Tiering as a Stochastic Submodular Optimization Problem

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ABSTRACT
Tiering is an essential technique for building large-scale information retrieval systems. While the selection of documents for high priority tiers critically impacts the efficiency, past work focuses on optimizing it with respect to a static set of queries in the history, and generalizes poorly to the future traffic. Instead, we formulate the optimal tiering as a stochastic optimization problem, and follow the methodology of regularized empirical risk minimization to maximize the generalization performance of the system. We also show that the optimization problem can be cast as a stochastic submodular optimization problem with a submodular knapsack constraint, and we develop efficient optimization algorithms by leveraging this connection.

CCS CONCEPTS
• Information systems → Search engine indexing.

KEYWORDS
information retrieval, tiering, submodular optimization, submodular knapsack

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1 INTRODUCTION
Tiering [21] is a classic method for scaling information retrieval systems to large corpora by restricting searches on a subset of documents. In this paper, we focus on two-tier systems because of their popularity and simplicity, but ideas in this paper can be applied to more than two tiers by iteratively splitting a tier into two. In two-tier systems, Tier 1 indexes and serves only a selected set of documents in the corpus, whereas Tier 2 covers all documents. For every incoming query, a query classifier checks the eligibility of the query to Tier 1. If the query is eligible, it is handled by Tier 1, which is more efficient than Tier 2 in serving the query because Tier 1 indexes much fewer documents than Tier 2 does. However, there are queries which documents in Tier 1 are not sufficient to serve, and they are handled by Tier 2.

The selection of documents for Tier 1 critically impacts the efficiency of tiering. The set of documents indexed in Tier 1 should be small enough to make searches on the tier more efficient than searches on Tier 2, while still being able to faithfully serve a good fraction of traffic in order to justify the opportunity cost of adding more compute power to Tier 2 instead. While early approaches used heuristic rules [4, 21] for the selection, Leung et al. [17] formulated the selection problem as a max-flow problem, and developed an efficient optimization algorithm for it.

However, these approaches optimize their selection with respect to a static set of queries from the past. Due to the heavy-tailed nature of query distributions in information retrieval, a large fraction of queries in the incoming traffic are novel ones which never appeared in the data from the past [3]. Therefore, these approaches can overfit to the “training data” used for the optimization, and generalize poorly to the incoming traffic.

In order to optimize the tiered architecture in terms of its generalization performance on the future traffic, we first propose a novel method of tiering which a tier is defined with the set of clauses it comprehensively indexes. Then, we formulate the task of finding the best selection of clauses as a stochastic optimization problem, and follow the methodology of regularized empirical risk minimization to optimize the generalization performance. While this involves a combinatorial optimization, we also show that the problem can be cast as a stochastic submodular maximization problem with a submodular knapsack constraint, which is called Submodular Cost Submodular Knapsack (SCSK) [14]. We also develop efficient optimization algorithms for the SCSK problem which make it possible to apply the proposed tiering method on large-scale corpora in the real world.

This paper makes contributions to both machine learning and information retrieval community. For the machine learning research, we introduce a novel application of large-scale stochastic submodular optimization, orders of magnitudes larger than applications considered in the literature. We also develop efficient algorithms for the SCSK problem, which can be used in other applications and thus are of independent interest. For the information retrieval community, we show the importance of formulating the tiering as a learning problem, and develop an efficient tiering method. We demonstrate the practical utility of our approach by applying our method to real data from a large-scale commercial search engine.
2 FORMULATION AND PREVIOUS WORK

2.1 Matching

In this paper, we focus on the task of finding the set of documents that match every term in a query. This is called matching, and matching algorithm is often the most computationally challenging component in scaling a search engine to large corpora [10]. To be specific, let \( V \) be the possibly infinite vocabulary which contains all terms that appear in the corpus or a query. Let \( D \) be the corpus to search on. Each document \( d \in D \) is represented as a set of terms in it, i.e., \( d \subseteq V \). Let \( Q \) be the probability distribution of queries, and each query \( q \) drawn from \( Q \) is also represented as a subset of \( V \). Then, the match set of a query \( q \) is defined as the set of documents which contain all terms in the query:

\[
m(q) := \{ d \in D; q \subseteq d \} = \bigcap_{v \in q} \{ d \in D; v \in d \}.
\]

The goal of a matching algorithm is to efficiently compute the match set (1) over an intimidatingly large corpus. Then, documents in the match sets are iteratively sorted and filtered by more sophisticated relevance algorithms to determine the final presentation to the user. Since these post-processing algorithms are too expensive to apply to the entire corpus, they rely on matching algorithms for drastically reducing the scope of analysis. This is why the efficiency of a matching algorithm is a critical determinant of the scalability of the information retrieval system [10, 22].

To illustrate the concept of matching with examples, consider a corpus with six documents, shown in Table 1. The match set of a query “red shirt” would be the intersection of postings list of the word red and that of shirt: \( m(\{\text{red, shirt}\}) = \{D_1, D_3, D_4\} \cap \{D_1, D_2, D_3\} = \{D_1, D_3\} \). Similarly, \( m(\{\text{blue, pants, striped}\}) = \{D_2, D_3, D_4\} \cap \{D_4, D_5, D_6\} \cap \{D_1, D_2, D_4, D_5\} = \{D_5\} \). Note that documents in the match set are not yet ordered. The ordering of documents presented to a user is determined by a ranking algorithm which takes the match set as an input.

