Effects of Thermal Fluctuations on Non-minimal Regular Magnetic Black Hole

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Abstract

We analyze the effects of thermal fluctuations on a regular black hole (RBH) of non-minimal Einstein-Yang-Mill theory with gauge field of magnetic Wu-Yang type and a cosmological constant. We consider the logarithmic corrected entropy in order to analyze the thermal fluctuations corresponding to non-minimal RBH thermodynamics. In this scenario, we develop various important thermodynamical quantities such as entropy, pressure, specific heats, Gibb’s free energy and Helmholtz free energy. We investigate first law of thermodynamics in the presence of logarithmic corrected entropy and non-minimal RBH. We also discuss the stability of this RBH using various frameworks such as \( \gamma \) factor (comprises of ratio of heat capacities), phase transition, grand canonical ensemble and canonical ensemble. It is observed that the non-minimal RBH becomes more globally and locally stable if we increase the value of cosmological constant.

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1 Introduction

Black hole (BH) solution is one of the interesting phenomena in general relativity. Although, its existence is vivid, so it is an open problem to understand the interior nature of BH in quantitative detail; the main aspects comes from the fact that a perfect theory of quantum gravity does not exist [1]. Since the discovery of Hawking radiation is that, the BH have temperature. Hence, the concept of BH entropy is no longer a mystery which is proposed by Bekenstein. Not only that, the work of Hawking proposed the famous formula of entropy $S = \frac{A}{4}$, where $A$ represents area of event horizon [2]. BHs have more entropy rather than any other object of same volume [3, 4]. Maximum entropy of BHs is expected to correct due to quantum fluctuations which leads to the development of holographic principle [5, 6]. As the BH reduces its size due to Hawking radiation, these fluctuation becomes very important and are expected to correct the standard relation between entropy and area [7].

There are several approaches to evaluate such corrections. Using non-pertubative quantum general relativity, one can calculate the density of microstates for asymptotically flat BHs which leads to the construction of logarithmic correction terms to standard Bekenstein entropy area relation [8]. One can also use Cardy formula to generate logarithmic correction terms for all BHs whose microscopic degrees of freedom are explained by conformal field theory [9, 10]. Ashtekar has obtained such logarithmic corrections for BTZ BHs by calculating the exact partition function [8]. These terms could also be generated by the effect of string theory to the entropy of BH. The analysis of matter fields in the presence of BH has also generated them for the Bekenstein entropy area formula [11, 12]. In fact, the corrections to the entropy of dilaton BHs are obtained which turns out to be logarithmic corrections [13]. Rademacher expansion of partition function can also generate such correction terms [14]. Recently, the effects of thermal fluctuations on charged ADS BH and modified Hayward BH has also been investigated [7, 15].

On the other hand, one of the major challenge in general relativity is the existence of essential singularities (which leads to various BHs) and it looks like the common property in most of the solutions of Einstein field equations. Hence, regular black holes (RBHs) have been constructed to resolve this problem. Since its metric is regular everywhere, so essential singularities could be avoided in the solution of Einstein equations of BHs physics [16]. Weak energy condition is satisfied by these RBHs while some of these violate
the strong energy conditions somewhere in space time \([17, 18]\). Since Penrose cosmic censorship conjecture claims that singularities predicted by GR occur and they must be discussed by event horizon \([19, 20]\). Hence, Bardeen \([21]\) was the pioneer who obtained a BH solution without any essential singularity at origin enclosed by event horizon known as 'Bardeen Black Hole' which satisfy weak energy conditions. Later on, many authors found similar solution \([22, 23, 24]\). The coupling of general relativity to non-linear electromagnetic theory has brought to new sets of charged BHs which came into the range of RBHs solution. Ayon-Beato and Garcia \([22]\) also found such RBH solution. Hayward \([25]\) and Berej et. al \([26]\) found different kinds of RBH solutions. Recently, Leonardo et. al \([27]\) used many distribution function in order to obtain charged RBH.

