HARMONIC PROJECTIONS IN NEGATIVE CURVATURE

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Abstract. In this paper we construct harmonic maps that are at a bounded distance from nearest-point retractions to convex sets, in negatively curved manifolds. Specifically, given a quasidisk $Q$ in hyperbolic space, we construct a harmonic map to the hyperbolic plane that corresponds to the nearest-point retraction to the convex hull of $Q$. If $M$ is a pinched Hadamard manifold so that its isometry group acts with cobounded orbits, and if $S$ is a set in the boundary at infinity of $M$, with the property that all elements of its orbit under the isometry group of $M$ have dimension less than $\frac{n-1}{2}$, we show that the nearest-point retraction to the convex hull of $S$ is a bounded distance away from some harmonic map.

1. Introduction

Yau conjectured in [19] Question 38] that any simply connected, complete Kähler manifold with sectional curvature at most $-1$ admits a holomorphic map onto the disk. A natural analogue of this question for general (not necessarily Kähler) Riemannian manifolds is as follows.

Question 1.1. Does any pinched Hadamard manifold admit a non-constant harmonic map to the hyperbolic plane $\mathbb{H}^2$?

A manifold is pinched Hadamard if it is simply connected, complete and with sectional curvature bounded between two negative constants. We give a partial positive answer to this question. Our basic strategy is as follows.

(1) We start with some quasi-isometric embedding $\iota : \mathbb{H}^2 \to M$. This defines a quasicircle $S$ in the boundary at infinity of $M$. A modification of the nearest-point retraction onto the convex hull of $S$ gives a map $r : M \to \mathbb{H}^2$.

(2) We deform the map $r$ to a harmonic map.

We conjecture that this strategy works in general.

Conjecture 1.2. Let $M$ be a pinched Hadamard manifold, and let $\iota : \mathbb{H}^2 \to M$ be a quasi-isometric embedding. There exists a harmonic map $h : M \to \mathbb{H}^2$ such that $\sup_{x \in \mathbb{H}^2} \text{dist}(x, h \circ \iota(x)) < \infty$. 
It was shown in [2] by Benoist and Hulin that any quasi-isometry between pinched Hadamard manifolds is at a bounded distance from a harmonic map, building on the work of Marković in [15]. Note that the nearest-point retraction onto convex subsets is not a quasi-isometry, so the nearest-point retraction is outside the scope of their work.

We solve Conjecture 1.2 for all hyperbolic spaces $\mathbb{H}^n$.

**Theorem 1.3.** Let $\iota : \mathbb{H}^2 \to \mathbb{H}^n$ be a quasi-isometric embedding. Then there exists a harmonic map $h : \mathbb{H}^n \to \mathbb{H}^2$ such that

$$\sup_{x \in \mathbb{H}^2} \text{dist}(x, h(\iota(x))) < \infty.$$ 

Note that $\iota$ as in Theorem 1.3 defines a quasi-circle $S$ in the boundary at infinity $\partial_\infty \mathbb{H}^n$. An essential ingredient in the proof of Theorem 1.3 is the existence of a constant $\beta < n - 1$ such that for any isometry $\gamma$ of $\mathbb{H}^n$, the Minkowski dimension of $\gamma S$ is at most $\beta$, which essentially follows from the work of Gehring [10]. More precisely, we use the fact that the invariant upper Minkowski dimension of $S$ is less than $n - 1$ (for an exact definition see §2).

**Theorem 1.4.** Let $S$ be a set in the sphere at infinity of $\mathbb{H}^n$ with invariant upper Minkowski dimension less than $n - 1$. Then there exists a harmonic map $h : \mathbb{H}^n \to \mathbb{H}^n$ that is a bounded distance away from the nearest-point retraction to the convex hull of $S$.

In the general case of a pinched Hadamard $n$-manifold $M$, we are only able to deal with nearest-point retractions onto convex hulls of subsets of the boundary of dimension less than $\frac{n-1}{2}$. Given a metric space $X$, we call a subset $S \subseteq X$ cobounded if for some $C > 0$, the $C$-neighborhood of $S$ is $X$.

**Theorem 1.5.** Let $M$ be a pinched Hadamard manifold of dimension $n$, such that the isometry group of $M$ has cobounded orbits. Let $S \subseteq \partial_\infty M$ be a closed set in the boundary at infinity of $M$, with the invariant upper Minkowski dimension less than $\frac{n-1}{2}$. Then there exists a harmonic map $h : M \to M$ at a bounded distance from the nearest-point retraction to the convex hull of $S$.

In particular, when $M$ is the universal cover of a closed negatively curved manifold, the isometry group of $M$ has cobounded orbits.

In the Theorem 1.5 above, we look for harmonic maps $M \to M$. From the proofs it will be clear that we could also construct harmonic maps $M \to Y$ when $Y \hookrightarrow M$ is quasi-isometrically embedded pinched Hadamard manifold with a similar condition on the invariant upper Minkowski dimension of the image of the boundary at infinity of $Y$. 
1.1. Outline. The reason we are able to prove the stronger Theorem 1.4 for hyperbolic spaces is that in this case we have a precise estimate on the heat kernel. Apart from this, the proofs of Theorems 1.4 and 1.5 are very similar, so the entire paper apart from §6 deals with the more general setting of Theorem 1.5. Hence let $S$ be a set in $\partial_\infty M$, and let $K$ be the convex hull of $S$. We let $r : M \to K$ be the nearest-point retraction. For the purposes of this outline, assume that $M$ has sectional curvature at most $-1$.

The proof consists of four steps.

(1) We construct a smooth map $\tilde{r} : M \to M$ that is at a bounded distance from $r$, so that its derivative and Hessian have the property $\| \nabla_x \tilde{r} \|, \| H(\tilde{r})_x \| \leq C e^{-\operatorname{dist}(x,K)}$, for some constant $C$. This $\tilde{r}$ is the result of a construction of Benoist and Hulin in [2, §2.2]. Their exact statements do not apply here since $\tilde{r}$ is not a quasi-isometry, and moreover we need slightly stronger conclusions than they do. We summarize their construction in §3 and explain how it applies to $r$.

(2) Let $N_C(K)$ be the $C$-neighborhood of $K$, for some $C$ large. We show that assuming that the integral $\int_{N_C(K)} G(x, y) d\operatorname{vol}(y)$ of the Green’s function $G(\cdot, \cdot)$ is bounded in $x$, there exists a bounded map $\Phi : M \to \mathbb{R}$ so that

$$\Delta \Phi > \| \tau(\tilde{r}) \|,$$

where $\tau(\tilde{r})$ denotes the tension field of $\tilde{r}$ (the exact definition will be given in §2). This is the content of Lemma 4.6. We use the assumption that the integral of the Green’s function is bounded to construct $\Phi$ on $N_C(K)$. On $M \setminus N_C(K)$, we construct $\Phi$ as a suitable function of the distance $\operatorname{dist}(x, \tilde{r}(x))$.

(3) Denote the ball centered at $x$ of radius $d$ by $B(x, d)$. We construct harmonic maps $h_d : B(x, d) \to M$ that agree with $\tilde{r}$ on $\partial B(x, d)$. Then by an estimate of Schoen and Yau [18] on the Laplacian of the distance between smooth maps, we get

$$\Delta \left( \operatorname{dist}(h_d(x), \tilde{r}(x)) + \Phi \right) > 0.$$

By the maximum principle we see that $\operatorname{dist}(h_d(x), \tilde{r}(x)) \leq 2 \sup_x |\Phi(x)|$. Since this bound is uniform in $d$, a compactness argument shows that we can take a limit of $h_d$ as $d \to \infty$ to get a harmonic map that is at a bounded distance from $\tilde{r}$. This argument is in the proof of Corollary 1.7. This step essentially appears in the work of Donnelly [9, Lemma 3.1].
To finish the proof of Theorems 1.4 and 1.5, we only need to verify that 
\[ \int_{N_C(K)} G(x, y) dv(y) \] is bounded in \( x \) for arbitrarily large \( C \). We first show the bound 
\[ \text{vol}(N_C(K) \cap B(x, \rho)) \leq C' \exp\left( (\text{dim} S + \varepsilon) \rho \right), \]
for all \( x \in M, \varepsilon > 0 \), and some constant \( C' \) that does not depend on \( x \) or \( \rho \). This estimate is shown in Lemma 5.1. We then use some classical estimates on the heat kernel \( H : M \times M \times [0, \infty) \to \mathbb{R} \) (see Proposition 6.1) and the fact that \( G(x, y) = \int_0^\infty H(x, y, t) dt \). The computations combining these estimates are in §6.2 and §6.4.

The assumptions that \( \text{dim} S < n - 1 \) and \( \overline{\text{dim}} S < \frac{n-1}{2} \) from Theorems 1.4 and 1.5 respectively, are only used when applying Lemma 5.1 to obtain an estimate on the integral of the Green’s function.

The proof of Theorem 1.3 also follows the outline above. The only difference is that we need to construct an initial map \( \mathbb{H}^n \to \mathbb{H}^2 \), and show that quasicircles have invariant upper Minkowski dimension less than \( n - 1 \). All of this is done in §6.3.

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2. Preliminaries and notation

Let \( M \) be a pinched Hadamard manifold, that is a simply connected complete Riemannian manifold of dimension \( n \) with sectional curvatures \( K_M \) with \( -b^2 \leq K_M \leq -a^2 \), for some fixed constants \( 0 < a \leq b \). We assume that the isometry group of \( M \) has cobounded orbits. Recall that the group action \( G \) on \( X \) has cobounded orbits if for any point \( x \in X \), there is a constant \( C > 0 \) such that the \( C \)-neighborhood of the orbit \( G \cdot x \) is \( X \).

