The Firefighter Problem: A Structural Analysis

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Abstract

We consider the complexity of the firefighter problem where \( b \geq 1 \) firefighters are available at each time step. This problem is proved \( \text{NP} \)-complete even on trees of degree at most three and budget one \([11]\) and on trees of bounded degree \( b + 3 \) for any fixed budget \( b \geq 2 \).[3]

In this paper, we provide further insight into the complexity landscape of the problem by showing that the pathwidth and the maximum degree of the input graph govern its complexity. More precisely, we first prove that the problem is \( \text{NP} \)-complete even on trees of pathwidth at most three for any fixed budget \( b \geq 1 \). We then show that the problem turns out to be fixed parameter-tractable with respect to the combined parameter “pathwidth” and “maximum degree” of the input graph.

Keywords: firefighter problem, trees, pathwidth, cutwidth, bandwidth, parameterized complexity

1. Introduction

The firefighter problem was introduced by Hartnell \([14]\) and received considerable attention in a series of papers \([1, 5, 9, 11, 15, 16, 17, 19, 20, 7]\). In its original version, a fire breaks out at some vertex of a given graph. At each time step, one vertex can be protected by a firefighter and then the fire spreads to all unprotected neighbors of the vertices on fire. The process ends when the fire can no longer spread. At the end all vertices that are not on fire are considered as saved. The objective is at each time step to choose a vertex which will be protected by a firefighter such that a maximum number of vertices in the graph is saved at the end of the process. In this paper, we consider a more general version which allows us to protect \( b \geq 1 \) vertices at each step (the value \( b \) is called budget).

The original firefighter problem was proved to be \( \text{NP} \)-hard for bipartite graphs \([19]\), cubic graphs \([17]\) and unit disk graphs \([12]\). Finbow et al. \([11]\) showed that the problem
is NP-hard even on trees. More precisely, they proved the following dichotomy theorem: the problem is NP-hard even for trees of maximum degree three and it is solvable in polynomial-time for graphs with maximum degree three, provided that the fire breaks out at a vertex of degree at most two. Furthermore, the problem is polynomial-time solvable for caterpillars and so-called P-trees [19]. Later, Bazgan et al. [3] extended the previous results by showing that the general firefighter problem is NP-hard even for trees of maximum degree $b + 3$ for any fixed budget $b \geq 2$ and polynomial-time solvable on $k$-caterpillars. From the approximation point of view, the problem is \( \frac{2}{k+1} \)-approximable on trees (\( \frac{2}{k+1} \approx 1.5819 \)) [5] and it is not \( n^{1-\epsilon} \)-approximable on general graphs for any \( \epsilon > 0 \) unless $P = NP$ [1]. Moreover for trees in which each non-leaf vertex has at most three children, the firefighter problem is 1.3997-approximable [16]. Very recently, Costa et al. [7] extended the \( \frac{2}{k+1} \)-approximation algorithm on trees to the case where the fire breaks out at $f > 1$ vertices and $b > 1$ firefighters are available at each step. From a parameterized perspective, the problem is $W[1]$-hard with respect to the natural parameters “number of saved vertices” and “number of burned vertices” [8, 2]. Cai et al. [5] gave first fixed-parameter tractable algorithms and polynomial-size kernels for trees for each of the following parameters: “number of saved vertices”, “number of burned vertices”, “number of saved leaves”, “number of protected vertices”, “number of burned leaves”, and “number of protected vertices”.

In this paper we show that the complexity of the problem is governed by the maximum degree and the pathwidth of the input graph. In Section 2, we first provide the formal definition of the problem as well as some preliminaries. In Section 3, we complete the hardness picture of the problem on trees by proving that it is also NP-complete on trees of pathwidth three. We note that the given proof is also a simpler proof of the NP-completeness of the problem on trees. In Section 4, we devise a parameterized algorithm with respect to the combined parameter “pathwidth” and “maximum degree” of the input graph. The conclusion is given in Section 5.

2. Preliminaries

Graph terminology. Let $G = (V, E)$ be an undirected graph of order $n$. For a subset $S \subseteq V$, $G[S]$ is the induced subgraph of $G$. The neighborhood of a vertex $v \in V$, denoted by $N(v)$, is the set of all neighbors of $v$. We denote by $N^k(v)$ the set of vertices which are at distance at most $k$ from $v$. The degree of a vertex $v$ is denoted by $\text{deg}_G(v)$ and the maximum degree of the graph $G$ is denoted by $\Delta(G)$.

A linear layout of $G$ is a bijection $\pi : V \rightarrow \{1, \ldots, n\}$. For convenience, we express $\pi$ by the list $L = (v_1, \ldots, v_n)$ where $\pi(v_i) = i$. Given a linear layout $L$, we denote the distance between two vertices in $L$ by $d_L(v_i, v_j) = |i - j|$.

