Quadratic Forms and their Number Systems and Geometry

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April 12, 2022

Abstract

Quaternions, split quaternions and hybrid number are very well known number systems. These number systems are used to make geometry in Euclidean and Lorentz spaces. These number systems can be obtained with the help of a quadratic form. This article examines how to find the corresponding number system for any quadratic form. As a result of this examination, bilinear form, vector product, skew symmetric matrix, and rotation matrices corresponding to any number system were obtained.

Keywords: Quaternions, Split Quaternions, Lorentzian geometry, Rotation maps, Coquaternions

Mathematics Subject Classification: 11R52 15A66 58B34

1 Introduction

Quadratic forms are a widely studied topic. Complex, hyperbolic, and dual numbers are well-known two-dimensional number systems. By associating these number systems with quadratic forms, the geometric structure of these number systems has been examined. Rotation transformations on the unit circle, hyperbola and dual circle in \( \mathbb{R}^2 \) are examined in detail in the article [7]. In this study, rotational transformation on three-dimensional objects such as ellipsoid, hyperboloid and cone in \( \mathbb{R}^3 \) will be examined. For this, firstly, a number system of any given quadratic form will be created. Then, using this number system, a rotational transformation will be performed on the quadric surface corresponding to a quadratic form, just like the effect of quaternions on the sphere and the effect of split quaternions on the hyperboloid. Now, let’s introduce the general properties of quadratic forms and their forms in \( \mathbb{R}^3 \).

Definition 1 [14] A quadratic form \( q: V \rightarrow F \) on a finite dimensional vector space \( V \) over \( F \) is a map satisfying:

i. \( Q(\lambda v) = \lambda^2 Q(v) \) for \( v \in V, \lambda \in F \).
The map

\[ B_Q(v, w) = \frac{1}{2} [Q(v + w) - Q(v) - Q(w)] \] (1)

is bilinear.

We denote a quadratic form by \((V, Q)\), or simply as \(Q\). The bilinear form \(B_Q\) is symmetric; \(Q\) determines \(B_Q\) and for all \(v \in V\),

\[ Q(v) = B_Q(v, v) \]

where \(F\) is a field with char\(F \neq 2\).

**Remark 2** Since \(F = \mathbb{R}\) will be taken in this article, the character of the field \(F\) will not be mentioned in the rest of the article.

Given a bilinear form on \(\mathbb{R}^3\), there exists a unique \(I^* \in \mathbb{R}^{3 \times 3}\) square matrix such that for all \(u, v \in \mathbb{R}^3\), \(B_Q(u, v) = u I^* v\). \(I^*\) is called “the matrix associated with the form” with respect to the standard basis. A bilinear form is said skew-symmetric if \(B_Q(u, v) = -B_Q(v, u)\), respectively. Also, the matrix associated with a symmetric bilinear form is symmetric, and similarly, the associated matrix of a skew-symmetric bilinear form is skew-symmetric.

Due to (1) symmetric bilinear forms and quadratic forms are in a one-to-one correspondence over \(\mathbb{R}\). The kernel of a symmetric bilinear form \(B\) on \(V\) is

\[ \ker B = \{v \in V : B(v, w) = 0 \text{ for all } w \in V\} . \] (2)

As can be seen from the definition (2) kernel of a bilinear form is a linear subspace of \(V\). The kernel of a quadratic form is defined as the kernel of corresponding bilinear form. A symmetric bilinear form \(B\) is called

- degenerate, if \(\ker B \neq \{0\}\),
- non-degenerate, if \(\ker B = \{0\}\).

A symmetric bilinear form is degenerate if and only if its matrix with respect to one basis has determinant 0. A quadratic form is called degenerate or non-degenerate if the corresponding bilinear form is degenerate or nondegenerate. The vector space \(\mathbb{R}^3\) over \(\mathbb{R}\), a bilinear form \(B\) (and the corresponding quadratic form) is called

- positive semidefinite, if \(B_Q(v, v) \geq 0\), for \(\forall v \in \mathbb{R}^3\)
- positive definite if \(B_Q(v, v) > 0\), for \(\forall v \in \mathbb{R}^3 - \{0\}\)
- indefinite if \(B_Q(v, v) < 0\), \(B_Q(w, w) > 0\), for \(\exists v, w \in \mathbb{R}^3\).

The conditions negative semidefinite and negative definite are defined similarly. But, a real non-degenerate symmetric bilinear form is either positive definite, negative definite, or indefinite. The character of a bilinear form as positive or negative definite (or neither) can also be determined by the determinant test. But these details will not be discussed here.
A scalar (bilinear) product on a real vector space $V$ is a non-degenerate symmetric bilinear form. We will represent the scalar product as $\langle \cdot, \cdot \rangle$. For any vectors $u, v$ in an inner product space are called orthogonal if $\langle u, v \rangle = 0$, also if $\langle u, u \rangle = \langle v, v \rangle = 1$, then these vectors are called orthonormal.

The Euclidean scalar product is a scalar product that is positive definite. The subject of this article the vector space $\mathbb{R}^3$ equipped with a Euclidean scalar product is called a Euclidean vector space. The Euclidean norm of a vector $x$ is

$$\|x\| = \sqrt{\langle x, x \rangle_E}.$$ 

A Lorentz scalar product is an indefinite scalar product. The vector space $\mathbb{R}^3$ equipped with a Lorentz scalar product with signature $(-, +, +)$ is called a Lorentz vector space. A vector $v$ in a vector space with indefinite scalar product is called

- spacelike if $\langle x, x \rangle_L > 0$,
- timelike if $\langle x, x \rangle_L < 0$
- lightlike if $\langle x, x \rangle_L = 0, x \neq 0$.

The Euclidean norm of a vector $x$ is

$$\|x\| = \sqrt{|\langle x, x \rangle_L|}.$$ 

The set of lightlike vectors is called the light cone and zero is spacelike vector.

Let $V$ be a finite dimensional real vector space equipped with a scalar product $\langle \cdot, \cdot \rangle$. An orthogonal map on $V$ is a linear map $F: V \rightarrow V$ that satisfies

$$\langle Fu, Fv \rangle = \langle u, v \rangle$$

for all $u, v \in V$. (3)

Since the scalar product is nondegenerate, this suggests that $F$ is injective. In this study, since dim $V = 3$ was assumed to be finite dimensional, it follows that $F$ is surjective, thus $F$ is invertible. The orthogonal transformations of a inner product space $(V, \langle \cdot, \cdot \rangle)$ constitute a group, this group is denoted by $O(V, \langle \cdot, \cdot \rangle)$. If the determinate of the matrix $A$ corresponding to the linear map $F$ is $+1$ or $-1$, then this group is called a special orthogonal group and this group is denoted by $SO(3)$ in Euclidean space and $SO(3,1)$ in Lorentz space [13, 3, 19].

