Simultaneous Code/Error-Trellis Reduction for Convolutional Codes Using Shifted Code/Error-Subsequences

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Abstract—In this paper, we show that the code-trellis and the error-trellis for a convolutional code can be reduced simultaneously, if reduction is possible. Assume that the error-trellis can be reduced using shifted error-subsequences. In this case, if the identical shifts occur in the subsequences of each code path, then the code-trellis can also be reduced. First, we obtain pairs of transformations which generate the identical shifts both in the subsequences of the code-path and in those of the error-path. Next, by applying these transformations to the generator matrix and the parity-check matrix, we show that reduction of these matrices is accomplished simultaneously, if it is possible. Moreover, it is shown that the two associated trellises are also reduced simultaneously.

I. INTRODUCTION

In this paper, we always assume that the underlying field is $F = GF(2)$. Let $G(D)$ and $H(D)$ be the generator matrix and the parity-check matrix of an $(n, n - m)$ convolutional code $C$, respectively. Ariel and Snyders [1] presented a construction of error-trellises based on the scalar check matrix derived from $H(D)$. They showed that when some $(j)$th “column” of $H(D)$ has a factor $D^l$, there is a possibility that state-space reduction can be realized. Being motivated by their work, we also examined the same case. The time-$k$ error $e_k = (e_k(1), \ldots, e_k(n))$ and syndrome $\zeta_k = (\zeta_k(1), \ldots, \zeta_k(m))$ are connected with the relation $\zeta_k = e_kH^T(D)$ ($T$ means transpose). From this relation, we noticed [9] that the transformation $e_k^{(j)} \rightarrow D^l e_k^{(j)} = e_k^{(j)}$ is equivalent to dividing the $j$th column of $H(D)$ by $D^l$. That is, reduction can be realized by shifting the “subsequence” $\{e_k^{(j)}\}$ of the original error-path $e$. It is stated [1] that their construction can be used also to obtain code-trellises. However, it is not described in the paper. On the other hand, our construction is based on an equivalent modification of the relation $\zeta_k = e_kH^T(D)$. Hence, our method can be directly extended to code-trellises. That is, in the case of code-trellises, the construction is based on the relation $y_k = u_kG(D)$ and its equivalent modifications, where $u_k$ and $y_k$ are the time-$k$ information and code symbols, respectively. Note that there exists a one-to-one correspondence between the code-paths in a code-trellis and the error-paths in the corresponding error-trellis. Accordingly, it is reasonable to think that the two trellises can be reduced simultaneously, if reduction is possible. Here, consider the situation that the identical shifts occur both in the components of $y_k$ and in those of $e_k$. In this case, if one trellis is reduced, then the other trellis should be equally reduced. In this paper, based on this idea, we discuss the simultaneous reduction of a code-trellis and the corresponding error-trellis. First, we obtain the general transformations which generate the identical shifts both in the subsequences of $y$ and in those of $e$. Next, we show that these transformations preserve the relation that one is a generator matrix and the other is the corresponding parity-check matrix. (In this paper, we call this relation the “GH Relation” and if $G(D)$ and $H(D)$ have this relation, then it is denoted as $G(D) \Leftrightarrow H(D)$). Using this property, it is shown that $G(D)$ and $H(D)$ are reduced simultaneously, if reduction is possible. Moreover, it is shown that the corresponding two trellises are also reduced simultaneously. These results again imply that a code/error-trellis construction using shifted code/error-subsequences is very effective.

II. TRELLIS CONSTRUCTION USING SHIFTED PATH-SUBSEQUENCES

A. Error-trellis construction using shifted error-subsequences

Let $H(D)$ be the parity-check matrix for an $(n, n - m)$ convolutional code $C$. Consider the error-trellis based on the syndrome former $H^T(D)$. In this case, the adjoint-obvious realization of $H^T(D)$ is assumed unless otherwise specified. Assume that the $j$th column of $H(D)$ has the form

$$
(D^l h_1^{(j)}(D) \quad D^l h_2^{(j)}(D) \quad \ldots \quad D^l h_m^{(j)}(D))^T,
$$

where $l_j \geq 1$. Let $H'(D)$ be the modified version of $H(D)$ with the $j$th column being replaced by

$$
(h_1^{(j)}(D) \quad h_2^{(j)}(D) \quad \ldots \quad h_m^{(j)}(D))^T.
$$

(2)

