Spin Superfluidity and Magnon Bose-Einstein Condensation

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Abstract

The spin superfluidity – superfluidity in the magnetic subsystem of a condensed matter – is manifested as the spontaneous phase-coherent precession of spins first discovered in 1984 in $^3$He-B. This superfluid current of spins – spin supercurrent – is one more representative of superfluid currents known or discussed in other systems, such as the superfluid current of mass and atoms in superfluid $^4$He; superfluid current of electric charge in superconductors; superfluid current of hypercharge in Standard Model of particle physics; superfluid baryonic current and current of chiral charge in quark matter; etc. Spin superfluidity can be described in terms of the Bose condensation of spin waves – magnons. We discuss different phases of magnon superfluidity, including those in magnetic trap; and signatures of magnons superfluidity: (i) spin supercurrent, which transports the magnetization on a macroscopic distance more than 1 cm long; (ii) spin current Josephson effect which shows interference between two condensates; (iii) spin current vortex – a topological defect which is an analog of a quantized vortex in superfluids, of an Abrikosov vortex in superconductors, and cosmic strings in relativistic theories; (iv) Goldstone modes related to the broken $U(1)$ symmetry – phonons in the spin-superfluid magnon gas; etc. We also touch the topic of spin supercurrent in general including spin Hall and intrinsic quantum spin Hall effects.

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I. INTRODUCTION

Nature knows different types of ordered states.

One major class is represented by equilibrium macroscopic ordered states exhibiting spontaneous breaking of symmetry. This class contains crystals; nematic, cholesteric and other liquid crystals; different types of ordered magnets (antiferromagnets, ferromagnets, etc.); superfluids, superconductors and Bose condensates; all types of Higgs fields in high energy physics; etc. The important subclasses of this class contain systems with macroscopic quantum coherence exhibiting off-diagonal long-range order (ODLRO), and/or nondissipative superfluid currents (mass current, spin current, electric current, hypercharge current, etc.).

The class of ordered systems is characterized by rigidity, stable gradients of order parameter (non-dissipative currents in quantum coherent systems), and topologically stable defects (vortices, solitons, cosmic strings, monopoles, etc.).

A second large class is presented by dynamical systems out of equilibrium. Ordered states may emerge under external flux of energy. Examples are the coherent emission from lasers; water flow in a draining bathtub; pattern formation in dissipative systems; etc.

Some of the latter dynamic systems can be close to stationary equilibrium systems of the first class. For example, ultra-cold gases in optical traps are not fully equilibrium states since the number of atoms in the trap is not conserved, and thus the steady state requires pumping. However, if the decay is small then the system is close to an equilibrium Bose condensate, and experiences all the corresponding superfluid properties.

A. BEC of quasiparticles

Bose-Einstein condensation (BEC) of quasiparticles whose number is not conserved is presently one of the debated phenomena of condensed matter physics. In thermal equilibrium the chemical potential of excitations vanishes and, as a result, their condensate does not form. The only way to overcome this situation is to create a dynamic steady state with a conserved number of excitations as a non-equilibrium system. Thus the Bose condensation of quasiparticles belongs to the phenomenon of second class, when the emerging steady state
of the system is not in a full thermodynamic equilibrium. It is decaying, but the loss of quasiparticles owing to their decay can be compensated by pumping of energy.

Formally BEC requires conservation of charge or particle number. However, condensation can still be extended to systems with weakly violated conservation. For sufficiently long-lived quasiparticles their distribution may be close to the thermodynamic equilibrium with a well defined finite chemical potential, which follows from the quasi-conservation of number of quasiparticles, and the Bose condensation becomes possible. Several examples of Bose condensation of quasiparticles have been observed or suggested, including phonons [1], excitons [2], exciton-polaritons [3, 4]. The BEC of quasi-equilibrium magnons – spin waves – in ferromagnets has been discussed in Ref. [5] and investigated in [6, 7].

The coherent spin precession observed in superfluid phases $^3$He also can be described in terms of magnon BEC. The coherent spin precession was discovered first in $^3$He-B, where it is was called the Homogeneously Precessing Domain (HPD) [8, 9]. This is the spontaneously emerging steady state of precession, which preserves the phase coherence across the whole sample even in the absence of energy pumping. The crucial property of HPD is that precession acquires a coherent phase throughout the whole sample even in an inhomogeneous external magnetic field. This is equivalent to the appearance of a coherent superfluid Bose condensate.

In the absence of energy pumping this HPD state slowly decays, but during the decay the system remains in the coherent state of BEC: the volume of the Bose condensate (the volume of HPD) gradually decreases with time without violation of the observed properties of the spin-superfluid phase-coherent state. A steady state of phase-coherent precession can be supported by pumping. But the pumping need not be coherent – it can be chaotic: the system chooses its own (eigen) frequency of coherent precession, which emphasizes the spontaneous emergence of coherence from chaos.

HPD is very close to the thermodynamic equilibrium of the magnon Bose condensate and exhibits all the superfluid properties which follow from the off-diagonal long-range order (ODLRO) of the coherent precession. After discovery of HPD, several other states of magnon BEC have been observed in superfluid phases of $^3$He, which we discuss in this review, including finite magnon BEC states in magnetic traps.

B. Spin superfluidity vs superfluidity of mass and charge

Last decade was marked by the fundamental studies of mesoscopic quantum states of dilute ultra cold atomic gases in the regime where the de Broglie wavelength of the atoms is comparable with their spacing, giving rise to the phenomenon of Bose-Einstein condensation (see reviews [10, 11]). The formation of the Bose-Einstein condensate (BEC) – accumulation of the macroscopic number of particles in the lowest energy state – was predicted by Einstein in 1925 [12]. In ideal gas, all atoms are in the lowest energy state in the zero temperature limit. In dilute atomic gases, weak interactions between atoms produces a small fraction of the non-condensed atoms.

In the only known bosonic liquid $^4$He which remains liquid at zero temperature, the BEC is strongly modified by interactions. The depletion of the condensate due to interactions is very strong: in the limit of zero temperature only about 10% of particles occupy the state with zero momentum. Nevertheless, BEC still remains the key mechanism for the phenomenon of superfluidity in liquid $^4$He: due to BEC the whole liquid (100% of $^4$He atoms) forms a coherent quantum state at $T = 0$ and participates in the non-dissipative
superfluid flow.

Superfluidity is a very general quantum property of matter at low temperatures, with a variety of mechanisms and possible nondissipative superfluid currents. These include supercurrent of electric charge in superconductors and mass supercurrent in superfluid \(^3\)He, where the mechanism of superfluidity is the Cooper pairing; hypercharge supercurrent in the vacuum of Standard Model of elementary particle physics, which comes from the Higgs mechanism; supercurrent of color charge in a dense quark matter in quantum chromo-dynamics; etc. All these supercurrents have the same origin: the spontaneous breaking of the \(U(1)\) or higher symmetry related to the conservation of the corresponding charge or particle number, which leads to the so-called off-diagonal long-range order. This spin supercurrent—the superfluid current of spins—is one more representative of superfluid currents. Here the \(U(1)\) symmetry is the approximate symmetry of spin rotation, which is related to the quasi-conservation of spin.

It appears that the finite life-time of magnons, and non-conservation of spin due to the spin-orbital coupling do not prevent the coherence and superfluidity of magnon BEC. The non-conservation leads to a decrease of the number of magnons in the Bose gas until it disappears completely, but during this relaxation, the coherence of BEC is preserved with all the signatures of spin superfluidity: (i) spin supercurrent, which transports the magnetization on a macroscopic distance more than 1 cm long; (ii) spin current Josephson effect which shows interference between two condensates; (iii) phase-slip processes at the critical current; (iv) spin current vortex—a topological defect which is an analog of a quantized vortex in superfluids, of an Abrikosov vortex in superconductors, and cosmic strings in relativistic theories; (v) Goldstone modes related to the broken \(U(1)\) symmetry—phonons in the spin-superfluid magnon gas; etc.

C. Magnon BEC vs equilibrium magnets

The magnetic \(U(1)\) symmetry is spontaneously broken also in some static magnetic systems. Sometimes this phenomenon is also described in terms of BEC of magnons [13–16]. Let us stress from the beginning that there is the principal difference between the magnetic ordering in equilibrium and the BEC of quasiparticles which we are discussing in this review.

In some magnetic systems, the symmetry breaking phase transition starts when the system becomes softly unstable towards growth of one of the magnon modes. The condensation of this mode can be used for the description of the soft mechanism of formation of ferromagnetic and antiferromagnetic states (see e.g. [17]). However, the condensation of this mode leads finally to the formation of the true equilibrium ordered state. In the same manner, the Bose condensation of phonon modes may serve as a soft mechanism of formation of the equilibrium solid crystals [18]. But this does not mean that the final crystal state is the Bose condensate of phonons.

On the contrary, BEC of quasiparticles is in principle a non-equilibrium phenomenon, since quasiparticles (magnons) have a finite life-time. In our case magnons live long enough to form a state very close to thermodynamic equilibrium BEC, but still it is not an equilibrium. In the final equilibrium state at \(T = 0\) all the magnons will die out. In this respect, the growth of a single mode in the non-linear process after a hydrodynamic instability [21], which has been discussed in terms of the Bose condensation of the classical sound or surface waves [22], is more close to magnon BEC than equilibrium magnets.

The other difference is that the ordered magnetic states are states with diagonal long-
range order. The magnon BEC is a dynamic state characterized by the off-diagonal long-range order (see below), which is the main signature of spin superfluidity.

II. COHERENT PRECESSION AS MAGNON SUPERFLUID

A. Spin precession

The magnetic subsystem which we discuss is the precessing magnetization. In a full correspondence with atomic systems, the precessing spins can be either in the normal state or in the ordered spin-superfluid state. In the normal state, spins of atoms are precessing with the local frequency determined by the local magnetic field and interactions. In the ordered state the precession of all spins is coherent: they spontaneously develop the common global frequency and the global phase of precession.

\[
\mathcal{M}_x + i\mathcal{M}_y = \mathcal{M}_\perp e^{i\omega t + i\alpha}, \quad \mathcal{M}_\perp = \chi HV \sin \beta .
\]

FIG. 1: The stroboscopic record of the induction decay signal on a frequency about 1 MHz. \textit{left}: During the first stage of about 0.002 s the induction signal completely disappears due to dephasing. Then, during about 0.02 s, the spin supercurrent redistributes the magnetization and creates the phase coherent precession, which is equivalent to the magnon BEC state. Due to small magnetic relaxation, the number of magnons slowly decreases but the precession remains coherent. \textit{right}: The initial part of the magnon BEC signal.

In pulsed NMR experiments the magnetization is created by an applied static magnetic field: \( \mathbf{M} = \chi \mathbf{H} \), where \( \chi \) is magnetic susceptibility. Then a pulse of the radio-frequency (RF) field \( \mathbf{H}_{\text{RF}} \perp \mathbf{H} \) deflects the magnetization by an angle \( \beta \), and after that the induction signal from the free precession is measured. In the state of the disordered precession, spins almost immediately loose the information on the original common phase and frequency induced by the RF field, and due to this decoherence the measured induction signal is very small. In the ordered state, all spins precess coherently, which means that the whole macroscopic magnetization of the sample of volume \( V \) is precessing

\[
\mathcal{M}_x + i\mathcal{M}_y = \mathcal{M}_\perp e^{i\omega t + i\alpha}, \quad \mathcal{M}_\perp = \chi HV \sin \beta .
\]
This coherent precession is manifested as a huge and long-lived induction signal, Fig. 1. It is important that the coherence of precession is spontaneous: the global frequency and the global phase of precession are formed by the system itself and do not depend on the frequency and phase of the initial RF pulse.

B. Off-diagonal long-range order

The superfluid atomic systems are characterized by the off-diagonal long-range order (ODLRO) [23]. In superfluid $^4$He and in the coherent atomic systems the operator of annihilation of atoms with momentum $p = 0$ has a non-zero vacuum expectation value:

$$\langle \hat{a}_0 \rangle = N_0^{1/2} e^{i\mu t + ia} ,$$

where $N_0$ is the number of particles in the Bose condensate, which in the limit of weak interactions between the atoms coincides at $T = 0$ with the total number of atoms $N$. Eq. (2.1) demonstrates that in the coherent precession the ODLRO is manifested by a non-zero vacuum expectation value of the operator of creation of spin:

$$\langle \hat{S}_+ \rangle = S_x + iS_y = \frac{M_+}{\gamma} e^{i\omega t + ia} ,$$

where $\gamma$ is the gyromagnetic ratio, which relates magnetic moment and spin. This analogy suggests that in the coherent spin precession the role of the particle number $N$ is played by the projection of the total spin on the direction of magnetic field $S_z$. The corresponding symmetry group $U(1)$ in magnetic systems is the group of the $O(2)$ rotations about the direction of magnetic field. This quantity $S_z$ is conserved in the absence of the spin-orbit interactions.

The spin-orbit interactions transform the spin angular momentum of the magnetic subsystem to the orbital angular momentum, which causes the losses of spin $S_z$ during the precession. In our system of superfluid $^3$He, the spin-orbit coupling is relatively rather small, and thus $S_z$ is quasi-conserved. Because of the losses of spin the precession will finally decay, but during its long life time the precession remains coherent, Fig. 1. This is similar to the non-conservation of the number of atoms in the laser traps, where the number of atoms decreases with time but this does not destroy the coherence of the atomic BEC.

