INVERSE NODAL PROBLEMS FOR STURM-LIOUVILLE EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this paper, a Sturm-Liouville problem with some nonlocal boundary conditions of the Bitsadze-Samarskii type is studied. We show that the coefficients of the problem can be uniquely determined by a dense set of nodal points. Moreover, we give an algorithm for the reconstruction of the potential function and some other coefficients in the boundary conditions.

1. Introduction

The inverse nodal problem for a Sturm-Liouville operator consists in reconstructing the operator from zeros of its eigenfunctions, namely nodal points. This problem was studied firstly by McLaughlin in 1988 [1]. She showed that the potential of a Sturm-Liouville problem with Dirichlet boundary conditions can be determined by a given dense subset of nodal points. Immediately after, Hald and McLaughlin give some numerical schemes for the reconstruction of the potential [2]. In 1997, X.F. Yang gave a solution algorithm of an inverse nodal problem for the Sturm-Liouville operator with separated boundary conditions [3]. Inverse nodal problems for Sturm-Liouville operators with the classical boundary conditions have been studied in the papers ([4]-[31]).

Nonlocal boundary conditions appear when we cannot measure data directly at the boundary. This kind conditions arise in various applied problems of biology, biotechnology, physics and etc. As it is known there are two kinds of nonlocal boundary conditions. One class of them is called integral type conditions, and the other is the Bitsadze-Samarskii-type conditions. Bitsadze and Samarskii are considered the originators of such conditions. Nonlocal boundary conditions of the Bitsadze-Samarskii type were first applied to elliptic equations by them [32]. Some important results on the properties of eigenvalues and eigenfunctions of nonlocal boundary value problems for Sturm-Liouville type operators have been published in various publications (see, for example, [33, 34] and the references therein).

Some inverse problems for a class of Sturm-Liouville operators with nonlocal boundary conditions are investigated in [35, 36]. In the literature, there are only a few studies about inverse nodal problems with nonlocal boundary conditions. Moreover all of them include integral type conditions. Inverse nodal problems for this-type operators with different nonlocal integral boundary are studied in ([37], [10]). Especially, C.F. Yang et all. solved inverse-nodal Sturm-Liouville problems with nonlocal integral-type boundary conditions at only one or both end-points (see [38] and [10]).

In the present paper, we consider Sturm-Liouville problems under some the Bitsadze-Samarskii type nonlocal boundary conditions and obtain the uniqueness
of coefficients of the problem according to a set of nodal points. Moreover, we give an algorithm for the reconstruction of these coefficients.

Let us consider the following boundary value problem

\[ \ell y := -y'' + q(x)y = \lambda y, \quad x \in \Omega = (0, 1) \]

\[ U(y) := y'(0) + hy(0) - \gamma_0 y(\xi_0) = 0, \]

\[ V(y) := y'(1) + Hy(1) - \gamma_1 y(\xi_1) = 0, \]

where \( q(x) \) is a real valued continuously differentiable function; \( h, H \in \mathbb{R} \cup \{ \infty \} \) and \( \gamma_i \neq 0 \) are real numbers for \( i = 0, 1 \); \( \xi_i \) are rational numbers in \( (0, 1) \) for \( i = 0, 1 \) and \( \lambda \) is the spectral parameter.

(2) and (3) are nonlocal conditions of a Bitsadze-Samarskii type. It is clear that if \( \xi_0 = 0 \) and \( \xi_1 = 1 \), (2) and (3) are not other than the classical separated boundary conditions. On the other hand, while \( \xi_0 = 1 \) and \( \xi_1 = 0 \) (2) and (3) turn into non-separated conditions. Inverse nodal problems for this type of boundary conditions are studied by C.F. Yang [41]. Therefore, we focus on the case \( \xi_i \in (0, 1) \) in our investigation. In fact, since \( \xi_0 \) and \( \xi_1 \) are arbitrary rational numbers, the problem we are considering involves a fairly large class of nonlocal boundary conditions.

The main goal of this paper is to solve inverse nodal problems for (1)-(3) in each of the following cases

i) \( h, H \in \mathbb{R} \),
ii) \( h = \infty, H \in \mathbb{R} \),
iii) \( h \in \mathbb{R}, H = \infty \).

We note that if \( h = \infty, H \in \mathbb{R} \), and \( h \in \mathbb{R}, H = \infty \) the boundary conditions can be written as

\[ y(0) = 0, \]
\[ y'(1) + Hy(1) = \gamma_1 y(\xi_1) \]

and

\[ y'(0) + hy(0) = \gamma_0 y(\xi_0), \]
\[ y(1) = 0, \]

respectively.

2. Spectral properties of the problem

Let \( S(x, \lambda) \) and \( C(x, \lambda) \) be the solutions of (1) under the initial conditions

\[ S(0, \lambda) = 0, S'(0, \lambda) = 1 \]
\[ C(0, \lambda) = 1, C'(0, \lambda) = 0 \]

respectively. It can be calculated that \( C(x, \lambda) \) and \( S(x, \lambda) \) satisfy the following asymptotic relations for \( |\lambda| \to \infty \) (see [20] and [38])

\[ C(x, \lambda) = \cos kx + \frac{\sin kx}{k} Q(x) + \frac{\cos kx}{k^2} q_1(x) + O \left( \frac{1}{k^3} \exp |\tau| x \right), \]

\[ S(x, \lambda) = \frac{\sin kx}{k} - \frac{\cos kx}{k^2} Q(x) + O \left( \frac{1}{k^3} \exp |\tau| x \right), \]

where \( \sqrt{\lambda} = k, \tau = |\text{Im} k|, Q(x) = \frac{1}{2} \int_0^x q(t) dt \) and \( q_1(x) = \frac{q(x) - q(0)}{4} - \frac{1}{8} \left( \int_0^x q(t) dt \right)^2. \)
The characteristic function of problem (1)-(3)

\[ \Delta(\lambda) = \det \begin{pmatrix} U(C) & U(S) \\ V(C) & V(S) \end{pmatrix} \]

and the zeros of the function \( \Delta(\lambda) \) coincide with the eigenvalues of the problem (1)-(3). Clearly, \( \Delta(\lambda) \) is entire function and so the problem has a discrete spectrum.