Also, let $e_k^{(j)} \triangleq (e_k(1), \ldots, e_k(j), \ldots, e_k(n))$, where $e_k^{(j)} \triangleq D^l e_k^{(j)} = e_k^{(j)}$. Then we have

$$
\zeta_k = e_k' H'^T(D).
$$

(3)
Hence, in the case where the $j^{th}$ column of $H(D)$ has a factor $D^{j}$, there is a possibility that an error-trellis with reduced number of states can be constructed by shifting the $j^{th}$ error-subsequence by $l_{j}$ time units [9]. Assume that the corresponding code-trellis is terminated in the all-zero state at $t = N$. Then $e_{k}^{(j)} = e_{k-l_{j}}^{(j)}$ is modified as $e_{k-l_{j}}^{(j)} = e_{k-l_{j}+N}^{(j)}$, where $< t >$ denotes $t \mod (N + l_{j})$ (i.e., “cyclic shift”).

B. Error-trellis construction using backward-shifted error-subsequences

The construction using shifted error-subsequences is further extended [9], [10]. That is, a reduced error-trellis can be equally constructed using “backward-shifted” error-subsequences. Consider the transformation $e_{k}^{(j)} \rightarrow D^{-l_{j}}e_{k}^{(j)} = e_{k+l_{j}}^{(j)}$, We see this is equivalent to “multiplying” the $j^{th}$ column of $H(D)$ by $D^{j}$. Let $H'(D)$ be the parity-check matrix after modification. If $H'(D)$ is reduced to an equivalent $H''(D)$ with overall constraint length less than that of $H(D)$, then reduction can be realized. We remark that the power $l_{j}$ of $D$ has to be determined properly for each $j$. For the purpose, we can use the reciprocal dual encoder [6] $H(D)$ associated with $H(D)$.

Example 1 ([9]): Consider the canonical parity-check matrix

$$H_{1}(D) = \begin{pmatrix} D & D & 0 \\ 1 & 1 + D & 0 \\ D & D & 1 \end{pmatrix}. \quad (4)$$

Since all the columns of $H_{1}(D)$ are delay free, any further reduction seems to be impossible. In fact, it follows from Theorem 1 of [1] that the dimension $d_{1}$ of the state space of the error-trellis based on $H_{1}^{T}(D)$ is 4. However, a corresponding generator matrix is given by $G_{1}(D) = (1 + D + D^{2}, 1, D^{3} + D^{4})$. Observe that the third “column” of $G_{1}(D)$ has a factor $D^{2}$. (Remark: It suffices to divide the third column by $D^{2}$ in order to obtain a reduced code-trellis.) This fact implies that a reduced error-trellis can be constructed [1], [9]. Then consider the reciprocal dual encoder

$$\hat{H}_{1}(D) = \begin{pmatrix} 1 \\ D^{2} & 1 + D & D^{2} \\ D & D & 0 \end{pmatrix}. \quad (5)$$

Note that the third column of $\hat{H}_{1}(D)$ has a factor $D^{2}$. Accordingly, dividing the third column of $\hat{H}_{1}(D)$ by $D^{2}$, we can construct an error-trellis with 4 states (i.e., $\bar{d}_{1} = 2$) [1], [9]. Here, notice that each error-path in the error-trellis based on $H_{1}^{T}(D)$ can be represented in time-reversed order using the error-trellis based on $H_{2}^{T}(D)$. Hence, a factor $D^{2}$ in the column of $H_{1}(D)$ corresponds to backward-shifting by two time units (i.e., $D^{-2}$) in terms of the original $H_{1}(D)$. Hence, multiply the third column $H_{1}(D)$ by $D^{2}$. Then we have

$$H_{1}'(D) = \begin{pmatrix} D^{2} & D^{2} & D^{2} \\ 1 & 1 + D & D^{2} \\ D^{2} & D^{2} & D^{2} \end{pmatrix}. \quad (6)$$

We see that this matrix can be reduced to an equivalent canonical parity-check matrix

$$H_{1}''(D) = \begin{pmatrix} 1 & 1 \\ 1 & 1 + D & D^{2} \\ 1 & 1 \end{pmatrix}. \quad (7)$$

by dividing the first “row” by $D^{2}$. Hence, the dimension $d_{1}$ can be reduced to 2.

