ESTIMATION OF RADIATIVE CORRECTIONS TO THE PROCESS OF MUON-ELECTRON CONVERSION

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Abstract

In detection of electrons from $\mu \to e$ conversion process the monochromatic electron spectrum is transformed due to a photon emission and fluctuations of energy loss in a target. The selection criterion of $\mu \to e(\gamma)$ conversion events is an electron momentum above the threshold momentum of 103.5 MeV/c, which corresponds to the maximum energy loss of 1.5 MeV. Radiative corrections including a virtual photon correction and soft photon emission below 1.5 MeV lead to a reduction by about 10% in the probability of $\mu \to e$ conversion process calculated without radiative corrections.

The soft photons emission below 1.5 MeV contributes to a change of electron spectrum from monoenergetic one at 105 MeV to a spectrum with a low energy tail for the process of $\mu \to e(\gamma)$ conversion. However the effect of smearing of the initial momentum distribution due to the soft photon emission is small in comparison with a smearing due to energy loss fluctuations in a target. The average energy of soft photons emitted below 1.5 MeV is found to be 40 keV. The soft photon approximation is a good description for $\mu \to e(\gamma)$ conversion process with photons emitted below 1.5 MeV.

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Introduction

In this work we study radiative corrections to the process of $\mu \rightarrow e$ conversion, i.e. we consider the process of $\mu \rightarrow e(\gamma)$ conversion including the elastic process of $\mu \rightarrow e$ conversion and the process of $\mu \rightarrow e$ conversion with a photon emission below a certain energy $\omega_{max}$. The $\mu \rightarrow e$ conversion is the process of a transformation of a muon into an electron in the presence of a nucleus. The clear signature of $\mu \rightarrow e$ conversion is an appearance in the final state of a monochromatic electron of the energy

$$E_{\text{max}} = E_\mu - \frac{E_\mu^2}{2 \cdot M_A}, \quad E_\mu = m_\mu \cdot (1 - (\alpha \cdot Z)^2/2)$$

where $M_A$ is a nuclear mass, $m_\mu$ is a muon mass, $\alpha$ is the fine structure constant, $Z$ is a nuclear charge. In particular, $E_{\text{max}}$ is 104.963 MeV for aluminum.

In an experimental setup a monochromatic spectrum of $\mu \rightarrow e$ conversion electrons at 105 MeV is transformed to a distribution with a low energy tail. This transformation in the electron spectrum can be either due to fluctuations of energy loss of electrons in a target or because of a photon emission in the process of $\mu \rightarrow e(\gamma)$ conversion. The selection criterion of $\mu \rightarrow e(\gamma)$ conversion events is an electron momentum above the threshold momentum of 103.5 MeV/c, which corresponds to the maximum energy loss of 1.5 MeV.

It is important to note that in real experiments a pure elastic process is not physically observable since due to the presence of charged particles in the initial, intermediate or final states the process can be accompanied by an emission of a photon. Therefore the measured probability of the process of $\mu \rightarrow e(\gamma)$ conversion is a sum of probabilities of the pure elastic process corrected by virtual photon exchange and of a real soft photon emission below a maximum energy $\omega_{max}$, which in our case is 1.5 MeV. In fact if the emitted real soft photon is not detected, one can not distinguish a bremsstrahlung process from the elastic one.

Because the energy release in the process of $\mu \rightarrow e(\gamma)$ conversion is about $\Delta E = 105$ MeV one could expect that the virtual photon corrections and the soft photon emission can be enhanced by a large logarithmic factor $ln(m_\mu/m_e)$ where $m_e$ is an electron mass. Radiative corrections lead to a change in the probability of $\mu \rightarrow e$ conversion process calculated without radiative corrections.

The real soft photons emission contributes to a change of electron spectrum from monoenergetic one at 105 MeV to a spectrum with a low energy
Virtual photon corrections

Let us consider $\mu \to e$ conversion process in which a virtual photon emission is involved. The Feynman diagrams describing virtual photon emission are presented in Figure 1.