Let \( \{\lambda_n\}_{n \geq 0} \) be the set of eigenvalues and \( \varphi(x, \lambda_n) \) be the eigenfunction corresponding to the eigenvalue \( \lambda_n \). Some asymptotic formulas of \( \lambda_n \) and \( \varphi(x, \lambda_n) \) are given in the following Lemmas.

**Lemma 1.** The numbers \( \{\lambda_n\}_{n \geq 0} \) are real for sufficiently large \( n \) and they satisfy the following asymptotic relation for \( n \to \infty \):

\[ \sqrt{\lambda_n} = k_n = k_0^0 + \frac{\kappa_n}{n\pi} + o\left(\frac{1}{n}\right) \]

where \( k_0^0 = \begin{cases} n\pi, & \text{if } h, \ H \in \mathbb{R}, \\ (n + \frac{1}{2})\pi, & \text{if } h = \infty, \ H \in \mathbb{R}, \quad \text{or } H = \infty, \ h \in \mathbb{R}, \end{cases} \)

\[ \kappa_n = \begin{cases} Q(1) + H - h - (-1)^n [\gamma_1 \cos (n\pi k_1) - \gamma_0 \cos (n\pi (1 - \xi_0))], & \text{if } h, \ H \in \mathbb{R}, \\ H + Q(1) - (-1)^n \gamma_1 \sin \left((n + \frac{1}{2})\pi k_1\right), & \text{if } h = \infty, \ H \in \mathbb{R}, \\ Q(1) - h + \gamma_0 \cos \left((n + \frac{1}{2})\pi \xi_0\right), & \text{if } h \in \mathbb{R}, \ H = \infty. \end{cases} \]

**Proof.** We give the proof for the case: \( h, \ H \in \mathbb{R} \); the other cases are similar. From (4), we have that

\[ \Delta(\lambda) = hS'(1, \lambda) - \gamma_0 C(\xi_0, \lambda)S'(1, \lambda) + HhS(1, \lambda) - \gamma_0 HC(\xi_0, \lambda)S(1, \lambda) 
- h\gamma_1 S(\xi_1, \lambda) + \gamma_1 \gamma_0 C(\xi_0, \lambda)S(\xi_1, \lambda) - C'(1, \lambda) + \gamma_0 C'(1, \lambda)S(\xi_0, \lambda) 
- HC(1, \lambda) + H\gamma_0 S(\xi_0, \lambda)C(1, \lambda) + \gamma_1 C(\xi_1, \lambda) - \gamma_1 \gamma_0 S(\xi_0, \lambda)C(\xi_1, \lambda). \]

Using (4) and (5), we obtain the following asymptotic formula for \( \Delta(\lambda) \) as \( k \to \infty \):

\[ \Delta(\lambda) = k \sin k + \cos k Q(1) + \frac{\sin k}{k} q_1(1) + \gamma_0 [C'(1, \lambda)S(\xi_0, \lambda) - C(\xi_0, \lambda)S'(1, \lambda)] 
+ h \left[ \cos k + \frac{\sin k}{k} Q(1) \right] + Hh \left[ \frac{\sin k}{k} - \cos k Q(1) \right] 
- H \left[ \cos k + \frac{\sin k}{k} Q(1) - \frac{\cos k}{k^2} q_1(x) \right] + H\gamma_0 [S(\xi_0, \lambda)C(1, \lambda) - C(\xi_0, \lambda)S(1, \lambda)] 
+ \gamma_1 \gamma_0 [C(\xi_0, \lambda)S(\xi_1, \lambda) - S(\xi_0, \lambda)C(\xi_1, \lambda)] 
+ \gamma_1 \left[ \cos k q_1 + \frac{\sin k q_1}{k} Q(1) + \frac{\cos k q_1}{k^2} q_1(x) \right] 
- \gamma_1 \left[ \frac{\sin k q_1}{k} - \frac{\cos k q_1}{k^2} Q(1) \right] + O \left( \frac{1}{k^3 \exp |\tau|} \right) \]

and so

\[ \Delta(\lambda) = k \sin k + w \cos k + \gamma_1 \cos (k \xi_1) - \gamma_0 \cos k(1 - \xi_0) + O (\exp |\tau|), \]

where \( w = h - H - Q(1) \). Let \( G_n(\varepsilon) = \{ k : |k - n\pi| < \varepsilon \} \) for \( n = 1, 2, \ldots \). It follows from (7) that there exist some \( M(\varepsilon) > 0 \) such that \( |\Delta(\lambda)| \geq M(\varepsilon) |k| \exp |\tau| \) for sufficiently large \( |k| \) in \( G_n(\varepsilon) \). Therefore \( \lambda_n \) must be a real number for sufficiently large \( n \).
Moreover, if we apply Rouché theorem to $h_1(\lambda) = k \sin k$ and $h_2(\lambda) = w \cos k + \gamma_1 \cos (k \xi_1) - \gamma_0 \cos k (1 - \xi_0) + O(\exp |\tau|)$ on $\partial G_\varepsilon$ for sufficiently small $\varepsilon$, we can see that zeros of $\Delta(\lambda)$ satisfy

