ON THE TEMPORAL DECAY FOR THE 2D NON-RESISTIVE INCOMPRESSIBLE MHD EQUATIONS

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Abstract. Califano-Chiuderi [4] gave the numerical observation that the energy of the MHD equations is dissipated at a rate independent of the ohmic resistivity, which was first proved by [13] (Ren et al., J. Funct. Anal., 2014) (the initial data near \((0, \mathbf{e}_1), \mathbf{e}_1 = (1, 0)\)). Precisely, they showed some explicit decay rates of solutions in \(L^2\) norm. So a nature question is whether the obtained decay rates in [13] is optimal. In this paper, we aim at giving the explicit decay rates of solutions in both \(L^2\) norm and \(L^\infty\) norm. In particular, our decay rate in terms of \(L^2\) norm improves the previous work [13].

1. Introduction

In this paper, we are concerned with the Cauchy problem for the two-dimensional (2D) non-resistive incompressible MHD equations given by

\[
\begin{align*}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p &= \mathbf{H} \cdot \nabla \mathbf{H}, \quad (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \\
\partial_t \mathbf{H} + \mathbf{u} \cdot \nabla \mathbf{H} &= \mathbf{H} \cdot \nabla \mathbf{u}, \\
\text{div} \mathbf{u} = \text{div} \mathbf{H} &= 0, \\
\mathbf{u} \big|_{t=0} &= \mathbf{u}_0(x, y), \quad \mathbf{H} \big|_{t=0} = \mathbf{H}_0(x, y),
\end{align*}
\]

where \(\mathbf{u} = (u, v) \in \mathbb{R}^2\) stand for the 2D velocity field, \(p\) the pressure and \(\mathbf{H} = (H_1, H_2) \in \mathbb{R}^2\) the magnetic field. (1.1) can be applied to model plasmas when the plasmas are strongly collisional, or the resistivity since these collisions are extremely small, see [3] for more explanations to this model. For the MHD equations with both velocity dissipation and magnetic diffusion, [7] and [14] obtained the local and global well-posedness of solutions to that model, respectively. In both 2D and 3D, Chemin et al [5] showed the local existence of solutions to (1.1) with the initial data in critical Besov space (see [15] for the uniqueness of solutions in 2D). However, since there is no dissipation or damping in the equation for \(\mathbf{H}\), global well-posedness of smooth solutions to (1.1) even under small assumption of the initial data has become an issue that needed to be resolved.

Based on Lagrangian coordinates and the techniques on anisotropic Besov spaces, Lin, Xu and Zhang [8] first established the global well-posedness of small solutions after translating the magnetic field by a constant vector and assuming that the initial magnetic field satisfies sort of admissible condition. If we set \(\mathbf{H} = \mathbf{b} + \mathbf{e}_1\), where \(\mathbf{b} = (b, B)\), then the investigated model in [8] is

\[
\begin{align*}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p &= \mathbf{b} \cdot \nabla \mathbf{b} + \partial_x \mathbf{b}, \quad (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \\
\partial_t \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} &= \mathbf{b} \cdot \nabla \mathbf{u} + \partial_x \mathbf{u}, \\
\text{div} \mathbf{u} = \text{div} \mathbf{b} &= 0, \\
\mathbf{u} \big|_{t=0} &= \mathbf{u}_0(x, y), \quad \mathbf{b} \big|_{t=0} = \mathbf{b}_0(x, y).
\end{align*}
\]

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Meanwhile,
\[
\int (\tilde{b}_0 - \epsilon_1) (Z(t, \alpha)) dt = 0 \quad \text{for all } \alpha \in \mathbb{R}^2 \times \{0\}
\]
is the admissible condition, where $Z(t, \alpha)$ is determined by
\[
d\frac{d}{dt} Z(t, \alpha) = \tilde{b}_0 (Z(t, \alpha)), \quad Z(t, \alpha)|_{t=0} = \alpha.
\]
Later, by carefully exploiting the divergence structure of the velocity, Ren, Wu, Xiang and Zhang \[13\] removed (1.3) and obtained some decay estimates of solutions as follows:
\[
\|\partial_x^k \tilde{u}(t)\|_{L^2(\mathbb{R}^2)} + \|\partial_x^k \tilde{b}(t)\|_{L^2(\mathbb{R}^2)} \lesssim \langle t \rangle^{-\frac{5}{4} + \epsilon},
\]
where $\epsilon \in (0, 1/2)$ and $k = 0, 1, 2$. (1.4) confirms the numerical observation that the energy of the MHD equations is dissipated at a rate independent of the ohmic resistivity, see \[4\].
Zhang \[19\] gave a more elementary proof for the global existence and uniqueness of solutions. Motivated by \[2\], Zhang \[20\] also proved global well-posedness with large background magnetic field using the techniques in \[19\] and the classical method for the oscillatory integrals.

Global well-posedness and large time behavior of solutions to the 3D case have been recently treated in Abidi-Zhang \[11\] and Deng-Zhang \[6\], where the method also works for the 2D case. As a matter of fact, \[11\] showed
\[
\|\tilde{u}(t)\|_{H^2(\mathbb{R}^3)} + \|\tilde{b}(t)\|_{H^2(\mathbb{R}^3)} \lesssim \langle t \rangle^{-\frac{3}{4}},
\]
which corresponds to the case $\epsilon = 0$ in (1.3). By exploiting Hörmander’s version of Nash-Moser iteration scheme, \[6\] derived the decay rate of solutions in both $L^\infty$ and $L^2$ norms. In particular, the decay rate in the $L^2$ norm is optimal in sense that it coincides with that of the linear system. Indeed, the decay rate of solutions in \[6\] can be given as follows:
\[
\|\tilde{u}(t)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C \kappa \langle t \rangle^{-\frac{7}{4} + \kappa}, \quad \|\tilde{b}(t)\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C \kappa \langle t \rangle^{-\frac{7}{4} + \kappa},
\]
\[
\|\tilde{u}(t)\|_{H^2(\mathbb{R}^3)} + \|\tilde{b}(t)\|_{H^2(\mathbb{R}^3)} \leq C \langle t \rangle^{-\frac{1}{2}}, \quad \|\nabla \tilde{u}(t)\|_{L^2(\mathbb{R}^3)} \leq C \langle t \rangle^{-1}, \quad \|\nabla \tilde{b}(t)\|_{L^2(\mathbb{R}^3)} \leq C \langle t \rangle^{-1},
\]
where the positive constant $\kappa$ is sufficiently small provided that the regularity of solutions is large enough. We refer the interested reader to \[12\, 16\, 17\, 18\] and references therein for other related works.

Let
\[
V = (u, v, b, B), \quad V_0 = (u_0, v_0, b_0, B_0).
\]
We define $S_1 \times S_2$ with their norm as follows:
\[
\|\tilde{u}\|_{S_1} = \sup_{t \geq 0} \left\{ \|\tilde{u}\|_{H^N} + \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^N)} + \langle t \rangle^{\frac{1}{2}} \|\tilde{u}\|_{L^2} + \langle t \rangle^{\frac{3}{2}} \|\partial_y u\|_{L^2} + \langle t \rangle^{\frac{5}{2}} \|\partial_y u\|_{F_{L^1}} \right\};
\]
\[
\|\tilde{b}\|_{S_2} = \sup_{t \geq 0} \left\{ \|\tilde{b}\|_{H^N} + \|\nabla \tilde{b}\|_{L^2(\mathbb{R}^{N-1})} + \langle t \rangle^{\frac{1}{4}} \|b\|_{L^2} + \langle t \rangle^{\frac{1}{2}} \|\nabla^{-1} \nabla b\|_{F_{L^1}} + \|B\|_{L^2} \right\};
\]
\[
\|V\|_3 = \|\tilde{u}\|_{S_1} + \|\tilde{b}\|_{S_2}, \quad \|V_0\|_3 = \|V_0\|_{H^N} + \|V_0\|_{W^{5,1}},
\]
where $\|f\|_{F_{L^1}} = \|\tilde{f}\|_{L^1}, \mathcal{R}_1$ stands for the Riesz transform, the operators $|\nabla|$ and $\langle \nabla \rangle$ are standard.

Now, we give the main result of this paper.
Remark 1.2. (1) Due to which obviously improves (1.4). In fact, our method works for the case fulfill the decay rates in \(L^2\) then (1.2) has a unique global solution (Theorem 1.1. Let \(N \geq 8\) and \((\vec{u}_0, \vec{b}_0) \in H^N(\mathbb{R}^2) \cap W^{5,1}(\mathbb{R}^2)\) satisfying \(\text{div}\vec{u}_0 = \text{div}\vec{b}_0 = 0\). Then there exists a sufficiently small positive constant \(c_0\) such that if
\[
\|V_0\|_3 \leq c_0,
\]
then (1.2) has a unique global solution \((\vec{u}, \vec{b}) \in S_1 \times S_2\). Moreover,
\[
\|V\|_3 \lesssim c_0. \tag{1.6}
\]

Remark 1.2. (1) Due to \(\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}\), we can get the decay rates in \(L^\infty\) norm, and then fulfill the decay rates in \(L^p\) (\(2 \leq p \leq \infty\)) norm by interpolation. In particular,
\[
\|\partial_x^k \vec{u}(t)\|_{L^2(\mathbb{R}^2)} + \|\partial_x^k B(t)\|_{L^2(\mathbb{R}^2)} \lesssim c_0(t)^{-\frac{1+k}{2}}, \quad \|\partial_x^k b(t)\|_{L^2(\mathbb{R}^2)} \lesssim c_0(t)^{-\frac{1+2k}{4}}, \quad k = 0, 1, \tag{1.7}
\]
which obviously improves (1.4). In fact, our method works for the case \(k \geq 2\) in (1.7), but we choose not to pursue on this direction here.
(2) Here the \(L^p\) (\(2 \leq p \leq \infty\)) norm of \(B\) decays faster than the associated norm of \(b\), whereas this type result is not proved in [13].
(3) Our idea seems hard to be applied for the 3D case, since the divergence structure of the velocity field in this case can not be effectively used.

Remark 1.3. Comparing with the result on the 2D compressible MHD equations [17], we can see that the estimates in \(H^N\) norm do not grow over time is the principal difference.

Formally, the approach in the present work is similar to the works [16] and [17], but there are many differences in the proof. Let us now outline some principal differences between [16] [17] and the present work.

In [16], Wu-Wu-Xu considered 2D incompressible MHD equations with only a velocity damping term when the initial data is close to \((0, \vec{e}_1)\). Not only did they obtain the global well-posedness of solutions, but also some decay estimates of solutions. The part on the velocity (i.e., \(\vec{b} = 0\)) is the 2D incompressible damped Euler equations, whose solution even in \(L^2\) norm has exponential decay estimate by only using energy method provided that the initial data is sufficiently small. Here the part on the velocity (i.e., \(\vec{b} = 0\) in [11]) is the 2D incompressible Navier-Stokes equations. However, the decay rate of solutions to this model are polynomial. In particular, the decay rate of solution in \(L^2\) norm is slower than \(\langle t \rangle^{-1}\). By the virtue that it is not integrable, it seems more difficult than [16] to obtain global well-posedness of solutions.

Besides the way dealing with the pressure, there are some other differences between our work and [17]. In our work, we can make use of the structure of incompressibility of the fluid to control the \(H^N\) estimate of solutions by some special norms, the decay rate of which is integrable, and then achieve the goal that the \(H^N\) estimates of solutions do not grow over time. In this process, it is the new decay estimate of \(\partial_x u\) in \(\mathcal{F}L^1\) norm (or \(L^\infty\) norm) that plays an important role, while this idea also works for the model studied in [10]. On the other hand, we need to establish some other new decay estimates including
\[
\|\|\nabla\|^{-1}\langle \nabla \rangle b(t)\|_{\mathcal{F}L^1} \lesssim \langle t \rangle^{-\frac{1}{2}}, \quad \|B(t)\|_{\mathcal{F}L^1} \lesssim \langle t \rangle^{-1}. \tag{1.8}
\]

However, this type goal seems hard to be fulfilled for the compressible model in [17].

Let us do some comments on the proof and our idea. Firstly, all previous works dealing with the 2D incompressible or compressible cases applied the magnetic potential equation for \(\phi\) defined by \(\vec{H} = (\partial_y \phi, -\partial_x \phi)\), here we do not introduce this magnetic potential equation any more. Our idea is considering (1.2) as two subsystems (3.1) and (3.2), and then using the method of diagonalization via the eigenvalues and eigenvectors to these subsystems. Secondly,
by using the structure of system and integration by parts many times, we can use the integral
\[
\int_0^t \left\| v \right\|_{L^\infty}^2 + \left\| B \right\|_{L^\infty}^2 + \left\| \partial_x u \right\|_{L^\infty} d\tau
\]
to control the \( H^N \) estimate of solutions. Thanks to the fast decay rate of these special norms: \( \left\| v \right\|_{L^\infty}, \left\| B \right\|_{L^\infty} \) and \( \left\| \partial_x u \right\|_{L^\infty} \), we can obtain the \( H^N \) estimate of solutions (uniformly in time). Thirdly, we shall establish some new decay estimate like (1.8) to get the different large-time behavior of \( b \) and \( B \) in \( \mathcal{F}L^1 \) norm (or \( L^\infty \) norm), which is not obtained for the compressible MHD equations in [17].

**Proof of Theorem 1.1.** Thanks to [5] and [15], one can easily get the local well-posedness of solutions to (1.2). Claim :
\[
\left\| V \right\|_3 \lesssim \left\| V_0 \right\|_3 + \left\| V_0 \right\|_3^2 + \left\| V \right\|_3^3 + \left\| V^3 \right\|_3,
\]
the proof of which is provided at the end of the ninth section, then we can conclude the proof of Theorem 1.1 by the standard continuity argument.

The present paper is structured as follows:
In the second section, we provide the definitions of some operators and some spaces. The third section devotes to giving the integral representation of solutions. The fourth section gives several decay estimates on some operators and nonlinear decay estimates, which is an essential part in this paper. From the fifth section to the ninth section, we devote to showing (1.9). In the Appendix, we give the proofs of some lemmas which are used in the previous parts.

Let us complete this section by describing the notations we shall use in this paper.

**Notations** We use \( A \lesssim B \) to denote the statement that \( A \leq CB \) for some absolute constant \( C > 0 \). \( A \approx B \) means \( A \lesssim B \) and \( B \lesssim A \). \( \langle t \rangle \) means \( \sqrt{1 + t^2} \). We use \( \mathcal{R}_{ij} = \mathcal{R}_i \mathcal{R}_j \), where \( \mathcal{R}_i \) and \( \mathcal{R}_j \) stand for the Riesz transform. We shall denote by \( (a|b) \) the \( L^2 \) inner product of \( a \) and \( b \), and
\[
(a|b)_{H^s} \overset{def}{=} (|\nabla|^sa|\nabla|^sb), \quad (a|b)_{H^m} \overset{def}{=} (\partial^m a|\partial^m b) \quad (m \text{ is an integer}), \quad (a|b)_{H^s} \overset{def}{=} (a|b) + (a|b)_{H^s}.
\]

2. Preliminaries

The fractional Laplacian operator \( |\nabla|^a = (-\Delta)^{a/2} \) is defined through the Fourier transform, namely,
\[
\hat{|\nabla|^a f}(\xi, \eta) = |\xi|^a \hat{f}(\xi, \eta),
\]
where \( \xi = (\xi, \eta) \) and the Fourier transform is given by
\[
\hat{f}(\xi, \eta) \overset{def}{=} \int_{\mathbb{R}^2} e^{-i(x\xi + y\eta)} f(x, y) d\xi d\eta.
\]
We also use \( \mathcal{F}\{f\} \) to stand for the Fourier transform for some convenience. We define
\[
\|f\|_{\mathcal{F}L^p} \overset{def}{=} \|\hat{f}\|_{L^p}, \quad 1 \leq p \leq \infty.
\]

Let \( \psi(\xi) \) be a smooth bump function adapted to \( \{\xi \leq 2\} \) and equal to 1 on \( \{\xi \leq 1\} \). For \( N > 0 \), we define the Fourier multipliers
\[
\hat{P}_{\leq N} f(\xi, \eta) = \psi(\xi/N, \eta/N) \hat{f}(\xi, \eta), \quad \hat{P}_{> N} f(\xi, \eta) = (1 - \psi(\xi/N, \eta/N)) \hat{f}(\xi, \eta), \quad \hat{P}_N f(\xi, \eta) = (\psi(\xi/N, \eta/N) - \psi(2\xi/N, 2\eta/N)) \hat{f}(\xi, \eta),
\]
and $P_{<N}$ and $P_{\geq N}$ can be defined similarly. We also define

$$P_{M<\leq N} = P_{<N} - P_{\leq M}$$

when $M < N$. We will usually apply these multipliers when $N$ and $M$ are dyadic numbers (i.e., of the form $2^k$ in general). In particular, all summation over $N$ are understood to be over dyadic numbers.

When $1 \leq p \leq \infty$, we define

$$\|f\|_{W^{k,p}(\mathbb{R}^2)} \overset{def}{=} \|f\|_{L^p(\mathbb{R}^2)} + \|\nabla |f|\|_{L^p(\mathbb{R}^2)} \quad (k > 0), \quad \|f\|_{W^{k,p}(\mathbb{R}^2)} \overset{def}{=} \|\nabla |f|\|_{L^p(\mathbb{R}^2)} \quad (k \in \mathbb{Z}).$$

For the special case $p = 2$, $W^{k,p}(\mathbb{R}^2)$ and $W^{k,p}(\mathbb{R}^2)$ reduces to $H^k(\mathbb{R}^2)$ and $H^k(\mathbb{R}^2)$, respectively.

The following two lemmas provide Bernstein’s inequality and product estimate.

**Lemma 2.1.** For $1 \leq p < q < \infty$ and $N > 0$, then

$$\|\nabla |f|\|_{L^p(\mathbb{R}^2)} \approx N^{s} \|f\|_{L^p(\mathbb{R}^2)},$$

$$\|P_N P_{\leq N} f\|_{L^p(\mathbb{R}^2)} \lesssim N^{s} \|N f\|_{L^p(\mathbb{R}^2)}.$$

**Lemma 2.2** (Product estimate [11]). Let $n \geq 1$, $s > 0$, $1 \leq p, r \leq \infty$, then

$$\|\nabla |f|\|_{L^p(\mathbb{R}^n)} \lesssim C \left\{ \|f\|_{L^p(\mathbb{R}^n)} \|\nabla |g|\|_{L^r(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)} \|\nabla |f|\|_{L^r(\mathbb{R}^n)} \right\},$$

(2.1) where $1 \leq p_1, r_1 \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$.

At last, we list some basic inequalities including classical estimates of solution to the Heat equation. Since the proof is easy, we omit it.

**Lemma 2.3.** Let $n = 1, 2$. (1) If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then

$$\| e^{t\Delta} f \|_{L^2(\mathbb{R}^n)} \lesssim t^{-\frac{n}{2}} \| f \|_{L^1(\mathbb{R}^n)} \quad \| e^{t\Delta} f \|_{F^1(\mathbb{R}^n)} \lesssim \min \{ t^{-\frac{n}{2}} \| f \|_{L^1(\mathbb{R}^n)}, t^{-\frac{n}{4}} \| f \|_{L^2(\mathbb{R}^n)} \}. \quad (2.2)$$

(2) Let $\varepsilon > 0$, $\sigma > \frac{n}{2}$ and $1 \leq r \leq 2$, then

$$\| f \|_{L^{\sigma,r}(\mathbb{R}^n)} \lesssim \| \hat{f} \|_{L^r(\mathbb{R}^n)} \quad \| f \|_{L^2(\mathbb{R}^n)} \approx \| \hat{f} \|_{L^2(\mathbb{R}^n)},$$

$$\|fg\|_{H^s(\mathbb{R}^n)} \lesssim \| f \|_{H^s(\mathbb{R}^n)} \| g \|_{H^s(\mathbb{R}^n)}, \quad \|f\|_{L^1(\mathbb{R}^n)} \| g \|_{L^1(\mathbb{R}^n)} \| f \|_{L^1(\mathbb{R}^n)} \| g \|_{L^1(\mathbb{R}^n)},$$

$$\| \hat{f} \|_{L^r(\mathbb{R}^n)} \lesssim \| \hat{\nabla} \|^{\frac{n}{2} + 1} f \|_{L^r(\mathbb{R}^n)}.$$