C. ODLRO and magnon BEC

The ODLRO in (2.3) can be represented in terms of magnon condensation. To view that let us use the Holstein-Primakoff transformation, which relates the spin operators with the operators of creation and annihilation of magnons

$$\hat{a}_0 \sqrt{1 - \frac{\hbar a_0^\dagger a_0}{2S}} = \frac{\hat{S}_+}{\sqrt{2Sh}} ,$$

$$\sqrt{1 - \frac{\hbar a_0^\dagger a_0}{2S}} \hat{a}_0^\dagger = \frac{\hat{S}_-}{\sqrt{2Sh}} ,$$

$$\hat{N} = \hat{a}_0^\dagger \hat{a}_0 = \frac{S - \hat{S}_z}{\hbar} .$$
Eq. (2.6) relates the number of magnons $N$ to the deviation of spin $S_z$ from its equilibrium value $S_z^{\text{(equilibrium)}} = S = \chi HV/\gamma$. In the full thermodynamic equilibrium, magnons are absent. Each magnon has spin $-\hbar$, and thus the total spin projection after pumping of $N$ magnons into the system by the RF pulse is reduced by the number of magnons, $S_z = S - \hbar N$. The ODLRO in magnon BEC is given by Eq. (2.2), where $N_0 = N$ is the total number of magnons (2.6) in the BEC:

$$\langle \hat{a}_0 \rangle = N^{1/2} e^{i \omega t + i \alpha} = \sqrt{\frac{2S}{\hbar}} \sin \frac{\beta}{2} e^{i \omega t + i \alpha}.$$ (2.7)

Comparing (2.7) and (2.2), one can see that the role of the chemical potential in atomic systems $\mu$ is played by the global frequency of the coherent precession $\omega$, i.e. $\mu \equiv \omega$. This demonstrates that this analogy with the phenomenon of BEC in atomic gases takes place only for the dynamic states of a magnetic subsystem—the states of precession. The ordered magnetic systems discussed in Refs. [13–16] are static, and for them the chemical potential of magnons is always zero.

There are two approaches to study the thermodynamics of atomic systems: at fixed particle number $N$ or at fixed chemical potential $\mu$. For the magnon BEC, these two approaches correspond to two different experimental arrangement: the pulsed NMR and continuous NMR, respectively. In the case of free precession after the pulse, the number of magnons pumped into the system is conserved (if one neglects the losses of spin). This corresponds to the situation with the fixed $N$, in which the system itself will choose the global frequency of the coherent precession (the magnon chemical potential). The opposite case is the continuous NMR, when a small RF field is continuously applied to compensate the losses. In this case the frequency of precession is fixed by the frequency of the RF field, $\mu \equiv \omega = \omega_{\text{RF}}$, and now the number of magnons will be adjusted to this frequency to match the resonance condition.

Finally let us mention that in the approach in which $N$ is strictly conserved and has quantized integer values, the quantity $\langle \hat{a}_0 \rangle = 0$ in Eq. (2.2). In the same way the quantity $\langle \hat{S}_z \rangle = 0$ in Eq. (2.3), if spin $S_z$ is strictly conserved and takes quantized values. This means that formally there is no precession if the system is in the quantum state with fixed spin quantum number $S_z$. However, this does not lead to any paradox in the thermodynamic limit: in the limit of infinite $N$ and $S_z$, the description in terms of the fixed $N$ (or $S_z$) is equivalent to the description in terms of with the fixed chemical potential $\mu$ (or frequency $\omega$).

III. PHENOMENOLOGY OF MAGNON SUPERFLUID

A. Magnon spectrum and magnon mass

Let us neglect for a moment the anisotropy of spin wave velocity $c$ and the spin-orbit interaction. Then the magnon spectrum in $^3$He-B has the following form:

$$\omega(k) = \frac{\omega_L}{2} + \sqrt{\frac{\omega_L^2}{4} + k^2c^2},$$ (3.1)

where $\omega_L = \gamma H$. At large momentum, $ck \gg \omega$, this spectrum transforms to the linear spectrum $\omega = ck$ of spin waves propagating with velocity $c$ which is on the order of the
Fermi velocity $v_F$. At small $k$, $ck \ll \omega$, this is the spectrum of massive particle

$$E_k = \hbar \omega(k), \quad \omega(k) = \omega_L + \frac{\hbar k^2}{2m},$$

(3.2)

where the magnon mass is:

$$m = \frac{\hbar \omega_L}{2c^2}.$$  

(3.3)

Since $c \sim v_F$, the relative magnitude of the magnon mass compared to the bare mass $m_3$ of the $^3$He atom is

$$\frac{m}{m_3} \sim \frac{\hbar \omega_L}{E_F},$$  

(3.4)

where the Fermi energy $E_F \sim m_3 v_F^2 \sim p_F^2/m_3$. With the magnon gap $\hbar \omega_L \sim 50 \mu K$ at $\omega_L \sim 1 MHz$, and $E_F \sim 1 K$, one has $m \sim 10^{-4} m_3$. Small mass of these bosons favors the Bose condensation. The opposite factor is the small density $n$ of the bosons. From (2.6) it follows that the magnon density is

$$n = \frac{S - S_z}{\hbar} = \frac{\chi H}{\hbar \gamma} (1 - \cos \beta).$$  

(3.5)

In the typical precessing state of $^3$He-B the magnon density $n$ is by the same factor $\hbar \omega_L/E_F$ smaller than the density of $^3$He atoms $n_3 = p_F^3/3\pi^2 h^3$ in the liquid

$$\frac{n}{n_3} \sim \frac{\hbar \omega_L}{E_F}.$$  

(3.6)

Nevertheless, one can easily estimate that at temperatures, at which superfluid $^3$He-B exists, $T \sim 10^{-3} E_F$, practically all the magnons must be condensed in the ground state with $k = 0$, i.e. the Bose condensation of magnons must be complete.

In $^3$He-B, the typical temperature is big compared to the magnon gap, $T \gg \hbar \omega_L$. This means that according to (3.1), thermal magnons are mostly the spin waves with linear spectrum $\omega(k) = ck$ and with characteristic momenta $k_T \sim T/\hbar c$. The density of thermal magnons is $n_T \sim k_T^3$. At $^3$He-B temperatures, this density is much smaller then the density of condensed magnons and can be neglected.

The smallness of $\omega_L \ll T$ modifies the estimate the temperature of the Bose condensation, compared to the atomic gases. Before we start pumping magnons, we have an equilibrium system of thermal magnons with $\mu = 0$. After pumping of extra magnons with density $n$ we obtain the quasi-equilibrium state in which the number of magnons is temporarily conserved and thus the magnon system acquires a non-zero chemical potential, $\mu \neq 0$. The number of extra magnons which can be absorbed by thermal distribution without formation of BEC is thus the difference of the distribution function at $\mu = 0$ and $\mu \neq 0$ at the same temperature:

$$n = \sum_k (f(E_k) - f(E_k - \mu)).$$  

(3.7)

This quantity reaches its maximum value when $\mu = \omega_L$. Since $\omega_L \ll T$ one has:

$$n_{\text{max}} = \omega_L \sum_k \frac{df}{dE} \sim \frac{T^2 \omega_L}{c^3}.$$  

(3.8)
This gives the dependence of BEC transition temperature on the number of pumped magnons

\[ T_{\text{BEC}} \sim \left( \frac{nc^3}{\omega_L} \right)^{1/2}. \]  

(3.9)

At \( T < T_{\text{BEC}} \) the magnon BEC with \( k = 0 \) must be formed. In \(^3\text{He}-\text{B}, T_{\text{BEC}} \sim E_F \) which is by 5 to 6 orders of magnitude higher than the temperature at which superfluid \(^3\text{He} \) exists.

The above estimate also demonstrates that in solid state magnetic systems the BEC may occur even at room temperature, see Ref. [6, 7].

**B. Order parameter and Gross-Pitaevskii equation**

As in the case of the atomic Bose condensates the main physics of the magnon BEC can be found from the consideration of the Gross-Pitaevskii equation for the complex order parameter. The local order parameter is obtained by extension of Eq. (2.7) to the inhomogeneous case and is determined as the vacuum expectation value of the magnon field operator:

\[ \Psi(r, t) = \left\langle \hat{\Psi}(r, t) \right\rangle, \quad n = |\Psi|^2, \quad \mathcal{N} = \int d^3r \; |\Psi|^2. \]  

(3.10)

where \( n \) is magnon density. To avoid the confusion, let us mention that this order parameter (3.10) describes the coherent precession in any system, superfluid or non-superfluid. It has nothing to do with the multi-component order parameter which describes the underlying systems – superfluid phases of \(^3\text{He} \) [24].

If the dissipation and pumping of magnons are ignored (on relaxation and pumping terms in magnon BEC see Ref. [25]), the corresponding Gross-Pitaevskii equation has the conventional form (\( \hbar = 1 \)):

\[ -i \frac{\partial \Psi}{\partial t} = \frac{\delta \mathcal{F}}{\delta \Psi^*}, \]  

(3.11)

where \( \mathcal{F}\{\Psi\} \) is the free energy functional. In the coherent precession, the global frequency is constant in space and time (if dissipation is neglected)

\[ \Psi(r, t) = \Psi(r)e^{i\omega t}, \]  

(3.12)

and the Gross-Pitaevskii equation transforms into the Ginzburg-Landau equation with \( \omega = \mu \):

\[ \frac{\delta \mathcal{F}}{\delta \Psi^*} - \mu \Psi = 0. \]  

(3.13)

The important feature of the magnon systems is that that their number density is limited

\[ n < n_{\text{max}} = \frac{2S}{\hbar}. \]  

(3.14)

For small \( n \ll n_{\text{max}} \) the Ginzburg-Landau free energy functional has the conventional form

\[ \mathcal{F} - \mu \mathcal{N} = \int d^3r \left( \frac{|\nabla \Psi|^2}{2m} + \left( \omega_L(r) - \omega \right)|\Psi|^2 + F_{\text{so}} \right). \]  

(3.15)
Here $\omega_L(r) = \gamma H(r)$ is the local Larmor frequency, which plays the role of external potential $U(r)$ in atomic condensates. The last term $F_{so}(|\Psi|^2)$ contains nonlinearity which come from the spin-orbit interaction. It is analogous to the 4-th order term in the atomic BEC, which describes the interaction between the atoms.

C. Spin-orbit interaction as interaction between magnons

In the magnetic subsystem of superfluid $^3$He, the interaction term in the Ginzburg-Landau free energy is provided by the spin-orbit interaction – interaction between the spin and orbital degrees of freedom. Though the structure of superfluid phases of $^3$He is rather complicated and is described by the multi-component superfluid order parameter [24]), the only output needed for investigation of the coherent precession is the structure of the spin-orbit interaction term $F_{so}(|\Psi|^2)$, which appears to be rather simple. The spin-orbit interaction provides the effective interaction between magnons, which can be attractive and repulsive, depending on the orientation of spin and orbital orbital degrees with respect to each other and with respect to magnetic field. The orbital degrees of freedom in the superfluid phases of $^3$He are characterized by the direction of the orbital momentum of the Cooper pair $\hat{l}$, which also marks the axis of the spatial anisotropy of these superfluid liquids. By changing the orientation of $\hat{l}$ with respect to magnetic field one is able to regulate the interaction term in experiments.

In superfluid $^3$He-B, the spin-orbit interaction has a very peculiar properties. The microscopic derivation (see (9.13)) leads to the following form [26]:

$$F_{so}(s, l, \gamma) = \frac{2}{15} \frac{\chi}{\gamma^2} \Omega_L^2 [(sl - \frac{1}{2} + \frac{1}{2} \cos \gamma(1 + s)(1 + l))^2 +$$

$$\frac{1}{8}(1 - s)^2(1 - l)^2 + (1 - s^2)(1 - l^2)(1 + \cos \gamma)] .$$

(3.16)

It is obtained by averaging of the spin-orbit energy over the fast precession of spins. Here $s = \cos \beta$, while $l = \hat{l} \cdot \hat{H}$ describes the orientation of the unit vector $\hat{l}$ with respect to the direction $\hat{H}$ of magnetic field. The parameter $\Omega_L$ is the so-called Leggett frequency, which characterizes the magnitude of the spin-orbit interaction and thus the shift of the resonance frequency from the Larmor value caused by spin-orbit interaction. In typical experimental situations, $\Omega_L^2 \ll \omega^2$, which means that the frequency shift is relatively small. Finally $\gamma$ is another angle, which characterizes the mutual orientation of spin and orbital degrees of freedom. It is a passive quantity: it takes the value corresponding to the minimum of $F_{so}$ for given $s$ and $l$, i.e. $\gamma = \gamma(s, l)$.

To obtain $F_{so}(|\Psi|^2)$ in (3.15) at fixed $\hat{l}$, one must express $s$ via $|\Psi|^2$:

$$1 - s = 1 - \cos \beta = \frac{\hbar|\Psi|^2}{S} ,$$

(3.17)

where $S = \chi H/\gamma$ is spin density. Since Eq. (3.16) is quadratic in $s$, the spin-orbit interaction contains quadratic and quartic terms in $|\Psi|$. While the quadratic term modifies the potential $U$ in the Ginzburg-Landau free energy, the quartic term simulates the interaction between magnons.

The profile of the spin-orbit interaction $F_{so}(s, l, \gamma(s, l))$ shown in Fig. [2] determines different states of coherent precession and thus different types of magnon BEC in $^3$He-B, which
FIG. 2: The profile of the spin-orbit energy as a function of \( s = \cos \beta \) and orbital variable \( l \), where \( \beta \) is the tipping angle of precession and \( l \) is the projection of the orbital angular momentum of a Cooper pair on the direction of magnetic field. Spontaneous phase-coherent precession emerging at \( l = 1 \) and \( s \approx -1/4 \) is called HPD (Homogeneously Precessing Domain).

depend on the orientation of the orbital vector \( \hat{l} \). The most important of them, which has got the name Homogeneously Precessing Domain (HPD), has been discovered 20 years ago [8, 9].

