MARTINGALE SOLUTIONS FOR THE COMPRESSIBLE MHD SYSTEMS WITH STOCHASTIC EXTERNAL FORCES

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Abstract. In this paper we consider the three-dimensional compressible MHD system with stochastic external forces in a bounded domain. We obtain the existence of martingale solution which is a weak solution for the fluid variables, the Brownian motion on a probability space. The construction of the solution is based on the Galerkin approximation method, stopping time, the compactness method and Jakubowski Skorokhod theorem, etc.

1. Introduction

In this paper, we consider the following three-dimensional stochastic compressible Magnetohydrodynamic (MHD) system:

\[
\begin{align*}
  d\rho + \text{div}(\rho u)dt &= 0, \\
  d(\rho u) + [\text{div}(\rho u \otimes u) + \nabla p - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u - (\nabla \times B) \times B]dt &= \sum f_k(\rho, \rho u, x)d\beta^1_k(t), \\
  dB - [\nabla \times (u \times B) + \nu \Delta B]dt &= \sum g_k(B, x)d\beta^2_k(t), \\
  \text{div} B &= 0,
\end{align*}
\]

where \(\rho\) is the density, \(u\) is the velocity field of the fluid, \(p\) is the scalar pressure and \(p = a\rho^\gamma\) with a positive constant \(a\) and adiabatic index \(\gamma \geq 1\), \(B\) is the magnetic field induced by the charged fluid, \(\mu\) and \(\lambda\) are two viscosity constants satisfying \(2\mu + 3\lambda \geq 0\), \(\nu > 0\) is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, and all these kinetic coefficients and the magnetic diffusivity are independent of the magnitude and direction of the magnetic field, the symbol \(\otimes\) denotes the tensor product, \(f_k(\rho, \rho u, x)d\beta^1_k\) and \(g_k(B, x)d\beta^2_k\) are stochastic external forces, where \(\beta^1_k, \beta^2_k, k = 1, 2, \cdots\) are two sequences of independent one-dimensional \(\mathbb{R}\)-valued Brownian motions. Denote \(\beta_k = (\beta^1_k, \beta^2_k)\).

System (1.1) models the interaction between a conductor fluid and the magnetic field in the presence of random perturbations. The inclusion of the stochastic terms in the governing equations is widely used to account for random fluctuations and past history of the system.

Notice that we have followed the usual convention of including the divergence-free condition on the magnetic field \(B\). This is really a condition on the initial data, as this property is preserved by the dynamics of (1.1). To be more precise, we impose the assumptions on the initial data

\[
\begin{align*}
  \rho|_{t=0} &= \rho_0, \quad \rho u|_{t=0} = m_0, \quad B|_{t=0} = B_0, \quad \text{div} B_0 = 0, \\
  \text{and the boundary conditions} \quad u|_{\partial D} &= 0, \quad B|_{\partial D} = 0,
\end{align*}
\]

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where $D \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial D$, and $\mathbf{n}$ is the unit outer normal at the boundary.

The deterministic version of system (1.1) was extensively studied in the literature. For instance, the existence of local strong solutions with large initial data has proved by Fan-Yu [11]; the existence of global weak and variational solution have obtained by Ducomet-Feireisl [10], Fan-Yu [12], Hu-Wang [17] [18]; the time decay of the compressible MHD systems with small initial data has established in [8, 23, 31, 36, 37]; the low Mach number limit problem was studied in [19, 21].

When $\rho = 1$, system (1.1) reduces to the stochastic incompressible MHD system, which has also received a lot of attention [1, 32, 35]. On the other hand, when $\rho$ is not a constant, system (1.1) becomes the stochastic compressible Navier-Stokes equations, to which, to the best of our knowledge, much less is known. Feireisl-Maslowski-Novotný in [14] considered the three-dimensional compressible Navier-Stokes equations driven by the stochastic external forces in Sobolev spaces and obtained the existence of solutions by using the abstract measurability theorem ([14]) to prove that the weak solution generates a random variable. For the existence of martingale solutions to the three-dimensional compressible Navier-Stokes equations driven by the multiplicative noise, see [7, 34, 38]. Recently, Breit-Feireisl-Hofmanová considered the low Mach number limit problem in [6].

1.1. Definition of solutions and assumptions. We will seek a particular type of solution to system (1.1), namely the martingale solution. The definition of such solutions is given as follows.

**Definition 1.1.** A martingale solution of (1.1)-(1.3) is a system $((\Omega, \mathcal{F}, \mathcal{F}_t, P), \beta_k, \rho, u, B)$ with the following properties:

1. $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a stochastic basis, where $\mathcal{F}_t$ is a filtration on the probability space $(\Omega, \mathcal{F}, P)$, i.e., a nondecreasing family $\{\mathcal{F}_t : t \geq 0\}$ of sub $\sigma$-fields of $\mathcal{F} : \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 \leq s < t < \infty$;
2. $\{\beta_k^1\}_{k \geq 1}, \{\beta_k^2\}_{k \geq 1}$ are two sequences of independent $\mathbb{R}$-valued Brownian motions;
3. for any $\varphi, \psi \in C_0^\infty(D)$, $\langle \rho, \varphi \rangle$, $\langle \rho u, \psi \rangle$ and $\langle B, \psi \rangle$ are progressively measurable;
4. for all $1 \leq p < \infty$, $\rho \geq 0$, $\rho \in L^p(\Omega, L^\infty(0, T; L^p(D)))$, $u \in L^p(\Omega, L^2(0, T; H_0^1(D)))$,
   
   $B \in L^p(\Omega, L^\infty(0, T; L^2(D))) \cap L^p(\Omega, L^2(0, T; H_0^1(D)))$ such that for all $t \in [0, T]$, $\varphi \in C_0^\infty([0, T] \times D)$ and $\psi \in C_0^\infty([0, T] \times D)$, it holds that $P$-a.s.

\[
\int_D \rho(t) \varphi dx - \int_D \rho_0 \varphi dx = \int_0^t \int_D \rho u \cdot \nabla \varphi dx dt,
\]

\[
\int_D (\rho u)(t) \cdot \psi dx - \int_0^t \int_D [\rho u \cdot \nabla \psi - \mu \nabla u \cdot \nabla \psi - (\lambda + \mu) \text{div} u \cdot \text{div} \psi + a \rho^\gamma \cdot \text{div} \psi] \, dx dt
\]

\[
= \int_D \rho_0 u_0 \cdot \psi dx + \int_0^t \int_D (\nabla \times (u \times B)) \cdot B \cdot \psi dx dt + \sum_{k=1}^\infty \int_0^t \int_D f_k(\rho, \rho u, x) \cdot \psi dx \beta_k^1,
\]

\[
\int_D B(t) \cdot \psi dx - \int_0^t \int_D [\nabla \times (u \times B) + \nu \Delta B] \cdot \psi dx dt = \int_D B_0 \cdot \psi dx + \sum_{k=1}^\infty \int_0^t \int_D g_k(B, x) \cdot \psi d\beta_k^2;
\]

5. the equation (1.1) is satisfied in sense of renormalized solutions, that is,

\[
'b(\rho) \psi + \text{div}(b(\rho) u) + (b'(\rho) \rho - b(\rho)) \text{div} u = 0
\]

holds in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$ $P$-a.s. for any $b \in C^1(\mathbb{R})$ with $b'(z) \equiv 0$ for all $z \in \mathbb{R}$ large enough.
Notations. Throughout this paper, we drop the parameter $\omega \in \Omega$ for simplicity of notations. \{\mathcal{F}_t\} is a right continuous filtration over the probability space (\(\Omega, \mathcal{F}, \mathbb{P}\)) such that \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-negligible subsets of \(\Omega\). We use \(\mathbb{E}X = \int_\Omega XD\mathbb{P}\) to denote the expectation of the random variable \(X(\omega, t)\) for fixed \(t\). All stochastic integrals are defined in the sense of Itô; see [9, 23]. Moreover, \(C\) denotes a generic constant which may vary in different estimates. For simplicity, we will write \(A \lesssim B\) if \(A \leq CB\).

We use \(C([0, T]; V_\omega)\) to denote the subspace of \(L^\infty([0, T]; V)\) consisting of those functions which are a.e. equal to weakly continuous functions with values in \(V\). We denote \(L_2(U, Y)\) the set of all Hilbert-Schmidt operators from \(U\) to \(Y\).

**Assumptions on the stochastic forces.** We assume that the stochastic forces \(f_k, g_k\) satisfy the following conditions:

(A) \(f_k(\rho, \rho u, x), g_k(B, x)\) are progressively measurable and \(C^1\) continuous in \(\rho, \rho u, x; B, x\), respectively, and

\[
\begin{align*}
&f_k(\rho, \rho u, x) = f_{k,1}(x, \rho) + f_{k,2}(x)\rho u, \\
&\sum_{k=1}^\infty |f_{k,1}|^2 \leq C|\rho|^{\gamma+1}, \quad \sum_{k=1}^\infty |\partial_{\rho} f_{k,1}|^2 \leq C|\rho|^{\gamma-1}, \quad \sum_{k=1}^\infty |f_{k,2}|^2 \leq C, \\
&\sum_{k=1}^\infty |g_k(B, x)|^2 \leq C|B|^2, \quad \sum_{k=1}^\infty |\partial_B g_k| \leq C.
\end{align*}
\]

(1.5)

The data \(\rho_0, m_0, B_0\) are assumed to satisfy the following conditions:

\[
\rho_0 \in L^{2p}(\Omega, L^1(D)), \quad \rho_0 \geq 0; \quad m_0 = 0 \quad \text{if} \quad \rho_0 = 0, \\
\frac{|m_0|^2}{\rho_0} \in L^p(\Omega, L^1(D)); \quad B_0 \in L^{2p}(\Omega, L^2(D)), \quad \text{div} B_0 = 0.
\]

(1.6)

Denote

\[
F(\rho, \rho u)dW_1 := \sum_{k=1}^\infty f_k(\rho, \rho u, x)d\beta_k^1, \quad G(B)dW_2 := \sum_{k=1}^\infty g_k(B, x)d\beta_k^2.
\]

By (130) and Hölder’s inequality, we have for \(\ell > \frac{3}{2}\),

\[
\|F(\rho, \rho u)^2\|_{L_2(U, W^{-t,2}(D))}^2 = \sum_{k=1}^\infty \|f_k(\rho, \rho u, x)^2\|_{W^{-t,2}(D)}^2 \lesssim \sum_{k=1}^\infty \|f_k(\rho, \rho u, x)^2\|_{L^1(D)}^2 \lesssim \left( \int_D \sqrt{\rho} \sum_{k=1}^\infty \frac{f_{k,1}(\rho, x)}{\sqrt{\rho}} \sqrt{\rho u} \sum_{k=1}^\infty f_{k,2}(x) dx \right)^2 \lesssim \|\rho\|_{L^1} \|\rho\|_{L^1(D)}^{\gamma} + \|\sqrt{\rho u}\|_{L^2(D)}^{\gamma} \lesssim 1,
\]

\[
\|G(B)^2\|_{L_2(U, W^{-t-2,2}(D))}^2 = \sum_{k=1}^\infty \|g_k(B, x)^2\|_{W^{-t-2,2}(D)}^2 \lesssim \sum_{k=1}^\infty \|g_k(B, x)^2\|_{L^2(D)}^2 \lesssim \|B\|_{L^2(D)}^{\gamma} \lesssim 1.
\]

Then the Itô’s integrals \(\int_0^T F(\rho, \rho u)dW_1\) and \(\int_0^T G(B)dW_2\) are well-defined martingales in \(W^{-t,2}(D)\) and \(W^{-t-2,2}(D)\) respectively. Since \(W_1\) and \(W_2\) don’t converge on \(U\), we define \(U_0 \supset U\) by \(U_0 = \{v = \sum_{k \geq 1} \alpha_k e_k : \sum_{k \geq 1} \frac{\alpha_k^2}{k^2} < \infty\}\) with the norm \(\|v\|_{U_0}^2 = \sum_{k \geq 1} \frac{\alpha_k^2}{k^2}\).
\[ v = \sum_{k \geq 1} \alpha_k e_k. \] Note that the embedding \( U \hookrightarrow U_0 \) is Hilbert-Schmidt and \( \{e_k\}_{k \geq 1} \) is a complete orthonormal basis of \( U \). Moreover, we have \( W_1, W_2 \in C([0, T], U_0) \) \( \mathbb{P} - \text{a.s.} \).

1.2. Main results. Our main result is stated in the following theorem:

**Theorem 1.2.** Suppose that the initial data \((\rho_0, m_0, B_0)\) satisfies \( (\ref{1.6}) \) and \( (\ref{1.5}) \) holds. Let \( \gamma > 3/2 \). If \( D \subset \mathbb{R}^3 \) is a bounded domain of class \( C^{2+\alpha} \) with \( \alpha > 0 \), then there exists a martingale solution \( ((\tilde{\Omega}, \mathcal{F}, \mathbb{P}), \mathcal{F}_t, \beta_k, \rho, u, B) \) in the sense of Definition \( (\ref{1.1}) \) with

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathcal{E}(t) + \int_0^T \int_D (\mu |\nabla u|^2 + (\lambda + \mu) |\text{div} u|^2 + \nu |\nabla B|^2) \, dx \, dt \right]^p \lesssim 1
\]

to the problem \( (\ref{1.1}), (\ref{1.3}) \) for any given \( T > 0 \) and for all \( 1 \leq p < \infty \).

The basic idea to prove Theorem 1.2 is to first construct the approximate schemes to the problem \( (\ref{1.1}), (\ref{1.3}) \), and then establish tightness estimates of the approximate solutions and pass to the limits in the spirit of Feireisl \( [13, 15] \). Of course, the apparent challenge stems from the stochastic term and the Itô integral, which is the primary focus of the work.

More precisely, as a first step we use Faedo-Galerkin method to construct the approximate system involving the artificial viscosity and artificial pressure, and apply the standard fixed point argument to obtain an \( n \)-dimensional truncated local solution \((\hat{\beta}_k, \rho_n, u_n, B_n)\) in the time interval \([0, \tau_{n,N})\). The existence of solutions in the whole time \([0, T]\) will be achieved by the energy estimates. Additional challenges arise due to the probabilistic nature of the system. Our method relies on a delicate stopping time argument with the use of the Burkholder-Davis-Gundy inequality. In particular, this approach leads to the convergence of the approximate solutions on \([0, T]\).

However the convergence is too weak to guarantee that the limit is a solution on \([0, T]\). In the two-dimensional case, it can be shown by using certain monotonicity principle that the nonlinear terms converge to the right limit and hence a global strong solution can be obtained \( [26] \). But when the space dimension is three the monotonicity does not hold and to the best of our knowledge there is no result on the global strong solutions. This is why we pursue instead the Martingale solutions. As is explained, the main issue is the convergence of the nonlinear terms. To this end, we need to derive the tightness and show the convergence as \( n \to \infty \) of the approximate finite-dimensional solutions in the Faedo-Galerkin method in three steps:

1. We apply the deterministic compactness criterion and Arzela-Ascoli’s Theorem to get the tightness property of the approximate solution \((\hat{\beta}_k, \rho_n, u_n, B_n)\). Moreover in order to analyze the nonlinear term, we prove the tightness of \( \rho_n u_n \) as well.

Then from the Jakubowski-Skorokhod Theorem, there exist a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) and random variables \((\hat{\beta}_{k,n}, \hat{\rho}_n, \hat{u}_n, \hat{\rho}_n \hat{u}_n, \hat{B}_n) \to (\beta_k, \rho, u, h, B) \) \( \tilde{\mathbb{P}} - \text{a.s.} \). Denote \( m_n = \rho_n u_n \), since they have same distribution, that is, for \( A \in \tilde{\mathcal{F}} \), \( \tilde{\mathbb{P}}(\tilde{m}_n \in A) = \tilde{\mathbb{P}}(\rho_n u_n \in A) = \tilde{\mathbb{P}}(\hat{\rho}_n \hat{u}_n \in A) \) we have \( \tilde{m}_n = \hat{\rho}_n \hat{u}_n \) almost surely \( \). With the the property that the probability distribution of \((\hat{\beta}_{k,n}, \hat{\rho}_n, \hat{u}_n, \hat{\rho}_n \hat{u}_n, \hat{B}_n)\) is the same as that of \((\hat{\beta}_k, \rho_n, u_n, \rho_n u_n, B_n)\). By using a cut-off function we can also show that the random variable \((\hat{\beta}_{k,n}, \hat{\rho}_n, \hat{u}_n, \hat{\rho}_n \hat{u}_n, \hat{B}_n)\) satisfy the approximate equations in \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}); \)

2. In view of the uniformly integrable criterion and Vitali’s convergence Theorem, together with the almost sure convergence on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), \) we can obtain the limit process of the stochastic parts, it’s quadratic variation and cross variation are martingale;

3. Finally, by the new method in \( [3, 28] \) on the martingale problem of Stroock and Varadhan, we can prove that the system \((\tilde{\Omega}, \tilde{\mathcal{F}}, \hat{\beta}_k, \rho, u, B)\) is a martingale solution of the approximation scheme. After obtaining the foregoing solutions to the approximate
scheme by the Faedo-Galerkin method, we shall follow the idea of [15, 30] to deal with the artificial viscosity and artificial pressure using the similar convergence framework for $n \to \infty$ in the Faedo-Galerkin solutions. Due to insufficient integrability of the density, the effective viscous flux and the cut-off function technique will be used to obtain the strong convergence of the density. Then we pass to the limits of vanishing artificial viscosity and vanishing artificial pressure, and the limit of the solutions to the approximate scheme is a martingale solution with finite energy to the initial-boundary value problem of (1.1)-(1.3). Note that here we shall use the property of martingale.

The rest of the paper is organized as follows. We shall recall some basic theory of stochastic analysis in Section 2. In Section 3 we shall construct the solutions to an approximate scheme by the Faedo-Galerkin method. In Section 4, we shall pass to the limit as the artificial viscosity goes to zero, and in Section 5 we shall pass to the limit as the artificial pressure goes to zero.

2. Preliminaries

In this section, we first introduce some function spaces. Let $D \subset \mathbb{R}^3$ be an open subset with smooth boundary $\partial D$. Let $L^p(D) (1 \leq p < \infty)$ denote the Banach space of Lebesgue measurable $\mathbb{R}^3$ valued integrable functions on the set $D$ with the standard norm $\|u\|_{L^p(D)}$, and $L^\infty(D)$ the Banach space of Lebesgue measurable essentially bounded $\mathbb{R}^3$ valued functions defined on $D$ with the standard norm $\|u\|_{L^\infty(D)}$. If $p = 2$, then $L^2(D)$ is a Hilbert space with the scalar product denoted by

$$\langle u, v \rangle_{L^2(D)} = \int_D u(x)v(x)dx, \quad \text{for } u, v \in L^2(D).$$

Let $H^1(D)$ stand for the Sobolev space of all $u \in L^2(D)$ for which there exist weak derivatives $\frac{\partial u}{\partial x_i} \in L^2(D), i = 1, 2, 3$. It is a Hilbert space with the scalar product denoted by

$$\langle u, v \rangle_{H^1(D)} = \langle u, v \rangle_{L^2(D)} + \langle \nabla u, \nabla v \rangle_{L^2(D)}, \quad \text{for } u, v \in H^1(D).$$

For a probability space $(\Omega, \mathcal{F}, P)$ and a Banach space $X$, denote by $L^p(\Omega, L^q(0,T;X))(1 \leq p, q < \infty)$ the space of random functions defined on $\Omega$ with value in $L^q(0,T;X)$, endowed with the norm:

$$\|u\|_{L^p(\Omega, L^q(0,T;X))} = \left( \mathbb{E} \|u\|_{L^q(0,T;X)}^p \right)^{\frac{1}{p}}.$$

If $q = \infty$, we write

$$\|u\|_{L^p(\Omega, L^\infty(0,T;X))} = \left( \mathbb{E} \text{ess sup}_{0 \leq t \leq T} \|u\|_{L^\infty(X)}^p \right)^{\frac{1}{p}}.$$

We now recall some preliminaries of stochastic analysis and useful tools for the sake of convenience and completeness. For details, we refer the reader to [9] 23 and the references therein.

Let us first recall the Itô formula for a local martingale.

**Lemma 2.1.** Suppose that $M_t = (M^1_t, M^2_t, \ldots, M^n_t)$ is a vector valued continuous local martingale, that is $(M^i_t, \mathcal{F}_t)$ is a local martingale for each $i = 1, 2, \ldots, n$ and $t \in \mathbb{R}_+$. Let $A_t = (A^1_t, A^2_t, \ldots, A^n_t)$ be a vector valued continuous processes adapted to the same filtration $\{\mathcal{F}_t\}$ such that the total variation of $A^i_t$ on each finite interval is bounded almost surely, and
and the initial data: Let $X_t = (X_t^1, X_t^2, \ldots, X_t^n)$ be a vector valued adapted processes such that $X_t = X_0 + M_t + A_t$, and let $\Phi \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$. Then, for any $t \geq 0$, the equality

$$
\Phi(t, X_t) = \Phi(0, X_0) + \sum_{i=1}^{n} \int_{0}^{t} \frac{\partial}{\partial x_i} \Phi(s, X_s) dM_s^{i} + \sum_{i=1}^{n} \int_{0}^{t} \frac{\partial}{\partial x_i} \Phi(s, X_s) dA_s^{i} + \frac{1}{2} \sum_{i,j=1}^{n} \int_{0}^{t} \frac{\partial^2}{\partial x_i x_j} \Phi(s, X_s) d\langle M^{i}, M^{j} \rangle_s + \int_{0}^{t} \frac{\partial}{\partial s} \Phi(s, X_s) ds
$$

holds almost surely. Here $\langle \cdot, \cdot \rangle_t$ is the cross variation process defined by $\langle X, Y \rangle_t = \frac{1}{4} \{(X + Y)_t - (X - Y)_t \}$, and $\langle X \rangle_t$ denotes the quadratic variation of $X$ on $[0, t]$.