The cutwidth $\text{cw}(G)$ of $G$ is the minimum $k \in \mathbb{N}$ such that the vertices of $G$ can be arranged in a linear layout $L = (v_1, \ldots, v_n)$ in such a way that, for every $i \in \{1, \ldots, n - 1\}$, there are at most $k$ edges between $\{v_1, \ldots, v_i\}$ and $\{v_{i+1}, \ldots, v_n\}$.

The bandwidth $\text{bw}(G)$ of $G$ is the minimum $k \in \mathbb{N}$ such that the vertices of $G$ can be arranged in a linear layout $L = (v_1, \ldots, v_n)$ so that $d_L(v_i, v_j) \leq k$ for every edge $v_iv_j$ of $G$.

A path decomposition $\mathcal{P}$ of $G$ is a pair $(P, \mathcal{H})$ where $P$ is a path with node set $X$ and $\mathcal{H} = \{H_x : x \in X\}$ is a family of subsets of $V$ such that the following conditions are met:

1. $\bigcup_{x \in X} H_x = V$.  

2. For each \( uv \in E \) there is an \( x \in X \) with \( u, v \in H_x \).

3. For each \( v \in V \), the set of nodes \( \{x : x \in X \text{ and } v \in H_x\} \) induces a subpath of \( P \).

The width of a path decomposition \( \mathcal{P} \) is \( \max_{x \in X} |H_x| - 1 \). The pathwidth \( pw(G) \) of a graph \( G \) is the minimum width over all possible path decompositions of \( G \).

We may skip the argument of \( pw(G) \), \( cw(G) \), \( bw(G) \) and \( \Delta(G) \) if the graph \( G \) is clear from the context.

A **star** is a tree consisting of one vertex, called the **center** of the star, adjacent to all the other vertices.

**Parameterized complexity.** Here we only give the basic notions on parameterized complexity used in this paper, for more background the reader is referred to [10, 21]. The parameterized complexity is a framework which provides a new way to express the computational complexity of problems. A decision problem parameterized by a problem-specific parameter \( k \) is called **fixed-parameter tractable** if there exists an algorithm that solves it in time \( f(k) \cdot n^{O(1)} \) where \( n \) is the instance size. The function \( f \) is typically super-polynomial and only depends on \( k \). In other words, the combinatorial explosion is confined into \( f \). We may sometimes say that a problem is fixed-parameter tractable with respect the **combined parameter** \( k_1, k_2, \ldots, k_p \), meaning that the problem can be solved in time \( f(k_1, k_2, \ldots, k_p) \cdot n^{O(1)} \). 

**Problems definition.** We start with an informal explanation of the propagation process for the firefighter problem. Let \( G = (V, E) \) be a graph of order \( n \) with a vertex \( s \in V \), let \( b \in \mathbb{N} \) be a budget. At step \( t = 0 \), a fire breaks out at vertex \( s \) and \( s \) starts burning.

At any subsequent step \( t > 0 \) the following two phases are performed in sequence:

1. **Protection phase**: The firefighter protects at most \( b \) vertices not yet on fire.
2. **Spreading phase**: Every unprotected vertex which is adjacent to a burned vertex starts burning.

Burned and protected vertices remain burned and protected until the propagation process stops, respectively. The propagation process stops when in a next step no new vertex can be burned. We call a vertex saved if it is either protected or if all paths from any burned vertex to it contains at least one protected vertex. Notice that, until the propagation process stops, there is at least one new burned vertex at each step. This leads to the following obvious lemma.

**Lemma 1.** The number of steps before the propagation process stops is less or equal to the total number of burned vertices.

A **protection strategy** (or simply **strategy**) \( \Phi \) indicates which vertices to protect at each step until the propagation process stops. Since there can be at most \( n \) burned vertices, it follows from Lemma 1 that the propagation unfolds in at most \( n \) steps. We are now in position to give the formal definition of the investigated problem.

**The Firefighter problem:**

**Input:** A graph \( G = (V, E) \), a vertex \( s \in V \), and positive integers \( b \) and \( k \).

**Question:** Is there a strategy for an instance \((G, s, b, k)\) with respect to budget \( b \) such that at most \( k \) vertices are burned if a fire breaks out at \( s \)?

When dealing with trees, we use the following observation which is a straightforward adaptation of the one by MacGillivray and Wang for the case \( b > 1 \) [19, Section 4.1].
Lemma 2. Among the strategies that maximizes the number of saved vertices (or equivalently minimizes the number of burned vertices) for a tree, there exists one that protects vertices adjacent to a burned vertex at each time step.