3. The integral representation of solutions

In this section, we shall obtain the integral representation of solutions to (1.2). Let us investigate the spectrum properties to the following two systems:

$$\begin{cases}
\partial_t u - \Delta u = \partial_x b + F^1, \\
\partial_t b = \partial_x u + G^1,
\end{cases} \quad (3.1)$$

and

$$\begin{cases}
\partial_t v - \Delta v = \partial_x b + F^2, \\
\partial_t B = \partial_x v + G^2,
\end{cases} \quad (3.2)$$

where

$$p = -\Delta^{-1} \text{div}(\vec{u} \cdot \nabla \vec{u} - \vec{b} \cdot \nabla \vec{B}).$$
Denote \( \tilde{\xi} = (\xi, \eta) \) and
\[
A = \begin{pmatrix}
-|\tilde{\xi}|^2 & -i\xi \\
-i\xi & 0
\end{pmatrix},
\]
then the eigenvalues of the matrix \( A \) can be given by
\[
\lambda_{\pm} = \begin{cases}
-|\tilde{\xi}|^2 + \sqrt{|\tilde{\xi}|^4 - 4|\xi|^2}, & \text{when } |\xi| < \frac{|\tilde{\xi}|^2}{2}, \\
-|\tilde{\xi}|^2 - \sqrt{4|\xi|^2 - |\tilde{\xi}|^4}, & \text{when } |\xi| \geq \frac{|\tilde{\xi}|^2}{2},
\end{cases}
\]
where \( i = \sqrt{-1} \). After using Fourier transform, (3.1) and (3.2) reduces to
\[
\partial_t \begin{pmatrix}
\hat{u} \\
\hat{v}
\end{pmatrix} (\hat{\xi}) = A \begin{pmatrix}
\hat{u} \\
\hat{v}
\end{pmatrix} (\hat{\xi}) + \begin{pmatrix}
\hat{F}^1 \\
\hat{G}^1
\end{pmatrix} (\hat{\xi}) \quad (3.3)
\]
and
\[
\partial_t \begin{pmatrix}
\hat{v} \\
\hat{B}
\end{pmatrix} (\hat{\xi}) = A \begin{pmatrix}
\hat{v} \\
\hat{B}
\end{pmatrix} (\hat{\xi}) + \begin{pmatrix}
\hat{F}^2 \\
\hat{G}^2
\end{pmatrix} (\hat{\xi}). \quad (3.4)
\]
It follows by using the method of diagonalization via the eigenvalues and eigenvectors, (3.3) and (3.4) that
\[
\tilde{b}(t) = M_1(\partial, t)\bar{u}_0 + M_2(\partial, t)\bar{v}_0 + \int_0^t M_1(\partial, t - \tau)\hat{F}d\tau + \int_0^t M_2(\partial, t - \tau)\hat{G}d\tau \quad (3.5)
\]
and
\[
\tilde{u}(t) = M_3(\partial, t)\bar{u}_0 + M_1(\partial, t)\bar{v}_0 + \int_0^t M_3(\partial, t - \tau)\hat{F}d\tau + \int_0^t M_1(\partial, t - \tau)\hat{G}d\tau, \quad (3.6)
\]
where
\[
\hat{F} = (F^1, F^2), \quad \hat{G} = (G^1, G^2), \quad \hat{M}_i(\tilde{\xi}, t) = \hat{M}_i(\xi, t) \hat{f}(\tilde{\xi}), \quad i = 1, 2, 3
\]
and
\[
(\hat{M}_1(\tilde{\xi}, t), \hat{M}_2(\tilde{\xi}, t), \hat{M}_3(\tilde{\xi}, t)) \overset{\text{def}}{=} (i\xi \frac{\lambda_+ - \lambda_{-t}}{\lambda_+ - \lambda_-}, \frac{\lambda_+ e^{\lambda_{-t} t} - \lambda_{-t} e^{\lambda_+ t}}{\lambda_+ - \lambda_-}, \frac{\lambda_{-t} e^{\lambda_+ t} - \lambda_+ e^{\lambda_{-t} t}}{\lambda_+ - \lambda_-}). \quad (3.7)
\]
Notice that
\[
\partial_t M_2 = \partial_{\xi} M_1, \quad \partial_t M_1 = \partial_{\xi} M_3, \quad (3.8)
\]
which is useful in the following context. To bound \( M_i(\partial, t) \), we split the whole space \( \mathbb{R}^2 \) into four regions:
\[
D_1 \overset{\text{def}}{=} \{ \xi \in \mathbb{R}^2 : |\xi| \geq |\tilde{\xi}|^2 \},
\]
\[
D_2 \overset{\text{def}}{=} \{ \xi \in \mathbb{R}^2 : \frac{|\tilde{\xi}|^2}{2} \leq |\xi| < |\tilde{\xi}|^2 \},
\]
\[
D_3 \overset{\text{def}}{=} \{ \xi \in \mathbb{R}^2 : \frac{|\tilde{\xi}|^2}{4} \leq |\xi| < \frac{|\tilde{\xi}|^2}{2} \},
\]
\[
D_4 \overset{\text{def}}{=} \{ \xi \in \mathbb{R}^2 : |\xi| < \frac{|\tilde{\xi}|^2}{4} \}. \quad (3.9)
\]
In order to help us establish some estimates of solutions, \( D_4 \) will be seen as two sets in many times, namely,
\[
D_4 = D_{41} \cup D_{42}, \quad D_{41} = D_4 \cap \{ \xi : |\xi| \geq 1 \}, \quad D_{42} = D_4 \cap \{ \xi : |\xi| < 1 \}. \quad (3.10)
\]
Due to the definition in (3.9), it is easy to get
\[
|\bar{\xi}| \leq 1 \text{ when } \tilde{\xi} \in D_1 \cup D_2 \cup D_3, \quad (3.11)
\]
so that we can bound some estimates on $D_i$ $(i = 1, 2, 3)$ under low regularity assumption.

Next, a proposition devoting to the estimates of $\tilde{M}_i(\xi, t)$ is given.

**Proposition 3.1.** $\tilde{M}_i(\xi, t)$ $(i = 1, 2, 3)$ defined by (3.7) satisfies the following estimates:

1. If $\xi \in D_1$,
   \begin{equation}
   |\tilde{M}_1(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}}, \quad |\tilde{M}_2(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}}, \quad |\tilde{M}_3(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}}; \quad (3.12)
   \end{equation}
2. If $\xi \in D_2$,
   \begin{equation}
   |\tilde{M}_1(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}}, \quad |\tilde{M}_2(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}}, \quad |\tilde{M}_3(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}};
   \end{equation}
3. If $\xi \in D_3$,
   \begin{equation}
   |\tilde{M}_1(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}}, \quad |\tilde{M}_2(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}}, \quad |\tilde{M}_3(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}};
   \end{equation}
4. If $\xi \in D_4$,
   \begin{equation}
   |\tilde{M}_1(\xi, t)| \lesssim \frac{1}{|\xi|} e^{|\xi|} e^{-\frac{\xi^2}{4t}}, \quad |\tilde{M}_2(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}}, \quad |\tilde{M}_3(\xi, t)| \lesssim e^{-\frac{\xi^2}{4t}} + \frac{\xi^2}{|\xi|^4} e^{-\frac{\xi^2}{4t}}. \quad (3.13)
   \end{equation}

**Proof.** (1) Since $\xi \in D_1$, we have
   \[ |\lambda_+ - \lambda_-| \approx |\lambda_\pm| \approx |\xi|, \quad |e^{\lambda \pm t}| \lesssim e^{-\frac{\xi^2}{4t}}, \]
   which yields the desired estimate (3.12) by some basic computations.

(2) In $D_2$, we have $|\lambda_\pm| = |\xi|$ and $|\xi| \approx |\xi|^2$. For $\tilde{M}_1(\xi, t)$, using $|\sin x| \leq |x|$, we have
   \[ |\tilde{M}_1(\xi, t)| \lesssim |\xi| e^{-\frac{\xi^2}{4t}} \|e^{\lambda - t} + \frac{|\lambda_-|}{|\xi|} e^{-\frac{\lambda_\pm t}{|\xi|^2}}\| \lesssim |\xi| e^{-\frac{\xi^2}{4t}} \lesssim \frac{|\xi|}{|\xi|^2} e^{-\frac{\xi^2}{4t}} \lesssim e^{-\frac{\xi^2}{4t}}, \]
   which, together with
   \[ |\tilde{M}_2(\xi, t)| \leq |e^{\lambda - t}| + \frac{|\lambda_-|}{|\xi|} |\tilde{M}_1(\xi, t)|, \quad |\tilde{M}_3(\xi, t)| \leq |e^{\lambda - t}| + \frac{|\lambda_+|}{|\xi|} |\tilde{M}_1(\xi, t)|, \quad (3.14) \]
   yields the desired result.

(3) In $D_3$, one has
   \[ |\xi| \approx |\xi|^2, \quad \lambda_- \in (-|\xi|^2, -\frac{|\xi|^2}{2}), \quad \lambda_+ \geq \lambda_- \]
   and
   \[ e^{\lambda_+ t} - e^{\lambda_- t} \leq e^{\lambda_+ t}(\lambda_+ - \lambda_-) t, \quad \lambda_+ = -\frac{2\xi^2}{|\xi|^2 + \sqrt{|\xi|^4 - 4\xi^2}} \in (-\frac{|\xi|^2}{2}, -\frac{|\xi|^2}{16}], \]
   which follows
   \[ |\tilde{M}_1(\xi, t)| \lesssim |\xi| \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \lesssim |\xi| e^{\lambda_- t} \lesssim \frac{|\xi|}{|\xi|^2} e^{-\frac{\xi^2}{4t} t} \lesssim e^{-\frac{\xi^2}{4t} t}. \]
   Thanks to (3.14), we can get the desired estimates.

(4) In $D_4$, we have
   \[ \lambda_+ - \lambda_- \in (\frac{\xi^2}{\sqrt{2}}, |\xi|^2), \quad \lambda_- \in (-|\xi|^2, -\frac{|\xi|^2}{2}), \quad \lambda_+ \in (-\frac{2\xi^2}{|\xi|^2}, -\frac{\xi^2}{|\xi|^2}), \quad \lambda_- \geq -\frac{|\xi|^2}{2}, \]

yielding $e^{\lambda t} \leq e^{-\frac{c^2}{|\xi|^2} t}$. So we have

$$|\hat{M}_1(\xi, t)| \lesssim \frac{|\xi|}{|\xi|^2} (e^{\lambda^* t} + e^{-\lambda t}) \lesssim \frac{|\xi|}{|\xi|^2} e^{-\frac{c^2}{|\xi|^2} t},$$

$$|\hat{M}_2(\xi, t)| \lesssim e^{-\lambda t} + |\lambda_-| e^{\lambda^* t - e^{-\lambda t}} \lesssim e^{-\frac{c^2}{|\xi|^2} t},$$

$$|\hat{M}_3(\xi, t)| \lesssim e^{-\lambda t} + |\lambda_+| e^{\lambda^* t - e^{-\lambda t}} \lesssim e^{-\frac{c^2}{|\xi|^2} t} + \frac{\xi^2}{|\xi|^4} e^{-\frac{c^2}{|\xi|^2} t}.$$

\[ \square \]

**Remark 3.2.** Since the estimates of $\hat{M}_i(\xi, t)$ ($i = 1, 2, 3$) in $D_1, D_2$ and $D_3$ are similar, we will give the detailed estimate on $D_1$ and omit the details of the estimate on $D_2 \cup D_3$.

4. Decay estimates and Nonlinear estimates

4.1. Decay estimate I.

**Lemma 4.1.** Let $k \geq 0$, $c > 0$ and $0 < \alpha \leq 1$, there holds

1) \( \| \nabla^k e^{ct\Delta} f \|_{L^2(D_1)} \lesssim \min \{ \langle t \rangle^{-\frac{k+1}{2}} \| f \|_{L^1}, \langle t \rangle^{-\frac{k+1}{2}} \| f \|_{L^2}, \langle t \rangle^{-\frac{1}{2}} \| f \|_{L^2} \}; \)

2) \( \| \nabla^k e^{ct\Delta} f \|_{L^2(D_1)} \lesssim \min \{ \langle t \rangle^{-\frac{k}{2}} \| f \|_{H^k}, \langle t \rangle^{-\frac{k+1}{2}} (\| f \|_{L^1} + \| \nabla^k f \|_{L^2}) \}; \)

3) \( \| \nabla^k e^{ct\Delta} f \|_{L^1(D_1)} \lesssim \min \{ \langle t \rangle^{-\frac{k+1}{2}} \| f \|_{L^1}, \langle t \rangle^{-\frac{k+1}{2}} \| \nabla f \|_{L^2}, \langle t \rangle^{-\frac{1}{2}} \| f \|_{L^2} \}; \)

4) \( \| \nabla^k e^{ct\Delta} f \|_{L^1(D_1)} \lesssim \langle t \rangle^{-\frac{1}{2}} \min \{ \langle t \rangle^{-1} \| f \|_{L^1} + \| \nabla f \|_{L^1}, \langle t \rangle^{-\frac{1}{2}} \| f \|_{L^2} \}; \)

5) \( \| \nabla^{-\alpha} e^{ct\Delta} f \|_{L^1(D_1)} \lesssim \langle t \rangle^{-\frac{2\alpha}{2}} \| f \|_{L^1}. \)

where $D_1$ and $D_4$ are defined by (2.9).

**Proof.** It suffices to consider the case $k = 0$, since we can easily get

\( \| \nabla^k e^{ct\Delta} f \|_{L^r(D_1)} \lesssim \langle t \rangle^{\frac{1}{2}} \| e^{ct\Delta} f \|_{L^r(D_1)}, \)

\( \| \nabla^k e^{ct\Delta} f \|_{L^r(D_1)} \lesssim \langle t \rangle^{\frac{1}{2}} (\| e^{ct\Delta} f \|_{L^r(D_1)} + \| \nabla^k f \|_{L^r}). \)

If we can show (4.1) and

\( \| e^{ct\Delta} f \|_{L^r(D_1)} \lesssim \langle t \rangle^{\frac{1}{2}} \| f \|_{L^r(D_1)}, \)

then other estimates can be proved by using (2.2), Hölder’s inequality, Plancherel’s identity and (4.2). Next, we focus on the estimates of (4.1) and (4.2). Using polar coordinate

\( \xi = r \cos \theta, \ \eta = r \sin \theta, \ \theta \in [0, 2\pi], \)

we have

\( \| \nabla^{-\alpha} e^{ct\Delta} f \|_{L^1(D_1)} \lesssim \int_{\theta \in [0, 2\pi]} d\theta \int_{0 \leq r \leq 1} e^{-c\alpha r^2} r^{1-\alpha} |\tilde{f}(r, \theta)| dr \)

\( \lesssim \min \{ 1, t^{\frac{1-\alpha}{2}} \int_{0 \leq r \leq 1} e^{-\frac{r^2}{2}} dr \} \| \tilde{f} \|_{L^\infty} \)

\( \lesssim \langle t \rangle^{-\frac{2\alpha}{2}} \| f \|_{L^1}. \)
By $|\xi|^2 = \xi^2 + \eta^2$ and Hölder’s inequality,

$$\|e^{ct\Delta} f\|_{L^r(D_1)} \lesssim \|e^{-ct\xi^2}e^{-ct\xi^2 \hat{f}}\|_{L^r(D_1)} \lesssim \|e^{-ct\xi^2}\|_{L^4(\eta \leq 1)} \|e^{-ct\xi^2}\|_{L^{\frac{8}{1}}(\xi \leq 1)} \|\hat{f}\|_{L^2(L^\infty)} \lesssim (t)^{\frac{1}{2} - \frac{1}{r}} \|f\|_{L^1(L^2)}.$$  

Thanks to the above estimates, we conclude the estimates of (4.1) and (4.2). □

4.2. Decay estimate II.

**Lemma 4.2.** Let $1 \leq r \leq 2$, $1 \leq q \leq 2$ and $1/p + 1/q > 1$. For all $k \geq 0$ and $\delta > 0$, there hold

1) $I_1 = \|G_{k,e^{-G_2 z t}}\|_{L^r(D_4)} \lesssim \langle t \rangle^{\frac{1}{2}} \min \{\|\hat{f}\|_{L^r}, \langle t \rangle^{\frac{1}{2}} \|\langle \nabla \rangle^{\frac{1}{2} + \delta} f\|_{L^4(L^p)}\}$;

2) $I_2 = \|G_{k+1,e^{-G_2 z t}}\|_{L^1(D_4)} \lesssim \langle t \rangle^{\frac{1}{2} - \frac{1}{2}} \|\langle \nabla \rangle^{\frac{1}{2} + \delta} f\|_{L^1(L^p)} + \|f\|_{L^2}$;

3) $I_3 = \|G_{k+1,e^{-G_2 z t}}\|_{L^1(D_4)} \lesssim \langle t \rangle^{\frac{1}{2} - \frac{1}{2}} \|\hat{f}\|_{L^1} + \|f\|_{L^2}$;

4) $I_4 = \|G_{k+1,e^{-G_2 z t}}\|_{L^1(D_4)} \lesssim \langle t \rangle^{\frac{1}{2} - \frac{1}{2}} \|\hat{f}\|_{L^1} + \|f\|_{L^2}$;

5) $I_5 = \|G_{k+1,3e^{-G_2 z t}}\|_{L^1(D_4)} \lesssim \langle t \rangle^{\frac{1}{2} - \frac{1}{2}} \|\hat{f}\|_{L^1} + \|f\|_{L^2}$;

6) $I_6 = \|G_{k+1,3e^{-G_2 z t}}\|_{L^1(D_4)} \lesssim \min\{t^{\frac{1}{2} + \frac{1}{2} - \frac{1}{2}} \|f\|_{L^4(L^p)}; \|f\|_{L^1}, \langle t \rangle^{\frac{1}{2} - \frac{1}{2}} \|\langle \nabla \rangle^{\frac{1}{2} + \delta} f\|_{L^1}\}$;

7) $I_7 = \|G_{k+1,3e^{-G_2 z t}}\|_{L^1(D_4)} \lesssim \langle t \rangle^{\frac{1}{2} - \frac{1}{2}} \|\hat{f}\|_{L^1} + \|f\|_{L^2}$.

where $G_{k,t} = \frac{e^{\xi k}}{L^r}$, $D_4$ and $D_{42}$ are defined by (3.3) and (3.10).

**Proof.** Using $A^k e^{-A^2 t} \lesssim t^{-\frac{1}{2}}$, $\forall \ A > 0$, it is easy to get

$$I_1 \lesssim \langle t \rangle^{-\frac{1}{2}} \|e^{-\xi^2/2 t^2} f\|_{L^r(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}} \|\hat{f}\|_{L^r}.$$  

(4.4)

Using dyadic decomposition and Hölder’s inequality, we have

$$\|e^{-\xi^2/2 t^2} f\|_{L^r(D_4)} \lesssim \sum_{M \geq 0} \|e^{-\xi^2 M^{-2} t \hat{M} f}\|_{L^r} \lesssim \sum_{M \geq 0} \|e^{-\xi^2 M^{-2} t \hat{M} f}\|_{L^r} \|P_M f\|_{L^6(L^\infty)}$$

$$\lesssim t^{-\frac{1}{2r}} \sum_{M \geq 0} M^{-\frac{2}{r}} \|P_M f\|_{L^6(L^\infty)} \left( \int_{|\eta| \leq M} d\eta \right)^{\frac{1}{2r}}$$

$$\lesssim t^{-\frac{1}{2r}} \sum_{M \geq 0} M^{-\frac{2}{r} - \frac{1}{2}} \|P_M f\|_{L^6(L^\infty)}$$

$$\lesssim t^{-\frac{1}{2r}} \left( \sum_{M \geq 1} M^{-\frac{2}{r} - \frac{1}{2}} \|\hat{f}\|_{L^6(L^\infty)} \right)$$

$$+ \sum_{0 \leq M < 1} M^{-\frac{2}{r} - \frac{1}{2}} \|\hat{f}\|_{L^6(L^\infty)}$$

$$\lesssim t^{-\frac{1}{2r}} \|\langle \nabla \rangle^{\frac{1}{2} - \frac{1}{2}} f\|_{L^6(L^\infty)} \left( \sum_{M \geq 1} M^{-2} + \sum_{0 \leq M < 1} M^{-\frac{2}{r} - \frac{1}{2}} \right)$$

$$\lesssim t^{-\frac{1}{2r}} \|\langle \nabla \rangle^{\frac{1}{2} - \frac{1}{2}} f\|_{L^6(L^\infty)}.$$
So
\[ I_1 \lesssim t^{-\frac{2}{3} - \frac{1}{2} + \frac{k}{3}} \left\| \langle \nabla \rangle^{\frac{1}{2} - \frac{1}{3} + \delta} f \right\|_{L^1_x(L^2_y)}, \]
which, along with (3.14) and \( \| \hat{f} \|_{L^1} \lesssim \| \langle \nabla \rangle^{\frac{1}{2} - \frac{1}{3} + \delta} f \|_{L^1_x(L^2_y)} \) yields (3.3). Thanks to (3.10), we have
\[ I_2 \leq \| G_{k,k+1} e^{-\frac{\xi^2}{2t^2}} \hat{f} \|_{L^1(D_{41})} + \| G_{k,k+1} e^{-\frac{\xi^2}{2t^2}} \hat{f} \|_{L^1(D_{42})} = I_{21} + I_{22}. \]
By (3.3) for \( r = 1 \), one has
\[ I_{21} \lesssim \| G_{k,k} e^{-\frac{\xi^2}{2t^2}} \hat{f} \|_{L^1(D_{41})} \lesssim \langle t \rangle^{-\frac{k+1}{2}} \left\| \langle \nabla \rangle^{\frac{1}{2} + \delta} f \right\|_{L^1_x(L^2_y)}. \]
By dyadic decomposition and Hölder’s inequality, we infer
\begin{align*}
I_{22} &\lesssim \sum_{M \leq 1} \left\| e^{\frac{-\xi^2}{2t^2}} \langle \xi \rangle^k M^{-k-1} e^{-c \xi^2 M^{-2t}} P_M f \right\|_{L^1(D_{42})} \\
&\lesssim t^{-\frac{k}{2}} \sum_{M \leq 1} M^{-1} e^{-\frac{\xi^2}{2t^2} M^{-2t}} \| \hat{P}_M f \|_{L^1(D_{42})} \\
&\lesssim t^{-\frac{k}{2}} \sum_{M \leq 1} M^{-1} e^{-\frac{\xi^2}{2t^2} M^{-2t}} \| \hat{P}_M f \|_{L^2} \int_{|\eta| \leq M} d\eta \\
&\lesssim t^{-\frac{k+1}{2}} \left\| \hat{f} \right\|_{L^2} \sum_{M \leq 1} M \lesssim t^{-\frac{k+1}{2}} \| f \|_{L^1},
\end{align*}
which, together with \( I_{22} \lesssim \| \hat{f} \|_{L^2} \| \langle \xi \rangle^{-1} \|_{L^1(|\xi| < 1)} \lesssim \| f \|_{L^1} \) leads
\[ I_{22} \lesssim \langle t \rangle^{-\frac{k+1}{2}} \| f \|_{L^1}. \]
Combining with the estimates of \( I_{21} \) and (4.5) can yield the desired result.
Using (3.10) again,
\[ I'_2 \leq \| G_{k+1,k+2} e^{-\frac{\xi^2}{2t^2}} \hat{f} \|_{L^1(D_{41})} + \| G_{k+1,k+2} e^{-\frac{\xi^2}{2t^2}} \hat{f} \|_{L^1(D_{42})} = I'_{21} + I'_{22}. \]
It is easy to obtain
\[ I'_{21} \lesssim \| G_{k+1,k+2} e^{-\frac{\xi^2}{2t^2}} \hat{f} \|_{L^1(D_{41})} \lesssim \langle t \rangle^{-\frac{k+1}{2}} \left\| \hat{f} \right\|_{L^1}. \]
Using dyadic decomposition, \( |\xi| \lesssim |\xi|^2 \) in \( D_4 \) and Hölder’s inequality, we have
\begin{align*}
I'_{22} &\lesssim \sum_{M \leq 1} \left\| e^{\frac{-\xi^2}{2t^2}} \langle \xi \rangle^{k+1} M^{-k-2} e^{-c \xi^2 M^{-2t}} P_M f \right\|_{L^1(D_{42})} \\
&\lesssim t^{-\frac{k+1}{2} - \frac{1}{2} + \frac{k}{3}} \sum_{M \leq 1} \left\| e^{\frac{-\xi^2}{2t^2}} \langle \xi \rangle^{k+1} M^{-k-2} e^{-c \xi^2 M^{-2t}} P_M f \right\|_{L^1(D_{42})} \\
&\lesssim t^{-\frac{k+1}{2} - \frac{1}{2} + \frac{k}{3}} \sum_{M \leq 1} M^{-2} \left\| e^{-\frac{\xi^2}{2t^2} M^{-2t}} \| \hat{P}_M f \|_{L^2} (\int_{|\eta| \leq M} d\eta)^{\frac{1}{2}} \right\|_{L^2} \\
&\lesssim t^{-\frac{k+1}{2} - \frac{1}{2} + \frac{k}{3} + \frac{1}{2}} \sum_{M \leq 1} M^{-\frac{3}{2} + \frac{1}{2} + \frac{k}{3}} \| P_M f \|_{L^2} \\
&\lesssim t^{-\frac{k+1}{2} - \frac{1}{2} + \frac{k}{3} + \frac{1}{2}} \| f \|_{L^2},
\end{align*}
Due to \( I'_{22} \lesssim \| \hat{f} \|_{L^1} \), one has
\[ I'_{22} \lesssim \langle t \rangle^{-\frac{k+1}{2}} (\| f \|_{L^2} + \| \hat{f} \|_{L^1}). \]
As a result, we deduce the desired bound by combining the estimate of $I_2$.

