DOPPLER EFFECTS FROM BENDING OF LIGHT RAYS IN CURVED SPACE-TIMES

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We study Doppler effects in curved space-time, i.e. the frequency shifts induced on electromagnetic signals propagating in the gravitational field. In particular, we focus on the frequency shift due to the bending of light rays in weak gravitational fields. We consider, using the PPN formalism, the gravitational field of an axially symmetric distribution of mass. The zeroth order, i.e. the sphere, is studied then passing to the contribution of the quadrupole moment, and finally to the case of a rotating source. We give numerical estimates for situations of physical interest, and by a very preliminary analysis, we argue that analyzing the Doppler effect could lead, in principle, in the foreseeable future, to the measurement of the quadrupole moment of the giant planets of the Solar System.

Keywords: Doppler Effect; General Relativity.

1. Introduction

Einstein’s theory of gravity, General Relativity (GR), has passed all observational tests with excellent results, at least at the scale of the Solar System and in many binary compact systems; however as it is well known, some problems arise with the recent cosmological observations. In fact, the evidence of the acceleration of the Universe is supported by experimental data deriving from different tests: i.e.,
from Type-Ia Supernovae, from CMWB and from the large scale structure of the universe. However, GR is not able to provide a theoretical explanation of these experimental facts unless some *exotic and invisible energy* is admitted to exist in the universe (*Dark Energy*). Also for these reasons, the reliability of the theories of gravity alternative to GR is of great interest today. Of course, all these theories must agree with the known tests of gravity performed in the weak field and slow motion limit, in the Solar System: the standard tool used for formulating (metric) gravity theories in this context is the Parameterized Post-Newtonian (PPN) formalism. Different theories are characterized by different values of the coefficients appearing in front of the post-Newtonian metric potentials, and the formalism determines the values of these parameters for each theory under consideration. Consequently, the formalism allows for a direct comparison of the competing theories with each other, and with the results of the experiments and observations. In particular, Solar System tests limit the range of the values that the PPN parameters may assume, thus excluding some theories of gravity and quantitatively limiting the deviations from GR.

The study of the Doppler effects in the space-time around a gravitating body was carried out by many authors in the past (see, for instance, and references therein), both for pedagogical purposes and for studying the feasibility of actual experiments aimed at detecting the effects themselves. In particular, in, Kopeikin and Schäfer carefully studied the covariant theory of propagation of light in very general gravitational fields, and the various phenomena related, among which, the frequency shift. Indeed, as it is clearly explained in, the Doppler frequency shift, the deflection of light and the time delay of electromagnetic signals are related to each other, in the sense that they might be thought of as "different manifestation of the same aspect of the [...] gravitational field". However, their measurements require different experimental techniques and, moreover, the interpretation of the measured quantities is quite different. In particular, the Doppler frequency shift is a relativistic invariant, which, roughly speaking, corresponds to the projection of the photons' four-momentum along the world-line of the observer, and, consequently, it can be defined without reference to any particular coordinate frame.

As for the measurement technique, in actual Doppler experiments a radio beam is transmitted from the Earth to the receiver located aboard a spacecraft; the radio beam is coherently transponded and then sent back to the Earth, where the received frequency is measured with great accuracy, usually by means of hydrogen masers. The comparison of the transmitted and received frequencies gives the measurement of the Doppler shift.

In the past, there have been different proposals and different experimental tests of the relativistic frequency shift (see and references therein), and, indeed, very recently, a test of GR was performed by studying the Doppler effect from the Cassini spacecraft. In particular, a measurement of the post-Newtonian parameter $\gamma$, which is equal to 1 in GR, was carried out: as a result of the experiment, $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$, thus sensibly limiting the deviations from GR.
Doppler Effects from Bending of Light Rays in Curved Space-Times

Here we apply the study of the Doppler frequency shift to some situations that may be of interest, both at the Solar System and astrophysical scale. In particular, we focus on Doppler effects originating from the bending of light rays in curved space-times (which will be measured, with great accuracy by the LATOR mission\(^{15}\)), in order to see if these effects could be used to constrain the PPN parameters and, on the other hand, we argue that they can be used as probes to determine some astrophysical properties of interest of the celestial bodies. Numerical estimates of the magnitude of these effects are given, and they are compared to the current accuracies; furthermore, the characteristics of the effects are examined, in order to suggest possible detection techniques.

The paper is organized as follows: in Section 2 we briefly outline the general framework used to study the Doppler frequency shift, according to previous works; in Section 3 we apply the general approach to some metrics of physical interest; in Section 4 we consider some prototypical situations and give numerical estimates for them; finally, the possibility of measuring the effects is discussed in Section 5.