These will be standing assumptions throughout the paper. In particular this holds whenever \( M \) is the universal cover of some closed negatively curved manifold.

We denote by dist(\( \cdot, \cdot \)) the path metric on \( M \), and by \([x, y]\) the geodesic segment connecting \( x \) to \( y \). We denote by \( B(x, r) \) the ball centered at \( x \) of radius \( r \) in \( M \), and by \( N_d(Y) = \bigcup_{y \in Y} B(y, d) \) the \( d \)-neighborhood of the set \( Y \subseteq X \). For a set \( S \subseteq M \), we denote by \( \text{CH}(S) \) its convex hull, that is the intersection of all convex sets containing \( S \).
We write $f \lesssim g$ when there exists a constant $C > 0$ such that $f \leq Cg$. When it is not clear from context, we will specify what $C$ is allowed to depend on. We write $f \gtrsim g$ for $g \lesssim f$ and $f \approx g$ for $f \lesssim g \lesssim f$.

2.1. **Harmonic maps, Green’s function and the heat kernel.** For a smooth map $h : X \to Y$ between Riemannian manifolds, we denote by $\nabla h$ its derivative, and by $H(h)$ its Hessian. Note that $\nabla h$ is a $h^*TY$-valued 1-form on $X$, and that $H(h)$ is a $h^*TY$-valued symmetric bilinear form on $X$.

**Definition 2.1.** For a smooth map $h : X \to Y$ between Riemannian manifolds, we define its tension field to be $\tau(h) = \text{tr} H(h)$. The function $h$ is harmonic if $\tau(h) = 0$. When $N = \mathbb{R}$, we denote $\Delta h = \tau(h)$.

We now recall the definitions of the heat kernel and Green’s function. For more detailed information, the reader can consult the book by Grigor’yan [11, Chapters 7, 11].

**Definition 2.2.** The heat kernel $H : \mathbb{R} \times M \times M \to \mathbb{R}$ is the unique smooth function such that for any smooth compactly supported function $f_0 : M \to \mathbb{R}$, the function

$$f_t(x) = \int_M H(t, x, y)f_0(y)d\text{vol}(y)$$

is the solution to $\frac{\partial f_t}{\partial t} = \Delta f_t$, and has the property that $f_t \to f_0$ as $t \to 0$.

**Definition 2.3.** The Green’s function is defined as

$$G(x, y) = \int_0^\infty H(t, x, y)dt,$$

whenever the right-hand side converges.

By [11] Theorem 13.17, $G$ is the fundamental solution to the Laplace’s equation whenever it is finite. It is well-known that Green’s functions exist on all complete noncompact manifolds without boundary (e.g. [14]). We will in particular show in §6.1 that in our setting, the integral in (2.1) converges.

2.2. **Visual metrics and upper invariant Minkowski dimension.** Denote by $\partial_\infty M$ the boundary at infinity of $M$, that is the set of geodesic rays in $M$ up to the equivalence relation of having finite Hausdorff distance (for a more detailed account of the theory of boundaries of negatively curved spaces, the reader may wish to consult [12]). We set
$\overline{M} = M \cup \partial_\infty M$, and extend the notation $\text{CH}(S)$ and $[x,y]$ for $S \subseteq \overline{M}$ and $x, y \in \overline{M}$.

We equip $\partial_\infty M$ with the family of visual metrics $\text{dist}_x^{\text{vis}}(\cdot, \cdot)$ indexed by $x \in M$, given by

$$\text{dist}_x^{\text{vis}}(y, z) \approx e^{-\alpha \text{dist}(x, [y,z])}.$$

**Remark 2.4.** Note that for general Gromov hyperbolic metric spaces, the visual metrics can only be defined as $\text{dist}_x^{\text{vis}}(y, z) \approx e^{-\kappa \text{dist}(x, [y,z])}$, for some $\kappa > 0$ small enough. However since $M$ is a CAT($-a^2$) space, such a metric exists whenever $0 < \kappa \leq a$. [4 §2.4].

The appropriate notion of dimension we will use for subsets of $\partial_\infty M$ is defined below. For a subset $S$ of some metric space $(X, d)$, we denote by $N_d(S, \varepsilon)$ the smallest number of $\varepsilon$-balls needed to cover $S$.

**Definition 2.5.** If $M$ is a pinched Hadamard manifold, for $S \subseteq \partial_\infty M$, the invariant upper Minkowski dimension of $S$, denoted $\dim S$, is the infimum of all $d \geq 0$ with the property that there exists a constant $C$ such that

$$N_{\text{dist}_x^{\text{vis}}}(S, \varepsilon) \leq C\varepsilon^{-d},$$

for all $x \in M$ and $\varepsilon > 0$.

If we fix some arbitrary base point $o \in M$, and write $\text{dist}^{\text{vis}} = \text{dist}_o^{\text{vis}}$, the Definition 2.5 is equivalent to the definition below.

**Definition 2.6.** For a set $S \subseteq \partial_\infty M$, the upper-invariant Minkowski dimension is the infimum of all $d$ such that there exists a constant $C > 0$ with the property that

$$N_{\text{dist}^{\text{vis}}}(S, \varepsilon) \leq C\varepsilon^{-d}$$

for all $\gamma \in \text{Isom}(M)$ and $\varepsilon > 0$.

3. **Deforming the nearest-point retraction to a smooth map**

In this section we deform the nearest-point retraction $r : M \to K$ to a convex set $K$ to a smooth map $\tilde{r}$ with $\sup_{x \in M} \text{dist}(r(x), \tilde{r}(x)) < \infty$, so that

$$\|\nabla \tilde{r}\| \lesssim e^{-a \text{dist}(\cdot, K)},$$

$$\|H(\tilde{r})(X, X)\| \lesssim e^{-a \text{dist}(\cdot, K)}\|X\|^2.$$
show that the local Lipschitz constant of the nearest-point retraction decays exponentially with the distance from the convex set.

3.1. Deforming Lipschitz maps to smooth maps. We do this by the methods of Benoist and Hulin in [2, §2.2]. We collect their results as Lemma 3.1.

**Lemma 3.1.** Let \( f : X \to Y \) be a Lipschitz map between pinched Hadamard manifolds \( X \) and \( Y \). Then there exists a smooth map \( \tilde{f} : X \to Y \) at a bounded distance from \( f \) and a polynomial \( P \) with non-negative coefficients and \( P(0) = 0 \) such that whenever \( x \in X \) and \( f \) is \( L \)-Lipschitz in a neighborhood of \( x \), we have

\[
\left\| \nabla_x \tilde{f} \right\| \leq P(L) \quad \text{and} \quad \left\| H(\tilde{f})_x \right\| \leq P(L).
\]

**Proof.** This is the result of [2, Lemma 2.8] and a slight strengthening of [2, Lemma 2.7]. We state these results as propositions below, and combine them as in [2, Proof of Proposition 2.4. Second step].

We say that a subset \( S \subset X \) is \( r \)-separated if for all \( x, y \in S \), we have \( \text{dist}(x, y) \geq r \).

**Proposition 3.2** (Lemma 2.8 in [2]). There exist constants \( r_0 > 0 \) and \( N_0 \in \mathbb{Z}_{>0} \) such that for each \( r < r_0 \), any \( r \)-separated subset of \( X \) can be decomposed into at most \( N_0 \) disjoint subsets, each of which is \( 4r \)-separated.

**Proposition 3.3** (Strengthening of Lemma 2.7 in [2]). Let \( g : X \to Y \) be a map between pinched Hadamard manifolds. Then for all \( r > 0 \) small enough, there exists a family of maps \( g_{r,x} : X \to Y \) indexed by \( x \in X \), such that

\[
g_{r,x}(z) = g(x) \quad \text{when} \quad \text{dist}(x, z) \leq \frac{r}{2},
\]

\[
g_{r,x} = g \quad \text{on} \quad X \setminus B(x, r).
\]

Moreover

\[
\text{Lip}(g_{r,x}|_{B(x, r)}) \lesssim \text{Lip}(g|_{B(x, r)}),
\]

and if \( g \) is \( C^2 \) in some neighborhood of \( x \), then so is \( g_{r,x} \) with

\[
\|H(g_{r,x})_z\| \lesssim \text{Lip}(g|_{B(x, r)}) + \text{Lip}(g|_{B(x, r)})^2 + \|H(g)_z\|,
\]

for each \( z \) in that neighborhood, where the implied constant depends on \( r \).