2.2 Tiering

Tiering [21] improves the efficiency of finding match sets of queries by selectively reducing the scope of search into a subset of documents in the corpus. In this paper, we focus on the case of two tiers, which is standard and simple to illustrate. Designing a tiered architecture requires defining two components: a document classifier and a query classifier.

| \( D_1 \) (red shirt striped) | red | blue | shirt | pants | striped |
| ----------------------------- |-----|------|-------|-------|--------|
| \( D_2 \) (blue shirt striped) | X   | X    | X     | X     |        |
| \( D_3 \) (red shirt)        | X   |      |       | X     | X      |
| \( D_4 \) (red pants striped) | X   | X    | X     |       |        |
| \( D_5 \) (blue pants striped) | X   | X    |       | X     |        |
| \( D_6 \) (blue pants)       | X   | X    |       |       |        |

Table 1: An Example Corpus \( D = \{D_1, D_2, \ldots, D_6\} \). X denotes that the document in the corresponding row contains the word in the corresponding column.

Figure 1: Illustration of Tiering. At the indexing time, all documents are indexed in Tier 2, and only documents with \( \phi(d) = 1 \) are additionally indexed in Tier 1, which is thus smaller than Tier 2 and more effective in serving queries. On the other hand, an incoming query is routed to either Tier 1 or Tier 2, depending on its query classification result \( \psi(q) \).

Let \( \phi : D \rightarrow \{1, 2\} \) be the document classifier, which determines the lowest tier an input document should be indexed in. Tier 1 indexes \( D^1 := \{ d \in D; \phi(d) \leq 1 \} \) only, whereas Tier 2 indexes every document; \( D^2 := \{ d \in D; \phi(d) \leq 2 \} = D \). Let \(|\cdot|\) be the cardinality of a set. In order for queries to be more efficiently executed in Tier 1 than in Tier 2, \(|D^1|\) should be substantially smaller than \(|D|\). For example, documents in a large corpus are often partitioned into shards [6] such that each partition of the index assigned to a shard fits into the memory capacity of a single machine. In such case, half-sized Tier 1 index (i.e., \(|D^1| < |D^2|\) ) would require half the number of machines needed by Tier 2 in processing each query.

On the other hand, let \( \psi : Q \rightarrow \{1, 2\} \) be the query classifier, which determines the tier the query should be executed in. Note that Tier 1 is unable to return the entire match set of a query \( q \) unless \( m(q) \subseteq D^1 \); whenever a query is classified into Tier 1 but incomplete match set was found, i.e., \( \psi(q) = 1 \) but \( m(q) \not\subseteq D^1 \), we consider this as an incorrect query classification. Some tiered systems don’t guarantee their query classifications to be always correct [4, 21].

Since any document missed by a matching algorithm will also be missed in the final search result presented to the user, it is critical for a matching algorithm to have a nearly perfect recall [10, 20]. Incorrect query classifications to Tier 1, however, incur false negative errors. In applications which the searchability of a document should be guaranteed, these errors can be intolerable. Therefore, we focus on developing methods which always make correct query classifications and return the comprehensive match set for every query. See Figure 1 for a graphical illustration.

2.3 Maximum Flow-Based Optimization

In order to maximize the efficiency of a tiered architecture, \( D^1 \), documents for Tier 1, should be carefully chosen so that a good fraction of traffic can be efficiently handled by Tier 1. We formulate this as the following stochastic optimization problem, which
objective is to maximize the fraction of traffic covered by Tier 1:

$$\max_{\psi, \phi} \frac{|X|}{n} \text{ subject to } \left| \bigcup_{q \in X} m(q) \right| \leq B.$$  

(5)

Although solving the original problem (5) requires an integer programming and thus it is practically infeasible even at moderate scale, they develop efficient solvers by replacing the constraint with a partial Lagrangian, and optimizing the convex relaxation of the objective. Since this approach is equivalent to solving a maximum flow problem over a directed graph, we call this method flow whenever an abbreviation is needed.

With a solution of (5), which we denote as $X^{\text{flow}}$, the query classifier and the document classifier of flow can be defined as follows:

$$\psi^{\text{flow}}(q) = \begin{cases} 1, & \text{if } q \in X^{\text{flow}} \\ 2, & \text{otherwise} \end{cases}$$  

(6)

$$\phi^{\text{flow}}(d) = \begin{cases} 1, & \text{if } d \in m(q) \text{ for some } q \in X^{\text{flow}} \\ 2, & \text{otherwise} \end{cases}$$  

(7)

It is easy to see that if $X^{\text{flow}}$ is a solution of (5), then induced query classifier $\psi^{\text{flow}}(q)$ and document classifier $\phi^{\text{flow}}(d)$ satisfy constraints (3) and (4). However, the objective (5) is at the risk of overfitting to the training data $Q_n$.

Indeed, there are two clear limitations of this approach when its generalization performance to incoming traffic is concerned. First, any query which did not appear in the training data, i.e., $q \not\in Q_n$, will be routed to Tier 2 because by design, the method can select queries in the training data only; $X^{\text{flow}} \subseteq Q_n$. Due to the heavy-tailed nature of query distributions, no matter how large the training data $Q_n$ is, a considerable fraction of new samples from $Q$ would not overlap with $Q_n$ [3], all of which are missed opportunities for $\text{flow}$. On the other hand, very specific queries in the training data which are very unlikely to reappear, for example a query like “book on submodular optimization with red cover”, are more likely to be selected because their match sets are small and therefore it is easy to meet the correctness constraint with them. Therefore, a tiering method which allows more effective optimization of generalization performance is called for.

