1. INTRODUCTION

We shall describe an approach to the functional integration on manifolds described by a finite number of parameters; the first part will provide a general treatment while in the second part we shall specialize to the 2-dimensional Regge case where exact results can be expressed in closed form. We won't delve in technical details, for which we refer to the original papers [2–4] but we shall mainly concentrate on the general setting and on conceptual issues.

In 1961 Regge [1] proposed a formulation of classical gravity in which the continuous geometry is replaced by a geometry which is piecewise flat, i.e. in which the curvature is concentrated on $D-2$ dimensional simplices being $D$ the dimension of spacetime. It is remarkable that Einstein's equations derived by the discrete analogue of the Einstein-Hilbert action assume a very simple form. The relation of such a formulation to the classical continuum formulation, obtained when the number of simplices goes to infinity has been thoroughly studied in [5,6]. In classical gravity such an approach can be used as an intrinsic geometric approximation scheme; at the quantum level the Regge formulation has also been proposed as a way to introduce a fundamental length into the theory [5]. Here we shall understand the scheme at the quantum level as a mean to break down the geometric degrees of freedom to a finite number, with the idea that the continuum limit is obtained (or defined) when the number of such degrees of freedom is allowed to go to infinity. Throughout the treatment we shall refer to euclidean closed manifolds.

2. GENERAL FORMULATION

We shall consider in this section a general situation in which the class of geometries described by a finite number of parameters is not necessarily the Regge model. Diffeomorphisms play a key role in the formulation of gravity and the viewpoint we shall adopt is to treat them exactly at every stage. The class of geometries will be parameterized by a finite number $M$ of invariants $l_i$ and described by a gauge fixed metric $\bar{g}_{\mu\nu}(x, l)$. The functional integration will be performed on the entire class of metrics $[f^*\bar{g}_{\mu\nu}(l)](x)$ with $f$ denoting the diffeomorphisms [7]. The introduction of a metric for us is crucial if we want to follow the analogy with gauge theory, being the metric field $g_{\mu\nu}$ the analogue of the gauge field $A_{\mu}$. The analogue of the gauge invariant metric of gauge theories is the De Witt supermetric [8]

$$ (\delta g, \delta g) = \int \sqrt{g(x)} d^Dx \delta g_{\mu\nu}(x) G^{\mu\nu\mu'\nu'}(x) \delta g_{\mu'\nu'}(x) $$

(1)

where

$$ G^{\mu\nu\mu'\nu'} = g^{\mu\mu'} g^{\nu\nu'} + g^{\mu\nu'} g^{\mu\nu} - \frac{2}{D} g^{\mu\nu'} g^{\mu'\nu'} + C g^{\mu\nu} g^{\mu'\nu'} $$

(2)

which is the most general ultra–local distance in the space of the metrics, invariant under diffeomorphisms. With regard to the reduction of the degrees of freedom we notice that such a reduction will involve only the geometries not the diffeomorphisms. Since the integration on the latter is infinite dimensional the related contribution
will be a true functional integral (the Faddeev–Popov determinant). As usual when dealing with the differential structure of a manifold, the charts and transition functions are to be given before imposing on the differential manifold the metric structure; in other words if we consider families of metrics on the same differential manifold the transition functions have to be independent of the metric itself. Such a feature is essential if we want that the variations of the metric tensor appearing in the De Witt distance are to be tensors under diffeomorphisms. This is the direct supermetric, it induces a volume element on the tangent space of the metrics. This is the direct

\[ \delta g_{\mu\nu} = [(F \xi)_{\mu\nu} + F(F^\dagger F)^{-1}F^\dagger \frac{\partial g_{\mu\nu}}{\partial \tau_i}] + [1 - F(F^\dagger F)^{-1}F^\dagger] \frac{\partial g_{\mu\nu}}{\partial \tau_i} \delta \tau_i \]  

(3)

where \( \xi \) is the vector field representing the infinitesimal diffeomorphism and \( F^\dagger F \) is the Lichnerowicz-De Rahm operator being \( F \) defined by \( (F \xi)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \). It can be shown that the inverse of \( F^\dagger F \) is well defined from \( \text{Im} F^\dagger \) to \( \text{Im} F^\dagger \). Obviously great simplifications would occur if the gauge fixed metric could be chosen such as \( \frac{\partial g_{\mu\nu}}{\partial \tau_i} \in \text{Ker}(F^\dagger) \) however in general such a choice cannot be accomplished \( \dagger \). We recall that given a distance, in our case the De Witt supermetric, it induces a volume element on the tangent space of the metrics. This is the direct generalization of how one computes the volume element in the case of curved finite dimensional space.