**Proof.** We use the coordinates given by the following Proposition to construct \( g_{r,x} \).
Proposition 3.4 (Lemma 2.6 in [2]). There exist constants \( r_0 > 0 \) and \( c_0 > 1 \) such that for any \( y \in Y \), there exists a chart \( \Phi_y : B(y, r_0) \to U_y \subseteq \mathbb{R}^{\dim Y} \) such that \( \Phi_y(y) = 0 \) and

\[
\|\nabla \Phi_y\|, \|\nabla \Phi_y^{-1}\|, \|H(\Phi_y)\|, \|H(\Phi_y^{-1})\| \leq c_0.
\]

Here \( U_y \subseteq \mathbb{R}^{\dim Y} \) is given the standard Euclidean metric. In particular, we have for \( r < r_0 \),

\[
B \left( 0, \frac{r}{c_0} \right) \subseteq \Phi_y(B(y, r)) \subseteq B(0, c_0r).
\]

Let \( \chi : \mathbb{R} \to [0, 1] \) be a smooth function with \( \chi\rvert_{[-\frac{1}{2}, \frac{1}{2}]} = 0 \), and \( \chi\rvert_{[\mathbb{R}\setminus[-1, 1]} = 1 \). Write \( \chi_r(x) = \chi(\frac{x}{r}) \), and let \( \Phi \), be the coordinates given by Proposition 3.4. We define

\[
g_{r,x}(z) = \begin{cases} 
  g(x) & \text{if } \text{dist}(x, z) \leq \frac{r}{2} \\
  \Phi^{-1}_x (\chi_r(\text{dist}(x, z)) \Phi_x(g(z))) & \text{if } \frac{r}{2} \leq \text{dist}(x, z) \leq r \\
  g(z) & \text{otherwise}
\end{cases}
\]

Note that this is the exact same function as in [2] Lemma 2.7, and is well-defined when \( c_0^2 r \Lip(g) < r_0 \). In this proof, denote by \( d(z) = \text{dist}(x, z) \), for ease of notation. We have for \( \frac{r}{2} \leq d(z) \leq r \),

\[
\nabla_z g_{r,x} = \nabla_{\chi_r(d(z)) \Phi_x(g(z))} \Phi^{-1}_x (\chi'(d(z)) (\nabla_z d) \Phi_x(g(z)) + \chi_r(d(z)) \nabla g_z \Phi_x \nabla z g),
\]

so

\[
\|\nabla_z g_{r,x}\| \lesssim \|\Phi_x(g(z))\| + \|\nabla z g\| \lesssim \|\nabla z g\|.
\]

Taking one more derivative, we see that

\[
\|H(g_{r,x})(z)\| \lesssim \|\nabla z g\|^2 + \|\nabla z g\| + \|H(g)(z)\|.
\]

The result now follows from \( \|\nabla z g\| \lesssim \Lip(g(\Omega_{B(x, r)})) \). \( \square \)

The rest of the proof is completely analogous as [2] Proof of Proposition 2.4. Second step]. Let \( r > 0 \) be small enough to be chosen later. Let \( X_0 \) be a maximal \( \frac{r}{2} \)-separated subset of \( X \). By Proposition 3.2 we can write \( X_0 = X_1 \cup X_2 \cup \cdots \cup X_{N_0} \) where each \( X_i \) is \( 2r \)-separated. Define \( f_0 = f \), and set

\[
f_i(z) = \begin{cases} 
  (f_{i-1})_{r,x}(z) & \text{if } z \in B(x, r) \text{ for some } x \in X_i \\
  f_{i-1}(z) & \text{otherwise}
\end{cases}
\]

Set \( \tilde{f} = f_{N_0} \). Then since each point \( z \in X \) is in \( B(x, \frac{r}{2}) \) for some \( x \in X_0 \) (by maximality of \( X_0 \)), some \( f_i \) is locally constant near \( x \) and hence \( f_{N_0} \) is smooth at \( x \).
By Proposition 3.3 whenever \( f \) has Lipschitz constant at most \( L \) near \( x \in X \), we have

\[
\left\| \nabla_x \tilde{f} \right\| \lesssim L,
\]

\[
\left\| H(\tilde{f})_x \right\| \leq \sum_{i=0}^{N_0} P^i(L),
\]

where \( P(z) = \Lambda(z+z^2) \) for some large enough constant \( \Lambda > 0 \). The result follows. \( \square \)

### 3.2. Local Lipschitz constant of the nearest-point retraction.

We now show that the local Lipschitz constant of the nearest-point retraction to \( K \) decays exponentially with the distance from \( K \). This is a basic result in \( \text{CAT}(-a^2) \) geometry, and is probably not new.

**Proposition 3.5.** Let \( K \) be a closed convex subset of \( M \), and let \( r : M \to K \) be the nearest-point retraction. Then its restriction \( r : M \setminus N_s(K) \to K \) has Lipschitz constant at most \( Ce^{-as} \), where we equip \( M \setminus N_s(K) \) with its induced path metric, denoted \( \text{dist}_{M \setminus N_s(K)}(\cdot, \cdot) \), for some constant \( C \) depending only on \( M \).

**Proof.** Let \( x, y \in M \setminus N_s(K) \) be such that \( \text{dist}(r(x), r(y)) \lesssim s \), where the implicit constant will be chosen later. Let \( x = x_0, x_1, x_2, \ldots, x_n = y \) be points in \( M \) on the shortest path between \( x \) and \( y \) in \( M \setminus N_s(K) \), such that

1. \( [x_i, x_{i+1}] \cap N_s(K) = \emptyset \) for \( i = 0, 1, \ldots, n-1 \), and
2. \( \sum_{i=0}^{n-1} \text{dist}(x_i, x_{i+1}) < 2\text{dist}_{M \setminus N_s(K)}(x, y) \).

Note in particular that \( \text{dist}([x_i, x_{i+1}], r(x)) \geq s \). We note that since \( K \) is convex, we have \( [r(x), r(y)] \subseteq K \). Since \( r \) is the nearest-point retraction, we have \( \angle([r(x), x], [r(x), r(y)]), \angle([r(y), y], [r(y), r(x)]) \geq \pi/2 \).

Pick comparison triangles for \( r(x)x_i x_{i+1} \) for \( i = 0, 1, \ldots, n-1 \) and for \( r(x)yr(y) \) in the 2-dimensional space \( \mathbb{H}(-a^2) \) of constant curvature \( -a^2 \) (these exist by [13]). We glue them appropriately to get a hyperbolic polygon in \( \mathbb{H}(-a^2) \). We work in the disk model, and we can suppose without loss of generality that \( r(x) \) corresponds to the origin. Suppose that \( x, y \) and \( r(y) \) correspond to \( A, B, C \in \mathbb{D} \), respectively. We know that

1. the angle between \([0, A]\) and \([0, C]\) is at least \( \pi/2 \),
2. the angle between \([C, 0]\) and \([C, B]\) is at least \( \pi/2 \),
3. \( \text{dist}(0, C) = \text{dist}(r(x), r(y)), \text{dist}(0, A), \text{dist}(0, B) \geq s \), and

\[
\text{dist}_{M \setminus N_s(K)}(x, y) \leq \sum_{i=0}^{n-1} \text{dist}(x_i, x_{i+1}).
\]
\[
(4) \sum_{i=0}^{n-1} \text{dist}([x_i, x_{i+1}]) \geq \text{dist}_{\mathbb{D}\setminus B(0,s)}(A, C) \text{ and hence } \text{dist}_{\mathcal{M}\setminus \mathcal{N}_s(K)}(x, y) \gtrsim \text{dist}_{\mathbb{D}\setminus B(0,s)}(A, B).
\]

**Claim 3.6.** We have \(\text{dist}_{\mathbb{D}\setminus B(0,s)}(A, C) \gtrsim e^{as}\text{dist}(0, C)\).

**Proof.** We can rescale the metric so that \(a = 1\) and we are working in \(\mathbb{H}^2(-1) = \mathbb{H}^2\). We can suppose without loss of generality that \(\text{dist}(0, A) = \text{dist}(0, B) = s\), and that the angles \(\angle([0, A], [0, C]) = \angle([C, 0], [C, B]) = \frac{\pi}{2}\). We note that \(\angle([0, A], [0, B]) \approx \text{dist}(0, C)\) for \(\text{dist}(0, C)\) small enough, and since the metric on \(\mathbb{D}\) is \(4s^2 + r^2d\theta^2\) in polar coordinates, we have

\[
\text{dist}_{\mathbb{D}\setminus B(0,s)}(A, C) \geq \frac{2\tanh \frac{s}{2}}{1 - \tanh^2 \frac{s}{2}} \angle([0, A], [0, B])
\approx \text{dist}(0, C) \sinh s \approx e^s\text{dist}(0, C).
\]

□

From the Claim we have

\[
\text{dist}_{\mathcal{M}\setminus \mathcal{N}_s(K)}(x, y) \gtrsim \text{dist}_{\mathbb{D}\setminus B(0,s)}(A, B) \gtrsim e^{as}\text{dist}(0, C) = e^{as}\text{dist}(r(x), r(y)).
\]

It follows that \(\text{dist}(r(x), r(y)) \lesssim e^{-as}\text{dist}_{\mathcal{M}\setminus \mathcal{N}_s(K)}(x, y)\). □

Applying Lemma 3.1 to the nearest-point retraction gives the following corollary.

**Corollary 3.7.** For any convex subset \(K \subseteq M\) there exists a map \(\tilde{r} : M \to K\) with \(\sup_{x \in M} \text{dist}(r(x), \tilde{r}(x)) < \infty\) so that

\[
\|\nabla \tilde{r}\| \lesssim e^{-\text{dist}(\cdot, K)},
\]

\[
\|H(\tilde{r})(X, X)\| \lesssim e^{-\text{dist}(\cdot, K)}\|X\|^2,
\]

for any vector field \(X\).

**Remark 3.8.** Note that the constants in this corollary depend only on \(M\), and are in particular independent of \(K\).

