Differential operators for Schur and Schubert polynomials

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Part 1: Schubert
Schubert Polynomials

The divided differences operators is given by

$$\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}.$$ 

**Definition**

For a permutation $w_0 = (n, n-1, \ldots, 1) \in S_n$, we define its Schubert polynomial as

$$\mathcal{S}_{w_0} = x_1^{n-1}x_2^{n-2} \cdots x_{n-1}^1 \in \mathbb{Q}[x_1, x_2, \ldots].$$

For a permutation $w \in S_n$,

$$\partial_i \mathcal{S}_w = \begin{cases} 
\mathcal{S}_{ws_i} & \text{if } \ell(ws_i) = \ell(w) - 1 \\
0 & \text{if } \ell(ws_i) = \ell(w) + 1.
\end{cases}$$
Definition
Given a reduced decomposition \( h = (h_1, h_2, \ldots, h_{\ell(w)}) \). Let \( C(h) \) be the set of all \( \ell(w) \)-tupels \((\alpha_1, \ldots, \alpha_{\ell(w)})\) of positive integers such that

- \( 1 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{\ell(w)}; \)
- \( \alpha_j \leq h_j; \)
- \( \alpha_j < \alpha_{j+1} \) if \( h_j < h_{j+1} \).

Theorem (Billey-Jockusch-Stanley, Fomin-Stanley)

For any permutation \( w \in S_N \), its Schubert polynomial is given by

\[
S_w = \sum_{h \in R(w)} \sum_{\alpha \in C(h)} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_{\ell}}.
\]
Proposition (Fomin-Kirillov)

For any permutation $w \in S_N$, its Schubert polynomial is given by

$$ S_w = \sum_{g \in RC(w)} m(g). $$
Theorem

There are unique constants $c_{u,v}^w$, $u, v, w \in S_N$ such that

$$S_u S_v = \sum_{w \in S_N} c_{u,v}^w S_w.$$  

Furthermore, $c_{u,v}^w$, $u, v, w \in S_N$ are non-negative integers.

Problem

Give a combinatorial interpretation of $c_{u,v}^w$.  

Operator $\nabla$

\[
\nabla := \sum_{i \in \mathbb{N}} \frac{\partial}{\partial x_i}
\]

**Theorem (Hamaker-Pechenik-Speyer-Weigandt)**

For any $u \in S_N$,

\[
\nabla S_u = \sum_{k \in \mathbb{N}: \ell(s_k u) = \ell(u) - 1} k S_{s_k u}.
\]
Stabilities

Let $\tau$ be a shift defined by

$$\tau w(i + 1) = w(i) + 1, \ i \in \mathbb{Z},$$

where $w \in S_{\mathbb{Z}}$ is a permutation of $\mathbb{Z}$ fixing all but finitely many elements.
Let \( \tau \) be a shift defined by

\[
\tau w(i + 1) = w(i) + 1, \quad i \in \mathbb{Z},
\]

where \( w \in S_{\mathbb{Z}} \) is a permutation of \( \mathbb{Z} \) fixing all but finitely many elements.

Stanley symmetric function for \( w \in S_{\mathbb{N}} \) is given by

\[
F_w(x_1, x_2, \ldots) := \lim_{k \to +\infty} S_{\tau^k w}(x_1, x_2, \ldots) \in \Lambda[x_i, i \geq 1].
\]
Stabilities

Let $\tau$ be a shift defined by

$$\tau w(i + 1) = w(i) + 1, \quad i \in \mathbb{Z},$$

where $w \in S_\mathbb{Z}$ is a permutation of $\mathbb{Z}$ fixing all but finitely many elements.

Stanley symmetric function for $w \in S_\mathbb{N}$ is given by

$$F_w(x_1, x_2, \ldots) := \lim_{k \to +\infty} \mathcal{S}_{\tau^k w}(x_1, x_2, \ldots) \in \Lambda[x_i, i \geq 1].$$

Back stable polynomial for $w \in S_\mathbb{Z}$ is given by

$$\mathcal{S}_w(x_i, i \in \mathbb{Z}) := \lim_{k \to +\infty} \mathcal{S}_{\tau^k w}(x_{1-k}, x_{2-k}, \ldots) \in \Lambda[x_i, i \leq 0] \oplus \mathbb{Q}[x_i, i \in \mathbb{Z}].$$
Theorem (Edelman-Greene)

\[ \mathcal{F}_w(x_1, x_2, \ldots) = a_{w, \lambda} s_\lambda(x_1, x_2, \ldots), \]

where \( a_{w, \lambda} \) are non-negative.
Theorem (Lam-Lee-Shimozono)