C. Code-trellis construction using shifted code-subsequences

Note that the relation $y_{k} = u_{k}G(D)$ holds with respect to a generator matrix $G(D)$, where $u_{k} = (u_{k}^{(1)}, \cdots, u_{k}^{(n-m)})$ and $y_{k} = (y_{k}^{(1)}, \cdots, y_{k}^{(m)})$ are the time-regular and code symbols, respectively. In the same way as for $H(D)$, by dividing the $j^{th}$ column of $G(D)$ by $D^{j}$ or by multiplying the $j^{th}$ column of $G(D)$ by $D^{j}$, reduction of $G(D)$ can be realized. We see that the former corresponds to the backward-shift $y_{k}^{(j)} \rightarrow y_{k+j}^{(j)}$, whereas the latter corresponds to the forward-shift $y_{k}^{(j)} \rightarrow y_{k-j}^{(j)}$. Note that the shift directions are reversed compared to $H(D)$.

III. TRANSFORMATIONS GENERATING THE IDENTICAL SHIFTS BOTH IN $y$ AND IN $e$

A. General case

Consider the transformations which generate the identical shifts both in the components of $y_{k}$ and in those of $e_{k}$. Now, assume that the relation $G(D) \Leftrightarrow H(D)$ holds. Consider a pair of transformations:

1) divide the $j^{th}$ column of $G(D)$ by $D^{j}$ and multiply the same column by $D^{j}$.
2) divide the $j^{th}$ column of $H(D)$ by $D^{j}$ and multiply the same column by $D^{j}$.

Then

1) the $j^{th}$ component of $y_{k}$ becomes

$$y_{k}^{(j)} \rightarrow y_{k+j}^{(j)} \rightarrow e_{k}^{(j)} \rightarrow e_{k-l_{j}}^{(j)}, \quad (8)$$

2) the $j^{th}$ component of $e_{k}$ becomes

$$e_{k}^{(j)} \rightarrow e_{k-l_{j}}^{(j)} \rightarrow y_{k-l_{j}+D^{j}}^{(j)}, \quad (9)$$

After shifting $e_{k-l_{j}}^{(j)} \rightarrow y_{k-l_{j}+D^{j}}^{(j)}$ by $l$ time units ($l$ is independent of $j$), compare the time-index of $e_{k-l_{j}}^{(j)}$ and that of $y_{k-l_{j}+D^{j}}^{(j)}$. If the two time-indices coincide, then $y_{k}^{(j)}$ and $e_{k}^{(j)}$ have “relatively” the identical shift. This condition is written as

$$l = (l_{j}^{(d)} + \tilde{l}_{j}^{(d)}) - (l_{j}^{(m)} + \tilde{l}_{j}^{(m)}) \quad (1 \leq j \leq n), \quad (10)$$

where $l$ is a constant independent of $j$ ($1 \leq j \leq n$). (In the following, this condition is denoted as “CST”).

B. Special cases

Case 1: Only division is applied both to the columns of $G(D)$ and to those of $H(D)$.

From the assumption, $l_{j}^{(m)} = \tilde{l}_{j}^{(m)} = 0$. Hence, we have

$$l = l_{j}^{(d)} + \tilde{l}_{j}^{(d)}. \quad (11)$$
Here, assume that either \( l_j^{(d)} \) or \( l_j^{(m)} \) is 0. Define the sets \( L_G \) and \( L_H \) as

\[
L_G \triangleq \{ j : l_j^{(d)} = l \} = \{ j : l_j^{(d)} = 0 \} \quad \text{(12)}
\]

\[
L_H \triangleq \{ j : l_j^{(m)} = l \} = \{ j : l_j^{(m)} = 0 \}. \quad \text{(13)}
\]

In words, \( L_G \) is the set of columns of \( G(D) \) from which \( D^l \) is factoring out, whereas \( L_H \) is the set of columns of \( H(D) \) from which \( D^l \) is factoring out. Note that \( L_G \) and \( L_H \) are disjoint and the relation

\[
L_G \cup L_H = \{1, 2, \ldots, n\} \quad \text{(14)}
\]

holds. In the following, we call this kind of transformations “type-1”.

