MULTI-SOLITON DYNAMICS IN THE SKYRME MODEL

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Abstract

We exhibit the dynamical scattering of multi-solitons in the Skyrme model for configurations with charge two, three and four. First, we construct maximally attractive configurations from a simple profile function and the product ansatz. Then using a sophisticated numerical algorithm, initially well-separated skyrmions in approximately symmetric configurations are shown to scatter through the known minimum energy configurations. These scattering events illustrate a number of similarities to BPS monopole configurations of the same charge. A simple modification of the dynamics to a dissipative regime, allows us to compute the minimal energy skyrmions for baryon numbers one to four to within a few percent.
1 Introduction

The Skyrme model was first proposed to describe the strong interactions of hadrons in 1962\cite{9}, well before the advent of QCD. In recent years interest in the model has been rekindled with the realization that it is in fact a low-energy effective theory for QCD in the large $N_c$ limit\cite{24}. Within the model, topological solitons, known as skyrmions, are identified with nucleons and the topological charge with baryon number, $B$. Hence, the dynamics of multi-soliton configurations are of considerable physical interest since they correspond almost directly to the low-energy excitations of nucleons within atoms. This approach has many advantages over the full theory of QCD, the most important of all being that it is computational much more accessible to current computer technology than the full theory. It is also of considerable mathematical interest due to the inherent nonlinear dynamics and the significant numerical problems which have to be overcome before realistic comparisons with the properties of nuclear matter can be made.

The dynamics of two skyrmions has been studied in previous work using analytic methods\cite{2,16} and also numerical simulations on very small grids\cite{2}. It is well understood that two initially well-separated skyrmions will scatter at right angles, through the axially symmetric, toroidal minimum-energy configuration\cite{22}. This seems to be a geometric property of generic two-soliton configurations, for example, BPS monopoles\cite{3} and vortices\cite{17,18}. Here, we investigate the dynamics of higher charge configurations up to $B=4$ using a sophisticated numerical algorithm. We shall see that symmetric configurations of higher charge seem to follow a picture similar to that of $B=2$, scattering — almost elastically — through the already known minimum energy configurations\cite{7,13}, which have tetrahedral and cubic symmetries for $B=3$ and $B=4$ respectively. Since the collisions are almost elastic, very little energy is lost to radiation. Hence, we introduce a phenomenological dissipation term which allows us to get to the global minimum energy configuration without the need for excessive amounts of computer time. The numerical methods and higher charge configurations with $B > 4$ will be discussed in a more detailed publication on this subject\cite{6}.

The Lagrangian of the Skyrme model may be written in terms of the SU(2) valued right currents $R_\mu = (\partial_\mu U) U^\dagger$ as

$$12\pi^2 \mathcal{L} = -\frac{1}{2} \text{Tr}(R_\mu R^\mu) - \frac{1}{16} \text{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]).$$

(1.1)

where we have used scaled units of energy and length, and a $(-\cdots)$ signature for the spacetime metric. For the purposes of this letter, we have also ignored the pion mass term, since the resulting chiral sigma model represents a more geometrical model with a connection to instantons.

The baryon density $\mathcal{B}$, whose spatial integral gives the integer-valued baryon number $B$, is given by

$$24\pi^2 \mathcal{B} = -\epsilon_{ijk} \text{Tr}(R_i R_j R_k),$$

(1.2)

where, as throughout this letter, we take latin indices to run over the spatial values 1, 2, 3. The above units are chosen so that the Fadeev-Bogomolny bound on the energy $E$ is simply $E \geq |B|$.
In order to convert to the sigma model notation, we combine the sigma field and triplet of pion fields by writing \( U = \sigma + i \pi \cdot \tau \) where \( \tau \) denote the Pauli matrices, and the normalization constraint which must be imposed is \( \sigma^2 + \pi \cdot \pi = 1 \). Collecting these fields together in a four component unit vector \( \phi = (\sigma, \pi_1, \pi_2, \pi_3) \), the Lagrangian (1.1) becomes

