Comments on the Tetrad (Vielbeins).

Takeshi FUKUYAMA

Department of Physics and R-GIRO, Ritsumeikan University,
Kusatsu, Shiga, 525-8577 Japan

Abstract

We want to correct the misunderstandings on the tetrad (or vielbeins in general) appeared in many text books or review articles. The tetrad should be defined without any condition. $e_{\mu a} = \partial_\mu X_a$ with local Lorentz coordinates $X_a$ is wrong in many senses: it gives the condition $\partial_\mu e_{\nu a} = \partial_\nu e_{\mu a}$, which leads us to the trivial result that the cyclic coefficients vanish identically and to the null Riemannian tensor. Also $e_{\mu a} e^a_{\nu} = g_{\mu \nu}$ is not scalar under the local Lorentz transformation etc. We show how these deficits are remedied by the correct definition, $e_{\mu a} = D_\mu Z_a$ with local (Anti) de Sitter coordinates $Z_A$.

From the gauge theoretical point of view [1], one of the most important characters of gravity is the soldering of internal space of gauge symmetry of gravity with the external space. The other one is that gravity, at least the leading term at low energy, is linear in the Riemannian tensor unlike the other gauge theories. So the gauge theory of gravity must reflects and explains these peculiarities.

In these processes, we can understand that the problem lies in the ambiguous situation of tetrad or metric in the gauge theoretical framework. There seems to exist some prejudice that the local symmetry of gravitation is Lorentz group (as the symmetry before the breaking) and misunderstanding that the tetrad is defined by

$$e^a_{\mu} = \partial_\mu X^a(x) \equiv X^a_{\cdot \mu}.$$  \hspace{1cm} (0.1)

Here $X^a$ are the local Lorentz coordinates. The metric tensor is defined by

$$g_{\mu \nu} = e_{\mu}^{a} e^{b}_{\nu} \eta_{ab},$$ \hspace{1cm} (0.2)

\hspace{1cm} 1E-mail:fukuyama@se.ritsumei.ac.jp
where $\eta_{ab}$ is the Minkowski metric. Since there is no reason why the integrability condition does not hold, (0.1) forces $e_{\mu a}$ to be conditional,

$$e_{\nu a,\mu} = e_{\mu a,\nu}. \quad (0.3)$$

(0.1) is often seen in many textbooks and popular review articles. Unfortunately, it is absolutely wrong since tetrad defined by (0.1) is not vector from the local Lorentz indicies and, therefore, $g_{\mu \nu}$ is not a scalar w.r.t. Lorentz transformation:

$$(X^U)^a = \left(U^{-1}(x)X(x)U(x)\right)^a \quad (0.4)$$

and

$$(\partial_\mu X^U)^a \neq \left(U^{-1}\partial_\mu XU\right)^a \quad (0.5)$$

Therefore, the right-hand side of (0.2) is not a scalar under the local Lorentz transformation.

The pathology is easily seen by counting the physical degrees of freedom of $e_{\mu a}$: The tetrad should be subject only to the condition (0.2) and the degrees of freedom of $e_{\mu a}$ are 16. If (0.3) was correct, it forces us 10 constrains and $16 - 10 = 6$ degrees are left, leaving no room to invoke the Lorentz and diffeomorphism symmetries. The correct counting should be as follows. The invariance of Lorentz transformation gives 6 second class constraints and the diffeomorphism invariance does 4 first class constraints [2][3] and the physical degrees of freedom are $16 - 6 - 4 \times 2 = 2$.

For pure gravity case, torsion free condition enables us to solve $\omega_{\mu ab}$ in terms of the tetrad,

$$\omega_{\mu ab} = -\frac{1}{2} e^c_\mu \left(\lambda_{abc} + \lambda_{bca} - \lambda_{cab}\right), \quad (0.6)$$

where

$$\lambda_{cab} = (e_{pc,\sigma} - e_{sc,\rho}) e^p_\mu e^\sigma_b. \quad (0.7)$$

If (0.3) was imposed on the tetrad, the above $\lambda_{cab}$ and therefore $\omega_{\mu ab}$ is identically vanish and spacetime must be flat. This point will be shown more explicitly soon later.