There is an interesting non-minimal theory that couple the gravitational field to other fields using cross terms of curvature tensor started to rise long time ago as alternative theories of gravity. There are five classes of non-minimal field theories divided accordingly to the types of fields that couple gravitation to non-minimality, for detail \([28, 29]\). These non-minimal theories construct exact solutions of stars \([30, 31]\), wormholes \([32, 33]\), BHs \([34, 35]\) and regular magnetic BHs \([36]\) with Wu-Yang anstaz \([37, 38]\). New regular exact spherically symmetric solutions of a non-minimal Einstein-Yang-Mills theory with a gauge field of magnetic Wu-Yang and cosmological constant is presented by Balakin, Lemos and Zayats \([28, 29]\). They found the most interesting solutions of BHs with metric and curvature invariant are regular everywhere. BH thermodynamics enable us to study various important thermodynamical quantities of solutions.

One of the most important thermodynamical quantity is thermal stability of BH. BHs should be stable in dynamical and thermodynamical frameworks due to their physical nature. The instability of BHs means whether it may have phase transition or it is completely unphysical. In this work, we analyze the effects of thermal fluctuations on RBH of non-minimal Einstein-Yang-Mill theory with gauge field of magnetic Wu-Yang type and a cosmological constant. We will use logarithmic correction terms to discuss various thermodynamical quantities such as pressure, entropy, specific heats, Gibb’s free energy and Helmothz free energy of non-minimal RBH. The outline of paper is as follows: In section 2, we discuss RBH of non-minimal Einstein-Yang-Mill theory with gauge field of magnetic Wu-Yang type and a cosmological constant, furthermore, we will find logarithmic correction terms which produces various thermodynamical quantities. In section 3, we investigate the
stability of non-minimal RBH using corrected value of specific heat to an-
alyze the phase transition. In further subsection, we also demonstrate the
grand canonical and canonical ensembles. Conclusion and observations are
given in the last section.

2 Non-minimal RBH

New exact regular spherically symmetric solution of non-minimal Einstein-
Yang-Mills theory with magnetic charge of Wu-Yang gauge field and the
cosmological constant is presented by Balakin, Lemos and Zayats [28]. Now
considering their static spherically symmetric space-time with line elemen-
tds

\[ ds^2 = f(r)dt^2 - (f(r))^{-1}dr^2 - r^2(d\theta^2 + \sin \theta d\phi^2), \]  
(1)

where

\[ f(r) = 1 + \left( \frac{r^4}{r^4 + 2Q_m^2q} \right) \left( - \frac{2M}{r} + \frac{Q_m^2}{r^2} - \frac{\Lambda r^2}{3} \right), \]  
(2)

is the exact solution to gravitational field equations. This contains four
important parameters such as \( \Lambda, q, Q_m \) and \( M \) which represent the cosmo-
logical constant, non-minimal parameter of the theory, magnetic charge of
gauge field Wu-Yang type and mass of the object, respectively. In this work,
we consider \( q > 0, \Lambda > 0, \Lambda \leq 0, Q_m^2 > 0 \) and \( M \geq 0 \), which have the
following reason. The limiting case \( q = 0 \) gives the magnetic RN solution
with cosmological constant,

\[ f(r) = 1 + \frac{2M}{r} - \frac{Q_m^2}{r^2} - \frac{\Lambda r^2}{3}. \]  
(3)

At \( r = 0 \), we find the curvature singularities. For \( q < 0 \) with finite positive
\( r \), we have space-time curvature singularities.

