Commutativity of quantization and reduction for quiver representations

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Abstract
Given a finite quiver, its double may be viewed as its non-commutative “cotangent” space, and hence is a non-commutative symplectic space. Crawley-Boevey, Etingof and Ginzburg constructed the non-commutative reduction of this space while Schedler constructed its quantization. We show that the non-commutative quantization and reduction commute with each other. Via the quantum and classical trace maps, such a commutativity induces the commutativity of the quantization and reduction on the space of quiver representations.

Keywords  Necklace Lie algebra · Quiver variety · Quantization · Reduction · Differential operator

Mathematics Subject Classification  16G20 · 53D55 · 81R60

Contents

1 Introduction ............................................. 3525
2 Non-commutative bi-symplectic spaces ............................... 3528
3 Non-commutative reduction and quantization ............................ 3531
4 Quantization of quiver representations ................................ 3540
5 Trace maps and proof of the main theorem ............................. 3545
References ................................................ 3553

1 Introduction

In 2000 Kontsevich and Rosenberg proposed in [15] a heuristic principle in the study of non-commutative geometry. It says that a non-commutative geometric structure on a non-commutative space (in this article, we mean an associative algebra), if it exists, should induce its classical counterpart on its representation schemes. This principle has achieved great success in the study of non-commutative Poisson geometry [6, 21] and non-commutative
symplectic geometry [3, 7, 10]. The purpose of this paper is to study the “quantization commutes with reduction” problem for quiver varieties, with the guidance of the Kontsevich–Rosenberg principle.

In the 1980s, Guillemin and Sternberg conjectured in [11] that for a symplectic manifold, its geometric quantization commutes with its reduction and proved this conjecture for the case of compact Kähler manifolds. In the 1990s, Fedosov proved in [8] that the deformation quantization commutes with reduction for symplectic manifolds. At about the same time, Kontsevich, inspired by the mirror symmetry from physics, initiated the study of “non-commutative” symplectic geometry (see [13, 14]). The above problem of “deformation quantization commutes with reduction” makes sense for non-commutative symplectic manifolds, too, as mathematicians have made much progress in this direction. In what follows, we focus on the case of quiver algebras, which already have ample non-commutative symplectic/Poisson structures.

Let \( Q \) be a finite quiver. Let \( \overline{Q} \) be the double of \( Q \); that is, the quiver obtained from \( Q \) by adding a new edge for each edge of \( Q \) but with the direction reversed. According to Kontsevich [13] (see also [7, 10]), \( \overline{Q} \) is the “non-commutative” cotangent space of \( Q \), and hence is a “non-commutative” symplectic space. Denote by \( \text{Rep}(\overline{Q}, d) \) and \( \mathcal{M}_d(Q) \) the space of \( d \)-dimensional quiver representations of \( Q \) and the associated quiver variety. The former is a symplectic space and the latter is a Hamiltonian reduction of the former. We have the following results:

(1) Holland in [12] showed that for \( \text{Rep}(\overline{Q}, d) \) and \( \mathcal{M}_d(Q) \), their quantizations commute with the reduction procedure in the sense of Fedosov.

(2) Ginzburg in [10], and simultaneously, Bocklandt–Le Bruyn in [3], showed that there is a Lie algebra structure on \((\mathbb{K}\overline{Q})_2\), called the necklace Lie algebra, such that the canonical trace map

\[
\text{Tr} : (\mathbb{K}\overline{Q})_2 \to \mathbb{K}[\text{Rep}(\overline{Q}, d)], \quad a \mapsto \{\rho \mapsto \text{trace}(\rho(a))\}
\]

is a map of Lie algebras, where the Lie bracket on the latter is the Poisson bracket. Here \( \mathbb{K}\overline{Q} \) is the path algebra of \( \overline{Q} \) over a base field \( \mathbb{K} \) of characteristic zero, and \((\mathbb{K}\overline{Q})_2\) is its commutator quotient space.

(3) Later, Schedler constructed in [19] a quantization of \((\mathbb{K}\overline{Q})_2\); such a quantization, denoted by \( \mathbb{N}(Q)_\hbar \), under the quantum trace map, is mapped to the differential operators on \( \text{Rep}(\overline{Q}, d) \), and hence gives a quantization of the latter.

(4) Crawley-Boevey, Etingof and Ginzburg showed in [7] that \( \mathbb{K}\overline{Q} \), in fact, has a bi-symplectic structure, which naturally induces the symplectic structure on \( \text{Rep}(\overline{Q}, d) \), and thus makes the “non-commutative symplectic structure” more precise. In loc. cit, they also introduced the procedure of non-commutative Hamiltonian reduction for bi-symplectic spaces; for the quiver algebra \( \mathbb{K}\overline{Q} \), the non-commutative reduction is nothing but a preprojective algebra, which is denoted by \( \Pi Q \).

Based on these results, we have the following two natural questions:

**Question 1.1**

1. Do we have the non-commutative version of “quantization commutes with reduction” for bi-symplectic spaces?
2. Does the non-commutative “quantization commutes with reduction” fit the Kontsevich–Rosenberg principle, that is, does it induce the classical one on the corresponding representation spaces?

This paper tries to answer to these two questions for quiver algebras. Our main theorem is:
Theorem 1.2 Suppose \( Q \) is a finite quiver, \( d \) is a dimension vector such that moment map \( \mu \) is a flat morphism. Then:

1. (Non-commutative quantization commutes with reduction) There exists a non-commutative reduction \( R_q(N(Q)_\hbar, \hat{w}) \) of \( N(Q)_\hbar \), which quantizes preprojective algebra \( \Pi Q \). Moreover, the following diagram

\[
\begin{array}{c}
N(Q)_\hbar \\
\downarrow \uparrow \\
(KQ)^\natural
\end{array} 
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array} 
\begin{array}{c}
R_q(N(Q)_\hbar, \hat{w}) \\
\downarrow \uparrow \\
(\Pi Q)^\natural
\end{array}
\] (1)

commutes.

2. (Existence of quantum trace) There is a quantum trace map \( \text{Tr}^q \), which maps \( R_q(N(Q)_\hbar, \hat{w}) \) to the quantum Hamiltonian reduction \( (\mathcal{D}_h(\text{Rep}(Q, d)))^{\text{GLd}} \) such that the following diagram

\[
\begin{array}{c}
N(Q)_\hbar \\
\downarrow \uparrow \\
R_q(N(Q)_\hbar, \hat{w})
\end{array} 
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array} 
\begin{array}{c}
\mathcal{D}_h(\text{Rep}(Q, d)) \\
\downarrow \uparrow \\
\mathcal{D}_h(\text{Rep}(Q, d))^{\text{GLd}}
\end{array}
\] (2)

commutes.

\( \mathcal{D}_h(-) \) in the theorem are the homogeneous differential operators on the corresponding spaces, which are \( \mathbb{K}[\hbar] \)-algebras generated by functions and vector fields. See Definition 4.10 for explicit construction.

In the above theorem, the curved arrow means quantization, the dotted arrow means the reduction procedure, and the “commutativity” of the diagram is understood as in Fedosov [8, Theorem 3]. Thus combining the above theorem with the above cited works, we, in fact, get the following commutative diagram

\[
\begin{array}{c}
N(Q)_\hbar \\
\downarrow \uparrow \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array} 
\begin{array}{c}
\text{Tr}^q \\
\text{Tr}^q \\
\text{Tr}
\end{array} 
\begin{array}{c}
\mathcal{D}_h(\text{Rep}(Q, d)) \\
\downarrow \uparrow \\
\mathcal{D}_h(\text{Rep}(Q, d))^{\text{GLd}}
\end{array}
\] (3)

This diagram exactly says that “quantization commutes with reduction” fits the Kontsevich–Rosenberg principle and hence gives an affirmative answer to Question 1.1.

The rest of the paper is devoted to the proof of the above theorem. It is organized as follows. In Sect. 2 we recall the bi-symplectic structure introduced by Crawley-Boevey, Etingof and Ginzburg; in Sect. 3 we show that the non-commutative quantization commutes with the non-commutative reduction for quiver algebras; in Sect. 4 we collect the results on the...
commutativity of quantization and reduction for quiver representations, which is mainly due to Holland; in Sect. 5 we show that under the quantum and the classical trace maps, the non-commutative version of “quantization commutes with reduction” induces the usual one on quiver representation spaces, and hence proves Theorem 1.2.

## 2 Non-commutative bi-symplectic spaces

Crawley-Boevey, Etingof and Ginzburg introduced in [7] a version of non-commutative symplectic spaces, which they called bi-symplectic spaces, and studied their non-commutative Hamiltonian reduction. They also showed, as an important example, that the double of a quiver is bi-symplectic.

In this section, we briefly go over their results.

### 2.1 Bi-symplectic spaces

In this paper, $K$ is an algebraically closed field of characteristic zero, $R$ is the commutative semisimple $K$-algebra $R = \bigoplus_{i,j} K e_{ij}$ with $e_i e_j = \delta_{ij} e_i$. Let $A$ be an $R$-algebra and $m : A \otimes_R A \to A$ be the multiplication of $A$.

**Definition 2.1** Let $A$ be an $R$-algebra. The set of non-commutative 1-forms of $A$ is $\Omega^1_R A := \ker m$.

Equivalently, $\Omega^1_R A$ is the $A$-bimodule generated by $da$ for $a \in A$, subject to the following relation:

$$d(ab) = (da)b + a(db), \text{ for any } a, b \in A.$$ 

Here, $d$ is considered as an $R$-linear map from $A$ to $\Omega^1_R A$.

**Definition 2.2** An $R$-algebra $A$ is called smooth if it is finitely generated as an $R$-algebra and $\Omega^1_R A$ is projective as an $A$-bimodule.

From now on, all algebras are assumed to be smooth.

**Definition 2.3** Let $A$ be an $R$-algebra. The set of non-commutative differential forms of $A$, denoted by $\Omega^\bullet_R A$, is the tensor algebra $T_A(\Omega^1_R A) = \bigoplus_{n \geq 0} T^n_A(\Omega^1_R A)$ equipped with differential $d : \Omega^{\bullet-1}_R A \to \Omega^\bullet_R A$, which is the extension of $d : A \to \Omega^1_R A$ by derivation and by setting $d^2 = 0$.

In practice, we write an $n$-form in the form $a_0 da_1 \cdots da_n$ for $a_i \in A$.

**Definition 2.4** Let $A$ be an $R$-algebra, and let $\Omega^\bullet_R A$ be the non-commutative differential forms of $A$. Then the Karoubi–de Rham complex of $A$ over $R$ is the complex

$$\text{DR}^\bullet_R A := \Omega^\bullet_R A / [\Omega^\bullet_R A, \Omega^\bullet_R A]_s$$

with the differential induced from $\Omega^\bullet_R A$.

Here $[\cdot, \cdot]_s$ means taking the super commutators: for $\alpha \in \Omega^i_R A, \beta \in \Omega^j_R A [\alpha, \beta]_s = \alpha \beta - (-1)^{ij} \beta \alpha$. By definition, we have $\text{DR}^0_R A \cong A$, $\text{DR}^1_R A \cong \Omega^1_R A / [A, \Omega^1_R A]_s$.

Let $A$ be an $R$-algebra; there exist two $A$-bimodule structures on $A \otimes A$: one is the outer and the other is the inner, given by $a(x \otimes y)b := (ax) \otimes (yb)$ and $a*(x \otimes y)*b := (xb) \otimes (ay)$, respectively, for any $a, b, x, y \in A$. 

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Definition 2.5  Let $A$ be an $R$-algebra. The set of double derivations on $A$, denoted by \( \mathbb{D}er_R A \), is the set of $R$-derivations from $A$ to the outer $A$-bimodule $A \otimes A$.

Due to the inner $A$-bimodule structure on $A \otimes A$, $\mathbb{D}er_R A$ is also an $A$-bimodule. Next, we recall the non-commutative version of contractions and Lie derivatives.

Definition 2.6  Let $A$ be an $R$-algebra and $\Theta \in \mathbb{D}er_R A$. The contraction of the non-commutative forms of $A$ with $\Theta$ is the $A$-linear map $i_\Theta : \Omega^*_R A \to \Omega^*_R A \otimes \Omega^*_R A$ given by

$$d\alpha_1 \cdots d\alpha_n \mapsto \sum_{k=1}^n (-1)^{k-1} (d\alpha_1 \cdots d\alpha_{k-1} \Theta'(\alpha_k)) \otimes (\Theta''(\alpha_k)d\alpha_{k+1} \cdots d\alpha_n).$$

Here for a 1-form $d\alpha \in \Omega^1_R A$, $i_\Theta(d\alpha) = \Theta(\alpha) = \Theta'(\alpha) \otimes \Theta''(\alpha)$.