A rather standard procedure allows now to factorize the infinite volume of the diffeomorphism group and one reaches the integration measure

\[ \Pi_k \delta t_k \det(t^i, t^j) \frac{1}{2} \text{Det}(F^\dagger F) \frac{1}{2} \]  

(4)

with

\[ t^i_{\mu\nu} = [1 - F(F^\dagger F)^{-1}F^\dagger] \frac{\partial g_{\mu\nu}}{\partial \tau_i} \]  

(5)

In words, the first determinant represents the density of the different geometries parameterized by the parameters \( \tau_i \) while the second determinant represents the gauge volume of such geometries (the Faddeev-Popov determinant); it counts the number of ways different metrics can be chosen to describe the same geometry. The source of this second term is the fact that following the analogy with gauge theories \( \dagger \) we chose the metric as the fundamental variable. Both term are invariant not only under \( l \)-independent diffeomorphisms but also under \( l \)-dependent diffeomorphisms \( \dagger \) and as such are both true geometric invariants. In addition measure (4) is invariant in form under a change of the \( M \) parameters which describe the geometry; i.e. the result does not change whether to describe the geometries we use a complete set of geodesic lengths, or a collection of angles, areas etc. A property of the above measure is to be dependent on the arbitrary constant \( C \) which appears in the De Witt metric. In fact both determinants in eq.(4) depend on \( C \) and one can show that the dependence in the general case cannot cancel \( \dagger \). This is not a surprise as such a constant disappears only under integration over the conformal factor, to which we shall now turn.

### 2.1. Integration over the conformal factor

We want to enlarge the treatment by replacing the integration variables \( l_k \) by a conformal factor \( \sigma(x) \) \( \dagger \) and a finite number of other parameters \( \tau_i \) describing geometric deformations transverse (i.e. non collinear) both to the diffeomorphism and to the Weyl group.

Thus the set of metrics we shall integrate on is given by

\[ g_{\mu\nu}(x, \tau, \sigma, f) = [f^* e^{2\sigma} \hat{g}_{\mu\nu}(\tau)](x) \]  

and now we have to compute the Jacobian \( J(\sigma, \tau) \) such that

\[ \mathcal{D}[g] = J(\sigma, \tau) \mathcal{D}[f] \mathcal{D}[\sigma] \prod_i d\tau_i \]  

(6)

being \( \mathcal{D}[\sigma] \) the measure induced by the distance

\[ (\delta \sigma, \delta \sigma) = \int \sqrt{g(x) d^D x} \delta \sigma(x) \delta \sigma(x) \]  

(7)
Proceeding as in the previous subsection the general variation of the metric can be written as

\[ \delta g_{\mu\nu}(x, \tau, \sigma, f) = (F\xi)_{\mu\nu}(x) + 2[f^* \delta\sigma \tilde{g}_{\mu\nu}(x) + f^* \frac{\partial g_{\mu\nu}}{\partial \tau_i} \delta \tau_i(x)] , \quad (8) \]

with \( \tilde{g}_{\mu\nu}(x, \tau, \sigma) = e^{2\sigma} \hat{g}_{\mu\nu}(x, \tau) \). It is useful at this stage to introduce the traceless part \( P \) of \( F \)

\[ (P\xi)_{\mu\nu} = (F\xi)_{\mu\nu} - \frac{g_{\mu\nu}}{D} g^{\alpha\beta} (F\xi)_{\alpha\beta} , \quad (9) \]

the analogue \( P^1 P \) of the Lichnerowicz-De Rahm operator and the traceless tensor

\[ k^i_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \tau_i} - \frac{g_{\mu\nu}}{D} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial \tau_i} . \quad (10) \]