$e_{\mu a}$ in the absence of Fermions is inversely described in terms of $g_{\mu \nu}$ as

$$e^0_\mu = (-\sqrt{h}, \sqrt{h} g_i) \quad e^\beta_\mu = (0, e^\beta_i). \quad (0.8)$$

Here

$$h = g_{00}, \quad g_i = \frac{g_{0i}}{g_{00}}, \quad e^\alpha_i e_{\bar{a}j} = g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}. \quad (0.9)$$

$\bar{a}$ and $(i, j)$ are the spatial parts of the Lorentz and world indicies, respectively.

The tetrad formulation is very powerful for the classification of homogeneous space [4].

As we have said above, (0.3) gives rise to the problem that it leads us to the trivial results as follows [5]. In the tetrad formalism of gravitation, the Ricci rotation coefficients play essential roles

$$\gamma_{abc} \equiv e_{\mu a,\nu} e^\mu_\nu e^\nu_c. \quad (0.10)$$
\[ e_{\mu\nu} \equiv \partial_\nu e_\mu - \Gamma^\rho_{\nu\mu} e_\rho \] (0.11)

and is distinguished from the covariant derivative w.r.t. the Lorentz coordinates \( D_\nu e_\mu \). The relation between \( \Gamma^\rho_{\nu\mu} \) and the spin connection will be given in (0.39).

The linear combinations of the Ricci coefficients satisfies

\[ \gamma_{abc} - \gamma_{acb} = \lambda_{abc} \] (0.12)

or inverse relations,

\[ \gamma_{abc} = \frac{1}{2} (\lambda_{abc} + \lambda_{bcd} - \lambda_{cab}) \] (0.13)

are very important since the Riemannian tensor

\[ R_{abcd} = (e_{\mu\nu;\rho} - e_{\mu\rho;\nu}) e^\mu_b e^\nu_c e^\rho_d \]

\[ = \gamma_{abc,d} - \gamma_{abd,c} + \gamma_{abf}(\gamma^f_{cd} - \gamma^f_{dc}) + \gamma_{afc} \gamma^f_{bd} - \gamma_{afd} \gamma^f_{bc} \] (0.14)

is described by \( \lambda_{abc} \). So if (0.3) was valid, \( \lambda_{abc} \) and \( \gamma_{abc} \) vanish identically and, therefore, the Riemannian tensors also vanish. It should be remarked that we have assumed \( \Gamma^\rho_{\nu\mu} \) to be symmetric on the lower indices. This is not the necessary condition in every case but is valid in the case where we do not consider fermions in the first order formalism. This point is discussed later in more detail.

Of course, if we consider the spinor, torsion may appear (in the first order formalism) and antisymmetric part of \( \Gamma^\rho_{\nu\mu} \) may survive. However, the tetrad formulation must be valid irrespectively to the presence of spinor, and this pathology is very serious and (0.3), and therefore (0.1), should be discarded.

Then, how to define the tetrad in place of (0.1)? In order to answer to this question, it is necessary to consider the tetrad (vielbein in general) in the gauge framework \(^2\). Saying first the result, it is given by (0.22) in the framework of SO(1,4) or SO(2,3) gauge theory \([7]\ [8]\ [9]\). Let us explain it. In this theory, we consider, in place of the Lorentz group, the following local coordinates,

\[ \sum_{A=1}^{d+1} Z_A^2 = -l^2 \text{ for SO}(1,d) \] (0.15)

\[ \sum_{A=1}^{d+1} Z_A^2 = l^2 \text{ for SO}(2,d-1) \] (0.16)

Here we have described for general \( d \) dimensional spacetime for later use. Real \( l \) measures the scale breaking from SO(2,d). Hereafter we discuss for SO(2,3), though this formalism is equally valid in case of SO(1,4).