Definition 2.7  Let $A$ be an $R$-algebra and let $\Theta$ be a double derivation on $A$. For an $n$-form $a_0 d\alpha_1 \cdots d\alpha_n \in \Omega^n_R A$, the Lie derivative of $a_0 d\alpha_1 \cdots d\alpha_n$ with respect to $\Theta$ is given by

$$L_\Theta(a_0 d\alpha_1 \cdots d\alpha_n) = \Theta'(a_0) \otimes \Theta''(a_0)d\alpha_1 \cdots d\alpha_n$$

$$+ \sum_{k=1}^n (a_0 d\alpha_1 \cdots d\alpha_{k-1} d\Theta'(a_k)) \otimes (\Theta''(a_k)d\alpha_{k+1} \cdots d\alpha_n)$$

$$+ (a_0 d\alpha_1 \cdots d\alpha_{k-1} \Theta'(a_k)) \otimes (d\Theta''(a_k)d\alpha_{k+1} \cdots d\alpha_n).$$

Now for $\alpha \otimes \beta \in \Omega^k_R A \otimes \Omega^l_R A$, we write

$$(\alpha \otimes \beta)^\circ := (-1)^{kl} \beta \alpha \in \Omega^{k+l}_R A.$$

Definition 2.8  Let $A$ be an $R$-algebra and let $\Theta$ be a double derivation on $A$. The reduced contraction and the reduced Lie derivative with respect to $\Theta$ are given by

$$\iota_\Theta : \Omega^*_R A \to \Omega^{*-1}_R A, \quad \alpha \mapsto \iota_\Theta \alpha = (i_\Theta \alpha)^\circ$$

and

$$\mathcal{L}_\Theta : \Omega^*_R A \to \Omega^*_R A, \quad \alpha \mapsto \mathcal{L}_\Theta \alpha = (L_\Theta \alpha)^\circ.$$ 

Proposition 2.9  [7, Lemmas 2.8.6 and 2.8.8] (1) The Cartan formulas hold:

$$i_\Theta \circ d - d \circ i_\Theta = L_\Theta \quad \text{and} \quad d \circ \iota_\Theta - \iota_\Theta \circ d = \mathcal{L}_\Theta, \quad \text{for any} \ \Theta \in \mathbb{D}er_R A.$$

(2) Suppose $A$ is an $R$-algebra and $\Theta \in \mathbb{D}er_R A$. Then

(i) for any $\omega \in \Omega^n_R A$, the map $\omega \mapsto \iota_\Theta \omega$ descends to a well-defined map

$$\iota_\Theta : \mathcal{D}er^n_R A \to \Omega^{n-1}_R A;$$

(ii) for a fixed $\omega \in \mathcal{D}er^n_R A$, there exists a homomorphism of $A$-bimodules:

$$\iota_\omega : \mathbb{D}er_R A \to \Omega^{n-1}_R A, \ \Phi \mapsto \iota_{\Phi} \omega;$$
(iii) for any $\omega \in \Omega^n_R A$, the following diagram commutes:

$\begin{array}{ccc}
(\mathbb{D}er_R A)_{\sharp} & \xrightarrow{m_{\cdot}} & \text{Der}_R A \\
(i(\omega))_{\sharp} \downarrow & & \downarrow i: \theta \mapsto \rightarrow i\theta \omega \\
(\Omega_R^{n-1} A)_{\sharp} & \xrightarrow{\cdot} & \text{DR}_R^{n-1} A.
\end{array}$

Recall that $R = \oplus_{i \in Q_0} \mathbb{K} e_i$, those idempotents give an important double derivation $\Delta_1: A \to A \otimes A$, $a \mapsto \sum i e_i \otimes e_i - e_i \otimes e_i a$.

Now, we recall a version of non-commutative symplectic structure introduced in [7].

**Definition 2.10** Let $A$ be an $R$-algebra. A 2-form $\omega \in \text{DR}_R^2 A$ is called bi-symplectic if it satisfies

1. $\omega$ is closed, that is $d \omega = 0$;
2. $i\omega: \mathbb{D}er_R A \to \Omega_R^1 A$ is a bijection of $A$-bimodules.

### 2.2 Representation spaces and trace maps

Let $R$ be the semisimple $\mathbb{K}$-algebra $\oplus_{i \in I} \mathbb{K} e_i$, where $I = \{1, 2, 3, \ldots, n\}$ is a finite set and let $A$ be an $R$-algebra. For a $\mathbb{K}$-vector space $V = \oplus_{i \in I} V_i$, where $V_i$ is a subspace of $V$. $V$ is canonically a left $R$-module: for any $k \in \mathbb{K}$ and $e_i$, $ke_i \in R$ acts on $V$ as the scalar multiplication of $k$ on subspace $V_i$. Thus $\text{End}_R(V)$ is an $R$-algebra.

**Definition 2.11** Let $A$ be an $R$-algebra, and let $V = \oplus_{i \in I} V_i$ be a $\mathbb{K}$-vector space. The representation space $\text{Rep}(A, V)$ of $A$ on $V$ is

$$\text{Rep}(A, V) := \text{Hom}_{\text{Alg}_R}(A, \text{End}_R(V)),$$

where $\text{Alg}_R$ is the category of $R$-algebras.

$\text{Rep}(A, V)$ is equipped with a $\text{GL}(V)$-action by conjugation, which makes $\text{Rep}(A, V)$ into a $\text{GL}(V)$-variety.

In practice, we usually consider $V = \oplus_{i \in I} \mathbb{K}^d_i$ for a dimension vector $d \in \mathbb{N}^I$; in this case, we denote $\text{Rep}(A, V)$ by $\text{Rep}(A, d)$. As an affine variety, the coordinate ring of $\text{Rep}(A, d)$ is characterized by the following:

**Proposition 2.12** [21, Section 7.1] For any $A \in \text{Alg}_R$ and $d \in \mathbb{N}^I$, the coordinate ring $A_d$ of $\text{Rep}(A, d)$ is generated by $\{(a)_{ij} \mid 1 \leq i, j \leq |d| := \sum_i d_i\}$, subject to following relations:

1. $(e_k)_{ij} = \delta_{\phi(i),k}\delta_{ij}$;
2. $(ka)_{ij} = k(a)_{ij}$, $(a + b)_{ij} = (a)_{ij} + (b)_{ij}$;
3. $(ab)_{ij} = \sum_n (a)_{in}(b)_{nj}$.

Here $\phi: \{1, 2, \ldots, |d|\} \to I$ is defined by the property

$\phi(i) = k$ if and only if $d_1 + d_2 + \cdots + d_{k-1} + 1 \leq i \leq d_1 + \cdots + d_k$. 

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Similarly, for an $n$-form $\omega = a_0 da_1 \cdots da_n \in \Omega^n_R A$,
\[
(\omega)_{ij} = \sum_{k_1 \cdots k_{n-1}} (a_1)_{i,k_1} d(a_1)_{k_1,k_2} \cdots d(a_n)_{k_n-1,j}.
\]

This formula can be rewritten in terms of products of matrices:
\[
(\omega) = (a_0) d(a_1) \cdots d(a_n)
\]
where $d(a_i)$ is the matrix with $(u, v)$-entity being $d(a_i)_{uv}$.

For $\Theta \in \mathbb{D}er_R(A)$, the induced vector field, unlike the case of differential forms, is given in the following form. For $1 \leq i, j, u, v \leq |d|$ and $a \in A$,
\[
\Theta_{ij}(a_{uv}) := \Theta'(a)_{u,j} \Theta(a)''_{i,v}.
\]

**Definition 2.13** Suppose $A$ is an $R$-algebra, and $d \in \mathbb{N}^I$ is a dimension vector. The (classical) trace map $\text{Tr} : A \to A_d$ is defined to be
\[
A \to A_d, \ a \mapsto \{ \rho \mapsto \text{tr}(\rho(a)) \}.
\]

Here $\text{tr}(-)$ means taking the trace of the matrices. Similarly, one can define the trace map $\text{Tr}$ for differential forms.

**Proposition 2.14** [7, Theorem 6.4.3] Given an $R$-algebra $A$ with bi-symplectic form $\omega$, $\text{Rep}(A, V)$ is a symplectic manifold with symplectic form $\text{Tr}(\omega)$.

### 2.3 Quivers and quiver representations

Now let $Q$ be a finite quiver, and let $\mathbb{K}Q$ be the path algebra of $Q$. Then a representation of $\mathbb{K}Q$ has an alternative description which is given as follows.

Let $Q_0$ and $Q_1$ the sets of vertices and arrows respectively. For $a \in Q_1$, denote by $s(a)$ and $t(a)$ the source and the target of $a$ respectively. Then a representation of $Q$ consists of the following collection of data:

- to each vertex $i$, we assign a $\mathbb{K}$-vector space $V_i$ with $\dim V_i = d_i$;
- to each arrow $a$, we assign a linear operator $f_a : V_{s(a)} \to V_{t(a)}$.

In what follows, we also write $\text{Rep}(\mathbb{K}Q, d)$ as $\text{Rep}(Q, d)$.

Let $\overline{Q}$ be the double of $Q$, which is obtained from $Q$ by adding opposite arrow $a^*$ for each $a \in Q_1$. Then we have:

**Proposition 2.15** [7, Proposition 8.1.1] Let $\mathbb{K}\overline{Q}$ be the path algebra of the double of $Q$. Then $\mathbb{K}\overline{Q}$ is smooth and the $2$-form $\omega = \sum_{a \in Q} dada^*$ is bi-symplectic. In particular, $\text{Rep}(\overline{Q}, d)$ endows an induced symplectic structure.

### 3 Non-commutative reduction and quantization

In this section, we study the non-commutative Hamiltonian reduction and quantization for the doubled quiver, viewed as a bi-symplectic space. The main result is Theorem 3.21. It states that its non-commutative quantization commutes with its non-commutative reduction for such a bi-symplectic space.
3.1 The non-commutative Hamilton operator

Let \( g \) be a Lie algebra and \((C^\bullet, d)\) be a cochain complex of \( \mathbb{K} \)-vector spaces; let \( \text{Com}_\mathbb{K} \) be the category of cochain complexes of \( \mathbb{K} \)-vector spaces.

A \( g \)-equivariant structure on \((C^\bullet, d)\) is a pair of linear maps

\[
L : g \to \text{Hom}_{\text{Com}_\mathbb{K}}(C^\bullet, C^\bullet), \quad x \mapsto L_x
\]

and

\[
i : g \to \text{Hom}_{\text{Com}_\mathbb{K}}(C^\bullet, C^{\bullet-1}), \quad x \mapsto i_x
\]

which satisfies that, for any \( x, y \in g \),

\[
[L_x, L_y] = L_{[x, y]}, \quad L_x = d \circ i_x + i_x \circ d, \quad i_x \circ i_y + i_y \circ i_x = 0, \quad [L_x, i_y] = i_{[x, y]}.
\]

**Definition 3.1** Let \((C^\bullet, d)\) be a \( g \)-equivariant complex. A linear map \( H : C^1 \to g \) is called a Hamilton operator if it satisfies the following conditions: for any \( \alpha, \beta \in C^1 \),

1. \( i_H(\alpha) \beta + i_H(\beta) \alpha = 0 \);
2. \( [H(\alpha), H(\beta)] = H(i_H(\alpha) \circ d \beta - i_H(\beta) \circ d \alpha + d \circ i_H(\alpha) \beta) \).

**Proposition 3.2** [7, Proposition 4.3.5] Let \((C^\bullet, d)\) be a \( g \)-equivariant complex and let \( H : C^1 \to g \) be a Hamilton operator. Then there exists a Lie bracket on \( C^0 \) induced from \( H \) by the formula

\[
\{x, y\} := i_H(dx) dy, \quad \text{for any } x, y \in C^1.
\]

From Definition 2.10 (2), we have a bijection

\[
(\iota_\omega) : (\Omega^1_R A)_{\#} \to (\Omega^1_R A)_{\#} = \text{DR}_R^1 A.
\]

On the other hand, the multiplication map \( m : A \otimes A \to A \) induces a map

\[
m_* : \text{Der}_R A \to \text{Der}_R A, \quad \Theta \mapsto m \circ \Theta.
\]

Hence, a bi-symplectic structure gives a Hamilton operator; that is, we have:

**Proposition 3.3** [7, Theorem 7.2.3] Let \( A \) be an \( R \)-algebra with a bi-symplectic form \( \omega \). Then we have the following.