The general variation of the metric can be written as [11–13]

\[ \delta g_{\mu\nu}(x, \tau, \sigma, f) = (P\xi')_{\mu\nu} + f^* 2\delta\sigma' \tilde{g}_{\mu\nu}(\tau, \sigma) + [1 - P(P^1 P)^{-1} P^1] k^i_{\mu\nu} \delta \tau_i . \quad (11) \]

where \( \sigma' \) and \( \xi' \) are proper translations of \( \sigma \) and \( \xi \). The three terms are mutually orthogonal and exploiting the invariance of the integrals under translations on the tangent space we have, apart for a constant multiplicative factor

\[ J(\sigma, \tau) = \text{Det}(\bar{P}^1 \bar{P})^\frac{1}{2} \times \]

\[ \left[ \text{det} \left( k^i, (1 - P(P^1 P)^{-1} P^1) \right) \right]^\frac{1}{2} . \quad (12) \]

The dependence on \( f \) has disappeared due to the invariance of the De Witt metric under diffeomorphisms and thus in eq.\,(12) the infinite volume of the diffeomorphisms can be factorized away; moreover in eq.\,(12), the dependence on \( C \) has been absorbed in an irrelevant multiplicative constant, as it happens in two dimensions [14]. This is the result of having integrated over all the conformal deformations. Again the first determinant appearing in eq.\,(12) is a true functional determinant while the second is an \( M \)-dimensional determinant.

We notice that, as we work with euclidean signature and on closed manifolds i.e. compact manifolds without boundaries, the functional determinants appearing in eq.\,(11) and eq.\,(12) are well defined through the usual \( Z \)-function regularization. This is due to the fact that as it is easily checked both operators \( F^1 F \) and \( P^1 P \) are elliptic for all \( D \geq 2 \). On the other hand it is difficult for \( D > 2 \) to extract the dependence of the two determinants in \( (12) \) on the conformal factor \( \sigma \). The reason is that the usual procedure which works in two dimensions of taking a variation with respect to \( \sigma \) and then integrating back stumbles into the appearance of the operator \( PP^1 \) which in \( D > 2 \) is not elliptic and thus the usual heat kernel technology is no longer available.

A finite dimensional approximation to eq.\,(12) is obtained by restricting to a family of conformal factors parameterized by a finite numbers of parameters \( s = \{ s_i \} \). Thus to the family \( f^* e^{2\sigma(s)} \hat{g}_{\mu\nu}(\tau) \) it is associated the measure

\[ \prod_k d\tau_k \prod_i d s_i \left[ \text{det} (J_{ij}^s) \right]^\frac{1}{2} \times \]

\[ \left[ \text{det} \left( k^i, (1 - P(P^1 P)^{-1} P^1) k^i \right) \right] \text{Det}(P^1 P)^\frac{1}{2} \]

where \( J_{ij}^s = \int d^D x \sqrt{g} e^{2\sigma(s)} \frac{\partial \sigma}{\partial s_i} \frac{\partial \sigma}{\partial s_j} \).

We now turn to the \( D = 2 \) case where explicit results can be obtained.

3. THE D=2 CASE: TWO DIMENSIONAL REGGE GRAVITY IN THE CONFORMAL GAUGE

We need now to specialize eq.\,(13) to the two dimensional case. Here the role of the \( \tau_i \) is played by the Teichmüller parameters which are absent for genus 0, are two in number for genus 1 and \( 6h - 6 \) for higher genus; in addition, contrary to what happens in higher dimensions were the generic geometry has no conformal Killing vector field, in two dimensions every topology carries its own conformal Killing vector fields which are 6 for genus 0 (sphere topology), 2 for genus 1 (torus topology) and are simply absent for higher genus. In presence of conformal Killing vectors
eq. (13) goes over to [11–13]

\[ D[\sigma] \frac{d\tau}{v(\tau)} \sqrt{\frac{\det'(P^1P)}{\det(\phi_a, \phi_b) \det(\psi_k, \psi_l)}} \times \det(\frac{\partial g}{\partial \tau}, \psi_m). \] (14)

\[ D[\sigma] \] is the functional integration measure induced by the metric

\[ (\delta\sigma^{(1)}, \delta\sigma^{(2)}) = \int \sqrt{g} e^{2\sigma} \delta\sigma^{(1)} \delta\sigma^{(2)}. \] (15)

φ_a and ψ_k are respectively the zero modes of P and P^1; v(τ) represents the volume of the conformal transformations, det'(P^1P) stays for the determinant from which the zero modes have been excluded. It is well known that P acts diagonally on the column vector (ξ_ω, ξ_ω) by transforming it into (h_{ω0}, h_{ωω}) which represent a traceless symmetric tensor in two dimensions. In the conformal gauge the operator L which takes ξ_ω into h_{ω0} and its adjoint are given by

\[ L = e^{2\sigma} \frac{\partial}{\partial \omega} e^{-2\sigma} \quad \text{and} \quad L^1 = -e^{-2\sigma} \frac{\partial}{\partial \omega} \] (16)

and we have det'(P^1P) = [det'(L^1L)]^2.