There are unique constants $c_{u,v}^w$, $u, v, w \in S_Z$ such that

$$\mathcal{G}_u \mathcal{G}_v = \sum_{w \in S_Z} c_{u,v}^w \mathcal{G}_w.$$
Theorem (Lam-Lee-Shimozono)

There are unique constants $c_{u,v}^w$, $u, v, w \in S_\mathbb{Z}$ such that

$$
\Sigma_u \Sigma_v = \sum_{w \in S_\mathbb{Z}} c_{u,v}^w \Sigma_w.
$$

Theorem

Given a pair of permutations $u, v \in S_\mathbb{Z}$, the following holds:

$$
\left( \ell(u) + \ell(v) \right) |\mathcal{R}(u)||\mathcal{R}(v)| = \sum_{w \in S_\mathbb{Z}} c_{u,v}^w |\mathcal{R}(w)|,
$$

where $\mathcal{R}(u)$ is the set of reduced words of $u$. 
Figure: Merge of reduced decompositions,

\[ \mathcal{G} (01324) \mathcal{G} (02314) = \mathcal{G} (12304) + \mathcal{G} (02413). \]
Operator $\xi$ on $s$ 

Define $\xi$ as

$$
\xi(f) := \sum_{\gamma \in \mathbb{Z}_{\geq 0}^\mathbb{Z}} (\lim_{k \to -\infty} \text{coef. of } x^\gamma x_k \text{ in } f) \cdot x^\gamma = \lim_{k \to -\infty} \frac{\partial f}{\partial x_k}.
$$

For Back stable Schubert polynomials, we have

$$
\xi_{\mathcal{S}} u = \sum_{k: \ell(s_k u) = \ell(u) - 1} \xi_{\mathcal{S}} s_k u.
$$
Operators $\xi$ and $\nabla$ on $\mathfrak{S}$

$\xi_{\mathfrak{S}} u := \sum_{k: \ell(s_k u) = \ell(u) - 1} \mathfrak{S}_{s_k u}$;

$\nabla_{\mathfrak{S}} u := \sum_{k: \ell(s_k u) = \ell(u) - 1} k \mathfrak{S}_{s_k u}$. 

Proposition (N.)

For any $u, v \in \mathfrak{S}$, we have $\xi_{\mathfrak{S}} (\nabla_{\mathfrak{S}} u) = (\xi_{\mathfrak{S}} u) \nabla_{\mathfrak{S}} v + \nabla_{\mathfrak{S}} u (\xi_{\mathfrak{S}} v)$. 

$\nabla_{\mathfrak{S}} (\nabla_{\mathfrak{S}} u) = (\nabla_{\mathfrak{S}} u) \nabla_{\mathfrak{S}} v + \nabla_{\mathfrak{S}} u (\nabla_{\mathfrak{S}} v)$. 

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Operators $\xi$ and $\nabla$ on $\mathfrak{S}$

\[
\xi \mathfrak{S}_u := \sum_{k: \ell(s_k u) = \ell(u) - 1} \mathfrak{S}_s k u;
\]

\[
\nabla \mathfrak{S}_u := \sum_{k: \ell(s_k u) = \ell(u) - 1} k \mathfrak{S}_s k u.
\]

Proposition (N.)

For any $u, v \in S_\mathbb{Z}$, we have

\[
\xi(\mathfrak{S}_u \mathfrak{S}_v) = (\xi \mathfrak{S}_u) \mathfrak{S}_v + \mathfrak{S}_u (\xi \mathfrak{S}_v);
\]

\[
\nabla(\mathfrak{S}_u \mathfrak{S}_v) = (\nabla \mathfrak{S}_u) \mathfrak{S}_v + \mathfrak{S}_u (\nabla \mathfrak{S}_v).
\]
Theorem (N.)

