Transition amplitudes and sewing properties for bosons on the Riemann sphere

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Abstract

We consider scalar quantum fields on the sphere, both massive and massless. In the massive case we show that the correlation functions define amplitudes which are trace class operators between tensor products of a fixed Hilbert space. We also establish certain sewing properties between these operators. In the massless case we consider exponential fields and have a conformal field theory. In this case the amplitudes are only bilinear forms but still we establish sewing properties. Our results are obtained in a functional integral framework.
1 Introduction

A conformal field theory is specified by a family of correlation functions defined on a Riemann surface. These correlation functions can be interpreted as transition amplitudes between various Hilbert spaces all built up as tensor products of a fixed Hilbert space. If the conformal field theory is describing a statistical mechanics model at the critical point then these amplitudes are connected with scattering amplitudes. The expected mathematical structure of these amplitudes was developed in a series of axioms due to G. Segal [14], [15], [5]. Verification of the axioms has been slow with the best results obtained for fermions [15], [8]. In the present paper we make some progress on verification for bosons in the case where the Riemann surface is a sphere.

We work in a functional integral formulation. Because conformal field theories are massless the functional integrals are somewhat singular and the manipulations one would like to make are awkward. Things are better for massive fields and so we start with this case. Then the fields satisfy a Markov property [12], [3] which makes it possible to reduce certain integrals over fields on the whole sphere to integrals over fields on one-dimensional submanifolds. This property facilitates the definition of the amplitudes and the sewing properties. We develop this massive case at length.

In the massless case one does not have a Markov property, at least not in the same strong sense as in the massive case. The original idea was to carry over results from the massive case by taking the limit as the mass goes to zero. Unfortunately many of the massive results do not hold for small mass, let alone uniformly in the mass. So for the moment at least this strategy is not as rewarding as one might have hoped.

One property that does carry over to the massless case is a reflection positivity result. Taking advantage of this and using some simplifications due to the conformal symmetry we are able to define amplitudes and establish sewing properties in this case as well. The results are somewhat weaker than in the massive case. The results for both cases are described in more detail in section 3.3.

2 Preliminaries

2.1 metrics

We work on the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Complex coordinates are the standard $z = x_1 + ix_2$ on $\mathbb{C}$ and $\zeta$ on $\mathbb{C}_\infty - \{0\}$ which is $\zeta = 1/z$ on the overlap. We also label points in $\mathbb{C}$ by $x = (x_1, x_2)$ when we want to ignore the complex structure. We will need to refer to the unit circle $C_0$ and the closed regions that it bounds:

$$C_0 = \{ z \in \mathbb{C}_\infty : |z| = 1 \}$$
$$D_0 = \{ z \in \mathbb{C}_\infty : |z| \leq 1 \}$$
$$D'_0 = \{ z \in \mathbb{C}_\infty : |z| \geq 1 \}$$

We consider conformal metrics on $\mathbb{C}_\infty$ which have the form in $\mathbb{C}$

$$\gamma = \rho(z)|dz|^2 = \rho(x)(dx_1^2 + dx_2^2)$$

for some smooth positive function $\rho$. In the other patch $\zeta = 1/z$ it has the form $\gamma = |\zeta|^{-4}\rho(1/\zeta)|d\zeta|^2$ so $|\zeta|^{-4}\rho(1/\zeta)$ should be smooth and positive at $\zeta = 0$.

For a Hilbert space structure in our field theory we will want to consider metrics $\gamma$ which are invariant under radial reflection through $C_0$. Radial reflection is defined by

$$\theta(z) = \frac{z}{|z|^2}$$
which preserves $C_0$ and exchanges $D_0$ and $D'_0$. We want $\theta^* \gamma = \gamma$ and if $\gamma = \rho|dz|^2$ the condition is that
\[
|z|^{-4} \rho(\bar{z}^{-1}) = \rho(z)
\] (4)
A reflection invariant metric is the round metric
\[
\gamma = \frac{4}{(1 + |z|^2)^2}|dz|^2
\] (5)
Another reflection invariant metric is the cylindrical metric
\[
\gamma = \frac{1}{|z|^2}|dz|^2
\] (6)
This is actually not a metric on the whole sphere but only on $\mathbb{C} - \{0\}$. Under the mapping $z = e^{iw}$ this is identified with the flat metric $|dw|^2$ on the cylinder $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$.

As a point of reference we will pick a standard metric. A metric $\gamma_0 = \rho_0(z)|dz|^2$ is defined to be a standard metric if it is invariant under radial reflections and also rotations (i.e. $\rho_0(e^{i\theta}z) = \rho_0(z)$) and if there is a constant $d$ such that that it has the toroidal form $|z|^{-2}|dz|^2$ for $e^{-d} < |z| < e^d$. The last requirement is to keep things simple in a neighborhood of $C_0$. Note that with this metric the strip $e^{-d} < |z| < e^d$ has width $2d$.

2.2 Laplacians

Associated with any conformal metric $\gamma = \rho(z)|dz|^2$ on $\mathbb{C}_\infty$ we have a measure ($|\gamma| = \det \gamma$)
\[
d\mu_\gamma(x) = |\gamma(x)|^{1/2} dx = \rho(x) dx
\] (7)
and the Hilbert space $L^2(\mathbb{C}_\infty, \mu_\gamma)$ with the inner product $(f, h)_\gamma = \int f \overline{h} d\mu_\gamma$. The Laplacian for this metric is
\[
\Delta_\gamma = \frac{4}{\rho(z)} \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{\rho(x)} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)
\] (8)
The negative Laplacian $-\Delta_\gamma$ is naturally a positive self-adjoint operator on $L^2(\mathbb{C}_\infty, d\mu_\gamma)$ and has purely discrete spectrum. The lowest eigenvalue is 0 and the eigenfunctions are the constants. For $\mu > 0$ the operator $(-\Delta_\gamma + \mu)^{-1}$ exists and is Hilbert-Schmidt. This is true on any compact two-dimensional manifold, see for example [16], p. 113.

We also need Sobolev spaces. For any real number $s$ the Sobolev space $H^s$ is a space of distributions on $\mathbb{C}_\infty$ defined by the requirement that in local coordinates they be in the corresponding Sobolev space on $\mathbb{R}^2$. We can give $H^s$ a norm and regard it as a Hilbert space by an alternate definition. For any $\gamma, \mu$ define $H^s_{\gamma, \mu}$ to be the completion of $C^\infty(\mathbb{C}_\infty)$ in the norm
\[
\|f\|_{s, \gamma, \mu} = (f, (-\Delta_\gamma + \mu)^s f)^{1/2}
\] (9)
Then $H^s = H^s_{\gamma, \mu}$ as a vector spaces. We have $H^0_{\gamma, \mu} = L^2(\mathbb{C}_\infty, d\mu_\gamma)$ and we have the inclusions $H^1_{\gamma, \mu} \subset H^0_{\gamma, \mu} \subset H^{-1}_{\gamma, \mu}$. The inner product $(f, h)_\gamma$ extends to a bilinear form on $H^1 \times H^{-1}$ and to emphasize this interpretation we sometimes write it as $(f, h)_{+1, -1}$. 

3
3 Massive fields

3.1 fields

Now we define massive scalar fields on the sphere \((\mathbb{C}_\infty, \gamma)\) with an arbitrary metric \(\gamma\). As the test function space we take the real Sobolev space \(H^{-1}_{\gamma,\mu}\). Let \(\{\phi(f)\}\) with \(f \in H^{-1}_{\gamma,\mu}\) be a family of Gaussian random variables with covariance given by the inner product. These are functions on an underlying probability space \((Q, \Sigma, m_{\gamma,\mu})\). Expectations are denoted by \(<\cdots>_{\gamma,\mu}\) so we have the characteristic function

\[
< e^{i\phi(f)} >_{\gamma,\mu} = \int_Q e^{i\phi(f)} dm_{\gamma,\mu} = \exp \left( -\frac{1}{2} \|f\|_2^2 \right) \tag{10}
\]

The family \(\{\phi(f)\}\) is our quantum field theory with mass \(\sqrt{\mu} > 0\).

We introduce Wick-ordered products in the standard way defining : \(\phi(f_1) \cdots \phi(f_n) :\) to be the projection in \(L^2(Q, \Sigma, m_{\gamma,\mu})\) of \(\phi(f_1) \cdots \phi(f_n)\) onto the orthogonal complement of polynomials in \(\phi(f)\) of degree \(n - 1\). Then \(\phi(f_1) \cdots \phi(f_n)\) : is a polynomial of degree \(n\) and all such polynomials span a dense set in \(L^2\). Any contraction \(T\) on \(H^{-1}_{\gamma,\mu}\) induces a contraction \(\Gamma(T)\) on \(L^2(Q, \Sigma, m_{\gamma,\mu})\) which satisfies \(\Gamma(T)1 = 1\) and

\[
\Gamma(T) : \phi(f_1) \cdots \phi(f_n) := \phi(Tf_1) \cdots \phi(Tf_n) : \tag{11}
\]

We have \(\Gamma(T)\Gamma(S) = \Gamma(TS)\) and \(\Gamma(A)^* = \Gamma(A^*)\). If \(U\) is unitary then \(\Gamma(U)\phi(f)\Gamma(U^{-1}) = \phi(Uf)\), but not in general.

For any closed subset \(A \subset \mathbb{C}_\infty\) let \(\Sigma_A\) be the \(\sigma\)-algebra of measurable subsets generated by the random variables \(\{\phi(f)\}\) with \(\text{supp} f \subset A\). Let \(E_{\gamma,\mu}^A\) denote the conditional expectation with respect to \(\Sigma_A\) for the measure \(m_{\gamma,\mu}\). As an operator on \(L^2(Q, \Sigma, m_{\gamma,\mu})\), \(E_{\gamma,\mu}^A\) can be characterized as the projection \(E_{\gamma,\mu}^A = \Gamma(e_{\gamma,\mu}^A)\) where \(e_{\gamma,\mu}^A\) is the projection in \(H^{-1}_{\gamma,\mu}\) onto elements with support in \(A\). \[17\]

The fields have the the Markov property \([11], [3]\) which states that for an open set \(\Omega \subset \mathbb{C}_\infty\)

\[
E_{\Omega_1}^{\gamma,\mu} E_{\Omega_2}^{\gamma,\mu} = E_{\Omega_1 \Omega_2}^{\gamma,\mu} \tag{12}
\]

Another way to put it is that if \(F\) is measurable with respect to \(\Sigma_\Omega\) then \(E_{\Omega_1}^{\gamma,\mu} F = E_{\Omega_1 \Omega_2}^{\gamma,\mu} F\).

3.2 a standard Hilbert space

We now pick a fixed standard metric \(\gamma_0\) as defined in section 2.1. The invariance under radial reflections gives a reflection positivity property for the fields. The latter is used to create a standard Hilbert space. Reflection positivity has long played a key role in field theory on Riemannian manifolds \([13]\), \([12]\), \([1]\), \([3]\), \([9]\).

Because radial reflection \(\theta\) is an isometry it induces an unitary operator \(\theta^*\) on \(H^{-1}_{\gamma_0,\mu}\) and hence a unitary operator \(\Gamma(\theta^*)\) on \(L^2(Q, \Sigma, m_{\gamma_0,\mu})\). We define

\[
\Theta \Psi = \Gamma(\theta^*) \Psi \tag{13}
\]

Then \(\Theta\) is anti-unitary and satisfies \(\Theta^2 = 1\). Now let \(\Psi \in L^2(Q, \Sigma, dm_{\gamma_0,\mu})\) be measurable with respect to \(\Sigma_{D_0}\), written \(\Psi \in L^2(Q, \Sigma_{D_0}, m_{\gamma_0,\mu})\). Then \(\Theta \Psi\) is measurable with respect to \(\Sigma_{D_0}^c\) and using the Markov property we find the reflection positivity result \([12], [3]\)

\[
< (\Theta \Psi) \Psi >_{\gamma_0,\mu} = \int |E_{C_0} \Psi|^2 dm_{\gamma_0,\mu} \geq 0 \tag{14}
\]

Using this we give three equivalent constructions of a standard Hilbert space \(\mathcal{H}\).
1. **first construction.** The first construction is analogous to the Osterwalder-Schrader reconstruction theorem for Euclidean field theory. Consider the multilinear functional from $H^{-1} \times \cdots \times H^{-1}$ to random variables on $(Q, \Sigma, m_{\gamma,\mu})$ which sends $(f_1, \ldots, f_n) \mapsto \phi(f_1) \cdots \phi(f_n)$. This induces a map $\Phi_n$ from the algebraic tensor product $\otimes_{i=1}^n H^{-1}$ to random variables such that

$$\Phi_n(f_1 \otimes \cdots \otimes f_n) = \phi(f_1) \cdots \phi(f_n) \quad (15)$$

This is the universal property of the algebraic tensor product [7]. Next consider the vector space of all finite sequences

$$F = (F_0, F_1, F_2, \ldots) \quad F_n \in \otimes_{i=1}^n H^{-1} \quad (16)$$

and let $S$ be the complexification. Define a linear map from $S$ to random variables by

$$\Phi(F) = \sum_n \Phi_n(F_n) \quad (17)$$

where $\Phi(F_0) = F_0 \in \mathbb{C}$. The map $\Theta^*$ on $H^{-1}$ induces a map on sequences $F$ and we let $\Theta$ on $S$ be this map followed by complex conjugation. The new map $F \mapsto \Theta F$ is related to the previous definition in [13] by $\Phi(\Theta F) = \Theta(\Phi(F))$.

Next for any closed subset $A \subset \mathbb{C}$ we let $S_A$ be sequences in which all functions have support in $A$. We want to define a norm on $S_{D_0}$ by

$$\|F\|^2 = \langle \Phi(\Theta F) \Phi(F) \rangle_{\gamma_0,\mu} = \sum_{n,m} \langle \Phi_n(\Theta F_n) \Phi_m(F_m) \rangle_{\gamma_0,\mu} \quad (18)$$

Since $\langle \Phi(\Theta F) \Phi(F) \rangle_{\gamma_0,\mu} = \langle \Theta(\Phi(F)) \Phi(F) \rangle_{\gamma_0,\mu}$ this is non-negative by [14]. But it is not definite. We divide by the null space $\mathcal{N} = \{ F \in S_{D_0} : \|F\| = 0 \}$ to get a pre-Hilbert space $\mathcal{H}_0 = S_{D_0}/\mathcal{N}$. Then complete it to get a Hilbert space (depending on $\gamma_0,\mu$)

$$\mathcal{H} = \overline{\mathcal{H}_0} = S_{D_0}/\mathcal{N} \quad (19)$$

Let $\nu$ map an element of $S_{D_0}$ to its equivalence class in $\mathcal{H}_0 \subset \mathcal{H}$. Then

$$\langle \nu(F_1), \nu(F_2) \rangle = \langle \Phi(\Theta F_1 \Phi(F_2) \rangle_{\gamma_0,\mu} \quad (20)$$

2. **second construction.** A second construction is a variation of this in which the $L^2$ space plays a more prominent role. Consider the Hilbert space $L^2(Q, \Sigma_{D_0}, m_{\gamma_0,\mu})$ but now supplied with the norm $\|\Psi\|^2 = \langle \Theta(\Psi) \Psi \rangle_{\gamma,\mu}$ and denoted $L^2_0(Q, \Sigma_{D_0}, m_{\gamma_0,\mu})$. Divide by the null space $\mathcal{N} = \{ \Psi : \|\Psi\| = 0 \}$ and get $\mathcal{H}_0' = L^2_0(Q, \Sigma_{D_0}, m_{\gamma_0,\mu})/\mathcal{N}$. Then complete it to obtain the Hilbert space

$$\mathcal{H} = \overline{\mathcal{H}_0'} = L^2_0(Q, \Sigma_{D_0}, m_{\gamma_0,\mu})/\mathcal{N} \quad (21)$$

Let $\nu$ map an element of $L^2_0(Q, \Sigma_{D_0}, m_{\gamma_0,\mu})$ to its equivalence class in $\mathcal{H}_0' \subset \mathcal{H}$. Then

$$\langle \nu(\Psi_1), \nu(\Psi_2) \rangle = \langle \Theta(\Psi_1) \Psi_2 \rangle_{\gamma_0,\mu} \quad (22)$$

To see that this construction is equivalent to the first note that the map $F \mapsto \Phi(F)$ from $S_{D_0}$ to $L^2_0(Q, \Sigma_{D_0}, m_{\gamma_0,\mu})$ is norm preserving and so determines a norm preserving map from $\mathcal{H}_0$ to $\mathcal{H}_0'$ which takes $\nu(F)$ to $\nu(\Phi(F))$. We argue that the range is dense hence the map extends to a unitary from $\mathcal{H}$ as defined in [15] in to $\mathcal{H}$ as defined in [21].