Since the estimate of $I_3$ is similar to the estimate of $I_2$, it suffices to prove

$$I_3 \lesssim (t)^{-\frac{k+2}{2}} \|f\|_{L^1}, \quad k \geq -1. \quad (4.6)$$

Using dyadic decomposition and Hölder’s inequality, we have

$$I_3 \lesssim t^{-\frac{k+1}{2}} \sum_{M \leq 1} M^{-1} \left\| e^{-\frac{\varepsilon^2 t}{|M|^2}} \hat{P}_M f \right\|_{L^1(D_{42})}$$

which, along with $I_3 \lesssim \|\hat{f}\|_{L^\infty} \lesssim \|f\|_{L^1}$ can lead (4.6).

Likewise, the previous arguments, we have

$$I_4 \leq \|G_{k+2,k+4} \hat{e} \|_{L^1(D_{41})} + \|G_{k+2,k+4} \hat{e} \|_{L^1(D_{42})} = I_{41} + I_{42}. \quad (4.7)$$

By $|\xi| \lesssim |\tilde{\xi}|^2$ in $D_4$, dyadic decomposition, Hölder’s inequality, $1 \leq q \leq 2$ and $1/p + 1/q > 1$, one has

$$I_{42} \lesssim \sum_{M \leq 1} \left\| |\xi|^{k+2} M^{-k-4} e^{-\varepsilon^2 t} \hat{P}_M f \right\|_{L^1(D_{42})}$$

which, together with $I_{42} \leq \|\hat{f}\|_{L^1}$, yield

$$I_{42} \leq \min\{\|\hat{f}\|_{L^1}, t^{-\frac{k+2}{2} + \frac{1}{2p} - \frac{1}{2q}} f \|_{L^1(D_{42})} \} \quad (4.8)$$

Using (4.8) for $p = q = \frac{4}{3}$ and the estimate of $I_4$, we can get the desired estimate (4.3). It follows from using $|\xi| \lesssim |\tilde{\xi}|^2$ that $I_5 \lesssim \|\tilde{\xi}^{-1} \hat{f}\|_{L^1(D_{42})} \lesssim \|f\|_{L^1}$. Notice that (4.7) holds for $k \geq -1$, so we can obtain

$$I_5 \lesssim t^{-\frac{k+2}{2} + \frac{1}{2p} - \frac{1}{2q}} \|f\|_{L^1(D_{42})}, \quad k \geq 0. \quad (4.9)$$

Finally, to complete the estimate of (4.3), it suffices to prove

$$I_5 \lesssim \langle t \rangle^{-\frac{k+2}{2}} \|\nabla |\xi|^{-\delta} \hat{f}\|_{L^1}. \quad (4.9)$$
It is easy to get
\[
\mathcal{I}_5 \lesssim \langle t \rangle^{-\frac{k+1}{2}} \| e^{-\frac{\xi^2}{2\xi^2} t} \hat{f} \|_{L^1(D_{4t})}.
\] (4.10)

Using dyadic decomposition and Hölder’s inequality, one obtains
\[
\| \frac{1}{|\xi|^2} e^{-\frac{\xi^2}{2|\xi|^2} t} \hat{f} \|_{L^1(D_{4t})} \lesssim \sum_{M \leq 1} M^{-2} \| e^{-\frac{\xi^2}{2|\xi|^2} M^{-2} t} \hat{f} \|_{L^1(D_{4t})}
\]
\[
\lesssim \sum_{M \leq 1} M^{-2} \| e^{-\frac{\xi^2}{2|\xi|^2} M^{-2} t} \|_{L^2_\xi} \| \hat{f} \|_{L^\infty_\eta} \int_{|\eta| \leq M} d\eta
\]
\[
\lesssim t^{-\frac{1}{2}} \sum_{M \leq 1} \| P_M f \|_{L^1} \lesssim t^{-\frac{1}{2}} \| |\nabla|^{-\delta} f \|_{L^1},
\]
which, along with
\[
\| \frac{1}{|\xi|^2} e^{-\frac{\xi^2}{2|\xi|^2} t} \hat{f} \|_{L^1(D_{4t})} \lesssim \| \frac{1}{|\xi|^2} \hat{f} \|_{L^1(D_{4t})} \lesssim \| \frac{1}{|\xi|^2 - \delta} \hat{f} \|_{L^1(D_{4t})} \lesssim \| |\nabla|^{-\delta} f \|_{L^1}
\]
follows
\[
\| \frac{1}{|\xi|^2} e^{-\frac{\xi^2}{2|\xi|^2} t} \hat{f} \|_{L^1(D_{4t})} \lesssim \langle t \rangle^{-\frac{1}{2}} \| |\nabla|^{-\delta} f \|_{L^1}.
\]

Hence by (4.10), we can get the desired estimate. (4.3) Like the previous process, we have
\[
\mathcal{I}_6 \leq \| G_{k+1,k+1} e^{-\frac{\xi^2}{2|\xi|^2} t} \hat{f} \|_{L^2(D_{4t})} + \| G_{k+1,k+1} e^{-\frac{\xi^2}{2|\xi|^2} t} \hat{f} \|_{L^2(D_{4t})} = \mathcal{I}_{61} + \mathcal{I}_{62}.
\]

\(\mathcal{I}_{61}\) can be easily bounded by \(C \langle t \rangle^{-\frac{k+1}{2}} \| f \|_{L^2}\). For \(\mathcal{I}_{62}\), using dyadic decomposition and Hölder’s inequality, we have
\[
\mathcal{I}_{62} \lesssim \sum_{M \leq 1} \| |\xi|^{k+1} M^{k-2} e^{-\frac{\xi^2}{2|\xi|^2} M^{-2} t} \hat{P}_M f \|_{L^2(D_{4t})}
\]
\[
\lesssim t^{-\frac{k+1}{2}} \sum_{M \leq 1} \| (|\xi| M^{-1}) \|_{L^2_\xi} \| (|\xi| M^{-3}) \|_{L^2_\xi} \| e^{-\frac{\xi^2}{2|\xi|^2} M^{-2} t} \hat{f} \|_{L^2(D_{4t})}
\]
\[
\lesssim t^{-\frac{2k+1}{2}} \sum_{M \leq 1} M^{-\frac{1}{2}} \| e^{-\frac{\xi^2}{2|\xi|^2} M^{-2} t} \|_{L^2_\xi} \| \hat{P}_M f \|_{L^\infty_\eta} \int_{|\eta| \leq M} d\eta \|_{L^2_\xi}
\]
\[
\lesssim t^{-\frac{k+1}{2}} \sum_{M \leq 1} M^{-\frac{1}{2}} \| P_M f \|_{L^1} \lesssim t^{-\frac{k+1}{2}} \| f \|_{L^1},
\]
which, together with the estimate of \(\mathcal{I}_{61}\) implies
\[
\mathcal{I}_6 \lesssim \langle t \rangle^{-\frac{k+1}{2}} (\| f \|_{L^1} + \| f \|_{L^2}).
\]

In addition, \(\mathcal{I}_6 \lesssim \| f \|_{L^2}\). So we can get the desired estimate. \(\square\)

4.3. Decay estimate III. Thanks to Lemma 4.1 and Lemma 4.2, we can get the following lemmas, the detailed proofs of which are showed in the Appendix.
Lemma 4.4. Let $M_1(\partial, t)$ be the operator defined by \eqref{3.7}, $1 \leq r \leq 2$, $\delta > 0$, then there holds

\begin{align*}
1) & \quad \|M_1(\partial, t)f\|_{L^2(D_4)} \lesssim \langle t \rangle^{-\frac{3}{2}} \min \{\|f\|_{L^1 L^2}, \|\nabla^{-1}f\|_{L^2}\}; \\
2) & \quad \|\nabla M_1(\partial, t)f\|_{L^r(D_4)} \lesssim \langle t \rangle^{-\frac{3}{2}} \min \{\|\bar{f}\|_{L^r}, \langle t \rangle^{-\frac{1}{2}} \|\nabla \|_{L^2(L^2)}\}; \\
3) & \quad \|\partial_2 M_1(\partial, t)f\|_{L^r(D_4)} \lesssim \langle t \rangle^{-1} \min \{\|\bar{f}\|_{L^r}, \langle t \rangle^{-\frac{1}{2}} \|\nabla \|_{L^2(L^2)}\}; \\
4) & \quad \|M_1(\partial, t)f\|_{L^1(D_4)} \lesssim \langle t \rangle^{-1} (\|\nabla \|_{L^2(L^2)} + \|f\|_{L^1}); \\
5) & \quad \|\Delta^{-1} M_1(\partial, t)f\|_{L^r(D_4)} \lesssim \langle t \rangle^{-\frac{3}{2}} \min \{\|f\|_{L^1} + \|f\|_{L^2(L^2)}, \langle t \rangle^{-\frac{1}{2}} \|\nabla \|_{L^2}\}; \quad (4.11) \\
6) & \quad \|\mathcal{R}_1 M_1(\partial, t)f\|_{L^1(D_4)} \lesssim \langle t \rangle^{-\frac{3}{2}} \|f\|_{L^1}; \\
7) & \quad \|\mathcal{R}_1 M_1(\partial, t)f\|_{L^2(D_4)} \lesssim \langle t \rangle^{-1} (\|\bar{f}\|_{L^1} + \|f\|_{L^2}); \\
8) & \quad \|\nabla \mathcal{R}_1 M_1(\partial, t)f\|_{L^1(D_4)} + \|\mathcal{R}_{11} M_1(\partial, t)f\|_{L^1(D_4)} \lesssim \langle t \rangle^{-1} \|\bar{f}\|_{L^1}; \\
9) & \quad \|\partial_2 \mathcal{R}_1 M_1(\partial, t)f\|_{L^1(D_4)} + \|\nabla \mathcal{R}_{11} M_1(\partial, t)f\|_{L^1(D_4)} \lesssim \langle t \rangle^{-\frac{3}{2}} \|\bar{f}\|_{L^1},
\end{align*}

where $D_4$ and $D_{42}$ are defined by \eqref{3.7} and \eqref{3.10}.

Lemma 4.4. Let $M_2(\partial, t)$ be the operator defined by \eqref{3.7}, $1 \leq r \leq 2$, $\delta > 0$, then there holds

\begin{align*}
1) & \quad \|M_2(\partial, t)f\|_{L^r(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}} \|\nabla \|_{L^2(L^2)}; \\
2) & \quad \|\partial_2 M_2(\partial, t)f\|_{L^r(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}} \min \{\|\nabla f\|_{L^r}, \langle t \rangle^{-\frac{1}{2}} \|\nabla \|_{L^2(L^2)}\}; \\
3) & \quad \|\partial_2^2 M_2(\partial, t)f\|_{L^2(D_4)} \lesssim \langle t \rangle^{-1} \|\nabla^2 f\|_{L^2}; \\
4) & \quad \|\nabla^{-1} M_2(\partial, t)f\|_{L^1(D_{42})} \lesssim \langle t \rangle^{-\frac{1}{2}} \|f\|_{L^1}; \\
5) & \quad \|\mathcal{R}_1 M_2(\partial, t)f\|_{L^1(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}} \min \{\|f\|_{L^1}, \langle t \rangle^{-\frac{1}{2}} \|\nabla \|_{L^2(L^2)}\} \quad (l = 1, 2); \\
6) & \quad \|\partial_2 \mathcal{R}_1 M_2(\partial, t)f\|_{L^1(D_4)} \lesssim \langle t \rangle^{-1} \|\nabla f\|_{L^1},
\end{align*}

where $\mathcal{R}_1 = \mathcal{R}_1, \mathcal{R}_2 = \mathcal{R}_{11}$ and $D_4$ and $D_{42}$ are defined by \eqref{3.7} and \eqref{3.10}. 
Lemma 4.5. Let $M_3(\partial, t)$ be the operator defined by (3.7), $1 \leq r \leq 2$, then there holds

\begin{align*}
1) & \quad \|M_3(\partial, t)f\|_{\mathcal{L}^2(D_4)} \lesssim\langle t\rangle^{-\frac{1}{2}} \min\{\|\nabla \partial f\|_{H^1}, \|f\|_{L^1\cap L^2}\}; \\
2) & \quad \|M_3(\partial, t)f\|_{\mathcal{L}^1(D_4)} \lesssim\langle t\rangle^{-1} \min\{\|f\|_{L^1\cap L^1_{\beta}(L^{3}_{\gamma})} + \|\hat{f}\|_{L^1}, \frac{1}{\langle t\rangle} \min\{\|\nabla \partial f\|_{L^2\cap L^2}, \|\hat{f}\|_{L^1}\}; \\
3) & \quad \|\partial_x M_3(\partial, t)f\|_{\mathcal{L}^2(D_4)} \lesssim \min\{\langle t\rangle^{-\frac{1}{2}} \|f\|_{H^1}, \langle t\rangle^{-1} \|\nabla^{-1} f\|_{H^3}, \langle t\rangle^{-1} \|f\|_{L^1\cap H^3}\}; \\
4) & \quad \|\partial_x M_3(\partial, t)f\|_{\mathcal{L}^1(D_4)} \lesssim \langle t\rangle^{-1} \min\{\langle t\rangle^{-\frac{1}{2}} \|\nabla^{-1} f\|_{H^3}, \langle t\rangle^{-1} \|f\|_{L^1\cap H^3}\}; \\
5) & \quad \|\partial_x M_3(\partial, t)f\|_{\mathcal{L}^1(D_4)} \lesssim \min\{\langle t\rangle^{-\frac{1}{2}} \|\nabla^{-1} f\|_{H^3}, \langle t\rangle^{-1} \|f\|_{L^1\cap H^3}\}; \\
6) & \quad \|\Delta M_3(\partial, t)f\|_{\mathcal{L}^2(D_4)} \lesssim \langle t\rangle^{-1} \|f\|_{H^2} \\
7) & \quad \|\partial_x^2 \mathcal{R}_1 M_3(\partial, t)f\|_{\mathcal{L}^1(D_4)} \lesssim \langle t\rangle^{-\frac{3}{2}} \|f\|_{\mathcal{L}^1} + \|\partial_x^2 f\|_{\mathcal{L}^1}\}; \\
8) & \quad \|\nabla \partial_x M_3(\partial, t)f\|_{\mathcal{L}^1(D_4)} \lesssim \langle t\rangle^{-\frac{1}{2}} \|f\|_{\mathcal{L}^1} + \|\partial_x \nabla f\|_{\mathcal{L}^1}\}; \\
9) & \quad \|\partial_x^2 M_3(\partial, t)f\|_{\mathcal{L}^1(D_4)} \lesssim \langle t\rangle^{-\frac{3}{2}} \|f\|_{\mathcal{L}^1} + \|\partial_x^2 f\|_{\mathcal{L}^1}\}; \\
10) & \quad \|\mathcal{R}_1 M_3(\partial, t)f\|_{\mathcal{L}^1(D_4)} \lesssim \langle t\rangle^{-1} \|\nabla^{-1} f\|_{\mathcal{L}^1} + \|\hat{f}\|_{\mathcal{L}^1}\}; \\
11) & \quad \|\partial_x \mathcal{R}_1 M_3(\partial, t)f\|_{\mathcal{L}^1(D_4)} \lesssim \langle t\rangle^{-\frac{3}{2}} \|\partial_x f\|_{\mathcal{L}^1} + \|\hat{f}\|_{\mathcal{L}^1}\},
\end{align*}

where $D_4$ is defined by (3.2).

4.4. **Nonlinear decay estimate.** In this subsection, we give some Lemmas devoting to estimating the nonlinear part, the proofs of which are given in the Appendix. Let $\|V\|_3$ be the norm defined in section 3. $\mathcal{P}_\circ$, $\mathcal{P}_A$ and $\tilde{R}'$ be the operator defined by

\begin{align*}
\mathcal{P}_\circ f &= P_{(\mathcal{L}_\circ)^{-0.05}} f, \\
\mathcal{P}_A f &= P_{(\mathcal{L}_A)^{-0.05}} f,
\end{align*}

\begin{equation}
\mathcal{R}'_1 = -\partial_x (-\Delta)^{-1}, \quad \mathcal{R}'_2 = \partial_y (-\Delta)^{-1}, \quad \tilde{R}' = (\mathcal{R}'_2, \mathcal{R}'_1),
\end{equation}

respectively.

Lemma 4.6. Let $(u, v, b, B)$ be sufficiently smooth solution solving (1.2), then

\begin{align*}
\|\nabla \mathcal{P}_\circ (\tilde{R}' \cdot \tilde{b}) b\|_{L^1} + \|\tilde{b} b\|_{L^1} \lesssim \langle t\rangle^{-\frac{1}{2}} \|V\|_3^2, \quad \|\partial_x \beta^1 (b b)\|_{L^1} \lesssim \langle t\rangle^{-\frac{1}{2}} \|V\|_3^3; \\
\|\partial_y v_{<\langle t\rangle^{-1}} b\|_{L^1} + \|\partial_y v_{>2(t)^{-0.05}} b\|_{L^1} + \|\partial_x \mathcal{P}_\circ (\tilde{R}' \cdot \tilde{b}) \partial_y u\|_{L^1} \lesssim \langle t\rangle^{-\frac{1}{2}} \|V\|_3^2; \\
\|\nabla \mathcal{F} \{\partial_x \beta^1 f\} = |\xi| \hat{f} \text{ and } 0 < \beta \leq 1.
\end{align*}

(14.17)
Lemma 4.7. Under the conditions in Lemma 4.6, let $1 \leq p \leq 2$, then
\[
\|b \partial_x b\|_{L_p^1(L_3^\infty)} \lesssim \langle t \rangle^{-\frac{3}{2} + \frac{1}{p}} \|V\|_3^2;
\]
\[
\langle \langle \nabla \rangle \rangle^3 (u \partial_x b, \vec{b} \cdot \nabla u)\|_{L_p^1(L_3^\infty)} + \langle \langle \nabla \rangle \rangle^3 (\partial_x P_\tau (\vec{R} \cdot \vec{b}) \partial_y u)\|_{L_p^1(L_3^\infty)}
\]
\[
\|\langle \langle \nabla \rangle \rangle^2 (\partial_x P_\tau ub)\|_{L_p^1(L_3^\infty)} + \|\langle \langle \nabla \rangle \rangle^2 (\partial_x P_\tau (\vec{R} \cdot \vec{b}) u)\|_{L_p^1(L_3^\infty)} \lesssim \langle t \rangle^{1-0.01} \|V\|_3^2;
\]
\[
\langle \langle \nabla \rangle \rangle^3 (v_{< (t) - s} b, v_{> 2(t) - 0.05} b)\|_{L_p^1(L_3^\infty)} + \|\langle \langle \nabla \rangle \rangle^3 (v_{< (t) - s} \partial_y b, v_{> 2(t) - 0.05} \partial_y b)\|_{L_p^1(L_3^\infty)}
\]
\[
\|\langle \langle \nabla \rangle \rangle^2 (\partial_x P_\tau ub)\|_{L_p^1(L_3^\infty)} \lesssim \langle t \rangle^{1-0.05} \|V\|_3^2.
\]

Lemma 4.8. Under the conditions in Lemma 4.6, there holds
\[
\|\vec{u} \cdot \vec{b}\|_{L_2} + \|\vec{b} B\|_{L_2} + \|u \partial_x b\|_{L_2} + \|b \partial_x b\|_{L_2} + \|\vec{b} \cdot \nabla u\|_{L_2} \lesssim \langle t \rangle^{-\frac{1}{2}} \|V\|_3^2;
\]
\[
\|\vec{u} \cdot \vec{u}\|_{L_2} \lesssim \langle t \rangle^{-\frac{3}{2}} \|V\|_3^2, \|\vec{b} B\|_{L_2} + \|\partial_y P_\tau (\vec{R} \cdot \vec{b}) B\|_{L_2} \lesssim \langle t \rangle^{-\frac{1}{2}} \|V\|_3^2;
\]
\[
\|\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \vec{b}\|_{H_2} + \|(\vec{u} \cdot \nabla b, \vec{b} \cdot \nabla u)\|_{H_2} \lesssim \langle t \rangle^{-1} \|V\|_3^2;
\]
\[
\|v_{< (t) - s} b\|_{H_1} + \|v_{> 2(t) - 0.05} b\|_{H_1} + \|u B\|_{H_2} + \|b v\|_{H_2} + \|B \vec{b}\|_{H_2} + \|v_{< (t) - s} \partial_y b\|_{L_2} + \|v_{> 2(t) - 0.05} \partial_y b\|_{L_2} + \|\partial_x P_\tau (\vec{R} \cdot \vec{b}) \partial_y b\|_{L_2} + \|\partial_x P_\tau (\vec{R} \cdot \vec{b}) B\|_{H_2} + \|\partial_x \partial_y P_\tau (\vec{R} \cdot \vec{b}) B\|_{H_2} \lesssim \langle t \rangle^{-1} \|V\|_3^2;
\]
\[
\|\partial_x P_\tau (\vec{R} \cdot \vec{b}) B\|_{H_2} \lesssim \langle t \rangle^{-1} \|V\|_3^2, \|\partial_x (\partial_y P_\tau (\vec{R} \cdot \vec{b}) b)\|_{H_1} + \|\partial_x P_\tau (\vec{R} \cdot \vec{b}) B\|_{H_2} \lesssim \langle t \rangle^{-1} \|V\|_3^2;
\]
\[
\|\partial_x (\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla \vec{b})\|_{H_2} \lesssim \langle t \rangle^{-1} \|V\|_3^2, \|\partial_x \partial_y P_\tau (\vec{R} \cdot \vec{b}) B\|_{H_2} \lesssim \langle t \rangle^{-0.6} \|V\|_3^2.
\]

