to the influence of terrain height difference, water resources are seriously lacking. In order to alleviate the shortage of agricultural irrigation water, large-scale water storage projects for rainwater collection irrigation have been carried out all over the country. Because there are many independent hills and the utilization rate of water resources is not high, the problem of agricultural irrigation has not been fundamentally solved. Therefore, according to the characteristics of hilly terrain, the development of small-scale rainwater harvesting and water storage project meets the needs of agricultural irrigation, and requires less funds. Because of the rugged terrain in hilly area, using the advantage of height difference can save time, labor and cost of agricultural irrigation, so it is very important to choose the right place to
build a small water storage pool.

Because the terrain of the hilly area is like a ladder and the ground of the same layer is uneven, it is great significant to find the low and high position of the local area of each layer in order to make rational use of the artesian irrigation. In the past ten years, most of the achievements in the optimal location of water storage projects in hilly areas have focused on the research of optimal allocation and capacity. Li zhe et al. [1] studied the feasibility evaluation of various projects of land resource consolidation through the establishment of multiple index evaluation models; Wang Hong lei et al. [2] studied how to select the appropriate location of multiple factors such as site, soil, vegetation and slope through the establishment of constraint sets; Liu Juan [3] and others focused on how to ensure the supply of water. However, these studies ignore the influence of terrain height difference on the location of reservoir, and there are few studies on this aspect. This paper proposes the Newton method, which can quickly and accurately find the extreme points of local geographical areas.

There are many research results on Newton method, such as Newton double objective optimization method [4], inexact Newton method to solve smooth and non smooth conditions [5, 6], generalized Newton method [7], discretized Newton element heuristic algorithm [8], improved Newton method [9] for convex unconstrained problems, Newton algorithm [10] for updated critical conditions based on singular value decomposition, Newton improved algorithm [11] for weak or under determined conditions, etc. The latest research on topography includes sun Wenzhou's "estimation of transponder's mutual ranging ability considering the submarine terrain slope" [12], Zhao Haibing's map based sensor node positioning method [13], Liu Weiping's reservoir research [14], network node positioning research [17-18], and few studies on positioning based on the characteristics of the terrain elevation difference of hills by Newton's method.

2. Newton method

Newton's method is a method proposed by Newton in the 17th century to solve equations approximately in the real and complex fields. The principle of Newton's method is to find the root of equation \( f(x) = 0 \) by using the previous terms of Taylor series of function \( f(x) \). Newton method is divided into basic Newton method and damping Newton method.

2.1. Basic newton method

2.1.1 Basic newton method theory

Newton method is also a classical method to solve unconstrained optimization problems. Its search direction is based on the negative gradient of the objective function and the second derivative matrix, which is called Newton direction.

For unconstrained optimization problem: \( \min f(X) \), the solution method:

The second-order Taylor formula is used to expand \( X_k \) near the iteration point.

\[
f(X) \approx f(X_k) + [\nabla f(X_k)]^T [X - X_k] + \frac{1}{2} [X - X_k]^T \nabla^2 f(X_k) [X - X_k]
\]

(1)

According to the necessary condition of the minimum point, the gradient of \( f(X) \) is equal to zero, that is:

\[
\nabla f(X_{k+1}) \approx \nabla f(X_k) + \nabla^2 f(X_k) [X_{k+1} - X_k] = 0
\]

(2)

\[
X_{k+1} = X_k - [\nabla^2 f(X_k)]^{-1} \nabla f(X_k)
\]

(3)
\[ H(k) = \nabla^2 f(X_k) \] is called Hessian matrix.