IV. MAGNON SUPERFLUIDS IN \(^3\)HE-B

A. HPD as unconventional magnon superfluid

In the right corner of Fig. 2, the minimum of the free energy occurs for \( l = 1 \), i.e. for the orbital vector \( \hat{l} \) oriented along the magnetic field. This means that if there is no other orientational effect on the orbital vector \( \hat{l} \), the spin-orbit interaction orients it along the magnetic field, and one automatically obtains \( l = \cos \beta_L = 1 \). The most surprising property emerging at such orientation is the existence of the completely flat region in Fig. 2. The spin-orbit interaction is identically zero in the large range of the tipping angle \( \beta \) of precession, for \( 1 > s = \cos \beta > -\frac{1}{4} \):

\[
F_{so}(\beta)_{l=1} = 0, \quad \cos \beta > -\frac{1}{4}, \tag{4.1}
\]

\[
F_{so}(\beta)_{l=1} = \frac{8}{15} \frac{\chi}{\gamma^2} \Omega_L^2 \left( \cos \beta + \frac{1}{4} \right)^2, \quad \cos \beta < -\frac{1}{4}. \tag{4.2}
\]
Using Eq. (3.17) one obtains the Ginzburg-Landau potential in (3.15) with:

\[
F_{so}(|\Psi|) = 0 , \quad |\Psi|^2 < n_c = \frac{5}{4} S, \quad (4.3)
\]

\[
F_{so}(|\Psi|) = \frac{8}{15} \chi \gamma^2 \Omega^2_L \left( \frac{|\Psi|^2}{S} - \frac{5}{4} \right)^2, \quad |\Psi|^2 > n_c = \frac{5}{4} S. \quad (4.4)
\]

Eqs. (4.3) and (4.4) demonstrate that when the orbital momentum is oriented along the magnetic field, magnons are non-interacting for all densities \( n \) below the threshold value \( n_c = (5/4)S/\hbar \). This is a really unconventional gas.

FIG. 3: \( F - \mu n \) for different values of the chemical potential \( \mu \equiv \omega \) in magnon BEC in \( ^3\)He-B. For \( \mu < U \), i.e. for \( \omega < \omega_L \), the minimum of \( F - \mu n \) corresponds to zero number of magnons, \( n = 0 \). It is the static state without precession. For \( \mu = U \), i.e. for \( \omega = \omega_L \), the energy is the same for all densities in the range \( 0 \leq n \leq n_c \). For \( \mu > U \), i.e. for \( \omega > \omega_L \), the minimum of \( F - \mu n \) corresponds the magnon BEC with density \( n \geq n_c \). This corresponds to the coherent precession of magnetization with tipping angle \( \beta > 104^\circ \).

The energy profile of \( F - \mu n \) is shown in Fig. 3 for different values of the chemical potential \( \mu \equiv \omega \). For \( \mu \) below the external potential \( U \), i.e. for \( \omega < \omega_L \), the minimum of \( F - \mu n \) corresponds to zero number of magnons, \( n = 0 \). It is the static state of \(^3\)He-B without precession. For \( \mu > U \), i.e. for \( \omega > \omega_L \), the minimum of \( F - \mu n \) corresponds to the finite value of the magnon density:

\[
n = n_c \left( 1 + \frac{3}{4} \frac{(\omega - \omega_L)\omega_L}{\Omega^2_L} \right). \quad (4.5)
\]

This shows that the formation of HPD starts with the discontinuous jump to the finite density \( n_c = 5S/4\hbar \), which corresponds to coherent precession with the large tipping angle – the so-called magic Leggett angle, \( \beta_c \approx 104^\circ \) (\( \cos \beta_c = -1/4 \)). This is distinct from the
standard Ginzburg-Landau energy functional (see Eq. (5.4) for magnon BEC in $^3$He-A-phase below), where the Bose condensate density smoothly starts growing from zero and is proportional to $\mu - U$ for $\mu > U$.

B. Two-domain precession

FIG. 4: Two domains in the coherent precession in $^3$He-B. left: Incoherent spin precession after the pulse of the RF field deflects magnetization from its equilibrium value. The total number of magnons pumped into the system is $N = (S - S_z)/\hbar$ magnons. middle: Formation of two domains. All $N$ magnons are concentrated in the lower part of the cell, forming the BEC state there. The volume of this domain is determined by the magnon density in BEC, $V = N/n$, where $n \approx n_c$. This volume determines the position $z_0$ of the domain boundary $z_0 = V/A$, where $A$ is the area of the cross-section of the cylindrical cell. The position of the interface in turn determines the global frequency of precession, which is equal to the local Larmor frequency at the phase boundary, $\mu \equiv \omega = \omega_L(z_0)$. right Decay of magnon BEC. The number of magnons decreases due to spin and energy losses. Since the magnon density in BEC is fixed (it is always close to $n_c$), the relaxation leads to the decrease of the volume of the BEC domain. However within this domain the precession remains fully coherent. While the phase boundary slowly moves down the frequency of the global precession gradually decreases, Fig. 5 left.

The two states with zero and finite density of magnons, resemble the low-density gas state and the high density liquid state of water, respectively. Gas and liquid can be separated in the gravitational field: the heavier liquid state will be concentrated in the lower part of the vessel. For the magnon BEC, the role of the gravitational field is played by the gradient of
magnetic field:
\[ \nabla U \equiv \nabla \omega_L = \gamma \nabla H. \]

Thus applying the gradient of magnetic field along the axis \( z \), one enforces phase separation, Fig. 4. The static thermodynamic equilibrium state with no magnons is concentrated in the region of higher field, where \( \omega_L(z) > \omega \), i.e. \( U(z) > \mu \). The magnon superfluid – the coherently precessing state – occupies the low-field region, where \( \omega_L(z) < \omega \), i.e. \( U(z) < \mu \). This is the so-called Homogeneously Precessing Domain (HPD), in which all spins precess with the same frequency \( \omega \) and the same phase \( \alpha \). In typical experiments the gradient is small, and magnon density is close to the threshold value \( n_c \).

The interface between the two domains is situated at the position \( z_0 \) where \( \omega_L(z_0) = \omega \), i.e. \( U(z_0) = \mu \). In the continuous NMR, the chemical potential is fixed by the frequency of the RF field: \( \mu = \omega_{RF} \), this determines the position of the interface in the experimental cell.

In the pulsed NMR, the two-domain structure spontaneously emerges after the magnetization is deflected by the RF pulse (Fig. 4, left and middle). The position of the interface between the domains is determined by the number of magnons pumped into the system: \( N = (S - S_z)/\hbar \). The number of magnons is quasi-conserved, i.e. it is well conserved during the time of the formation of the two-domain state of precession. That is why the volume of the domain occupied by the magnon BEC after its formation is \( V = N/n_c \). This determined the position \( z_0 \) of the interface, and the chemical potential \( \mu \) will be adjusted to this position: \( \mu = \omega_L(z_0) \).

In the absence of the RF field, i.e. without continuous pumping of magnons, the magnon BEC decays due to losses of spin. But the precessing domain (HPD) remains in the fully coherent Bose condensate state, while the volume of the magnon superfluid gradually decreases due to losses and the domain boundary slowly moves down (Fig. 4, right). The frequency \( \omega \) of spontaneous coherence as well as the phase of precession remain homogeneous across the whole Bose condensate domain, but the magnitude of the frequency changes with time, since it is determined by the Larmor frequency at the position of the interface, \( \mu(t) \equiv \omega_L(z_0(t)) \). The change of frequency during the decay is shown in Fig. 5 left. This frequency change during the relaxation was the main observational fact that led Fomin to construct the theory of the two-domain precession [9].

The details of formation of the magnon BEC are shown in Fig. 1 where the stroboscopic record of induction decay signal is shown. During the first stage of about 0.002 s the induction signal completely disappears due to dephasing. Then, during about 0.02 s, the phase coherent precession spontaneously emerges, which is equivalent to the magnon BEC state. Due to a weak magnetic relaxation, the number of magnons slowly decreases but the precession remains coherent during the whole process of relaxation. The time of formation of magnon BEC is essentially shorter than the relaxation time, as clearly shown in Fig. 1 right.

C. London limit: spin and mass supercurrents

In HPD state of magnon BEC, the magnon density is comparable with the limiting value, \( n \approx (5/8)n_{max} \), as a result the kinetic term in Ginzburg-Landau energy becomes complicated. Situation becomes simple in the London limit, where the magnon superfluidity is described by the hydrodynamic energy functional written in terms of density \( n \) and superfluid velocity \( v_s \).
The condensate occupies the domain where the chemical potential $\mu > U$ and radiates the signal corresponding to the Larmor frequency at the domain boundary of condensate. With relaxation the number of magnons decreases, and the chemical potential moves to the region with a lower Larmor frequency. The spectroscopic distribution of magnons. Immediately after the RF pulse each spin precesses with the local Larmore frequency. After the BEC formation, all the spins precess with the common frequency $\omega$ and spontaneously emergent common phase $\alpha$. Due to relaxation the number of magnons decreases, leading to the continuously decreasing frequency. The small broadening of BEC state is due to relaxation. By comparing the initial broadening of the NMR line of about 600 Hz and final broadening of about 0.5 Hz we can estimate that about 99.9% of the pumped magnons are in the condensate.

Superfluidity is phenomenon arising due to spontaneous breaking of $U(1)$ symmetry, which in our case is represented by the symmetry group $SO(2)$ of spin rotations about direction of magnetic field. In atomic BEC and in helium superfluids such symmetry breaking leads to a non-zero value of the superfluid rigidity – the superfluid density $\rho_s$ which enters the non-dissipative supercurrent of particles and thus to mass supercurrent. The same takes place for magnon BEC. But since magnons has both mass $m$ and spin $-\hbar$, they carry both the mass and spin supercurrents. The corresponding Goldstone phonon mode of the magnon BEC has been experimentally observed: it is manifested as twist oscillations of the precessing domain in $^3$He-B [27] (see the Section below).

For $n \ll n_{\text{max}}$ the mass current of magnons is given by the traditional equation:

$$J = \rho_s v_s, \quad v_s = \frac{\hbar}{m} \nabla \alpha, \quad \rho_s(T = 0) = nm.$$  
(4.7)

As in conventional superfluids, the superfluid density of the magnon liquid is determined by the magnon density $n$ and magnon mass $m$. To avoid the confusion let us mention that this superfluid density describes the coherent precession in magnetic subsystem and has nothing to do with the superfluid density of the underlying system – the superfluid $^3$He.

Since each magnon carries spin $-\hbar$, the magnon mass supercurrent is necessarily accompanied by the magnetization supercurrent – the supercurrent of the $z$-component of spin:
\[ J^z_i = -\frac{\hbar}{m} \quad J_i = -\frac{\hbar^2}{m} \nabla_i \alpha \]  

(4.8)

The nonzero superfluid density \( \rho_s > 0 \) is the main condition for superfluidity. For the HPD state of magnon BEC in \(^3\)He-B this condition is fulfilled.

In the London limit the magnon superfluidity is described by the hydrodynamic energy functional. It is similar to that in superfluid liquids and atomic BEC. However, there are important differences. One of the is the anisotropy of magnon mass, which leads to anisotropy of superfluid density even at \( T = 0 \). Another one is the presence of the symmetry breaking term which depends explicitly on \( \alpha \). The hydrodynamic energy functional is expressed in terms of the canonically conjugated variables – number density \( n \) (magnon density) and the superfluid velocity \( \mathbf{v}_s \), which is expressed via the gradient of the phase of the Bose condensate \( \alpha \) (the phase of the coherent precession of magnetization). It has the following general form:

\[ F = \frac{1}{2} \rho_{sij}(n)v_{si}v_{sj} + \epsilon(n) - \mu n + F_{sb}(\alpha, n) \]  

(4.9)

Here \( \mu \) as before is the chemical potential; and \( m_{ij}(n) \) is the matrix of magnon masses, which in general depends on magnon density \( n \). In \(^3\)He-B when the density of magnon increases the interaction modifies the magnon mass. It becomes anisotropic. For the magnons propagating along the field and in the transverse directions, their mass depends on the tilting angle in the following way:

\[ \frac{1}{m^\parallel(n)} = 2c_\parallel^2 \cos \beta + c_\perp^2 (1 - \cos \beta) \frac{\hbar \omega_L}{} \]

(4.10)

\[
\frac{1}{m^\perp(n)} = \frac{c_\parallel^2 (1 + \cos \beta) + c_\perp^2 (1 - \cos \beta)}{\hbar \omega_L},
\]

(4.11)

where the parameters \( c_\parallel \) and \( c_\perp \) are on the order of the Fermi velocity \( v_F \).

It is important that anisotropy of magnon mass leads to anisotropic superfluid density and superfluid velocity of magnon superfluid:

\[ \rho_{sij}(n) = nm_{ij}, \quad v_{si} = \hbar \left( m^{-1}\right)_{ij} \nabla_j \alpha \]  

(4.12)

The mass supercurrent becomes isotropic, when it expressed via \( \alpha \):

\[ J_i = \frac{dF}{dv_{si}} = \hbar n \nabla_i \alpha \]  

(4.13)

while the spin supercurrent becomes isotropic, when it is expressed in terms of superfluid velocity:

\[ J^z_i = \hbar \left( m^{-1}\right)_{ij} J_j = \hbar n v_{si} \]  

(4.14)

Finally \( F_{sb} \) is the symmetry breaking term which depends explicitly on \( \alpha \). It arises only in continuous wave NMR, when the relaxation of magnon BEC is compensated by RF field. It gives rise to the mass of Goldstone boson which we discuss later on.
D. Supercurrent and linear momentum

The magnon superfluidity is very similar to superfluidity of the A\textsubscript{1} phase of \textsuperscript{3}He where only one spin component is superfluid\textsuperscript{24}: as a result the superfluid mass current is accompanied by the superfluid spin current.

