Analogy between dynamics of thermo-rheological and piezo-rheological pendulums

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Abstract. The constitutive stress-strain relations of the standard thermo-rheological and piezo-rheological hereditary element in differential form as well as in two different integro-differential forms are defined. The considered problem of a thermo-rheological hereditary discrete system nonlinear dynamics in the form of thermo-rheological double pendulum system with coupled pendulums gets the significance of two constrained bodies in plane motion problem, as a problem important for studying a sensor dynamics or actuator dynamics in active structure dynamics. System of the averaged equations in the first approximation for amplitudes and phases are derived and qualitatively analyzed. Analogy between nonlinear dynamics of the double pendulum systems with thermo-rheological and piezo-rheological properties between pendulums is pointed out.

1. Introduction

Research results in area of mechanics of hereditary discrete systems, obtained by Goroshko and Hedrih are generalized and presented in the monograph [1] and in the Reference [2] which contains first presentation of the analytical dynamics of the hereditary discrete systems. We can conclude that this monograph contains complete foundation of the analytical dynamics theory of discrete hereditary systems and by using these results, numerous examples are obtained and solved.

In current literature term “hereditary” and “rheological” systems are equivalent. In opinion of Работнов Ю.Н. [1], the name “hereditary” system or continuum proposed by В.Вольтерра, is more precise as well as suitable. The *discrete hereditary system* is a system of discrete material particles interconnected by standard constraint light hereditary elements The double pendulum system is hereditary if material particles are interconnected by one or more standard hereditary elements.

In the References [1], [2], [3], [8], [10] and [11] dynamics of a thermo-rheological hereditary pendulum is mathematically described.

1.1. Light standard thermo-rheological hereditary element.

For stress strain state of the standard hereditary element, the relation between generalized coordinate of hereditary element deformation $\rho - \rho_0$ and corresponding force of deformation $P(t)$ is (seeRefs. [1-12],[2]):

$$n\dot{P}(t) + P(t) = n\dot{\rho}(t) + c[\rho(t) - \rho_0]$$

(1)
When standard hereditary element is modified by two temperatures $T_K(t)$ and $T_M(t)$, which are introduced by thermo-modification of visco-elastic properties by temperature $T_K(t)$, and by thermo-modification of elasto-viscous properties by temperature $T_M(t)$, than constitutive relation between stress and strain state of the thermo-rheological hereditary element is:

$$n\dot{P}(t) + P(t) + n\dot{F}_M(t) + F_K(t) = n c \rho(t) \dot{c} [\rho(t) - \rho_0]$$

where (see Refs. [1] and [2]) $F_M(t) = c_M \alpha_M T_K(t)$ and $F_K(t) = c_K \alpha_K T_K(t)$ are thermo-elastic forces, and $\rho(t)$ is rheological coordinate, $c_M, c_K$ are coefficients of thermo-elastic rigidity, $\alpha_M, \alpha_K$ are coefficients of thermo-elastic dilatations, $n$ is time of relaxation, and $c, \tilde{c}$ an instantaneous rigidity and a prolonged one of an element. Constitutive relation (2) of the thermo-rheological hereditary element from differential form we can rewrite in two integro-differential following forms:

a* explicit with respect to the force $P(t)$

$$P(t) = c [\rho(t) - \rho_0] - \int_0^t \left[ \dot{\rho}(\tau) - \rho_0 \right] R(t-\tau) d\tau - F_M(t) + n c \tilde{c} \int_0^t \left[ F_M(\tau) - F_K(\tau) \right] R(t-\tau) d\tau$$

or b* explicit with respect to the coordinate $\rho(t)$

$$\rho(t) - \rho_0 = \frac{1}{c} \left[ \dot{P}(t) + \int_0^t \left[ \dot{P}(\tau) K(t-\tau) d\tau \right] \right] + F_M(t) - \frac{\tilde{c}}{c-\tilde{c}} \int_0^t \left[ F_M(\tau) K(t-\tau) d\tau \right] + \frac{c}{c-\tilde{c}} \int_0^t \left[ F_K(\tau) K(t-\tau) d\tau \right]$$

where $R(t-\tau) = \frac{e^{\frac{-c}{nc}(t-\tau)}}{e^{\frac{-c}{nc}}} \tilde{c}$ is a kernel of relaxation and $K(t-\tau) = \frac{e^{\frac{-c}{nc}}}{e^{\frac{-c}{nc}}} \tilde{c}$ is a kernel of rheology.

