Relativistic effects and angular dependence in the reaction $\bar{p}p \to \pi^-\pi^+$

B. El-Bennich and W.M. Kloet
Department of Physics and Astronomy, Rutgers University,
136 Frelinghuysen Road, Piscataway, New Jersey 08854-8019, USA
(Dated: March 30, 2022)

We present a new fit to the LEAR data on $pp \to \pi^-\pi^+$ differential cross sections and analyzing powers motivated by relativistic considerations. Within a quark model describing this annihilation we argue, since the pions are highly energetic, that relativistic effects cannot be neglected. The intrinsic pion wave functions are Lorentz transformed to the center of mass frame. This change in quark geometry gives rise to additional angular dependence in the transition operators and results in a relative enhancement of higher $J \geq 2$ partial wave amplitudes. The fit to the data is improved significantly.

PACS numbers: 12.39.Jh, 13.75.Cs, 21.30.Fe, 25.43.+t

I. INTRODUCTION

We recently studied effects of particle size and final-state interactions in the reaction $\bar{p}p \to \pi^-\pi^+$ within the framework of a constituent quark model [1]. The aim was to improve previous attempts [2, 3] to describe the LEAR framework of a constituent quark model [1]. The aim was as those on asymmetries point to a significant improvement on differential cross sections as well as final-state interactions, improves the quality of the fit in Ref. [1] dramatically – the forward and backward peaks of $d\sigma/d\Omega$ are very well reproduced and so are the characteristic double-dip structures of $A_{0n}$. It should come as no surprise that the relative contribution of the higher $J \geq 2$ amplitudes in Ref. [1] turns out to be significantly larger than in Ref. [2].

In this paper, we report an alternative approach that also leads to enhancement of the higher partial waves. It addresses the relativistic effects due to Lorentz transformed intrinsic pion wave functions in the reaction $\bar{p}p \to \pi^-\pi^+$. The reason for this rather different approach stems from the observation that at the center-of-mass (c.m.) energies $\sqrt{s}$ considered in the LEAR experiment $\bar{p}p \to \pi^-\pi^+$, the produced pions are highly energetic. The relativistic factor in this energy range is $\gamma = E_{cm}/2m_\pi \approx 6.8 - 8.0$, which means the pions are ejected at speeds $v \approx 0.98c$. One therefore expects considerable relativistic effects due to their modified internal structure. In the c.m. frame in which the transition amplitudes are calculated, the intrinsic pion wave functions employed must be Lorentz transformed from the pion rest frame. In our model [1, 2, 3] the intrinsic pion as well as the proton and antiproton wave functions are of Gaussian form in their respective rest frames. We neglect the distortions of the proton and antiproton intrinsic wave function due to their much smaller kinetic energy and larger mass.

The relativistic effects have several consequences. First, the intrinsic geometry of the pions is modified — instead of spherical particles one deals now with highly flattened ellipsoids in the c.m. frame. Obviously, the reaction geometry is altered also since the boosted pion

*Electronic address: bennich@physics.rutgers.edu*
wave functions enter the computation of the transition operators, which are obtained from an overlap integral of initial antiproton-proton, final pion-pion wave functions and the $^3S_1$ or $^3P_0$ annihilation mechanism. Details of this calculation were presented in Ref. [1]. Secondly, due to less overlap of the pion and antiproton-proton intrinsic wave functions, the annihilation range actually decreases but this effect is far from isotropic. As a result the transition operators exhibit significant additional angle dependence. Finally, the annihilation amplitudes, already non-local in the case of non-relativistic wave functions, become explicitly dependent on the c.m. energy $\sqrt{s}$ via the boost factor $\gamma$. In Ref. [1] it was shown that the relativistic transformation of the spatial part of the intrinsic pion wave functions introduces new angle dependent terms in the transition operators. These new terms also depend on the boost factor $\gamma$. In this paper, we present a new fit to the LEAR data on $\bar{p}p \rightarrow \pi^-\pi^+$ using the $R^2$ transition operators of Ref. [1], which supports our claim that relativistic considerations lead to a strongly modified angular momentum content of the helicity amplitudes for $0 \leq J \leq 4$. Especially the amplitudes for $J \geq 2$ are amplified considerably. Even though we keep the original particle radii as in [2], i.e. smaller than the corresponding charge distribution radii, we achieve a very good reproduction of the LEAR observables $d\sigma/d\Omega$ and $A_{0n}$, comparable to the one in Ref. [1].