In D = 2 the singularities of the Regge geometry are confined to isolated points where conical defects (positive or negative) are present. Our problem therefore will be to compute the determinant appearing in eq. (14) on a two dimensional surface which is everywhere flat except for isolated conical singularities. The determinant of L^1L will be defined through the Z-function technique where

\[ Z_K(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{Tr} (e^{-tlL}) \] (17)

and

\[ -\log(\det'(L^1L)) = \hat{Z}_K(0) = \gamma E Z_K(0) + \text{Finite}_{\epsilon \to 0} \int_0^\infty d\epsilon \frac{t}{t} \text{Tr} (e^{-tlL}). \] (18)

The standard procedure is to compute the change of \( \hat{Z}_K(0) \) under a variation of the conformal factor

\[ -\delta \log \left[ \frac{\det'(L^1L)}{\det(\Phi_a, \Phi_b) \det(\Psi_i, \Psi_m)} \right] = \gamma E \delta c^K_0 + \text{Finite}_{\epsilon \to 0} \text{Tr} [4 \delta\sigma K(\epsilon) - 2\delta\sigma H(\epsilon)], \] (19)

and then integrating back the result. In the previous equation \( K = L^1L, H = LL^1, K \) is the heat kernel of \( K \) and \( H \) is the heat kernel of \( H \); \( c^K_0 \) is the constant term in the asymptotic expansion of the trace of the heat kernel \( K(t) \) and is related to \( Z_K(0) \) by

\[ c^K_0 = Z_K(0) + \dim(\text{Ker} K). \] (20)

\( \Phi_a \) and \( \Psi_i \) are the zero modes of \( K \) and \( H \) respectively. The central point in the evaluation of the r.h.s. of eq. (19) will be the knowledge of \( c^K_0 \) and of \( K(t) \) and \( H(t) \) on the Regge manifold for small \( t \). As is well known such quantities are local in nature and thus we shall start by computing them on a single cone.

### 3.1. Self-adjoint extension of the Lichnerowicz-De Rahm operator

In order to compute the small time behavior of the heat kernel around a conical singularity we need to solve the eigenvalue equation \( (L^1L)\xi = \lambda \xi \) on a cone. This is done as usual by decomposing \( \xi \) in circular harmonics. A peculiar aspect of the problem is that for a generic opening \( \alpha \) of the cone (\( \alpha = 1 \) is the plane) there is a number of partial waves for which both solutions (the “regular” and the “irregular” at the origin) are square integrable in the invariant metric; as well known such a circumstance poses the problem of the correct self-adjoint extension of the Lichnerowicz-De Rahm operator. Originally [3] the problem was solved by regularizing the tip of the cone by a segment of sphere or Poincaré pseudosphere and then taking the regulator to zero. A more general approach was given subsequently in [3] by considering all self-adjoint extension of \( L^1L \) and imposing on them the restrictions due
to the Riemann–Roch relation
\[ \dim (\ker (P^\dagger P)) - \dim (\ker (PP^\dagger)) = 3\chi \]
being \( \chi = 2 - 2h \) the Euler characteristic of the surface of genus \( h \). The two procedures give exactly the same results; in particular
\[ c_0^K = \frac{1 - \alpha^2}{12\alpha} + \frac{(\alpha - 1)(\alpha - 2)}{2\alpha}. \]
and
\[ c_0^H = \frac{1 - \alpha^2}{12\alpha} + \frac{(2\alpha - 1)(2\alpha - 2)}{2\alpha}. \]

A posteriori the agreement between the two methods is not surprising since the Riemann–Roch relation is a statement related to the topology of the surface which is correctly provided by the smoothing process. In different words the compactness of the manifold. Outside the interval \( \frac{1}{2} < \alpha < 2 \) is not possible to satisfy the Riemann–Roch relation within the realm of \( L^2 \)-functions.

We now turn to the explicit computation of the determinants referring to the simpler cases of \( h \) equal 0 and 1.