One can obtain no singularities for \( q > 0 \) i.e. \( f(r) \) near the center behaves

\[ f(r) = 1 + \frac{r^2}{2q^2} - \frac{Mr^3}{Q_m^2q} + .... \]  
(4)

One can see \( f(0) = 1, f'(0) = 0 \) and \( f''(0) = \frac{1}{r^2} \). Hence \( r = 0 \) is the
minimum of the regular function \( f(r) \) which is independent of cosmological
constant and the mass of black hole. Since \( f(0) = 1 \) and \( R(0) = \frac{6}{q} \) shows the
curvature scalar is regular at center. For \( q > 0 \), other curvatures invariants
and quadratic scalar \( R_{\mu\nu}R^{\mu\nu} = \frac{9}{q} \) are also finite at center \[29\]. Thus due to non-minimality of the model, space-time is truly regular in center.

The metric function \( \mathbf{(2)} \) is described by four parameters with different units: \( \Lambda, M, Q_m \) and \( q \). We can rewrite the metric function \( \mathbf{(2)} \) in dimensionless form by introducing the following dimensionless quantities \[28\]

\[
\begin{align*}
\gamma_\Lambda &= \sqrt{\frac{3}{|\Lambda|}}, \quad \gamma_g = 2M, \quad \gamma_q = \left(2Q_m^2 q\right)^{\frac{1}{4}}, \quad \gamma_Q = Q_m \\
\sigma &= \frac{\gamma_g}{\gamma_\Lambda}, \quad \eta = \left(\frac{\gamma_Q}{\gamma_\Lambda}\right)^2, \quad \zeta = \left(\frac{\gamma_q}{\gamma_\Lambda}\right)^4, \quad \rho = \left(\frac{r_+}{\gamma_\Lambda}\right)^2.
\end{align*}
\] (5)

In terms of these variable the metric function \( f(r) \) in \( \mathbf{(2)} \) can be rewritten in \( f(\rho) \) as follows

\[
f(\rho) = 1 + \left(\frac{\rho^2(\rho^4 - \sigma \rho + \eta)}{\rho^4 + \zeta}\right).
\] (6)

The most important feature of metric function \( \mathbf{(2)} \) is horizon radius which depend upon different values of parameters. The horizon radius of non-minimal RBH could be obtained by considering real roots of the following equation

\[
-\frac{\Lambda r_+^6}{3} + r_+^4 - 2Mr_+^3 + Q_m^2 r_+^2 + 2Q_m^2 q = 0.
\] (7)

For \( \Lambda > 0 \), the non-minimal RBH solution can have three horizons depending upon different values of parameters, the cosmological horizon, Cauchy horizon and event horizon. On the other hand, for \( \Lambda \leq 0 \), it can have two horizons depending upon different values of parameters, i.e. Cauchy and event horizons. There is no cosmological horizon in this case \[28\]. Since, we want to discuss the thermal quantities on outer horizon that is why we refer \( r_+ \) as outer horizon in the present work.

One can obtain the mass of the non-minimal RBH in horizon radius and other parameters as follows

\[
M = \frac{-\Lambda r_+^6 + 3Q_m^2 r_+^2 + 3r_+^4 + 6Q_m^2 q}{6r_+^3},
\] (8)

which implies that \( r_+ \neq 0 \). The entropy of non-minimal RBH which is related to area of BH horizon is

\[
S_0 = \pi r_+^2,
\] (9)
and volume is 
\[ V = \frac{4}{3} \pi r_+^3 \]  
(10)

The temperature of non-minimal RBH can be written as
\[ T = \frac{f'(r)}{4\pi} \]  
(11)
\[ = \frac{-\Lambda r_+^6 + 3 M r_+^6 + 6 Q_m^2 q r_+^2 - 18 M Q_m^2 q r_+^2 - (6 q \Lambda + 3) Q_m^2 r_+^5}{6 \pi (r_+^4 + 2 Q_m^2 q)^2} \]  
(12)

where the mass \( M \) is given in Eq. (8). One can examine the thermodynamics of non-minimal RBH in terms of mass \( M \), horizon radius \( r_+ \), non-minimal parameter \( q \), cosmological constant \( \Lambda \) and magnetic charge \( Q_m \). By utilizing Eq.(8) in above expression, the temperature reduces to
\[ T = \frac{r_+^4 - \Lambda r_+^6 - Q_m^2 q r_+^2 - 6 Q_m^2 q}{4 \pi r_+ (r_+^4 + 2 Q_m^2 q)}. \]  
(13)

For modeling of the metric function (2) and of the thermodynamic quantities, the dimensionless parameters (5) are used, but for visibility of the graphs presentation, we use the following explicit values of different parameters.