Now polynomials $\Phi(F), F \in \mathcal{S}$ are dense in $L^2(Q, \Sigma, m_{\gamma_0,\mu})$. (More precisely we can choose the measure space so this is true.) Hence polynomials $\Phi(F), F \in \mathcal{S}_{D_0}$ are dense in $L^2(Q, \Sigma_{D_0}, m_{\gamma_0,\mu})$ and hence they are dense in $L^2_0(Q, \Sigma_{D_0}, m_{\gamma_0,\mu})$. Hence vectors $\nu(\Phi(F))$ are dense in $\mathcal{H}_0'$ as required.
3. third construction. The third construction is just to take
\[ \mathcal{H} = L^2(Q, \Sigma_{C_0}, m_{\gamma_0, \mu}) \] (23)
as the standard space. To see that this is equivalent note that the identity \[14\] shows that
the map \( F \to \mathcal{E}_{C_0}F \) from \( L^2(Q, \Sigma_{D_0}, m_{\gamma_0, \mu}) \) to \( L^2(Q, \Sigma_{C_0}, m_{\gamma_0, \mu}) \) is norm preserving. Hence it
defines an isometry from \( \mathcal{H}_0^\prime \) to \( L^2(Q, \Sigma_{C_0}, m_{\gamma_0, \mu}) \) which takes \( \nu(\Psi) \) to \( \mathcal{E}_{C_0}\Psi \). But the map is
also onto. Hence \( \mathcal{H}_0^\prime \) is complete (i.e. the completion in \( \mathcal{H}_0 \) was unnecessary). Thus we have
a unitary operator from \( \mathcal{H} \) defined in (21) to the new space \( L^2(Q, \Sigma_{C_0}, m_{\gamma_0, \mu}) \).

3.3 the problem

We are concerned with the following situation. Let \( D_1, \ldots, D_n \) be a collection of disjoint discs in \( \mathbb{C}_\infty \). For each \( i \) we pick a Mobius transformation \( \alpha_i \) which takes \( D_i \) to the unit disc \( D_0 \). We have
\[ \alpha_i(z) = \begin{cases} (z - a_i)/r_i & \text{if } D_i = \{z : |z - a_i| \leq r_i\} \\ r_i/(z - a_i) & \text{if } D_i = \{z : |z - a_i| \geq r_i\} \end{cases} \] (24)
A third possibility, which we have not written explicitly, is that \( D_i \) is a half plane. The pull back of
our standard metric \( \gamma_0 = \rho_0(z)|dz|^2 \) is metric \( \gamma_i = \alpha_i^\ast(\gamma_0) \) which in coordinates \( z_i = \alpha_i(z) \) is given by
\[ \gamma_i(z) = \rho_0(z_i)|dz_i|^2 \] (25)

Now using a partition of unity construct metrics \( \gamma \) with the property that \( \gamma = \gamma_i \) on a neighborhood
of \( D_i \) and is arbitrary elsewhere. Specifically define
\[
\begin{align*}
D_0 &= \{z : |z| \leq 1\} & C_0 &= \{z : |z| = 1\} & D_0' &= \{z : |z| \geq 1\} \\
D_{0+} &= \{z : |z| \leq e^{d/2}\} & C_{0+} &= \{z : |z| = e^{d/2}\} & D_{0+}' &= \{z : |z| \geq e^{d/2}\} \\
D_{0++} &= \{z : |z| \leq e^d\} & C_{0++} &= \{z : |z| = e^d\} & D_{0++}' &= \{z : |z| \geq e^d\}
\end{align*}
\] (26)

and let \( D_i, D_{i+}, D_{i++}, C_i, C_{i+}, C_{i++}, D_i', D_{i+}', D_{i++}' \) etc. be the image of these sets under \( \alpha_i^{-1} \). Then
\( D_{i+}, D_{i++} \) are enlargements of \( D_i \). The condition is that the \( D_{i+} \) are disjoint and that \( \gamma = \gamma_i \) on
\( D_{i++} \). For such a metric \( \gamma \) we call \((\mathbb{C}_\infty, \gamma)\) a sphere with standard discs \( \{D_i\} \).

We also suppose we have a specific parametrization of \( D_i \). Each disc is labeled as either an
in-disc or an out-disc. If \( D_i \) is an in-disc then we define
\[ j_i(z) = e^{i\theta_i} \alpha_i(z) \] (27)
which maps \( D_i \) to \( D_0 \). If \( D_i \) is an out-disc then we define
\[ j_i'(z) = e^{i\theta_i} \alpha_i(z)^{-1} \] (28)
which maps \( D_i \) to \( D_i' \). In either case we have allowed a twist with the phase factor \( e^{i\theta_i} \). Even with
the twist we still have \( j_i^\ast \gamma_0 = \gamma_i \) and \( (j_i^\ast) \gamma_0 = \gamma_i \) thanks to the rotation invariance . When supplied
with a choice of maps \( j_i, j_i' \) a sphere with standard discs \( \{D_i\} \) said to be parametrized.

Note that the map \( j_i \) induces a pull back \( (j_i^\ast f)(z) = f(j_i z) \) on functions or on \( H^{-1} \). This induces a
map \( J_i \) on \( S \). Similarly \( j_i' \) induces a map \( J_i' \) on \( S \). We have that
\[ J_i : S_{D_0} \to S_{D_i} \]
\[ J_i' : S_{D_0}' \to S_{D_i} \] (29)

Now we can state the problem, more or less as posed by Gawedski [5]. Let \((\mathbb{C}_\infty, \gamma)\) be a sphere with
standard discs parametrized so that \( \{D_i\}_{i \in I} \) are in-discs and \( \{D_i'\}_{i' \in I'} \) are out-discs. Let \( \{F_i\}_{i \in I \cup I'} \)
be a collections of elements in \( S_{D_0} \). Then \( J_i F_i \in S_{D_i} \) for \( i \in I \) and \( J_i' \Theta F_i \in S_{D_i} \) for \( i \in I' \) and we can consider
\[ \prod_{i \in I'} \Phi(J_i' \Theta F_i) \prod_{i \in I} \Phi(J_i F_i) >_{\gamma, \mu} \] (30)

We would like to establish the following:
1. The expectation (30) depends only on the equivalence class of $F_i$ and so defines a multilinear functional on $H_0 \times \cdots \times H_0$.

2. The expectation extends by continuity to a multilinear functional on $H \times \cdots \times H$.

3. The multilinear functional defines an operator $A^I_{\gamma', \mu}$ from $[\otimes_{i \in I} H] \to [\otimes_{i \in I'} H]$ such that if $F_i = \nu(F_i)$, etc.

$$\left( [\otimes_{i \in I} F_i], A^I_{\gamma', \mu} [\otimes_{i \in I} F_i] \right) = Z_{\gamma', \mu} \prod_{i \in I'} \Phi(J_i' \Theta F_i) \prod_{i \in I} \Phi(J_i F_i) >_{\gamma', \mu}$$

with a constant $Z_{\gamma', \mu}$ to be specified.

4. The constants $Z_{\gamma, \mu}$ can be chosen so certain sewing properties hold, for example

$$A^I_{\gamma', 1} A^I_{1, \mu} = A^I_{\gamma', \mu}$$

We will be able to establish all this is the massive case. In the massless case the formulation of the problem is somewhat different, but still there are questions analogous to these four. In this case we establish (1.) and weaker versions of (2.), (3.), (4.).

### 3.4 Measure Theory Results

We start with some preliminary results. Consider conformal metrics $\gamma' = \rho' |dz|^2$ and $\gamma = \rho |dz|^2$. Then $\gamma' = \lambda \gamma$ where $\lambda = \rho'/\rho$ is a smooth function on $C_\infty$. By the compactness the positive functions $\rho, \rho'$ are bounded above and below and hence so is $\lambda$. Any function $\lambda$ on the sphere gives a map on functions $f \to \lambda f$ and this induces a map $F \to F_\lambda$ on $S$.

**Lemma 1** If $\gamma' = \lambda \gamma$ then for any $F \in S$

$$< \Phi(F) >_{\gamma', \mu} = < \Phi(F_\lambda) >_{\gamma, \lambda \mu}$$

**Proof.** Using $\Delta_{\gamma'} = \lambda^{-1} \Delta_{\gamma}$ and $d\mu_{\gamma'} = \lambda d\mu_{\gamma}$ we have for smooth functions $f$

$$(f, (-\Delta_{\gamma'} + \mu)^{-1} f)_{\gamma'} = (\lambda f, (-\Delta_{\gamma} + \lambda \mu)^{-1} \lambda f)_{\gamma}$$

Hence the map $f \to \lambda f$ extends to an unitary from $H^{-1}_{\gamma', \mu}$ to $H^{-1}_{\gamma, \lambda \mu}$. Then we have

$$< e^{i\phi(f)} >_{\gamma', \mu} = \exp \left( -\frac{1}{2} \|f\|_{1, \gamma', \mu}^2 \right) = \exp \left( -\frac{1}{2} \|\lambda f\|_{1, \gamma, \lambda \mu}^2 \right) = < e^{i\phi(\lambda f)} >_{\gamma, \lambda \mu}$$

Taking derivatives of the characteristic functions gives a result for polynomials which is what we want.

For the next result consider functions of the form

$$\phi^2 : (g) \equiv \int : \phi(x)^2 : g(x) \, d\mu_{\gamma}(x)$$

for some smooth function $g$. Here $\phi(x) = \phi(\delta_x)$ is defined with the delta function $\delta_x$. Since $\delta_x$ is not in $H^{-1}$, the expression $\phi^2 : (g)$ is not obviously well-defined. Nevertheless it does define a function in $L^2(Q, \Sigma, m_{\gamma', \mu})$ (and more generally in $L^p$ for $p < \infty$). This is a standard result in the plane and we give a treatment for the sphere in Appendix A.
Lemma 2  For any smooth positive function \( \lambda \) on \( C_\infty \)

1. \( \int \exp \left( -\frac{1}{2} : \phi^2 : (\lambda \mu - \mu) \right) \, dm_{\gamma,\mu} \) is finite and non-zero.

2. \( m_{\gamma,\lambda \mu} \) is absolutely continuous with respect to \( m_{\gamma,\mu} \) with Radon-Nikodym derivative

\[
\frac{dm_{\gamma,\lambda \mu}}{dm_{\gamma,\mu}} = \frac{\exp \left( -\frac{1}{2} : \phi^2 : (\lambda \mu - \mu) \right)}{\int \exp \left( -\frac{1}{2} : \phi^2 : (\lambda \mu - \mu) \right) \, dm_{\gamma,\mu}} \tag{37}
\]

3. Let \( \delta = \inf_x \lambda(x) \).
   
   (a) If \( \delta \geq 1 \) then \( [dm_{\gamma,\lambda \mu}/dm_{\gamma,\mu}] \in L^q \) for \( 1 \leq q < \infty \).
   
   (b) If \( \delta < 1 \) then \( [dm_{\gamma,\lambda \mu}/dm_{\gamma,\mu}] \in L^q \) for \( 1 \leq q < 1/(1 - \delta) \)

4. If \( A \) is a closed set and \( \lambda = 1 \) on \( A^c \) then \( [dm_{\gamma,\lambda \mu}/dm_{\gamma,\mu}] \) is \( \Sigma_A \) measurable.

Remark. We have not insisted that \( m_{\gamma,\lambda \mu} \) and \( m_{\gamma,\mu} \) are defined on the same space so the second statement needs some explanation. The claim is that if \( \{ \phi(f) \} \) is a Gaussian family with covariance \( (\lambda + \mu)^{-1} \) on a measure space \( (Q, \Sigma, m_{\gamma,\mu}) \) then changing the measure to \( [dm_{\gamma,\lambda \mu}/dm_{\gamma,\mu}] \) as specified by (37) makes \( \{ \phi(f) \} \) into a Gaussian family with covariance \( (\lambda + \mu)^{-1} \).

Proof. As explained in appendix B the first point is true if

\[
(-\Delta_{\gamma} + \mu) + (\lambda \mu - \mu) = -\Delta_{\gamma} + \lambda \mu > 0 \tag{38}
\]

which is clear since \( \lambda \) is positive. Also from Appendix B the new measure \( [dm_{\gamma,\lambda \mu}/dm_{\gamma,\mu}] \) has the claimed covariance

\[
((-\Delta_{\gamma} + \mu) + (\lambda \mu - \mu))^{-1} = (-\Delta_{\gamma} + \lambda \mu)^{-1} \tag{39}
\]

The third point is also an integrability question. Now we need

\[
(-\Delta_{\gamma} + \mu) + q(\lambda \mu - \mu) > 0 \tag{40}
\]

This is clear if \( \delta \geq 1 \) and if \( \delta < 1 \) it follows from \( \lambda > 1 - 1/q \) which is implied by our condition \( q < 1/(1 - \delta) \). The fourth point follows from the fact that if \( \text{supp } g \subset A \) then : \( \phi^2 : (g) \) is \( \Sigma_A \) measurable; see lemma 12 in Appendix A. This completes the proof.

Lemma 3  Suppose that \( \gamma' = \lambda \gamma \) with \( \lambda = 1 \) on \( \Omega \subset C_\infty \) open. Then for \( F \in \mathcal{S}_\Omega \)

\[
\mathcal{E}_{\partial \Omega}^{\gamma',\mu} \Phi(F) = \mathcal{E}_{\partial \Omega}^{\gamma,\lambda \mu} \Phi(F) = \mathcal{E}_{\partial \Omega}^{\gamma,\mu} \Phi(F) \tag{41}
\]

Remark. Thus the conditional expectation does not depend on the metric or the mass outside \( \Omega \).

Note that \( \Phi(F) \) is \( \Sigma_\Omega \) measurable and so by the Markov property\(^1\) an equivalent statement is

\[
\mathcal{E}_{\Omega}^{\gamma',\mu} \Phi(F) = \mathcal{E}_{\Omega}^{\gamma,\lambda \mu} \Phi(F) = \mathcal{E}_{\Omega}^{\gamma,\mu} \Phi(F) \tag{42}
\]

\(^1\) The Markov property is also true with variable mass.
Proof. We prove (42). For any $G \in \mathcal{S}_{\Omega}$ we have that $\Phi(G)$ is $\Sigma_{\Omega}$ measurable and so by lemma 1 we compute

$$\int \Phi(G) \left( \mathcal{E}_{\Omega}^{\gamma,\lambda} \Phi(F) \right) dm_{\gamma,\mu} = \int \Phi(G) \Phi(F) dm_{\gamma,\mu}$$

$$= \int \Phi(G) \Phi(F) dm_{\gamma,\lambda}$$

$$= \int \Phi(G) (\mathcal{E}_{\Omega}^{\gamma,\lambda} \Phi(F)) dm_{\gamma,\lambda}$$

$$= \int \Phi(G) (\mathcal{E}_{\Omega}^{\gamma,\lambda} \Phi(F)) dm_{\gamma,\mu}$$ (43)

In the last step we use again that $\mathcal{E}_{\Omega}^{\gamma,\lambda} \Phi(F) = \mathcal{E}_{\Omega}^{\gamma,\lambda} \Phi(F)$ to conclude that the term is unaffected.

