ON A SPECTRAL TOPOLOGY IN POSITIVE MODEL THEORY

JEAN BERTHET

Abstract. We introduce a dual "spectral topology" on the spaces of positive types of an h-inductive theory $T$. First, we show that it is finer than the definable topology and Hausdorff, and that its "global" compactness characterises positive model completeness; this means that in general, we can only count on an "infinite compactness", the index of which is given by the cardinality of the language. We associate to $T$ a concrete, small and finitely complete "spectral category" $SC(T)$, which is closely linked to the spectral topology, and on which left exact functors are equivalent to the models of the Kaiser hull of $T$. Defining a "spectral" Grothendieck topology $G(T)$ on $SC(T)$, we refine this result in showing that the positive (existentially closed) models of $T$, the objects of study of positive model theory, are essentially the continuous left exact functors on $SC(T)$ for $G(T)$.

1. Introduction and preliminaries

Positive model theory was introduced in [2] and revisited in [3] in terms of the study of positively existentially closed models of an h-inductive theory. In this context, which may be construed as a generalisation of classical first order model theory by positive Morleyisation, the spaces of types still play an essential role. Contrary to the classical case, the definable topology on these spaces, however compact, is not Hausdorff, and the fact that only positive formulas are considered implies that some properties which are topologically linked to the definable topology in the classical setting, like the existence of coheirs, fail to be reproducible in the positive context.

In this paper we study a topology which is in some sense "dual" to the definable topology, and intrinsically linked to the positively existentially closed models of an h-inductive theory; we call it the "spectral" topology. It turns out that this topology is Hausdorff and finer than the definable topology, though not compact in general, and contains some model-theoretic information; this is the content of section 2.1. The information contained in the positive types may be used to construct a "spectral category", which is done in section 2.2 and which enables to classify the models of the Kaiser hull of an h-inductive theory. However, this is not enough to classify the positively existentially closed models, and translating the information about the spectral topology as a Grothendieck topology on the spectral category, we bridge the gap in section 2.3, building eventually a classifying topos for the intended models of positive logic.

In the rest of the introduction we introduce the conventions, notations and preliminary facts which are needed, concerning positive model theory, categories and Grothendieck topologies.

Model theoretic conventions and notations. Our general reference for model theory is [4]. We work in a many-sorted first order language $L$ (as for instance in [6], 2.1), the
subset of sort symbols of which we note $S$. Relation symbols have a *sorting* and function symbols have an *arity*, which are both a *finite* strings (or "tuples") of sort symbols; in addition function (and constant) symbols have a *sort*, which is a sort symbol; languages are interpreted in the classical way. We fix the presence of two $0$-ary relation symbols $\top$ (true) and $\bot$ (false), interpreted in the obvious way in each $L$-structure. We work with a set $V$ of variables, each coming with a sort, and each sort having a countable supply of variables; we may take $V = S \times \omega$, with a variable $(s, i) \in V$ having sort $s$.

Formulas are considered only in $L_\infty\omega$ and are noted $\varphi(x)$; strictly speaking this denotes a *couple* $(\varphi, x)$, where $\varphi$ is a formula which free variables are among the finite tuple $x$ of variables. We use the same convention for terms $t(x)$. We note $V^*$ the set of all finite tuples of variables. Such tuples are noted by single letters $x, y, z, \ldots$; the expression $x \cap y = \emptyset$ is intended to mean that $x$ and $y$ have no common variables, while the notation $|x| = |y|$ mean that $x$ and $y$ have the same *sorting*, i.e. the sorts of the variables appearing in order in $x$ and $y$ are the same. A formula is positive (existential) if is is finitary and mentions only finite conjunctions, disjunctions and existential quantifications; we note $L^+$ the set of positive formulas.

A *sorted map* of $L$-structures $f : A \to B$ is a family $(f_s)_{s \in S}$ of maps $f_s : A_s \to B_s$ for each sort symbol $s \in S$; it is a ($L$-)homomorphism if for every atomic sentence $\varphi(a)$ with parameters in $A$ such that $A \models \varphi(a)$, we have $B \models \varphi(fa)$. A homomorphism $f : A \to B$ is an *immersion* if for every positive sentence $\varphi(a)$ with parameters in $A$, if $B \models \varphi(fa)$ then $A \models \varphi(a)$.

**Positive model theory.** Our references for positive model theory are [2] and [3], but our exposition follows more closely [3]. We review the basic elements in a many-sorted context and provide some special notation and terminology.

**h-Inductive theories and positive models.** We recall from [3] that a finitary first order $L$-sentence is $h$-inductive, if it is a finite conjunct of basic $h$-inductive sentences, which have the form $\forall x (\varphi(x) \Rightarrow \psi(x))$, where $\varphi(x)$ and $\psi(x)$ are positive (existential) formulas, which means they are obtained from atomic formulas by finite conjuncts, disjuncts or existential quantification over finite strings of variables. A first order $L$-theory $T$ is $h$-inductive, if it consists of $h$-inductive sentences. The models of such a theory $T$ form an inductive class, i.e. closed under directed colimits of $L$-homomorphisms. If $C$ is a class of $L$-structures and $A \in C$, $A$ is positively existentially closed in $C$ if every $L$-homomorphism $f : A \to B$, with $B \in C$, is an immersion.

