How the Scalar Field of Unified Dark Matter Models Can Cluster

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Abstract. We use scalar-field Lagrangians with a non-canonical kinetic term to obtain unified dark matter models where both the dark matter and the dark energy, the latter mimicking a cosmological constant, are described by the scalar field itself. In this framework, we propose a technique to reconstruct models where the effective speed of sound is small enough that the scalar field can cluster. These models avoid the strong time evolution of the gravitational potential and the large Integrated Sachs-Wolfe effect which have been a serious drawback of previously considered models. Moreover, these unified dark matter scalar field models can be easily generalized to behave as dark matter plus a dark energy component behaving like any type of quintessence fluid.
1. Introduction

If we assume that General Relativity correctly describes the phenomenology of our universe, astronomical observations provide compelling evidence that (1) the dynamics of cosmic structures is dominated by dark matter (DM), a cold collisionless component mostly made of hypothetical elementary particles, and (2) the expansion of the universe is currently accelerating because of the presence of a positive cosmological constant or a more general Dark Energy (DE) component. The DM particles have not yet been detected and there is no theoretical justification for the tiny cosmological constant (or more general DE component) implied by observations (see, e.g. Refs. [1, 2]). Therefore, over the last decade, the search for extended or alternative theories of gravity has flourished.

In this paper we focus on unified models of DM and DE (UDM), in which a single scalar field provides an alternative interpretation to the nature of the dark components of the universe. Compared with the standard DM + DE models (e.g. ΛCDM), these models have the advantage that one can describe the dynamics of the universe with a single dark fluid which triggers both the accelerated expansion at late times and the large-scale structure formation at earlier times. Moreover, for these models, we can use Lagrangians with a non-canonical kinetic term, namely a term which is an arbitrary function of the square of the time derivative of the scalar field, in the homogeneous and isotropic background. These models are known as “k-essence models” [3, 4, 5, 6, 7, 8] (see also [9, 10, 11, 12, 13, 14]) and have been inspired by earlier studies of k-inflation [15, 16] (a complete list of dark energy models can be found in the review [17]).

Most UDM models studied so far in the literature require non-trivial fine tunings. Moreover, the viability of UDM models strongly depends on the value of the effective speed of sound \( c_s \) [18, 16, 19], which has to be small enough to allow structure formation [20, 21] and to reproduce the observed pattern of Cosmic Microwave Background (CMB) temperature anisotropies [18, 22, 20, 23, 24]. The prospects for a unified description of DM/DE (and inflation) through a single scalar field has been addressed also in Ref. [25].

Several adiabatic or, equivalently, purely kinetic models have been investigated in the literature: for example, the generalized Chaplygin gas [26, 27, 28] (see also Refs. [29, 30, 31, 32, 33, 34]), the Modified Chaplygin gas [35], the Scherrer [36] and generalized Scherrer solutions [22], the single dark perfect fluid with a simple 2-parameter barotropic equation of state [37], or the homogeneous scalar field deduced from the galactic halo space-time [38] (see also Ref. [39]).

Moreover, one can build up scalar field models where the constraint that the Lagrangian is constant along the classical trajectories, namely the solutions of the equations of motion, allows to describe a UDM fluid whose average behaviour is that of dark matter plus a cosmological constant [22] (see also Ref. [40, 41, 42], for a different approach). Alternative approaches to the unification of DM and DE have been proposed.
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in Ref. [43], in the framework of supersymmetry, in Ref. [44] in connection with chaotic scalar field solutions in Friedmann-Robertson-Walker cosmologies and in Ref. [45], in connection with the solution to the strong CP problem. One could also easily reinterpret UDM models based on a scalar field Lagrangian in terms of generally non-adiabatic fluids [46, 47] (see also [48]).

Here we choose to investigate the class of scalar-field Lagrangians with a non-canonical kinetic term to obtain UDM models. In Ref. [22], the authors require that the Lagrangian of the scalar field is constant along the classical trajectories. Specifically, by requiring that $\mathcal{L} = -\Lambda$ on cosmological scales, the background they obtain is identical to the background of the $\Lambda$CDM model. In this case the limited number of degrees of freedom does not leave any room for choosing the evolution of the effective speed of sound $c_s^2$ in agreement with observations [22].

Moreover, one of the main issues of these UDM models is to see whether their single dark fluid is able to cluster and produce the cosmic structures we observe in the universe today. In fact, the effective speed of sound can be significantly different from zero at late times; the corresponding Jeans length (or sound horizon), below which the dark fluid can not cluster, can be so large that the gravitational potential first strongly oscillates and then decays [24], thus preventing structure formation. Previous work attempted to solve this problem by a severe fine-tuning of the parameters appearing in the Lagrangian (see for example [30, 31, 32, 33, 36, 21]).

In Section 2, we layout the basic equations; in Sections 3, 4 and 5, we suggest a reconstruction technique to find models where the effective speed of sound is small enough that the scalar field can cluster. Specifically, in Section 4 we consider a model with kinetic term of Born-Infeld type [42, 49, 50, 51, 41, 40] that does not allow a strong time evolution of the gravitational potential and the large Integrated Sachs-Wolfe (ISW) effect which have been a serious drawback of previous models. In Section 6, we consider a more general class of UDM Lagrangians, with a non-canonical kinetic term, whose equations of motion are dynamically equivalent to those of the previous models. Finally, in Section 7 we study a possible way to generalize UDM models so that they can mimic dark matter and dark energy in the form of a general quintessence fluid.

2. Basic equations

The action describing the dark matter unified models can be written as

$$ S = S_G + S_{\phi} = \int d^4x \sqrt{-g} \left[ \frac{R}{2} + \mathcal{L}(\phi, X) \right], $$

where

$$ X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi. $$

We use units such that $8\pi G = c^2 = 1$ and signature $(-, +, +, +)$ (greek indices run over spacetime dimensions, whereas latin indices label spatial spatial coordinates).
The energy-momentum tensor of the scalar field $\varphi$ is
\[ T_{\mu\nu}^\varphi = -\frac{2}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}} = \frac{\partial \mathcal{L}(\varphi, X)}{\partial X} \nabla_\mu \varphi \nabla_\nu \varphi + \mathcal{L}(\varphi, X)g_{\mu\nu}. \] (3)

If $X$ is time-like, $S_\varphi$ describes a perfect fluid with $T_{\mu\nu}^\varphi = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$, with pressure
\[ \mathcal{L} = p(\varphi, X), \] (4)
and energy density
\[ \rho = \rho(\varphi, X) = 2X \frac{\partial p(\varphi, X)}{\partial X} - p(\varphi, X) \] (5)
where
\[ u_\mu = \frac{\nabla_\mu \varphi}{\sqrt{2X}} \] (6)
is the four-velocity.

Now we assume a flat, homogeneous Friedmann-Robertson-Walker background with scale factor $a(t)$. With this metric, when the energy density of the radiation becomes negligible, and disregarding also the small baryonic component, the background evolution of the universe is completely described by the following equations
\[ H^2 = \frac{1}{3} \rho, \] (7)
\[ \dot{H} = -\frac{1}{2} (p + \rho), \] (8)
where the dot denotes differentiation w.r.t. the cosmic time $t$ and $H = \dot{a}/a$. In these equations, the energy density and pressure of our scalar field $\varphi$, are supposed to describe both the dark matter and dark energy fluids.

On the background, the kinetic term becomes $X = \frac{1}{2} \dot{\varphi}^2$, and the equation of motion for the homogeneous mode $\varphi(t)$ reads
\[ \left( \frac{\partial p}{\partial X} + 2X \frac{\partial^2 p}{\partial X^2} \right) \ddot{\varphi} + \frac{\partial p}{\partial X} (3H \dot{\varphi}) + \frac{\partial^2 p}{\partial \varphi \partial X} \dot{\varphi}^2 - \frac{\partial p}{\partial \varphi} = 0. \] (9)
The two relevant relations for the dark energy problem are the equation of state $w \equiv p/\rho$, which, in our case, reads
\[ w = \frac{p}{2X \frac{\partial p}{\partial X} - p}, \] (10)
and the effective speed of sound
\[ c_s^2 \equiv \frac{(\partial p/\partial X)}{(\partial p/\partial \varphi)} = \frac{\frac{\partial p}{\partial X}}{\frac{\partial X}{\partial \varphi} + 2X \frac{\partial^2 p}{\partial X^2}}. \] (11)
The latter relation plays a major role in the evolution of the scalar field perturbations $\delta \varphi$ and in the growth of the overdensities $\delta \rho$. In fact, we start from small inhomogeneities of the scalar field $\varphi(t, x) = \varphi_0(t) + \delta \varphi(t, x)$, and write the metric in the longitudinal gauge,
\[ ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Phi)\delta_{ij}dx^i dx^j, \] (12)
(where $\delta_{ij}$ is the Kronecker symbol), having used the fact that $\delta T^i_j = 0$ for $i \neq j$ \cite{52}; here $\Phi$ is the peculiar gravitational potential. When we linearize the $(0-0)$ and $(0-i)$ components of Einstein equations (see Ref. \cite{16} and Ref. \cite{19}), we obtain the second order differential equation \cite{19} \cite{24}

$$u'' - c_s^2 \nabla^2 u - \frac{\theta''}{\theta} u = 0$$ \hspace{1cm} (13)

where primes indicate derivatives w.r.t. the conformal time $\eta$, defined through $d\eta = dt/a$; $u \equiv 2\Phi/(p + \rho)^{1/2}$ and $\theta \equiv (1 + p/\rho)^{-1/2}/(\sqrt{3a})$ \cite{19}.