In $\mathbb{R}^3$, a quadric can be given by the equation

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + Gx + Dy + Iz + J = 0.$$ (4)

As can be seen, when the first six coefficients are taken as zero, the equation (4) indicates a plane. In the most general form, the equation (4) denote ellipsoid, hyperboloid, and paraboloid. A quadratic form a set of points $(x, y, z) \in \mathbb{R}^3$ satisfying

$$Q(x, y, z) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz$$
where \( A, B, C, D, E, F \in \mathbb{R} \) with \( A, B, C, D, E, F \) not all zero and by \( M_Q \) its matrix
\[
M_Q = \begin{bmatrix}
A & D & E \\
D & B & F \\
E & F & C
\end{bmatrix}.
\]

The signature of a matrix \( M_Q \in \mathbb{R}^{3 \times 3} \) is \((k, l, m)\) where \( k \) is the number of strictly positive eigenvalues of \( M_Q \), \( l \) is the number of strictly zero eigenvalues of \( M_Q \), \( m \) is the number of strictly negative eigenvalues of \( M_Q \). Namely, the rank of \( M_Q \) is \( k + m \). For \( \text{rank} M_Q = 3 \), classifies quadratic form in the following table:

| Signature of \( M_Q \) | Type of quadratic form |
|------------------------|------------------------|
| \((3, 0, 0)\) or \((0, 0, 3)\) | Ellipsoid |
| \((2, 0, 1)\) or \((1, 0, 2)\) | cone, 1 or 2-sheeted hyperboloid |

The most famous geometric object in which quadratic forms are applied to geometry is the sphere. If we take the radius of the sphere as the unit of length, we can do the geometry on the 3-dimensional unit sphere,
\[
S^2 = \{ x \in \mathbb{R}^3 : \| x \| = 1 \} \subset \mathbb{R}^3,
\]
where \( \| x \| = \sqrt{\langle x, x \rangle_E} \), and \( \langle x, y \rangle_E = \sum_{i=1}^{3} x_i y_i \) is the standard Euclidian scalar product. Another famous sphere is the 3-dimensional Lorentz-Minkowski space sphere
\[
H^2 = \{ x \in \mathbb{R}^3 : \langle x, x \rangle_L = -1 \} \subset \mathbb{R}^3
\]
where \( \langle \cdot, \cdot \rangle_L \) denotes the Lorentz scalar product
\[
\langle x, y \rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3.
\]

Generalized scalar product can be found in detail in Özdemir and Şimşek articles [16], [20], [21]. In addition, the authors also examined linear transformations for generalized scalar products in the same papers.

1.1 Quaternions

Number sets such as quaternions, split quaternions, hybrid numbers is defined through quadratic forms. These number systems form a geometry according to the quadratic form in which they are defined. Three-dimensional Euclidean space and Lorentz-Minkowski space are associated with quaternions and split quaternions, respectively. The geometric structure of hybrid numbers is newly studied. These number systems have provided many conveniences in the geometry to which they are associated. Let’s briefly explain the quaternion, split quaternion and hybrid number systems:

In 1843, Sir William Rowan Hamilton invented the quaternion algebra, which is familiar denoted \( \mathbb{H} \) in his honor. The set of quaternions can be stated as follows:
\[
\mathbb{H} = \{ a + bi + cj + dk : a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1 \}.
\]
It was soon realized that the quaternion product could be used to represent the rotational transformation in $\mathbb{E}^3$. In 1855, Arthur Cayley explored that quaternions could also be used to represent rotations in $\mathbb{E}^4$. Quaternion algebra has accepted a significant role lately in different areas of physical science; for example, synthesis of mechanism and machines, in differential geometry, simulation of particle motion in molecular physics, in analysis, and quaternionic formulation of the equation of motion in the theory of relativity. Additionally, quaternions have been indicated in terms of $4 \times 4$ matrices by means of left and right multiplication operators. If the literature is examined, it can be clearly seen that the geometric properties of quaternions are a number system designed to fit the orthonormal frame of $\mathbb{E}^3$ [6], [5], [2].

The set of split quaternions presented by Cockle [1] in 1849 is as follows:

$$\mathcal{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}, i^2 = -1, j^2 = k^2 = ijk = 1\}.$$  

Lately, since split quaternions are used to state Lorentzian rotations, there are studies on geometric and physical applications of split quaternions that call for solving split quaternionic equations. In general, split quaternions are used to express rotation and reflection transformations for timelike and spacelike vectors in Minkowski 3-space. Because the units of split quaternions are compatible with the unit base vectors in Minkowski 3-space [3], [12], [13], [15]. Rotations around the lightlike axis were also investigated in [4], [10].

In 2018, Özdemir invented the set of hybrid numbers [17], which is a non-commutative ring, is a generalization of complex, hyperbolic and dual number sets and it is defined as

$$\mathbb{K} = \{a + bi + c\varepsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i\}.$$  

Generalized quaternions and their algebraic properties are discussed in detail in [8], [18].

### 2 Generalized Quadratic Number System

Quadratic forms are geometric structures whose geometry is studied by associating them with a number system. Now we will present here a way to give you all the number systems that can be constructed for all quadratic forms. Let’s define the generalized quadratic number system as follows:

**Definition 3** The set of generalized quadratic number system, denoted by $\Omega$, is defined as

$$\Omega = \left\{ \begin{array}{l}
q = q_0 + q_1i + q_2j + q_3k : i^2 = A, j^2 = B, k^2 = C, \\
i = D + \alpha_1i + \alpha_2j + \alpha_3k, ik = E + \beta_1i + \beta_2j + \beta_3k, jk = F + \lambda_1i + \lambda_2j + \lambda_3k, \\
l = G - \alpha_1i - \alpha_2j - \alpha_3k, kl = H - \beta_1i - \beta_2j - \beta_3k, \\
n = I - \lambda_1i - \lambda_2j - \lambda_3k,
\end{array} \right\},$$

where $A, B, C, D, E, F \in \mathbb{R}$,

for $\alpha_1 = \lambda_3$, $\beta_1 = -\lambda_2$, $\beta_3 = -\alpha_2$;
\[ A = \alpha_2^2 + \alpha_3 \beta_2, \quad B = \alpha_2^2 - \lambda_1 \alpha_3, \quad C = \beta_1^2 + \lambda_1 \beta_2, \]
\[ D = -(\alpha_1 \alpha_2 + \alpha_3 \beta_1), \quad E = -(\beta_2 \alpha_1 - \alpha_2 \beta_1), \quad F = \alpha_1 \beta_1 + \lambda_1 \alpha_2. \]