1. There exists a Hamilton operator

\[
H_\omega : \text{DR}_R^1 A \to \text{Der}_R A
\]

which is the composition

\[
\text{DR}_R^1 A \xrightarrow{(\iota_\omega)^{-1}_{\#}} (\text{Der}_R A)_{\#} \xrightarrow{(m_*)_{\#}} \text{Der}_R A.
\]

2. \( H_\omega \) induces a Lie bracket on \( A_{\#} \), explicitly,

\[
\{a, b\} := i_{H_\omega(da)}(db), \quad \text{for any } a, b \in A_{\#}.
\]

The Lie bracket in Proposition 3.3 (2) was first found by Kontsevich, and coincides with the \( H_0 \)-Poisson structure introduced by Crawley-Boevey in [6], whose definition is the following:
**Definition 3.4** [6, Definition 1.1] Let $A$ be an $R$-algebra. A linear map

$$p : A_\natural \to \frac{\text{Der}_R A}{\text{Inn}_R A}$$

is called an $H_0$-Poisson structure if for any $a, b \in A$ with $\bar{a}, \bar{b} \in A_\natural$ respectively,

$$\{\bar{a}, \bar{b}\}_p := p(a)(b)$$

is a Lie bracket on $A_\natural$.

**3.2 Non-commutative Hamiltonian reduction**

Suppose $R$ is the semisimple $\mathbb{K}$-algebra $\bigoplus_{i=1}^n \mathbb{K}e_i$. Suppose $A$ is a bi-symplectic $R$-algebra with bi-symplectic form $\omega$. For any $a \in A$, by Definition 2.10, there exists $\mathcal{H}_a \in \text{Der}_R A$ such that

$$\iota_{\mathcal{H}_a} \omega = da \in \Omega^1_R A.$$

**Definition 3.5** ([7, Section 4.1] and [21, Definition 2.6.4]) Let $A$ be as above. A non-commutative moment map is an element $w \in \bigoplus_i e_i Ae_i$ such that

$$\mathcal{H}_w(a) = \Delta(a) \in A \otimes A,$$

for any $a \in A$.

**Definition 3.6** Suppose $(A, \omega)$ is a bi-symplectic space over $R$ and $w$ is a non-commutative moment map. Then the non-commutative Hamiltonian reduction of $A$ with respect to $w$ is

$$\mathcal{R}(A, w) := \frac{A}{AwA},$$

where $AwA$ is the two-sided ideal generated by $w$.

**Proposition 3.7** [7, Proposition 4.4.3 and Theorem 7.2.3] Given an $R$-algebra $A$ with bi-symplectic form $\omega$ and non-commutative moment map $w$, we have:

1. For any $f \in A_\natural$, $\theta_f := H_\omega(df)$ preserves $AwA$, and thus descends to a well-defined derivation

$$\overline{\theta_f} \in \text{Der}_R(\mathcal{R}(A, w)).$$

2. The Lie bracket on $A_\natural$ descends to a well-defined Lie bracket on $(\mathcal{R}(A, w))_\natural$.

The following proposition says the non-commutative Hamiltonian reduction introduced above fits the Kontsevich–Rosenberg principle, which justifies its definition.

**Proposition 3.8** [7, Theorem 6.4.3]

Given an $R$-algebra $A$ with bi-symplectic form $\omega$ and non-commutative moment map $w$, and a $\mathbb{K}$-vector space $V$ of finite dimension, we have:

1. Let $\mu := (w) \in \mathbb{K}[\text{Rep}(A, V)] \otimes \text{End}_\mathbb{K}(V)$, then $\mu$ is a moment map for symplectic space $\text{Rep}(A, V)$;
2. $\text{Rep}(\mathcal{R}(A, w), V)$ is a subscheme of $\text{Rep}(A, V)$ and $\text{Rep}(\mathcal{R}(A, w), V) = \mu^{-1}(0)$.  

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Proposition 3.9 [7,Proposition 8.1.1] Let $\mathbb{K}Q$ be the path algebra of doubled quiver $Q$. Then the following element

$$w := \sum_{a \in Q} (aa^* - a^*a)$$

is a non-commutative moment map.

It is also direct to see that, if $w$ is a non-commutative moment map, then for any $\lambda \in R$, so is $w - \lambda$.

Corollary 3.10 Suppose $Q$ is a finite quiver. Then for any $\lambda \in R$,

1. the following algebra, $\Pi^\lambda Q := \frac{\mathbb{K}Q}{(w - \lambda)}$, called the deformed preprojective algebra, is the non-commutative Hamiltonian reduction of $\mathbb{K}Q$ at $w - \lambda$. In particular, $\Pi^\lambda Q$ is equipped with an $H_0$-Poisson structure;
2. suppose $V$ is a representation of $Q$ with dimension vector $d \in \mathbb{N}^Q_0$, and $\sum_{i \in Q_0} d_i \lambda_i = 0$. Then $\text{Rep}(\Pi^\lambda Q, V) = \mu^{-1}(\lambda)$.

Proof The proof is given in [21, Propositions 6.8.1 and 7.11.1]; see also [7, Theorem 6.4.3].

From now on, the preprojective algebra $\Pi^0 Q$ will be denoted by $\Pi Q$, and the quiver variety $\mathcal{M}_d(Q)$ is the categorical quotient $\mu^{-1}(\lambda)/\text{GL}_d$.

In summary, suppose $R$ is a commutative semisimple $\mathbb{K}$-algebra $\bigoplus_{i=1}^n \mathbb{K}e_i$, $A$ is an $R$-algebra equipped with a bi-symplectic form $\omega \in \text{DR}^2_R A$ and non-commutative moment map $w$. One constructs the non-commutative reduction $R(A, w)$ of $A$ at $w$. This procedure is denoted by

$$A \rightsquigarrow R(A, w).$$

### 3.3 Quantization of the necklace Lie algebra

In this subsection, we go over the quantization of the non-commutative Poisson structure on $\mathbb{K}Q$, which is due to Schedler [19]; the DG algebras case has recently been studied by Chen and Eshmatov in [4]. In this article, by quantization of a Lie algebra $g$, we always mean the quantization of the Poisson algebra $\text{Sym}(g)$, the symmetric algebra of $g$.

By Definition 3.6 and Proposition 3.7, there exists a Lie bracket on

$$([\mathbb{K}Q]_\mathbb{K}) = \frac{\mathbb{K}Q}{[\mathbb{K}Q, \mathbb{K}Q]}.$$

More precisely, it is given as follows: for any $a \in Q$, set $\{a, a^*\} = 1$ and $\{a^*, a\} = -1$; set also $\{f, g\} = 0$ for any $f, g \in \mathbb{Q}$ with $f \neq g^*$. For cyclic paths, $a_1 a_2 \cdots a_k, b_1 b_2 \cdots b_l \in ([\mathbb{K}Q]_\mathbb{K})$ with $a_i, b_j \in \mathbb{Q}$, we have

$$\{a_1 a_2 \cdots a_k, b_1 b_2 \cdots b_l\} = \sum_{1 \leq i \leq k, 1 \leq j \leq l} \{a_i, b_j\}(a_{i+1} a_{i+2} \cdots a_k a_1 \cdots a_{i-1} b_{j+1} \cdots b_l b_1 \cdots b_{j-1});$$

we also set $\{b, x\} = \{x, b\} = 0$ for any $b \in R$. 

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Definition 3.11 Suppose \( Q \) is a finite quiver, \((\mathbb{K}Q)_{\mathbb{Z}}\) equipped with above bracket is called the necklace Lie algebra of \( Q \).

We next recall Schedler’s construction of the quantization of necklace Lie algebras. Let us first recall some notations.

**Notation 3.12** Suppose \( Q \) is a finite quiver and \( R \) is the semisimple algebra \( \oplus_{i \in Q_0} \mathbb{K} e_i \). We recall the following notations introduced by Schedler in [19].

1. Let \( AH := Q \times \mathbb{N} \), which is called the space of arrows with heights.
2. Let \( E_{Q,h}^- \) be the \( \mathbb{K} \)-vector space spanned by \( AH \).
3. Let \( LH := (T_R E_{Q,h})^\mathbb{Z} \), which is called the generalized cyclic path algebra with heights.
4. Let \( SLH[h] := SLH \otimes \mathbb{K}[h] \), which is the symmetric algebra generated by \( LH \).

In what follows all symmetric products, not just on \( SLH[h] \), are denoted by &.

Consider the \( \mathbb{K}[h] \)-submodule \( SLH' \) spanned by elements of the form
\[
(a_{1,1}, h_{1,1}) \cdots (a_{1,l_1}, h_{1,l_1}) \& (a_{2,1}, h_{2,1}) \cdots (a_{2,l_2}, h_{2,l_2}) \\
& \cdots \& (a_{k,1}, h_{k,1}) \cdots (a_{k,l_k}, h_{k,l_k}) \& v_1 \& v_2 \& \cdots \& v_m.
\]
(4)

where the \( h_{i,j} \) are all distinct, \( a_{i,j} \in Q \) and \( v_i \in Q_0 \). Let \( \tilde{A} \) be the quotient of \( SLH' \) where two elements in \( SLH' \) are identified if and only if the order of heights in each element are preserved when we exchange each height in corresponding places.

Next, consider the \( \mathbb{K}[h] \)-submodule \( \tilde{B} \) of \( \tilde{A} \) generated by the following forms:

- \( X - X'_{i,j,i',j'} - X''_{i,j,i',j'} \), where \( i \neq i', h_{i,j} < h_{i',j'}, \tilde{\mathbb{P}}(i'', j'') \) with \( h_{i,j} < h_{i'',j''} < h_{i',j'} \);
- \( X - X'_{i,j,i',j'} - hX''_{i,j,i',j'} \), where \( h_{i,j} < h_{i',j'}, \tilde{\mathbb{P}}(i'', j'') \) with \( h_{i,j} < h_{i'',j''} < h_{i',j'} \).

In the above, \( X' \) and \( X'' \) are defined as follows:

\( X'_{i,j,i',j'} \) is same as \( X \) but with heights \( h_{i,j} \) and \( h_{i',j'} \) interchanged; \( X''_{i,j,i',j'} \) replaces the components \((a_{i,1}, h_{i,1}) \cdots (a_{i,l_i}, h_{i,l_i})\) and \((a_{i',1}, h_{i',1}) \cdots (a_{i',l_{i'}}, h_{i',l_{i'}})\) with the single component

\[
(a_{i,j}, a_{i',j'}) (a_{i,j+1} h_{i,j+1}) \cdots (a_{i,j-1} h_{i,j-1}) (a_{i',j'+1} h_{i',j'+1}) \cdots (a_{i',j'-1} h_{i',j'-1}).
\]

\( X'_{i,j,i',j'} \) is similar to \( X'_{i,j,i',j'} \), but with heights \( h_{i,j} \) and \( h_{i',j'} \) interchanged; \( X''_{i,j,i',j'} \), is given by replacing the component \((a_{i,1}, h_{i,1}) \cdots (a_{i,l_i}, h_{i,l_i})\) with

\[
((a_{i,j}, a_{i,j'})) (a_{i,j'+1} h_{i,j'+1}) \cdots (a_{i,j-1} h_{i,j-1}) \\
& \& (a_{i,j+1} h_{i,j+1}) \cdots (a_{i,j'-1} h_{i,j'-1}).
\]

Let \( N(Q)_h := \tilde{A} \tilde{B} \). For any \( X, Y \in N(Q)_h \), the product of \( X \) and \( Y \), denoted by \( X \ast Y \), is defined to be “placing \( Y \) above \( X \)”. More precisely, take two elements \( X, Y \) in the form (4). Without loss of generality, we assume the heights of the edges in \( Y \) are all greater than those in \( X \). Then \( X \ast Y \) is represented by \( X \& Y \).

**Remark 3.13** For an arbitrary element \( x \in N(Q)_h \), \( x \) can be chosen to be represented by a linear combination of elements in the form (4) whose heights all increase from left to right; that is, \( h_{i,j} < h_{i',j'} \) whenever \((i, j) < (i', j')\) in the lexicographical order.
Proposition 3.14 [19, Section 3.6] Suppose \( Q \) is a finite quiver. Then the \( \mathbb{K}[\hbar] \)-algebra \( \mathbf{N}(Q)_\hbar \) constructed as above quantizes the necklace Lie algebra \( (\mathbb{K}Q)^\# \).