2.4 Other Related Work

There are other methods for optimizing the size of index for Tier 1. Index pruning methods [7] reduce the size of the index by removing postings which minimally impact the quality of the search results. While these methods can reduce the size of the backward index, they don’t have a direct control for the size of the forward index. On the other hand, we focus on controlling the size of the forward index by constraining the number of documents added to the index, without directly optimizing the size of the backward index. This is because commercial search engines need to return rich metadata associated with each document in the search result, and therefore the size of the forward index dominates the size of the backward index. Conceptually, however, these methods are complementary and can be combined together. Most index pruning methods do not guarantee the correctness of query classifications, but Ntoulas and Cho [20] is a notable exception which provides similar guarantees as ours; while they propose simple heuristics for finding the optimal set of terms for Tier 1, their optimization problem can also be formulated and approached in the way similar to the method we discuss in Section 3.

Anagnostopoulos et al. [2] is also a relevant work which formulates caching as a stochastic optimization problem. While we focused on the generalization of query classifiers to the future traffic, their work was concerned about stochastically modeling the size of the cache needed to run their algorithm, and used the same query classifier as in Leung et al. [17] which cannot generalize to queries unseen in the past.

3 TIERING WITH CLAUSE SELECTION

In order to build a tiered architecture which can generalize to queries unseen in the training data, we propose a novel method of tiering which query and document classification decisions are made with clauses, which are sets of terms and can be more granular than full queries. This makes it possible for queries unseen in the training data to be classified into Tier 1, as long as some clauses in the query were observed.

3.1 Query and Document Classifier

We parameterize query and document classifiers with $X^{\text{clause}} \subseteq 2^V$, which is a subset of all possible clauses in the vocabulary. Then, the query classifier and the document classifier identically check whether any of the clauses in a query or a document is included in the selection:

$$\psi^{\text{clause}}(q) = \begin{cases} 1, & \text{if } c \subseteq q \text{ for some } c \in X^{\text{clause}} \\ 2, & \text{otherwise} \end{cases}$$  

(8)

$$\phi^{\text{clause}}(d) = \begin{cases} 1, & \text{if } c \subseteq d \text{ for some } c \in X^{\text{clause}} \\ 2, & \text{otherwise} \end{cases}$$  

(9)

Efficient algorithms for computing these “subset query” functions exist, e.g., Charikar et al. [8] or Savnik [23], which make it possible for these classifiers to be used in applications with low latency requirements.
To illustrate these classifiers, consider the example corpus in Table 1 and suppose we chose two clauses $X_{\text{clause}} = \{\{\text{red}\}, \{\text{blue}, \text{shirt}\}\}$. Then, documents which contain a single-word clause "red" or a double-word clause "blue shirt" will be classified into Tier 1: $D^1 = \{D_1, D_2, D_3, D_4\}$. With these documents, Tier 1 shall serve queries such as "red", "red shirt", "red pants", or "blue shirt stripped", but not "blue pants", because neither {red} nor {blue, shirt} is a subset of {blue, pants}. Indeed, $D_6$ (blue pants) is not included in $D^1$.

Clearly, the query classifier is always correct:

**Theorem 3.1 (Correctness).** For any $q \subseteq V$ with $\psi_{\text{clause}}(q) = 1$, we have $\phi_{\text{clause}}(d) = 1$ for any $d \in m(q)$. This implies $m(q) \subseteq D_1$.

**Proof.** Suppose $\psi_{\text{clause}}(q) = 1$ for some $q \subseteq V$. Then, from the definition of $\psi_{\text{clause}}(q)$, there exists some $c \in X_{\text{clause}}$ which satisfies $c \subseteq q$. Now pick any $d \in m(q)$. By definition (1), we have $q \subseteq d$. Therefore, $c \subseteq d$ and $\phi_{\text{clause}}(d) = 1$. □

### 3.2 Stochastic Submodular Optimization

In order to maximize the efficiency of the tiered architecture, now we need to solve (2) parameterized by $X_{\text{clause}}$. Since Theorem 3.1 guarantees the correctness, the corresponding constraint can be dropped and the problem can be simplified as follows:

$$
\max_{X_{\text{clause}} \subseteq 2^V} \mathbb{P}_{Q \sim G} \left[ c \subseteq q \text{ for some } c \in X_{\text{clause}} \right],
$$

subject to

$$
\left| \left\{ d \in D; c \subseteq d \text{ for some } c \in X_{\text{clause}} \right\} \right| \leq B. \tag{11}
$$

While this may seem as an intractable combinatorial problem as every subset of the power set of $V$ should be considered as a potential solution, we argue that there is a strong structure in the problem we can exploit for developing efficient optimization algorithms. Specifically, we show that this is a stochastic version of the Submodular Cost Submodular Knapsack (SCSK) problem [14], which aims to maximize a stochastic submodular function under an upper bound constraint on another submodular function. This observation makes it promising to develop practical algorithms for the problem, as optimization problems formulated with submodular functions often allow practical algorithms with theoretical guarantees.

To prove that (10) is a SCSK problem, we first remind readers basic notions of submodularity [9]:

**Definition 3.2 (Gain, Monotonicity, and Submodularity).** Let $G$ be a set, and $f : 2^G \to \mathbb{R}$ be a set function. The gain of $j \in G$ at $Y$ is defined as $f(j \mid Y) := f(Y \cup \{j\}) - f(Y)$. If $f(j \mid Y) \geq 0$ for any $j \notin Y$, then $f(\cdot)$ is called monotone. Submodular functions have diminishing gains; that is, $f(\cdot)$ is called submodular if $f(j \mid Y) \geq f(j \mid Z)$ for any $Y, Z$ with $Y \subseteq Z$ and $j \notin Y$.

Now we prove that the objective function is submodular.

**Theorem 3.3 (Submodularity of the Objective).** The objective function (10), $f(X) := \mathbb{P}_{Q \sim G} \left[ c \subseteq q \text{ for some } c \in X \right]$, is a monotone submodular function of $X$.