For real positive temperature, the following three conditions can be obtained

1. For \( \Lambda = 0.01 \) (positive), \( Q_m = 1, q = 0.1 \), we have \( 1.2 \leq r_+ \leq 9.95 \).

2. For \( \Lambda = 0, Q_m = 2, q = 0.1 \), we have \( r_+ \geq 2.1283 \).

3. For \( \Lambda = -0.01 \) (negative), \( Q_m = 3, q = 0.1 \), we have \( r_+ \geq 2.9713 \).

If we variate the values of \( \Lambda \) then the range of horizon also change for real positive temperature. For simplicity, we discuss above three special cases throughout the paper.

The first law of thermodynamics can be defined as [39, 40],
\[ dM = T dS + V dP + \ldots \]  
(14)

One can easily check the above relation is violated. For obtaining the thermodynamic quantities which have to satisfy the above relation, we will use the logarithmic corrections in the following subsections.
2.1 Logarithmic correction and thermodynamical relations

In this section, we discuss the effect of thermal fluctuations on non-minimal RBH thermodynamics. It is done by using the formalism of Euclidean quantum gravity, where temporal coordinate is rotated on complex plane. Hence, one can write the partition function for non-minimal RBH \[ Z = \int DgDA \exp(-I), \] (15)

where \( I \rightarrow iI \) is Euclidean action for this system. One can relate the statistical mechanical partition function \([46, 47]\) as

\[ Z = \int_0^\infty DE\eta(E) \exp(-\alpha E), \] (16)

where \( \alpha = T^{-1} \). We can calculate the density of states by using

\[ \eta(E) = \frac{1}{2\pi i} \int_{\alpha_0-i\infty}^{\alpha_0+i\infty} d\alpha e^{S(\alpha)}, \] (17)

where \( S = \alpha E + \ln Z \). This entropy can be obtained around the equilibrium temperature \( \alpha \) by neglecting all thermal fluctuations which becomes \( S_0 = \pi r_+^2 \). However, if thermal fluctuations are taken into account, then \( S(\alpha) \) becomes \([7]\)

\[ S = S_0 + \frac{1}{2} (\alpha - \alpha_0) \left( \frac{\partial^2 S(\alpha)}{\partial \alpha^2} \right)_{\alpha=\alpha_0}. \] (18)

So, one can write density of states as

\[ \eta(E) = \frac{1}{2\pi i} \int_{\alpha_0-i\infty}^{\alpha_0+i\infty} d\alpha e^{\frac{1}{2} (\alpha - \alpha_0) \left( \frac{\partial^2 S(\alpha)}{\partial \alpha^2} \right)_{\alpha=\alpha_0}}, \] (19)

which leads to

\[ \eta(E) = \frac{e^{S_0}}{\sqrt{2\pi}} \left[ \left( \frac{\partial^2 S(\alpha)}{\partial \alpha^2} \right)_{\alpha=\alpha_0} \right]^{\frac{1}{2}}. \] (20)