This set of numbers can be thought of as a set of quadruplets defined below:

\[ 1 \leftrightarrow (1, 0, 0, 0), \quad i \leftrightarrow (0, 1, 0, 0), \quad j \leftrightarrow (0, 0, 1, 0), \quad k \leftrightarrow (0, 0, 0, 1). \]

For a quadratic number \( q = q_0 + q_1 i + q_2 j + q_3 k \), a real number \( q_0 \) is called the scalar part and is denoted by \( S(q) \) and the part \( q_1 i + q_2 j + q_3 k \) is called the vector part, and is denoted by \( \mathbf{v}_q \). Now the matrix (5) can be written as:

\[ M_Q = \begin{bmatrix}
\alpha_2^2 + \alpha_3 \beta_2 & -(\alpha_1 \alpha_2 + \alpha_3 \beta_1) & \alpha_2 \beta_1 - \beta_2 \alpha_1 \\
-(\alpha_1 \alpha_2 + \alpha_3 \beta_1) & \alpha_1^2 - \lambda_1 \alpha_3 & \lambda_1 \alpha_2 + \alpha_1 \beta_1 \\
\alpha_2 \beta_1 - \beta_2 \alpha_1 & \lambda_1 \alpha_2 + \alpha_1 \beta_1 & \beta_1^2 + \lambda_1 \beta_2
\end{bmatrix}. \quad (9) \]

### 2.1 Operation in the Quadratic Numbers

Two quadratic numbers are equal if all their components are equal, one by one. The sum of two quadratic numbers is defined by summing their components. Addition operation in the quadratic numbers is both commutative and associative. Zero is null element. Regarding the addition operation, the inverse element of \( q \) is \(-q\), which is defined as having all components of \( q \) changed in their signs. This requires that \((\mathbb{Q}, +)\) is an Abelian group.

The quadratic numbers product

\[ q \mathbf{p} = (q_0 + q_1 i + q_2 j + q_3 k) (p_0 + p_1 i + p_2 j + p_3 k) \]

is obtained by distributing the terms on the right as ordinary algebra. Given the definition of the set of quadratic numbers, the following multiplication table can be constructed.

|   | 1  | i  | j  | k  |
|---|----|----|----|----|
| 1 | 1  | 1  | j  | k  |
| i | i  | A  | B  | C  |
| j | j  | D  | E  | F  |
| k | k  | F  | -\lambda_1^2 - \lambda_2^2 - \lambda_3^2 | C  |

This table will be used to multiplication any two quadratic numbers. The table shows us that the multiplication operation in the quadratic numbers is not commutative. But it has the property of associativity. The conjugate of a quadratic number \( q = q_0 + q_1 i + q_2 j + q_3 k \), denoted by \( \overline{q} \), is defined as

\[ \overline{q} = q_0 - q_1 i - q_2 j - q_3 k. \]

The conjugate of the sum of quadratic numbers is equal to the sum of their conjugate:

\[ \overline{q + \mathbf{p}} = \overline{q} + \overline{\mathbf{p}}. \]
Besides, according to quadratic numbers product, we have \( \overline{qq} = q\overline{q} \) and its value is

\[ \overline{qq} = q\overline{q} = q^2 - Ax^2 - By^2 - Cz^2 - 2Dxy - 2Eyz. \]

The real number \( \sqrt{\overline{qq}} \) will be called the norm of the quadratic number \( q \) and will be denoted by \( \|q\| \).

Let's take the quadratic numbers \( q = q_0 + q_1i + q_2j + q_3k \) and \( p = p_0 + p_1i + p_2j + p_3k \). The scalar product of \( q \) and \( p \) is defined as

\[ \langle q, p \rangle \Omega = \frac{q^\mathbf{T} + p^\mathbf{T}}{2} = p_0q_0 - Ap_1q_1 - Bp_2q_2 - Cp_3q_3 - Fp_2q_3 - Fp_3q_2 - Dp_1q_2 - Dp_2q_1 - Ep_1q_3 - Ep_3q_1 \]

\[ = p_0q_0 - (Ap_1q_1 + Bp_2q_2 + Cp_3q_3 + D(p_1q_2 + p_2q_1) + E(p_1q_3 + p_3q_1) + F(p_2q_3 + p_3q_2)). \]

So, scalar product of quadratic vectors \( v = (v_1, v_2, v_3) \) and \( u = (u_1, u_2, u_3) \) is

\[ \langle u, v \rangle \Omega = \Delta (Au_1v_1 + Bu_2v_2 + Cu_3v_3 + D(u_1v_2 + u_2v_1) + E(u_1v_3 + u_3v_1) + F(u_2v_3 + u_3v_2)) \]

where \( \Delta = -1 \) if the quadratic form corresponding to \( \langle u, v \rangle \Omega \) is an ellipsoid, and \( \Delta = 1 \) if the quadratic form corresponding to \( \langle u, v \rangle \Omega \) is hyperboloid. Thus, the matrix associated with the bilinear form \( \langle u, v \rangle \Omega \) becomes

\[ \mathcal{M}_\Omega = \Delta M_\Omega = \Delta \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}. \] (10)

While obtaining this matrix (10), the type of quadratic form can be determined by looking at the sign of the eigenvalues of the matrix (9). After that, we'll work with vectors in the quadratic 3-space and find the norm of the vectors with the help of this scalar product. Also, the vector product of two quadratic numbers \( q \) and \( p \) defined as

\[ q \times \Omega p = \frac{q^\mathbf{T} - p^\mathbf{T}}{2} \]

\[ = (p_0q_1 - p_1q_0 + \alpha_1 (p_1q_2 - p_2q_1) + \beta_1 (p_2q_3 - p_3q_2))i + \\
+ (p_0q_2 - p_2q_0 + \alpha_2 (p_1q_2 - p_2q_1) + \beta_1 (p_3q_3 - p_3q_2))j + \\
+ (p_0q_3 - p_3q_0 + \alpha_3 (p_2q_3 - p_3q_2) + \alpha_2 (p_3q_1 - p_1q_3))k. \]

**Definition 4** For \( u = (u_1, u_2, u_3) \), \( v = (v_1, v_2, v_3) \), the vector product in quadratic 3-space is

\[ u \times v = \begin{vmatrix} \lambda_1i - \beta_1j + \alpha_1k & -\beta_1i - \beta_2j + \alpha_2k & \alpha_1i + \alpha_2j + \alpha_3k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \]