One of the main issues in the framework of UDM model building is to see whether the single dark fluid is able to cluster and produce the cosmic structures we observe in the universe. In fact, the sound speed appearing in Eq. (13) can be significantly different from zero at late times; the corresponding Jeans length (or sound horizon), below which the dark fluid can not cluster, can be so large that the gravitational potential first strongly oscillates and then decays \cite{24}, thus preventing structure formation.

Previous work attempted to solve this problem by a severe fine-tuning of the parameters appearing in the Lagrangian (see for example \cite{30, 31, 32, 33, 36, 21, 22}). Here, we propose a class of UDM models where, at all cosmic times, the sound speed is small enough that cosmic structure can form. To do so, a possible approach is to consider a scalar field Lagrangian $L$ of the form

$$L = p(\varphi, X) = f(\varphi)g(X) - V(\varphi)$$ \hspace{1cm} (14)

In particular, by introducing the two potentials $f(\varphi)$ and $V(\varphi)$, we want to decouple the equation of state parameter $w$ and the sound speed $c_s$. This condition does not occur when we consider either Lagrangians with purely kinetic terms or Lagrangians like $L = g(X) - V(\varphi)$ or $L = f(\varphi)g(X)$ (see for example \cite{22}).

Actually, we could start from a more general Lagrangian where $g = g(h(\varphi)X)$. However, by defining a new kinetic term $Y = h(\varphi)X$, $h(\varphi)$ disappears and we need to recast $w$ and $c_s$ in terms of the new kinetic term. Therefore, this generalization does not describe a kinematics different from that generated by Eq. (14) (see Section 6, Appendix C and Appendix D).

In the following sections we will describe how to construct UDM models based on Eq. (14).

3. How to construct UDM models

Let us consider the scalar field Lagrangian of Eq. (14). The energy density $\rho$, the equation of state $w$ and the speed of sound $c_s$ are

$$\rho(X, \varphi) = f(\varphi) \left[ 2X \frac{\partial g(X)}{\partial X} - g(X) \right] - V(\varphi),$$ \hspace{1cm} (15)

$$w(X, \varphi) = \frac{f(\varphi)g(X) - V(\varphi)}{f(\varphi) \left[ 2X(\partial g(X)/\partial X) - g(X) \right] - V(\varphi)}.$$ \hspace{1cm} (16)
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\[ c_s^2(X) = \frac{(\partial g(X)/\partial X)}{(\partial g(X)/\partial X) + 2X (\partial^2 g(X)/\partial X^2)} , \]

respectively. The equation of motion becomes

\[ \left( \frac{\partial g}{\partial X} + 2X \frac{\partial^2 g}{\partial X^2} \right) \frac{dX}{dN} + 6X \frac{\partial g}{\partial X} + \frac{d\ln f}{dN} \left( 2X \frac{\partial g}{\partial X} - g \right) \frac{1}{f} \frac{dV}{dN} = 0 , \]

where \( N = \ln a \).

Unlike models with a Lagrangian with purely kinetic terms, here we have one more degree of freedom, the scalar field configuration itself. Therefore this allows to impose a new condition to the solutions of the equation of motion. In Ref. [22], the scalar field Lagrangian was required to be constant along the classical trajectories. Specifically, by requiring that \( L = -\Lambda \) on cosmological scales, the background is identical to the background of \( \Lambda \)CDM. In general this is always true. In fact, if we consider Eq. (9) or, equivalently, the continuity equations \( (d\rho/dN) = -3(p + \rho) \), and if we impose that \( p = -\Lambda \), we easily get

\[ \rho = \rho_{DM}(a = 1) a^{-3} + \Lambda = \rho_{DM} + \rho_{\Lambda} , \]

where \( \rho_{\Lambda} \) behaves like a cosmological constant “dark energy” component (\( \rho_{\Lambda} = \text{const.} \)) and \( \rho_{DM} \) behaves like a “dark matter” component (\( \rho_{DM} \propto a^{-3} \)). This result implies that we can think the stress tensor of our scalar field as being made of two components: one behaving like a pressure-less fluid, and the other having negative pressure. In this way the integration constant \( \rho_{DM}(a = 1) \) can be interpreted as the “dark matter” component today; consequently, \( \Omega_m(0) = \rho_{DM}(a = 1)/(3H^2(a = 1)) \) and \( \Omega_{\Lambda}(0) = \Lambda/(3H^2(a = 1)) \) are the density parameters of “dark matter” and “dark energy” today.

Let us now describe the procedure that we will use in order to find UDM models with a small speed of sound. By imposing the condition \( L(X, \varphi) = -\Lambda \), we constrain the solution of the equation of motion to live on a particular manifold \( M_{\Lambda} \) embedded in the four dimensional space-time. This enables us to define \( \varphi \) as a function of \( X \) along the classical trajectories, i.e. \( \varphi = L^{-1}(X, \Lambda)|_{M_{\Lambda}} \). Notice that therefore, by using Eq. (18) and imposing the constraint \( p = -\Lambda \), i.e. \( V(\varphi) = f(\varphi)g(X) + \Lambda \), we can obtain the following general solution of the equation of motion on the manifold \( M_{\Lambda} \)

\[ 2X \frac{\partial g(X)}{\partial X} f(\varphi(X)) = \Lambda \nu a^{-3} , \]

where \( \nu \equiv \Omega_m(0)/\Omega_{\Lambda}(0) \). Here we have constrained the pressure to be \( p = -\Lambda \). In Section [7] we will describe an even more general technique to reconstruct UDM models where the pressure is a free function of the scale factor \( a \).

If we define the function \( g(X) \), we immediately know the functional form of \( c_s^2 \) with respect to \( X \) (see Eq. (17)). Therefore, if we have a Lagrangian of the type \( \mathcal{L} = f(\varphi)g(X) \) or \( \mathcal{L} = g(X) - V(\varphi) \), we are unable to decide the evolution of \( c_s^2(X) \) along the solutions of the equation of motion [22] because, once \( g(X) \) is chosen, the constraint \( \mathcal{L} = -\Lambda \) fixes immediately the value of \( f(\varphi) (V(\varphi)) \). On the contrary, in the case of Eq. (14), we can do it through the function \( f(\varphi(X)) \). In fact, by properly
defining the value of \( f(\varphi(X)) \) and using Eq. (18), we are able to fix the slope of \( X \) and, consequently (through \( g(X) \)), the trend of \( c_s^2(X) \) as a function of the scale factor \( a \).

Finally, we want to emphasize that this approach is only a method to reconstruct the explicit form of the Lagrangian (14), namely to separate the two variables \( X \) and \( \varphi \) into the functions \( g, f \) and \( V \).

Now we give some examples where we apply this prescription. In the following subsection, we consider the explicit solutions when we assume a kinetic term of Born-Infeld type [42, 49, 50, 51, 41, 40]. Other examples (where we have the kinetic term \( g(X) \) of the Scherrer model [36] or where we consider the generalized Scherrer solutions [22]) are reported in Appendix A.

### 3.1. Lagrangians with Born-Infeld type kinetic term

Let us consider the following kinetic term

\[
g(X) = -\sqrt{1 - 2X/M^4},
\]

with \( M \) a suitable mass scale. We get

\[
\frac{2X/M^4}{\sqrt{1 - 2X/M^4}} f(\varphi(X)) = \Lambda \nu a^{-3},
\]

and

\[
c_s^2(X) = 1 - 2X/M^4.
\]

At this point it is useful to provide two explicit examples where we show the power of this approach. In the next section, we give the example par excellence: a Lagrangian where the sound speed can be small. It is important to emphasize that the models described here and in the next section satisfy the weak energy conditions \( \rho \geq 0 \) and \( p + \rho \geq 0 \).

- **Example 1)**

  By defining \( f \) as

  \[
f(\varphi(X)) = \Lambda \left(1 - 2X/M^4\right)^{3/2},
  \]

  we get

  \[
  X(a) = \frac{M^4/2}{1 + \nu a^{-3}}.
  \]

  In order to obtain an expression for \( \varphi(a) \), we use Eq. (7) and find

  \[
  \varphi(a) = \left(\frac{M^2}{3\Lambda}\right)^{1/2} \ln \left(\frac{1 + \nu a^{-3}}{\nu a^{-3}}\right).
  \]

  Now using Eq. (21) and our initial ansatz \( p = -\Lambda \) we obtain

  \[
f(\varphi) = \frac{\Lambda}{4} \left\{ \sinh \left[-\left(\frac{3\Lambda}{4M^4}\right)^{1/2} \varphi\right] + \cosh \left[-\left(\frac{3\Lambda}{4M^4}\right)^{1/2} \varphi\right] \right\}^2.
  \]
and

\[ V(\varphi) = \frac{\Lambda}{4} \left[ \sinh \left( \frac{3\Lambda}{M^4} \right)^{1/2} \varphi \right] + \cosh \left( \frac{3\Lambda}{M^4} \right)^{1/2} \varphi - 2 \sinh \left[ - \left( \frac{3\Lambda}{M^4} \right)^{1/2} \varphi \right] \right]. \]  

(28)

We can immediately see that \( dX/dN > 0 \). Therefore, when \( a \to 0 \) we have \( c_s^2 \to 1 \), whereas when \( a \to \infty \), \( c_s^2 \to 0 \). In other words, this model describes a unified fluid of dark matter and cosmological constant which is unavoidably in conflict with cosmological structure formation.

