Cluster densities at 2D critical points in rectangular geometries

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Abstract
Making use of the complete calculation [1] of the chiral six-point correlation function
\[ C(z) = \langle \phi_{1,2} \phi_{1,2} \Phi_{1/2,0}(z, \bar{z}) \phi_{1,2} \phi_{1,2} \rangle, \]
with the four \( \phi_{1,2} \) operators at the corners of an arbitrary rectangle and the point \( z = x + iy \) in the interior, for arbitrary central charge (equivalently, SLE parameter \( \kappa > 0 \)), we calculate various quantities of interest for percolation (\( \kappa = 6 \)) and many other two-dimensional critical points. In particular, we use \( C \) to specify the density at \( z \) of critical clusters conditioned to touch either or both vertical sides of the rectangle, with these sides ‘wired’, i.e. constrained to be in a single cluster, and the horizontal sides free. These quantities probe the structure of various cluster configurations, including those that contribute to the crossing probability. We first examine the effects of boundary conditions on \( C \) for the critical \( O(n) \) loop models in both high- and low-density phases and for both Fortuin–Kasteleyn (FK) and spin clusters in the critical \( \mathcal{Q} \)-state Potts models. A Coulomb gas analysis then allows us to calculate the cluster densities with various conditionings in terms of the conformal blocks calculated in [1]. Explicit formulas generalizing Cardy’s horizontal crossing probability to these models (using previously known results) are also presented. These solutions are employed to generalize previous results demonstrating factorization of higher order correlation functions to the critical systems mentioned. An explicit formula for the density of critical percolation clusters that cross a rectangle horizontally with free boundary conditions is also given. Simplifications of the hypergeometric functions in our solutions for various models are presented. High-precision simulations verify these predictions for percolation and for the \( \mathcal{Q} = 2 \)- and \( 3 \)-state Potts models, including both FK and spin clusters. Our
1. Introduction

The conformal invariance of crossing probabilities in critical percolation was first suggested by Aizenman, on the basis of numerical results by Langlands et al [2]. This prompted Cardy [3] to apply the methods of boundary conformal field theory (CFT) to derive his celebrated formula for the horizontal crossing probability in a rectangle of arbitrary aspect ratio. This formula is an excellent example of the predictive power of CFT.

Recently, there has been a renewed interest in crossing probabilities as prototype non-local observables; rigorous proofs have appeared [4], and progress has been made in exploiting CFT and other methods to capture non-local phenomena [5, 6]. This includes our previous crossing results for percolation [7]. Much of the contemporary research has been motivated by the development of Schramm–Loewner evolution (SLE) [8], which provides a rigorous framework for conformal symmetry but with a perspective contrasting CFT. Work on crossing probabilities has also stimulated new research in number theory, in particular the development of the theory of higher order modular forms (see [9] and [10] for details and references).

In [11] and [12] we established, by use of CFT and high-precision simulation, exact and universal factorizations of certain higher order correlation functions in terms of lower order correlation functions for percolation clusters in two dimensions at the percolation point. In that work, the correlation functions involved the density of critical percolation clusters constrained to touch one or two isolated boundary points, or single boundary intervals. The question of how far this factorization can be generalized to other conformal correlation functions was then examined in [13].

A more recent paper [14] considers percolation densities in a rectangle conditioned to touch one or both vertical sides, with the sides ‘wired’, i.e. constrained to belong to a single cluster. Evaluating these quantities via CFT requires calculation of the correlation function (1)

\[ C(z) = \langle \phi_{1/2}^1(0)\phi_{1/2}^2(i)\Phi_{1/2,0}(z, \bar{z})\phi_{1/2}^1(R)\phi_{1/2}^2(R+i) \rangle_R \]

in a rectangular geometry \( R \) with the boundary operators at the corners. This is tantamount to the challenging task of evaluating a chiral six-point function in the complex plane. In [1], a complete set of solutions of the PDEs governing this correlation function are determined, for arbitrary central charge (or equivalently, SLE parameter \( \kappa > 0 \)) in terms of algebraic and Appell hypergeometric functions.

The solutions found in [1] are employed here to give variously conditioned cluster weights and densities in a range of critical models. (We use the term ‘cluster density’ in a particular way; the definition is given in subsection 2.2.) We index the models via the SLE parameter \( \kappa \), as it includes both branches of the \( O(n) \) model and thus both the FK and Potts spin cluster models. The interpretation of \( \Phi_{1/2,0} \) as a density operator holds as long as its weight is positive, or when \( 8/3 \leq \kappa \leq 8 \) (corresponding to \( 0 \leq n \leq 2 \)). Outside of this region, we do not offer a physical interpretation for \( C(z) \), though the solutions found in [1] remain valid.

We also employ the results for \( C \) to obtain generalized factorization of correlations. In [14], which treats percolation only, a certain universal ratio of cluster densities within a rectangular
region is considered, involving the densities of clusters touching the left and/or right sides of the rectangle with wired, or fixed, boundary conditions on those sides. The ratio is nearly constant everywhere in the rectangle, varying by less than 3%, and in addition does not depend on the vertical position \( y \) in the rectangle. In this paper, we generalize the percolation results to the range of critical models mentioned. How exactly the factorization holds depends on the model, but the independence from \( y \) is always valid.

From a broader perspective, \( C(z) \) provides, via the densities, information on the structure of crossing and related cluster configurations (or equivalently loop configurations) in the critical models mentioned. In this sense it generalizes and deepens Cardy’s analysis of crossing in percolation [3].

In subsection 2.1, we recall the results for the correlation function (1) obtained in [1]. The interesting independence from one coordinate mentioned (see (16)) appears here. Subsection 2.2 then begins the application of our results to the specific critical models mentioned. In particular, we examine the effects of boundary conditions, and obtain formulas for properly normalized densities. In addition, we give explicit formulas for a certain generalization of Cardy’s horizontal percolation crossing probability [3]. Subsection 2.3 identifies the solutions for the densities of various cluster configurations of interest by use of Coulomb gas methods.

In section 3, we analyze a certain universal ratio \( \rho \) of densities, which (in the scaling limit) is equal to a ratio involving critical cluster densities and crossing probabilities. This quantity is universal, and almost constant everywhere in the rectangle, regardless of the aspect ratio, in many models. This implies an almost exact factorization of the corresponding correlation functions (or densities). This work generalizes the ratio examined in [14] for percolation.

Section 4 returns to the case of percolation. Here, we take advantage of the ‘locality’ property of percolation, as it is referred to in SLE (it is equivalent to the independence of local sites or bonds in lattice models) to derive a formula for the density of horizontal crossing clusters \( P_{Ax} \) in rectangles with free boundary conditions on all sides.

In section 5, we compare our predictions for the ratio \( \rho \), which governs factorization, and our formula for the density of percolation clusters crossing a rectangle with high-precision simulations. The agreement is very good.

Section 6 contains a detailed summary of our results.

The correlation function \( C(z) \) determines various densities; a similar correlation function without \( \Phi_{1/2,0} \) determines crossing weights. These in turn depend on the Appell or \( _2F_1 \) hypergeometric functions, respectively. The appendix presents formulas for these quantities in various critical models where the hypergeometric functions simplify.

2. Theory

2.1. Results for the correlation function

In this subsection, we recall results obtained in [1] for the six-point correlator

\[
C(z) = \langle \phi_{1,2}^e(0)\phi_{1,2}^e(i)\Phi_{1/2,0}(z, \bar{z})\phi_{1,2}^e(R)\phi_{1,2}^e(R + i) \rangle_{R},
\]

in the rectangular geometry \( R := \{z = x + iy \in \mathbb{C} | 0 < x < R, 0 < y < 1 \} \). The aspect ratio and the elliptic parameter \( m \) are related by

\[
R = \frac{K(m)}{K'(m)} \quad \text{and} \quad m = \frac{\partial_4^4(0, e^{-\pi R})}{\partial_3^4(0, e^{-\pi R})},
\]

where \( K'(m) := K(1 - m) \), with \( K \) the complete elliptic integral. Note that \( m \) differs from the standard modular lambda parameter, which is \( 1 - m \) here.
The conformal dimensions and central charge are
\[ h_{1,2} = \frac{6 - \kappa}{2\kappa}, \quad h_{1/2,0} = \tilde{h}_{1/2,0} = \frac{(8 - \kappa)(3\kappa - 8)}{64\kappa}, \]
\[ h_{1,3} = \frac{8 - \kappa}{\kappa}, \quad c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}, \]
where \( \kappa \) is the SLE parameter and \( h_{1,3} \) is used below. Making use of the coordinates
\[ \xi := \text{sn}(xK'[m]^2), \quad \text{and} \quad \psi := \text{sn}(yK'[1 - m]^2), \]
we find that any solution that is a single conformal block can be written either in the form
\[ C(z) = f(\xi, \psi, m)G(\xi, m), \]
where the algebraic prefactor \( f \) is given by
\[ f(\xi, \psi, m) = c(m) \left[ \frac{(1 - m\xi^2)^2}{\xi(1 - \xi)(1 - m\xi)} + \frac{(1 - (1 - m)\psi^2)^2}{\psi(1 - \psi)(1 - (1 - m)\psi)} - 4 \right]^{-h_{1,3}/2+h_{1/2,0}}, \]
with
\[ c(m) = 2^{h_{1,3}}(K'^4)^{2h_{1,3}/2h_{1/2,0}}(m(1 - m))^{2h_{1/2},} \]
or as in (5) with \( G(\xi, m) \rightarrow G(\psi, 1 - m). \)

The algebraic prefactor \( f \) is independent of boundary conditions. The conformal block \( G \), on the other hand, is strongly dependent on them, as discussed below. This plays an important role in our results. Now \( G \) may be given by one of the five solutions
\[ G_I(\xi, m) = \frac{\Gamma(2 - 8/\kappa)\Gamma(16/\kappa - 1)}{\Gamma(12/\kappa)\Gamma(1 - 4/\kappa)} \frac{[m(1 - m)]^{2/\kappa} \xi^{8/\kappa-1/2}}{(1 - \xi)(1 - m\xi)^{1/\kappa-1/2}} \times F_1 \left( 1 - \frac{4}{\kappa}; \frac{4}{\kappa}, \frac{4}{\kappa} - \frac{12}{\kappa}, \frac{m\xi}{\kappa} \right), \]
\[ G_{II}(\xi, m) = \frac{(1 - m)^{2/\kappa}}{m^{6/\kappa-1} \xi(1 - \xi)(1 - m\xi)^{1/\kappa-1/2}} \times F_1 \left( 1 - \frac{4}{\kappa}; \frac{4}{\kappa}, \frac{4}{\kappa} - \frac{2}{\kappa} - \frac{16}{\kappa}; 2 - \frac{8}{\kappa} \right), \]
\[ G_{III}(\xi, m) = \frac{(1 - m)^{2/\kappa}}{m^{6/\kappa-1} \xi(1 - \xi)(1 - m\xi)^{1/\kappa-1/2}} \times F_1 \left( 1 - \frac{4}{\kappa}; \frac{4}{\kappa}, \frac{4}{\kappa} - \frac{16}{\kappa}; \frac{2}{\kappa} - \frac{8}{\kappa} \right), \]
\[ G_{IV}(\xi, m) = \frac{(1 - m\xi)^{12/\kappa-3/2}}{[m(1 - m)]^{6/\kappa-1} \xi(1 - \xi)^{3/\kappa-1/2}} \times F_1 \left( 1 - \frac{4}{\kappa}; \frac{4}{\kappa}, \frac{4}{\kappa} - \frac{2}{\kappa} - \frac{16}{\kappa}; 1 - \frac{1 - m}{1 - m\xi} \right) \]
and
\[ G_{V}(\xi, m) = \frac{\Gamma(2 - 8/\kappa)\Gamma(16/\kappa - 1)}{\Gamma(12/\kappa)\Gamma(1 - 4/\kappa)} \frac{m^{2/\kappa} (1 - \xi)^{8/\kappa-1/2}}{(1 - m\xi)^{4/\kappa-1/2} (1 - m\xi)^{1/2}} \times F_1 \left( 1 - \frac{4}{\kappa}; \frac{4}{\kappa}, \frac{4}{\kappa}; \frac{4}{\kappa} - \frac{12}{\kappa}, \frac{1 - \xi}{m(1 - \xi)} - \frac{1}{1 - m\xi} \right). \]
where \( F_1 \) is the first Appell function

\[
F_1(a; b_1, b_2; c|z_1, z_2) := \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b_1)_i(b_2)_j}{i!j!(c)_{i+j}} z_1^i z_2^j,
\]

defined using the Pochhammer symbol \((z)_n := \Gamma(z + n)/\Gamma(z)\).

For our ranges of \( \xi \) and \( m \) values \((0 \leq \xi, m \leq 1), (8)–(12) \) exhaust the convergent Frobenius series solutions to the differential equations that can be expressed with a single \( F_1 \). One can also use one of these solutions (or the five mentioned below) with \(|\xi| \rightarrow \psi, m \rightarrow 1 - m\). Each of these is a single conformal block. However, only three of them are independent, as discussed in section II A of [1].