The spin current in magnon superfluids can be obtained directly from the definition of linear momentum density in spin systems:

$$\mathbf{P} = (S - S_z) \nabla \alpha = n \hbar \nabla \alpha . \quad (4.15)$$

Here we used the fact that $S - S_z$ and $\alpha$ are canonically conjugated variables. This superfluid mass current is accompanied by the superfluid current of spins transferred by the magnon condensate. It is determined by the spin to mass ratio for the magnon, and because the magnon mass is anisotropic, the spin current transferred by the coherent spin precession is anisotropic too:

$$J^z = -\frac{\hbar^2}{m^\parallel(n)} n \nabla^2 \alpha , \quad (4.16)$$

$$J^\perp = -\frac{\hbar^2}{m^\perp(n)} n \nabla \perp \alpha . \quad (4.17)$$

The anisotropy of the current in Eqs. (4.16-4.17) is an important modification of the conventional Bose condensation. This effect is absent in the atomic Bose condensates.

Let us note that the density of linear momentum of the spin subsystem is not well defined globally. While the total momentum of the system is conserved, the canonical momenta of the spin and orbital subsystems are not conserved separately\textsuperscript{28 –30}. For the particular choice of the linear momentum density in (4.15), $\mathbf{P}$ is not defined at the points where $\beta = \pi$, because at $\beta = \pi$ spins are stationary and thus the spin precession angle $\alpha$ is ill defined. This is another interesting feature of the magnon superfluids, which becomes important for the magnon BEC emerging in normal (non-superfluid) \textsuperscript{3}He. The latter represents a coherently precessing structure at the interface between the equilibrium domain with $\beta = 0$ and the domain with the reversed magnetization, i.e. with $\beta = \pi$\textsuperscript{31}.

E. Goldstone mode of coherent precession – sound in magnon BEC

In atomic superfluids, sound is the Goldstone mode of the spontaneously broken $U(1)$ symmetry. The same sound mode exists in magnon BEC. The sound in magnon subsystem has been calculated in Ref.\textsuperscript{32} and identified experimentally in Ref.\textsuperscript{27}.

There are several conditions required for the existence and stability of the magnon BEC. Some of them are the same as for the conventional atomic BEC, but there are also important differences, which we discuss later. One of the conditions is that the compressibility $\beta_M$ of the magnon gas must be positive:

$$\beta_M^{-1} = n \frac{dP}{dn} = n^2 \frac{d^2 \epsilon}{dn^2} > 0 . \quad (4.18)$$

This condition means that the fourth order term in the Ginzburg-Landau free energy should be positive, i.e. the interaction between magnons should be repulsive. The magnon interaction energy $\epsilon(n)$ is provided by spin-orbit (dipole-dipole) interaction. It has a very peculiar
form for HPD in $^3$He-B:

$$\epsilon(n) \equiv E_{so}(n) = \frac{8 \chi \Omega_L^2}{15 \gamma^2} \left( \frac{\hbar n}{S} - \frac{5}{4} \right)^2 \Theta \left( \frac{\hbar n}{S} - \frac{5}{4} \right),$$  \hspace{1cm} (4.19)

where $\Theta(x)$ is Heaviside step function; $\Omega_L$ is Leggett frequency (we assume that $\Omega_L \ll \omega_L$). This means that in this state of magnon BEC a stable coherent precession occurs only at large enough magnon density $n > 5S/4\hbar$, where $d^2E_{so}/dn^2 > 0$. This corresponds to $\cos \beta < -1/4$. The magnon BEC state in (4.30) also satisfy the condition (4.18), while the magnon condensates in (4.29) and in bulk $^3$He-A are unstable.

The compressibility of the magnon gas determines the speed of sound propagating in the magnon gas. Since the magnons mass is anisotropic the phonon spectrum is also anisotropic:

$$(c_s^2)^{ij} = (m^{-1})^{ij} \frac{dP}{dn} = n \frac{d^2E_{so}}{dn^2} (m^{-1})^{ij}.$$  \hspace{1cm} (4.20)

In the typical experiments with HPD, $\cos \beta$ is close to $-1/4$, i.e. $\cos \beta = -1/4 - 0$. For such $\beta$ one has:

$$c_s^2 = \frac{n}{m_s} \frac{d^2E_{so}}{dn^2} = \frac{2 \Omega_L^2}{3 \omega_L^2} (5c_{\perp}^2 - c_{\parallel}^2),$$  \hspace{1cm} (4.21)

$$c_{s\perp}^2 = \frac{n}{m_{s\perp}} \frac{d^2E_{so}}{dn^2} = \frac{1 \Omega_L^2}{3 \omega_L^2} (5c_{\perp}^2 + 3c_{\parallel}^2).$$  \hspace{1cm} (4.22)

F. Mass of phonons in magnon superfluid

As distinct from the conventional superfluids, in magnon BEC one may introduce experimentally the symmetry breaking field which smoothly violates the $U(1)$ symmetry and induces a small gap (mass) in the phonon spectrum. This mass has been measured.

The symmetry-breaking term appears in continuous wave NMR, when the relaxation of magnon BEC is compensated by RF field. It describes the interaction $F_{sb}(\alpha, n) = -\gamma H_{RF} \cdot S$ of the precessing magnetization with the RF field $H_{RF}$, which is transverse to the applied constant field $H$. In continuous wave NMR experiments the RF field prescribes the frequency of precession, $\omega = \omega_{RF}$, and thus fixes the chemical potential $\mu$; while in the state of free precession the chemical potential $\mu$ is determined by the number of pumped magnons. The symmetry-breaking term depends explicitly on the phase of precession $\alpha$ with respect to the direction of the RF-field in the precessing frame:

$$F_{sb} = -\gamma H_{RF} S_{\perp} \cos \alpha = -\gamma H_{RF} S \sin \beta \left( 1 - \frac{\alpha^2}{2} \right).$$  \hspace{1cm} (4.23)

Due to explicit dependence on $\alpha$, this term generates the mass of the Goldstone boson (phonon) \cite{33}. For $\cos \beta = -1/4$ the phonon spectrum becomes:

$$\omega_s^2(k) = (c_s^2)^{ij} k_i k_j + m_s^2, \quad m_s^2 = \frac{4}{\sqrt{15}} \gamma H_{RF} \Omega_L^2 \omega_L.$$  \hspace{1cm} (4.24)

Two experiments with HPD \cite{34,35} reported the gap in the spectrum of the collective mode of the coherent precession. The measured gap is proportional to $H_{RF}^{1/2}$ in agreement with (4.21).
FIG. 6: Phonon mass in magnon BEC as a function of the symmetry breaking field. From [35].

G. Spin-current Josephson effect

The phase coherent precession of magnetization in superfluid $^3$He has all the properties of the coherent Bose condensate of magnons. The main spin-superfluid properties of HPD have been verified already in the early experiments 20 years ago. These include spin supercurrent which transports the magnetization (analog of the mass current in conventional superfluids); spin current Josephson effect and phase-slip processes at the critical current [36, 37] (Fig. 7).

H. Spin vortex– topological defect of magnon BEC

Later on the spin current vortex has been observed [38] – a topological defect which is an analog of a quantized vortex in superfluids and of an Abrikosov vortex in superconductors. The condensate phase $\alpha$ has $2\pi$ winding around the vortex core (see Fig. 8). The mass current around the vortex core is accompanied by spin current.

In HPD with a single vortex in the central part of cylindrical cell the phase $\alpha$ of precession changes by $2\pi$ around the center, i.e. $\alpha$ is opposite on the opposite sides of the cell. In the central part of the cell, i.e. in the vortex core, the magnetization remains vertical and does not precess. The magnon BEC with a spin vortex is created by applying the quadrupole RF field. For this purpose two parts of the saddle NMR coil are connected in opposite directions, so that the phase of RF field (and consequently the phase $\alpha$) was opposite at the opposite sides of the cell. By these NMR coils practically the same HPD signal was observed as in the conventional arrangement with the parallel connection of the coils, though with a slightly reduced amplitude. This shows that HPD is created with opposite $\alpha$ on opposite sides of the cell. To verify this a pair of small pick-up coils are installed at the top of the cell connected in usual way. When the RF field is switched off, the pick up coils received a very small RF signal from HPD, while the frequency of this signal corresponded to the
FIG. 7: Josephson effect for magnon BEC in $^3$He-B which demonstrates the interference between two condensates. *left:* Spin current as a function of the phase difference across the junction, $\alpha_2 - \alpha_1$, where $\alpha_1$ and $\alpha_2$ are phases of precession in two coherently precessing domains connected by a small orifice (*right*). Different experimental records correspond to a different ratio between the diameter of the orifice and the stiffness (magnetic coherence length) of the HPD state.

full HPD signal. This means that HPD generated the signal with the opposite phase at the two sides of the pickup coils, which nearly compensated each other (see Fig. 9). This corresponds to HPD with a circular gradient of $\alpha$, as shown in Fig. 8. The magnetization is oriented vertically in the vortex core. On the periphery of the cell it precesses with tipping angle $104^\circ$, and with $2\pi$ phase winding around the center. This type of HPD should radiate at frequency, which corresponds to the Larmor field on the boundary of HPD, but should not produce any signal in the pick-up coil. A small signal appears due to asymmetry of the pick-up coil; oscillations of this signal may correspond to nutations of the vortex core.

I. Critical velocities, coherence length and the vortex core radius

The phonon spectrum in magnon gas determines the Landau critical velocity of the counterflow at which phonons are created. For the case of isotropic sound one has (the Landau criterion for the anisotropic case has been derived in [39]):

$$v_L = c_s.$$  \hfill (4.25)

For conventional superfluids this suggests that the coherence length and the size of the vortex core should be on the order of:

$$\frac{\hbar}{mc_s} \sim \frac{c}{\Omega_L}.$$ \hfill (4.26)

However, for magnon superfluid in the HPD state this gives only the lower bound on the core size. The core is larger due to specific profile of the Ginzburg-Landau (dipole) energy in Eq. 4.19 which is strictly zero for $\cos \beta > -1/4$. This leads to the special topological
FIG. 8: Spin-mass vortex in magnon BEC. The spin current around the vortex core is accompanied by mass current. According to Eq. (4.26) the size of the vortex core diverges when the HPD domain boundary is approached where the local Larmor frequency $\omega_L = \omega$.

Properties of coherent precession (see Ref. [40]). As a result the spin vortex created and observed in Ref. [38] has a continuous core with broken symmetry, similar to vortices in superfluid $^3$He-A [41]. The size of the continuous core is determined by the proper coherence length [42] which can be found from the competition between the first two terms in the Ginzburg-Landau free energy in Eq. (3.15):

$$r_{core} \sim \frac{\hbar}{\sqrt{m(\mu - \omega_L)}} \sim \frac{c}{\sqrt{\omega_L(\omega - \omega_L)}}.$$  \hspace{1cm} (4.27)

This coherence length determines also the critical velocity for creation of vortices:

$$v_c \sim \frac{\hbar}{m r_{core}}.$$ \hspace{1cm} (4.28)

For large tipping angles of precession the symmetry of the vortex core is restored: the vortex becomes singular with the core radius $r_{core} \sim c/\Omega_L$ [43].

Other topological defects possible in the coherent precession beyond the Ginzburg-Landau model of magnon BEC are discussed in [40]. In BEC of excitations, the topological defects have been detected in exciton-polariton condensate [44]. Among them the half quantum vortices – vortices with half of the circulation quantum. Such vortices are topologically stable in the superfluid $^3$He-A [45], but they still remain elusive there.
FIG. 9: NMR signature of the spin vortex in magnon BEC (HPD). The frequency decays with time in the same way as in conventional vortex-free precession, while the amplitude of the HPD signal is nearly zero because of compensation of signals from opposite sides of the cells where the phase $\alpha$ of precession differs by $\pi$.

J. Other states of magnon BEC in $^3$He-B

Recent experiments in $^3$He-B allowed to probe the BEC states that emerge in the valley on the other side of the energy barrier in Fig. 2. This became possible by immersing the superfluid $^3$He in a very porous material called aerogel. By squeezing or stretching the aerogel sample, one creates the global anisotropy which captures the orbital vector $\hat{l}$. This allows to orient the orbital vector $\hat{l}$ in the desirable direction with respect to magnetic field [46, 47].

For the transverse orientation of $\hat{l}$, i.e. for $l = 0$, two new BEC states have been identified. One of them exists at $|\Psi|^2 < S/\hbar$ and has the following form of spin-orbit interaction obtained from Eq. (3.16) (we omit for simplicity the constant term):

$$F_{so}(\Psi)_{l=0} = -\frac{\chi}{4\gamma^2} \Omega_L^2 \left( \frac{|\Psi|^2}{S} - \frac{4}{5} \right)^2, \quad |\Psi|^2 < \frac{S}{\hbar}. \quad (4.29)$$

This state has an attractive interaction between magnons, and is unstable since the compressibility $\beta_M$ of the magnon gas in (1.18) is negative: $d^2\epsilon/dn^2 < 0$. 

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The other state exists at $|\Psi|^2 > S/\hbar$ and has the following form of spin-orbit interaction

$$F_{so} (|\Psi|_{l=0}) = \frac{\chi \Omega^2}{20 \gamma^2} \Omega_L^2 \left( \frac{|\Psi|^2}{S} - 2 \right)^2, \quad |\Psi|^2 > \frac{S}{\hbar}. \quad (4.30)$$

This state has repulsive interaction between magnons and is stable.