**Proof.** First, pick any $q \subseteq V$, and define $f_q(X) := 1 \{ c \subseteq q \text{ for some } c \in X \}$, where $1 \{ \cdot \}$ is an indicator function. We start by showing monotone submodularity of this function.

The monotonicity of the function can be shown by noting that:

$$
f_q(j \mid Y) = \begin{cases} 0, & \text{if } c \subseteq q \text{ for some } c \in Y \\ 0, & \text{if } c \not\subseteq q \text{ for any } c \in Y \cup \{j\} \\ 1, & \text{if } c \not\subseteq q \text{ for any } c \in Y \text{ but } j \subseteq q \end{cases}
$$

and thus $f_q(j \mid Y) \geq 0$ for every case.

Now suppose $Y, Z$ are given, with $Y \subseteq Z \subseteq 2^V$. We have three cases:

- Case 1: There exists some $c \in Y$ which $c \subseteq q$. Then, $f_q(Y) = f_q(Z) = 1$, and $f_q(j \mid Y) = f_q(j \mid Z) = 0$ as indicator functions are bounded above by 1.
- Case 2: There does not exist any $c \in Z$ which $c \subseteq q$. In this case, $f_q(Y) = f_q(Z) = 0$, and $f_q(j \mid Y) = f_q(j \mid Z) = 1$ if $j \subseteq q$, and $f_q(j \mid Y) = f_q(j \mid Z) = 0$ otherwise.
- Case 3: There does not exist any $c \in Y$ with $c \subseteq q$, but there exists $c' \in Z$ which $c' \subseteq q$. Then, $f_q(Y) = 0$ and $f_q(Z) = 1$. Since $f_q(\cdot)$ is bounded above by 1, $f_q(j \mid Z) = 0$. Because $f_q(\cdot)$ is monotonic, $f_q(j \mid Y) \geq f_q(j \mid Z)$.

Therefore, $f_q(\cdot)$ is monotone submodular.

In order to prove the monotone submodularity of $f(\cdot)$, note that $f(X) = \mathbb{E}_{Q \sim G} f_q(X)$. A convex combination of monotone submodular functions is again monotone submodular. □

**Theorem 3.4 (Submodularity of the Constraint).** $g(X) := |\{d \in D; c \subseteq d \text{ for some } c \in X\}|$ is monotone submodular.

**Proof.** Observe that

$$
\{d \in D; c \subseteq d \text{ for some } c \in X\} = \bigcup_{c \in X} \{d \in D; c \subseteq d\} = \bigcup_{c \in X} m(c).
$$

Therefore, $g(X)$ is a set covering function, which is known to be monotone submodular [27]. □

### 3.3 Regularized Empirical Risk Minimization

Since we typically don’t have an access to the true query distribution $Q$, we need to use our training data $Q_n$ and optimize (10) with respect to the induced empirical distribution $Q_n$ instead of $Q$. In order to avoid overfitting to the training data, we follow the methodology of regularized risk minimization, and control the capacity of the function we learn [25]. Specifically, we restrict the ground set of the objective function to be $X := \{c \in 2^V; \mathbb{P}_{Q \sim G} [c \subseteq q] \geq \lambda\}$, so that only clauses which frequency of appearance in the training data is at least $\lambda$ are considered. The optimization problem then becomes

$$
\max_{X \subseteq X} f(X) := \mathbb{P}_{Q \sim G_n} [c \subseteq q \text{ for some } c \in X], \tag{12}
$$

subject to $g(X) := \left| \bigcup_{c \in X} m(c) \right| \leq B$.

Not only does this regularize the solution for improved generalization, it also has a computational benefit because the number of clauses to be considered is drastically reduced. The time complexity of most submodular optimization algorithms are at least linear to the cardinality of the ground set. Also note that $X$ can be efficiently
computed from $Q_n$ using frequent pattern mining algorithms, and we use FPGrowth [11] in our experiments.

4 OPTIMIZATION ALGORITHMS FOR SCSK

Solving (12) for large-scale information retrieval systems is a computational challenge, since the number of documents to be indexed $|D|$ and the number of query logs available $|Q_n|$ in such systems are often at formidable scale ($10^6 \sim 10^2$) [22]. In order to achieve a high coverage of traffic with Tier 1, we also consider a large number of clauses $|\bar{X}|$ at the scale of $10^4 \sim 10^6$, orders of magnitude higher than problems considered to be large-scale in submodular optimization research (e.g., [14, 15]). Therefore, in order to apply the proposed tiering method to real-world problems, it is critical to develop efficient and scalable optimization algorithms for (12). In this section, we develop multiple algorithms for the problem which can be broadly applicable to other SCSK problems, and therefore of interest on their own.

4.1 Greedy

Greedy algorithms are often very competitive at solving a wide range of submodular maximization problems (e.g., [5, 19, 24, 26]). Because of the submodular upper bound constraint in the problem (12), however, most of existing greedy algorithms for submodular maximization problems do not directly apply. A notable exception is the greedy algorithm from Iyer and Bilmes [14], but it ignores the constraint when comparing candidates to add to the solution; while this simplicity facilitates the mathematical analysis of the algorithm, it is clearly important to take the cost of each clause into consideration, and we empirically validate this in Section 5.1.

To this end, we propose a novel greedy algorithm for SCSK, which starts with an empty solution $X^0 = \emptyset$, and iteratively adds a new clause with the highest utility ratio:

$$ X^{t+1} \leftarrow X^t \cup \left\{ j^t := \arg \max_{j \in \bar{X}, g(j \mid X^t) \leq B} f(j \mid X^t) \right\}. $$

Computing $f(j \mid X^t)$ and $g(j \mid X^t)$ for every $j \in \bar{X}$ at each iteration, however, is clearly not feasible at the scale of problems we consider. This is because computing $g(j \mid X^t)$ involves calculating intersections between $m(j)$ and $\bigcup_{c \in \bar{X}} m(c)$, which can be both large sets with millions of elements, and computing $f(j \mid X^t)$ is also expensive for the same reason with large scale training data.