1.2. Light standard piezo-rheological hereditary element

When standard hereditary element with piezo-ceramic element is modified by two polarization voltages $U_K(t)$ and $U_M(t)$, which are introduced by piezo-modification of visco-elastic properties of piezo-ceramic component of the hereditary element by $U_K(t)$ and by piezo-modification of elasto-viscous properties by $U_M(t)$, and thermo-modified by two temperatures $T_K(t)$ and $T_M(t)$, than constitutive relation between stress and strain state of the thermo-piezorheological hereditary element is in the form (3) or (4) in which [1-2]

$$F_M(t) = c_{uM} \alpha_{uM} U_M(t) + c_{TM} \alpha_{TM} T_M(t) \quad F_K(t) = c_{uK} \alpha_{uK} U_K(t) + c_{TK} \alpha_{TK} T_K(t)$$

are thermo-elastic forces and piezo-elastic forces, and $\rho(t)$ is rheological coordinate, $c_{uM}, c_{uK}$ are coefficients of thermo-elastic rigidity, $\alpha_{TM}, \alpha_{TK}$ are coefficients of thermo-elastic dilatations, $c_{uM}, c_{uK}$ are coefficients of piezo-elastic rigidity, $\alpha_{TM}, \alpha_{TK}$ are coefficients of piezo-elastic dilatations $n$ is time of relaxation, and $c, \tilde{c}$ an instantaneous rigidity and a prolonged one of an element. Parametrical parts, as small are neglected.

2. Thermo-rheological coupled pendulums – Nonlinear Approach

In Figures 1. a thermo-rheological system, containing two coupled pendulums (see Refs. [1], [2] and [8]), is presented. We take into consideration two coupled mathematical pendulums presented in Figure 2., both with material particles of mass $m$, with length $\ell$ and with two degrees of freedom defined by generalized coordinates $\varphi_1$ and $\varphi_2$, and a standard light thermo-visco-elastic element thermo-modified by temperature $T(t)$, coupling pendulum at distance $\ell$ and parallel coupled, but temperature isolated, and with one standard light nonlinear spring with coefficients of the linear and nonlinear rigidity respectively denoted by $c_T$ and $\hat{c} = \varepsilon \chi c$, where $\varepsilon$ is small parameter.
Figure 1. System with two pendulums interconnected by standard light thermo-modified hereditary element

Figure 2. Homogeneous system with two pendulums interconnected by standard light thermo-modified hereditary element
Now, we take into account that this standard light thermo-visco-elastic element thermo-modified by temperature $T(t)$ is in the dynamic state, and that we didn’t neglect thermo-modification of the element strain, then we can write that is

$$\Delta \ell_0 = \alpha_r T(t)(\ell_0 + x),$$

and that the constitutive relations of the thermo-visco-elastic stress-strain state is in the following form:

$$P_{\text{ther}}(t) = -c_r \left[ \Delta \ell_0 + \ell_r (\phi_2 - \phi_1) \right] = -c_r \ell_r (\phi_2 - \phi_1) \left[ 1 + \alpha_r T(t) \right] - c_r \alpha_r \ell_0 T(t)$$

$$P_{\text{nolinear}}(t) = -c \left[ \ell_c (\phi_2 - \phi_1) + \epsilon \chi \ell (\phi_2 - \phi_1)^3 \right]$$

and

$$P_{\text{damp}}(t) = -b \ell_0 (\phi_2 - \phi_1)$$

Differential equations of the thermo-rheological coupled pendulums presented in Figure 2. are in the form:

$$\ddot{\phi}_1 + \omega_0^2 \phi_1 - \omega_0^2 (\phi_2 - \phi_1) + \omega_{0\gamma}^2 (\phi_1 - \phi_2) \left[ 1 + \gamma \tilde{T}(t) \right] + 2 \delta (\phi_1 - \phi_2) =$$