II. RELATIVISTIC MODIFICATIONS

We summarize the effects of Lorentz transforming intrinsic pion wave functions on $\bar{p}p \rightarrow \pi^-\pi^+$ annihilation operators within the constituent quark model. The actual details may be found in Ref. [1]. In the pion rest frame, the radial part of its intrinsic wave function is described by a Gaussian of the form

$$\psi_\pi(r_i, r_j) = N_\pi \exp \left\{ -\beta \sum_{\alpha=i,j} (r_\alpha - R_\pi)^2 \right\},$$  \hspace{1cm} (1)

which reads in the c.m. frame

$$\psi_\pi(l^{-1}r_i, l^{-1}r_j) = \tilde{N}_\pi \exp \left\{ -\frac{\beta}{2} \sum_{\alpha=i,j} \left[ (r_\alpha - R_\pi)^2 + \gamma^2 (r_\alpha - R_\pi)^2 \right] \right\}. $$  \hspace{1cm} (2)

The vectors $r_i$ and $r_j$ are the quark and antiquark coordinates, $R_\pi = \frac{1}{2}(r_i + r_j)$ is the pion coordinate, and $\beta$ is the size parameter. As mentioned above, in previous work [1] we varied the value of $\beta$ in order to obtain an increase in the annihilation range. In this paper we use throughout the fixed value $\beta = 3.23 \text{ fm}^{-2}$, which corresponds to a pion radius of $0.48 \text{ fm}$ [2, 3, 16]. The new normalization factor $\tilde{N}_\pi = \sqrt{7} N_\pi$ is due to the condition that $\psi_\pi$ be normalized to unity in the c.m.

The c.m. wave functions of Eq. (2) are used to compute the transition operators for the $^3P_0$ and $^3S_1$ annihilation amplitudes. It is shown in Ref. [1] that these operators acquire new terms if compared with the non-relativistic expressions [2] and which manifestly introduce additional angular dependence. We here briefly recapitulate the results for the $R^2$ diagrams from [1]. The complete form of the $R^2$ transition operators for the vacuum $^3P_0$ amplitude is

$$\hat{T}(^3P_0) = iN \left[ A_V \sigma \cdot R' \sinh(C R \cdot R') + B_V \sigma \cdot R \cosh(C R \cdot R') + C_V (\sigma \cdot \hat{R}') R \cos \theta \cosh(C R \cdot R') \right] \times \exp\{AR^2 + BR^2 + DR^2 \cos^2 \theta \}. $$  \hspace{1cm} (3)

The total $^3S_1$ amplitude for the longitudinal component is given by

$$\hat{T}(^3S_1^L) = iN \left[ A_L \sigma \cdot R' \sinh(C R \cdot R') + B_L \sigma \cdot R \cosh(C R \cdot R') + C_L (\sigma \cdot \hat{R}') R \cos \theta \cosh(C R \cdot R') \right] \times \exp\{AR^2 + BR^2 + DR^2 \cos^2 \theta \} $$  \hspace{1cm} (4)

and

$$\hat{T}(^3S_1^T) = N \left[ A_T \sigma \cdot R' \cosh(C R \cdot R') + B_T \sigma \cdot R \sinh(C R \cdot R') + C_T (\sigma \cdot \hat{R}') R \cos \theta \sinh(C R \cdot R') \right] \times \exp\{AR^2 + BR^2 + DR^2 \cos^2 \theta \} $$  \hspace{1cm} (5)

for the transversal component. The factor $N$ is an overall normalization and $R' = R_\pi - R_{\pi^+}$ and $R = R_\bar{p} - R_p$ are the relative $\pi^-\pi^+$ and antiproton-proton coordinates, respectively. The angle $\theta$ is between $R'$ and $R$ in the c.m. frame. In the experiment it is the c.m. angle between the antiproton beam direction and the outgoing $\pi^-$ direction.
The new terms mentioned previously are \( C_V, C_L, C_T, \) and \( D. \) The selection rules for the \( R2 \) diagrams discussed in Ref. 2 are unchanged. More precisely, \( \hat{T}(3P_0) \) and \( \hat{T}(3S_1^T) \) act in \( pp \) states with \( J^\pi = 0^+, 2^+, 4^+, \ldots \) waves while \( \hat{T}(3S_1^T) \) contributes only to \( J^\pi = 1^-, 3^-, 5^-, \ldots \) waves. The explicit expressions for the coefficients of Eqs. 3–5 are (with \( \alpha \) being the proton size parameter defined similarly to \( \beta \) in Eq. 1):