Lemma 4.9. Under the conditions in Lemma 4.6, there holds
\[
\|\vec{u} \cdot \vec{b} \cdot \vec{b} \cdot \nabla P_\tau (\vec{R} \cdot \vec{b}) B, \partial_x (\partial_y P_\tau (\vec{R} \cdot \vec{b}) B), \nabla P_\tau (\vec{R} \cdot \vec{b}) u\|_{F_1,1} \lesssim \langle t \rangle^{-\frac{3}{2}} \|V\|_3^3;
\]
\[
\|\partial_x (\nabla b u)\|_{F_1,1} + \|\partial_y (b \partial_x v, v \partial_x b)\|_{F_1,1} \lesssim \langle t \rangle^{-\frac{1}{2}} \|V\|_3^3;
\]
\[
\|P_\tau (\vec{R} \cdot \vec{b}) b\|_{F_1,1} + \|P_\tau (\vec{R} \cdot \vec{b}) B\|_{F_1,1} \lesssim \langle t \rangle^{-0.99} \|V\|_3^3;
\]
\[
\|\partial_x \partial_y P_\tau (\vec{R} \cdot \vec{b}) B\|_{F_1,1} + \|\partial_x \partial_y P_\tau (\vec{R} \cdot \vec{b}) \partial_y u\|_{F_1,1} \lesssim \langle t \rangle^{-0.09} \|V\|_3^3;
\]
\[
\|\langle \langle \nabla \rangle \rangle^2 (\partial_x P_\tau (\vec{R} \cdot \vec{b}) \partial_y b)\|_{F_1,1} \lesssim \langle t \rangle^{-1.01} \|V\|_3^3;\]
\[
\|\vec{u} \cdot \nabla \vec{u}\|_{F_1,1} + \|\vec{b} \cdot \nabla \vec{b}\|_{F_1,1} + \|\partial_x (\partial_y b, b \partial_x b)\|_{F_1,1} \lesssim \langle t \rangle^{-1.3} \|V\|_3^3;
\]
\[
\|\nabla (B \vec{b})\|_{F_1,1} + \|\langle \langle \nabla \rangle \rangle (\vec{u} \cdot \nabla \vec{u})\|_{F_1,1} + \|\langle \langle \nabla \rangle \rangle (\vec{u} \cdot \nabla \vec{b}, \vec{b} B)\|_{F_1,1} \lesssim \langle t \rangle^{-1.2} \|V\|_3^3;
\]
\[
\|\langle \langle \nabla \rangle \rangle^2 (\vec{b} \cdot \nabla \vec{b})\|_{F_1,1} \lesssim \langle t \rangle^{-1.1} \|V\|_3^3, \|P_\tau (\vec{R} \cdot \vec{b}) \partial_y b\|_{F_1,1} \lesssim \langle t \rangle^{-0.9} \|V\|_3^3.
\]

Remark 4.10. We do not focus on the optimal decay rate for the nonlinear terms like (4.18), since it is sufficient to help us achieve the final goal in the present paper.
5. Energy estimate in $H^N$

In this section, we show the following energy estimate of solutions:

$$
\|\tilde{u}(t)\|_{H^N}^2 + \|\tilde{b}(t)\|_{H^N}^2 + \|\nabla \tilde{u}\|_{L^2_t(H^N)}^2 + \|\partial_x \tilde{b}\|_{L^2_t(H^{N-1})}^2 \\
\lesssim \|V_0\|_3^2 + \|V\|_4^4 + \|V\|_3^3, \ \forall \ t > 0.
$$

(5.1)

Using

$$
(\tilde{u} \cdot \nabla \tilde{u})\tilde{u} = (\tilde{b} \cdot \nabla \tilde{b})\tilde{u} + (\tilde{b} \cdot \nabla \tilde{u})\tilde{b} = (\tilde{u} \cdot \nabla \tilde{b}) = 0,
$$

we get the $L^2$ energy estimate:

$$
\frac{d}{dt}(\|\tilde{u}\|_{L^2}^2 + \|\tilde{b}\|_{L^2}^2) + \|\nabla \tilde{u}\|_{L^2}^2 = 0.
$$

(5.2)

The $\dot{H}^N$ estimate of the solution reads:

$$
\frac{d}{dt}(\|\tilde{u}\|_{\dot{H}^N}^2 + \|\tilde{b}\|_{\dot{H}^N}^2) + \|\nabla \tilde{u}\|_{\dot{H}^N}^2 = -\langle \tilde{u} \cdot \nabla \tilde{u}, \tilde{u} \rangle_{H^N} + \langle \tilde{b} \cdot \nabla \tilde{b}, \tilde{u} \rangle_{H^N} \\
+ \langle \tilde{b} \cdot \nabla \tilde{u} \tilde{b}, \tilde{u} \rangle_{H^N} - \langle \tilde{u} \cdot \nabla \tilde{b} \tilde{b}, \tilde{u} \rangle_{H^N} = I_1 + I_2 + I_3 + I_4.
$$

(5.3)

From (1.2), using $(\nabla p|\partial_x \tilde{b}) = 0$, we have

$$(\partial_x \tilde{b}|\partial_x \tilde{b})_{H^{N-1}} = -(\Delta \tilde{u}|\partial_x \tilde{b})_{H^{N-1}} - \langle \tilde{b} \cdot \nabla \tilde{u} \tilde{b}, \tilde{u} \rangle_{H^{N-1}} + \langle \tilde{u} \cdot \nabla \tilde{u} \tilde{b}, \tilde{b} \rangle_{H^{N-1}} + (\partial_x \tilde{u}|\partial_x \tilde{b})_{H^{N-1}}.
$$

Using the formation of (1.2) and integration by parts, we can get

$$
(\partial_x \tilde{u}|\partial_x \tilde{b})_{H^{N-1}} = \frac{d}{dt}(\tilde{u}|\partial_x \tilde{b})_{H^{N-1}} + (\partial_x \tilde{u}|\partial_x \tilde{b})_{H^{N-1}} \\
= \frac{d}{dt}(\tilde{u}|\partial_x \tilde{b})_{H^{N-1}} + \|\partial_x \tilde{u}\|_{H^{N-1}}^2 - (\partial_x \tilde{u}|\tilde{u} \cdot \nabla \tilde{b})_{H^{N-1}} + (\partial_x \tilde{u}|\tilde{b} \cdot \nabla \tilde{u})_{H^{N-1}}.
$$

So

$$
\|\partial_x \tilde{b}\|_{H^{N-1}}^2 = -(\Delta \tilde{u}|\partial_x \tilde{b})_{H^{N-1}} - \langle \tilde{b} \cdot \nabla \tilde{u} \tilde{b}, \tilde{u} \rangle_{H^{N-1}} + \langle \tilde{u} \cdot \nabla \tilde{u} \tilde{b}, \tilde{b} \rangle_{H^{N-1}} \\
+ \frac{d}{dt}(\tilde{u}|\partial_x \tilde{b})_{H^{N-1}} + \|\partial_x \tilde{u}\|_{H^{N-1}}^2 - (\partial_x \tilde{u}|\tilde{u} \cdot \nabla \tilde{b})_{H^{N-1}} + (\partial_x \tilde{u}|\tilde{b} \cdot \nabla \tilde{u})_{H^{N-1}},
$$

together with the application of Young's inequality

$$
|\langle \Delta \tilde{u}|\partial_x \tilde{b}\rangle_{H^{N-1}}| \leq \frac{1}{2}\|\partial_x \tilde{b}\|_{H^{N-1}}^2 + \frac{1}{2}\|\Delta \tilde{u}\|_{H^{N-1}}^2,
$$

yields

$$
- \frac{d}{dt}(\tilde{u}|\partial_x \tilde{b})_{H^{N-1}} + \frac{1}{2}\|\partial_x \tilde{b}\|_{H^{N-1}}^2 - \|\partial_x \tilde{u}\|_{H^{N-1}}^2 \\
\leq \frac{1}{2}\|\Delta \tilde{u}\|_{H^{N-1}}^2 + \|\tilde{b} \cdot \nabla \tilde{u}|\partial_x \tilde{b}\|_{H^{N-1}}^2 + \|\tilde{u} \cdot \nabla \tilde{u}|\partial_x \tilde{b}\|_{H^{N-1}}^2 \\
+ |\langle \partial_x \tilde{u}|\tilde{u} \cdot \nabla \tilde{b}\rangle_{H^{N-1}}| + |\langle \partial_x \tilde{u}|\tilde{b} \cdot \nabla \tilde{u}\rangle_{H^{N-1}}| \\
= \frac{1}{2}\|\Delta \tilde{u}\|_{H^{N-1}}^2 + I_5 + I_6 + I_7 + I_8.
$$

(5.4)

Multiplying (5.4) by $\frac{1}{2}$, and adding the resulting inequality, (5.2) and (5.3) together, we have

$$
\frac{d}{dt}\left(\|\tilde{u}\|_{H^N}^2 + \|\tilde{b}\|_{H^N}^2 - \frac{1}{4}(\tilde{u}|\partial_x \tilde{b})_{H^{N-1}}\right) + \frac{1}{2}\|\nabla \tilde{u}\|_{H^N}^2 + \frac{1}{8}\|\partial_x \tilde{b}\|_{H^{N-1}}^2 \leq \sum_{i=1}^{4} I_i + \sum_{i=5}^{8} I_i.
$$
Using product estimate and Young's inequality, we have
\begin{align*}
I_1 &\leq \| \vec{u} \cdot \nabla \vec{u} \|_{L^2} \| \vec{u} \|_{H^N} \lesssim \| \vec{u} \|_{H^N} \| \nabla \vec{u} \|_{H^N} \| \nabla \vec{u} \|_{H^{N-1}} \\
&\leq C \| \vec{u} \|_{H^N}^2 \| \nabla \vec{u} \|_{H^{N-1}}^2 + 0.001 \| \nabla \vec{u} \|_{H^N}^2,
\end{align*}
\begin{align*}
I_6 &\leq \| \vec{u} \cdot \nabla \vec{u} \|_{H^N-1} \| \partial_x \vec{b} \|_{H^N-1} \lesssim \| \vec{u} \|_{H^N-1} \| \nabla \vec{u} \|_{H^{N-1}} \| \partial_x \vec{b} \|_{H^{N-1}} \\
&\leq C \| \vec{u} \|_{H^N-1}^2 \| \nabla \vec{u} \|_{H^{N-1}}^2 + 0.001 \| \partial_x \vec{b} \|_{H^{N-1}}^2,
\end{align*}
\begin{align*}
I_8 &\leq \| \vec{b} \cdot \nabla \vec{u} \|_{H^N-1} \| \partial_x \vec{u} \|_{H^N-1} \lesssim \| \vec{b} \|_{H^N-1} \| \nabla \vec{u} \|_{H^{N-1}} \| \partial_x \vec{u} \|_{H^{N-1}} \\
&\leq C \| \vec{b} \|_{H^N-1}^2 \| \partial_x \vec{u} \|_{H^{N-1}}^2 + 0.001 \| \nabla \vec{u} \|_{H^{N-1}}^2.
\end{align*}

Since
\begin{align*}
\| \partial^{N-1} (B \partial_y \vec{b}) \|_{L^2} &\leq \| \partial_x^{N-1} (B \partial_y \vec{b}) \|_{L^2} + \| \partial_y^{N-1} (B \partial_y \vec{b}) \|_{L^2} \\
&\lesssim \| \|B\|_{L^\infty} \| \partial_y \partial_x^{N-1} \vec{b} \|_{L^2} + \| \partial_y \vec{b} \|_{L^\infty} \| \partial_x^{N-1} B \|_{L^2} \| \vec{b} \|_{L^2} \\
&\quad + \| \|B\|_{L^\infty} \| \partial_y^{N-1} \vec{b} \|_{L^2} + \| \partial_y \vec{b} \|_{L^\infty} \| \partial_y^{N-1} B \|_{L^2} \| \vec{b} \|_{L^2} \\
&\lesssim \| \vec{b} \|_{H^3} \| \partial_x \vec{b} \|_{H^{N-1}} + \| \|B\|_{L^\infty} \| \vec{b} \|_{H^N} \| \partial_y \vec{b} \|_{H^{N-1}}
\end{align*}

with \( \| \partial_y \vec{b} \|_{L^2} \leq \| B \|_{L^\infty} \| \vec{b} \|_{H^1} \) yields
\begin{align*}
\| \partial_y \vec{b} \|_{H^{N-1}} &\lesssim \| \vec{b} \|_{H^3} \| \partial_x \vec{b} \|_{H^{N-1}} + \| \|B\|_{L^\infty} \| \vec{b} \|_{H^N}.
\end{align*}

Thanks to \( (5.5) \), we have
\begin{align*}
I_5 &\leq (\| \vec{b} \|_{H^N-1} + \| \|B\|_{H^{N-1}} \| \partial_x \vec{b} \|_{H^{N-1}} \\
&\lesssim (\| \vec{b} \|_{H^N-1} \| \partial_x \vec{b} \|_{H^{N-1}} + \| \|B\|_{L^\infty} \| \vec{b} \|_{H^N} \| \partial_x \vec{b} \|_{H^{N-1}} \\
&\leq C(\| \vec{b} \|_{H^N-1}^2 \| \partial_x \vec{b} \|_{H^{N-1}}^2 + \| \|B\|_{L^\infty} \| \vec{b} \|_{H^N}^2 + 0.001 \| \partial_x \vec{b} \|_{H^{N-1}}^2)
\end{align*}

Following the arguments yielding \( (5.5) \), we can also get
\begin{align*}
\| v \partial_y \vec{b} \|_{H^{N-1}} &\lesssim \| \vec{b} \|_{H^3} \| \partial_x \vec{u} \|_{H^{N-1}} + \| v \|_{L^\infty} \| \vec{b} \|_{H^N},
\end{align*}
which deduce that
\begin{align*}
I_7 &\leq (\| \vec{u} \|_{H^N-1} + \| v \partial_y \vec{b} \|_{H^{N-1}}) \| \partial_x \vec{u} \|_{H^{N-1}} \\
&\lesssim (\| \vec{u} \|_{H^N-1} \| \partial_x \vec{b} \|_{H^{N-1}} + \| \vec{v} \|_{L^\infty} \| \vec{b} \|_{H^N} + \| \vec{b} \|_{H^3} \| \partial_x \vec{u} \|_{H^{N-1}}) \| \partial_x \vec{u} \|_{H^{N-1}} \\
&\leq C(\| \vec{u} \|_{H^N-1}^2 \| \partial_x \vec{b} \|_{H^{N-1}}^2 + \| \vec{b} \|_{H^3}^2 \| \partial_x \vec{u} \|_{H^{N-1}}^2 + \| \vec{v} \|_{L^\infty}^2 \| \vec{b} \|_{H^N}^2 + 0.001 \| \partial_x \vec{u} \|_{H^{N-1}}^2).
\end{align*}

For \( I_4 \), we have
\begin{align*}
I_4 &= -\int \partial^N (\vec{u} \cdot \nabla \vec{b}) \cdot \partial^N \vec{b} \, dx \, dy = -\sum_{\iota+\kappa=N,0\leq\kappa\leq\iota+N-1} C^N_N \int \partial^\iota \vec{u} \cdot \nabla \partial^\kappa \vec{b} \cdot \partial^N \vec{b} \, dx \, dy \\
&= -\sum_{\iota+\kappa=N,0\leq\kappa\leq\iota+N-1} C^N_N \int \partial^\iota \vec{u} \partial_\kappa \vec{b} \cdot \partial^N \vec{b} \, dx \, dy - \sum_{\iota+\kappa=N,0\leq\kappa\leq\iota+N-1} C^N_N \int \partial^\iota v \partial_\kappa \vec{b} \cdot \partial^N \vec{b} \, dx \, dy
\end{align*}
The first term can be bounded by
\begin{align*}
C \| \nabla \vec{u} \|_{H^N} \| \partial_x \vec{b} \|_{H^{N-1}} \| \vec{b} \|_{H^N} \leq C \| \vec{b} \|_{H^N}^2 \| \nabla \vec{u} \|_{H^N} + 0.001 \| \partial_x \vec{b} \|_{H^{N-1}}^2.
\end{align*}
For the second integral, we consider it as two types based on whether \( \partial^N \) contains \( \partial_x \). If \( \partial^N = \partial^i \partial_j^j \), \( i + j = N \) and \( i \geq 1 \), we can bound this case by
\begin{align*}
C \| \nabla v \|_{H^N} \| \vec{b} \|_{H^N} \| \partial_x \vec{b} \|_{H^{N-1}} \leq C \| \nabla v \|_{H^N}^2 \| \vec{b} \|_{H^N}^2 + 0.001 \| \partial_x \vec{b} \|_{H^{N-1}}^2.
\end{align*}
Otherwise, we need to estimate \( \int \partial_y v \partial_y^{\kappa+1} \cdot \partial_y^N b \, dx \, dy \). In fact, when \( 0 \leq \kappa \leq N - 2 \), using \( \partial_y v = -\partial_x u \) and integrating by parts twice, we have

\[
\int \partial_y v \partial_y^{\kappa+1} \cdot \partial_y^N b \, dx \, dy = - \int \partial_y^{\kappa-1} \partial_x u \partial_y^{\kappa+1} \cdot \partial_y^N b \, dx \, dy
\]

\[
= \int \partial_y^{\kappa-1} u \partial_x \partial_y^{\kappa+1} \cdot \partial_y^N b \, dx \, dy + \int \partial_y^{-1} u \partial_y^{\kappa+1} \cdot \partial_x \partial_y^N b \, dx \, dy
\]

\[
= \int \partial_y^{-1} u \partial_x \partial_y^{\kappa+1} \cdot \partial_y^N b \, dx \, dy - \int \partial_y^{\kappa-1} u \partial_y^{\kappa+1} \cdot \partial_x \partial_y^N b \, dx \, dy
\]

\[
- \int \partial_y^{\kappa-1} u \partial_y^{\kappa+2} \cdot \partial_x \partial_y^{N-1} b \, dx \, dy
\]

\[
\lesssim C\|\nabla \bar{u}\|_{H^N} \|\bar{b}\|_{H^N} \|\partial_x \bar{b}\|_{H^{N-1}}
\]

\[
\leq C\|\bar{b}\|_{H^N}^2 \|\nabla \bar{u}\|_{H^N}^2 + 0.001 \|\partial_x \bar{b}\|_{H^{N-1}}^2.
\]

When \( \kappa = N - 1 \), we can bound this integral by \( \|\partial_x u\|_{L^\infty} \|\bar{b}\|_{H^N}^2 \). Hence, we have

\[
I_4 \leq C\|\bar{b}\|_{H^N}^2 \|\nabla \bar{u}\|_{H^N}^2 + \|\partial_x u\|_{L^\infty} \|\bar{b}\|_{H^N}^2 + 0.01 \|\partial_x \bar{b}\|_{H^{N-1}}^2.
\]

At last, we bound \( I_2 + I_3 \). Applying the cancelation property

\[
\int \bar{b} : \nabla \partial^N \bar{u} \cdot \partial^N \bar{b} \, dx = 0,
\]

we have

\[
I_2 + I_3 = \sum_{\iota + \kappa = N, 0 \leq \kappa \leq N - 1} C_{\iota}^\kappa \int \partial^\iota \bar{b} : \nabla \partial^\kappa \bar{u} \cdot \partial^\kappa \bar{b} \, dx \, dy + \sum_{\iota + \kappa = N, 0 \leq \kappa \leq N - 1} C_{\iota}^\kappa \int \partial^\iota \bar{b} : \nabla \partial^\kappa \bar{u} \cdot \partial^\kappa \bar{b} \, dx \, dy.
\]

By a similar analysis of \( I_4 \), we only need to bound the integral:

\[
F_1 = \int \partial_y^{\kappa} b \partial_x \partial_y^{\kappa} u \partial_y^N b \, dx \, dy,
\]

since other cases can be bounded by the left hand side of (5.76). When \( 1 \leq \kappa \leq N - 1 \), integrating by parts twice, we have

\[
F_1 = - \int \partial_x \partial_y^{\kappa} b \partial_y^{\kappa} u \partial_y^N b \, dx \, dy - \int \partial_y^{\kappa} b \partial_y^{\kappa} u \partial_x \partial_y^N b \, dx \, dy
\]

\[
- \int \partial_x \partial_y^{\kappa} b \partial_y^{\kappa} u \partial_y^N b \, dx \, dy + \int \partial_y^{\kappa+1} b \partial_y^{\kappa+1} u \partial_x \partial_y^{N-1} b \, dx \, dy
\]

\[
+ \int \partial_y^{\kappa} b \partial_y^{\kappa+1} u \partial_x \partial_y^{N-1} b \, dx \, dy \lesssim \|\bar{b}\|_{H^N} \|\nabla \bar{u}\|_{H^N} \|\partial_x \bar{b}\|_{H^{N-1}}
\]

\[
\leq C\|\bar{b}\|_{H^N}^2 \|\nabla \bar{u}\|_{H^N}^2 + 0.001 \|\partial_x \bar{b}\|_{H^{N-1}}^2.
\]

When \( \kappa = 0 \), we have

\[
F_1 \lesssim \|\partial_x u\|_{L^\infty} \|\bar{b}\|_{H^N}^2.
\]

So

\[
I_2 + I_3 \lesssim C\|\partial_x u\|_{L^\infty} + \|\nabla \bar{u}\|_{H^N}^2 \|\bar{b}\|_{H^N}^2 + 0.01 \|\partial_x \bar{b}\|_{H^{N-1}}^2.
\]

Collecting the above estimates of \( I_i \) \( i = 1, 8 \), we can get

\[
\frac{d}{dt} (\|\bar{u}\|_{H^N}^2 + \|\bar{b}\|_{H^N}^2 - \frac{1}{4} \|\partial_x \bar{b}\|_{H^{N-1}}^2) + \frac{1}{2} \|\nabla \bar{u}\|_{H^N}^2 + 0.05 \|\partial_x \bar{b}\|_{H^{N-1}}^2 \lesssim (\|\bar{u}\|_{H^N}^2 + \|\bar{b}\|_{H^N}^2) (\|\nabla \bar{u}\|_{H^N}^2 + \|\partial_x \bar{b}\|_{H^{N-1}}^2 + \|v\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 + \|\partial_x u\|_{L^\infty}^2).
Integrating in time, using
\[ \|\bar{u}\|_{H^N}^2 + \|\bar{b}\|_{H^N}^2 - \frac{1}{4}(\bar{u}\partial_x \bar{b})_{H^{N-1}} \approx \|\bar{u}\|_{H^N}^2 + \|\bar{b}\|_{H^N}^2 \]
and
\[ \int_0^t (\|v\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 + \|\partial_x u\|_{L^\infty}) d\tau \leq \int_0^t (\tau)^{-\frac{5}{4}} d\tau (\|V\|_3^2 + \|V\|_3) \leq C(\|V\|_3^2 + \|V\|_3), \]
we conclude the proof of (5.1).