2. The Problem

In this section we outline the theoretical framework which allows to study the frequency shifts of electromagnetic signals propagating in a gravitational field. In doing so, we follow the approach of Bertotti and Giampieri\(^9\), which is in agreement for the cases under consideration, with the very general approach developed by Kopeikin and Schäfer\(^{11}\).

Let us consider an emitter of monochromatic electromagnetic waves, which emits with proper frequency \(\nu_E\). The emitter is moving in the gravitational field of a massive source; let us call \(u_E\) its four-velocity, evaluated at the (coordinate) time of emission of a signal when the emitter is at the position \(\vec{x}_E\) (the massive source is located at the origin of our reference frame). The problem is to evaluate the frequency \(\nu_R\) measured by a receiver at \(\vec{x}_R\), whose four-velocity is \(u_R\)\(^a\).

We may write the four-velocities in terms of the world-lines of the emitter (\(x_E\)) and receiver (\(x_R\))

\[
\begin{align*}
u_E &= \frac{dx_E}{ds_E} \quad \nu_R = \frac{dx_R}{ds_R}. \quad (1)
\end{align*}
\]

Of course, in terms of the parameters \(s_E, s_R\) along the world-lines:

\[
\begin{align*}
\vec{x}_E(s_E) &\equiv (x^0_E(s_E), \vec{x}_E(s_E)) \quad \vec{x}_R(s_R) \equiv (x^0_R(s_R), \vec{x}_R(s_R)). \quad (2)
\end{align*}
\]

Furthermore let \(\xi(\lambda) = \xi_0(\lambda) + \delta \xi(\lambda)\) be the world-line of the emitted photon, where \(\xi_0\) is the trajectory that the photon would have if it were emitted in flat space-time

\(^a\)Greek indices run from 0 to 3, Latin indices run from 1 to 3; the space-time metric has signature \((-1, 1, 1, 1)\); and we use units such that \(G=c=1\); nonetheless, for the sake of clarity, we re-introduce physical units in the final formulæ; boldface arrowed letters, refer to vectors in the three-dimensional space; boldface letters refer to four-vectors.
(i.e. a straight line) and $\delta \xi$ is the perturbation induced by the gravitational field. This world-line is parameterized in such a way that the photon is emitted at $\vec{x}_E(s_E)$ when $\lambda = -l_E$, and it is received at $\vec{x}_R(s_R)$ when $\lambda = l_R$.

Let us define the four-vector tangent to the photons’ world-line

$$p = \frac{d\xi(\lambda)}{d\lambda} = \frac{d\xi_0(\lambda)}{d\lambda} + \frac{d\delta \xi(\lambda)}{d\lambda} \equiv p_0 + \delta p.$$  \hfill (3)

where $p_0$ is the flat space-time term and $\delta p$ is the perturbation induced by the gravitational field. If we set $l = l_E + l_R$ the flat space-time term $p_0$ can be written in the form

$$p_0 = \left(1, \frac{\vec{x}_R - \vec{x}_E}{l}\right) \equiv (1, \hat{n}),$$  \hfill (4)

where $\hat{n}$ is a unit vector along the spatial path of the photons in flat space-time.

The frequencies in the receiver and emitter reference frames are related by the following formula

$$\nu_R \nu_E = \left(\frac{u_R \cdot p_0}{u_E \cdot p_0}\right) \left(1 + \frac{u_R \cdot \delta p_R}{u_E \cdot \delta p_E}\right),$$  \hfill (5)

where $k$ is the photons’ wave vector. Indeed, in general, the frequency $\nu_p$ of a photon measured by an observer at a point $P$ is found by projecting $k$ onto the observer’s four-velocity:

$$2\pi \nu_p = k \cdot u_p.$$  \hfill (6)

Now, we can expand the components of the wave vector about their flat space-time values $^{10}$:

$$k = \omega \left(p_0 + \delta p\right) = \left[-\omega \left(1 + \delta p^0\right), \omega \left(\hat{n} + \delta \vec{p}\right)\right].$$  \hfill (7)

Hence, eq. (5) can be written as

$$\frac{\nu_R}{\nu_E} = \left(\frac{u_R \cdot p_0}{u_E \cdot p_0}\right) \left(1 + \frac{u_R \cdot \delta p_R}{u_E \cdot \delta p_E}\right).$$  \hfill (8)

where the fact that $p_0|E = p_0|R$ has been exploited. From (8) we then obtain

$$\ln \left(\frac{\nu_R}{\nu_E}\right) = \ln \left(\frac{u_R \cdot p_0}{u_E \cdot p_0}\right) + \ln \left(1 + \frac{u_R \cdot \delta p_R}{u_E \cdot \delta p_E}\right).$$  \hfill (9)

We see that the frequency shift (9) is made of two contributions. The first one depends on the velocities of the emitter and receiver and, also, on the gravitational field at the emission and reception points. In particular, it is possible to show $^{10}$ that the ordinary Doppler effect (transversal and longitudinal) and the gravitational frequency shift come from this term.