Since polynomials $\Phi(G)$ are dense in $L^2(Q, \Sigma_{\Omega}, m_{\gamma,\mu})$ the first identity follows.

For the second point we again take $G \in \mathcal{S}_{\Omega}$ and compute by lemma 2

$$\int \Phi(G) \left( \mathcal{E}_{\Omega}^{\gamma,\lambda} \Phi(F) \right) dm_{\gamma,\mu} = \int \Phi(G) \Phi(F) dm_{\gamma,\mu}$$

$$= \int \Phi(G) \Phi(F) \left[ \frac{dm_{\gamma,\lambda}}{dm_{\gamma,\mu}} \right] dm_{\gamma,\mu}$$

$$= \int \Phi(G) (\mathcal{E}_{\Omega}^{\gamma,\lambda} \Phi(F)) \left[ \frac{dm_{\gamma,\lambda}}{dm_{\gamma,\mu}} \right] dm_{\gamma,\mu}$$

$$= \int \Phi(G) (\mathcal{E}_{\Omega}^{\gamma,\lambda} \Phi(F)) dm_{\gamma,\lambda}$$ (44)

Here in the third step we have used that $\lambda = 1$ on $\Omega$ and lemma 2 to conclude that $[dm_{\gamma,\lambda}/dm_{\gamma,\mu}]$ is $\Sigma_{\Omega}$-measurable. Since polynomials $\Phi(G)$ are dense in $L^2(Q, \Sigma_{\Omega}, m_{\gamma,\lambda})$ the second identity follows.

3.5 amplitudes

We now return to the main problem and consider a sphere $(\mathcal{C}_\infty, \gamma)$ with standard discs $D_i$ where $\gamma = \gamma_i$. At first there is no parametrization and we just study the functions $\langle \prod_{i=1}^n \Phi(F_i) \rangle_{\gamma,\mu}$ with $F_i \in \mathcal{S}_{D_i}$.

First note that $\Phi(F_1)$ is measurable with respect to $\Sigma_{D_1} \Phi(F_2) \cdots \Phi(F_n)$ is measurable with respect to $\Sigma_{D'_1}$. Thus by the Markov property

$$\langle \prod_{i=1}^n \Phi(F_i) \rangle_{\gamma,\mu} = \langle \mathcal{E}_{D_1}^{\gamma,\mu} \Phi(F_1) \mathcal{E}_{D'_1}^{\gamma,\mu} (\Phi(F_2) \cdots \Phi(F_n)) \rangle_{\gamma,\mu}$$

$$= \langle \mathcal{E}_{C_1}^{\gamma,\mu} \Phi(F_1) \cdots \Phi(F_n) \rangle_{\gamma,\mu}$$ (45)

The same argument works with $\mathcal{E}_{C_{1}++}^{\gamma,\mu} \Phi(F_1)$ and also by the same argument we can successively replace each $\Phi(F_i)$ by $\mathcal{E}_{C_{1}++}^{\gamma,\mu} \Phi(F_1)$ Thus we have

$$\langle \prod_{i=1}^n \Phi(F_i) \rangle_{\gamma,\mu} = \langle \prod_{i=1}^n \mathcal{E}_{C_{1}++}^{\gamma,\mu} \Phi(F_1) \rangle_{\gamma,\mu}$$ (46)

By Holder’s inequality

$$\langle \prod_{i=1}^n \Phi(F_i) \rangle_{\gamma,\mu} \leq \prod_{i=1}^n \langle \mathcal{E}_{C_{1}++}^{\gamma,\mu} \Phi(F_1) \rangle_{\gamma,\mu}$$ (47)
where the norm is in $L^n(Q, \Sigma, dm_{\gamma, \mu})$. Thus we study the norms $\|E^\gamma_{C_i++} \Phi(F_i)\|_{n, \gamma, \mu}$. We would like to replace the metric $\gamma$ by the standard $\gamma_i$ and the $L^n$ norm by the $L^2$ norm.

Define $\lambda_i$ by $\gamma = \lambda_i \gamma_i$. Since $\gamma = \gamma_i$ on $D_{i++}$ we have $\lambda_i = 1$ on $D_{i++}$.

**Lemma 4** Let $F_i \in S_{D_i}$ and suppose
\[
\frac{1}{p} < \delta \equiv \inf_{i,x} \lambda_i(x) \quad (48)
\]
Then for each $n$ there is a constant $C$ such that
\[
| \prod_{i=1}^n \Phi(F_i) | \leq C \prod_i \|E^\gamma_{C_i++} \Phi(F_i)\|_{np, \gamma_i, \mu} \quad (49)
\]

**Proof.** By (47) this reduces to an estimate on $\|E^\gamma_{C_i++} \Phi(F_i)\|_{n, \gamma, \mu}$. We compute with
\[
\Delta_i = \left[ \frac{dm_{\gamma_i, \lambda_i \mu}}{dm_{\gamma_i, \mu}} \right] \quad (50)
\]
that
\[
\|E^\gamma_{C_i++} \Phi(F_i)\|_{n, \gamma, \mu} = \int |E^\gamma_{C_i++} \Phi(F_i)|^n \, dm_{\gamma, \mu} = \int |E^\gamma_{C_i++} \Phi(F_i)|^n \, dm_{\gamma_i, \lambda_i \mu} \quad \text{(by lemma 1)}
\]
\[
= \int |E^\gamma_{C_i++} \Phi(F_i)|^n \, dm_{\gamma_i, \lambda_i \mu} \quad \text{(by lemma 2)}
\]
\[
= \int |E^\gamma_{C_i++} \Phi(F_i)|^n \Delta_i \, dm_{\gamma_i, \mu} \quad \text{(by lemma 2)}
\]
\[
\leq \| \left[ E^\gamma_{C_i++} \Phi(F_i) \right]^n_{n, \gamma_i, \mu} \|_{\Delta_i, \gamma_i, \mu} \quad (51)
\]
where $1/p + 1/q = 1$. Equivalently
\[
\|E^\gamma_{C_i++} \Phi(F_i)\|_{n, \gamma, \mu} \leq \|E^\gamma_{C_i++} \Phi(F_i)\|_{np, \gamma_i, \mu} \|\Delta_i\|_{q, \gamma_i, \mu}^{1/n} \quad (52)
\]
Since $1 - 1/q < \delta$ we have $1 - 1/q < \delta_i \equiv \inf_x \lambda_i(x)$, hence $q < (1 - \delta_i)^{-1}$, and hence the factor $\|\Delta_i\|_{q, \gamma_i, \mu}$ is finite by lemma 2. This completes the proof.

**Remarks.**

1. Hereafter we use the abbreviated notation $E^\gamma_A = E^\gamma_{C_i++}$.

2. For the next result we use a hypercontractivity estimate. The general result is the following. Let $T$ be a bounded operator on a real Hilbert space $H$ and suppose for $s < t$
\[
\|T\| \leq \sqrt{s - 1 \over t - 1} \quad (53)
\]
Let $\phi(h)$ be associated Gaussian process on $(Q, \Sigma, m)$. Then $\Gamma(T)$ defined as in (11) is a contraction from $L^s(Q, \Sigma, m)$ to $L^t(Q, \Sigma, m)$, i.e.
\[
\|\Gamma(T)\psi\|_t \leq \|\psi\|_s \quad (54)
\]
This result is due to Nelson [11], [12], [17].
Lemma 5 Let $F_i \in S_{D_i^+}$. Then for $\mu$ sufficiently large

$$| < \prod_{i=1}^{n} \Phi(F_i) >_{\gamma,\mu} | \leq C \prod_{i} \| e_{C_{1+i}}^i \Phi(F_i) \|_{2, \gamma_i, \mu}$$  \hspace{1cm} (55)

Proof. Applying the Markov property twice we have

$$e_{C_{1+i}}^i \Phi(F_i) = e_{D_{1+i}^+}^i \Phi(F_i) = e_{D_{1+i}^+}^i e_{C_{1+i}}^i \Phi(F_i) = e_{D_{1+i}^+}^i e_{D_{1+i}^+}^i \Phi(F_i)$$  \hspace{1cm} (56)

Thus from (49)

$$| < \prod_{i=1}^{n} \Phi(F_i) >_{\gamma,\mu} | \leq C \prod_{i} \| e_{D_{1+i}^+}^i e_{C_{1+i}}^i \Phi(F_i) \|_{np, \gamma_i, \mu}$$  \hspace{1cm} (57)

In a following lemma we show that as $\mu \to \infty$

$$\| e_{D_{1+i}^+}^i e_{D_{1+i}^+}^i \| \leq O(\mu^{-1/2+\epsilon})$$  \hspace{1cm} (58)

For $\mu$ sufficiently large (depending on $\gamma, n$) this implies that

$$\| e_{D_{1+i}^+}^i e_{D_{1+i}^+}^i \| \leq \frac{1}{np - 1}$$  \hspace{1cm} (59)

Then by the hypercontractive bound with $s = 2$ and $t = np$ we conclude that

$$\| \Gamma(e_{D_{1+i}^+}^i e_{D_{1+i}^+}^i) \|_{2, \gamma_i, \mu} \leq \| e_{C_{1+i}}^i \Phi(F_i) \|_{np, \gamma_i, \mu} \leq \| e_{C_{1+i}}^i \Phi(F_i) \|_{2, \gamma_i, \mu}$$  \hspace{1cm} (60)

whence the result.

Lemma 6 Let $(C_\infty, \gamma)$ be the sphere with conformal metric and let $\Lambda_1, \Lambda_2$ be disjoint closed subsets. Then in the Sobolev space $H^{-1}_{\gamma, \mu}$

1. For any $\epsilon > 0$ we have as $\mu \to \infty$

$$\| e_{\Lambda_1} e_{\Lambda_2} \|_{-1} = O(\mu^{-1/2+\epsilon})$$  \hspace{1cm} (61)

2. $e_{\Lambda_1} e_{\Lambda_2}$ is Hilbert Schmidt

Remark. Similar results are known on $\mathbb{R}^2$, see Simon [17] who attributes the idea to E. Stein. Our proof is a straightforward adaptation to the sphere.

Proof. The norm in $H^{-1}_{\gamma, \mu}$ can be written in terms of the exterior derivative and the $L^2(C_\infty, \mu_\gamma)$ norm as

$$\| f \|_{-1}^2 = \| df \|^2 + \mu \| f \|^2$$  \hspace{1cm} (62)

Choose a smooth function $g$ so that $g = 1$ on $\Lambda_1$ and $g = -1$ on $\Lambda_2$ and $\| g \|_\infty = 1$. Then for $\alpha > 0$

$$\| gf \|_{-1}^2 = \| df/f \|^2 + \mu \| g \|^2$$

$$\leq (\| df \|^2 + \| dg \|_\infty \| f \|)^2 + \mu \| f \|^2$$

$$\leq (1 + \alpha^{-1}) \| df \|^2 + (1 + \alpha) \| dg \|_\infty \| f \|^2 + \mu \| f \|^2$$

$$\leq (1 + \alpha^{-1}) \| f \|_{-1}^2$$  \hspace{1cm} (63)
where the last step holds provided \((1 + \alpha)\|dg\|_\infty^2 \leq \mu a^{-1}\). Now choose \(\alpha = \mu^{1/2+\epsilon}\) and \(\mu\) sufficiently large so the inequality holds.\(^2\) Then with \(\beta^2 = 1 + \alpha^{-1} = 1 + \mu^{-1/2+\epsilon}\) have

\[
\|gf\|_{+1} \leq \beta \|f\|_{+1} \tag{64}
\]

Referring to the \(H^{-1}\), \(H^{-1}\) pairing the dual operator to multiplication by \(g\) on \(H^{-1}\) is multiplication by \(g\) in \(H^{-1}\). It has the same norm and so

\[
\|gf\|_{-1} \leq \beta \|f\|_{-1} \tag{65}
\]

Now suppose \(\text{supp } f \subset \Lambda_1\) and \(\text{supp } h \subset \Lambda_2\) so that \(f + h = g(f - h)\). Then we have

\[
4(f, h)_{-1} = \|f + h\|_{-1}^2 - \|f - h\|_{-1}^2 \leq (\beta - 1)\|f - h\|_{-1}^2 \tag{66}
\]

Expanding \(\|f - h\|_{-1}^2 = \|f\|_{-1}^2 - 2(f, h)_{-1} + \|h\|_{-1}^2\) we can rewrite this as

\[
(f, h)_{-1} \leq \left(\frac{\beta - 1}{\beta + 1}\right) \left(\frac{\|f\|_{-1}^2 + \|h\|_{-1}^2}{2}\right) \tag{67}
\]

Replacing \(f\) by \(f/\|f\|_{-1}\), etc. we obtain the same bound but with \(\|f\|_{-1}\|h\|_{-1}\) on the right. Replacing \(f\) by \(-f\) we get the same bound with a minus sign on the left. Hence still with \(\text{supp } f \subset \Lambda_1\) and \(\text{supp } h \subset \Lambda_2\) we have

\[
|(f, h)_{-1}| \leq \left(\frac{\beta - 1}{\beta + 1}\right) \|f\|_{-1}\|h\|_{-1} \tag{68}
\]

Since \(\|e_A\| \leq 1\) we have for any \(f, h\)

\[
|(e_{A_1} f, e_{A_2} h)_{-1}| \leq \left(\frac{\beta - 1}{\beta + 1}\right) \|f\|_{-1}\|h\|_{-1} \tag{69}
\]

and so

\[
\|e_{A_1} e_{A_2}\|_{-1} \leq \frac{\beta - 1}{\beta + 1} \leq \frac{\beta^2 - 1}{\beta^2 + 1} = \frac{\mu^{-1/2+\epsilon}}{2 + \mu^{-1/2+\epsilon}} \tag{70}
\]

This proves the first result.

For the second result let \(\omega = \sqrt{-\Delta_\gamma + \mu}\) regarded as a unitary from from \(H^{-1}\) to \(L^2\) or from \(L^2\) to \(H^{-1}\). Let \(\zeta_1\) be smooth and equal to 1 on \(\Lambda_1\), let \(\zeta_2\) be smooth and equal to 1 on \(\Lambda_2\), and let \(\zeta_1\zeta_2 = 0\). Then we have

\[
(e_{A_1} f, e_{A_2} h)_{-1} = (\zeta_1 e_{A_1} f, \omega^{-\Delta_\gamma} \zeta_2 e_{A_2} h)_{-1} = (\omega^{-1} \zeta_1 e_{A_1} f, \omega^{-\Delta_\gamma} \zeta_2 e_{A_2} h)_{\gamma} = (\omega^{-1} e_{A_1} f, A \omega^{-1} e_{A_2} h)_{\gamma} \tag{71}
\]

where \(A\) is the operator on \(L^2(\mathbb{C}^\infty, \mu_{\gamma})\)

\[
A = \omega \zeta_1 \omega^{-1} \zeta_2
\]

Therefore

\[
e_{A_1, A_2} = [\omega^{-1} e_{A_1}]^* A [\omega^{-1} e_{A_2}] \tag{73}
\]

But \([\omega^{-1} e_{A_2}]\) is bounded and so is \([\omega^{-1} e_{A_1}]^* = [e_{A_1}, \omega]\). Thus it suffices to show that \(A\) is Hilbert-Schmidt.