6. The estimates on \(v\)

In this section, we shall prove
\[
\begin{align*}
\|v(t)\|_{L^2} &\lesssim (t)^{-\frac{7}{4}} (\|V_0\|_3 + \|V\|_3^2); \\
\|\partial_x v(t)\|_{H^2} &\lesssim (t)^{-1} (\|V_0\|_3 + \|V\|_3^2); \\
\|v(t)\|_{F^1_{L^1}} &\lesssim (t)^{-1} (\|V_0\|_3 + \|V\|_3^2).
\end{align*}
\]
(6.1)

Thanks to the Plancherel’s identity, let us turn to the estimate of \(\|v\|_{F^2_{L^2}}, \|\partial_x (\nabla)^2 v\|_{F^2_{L^2}}\) and \(\|v\|_{F^1_{L^1}}\), respectively. By (3.6), the expression of \(v\) can be given by
\[
v(t) = M_3(\partial, t)v_0 + M_1(\partial, t)B_0 + \int_0^t M_3(\partial, t - \tau)F^2 d\tau + \int_0^t M_1(\partial, t - \tau)G^2 d\tau, \tag{6.2}
\]
where
\[
F^2 = -\bar{u} \cdot \nabla v - \partial_y \Delta^{-1} \text{div}(\bar{b} \cdot \nabla \bar{b} - \bar{u} \cdot \nabla \bar{u}) + \bar{b} \cdot \nabla B = \mathcal{R}_{12} (\bar{b} \cdot \nabla \bar{b} - \bar{u} \cdot \nabla \bar{u}) - \mathcal{R}_{11} (\bar{b} \cdot \nabla B - \bar{u} \cdot \nabla v), \tag{6.3}
\]
and
\[
G^2 = -\bar{u} \cdot \nabla B + \bar{b} \cdot \nabla v = -\partial_x (u B - b v). \tag{6.4}
\]
The nonlinear term \(\bar{b} \cdot \nabla \bar{b}\) can be rewritten as
\[
\bar{b} \cdot \nabla \bar{b} = (b \partial_x b + B \partial_y b, \bar{b} \cdot \nabla B) = (2b \partial_x b + \partial_y (b B), \bar{b} \cdot \nabla B) = (\partial_x (bb) + \partial_y (bB), \bar{b} \cdot \nabla B). \tag{6.5}
\]

6.1. The estimate of (6.1). Using (3.12) and (4.1) for \(k = 0\), one can get
\[
\|L_v\|_{F^2_{L^2(D_1)}} \lesssim \|M_3(\partial, t)v_0\|_{F^2_{L^2(D_1)}} + \|M_1(\partial, t)B_0\|_{F^2_{L^2(D_1)}} \lesssim (t)^{-\frac{7}{4}} (\|v_0\|_{L^1} + \|B_0\|_{L^1}).
\]
By (3.12), (6.5), (4.1) for \( k = 1 \) and \( k = 1/2 \), (4.19), (4.19) and (4.17), we infer
\[
\sum_{i=1,2} \| NL_{vi} \|_{F \mathcal{L}^2(D_1)} \lesssim \int_0^t \| M_3(\partial, t - \tau) F^2 \|_{F \mathcal{L}^2(D_1)} d\tau + \int_0^t \| M_1(\partial, t - \tau) G^2 \|_{F \mathcal{L}^2(D_1)} d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| \nabla \|^{-1} \left( \bar{u} \cdot \nabla \bar{u}, \partial_y (Bb), \bar{b} \cdot \nabla B, \bar{u} \cdot \nabla \bar{b}, \bar{b} \cdot \nabla \bar{u} \right) \|_{L^2} d\tau
\]
\[
+ \int_0^t \| M_3(\partial, t - \tau) \partial_x (bb) \|_{F \mathcal{L}^2(D_1)} d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| \nabla \|^{-1} \left( \bar{u} \otimes \bar{u}, \bar{b} \right) \|_{L^2} d\tau + \| \bar{b} B \|_{L^2} + \| \bar{u} \otimes \bar{b} \|_{L^2} d\tau
\]
\[
+ \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| \partial_x (bb) \|_{L^1} d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \tau^{-1.1} \langle t - \tau \rangle^{-\frac{3}{4}} \tau^{-\frac{3}{4}} d\tau \| \nabla \|_{L^2}^2 \]
\[
\lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_{L^2}^3.
\]
So
\[
\| \bar{u} \|_{L^2(D_1)} \lesssim \langle t \rangle^{-\frac{1}{2}} \left( \| V_0 \|_{L^2} + \| V \|_{L^2}^3 \right). \tag{6.6}
\]

By (4.13), (4.13) \( 1 \), (4.13) \( 2 \), we have
\[
\| L_v \|_{F \mathcal{L}^2(D_4)} \lesssim \| M_3(\partial, t) v_0 \|_{F \mathcal{L}^2(D_4)} + \| M_1(\partial, t) B_0 \|_{F \mathcal{L}^2(D_4)}
\]
\[
\lesssim \langle t \rangle^{-\frac{1}{2}} \left( \| v_0 \|_{L^2} + \| B_0 \|_{L^1} \right).
\]

It follows from (3.13), (6.5), (4.13) \( 1 \) and (4.13) \( 4 \) for \( r = 2 \) that
\[
\| NL_{v1} \|_{F \mathcal{L}^2(D_4)} \lesssim \int_0^t \| M_3(\partial, t - \tau) F^2 \|_{F \mathcal{L}^2(D_4)} d\tau
\]
\[
\lesssim \int_0^t \| M_3(\partial, t - \tau) (\bar{u} \cdot \nabla \bar{u}, \bar{b} \cdot \nabla \bar{b}) \|_{F \mathcal{L}^2(D_4)} d\tau
\]
\[
\lesssim \int_0^t \| M_3(\partial, t - \tau) (\bar{u} \cdot \nabla \bar{u}, \bar{b} \cdot \nabla B, \partial_y (\bar{b} B)) \|_{F \mathcal{L}^2(D_4)} d\tau
\]
\[
+ \int_0^t \| \partial_x M_3(\partial, t - \tau) (bb) \|_{F \mathcal{L}^2(D_4)} d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| \nabla \|^{-1} \left( \bar{u} \cdot \nabla \bar{u}, \bar{b} \cdot \nabla B, \partial_y (\bar{b} B) \right) \|_{H^1} d\tau
\]
\[
+ \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \left( \| bb \|_{F \mathcal{L}^2} + \| b \partial_x b \|_{F \mathcal{L}^2} \right) d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \left( \| \bar{u} \otimes \bar{u}, \bar{u} \cdot \nabla \bar{u} \|_{L^2} + \| \bar{b} B \|_{H^1} \right) d\tau
\]
\[
+ \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \left( \| bb \|_{L^2} + \| b \partial_x b \|_{L^2} \right) d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{4}} \tau^{-1.1} \langle t - \tau \rangle^{-\frac{3}{4}} \tau^{-\frac{3}{4}} d\tau \| V \|_{H^1}^2 \]
\[
\lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_{L^2}^3.
\]
Using (3.13), (4.11)1 and (4.19), we can deduce
\[
\|NLv2\|_{F^{L^2}(D_1)} \lesssim \int_0^t \|M_1(\partial, t - \tau)G^2\|_{F^{L^2}(D_1)}d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \|\nabla|^{-1}(\bar{u} \cdot \nabla \bar{b}, \bar{b} \cdot \nabla \bar{u})\|_{L^2}d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \|\bar{u} \otimes \bar{b}\|_{L^2}d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} (\tau)^{-1,1}d\tau \|V\|_3^2 \lesssim \langle t \rangle^{-\frac{3}{2}} \|V\|_3^2.
\]

Then we have
\[
\|\tilde{v}\|_{L^2(D_1)} \lesssim \langle t \rangle^{-\frac{1}{2}} (\|V_0\|_3 + \|V\|_3^2).
\tag{6.7}
\]

Due to Remark 6.2, combining with (6.6) and (6.7) leads to the estimate (6.1).1

**Remark 6.1.** In the estimate of \(\|v\|_{L^2}\), \(\hat{F}^2\) and \(\hat{G}^2\) are bounded as follows:
\[
|\hat{F}^2| \leq |\tilde{\xi}|F\{\bar{u} \otimes \bar{u}, \bar{b}B\} + |\xi|\|F\{bb\}|, \quad |\hat{G}^2| \leq |\tilde{\xi}|F\{\bar{u} \otimes \bar{b}\}.
\]

Since \(F^1\) and \(G^1\) can be bounded by the same way, we can also obtain the similar estimate of \(\|u\|_{L^2}\), see (6.1).1

6.2. The estimate of (6.1)2. Due to (3.11), we have
\[
\|\partial_x(\nabla)^2v\|_{F^{L^2}(D_1)} \lesssim \|\partial_xv\|_{F^{L^2}(D_1)}.
\]

By (3.12), (4.1)1 for \(k = 1\), we have
\[
\|\partial_xL_v\|_{F^{L^2}(D_1)} \lesssim \|\partial_xM_3(\partial, t)v_0\|_{F^{L^2}(D_1)} + \|\partial_xM_1(\partial, t)B_0\|_{F^{L^2}(D_1)} \lesssim \langle t \rangle^{-1}(\|v_0\|_{L^1} + \|B_0\|_{L^1}).
\]

By using (3.12), (6.5), (4.1)1 for \(k = 2\), we can also obtain
\[
\sum_{i=1,2} \|\partial_xNL\|_{F^{L^2}(D_1)} \\
\lesssim \int_0^t \|\partial_xM_3(\partial, t - \tau)F^2\|_{F^{L^2}(D_1)}d\tau + \int_0^t \|\partial_xM_1(\partial, t - \tau)G^2\|_{F^{L^2}(D_1)}d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \|\nabla|^{-1}(\bar{u} \cdot \nabla \bar{u}, \partial_y(Bb), \bar{b} \cdot \nabla B, \bar{u} \cdot \nabla \bar{b}, \bar{b} \cdot \nabla \bar{u})\|_{L^2}d\tau \\
+ \int_0^t \|\partial_x^2M_3(\partial, t - \tau)(bb)\|_{F^{L^2}(D_1)}d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-1} (\|\bar{u} \otimes \bar{u}, B\bar{b}^2, \bar{u} \otimes \bar{b}\|_{L^2}d\tau + \int_0^t \|\partial_x^2M_3(\partial, t - \tau)(bb)\|_{F^{L^2}(D_1)}d\tau.
\]

By (4.19), the first integral on the right hand side can be bounded by
\[
C \int_0^t \langle t - \tau \rangle^{-1,1}d\tau \|V\|_3^2 \lesssim \langle t \rangle^{-1} \|V\|_3^2.
\tag{6.8}
\]
For the second integral, we shall split it into two integrals. Using (3.12), (4.11) for $k = 2$ and $k = 1$, (4.17), (4.18) for $p = 2$, one can get

\[
\int_0^t \| \partial_x^2 M_3(\partial, t - \tau)(bb) \|_{L^2(D_1)} d\tau \\
\lesssim \int_0^{t/2} \| \partial_x^2 M_3(\partial, t - \tau)(bb) \|_{L^2(D_1)} d\tau + \int_{t/2}^t \| \partial_x M_3(\partial, t - \tau)(b\partial_x b) \|_{L^2(D_1)} d\tau \\
\lesssim \int_0^{t/2} \langle t - \tau \rangle^{-\frac{3}{2}} \| bb \|_{L^1} d\tau + \int_{t/2}^t \langle t - \tau \rangle^{-\frac{1}{2}} \| b\partial_x b \|_{L^1(D_2)} d\tau \\
\lesssim \left( \int_0^{t/2} \langle t - \tau \rangle^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{1}{2}} d\tau + \int_{t/2}^t \langle t - \tau \rangle^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{1}{2}} d\tau \right) \| V \|_3^2 \\
\lesssim \langle t \rangle^{-1} \| V \|_3^2.
\]

So

\[
\sum_{i=1,2} \| \partial_x N L v_i \|_{L^2(D_1)} \lesssim \langle t \rangle^{-1} \| V \|_3^2.
\]

Hence

\[
\| \partial_x (\nabla)^2 v \|_{L^2(D_1)} \lesssim \langle t \rangle^{-1} (\| V_0 \|_3 + \| V \|_3^2).
\]  

Applying (4.13), (4.11) and (4.13) for $r = 2$, (4.19), we have

\[
\| \partial_x (\nabla)^2 L v \|_{L^2(D_2)} \lesssim \| \partial_x (\nabla)^2 M_3(\partial, t)v_0 \|_{L^2(D_2)} + \| \partial_x (\nabla)^2 M_1(\partial, t)B_0 \|_{L^2(D_2)} \\
\lesssim \langle t \rangle^{-1} (\| v_0 \|_{H^3} + \| B_0 \|_{H^2}).
\]

We deduce from (4.13), (4.11), (4.13) for $r = 2$, (4.13) for $r = 2$ and (4.19) that

\[
\sum_{i=1,2} \| \partial_x (\nabla)^2 NL v_i \|_{L^2(D_4)} \\
\lesssim \int_0^t \left( \| \partial_x M_3(\partial, t - \tau)F^2 \|_{L^2(D_4)} + \| \partial_x M_3(\partial, t - \tau)F^2 \|_{L^2(D_4)} \\
+ \| \partial_x (\nabla)^2 M_1(\partial, t - \tau)G^2 \|_{L^2(D_4)} \right) d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-1} (\| F^2 \|_{H^1} + \| G^2 \|_{H^2}) d\tau + \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \langle \tau \rangle^{-1} (\| F^2 \|_{H^1} + \| \partial_x F^2 \|_{H^2}) d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \langle \tau \rangle^{-1} (\tau)^{-1} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-1} \| V \|_3^2.
\]

Thus we get

\[
\| \partial_x (\nabla)^2 v \|_{L^2(D_1)} \lesssim \langle t \rangle^{-1} (\| V_0 \|_3 + \| V \|_3^2).
\]  

Due to Remark 3.2, we can get the desired result (6.12) by combining (6.9) with (6.10).

6.3. The estimate of (6.13). Using (3.12) and (4.13) for $k = 0$, we can infer

\[
\| L v \|_{L^1(D_1)} \lesssim \| M_3(\partial, t)v_0 \|_{L^1(D_1)} + \| M_1(\partial, t)B_0 \|_{L^1(D_1)} \lesssim \langle t \rangle^{-1} (\| v_0 \|_{L^1} + \| B_0 \|_{L^1}).
\]

It follows from (3.12), (6.5) and (4.13) for $k = 0$ that
\[
\sum_{i=1,2} \|NL_{vi}\|_{L^1_t(D_1)}
\]
\[
\lesssim \int_0^t \|M_3(\partial, t - \tau)F^2\|_{L^1_t(D_1)}d\tau + \int_0^t \|M_1(\partial, t - \tau)G^2\|_{L^1_t(D_1)}d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \|\nabla\|^{-1} \left( \tilde{u} \cdot \nabla \tilde{u}, \partial_y(Bb), \tilde{b} \cdot \nabla \tilde{b}, \tilde{b} \cdot \nabla \tilde{u} \right) \|L^2\|d\tau
\]
\[
+ \int_0^t \|\partial_x M_3(\partial, t - \tau)(bb)\|_{L^1_t(D_1)}d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \|\left( \tilde{u} \otimes \tilde{u}, Bb, \tilde{b} \otimes \tilde{b} \right) \|L^2\|d\tau + \int_0^t \|\partial_x M_3(\partial, t - \tau)(bb)\|_{L^1_t(D_1)}d\tau,
\]
where the first integral can be bounded by (6.8). If we use (4.13)_4, the second integral can be bounded by
\[
C \int_0^t \langle t - \tau \rangle^{-1} \|bb\|_{L^1_t}d\tau \lesssim \int_0^t \langle t - \tau \rangle^{-1} \|V\|_3 \lesssim \langle t \rangle^{-1+\epsilon} \|V\|_3^2
\]
for sufficiently small $\epsilon > 0$. However, there is an $\epsilon$-loss in the decay rate. To overcome this difficulty, we split the interval $[0, t]$ into $[0, t/2]$ and $[t/2, t]$ again. In fact, using (3.12), (4.11)_3 for $k = 1$ and $k = 0$, (4.17) and (4.18), we can bound the second integral by
\[
C \left( \int_0^{t/2} \langle t - \tau \rangle^{-3} \|bb\|_{L^1_t}d\tau + \int_{t/2}^t \langle t - \tau \rangle^{-3} \|b\|_{L^1_t}d\tau \right)
\]
\[
\lesssim \left( \int_0^{t/2} \langle t - \tau \rangle^{-3} \|\tilde{V}\|_2^2 d\tau + \int_{t/2}^t \langle t - \tau \rangle^{-3} \|\tilde{V}\|_2^2 d\tau \right) \|V\|_3^2
\]
\[
\lesssim \langle t \rangle^{-1} \|V\|_3^2.
\]
These estimates follow
\[
\sum_{i=1,2} \|NL_{vi}\|_{L^1_t(D_1)} \lesssim \langle t \rangle^{-1} \|V\|_3^2.
\]
So
\[
\|\tilde{v}\|_{L^1_t(D_1)} \lesssim \langle t \rangle^{-1} (\|V_0\|_3 + \|V\|_3^2).
\] (6.11)

Using (3.13), (4.13)_2, (4.11)_4 for $r = 1$, we can deduce
\[
\|L_v\|_{L^1_t(D_1)} \lesssim \|M_3(\partial, t)v_0\|_{L^1_t(D_1)} + \|M_1(\partial, t)B_0\|_{L^1_t(D_1)}
\]
\[
\lesssim \langle t \rangle^{-1} (\|v_0\|_{L^1_t \cap L^1_t(\|u\|_{L^4_t})} + \|B_0\|_{L^1} + \|\|\nabla\|^{1.5}B_0\|_{L^1_t(\|u\|_{L^4_t})})
\]
\[
\lesssim \langle t \rangle^{-1} \|V_0\|_3.
\]
Using (3.13), (6.3), (6.5), (4.13)_{10}, and (4.13)_{11}, we infer

\[ NL_{v1} \lesssim \int_0^t \| M_3(\partial, t - \tau) F^2 \|_{FL^1(D_4)} d\tau \]

\[ \lesssim \int_0^t \| R_1 M_3(\partial, t - \tau)(\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla \vec{b}) \|_{FL^1(D_4)} d\tau \]

\[ \lesssim \int_0^t \| R_1 M_3(\partial, t - \tau)(\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla B, \partial_y(\vec{b} B)) \|_{FL^1(D_4)} d\tau \]

\[ + \int_0^t \| \partial_x R_1 M_3(\partial, t - \tau)(\vec{b}) \|_{FL^1(D_4)} d\tau \]

\[ \lesssim \int_0^t (t - \tau)^{-\frac{5}{2}} \| (\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla B, \partial_y(\vec{b} B)) \|_{FL^1} d\tau \]

\[ + \int_0^t (t - \tau)^{-\frac{3}{2}} \| (bb, b\partial_x b) \|_{FL^1} d\tau \]

\[ \lesssim \int_0^t (t - \tau)^{-1} + (t - \tau)^{-\frac{3}{2}} d\tau \| V \|_3^2 \]

\[ \lesssim (t)^{-1} \| V \|_3^2 \]

and, by (6.4), (4.11)_{3} and (4.20), we can also get

\[ \| NL_{v2} \|_{FL^1(D_4)} \lesssim \int_0^t \| \partial_x M_1(\partial, t - \tau)(uB - bv) \|_{FL^1(D_4)} d\tau \]

\[ \lesssim \int_0^t (t - \tau)^{-\frac{3}{2}} \| v \|_{FL^1} d\tau \]

\[ \lesssim \int_0^t (t - \tau)^{-1} \| V \|_3^2 \lesssim (t)^{-1} \| V \|_3^2. \]

Thus

\[ \| \vec{v} \|_{L^1(D_4)} \lesssim \langle t \rangle^{-1}(\| V_0 \|_3 + \| V \|_3^2). \] (6.12)

Due to Remark 3.2, collecting (6.11) and (6.12) can yield (6.13). This completes the proof of (6.1).

7. The estimate on \( u \)

In this section, we will prove

\[
\begin{align*}
\| u(t) \|_{L^2} & \lesssim \langle t \rangle^{-\frac{1}{2}} (\| V_0 \|_3 + \| V \|_3^2); \\
\| \partial_x u(t) \|_{L^2} & \lesssim \langle t \rangle^{-\frac{1}{2}} (\| V_0 \|_3 + \| V \|_3^2 + \| V \|_3^3); \\
\| \partial_x u(t) \|_{H^2} & \lesssim \langle t \rangle^{-1} (\| V_0 \|_3 + \| V \|_3^2); \\
\| u(t) \|_{FL^1} & \lesssim \langle t \rangle^{-1} (\| V_0 \|_3 + \| V \|_3^2 + \| V \|_3^3); \\
\| \partial_x u(t) \|_{FL^1} & \lesssim \langle t \rangle^{-\frac{3}{2}} (\| V_0 \|_3 + \| V \|_3^2).
\end{align*}
\] (7.1)
7.1. The expression of $u$. By (3.6), we can obtain

$$\begin{aligned}
    u &= M_3(\partial, t)u_0 + M_1(\partial, t)b_0 + \int_0^t M_3(\partial, t - \tau)F^1 d\tau + \int_0^t M_1(\partial, t - \tau)G^1 d\tau \\
    &= M_3(\partial, t)u_0 + M_1(\partial, t)b_0 + \int_0^t M_3(\partial, t - \tau)F^1 d\tau + \int_0^t M_1(\partial, t - \tau)G^1 d\tau \\
    &- \int_0^t M_1(\partial, t - \tau)v\partial_y b d\tau,
\end{aligned}$$

where $\mathcal{N}L_u = \mathcal{N}L_{u2} - \mathcal{N}L_{u3}$ and

$$F^1 = -\vec{u} \cdot \nabla u + \partial_x p + \vec{b} \cdot \nabla b, \quad G^1 = G^{11} - v\partial_y b, \quad G^{11} = -u\partial_x b + \vec{b} \cdot \nabla u.$$

As Remark 6.1, we can prove (7.1) by the previous arguments yielding (6.1). Since the estimates of $M_i (i = 1, 2, 3)$ on $D_1$ are similar, following the proof of $\|\partial_x v\|_{L^2(D_1)}$ and $\|\hat{v}\|_{L^1(D_1)}$ yields

$$\|\nabla (\nabla)^2 u\|_{L^2(D_1)} + \|\hat{u}\|_{L^1(D_1)} \lesssim (t)^{-1}(\|V_0\|_3 + \|V\|_2^2). \quad (7.2)$$

Next, we turn to these estimates on $D_4$. When we use the arguments in the section 6, the integral on $v\partial_y b$ seems hard to be controlled. It is difficult to get the desired decay rate at least. So we shall seek some new approaches to overcome this difficulty. By frequency decomposition technique, we first obtain

$$\begin{aligned}
    \mathcal{N}L_{u3} &= \int_0^t M_1(\partial, t - \tau)(v_{<\langle r \rangle} - s\partial_y b + v_{>2\langle r \rangle} - 0.05 \partial_y b)d\tau \\
    &\quad + \int_0^t M_1(\partial, t - \tau)(P_\prec v\partial_y b)d\tau \\
    &: = \mathcal{N}L'_{u3} + \mathcal{N}L''_{u3},
\end{aligned}$$

where $P_\prec$ is given by (4.14). To obtain the desired estimate of $\mathcal{N}L''_{u3}$, we need rewriting its expression. Thanks to (1.16), then $v = \vec{R} \cdot \partial_x \vec{u}$. Using the formations of (1.2) and (1.2),
Remark 7.1. Here "\( \vec{b} \)" where unknowns. The process that acts on the term in which \( \partial_x \) is used to prove the estimate of \( \| \vec{b} \| \). Hence, we can get the new expression of \( J_1 = 3 - \int_0^t (\partial_y M_1(\partial, t - \tau)) \bigl[ P_x(\vec{b} \cdot \vec{b}) \bigr] d\tau \). Using (7.4), we have \( J_2 = J_2 + J_3 + \text{"other good parts"}, \)

where \( \vec{b} = H(\vec{b}) \) and \( P_x f = P_\leq 2 f \).