3. The Faedo-Galerkin approximation and a priori estimates

3.1. Approximation scheme and a priori estimates. For fixed $\varepsilon, \delta > 0$, we consider the following approximation problem with $\beta > \max\{4, \gamma\}$:

$$
\begin{align*}
\begin{cases}
    d\rho + \text{div}(\rho u) dt = \varepsilon \Delta \rho dt, \\
    d(\rho u) + [\text{div}(\rho u \otimes u)] + a \nabla \rho^{\gamma} + \delta \nabla \rho^{\beta} + \varepsilon \nabla u \cdot \nabla \rho - (\lambda + \mu) \nabla \text{div} u - \mu \Delta u] dt \\
    dB - \{\nabla \times (u \times B) + \nu \Delta B\} dt = \sum_{k \geq 1} f_k(\rho, \rho u, x) d\beta_k^2(t),
\end{cases}
\end{align*}
$$

(3.1)

with the boundary conditions:

$$
\nabla \rho \cdot n|_{\partial D} = 0, \quad u|_{\partial D} = 0, \quad B|_{\partial D} = 0
$$

(3.2)

and the initial data:

$$
\rho|_{t=0} = \rho_{0, \delta} \in C^{2+\varepsilon}(\overline{D}), \quad \nabla \rho_{0, \delta} \cdot n|_{\partial D} = 0, \\
(\rho u)|_{t=0} = m_{0, \delta} \in C^{2}(\overline{D}), \quad B|_{t=0} = B_{0, \delta} \in C^{2}(\overline{D}), \quad \text{div} B_{0, \delta} = 0.
$$

(3.3)

Here we take the initial data that satisfies the following conditions:

$$
\rho_{0, \delta} \to \rho_0 \text{ in } L^{\gamma}(D), \quad B_{0, \delta} \to B_0 \text{ in } L^{2}(D) \text{ as } \delta \to 0; \\
0 < \delta \leq \rho_{0, \delta}(x) \leq \delta^{-\frac{1}{\beta}}, \quad m_{0, \delta} = h_\delta \sqrt{\rho_{0, \delta}},
$$

almost surely and $h_\delta$ is defined as follows. First, take

$$
\tilde{m}_{0, \delta}(x) = \begin{cases}
    m_0(x) \sqrt{\frac{\rho_{0, \delta}(x)}{\rho_0(x)}}, & \text{if } \rho_0(x) > 0, \\
    0, & \text{if } \rho_0(x) = 0.
\end{cases}
$$

By using (1.6), we have

$$
\frac{|\tilde{m}_{0, \delta}|^2}{\rho_{0, \delta}} \text{ is bounded in } L^p(\Omega, L^1(D)) \text{ independently of } \delta > 0, \ \forall \ p \in [1, \infty).
$$

Since $C^{2}(\overline{D})$ is dense in $L^{2}(D)$, then we can find $h_\delta \in C^{2}(\overline{D})$ such that

$$
\left\| \frac{\tilde{m}_{0, \delta}}{\sqrt{\rho_{0, \delta}}} - h_\delta \right\|_{L^p(\Omega, L^2(D))} < \delta.
$$

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with two given Brownian motions $\beta_1^k$ and $\beta_2^k$, $k \geq 1$, denote $\hat{\beta}_k = (\beta_1^k, \beta_2^k)_{k \geq 1}$. 

In order to solve (3.1)-(3.3), we first consider a suitable orthogonal system formed by a family of smooth functions $\varphi_n$ vanishing on $\partial D$. One can take the eigenfunctions of the Dirichlet problem for the Laplacian operator:

$$-\Delta \varphi_n = \lambda_n \varphi_n \quad \text{on} \quad D, \quad \varphi_n|_{\partial D} = 0.$$ 

Now, we consider a sequence of finite dimensional spaces

$$X_n = \text{span}\{\varphi_j\}_{j=1}^n, \quad n = 1, 2, \ldots.$$ 

First, we introduce a family of operators

$$\mathcal{M}[\rho] : X_n \rightarrow X_n; \quad \langle \mathcal{M}[\rho]v, w \rangle = \int_D \rho v \cdot w dx, \quad \forall \ v, w \in X_n.$$ 

If $\inf_{x \in \partial D} \rho > 0$, then these operators are invertible and we have

$$\|\mathcal{M}^{-1}[\rho]\|_{\mathcal{L}(X_n, X_n)} \leq \left( \inf_{x \in \partial D} \rho(x) \right)^{-1},$$

where $\mathcal{M}^{-1}[\rho]$ is the inverse of $\mathcal{M}[\rho]$. Moreover, by using the following identity

$$\mathcal{M}^{-1}[\rho_1] - \mathcal{M}^{-1}[\rho_2] = \mathcal{M}^{-1}[\rho_2](\mathcal{M}[\rho_2] - \mathcal{M}[\rho_1])\mathcal{M}^{-1}[\rho_1],$$

we can deduce that the map

$$\rho \mapsto \mathcal{M}^{-1}[\rho]$$

mapping $L^1(D)$ into $\mathcal{L}(X_n, X_n)$ is well-defined and satisfies

$$\|\mathcal{M}^{-1}[\rho_1] - \mathcal{M}^{-1}[\rho_2]\|_{\mathcal{L}(X_n, X_n)} \leq C(n, \eta)\|\rho_1 - \rho_2\|_{L^1(D)} \quad (3.4)$$

for any $\rho_1, \rho_2$ belonging to the set

$$N_\eta = \left\{ \rho \in L^1 : \inf_{x \in \partial D} \rho \geq \eta > 0 \right\}.$$ 

Now we will try to solve (3.1) on $X_n$. For this, let $\mathbb{P}$ be the projection from $L^2(D)$ to $X_n$. $\mathbb{P}$ is also a linear projection from $H^1(D)$ to $X_n$. In fact,

$$\mathbb{P} h = \sum_{i=1}^n \langle h, e_i \rangle_{H^1 - H^1} e_i, \quad h \in H^1(D), e_i \in X_n.$$ 

We shall look for the sequence of pairs $(u_n, B_n) \in C([0, T]; X_n)$ satisfying the integral equations:

$$\int_D \rho(t)u_n(t) \cdot \varphi dx + \int_0^t \int_D \left[ \text{div}(\rho u_n \otimes u_n) - \mu \Delta u_n - (\nabla \times B_n) \times B_n \right] \cdot \varphi dxds = \int_D m_0 \cdot \varphi dx + \int_0^t \int_D \left[ \nabla \left( \lambda + \mu \right) \text{div} u_n - a \rho^\gamma - \delta \rho^\beta \right] - \varepsilon \nabla u_n \cdot \nabla \varphi \right] \cdot \varphi dxds + \int_0^t \sum_{k \geq 1} f_k^\varepsilon (\rho, \rho u_n, x, \varphi) d\beta_k^1,$$

$$\int_D B_n(t) \cdot \varphi dx - \int_D B_0 \cdot \varphi dx = \int_0^t \int_D \left[ \nabla \times (u_n \times B_n) + \nu \Delta B_n \right] \cdot \varphi dxds + \int_0^t \sum_{k \geq 1} g_k(B_n, x) \cdot \varphi dx d\beta_k^2,$$
for all \( t \in [0, T] \) and any function \( \varphi \in X_n \), where
\[
\sum_{k \geq 1} f_k^n(\rho, \rho u_n, x) := M_2^n[\rho] \left( \sum_{k \geq 1} f_k(\rho, \rho u_n, x)/\sqrt{D} \right), \quad \text{with} \quad \langle M_2^n[\rho] v, w \rangle = \int_D \sqrt{\rho v} \cdot w dx.
\]
Then, we can rewrite the above integral equations as
\[
(u_n(t), B_n(t)) = \left( \mathcal{M}^{-1}[\rho(t)] \left[ m_0^* + \int_0^t N_1[\rho(s), u_n(s), B_n(s)] ds + \int_0^t \sum_{k \geq 1} f_k^n(\rho, \rho u_n, x) d\beta_k^1 \right], \right.
\]
\[
B_0^* + \int_0^t N_2[u_n(s), B_n(s)] ds + \int_0^t \sum_{k \geq 1} g_k(B_n, x) d\beta_k^2 \right),
\]
where
\[
\langle m_0^*, \varphi \rangle = \int_D m_0 \cdot \varphi dx, \quad \langle B_0^*, \varphi \rangle = \int_D B_0 \cdot \varphi dx,
\]
\[
\langle N_1[\rho, u_n, B_n], \varphi \rangle = \int_D \left[ \mu \Delta u_n - \text{div}(\rho u_n \otimes u_n) + \nabla \left( (\lambda + \mu) \text{div} u_n - a \rho^\gamma - \delta \rho^2 \right) - \varepsilon \nabla u_n \cdot \nabla \rho + (\nabla \times B_n) \times B_n \right] \cdot \varphi dx,
\]
\[
\langle N_2[u_n, B_n], \varphi \rangle = \int_D \left[ \nabla \times (u_n \times B_n) + \nu \Delta B_n \right] \cdot \varphi dx.
\]

We will take the following steps to solve (3.5).

**Step 1:** Solve \( \rho \) in terms of \( u_n \) and derive a fixed-point problem. Note that \( \rho \) is determined as the solution of the following Neumann initial-boundary value problem (for e.g., see Lemma 2.2 [13]):
\[
\begin{cases}
\rho_t + \text{div}(\rho u) = \varepsilon \Delta \rho, \\
\nabla \rho \cdot n|_{\partial D} = 0, \\
\rho|_{t=0} = \rho_0, \delta (x).
\end{cases}
\]

From [15] Lemma 2.1, Lemma 2.2 we know that there exists a mapping \( S : C([0, T]; C^2(\overline{D})) \to C([0, T]; C^{2+\alpha}(\overline{D})) \) such that

1. \( \rho = S[u] \) is the unique classical solution of (3.6);
2. \( \rho \exp \left( -\int_0^t \|\text{div}(u)\|_{L^\infty(D)} ds \right) \leq S[u](t, x) \leq \bar{\rho} \exp \left( -\int_0^t \|\text{div}(u)\|_{L^\infty(D)} ds \right) \) for all \( t \in [0, T] \), where \( 0 < \bar{\rho} \leq \rho_0, \delta \leq \bar{\rho} \);
3. \( \|S[u_1] - S[u_2]\|_{C([0, T]; W^{1,2}(D))} \leq TC(K, T) \|u_1 - u_2\|_{C([0, T]; W^{1,2}(D))} \) for any \( u_1, u_2 \) in \( M_K := \{ u \in C([0, T]; W^{1,2}(D)) : \|u(t)\|_{L^\infty(D)} + \|\nabla u(t)\|_{L^\infty(D)} \leq K \text{ for all } t \} \).

Thus on \( X_n \) we can write \( \rho = S[u_n] \), and construct the approximate solutions for (3.1)-(3.3) by means of (3.5). That is, we can get the approximate solutions by solving the following integral equations:
\[
(u_n(t), B_n(t)) = \left( \mathcal{M}^{-1}[S[u_n]](t) \left[ m_0^* + \int_0^t N_1[S[u_n]], u_n(s), B_n(s)] ds + \int_0^t \sum_{k \geq 1} f_k^n(S[u_n], S[u_n] u_n, x) d\beta_k^1(s) \right], \right.
\]
\[
B_0^* + \int_0^t N_2[u_n(s), B_n(s)] ds + \int_0^t \sum_{k \geq 1} g_k(B_n, x) d\beta_k^2 \right),
\]
\[ B_0^* + \int_0^t N_2[u_n(s), B_n(s)] ds + \int_0^t \sum_{k \geq 1} g_k(B_n, x) d\beta_k^2(s). \]

**Step 2.** Solve a cut-off problem. For \( N > 0 \), choose a \( C^\infty \) smooth cut-off function \( \theta_N : [0, \infty) \to [0, 1] \) such that

\[
\theta_N(x) := \begin{cases} 
1, & \text{for } |x| \leq N, \\
0, & \text{for } |x| \geq N + 1.
\end{cases}
\]

Note that we can consider the following cut-off problem of (3.5) for a fixed \( n \):

\[(u_n^N(t), B_n^N(t)) = \left( \mathcal{M}^{-1}[S[u_n^N](t)] \left[ m_0 + \int_0^t \theta_N^u B_n^N(s) N_1[S[u_n^N](s), u_n^N(s), B_n^N(s)] ds \right. \right.
\]
\[ + \int_0^t \sum_{k \geq 1} \theta_N^{u,S,B_n^N}(s) f_k^u(S[u_n^N], S[u_n^N], x) ds + \sum_{k \geq 1} \theta_N^{u,S,B_n^N}(s) \left. \right) \]
\[ B_0^* + \int_0^t \theta_N^{u,S,B_n^N}(s) N_2[u_n^N(s), B_n^N(s)] ds + \sum_{k \geq 1} \theta_N^{u,S,B_n^N}(s) g_k(B_n^N, x) d\beta_k^2(s), \]

where \( \theta_N^{u,S,B_n^N}(s) = \theta_N \left( \max\{||u_n^N(s)||_{W^{1, \infty}}, ||B_n^N(s)||_{W^{1, \infty}} \} \right) \). Let \( N \) and \( n \) be fixed. By means of the standard fixed point argument on the Banach space \( C([0, T]; X_n) \), we can solve the integral equations (3.7), at least on a short time interval \([0, T_{n,N}]\), \( T_{n,N} \leq T \). For simplicity, we denote \((u_n^N, B_n^N) := (u_n^N, B_n^N)\).

**Proposition 3.1.** Given a \( T > 0 \), for each fixed \( n \) and \( N \), there exists a \( T_{n,N} \in [0, T] \) such that the equation (3.7) admits a unique solution \((u_n^N, B_n^N) \in L^2(\Omega, C([0, T_{n,N}]; X_n))^2\).

**Proof.** Define

\[ B_{n,T_{n,N}} = \left\{ U^N := (u^N, h^N) \in L^2(\Omega, C(I_{n,N}; X_n))^2 : \ ||(u^N, h^N)||_{C(I_{n,N}; W^{1, \infty}(D))) \leq N \right\}, \]

with the norm \( ||U^N||^2_{B_{n,T_{n,N}}} = \operatorname{E} \sup_{I_{n,N}} ||U^N||^2_{X_n} \), where \( I_{n,N} = [0, T_{n,N}] \) and \( ||U^N||^2_{X_n} = ||u^N||^2_{X_n} + ||h^N||^2_{X_n} \). Introduce a map

\[ \mathcal{T} : L^2(\Omega, C(I_{n,N}; X_n))^2 \to L^2(\Omega, C(I_{n,N}; X_n))^2, \]

defined by

\[ \mathcal{T}(U^N) := q_0^* + \int_0^t \theta_N^{u,S}(s) N ds + \int_0^t \sum_{k \geq 1} \theta_N^{u,S}(s) F_k d\beta_k(s), \]

where

\[
q_0^* = (\mathcal{M}^{-1}[S(u^N)](t)) m_0, \quad B_0^* = (\mathcal{M}^{-1}[S(u^N)](t)) N_1[S(u^N), u^N, h^N], \quad N_2[u^N, h^N], \quad F_k = (\mathcal{M}^{-1}[S(u^N)](t) f_k^u(S(u^N), S(u^N), u^N, x), g_k(h^N, x))).
\]

Similar to [27], by using the properties of \( \mathcal{M}[\rho], S[u] \), Hölder’s inequality and the Burkholder-Davis-Gundy inequality, we can infer that \( \mathcal{T} \) maps \( B_{n,T_{n,N}} \) into itself for some suitable \( T_{n,N} \).
We will prove that $\mathcal{T}$ is a contraction on $L^2(\Omega, C(I_{n,N}; X_n))^2$ for some (small) $T_{n,N} > 0$. Since the deterministic parts are similar to [27, Section 7], we only need to consider the stochastic parts

$$\mathcal{T}^s(U^N) := \int_0^t \sum_{k \geq 1} \theta_{n,N}^{U_k} (s) F_k \, d\beta_k(s).$$

For $U = (u, h), V = (v, \ell)$, one can write

$$\mathcal{T}^s(U) - \mathcal{T}^s(V) = (\mathcal{M}^{-1}[S(u)(t)] - \mathcal{M}^{-1}[S(v)(t)]) \int_0^t \sum_{k \geq 1} \theta_{n,N}^{U_k} (s) f_k^n(S(u), S(u)u, x) \, d\beta_k^1$$

$$+ \mathcal{M}^{-1}[S(v)(t)] \int_0^t \sum_{k \geq 1} (\theta_{n,N}^{U_k}(s) f_k^n(S(u), S(u)u, x) - \theta_{n,N}^{V_k}(s) f_k^n(S(v), S(v)v, x)) \, d\beta_k^1$$

$$+ \int_0^t \sum_{k \geq 1} (\theta_{n,N}^{U_k}(s) g_k(h, x) - \theta_{n,N}^{V_k}(s) g_k(\ell, x)) \, d\beta_k^2.$$

Then

$$\mathbb{E} \| \mathcal{T}^s(U) - \mathcal{T}^s(V) \|^2_{X_n} \leq \mathbb{E} \sup_{0 \leq t \leq T_{n,N}} \left\| \int_0^t (\mathcal{M}^{-1}[S(u)(t)] - \mathcal{M}^{-1}[S(v)(t)]) \sum_{k \geq 1} f_k^n(S(u), S(u)u, x) \, d\beta_k^1 \right\|^2_{X_n}$$

$$+ \mathbb{E} \sup_{0 \leq t \leq T_{n,N}} \left\| \int_0^t \mathcal{M}^{-1}[S(v)(t)] \sum_{k \geq 1} [f_k^n(S(u), S(u)u, x) - f_k^n(S(v), S(v)v, x)] \, d\beta_k^1 \right\|^2_{X_n}$$

$$+ \mathbb{E} \sup_{0 \leq t \leq T_{n,N}} \left\| \int_0^t \sum_{k \geq 1} (g_k(h, x) - g_k(\ell, x)) \, d\beta_k^2 \right\|^2_{X_n} =: I_1 + I_2 + I_3.$$

For the term $I_1$, using the Burkholder-Davis-Gundy inequality, (1.3), (3.4) and the fact that $S(u)$ is a contraction map, we have

$$I_1 \leq \mathbb{E} \left( \int_0^{T_{n,N}} \left\| (\mathcal{M}^{-1}[S(u)(t)] - \mathcal{M}^{-1}[S(v)(t)]) \sum_{k \geq 1} f_k^n(S(u), S(u)u, x) \right\|_{X_n}^2 \, ds \right)$$

$$\leq \mathbb{E} \left( \left\| (\mathcal{M}^{-1}[S(u)(t)] - \mathcal{M}^{-1}[S(v)(t)]) \sum_{k \geq 1} f_k^n(S(u), S(u)u, x) \right\|_{L^2(X_n, X_n)} \int_0^{T_{n,N}} \left\| \sum_{k \geq 1} f_k^n(S(u), S(u)u, x) \right\|_{X_n}^2 \, ds \right)$$

$$\leq \mathbb{E} \left( \| S(u) - S(v) \|^2_{L^1(D)} \int_0^{T_{n,N}} \left\| \sum_{k \geq 1} f_k^n(S(u), S(u)u, x) \right\|_{X_n}^2 \, ds \right)$$

$$\leq CT_{n,N} \mathbb{E} \left( \| u - v \|^2_{X_n} \right) \leq CT_{n,N} \mathbb{E} \left( \| U - V \|^2_{X_n} \right).$$

Similarly, for the terms $I_2$ and $I_3$, by (1.3), we obtain

$$I_2 \leq \mathbb{E} \left( \int_0^{T_{n,N}} \left\| (\mathcal{M}^{-1}[S(u)(t)]) \sum_{k \geq 1} (f_k^n(S(u), S(u)u, x) - f_k^n(S(v), S(v)v, x)) \right\|_{X_n}^2 \, ds \right)$$
Then sufficiently small $M$ and the proposition.

3.2 Remark

Step 3. Derive uniform a priori estimates to extend $\tau_{n,N}$ to $T$. Set $U^N_n = (u^N_n, B^N_n)$ be the solution to equation (3.5) on $[0, \tau_{n,N})$ obtained from Proposition 3.1. Let us introduce the following stopping times:

$$
\tau_{n,N} = \left\{ \begin{array}{ll}
\inf \{ t \geq 0 : \|U^N_n(t)\|_{L^2(D)} \geq N \} \land \inf \{ t \geq 0 : \left\| \int_0^t F^N_n dW \right\|_{L^2(D)} ds \geq N \}, \\
T, \text{ otherwise.}
\end{array} \right.
$$

We denote $(u_N, B_N) := (u^N_n, B^N_n)$. Let $\rho_N := S(u^N_n)$. Our next goal is to prove

Lemma 3.3. For any fixed $n$,

$$
\lim_{N \to \infty} P(\tau_{n,N} = T) = 1. \tag{3.8}
$$

To prove the above lemma, we need the following energy estimates.

Proposition 3.4. For any $(u_N, B_N)$ solving equation (3.5), for $1 \leq p < \infty$, we have

$$
E \left[ \sup_{0 \leq t \leq T} \mathcal{E}_t + E \int_0^T \left( \mu \|\nabla u_N(t)\|^2_{L^2(D)} + (\lambda + \mu) \|\text{div} u_N(t)\|^2_{L^2(D)} + \nu \|\nabla B_N(t)\|^2_{L^2(D)} \right) dt \\
+ \varepsilon E \int_0^T \int_D \left( a \gamma \rho_N^{-2} + \delta \beta \rho_N^{-2} \right) |\nabla \rho_N(t)|^2 dx dt \right] ^p \leq C E (\mathcal{E}_0)^p. \tag{3.9}
$$
Proof. Define the function $\Phi(\rho, m) := \int_D \frac{|m|^2}{\rho} dx$. Note that

$$\nabla_m \Phi(\rho, m) = \int_D \frac{2m}{\rho} dx, \quad \nabla^2_m \Phi(\rho, m) = \int_D \frac{2}{\rho^2} dx, \quad \partial_\rho \Phi(\rho, m) = -\int_D \frac{|m|^2}{\rho^2} dx.$$

Here $\mathbb{I}$ is the identity matrix.