We give an example of the quantization procedure.

Example 3.15 Assume \( \overline{Q} \) to be

\[
\begin{array}{ccc}
  & a & \\
1 & \rightarrow & 2 \\
& b^* & \leftarrow
\end{array}
\]

and choose cyclic paths with height to be

\[
X = (a^*, 1)(a, 2)(b^*, 3), \quad Y = (b, 1)(b^*, 2).
\]

Also we assume that \( a^*ab^* < bb^* \). Thus,

\[
X \ast Y - Y \ast X = X \ast Y - (b, 1)(b^*, 2)&(a^*, 3)(a, 4)(b^*, 5) \\
= X \ast Y - (b, 1)(b^*, 3)&(a^*, 2)(a, 4)(b^*, 5) \\
= X \ast Y - (b, 1)(b^*, 4)&(a^*, 2)(a, 3)(b^*, 5) \\
= X \ast Y - (b, 2)(b^*, 5)&(a^*, 1)(a, 3)(b^*, 4) \\
= X \ast Y - (b, 3)(b^*, 5)&(a^*, 1)(a, 2)(b^*, 4) \\
= X \ast Y - (b, 4)(b^*, 5)&(a^*, 1)(a, 2)(b^*, 3) - \{b, b^*\}(b^*, 5)(a^*, 1)(a, 2) \\
= \{b, b^*\}(b^*, 5)(a^*, 1)(a, 2) \\
= (b^*, 5)(a^*, 1)(a, 2).
\]

On the other hand, we have

\[
\{a^*ab^*, bb^*\} = b^*a^*a.
\]

In fact, Schedler actually showed more, namely, \( (\mathbb{K}\overline{Q})^\# \) is not only a Lie algebra, but also a Lie bialgebra, and \( \mathbf{N}(Q)_\hbar \) is a Hopf algebra which quantizes this Lie bialgebra.

From now on, if an algebra \( A \) is a quantization of another algebra \( B \), we will denote quantization as follows:

\[
B \rightsquigarrow A.
\]

3.4 Non-commutative reduction at quantum level

In this subsection, we introduce the non-commutative quantum reduction. In the previous subsection, we already obtain the quantization and reduction of the non-commutative cotangent bundle of \( \mathbb{K}Q \) for a finite quiver \( Q \). To complete the non-commutative “quantization commutes with reduction”, we have to develop the non-commutative reduction at the quantum level; such a construction should fit the Kontsevich–Rosenberg principle, that is to say,
the non-commutative quantum reduction must induce the classical quantum reduction on the quiver representations.

First, we recall a result in [19].

**Proposition 3.16** [19, Corollary 4.2] Let $Q$ be a finite quiver and fix an ordering on the set $\{x_i\}$ of cyclic words in $\tilde{Q}$ and idempotents in $Q_0$. We have:

1. The projection $\tilde{A} \rightarrow SL[\hbar]$ obtained by forgetting the heights descends to an isomorphism $N(Q)_\hbar \rightarrow SL[\hbar]$ of free $K[\hbar]$-modules;

2. A basis of $N(Q)_\hbar$ as a free $K[\hbar]$-module is given by choosing one element of the form (4) which projects to each element of the basis $\{x_i, \cdots, x_k\}$ for any $k \in \mathbb{Z}_{\geq 0}$ and $i_1 < i_2 < \cdots < i_k$.

Following the idea in [19], we fix the basis as following: Fix the ordering $\{x_i\}$ of the set of cyclic words in $\tilde{Q}$ and idempotents in $Q_0$; choose a particular “first element” of each $x_i$, and for any $x_i, \cdots, x_k$ with $i_j$ non-decreasing order, heights can be assigned by starting with the first element to the last. We simply denote the lifting of $x_i, \cdots, x_k$ by $\tilde{x}_{i_1} \cdots \tilde{x}_{i_k}$. For example, given two cyclic paths $a_1 a_2 a_3, b_1 b_2$ with $a_1 a_2 a_3 < b_1 b_2$, the lifting of $a_1 a_2 a_3 & b_1 b_2$ is $(a_1, 1)(a_2, 2)(a_3, 3) \ast (b_1, 4)(b_2, 5)$. Sometimes, for a long word like $(a_1 a_2 a_3 \cdots$ representing a cyclic path, we also denote the lifting of $a_1 a_2 a_3 \cdots$ by $(a_1 a_2 a_3 \cdots)$. Notice that $\&$ in $x_i, \cdots, x_k$ is the symmetric product in the symmetric algebra $SL[\hbar]$.

In fact, in [19], Schedler constructed an isomorphism of $K$-modules,

$$l : U((K\tilde{Q})_\hbar) \rightarrow \frac{N(Q)_\hbar}{hN(Q)_\hbar}.$$ 

In what follows let $\boxtimes$ be the multiplication in $U((K\tilde{Q})_\hbar)$ induced from the tensor product in the tensor algebra $T_K(K\tilde{Q})_\hbar$. For an arbitrary element $\sum_i x_{i,1} \boxtimes x_{i,2} \boxtimes \cdots \boxtimes x_{i,r} \in U((K\tilde{Q})_\hbar)$, $l(\sum_i x_{i,1} \boxtimes x_{i,2} \boxtimes \cdots \boxtimes x_{i,r})$ is represented by $\sum_i (\tilde{x}_{i,1} \ast \tilde{x}_{i,2} \ast \cdots \ast \tilde{x}_{i,r})$.

**Definition 3.17** Let $Q$ be a finite quiver, and let $N(Q)_\hbar$ be the quantized necklace Lie algebra. The non-commutative quantum moment map is defined to be the element in $N(Q)_\hbar$ represented by

$$\tilde{\mathbf{w}} := \sum_{a \in Q_1} (a, 1)(a^* , 2) - (a^* , 1)(a, 2).$$

Proposition 5.4 below justifies the above definition from the Kontsevich–Rosenberg principle point of view.

**Definition 3.18** Suppose $Q$ is a finite quiver and $N(Q)_\hbar$ is the quantized necklace Lie algebra. The non-commutative quantum reduction of $N(Q)_\hbar$ at $\tilde{w}$ is defined to be

$$R_q(N(Q)_\hbar, \tilde{w}) := \frac{N(Q)_\hbar}{N(Q)_\hbar(\overline{KQ\tilde{w}})_\hbar N(Q)_\hbar},$$

where $(\overline{KQ\tilde{w}})_\hbar$ are liftings of elements in $(\overline{KQ\tilde{w}})_\hbar$, and $N(Q)_\hbar(\overline{KQ\tilde{w}})_\hbar N(Q)_\hbar$ is the two-side ideal generated by $(\overline{KQ\tilde{w}})_\hbar$. 

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Now we fix some notations. Given a Lie algebra \( \mathfrak{g} \), denote its universal enveloping algebra by \( \mathcal{U}\mathfrak{g} \). Define a map
\[
v : \mathcal{U}(\prod Q) \rightarrow \frac{\mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w})}{h\mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w})}
\]
by
\[
\sum_i c_i pr(x_{i1}) \otimes \cdots \otimes pr(x_{ik_i}) \mapsto \sum_i c_i [x_{i1} \cdots x_{ik_i}],
\]
where \( pr : (\mathbb{K} \overline{Q})_z \rightarrow (\prod Q)_z \) is the canonical homomorphism induced from the projection of algebras \( \mathbb{K} \overline{Q} \rightarrow \prod Q, \overline{x}_{i1} \cdots \overline{x}_{ik_i} \) is the class in \( \mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w}) \) represented by \( \overline{x}_{i1} \cdots \overline{x}_{ik_i} \).

Lemma 3.19 Let \( Q \) be a finite quiver and \( w = \sum_{a \in Q_1} [a, a^*] \). For any cyclic path \( x \in (\mathbb{K} \overline{Q})_z \) and any element \( \sum_i y_i w \in (\mathbb{K} \overline{Q}w)_z \), we have
\[
\left\{ x, \sum_i y_i w \right\} \in (\mathbb{K} \overline{Q}w)_z.
\]

Proof See [7, Proposition 4.4.3].

Theorem 3.20 Suppose \( Q \) is a finite quiver and \( \mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w}) \) is the non-commutative quantum reduction of \( \mathcal{N}(Q)_h \). Then
\[
v : \mathcal{U}(\prod Q) \cong \frac{\mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w})}{h\mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w})}
\]
as \( \mathbb{K} \)-modules.

Proof First, we check \( v \) is well-defined. We choose an arbitrary element \( \sum_i c_i pr(x_{i1}) \otimes \cdots \otimes pr(x_{ik_i}) \), and since \( pr \) is not an isomorphism, we need to make sure the value of \( v \) is independent of \( \ker pr \). For each \( i, j \), \( y_{ij} - x_{ij} \in \ker pr = (\mathbb{K} \overline{Q}w)_z \), by definition of \( \mathcal{R}_q(\mathcal{N}(Q)_h, w) \), we have \( \overline{x}_{i1} \cdots \overline{x}_{ik_i} = \overline{y}_{i1} \cdots \overline{y}_{ik_i} \) for each \( i \). Thus \( v(\sum_i c_i pr(x_{i1}) \otimes \cdots \otimes pr(x_{ik_i})) = v\left( \sum_i c_i pr(y_{i1}) \otimes \cdots \otimes pr(y_{ik_i}) \right) \).

Second, we check \( v \) is surjective. On one hand, we have a surjective morphism
\[
\frac{\mathcal{N}(Q)_h}{h\mathcal{N}(Q)_h} \rightarrow \frac{\mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w})}{h\mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w})}
\]
which is induced from canonical surjective morphism \( \mathcal{N}(Q)_h \rightarrow \mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w}) \). On the other hand, by Proposition 3.14 we know \( \mathcal{U}(\prod \overline{Q})_z \cong \frac{\mathcal{N}(Q)_h}{h\mathcal{N}(Q)_h} \). Combine these two morphisms, we have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{N}(Q)_h & \xrightarrow{=} & \mathcal{U}(\prod \overline{Q})_z \\
\frac{h\mathcal{N}(Q)_h}{h\mathcal{N}(Q)_h} & \xrightarrow{pr} & \frac{\mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w})}{h\mathcal{R}_q(\mathcal{N}(Q)_h, \hat{w})} \\
& \xrightarrow{v} & \mathcal{U}(\prod \overline{Q}).
\end{array}
\]
which implies that \( v \) is surjective.

Finally, we check \( v \) is injective. Suppose we have an arbitrary element \( \sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes \text{pr}(x_{i_k}) \) such that \( v \left( \sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes \text{pr}(x_{i_k}) \right) = 0 \). We need to show this element is zero.

There are only two cases to check. The first case is that \( \sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes x_{i_k} = 0 \) in \( \mathcal{R}_q(\mathbb{N}(Q)_h, \hat{\mathcal{W}}) \); that is to say, \( \sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes x_{i_k} \in \mathbb{N}(Q)_h(\mathcal{K}\mathcal{Q}\mathcal{W})_2 \mathbb{N}(Q)_h \). Then we can rewrite this element as

\[
\sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes x_{i_k} = \sum_i \alpha_i \cdot \cdots \cdot \alpha_i \cdot \alpha_i \cdot (\sum b_1 \cdots b_s \sum_{a \in Q_1} [a, a^*]) * \beta_i \cdot \cdots \cdot \beta_i, \]

here all \( \alpha_i, \beta_i \) are elements in \( (\mathcal{K}\mathcal{Q})_2 \), and \( b_1 \cdots b_s \) is a cyclic path in \( \mathcal{Q} \). Here, we use \( \sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes x_{i_k} \) to represent the equivalence class in \( \mathbb{N}(Q)_h(\mathcal{K}\mathcal{Q}\mathcal{W})_2 \mathbb{N}(Q)_h \). Since the lifting

\[
l : \mathcal{U}(\mathcal{K}\mathcal{Q})_2 \rightarrow \mathbb{N}(Q)_h / h\mathbb{N}(Q)_h
\]

is an isomorphism, we have that

\[
l^{-1} \left( \sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes x_{i_k} \right) = l^{-1} \left( \sum_i \alpha_i \cdot \cdots \cdot \alpha_i \cdot \alpha_i \cdot (\sum b_1 \cdots b_s \sum_{a \in Q_1} [a, a^*]) * \beta_i \cdot \cdots \cdot \beta_i \right)
\]

implies

\[
\sum_i c_i x_{i_1} \boxtimes \cdots \boxtimes x_{i_k} = \sum_i l^{-1} \left( \alpha_i \cdot \cdots \cdot \alpha_i \cdot \alpha_i \cdot (\sum b_1 \cdots b_s \sum_{a \in Q_1} [a, a^*]) * \beta_i \cdot \cdots \cdot \beta_i \right).
\]

If we apply \( \text{pr} \) to the left hand side, we get \( \sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes \text{pr}(x_{i_k}) \). To show this is zero we need to show the right hand side is zero.