**Remark 7.1.** Here "other good parts" include two kinds of "good" cases: (1) the integral on the term in which \( \partial_x \) acts on \( P_x \); (2) the integral on the nonlinear term consisting of three unknowns. The process that \( \partial_x \) hits \( P_x \) brings the decay rate \((\tau)^{-1}\), while three unknowns shall bring the faster decay rate than two unknowns. So we can easily bound these cases, and omit the details in the following context.

Hence, we can get the new expression of \( u \):

\[
U = L_u + NL_{u1} + NL_{u2} + NL_{u3} - \sum_{i=1}^3 J_i - \text{"other good parts"}, \quad (7.4)
\]

where, by using (7.5), \( J_2 \) and \( J_3 \) can be rewritten as follows

\[
J_2 = \int_0^t (\partial_y M_1(\partial, t - \tau)) \bigl[ P_x(\vec{b} \cdot \vec{b}) \bigr] d\tau - \int_0^t \partial_y P_x(\vec{b} \cdot \vec{b}) d\tau
\]

and

\[
J_3 = \int_0^t \partial_x M_1(\partial, t - \tau) \bigl[ P_x(\vec{b} \cdot \vec{b}) \bigr] d\tau + \int_0^t M_1(\partial, t - \tau) \bigl[ \partial_x P_x(\vec{b} \cdot \vec{b}) \bigr] d\tau. \quad (7.6)
\]

(7.5) is used to prove the estimate of \( \| \partial_y u \|_{L^2_1(D_4)} \), but seems useless for the estimate of \( \| \vec{u} \|_{L^2_1(D_4)} \). Motivated by this fact, we give another expression of \( NL''_{u3} \). Using \( \partial_y v = -\partial_x u, \)
we have
\[
NL''_{u3} = \int_0^t M_1(\partial, t - \tau)(\partial_x P_r ub) d\tau + \int_0^t \partial_y M_1(\partial, t - \tau)(P_r vb) d\tau
\]
\[
= \int_0^t M_1(\partial, t - \tau)(\partial_x P_r ub) d\tau + \int_0^t \partial_y M_1(\partial, t - \tau)(P_r vb) d\tau
\]
\[
+ \int_0^t \partial_y M_1(\partial, t - \tau)(P_r vb) d\tau = O_1 + O_2 + O_3.
\]
Applying the similar techniques yielding the previous expression of \(NL''_{u3}\) to \(O_3\), we can get
\[
O_3 = \partial_y M_1(\partial, t - \tau)(P_r(\vec{R}' \cdot \vec{b}) b) \bigg|_{\tau = \frac{t}{2}} - \int_0^{t/2} (\partial_x \partial_y M_1(\partial, t - \tau))[P_r(\vec{R}' \cdot \vec{b}) b] d\tau
\]
\[
- \int_0^{t/2} \partial_y M_1(\partial, t - \tau)[P_r(\vec{R}' \cdot \vec{b}) \partial_x u] d\tau + \text{“other good parts”}
\]
\[
= O_{31} + O_{32} + O_{33} + \text{“other good parts”}.
\]
Thus we have
\[
u = L_u + NL_{u1} + NL_{u2} - NL'_{u3} - O_1 - O_2 - \sum_{i=1}^3 O_{3i} - \text{“other good parts”}, \quad (7.7)
\]
where, by using \(3.8\), \(O_{32}\) and \(O_{33}\) can be rewritten by
\[
O_{32} = \int_0^{t/2} (\partial_x \partial_y M_1)(\partial, t - \tau)[P_r(\vec{R}' \cdot \vec{b}) b] d\tau = \int_0^{t/2} (\partial_x \partial_y M_3)(\partial, t - \tau)[P_r(\vec{R}' \cdot \vec{b}) b] d\tau
\]
and
\[
O_{33} = - \int_0^{t/2} \partial_x \partial_y M_1(\partial, t - \tau)[P_r(\vec{R}' \cdot \vec{b}) u] d\tau + \int_0^{t/2} \partial_y M_1(\partial, t - \tau)[\partial_x P_r(\vec{R}' \cdot \vec{b}) u] d\tau
\]
\[
= - \int_0^{t/2} \partial_x M_1(\partial, t - \tau)[\partial_y P_r(\vec{R}' \cdot \vec{b}) u] d\tau - \int_0^{t/2} \partial_x M_1(\partial, t - \tau)[P_r(\vec{R}' \cdot \vec{b}) \partial_y u] d\tau
\]
\[
+ \int_0^{t/2} \partial_y M_1(\partial, t - \tau)[\partial_x P_r(\vec{R}' \cdot \vec{b}) u] d\tau.
\]

7.2. The estimate of \((7.1)_2\). Due to \((7.2)\), it suffices to bound \(\|\partial_y L_u\|_{F L^2(D_4)}\). We shall use \((7.4)\) to achieve the goal. Thanks to \(3.13\), \(4.13\) and \(4.14\) for \(r = 2\) and \(\delta = 0.01\), we have
\[
\|\partial_y L_u\|_{F L^2(D_4)} \lesssim \|\partial_y M_3(\partial, t) u_0\|_{F L^2(D_4)} + \|\partial_y M_1(\partial, t) b_0\|_{F L^2(D_4)}
\]
\[
\lesssim \langle t \rangle^{-1} \|u_0\|_{L^1 \cap H^1} + \langle t \rangle^{-\frac{5}{2}} \|\nabla\|^0.5 b_0\|_{L^1_1(L^3)}
\]
\[
\lesssim \langle t \rangle^{-\frac{5}{2}} \|V_0\|_3.
\]
We deduce from (3.13), (4.11), (4.17) and (4.19) that

\[
\|\partial_yNL_{u1}\|_{L^2(D_4)} \lesssim \int_0^t \|\partial_y M_3(\partial, t - \tau)(\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla B, \partial_y(B\vec{b}))\|_{L^2(D_4)} d\tau \\
+ \int_0^t \|\partial_y M_3(\partial, t - \tau)(\vec{b}_x b)\|_{L^2(D_4)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \|\vec{b} \cdot \nabla B\|_{H^2} + \|\vec{u} \otimes \vec{u}\|_{H^2} d\tau \\
+ \int_0^t \langle t - \tau \rangle^{-1} \|\vec{b}_x b\|_{H^1 \cap L^1} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \|\langle \tau \rangle^{-1.01} + \langle t - \tau \rangle^{-1} \|V\|_2^2 d\tau \\
\lesssim \langle t \rangle^{-0.89} \|V\|_3^2.
\]

By (3.13), (4.11) for \( r = 2 \) and \( \delta = 0.01 \), (4.18), one has

\[
\|\partial_y NL_{u2}\|_{L^2(D_4)} + \|\partial_y NL_{\omega_3}\|_{L^2(D_4)} \\
\lesssim \int_0^t \|\partial_y M_3(\partial, t - \tau)(u\partial_y b, \vec{b} \cdot \nabla u, v_{<(\tau)} - s\partial_y b, v_{>2(\tau)} - 0.05 \partial_y b)\|_{L^2(D_4)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{4}{3}} \|\langle \nabla \rangle^{0.51} (u\partial_y b, \vec{b} \cdot \nabla u, v_{<(\tau)} - s\partial_y b, v_{>2(\tau)} - 0.05 \partial_y b)\|_{L^2(L^2)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \|\langle \tau \rangle^{-1.01} \|V\|_3^2 d\tau \\
\lesssim \langle t \rangle^{-\frac{4}{3}} \|V\|_3^2.
\]

Similarly, by (3.13), (4.11) for \( r = 2 \) and \( \delta = 0.01 \), we infer

\[
\|\partial_y J_1\|_{L^2(D_4)} \lesssim \langle t \rangle^{-\frac{4}{3}} \|\langle \nabla \rangle^{0.51} \{P_{<1}(\vec{R}' \cdot \vec{b}_0)\partial_y b_0\}\|_{L^2(L^2)} \\
\lesssim \langle t \rangle^{-\frac{4}{3}} \|\langle \nabla \rangle^2 \{P_{<1}(\vec{R}' \cdot \vec{b}_0)\partial_y b_0\}\|_{L^2} \\
\lesssim \langle t \rangle^{-\frac{4}{3}} \|P_{<1}(\vec{R}' \cdot \vec{b}_0)\|_{H^2} \|\partial_y b_0\|_{H^2} \\
\lesssim \langle t \rangle^{-\frac{4}{3}} \|\vec{b}_0\|_{H^2} \|\partial_y b_0\|_{H^2} \lesssim \langle t \rangle^{-\frac{4}{3}} \|V_0\|_3^2.
\]

For the estimate on \( J_2 \), it is sufficient to bound \( J_{21}, J_{22} \) and \( J_{23} \). We obtain, by using (3.13), (4.11)\_6, and (4.19), that

\[
\|\partial_y J_{21}\|_{L^2(D_4)} + \|\partial_y J_{22}\|_{L^2(D_4)} \\
\lesssim \int_0^t \|\partial_y^2 M_3(\partial, t - \tau)(\partial_x P_{<}(\vec{R}' \cdot \vec{b}) b, P_{<}(\vec{R}' \cdot \vec{b}) \partial_x b)\|_{L^2(D_4)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \|\partial_x P_{<}(\vec{R}' \cdot \vec{b}) b\|_{H^2} + \|P_{<}(\vec{R}' \cdot \vec{b}) \partial_x b\|_{H^2} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \|\langle \tau \rangle^{-1.01} \|V\|_3^2 d\tau \\
\lesssim \langle t \rangle^{-1} \|V\|_3^2.
\]
For \( J_{23} \), using (3.13), (4.13) and (4.19), we have

\[
\|\partial_y J_{23}\|_{L^2(D_4)} \lesssim \int_0^t \|\partial_x \partial_y M_3(\partial, t - \tau)(\partial_y P_\cdot(\vec{R}' \cdot \vec{b})b)\|_{L^2(D_4)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{4}}(\|\partial_x (\partial_y P_\cdot(\vec{R}' \cdot \vec{b})b)\|_{L^1} + \|\partial_y P_\cdot(\vec{R}' \cdot \vec{b})b\|_{L^2}) d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{4}} \langle \tau \rangle^{-\frac{4}{9}} d\tau \lesssim \langle t \rangle^{-\frac{4}{9}} \|V\|_3^2.
\]

It follows from (3.13), (4.11) for \( r = 2 \), (4.11) for \( r = 2 \) and \( \delta = 0.01 \), (4.19) and (4.18) that

\[
\|\partial_y J_3\|_{L^2(D_4)} \lesssim \int_0^t \|\partial_x \partial_y M_1(\partial, t - \tau)(P_\cdot(\vec{R}' \cdot \vec{b})\partial_y u)\|_{L^2(D_4)} d\tau \\
+ \int_0^t \|\partial_y M_1(\partial, t - \tau)(\partial_x P_\cdot(\vec{R}' \cdot \vec{b})\partial_y u)\|_{L^2(D_4)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \|P_\cdot(\vec{R}' \cdot \vec{b})\partial_y u\|_{L^1} d\tau \\
+ \int_0^t \langle t - \tau \rangle^{-\frac{2}{3}} \|\langle \nabla \rangle^{0.51} (\partial_x P_\cdot(\vec{R}' \cdot \vec{b})\partial_y u)\|_{L^1} d\tau \\
\lesssim \int_0^t \langle (t - \tau)^{-1} \rangle^{-1.01} \langle t - \tau \rangle^{-\frac{3}{4}} \langle \tau \rangle^{-1.01} d\tau \|V\|_3^2 \\
\lesssim \langle t \rangle^{-\frac{4}{9}} \|V\|_3^2.
\]

“Other good parts” on \( D_4 \) can be bounded by \( C \langle t \rangle^{-\frac{3}{4}}(\|V\|_3 + \|V\|_3^2) \) by using similar arguments, so

\[
\|\partial_y u\|_{L^2(D_4)} \lesssim \langle t \rangle^{-\frac{4}{9}}(\|V\|_3 + \|V\|_3^2 + \|V\|_3^2).
\]

Hence, we have proved (7.12).

7.3. The estimate of (7.13). Thanks to (7.2), it is sufficient to estimate \( \|\partial_x \langle \nabla \rangle^2 u\|_{L^2(D_4)} \). Using (3.13), (4.13) and (4.11) for \( r = 2 \), we get

\[
\|\partial_x \langle \nabla \rangle^2 u\|_{L^2(D_4)} \lesssim \|\partial_x \langle \nabla \rangle^2 M_3(\partial, t)u_0\|_{L^2(D_4)} + \|\partial_x \langle \nabla \rangle^2 M_1(\partial, t)b_0\|_{L^2(D_4)} \\
\lesssim \langle t \rangle^{-1} \|u_0\|_{H^3} + \|b_0\|_{H^2} \lesssim \langle t \rangle^{-1} \|V\|_3.
\]

Using (3.13), (4.13) for \( r = 2 \) and (4.13) for (4.13), and (4.19), we can obtain

\[
\|\partial_x \langle \nabla \rangle^2 NLu_1\|_{L^2(D_4)} \lesssim \|\partial_x NLu_1\|_{L^2(D_4)} + \|\partial_x NLu_1\|_{L^2(D_4)} \\
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \|\partial_x (\vec{u} \cdot \nabla \vec{u} \cdot \nabla \vec{b})\|_{H^1} d\tau \\
+ \int_0^t \langle t - \tau \rangle^{-\frac{3}{4}} \|\partial_x (\vec{u} \cdot \nabla \vec{u} \cdot \nabla \vec{b})\|_{H^2} d\tau \\
\lesssim \int_0^t \langle (t - \tau)^{-1} \rangle^{-1.01} \langle t - \tau \rangle^{-\frac{3}{4}} \langle \tau \rangle^{-1.01} d\tau \|V\|_3^2 \\
\lesssim \langle t \rangle^{-1} \|V\|_3^2.
\]
We can deduce from (3.13), (4.13) for \( r = 2 \) and (4.19) that
\[
\| \partial_x (\nabla)^2 NL u \|_{L^2(D_4)} \lesssim \int_0^t \| \partial_x (\nabla)^2 M_1(\partial_x, t - \tau)(\vec{u} \cdot \nabla b, \vec{b} \cdot \nabla u) \|_{L^2(D_4)} d\tau
\]
\[
\lesssim \int_0^t (t - \tau)^{-1} (\| \vec{u} \cdot \nabla b \|_{H^2} + \| \vec{b} \cdot \nabla u \|_{H^2}) d\tau
\]
\[
\lesssim \int_0^t (t - \tau)^{-1} (\tau)^{-1} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-1} \| V \|_3^2.
\]
As a consequence, we have
\[
\| \partial_x (\nabla)^2 u \|_{L^2(D_4)} \lesssim \langle t \rangle^{-1} (\| V \|_3 + \| V \|_3^2).
\]
Then we complete the proof of (7.1)_3.

7.4. The estimate of (7.1)_4. In this subsection, we shall use (7.7) to bound \( \| \vec{u} \|_{L^1(D_4)} \). Like the derivation of the estimate of \( \| L_v \|_{L^1(D_4)} \), one can get
\[
\| L_u \|_{L^1(D_4)} \lesssim \langle t \rangle^{-1} \| V \|_3.
\]
Using (6.5), we have
\[
\| NL u_1 \|_{L^1(D_4)} \lesssim \int_0^t \| M_3(\partial_x, t - \tau)P^1 \|_{L^1(D_4)} d\tau
\]
\[
\lesssim \int_0^t \| M_3(\partial_x, t - \tau)(\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla B, \partial_y(Bb)) \|_{L^1(D_4)} d\tau
\]
\[
+ \int_0^t \| \partial_x M_3(\partial_x, t - \tau)(bb) \|_{L^1(D_4)} d\tau.
\]
Using (3.13), (4.13)\_2, (4.13)\_5, (4.17) - (4.20), the first integral can be bounded by
\[
C \int_0^t (t - \tau)^{-1} (\| \nabla \|^{-1}(\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla B, \partial_y(Bb)) \|_{L^2(D_4)} + \| (\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla B, \partial_y(Bb)) \|_{L^1(D_4)}) d\tau
\]
\[
\lesssim \int_0^t (t - \tau)^{-1} (\tau)^{-1} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-1} \| V \|_3^2,
\]
while the second integral can be bounded by
\[
C \left( \int_0^t (t - \tau)^{-\frac{3}{2}} (\| bb \|_{L^2(D_4)} + \| b \partial_x b \|_{L^1(D_4)}) d\tau
\]
\[
+ \int_{\frac{t}{2}}^t (t - \tau)^{-\frac{3}{2}} (\| b \partial_x b \|_{L^1(D_4)} + \| b \partial_x b \|_{L^1(D_4)}) d\tau
\]
\[
\lesssim \left( \int_0^t (t - \tau)^{-\frac{3}{2}} (t - \tau)^{-\frac{3}{2}} d\tau + \int_{\frac{t}{2}}^t (t - \tau)^{-\frac{3}{2}} (t - \tau)^{-\frac{3}{2}} d\tau \right) \| V \|_3^2
\]
\[
\lesssim \langle t \rangle^{-1} \| V \|_3^2.
\]
Thus
\[
\| NL u_1 \|_{L^1(D_4)} \lesssim \langle t \rangle^{-1} \| V \|_3^2.
\]
It follows from \((3.13), (4.11)\) and \((4.17), (4.18)\) that

\[
\|NL_{u2}\|_{\mathcal{L}^1(D_4)} \lesssim \int_0^t \|M_1(\partial, t - \tau)(u \partial_x b, \vec{b} \cdot \nabla u)\|_{\mathcal{L}^1(D_4)} d\tau
\]

\[
\lesssim \int_0^t (t - \tau)^{-1} \left( \|\langle \nabla \rangle^{1.51} (u \partial_x b, \vec{b} \cdot \nabla u)\|_{L^1_t(L^2_y)} + \|(u \partial_x b, \vec{b} \cdot \nabla u)\|_{L^1_t(L^2_y)} \right) d\tau
\]

\[
\lesssim \int_0^t (t - \tau)^{-1} \langle \tau \rangle^{-1.01} d\tau \|V\|_3^2
\]

\[
\lesssim (t)^{-1} \|V\|_3^2.
\]

Using \((3.13), (4.11)\) for \(r = 1\) and \(\delta = 0.01\), \((4.11)\), \((4.17)\) and \((4.18)\) that

\[
\|NL_{u3}'\|_{\mathcal{L}^1(D_4)} \lesssim \int_0^t \|M_1(\partial, t - \tau)(v_{<\langle \tau \rangle} - \partial_y b, v_{>2\langle \tau \rangle} - 0.05 \partial_y b)\|_{\mathcal{L}^1(D_4)} d\tau
\]

\[
\lesssim \int_0^t \|\partial_y M_1(\partial, t - \tau)(v_{<\langle \tau \rangle} - \partial_y b, v_{>2\langle \tau \rangle} - 0.05 b)\|_{\mathcal{L}^1(D_4)} d\tau
\]

\[
+ \int_0^t \|M_1(\partial, t - \tau)(\partial_y v_{<\langle \tau \rangle} - \partial_y b, \partial_y v_{>2\langle \tau \rangle} - 0.05 b)\|_{\mathcal{L}^1(D_4)} d\tau
\]

\[
\lesssim \int_0^t (t - \tau)^{-1} \|\langle \nabla \rangle^{1.51} (v_{<\langle \tau \rangle} - \partial_y b, v_{>2\langle \tau \rangle} - 0.05 b)\|_{L^1_t(L^2_y)} d\tau
\]

\[
+ \int_0^t (t - \tau)^{-1} \left( \|\langle \partial_x v_{<\langle \tau \rangle} - \partial_x b, \partial_x v_{>2\langle \tau \rangle} - 0.05 b\|_{L^1_t(L^2_y)} \right) d\tau
\]

\[
\lesssim \int_0^t (t - \tau)^{-1} \langle \tau \rangle^{-1.01} d\tau \|V\|_3^2 \lesssim (t)^{-1} \|V\|_3^2.
\]

We can get by using \((3.13), (4.11)\),

\[
\|\partial_x P_{\nu} ub\|_{L^1} \lesssim \|\partial_x u\|_{L^2} \|b\|_{L^2} \lesssim (t)^{-\frac{5}{2}} \|V\|_3^2
\]

and \((4.18)\) that

\[
\|\widehat{O}_1\|_{L^1(D_4)} \lesssim \int_0^t \|M_1(\partial, t - \tau)(\partial_x P_{\nu} ub)\|_{\mathcal{L}^1(D_4)} d\tau
\]

\[
\lesssim \int_0^t (t - \tau)^{-1} \left( \|\partial_x P_{\nu} ub\|_{L^1_t} + \|\langle \nabla \rangle^{1.51} (\partial_x P_{\nu} ub)\|_{L^1_t(L^2_y)} \right) d\tau
\]

\[
\lesssim \int_0^t (t - \tau)^{-1} \langle \tau \rangle^{-1.01} d\tau \|V\|_3^2 \lesssim (t)^{-1} \|V\|_3^2.
\]