There are also five other convergent solutions that can be expressed with a single second Appell function \( F_2 \) (see [1] for details), of which we use only one here:

\[
G_{VI}(\xi, m) := G_{II} - nG_I = G_{IV} - nG_V
\]

\[
= \frac{\Gamma(2 - 8/\kappa)\Gamma(16/\kappa - 1)\Gamma(4/\kappa)}{\Gamma(4/\kappa)\Gamma(8/\kappa)^2} \left[ m(1 - m) \right]^{2/\kappa} \left[ \frac{(1 - \xi)}{1 - m \xi^2} \right]^{8/\kappa-1/2}
\]

\[
\times F_2 \left( \frac{16}{\kappa} - 1; \frac{4}{\kappa}, \frac{4}{\kappa}, \frac{8}{\kappa}, \frac{8}{\kappa}; 1 - \xi, \frac{\xi(1 - m)}{1 - m \xi^2} \right),
\]

where \( n \) is the parameter of the \( O(n) \) loop models, given by

\[
n = -2 \cos \left( \frac{4\pi}{\kappa} \right),
\]

and with \( F_2 \) defined by

\[
F_2(a; b_1, b_2; c_1, c_2|z_1, z_2) := \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b_1)_i(b_2)_j}{i!j!(c_1)_i(c_2)_j} z_1^i z_2^j.
\]

The fact that these solutions for \( G \) only depend on two variables rather than three follows on combining the CFT null-state PDEs. One finds that

\[
\partial_\xi \partial_\psi G(\xi, \psi, m) = 0.
\]

This equation is the basis for the \( \psi \)-independence of the factorization behavior discussed below. It indicates the presence of an unknown symmetry.

For use below, we also recall the behavior of \( \xi, \psi \) and \( m \) under the symmetries of the rectangle: mirror symmetries about \( x = R/2, y = 1/2 \) and a mirror symmetry about \( x = y \) with a concurrent scaling by \( 1/R \) to preserve the height. These symmetry operations translate, respectively, into

\[
(\xi, \psi, m) \rightarrow \left( \frac{1 - \xi}{1 - m \xi}, \psi, m \right), \left( \xi, \frac{1 - \psi}{1 - (1 - m)\psi}, m \right), \left( \psi, \xi, 1 - m \right).
\]

2.2. Physical interpretation of equation (1) and boundary conditions

The correlation function (1) describes cluster densities for a variety of critical statistical mechanics models; in this section we detail this correspondence. We focus in particular on the Potts models and their boundary conditions in order to obtain explicit formulas. The results allow us to implement simulations (see section 5) to test our predictions.

We begin with critical \( O(n) \) loop models (in the continuum limit) because they furnish a clear picture for these densities, and describe a continuum of critical points with \( 0 \leq n \leq 2 \). There are many realizations of these models but in each case the degrees of freedom are closed fractal loops, each contributing to the partition function with a fugacity \( n \). There are two
critical branches depending on the energy cost per loop length, representing dilute and dense loop phases. As noted, we use the SLE parameter $\kappa$ to index our models. This is convenient as $\kappa$ nicely parameterizes both critical branches of the $O(n)$ loop model; the two parameters are related by (15) with $\kappa \leqslant 4$ (respectively $\kappa > 4$) corresponding to the dilute (respectively dense) phase.

We can condition the loop ensembles so that open loop segments emerge from pairs of points on the boundary. We call these loops ‘boundary arcs’ to distinguish them from the closed loops in the bulk. These boundary arcs are equivalent to SLE traces, and in CFT are implemented by placing $\phi_{1,2}$ Kac operators at their endpoints on the boundary. Because they are distinguished from bulk loops, we can give the boundary arcs a fixed weight of 1. The correlation functions we consider correspond to an $O(n)$ model with boundary arcs attached to the four corners of an arbitrary rectangle.

In the $O(n)$ model, bulk points are said to be adjacent to a given boundary segment whenever a path can be drawn from the point to the segment without crossing any loops or boundary arcs. It has been argued using Coulomb gas techniques that the density operator for these adjacent points is $\Phi_{1/2,0}(z, \bar{z})$ [15].

In order to proceed, we also need to consider generalizations of Cardy’s crossing probability. They provide a necessary normalization.

In applying our results to model systems, three kinds of quantities arise. Unnormalized weights, which are denoted by $\Pi$, crossing probabilities denoted by $P$, and cluster densities, also denoted by $P$. The crossing probabilities are true probabilities, and can be completely determined from our CFT results. For a given model, they depend on the aspect ratio $R$ only. A cluster density (or more simply, ‘density’) is a bit more complicated. It is the scaled probability in a small region around a point with some specified conditioning, e.g. the presence of a cluster connected to a specific boundary segment. This is the probability, on a lattice with mesh size $\delta$, that a neighborhood of $z$ with characteristic length $\epsilon$ includes points with the specified conditioning, divided by the scaling factor $\epsilon^{2h_{1/2,0}}$ and taken in the limit $\delta \ll \epsilon \rightarrow 0$. These densities are proportional to the correlation function (1), divided by a partition function, which is a four-point function of $\phi_{1,2}$ operators. (The partition function depends on $R$ but not $z$.) This determines the density up to a non-universal factor that depends on the details of how the $\Phi_{1/2,0}$ operator in (1) is regularized. This factor depends on the particular model and is thus not given by CFT. We will ignore it in the formulas below. However, in the case of a ratio such as $\rho$ (see section 3) the nonuniversal factors divide out, so the conformal result is complete.

2.2.1. Crossing probabilities. We consider the crossing probabilities for the $O(n)$ model in a rectangle with boundary arcs attached to the corners. Inside the rectangle, the boundary arcs join in one of two ways: they may connect the two top and two bottom corners, which we call configuration $H$, or they may connect the two left and two right corners, which we call configuration $V$ as shown in figure 1. We use $\Pi_H$ (respectively $\Pi_V$) to denote the weight of the corresponding configurations for the $O(n)$ model, with (as mentioned) the boundary arcs having weight 1.
We now consider the dense $O(n)$ model phase. Recall that the Potts model can be exactly mapped onto a representation known as the Fortuin–Kastelyn (FK) random cluster model, which replaces the sum over spin configurations with a sum over nearest-neighbor bond configurations [16]. This is the basis of the Swendsen–Wang (SW) algorithm frequently used to simulate the Potts model [17]. At the critical point, replacing the sum over bond configurations in the FK model partition function with a sum over loops following the cluster boundaries gives a mapping to the dense phase of the $O(n)$ model, where each loop contributes a weight $n = \sqrt{Q}$ to the partition function.

In the FK representation, the $\phi_{1,2}$ operators that mark the ends of SLE traces are equivalent to changes between free and wired boundary conditions. On a free boundary there is no conditioning and the bond configurations are summed over freely as in the bulk, while on a wired boundary all bonds are occupied, so that the boundary sites all belong to the same FK cluster. Thus, there are two possible interpretations for the correlation function (1): either the pairs of operators $\phi_{1,2}(0)\phi_{1,2}(i)$ and $\phi_{1,2}(R)\phi_{1,2}(R + i)$ represent the left and right wired sides of the rectangle, respectively, or $\phi_{1,2}(0)\phi_{1,2}(R)$ and $\phi_{1,2}(i)\phi_{1,2}(R + i)$ represent wired bottom and top sides. We will adopt the former interpretation (unless explicitly stated otherwise), so the SLE curves follow the boundaries of the FK clusters anchored to the left and/or right sides.

For $O(n)$ model configurations of type $V$, the boundary arcs close into two loops that surround the clusters connected to the left and right sides. Thus, the contribution of each such configuration to the FK partition function gains a factor $n^2$ and one has

$$Z_V = n^2 \Pi_V.$$  

For configurations of type $H$, the boundary arcs close into a single loop around the crossing cluster. Thus, the contribution to the FK partition function gains a single factor of $n$ and

$$Z_H = n \Pi_H.$$  

The total FK partition function is therefore

$$Z = n \Pi_H + n^2 \Pi_V. \quad (18)$$

The corresponding result when the top and bottom sides are wired follows on interchanging $H$ and $V$.

At this point we notice a subtlety that will become important when we consider spin cluster densities, namely whether the fixed spins on the right side match the fixed spins on the left side. We next construct an FK model partition function $\tilde{Z}$ that only samples configurations where the left and right sides are mutually fixed to the same spin. This is achieved by adding, to all configurations, a single occupied bond between the left- and right-hand wired sides. Thus, the left and right sides are always wired to the same spin. Now for the $O(n)$ model, the boundary arcs must follow the new bond between the left and right edges, so the boundary arcs close along the top and bottom sides. The consequences for the FK partition function are as follows. For type $V$ configurations, the boundary arcs close into a single loop. Thus

$$\tilde{Z}_V = n \Pi_V,$$

where $\tilde{Z}$ indicates an FK partition function with the extra occupied bond included. For type $H$ configurations, the left and right sides are already connected, so the boundary arcs close into two loops. They can be envisaged by placing the new bond above the top edge. Then, one extra loop follows the top edge of the horizontal crossing cluster but is below the new bond, and the other follows the bottom edge of the horizontal crossing cluster, and goes around both left and right sides and above the extra bond. Thus

$$\tilde{Z}_H = n^2 \Pi_H.$$
The net result for the extended FK partition function is therefore
\[ \tilde{Z} = n^2 \Pi_H + n \Pi_V. \] (19)

Note that the forms \( Z \) and \( \tilde{Z} \) are symmetric under exchanges of the labels \( H \) and \( V \). This is a consequence of the duality transformation that maps the rectangle with left and right sides wired and an extra bond onto a rectangle with top and bottom wired and a similar extra bond that is dual to the original bond.

Now the rectangle with an extra bond is not physical. Thus, we need to adjust \( \tilde{Z} \). The necessary factor is determined from the explicit form of the critical FK partition function, where each occupied bond contributes a factor of \( n \). Adding an extra occupied bond multiplies each term in the partition function by an extra factor of \( n \). Dividing \( \tilde{Z} \) by \( n \), we find the FK partition function for mutually fixed left and right sides:
\[ \bar{Z} = n \Pi_H + \Pi_V. \] (20)

As above, the corresponding result when the top and bottom sides are mutually wired follows on interchanging \( H \) and \( V \).

The result (20) is what we expect. Comparing (18) and (20) we see that the weight of an \( H \) configuration is insensitive to mutual wiring, while the \( V \) weights loose a factor of \( Q = n^2 \).

We now consider the dilute phase of the \( O(n) \) model. This can be related to spin clusters in the \( Q \)-state Potts model. In contrast to the FK representation, this association does not follow from direct manipulation of the Potts and \( O(n) \) models. Instead, one invokes a result known as SLE duality [18], which states that for \( \kappa > 4 \) the outer hull of an SLE with \( \kappa \) has the same fractal dimension as an SLE with \( \kappa' = 16/\kappa = (4/\pi) \arccos(-\sqrt{Q}/2) \), and furthermore that these related values of \( \kappa \) correspond to CFTs with the same central charge. If the spin clusters can be described by an SLE process, then it must have the parameter \( \kappa' \).

The boundaries of spin clusters separate regions with a single spin value, say \( s = 1 \), from neighboring regions with \( s \neq 1 \). An SLE that describes these domain walls should therefore separate boundary regions with \( s = 1 \) and \( s \neq 1 \). For the Ising model, the \( \kappa = 3 \) SLE trace represents a change between + and − boundary conditions [19]. For the 3-state Potts model it has been convincingly argued that the change between \( s = 1 \) and \( s = 2 \) or 3 (boundary spins freely summed over 2 and 3) corresponds to an SLE with \( \kappa = 10/3 \) [20].

In section 5, we simulate these dilute phases in a rectangle with \( s = 1 \) boundary conditions on the left and right sides, freely summing over \( s \neq 1 \) spins on the top and bottom edges. Unlike the FK cluster model, there is only one way to fix the boundaries, by fixing both edges to \( s = 1 \). This can be implemented by adding an extra nearest-neighbor bond between the left and right sides exactly as with the FK clusters; the factors found for the mutually fixed FK partition function carry through in exactly the same form. Because there is no exact mapping from the spin cluster representation to the dilute \( O(n) \) model we cannot fix their absolute normalization. Instead, we adopt the \( \bar{Z} \) normalization for the spin clusters.

From the partition functions (18) and (20), we can calculate the probability \( P_H \) (respectively \( \bar{P}_H \)) of a horizontal crossing FK (respectively either FK or spin) cluster by considering a rectangle with independently (respectively mutually) wired left and right sides. This is simply the ratio of the weight of \( H \) configurations with the chosen boundary conditions to the appropriate partition function. From the above, the weight of \( H \) configurations is \( n \Pi_H \) for independently or mutually wired sides, and the weight of \( V \) configurations is \( n^2 \Pi_V \) (respectively \( \Pi_V \)) for independently (mutually) wired sides. Thus
\[ P_H = \frac{\Pi_H}{\Pi_H + n \Pi_V}, \quad \bar{P}_H = \frac{n \Pi_H}{n \Pi_H + \Pi_V}. \] (21)
For \( n = 1 \), which is realized by percolation (dense phase) or Ising spins (dilute phase), the distinction between mutually wired and independently wired boundary conditions vanishes. For Ising spin clusters, sides have alternating \( +/− \) fixed boundary conditions, and the symmetry of the two types of wiring is evident from spin duality. For percolation, the partition function for either type of boundary condition is constant (i.e. independent of the rectangle aspect ratio) and normalized to 1. In this case, \( \Pi_H \) and \( \Pi_V \) become Cardy’s formula [3] for horizontal and vertical crossings, respectively, with the property \( \Pi_H + \Pi_V = 1 \).

Explicit formulas for \( \Pi_H \) and \( \Pi_V \) (which use results from [21]) follow from (31), (32), (33e) and (33f). A result for crossing on Ising model spin clusters in a circle, which is conformally equivalent to \( \tilde{\Pi}_H \), was obtained in [22]. Note that there are other generalizations of Cardy’s formula using different boundary conditions (see [8, 18]).

2.2.2. Cluster densities. The correlation function (1) involves a density operator at the point \( z \), and has contributions from six types of configurations \( \{ A, B_1, B_2 \} \) and \( \{ \bar{A}, B_1, B_2 \} \) that depend on the location of \( z \) relative to the boundary arcs, as illustrated in figure 2.