4.1.1 Lazy Greedy: The lazy evaluation technique has been essential to the success of many large-scale submodular maximization algorithms (e.g., [1, 16, 18]), as they can effectively avoid the costly evaluation of gain on less promising candidates. The submodularity of the constraint $g(\cdot)$, however, requires us to be more careful in applying the technique.

To illustrate, consider a simple case where $g(\cdot)$ is a modular function; then, there exists $w_0 \in \mathbb{R}$ and $\{ w_c \in \mathbb{R} \}_{c \in \bar{X}}$ such that $g(X) = w_0 + \sum_{j \in \bar{X}} w_j$. In this case, the problem reduces to submodular knapsack, and the greedy procedure (13) acquires strong guarantees [24]. The utility ratio $f(j \mid X^t) / g(j \mid X^t)$ is nonincreasing in $t$, because $f(\cdot)$ is submodular and $g(j \mid X^t) = w_j$ is a constant. The classic lazy greedy algorithm [18] exploits this by maintaining a max heap of candidates sorted by the most recent evaluation of the utility ratio, and re-evaluate the ratio for candidates which are promising enough to be placed at the top of the heap. Since $g(\cdot)$ is also submodular in SCSK, however, the ratio can also increase over iterations, and thus this technique is not directly applicable.

In order to leverage lazy evaluations for SCSK, we maintain the lower bound of $g(j \mid X^t)$ as $\bar{g}(j \mid X^t)$ with the following update rule:

$$ \bar{g}(j \mid X^{t+1}) \leftarrow \max \left( 0, \bar{g}(j \mid X^t) - g(j^t \mid X^t) \right). \quad (14) $$

This allows us to efficiently update the lower bound when adding a new clause to the solution. Only when the optimistic estimate of the utility of a clause is good enough for consideration, we re-compute the function to make the bound tight. We prove the correctness of this update:

**Theorem 4.1 (Correctness of Updated Lower Bound).** Suppose $g(\cdot)$ is a monotone submodular function, and $\bar{g}(j \mid X^t) \geq g(j \mid X^t)$. With the update rule (14), we have $g(j \mid X^{t+1}) \geq \bar{g}(j \mid X^{t+1})$.

**Proof.**

$$ g(j \mid X^{t+1}) = g(X^{t+1} \cup \{ j \}) - g(X^t) $$
$$ = \left( g(X^t) + g(j \mid X^t) + g(j^t \mid X^t \cup \{ j \} \right) - \left( g(X^t) + g(j^t \mid X^t) \right) $$
$$ = g(j \mid X^t) - g(j^t \mid X^t) + g(j^t \mid X^t \cup \{ j \} \right) $$
$$ \geq g(j \mid X^t) - g(j^t \mid X^t) \geq \bar{g}(j \mid X^t) - g(j^t \mid X^t), $$

using the definition of gain, the monotonicity of $g(\cdot)$ for the first inequality, and the assumption for the second inequality. Combining this with the fact that $g(j \mid X^{t+1}) \geq 0$ due to monotonicity of $g(\cdot)$, the proof is completed. □

Algorithm 1 shows the pseudo-code of the lazy greedy algorithm we propose. $\bar{g}(j \mid X^t)$ is potentially outdated as it equals to $f(j \mid X^t)$ for some $0 \leq s \leq t$, and serves as an upper bound due to submodularity. We then maintain a max heap of every feasible candidate, exactly compute the utility ratio of the best candidate at the moment, and add it to the solution if it is better than the second candidate in the heap. This allows us to avoid computing the utility ratio of candidates which even optimistic estimate of the ratio is lower than $f^t$, the clause selected in the $t$-th iteration.

4.1.2 Optimistic-Pessimistic Parallel Greedy: While the lazy evaluation procedure of Algorithm (14) reduces the number of evaluations of $f(\cdot \mid X^t)$ and $g(\cdot \mid X^t)$, the procedure is inherently sequential. In order to leverage the compute power for parallel processing available in modern computers, we propose an extension of the lazy evaluation algorithm. In addition to optimistic estimates $\bar{g}(j \mid X^t)$ in the lazy greedy algorithm, we also maintain pessimistic estimates $\bar{g}(j \mid X^t)$ in the lazy greedy algorithm, we also maintain pessimistic estimates $\bar{g}(j \mid X^t)$ in the lazy greedy algorithm, we also maintain pessimistic estimates $\bar{g}(j \mid X^t)$ in the lazy greedy algorithm, we also maintain pessimistic estimates $\bar{g}(j \mid X^t)$ in the lazy greedy algorithm, we also maintain pessimistic estimates $\bar{g}(j \mid X^t)$ in the lazy greedy algorithm, we also maintain pessimistic estimates $\bar{g}(j \mid X^t)$ in the lazy greedy algorithm, we also maintain pessimistic estimates $\bar{g}(j \mid X^t)$ in the lazy greedy algorithm, we also maintain pessimistic estimates $\bar{g}(j \mid X^t)$ in the lazy greedy algorithm. This allows us to update the estimate of every candidate which optimistic estimate is better than the best pessimistic estimate. Algorithm 2 illustrates the pseudo-code.