The second term depends, again, on the velocities and on the gravitational field, but, furthermore, it depends also on the gravitational perturbations of the world-lines of the photons. In the following, we focus on this term only. Also, we
assume that space-time is flat at infinity, which amounts to say that, in the limit for \( l_E, l_R \to \infty \), we may neglect the gravitational field at the emission and reception point. Consequently the first contribution accounts for the special relativistic effects (and, to lowest order, it is \( O(v^b) \), where \( v \) is here the relative velocity of the emitter and the receiver), while the second term takes into account the gravitational effects along the world line.

On neglecting the gravitational field both at the emission and reception points (indices are raised and lowered by means of the Minkowski tensor \( \eta_{\mu\nu} \)), we may write the four-velocities of the emitter and the receiver

\[
\mathbf{u}_{E/R} = \gamma_{E/R} \left( 1, \mathbf{\bar{v}}_{E/R} \right),
\]

where

\[
\gamma_{E/R} = \frac{1}{\sqrt{1 - v_{E/R}^2}}.
\]

So, the contribution due to the perturbation of the photons’ paths ("gravitational" contribution) reads

\[
\ln \left( \frac{\nu_R}{\nu_E} \right)_{gr} = \left[ \ln \left( 1 + \frac{\delta p^0 - \mathbf{\bar{v}} \cdot \delta \mathbf{p}}{(1 - \mathbf{\bar{v}} \cdot \mathbf{\hat{n}})} \right) \right]_{E}^{R},
\]

where \([z]_E^R \) means \( z|_R - z|_E \) for an arbitrary function \( z \). If we approximate the logarithm on the right hand side of (12), we may write

\[
\ln \left( \frac{\nu_R}{\nu_E} \right)_{gr} = \left[ (\delta p^0 - \mathbf{\bar{v}} \cdot \delta \mathbf{p}) (1 + \mathbf{\bar{v}} \cdot \mathbf{\hat{n}}) \right]_{E}^{R}.
\]

In order to calculate (13), we must evaluate the perturbations of the photons’ momentum, at the emission and reception points, which can be done by integrating the geodesic equation. We reproduce here, without details, the results of 9.

The perturbations in the photons’ four-momentum can be evaluated by solving the geodesic equations:

\[
\frac{d\delta p^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} (p^\beta + \delta p^\beta) \left( p^\gamma + \delta p^\gamma \right) = 0.
\]

We assume that the gravitational field is everywhere weak, so that it can be written in the form \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), where \( h_{\mu\nu} \) is treated as a perturbation of the Minkowski background \( \eta_{\mu\nu} \). Consequently, on using a linear approximation and supposing that the perturbations originate only in a small region of space around the source of gravitational field (this is the case if the impact parameter \( b \) is much smaller than

\[\text{b} \text{Since in our units } c = 1, \ O(v^n) \text{ means } O(v^n/c^n), \text{ in other words we use approximations with respect to the small parameter } v/c, \text{ where } v \text{ is the scalar velocity.}\]
Fig. 1. Because of the presence of the massive source, the line linking the emission and reception points $E, R$ bends, and does not follow a straight line.

the distances of the emitter and receiver: $b \ll l_E, l_R$, we can write:

$$
\delta p_\mu|_E = -\frac{l_R}{2l} \int_{-\infty}^{\infty} h_{\alpha\beta,\mu}(\lambda)p_0^\alpha p_0^\beta d\lambda, \quad (15)
$$

$$
\delta p_\mu|_R = \frac{l_E}{2l} \int_{-\infty}^{\infty} h_{\alpha\beta,\mu}(\lambda)p_0^\alpha p_0^\beta d\lambda. \quad (16)
$$

The integrals (15), (16) must be evaluated along the unperturbed photons’ path, i.e. along the straight line connecting the emission point and the reception point.

Eq. (13), together with eqs. (15) and (16) are all we need to evaluate, to the desired order, the gravitational contribution to the Doppler frequency shift.

3. Applications

After having determined the perturbations of the photons’ momentum, we may apply eq. (13) for evaluating the gravitational frequency shift. In the following, we study the frequency shift induced by a spherical distribution of mass, then we take into account a quadrupole contribution to the gravitational field and study the corresponding frequency shift and, finally, we study the effect of the rotation of the source of the gravitational field.