- **Example 2)**

  Let us define

  \[ f(\varphi(X)) = \frac{\Lambda}{(1 - 2X/M^4)^{1/2}}; \]  

  (29)

  then we get

  \[ X(a) = \frac{M^4}{2} \frac{\nu a^{-3}}{1 + \nu a^{-3}}. \]  

  (30)

  Following the same procedure adopted in the previous example, we obtain

  \[ \varphi(a) = \frac{2M^2}{\sqrt{3}\Lambda} \left\{ \arctan \left[ \left( \nu a^{-3} \right)^{-1/2} \right] - \frac{\pi}{2} \right\}. \]  

  (31)

  We immediately recover the same model studied in Ref. [22]:

  \[ f(\varphi) = \frac{\Lambda}{\cos \left( \frac{3\Lambda}{4M^4} \right)^{1/2} \varphi}, \quad V(\varphi) = 0. \]  

  (32)

  In this case, the \( c_s^2 \) dependence on the scale factor \( a \) is exactly opposite to the previous example: we have \( c_s^2 \to 0 \) when \( a \to 0 \), and \( c_s^2 \to 1 \) when \( a \to \infty \). In this model, as explained in Ref. [24], the non-negligible value of the sound speed today gives a strong contribution to the ISW effect and produces an incorrect ratio between the first peak and the plateau of the CMB anisotropy power-spectrum \( l(l + 1)C_l/(2\pi) \). In Appendix B, we study the kinematic behavior of this UDM fluid during the radiation-dominated epoch and we investigate for what values of \( \varphi \) the kinetic term \( X \) generates an appropriate basin of attraction.

4. **UDM models with Born-Infeld type kinetic term and a low speed of sound**

Following the study of the second example of the previous section, we now improve the dependence of \( c_s^2 \) on \( a \) when \( a \to \infty \). Let us consider for \( f \) the following definition

\[ f(\varphi(X)) = \frac{\Lambda}{\mu} \frac{2X/M^4 - h}{2X/M^4 (1 - 2X/M^4)^{1/2}}, \]  

(33)
where \( h \) and \( \mu \) are appropriate positive constants. Moreover, we impose that \( h < 1 \). Thus we get
\[
X(a) = \frac{M^4 h + \mu \nu a^{-3}}{2} \quad \text{or} \quad \left( \frac{d\varphi}{dN} \right)^2 = \frac{3M^4}{\Lambda} \frac{h + \mu \nu a^{-3}}{(1 + \nu a^{-3})(1 + \mu \nu a^{-3})},
\]
and, for \( c_s^2 \), we obtain the following relation
\[
c_s^2(a) = \frac{1 - h}{1 + \mu \nu a^{-3}}.
\]
Therefore, with the definition (33) and using the freedom in choosing the value of \( h \), we can shift the value of \( c_s^2 \) for \( a \to \infty \). Specifically,
\[
h = 1 - c_s^\infty \quad \text{where} \quad c_s^\infty = c_s(a \to \infty).
\]
At this point, by considering the case where \( h = \mu \) (which makes the equation analytically integrable), we can immediately obtain the trajectory \( \varphi(a) \), namely
\[
\varphi(a) = \left( \frac{4hM^4}{3\Lambda} \right)^{1/2} \arcsinh \left( \nu h a^{-3} \right)^{-1/2}.
\]
Finally, we obtain
\[
f(\varphi) = \frac{\Lambda (1 - h)^{1/2}}{h} \cosh \left[ \left( \frac{3\Lambda}{4 \Lambda M^4} \right)^{1/2} \varphi \right] \sinh \left[ \left( \frac{3\Lambda}{4 \Lambda M^4} \right)^{1/2} \varphi \right] \left\{ 1 + h \sin^2 \left[ \left( \frac{3\Lambda}{4 \Lambda M^4} \right)^{1/2} \varphi \right] \right\},
\]
and
\[
V(\varphi) = \frac{\Lambda}{h} \left\{ h^2 \sin^2 \left[ \left( \frac{3\Lambda}{4 \Lambda M^4} \right)^{1/2} \varphi \right] + 2h - 1 \right\}.
\]
This result implies that in the early universe \( \sqrt{3\Lambda/(4hM^4)} \varphi \ll 1 \) and \( 2X/M^4 \approx 1 \), and we obtain
\[
f(\varphi) \approx \left( \frac{4hM^4}{3\Lambda} \right)^{1/2} \frac{\Lambda \sqrt{1 - h}}{h} \varphi \propto a^{3/2} \quad \text{and} \quad |g(X)| = \sqrt{1 - 2X/\Lambda} \propto a^{-3/2},
\]
\[
|V(\varphi)| \to \left| \frac{\Lambda (2h - 1)}{h} \right| \ll f(\varphi) \left( 2X \frac{\partial g(X)}{\partial X} - g(X) \right) \propto a^{-3}.
\]
In other words, we find, for \( f(\varphi) \) and \( g(X) \), a behaviour similar to that of Example 2), as also obtained in Ref. [22] for a UDM Lagrangian of the type \( \mathcal{L} = f(\varphi)g(X) \).

When \( a \to \infty \), we have \( \varphi \to \infty \) and \( 2X/M^4 \to h \). Therefore
\[
f(\varphi)g(X) \to 0 \quad \text{and} \quad V(\varphi) \to \Lambda,
\]
that is, for \( a \to \infty \), the dark fluid of this UDM model will converge to a Cosmological Constant.

Because the dark fluids described by this Lagrangian and the Lagrangian defined in Example 2) behave similarly at early times, we conclude that the relative amounts of DM and DE that characterize the present universe are fully determined by the value
of $\varphi(a \sim 0)$. In other words, to reproduce the present universe, one has to tune the value of $f(\varphi)$ in the early universe. However, as we analytically show in Appendix B, once the initial value of $\varphi$ is fixed, there is still a large basin of attraction in terms of the initial value of $d\varphi/dt$, which can take any value such that $2X/M^4 \ll 1$. Moreover, in Appendix B, we analytically investigate the kinematic behavior of this UDM fluid during the radiation-dominated epoch.

Finally, we can conclude that, once it is constrained to yield the same background evolution as $\Lambda$CDM and we set an appropriate value of $c_\infty$, this UDM model provides a sound speed small enough that i) the dark fluid can cluster and ii) the Integrated Sachs-Wolfe contribution to the CMB anisotropies is compatible with observations. Figure 1 shows an example of the dependence of $c_s^2$ on $a$ for different values of $c_\infty$.

5. Prescription for UDM Models with a generic kinetic term

We now describe a general prescription to obtain a collection of models that reproduce a background similar to $\Lambda$CDM and have a suitable sound speed. Some comments about the master equation (20) are first necessary. The relation (20) enables to determine a connection between the scalar factor $a$ and the kinetic term $X$ on the manifold $\mathcal{M}_\Lambda$ and
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therefore a mapping between the cosmic time and the manifold $\mathcal{M}_\Lambda$.

Now it is easy to see that the LHS of Eq. (20), seen as a single function of $X$, must have at least a vertical asymptote and a zero, and the function must be continuous between the two. In particular, when $X$ is near the vertical asymptote the universe approaches the cosmological constant regime, whereas when $X$ is close to the zero of the function, the dark fluid behaves like dark matter. Therefore, if we define

$$f(\varphi(X)) = \frac{\mathcal{F}(X)}{2X(\partial g(X)/\partial X)}$$

(40)

where, for example,

$$\mathcal{F}(X) = \frac{1}{\mu} \frac{X_f - X}{X - X_i},$$

(41)

(where $\mu$ is an appropriate positive constant) the value of $X_f$ and $X_i$ are the zero and the asymptote mentioned above, namely, when $a \to 0$ we have $X \to X_i$ and when $a \to \infty$ we have $X \to X_f$. Moreover, if $X_f > X_i$ we have $dX/dN > 0$, whereas if $X_f < X_i$ we have $dX/dN < 0$. In other words, according to Eq. (20),

$$X(a) = X_f \frac{1 + (X_i/X_f)\Lambda \mu \nu a^{-3}}{1 + \Lambda \mu \nu a^{-3}}.$$  

(42)

Let us emphasize that the values of $X_i$ and $X_f$ are very important because they automatically set the range of values that the sound speed can assume at the various cosmic epochs.

Let us finally make another important comment. One can use this reconstruction of the UDM model in the opposite way. In fact, by imposing a cosmological background identical to $\Lambda$CDM, the observed CMB power spectrum, and the observed evolution of cosmic structures, we can derive the evolution of the sound speed $c_s^2$ vs. cosmic time. In this case, by assuming an appropriate kinetic term $g(X)$ through Eq. (17), we can derive $X(a)$ and, consequently, $\varphi(a)$ and $X(a(\varphi)) = X(\varphi)$. Therefore, by using the relations (20) and $V(\varphi) = f(\varphi)g(X) + \Lambda$, we can determine the functional form of $f(\varphi)$ and $V(\varphi)$.