Therefore, the quadratic product of \( q = S_q + v_q \) and \( p = S_p + v_p \) can be written as

\[ qp = S_qS_p + S_qv_p + S_pv_q + \Delta \langle v_q, v_p \rangle + v_q \times v_p. \] (11)
Actually,
\[
q\mathbf{p} = \begin{bmatrix}
q_0 & Aq_1 + Dq_2 + E_{q_3} & Bq_2 + Dq_1 + F_{q_3} & C_{q_3} + E_{q_1} + F_{q_2} \\
q_1 & q_0 - q_2\alpha_1 - q_3\beta_1 & q_1\alpha_1 - q_3\lambda_1 & \lambda_1q_2 + q_1\beta_1 \\
q_2 & -q_2\alpha_2 - \beta_2q_3 & q_0 + q_1\alpha_2 + q_3\beta_1 & \beta_2q_1 - \beta_1q_2 \\
q_3 & -\alpha_3q_1 - q_3\alpha_1 & q_0 - q_1\alpha_2 + q_2\alpha_1 & p_3
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p_2 \\
p_3
\end{bmatrix}
\]
and also,
\[
(u \times v) \times w = \Delta(v, w) u - \Delta(u, w) v.
\]

This scalar and vector products on \(\mathbb{R}^3\) allows us to define a new metric space. We will call it the quadratic metric.

3 Geometry of Non-Degenerate Quadratic Metrics

This section will focus on two types of quadratic spaces derived from quadratic metrics. These are the ellipsoid quadratic space and the hyperboloid quadratic space.

4 Ellipsoid Quadratic Spaces

Definition 5 For a quadratic number \(X = q + xi + yj + zk\), the vector,
\[
v_X = (x, y, z) \in \mathbb{R}^3
\]
is called the quadratic vector of \(X\) and a pure quadratic number \(xi + yj + zk\) will be taken as a vector of quadratic 3-space.

Definition 6 The unit ellipsoid sphere is the set of all ellipsoid vector of \(\mathbb{R}^3\) is
\[
\mathcal{E} = \{(x, y, z) \in \mathbb{R}^3 : Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz = -1\} \cup \{(0, 0, 0)\}.
\]

Definition 7 If the signature of the matrix \((10)\) associated with the bilinear form \((\cdot, \cdot)\) is \((3, 0, 0)\), then \((\cdot, \cdot)\) is a positive definite bilinear form. Therefore, the quadratic form corresponding to the bilinear form is ellipsoid. So, the quadratic number \(X = q + xi + yj + zk\) is called the ellipsoid number, and \(v_X = (x, y, z)\) is called the ellipsoid vector. We will denote the set of ellipsoid quadratic number by \(\mathcal{E} \Omega\).
A scalar product on a real vector space $V$ is a non-degenerate symmetric bilinear form. A positively definite scalar product is a Euclidean scalar product and its sign is $(3,0)$. The space equipped with a Euclidean scalar product is called a Euclidean vector space $\mathbb{R}^3$.

An Euclidean scalar product is a scalar product that is positive definite, and its one with signature $(3,0)$. A vector space equipped with a Euclidean scalar product is called a Euclidean vector space.

**Definition 8** The ellipsoid quadratic space is the vector space $\mathbb{R}^3$ equipped with the positive definite symmetric bilinear form

$$\langle u, v \rangle_{\mathbb{O}} = \Delta \left( A u_1 v_1 + B u_2 v_2 + C u_3 v_3 + D (u_1 v_2 + u_2 v_1) + E (u_1 v_3 + u_3 v_1) + F (u_2 v_3 + u_3 v_2) \right)$$

for the vectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$. The matrix associated with the symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{O}}$ is $-M_{\mathbb{O}}$. Throughout the article, a pure quadratic number will be taken as a vector in the quadratic 3-space. If $\langle u, v \rangle_{\mathbb{O}} = 0$, we call them $u$ and $v$ are orthogonal to each other. Also, norm of the vector $u$ is

$$\|u\|_{\mathbb{O}} = \sqrt{\langle u, u \rangle_{\mathbb{O}}}.$$

It is also very obvious that the ellipsoid quadratic number forms a group by multiplication. We can denote this group with $\mathbb{E} \mathbb{O} = \{ q \in \mathbb{O} : C(q) > 0 \}$.

**Proposition 9** The polar form of the ellipsoid quadratic number $q = q + x i + y j + z k$ is

$$q = \|q\| (\cos \theta + v \sin \theta)$$

where $v$ is an ellipsoid vector, $v = \frac{v_1}{\|v_1\|}$, $v^2 = -1$, and $\theta$ is the argument of $q$ defined as

$$\theta = \begin{cases} \pi - \arctan \left( \frac{\|v_1\|}{|q|} \right), & q < 0; \\ \arctan \left( \frac{\|v_1\|}{|q|} \right), & q > 0; \end{cases}.$$
The real numbers
\[ C(q) = qr = qr = q^2 - Ax^2 - By^2 - Cz^2 - 2Dxy - 2Exz - 2Fyz \]

and
\[ V(q) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz > 0 \]

are called the character of the indefinite number \( q = x + yi + zj + 2k \) and the type of the indefinite vector \( xi + yj + zk \), respectively. Accordingly, we can give the following definitions.

**Definition 11** If the matrix \( [17] \) associated with the bilinear form \( \langle u, v \rangle_{Q} \) is indefinite form, then the quadratic number \( X \) is called indefinite number. So, the quadratic number \( X = q + xi + yj + zk \) is called the indefinite number, and \( v_X = (x, y, z) \) is called the indefinite vector. We will denote the set of indefinite quadratic number by \( Q \). A indefinite number is called spacelike, lightlike or time-like, depending on whether \( C(X) < 0, C(X) = 0 \) or \( C(X) > 0 \), respectively. The indefinite vector is called timelike (2-sheeted hyperbolic vector), lightlike (cone vector) or spacelike (1-sheeted hyperbolic vector), if \( V_{v}(X) < 0, V_{v}(X) = 0, V_{v}(X) > 0 \), respectively. We will denote the set of indefinite quadratic number by \( Q \).

A indefinite scaler product with the signature \( (2, 0, 1) \), with 1 negative index is a Lorentz scalar product. A vector space equipped with a Lorentz scalar product is called a Lorentz vector space [19].