Applying \((3.13), (4.11)\), we have

\[
\|\widehat{O}_2\|_{L^1(D_4)} \lesssim \int_{t/2}^t \|\partial_y M_1(\partial, t - \tau)(P_{\nu} v b)\|_{\mathcal{L}^1(D_4)} d\tau
\]

\[
\lesssim \int_{t/2}^t (t - \tau)^{-\frac{1}{2}} \|\widehat{v}\|_{L^1_t} \|\widehat{b}\|_{L^1_t} d\tau
\]

\[
\lesssim \int_{t/2}^t (t - \tau)^{-\frac{3}{2}} \langle \tau \rangle^{-\frac{1}{2}} d\tau \|V\|_3^2
\]

\[
\lesssim (t)^{-1} \|V\|_3^2.
\]
It follows from (3.13), (4.11)$_2$, and (4.20), (4.18) that

$$\|\widehat{O}_{31}\|_{L^1(D_4)} \lesssim \|\partial_y M_1(\partial, t, \frac{t}{2})(P_{\infty}^{\tau} \cdot \vec{b})b(x)\|_{L^1(D_4)} + \|\partial_y M_1(\partial, t)(P_{\infty}^{\tau} \cdot \vec{b})b_0\|_{L^1(D_4)}$$

$$\lesssim \langle t \rangle^{-\frac{3}{2}} \|\partial_y M_1(\partial, t, \frac{t}{2})(P_{\infty}^{\tau} \cdot \vec{b})b(x)\|_{L^1(D_4)} + \langle t \rangle^{-1} \|\langle \nabla \rangle^{1.51}(P_{\infty}^{\tau} \cdot \vec{b})b_0\|_{L^1_2(L_3^2)}$$

$$\lesssim \langle t \rangle^{-1}(\|V\|_3^3 + \|V_0\|_3^3).$$

Using (3.13), (4.13)$_8$, and (4.20), we can deduce

$$\|\widehat{O}_{32}\|_{L^1(D_4)} \lesssim \int_0^{t/2} \|\partial_x \partial_y M_3(\partial, t - \tau)(P_{\infty}^{\tau} \cdot \vec{b})b\|_{L^1(D_4)} d\tau$$

$$\lesssim \int_0^{t/2} \langle t - \tau \rangle^{-\frac{3}{2}}(\|\partial_x \partial_y M_3(\partial, t, \tau)(P_{\infty}^{\tau} \cdot \vec{b})b\|_{L^1(D_4)}) d\tau$$

$$\lesssim \int_0^{t/2} \langle t - \tau \rangle^{-\frac{3}{2}}(\|\partial_x \partial_y (P_{\infty}^{\tau} \cdot \vec{b})b\|_{L^1(D_4)}) d\tau$$

$$\lesssim \int_0^{t/2} \langle t - \tau \rangle^{-0.99} t^2 \langle t \rangle^{-1} \|V\|_3^2 \lesssim \langle t \rangle^{-1.4} \|V\|_3^2.$$ 

Thanks to (3.13), (4.11)$_3$ for $r = 1$ and (4.11)$_2$ for $r = 1$, one has

$$\|\widehat{O}_{33}\|_{L^1(D_4)} \lesssim \int_0^{t/2} \|\partial_x M_1(\partial, t - \tau)(\partial_y P_{\infty}^{\tau} \cdot \vec{b})u\|_{L^1(D_4)} d\tau$$

$$+ \int_0^{t/2} \|\partial_x M_1(\partial, t - \tau)(P_{\infty}^{\tau} \cdot \vec{b})\partial_y u\|_{L^1(D_4)} d\tau$$

$$+ \int_0^{t/2} \|\partial_y M_1(\partial, t - \tau)(\partial_x P_{\infty}^{\tau} \cdot \vec{b})u\|_{L^1(D_4)} d\tau$$

$$\lesssim \int_0^{t/2} \langle t - \tau \rangle^{-1}(\|\partial_y P_{\infty}^{\tau} \cdot \vec{b})u\|_{L^1(D_4)} + \|P_{\infty}^{\tau} \cdot \vec{b})\partial_y u\|_{L^1(D_4)}$$

$$+ \|\langle \nabla \rangle^{1.51}(\partial_x P_{\infty}^{\tau} \cdot \vec{b})u\|_{L^2_2(L_3^2)}) d\tau$$

$$\lesssim \int_0^{t/2} \langle t - \tau \rangle^{-1} \langle t \rangle^{-1.01} t^2 \langle t \rangle^{-1} \|V\|_3^2 \lesssim \langle t \rangle^{-1} \|V\|_3^2.$$ 

The estimate on “other good parts” can be bounded by $C(t)^{-1}(\|V\|_3^3 + \|V_0\|_3^3)$. Hence, we can obtain

$$\|u\|_{L^1_1(D_4)} \lesssim \langle t \rangle^{-1}(\|V_0\|_3 + \|V_0\|_3^2 + \|V\|_3^2 + \|V\|_3^3),$$

which completes the proof of (7.1)$_4$.

7.5. The estimate of (7.1)$_5$. Using (3.12), $|\vec{\xi}| \lesssim 1$ in $D_1$, (4.11)$_3$ for $k = 1$, we have

$$\|\partial_x L_u\|_{L^1_1(D_1)} \leq \|\partial_x M_3(\partial, t)u_0\|_{L^1_1(D_1)} + \|\partial_x M_1(\partial, t)b_0\|_{L^1_1(D_1)} \lesssim \langle t \rangle^{-\frac{3}{2}}(\|u_0\|_{L^1} + \|b_0\|_{L^1}).$$
Thanks to (3.12), (4.1) for $k = 1$, (4.18) and (4.19), one can get
\[
\| \partial_x N L u_1 \|_{F_{L^1,1}(D_1)} \leq \int_0^t \| \partial_x M_3(\partial, t - \tau) F^1 \|_{F_{L^1,1}(D_1)} d\tau
\]
\[
\lesssim \int_0^t \| \partial_x M_3(\partial, t - \tau) (\bar{u} \cdot \nabla \bar{u}, \bar{b} \cdot \nabla B, \partial_y (Bb)) \|_{F_{L^1,1}(D_1)} d\tau
\]
\[
+ \int_0^t \| \partial_x M_3(\partial, t - \tau) (b \partial_x b) \|_{F_{L^1,1}(D_1)} d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \| \nabla^{-1} (\bar{u} \cdot \nabla \bar{u}, \bar{b} \cdot \nabla B, \partial_y (Bb)) \|_{L^2,1} d\tau
\]
\[
+ \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \| b \partial_x b \|_{L^1,1} d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \langle \tau \rangle^{-\frac{5}{2}} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-\frac{5}{2}} \| V \|_3^2.
\]
It follows from (3.12), (4.1) for $k = 1$ and
\[
\| \partial_x (b \bar{u} - u \bar{b}) \|_{L^1} \lesssim \| \bar{b} \|_{L^2} \| \partial_x \bar{u} \|_{L^2} + \| \bar{u} \|_{L^2} \| \partial_x \bar{b} \|_{L^2} \lesssim \langle t \rangle^{-\frac{5}{2}} \| V \|_3^2,
\]
that
\[
\| \partial_x N L u \|_{F_{L^1,1}(D_1)} \leq \int_0^t \| \partial_x M_1(\partial, t - \tau) \text{div}(b \bar{u} - u \bar{b}) \|_{F_{L^1,1}(D_1)} d\tau
\]
\[
\lesssim \int_0^t \| M_1(\partial, t - \tau) \text{div} [\partial_x (b \bar{u} - u \bar{b})] \|_{F_{L^1,1}(D_1)} d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \| \partial_x (b \bar{u} - u \bar{b}) \|_{L^1} d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \langle \tau \rangle^{-\frac{5}{2}} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-\frac{5}{2}} \| V \|_3^2.
\]
Thus
\[
\| \partial_x u \|_{F_{L^1,1}(D_1)} \lesssim \langle t \rangle^{-\frac{5}{2}} (\| V_0 \|_3 + \| V \|_3^2).
\tag{7.8}
\]
Next, we bound the estimate on $D_4$. Using (3.13), (4.13) for $r = 1$, and (4.11) for $r = 1$, we find
\[
\| \partial_x L u \|_{F_{L^1,1}(D_1)} \lesssim \| \partial_x M_3(\partial, t) u_0 \|_{F_{L^1,1}(D_4)} + \| \partial_x M_1(\partial, t) b_0 \|_{F_{L^1,1}(D_1)}
\]
\[
\lesssim \langle t \rangle^{-\frac{5}{2}} (\| u_0 \|_{L^2,1} + \| \partial_x u \|_{F_{L^1,1}} + \| b_0 \|_{L^1})
\]
\[
\lesssim \langle t \rangle^{-\frac{5}{2}} \| V_0 \|_3.
\]
Applying (3.13), (4.13) for \( r = 1 \), (4.13) for \( r = 1 \), and (4.20), one has

\[
\|\partial_t N u_1\|_{L_1^1(D_4)} \lesssim \int_0^t \|\partial_t M_3(\partial, t - \tau) F_1\|_{L_1^1(D_4)} d\tau \\
\lesssim \int_0^t \|\partial_x M_3(\partial, t - \tau)(\vec{u} \cdot \nabla \vec{u})\|_{L_1^1(D_4)} d\tau \\
+ \int_0^t \|\partial_x^2 M_3(\partial, t - \tau) (bb)\|_{L_1^1(D_4)} d\tau \\
\lesssim \int_0^t (t - \tau)^{-\frac{3}{2}} (\|\vec{u} \otimes \vec{u}\|_{L_1^1} + \|\partial_x (\vec{u} \cdot \nabla \vec{u})\|_{L_1^1} \\
+ \|B\vec{u}\|_{L_1^1} + \|\partial_x \nabla (\vec{b} B)\|_{L_1^1} + \|b\|_{L_1^1} + \|\partial_x (b \partial_x b)\|_{L_1^1}) d\tau \\
+ \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} (\|b\partial_x b\|_{L_1^1} + \|\partial_x (b \partial_x b)\|_{L_1^1}) d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \langle \tau \rangle^{-\frac{1}{3}} d\tau + \int_0^t \langle t - \tau \rangle^{-\frac{1}{3}} d\tau \\
+ \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \langle \tau \rangle^{-\frac{1}{3}} d\tau \lesssim \langle t \rangle \langle \frac{3}{2} \rangle \|V\|_3^2.
\]

We can infer from (3.13), (4.11) for \( r = 1 \) and \( \delta = 0.01 \) that

\[
\|\partial_x N L u\|_{L_1^1(D_4)} \lesssim \int_0^t \|\partial_x M_1(\partial, t - \tau)(\vec{u} \cdot \nabla \vec{u} + \vec{b} \cdot \nabla \vec{u})\|_{L_1^1(D_4)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \langle \tau \rangle^{-\frac{1}{3}} d\tau + \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \langle \tau \rangle^{-\frac{1}{3}} d\tau + \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \langle \tau \rangle^{-0.75} d\tau \|V\|_3^2 \\
\lesssim \langle t \rangle \langle \frac{3}{2} \rangle \|V\|_3^2.
\]

Therefore, it comes out

\[
\|\partial_x u\|_{L_1^1(D_4)} \lesssim \langle t \rangle^{-\frac{3}{2}} (\|V_0\|_3 + \|V\|_3^2). \tag{7.9}
\]

Collecting the above estimates (7.8) and (7.9) can yield (7.10).
8. The estimate on $b$

In this section, we will prove

\[
\left\{ \begin{align*}
\| b(t) \|_{L^2} & \lesssim \langle t \rangle^{-\frac{3}{2}} (\| V_0 \|_3 + \| V_0 \|_3^2 + \| V_\beta \|_3^3); \\
\| \partial_x b(t) \|_{H^1} & \lesssim \langle t \rangle^{-\frac{3}{2}} (\| V_0 \|_3 + \| V_0 \|_3^2 + \| V_\beta \|_3^3); \\
\| |\nabla|^{-1} (\nabla) b(t) \|_{L^1} & \lesssim \langle t \rangle^{-\frac{3}{2}} (\| V_0 \|_3 + \| V_0 \|_3^2 + \| V_\beta \|_3^3); \\
\| \mathcal{R}_1 (\nabla) b(t) \|_{L^1} & \lesssim \langle t \rangle^{-1} (\| V_0 \|_3 + \| V_0 \|_3^2 + \| V_\beta \|_3^3).
\end{align*} \right. \tag{8.1}
\]

8.1. The expression of $b$. Recalling (3.5), it is easy to get

\[
b = M_1(\partial, t) u_0 + M_2(\partial, t) b_0 + \int_0^t M_1(\partial, t - \tau) F^1 d\tau + \int_0^t M_2(\partial, t - \tau) G^1 d\tau
\]

\[
= M_1(\partial, t) u_0 + M_2(\partial, t) b_0 + \int_0^t M_1(\partial, t - \tau) F^1 d\tau
\]

\[
+ \int_0^t M_2(\partial, t - \tau) G^{11} d\tau - \int_0^t M_2(\partial, t - \tau) v \partial_y b d\tau.
\]

As the same reason in the previous section, we need some new expression of $b$ to overcome the difficulty coming from the estimate of $v \partial_y b$. Similar to the derivation of the expression of $NL_{u3}$, we can get the expression of $NL_{b3}$. As a matter of fact,

\[
NL_{b3} = NL_{b3}' + NL_{b3}'';
\]

where

\[
NL_{b3}' = \int_0^t M_2(\partial, t - \tau)(v_{<\tau} - s \partial_y b) d\tau + \int_0^t M_2(\partial, t - \tau)(v_{>2(\tau) - 0.05} \partial_y b) d\tau
\]

\[
= \int_0^t \partial_y M_2(\partial, t - \tau)(v_{<\tau} - s b + v_{>2(\tau) - 0.05} b) d\tau
\]

\[
- \int_0^t M_2(\partial, t - \tau)(\partial_y v_{<\tau} - s b + \partial_y v_{>2(\tau) - 0.05} b) d\tau,
\]

and

\[
NL_{b3}'' = - M_2(\partial, t)(P_{-1}(\tilde{R} \cdot \tilde{b}) \partial_y b_0) - \int_0^t \left( \partial_x M_2(\partial, t - \tau) \right) \left[ P_{-1}(\tilde{R} \cdot \tilde{b}) \partial_y b \right] d\tau
\]

\[
- \int_0^t M_2(\partial, t - \tau) \left[ P_{-1}(\tilde{R} \cdot \tilde{b}) \partial_y \partial_x u \right] d\tau + \text{“other good parts”}
\]

\[
=L_1 + L_2 + L_3 + \text{“other good parts”}.
\]

Thus we have

\[
b = L_b + NL_{b1} + NL_{b2} - NL_{b3}' - \sum_{i=1}^3 L_i - \text{“other good parts”},
\]
where, by (3.8), $L_2$ and $L_3$ can be rewrite as follows

$$L_2 = \int_0^t \left( \partial_y M_1 \right) (\partial, t - \tau) \left[ P_\omega (\vec{R} \cdot \vec{b}) \partial_y \vec{b} \right] d\tau$$

$$= \int_0^t \left( \partial_y \partial_x M_1 \right) (\partial, t - \tau) \left[ P_\omega (\vec{R} \cdot \vec{b}) \partial_y \vec{b} \right] d\tau - \int_0^t \left( \partial_x M_1 \right) (\partial, t - \tau) \left[ \partial_y P_\omega (\vec{R} \cdot \vec{b}) \partial_y \vec{b} \right] d\tau$$

$$= \int_0^t \left( \partial_y M_1 (\partial, t - \tau) \right) \partial_x \left[ P_\omega (\vec{R} \cdot \vec{b}) \partial_y \vec{b} \right] d\tau - \int_0^t \left( \partial_x M_1 (\partial, t - \tau) \right) \left[ \partial_y P_\omega (\vec{R} \cdot \vec{b}) \partial_y \vec{b} \right] d\tau$$

and

$$L_3 = - \int_0^t \partial_x M_2 (\partial, t - \tau) \left[ P_\omega (\vec{R} \cdot \vec{b}) \partial_y \vec{b} \right] d\tau + \int_0^t M_2 (\partial, t - \tau) \left[ \partial_x P_\omega (\vec{R} \cdot \vec{b}) \partial_y \vec{b} \right] d\tau.$$ 

Following the idea dealing with $\| \vec{v} \|_{L^2(D_1)}$ and $\| \partial_x v \|_{F^L L^2(D_1)}$, one can easily obtain

$$\| \vec{b} \|_{L^2(D_1)} \lesssim \langle t \rangle^{-\frac{1}{2}} (\| V_0 \|_3 + \| V \|_3^2),$$

$$\| \partial_x \nabla b \|_{F^L L^2(D_1)} \lesssim \| \partial_x b \|_{F^L L^2(D_1)} \lesssim \langle t \rangle^{-1} (\| V_0 \|_3 + \| V \|_3^2).$$

### 8.2. The estimate of (8.1). Thanks to (8.2), it suffices to give the estimate of $\| \vec{b} \|_{L^2(D_4)}$.

Using (3.13), (4.11), (4.12) for $r = 2$, we can get

$$\| L_0 \|_{F^L L^2(D_4)} \lesssim \| M_1 (\partial, t) u_0 \|_{F^L L^2(D_4)} + \| M_2 (\partial, t) b_0 \|_{F^L L^2(D_4)}$$

$$\lesssim \langle t \rangle^{-\frac{1}{2}} (\| u_0 \|_{L^1 \cap L^2} + \| \nabla b_0 \|_{L^1(L^2)})$$

$$\lesssim \langle t \rangle^{-\frac{1}{2}} \| V_0 \|_3.$$ 

One can get from (3.13), (4.11), (4.12) for $r = 2$, (4.18) and (4.19) that

$$\sum_{i=1,2} \| NL_{b,i} \|_{F^L L^2(D_4)} \lesssim \int_0^t \langle \| M_1 (\partial, t - \tau) F^1 \|_{F^L L^2(D_4)} + \| M_2 (\partial, t - \tau) G^{11} \|_{F^L L^2(D_4)} \rangle d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| \nabla \|^{-1} F^1 \|_{L^2} + \langle t - \tau \rangle^{-\frac{1}{2}} \| \nabla \|^0.51 G^{11} \|_{L^1(L^2)} \rangle d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| (\vec{u} \otimes \vec{u}, \vec{b} \otimes \vec{b}) \|_{L^2} d\tau$$

$$+ \int_0^t \langle t - \tau \rangle^{-\frac{3}{2}} \| \nabla \|^{0.51} (u \partial_x b, \vec{b} \cdot \nabla u) \|_{L^1(L^2)} d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| \nabla \|^{-1} (\tau)^{-\frac{1}{4}} \| (t - \tau)^{-\frac{1}{4}} (\tau)^{-1.01}) d\tau \| V \|_3^2$$

$$\lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2.$$ 

Thanks to (3.13), (4.12) for $r = 2$, and (4.18), we can obtain

$$\| NL_{b,3} \|_{F^L L^2(D_4)} \lesssim \int_0^t \langle \| M_2 (\partial, t - \tau) (v_{\langle \tau \rangle} - s \partial_y b, v_{\langle \tau \rangle} - s \partial_y b) \|_{F^L L^2(D_4)} \rangle d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{4}} \| \nabla \|^{0.51} (v_{\langle \tau \rangle} - s \partial_y b, v_{\langle \tau \rangle} - s \partial_y b) \|_{L^1(L^2)} d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{4}} (\tau)^{-1.01} d\tau \| V \|_3^2$$

$$\lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2.$$
and

\[ \|L_1\|_{F^2(L^2(D_4))} \lesssim \|M_2(\partial, t)(P_\leq 1(\vec{R} \cdot \vec{b})\partial_y b_0)\|_{F^2(L^2(D_4))} \lesssim \langle t \rangle^{-\frac{1}{4}}\|\nabla\|^{0.51}(P_\leq 1(\vec{R} \cdot \vec{b})\partial_y b_0)\|_{L^2_y(L^2_3)} \]
\[ \lesssim \langle t \rangle^{-\frac{1}{4}}\|(\nabla)^2(P_\leq 1(\vec{R} \cdot \vec{b})\partial_y b_0)\|_{L^1} \lesssim \langle t \rangle^{-\frac{1}{4}}\|P_\leq 1(\vec{R} \cdot \vec{b})\|_{H^2}\|\partial_y b_0\|_{H^2} \]
\[ \lesssim \langle t \rangle^{-\frac{1}{4}}\|V_0\|^2. \]

It follows from (3.13), (4.11) for \( r = 2 \) and (4.19) that

\[ \|L_2\|_{F^2(L^2(D_4))} \lesssim \int_0^t \|\partial_x M_1(\partial, t - \tau)(P_\leq 1(\vec{R} \cdot \vec{b})\partial_y b)\|_{F^2(L^2(D_4))} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-1}\|P_\leq 1(\vec{R} \cdot \vec{b})\partial_y b\|_{L^2} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-1}\langle \tau \rangle^{-0.6} d\tau\|V\|^2 \]
\[ \lesssim \langle t \rangle^{-0.39}\|V\|^2. \]

Using (3.13), (4.12) for \( r = 2 \), (4.12) for \( r = 2 \), (4.19) and (4.18), one can deduce

\[ \|L_3\|_{F^2(L^2(D_4))} \lesssim \int_0^t \|\partial_x M_2(\partial, t - \tau)(P_\leq 1(\vec{R} \cdot \vec{b})\partial_y u)\|_{L^2(D_4)} d\tau \]
\[ + \int_0^t \|M_2(\partial, t - \tau)(\partial_x P_\leq 1(\vec{R} \cdot \vec{b})\partial_y u)\|_{L^2(D_4)} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}}\|\nabla(P_\leq 1(\vec{R} \cdot \vec{b})\partial_y u)\|_{L^2} d\tau \]
\[ + \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}}\|\nabla\|^{0.51}(\partial_x P_\leq 1(\vec{R} \cdot \vec{b})\partial_y u)\|_{L^2_y(L^2_3)} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}}\langle \tau \rangle^{-1.1} + \langle t - \tau \rangle^{-\frac{1}{2}}\langle \tau \rangle^{-1.1} d\tau\|V\|^2 \]
\[ \lesssim \langle t \rangle^{-\frac{1}{2}}\|V\|^3. \]

Like the previous arguments, the associated estimate of “other good parts” on \( D_4 \) can be bounded by \( C(t)^{-\frac{1}{2}}(\|V\|^2_3 + \|V\|^3_3) \). Collecting the above estimate yields

\[ \|\vec{b}\|_{L^2(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}}(\|V_0\|_3 + \|V_0\|^2_3 + \|V\|^2_3 + \|V\|^3_3). \]

which completes the estimate of (8.1)1.