**Example 2:** Consider the relation

\[
G_2(D) = (D + D^2, D^2, 1 + D) \quad \Leftrightarrow \quad H_2(D) = \begin{pmatrix} 1 & 0 & D \\ D & 1 + D & 0 \end{pmatrix}. \quad \text{(15)}
\]

Choosing \( l = 1 \), \( L_G = \{1, 2\} \), and \( L_H = \{3\} \), we have

\[
G_2'(D) = (1 + D, D, 1 + D) \quad \Leftrightarrow \quad H_2'(D) = \begin{pmatrix} 1 & 0 & 1 \\ D & 1 + D & 0 \end{pmatrix}. \quad \text{(16)}
\]

**Case 2:** Division and multiplication are separately applied either to the columns of \( G(D) \) or to the columns of \( H(D) \).

Without loss of generality, assume that division is applied to the columns of \( G(D) \), whereas multiplication is applied to the columns of \( H(D) \). From the assumption, \( l_j^{(m)} = l_j^{(d)} = 0 \). Hence, we have

\[
l = l_j^{(d)} - l_j^{(m)}. \quad \text{(17)}
\]

In particular, set \( l = 0 \). Then we have

\[
l_j^{(d)} = l_j^{(m)} \quad (\triangleq l_j). \quad \text{(18)}
\]

This is equivalent to dividing the \( j \)th column of \( G(D) \) by \( D^l_j \) and multiplying the \( j \)th column of \( H(D) \) by \( D^l_j \). In the following, we call this kind of transformations “type-2”.

**Example 3:** Consider the relation

\[
G_3(D) = (1 + D, 1, D + D^2) \quad \Leftrightarrow \quad H_3(D) = \begin{pmatrix} D & 0 & 1 \\ 1 & 1 + D & 0 \end{pmatrix}. \quad \text{(19)}
\]

Choosing \( l_3^{(d)} = l_3^{(m)} = 1 \), we have

\[
G_3'(D) = (1 + D, 1, 1 + D) \quad \Leftrightarrow \quad H_3'(D) = \begin{pmatrix} D & 0 & D \\ 1 & 1 + D & 0 \end{pmatrix}. \quad \text{(20)}
\]

Note that \( H_3'(D) \) can be reduced to

\[
H_3''(D) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 + D & 0 \end{pmatrix}. \quad \text{(21)}
\]

C. Property of transformations

Observe that in Example 2 and Example 3, the GH Relation is preserved after type-1 and type-2 transformations. It is shown that this property holds in general. Assume that the relation \( G(D) \Leftrightarrow H(D) \) holds. Also, assume that a pair of transformations which satisfies the condition \( C_{SR} \) is applied to \( G(D) \) and \( H(D) \). Let \( G'(D) \) and \( H'(D) \) be the resulting matrices, respectively. Then we have the following.

**Proposition 1:** The relation \( G'(D) \Leftrightarrow H'(D) \) holds.

**Proof:** Fix \( p, q \) \((1 \leq p \leq n-m, 1 \leq q \leq m)\) arbitrarily. Let

\[
(g_{p1}(D), \ldots, g_{pj}(D), \ldots, g_{pm}(D)) \quad \text{(22)}
\]

be the \( p \)th row of \( G(D) \). Then the \((p, j)\) element of \( G'(D) \) is given by

\[
g_{pj}(D) \frac{D'}{D_j^{(m)}}. \quad \text{(23)}
\]

Similarly, defining the \( q \)th row of \( H(D) \) as

\[
(h_{q1}(D), \ldots, h_{qj}(D), \ldots, h_{qn}(D)), \quad \text{(24)}
\]

the \((q, j)\) element of \( H'(D) \) is given by

\[
h_{qj}(D) \frac{D_j^{(m)}}{D_j^{(d)}}. \quad \text{(25)}
\]

Then the \((p, q)\) element \( h'_{pq} \) of \( G'(D)H'^T(D) \) is given by

\[
h'_{pq} = \sum_{j=1}^{n} g_{pj}(D)D_j^{(m)}h_{qj}(D) \frac{D_j^{(m)}}{D_j^{(d)}}
= \sum_{j=1}^{n} g_{pj}(D)h_{qj}(D)D_j^{(m)} + (l_j^{(m)} + l_j^{(d)}) - (l_j^{(d)} + l_j^{(d)}) = \frac{1}{D} \sum_{j=1}^{n} g_{pj}(D)h_{qj}(D). \quad \text{(26)}
\]

Since \( G(D) \Leftrightarrow H(D) \), \( \sum_{j=1}^{n} g_{pj}(D)h_{qj}(D) = 0 \). Hence, we have \( h'_{pq} = 0 \).

