Eigenvalue gaps for the Laplacian on hypersurfaces of the sphere

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Abstract

We provide lower estimates for the eigenvalues of the laplacian for hypersurfaces of the round sphere.

Introduction

The laplacian acting on functions on a compact riemannian manifold exhibits a discrete spectrum of positive eigenvalues counted with multiplicity:

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \]

We provide an estimate for the gaps between eigenvalues:

\[ \lambda_{i+1} - \lambda_i \geq C_i > 1 \]

in the case of a convex analytic hypersurface \( H \subset (S^{n+1}, e) \), \( n > 1 \) in the round sphere with constant depending on its second fundamental form. The problem appears interesting in the construction of minimal hypersurfaces with singularities and \( H \) is then a minimal hypersurface in \((S^{n+1}, e)\).

The harmonic extension method

We will apply the method of harmonic extension from \([CW]\) or \([SY]\) Let \( f : H \to \mathbb{R} \) be a function defined on a smooth hypersurface \( H \subset (S^{n+1}, e) \), dividing \((S^{n+1}, e)\) in two regions \( S_1, S_2 \) that we denote simply by \( S \). Let \( U : S \to \mathbb{R} \) be the solution of the following boundary value problem:

\[ \Delta_S U = 0, \quad U|_H = f \]

Then we have that if \( N \) is chosen as the unit normal to \( H \) and \( H \) its mean curvature then

\[ 0 = \Delta_S U = N(NU) + \Delta_H f + nHNU \]

Hence we have that by an integration by parts

\[ \int_S \Delta_S |\nabla U|^2 = -\int_H w(4\Delta_H f + 2nHw) - 2\int_H h(\nabla f, \nabla f) \] (1)

where \( w(x) = N(U) \). This identity will be the basis of our considerations. We assume that \( f \) possesses the unique continuation property on \( H \) while \( u \) is analytic in \( S \). Then \( N \) the normal vector field to \( H \) that satisfies the unique continuation property (e.g. for a minimal variety) then the set the analytic weight function \( w(x) \) and

\[ N = \{ x \in H/w(x) = 0 \} \]

is a recitifiable set of finite multiplicity in the hypersurface \( H \subset (S^{n+1}, e) \). Actually Sard’s lemma asserts that for almost all \( \eta \in (0, \eta_0) \) the sets

\[ N_\eta = \{ x \in H/|w(x)| \leq \eta \} \]

are smooth. Then we have the subsets

\[ H_\pm = \{ x \in H/ \pm w(x) > 0 \} \]

where we need to choose \( f \) in order to estimate the integrals \( \int_{H_\pm} w \Delta_{H_\pm} f \). We will glue the two functions \( f_\pm \) with a partition of unity \( \chi_\pm \) and transition area a tubular neighbourhood \( N_\eta \) of \( N \) of thickness \( \eta \) after the Lojasiewicz inequality for analytic functions, adapted to the case of an analytic submanifold of the sphere \((S^{n+1}, e)\). Indeed we have that the localizing functions satisfy

\[ \text{supp}(\chi_\pm) \subset H_\pm, \quad \chi_\pm = 1 \text{ in } H_\pm \backslash N_\eta = H_{\pm, \eta}, \quad |\nabla \chi_\pm| + \tau|\nabla^2 \chi_\pm| \leq \frac{C}{\tau} \]

for the parameter \( 0 < \tau < 1 \) controlling the transition regions.
The choice of \( f \). We select as
\[
f = \frac{(u^2 + \alpha^2)}{(v^2 + \beta^2)}
\]
where \( u, v \) is the eigenfunction corresponding to the eigenvalue \( \lambda, \mu \) and the \( \pm \) sign depends on the \( \mathcal{H}_\pm \) region.