Corresponding to SO(2,3), the covariant derivative is defined by

\[ D_\mu \psi = (\partial_\mu - i \omega_{\mu AB} S_{AB} / 2) \psi \ (A, B = 1, ..., 4, 5). \] (0.17)

\(^2\)We can consider the metric as the gauge theory of diffeomorphism \([6]\) though it can not incorporate Fermion.
Here $\omega_{\mu AB}$ are $4 \times 10$ connection fields and $S_{AB}$ are the generators of anti de Sitter group. You will see the differences of Poincare gauge theories [10] in the subsequent arguments. The field strength is derived from the commutation relation

$$i[D_\mu, D_\nu] \psi = -\frac{1}{2} R_{\mu\nu AB} S_{AB} \psi.$$  \hspace{1cm} (0.18)

$$R_{\mu\nu AB} = \partial_\mu \omega_{\nu AB} - \partial_\nu \omega_{\mu AB} - \omega_{\mu AC} \omega_{\nu CB} + \omega_{\nu AC} \omega_{\mu CB}.$$  \hspace{1cm} (0.19)

The Einstein’s action is written as

$$I = \int d^4 x \epsilon^{ABCDE} \epsilon^{\mu\nu\rho\sigma} (Z_A/l) \left[ R_{\mu\nu BC} R_{\rho\sigma DE} / (16g^2) \right] + D_\mu Z_B D_\nu Z_C D_\rho Z_D D_\sigma Z_E \sigma(x) \left( (Z_A'^2/l^2) - 1 \right).$$  \hspace{1cm} (0.20)

Here $\epsilon^{\mu\nu\lambda\sigma}$ and $\epsilon^{ABCDE}$ are fully antisymmetric tensors with $\epsilon^{0123} = 1$ and $\epsilon^{12345} = 1$, respectively. It should be remarked that this action is a geometrical invariant and that we do not introduce metric ad hoc. After the gauge choice

$$Z_A^A = (0, 0, 0, 0, l),$$  \hspace{1cm} (0.21)

$$D_\mu Z_A = (\partial_\mu \delta_{AB} - \omega_{\mu AB}) Z_B = \{ -\omega_{\mu a 5 l} \equiv e_{\mu a} \quad \text{if} \quad A = a \\
0 \quad \text{if} \quad A = 5 \}.\hspace{1cm} (0.22)$$

It is important that $e_{\mu a}$ transforms covariantly under the remaining 4-dim Lorentz rotation. Generalized Riemannian tensor $R_{\mu\nu ab}$ is divided into two terms

$$R_{\mu\nu ab} = \hat{R}_{\mu\nu ab} - e_{[\mu a e_{\nu b]/l^2.}$$  \hspace{1cm} (0.23)

Here $\hat{R}_{\mu\nu ab}$ is the conventional Riemannian tensor defined by

$$\hat{R}_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \omega_{\mu ac} \omega_{\nu cb}$$  \hspace{1cm} (0.24)

and $e_{[\mu a e_{\nu b]} \equiv e_{\mu a} e_{\nu b} - e_{\nu a} e_{\mu b}.

$L_{grav}$ takes the form of Euler class

$$L_{grav} = \mathcal{E}_A(\text{gravity}) = \epsilon^{abcd} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu ab} R_{\rho\sigma cd} / (16g^2)$$

$$= \partial_\mu C^\mu - e \left( \hat{R} - \frac{6}{l^2} \right) \left/ (16\pi G) \right.,$$  \hspace{1cm} (0.25)

where

$$16\pi G \equiv g^2 l^2,$$  \hspace{1cm} (0.26)

$$e = \det e_{\mu a}, \quad \hat{R}_{\mu a} = e^{\nu b} \hat{R}_{\nu ab}, \quad \hat{R} = e^{\mu a} \hat{R}_{\mu a},$$  \hspace{1cm} (0.27)

and use has been made of

$$\epsilon^{abcd} \epsilon^{\mu\nu\rho\sigma} e_{\mu a} e_{\nu b} e_{\rho c} e_{\sigma d} = 4! e,$$

$$\epsilon^{abcd} \epsilon^{\mu\nu\rho\sigma} e_{\mu a} e_{\nu b} = 2 e e^{[\rho e] d} \text{ etc.}$$  \hspace{1cm} (0.28)
Here $\epsilon^{\mu a}e_{\mu b} = \delta_{ab}$, $\epsilon^{\mu a}e_{\nu a} = \delta_{\nu}^{\mu}$. The quadratic term in $\hat{R}_{\mu\nu ab}$ is total derivative $\partial_{\mu}C_{4}^{\mu}$ (the Gauss-Bonnet term),

$$\partial_{\mu}C_{4}^{\mu} = e \left( \hat{R}^{2} - 4\hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \hat{R}_{\mu\nu\rho\sigma} \hat{R}^{\mu\nu\rho\sigma} \right) / (4g^{2}).$$ (0.29)