\[
12\pi^2 \mathcal{L} = \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} (\partial_\mu \phi \cdot \partial_\nu \phi) (\partial^\mu \phi \cdot \partial^\nu \phi) + \lambda (\phi \cdot \phi - 1), \tag{1.3}
\]

where \( \lambda \) is the Lagrange multiplier which imposes the chiral constraint \( \phi \cdot \phi = 1 \). Using the Euler-Lagrange equations, the field equations are

\[
(1 - \partial_\mu \phi \cdot \partial_\mu \phi) \Box \phi - (\partial^\rho \phi \cdot \partial_\rho \partial_\rho \phi - \partial_\rho \phi \cdot \Box \phi) \partial^\rho \phi + (\partial^\mu \phi \cdot \partial^\nu \phi) \partial_\mu \partial_\nu \phi + [(\partial_\mu \phi \cdot \partial_\nu \phi)(\partial^\mu \phi \cdot \partial^\nu \phi) + (1 - \partial_\mu \phi \cdot \partial_\mu \phi) \partial_\rho \phi \cdot \partial_\rho \phi] \phi = 0, \tag{1.4}
\]

where \( \Box \) denotes the wave operator in \((3 + 1)\)-dimensions.

This equation has a number of numerical problems associated with it\[8\]. The first is a standard instability of the model itself. When the kinetic energy is locally greater than the potential energy the assumptions under which the model is deduced from QCD are invalidated and the equation becomes elliptic, rather than hyperbolic. Therefore any numerical method designed to solve the equations in the hyperbolic regime will become unstable in the elliptic regime and vice-versa. Since this is an instability of the model, rather than one of any numerical scheme which may be employed, we will be restricted to studying configurations which do not have large amounts of kinetic energy. In particular, we will not be able to investigate those Lorentz boosted to ultra-relativistic velocities and those which emit large amounts of radiation.

The other more numerical problems are related to the overall Courant stability and also the imposition of the chiral constraint. The precise resolution of these problems will be discussed in later work, but it suffices to say that it is possible to construct an algorithm which is stable for many time steps (> 1000) with values of the spatial step size \( \Delta x \) and the time step \( \Delta t \) within half an order of magnitude of the standard Courant criterion \( \Delta x = \sqrt{3} \Delta t \). The simulations presented in this letter used fourth order spatial differences on grids which range in size from \( 70^3 \) points to \( 100^3 \) points, depending upon the particular simulation, with \( \Delta x = 0.1 \) and the range \( 0.01 \leq \Delta t \leq 0.02 \).

The field of a static single skyrmion centred at the origin has the hedgehog form,

\[
U(\mathbf{x}) = \exp[i f(r) \hat{\mathbf{x}} \cdot \tau], \tag{1.5}
\]

where \( r = |\mathbf{x}| \) and the profile function \( f(r) \) has boundary conditions \( f(0) = \pi \) and \( f(\infty) = 0 \). Solving numerically for this profile function\[1\] determines the energy of a single skyrmion to be \( E = 1.232 \). The centre of the skyrmion can be placed at any point in space by a translation, and its isospin orientation can be changed by conjugation of the \( U \) field by any fixed element of \( SU(2) \). It can also be made to move with a fixed velocity by Lorentz boosting the static solution.

Instead of using the numerical profile function, we shall use a simple analytic approximation based on sine-Gordon kinks\[20\],

\[
f(r) = 4 \tan^{-1}(e^{-r}), \tag{1.6}
\]
which gives a skyrmion with energy less than 1% above the true skyrmion energy. Since we are using this configuration only for our initial conditions, it will be an adequate approximation. Note that (1.6) has an exponential decay, whereas in the massless pion limit the true profile function has a power-law decay. An approximate profile function with the correct decay behaviour can be obtained by computing the holonomy of instantons\[4\]. However, this approximation has slightly more energy than (1.6), which implies that the kink approximation is better in the region of interest near the core of the skyrmion.