In fact, by Proposition 3.16, we can rewrite

\[
\alpha_i \cdot \cdots \cdot \alpha_i \cdot \alpha_i \cdot (\sum b_1 \cdots b_s \sum_{a \in Q_1} [a, a^*]) * \beta_i \cdot \cdots \cdot \beta_i
\]

to be the linear combination of the basis, and due to Lemma 3.19, we see that each summand must contain an element in \( (\mathcal{K}\mathcal{Q}\mathcal{W})_2 \). Thus by definition of \( \text{pr} \), the right hand side is zero, and hence we do have \( \sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes \text{pr}(x_{i_k}) = 0 \).

For the second case, that is

\[
\sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes x_{i_k} \neq 0
\]

but

\[
\sum_i c_i \text{pr}(x_{i_1}) \boxtimes \cdots \boxtimes x_{i_k} = 0,
\]

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this means \( \sum_i c_i \overline{x_{i1}} \cdots \overline{x_{ik_i}} \in \hbar R_q(N(Q), \hat{\mathbb{w}}) \), which contradicts to the fact \( \sum_i c_i \overline{x_{i1}} \cdots \overline{x_{ik_i}} \) does not contain \( \hbar \).

Combining the above three facts, we get an isomorphism

\[
\mathcal{U}((/\Pi Q)_{\zeta}) \cong \frac{R_q(N(Q)_{\hbar}, \hat{\mathbb{w}})}{\hbar R_q(N(Q)_{\hbar}, \hat{\mathbb{w}})}.
\]

\( \square \)

In summary, we construct in Definition 3.18 the non-commutative quantum reduction \( R_q(N(Q)_{\hbar}, \hat{\mathbb{w}}) \) of \( N(Q)_{\hbar} \), and prove in Theorem 3.20 that this reduction is the quantization of the corresponding preprojective algebra. In other words, we now have the non-commutative version of “quantization commutes with reduction” for quiver algebras, which is summarized as follows.

**Theorem 3.21** (Non-commutative quantization commutes with reduction) Suppose \( Q \) is a finite quiver. Then we have the following commutative diagram:

\[
\begin{array}{ccc}
N(Q)_{\hbar} & \overset{\leftarrow}{\rightarrow} & R_q(N(Q)_{\hbar}, \hat{\mathbb{w}}) \\
\downarrow & & \downarrow \\
(\mathbb{K}Q)_{\zeta} & \overset{\leftarrow}{\rightarrow} & (\Pi Q)_{\zeta}.
\end{array}
\]

\( 5 \)

### 4 Quantization of quiver representations

In this section, we collect the results of Holland [12] on the commutativity of quantization and reduction for quiver representations. Since we will heavily use his results, in what follows we give some details.

#### 4.1 Linear differential operators

Let us start by recalling the following.

**Definition 4.1** Let \( X \) be a smooth affine variety with the coordinate ring \( T := \mathbb{K}[X] \). The algebra of linear differential operators on \( X \) is the subalgebra of \( \text{End}_\mathbb{K} T \) generated by \( \text{End}_T T \cong T \) and \( \text{Der}_\mathbb{K}(T) \).

Let \( T \) be as above. Suppose \( V \) is a finite dimensional \( \mathbb{K} \)-vector space. Let \( E_V := V \otimes T \); then \( \text{End}_\mathbb{K} V \otimes \mathcal{D}(T) \) embeds naturally in \( \text{End}_\mathbb{K} E_V \), which is denoted by \( \mathcal{D}(E_V) \). By definition, \( \mathcal{D}(E_V) = \mathcal{D}(T) \).

The following two propositions will be used later, whose proof can be found in Holland [12, Section 2].

**Proposition 4.2** Suppose \( V \) is a finite dimensional \( \mathbb{K} \)-vector space. Then \( \mathcal{D}(E_V) \) is equipped with a natural filtration:

\[
\mathcal{D}^0(E_V) \subset \mathcal{D}^1(E_V) \subset \cdots \subset \mathcal{D}^n(E_V) \subset \cdots
\]

where

\[
\mathcal{D}^0(E_V) = \text{End}_\mathbb{K} V \otimes T, \quad \mathcal{D}^1(E_V) = \text{End}_\mathbb{K} V \otimes (T + \text{Der}_\mathbb{K} T)
\]

and \( \mathcal{D}^n(E_V) = \mathcal{D}^{n-1}(E_V) \mathcal{D}^1(E_V) \) for \( n \geq 2 \).
Proposition 4.3 Suppose $X$ is an smooth affine variety and $V$ is a finite dimensional vector space. Then

1. for $n > 1$, $\frac{\mathcal{D}^n(E_V)}{\mathcal{D}^{n-1}(E_V)} \cong \text{End}_K V \otimes \text{Sym}^n(\text{Der}_K T)$;
2. $\text{gr} \mathcal{D}(E_V) \cong \text{End}_K V \otimes K[\mathcal{T}^* X]$. In particular, $\text{gr}(\mathcal{D}(T)) \cong K[\mathcal{T}^* X]$. Here $\text{gr}(-)$ means the associated graded algebra.

Let $G$ be a reductive algebraic group with Lie algebra $\mathfrak{g}$. Suppose $G$ acts on $X$, then we get a Lie algebra map $\tau : \mathfrak{g} \to \text{Der}_K T$. Furthermore, one can extend $\tau$ to be an algebra map $\tau : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}(T)$. Define a $G$-action on $\mathcal{D}(T)$ by

$$g.D := g \circ D \circ g^{-1}, \quad \text{for any } g \in G, \ D \in \mathcal{D}(T).$$  \hfill (6)

This $G$-action induces a $\mathfrak{g}$-action on $\mathcal{D}(T)$ by:

$$x.D := \tau(x)D - D\tau(x) = [\tau(x), D], \quad \text{for any } x \in \mathfrak{g}, \ D \in \mathcal{D}(T).$$  \hfill (7)

Now consider the following function:

$$p : \mathbb{N}^{Q_0} \to \mathbb{Z}, \alpha \mapsto 1 + \sum_{a \in Q_1} \alpha_s(a)\alpha_t(a) - \sum_{i \in Q_0} \alpha_i^2.$$ 

The following proposition, due to Crawley-Boevey, gives several equivalent descriptions of a flat moment map:

Proposition 4.4 [5, Theorem 1.1] Suppose $Q$ is a finite quiver, $\mathbf{d} \in \mathbb{N}^{Q_0}$ is a dimension vector, and $\mu : T^* \text{Rep}(Q, \mathbf{d}) \to \mathfrak{g}_d$ is the moment map. Then the following statements are equivalent:

1. $\mu$ is a flat morphism;
2. $\mu^{-1}(0)$ is of dimension $\sum_{i \in Q_0} d_i^2 - 1 + 2p(\mathbf{d})$;
3. $\dim \mu^{-1}(0) = 2 \dim \text{Rep}(Q, \mathbf{d}) - \dim \text{GL}_d$.

Now take a character $\chi : \mathfrak{g} \to \mathbb{K}$ which does not vanish on $\text{ker} \tau$. The following two propositions are obtained by Holland in [12], which will be used later:

Proposition 4.5 [12, Proposition 2.4] Suppose that the moment map $\mu : T^* X \to \mathfrak{g}^*$ is flat. Then

$$\text{gr} \left( \frac{\mathcal{D}(E_V)}{\mu^{-1}(0)} \right) \cong \text{End}_K V \otimes K[\mu^{-1}(0)]$$

as $K[T^* X]$-modules. In particular,

$$\text{gr} \left( \frac{(\mathcal{D}(E_V))^G}{(\mathcal{D}(E_V)(\mathcal{T} - \chi)(\mathfrak{g}))^G} \right) \cong (\text{End}_K V \otimes K[\mu^{-1}(0)])^G$$

as $\mathbb{K}$-algebras, where $\tau$ is described after Proposition 4.3 and $\chi : \mathfrak{g} \to \mathbb{K}$ is a character of $\mathfrak{g}$, which does not vanish on ker $\tau$.

Proposition 4.6 [12, Proposition 2.4] With notations as above, there exists an isomorphism

$$\text{gr} \left( \frac{(\mathcal{D}(T))^G}{(\mathcal{T} - \chi)(\mathfrak{g}))^G} \right) \cong (K[\mu^{-1}(0)])^G.$$
4.2 Quantization of quiver representations

Now we are ready to study the quantization and reduction of quiver representations.

**Definition 4.7** Suppose \( A \) is commutative \( \mathbb{Z}_{\geq 0} \)-graded \( \mathbb{K} \)-algebra, equipped with a Poisson bracket whose degree is \(-1\). A *filtered quantization* of \( A \) is a filtered algebra \( A_{\leq \bullet} \), such that \( \text{gr} A_{\leq \bullet} \) is isomorphic to \( A \) as graded Poisson algebras.

Notice that the Poisson bracket on \( \text{gr} A_{\leq \bullet} \) is given as follows: for any \( \overline{a}, \overline{b} \in \text{gr} A_{\leq \bullet} \),

\[
\{ \overline{a}, \overline{b} \} := \overline{[a, b]}
\]

Here \([a, b]\) is the commutator with respect to the product in \( A_{\leq \bullet} \).

By the above definition and Proposition 4.6, we obtain the following:

**Corollary 4.8** Suppose \( Q \) is a finite quiver, \( d \) is a dimension vector and \( \chi \) is a character of \( \mathfrak{gl}_d \) such that the moment map \( \mu : T^* \text{Rep}(Q, d) \to \mathfrak{gl}_d \) is a flat morphism and \( \chi \) does not vanish on \( \text{Ker} \tau \). Then

1. \( D(\text{Rep}(Q, d)) \) quantizes \( \mathbb{K}[T^* \text{Rep}(Q, d)] \), and
2. \( (D(\text{Rep}(Q, d)) (\tau - \chi)(\mathfrak{gl}_d))_{\text{GL}_d} \) quantizes \( \mu^{-1}(0) / \text{GL}_d \).

For the convenience of our argument below, we also need the *graded quantization*, which we now recall.

**Definition 4.9** Suppose \( A \) is a \( \mathbb{Z}_{\geq 0} \)-graded commutative \( \mathbb{K} \)-algebra equipped with Poisson bracket \( \{ , \} \) of degree \(-1\). A *graded quantization* of \( A \) is a graded \( \mathbb{K}[\hbar] \)-algebra (deg \( \hbar = 1 \)) which is free as a \( \mathbb{K}[\hbar] \)-module, equipped with an isomorphism of \( \mathbb{K} \)-algebras:

\[
\Phi : \frac{A_\hbar}{\hbar A_\hbar} \to A,
\]

such that for any \( a, b \in A_\hbar \), if we denote their images in \( \frac{A_\hbar}{\hbar A_\hbar} \) by \( \overline{a}, \overline{b} \), then

\[
\Phi \left( \frac{1}{\hbar}(ab - ba) \right) = \{ \Phi(\overline{a}), \Phi(\overline{b}) \}.
\]

Now let us recall the notion of the homogeneous differential operators (see Losev [16]):

**Definition 4.10** Let \( X \) be a smooth affine variety, and \( T := \mathbb{K}[X] \) be its coordinate ring. The homogeneous differential operators on \( X \) is defined to be the graded \( \mathbb{K}[\hbar] \)-algebra \( D_\hbar(X) \) generated by \( T \) equipped with degree 0, \( \text{Der}_\mathbb{K}(T) \) equipped with degree 1 and subject to the following relation:

1. for any \( f, g \in T \), \( f \star g = fg \);
2. for \( X \in \text{Der}_\mathbb{K}T, f \in T, f \star X = fX \);
3. \( X \in \text{Der}_\mathbb{K}T, f \in T, X \star f = fX + \hbar X(f) \);
4. for \( X, Y \in \text{Der}_\mathbb{K}T, X \star Y - Y \star X = \hbar [X, Y] \).