$$k_n = n \pi + \mu_n, \quad \mu_n = o(1), \quad n \to \infty.$$ 

It follows from (7) that

$$\sin (n \pi + \mu_n) + O\left(\frac{1}{n}\right) = 0.$$ 

Hence $\sin (\mu_n) = O\left(\frac{1}{n}\right)$ and so $\mu_n = O\left(\frac{1}{n}\right)$. Thus

$$k_n = n \pi + O\left(\frac{1}{n}\right), \quad n \to \infty.$$ 

Using (7) and (8) together, we get

$$\sin k_n + \frac{w}{n \pi} \cos k_n + \frac{\gamma_1}{n \pi} \cos (k_n \xi_1) - \frac{\gamma_0}{n \pi} \cos (k_n (1 - \xi_0)) + o\left(\frac{1}{n}\right) = 0.$$ 

Therefore, we obtain

$$\tan k_n = \frac{\frac{w}{n \pi} - \frac{\gamma_1}{n \pi} \cos (k_n \xi_1)}{\frac{\gamma_0}{n \pi} \cos (k_n (1 - \xi_0)) + o\left(\frac{1}{n}\right)}.$$ 

On the other hand, we have

$$\cos \left(\frac{k_n \xi_1}{n \pi \cos k_n}\right) = (-1)^n \frac{\left(\frac{\gamma_0}{n \pi} \cos (k_n (1 - \xi_0)) + o\left(\frac{1}{n}\right)\right)}{\left(\frac{n \pi \xi_1}{n \pi \cos k_n}\right)} + o\left(\frac{1}{n}\right)$$

and

$$\cos \left(\frac{k_n (1 - \xi_0)}{n \pi \cos k_n}\right) = (-1)^n \frac{\left(\frac{\gamma_0}{n \pi} \cos (k_n (1 - \xi_0)) + o\left(\frac{1}{n}\right)\right)}{\left(\frac{n \pi (1 - \xi_0)}{n \pi \cos k_n}\right)} + o\left(\frac{1}{n}\right).$$

Using (10) and (11) in (9), we get

$$\tan k_n = \frac{\frac{w}{n \pi} - \frac{\gamma_1}{n \pi} \cos (n \pi \xi_1)}{\frac{\gamma_0}{n \pi} \cos (n \pi (1 - \xi_0)) + o\left(\frac{1}{n}\right)}.$$

Using Taylor’s expansion of Arctangent, the proof can be concluded. \hfill \Box

It is clear that

$$\varphi(x, \lambda_n) = U(S(x, \lambda_n))C(x, \lambda_n) - U(C(x, \lambda_n))S(x, \lambda_n)$$

From (12) and Lemma 1, we can prove easily the following lemma:

**Lemma 2.** The asymptotic formula

$$\varphi(x, \lambda_n) = \begin{cases}
\cos k_n x + \frac{(Q(x)-h)}{k_n} \sin k_n x + \frac{\gamma_0}{k_n} \sin k_n (x - \xi_0) + O\left(\frac{1}{k_n^2} \exp |\tau| x\right), & \text{for } h, H \in \mathbb{R},
\
\sin k_n x + \frac{\gamma_0}{k_n} Q(x) + O\left(\frac{1}{k_n^3} \exp |\tau| x\right), & \text{for } h = \infty, H \in \mathbb{R},
\
\sin k_n (1-x) + \frac{\gamma_0}{k_n} Q(x) + O\left(\frac{1}{k_n^3} \exp |\tau| x\right), & \text{for } h \in \mathbb{R}, H = \infty
\end{cases}$$

is valid for sufficiently large $n$. 
3. Inverse nodal problems: Uniqueness and reconstruction

We can see from Lemma 2 that $\varphi(x, \lambda_n)$ has exactly $n - 1$ nodal points in $(0, 1)$. Let $X = \{x_n^j : n = 0, 1, 2, \ldots \text{ and } j = 1, 2, \ldots, n - 1\}$ be the set of nodal points. We assume that $\int_0^1 q(x) dx = 0$. Otherwise, the term $q(x) - \int_0^1 q(x) dx$ is determined uniquely, instead of $q(x)$.

**Lemma 3.** The elements of $X$ satisfy the following asymptotic formulas for sufficiently large $n$,

$$
x_n^j = \begin{cases} 
\frac{j + 1/2}{n} + \frac{h - H - (-1)^n A_n (j + 1/2)}{n^2 \pi^2} + \frac{(Q(x_n^j) - h)}{n^2 \pi^2} + \frac{\gamma_0}{n^2 \pi^2} \cos (n\pi \xi_0) + o\left(\frac{1}{n}\right), & \text{if } h, H \in \mathbb{R} \\
\frac{j}{n + \frac{1}{2}} - \frac{H - (-1)^n \gamma_1 \sin \left(\left(n + \frac{1}{2}\right) \pi \xi_1\right)}{(n + \frac{1}{2})^2 \pi^2} - \frac{Q(x_n^j)}{(n + \frac{1}{2})^2 \pi^2} + \frac{\gamma_0}{(n + \frac{1}{2})^2 \pi^2} \cos \left(\left(n + \frac{1}{2}\right) \pi \xi_0\right) + o\left(\frac{1}{n^2}\right), & \text{if } h = \infty, H \in \mathbb{R} \\
\frac{j + \frac{1}{2}}{n + \frac{1}{2}} + \left[h - \gamma_0 \cos \left(\left(n + \frac{1}{2}\right) \pi \xi_0\right)\right] - \frac{j + \frac{1}{2}}{(n + \frac{1}{2})^2 \pi^2} - \frac{Q(x_n^j)}{(n + \frac{1}{2})^2 \pi^2} + \frac{\gamma_0}{(n + \frac{1}{2})^2 \pi^2} \cos \left(\left(n + \frac{1}{2}\right) \pi \xi_0\right) + o\left(\frac{1}{n^2}\right), & \text{if } H = \infty, h \in \mathbb{R}
\end{cases}
$$

where $A_n = [\gamma_1 \cos (n\pi \xi_1) - \gamma_0 \cos (n\pi (1 - \xi_0))]$.