This term with the definite coefficient $1/(4g^{2})$ are both indispensable for the conservation of, for instance, mass and angular momentum of AdS Kerr Black hole [11].

So far we imposed (0.21) before Euler variation of (0.20). This is not equal in general to imposing gauge condition after Euler variation which is the natural approach. This is easily understood if we replaced $\sigma$ term as $\sigma(x) \{ (Z_{A}^{2}/l^{2}) - 1 \}^{2}$ from linear $\sigma(x) \{ (Z_{A}^{2}/l^{2}) - 1 \}$ (See (0.33)). Let us explain in more detail [3]. We may express (0.20) in a coordinate-free form.

$$I = \int \left[ \frac{1}{2g^{2}} \tilde{\Theta}^{AB} \wedge \Theta^{AB} + \tilde{\sigma} \left( \frac{Z_{A}^{2}}{l^{2}} - 1 \right) \right].$$ (0.30)

Here

$$\Theta^{AB} \equiv d\Omega^{AB} - (\Omega \wedge \Omega)_{AB} = \frac{1}{2} \hat{R}_{\mu\nu AB} dx^{\mu} \wedge dx^{\nu},$$

$$\tilde{\Theta}^{AB} \equiv \frac{1}{2} \epsilon^{ABCDE} \frac{Z_{C}}{l} \Theta^{DE},$$

$$\tilde{\sigma} = \epsilon^{ABCDE} \epsilon^{\mu\nu\rho\sigma} \left( Z_{A}/l \right) D_{\mu}Z_{B}D_{\nu}Z_{C}D_{\rho}Z_{D}D_{\sigma}Z_{E} \sigma d^{4}x.$$ (0.31)

The Euler equations are derived by taking the variation of the action with respect to $\Omega, Z$ and $\sigma$:

$$d\tilde{\Theta}^{AB} - [\Omega, \tilde{\Theta}]^{AB} = 0,$$ (0.32)

$$\epsilon^{ABCDE} \frac{1}{4g^{2}l} \tilde{\Theta}^{BCDE} \Theta^{DE} + \frac{2}{l^{2}} Z^{A} \tilde{\sigma} = 0,$$ (0.33)

$$\frac{Z_{A}^{2}}{l^{2}} - 1 = 0.$$ (0.34)

(0.33) can be regarded as the equation determining $\sigma$ in terms of connections. In the gauge (0.21), (0.32) becomes

$$d\tilde{\Theta}^{ab} - [\Omega, \tilde{\Theta}]^{ab} = 0 \quad \text{for (AB) = (ab)},$$ (0.35)

$$\tilde{\Theta}^{ab} \Omega^{55} = 0 \quad \text{for (AB) = (a5)}.$$ (0.36)

In (0.35) summation is taken among small Latin. It should be remarked that (0.35) and (0.36) are dual to the Bianchi and cyclic identities, respectively. (0.35) correspond to the first order formalism of Palatini [12]. (0.36) is the Einstein equation in vacuum. Thus the arguments (0.25)-(0.29) are remained valid.