4. **Reduction to an integral estimate of Green’s function**

In this section we show that, assuming \(x \to \int_U G(x, y)d\text{vol}(y)\) is bounded for a fixed large neighborhood \(U\) of \(K\), there exists a bounded subharmonic map \(\Phi : M \to \mathbb{R}\) with the bound on the Laplacian

\[
\Delta \Phi \geq e^{-\text{dist}(\cdot, K)}.
\]

To construct \(\Phi\) on \(U\), we use the assumption on the integral of Green’s function. On \(M \setminus U\), we construct \(\Phi\) as some function of the distance \(\delta(x) = \text{dist}(x, \tilde{r}(x))\). Hence to bound \(\Delta \Phi\), we need bounds on the
Laplacian and derivative of $\delta$. This essentially follows from the work of Benoist and Hulin [11, Remark 4.6], but we include a different proof in §4.1 as Proposition 4.1 for completeness. We then finish the construction of $\Phi$ in §4.2.

Suppose now we are given a bounded subharmonic function $\Phi : M \to \mathbb{R}$ with $\Delta \Phi \geq e^{-\text{dist}(\cdot,K)}$. Suppose we are given a function $f : M \to N$ with $\|\tau(f)\| \lesssim e^{-\text{dist}(\cdot,K)}$. On any ball $B(x, R)$, we can construct a harmonic map $h_R : B(x, R) \to N$ with $h_R = f$ on $\partial B(x, R)$. An estimate by Schoen and Yau from [18] shows that $\Delta \text{dist}(f,g) \geq -\|\tau(f)\| - \|\tau(g)\|$, for any smooth functions $f,g : M \to N$. In fact we use a general formula from which Schoen and Yau derive this estimate in §4.1 to bound the Laplacian of $\delta(x) = \text{dist}(x, \tilde{r}(x))$.

Using this formula we see that $\Delta (\text{dist}(h_R,f)+C\Phi) > 0$ for some large constant $C$. Therefore $\text{dist}(h_R,f) + C\Phi \leq \text{sup} \Phi$ by the maximum principle. It follows that $\text{dist}(h_R,f)$ is bounded uniformly in $R$. A classical argument combining Cheng’s lemma, Schauder elliptic estimates and Arzela-Ascoli theorem then shows that we can take the limit of such harmonic maps as $R \to \infty$, to get a harmonic map $h_\infty : M \to N$ at a bounded distance from $f$. The details of this argument are in §4.3.

This part of the argument is similar to something that appears in the work of Donnelly [9]. Specifically, it is shown in [9, Lemma 3.1] that given a function $f : M \to N$ and a bounded non-negative map $\Phi : M \to \mathbb{R}$ with $\Delta \Phi > \|\tau(f)\|$, there exists a harmonic map $h : M \to N$ such that $\text{dist}(h(x),f(x)) \leq \text{sup} \Phi(x)$.

In [9], $\Phi$ is constructed using an assumption on the integral of the Green’s function, in a way completely analogous to how we construct $\Phi$ on a large neighborhood of $K$. However here we are able to construct $\Phi$ far away from $K$ without any assumptions on $K$ or the Green’s function, by Proposition 4.4.

4.1. Properties of the distance function. We set $\delta(x) = \text{dist}(x, \tilde{r}(x))$. We will consider $\delta$ on $M \setminus N_C(K)$, for some $C$ large to be chosen later. In particular, we let $C$ be large enough so that $x \neq \tilde{r}(x)$ for all $x \in M \setminus N_C(K)$.

**Proposition 4.1.** For some $C > 0$ large enough, the distance function $\delta$ is smooth on $M \setminus N_C(K)$ and has

$$\Delta \delta \gtrsim 1 \text{ and } \|\nabla \delta\| \lesssim 1.$$
Proof. It is well-known that dist(·, ·) : M^2 → ℝ is smooth away from the diagonal, so the smoothness of δ follows from that of ˜r.

Since ∥∇ ˜r∥ is finite, ˜r is Lipschitz, and hence so is δ(x) = dist(x, ˜r(x)). Therefore ∥∇δ∥ < ∞.

The rest of this proof is estimating ∆δ. We compare δ near some arbitrary point x_0 to the function dist(·, ˜r(x_0)), using standard estimates on the Hessian of the distance function. The computation is analogous to that of Schoen and Yau [18].

We fix a large C > 0 such that ˜r(x) ̸= x for x ∈ M \ NC(K). Fix a point x_0 ∈ M \ NC(K) and set p_0 = ˜r(x_0). We first state a calculus claim, proved by a straightforward computation that we omit.

Claim 4.2. Let f : A → B be a smooth map between Riemannian manifolds, and let g : B → ℝ be a smooth function. Then for x ∈ A,

\[ \Delta_x(g \circ f) = dg_{f(x)}(\tau(f)_x) + \text{tr}((dx_f)^*H(g)_{f(x)}). \]

Remark 4.3. We note that here H(g)_{f(x)} is a symmetric bilinear form on T_{f(x)}B, that we pull back by d_x f : T_x A → T_{f(x)}B to get a symmetric bilinear form on T_x A. The trace then refers to the Riemannian metric on A.

We apply Claim 4.2 to the maps (id_X, ˜r) : M \ NC(K) → M^2 and dist : M^2 → ℝ. This yields

\[ \Delta_{x_0}\delta = \text{dist}(\tau(1), x_0) + \sum_{\alpha} H(\text{dist})_x(x_0, p_0)(e_\alpha \oplus \delta_\alpha, e_\alpha \oplus \delta_\alpha), \]

where e_α is some orthonormal basis for T_{x_0}M. Note that e_α \oplus \delta_\alpha \in T_{x_0}M \oplus T_{p_0}M \cong T_{(x_0, p_0)}(M^2). Applying Claim 4.2 once again to the map id_X × const_{p_0} : M → M × {p_0} ⊂ M × M (here by const_{p_0} we denote the constant map M → {p_0} ⊂ M), we get

\[ \Delta_{x_0}\text{dist}(\cdot, p_0) = \sum_{\alpha} H(\text{dist})_x(x_0, p_0)(e_\alpha \oplus 0, e_\alpha \oplus 0). \]

Note that from [2] Lemma 2.3, we see that

\[ \|H(\text{dist})_x(x_0, p_0)\| \leq b \coth(bC). \]

Here the norm is defined relative to the Riemannian metric on T_{x_0}M \oplus T_{p_0}M by \|H\| = \sup_{|X| = 1}|H(X, X)|. Subtracting (4.1) from (4.2), using (4.3) with the fact that ∥\delta_\alpha\∥ ≤ e^{-aC}, we see that

\[ \|\Delta_{x_0}\delta - \Delta_{x_0}\text{dist}(p_0, \cdot)\| \lesssim e^{-aC}(1 + b^2 \coth^2(bC)) \to 0 \text{ as } C \to \infty. \]

It is well-known that \( \Delta_{x_0}\text{dist}(p_0, \cdot) \geq a \) (see e.g. [3] Lemma 2.5), so for C > 0 large enough, we have \( \Delta_{x_0}\delta > a/2 \) for \( x_0 \in M \setminus NC(K) \).
4.2. Constructing bounded subharmonic functions. We use the following Proposition to construct $\phi$ on $N_{d+1}(K) \setminus N_d(K)$.

**Proposition 4.4.** Let $S \subseteq \partial_{\infty} M$ and let $K = CH(S)$. For all $d > d_0$, where $d_0 = d_0(M)$ is some constant depending only on $M$, there exists a bounded subharmonic function $\phi_d : M \to \mathbb{R}$ such that

$$\Delta \phi_d \geq 1 \text{ on } N_{d+1}(K) \setminus N_d(K),$$

so that $\sup_x |\phi_d(x)|$ does not depend on $d$.

**Proof.** Let $f : [0, \infty) \to [0, \infty)$ be a $C^2$ function. Then

$$\Delta (f \circ \delta) = f'(\delta) \Delta \delta + f''(\delta) \|\nabla \delta\|^2.$$

By Proposition 4.1 we can suppose $\Delta \delta \geq A$ and $\|\nabla \delta\|^2 \leq B$, for some positive constants $A, B$. We will construct $\phi_d$ as $f \circ \delta$, for a suitable function $f$.

**Claim 4.5.** There exists a $C^1$ function $u : \mathbb{R} \to [0, \infty)$ be a $C^1$ function with the following properties:

1. We have $u = 0$ on $(-\infty, -\frac{1}{2}]$ and $u = 1$ on $[0, 1]$.
2. On $(-\infty, 1]$, $u$ is non-decreasing, and on $[1, \infty)$, $u$ is decreasing.
3. We have $u \approx e^{-\varepsilon x}$ for $x$ large enough, for some $\varepsilon > 0$.
4. We have $Au + B \min(u', 0) \geq 0$.

**Proof.** Fix $\varepsilon < \frac{A}{B}$. Let $v : [-1/2, \infty) \to \mathbb{R}$ be a $C^1$ function with the following properties:

1. For some small $\lambda > 0$, we have

$$v(x) = v_{\text{lower}}(x) = -\frac{1}{(2x - 1)^2} \text{ for } x \in \left(-\frac{1}{2}, -\frac{1}{2} + \lambda\right)$$

2. In the interval $(-\frac{1}{2} + \lambda, 2)$, the function is defined by $v(x) = v_{\text{middle}}(x)$, where

$$v'_{\text{middle}}(x) = \begin{cases} \frac{1-x}{2-\varepsilon} \cdot v'_{\text{lower}}(-1/2 + \lambda) & \text{for } -\frac{1}{2} + \lambda < x \leq 1 \\ -\varepsilon(x-1) & \text{for } 1 < x < 2 \end{cases}$$

and $v_{\text{middle}}(-1/2 + \lambda) = v_{\text{lower}}(-1/2 + \lambda)$.
3. For $x \geq 2$, we have $v(x) = v_{\text{middle}}(2) - \varepsilon(x-2)$.

