Making Pulsed-Beam Wavelets
Gerald Kaiser
The Virginia Center for Signals and Waves
kaiser@wavelets.com • www.wavelets.com

Abstract
Point sources in complex spacetime, which generate acoustic and electromagnetic pulsed-beam wavelets, are rigorously defined and computed with a view toward their realization.

1. Introduction
Acoustic and EM waves are natural candidates for multiscale analysis: the wave/Maxwell equations have no a priori scale.

This motivated the idea of physical wavelets [K94a, K02a], localized solutions $W_z$ giving frames in solution spaces.

$z = x + iy$ is a complex spacetime point generalizing the time and scale parameters in 1D wavelets.

$W_z$ propagates and can’t have bounded spatial support.

But its source does, giving $z$ a simple interpretation:

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• $x = (x, t)$ is the **source center** in space and time

• $y = (y, u)$ gives its **extension** around $x$:
  - $y$ gives radius and orientation of **launching disk**
  - $u$ gives **pulse duration**

• $(x, y) \sim 4D$ version of time & scale

• **Stability** $\Rightarrow y$ is timelike $\Rightarrow z \in \mathcal{T} = \text{causal tube}$.

Thus $W_z$ is a **pulsed beam (PB)** with origin, direction, sharpness and duration controlled by $y$.

**Folklore:** In some sense,

$$W_z \sim \text{wave emitted by a complex sourcepoint } z.$$ 

Since the 1970s, **complex-source beams** have been used in engineering; see [HF01] for a comprehensive review.

Related ideas have circulated in relativity since the 1960s, *e.g.*, the **Kerr-Newman** solution [N65] representing charged, spinning black holes; see also [K01a, N02].

My motivation came from a long-time project combining QM with relativity via coherent-state representations, with $\mathcal{T}$ as **extended phase space** [K77, K78, K87, K90].
• **PB wavelet analysis of waves** - local alternative to Fourier.

• Associated **complex sourcepoint analysis** of sources.

• Applications to radar and communications: [K96, K97, K01].

*Can platforms be made to launch and detect $W_z$’s?*

Must understand the nature of the **sources**, making precise the notion of a wave emitted “from” $x + iy$. See also [HLK00].

## 2. Point sources in complex space

First define a point source in complex space $\mathbb{C}^n, n \geq 3$.

($n = 2$ is special and must be treated separately.)

The point source at $x = 0 \in \mathbb{R}^n$ can be defined as the source of the Newtonian potential $G_n$:

$$\delta_n(x) = \Delta_n G_n(x), \quad G_n(x) = \frac{1}{\omega_n} \frac{r^{2-n}}{2-n},$$

where $\omega_n$ is the area of the unit sphere in $\mathbb{R}^n$ and

$$r(x) = \sqrt{x \cdot x} = \sqrt{x^2}$$

is the Euclidean distance.

Define the **complex-distance** function

$$\tilde{r}(x + iy) = \sqrt{(x + iy)^2} = \sqrt{r^2 - a^2 + 2ix \cdot y}, \quad a = |y| > 0.$$
Fixing $\mathbf{y} \neq \mathbf{0}$, the branch points form an $(n-2)$-sphere

$$\mathcal{B}(\mathbf{y}) = \{ \mathbf{x} \in \mathbb{R}^n : r = a, \mathbf{x} \cdot \mathbf{y} = 0 \}.$$  

The simplest branch cut for which $\tilde{r}(\mathbf{x}) = +r(\mathbf{x})$ is

$$\text{Re } \tilde{r} \geq 0,$$  

which implies $-a \leq \text{Im } \tilde{r} \leq a$.

The branch cut is the $(n-1)$-disk spanning $\mathcal{B}(\mathbf{y})$,

$$\mathcal{D}(\mathbf{y}) = \{ \mathbf{x} : r \leq a, \mathbf{x} \cdot \mathbf{y} = 0 \}, \quad \partial \mathcal{D} = \mathcal{B}.$$  

We now define the point source at $\mathbf{x} = -i\mathbf{y}$ by

$$\tilde{\delta}_n(\mathbf{x} + i\mathbf{y}) = \Delta_n G_n(\mathbf{x} + i\mathbf{y}), \quad G_n(\mathbf{z}) \equiv \frac{1}{\omega_n} \frac{\tilde{r}^{2-n}}{2-n},$$

where $\Delta_n$ is the distributional Laplacian in $\mathbf{x}$.