6. A Particular Equivalence Class of UDM models

In this section we investigate different UDM Lagrangians that have the same equation of state parameter $w$ and speed of sound $c_s$. We show that a class of equivalent Lagrangians that have similar kinematical properties exists. Appendix C gives the most general derivation of this class. Here, we describe a restricted class to emphasize the general procedure. Let us begin with the Lagrangian

$$\mathcal{L} = \mathcal{L}(h(\varphi)X, \varphi),$$

(43)

with $h(\varphi) > 0$. It is very easy to show that, if $h(\varphi) \neq 0$, a field-redefinition $\phi \to \psi$ exists such that

$$Y = \frac{\dot{\psi}^2}{2} = h(\varphi)X \quad \text{and} \quad \psi = \pm \int^\varphi [h(\hat{\varphi})^{1/2}d\hat{\varphi}] + K,$$

(44)
where $K$ is an appropriate integration constant. Without any loss of generality, consider the case with the $+\,$ sign in front of the integral above.

By performing this coordinate transformation, the Lagrangian becomes

$$L(h(\varphi)X, \varphi) = L(Y, \psi),$$

and the equation of motion (9) becomes

$$\left( \frac{\partial p}{\partial Y} + 2Y \frac{\partial^2 p}{\partial Y^2} \right) \psi + \frac{\partial p}{\partial Y}(3H \dot{\psi}) + \frac{\partial^2 p}{\partial \psi \partial Y} \dot{\varphi}^2 - \frac{\partial p}{\partial \psi} = 0. \quad (46)$$

The most important property of this transformation is that the dependences of the equation of state and the effective speed of sound on the scale factor $a$ remain the same. In terms of the new variables $\psi$ and $Y$, one obviously has

$$w(\psi, Y) = \frac{p}{2Y \frac{\partial p}{\partial Y} - p}, \quad (47)$$

and

$$c_s^2(\psi, Y) = \left( \frac{\partial p / \partial Y}{\partial p / \partial Y} \right) = \frac{\frac{\partial p}{\partial Y}}{\frac{\partial p}{\partial Y} + 2Y \frac{\partial^2 p}{\partial Y^2}}. \quad (48)$$

It is easy to see that the transformations to the new variables used in Ref. [53] to study scaling solutions are a particular case of this general prescription (see also Appendix C).

Obviously, we can make the reverse reasoning (see Appendix C): namely, by starting from the Lagrangian dependent on $\psi$ and $Y$, we can obtain several Lagrangians of type $L(\mathcal{R}(\theta)Z, \theta)$ with $Y = \mathcal{R}(\theta)Z$, where

$$Z = \dot{\theta}^2/2 \quad \text{and} \quad \theta = \int^\psi \left[ \mathcal{R}(\tilde{\psi})^{-1/2} d\tilde{\psi} \right] + \mathcal{K}, \quad (49)$$

where $\mathcal{K}$ is an appropriate integration constant and where we have used the fact that $\mathcal{R}(\theta)$ becomes a function of $\psi$, $\mathcal{R}[\theta(\psi)]$, thanks to the above coordinate transformation. Therefore, by considering the models obtained in the previous section and in Appendix A, we can get different Lagrangians that have the same $w$ and $c_s^2$ evolution but have different kinematical properties. For instance, if we start from Eq. (14), we get

$$L = f(\varphi)g(X) - V(\varphi) = f(\theta)g(\mathcal{R}(\theta)Z) - V(\theta) \quad (50)$$

with $X = \varphi^2/2 = \mathcal{R}(\theta)Z = \mathcal{R}(\theta)\dot{\theta}^2/2$ and for simplicity we write $f(\theta) \equiv f(\varphi(\theta))$ and $V(\theta) \equiv V(\varphi(\theta))$.

Now we describe some cases obtained starting from the model of Section 4. In this way we can obtain Lagrangians with kinetic term of Dirac-Born-Infeld type [51]. First of all, we consider an appropriate variable that simplifies the functions $f(\varphi)$ and $V(\varphi)$.\[\text{\textit{‡}}\]

\[\text{\textit{‡}}\text{ It has been shown that models with the pure kinetic Lagrangian } L(Y) \text{ (see for example Ref. [22]) can be described as an adiabatic perfect fluid with pressure } p \text{ uniquely determined by the energy density, because both the pressure and the energy density depend on a single degree of freedom, the kinetic term } Y. \text{ Thus, through this transformation, we can extend the adiabatic fluid Lagrangians studied in Ref. [22] to a more general class of equivalent models.}\]
In fact, if $R[\theta(\varphi)]^{-1/2} = \cosh(\gamma \varphi)$ where $\gamma = [(3\Lambda)/(4hM^4)]^{1/2}$, we get the following simplified Lagrangian

$$\mathcal{L}(\theta, Z) = -\frac{\Lambda c_\infty}{1 - c_\infty^2} \frac{1 + (\gamma^2 \theta^2)^{1/2}}{\gamma \theta} \left[ \frac{1 - 2Z/M^4}{1 + (1 - c_\infty^2)\gamma^2 \theta^2} \right]^{1/2} +$$

$$-\frac{\Lambda}{1 - c_\infty^2} \frac{(1 - c_\infty^2)^2 \gamma^2 \theta^2 + 1 - 2c_\infty^2}{[1 + (1 - c_\infty^2)\gamma^2 \theta^2]}.$$ (51)

Thus, by using the coordinate transformation, we obtain the following relations

$$\theta(a) = \frac{1}{\gamma \left[ (1 - c_\infty^2)\nu a^{-3} \right]^{1/2}},$$ (52)

and

$$Z(a) = \frac{M^4}{2} \frac{1 + \nu a^{-3}}{\nu a^{-3}}.$$ (53)

Another possibility to obtain a simpler Lagrangian is to define $R[\theta(\varphi)]^{-1/2} = 2 \cosh(\gamma \varphi) \sinh(\gamma \varphi)$. In Appendix D, we give more examples of how coordinate transformations of Lagrangians with a Born-Infeld kinetic term can yield Lagrangians with different kinematical properties.

7. Generalized UDM Models

In this Section we consider several possible generalizations of the technique introduced in Section 3, with the aim of studying models where the background does not necessarily mimic the $\Lambda$CDM background. Finally, we want to emphasize that the Lagrangians we obtain here can also be generalized by means of the field redefinition defined above and further detailed in Appendix C. We can write the Lagrangian with two different simple approaches:

1) By choosing $p(N)$. Indeed we get

$$\frac{d\rho}{dN} + 3\rho = -3p(N), \quad \text{i.e.} \quad \rho(N) = e^{-3N} \left[ -3 \int^N \left( e^{3N'} p(N')dN' \right) + K \right],$$ (54)

where $K$ is an integration constant. By imposing the condition $\mathcal{L}(X, \varphi) = p(N)$ along the classical trajectories, we obtain $\varphi = \mathcal{L}^{-1}(X(N), p(N))|_{\mathcal{M}_p(N)}$. Thus, starting from a generic Lagrangian $\mathcal{L} = f(\varphi)g(X) - V(\varphi)$ we get

$$2X(N) \left[ \frac{\partial q(X)}{\partial X} \right] (N) f(\varphi(X, N)) = p(N) + e^{-3N} \left[ -3 \int^N \left( e^{3N'} p(N')dN' \right) + K \right].$$ (55)

For example, if $p = -\Lambda$, $K = \rho(a = 1)$. The freedom provided by the choice of $K$ is particularly relevant. In fact, by setting $K = 0$, we can remove the term $\rho \propto a^{-3}$. Alternatively, when $K \neq 0$, we always have a term that behaves like pressure-less matter. We thus show that the single fluid of UDM models can mimic not only a cosmological constant but also any quintessence fluid.
Thus, using Eq. (55) and by following the argument described in Section 3, we can get the relations $X \equiv G_p(N)$, and consequently

$$\varphi \equiv Q_p(N) = \varphi_0$$

$$+ \int^{N} \left\{ G_p(N')^{1/2} \left[ -3e^{-3N} \int^{N'} \left( e^{3N'} p(N') dN' \right) + Ke^{-3N} \right]^{-1/2} dN' \right\}. \quad (56)$$

Therefore, with the functions $G_p(N)$ and $Q_p(N)$, we can write $f(X, N) = f(G_p(N), N) = f(G_p(Q^{-1}_p(\varphi)), Q^{-1}_p(\varphi)) = f(\varphi)$. Thus, by starting from a Lagrangian whose behavior is given by $p(N)$, the speed of sound is determined by the appropriate choice of $g(X)$.

2) By choosing the equation of state $w(N)$. Indeed

$$\rho(N) = \rho_0 e^{-3} f^{N}(w(N')+1)dN', \quad (57)$$

where $\rho_0$ is a positive integration constant, and

$$p(N) = \rho_0 w(N) e^{-3} \int^{N}(w(N')+1)dN'. \quad (58)$$

Therefore, still by imposing the condition $\mathcal{L}(X, \varphi) = p[w(N), N]$ along the classical trajectories, i.e. $\varphi = \mathcal{L}^{-1}[X(N), p(w(N), N)]|_{\mathcal{M}_{w(N)}}$, we get

$$2X\frac{\partial g(X)}{\partial X} f(X, N) = \rho_0[w(N) + 1]e^{-3} \int^{N}(w(N')+1)dN'. \quad (59)$$

Therefore, on the classical trajectory we can impose, by using $w(N)$, a suitable function $p(N)$ and thus the function $\rho(N)$. The master equation Eq. (59) generalizes Eq. (20). Also in this case, by Eq. (59) and by following the argument described in Section 3, we can get the relations $X \equiv G_w(N)$, and consequently

$$\varphi \equiv Q_w(N) = \pm \int^{N} \left\{ G_w(N')^{1/2} \left[ \rho_0 e^{-3} \int^{N'}(w(N'')+1)dN'' \right]^{-1/2} dN' \right\} + \varphi_0. \quad (60)$$

Thus, with the functions $G_w(N)$ and $Q_w(N)$, we can write $f(X, N) = f(G_w(N), N) = f(G_w(Q^{-1}_w(\varphi)), Q^{-1}_w(\varphi)) = f(\varphi)$. Then we can find a Lagrangian whose behavior is determined by $w(N)$ and whose speed of sound is determined by the appropriate choice of $g(X)$.

