Magnetic reconnection in terms of catastrophe theory

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Abstract. Magnetic field line reconnection (magnetic reconnection) is a phenomenon that occurs in space and laboratory plasma. Magnetic reconnection allows both the change of the magnetic topology and the conversion of the magnetic energy into energy of fast particles. The critical point (critical line or plane in higher dimensional cases) of the magnetic field play an important role in process of magnetic reconnection, as in its neighborhood occurs a change of its topology of a magnetic field and redistribution of magnetic field energy. A lot of literature is devoted to the analytical and numerical investigation of the reconnection process. The main result of these investigations as the result of magnetic reconnection the current sheet is formed and the magnetic topology is changed. While the studies of magnetic reconnection in 2D and 3D configurations have led to several important results, many questions remain open, including the behavior of a magnetic field in the neighborhood of a critical point of high order. The magnetic reconnection problem is closely related to the problem of the structural stability of vector fields. Since the magnetic field topology changes during both spontaneous and induced magnetic reconnection, it is natural to expect that the magnetic field should evolve from a structurally unstable into a structurally stable configuration. Note that, in this case, the phenomenon under analysis is more complicated since, during magnetic reconnection in a highly conducting plasma, we deal with the non-linear interaction between two vector fields: the magnetic field and the field of the plasma velocities. The aim of our article is to consider the process of magnetic reconnection and transformation of the magnetic topology from the viewpoint of catastrophe theory. Bifurcations in similar configurations (2D magnetic configuration with null high order point) with varying parameters were thoroughly discussed in a monograph by Poston and Stewart.

1. Introduction
Magnetic field line reconnection (magnetic reconnection) is a phenomenon in space and laboratory plasmas. Magnetic reconnection allow both the change of the magnetic topology and the conservation of the magnetic energy into energy of fast particles [3], [1], [4].

The critical point (critical line or plane in higher dimensional case) of the magnetic field play an important role in process of magnetic reconnection, because in their neighborhood there is an evolution of topology of a magnetic field and redistribution of magnetic field energy.

A lot of literature is devoted to the analytical and numerical investigation of the reconnection process [5]-[8]. The main result of these investigations is that the current sheet is formed as the result of magnetic reconnection and the magnetic topology is changed.

Although the studies of magnetic reconnection in 3D and 2D configurations have a led to a number of important results, many questions remain open. One of these questions is the behavior of a magnetic field in the vicinity of a critical point of a high order at the selfevolution.
The magnetic reconnection problem is closely related to the problem of the structural stability of vector fields. The mathematical aspects of the latter problem were discussed by Arnold [9], who defined a structurally stable dynamic system as a system whose state remains topologically equivalent to the initial (unperturbed) state for any sufficiently small perturbation of the vector field. Since the magnetic field topology changes during both spontaneous and induced magnetic reconnection, it is natural to expect that the magnetic field should evolve from a structurally unstable into structurally stable configuration. Note that, in this case, the phenomenon under analysis is more complicated because, during magnetic reconnection in a highly conducting plasma, we deal with the nonlinear interaction between two vector fields: the magnetic field and the field of the plasma velocities. This circumstance significantly complicates the problem in comparison with the case of one vector field. The simplest example of a structurally unstable solenoidal vector field is a magnetic field with third-order null lines in 2D space

\[
B = 2xyi + (x^2 + y^2)j + 0k
\]  

(1)

The magnetic configuration described by the vector potential (1) is structurally unstable because the critical point \((x = 0, y = 0)\) of the vector field is degenerate. Small perturbations of the magnetic field described by the vector potential (1) cause the degenerate singular point to bifurcate or disappear, in which case the perturbed magnetic field can be expressed in terms of the perturbed magnetic field

\[
B = (2xy + 2\gamma y + \delta x + \mu)i + (x^2 + y^2 + 2\beta + \delta y + \epsilon)j + 0k
\]  

(2)

where the parameters \(\beta, \gamma, \epsilon, \mu, \delta\) are assumed to be small. Bifurcations in similar configurations with varying parameters were thoroughly discussed in a monograph by Poston and Stewart [10].

By choosing the appropriate parameters, one can make the magnetic configuration (2) physically stable but structurally unstable. Consequently, this configuration is suitable for analyzing magnetic reconnection in structurally unstable systems. In formulating the problem, we assume that the magnetic field topology changes under the action of the perturbations excited at the boundary of the region under consideration. In other words, we consider the regime of driven magnetic reconnection.

![Figure 1. The magnetic configuration with third-order critical point (1)](image-url)
In this paper we present the results of 3D and 2D computing simulation of the process of transformation of structurally unsta\ble into structurally one and formation of the regular current structures.

The paper is organized as follows. The mathematical model are described in Section 2. In section 3 we presented our numerical results. Comparison of results of the accidents received by means of computing experiment with catastrophe theory will be given in section 4. The numerical methods are defined in the Appendix Section.