**Definition 12** The quadratic 3-space is the vector space \( \mathbb{R}^3 \) equipped with the indefinite symmetric bilinear form
\[ \langle u, v \rangle_{2\Omega} = A_{u}v_1 + B_{u}v_2 + C_{u}v_3 + D(u_1v_2 + u_2v_1) + E(u_1v_3 + u_3v_1) + F(u_2v_3 + u_3v_2) \]

for the vectors \( u = (u_1, u_2, u_3) \), \( v = (v_1, v_2, v_3) \). The matrix associated with the symmetric bilinear form \( \langle \cdot, \cdot \rangle_{2\Omega} \) is \( M_{\Omega} \). If \( \langle u, v \rangle_{2\Omega} = 0 \), we call them \( u \) and \( v \) are pseudo-orthogonal. Also, norm of the vector \( u \) is
\[ ||u||_{2\Omega} = \sqrt{\langle u, u \rangle_{2\Omega}}. \]

The indefinite vector of a indefinite quadratic number can be timelike, spacelike or lightlike. Also, the indefinite vector of a spacelike quadratic number will definitely be spacelike. We can easily see that
\[ C_v(q) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz > 0 \]

from the inequality
\[ C(q) < 0 \Rightarrow q^2 - Ax^2 - By^2 - Cz^2 - 2xyD - 2xzE - 2Fyz < 0 \]
\[ \Rightarrow Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz > q^2 > 0. \]
It means that the indefinite vector type of a spacelike quadratic number is certain- 
ly spacelike. Similarly, the indefinite vector of a lightlike quadratic number is de-

definitely a spacelike vector if the scalar part of lightlike quadratic number is non-zero, 
and lightlike vector if scalar part is zero. So, the type of a lightlike indefinite quadratic number is either spacelike or lightlike. Therefore, we can give the following table.

| q    | Spacelike | Lightlike | Timelike |
|------|-----------|-----------|----------|
| v_q  | Spacelike | Spacelike | Spacelike |
|      | Lightlike | Lightlike | Timelike  |

Also, using the product of the quadratic number, we can obtain the following equation:
\[ C(pq) = C(p)C(q). \]
Therefore, the timelike indefinite quadratic number forms a group by multiplication. We can denote this group with
\[ \mathfrak{T}Q = \{ q \in \mathfrak{H}Q : C(q) > 0 \}. \]

The following table can be created according to the timelike indefinite quadratic number multiplication operation.

| ·    | Spacelike | Timelike | Lightlike |
|------|-----------|----------|-----------|
| Spacelike | Timelike | Spacelike | Lightlike |
| Timelike | Spacelike | Timelike | Lightlike |
| Lightlike | Lightlike | Lightlike | Lightlike |

**Proposition 13** The polar form of the spacelike quadratic number \( q = qx + yj + zk \) is
\[
q = \|q\| (\sinh \theta + v \cosh \theta)
\]
where \( v \) is a spacelike vector, \( v = \frac{v_q}{\|v_q\|}, v^2 = 1. \)

**Proposition 14** The polar form of the timelike quadratic number \( q = qx + yj + zk \) is
\[
q = \|q\| (\epsilon \cosh \theta + v \sinh \theta), \quad \text{if } v \text{ is spacelike}
q = \|q\| (\cos \theta + v \sin \theta), \quad \text{if } v \text{ is timelike,}
q = |q| (\epsilon + v), \quad \text{if } v \text{ is lightlike,}
\]
where \( v = \frac{v_q}{\|v_q\|} \) and \( \epsilon = \text{sign}q. \)

**Proposition 15** The polar form of the lightlike quadratic number \( q = qx + yj + zk \) is
\[
q = |q| (\epsilon + v), \quad \text{if } v \text{ is spacelike,}
\]
where \( v = \frac{v_q}{|q|} \), \( \epsilon = \text{sign}q \) and
\[
q = v_q, \quad \text{if } v \text{ is lightlike.} \]
Proposition 16 The argument of a non-lightlike quadratic number \( q = q + xi + yj + zk \) is defined as

\[
\theta = \begin{cases} 
\pi - \arctan \frac{\|v_q\|}{|q|}, & q_1 < 0 \\
\arctan \frac{\|v_q\|}{|q|}, & q_1 > 0
\end{cases}
\]

and

\[\theta = \ln \left( \frac{|q| + \|v_q\|}{\|q\|} \right), \quad \text{and} \quad \theta = 1\]

for timelike, spacelike and lightlike quadratic numbers, respectively.

6 Rotation with Quadratic Numbers

Before moving on to the geometric properties of quadratic form multiplication, let’s introduce the left multiplication and right multiplication matrices. Let the left multiplication and right multiplication matrices be \( L \) and \( R \), respectively.

For \( q = q + xi + yj + zk \), these matrices are as follows:

\[
L(q) = \begin{bmatrix} 
q & yD + zE + Ax & xD + By + Fz & xE + Cz + Fy \\
x & q - y\alpha_1 - z\beta_1 & x\alpha_1 - z\lambda_1 & x\beta_1 + y\lambda_1 \\
y & -y\alpha_2 - z\beta_2 & q + x\alpha_2 + z\beta_1 & x\beta_2 - y\beta_1 \\
z & z\alpha_2 - y\alpha_3 & x\alpha_3 - z\alpha_1 & q - x\alpha_2 + y\alpha_1 \\
\end{bmatrix}, \quad (12)
\]

\[
R(q) = \begin{bmatrix} 
q & yD + zE + Ax & xD + By + Fz & xE + Cz + Fy \\
x & q + y\alpha_1 + z\beta_2 & z\lambda_1 - x\alpha_1 & -x\beta_1 - y\lambda_1 \\
y & y\alpha_2 + z\beta_2 & q - x\alpha_2 - z\beta_1 & y\beta_1 - x\beta_2 \\
z & y\alpha_3 - z\alpha_2 & z\alpha_3 - x\alpha_1 & q + x\alpha_2 - y\alpha_1 \\
\end{bmatrix}, \quad (13)
\]

and their eigenvectors are

\[q + \sqrt{Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz},\]

\[q - \sqrt{Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz}.
\]

At the same time, the skew symmetric (or semi skew symmetric) matrix corresponding to the unit pure quadratic number \( v_q = (x, y, z) \) is

\[
\mathbf{\mathcal{S}} = \begin{bmatrix} 
-\alpha_1 y - \beta_1 z & \alpha_1 x - \lambda_1 z & \beta_1 x + \lambda_1 y \\
-\beta_2 z - \alpha_2 y & \alpha_2 x + \beta_1 z & \beta_2 x - \beta_1 y \\
\alpha_2 z - \alpha_3 y & \alpha_3 x - \alpha_1 z & \alpha_1 y - \alpha_2 x
\end{bmatrix}.
\]

This matrix will be used to obtain rotation matrices.