Throughout the paper, we assume all graphs to be connected since otherwise we can simply consider the component where the initial burned vertex $s$ belongs to.

3. Firefighting on path-like graphs

Finbow et al. [11] showed that the problem is NP-complete even on trees of degree at most three. However, the constructed tree in the proof has an unbounded pathwidth. In this section, we show that the Firefighter problem is NP-complete even on trees of pathwidth three. For that purpose we use the following problem.

The Cubic Monotone 1-In-3-Sat problem:

**Input:** A CNF formula in which every clause contains exactly and only three positive literals and every variable appears in exactly three clauses.

**Question:** Is there a satisfying assignment (a truth assignment such that each clause has exactly one true literal) for the formula?

The NP-completeness of the above problem is due to its equivalence with the NP-complete Exact Cover by 3-Sets problem [13].

**Theorem 1.** The Firefighter problem is NP-complete even on trees of pathwidth three and budget one.

**Proof.** Clearly, Firefighter belongs to NP. Now we provide a polynomial-time reduction from Cubic Monotone 1-In-3-Sat.

In the proof, a guard-vertex is a star with $k$ leaves where the center is adjacent to a vertex of a graph. It is clear that if at most $k$ vertices can be burned then the guard-vertex has to be saved.

Let $\phi$ be a formula of Cubic Monotone 1-In-3-Sat with $n$ variables $\{x_1, \ldots, x_n\}$ and $m$ initial clauses $\{c_1, \ldots, c_m\}$. Notice that a simple calculation shows that $n = m$.

First, we extend $\phi$ into a new formula $\phi'$ by adding $m$ new clauses as follows. For each clause $c_j$ we add the clause $\overline{c}_j$ by taking negation of each variable of $c_j$. A satisfying assignment for $\phi'$ is then a truth assignment such that each clause $c_j$ has exactly one true literal and each clause $\overline{c}_j$ has exactly two true literals. It is easy to see that $\phi$ has a satisfying assignment if and only if $\phi'$ has one.

Now we construct an instance $I' = (T, s, 1, k)$ of Firefighter from $\phi'$ as follows (see Figure 1). We start with the construction of the tree $T$, the value of $k$ will be specified later.

- Start with a vertex set $\{s = u_1, u_2, \ldots, u_p\}$ and edges of $\{su_2, u_2u_3, \ldots, u_{p-1}u_p\}$ where $p = 2n - 1$ and add two degree-one vertices $v_{x_i}$ and $v_{\overline{x}_i}$ adjacent to $u_{2i-1}$ for every $i \in \{1, \ldots, n\}$.

Then for each $i \in \{1, \ldots, n\}$ in two steps:

- Add a guard-vertex $g_i$ (resp. $\overline{g}_i$) adjacent to $v_{x_i}$ (resp. $v_{\overline{x}_i}$).
• At each vertex \( v_{x_i} \) (resp. \( v_{x_i}^- \)) root a path of length \( 2 \cdot (n - i) \) at \( v_{x_i} \) (resp. \( v_{x_i}^- \)) in which the endpoint is adjacent to three degree-one vertices (called literal-vertices) denoted by \( \ell_1^{x_i}, \ell_2^{x_i}, \) and \( \ell_3^{x_i} \) (resp. \( \ell_1^{x_i}', \ell_2^{x_i}', \) and \( \ell_3^{x_i}' \)). Each literal-vertex corresponds to an occurrence of the variable \( x_i \) in an initial clause of \( \phi \). Analogously, the literal-vertices \( \ell_1^{x_i}, \ell_2^{x_i}, \ell_3^{x_i} \) represent the negative literal \( \overline{x}_i \) that appears in the new clauses of \( \phi' \).

Notice that each leaf of the constructed tree so far is at distance exactly \( p + 1 \) from \( s \).

• For each variable \( x_i \) (resp. \( \overline{x}_i \)), \( i \in \{1, \ldots, n\} \), there are exactly three clauses containing \( x_i \) (resp. \( \overline{x}_i \)). Let \( c_j \) (resp. \( \overline{c}_j \)), \( j \in \{1, \ldots, m\} \), be the first one of them. Then root a path \( Q_j^{c_j} \) (resp. \( Q_j^{\overline{c}_j} \)) of length \( 3 \cdot (j - 1) \) at \( \ell_1^{c_j} \) (resp. \( \ell_1^{\overline{c}_j} \)), and add a guard-vertex \( g^{c_j}_{j_1} \) adjacent to the endpoint of \( Q_j^{c_j} \). To the endpoint of \( Q_j^{c_j} \):
  (i) add a degree-one vertex \( d_{j_1}^{c_j} \) (a dummy-vertex) and (ii) root a path of length 3 where the last vertex of the path is a guard vertex \( g^{c_j}_{j_1} \). Repeat the same for two other clauses with \( x_i \) (resp. \( \overline{x}_i \)) and \( \ell_2^{c_j}, \ell_3^{c_j} \) (resp. \( \ell_2^{\overline{c}_j}, \ell_3^{\overline{c}_j} \)).