$\tilde{\delta}_n(\mathbf{x} + i\mathbf{y})$ is a distribution supported in $\mathbf{x} \in \mathcal{D}(\mathbf{y})$ [K00].

For even $n$, $G_n(\mathbf{z})$ is analytic whenever $\mathbf{z}^2 \neq 0$, hence

$$\text{supp}_\mathbf{x} \tilde{\delta}_n(\mathbf{x} + i\mathbf{y}) = \mathcal{B}(\mathbf{y}) \text{ for even } n \geq 4.$$  

But for odd $n$, $G_n$ inherits the branch cut and

$$\text{supp}_\mathbf{x} \tilde{\delta}_n(\mathbf{x} + i\mathbf{y}) = \mathcal{D}(\mathbf{y}) \text{ for odd } n \geq 3.$$  

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The distribution $\tilde{\delta}_3(z)$ will be computed along with the time-dependent one for pulsed beams in $\mathbb{R}^{3,1}$.

3. Complex spacetime sourcepoints

Begin with Euclidean $\mathbb{R}^4$ and complexify:

$$x_E = (x, -u), \quad y_E = (y, t) \in \mathbb{R}^4$$

$$z = x_E + iy_E = (x + iy, -u + it) \in \mathbb{C}^4.$$ 

Rewrite this as a complex Minkowski vector:

$$z = x + iy = (z, i\tau), \quad z = x + iy, \quad \tau = t + nu$$

$$x = (x, it), \quad y = (y, iu) \in \mathbb{R}^{3,1}$$

$$x^2 = x^2 - t^2, \quad y^2 = y^2 - u^2, \quad z^2 = z^2 - \tau^2.$$ 

Consider the Newtonian potential in $\mathbb{R}^4$ and its extension,

$$G_4(x_E) = -\frac{1}{4\pi^2x_E^4}, \quad G_4(z) = -\frac{1}{4\pi^2z^4}.$$ 

$G_4(x_E)$ is a fundamental solution for the Laplacian,

$$\Delta_4G_4(x, u) = (\Delta_x + \partial^2_{\tau})G_4(x, u) = \delta_4(x, u). \quad (\mathcal{E})$$
Naive question: Does \( u \rightarrow it \) gives a fundamental solution for the wave operator, i.e.,

\[
\Box G_4(x, it) \equiv (\Delta_x - \partial_t^2) G_4(x, it) \sim \delta(x, t) ??
\]

Answer: No!

- \( G_4 \) is singular on \( x^2 = 0 \), so it must be defined in \( \mathbb{R}^{3,1} \).
- When properly defined, it will be sourceless.
- Causality makes sense in \( \mathbb{R}^{3,1} \) but not in \( \mathbb{R}^4 \).

To find the correct definition, note that

\[
-z^2 = r^2 - z^2 = r^2 - \tilde{r}^2 = (\tau + \tilde{r})(\tau - \tilde{r}),
\]

which gives the partial-fractions decomposition

\[
G_4(z) = G^+(z) - G^-(z), \quad G^\pm(z) = \frac{1}{8\pi^2 \tilde{r}} \cdot \frac{1}{\tau \mp \tilde{r}} \quad (\spadesuit)
\]

Since formal differentiation gives

\[
\Box G^\pm(z) = 0,
\]

\( G^\pm(z) \) can have sources only at \( x \in D \) or \( \tau = \pm \tilde{r} \).
Define the time-dependent radiation pattern

\[ 2\pi R^\pm = \frac{i}{\tau \mp \tilde{r}} = \frac{i}{(t \mp p) + i(u \mp q)} \]  \hspace{1cm} (R)

\[ \tilde{r} = p + iq, \quad p > 0, \quad -a < q < a. \]

Now \((p, q)\) are oblate spheroidal coordinates in \(\mathbb{R}^3\) whose level surfaces are \(\mathcal{B}\)-confocal ellipsoids and hyperboloids.

In cylindrical coordinates with \(x_3 = \hat{y} \cdot x\) and \(\rho = \sqrt{r^2 - x_3^2}\),

\[ E_p \equiv \{\text{constant } p > 0\} = \left\{ x : \frac{\rho^2}{a^2 + p^2} + \frac{x_3^2}{p^2} = 1 \right\} \]

\[ H_q \equiv \{\text{constant } q^2 < a^2\} = \left\{ x : \frac{\rho^2}{a^2 - q^2} - \frac{x_3^2}{q^2} = 1 \right\}. \]

\(E_p\)'s are wave fronts, and \(H_q\)'s give orthogonal flow.