If we perform exterior derivative $d$ on (0.36) and use (0.36) again, we obtain

$$\tilde{\Theta}^{ab} (d\Omega^{55} - \Omega^{bc} \hat{\epsilon}^{c5}) = 0.$$ (0.37)
As is easily seen antisymmetric $\tilde{\Theta}^{\alpha\beta}$ is nondegenerate in (0.36) and we obtain torsion-free condition $\Theta^{\alpha5} = 0$ or explicitly

$$R_{\mu\nu\alpha5} = -\frac{1}{l} \left( \partial_\mu e_{\nu\alpha} - \partial_\nu e_{\mu\alpha} - \omega_{\mu\alpha\epsilon} e^\epsilon_\nu + \omega_{\nu\alpha\epsilon} e^\epsilon_\mu \right) = 0.$$  

(0.38)

Furthermore, we use the metric condition of $e_{\mu\alpha}$

$$e_\alpha^{\mu\nu} \equiv \partial_\nu e_{\mu\alpha} - \omega_{\mu\alpha\epsilon} e^\epsilon_\nu - \Gamma^\rho_{\nu\mu} e_\rho = 0.$$  

(0.39)

(0.38) represents $\Gamma^\rho_{\nu\mu} = \Gamma^\rho_{\nu\mu}$.

Fermions (Dirac, Weyl, Majorana) are studied in the framework of (Anti) de Sitter gravity in [13].

(0.25) shows the reason why the gravitational action is linear on the Riemannian tensor, and (0.22) indicates that it transforms as vector under the local Lorentz transformation. The tetrad is not subject to any condition since it comes from the additional gauge freedom of $\omega_{5\alpha}$. Hamilton formulation of full (Anti) deSitter gravity in terms of Dirac prescription [2] was performed in [3] and physical degrees of freedom are correctly 2. That is, the total degrees of freedom of the system are 92 (40 $\omega_{\mu\alpha\beta}$, 5 $Z_A$, $\sigma$ and their conjugate momenta). The number of first class constraints is 20,

$$\pi^{0BC} = 0,$$

(0.41)

$$D_i \pi^{iBC} = 0,$$

(0.42)

and that of second class constraints is 48

$$\pi^{ibc} - \frac{1}{2g^2} \epsilon^{ijk} c^{bcde} R_{jkde} = 0,$$

(0.43)

$$\pi^{ia5} = 0,$$

(0.44)

$$p^A = 0,$$

(0.45)

$$Z_a = 0, \quad Z_5 = l,$$

(0.46)

$$p_\sigma = 0,$$

(0.47)

$$\sigma - \frac{1}{8g^2} \epsilon^{bcd} \epsilon^{ijk} R_{jkde} \lambda_{ibc}^0 = 0,$$

(0.48)

$$R_{jka5} = 0,$$

(0.49)

where $\pi^{\mu\alpha\beta}$, $p^A$, $p_\sigma$ are canonical momenta of $\omega_{\mu\alpha\beta}$, $Z_A$, $\sigma$, respectively. $\lambda_{ibc}^0$ are the Lagrange multiplier of $\pi^{ibc}$. Only 6 equations of (0.49) are independent (See the detail [3]). So we have 4 (= 92 - 2 x 20 - 48) physical degrees of freedom in Hamilton formalism in agreement with the gravitational field.

Thus the problems discussed above have been all solved.

Finally we comment on the vielbein in the other dimensions than four. Our formulation is straightforwardly applied to 3,5,6,... dimensions [14] 3. It is enough to discuss five