Apply Itô’s formula in Lemma 2.11 to the above function $\Phi$ with $(\rho, m) = (\rho_n, \rho_N u_N)$, and then apply Itô’s formula again with $\Phi(B) := \int_D |B|^2 dx$. From equation (3.11), one deduces that

$$d \int_D \left( \frac{1}{2} \sqrt{\rho_N u_N}^2 + \frac{a}{\gamma - 1} \rho_N + \frac{\delta}{\beta - 1} \rho_N^\beta + \frac{1}{2} |B|^2 \right) dx + \int_D \left( \mu \nabla u_N^2 + (\lambda + \mu) \text{div} u_N^2 \right) dx$$

$$+ \int_D \nu |\nabla B|^2 dx + \varepsilon \int_D \left( a \gamma \rho_N^\gamma - \delta \beta \rho_N^\beta \right) |\nabla \rho_N|^2 dx$$

$$= \int_D \sum_{k=1}^N \int_D \left( \frac{1}{2} \rho_{0\delta}^{|u_0|} \right)^2 + \frac{a}{\gamma - 1} \rho_{0\delta}^\gamma + \frac{\delta}{\beta - 1} \rho_{0\delta}^\beta + \frac{1}{2} |B_{0\delta}|^2 dx$$

$$+ \int_D \sum_{k=1}^N \int_D \left( \frac{1}{2} g_k(B_n, x) \cdot B_N dx \right)^2 + \frac{1}{2} \sum_{k=1}^N \int_D \left( g_k(B_n, x) \right)^2 dx,$$

where $s \in [0, t \land \tau_n, \tau_n]$, $t \land \tau_n, \tau_n = \min\{t, \tau_n, N\}$ and $t \in [0, T]$. Let

$$\mathcal{E}_\delta = \int_D \left( \frac{1}{2} \rho_N |u_N|^2 + \frac{a}{\gamma - 1} \rho_N + \frac{\delta}{\beta - 1} \rho_N^\beta + \frac{1}{2} |B|^2 \right) dx,$$

$$\mathcal{E}_{0, \delta} = \int_D \left( \frac{1}{2} \rho_{0\delta} |u_{0\delta}|^2 + \frac{a}{\gamma - 1} \rho_{0\delta}^\gamma + \frac{\delta}{\beta - 1} \rho_{0\delta}^\beta + \frac{1}{2} |B_{0\delta}|^2 \right) dx,$$

where $m_N = \rho_N u_N$. Integrating (3.10) on $[0, s]$ for all $s \in [0, t \land \tau_n, \tau_n]$ yields

$$\mathcal{E}_\delta + \int_0^s \left[ \mu \|\nabla u_N(r)\|_{L^2(D)}^2 + (\lambda + \mu) \|\text{div} u_N(r)\|_{L^2(D)}^2 + \nu \|\nabla B_N(r)\|_{L^2(D)}^2 \right] dr$$

$$+ \varepsilon \int_0^s \int_D \left( a \gamma \rho_N^\gamma - \delta \beta \rho_N^\beta \right) |\nabla \rho_N|^2 dx + \int_0^s \left( u_N(r), \sum_{k=1}^N f_k^n(\rho_N, \rho_N u_N, x) \right) dx$$

$$\leq \mathcal{E}_{0, \delta} + \frac{1}{2} \int_0^s \sum_{k=1}^N \left\| f_k^n(\rho_N, \rho_N u_N, x) \right\|_{L^2(D)}^2 dx + \int_0^s \left( u_N(r), \sum_{k=1}^N f_k^n(\rho_N, \rho_N u_N, x) \right) dx$$

$$+ \frac{1}{2} \int_0^s \sum_{k=1}^N \left( g_k(B_n, x) \right)_{L^2(D)}^2 dx + \int_0^s \left( B_N(r), \sum_{k=1}^N g_k(B_n, x) \right)_{L^2(D)}^2,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(D)$. Taking the sup over $[0, t \land \tau_n, \tau_n]$ and the mathematical expectation in the above inequality, one easily deduces that

$$\mathbb{E} \sup_{0 \leq s \leq t \land \tau_n, \tau_n} \mathcal{E}_\delta + \mathbb{E} \int_0^{t \land \tau_n, \tau_n} \left( \mu \|\nabla u_N(s)\|_{L^2(D)}^2 + (\lambda + \mu) \|\text{div} u_N(s)\|_{L^2(D)}^2 + \nu \|\nabla B_N(s)\|_{L^2(D)}^2 \right) ds$$

$$+ \varepsilon \mathbb{E} \int_0^{t \land \tau_n, \tau_n} \int_D \left( a \gamma \rho_N^\gamma - \delta \beta \rho_N^\beta \right) |\nabla \rho_N(s)|^2 dx ds$$

$$\leq \mathbb{E} \mathcal{E}_{0, \delta} + \frac{1}{2} \mathbb{E} \int_0^{t \land \tau_n, \tau_n} \sum_{k=1}^N \left\| f_k^n(\rho_N, \rho_N u_N, x) \right\|_{L^2(D)}^2 ds + \mathbb{E} \sup_s \int_0^s \left( u_N, \sum_{k=1}^N f_k^n(\rho_N, \rho_N u_N, x) \right) dx.$$
and the Cauchy-Schwarz inequality imply that
We now estimate the terms $I_i$, $i = 1, 2, 3, 4$. For the first term $I_1$, the assumptions \textbf{(1.5)} on $f$ and the Cauchy-Schwarz inequality imply that

$$I_1 \lesssim E \int_0^{\tau_{n,N}} \int_D \left( \sqrt{\rho_N u_N} \right)^2 \, dx \, ds$$

$$\lesssim E \int_0^{\tau_{n,N}} \left( \|\sqrt{\rho_N u_N}\|^2_{L^2(D)} + \|\rho_N\|^2_{L^\gamma(D)} \right) \, ds,$$

For $I_2$, by the Burkholder-Davis-Gundy, Hölder and Young inequalities, and \textbf{(1.5)}, for small $\eta > 0$, we have

$$I_2 \lesssim E \left[ \int_0^{\tau_{n,N}} \sum_{k \geq 1} \langle f_k(\rho_N, \rho_N u_N, x), u_N \rangle^2 \, ds \right]^{\frac{1}{2}}$$

$$\lesssim E \left[ \int_0^{\tau_{n,N}} \|\sqrt{\rho_N u_N}\|^2_{L^2(D)} \left( \|\sqrt{\rho_N u_N}\|^2_{L^2(D)} + \|\rho_N\|^2_{L^\gamma(D)} \right) \, ds \right]^{\frac{1}{2}}$$

$$\leq \eta E \left[ \sup_{0 \leq s \leq \tau_{n,N}} \|\sqrt{\rho_N u_N(s)}\|^2_{L^2(D)} + C_\eta E \int_0^{\tau_{n,N}} \int_D \left( \|\sqrt{\rho_N u_N}\|^2 + \|\rho_N\|^\gamma \right) \, dx \, ds. \right.$$}

Similarly, from \textbf{(1.5)}, one has

$$I_3 \leq CE \int_0^{\tau_{n,N}} \|B_N\|^2_{L^2(D)} \, ds,$$

and

$$I_4 \leq \eta E \left[ \sup_{0 \leq s \leq \tau_{n,N}} \|B_N(s)\|^2_{L^2(D)} + C_\eta E \int_0^{\tau_{n,N}} \|B_N\|^2_{L^2(D)} \, ds. \right.$$}

When $p > 2$, we obtain as $I_2$ that

$$|I_2|^p \leq CE \left[ \int_0^{\tau_{n,N}} \sum_{k \geq 1} \langle f_k(\rho_N, \rho_N u_N, x), u_N \rangle^2 \, ds \right]^{\frac{p}{2}}$$

$$\leq \eta E \left( \sup_{0 \leq s \leq \tau_{n,N}} \|\sqrt{\rho_N u_N(s)}\|^p_{L^2(D)} \right) + C_\eta E \int_0^{\tau_{n,N}} \left( \int_D \|\sqrt{\rho_N u_N}\|^2 + \|\rho_N\|^\gamma \, dx \right)^p \, ds,$$

and

$$|I_4|^p \leq CE \left[ \int_0^{\tau_{n,N}} \sum_{k \geq 1} \langle g_k(B_N, x), B_N \rangle^2 \, ds \right]^{\frac{p}{2}}$$

$$\leq \eta E \left( \sup_{0 \leq s \leq \tau_{n,N}} \|B_N(s)\|^2_{L^2(D)} \right) + C_\eta E \int_0^{\tau_{n,N}} \left( \int_D \|B_N\|^2 \, dx \right)^p \, ds.$$
Plugging the estimates on $I_1 - I_4$ into (3.11), one has, for small enough $\eta > 0$,

$$E \sup_{0 \leq s \leq t} \mathcal{E}_\delta + E \int_0^{t \wedge \tau_{n,N}} \left( \int_D \left( a\gamma \rho_N^{-2} + \delta \beta \rho_N^{-2} \right) |\nabla \rho_N(s)|^2 \, dx \right) \, ds \leq \epsilon E \mathcal{E}_{0,\delta} + C \eta E \int_0^{t \wedge \tau_{n,N}} \left( \int_D \left( |\sqrt{\rho_N} u_N|^2 + |B_N|^2 \right) \, dx \right) \, ds.$$

Then by Gronwall’s inequality, we have

$$E \sup_{0 \leq s \leq t} \mathcal{E}_\delta + E \int_0^{t \wedge \tau_{n,N}} \left( \mu \|\nabla u_N(s)\|^2_{L^2(D)} + (\lambda + \mu) \|\text{div} u_N(s)\|^2_{L^2(D)} + \nu \|\nabla B_N(s)\|^2_{L^2(D)} \right) \, ds \leq CE \mathcal{E}_{0,\delta},$$

from which we get the following $L^2$ energy estimates for any $t \in [0, T]$:

$$E \sup_{0 \leq s \leq t} \mathcal{E}_\delta + E \int_0^t \left( \mu \|\nabla u_N(s)\|^2_{L^2(D)} + (\lambda + \mu) \|\text{div} u_N(s)\|^2_{L^2(D)} + \nu \|\nabla B_N(s)\|^2_{L^2(D)} \right) \, ds \leq CE \mathcal{E}_{0,\delta}.$$

In the same way, for $p > 2$, by virtue of the estimates on $|I_3|^p$ and $|I_4|^p$, we obtain the $L^p$ energy estimates for $t \in [0, T]$:

$$E \sup_{0 \leq s \leq t} \mathcal{E}_\delta + E \int_0^t \left( \mu \|\nabla u_N(s)\|^2_{L^2(D)} + (\lambda + \mu) \|\text{div} u_N(s)\|^2_{L^2(D)} + \nu \|\nabla B_N(s)\|^2_{L^2(D)} \right) \, ds \leq CE \mathcal{E}_{0,\delta}^p,$$

which completes the proof of the proposition. \(\square\)

Now, we are ready to prove Lemma 3.3.

**Proof of Lemma 3.3** It follows from (3.12) that

$$E \int_0^T \|\bar{\nabla} u_N\|^2_{L^2(D)} \, dt \leq CE \mathcal{E}_{0,\delta}, \quad E \sup_{t \in [0, T]} \|\sqrt{\rho} u_N(t)\|^2_{L^2(D)} \leq CE \mathcal{E}_{0,\delta}.$$

Since $\text{dim} X_n$ is finite, the $L^\infty$, $C^2$ and $L^2$ norms are equivalent on $X_n$. Moreover, $\rho = S[u]$ is bounded. It follows that

$$\rho \exp \left( - \int_0^T \|\bar{\nabla} u_N(s)\|^2_{L^2(D)} \, ds \right) \lesssim \rho(t, x) \lesssim \overline{\rho} \exp \left( \int_0^T \|\nabla u_N(s)\|^2_{L^2(D)} \, ds \right),$$

which yields that

$$E \left[ \exp \left( - \int_0^T \|\nabla u_N(s)\|^2_{L^2(D)} \, ds \right) \sup_{t \in [0, T]} \|u_N\|_{L^2(D)} \right] \leq C,$$

where $C$ is independent of $N$. It follows from (3.9), (3.13) and (3.14) that $\lim_{N \to \infty} P(\tau_{n,N} = T) = 1$. \(\square\)
Moreover, it follows from (3.9) that
\[ \sqrt{\varepsilon_0 \rho_N^2} \] is bounded in \( L^p(\Omega, L^2(0, T; H^1(D))) \).

The Sobolev embedding \( H^1(D) \hookrightarrow L^6(D) \) yields that
\[ \rho_N^2 \in L^p(\Omega, L^2(0, T; L^6(D))) \]
that is,
\[ \mathbb{E} \left( \left\| \rho_N^2 \right\|_{L^1(0,T;L^3(D))} \right)^{2p} \leq C \]
where \( C \) is independent of \( N \).

In view of (3.9), we deduce that
\[ \mathbb{E} \left( \sup_{t \in [0,T]} \left\| \rho_N^2 \right\|_{L^1(D)} \right)^p \leq C \]
this together with the interpolation inequality
\[ \left\| \rho_N^2 \right\|_{L^2(D)} \lesssim \left\| \rho_N^2 \right\|_{L^1(D)}^{\frac{1}{3}} \left\| \rho_N^2 \right\|_{L^3(D)}^{\frac{2}{3}} \]
yields that
\[ \mathbb{E} \left( \int_0^T \left\| \rho_N^2 \right\|_{L^2(D)} dt \right)^p \leq C. \]

By the Hölder inequality, we have
\[ \mathbb{E} \left( \int_0^T \int_D \frac{4}{3} dx dt \right)^p \leq \mathbb{E} \left( \int_0^T \left( \int_D \rho_N^2 dx \right)^\frac{4}{3} \left( \int_D dx \right)^\frac{1}{3} dt \right) \leq C. \]

Thus
\[ \mathbb{E} \left( \left\| \rho_N^2 \right\|_{L^{\beta+1}((0,T) \times D)} \right)^p \leq C \text{ if } \beta \geq 3. \]

If \( \beta \geq 4 \), multiply (3.1) by \( \rho_N \) and integrate by parts to get
\[ \mathbb{E} \left( \varepsilon \int_0^T \left\| \nabla \rho_N \right\|_{L^2(D)}^2 dt \right)^p \leq \mathbb{E} \left( \left\| \rho_N(0) \right\|_{L^2(D)}^2 + \int_0^T \int_D \rho_N^2 \text{div} u_N dx dt \right)^p \]
\[ \lesssim \mathbb{E} \left( \left\| \rho_N(0) \right\|_{L^2(D)}^2 + \left\| \rho_N \right\|_{L^4(D)}^4 + \left\| \nabla u_N \right\|_{L^2((0,T) \times D)}^2 \right)^p \]
\[ \lesssim C. \]

For each \( t \in [0,T] \), we define \( (\rho_n, u_n, B_n) := \lim_{N \to \infty} (\rho_N, u_N, B_N) \). By the same argument as above, we can infer that \( (\rho_n, u_n, B_n) \) satisfies the corresponding a priori estimate uniformly in \( n \). More precisely, for any \( 1 \leq p < \infty \), we have the following lemma:
Lemma 3.5. Let \((\rho_n, u_n, B_n)\) be the solution of (3.1)–(3.2) on \(\Omega \times (0, T) \times D\) constructed above. Then, for \(\beta \geq 4\), we have,

\[
\begin{align*}
&\mathbb{E}\left( \sup_{t \in [0, T]} \|\rho_n(t)\|_{L^2(D)}^p \right) \leq C,
&\mathbb{E}\left( \sup_{t \in [0, T]} \|\rho_n(t)\|_{L^3(D)}^p \right) \leq C,
&\mathbb{E}\left( \sup_{t \in [0, T]} \|B_n(t)\|_{L^2(D)}^p \right) \leq C,
&\mathbb{E}\left( \sup_{t \in [0, T]} \|u_n(t)\|_{L^2([0, T], H^1(D))}^p \right) \leq C,
&\mathbb{E}\left( \sup_{t \in [0, T]} \|B_n(t)\|_{L^2([0, T], H^1(D))}^p \right) \leq C,
\end{align*}
\]

(3.15)

where the constant \(C\) is independent of \(n\).

3.2. Tightness Property. In this subsection, we shall show the tightness property for the approximation solution in the following lemma.

Lemma 3.6. Define

\[
S := C(0, T; \mathbb{R}) \times \left(C([0, T]; L^2(D)) \cap L^2(0, T; L^2(D)) \cap L^2(0, T; H^1_w(D))\right)
\times L^2(0, T; H^1_w(D)) \times C([0, T]; L^2_w(D)) \times \left(L^2(0, T; H^1_w(D)) \cap L^2(0, T; L^2(D))\right)
\]

equipped with its Borel \(\sigma\)-algebra. Let \(\Pi_n\) be the probability on \(S\) which is the image of \(P\) on \(\Omega\) by the map: \(\omega \mapsto (\hat{\beta}_k(\omega, \cdot), \rho_n(\omega, \cdot), u_n(\omega, \cdot), \rho_n u_n(\omega, \cdot), B_n(\omega, \cdot))\), that is, for any \(A \subseteq S\),

\[
\Pi_n(A) = P \left\{ \omega \in \Omega : (\hat{\beta}_k(\omega, \cdot), \rho_n(\omega, \cdot), u_n(\omega, \cdot), \rho_n u_n(\omega, \cdot), B_n(\omega, \cdot)) \in A \right\}.
\]

Then the family \(\Pi_n\) is tight.

Proof. We want to check the tightness of the family of \(\Pi_n\) in the following five steps:

1° (the tightness of \(\hat{\beta}_k\)) First, we will check the tightness of \(\hat{\beta}_k\), that is, for \(\varepsilon > 0\), we now need to find the compact subset \(\Sigma_{\varepsilon} \subseteq C(0, T; \mathbb{R})\) such that \(P(\hat{\beta}_k \notin \Sigma_{\varepsilon}) \leq \frac{\varepsilon}{5}\). For \(\Sigma_{\varepsilon}\) we rely on classical results concerning the Brownian motion. For a constant \(L_{\varepsilon}\) to be chosen later, we consider the set

\[
\Sigma_{\varepsilon} = \left\{ \hat{\beta}_k(\cdot) \in C(0, T; \mathbb{R}) : \sup_{t_1, t_2 \in [0, T], |t_1 - t_2| \leq \frac{L_{\varepsilon}}{m}} m|\hat{\beta}_k(t_2) - \hat{\beta}_k(t_1)| \leq L_{\varepsilon}, \forall m \in \mathbb{N} \right\}.
\]

\(\Sigma_{\varepsilon}\) is relatively compact in \(C(0, T; \mathbb{R})\) by Arzela-Ascoli’s Theorem. Furthermore \(\Sigma_{\varepsilon}\) is closed in \(C(0, T; \mathbb{R})\). Therefore \(\Sigma_{\varepsilon}\) is a compact subset of \(C(0, T; \mathbb{R})\). We can show that \(P(\hat{\beta}_k \notin \Sigma_{\varepsilon}) \leq \frac{\varepsilon}{5}\). In fact, by Chebyshev’s inequality \(P(\omega : \xi(\omega) \geq r) \leq \frac{1}{r^2} \mathbb{E}[|\xi(\omega)|^2]\), one has

\[
P\left\{ \omega : \hat{\beta}_k(\omega, \cdot) \notin \Sigma_{\varepsilon} \right\}
\leq P\left[ \bigcup_{m=1}^{\infty} \left\{ \omega : \sup_{t_1, t_2 \in [0, T], |t_1 - t_2| \leq \frac{L_{\varepsilon}}{m}} |\hat{\beta}_k(t_1) - \hat{\beta}_k(t_2)| > \frac{L_{\varepsilon}}{m} \right\} \right]
\leq \sum_{m=1}^{\infty} m^6 \sum_{i=0}^{m^6-1} \left( \frac{m}{L_{\varepsilon}} \right)^4 E \left[ \sup_{T m^6 \leq t \leq (i+1)T m^6} |\hat{\beta}_k(t) - \hat{\beta}_k(iT m^6)|^4 \right].
\]
\[ \leq C \sum_{m=1}^{\infty} \left( \frac{m}{L_\varepsilon} \right)^4 (Tm^{-6})^2 m^4 = \frac{C}{L_\varepsilon^4} \sum_{m=1}^{\infty} \frac{1}{m^2}. \]

We choose \( L_\varepsilon = \frac{1}{5C\varepsilon} \left( \sum_{m=1}^{\infty} \frac{1}{m} \right)^{-1} \) to obtain \( P(\beta_k \notin \Sigma_\varepsilon) \leq \frac{\varepsilon}{5}. \)

2° (the tightness of \( \rho_n \)) In this step, we want to find \( X_\varepsilon \subset C([0, T]; L^b_w(D)) \cap L^2(0, T; L^2(D)) \cap L^2(0, T; H^1_w(D)) \) such that \( P(\rho_n \notin X_\varepsilon) \leq \frac{\varepsilon}{5} \). For this, we define a function space \( \mathcal{X} \) with the norm

\[ \| f \|_{\mathcal{X}} = \sup_{0 \leq t \leq T} \| f(t) \|_{L^2(D)} + \| f(t) \|_{L^2([0, T]; H^1(D))} + \sup_{0 \leq t \leq T} \| \partial_t f \|_{W^{-1, \frac{2n}{n+1}}(D)} + \| \partial_t f \|_{L^2([0, T]; H^{-1}(D))}. \]

Choose \( X_\varepsilon \) to be a closed ball of radius \( r_\varepsilon \) centered at 0 in \( \mathcal{X} \). By Aubin-Lions Lemma, we know that \( X_\varepsilon \) is compact in \( C(0, T; L^w(D)) \cap L^2(0, T; H^1_w(D)) \cap L^2(0, T; L^2(D)) \). It follows from (3.15) in Lemma 3.3 that

\[ P(\rho_n \notin X_\varepsilon) = P(\| \rho_n \|_{\mathcal{X}} > r_\varepsilon) \leq \frac{1}{r_\varepsilon} E(\| \rho_n \|_{\mathcal{X}}) \leq \frac{C}{r_\varepsilon}. \]

Choosing \( r_\varepsilon = 5C\varepsilon^{-1} \), we have \( P(\rho_n \notin X_\varepsilon) \leq \frac{\varepsilon}{5}. \) Then \( P\{\omega : \rho_n(\omega, \cdot) \notin X_\varepsilon \} \geq 1 - \frac{\varepsilon}{5}. \)

3° (the tightness of \( u_n \)) In this step, we find \( Y_\varepsilon \subset L^{2}(0, T; H^1(D)) \) such that \( P(u_n \notin Y_\varepsilon) \leq \frac{\varepsilon}{5}. \) To this end, we choose \( Y_\varepsilon \) as a closed ball of radius \( \tilde{r}_\varepsilon \) centered at 0 in \( L^{2}(0, T; H^1(D)) \). Then \( Y_\varepsilon \) is compact in \( L^2(0, T; H^1_w(D)) \). (3.15) in Lemma 3.3 implies that

\[ P(u_n \notin Y_\varepsilon) = P(\| u_n \|_{L^2(0, T; H^1(D))} > \tilde{r}_\varepsilon) \leq \frac{1}{\tilde{r}_\varepsilon} E(\| u_n \|_{L^2(0, T; H^1(D))}) \leq \frac{C}{\tilde{r}_\varepsilon}. \]

Choosing \( \tilde{r}_\varepsilon = 5C\varepsilon^{-1} \), we have \( P(u_n \notin Y_\varepsilon) \leq \frac{\varepsilon}{5}. \) Then \( P\{\omega : u_n(\omega, \cdot) \notin Y_\varepsilon \} \geq 1 - \frac{\varepsilon}{5}. \)

4° (the tightness of \( \rho_n u_n \)) In this step, we find \( Z_\varepsilon \subset C([0, T]; L^{2\delta}(D)) \) such that \( P(\rho_n u_n \notin Z_\varepsilon) \leq \frac{\varepsilon}{5}. \) For this, define a function space \( Z = L^2([0, T]; L^{2\delta}(D)) \cap C^{0, \alpha} (0, T; W^{0, \ell, 2}(D)) \), \( \ell > 3 \), \( 0 < \alpha < \frac{1}{2} \) with the norm

\[ \| f \|_Z = \sup_{0 \leq t \leq T} \| f(t) \|_{L^{2\delta}(D)} + \| f(t) \|_{C^{0, \alpha}(0, T; W^{0, \ell, 2}(D))}. \]

Because the Brownian motion is not differentiable in time, we cannot estimate \( \partial_t(\rho_n u_n) \) directly. To overcome the difficulty, we consider the momentum equation directly. From (3.1), we have

\[
P[\rho_n u_n(t)] = P[\rho_0 u_0] - \int_0^t P[\operatorname{div}(\rho_n u_n \otimes u_n) - \mu \Delta u_n - (\lambda + \mu) \operatorname{div} u_n - a \nabla \rho_n^\alpha - \delta \nabla \rho_n^\beta] \, ds
+ \int_0^t P[(\nabla \times B_n) \times B_n - \varepsilon \nabla u_n \cdot \nabla ds] - \int_0^t \sum_{k \geq 1} f_k^\varepsilon(\rho_n, \rho_n u_n, x) d\beta_k^1,
\]

(3.16)

where \( P : L^2(D) \rightarrow X_n \) is the projection onto \( X_n \). First, from (3.15), we have

\[ \mathbb{E} \int_0^T \| P[\operatorname{div}(\rho_n u_n \otimes u_n)] \|^2_{L^{2\delta}(D)} \, ds \leq \mathbb{E} \int_0^T \| \rho_n u_n \otimes u_n \|^2_{L^{2\delta}(D)} \, ds \leq C. \]
Similarly, from Lemma 3.5, we can handle the other deterministic terms as:

\[
\begin{align*}
\mathbb{E} \int_0^T \| \mathbb{P} [\mu \Delta u_n + (\lambda + \mu) \nabla \text{div} u_n] \|_{W^{-1,2}(D)}^2 ds & \leq C; \\
\mathbb{E} \int_0^T \| \mathbb{P} (\varepsilon \nabla u_n \cdot \nabla \rho_n) \|_{W^{-1,2}(D)}^{2\sigma} ds & \leq C, \\
\mathbb{E} \int_0^T \| \mathbb{P} (a \nabla \rho_n^\alpha + \delta \nabla \rho_n^\beta) \|_{W^{-1,2}(D)}^{\delta + 1} ds & \leq C, \\
\mathbb{E} \int_0^T \| \mathbb{P} [(\nabla \times B_n) \times B_n] \|_{W^{-1,2}(D)}^2 ds & \leq C. 
\end{align*}
\]

(3.18)

Here we have used \( \nabla \rho_n \in L^q([0,T]; L^2(D)), q > 2 \) (see [14, Lemma 2.4]). For the stochastic terms, by Lemma 3.5, one has for \( \sigma > 1 \)

\[
\begin{align*}
\mathbb{E} \left( \int_s^t \sum_{k \geq 1} \| f_k^n(\rho_n, \rho_n u_n, x) \|_{W^{-1,2}(D)} \frac{d^\beta_k}{L^2(s)} \right)^{2\sigma} & \leq \mathbb{E} \left( \int_s^t \sum_{k=1}^{n} \| f_k^n(\rho_n, \rho_n u_n, x) \|_{W^{-1,2}(D)}^2 ds \right)^{\sigma} \\
& \leq \mathbb{E} \left( \int_s^t \sum_{k=1}^{n} \| \mathbb{P}(f_k(\rho_n, \rho_n u_n, x)/\sqrt{\rho_n}) \|_{L^2(D)}^2 \sup_{\varphi \in \mathcal{M}^2 \{[0,T]; L^2(D)\}} \| \mathcal{M}^2 \{[0,T]; L^2(D)\} ds \right)^{\sigma} \\
& \leq C(t-s)^{\sigma}.
\end{align*}
\]

Thanks to the Kolmogorov continuity Theorem, we can infer that the stochastic term is almost surely \( \alpha \)-Hölder continuous for every \( 0 < \alpha < \frac{1}{2} - \frac{1}{2\sigma} \). Note that when \( \ell > 3 \), we have \( W^{-1,\frac{6\ell}{\ell+3}}(D) \hookrightarrow W^{-\ell,2}(D) \) and \( W^{-1,\frac{2\ell+1}{\ell+2}}(D) \hookrightarrow W^{-\ell,2}(D) \).