![Figure 1: Virtual photon corrections to lepton vertex and ingoing and outgoing leptons in the process of muon-electron conversion.](image)

It can be shown that an integral describing the radiative correction to the lepton vertex is infrared divergent. This integral can be artificially well-defined with an introduction of an infinite small mass $\lambda$ of a photon. With this modification the leading term in leading logarithm approximation is given by
\[
\Gamma_{\text{virt}} = \Gamma_{\text{el}} \cdot \left[ 1 - \frac{2\alpha}{\pi} \frac{m_\mu}{m_e} \ln \frac{m_\mu}{m_e} \right] \tag{2}
\]

where \(\Gamma_{\text{el}}\) is the \(\mu \to e\) conversion rate calculated without radiative corrections.

**Soft photon emission**

Feynman diagrams describing the process of inner bremsstrahlung are presented in Figure 2.

![Feynman diagrams](image)

**Figure 2:** Inner bremsstrahlung in the process of muon-electron conversion.

Let’s consider the case of emission of low energy photons. In this limit we assume that a photon energy is much less than energies of particles taking part in muon-electron conversion and momentum transferred to a nucleus. On the other hand we assume the validity of the perturbation theory, i.e. the smallness of a parameter \(\alpha \cdot \ln(m_\mu/m_e) \ll 1\) which provides a suppression of multi-photon emission.

Following a standard technique (see e.g. [1]) in the limit of photon low energies the photon emission can be taken into account by adding an external photon line to a diagram of the elastic process. In this case the most important are diagrams with an emission from external charged particles lines, in all other parts of diagrams one can neglect changes in momenta due to the soft photon emission. Therefore Dirac spinors of initial muon and outgoing electron are modified in the following way

\[
u_\mu \to e \cdot \frac{\hat{k} - \hat{q} + m_\mu}{(\hat{k} - \hat{q})^2 - m_\mu^2} \cdot \hat{\varepsilon}^* u_\mu \approx -e \cdot \frac{\hat{k} + m_\mu}{2(kq)} \cdot \hat{\varepsilon}^* u_\mu ,
\]
\[ \bar{u}_e \to e \cdot \bar{u}_e \varepsilon^* \frac{k_1 + q + m_e}{(k_1 + q)^2 - m_e^2} \approx e \cdot \bar{u}_e \varepsilon^* \frac{k_1 + m_e}{2(k_1 q)} , \]

where \( k, k_1, q \) are the four-vectors of muon, electron and photon, respectively, \( \varepsilon \) is the polarization of a photon. A photon emission from a non-relativistic nucleus is neglected.

By using Dirac equations these relations can be simplified

\[ u_\mu \to -e \cdot \frac{(k \varepsilon^*)}{(k q)} u_\mu , \]
\[ \bar{u}_e \to e \cdot \bar{u}_e \cdot \frac{(k_1 \varepsilon^*)}{(k_1 q)} . \]

As a result amplitude \( M \) of muon-electron conversion with the soft photon emission can be factorized according to

\[ M = M_{el} \cdot e \left[ \frac{(k_1 \varepsilon^*)}{(k_1 q)} - \frac{(k \varepsilon^*)}{(k q)} \right] , \quad (3) \]

where \( M_{el} \) is the amplitude of the same process of \( \mu \to e \) conversion without photon emission. This is well-know general result which is process independent.