Next note that \([-\Delta_\gamma, \xi] = B\) where

\[
(Bf)(x) = (-\Delta_\gamma \xi)(x) f(x) - 2(d\xi(x), df(x)) \tag{74}
\]

\(^2\) For \(\mu \geq 1\) it suffices that \(\|dg\|_\infty \leq \mu^{2\epsilon}\). Note that the larger the distance between the sets, the smaller one can take \(\|dg\|_\infty\) and hence the weaker the restriction on \(\mu\).
Using this and $\zeta_1\zeta_2 = 0$ we can rewrite $A$ as

$$A = -[\omega^{-1}B]\omega^{-2}[B\omega^{-1}]$$

But $\|Bf\| \leq \text{const}\|f\|_{+1}$. Hence $B\omega^{-1}$ is bounded on $L^2$ and so is the adjoint $\omega^{-1}B$. Finally since $\omega^{-2} = (-\Delta_\gamma + \mu)^{-1}$ is Hilbert-Schmidt as noted earlier, we conclude that $A$ is Hilbert-Schmidt to complete the proof.

To state our first main result we create Hilbert spaces based on the discs $D_i$ analogous to the construction of section 3.2 on $D_0$. Consider $\mathcal{S}_{D_i}$ with the norm $\|F\|^2 = <(\Phi(\Theta_i)F)\Phi(F) >_{\gamma_i,\mu}$ where $\Theta_i$ is the map induced by radial reflection through $C_i$. Let $\mathcal{N}_i$ be the null space, form the quotient space $\mathcal{H}_{i,0} = \mathcal{S}_{D_i}/\mathcal{N}_i$ and then take the completion

$$\mathcal{H}_i = \overline{\mathcal{H}_{i,0}}$$

If $(\cdot,\cdot)_i$ is the inner product in $\mathcal{H}_i$ and $\nu_i$ maps elements of $\mathcal{S}_{D_i}$ to equivalence classes in $\mathcal{H}_i$ then

$$(\nu_i(F), \nu_i(F'))_i = <\Phi(\Theta_i)F\Phi(F') >_{\gamma_i,\mu}$$

There are also alternate constructions of $\mathcal{H}_i$ analogous to the second and third constructions in section 3.2. In the third construction $\mathcal{H}_i = L^2(Q, \sigma_{C_i}, \mu_{\gamma_i,\mu})$.

**Theorem 1** Let $(C,\gamma)$ be a sphere with standard discs $\{D_i\}$. For $F_i \in \mathcal{S}_{D_i}$ and $\mu$ sufficiently large:

1. The expectation $<\prod_{i=1}^n \Phi(F_i)>_{\gamma_i,\mu}$ depends on $F_i$ only through the equivalence class in $\mathcal{H}_{i,0}$ and satisfies

$$|<\prod_{i=1}^n \Phi(F_i)>_{\gamma_i,\mu}| \leq C \prod_{i=1}^n \|\nu_i(F_i)\|_i$$

Hence it extends to a bounded multilinear functional on $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$.

2. Given a constant $Z_{\gamma_i,\mu}$ there is a unique linear functional $A_{\gamma_i,\mu}: \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \to \mathbb{C}$ such that if $F_i = \nu_i(F_i) \in \mathcal{H}_i$ then

$$A_{\gamma_i,\mu}(F_1 \otimes \cdots \otimes F_n) = Z_{\gamma_i,\mu} <\prod_{i=1}^n \Phi(F_i)>_{\gamma_i,\mu}$$

**Proof.** Again by the Markov property twice we have

$$\mathcal{E}_{C_i}^i \Phi(F_i) = \mathcal{E}_{D_i}^i \Phi(F_i) = \mathcal{E}_{D_i}^i \mathcal{E}_{C_i}^i \Phi(F_i) = \mathcal{E}_{D_i}^i \mathcal{E}_{C_i}^i \mathcal{E}_{C_i}^i \Phi(F_i) = \mathcal{T}_i \mathcal{E}_{C_i}^i \Phi(F_i)$$

where $\mathcal{T}_i = \mathcal{E}_{D_i}^i \mathcal{E}_{C_i}^i$. Thus lemma 5 can be rewritten as

$$|<\prod_{i=1}^n \Phi(F_i)>_{\gamma_i,\mu}| \leq C \prod_{i=1}^n \|\mathcal{T}_i \mathcal{E}_{C_i}^i \Phi(F_i)||_{2,\gamma_i,\mu}$$

At first we ignore the $\mathcal{T}_i$ using $\|\mathcal{T}_i\| \leq 1$ to write

$$|<\prod_{i=1}^n \Phi(F_i)>_{\gamma_i,\mu}| \leq C \prod_{i=1}^n \|\mathcal{E}_{C_i}^i \Phi(F_i)||_{2,\gamma_i,\mu}$$
Lemma 7. Let $L$ be a real Hilbert space and let $\Phi(F_i)$ be the Gaussian process indexed by $H$ on $(Q, \Sigma, m).$ If $T \geq 0$ is a trace class contraction on $H,$ then $\Gamma(T) \geq 0$ is a trace class contraction on $L^2(Q, \Sigma, m).$

Remark. This lemma is familiar from the proof that the partition function for the free boson gas is finite.

Proof. Let $e_k$ be a basis of eigenfunctions for $T$ with $Te_k = \lambda_k e_k$ and $0 \leq \lambda_k < 1$ and $\sum_k \lambda_k < \infty.$ Then there is an associated basis for the $L^2$ space indexed by finite sequences of non-negative integers $\{n_k\}_{k=1}^N$ and given by

$$\Phi_{\{n_k\}} = \prod_k \frac{1}{\sqrt{n_k!}} : \phi(e_1^{n_1}) \cdots \phi(e_N^{n_N}) :$$

We have

$$\Gamma(T) \Phi_{\{n_k\}} = \left( \prod_k \lambda_k^{n_k} \right) \Phi_{\{n_k\}}$$

Now we can compute

$$\text{Tr} \left( \Gamma(T) \right) = \sum_{\{n_k\}} \prod_k \lambda_k^{n_k} = \prod_k \sum_n \lambda_k^n = \prod_k \frac{1}{1 - \lambda_k}$$

The product converges since $\log(1 - \lambda_k) = O(\lambda_k)$ and $\sum_k \lambda_k < \infty.$ This completes the proof of the lemma and the theorem.

Now suppose we have a parametrized sphere as explained in section 3.3. Then we can refer everything to our standard Hilbert space $H$ based on $D_0.$
Theorem 2 Let \((C_\infty, \gamma)\) be a parametrized sphere with in-discs \(\{D_i\}_{i \in I}\) and out-discs \(\{D_i\}_{i \in I'}\). Let \(\mu\) be sufficiently large. Then there is a unique operator

\[
A^{I'}_{\gamma, \mu} : \bigotimes_{i \in I'} H \to \bigotimes_{i \in I'} H
\]

such that if \(F_i \in S_{D_0}\) and \(F = \nu(F_i) \in H\) then

\[
\left( [\bigotimes_{i \in I'} F_i], A^{I'}_{\gamma, \mu}[\bigotimes_{i \in I'} F_i] \right) = Z_{\gamma, \mu} \prod_{i \in I'} \Phi(J_i \Theta F_i) \prod_{i \in I} \Phi(J_i F_i) >_{\gamma, \mu}
\]

Furthermore \(A^{I'}_{\gamma, \mu}\) is Hilbert-Schmidt.

Proof. The map \(j_i\) defined in (27) satisfies \(j_i^* \gamma_0 = \gamma_i\) and so the pullback \(j_i^*\) on smooth functions extends to a unitary from \(L^2(C_\infty, \mu_0)\) to \(L^2(C_\infty, \mu_\gamma)\). We also have \(j_i^* \Delta_0 = \Delta_{\gamma, i} j_i^*\) and so \(j_i^*\) also determines a unitary map from \(H_{\gamma_0}^{-1, \mu}\) to \(H_{\gamma_i}^{-1, \mu}\) which takes elements with support in \(D_0\) to elements with support in \(D_i\). Furthermore \(\theta j_i = j_i \theta\) hence \(j_i^* \theta^* = \theta^* j_i^*\) and combining these facts

\[
< \Phi(\Theta F) \Phi(F) >_{\gamma_0, \mu} = < \Phi(J_i \Theta F) \Phi(J_i F) >_{\gamma_i, \mu} = < \Phi(\Theta i J_i \Theta F) \Phi(J_i F) >_{\gamma_i, \mu}
\]

Thus the map \(J_i : S_{D_0} \to S_{D_i}\) is norm preserving and so determines a unitary \(U_i : H \to H_i\) such that

\[
U_i \nu(F) = \nu_i(J_i F)
\]

The operator \((j_i^*)^*\) is also unitary from \(H_{\gamma_0, \mu}^{-1}\) to \(H_{\gamma_i, \mu}^{-1}\) and \((j_i^*)^* \theta^* = \theta^* j_i^*\) and so

\[
< \Phi(\Theta F) \Phi(F) >_{\gamma_0, \mu} = < \Phi(J_i' \Theta F) \Phi(J_i' F) >_{\gamma_i, \mu} = < \Phi(J_i' \Theta F) \Phi(J_i' F) >_{\gamma_i, \mu}
\]

Thus the map \(J_i' \Theta : S_{D_0} \to S_{D_i}\) is norm preserving and determines an anti-unitary \(V_i : H \to H_i\) such that

\[
V_i \nu(F) = \nu_i(J_i' \Theta F)
\]

Let \(H_L = \bigotimes_{i \in I} H\) and \(H_{I'} = \bigotimes_{i \in I'} H\). We define first a bounded linear functional \(A^{I'}_{I', \mu} : H_{I'} \otimes H_L \to \mathbb{C}\) (anti-linear in \(H_{I'}\)) by

\[
A^{I'}_{I', \mu} = A^{I'}_{\gamma, \mu} \circ \left( (\bigotimes_{i \in I'} V_i) \otimes (\bigotimes_{i \in I} U_i) \right)
\]

Then by (79)

\[
A^{I'}_{I', \mu} \left( [\bigotimes_{i \in I'} F_i] \otimes [\bigotimes_{i \in I} F_i] \right) = A_{\gamma_i, \mu} \left( [\bigotimes_{i \in I'} V_i F_i] \otimes [\bigotimes_{i \in I} U_i F_i] \right)
\]

\[
= A_{\gamma_i, \mu} \left( [\bigotimes_{i \in I'} \nu_i(J_i' \Theta F_i)] \otimes [\bigotimes_{i \in I} \nu_i(J_i F_i)] \right)
\]

\[
= Z_{\gamma, \mu} \prod_{i \in I'} \Phi(J_i' \Theta F_i) \prod_{i \in I} \Phi(J_i F_i) >_{\gamma, \mu}
\]

If \(\Phi_\alpha\) is an orthonormal basis for \(H_L\) and \(\Phi'_\beta\) is an orthonormal basis for \(H_{I'}\) then \(\Phi'_\beta \otimes \Phi_\alpha\) is an orthonormal basis for \(H_{I'} \otimes H_L\). Since \(A^{I'}_{\gamma, \mu}\) is a bounded linear functional on this space

\[
\sum_{\alpha, \beta} |A^{I'}_{\gamma, \mu}(\Phi'_\beta \otimes \Phi_\alpha)|^2 < \infty
\]

The bounded linear functional \(A^{I'}_{\gamma, \mu}\) determines a bounded bilinear form \(A^{I'}_{\gamma, \mu}\) on \(H_{I'} \times H_L\) (anti-linear in \(H_{I'}\)) such that \(A^{I'}_{\gamma, \mu}(\Phi' \otimes \Phi) = A^{I'}_{\gamma, \mu}(\Phi, \Phi')\). The bilinear form determines a bounded operator \(A^{I'}_{\gamma, \mu}\) from \(H_L\) to \(H_{I'}\) such that \(A^{I'}_{\gamma, \mu}(\Phi', \Phi) = (\Phi', A^{I'}_{\gamma, \mu}(\Phi))\). Then (97) says that the operator satisfies (91) and (98) says that the operator is Hilbert-Schmidt.
3.6 sewing

We now establish a sewing property in a simple configuration. This is facilitated by a special choice of the constant \( Z_{\gamma,\mu} \). If \( \gamma = \lambda \gamma_0 \) we take

\[
Z_{\gamma,\mu} = \int \exp \left( -\frac{1}{2} : \phi^2 : (\lambda \mu - \mu) \right) \ dm_{\gamma_0,\mu}
\] (99)

We start by finding a more explicit representation of the operators \( A^{I';I}_{\gamma,\mu} \) in the case where there is one out-disc. Consider a parametrized sphere \((C,\gamma)\) with out-disc \( D'_0 \) (with the identity parametrization) and in-discs \( \{ D_i \}_{i \in I} \) in \( D_0 \). Then \( \gamma = \gamma_0 \) on a neighborhood of \( D'_0 \) and so if \( \gamma = \lambda \gamma_0 \) then \( \lambda = 1 \) on a neighborhood of \( D'_0 \). We consider the corresponding amplitude denoted \( A^{I';I}_{\gamma,\mu} \). With \( \mathcal{F} = \nu(F), \mathcal{F}_j = \nu(F_j) \) we compute

\[
\left( \mathcal{F}, A^{I';I}_{\gamma,\mu} [\otimes_{i \in I} F_i] \right) = Z_{\gamma,\mu} \int \Phi(\Theta F) \prod_{i \in I} \Phi(\mathcal{J}_i F_i) \ dm_{\gamma,\mu}
\]

\[
= Z_{\gamma,\mu} \int \Phi(\Theta F) \prod_{i \in I} \Phi((\mathcal{J}_i F_i)_0) \ dm_{\gamma_0,\mu}
\]

\[
= Z_{\gamma,\mu} \int \Phi(\Theta F) \prod_{i \in I} \Phi((\mathcal{J}_i F_i)_\lambda) \left[ \frac{dm_{\gamma_0,\lambda \mu}}{dm_{\gamma_0,\mu}} \right] \ dm_{\gamma_0,\mu}
\] (100)

\[
= \int \Phi(\Theta F) \prod_{i \in I} \Phi((\mathcal{J}_i F_i)_\lambda) e^{-\phi^2 : (\lambda \mu - \mu)/2} \ dm_{\gamma_0,\mu}
\]

\[
= \left( \nu(\Phi(F)), \nu \left( \prod_{i \in I} \Phi((\mathcal{J}_i F_i)_\lambda) e^{-\phi^2 : (\lambda \mu - \mu)/2} \right) \right)
\]

Here we have used lemma [1] and lemma [2]. In the fourth step we use the constant \( Z_{\gamma,\mu} \) to cancel the denominator in the expression (97) for \( [dm_{\gamma_0,\lambda \mu}/dm_{\gamma_0,\mu}] \). In the last step we take \( \Phi(\Theta F) = \Theta(\Phi(F)) \) and use the second construction of \( \mathcal{H} \), taking into account that \( \lambda = 1 \) on \( D'_0 \) so \( \exp(- : \phi^2 : (\lambda \mu - \mu)/2) \) is measurable with respect to \( \Sigma_{D_0} \) by lemma [2]. Since \( \nu(\Phi(F)) \leftrightarrow \nu(F) = \mathcal{F} \) under the equivalence of constructions we conclude that

\[
A^{I';I}_{\gamma,\mu} [\otimes_{i \in I} F_i] = \nu \left( \prod_{i \in I} \Phi((\mathcal{J}_i F_i)_\lambda) e^{-\phi^2 : (\lambda \mu - \mu)/2} \right)
\] (101)