Then it is immediate that the function

$$u = \begin{cases} e^v & \text{for } x > -\frac{1}{2} \\ 0 & \text{for } x \leq -\frac{1}{2} \end{cases}$$

has properties 1), 2), 3). Note that when $x \geq 1$, we have

$$v'(x) = \max(-\varepsilon, \varepsilon(1-x)) \geq -\varepsilon,$$
and hence \( \frac{\nu}{n} \geq -\varepsilon > -\frac{4}{B} \). Therefore \( Bu' + Au > 0 \) when \( x \geq 1 \). When \( x < 1 \), we have \( B \min(u', 0) + Au = Au > 0 \), so 4) is shown.

We set \( f(x) = \int_0^x u(t - d)dt \). By exponential decay of \( u \) at infinity, \( f \) is bounded. When \( d \geq C \) from Proposition \[4.1\], we have

\[
\Delta(f \circ \delta) \geq Au(\delta - d) + B \min(u'(\delta - d), 0) \geq 0,
\]
so \( f \circ \delta \) is subharmonic, and moreover \( \Delta(f \circ \delta) = \Delta \delta \geq 1 \) whenever \( d \leq \delta(x) \leq d + 1 \), or equivalently \( x \in N_{d+1}(K) \setminus N_d(K) \). We rescale \( f \circ \delta \) to get \( \phi_d \).

The following lemma uses the assumption on the integral of the Green’s function to construct \( \Phi \) on \( N_C(K) \) for \( C \) large enough, and hence finishes the construction of \( \Phi \).

**Lemma 4.6.** Let \( S \subseteq \partial_\infty M \) and let \( K = \text{CH}(S) \). Then if

\[
\sup_x \int_{N_{d_0}(M) + 2(K)} G(x, y) d\text{vol}(y) < \infty,
\]
there exists a bounded map \( \Phi : M \to \mathbb{R} \) with \( \Delta \Phi \geq e^{-a \text{dist}(\cdot, K)} \).

**Proof.** Let \( \chi : M \to [0, 1] \) be a smooth map with \( \chi = 1 \) on \( N_{d_0+1}(K) \) and \( \chi = 0 \) on \( M \setminus N_{d_0+2}(K) \). Then we construct \( \Phi \) as

\[
\Phi(x) = \sum_{n = [d_0]}^\infty e^{-an} \phi_n - \int_M \chi(y) G(x, y) d\text{vol}(y).
\]

We have \( \int_M \chi(y) G(x, y) d\text{vol}(y) \leq \int_{N_{d_0+2}(K)} G(x, y) d\text{vol}(y) \) which is bounded by assumption, so \( \Phi \) is bounded. Note that

\[
-\Delta \int_M \chi(y) G(x, y) d\text{vol}(y) = \chi(x) \geq \begin{cases} 1 & \text{on } N_{d_0+1}(K), \\ 0 & \text{on } M. \end{cases}
\]
Therefore \( \Delta \Phi \gtrsim e^{-a \text{dist}(\cdot, K)} \), so after rescaling we can take \( \Delta \Phi \geq e^{-a \text{dist}(\cdot, K)}. \)

4.3. Using bounded subharmonic functions to finish the proof.

We derive all our Theorems from the following corollary of Lemma 4.6.

**Corollary 4.7.** Let \( S \subseteq \partial_\infty M \) and let \( K = \text{CH}(S) \). Let \( f : M \to N \) be a smooth map between pinched Hadamard manifolds such that

\[
\|\tau(f)\| \lesssim e^{-a \text{dist}(\cdot, K)}.
\]
There exists a constant \( C = C(M) > 0 \) such that if

\[
\sup_x \int_{N_C(K)} G(x, y) d\text{vol}(y) < \infty,
\]
there exists a harmonic map $h : M \to N$ at a bounded distance from $f$.

Proof. Set $C = d_0(M) + 2$, and let $\Phi : M \to \mathbb{R}$ be the function from Lemma 4.6. We fix an arbitrary $x_0 \in M$. For all $d > 0$, we let $h_d : B(x_0, d) \to N$ be the harmonic map such that $h_d = f$ on $\partial B(x_0, d)$. Then by [18], we have

$$\Delta \text{dist}(h_d, f) \geq -\| \tau(f) \| \geq -e^{-\text{dist}(\cdot, K)}.$$

Hence for a suitable constant $C' > 0$ that does not depend on $d$, we have

$$\Delta \left( \text{dist}(h_d, f) + C'\Phi \right) > 0.$$

By the maximum principle, we have

$$\sup_{B(x_0, d)} \text{dist}(h_d, f) \leq 2C' \sup_M |\Phi| =: D.$$

For any fixed $x \in M$, for arbitrarily large $n$, $h_n$ maps the ball $B(x, 2)$ to the fixed bounded set $N_D(f(B(x, 2)))$. As $h_n$ is harmonic, by Cheng’s lemma (see [6]), we have

$$\sup_n \sup_{y \in B(x, 1)} \| \nabla_y h_n \| < \infty,$$

for any $x \in M$. It follows by the Arzela-Ascoli theorem that there is a sequence $k_n \to \infty$ such that $h_{k_n} \to h_\infty$ as $n \to \infty$, uniformly on compact sets. By the classical elliptic estimates (that can be found in [17, Theorem 70, p. 303])

$$\| h_n \|_{C^{2, \alpha}(B(x, \varepsilon))} \lesssim \| h_n \|_{C^{\alpha}(B(x, \varepsilon))},$$

for $\alpha < 1$, so by Arzela-Ascoli applied again, there exists a further subsequence, that we also denote $k_n$, such that $H(h_{k_n}) \to H(h_\infty)$ uniformly on compact sets. Therefore $h_\infty$ is also harmonic, is defined everywhere and

$$\text{dist}(h_\infty(x), f(x)) \leq \limsup_{n \to \infty} \text{dist}(h_{k_n}(x), f(x)) \leq D$$

for any $x \in M$. \qed

Remark 4.8. Note that the assumption that $x \to \int_{N_C(K)} G(x, y) \text{dvol}(y)$ is bounded is only used to construct $\Phi$ on $N_C(K)$. The only place we use dimension bounds on $S$ is to verify this assumption. In particular, if there exists a bounded subharmonic map $\phi : M \to \mathbb{R}$ with $\Delta \phi \geq 1$ on $N_C(K)$, then there exists a harmonic map $h : M \to M$ at a bounded distance from the nearest-point retraction $r : M \to \text{CH}(S) = K$ (with no assumptions on $\dim S$).
5. UPPER BOUND ON THE VOLUME OF THE CONVEX HULL WITHIN A LARGE BALL

In this section we show that given an upper bound on the invariant upper Minkowski dimension of a set \( S \subseteq \partial_\infty M \), we get an upper bound on \( \text{vol}(B(x, \rho) \cap N_d(\text{CH}(S))) \) for any \( d > 0 \).

Recall that the invariant upper Minkowski dimension is defined using the visual metric \( \text{dist}_{\text{vis}}^x(\cdot, \cdot) \), where \( x \in M \) is some fixed basepoint, satisfying
\[
A^{-1}e^{-a \text{dist}^x(y, z)} \leq \text{dist}^x_{\text{vis}}(y, z) \leq Ae^{-a \text{dist}^x(y, z)},
\]
for all \( y, z \in \partial_\infty M \).

**Lemma 5.1.** Let \( S \) be a set in the boundary \( \partial_\infty M \). Then for all \( x \in M \), we have
\[
\text{vol}(B(x, \rho) \cap N_d(\text{CH}(S))) \lesssim e^{a \alpha \beta},
\]
for any \( \beta > \text{dim} S, d > 0 \), where the implicit constant depends only on \( M, d \) and \( \beta \).

We first outline the proof of Lemma 5.1. We will estimate the volume of the intersection of \( N_d(\text{CH}(S)) \) with annuli \( \text{An}(R) = B(x, R + 1) \setminus B(x, R) \). To achieve this, we cover the set \( S \) with balls \( B_1, B_2, ..., B_N \) of radius \( e^{-aR} \) in the visual metric \( d^x_{\text{vis}} \). The proof has three ingredients, sketched below.

1. We first show that \( \text{CH}(S) \subseteq N_{C'}(\text{Cone}(x, S)) \) for some absolute constant \( C' > 0 \), where \( \text{Cone}(x, S) = \bigcup_y [x, y] \).
2. We next show that \( N_{C}(\text{Cone}(x, B_i)) \setminus B(x, R) \subseteq \text{Cone}(x, \tilde{B}_i) \), where \( \tilde{B}_i \) is the ball with the same center as \( B_i \), but has radius larger by a bounded factor. This result explicitly uses that \( B_i \) has radius \( e^{-aR} \).
3. Finally, we show that \( \text{Cone}(x, \tilde{B}_i) \cap \text{An}(R) \) has bounded diameter independent of \( R \), and hence bounded volume.

Combining these three ingredients, we see that \( \text{vol}(B(x, \rho) \cap N_d(\text{CH}(S))) \lesssim N = N(R) \lesssim e^{a \alpha \text{dim} S} \) by assumption. Uniformity follows from the fact that \( \beta \) is required to be strictly larger than the invariant upper Minkowski dimension. The rest of this section is devoted to proving Lemma 5.1.