In the far zone \(r \gg a\), they become spheres and cones:

\[ \tilde{r} = \sqrt{r^2 - a^2} + 2iar \cos \theta \approx r + ia \cos \theta \]

\[ E_p \rightarrow \{ r = p \}, \quad H_q \rightarrow \{ \cos \theta = q/a \} \]

\[ 2\pi R^\pm \approx \frac{i}{(t \mp r) + i(u \mp a \cos \theta)} \frac{1}{u \mp a \cos \theta}. \]  \hspace{1cm} (♣)

\(I\) \(y\) is timelike: \(|u| > a\), \(G^\pm\) is a smooth pulse around \(\pm y\) peaking at \(t = \pm r\), with duration \(T(x) = |u \mp a \cos \theta|\) and elliptical radiation pattern of eccentricity \(a/|u|\).
(II) \( y \) is spacelike: \(|u| < a\), \( G^\pm \) is singular on the cone \( \cos \theta = \pm u/a \) at \( t = \pm r \), with a hyperbolic radiation pattern.

(III) \( y \) is lightlike: \(|u| = a\), \( G^\pm \) is singular on ray \( \cos \theta = \pm 1 \) at \( t = \pm r \), with a parabolic radiation pattern.

Only (I) gives a reasonable PB, though (II, III) should be of interest otherwise since \( G_4(z) \) is holomorphic for all \( z^2 \neq 0 \).

We therefore assume

\[
y \in V_\pm = \{(y, iu) : \pm u > |y|\} = \text{future/past cone},
\]

so that \( z \) belongs to the causal tube \( \mathcal{T} = \mathcal{T}_+ \cup \mathcal{T}_- \), where

\[
\mathcal{T}_\pm = \{x + iy : y \in V_\pm\} = \text{future/past tube}.
\]

\( \mathcal{T}_\pm \) are famous in physics (quantum field theory, twistors) as well as mathematics (Lie groups, harmonic analysis).

**Relation to propagators:** Fix \( y \in V_\pm \) and write

\[
G^\kappa(x \pm i0) = \lim_{\varepsilon \to 0^+} G^\kappa(x \pm i\varepsilon y), \quad \kappa = +, -.
\]

Then (♠) gives (with \( \mathcal{P}= \) principal value)

\[
G^\pm(x + i0) = \frac{1}{8\pi^2 r} \mathcal{P} \frac{1}{t \mp r} - \frac{i\delta(t \mp r)}{8\pi r},
\]

\[
G^\pm(x - i0) = \frac{1}{8\pi^2 r} \mathcal{P} \frac{1}{t \mp r} + \frac{i\delta(t \mp r)}{8\pi r}.
\]
Huygens’ principle requires the Minkowskian limit

\[ G^\pm_M(x) = G^\pm(x - i0) - G^\pm(x + i0) = \frac{i\delta(t \mp r)}{4\pi r}. \quad (M) \]

These are the retarded and advanced propagators, fundamental solutions vanishing for \( \pm t < 0 \):

\[ \Box G^\pm_M(x) = -i\delta(x)\delta(t) = \delta(x), \quad (H) \]

where the last equality comes from our volume element,

\[ x = (x, it) \in \mathbb{R}^{3,1} \Rightarrow dx = idx dt. \]

- \((H)\) is the desired hyperbolic counterpart of \((E)\).
- \(G_M = G^+_M - G^-_M\) is the Riemann function:

\[ G_M(x) = \frac{i\delta(t - r)}{4\pi r} - \frac{i\delta(t + r)}{4\pi r} \]

\[ \Box G_M(x) = 0, \quad G_M(x, 0) = 0, \quad \partial_t G_M(x, 0) = \delta(x). \]

- Causality comes with a choice of branch and has no meaning for \(G_4(z)\): \( \tilde{r} \rightarrow -\tilde{r} \Rightarrow G^+ \rightarrow -G^+ \).

\((M)\) is typical of hyperfunction theory, representing distributions as limits of differences of local holomorphic functions (in general, sheaf cohomology classes) [K88].
We now define the point source at \( x = -iy \) as

\[
\tilde{\delta}(z) = \Box G^\pm(z), \quad z = x + iy. \tag{\star}
\]

- This is identical for \( G^+ \) and \( G^- \):

\[
z^2 = 0 \Rightarrow x^2 = y^2 < 0 \text{ and } x \cdot y = 0,
\]

but \( x \cdot y \neq 0 \) since both vectors are timelike.