We can write

\[ S = S_0 - \frac{1}{2} \ln \left[ \left( \frac{\partial^2 S(\alpha)}{\partial \alpha^2} \right)_{\alpha=\alpha_0} \right]^{\frac{1}{2}}. \] (21)
One can notice that this second derivative of entropy is a fluctuation squared of energy. It is possible to simplify this expression by using the relation between the conformal field theory and the microscopic degrees of freedom of a BH [48]. Thus, we can consider the entropy of the form

\[ S = m_1 \alpha^{n_1} + m_2 \alpha^{-n_2}, \]

where \( m_1, m_2, n_1, n_2 \) are all positive constants [49]. This has an extremum at \( \alpha_0 = \left( \frac{m_2 n_2}{m_1 n_1} \right)^{n_1+n_2} = T^{-1} \) and expanding entropy around this extremum, we can determine [50, 51]

\[ \left( \frac{\partial^2 S(\alpha)}{\partial \alpha^2} \right)_{\alpha = \alpha_0} = S_0 \alpha_0^{-2}. \]  

(22)

Thus, the corrected form for the entropy by neglecting higher order correction terms can be written as

\[ S = S_0 - \frac{1}{2} \ln S_0 T^2. \]  

(23)

Moreover, the quantum fluctuation in the geometry of BH give rise to the very important problem of thermal fluctuations in the thermodynamics of BH. When the size of BH is small and its temperature is large then it is sufficient to contribute this correction term. Hence we can avoid the quantum fluctuations for large BH. It is evident that thermal fluctuation only become significant for BHs with large temperature and if the size of BH reduces then its temperature increases. Hence we can conclude that this corrected terms will only come for sufficiently small BHs which temperature is large [7].

Next, we can write the general expression for entropy by neglecting higher order correction terms

\[ S = S_0 - \frac{b}{2} \ln S_0 T^2, \]  

(24)

where \( b \) is added as constant parameter to handle the logarithmic correction terms coming from thermal fluctuations. One can recover the entropy without any correction terms by setting \( b = 0 \). As mention before, one can take \( b \to 0 \), for large BHs which temperature is very small and one can consider \( b \to 1 \), for small BHs which temperature is sufficiently large. By using Eqs. (13) and (24), we can obtain the following corrected entropy:

\[ S = \pi r_+^2 - \frac{b}{2} \ln \left( \frac{(r_+^6 + \Lambda + Q_m^2 r_+^2 - r_+^4 + 6 Q_m^2 q)^2}{16 \pi (r_+^4 + 2 Q_m^2 q)^2} \right). \]  

(25)

It is suggested that the presence of logarithmic correction causes the reduc-
tion of entropy of BH. We can calculate the pressure using Eqs. (10), (13), (24), and the following relation:

\[ P = T \left( \frac{\partial S}{\partial V} \right)_V. \]  

(26)

which turns out to be

\[
P = \frac{1}{8\pi^2r_+^2(r_+^4 + 2qQ_m^2)^2} \left( (b\Lambda + \pi)r_+^8 + Q_m^2r_+^4(bq\Lambda - 4\pi q - b) \right) \]  

(27)

\[ + \ 2qQ_m^4(b - 6\pi q) - \pi r_+^{16}\Lambda - \pi Q_m^2r_+^6(2q\Lambda + 1) - 2r_+^2qQ_m^2(\pi Q_m^2 + 8b) \] .

In Figure 1, we discuss the behavior of pressure for three cases of \( \Lambda \). If we compare \( b = 0 \) and \( b = 1 \), we see that the pressure decreases due to logarithmic correction. Further, from Fig. 1, we observe that pressure is maximum for positive cosmological constant but it becomes negative when \( r \geq 9.945 \), which is evident in case 1. It means that when we increase the cosmological constant then the pressure also increases but the range of horizon for positive pressure decreases. Same behavior could be observed for temperature. We observe that the pressure is high for \( \Lambda = 0 \) as compare
to negative cosmological constant. Hence we conclude that the pressure will decrease due to logarithmic correction and the lower values of cosmological constant.