3.1. The weak field around a spherical distribution of mass

Let us consider a spherical mass distribution, and let us suppose that its gravitational field is weak. The line element can be written in the PPN form (up to $O(v^2)$)
\[
\begin{align*}
\frac{ds^2}{c^2} &= \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2\gamma M}{r}\right) \left(dx^2 + dy^2 + dz^2\right), \\
&= -\left(1 - \frac{2M}{r}\right) \left(1 + \frac{2\gamma M}{r}\right) \left(dx^2 + dy^2 + dz^2\right),
\end{align*}
\]

where \( r = \sqrt{x^2 + y^2 + z^2} \), and \( \gamma \) is a PPN parameter. Let us suppose that photons propagate in the plane \( z = 0 \), and that their unperturbed path is parallel to the \( x \) axis, with impact parameter \( b \) \(^c\) (see Figure 1). In other words, we have
\[
p_0 = (1, \hat{n}) \parallel \hat{x}.
\]

Considering that we have no change of the time component of the photons’ momentum, since the metric does not depend on time (“conservation of energy”), eq. (13), up to the lowest order in \( v \) becomes
\[
\ln \left(\frac{\nu_E}{\nu_R}\right) = [(\tilde{v} \cdot \delta \vec{p})]^R_{E} = -\tilde{v} \cdot \delta \vec{p}_R + \tilde{v} \cdot \delta \vec{p}_E + O(v^4).
\]

Then, according to eqs. (15) and (16)
\[
\delta p_y |_E = \delta p_y |_R = 0, \quad \delta p_z |_E = \delta p_z |_R = 0,
\]
and the only non null perturbations in the photons’ world-line are given by
\[
\begin{align*}
\delta p_y |_E &= -\frac{l_R}{2l} \int_{-\infty}^{\infty} 2h_{xx,y} \, dx, \\
\delta p_y |_R &= \frac{l_E}{2l} \int_{-\infty}^{\infty} 2h_{xx,y} \, dx.
\end{align*}
\]

On integrating (21) and (22) we obtain
\[
\delta p_y |_E = \frac{2(1 + \gamma)M l_R}{b} l, \quad \delta p_y |_R = -\frac{2(1 + \gamma)M l_E}{b} l.
\]

As a consequence, the frequency shift due to a spherical distribution of mass, turns out to be
\[
\ln \left(\frac{\nu_E}{\nu_R}\right)_M = \frac{2(1 + \gamma)M}{b} \left[ \frac{(v_R y)_y L_E + (v_E y)_y L_R}{l} \right] + O(v^4).
\]

3.1.1. On the physical meaning of the gravitational frequency shift

What we have seen so far, allows us to give a simple interpretation of the gravitationally induced frequency shift.

Let us consider an emitter moving with velocity \( \vec{v} \) nearby a massive source (see Figure 2). Let us suppose that the emitter’s velocity is orthogonal to the line connecting it to the receiver. If we forget the gravitational field, we know that, in this case, there is no longitudinal Doppler effect (or first order Doppler effect, since it is \( O(v) \)), but there is only a transverse Doppler effect (which is \( O(v^2) \)), i.e. of second

\(^c\)In fact \( b \) is the ratio between two constants of the motion and coincides with the closest approach distance from the origin in the case of a flat space-time.
order). This depends on the fact that the velocity of the emitter has no components along the line of sight. However, we know that one of the effects of the gravitational field is the bending of the light rays: this means that, roughly speaking, the light ray propagates along the curve 2 and not along the straight line 1. As a consequence, there is a component of the emitter’s velocity along the actual direction of propagation of the signal, and its value can easily be calculated. In fact, the deflection angle $\theta$ in GR (see, for instance, $^1$) is given by

$$\theta = \frac{2M}{b}. \quad (25)$$

From Figure 2, it is easy to recognize that the component of the emitter’s velocity along the propagation line is approximately given by

$$v_p \simeq v 2\theta = \frac{4M}{b}. \quad (26)$$

To this longitudinal velocity it corresponds a first order Doppler effect

$$\left|\frac{\delta \nu}{\nu_E}\right| = v_p, \quad (27)$$

or, reintroducing physical units,

$$\left|\frac{\delta \nu}{\nu_E}\right| = \frac{v 4GM}{c^2 b}, \quad (28)$$

which is in agreement with (24) if we set $\gamma = 1$, as in GR, and let $l_E \ll l_R$ (see below). So, this frequency shift may be explained in terms of the bending of the

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Fig. 2. The combined effect of the emitter’s motion and of the gravitational bending of light rays produces a gravitational Doppler effect.
light rays due to the gravitational field, and it is a third order \( O(v^3) \) Doppler effect.