3For two dimensional case we need the special treatment concerning with conformal invariance [15]
dimensions for the present paper’s purpose. Usually, Euler class can be defined only in even dimensions. However, in our formulation there is no essential difference between odd and even dimensions. Indeed, this formulation is easily extended to five dimensional spacetime. That is
\[ I = \int d^5x \epsilon^{ABCDEF} \epsilon^{\mu\nu\rho\sigma\lambda}(Z(A)/l)D_\mu Z_B \left[ R_{\nu\rho\sigma\lambda}R^{\nu\rho\sigma\lambda}/(48g^2l) \right] \]
+ \[ D_\nu Z_C D_\rho Z_D D_\sigma Z_E D_\lambda Z_F \sigma(x) \sum_{A=1}^{6} \left\{ (Z_A^2/l^2) - 1 \right\} \] (0.50)
with
\[ Z_A = (0, 0, 0, 0, 0, l). \] (0.51)
In this case
\[ D_\mu Z_A = (\partial_\mu \delta_{AB} - \omega_{\mu AB})Z_B = \left\{ \begin{array}{ll} -\omega_{\mu a} l = e_{\mu a} & \text{if } A = a \\ 0 & \text{if } A = 6 \end{array} \right. \] (0.52)
Here \( \mu \) and \( a \) run over 1,...,5 in world and local Lorentz coordinates, respectively. Consequently (0.50) is reduced to
\[ L_{grav} = \epsilon^{abcde} \epsilon^{\mu\nu\rho\sigma\lambda} e_{\mu a} R_{\nu\rho\sigma\lambda}/(48g^2l) \]
= \[ \epsilon^{abcde} \epsilon^{\mu\nu\rho\sigma\lambda} e_{\mu a} \hat{R}_{\nu\rho\sigma\lambda}/(48g^2l) - e \left( \hat{R} - \frac{10}{l^2} \right)/(16\pi G_5), \] (0.53)
where \( G_5 \) is five dimensional Newton constant. Thus we obtain \( AdS_5 \) in low energy scale. In this case, however, higher derivative terms (the first term of (0.53)) are not total derivatives and change the equation of motion in high energy region and do therefore Black Hole solution and its near horizon property. The explicit representation of the first term of (0.53) is
\[ \epsilon^{abcde} \epsilon^{\mu\nu\rho\sigma\lambda} e_{\mu a} R_{\nu\rho\sigma\lambda}/(48g^2l) = e \left( \hat{R}^2 - 4\hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \hat{R}_{\mu\nu\sigma\tau} \hat{R}^{\mu\nu\sigma\tau} \right)/(12g^2l). \] (0.54)
The coefficient \( 12g^2l \) is fixed by five dimensional gravitational constant \( G_5 \) and cosmological constant \( \Lambda_5 \),
\[ 16\pi G_5 = g^2 l^3, \quad \Lambda_5 = -5/l^2. \] (0.55)
For six and higher dimensions there appears cubic term on the Riemannian tensor. Even in that case all coefficients are definitely given in terms of \( G \) and \( l \). This is quite different from the conventional approaches of Lovelock Lagrangian [16]. The coefficients of quadratic and cubic terms are each left free parameters there. However, as I have emphasized, the special coefficient determined by our theory has the special merit for four dimensions and we may expect the same thing occurs in the other dimensions, which is now under investigation.

**Acknowledgments**

We would like to thank A. Randono for very useful conversations. This work is supported in part by the grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan (No. 20540282).
References

[1] R.Utiyama, Phys. Rev. 101 1597 (1956).

[2] P.A.M.Dirac, Can.J.Math. 2 129 (1950); Lecture on Quantum Mechanics (Yeshiva University, NY 1964).

[3] T.Fukuyama and K.Kamimura, Nuovo Cimento 74 93 (1983).

[4] L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields (Pergamon Press 1973) section 98.

[5] The reference [4] section 116.

[6] R. Utiyama and T. Fukuyama, Prog. Theor. Phys. 45 612 (1970); T.Fukuyama, Gravitational Field as a Generalized Gauge Field Revisited. arXiv:0902.3283 [gr-qc].

[7] T.Fukuyama, Ann. Phys. 157 321 (1984).

[8] S.W.MacDowell and S.Mansouri, Phys.Rev.Lett. 38 739 (1977).

[9] K.S.Stelle and P.C.West, Phys.Rev. D21 1466 (1980).

[10] T.W.B.Kibble, J.Math.Phys. 2 212 (1961).

[11] R.Aros, M.Contreras, R.Olea, R.Troncoso and J.Zanelli, Phys.Rev.Lett. 84 1647 (2000).

[12] A.Palatini, R.C.Circ.mat.Palermo 43 203 (1919).

[13] N.Ikeda and T.Fukuyama, Fermions in (Anti) de Sitter Gravity in Four Dimensions.arXiv:0904.1936 [hep-th] to appear in Prog.Theor.Phys. 122 No.2 (2009).

[14] T.Fukuyama, SO(2,d-1) Gauge Theory of Gravity in d Dimensional Spacetime and AdS_d/CFT_{d-1} Correspondence. arXiv:0902.2820 [hep-th].

[15] T.Fukuyama and K.Kamimura, Phys.Lett. 160B 259 (1985).

[16] D.Lovelock, J. Math. Phys. 12 498 (1971).