Moreover, we can investigate the first law of thermodynamics by rewriting Eq. (14) as follows

\[
dM - TdS - VdP = 0,
\]

We construct the table for special three cases and find the horizon at which the first law of thermodynamics is satisfied. We also compare our results with respect to \( b = 0 \) and \( b = 1 \). From Table 1, we observe that the location of horizon on which first law of thermodynamics satisfied is more for \( b = 1 \) as compare to \( b = 0 \). For positive cosmological constant, location of horizons are equal on \( b = 0 \) and 1 but for negative cosmological constant the location of horizons increase for \( b = 1 \) as compare to \( b = 0 \). We obtain more number of location of horizons on which first law of thermodynamics is satisfied in the absence of cosmological constant for \( b = 1 \) as compare to \( b = 0 \). Hence, we can conclude that logarithmic correction term increases the chance of first law of thermodynamics to satisfy.

### 3 Stability of non-minimal RBH

In this section, we will analyze the thermodynamical stability of non-minimal RBH due to the effect of thermal fluctuations. For this purpose we use the
well-known relation

$$E = \int T dS.$$  \hfill (29)

We can find the internal energy and observe that it decreases dramatically due to logarithmic corrections. An important measurable physical quantity in BH thermodynamics is thermal capacity or heat capacity. It identifies the amount of heat required to change the temperature of a BH. The nature of heat capacity (positivity or negativity) represents the stability or instability of a BH. There are two different heat capacities associated with a system. \(C_p\): measures the specific heat when the heat is added at constant pressure and \(C_v\): measures the specific heat when the heat is added to the system by keeping the volume constant. We obtain the specific heat with constant volume by using the \(T\) and \(S\) as follows

$$C_v = T \left( \frac{\partial S}{\partial T} \right)_V.$$  \hfill (30)

Using Eqs. (10), (13) and (24), we have
Using Eqs. (10), (13), (27) and (29), we have

\[ C_v = \left( 2\pi r_+^{12}\Lambda + 4\pi Q_m^2 q r_+^8\Lambda - 2b r_+^{10}\Lambda + 2\pi Q_m^2 r_+^8 - 12Q_m b q r^6_+ \right) \]  \quad (31)

Moreover, the specific heat at constant pressure can be obtained using the relation of \( \gamma \) denoted by

\[ \gamma \]

The above two specific heat relations can be comprised into a ratio that is

\[ C_p = \left( \frac{\partial(E + PV)}{\partial T} \right)_p. \]  \quad (32)

Using Eqs. (10), (13), (27) and (29), we have

\[ C_p = \left( 64r_+^{12}\pi Q_m^2 q\Lambda + 80r_+^8\pi Q_m^2 q^2\Lambda + 4r_+^{12}\pi Q_m^2 - 48r_+^{10}Q_m b q\Lambda \right) \]  \quad (33)

The above two specific heat relations can be comprised into a ratio that is denoted by \( \gamma = C_p/C_v \) and its plot is given in Fig. 2 for special three cases. We observe that due to logarithmic correction the value of \( \gamma \) increases. We obtain the maximum value of \( \gamma \) for negative cosmological constant and \( \gamma \to 1.7 \) for large horizon. Further, we observe that \( \gamma \to 1.4 \) when \( \Lambda = 0 \) for large horizon. It is interesting to note that the value of \( \gamma \) decreases drastically and becomes zero at \( r_+ = 9.92 \) and the value of \( \gamma \) becomes negative for large horizon. We can say that the value of \( \gamma \) shows stable behavior for negative and zero values of cosmological constant but represents unstable behavior for positive cosmological constant. Hence, we can conclude that if the value of cosmological constant decreases then the value of \( \gamma \) is higher and exhibits the more stable behavior but it experiences the unstable behavior for higher values of cosmological constant.
3.1 Phase Transition

Another way to find the thermodynamical stability of BH locally is to investigate the sign of specific heat given in Eq. (32). The BH is locally stable for $C_v > 0$, one can find the point of phase transition at $C_v = 0$ and BH is locally unstable for $C_v < 0$. We can find the range of horizon radius of BH stability for three specific cases in Fig. 3.