**Proof.** As before, we consider only the first case. One can obtain similarly desired formulas for the other cases. Use the asymptotic formula (13) to get

$$0 = \varphi(x_n^j, \lambda_n) = \cos k_n x_n^j + \left(\frac{Q(x_n^j) - h}{k_n} \right) \sin k_n x_n^j + \frac{\gamma_0}{k_n} \sin k_n (x_n^j - \xi_0) + o\left(\frac{1}{k_n}\right)$$

and so

$$\tan \left(\frac{k_n x_n^j - \pi}{2}\right) = \left(\frac{Q(x_n^j) - h}{k_n} \right) \frac{\gamma_0}{k_n} \frac{\sin k_n (x_n^j - \xi_0)}{\sin k_n x_n^j} + o\left(\frac{1}{k_n}\right).$$

This yields

$$x_n^j = \left(\frac{j + 1/2}{k_n}\right) \pi + \left(\frac{Q(x_n^j) - h}{k_n^2} \right) \gamma_0 \frac{\sin k_n (x_n^j - \xi_0)}{\sin k_n x_n^j} + o\left(\frac{1}{k_n^2}\right).$$

Using $k_n x_n^j = (j + 1/2) \pi + O\left(\frac{1}{n}\right)$, $n \to \infty$ we can show

$$\sin k_n (x_n^j - \xi_0) = \frac{\cos (n\pi \xi_0)}{n^2 \pi^2} + o\left(\frac{1}{n^2}\right)$$

On the other hand, we have

$$\frac{1}{k_n} = \frac{1}{n \pi} \left(1 + \frac{w}{n^2 \pi^2} + \frac{(-1)^n}{n^2 \pi^2} A_n + o\left(\frac{1}{n}\right)\right)$$

$$\frac{1}{k_n^2} = \frac{1}{n^2 \pi^2} + o\left(\frac{1}{n^3}\right)$$


using by Lemma 1. Therefore, it is concluded that,

\[
x_n' = \frac{j + 1/2}{n} + \frac{h - H + (-1)^n A_n (j + 1/2)}{n^2 \pi^2} + \frac{(Q(x_n') - h)}{n^2 \pi^2} + \frac{\gamma_0}{n^2 \pi^2} \cos (n \pi \xi_0) + o \left( \frac{1}{n^2} \right).
\]

According to Lemma 3 the existence of a dense subset \( X_0 \) of \( X \) is obvious.

3.1. The Case \( h, H \in \mathbb{R} \). Consider the problem \( \tilde{L} = \tilde{L} \left( \tilde{q}, \tilde{h}, \tilde{H}, \tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\xi}_0, \tilde{\xi}_1 \right) \) under the same assumptions with \( L \). It is assumed in what follows that if a certain symbol \( s \) denotes an object related to the problem \( L \) then \( \tilde{s} \) denotes the corresponding object related to the problem \( \tilde{L} \).

The following theorem is the first of our main results in this article.

**Theorem 1** (Uniqueness). If \( X_0 = \tilde{X}_0 \) then \( q(x) = \tilde{q}(x) \) a.e. in \((0, 1)\), \( h = \tilde{h}, H = \tilde{H}, \gamma_0 = \tilde{\gamma}_0, \gamma_1 = \tilde{\gamma}_1 \). Thus, the potential \( q(x) \), a.e. in \((0, 1)\), the coefficients \( \gamma_0, \gamma_1 \), \( h \) and \( H \) are uniquely determined by \( X_0 \).

**Proof.**  

**Step 1.** Put \( \xi_0 = \frac{p_0}{r_0} \) and \( \xi_1 = \frac{p_1}{r_1} \), where \( p_i, r_i \in \mathbb{Z} \) for \( i = 0, 1 \). For each fixed \( x \in [0, 1] \), there exists a sequence \( \{x_m'\} \) converges to \( x \). Clearly the subsequence \( \{x_m'\} \) converges also to \( x \) for \( m = 2r_0 r_1 n \). On the other hand, \( \lim_{m \to \infty} A_m = \gamma_1 - \gamma_0 \). Therefore we can see from Lemma 3 the following limit exists and given equality holds:

\[
\lim_{m \to \infty} n^2 \pi^2 \left( x_m' - \frac{j}{m} \right) = f(x) = (h - H + \gamma_1 - \gamma_0) x + Q(x) - h + \gamma_0.
\]

Direct calculations in (15) yield

\[
\gamma_0 - h = f(0),
\gamma_1 - H = f(1),
q(x) = 2 \left( f'(x) + f(0) - f(1) \right).
\]

Since \( X_0 = \tilde{X}_0 \) then \( f(x) = \tilde{f}(x) \) and so \( q(x) = \tilde{q}(x) \), a.e. in \((0, 1)\).