2. Mathematical model

We investigat\e the plasma evolution using the set of MHD equations. We write the MHD equations in a form used in Ref. [8]

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]  

\[ \rho \frac{\partial \mathbf{v}}{\partial t} = -\frac{\beta}{2} \nabla p + [\nabla \times \mathbf{B}] \times \mathbf{B} \]  

\[ \frac{\rho}{\gamma - 1} \frac{dT}{dt} + \rho \mathbf{v} \cdot \nabla = -k\Delta T + \frac{2\nu_m}{\beta} (\nabla \times \mathbf{B})^2 \]  

\[ p = \rho T \]  

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nu_m \nabla \times \nabla \times \mathbf{B} \]

Here \( \rho, p, T \) correspond to density, pressure, and temperature of plasma. We probe the following value of parameters ab\iabatic index \( \gamma = 5/3 \), magnetic velocity \( \nu_m = c^2/4\pi \sigma v_a \sigma = 0.006 \), \( \sigma \) - \plas\ma\ conductivity, \( \beta = 8\pi \rho_0 / B_0^2 = 0.012 \) is the ratio of the plasma pressure to the magnetic pressure on the boundary.

At the initial times the electric current density is zero and plasma is equilibrium state \( \mathbf{v}(0) = 0 \) with th\ constant \density \( \rho(0) = 1 \), \pressure \( p(0) = 1 \).

We consider the initially magnetic configuration with third-order critical point

\[ \mathbf{B} = 2xy \mathbf{i} + (x^2 + y^2) \mathbf{j} + 0\mathbf{k} \]
in 2D case and

$$\mathbf{B} = yz \mathbf{i} + zx \mathbf{j} + xy \mathbf{k}$$  \hspace{1cm} (9)

in 3D ones.

The system of the MHD equation is solved in a cubic computation box \((-1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1\) in case of three dimensional, and a square region \((-1 \leq x \leq 1, -1 \leq y \leq 1\).

In our simulation the boundary conditions correspond to an electric current perturbation imposed at the boundary of the computational region. We considered the cylindrical MHD wave converging toward the null points \((x \pm 1, y \pm 1)\). The boundaries \(z \pm 1\) correspond to the free inflow and outflow of the plasma. As for other parameters of plasma, there where plasma is entered into the computational area we set \(p = 1, T = 1, \rho = 1\), in other boundary we assume free outflow conditions.

3. Simulation results

We present the result of two series of numerical calculation for the evolution of structurally unstable 2D and 3D magnetic fields with high-order critical point.

At first we consider the structurally unstable 2D magnetic field (8). In figure 4 the quasisteady regime is reached. We can see the appearance of the finite -size current region bounded by the three separatrices of the magnetic field. There are three null lines of the X-types at the corners of this region and a null line of the O-type at the center of this region. Current sheet similar to those that appear in the vicinity of null lines of the X-type are observed at the corners. The evolution of the magnetic field corresponds to a transformation of the initial magnetic field configuration to the structurally stable one.

We will consider the process of a transformation of this magnetic field from the point of view of the catastrophe theory in the following section.

In the second series of calculations we analyzed the structurally unstable 3D magnetic field with high-order critical point (9). To projections of power lines to the coordinate planes hyperboles are. In each octant in process of removal on infinity power lines of the field asymptotically aspire to the straight line passing through this central part. As a result in each of eight the octant forms to about one thorn.
The field has high degree of symmetry: in all octants it behaves equally. Therefore orientation of the field concerning the axis z which is an axis of symmetry of indignation isn’t essential. At a quasistationary stage ”disorder” of a critical point of the higher order of a magnetic field on points of the first order is clearly visible. Taking into account an initial configuration the triangular structure of disintegration is observed (fig. 6). Such thin nonlinear structures as current locks and current layers in numerical experiment in this case weren’t observed.

4. Structurally unstable magnetic field and catastrophe theory
In this section we try to consider our magnetic field (8) from the view point of the catastrophe theory.

The catastrophe theory is a program. [12] The purpose of the program is to determine how the qualitative properties of solutions of equations change as the parameters that appear in the equations change. Elementary catastrophe theory is study how the critical points of a potential \( V(x, c) \) move about, coalesce and annihilate each other or are created and disperse from each other in state space \( x \in R^n \) as the control parameters \( c \in R^k \) are varied.

The elementary catastrophe function \( Cat(l, k) \) is the sum of two terms:

\[
Cat(l, k) = CG(l) + Pert(l, k). \tag{10}
\]

The catastrophe germ \( CG(l) \) depens only on the \( l \) state variables. All its second partial derivatives vanish at the critical point. The universal perturbations depends on the \( k \) control parameterers as well as the \( l \) state variables. The dependence of \( Pert(l, k) \) on the control-parameter values is linear. For ”most” choices of control-parameter values (all but a set of measure zero) the function \( Cat(l, k) \) has isolated critical points. The list of canonical catastrophe function is present in book [12].