Also consider a parametrized sphere \((C,\gamma)\) with in-disc \( D_0 \) (with the identity parametrization) and out-discs \( \{ D_i \} \) in \( D'_0 \). Then \( \gamma = \gamma_0 \) on a neighborhood of \( D_0 \) and so if \( \gamma = \lambda \gamma_0 \) then \( \lambda = 1 \) on a neighborhood of \( D_0 \). We consider the corresponding amplitude denoted \( A^{I';I}_{\gamma,\mu} \). Then

\[
\left( [\otimes_{i \in I} F_i], A^{I';I}_{\gamma,\mu} \mathcal{F} \right) = Z_{\gamma,\mu} \int \prod_{i \in I'} \Phi((\mathcal{J}_i' \Theta F_i)) \Phi(F) \ dm_{\gamma',\mu}
\]

\[
= Z_{\gamma,\mu} \int \prod_{i \in I'} \Phi((\mathcal{J}_i' \Theta F_i)_\lambda) \Phi(F) \ dm_{\gamma_0,\lambda \mu}
\]

\[
= Z_{\gamma,\mu} \int \prod_{i \in I'} \Phi((\mathcal{J}_i' \Theta F_i)_\lambda) \Phi(F) \left[ \frac{dm_{\gamma_0,\lambda \mu}}{dm_{\gamma_0,\mu}} \right] \ dm_{\gamma_0,\mu}
\] (102)

\[
= \int \prod_{i \in I'} \Phi((\mathcal{J}_i' \Theta F_i)_\lambda) \Phi(F) e^{-\phi^2 : (\lambda \mu - \mu)/2} \ dm_{\gamma_0,\mu}
\]

\[
= \left( \Theta e^{-\phi^2 : (\lambda \mu - \mu)/2} \prod_{i \in I'} \Phi((\mathcal{J}_i' \Theta F_i)_\lambda), \nu(\Phi(F)) \right)
\]
Then we have
\[
(A_{\gamma,\mu}^{1,1})^*[(\otimes_{i\in I'}F_i)] = \nu \left( \Theta e^{-\lambda(\gamma_0 - \mu)/2} \prod_{i \in I'} \Phi((J_i^*\Theta F_i)_{\lambda_i}) \right)
\] (103)

We sew together two such amplitudes by composition. Here is the result for a simple configuration:

**Theorem 3** Let \((C, \gamma)\) be a parametrized sphere with in-discs \(\{D_i\}_{i \in I}\) contained in \(D_0\) and out-discs \(\{D_i'\}_{i \in I'}\) contained in \(D'_0\). Let \(\gamma_1\) be a metric with \(\gamma_1 = \gamma\) on \(D_0\) and \(\gamma_1 = \gamma_0\) on \(D'_0\). Let \(\gamma_2\) be a metric with \(\gamma_2 = \gamma_0\) on \(D_0\) and \(\gamma_2 = \gamma\) on \(D'_0\). Then

\[
A_{\gamma_2,\mu}^{1,1}A_{\gamma_1,\mu}^{1,1} = A_{\gamma,\mu}^{1,1}
\] (104)

**Proof.** We have \(\gamma_1 = \lambda_1 \gamma_0\), \(\gamma_2 = \lambda_2 \gamma_0\), \(\gamma = \lambda \gamma_0\). Then \(\lambda_1 = \lambda\) on \(D_0\) and \(\lambda_1 = 1\) on \(D'_0\). Also \(\lambda_2 = 1\) on \(D_0\) and \(\lambda_2 = \lambda\) on \(D'_0\). Then we compute using (101) and (103)

\[
\begin{align*}
&\left( [\otimes_{i\in I'}F_i], A_{\gamma_2,\mu}^{1,1}A_{\gamma_1,\mu}^{1,1}[\otimes_{i\in I'}F_i] \right) \\
&= \left( \nu \left( \Theta e^{-\lambda(\gamma_0 - \mu)/2} \prod_{i \in I'} \Phi((J_i^*\Theta F_i)_{\lambda_i}) \right) , \nu \left( \prod_{i \in I} \Phi((J_i F_i)_{\lambda_i}) e^{-\lambda(\gamma_0 - \mu)/2} \right) \right) \\
&= \int \prod_{i \in I'} \Phi((J_i^*\Theta F_i)_{\lambda_i}) \prod_{i \in I} \Phi((J_i F_i)_{\lambda_i}) \left( \frac{dm_{\gamma_0,\lambda_i}}{dm_{\gamma_0,\mu}} \right) dm_{\gamma_0,\mu} \\
&= Z_{\gamma,\mu} \int \prod_{i \in I'} \Phi((J_i^*\Theta F_i)_{\lambda_i}) \prod_{i \in I} \Phi((J_i F_i)_{\lambda_i}) \left( \frac{dm_{\gamma_0,\lambda_i}}{dm_{\gamma_0,\mu}} \right) dm_{\gamma_0,\lambda_i} \\
&= Z_{\gamma,\mu} \int \prod_{i \in I'} \Phi((J_i^*\Theta F_i)_{\lambda_i}) \prod_{i \in I} \Phi((J_i F_i)_{\lambda_i}) dm_{\gamma_0,\lambda_i} \\
&= Z_{\gamma,\mu} \int \prod_{i \in I'} \Phi((J_i^*\Theta F_i)_{\lambda_i}) \prod_{i \in I} \Phi((J_i F_i)_{\lambda_i}) dm_{\gamma_0,\lambda_i} \left( \otimes_{i\in I'}F_i, A_{\gamma_1,\mu}^{1,1}[\otimes_{i\in I'}F_i] \right) \\
\end{align*}
\] (105)

Here we have used \((\lambda_1 - 1) + (\lambda_2 - 1) = \lambda - 1\) in the second step. This completes the proof.

Since the product of two Hilbert-Schmidt operators is trace class we have

**Corollary 1** \(A_{\gamma,\mu}^{1,1}\) is trace class

**Remarks.**

1. The fact that we are sewing together the out-disc \(D'_0\) with a the in-disc \(D_0\) was just a convenience. More general configurations can be treated by the same methods.

2. Because all our amplitudes refer to spheres we have managed to avoid any actual sewing of manifolds \(A\) somewhat different approach to sewing was developed in [3]. It had the advantage of working for any compact Riemann surface, but the disadvantage that the identities like (102) did not hold.

3. Another way to characterize our sewing theorem is to take an orthonormal basis \(\Phi_\alpha\) for \(\mathcal{H}\) and then with \(\Psi' = \otimes_{i \in I'}F_i\) and \(\Psi = \otimes_{i \in I'}F_i\) we have

\[
\sum_\alpha (\Psi, A_{\gamma_2,\mu}^{1,1} \Phi_\alpha)(\Phi_\alpha, A_{\gamma_1,\mu}^{1,1} \Psi) = (\Psi', A_{\gamma,\mu}^{1,1} \Psi)
\] (106)
In the same way we could sew together two legs on the same sphere forming for example
\[ \sum_\alpha ((\Psi \otimes \Phi_\alpha), A_{\tau,\mu}^{I_1,J_1} (\Phi_\alpha \otimes \Psi')) \] (107)

Our estimates are good enough to show that the sum converges. However this should be identified with an amplitude on a torus which is outside the scope of this paper.

4 Massless fields

4.1 fields

Now consider massless fields. In this case we have a conformal field theory which we develop following Gawedski [5]. The interesting fields are now the exponential fields \( e^{ik\phi(x)} \). If we restrict to integer \( k \) the these are the fields that occur in circle valued compactification.

The fields \( e^{ik\phi(x)} \) are singular objects and we will need a regularized version denoted \([e^{ik\phi(x)}]_r\). Our first task is to give a meaning to expressions like
\[ \langle [e^{ik\phi(x_1)}]_r \cdots [e^{ik\phi(x_n)}]_r \rangle_\gamma \] (108)

We approach the problem by starting again with massive fields and then taking the limit as the mass goes to zero (see also [2]). So as in section 3.1 let \( \{\phi(f)\} \) with be a family of Gaussian random variables indexed by \( H_{\gamma,\mu}^{-1} \) with covariance given by the inner product \((\Delta_\gamma + \mu)^{-1}\) and let \( \langle \cdots \rangle_\gamma,\mu \) be the expectation. We define first for smooth \( f \)
\[ [e^{i\phi(f)}]_r = e^{i\phi(f)} e^{(f,G^\#_\gamma f)/2} \] (109)
where \( G^\#_\gamma \) is an operator on \( L^2(\mathbb{C}_\infty,\mu_\gamma) \) with kernel \( G^\#_\gamma(x,y) \) satisfying
\[ (f,G^\#_\gamma h) = \int f(x)G^\#_\gamma(x,y)h(y)d\mu_\gamma(x)d\mu_\gamma(y) \] (110)
and where the kernel is chosen to have a specific singularity at \( x = y \). One possible choice is to take \( G^\#_\gamma \) to be \((-\Delta_\gamma + \mu)^{-1}\) in which case the regularization is Wick ordering. However the \( \mu \) dependence leads to problems. Instead we take something which is independent of \( \mu \), has the same singularity, and is still covariant, namely
\[ G^\#_\gamma(x,y) = -\frac{1}{2\pi} \log(d_\gamma(x,y)) \] (111)
where \( d_\gamma(x,y) \) is the distance.

We want to take \( f = \delta_x \) the delta function at \( x \). Instead let \( \delta_\gamma(\cdot - x) \) be an approximate delta function in the plane satisfying \( \int \delta_\gamma(y-x)dy = 0 \). Then an approximate delta function on \( \mathbb{C}_\infty \) is given (in local coordinates) by \(|\gamma|^{-1/2}\delta_\gamma(\cdot - x)\). Indeed we have for any continuous \( h \)
\[ \lim_{\kappa \to \infty} (|\gamma|^{-1/2}\delta_\gamma(\cdot - x), h)_\gamma = h(x) \equiv \delta_x(h) \] (112)
We define a regularized field
\[ \phi_\gamma(z) = \phi(\gamma|^{-1/2}\delta_\gamma(\cdot - z)) \] (113)
and then
\[ [e^{ik\phi_\gamma(z)}]_r = e^{ik\phi_\gamma(z)} \exp \left( \frac{k^2}{2} \int \delta_\gamma(z-x)G^\#_\gamma(x,y)\delta_\gamma(y-z)dx dy \right) \] (114)
Proof
We compute be the fundamental solution for \( \Delta \in M \).

**Theorem 4** Let

\[ Z = [k_1, z_1, \ldots, k_n, z_n] \tag{115} \]

be a sequence of integers \( k_i \) and points \( z_i \in \mathbb{C}_\infty \). If the \( z_i \) are distinct then the limit

\[ < Z >_\gamma = \lim_{\kappa \to \infty} \lim_{\mu \to 0} < [e^{ik_1 \phi_\gamma(z_1)}] \ldots [e^{ik_n \phi_\gamma(z_n)}] >_{\gamma, \mu} \tag{116} \]

exists. If \( \gamma = e^{\sigma |dz|^2} \) the limit is

\[ < Z >_{e^{\sigma \gamma}} = \begin{cases} 0 & \exp \left( -\frac{1}{8\pi} \sum_i k_i^2 \sigma(z_i) \right) \prod_{i<j} |z_i - z_j|^{k_i k_j / 2\pi} \sum_i k_i 
eq 0 \sum_i k_i = 0 \end{cases} \tag{117} \]

The expression \( (117) \) gives a precise meaning to \( (108) \) with \( Z = [k_1, z_1, \ldots, k_n, z_n] \) standing for the formal expression \( \prod [e^{ik_0 \phi(z_i)}]_{\gamma} \). Note that \( < Z >_{\gamma} \) is a symmetric function of the \( (k_i, z_i) \). The theorem has the immediate Corollary

**Corollary 2** For any conformal metric \( \gamma \)

\[ < Z >_{e^{\sigma \gamma}} = \exp \left( -\sum_i \frac{k_i^2}{8\pi} \sigma(z_i) \right) < Z >_{\gamma} \tag{118} \]

Before proving the theorem we get a preliminary result. The negative Laplacian is not invertible on all of \( L^2(\mathbb{C}_\infty, \mu_\gamma) \). But it is invertible if we restrict to the orthogonal complement of the constants denoted \( L^{2,\perp}(\mathbb{C}_\infty, \mu_\gamma) \). We denote the inverse by \( (-\Delta_\gamma)^{-1} \). Also let

\[ G^\#(z, z') = \frac{-1}{2\pi} \log |z - z'| \tag{119} \]

be the fundamental solution for \( -\Delta \) in the plane.

**Lemma 8**

1. For smooth \( f \in L^{2,\perp} \)

\[ g(z) = \int G^\#(z, z') f(z') d\mu_\gamma(z') \tag{120} \]

defines a function on \( \mathbb{C}_\infty \) which satisfies \( (-\Delta_\gamma) g = f \)

2. For smooth \( f, h \in L^{2,\perp} \)

\[ (h, (-\Delta_\gamma)^{-1} f) = \int h(z) G^\#(z, z') f(z') d\mu_\gamma(z) d\mu_\gamma(z') \tag{121} \]

**Proof.** The function \( g(z) \) is well-defined for \( z \in \mathbb{C} \) since the measure \( d\mu_\gamma(z') \) is \( \mathcal{O}(|z'|^{-4}) \) as \( z \to \infty \). We compute

\[ (-\Delta_\gamma g)(z) = |\gamma(z)|^{-1/2} (-\Delta_\gamma) \int G^\#(z, z') f(z') d\mu_\gamma(z') = |\gamma(z)|^{-1/2} |\gamma(z)|^{1/2} f(z) = f(z) \tag{122} \]

To include the point at infinity we go to the other coordinate patch. First write

\[ g(z) = \int (G^\#(z, z') - G^\#(z, 0)) f(z') d\mu_\gamma(z') = \frac{-1}{2\pi} \int \log \left| 1 - \frac{z'}{z} \right| f(z') d\mu_\gamma(z') \tag{123} \]
and then with \( \tilde{g}(\zeta) = g(1/\zeta) \) and \( \hat{\mu}_\gamma \) the measure in the new coordinates we have

\[
\tilde{g}(\zeta) = -\frac{1}{2\pi} \int \log \left| 1 - \frac{\zeta}{\zeta'} \right| \hat{f}(\zeta')d\hat{\mu}_\gamma(\zeta')
\]  

(124)

which is finite at \( \zeta = 0 \). Then \(( -\Delta_\gamma )\tilde{g} = \tilde{f} \) as before. This proves the first point.

For the second point define \( g \) as above and compute

\[
(( -\Delta_\gamma )^{-1}h, f) = (( -\Delta_\gamma )^{-1}h, ( -\Delta_\gamma )g) = (h, g)
\]  

(125)

This completes the proof.