If an operator $\zeta$ satisfies:

1. $\zeta \overset{\leftarrow}{s} u = \sum_{k: \ell(s_k u) = \ell(u) - 1} b_{u,k} \overset{\leftarrow}{s} s_k u, \quad b_{u,k} \in \mathbb{Q}$;
2. $\zeta(\overset{\leftarrow}{s} u \overset{\leftarrow}{s} v) = (\zeta \overset{\leftarrow}{s} u) \overset{\leftarrow}{s} v + \overset{\leftarrow}{s} u(\zeta \overset{\leftarrow}{s} v)$,

then $\zeta$ is a linear combination of $\xi$ and $\nabla$. 
Define the vector space $\mathbb{Q}S_{\mathbb{Z}}$ as formal finite sums of permutations with rational coefficients, i.e.,

$$\mathbb{Q}S_{\mathbb{Z}} := \left\{ \sum_{i=1}^{k} a_i w^{(i)} : k \in \mathbb{N}, a_i \in \mathbb{Q}, w^{(i)} \in \mathbb{Q}S_{\mathbb{Z}} \right\}.$$
Main theorem (weak form)

A descent of \( u \in S_\mathbb{Z} \) is a position \( k \in \mathbb{Z} \) with \( u(k) > u(k + 1) \).

Theorem (N.)

Let \( f : \mathbb{Q}S_\mathbb{Z} \times \mathbb{Q}S_\mathbb{Z} \to \mathbb{Q}S_\mathbb{Z}, \ f(u, v) = \sum_{w \in \mathbb{Q}S_\mathbb{Z}} b^w_{u,v} w \) be a linear map, such that

1. \( b^w_{u,v} = 0 \) if \( \ell(w) \neq \ell(u) + \ell(v) \);
2. \( b^w_{u,v} = 0 \) if \( k \) is a descent of \( u \) and \( w(a) \leq k \) for all \( a \leq k \);
3. \( f(id, v) = v \);
4. \( \xi f(u, v) = f(\xi u, v) + f(u, \xi v) \);
5. \( \nabla f(u, v) = f(\nabla u, v) + f(u, \nabla v) \).

Then \( b^w_{u,v} = c^w_{u,v} \) for all \( u, v, w \in S_\mathbb{Z} \).
Main theorem (weak form; symmetric)

**Theorem (N.)**

Let \( f : \mathbb{Q}S_{\mathbb{Z}} \times \mathbb{Q}S_{\mathbb{Z}} \to \mathbb{Q}S_{\mathbb{Z}} \), \( f(u, v) = \sum_{w \in \mathbb{Q}S_{\mathbb{Z}}} b_{u,v}^w w \) be a linear map, such that

1. \( b_{u,v}^w = 0 \) if \( \ell(w) \neq \ell(u) + \ell(v) \);
2. \( b_{u,v}^w = 0 \) if \( k \) is a descent of \( u \) or \( v \) and \( w(a) \leq k \) for all \( a \leq k \);
3. \( f(id, id) = id \);
4. \( \xi f(u, v) = f(\xi u, v) + f(u, \xi v) \);
5. \( \nabla f(u, v) = f(\nabla u, v) + f(u, \nabla v) \).

Then \( b_{u,v}^w = c_{u,v}^w \) for all \( u, v, w \in S_{\mathbb{Z}} \).
Main theorem (weak form; positive)

Theorem (N.)

Let \( f : \mathbb{Q}S_{\mathbb{Z}} \times \mathbb{Q}S_{\mathbb{Z}} \to \mathbb{Q}S_{\mathbb{Z}} \), \( f(u, v) = \sum_{w \in \mathbb{Q}S_{\mathbb{Z}}} b_{u, v}^w w \) be a linear map, such that

1. \( b_{u, v}^w = 0 \) if \( \ell(w) \neq \ell(u) + \ell(v) \);
2. \( b_{u, v}^w \geq 0 \);
3. \( f(id, id) = id \);
4. \( \xi f(u, v) = f(\xi u, v) + f(u, \xi v) \);
5. \( \nabla f(u, v) = f(\nabla u, v) + f(u, \nabla v) \).

Then \( b_{u, v}^w = c_{u, v}^w \) for all \( u, v, w \in S_{\mathbb{Z}} \).

Remark

My proof of this theorem is different from proofs of the previous two theorems.
Define the sequence of \textit{bosonic} operators

\begin{itemize}
  \item $\rho^{(1)} := \xi$
  \item $\rho^{(k+1)} := \frac{[\rho^{(k)}, \nabla]}{k} = \frac{\rho^{(k)} \cdot \nabla - \nabla \cdot \rho^{(k)}}{k}$
\end{itemize}
Bosonic operators

Define the sequence of *bosonic* operators

- \( \rho^{(1)} := \xi \);
- \( \rho^{(k+1)} := \frac{[\rho^{(k)}, \nabla]}{k} = \frac{\rho^{(k)} \cdot \nabla - \nabla \cdot \rho^{(k)}}{k} \).