We prove that this algorithm is consistent with the original greedy update (13):
Algorithm 1: Lazy Greedy Algorithm for SCK

Function LazyGreedy(f, g, B, X)

\[ t = 0, X^0 = \emptyset; \]
\[ \overline{f}(j | X^0) = f(j), g(j | X^0) = g(j) \text{ for every } j \in X; \]
while \( g(X^t) \leq B \) do
heap \( \leftarrow \) Max heap with items \[ \left\{ j \in X; g(X^t \cup \{j\}) \leq B \right\}, \] and score \[ \overline{f}(j | X^t) \]
while heap.size() > 0 do
\[ j \leftarrow \text{heap.pop}(); \]
// Tighten bounds
\[ \overline{f}(j | X^t) \leftarrow f(j | X^t); \]
\[ g(j | X^t) \leftarrow g(j | X^t); \]
if \( g(X^t) + g(j | X^t) > B \) then
1 continue
end
if heap.isEmpty() or \[ \overline{f}(j | X^t) \geq \overline{f}(j | X^t) \] with
\[ k \leftarrow \text{heap.peek}(); \]
\[ X^{t+1} \leftarrow X^t \cup \{j\}; \]
\[ \overline{f}(i | X^{t+1}) = \overline{f}(i | X^t), \]
\[ g(i | X^{t+1}) = \max(0, g(i | X^t) - g(j | X^t)) \text{ for every } i \in X; \]
\[ t \leftarrow t + 1; \]
end
end
end
end

Algorithm 2: Optimistic-Pessimistic Greedy Algorithm

Function OptimisticPessimisticGreedy(f, g, B, X)

\[ t = 0, X^0 = \emptyset; \]
\[ \overline{f}(j | X^0) \leftarrow f(j | X^0) \leftarrow f(j), \]
\[ \overline{g}(j | X^0) \leftarrow g(j | X^0) \leftarrow g(j) \text{ for every } j \in X; \]
while \( g(X^t) \leq B \) do
\[ C \leftarrow \left\{ j \in X; g(X^t \cup \{j\}) \leq B, \overline{f}(j | X^t) \geq \overline{f}(j | X^t) \right\}; \]
// Tighten bounds
for \( j \in C \text{ in parallel} \) do
\[ \overline{f}(j | X^t) \leftarrow f(j | X^t); \]
\[ \overline{g}(j | X^t) \leftarrow g(j | X^t); \]
end
\[ j^t \leftarrow \arg \max_{j \in C} \overline{f}(j | X^t); \]
\[ X^{t+1} \leftarrow X^t \cup \overline{\{j^t\}}; \]
\[ \overline{f}(i | X^{t+1}) = \overline{f}(i | X^t), \overline{g}(i | X^{t+1}) = \overline{g}(i | X^t), \]
\[ g(i | X^{t+1}) = \max(0, g(i | X^t) - g(j^t | X^t)), \]
\[ \overline{f}(i | X^{t+1}) = \max(0, \overline{f}(i | X^t) - f(j^t | X^t)) \text{ for every } \]
\[ t \leftarrow t + 1; \]
end
end
end

4.2 Iterative Submodular Knapsack

Iterative Submodular Knapsack is proposed by Iyier and Bilmes [14]. Inspired by the minorization-maximization procedure from Iyier et al. [13], \( g(.) \) is iteratively approximated by a modular upper bound \( \overline{g}^{t+1}(\cdot) \), which is exact at the current solution \( X_t \); i.e., \( g(X_t) = \overline{g}^{t+1}(X_t) \), and \( g(X) \leq \overline{g}^t(X) \) for every \( X \) and \( t \). They suggest two choices for the bound:

\[ \overline{g}_1^{t+1}(X) := g(X_t) - \sum_{j \in X^t \setminus X} g(j | X_t \setminus j) + \sum_{j \in X \setminus X_t} g(j), \]
\[ \overline{g}_2^{t+1}(X) := g(X_t) - \sum_{j \in X \setminus X_t} g(j | X \setminus j) + \sum_{j \in X \setminus X_t} g(j | X_t). \]  

5 EXPERIMENTS

We empirically evaluate the effectiveness of methods we propose in this paper on real-world data. We use a corpus of about 8 million documents in a particular category of a commercial search engine for \( D \), and we collected about 2 million queries uniformly sampled over three days for the training data \( Q_n \). We additionally sampled about 700,000 queries in the following day for the test data.

We implemented every algorithm in Java, using its standard library for multi-threading. We leveraged multi-threading in most of straightforward opportunities for parallelization, most importantly...
the computation of marginal gains and losses. In order to efficiently compute set operations, we leveraged utilities provided by Apache Lucene\(^2\). We also adopted techniques suggested in Iyer and Bilmes [12] for incrementally computing marginal gains and losses of submodular functions. All experiments were ran on a machine with 16 Intel Xeon 2.50GHz CPUs and 64GBs of RAM.

5.1 Submodular Optimization Algorithms

In this experiment, we focus on evaluating the performance of submodular optimization algorithms we proposed in Section 4 for the SCSK problem. We set \( B = \frac{|D|}{2} \), so that Tier 1 shall choose up to 50% documents of the whole corpus. We consider following algorithms:

**Constraint-Agnostic Greedy**: Greedy algorithm proposed in Iyer and Bilmes [14] for solving SCSK, which ignores the constraint function when comparing candidates. We implemented a lazy version of the algorithm [18].

**Greedy**: Greedily selects clauses according to the procedure (13) we propose. Gains \( f(j \mid X^t) \) and \( g(j \mid X^t) \) are re-computed at every iteration.

**Lazy Greedy**: Algorithm 1, which uses the same greedy procedure (13) but improves efficiency by lazily evaluating candidates with max heap and optimistic estimates.

**Opt./Pes. Greedy**: Algorithm 2, which recomputes gains of clauses which optimistic estimate is better than the best pessimistic estimate.

**ISK**: Algorithm 3 from 4.2. We use two variants ISK\(_1\) and ISK\(_2\), using each of the modular upper bound in (15), respectively.