Hence \( G_4 \) is holomorphic in \( \mathcal{T} \) and

\[
\Box G^+(z) - \Box G^-(z) = \Box G_4(z) = 0.
\]

- By (\( \mathcal{H} \)), the Minkowskian limit of \( \tilde{\delta} \) is \( \delta \):

\[
\delta_M(x) \equiv \tilde{\delta}(x - i0) - \tilde{\delta}(x + i0) = \delta(x). \tag{\( \mathcal{M}' \)}
\]

- Since \( \tau \mp \tilde{\tau} \neq 0 \) in \( \mathcal{T} \), \( G^\pm \) is singular only when \( x \in \mathcal{D} \) and

\[
\text{supp } \tilde{\delta}(x + iy) = \{(x, it) : x \in \mathcal{D}(y)\} \equiv \tilde{\mathcal{D}}(y) \quad \forall y \in V_\pm.
\]

\( \tilde{\mathcal{D}}(y) \) is the world tube swept out in \( \mathbb{R}^{3,1} \) by \( \mathcal{D}(y) \) at rest.
4. Computing the source distributions

Consider the function on $\mathcal{T}$ defined by

$$W(z, i\tau) = \frac{g(\tau - \kappa \tilde{r})}{4\pi \tilde{r}}, \quad \kappa = \pm \quad (W)$$

with $g(\tau)$ an **analytic signal**, holomorphic for $\mathrm{Im} \ \tau \neq 0$.

We want to compute the source distribution

$$S(z) = \Box_x W(z).$$

Note that

$$g(\tau) = \frac{1}{2\pi \tau} \Rightarrow W(z) = G^\kappa(z) \Rightarrow S(z) = \delta(z)$$

$$g(\tau) \equiv -1 \Rightarrow W(z) = G_3(z) \Rightarrow S(z) = \tilde{\delta}(z).$$

- $\mathrm{Im} \ (\tau \pm \tilde{r}) \neq 0$ in $\mathcal{T} \Rightarrow \supp S \subset \tilde{\mathcal{D}}(y)$.
- $W$ is **doubly singular** on $\tilde{\mathcal{D}}$, where $g(\tau - \kappa \tilde{r})$ has a jump.

Regularize $W$ with the Heaviside function $\Theta$. Let $\varepsilon > 0$ and

$$W_\varepsilon(z) = \Theta(p - \varepsilon) W(z) = \begin{cases} W(z) & \text{outside } E_\varepsilon \\ 0 & \text{inside } E_\varepsilon. \end{cases}$$

Then the **regularized source**, defined by

$$S_\varepsilon(z) \equiv \Box W_\varepsilon(z),$$
is supported on the world tube $\tilde{E}_\varepsilon \subset \mathbb{R}^{3,1}$ of $E_\varepsilon$.

A computation gives

$$|\tilde{r}|^2 S_\varepsilon = \frac{\varepsilon^2 + a^2}{4\pi\tilde{r}} + \frac{(\varepsilon^2 + a^2)(\tilde{r}g_p - \tilde{g})}{2\pi\tilde{r}^2} + \frac{\varepsilon\tilde{g}}{2\pi\tilde{r}},$$

where

$$\delta(p - \varepsilon) = \Theta'(p - \varepsilon), \quad g_p \equiv \partial_pg = -\kappa g'(\tau - \kappa\tilde{r}).$$

$S_\varepsilon$ acts on a test function $f(x)$ (no smearing needed in $t$) by

$$\langle S_\varepsilon, f \rangle \equiv \int dx \ S_\varepsilon(x + iy, i\tau)f(x)$$

$$\text{with} \quad dx = \frac{p^2 + q^2}{a} dp dq d\phi. \quad \text{(Vol)}$$

Integrating by parts in $p$ and simplifying gives

$$\langle S_\varepsilon, f \rangle = \frac{\varepsilon^2 + a^2}{2a} \int_{-a}^{a} dq \left[ \frac{g_p \hat{f}}{\tilde{r}} - \frac{g \hat{f}}{\tilde{r}^2} - \frac{g \hat{f}_p}{\tilde{r}} \right]_{p=\varepsilon}$$

where $\hat{f}(p, q)$ is the mean of $f(p, q, \phi)$ over $\phi$.