**Step 2.** To show \( \tilde{h} = h \) and \( \gamma_0 = \tilde{\gamma}_0 \) consider a sequence \( \{x_m'\} \subset X_0 \) converges to \( \xi_0 \) and write the equation (11) for \( \varphi(x, \lambda_n) \) and \( \tilde{\varphi}(x, \tilde{\lambda}_n) \):

\[
-\tilde{\varphi}'' (x, \tilde{\lambda}_n) + q(x) \tilde{\varphi} (x, \tilde{\lambda}_n) = \tilde{\lambda}_n \tilde{\varphi} (x, \tilde{\lambda}_n),
-\varphi'' (x, \lambda_n) + q(x) \varphi (x, \lambda_n) = \lambda_n \varphi (x, \lambda_n).
\]

If we apply the procedure:

(i): multiplied by \( \varphi (x, \lambda_n) \) and \( \tilde{\varphi} (x, \tilde{\lambda}_n) \), respectively; (ii): subtracted from each other and (iii): integrated over the interval \((\xi_0, x_m')\) the equality

\[
\varphi' (\xi_0, \lambda_n) \tilde{\varphi} (\xi_0, \tilde{\lambda}_n) - \tilde{\varphi}' (\xi_0, \tilde{\lambda}_n) \varphi (\xi_0, \lambda_n) = (\tilde{\lambda}_n - \lambda_n) \int_{\xi_0}^{x_m'} \tilde{\varphi} (x, \tilde{\lambda}_n) \varphi (x, \lambda_n) \ dx
\]

is obtained. From Lemma 1 the following estimate holds for sufficiently large \( n \)

\[
\varphi' (\xi_0, \lambda_n) \tilde{\varphi} (\xi_0, \tilde{\lambda}_n) - \tilde{\varphi}' (\xi_0, \tilde{\lambda}_n) \varphi (\xi_0, \lambda_n) = o(1), \ n \to \infty.
\]

Using (10) and Lemma 2 we get

\[
\left[ \varphi' (\xi_0, \lambda_n) - \tilde{\varphi}' (\xi_0, \tilde{\lambda}_n) \right] \cos n \pi \xi_0 = o(1), \ n \to \infty.
\]
The last equality yields
\[ (\tilde{h} - h) \cos n\pi \xi_0 + (\gamma_0 - \tilde{\gamma}_0) \cos n\pi \xi_0 = o(1), \ n \to \infty. \]

Therefore, we conclude that \( \tilde{h} = h \) and \( \gamma_0 = \tilde{\gamma}_0 \).

**Step 3.** Finally let us prove \( \tilde{\gamma}_1 = \gamma_1 \) and \( H = \tilde{H} \). Consider another sequence \( \{x_n^i\} \subset X_0 \) converges to \( \xi_1 \). If we apply above procedure but take the integral from \( \xi_1 \) to \( x_n^i \), we get
\[
\tilde{\varphi}'(\xi_1, \tilde{\lambda}_n) \varphi(\xi_1, \lambda_n) - \varphi'(\xi_1, \lambda_n) \tilde{\varphi}(\xi_1, \tilde{\lambda}_n) = o(1), \ n \to \infty
\]
instead of (10). From 3, we have
\[
\varphi'(\xi_1, \tilde{\lambda}_n) \left[ \frac{\varphi'(1, \lambda_n) + H \varphi'(1, \lambda_n)}{\gamma_1} \right] - \varphi'(\xi_1, \lambda_n) \left[ \frac{\varphi'(1, \tilde{\lambda}_n) + \tilde{H} \varphi'(1, \tilde{\lambda}_n)}{\tilde{\gamma}_1} \right] = o(1), \ n \to \infty.
\]

Using Lemma 1 and Lemma 2, it can be calculated that
\[
\sin n\pi \xi_1 \left[ \left( \frac{H - h}{\gamma_1} \right) \left( \frac{\tilde{H} - h}{\tilde{\gamma}_1} \right) \left( \frac{-1)^n}{n\pi} + \frac{\gamma_0}{n\pi} \left( \frac{1}{\gamma_1} - \frac{1}{\tilde{\gamma}_1} \right) \cos n\pi (1 - \xi_0) \right] = o(1), \ n \to \infty.
\]
This yields
\[
\sin n\pi \xi_1 \left[ (-1)^n \left( \frac{H - h}{\gamma_1} \frac{\tilde{H} - h}{\tilde{\gamma}_1} \right) + \gamma_0 \left( \frac{1}{\gamma_1} - \frac{1}{\tilde{\gamma}_1} \right) \cos n\pi (1 - \xi_0) \right] = o(1).
\]
Hence \( \gamma_1 = \tilde{\gamma}_1 \) and \( H = \tilde{H} \). This completes the proof.

**Corollary 1** (Reconstruction algorithm). Let \( X_0, \xi_0 = \frac{p}{r_0} \) and \( \xi_1 = \frac{p}{r_1} \) be given.

Then \( q(x), \gamma_0 - h \) and \( \gamma_1 - H \) can be reconstructed by the following algorithm:

i) Denote \( m = 2r_0 r_1 n \);

ii) Find \( f(x) \) by (13);

iii) Find \( q(x), \gamma_0 - h \) and \( \gamma_1 - H \) by the formulas
\[
q(x) = 2 (f'(x) + f(0) - f(1))
\]
\[
\gamma_0 - h = f(0),
\]
\[
\gamma_1 - H = f(1).
\]

Note that if one of the pairs \((h, H)\) and \((\gamma_0, \gamma_1)\) is given, we can find the other pair.