IV. Simultaneous reduction of \( G(D) \) and \( H(D) \)

The discussion in the previous section implies that \( G(D) \) and \( H(D) \) can be reduced simultaneously, if reduction is possible. Assume that the relation \( G(D) \Leftrightarrow H(D) \) holds. Let \( \nu \) and \( \nu' \) be the overall constraint lengths of \( G(D) \) and \( H(D) \), respectively. If both \( G(D) \) and \( H(D) \) are canonical \([4],[5]\), then we have \( \nu = \nu' \). Here, apply a pair of transformations which satisfies the condition \( C_{SR} \) to \( G(D) \) and \( H(D) \). Denote by \( \nu' \) and \( \nu'^\perp \) the overall constraint lengths of the modified matrices \( G'(D) \) and \( H'(D) \), respectively. Note that the relation \( G'(D) \Leftrightarrow H'(D) \) still holds from Proposition 1. Hence, if necessary, by modifying equivalently, we have \( \nu' = \nu'^\perp \). Therefore, if the strict inequality \( \nu' < \nu \) \((\nu'^\perp < \nu'^\perp)\) holds, then \( G(D) \) and \( H(D) \) are reduced simultaneously. That is, we have the following.

**Proposition 2:** Assume that the relation \( G(D) \Leftrightarrow H(D) \) holds. Also, assume that a pair of transformations which
satisfies the condition $C_{SR}$ is applied to $G(D)$ and $H(D)$. In this case, if $G(D)$ is reduced, then $H(D)$ is equally reduced, and vice versa.

**Example 4:** Assume that

$$G_4(D) = (1 + D + D^2, D, D^4 + D^5)$$

$$\iff H_4(D) = \begin{pmatrix} D^3 & D^2 \\ D & 1 + D + D^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (27)

Note that both $G_4(D)$ and $H_4(D)$ are canonical and the equality $\nu = \nu' = 5$ holds. Choosing $l = 1$, $L_G = \{2, 3\}$, and $L_H = \{1\}$, let us apply a type-1 transformation. Then we have

$$G'_4(D) = (1 + D + D^2, 1, D^3 + D^4)$$

$$\iff H'_4(D) = \begin{pmatrix} D^2 & D^2 \\ 1 & 1 + D + D^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (28)

Also, let us apply a type-2 transformation with $l_3^{(d)} = l_3^{(m)} = 3$. Then we have

$$G''_4(D) = (1 + D + D^2, 1, D + D^2)$$

$$\iff H''_4(D) = \begin{pmatrix} D^2 & D^2 \\ 1 & 1 + D + D^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (29)

Since $H''_4(D)$ is reduced to

$$H''_4(D) = \begin{pmatrix} 1 & 0 \\ 1 & 1 + D + D^2 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

we finally have

$$G_4(D) = (1 + D + D^2, 1, D + D^2)$$

$$\iff H''_4(D) = \begin{pmatrix} 1 & 0 \\ 1 & 1 + D + D^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (30)

In this example, the overall constraint lengths are reduced from $\nu = \nu' = 5$ to $\nu'' = \nu''' = 2$.

**Remark:** The reduction process is not unique. In the above example, if a type-2 transformation is applied to $G_4(D)$ and $H_4(D)$ with $l_3^{(d)} = l_3^{(m)} = 3$, then we have

$$G''_4(D) = (1 + D + D^2, 2, D + D^2)$$

$$\iff H''_4(D) = \begin{pmatrix} 2 & 1 \\ 2 & 1 + D + D^2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (31)

where “$\simeq$” means equivalent. Here, choosing $l = 1$, $L_G = \{2\}$, and $L_H = \{1, 3\}$, let us apply a type-1 transformation. Then we have $G''_4(D) \iff H''_4(D)$.  

**V. Simultaneous code/error-trellis reduction**

Assume that the relation $G(D) \iff H(D)$ holds. Let $T_c$ be the code-trellis associated with $G(D)$. It is assumed that $T_c$ is terminated in the all-zero state at $t = N$. Denote by $T_e$ the corresponding error-trellis. Note that each code-path $y$ in $T_c$ corresponds to the unique error-path $e$ in $T_e$ by way of the received data $z$. Here, apply a pair of transformations which satisfies the condition $C_{SR}$ to $G(D)$ and $H(D)$. (Let $G'(D)$ and $H'(D)$ be the resulting matrices.) Then from Proposition 2, it is reasonable to think that $T_c$ and $T_e$ are reduced simultaneously. In fact, we have the following.