**Proposition**

For any \( k \in \mathbb{N} \) and \( u, v \in S_{\mathbb{Z}} \), we have

\[
\rho^{(k)}(\tilde{\mathcal{G}}_u \tilde{\mathcal{G}}_v) = (\rho^{(k)} \tilde{\mathcal{G}}_u) \tilde{\mathcal{G}}_v + \tilde{\mathcal{G}}_u (\rho^{(k)} \tilde{\mathcal{G}}_v).
\]
Bosonic operators

Define the sequence of *bosonic* operators

- $\rho^{(1)} := \xi$;
- $\rho^{(k+1)} := \frac{[\rho^{(k)}, \nabla]}{k} = \frac{\rho^{(k)} \cdot \nabla - \nabla \cdot \rho^{(k)}}{k}$.

**Proposition**

For any $k \in \mathbb{N}$ and $u, v \in S_{\mathbb{Z}}$, we have

$$\rho^{(k)}(\overleftarrow{G}_u \overleftarrow{G}_v) = (\rho^{(k)} \overleftarrow{G}_u) \overleftarrow{G}_v + \overleftarrow{G}_u (\rho^{(k)} \overleftarrow{G}_v).$$

**Theorem (N.)**

Operators $\rho^{(k)}$, $k \in \mathbb{N}$ commute pairwise.
For a partition $\lambda$ we define operator $\xi^\lambda$ as

$$\xi^\lambda := \sum_\mu \frac{\chi_\mu^\lambda}{z_\mu} \rho(\mu_1) \cdots \rho(\mu_k).$$

**Proposition**

For any $u, v \in S_\mathbb{Z}$ and $\lambda$,

$$\xi^\lambda(\xi^u \xi^v) = \sum_{\mu, \nu} c^\lambda_{\mu, \nu} (\xi^\mu \xi^u)(\xi^\nu \xi^v),$$

where $c^\lambda_{\mu, \nu}$ are Littlewood-Richardson coefficients.
Theorem (N.)

For a permutation $w$ and a partition $\lambda$, we have

$$\xi^\lambda \overleftarrow{G}_w = \sum_{\ell(u) = |\lambda|, \ell(u^{-1}w) = \ell(w) - |\lambda|} a_{\lambda,u} \overleftarrow{G}_{u^{-1}w},$$

where $a_{\lambda,u}$ are coefficients in the expressions of Stanley symmetric functions in terms of Schur functions.
Main theorem

Theorem (N.)
Let \( f : \mathbb{Q} S_{\mathbb{Z}} \times \mathbb{Q} S_{\mathbb{Z}} \to \mathbb{Q} S_{\mathbb{Z}} \), \( f(u, v) = \sum_{w \in \mathbb{Q} S_{\mathbb{Z}}} b_{u,v}^w w \) be a linear map, such that

1. \( b_{u,v}^w = 0 \) if \( \ell(w) \neq \ell(u) + \ell(v) \);
2. \( b_{u,v}^w = 0 \) if \( k \) is a descent of \( u \) and \( w(a) \leq k \) for all \( a \leq k \);
3. \( f(id, v) = v \);
4. for any \( d \in \mathbb{N} \), \( \xi^{(d)} f(u, v) = \sum_{i=0}^{d} f(\xi^{(i)} u, \xi^{(d-i)} v) \).

Then \( b_{u,v}^w = c_{u,v}^w \) for all \( u, v, w \in S_{\mathbb{Z}} \).

Remark
We can replace conditions (2) and (3) with symmetric or positive conditions.
Main theorem

Theorem (N.)

Let \( f : \mathbb{Q}S_{\mathbb{Z}} \times \mathbb{Q}S_{\mathbb{Z}} \to \mathbb{Q}S_{\mathbb{Z}} \), \( f(u, v) = \sum_{w \in \mathbb{Q}S_{\mathbb{Z}}} b_{u,v}^w w \) be a linear map, such that

1. \( b_{u,v}^w = 0 \) if \( \ell(w) \neq \ell(u) + \ell(v) \);
2. \( b_{u,v}^w = 0 \) if \( k \) is a descent of \( u \) and \( w(a) \leq k \) for all \( a \leq k \);
3. \( f(id, v) = v \);
4. for any \( d \in \mathbb{N} \), \( \rho^d f(u, v) = f(\rho^d u, v) + f(u, \rho^d v) \).

Then \( b_{u,v}^w = c_{u,v}^w \) for all \( u, v, w \in S_{\mathbb{Z}} \).