By taking a product ansatz $U = U_1 U_2$ of two well-separated single skyrmion fields $U_1$ and $U_2$, a $B=2$ configuration can be constructed which models two well-separated and undistorted skyrmions. The interaction between the two skyrmions can be repulsive or attractive depending upon the relative isospin orientation of the two skyrmions\[12\]. This attraction is maximal if one skyrmion is rotated relative to the other through an angle of $180^\circ$ about an axis orthogonal to the line joining the two skyrmions. Two skyrmions which have such a relative orientation are said to be in the attractive channel.

## 2 Multi-soliton dynamics

The simplest scattering event involves the head-on collision of two skyrmions in the attractive channel. As in all our simulations we begin with a configuration of well-separated skyrmions obtained from the product ansatz. The precise configuration used consists of two skyrmions with positions

$$X_1 = (0, 0, a), \quad X_2 = (0, 0, -a)$$

(2.1)

where $a=1.5$, the relative orientation given by a $180^\circ$ rotation around the $y$-axis and each skyrmion boosted to a velocity of $v=0.3$. Fig. 1 shows an isosurface plot of baryon density $B$ at regular intervals\[5\]. As the initially well-separated skyrmions come together they distort, eventually reaching the expected torus followed by $90^\circ$ scattering. The interaction is almost elastic with very little radiation involved in the scattering process. The skyrmions do, of course, attract once more and pass through the torus again with a $90^\circ$ scattering. This process repeats, with multiple right angle scatterings and the amplitude of oscillation around the torus decreasing with time. At first sight it may seem surprising that so little radiation is observed in the skyrmion scattering event, particularly since we are considering the model without a pion mass term. However, from studies of baby skyrmions\[13\], it appears that the production of radiation is counter-intuitive; less radiation is observed in the case where it is massless. Superficially at least, our results appear to support these findings.

For the case of charge three, we choose an initial configuration which will scatter close to the minimal energy tetrahedral $B=3$ skyrmion. It turns out that in choosing skyrmion initial configurations it is a good guide to first consider the analogous case in the context of

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1 We display all our results in the form of isosurfaces of baryon density, but the isosurfaces of constant energy density are almost indistinguishable.
BPS monopoles. It is known that a tetrahedral 3-monopole exists\[^9, 10\], which is formed during the scattering of three monopoles with cyclic $C_3$ symmetry\[^9, 21\]. The monopoles are initially well-separated on the vertices of a large contracting equilateral triangle. We therefore take three well-separated skyrmions in such a configuration. In the skyrmion case, we must also choose the initial orientations. We do this so as to maximize the amount of attraction, which in this case can be done by choosing each pair of skyrmions to be in the attractive channel. Take the positions of the skyrmions to be $X_i$, and define the relative position vectors $X_{ij} = X_i - X_j$. Let $n_{ij}$ be the unit vector such that the relative orientation between the skyrmion at $X_i$ and the one at $X_j$ is given by a rotation by $180^\circ$ around the axis $n_{ij}$. Explicitly, we take

$$X_1 = (-a, -a, -a), \ X_2 = (-a, a, a), \ X_3 = (a, -a, a), \ n_{12} = (1, 0, 0), \ n_{13} = (0, 1, 0).$$

This implies that $n_{23} = (0, 0, 1)$ and it is easily checked that this choice satisfies the requirement of all pairs being in the attractive channel i.e. $X_{ij} \cdot n_{ij} = 0$ for all $i \neq j$.

Again, we choose $a = 1.5$, but this time each skyrmion is boosted to have an initial velocity of $v = 0.1\sqrt{3}$ towards the centre of the triangle. The evolution of this configuration is shown in fig. 2. We should point out that, although we have constructed a configuration with cyclic symmetry, this symmetry is broken by the product ansatz implementation $U = U_1U_2U_3$, which is clearly asymmetric under permutations of the indices.