Choose \( Z_\varepsilon \) to be a closed ball of radius \( r'_\varepsilon \) centered at 0 in \( Z \). By [28, Corollary B.2] and [33], we know that \( Z_\varepsilon \) is compact in \( C([0,T]; L^0_w(D)) \). Moreover, from (3.16)–(3.19), by the Chebyshev inequality, we have

\[
P(\rho_n u_n \notin Z_\varepsilon) = P(\| \rho_n u_n \|_Z > r'_\varepsilon) \leq \frac{1}{r'_\varepsilon^\alpha} E(\| \rho_n u_n \|_Z) \leq \frac{C}{r'_\varepsilon^\alpha}. \tag{3.20}
\]

Let \( r'_\varepsilon = 5C\varepsilon^{-1} \) in (3.20), one deduces that

\[
P(\rho_n u_n \notin Z_\varepsilon) \leq \frac{\varepsilon}{5}.
\]

Then \( P(\omega: \rho_n u_n(\omega, \cdot) \in Z_\varepsilon) \geq 1 - \frac{\varepsilon}{5} \).

**Remark 3.7** Here we can also use the following tightness criterion: Let \( \{ f_n \}_{n=1}^\infty \) be a collection of \( C([0,T], L^0_w(D)) \)-valued random variables defined on a probability space \( (\Omega_n, \mathcal{F}_n, P_n) \) such that \( \sup_n E(\| f_n \|_{L^\infty([0,T]; L^p(D))}) < \infty \) and for any \( \varphi \in C_0^\infty(D) \), there exist an integer \( k, \sigma > 0, q > \frac{1}{\sigma} \) such that \( \sup_n E(\| f_n(t) - f_n(s) - \varphi \|_q^q) \leq \| \varphi \|_{L^\infty}^q |t-s|^\sigma q \) for all \( 0 \leq s, t \leq T \).

Then the sequence of the induced measures \( P \circ f_n^{-1} \) are tight on \( C([0,T], L^0_w(D)) \).

For example, by Hölder’s inequality and Lemma 3.5, we have

\[
\mathbb{E} \left| \int_s^t \langle \text{div}(\rho_n u_n \otimes u_n), \varphi \rangle dr \right|
\]
Choosing \( \varepsilon \) Here we have used \( r \). First, we choose \( \varepsilon \) is compact in \( \varepsilon \). On the other hand, it follows from (3.1) that
\[
5 \text{ (the tightness of } B_n) \text{ First, we will find } R_c \subset L^2(0, T; H^1_w(D)) \text{ such that } P(B_n \notin R_c) \leq \frac{\varepsilon}{5}. \text{ First, we choose } R_c \text{ as a closed ball of radius } \hat{r}_c \text{ centered at } 0 \text{ in } L^2(0, T; H^1(D)). \text{ Then } R_c \text{ is compact in } L^2(0, T; H^1_w(D)). \text{ From } [3, 15], \text{ we have}
\]
\[
P(B_n \notin R_c) = P(\|B_n\|_{L^2(0, T; H^1(D))} > \hat{r}_c) \leq \frac{1}{r_c} E \left( \|B_n\|_{L^2(0, T; H^1(D))} \right) \leq \frac{C}{r_c}.
\]
Choosing \( \hat{r}_c = 5C \varepsilon^{-1} \), we have \( P(B_n \notin R_c) \leq \frac{\varepsilon}{5}. \) Then \( P\{\omega : B_n(\omega, \cdot) \in R_c\} \geq 1 - \frac{\varepsilon}{5}. \)

On the other hand, it follows from (3.1) that
\[
E \int_0^{T-\theta} \|B_n(t + \theta) - B_n(t)\|_{H^{-1}(D)}^2 dt = E \int_0^{T-\theta} \left( \int_{t+\theta}^{t} dB_n(s) \right)_{H^{-1}(D)}^2 dt \\
\leq E \int_0^{T-\theta} (J_1 + J_2 + J_3) dt,
\]  
(3.21)

where
\[
J_1(t) = \left\| \int_t^{t+\theta} \nabla \times (u_n \times B_n) ds \right\|_{H^{-1}(D)}^2, \quad J_2(t) = \left\| \int_t^{t+\theta} \nu \Delta B_n ds \right\|_{H^{-1}(D)}^2,
\]
\[ J_3(t) = \left\| \int_t^{t+\theta} \sum_{k \geq 1} g_k(B_n, x) d\beta_k^n \right\|^2_{H^{-1}(D)}. \]

For the term \( J_1(t) \), one has
\[
J_1^{1/2} = \sup_{\varphi \in H^1_0(D), \|\varphi\|_{H^1(D)} = 1} \left\{ \int_D \left( \int_t^{t+\theta} \nabla \times (u_n \times B_n) ds \right) \varphi(x) dx \right\}
\leq C \int_t^{t+\theta} \|u_n\|_{L^4(D)} \|B_n\|_{L^4(D)} ds.
\]

By Hölder’s inequality and Lemma 3.5, then
\[
E \int_0^{T-\theta} J_1(t) dt \lesssim \theta^{1/4} E \left( \|u_n\|^2_{L^2(0,T;H^1(D))} \|B_n\|^2_{L^4(0,T;L^4(D))} \right) \lesssim \theta^{1/4}.
\] (3.22)

For the term \( J_2(t) \), similar to (3.22), we have
\[
E \int_0^{T-\theta} J_2(t) dt \lesssim E \int_0^{T-\theta} \left( \int_t^{t+\theta} \|\nabla B_n\|_{L^2(D)}^2 ds \right) dt \lesssim \theta E \int_0^T \|\nabla B_n\|_{L^2(D)}^2 dt \lesssim \theta.
\] (3.23)

For the term \( J_3 \), using the Burkholder-Davis-Gundy inequality, Hölder’s inequality and (1.5), we obtain
\[
E \int_0^{T-\theta} J_3(t) dt \leq \int_0^T E \left( \sup_{\varphi \in H^1_0(D), \|\varphi\|_{H^1(D)} = 1} \int_t^{t+\theta} \left( \sum_{k \geq 1} g_k(B_n, x) \varphi dx \right)^2 ds \right) dt
\leq \int_0^T \left( E \int_t^{t+\theta} \left( \sum_{k \geq 1} \|g_k(B_n, x)\|^2_{L^2(D)} ds \right) \right) dt
\leq \int_0^T \left( E \int_t^{t+\theta} \left( \sum_{k \geq 1} \|g_k(B_n, x)\|^2_{L^2(D)} ds \right) \right) dt
\lesssim \theta.
\] (3.24)

Combining (3.21), (3.24), and then by Aubin-Simon Lemma in [33], we know that \( B_n \) is compact in \( L^2(0,T;L^2(D)) \).

From the above argument, it follows from [24] pp.16, Corollary 1.3 that the distribution of the joint processes \( (\tilde{\beta}_k, \rho_n, u_n, \rho_n u_n, B_n) \) are tight. Hence, the proof of the lemma is completed. \( \square \)

3.3. Application of Jakubowski-Skorokhod Theorem. From the tightness property and Jakubowski-Skorokhod’s Theorem, there exists a subsequence, still indexed by \( n \), such that \( \Pi_n \to \Pi \) weakly, where \( \Pi \) is a probability on \( S \), a probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \); and there exist random variables \( (\tilde{\beta}_{k,n}, \tilde{\rho}_n, \tilde{u}_n, \tilde{\rho}_n \tilde{u}_n, \tilde{B}_n) \) with distribution \( \Pi_n, (\tilde{\beta}_k, \rho, u, h, B) \) with values in \( S \) such that
\[
(\tilde{\beta}_{k,n}, \tilde{\rho}_n, \tilde{u}_n, \tilde{\rho}_n \tilde{u}_n, \tilde{B}_n) \to (\tilde{\beta}_k, \rho, u, h, B) \text{ in } S \tilde{\mathbb{P}} \text{ a.s.} \] (3.25)
We set
\[ \tilde{\mathcal{F}}_t = \sigma\{\beta_k(s), \rho(s), u(s), B(s)\}_{s \in [0,t]}, \]
which is the union \( \sigma \)-algebra generated by random variables \( (\beta_k(s), \rho(s), u(s), B(s)) \) for \( s \in [0,t] \). It's easy to see that \( \beta_k(t) \) is a \( \mathcal{F}_t \) standard Brownian motion.

Now, we need to prove that \((\tilde{\beta}_k, \tilde{\rho}, \tilde{\nu}, \tilde{B})\) satisfies the equation (3.1), that is,
\[ \mathbb{P}[\tilde{\rho}(t) - \tilde{\rho}(0)] = \int_0^t \mathbb{P}[\varepsilon \Delta \tilde{\rho} - \text{div}(\tilde{\rho} \tilde{\nu})] \, ds, \]
\[ \mathbb{P}[\tilde{\rho} \tilde{\nu}(t)] + \int_0^t \mathbb{P}\left[\varepsilon \Delta \tilde{\rho} - \mu \Delta \tilde{\nu} - (\lambda + \mu) \nabla \nabla \tilde{w} + a \nabla \tilde{\nu} + \delta \nabla \tilde{\nu}^2\right] \, ds \]
\[ = \mathbb{P}[\tilde{\rho} \tilde{\nu}(0)] + \int_0^t \mathbb{P}\left((\nabla \times \tilde{B}) \times \tilde{B} - \varepsilon \nabla \tilde{\nu} \cdot \nabla \tilde{\rho}\right) \, ds + \int_0^t \sum_{k \geq 1} f_k^0(\tilde{\rho}, \tilde{\rho} \tilde{\nu}(x)) \, d\tilde{\beta}_k, \]
and
\[ \mathbb{P}[\tilde{B}(t)] - \int_0^t \mathbb{P}\left(\nabla \times (\tilde{u} \times \tilde{B}) + \nu \Delta \tilde{B}\right) \, ds = \mathbb{P}[\tilde{B}(0)] + \int_0^t \mathbb{P}\left[g_k(\tilde{B}, x)\right] \, d\tilde{\beta}_k. \]

Here \( \mathbb{P} \) is a projection from \( L^2 \) to \( X_n \). To this end, we define
\[ \chi_n(t) = \mathbb{P}[\rho_n(t) - \rho(0)] - \int_0^t \mathbb{P}[\varepsilon \Delta \rho_n - \text{div}(\rho_n \nu_n)] \, ds, \]
\[ \xi_n(t) = \mathbb{P}[\rho_n \nu_n(t)] + \int_0^t \mathbb{P}\left[\varepsilon \Delta \rho_n - (\lambda + \mu) \nabla \nabla \rho_n + a \nabla \rho_n + \delta \nabla \rho_n^2\right] \, ds \]
\[ - \mathbb{P}[\rho_n \nu_n(0)] - \int_0^t \mathbb{P}\left((\nabla \times B_n) \times B_n - \varepsilon \nabla \nu_n \cdot \nabla \rho_n\right) \, ds + \int_0^t \sum_{k \geq 1} f_k^0(\rho_n, \rho_n \nu_n) \, d\tilde{\beta}_k, \]
\[ \zeta_n(t) = \mathbb{P}[B_n(t) - B_n(0)] - \int_0^t \mathbb{P}\left[\nabla \times (u \times B_n) + \nu \Delta B_n\right] \, dt - \int_0^t \mathbb{P}\left[g_k(B, x)\right] \, d\tilde{\beta}_k, \]
\[ G_n = \int_0^T \| \chi_n(t) \|^2_{H^{-1}(D)} \, dt, \quad Z_n = \int_0^T \| \xi_n(t) \|^2_{W^{-\ell,2}(D)} \, dt, \quad H_n = \int_0^T \| \zeta_n(t) \|^2_{H^{-1}(D)} \, dt. \]

Since \( (\rho_n, u_n, B_n) \) is a solution, we have
\[ G_n = 0, \quad Z_n = 0, \quad H_n = 0, \quad \mathbb{P} - \text{a.s.} \quad (3.26) \]

Let
\[ \tilde{\chi}_n(t) = \mathbb{P}[\tilde{\rho}(t) - \tilde{\rho}(0)] - \int_0^t \mathbb{P}[\varepsilon \Delta \tilde{\rho} - \text{div}(\tilde{\rho} \tilde{\nu})] \, ds, \]
\[ \tilde{\xi}_n(t) = \mathbb{P}[\tilde{\rho} \tilde{\nu}(t)] - \int_0^t \mathbb{P}\left[\varepsilon \Delta \tilde{\rho} - (\lambda + \mu) \nabla \nabla \tilde{w} + a \nabla \tilde{\nu} + \delta \nabla \tilde{\nu}^2\right] \, ds \]
\[ - \mathbb{P}[\tilde{\rho} \tilde{\nu}(0)] + \int_0^t \mathbb{P}\left(\varepsilon \nabla \tilde{\nu} \cdot \nabla \tilde{\rho} + (\nabla \times \tilde{B}) \times \tilde{B}\right) \, ds - \int_0^t \sum_{k \geq 1} f_k^0(\tilde{\rho}, \tilde{\rho} \tilde{\nu}(x)) \, d\tilde{\beta}_k, \]
\[ \mathcal{Z}_n(t) = \mathbb{P} [ \hat{B}_n(t) - \hat{B}_n(0)] - \int_0^t \mathbb{P} [ \nabla \times (\vec{u}_n \times \hat{B}_n) + \nu \Delta \hat{B}_n] dt - \int_0^t \sum_{k \geq 1} \mathbb{P} [ g_k(\hat{B}_n, x) ] d\hat{\beta}_{k,n}^2, \]

and

\[ \mathcal{G}_n = \int_0^T \| \mathcal{E}_n(t) \|_{H^{-1}([D])} dt, \quad \mathcal{Z}_n = \int_0^T \| \mathcal{E}_n(t) \|_{W^{-1,2}([D])} dt, \quad \mathcal{H}_n = \int_0^T \| \mathcal{E}_n(t) \|_{H^{-1}([D])} dt. \]

We want to verify that

\[ \hat{E} \mathcal{G}_n = 0, \quad \hat{E} \mathcal{Z}_n = 0, \quad \hat{E} \mathcal{H}_n = 0. \]

Here \( \hat{E} \) denotes the mathematical expectation with respect to the probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \).

To this end, we have the following proposition:

**Proposition 3.8.** \( \mathcal{G}_n = 0, \mathcal{Z}_n = 0, \mathcal{H}_n = 0 \) \( \hat{\mathbb{P}} \) – a.s., that is, \( (\hat{\beta}_{k,n}, \hat{\rho}_n, \hat{u}_n, \hat{B}_n) \) solves the equation \( (3.1) \).

**Proof.** The difficulty comes from the fact that \( (Z_n, H_n) \) is not expressed as a deterministic function of \( (\hat{\beta}_k, \rho_n, u_n, B_n) \) due to the presence of the stochastic integrals. By Theorem 2.4 and \cite{2} Corollary 2.5, we can infer that

\[ \mathcal{L}(\hat{\beta}_k, \rho_n, u_n, B_n, Z_n, H_n, \xi_n, \zeta_n) = \mathcal{L}(\hat{\beta}_{k,n}, \hat{\rho}_n, \hat{u}_n, \hat{B}_n, \hat{\xi}_n, \hat{\zeta}_n). \tag{3.27} \]

Here \( \mathcal{L}(f) \) is the probability distribution of \( f \). Note that \( (\hat{Z}_n, \hat{H}_n) \) is continuous as a function of \( (\hat{\xi}_n, \hat{\zeta}_n) \). From (3.27), one deduces that the distribution of \( (\hat{Z}_n, \hat{H}_n) \) is equal to the distribution of \( (Z_n, H_n) \) on \( \mathbb{R}_+ \), which yields that

\[ \hat{E} \phi(\hat{Z}_n) = E \phi(Z_n), \quad \hat{E} \phi(\hat{H}_n) = E \phi(H_n), \quad \text{for any } \phi \in C_b(\mathbb{R}_+), \tag{3.28} \]

where \( C_b(X) \) is the space of continuous bounded functions defined on \( X \). Now, define \( \phi_\varepsilon \in C_b(\mathbb{R}_+) \) by

\[ \phi_\varepsilon = \begin{cases} \frac{y}{\varepsilon}, & 0 \leq y < \varepsilon; \\ 1, & y \geq \varepsilon. \end{cases} \]

One can check that

\[ \hat{P}(\hat{Z}_n \geq \varepsilon) = \int_\Omega 1_{[\varepsilon, \infty]} \hat{Z}_n d\hat{\mathbb{P}} \leq \int_\Omega 1_{[0, \varepsilon]} \frac{\hat{Z}_n}{\varepsilon} d\hat{\mathbb{P}} + \int_\Omega 1_{[\varepsilon, \infty]} \hat{Z}_n d\hat{\mathbb{P}}, \]

\[ \hat{P}(\hat{H}_n \geq \varepsilon) = \int_\Omega 1_{[\varepsilon, \infty]} \hat{H}_n d\hat{\mathbb{P}} \leq \int_\Omega 1_{[0, \varepsilon]} \frac{\hat{H}_n}{\varepsilon} d\hat{\mathbb{P}} + \int_\Omega 1_{[\varepsilon, \infty]} \hat{H}_n d\hat{\mathbb{P}}, \]

Hence by the definition of \( \hat{E} \phi_\varepsilon(\hat{Z}_n) \) and \( \hat{E} \phi_\varepsilon(\hat{H}_n) \), we can infer that

\[ \hat{P}(\hat{Z}_n \geq \varepsilon) \leq \hat{E} \phi_\varepsilon(\hat{Z}_n), \quad \hat{P}(\hat{H}_n \geq \varepsilon) \leq \hat{E} \phi_\varepsilon(\hat{H}_n), \]

which together with (3.28) imply that

\[ \hat{P}(\hat{Z}_n \geq \varepsilon) \leq E \phi_\varepsilon(Z_n), \quad \hat{P}(\hat{H}_n \geq \varepsilon) \leq E \phi_\varepsilon(H_n). \]

From (3.29), it holds that

\[ \hat{P}(\hat{Z}_n \geq \varepsilon) \leq E \phi_\varepsilon(Z_n) = 0, \quad \hat{P}(\hat{H}_n \geq \varepsilon) \leq E \phi_\varepsilon(H_n) = 0. \tag{3.29} \]

Since \( \varepsilon > 0 \) is arbitrary, from (3.29), we can infer that

\[ \hat{Z}_n = 0, \quad \hat{H}_n = 0 \quad \hat{\mathbb{P}} - \text{a.s.} \]

Similarly, we can prove that \( \mathcal{G}_n = 0, \quad \hat{\mathbb{P}} - \text{a.s.} \).
3.4. Passage to the limit for $n \to \infty$. By Proposition 3.8 we know that $(\hat{\rho}_n, \hat{u}_n, \hat{\beta}_n \hat{u}_n, \hat{B}_n)$ satisfies the same estimates as $(\rho_n, u_n, \rho_n u_n, B_n)$. In this subsection, to simplify notations, we will drop the tildes on the random variables. In order to obtain a solution of the problem \([3.1]-[3.3]\), we will use the estimates obtained in Lemma 3.5 to pass to the limit for $n \to \infty$ in the sequence $(\rho_n, u_n, B_n)$ in the sample space. For this, we need the following proposition and lemma in [22, Chapter 3].

Proposition 3.9 (Uniformly integrability). If there exists a nonnegative measurable function $f$ in $\mathbb{R}^+$, such that $\lim_{x \to \infty} \frac{f(x)}{x} = \infty$ and $\sup_{t \in \mathcal{T}} E[|f(X_t)|] < \infty$. Then $X_t$ is a set of uniformly integrable.

Lemma 3.10 (Vitali’s convergence Theorem). Suppose $p \in [1, \infty)$, $X_n \in L^p$ and $X_n$ converges to $X$ in probability. Then the following are equivalent:

1. $X_n \overset{L^p}{\to} X$;
2. $|X_n|^p$ is uniformly integrable;
3. $E(|X_n|^p) \to E(|X|^p)$.

The estimate \([3.15]\) along with the fact that $\partial_t \rho_n$ satisfies \([3.1]\) makes it possible to use \([3.25]\) to infer that there exists a function $\rho$ such that

$$\rho_n \to \rho \text{ in } L^4((0,T) \times D) \quad \tilde{P} - \text{a.s..} \quad (3.30)$$

In fact, \([3.25]\) implies that $\rho_n \to \rho$ in $L^2((0,T) \times D) \tilde{P}$-a.s.. Since $\rho_n \in L^\infty(0,T; L^\beta(D))$, $\beta > 4$, using the interpolation of $L^2((0,T) \times D)$ and $L^\infty(0,T; L^\beta(D))$, we have \([3.30]\).