### 3.2. Sphere topology

The conformal factor describing a Regge geometry with the topology of the sphere is given by
\[ e^{2\sigma} = e^{2\lambda_0} \prod_{i=1}^N |\omega - \omega_i|^{2(\alpha_i - 1)} \]
with 0 < \( \alpha_i \) and \( \sum_{i=1}^N (1 - \alpha_i) = 2 \). As it happens on the continuum such a conformal factor is unique up to the 6 parameter \( SL(2,C) \) transformations corresponding to the six conformal Killing vectors of the sphere
\[ \omega_i' = \frac{a\omega_i + b}{a\omega_i + d}, \quad \alpha_i' = \alpha_i \]
\[ \lambda_0' = \lambda_0 + \sum_{i=1}^N (\alpha_i - 1) \log |\omega_i c + d|, \]
with the complex parameters satisfying \( ad - bc = 1 \). Under the written transformation the conformal factor
\[ \sigma \equiv \sigma(\omega; \lambda_0, \omega_i, \alpha_i) = \lambda_0 + \sum_i (\alpha_i - 1) \log |\omega - \omega_i| \]

shows over to
\[ \sigma'(\omega; \lambda_0, \omega_i, \alpha_i) = \sigma(\omega; \lambda_0', \omega_i', \alpha_i') \]
where \( \omega_i', \alpha_i' \) and \( \lambda_0' \) are given by eq.(25). The area \( A \)
\[ A = e^{2\lambda_0} \int d^2 \omega |\omega - \omega_i|^{2(\alpha_i - 1)} \]
being a geometric invariant is left unchanged. It is a remarkable feature of the family of Regge conformal factors to be closed under such \( SL(2,C) \) transformations. Such a description is equivalent to usual one in terms of link lengths; in fact from the Euler relation \( F + V = H + 2 \) with \( H = \frac{2}{3}F \) we get \( H = 3V - 6 \), where \( -6 \) corresponds to the 6 conformal Killing vectors of the sphere.

Substituting now the variation of the conformal factor eq.(26) into eq.(19) and integrating back the result, one obtains
\[ \log \sqrt{\frac{\det(P^\dagger P)}{\det(\phi_\alpha, \phi_\beta)}} = \frac{26}{12} \sum_{i,j \neq i} (1 - \alpha_i)(1 - \alpha_j) \frac{1}{\alpha_i} \log |\omega_i - \omega_j| + \lambda_0 \sum_i (\alpha_i - \frac{1}{\alpha_i}) - \sum_i F(\alpha_i). \]
$N \to \infty$, the $\omega_i$ become dense and the $\alpha_i \to 1$, always with $\sum_{i=1}^{N}(1 - \alpha_i) = 2$. In such a limit $\sum_{i=1}^{N}F(\alpha_i)$ goes over to the topological invariant $N F(1) - \chi F'(1)$, while the remainder goes over to the well known continuum expression. In fact we have

$$\frac{1}{2\pi} \log |\omega - \omega'| = \frac{1}{\Box}(\omega, \omega')$$

(30)

and for any region $V$ of the plane $\omega$

$$\int_{V} d^2\omega e^{2\sigma} R = -2 \int_{V} d^2\omega \Box \sigma = 4\pi \sum_{\omega_i \in V} (1 - \alpha_i).$$

(31)

Thus the r.h.s. of eq.(29) goes over to

$$\frac{26}{96\pi} \int d^2\omega d^2\omega' (\sqrt{g}R) = \frac{1}{\Box}(\omega, \omega')(\sqrt{g}R)_{\omega'} - 2(\log \frac{A}{A_0}) \int d^2\omega \sqrt{g}R$$

(32)

where $A_0$ is the value of the area for $\lambda_0 = 0$; eq.(32) is the correct continuum result.

We notice that as it happens on the continuum, the F.P. contribution eq.(29) is not the result of integrating on the fluctuations of the geometry but of integrating on the diffeomorphisms, while keeping the geometry (in our case described by the conformal factor) exactly fixed. One should not confuse the diffeomorphisms with the zero modes of the action i.e. changes in the geometry which leave the action invariant.