**Proof.** (of theorem 4) Define

\[
f^\gamma_i = \sum_k f^\gamma_{i,k} \quad f^\gamma_{i,k}(y) = k_i |\gamma(y)|^{-1/2} \delta_\gamma(y-z_i)
\]  

(126)

We have

\[
< e^{i k_1 \phi_\gamma(z_1)} \cdots e^{i k_n \phi_\gamma(z_n)} >_{\gamma,\mu} = \exp(i \phi(f^\gamma_i)) >_{\gamma,\mu} \exp \left( \frac{1}{2} \sum_i (f^\gamma_i, G^\gamma_i f^\gamma_i) \right)
\]  

(127)

Note that

\[
\int f^\gamma_i(y) d\mu_\gamma(y) = \sum_i k_i \int \delta_\gamma(y-z_i)dy = \sum_i k_i
\]  

(128)

If \( \sum_i k_i \neq 0 \) then \( f^\gamma_i \) has a constant component in \( L^2(C_\infty, d\mu_\gamma) \) and hence

\[
\lim_{\mu \to 0} (f^\gamma_i, (-\Delta_\gamma + \mu)^{-1} f^\gamma_i) = +\infty
\]  

(129)

which shows that the expression \( \text{(127)} \) goes to zero. Thus we can restrict attention to the case \( \sum_i k_i = 0 \) in which case \( f^\gamma_i \) is orthogonal to constants and \( (-\Delta_\gamma)^{-1} f^\gamma_i \) is well-defined and

\[
\lim_{\mu \to 0} (f^\gamma_i, (-\Delta_\gamma + \mu)^{-1} f^\gamma_i) = (f^\gamma_i, (-\Delta_\gamma)^{-1} f^\gamma_i)
\]  

(130)

The latter is evaluated by lemma 5 and so

\[
\lim_{\mu \to 0} < e^{i k_1 \phi_\gamma(z_1)} \cdots e^{i k_n \phi_\gamma(z_n)} >_{\gamma,\mu} = \exp \left( -\frac{1}{2} \sum_{ij} k_i k_j \int \delta_\gamma(z_i-x) G^\gamma(x,y) \delta_\gamma(y-z_j) dxdy \right) \exp \left( \frac{1}{2} \sum_i k_i^2 \int \delta_\gamma(z_i-x) G^\gamma(x,y) \delta_\gamma(y-z_i) dxdy \right)
\]  

(131)

Since \( G^\gamma(x,y) \) is continuous away from \( x = y \), the terms with \( i \neq j \) have a limit which is

\[
\exp \left( -\sum_{i < j} k_i k_j G^\gamma(z_i, z_j) \right) = \prod_{i < j} |z_i - z_j|^k_{i,j}/2\pi
\]  

(132)
It remains to study the contribution from terms with \( i = j \) which now have the form

\[
\exp \left( \frac{1}{4\pi} \sum_i k_i^2 \int \delta_n(z_i - x) \left[ \log |x - y| - \log d_n(x, y) \right] \delta_n(y - z_i) dx dy \right)
\]

(133)

However \( \gamma = e^\sigma |dz|^2 \) and in Appendix C we show that

\[
\lim_{y \to x} \log d_n(x, y) - \log |x - y| = \frac{\sigma(x)}{2}
\]

(134)

With this definition at coinciding points \( [\log |x - y| - \log d_n(x, y)] \) is continuous. Then (133) has the limit as \( \kappa \to \infty \)

\[
\exp \left( -\frac{1}{8\pi} \sum_i k_i^2 \sigma(z_i) \right)
\]

(135)

to complete the proof.

Next we exhibit the covariance of our expectations under Mobius transformations

\[
\alpha(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0
\]

(136)

These are biholomorphic on \( \mathbb{C}_\infty \) and preserve the class of conformal metrics

**Lemma 9** Let \( \alpha \) be a Mobius transformation and \( z_i = \alpha(w_i) \). Then

\[
< [k_1, z_1, \ldots, k_n, z_n] >\gamma =< [k_1, w_1, \ldots, k_n, w_n] >\alpha^* (\gamma)
\]

(137)

**Proof.** We have for the covariance

\[
(f_i, (\Delta_\gamma + \mu)^{-1} f)\gamma = (\alpha^* f, (\Delta_{\alpha^* \gamma} + \mu)^{-1} \alpha^* f)_{\alpha^* \gamma}
\]

(138)

and similarly for the operator \( \gamma^\# \). Thus we have the identity

\[
\exp \left( -\frac{1}{2} \sum_{ij} (f_{ik}^\gamma, (\Delta_\gamma + \mu)^{-1} f_{jk}^\gamma)_{\gamma} + \frac{1}{2} \sum_i (f_{ik}^\gamma, \gamma^\# f_{ik}^\gamma)_{\gamma} \right)
\]

\[
= \exp \left( -\frac{1}{2} \sum_{ij} (\alpha^* f_{ik}^\gamma, (\Delta_{\alpha^* \gamma} + \mu)^{-1} \alpha^* f_{jk}^\gamma)_{\alpha^* \gamma} + \frac{1}{2} \sum_i (\alpha^* f_{ik}^\gamma, \gamma^\# \alpha^* f_{ik}^\gamma)_{\alpha^* \gamma} \right)
\]

(139)

As we have seen the left side converges to \( < [k_1, z_1, \ldots, k_n, z_n] >\gamma \) as \( \mu \to 0, \kappa \to \infty \) since \( f_{ik}^\gamma \) converges to \( \kappa \delta_{z_i} \) in the metric \( \gamma \) when integrated against continuous functions. For the right side first take \( \mu \to 0 \) as before. Then as \( \kappa \to \infty \) we have that \( \alpha^* f_{ik}^\gamma \) converges to \( \kappa \delta_{w_i} \) in the metric \( \alpha^* \gamma \) since for a continuous function \( h \)

\[
(\alpha^* f_{ik}^\gamma, h)_{\alpha^* \gamma} = (f_{ik}^\gamma, \alpha_* h)_\gamma \to \kappa \delta_{z_i} (\alpha_* h) = \kappa h(w_i)
\]

(140)

Here \( \alpha_* \) is the push forward \( (\alpha_* h)(z) = h(\alpha^{-1}(z)) \). Using this fact one shows in the same way that the right side converges to \( < [k_1, w_1, \ldots, k_n, w_n] >\alpha^* (\gamma) \) to complete the proof.

**Remarks.** Given \( Z = [k_1, z_1, \ldots, k_n, z_n] \) suppose we take a metric \( \gamma = |dz|^2 \) near \( z_i \). Then

\[
< Z >\gamma = \prod_{i<j} |z_i - z_j|^{k_i k_j / 2\pi}
\]

(141)
The expectation is independent of the metric and we write it as \( <Z> \).

If \( \gamma = |dz|^2 \) in some region \( \Omega \) and \( z = \alpha(w) \) is a Mobius transformation then

\[
\alpha^* (\gamma) = \frac{\partial z}{\partial w} \frac{\partial \bar{z}}{\partial \bar{w}} |dw|^2 = \left| \frac{\partial z}{\partial w} \right|^2 |dw|^2
\]

in \( \alpha^{-1}(\Omega) \). If we specialize \( \lbrack 37 \rbrack \) to a metric \( \gamma \) flat on \( Z \) and use \( \lbrack 13 \rbrack \) on the right side we get the familiar

\[
< [k_1, z_1, \ldots, k_n, z_n] > = \prod_i \left[ \frac{\partial z}{\partial w}(w_i) \right]^{-k_i^2/4\pi} < [k_1, w_1, \ldots, k_n, w_n] >
\]

4.2 an algebra of symbols

We next reformulate some of these results in a more algebraic language. Define the product of \( Z = [k_1, z_1, \ldots, k_n, z_n] \) and \( Z' = [k'_1, z'_1, \ldots, k'_m, z'_m] \) to be

\[
ZZ' = [k_1, z_1, \ldots, k_n, z_n, k'_1, z'_1, \ldots, k'_m, z'_m]
\]

(144)

Then the space of sequences is a monoid and we let \( \Upsilon \) be the free algebra generated by this monoid. The elements of \( \Upsilon \) are functions \( F \) from sequences \( Z \) to \( F(Z) \in \mathbb{C} \) written

\[
F = \sum Z F(Z) \tag{145}
\]

such that \( F(Z) = 0 \) for all but finitely many \( Z \). Scalar multiplication, addition, and multiplication satisfy

\[
\alpha F = \sum Z \alpha F(Z) \quad \alpha \in \mathbb{C}
\]

\[
F + F' = \sum Z (F(Z) + F'(Z))Z
\]

\[
FF' = \sum Z Z' F(Z)F'(Z')ZZ'
\]

(146)

We define the subspace

\[
\Upsilon_0 = \{ F \in \Upsilon : F(Z) = 0 \text{ if } Z \text{ has coinciding points } \}
\]

(147)

and for \( A \subset \mathbb{C}_\infty \) the subalgebra localized in \( A \)

\[
\Upsilon_A = \{ F \in \Upsilon : F(Z) = 0 \text{ if } Z \text{ has points outside } A \}
\]

(148)

Also let \( \Upsilon_{0,A} = \Upsilon_0 \cap \Upsilon_A \).

Define expectations as a linear functional on \( \Upsilon_0 \) by

\[
< F >_\gamma = \sum Z F(Z) < Z >_\gamma
\]

(149)

We collect some properties of these expectations.

1. For any Mobius transformation \( \alpha \) define an isomorphism \( \tau_\alpha \) on \( \Upsilon \) by setting

\[
\tau_\alpha(Z) = \tau_\alpha[k_1, z_1, \ldots, k_n, z_n] = [k_1, \alpha(z_1), \ldots, k_n, \alpha(z_n)]
\]

(150)

and then extending by linearity to the full algebra. Then for \( F \in \Upsilon_0 \) we have by \( \lbrack 47 \rbrack \)

\[
< \tau_\alpha F >_\gamma = < F >_{\alpha^*(\gamma)}
\]

(151)
2. Given two conformal metrics $\gamma', \gamma$ with $\gamma' = e^{\sigma} \gamma$ we define an isomorphism $\tau_{\gamma', \gamma}$ on $\Upsilon$ by

$$\tau_{\gamma', \gamma}(Z) = \tau_{\gamma', \gamma}[k_1, z_1, \ldots, k_n, z_n] = \exp \left( \frac{1}{8\pi} \sum_i k_i^2 \sigma(z_i) \right) [k_1, z_1, \ldots, k_n, z_n]$$

and then extending by linearity. Then for $F \in \Upsilon_0$ we have by (118)

$$< F >_{\gamma} = < \tau_{\gamma', \gamma} F >_{\gamma'}$$

3. Define a homomorphism $\tau_\kappa$ from $\Upsilon$ to the polynomial algebra generated by $[e^{ik\phi_\kappa(x)}]_r$ by

$$\tau_\kappa(Z) = \tau_\kappa([k_1, z_1, \ldots, k_n, z_n]) = \prod_i [e^{ik_i \phi_\kappa(z_i)}]_r$$

and then extended by linearity. For $F \in \Upsilon_0$ we have by theorem 4

$$< F >_{\gamma} = \lim_{\kappa \to \infty} \lim_{\mu \to 0} < \tau_\kappa(F) >_{\gamma, \mu}$$

4. We also have a reflection positivity result. Our radial reflection mapping $\theta$ induces a reflection $\Theta$ on sequences by

$$\Theta Z = \Theta[k_1, z_1, \ldots, k_n, z_n] = [-k_1, \theta z_1, \ldots, -k_n, \theta z_n]$$

This induces an anti-linear mapping $\Theta$ on $\Upsilon$ by

$$\Theta \left( \sum_{Z} F(Z) Z \right) = \sum_{Z} F(Z) \Theta Z$$

We work with open discs

$$B_0 = \{ z : |z| < 1 \} \quad B_0' = \{ z : |z| > 1 \}$$

rather than the closed discs $D_0, D_0'$ used in the massive case. If $F \in \Upsilon_{0,B_0}$ then $\Theta F \in \Upsilon_{0,B_0'}$ and $(\Theta F) F \in \Upsilon_0$.

**Lemma 10** Let $\gamma$ be reflection invariant $\theta^* \gamma = \gamma$. If $F \in \Upsilon_{0,B_0}$ then

$$< (\Theta F) >_{\gamma} \geq 0$$

**Proof.** First consider $\Theta$ on functions in $L^2(Q, \Sigma, m_{\gamma, \mu})$ as defined in (13). Since the power series for $e^{i\phi(f)}$ converges in $L^2$ and since $\Theta \phi(f) \Theta = \phi(\theta^* f)$ we have $\Theta e^{i\phi(f)} \Theta = e^{-i\phi(\theta^* f)}$. Since $\theta$ is an isometry we also have $(f, G^0_{\gamma} f) = (\theta^* f, G^0_{\gamma} \theta^* f)$ and hence $\Theta [e^{i\phi(f)}]_r \Theta = [e^{-i\phi(\theta^* f)}]_r$.

Now we compute for $Z, Z' \in \Upsilon_{B_0}$

$$< (\Theta Z) Z' >_{\gamma} = < [-k_1, \theta z_1, \ldots, -k_n, \theta z_n, k_1, z_1', \ldots, k_n, z_n'] >_{\gamma}$$

$$= \lim_{\kappa \to \infty} \lim_{\mu \to 0} \prod_{i=1}^n [e^{-ik_i \phi(\theta^* \gamma)^{-1/2} \delta_\kappa(\cdot - z_i)}]_r \prod_{j=1}^m [e^{ik_j \phi(\gamma)^{-1/2} \delta_\kappa(\cdot - z_j)}]_r >_{\gamma, \mu}$$

$$= \lim_{\kappa \to \infty} \lim_{\mu \to 0} \Theta \left( \prod_{i=1}^n [e^{-ik_i \phi(\theta^* \gamma)^{-1/2} \delta_\kappa(\cdot - z_i)}]_r \right) \prod_{j=1}^m [e^{ik_j \phi(\gamma)^{-1/2} \delta_\kappa(\cdot - z_j)}]_r >_{\gamma, \mu}$$

$$= \lim_{\kappa \to \infty} \lim_{\mu \to 0} < (\Theta \tau_\kappa Z) \tau_\kappa Z' >_{\gamma, \mu}$$

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The second step we put $\theta^* |\gamma|^{-1/2} \delta_e(\cdot - z_i)$ rather than the usual $|\gamma|^{-1/2} \delta_e(\cdot - \theta z_i))$. This still works since $\theta^* |\gamma|^{-1/2} \delta_e(\cdot - z_i)$ still converges to the delta function at $\theta z_i$ as in (140).

It follows that

$$\langle (\Theta F) \rangle_{\gamma^\prime} = \lim_{\kappa \to \infty} \lim_{\mu \to 0} \langle \Theta(\tau_\kappa F) \tau_\kappa F \rangle_{\gamma^\prime,\mu}$$

(161)

For $\kappa$ sufficiently large $\tau_\kappa F$ is $\Sigma_{D_0}$ measurable and so $\langle \Theta(\tau_\kappa F) \tau_\kappa F \rangle_{\gamma^\prime,\mu} \geq 0$ by the massive result (14). We conclude $\langle (\Theta F) \rangle_{\gamma^\prime} \geq 0$.

### 4.3 a standard Hilbert space

As in section 2.1 we choose a standard metric, but now we want it to be as flat as possible. Well inside $B_0$ we want it to be $|dz|^2$. We could try to make it reflection invariant by restricting to $B_0$ and then reflecting to get the metric $|z|^{-4} |dz|^2 = |d\zeta|^2$ in $B'_0$. However the result would not be smooth at the boundary. This is corrected by the requirement that the metric be $|z|^{-2} |dz|^2$ on a neighborhood of the boundary. Thus we define for some constant $\gamma$

$$\gamma_0 = \rho_0(z) |dz|^2$$

$$\rho_0(z) = \begin{cases} 1 & |z| < e^{-2d} \\ |z|^{-2} & e^{-d} < |z| < e^d \\ e^{2d} & |z| > e^d \end{cases}$$

(162)

In the regions $e^{-2d} \leq |z| \leq e^{-d}$ and $e^d \leq |z| \leq e^{2d}$ the function $\rho_0(z)$ is a smooth interpolation that preserves the reflection invariance.