On the other hand, similar to \([3.30]\), in view of \([3.15]\), and $\beta > \gamma$, one has

$$\rho_n^\gamma \to \rho^\gamma, \quad \rho_n^\beta \to \rho^\beta \text{ in } L^1((0,T) \times D) \quad \tilde{P} - \text{a.s.,}$$

which together with integration by parts yields that for $\phi \in C_0^\infty([0,T] \times D)$

$$\langle \nabla \rho_n^\gamma, \phi \rangle \to \langle \nabla \rho^\gamma, \phi \rangle, \quad \langle \nabla \rho_n^\beta, \phi \rangle \to \langle \nabla \rho^\beta, \phi \rangle \quad \tilde{P} - \text{a.s. as } n \to \infty. \quad (3.31)$$

Now, we pass to the limit in the products $\rho_n u_n$ and $\rho_n u_n \otimes u_n$. From \([3.25]\), we know that

$$\rho_n u_n \to h \text{ in } C([0,T]; L^{\frac{2\beta}{\beta+1}}(D)) \quad \tilde{P} - \text{a.s.}$$

Then, by \([3.25]\)($\rho_n \to \rho$ in $L^2(0,T; L^2(D))$, $u_n \to u$ in $L^2(0,T; H^1(D))$), \([3.25]\) holds, one has

$$\rho_n u_n \to \rho u \text{ in } L^1(0,T; L^1(D)) \quad \tilde{P} - \text{a.s.} \quad (3.32)$$

which together with \([15]\) Lemma 2.4 yields that $h = \rho u$. Then $\rho_n u_n \to \rho u$ in $C([0,T]; L^{\frac{2\beta}{\beta+1}}(D)) \tilde{P}$-a.s.. Since $L^{\frac{2\beta}{\beta+1}}(D) \hookrightarrow H^{-1}(D)$, by Aubin-Lions Lemma, one has

$$\rho_n u_n \to \rho u \text{ in } C([0,T]; H^{-1}(D)) \quad \tilde{P} - \text{a.s..} \quad (3.32)$$

Therefore, from \([3.25]\)($u_n \to u$ in $L^2(0,T; H^1(D)) \tilde{P}$-a.s.) and \([3.32]\), we can deduce that

$$\rho_n u_n \otimes u_n \to \rho u \otimes u \text{ in } D'([0,T] \times D) \quad \tilde{P} - \text{a.s..} \quad (3.33)$$

Now, we turn to the magnetic terms. By (3.25), we know that

$$B_n \to B \text{ in } L^2(0,T; L^2(D)) \text{ and } B_n \to B \text{ in } L^2(0,T; H^1(D)) \quad \tilde{P} - \text{a.s..} \quad (3.34)$$

Moreover, from \([3.13]\) and \([3.15]\), by Aubin-Lions Lemma, one has

$$B_n \to B \text{ in } C([0,T]; H^{-1}(D)) \quad \tilde{P} - \text{a.s..} \quad (3.34)$$
Then we have
\[ (\nabla \times B_n) \times B_n \to (\nabla \times B) \times B \quad \text{in } D'([0,T] \times D) \quad \tilde{P} - \text{a.s.} \quad (3.35) \]
Similarly, it follows from \[3.25\] \((u_n \to u) \quad \text{in } L^2(0,T; H^1(D)) \) \(\tilde{P} - \text{a.s.}\) and \[3.34\] that \[ \nabla \times (u_n \times B_n) \to \nabla \times (u \times B) \quad \text{in } D'([0,T] \times D) \quad \tilde{P} - \text{a.s.} \quad (3.36) \]
Finally, we will treat the force terms. We claim that
\[ \sum_{k \geq 1} f_k^n(\rho_n, \rho_n u_n, x, \phi) \to \sum_{k \geq 1} f_k(\rho, \rho u, x, \phi) \quad \text{in } L^1([0,T]) \quad \tilde{P} - \text{a.s.}, \quad (3.37) \]
Prove of the claim. By definition and the symmetry of \(M[\rho]\) we have
\[ \langle f_k^n(\rho_n, \rho_n u_n, x, \phi) \rangle = \left\langle M[\rho_n]^\perp \left( \frac{f_k(\rho_n, \rho_n u_n, x)}{\sqrt{\rho_n}} \right), \phi \right\rangle = \left\langle P \left( \frac{f_k(\rho_n, \rho_n u_n, x)}{\sqrt{\rho_n}} \right), M[\rho_n]^\perp \phi \right\rangle. \]
By using \[1.5, 3.15\] and the strong convergence in \[3.25, 3.30, 3.32\], we have
\[ \frac{f_k(\rho_n, \rho_n u_n, x)}{\sqrt{\rho_n}} \to \frac{f_k(\rho, \rho u, x)}{\sqrt{\rho}} \quad \text{in } L^2(D) \quad \tilde{P} \otimes \tilde{L} - \text{a.e.}, \]
where \(\tilde{L}\) is the Lebesgue measure in time.
Next we extend the operator \(M[\rho]\) to \(\tilde{M}[\rho] : H^1(D) \to L^2(D)\), \[ \left\langle \tilde{M}[\rho] v, w \right\rangle = \left\langle M[\rho] v, w \right\rangle \]
for \(v \in H^1(D), \quad w \in L^2(D)\). It is easy to see that as \(n \to \infty, \)
\[ \left| \left\langle \tilde{M}[\rho_n] v, w \right\rangle - \left\langle \rho v, w \right\rangle \right| \leq \|\rho_n\|_{L^\infty} \|v - P v\|_{L^2} \|w - P w\|_{L^2} + \|\rho_n\|_{L^\infty} \|v\|_{L^2} \|w - P w\|_{L^2} \]
\[ + \|\rho_n - \rho\|_{L^1} \|v\|_{L^1} \|w - P w\|_{L^2} \to 0, \]
which implies that
\[ \left\langle \tilde{M}[\rho_n] v, \cdot \right\rangle \to \left\langle P v, \cdot \right\rangle \quad \tilde{P} \otimes \tilde{L} - \text{a.e.}, \]
Next we look at \(\tilde{M}[\rho_n]^\perp\). Note that \(\tilde{M}[\rho_n]\) and \(\rho\) are both symmetric positive definite operators with a lower bound (independent of \(n\)) strictly away from 0, and hence they are both invertible. Moreover we know that they also commute. This way we have
\[ \tilde{M}[\rho_n]^\perp - \sqrt{\rho} = \left( \tilde{M}[\rho_n]^\perp + \sqrt{\rho} \right)^{-1} \left( \tilde{M}[\rho_n] - \rho \right), \]
and hence
\[ \left\langle \tilde{M}[\rho_n]^\perp v, \cdot \right\rangle \to \left\langle \sqrt{\rho} v, \cdot \right\rangle \quad \tilde{P} \otimes \tilde{L} - \text{a.e.}. \quad (3.39) \]
This way \[3.37\] follows from \[3.38\] and \[3.39\].
Next, we prove \[3.37\] 2. Direct computation yields that
\[
\left| \sum_{k \geq 1} \left( f_k^n(\rho_n, \rho_n u_n, x, \phi) \right)^2 - \sum_{k \geq 1} \left( f_k(\rho_n, \rho_n u_n, x, \phi) \right)^2 \right|
\leq \sum_{k \geq 1} \left| \left( f_k^n(\rho_n, \rho_n u_n, x, \phi) - f_k(\rho_n, \rho_n u_n, x, \phi) \right) \right| \left| \left( f_k^n(\rho_n, \rho_n u_n, x, \phi) + f_k(\rho_n, \rho_n u_n, x, \phi) \right) \right|.
\]
By (3.25) and Lemma 3.5, we can bound
\[ \sum_{k \geq 1} \left| \langle f_k(n, \rho_n u_n, x), \phi \rangle \right| \leq \sum_{k \geq 1} \| f_k \|_{L^2} \lesssim \| \rho_n \|_{L^{(\gamma+1)/2}} + \| \rho_n u_n \|_{L^2} \leq C. \]

Similarly, we can bound \( \sum_{k \geq 1} \left| \langle f_k^+(n, \rho_n u_n, x), \mathbb{P}\phi \rangle \right| \). Then from (3.37) we get (3.37)2.

Thanks to (3.34) and (1.5), we can similarly obtain
\[ \langle g_k(B_n, x), \phi \rangle \to \langle g_k(B, x), \phi \rangle \quad \text{in} \quad L^1([0, T]) \quad \tilde{P} \text{-a.s.}, \]
\[ \langle \sum_{k \geq 1} g_k(B_n, x), \phi \rangle^2 \to \langle \sum_{k \geq 1} g_k(B, x), \phi \rangle^2 \quad \text{in} \quad L^1([0, T]) \quad \tilde{P} \text{-a.s.} \quad (3.40) \]

Consequently, by (3.25), (3.31)–(3.36), Lemma 3.5, Proposition 3.9 and Lemma 3.10, we can pass to the limit in the continuity equation (3.1) and have the following propositions.

**Proposition 3.11.** \(((\tilde{\Omega}, \tilde{\beta}, \tilde{P}), \beta_k, \rho, u, B)\) is a weak solution of (3.1). Moreover, \( \nabla \rho_n \cdot \nabla u_n \to \nabla \rho \cdot \nabla u \) in \( \mathcal{D}'((0, T) \times D) \), \( \tilde{P} \)-a.s.

**Proof.** For all \( t \in [0, T] \) and any \( \varphi \in C_0^\infty(D) \), denote
\[ m_n(t) = \langle \rho_n(t), \varphi \rangle - \langle \rho_0, \varphi \rangle + \int_0^t \langle \rho_n u_n, \nabla \varphi \rangle ds - \int_0^t \langle \varepsilon \nabla \rho_n, \nabla \varphi \rangle ds. \]

By Lemma 3.5, we know that \( \sup_{0 \leq t \leq T} \tilde{E}(|m_n(t)|^2) \leq C. \) Then \( m_n(t) \) is uniformly integrable. By (3.25), (3.32), and Lemma 3.5, we know that \( m_n(t) \to m(t) \) \( \tilde{P} \)-a.s. Hence applying Proposition 3.9 and Lemma 3.10, we have \( \tilde{E}(|m(t)|^2) = \lim_{n \to \infty} \tilde{E}(|m_n(t)|^2) = 0. \)

Next, we shall show the convergence of the term \( \nabla \rho_n \cdot \nabla u_n \). Multiplying (3.1) by \( \rho_n \), integrating by parts, we obtain \( \tilde{P} \)-a.s.
\[ \| \rho_n(t) \|_{L^2(D)}^2 + 2\varepsilon \int_0^t \| \nabla \rho_n \|_{L^2(D)}^2 dt = -\int_0^t \int_D \rho_n^2 \text{div} u_n dx dt + \| \rho_0 \|_{L^2(D)}^2. \quad (3.41) \]

Sending \( n \to \infty \) in equation (3.41) and then multiplying the result by \( \rho \) (since from [15] Lemma 2.4) we know \( \rho \) enjoys enough regularity due to parabolic smoothing), one has \( \tilde{P} \)-a.s.,
\[ \| \rho(t) \|_{L^2(D)}^2 + 2\varepsilon \int_0^t \| \nabla \rho \|_{L^2(D)}^2 dt = -\int_0^t \int_D \rho^2 \text{div} u dx dt + \| \rho_0 \|_{L^2(D)}^2. \quad (3.42) \]

From [15] Lemma 2.4, (3.25), (3.31), (3.41), and (3.42), for any \( t \), we deduce that \( \tilde{P} \)-a.s.
\[ \| \nabla \rho_n \|_{L^2((0, T) \times D)}^2 \to \| \nabla \rho \|_{L^2((0, T) \times D)}^2, \quad \| \rho_n(t) \|_{L^2(D)}^2 \to \| \rho(t) \|_{L^2(D)}^2, \]
which together with (3.25) imply \( \nabla \rho_n \to \nabla \rho \) in \( L^2(0, T; L^2(D)) \) \( \tilde{P} \)-a.s.. Then, by the fact \( u_n \to u \in L^2(0, T; H^1(D)) \) \( \tilde{P} \)-a.s. (3.25), one has
\[ \nabla \rho_n \cdot \nabla u_n \to \nabla \rho \cdot \nabla u \quad \text{in} \quad \mathcal{D}'((0, T) \times D) \quad \tilde{P} \text{-a.s.}, \quad i = 1, 2, 3. \]

**Proposition 3.12.** The system \(((\tilde{\Omega}, \tilde{\beta}, \tilde{P}), \beta_k, \rho, u, B)\) is a martingale solution of (3.1)2 and (3.1)3.
Proof. It follows from Proposition 3.8 that
\[
(M_n(t), \phi) = \langle \rho_n u_n(t) - \rho_n u_n(0), \phi \rangle + \int_0^t \langle \text{div}(\rho_n u_n) + \delta \nabla \rho_n^\beta - \mu \Delta u_n, \phi \rangle ds \\
+ \int_0^t \langle (\nabla \times B_n) \times B_n + \varepsilon \nabla \rho_n \cdot \nabla u_n - (\lambda + \mu) \nabla \text{div} u_n + a \nabla \rho_n^\gamma, \phi \rangle ds,
\]
where \( (\rho_n u_n(t), \phi) = \int_0^t \sum_{k \geq 1} \langle f_k(\rho_n, \rho_n u_n, x), \phi \rangle d\beta_{k,n}^1 \) and \( (\nabla \times \mathbf{B}_n, \phi) = \int_0^t \sum_{k \geq 1} \langle g_k(\mathbf{B}_n, x), \phi \rangle d\beta_{k,n}^2 \) are martingales under \( \tilde{P} \). Therefore, with \( 0 \leq s < t < \infty \) and a bounded, continuous \( \mathcal{F}_s \)-measurable function \( \varphi_n = \varphi(\beta_{k,n}, \rho_n, u_n, B_n) \), we have
\[
\tilde{E} \left[ (M_n(t) - M_n(s), \phi) \varphi_n \right] = 0, \quad \tilde{E} \left[ (M_n(t) - \tilde{M}_n(s), \phi) \varphi_n \right] = 0,
\]
\[
\tilde{E} \left[ \left( (M_n(t) - M_n(s), \phi) \beta_{k,n}^1 \right) - \int_s^t \langle f_k(\rho_n, \rho_n u_n, x), \phi \rangle d\tau \right] \varphi_n = 0,
\]
\[
\tilde{E} \left[ \left( \tilde{M}_n(t) - \tilde{M}_n(s), \phi \beta_{k,n}^2 \right) - \int_s^t \langle g_k(\mathbf{B}_n, x), \phi \rangle d\tau \right] \varphi_n = 0,
\]
\[
\tilde{E} \left[ (M_n(t), \phi)^2 - (M_n(s), \phi)^2 \right] = \int_s^t \sum_{k \geq 1} \langle f_k(\rho_n, \rho_n u_n, x), \phi \rangle^2 d\tau \varphi_n = 0,
\]
\[
\tilde{E} \left[ (\tilde{M}_n(t), \phi)^2 - (\tilde{M}_n(s), \phi)^2 \right] = \int_s^t \sum_{k \geq 1} \langle g_k(\mathbf{B}_n, x), \phi \rangle^2 d\tau \varphi_n = 0.
\]
By (3.25), (3.30)–(3.36), one deduces that
\[
\langle \rho_n u_n(t) - \rho_n u_n(s), \phi \rangle + \int_s^t \langle \Lambda_n, \phi \rangle d\tau, \quad \langle B_n(t) - B_n(s), \phi \rangle + \int_s^t \langle \tilde{\Lambda}_n, \phi \rangle d\tau
\]
converges to
\[
\langle pu(t) - pu(s), \phi \rangle - \int_s^t \langle \Lambda, \phi \rangle d\tau, \quad \langle B(t) - B(s), \phi \rangle + \int_s^t \langle \tilde{\Lambda}, \phi \rangle d\tau \quad \text{almost surely on } \tilde{\Omega},
\]
where
\[
\langle \Lambda, \phi \rangle = \langle \text{div}(\rho u) - \mu \Delta u - (\lambda + \mu) \text{div} u + a \nabla \rho^\gamma + \delta \nabla \rho^\beta + \varepsilon \nabla u \cdot \nabla \rho + (\nabla \times B) \times B, \phi \rangle,
\]
\[
\langle \tilde{\Lambda}, \phi \rangle = \langle \nabla \times (u \times B) + \nu \Delta B, \phi \rangle.
\]
Next, we prove that the functions \( \tilde{T}_n := (M_n(t) - M_n(s), \phi) \varphi(\beta_{k,n}, \rho_n, u_n, B_n) \) are uniformly integrable. Indeed, by the Schwarz inequality and the Burkholder-Davis-Gundy inequality, for each \( n \), we have
\[
\tilde{E} \left[ (\tilde{T}_n)^2 \right] \leq \tilde{E} \left[ \|M_n(t)\|^2_{W^{-2,2}(D)} + \|M_n(s)\|^2_{W^{-2,2}(D)} \right]
\]
In view of \((3.37)\), \((3.40)\), \((3.43)\), \((3.45)-(3.46)\), together with Proposition \(3.9\) and Lemma \(3.10\), we have

$$\hat{E} \left( \int_0^T \sum_{k=1}^n \| f_k^n(\rho_n, \rho_n u_n, x) \|^2_{L^1(D)} dt \right) \leq C.$$ \((3.44)\)

Then the sequence \(f_n\) is uniformly integrable. For some \(r > 1\), we obtain as \((3.44)\)

$$\hat{E} \left( \int_0^T \sum_{k=1}^n \| f_k^n(\rho_n, \rho_n u_n, x) \|^2_{L^1(D)} dt \right)^r \leq C.$$ \((3.45)\)

Similarly, we have

$$\hat{E} \left( \int_0^T \sum_{k=1}^n \| g_k(B, x) \|^2_{H^{-1}(D)} dt \right)^r \leq C.$$ \((3.46)\)

In view of \((3.37), (3.40), (3.43), (3.45)-(3.46)\), together with Proposition \(3.9\) and Lemma \(3.10\), one has

$$\hat{E} \left[ \langle M(t) - M(s), \phi \rangle \varphi(\beta_k, \rho, u, B) \right] = \lim_{n \to \infty} \hat{E} \left[ \langle M_n(t) - M_n(s), \phi \rangle \varphi_n \right] = 0,$$

$$\hat{E} \left[ \langle \tilde{M}(t) - \tilde{M}(s), \phi \rangle \varphi(\beta_k, \rho, u, B) \right] = \lim_{n \to \infty} \hat{E} \left[ \langle \tilde{M}_n(t) - \tilde{M}_n(s), \phi \rangle \varphi_n \right] = 0,$$

$$\hat{E} \left[ \left( \langle M(t) - M(s), \phi \rangle \beta_k^1 - \int_s^t \langle f_k(\rho, \rho u, x), \phi \rangle d\tau \right) \varphi(\beta_k, \rho, u, B) \right]$$

$$= \lim_{n \to \infty} \hat{E} \left[ \left( \langle M_n(t) - M_n(s), \phi \rangle \beta_k^1 - \int_s^t \langle f_k^n(\rho_n, \rho_n u_n, x), \phi \rangle d\tau \right) \varphi_n \right] = 0,$$

and

$$\hat{E} \left[ \left( \langle M(t), \phi \rangle^2 - \langle M(s), \phi \rangle^2 - \int_s^t \sum_{k=1}^n \langle f_k(\rho, \rho u, x), \phi \rangle^2 d\tau \right) \varphi(\beta_k, \rho, u, B) \right]$$

$$= \lim_{n \to \infty} \hat{E} \left[ \left( \langle M_n(t), \phi \rangle^2 - \langle M_n(s), \phi \rangle^2 - \int_s^t \sum_{k=1}^n \langle f_k^n(\rho_n, \rho_n u_n, x), \phi \rangle^2 d\tau \right) \varphi_n \right] = 0,$$

$$\hat{E} \left[ \langle \tilde{M}(t), \phi \rangle^2 - \langle \tilde{M}(s), \phi \rangle^2 - \int_s^t \sum_{k=1}^n \langle g_k(B, x), \phi \rangle^2 d\tau \right) \varphi(\beta_k, \rho, u, B) \right]$$

$$= \lim_{n \to \infty} \hat{E} \left[ \langle \tilde{M}_n(t), \phi \rangle^2 - \langle \tilde{M}_n(s), \phi \rangle^2 - \int_s^t \sum_{k=1}^n \langle g_k(B_n, x), \phi \rangle^2 d\tau \right) \varphi_n \right] = 0.$$

Then we can deduce that \(\langle M(t), \phi \rangle, \langle M(t) - M(s), \phi \rangle \beta_k^1 - \int_s^t \langle f_k(\rho, \rho u, x), \phi \rangle d\tau, \langle M(t), \phi \rangle^2 - \int_s^t \sum_{k=1}^n \langle f_k(\rho, \rho u, x), \phi \rangle^2 d\tau, \langle \tilde{M}(t), \phi \rangle, \langle \tilde{M}(t) - \tilde{M}(s), \phi \rangle \beta_k^2 - \int_s^t \langle g_k(B, x), \phi \rangle d\tau \rangle \) and \(\langle \tilde{M}(t), \phi \rangle^2 - \int_s^t \sum_{k=1}^n \langle g_k(B_n, x), \phi \rangle^2 d\tau \) are martingales.
Applying the method in [3] [28], we can infer that
\[ \langle M(t), \phi \rangle = \int_0^t \langle f_k(\rho, \rho u, x), \phi \rangle d\beta_k^1, \quad \langle M(t), \phi \rangle = \int_0^t \langle g_k(B, x), \phi \rangle d\beta_k^2. \]
Therefore, ((Ω, \text{F}, \mathbb{P}), \beta_k, \rho, u, B) is a martingale solution of (3.1)-(3.3). □

4. The Vanishing Viscosity Limit

In this section, we shall consider the limit of (3.1)-(3.3) as ε → 0 following the idea of [13, 15, 30]. Similar to Lemma 3.3 we can get the following estimates:
\[ \rho_ε \in L^p(Ω, L^∞(0, T; L^7(D))), \]
\[ \sqrt{\varepsilon} \nabla u_ε \in L^p(Ω, L^∞(0, T; L^2(D))), \]
\[ \rho_ε u_ε \in L^p(Ω, L^∞(0, T; L^{2,0}(D))), \]
\[ u_ε \in L^p(Ω, L^2(0, T; H^1(D))), \]
\[ \sqrt{\varepsilon} \nabla \rho_ε \in L^p(Ω, L^2(0, T; L^2(D))), \]
\[ \rho_ε u_ε^ε \in L^p(Ω, L^2(0, T; L^{2,0}(D))), \]
\[ B_ε \in L^p(Ω, L^2(0, T; H^1(D))) \text{ and } L^p(Ω, L^∞(0, T; L^2(D))), \]
for 1 ≤ p < ∞, β > max{4, γ}. It follows from (4.1) that εL∞(0, T; L^β(D)), i.e., εL∞(0, T; L^β(D)), and hence we can only conclude that εL∞(0, T; L^β(D)) converges to a Radon measure. To gain some better convergence of this term we need to improve the integrability of the density ρ as follows.

4.1. On the equation \text{div} v = f. We introduce a linear operator
\[ \mathcal{B} : \left\{ f \in L^p(D) : \int_D f \ dx = 0 \right\} \rightarrow [W_0^{1,p}(D)]^3, \]
which is a bounded linear operator, that is,
\[ \|\mathcal{B}[f]\|_{W^{1,p}(D)} \leq C(p)\|f\|_{L^p(D)}, \quad 1 < p < \infty. \]
and \mathcal{B}(f) solves
\[ \text{div} \mathcal{B}(f) = f \text{ in } D, \quad \mathcal{B}(f)|_{\partial D} = 0. \]
Moreover, if g ∈ L^p(D) and g \cdot n|_{\partial D} = 0, then
\[ \|\mathcal{B}[\text{div} g]\|_{L^p(D)} \leq C(p)\|g\|_{L^p(D)} \text{ for } 1 < p < \infty. \]
The operator \mathcal{B} was first constructed by Bogovskii [5]. The existence and the above properties of the operator \mathcal{B} have been proved in many papers, and we refer the reader to [16] and the references therein for details.

4.2. Uniform (on viscosity) estimates of density. Next, we shall use the operator \mathcal{B} to improve the integrability of the density. First, let ψ ∈ D(0, T), 0 ≤ ψ ≤ 1 and \bar{m} = \frac{1}{|D|} \int_D \rho(t)dx, we note that \bar{m} is conserved. Taking \psi(t) \mathcal{B}[\rho_ε - \bar{m}] as a test function for (3.1)2, integrating over Ω × (0, T) × D, we arrive at
\[ \mathbb{E} \int_0^T \int_D \left[ \psi(t) \mathcal{B}[\rho_ε - \bar{m}](\rho_ε u_ε)_t + \psi(t) \mathcal{B}[\rho_ε - \bar{m}] \text{div}(\rho_ε u_ε \otimes u_ε) \right] \ dx \ dx \ dt \]
\[ + a \mathbb{E} \int_0^T \int_D \psi(t) \mathcal{B}[\rho_ε - \bar{m}] \nabla \rho_ε^3 \ dx \ dx \ dt + \delta \mathbb{E} \int_0^T \int_D \psi(t) \mathcal{B}[\rho_ε - \bar{m}] \nabla \rho_ε^3 \ dx \ dx \ dt \]
Then the family $\Pi$.