On the numerical front accurate simulation have been given of two dimensional gravity, both pure and coupled with Ising spins, by adopting the measure $\prod_i dl_i f(l_i)$. The results are consistent with the Onsager exponents and in definite disagreement with the KPZ exponents [18] while the situation for the string susceptibility is still unclear [18][19]. That measures of type $\prod_i dl_i f(l_i)$ fail to reproduce the Liouville action can be understood by the following argument [8]: on the continuum for geometries which deviate slightly from the flat space one can compute approximately the Liouville action by means of a one loop calculation. If one tries to repeat a similar calculation for the Regge model with the measure $\prod_i dl_i f(l_i)$ one realizes that being the Einstein action in two dimensions a constant, the only dynamical content of the theory is played by the triangular inequalities. But at the perturbative level triangular inequalities do not play any role and thus one is left with a factorized product of independent differentials which bears no dynamics and thus no Liouville action.

3.3. Integration measure for the conformal factor

It is not enough to give the Faddeev- Popov determinant eq.(29); one must also give the explicit form of the measure $D[\sigma]$ in the Regge case. According to eq.(34) the distance between two nearby configurations $\sigma$ and $\sigma + \delta \sigma$ of the conformal factor is given by

$$(\delta \sigma, \delta \sigma) = \int d^2 \omega e^{2\sigma} \delta \sigma \delta \sigma.$$ (33)

Such an expression is a direct outcome of the original De-Witt measure [6]. From eq.(34) it follows that having parameterized the Regge surface by means of the $3N$ variables $p_i$

$$\{p_1, \ldots, p_{3N}\} \equiv \{\omega_{1,x}, \omega_{1,y}, \omega_{2,x}, \omega_{2,y}, \ldots, \omega_{N,x}, \omega_{N,y}, \lambda_0, \alpha_1, \alpha_2, \ldots, \alpha_{N-1}\}$$

(34)

$D[\sigma]$ is given by

$$D[\sigma] = \sqrt{\det J} \prod_{k=1}^{N} d^2\omega_k \prod_{i=1}^{N-1} d\alpha_i d\lambda_0$$

(35)

being $J$ the $3N \times 3N$ matrix

$$J_{ij} = \int d^2 \omega e^{2\sigma} \frac{\partial \sigma}{\partial p_i} \frac{\partial \sigma}{\partial p_j}.$$ (36)

with $\alpha_N = \sum_{i=1}^{N-1} (1 - \alpha_i) - 1$.

We notice that all $J_{ij}$ are given by convergent integrals except those involving two $\omega_i$ with the same indexes, which converge only for $\alpha_i > 1$. For example we have

$$J_{\omega_{i,x}, \omega_{i,x}} = (\alpha_i - 1)^2 \int d^2 \omega e^{2\sigma} \frac{(\omega_{i,x} - \omega_j)^2}{|\omega - \omega_i|^4}$$

(37)
For $\alpha_i - 1 \to 0$ $J_{ij}$ vanishes and for $\alpha_i < 1$ it has a well defined analytic continuation [3]. As a result of the factor $\delta_i = 1 - \alpha_i$ appearing in front of all rows of the type $J_{\alpha_i \alpha_j}$, det $J$ vanishes whenever an $\alpha_i$ equals 1, as expected from the fact that in such a situation the position of the vertex $i$ is irrelevant in determining the metric. A remarkable property of $D[\sigma]$ is to be invariant under the $SL(2, C)$ group with the result that the whole theory is invariant under such transformations. The measure can also be written as

$$\prod_{l=1}^{3N} dp_r e^{3N\lambda_0} \prod_{k=1}^{N} |\alpha_i - 1| \prod_{i,j > 1} |\omega_i - \omega_j|^4 \beta_{ij} Y$$

(38)

where

$$\beta_{ij} = \frac{3}{2} \frac{N}{N-2} \left( \frac{2}{N-1} - \delta_i - \delta_j \right) - \frac{2}{N-1} (39)$$

and $Y$ is a function only of the $\alpha_i$ and the harmonic ratios of the $\omega_i$.

The invariance under the finite dimensional group $SL(2, C)$ is sufficient to prove that the field $\exp(2q\sigma(x))$ maintains its canonical dimension $q$ which is in agreement with the analysis of the continuum theory in presence of the Weyl covariant measure [4].