Theorem 17 \( S_q(p) = qp\overline{q} \) is a rotation transformation where \( q = q + xi + yj + zk \) is the unit ellipsoid (or unit timelike indefinite quadratic number) and \( p \) is ellipsoid (or unit timelike indefinite quadratic number).
Proof. Using (12) and (13), the following equation can be written:

\[ S_q(p) = qpq = L(q)R(q)p. \]

We have the matrix \( R^\theta_v = L(q)R(q) \) that is

\[
R^\theta_v = \begin{bmatrix}
1 & 0 \\
0 & M^\theta_v
\end{bmatrix}
\]

where

\[
M^\theta_v = \begin{bmatrix}
-Aq_1^2 + Bq_2^2 + Cq_3^2 & -2Dq_1^2 - 2Bq_1q_2 & -2Eq_1^2 - 2Cq_1q_3 \\
+2Fq_2q_3 + q_5^2 & -2Fq_1q_3 + 2a_1q_0q_1 & -2Fq_1q_2 + 2b_1q_0q_1 \\
-2a_1q_0q_2 - 2b_1q_0q_3 & -2l_1q_0q_3 & +2l_1q_0q_2 \\
-2Dq_1q_3 - 2a_2q_0q_2 & +2E_lq_1q_3 + q_5^2 & -2E_q_1q_2 - 2b_1q_0q_2 \\
-2l_2q_0q_3 & -2Fq_3^2 - 2Dq_3q_3 & q_0^2 A_q_1^2 + B_q_2^2 - C_q_3^2 \\
-2Dq_2q_3 + 2a_2q_0q_3 & -2Dq_3q_3 + 2a_3q_0q_1 & +2Dq_3q_2 + 2a_1q_0q_2 \\
-2a_3q_0q_2 & -2a_3q_0q_3 & -2a_2q_0q_2
\end{bmatrix}.
\]

For any unit ellipsoid number or unit non lightlike number \( q \), one of the eigenvectors of the matrix (14) is \( x_i + y_j + z_k \) and the corresponding eigenvalue is 1. At the same time, \( \det R = 1 \).

6.1 Rodrigues Rotation Formula

The exponential map is defined by the matrix exponential series \( e^\mathcal{G} \). For any skew-symmetric matrix \( \mathcal{G} \), the matrix exponential \( e^\mathcal{G} \) always gives a rotation matrix. This method is known as the Rodrigues formula. The Rodrigues rotation formula is a benefit procedure for generating a rotation matrix [3, 10], [16]. We can use it to obtain a rotation matrix in the quadratic 3-space. For this, the skew-symmetric matrix with rotation axis \( v = x_i + y_j + z_k \) in the quadratic 3-space is

\[
\mathcal{G} = \begin{bmatrix}
-\alpha_1 y - \beta_1 z & \alpha_1 x - \lambda_1 z & \beta_1 x + \lambda_1 y \\
-\beta_2 z - \alpha_2 y & \alpha_2 x + \beta_2 z & \beta_2 x - \beta_2 y \\
\alpha_2 z - \alpha_3 y & \alpha_3 x - \alpha_1 z & \alpha_1 y - \alpha_2 x
\end{bmatrix}.
\]

Theorem 18 Let \( \mathcal{G} \) be a skew symmetric matrix in the form (15) and \( v = x_i + y_j + z_k \) is a quadratic vector and on the quadric surface \( Q \). Then the matrix exponential

\[
\mathcal{R}^\theta_v = e^{\theta \mathcal{G}} = I_3 + (\sin \theta) \mathcal{G} + (1 - \cos \theta) \mathcal{G}^2, \theta \in [0, 2\pi) \quad \text{if} \ v \ \text{is a unit ellipsoid or timelike}
\]

\[
\mathcal{R}^\theta_v = e^{\theta \mathcal{G}} = I_3 + (\sinh \theta) \mathcal{G} - (1 - \cosh \theta) \mathcal{G}^2, \quad \text{if} \ v \ \text{is a unit spacelike}
\]

\[
\mathcal{R}^\theta_v = e^{-\theta \mathcal{G}} = I_3 - \theta \mathcal{G} + \frac{\theta^2 \mathcal{G}^2}{2!}, \quad \text{if} \ v \ \text{is a lightlike vector}
\]
gives a rotation on the quadric surface $\Omega$ where $v = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is a rotation axis, and $I_3$ is the identity matrix. These matrices are as follows, respectively.

\[
\begin{bmatrix}
(1 - \cos \theta) \sigma_{11} & (\cos \theta - 1) \sigma_{12} & (\cos \theta - 1) \sigma_{13} \\
-(\sin \theta) (y\alpha_1 + z\beta_1) + 1 & x\alpha_1 \sin \theta - z\lambda_1 \sin \theta & +x\beta_1 \sin \theta + y\lambda_1 \sin \theta \\
\sigma_{21} (\cos \theta - 1) & -x\alpha_2 \sin \theta - z\beta_1 \sin \theta + 1 & +x\beta_2 \sin \theta - y\beta_1 \sin \theta \\
-(\sin \theta) (y\alpha_2 + z\beta_2) & \cos \theta - 1) \sigma_{31} & (\cos \theta - 1) \sigma_{32} \\
+(z\alpha_2 - y\alpha_3) \sin \theta & + (x\alpha_3 - z\alpha_1) \sin \theta & + (y\alpha_1 - x\alpha_2) \sin \theta + 1 \\
\end{bmatrix}
\]

(16)

\[
\begin{bmatrix}
\sigma_{11} (\cosh \theta - 1) & \sigma_{12} (1 - \cosh \theta) & \sigma_{13} (1 - \cosh \theta) \\
-(\sinh \theta) (y\alpha_1 + z\beta_1) + 1 & x\alpha_1 \sinh \theta - z\lambda_1 \sinh \theta & +x\beta_1 \sinh \theta + y\lambda_1 \sinh \theta \\
\sigma_{21} (1 - \cosh \theta) & -x\alpha_2 \sinh \theta - z\beta_1 \sinh \theta + 1 & +x\beta_2 \sinh \theta - y\beta_1 \sinh \theta \\
-(y\alpha_2 + z\beta_2) \sinh \theta & \sigma_{31} (1 - \cosh \theta) & (\cos \theta - 1) \sigma_{32} \\
+(z\alpha_2 - y\alpha_3) \sinh \theta & + (x\alpha_3 - z\alpha_1) \sinh \theta & + (y\alpha_1 - x\alpha_2) \sinh \theta + 1 \\
\end{bmatrix}
\]