If $Z, Z'$ are monoids in $\mathcal{Y}_{0,B_0}$ with $Z = \{k_1, z_1, \ldots, k_n, z_n\}$ and if $d$ is sufficiently small, then by (118)

$$\langle (\Theta Z) Z' \rangle_{\gamma_0} = \prod_i |\theta z_i|^{k_i^2/2\pi} \langle (\Theta Z) Z' \rangle \geq \langle (\tilde{\Theta} Z) Z' \rangle$$

(163)

where we define

$$\tilde{\Theta} Z = \prod_i |\theta z_i|^{k_i^2/2\pi} \Theta Z$$

(164)

Extend $\tilde{\Theta}$ to be anti-linear on the whole algebra and then for $F \in \mathcal{Y}_{0,B_0}$

$$\langle (\tilde{\Theta} F) \rangle \geq \langle (\Theta F) \rangle_{\gamma_0} \geq 0$$

(165)

by lemma 10. This is positivity for the flat expectation. For another way to derive it see [4].

Now start with with the vector space $\mathcal{Y}_{0,B_0}$ and give it the norm $\|F\|^2 = \langle (\tilde{\Theta} F) F \rangle$. Divide by the null space $\mathcal{N} = \{F : \|F\| = 0\}$ and get an inner product space $\mathcal{H}_0 = \mathcal{Y}_{0,B_0}/\mathcal{N}$. Then take the completion to get the standard Hilbert space

$$\mathcal{H} = \overline{\mathcal{H}_0} = \overline{\mathcal{Y}_{0,B_0}/\mathcal{N}}$$

(166)

If $\nu$ is the mapping from $\mathcal{Y}_{0,B_0}$ to $\mathcal{H}_0$, then

$$\langle \nu(F), \nu(F') \rangle = \langle (\tilde{\Theta} F) F' \rangle$$

(167)

### 4.4 amplitudes

As in section 2.3 we now suppose we are given open discs $B_i \subset \mathbb{C}_\infty$ with disjoint closures $D_i$. These will have the form $B_i = \{z : |z - a_i| < r_i\}$ or $B_i = \{z : |z - a_i| > r_i\}$ or else be a half plane. Let $\alpha_i$ be the Mobius transformation which takes $B_i$ to the unit disc $B_0$ as in (24) and let $\gamma_i = \alpha_i^* \gamma_0$ be a standard flat metric based on $B_i$. We consider metrics $\gamma$ such that $\gamma = \gamma_i$ on a neighborhood of $B_i$. Then $(\mathbb{C}_\infty, \gamma)$ is a sphere with standard flat discs $B_i$. 

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Also as in section [3] we parametrize the discs. They are divided into in-discs \( \{ B_i \}_{i \in I} \) and out-discs \( \{ B'_i \}_{i \in I'} \). For each in-disc \( B_i \) let \( j_i(z) = e^{i \alpha_i}(z) \) which takes \( B_i \) to \( B_0 \) and satisfies \( j_i^* \gamma_0 = \gamma_i \). For each out-disc \( B'_i \) let \( j'_i = e^{i \alpha_i}(z)^{-1} \) which takes \( B'_i \) to \( B'_0 \) and satisfies \( (j'_i)^* \gamma_0 = \gamma_i \). These are Mobius transformations and we define the isomorphisms on \( \mathcal{Y} \) by

\[
\mathcal{J}_i = \tau_j^{-1} \quad \mathcal{J}'_i = \tau_{j'_i}^{-1}
\]

Then

\[
\mathcal{J}_i : \mathcal{Y}_{B_0} \to \mathcal{Y}_{B_i} \quad \mathcal{J}'_i : \mathcal{Y}_{B'_0} \to \mathcal{Y}_{B'_i}
\]

Again we want to study the amplitudes \( Z_\gamma < \prod_{i \in I} \mathcal{J}_i' \Theta F_i \prod_{i \in I} \mathcal{J}_i F_i > \gamma \) with \( F_i \in \mathcal{Y}_{0,B_0} \). For our purposes it is enough to take \( Z_\gamma = 1 \), although there there are other possibilities [5]. Also for given \( F_i \) we can choose the parameter \( d \) in \( \gamma_0 \) sufficiently small so that the points of \( F_i \) are entirely in the flat region. Then the points of \( \mathcal{J}_i F_i \) and \( \mathcal{J}_i' \Theta F_i \) in \( B_i \) are entirely in the flat region for \( \gamma_i \), and since \( \gamma = \gamma_i \) on \( B_i \) the points are in the flat region for \( \gamma \). Then by [153] the expectations \( < \prod_{i \in I} \mathcal{J}_i' \Theta F_i \prod_{i \in I} \mathcal{J}_i F_i > \gamma \) are independent of the particular \( \gamma_0 \) or \( \gamma \) that we choose.

**Theorem 5** Let \( (\mathbb{C}_\infty, \gamma) \) be a parametrized sphere with in-discs \( \{ B_i \}_{i \in I} \) and out-discs \( \{ B'_i \}_{i \in I'} \). Then there is a bilinear form on the algebraic tensor products

\[
A'' : (\otimes_{i \in I'} \mathcal{H}_0) \times (\otimes_{i \in I} \mathcal{H}_0) \to \mathbb{C}
\]

anti-linear in the first factor, such that for \( F_i = \nu(F_i), F_i \in \mathcal{Y}_{0,B_0} \)

\[
A'' ( [\otimes_{i \in I} F_i], [\otimes_{i \in I'} F_i] ) = < \prod_{i \in I'} \mathcal{J}_i' \Theta F_i \prod_{i \in I} \mathcal{J}_i F_i > \gamma
\]

**Proof.** For any \( k \in I \) let \( G_k = \prod_{i \in I'} \mathcal{J}_i' \Theta F_i \prod_{i \in I, i \neq k} \mathcal{J}_i F_i \). Then we have

\[
< \prod_{i \in I'} \mathcal{J}_i' \Theta F_i \prod_{i \in I} \mathcal{J}_i F_i > \gamma = < G_k \mathcal{J}_k F_k > \gamma
\]

\[
= < \tau_{\tau_k, \gamma} (G_k \mathcal{J}_k F_k) > \gamma = < (\tau_{\tau_k, \gamma} G_k) \mathcal{J}_k F_k > \gamma = < (\mathcal{J}_k^{-1} \tau_{\tau_k, \gamma} G_k) F_k > \gamma_0
\]

Here we have used [153], then \( \tau_{\tau_k, \gamma} G_k = \tau_{\gamma, k} \) since \( \gamma = \gamma_k \) on \( B_k \), then [151]. Note that \( \tau_{\tau_k, \gamma} G_k \in \mathcal{Y}_{0,B'_k} \) so \( \mathcal{J}_k^{-1} \tau_{\tau_k, \gamma} G_k \in \mathcal{Y}_{0,B'_0} \). Then by the Schwarz inequality for the bilinear form \( < (\Theta F)^{I'} > \gamma_0 \) we have

\[
| < \prod_{i \in I'} \mathcal{J}_i' \Theta F_i \prod_{i \in I} \mathcal{J}_i F_i > \gamma | \leq \| F_k \| \| \mathcal{J}_k^{-1} \tau_{\tau_k, \gamma} G_k \|
\]

where \( \| F \|^2 = < (\Theta F)^{I'} > \gamma_0 = < (\tilde{\Theta} F) > \). Hence the expression only depends on the equivalence class of each \( F_k \) and we have a linear functional on \( \mathcal{H}_0 \). The argument is similar for \( k \in I' \) but we get an anti-linear functional on \( \mathcal{H}_0 \).

Now we have a multilinear functional on \( (\otimes_{i \in I'} \mathcal{H}_0) \times (\otimes_{i \in I} \mathcal{H}_0) \), anti-linear in the first group of factors, and this gives a mapping \( (\otimes_{i \in I'} \mathcal{H}_0) \otimes (\otimes_{i \in I} \mathcal{H}_0) \to \mathbb{C} \) by the universal property of the tensor product [7]. Hence it also determines a bilinear form on \( (\otimes_{i \in I'} \mathcal{H}_0) \times (\otimes_{i \in I} \mathcal{H}_0) \). This completes the proof.

**Remarks.** This result is considerably weaker than the massive result. The basic space is the pre-Hilbert space \( \mathcal{H}_0 \) not the completion \( \mathcal{H} \). The tensor product is the algebraic tensor product not the Hilbert space tensor product. And \( A'' \) is a bilinear form rather than an operator.

Note that as a consequence of [173] the functional is continuous in any particular variable \( F_k \) and one can extend the definition from \( \mathcal{H}_0 \) to the completion \( \mathcal{H} \). But it is not proved that one can do
it for all $F_k$ at once and so we do not have a bounded linear functional on $\mathcal{H} \times \cdots \times \mathcal{H}$, let alone a Hilbert-Schmidt functional. These are the obstacles to obtaining the stronger results of the massive case.

Since the algebraic tensor product is not complete the bilinear form does not necessarily define an operator. However it does define an operator if there is only one in-disc or only one out-disc. For example $A^{1I}(\mathcal{F}, \otimes_{i \in I} \mathcal{F}_i)$ is defined and continuous in $\mathcal{F} \in \mathcal{H}$ as noted above, and so by the Riesz theorem there is a linear operator $A^{1I} : \otimes_{i \in I} \mathcal{H}_i \to \mathcal{H}$ such that

$$(\mathcal{F}, A^{1I}[\otimes_{i \in I} \mathcal{F}_i]) = A^{1I}(\mathcal{F}, [\otimes_{i \in I} \mathcal{F}_i])$$

Similarly there is a linear operator $[A^{1I}]^* : \otimes_{i \in I'} \mathcal{H}_i \to \mathcal{H}$ such that

$$([A^{1I}]^*[\otimes_{i \in I'} \mathcal{F}_i], \mathcal{F}) = A^{1I}(\otimes_{i \in I'} \mathcal{F}_i, \mathcal{F})$$

But $A^{1I}_1$ itself is not defined.

4.5 sewing

We study the amplitudes $A^{1I}$ with one out disc. For simplicity we assume the in-discs $\{B_i\}_{i \in I}$ are all in $B_0$ and the out disc is $B_0'$ with the identity parametrization. Then our metric will satisfy $\gamma = \gamma_0$ on a neighborhood of $B_0'$. Take $\mathcal{F} = \nu(F)$ and $\mathcal{F}_i = \nu(F_i)$ and compute

$$(\mathcal{F}, A^{1I}[\otimes_{i \in I} \mathcal{F}_i]) = <\Theta F \prod_{i \in I} \mathcal{J}_i F_i >_{\gamma}$$

$$= <\Theta F \prod_{i \in I} \tau_{\gamma_0, \gamma} \mathcal{J}_i F_i >_{\gamma_0}$$

$$= (\mathcal{F}, \nu(\prod_{i \in I} \tau_{\gamma_0, \gamma} \mathcal{J}_i F_i))$$

where we use $\tau_{\gamma_0, \gamma} \Theta F = \Theta F$. Since the $\mathcal{F} = \nu(F)$ are dense we conclude

$$A^{1I}[\otimes_{i \in I} \mathcal{F}_i] = \nu \left( \prod_{i \in I} \tau_{\gamma_0, \gamma} \mathcal{J}_i F_i \right)$$

We also study amplitudes $A^{1I'}$ with one in-disc. We assume the out-discs are all in $B_0'$ and the in-disc is $B_0$ with identity parametrization. Then $\gamma = \gamma_0$ on a neighborhood of $B_0$. Take $\mathcal{F} = \nu(F)$ and $\mathcal{F}_i = \nu(F_i)$, and compute

$$([A^{1I'}]^*[\otimes_{i \in I'} \mathcal{F}_i], \mathcal{F}) = <\prod_{i \in I'} \mathcal{J}_i' \Theta F_i >_{\gamma}$$

$$= <\prod_{i \in I'} \tau_{\gamma, \gamma_0} \mathcal{J}_i' \Theta F_i >_{\gamma_0}$$

$$= (\nu(\Theta \prod_{i \in I'} \tau_{\gamma, \gamma_0} \mathcal{J}_i' \Theta F_i), \mathcal{F})$$

and so

$$(A^{1I'})^*[\otimes_{i \in I'} \mathcal{F}_i] = \nu \left( \Theta \prod_{i \in I'} \tau_{\gamma, \gamma_0} \mathcal{J}_i' \Theta F_i \right)$$

Now we state the sewing result with a simple configuration. (Compare theorem 3).
Theorem 6 Let $A^I$ be the amplitude for transitions from $\{B_i\}_{i \in I}$ in $B_0$ to $B'_0$, the latter with identity parametrization. Let $A'^I$ be the amplitude for transitions from $B_0$ with identity parametrization to $\{B_i\}_{i \in I'}$ in $B'_0$. Finally let $A''^I$ be the amplitude for transitions from $\{B_i\}_{i \in I}$ to $\{B_i\}_{i \in I'}$ with the same parametrizations. Then $A''^I A^I = A'^I$ in the sense that

$$\left( [A'^I]^*[\otimes_{i \in I'} F_i], A^I [\otimes_{i \in I} F_i] \right) = A'^I ([\otimes_{i \in I'} F_i], [\otimes_{i \in I} F_i])$$

(180)

Proof. Let $\gamma_1$ be a metric suitable for $A^I$ so that $\gamma_1 = \gamma_0$ on a neighborhood of $B'_0$ and $\gamma_1 = \gamma_i$ on $B_i, i \in I$. Let $\gamma_2$ be a metric suitable for $A'^I$ so that $\gamma_2 = \gamma_0$ on a neighborhood of $B_0$ and $\gamma_2 = \gamma_i$ on $B_i, i \in I'$. Define a smooth metric $\gamma$ by $\gamma = \gamma_1$ in $B_0$ and $\gamma = \gamma_2$ in $B'_0$. Then $\gamma$ is a suitable metric for $A''^I$ and we compute using (177), (179)

$$\left( [A'^I]^*[\otimes_{i \in I'} F_i], A^I [\otimes_{i \in I} F_i] \right) = \left( \nu(\Theta \prod_{i \in I'} \tau_{\gamma_0, \gamma_2} F_i \Theta F_i), \nu(\prod_{i \in I} \tau_{\gamma_0, \gamma_1} F_i F_i) \right)$$

$$= \left( \prod_{i \in I'} \tau_{\gamma_0, \gamma_2} F_i \prod_{i \in I} \tau_{\gamma_0, \gamma_1} F_i F_i > \gamma_0 \right)$$

$$= \left( \prod_{i \in I'} \tau_{\gamma_0, \gamma_2} F_i \prod_{i \in I} \tau_{\gamma_0, \gamma_1} F_i F_i > \gamma_0 \right)$$

(181)

$$= \left( \nu(\Theta \prod_{i \in I} \tau_{\gamma_0, \gamma} F_i \Theta F_i), \nu(\prod_{i \in I} (\tau_{\gamma} F_i F_i)) \right)$$

$$= A''^I ([\otimes_{i \in I'} F_i], [\otimes_{i \in I} F_i])$$

Remarks.

1. Much more general configurations are possible using the same basic ideas.

2. Somewhat similar results have been obtained by Tsukada [18], however in this work the formulation of the problem is rather different.

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A Wick monomials

We state some results about Wick monomials on the sphere. These are generalizations of standard results in the plane [17, 6]. However we avoid regularizing the field with approximate delta functions.

Let $m_C = m_{\gamma, \mu}$ be the Gaussian measure with covariance $C = (-\Delta_\gamma + \mu)^{-1}$ as in the text. We want to consider expressions of the form

$$V = \frac{1}{2} \int : \phi(x)\phi(y) : v(x, y)d\mu_\gamma(x)d\mu_\gamma(y)$$

(182)

where formally $\phi(x) = \phi(\delta_x)$. We take $v(x, y)$ to be the distribution kernel of a bounded symmetric bilinear form operator $v$ on $H^{+1} \times H^{+1}$, i.e.

$$v(f, f') = \int v(x, y)f(x)f'(x)d\mu_\gamma(x)d\mu_\gamma(y)$$

(183)

If $e_i$ is an orthonormal basis for $H^{+1}$ and $\chi^i = C^{-1}e_i$ is the dual basis for $H^{-1}$ then $f = \sum_i e_i(\chi^i, f)$ and so we have

$$v(f, f') = \sum_{i, j} v(e_i, e_j)(\chi^i, f)(\chi^j, f')$$

(184)

Thus what we seek to define is

$$V = \frac{1}{2} \sum_{i, j} : \phi(\chi^i)\phi(\chi^j) : v(e_i, e_j)$$

(185)

As an approximation we consider

$$V_N = \frac{1}{2} \sum_{1 \leq i, j \leq N} : \phi(\chi^i)\phi(\chi^j) : v(e_i, e_j)$$

(186)

which is well-defined.