3.4. Torus topology

The most general metric, modulo diffeomorphisms, is given by a flat metric $g_{\mu\nu}(\tau_1, \tau_2)$ times a conformal factor $e^{2\sigma}$. $\tau_1$ and $\tau_2$ are the two Teichmüller parameters in terms of which, with $\tau = \tau_1 + i\tau_2$,

$$ds^2 = dx^2 + 2\tau_1 dx dy + |\tau|^2 dy^2$$

(40)

and the fundamental region has been taken the square $0 \leq x < 1, 0 \leq y < 1$. The conformal factor for the torus can be expressed in terms of the torus Green’s function

$$G(\omega - \omega'|\tau) = \frac{1}{2\pi} \log \left| \frac{\theta_1(\omega - \omega'|\tau)}{\eta(\tau)} \right| - \frac{(\omega_y - \omega'_y)^2}{2\tau_2}$$

(41)

being $\theta_1(\omega|\tau)$ the Jacobi $\theta$-function and

$$\eta(\tau) = e^{\frac{i\pi}{\tau_2}} \prod_{n=1}^{\infty} [1 - e^{2in\pi\tau}]$$

(42)

From the relation

$$R(e^{2\sigma} \hat{g}) = e^{-2\sigma}(R(\hat{g}) - 2\Box \sigma)$$

(43)

we have that the conformal factor for the Regge surface with the topology of the torus is given by

$$\sigma(\omega) = \lambda_0 + \sum_{i=1}^{N} (\alpha_i - 1) \left[ \log \left| \frac{\theta_1(\omega - \omega_i|\tau)}{\eta(\tau)} \right| - \frac{\pi}{\tau_2} (\omega_y - \omega_{i,y})^2 \right]$$

(44)

Thus the physical degrees of freedom are $3N$: in fact in addition to the $2N$ $x_i, y_i$ we have $N - 1$ independent angular deficits ($\sum_{i=1}^{N} (\alpha_i - 1) = 0$), two Teichmüller parameters and $\lambda_0$, to which we must subtract the two conformal Killing vectors of the torus. We have the same number of physical degrees of freedom as the number of bones in a Regge triangulation of the torus with $N$ vertices as it can be easily checked through the Euler relation for a torus ($F + V = H = 3F/2$, from which $H = 3V$). The derivation of the Liouville action proceeds similarly as for the sphere topology with the final result for the partition function

$$\int D[\sigma] \frac{d^2 \tau}{\tau_2} |\eta(\tau)|^4 e^{\frac{i\pi}{\tau_2} S_l}$$

(45)

where

$$S_l = \sum_{i,j \neq i} \frac{(1 - \alpha_j)(1 - \alpha_j)}{\alpha_i} \left[ \log \left| \frac{\theta_1(\omega_j - \omega_i|\tau)}{\eta(\tau)} \right| - \frac{\pi}{\tau_2} (\omega_{i,y} - \omega_{j,y})^2 \right] + \lambda_0 \log[2\pi\eta^2] \times \sum_{i}(\alpha_i - 1) - \sum_{i} F(\alpha_i)$$

(46)

In the continuous limit eq. (46) goes over to the well known expression

$$\frac{1}{8\pi} \int d^2 \omega d^2 \omega' \sqrt{g} R \omega \frac{1}{\Box}(\omega, \omega') \sqrt{g} R \omega$$

(47)
3.5. Modular invariance

It is possible now to give an explicit, non formal proof of the modular invariance of the theory. In eq. (43) it is well known that $d^2 \tau |\eta(\tau)|^4 / \tau_2$ is invariant under the modular transformation

$$\tau \rightarrow \tau' = \frac{\tau a + b}{\tau c + d}$$  \hspace{1cm} (48)

with $(a, b, c, d) \in \mathbb{Z}$ and $ad - bc = 1$. Thus we are left to prove the modular invariance of $\int D[\sigma] e^{\frac{\sqrt{-1}}{2} S}$.

This is achieved by accompanying the change in $\tau$ by a proper change in the integration variables $\omega_1, \lambda_0$ given by

$$\omega' = \frac{\omega}{\tau c + d} \quad \lambda_0' = \lambda_0 + \log |\tau c + d|$$  \hspace{1cm} (49)

where the transformation of $\lambda_0$ follows from the transformation of $\sigma$ and the modular invariance of $G$, i.e. $G(\omega - \omega_1 | \tau) = G(\omega' - \omega'_1 | \tau')$. $S_1$, as given by eq. (10), is invariant under transformations (48), (49) because of the just cited modular invariance of the Green function and because

$$\eta \left( \frac{a \tau + b}{c \tau + d} \right) = e^{i \phi (c \tau + d) \frac{1}{2} \eta(\tau)}$$  \hspace{1cm} (50)

compensates the change in $\lambda_0$.