With the above definition, we induce the following corollary from Proposition 4.5 and Corollary 4.8.

**Corollary 4.11** [16, Section 3 and Section 4.2] Suppose \( Q \) is a finite quiver, \( d \) is a dimension vector and \( \chi \) is a character of \( \mathfrak{gl}_d \) such that the moment map \( \mu : T^* \text{Rep}(Q, d) \to \mathfrak{gl}_d \) is a flat morphism and \( \chi \) does not vanish on \( \text{Ker} \tau \). Then
(1) \( D_h(\text{Rep}(Q, d)) \) is a graded quantization of \( \mathbb{K}[T^*\text{Rep}(Q, d)] \), and
\[
\left( \frac{D_h(\text{Rep}(Q, d))}{hD_h(\text{Rep}(Q, d))} \right)_{\text{GL}_d}^{\text{GL}_d} \text{ is a graded quantization of } \mathbb{K}[\mathcal{M}_d(Q)].
\]

we give a sketchy proof:
1. Since \( T^*\text{Rep}(Q, d) \) is a vector space, \( D_h(\text{Rep}(Q, d)) \) is the Rees algebra of the filtered algebra \( D(\text{Rep}(Q, d)) \); this ensures
\[
\frac{D_h(\text{Rep}(Q, d))}{hD_h(\text{Rep}(Q, d))} \cong \text{gr}(D(\text{Rep}(Q, d))) \cong \mathbb{K}[T^*\text{Rep}(Q, d)]
\]
as \( \mathbb{K} \)-algebras. Therefore, we have the \( \Phi \) in Definition 4.9 and \( \Phi\left( \frac{1}{h}(ab - ba) \right) = \{ \Phi(\bar{a}), \Phi(\bar{b}) \} \) is guaranteed by the second isomorphism; noticed that the second isomorphism is an isomorphism of Poisson algebras.
2. The second claim holds since the \( \Phi \) descends to be a well-defined
\[
\Phi' : \frac{D_h(\text{Rep}(Q, d))}{(D_h(\text{Rep}(Q, d))(\tau - h\chi)(g_d))_{\text{GL}_d}^{\text{GL}_d}} \to \mathbb{K}[\mathcal{M}_d(Q)].
\]
\((h)\) is the ideal generated by \( h \). Noticed that the image of
\[
\frac{D_h(\text{Rep}(Q, d))(\tau - h\chi)(g_d))_{\text{GL}_d}^{\text{GL}_d}}{hD_h(\text{Rep}(Q, d))} \to \mathbb{K}[T^*\text{Rep}(Q, d)]
\]
via \( \Phi : \frac{D_h(\text{Rep}(Q, d))}{hD_h(\text{Rep}(Q, d))} \to \mathbb{K}[T^*\text{Rep}(Q, d)] \) is
\[
\text{gr}\left( \frac{(D_h(\text{Rep}(Q, d))(\tau - h\chi)(g_d))_{\text{GL}_d}^{\text{GL}_d}}{D(\text{Rep}(Q, d))(\tau - \chi)(g_d))_{\text{GL}_d}^{\text{GL}_d}} \right).
\]
Now, apply Holland’s result (see Proposition 4.5 and Corollary 4.8) which is
\[
\mathbb{K}[\mathcal{M}_d(Q)] \cong \text{gr}\left( \frac{(D(\text{Rep}(Q, d)))_{\text{GL}_d}^{\text{GL}_d}}{D(\text{Rep}(Q, d))(\tau - \chi)(g_d))_{\text{GL}_d}^{\text{GL}_d}} \right) \cong \frac{\text{gr}(D(\text{Rep}(Q, d))_{\text{GL}_d}^{\text{GL}_d})}{\text{gr}(D(\text{Rep}(Q, d))(\tau - \chi)(g_d))_{\text{GL}_d}^{\text{GL}_d}},
\]
we have a well-defined \( \Phi' \); (1) ensures that \( \Phi' \) is an isomorphism and \( \Phi'\left( \frac{1}{h}(ab - ba) \right) = \{ \Phi'(\bar{a}), \Phi'(\bar{b}) \} \).

**Definition 4.12** (Quantum moment map) Let \( G \) be an algebraic group with Lie algebra \( g \); let \( A_h \) be a graded \( \mathbb{K}[h] \)-algebra equipped with a \( g \)-action. The map \( \mu_h^\# : U_h g \to A_h \) is called a quantum moment map if \( \mu_h^\#(g) \subset (A_h)_1 \) and for any \( v \in g \),
\[
A_h \to A_h, \ a \mapsto \frac{1}{h} [\mu_h^\#(v), a]
\]
is the \( g \)-action of \( v \).

Here
\[
U_h g := T_{\mathbb{K}[h]}(\mathbb{K}[h] \otimes g)/(x \otimes y - y \otimes x - h[x, y] \text{ for any } x, y \in g).
\]
The above definition is very close to the quantum moment map introduced by Lu in [17] (see also Xu [23] and Losev [16]). Let us briefly recall the main idea in loc. cit. Let \( H \) be a Hopf
algebra and let $V$ be an algebra with a compatible $H$-action. Lu proposed that $\Phi : H \to V$ is called the quantum moment map if
\[ a.v = \Phi(S(a^{(1)}))v\Phi(a^{(2)}), \]
where $a \in H$, $v \in V$, $S$ is the antipode of $H$ and the coproduct of $a$ is written as $a^{(1)} \otimes a^{(2)}$.

Definition 4.12 above is just the infinitesimal version of the one of Lu’s.

Now, fix a dimension vector $\mathbf{d}$; suppose $Q$ is a finite quiver with a flat moment map $\mu : T^*\text{Rep}(Q, \mathbf{d}) \to \mathfrak{gl}_d$. Differentiating the $\text{GL}_d$-action on $\text{Rep}(Q, \mathbf{d})$, one has (see also Holland [12, Lemma 3.1]):

**Lemma 4.13** The $\mathfrak{gl}_d$-action on $\text{Rep}(Q, \mathbf{d})$ is given by the Lie algebra homomorphism $\tau : \mathfrak{gl}_d \to \mathcal{D}(\text{Rep}(Q, \mathbf{d}))$, which maps
\[ e^i_{pq} \mapsto \sum_{a \in Q, s(a) = i} \sum_{j=1}^{d_{i(a)}} [a]_{jp} \frac{\partial}{\partial(a)_{jq}} - \sum_{a \in Q, t(a) = i} \sum_{j=1}^{d_{i(a)}} [a]_{qj} \frac{\partial}{\partial(a)_{pj}}, \]
where $e^i_{pq}$ is the $(q, p)$-th elementary matrix in the $i$-th summand of $\mathfrak{gl}_d$.

By this lemma, the quantum moment map in the case of quiver varieties is given by
\[ \mu^\#_{\hbar} = \tau - \hbar \chi \]
for some characters $\chi$ not vanishing on $\ker \tau$.

Thus according to [9] and [16],
\[ \frac{\mathcal{D}_\hbar(\text{Rep}(Q, \mathbf{d}))^{\text{GL}_d}}{(\mathcal{D}_\hbar(\text{Rep}(Q, \mathbf{d}))(\tau - \hbar \chi)(\mathfrak{gl}_d))^{\text{GL}_d}} \]
obtained from $\mathcal{D}_\hbar(\text{Rep}(Q, \mathbf{d}))$ is a quantum Hamiltonian reduction.

That such a reduction is called the quantum counterpart of classical Hamiltonian reduction, is based on the following observations. In classical reduction, a moment map for a Hamiltonian space gives the infinitesimal description of the Hamiltonian action; while a quantum moment map gives the description of the quantum action. Also, as the classical reduction gives a new symplectic space out of a given symplectic space, the quantum Hamiltonian should give the quantization of the corresponding space in the classical reduction. One can find more explanations in [9, 17].

Summarizing the above argument, we have the following theorem, due to Holland [12] (see also Losev [16, Lemma 3.3.1] for more details), saying that on quiver representations “the quantization commutes with the reduction”.

**Theorem 4.14** (Quantization commutes with reduction) Let $Q$ be a finite quiver and $\mathbf{d}$ a dimension vector such that the moment map $\mu$ is flat. Let $\tau$ be the Lie algebra homomorphism in Lemma 4.13. For a character $\chi : \mathfrak{gl}_d \to \mathbb{K}^*$ not vanishing on $\ker \tau$, we have a commutative diagram
\[ \begin{array}{ccl}
\mathcal{D}_\hbar(\text{Rep}(Q, \mathbf{d})) & \to & \mathcal{D}_\hbar(\text{Rep}(Q, \mathbf{d}))^{\text{GL}_d} \\
\downarrow & & \downarrow \\
\mathbb{K}[T^*\text{Rep}(Q, \mathbf{d})] & \to & \mathbb{K}[\mathcal{M}_d(Q)].
\end{array} \]
5 Trace maps and proof of the main theorem

In this section, we study two trace maps, at the classical and the quantum level respectively, connecting the previously obtained two commutative diagrams (5) and (8), from which we obtain that “the quantization commutes with the reduction” of quiver algebras fits the Kontsevich–Rosenberg principle.

5.1 Quantum trace maps

The classical trace map on the representation spaces is introduced in Sect. 2.2. In this subsection we recall the quantum trace map, due to Schedler, from the quantized necklace Lie algebras to the quantization of quiver representation spaces.

Definition 5.1 [19, Section 3.4] Suppose $Q$ is a finite quiver, $d$ is a dimension vector, and $N(Q)_\hbar$ is the quantized necklace Lie algebra. Then quantum trace map, denoted by $\text{Tr}^q$, is a $K[\hbar]$-linear map from $N(Q)_\hbar$ to $\mathcal{D}_\hbar(\text{Rep}(Q, d))$ such that for any element in the form (4), its image is

$$d_{v_1} \cdots d_{v_m} \prod_{i,j=1}^{N} \left[ a_{\phi^{-1}(h)} ]_{h=1}^{\phi^{-1}(h)+1} \right],$$

where $\{h_{i,j}\} = \{1, 2, \ldots, N\}$, $\phi$ is the map such that $\phi(i,j) = h_{i,j}$ and for $x \in Q$, $[x]_{i,j}$ is the $(i,j)$-th coordinate function and $[x^*]_{i,j}$ is the differential operator $\frac{\partial}{\partial [x]_{j,i}}$.

It is proved in Schedler [19, Section 3.4] that $\text{Tr}^q$ in the above definition does not depend on choice of the representatives of the elements in $N(Q)_\hbar$.

From the above definition, if the heights in (4) increase from the left to the right, then its image under $\text{Tr}^q$ is

$$d_{v_1} \cdots d_{v_m} \prod_{i=1}^{m} \text{tr} \left( \prod_{j=1}^{N} [a_{ij}] \right).$$

With this formula, we also see that the images of quantum trace map are $\text{GL}_d$-invariant; in other words, we have the following:

Proposition 5.2 Let $Q$ be a finite quiver, and $d$ is a dimension vector. Then we have

$$\text{Tr}^q(N(Q)_\hbar) \subset (\mathcal{D}_\hbar(\text{Rep}(Q, d)))^{\text{GL}_d}.$$
For example, for \( a \) as above and \( x \in Q \),

\[
a[x] = \begin{cases} 
    a^{-1}_t[x]a_{s(x)}, & \text{if } x \in Q_1, \\
    a^{-1}_s[x]a_t(x), & \text{if } x \notin Q_1 \setminus Q_1.
\end{cases}
\] (11)

On the right hand side of (11), all products are matrix products. Consequently, we have \( a.\text{Tr}^q(X) = \text{Tr}^q(X) \).

**Lemma 5.3** Suppose \( Q \) is a finite quiver, \( \mathbf{d} \) is a dimension vector. Then there is a unique character \( \chi_0 \) of \( \mathfrak{gl}_d \) such that

\[
\text{Tr}^q \left( N(Q)\hbar(\mathbb{K} \overline{Q} \sum_{a \in Q} [a, a^*]_\gamma N(Q)\hbar) \right) \subset (\mathcal{D}_\hbar(\text{Rep}(Q, \mathbf{d})) (\tau - \hbar \chi_0)(\mathfrak{gl}_d))^{\text{GL}_d}.
\] (12)

**Proof** The character \( \chi_0 \) is obtained as follows. Since \( \text{Tr}^q \) is linear, without loss of generality, we choose \( X = x_1 \cdots x_r \sum_{a \in Q} (aa^* - a^*a) \in (\mathbb{K} \overline{Q} \sum_{a \in Q} [a, a^*]_\gamma) \) and assume \( s(x_r) = k \).