Summing up over the photon polarization and by applying standard technique of calculation of process probability the differential probability to emit a single soft photon with momentum \( q \) in the process of \( \mu \to e \) conversion is given by

\[ d^9 \Gamma_{soft} = d^6 \Gamma_{el} \cdot 4\pi\alpha \left[ \frac{2(k_1 k)}{(k_1 q)(k q)} - \frac{(m_e^2)}{(k_1 q)^2} - \frac{(m_\mu^2)}{(k q)^2} \right] \frac{d^3 q}{(2\pi)^3 2\omega} . \quad (4) \]

Integration of Eq. (4) over an electron momentum within kinematic limits gives the spectrum of soft photons in the process of muon-electron conversion

\[ \frac{1}{\Gamma_{el}} \frac{d\Gamma_{soft}}{d\omega} = \frac{2\alpha}{\pi} \cdot ln \left( \frac{m_\mu}{m_e} \right) \cdot \frac{1}{\omega} . \quad (5) \]

Note that the same expression for the spectrum of soft photon emission can be obtain by applying the classical consideration \[2\] instead of Feynman diagram technique.

The limit \( \omega \to 0 \) in Eq. (5) represents a logarithmic divergence meaning that the total probability of emitting a single very soft photon by a charge particle is infinite, this is well-known infrared catastrophe. By introducing a small photon mass \( \lambda \) the integral over \( \omega \) becomes well-defined and the
probability of the soft photon emission is derived from Eq. (5) by integrating over the photon energy from the small photon mass to the maximum energy \( \omega_{\text{max}} \) of emitted soft photons

\[
\Gamma_{\text{soft}} = \Gamma_{\text{el}} \cdot \frac{2\alpha}{\pi} \ln \frac{m_{\mu}}{m_e} \ln \frac{\omega_{\text{max}}}{\lambda}.
\]  

(6)

**Probability of radiative muon - electron conversion**

The inner bremsstrahlung spectrum diverges for low energy photons but this divergence in the total rate is canceled by virtual photon corrections. Therefore physically observable is the sum of probabilities of the elastic process corrected by the virtual photon emission and of the soft photon emission.

By adding the probabilities \( \Gamma_{\text{virt}} \) and \( \Gamma_{\text{soft}} \) the infrared divergences from the soft bremsstrahlung and from the lepton vertex corrections cancel each other producing a finite probability of radiative \( \mu \rightarrow e(\gamma) \) conversion with the emission of soft photon with the energy below \( \omega_{\text{max}} \):

\[
\Gamma = \Gamma_{\text{el}} \cdot \left[ 1 - \frac{2\alpha}{\pi} \ln \frac{m_{\mu}}{m_e} \ln \frac{m_{\mu}}{\omega_{\text{max}}} \right].
\]  

(7)

Note that this is the probability of radiative \( \mu \rightarrow e(\gamma) \) conversion calculated in the first order in electromagnetic constant \( \alpha \).

Table 1 shows the ratio of probabilities \( \Gamma/\Gamma_{\text{el}} \) of \( \mu \rightarrow e \) conversion calculated with and without radiative corrections. Radiative corrections include the virtual photon emission and soft photon emission with energy below \( \omega_{\text{max}} \).

| \( \omega_{\text{max}} \) (MeV) | 0.1 | 0.5 | 1   | 1.5  | 2    | 2.5  |
|-------------------------------|-----|-----|-----|------|------|------|
| \( \Gamma/\Gamma_{\text{el}} \) | 0.828 | 0.867 | 0.885 | 0.895 | 0.902 | 0.907 |

Table 1: The ratio of probabilities of \( \mu \rightarrow e \) conversion calculated with and without radiative corrections versus maximum soft photon energy \( \omega_{\text{max}} \).

One can see that radiative corrections reduce the probability calculated without radiative corrections. For \( \omega_{\text{max}} = 1.5\text{MeV} \) this corrections is about
Spectrum of conversion electrons

In this section we consider the process of soft photon emission with energy below \( \omega_{\text{max}} = 1.5 \text{MeV} \) because as it was discussed above the selection criterion of \( \mu \rightarrow e(\gamma) \) conversion events is an electron momentum above the threshold momentum of 103.5 MeV/c, which corresponds to the maximum energy loss of 1.5 MeV.