### 3.2. Quadrupole contribution in weak field approximation

If the mass distribution is not perfectly spherical (still preserving an axial symmetry), the quadrupole contribution comes into play. In this case, the gravitational potential is written in the form

\[
\phi(x) = -\frac{M}{r} \left[ 1 - J_2 \left( \frac{R}{r} \right)^2 3 \cos^2 \theta - 1 \right],
\]

(29)

where \( R \) is the average radius of the mass distribution, and \( J_2 = -\frac{Q_{zz}}{MR^2} \), contains the components of the quadrupole tensor \( Q_{ij} \).

In particular, the quadrupole contribution to the gravitational potential is

\[
\phi(x)_Q = \frac{M}{r} J_2 \left( \frac{R}{r} \right)^2 3 \cos^2 \theta - 1.
\]

(30)

Let us work out the quadrupole contribution to the gravitational Doppler effect. The line element is written in the form

\[
ds^2 = -(1 + 2\phi) dt^2 + (1 - 2\gamma \phi) \left( dx^2 + dy^2 + dz^2 \right).
\]

(31)

As before, we consider photons propagating in the equatorial plane \( (z = 0 \text{ or } \theta = \pi/2) \) of the source, parallel to the \( x \) axis. This amounts to writing the metric perturbation \( h_{xx} \) in the form

\[
h_{xx} = \frac{2\gamma M}{r} + J_2 \frac{\gamma M}{r} \left( \frac{R}{r} \right)^2 = (h_{xx})_{M} + (h_{xx})_{Q}.
\]

(32)

The first term, giving the monopole contribution to the frequency shift, has been calculated before; now we focus on the second term, expressing the quadrupole effect.

Proceeding as before, we obtain that the only non null perturbations are given by

\[
\delta p^\nu |_{E} = \frac{2J_2(1 + \gamma)M}{b} \left( \frac{R}{b} \right)^2 \frac{l_R}{l}, \quad \delta p^\nu |_{R} = -\frac{2J_2(1 + \gamma)M}{b} \left( \frac{R}{b} \right)^2 \frac{l_E}{l}.
\]

(33)

Then, from

\[
\ln \left( \frac{\nu_R}{\nu_E} \right) = \left[ (-\vec{v} \cdot \delta \vec{p}) |_{E}^R = -\vec{v} \cdot \delta \vec{p} |_{R} + \vec{v} \cdot \delta \vec{p} |_{E} + O(v^4), \right.
\]

(34)

taking into account the perturbations (33), we obtain for the quadrupole contribution

\[
\ln \left( \frac{\nu_R}{\nu_E} \right)_Q = \frac{2J_2(1 + \gamma)M}{b} \left( \frac{R}{b} \right)^2 \left[ (v_R)_b l_E + (v_E)_b l_R \right] + O(v^4).
\]

(35)
3.3. The weak field around a rotating distribution of mass

In the case of a rotating source and weak field approximation the line element, to
the lowest meaningful order in PPN form, is
\[ ds^2 = - \left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2\gamma M}{r}\right)(dx^2 + dy^2 + dz^2) + \frac{(4\gamma + 4 + \alpha_1)}{2r^3}(\vec{x} \times \vec{S})_i dx_i dt, \]
where \( \vec{S} \) is the angular momentum and \( \alpha_1 \) is another PPN parameter. As it is
well known, the angular momentum contribution to the gravitational field, in this
weak field context, is usually referred to as gravito-magnetic field.

Since the angular momentum contribution is of the order \( O(v^3) \), we do expect
that the gravito-magnetic contribution to the gravitational dopp ler shift is \( O(v^4) \).

In particular, it is possible to show that this contribution is expressed by
\[ \ln \left( \frac{\nu_R}{\nu_E} \right)_{GM} = \left[ -\vec{v} \cdot (^{(3)} \delta\vec{p}) \right]_E^R, \] where \( ^{(3)} \delta\vec{p} \) means that we have to evaluate the perturbations up to \( O(v^3) \).

We consider, as before, photons propagating in the equatorial plane \( (z = 0 \text{ or } \theta = \pi/2) \) of the source, parallel to the \( x \) axis; furthermore, we assume that the
angular momentum of the source is parallel to the \( z \) axis: consequently there are no
perturbations of the momentum along the \( x \) axis, and the non null perturbations
are given by (see \( ^{9}9 \))
\[ ^{(3)} p^y_E = \frac{l_R S}{l b^2} \left( 2(\gamma + 1) + \frac{\alpha_1}{2} \right), \]
\[ ^{(3)} p^y_R = -\frac{l_E S}{l b^2} \left( 2(\gamma + 1) + \frac{\alpha_1}{2} \right). \]