**Example 1.** Consider the nonlinear BVP
\[
L : \begin{cases}
\ell y := -y'' + q(x)y = \lambda y, & x \in (0, 1) \\
U(y) := y'(0) + hy(0) = \gamma_0 y(\frac{p}{r_0}), \\
V(y) := y'(1) + H y(1) = \gamma_1 y(\frac{p}{r_1}),
\end{cases}
\]
where \( q(x) \in C^1 [0, 1] \), \( \gamma_0, \gamma_1, h, \) and \( soln H \in \mathbb{R} \) are unknown coefficients. Let \( X_0 = \{x_n^i\} \) be the given subset of nodal points which satisfy the following asymptotics
\[
x_n^i = \frac{(j + 1/2)}{n} + \frac{-1 + (-1)^n \left[ 6 \cos \left( \frac{3n\pi}{2} \right) - 3 \cos \left( \frac{5n\pi}{2} \right) \right]}{n^2 \pi^2} \left( j + 1/2 \right) + \frac{\left( \frac{\sin (j+1/2)\pi}{n} - 2\pi \right)}{2n^2 \pi^2} + o \left( \frac{1}{n^2} \right).
\]
Let \( m := 70n. \) One can calculate that,

\[
\lim_{m \to \infty} m^2 \pi^2 \left( x_m^2 - \frac{j}{m} \right) = f(x) = 2x + \left( \frac{\sin \pi x - 2\pi}{2\pi} \right) + 3
\]

According to Theorem 1, we find

\[
q(x) = 2 \left( f'(x) + f(0) - f(1) \right) = \cos \pi x.
\]

and

\[
\gamma_0 - h = f(0) = 2,
\]

\[
\gamma_1 - H = f(1) = 4.
\]

If the pair \((h,H)\) is given as, for example, \( h = 1 \) and \( H = 2 \) then we find \( \gamma_0 = 3 \) and \( \gamma_1 = 6. \)

3.2. The Case \( \infty, H \in \mathbb{R}. \) In this subsection, we consider the equation (11) with one Dirichlet boundary condition (17)

\[
U(y) := y(0) = 0
\]

and with the nonlocal boundary condition (3).

Let \( X_0 \) be a dense nodal points-set. For each fixed \( x \) in \((0,1)\), it can be chosen a sequence \( \{x_n^j\} \subset X_0 \) which converges to \( x. \) Therefore we can show from Lemma 3 that the following limit exists and finite for \( m = 2n \):

\[
\lim_{m \to \infty} \left( m + \frac{1}{2} \right)^2 \pi^2 \left( x_m^j - \frac{j}{m + \frac{1}{2}} \right) = g(x)
\]

\[
= \left( \gamma_1 \sin \frac{\pi}{2}\xi_1 - H \right)x + \frac{1}{2} \int_0^x q(t)dt.
\]

Thus, we can prove the following theorem using methods similar to one in the proof of Theorem 1.

**Theorem 2.** If \( X_0 = \tilde{X}_0 \) then \( q(x) = \tilde{q}(x) \) a.e. in \((0,1)\), \( H = \tilde{H}, \) and \( \gamma_1 = \tilde{\gamma}_1. \)

Moreover if \( X_0 \) and \( \xi_1 = \frac{p_1}{r_1} \) is given, \( q(x) \) and \( \gamma_1 \sin \frac{\pi}{2}\xi_1 - H \) can be reconstructed by the following formulas:

\[
q(x) = 2 \left( g'(x) - g(1) \right),
\]

\[
\gamma_1 \sin \frac{\pi}{2}\xi_1 - H = g(1).
\]

**Example 2.** Consider the nonlocal BVP

\[
L: \begin{cases} -y'' + q(x)y = \lambda y, & x \in (0,1) \\
 y(0) = 0, \\
 y'(1) + 2y(1) - \gamma_1 y\left(\frac{1}{2}\right) = 0, 
\end{cases}
\]

where \( q(x) \in C^1[0,1] \) and \( \gamma_1 \) are unknown real coefficients. Let \( X_0 = \{x_n^j\} \) be the given subset of nodal points which satisfy the following asymptotics

\[
x_n^j = \frac{j}{n + \frac{1}{2}} - \frac{2 - (-1)^n \sin \left( \frac{2n + 1}{2} \pi \right)}{(n + \frac{1}{2})^2 \pi^2} \left( \frac{j}{n + \frac{1}{2}} \right) + \frac{1}{2 \left( n + \frac{1}{2} \right)^3 \pi^3} \left( \sin \frac{j\pi}{n + \frac{1}{2}} + \frac{j\pi}{n + \frac{1}{2}} \left( \frac{j}{2n + 1} - \frac{1}{2} \right) \right) + o \left( \frac{1}{n^2} \right).
\]
To find \( q(x) \) and \( \gamma_1 \) we take \( m = 10n \) and calculate the following limit

\[
\lim_{m \to \infty} \left( m + \frac{1}{2} \right)^2 \pi^2 \left( x^j_m - \frac{j}{m + \frac{1}{2}} \right) = g(x) = \left( 3 \sin \frac{\pi}{3} - 2 \right) x + \frac{\sin \pi x}{2\pi} + \frac{x}{2} \left( \frac{x}{2} - \frac{1}{2} \right)
\]

Thus, we find

\[
q(x) = 2 \left( g'(x) - g(1) \right) = \cos \pi x + x - \frac{1}{2}
\]

\[
\gamma_1 = \frac{g(1) + 2}{\sin \frac{\pi}{2}} = 3
\]

3.3. The Case \( H = \infty, h \in \mathbb{R} \). In this subsection, we consider the equation (11) with nonlocal boundary condition (2) and one Dirichlet boundary condition (18)

\[
V(y) := y(1) = 0,
\]

Let \( m := 2r_0n \). Here, \( r_0 \) denotes the denominators of \( \xi_0 \).