5.1. **Notation.** For \( S \subseteq \partial_\infty M \), denote by \( \text{Cone}(x, S) \) the union of geodesic rays with one endpoint \( x \) and the other endpoint (at infinity) in \( S \). We denote by \( \pi_x : M \setminus \{x\} \to \partial_\infty M \) the projection that maps \( y \in M \setminus \{x\} \) to the unique point \( z \in \partial_\infty M \) so that \( y \in [x, z] \). We also
write, for the duration of this proof $\text{An}(R) = B(x, R + 1) \setminus B(x, R)$. We also remind the reader that $[a, b]$ denotes the geodesic segment connecting $a, b \in \overline{M}$ (potentially infinite on one or both sides).

### 5.2. Estimating the convex hull with the cone

The purpose of this subsection is to show the proposition below.

**Proposition 5.2.** There exists a constant $C$ such that for all $S \subseteq \partial_\infty M$ and $x \in M$, we have $\text{CH}(S) \subseteq N_C(\text{Cone}(x, S))$.

**Proof.** Denote by $\text{GH}(S)$ the union of all geodesics with both endpoints in $S$. Clearly $\text{GH}(S) \subseteq \text{CH}(S)$.

**Claim 5.3.** For some constant $C'$ depending only on $M$, we have $\text{CH}(S) \subseteq N_{C'}(\text{GH}(S))$.

**Proof.** Suppose not, so that we have a sequence $S_n$ of subsets of $\partial_\infty M$ with points $x_n \in \text{CH}(S_n)$ such that $\text{dist}(x_n, \text{GH}(S_n)) \rightarrow \infty$. Since the action of $\text{Isom}(M)$ on $M$ has cobounded orbits, let $\Phi$ be a compact subset of $M$ that intersects every orbit. Without loss of generality, we can modify $S_n, x_n$ by an isometry so that $x_n \in \Phi$, for each $n$. We pass to a subsequence of $x_n$ such that $x_n \rightarrow x$. Then $\text{dist}(x, \text{GH}(S_n)) \rightarrow \infty$ and $\text{dist}(x, \text{CH}(S_n)) \rightarrow 0$, as $n \rightarrow \infty$. We equip $\partial_\infty M$ with the visual metric based at $x$. Since $\text{dist}(x, \text{GH}(S_n)) \rightarrow \infty$, we have

$$\sup_{y, z \in S_n} d^\text{vis}_x(y, z) \approx e^{-a \text{dist}(x, [y, z])} \rightarrow 0$$

as $n \rightarrow \infty$. But then $\text{diam}(S_n) \rightarrow 0$ and hence $\text{dist}(x, \text{CH}(S_n)) \rightarrow \infty$, which is a contradiction. \qed

Note that for any $a, b \in S$, since $M$ is $\delta$-hyperbolic (as a metric space) for some $\delta > 0$, we have $[a, b] \subseteq N_\delta([x, a] \cup [x, b])$, and therefore

$$\text{GH}(S) \subseteq N_\delta(\text{Cone}(x, S)),$$

for all $x \in M$ and $S \subseteq \partial_\infty M$. In particular $\text{CH}(S) \subseteq N_{C'}(\text{Cone}(x, S))$, so we set $C = C' + \delta$. \qed

### 5.3. Estimating neighborhood of a cone

In this subsection, we show the following proposition.

**Proposition 5.4.** For any $R > 0$ and constant $C > 0$, there exists a constant $\tilde{C} = \tilde{C}(C, M)$, such that for all sets $S \subseteq \partial_\infty M$, we have

$$N_C(\text{Cone}(x, S)) \setminus B(x, R) \subseteq \text{Cone}(x, N_{\tilde{C} - aR}(S)).$$

**Proof.** This is essentially equivalent to the following claim. We remind the reader that $\pi_x(y)$ is the unique point of intersection of the half-ray $xy$ with the boundary at infinity $\partial_\infty M$. 


Claim 5.5. For any $C > 0$, there exists $D = D(C)$ such that for all $x, y, z \in M$ with $\text{dist}(y, z) \leq C$, we have

$$\text{dist}(x, y) - \text{dist}(x, [\pi_x(y), \pi_x(z)]) \leq D(C).$$

Proof. Suppose there exist sequences $x_n, y_n, z_n$ with $\text{dist}(y_n, z_n) \leq C$ and

$$\text{dist}(x_n, y_n) - \text{dist}(x_n, [\pi_{x_n}(y_n), \pi_{x_n}(z_n)]) \to \infty \text{ as } n \to \infty.$$ 

Let $w_n$ be the nearest point on $[\pi_{x_n}(y_n), \pi_{x_n}(z_n)]$ to $x_n$. After applying an appropriate isometry of $M$, we can suppose $y_n$ lies in a fixed compact set for all $n$. Since $\text{dist}(y_n, z_n) \leq C$, all $z_n$ also lie in a fixed compact set. Therefore we can pass to a subsequence so that $y_n \to y \in M$ and $z_n \to z \in M$.

Note that

$$\text{dist}(x, y_n) - \text{dist}(x, [\pi_{x_n}(y_n), \pi_{x_n}(z_n)]) \leq \min(\text{dist}(x, y_n), \text{dist}(y_n, w_n))$$

so the sequences $(x_n)$ and $(w_n)$ converge to some points on the boundary at infinity $\partial\infty M$. Denote these points $x$ and $w$, respectively. Note that $\pi_{x_n}(y_n)$ converges to the unique point $\pi_x(y) \in \partial\infty M$ such that $y \in [x, \pi_x(y)]$. We define $\pi_x(z)$ analogously, and observe that $\pi_{x_n}(z_n) \to \pi_x(z)$. Since $w_n$ lies on the geodesic $[\pi_{x_n}(y_n), \pi_{x_n}(z_n)]$, in the limit we have $w = \pi_x(y)$ or $w = \pi_x(z)$. The claim below then implies that $\pi_x(y) = \pi_x(z)$, i.e. $x, y, z$ all lie on the same geodesic.

Claim 5.6. Let $\alpha, \beta, \gamma$ be distinct points in $\partial\infty M$. Suppose that $\alpha_n, \beta_n, \gamma_n$ are sequences of points in $M$ that converge to $\alpha, \beta, \gamma$, respectively. If $\omega_n$ is the nearest point on $[\beta_n, \gamma_n]$ to $\alpha_n$, then the set $\{\omega_n : n = 1, 2, \ldots\}$ is bounded.

Proof. This follows from the basic properties of horofunctions [5 Chapter II.8], but we sketch the proof for completeness. Let $f_n : (-L_n, R_n) \to M$ be the arc-length parameterization of $[\beta_n, \gamma_n]$, appropriately shifted so that $f_n \to f$, where $f : \mathbb{R} \to M$ is the arc-length parameterization of $[\beta, \gamma]$. We fix an arbitrary basepoint $o \in M$, and consider the functions $d_n : M \to \mathbb{R}$ given by

$$d_n(x) = \text{dist}(x, \alpha_n) - \text{dist}(\alpha_n, o).$$

Then $d_n \to d_{\infty}$ uniformly on compact sets. In particular, $d_n \circ f_n \to d_{\infty} \circ f$ uniformly on compact sets. These are all convex functions, and $d_{\infty} \circ f$ has a unique minimum, so the minima of $d_n \circ f_n$ remain bounded. □
Since \( w_n \) are unbounded, by Claim 5.6 we see that \( \pi_x(y) = \pi_x(z) = w \). Therefore \( x, y, z \) lie on the same geodesic. For ease of notation, we assume that \( y \) lies between \( x \) and \( z \).

Denote the angle at \( y_n \) between \([y_n, x_n]\) and \([y_n, w_n]\) by \( \theta_n \). Write

\[
A_n = \text{dist}(y_n, x_n), \quad B_n = \text{dist}(y_n, w_n) \quad \text{and} \quad C_n = \text{dist}(x_n, w_n).
\]

By the arguments above we see that \( \theta_n \to \pi \) as \( n \to \infty \). Since \( M \) is a \( \text{CAT}(-a^2) \) metric space, we see that for a triangle with sides \( aA_n, aB_n, aC_n \) in \( \mathbb{H}^2 \), the corresponding angle opposite \( C_n \) is at least \( \theta_n \). By the hyperbolic law of cosines

\[
cosh aC_n \geq \cosh aB_n \cosh aA_n + (-\cos \theta_n) \sinh aB_n \sinh aA_n.
\]

Therefore \( \frac{\cosh aC_n}{\cosh a(A_n+B_n)} \to 1 \) as \( n \to \infty \) and hence \( B_n + (A_n - C_n) \to 0 \). However by assumption \( A_n - C_n \to \infty \), and we get a contradiction since \( B_n \geq 0 \).

Let \( z \in N_C(\text{Cone}(x, S)) \setminus B(x, R) \), so that for some \( y \in \text{Cone}(x, S) \) we have \( \text{dist}(y, z) \leq C \). By Claim 5.5

\[
\text{dist}(x, [\pi_x(y), \pi_x(z)]) \geq \text{dist}(x, y) - D,
\]

so that \( \text{dist}^{\text{vis}}(\pi_x(y), \pi_x(z)) \leq A e^{aD} e^{-a\text{dist}(x,y)} \). But \( \pi_x(y) \in S \) since \( y \in \text{Cone}(x, S) \), and hence \( \text{dist}^{\text{vis}}(\pi_x(z), S) \leq A e^{aD} e^{-a\text{dist}(x,y)} \). Since \( \text{dist}(x, z) \geq R \), we have \( \text{dist}(x, y) \geq R - \text{dist}(y, z) \geq R - C \) and hence \( \text{dist}^{\text{vis}}(\pi_x(z), S) \leq A e^{a(C+D)} e^{-aR} \). We thus let \( \tilde{C} = A e^{a(C+D)} \), and see that \( \pi_x(z) \in \tilde{N}_{\tilde{C}}(S) \), and hence \( z \in \text{Cone}(x, \tilde{N}_{\tilde{C}}(S)) \). Since \( \tilde{C} \) or this argument do not depend on the choice of \( z \), we are done. \( \square \)

5.4. **Decomposition.** In this and the next two subsections, we show Lemma 5.1 using Propositions 5.2 and 5.3.