It is important to note that one does not need to simulate the photon spectrum down to very small photon energies. A straggling in a target leads to a smearing of the electron momentum. To get the electron momentum distribution from a target a pattern recognition and track reconstruction procedure [3] based on the Kalman filter technique [4] was applied taking into account backgrounds, delta-rays and straw inefficiency. It will be shown below that an electron momentum distribution from a target can be fitted by a Gaussian with a standard deviation \( \sigma = 200 \text{ keV/c} \). Therefore a photon emission below 200 keV is indistinguishable from the elastic process of muon-electron conversion and it is not required to simulate this photon emission. In the following we will simulate photon emission in the range from \( \omega_{\text{min}} = 100 \text{keV} \) to \( \omega_{\text{max}} = 1.5 \text{MeV} \). The particular choice of \( \omega_{\text{min}} \) is not important because as we mentioned above for photon energies below 200 keV the process of \( \mu \rightarrow e(\gamma) \) conversion will look as the elastic process. Moreover a formula for the total probability of \( \mu \rightarrow e(\gamma) \) process contains logarithm of \( \omega_{\text{min}} \) and therefore it is not very sensitive to \( \omega_{\text{min}} \).

The relative probability \( \Gamma_{\text{soft}}/\Gamma_{\text{el}} \) of photon emission in the range from \( \omega_{\text{min}} \) to \( \omega_{\text{max}} \) with respect to the probability of the corresponding elastic process can be calculated by integrating the photon spectrum Eq.(5) of radiative \( \mu \rightarrow e \) conversion over photon energy:

\[
\frac{\Gamma_{\text{soft}}}{\Gamma_{\text{el}}} = \frac{2\alpha}{\pi} \cdot \ln\left(\frac{m_\mu}{m_e}\right) \cdot \ln\left(\frac{\omega_{\text{max}}}{\omega_{\text{min}}}\right).
\]  

(8)

Figure 3 shows the relative probability to emit a photon for the radiative \( \mu \rightarrow e \) conversion.

Table 2 shows the relative probability to emit a photon in the range from \( \omega_{\text{min}} \) to \( \omega_{\text{max}} \) for radiative muon-electron conversion calculated by Eq.(8).

Let’s consider two subranges of the photon energy: below 100 keV and from 100 keV to 1.5 MeV. Below 100 keV we consider the process as the
Figure 3: Relative probability to emit a photon in the range from $\omega_{\text{min}}$ to $\omega_{\text{max}}$ for radiative muon-electron conversion in the approximation of soft photon emission. Minimal photon energy is $\omega_{\text{min}} = 100$ keV.

| $\omega_{\text{max}}$ (MeV) | 0.5 | 1 | 1.5 | 2 | 3 | 4 | 5 |
|-----------------------------|-----|---|-----|---|---|---|---|
| probability (%)             | 4   | 5.7 | 6.7 | 7.4 | 8.4 | 9.1 | 9.7 |

Table 2: Relative probability to emit a photon in the range from 100 keV to $\omega_{\text{max}}$.

elastic one occurring according to Table 1 for $\omega_{\text{max}} = 100$ keV with the relative rate $\Gamma / \Gamma_{\text{el}} = 82.8\%$.

Photons emitted in the subrange 100 keV - 1.5 MeV carry away a part of energy released in $\mu \rightarrow e(\gamma)$ conversion. This leads to the appearance of low energy tail in the electron distribution. According to Table 2 with the relative rate $\Gamma_{\text{soft}} / \Gamma_{\text{el}} = 6.7\%$ in the process of $\mu \rightarrow e(\gamma)$ conversion the
electron energy is distributed in the range 103.5 - 104.863 MeV.