Let \( X_0 \) be a dense nodal points-set. For each fixed \( x \) in \( (0, 1) \), it can be chosen a sequence \( (x^j_n) \subset X_0 \) which converges to \( x \). Therefore we can show from Lemma 3 that

\[
\lim_{m \to \infty} \left( m + \frac{1}{2} \right)^2 \pi^2 \left( x^j_m - \frac{j}{m + \frac{1}{2}} \right) = \psi(x)
\]

\[
= (h - \gamma_0 \cos \frac{\pi}{2} \xi_0)x - h + \gamma_0 \cos \frac{\pi}{2} \xi_0 + \frac{1}{2} \int_0^x q(t)dt.
\]

Thus, we can give the following theorem.

**Theorem 3.** If \( X_0 = \hat{X}_0 \) then \( q(x) = \tilde{q}(x) \) a.e. in \( (0, 1) \), \( h = \tilde{h} \) and \( \gamma_0 = \tilde{\gamma}_0 \). Moreover if \( X_0 \) and \( \xi_0 = \frac{p_a}{r_0} \) are given, \( q(x) \) and \( \tilde{\gamma}_0 \cos \frac{\pi}{2} \xi_0 - h \) can be reconstructed by the following formulae:

\[
q(x) = 2 \left( \psi'(x) + \psi(0) \right),
\]

\[
\gamma_0 \cos \frac{\pi}{2} \xi_0 - h = \psi(0)
\]

**Example 3.** Consider the nonlocal BVP

\[
L : \begin{cases}
\dot{y} := -y'' + q(x)y = \lambda y, & x \in (0, 1) \\
U(y) := y'(0) + y(0) = \gamma_0 \theta(\frac{\pi}{2}), \\
V(y) := y(1) = 0,
\end{cases}
\]

where \( q(x) \in C^1[0, 1] \) and \( \gamma_0 \) are unknown coefficients. Let \( X_0 = \{ x^j_n \} \) be the given subset of nodal points which satisfy the following asymptotics

\[
x^j_n = \frac{j + \frac{1}{2}}{n + \frac{1}{2}} + \left[ 1 - 2 \cos \left( \frac{2n + 1}{3} \pi \right) \right] \frac{j + \frac{1}{2}}{(n + \frac{1}{2})^2 \pi^2} - \frac{1}{(n + \frac{1}{2})^2 \pi^2} + \frac{2 \cos \left( \frac{2n + 1}{3} \pi \right)}{(n + \frac{1}{2})^2 \pi^2} - \frac{\cos \left( \frac{j + \frac{1}{2}}{n + \frac{1}{2}} \right) \pi}{2 (n + \frac{1}{2})^2 \pi^3} - \frac{j + \frac{1}{2}}{(n + \frac{1}{2})^3 \pi^3} + o \left( \frac{1}{n^2} \right).
\]

Let \( m := 6n \). One can calculate that,

\[
\lim_{m \to \infty} \left( m + \frac{1}{2} \right)^2 \pi^2 \left( x^j_m - \frac{j}{m} \right) = \psi(x) = \left( 1 - 2 \cos \frac{\pi}{3} \right) x - 1 + 2 \cos \frac{\pi}{3} \frac{\cos \pi x}{2\pi} - \frac{1}{2} \frac{x}{\pi}.
\]
According to Theorem 3, we find
\[ q(x) = 2 \left( \psi''(x) + \psi(0) \right) = \sin \pi x - \frac{2}{\pi}, \]
\[ \gamma_0 = 2\psi(0) + 2 = 2. \]