Let $\Pi$.

For convenience, we write the above equation as

$$I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = J_1 + J_2 + J_3.$$ 

We will just discuss the term $I_6$. Other terms can be handled as in \[34\] \[38\]. For the term $I_6$, using Hölder’s inequality and the fact that $\beta > 4$, from \[41\] and \[42\], we obtain

$$I_6 = -E \int_0^T \int_D \mathcal{B}[\rho_\varepsilon - \bar{m}] (\nabla \times \mathbf{B}) \times \mathbf{B}_\varepsilon \, dx \, dt$$

$$\leq E \left( \|B_\varepsilon\|_{L^2(0,T;L^2(D))} \|\nabla B_\varepsilon\|_{L^2(0,T;L^2(D))} \|B[\rho_\varepsilon - \bar{m}]\|_{L^\infty(0,T;L^\infty(D))} \right)$$

$$\leq \left( E \|\nabla B_\varepsilon\|_{L^2(0,T;L^2(D))}^4 \right)^{\frac{3}{4}} \left( E \|ho_\varepsilon\|_{L^\infty(0,T;L^3(D))}^4 \right)^{\frac{1}{4}} \leq C.$$ 

Summing up the estimates for $I_1, I_2, \ldots, I_6$ and $J_1, J_2, J_3$, we obtain the following results:

**Lemma 4.1.** Let $(\beta_{k,\varepsilon}, \rho_\varepsilon, u_\varepsilon, B_\varepsilon)$ be a weak solution of the problem \[35\]-\[39\]. Then

$$E \left( \|\rho_\varepsilon\|_{L^{\gamma+1}(0,T) \times D} + \|\rho_\varepsilon\|_{L^{\beta+1}(0,T) \times D} \right) \leq C,$$  

(4.3)

where $C = C(\delta, \rho_{0,\delta}, m_{0,\delta}, B_{0,\delta})$ is a constant independent of $\varepsilon$.

4.3. **Tightness property.**

**Lemma 4.2.** Define

$$S = C(0,T;\mathbb{R}) \times C([0,T];L^6_w(D)) \times L^2(0,T;H^1_w(D)) \times C([0,T];L^{\frac{24}{11}}_w(D)) \times \left( L^2(0,T;H^1_w(D)) \cap L^2(0,T;L^2(D)) \right)$$

equipped with its Borel $\sigma$-algebra. Let $\Pi_\varepsilon$ be the probability on $S$ which is the image of $\mathbb{P}$ on $\Omega$ by the map: $\omega \mapsto (\beta_{k,\varepsilon}(\omega,\cdot), \rho_\varepsilon(\omega,\cdot), u_\varepsilon(\omega,\cdot), \rho_\varepsilon u_\varepsilon(\omega,\cdot), B_\varepsilon(\omega,\cdot))$, that is, for any $A \subseteq S$,

$$\Pi_\varepsilon(A) = \mathbb{P} \{ \omega \in \Omega : (\beta_{k,\varepsilon}(\omega,\cdot), \rho_\varepsilon(\omega,\cdot), u_\varepsilon(\omega,\cdot), \rho_\varepsilon u_\varepsilon(\omega,\cdot), B_\varepsilon(\omega,\cdot)) \in A \}.$$ 

Then the family $\Pi_\varepsilon$ is tight.

**Proof.** Let $\Pi_\varepsilon^1$ be the probability on

$$S_1 = C(0,T;\mathbb{R}) \times C([0,T];L^6_w(D)) \times L^2(0,T;H^1_w(D)) \times \left( L^2(0,T;H^1_w(D)) \cap L^2(0,T;L^2(D)) \right)$$

which is the image of $\mathbb{P}$ on $\Omega$ by the map $\omega \mapsto (\beta_{k,\varepsilon}(\omega,\cdot), \rho_\varepsilon(\omega,\cdot), u_\varepsilon(\omega,\cdot), B_\varepsilon(\omega,\cdot))$. Similar to Section \[3.2\] we can obtain the tightness of the family of $\Pi_\varepsilon^1$:

$$\Pi_\varepsilon^1(\Sigma_{\eta} \times X_{\eta} \times Y_{\eta} \times R_{\eta}) \geq \left( 1 - \frac{\eta}{5} \right)^4.$$  

(4.4)
Hence we only need to check the tightness of $\Pi_c^2$, which is defined as: for any $A \subseteq S_2 = C([0, T]; \mathbb{L}_{\infty}^2(D))$,

$$\Pi_c^2(A) = P(\omega \in \Omega : \rho_c u_\varepsilon(\omega, \cdot) \in A).$$

From \textbf{[3.1]}, we have

$$\rho_c u_\varepsilon(t) = \rho_0 u_0 - \int_0^t \left[ \text{div}(\rho_c u_\varepsilon \otimes u_\varepsilon) - \mu \Delta u_\varepsilon - (\lambda + \mu) \text{div} u_\varepsilon + a \nabla \rho_c^\gamma + \delta \nabla \rho_c^\beta \right] ds - \int_0^t [\varepsilon \nabla u_\varepsilon \cdot \nabla \rho_\varepsilon - (\nabla \times B_\varepsilon) \times B_\varepsilon] ds + \int_0^t f_k(\rho_\varepsilon, \rho_c u_\varepsilon, x) d\beta_{k, \varepsilon}.$$

For $\phi \in C_0^\infty(D)$, denote $\Gamma_\varepsilon := \{\rho_c u_\varepsilon, \phi\} + \varepsilon \int_0^t (\nabla u_\varepsilon \cdot \nabla \rho_\varepsilon, \phi) ds$. By virtue of \textbf{(4.1)}, for some $\alpha > 1$, one has P-a.s.

$$\partial_t \Gamma_\varepsilon = \langle \partial_t (\rho_c u_\varepsilon) + \varepsilon \nabla u_\varepsilon \cdot \nabla \rho_\varepsilon, \phi \rangle = \langle -\text{div}(\rho_c u_\varepsilon \otimes u_\varepsilon) - a \nabla \rho_c^\gamma - \delta \nabla \rho_c^\beta + \mu \Delta u_\varepsilon + (\lambda + \mu) \text{div} u_\varepsilon, \phi \rangle + (\nabla \times B_\varepsilon) \times B_\varepsilon, \phi \rangle \in L^\alpha(0, T).$$

On the other hand, $\Gamma_\varepsilon$ is uniformly bounded in $L^\alpha(0, T)$. Hence $\Gamma_\varepsilon$ is equi-continuous in $C[0, T]$. We can infer that $\int_0^T \varepsilon \nabla u_\varepsilon \cdot \nabla \rho_\varepsilon dt \to 0$ in $C([0, T]; L^1(D))$ almost surely on $\Omega$. Since $L^1(D) \to H^{-\ell}(D)(\ell \geq 3)$ is compact, by \textbf{[22] Lemma 3.9}, the distribution of $\int_0^T \varepsilon \nabla u_\varepsilon \cdot \nabla \rho_\varepsilon dt$ is tight on $H^{-\ell}(D)$. That is, for any $\eta > 0$, there exists a compact set $K \subset C([0, T]; H^{-\ell}(D))$ such that $P(\omega \in \Omega : \int_0^T \varepsilon \nabla u_\varepsilon \cdot \nabla \rho_\varepsilon dt \notin K) \leq \frac{\eta}{10}$. Set

$$\mathcal{H} = L^\infty([0, T]; L_{\infty, \varepsilon}^{\frac{2}{\alpha}}(D)) \cap C^{0, \alpha'}([0, T]; H^{-\ell}(D)), \ \ell \geq 3, \ 0 < \alpha' < \frac{1}{2} - \frac{1}{2\sigma},$$

with the norm $\|f\|_\mathcal{H} = \sup_{0 \leq t \leq T} \|f(t)\|_{L_{\infty, \varepsilon}^{\frac{2}{\alpha}}(D)} + \|f(t)\|_{C^{0, \alpha'}([0, T]; H^{-\ell}(D))}.$

We choose $Z_\eta$ as a closed ball of radius $r_\eta$ centered at 0 in $\mathcal{H} \cap K$. Similar to Section \textbf{3.2}, choosing $r_\eta = 10C\eta^{-1}$, we have

$$\Pi_c^2(Z_\eta) = 1 - \Pi_c^2(Z_\eta^c) = 1 - P(\omega \in \Omega : \rho_c u_\varepsilon(\omega, \cdot) \in Z_\eta^c) = 1 - P(\omega \in \Omega : \rho_c u_\varepsilon(\omega, \cdot) \in \mathcal{H}^c \cup K^c) \geq 1 - P(\omega \in \Omega : \rho_c u_\varepsilon(\omega, \cdot) \notin K) \left(\int_0^T \varepsilon \nabla \rho_c \cdot \nabla u_\varepsilon(\omega, \cdot) dt \notin K\right) \geq 1 - \frac{1}{r_\eta} \mathbb{E}(\|\rho_c u_\varepsilon\|_H) - \frac{\eta}{10} \geq 1 - \frac{C}{r_\eta} - \frac{\eta}{10} = 1 - \frac{\eta}{5},$$

where $Z^c$ is the complement of $Z$ in $\mathcal{H} \cap K$. Therefore, \textbf{(4.3)} and \textbf{(4.5)} imply that

$$\Pi_\varepsilon(\Sigma_\eta \times X_\eta \times Y_\eta \times Z_\eta \times R_\eta) = \Pi_c^1(\Sigma_\eta \times X_\eta \times Y_\eta \times R_\eta) \times \Pi_c^2(Z_\eta) \geq 1 - \eta.$$

Then $\Pi_\varepsilon$ is tight.

According to Jakubowski-Skorohod Theorem, there exists a subsequence such that $\Pi_\varepsilon \to \Pi$ weakly, where $\Pi$ is a probability on $S$. Moreover, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.
and two random variables \((\tilde{\beta}_{k,\varepsilon}, \tilde{\rho}_{\varepsilon}, \tilde{u}_{\varepsilon}, \tilde{\rho}_e \tilde{u}_{\varepsilon}, \tilde{B}_e)\) with distribution \(\Pi_{\varepsilon}\), \((\beta_k, \rho, u, h, B)\) with values in \(S\) such that as \(\varepsilon \to 0\)

\[
(\tilde{\beta}_{k,\varepsilon}, \tilde{\rho}_{\varepsilon}, \tilde{u}_{\varepsilon}, \tilde{\rho}_e \tilde{u}_{\varepsilon}, \tilde{B}_e) \to (\beta_k, \rho, u, h, B) \quad \text{in} \quad S \quad \tilde{P} \quad \text{a.s.}
\]  

(4.6)

4.4. The vanishing viscosity limit. In this subsection, we shall pass to the limit as \(\varepsilon \to 0\) in (3.1)-(3.3). Note that the parameter \(\delta\) is kept fixed throughout this section and then we can use the previously derived estimates. Similarly, from Proposition 3.8, we can deduce that

\[
(\tilde{\beta}, \tilde{u}, \tilde{\rho}_e \tilde{u}, \tilde{B}_e)\]

in (3.1)-(3.3). Note that the parameter \(\varepsilon\) is kept fixed throughout this section and then we can use the previously derived estimates. Similarly, from Proposition 3.8, we can deduce that

\[
(\tilde{\beta}, \tilde{u}, \tilde{\rho}_e \tilde{u}, \tilde{B}_e)\]

satisfies the same estimates as \((\rho_e, u_e, \rho_e u_e, B_e)\). For simplicity of notations, we will again drop the tildes on the random variables. It follows from (4.1) that

\[
\varepsilon \nabla u_\varepsilon \to 0 \quad \text{in} \quad L^1(\tilde{\Omega} \times (0, T) \times D).
\]  

(4.7)

Similarly, we have \(\varepsilon \Delta \rho_\varepsilon \to 0\) in \(L^2(\tilde{\Omega}, L^2(0, T; H^{-1}(D)))\). From (4.3), we have

\[
\alpha \rho_\varepsilon^\gamma + \delta \rho_\varepsilon^\beta \to \bar{P} \quad \text{weakly in} \quad L^{\frac{\beta+1}{\alpha}}(\tilde{\Omega} \times (0, T) \times D).
\]  

(4.8)

We can obtain as in Section 3.4

\[
\rho_\varepsilon \to \rho \quad \text{in} \quad C([0, T]; H^{-1}(D)) \quad \tilde{P} \quad \text{a.s.}
\]  

(4.9)

Furthermore, (4.6) implies that

\[
u_\varepsilon \to u \quad \text{in} \quad L^2(0, T; H^1(D)) \quad \tilde{P} \quad \text{a.s.}
\]  

(4.10)

It follow from (4.10) and (4.9) that

\[
\rho_e u_\varepsilon \to \rho u \quad \text{in} \quad \mathcal{D}'((0, T) \times D) \quad \tilde{P} \quad \text{a.s.}
\]  

Since \(C_0^\infty(D)\) is density in \(L^\frac{2\alpha}{\gamma+1}(D)\), we know that

\[
\rho_e u_\varepsilon \to \rho u \quad \text{in} \quad C([0, T]; L^\frac{2\alpha}{\gamma+1}(D)) \quad \tilde{P} \quad \text{a.s.},
\]

which together with the fact that \(\frac{2\alpha}{\gamma+1} \geq \frac{6}{7} \) \(L^\frac{2\alpha}{\gamma+1}(D) \hookrightarrow H^{-1}(D)\) and Aubin-Lions Lemma implies that

\[
\rho_e u_\varepsilon \to \rho u \quad \text{in} \quad C([0, T]; H^{-1}(D)) \quad \tilde{P} \quad \text{a.s.}
\]  

(4.11)

Therefore, by (4.10) and (4.11) one has

\[
\rho_e u_i^e u_j^e \to \rho u_i^u u_j^u \quad \text{in} \quad \mathcal{D}'((0, T) \times D), i, j = 1, 2, 3 \quad \tilde{P} \quad \text{a.s.}
\]  

(4.12)

By (4.6), we know that

\[
B_\varepsilon \to B \quad \text{in} \quad L^2(0, T; L^2(D)) \quad \text{and} \quad B_\varepsilon \to B \quad \text{in} \quad L^2(0, T; H^1(D)) \quad \tilde{P} \quad \text{a.s.}
\]  

(4.13)

Moreover, from (3.1) and (4.2), by Aubin-Lions Lemma, one has

\[
B_\varepsilon \to B \quad \text{in} \quad C([0, T]; H^{-1}(D)) \quad \tilde{P} \quad \text{a.s.}
\]  

which together with (4.13) yields

\[
(\nabla \times B_\varepsilon) \times B_e \to (\nabla \times B) \times B \quad \text{in} \quad \mathcal{D}'([0, T] \times D) \quad \tilde{P} \quad \text{a.s.}
\]  

(4.14)

Similarly, it follows from (4.10) and (4.13) that

\[
\nabla \times (u_\varepsilon \times B_\varepsilon) \to \nabla \times (u \times B) \quad \text{in} \quad \mathcal{D}'([0, T] \times D) \quad \tilde{P} \quad \text{a.s.}
\]  

(4.15)

By (4.6), Hölder’s inequality, (4.6), (4.11), (4.13), we have

\[
\langle f_k(\rho_e, \rho_e u_e, x), \phi \rangle \to \langle f_k(\rho, \rho u, x), \phi \rangle \quad \text{in} \quad L^1([0, T]) \quad \tilde{P} \quad \text{a.s.}
\]

\[
\langle g_k(B_\varepsilon, x), \phi \rangle \to \langle g_k(B, x), \phi \rangle \quad \text{in} \quad L^1([0, T]) \quad \tilde{P} \quad \text{a.s.}
\]  

(4.16)
For $\phi \in C_0^\infty(D)$, we set $\langle M_\varepsilon(t), \phi \rangle := \int_0^t \sum_{k \geq 1} \langle f_k(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon, x), \phi \rangle d\beta_{k,\varepsilon}^1$ and $\langle \dot{M}_\varepsilon(t), \phi \rangle := \int_0^t \sum_{k \geq 1} \langle g_k(B_\varepsilon, x), \phi \rangle d\beta_{k,\varepsilon}^2$, which are martingales under $\tilde{P}$. It follows from Proposition 3.8 that

$$\langle M_\varepsilon(t), \phi \rangle = \langle \rho_\varepsilon u_\varepsilon(t) - \rho_\varepsilon u_\varepsilon(0), \phi \rangle + \int_0^t \langle \text{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \varepsilon \nabla \rho_\varepsilon \cdot \nabla u_\varepsilon - (\lambda + \mu) \nabla u_\varepsilon + a \nabla \rho_\varepsilon^2, \phi \rangle ds$$

$$+ \int_0^t \langle (\nabla \times B_\varepsilon) \times B_\varepsilon + \varepsilon \nabla \rho_\varepsilon \cdot \nabla u_\varepsilon - (\lambda + \mu) \nabla u_\varepsilon + a \nabla \rho_\varepsilon^2, \phi \rangle ds$$

$$:= \langle \rho_\varepsilon u_\varepsilon(t) - \rho_\varepsilon u_\varepsilon(0), \phi \rangle + \int_0^t \langle \Lambda_\varepsilon, \phi \rangle ds,$$

$$\langle \dot{M}_\varepsilon(t), \phi \rangle = \langle B_\varepsilon(t) - B_\varepsilon(0), \phi \rangle + \int_0^t \langle \nabla \times (u_\varepsilon \times B_\varepsilon) + \nu \Delta B_\varepsilon, \phi \rangle ds$$

$$:= \langle B_\varepsilon(t) - B_\varepsilon(0), \phi \rangle + \int_0^t \langle \dot{\Lambda}_\varepsilon, \phi \rangle ds,$$

We define $\langle M(t), \phi \rangle$ and $\langle \dot{M}(t), \phi \rangle$ by dropping the subscript $\varepsilon$ on the right-hand side of the above two equalities.

**Proposition 4.3.** The system $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), \beta_k, \rho, u, B)$ is a martingale solution of the following equations:

$$\begin{cases}
dp + \text{div}(pu)dt = 0, \\
d\langle pu \rangle + \left[ \text{div}(pu \otimes u) + \nabla \tilde{P} - \mu \Delta u + (\lambda + \mu) \nabla u - (\nabla \times B) \times B \right] dt = dM, \\
dB - [\nabla \times (u \times B) + \nu \Delta B] dt = d\tilde{M},
\end{cases}$$

in $\mathcal{D}'((0, T) \times D)$ $\tilde{P}$-a.s.

**Proof.** Similar to Proposition 3.11 we can easily prove that $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), \beta_k, \rho, u, B)$ satisfies the equation (4.17). Now, we only need to prove that $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), \beta_k, \rho, u, B)$ satisfies the equations (4.17)$_2$ and (4.17)$_3$.

For $0 \leq s < t < \infty$ and a bounded, continuous $\mathcal{F}_s$-measurable function $\varphi_\varepsilon = \varphi(\beta_{k,\varepsilon}, \rho_\varepsilon, u_\varepsilon, B_\varepsilon)$, we have

$$\tilde{E} \left[ \langle M_\varepsilon(t) - M_\varepsilon(s), \phi \varphi_\varepsilon \rangle \right] = 0, \quad \tilde{E} \left[ \langle \dot{M}_\varepsilon(t) - \dot{M}_\varepsilon(s), \phi \varphi_\varepsilon \rangle \right] = 0,$$

$$\tilde{E} \left[ \langle (M_\varepsilon(t) - M_\varepsilon(s), \phi) \beta_{k,\varepsilon}^1 \rangle - \int_s^t \langle f_k(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon, x), \phi \rangle d\tau \right] \varphi_\varepsilon = 0,$$

$$\tilde{E} \left[ \langle (\dot{M}_\varepsilon(t) - \dot{M}_\varepsilon(s), \phi) \beta_{k,\varepsilon}^2 \rangle - \int_s^t \langle g_k(B_\varepsilon, x), \phi \rangle d\tau \right] \varphi_\varepsilon = 0.$$

By (4.6)-(4.8), (4.10)-(4.16), we can infer that

$$\langle \rho_\varepsilon u_\varepsilon(t) - \rho_\varepsilon u_\varepsilon(s), \phi \rangle + \int_s^t \langle \Lambda_\varepsilon, \phi \rangle d\tau, \quad \langle B_\varepsilon(t) - B_\varepsilon(s), \phi \rangle + \int_s^t \langle \dot{\Lambda}_\varepsilon, \phi \rangle d\tau$$

converges to

$$\langle pu(t) - pu(s), \phi \rangle - \int_s^t \langle \Lambda, \phi \rangle d\tau, \quad \langle B(t) - B(s), \phi \rangle + \int_s^t \langle \dot{\Lambda}, \phi \rangle d\tau$$

almost surely on $\tilde{\Omega}$,
where

\[ \langle \Lambda, \phi \rangle = \langle \text{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\text{div}\nabla u + (\nabla \times B) \times B + \overline{P}, \phi \rangle \]

\[ \langle \tilde{\Lambda}, \phi \rangle = \langle \nabla \times (u \times B) + \nu \Delta B, \phi \rangle. \]

We can prove that the functions \( f_\varepsilon := \langle M_\varepsilon(t) - M_\varepsilon(s), \phi \rangle \varphi(\beta_{k,\varepsilon}, \rho_\varepsilon, u_\varepsilon, B_\varepsilon) \) are uniformly integrable as (3.44), (3.45) and (4.10). By the uniform integrability, (4.18), and Lemma 3.10 we have

\[ \lim_{\varepsilon \to 0} \tilde{E} \left[ \langle M(t) - M(s), \phi \rangle \varphi(\beta_k, \rho, u, B) \right] = 0, \]

\[ \lim_{\varepsilon \to 0} \tilde{E} \left[ \langle \tilde{M}(t) - \tilde{M}(s), \phi \rangle \varphi(\beta_k, \rho, u, B) \right] = 0, \]

\[ \lim_{\varepsilon \to 0} \tilde{E} \left[ \langle M(t) - M(s), \phi \beta_k^2 - \int_s^t f_k(\rho, \rho u, x, \phi) d\tau \rangle \varphi(\beta_k, \rho, u, B) \right] = 0. \]

Our next goal is to prove

\[ \overline{P} = \alpha \rho^\gamma + \delta \rho^\beta \quad (\overline{P} \text{ appears in the limit in (4.8)}) \]

which is equivalent to the strong convergence of \( \rho_\varepsilon \) in \( L^1(\tilde{\Omega} \times (0, T) \times D) \).

4.5. The effective viscous flux. We introduce the quantity \( \alpha \rho^\gamma + \delta \rho^\beta - (\lambda + 2\mu)\text{div}u \) that is usually called the effective viscous flux. This quantity satisfies many properties for which we refer to [15, 30] for details. As in [31, 38], we can obtain the following lemma:

**Lemma 4.4.** Assume that \( \beta > \max\{\gamma, 4/3\} \). Let \((\rho_\varepsilon, u_\varepsilon, B_\varepsilon)\) be the sequence of approximate solutions obtained in Lemma 3.3 and let \( \rho, u, \overline{P} \) be the limits appearing in (4.9), (4.10), and (4.8) respectively. Then we have

\[ \lim_{\varepsilon \to 0^+} \tilde{E} \int_0^T \int_D \phi \left( \alpha \rho_\varepsilon^\gamma + \delta \rho_\varepsilon^\beta - (\lambda + 2\mu)\text{div}u_\varepsilon \right) \rho_\varepsilon dx dt \]

\[ = \tilde{E} \int_0^T \int_D \phi \left( \overline{P} - (\lambda + 2\mu)\text{div}u \right) \rho dx dt, \]

for any \( \phi \in D(0, T) \) and \( \phi \in D(D) \).