According to the basic formula of numerical iteration method:

\[ X_{k+1} = X_k + \alpha_k s_k \]  

(4)

In the basic Newton method, if the value of \( \alpha_k \) is 1 and \( s_k \) is the search direction, then:

\[ s_k = H(k)^{-1} \nabla f(X_k) = \left[ \nabla^2 f(X_k) \right]^{-1} \nabla f(X_k) \]  

(5)

Therefore:

\[ X_{k+1} = X_k + s_k \]  

(6)

For positive definite quadratic functions:

\[ f(X) = \frac{1}{2} X^T HX + BX^T + c \]  

(7)

As the form of Taylor's quadratic expansion, the \( X_{k+1} \) point obtained from the extremum condition is the exact minimum point of the function. No matter from which initial point, the minimum point can be directly reached along the Newton direction of the point.

2.1.2. Basic newton algorithm

The iterative steps of the basic Newton method are as follows:

1. Given the initial point \( X_0 \) and convergence accuracy \( \epsilon > 0 \), set \( k = 0 \).

2. Calculating function gradient, Hessian matrix and its inverse, constructing search direction, 

\[ s_k = H(k)^{-1} \nabla f(X_k) = \left[ \nabla^2 f(X_k) \right]^{-1} \nabla f(X_k) \]  

(3)

3. Find a new iteration point. \( X_{k+1} = X_k + s_k \).

4. Convergence judgment, meet \( \|f(X_{k+1})\| \leq \epsilon \), the iteration is terminated, otherwise, \( k = k + 1 \), turn to (2).

Because the quadratic Taylor expansion is only the approximation of the original function, the point obtained is only the approximate minimum point. Taking this point as the starting point of the next iteration can accelerate the convergence speed to the minimum point.

2.2. Damped newton method

2.2.1. Damped newton method theory

The damped Newton method, also known as the modified Newton method, is an optimization method improved by the basic Newton method. It retains the Newton direction, but the step size is determined by one-dimensional search.

\[
\begin{cases}
S_k = - \left[ \nabla^2 f(X_k) \right]^{-1} \nabla f(X_k) \\
\min f(X_k + \alpha S_k) \rightarrow \alpha_k \\
X_{k+1} = X_k + \alpha_k S_k
\end{cases}
\]  

(8)
The convergence rate of the modified Newton method is very fast. When the second derivative of \( f \) and the inverse matrix of Hessian matrix are easy to calculate, this method is very effective.

2.2.2. **damped Newton algorithm**

The iterative steps of damped Newton method are as follows:

1. The gradient of function, Hessian matrix and its inverse are calculated, and the search direction is constructed.

2. Given the initial point \( x_0 \) and convergence accuracy \( \varepsilon > 0 \), set \( k=0 \).

3. The optimal step size \( \alpha_k \) is searched in one dimension, and a new iteration point

\[
\begin{align*}
\min f(X_k + \alpha S_k) \rightarrow \alpha_k, \\
X_{k+1} = X_k + \alpha S_k
\end{align*}
\]

4. Convergence judgment, meet the \( \|\nabla f(X_{k+1})\| \leq \varepsilon \), the iteration is terminated, otherwise, \( k=k+1 \), turn to (2).

3. **Verification**

Simulate a part of hilly area and find the objective function:

\[
f(x) = x_1^2 + 4x_2^2 + 9x_3^2 - 2x_1 - 18x_3
\]

The initial point \( X_0 \) is set to \([1 \ 2 \ 1]^T\), seek \( \min f(x) = x_1^2 + 4x_2^2 + 9x_3^2 - 2x_1 - 18x_3 \).

\[
\nabla f(x) = \begin{bmatrix} 2x_1 - 2 \\ 8x_2 \\ 18x_3 - 18 \end{bmatrix}, \quad \nabla^2 f(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 18 \end{bmatrix}
\]

\[
\nabla f(x_0) = \begin{bmatrix} 0 \\ 16 \end{bmatrix}, \quad [\nabla^2 f(x)]^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{18} \end{bmatrix}
\]

\[
s_0 = -[\nabla^2 f(x)]^{-1} \nabla f(x_0) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{18} \end{bmatrix} \begin{bmatrix} 0 \\ 16 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}
\]

\[
x_1 = x_0 + s_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
f(x_1) = 1^2 + 4 \times 0 + 9 \times 1^2 - 2 \times 1 - 18 \times 1 = -10
\]

\[
\nabla f(x_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \|\nabla f(x_1)\| = 0
\]
\[ x^* = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad f^* = -10 \]

That is to say, the minimum value at \( x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \) is -10.