$$= -\omega_{0\gamma}^2 h_0 \tilde{T}(t) + \omega_0^2 \left( \frac{\phi_1^3}{3!} - \frac{\phi_1^5}{5!} + \frac{\phi_1^7}{7!} - \frac{\phi_1^9}{9!} + \ldots \right) + \omega_0^2 \epsilon \chi (\phi_2 - \phi_1)^3$$

$$\ddot{\phi}_2 + \omega_0^2 \phi_2 - \omega_0^2 (\phi_2 - \phi_1) + \omega_{0\gamma}^2 (\phi_2 - \phi_1) \left[ 1 + \gamma \tilde{T}(t) \right] + 2 \delta (\phi_2 - \phi_1) =$$

$$= \omega_{0\gamma}^2 h_0 \tilde{T}(t) + \omega_0^2 \left( \frac{\phi_2^3}{3!} - \frac{\phi_2^5}{5!} + \frac{\phi_2^7}{7!} - \frac{\phi_2^9}{9!} + \ldots \right) - \omega_0^2 \epsilon \chi (\phi_2 - \phi_1)^3$$

where
oscillate with same eigen frequency, \( \omega_1 = \omega_2 \) and with solution: \( \phi_1(t) = C_1 \cos(\omega_1 t + \alpha_1) \pm C_2 \cos(\omega_2 t + \alpha_2) \), where \( C_1, C_2, \alpha_1 \) and \( \alpha_2 \) are constant.

By the change the generalized coordinates in the normal coordinates by the following ways \( \xi_1 = \varphi_1 + \varphi_2 \) and \( \xi_2 = \varphi_1 - \varphi_2 \) previous system (7)-(8) of the non-linear equations, take form:

\[
\ddot{\xi}_1 + 2 \omega_0^2 \xi_1 = \ddot{\omega}_0^2 \xi_1 = \ddot{\omega}_0^2 \bigg[ \left( \left( \frac{\xi_1 + \xi_2}{3!} \right) + \left( \frac{\xi_1 - \xi_2}{5!} \right) \right)^2 + \left( \frac{\xi_1 + \xi_2}{3!} \right) - \left( \frac{\xi_1 - \xi_2}{5!} \right) + \cdots \bigg]
\]

(10)

\[
\ddot{\xi}_2 + 2 \omega_0^2 \ddot{\xi}_2 = \ddot{\omega}_0^2 \ddot{\xi}_2 = \ddot{\omega}_0^2 \bigg[ \left( \left( \frac{\xi_1 + \xi_2}{3!} \right) - \left( \frac{\xi_1 - \xi_2}{5!} \right) \right)^2 + \left( \frac{\xi_1 + \xi_2}{3!} \right) - \left( \frac{\xi_1 - \xi_2}{5!} \right) + \cdots \bigg]
\]

(11)

If we take into account only first member of the expansion of the pendulum no linearity we obtain:

\[
\ddot{\xi}_1 + 2 \omega_0^2 \ddot{\xi}_1 = \ddot{\omega}_0^2 \ddot{\xi}_1 = \ddot{\omega}_0^2 \left( \xi_1 + 3 \xi_1 \xi_2^2 \right)
\]

(12)

\[
\ddot{\xi}_2 + 2 \omega_0^2 \ddot{\xi}_2 = \ddot{\omega}_0^2 \ddot{\xi}_2 = \ddot{\omega}_0^2 \left( \xi_2 + 3 \xi_1^2 \xi_2 \right)
\]

(13)

and for corresponding rheolinear system:

\[
\ddot{\xi}_1 + \ddot{\omega}_0^2 \xi_1 = 0
\]

(14)

\[
\ddot{\xi}_2 + (\ddot{\omega}_0^2 + 2 \omega_0^2) \ddot{\xi}_2 = (\ddot{\omega}_0^2 + 2 \omega_0^2) \left[ 1 + g \ddot{T}(t) \right] \ddot{\xi}_2 + 4 \alpha \ddot{\xi}_2 = -2 \omega_0^2 h_0 \ddot{T}(t) + 2 \omega_0^2 e \ddot{\xi}_2
\]

(15)

First equation (14) of the rheolinear system represents partial pure harmonic oscillator, presented in Figure 3, with frequency \( \omega_0^2 = \ddot{\omega}_0^2 = \frac{g}{\ell} \) of the free vibrations. This case is when both pendulum oscillate with same eigen frequency, \( \ddot{\omega}_0^2 = \frac{g}{\ell} \), as decoupled pendulums, as single mathematical pendulum, and then standard light thermo-visco-elastic element thermo-modified by temperature \( T(t) \) haven’t influence to this normal; coordinate composed by sum \( \xi_1 = \varphi_1 + \varphi_2 \). On this coordinate oscillation are free, without temperature influence.