The idea of the proof of the above lemma is based on the Div-Curl Lemma. First, we introduce the operator \( \mathcal{A} \) by

\[ \mathcal{A}_i v = \Delta^{-1} [\partial_i v], \quad i = 1, 2, 3. \]

Here \( \Delta^{-1} \) stands for the inverse of the Laplace operator in \( \mathbb{R}^3 \). Furthermore

\[ \mathcal{A}_j(\xi) = \frac{-i\xi_j}{|\xi|^2}, \quad \partial_x \mathcal{A}_i[v] = v. \]
We have the following lemma:

The classical Mikhlin multiplier theorem implies that

$$\|A_i v\|_{W^{1,s}(D)} \leq C(s,D)\|v\|_{L^q(\mathbb{R}^3)}, \quad 1 < s < \infty,$$

$$\|A_i v\|_{L^q(D)} \leq C(q,s,D)\|v\|_{W^{1,s}(D)} \leq C(q,s,D)\|v\|_{L^q(\mathbb{R}^3)}, \quad \frac{1}{q} \geq \frac{1}{s} - \frac{1}{3},$$

and

$$\|A_i v\|_{L^\infty(D)} \leq C(s,D)\|v\|_{L^\infty(\mathbb{R}^3)}, \quad s > 3.$$
Lemma 4.5. Assume that ρ ∈ Lp(Ω, L2((0, T) × D)) and u ∈ Lp(Ω, L2(0, T; H01(D))) solve \((4.17)\) in \(D'((0, T) × D)\) almost surely. Then we can extend \((ρ, u)\) to be zero on \(\mathbb{R}^3 \setminus D\). Moreover, the equation \((4.17)\) holds in \(D'((0, T) × \mathbb{R}^3)\) almost surely.

Next, we consider the test functions
\[φ_i(t, x) = ψ(t, x)φ(x,A_i[ρ]), \quad i = 1, 2, 3,\]
where ρ is zero outside D. Similar to \((4.19)\), taking φ as test functions for \((4.17)_2\), integrating over \(Ω × (0, T) × D\), we obtain
\[
\tilde{E} \int_0^T \psi \int_D φ [T − (λ + 2μ)\text{div}u] ρ dxdt
= (λ + μ)\tilde{E} \int_0^T \psi \int_D \text{div}u φ_x ρ dxdt - \tilde{E} \int_0^T \int_D ρφ x φ_x ρ dxdt
+ μ\tilde{E} \int_0^T \psi \int_D ρφ_x u_t ρ dxdt − \tilde{E} \int_0^T \psi \int_D ρu_t φ_x ρ dxdt
− \tilde{E} \int_0^T \psi \int_D ρu_t φ_x ρ dxdt
+ \tilde{E} \int_0^T \psi \int_D μψ(ρR_{i,j}[ρu_j] − ρu_j R_{i,j}[ρ]) dxdt.
\]
Here we have used the following property: Recall that \(M(t) = ρu(t) − ρu(0) + \int_0^t \text{div}(ρu ⊗ u) + ∇P − μ∆u − (λ + μ)\text{div}u − (∇ × B) × B)ds\) and \(M\) is a martingale. So we can prove that \(\int_0^t ψ(t)φ(x,A_i[ρ])dM = 0\), where \(ψ \in C_0^∞([0, T]), φ(x) \in C_0^∞(D)\). In fact, integrating by parts, we have
\[
\int_0^t ψ(t)φ(x,A_i[ρ])dM = \psi(t)φ(x,A_i[ρ])M − \int_0^t Mddψ(t)φ(x,A_i[ρ]).
\]
We denote the term \(\int_0^t Mddψ(t)φ(x,A_i[ρ])\) by \(\int_0^t Mdn(ψ)\). Let \(Δ_n := \{0 = t_0 ≤ t_1 ≤ \ldots ≤ t_k ≤ \ldots ≤ t_n = t\}\) be a subdivision of the time interval and denote \(∥Δ_n∥ := \max_{1 ≤ k ≤ n} |t_k − t_{k−1}|\). Then we have
\[
\int_0^t Mddψ(t)φ(x,A_i[ρ]) = \sum_{k=1}^n \sum_{k=1}^{n-1} M(t_k)[N(t_k) − N(t_k−1)]
= \sum_{k=1}^n N(t_k)M(t_k) − \sum_{k=1}^{n-1} N(t_k)M(t_k) + \sum_{k=0}^{n-1} N(t_k)M(t_k) − \sum_{k=0}^{n-1} M(t_{k+1})N(t_k)
= N(t)M(t) − \sum_{k=0}^{n-1} N(t_k)[M(t_{k+1}) − M(t_k)].
\]
Let \(∥Δ_n∥ → 0\), since \(N(t_k)\) is independent on \(M(t_{k+1}) − M(t_k)\) and \(M\) is a martingale, we have \(E\int_0^t Mddψ(t)φ(x,A_i[ρ]) = E[N(t)M(t)].\) Then \(E\int_0^t ψ(t)φ(x,A_i[ρ])dM = 0.\)

Now following [34, 38] we can prove that the right-hand side of \((4.19)\) converges to the right-hand side of \((4.20)\) as \(ε → 0\), which proves Lemma 4.4.
4.6. Strong convergence of the density. In this subsection, we will follow the idea of \[15, 30\] to prove the strong convergence of the sequence $\rho_\varepsilon$. Our goal is to show $\overline{T} = a \rho^\gamma + \delta \rho^\beta$.

Lemma 4.3 implies that we can extend $(\rho, u)$ to be zero on $\mathbb{R}^3 \setminus D$, and

$$\rho \in L^p(\tilde{\Omega}, L^2(0, T; L^2(D))), \quad u \in L^p(\tilde{\Omega}, L^2(0, T; H_0^1(D)))$$

solve the continuity equation (4.17) in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$ almost surely. Taking a regularizing sequence $\vartheta_m = \vartheta_m(x)$ and then using $S_m$ to test (4.17), we have

$$\partial_t S_m[\rho] + \text{div}[S_m(\rho)u] = r_m \quad \text{on } (0, T) \times \mathbb{R}^3. \quad (4.21)$$

Here $S_m[v] = \vartheta_m * v$ is the standard smoothing operator. We note that [29] Lemma 2.3 implies that the remainder term $r_m \to 0$ in $L^1(\tilde{\Omega} \times (0, T) \times \mathbb{R}^3)$ as $m \to \infty$.

For any $b$ satisfying the conditions in Definition 1.1 multiplying (4.21) by $b'(S_m[\rho])$ and passing to the limit as $m \to \infty$, we can infer that $(\rho, u)$ solves (4.17) in the sense of renormalized solutions. Furthermore, taking $b(z) = z \ln z$ and then integrating (1.4) over $\tilde{\Omega} \times (0, T) \times D$, one has

$$\tilde{E} \int_0^T \int_D \rho \text{div} u dx dt = \tilde{E} \int_D \rho_0 \ln \rho_0 dx - \tilde{E} \int_0^T \rho(T) \ln \rho(T) dx. \quad (4.22)$$

Since $(\rho_\varepsilon, u_\varepsilon)$ satisfies the equation (3.1) a.e. on $\tilde{\Omega} \times (0, T) \times D$, multiplying (3.1) with $b'(\rho_\varepsilon)$, then for any $b$ convex and globally Lipschitz on $\mathbb{R}^+$, we have

$$\partial_t b(\rho_\varepsilon) + \text{div}(b(\rho_\varepsilon)u_\varepsilon) + [b'(\rho_\varepsilon)\rho_\varepsilon - b(\rho_\varepsilon)] \text{div} u_\varepsilon - \varepsilon \Delta b(\rho_\varepsilon) = -\varepsilon b''(\rho_\varepsilon)|\nabla \rho_\varepsilon|^2 \leq 0.$$ 

Integrating the above on $\tilde{\Omega} \times (0, T) \times D$, the boundary condition yields

$$\tilde{E} \int_0^T \int_D [b'(\rho_\varepsilon)\rho_\varepsilon - b(\rho_\varepsilon)] \text{div} u_\varepsilon dx dt \leq \tilde{E} \int_D b(\rho_0) dx - \tilde{E} \int_D b(\rho_\varepsilon(T)) dx.$$ 

Let $b(z) = z \ln z$ in the above inequality, then

$$\lim_{\varepsilon \to 0} \tilde{E} \int_0^T \int_D \rho_\varepsilon \text{div} u_\varepsilon dx dt \leq \tilde{E} \int_D \rho_0 \ln \rho_0 dx - \lim_{\varepsilon \to 0} \tilde{E} \int_D \rho_\varepsilon(T) \ln \rho_\varepsilon(T) dx \quad (4.23)$$

Taking two nondecreasing sequences $0 \leq \psi_n \in \mathcal{D}(0, T)$, $0 \leq \phi_n \in \mathcal{D}(D)$ satisfying

$$\psi_n \to 1, \quad \phi_n \to 1 \quad \text{as } n \to \infty; \quad \psi_n(t) = 1 \quad \text{for } t \geq \frac{1}{n} \quad \text{or } t \leq T - \frac{1}{n};$$

$$\phi_n(x) = 1 \quad \text{for } x \in D, \quad \text{dist}(x, \partial D) \geq \frac{1}{n}.$$ 

Combining Lemma 4.4, (4.22) and (4.23), for all $m \leq n$, we obtain

$$\lim_{\varepsilon \to 0} \tilde{E} \int_0^T \psi_n \int_D \phi_m \left( a \rho_\varepsilon^\gamma + \delta \rho_\varepsilon^\beta \right) \rho_\varepsilon dx dt \leq \lim_{\varepsilon \to 0} \tilde{E} \int_D \phi_n \int_0^T \left( a \rho_\varepsilon^\gamma + \delta \rho_\varepsilon^\beta \right) \rho_\varepsilon dx dt$$

$$\leq \lim_{\varepsilon \to 0} \tilde{E} \int_0^T \psi_n \int_D \phi_n \left[ a \rho_\varepsilon^\gamma + \delta \rho_\varepsilon^\beta - (\lambda + 2\mu) \text{div} u_\varepsilon \right] \rho_\varepsilon dx dt$$

$$+ (\lambda + 2\mu) \lim_{\varepsilon \to 0} \tilde{E} \int_0^T \psi_n \int_D \phi_n \rho_\varepsilon \text{div} u_\varepsilon dx dt.$$
lim \lim_{\varepsilon \to 0} \tilde{E} \int_0^T \psi_n \int_D \phi_n [\tilde{\mathcal{P}} - (\lambda + 2\mu) \text{div} u] \rho dx dt + (\lambda + 2\mu) \tilde{E} \int_0^T \int_D \rho_c |1 - \psi_n \phi_n| |\text{div} u| dx dt \\
+ (\lambda + 2\mu) \tilde{E} \int_0^T \int_D \rho_c |\text{div} u| dx dt \\
\leq \tilde{E} \int_0^T \int_D \tilde{\mathcal{P}} \rho dx dt + (\lambda + 2\mu) \tilde{E} \int_0^T \int_D \rho \text{div} u dx dt - (\lambda + 2\mu) \tilde{E} \int_0^T \int_D \rho \text{div} u dx dt + \eta(n) \\
\leq \tilde{E} \int_0^T \int_D \tilde{\mathcal{P}} \rho dx dt + \eta(n),

where \eta(n) = \tilde{E} \int_0^T \int_D (\psi_n \phi_n - 1) [\tilde{\mathcal{P}} - (\lambda + 2\mu) \text{div} u] \rho dx dt + (\lambda + 2\mu) \tilde{E} \int_0^T \int_D \rho_c |1 - \psi_n \phi_n| |\text{div} u| dx dt \to 0 \text{ as } n \to \infty. \text{ Passing to the limit in the above inequality as } n \to \infty, \text{ one deduces that}

\lim_{\varepsilon \to 0^+} \tilde{E} \int_0^T \psi_m \int_D \phi_m (a \rho_e^\gamma + \delta \rho_e^\beta) \rho_e dx dt \leq \tilde{E} \int_0^T \int_D \tilde{\mathcal{P}} \rho dx dt \text{ for all } m = 1, 2, \ldots

Since \( P(z) = az^\gamma + \delta z^\beta \) is monotone, then we have

\tilde{E} \int_0^T \psi_m \int_D \phi_m [P(\rho_e) - P(v)] (\rho_e - v) dx dt \geq 0.

Furthermore, the facts \( \rho_e \to \rho \) and \( P(\rho_e) \to \tilde{\mathcal{P}} \) imply that

\tilde{E} \int_0^T \psi_m \int_D \phi_m \tilde{\mathcal{P}} \rho dx dt + \tilde{E} \int_0^T \psi_m \int_D \phi_m P(v) v dx dt - \tilde{E} \int_0^T \psi_m \int_D \phi_m [\tilde{\mathcal{P}} v + P(v) \rho] dx dt \geq 0.

Now, letting \( m \to \infty \) in the above inequality, one has

\tilde{E} \int_0^T \int_D [\tilde{\mathcal{P}} - P(v)] (\rho - v) dx dt \geq 0.

Choosing \( v = \rho + \alpha \varphi \), for an arbitrary \( \varphi \), and then letting \( \alpha \to 0 \), we have

\( \mathcal{P} = a \rho^\gamma + \delta \rho^\beta \).

That is,

\( a \rho_e^\gamma + \delta \rho_e^\beta \to a \rho^\gamma + \delta \rho^\beta \) in \( L^{s+\frac{1}{s}}(\hat{\Omega} \times (0, T) \times D) \).

Then we have

\( \rho_e \to \rho \) in \( L^s(\hat{\Omega} \times (0, T) \times D) \), \( \forall \ 1 \leq s < \beta + 1 \).

(4.24)

Similar to the proof of (3.40), by (4.10), Hölder’s inequality, (4.11), (4.13), (4.24), we can deduce that

\begin{align*}
\left\langle \sum_{k \geq 1} f_k(\rho_e, \rho_e u_e, x), \phi \right\rangle^2 & \to \left\langle \sum_{k \geq 1} f_k(\rho, \rho u, x), \phi \right\rangle^2 \text{ in } L^1([0, T]) \tilde{P} \text{ a.s.,} \\
\left\langle \sum_{k \geq 1} g_k(B_e, x), \phi \right\rangle^2 & \to \left\langle \sum_{k \geq 1} g_k(B, x), \phi \right\rangle^2 \text{ in } L^1([0, T]) \tilde{P} \text{ a.s.}
\end{align*}

(4.25)
Note that
\[
\mathbb{E} \left[ \left( \langle M(t), \phi \rangle^2 - \langle M(s), \phi \rangle^2 - \int_s^t \sum_{k \geq 1} \langle f_k(\rho, \rho u, x), \phi \rangle^2 d\tau \right) \varphi_\varepsilon \right] = 0,
\]
(4.26)
\[
\mathbb{E} \left[ \langle \tilde{M}(t), \phi \rangle^2 - \langle \tilde{M}(s), \phi \rangle^2 - \int_s^t \sum_{k \geq 1} \langle g_k(B, x), \phi \rangle^2 d\tau \right] \varphi_\varepsilon = 0.
\]

Then, it follows from (4.25), (4.26), the uniform integrability, Proposition 3.9 and Lemma 3.10 that
\[
\mathbb{E} \left[ \left( \langle M(t), \phi \rangle^2 - \langle M(s), \phi \rangle^2 - \int_s^t \sum_{k \geq 1} \langle f_k(\rho, \rho u, x), \phi \rangle^2 d\tau \right) \varphi(\beta_k, \rho, u, B) \right]
= \lim_{\varepsilon \to 0} \mathbb{E} \left[ \left( \langle \tilde{M}(t), \phi \rangle^2 - \langle \tilde{M}(s), \phi \rangle^2 - \int_s^t \sum_{k \geq 1} \langle g_k(B, x), \phi \rangle^2 d\tau \right) \varphi(\beta_k, \rho, u, B) \right] = 0,
\]
(4.27)
\[
\mathbb{E} \left( \langle M(t), \phi \rangle^2 - \langle \tilde{M}(t), \phi \rangle^2 - \int_0^t \sum_{k \geq 1} \langle f_k(\rho, \rho u, x), \phi \rangle d\beta_k^1 \right) \varphi(\beta_k, \rho, u, B) = 0.
\]

Hence we deduce that \( \langle M(t), \phi \rangle^2 - \int_0^t \sum_{k \geq 1} \langle f_k(\rho, \rho u, x), \phi \rangle^2 ds; \langle \tilde{M}(t), \phi \rangle \) and \( \langle \tilde{M}(t), \phi \rangle^2 - \int_0^t \sum_{k \geq 1} \langle g_k(B, x), \phi \rangle^2 ds \) are continuous martingales. Using again the method in [3, 28] we can also infer that
\[
\langle M(t), \phi \rangle = \int_0^t \langle \sum_{k \geq 1} f_k(\rho, \rho u, x), \phi \rangle d\beta_k^1, \quad \langle \tilde{M}(t), \phi \rangle = \int_0^t \langle \sum_{k \geq 1} g_k(B, x), \phi \rangle d\beta_k^2.
\]

Summing up the above results, we have the following proposition:

**Proposition 4.6.** Assume that \( D \in C^{2+\alpha} \) is a bounded domain. If \( \beta > \max\{\frac{6\gamma}{2\gamma-3}, \gamma, 4\} \), then there exists a finite energy martingale solution \((\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\beta}_k, \rho, u, B)\) of the problem:

\[
\begin{align*}
\frac{d\rho}{dt} + \text{div}(\rho u) &= 0, \\
d(\rho u) + \left[ \text{div}(\rho u \otimes u) + a\nabla \rho^\gamma + \delta \nabla \rho^\beta - \mu \Delta \rho + (\lambda + \mu) \nabla \text{div} u \right] dt &= (\nabla \times B) \times B dt + \sum_{k \geq 1} f_k(\rho, \rho u, x) d\beta_k^1, \\
\frac{d\rho u}{dt} + \nabla \times (u \times B) + \nu \Delta B dt &= \sum_{k \geq 1} g_k(B, x) d\beta_k^2.
\end{align*}
\]

\( \tilde{\beta} \)-a.s. and \((\rho, u, B)\) satisfies the initial-boundary valued conditions (3.2), (3.3). Moreover, if \((\rho, u)\) is prolonged to be zero on \( \mathbb{R}^3 \setminus D \), then the equation (4.27) holds in sense of renormalized solutions on \( \mathcal{D}'(\mathbb{R}^3 \times (0, T)) \) \( \tilde{\beta} \)-a.s. Finally, \((\rho, u, B)\) satisfies the following estimates:
\[
\mathbb{E} \left( \sup_{t \in [0, T]} ||\rho(t)||_{L^\gamma(D)} \right)^p \leq C, \quad \mathbb{E} \left( \delta \sup_{t \in [0, T]} ||\rho(t)||_{L^\beta(D)}^{\beta} \right)^{\beta p} \leq C,
\]
δ > \text{in of as test functions for the equation (4.27)}

5.1. On integrability of the density. First, we shall estimate the density $\rho_\delta$, uniformly in $\delta > 0$. Recall

$$
\partial_t S_m[b(\rho_\delta)] + \text{div}(S_m[b(\rho_\delta)]u_\delta) + S_m[(b'(\rho_\delta)\rho_\delta - b(\rho_\delta))]\text{div}u_\delta = r_m,
$$

where

$$
r_m \to 0 \text{ in } L^2(\Omega, L^2(0, T; L^2(\mathbb{R}^3))) \quad \text{as } m \to \infty,
$$

and $b$ is uniformly bounded.

Denote $\int_D f dx = \frac{1}{|D|} \int_D f dx$. Similar to Section 4.2, we use the operator $B$ introduced in Section 4.1 to construct the following test functions:

$$
\varphi_i(t, x) = \psi(t) \left[ S_m[b(\rho_\delta)] - \int_D S_m[b(\rho_\delta)] dx \right], \quad i = 1, 2, 3, \quad \psi \in \mathcal{D}(0, T).
$$

Note that $\varphi_i|_{\partial D} = 0$, $\varphi_i \in L^\infty(0, T; H^1_0(D))$, $\partial_t \varphi_i \in L^2(0, T; H^1_0(D))$. Thus, we can use $\varphi_i$ as test functions for the equation (4.27). Using (5.1), we obtain

$$
\begin{align*}
\mathbb{E} \left( \sup_{t \in [0, T]} \| \sqrt{\rho_\delta} u_\delta \|_{L^2(D)}^2 \right)^p &\leq C, \quad \mathbb{E} \left( \| \rho_\delta \|_{L^{p+1}(0, T) \times D}^p \right) \leq C, \\
\mathbb{E} \left( \| u_\delta \|_{L^2(0, T; H^1_0(D))}^p \right) &\leq C, \quad \mathbb{E} \left( \| B \|_{L^{\infty}(0, T; L^2(D))} + \| B \|_{L^2(0, T; H^1_0(D))}^p \right) \leq C,
\end{align*}
$$

where the constant $C$ is independent of $\delta > 0$.

5. PASSING TO THE LIMIT IN THE ARTIFICIAL PRESSURE TERM

In this section, we shall follow the idea of [15, 30] to pass to the limit in (4.27) as $\delta \to 0$ for the artificial pressure term and relax the hypotheses on the initial data $(\rho_{0,\delta}, m_{0,\delta}, B_{0,\delta})$, that is, (3.2) and (3.3). First, from (3.1), we have $\rho_{0,\delta} \to \rho_0$ in $L^\gamma(D)$, $m_{0,\delta} \to m_0$ in $L^1(D)$, $B_{0,\delta} \to B_0$ in $L^2(D)$ $\tilde{P}$-a.s. as $\delta \to 0$.

Next, we consider the problem (4.27) with the initial data $(\rho_{0,\delta}, m_{0,\delta}, B_{0,\delta})$. From Proposition 4.6, we obtain the existence of a martingale solution, denoted by $(\beta_{k,\delta}, \rho_\delta, u_\delta, B_\delta)$. From (5.1), $\mathcal{E}_{0,\delta}$ is bounded uniformly in $\delta$. Hence, the estimates in Proposition 4.6 hold independently of $\delta$.
\[ E \int_0^T \psi \int_D \mathcal{B}_i \left[ S_m[b(\rho_\delta)] - \int_D S_m[b(\rho_\delta)] \right] (\nabla \times B_\delta) \times B_\delta \, dx \, dt. \] (5.3)

Using (5.2), we can pass to the limit in (5.3) for \( m \to \infty \). Moreover, we can apply the function \( z^\theta \approx b(z) \) to obtain

\[ E \int_0^T \psi \int_D \left(a \rho_\delta^{3+\theta} + \delta \rho_\delta^{3+\theta}\right) \, dx \, dt := \sum_{i=1}^8 J_i, \]

for some \( \theta > 0 \) to be determined below. Now, we just estimate the term \( J_8 \) on the right hand side of the above identity. Other terms can be bounded as in \([38]\). For the term \( J_8 \), if \( \theta < \frac{\gamma}{3} \), using Proposition 4.6 and (4.2), together with the embedding \( W^{1,p}(D) \subset L^\infty(D) \) for \( p > 3 \), we have

\[
J_8 \leq E \left| \int_0^T \int_D \psi \mathcal{B}_i \left[ \rho_\delta^{\theta} - \int_D \rho_\delta^{\theta} \right] (\nabla \times B_\delta) \times B_\delta \, dx \, dt \right| \\
\leq E \int_0^T |\psi||\nabla B_\delta||L^2(D)||B_\delta||L^2(D)| \left\| \mathcal{B}_i \left[ \rho_\delta^{\theta} - \int_D \rho_\delta^{\theta} \right] \right\|_{L^\infty(D)} \, dt \\
\leq E \int_0^T |\psi||\nabla B_\delta||L^2(D)||B_\delta||L^2(D)| \left\| \mathcal{B}_i \left[ \rho_\delta^{\theta} - \int_D \rho_\delta^{\theta} \right] \right\|_{W^{1,\frac{3}{2}}(D)} \, dt \\
\leq E \left( \|\nabla B_\delta\|_{L^2(0,T;L^2(D))} \right) \left( \|B_\delta\|_{L^2(0,T;L^2(D))} \right)^{\frac{3}{2}} \left( \|\rho_\delta\|_{L^\infty(D)} \right)^{\frac{1}{2}} \leq C.
\]

Here the constant \( C \) is independent of \( \delta \).