In Monte Carlo simulations the elastic process is simulated with the probability $82.8/(82.8 + 6.7)\% \approx 92.5\%$ and the process of soft photon emission in the range from 100 keV to 1.5 MeV with the probability $6.7/(82.8 + 6.7)\% \approx 7.5\%$. The total number of useful $\mu \to e(\gamma)$ conversion events including the elastic process and one photon emission is $N = (0.828 + 0.067)N_0 = 0.895N_0$ where $N_0$ is the number of elastic events calculated without radiative corrections. We take into account the photon energy distribution in the simulation by sampling the photon spectrum \cite{5} according to the equation $\omega = \omega_{\text{min}} \cdot \left( \frac{\omega_{\text{max}}}{\omega_{\text{min}}} \right)^r$ where $r$ is distributed uniformly from 0 to 1.

The spectrum of electrons produced in the process of radiative $\mu \to e$ conversion is shown in Figure 4 in linear (left) and logarithmic (right) scale. Due to the photon emission a low energy tail in the electron spectrum appears.

![Figure 4: Momentum distribution of electrons produced in the process of $\mu \to e(\gamma)$ conversion in linear and logarithmic scale. Soft photon emission is taking into account.](image)

The momentum distribution of electrons from a target smeared by straggling is shown in Figure 5 in linear (left) and logarithmic (right) scale by neglecting the photon emission.

By comparing of Figures 4 and 5 one can see that the effect of the photon emission is small in comparison with the effects of straggling in the target.
In the soft photon approximation the average energy of photons emitted in the range from $\omega_{\text{min}}$ to $\omega_{\text{max}}$

$$\bar{\omega} = \Gamma^{-1} \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \omega \frac{d\Gamma_{\text{soft}}}{d\omega} d\omega = \frac{2\alpha \ln \frac{m_\mu}{m_e} \cdot (\omega_{\text{max}} - \omega_{\text{min}})}{1 - 2\alpha \ln \frac{m_\mu}{m_e} \ln \frac{m_\mu}{\omega_{\text{max}}}}. \quad (9)$$

For $\omega_{\text{min}} \ll \omega_{\text{max}} = 1.5$ MeV this equation gives $\bar{\omega} \approx 0.04$ MeV.

The classical formula from [2] also describes correctly the soft photon emission but the extension of this formula to high energy photons is a rough approximation. This is especially important in calculations of the average radiated energy because this quantity depends crucially on the high energy behavior of the spectrum. In order to describe the high energy end of the photon spectrum specific particle physics models are required.

### Inner bremsstrahlung in the radiative pion decay

In order to understand qualitatively a hard photon emission and to validate the range of application of soft photon approximation the radiative pion decay is considered. In previous section the spectrum of soft photons was found for the process of $\mu \rightarrow e(\gamma)$ conversion. In order to get predictions for the high energy part of the photon spectrum one has to rely on a variety of specific models for the $\mu - e$ vertex. In order to avoid complications in analysis of the high energy photon spectrum we just consider the inner
bremsstrahlung in the process of radiative pion decay that possesses similar kinematic behavior and similar mass scale as $\mu \to e(\gamma)$ conversion. Equations describing the photon and electron spectra for the radiative pion decay are given in Appendix.

Figure 6 shows the differential photon spectrum in relative photon energy $x = 2\omega/m_\pi$ for the inner bremsstrahlung in the radiative pion decay. Lower curve is the result of calculations based on the exact formula of Appendix. Upper curve is obtained in the approximation of soft photon emission where only a term proportional to $x^{-1}ln(m_\pi/m_e)$ contributes.

Figure 6: Differential spectrum in relative photon energy $x = 2\omega/m_\pi$ for the inner bremsstrahlung in the radiative pion decay. Lower curve is based on the exact formula, upper curve is obtained in the approximation of soft photon emission.

It follows from this plot that up to the photon energy $\omega = 5$ MeV the exact photon spectrum can be approximated by soft photon emission $1/\omega$ with the precision better than 10%. At high energies the spectra differ significantly because as $x$ tends to the kinematic boundary $x = 1 - \Delta^2$ where $\Delta = m_e/m_\pi$ the exact spectrum tends to 0.