Thus,

\[
X = x_1 \cdots x_r \sum_{a \in Q} (aa^* - a^*a) = \sum_{a \in Q, t(a) = k} x_1 \cdots x_r aa^* - \sum_{a \in Q, s(a) = k} x_1 \cdots x_r a^*a.
\]

Now, lift \( X \) to \( N(Q)\hbar \) (see the discussion after Proposition 3.16), and suppose the lifting \( \tilde{X} = \sum_{a \in Q, t(a) = k} x_1 \cdots x_r aa^* - \sum_{a \in Q, s(a) = k} x_1 \cdots x_r a^*a \).

Applying \( \text{Tr}^q \) to \( \tilde{X} \), we get

\[
\text{Tr}^q(\tilde{X}) = \sum_{a \in Q, t(a) = k} \sum_{l_1, \ldots, l_{r+2}} [x_1]_{l_1, l_2} \cdots [x_r]_{l_r, l_{r+2}} [a]_{l_{r+1}, l_{r+2}} [a^*]_{l_{r+2}, l_1} \\
- \sum_{a \in Q, s(a) = k} \sum_{l_1, \ldots, l_{r+2}} [x_1]_{l_1, l_2} \cdots [x_r]_{l_r, l_{r+2}} [a^*]_{l_{r+1}, l_{r+2}} [a]_{l_{r+2}, l_1}
\]

\[
= \sum_{l_1, \ldots, l_{r+1}} [x_1]_{l_1, l_2} \cdots [x_r]_{l_r, l_{r+1}} \sum_{a \in Q, t(a) = k} \sum_{l_{r+2}} [a]_{l_{r+1}, l_{r+2}} [a^*]_{l_{r+2}, l_1} \\
- \sum_{l_1, \ldots, l_{r+1}} [x_1]_{l_1, l_2} \cdots [x_r]_{l_r, l_{r+1}} \sum_{a \in Q, s(a) = k} \sum_{l_{r+2}} [a]_{l_{r+2}, l_1} [a^*]_{l_{r+1}, l_{r+2}} h_{l_{r+1}}^{l_1} + \sum_{a \in Q, s(a) = k} [a]_{l_{r+2}, l_1} [a^*]_{l_{r+1}, l_{r+2}} h_{l_{r+1}}^{l_1}
\]

\[
= \sum_{l_1, \ldots, l_{r+1}} [x_1]_{l_1, l_2} \cdots [x_r]_{l_r, l_{r+1}} \left( \sum_{a \in Q, t(a) = k} \sum_{l_{r+2}} [a]_{l_{r+1}, l_{r+2}} \frac{\partial}{\partial [a]_{l_1, l_{r+2}}} \\
- \sum_{a \in Q, s(a) = k} \sum_{l_{r+2}} [a]_{l_{r+2}, l_1} \frac{\partial}{\partial [a]_{l_{r+2}, l_{r+1}}} - \sum_{a \in Q, s(a) = k} d_{l(a)} \delta_{l_{r+1}} \frac{\partial}{\partial [a]_{l_{r+2}, l_{r+1}}} \right)
\]

\[
= \sum_{l_1, \ldots, l_{r+1}} [x_1]_{l_1, l_2} \cdots [x_r]_{l_r, l_{r+1}} \left( -\tau(e_{l_1, l_{r+1}}^{k}) - \sum_{a \in Q, s(a) = k} d_{l(a)} \delta_{l_{r+1}} \frac{\partial}{\partial [a]_{l_{r+2}, l_{r+1}}} \right)
\]

From the above identity, we see that the character is

\[
\chi_0 = -\sum_{k \in \mathcal{Q}_0} \left( \sum_{a \in Q, s(a) = k} d_{l(a)} \right) \text{tr}_k,
\]
where $\text{Tr}^q_k$ is taking the trace of the $k$-th matrix. Also, this $\chi_0$ is the unique character satisfying (12).

By (10) and the definition of the product on $\mathcal{N}(Q)_\hbar$, we have that

$$\text{Tr}^q \left( \mathcal{N}(Q)_\hbar \left( \mathbb{K} \mathbb{Q} \sum_{a \in Q} [a, a^*] \right) \right) \subset (\mathcal{D}_\hbar(\text{Rep}(Q, d))) (\tau - \hbar \chi_0)(g_{\mathfrak{d}})^{GL_d}.$$

We next show the quantum trace of $(\mathbb{K} \mathbb{Q} \sum_{a \in Q} [a, a^*]) \mathcal{N}(Q)_\hbar$ also lies in $(\mathcal{D}_\hbar(\text{Rep}(Q, d))) (\tau - \hbar \chi_0)(g_{\mathfrak{d}})^{GL_d}$. Without loss of generality, we prove this statement for $\hat{X} & (y_1, g_1) \cdots (y_s, g_s)$ with $g_1 < \cdots < g_s$.

In fact, by (10) we have that

$$\text{Tr}^q \left( \hat{X} & (y_1, g_1) \cdots (y_s, g_s) \right) = \text{Tr}^q \left( \hat{X} \right) \text{Tr}^q \left( (y_1, g_1) \cdots (y_s, g_s) \right)$$

$$= \sum_{l_1, \ldots, l_{r+1}} [x_1]_{l_1, l_2} \cdots [x_r]_{l_r, l_{r+1}} \left( - \tau(e^k_{l_1, l_{r+1}}) - \sum_{a \in Q, s(a) = k} d_{s(a)} g_{l_{r+1}}^k \right) \text{tr}([y_1] \cdots [y_s])$$

$$= \sum_{l_1, \ldots, l_{r+1}} [x_1]_{l_1, l_2} \cdots [x_r]_{l_r, l_{r+1}} \left( - \tau(e^k_{l_1, l_{r+1}}) \text{tr}([y_1] \cdots [y_s]) \right)$$

$$- \sum_{a \in Q, s(a) = k} d_{s(a)} g_{l_{r+1}}^k \text{tr}([y_1] \cdots [y_s]) \hbar.$$

Since $\text{tr}([y_1] \cdots [y_s])$ is $GL_d$- and hence also $\mathfrak{g}_d$-invariant, $\tau(e^k_{l_1, l_{r+1}})$ and $\text{tr}([y_1] \cdots [y_s])$ commute with each other. Therefore

$$\text{Tr}^q \left( \hat{X} & (y_1, g_1) \cdots (y_s, g_s) \right)$$

$$= \sum_{l_1, \ldots, l_{r+1}} [x_1]_{l_1, l_2} \cdots [x_r]_{l_r, l_{r+1}} \left( - \tau(e^k_{l_1, l_{r+1}}) - \sum_{a \in Q, s(a) = k} d_{s(a)} g_{l_{r+1}}^k \hbar \right) \text{tr}([y_1] \cdots [y_s])$$

$$= \sum_{l_1, \ldots, l_{r+1}} [x_1]_{l_1, l_2} \cdots [x_r]_{l_r, l_{r+1}} \text{tr}([y_1] \cdots [y_s]) \left( - \tau(e^k_{l_1, l_{r+1}}) - \sum_{a \in Q, s(a) = k} d_{s(a)} g_{l_{r+1}}^k \hbar. \right).$$

Thus $\text{Tr}^q \left( \mathcal{N}(Q)_\hbar \left( \mathbb{K} \mathbb{Q} \sum_{a \in Q} [a, a^*] \right) \mathcal{N}(Q)_\hbar \right) \subset (\mathcal{D}_\hbar(\text{Rep}(Q, d))) (\tau - \hbar \chi_0)(g_{\mathfrak{d}})^{GL_d}$.

Next, we show the non-commutative quantum moment map is well-defined in the sense of Kontsevich–Rosenberg principle. Recall from Definition 3.17 that for the quantized necklace Lie algebra $\mathcal{N}(Q)_\hbar$, the non-commutative quantum moment map is

$$\sum_{a \in Q} (a, 1)(a^*, 2) - (a^*, 1)(a, 2).$$

**Proposition 5.4** Let $Q$ be a finite quiver and let $\hat{W}$ be the non-commutative quantum moment map. For a fixed dimension vector $d \in \mathbb{N}^{Q_0}$, the map $\mu^\sharp_{\hat{W}} : U_{\hbar}(\mathfrak{gl}_d) \rightarrow \mathcal{D}_\hbar(\text{Rep}(Q, d))$ sending an arbitrary $(g_i) \in \mathfrak{gl}_d$ to $-tr(\hat{W}(g_i))$ is a quantum moment map.
Taking the trace on both sides of the above equality, we have

\[
\begin{align*}
\text{tr}([\hat{\chi}]e_{p,q}^i) &= \text{tr} \left( \sum_{t(a)=i} \sum_{k,l} [a]_{k,l}^{a*} [a]_{l,p}^i - \sum_{s(a)=i} \sum_{k,l} [a]^{a*}_{s,a} [a]_{l,p}^i \right) \\
&= \sum_{t(a)=i} \sum_{l} [a]_{q,l}^{a*} [a]_{l,p} - \sum_{s(a)=i} \sum_{l} [a]^{a*}_{s,a} [a]_{l,p} \\
&= \sum_{t(a)=i} \sum_{l} \frac{\partial}{\partial (a)_{p,l}} - \sum_{s(a)=i} \sum_{l} \frac{\partial}{\partial (a)_{l,q}} \left( [a]_{l,p}^i \delta^p_q + \hbar \right) \\
&= \sum_{t(a)=i} \sum_{l} \frac{\partial}{\partial (a)_{p,l}} - \sum_{s(a)=i} \sum_{l} \left( [a]_{l,p}^i \frac{\partial}{\partial (a)_{l,q}} - \hbar \sum_{l} \delta^p_q \sum_{s(a)=i} d_{t(a)\delta^p_q} \right).
\end{align*}
\]

Therefore \( \mu_{\hat{\chi}}^i(e_{p,q}^i) \) is

\[
\sum_{s(a)=i} \sum_{l} [a]_{l,p}^i \frac{\partial}{\partial (a)_{l,q}} - \sum_{t(a)=i} \sum_{l} [a]_{q,l}^{a*} \frac{\partial}{\partial (a)_{p,l}} + \hbar \sum_{s(a)=i} \sum_{l} d_{t(a)\delta^p_q},
\]

which exactly equals \((\tau - \hbar \chi_0)(e_{p,q}^i)\) (see Lemma 4.13 for the definition of \(\tau\) and Lemma 5.3 for definition of \(\chi_0\)). This gives the infinitesimal action of \(e_{p,q}^i\), and by Definition 4.12, \(\mu_{\hat{\chi}}^i\) is the quantum moment map.

\[\square\]

5.2 Reduction commutes with the trace map

This subsection is to show that the classical and quantum trace maps preserve the classical and quantum reductions respectively.

First, it is proved by Crawley-Boevey, Etingof and Ginzburg in [7] that the non-commutative reduction induces the Hamiltonian reduction, which we state as follows.
**Proposition 5.5** (Reductions commute with trace maps) Suppose \( Q \) is a finite quiver, \( d \in \mathbb{N}^{Q_0} \) is a dimension vector. Then we have the following commutative diagram

\[
\begin{array}{ccc}
(KQ)_{\natural} & \xrightarrow{\text{Tr}} & \mathbb{K}[T^*\text{Rep}(Q, d)] \\
\downarrow & & \downarrow \\
(\Pi Q)_{\natural} & \xrightarrow{\text{Tr}} & \mathbb{K}[\mathcal{M}_d(Q)].
\end{array}
\] (13)

**Proof** See [7, Theorem 6.4.3] for a proof. \( \square \)

At the quantum level, we have a similar result, which is stated as follows.

**Theorem 5.6** (Quantum reductions commute with quantum trace maps) Suppose \( Q \) is a finite quiver, \( d \) is a dimension vector such that the moment map \( \mu \) is flat. Then we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{N}(Q)_h & \xrightarrow{\text{Tr}^q} & \mathcal{D}_h(\text{Rep}(Q, d)) \\
\downarrow & & \downarrow \\
\mathcal{R}_q(\mathcal{N}(Q)_h, \widehat{w}) & \xrightarrow{\text{Tr}^q} & (\mathcal{D}_h(\text{Rep}(Q, d)) (\tau - \hbar \chi_0)(\mathfrak{gl}_d))^{\text{GL}_d}.
\end{array}
\] (14)

**Proof** First, by Lemma 5.3 the quantum trace map

\[
\mathcal{R}_q(\mathcal{N}(Q)_h, \widehat{w}) \rightarrow \frac{(\mathcal{D}_h(\text{Rep}(Q, d))^{\text{GL}_d}}{(\mathcal{D}_h(\text{Rep}(Q, d)) (\tau - \hbar \chi_0)(\mathfrak{gl}_d))^{\text{GL}_d}
\]

is well-defined. According to Remark 3.13, we prove the theorem for elements in \( \mathcal{N}(Q)_h \) like \( (x_1, 1)(x_2, 2) \cdots (x_r, r) \) without loss of generality.