Set \( \varepsilon = e^{-aR} \), and cover the set \( S \) by \( N = N(\varepsilon) \) balls of radius \( \varepsilon \), centered at \( y_1, y_2, \ldots, y_N \). Since \( \text{CH}(S) \subseteq N_C(\text{Cone}(x, S)) \) by Proposition 5.2 and \( S \subseteq \bigcup_{i=1}^N B(y_i, \varepsilon) \), we have

\[
N_d(\text{CH}(S)) \cap \text{An}(R) \subseteq \text{An}(R) \cap \bigcup_{i=1}^N N_{C+d}(\text{Cone}(x, B(y_i, \varepsilon))) \setminus B(x, R)
\]

\[
(5.1) \quad \subseteq \text{An}(R) \cap \bigcup_{i=1}^N \text{Cone}(x, B(y_i, \varepsilon + \tilde{C} e^{-aR})),
\]

where we used Proposition 5.3 in going from the first to the second line.
5.5. **Volume bound on cones over visual balls.** In this subsection, we show that each piece in the decomposition [5.1] has bounded volume.

**Claim 5.7.** Fix a constant $C$. Then for points $x \in M, y \in \partial_\infty M$, define the set

$$S_{R,C}(x, y) = \text{Cone} \left( x, \{ z \in \partial_\infty M : \text{dist}(x, [y, z]) \geq R \} \right) \cap B(x, R + C) \setminus B(x, R).$$

Then the diameter of $S_{R,C}$ is bounded by a constant $D(C)$ depending only on $C$ and $M$.

**Proof.** Suppose the diameter of $S_{R,C}(x, y)$ is unbounded. Then there exist sequences $x_n, w_n \in M, y_n, z_n \in \partial_\infty M$ and $R_n > 0$ such that

$$\text{dist}(x_n, [y_n, z_n]) \geq R_n,$$

$$w_n \in [x_n, z_n],$$

$$R_n \leq \text{dist}(x_n, w_n) \leq R_n + C,$$

$$\text{dist}(w_n, p_n) \to \infty,$$

where $p_n$ is the point on $[x_n, y_n]$ at a distance $R_n$ from $x_n$. By applying an isometry of $M$, we can suppose that $p_n$ is in a fixed compact set. Pass to a subsequence so that $p_n \to p \in M$. Note that the diameter of $S_{R,C}(x, y)$ is at most $2(R + C)$, so in particular $R_n \to \infty$. Hence $\text{dist}(x_n, p_n) = R_n \to \infty$, so we have $x_n \to x \in \partial_\infty M$ along some subsequence. Then $y_n \to y$ with $p \in [y, x]$ and pass to a further subsequence so that $z_n \to z \in \partial_\infty M$ and $w_n \to w \in M$.

Note that since $[y_n, z_n]$ is disjoint from the ball centered at $x_n$ through $p_n$, it follows that $[y, z]$ is disjoint from the horoball $H$ based at $x$ passing through $p$. In particular $y, z \neq x$. We also have $w \in [x, z] \cap N_C(H) \setminus H$ which is a bounded set in $M$, so $\text{dist}(p, w) < \infty$. This is a contradiction. $\square$

Denote by $V(C)$ the maximal volume of a ball of radius $D(C)$. Note that for $z \in B(y_i, \varepsilon + \tilde{C}e^{-aR})$, we have

$$\varepsilon + \tilde{C}e^{-aR} = e^{-aR}(1 + \tilde{C}) \geq \text{dist}^{\text{vis}}(y_i, z) \geq A^{-1}e^{-a\text{dist}(x,[y_i,z])},$$

and hence $\text{dist}(x, [y_i, z]) \geq R - \frac{1}{a} \log A(1 + \tilde{C})$. Therefore

$$B(x, R + 1) \cap \text{Cone}(x, B(y_i, \varepsilon(1 + A\tilde{C}^\prime))) \subseteq S_{R - \frac{1}{a} \log A(1 + \tilde{C})}(x, y_i),$$

and hence

$$\text{vol}(\text{An}(R) \cap N_{C+d}(\text{Cone}(x, B(y_i, \varepsilon)))) \leq V(1 + a^{-1} \log A(1 + \tilde{C})) =: V_0(d).$$
5.6. **Finishing the proof.** We now combine previous results to show the main volume estimate. We have

$$\text{vol}(N_d(CH(S)) \cap \text{An}(R)) \leq N(e^{-aR})V_0 \lesssim e^{aR\beta}V_0.$$ 

Hence we have

$$\text{vol}(N_d(CH(S)) \cap B(x, \rho)) \leq \sum_{r=0}^{\lfloor \rho \rfloor} \text{vol}(CH(K) \cap \text{An}(r)) \lesssim e^{a\rho\beta}V_0(d),$$

where the implicit constant depends only on $M$ and $\beta$.

6. **Proof of the main results**

6.1. **Estimates on the heat kernel.** We collect some estimates on the heat kernel in pinched Hadamard manifolds and hyperbolic spaces we will use to bound $\int_{N_d(K)} G(x, y) d\text{vol}(y)$.

Recall that $H(x, y, t)$ denotes the heat kernel on $M$, for distinct $x, y \in M$ and $t \geq 0$. The connection to Green’s function is through the identity

$$G(x, y) = \int_0^\infty H(x, y, t) dt.$$

**Proposition 6.1.** Assume $t \geq 1$, and denote $\rho = \text{dist}(x, y)$.

1. On a pinched Hadamard manifold with sectional curvature at most $-a^2$, we have

$$H(x, y, t) \lesssim (1 + \rho^n)e^{-\frac{\rho^2}{4t} - \frac{(n-1)a^2}{4t}}.$$

2. On $\mathbb{H}^n$, we have

$$H(x, y, t) \lesssim (1 + \rho^n)e^{-\frac{\rho^2}{4t} - \frac{(n-1)a^2}{4t} - \frac{n-1}{2}\rho}.$$

**Proof.** 

(1) Let $\lambda_1(M)$ be the bottom of the spectrum of the negative Laplacian $-\Delta$ on $M$. Note that $\Delta$ is elliptic, so in particular $\lambda_1(M) \geq 0$. We have the bound derived by Davies in [7]

$$H(x, y, t) \lesssim \frac{1}{\sqrt{\text{vol}(B(x, r))\text{vol}(B(y, r))}} e^{-\frac{\rho^2}{4t} - \lambda_1(M)t} = \overline{H}(\rho, t),$$

where $r = \min\left(1, \sqrt{t}, \frac{t}{\rho}\right)$. Since $t \geq 1$, we have $r = \min\left(1, \frac{t}{\rho}\right)$. Note that since $r \leq 1$, we have $\text{vol}(B(x, r)) \approx r^n$ by Bishop’s
volume estimates. Therefore
\[
\bar{H}(\rho, t) \approx r^{-n} e^{-\frac{\rho^2}{4t} - \lambda_1(M) t} = \max \left(1, \frac{\rho}{t} \right)^n e^{-\frac{\rho^2}{4t} - \lambda_1(M) t}
\]
\[
\approx \left(\frac{\rho + t}{t}\right)^n e^{-\frac{\rho^2}{4t} - \lambda_1(M) t} \lesssim (1 + \rho^n)e^{-\frac{\rho^2}{4t} - \frac{(n-1)^2a^2}{4} t},
\]
where in the last inequality we used \(\lambda_1(M) \geq \frac{(n-1)^2a^2}{4}\), as shown by McKean in [16].

(2) By [8, Theorem 3.1], we have
\[
H(x, y, t) \approx (1 + \rho)(1 + \rho + t)^{\frac{n-1}{2}} e^{-\frac{\rho^2}{4t} - \frac{(n-1)^2a^2}{4} t - \frac{n-1}{2} \rho}
\]
\[
\approx \frac{1 + \rho}{(2 + \rho)^{\frac{n}{2}}} \left(\frac{1 + \rho}{t} + 1\right)^{\frac{n-1}{2}} e^{-\frac{\rho^2}{4t} - \frac{(n-1)^2a^2}{4} t - \frac{n-1}{2} \rho}
\]
\[
\lesssim (2 + \rho)^{\frac{n-1}{2}} e^{-\frac{\rho^2}{4t} - \frac{(n-1)^2a^2}{4} t - \frac{n-1}{2} \rho}
\]
\[
\lesssim (1 + \rho^n)e^{-\frac{\rho^2}{4t} - \frac{(n-1)^2a^2}{4} t - \frac{n-1}{2} \rho}.
\]
\[
\square
\]

6.2. Proof of Theorem [1,4] Let \(r : \mathbb{H}^n \to CH(S) = K\) be the nearest-point retraction, and let \(\tilde{r} : \mathbb{H}^n \to \mathbb{H}^n\) be the smooth map from Corollary 3.7.