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**References**

[1] McLaughlin J.R., (1988) Inverse spectral theory using nodal points as data – a uniqueness result, J. Diff. Eq. 73 354–362.
[2] Hald O.H. and McLaughlin J.R., (1989) Solutions of inverse nodal problems, Inv. Prob. 5 307–347.
[3] Yang X.F., (1997) A solution of the nodal problem, Inverse Problems, 13 203-213.
[4] Yang X.F., (2001) A new inverse nodal problem, J. Differ. Equ. 169 633–653.
[5] Buterin S.A. and Shieh C.T., (2009) Inverse nodal problem for differential pencils, Appl. Math. Lett. 22 1240–1247.
[6] Buterin, S.A, Shieh, C.T., (2012) Incomplete inverse spectral and nodal problems for differential pencil. Results Math. 62 167-179.
[7] Cheng Y.H., Law C-K. and Tsay J., (2000) Remarks on a new inverse nodal problem, J. Math. Anal. Appl. 248 145–155.
[8] Currie S. and Watson B.A., (2007) Inverse nodal problems for Sturm–Liouville equations on graphs, Inv. Prob. 23 2029–2040.
[9] Guo Y.X., and Wei G.S., (2013) Inverse problems: Dense nodal subset on an interior subinterval, J. Differential Equations, 255(7) 2002–2017.
[10] Law C.K. and Tsay J., (2001) On the well-posedness of the inverse nodal problem, Inv. Prob. 17 1493–1512.
[11] Law C.K., Shen, C. L., and Yang C.F., (1999). The inverse nodal problem on the smoothness of the potential function. Inverse Problems, 15(1) 253.
[12] Law C. K. and Yang C. F., (1998) Reconstructing the potential function and its derivatives using nodal data, Inverse Problems 14 299–312.
[13] Ozkan, A. S. and Keskin, B. (2015). Inverse nodal problems for Sturm–Liouville equation with eigenparameter-dependent boundary and jump conditions. Inverse Problems in Science and Engineering, 23(8), 1306-1342.
[14] Keskin, B. and Ozkan, A. S. (2017). Inverse nodal problems for Dirac-type integro-differential operators. Journal of Differential Equations, 263(12), 8838-8847.
[15] Keskin, B. and Ozkan, A. S. (2017). Inverse nodal problems for impulsive Sturm-Liouville equation with boundary conditions depending on the parameter. Advances in Analysis, 2(3), 151-156.
[16] Shieh and Yurko V. A., (2008) Inverse nodal and inverse spectral problems for discontinuous boundary value problems, J. Math. Anal. Appl. 347 266-272.
[17] Shen C.L. and Shieh C.T., (2000) An inverse nodal problem for vectorial Sturm–Liouville equation, Inv. Probl. 16 349–356.
[18] Yang C.F and Yang X.P., (2011),Inverse nodal problems for the Sturm-Liouville equation with polynomially dependent on the eigenparameter, Inverse Problems in Science and Engineering, 19(7) 951-961.
[19] Yang C.F., (2013) Inverse nodal problems of discontinuous Sturm–Liouville operator, J. Differential Equations, 254 1992–2014.
[20] Yurko V. A., Inverse Spectral Problems for Differential Operators and Their Applications, Gordon and Breach, Amsterdam, 2000.
[21] Hu Y.T., Bondarenko, N.P., and Yang C.F., (2020) Traces and inverse nodal problem for Sturm–Liouville operators with frozen argument. Applied Mathematics Letters, 102, 106096.
[22] Hu Y.T., Bondarenko, N.P., Shieh, C.T., and Yang C.F. (2019) Traces and inverse nodal problems for Dirac-type integro-differential operators on a graph. Applied Mathematics and Computation, 363, 124606.
[23] Wang Y. P., and Yurko, V. A., (2016) On the inverse nodal problems for discontinuous Sturm–Liouville operators. Journal of Differential Equations, 260(5), 4086-4109.
[24] Yang C.F., (2012) Inverse nodal problems for the Sturm-Liouville operator with eigenparameter dependent boundary conditions. Operators and Matrices, 6(1), 63-77.
[25] Yurko V. A., (2008) Inverse nodal problems for Sturm–Liouville operators on star-type graphs, J. Inverse Ill-Posed Probl. 16 715–722.
[26] Freiling G., and Yurko V. A., (2010) Inverse nodal problems for differential operators on graphs with a cycle. Tamkang Journal of Mathematics, 41(1), 15-24.

[27] Koyunbakan H., and Mosazadeh S., (2021) Inverse nodal problem for discontinuous Sturm–Liouville operator by new Prüfer Substitutions. Mathematical Sciences, 1-8.

[28] Goktas S., Koyunbakan H., and Gulsen T., (2018) Inverse nodal problem for polynomial pencil of Sturm-Liouville operator. Mathematical Methods in the Applied Sciences, 41(17), 7576-7582.

[29] Yılmaz, E., and Koyunbakan, H. (2010) Reconstruction of potential function and its derivatives for Sturm–Liouville problem with eigenvalues in boundary condition. Inverse Problems in Science and Engineering, 18(7), 935-944.

[30] Akkarpooor S., Koyunbakan H., and Dabbaghian, A., (2019) Solving inverse nodal problem with spectral parameter in boundary conditions. Inverse Problems in Science and Engineering, 27(12), 1790-1801.

[31] Guo Y., and Wei G. (2013). Inverse problems: dense nodal subset on an interior subinterval. Journal of Differential Equations, 255(7), 2002-2017.

[32] Bitsadze A. V. and Samarskii A. A., (1969) Some elementary generalizations of linear elliptic boundary value problems. Dokl. Akad. Nauk SSSR 185 739–740.

[33] Stikonas A. and Stikoniene O., (2009) Characteristic functions for Sturm–Liouville problems with nonlocal boundary conditions, Math. Model. Anal. 14 229–246.

[34] Şen, E., and Stikonas, A. (2021). Asymptotic distribution of eigenvalues and eigenfunctions of a nonlocal boundary value problem. Mathematical Modelling and Analysis, 26(2), 253-266.

[35] Albeverio S., Hryniv R. O., and Nizhnik, L. P. (2007). Inverse spectral problems for non-local Sturm–Liouville operators. Inverse problems, 23(2), 523.

[36] Nizhnik L. (2010). Inverse nonlocal Sturm–Liouville problem. Inverse problems, 26(12), 125006.

[37] Xu X.J., and Yang C.F., (2019) Inverse nodal problem for nonlocal differential operators. Tamkang Journal of Mathematics 50.3 337-347.

[38] Hu Y.T., Yang C.F., and Xu X.C., (2017) Inverse nodal problems for the Sturm–Liouville operator with nonlocal integral conditions. Journal of Inverse and Ill-Posed Problems 25.6 799-806.

[39] Qin, X., Gao Y., and Yang C., (2019) Inverse Nodal Problems for the Sturm-Liouville Operator with Some Nonlocal Integral Conditions. Journal of Applied Mathematics and Physics 7.01 111.

[40] Yang C.F., (2010) Inverse nodal problem for a class of nonlocal Sturm-Liouville operator. Mathematical Modelling and Analysis 15.3 383-392.

[41] Yang C.F., (2014) An inverse problem for a differential pencil using nodal points as data. Israel Journal of Mathematics, 204(1), 431-446.

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