Finally, we conclude that the $p(N)$ constraint on the equation of motion is actually a weaker condition than the $w(N)$ constraint. The larger freedom that the $p(N)$ constraint provides naturally yields an additive term in the energy density that decays like $a^{-3}$, i.e. like a matter term in the homogeneous background. Let us emphasize that this important result is a natural consequence of the $p(N)$ constraint and is not imposed a priori.

8. Conclusions

A general severe problem of many UDM models considered so far is that their large effective speed of sound causes a strong time evolution of the gravitational potential
and generates an ISW effect much larger than current observational limits. In this paper we have outlined a technique to reconstruct UDM models such that the effective speed of sound is small enough that these problems are removed and the scalar field can cluster.

We have also considered a more general class of UDM Lagrangians with a non-canonical kinetic term. Specifically, we have studied some invariance properties of general Lagrangians of the form $\mathcal{L} = \mathcal{L}(h(\varphi)X, \varphi)$ which allows to define different models whose equations of motion are dynamically equivalent.

Finally, we have studied a possible way to generalize UDM models that can mimic a fluid of dark matter and quintessence-like dark energy.

The Lagrangians that we obtained appear rather contrived. Indeed, these models should be understood as examples which show that the mechanism itself can work. These models can however help to search for physically motivated models with the desired properties. Moreover many previous work attempted to solve this problem by a severe fine-tuning of the parameters appearing in the Lagrangian. This drawback does not belong to our models.

In future work, we will consider models with Lagrangians $\mathcal{L} = \mathcal{L}(X, \varphi)$ to estimate astrophysical observables, like the cross-correlation of CMB anisotropies and large-scale structure or the weak lensing shear signal power-spectrum.

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Appendix A. Other UDM model examples

In this Appendix we will give further examples of UDM models with small sound speed $c_s$.

Appendix A.1. Others possible Lagrangians with a kinetic term of Born-Infeld type

Let us define $f(\varphi(X))$ in the following way

$$f(\varphi(X)) = \frac{\Lambda}{\mu} \frac{(1 - 2X/M^4)^{3/2}}{2X/M^4 (2X/M^4 - h)} ,$$

where $0 < h < 1$ and $\mu > 0$. Therefore, with Eq. (22), we obtain

$$X(a) = \frac{M^4}{2} \frac{1 + h\mu a^{-3}}{1 + \mu a^{-3}} \quad \text{or} \quad \left( \frac{d\varphi}{dN} \right)^2 = \frac{3M^4}{\Lambda} \frac{1 + h\mu a^{-3}}{(1 + \nu a^{-3})(1 + \mu a^{-3})} .$$
As it is easy to see, this Lagrangian has opposite properties to those of the model in Section 4. Indeed, here we have \( dX/dN > 0 \) and, consequently, \( 2X/M^4 \to h \) at early times and approaches 1 when \( a \to \infty \). Therefore, using Eq. (23), we can conclude that \( c_s^2 \to 1 - h \) when \( a \to 0 \) and zero when \( a \to \infty \).

A possible simple analytical solution can be obtained if we define \( \mu = 1/h \). In fact, in this case we get

\[
\varphi(a) = \left( \frac{4M^4}{3A} \right)^{1/2} \arcsinh \left( \frac{\nu a}{3} \right)^{-1/2}.
\]

This gives

\[
f(\varphi) = \Lambda h \sqrt{1-h} \frac{\cosh \left[ \left( \frac{3A}{4M^4} \right)^{1/2} \varphi \right]}{\sinh^2 \left[ \left( \frac{3A}{4M^4} \right)^{1/2} \varphi \right]} \left\{ h + \sinh^2 \left[ \left( \frac{3A}{4M^4} \right)^{1/2} \varphi \right] \right\},
\]

and

\[
V(\varphi) = \Lambda \left\{ \sinh^4 \left[ \left( \frac{3A}{4M^4} \right)^{1/2} \varphi \right] + h \sinh^2 \left[ \left( \frac{3A}{4M^4} \right)^{1/2} \varphi \right] - h(1-h) \right\} \sinh^2 \left[ \left( \frac{3A}{4M^4} \right)^{1/2} \varphi \right] \left\{ h + \sinh^2 \left[ \left( \frac{3A}{4M^4} \right)^{1/2} \varphi \right] \right\}.
\]

One can see that the speed of sound depends on the scale factor \( a \) as follows

\[
c_s^2 = (1-h) \frac{(\nu/h)a^{-3}}{1+(\nu/h)a^{-3}}.
\]

Finally, one can generalize these models by imposing that the sound speed is zero neither when \( a \to 0 \) nor when \( a \to \infty \). Consider the following relation

\[
f(\varphi(X)) = \frac{\Lambda}{\mu} \frac{(1-2X/M^4)^{1/2}(h_\infty - 2X/M^4)}{2X/M^4(2X/M^4 - h_0)},
\]

where \( 0 < h_0 < 1 \), \( 0 < h_\infty < 1 \) and \( \mu > 0 \). Now \( c_s^2 \to 1 - h_0 \) in the early universe and \( c_s^2 \to 1 - h_\infty \) when \( a \to \infty \). In fact,

\[
c_s^2(a) = \frac{(1-h_\infty) + (1-h_0)\mu a^{-3}}{1 + \mu a^{-3}}.
\]

Then

\[
X(a) = \frac{M^4 h_\infty + h_0\mu a^{-3}}{2} \frac{1}{1 + \mu a^{-3}} \quad \text{or} \quad \left( \frac{d\varphi}{dN} \right)^2 = \frac{3M^4}{\Lambda} \frac{h_\infty + h_0\mu a^{-3}}{(1+\nu a^{-3})(1+\mu a^{-3})}.
\]

Therefore if \( \mu = h_\infty/h_0 \) we obtain

\[
\varphi(a) = \left( \frac{4h_\infty M^4}{3A} \right)^{1/2} \arcsinh \left( \frac{h_\infty \nu a^{-3}}{h_0} \right)^{-1/2},
\]

which gives

\[
f(\varphi) = \Lambda h_0 \frac{\cosh \left[ \left( \frac{3A}{4h_\infty M^4} \right)^{1/2} \varphi \right]}{\sinh^2 \left[ \left( \frac{3A}{4h_\infty M^4} \right)^{1/2} \varphi \right]} \left\{ (1-h_0) + (1-h_\infty) \sinh^2 \left[ \left( \frac{3A}{4h_\infty M^4} \right)^{1/2} \varphi \right] \right\}^{1/2},
\]

\[
(A.11)
\]
and

\[ V(\varphi) = \frac{\Lambda}{h_\infty} h_\infty^2 \sinh^4 \left[ \left( \frac{3\Lambda}{4h_\infty M^4} \right)^{1/2} \varphi \right] + h_0 (2h_\infty - 1) \sinh^2 \left[ \left( \frac{3\Lambda}{4h_\infty M^4} \right)^{1/2} \varphi \right] - h_0 (1 - h_0) \].

(A.12)

It is easy to see that these relations can be used both when \( h_0 < h_\infty \) (i.e. \( dX/dN > 0 \)) and when \( h_\infty < h_0 \) (i.e. \( dX/dN < 0 \)).

**Appendix A.2. Lagrangian with kinetic term of the generalized Scherrer solutions type**

Consider the following kinetic term \[ g(X) = g_n (X/M^4 - \hat{\chi})^n. \] (A.13)

with \( n > 1 \) and with \( \hat{\chi} > 0 \). In this case the sound speed becomes

\[ c_s^2 = \frac{(X/M^4 - \hat{\chi})}{2(n - 1)\hat{\chi} + (2n - 1)(X/M^4 - \hat{\chi})}. \] (A.14)

Moreover, if we set \( \epsilon = [(X/M^4 - \hat{\chi})/\hat{\chi}] \ll 1 \) we easily obtain

\[ c_s^2 \approx \frac{1}{2(n - 1)} \epsilon. \] (A.15)

Now Eq. (A.16) takes the form

\[ 2ng_n(X/M^4)(X/M^4 - \hat{\chi})^{n-1} f(\varphi(X)) = \Lambda \nu a^{-3}. \] (A.16)

Below we provide an example of a UDM model where \( dX/dN < 0 \) (i.e. for \( dc_s^2/dN < 0 \)) and, finally, we give an example that generalizes the Lagrangians with the kinetic term of the generalized Scherrer solutions both for \( dX/dN < 0 \) (\( dc_s^2/dN < 0 \)) and for \( dX/dN > 0 \) (\( dc_s^2/dN < 0 \)).