We now apply our results to find formulas for the cluster densities in the FK and spin cluster Potts models in terms of the weights \( \Pi_H, \Pi_V, \Pi_A, \Pi_B, \) and \( \Pi_B \). These weights are defined for the \( O(n) \) model with boundary arcs having weight 1, as for \( \Pi_H \) and \( \Pi_V \). In the FK cluster case our results give the density of sites that belong to the same FK bond cluster as the specified wired edge. In the spin cluster case they give the density of \( s = 1 \) sites that belong to clusters attached to the specified \( s = 1 \) spin edges.

We now calculate the density \( P_r(z) \) (respectively \( P_r(z), P_{cr}(z) \)) of clusters anchored to the left (respectively right, left and right) side(s) of a rectangle with either independently or mutually wired left and right sides. This is equal to some linear combination of the weights \( \{ \Pi_A, \Pi_B \} \) (respectively \( \{ \Pi_A, \Pi_B, \Pi_{cl} \} \)) divided by the partition function (18) or (20). The coefficients of this linear combination are determined by the boundary conditions in the same way as above. The boundary arcs of \( A \) (respectively \( B_1, B_2 \)) connect as in \( H \) (respectively \( V \)), so the boundary conditions introduce the same factors for \( \Pi_A \) as they did for \( \Pi_H \) (respectively the same for \( \Pi_B, \Pi_B \), as for \( \Pi_V \)). Therefore,

\[
P_r(z) = \frac{\Pi_A + n\Pi_B}{\Pi_H + n\Pi_V}, \quad \tilde{P}_r(z) = \frac{n\Pi_A + \Pi_B}{n\Pi_H + \Pi_V}, \quad (22a)
\]

\[
P_r(z) = \frac{\Pi_A + n\Pi_B}{\Pi_H + n\Pi_V}, \quad \tilde{P}_r(z) = \frac{n\Pi_A + \Pi_B}{n\Pi_H + \Pi_V}, \quad (22b)
\]

\[
P_{cr}(z) = \frac{\Pi_A}{\Pi_H + n\Pi_V}, \quad \tilde{P}_{cr}(z) = \frac{n\Pi_A}{n\Pi_H + \Pi_V}. \quad (22c)
\]
As before, the $P$ expressions apply to the independently wired FK configurations, and the $\bar{P}$ expressions apply to both the mutually wired FK configurations and the spin cluster configurations.

We could also assume that the top and bottom sides are wired, which would result in the same expressions with $[A, \ell, r, V, H]$ replaced with $[\bar{A}, b, t, H, V]$, reflecting the $(\xi, \psi m) \rightarrow (\psi, \xi, 1 - m)$ symmetry noted in section 2.1.

In section 5, we compare the densities calculated in this section with simulations that measure $P_\ell(z)$, $P_r(z)$, $P_\ell r(z)$ for $Q = 1$, 2- and 3-state FK clusters and $Q = 2$, 3-state spin clusters.

In the next section, we identify the conformal blocks of the six-point function (1) that contribute to the weights $\Pi_A$, $\Pi_B$, and $\Pi_F$ (see (26a)–(26c)).

2.3. Identifying cluster configurations

We have solutions to the PDEs that govern the correlation function (1) and the prefactor $f$, given in (6). To determine physical content we must identify how the weights $\Pi_A$, $\Pi_B$, and $\Pi_F$ contribute to the various conformal blocks $G$. This can be done in an elegant way using Coulomb gas vertex operators, which we will now briefly review; for a more complete treatment see, for example, [23, 24].

The Coulomb gas representation makes use of the chiral bosonic variable $\phi(z, \bar{z})$ with action $S = S_O + S_C + S_D$:

$$S_O = \frac{g}{4\pi} \int (\nabla \phi)^2 d^2x, \quad S_C = \frac{i\alpha_0}{8\pi} \int R \phi d^2x, \quad S_D = a \int \cos 2\phi d^2x.$$ 

Starting from the free boson action $S_O$, adding the complex term $S_C$ that couples the field to the scalar curvature $R$ modifies the stress tensor. This reduces the central charge of the theory from $c = 1$ to $c = 1 - 24\alpha_0^2$. The field takes discrete values, $\phi \in \pi \mathbb{Z}$ almost everywhere, due to the term $S_D$. This suppresses any rescaling of the parameter $g$. At short distances, the boundaries between these discrete values naturally form a loop ensemble, but at long distances, the behavior of the system should still be dominated by $S_O$. Reconciling these two scales is only possible if $S_D$ is a marginal perturbation, and this condition lets us relate $g$ and $\alpha_0$, leaving a single free parameter. Thus

$$c = 1 - 24\alpha_0^2 = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}$$

(23)

gives $g = 4/\kappa$ and $2\alpha_0 = (1 - g)g^{-1/2}$.

Ignoring a subtlety of the zero mode, which is unimportant here, we decompose the boson into holomorphic and antiholomorphic components $\phi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$ in order to write chiral vertex operators:

$$V_\alpha(z) = e^{i\sqrt{2}\phi(z)}.$$ 

Chiral operators are sufficient because our problem includes a boundary. The energy flux leaving the system is zero, which means $\phi(z)$ and $\bar{\phi}(\bar{z})$ are not independent on the boundary nor, by analytic continuation, anywhere else. Specifically, in the upper half-plane $\bar{\phi}$ is the analytic continuation of $\phi$ from the lower half-plane [25].

The parameter $\alpha$ is called the charge of the vertex operator, by analogy with two-dimensional electrostatics. Non-trivial correlation functions must obey a charge neutrality condition. Since $S_C$ effectively adds a non-local background charge $-2\alpha_0$ to the system, the
total vertex charges must satisfy $\sum \alpha_i = 2\alpha_0$. The form of charge neutral correlation functions is still determined by the original Gaussian action $S_0$:

$$\left< \prod_i \mathcal{V}_{\alpha_i}(z_i) \right> = \prod_{j>i} (z_j - z_i)^{2\alpha_j/\alpha_i},$$

and the charge neutrality condition is entirely responsible for the differences between the Coulomb gas and free boson correlation functions.

The form of the two-point function then implies that a given scaling operator has two possible vertex representations with charges, $\alpha$ or $2\alpha_0 - \alpha$, related to the conformal weight by

$$h = \alpha(\alpha - 2\alpha_0).$$

The condition of charge neutrality severely limits the set of computable correlation functions. However, screening operators allow one to extend this set of correlations by adding charge without changing the conformal properties of the system. Screening operators are non-local with zero weight, and are formed by integrating weight 1 vertex operators around the singular points of the correlator in a non-trivial way. There are two types of screening operators:

$$Q_{\pm} = \oint dz \mathcal{V}_{\alpha_\pm}(z).$$

The charges $\alpha_+$ and $\alpha_-$ are the positive and negative solutions of $1 = \alpha(\alpha - 2\alpha_0)$, respectively. Using (23) these are

$$\alpha_+ = \sqrt{\frac{\kappa}{4}}$$

and

$$\alpha_- = -\sqrt{\frac{4}{\kappa}}.\tag{24a}$$

We parameterize all other charges as

$$\alpha_{r,s}^\pm = \frac{1 \pm r}{2} \alpha_+ + \frac{1 \pm s}{2} \alpha_-\tag{24b}$$

When $r, s \in \mathbb{Z}^+$ the charge corresponds to a Kac operator $\phi_{1/2,0}$. We can generalize to $r, s \in \mathbb{Z}/2$, so that the density operator has fixed $r$ and $s$ independent of the model parameter $\kappa$.

The vertex operator method aids us in identifying the configurations contributing to various conformal blocks. The charge neutrality condition for a correlation function with two chiral $\Phi_{1/2,0}$ and four $\phi_{1/2}$ operators requires a unique set of charges and screening operators: one $\alpha_{1/2,0}^+$, one $\alpha_{1/2,0}^-$, four $\alpha_{1/2}^\pm$s and two $Q_-$s. Given (16) we can calculate functions such as (8)–(12) with the method used in [14], which exploits the common $y$ dependence (assumed in [14]) by moving the density operator to the lower boundary, and replacing it with its limiting boundary operator. Thus, the bulk $\Phi_{1/2,0}$ operator becomes a boundary magnetization operator (see figure 3) and we are left with a five-point boundary correlation function made up of one $\alpha_{1/2,0}^+$, four $\alpha_{1/2}^\pm$s and one $Q_-$.  

Figure 3. The fusion of a bulk magnetization operator to a free boundary has a unique vertex/screening operator representation.
We choose integration paths for the $Q_-$ that entwine neighboring pairs of boundary operators because, with an appropriately chosen phase, they yield real solutions for arbitrary values of the aspect ratio and model parameter.

In general, we integrate along non-contractible closed paths such as the one on the left-hand side of figure 4(a). These paths can be deformed to run along the real axis except for small circles surrounding the operators. If $4 < \kappa$, the contribution to the integral from the small circles vanishes with the circle’s radius, so we can replace the path with its component along the real axis. Choosing the paths as in figure 4(b) reproduces the indicated conformal blocks from (8)–(12). The integration is defined in the upper half-plane, and must be transformed into the rectangle as shown in figure 4(c). For a given correlation function, a conformal block is determined in part by specifying an order in which to fuse the operators, retaining a single conformal family from the OPE under each fusion. Mathematically, conformal blocks are multiple Frobenius series in small parameters implied by the fusion order with the leading exponents determined by the specified conformal families. In the vertex operator formalism pairs of vertices that are not entwined with distant vertices have a definite charge when fused together. Such a pair of operators may or may not be entwined by its own screening operator circulating around just the pair. In either case, such pairs fuse to a single conformal family, which is easily identified by the sum of the fused charges and any entwining screening charges. Thus, a given set of screening charge integration paths in an arbitrary correlation function is simply identified with a conformal block.

Next, as was done in [14] for percolation, we use the vertex operator formulation to determine the contribution of the conformal blocks to the configurations of interest. First, for the integration paths in figure 4 we deduce the fusion structure of each block to determine the contributions from the three configurations illustrated in the first row of figure 2. Recall that the corresponding weights are defined for the $O(n)$ models, with boundary arcs having weight 1. These conformal blocks can be identified with sets of boundary conditions similar to those in section 2.2, except that we may leave unattached arcs instead of boundary loops.

First consider the block $G_1$ in the limit where the magnetization operator goes to the lower left-hand corner. This traps the screening operator as well, for a total charge of $\alpha_{1,2}^- + \alpha_{1,3}^- + \alpha_- = \alpha_{1,4}^-$, which is the charge for a three-leg operator. An $n$-leg operator, associated with $\phi_{1,1+n}$, represents a point where at least $n$ non-contractible boundary arcs (arcs
that cannot be smoothly deformed to a point because of some constraint on their behavior) each with weight 1 are attached to the boundary. The only configuration with three or more non-contractible boundary arcs emanating from this corner in that limit is $B_r$; therefore, this configuration must be the sole contribution and $G_I \sim \Pi_{B_r}$. (Note that this proportionality disregards all $y$ dependence, which is contained entirely in the prefactor $f(\xi, \psi, m)$.)

The block $G_{II}$ is more subtle. Fusing the right-hand corners together yields a charge of $2\alpha_{1,2} = a_{1,3}$, implying a two-leg operator; this excludes $B_\ell$ configurations which have contractible boundary arcs attached to the right side.

The remaining contributions to $G_{II}$ are fixed by considering the left-hand edge, which has a net charge of $2\alpha_{1,2} + \alpha_- = 0$ indicating that the one-leg operators at the corners fuse to the identity family. If the arcs attached to this edge are non-contractible (as with $A$ configurations), then they must be attached to other one-leg operators. These operators will fix the arc weight at 1, so the total weight does not change. If the attached arc is contractible, as with $B_r$ configurations, then bringing the operators together would form a loop with weight 1, and a compensating factor of $n$ is needed. The net result is that $G_{II} \sim \Pi_A + n\Pi_{B_r}$. The same result follows if we add an external arc connecting open boundary arc ends that fuse to the identity.

When we consider the block $G_{III}$ the top edge has net charge $2\alpha_{1,2} + \alpha_- = 0$ and fusion is through the identity. Since there are no other restrictive fusions this block contains contributions from all three configuration types. The top edge of type $A$ configurations has a contractible loop, while $B_r$ and $B_\ell$ have non-contractible loops attached to the top edge; thus, $G_{III} \sim n\Pi_A + \Pi_{B_r} + \Pi_{B_\ell}$. Again, this result can be obtained via an external arc; this time connecting the top corners.

The associations for $G_{IV}$ and $G_V$ follow from mirror symmetry and for $G_{VI}$ by definition. The complete list is

$$G_I \sim \Pi_{B_r}, \quad (25a)$$
$$G_{II} \sim \Pi_A + n\Pi_{B_r}, \quad (25b)$$
$$G_{III} \sim n\Pi_A + \Pi_{B_r} + \Pi_{B_\ell}, \quad (25c)$$
$$G_{IV} \sim \Pi_A + n\Pi_{B_\ell}, \quad (25d)$$
$$G_V \sim \Pi_{B_\ell}, \quad (25e)$$
$$G_{VI} \sim \Pi_A. \quad (25f)$$

The relative normalization in (8)–(12) emerges naturally in the vertex operator formalism, since the integral expressions for the different conformal blocks have different integration paths but the same integrand. Hence, the proportionality in equations (25a)–(25f) all omit the same factor. This is an advantage of using vertex operator over the differential equation analysis of [1], which would require comparing several limiting cases to fix the relative normalizations.