To finish the construction, set \( k = p + \frac{3}{2}(11n + 7) \).

In what follows, we use Lemma 2 and thus we only consider strategies that protect a vertex adjacent to a burned vertex at each time step. Recall that the budget is set to one in the instance \( I' \). Now we show that there is a satisfying assignment for \( \phi' \) if and only if there exists a strategy for \( I' \) such that at most \( k \) vertices in \( T \) are burned.

\[ \Rightarrow \] Suppose that there is a satisfying assignment \( \tau \) for \( \phi' \). We define the following strategy \( \Phi_\tau \) from \( \tau \). At each step \( t \) from 1 to \( p + 1 \), if \( t \) is odd then protect \( v_{x_{(t/2)}} \) if \( x_{(t/2)} \) is true otherwise protect \( v_{x_{(t/2)}}^- \). If \( t \) is even then protect the guard-vertex \( g_{(t/2)} \) if \( v_{x_{(t/2)}} \) has been protected, otherwise protect \( \overline{g}_{(t/2)} \). At the end of time step \( p + 1 \), the number of burned vertices is exactly \( p + \sum_{i=1}^{n} (3 \cdot 2(n - i) + 1) = p + 3n + n^2 \). Moreover, the literal-vertices that are burned in \( T \) correspond to the true literals in \( \phi' \). Thus, by construction and since \( \tau \) satisfies \( \phi' \), the vertices adjacent to a burning vertex are exactly one guard-vertex \( g_{j_1}^{c_j} \), two dummy vertices \( d_{j_1}^{c_j}, d_{j_1}^{\overline{c}_j} \) and \( 3n - 1 \) other vertices where \( x_a \vee x_b \vee x_c \) is the first clause, \( a, b, c \in \{1, \ldots, n\} \). At step \( p + 2 \), we must protect the guard vertex \( g_{j_1}^{c_j} \). During the steps \( p + 3 \) and \( p + 4 \), the strategy must protect one vertex lying on the path \( D_{j_1}^{c_j} \) and \( D_{j_1}^{\overline{c}_j} \), respectively. Thus \( 3(3n - 3) + 5 = 9n - 4 \) more vertices are burned at the end of step \( p + 4 \). More generally, from time step \( p + 3(j - 1) + 2 \) to \( p + 3(j - 1) + 4 \), for some \( j \in \{1, \ldots, m\} \), the strategy \( \Phi_\tau \) must protect a guard-vertex \( g_{j_1}^{c_j} \) and one vertex of each path \( D_{j_1}^{c_j} \) and \( D_{j_1}^{\overline{c}_j} \), where \( x_a, x_b, x_c \) appear in the clause \( c_j \). Thus \( 9(n - (j - 1)) - 4 \) vertices get burned. It follows that the number of burned vertices from step \( p + 2 \) to \( p + 3m + 1 \) is \( \sum_{j=1}^{m} [9(n - (j - 1)) - 4] = \frac{3}{2}m(m + 1) - 4m \). Putting all together, we arrive at a total of \( p + 3n + n^2 + \frac{9}{2}m(m + 1) - 4m = p + \frac{3}{2}(11n + 7) = k \) burned vertices.

\[ \Leftarrow \] Conversely, assume that there is no satisfying assignment for \( \phi' \). Observe first that any strategy \( \Phi \) for \( I' \) protects either \( v_{x_i} \) or \( v_{x_i}^- \) for each \( i \in \{1, \ldots, n\} \). As a contradiction, suppose that there exists \( i \in \{1, \ldots, n\} \) such that \( \Phi \) does not protect \( v_{x_i} \) and \( v_{x_i}^- \). Then in some time step both \( v_{x_i} \) and \( v_{x_i}^- \) get burned. Hence, it is not possible to protect both \( g_i \) and \( \overline{g}_i \) and at least one will burn implying that more than \( k \) vertices would burn, a contradiction. Furthermore, \( v_{x_i} \) and \( v_{x_i}^- \) cannot be both protected otherwise we would have protected a vertex not adjacent to a burned vertex at some step. Now consider
the situation at the end of step $p+1$. By the previous observation, the literal-vertices that are burned in $T$ can be interpreted as being the literals in $\phi'$ set to true. As previously, the number of burned vertices so far is exactly $p + \sum_{i=1}^{n}(3+2(n-i)+1) = p + 3n + n^2$. Let $n_g$ and $n_d$ be the number of guard-vertices and dummy-vertices adjacent to a burned vertex, respectively. As it follows from the previous construction, we know that $n_g = 3 - n_d$ with $0 \leq n_g \leq 3$ and $0 \leq n_d \leq 3$. We have the following possible cases:

1. $n_g > 1$. In this case, a guard-vertex gets burned and hence more than $k$ vertices would burn.