**Fact 1.1 ([3], Theorem 1).** In an inductive class, every structure continues into a positively existentially closed one.

If $T$ is $h$-inductive, we note $\mathcal{M}(T)$ the full subcategory of its models and $\mathcal{M}^+(T)$ the full subcategory of positively existentially closed models of $T$. In order to lighten the terminology and avoid confusion with classical existential completeness, we suggest to call the objects of $\mathcal{M}^+(T)$ the *positive models* of $T$. If every model of $T$ is positive (i.e. if $\mathcal{M}(T) = \mathcal{M}^+(T)$) we say that $T$ is positively model complete.

**Fact 1.2 ([3], Lemma 15).** $T$ is positively model complete if and only if for every positive formula $\varphi(x)$, there exists a positive formula $\psi(x)$ such that $T \models \forall x (\varphi(x) \land \psi(x) \Rightarrow \bot)$ and $T \models \forall x (\top \Rightarrow \varphi(x) \lor \psi(x))$. 
We define as in ([1], Definition 9), the resultant of a positive formula \( \varphi(x) \), which is the set \( \text{Rest}_T(\varphi(x)) \) of all positive formulas \( \psi(x) \) such that \( T \models \forall x (\varphi(x) \land \psi(x)) \Rightarrow \bot \). Using Lemma 14 of [3], it is possible to characterise the positive models of \( T \) as follows.

**Fact 1.3.** A model \( M \) of \( T \) is positive if and only if for every positive formula \( \varphi(x) \), we have \( M^* - \varphi(x)^M = \bigcup \{ \psi(x)^M : \psi \in \text{Rest}_T(\varphi) \} \).

**h-Universal sentences and companions.** A (basic) h-inductive sentence is h-universal if it has the form \( \forall x (\varphi(x) \Rightarrow \bot) \). We note \( T_u \) the set of h-universal consequences of \( T \).

**Fact 1.4 ([3], Lemma 5).** An \( L \)-structure \( A \) is a model of \( T_u \) if and only if there exists an \( L \)-homomorphism \( f : A \rightarrow M \) into a model \( M \) of \( T \).

If \( A \) is an \( L \)-structure, we note \( L(A) \) the expansion of \( L \) by the elements of \( A \) naming themselves, and \( D^+A \) the set of atomic sentences, with parameters in \( A \) and true in \( A \). A model \( B \) of \( D^+A \) is essentially the same thing as an \( L \)-homomorphism \( f : A \rightarrow B \), preserving the canonical interpretation of \( A \) in itself. By the fact, \( L \)-homomorphisms into a model of \( T \) are essentially the models of \( T_u \cup D^+A \) in the language \( L(A) \).

If \( T' \) is another h-inductive theory in the same language, we say that \( T \) and \( T' \) are positive companions if they have the same positive models, i.e. \( M^+(T) = M^+(T') \). The Kaiser hull of \( T \), noted \( T_k \), is the class of all h-inductive sentences which are satisfied in every positive model of \( T \).

**Fact 1.5 ([3], Lemma 7).** \( T_u \) is the smallest positive companion of \( T \) and \( T_k \) is the largest positive companion of \( T \).

**Spaces of positive types.** If \( x \) is a (possibly infinite) tuple of variables (or of new type constants), a positive type in \( T \) in variables \( x \) is a set \( p \) of positive formulas \( \varphi(x) \), such that \( T \cup p(x) \) is consistent, and \( p(x) \) is maximal with this property. We write \( S_x(T) \) the set of all such positive types. The sets of the form \( [\varphi(x)] = \{ p \in S_x(T) : \varphi \in \varphi \} \), for positive formulas \( \varphi(x) \), are closed under finite unions and intersections, and are the basic closed sets of the definable topology on \( S_x(T) \), which we will note \( \mathcal{D} \). If \( A \) is an \( L \)-structure, \( x \) is a finite tuple of variables and \( a \in M_x \), the positive type of \( a \) in \( M \), noted \( tp_A^+(a) \), is the set of all positive formulas \( \varphi(x) \) such that \( A \models \varphi(a) \).

**Fact 1.6 ([3], Lemma 13).** For every finite tuple of variables \( x \), the positive types of \( S_x(T) \) are the positive types of corresponding tuples in positives models of \( T \), and we have \( S_x(T_u) = S_x(T) = S_x(T_k) \).

**Fact 1.7 ([3], Lemma 16).** The definable topology is compact, though not Hausdorff in general.

**Categories and Grothendieck topologies.** For background on Grothendieck topologies and classifying topoi, the reader is referred to [5] and [6].

A category \( \mathcal{C} \) has finite (left) limits if it has a terminal object, any product of two objects and an equaliser for any pair of parallel arrows. This is equivalent to \( \mathcal{C} \) having a terminal object and pullbacks. A functor \( F : \mathcal{C} \rightarrow \mathcal{D} \) is left exact if it preserves finite left limits; for this it suffices for \( F \) to preserve finite products and equalisers or the terminal object and pullbacks. We recall the following

**Fact 1.8 ([3], Corollary VII.6.4).** If \( \mathcal{C} \) has finite left limits, a functor \( F : \mathcal{C} \rightarrow \text{Sets} \) is left exact if and only if it is flat.
If $G$ is a Grothendieck topology on $C$, we say (see [5] VII.5) that a functor $F : G \to \text{Sets}$ is \textit{continuous} for $J$ if $F$ sends covering sieves for $J$ to colimit diagrams in $\text{Sets}$.