This behavior of the photon spectrum is of general nature. In particular
in the case of $\mu \rightarrow e(\gamma)$ conversion low energy spectrum is exactly the same if one uses the proper mass scale. At high energies one could expect qualitatively similar photon spectra for $\mu \rightarrow e(\gamma)$ conversion and radiative pion decay.

Figure 7 shows the differential electron spectrum in relative electron energy $y = 2E/m_\pi$ for the inner bremsstrahlung in the radiative pion decay. Lower curve is based on the exact formula of Appendix, upper curve is obtained in the approximation of soft photon emission. This spectrum is divergent as $y$ tends to $1 + \Delta^2$ because this limit corresponds to the emission of low energy photons.

![Electron Spectrum of $\pi \rightarrow e^+ \nu \gamma$](image)

**Figure 7:** Differential spectrum in relative electron energy $y = 2E/m_\pi$ for the inner bremsstrahlung in the radiative pion decay. Lower curve is based on the exact formula, upper curve is obtained in the approximation of soft photon emission.

By integrating the photon spectrum of radiative pion decay over photon energy one gets a relative probability $\Gamma/\Gamma_{el}$ of photon emission in the range from $\omega_{min}$ to $\omega_{max}$ with respect to the probability of the corresponding elastic process (see Appendix). Figure 8 shows the relative probability of photon emission for the radiative pion decay. Minimal photon energy is
\( \omega_{\text{min}} = 100 \text{ keV} \). Lower curve is based on the exact formula, upper curve is obtained in the approximation of soft photon emission.

![Relative Probability of \( \pi \to e \nu \gamma \)](image)

Figure 8: Relative probability to emit a photon in the range from \( \omega_{\text{min}} \) to \( \omega_{\text{max}} \) for the radiative pion decay. Minimal photon energy is \( \omega_{\text{min}} = 100 \text{ keV} \). Lower curve is based on the exact formula, upper curve is obtained in the approximation of soft photon emission.

In particular, it follows from Figure 8 that the relative probability to produce a photon in the energy range 100 keV - 1.5 MeV is 7% which is comparable, due to an increase in a rate of photon emission at low energies, with the probability of 7.8% to produce a high energy photon in much broader range 1.5 MeV - 70 MeV. Note that the range 100 keV - 1.5 MeV is described with high precision (better than 1%) by the approximate formula of soft photon emission.

A comparison of this plot with the corresponding Figure 3 for \( \mu \to e(\gamma) \) conversion, which was obtained in the approximation of soft photon emission by Eq. 8 shows that the relative probabilities \( \Gamma_{el} \) of these processes are quite close even though the energies released in these processes are somewhat different.
Conclusion

This study has shown that radiative corrections including a virtual photon correction and soft photon emission below 1.5 MeV lead to a reduction by about 10% in the probability of $\mu \rightarrow e$ conversion process calculated without radiative corrections. It is not necessary to consider a photon emission above 1.5 MeV because these events do not survive a cut in the electron momentum of 103.5 MeV.

The soft photons emission below 1.5 MeV contributes to a change of electron spectrum from monoenergetic one at 105 MeV to a spectrum with a low energy tail. The effect of smearing of the initial momentum distribution due to the soft photon emission is small in comparison with a smearing due to energy loss fluctuations in the target. The average energy of photons emitted below 1.5 MeV is found to be small of about 40 keV.

By analyzing the radiative pion decay which possesses the same kinematic properties and a comparable energy scale as $\mu \rightarrow e(\gamma)$ conversion it was found that up to the photon energy $\omega = 5$ MeV the exact photon spectrum can be approximated by soft photon emission $dN/d\omega \sim 1/\omega$ with the precision better than 10%.