\[
A = -\frac{\alpha(5\alpha + 4\beta \gamma^2)}{2(4\alpha + 3\beta \gamma^2)} \tag{6}
\]

\[
B = -\frac{3(7\alpha^2 + 18\alpha \beta \gamma^2 + 9\beta^2 \gamma^4)}{8(4\alpha + 3\beta \gamma^2)} - D \tag{7}
\]

\[
C = -\frac{3\alpha(\alpha + 3\beta \gamma^2)}{2(4\alpha + 3\beta \gamma^2)} \tag{8}
\]

\[
D = -\frac{9\beta \gamma^2 - 1}{8} \left\{ 1 + \frac{\alpha^2}{(5\alpha + 4\beta \gamma^2)(5\alpha + 4\beta \gamma^2)} \right\} \tag{9}
\]

\[
A_V = \frac{\alpha(\alpha + \beta \gamma^2)}{4\alpha + 3\beta \gamma^2} \tag{10}
\]

\[
B_V = \frac{3(\alpha + \beta \gamma^2)(5\alpha + 3\beta \gamma^2)}{2(4\alpha + 3\beta \gamma^2)} - C_V \tag{11}
\]

\[
C_V = \frac{3\beta \gamma^2 - 1}{2} \left\{ 1 + \frac{\alpha^2}{(5\alpha + 4\beta \gamma^2)(5\alpha + 4\beta \gamma^2)} \right\} \tag{12}
\]

\[
A_L = -A_V \tag{13}
\]

\[
B_L = \frac{9(\alpha + \beta \gamma^2)^2}{2(4\alpha + 3\beta \gamma^2)} - C_L \tag{14}
\]

\[
C_L = \frac{3\beta \gamma^2 - 1}{2} \left\{ 1 - \frac{\alpha^2}{(5\alpha + 4\beta \gamma^2)(5\alpha + 4\beta \gamma^2)} \right\} \tag{15}
\]

\[
A_T = -2A_V \tag{16}
\]

\[
B_T = \frac{3\alpha(\alpha + \beta \gamma^2)}{4\alpha + 3\beta \gamma^2} - C_T \tag{17}
\]

\[
C_T = \frac{3\beta \gamma^2(\gamma^2 - 1)}{(5\alpha + 4\beta)(5\alpha + 4\beta \gamma^2)} \tag{18}
\]

Clearly, \( C_V, C_L, C_T, \) and \( D \) vanish in the non-relativistic limit \( \gamma \to 1. \) A noticeable feature is that some of the new coefficients become relatively large for large \( \gamma \) values. In particular, the \( DR^2 \cos^2 \theta \) term dominates the exponential part of Eqs. 4–5 and, for example, \( C_V \gg A_V, B_V \) and \( C_L \gg A_L, B_L. \) On the other hand, one finds that \( C_T \approx A_T \approx B_T. \)

At this stage we will neglect additional relativistic corrections due to the (anti)quark spinors in the final state pions since it was argued in Ref. [15] that boosting of spin wave functions results in relatively smaller corrections.

### III. RESULTS

The LEAR data for the \( d\sigma/d\Omega \) and \( A_{0n} \) have been fitted using the transition operators in Eqs. 3–5, and 6. As before, taking only the \( R2 \) topology into account, we use a distorted wave approximation (DWA)

\[
T^J = \int dRdR' \phi^{J}_{\pi \pi}(R') \hat{T}_{R2}(R, R') \Psi_{\bar{p}p}^{J=J^\pm 1}(R) \tag{19}
\]

to obtain the scattering matrix elements and introduce final-state interactions as in Ref. 1. The pion wave function \( \phi^{J}_{\pi \pi}(R') \) is obtained from the coupled-channel model of Ref. 17 and can be parameterized with phases \( \delta_J \) and inelasticities \( \eta_J. \) The initial \( \bar{p}p \) wave
FIG. 2: As in Fig. 1 but for $T_{\text{lab}} = 123.5$ MeV ($p_{\text{lab}} = 497$ MeV/c).

FIG. 3: As in Fig. 1 but for $T_{\text{lab}} = 219.9$ MeV ($p_{\text{lab}} = 679$ MeV/c).

function $\Psi_{J,l}^{J\pm 1}(R)$ is taken from an optical potential model. Both $\phi_{\pi\pi}^J(R')$ and $\Psi_{pp}^{J\pm 1}(R)$ still contain the angle dependence appropriate for the values of $J$ and $l$.