2. $n_g = 1$. Let $g_1^k$ be that guard-vertex and let $d^x_0, d^x_1$ be the $n_d = 3 - n_g = 2$ dummy-vertices where $x_0, x_1, x_2$ are variables of the first clause. At time step $i = p + 2$, we must protect $g_1^k$. Furthermore, during the step $i = p + 3$ (resp. $i = p + 4$), any strategy must protect a vertex lying on the path $D_1^k$ (resp. $D_1^k$). Indeed, if a strategy does otherwise then at least one guard-vertex $g_1^k$ or $g_1^k$ gets burned. Thus 2 dummy-vertices are burned.

3. $n_g = 0$. Hence we have exactly $n_d = 3 - n_g = 3$ dummy-vertices $d^x_0, d^x_1, d^x_2$ adjacent to burned vertices. Using a similar argument as before, we know that during the step $i = p + 2$ (resp. $i = p + 3$, $i = p + 4$), a strategy must protect a vertex lying on the path $D_1^k$ (resp. $D_1^k$, $D_1^k$). Thus 3 dummy-vertices are burned.

Notice that at step $p + 5$, we end up with a similar situation as in step $p + 2$. Now consider an assignment for $\phi'$. Since $\phi'$ is not satisfiable, therefore $\phi$ is not satisfiable as well. There are two possibilities:

- There exists a clause $c_j$ in $\phi$ with more than one true literal. Thus, we end up with case (1) and there is no strategy for $I'$ such that at most $k$ vertices are burned.

- There is a clause $c_j$ in $\phi$ with only false literals. This corresponds to the case (3) and the number of burned vertices would be at least $1 + p + \frac{2}{3}(11n + 7)$ (at least one extra dummy-vertex gets burned) giving us a total of at least $k + 1$ burned vertices. Hence there is no strategy for $I'$ where at most $k$ vertices are burned.

It remains to prove that the pathwidth of $T$ is at most three. To see this, observe that any subtree rooted at $v_{x_1}$ or $v_{x_2}$ has pathwidth two. Let $P_{x_1}$ and $P_{x_2}$ be the paths of the path-decompositions of these subtrees, respectively. We construct the path-decomposition for $T$ as follows. For every $i \in \{1, \ldots, n - 1\}$, define the node $B_i = \{u_{2i-1}, u_{2i}, u_{2i+1}\}$. Extend all nodes of the paths $P_{x_1}$ and $P_{x_2}$ to $P'_{x_1}$ and $P'_{x_2}$ by adding the vertex $u_{2i-1}$ inside it. Finally, connect the paths $P'_{x_1}$, $P'_{x_2}$ and the node $B_i$ to form a path and continue in this way with $P'_{x_2}, B_2$, $P'_{x_3}, P'_{x_2}, B_3$, $\ldots$, $B_{n-1}, P'_{x_n}, P'_{x_1}$. This completes the proof.

We can generalize the previous result to any fixed budget $b \geq 1$ as follows.

**Corollary 1.** For any fixed budget $b \geq 1$, the Firefighter problem is NP-complete even on trees of pathwidth three.

**Proof.** We start from the reduction of Theorem 1 and alter the tree $T$ as follows. Let $w_1$ be the vertex $s$ (corresponds also to $u_1$). Add a path $\{w_1w_2, w_2w_3, \ldots, w_{5n}w_{5n+1}\}$ to $T$. 

together with $b - 1$ guard-vertices added to each $w_i$. First, one can easily check that the pathwidth remains unchanged since the added component has pathwidth two and is only connected to the root $s$. Second, it can be seen that at each time step, only one firefighter can be placed “freely” as the other $b - 1$ firefighters must protect $b - 1$ guard-vertices. It follows that we end up to a similar proof as for Theorem 1. This completes the proof. □

4. Path-like graphs of bounded degree

As previously shown, for any fixed budget $b \geq 1$, the Firefighter problem is NP-complete on trees of bounded degree $b + 3$ [11, 3] and on trees of bounded pathwidth three (Theorem 1). It is thus natural to ask for the complexity of the problem when both the degree and the pathwidth of the input graph are bounded. In what follows, we answer this question positively. A first step toward this goal is to use the following combinatorial characterization of the number of burned vertices in a graph.