Using expression (6) for the common pre-factor $f(\xi, \psi, m)$ we thus find explicit expressions for the weights of the three configuration types $A$, $B_r$ and $B_\ell$ as

$$\Pi_A = f(\xi, \psi, m)G_{VI}(\xi, m), \quad (26a)$$
$$\Pi_{B_r} = f(\xi, \psi, m)G_I(\xi, m) \quad \text{and}$$
$$\Pi_{B_\ell} = f(\xi, \psi, m)G_V(\xi, m), \quad (26c)$$
formulas for \( G_I, G_V \) and \( G_{IV} \) in (8), (12) and (14), respectively. Recall that \( \xi \) is independent of \( y \) and the common factor \( f(\xi, \psi, m) \) contains (via \( \psi \)) all of the \( y \)-dependence in these weights. This plays an important role in section 3.

Returning to the functions \( G_I \)–\( G_{IV} \), we can also derive relations analogous to the four-point crossing matrix directly from the vertex operator formalism. The integral of the screening charge around a small loop in the bulk containing no operators is zero. Deforming the contour to run along the boundary gives the blocks as described above, while picking up a complex argument of \( 2\pi \alpha - \alpha \) each time the contour passes a boundary charge \( \alpha \). Thus,

\[
0 = e^{8\pi i/\kappa} G_I + e^{4\pi i/\kappa} G_{II} + G_{III} + e^{-4\pi i/\kappa} G_{IV} + e^{-8\pi i/\kappa} G_{V}.
\]

Making use of (15) we find

\[
2G_{III} = (2 - n^2)G_I + nG_{II} + nG_{IV} + (2 - n^2)G_V, \quad \text{and}
\]

\[
nG_I + G_{IV} = G_{II} + nG_V,
\]

which also follow from (25a)–(25e). These two equations reflect the fact that the five solutions live in a three-dimensional solution space, which is consistent with the number of configurations in the top row of figure 2.

In the foregoing, we have simplified our figures and discussion by considering linear integration paths along the real line only. However, our results also hold for the more general paths entwining the operators required for \( \kappa \leq 4 \). Thus, our solutions and the identifications of their fusion channels extend automatically to the dilute phase of the \( O(n) \) model, and equivalently to the spin cluster behavior of the critical \( Q \)-state Potts models.

While the Coulomb gas formalism has considerable predictive power, our differential equation analysis has the advantage of specifying the form of the prefactor (6), and determining the dimensionality of the entire solution space without making \textit{a priori} assumptions about the properties of the density operator. In addition, (16) does not follow simply in vertex operator formalism.

The three cluster configurations illustrated in the top row of figure 2 account for the three-dimensional solution space when \( F(\xi, \psi, m) = G(\xi, m) \) as discussed above. In addition, there are three solutions that occur when the top and bottom rather than left and right sides are wired, as in the bottom row of figure 2. These additional solutions are implied by the \( \{\xi, \psi, m\} \leftrightarrow \{\psi, \xi, 1 - m\} \) symmetry (17). This exhausts the set of solutions that are consistent with the conformal weights and null-state conditions of the correlation function (1).

3. Factorization behavior

In this section, we consider a universal ratio of correlation functions that demonstrates an approximate factorization of certain higher order correlation functions (or the equivalent densities) in terms of less complicated correlation functions (or densities). The ratio was originally defined at the percolation point (\( \kappa = 6 \)), and is discussed in [11–14]. Here, we generalize it to a set of two-dimensional critical points. The generalized ratio is universal, constant (except near the sides of the rectangle), so that factorization occurs, and independent of the vertical coordinate \( y \).

In [14], the universal quantity

\[
\rho_{\text{perc}}(x, R) = \frac{P_{xR}(z, R)}{\sqrt{P_{xR}(z, R)P_{(x, R)x}(R)P_{xR}(R)}}
\]

was considered for critical percolation clusters. Here, \( P_{xR}(R) \) is the horizontal crossing probability given by Cardy, and \( P_{xR}(z, R) \) (respectively \( P_{xR}(z, R) \), \( P_{xR}(z, R) \)), already introduced in subsection 2.2.2, are the densities of clusters connecting to both (respectively right, left)
vertical sides of the rectangle. (These densities are given in terms of \(\Pi_A, \Pi_B\), and \(\Pi_C\) by (22d)–(22e) with \(n = 1\). Note that in this case \(\Pi_H + \Pi_V = 1\). Note also that all nonuniversal factors divide out of the ratio, so it has the same value if cluster densities in a specific model of percolation are used in place of the CFT results.)

The \(\gamma\)-independence of \(\rho_{\text{perc}}\) is discussed in [14] and herein.

In the limit \(R \to \infty\), \(\rho_{\text{perc}}\) is constant and \(P_i\) exactly factorizes in terms of two- and three-point functions. We showed that in fact \(\rho_{\text{perc}}(\infty) = 1.0299 \ldots\) is the fusion coefficient for three two-leg operators. See [11–13] for more details.

In a finite rectangle \(\rho_{\text{perc}}(x, R)\) is not quite constant, though it is independent of the vertical coordinate, as implied by (16). Instead \(\rho_{\text{perc}}(x, R)\) equals 1 at \(x = 0\) and \(R\), and decays exponentially, on a distance scale of the height of the rectangle, toward \(\rho_{\text{perc}}(\infty)\) within the bulk. Thus, \(\rho_{\text{perc}}\) is constant to within 3%, and the factorization holds to good approximation for arbitrary \(x\) and \(R\) at the percolation point.

The results of the previous sections allow us to generalize \(\rho_{\text{perc}}\) to arbitrary \(\kappa\). To begin, we introduce the notation \(I_r := \phi^r_{1,2}(0)\phi^r_{1,2}(i), I_r := \phi^r_{1,2}(R)\phi^r_{1,2}(R + i)\), and \(\sigma(z, \bar{z}) := \Phi_{1/21}(z, \bar{z})\) for the wired left and right sides of the rectangle and the magnetization operator at interior point \(z\), respectively. We then generalize the definition from [14] as

\[
\rho(z, \bar{z}, R) = \frac{\langle I_{[1,3]} \sigma(z, \bar{z}) \rangle}{\langle \sigma(z, \bar{z}) \rangle};
\]

the motivation for this particular choice is explained below. The subscripts index the propagating channel between sets of operators, thereby identifying the conformal block. The new correlations introduced here specify weights. Dividing by the appropriate normalization these weights become densities (if \(z\) dependent) or probabilities (if not).

The correlation functions \(I_r, I_r\) with four \(\phi^{1,2}\) operators, in the cases where they can be understood in terms of critical clusters, generalize Cardy’s horizontal percolation crossing probability, as discussed in subsection 2.2. The configurations in figure 1 determine the blocks (see (33d) and (33e) below), which are given by

\[
\langle I_{[1]} I_r \rangle = \langle \phi^r_{1,2}(i) \phi^r_{1,2}(0) \phi^r_{1,2}(R) \phi^r_{1,2}(R + i) \rangle = \lim_{\varepsilon_i \to 0} \left| \frac{w'(\varepsilon_1)w'(R - \varepsilon_2)w'(R + i - \varepsilon_3)w'(i + \varepsilon_4)}{16\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4} \right|^{h_{\text{bc}}}
\]

\[
\times \left[ \phi^{1,2}_{1,2}(K^{1,2}\varepsilon_3^2) \phi^{1,2}_{1,2}(0) \phi^{1,2}_{1,2}(m) \phi^{1,2}_{1,2}(1) \right] = (K')^{h_{\text{bc}}} G_F(1 - m).
\]

The particular form of the covariance factor in (30), due to the corner operators [25], compensates for the divergence of \(w'(z)\) in the corners of the rectangle. The details of this analysis are described in [1]. The conformal blocks for this four-point function were given in [21] for a multiple SLE process. Due to the factors arising from the corner operators we use the slightly modified form

\[
G_{1,1}(m) = \binom{2 - \frac{12}{\kappa}, 1 - \frac{4}{\kappa}; 2 - \frac{8}{\kappa}}{m}
\]

\[
G_{1,3}(m) = \Gamma(12/\kappa - 1)\Gamma(2 - 8/\kappa)\Gamma(8/\kappa)\Gamma(1 - 4/\kappa)\Gamma(2 \kappa - 1) m^{8/\kappa - 1} \binom{1 - \frac{4}{\kappa}, \frac{4}{\kappa}; \frac{8}{\kappa}}{m}. \]

We next deduce the form of the conformal blocks for the six-point function using the methods from section 2.3 and (26a)–(26c). Now two \(\phi^{1,2}\) operators that fuse in the \(\phi^{1,3}\) channel cannot represent the two ends of a single boundary arc. Also, while two \(\phi^{1,2}\) operators that fuse in the identity channel do not place any conditions on which configurations contribute to the block,
they do imply an extra factor of \( n \) for configurations in which they are connected by a single arc. It follows that

\[
\langle \ell_{\{1,3\}} \sigma(z, \bar{z}) \rangle_{\ell_{1,3}} = \Pi_A = f(\xi, \psi, m)G_{V1}(\xi, m)
\]

(33a)

\[
\langle \ell_{\{1,1\}} \sigma(z, \bar{z}) \rangle_{\ell_{1,1}} = \Pi_A + n\Pi_B = f(\xi, \psi, m)G_{II}(\xi, m)
\]

(33b)

\[
\langle \ell_{\{1,3\}} \sigma(z, \bar{z}) \rangle_{\ell_{1,3}} = \Pi_A + n\Pi_B = f(\xi, \psi, m)G_{IV}(\xi, m)
\]

(33c)

\[
\langle \ell_{\{1,1\}} \rangle = \Pi_H + n\Pi_V = (K')^{h_{12}}G_{1,1}(1 - m)
\]

(33d)

\[
\langle \ell_{\{1,3\}} \rangle = \Pi_H = (K')^{h_{12}}G_{1,3}(1 - m).
\]

(33e)

One may also write

\[
\Pi_V = (K')^{h_{12}}G_{1,3}(m).
\]

(33f)

The method of determining these contributions via \( \Pi_B \) uses \( O(n) \) loop model concepts, and thus applies when \( 8/3 \leq \kappa \leq 8 \). But conformal blocks are uniquely identified by the exponents in their various functional expansions, so once identified the results extend to all \( \kappa > 0 \).

Expression (29) for \( \rho(z, \bar{z}, R) \) thus becomes

\[
\rho(x, R) = \sqrt{\frac{\Pi_A^2(\Pi_H + n\Pi_V)}{(\Pi_A + n\Pi_B)(\Pi_A + n\Pi_B)\Pi_H}}
\]

(34)

\[
= \sqrt{\frac{G_{V1}(\xi, m)^2G_{1,1}(1 - m)}{G_{II}(\xi, m)G_{IV}(\xi, m)G_{1,3}(1 - m)}}.
\]

(35)

Note that the independence of \( \rho(x, R) \) from the vertical coordinate \( y \) is manifest. This feature may be traced back to (5).

The particular choice (29) for \( \rho(x, R) \) ensures that each conformal operator in the numerator has a counterpart in the denominator. This guarantees that we may replace each correlation function by its corresponding density, since all the nonuniversal scaling factors cancel regardless of the regularization scheme, so that the resulting ratio of densities is a universal conformally invariant quantity. We emphasize that this generalization is not unique. It does, however, minimize the number of conformal blocks that appear in the definition. Other reasons for why this generalization is natural become clear in the following discussion.

The left-hand sides of equations (22a)–(22c) and (21) then allow us to write the universal ratio \( \rho \) in terms of the densities \( P_1 \) and the crossing probability \( P_H \) for FK clusters in rectangles with independently wired vertical sides as

\[
\rho(x, R) = \frac{P_{12}(z, R)}{\sqrt{P_{1}(z, R)P_{1}(z, R)P_{2}(R)}},
\]

(36)

which naturally generalizes \( \rho_{\text{perc}} \) in (28) to \( 0 \leq Q \leq 4 \).

For spin or FK clusters in rectangles with mutually wired vertical sides one may write \( \rho \) in terms of the densities \( \tilde{P}_{1}(z, R), \tilde{P}_{1}(z, R), \tilde{P}_{1}(z, R) \) and crossing probability \( \tilde{P}_{H}(R) \) by using the right-hand sides of (22a)–(22c) and (21), respectively. By inverting these equations, we find

\[
\rho(x, R) = \sqrt{\frac{\tilde{P}_{1}(z, R)^2(n^2 - (n^2 - 1)\tilde{P}_{H}(R))}{[(n^2 - 1)\tilde{P}_{12}(z, R) - n^2\tilde{P}_{1}(z, R)][(n^2 - 1)\tilde{P}_{12}(z, R) - n^2\tilde{P}_{1}(z, R)\tilde{P}_{H}(R)]},
\]

(37)
As mentioned in section 2.2.1, when \( n = 1 \) the crossing probabilities in (21) and densities in (22a)–(22c) are identical for independent and mutually wired boundaries. Thus, in that case \( \rho \) has the same form for both boundary conventions.

One could define \( \rho \) differently so that (37) was simpler, e.g. by (36), with each \( P \) replaced with a \( \tilde{P} \). However, doing so results in an expression that does not reduce to a single OPE coefficient in the limit \( R \to \infty \) (see (38) below).

The ratio \( \rho(x, R) \) is normalized so that \( \rho = 1 \) when we take \( x \to 0 \) or \( R \). In terms of critical clusters, as \( z \) goes to the left-hand side it becomes vanishingly likely that \( z \) is not connected to the adjacent wired boundary and \( \Pi_{R_x} \to 0 \). In addition, \( \Pi_A \to \Pi_H \) and \( \Pi_{R_x} \to \Pi_V \), so that indeed \( \rho \to 1 \). An analogous argument holds as \( z \) goes to the right-hand side.