\textbf{Fact 1.9 ([5], Corollary VII.5.4).} If $(C, J)$ is a site, the category of points of $\text{Sh}(C, J)$ is equivalent to the category of continuous flat functors from $C$ to $\text{Sets}$.

Alternatively, we will say that a collection $G_0$ of covers in $C$ (a cover is a family of morphisms with the same codomain) is a \textit{Grothendieck pretopology} (a basic covering family in [5] 6.1) and that a functor $F : C \to \text{Sets}$ is \textit{continuous} for $G_0$ if for every cover $(f_i : c_i \to c)_I$ in $G_0$, we have $Fc = \bigcup_{I \in I} \exists F f, \text{ where } \exists F f$ denotes the set-theoretic image of $F f$.

2. The spectral topology

Throughout this section, $T$ denotes an h-inductive theory in a first order language $L$, $x$ a (possibly infinite) tuple of variables, $S_x(T)$ the set of positive types in the tuple $x$. If $\varphi(x) \in L^+$, we recall that $[\varphi(x)] = \{ p \in S_x(T) : \varphi \in p \}$.

2.1. Another topology on positive type spaces. Positive formulas are closed under finite conjuncts and disjuncts. We use this fact in order to define a Hausdorff topology $\mathcal{S}$ we call "spectral" on the positive type spaces over $T$. In some sense it is "dual", from its definition, to the definable topology $\mathcal{D}$, however we show in Proposition 2.2 that it is actually finer than $\mathcal{D}$, due to the particularity of positive type spaces: they are the set of positive types of tuples in positive models of $T$, so the resultant of a formula defines its complement in a space of type. The spectral topology has a further model theoretic significance, connected to these considerations: its compactness is equivalent to positive model completeness, which is the content of Theorem 2.3. As we expect positive model theory to be of interest in non axiomatisable contexts, this means we cannot count on compactness of type spaces for $\mathcal{S}$, in contrast with $\mathcal{D}$; this leads us to a form of "infinitary compactness" for $\mathcal{S}$, an index of which is intrinsic to the language $L$ (Proposition 2.6).

\textbf{Definition 2.1.} The \textit{spectral topology} on $S_x(T)$ is the topology $\mathcal{S}$ which basis of open sets are the sets $[\varphi]$, for $\varphi(x) \in L^+$.

\textbf{Proposition 2.2.} The spectral topology is Hausdorff and finer than the definable topology.

\textit{Proof.} First we show that $\mathcal{S}$ is Hausdorff. If $p \neq q \in S_x(T)$, by maximality of positive types there are $\varphi(x) \in p - q$ and $\psi(x) \in q - p$, so $p \in [\varphi] - [\psi]$ and $q \in [\psi] - [\varphi]$, i.e. $[\varphi]$ and $[\psi]$ separate $p$ and $q$.

Now suppose that $p \in S_x(T)$ and $\varphi(x) \in L^+$. If $p \notin [\varphi]$, let $M \models^+ T$ and $a \in M_\varphi$ a realisation of $p$ in $M$: as $M$ is positive, there exists $\psi \in \text{Res}_T(\varphi)$ such that $M \models \psi(a)$, and as $p$ is maximal, we have $\psi \in p$, i.e. $p \in [\psi]$. Reciprocally, if $\psi \in \text{Res}_T(\varphi)$ and $p \in [\psi]$, by consistency of $p$ we have $\varphi \notin p$, i.e. $p \notin [\varphi]$. This means we have $S_x(T) - [\varphi] = \bigcup \{ [\psi] : \psi \in \text{Res}_T(\varphi) \}$ and this last is open for the spectral topology, so $[\varphi]$ itself is spectrally closed. The basic definably closed sets $[\varphi]$ are closed for $\mathcal{S}$, hence $\mathcal{S}$ is finer than $\mathcal{D}$. \hfill $\square$
Example 2.3. Let $T$ be the $h$-inductive theory of (strict) linear orders in the language $\{<\}$: its positive models are the dense linear orders, which is a consequence of quantifier elimination for this last theory, or may be checked directly. The rational order $(\mathbb{Q},<)$ is a positive model, and the assignation to every element of $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$ of its (positive) type over $\mathbb{Q}$ is a homeomorphism between $\mathbb{R}$ with the "constructible topology" (i.e. the full definable topology, with as basis the intervals) and $S_1(\mathbb{Q})$ endowed with the spectral topology, which happens here to be the definable one, because the theory $T_\kappa$ is positively model complete.

Proposition 2.4. $T_\kappa$ is positively model complete if and only if for every finite tuple $x \in V^*$ the space $S_x(T)$ is compact for the spectral topology, and in this case the two topologies coincide with the classic definable topology (and $\mathcal{M}^+(T) = \mathcal{M}(T_\kappa)$).

Proof. If $T_\kappa$ is positively model complete, every $L$-formula is equivalent modulo $T_\kappa$ to a positive formula, hence the positive types spaces $S_x(T) = S_x(T_\kappa)$ are homeomorphic to the classic type spaces (for the spectral topology), hence the positive type spaces are compact Hausdorff.