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Appendix
Spectra of the radiative pion decay

According to [5] the double differential rate of pion decay including the inner bremsstrahlung with respect to photon energy $\omega$ and electron energy $E$ is given by

$$\frac{d^2\Gamma}{dx dy} = \frac{\alpha}{2\pi} \cdot \frac{\Gamma(\pi \rightarrow e\nu)}{1 - \Delta^2} \cdot I_{IB}$$

(10)

where $\Gamma(\pi \rightarrow e\nu)$ is the decay rate for non-radiative pion decay, $\Delta = m_e/m_\pi$.

In Eq.(10) $I_{IB}$ is

$$I_{IB} = \left[ \frac{1 - y + \Delta^2}{x^2(x + y - 1 - \Delta^2)} \right] \cdot \left[ x^2 + 2(1 - x)(1 - \Delta^2) + \frac{2x\Delta^2(1 - \Delta^2)}{x + y - 1 - \Delta^2} \right]$$

(11)

where $x = 2\omega/m_\pi$ and $y = 2E/m_\pi$ are relative photon and electron energies, respectively. These energies are directly measurable in radiative pion decay.

Differential distributions in photon energy $\omega$ and electron energy $E$ are obtained from Eq.(10) by integrating over corresponding kinematic regions.

It follows from the conservation of four-momenta $p_\pi = k + q + p_e$, where $p_\pi, k, q, p_e$ are the four-momenta of pion, neutrino, photon, and electron, respectively, that

$$k^2 = (p_\pi - q - p_e)^2 = m_\pi^2 - 2m_\pi(\omega + E) + 2\omega(E - |p_e|z) + m_e^2 = 0$$

where $z = \cos\theta$, $\theta$ is the angle between the electron and photon momenta.

Since $|z| \leq 1$ the previous condition implies that

$$(1 + \Delta^2 - x - y + xy/2)^2 - x^2(y^2/4 - \Delta^2) \leq 0.$$ 

The ranges of definition of $x$ and $y$ are easily established from this condition:

$$2\Delta \leq y \leq 1 + \Delta^2, 1 - y/2 - (\sqrt{y^2 - 4\Delta^2})/2 \leq x \leq 1 - y/2 + (\sqrt{y^2 - 4\Delta^2})/2$$

or
\[ 0 \leq x \leq 1 - \Delta^2, (1 + \Delta^2 - 2x + x^2)/(1 - x) \leq y \leq 1 + \Delta^2. \]

Additional conditions were used to get ranges above: \( y^2 - 4\Delta^2 \geq 0, \) \( (p_\pi - p_\nu)^2 \geq 0, \) \( (1 + \Delta^2 - 2x + x^2)/(1 - x) \leq 1 + \Delta^2. \)

By integrating the double differential rate Eq. (10) in corresponding limits one gets for the photon spectrum

\[
\frac{d\Gamma}{dx} = \frac{\alpha}{2\pi} \cdot \frac{\Gamma(\pi \rightarrow e\nu)}{1 - \Delta^2} \cdot \left[ (x - 2(1 - \Delta^2) + 2(1 - \Delta^2)^2/x)ln\left(\frac{1 - x}{\Delta^2}\right) - x(1 - \Delta^2/(1 - x)) \right]
\]

and for the electron spectrum

\[
\frac{d\Gamma}{dy} = \frac{\alpha}{2\pi} \cdot \frac{\Gamma(\pi \rightarrow e\nu)}{1 - \Delta^2} \cdot \frac{1}{1 + \Delta^2 - y} \cdot (A + B)
\]

\[
A = 2(1 - \Delta^2)(2\Delta^2 - y)ln \left[ \frac{(1 - y/2 + \sqrt{y^2 - 4\Delta^2})/2^2}{1 + \Delta^2 - y} \right]
\]

\[
B = (1 + \Delta^2(-2 + 5\Delta^2) - 4\Delta^2 y + y^2)ln \left[ \frac{(y/2 - \Delta^2 + \sqrt{y^2 - 4\Delta^2})/2^2}{\Delta^2(1 + \Delta^2 - y)} \right].
\]