We can also take the long rectangle limit \( R \to \infty \) with \( 0 \ll x \ll R \). Then, the existence of a horizontal spanning cluster is exponentially unlikely so that \( \Pi_A \ll \Pi_{R_x}, \Pi_{R_x}, \text{and} \Pi_H \ll \Pi_V \). Furthermore, we can replace the distant left and right sides with point operators, and due to the invariance with respect to the vertical direction we can assume the magnetization operator sits on the boundary. Thus,

\[
\rho(x, R \to \infty) = \frac{\Pi_A}{\sqrt{n\Pi_{R_x}\Pi_H}} = \frac{\langle \phi_1(0)\phi_1(x)\phi_1(R) \rangle}{\sqrt{n\langle \phi_1(0)\phi_1(R) \rangle \langle \phi_1(x) \rangle \langle \phi_1(0)\phi_1(R) \rangle}} = \frac{C_{1,3,1,3}^3}{\sqrt{n}} \frac{\Gamma \left( \frac{16-\kappa}{\kappa} \right) \Gamma \left( \frac{12-\kappa}{\kappa} \right) \Gamma \left( \frac{8-\kappa}{\kappa} \right) \Gamma \left( \frac{6-\kappa}{\kappa} \right)}{n(\kappa) \Gamma \left( \frac{12-\kappa}{\kappa} \right) \Gamma \left( \frac{12-\kappa}{\kappa} \right) \Gamma \left( \frac{12-\kappa}{\kappa} \right)},
\]

where the final line uses the known expression for the OPE coefficient, given in [26]. (Recall that \( n(\kappa) \) is given in (15).) Note that the value of \( C_{1,3,1,3}^3 \) increases monotonically from 0 to \( 5\sqrt{2} \) as \( \kappa \) decreases from 8 to 2. Some values of the limit (38) for specific models are given in the captions to figures 6–10. For percolation \( (\kappa = 6) \) \( C_{1,3,1,3}^3 \) is close to 1, and the factorization is similar anywhere in the rectangle, since \( \rho = 1 \) at the edges. When \( C_{1,3,1,3}^3 \) is far from 1, however, the nature of the factorization is different near the edges and far from them.

Note that the long rectangle limit, as described above, fails when \( \kappa = 8/3 \). This value of \( \kappa \) corresponds to the dilute phase with \( n = 0 \) and from (34) it follows that \( \rho(x, R) = 1 \); in apparent contradiction to (38). This is because the two limits \( \kappa \to 8/3 \) and \( R \to \infty \) do not commute; vertical crossings are suppressed as \( n \to 0 \), while horizontal crossings are suppressed as \( R \to \infty \). This problem does not occur in the dense phase with \( n = 0 \) \( (\kappa = 8) \). Here, the limits do commute and \( \rho = 1 \), because the curves are space filling, so there is no suppression of horizontal crossings for large \( R \).

The form of \( \rho \) for the \( Q = 1 \), 2- and 3-state Potts models (for either FK or spin clusters) is given in figures 6–10 (solid curves) for various values of the aspect ratio \( R \). Note that for \( R = 2 \) and 3 there is virtually no difference between the \( \rho(x, R) \) curves for a given \( Q \) value when \( 0 < x < 1 \), although the spin clusters deviate slightly more. This is consistent with the observation made in [14] for percolation \( (Q = 1) \) that the semi-infinite strip result \( \rho(x, \infty) \) is a good approximation to \( \rho(x, R) \) for \( 0 < x < R/2 \) and \( R \gg 1 \).

Note that taking the limit \( R \to \infty \) with \( 0 \ll x \ll R \) (so that \( \rho(x, R) \) becomes a constant) is equivalent to bringing the pair of \( \phi_{i,2} \) operators in (1) on the left-hand side of the rectangle together, and likewise the pair on the right. Applying this to \( \langle I_{1,1,3}\sigma(z)I_{1,1,3}^{\dagger} \rangle \) one obtains a four-point correlation function with two \( \phi_1 \) operators; the other correlation functions in (29) can be treated similarly. The result generalizes our exact percolation factorization formula for the densities of clusters anchored to two points [12] to all the critical models mentioned here.
Figure 5. Percolation configurations discussed in this section. We use fixed boundary conditions on both (respectively right, left) vertical sides to calculate the densities corresponding to $B_r$, $B_l$, $A$ (respectively $C_r$, $C_l$). Due to locality, however, this decomposition of the densities holds regardless of boundary conditions. The top two rows represent the disjoint subsets that span the configuration space. The bottom row represents quantities that we calculate here which are composites, as indicated by the paths from their constituents in the upper rows.

By keeping $x$ near 0, one could similarly obtain results for the densities of clusters anchored to one interval and one point. We will examine the consequences of these observations elsewhere.

4. Density of percolation crossing clusters

We now restrict our attention to $\kappa = 6$, or critical percolation. We return to percolation, not just for its intrinsic interest, but because we are now in a position to exploit the property known as locality in the SLE literature [8]. This property allows us to use the results from subsection 2.3, with wired boundary conditions on the vertical sides, to derive an explicit expression for the density of clusters that cross a rectangle with free boundary conditions.

With respect to a growing SLE hull, ‘locality’ means that as the curve grows in time, its statistics are completely independent of boundaries up until the hull actually touches 1. Percolation exhibits this property because the decision to include or exclude a link is determined by a purely local random variable and not by the state of neighboring sites. This implies that the bulk sites all contribute in the same way to the overall weight of a configuration regardless of the state of the boundary sites.

It is this insensitivity to the boundary conditions that allows us to generalize the idea of wiring. Suppose we condition boundary sites in some way, and find the density of clusters attached to certain segments, as above. If we change the conditioning and measure the density again, then the two densities are directly comparable due to the fixed contribution of the bulk sites to the overall weight.

This is intrinsically different from a system without locality. In general, conditioning a set of boundary sites changes the degrees of freedom of the attached clusters, so boundary conditions are a defining characteristic of the system. For a system with locality we need not think of boundary conditions as fixed parameters; instead, they provide a variable means of probing the connectivities of clusters within the system.

We begin with the three configurations illustrated in figure 2, and proceed to decompose the contributions to $\Pi_A$, as in figure 5.
Figure 6. The upper graph compares numerical results (on lattices of about $10^6$ sites) for $\bar{\rho}(x,R)$ (points) against the CFT prediction (solid curve) for percolation ($\kappa = 6$). The plateau value agrees with 1.02993 (dashed line) as per (38). The standard deviation of $\rho(x,y,R)$ from $\bar{\rho}(x,R)$ is plotted in the lower graph.

Figure 7. Same as figure 6, but for $Q = 2$ FK cluster densities ($\kappa = 16/3$). The plateau value agrees with 1.07871 (dashed line), as per (38).

We distinguish configurations according to whether the point $z$ actually belongs to a crossing cluster, as in configuration type $A_x$, or whether it connects to the crossing cluster only through the wired boundaries on the right- or left-hand side, as in type $A_r$ and $A_\ell$, respectively. Thus

$$P_A = P_{A_r} + P_{A_\ell} + P_{A_x}.$$ (39)

Our goal is to determine the density for configurations of type $A_x$, since these clusters contribute to the horizontal crossing regardless of boundary conditions and represent the sole crossing clusters when the rectangle has free boundary conditions on all sides.
Figure 8. Same as figure 6, but for $Q = 3$ FK cluster densities ($\kappa = 24/5$). We increased the lattice size to about $4 \times 10^6$ sites for $R \geq 1$. The plateau value agrees with 1.15470 (dashed line) as per (38).

Figure 9. Results, for $\rho(x, R)$ (points) from densities of $Q = 2$ Potts spin clusters ($\kappa = 3$) on lattices of about $10^6$ sites, compared with theory (solid curve). The plateau agrees reasonably with the value 3.15123 (dashed line), as per (38). Data for $R = 1/2$ is not visible on this scale and is presented separately in figure 11.

Figure 10. Same as figure 9, but for $Q = 3$ spin cluster densities ($\kappa = 10/3$) with the lattice size increased to about $4 \times 10^6$ sites for $R \geq 1$. The plateau agrees with the value 2.08229 (dashed line), as per (38). See figure 11 for details of the $R = 1/2$ data.

For percolation, the expressions for the conformal blocks $G$ in (8)–(12) are equivalent to those calculated in [14]; however, (6) gives us an expression for the prefactor $f(\xi, \psi, m)$, which contains the explicit vertical dependence not previously available, thus allowing a complete calculation of the densities.
Now we consider the correlation functions given by wiring either the left- or right-hand side of the rectangle. In [11] we calculated the density of percolation clusters at a point \( w = u + i v \) in the upper half-plane that are attached to an interval on the real line \( I = (u_1, u_2) \):

\[
P_I(w) \sim |w - \bar{w}|^{-5/48} (2 \text{Im}[\eta^{1/4}])^{1/3}, \quad \eta = \frac{\bar{w} - u_1}{(w - u_1)(\bar{w} - u_2)}.
\]  

(40)

Note that for percolation the normalization factor \( \Pi_H + n \Pi_V = 1 \), so the weights found in [11] and also those from (1) are in fact densities.

We transform \( P_I(w) \) into the rectangle \( \mathcal{R} \) using the map \( w(z) = m \text{sn}(zK'|m)^2 \) from [1]. The interval \([-\infty, 0]\) will map to the left side and \([m, 1]\) maps to the right side; thus, the corresponding cross-ratios are

\[
\eta_L = \frac{w}{\bar{w}} = \exp(4i \text{arg}(\text{sn}(zK'|m)))
\]

\[
\eta_R = \frac{(\bar{w} - m)(w - 1)}{(w - u_1)(\bar{w} - u_2)} = \exp(4i \text{arg}(\text{dc}(zK'|m))).
\]  

(41)

Combining the half-plane densities with the conformal transformation factor \( |w'(z)|^{5/48} \) gives

\[
P_C(z, \bar{z}) = 2^{1/3} (K')^{5/48} \sin^{1/3}[\text{arg}(\text{dc}(zK'|m))] \left( \frac{\text{Im}[\text{sn}(zK'|m)^2]}{\text{sn}(zK'|m) \text{cn}(zK'|m) \text{dn}(zK'|m)} \right)^{5/48}.
\]  

\[P_C(z, \bar{z}) = 2^{1/3} (K')^{5/48} \sin^{1/3}[\text{arg}(\text{sn}(zK'|m))] \left( \frac{\text{Im}[\text{sn}(zK'|m)^2]}{\text{sn}(zK'|m) \text{cn}(zK'|m) \text{dn}(zK'|m)} \right)^{5/48},
\]  

when the right or left side of the rectangle is wired, respectively.

Using identities for elliptic functions and (6) we can rewrite these quantities in terms of the real coordinates (4):

\[
P_C(z, \bar{z}) = f(\xi, \psi, m)(\text{dc}^2(xK'|m) + m \text{cd}^2(xK'|m) - \text{dn}^2(yK'|1 - m))
\]

\[-m \text{nd}^2(yK'|1 - m)]^{1/6}
\]

\[= 2^{1/3} (K')^{5/48} \left[ \frac{1 - m \xi^2}{\xi(1 - \xi)(1 - m \xi)} + \frac{1 - (1 - m) \psi^2}{\psi(1 - \psi)(1 - (1 - m) \psi)} - 4 \right]^{11/96}
\]

\[\times \left[ 1 - m \xi + m \xi - \frac{(1 - (1 - m) \psi)^2 + m}{1 - (1 - m) \psi} \right]^{1/6}.
\]  

(42)

\[
P_C(z, \bar{z}) = f(\xi, \psi, m)(\text{ns}^2(xK'|m) + m \text{ns}^2(xK'|m) - \text{dn}^2(yK'|1 - m))
\]

\[-m \text{nd}^2(yK'|1 - m)]^{1/6}
\]

\[= 2^{1/3} (K')^{5/48} \left[ \frac{1 - m \xi^2}{\xi(1 - \xi)(1 - m \xi)} + \frac{1 - (1 - m) \psi^2}{\psi(1 - \psi)(1 - (1 - m) \psi)} - 4 \right]^{11/96}
\]

\[\times \left[ \frac{1}{\xi} + m \xi - \frac{(1 - (1 - m) \psi)^2 + m}{1 - (1 - m) \psi} \right]^{1/6}.
\]  

(43)

The coefficient \( 2^{1/3} = 2^{h_{13}} \) indicates the presence of a corner operator, as in (7). Note that under (17) and the invariances of \( f(\xi, \psi, m) \) [1], the expressions for \( P_C \) and \( P_C \) transform properly under mirror symmetry about \( x = R/2 \) or \( y = 1/2 \).

With these expressions for the densities \( P_C \) and \( P_C \), we can isolate the density of crossing clusters \( P_{Ac} \). Now

\[
P_C = P_{Bc} + P_{Ac} + P_{Ac}, \quad \text{and}
\]

\[
P_C = P_{Bc} + P_{Ac} + P_{Ac}.
\]  

(44)

(45)
Note that in (44) and (45) the left- and right-hand sides involve quantities originally defined with different boundary conditions. Only when there is locality does this difference become irrelevant.

We now check that the normalizations of the various quantities are consistent. Using (33c), (45) and figure 5, we find \( G_{IV} = P_{c_1} = P_{A_1} \). The latter is the density of points \( z \) that connect to the right-hand side but not to the left, in configurations with a crossing cluster. As \( x \to 0 \), our expression for this vanishes, as it must. Using (33b) and (44) we come to the same conclusion for \( G_{II} = P_{c_2} = P_{B_2} \) as \( x \to R \).