**4. Conclusion**

The basic Newton method is usually used to solve unconstrained optimization problems. It is an optimization iterative method with less iterations and without step factor. The key of basic Newton method is to find the initial approximate point \( x_0 \) near the extreme point, Search in the given direction to find the extreme value of the function. Because the basic Newton method does not need one-dimensional search and is the iterative operation with the fastest convergence speed in a given region, in the solution of the positive definite quadratic function, no matter where \( X_k \) starts from and searches along the Newton direction of \( X_k \), the extreme point of the function can be reached and the calculation accuracy is high. However, the basic Newton method does not need step factor and one-dimensional search. In the calculation of non-linear function problems, the function may not have monotonically or convergence, which may lead to the point \( X_k \) that we seek does not have the characteristic of keeping the function descending all the time, and may lead to the failure of the basic Newton method.

The damped Newton method, also known as the modified Newton method, is a more widely used optimization algorithm to ensure the descent of the iterative point \( X_k + 1 \). Compared with the basic Newton method, the modified Newton method incorporates one-dimensional search. When constructing the search direction, it makes full use of the information of the first derivative and the second derivative of the function at the initial point, so that the search direction of the minimum value of the direct function can be found. For positive definite quadratic matrix, the minimum point of function can be found by one iteration. Because the first derivative is added to the second derivative and negative gradient of the function, the damped Newton method is stricter in finding the search direction, and the accuracy of the extremum is higher. But at the same time, it also increases the amount of calculation, and the calculation process is more lengthy.

Basic Newton method and damped Newton method are called Newton method. The key is to use the search direction of negative gradient and second derivative matrix to solve the objective function (usually positive definite quadratic function) by iteration. The key of Newton method to solve the equation is to find the initial point \( x_0 \). The accuracy of the extreme point is usually determined by the distance between the initial point and the extreme point. In the calculation, we determine the initial point \( x_0 \) by prediction, and then search near the initial point.

The innovation of Newton's method is that Newton's method uses the search direction composed of negative gradient and second derivative to search the objective function. The search method of negative gradient is adopted, that is, search along the inverse line of gradient, and the direction of negative gradient is usually in the local area, the direction of function local decline is the fastest, and the extreme point can be obtained by one iteration. Compared with other methods such as gradient method, for example,

Newton method needs less calculation times and is more convenient. At the same time, Newton method uses the second-order matrix which is transformed from the constraints to calculate the constraints, which is more intuitive. In practical application, Newton method greatly reduces the number of iterations required by the objective function and reduces the amount of calculation.
However, in the process of its operation, it is often necessary to use matrix inversion and other more difficult operations, which may have some limitations. In order to solve the optimization problem of extremum, it should be different from problem to problem. For the simple one-dimensional extremum problem, gradient method and one-dimensional derivative can be used to solve it directly. The calculation is simple and convenient. For quadratic function, we usually use Newton method. Newton method has second-order convergence, and it is the algorithm with the fastest second-order convergence. It is more convenient and has less iterations for solving extreme value problems.

5. Summary

According to the characteristics of the height difference of the hilly terrain, this paper introduces the Newton method which is suitable for the location of the height difference of the hills. Newton method is divided into basic Newton method and damping Newton method. The theoretical knowledge and algorithm of these two Newton methods are introduced respectively. The application of Newton method is described in detail through simulation data. Finally, the algorithm idea of Newton method is summarized and compared with other algorithms. The advantages of Newton method in calculation convergence are obtained.

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