But $g_p = -ig_q$ as $g$ is holomorphic. Integrating by parts in $q$,

$$\langle S_\varepsilon, f \rangle = \frac{\alpha\bar{\alpha}}{2ia} \left[ \frac{g \hat{f}}{\tilde{r}} \right]_{\tilde{r}=\alpha}^{\tilde{r}=\bar{\alpha}} - \frac{\alpha\bar{\alpha}}{a} \int_{-a}^{a} dq \frac{g \hat{f}_{\tilde{r}}}{\tilde{r}} \bigg|_{p=\varepsilon} \quad (\diamond)$$
with \( \alpha = \varepsilon + ia \) and \( \bar{\alpha} \) the **north and south poles** of \( E_\varepsilon \),

\[
\dot{f}(\bar{r}) \equiv \dot{f}(p, q) \quad \text{(no analyticity implied in } p + iq \text{)}
\]

and

\[
\dot{f}(\bar{r}) = \partial_r \dot{f}(\bar{r}) \equiv \frac{1}{2} (\partial_p - i \partial_q) \dot{f}(p, q).
\] (\( \dot{f} \))

Taking the limit \( \varepsilon \to 0 \) now gives the action of \( S \):

\[
\langle S, f \rangle = -\hat{g}(\tau, a) f(0) + 2ia \int_0^a \frac{dq}{q} \hat{g}(\tau, q) \dot{f}(iq) \bigg|_{p=\varepsilon}
\] (\( \bigvee \))

where

\[
\hat{g}(\tau, q) = \frac{1}{2} [g(\tau - iq) + g(\tau + iq)]
\]

is the average of \( g(\tau - k\bar{r}) \) over the jump at \( \bar{r} = \pm iq \in \mathcal{D} \) and we used the continuity of \( f \) and its derivatives across \( \mathcal{D} \),

\[
\begin{align*}
\dot{f}(-iq) &= \dot{f}(iq), & \dot{f}_\bar{r}(-iq) &= -\dot{f}_\bar{r}(iq) \\
\dot{f}(\pm ia) &= f(0), & \dot{f}_\bar{r}(0) &= 0,
\end{align*}
\]

which ensures that the integral in (\( \bigvee \)) is defined.

As a check, note that (\( \bigvee \)) gives the correct value as \( y \to 0 \):

\[
\langle S, f \rangle \to -g(\tau) f(0) \quad \Rightarrow \quad S(x, i\tau) = -g(\tau) \delta(x).
\]
5. Interpretation

Let us examine the meaning of the right side in

\[ \langle S_\varepsilon, f \rangle = \frac{\alpha \bar{\alpha}}{2ia} \left[ \frac{g f}{\bar{r}} \right]_{\bar{r}=\bar{\alpha}} - \frac{\alpha \bar{\alpha}}{a} \int_{-a}^{a} dq \frac{g f}{\bar{r}} \bigg|_{p=\varepsilon}. \]  \hfill (\diamond)

- The boundary term is a pair of real sourcepoints at the poles \( \bar{r} = \alpha, \bar{\alpha} \) \((\rho = 0, \quad x_3 = \pm \varepsilon)\) of \( E_\varepsilon \) modulated by \( g \).
- Since \( \partial_p \) is outward-orthogonal to \( E_\varepsilon \), the term with \( \partial_p f \) is a double layer on \( E_\varepsilon \), modulated by \( g \); see Eq. \((f_{\bar{r}})\).
- \( \partial_q \) is tangent to \( E_\varepsilon \), representing a flow from \( \bar{\alpha} \) to \( \alpha \). Hence the term with \( \partial_q f \) is a flow on \( E_\varepsilon \) modulated by \( g \).
- Since \( g(\tau) \) is an analytic signal [K94], the modulation by

\[ g = g(\tau \mp \bar{r}) = g(t \mp \varepsilon + i(u \mp q)) \]

is strongly amplified if \( \pm qu > 0 \) and diminished if \( \pm qu < 0 \).

This makes \( W(z) \) a unidirectional beam.

Taking into account (Vol) gives the unsmeared form

\[ S_\varepsilon(z) = W(z) \left[ i \delta(\bar{r} - \bar{\alpha}) - i \delta(\bar{r} - \alpha) - 2 \frac{\alpha}{\bar{r}} \right]^{2} \delta(p - \varepsilon) \partial_{\bar{r}} \]

where

\[ \delta(\bar{r} - \varepsilon \pm ia) = \delta(p - \varepsilon) \delta(q \pm a). \]
Similarly, $(\bigtriangledown)$ gives the unsmeared form

$$S(z) = -\tilde{g}(\tau, a)\delta(x) - \hat{g}(\tau, q) \frac{a^2 \delta(p)}{2\pi i q^3} \partial_\tau.$$  