Thus, we arrive at an explicit formula for the density of clusters that cross the rectangle horizontally regardless of wiring at the sides (equivalently, cross a rectangle horizontally with open boundaries):

\[
\rho(x) = P_{c_1} + P_{c_2} - P_{A_1} - P_{B_1},
\]

\[
= 2^{1/3} (K')^{5/48} \left[ \frac{(1 - m \xi^2)^2}{\xi (1 - \xi) (1 - m \xi)} + \frac{(1 - (1 - m) \psi^2)^2}{\psi (1 - \psi) (1 - (1 - m) \psi)} - 4 \right]^{11/96} \times \left[ \frac{1 - m \xi}{1 - \xi} + m \frac{1 - \xi}{1 - m \xi} - \frac{(1 - (1 - m) \psi^2 + m)}{1 - (1 - m) \psi} \right]^{1/6} + \left( \frac{1}{\xi} + m - \frac{(1 - (1 - m) \psi^2 + m)}{1 - (1 - m) \psi} \right)^{1/6} - \frac{(1 - m)^{1/3}}{\xi (1 - \xi) (1 - m \xi)} \right].
\]

(The invariance of the last term within the bracket in (47) under mirror symmetry about \( x = R/2 \) follows from (17) and standard results for the Appell functions.)

5. Simulation results for \( \rho(x, R) \) and \( P_{A_1} \)

In this section, we present numerical results that verify the predictions (26a)–(26c) via calculation of the universal ratio \( \rho(x, R) \) discussed in section 3. Here, we simulated the \( Q = 1, 2, \) and 3-state Potts models on a square lattice in rectangles \( R \) of aspect ratios \( R = 1/2, 1, 2 \) and 3 using the SW algorithm [17] at the critical bond activation probability \( p_c(Q) = \sqrt{Q}/(1 + \sqrt{Q}) \). Both FK and spin clusters were considered, and we found good agreement between the predictions of this paper and simulation. We also verify (47), the density \( P_{A_1} \) of percolation clusters that cross horizontally with free boundary conditions via simulations.

5.1. Percolation: the universal ratio \( \rho(x, R) \)

When \( Q = 1 \), the rectangle is trivially filled by a single spin cluster, but the FK clusters give bond percolation. The left and right sides are wired leaving the top and bottom free, as described above. All the bonds on the wired edges are activated with each update, while those on the free edge are activated with critical probability \( p_c(1) = 1/2 \), like those in the bulk of \( R \).

In our simulations, we found the densities \( P_l(z, R), P_r(z, R) \) and \( P_{cr}(z, R) \) (see (22a)–(22c)) numerically by dividing the number of samples where \( z \) is connected to the left (respectively, right, left and right) side(s) through activated bonds by the total number of samples. We also measured the probability of horizontal cluster crossings \( P_h(R) \), given by (21). We considered
aspect ratios $R = 1/2, 1, 2, 3$, and for each $R$, the length and width of the rectangle were chosen to keep its area equal to approximately $10^6$ square lattice spacings.

The number of samples generated for each aspect ratio are shown in the top row of table 1.

Using our numerical densities and crossing probabilities we calculate $\rho_{\text{perc}}(x, y, R)$ in (28). Theory predicts that this quantity is independent of $y$, which is not strictly true of the numerical result due to finite size effects and finite sample sizes. For the purpose of comparison we define $\bar{\rho}_{\text{perc}}(x, R)$ to be the average of $\rho_{\text{perc}}(x, y, R)$ over $y$.

In figure 6, we plot simulation results for $\bar{\rho}_{\text{perc}}(x, R)$ and the CFT prediction (35). The agreement is excellent. As we expect, when $x$ approaches $0$ or $R$, $\rho_{\text{perc}}$ approaches $1$, and when $0 \ll x \ll R$ for large $R$, it reaches the plateau predicted in (38). The lower plot in figure 6 is the standard deviation of $\rho_{\text{perc}}(x, R)$ from $\bar{\rho}_{\text{perc}}(x, R)$ for fixed $x$ and $R$. Note that the standard deviation is at least three orders of magnitude smaller than $\rho_{\text{perc}}$, verifying the independence of $\rho_{\text{perc}}$ from $y$.

Some of our percolation results overlap with the simulations in [14]. However, we include them here for ease of comparison with our results for Potts models at $Q = 2$ and $3$, which are new.

5.2. $Q = 2$ and $3$ FK clusters: the universal ratio $\rho(x, R)$

For the $Q = 2$ and $3$ Potts model we consider two different types of clusters. In this section, we consider FK or critical clusters of Potts spins, while in section 5.3 we consider geometric clusters of Potts spins. In the continuum limit, boundaries of FK clusters attached to the wired sides of $\mathcal{R}$ correspond to dense SLEs with parameter $\kappa$ related to $Q = n^2$ via (15), thus when $Q = 2$ (respectively $3$), $\kappa = 16/3$ (respectively $24/5$).

We simulate FK clusters of the $Q$-state Potts model in a rectangle with independently wired left and right sides and free top and bottom sides. As discussed in section 2.2, in the FK representation all bonds are activated along a wired boundary and bonds along a free boundary are activated with the critical bulk probability $p_c(Q)$. From these simulations we measure the densities $P_t(z, R)$, $P_r(z, R)$ and $P_{t,r}(z, R)$ and the crossing probability $P_c(R)$ by the same method described in section 5.1 for $Q = 1$ FK clusters. We then use the results and (36) to calculate $\rho(x, y, R)$.

The system size and sample size were chosen differently for $Q = 2$ and $Q = 3$. When $Q = 2$, the length and width of $\mathcal{R}$ were chosen so that the area is approximately equal to $10^6$ square lattice spacings. When $Q = 3$, we noted that the numerical results for $\rho(x, y, R)$ in a rectangle of $10^6$ square lattice spacings were noticeably lower than the theory prediction, so in this case we quadrupled the area of $\mathcal{R}$ to about $4 \times 10^6$ square lattice spacings when $R = 1, 2$ or $3$, which improved the agreement. The number of samples generated for each aspect ratio are shown in table 1.

| $R$  | $Q = 1$   | $Q = 2$   | $Q = 3$   |
|------|-----------|-----------|-----------|
|      | $5 \times 10^6$ | $5 \times 10^6$ | $5 \times 10^6$ |
| FK   |           |           |           |
| FK   |           |           |           |
| FK   |           |           |           |
| Spin |           |           |           |
| Spin |           |           |           |

Table 1. The number of samples generated in simulations of the Potts model in a rectangle with the given aspect ratio.
In figure 7, we plot values of $\tilde{\rho}(x, R)$ (again we have averaged $\rho(x, y, R)$ over $y$) calculated from our simulations for $Q = 2$ and observe good agreement with the theoretical prediction (35). In figure 8, we plot values of $\tilde{\rho}(x, R)$ computed from our $Q = 3$ simulations. Again, the agreement with theory is good. In both cases we see that when $x$ is sufficiently far from the endpoints and $R$ is large ($R = 3$), $\tilde{\rho}(x, R)$ roughly plateaus at the appropriate value given by (38), which is marked in the figures with the dashed line. When $x$ nears 0 or $R$, $\tilde{\rho}(x, R)$ decays to 1.

In both cases we found the standard deviation in $y$ of our numerical $\rho(x, y, R)$ from $\tilde{\rho}(x, R)$ as a function of $x$ for fixed $R$ as we did with our percolation simulations. When plotted as a function of $x$ the standard deviation resembles the lower graph in figure 6 but with greater magnitudes. To save space we do not plot the deviations, but instead we give their average over $x$ for fixed $R$ in table 2. For small $R$, these deviations are less than 0.018%, and for larger $R$, they are less than 0.37%.

### Table 2. The averages over $x$ of the standard deviation in $y$ of our numerical values of $\rho(x, y, R)$ for fixed $R$ for our various simulations.

| $R$ | FK $Q = 1$ | FK $Q = 2$ | FK $Q = 3$ | Spin $Q = 2$ | Spin $Q = 3$ |
|-----|-------------|-------------|-------------|-------------|-------------|
| 1/2 | $1.0 \times 10^{-4}$ | $3.8 \times 10^{-4}$ | $7.4 \times 10^{-4}$ | $5.8 \times 10^{-5}$ | $1.1 \times 10^{-4}$ |
| 1   | $2.4 \times 10^{-4}$ | $7.9 \times 10^{-4}$ | $2.2 \times 10^{-4}$ | $1.0 \times 10^{-3}$ | $2.3 \times 10^{-2}$ |
| 2   | $3.8 \times 10^{-4}$ | $7.9 \times 10^{-4}$ | $2.2 \times 10^{-4}$ | $1.0 \times 10^{-3}$ | $2.3 \times 10^{-2}$ |
| 3   | $7.4 \times 10^{-4}$ | $1.6 \times 10^{-3}$ | $3.7 \times 10^{-3}$ | $7.2 \times 10^{-2}$ | $4.0 \times 10^{-2}$ |

5.3. $Q = 2$ and 3 Potts spin clusters: the universal ratio $\rho(x, R)$

In the continuum limit, the boundaries of spin clusters attached to a fixed boundary are dilute SLE curves with parameter $\kappa = 16/\kappa_{\text{FK}}$, dual to the FK cluster parameter as described in section 2.2. When $Q = 2$, $\kappa = 3$ and when $Q = 3$, $\kappa = 10/3$. As mentioned in subsection 2.2, the spins on the left and right hand sides of $\mathcal{R}$ must be spin $A$ while the top and bottom are summed freely over all spins other than $A$. Thus, the left and right sides are necessarily mutually wired.

Here, we proceed as in subsection 5.2, but because of the boundary conditions that we must use, we will measure $\tilde{P}_t(z, R)$, $\tilde{P}_z(z, R)$, $\tilde{P}_{z,t}(z, R)$, $\tilde{P}_{t R}(R)$ instead of the corresponding un-barred quantities.

Because clusters of spin $A$ cannot touch the top or bottom of the rectangle, maintaining the boundary condition change at the corners with each update is different for the spin clusters than for FK clusters. Suppose we start with some arbitrary spin configuration inside $\mathcal{R}$ that satisfies the necessary boundary conditions. To update the system, we activate all bonds on the left and right sides and grow the FK cluster anchored to the left side via the SW algorithm. Note that this FK cluster will not touch the top and bottom since these sides contain no $A$ spins. If the left cluster finishes its growth without hitting the right edge, then we repeat this process from the right side. We assign spin $A$ to the FK clusters anchored to either side. Next, we grow the FK clusters anchored to the top and bottom, and assign them a spin other than $A$ with uniform probability. Finally, we grow FK clusters from sites in the interior of $\mathcal{R}$ whose spin is not yet updated, placing no restriction on the new spin.

For given $Q$ and $R$, the rectangle size is the same as for the respective FK cluster simulation. Our original simulations on $10^6$ site rectangles for $Q = 3$ when $R \geq 1$ produced results that
were higher than the prediction of (37), which contrasted with the FK case where the numerics originally seemed too low. Again, increasing the system size alleviated this effect.

In figure 9 (respectively 10), we plot values of $\bar{\rho}(x, R)$ measured in our $Q = 2$ (respectively $Q = 3$) simulations. The data at $R = 1/2$, too small to see, is plotted on a more visible scale in figure 11. The agreement is reasonably good. In both cases, when $x$ approaches 0 or $R$, $\bar{\rho}(x, R)$ goes to 1 and when $R = 3$ and $x$ is away from the endpoints, $\bar{\rho}(x, R)$ roughly plateaus at the value given by (38).

Again, standard deviations of $\rho(x, y, R)$ from $\bar{\rho}(x, R)$ are shown in table 2. For small $R$, these deviations are less than 0.011%, and for larger $R$, they are less than 7.2%.

5.4. Density of percolation crossing clusters with free boundary conditions

The density $P_{A_x}$ of horizontal percolation crossing clusters with free boundary conditions on all sides of $R$ is predicted by (47). The density $P_{A_x}$ is not universal and depends on the regularization of our density operator. We remedy this by normalizing the theory and simulation data so that both equal 1 in the center of the rectangle. In order to simulate $P_{A_x}$, we preceded similarly to what was done in [14], in which we grew clusters from all the sites at one edge of the rectangle. For those sites in clusters that reached the opposite edge, we added 1 to their element in a raw density array. Dividing this array by its value for the center site in the rectangle produced the normalized density. This differed from the simulations of section 5.1, as we did not include contributions from the non-crossing clusters attached to the left and right sides in our raw density. We simulated bond percolation on a square lattice of $128 \times 256$ sites, carrying out about $10^8$ samples.

In figure 12, we show a comparison between the theory and the measured values. The contour lines are at heights 0.5, 0.6, 0.7, 0.8, 0.87, 0.93, 0.96, 0.98, 0.994, 0.999, from the outside inward. The deviations are quite small. For various fixed $x$ values, we found the following relative errors (absolute value of the difference between predicted and measured value divided by the predicted value) averaged along the $y$ coordinate: $(x, \text{error}) = (1, 0.0094), (0.5, 0.0057), (0.25, 0.0038), (0.125, 0.0036)$. And for various fixed $y$ values: $(y, \text{error}) = (0.5, 0.011), (0.25, 0.0055), (0.125, 0.0030), (0.0625, 0.0021).