V. MAGNON BEC IN $^3$He-A

As in the case of $^3$He-B, all the information on the $^3$He-A order parameter needed to study the coherent precession is encoded in the spin-orbit interaction.

A. Instability of magnon BEC in bulk $^3$He-A

For $^3$He-A, the spin-orbit interaction averaged over the fast precession has the following form [48]:

$$F_{so} (|\Psi|) = \frac{\chi \Omega^2}{4 \gamma^2} \times
\left[ -2 \frac{|\Psi|^2}{S} + \frac{|\Psi|^4}{S^2} + \left( -2 + 4 \frac{|\Psi|^2}{S} - \frac{7}{4} \frac{|\Psi|^4}{S^2} \right) \left( 1 - l^2 \right) \right]. \quad (5.1)$$

In a static bulk $^3$He-A, when $\Psi = 0$, the spin-orbit energy $F_{so}$ in Eq. (5.1) is minimized when the orbital vector $\hat{l}$ is perpendicular to magnetic field, i.e. for $l = 0$. Then one has

$$F_{so} (|\Psi|, l = 0) = \frac{\chi \Omega^2}{4 \gamma^2} \left[ -2 + 2 \frac{|\Psi|^2}{S} - \frac{3}{4} \frac{|\Psi|^4}{S^2} \right], \quad (5.2)$$

with a negative quartic term. The attractive interaction between magnons destabilizes the BEC, which means that homogeneous precession of magnetization in $^3$He-A becomes unstable. This instability predicted by Fomin [49] was experimentally confirmed [50].

However, as follows from (5.1), at sufficiently large magnon density $n = |\Psi|^2$,

$$\frac{8 + \sqrt{8}}{7} S > n > \frac{8 - \sqrt{8}}{7} S, \quad (5.3)$$

the factor in front of $l^2$ becomes negative. Therefore it becomes energetically favorable to orient the orbital momentum $\hat{l}$ along the magnetic field, $l = 1$. For this orientation one obtains the Ginzburg-Landau free energy with

$$F_{so} (|\Psi|, l = 1) = \frac{\chi \Omega^2}{4 \gamma^2} \left[ -2 \frac{|\Psi|^2}{S} + \frac{|\Psi|^4}{S^2} \right]. \quad (5.4)$$

It corresponds to the conventional Ginzburg-Landau free energy in atomic BEC. The quadratic term modifies the potential $U$; the quartic term is now positive.

In the language of BEC, this means that, with increasing the density of Bose condensate, the originally attractive interaction between magnons should spontaneously become repulsive when the critical magnon density $n_c = S(8 - \sqrt{8})/7$ is reached. If this happens,
FIG. 10: $F - \mu n$ for different values of the chemical potential $\mu \geq U$ in magnon BEC in $^3$He-A. Magnon BEC in $^3$He-A is similar to BEC in atomic gases.

the magnon BEC becomes stable and in this way the state with spontaneous coherent precession could be formed [48]. This self-stabilization effect is similar to the effect of $Q$-ball, where bosons create the potential well in which they condense (we shall discuss the $Q$-ball phenomenon in magnon BEC later on). However, such a self-sustaining BEC with originally attractive boson interaction has not been achieved experimentally in bulk $^3$He-A, most probably because of the large dissipation, due to which the threshold value of the condensate density has not been reached.

B. Magnon BEC of $^3$He-A in deformed aerogel

Finally the fixed orientation of the orbital vector $\hat{l}$ has been achieved in $^3$He-A confined in aerogel – the material with high porosity, which is about 98% of volume. Silicon strands of aerogel play the role of impurities with local anisotropy along the strands. According to the Larkin-Imry-Ma effect, the random anisotropy suppresses the orientational long-range order of the orbital vector $\hat{l}$; however, when the aerogel sample is deformed the long-range order of $\hat{l}$ is restored [51]. Experiments with globally squeezed aerogel [52] demonstrated that a uni-axial deformation by about 1% is sufficient for global orientation of the vector $\hat{l}$ along the anisotropy axis. When magnetic field is also oriented along the anisotropy axis one obtains the required geometry with $l = 1$, at which the magnon BEC in $^3$He-A becomes stable. The first indication of coherent precession in $^3$He-A has been reported in [53, 54] and confirmed in [55]. Contrary to the unconventional magnon BEC in the form of HPD in $^3$He-B, the magnon BEC emerging in the superfluid $^3$He-A is in one-to-one correspondence with the atomic BEC, see Fig. 10. For $\mu > U$, the condensate density determined from equation $dF/dn = \mu$ continuously grows from zero as $n \propto \mu - U$. 

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For \( l = 1 \) the Ginzburg-Landau free energy acquires the standard form:

\[
F = \int d^3r \left( \frac{\left| \nabla \Psi \right|^2}{2m} + (\omega_L(r) - \mu)|\Psi|^2 + \frac{1}{2}b|\Psi|^4 \right),
\]

(5.5)

where we modified the chemical potential by the constant frequency shift:

\[
\mu = \omega + \frac{\Omega^2}{2\omega},
\]

(5.6)

and the parameter \( b \) of repulsive magnon interaction is

\[
b = \frac{\Omega^2}{2\omega S}
\]

(5.7)

At \( \mu > \omega_L \), magnon BEC must be formed with density

\[
|\Psi|^2 = \frac{\mu - \omega_L}{b}.
\]

(5.8)

This is distinct from \( ^3\text{He-B} \), where condensation starts with finite condensate density. Eq. (5.8) corresponds to the following dependence of the frequency shift on tipping angle \( \beta \) of coherence precession:

\[
\omega - \omega_L = -\frac{\Omega^2}{2\omega} \cos \beta.
\]

(5.9)

The final proof of the coherence of precession in \( ^3\text{He-A} \) in aerogel was the observation of the free precession after a pulsed NMR and also after a switch off the CW NMR [55]. In conclusion, in contrast to the homogeneously precessing domain (HPD) in \( ^3\text{He-B} \), the magnon Bose condensation in \( ^3\text{He-A} \) obeys the standard Gross-Pitaevskii equation. In bulk \( ^3\text{He-A} \), the Bose condensate of magnons is unstable because of the attractive interaction between magnons. In \( ^3\text{He-A} \) confined in aerogel, the repulsive interaction is achieved by the proper deformation of the aerogel sample, and the Bose condensate becomes stable.

VI. MAGNON BEC IN MAGNETIC TRAP AND Q-BALL

There are many new physical phenomena related to the Bose condensation of magnons, which have been observed after the discovery of HPD. These include in particular compact objects with finite number of the Bose condensed magnons. At small number \( N \) of the pumped magnons, the system is similar to the Bose condensate of the ultracold atoms in harmonic traps, while at larger \( N \) the analog of the \( Q \)-ball in particle physics develops [56].

A \( Q \)-ball is a non-topological soliton solution in field theories containing a complex scalar field \( \Psi \). \( Q \)-balls are stabilized due to the conservation of the global \( U(1) \) charge \( Q \) [57]: they exist if the energy minimum develops at nonzero \( \phi \) at fixed \( Q \). At the quantum level, \( Q \)-ball is formed due to suitable attractive interaction that binds the quanta of \( \Psi \)-field into a large compact object. In some modern SUSY scenarios \( Q \)-balls are considered as a heavy particle-like objects, with \( Q \) being the baryon and/or lepton number. For many conceivable alternatives, \( Q \)-balls may contribute significantly to the dark matter and baryon contents of
The trapping potential Eq. (6.2) is formed in the cylindrically symmetric “flare-out” texture of the orbital angular momentum \( \mathbf{l} \) (dashed lines) in a shallow minimum of the vertical magnetic field (right). The arrows represent the precessing magnetization \( \mathcal{M} \), which precesses coherently within the condensate droplet (dark blue). From \([63]\).

A. Magnetic trap in \( ^3\text{He-B} \)

A cylindrically symmetric magnetic trap for magnon BEC in \( ^3\text{He-B} \) is schematically shown in Fig. \( 11 \) \([63]\). The confinement potential \( U_\parallel(z) \) in the axial direction is produced by a small pinch coil, while in the radial direction the well \( U_\perp(r) \) is formed by the cylindrically symmetric flare-out texture of the orbital vector \( \mathbf{l} \):

\[
U(\mathbf{r}) = U_\parallel(z) + U_\perp(r) = \omega_L(z) + \frac{2\Omega_L^2}{5\omega_L}(1 - l(r)) \tag{6.1},
\]

where as before \( l = \hat{l} \cdot \hat{H} \) describes the orientation of the unit vector \( \hat{l} \) with respect to the direction \( \hat{H} \) of magnetic field. On the side wall of the cylindrical container the orbital...
momentum \( \mathbf{l} \) is normal to the wall, while in the center it is parallel to the axially oriented applied magnetic field. This produces a minimum of the potential \( U_L(r) \) on the cylinder axis. Close to the axis the angle \( \beta_L \) varies linearly with distance \( r \) from the axis [41]. As a result, the potential \( U(r) \) reduces to that of a usual harmonic trap used for the confinement of dilute Bose gases [11]:

\[
U(r) = U(0) + \frac{m_M}{2} \left( \omega_z^2 z^2 + \omega_r^2 r^2 \right),
\]

(6.2)

The oscillator frequency \( \omega_r \) of the well in the radial direction can be adjusted by applying rotation, since adding vortex-free superfluid flow and/or rectilinear vortex lines modifies the flare-out texture. Low-amplitude standing spin waves in the trap have the conventional spectrum

\[
\omega_{mn} = \omega_L(0) + \omega_r(m + 1) + \omega_z(n + 1/2).
\]

(6.3)

FIG. 12: Numerical simulation of the \( Q \)-ball growing in the center of the cell. At small \( Q \), when the \( \beta \)-angle of spin \( \mathbf{S} \) is less than about \( 10^\circ \), the l-texture is practically \( Q \)-independent (see \( \beta_L(x) \) marked by solid line). At larger \( Q \) the potential well for magnons becomes wider: the dashed line corresponds to the l-texture formed by \( Q \)-ball with \( \beta = 20^\circ \) in the middle of the \( Q \)-ball; and the dotted line – with \( \beta = 33^\circ \). For details see [65].

B. Ground-state condensate and self-localization

Typically the condensate is formed in the ground state \((0,0)\) of the trap. When the number of magnons \( N \) in the ground state increases, the interactions between them become important and the condensate wave function starts deviating from the Gaussian form of an ideal gas. As distinct from a system of cold atoms, the peculiarity of the magnon Ginzburg-Landau functional in Eq. [3.16] is that the prefactor of the quartic term, which describes
the interactions between the magnons, is not a constant, but \( \propto \omega^4 r^4 \) and thus is small in the region of the trap and can be neglected.

Under conditions of our experiment the main effect the self-localization discussed in [56]. At high density of magnons they start to influence the radial \( l \)-texture. According to (3.16) the condensate leads to the preferable orientation of \( l \) parallel to magnetic field, \( l = 1 \), in the region of the trap. As a result the potential well becomes wider and the energy of the level in the trap decreases. This allows to incorporate more magnons at this same level by sweeping the frequency up. This is equivalent to effective attractive interaction induced by the exchange of the quanta of the \( l \)-field.

In the language of relativistic quantum fields, the self-localization is caused by interaction between the charged and neutral fields, where the neutral field provides the potential for the charged one. In the process of self-localization the charged field modifies locally the neutral field so that the potential well is formed in which the charge is condensed. Such a trapped condensate resembles a droplet of quantum field with a fixed value of field quanta \( Q \) – the \( Q \)-ball [57]. The charge \( Q \) corresponds to the spin \( S_z \) or, equivalently, to the magnon number \( N \). Experimentally the \( Q \)-ball in \(^3\)He-B [56] is manifested as long-lived ringing (of up to an hour!) of the free induction decay after a NMR tipping pulse [64]. In steady state it can be maintained by CW RF pumping (Fig. 13), and even by off-resonance excitation [65, 66].

One can estimate the radius of the self-localization as a function of the magnon number at large \( N \) [63]:

\[
R(N) \sim a_r \left( \frac{N}{N_c} \right)^p, \quad N \gg N_c, \tag{6.4}
\]

where \( a_r \) is the harmonic oscillator length in the original (non-modified) radial trap, and \( N_c \) is the characteristic number at which interaction becomes important. Assuming that the trap is flat in the region of the condensate, \( \beta_L = 0 \), one obtains \( p = 1/2 \). For comparison, for an atomic BEC with repulsive inter-particle interaction the exponent in the Thomas-Fermi limit is \( p = 1/5 \) [11]. Compare also the change in chemical potential due to interaction in the atomic BEC in the Thomas-Fermi regime, and the frequency shift in magnon BEC:

\[
\mu - U(0) \approx \omega_r \left( \frac{N}{N_c} \right)^{2/5}, \quad \text{atomic BEC}, \tag{6.5}
\]

\[
\omega - \omega_L(0) \sim \omega_r \left( \frac{N}{N_c} \right)^{-1}, \quad \text{magnon BEC}. \tag{6.6}
\]

With the growing \( Q \)-ball, its frequency \( \omega \) decreases approaching the Larmor frequency asymptotically. As distinct from the atomic BEC, the magnon BEC droplet has \( d\mu/d\mathcal{N} < 0 \). This determines the way in which the magnon condensate is grown in a cw NMR measurement, as seen in Fig. 13 for the ground state of the trap in Fig. 11. The magnons are created when the frequency \( \omega \) of the applied RF field is swept down and crosses the ground state level \( \omega_{00} \). When \( \omega \) is reduced further, the number of magnons \( N \) follows from Eq. (6.6).