**Theorem 2.** Consider a graph of pathwidth $pw$ and maximum degree $\Delta$.
If the number of initially burned vertices is bounded by $f_1(pw, \Delta)$ for some function $f_1$ then there exists a protection strategy such that at most $f_2(pw, \Delta) \geq f_1(pw, \Delta)$ vertices are burned for some function $f_2$. 

Figure 1: An example of part of a tree constructed from the formula $\phi = (x_1 \lor x_3 \lor x_6) \land (x_1 \lor x_3 \lor x_2) \land (x_3 \lor x_4 \lor x_5) \land (x_1 \lor x_4 \lor x_6) \land (x_2 \lor x_4 \lor x_5)$. Guard vertices are represented by a dot within a circle.
Proof. First we prove the following claim: Consider a graph of cutwidth \(cw\). If the number of initially burned vertices is bounded by \(g_1(cw)\) for some function \(g_1\) then there exists a protection strategy such that at most \(g_2(cw)\) vertices are burned for some function \(g_2\). We will prove this by induction on \(cw\).

The claim is obviously true when the cutwidth is 0 since the graph cannot contain any edge. Suppose now that the claim is true for any graph of cutwidth at most \(k\), \(k > 0\). We show that it also holds for a graph of cutwidth \(k + 1\). Let \(H = (V, E)\) be such a graph and \(F \subseteq V\) be the set of initially burned vertices with \(|F| \leq g_1(cw(H))\) for some function \(g_1\). Consider a linear layout \(L = (v_1, \ldots, v_n)\) of \(H\) such that for every \(i = 1, \ldots, n - 1\), there are at most \(k + 1\) edges between \(\{v_1, \ldots, v_i\}\) and \(\{v_{i+1}, \ldots, v_n\}\). For every \(s \in F\) and \(i \geq 0\), we define inductively the following sets, where \(R_i(s) = L_0(s) = \{s\}\):

\[
R_i(s) = \begin{cases} 
\{s = v_k, v_{k+1}, \ldots, v_{k'}\} & \text{if } \exists v_{k'} \in N^i(s) : v_{k'} = \arg\max_{v \in N^i(s)} d_L(s, v) \\
R_{i-1}(s) & \text{otherwise}
\end{cases}
\]

\[
L_i(s) = \begin{cases} 
\{s = v_k, v_{k-1}, \ldots, v_{k'}\} & \text{if } \exists v_{k'} \in N^i(s) : v_{k'} = \arg\min_{v \in N^i(s)} d_L(s, v) \\
L_{i-1}(s) & \text{otherwise}
\end{cases}
\]

We are now in position to define the set \(B_i(s)\), called a bubble, by \(B_i(s) = L_i(s) \cup R_i(s)\) for all \(i \geq 0\). Informally speaking, the bubble \(B_i(s)\) corresponds to the effect zone of \(s\) after \(i\) steps of propagation i.e. every burned vertex inside the bubble is due to the vertex \(s\). The idea of the proof is to show that every bubble can be “isolated” from the rest of the graph in a bounded number of steps by surrounding it with firefighters (see Figure 2). We then show that the inductive hypothesis can be applied on each bubble which will prove the theorem.

Let \(s_1, s_2 \in F\). We say that two bubbles \(B_i(s_1)\) and \(B_j(s_2)\) for some \(i, j \geq 0\) overlap if there exists an edge \(uv \in E\) with \(u \in B_i(s_1)\) and \(v \in B_j(s_2)\). In this case, we can merge two bubbles into one i.e. we create a new bubble which is the union of \(B_i(s_1)\) and \(B_j(s_2)\).