**Proposition 3:** Assume that a pair of transformations which satisfies the condition $C_{SR}$ is applied to $G(D)$ and $H(D)$. In this case, if the code-trellis associated with $G(D)$ is reduced, then the error-trellis based on $H^T(D)$ is equally reduced, and vice versa.

**Proof:** Denote by $e'$ the shifted version of $e$. Assume that the set of shifted error-paths $\{e'\}$ is represented using the reduced error-trellis $T'_e$ based on $H'^T(D)$. Note that there exists a one-to-one correspondence between the code-paths $\{y\}$ and the error-paths $\{e\}$. Also, from the assumption of the transformations, the identical shifts are generated both in the subsequences of a code-path $y$ and in those of the corresponding error-path $e$. Hence, the set of shifted code-paths $\{y'\}$ is also represented using the reduced code-trellis $T'_c$ associated with $G'(D)$. That is, if one trellis is reduced, then the other trellis is equally reduced.

**Example 5:** Consider the relation $G_2(D) \iff H_2(D)$. Fig.1 shows the code-trellis associated with $G_2(D)$. Note that the trellis is terminated in the all-zero state $(00)$ at $t = 4$. The corresponding error-trellis based on $H^T_2(D)$ is shown in Fig.2. A received data $z$ is assumed to be

$$z = z_1 z_2 z_3 z_4 z_5 = 001 000 011 010 000,$$  \hspace{1cm} (33)
where \( z_5 = 000 \) is the “imaginary” received data at \( t = 5 \).

The syndrome sequence is given as

\[
\zeta = \zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_5 = 0010101001.
\]  

(34)

As we have already seen in Example 2, if the first and second components of \( y_k \) and \( e_k \) are shifted left by the unit time and if the third component of \( e_k \) is shifted right by the unit time, then \( G_2(D) \) and \( H_2(D) \) are reduced simultaneously. Denote by \( G'_2(D) \) and \( H'_2(D) \) the modified generator and parity-check matrices after transformation, respectively. The corresponding code and error-trellises are shown in Fig.3 and Fig.4, respectively.

First, consider the reduced error-trellis in Fig.4. In this example, it is defined as \( e_k^{(3)} = e_{k-1}^{(3)} \), where \( t \) denotes \( t \mod 5 \). Since \( e_5 = 000 \), we have \( e_1^{(3)} = e_4^{(3)} = e_5^{(3)} = 0 \), and in the components of \( e_k \), then the two trellises are reduced simultaneously, if reduction is possible. We have obtained the general transformations which generate the identical shifts both in the subsequences of \( y \) and in those of \( e \). We have shown that these transformations preserve the GH Relation. Using this property, we have shown that reduction of \( G(D) \) and \( H(D) \) is accomplished simultaneously, if it is possible. Moreover, we have shown that the corresponding two trellises are also reduced simultaneously. These results again imply that a code/error-trellis construction using shifted code/error-subsequences is very effective. We remark that a parity-check matrix with the form described in the paper appears in [11] in connection with a class of LDPC convolutional codes. We think [10] that the proposed method is useful for reducing the state complexity of the code/error-trellis for such an LDPC convolutional code.

VI. Conclusion

We have shown that the code-trellis and the error-trellis for a convolutional code can be reduced simultaneously. The proposed method is based on the fact that if the identical shifts occur both in the components of \( y_k \) and in the components of \( e_k \), then the two trellises are reduced simultaneously, if reduction is possible. We have obtained the general transformations which generate the identical shifts both in the subsequences of \( y \) and in those of \( e \). We have shown that these transformations preserve the GH Relation. Using this property, we have shown that reduction of \( G(D) \) and \( H(D) \) is accomplished simultaneously, if it is possible. Moreover, we have shown that the corresponding two trellises are also reduced simultaneously. These results again imply that a code/error-trellis construction using shifted code/error-subsequences is very effective. We remark that a parity-check matrix with the form described in the paper appears in [11] in connection with a class of LDPC convolutional codes. We think [10] that the proposed method is useful for reducing the state complexity of the code/error-trellis for such an LDPC convolutional code.

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