\( Q \)-balls can be also formed in pulsed NMR measurements. As distinct from cw NMR, where the \( Q \)-balls are generated starting continuously from \( \mathcal{N} = 0 \), in pulsed NMR a \( Q \)-ball is formed after a large \( \mathcal{N} \) is pumped to the cell. In this case the 3D \( Q \)-ball is often formed on the axis of the flared-out texture, away from the horizontal walls [88]. This clearly demonstrates the effect of self-localization: the main part of the pumped charge relaxes but the rest of \( \mathcal{N} \) starts to concentrate at some place on the axis, digging a potential well there.
and attracting the charge from other places of the container. Moreover, this also shows that Q-balls are not necessarily formed at the bottom of the original potential: Q-ball may dig the potential well in a different place. In the formation of Q-ball with off-resonance excitation [65, 66], the effect of self-localization also plays a crucial role.

In conclusion, Q-balls represent a new phase coherent state of Larmor precession. They emerge at low $T$, when the homogeneous bulk BEC of magnons (HPD) becomes unstable. These Q-balls are compact objects which exist due to the conservation of the global $U(1)$ charge $Q = S_z$. At small $Q$ they are stabilized in the potential well, while at large $Q$ the effect of self-localization is observed. In terms of relativistic quantum fields the localization is caused by the peculiar interaction between the charged and neutral fields [67]. The neutral field $l$ provides the potential for the charged field $\Psi$; the charged field modifies locally the neutral field so that the potential well is formed in which the charge $Q$ is further condensed.

C. Non-ground-state condensates

The condensate can be also formed when one starts filling magnons to one of the levels $(m,n)$ in Eq. (6.3). Then one obtains the non-ground-state condensate [63]. The formation of non-ground-state condensates has been proposed for cold atoms [68], but as a dynamic mixture of the ground state and an excited level. It was suggested to use resonant modulation of either the trap potential or the atomic scattering length, eg. by applying a temporal modulation of the atomic interactions via the Feshbach resonance technique. In contrast to such schemes with atomic condensates in optical traps, the properties of magnon condensates make it possible to populate different excited trap levels by pumping magnons resonantly directly to these levels. This fact shows that condensates formed from excitations lead to interesting new consequences.

FIG. 13: Formation of the magnon condensate droplet in the ground state of the trap in a cw NMR measurement. $M_\perp$ grows while the frequency of applied RF field $f = \omega/(2\pi)$ is swept down (or equivalently, the steady magnetic field $H$ is swept up), in agreement with Eq. (6.6). The condensate is destroyed (vertical lines), when energy dissipation exceeds the RF pumping and therefore the maximum volume of the condensate droplet grows with increasing RF excitation.
FIG. 14: Twisted core of non-axisymmetric vortex in $^3$He-B. The gradient of the Goldstone field $\nabla \phi$ along the string corresponds to the superconducting current along the superconducting cosmic string. Such twisted core has been obtained and detected using coherent precession of magnetization [71].

The excited states $(m,n)$ can be populated [63], because their frequencies $\omega_{mn}$ also decrease with increasing the magnon number $N$. The condensate in the state $(m,n)$ grows when the frequency of RF field is swept down and crosses the level $\omega_{mn}(0)$ from above [63]. An excited state is populated only if magnons are pumped directly to this level by applying an RF field with the exact frequency $\omega_{mn}(N)$. The condensate in the excited state is metastable: it is supported by continuous pumping at $\omega_{mn}(N)$ or otherwise decays to the ground state $\omega_{00}$. Thus in pulsed NMR the long-lived free precession, which is still ringing well after the RF tipping pulse, is produced by the ground state population. The spectrum (6.3) and its evolution due to self-localization can be conveniently examined experimentally by modifying the trap in Fig. 11 by applying rotation or by changing the depth of the axial field minimum.

VII. EXPLOITING BOSE CONDENSATE OF MAGNONS

The Bose condensation of magnons in superfluid $^3$He-B has many practical applications. In Helsinki, owing to the extreme sensitivity of the Bose condensate to textural inhomogeneity, the phenomenon of Bose condensation has been applied to studies of supercurrents and topological defects in $^3$He-B in rotating cryostat. The measurement technique was called HPD spectroscopy [69, 70].

A. Observation of Witten string

In particular, HPD spectroscopy provided direct experimental evidence for broken axial symmetry in the core of quantized vortices in $^3$He-B [71].

The dominating area of the phase diagram of the vortex states in $^3$He-B is occupied by
vortices with non-axisymmetric cores, i.e. vortices with the spontaneously broken rotational $SO(2)$ of the core calculated in Refs. [72, 73]. The core with broken rotational symmetry can be considered as a pair of half-quantum vortices, connected by a non-topological soliton wall (see Fig. 14). The separation of the half-quantum vortices increases with decreasing pressure and thus the double-core structure is most pronounced at zero pressure.

In the physics of cosmic strings, an analogous breaking of continuous symmetry in the core was first discussed by Witten [74], who considered the spontaneous breaking of the electromagnetic gauge symmetry $U(1)$. Since the same symmetry group is broken in the condensed matter superconductors, one can say that in the core of the cosmic string there appears the superconductivity of the electric charges, hence the name ‘superconducting cosmic strings’.

For the $^3$He-B vortices, the spontaneous breaking of the $SO(2)$ symmetry in the core leads to the Goldstone bosons – the mode in which the degeneracy parameter, the axis of anisotropy of the vortex core, is oscillating. The homogeneous magnon condensate, the HPD state, has been used to study the structure and twisting dynamics of the non-axisymmetric core of the low-temperature vortex in $^3$He-B [71]. This is because the coherent precession of magnetization excites the vibrational Goldstone mode via spin-orbit interaction. Moreover, due to spin-orbit interaction the precessing magnetization rotates the core around its axis with constant angular velocity. In addition, since the core was pinned on the top and the bottom of the container, it was possible even to screw the core (see Fig. 14). Such a twisted core corresponds to the Witten superconducting string with the electric supercurrent along the core. The rigidity of twisted core differs from that of the straight core. This allowed a detailed study of the Goldstone mode of the vortex core resulting from the spontaneous violation of rotational $U(1)$ symmetry in the core [71].

B. Observation of spin-mass vortex

There are different types of the topological defects in the (non-precessing) $^3$He-B. Among them there is a $Z_2$ spin vortex – topological defect of the order parameter matrix $R_{ai}$ in (9.8). Due to spin-orbit coupling this defect serves as the termination line of the topological soliton wall, and due to the soliton tension it cannot be stabilized in the rotating vessel. However, spin and mass vortices attract each other and form the combined defect, the so-called spin-mass vortex which can be stabilized under rotation (see Fig. 15). The spin-mass vortices also form molecules where the soliton serves as chemical bond. These defects – spin-mass vortex connected with the wall by soliton and bound pairs of spin-mass vortices – have been observed and studied using HPD spectroscopy [75].

C. Magnon condensates in aerogel

HPD spectroscopy proved to be extremely useful for the investigation of the superfluid order parameter in a novel system – superfluid $^3$He confined in aerogel [53, 76–80].

D. Towards observation of Majorana fermions

The condensate in the magnetic trap can be used to continue measurements down to $0.1 – 0.2 T_c$. Of great current interest are the fermion states in the vortex core, especially the still
elusive Majorana fermions. Recently the topologically nontrivial gapless and gapped phases of matter, topological insulator states in semimetals, superconductors, and superfluids have attracted attention. $^3$He-B is the best representative of a 3-dimensional coherent quantum system with time reversal symmetry. Its nontrivial topology gives rise to gapless Andreev-Majorana fermions as surface states [81, 83]. There is now experimental evidence for Andreev surface states in $^3$He-B at a solid wall [84, 85], but the Majorana signature of these fermions – the linear ‘relativistic’ energy spectrum at low energy – can be observed only at extreme low temperatures, when the thermal quasiparticles in bulk $^3$He-B are exponentially depleted. Majorana fermions, both in the vortex core and at surfaces, are then expected to give the main contribution to thermodynamics and dissipation, with a power-law dependence of the physical quantities on $T$. In rotation or by moving the magnon condensate droplet next to the wall, one will be able to probe Majorana fermion states.

VIII. MAGNON BEC IN OTHER SYSTEMS

A. Magnon BEC in normal $^3$He

A very long lived induction signal was observed in normal Fermi liquids: in spin-polarized $^3$He-$^4$He solutions [86] and in normal liquid $^3$He [87]. It was explained as a coherently precessing structure at the interface between the equilibrium domain and the domain with the reversed magnetization [31]. It would be interesting to treat this type of dynamic
magnetic ordering as a new mode of magnon BEC.

B. Magnons condensation in yttrium-iron garnet

In all the magnon systems under consideration, the typical temperature is big compared to the magnon gap, $T \gg \hbar \omega_L$. In $^3$He-B this means that according to (3.1) thermal magnons are spin waves with linear spectrum $\omega(k) = ck$, with characteristic momenta $k_T \sim T/\hbar c$. The density of thermal magnons $n_T \sim k_T^3T$ is much smaller than the density of condensed magnons and in $^3$He-B can be neglected.

In yttrium-iron garnet (YIG) situation is opposite: the temperature is high and number density of the magnons concentrated at small momentum is small compared to the thermal magnons, $n \ll n_T$ [6, 7]. Magnons in yttrium-iron garnet (YIG) films have the quasi 2D spectrum: 

$$\omega_n(k_x, k_y) = \Delta_0 + \frac{k_y^2}{2m_y} + \frac{(k_x - k_0)^2}{2m_x},$$

(8.1)

where magnetic field is along $x$; the gap in the lowest branch $\Delta_0 = 2.1$ GHz $\equiv$ 101 mK at $H = 700$ Oe [6] and $\Delta_0 = 2.9$GHz at $H = 1000$ Oe [7]; the position of the minimum $k_0 \approx 5 \cdot 10^4$ 1/cm [6]; the anisotropic magnon mass can be probably estimated as $m_x \sim k_0^2/\Delta$ with $m_y$ being somewhat bigger, both are of order of electron mass.

In two-dimensional systems the number of extra magnons – the difference of the distribution function at $\mu = 0$ and $\mu \neq 0$ – is determined by low energy Rayleigh-Jeans part of the spectrum:

$$n = \sum_k \left( \frac{T}{E_k - \mu} - \frac{T}{E_k} \right),$$

(8.2)

If one neglects the contribution of the higher levels and consider the 2D gas, the Eq.(8.2) becomes

$$n_{\text{extra}} = \frac{T}{2\pi\hbar \sqrt{m_x m_y}} \ln \frac{\Delta_0}{\Delta_0 - \mu},$$

(8.3)

In 2D, all extra magnons can be absorbed by thermal distribution at any temperature without formation of Bose condensate. The larger is the number $n$ of the pumped magnons the closer is $\mu$ to $\Delta_0$, but $\mu$ never crosses $\Delta_0$. At large $n$ the chemical potential exponentially approaches $\Delta_0$ from below and the width of the distribution becomes exponentially narrow:

$$\frac{(\delta k_y)^2}{m_y \Delta_0} \sim \frac{(\delta k_x)^2}{m_x \Delta_0} \sim \frac{\Delta_0 - \mu}{\Delta_0} \sim \exp \left(-\frac{2\pi \hbar n_M}{T \sqrt{m_x m_y}}\right).$$

(8.4)

If one uses the 2D number density $n = \delta N d$ with the film thickness $d = 5 \mu m$ and 3D number density $\delta N \sim 5 \cdot 10^{18}$ cm$^{-3}$, one obtains that at room temperature the exponent is

$$\frac{2\pi \hbar n_M}{T \sqrt{m_x m_y}} \sim \frac{2\pi \Delta_0 \delta N d}{T \cdot 10 k_0^2} \sim 10^2.$$

(8.5)

If this estimation is correct, the peak should be extremely narrow, so that all extra magnons are concentrated at the lowest level of the discrete spectrum. However, there are other
contributions to the width of the peak due to: finite resolution of spectrometer, magnon interaction, finite life time of magnons and the influence of the higher discrete levels \( n \neq 0 \).

In any case, the process of the concentration of extra magnons in the states very close to the lowest energy is the signature of the BEC of magnons. The main property of the room temperature BEC in YIG is that the transition temperature \( T_c \) is only slightly higher than temperature, \( T_c - T \ll T \); as a result the number of condensed magnons is small compared to the number of thermal magnons: \( n \ll n_T \). Situation with magnon BEC in \(^3\)He is the opposite, one has \( T \ll T_c \) and thus \( n \gg n_T \).

### C. Magnon BEC vs planar ferromagnet

The spin density in coherently precessing state HPD state of \(^3\)He-B and in YIG are correspondingly

\[
S_x + i S_y = S_\perp e^{i\omega t + i\alpha},
\]  

(8.6)

and

\[
S_x + i S_y = S_\perp \cos(k_0 x) e^{i\omega t + i\alpha}.
\]  

(8.7)

On the other hand in the equilibrium planar ferromagnet one has \(^{17, 89}\)

\[
S_x + i S_y = S_\perp e^{i\alpha}.
\]  

(8.8)

This means that in principle, the equilibrium planar ferromagnet can be also described in terms of the ODLRO.

Magnons were originally determined as second quantized modes in the background of stationary state with magnetization along \( z \) axis. Both the static state in Eq. (8.8) and precessing states (8.6) and (8.7) can be interpreted as BEC of these original magnons. On the other hand, the stationary and precessing states can be presented as new vacuum states, the time independent and the time dependent vacua respectively. The excitations – phonons – are the second quantized modes in the background of a new vacuum. What is the principle difference between the stationary vacuum of planar ferromagnet and the time dependent vacuum of coherent precession?