6. Summary

In this section, we summarize our main results in some detail, and reproduce the main formulas. The research done here exploits, for a variety of two-dimensional critical models, the solutions for a chiral six-point correlation function in order to specify the density of critical clusters anchored to one or both vertical sides of a rectangle with wired boundary conditions on those sides and free boundary conditions on the horizontal sides. These models include the critical
Subsection 2.1 reviews the solutions found in [1] for the correlation function (1)
\[ C(\zeta) = \langle \phi_{1,2}^c(0)\phi_{1,2}^c(\zeta)\phi_{1,2}^c(R)\phi_{1,2}^c(R+i) \rangle_{\mathcal{R}}, \]
in the rectangular geometry \( \mathcal{R} := \{ \zeta = x+iy \in \mathbb{C} | 0 < x < R, 0 < y < 1 \}. \) Here, \( C(\zeta) \) also depends on the parameter \( m \) that specifies the aspect ratio of the rectangle (see (2)), and the solutions hold for an arbitrary SLE parameter \( \kappa > 0. \) They appear in the form (5)
\[ C(\zeta) = f(\xi, \psi, m)G(\xi, m), \]
where the algebraic prefactor \( f \) is given by (6) and (7). It is independent of boundary conditions, and in fact a function of \( x \) and \( y \) via the coordinates \( \xi = \xi(x, m) \) and \( \psi = \psi(y, m) \), given by elliptic functions (see (4)). \( \xi \) and \( \psi \) are the natural coordinates for this problem, as explained in [1]. The factor \( G \) is a single conformal block, given by an algebraic factor and an Appell hypergeometric function (see (8)–(12) and (14)). \( G \) depends on boundary conditions, and determining how the various \( G \)s contribute to the cluster densities for the different boundary conditions of interest is one of our main results. (\( G \) can depend either on \( \xi \) and \( m \), as written here, or alternatively on \( \psi \) and \( m \) as mentioned in subsection 2.1, depending, respectively, on whether the right and left or top and bottom sides of the rectangle are wired.)

Subsections 2.2 and 2.3 next examine the question of expressing cluster densities (respectively horizontal crossing probabilities) for the boundary conditions of interest (see figure 2 (respectively figure 1)) in terms of the \( G \)s (respectively hypergeometric functions), and also explains how the various cases are implemented in specific models. Making use of results in [21], an explicit form for a horizontal crossing probability that generalizes Cardy’s horizontal crossing probability for percolation to these models is given in (21) (see also the lines just below this equation). Expressions for the density \( P_\ell(z, R) \) (respectively \( P_r(z, R), P_{lr}(z, R) \) of clusters anchored to the left (respectively right, left and right) side(s) of a rectangle with independently wired left and right sides in terms of the weights \( \Pi_A, \Pi_B, \Pi_{Br}, \Pi_{H} \) and \( \Pi_V \) (and the corresponding densities \( \tilde{P}_\ell(z, R) \) (respectively \( \tilde{P}_r(z, R), \tilde{P}_{lr}(z, R) \)) for mutually wired sides) are given in (22a)–(22c). (Explicit formulas for \( \Pi_H \) and \( \Pi_V \) are found in (31), (32), (33e) and (33f).) Subsection 2.3 then employs Coulomb gas methods to find solutions for the weights \( \Pi_A, \Pi_B, \) and \( \Pi_{Br} \) (the first row of figure 2). These are given in the three equations

![Figure 12. Density of percolation crossing clusters at \( R = 2 \). The solid curves are predictions from (47), data points from simulations. Data deviate more from the prediction near the center since \( P_{\ell,A} \) is almost flat there. See section 5.4 for values.](image-url)
\[ (26a) - (26c): \]
\[
\Pi_A = f(\xi, \psi, m)G_{V1}(\xi, m)
\]
\[
\Pi_{B_1} = f(\xi, \psi, m)G_{f1}(\xi, m)
\]
\[
\Pi_{B_2} = f(\xi, \psi, m)G_{V1}(\xi, m).
\]

The corresponding expressions for the configurations in the bottom row of figure 2 follow by symmetry. This completes our determination of the densities for the models and boundary conditions of interest here.

Section 3 then discusses the factorization behavior implied by our solutions, which generalizes previous results for percolation to a variety of critical models. For percolation, the universal ratio (28)
\[
\rho_{\text{perc}}(x, R) = \frac{P_H(z, R)}{\sqrt{P_H(z, R)P_V(z, R)P_R(R)}}
\]
was considered [14], with \( P_H(R) \) the horizontal crossing probability given by Cardy. In that case, \( \rho \) is constant to within 3\% inside the rectangle, being equal to 1 when \( z \) is on the left or right side, and rising to about 1.03 when \( z \) is far from the sides. The numerator, a six-point correlation function, therefore factorizes into the lower order correlations in the denominator to very good approximation. In this section, we generalize \( \rho \) to an expression (29) defined in terms of correlation functions restricted to be single conformal blocks by specification of the propagating channels. This then leads to the expression for \( \rho \) (35) in terms of the weights \( \Pi_A, \Pi_{B_1}, \Pi_{B_2}, \Pi_H \) and \( \Pi_V \) or equivalently the conformal blocks \( G \) and the quantities \( G_{1,2} \) and \( G_{1,3} \), valid for all the critical models mentioned. In all cases, by definition \( \rho = 1 \) when \( z \) is on the left or right side. It is also constant when \( z \) is far from the sides, but how much that constant deviates from 1 depends on the model (more exactly, on \( \kappa \)). Thus, the factorization changes significantly with \( x \) in some models. However, in all cases \( \rho \) is independent of \( y \), as in percolation, because the algebraic prefactor \( f \), which also depends on \( y \) via \( \psi \), divides out of the ratio, leaving only conformal blocks \( G \) and other factors that are independent of \( y \). This \( y \)-independence is a consequence of the unusual symmetry of the conformal blocks \( G \) found in [1].

Next, section 4 makes use of the locality property of percolation to derive the explicit result (47) giving the density of percolation clusters that cross a rectangle horizontally with free boundary conditions on all sides:
\[
\begin{align*}
P_{\text{perc}} &= 2^{1/3}(K')^{5/48} \left[ \frac{(1 - m^2)}{\xi (1 - \xi)} + \frac{(1 - (1 - m)^2)}{\psi (1 - \psi)} - 4 \right]^{11/96} \\
&\times \left[ \frac{1 - m^2}{1 - \xi} + m \frac{1 - \xi}{1 - m^2} - (1 - (1 - m)^2)^{1/6} \psi (1 - \psi)^{1/6} \right] \\
&+ \left[ \frac{(1 - m^2)}{1 - \xi} + m \frac{1 - \xi}{1 - m^2} - (1 - (1 - m)^2)^{1/6} \psi (1 - \psi)^{1/6} \right] \\
&= \frac{(1 - m)^{1/3}}{\xi (1 - \xi)} \left[ F_1 \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3} | m, m^2 \right) \right],
\end{align*}
\]

with \( F_1 \) the Appell function (13) and \( K'(m) = K(1 - m) \), with \( K \) the complete elliptic integral.

In section 5 we compare our predictions for the ratio \( \rho \) (see (35)), which governs factorization, for the \( Q = 1-, 2- \) and 3-state Potts models, including both FK and spin clusters,
for various aspect ratios. The agreement is very good. The formula (47) for the density of crossing clusters in percolation is also verified.

The appendix presents forms for the densities and crossing weights in models where the Appell or Gaussian hypergeometric functions simplify.

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Appendix. Special parameter values

In this appendix, we consider special values of the parameter $\kappa$ for which any of the hypergeometric series that occur in the density conformal blocks or crossing weights (see (8)–(12) and (31)–(32), respectively) may be written in terms of simpler functions. In particular, we look at $\kappa$ values for which the functions have non-positive integer parameters, as this may imply that the series truncates into hypergeometric series in fewer variables, or even into a polynomial. There are five such quantities proportional to the Appell function $F_1$, appearing in (9)–(11) and (8), (12), respectively, or as one of the functions $G_{1,1}(m)$ and $G_{1,3}(m)$, defined in (31), (32), which are given by a hypergeometric function $\, _2F_1$. In what follows, we consider values $2 \leq \kappa \leq 8$ for which at least one of the three hypergeometric parameters $2 - 8/\kappa, 2 - 12/\kappa$ or $2 - 16/\kappa$ is a non-positive integer. There are ten such values: $\kappa \in \{8, 6, 16/3, 4, 16/5, 3, 8/3, 12/5, 16/7, 2\}$, which correspond to $n \in \{0, \pm 1, \pm \sqrt{2}, \pm 2\}$ in the critical $O(n)$ loop model dilute phase and $n \in \{0, 1, \sqrt{2}\}$ in the dense phase.

The common pre-factor given in (6) is already an algebraic function in $\xi$ and $\psi$. Thus, for brevity we will not always include it, and restrict our attention to the functions (8)–(12), (14), (31) and (32).

We also note that in some cases hypergeometric series with rational parameters of small denominator can be written in terms of relatively simple functions. The ten $\kappa$ values listed, as mentioned, are all those with $2 \leq \kappa \leq 8$ for which at least one of the three hypergeometric parameters $2 - 8/\kappa, 2 - 12/\kappa$ or $2 - 16/\kappa \in [0] \cup \mathbb{Z}^-$. In addition, the other hypergeometric parameters for all five of the functions mentioned are rational, with denominator of 4 or less. Because the denominator value has a pronounced effect on the type of functions that appear upon simplifying the hypergeometric series, and because the associated values of $n$ for each denominator of 4 or less all have the same magnitude, we group these values together.

\[ u_1(s, t) := F_1 \left( 1 - \frac{4}{\kappa}; \frac{4}{\kappa}, 2 - \frac{16}{\kappa}; 2 - \frac{8}{\kappa} \middle| s, t \right) \]  \hspace{1cm} (A.1)\]

\[ u_2(s, t) := \frac{\Gamma(2 - 8/\kappa)\Gamma(16/\kappa - 1)}{\Gamma(12/\kappa)\Gamma(1 - 4/\kappa)} F_1 \left( 1 - \frac{4}{\kappa}; \frac{4}{\kappa}, \frac{4}{\kappa}, \frac{12}{\kappa} \middle| s, t \right), \]  \hspace{1cm} (A.2)\]
A.1. Integer hypergeometric parameters: \(|n| = 2\)

A.1.1. Dilute \(n = 2\), Gaussian free field. The parameter value \(\kappa = 4\) corresponds to the critical Gaussian free field or to the \(Q = 4\) state Potts model. For this value the various hypergeometric functions are

\[
\begin{align*}
    u_1(s, t) &= \frac{2 - s - t(2 - t)}{2(1 - s)} \quad u_2(s, t) = \frac{1}{2}, \\
    G_{1,1}(m) &= \frac{2 - m}{2} \quad G_{1,3}(m) = \frac{m}{2}.
\end{align*}
\]

The conformal blocks are proportional to

\[
G_i(\xi, m) = \frac{m(1 - m)\xi^2}{\sqrt{4m(1 - m)\xi(1 - \xi)(1 - m\xi)}}.
\]

Expression (35) for \(\rho\) becomes

\[
\rho(x) = \frac{1 + m}{\sqrt{1 + mcn^2(2K'x|m)}}, \tag{A.3}
\]

where double argument identities for the Jacobi elliptic functions have been used.

A.1.2. Dilute \(n = -2\), loop-erased random walk. An SLE with parameter \(\kappa = 2\) has been shown to correspond to the loop erased random walk [27]. However, the loop fugacity is less than zero for that model, so the correlation function \(C(z)\) does not correspond to a set of densities. As \(\kappa \to 2\) the hypergeometric functions become

\[
\begin{align*}
    u_1(s, t) &= \frac{1 + s - 3t}{2} + \frac{(1 - t)^3}{2(1 - s)}(1 - 2s + 2t - st) \quad u_2(s, t) = \frac{s + t - 3}{2}, \\
    G_{1,1}(m) &= 1 - 2m + m^2 - \frac{m^4}{2} \quad G_{1,3}(m) = -m^3 + \frac{m^4}{2}.
\end{align*}
\]

And we identify a few key functions:

\[
\begin{align*}
    G_1(\xi, m) &= \frac{m(1 - m)\xi^3[3 - \xi - m\xi]}{2\xi(1 - \xi)(1 - m\xi)^{3/2}}, \\
    G_5(\xi, m) &= -\frac{m(1 - \xi)^3[2 - m + \xi - 2m\xi]}{2(1 - m)^2[\xi(1 - \xi)(1 - m\xi)]^{3/2}}, \\
    G_{11}(\xi, m) &= -\frac{(1 - m)[1 + m - 3m\xi + 3m^3\xi^5 - m^3\xi^6 - m^4\xi^6]}{2m^2[\xi(1 - \xi)(1 - m\xi)]^{3/2}}, \\
    &= -\frac{(1 - m)[4 + 4m - 3msn^2(2K'x|m)]}{m^2sn^2(2K'x|m)}.
\end{align*}
\]
A.2. Half-integer hypergeometric parameters: \( n = 0 \)

For fugacity \( n = 0 \) in either the dense (\( \kappa = 8 \)) or dilute (\( \kappa = 8/3 \)) phase \( \rho(x, R) = 1 \), as discussed. A peculiar property of \( n = 0 \) models is the absence of bulk loops. Therefore, every point (except for those that belong to the SLE hulls) is adjacent to some part of the boundary, so that the bulk density operator is a generalization of the zero weight indicator operator used by Schramm in [28].

Schramm’s paper considered a single SLE hull in a simply connected domain, and calculated the probability that the SLE passed to the left of a marked point in the bulk. This is equivalent to a correlation function with two SLE operators generating the SLE hull, and a bulk indicator at the marked point that returns 1 if the hull passes to its left and 0 otherwise. If it is possible (not possible) to draw a path from the point to the right boundary without crossing the hull, the hull passes left (right) of the marked point. Thus, when \( n = 0 \) and the only loop in the configuration is the hull itself the interpretation of the indicator and the density operator are identical, as evidenced by the weight \( 2h_{1/2,0} = 0 \), which is also the weight of Schramm’s operator.

In the current correlation function, the bulk operator acts as an indicator operator in the presence of the two SLE hulls generated by the four \( \phi_{1,2} \) operators. The indicator distinguishes between six cases: three correspond to the possible locations of a point relative to a horizontal crossing of hulls (configurations \( A, B_h, B_t \) in figure 2), and three correspond to the analogous case for vertical crossing (configurations \( \bar{A}, B_v, B_B \)).