Remark

We can replace conditions (2) and (3) with symmetric or positive conditions.
Part 2: Schur
A permutation is a Grassmannian permutation if and only if it has at most one descent.

\[ \lambda(w) = (w_k - k, w_{k-1} - k + 1, w_{k-2} - k + 2, \ldots) . \]
A permutation is a Grassmannian permutation if and only if it has at most one descent.

\[ \lambda(w) = (w_k - k, w_{k-1} - k + 1, w_{k-2} - k + 2, \ldots) \].

A reduced decomposition of \((2571346) \in S_7\) and the corresponding Young diagram \((4, 3, 1)\).
A descent of $w \in S_\mathbb{Z}$ is a position $k \in \mathbb{Z}$ with $w(k) > w(k + 1)$.
A permutation is a Grassmannian permutation if and only if it has at most one descent.

$$\lambda(w) = (w_k - k, w_{k-1} - k + 1, w_{k-2} - k + 2, \ldots).$$

**Theorem**

$$\mathcal{G}_w = s_{\lambda(w)}(x_i, \ i \leq k).$$
We denote by $\mathcal{Y}$ the set of Young diagrams (partitions), i.e.,
\[ \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{Y} \text{ s.t. } \lambda_1 \geq \ldots \geq \lambda_k \geq 0, \lambda_i \in \mathbb{Z}_{\geq 0}. \]
For example,
\[ (4,3,1) = \begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
& & & \\
\hline
\end{array} \]
We denote by $\mathcal{Y}$ the set of Young diagrams (partitions), i.e.,
$\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{Y}$ s.t. $\lambda_1 \geq \ldots \geq \lambda_k \geq 0$, $\lambda_i \in \mathbb{Z}_{\geq 0}$. For example,

$$(4, 3, 1) = \begin{array}{|c|c|c|}
\hline
& & \\
\hline
& \\
\hline
& & \\
\hline
\end{array}$$

Define the vector space $\mathbb{Q}\mathcal{Y}$ as formal finite sums of Young diagrams with rational coefficients, i.e.,

$$\mathbb{Q}\mathcal{Y} := \left\{ \sum_{i=1}^{k} a_i \lambda^{(i)} : k \in \mathbb{N}, \ a_i \in \mathbb{Q}, \ \lambda^{(i)} \in \mathcal{Y} \right\}.$$
Define two linear “differential” operators on $\mathbb{Q}Y$. For a Young diagram $\lambda \in Y$, we have

$$\xi(\lambda) := \sum_{(i,j) \in \mathbb{N}^2} \lambda' ;$$

and

$$\nabla(\lambda) := \sum_{(i,j) \in \mathbb{N}^2} (j - i) \lambda'.$$

and

$$\lambda' = \lambda \setminus (i,j) \in Y.$$
Schur functions and operations

\[ \xi \left( \begin{array}{c}
\end{array} \right) = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \]

\[ \nabla \left( \begin{array}{c}
\end{array} \right) = 3 \begin{array}{c}
\end{array} + 1 \begin{array}{c}
\end{array} - 2 \begin{array}{c}
\end{array} \]
For the empty diagram, we have $\xi(\emptyset) = \nabla(\emptyset) = 0$, therefore we associate the empty diagram with 1.
Key Lemma

For the empty diagram, we have $\xi(\emptyset) = \nabla(\emptyset) = 0$, therefore we associate the empty diagram with 1.

**Lemma (N.)**

An element from $\mathbb{Q} \mathbb{Y}$ is constant if and only if both operators give zero, i.e.,

$$x \in \mathbb{Q} \iff \xi(x) = \nabla(x) = 0.$$
We say that a map $\star : \mathbb{Q} \mathcal{V}^2 \rightarrow \mathbb{Q} \mathcal{V}$ is a multiplication if

- $n, m \in \mathbb{N}$ and $x_i \in \mathbb{Q} \mathcal{V}_i$, $i \in [0, n]$, $y_j \in \mathbb{Q} \mathcal{V}_j$, $j \in [0, m]$,

\[(x_0 + \ldots + x_n) \star (y_0 + \ldots + y_m) = \sum_{0 \leq i \leq n, 0 \leq j \leq m} x_i \star y_j,
\]

where $x_i \star y_j \in \mathbb{Q} \mathcal{V}_{(i+j)}$;
- for $a, b \in \mathbb{Q}$, $a \star b = ab$;
- for any $x, y \in \mathbb{Q} \mathcal{V}$, $\xi(x \star y) = (\xi x) \star y + x \star (\xi y)$;
- for any $x, y \in \mathbb{Q} \mathcal{V}$, $\nabla(x \star y) = (\nabla x) \star y + x \star (\nabla y)$.
We say that a map $\star : \mathbb{Q}Y^2 \rightarrow \mathbb{Q}Y$ is a multiplication if