Let us consider an initially burned vertex \(s \in F\) and its bubble \(B_{2\cdot cw(H)}(s)\). First, merge \(B_{2\cdot cw(H)}(s)\) with every other bubble \(B_{2\cdot cw(H)}(s')\) with \(s' \in F\) that possibly overlap into a new one \(B'_{2\cdot cw(H)}(s)\). By definition, we know that the number of edges with an endpoint in \(B'_{2\cdot cw(H)}(s)\) and the other one in \(V \setminus B'_{2\cdot cw(H)}(s)\) is less or equal to \(2 \cdot cw(H)\). Thus, we define the strategy that consists in protecting one vertex \(v \in V \setminus B_{2\cdot cw(H)}(s)\) adjacent to a vertex in \(B_{2\cdot cw(H)}(s)\) at each step \(t = 1, \ldots, 2 \cdot cw(H)\). Let \(F'\) be the set of vertices burned at step \(2 \cdot cw(H)\). Since \(\Delta(H) \leq 2 \cdot cw(H)\), we deduce that \(|F'|\) is less or equal to \(|F| \cdot \Delta(H)^{2\cdot cw(H)} \leq g_1(cw(H)) \cdot (2 \cdot cw(H))^{2\cdot cw(H)}\) hence bounded by a function of \(cw(H)\). Let us consider the subgraph \(H' = H[B'_{2\cdot cw(H)}(s)]\). Observe that we can safely remove every edge \(uv\) from \(H'\) for which \(u, v \in F'\). Indeed, such edge cannot have any influence during the subsequent steps of propagation. By the definition of a bubble, this implies that the cutwidth of \(H'\) is decreased by one and thus is now at most \(k\). Therefore, we can apply our inductive hypothesis to \(H'\) which tells us that there is a strategy for \(H'\) such that at most \(g_2(cw(H'))\) vertices are burned for some function \(g_2\). By Lemma 1, this strategy uses at most \(g_2'(cw(H'))\) steps to be applied. It follows that the number of burned vertices in \(H\) after applying this strategy is at
most the number of burned vertices from step 1 to the step $2 \cdot \text{cw}(H) + g'_2(\text{cw}(H'))$ which is $|F| \cdot \Delta(H)^2 \cdot \text{cw}(H) + g'_2(\text{cw}(H')) \leq g_1(\text{cw}(H)) \cdot (2 \cdot \text{cw}(H))^2 \cdot \text{cw}(H) + g'_2(\text{cw}(H'))$ which is bounded by a function of $\text{cw}(H)$. From now on, one can see that the previous argument can be applied iteratively to each bubble. Since the number of bubbles is bounded by $g_1(\text{cw}(H))$ (there is at most one bubble for each vertex initially on fire), we deduce that the total number of burned vertices is bounded by $g_2(\text{cw}(H))$ some function $g_2$. This concludes the proof of the claim.

We are now in position to prove the theorem. Let $G$ be a graph. Suppose that the number of initially burned vertices in $G$ is at most $f_1(\text{pw}(G), \Delta(G))$ for some function $f_1$. We know that $\text{pw}(G) \leq \text{cw}(G)$ and $\Delta(G) \leq 2 \cdot \text{cw}(G)$ [18]. Thus the number of burned vertices is at most $f'_1(\text{cw}(G))$ for some function $f'_1$. From the above claim we deduce that there exists a strategy such that at most $f'_2(\text{cw}(G))$ vertices get burned. Since $\text{cw}(G) \leq \text{pw}(G) \cdot \Delta(G)$ [6], it follows that the number of burned vertices is bounded by $f_2(\text{pw}(G), \Delta(G))$ for some function $f_2$. This completes the proof. □

Remark 1. Notice that Theorem 2 is still valid even if the number of firefighters available at each step is not the same (for example if there are $b_1$ firefighters at time step one, $b_2$ firefighters during the second time step, etc.).

In [2] the authors proved that the Firefighter problem is fixed-parameter tractable with respect to the combined parameter $k$ and the budget $b$. Let $(G, s, b, k)$ be an instance of Firefighter where $G$ has maximum degree $\Delta$. We can derive the following algorithm: If $b \geq \Delta$ then protect all the vertices in $N(s)$ at time step one; otherwise, apply the algorithm from [2]. We then easily obtain the following

Theorem 3. The Firefighter problem is fixed-parameter tractable with respect to the combined parameter $k$ and “maximum degree” of the input graph.

We are now in position to give the main result of this section.

Theorem 4. The Firefighter problem is fixed-parameter tractable with respect to the combined parameter “pathwidth” and “maximum degree” of the input graph.

Proof. Let $(G, s, b, k)$ be an instance of Firefighter where $G$ has maximum degree $\Delta$ and pathwidth $\text{pw}$. We design the following algorithm. For each value $k' = 1, \ldots, k$ run the $f(k', \Delta) \cdot n^{O(1)}$-time algorithm of Theorem 3: If the algorithm returns “yes” then return “yes”. If the algorithm has returned the answer “no” for all $k' = 1, \ldots, k$ then return “no”.

Using Theorem 2, we know that there exists a function $f'$ such that if $k' \geq f'(\text{pw}, \Delta)$ the algorithm will necessarily returns “yes” and stops. It follows that the algorithm is
called at most \( f'(pw, \Delta) \) times. The overall running time is then bounded by

\[
O(f'(pw, \Delta) \cdot f(k', \Delta) \cdot n^{O(1)}) = O(f'(pw, \Delta) \cdot f'(pw, \Delta) \cdot n^{O(1)}) = f''(pw, \Delta) \cdot n^{O(1)}
\]

for some function \( f'' \). This completes the proof.