So the gravito-magnetic contribution to the frequency shift turns out to be
\[ \ln \left( \frac{\nu_R}{\nu_E} \right)_{GM} = \frac{S}{b^2} \left( 2(\gamma + 1) + \frac{\alpha_1}{2} \right) \left[ \left( \frac{(v_R)_y l_E + (v_E)_y l_R}{l} \right) \right] + O(v^5). \]

4. Examples and Numerical Estimates

In order to evaluate the magnitude of the frequency shifts studied above, we consider
three cases corresponding to the different positions of the emitter and receiver with
respect to the source of the gravitational field:

1. \( l_E \ll l_R \)
2. \( l_R \ll l_E \)
3. \( l_E \approx l_R \)

The cases above correspond to different actual situations, that may prove useful
for detecting the effect. Namely, case 1 can occur, say, in the Solar System, when an
emitter is on board a spacecraft orbiting the Sun or one planet, and the receiver is
far away, for instance, on the Earth. Case 2 could correspond to a source of electromagnetic signals, which is very distant from the massive source of the gravitational field. For instance, one may think of a distant star, such as a pulsar, emitting electromagnetic signals, whose paths pass close to the Sun, and then are received on the Earth. Finally, case 3 may occur when both the emitter and the receiver are moving on similar orbits around a massive body. In the following, we shall evaluate the magnitude of the frequency shifts for some prototypical situations.

Before going on, we notice that if we assume that

$$\nu_R = \nu_E + \delta \nu,$$  \hspace{1cm} (41)

we may write

$$\ln \left( \frac{\nu_R}{\nu_E} \right) = \ln \left( \frac{\nu_E + \delta \nu}{\nu_E} \right) = \ln \left( 1 + \frac{\delta \nu}{\nu_E} \right) \approx \frac{\delta \nu}{\nu_E},$$  \hspace{1cm} (42)

since the expected frequency shift is small.

4.1. **Case 1.,** \( l_E \ll l_R \)

As we said above, this situation may occur when the emitter is orbiting a massive source, and the receiver is far away from it. To fix the ideas, we may think of an emitter on a planet around the Sun, or on board a spacecraft orbiting the Sun or some other planets of the Solar System. Provided that

$$l_E \ll l_R,$$  \hspace{1cm} (43)

we have \( l_R \approx l \). As a consequence, the orders of magnitude of the various contributions that we have studied above may be evaluated by considering the following expressions, where, here and henceforth, for the sake of simplicity, the PPN parameters are set equal to their GR values (i.e. \( \gamma = 1, \alpha_1 = 0 \)):

$$\left| \frac{\delta \nu}{\nu_E} \right|_M \approx \frac{GM}{c^2b} \frac{v_E}{c},$$  \hspace{1cm} (44)

$$\left| \frac{\delta \nu}{\nu_E} \right|_Q \approx \frac{J_2GM}{c^2b} \left( \frac{R}{b} \right)^2 \frac{v_E}{c},$$  \hspace{1cm} (45)

$$\left| \frac{\delta \nu}{\nu_E} \right|_{GM} \approx \frac{GS}{c^3b^2} \frac{v_E}{c}.$$  \hspace{1cm} (46)

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
Source & \( S \) (Kg m\(^2\) s\(^{-1}\)) & \( J_2 \) \\
\hline
Sun & \( 190.0 \times 10^{39} \) & \( 2 \times 10^{-7} \) \\
Earth & \( 5.86 \times 10^{33} \) & \( 1.08 \times 10^{-3} \) \\
Jupiter & \( 4.33 \times 10^{38} \) & \( 1.49 \times 10^{-2} \) \\
Saturn & \( 6.63 \times 10^{37} \) & \( 1.62 \times 10^{-2} \) \\
\hline
\end{tabular}
\end{table}
where $v_E$, considering Keplerian motion, can be approximately written as

$$v_E \simeq \sqrt{\frac{GM}{R_b}},$$

where $R_b$ is the order of magnitude of the semi(major)-axis of the orbit. Consequently, we may write

$$\left| \frac{\delta \nu}{\nu_E} \right|_M \simeq \frac{GM}{c^2b} \sqrt{\frac{GM}{c^2R_b}},$$

$$\left| \frac{\delta \nu}{\nu_E} \right|_Q \simeq \frac{J_2GM}{c^2b} \left( \frac{R}{b} \right)^2 \sqrt{\frac{GM}{c^2R_b}},$$

$$\left| \frac{\delta \nu}{\nu_E} \right|_{GM} \simeq \frac{GS}{c^3b^2} \sqrt{\frac{GM}{c^2R_b}}.$$