We see that initially there are three separated skyrmions on the vertices of an equilateral triangle, which deform as they coalesce. But each skyrmion behaves slightly differently, due to the symmetry breaking product ansatz. Clearly the greater the initial separation of the skyrmions then the closer the product ansatz configuration is to having cyclic symmetry. The dynamics is, nonetheless, remarkably similar to the monopole case[21], except for the influence of the potential and the approximate nature of the symmetry. The skyrmions form an approximately tetrahedral configuration, which then splits into a single skyrmion and a charge two torus. This may well be one of the important vibrational modes of the tetrahedral skyrmion which needs to be considered in a quantization of the classical solution.

In the monopole case, a second scattering process through the tetrahedral 3-monopole is also known\[^11\], which involves a twisted line scattering of three collinear monopoles. We now investigate whether a similar scattering process can occur for three skyrmions by taking the positions of the three collinear skyrmions to be

$$X_1 = (0, 0, a), \ X_2 = (0, 0, 0), \ X_3 = (0, 0, -a).$$

To put the first and second skyrmions in the attractive channel we may, without loss of generality, take $n_{12} = (1, 0, 0)$. For the second and third skyrmions to be in the attractive channel requires that the axis $n_{23}$ be in the $x_1x_2$-plane. Furthermore, if the relative orientation between the first and third skyrmions is to be given by a $180^\circ$ rotation about some axis then we must have that $n_{12} \cdot n_{23} = 0$, which determines that $n_{23} = (0, 1, 0)$. Thus, we deduce that $n_{13} = (0, 0, 1)$, but $X_{13} \cdot n_{13} \neq 0$. So the first and third skyrmions are not in
the attractive channel; in fact, they are in a repulsive channel. However, one may argue that the interaction between the first and third skyrmion should not be classified in this naive way, since the second skyrmion lies directly between the two and thus distorts the dipole interaction.

We therefore proceed with the above configuration, which incidentally has the twisted line symmetry (up to isospin rotations) of inversion plus 90° rotation, with the value $a=1.5$ and the two outer skyrmions boosted towards the one at the origin with a velocity $v=0.1$. The results are displayed in fig. 3. The initial configuration is interesting, since it does not resemble three well-separated skyrmions even though we used the product ansatz. If we increase the initial separation $a$, then three well-separated skyrmions do appear. However, at this separation we obtain a twisted figure-of-eight shape which is remarkably similar to the corresponding monopole configuration\[11\]. Once again, the similarity to monopole dynamics is quite striking: an approximate tetrahedral skyrmion is formed, before the motion continues towards a toroidal configuration.

It is known that a toroidal $B=3$ skyrmion has a relatively high energy, and with these initial conditions the combined energy of the skyrmions is not sufficient to carry it through to the torus. Therefore, it returns back through similar dynamics, up to a slight rotation. This contrasts sharply with the monopole case where all configurations have equal energy and the dynamics continues through the torus and forms the dual tetrahedron \[11\]. In the skyrmion case, the dynamics is influenced by the potential which prevents the formation of the toroidal configuration. This is an important difference between skyrmions and BPS monopoles. The above type of skyrmion scattering event has previously been conjectured in ref.\[23\] based upon JNR instanton holonomy calculations. However, it was not possible to obtain configurations of well-separated skyrmions due to the restriction to JNR instantons rather than the general instanton.

Four monopoles on the vertices of a contracting regular tetrahedron scatter through a cubic monopole\[14\]. Therefore, we now consider an analogous four skyrmion scattering process. The starting point is the $B=3$ system given by (2.2), where the skyrmions may be considered as positioned on three vertices of a regular tetrahedron. To this system we add a fourth skyrmion placed at the remaining vertex $X_4 = (a, a, -a)$ with orientation given by $n_{14} = (0, 0, 1)$. The additional relative orientations are then determined to be $n_{24} = (0, 1, 0)$ and $n_{34} = (1, 0, 0)$ so that again we have $X_{ij} \cdot n_{ij} = 0$ for all $i \neq j$, showing that all pairs of skyrmions are in the attractive channel. Again, we use $a=1.5$, but this time with no initial Lorentz boosts.