**Appendix A.2.1. dX/dN < 0.** Define

\[ f(\varphi(X)) = \frac{\Lambda}{\mu} \frac{1}{2ng_n(X/M^4)(X/M^4 - \hat{\chi})^{n-1}(\chi_i - X/M^4)}, \] (A.17)

where \( \chi_i > \hat{\chi} \). Then by Eq. (A.16) we get

\[ X(a)/M^4 = \hat{\chi} \frac{1 + (\mu \chi_i/\hat{\chi})\nu a^{-3}}{1 + \mu \nu a^{-3}} \quad \text{or} \quad \left( \frac{d\varphi}{dN} \right)^2 = \frac{6M^4\hat{\chi}}{\Lambda} \frac{1 + (\mu \chi_i/\hat{\chi})\nu a^{-3}}{(1 + \nu a^{-3})(1 + \mu \nu a^{-3})}. \] (A.18)

Now, if \( \mu = \hat{\chi}/\chi_i \), we obtain the following relations

\[ \varphi(a) = \left( \frac{8M^4\hat{\chi}}{3\Lambda} \right)^{1/2} \arcsinh \left( \frac{\hat{\chi}}{\chi_i} \nu a^{-3} \right)^{-1/2}, \] (A.19)

\[ f(\varphi) = \frac{\chi_i \Lambda}{2ng_n(\chi_i - \hat{\chi})^{n-1}} \cosh^{2n} \left[ \left( \frac{3\Lambda}{8M^4\chi} \right)^{1/2} \varphi \right] \left\{ \chi_i + \hat{\chi} \sinh^2 \left[ \left( \frac{3\Lambda}{8M^4\chi} \right)^{1/2} \varphi \right] \right\}, \] (A.20)
Consider the following relation

\[ V(\varphi) = \frac{\Lambda}{2n}\chi^2 \sinh^4 \left[ \left( \frac{3\Lambda}{8M^4} \right)^{1/2} \varphi \right] + 2n\chi \sinh^2 \left[ \left( \frac{3\Lambda}{8M^4} \right)^{1/2} \varphi \right] + \chi_i(\chi_i - \hat{\chi}) \] (A.21)

Then the sound speed is given by

\[ c_s^2(a) = \frac{1}{2(n-1)} \frac{(\varphi/\chi_i)^{1/2}}{1 + (\varphi/\chi_i)^{1/2}}. \] (A.22)

Therefore at early times \( c_s^2 \to (\chi_i - \hat{\chi})/(2n - 1) \chi_i - \hat{\chi} \) and when \( a \to \infty \) we have \( c_s^2 \to 0 \). Moreover, if \( \varepsilon \ll 1 \) (provided \( (\chi_i - \hat{\chi})/\chi_i \ll 1 \)) the sound speed takes the form

\[ c_s^2 \simeq \frac{1}{2(n-1)} \frac{(\varphi/\chi_i)^{1/2}}{1 + (\varphi/\chi_i)^{1/2}}. \] (A.23)

**Appendix A.2.2. General case.** Consider the following relation

\[ f(\varphi(X)) = \frac{\Lambda}{\mu} \frac{1}{2n g_n (X/M^4)(X/M^4 - \hat{\chi})} \frac{(X/M^4 - \chi_f)}{(\chi_i - X/M^4)}, \] (A.24)

where \( \chi_i > \hat{\chi} > 0 \) and \( \chi_f > \hat{\chi} \). Then if \( \mu = \chi_f/\chi_i \) we get the following relations

\[ X(a)/M^4 = \chi_f \frac{1 + \nu a^{-3}}{1 + (\chi_f/\chi_i)\nu a^{-3}}, \] (A.25)

\[ \varphi(a) = \left( \frac{8M^4\chi_f}{3\Lambda} \right)^{1/2} \arcsinh \left( \frac{\chi_f}{\chi_i} \nu a^{-3} \right)^{-1/2}, \] (A.26)

\[ f(\varphi) = \frac{\chi_i \Lambda}{2n g_n \chi_f \sinh^2 \left[ \left( \frac{3\Lambda}{8M^4} \right)^{1/2} \varphi \right]} \frac{\cosh^{2n} \left[ \left( \frac{3\Lambda}{8M^4} \right)^{1/2} \varphi \right]}{\left\{ \left( \chi_i - \hat{\chi} \right) + (\chi_f - \hat{\chi}) \sinh^2 \left[ \left( \frac{3\Lambda}{8M^4} \right)^{1/2} \varphi \right] \right\}^{1-n}}, \] (A.27)

\[ V(\varphi) = \frac{\Lambda}{2n\chi_f} \left\{ 2n\chi^2 \sinh^4 \left[ \left( \frac{3\Lambda}{8M^4} \right)^{1/2} \varphi \right] + \chi_i(2n+1)\chi_f - \hat{\chi} \right\} \sinh^{-2} \left[ \left( \frac{3\Lambda}{8M^4} \right)^{1/2} \varphi \right] \left\{ \chi_i + \chi_f \sinh^2 \left[ \left( \frac{3\Lambda}{8M^4} \right)^{1/2} \varphi \right] \right\}^{-1}. \] (A.28)

The sound speed is

\[ c_s^2(a) = \frac{1}{2(n-1)} \frac{(\varphi/\chi_i)^{1/2}}{1 + (\varphi/\chi_i)^{1/2}} \nu a^{-3}, \] (A.29)

We can immediately see that at early times \( c_s^2 \to (\chi_i - \hat{\chi})/(2n - 1) \chi_i - \hat{\chi} \) and when \( a \to \infty \) we have \( c_s^2 \to (\chi_f - \hat{\chi})/(2n - 1) \chi_f - \hat{\chi} \). Therefore, with this Lagrangian, the sound speed can both grow and decrease, depending on the value taken by \( \chi_i \) and \( \chi_f \). Moreover, if \( \varepsilon \ll 1 \) we obtain

\[ c_s^2 \simeq \frac{1}{2(n-1)} \frac{(\chi_f - \hat{\chi})/(\chi_i)^{1/2}}{1 + (\chi_f/\chi_i)^{1/2}} \nu a^{-3}. \] (A.30)
Appendix B. Study of UDM models when the universe is dominated by radiation

In general, when we do not neglect the radiation, the background evolution of the universe is completely characterized by the following equations

\[ H^2 = \frac{1}{3}(\rho + \rho_R), \] (B.1)

\[ \dot{H} = -\frac{1}{2}(p + \rho + p_R + \rho_R), \] (B.2)

where \( \rho_R \) and \( p_R \) are the radiation energy density and pressure, respectively.

In this Appendix we consider the universe dominated by radiation and we want to study analytically the behavior of UDM models with kinetic term of Born-Infeld type. In particular, we study the Lagrangian obtained in Example 2) and in the model of Section 4 (Example 3)). These Lagrangians have similar behavior at early times; thus, because Example 2) is simpler than Example 3), we investigate the former: the result will then also apply to Example 3). We proceed by defining some functions of the scale factor, which make simpler the study of the dynamics of these Lagrangians when the universe is not dominated by the UDM field.

Appendix B.1. Lagrangian of the type \( \mathcal{L}(\phi, X) = f(\phi)g(X) \)

Let us introduce appropriate functions of the scale factor. We write Eq. (9) as follows

\[ \frac{1}{f} \frac{df}{dN} = \lambda(N, X) \]

\[ \left( \frac{\partial g}{\partial X} + 2X \frac{\partial^2 g}{\partial X^2} \right) dX + \left[ 3 \left( 2X \frac{\partial g}{\partial X} \right) + \lambda(N, X, f(N, X)) \left( 2X \frac{\partial g}{\partial X} - g \right) \right] dN = 0. \] (B.3)

Eqs. (B.3) define the quantity \( \lambda \) as a generic function of \( N \).

Now, in order to get a second function of the scale factor, we find the set of scalar field trajectories where the second of Eq. (B.3) defines an exact differential form. To this aim, first of all we have to study the differential form \( P(X, N) dX + Q(X, N) dN = 0 \). One possible way to make it an exact differential form is to search for an integral factor \( I \), which is an explicit function of \( N \). In our situation \( P(X, N) = P(X) \), thus \( I(N) \) is

\[ \frac{dI}{I} = \frac{\partial Q(X,N)}{\partial X} \frac{dX}{P(X)} dN. \] (B.4)

In this case, we have to impose the integrability condition

\[ \frac{\partial Q(X,N)}{\partial X} = \alpha(N)P(X) \] (B.5)

so that \( I(N) = \exp \int_{N_0}^{N} dN' \alpha(N') \) only depends on \( N \).