Second equation (15) of the rheolinear system, on the normal coordinate \( \xi_2 = \varphi_1 - \varphi_2 \) is Mathieu-Hill type equation (see Refs. [4], [13], [15] and [16]) and represents mathematical description of the thermo-rheological oscillator, presented in Figure 4., with parallel coupled two light standard thermo-visco-elastic element thermo-modified by same temperature \( T(t) \) and one elastic spring with rigidity.
\( c_u = mg/t \), in the dynamic state. For this coordinate \( \xi_2 = \varphi_1 - \varphi_2 \), we can separate two main cases. 

For both cases, we take into consideration asymptotic approximation of the amplitude and phase of the dynamic process on this coordinate \( \xi_2 = \varphi_1 - \varphi_2 \) close around I* main resonance when
\[
\Omega \approx \omega_2 = \sqrt{\omega_{0}^2 + 2\left(\omega_0^2 + \omega_{0r}^2\right)}
\]
and II* around parametric resonance when
\[
\Omega \approx \frac{1}{2} \omega_2 = \frac{1}{2} \sqrt{\omega_{0}^2 + 2\left(\omega_0^2 + \omega_{0r}^2\right)}
\]. Then, we can conclude that on this coordinate is possible to appear under the corresponding kinetic parameters I* regimes closest to main resonant state, as well as one main resonant state, and II* regimes closest to parametric resonant state, as well as one resonant state under the thermo-viscoelastic temperature single frequency excitation.

For solving system of nonlinear differential equations (12) - (13), we take into account temperature excitation in the form:
\[
\tilde{T}(t) = \sin(\Omega t + \beta)
\]
with deterministic constant value frequency \( \Omega \) and constant deterministic phase \( \beta \) and we take into account the following form of the first approximation of solutions (see Ref. [14]):
\[
\xi_1(t) = C_1(t) \cos \Phi_1(t) \quad \text{and} \quad \xi_2(t) = C_2(t) \cos \Phi_2(t)
\]
where amplitudes \( C_1(t) \) and \( C_2(t) \), and phases \( \Phi_1(t) \) and \( \Phi_2(t) \) are unknown function of time \( t \), we obtain system of ordinary differential equations for these unknown amplitudes and phases, in the averaged form, along full phases \( \Phi_1(t) \) and \( \Phi_2(t) \), and for both cases we obtain the following:

\( a^* \) \( \Delta_{\text{det}(i)} = \omega_i - \Omega \)
\[
\Phi_i(t) = \Omega t + \phi_i \quad \Omega = \omega_i - \Delta_{\text{det}(i)} \quad \text{main resonance}
\]

\( b^* \) \( \Delta_{\text{det}(i)} = \omega_i - \Omega \)
\[
\Phi_i(t) = \omega_i - \Delta_{\text{det}(i)} \quad \text{parametric}
\]

3. Concluding Remarks
We can conclude that the basic system (rheolinear-unperturbed system) corresponding to rheolinear – thermo-rheological perturbed system, has two main normal coordinates, and that for thermo-rheological perturbed system result is in one free partial oscillator and one forced and parametrically perturbed on the second mode, when first mode is unperturbed. Non-linearities of pendulums explicitly appear in both phase equations for both modes, and non-linearity of the spring explicate appear only in the phase equation for second mode, for the both case of the main and parametric resonance. Also, on the basis of the analogy between models in the chapters 2.1 and 2.2., we can use all obtained models of double pendulum system with piezo-modifies rheological light element and all obtained approximation for amplitudes and phases of the nonlinear modes.

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