The evolution of this configuration is displayed in fig. 4. The mutual attractions cause the skyrmions to coalesce and form a cubic configuration, before scattering out to lie on the vertices of a tetrahedron dual to the initial one. Again the product ansatz implementation slightly distorts the above description, resulting in the symmetries being only approximately attained. Up to this technicality, however, the scattering event is once again a close copy of the monopole case\[10\]. Note that a one-parameter family of ADHM instanton generated skyrmions exists which is a good approximate description of this process\[13\]. By taking an initial configuration from this family, rather than using the product ansatz, the above approximate symmetries could be made exact.
3 Minimum energy configurations

If we were able to run the simulation for long enough, each of the attractive configurations would eventually settle down to the global minimal energy configuration. However, given the small amounts of radiation emitted in a single scattering event this is not computationally feasible. Instead, we modify the dynamics to a dissipative regime by the inclusion of a friction term in the equation of motion. That is, we add a phenomenological term $\epsilon \dot{\phi}$, with $\epsilon = 0.5$, to the equations of motion \([\mathcal{L}]\). This does not affect the static solutions of the model, but enables the configuration to settle to a static solution in a much shorter period of time. Of course there is no guarantee that this static solution will be the minimum energy configuration, but given the lack of symmetry imposed by the product ansatz it is likely to be. In fig. 5 we plot a $\mathcal{B}$ surface for the minimal energy skyrmions of charge one to four and we also compute their energy and baryon number, which are given in Table 1.

Numerical effects, particularly the fact that we work on a grid covering only a finite volume of space, mean that there will be some inaccuracies in calculating both $\mathcal{B}$ and $E$. For example, in the charge two case we obtain $\mathcal{B} = 1.971$ rather than the integer value 2. However, our results are more accurate than the previous numerical study of multiskyrmions\([7]\), with all our calculations of the baryon number being considerably closer to integer values. Moreover, our computations are the first to be performed for the massless pion case, which makes the calculations more difficult since the fields are only localized power-like rather than exponential. By calculating the energy per baryon $E/\mathcal{B}$ we should reduce the severity of these inaccuracies. The effects of the lost tails will not cancel out exactly — the baryon density contains three derivatives and the two terms in the energy density contain two and four derivatives — but it should correct the results in the right direction. For comparison we use our scheme to compute these quantities for the single skyrmion, where the result is accurately known to be $E/\mathcal{B} = 1.232$.

It is possible to obtain approximate skyrmions by computing the holonomies of instantons\([4]\). This provides a method of computing upper bounds for the energies of the minimal energy skyrmions of each baryon number. Results are available for $\mathcal{B} = 1, 2, 3, 4$ and the Skyrme crystal \([4, 5, 13, 14]\). For comparison we include in Table 1 the instanton computed values for $E/\mathcal{B}$ which we denote by $E_i$. These results seem to overestimate the energy by around 1% and in the case of $\mathcal{B}=1$ are significantly less accurate than our calculations. We suggest that this is likely to be true for the higher charge configurations.

| charge | $\mathcal{B}$ | $E$ | $E/\mathcal{B}$ | $E_i$ |
|--------|---------------|-----|----------------|------|
| 1      | 0.984         | 1.214 | 1.233       | 1.243 |
| 2      | 1.971         | 2.309 | 1.171       | 1.192 |
| 3      | 2.965         | 3.407 | 1.149       | 1.163 |
| 4      | 3.945         | 4.378 | 1.110       | 1.132 |

Table 1: Calculated values of the total energy for charge one to four in the massless pion limit. For comparison to previous work, use $E/\mathcal{B}$.
4 Conclusions

We have performed numerical simulations of skyrmion dynamics, with initial conditions created from a product ansatz of single skyrmions, using state of the art computer technology. By considering symmetric configurations, motivated by BPS monopoles, we have displayed scattering processes which form the minimal energy configurations for baryon numbers two to four. However, the analogy to monopoles is not exact, since in the skyrmion case the dynamics is influenced by the potential. Nonetheless, it seems possible to construct similar scattering events associated with platonic solids already seen in the monopole case.