From the proof of Theorem 2 and the fact that \( cw(G) \leq \frac{bw(G)(bw(G)+1)}{2} \) [4] for any graph \( G \), we easily deduce the following corollary.

**Corollary 2.** The Firefighter problem is fixed-parameter tractable with respect to the parameters “cutwidth” and “bandwidth”.

5. Conclusion

In this paper we showed that the Firefighter problem is \( \mathbf{NP} \)-complete even on trees of pathwidth three but fixed-parameter tractable with respect to the combined parameter “pathwidth” and “maximum degree” of the input graph. The combination of these two results with the \( \mathbf{NP} \)-completeness of the problem on trees of bounded degree [11] indicates that the complexity of the problem depends heavily on the degree and the pathwidth of the graph. We left as an open question whether the problem is polynomial-time solvable on graphs of pathwidth two.

References

[1] E. Anshelevich, D. Chakrabarty, A. Hate, and C. Swamy. Approximability of the firefighter problem. *Algorithmica*, 62(1–2):520–536, 2012.

[2] C. Bazgan, M. Chopin, and M. Fellows. Parameterized complexity of the firefighter problem. In *Proceedings of the 22nd International Symposium on Algorithms and Computation (ISAAC’11)*, LNCS 7074, pages 643–652, 2011.

[3] C. Bazgan, M. Chopin, and B. Ries. The firefighter problem with more than one firefighter on trees. *Discrete Applied Mathematics*, 161(7-8):899 – 908, 2013.

[4] H. L. Bodlaender. Classes of graphs with bounded tree-width. *Bulletin of EATCS*, (116-128), 1988.

[5] L. Cai, E. Verbin, and L. Yang. Firefighting on trees: (1 - 1/e)-approximation, fixed parameter tractability and a subexponential algorithm. In *Proceedings of the 19th International Symposium on Algorithms and Computation (ISAAC ’08)*, LNCS 5369, pages 258–269, 2008.

[6] F. R. Chung and P. D. Seymour. Graphs with small bandwidth and cutwidth. In *Graph Theory and combinatorics 1988 Proceedings of the Cambridge Combinatorial Conference in Honour of Paul Erdos*, volume 43 of *Annals of Discrete Mathematics*, pages 113–119, 1989.

[7] V. Costa, S. Dantas, M. C. Dourado, L. Penso, and D. Rautenbach. More fires and more fighters. *Discrete Applied Mathematics*, 155(17):2257–2268, 2007.

[8] M. Cygan, F. Fomin, and E. J. van Leeuwen. Parameterized complexity of firefighting revisited. In *Proceedings of the 6th International Symposium on Algorithms and Computation (IPEC’11)*, LNCS 7112, pages 13–26, 2012.

[9] M. Develin and S. G. Hartke. Fire containment in grids of dimension three and higher. *Discrete Applied Mathematics*, 155(17):2257–2268, 2007.

[10] R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Springer, 1999.

[11] S. Finbow, A. King, G. MacGillivray, and R. Rizzi. The firefighter problem for graphs of maximum degree three. *Discrete Mathematics*, 307(16):2094–2105, 2007.

[12] F. Fomin, P. Heggernes, and E. J. van Leeuwen. Making life easier for firefighters. In *Proceedings of the 6th International conference on Fun with Algorithms (FUN ’12)*, LNCS 7288, pages 177–188, 2012.
[13] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, 1979.

[14] B. Hartnell. Firefighter! an application of domination, Presentation. In *10th Conference on Numerical Mathematics and Computing, University of Manitoba in Winnipeg, Canada*, 1995.

[15] B. Hartnell and Q. Li. Firefighting on trees: how bad is the greedy algorithm? *Congressus Numerantium*, 145:187–192, 2000.

[16] Y. Iwaikawa, N. Kamiyama, and T. Matsui. Improved approximation algorithms for firefighter problem on trees. *IEICE Transactions on Information and Systems*, E94.D(2):196–199, 2011.

[17] A. King and G. MacGillivray. The firefighter problem for cubic graphs. *Discrete Mathematics*, 310(3):614–621, 2010.

[18] E. Korach and N. Solel. Tree-width, path-width, and cutwidth. *Discrete Applied Mathematics*, 43(1):97–101, 1993.

[19] G. MacGillivray and P. Wang. On the firefighter problem. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 47:83–96, 2003.

[20] K. L. Ng and P. Raff. A generalization of the firefighter problem on ZxZ. *Discrete Applied Mathematics*, 156(5):730–745, 2008.

[21] R. Niedermeier. *Invitation to Fixed-Parameter Algorithms*. Number 31 in Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, 2006.