We discuss the range of black hole horizon of locally thermodynamical stability for each case. We observe that when $\Lambda = 0.01$ (positive) the horizon radius for local stability $1.3047 < r_+ < 1.9$ and $r_+ > 9.965$, when $\Lambda = 0$ the horizon radius for local stability is $2.1972 < r_+ < 3.6$ and the horizon radius is $r_+ > 3.0232$ for negative cosmological constant. Hence we can conclude that for negative cosmological constant the range of horizon radius for local stability of BH is higher as compare to positive and zero cosmological constant, respectively. Furthermore, we find the critical point of horizon for phase transition in each case. We obtain two critical points of phase transition at $r_+ = 1.3047$ and $9.965$ for positive cosmological constant, the critical point for $\Lambda = 0$ is $r_+ = 2.1972$ and the critical point is $r_+ = 3.0232$ for negative cosmological constant. We notice that the phase transition for positive cosmological constant is near to BH as compare to zero and negative cosmological constant. Hence we can conclude that if we increase the value of cosmological constant the phase transition shifted towards the BH and vice versa.
3.2 Grand Canonical Ensemble

We may treat BH as a thermodynamical object by considering it as a grand canonical ensemble system where \( \mu = \frac{Q_m}{r_+} \) is a fix chemical potential. The corresponding temperature and entropy with logarithmic corrected term are

\[
T_g = \frac{1}{4\pi r_+} \left( \frac{r_+^2 - r_+^4 \Lambda - \mu^2 r_+^2 - 6\mu^2 q}{r_+^2 + 2\mu^2 q} \right), \tag{34}
\]

\[
S = \pi r_+^2 - \frac{b}{2} \ln \left( \frac{(r_+^6 \Lambda + \mu^2 r_+^4 + 6\mu^2 qr_+^2 - r_+^4)^2}{16\pi (r_+^4 + 2\mu^2 qr_+^2)^2} \right). \tag{35}
\]

The effect of chemical potential \( (\mu) \) decreases the temperature. The free energy in grand canonical ensemble also called Gibbs free energy can be defined as

\[
G = M - TS - \mu Q_m. \tag{36}
\]

which turns out to be

\[
G = \frac{-\Lambda r_+^6 + 3\mu^2 r_+^4 + 3r_+^4 + 6\mu^2 r_+^2 q}{6r_+^4} \left( \frac{r_+^2 - r_+^4 \Lambda - \mu^2 r_+^2 - 6\mu^2 q}{r_+^2 + 2\mu^2 q} \right) + \left( \frac{r_+^2}{r_+^2 + 2\mu^2 q} - \frac{6r_+^4 q}{8\pi r_+} \ln \left( \frac{(r_+^6 \Lambda + \mu^2 r_+^4 + 6\mu^2 qr_+^2 - r_+^4)^2}{16\pi (r_+^4 + 2\mu^2 qr_+^2)^2} \right) \right) - \mu^2 r_+. \tag{37}
\]

The effect of chemical potential reduces the free energy as we can see from the last term in Eq.(37). Fig. 4 represents the behavior of Gibbs free energy for special three cases. We observe that the Gibbs free energy is minimum at \( r_+ \simeq 0.8 \) due to the contribution of logarithmic correction term. It means logarithmic correction term also reduces the Gibbs free energy. From Figure, we notice that free energy is higher for positive cosmological constant as compare zero and negative cosmological constant respectively. It is interesting that free energy becomes negative at \( r_+ \simeq 15 \) for negative cosmological constant. It means free energy is globally thermodynamically unstable for negative cosmological constant. Hence, we can conclude that if the value of cosmological constant is higher, then free energy becomes more globally stable. Moreover, thermodynamical stability does not only depend on \( \Lambda \) and \( q \) but also on chemical potential \( \mu \).