Figure 2 shows the value of the objective function \( f(X) \) with respect to the elapsed wall clock time. It is noticeable that ISK algorithms are much faster than most of greedy algorithms. This is because first few iterations of ISK are much more effective than those of greedy ones; the very first iteration of ISK (16) adds 28% documents, whereas the greedy algorithm needs to iterate hundreds of thousands of times to add the same number of documents. While each iteration of ISK involves solving a new submodular knapsack problem from scratch, the lazy evaluation technique [18] seems to be more efficient with these submodular knapsack sub-problems than with the original SCSK because only one function \( f(\cdot) \) is being approximated in submodular knapsack, whereas we simultaneously bound two functions \( f(\cdot) \) and \( g(\cdot) \) in SCSK. On the other hand, the final objective function value of the greedy algorithm was 7.6% and 0.6% higher than that of ISK\(_1\) and ISK\(_2\), respectively. Therefore, the greedy algorithm seems to be more effective at refining the high-quality solution than ISK is, as each iteration of ISK relies on a rough approximation of \( g(\cdot) \).

Also note that the greedy algorithm has an advantage of finding the entire solution path for different values of the capacity parameter up to \( B \), as any intermediate solution \( X^t \) can be considered as the final solution of the problem (10) with the parameter \( B \) set as \( g(X^t) \). This is useful when the suitable size of Tier 1 is not given a priori, and we need to search for the optimal configuration of \( B \). Figure 3 shows the solution path of different algorithms. ISK

\(^2\)https://lucene.apache.org/
Algorithms have very few intermediate solutions which can be useful for determining B, whereas the greedy algorithm has a very continuous solution path.

While Constraint-Agnostic Greedy algorithm is much faster than other greedy algorithms because it ignores the constraint in the selection process, it converges to a clearly suboptimal solution for the same reason. Therefore, the greedy procedure we propose (13) seems to be more effective than Constraint-Agnostic Greedy from Iyer and Bilmes [14]. Among algorithms which implement the proposed greedy procedure, Opt./Pes. Greedy is the fastest because it leverages both parallelization and lazy evaluation. This is confirmed in Figure 4: when smaller number of CPUs are used, the efficiency gap between Lazy Greedy and Opt./Pes. Greedy becomes narrower.

5.2 Tiering Performance

We compare the performance of tiering methods with respect to the fraction of queries covered by Tier 1, which index is half the size of the full index (B = \( \frac{|D|}{2} \)). We consider following methods:

- **popularity** An intuitive baseline from Leung et al. [17] which selects B most frequently appearing documents in the training data. That is, the score for each document \( d \) is defined as \( \mathbb{P}_{q\sim Q_0} [d \in m(q)] \), and top B documents with respect to this score are chosen as \( D_B \). Then, queries in \( Q_B \) which match set is contained in these B documents are set as \( \mathcal{X}^{flow} \).

- **flow-max** The score for each document \( d \) is defined as
  \[
  \max_{q \in Q_0, d \in m(q)} \mathbb{P}_{q\sim Q_0} [q = q'],
  \]
  which is the maximum probability of the query which match set includes the document. Top B documents according to this score are chosen, and again queries in \( Q_0 \) which match set is contained in these B documents are set as \( \mathcal{X}^{flow} \). This rule is derived from the subgradient of the \( \mathcal{L}^{flow} \)'s objective function. In Leung et al. [17], this simple heuristic was very competitive to principled optimization algorithms.

- **flow-sgd** Convex relaxation of (5), which is a maximum flow problem, is optimized with stochastic gradient descent. In order to provide regularization to this method, we introduced a hyperparameter \( \lambda \), and removed any query which frequency is lower than the parameter, i.e., any \( q \in Q_0 \) with \( \mathbb{P}_{q\sim Q_0} [q = q'] < \lambda \) are removed from the training data, so that the method does not overfit to rare queries.

- **clause** The clause selection-based query and document classifiers we proposed in Section 3.1 optimized with Opt./Pes. Greedy.

Figure 5 simultaneously compares each method’s fit to the training data and their generalization to the future traffic. Simple heuristics popularity and flow-max had a very poor fit to the training data, although they were very competitive in Leung et al. [17]. This is probably because they only considered top few documents per query, whereas we aim to cover the entire match set \( m(q) \); the larger the number of documents per query \( |m(q)| \), the less likely it is for the heuristically chosen set of documents \( D_B \) to cover the entire match set. Since flow-sgd is a principled optimization algorithm, it is successful at enforcing the correctness of query classification, and achieves as high coverage in the training data as that of the clause method we propose. However, the generalization performance of flow-sgd was poorer than clause, and exploring different values of the regularization parameter did not help. This demonstrates the effectiveness of our regularized empirical risk minimization approach for optimizing the generalization performance.

6 CONCLUSION AND FUTURE WORK

We demonstrated that the clause selection-based tiering method is more effective than the query selection-based method in terms of the generalization performance to the future traffic. We proposed
multiple algorithms for optimizing the configuration of the tiered architecture, which can also be used for general submodular maximization problems with a submodular upper bound constraint. We showed that these algorithms can scale to practical problems by evaluating on real-world data.

In the future, it would be of interest to study how the proposed method can be generalized for an arbitrary number of tiers. Theoretical studies of the algorithms we proposed would also be insightful. Indeed, it is quite surprising that very different discrete optimization algorithms are converging to similar solutions; there is probably a structure in the problem stronger than SCSk, which helps algorithms to avoid being stuck in bad local minima.

REFERENCES

[1] Amr Ahmed, Choon Hui Teo, SVN Vishwanathan, and Alex Smola. 2012. Fair and balanced: Learning to present news stories. In Proceedings of the fifth ACM international conference on Web search and data mining. 333–342.