**Lemma 11** If $\sum_{ij} |v(e_i, e_j)|^2$ converges then

1. $V = \lim_{N \to \infty} V_N$ exists in $L^2(Q, \Sigma, m_C)$ and satisfies

$$\|V\|^2 = \frac{1}{2} \sum_{ij} |v(e_i, e_j)|^2$$

(187)

2. For $h_i \in H^{-1}$

$$\int : \phi(h_1) \cdots \phi(h_n) : V dm_C = \begin{cases} 0 & n \neq 2 \\ v(Ch_1, Ch_2) & n = 2 \end{cases}$$

(188)

3. The definition of $V$ is independent of basis.

**Remark.** Note that the bilinear form $v$ on $H^{+1} \times H^{+1}$ determines a bounded operator $v$ from $H^{+1}$ to $H^{-1}$ so that

$$v(f, g) = (f, v g)_{+1, -1}$$

(189)

Using also

$$(f, h)_{+1, -1} = (C^{-1} f, h)_{-1} = (f, Ch)_{+1}$$

(190)

---

3All the results in the appendices hold with the sphere replaced by a compact two dimensional manifold.
we have
\[ \sum_{ij} |v(e_i, e_j)|^2 = \sum_{ij} |(e_i, ve_j)_{+1, -1}|^2 = \sum_{ij} |(\chi^i, ve_j)_{-1}|^2 = \sum_{i} ||ve_j||^2_{-1} \quad (191) \]

Thus the condition \( \sum_{ij} |v(e_i, e_j)|^2 < \infty \) is the same as the condition that the operator \( v \) be Hilbert-Schmidt and we have
\[ \|V\|^2_2 = \frac{1}{2} \|v\|^2_{HS} \quad (192) \]

An equivalent condition is that \( C^{1/2}vC^{1/2} \) is Hilbert-Schmidt on \( H^0 \) \( (C^{1/2} \) is unitary from \( H^0 \) to \( H^{+1} \) and from \( H^{-1} \) to \( H^0 \)). Another equivalent condition is that \( vC \) is Hilbert-Schmidt on \( H^{-1} \) \( (C \) is unitary from \( H^{-1} \) to \( H^{+1} \)).

**Proof.** We compute for \( M > N \)
\[ \|V_M - V_N\|^2 = \frac{1}{4} \int \sum_{N \leq i,j \leq M} \phi(\chi^i)\phi(\chi^j) : v(e_i, e_j) : dm_C \]
\[ = \frac{1}{2} \sum_{N \leq i,j \leq M} |v(e_i, e_j)|^2 \quad (193) \]

This converges to zero as \( N, M \to \infty \) so \( V \) exists. The identity (187) is established similarly.

For the second point we note that \( V \) is in the closed subspace spanned by the quadratic Wick monomials \( : \phi(h)\phi(h') : \). Thus it is orthogonal to Wick monomials of any other degree. We compute
\[ \int : \phi(h)\phi(h') : V dm_C = \lim_{N \to \infty} \int : \phi(h)\phi(h') : V_N dm_C \]
\[ = \lim_{N \to \infty} \sum_{1 \leq i,j \leq N} (\chi^i, h)_{-1} (\chi^j, h')_{-1} v(e_i, e_j) \]
\[ = \lim_{N \to \infty} \sum_{1 \leq i,j \leq N} (\chi^i, Ch)_{-1, +1} (\chi^j, Ch')_{-1, +1} v(e_i, e_j) \]
\[ = v(Ch, Ch') \quad (194) \]

For the third point note that the inner products of \( V \) with Wick monomials are independent of basis by the previous result. Since the Wick monomials span a dense set it follows that \( V \) is independent of basis. This completes the proof.

In the text we are particularly concerned with the case where the bilinear form has the kernel (in local coordinates)
\[ v(x, y) = |\gamma(x)|^{-1/2} g(x) \delta(x - y) \quad (195) \]

for some smooth function \( g \) on \( C_\infty \). Thus the bilinear form is
\[ v(f, f') = \int f(x) f'(x) g(x) d\mu_\gamma(x) \quad (196) \]

The associated operator from \( H^{+1} \) to \( H^{-1} \) is just multiplication by \( g \) and it is Hilbert-Schmidt since is is bounded on \( H^{+1} \) and the injection \( H^{+1} \to H^{-1} \) is Hilbert-Schmidt. (The latter is equivalent to the statement that \( C \) is Hilbert-Schmidt on \( H^0 \).) Thus \( V \) exists in this case and we denote it by
\[ \frac{1}{2} : \phi^2 : (g) = \frac{1}{2} \int \phi(x)^2 : g(x) d\mu_\gamma(x) \quad (197) \]

**Lemma 12** If \( A \) is closed in \( C_\infty \) and \( \text{supp } g \subset A \) then \( : \phi^2 : (g) \) is \( \Sigma_A \) measurable.
Proof. It suffices to show $E_A(\phi^2 : (g)) =: \phi^2 : (g)$. This follows if the inner product with any Wick monomial is the same, and it suffices to consider quadratic monomials. Taking the conditional expectation onto the Wick monomial we must show

$$\int : \phi(e_A h) \phi(e_A h') : : \phi^2 : (g) dm_C = \int : \phi(h) \phi(h') : : \phi^2 : (g) dm_C$$

(198)

But by (188) and $e_A g = g$ the left side of (198) is computed as

$$(Ce_A h, g Ce_A h)_{+1,-1} = (e_A h, g Ce_A h)_{-1} = (h, g Ce_A h)_{-1} = (Ch, g Ce_A h)_{+1,-1}$$

(199)

Similarly the other $e_A$ is eliminated and we get $(Ch, g Ch)_{+1,-1}$ which is the evaluation of the right side of (198). This completes the proof.

B. Perturbations of Gaussian measures

In this section we consider quadratic perturbations of Gaussian measures. The treatment is a slightly different formulation of standard results [17], [6].

Again we consider the Gaussian measure $m_C = m_{\gamma,\mu}$ with covariance $C = (-\Delta_\gamma + \mu)^{-1}$ as in the text. We want to study measures of the form $e^{-V} dm_C$ where $V$ is a Wick monomial defined by a bilinear form $v$ on $H^{+1} \times H^{+1}$ as in (182). We continue to assume that $v$ satisfies a Hilbert-Schmidt condition (i.e. $v C$ on $H^{-1}$ is Hilbert-Schmidt) so that $V$ exists. We also need a positivity condition which is

$$\|f\|^2_{+1} + v(f,f) > 0 \quad f \in H^{+1}$$

(200)

or equivalently

$$-\Delta_\gamma + \mu + v > 0$$

(201)

Note that the operator $v C$ on $H^{-1}$ is self-adjoint since

$$(h', [v C] h)_{-1} = (Ch', v Ch)_{+1,-1} = v(Ch', Ch)$$

(202)

Furthermore the positivity condition implies

$$(h, [v C] h)_{-1} = v(Ch, Ch) > -\|Ch\|^2_{+1} = -\|h\|^2_{-1}$$

(203)

so we have $v C > -1$.

Lemma 13 Let $v C$ be Hilbert-Schmidt and suppose the positivity condition (201) is satisfied.

1. $e^{-V}$ is integrable with respect to $m_C$ and

$$\int e^{-V} dm_C = \det_2 (1 + v C)$$

(204)

2. For $h \in H^{-1}$

$$\frac{\int e^{i\phi(h)} e^{-V} dm_C}{\int e^{-V} dm_C} = \exp \left( -\frac{1}{2} (h, (1 + v C)^{-1} h)_{-1} \right)$$

(205)

Hence the measure $e^{-V} dm_C$, once normalized, is Gaussian with covariance $(1 + v C)^{-1}$

Remark. Note that the variance of the measure can also be characterized as

$$(h, (1 + v C)^{-1} h)_{-1} = (h, C (1 + v C)^{-1} h)_{-1,+1} = (h, (C^{-1} + v)^{-1} h)_{-1,+1}$$

(206)
which is the same as

\( (h, (\Delta_\gamma + \mu + v)^{-1} h) \)  \hspace{1cm} (207)

**Proof.** Pick an orthonormal basis \( \chi^i \) of \( H^{-1} \) consisting of eigenvectors for \( vC \) so \( [vC]\chi^i = \lambda_i \chi^i \) with \( \lambda_i > -1 \) and \( \sum_i \lambda_i^2 < \infty \). Let \( e_i = C\chi^i \) be the dual orthonormal basis for \( H^+ \). Then we have from (202)

\[ v(e_i, e_j) = (\chi_i, [vC]\chi_j) = \lambda_i \delta_{ij} \]  \hspace{1cm} (208)

We compute

\[
\int e^{-V} dm_C = \int \exp \left( -\frac{1}{2} \sum_{i=1}^{N} \lambda_i \phi(\chi_i)^2 \right) \, dm_C
\]

\[
= \prod_{i=1}^{N} \frac{\int \exp(-\frac{1}{2}\lambda_i (x_i^2 - 1) - \frac{1}{2} x_i^2) \, dx_i}{\int \exp(-\frac{1}{2} x_i^2) \, dx_i}
\]

\[
= \prod_{i=1}^{N} (1 + \lambda_i)^{-1/2} e^{\lambda_i/2}
\]

Since \( \log((1 + \lambda_i)^{-1/2} e^{\lambda_i/2}) = \mathcal{O}(\lambda_i^2) \) and since \( \sum_i \lambda_i^2 < \infty \) this has a limit as \( N \to \infty \) which is \( \det_2(I + vC)^{-1/2} \) where

\[ \det_2(I + vC) = \prod_{i=1}^{\infty} (1 + \lambda_i) e^{-\lambda_i} \]  \hspace{1cm} (210)

Thus we have

\[ \lim_{N \to \infty} \int e^{-V} dm_C = \det_2(I + vC)^{-1/2} \]  \hspace{1cm} (211)

Next we note that for \( M > N \)

\[
\|e^{-V_M/2} - e^{-V_N/2}\|^2_2 = \int (e^{-V_M} + e^{-V_N} - 2 e^{-(V_M + V_N)/2}) \, dm_C
\]

\[
= \prod_{i=1}^{M} (1 + \lambda_i)^{-1/2} e^{\lambda_i/2} + \prod_{i=1}^{N} (1 + \lambda_i)^{-1/2} e^{\lambda_i/2}
\]

\[-2 \prod_{i=1}^{N} (1 + \lambda_i)^{-1/2} e^{\lambda_i/2} \prod_{i=N+1}^{M} (1 + \lambda_i/2)^{-1/2} e^{\lambda_i/4}
\]

Hence it converges to zero as \( M, N \to \infty \). Then

\[ \|e^{-V_M} - e^{-V_N}\|_1 \leq \|e^{-V_M/2} - e^{-V_N/2}\|_2 \|e^{-V_M/2} + e^{-V_N/2}\|_2 \]  \hspace{1cm} (213)

goest to zero as well and hence \( e^{-V_N} \) converges pointwise almost everywhere to \( V \), we conclude that \( e^{-V} \in L^1 \) and that \( e^{-V} \to e^{-V} \) in \( L^1 \). Combined with (211) this establishes that \( \int e^{-V} dm_C = \det_2(I + vC)^{-1/2} \). This completes the first part.

For the second part expand for \( h \in H^{-1} \) in the orthonormal basis \( \chi^i \) we by \( h = \sum_i h_i \chi^i \) with \( \sum_i h_i^2 < \infty \). Also define the approximation \( h_N = \sum_{i=1}^{N} h_i \chi^i \). Then we have

\[
\frac{\int e^{i\phi(h_N)} e^{-V} dm_C}{\int e^{-V} dm_C} = \prod_{i=1}^{N} \frac{\int \exp(i h_i x_i - \frac{1}{2} \lambda_i (x_i^2 - 1) - \frac{1}{2} x_i^2) \, dx_i}{\int \exp(-\frac{1}{2} \lambda_i (x_i^2 - 1) - \frac{1}{2} x_i^2) \, dx_i}
\]

\[
= \prod_{i=1}^{N} \exp \left( -\frac{1}{2} h_i^2 (1 + \lambda_i)^{-1} \right)
\]

\[\]
Taking the limit $N \to \infty$ this converges to
\[
\int e^{i\phi(h)} e^{-V} dm_C \quad = \quad \exp \left( -\frac{1}{2} \sum_i h_i^2 (1 + \lambda_i)^{-1} \right) = \exp \left( -\frac{1}{2} (h, (1 + vC)^{-1} h)_{-1} \right)
\] (215)
as announced.

\section{distances in conformally equivalent metrics}

Let $\gamma, \gamma'$ be conformal metrics on the Riemann sphere. The following lemma compares distances as points come together.

\textbf{Lemma 14} If $\gamma' = e^\sigma \gamma$ then
\[
\lim_{y \to x} \frac{d_{\gamma'}(x, y)}{d_{\gamma}(x, y)} = e^{\sigma(x)/2}
\] (216)

\textbf{Proof}. For each $y$ near $x$ let $\alpha_{xy}(t)$ be the geodesic for $\gamma'$ with $\alpha_{xy}(0) = x$ and $\alpha_{xy}(1) = y$. In terms of the exponential map at $x$ it has the representation
\[
\alpha_{xy}(t) = \exp_x \left( t \exp_x^{-1}(y) \right)
\] (217)

If $\gamma = \rho |dz|^2$ then $\gamma' = e^\sigma \rho |dz|^2$ and we have
\[
d_{\gamma'}(x, y) = \int_0^1 e^{\sigma(\alpha_{xy}(t))/2} \sqrt{\rho(\alpha_{xy}(t))|\alpha'_{xy}(t)|} \, dt
\] (218)

Given $\epsilon > 0$ choose $\delta_0$ so that if $|x - y| < \delta_0$ then $|\sigma(x) - \sigma(y)| < \epsilon$. Then choose $\delta_1$ so that if $|x - y| < \delta_1$ then $|\alpha_{xy}(t) - x| < \delta_0$ for all $0 \leq t \leq 1$. Since $x = \alpha_{xx}(t)$ this is continuity of $\alpha_{xy}(t)$ as $y \to x$ uniformly in $t$. That one can accomplish this follows from the representation (217) and the continuity of the exponential map and its inverse. Now for $|x - y| < \delta_1$
\[
d_{\gamma'}(x, y) \geq e^{\sigma(x)/2} e^{-\epsilon/2} \int_0^1 \sqrt{\rho(\alpha_{xy}(t))|\alpha'_{xy}(t)|} \, dt
\] (219)

Reversing the roles of the metrics there is a $\delta_2$ such that if $|x - y| < \delta_2$
\[
d_{\gamma}(x, y) \geq e^{-\sigma(x)/2} e^{-\epsilon/2} d_{\gamma'}(x, y)
\] (220)

Hence if $\delta = \min(\delta_1, \delta_2)$ and $|x - y| < \delta$ then
\[e^{-\epsilon/2} \leq \frac{d_{\gamma'}(x, y)}{e^{\sigma(x)/2} d_{\gamma}(x, y)} \leq e^{\epsilon/2}
\] (221)

which is the result.
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