The sum of the weights including horizontal crossings equals the total weight of horizontal crossings. A similar result holds for vertical crossings. Thus,

\[
\Pi_H \propto \Pi_A + \Pi_B_h + \Pi_B_t \tag{A.4}
\]

\[
\Pi_V \propto \Pi_A + \Pi_B_v + \Pi_B_B \tag{A.5}
\]

The weights from the bottom row of figure 2 follow from (26a)–(26c) upon letting \( (\xi, m, A, B_h, B_t) \rightarrow (\psi, 1 - m, \bar{A}, B_v, B_B) \). To emphasize the relations between the densities and crossing weights, we include the pre-factor (6) in this section.

A.2.1. Dense \( n = 0 \), Peano curve. We first examine the dense phase with \( \kappa = 8 \). This case is equivalent to a space-filling Peano curve. The relevant hypergeometric functions reduce to elliptic integrals on applying well-known identities:

\[
u_1(x, y) = G_{1,1}(x) = G_{1,3}(x) = {}_2F_1 \left( \frac{1}{2}, 1; \frac{3}{2} \middle| x \right) = \frac{2}{\pi} K(x)
\]

\[
u_2(x, y) = \frac{2}{\pi} {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \middle| x, y \right) = \frac{2}{\pi} \sqrt{y} F \left( \sin^{-1} \sqrt{\frac{y}{x}} \right),
\]

where \( F(\cdot|m) \) is the incomplete elliptic integral of the first kind. This satisfies \( F(\sin^{-1} \sin(\xi|m)|m) = \xi \), i.e. it acts as an inverse elliptic function.

For \( \kappa = 8 \) the function \( f_{\kappa=8}(\xi, \psi, m) = (K')^{-1} \left[ m(1 - m) \right]^{-1/4} \) and we find

\[
\Pi_H = \frac{2K'}{\pi K'} = \frac{2}{\pi}
\]

\[
\Pi_V = \frac{2K}{\pi K'} = \frac{2R}{\pi}
\]

\[
\Pi_{B_v} = \frac{2}{\pi K'} F \left( \sin^{-1} \sqrt{\frac{1}{m}} \right) = \frac{2x}{\pi}
\]

\[
\Pi_{B_B} = \frac{2}{\pi K'} F \left( \sin^{-1} \sqrt{\xi} \right) = \frac{2x}{\pi}
\]
density of the trees, this means that clusters, one attached to the top edge and the other to the bottom. Along with the uniform
when the UST spans the rectangle from left to right there are two components to the dual
the left and right edges we prevent the connection of the top and bottom and vice versa. Thus,
throughout the rectangle.

Only one of the UST and its dual can be a single-component tree, since by connecting
the vertical sides and its dual, which is another UST anchored to the horizontal
operations in (17).

The Peano curve is equivalent to the boundary between the uniform spanning tree (UST) [27] anchored to the vertical sides and its dual, which is another UST anchored to the horizontal sides. As the curve is space filling, any infinitesimal neighborhood intersects both the UST and its dual, so the densities of spanning trees and dual spanning trees are equal and uniform throughout the rectangle.

A similar result holds when the left and right sides belong to different trees, which implies a vertical crossing by the dual tree.

A.2.2. Dilute n = 0, self-avoiding walks. The value of $\kappa = 8/3$ corresponds self-avoiding walks. For $\kappa = 8/3 + \epsilon$ the hypergeometric expressions all diverge with $\Gamma(2 - 8/\kappa) \sim -8/9\epsilon$, as $\epsilon \to 0$. We adjust the normalization of $u_1$, $u_2$, $G_{1,1}$ and $G_{1,3}$, dividing by a factor of $-\Gamma(2 - 8/\kappa)$ to eliminate this divergence:

$$
\tilde{G}_{1,1}(m) := \frac{G_{1,1}(m)}{-\Gamma(2 - 8/\kappa)} = \frac{15}{32} m^2 (1-m)^2 \text{F}_1 \left( \frac{7}{2}, \frac{3}{2}; \frac{3}{2}; m \right) = \left[ 1 - m + m^2 \frac{E(m)}{\pi} - \frac{2 - 3m + m^2}{2\pi} K(m) \right]
$$

$$
\tilde{G}_{1,3}(m) := \frac{G_{1,3}(m)}{-\Gamma(2 - 8/\kappa)} = \tilde{G}_{1,1}(m)
$$

$$
\tilde{u}_1(s,t) := \frac{u_1(s,t)}{-\Gamma(2 - 8/\kappa)} = \left[ \frac{(s - 2st + t^2)^2}{s^2(1-s)^2} - \frac{8t(s-t)(1-t)}{5s(1-s)} \frac{d}{ds} \right] \tilde{G}_{1,1}(s),
$$

where $E(m)$ is the complete elliptic integral of the second kind. Thus, similar to $\kappa = 8$, the Gaussian hypergeometric functions are replaced by complete elliptic integrals.

The crossing weights are

$$
\Pi_H = (K')^5 \left[ \frac{1 - m + m^2}{\pi} E(1-m) - \frac{m(1+m)}{2\pi} K(1-m) \right]
$$

$$
\Pi_V = (K')^5 \left[ \frac{1 - m + m^2}{\pi} E(m) - \frac{2 - 3m + m^2}{2\pi} K(m) \right].
$$
The density pre-factor is
\[
f_{κ=8/3}(x, y, m) = \frac{(K')^5 [m(1-m)]^{5/4}}{ds^2(2K'x|m) + ds^2(2K'y)[1-m]},
\]
which, along with the simplified expressions for the various \(G(ξ, m)\) functions, gives
\[
\begin{align*}
Π_{B_l} + Π_{B_r} &= \frac{4ds^2(2K'x|m)Π_V - (8/5)m(1-m)\partial_m Π_V}{ds^2(2K'x|m) + ds^2(2K'y)[1-m]} \\
Π_A &= \frac{4ds^2(2K'x|m)Π_H - (8/5)m(1-m)\partial_m Π_H}{ds^2(2K'x|m) + ds^2(2K'y)[1-m]} \\
Π_{B_l} + Π_{B_r} &= \frac{4ds^2(2K'y)[1-m]Π_V + (8/5)m(1-m)\partial_m Π_V}{ds^2(2K'x|m) + ds^2(2K'y)[1-m]} \\
Π_π &= \frac{4ds^2(2K'y)[1-m]Π_V + (8/5)m(1-m)\partial_m Π_V}{ds^2(2K'x|m) + ds^2(2K'y)[1-m]}
\end{align*}
\]
after simplifying with double argument identities for the elliptic functions. With the densities written in this way, relations (A.4) and (A.5) follow immediately.

A.3. Hypergeometric parameters with denominator 3: \(|n| = 1\)

For these particular parameters (κ = 12/5, 3 or 6) the hypergeometric functions do not simplify beyond the truncation of \(G_{1,1}(m)\) that occurs because \(2 - 12/κ \in [0]∪\mathbb{Z}\).

The truncated functions, which are related to the identity channel of the crossing weights (see (33d)), are
\[
G_{1,1}(m) = \begin{cases} 
1 & κ = 6 \\
1 - m + m^2 & κ = 3 \\
(1 - 2m)(1 + m)(1 - m/2) & κ = 12/5.
\end{cases}
\]

The SLE parameter κ = 6 corresponds to critical percolation, or equivalently the dense phase of the O(1) loop gas. This is a purely probabilistic limit. Here, the form of the identity channel crossing weight is such that the sum of all crossing probabilities is 1.

The SLE parameter κ = 3 corresponds to spin clusters in the critical Ising model, or equivalently the dilute phase of the O(1) loop gas. This is proportional to the Ising partition function in a rectangle with the horizontal and vertical edges fixed in opposite spins.

The SLE parameter κ = 12/5 corresponds to the dilute phase of the O(−1) loop gas.

A.4. Hypergeometric parameters with denominator 4: \(|n| = \sqrt{2}\)

For κ = 16/3 or 16/5, which correspond to \(n^2 = Q = 2\), our expressions simplify greatly.

A.4.1. Dense \(n = \sqrt{2}\), Ising FK clusters. An SLE with parameter κ = 16/3 corresponds to the boundary of the critical \(Q = 2\) FK cluster model. The hypergeometric functions become
\[
\begin{align*}
G_{1,1}(m) &= \sqrt{1 + \sqrt{1-m}/2} \\
G_{1,3}(m) &= \sqrt{1 - \sqrt{1-m}/2} \\
υ_1(s, t) &= \sqrt{1 + \sqrt{1-s}/2(1-s)} - t^{1/2} \sqrt{1 - \sqrt{1-s}/2s(1-s)}.
\end{align*}
\]
The conformal blocks that simplify become
\[ G_{II} = \frac{(1 - m)^{3/8}}{\sqrt{2m^{1/8} \sqrt{\xi (1 - \xi) (1 - m\xi)}}} \frac{1 + \sqrt{m\xi}}{1 + \sqrt{m}} \]
\[ G_{IV} = \frac{(1 - m)^{3/8}}{\sqrt{2m^{1/8} \sqrt{\xi (1 - \xi) (1 - m\xi)}}} \frac{1 - \sqrt{m\xi}}{1 - \sqrt{m}} \]
\[ G_{III} = [G_{II} + G_{IV}] / \sqrt{2}. \]

Combining these with
\[ G_{VI} = \frac{(1 - m)^{3/8}}{\sqrt{2m^{1/8} \sqrt{\xi (1 - \xi) (1 - m\xi)}}} \frac{1 + \sqrt{m\xi}}{1 + \sqrt{m}} \times \left[ \frac{1 + \sqrt{m\xi}}{\sqrt{1 + \sqrt{m}}} - \frac{32\sqrt{m\pi} \xi^{5/4}}{5(1/4)^2} F_1 \left( \frac{1}{4}; \frac{3}{4}; \frac{9}{4} \right| \xi, m\xi \right] \]
gives the relatively simple expression
\[ \rho(x) = \sqrt[3]{\frac{1 + \sqrt{m\xi}}{1 - \sqrt{m\xi}}} = \frac{32\xi^{5/4}}{5(1/4)^2} \sqrt{\frac{m(1 + \sqrt{m})\pi}{1 - m\xi^2}} F_1 \left( \frac{1}{4}; \frac{3}{4}; \frac{9}{4} \right| \xi, m\xi \right) \]

\[ (A6) \]

**A.4.2. Dilute n = \sqrt{2}**. An SLE with parameter \( \kappa = 16/5 \) corresponds to the dilute phase of the O(\( \sqrt{2} \)) loop gas. The hypergeometric functions become
\[ u_1(s, t) = \frac{1}{4} \left[ 2s^{3/2} - t^3 - \frac{(t - \sqrt{s})^3}{s (1 - \sqrt{s})^{3/2}} + 2s^{3/2} + t^3 - \frac{(t + \sqrt{s})^3}{s (1 + \sqrt{s})^{3/2}} \right] \]
\[ G_{1,1}(m) = [2 - 3\sqrt{1 - m} + 2(1 - m)] \left( \frac{1 + \sqrt{1 - m}}{2} \right)^{3/2} \]
\[ G_{1,3}(m) = [2 + 3\sqrt{1 - m} + 2(1 - m)] \left( \frac{1 - \sqrt{1 - m}}{2} \right)^{3/2} \]

This means that
\[ G_{II} = \left( 1 - \sqrt{m} \right)^{3/2}[2(1 + \sqrt{m})^2(1 + m^{3/2} \xi^3) - \sqrt{m}(1 + \sqrt{m\xi})^3] \frac{16m^{5/8}[\xi (1 - \xi) (1 - m\xi)]^{3/4}(1 - m)^{1/8}}{16m^{5/8}[\xi (1 - \xi) (1 - m\xi)]^{3/4}(1 - m)^{1/8}} \]
\[ G_{IV} = \left( 1 + \sqrt{m} \right)^{3/2}[2(1 - \sqrt{m})^2(1 - m^{3/2} \xi^3) + \sqrt{m}(1 - \sqrt{m\xi})^3] \frac{16m^{5/8}[\xi (1 - \xi) (1 - m\xi)]^{3/4}(1 - m)^{1/8}}{16m^{5/8}[\xi (1 - \xi) (1 - m\xi)]^{3/4}(1 - m)^{1/8}} \]
\[ G_{III} = [G_{II} + G_{IV}] / \sqrt{2}. \]

**A.4.3. Dilute n = -\sqrt{2}**. The final case corresponds to \( \kappa = 16/7 \), the dilute phase of the O(\( -\sqrt{2} \)) loop gas. The hypergeometric functions become
\[ G_{1,1}(m) = -\left( \frac{m}{2} \right)^{5/2} \frac{4(1 - m)^2 - 10(1 - m)^{3/2} + 11(1 - m) - 10(1 - m)^{1/2} + 4}{(1 - \sqrt{1 - m})^{5/2}} \]
\[ G_{1,3}(m) = -\left(\frac{m}{2}\right)^{5/2} \frac{4(1-m)^2 + 10(1-m)^{3/2} + 11(1-m) + 10(1-m)^{1/2} + 4}{(1 + \sqrt{1-m})^{5/2}} \]

\[ u_1(s, t) = \sum_{i=0}^{5} \binom{5}{i} (-t)^i \frac{2 F_1}{1 - 5t + 10t^2} \frac{\partial}{\partial s} G_{1,1}(s) \]

\[ = \left[ \frac{r^3(t^2 - 5s + 10 s^2)}{[s(1-s)]^{5/2}} - \frac{20r(1 - 4s)}{13[s(1-s)]^{1/2}} \frac{\partial}{\partial s} + \frac{100r^2}{117[s(1-s)]^{1/2}} \frac{\partial^2}{\partial s^2} \right] G_{1,3}(s). \]

In this case, the expressions being rather lengthy, we leave it to the interested reader to assemble the simplified functions and densities.

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