The major point which distinguishes the HPD state (8.6) in \(^3\)He-B and the coherent precession (8.7) in YIG from the equilibrium magnetic states is the conservation (or quasi-conservation) of the \( U(1) \) charge \( Q \). The charge \( Q \) is played by the spin projection \( S_z \) in magnetic materials (or the related number \( N \) of magnons) and by number \( N \) of atoms in atomic BEC. This conservation gives rise to the chemical potential \( \mu = dE/dN \), which is the precession frequency \( \omega \) in magnetic systems. On the contrary, the static state in Eq. (8.8) does not contain the chemical potential \( \omega \), i.e. the conservation is not in the origin of formation of the static equilibrium state; the chemical potential of magnons in a fully equilibrium state is always strictly zero, \( \mu = \omega = 0 \).

The spin-orbit interaction violates the conservation of \( S_z \), as a result the life time of magnons is finite. For the precessing states (8.6) and (8.7) this leads to the finite life time of the coherent precession. To support the steady state of precession the pumping of spin and energy is required. On the contrary, the spin-orbit interaction does not destroy the long-range magnetic order in the static state (8.8); this is fully equilibrium state which does
not decay and thus does not require pumping: the life time of static state is macroscopically large and thus by many orders of magnitude exceeds the magnon relaxation time. That is why a planar ferromagnet is just one more equilibrium state of quantum vacuum, in addition to the easy axis ferromagnetic state, rather than the magnon condensate.

The property of (quasi)conservation of the $U(1)$ charge distinguishes the coherent precession from the other coherent phenomena, such as optical lasers and standing waves. For the real BEC one needs the conservation of particle number or charge $Q$ during the time of equilibration. BEC occurs due to the thermodynamics, when the number of particles (or charge $Q$) cannot be accommodated by thermal distribution, and as a result the extra part must be accumulated in the lowest energy state. This is the essence of BEC.

Photons and phonons can also form the true BEC (in thermodynamic sense) under pumping, again if the lifetime is larger than thermalization time. These BEC states are certainly different from such coherent states as optical lasers and from the equilibrium deformations of solids.

IX. BEYOND MAGNON BEC: SUHL INSTABILITY OF COHERENT PRECESSION

A. Catastrophic relaxation of magnon BEC

The instability of BEC which we discuss here is applicable only to the BEC of magnon quasiparticles, and is irrelevant for atomic condensates. For quasiparticles the $U(1)$ symmetry is not strictly conserved. For magnetic subsystem, this is the $SO(2)$ symmetry with respect to spin rotations in the plane perpendicular to magnetic field, and it is violated by spin-orbit interactions. The magnon BEC is a time dependent process, and it may experience instabilities which do not occur in equilibrium condensates of stable particles. In 1989 it was found that the original magnon condensate – the HPD state – abruptly looses its stability below about 0.4 $T_c$ [61]. This was called the catastrophic relaxation. This phenomenon was left unexplained for a long time until the reason was established: in the low-temperature regime, where dissipation becomes sufficiently small, the Suhl instability destroys the homogeneous precession [90–92]. This is the parametric instability, which leads to decay of HPD due to the parametric amplification of spin wave modes. The latter modes are different from the magnons which we discuss here: they represent another branch of the collective modes of superfluid $^3$He-B. The instability occurs because the spin-orbit interactions violate the $U(1)$ symmetry.

For magnetically ordered systems the instability of homogeneous precession is a well known phenomenon. Suhl [93] explained it in terms of parametric instability of the mode of precession with respect to excitations of pairs of spin waves satisfying the condition of resonance:

$$n\omega_L = \omega_s(k) + \omega_s(-k), \quad (9.1)$$

where $\omega_L$ is the precession frequency and $n$ is integer (see also review [94]). All the magnetic systems, where the Suhl instability has been observed, are anisotropic. In particular, as quantum solids and liquids are concerned, Suhl instability has been observed in antiferromagnetic solid $^3$He [95], and has been predicted by Fomin for anisotropic superfluid liquid $^3$He-A [96] where it has been observed later [97]. Due to the extreme isotropy of
$^3\text{He-B}$ and due to the unique symmetry of the spin-orbit interaction, the Suhl instability was not expected there.

However, under conditions of the experiment, the boundary conditions on the wall of container induce the texture of the order parameter in which the orbital vector $\mathbf{l}$ deviates from its symmetric orientation along the magnetic field $\mathbf{H}$. The symmetry of the spin-orbit interaction is violated providing the additional term in the interaction between the modes, which is dominating in typical experiments with the catastrophic relaxation [98–100].

### B. Precessing states and their symmetry

To describe the interaction of magnon condensate with the other modes of superfluid $^3\text{He-B}$, we must go beyond the magnon BEC description and consider all the degrees of freedom of homogeneous free precession in external magnetic field $\mathbf{H}$. In liquid $^3\text{He}$ the spin-orbit (dipole-dipole) interaction is weak. If it is neglected, we can apply the powerful Larmor theorem, according to which, in the spin-space coordinate frame rotating with the Larmor frequency the effect of magnetic field on spins of the $^3\text{He}$ atoms is completely compensated. This follows from the observation that in dynamics the time derivative of a spin vector enters equations together with the Larmor frequency vector $\mathbf{\omega}_L = \gamma \mathbf{H}$.

$$D_t\mathbf{f} = \partial_t \mathbf{f} - \mathbf{\omega}_L \times \mathbf{f}. \quad (9.2)$$

In other words, the Pauli magnetic field acts on spin vectors as time component of the effective $SO(3)$ gauge field,

$$A_0 = \gamma \mathbf{H}. \quad (9.3)$$

We shall use this equation later in Sec. \[X\] for discussion of spin currents and spin-quantum Hall effect.

Since the magnetic field becomes irrelevant, the symmetry group of the physical laws in the precessing frame becomes the same as in the absence of the field. If the spin-orbit interaction is neglected, it is the product of the $SO_L^3$ group of orbital rotations and the $SO_S^3$ group of spin rotations:

$$G = SO_L^3 \times SO_S^3, \quad (9.4)$$

The only difference is that while $SO_L^3$ is the group of orbital rotations in the laboratory frame, the $SO_S^3$ rotations are in the precessing frame. The elements of the latter group $\tilde{g}(t)$ are constructed from the elements $g$ of conventional spin rotations in the laboratory frame:

$$\tilde{g}(t) = O^{-1}(\hat{z}, \omega_L t) \ g \ O(\hat{z}, \omega_L t). \quad (9.5)$$

Here the matrix $O_{\alpha\beta}(\hat{z}, \omega t)$ describes the transformation from the laboratory frame into the rotating frame - this is the rotation about the magnetic field axis $\hat{z}$ by angle $\omega_L t$. Now we can find all the degenerate coherent states of the Larmor precession applying the symmetry group $G$ to the simplest equilibrium state of the given superfluid phase: $\mathbf{A} = O^{-1}R^{(1)}O\mathbf{A}(0)(R^{(2)})^{-1}$, where $R^{(1)}$ is the arbitrary matrix describing spin rotations in the precessing frame and $R^{(2)}$ is another arbitrary matrix which describes the orbital rotations.
in the laboratory frame. In case of $^3$He-B this state corresponds to the state of Cooper pairs with $L = S = 1$ and the total angular momentum $J = 0$ \[24\]:

$$A_{\alpha i}^{(0)} = \Delta_B \delta_{\alpha i} \quad .$$

(9.6)

The action of elements of the group $G$ on this stationary state leads to the following general precession of $^3$He-B with the Larmor frequency (if the spin-orbit interaction is neglected):

$$A_{\alpha i}(t) = \Delta R_{\alpha i}(t) \quad ,$$

(9.7)

$$R_{\alpha i}(t) = O_{\alpha \beta} (\hat{z}, -\omega t) R^{(1)}_{\beta \gamma} O_{\gamma \mu} (\hat{z}, \omega t) (R^{(2)}_{\mu \iota})^{-1} \quad .$$

(9.8)

The matrix $R^{(1)}$ determines the direction of spin density in the precessing frame:

$$S_\alpha = \chi R^{(1)}_{\alpha \beta} H_\beta \quad ,$$

(9.9)

where $\chi$ is the spin susceptibility of $^3$He-B. This corresponds to the precession of spin with the tipping angle $\cos \beta_1 = R_{zz}^{(1)}$. The matrix $R^{(2)}$ determines the direction of orbital momentum density in the laboratory frame:

$$L_i = - R_{\alpha i}(t) S_\alpha(t) = - \chi R^{(2)}_{i\alpha} H_\alpha \quad ,$$

(9.10)

with the tipping angle $\cos \beta_2 = R_{zz}^{(2)}$.

C. Spin-orbit interaction as perturbation

The spin-orbit interaction couples the spin and orbital components of the matrix $A_{\alpha i}$. For $^3$He-B in \[9.7\] one obtains \[24\]:

$$F_D = \frac{2}{15} \chi \Omega_L^2 \left( R_{ii}(t) - \frac{1}{2} \right)^2 = \frac{8}{15} \chi \Omega_L^2 \left( \cos \theta(t) + \frac{1}{4} \right)^2 \quad ,$$

(9.11)

where $\Omega_L$ is the so called Leggett frequency – the frequency of the longitudinal NMR; $\theta$ is the angle of rotation in the parametrization of the matrix $R_{\alpha i}$ in terms of the angle and axis of rotation \[24\]; we shall use here the system of units in which the gyromagnetic ratio $\gamma$ for the $^3$He atom is 1, hence the magnetic field and the frequency will have same physical dimension.

In the general state of the Larmor precession \[9.7\], the spin-orbit interaction contains the time independent part and rapidly oscillating terms with frequencies $\omega_L$, $2\omega_L$, $3\omega_L$ and $4\omega_L$:

$$F_D(\gamma) = F_0 + \sum_{n=1}^{4} F_n \cos(n\omegaLt) \quad .$$

(9.12)

The time-independent part – the average over fast oscillations – gives the spin-orbit potential in \[3.16\], which determines the phases of magnon BEC in $^3$He-B:

$$F_0 = F_{so}(s, l, \gamma) =$$

$$\frac{2}{15} \chi \Omega_L^2 \{(s l - \frac{1}{2} + \frac{1}{2}(1+s)(1+l) \cos \gamma)^2 +$$

$$\frac{1}{8}(1 - s)^2(1 - l)^2 + (1 - s^2)(1 - l^2)(1 + \cos \gamma)\} \quad .$$

(9.13)
Here \( s = \cos \beta \) and \( l = \cos \beta_L \) are \( z \) projections of unit vectors \( \mathbf{s} = \mathbf{S}/S \) and \( \mathbf{l} = -\mathbf{L}/L \); and \( \gamma \) is another free parameter of the general precession. Altogether the free precession is characterized by 5 independent parameters coming from two matrices \( \mathbf{R}^{(1)} \) and \( \mathbf{R}^{(2)} \): two angles of spin \( \mathbf{S} \), two angles of the orbital momentum \( \mathbf{l} \), and the relative rotation of matrices by angle \( \gamma \). In the case of the stationary (non-precessing) magnetization, the \( \gamma \)-mode corresponds to the longitudinal NMR mode.

D. Parametric instability of HPD

In the simplest description, the dynamics of the \( \gamma \)-mode is determined by the following Lagrangian:

\[
\mathcal{L} = -\frac{1}{2} \chi (\dot{\gamma}^2 - c^2 (\nabla \gamma)^2) + F_D(\gamma). \tag{9.14}
\]

Here we used the approximation of an isotropic speed of spin waves \( c \). In the time-dependent part of \( F_D \) we only consider the first harmonic, i.e. according to Eq.(9.1) we discuss the parametric excitation of two \( \gamma \)-modes with \( c k \approx \omega_L/2 \). The amplitude of the first harmonic is:

\[
F_1 = \frac{4}{15} \chi \Omega_L^2 \sin \beta \sin \beta_L \cos(\gamma/2) \times \left( 2sl - 1 + \frac{(1-s)(1-l)}{2} + (1+s)(1+l) \cos \gamma \right). \tag{9.15}
\]

Further we assume that the system is in the minimum of the dipole energy \( F_0 \) as a function of \( \gamma \). The equilibrium value \( \gamma = \gamma_0 \) is

\[
\cos \gamma_0 = -\frac{(2sl - 1) + 2(1-s)(1-l)}{(1+s)(1+l)}, \tag{9.16}
\]

which is valid if the right hand side of Eq. (9.16) does not exceed unity, i.e. when \( s + l - 5sl < 2 \).