3.3 Canonical ensemble

On the other hand, BH could be considered as a closed system (canonical ensemble) if the charge transfer is prohibited. The mass and temperature is
Figure 4: Gibb’s free energy in terms of horizon radius with $\mu = 0.5$.

Figure 5: Helmhotz free energy in terms of horizon radius.
given by Eqs. (8) and (13) respectively and the corresponding entropy with logarithmic corrected term is given in Eq. (25). The free energy in canonical ensemble is known as Helmhotz free energy if the charge is fixed, which is

\[ F = M - TS, \]  

(38)

which turns out to be

\[ F = \frac{-\Lambda r_+^6 + 3Q_m^2r_+^2 + 3r_+^4 + 6Q_m^2q - r_+^4 - \Lambda r_+^6 - 6Q_m^q - 6Q_m^2q}{6r_+^3} - \frac{4\pi r_+ (r_+^4 + 2Q^2q)}{4\pi (r_+^4 + 2Q^2q)} \times \left( \pi r_+^2 - \frac{b}{2} \ln \left( \frac{(r_+^4 + 6Q_m^2q)^2}{16\pi (r_+^4 + 2Q_m^2q)^2} \right) \right). \]  

(39)

Fig. 5 represents the behavior of Helmhotz free energy for specific values of parameters. We observe that the logarithmic corrected term reduces the free energy in every case, which is evident. Initially, the free energy for negative cosmological constant is high till \( r_+ = 7 \) as compare to zero and positive cosmological constant but for large horizon, free energy is decreasing further at \( r_+ = 18 \) it becomes negative. The free energy is highest for positive cosmological constant as compare to \( \Lambda = 0 \) and \(-0.01\) for large horizon. Hence, we can conclude that BH is more thermodynamically stable if the value of cosmological constant increases in large horizon and it becomes thermodynamically unstable for lower values of cosmological constant. Moreover, thermodynamical stability do not depend only on \( \Lambda \) and \( q \) but also on magnetic charge \( Q_m \) rather than chemical potential.

4 Concluding Remarks

In this paper, we have discussed the new exact regular spherically symmetric solution of non-minimal Einstein Yang-Mill theory with magnetic charge of Wu-Yang gauge field and the cosmological constant. We only considered the positive non-minimal parameter \( q \) as zero and negative leads to space-time curvature singularities. After calculating the mass, entropy and temperature, we have discussed the effect of thermal fluctuation on non-minimal RBH. We have also utilized the logarithmic correction of entropy and discuss the behavior of pressure and specific heat. We observed that the pressure reduces due to the logarithmic correction when decreasing the value of cosmological constant.
We have also investigated the ratio of specific heat at constant pressure and volume ($\gamma$), we observe that due to logarithmic correction the value of $\gamma$ increases. We have also noticed that the values of $\gamma$ are higher and more stable for negative values of cosmological constant while it becomes unstable upon positive values of cosmological constant for large horizon. We observed that the first law of thermodynamics is satisfied for non-minimal RBH even in the presence of thermal fluctuations. It is mentioned here that the logarithmic correction term increases the chance of first law of thermodynamics to satisfy. We have also investigated the phase transition for non-minimal RBH and found its critical points. We observed that the range of horizon radius for local stability of BH is increased for negative cosmological constant as compare to positive and zero cosmological constant, respectively.

We have noticed that if we increase the value of cosmological constant, the phase transition shifted towards the BH and vice versa. We have also discussed the free energy in grand canonical (Gibb’s free energy) and canonical (Helmothz free energy) ensembles. We notice that free energy reduces in the presence of logarithmic correction. It is concluded that the non-minimal RBH becomes more stable globally as well as locally if we increase the value of cosmological constant and vice versa. We have also noticed that the thermodynamics of non-minimal RBH gets modified because of general uncertainty principle [53, 54]. Such correction terms are non-trivial which is evident from our results and they may lead to interesting consequences like existence of BH remnants.

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