We select from the specific energies where LEAR data are available the same set of five energies as in Ref. 1: $T_{\text{lab}} = 66.7, 123.5, 219.9, 499.2,$ and $803.1$ MeV corresponding to antiproton momenta of respectively $p_{\text{lab}} = 360, 497, 679, 1089,$ and $1467$ MeV/c. $T_{\text{lab}}$ is the laboratory kinetic energy of the antiproton beam. In fitting the data, we proceed as in Ref. 1. One starts with $\pi^-\pi^+$ final-state plane waves, i.e. with $\delta_J = 0$ and $\eta_J = 1$, and varies these parameters for $0 \leq J \leq 4$. The other parameters are the relative (complex) strength $\lambda$ as defined by

$$\hat{T}_{\text{tot}}(R', R) = N_0 \left[ \hat{T}(3P_0)(R', R) + \lambda \hat{T}(3S_1)(R', R) \right], \quad (20)$$

and the overall normalization $N_0$. The size parameters $\alpha$ and $\beta$ of the quark model are kept fixed at the original values $\alpha = 2.8$ fm$^{-2}$ and $\beta = 3.23$ fm$^{-2}$. The aim is to find out how much improvement is obtained solely from introducing final-state interactions.

At this initial stage, no relativistic effects have been included which implies $\gamma = 1$. The results of this fit are plotted in Figs. 1-5 as dashed lines. They will serve as reference point and were obtained previously in Ref. 1 where the model with and without final-state interactions was discussed. It was found that the improvement from
FIG. 4: As in Fig. 1 but for $T_{\text{lab}} = 499.2$ MeV ($p_{\text{lab}} = 1089$ MeV/c).

FIG. 5: As in Fig. 1 but for $T_{\text{lab}} = 803.1$ MeV ($p_{\text{lab}} = 1467$ MeV/c).

just final-state interactions is very modest — the double-dip structure in the analyzing power is somewhat more enhanced but the forward peaks in the differential cross sections at $T_{\text{lab}} = 66.7$ and 123.5 MeV ($p_{\text{lab}} = 360$ and 497 MeV/c) as well as the backward peaks at $T_{\text{lab}} = 499.2$ and 803.1 MeV ($p_{\text{lab}} = 1089$ and 1467 MeV/c) are poorly reproduced.

Next, we consider the effects of relativistic distortions of the intrinsic pion wave functions. Hence, the transition operators of Eqs. (3)–(5) must be used. The exponential part is dominated by the $D$ term (typically $D \simeq 200$ fm$^{-2}$, about two orders of magnitude larger than the $C$ term), which makes the partial wave amplitudes $T^J$ very sensitive to the value of the relative strength $\lambda = |\lambda| \exp(i\theta_\lambda)$ of the $^3S_1$ versus the $^3P_0$ mechanism and to the pion wave parameters $\eta_J$ and $\delta_J$. This is a consequence of the strong angle dependence of $\exp\{DR^2 \cos^2 \theta\}$. In order for the partial wave expansion of the scattering amplitude $T$ to converge, all ingredients must be fine-tuned, in particular the phases $\delta_J$ and inelasticities $\eta_J$ for $0 \leq J \leq 4$. It should be emphasized that $\alpha$ and $\beta$ remain fixed.

The improved new fit is illustrated in Figs. 4, 5, where the predictions that include relativistic effects and final-state interactions are presented as solid curves. The new fit is a significant improvement over the non-relativistic version of the $R2$ annihilation model. The differential cross sections $d\sigma/d\Omega$ are particularly well reproduced.
for the energies $T_{lab} = 66.7, 123.5, \text{ and } 219.9$ MeV ($p_{lab} = 360, 497, \text{ and } 679$ MeV/c). For $T_{lab} = 499.2$ and 803.1 MeV ($p_{lab} = 1089$ and 1467 MeV/c) the results for $d\sigma/d\Omega$ show improvement in forward and backward direction. At the two higher energies, the experimental double-well structure of the cross section is reproduced but some problems remain near $\theta = \pi/2$.