Using the explicit expressions of \( Q(X,N) \) and \( P(X) \), the condition (B.5) becomes

\[ 3 \frac{\partial \left( 2X \frac{\partial g}{\partial X} \right)}{\partial X} + \frac{\partial \lambda}{\partial X} \left( 2X \frac{\partial g}{\partial X} - g \right) + (\lambda - \alpha) \frac{\partial \left( 2X \frac{\partial g}{\partial X} - g \right)}{\partial X} = 0. \] (B.6)
It is easy to see that $\lambda - \alpha$ is a function of (at least) $X$; then, defining $G(X) \equiv \alpha - \lambda$, Eq. (B.6) becomes

$$3 \frac{\partial}{\partial X} \left( \frac{2X \frac{\partial g}{\partial X}}{\partial X} - \frac{G}{\partial X} \left( 2X \frac{\partial g}{\partial X} - g \right) - G \frac{\partial}{\partial X} \left( 2X \frac{\partial g}{\partial X} - g \right) \right) = 0$$

(B.7)

which can be trivially integrated to give

$$3 \left( 2X \frac{\partial g}{\partial X} \right) + K = G \left( 2X \frac{\partial g}{\partial X} - g \right)$$

(B.8)

with $K$ a generic constant. Without any loss of generality, we can set $K = 0$ so that

$$\alpha - \lambda = G = 3(w + 1).$$

(B.9)

By inserting Eq. (B.8) into the second of Eqs. (B.3) we find

$$\left( \frac{\partial g}{\partial X} + 2X \frac{\partial^2 g}{\partial X^2} \right) dX + \alpha(N) \left( 2X \frac{\partial g}{\partial X} - g \right) dN = 0 .$$

(B.10)

By multiplying both sides by $I(N)$, we finally obtain

$$\left( 2X \frac{\partial g}{\partial X} - g \right) = Ke^{-f^N dN'} \alpha(N')$$

(B.11)

where $K$ is a new integration constant. Using the general equation (B.11), we can express the energy density as

$$\rho = Ke^{-f^N dN' (\alpha(N') - \lambda(N'))} = Ke^{-3 \int^N dN' (w(N') + 1)} .$$

(B.12)

If $\lambda \to 0$ and $\alpha \to 0$, $w \to -1$ and $Kf \to \text{const}$. Therefore, the energy density $\rho$ tends to a constant $\rho_0$. It is interesting to note that, if $\alpha \geq 0$ the term $\exp \left(-f^N dN' \alpha(N')\right)$ determines $\rho_0$. In order to have $\rho > 0$ we have to require $K > 0$.

First of all it is worth to make some comments on $w$ and $c_s^2$. If we impose the conditions $w + 1 \geq 0$ and $c_s^2 \geq 0$, in terms of $\alpha$ and $w$, or, equivalently, of $\alpha$ and $\lambda$, the effective speed of sound, Eq. (11), reads

$$c_s^2 = -(w + 1) \frac{d \ln X}{dN} = -\frac{\alpha - \lambda}{6\alpha} \frac{d \ln X}{dN} \geq 0 .$$

(B.13)

If the universe is dominated by a fluid with equation of state $w_B = \text{const}$ then

$$H = \dot{N} \sim 2/[3(w_B + 1)t]$$

(B.14)

and, if $\alpha \neq 0$ and $f \sim \varphi^{-\beta}$, up to a multiplicative constant, we have

$$\alpha(t) = 3(w + 1) - \frac{3}{2} \beta(w_B + 1) \sqrt{2Xt} \varphi .$$

(B.15)

When $\alpha = 0$, we recover the scaling k-essence models [22, 54].

Now we want to describe some properties of the Lagrangian studied in Example 2) of Section 3.1 (see also Ref. [22]) when the universe is dominated by the radiation

§ For $\lambda = 0$, the Lagrangian $\mathcal{L}$ (i.e. the pressure $p$) depends only on $X$; in other words, we are obtaining the equations that describe the purely kinetic models, namely the Lagrangians $\mathcal{L} = \mathcal{L}(X)$.

∥ In purely kinetic models ($\lambda = 0$), we get $(1/\alpha) d \ln X / dN \leq 0$. Therefore if $\alpha > 0$, $X$ can only decrease with time to its minimum value [22].
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$(w_B = w_R = 1/3)$. Specifically, we want to investigate how our UDM fluid behaves during this epoch and for what values of $\varphi$ the kinetic term $X$ provides a basin of attraction. Moreover, for the sake of simplicity we impose $M^4 = \Lambda$ and we apply the field redefinition $\sqrt{3} \varphi/2 \rightarrow \sqrt{3} \varphi/2 - \pi/2$. Then, for $\varphi > 0$, the Lagrangian becomes

$$L = -\Lambda \frac{\sqrt{1 - 2X/\Lambda}}{\sin \left( \frac{\sqrt{2}}{2} \varphi \right)} . \quad (B.16)$$

At early times, $\sqrt{3} \varphi(a \sim 0)/2 \ll 1$ and $2X(a \sim 0)/\Lambda \approx 1$. Therefore we have $\beta = 1$ and $w \approx 0$. Therefore from Eq. (B.15) we get

$$\alpha(t) = 3 - 2 \frac{\sqrt{2Xt}}{\varphi} . \quad (B.17)$$

By recalling the definition of $\alpha$,

$$\alpha = \frac{d \ln[2X(\partial g/\partial X) - g]}{dN} , \quad (B.18)$$

from the Born-Infeld type kinetic term, we obtain

$$- \frac{2\dot{X}/\Lambda}{1 - 2X\Lambda} t = 3 - 2 \frac{\sqrt{2Xt}}{\varphi} . \quad (B.19)$$

Now it is easy to see that if $\varphi \simeq \sqrt{2Xt}$, at early times the variation of the kinetic term is slow, as required to obtain appropriate values of $X$ and $\varphi$ when the universe enters the UDM-dominated epoch. In fact, by solving the differential equation (B.19), we obtain

$$X = \frac{\Lambda}{2}(1 - \xi t) , \quad (B.20)$$

where $\xi$ is a positive integration constant. By hypothesis, we know that $2X/\Lambda \approx 1$ then $\xi \ll 1$ and $\varphi \simeq \sqrt{\Lambda t}$. Therefore $\varphi$ and $X$ vary slowly and the solution is sufficiently stable during the radiation-dominated epoch. We can thus determine the value of $\xi$ (i.e. of $X(a \sim 0)$) at early times.

Now we want to study some properties of the initial conditions of our UDM fluid. First of all we want to know for what values of $\varphi$ we can have a basin of attraction in $X$. By making explicit $w$ in terms of $g(X)$, we rewrite the relation (B.9) as

$$\dot{\chi} = (1 - \chi) \left( \frac{2\sqrt{\chi} \sqrt{\Lambda t}}{\varphi} - 3 \chi \right) . \quad (B.22)$$

where $\chi = 2X/\Lambda$. In this case $\dot{\chi} \gg 1$ because we are at early times and $\chi$ starts growing very fast. We reach the condition $\dot{\chi} \rightarrow 0$ at some later time $\hat{t} > t_{in}$. It is important to choose $\varphi_{in}$ such that $\chi = 2X(\hat{t})/\Lambda \simeq 1$ and $2\sqrt{\Lambda t}/\hat{\varphi} - 3 \geq 0$, where
\begin{align*}
\dot{\varphi} &= \sqrt{\Lambda} t - K \quad \text{and} \quad \sqrt{\Lambda} t \leq 3K < 3\sqrt{\Lambda} t. \quad \text{Obviously,} \quad K \text{ depends on } \varphi_{\text{in}}. \quad \text{Finally, for } t \geq \hat{t}, \quad \alpha \rightarrow 0 \text{ and then becomes positive. Consequently, } \dot{X} < 0 \text{ and we recover the solution studied previously, i.e. Eq. (B.20).}
\end{align*}

Appendix B.2. Lagrangian of the type \( \mathcal{L}(\varphi, X) = f(\varphi)g(X) - V(\varphi) \)

Starting from the scalar field Lagrangian considered in Eq. (14), the energy density \( \rho \), the equation of state \( w \) and the speed of sound \( c_s^2 \) are given by Eqs. (15), (16) and (17), respectively. We can write the equation of motion (18) as follows
\begin{align*}
\frac{1}{f} \frac{df}{dN} &= \lambda_1(N, X) \\
\frac{1}{f} \frac{dV}{dN} &= \lambda_2(N, X) \left( 2X \frac{\partial g}{\partial X} - g \right) \\
\left( \frac{\partial g}{\partial X} + 2X \frac{\partial^2 g}{\partial X^2} \right) dX + \left[ 3 \left( 2X \frac{\partial g}{\partial X} \right) + \lambda(N, X) \left( 2X \frac{\partial g}{\partial X} - g \right) \right] dN &= 0 , \quad (B.23)
\end{align*}
where also in this case \( \lambda = \lambda_1 + \lambda_2 \) is a generic function of \( N \) and \( X \) and \( \lambda_2 = (dV/dN)/(\rho - V) \). Now, following the same reasoning of Appendix B.1, we obtain
\begin{align*}
\alpha - \lambda = 3(w_V + 1) = 3 \frac{2X(\partial g/\partial X)}{2X(\partial g/\partial X) - g} , \quad (B.24)
\end{align*}
where \( \alpha \) is given by Eq. (B.18) and \( w_V = (p + V)/(\rho - V) \), and we recover Eq. (B.11). Then
\begin{align*}
V(N) &= V_0 + \hat{K} \int_{N_0}^N \left[ \lambda_2(N')e^{-\int N' dN''\alpha(N'')} dN' \right] \quad (B.25)
\end{align*}
and,
\begin{align*}
\rho &= V_0 + \hat{K} \left\{ e^{-\int N dN'(\alpha(N') - \lambda_1(N'))} + \int_{N_0}^N \left[ \lambda_2(N')e^{-\int N' dN''\alpha(N'')} dN' \right] \right\} . \quad (B.26)
\end{align*}
In other words, the quantities \( \alpha, \lambda_1 \) and \( \lambda_2 \) completely describe the dynamics of these models.