Reciprocally, suppose that every space of (positive) types is spectrally compact and let $x \in S_x(T)$ be such that $\phi$ models of $T$. Hence the positive type spaces are spectrally compact and let $x \in S_x(T)$ be such that $\phi$ is a positive type over $\mathbb{Q}$.

Reciprocally, suppose that every space of (positive) types is spectrally compact and let $x \in L^+$ be a finite tuple. In $S_x(T)$ the complement $S_x(T) - [\phi] = \bigcup_{\psi \in \text{Res}_{T^+}(\phi)} [\psi]$ is spectrally closed, hence compact, so we may find a finite subset of $\text{Res}_{T^+}(\phi)$, in fact a single $\psi \in \text{Res}_{T^+}(\phi)$, such that $S_x(T) - [\phi] = [\psi]$. Now if $M = T^+$ and $a \in M_x$ is such that $M \not\models \phi(a)$, we have $\text{tp}(a) = p \notin [\phi]$, hence $p \in [\psi]$, which means that $M \models \psi(a)$. In other words, $\psi(x)$ defines a complement of $\phi$ modulo $T_\kappa$, which is then positively model complete (and axiomatises the positive models of $T$).

Definition 2.5. If $\kappa$ is a cardinal, say that a topological space $X$ is $\kappa$-compact if for every open cover $X = \bigcup_{i \in I} O_i$, there exists a subset $J \subset I$, such that $|J| < \kappa$ and $X = \bigcup_{i \in J} O_i$.

Proposition 2.6. If $\kappa = |L|$, for every finite tuple $x$ of variables, the spectral topology on $S_x(T)$ is $\kappa^+$-compact.

Proof. Suppose that $S_x(T) = \bigcup_{i \in I} O_i$, with $O_i$ a spectral open for each $i$. By definition of the spectral topology, for every $i \in I$ there exists a family $(\phi^i_j(x))_{j \in J_i}$ of positive formulas such that $O_i = \bigcup_{j \in J_i} [\phi^i_j(x)]$. This means we have $S_x(T) = \bigcup_{i \in I} \{[\phi^i_j(x)] : (i, j) \in J_i \}$. Now the set of positive $L$-formulas with free variables among $x$ has cardinality $\kappa^\omega = \kappa$, hence we may choose a subset $K \subset \bigcup_{i \in I} \{i \} \times J_i$ such that $|K| \leq \kappa$ and $S_x(T) = \bigcup_{(i, j) \in K} [\phi^i_j]$. Let $I'$ be the set of all $i \in I$ such that $(i, j) \in K$: we have $|I'| \leq |K|$. For every $(i, j) \in K$, we have $[\phi^i_j] \subset O_i$, whence $S_x(T) = \bigcup_{(i, j) \in K} [\phi^i_j] = \bigcup_{i \in I'} O_i$, and the proof is complete.

2.2. The spectral category. In this section, we use the maximality of positive types in order to define some "formal" maps between finitary types in different type spaces, using the positive formulas and the relation of the Kaiser hull $T_k$ of $T$ with the positive models of $T$ (Lemma 2.7). We then define a "syntactic" equivalence relation on the global type space $S(T) = \bigcup_{x \in V^*} S_x(T)$, in order to build a small and finitely complete spectral category $\mathcal{S}(T)$, which "classifies" models of $T_k$ as set-valued left exact functors (Proposition 2.9).
Let then $\theta(x, y) \in L^+$ be such that $x \cap y = \emptyset$ and $\theta(x, y)^M$ is the graph of a map in every positive model $M$ of $T$. This is equivalent to saying that the h-inductive sentence $\forall x, y, y' \theta(x, y) \land \theta(x, y') \Rightarrow y = y'\), where $y'$ has the same type as $y$ and is disjoint from $x$ and $y$, is in $T_k$. If $\lambda(y)$ is a positive formula, we consider the following formula:

$$(\theta^* \lambda)(x) = \exists y \ theta(x, y) \land \lambda(y)$$

and if $p \in S_2(T)$ is a positive type such that $\exists y \theta(x, y) \in p$, the following set

$$(\theta_* \lambda)(y) = \{y \in L : \theta^* \lambda(x) \in p\}.$$  

Lemma 2.7. The set $\theta_*(p)$ is a positive type of $T$ in the variables $y$.

Proof. We set $q(y) = \theta_*(p)$. By definition of $p$, there exists a positive model $M$ of $T$ and a point $a \in M_\Sigma$ such that $M \models p(a)$. As $p \in [\exists y \theta(x, y)]$ and $\theta(x, y)^M$ is the graph of a map, there is a unique point $b \in M_\Sigma$ such that $M \models \theta(a, b)$. Suppose that $\lambda(y)$ is any positive formula : as $p$ is a positive type, the following properties are equivalent by maximality of $p$ :

- $\theta^* \lambda(x) \in p$
- $M \models \theta^* \lambda(a)$
- there is $b' \in M_\Sigma$ such that $M \models \theta(a, b') \land \lambda(b')$
- $M \models \lambda(b)$

the last equivalence coming from the unicity of $b$. In short, we have $q = tp^+_M(b)$, so $q$ is a complete type over $T$.