[2] Aris Anagnostopoulos, Luca Becchetti, Ilaria Bordino, Stefano Leonardi, Ida Mele, and Pietro Sankowski. 2015. Stochastic query covering for fast approximate document retrieval. ACM Transactions on Information Systems (TOIS) 33, 3 (2015), 1–35.

[3] Ricardo Baeza-Yates, Aritides Gionis, Flavio Junqueira, Vanessa Murdock, Vasilis Plachouras, and Fabrizio Silvestri. 2007. The impact of caching on search engines. In Proceedings of the 30th annual international ACM SIGIR conference on Research and development in information retrieval. ACM, 183–190.

[4] Ricardo Baeza-Yates, Vanessa Murdock, and Claudia Hauff. 2009. Efficiency trade-offs in two-tier web search systems. In Proceedings of the 32nd international ACM SIGIR conference on Research and development in information retrieval. ACM, 163–170.

[5] Wenruo Bai, Rishabh Iyer, Kai Wei, and Jeff Bilmes. 2016. Algorithms for optimizing the ratio of submodular functions. In International Conference on Machine Learning. 2751–2759.

[6] Luiz André Barroso, Jeffrey Dean, and Urs Hölze. 2003. Web search for a planet: The Google cluster architecture. IEEE micro 2 (2003), 22–28.

[7] David Carmel, Doron Cohen, Ronald Fagin, Eitan Farchi, Michael Herscovici, Yoelle S Maarek, and Aya Soffer. 2001. Static index pruning for information retrieval systems. In Proceedings of the 24th annual international ACM SIGIR conference on Research and development in information retrieval. ACM, 43–50.

[8] Moses Charikar, Piotr Indyk, and Rina Panigrahy. 2002. New algorithms for subset query, partial match, orthogonal range searching, and related problems. In International Colloquium on Automata, Languages, and Programming. Springer, 451–462.

[9] Satoru Fujishige. 2005. Submodular functions and optimization. Elsevier.

[10] Bob Goodwin, Michael Hopcroft, Dan Luu, Alex Clemmer, Mihaiuca Curmeta, Sameh Elhakety, and Yuxiong He. 2017. BitFunnel: Revisiting signatures for search. In Proceedings of the 40th International ACM SIGIR Conference on Research and Development in Information Retrieval. ACM, 605–614.

[11] Jure Leskovec, Andreas Krause, Carlos Guestrin, Christos Faloutsos, Jeanne VanBriesen, and Natalie Glance. 2007. Cost-effective outbreak detection in networks. In Proceedings of the 30th ACM SIGKDD international conference on Knowledge discovery and data mining. ACM, 420–429.

[12] Gilbert Leung, Novi Quadrianto, Kostas Tsotsouliakis, and Alex J Smola. 2010. Optimal web-scale tiering as a flow problem. In Advances in Neural Information Processing Systems. 1333–1341.

[13] Michel Minoux. 1978. Accelerated greedy algorithms for maximizing submodular set functions. In Optimization techniques. Springer, 234–243.

[14] George Nemhauser, Laurence A Wolsey, and Marshall L Fisher. 1978. An analysis of approximations for maximizing submodular set functions. At. Mathematical programming 14, 1 (1978), 265–294.

[15] Mohammad Karimi, Mario Lucic, Hamed Hassani, and Andreas Krause. 2017. Stochastic submodular maximization: The case of coverage functions. In Advances in Neural Information Processing Systems. 2436–2444.

[16] Jure Leskovec, Andreas Krause, Carlos Guestrin, Christos Faloutsos, Jeanne VanBriesen, and Natalie Glance. 2007. Cost-effective outbreak detection in networks. In Proceedings of the 13th ACM SIGKDD international conference on Knowledge discovery and data mining. ACM, 420–429.

[17] Knut Magne Rishvik, Yuje Aasheim, and Mathias Lidal. 2003. Multi-Tier Architecture for Web Search Engines. In LA-WE, Vol. 3. 132.

[18] Maxim Sviridenko. 2004. A note on maximizing a submodular set function subject to a knapsack constraint. Operations Research Letters 32, 1 (2004), 41–45.

[19] Vladimir Vapnik. 1992. Principles of risk minimization for learning theory. In Advances in neural information processing systems. 831–838.

[20] Alexandros Ntoulas and Junghoo Cho. 2007. Pruning policies for two-tiered inverted index with correctness guarantee. In Proceedings of the 30th annual international ACM SIGIR conference on Research and development in information retrieval. 191–198.

[21] Bob Goodwin, Michael Hopcroft, Dan Luu, Alex Clemmer, Mihaela Curmei, and Chris Anderson. 2013. Maguro, a system for indexing and searching over very large text collections. In Proceedings of the sixth ACM international conference on Web search and data mining. ACM, 727–736.

[22] Intok Savin. 2013. Index data structure for fast subset and superset queries. In International Conference on Availability, Reliability, and Security. Springer, 134–148.

[23] Maxim Sviridenko. 2004. A note on maximizing a submodular set function subject to a knapsack constraint. Operations Research Letters 32, 1 (2004), 41–45.

[24] Vladimir Vapnik. 1992. Principles of risk minimization for learning theory. In Advances in neural information processing systems. 831–838.

[25] Kai Wei, Rishabh K Iyer, Shengjie Wang, Wenruo Bai, and Jeff A Bilmes. 2015. Mixed robust/average submodular partitioning: Fast algorithms, guarantees, and applications. In Advances in Neural Information Processing Systems. 2233–2241.

[26] Laurence A Wolsey. 1982. An analysis of the greedy algorithm for the submodular set covering problem. Combinatorica 2, 4 (1982), 385–393.