As for the analyzing powers $A_{0n}$, we are able to reproduce their double-dip structure, already present in the LEAR data at the lowest energies. When the experimental errors are small, like for instance for $T_{lab} = 219.9$ and 499.2 MeV, the fit is particularly successful. At lowest energy, $T_{lab} = 66.7$ MeV, the error bars in $A_{0n}$ are rather large and we do not fit very well certain data points. Overall the improvement is clearly important and of similar quality as the results of Ref. II which were obtained from increasing the particle radii.

In Tab. II we give the values of the quark model parameters as a function of $T_{lab}$, while the $\pi\pi$ phases $\delta_J$ and inelasticities $\eta_J$ are conveniently listed as a function of c.m. energy $\sqrt{s}$ in Tab. III. In the quark model parameters of Tab. III one notes that the relative strength $|\lambda|$ of the $^3S_1$ versus the $^3P_0$ mechanism varies with energy and the average value $|\lambda| \simeq 1.17$. There is no a priori reason why $\lambda$ should be energy independent. The energy variation of the inelasticities $\eta_J$ in the $\pi\pi$ interaction observed in Tab. III may be associated with the presence of $\pi\pi$ resonances, in particular with the $f_2$ at 2010 MeV in the $J = 2$ amplitude.

### IV. SUMMARY AND CONCLUSION

We present a new fit to the LEAR $pp \rightarrow \pi^-\pi^+$ data based on relativistic effects in the annihilation. These effects are due to Lorentz transformations of intrinsic pion wave functions from the pion rest frame to the c.m. frame of the reaction. The impact from the modified geometry of the pions is considerable and carries with it a much richer angle dependence of the transition operators.

The quality of this new fit is very similar to the one of a previous fit II. However, the approaches are very different. While in Ref. II we achieved a good fit by means of particle radii increase, so the antiquarks and quarks do overlap more, in this work we keep the original particle sizes fixed but take into account the relativistic distortions of the pions in the c.m. frame of the reaction.

We conclude that even though observed data at LEAR on $d\sigma/d\Omega$ and $A_{0n}$ indicate strongly the presence of higher partial wave amplitudes and therefore seem to call for an increase of the annihilation range, one can obtain a similar improvement by additional angle dependence in the transition operators. This additional angle dependence arises naturally when boosted intrinsic pion wave functions are employed in the quark model calculations, as becomes clear from the present fit. Relativistic effects in the reaction $pp \rightarrow \pi^-\pi^+$ therefore should not be ignored.

---

[1] B. El-Bennich, W.M. Kloet, and B. Loiseau, Phys. Rev. C 68, 014003 (2003).
[2] G. Bathas and W.M. Kloet, Phys. Lett. B301, 155 (1993).
[3] G. Bathas and W.M. Kloet, Phys. Rev. C 47, 2207 (1993).
[4] A. Hasan et al., Nucl. Phys. B378, 3 (1992).
[5] B. Moussallam, Nucl. Phys. A407, 413 (1983); ibid. A429, 429 (1984).
[6] V. Mull, J. Haidenbauer, T. Hippen, and K. Holinde, Phys. Rev. C 44, 1337 (1991).
[7] V. Mull, K. Holinde, and J. Speth, Phys. Lett. B 275, 12 (1992).
[8] A.M. Green and G.Q. Liu, Nucl. Phys A486, 581 (1988).
[9] A. Muhm, T. Gutsche, R. Thierauf, Y. Yan, and Amand Faessler, Nucl. Phys. A598, 265 (1996).
[10] Y. Yan and R. Tegen, Phys. Rev. C 54, 1441 (1996); Nucl. Phys A648, 89 (1999).
[11] M.N. Oakden and M.R. Pennington, Nucl. Phys. A574, 731 (1994).
[12] A. Hasan and D.V. Bugg, Phys. Lett. B 334, 215 (1994).
[13] W.M. Kloet and F. Myhrer, Phys. Rev. D 53, 6120 (1996).
[14] B.R. Martin and G.C Oades, Phys. Rev. C 56, 1114 (1997); *ibid.* 57, 3492 (1998).
[15] B. El-Bennich and W.M. Kloet. \texttt{arXiv:nucl-th/0403050}
[16] A.M. Green and J.A. Niskanen, Nucl. Phys. A412, 448 (1984); *ibid.* A430, 605 (1984); Mod. Phys. Lett A 1, 441 (1986).
[17] W.M. Kloet and B. Loiseau, Eur. Phys. J. A 1, 337 (1998).
[18] B. El-Bennich, M. Lacombe, B. Loiseau, and R. Vinh Mau, Phys. Rev. C 59, 2313 (1999).