Again the invariance of the area leaves $\sqrt{\prod_{i=1}^{2N} d^2 \omega_i} d\lambda_0 \prod_{j=1}^{N-1} d\sigma_j$ invariant and this concludes the proof of modular invariance.

4. CONCLUSIONS

Starting from the De Witt distance among metrics we have derived, when we restrict ourselves to a subclass of geometries described by a finite number of parameters, the ensuing expression for the functional integration measure. The results eq. (4) and eq. (13) follow directly from the De Witt supermetric without any other additional input. They are mathematically well defined provided we work in the euclidean and on closed manifolds, i.e. compact manifolds without boundaries. In two dimensions we can give an exact expression for the F.P. both for the sphere and torus topology. Higher genus require the knowledge of the Green function on a surface of constant negative curvature and given Teichmüller parameters. The integration measure on the conformal factor appears in the form of a $3N \times 3N$ determinant which has the correct invariance properties as the F.P. term. The matrix elements of the determinant are given in terms of homogeneous integrals of dimension $\omega^{-2}$ of the type which appeared in the old conformal field theory. It would be of interest the closed evaluation of the determinant at least in some simple example or a rigorous estimate of the determinant for large $N$, which is of importance for the continuum limit.

The exact analytical evaluation of the functional determinants appearing in eq. (4) and eq. (13) in dimension higher that two has not yet been performed and we have discussed the main technical differences with respect to two dimensions. For $D > 2$ the extraction of the $\sigma$-dependence of the $\text{Det}$ appearing in eq. (13) is an old standing problem (see e.g. [21]). In the meantime the simpler approach with the measure (4) appears more viable in $D > 2$. According to the treatment of section (2.1), here one expects that for $M \rightarrow \infty$ the dependence on the parameter $C$ should disappear; actually such a dependence could be taken as a measure of the approach to the continuum limit. The $D = 3$ which appears simpler than the $D = 4$ case bears some relation with recent result of classical $2 + 1$ dimensional gravity in presence of point particle and progress in that field may also be helpful [22,23].

REFERENCES

1. T. Regge, Nuovo Cimento 19 (1961) 558.

2. P. Menotti, P.P. Peirano, Phys. Lett., B353 (1995) 444.

3. P. Menotti, P.P. Peirano, Nucl. Phys., B473 (1996) 426.

4. P. Menotti, P.P. Peirano, Preprint: IFUP-TH 38/96 [hep-th/9607071]. To be published in Nucl. Phys. B.

5. R. Friedberg, T.D. Lee, Nucl. Phys. B242 (1984) 145; G. Feinberg, R. Friedberg, T.D. Lee, Nucl. Phys. B245 (1984) 343.

6. J. Cheeger, W. Müller, R. Schrader, Comm. Math. Phys. 92 (1984) 405.

7. A. Jevicki, M. Ninomiya, Phys. Rev. D33 (1986) 1634.
8. B. S. De Witt, *Phys. Rev.* **160**, 1113.
9. J.S. Dowker, *J. Phys. A* **10** (1977) 115; *Phys. Rev. D* **36** (1987) 620.
10. P. O. Mazur, E. Mottola, *Nucl. Phys.* **B341** (1990) 187.
11. O. Alvarez, *Nucl. Phys.* **B216** (1983) 125.
12. G. Moore, P. Nelson, *Nucl. Phys.* **B266** (1986) 58.
13. J. Polchinski, *Comm. Math. Phys.* **104** (1986) 37.
14. A.M. Polyakov, *Phys. Lett.* **103B** (1984) 207.
15. D. Foerster, *Nucl. Phys.* **B283** (1987) 669.
16. E. Aurell, P. Salomonson, *Comm. Math. Phys.* **165** (1994) 233 and hep-th/9405140.
17. P. Ginsparg, Les Houches, Session XLIX, (1988), Elsevier Science Publishers (1989).
18. C. Holm, W. Janke, *Phys. Lett.*, **B375** (1996) 69; *Nucl. Phys.*, **B42** (Proc. Suppl.) (1995) 725.
19. W. Bock, J.C. Vink, *Nucl. Phys.*, **B438** (1995) 320.
20. E. D’Hoker, *Mod. Phys. Lett.*, **A6** (1991) 745.
21. S. Deser, contribution to this conference.
22. M. Ciafaloni, P. Valtancoli, *Nucl. Phys.*, **462** (1996) 453.
23. M. Welling, *Class. Quant. Gravity*, **13** (1996) 653.