(17)

\[
\begin{bmatrix}
\frac{1}{2} \theta^2 \sigma_{11} & -\frac{1}{2} \theta^2 \sigma_{12} & -\frac{1}{2} \theta^2 \sigma_{13} \\
(y\alpha_1 + z\beta_1) \theta + 1 & + (z\lambda_1 - x\alpha_1) \theta & -y\theta \beta_1 - x\theta \beta_1 \\
\frac{1}{2} \theta^2 \sigma_{21} & \frac{1}{2} \theta^2 \sigma_{22} & -\frac{1}{2} \theta^2 \sigma_{23} \\
+y (y\alpha_2 + z\beta_2) & - (x\alpha_2 + z\beta_1) \theta + 1 & +y\theta \beta_1 - x\theta \beta_2 \\
-\frac{1}{2} \theta^2 \sigma_{31} & \frac{1}{2} \theta^2 \sigma_{32} & \frac{1}{2} \theta^2 \sigma_{33} \\
+(z\alpha_3 - y\alpha_2) \theta & + (x\alpha_1 - x\alpha_2) \theta & + x\theta \alpha_2 - y\theta \alpha_1 + 1 \\
\end{bmatrix}
\]

(18)

where the coefficients $\sigma_{ij}$ is following in the table:

| $\sigma_{11}$ | $(By^2 + Cz^2 + Dx y + Ez x + 2Fyz)$ | $\sigma_{23}$ | $(Fy^2 + Czy + Ezx)$ | $\sigma_{23} = (Fy^2 + Czy + Ezx)$ |
| $\sigma_{12}$ | $(Bxy + Cxz)$ | $\sigma_{31}$ | $(Ey^2 + Fzy + Cxz)$ |
| $\sigma_{13}$ | $(Ex^2 + Fxy + Czx)$ | $\sigma_{32}$ | $(Fz^2 + Dxz + Byz)$ |
| $\sigma_{21}$ | $(Dy^2 + Eyz + Ayx)$ | $\sigma_{33}$ | $(Ax^2 + By^2 + 2xyD + zwE + Fyz)$ |
| $\sigma_{22}$ | $(Ax^2 + Cz^2 + xyD + ZwE + Fyz)$ | $\sigma_{33} = (Ax^2 + By^2 + 2xyD + 2zwE + Fyz)$ |

**Proof.** Let’s first find the characteristic polynomial of the matrix $\mathcal{S}$ required for the Rodrigues formula and the powers of the matrix $\mathcal{S}$. The characteristic polynomial of $\mathcal{S}$ is

$$X^3 - C_v(v)X = 0.$$ 

According to Cayley-Hamilton theorem, we have

$$\mathcal{S}^3 = C_v(v)\mathcal{S}.$$ 

The relation between the powers of $\mathcal{S}$ are

$$\mathcal{S}^4 = C_v(v)\mathcal{S}^2, \mathcal{S}^5 = C_v(v)\mathcal{S}^3, \mathcal{S}^6 = (C_v(v))^2\mathcal{S}^2, \mathcal{S}^7 = (C_v(v))^3\mathcal{S}...$$

and so we have

\[
\begin{align*}
\mathcal{S}^{2n} &= (C_v(v))^{n-1}\mathcal{S}^2 & \mathcal{S}^{2n-1} &= (C_v(v))^{n-1}\mathcal{S} & \text{if } v \text{ is a unit ellipsoid or timelike} \\
\mathcal{S}^{2n} &= \mathcal{S}^2 & \mathcal{S}^{2n-1} &= \mathcal{S} & \text{if } v \text{ is a unit spacelike} \\
\mathcal{S}^{2n} &= 0 & \mathcal{S}^{2n-1} &= 0 \text{ for } n > 2 & \text{if } v \text{ is a lightlike}
\end{align*}
\]
For unit ellipsoid or timelike vector $\mathbf{v}$, the rotation matrix is

\[
\mathcal{R}_\theta^0 = e^{\theta \mathbf{S}} = I_3 + \theta \mathbf{S} + \frac{\theta^2 \mathbf{S}^2}{2!} + \frac{\theta^3 \mathbf{S}^3}{3!} + \frac{\theta^4 \mathbf{S}^4}{4!} + \ldots
\]

\[
= I_3 + \theta \mathbf{S} + \frac{\theta^2 \mathbf{S}^2}{2!} - \frac{\theta^3 \mathbf{S}^3}{3!} - \frac{\theta^4 \mathbf{S}^4}{4!} + \ldots
\]

\[
= I_3 + \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \right) \mathbf{S} + \left( \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \ldots \right) \mathbf{S}^2
\]

\[
\mathcal{R}_\theta^0 = I_3 + (\sin \theta) \mathbf{S} + (1 - \cos \theta) \mathbf{S}^2.
\]

For unit spacelike vector $\mathbf{v}$, the rotation matrix is

\[
\mathcal{R}_\theta^0 = e^{\theta \mathbf{S}} = I_3 + \theta \mathbf{S} + \frac{\theta^2 \mathbf{S}^2}{2!} + \frac{\theta^3 \mathbf{S}^3}{3!} + \frac{\theta^4 \mathbf{S}^4}{4!} + \ldots
\]

\[
= I_3 + \theta \mathbf{S} + \frac{\theta^2 \mathbf{S}^2}{2!} - \frac{\theta^3 \mathbf{S}^3}{3!} - \frac{\theta^4 \mathbf{S}^4}{4!} + \ldots
\]

\[
= I_3 + \left( \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots \right) \mathbf{S} + \left( \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots \right) \mathbf{S}^2
\]

\[
\mathcal{R}_\theta^0 = I_3 + (\sinh \theta) \mathbf{S} - (1 - \cosh \theta) \mathbf{S}^2.
\]

For unit lightlike vector $\mathbf{v}$, the rotation matrix is

\[
\mathcal{R}_\theta^0 = e^{-\theta \mathbf{S}} = I_3 - \theta \mathbf{S} + \frac{\theta^2 \mathbf{S}^2}{2!}.
\]

Extending these equations gives the quadratic rotation matrices (10), (17), (18).

6.2 Cayley Representation for Quadratic numbers

For the skew-symmetric matrix $\mathbf{S}$ of the form (15), $(I_3 + \mathbf{S})$ have an inverse, and its Cayley map is

\[
\mathcal{R}_\mathbf{v} (\mathbf{S}) = (I_3 - \mathbf{S}) (I_3 + \mathbf{S})^{-1} = (I_3 + \mathbf{S})^{-1} (I_3 - \mathbf{S})
\]

where the unit vector $\mathbf{v}$ forming the entries of $\mathbf{S}$ is not a unit spacelike quadratic vector [3], [10], [16].