Summing up the estimates for (5.3), we have

\[ E \int_0^T \psi \int_D \left(a \rho_\delta^{3+\theta} + \delta \rho_\delta^{3+\theta}\right) \, dx \, dt \leq C. \] (5.4)

Taking \( \psi_n \in \mathcal{D}(0,T), 0 \leq \psi_n \leq 1 \) and \( T \int_0^T |\partial_t \psi_n| \, dt \leq C \). By approximation and Lebesgue convergence theorem, we can take \( \psi = 1 \) in (5.4). Then we have proved the following result:

**Lemma 5.1.** For \( \gamma > \frac{3}{2} \) and \( 0 < \theta < \min \left\{ 1, \frac{\gamma}{3}, \frac{2}{3} \gamma - 1 \right\} \), then there exists a constant \( C \) independently of \( \delta > 0 \), such that

\[ E \int_0^T \int_D \left(a \rho_\delta^{\gamma+\theta} + \delta \rho_\delta^{\gamma+\theta}\right) \, dx \, dt \leq C. \]

5.2. **Tightness Property.** Similar to Section 4.3, we can prove the following lemma:

**Lemma 5.2.** Define

\[ S = C(0,T;\mathbb{R}) \times C([0,T];L^2_w(D)) \times L^2(0,T;H^1_w(D)) \times C([0,T];L^2_w(D)^2) \times (L^2(0,T;H^1_w(D)) \cap L^2(0,T;L^2(D))). \]
5.3. The limit passage. In this subsection, we shall pass to the limit for $\delta \to 0$ following the idea of [15][30]. According to Jakubowski-Skorohod Theorem, there exists a subsequence such that $\Pi_{\delta} \to \Pi$ weakly, where $\Pi$ is a probability on $S$. Moreover, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables $(\tilde{\beta}_{k,\delta}, \tilde{\rho}_{\delta}, \tilde{u}_{\delta}, \tilde{\beta}_{k}\tilde{u}_{\delta}, \tilde{B}_{\delta})$ with distribution $\Pi_{\delta}$, $(\beta_{k,\rho}, u, h, B)$ with values in $S$ such that
\[
(\beta_{k,\delta}, \tilde{\rho}_{\delta}, \tilde{u}_{\delta}, \tilde{\beta}_{k}\tilde{u}_{\delta}, \tilde{B}_{\delta}) \to (\beta_{k,\rho}, u, h, B) \quad \text{in} \quad S \quad \tilde{\mathbb{P}} \quad \text{a.s.}, \tag{5.5}
\]
Since the distribution of $(\tilde{\beta}_{k,\delta}, \tilde{\rho}_{\delta}, \tilde{u}_{\delta}, \tilde{\beta}_{k}\tilde{u}_{\delta}, \tilde{B}_{\delta})$ is $\Pi_{\delta}$, we can deduce that $(\tilde{\rho}_{\delta}, \tilde{u}_{\delta}, \tilde{\beta}_{k}\tilde{u}_{\delta}, \tilde{B}_{\delta})$ satisfies the same estimate as $(\rho_{\delta}, u_{\delta}, \beta_{k}u_{\delta}, B_{\delta})$. To simplify notations, we drop the tilde on the random variables. First, we can apply Lemma 5.1 to obtain
\[
\delta\rho_{\delta}^{\gamma} \to 0 \quad \text{in} \quad L^\frac{\beta}{\beta+\delta}(\tilde{\Omega} \times (0, T) \times D),
\]
\[
\rho_{\delta}^{\gamma} \to \rho^\gamma \quad \text{weakly in} \quad L^\frac{\beta}{\beta+\delta}(\tilde{\Omega} \times (0, T) \times D). \tag{5.6}
\]
Similar to Section 4.2, (5.5) implies that
\[
\rho_{i} \to \rho \quad \text{in} \quad C([0, T]; L_w^2(D)) \cap C([0, T]; H^{-1}(D)) \quad \tilde{\mathbb{P}} \quad \text{a.s.},
\]
\[
u_{\delta} \to u \quad \text{in} \quad L^2(0, T; H^1(D)) \quad \tilde{\mathbb{P}} \quad \text{a.s.}, \tag{5.7}
\]
\[
\rho_{\delta}u_{\delta} \to \rho u \quad \text{in} \quad C([0, T]; L_w^{\frac{2\gamma+\theta}{\gamma+1}}(D)) \quad \tilde{\mathbb{P}} \quad \text{a.s.}.
\]
Since $\gamma > \frac{2}{3}$, then $\frac{2\gamma+\theta}{\gamma+1} > \frac{2}{3}$. By (5.7), one has
\[
\rho_{\delta}u_{\delta}^{i}u_{\delta}^{j} \to \rho^{i}u^{j} \quad \text{in} \quad \mathcal{D}'((0, T) \times D) \quad \tilde{\mathbb{P}} \quad \text{a.s.}. \tag{5.8}
\]
By (5.5), we know that
\[
B_{\delta} \to B \quad \text{in} \quad L^2(0, T; L^2(D)) \quad \text{and} \quad B_{\delta} \to B \quad \text{in} \quad L^2(0, T; H^1(D)) \quad \tilde{\mathbb{P}} \quad \text{a.s.}. \tag{5.9}
\]
Moreover, from 3.13 and Proposition 4.6, by Aubin-Lions Lemma, one has
\[
B_{\delta} \to B \quad \text{in} \quad C([0, T]; H^{-1}(D)) \quad \tilde{\mathbb{P}} \quad \text{a.s.}.
\]
Then we have
\[
(\nabla \times B_{\delta}) \times B_{\delta} \to (\nabla \times B) \times B \quad \text{in} \quad \mathcal{D}'([0, T] \times D) \quad \tilde{\mathbb{P}} \quad \text{a.s.}. \tag{5.10}
\]
Similarly, it follows from (5.7), and (5.9) that
\[
\nabla \times (u_{\delta} \times B_{\delta}) \to \nabla \times (u \times B) \quad \text{in} \quad \mathcal{D}'([0, T] \times D) \quad \tilde{\mathbb{P}} \quad \text{a.s.}. \tag{5.11}
\]
Now, it remains to show the strong convergence of $\rho_{\delta}$ in $L^1(\tilde{\Omega} \times (0, T) \times D)$, i.e., $\rho^{\gamma} = \rho^\gamma$ for proving Theorem 1.2. First, we introduce the cut off functions:
\[
T_{k}(z) = kT\left(\frac{z}{k}\right), k = 1, 2, 3, \ldots,
\]
where
\[
T(z) = \begin{cases} 
  z, & z \leq 1 \\
  2, & z \geq 3
\end{cases} \in C^\infty(\mathbb{R}^3), \text{ concave in } z \in \mathbb{R}.
\]
Since \((\rho_\delta, u_\delta)\) is a renormalized solution of (4.27) in \(D'((0, T) \times \mathbb{R}^3)\) \(\tilde{P}\)-a.s., choosing \(b(z) = T_k(z)\), one deduces that in \(D'((0, T) \times \mathbb{R}^3)\) \(\tilde{P}\)-a.s.

\[
\partial_t T_k(\rho_\delta) + \text{div}(T_k(\rho_\delta)u_\delta) + [T'_k(\rho_\delta)\rho_\delta - T_k(\rho_\delta)]\text{div} u_\delta = 0.
\]

(5.12)

Noting that for any fixed \(k\), \(T_k(\rho_\delta) \in L^p(\tilde{\Omega}, L^\infty((0, T) \times D))\) and \(T_k(\rho_\delta) \to \overline{T_k(\rho)}\) weakly star in \(L^p(\tilde{\Omega}, L^\infty((0, T) \times D))\) and \(\partial_t T_k(\rho_\delta)\) satisfies the equation (5.12), by Aubin-Lions Lemma, as \(\delta \to 0\), for all \(1 \leq p < \infty\), we can infer that \(\tilde{P}\)-a.s.

\[
T_k(\rho_\delta) \to \overline{T_k(\rho)} \quad \text{in} \quad C_w([0, T]; L^p(D)) \cap C([0, T]; H^{-1}(D)).
\]

(5.13)

Letting \(\delta \to 0+\) in (5.12) and by using (5.13), we have \(\tilde{P}\)-a.s.

\[
\partial_t \overline{T_k(\rho)} + \text{div}(\overline{T_k(\rho)}u) + [\overline{T'_k(\rho)}\rho - \overline{T_k(\rho)}]\text{div} u = 0,
\]

(5.14)

in \(D'((0, T) \times \mathbb{R}^3)\), where

\[
[T'_k(\rho_\delta)\rho_\delta - T_k(\rho_\delta)]\text{div} u_\delta \to [\overline{T'_k}(\rho)\rho - \overline{T_k}(\rho)]\text{div} u \quad \text{in} \quad L^2(\tilde{\Omega}, L^2((0, T) \times D)).
\]

Here \(\overline{h(\rho)}\) is the weak limit of \(h(\rho_\delta)\).

5.4. **The effective viscous flux.** Similar to the Section 4.3, we have the following lemma:

**Lemma 5.3.** Let \((\rho_\delta, u_\delta, B_\delta)\) be the sequence of approximate solutions constructed in Proposition 4.6. Then for any \(\psi \in D(0, T), \phi \in D(D),\)

\[
\lim_{\delta \to 0+} \tilde{E} \int_0^T \psi \int_D \phi \left[a\rho_\delta^2 - (\lambda + 2\mu)\text{div} u_\delta\right] T_k(\rho_\delta) dx dt = \tilde{E} \int_0^T \psi \int_D \phi \left[a\rho^2 - (\lambda + 2\mu)\text{div} \overline{T_k(\rho)}\right] dx dt.
\]

Here we have used (5.14) and the property under (1.20).

5.5. **The amplitude of oscillations.** In this subsection, we have the following lemma in the sense of expectation as in [38]:

**Lemma 5.4.** There exists a constant \(C\) independent of \(k \geq 1\) such that

\[
\lim_{\delta \to 0+} \tilde{E}\|T_k(\rho_\delta) - T_k(\rho)\|_{L^{\gamma+1}((0, T) \times D)}^{\gamma+1} \leq C.
\]

5.6. **The renormalized solutions.** In this subsection, we shall use Lemma 5.4 and the cut-off function technique to show the following crucial lemma as in [38]:

**Lemma 5.5.** Suppose \((\rho, u)\) be zero outside \(D\), then \((\rho, u)\) solves the continuity equation in the sense of renormalized solutions, that is,

\[
\partial_t b(\rho) + \text{div}(b(\rho)u) + [b'(\rho)\rho - b(\rho)] \text{div} u = 0 \quad \text{in} \quad D'((0, T) \times \mathbb{R}^3),
\]

\(\tilde{P}\)-a.s. for any \(b\) satisfying the conditions on \(b\).
5.7. Strong convergence of the density. This subsection is devoted to completing the proof of Theorem 1.2. First, we introduce a family of functions $L_k$ as

$$ L_k(z) = \begin{cases} 
  z \ln z, & \text{for } 0 \leq z < k, \\
  z \ln k + z \int_k^\infty \frac{T_k(s)}{s} ds, & \text{for } z \geq k,
\end{cases} \in C^1(\mathbb{R}_+) \cap C^0[0, \infty). $$

Then, for large enough $\delta$, as $u$ is renormalized solution of (4.27) and $b$ is large enough. Meanwhile, $b_k(z) = 0$ when $z$ is large enough. Meanwhile, $b_k(z) - b_k(z) = T_k(z)$. Taking $b(z) = b_k(z)$, the fact that $(\rho_\delta, u_\delta)$ is renormalized solution of (4.27) yields that

$$ \partial_t L_k(\rho_\delta) + \text{div}(L_k(\rho_\delta)u_\delta) + T_k(\rho_\delta)\text{div}u_\delta = 0. \quad (5.15) $$

As in (5.15), it follows from the continuity equation and Lemma 5.5 that

$$ \partial_t L_k(\rho) + \text{div}(L_k(\rho)u) + T_k(\rho)\text{div}u = 0 \quad \text{in } \mathcal{D}'((0, T) \times D). \quad (5.16) $$

Since $L_k(z)$ is a linear function when $z$ is large enough, by (5.15) and Aubin-Lions Lemma, then it holds that $\mathbb{P}$-a.s.

$$ L_k(\rho_\delta) \to \overline{L_k(\rho)} \quad \text{in } C_w([0, T]; L^\gamma(D)) \cap C^0([0, T]; H^{-1}(D)), \quad (5.17) $$

as $\delta \to 0$. Taking the difference of (5.15) and (5.16) and then taking $\phi \in \mathcal{D}(D)$ as a test function, we have

$$ \tilde{E} \int_D (L_k(\rho_\delta) - L_k(\rho))(t)\phi dx = \tilde{E} \int_D (L_k(\rho_{0, \delta}) - L_k(\rho_0))(t)\phi dx \quad (5.18) $$

$$ + \tilde{E} \int_0^t \int_D (L_k(\rho_\delta)u_\delta - L_k(\rho)u) \cdot \nabla \phi + (T_k(\rho)\text{div}u - T_k(\rho_\delta)\text{div}u_\delta)\phi dx dt. $$

Letting $\delta \to 0$ in (5.18), noting that $L_k(\rho_{0, \delta}) - L_k(\rho_0) \to 0$ as $\delta \to 0$, by (5.17), one has

$$ \tilde{E} \int_D (\overline{L_k(\rho)} - L_k(\rho))(t)\phi dx = \tilde{E} \int_0^t \int_D (\overline{L_k(\rho)}u - L_k(\rho)u) \cdot \nabla \phi dx dt \quad (5.19) $$

$$ + \lim_{\delta \to 0^+} \tilde{E} \int_0^t \int_D (T_k(\rho)\text{div}u - T_k(\rho_\delta)\text{div}u_\delta)\phi dx ds, \quad \forall \phi \in \mathcal{D}(D). $$

By approximating $\phi$ in the above inequality, similar to Lemma 4.5 taking $\phi = 1$ in (5.19), $u|_{\partial D} = 0$, Lemma 5.3 and Lemma 5.4 imply

$$ \tilde{E} \int_D (\overline{L_k(\rho)} - L_k(\rho))(t) dx $$

$$ = \tilde{E} \int_0^t \int_D T_k(\rho)\text{div}u dx dt - \lim_{\delta \to 0^+} \tilde{E} \int_0^T \int_D T_k(\rho_\delta)\text{div}u_\delta dx dt $$

$$ \leq \tilde{E} \int_0^T \int_D (T_k(\rho) - \overline{T_k(\rho)})\text{div}u dx dt $$

$$ \lesssim \tilde{E} \left( \|\text{div}u\|_{L^2(\{\rho \geq k\})} + \|T_k(\rho) - \overline{T_k(\rho)}\|_{L^2(\{\rho \leq k\})} \right). $$


On the one hand, since $T_k(\rho) \geq \bar{T}_k(\rho)$ and $T_k(\rho) \leq \rho$, by (5.20) in [38], we have

$$\bar{\mathbb{E}}\|T_k(\rho) - \bar{T}_k(\rho)\|_{L^1(\rho \leq k)} = \bar{\mathbb{E}}\|\rho - \bar{T}_k(\rho)\|_{L^1(\rho \leq k)} \leq \bar{\mathbb{E}}\|\rho - \bar{T}_k(\rho)\|_{L^1(\rho ; T, D)} \leq 2ck^{-\gamma - 1} \to 0,$$

as $k \to \infty$. By interpolation and Lemma 5.4, we have

$$\bar{\mathbb{E}}\|T_k(\rho) - \bar{T}_k(\rho)\|_{L^2(\rho \leq k)} \leq \left( \bar{\mathbb{E}}\|T_k(\rho) - \bar{T}_k(\rho)\|_{L^1(\rho \leq k)} \right)^{\gamma + 1} \to 0.$$

This and $u \in L^2(\Omega, L^2(0, T; H^1_0(D))$ imply that the right-hand side of (5.20) goes to zero as $k \to \infty$. Therefore,

$$\lim_{k \to \infty} \bar{\mathbb{E}} \int_D (\bar{T}_k(\rho) - L_k(\rho)) \, dx \leq 0, \quad t \in [0, T]. \quad (5.21)$$

On the other hand, since $L_k(\rho)$ is linear when $\rho$ is large enough, we have

$$\bar{\mathbb{E}}\|L_k(\rho) - \rho \ln \rho\|_{L^1(\rho ; L^1(D))} \leq \bar{\mathbb{E}} \int_0^T \int_{\rho \geq k} |L_k(\rho) - \rho \ln \rho| \, dx \, dt \leq \bar{\mathbb{E}} \int_0^T \int_{\rho \geq k} |\rho \ln \rho| \, dx \, dt \to 0,$$

as $k \to \infty$.

Similarly, for small enough $\varepsilon$, one has

$$\bar{\mathbb{E}}\|L_k(\rho_\delta) - \rho_\delta \ln \rho_\delta\|_{L^1(\rho ; L^1(D))} \leq \bar{\mathbb{E}} \int_0^T \int_{\rho_\delta \geq k} |L_k(\rho_\delta) - \rho_\delta \ln \rho_\delta| \, dx \, dt \leq \bar{\mathbb{E}} \int_0^T \int_{\rho_\delta \geq k} \frac{|L_k(\rho_\delta)| + |\rho_\delta \ln \rho_\delta|}{\rho_\delta^\gamma} \, dx \, dt \leq C(\varepsilon) \bar{\mathbb{E}} \int_0^T \int_{\rho_\delta \geq k} \rho_\delta^{\gamma - 1 - \varepsilon} \, dx \, dt \leq C(\varepsilon)k^{1 - \gamma + \varepsilon} \to 0 \quad \text{as} \quad k \to \infty,$$

uniformly in $\delta$. Let $k \to \infty$ in the above inequality, then the weak lower semi-continuity of norm implies that

$$\bar{\mathbb{E}}\|L_k(\rho) - \rho \ln \rho\|_{L^1(\rho ; L^1(D))} \to 0, \quad \text{uniformly in} \ \delta. \quad (5.23)$$

Combining (5.21) - (5.23), we have

$$\bar{\mathbb{E}} \int_D (\rho \ln \rho - \rho \ln \rho) (x, t) \, dx \leq 0,$$

this together with $\rho \ln \rho \leq \rho \ln \rho$ imply that

$$\rho \ln \rho(t) = \rho \ln \rho(t) \quad \text{for all} \ t \in [0, T] \ \text{a.e.}.$$

From this, by [13, Theorem 2.11], we can infer that

$$\rho_\delta \to \rho \quad \text{almost everywhere in} \ \bar{\Omega} \times [0, T] \times D.$$

It follows from Proposition 4.6 and [13, Proposition 2.1] that

$$\rho_\delta \to \rho \quad \text{weakly in} \ L^1(\bar{\Omega} \times (0, T) \times D),$$
By [13] Theorem 2.10, for any \( \eta > 0 \) and all \( \delta > 0 \), there exists \( \sigma > 0 \) such that 
\[
\int_F \rho_\delta(t,x) dxdtd\tilde{P} < \eta \text{ for any measurable } F \subset \tilde{\Omega} \times (0,T) \times D
\]
with \( \text{meas}\{F\} < \sigma \). On the other hand, by Egorov's Theorem, there exists a measurable set \( F_\sigma \subset \tilde{\Omega} \times (0,T) \times D \) such that \( \text{meas}\{F_\sigma\} < \sigma \) and \( \rho_\delta \to \rho \) uniformly in \( \tilde{\Omega} \times (0,T) \times D - F_\sigma \). Note that
\[
\tilde{E} \int_0^T \int_D |\rho_\delta - \rho| dxdt \leq \int_F \rho_\delta - \rho dxdtd\tilde{P} + \int_{\tilde{\Omega}\times(0,T)\times D - F_\sigma} |\rho_\delta - \rho| dxdtd\tilde{P} \quad (5.24)
\]
\[
\leq 2\eta + T|D| \sup_{\tilde{\Omega}\times(0,T)\times D - F_\sigma} |\rho_\delta - \rho|.
\]
Then \((5.24)\) tends to 0 as \( \delta \to 0 \) and \( \eta \to 0 \). This implies the strong convergence of the sequence \( \rho_\delta \) in \( L^1(\tilde{\Omega} \times (0,T) \times D) \), that is,
\[
\lim_{\delta \to 0} \rho_\delta = \rho^\gamma. \quad (5.25)
\]

Similar to the proof of \((5.27)\) and \((5.40)\) in Section 3.1 by using \((5.15)\), \((5.17)\), \((5.19)\), \((5.20)\)
and Hölder’s inequality, we have
\[
\langle f_k(\rho_\delta, \rho_\delta u, x), \phi \rangle \to \langle f_k(\rho, \rho u, x), \phi \rangle \text{ in } L^1([0,T]) \tilde{P} - \text{a.s.},
\]
\[
\langle g_k(B_\delta, x), \phi \rangle \to \langle g_k(B, x), \phi \rangle \text{ in } L^1([0,T]) \tilde{P} - \text{a.s.}, \quad (5.26)
\]
and
\[
\sum_{k \geq 1} |\langle f_k(\rho, \rho u, x), \phi \rangle|^2 \text{ in } L^1([0,T]) \tilde{P} - \text{a.s.},
\]
\[
\sum_{k \geq 1} |\langle g(B, x), \phi \rangle|^2 \text{ in } L^1([0,T]) \tilde{P} - \text{a.s.}. \quad (5.27)
\]
As in Section 3 by using \((5.6)\)-\((5.11)\), \((5.25)-(5.27)\), we deduce that \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \beta_k, \rho, u, B)\)
satisfies the following equations:
\[
\begin{cases}
  d\rho + \text{div}(\rho u) dt = 0, \\
  (\rho u)_t + \text{div}(\rho u \otimes u) + a \nabla \rho^\gamma - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u = (\nabla \times B) \times B + \sum_{k \geq 1} f_k(\rho, \rho u, x) \hat{\beta}_k^1, \\
  B_t - (\nabla \times u) \times B - \nu \Delta B = \sum_{k \geq 1} g_k(B, x) \hat{\beta}_k^2,
\end{cases}
\]
in \( D'(\Omega \times (0,T) \times D) \tilde{P}\)-a.s.. Here we formally denote \( \dot{W} := dW/dt \). The proof of Theorem 1.2
is thus completed.

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