The proof shows in particular that $\theta_* (p)$ does not depend on a realisation of $p$. Now let $\varphi(x)$ and $\psi(y)$ be two positive formulas such that $\varphi(x)^M$ is in the domain of $\theta(x, y)^M$ and $\psi(y)^M$ contains the codomain of $\theta(x, y)^M$ in every positive model $M$ of $T$. This is equivalent to saying that the following additional h-inductive sentences are in $T_k$

- $\forall x \varphi(x) \Rightarrow \exists y \theta(x, y)$
- $\forall x, y \theta(x, y) \Rightarrow \psi(y)$.

Now consider the global type space $S(T) = \bigcup_{x \in V} S_x(T)$ and define an equivalence relation $\sim$ on $S(T)$ by putting, for $p(z), q(w) \in S(T)$, $p(z) \sim q(w)$ if and only if $z$ and $w$ have the same sorting and $q(w) = p(w/z)$. On the quotient set $S(T)/\sim$, denoting by $p(z)_*$ the equivalence class of $p(z)$, the formula $\varphi(x)$ defines a subset $\varphi(x)_* = \{p(x)_* : \varphi(x) \in p\} = [\varphi(x)]/\sim$, and of course similarly for $\psi(y)$. By what precedes the formula $\theta(x, y)$ defines a map $\theta(x, y)_* : \varphi(x)_* \rightarrow \psi(y)_*$, if we set

$$\theta(x, y)_*(p(x)_*) = (\theta_* (p))(y)_*.$$  

We may compose the maps as follows : if $\chi(y, z)_* : \psi(y)_* \rightarrow \lambda(z)_*$ is another such map, we may suppose that $z$ is disjoint from $x$ as well and we easily see that $\chi(y, z)_* \circ \theta(x, y)_* = (\exists y \theta(x, y) \land \chi(y, z))_*$. If $z$ is a tuple of variables of the same type as and disjoint from $x$, the formula $z = x$ defines the identity map $1_\varphi(x)_* = (z = x)_* : \varphi(x)_* \rightarrow \varphi(x)_*$, so the sets of the form $\varphi(x)_*$ and the maps of the form $\theta(x, y)_*$ form a small category.

Definition 2.8. We say that the sets $\varphi(x)_*$ and maps $\theta(x, y)_*$ as before are the spectral sets and maps of $T$ and we call their category the spectral category of $T$, which we note $SC(T)$. 

Notice that \( SC(T) \) is a small subcategory of \( \text{Sets} \). There is only one positive type in \( S_0(T) = S_0(T) \), which is the set of positive sentences true in any positive model of \( T \); we have \( \top = S_0(T) \) and for every spectral set \( \varphi(x)_* \), the formula \( \varphi(x) \) itself defines a unique spectral map \( \varphi(x)_* : \varphi(x)_* \to \top = S_0(T)_* \). This leads us to the

**Proposition 2.9.** The spectral category \( SC(T) \) has all finite projective limits, and the category \( \mathcal{M}(T_k) \) of models of \( T_k \) is equivalent to the category \( \text{Lex}(SC(T), \text{Sets}) \) of left exact functors \( SC(T) \to \text{Sets} \).

**Proof.** First we show that \( SC(T) \) has finite projective limits. It suffices to show that it has a terminal object, products of any two objects and equalisers. The existence of a terminal object \( (\top(\theta))_* \) has just been treated, so let \( \varphi(x)_*, \psi(y)_* \) be two spectral sets: changing variables if necessary, we may suppose that \( x \cap y = \emptyset \), so the spectral maps \((\exists y \varphi(x) \land \psi(y))_* : (\varphi(x) \land \psi(y))_* \to \varphi(x)_*, \varphi(x) \land \psi(y))_* : (\varphi(x) \land \psi(y))_* \to \psi(y)_* \) define a product of \( \varphi(x)_* \) and \( \psi(y)_* \) in \( SC(T) \) (details are straightforward and left to the reader). Analogously, if \( \theta(x,y)_*, \theta'(x,y)_* : \varphi(x)_* \to \psi(y)_* \) are two parallel spectral maps \((x \cap y = \emptyset) \), it is easy to see that the spectral subset \((\exists y \theta(x,y) \land \theta'(x,y))_* \) of \( \varphi(x)_* \) is an equaliser of \( \theta_* \) and \( \theta'_* \) : \( SC(T) \) is finitely complete.

Secondly, we show the equivalence of categories, starting with a model \( M \) of \( T_k \), and defining a functor \( A^M : SC(T) \to \text{Sets} \) in the most natural way : we set \( A^M(\varphi(x)_*) = \varphi(x)^M \). This indeed defines a functor precisely because \( M \) is a model of \( T_k \) : spectral maps become maps in \( \text{Sets} \) through \( A^M \); that \( A^M \) is left exact is obvious, if again we restrict our attention to its preserving the terminal object, binary products and equalisers. If \( g : M \to N \) is an \( L \)-homomorphism and \( \theta(x,y)_* : \varphi(x)_* \to \psi(y)_* \) is a spectral map, we have the induced map \( g_x : M_x \to N_x \) and \( g_x(\varphi(x)^M) \subseteq \varphi(x)^N \), hence if we define \( A^g_{\varphi(x)_*} \) as the restriction of \( g_x \) to \( \varphi(x)^M \) the following diagram commutes:

\[
\begin{array}{ccc}
\varphi(x)^M = A^M\varphi(x)_* & \xrightarrow{A^M\theta(x,y)_*=\theta(x,y)^M} & A^M\psi(y)_* = \psi(y)^M \\
\downarrow^{A^g_{\varphi(x)_*}} & & \downarrow^{A^g_{\psi(y)_*}} \\
A^N\varphi(x)_* & \xrightarrow{A^N\theta(x,y)_*=} & A^N\psi(y)_*,
\end{array}
\]

i.e. \( A^g \) is a natural transformation from \( A^M \) into \( A^N \), and this defines a functor \( A : \mathcal{M}(T_k) \to \text{Lex}(SC(T), \text{Sets}) \), \((g : M \to N) \mapsto (A^g : A^M \to A^N) \). If \( g \neq g' : M \to N \) are two homomorphisms in \( \mathcal{M}(T_k) \), there is a point \( a \) from \( M \), say \( a \in M_x \), such that \( g_x(a) \neq g'_x(a) \) in \( N_x \). By definition of \( A^g \), we have \( A^g_{\tau(x)_*} = g_x \neq A^g_{\tau(x)_*} \), hence \( A^g \neq A^{g'} \), so \( A \) is faithful. If \( \tau : A^M \to A^N \) is a natural transformation, let \( s \) be a sort symbol and \( x \) a variable of sort \( s \) : we have \( \tau_{\top(x)_*} : A^M\top(x)_* = M_x \to A^N\top(x)_* = N_x \) and setting \( g_s = \tau_{\top(x)_*} \) defines an \( L \)-homomorphism \( g : M \to N \) such that \( A^g = \tau \), because \( \tau \) is a transformation (details are omitted) : \( A \) is full.

As for essential surjectivity, if \( F : SC(T) \to \text{Sets} \) is a left exact functor, for each sort symbol \( s \) we put \( M^F_s = A(\top(x)_s) \) with any \( x \) of sort \( s \). For every finite tuple of variables \( x \), we have a canonical bijection \( \tau_x \) between \( M_x \) and \( F\top(x)_* \) : the preimage of any subset \( S \) of \( F\top(x)_* \) under this isomorphism we call the transpose of \( S \). If \( f \) is a function symbol of arity \( s = (s_i : i < m) \) and sort \( t \) and \( r \) a relation symbol of sort \( s \), let \( x \) be an \( s \)-tuple of variables and \( y \) a distinct variable of sort \( t \) : we define \( M^F_{xy} \) as the map with
graph the "transpose" in $\prod_{i<m} M^F_i$ of the image of the injective map $F(y = f(x)_s)$, and $M^F_s$ similarly as the transpose of the image of $A(x' = x \land r(x)_s)$, where $x'$ is a new tuple of variables of type $s$, disjoint from $x$. For every positive formula $\varphi(x)$, we have in $SC(T)$ a canonical monomorphism $i_\varphi = (x' = x \land \varphi(x))_s : \varphi(x)_s \to \top(x)_s$ (with $|x'| = |x|$ and $x \cap x' = \emptyset$); by induction on the complexity of $\varphi$ one checks that $\varphi(x)^M = \tau_x^{-1}(Ai_\varphi(Ai_\varphi(x)_s))$. Now let $\chi = \forall x(\varphi(x)) \Rightarrow (\psi(x))$ be a basic h-inductive sentence of $T_k$ and $y$ a new distinct $|x|$-tuple of variables: we have the spectral map $\theta_y = (y = x \land \varphi(x))_s \in SC(T)$, and as $i_\psi \circ \theta_y = i_\varphi$, we have $Ai_\varphi(Ai_\psi(x)_s) \subset Ai_\psi(Ai_\varphi(x)_s)$, whence $\varphi(x)^{M^F} = \tau_x^{-1}(Ai_\varphi(Ai_\psi(x)_s)) \subset \tau_y^{-1}(Ai_\psi(Ai_\varphi(x)_s)) = \psi(x)^{M^F}$, and $M^F \models \chi$. As this is true for every basic $\chi \in T_k$, we have $M^F \models T_k$. By the "full and faithful" part of the proof, we may now consider the functor $A^{M^F} : SC(T) \to \mathbf{Sets}$, $\varphi(x)_s \mapsto \varphi(x)^{M^F}$; by what precedes $\tau^{-1}$ defines a natural isomorphism from $F$ into $A^{M^F}$, so $A$ is essentially surjective, and $A$ is an equivalence of categories. \hfill $\Box$

Remark 2.10. Any h-inductive theory may be construed as a "geometric theory" in the sense of [5], X.3. The category $SC(T)$ is a "concrete" equivalent to the syntactic category constructed in [5], X.5 and Proposition 2.3 is an analogue of Lemma X.5.1. However here we are interested in positive models of $T$ (or equivalently of $T_k$) so we will not reproduce a construction of a classifying topos of $T_k$ as in [5] X.6.

2.3. Positive models as continuous functors. We translate the information contained in the spectral topology into a Grothendieck (pre)topology $G(T)$ on $SC(T)$, and we establish in Theorem 2.11 that positive models of $T$ are essentially the $G(T)$-continuous left exact functors (Theorem 2.11). Standard considerations from categorical logic allow us to rephrase this result in the language of Grothendieck topoi: we show that $G(T)$ is in fact a (basis for a) Grothendieck topology and that the category of sheaves over the site $(SC(T), G(T))$ is a classifying topos for positive models of $T$.