In cylindrical coordinates, this becomes

$$S(z) = -\tilde{g}(\tau, 0)\delta(x) - \tilde{g}(\tau, \rho) \frac{\Theta(a - \rho) \delta(x_3)}{2\pi \sqrt{a^2 - \rho^2}} \left( \frac{a}{\rho} \partial_\rho - i \partial_3 \right)$$

where

$$\tilde{g}(\tau, \rho) = \frac{1}{2} \left[ g(\tau - i\sqrt{a^2 - \rho^2}) + g(\tau + i\sqrt{a^2 - \rho^2}) \right].$$

Letting $g \equiv -1$ gives the static complex sourcepoint

$$\delta(z) = \delta(x) + \frac{\Theta(a - \rho) \delta(x_3)}{2\pi \sqrt{a^2 - \rho^2}} \left( \frac{a}{\rho} \partial_\rho - i \partial_3 \right)$$

which is simpler than the form derived in [K00], being local while the latter had a subtraction.

Letting $g(\tau) = 1/2\pi \tau$ gives the complex spacetime point source,

$$\delta(z) = \frac{\tau \delta(x)}{2\pi z^2} + \frac{\tau \Theta(a - \rho) \delta(x_3)}{2\pi z^2 \sqrt{a^2 - \rho^2}} \left( \frac{a}{\rho} \partial_\rho - i \partial_3 \right),$$

where

$$z = (z, i\tau) \in \mathring{D} \Rightarrow z^2 = \rho^2 - a^2 - \tau^2 \neq 0.$$
• The above method works in \( \mathbb{R}^{n,1} \) for all odd \( n \geq 3 \).

• An interesting connection has been obtained between the Euclidean source distribution \( \tilde{\delta}_{n+1} \) and solutions of the homogeneous wave equation in \( \mathbb{R}^{n,1} \), based on the fact that the latter are spherical means over \((n-1)\)-spheres and \( \tilde{\delta}_{n+1} \) is supported precisely on these spheres for odd \( n \) [K00].

• These results extend to even \( n \geq 2 \) by a variant of Hadamard’s method of descent relating \( \tilde{\delta}_n \) to \( \tilde{\delta}_{n+1} \) and \( \tilde{\delta}_{n,1} \) to \( \tilde{\delta}_{n+1,1} \). The cases \( \tilde{\delta}_2 \) and \( \tilde{\delta}_{1,1} \) are special since \( G_2 \) is logarithmic, but the results are similar.

6. Making waves

Acoustic wavelets are generated from a ‘mother’ PB \( W^\pm \) by

\[
W_z^\pm(x') = W^\pm(x' - \bar{z}) = W^\pm(x' - x + iy).
\]

For suitable \( g \), subfamilies of \( W_z^\pm \)'s are frames, giving a pulsed-beam analysis-synthesis scheme for general waves as a local alternative to Fourier analysis.

In [K94], the \( W(z) \)'s are (fractional) time derivatives of \( G^\pm(z) \).

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Electromagnetic wavelets are frames of dyadic PB propagators \( G^\pm(x') = G^\pm(x' - \vec{z}) \) constructed as follows [K94, K02].

1. An electric & magnetic (E&M) dipole with dipole moments \( p_e, p_m \) at \( x = -iy \) gives an E&M polarization density

\[
P(z) \equiv P_e(z) + iP_m(z) = p \delta(z), \quad p = p_e + ip_m \in \mathbb{C}^3.
\]

2. This gives retarded/advanced E&M PB Hertz potentials

\[
Z^\pm(z) \equiv Z^\pm_e(z) + iZ^\pm_m(z) = p G^\pm(z)
\]

\[
\Box Z^\pm(z) = p \Box G^\pm(z) = P^\pm(z).
\]

3. These in turn generate EM fields \( F^\pm = E^\pm + iB^\pm \) given by

\[
F^\pm(z) = \nabla \times \nabla \times Z^\pm(z) - \partial_t \nabla \times Z^\pm(z).
\]

4. The dyadic ‘mother’ PBs \( G^\pm \) are now defined by

\[
G^\pm(z)p = F^\pm(z).
\]

- All the fields are real! The self-dual pairings \( E + iB \), etc. merely exhibit the (complex!) structure of EM.

- Gives local analysis/synthesis schemes for fields and sources.

- PB analysis seems more natural for EM than scalar waves since the real and imaginary parts have direct interpretations.
Applications to radar and communications have been proposed [K96, K97, K01] whose utility would be greatly enhanced if sources can be constructed to launch and detect EM wavelets.

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