- $n, m \in \mathbb{N}$ and $x_i \in \mathbb{Q}Y_i$, $i \in [0, n]$, $y_j \in \mathbb{Q}Y_j$, $j \in [0, m]$,

\[
(x_0 + \ldots + x_n) \star (y_0 + \ldots + y_m) = \sum_{0 \leq i \leq n} \sum_{0 \leq j \leq m} x_i \star y_j,
\]

where $x_i \star y_j \in \mathbb{Q}Y_{i+j}$;

- for $a, b \in \mathbb{Q}$, $a \star b = ab$;

- for any $x, y \in \mathbb{Q}Y$, $\xi(x \star y) = (\xi x) \star y + x \star (\xi y)$;

- for any $x, y \in \mathbb{Q}Y$, $\nabla(x \star y) = (\nabla x) \star y + x \star (\nabla y)$.

**Corollary**

*There is at most one multiplication map.*
**Theorem (N.)**

There is a unique multiplication map. Furthermore, this map is linear and satisfies commutative and associative properties and it is given by

\[ \lambda \ast \mu = \sum_{\nu} c_{\lambda,\mu}^{\nu}, \]

where \(c_{\lambda,\mu}^{\nu}\) are Littlewood-Richardson coefficients.
Theorem (Jacobi-Trudi identity)

For a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k \geq 0)$, we have

$$s_\lambda = \det \begin{bmatrix}
h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \cdots & h_{\lambda_1+k-1} \\
h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \cdots & h_{\lambda_2+k-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & h_{\lambda_k-k+3} & \cdots & h_{\lambda_k}
\end{bmatrix}$$
We prove it by induction by $|\lambda| = \lambda_1 + \ldots + \lambda_k$.

Base case: $|\lambda| = 0$. We have $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$, therefore $s_\lambda = 1 = det_\lambda$.

Induction step. It is enough to check $\xi(s_\lambda - det_\lambda) = \nabla(s_\lambda - det_\lambda) = 0$. 
Proof: Induction step

$s_\lambda \overset{?}{=} \det_\lambda := \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \cdots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \cdots & h_{\lambda_2+k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & h_{\lambda_k-k+3} & \cdots & h_{\lambda_k} \end{bmatrix}$

We have

$\xi(h_{\lambda_i-i+j}) = h(\lambda_i-1)-i+j,$

then after combining by rows we get

$\xi(\det_\lambda) = \sum_{\lambda'=\lambda\setminus(i,\lambda_i)\in\mathcal{Y}} \det_{\lambda'} = \sum_{\lambda'=\lambda\setminus(i,j)\in\mathcal{Y}} s_{\lambda'} = \xi(s_\lambda).$
Proof; Induction step

We have

\[ \nabla (h_{\lambda_i - i + j}) = (\lambda_i - i + j - 1)h_{\lambda_i - i + j - 1} = \]
\[ = (\lambda_i - i)h_{(\lambda_{i-1}) - i + j} + (j - 1)h_{\lambda_i - i + (j-1)}, \]

then

\[ \nabla (\det_\lambda) = \sum_{\lambda' = \lambda \setminus (i, \lambda_i) \in \mathcal{Y}} (\lambda_i - i)\det_{\lambda'} = \sum_{\lambda' = \lambda \setminus (i, j) \in \mathcal{Y}} (j - i)s_{\lambda'} = \nabla (s_\lambda). \]
\[ \rho^{(1)} := \xi; \]
\[ \rho^{(k+1)} := \left[ \frac{\rho^{(k)}, \nabla}{k} \right] = \frac{\rho^{(k)} \cdot \nabla - \nabla \cdot \rho^{(k)}}{k}. \]

**Theorem (N.)**

\[
\rho^{(k)} \lambda = \sum_{\mu: \mu \subset \lambda, \ |\mu| = |\lambda| - k, \ \lambda \setminus \mu \text{ is a border strip}} (-1)^{ht(\lambda \setminus \mu)-1} s_{\mu}.
\]
\[ s_\lambda p_k = \sum_{\mu: \mu \subset \lambda, \ |\mu|=|\lambda|+k, \ \mu \setminus \lambda \ is \ a \ border \ strip} (-1)^{ht(\mu \setminus \lambda)-1} s_\mu. \]
Thank You!