We define a Grothendieck pretopology $G(T)$ on $SC(T)$ by putting as covers those families $(\theta_i(x_i,y)_s : \varphi_i(x_i)_s \to \psi(y)_s)_{i \in I}$ such that $x_i \cap y = \emptyset$ for all $i \in I$ and such that for every positive model $M$ of $T$, we have $\psi(y)^M = \bigcup \{(\exists x_i \theta_i(x_i,y) \land \varphi_i(x_i))_s : i \in I\}$ (the subset $(\exists x_i \theta_i(x_i,y) \land \varphi_i(x_i))_s^M$ of $\psi(y)^M$ is of course the image of $\varphi_i(x_i)^M$ under $\theta_i(x_i,y)^M$). If one does not like this semantic definition, remember that for every $i \in I$ we have a well defined map $[\theta_i(x_i,y)] : [\varphi_i(x_i)] \to [\psi(y)]$ between basic spectral open sets, so we may rephrase the condition as $[\psi(y)] = \bigcup \{(\exists x_i \theta_i(x_i,y) \land \varphi_i(x_i))_s : i \in I\}$.

Theorem 2.11. The category $M^+(T)$ of positive models of $T$ is equivalent to the full subcategory of left exact functors $SC(T) \to \mathbf{Sets}$, which are continuous for $G(T)$.

Proof. Suppose that $M \models T$ is a positive model of $T$. As $M \models T_k$, by Theorem 2.10 $F(M) = A^M$ is at least a left exact functor and if $(\theta_i(x_i,y)_s : \varphi_i(x_i)_s \to \psi(y)_s)^F$ is a cover in $G(T)$, for every $i$ we have $A^M(\theta_i(x_i,y)_s) = \theta_i(x_i,y)^M$, $A^M(\varphi_i(x_i)_s) = \varphi_i(x_i)^M$ and $A^M(\psi(y)_s) = \psi(y)^M$, so $A^M$ is continuous by definition of $G(T)$.

Reciprocally, if $A : SC(T) \to \mathbf{Sets}$ is left exact and continuous for $G(T)$, let $\varphi(x) \in L^+$ : by Fact 1 in every positive model $M$ of $T$, we have $M_x = \varphi(x)^M \cup \{\psi(x)^M : \psi \in Rest_T(\varphi)\}$, so if $y$ is a new tuple of distinct variables with $|y| = |x|$, the set $\{(y = x \land \varphi(x))_s : \varphi(x)_s \to \top(y)_s\} \cup \{(y = x \land \psi(x))_s : \psi(x)_s \to \top(y)_s : \psi \in Rest_T(\varphi)\}$ is a cover in $G(T)$. By hypothesis, $A$ transforms this cover into a set cover, so $A^M - \varphi(x)^{M^A} = \varphi(x)^{M^F}$.
Suppose \( \psi \in \text{Res}_T(\varphi) \) : this is true for every \( \varphi(x) \in L^+ \), hence \( M^A \) is a positive model of \( T \), and the equivalence \( F \) restricts to an equivalence between \( \mathcal{M}^+(T) \) and continuous left exact functors on \( SC(T) \). \( \square \)

**Lemma 2.12.** The pretopology \( G(T) \) is a basis for a Grothendieck topology on \( SC(T) \).

**Proof.** Let \( \theta(x,y)_*: \varphi(x)_* \to \psi(y)_* \) be an isomorphism in \( SC(T) \). If \( M \models T \), then \( M \models T \), so the functor \( A^M \) of Proposition 2.9 defines a bijection \( \theta^M: \varphi(x)^M \to \psi(y)^M \), hence \( \psi(y)^M = (\exists x \varphi(x) \wedge \theta(x,y))^M \), and by definition \( \{ \theta_* \} \in G(\psi(y)_*) \); this takes care of isomorphisms.

Let \( X = (\theta_i(x_i,y)_*: \varphi_i(x_i)_* \to \psi(y)_*)_{i \in I} \in G(\psi(y)_*) \), \( \theta(x,y)_*: \varphi(x)_* \to \psi(y)_* \) and for every \( i \in I \), a pullback in \( SC(T) \) as in the following diagram:

\[
\begin{array}{ccc}
\chi_i(z_i)_* & \xrightarrow{\mu_i(z_i,x)_*} & \varphi(x)_* \\
\lambda_i(z_i,x)_* \downarrow & & \downarrow \theta(x,y)_* \\
\varphi_i(x_i)_* & \xrightarrow{\theta(x,y)_*} & \psi(y)_*.
\end{array}
\]

If \( M \models T \), as \( A^M \) is left exact by Proposition 2.9 the following diagram is a pullback in \( \text{Sets} \):

\[
\begin{array}{ccc}
\chi_i(z_i)^M & \xrightarrow{\mu_i(z_i,x)^M} & \varphi(x)^M \\
\lambda_i(z_i,x)^M \downarrow & & \downarrow \theta(x,y)^M \\
\varphi_i(x_i)^M & \xrightarrow{\theta(x,y)^M} & \psi(y)^M.
\end{array}
\]

Suppose \( a \in \varphi(x)^M \) : we have \( \theta^M(a) \in \psi(y)^M \), hence by hypothesis on \( X \) and definition of \( G(T) \) there exists \( i \in I \) and \( b \in \varphi(x_i)^M \) such that \( \theta_i^M(b) = \theta^M(a) \). As \( \chi(z_i)^M \) is a pullback, to exists a \( c \in \chi(z_i)^M \) such that \( \mu_i^M(c) = a \), hence \( a \in (\exists z_i \chi(z_i) \wedge \mu_i(z_i,x))^M \), and we get \( \varphi(x)^M = \bigcup ((\exists z_i \chi(z_i) \wedge \mu_i(z_i,x))^M : i \in I) \), which means that \( G(T) \) is "stable under pullbacks".