By inclusion of a dissipation term, we were able to accurately (within 1%) compute the energies of the known bound states. We see that the cubic $B=4$ configuration is the most tightly bound, reflecting the behaviour of low-energy baryons. This makes creating higher charge configurations from the product ansatz more difficult, since, for example, a general $B=5$ configuration has tendency to split up into a cubic $B=4$ configuration and a single skyrmion. In fact, it is not possible to arrange more than four well-separated skyrmions so that all pairs of interactions are in the attractive channel. However, by going beyond a simple product ansatz approach we have been able to construct skyrmion bound states for baryon numbers greater than four. These results, together with other saddle point configurations and scattering events, will be presented in an future publication[6].

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References

[1] G.S. Adkins, C.R. Nappi & E. Witten [1983], Nucl. Phys. B228, 552.
[2] A.E. Allder, S.E. Koonin, R. Seki & H.M. Sommermann [1987], Phys. Rev. Lett. 59, 2837.
[3] M.F. Atiyah & N.J. Hitchin [1988], The geometry and dynamics of magnetic monopoles (Princeton University Press).
[4] M.F. Atiyah & N.S. Manton [1989], Phys. Lett. 222B, 438.
[5] M.F. Atiyah & N.S. Manton [1993], Comm. Math. Phys. 153, 391.
[6] R.A. Battye & P.M. Sutcliffe [1996], ‘Bound States and Dynamics of Skyrmions in (3+1) dimensions’, in preparation.

[7] E. Braaten, S. Townsend & L. Carson [1990], Phys. Lett. 235B, 147.

[8] W.Y. Crutchfield & J.B. Bell [1991], J. Comp. Phys. 110, 234.

[9] N.J. Hitchin, N.S. Manton & M.K. Murray [1995], Nonlinearity 8, 661.

[10] C.J. Houghton & P.M. Sutcliffe [1996], Comm. Math. Phys. 180, 343.

[11] C.J. Houghton & P.M. Sutcliffe [1996], Nucl. Phys. B464, 59.

[12] A. Jackson, A.D. Jackson & V. Pasquier [1985], Nucl. Phys. A432, 567.

[13] R.A. Leese & N.S. Manton [1994], Nucl. Phys. A572, 575.

[14] N.S. Manton & P.M. Sutcliffe [1995], Phys. Lett. 342B, 196.

[15] B.M.A.G. Piette, B.J. Schroers & W.J. Zakrzewski [1995], Nucl. Phys. B439, 205.

[16] B.J. Schroers [1994], Z. Phys. C 61, 479.

[17] E.P.S. Shellard [1987], Nucl. Phys. B283, 624.

[18] E.P.S. Shellard & P.J. Ruback [1988], Phys. Lett. 209B, 262.

[19] T.R.H. Skyrme [1962], Nucl. Phys. 31, 556.

[20] P.M. Sutcliffe [1992], Phys. Lett. 292B, 104.

[21] P.M. Sutcliffe [1996], ‘Cyclic Monopoles’, hep-th/9610030.

[22] J.J.M. Verbaarschot [1987], Phys. Lett. 195B, 235.

[23] N.R. Walet [1996], ‘Quantizing the B=2 and B=3 skyrmion systems’, preprint UMIST-TP-96/1.

[24] E. Witten [1983], Nucl. Phys. B223, 422; B223 423.

Figure Captions

Fig. 1. Head-on collision of two skyrmions in the attractive channel illustrating the right-angled scattering.

Fig. 2. Three skyrmion scattering with cyclic symmetry to form tetrahedron.
Fig. 3. Twisted line scattering of three skyrmions through a tetrahedron. The configuration does not reach the charge three torus due to the potential, illustrating the difference between skyrmions and BPS monopoles.

Fig. 4. Four skyrmion with tetrahedral symmetry scattering through a configuration with approximate cubic symmetry.

Fig. 5. Minimal energy skyrmions for baryon numbers one to four.

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