In fact, \( \text{Tr}^q \left( (x_1, 1)(x_2, 2) \cdots (x_r, r) \right) = tr[x_1]tr[x_2] \cdots tr[x_r] \). Now, we denote the two projections

\[
\mathcal{N}(Q)_h \rightarrow \mathcal{R}_q(\mathcal{N}(Q)_h, \widehat{w}) \quad \text{and} \quad (\mathcal{D}(\text{Rep}(Q, d)))^{\text{GL}_d} \rightarrow \frac{(\mathcal{D}_h(\text{Rep}(Q, d))^{\text{GL}_d}}{(\mathcal{D}_h(\text{Rep}(Q, d)) (\tau - \hbar \chi_0)(\mathfrak{gl}_d))^{\text{GL}_d}
\]

by \( p_1 \) and \( p_2 \) respectively. Then on one hand,

\[
\text{Tr}^q \circ p_1 \left( (x_1, 1)(x_2, 2) \cdots (x_r, r) \right)
\]

\[
= \text{Tr}^q \left( (x_1, 1)(x_2, 2) \cdots (x_r, r) + \mathcal{N}(Q)_h(\mathbb{K}Q \sum_{a \in Q} [a, a^*])_{\natural} \mathcal{N}(Q)_h \right)
\]

\[
= \left( \prod_i tr[x_i] \right) + (\mathcal{D}_h(\text{Rep}(Q, d)) (\tau - \hbar \chi_0)(\mathfrak{gl}_d))^{\text{GL}_d}.
\]
On the other hand,
\[ p_2 \circ \text{Tr}^d \left( (x_1, 1)(x_2, 2) \cdots (x_r, r) \right) = p_2 \left( \prod_i \text{tr}[x_i] \right) = \left( \prod_i \text{tr}[x_i] \right) + \left( \mathcal{D}_h(\text{Rep}(Q,d)) \left( \tau - \hbar \chi_0 \right)(\mathfrak{g}(d))^{\text{GL}_d} \right) \]
These two terms are equal and thus the diagram commutes.

5.3 Quantization commutes with trace maps

In this subsection, we show that the trace map commutes with the quantization. This is implicit in Schedler’s paper [19], which we present below for completeness.

**Theorem 5.7** (Quantizations commute with trace maps: before reduction) Suppose \( Q \) is a finite quiver. Then for any \( x, y \in (\mathbb{K}Q)^\natural \), we have
\[
\Phi \left( \frac{1}{\hbar} \text{Tr}^d(\hat{x} \star \hat{y} - \hat{y} \star \hat{x}) \right) = \{ \text{Tr}(x), \text{Tr}(y) \},
\]
where
\[ \Phi : \frac{\mathcal{D}_h(\text{Rep}(Q,d))}{\hbar \mathcal{D}_h(\text{Rep}(Q,d))} \to \mathbb{K}[T^*\text{Rep}(Q,d)], [a]_{ij} \mapsto (a)_{ij}, [a^*]_{ij} \mapsto (a^*)_{ij}. \]
In other words, we have the following commutative diagram:
\[
\begin{array}{c}
N(Q) \quad \xrightarrow{\text{Tr}^q} \quad \mathcal{D}_h(\text{Rep}(Q,d)) \\
\downarrow \quad \downarrow \quad \downarrow \\
(\mathbb{K}Q)^\natural \quad \xrightarrow{\text{Tr}} \quad \mathbb{K}[T^*\text{Rep}(Q,d)]
\end{array}
\]
which means the trace maps commute with quantization.

**Proof** For any \( x = x_1 \cdots x_r, y = y_1 \cdots y_s \in (\mathbb{K}Q)^\natural \), without loss of generality, we assume their liftings are \( \hat{x} = (x_1, 1)(x_2, 2) \cdots (x_r, r) \), \( \hat{y} = (y_1, 1) \cdots (y_s, s) \). By (10), we have
\[
\text{Tr}^d(\hat{x} \star \hat{y} - \hat{y} \star \hat{x}) = \text{Tr}^d(\hat{x})\text{Tr}^d(\hat{y}) - \text{Tr}^d(\hat{y})\text{Tr}^d(\hat{x})
\]
\[
= \sum_{i, j_1, \ldots, u, v_1, \ldots} \left( [x_1]_{j_1} [x_2]_{j_1, j_2} \cdots [x_r]_{j_1, j_2, \ldots, j_{r-1}, i} [y_1]_{u, v_1} \cdots [y_s]_{v_1, \ldots, v_{r-1}, u} - [y_1]_{u, v_1} \cdots [y_s]_{v_1, \ldots, v_{r-1}, u} [x_1]_{j_1} [x_2]_{j_1, j_2} \cdots [x_r]_{j_1, j_2, \ldots, j_{r-1}, i} \right).
\]
Now, by repeatedly applying the four conditions in Definition 4.10, we can switch those terms containing \( x \) and those containing \( y \) in the first summand; for example, in the first step, we plug \( [x_r]_{j_{r-1}, i} [y_1]_{u, v_1} = [y_1]_{u, v_1} [x_r]_{j_{r-1}, i} + \hbar \{ (x_r)_{j_{r-1}, i}, (y)_{u, v_1} \} \) into the right hand side.
of the above equality, we get a switch of positions of \([x_r]_{j_{r-1},i}\) and \([y_1]_{u,v_1}\). Eventually, we get

\[
\text{Tr}^q(\widehat{x} \ast \widehat{y} - \widehat{y} \ast \widehat{x}) = \hbar \sum_{i,j_1 \ldots u,v_1} \left( (x_1)_{y,j_1, (y_1)_{u,v_1}} [x_2]_{j_1,j_2 \ldots [x_r]_{j_{r-1},i}} [y_2]_{v_1,v_2} \cdots [y_s]_{v_{s-1},u} + \cdots \right).
\]

On the other hand, we have

\[
\{\text{Tr}(x), \text{Tr}(y)\} = \sum_{i,j_1 \ldots u,v_1} \left( (x_1)_{i,j_1} \cdots (x_r)_{j_{r-1},i}, (y_1)_{u,v_1} \cdots (y_s)_{v_{s-1},u} \right)
\]

Therefore,

\[
\Phi \left( \frac{1}{\hbar} \text{Tr}^q(\widehat{x} \ast \widehat{y} - \widehat{y} \ast \widehat{x}) \right) = \{\text{Tr}(x), \text{Tr}(y)\}.
\]

\[\Box\]

In a similar way, we have

**Theorem 5.8 (Quantizations commute with trace maps: after reduction)** Let \(Q\) be a finite quiver and let \(d\) be a dimension vector such that the moment map \(\mu\) is flat. Then for any \(x, y \in (\Pi Q)_\sharp\), we have

\[
\Phi(\text{Tr}^q(\widehat{x} \ast \widehat{y} - \widehat{y} \ast \widehat{x})) = \{\text{Tr}(x), \text{Tr}(y)\}.
\]

In other words, the following diagram

\[
\begin{array}{ccc}
\mathcal{R}_q(N(Q)_\hbar, \hat{\mathbf{w}}) & \xrightarrow{\text{Tr}^q} & (D_\hbar(\text{Rep}(Q, d)))^{\text{GL}_d} \\
& \downarrow \text{Tr} & \downarrow \text{Tr} \\
(\Pi Q)_\sharp & \xrightarrow{\text{Tr}} & \mathbb{K}[\mathcal{M}_d(Q)]
\end{array}
\]

(16)

commutes, which means the trace map commutes with the quantization for the preprojective algebras.

**Proof** Without loss of generality, we prove this theorem for arbitrary elements

\[(x_1, 1)(x_2, 2) \cdots (x_r, r) + L_\hbar\]

and

\[(y_1, 1)(y_2, 2) \cdots (y_s, s) + L_\hbar\]

in \(\mathcal{R}_q(N(Q)_\hbar, \hat{\mathbf{w}})\), where \(x := x_1x_2 \cdots x_r\), \(y := y_1y_2 \cdots y_s\) are cyclic paths in \(\overline{Q}\) and

\[L_\hbar := N(Q)_\hbar(\mathbb{K}[\overline{Q}\mathbf{w}]) N(Q)_\hbar.\]

Here we denote the equivalent classes of \(x, y \in (\Pi Q)_\sharp\) by \(x + \ker pr\), \(y + \ker pr\) respectively.
Moreover, since the canonical projection

$$\text{Tr}^q((x_1, 1)(x_2, 2) \cdots (x_r, r) + L_h) = \left\langle tr \left( \prod_{i=1}^r [x_i] \right) \right\rangle,$$

where $\{tr(\prod_{i=1}^r [x_i])\}$ is the element in $(\mathcal{D}_h(\text{Rep}(Q, d)))^{G^{\mathcal{L}_d}}$ represented by the differential operator $tr(\prod_{i=1}^r [x_i])$. Similarly, we have

$$\text{Tr}^q((y_1, 1)(y_2, 2) \cdots (y_s, s) + L_h) = \left\langle tr \left( \prod_{i=1}^s [y_i] \right) \right\rangle.$$

By Proposition 5.2,

$$\left\langle tr \left( \prod_{i=1}^r [x_i] \right) \right\rangle \left\langle tr \left( \prod_{i=1}^s [y_i] \right) \right\rangle - \left\langle tr \left( \prod_{i=1}^s [y_i] \right) \right\rangle \left\langle tr \left( \prod_{i=1}^r [x_i] \right) \right\rangle = h \sum_{i, j_1, \ldots, u, v_1, \ldots} \left\langle \begin{array}{l}
\{ (x_1)_{y, j_1}, (y_1)_{u, v_1} \} [x_2]_{j_1, j_2} \cdots [x_r]_{j_{r-1}, i} [y_2]_{v_1, v_2} \cdots [y_s]_{v_{s-1}, u} + \ldots
\end{array} \right\rangle$$

$$+ \left\langle \begin{array}{l}
(x_2)_{j_1, j_2, \ldots} (y)_{u, v_1} \{ [x_1]_{i, j_1} \cdots [x_r]_{j_{r-1}, i} [y_2]_{v_1, v_2} \cdots [y_s]_{v_{s-1}, u} + \ldots
\end{array} \right\rangle$$

Moreover, since the canonical projection

$$\pi : (\mathbb{K}[T^*\text{Rep}(Q, d)])^{G^{\mathcal{L}_d}} \rightarrow \mathbb{K}[\mathcal{M}_d(Q)]$$

preserves Poisson brackets, we have

$$\left\{ \text{Tr}(x + \ker pr), \text{Tr}(y + \ker pr) \right\} = \left\{ \text{Tr} \circ pr(x), \text{Tr} \circ pr(y) \right\}$$

$$= \left\{ \pi \circ \text{Tr}(x), \pi \circ \text{Tr}(y) \right\}$$

$$= \pi \left\{ \text{Tr}(x), \text{Tr}(y) \right\}.$$
Proof of the commutativity of Diagram 3 The bottom and top diagrams are given by (13) and (14) respectively. The front and back diagrams are given by (15) and (16) respectively, and the left and right diagrams are given by (8) and (5) respectively.

Remark 5.9 In [1, 2], Alekseev, Kosmann-Schwarzbach, Malkin and Meinrenken introduced the notion of quasi-hamiltonian reduction for a symplectic space with a group-valued moment map. Furthermore, the (geometric) quantization commutes with the quasi-Hamiltonian reduction for these spaces (see [18] for details). In [21, 22], Van den Bergh showed there is a non-commutative version of quasi-hamiltonian reduction for bi-symplectic spaces (or more generally, for double Poisson spaces). It is very plausible that our result in this paper remains valid for quasi-hamiltonian reductions; that is, there exists a commutativity of the non-commutative quantization and the non-commutative quasi-hamiltonian reduction, at least for quiver algebras, which, via the trace maps, induces the commutativity of the quantization and the quasi-hamiltonian reduction on their representation spaces. As a potential application, we get a quantization of the character varieties from non-commutative geometry. We hope to address this problem in a future work.

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