Finally, with the same family \( X \) as before, let for every \( i \in I \), \( Y_i = (\lambda_i(z_i,x)_*: \chi_j(z_j)_* \to \varphi_i(x_i)_*)_{j \in J_i} \in G(\varphi_i(x_i)_*) \) a covering family. If \( M \models T \) and \( a \in \psi(y)^M \), there exists \( i \in I \) and \( b \in \varphi_i^M \) such that \( a = \theta_i^M(b) \), so there is a \( j \in J_i \) and a \( c \in \chi_j^M \) such that \( b = \lambda_j(c) \), thus we have \( a = (\theta_i^M \circ \lambda_j^M)(c) \), and this means that \( \psi(y)^M = \bigcup (\exists z_i \chi_j(z_i) \wedge (\exists x_i \theta_i(x_i) \wedge \lambda_j^M(z_j,x_i)) : i \in I, j \in J_i) \), which is equivalent to saying that the family \( \lambda_j(z_j)_*: \chi_j(z_j)_* \to \varphi_i(x_i)_*, i \in I, j \in J_i \) is a G-cover of \( \psi(y)_* \), and this takes care of "composition" of basic families : \( G(T) \) is a basis for a Grothendieck topology on \( SC(T) \). \( \square \)

**Lemma 2.13.** If \( C \) is a small category, \( K \) is a basis for a Grothendieck topology on \( C \) and \( J \) is the topology generated by \( K \), a functor \( F: C \to \text{Sets} \) is continuous for \( K \) if and only if \( F \) is continuous for \( J \).

**Proof.** First, if \( G \) is any Grothendieck topology on \( C \), \( F \) is continuous for \( G \) if and only if for every object \( c \) of \( C \) and every sieve \( S \in G(c) \), we have \( Fc = \bigcup \exists f \) \( f \in F(c) \), where \( \exists f \) denotes the image of \( Ff \).

Suppose \( F \) is continuous for \( K \) and let \( S \in J(c) \) : there exists a basic cover \( R \in K(c) \)
such that $R \subset S$, and by hypothesis we have $Fc = \bigcup_{f \in R} \exists Ff$, whence $Fc = \bigcup_{f \in S} \exists Ff$; as this is true for any covering sieve $S$ of $J$, $F$ is continuous for $J$.

Reciprocally, suppose $F$ is continuous for $J$ and let $R \in K(c)$ be a basic cover of $K$. We consider the sieve $(R) = \{ f \circ g : f \in R \}$ generated by $R$. As $F$ is continuous for $J$, we have $Fc = \bigcup_{f \in R} \exists Ff$. If $x \in Fc$, there exists $f \circ g \in (R)$ such that $x \in \exists Ff \circ g$. If $x \in Fc$, there exists $f \in R$ such that $x \in \exists Ff$. This is true for any basic cover $R \in K$, so $F$ is continuous for $K$. □

**Corollary 2.14.** The category $\text{Sh}(SC(T), G(T))$ of sheaves over the site $(SC(T), G(T))$ is a classifying topos for positive models of $T$ (i.e., $\mathcal{M}^+(T)$ is equivalent to the category of points of $\text{Sh}(SC(T), G(T))$).

**Proof.** Let $F : SC(T) \to \text{Sets}$ be a functor. As $SC(T)$ is finitely complete, by Fact 1.8 $F$ is left exact if and only if $F$ is flat. By Lemma 2.12, $G(T)$ is a basis for a Grothendieck topology on $SC(T)$ which we note $J(T)$; by Lemma 2.13, $F$ is continuous for $J(T)$ if and only if $F$ is continuous for $G(T)$ in the sense of section 1. We conclude that $F$ is left exact and continuous for $G(T)$ if and only if $F$ is flat and continuous for $J(T)$; by Fact 1.9, the category of points of $\text{Sh}(SC(T), G(T))$ is equivalent to the category of left exact continuous and set-valued functors from $C$ to $\text{Sets}$, which is equivalent by Theorem 2.11 to the category of positive models of $T$. □

**References**

[1] M. Belkacemi, *Positive model theory and amalgamations*, Notre Dame J. Form. Log., in press.
[2] I. Ben Yaacov, *Positive model theory and compact abstract theories*, J. Math. Log. 3, 85-118 (2003).
[3] I. Ben Yaacov and B. Poizat, *Fondements de la logique positive*, J. Symbolic Logic 72 (4), 1141-1162 (2007).
[4] W. Hodges, *Model theory*, Encyclopedia Math. Appl. 42, Cambridge University Press, 1993.
[5] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic*, Springer-Verlag, 1992.
[6] M. Makkaï and G. Reyes, *First order categorical logic*, Lecture notes in Math. 611, Springer-Verlag, 1977.