More explicitly

$$\left| \frac{\delta \nu}{\nu_E} \right|_M \simeq 3,06 \times 10^{-9} \left( \frac{M}{M_\odot} \right)^{3/2} \left( \frac{R_\odot}{b} \right) \left( \frac{R_\odot}{R_0} \right)^{1/2},$$

$$\left| \frac{\delta \nu}{\nu_E} \right|_Q \simeq 3,06 \times 10^{-9} J_2 \left( \frac{M}{M_\odot} \right)^{3/2} \left( \frac{R_\odot}{b} \right)^3 \left( \frac{R_\odot}{R_0} \right)^2 \left( \frac{R_\odot}{R_0} \right)^{1/2},$$

$$\left| \frac{\delta \nu}{\nu_E} \right|_{GM} \simeq 1,41 \times 10^{-15} \left( \frac{S}{S_\odot} \right) \left( \frac{R_\odot}{b} \right)^2 \left( \frac{M}{M_\odot} \right)^{1/2} \left( \frac{R_\odot}{R_0} \right)^{1/2}.$$

The orders of magnitude of the frequency shifts (51),(52),(53) are evaluated in Table 2, where we have considered the Sun and some planets of the Solar System as sources of the gravitational field, and we have chosen orbits such that $R_0 \simeq b \simeq R$.

### 4.2. Case 2., $l_R \ll l_E$

Another interesting situation corresponds to a source of electromagnetic signals, which is indeed very distant from the massive source of gravitational field. For instance, one may think of a distant star, such as a pulsar, emitting electromagnetic signals, whose paths pass close to the Sun, and then are received on the Earth. In this case, we have

$$l_R \ll l_E,$$
so that \( l_E \simeq l \). As a consequence, the velocity which plays a role is now the one of the receiver, i.e. the Earth-based observer. To fix the ideas, for a pulsar whose beam passes near the Sun, \( v_R \) is nothing but the apparent velocity of the Sun in the sky, i.e. \( v_R \simeq 10^4 \text{ m/s} \). Then

\[
|\frac{\delta \nu}{\nu_E}| \mathbf{M} \simeq \frac{GM}{c^2b} \frac{v_R}{c},
\]

(55)

\[
|\frac{\delta \nu}{\nu_E}| \mathbf{Q} \simeq \frac{J_2GM}{c^2b} \left( \frac{R}{b} \right)^2 \frac{v_R}{c},
\]

(56)

\[
|\frac{\delta \nu}{\nu_E}| \mathbf{GM} \simeq \frac{GS}{c^3b^2} \frac{v_R}{c}.
\]

(57)

On evaluating for \( M = M_\odot \), \( S = S_\odot \), we have

\[
|\frac{\delta \nu}{\nu_E}| \mathbf{M} \simeq 7.06 \times 10^{-11} \left( \frac{R_\odot}{b} \right),
\]

(58)

\[
|\frac{\delta \nu}{\nu_E}| \mathbf{Q} \simeq 7.06 \times 10^{-11} J_2 \left( \frac{R_\odot}{b} \right)^3,
\]

(59)

\[
|\frac{\delta \nu}{\nu_E}| \mathbf{GM} \simeq 3.24 \times 10^{-17} \left( \frac{R_\odot}{b} \right)^2.
\]

(60)

It is clear that the most favourable position is near to the opposition, where the impact parameter is as close as possible to the radius of the Sun. Near to the opposition, we may pose

\[
|v_R| = \left| \frac{db}{dt} \right|.
\]

(61)

As a consequence, the estimates of the frequency shifts can now be written as

\[
|\frac{\delta \nu}{\nu_E}| \mathbf{M} \simeq \frac{GM}{c^2b} \left| \frac{db}{cdt} \right|,
\]

(62)

\[
|\frac{\delta \nu}{\nu_E}| \mathbf{Q} \simeq \frac{J_2GM}{c^2b} \left( \frac{R}{b} \right)^2 \left| \frac{db}{cdt} \right|,
\]

(63)

\[
|\frac{\delta \nu}{\nu_E}| \mathbf{GM} \simeq \frac{GS}{c^3b^2} \left| \frac{db}{cdt} \right|.
\]

(64)

We point out that the frequency shifts (62) and (64) were already obtained in a previous work \(^{29}\), using a different approach based on the calculation of the gravitational induced time delay (“Shapiro time delay”); furthermore, (62) corresponds to the recent measurements performed by Bertotti and collaborators \(^{14}\) of the PPN \( \gamma \) parameter by means of radio ranging to the Cassini spacecraft.