In our case the magnetic configuration 8 can be described by the vector potential \( \mathbf{A} = A(x, y)\mathbf{e}_z \), where

\[
A(x, y) = \frac{1}{3}(x^3 - xy^2). \tag{11}
\]

The magnetic configuration described by the vector potential is structurally unstable, because the zero point of the vector field \( \mathbf{B} = \nabla \times A\mathbf{e}_z \) at \( x = 0, y = 0 \) is degenerate. This is
elementary catastrophe is named the elliptic umbilic catastrophe. In this case the \( k = 3, l = 2, \)
\[ CG(l) = y^2x - x^3 \]
\[ \text{Pert}(l, k) = 1y + a_2x + a_3y^2 + a_4x^2. \]

In this paper we demonstrate the simplest case where the values of control parameters are equal to zero. It should be noted that "device" of the catastrophe theory is developed generally for two-dimensional space.

5. Conclusion
In the present paper we have studied the transformation of structurally unstable magnetic configurations into structurally stable configurations. We have investigated the transformations induced by a finite amplitude electric current excited in the plasma under the action of perturbations imposed at the boundary. The transformation into configurations with a magnetic topology that differs from the initial one is forbidden in the framework of ideal magnetohydrodynamics, but can occur during magnetic field line reconnection. We have considered two different initial magnetic configurations: one 2D structurally unstable and the other 3D structurally unstable. In the case of the initially 2D structurally unstable configuration we have observed a global redistribution of the electric current and formation electric current "castel", and disappearance of a critical point of a high order on several critical points of the lowest order.

In another case when we consider 3D structurally unstable configuration. Such thin nonlinear structures as current castel and current sheets weren’t observed, but disappearance of the critical point was observed.

5.1. Appendices
In the present paper we use the semi-implicit method for solving the system of MHD equations. The semi-implicit method is similar to the well know predictor - corrector method. At first we made the explicit step

\[ \mathbf{v}^* = \mathbf{v}^n + \alpha \Delta t \mathbf{F}(\rho, \mathbf{v}, \mathbf{B}, p)^n/\rho^n, \]
\[ \rho^* = \rho^n - \alpha \Delta t \nabla (\rho \mathbf{v})^n, \]
\[ T^* = T^n + \alpha \Delta t F_1(\rho, \mathbf{B}, T)^n, \]
\[ \mathbf{B}^* = \mathbf{B}^n + \alpha \Delta t \nabla \times (\mathbf{v}^n \times \mathbf{B}^n). \]

Here \( 0.5 \leq \alpha \leq 1.0 \) is constant, \( \mathbf{F} \) - presents the right hand side the motion equation, the \( F_1 \) - the function is the right hand side of the energy equation. Then the semi-implicit term is corrected the equation of velocity

\[ \mathbf{v}^{n+1} - (\Delta t)^2 [\nabla \times \nabla \times (\mathbf{v}^{n+1} \times \mathbf{B}_0)] \times \mathbf{B}_0/\rho^* = \mathbf{v}^n + \Delta t \mathbf{F}(\rho, \mathbf{v}, \mathbf{B}, p)^*/\rho^* - (\Delta t)^2 [\nabla \times \nabla \times (\mathbf{v}^n \times \mathbf{B}_0)] \times \mathbf{B}_0/\rho^* \]

The \( \mathbf{v}^{n+1} \) is used in calculating the magnetic field, pressure, density of plasma

\[ \rho^{n+1} = \rho^n - \Delta t \nabla [\rho^*(\mathbf{v}^n + \mathbf{v}^{n+1})]/2, \]
\[ T^{n+1} = T^n + \frac{\Delta t \gamma - 1}{2 \rho^n} F_1(\rho, \mathbf{B}, T, \mathbf{v}^n + \mathbf{v}^{n+1})^* - \frac{\Delta t}{2} ((\mathbf{v}^{n+1} + \mathbf{v}^n) \nabla) T^*, \]
\[ \mathbf{B}^{**} = \mathbf{B}^n + \Delta t \nabla \times [(\mathbf{v}^n + \mathbf{v}^{n+1}) \times \mathbf{B}^n]/2, \]

At least, we are corrected the magnetic field term

\[ \mathbf{B}^{n+1} + \Delta t \nu \nabla \times \nabla \times \mathbf{B}^{n+1} = \mathbf{B}^{**} - \Delta t \nu_m \nabla \times \nabla \times \mathbf{B}^{**} + \Delta t \nu_0 \nabla \times \nabla \times \mathbf{B}^{**} \]
This is a second order accuracy method capable for carrying out computations with large time step. For achievement of necessary accuracy it is necessary to use the grid containing at least 200x200x200 points on space variables that results in need of development of the parallel computing algorithm described in paper [13].

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