Claim 6.2. There exists \(\varepsilon = \varepsilon(d, r) > 0\), so that
\[
\int_{N_d(K)} H(x, y, t) d\text{vol}(y) \lesssim e^{-\varepsilon t},
\]
uniformly in \(x\).

Proof. By Lemma 5.1, there exists some \(\beta < n - 1\) with
\[
\text{vol}(B(x, \rho) \cap K) \lesssim e^{\beta \rho},
\]
uniformly in \(x \in \mathbb{H}^n\).

We have by Proposition 6.1
\[
\int_{N_d(K)} H(x, y, t) d\text{vol}(y) \lesssim \sum_{\rho=1}^{\infty} (1 + \rho^n)e^{-\frac{\rho^2}{4t} - \frac{(n-1)^2a^2}{4} t - \frac{n-1}{2} \rho} \text{vol}(\text{An}(\rho) \cap N_d(K)),
\]
where \(\text{An}(\rho) = B(x, \rho) \setminus B(x, \rho - 1)\). Since
\[
\text{vol}(\text{An}(\rho) \cap N_d(K)) \leq \text{vol}(B(x, \rho) \cap N_d(K)) \lesssim e^{\beta \rho},
\]
Proof. Note that 
\[
\int_{N_d(K)} H(x, y, t) d\text{vol}(y) \lesssim \sum_{\rho = 1}^{\infty} (1 + \rho^n) e^{-\frac{(n-1)^2 t^2}{4t}} \exp \left( -\frac{\rho^2 + 2t(n-1-2\beta)\rho}{4t} \right)
\]
\[
\lesssim e^{-\frac{(n-1)^2 t_0^2}{4t}} \sum_{\rho = 1}^{\infty} (1 + \rho^n) \exp \left( -\frac{(\rho + (n-1-2\beta)t)^2}{4t} \right)
\]
\[
\lesssim e^{-\beta(n-1-\beta) t} \int_{1}^{\infty} (1 + \rho^n) \exp \left( -\frac{(\rho + (n-1-2\beta)t)^2}{4t} \right) d\rho.
\]

The final integral grows at most polynomially in $t$, so the claim holds for any $0 < \varepsilon < \beta(n-1-\beta)$. \hfill \Box

It follows that
\[
\int_{N_d(K)} \int_{0}^{\infty} H(x, y, t) \|\tau(\bar{r})(y)\| dtd\text{vol}(y) \lesssim 1,
\]
uniformly in $x$, as $\|\tau(\bar{r})\| \lesssim 1$. Since $G(x, y) = \int_{0}^{\infty} H(x, y, t) dt$, we are done by Corollary 4.7.

6.3. Proof of Theorem 1.3. Let $S$ be the image of $\partial t : S^1 \to \partial \mathbb{H}^n$, and let $K$ be the convex hull of $S$.

Claim 6.3. For some $d$ large enough, there exists a Lipschitz map $f : N_d(K) \to \mathbb{H}^2$ such that
\[
\sup_{x \in \mathbb{H}^2} \text{dist}(f \circ \iota(x), x) < \infty.
\]

Proof. Note that $\mathbb{H}^2 = \bigcup_{z \in S^1} [-z, z]$, so that
\[
\iota(\mathbb{H}^2) = \bigcup_{z \in S^1} \iota([-z, z]).
\]
Each $\iota([-z, z])$ is a quasigeodesic with the same constants, so Morse lemma implies that $\iota([-z, z]) \subseteq N_C([\iota(-z), \iota(z)])$ for some $C > 0$ that depends only on quasi-isometry constants of $\iota$. Therefore $\iota(\mathbb{H}^2) \subseteq N_C(K)$. Similarly we have $[\iota(z), \iota(w)] \subseteq N_C([z, w])$, so $\text{GH}(S) \subseteq N_C(\iota(\mathbb{H}^2))$. We have already shown as part of the first Claim of Lemma 5.1 that $K \subseteq N_{C'}(\text{GH}(S))$ for some constant $C' > 0$ that depends only on $n$, so that $K \subseteq N_{C+C'}(\iota(\mathbb{H}^2))$, and therefore the quasi-isometric embedding $\iota : \mathbb{H}^2 \to N_C(K)$ is quasirejective.

Therefore there exists a quasi-inverse $\tilde{f} : N_C(K) \to \mathbb{H}^2$, for all $C$ large enough, meaning $\sup_{x \in \mathbb{H}^2} \text{dist}(x, \tilde{f} \circ \iota(x)) < \infty$. We in fact construct a quasi-inverse on a larger set $\tilde{f} : N_{C+1}(K) \to \mathbb{H}^2$, so that by the construction of Benoist and Hulin from [2] Proposition
we can construct a Lipschitz map \( f : N_C(K) \to \mathbb{H}^2 \) so that \( \sup_x \text{dist}(\tilde{f}(x), f(x)) < \infty \). Then \( f \) is a Lipschitz quasi-inverse of \( \iota \), as claimed.

Now let \( r : \mathbb{H}^n \to K \) be the nearest-point retraction, and write \( \hat{r} = f \circ r \), for some \( f \) as in the Claim. Then \( \hat{r} : \mathbb{H}^n \to \mathbb{H}^2 \) is Lipschitz with

\[
\text{Lip} \left( \hat{r}|_{\mathbb{H}^n \setminus N_d(K)} \right) \lesssim e^{-ad},
\]

for all \( d > 0 \), by Proposition 3.5. By Lemma 3.1 there exists a smooth map \( \tilde{r} : \mathbb{H}^n \to \mathbb{H}^2 \) so that

\[
\sup_{x \in \mathbb{H}^n} \text{dist}(\tilde{r}(x), \hat{r}(x)) < \infty,
\]

\[
\| \nabla x \tilde{r} \| \lesssim e^{-ad \text{dist}(x, K)},
\]

\[
\| H(\tilde{r})_x(X, X) \| \lesssim e^{-ad \text{dist}(x, K)} \| X \|^2,
\]

for all \( x \in \mathbb{H}^n, X \in T_x \mathbb{H}^n \). It follows from the first inequality and the Claim that

\[
\sup_{x \in \mathbb{H}^2} \text{dist}(x, \tilde{r} \circ \iota(x)) < \infty.
\]

Note that each set in \( \{ \gamma S : \gamma \in \text{Isom}(\mathbb{H}^n) \} \) is a quasicircle with the same quasisymmetry constants as \( S \), so by [10, Theorem 18, Lemma 16] we see that

\[
\dim \text{Isom}( \mathbb{H}^n ) S < n - 1.
\]

Therefore we can apply the Claim from \( \S 6.2 \) to get

\[
\int_{N_d(K)} \int_0^\infty H(x, y, t) \| \tau(\tilde{r})(y) \| dt d\text{vol}(y) \lesssim 1,
\]

uniformly in \( x \), as \( \| \tau(\tilde{r}) \| \lesssim 1 \). Since \( G(x, y) = \int_0^\infty H(x, y, t) dt \), by Corollary 4.7 we are done.

6.4. Proof of Theorem 1.5. As always, let \( K \) be the convex hull of \( S \).

Claim 6.4. There exists \( \varepsilon = \varepsilon(K, d) \), so that

\[
\int_{N_d(K)} H(x, y, t) d\text{vol}(y) \lesssim e^{-\varepsilon t},
\]

uniformly in \( x \).
Proof. Let $\dim S < \beta < \frac{n-1}{2}$. Note that by Lemma 5.1 we have

$$\text{vol}(B(x, \rho) \cap N_d(K)) \lesssim e^{a \rho \beta},$$

uniformly in $x$. Therefore by Proposition 6.1, we have

$$\int_{N_d(K)} H(x, y, t) d\text{vol}(y) \lesssim \sum_{\rho=1}^{\infty} (1 + \rho^n) e^{-\frac{\rho^2}{4t} - \frac{\rho^2}{4t} \left(\frac{n-1}{4}\right)^2} \text{vol}(B(x, \rho) \cap N_d(K))$$

$$\lesssim \sum_{\rho=1}^{\infty} (1 + \rho^n) e^{-\frac{\rho^2}{4t} - \frac{\rho^2}{4t} \left(\frac{n-1}{4}\right)^2} \int_{\rho=1}^{\infty} \text{vol}(B(x, \rho) \cap N_d(K))$$

$$= e^{-a^2 t \left(\frac{n-1}{4}\right)^2 - \beta^2} \sum_{\rho=1}^{\infty} (1 + \rho^n) \exp \left(-\frac{\rho - 2ta\beta}{4t}\right)$$

$$\lesssim e^{-a^2 t \left(\frac{n-1}{4}\right)^2 - \beta^2} \int_{1}^{\infty} (1 + \rho^n) \exp \left(-\frac{\rho - 2ta\beta}{4t}\right).$$

The integral in the final line grows at most polynomially, so the Claim holds for any $0 < \varepsilon < a^2 \left(\frac{n-1}{4}\right)^2 - \beta^2$.

Let $r : M \to K$ be the nearest-point retraction, and let $\tilde{r} : M \to M$ be as in Corollary 3.7. Fix an arbitrary $d > 0$. We have

$$\int_{N_d(K)} \int_{0}^{\infty} H(x, y, t) \|\tau(\tilde{r})(y)\| dt d\text{vol}(y) \lesssim 1,$$

uniformly in $x$, as $\|\tau(\tilde{r})\| \lesssim 1$. Since $G(x, y) = \int_{0}^{\infty} H(x, y, t) dt$, by Corollary 4.7 we are done.

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