**Theorem 19** Let $\mathbf{S}$ is a skew-symmetric matrix with $\mathbf{v} = xi + yj + zk$ in the form (15). If $\mathbf{v}$ is not a unit spacelike vector, then

\[
\mathcal{R}_\mathbf{v} (\mathbf{S}) = (I_3 - \mathbf{S}) (I_3 + \mathbf{S})^{-1}
\]

is a quadratic rotation matrix where $\mathbf{v}$ is the rotation matrix. So, we get the following rotation matrix:

\[
\mathcal{R}_\mathbf{v} = \rho \mathcal{M}_\mathbf{v}
\]
where \( \rho = \frac{1}{\Delta(v, v)_{\Omega} - 1} \) and

\[
\mathcal{M} = \begin{pmatrix}
Ax^2 - By^2 - Cz^2 \\
-2Fyz - 2z\beta_1 \\
-2y\alpha_1 - 1 \\
2(\begin{array}{c}
Axy + Dy^2 + Eyz \\
-2\beta_2 - y\alpha_2 \\
2(\begin{array}{c}
Axz + Ez^2 + Dyz \\
+ \alpha_2 - y\alpha_3
\end{array})
\end{array})
\end{pmatrix}.
\]

**Proof.** We have

\[
(I + \mathcal{E})^T \mathcal{M}_\Omega = \mathcal{M}_\Omega (I - \mathcal{E}) \quad \text{and} \quad (I - \mathcal{E})^T \mathcal{M}_\Omega = \mathcal{M}_\Omega (I + \mathcal{E}).
\]

Using these equalities, it can be found that

\[
(R_v)^T \mathcal{M}_\Omega (R_v) = \left((I + \mathcal{E})(I - \mathcal{E})^{-1}\right)^T \mathcal{M}_\Omega (I + \mathcal{E})(I - \mathcal{E})^{-1} = \mathcal{M}_\Omega.
\]

Also, we have \( \det R_v = 1 \) because of \( \det (I + \mathcal{M}_\Omega) = 1 - \langle v, v \rangle_\Omega \) and \( \det (I - \mathcal{M}_\Omega)^{-1} = \frac{1}{1 - \langle v, v \rangle_\Omega} \). That is, \( R_v \) is a quadratic rotation matrix. Also, the eigenvalue matrix \( R_v \) are

\[
\mathcal{M}_\Omega = \begin{pmatrix}
\alpha_1, \alpha_2 \\
\alpha_3, \beta_1, \beta_2
\end{pmatrix}
\]

And the eigenvector corresponding to 1 is \( v \). The rotation angle \( \theta \) is given by

\[
\tan \theta = \frac{2\|v\|_\Omega}{1 - \langle v, v \rangle_\Omega} \quad \text{if } v \text{ is a unit ellipsoid or timelike},
\]

\[
\cosh \theta = \frac{1 + \langle v, v \rangle_\Omega}{1 - \langle v, v \rangle_\Omega} \quad \text{if } v \text{ is a unit spacelike},
\]

\[
\theta = 1 \quad \text{if } v \text{ is a lightlike}.
\]

\[
\mathcal{M}_{\Omega} = \begin{pmatrix}
M_{13} & M_{23} & M_{33} & M_{12} & M_{22} & M_{32} & M_{11}
\end{pmatrix}
\]

\[
\Delta \Gamma = |\mathcal{M}_\Omega|.
\]

**Remark 20** When constructing the number system of any quadratic form \( \mathcal{E} \), the following equations between the coefficients of the quadratic form and the coefficients \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \) and \( \lambda_1 \) of the number system can be used.

| Number | \( \alpha_1 \) | \( \alpha_2 \) | \( \alpha_3 \) | \( \beta_1 \) | \( \beta_2 \) | \( \lambda_1 \) |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| Value  | \( -\frac{M_{13}}{\Delta \Gamma} \) | \( \frac{M_{23}}{\Delta \Gamma} \) | \( -\frac{M_{33}}{\Delta \Gamma} \) | \( -\frac{M_{12}}{\Delta \Gamma} \) | \( -\frac{M_{22}}{\Delta \Gamma} \) | \( -\frac{M_{32}}{\Delta \Gamma} \) |

where \( M_{ij} \) is minors of the matrix \( \Delta \mathcal{M}_\Omega \) and \( |\Gamma| = \sqrt{\det \Delta \mathcal{M}_\Omega} \). Note that if some of the quadratic coefficients are zero, these numbers \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \) and \( \lambda_1 \) can be obtained differently.
Now let’s give examples of how the rotation transformation is realized using positive definite and indefinite quadratic numbers.

**Example 21** Let’s create a positive definite bilinear form with the following ellipsoid

\[
6x^2 + 6xy + 4xz + 2y^2 + 4yz + 3z^2 = 1
\]

and exemplify some of the geometric results obtained throughout the article. The coefficients of the bilinear form and quadratic number system corresponding to this ellipsoid are as follows:

\[
\langle \mathbf{u}, \mathbf{v} \rangle_{\Omega} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 6 & 3 & 2 \\ 3 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

(20)

| Number | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\beta_1$ | $\beta_2$ | $\lambda_1$ |
|--------|------------|------------|------------|------------|------------|------------|
| Value  | 2          | −6         | 3          | 5          | −14        | 2          |

Let’s take the vector $\mathbf{v} = (0, 0, 1/\sqrt{3})$ and the point $A = (0, 1/\sqrt{2}, 0)$ over this ellipsoid. Let us take the plane \[ \frac{2}{3}x + \frac{2}{3}y + z - \frac{1}{3}\sqrt{2} = 0, \]

which is orthogonal to the vector $\mathbf{v}$ and passes through the point $A$. The conic formed by the intersection of this quadric surface (20) and the given plane and its center are as follows:

\[
\frac{14}{3}x^2 + \frac{40}{3}xy + \frac{8}{3}y^2 + \frac{2}{3}z^2 = 1 , \quad C = (0, 0, \frac{1}{3}\sqrt{2})
\]

For any $\mathbf{u}, \mathbf{v}$, the vector product on this quadratic form is as follows:

\[
\begin{vmatrix}
2i - 5j + 2k & -5i + 14j - 6k & 2i - 6j + 3k \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3 \\
\end{vmatrix}
\]

Let’s take 3 points $A (0, 1/\sqrt{2}, 0), C, B = (1/\sqrt{2}, 0, 0)$ on the given plane and show that the normal of the plane is $\mathbf{v}$ with the help of the vector product.

\[
CA \times CB = \begin{vmatrix}
2i - 5j + 2k & -5i + 14j - 6k & 2i - 6j + 3k \\
0 & 1/\sqrt{2} & -\frac{1}{3}\sqrt{2} \\
1/\sqrt{2} & 0 & -\frac{1}{3}\sqrt{2} \\
\end{vmatrix} = (0, 0, -1/6)
\]

So, we get

\[
\mathbf{v} = (OA \times OB) / \|OA \times OB\|.
\]
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