Now, in the radiation-dominated epoch, \( \lambda_2(a \sim 0) \propto a^3 \) \( dV/dN(a \sim 0) \sim 0 \). Therefore, by considering the Lagrangian of the model in Section 4, by defining \( M^4 = \Lambda \) and knowing that \( \sqrt{3}\varphi(a \sim 0)/2 \ll (1 - c_s^2 \infty) \) and \( 2X(a \sim 0)/\Lambda \sim 1 \), we immediately recover the particular case investigated in Appendix B.1.

Appendix C. Proof of the equivalence of Lagrangians of type
\( \mathcal{L} = \mathcal{L}(h(\varphi)X, \varphi) \)

We briefly investigate some properties of invariance of the Lagrangians \( \mathcal{L} = \mathcal{L}(h(\varphi)X, \varphi) \). Write the equation of motion as follows
\begin{align*}
\left( \frac{\partial \mathcal{L}}{\partial X} + 2X \frac{\partial^2 \mathcal{L}}{\partial X^2} \right) \dot{X} + \dot{\varphi} \frac{\partial}{\partial \varphi} \left( 2X \frac{\partial \mathcal{L}}{\partial X} - \mathcal{L} \right) &= -6HX \frac{\partial \mathcal{L}}{\partial X} , \quad (C.1)
\end{align*}
where the RHS of Eq. (C.1) is an explicit function of time, through $H$, and has the meaning of “non-inertial force” in the equation of motion.

In particular we want to prove that we can always make the following change of field variable

$$h(\varphi)X = R(\theta)Z,$$  \hspace{1cm} (C.2)

Consider $h(\varphi) > 0$ and $R(\theta) > 0$ are continuous functions of $\varphi$ and $\theta$ respectively, with $h(\varphi) \neq R(\theta)$. Eq. (C.2) can be written in the following differential form

$$h(\varphi)^{1/2}d\varphi \mp R(\theta)^{1/2}d\theta = 0.$$  \hspace{1cm} (C.3)

Without any loss of generality, hereafter we consider the case with the minus sign.

We have indirectly constructed a map $\iota : \varphi \rightarrow \iota(\varphi)$ such that

$$\theta \equiv \iota(\varphi) = \int_{\varphi_0}^{\varphi} \left\{ \frac{h(\tilde{\varphi})}{R(\tilde{\varphi})} \right\}^{1/2} d\tilde{\varphi} + \theta_0,$$ \hspace{1cm} (C.4)

where $R(\theta) = R(\iota(\varphi)) \equiv R(\varphi)$. We necessarily have $h(\varphi) = h(\varphi^{-1}(\theta)) \equiv h(\theta)$. Therefore, the Lagrangian becomes $L(h(\varphi)X, \varphi) = L(R(\theta)Z, \varphi^{-1}(\theta)) = L(R(\theta)Z, \theta)$.

In order to study the change of field variables, we rewrite Eq. (C.2) in differential form

$$X \left( \frac{\partial h}{\partial \varphi} \right) d\varphi + h dX = Z \left( \frac{\partial R}{\partial \theta} \right) d\theta + R dZ.$$ \hspace{1cm} (C.5)

Then

$$\dot{X} = Z \left[ \frac{\partial}{\partial \theta} \left( \frac{R}{h} \right) \right] \dot{\theta} + \left( \frac{R}{h} \right) \dot{Z}.$$ \hspace{1cm} (C.6)

Finally, starting from this change of field variables, we are able to prove that their equations of motion are dynamically equivalent, namely Eq. (C.1) is identical to

$$\left( \frac{\partial L}{\partial Z} + 2Z \frac{\partial^2 L}{\partial Z^2} \right) \dot{Z} + \dot{\theta} \frac{\partial}{\partial \theta} \left( 2Z \frac{\partial L}{\partial Z} - L \right) = -6HZ \frac{\partial L}{\partial Z},$$ \hspace{1cm} (C.7)

and that they consequently have the same equation of state and effective speed of sound, i.e. Eqs. (10) and (11) are respectively equal to

$$w = \frac{L}{2Z \frac{\partial L}{\partial Z} - L},$$ \hspace{1cm} and \hspace{1cm} $$e_s^2 = \frac{(\partial L/\partial Z)}{(\partial \rho/\partial Z)} = \frac{\partial L}{\partial Z} + 2Z \frac{\partial^2 L}{\partial Z^2}.\hspace{1cm} (C.8)$$

The proof is a trivial consequence of Eqs. (C.2), (C.6) and the following relations

$$\frac{\partial L}{\partial X} = \left( \frac{h}{R} \right) \frac{\partial L}{\partial Z},$$ \hspace{1cm} and \hspace{1cm} $$\frac{\partial^2 L}{\partial X^2} = \left( \frac{h}{R} \right)^2 \frac{\partial^2 L}{\partial Z^2},$$ \hspace{1cm} (C.9)

$$\frac{\partial L}{\partial \varphi} = \left( \frac{h}{R} \right)^{1/2} \frac{\partial L}{\partial \theta} + Z \left( \frac{R}{h} \right)^{1/2} \left[ \frac{\partial}{\partial \theta} \left( \frac{h}{R} \right) \right] \frac{\partial L}{\partial Z},$$ \hspace{1cm} (C.10)

$$\frac{\partial^2 L}{\partial X \partial \varphi} = \left( \frac{h}{R} \right)^{1/2} \left\{ \left[ \frac{\partial}{\partial \theta} \left( \frac{h}{R} \right) \right] \left( \frac{\partial L}{\partial Z} + Z \frac{\partial^2 L}{\partial Z^2} \right) + \left( \frac{h}{R} \right) \frac{\partial^2 L}{\partial Z \partial \theta} \right\}.$$ \hspace{1cm} (C.11)

If $R(\theta) = 1$ (or $h(\varphi) = 1$) we can immediately recover the particular case investigated in Section 6.
Appendix D. Other examples of coordinate transformations of Lagrangians with a Born-Infeld kinetic term

Here, we give some examples of how a coordinate transformation of Lagrangians with a Born-Infeld kinetic term can yield Lagrangians with different kinematical properties. In fact, starting from the equality \( (50) \) and still with a Born-Infeld kinetic term, we can see that if \( Z = f(\varphi)X \), and \( W(\theta) = V(\varphi) + f(\varphi) \), we obtain

\[
\mathcal{L} = -f(\theta) \left[ 1 - \frac{2Z/M^4}{f(\theta)} \right]^{1/2} + f(\theta) - W(\theta),
\]

where we have assumed that \( W(\theta) > 0 \). In other words, it is possible to transform a Born-Infeld Lagrangian into a Dirac-Born-Infeld Lagrangian [51]. This is a particular case of a more general transformation. In fact, if \( X = Z/T(\theta) \) and \( V(\varphi) = W(\theta) - T(\theta) \) (with \( W(\theta) > 0 \) and \( T(\theta) > 0 \)), we get

\[
\mathcal{L} = -f(\theta) \left[ 1 - \frac{2Z/M^4}{T(\theta)} \right]^{1/2} + T(\theta) - W(\theta).
\]

Starting from the model of Section 4, we can obtain, for example, two similar Lagrangians that can be rewritten in the form \( (D.2) \). Define

\[
T^{1/2}[\theta(\varphi)] = \kappa_i^{1/2} \frac{\cosh (\gamma \varphi)}{[1 + (1 - c_{\infty}^2) \sinh^2 (\gamma \varphi)]^{1/2}},
\]

where \( i = 1, 2 \) and \( \kappa_1 = \Lambda c_{\infty}^2/(1 - c_{\infty}^2) \) and \( \kappa_2 = \Lambda/(1 - c_{\infty}^2) \). In this case,

\[
\theta(\varphi) = \frac{1}{\gamma} \left( \frac{\kappa_i}{1 - c_{\infty}^2} \right)^{1/2} \text{arc sinh} \left[ (1 - c_{\infty}^2)^{1/2} \sinh (\gamma \varphi) \right],
\]

and the various terms of Eq. \( (D.2) \) become

\[
f(\theta) = \frac{\Lambda c_{\infty}}{1 - c_{\infty}^2} \left[ 1 + (1 - c_{\infty}^2) \sinh^{-2} \left( \left( \frac{1 - c_{\infty}^2}{\kappa_i} \right)^{1/2} \frac{\gamma \theta}{\gamma} \right) \right]^{1/2},
\]

\[
T(\theta) = \frac{\kappa_i}{(1 - c_{\infty}^2)} \frac{1 - c_{\infty}^2 + \sinh^2 \left( \left( \frac{1 - c_{\infty}^2}{\kappa_i} \right)^{1/2} \frac{\gamma \theta}{\gamma} \right)}{\cosh^2 \left( \left( \frac{1 - c_{\infty}^2}{\kappa_i} \right)^{1/2} \frac{\gamma \theta}{\gamma} \right)},
\]

and

\[
W(\theta) = \frac{\Lambda}{1 - c_{\infty}^2} \frac{(1 - c_{\infty}^2) \cosh^2 \left( \left( \frac{1 - c_{\infty}^2}{\kappa_i} \right)^{1/2} \frac{\gamma \theta}{\gamma} \right) + (\kappa_i/\Lambda) \sinh^2 \left( \left( \frac{1 - c_{\infty}^2}{\kappa_i} \right)^{1/2} \frac{\gamma \theta}{\gamma} \right)}{\cosh^2 \left( \left( \frac{1 - c_{\infty}^2}{\kappa_i} \right)^{1/2} \frac{\gamma \theta}{\gamma} \right)}. \tag{D.7}
\]
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