4.3. **Case 3.,** \( l_E \simeq l_R \)

This situation may occur, for instance, when both the emitter and the receiver are orbiting the source of the gravitational field, and their orbits have similar dimensions. One might think of two satellites communicating with each other and orbiting
the Earth. To fix the ideas, one could consider two GPS satellites, even though in their present configuration they can communicate with the Earth-based stations only. However, the satellites of the forthcoming GALILEO positioning system, at least in their second generation, could be able to communicate with one another, so it is useful to evaluate the magnitude of the effect for this possible physical situation. In order to evaluate the magnitude of the effect, we consider two satellites orbiting the Earth, on the same circular orbit with radius $R_c \approx 2.66 \times 10^6 \text{ m}$; we also assume that the orbit lays in the equatorial plane of the Earth. Furthermore, we calculate the frequency shift in the most favourable position of the two satellites, i.e. when $b \approx R_\oplus$. As a consequence, we may write

$$\frac{\delta \nu}{\nu_E} \simeq \frac{GM_\oplus}{c^2 R_\oplus} \sqrt{\frac{GM_\oplus}{c^2 R_c}}, \quad (65)$$

$$\frac{\delta \nu}{\nu_E} \simeq \frac{J_2 GM_\oplus}{c^2 R_\oplus} \sqrt{\frac{GM_\oplus}{c^2 R_c}}, \quad (66)$$

$$\frac{\delta \nu}{\nu_E} \simeq \frac{GS_\oplus}{c^2 R_\oplus} \sqrt{\frac{GM_\oplus}{c^2 R_c}}. \quad (67)$$

Numerically evaluating the formulas, we obtain

$$\frac{\delta \nu}{\nu_E} \simeq 2.80 \times 10^{-16}, \quad (68)$$

$$\frac{\delta \nu}{\nu_E} \simeq 3.03 \times 10^{-19}, \quad (69)$$

$$\frac{\delta \nu}{\nu_E} \simeq 1.44 \times 10^{-22}. \quad (70)$$

As a general comment, we may say that, in all cases analyzed above, the gravitomagnetic contribution to the frequency shift is much lower than the other ones, so it would hardly be detected. Also, the value of the $J_2$ (Table 1) for the bodies of the Solar System, makes the quadrupole contribution much smaller than the monopole one.

5. Discussion and Conclusions

We have shown that the relative frequency shifts originating from the gravitational bending of light rays range from $10^{-9}$ to $10^{-22}$. The measurements of these frequency shifts, could allow, in principle, to estimate the PPN parameters that we have introduced in our formulas, or to measure the gravitational field of the celestial bodies. Actually, as we pointed out before, a recent estimate of the $\gamma$ parameter, was performed, by means of the measurement of the frequency shift of radio photons to and from the Cassini spacecraft as they passed near the Sun. We remember that, in GR, $\gamma = 1, \alpha_1 = 0$, and the most recent values of the PPN parameters are reported in.
Of course, in order to evaluate the reliability of actual measurements of these effects, a careful and detailed analysis of the error budget is required. In this work, however, we are just interested in pointing out the possibility of detection of some of them, on the bases of the knowledge of the current accuracy in frequency shift measurements.

We see that, in order to measure the quadrupole contribution, an accuracy of $10^{-14}$ at least is required, while the gravito-magnetic contribution requires an accuracy of $10^{-15}$. Nowadays, it is known that an atomic standard stable to 1 part in $10^{16}$ is available \cite{30}, so that such a fractional frequency stability is not unreasonable for future experiments with atomic clocks. Indeed, the giant planets, Jupiter and Saturn, are, at least in principle, candidates for such measurements: in fact, the estimates of their quadrupole moment seem to be in the actual or foreseeable accuracy of the instruments. Furthermore, they do not have an electromagnetic activity as intense as the one of the Sun: so, again, at least in principle, it would not bias the radio communications.

We must remember that in an actual measurement, besides the gravitationally induced terms, also the ordinary Doppler effect ($O(v)$), the second order Doppler effect ($O(v^2)$) and the gravitational redshift ($O(v^2)$) contribute to the Doppler signal (in particular, the latter two effects were tested using a space-borne hydrogen maser carried aboard a rocket \cite{31}). In order to measure the frequency shifts induced by the gravitational bending of light rays, these "ordinary" Doppler contributions must be subtracted: to this end, a very accurate knowledge of the orbits of the spacecrafts or planets is required. On the other hand, the different dependence on the geometric parameters (such as $b$) of the various contributions, may suggest a way of discriminating them; in the case of the gravito-magnetic contribution to the frequency shift, the antisymmetry of the effect with respect to the rotation axis of the source may help to disentangle the extremely weak signal from the remaining bigger shifts due to other causes \cite{29}.

In conclusion, we have shown that the Doppler effects induced by the gravitational bending of light rays, though small, deserve further analysis. If measured, these effects may allow us to put limits on the values of the PPN parameters and, on the other hand, they can be used as probes to understand the nature and explore the physical properties of the celestial bodies.

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