For the discussion of Suhl instability we need the time-dependent term which is quadratic in \( \gamma - \gamma_0 \). Then the Lagrangian (9.14) which describes the parametric instability towards decay of Larmor precession to two \( \gamma \)-modes with \( c k \approx \omega_L/2 \) is (after the shift \( \gamma - \gamma_0 \rightarrow \gamma \); neglecting \( \Omega_L \) compared to \( \omega_L \); and neglecting the anisotropy of the spin-wave velocity):

\[
\mathcal{L} = \frac{1}{2} \chi \left( -\dot{\gamma}^2 + c^2 (\nabla \gamma)^2 + a \Omega_L^2 \gamma^2 \cos \omega_L t \right), \tag{9.17}
\]

where, if \( s + l - 5sl < 2 \), the parameter \( a \) is

\[
a = \frac{4}{15} \sin \beta \sin \beta_L \left[ \frac{3(s + l - sl)}{2(1+s)(1+l)} \right]^{1/2} \times \left[ (1+s)(1+l) + 2(2sl - 1) + \frac{35}{8}(1-s)(1-l) \right]. \tag{9.18}
\]
Let us rewrite the Lagrangian (9.17) in terms of Hamiltonian as function of creation and annihilation operators $b_k$ and $b_k^*$:

$$\gamma_k = \frac{i}{\sqrt{2\chi\omega_s(k)}}(b_k - b_k^*), \quad \omega_s^2(k) = c^2k^2$$

(9.19)

$$p_k = \chi \dot{\gamma}_k = \sqrt{\chi\omega_s(k)/2}(b_k + b_k^*)$$

(9.20)

$$\mathcal{H} = \sum_k \omega_s(k)b_k^*b_k + \frac{a\Omega_L^2}{2\omega_s(k)}(e^{-i\omega_L/2t}b_kb_{-k} + e^{i\omega_L/2t}b_k^*b_{-k})$$

(9.21)

where we neglected $\Omega_L$ compared to $\omega_L$. The spectrum of the excited mode is

$$b_k(t) = \bar{b}_k e^{-i\omega_L/2t + i\nu t}$$

$$\nu_k = \sqrt{(\omega_s(k) - \omega_L/2)^2 - a^2\Omega_L^4/\omega_s^2(k)}$$

(9.22)

At the resonance, i.e. when $\omega_s(k) = \omega_L/2$, the mode grows exponentially:

$$b_k(t) \propto e^{\lambda t} \quad \lambda = 2a\Omega_L^2/\omega_L.$$  

(9.23)

At finite temperatures this growing is damped by dissipation, but at low temperature the dissipation becomes small and catastrophic relaxation occurs. Following Ref. [90] one may assume the spin diffusion mechanism of dissipation. In this case the equation for temperature $T_{\text{cat}}$ below which the instability of the homogeneous precession towards radiation of spin waves with $\omega_s(k) = ck = \omega_L/2$ starts to develop is [91, 92]:

$$D(T_{\text{cat}}) = 2\lambda c^2/\omega_L^2.$$  

(9.24)

Here $D(T)$ is the spin diffusion coefficient, which depends on temperature and decreases with decreasing $T$. This agrees with observations.

X. BEYOND MAGNON BEC: SPIN SUPERCURRENTS AND SPIN HALL EFFECTS

A. Microscopic theory of spin supercurrent in $^3$He-B

Here we give the ‘microscopic’ derivation for the spin supercurrent, which has been discussed on the phenomenological level of magnon BEC. The underlying microscopic physics is the BCS theory of $p$-wave spin-triplet superfluid $^3$He. Superfluid spin currents in $^3$He-B exist even in the absence of magnon BEC. They come from the spontaneous breaking of spin-rotation symmetry. The spin supercurrent in $^3$He-B is expressed in terms of the spin superfluid velocities:

$$\omega_{\alpha i} = \nabla_i \theta_\alpha = \frac{1}{2}e_{\alpha\beta\gamma}R_{\beta j}\nabla_i R_{\gamma j}.$$  

(10.1)

The corresponding gradient energy is

$$F_{\text{grad}} = \frac{1}{2}r_{\alpha i,\beta j}\omega_{\alpha i}\omega_{\beta j}.$$  

(10.2)
where $\rho_{\alpha i,\beta j}$ is the spin rigidity tensor with spin rigidity parameters
\[
\rho_{\alpha i,\beta j} = \frac{\chi B}{\gamma^2} \left[ \tilde{c}^2 \delta_{\alpha \beta} \delta_{ij} - (\tilde{c}_r^2 - \tilde{c}_\perp^2) (R_{\alpha i} R_{\beta j} + R_{\alpha j} R_{\beta i}) \right].
\] (10.3)

From these equations one obtains the spin supercurrent in $^3$He-B:
\[
J_{\alpha i} = -\frac{\partial F_{\text{grad}}}{\partial \omega_{\alpha i}} = -\rho_{\alpha i,\beta j} \omega_{\beta j}.
\] (10.4)

This spin current averaged over the fast precession determines the parameters of the phenomenological equations (4.16) and (4.17) for the spin supercurrent emerging in magnon BEC.

B. Spin current and electric field in $^3$He-B

The symmetry properties of the spin superfluid velocity, and thus of the spin supercurrent, allows to couple linearly the spin current with electric field even in the absence of the spin-orbital interaction. The following term in the action is possible [101]:
\[
F = -\beta e_{ijk} \omega_{\alpha i} R_{\alpha j} E_k.
\] (10.5)

The parameter $\beta$ is not well defined from the microscopic theory due to the unknown Fermi-liquid corrections involved. The estimate for $\beta$ reported in Ref. [102] is $\beta \sim 10^{-4}$ e/cm. Variation with respect to $\omega_{\alpha i}$ demonstrates that there exists a linear response of the spin supercurrent on the electric field:
\[
J_{\alpha i} = -\frac{\partial F}{\partial \omega_{\alpha i}} = \beta e_{ijk} R_{\alpha j} E_k.
\] (10.6)

This spin current is transverse to the electric field and thus represents the spin current Hall effect. As distinct from the spin Hall effect predicted by Dyakonov and Perel [103], this spin-Hall effect does occur in the absence of spin-orbit interaction.

C. Electric and magnetic fields as $SU(2)$ gauge fields

The interaction of the electric and magnetic fields with the order parameter in superfluid phases of $^3$He, can be also found using observation that $H$ and $E$ may be considered as temporal and spatial components of the $SU(2)$ gauge field, where $SU(2)$ is the group of the spin rotations. The auxiliary $SU(2)$ gauge field $A^\alpha_\mu$ is convenient for the description of the effects related to the spin current. In the spinor representation one has the following covariant derivatives coming from the auxiliary $SU(2)$ gauge field [101, 104]
\[
\mathbf{D} = \nabla - iA^i \frac{\sigma^i}{2}, \quad D_0 = \partial_t + iA^i_0 \frac{\sigma^i}{2}.
\] (10.7)

Some components of the field $A^\alpha_\mu$ are physical, being represented by the real physical quantities which couple to the fermionic charges. Example is provided by the Pauli magnetic field $H^i$, which play the role of the component $A^i_0$ of the $SU(2)$ gauge field, see [9.3] [40, 105].
while the spatial components are played by the electric field which enters the gradient energy via the covariant spatial derivative \[101, 104, 106\]:

\[ A_0^i = B^i, \quad A_j^i = \epsilon_{jik}E_k. \] (10.8)

The electric field enters due to the relativistic spin-orbit interaction of the spin of \(^3\)He atom with the electric field \(E\).

The spin current is obtained as variation of the action with respect to the fictitious \(SU(2)\) gauge field:

\[ \mathbf{J}^i = \frac{\delta S}{\delta A^i}. \] (10.9)

After the spin current is calculated the values of the auxiliary fields are made equal to zero or to the values of the corresponding physical fields which simulate the gauge fields.

For example, in the presence of electric field, equation (10.2) becomes

\[ F_{\text{grad}} = \frac{1}{2} \rho_{\alpha i,\beta j}(\omega_{\alpha i} - \gamma \epsilon_{\alpha ik}E_k)(\omega_{\beta j} - \gamma \epsilon_{\beta jl}E_l), \] (10.10)

which demonstrates that electric field enters as the \(SU(2)\) gauge field forming the covariant derivative. This gives another response of the spin supercurrent on the external electric field:

\[ J_{\alpha i} = - \frac{\partial F_{\text{grad}}}{\partial \omega_{\alpha i}} = - \rho_{\alpha i,\beta j}(\omega_{\beta j} - \gamma \epsilon_{\beta jl}E_l), \] (10.11)

As distinct from (10.6) this spin-Hall effect is governed by the spin-orbit interaction. According to Ref. [107] both spin Hall effects in \(^3\)He-B should modify the spin current Josephson effect in magnon BEC. The supercurrent, induced by electric field, leads to an additional phase shift proportional to electric field, which is to be measured.

### D. Quantum spin Hall effect

There are several types of responses of spin and electric currents to transverse forces which are quantized in 2+1 systems under appropriate conditions. The most familiar is the conventional quantum Hall effect (QHE). It is quantized response of the particle current to the transverse force, say to transverse gradient of chemical potential, \(\mathbf{J} = \sigma_{xy}\hat{z} \times \nabla \mu\). In the electrically charged systems this is the quantized response of the electric current \(\mathbf{J}^e\) to transverse electric field \(\mathbf{J}^e = e^2\sigma_{xy}\hat{z} \times \mathbf{E}\).

The other effects involve the spin degrees of freedom. An example is the mixed spin quantum Hall effect: quantized response of the particle current \(\mathbf{J}\) (or electric current \(\mathbf{J}^e\)) to transverse gradient of magnetic field interacting with Pauli spins (Pauli field in short) \[108, 109\]:

\[ \mathbf{J} = \sigma_{xy}^{\text{mixed}}\hat{z} \times \nabla (\gamma H^z), \quad \mathbf{J}^e = e\mathbf{J}. \] (10.12)

The related effect, which is determined by the same quantized parameter \(\sigma_{xy}^{\text{mixed}}\), is the quantized response of the spin current, say the current \(\mathbf{J}^z\) of the \(z\) component of spin, to the
gradient of chemical potential \[110\]. In the electrically charged systems this corresponds to the quantized response of the spin current to transverse electric field:

\[
J_z = \sigma_{xy}^{\text{mixed}} \mathbf{z} \times \nabla \mu = e \sigma_{xy}^{\text{mixed}} \mathbf{z} \times \mathbf{E}.
\]  

(10.13)

This kind of mixed Hall effect is now used in spintronics \[111\].

Finally there is a pure spin Hall effect – the quantized response of the spin current to transverse gradient of magnetic field \[108, 109, 112, 113\]:

\[
J_z = \sigma_{xy}^{\text{spin}} \mathbf{z} \times \nabla (\gamma H_z).
\]  

(10.14)

Let us consider the mixed spin Hall effects in (10.13) and (10.14). These two effects are related, since they described by the same topological Chern-Simons action \[108\] and thus by the same parameter \(\sigma_{xy}^{\text{mixed}}\). To see this, let us remind that the spin current is obtained as variation of the action over the fictitious \(\text{SU}(2)\) or \(\text{SO}(3)\) gauge field, see (10.9). For example, the current of the \(z\)-projection of spin is

\[
J_z = \frac{\delta S}{\delta A^z}.
\]  

(10.15)

The corresponding Chern-Simons term in the action is given by \[108\]

\[
F_{CS} = e \sigma_{xy}^{\text{mixed}} \epsilon^{\nu\alpha\beta} \int d^2x dt A^\nu_\beta \nabla_\alpha A^z_\beta,
\]  

(10.16)

where \(A_\beta\) is the vector potential of the conventional electromagnetic field, and \(A^z_\nu = (A^z_i, A^z_0)\) represent components of auxiliary (fictitious) \(\text{SU}(2)\) gauge field. Variation of the action with respect to the field \(A^z_i\) gives the spin current in (10.13). On the other hand, the variation of the action with respect to field \(A_i\) gives electric current in terms of the gradients of an auxiliary gauge field. However, from equation (9.3) or (10.8) it follows that the role of the auxiliary gauge field \(A^z_0\) is played by magnetic field \(H^z\). As a result one obtains equation (10.12) for electric current.

Equation (10.16) has been originally introduced for a thin film of the so-called planar phase of superfluid \(^3\)He \[108\]. However, it is better suited for the two-dimensional topological insulators with time reversal invariance (on topological insulators see review \[114\]). These materials have the same topological structure as the planar phase, which is also time reversal invariant, but the advantage of these materials is that they are insulating and thus the superconductivity does not mask the spin Hall quantization.

Discussion of the mixed Chern-Simons term can be found in Ref. \[115\]. For the related phenomenon of axial anomaly in particle physics, the mixed action in terms of different (real and fictitious) gauge fields has been introduced in Ref. \[116\].

XI. CONCLUSION

The superfluid phases of liquid \(^3\)He at extreme low temperatures are unique states of condensed matter with physical properties which can be compared to the vacuum of relativistic quantum field theories. The A phase \((^3\)He-A) belongs to the same symmetry and topology class as the vacuum of the Standard Model of particle physics in its massless \((i.e.\text{ gapless})\) phase and can also be described as a semi-metal-like system with non-trivial topology. The
B phase ($^3$He-B), in contrast, is similar to the vacuum of the Standard Model in its massive (or gapped) phase and to 3-dimensional topological insulators with time reversal symmetry. In addition to fermionic quasiparticle excitations, superfluid $^3$He has also bosonic quasiparticles, such as magnons – quanta of excitations of the magnetic subsystem. These magnon excitations can form long-lived Bose-Einstein condensates both in $^3$He-A and $^3$He-B, and these condensates experience their own superfluidity, which is not related to superfluidity of the underlying system.

Formally, the phenomenon of superfluidity requires the conservation of charge or particle number. However, the consideration can be extended to systems with a weakly violated conservation law, including a system of sufficiently long-lived quasiparticles - discrete quanta of energy that can be treated as real particles in condensed matter. The spin superfluidity – superfluidity in the magnetic subsystem of a condensed matter – is manifested as the spontaneous phase-coherent precession of spins first discovered in 1984 [8, 9]. This superfluid current of spins is one more representative of superfluid currents known or discussed in other systems, such as the superfluid current of mass and atoms in superfluid $^4$He; superfluid current of electric charge in superconductors; superfluid current of hypercharge in Standard Model; superfluid baryonic current and current of chiral charge in quark matter; etc. The analogy of the dynamical superfluid state of coherent precession with the non-perturbative dynamics of the physical vacuum is to be found.

Different condensates and thus different states of magnon superfluidity have been created by choosing different experimental arrangements. At low temperatures the condensate is confined in a magnetic trap which is formed by the order parameter texture of the superfluid state. This produces the analog of atomic BEC in laser traps, but adds some new features, such as formation of the non-ground-state condensate and the mass of the Goldstone bosons. The magnon condensates can be used to probe the quantum vacuum of $^3$He in the limit $T \to 0$, where conventional measuring signals become insensitive.

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