This function obeys the differential equation on \( \mathcal{H}, \delta = \lambda - \mu \)
\[
\Delta_H f = 2 [\delta + \Pi] f
\]
where
\[
\Pi = \frac{\lambda \alpha^2}{u + \alpha^2} - \frac{\mu \beta^2}{u^2 + \beta^2} - \frac{4uv \nabla u \cdot \nabla v}{(u^2 + \alpha^2)(v^2 + \beta^2)} + \frac{|\nabla u|^2}{u^2 + \alpha^2} - \frac{\zeta |\nabla v|^2}{v^2 + \beta^2}, \quad \zeta = \frac{b^2 - 5v^2}{v^2 + b^2}, \quad \zeta_0 \leq \zeta \leq 1
\]
where \( \zeta_0 \) is determined by \( \beta \).

A. The \( \mathcal{H}_{+, \eta} \) region. In order to bound \( \Pi \) we appeal to Harnack and Berstein inequalities proved in \( [P2] \). Specifically, exhausting the region \( \mathcal{H}_\eta \) through \( 0 < \theta < 1 \):
\[
\mathcal{H}_\eta = \bigcup_{j=1}^N G_j, \quad G_j = \{ x \in \mathcal{H}/\eta \theta^j \leq w(x) \leq \eta \theta^j \}
\]

We have that for numerical constants \( c_1, c_2 > 0, d \geq 1 \) and near \( |u| \geq \rho \) and \( |v| \geq \rho \):
\[
\sup_{\mathcal{H}_\eta} |\nabla u| \leq c_1 (\eta \theta^j)^{\frac{d}{d-2}} \lambda^{d+n+1} \rho
\]
\[
\sup_{\mathcal{H}_\eta} |\nabla v| \leq c_2 (\eta \theta^j)^{\frac{d}{d-2}} \mu^{d+n+1} \rho
\]

If we assume that the hypersurface is real analytic - it holds for the minimal case- then the \( \rho \) tubular neighbourhood of the nodal sets of \( u, v \) varies as \( \rho \lambda^{n-1} \), \( \rho \mu^{n-1} \), and hence selecting \( \rho \) analogously we can make this arbitrarily small. Young’s inequality allows us to write that:
\[
\Pi \geq \frac{\lambda \alpha^2 + (1 - 2 \epsilon) |\nabla u|^2}{u^2 + \alpha^2} - \frac{\mu \beta^2 + \left( \frac{2}{d} + \zeta \right) |\nabla v|^2}{v^2 + \beta^2}
\]

Careful consideration of the preceding bounds allows us to select \( \alpha, \beta, \theta \) so that for \( n > 2 \)
\[
\theta = \frac{1}{(2\mu)^{\frac{n-2}{d}} \mu^{n(n-2)}}, \quad \alpha = 2 \sqrt{c_1 (\eta \theta^j)^{\frac{d}{d-2}} \lambda^{d+n} \rho}, \quad \beta = c_3 \rho
\]

The \( n = 2 \) case requires that
\[
\theta = \frac{1}{\mu}, \quad \alpha = 2 \sqrt{c_1 (\eta \theta^j)^{d} \lambda^{2d} \rho}, \quad \beta = c_3 \rho
\]

We conclude that
\[
\Pi \geq \frac{1}{2} \delta
\]

B. The region \( \mathcal{H}_{-, \eta} \). In this part of the bounding hypersurface \( \mathcal{H} \) we choose \(-f\) and use the same tricks as before.

C. The region \( \mathcal{N}_\eta \) Selecting accordingly we have that for \( \chi_0 = 1 - \chi_- - \chi_+ \) we have that
\[
\int_{\mathcal{N}_\eta} \chi_0 w \Delta_H f \leq C \eta^\ell
\]
for suitable \( \ell \).
Gluing patches together  Finally we have that for appropriate choice of $\eta$ relative to $\epsilon$ we have that:

$$4\delta \int_{\mathcal{H}_{\pm,\eta}} wf - \eta - 2 \int_{\mathcal{H}} (2Hw^2 + h(\nabla f, \nabla f)) \geq 2n \int_{\mathcal{H}} wf$$

Therefore we have that for minimal hypersurfaces then $H = 0$ and selecting $\mathcal{S}_1, \mathcal{S}_2$ so that $h(\nabla f, \nabla f) > 0$ we conclude that:

$$\delta > \frac{n}{2}$$

References

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