8.3. The estimate of (8.1)2. Due to

\[ \|\partial_x b\|_{H^1} \approx \|\partial_x(\nabla)b\|_{L^2} \]

and Plancherel’s identity, it is sufficient to bound \( \|\partial_x(\nabla)b\|_{F^2(L^2(D_4))} \). Thanks to (8.2), we only aim at estimating \( \|\partial_x(\nabla)b\|_{F^2(L^2(D_4))} \). Using (3.13), (4.11) for \( r = 2 \), (4.12) for \( r = 2 \) and \( \delta = 0.01 \), we have

\[ \|\partial_x(\nabla)b\|_{F^2(L^2(D_4))} \lesssim \|\partial_x(\nabla)M_1(\partial, t)u_0\|_{F^2(L^2(D_4))} + \|\partial_x(\nabla)M_2(\partial, t)b_0\|_{F^2(L^2(D_4))} \]
\[ \lesssim \langle t \rangle^{-1}\|u_0\|_{H^1} + \langle t \rangle^{-\frac{3}{4}}\|\nabla\|^{2.51}b_0|_{L^2_y(L^2_3)} \]
\[ \lesssim \langle t \rangle^{-\frac{3}{4}}\|V_0\|_3. \]
We can deduce from (3.13), (1.11) for \( r = 2 \), and (4.19) that

\[
\| \partial_x \langle \nabla \rangle NLb_1 \|_{F_L^2(D_4)} \lesssim \int_0^t \| \partial_x M_1(\partial, t - \tau) \langle \nabla \rangle F_1 \|_{F_L^2(D_4)} d\tau \\
\lesssim \int_0^t (t - \tau)^{-1} \| \langle \nabla \rangle (\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla \vec{b}) \|_{L^2} d\tau \\
\lesssim \int_0^t (t - \tau)^{-1} (\tau)^{-1.01} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-1} \| V \|_3^2.
\]

Applying (3.13), (1.12) for \( r = 2 \) and \( \delta = 0.01 \), and (1.18), we get

\[
\| \partial_x \langle \nabla \rangle NLb_2 \|_{F_L^2(D_4)} + \| \partial_x \langle \nabla \rangle NLb_3 \|_{F_L^2(D_4)} \\
\lesssim \int_0^t \| \partial_x M_2(\partial, t - \tau) \langle \nabla \rangle (G^{11}, \nu_{<}(\tau)^{-s} \partial_y b, \nu_{>2}(\tau)^{-0.05} \partial_y b) \|_{F_L^2(D_4)} d\tau \\
\lesssim \int_0^t (t - \tau)^{-\frac{3}{2}} \| \langle \nabla \rangle^{2.51} (u \partial_x b, \vec{b} \cdot \nabla u, \nu_{<}(\tau)^{-s} \partial_y b, \nu_{>2}(\tau)^{-0.05} \partial_y b) \|_{L_2^3(L_2^3)} d\tau \\
\lesssim \int_0^t (t - \tau)^{-\frac{3}{2}} (\tau)^{-1.01} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-\frac{3}{2}} \| V \|_3^2.
\]

Similarly, by (3.13), (4.12) for \( r = 2 \) and \( \delta = 0.01 \), one has

\[
\| \partial_x \langle \nabla \rangle L_1 \|_{F_L^2(D_4)} \lesssim \| \partial_x \langle \nabla \rangle M_2(\partial, t) (P_{\gamma_1}(\vec{R} \cdot \vec{b}_0) \partial_y b_0) \|_{F_L^2(D_4)} \\
\lesssim \langle t \rangle^{-\frac{3}{2}} \| \langle \nabla \rangle^{2.51} (P_{\gamma_1}(\vec{R} \cdot \vec{b}_0) \partial_y b_0) \|_{L_2^3(L_2^3)} \\
\lesssim \langle t \rangle^{-\frac{3}{2}} \| \langle \nabla \rangle^4 (P_{\gamma_1}(\vec{R} \cdot \vec{b}_0) \partial_y b_0) \|_{L^4} \\
\lesssim \langle t \rangle^{-\frac{3}{2}} \| P_{\gamma_1}(\vec{R} \cdot \vec{b}_0) \|_{H^4} \| \partial_y b_0 \|_{H^4} \\
\lesssim \langle t \rangle^{-\frac{3}{2}} \| V_0 \|_3^2.
\]

By using (3.13), (1.11) for \( r = 2 \), (4.19), we obtain,

\[
\| \partial_x \langle \nabla \rangle L_2 \|_{F_L^2(D_4)} \lesssim \int_0^t \| \partial_x M_1(\partial, t - \tau) \langle \nabla \rangle \partial_x (P_{\gamma_1}(\vec{R} \cdot \vec{b}) \partial_y b) \|_{F_L^2(D_4)} d\tau \\
\lesssim \int_0^t (t - \tau)^{-1} \| \langle \nabla \rangle \partial_x (P_{\gamma_1}(\vec{R} \cdot \vec{b}) \partial_y b) \|_{L^2} d\tau \\
\lesssim \int_0^t (t - \tau)^{-1} (\tau)^{-1.01} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-1} \| V \|_3^2.
\]
One deduces from (3.13), (4.12), for \( r = 2 \) and \( \delta = 0.01 \) that
\[
\| \partial_x (\nabla) L_3 \|_{L^2(D_4)} \lesssim \int_0^t \| \partial_x^2 M_2 (\partial, t - \tau) (\nabla) (P_x (\tilde{R}^t \cdot \tilde{b}) \partial_y u) \|_{L^2(D_4)} d\tau
\]
\[
+ \int_0^t \| \partial_x M_2 (\partial, t - \tau) (\nabla) (\partial_x P_x (\tilde{R}^t \cdot \tilde{b}) \partial_y u) \|_{L^2(D_4)} d\tau
\]
\[
\lesssim \int_0^t (t - \tau)^{-1} \| \langle \nabla \rangle^3 \left( P_x (\tilde{R}^t \cdot \tilde{b}) \partial_y u \right) \|_{L^2} d\tau
\]
\[
+ \int_0^t (t - \tau)^{-\frac{3}{4}} \| \langle \nabla \rangle^{2.51} \left( \partial_x P_x (\tilde{R}^t \cdot \tilde{b}) \partial_y u \right) \|_{L^2}_{L^2} d\tau
\]
\[
\lesssim \int_0^t (t - \tau)^{-1} (\tau)^{-0.9} + (t - \tau)^{-\frac{3}{4}} (\tau)^{-1.01} d\tau \| V \|_3^2
\]
\[
\lesssim \langle t \rangle^{-\frac{3}{4}} \| V \|_3^2.
\]

“Other good parts” on \( D_4 \) can be bounded by \( C \langle t \rangle^{-\frac{3}{4}} (\| V \|_3^2 + \| V \|_3^2) \). Hence, there holds
\[
\| \partial_x (\nabla) b \|_{L^2(D_4)} \lesssim \langle t \rangle^{-\frac{3}{4}} (\| V \|_3 + \| V \|_3 + \| V \|_3^2).
\]

8.4. The estimate of (8.1). Using \( |\tilde{q}| \lesssim 1 \) in \( D_1 \), one can get
\[
\| \langle \nabla \rangle^{-1} \langle \nabla \rangle b \|_{L^1(D_1)} \lesssim \| \langle \nabla \rangle^{-1} b \|_{L^1(D_1)}.
\]

Thanks to (3.12) and (4.1) for \( \alpha = 1 \), we have
\[
\| \langle \nabla \rangle^{-1} L \|_{L^1(D_1)} \lesssim \| \langle \nabla \rangle^{-1} (M_1 (\partial, t) u_0, M_2 (\partial, t) b_0) \|_{L^1(D_1)} \lesssim \langle t \rangle^{-\frac{1}{2}} \| (u_0, b_0) \|_{L^1}.
\]

Applying (3.12), (4.1) for \( \alpha = 1, \) (6.5), we have
\[
\| \langle \nabla \rangle^{-1} N L_b \|_{L^1(D_1)} \lesssim \int_0^t \| \langle \nabla \rangle^{-1} (M_1 (\partial, t - \tau) F_1) \|_{L^1(D_1)} d\tau
\]
\[
+ \int_0^t \| \langle \nabla \rangle^{-1} (M_2 (\partial, t - \tau) G_1) \|_{L^1(D_1)} d\tau
\]
\[
\lesssim \int_0^t \| M_1 (\partial, t - \tau) (\tilde{u} \otimes \tilde{u}, \tilde{b} B) \|_{L^1(D_1)} d\tau
\]
\[
+ \int_0^t \| \langle \nabla \rangle^{-1} M_1 (\partial, t - \tau) (b \partial_x b) \|_{L^1(D_1)} d\tau
\]
\[
+ \int_0^t \| M_2 (\partial, t - \tau) (\tilde{u} \otimes \tilde{b}) \|_{L^1(D_1)} d\tau
\]

Due to (4.1) for \( k = 0, (4.1) \) for \( \alpha = 1/2, (4.17) \) and (4.19), the first and third integral can be bounded by
\[
C \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| (\tilde{u} \otimes \tilde{u}, \tilde{u} \otimes \tilde{b}, \tilde{b} B) \|_{L^2} d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} (\tau)^{-1.1} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2,
\]
while the second integral can be bounded by
\[
\begin{align*}
C \int_0^t \left\| |\nabla|^{-\frac{1}{2}} M_1(\partial, t - \tau) \partial_x \frac{1}{4} (bb) \right\|_{\mathcal{F}L^1(D_t)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{4}} \left\| \partial_x \frac{1}{4} (bb) \right\|_{L^1} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{4}} \langle \tau \rangle^{-\frac{3}{4}} d\tau \|V\|_3^2 \lesssim \langle t \rangle^{-\frac{1}{2}} \|V\|_3^2.
\end{align*}
\]
So
\[
\left\| |\nabla|^{-1} \langle \nabla \rangle b \right\|_{\mathcal{F}L^1(D_t)} \lesssim \langle t \rangle^{-\frac{1}{2}} (\|V_0\|_3 + \|V\|_3^2). 
\] (8.3)

Using
\[
\left\| |\nabla|^{-1} \langle \nabla \rangle b \right\|_{\mathcal{F}L^1(D_t)} \lesssim \|b\|_{\mathcal{F}L^1(D_t)} + \left\| |\nabla|^{-1} b \right\|_{\mathcal{F}L^1(D_t)},
\]
is is sufficient to show the estimates of the terms on the right hand side. Using (3.13), (4.11)\(_1\) and (4.12)\(_1\) for \(r = 1\), one can easily get
\[
\|L_b\|_{\mathcal{F}L^1(D_t)} \lesssim \langle t \rangle^{-\frac{1}{2}} \|V_0\|_3.
\]
One can infer by (3.13), (6.5), (4.11)\(_2\) for \(r = 1\) and (4.11)\(_3\) for \(r = 1\), (4.20) that
\[
\|NL_{b1}\|_{\mathcal{F}L^1(D_t)} \lesssim \int_0^t \|M_1(\partial, t - \tau) F^1\|_{\mathcal{F}L^1(D_t)} d\tau \\
\lesssim \int_0^t \|M_1(\partial, t - \tau) (\tilde{u} \cdot \nabla \tilde{u}, \tilde{b} \cdot \nabla B, \partial_y (Bb)) \|_{\mathcal{F}L^1(D_t)} d\tau \\
+ \int_0^t \|\partial_x M_1(\partial, t - \tau) (bb) \|_{\mathcal{F}L^1(D_t)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| (\tilde{u} \otimes \tilde{u}, \tilde{B} \tilde{b}) \|_{\mathcal{F}L^1} + \langle t - \tau \rangle^{-1} \|bb\|_{\mathcal{F}L^1} d\tau \|V\|_3^2 \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{1}{2}} + \langle t - \tau \rangle^{-1} \langle \tau \rangle^{-1} d\tau \|V\|_3^2 \\
\lesssim \langle t \rangle^{-\frac{1}{2}} \|V\|_3^2.
\]
Applying (3.13), (4.12)\(_1\) for \(r = 1\) and (4.18), we can get
\[
\|NL_{b2}\|_{\mathcal{F}L^1(D_t)} + \|NL_{b3}\|_{\mathcal{F}L^1(D_t)} \\
\lesssim \int_0^t \|M_2(\partial, t - \tau) \{u_0 \partial_x b, \tilde{b} \cdot \nabla u, v_{< \langle \tau \rangle - \delta} \partial_y b, v_{> \langle \tau \rangle -0.05} \partial_y b\} \|_{\mathcal{F}L^1(D_t)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \|\langle \nabla \rangle^{1.51} (u_0 \partial_x b, \tilde{b} \cdot \nabla u, v_{< \langle \tau \rangle - \delta} \partial_y b, v_{> \langle \tau \rangle -0.05} \partial_y b) \|_{L^1(L^2)} d\tau \\
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \langle \tau \rangle^{-1.01} d\tau \|V\|_3^2 \lesssim \langle t \rangle^{-\frac{1}{2}} \|V\|_3^2.
\]
Using (3.13) and (4.12)\(_1\) again, it is easy to get
\[
\|L_1\|_{\mathcal{F}L^1(D_t)} \lesssim \|M_2(\partial, t) (P_{>1}(\tilde{R} \cdot \tilde{b}_0) \partial_y b_0) \|_{\mathcal{F}L^1(D_t)} \lesssim \langle t \rangle^{-\frac{1}{2}} \|V_0\|_3.
\]
Using (3.13), (4.11) and (4.20), we can get

\[ \| L_2 \|_{L^1_t(D_{41})} \lesssim \int_0^t \| \partial_x M_1(\partial, t - \tau) (P_\omega (\vec{R} \cdot \vec{b}) \partial_y b) \|_{L^1_t(D_{41})} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| P_\omega (\vec{R} \cdot \vec{b}) \|_{L^1_t} \| \partial_y b \|_{L^1_t} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| \tau \|^{-0.9} d\tau \| V \|_3^2 \]
\[ \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2. \]

By (3.13), (4.12) for \( r = 1 \), (4.12) for \( r = 1 \) for \( r = 1 \), (4.13) and (4.20), it follows

\[ \| L_3 \|_{L^1_t(D_{41})} \lesssim \int_0^t \| \partial_x M_2(\partial, t - \tau) (P_\omega (\vec{R} \cdot \vec{b}) \partial_y u) \|_{L^1_t(D_{41})} d\tau \]
\[ + \int_0^t \| M_2(\partial, t - \tau)(\partial_x P_\omega (\vec{R} \cdot \vec{b}) \partial_y u) \|_{L^1_t(D_{41})} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| \nabla (P_\omega (\vec{R} \cdot \vec{b}) \partial_y u) \|_{L^1_t} d\tau \]
\[ + \int_0^t \langle t - \tau \rangle^{-\frac{1}{2}} \| \langle \nabla \rangle^{1.51} (\partial_x P_\omega (\vec{R} \cdot \vec{b}) \partial_y u) \|_{L^1_t(L^2)} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-\frac{1}{10}} (\tau)^{-1.01} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2. \]

“Other good parts” can be bounded by \( C \langle t \rangle^{-\frac{1}{2}} (\| V \|_3^2 + \| V \|_3^3). \) Finally,

\[ \| b \|_{L^1_t(D_{41})} \lesssim \langle t \rangle^{-\frac{1}{2}} (\| V_0 \|_3 + \| V_0 \|_3^3 + \| V \|_3^2 + \| V \|_3^3). \]  

(8.4)

By (3.13), (4.11) for \( \delta = 0.01 \), and (1.17) that

\[ \| \nabla^{-1} L b \|_{L^1_t(D_{42})} \lesssim \| \nabla^{-1} M_1(\partial, t) u_0 \|_{L^1_t(D_{42})} + \| \nabla^{-1} M_2(\partial, t) b_0 \|_{L^1_t(D_{42})} \]
\[ \lesssim \langle t \rangle^{-\frac{1}{2}} (\| u_0 \|_{L^1} + \| b_0 \|_{L^1(L^2)}). \]

It follows from (3.13), (6.5), (1.11) and (1.17) that

\[ \| \nabla^{-1} N b \|_{L^1_t(D_{42})} \lesssim \int_0^t \| \nabla^{-1} M_1(\partial, t - \tau) (\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla B, \partial_y (B \vec{b})) \|_{L^1_t(D_{42})} d\tau \]
\[ + \int_0^t \| \nabla^{-1} M_1(\partial, t - \tau) (b \partial_x b) \|_{L^1_t(D_{42})} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-1} \| \nabla \|^{0.99} (\vec{u} \otimes \vec{u}, \vec{b} \otimes \vec{B}) \|_{L^1} + \| \partial_x \|^{0.99} (\vec{b} \otimes \vec{b}) \|_{L^1} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-1} (\tau)^{-0.6} d\tau \| V \|_3^2 \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2. \]
Similarly, we can get by using (4.12) that
\[|||\nabla|||^{-1} F_{L^1(D_{42})} \lesssim \langle t \rangle^{-\frac{1}{2}} \left\| P_{\sigma}(\vec{R} \cdot \vec{b}) \right\|_{L^1} \lesssim \langle t \rangle^{-\frac{1}{2}} \| V_0 \|_3^2.\]

By (3.13), (4.11), (4.12), (4.17), and (4.20), we achieve
\[|||\nabla|||^{-1} L_2 \|_{F_{L^1(D_{42})}} \lesssim \int_0^t \langle t \rangle^{-\frac{1}{2}} \left\| P_{\sigma}(\vec{R} \cdot \vec{b}) \right\|_{L^1} \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2.\]

We can obtain from (3.13), (4.12), (4.12), (4.17), and (4.20) that
\[|||\nabla|||^{-1} L_3 \|_{F_{L^1(D_{42})}} \lesssim \int_0^t \langle t \rangle^{-\frac{1}{2}} \left\| P_{\sigma}(\vec{R} \cdot \vec{b}) \right\|_{L^1} \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2.\]

Hence, it comes out
\[|||\nabla|||^{-1} b \|_{F_{L^1(D_{42})}} \lesssim \langle t \rangle^{-\frac{1}{2}} (\| V_0 \|_3 + \| V_0 \|_3^2 + \| V \|_3^2 + \| V \|_3^2). \quad (8.5)\]

Combining with (8.3), (8.4) and (8.5) follows (8.1).  

8.5. The estimate of (8.1). Direct computations yield
\[\| R_1(\nabla) b \|_{F_{L^1(D_1)}} \lesssim \langle t \rangle^{-\frac{1}{2}} (\| V_0 \|_3 + \| V_0 \|_3^2 + \| V \|_3^2). \quad (8.6)\]
So it suffices to bounding $\|R_1(\nabla)b\|_{L^1_t(L^2_x)}$. By (3.13), (4.11) and (4.12) for $l = 1$, we have

$$
\|R_1(\nabla)L_b\|_{L^1_t(L^2_x)} \lesssim \|R_1(\nabla)M_1(\partial,t)u_0\|_{L^1_t(L^2_x)} + \|R_1(\nabla)M_2(\partial,t)b_0\|_{L^1_t(L^2_x)} \\
\lesssim \|\nabla(\nabla)M_1(\partial,t)u_0\|_{L^1_t(L^2_x)} + \|R_1(\nabla)M_2(\partial,t)b_0\|_{L^1_t(L^2_x)} \\
\lesssim (t)^{-1}(\|\nabla^4(u_0,b_0)\|_{L^2_t(L^2_x)} + \|\nabla^2u_0\|_{L^1_x}) \\
\lesssim (t)^{-1}\|V_0\|_3.
$$

It follows by (3.13), (4.11) and (4.20) that

$$
\|R_1(\nabla)NL_{b1}\|_{L^1_t(L^2_x)} \lesssim \int_0^t \|R_1(\nabla)M_1(\partial,t - \tau)(\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla \vec{B}, \partial_y (\vec{B} b))\|_{L^1_t(L^2_x)}d\tau \\
+ \int_0^t \|\partial_x R_1(\nabla)M_1(\partial,t - \tau)(\vec{b} b)\|_{L^1_t(L^2_x)}d\tau \\
+ \int_0^t \|\nabla R_1(\nabla)M_1(\partial,t - \tau)\partial_x (\vec{b} b)\|_{L^1_t(L^2_x)}d\tau \\
\lesssim \int_0^t (t - \tau)^{-1}(\|\nabla\|_{L^1_t(L^2_x)}(\vec{u} \cdot \nabla \vec{u}, \vec{b} \cdot \nabla \vec{B}, \partial_y (\vec{B} b))\|_{L^1_t(L^2_x)}d\tau \\
+ \int_0^t (t - \tau)^{-\frac{1}{2}}\|\vec{b} b\|_{L^1_t(L^2_x)} + \|t - \tau\|^{-\frac{3}{2}}\|\vec{b} b\|_{L^1_t(L^2_x)}d\tau \\
\lesssim \int_0^t (t - \tau)^{-1}(t^{-1.1} + t - \tau)^{-\frac{3}{2}}(t^{-1})d\tau \|V\|^2_3 \\
\lesssim (t)^{-1}\|V\|^2_3.
$$

Using (3.13), (4.12) for $l = 1$, (4.18), one has

$$
\|R_1(\nabla)L_1\|_{L^1_t(L^2_x)} \lesssim \int_0^t \|R_1(\nabla)M_2(\partial,t - \tau)(u \partial_x \vec{b}, \vec{b} \cdot \nabla \vec{u}, v_{<\tau} - s \partial_y \vec{b}, v_{>\tau} - 0.05 \partial_y \vec{b})\|_{L^1_t(L^2_x)}d\tau \\
\lesssim \int_0^t (t - \tau)^{-1}(\|\nabla\|_{L^1_t(L^2_x)}(u \partial_x \vec{b}, \vec{b} \cdot \nabla \vec{u}, v_{<\tau} - s \partial_y \vec{b}, v_{>\tau} - 0.05 \partial_y \vec{b})\|_{L^1_t(L^2_x)}d\tau \\
\lesssim \int_0^t (t - \tau)^{-1}(t^{-1.01})d\tau \|V\|^2_3 \lesssim (t)^{-1}\|V\|^2_3.
$$

Thanks to (3.13), (4.12) for $l = 1$, we deduce

$$
\|R_1(\nabla)L_2\|_{L^1_t(L^2_x)} \lesssim \|R_1(\nabla)M_2(\partial,t)\|_{L^1_t(L^2_x)}(P_{-1}(\vec{R} \cdot \vec{b}_0)\partial_y b_0)\|_{L^1_t(L^2_x)} \lesssim (t)^{-1}\|V_0\|^2_3.
$$

Using (3.13), (4.11) and (4.20), we achieve

$$
\|R_1(\nabla)L_2\|_{L^1_t(L^2_x)} \lesssim \int_0^t \|\partial_y R_1(\nabla)M_1(\partial,t - \tau)\partial_x (P_{-1}(\vec{R} \cdot \vec{b})bb)\|_{L^1_t(L^2_x)}d\tau \\
+ \int_0^t \|\partial_y R_1(\nabla)M_1(\partial,t - \tau)(\partial_y P_{-1}(\vec{R} \cdot \vec{b})b)\|_{L^1_t(L^2_x)}d\tau \\
\lesssim \int_0^t (t - \tau)^{-1}(\|\partial_x(\nabla)(P_{-1}(\vec{R} \cdot \vec{b})b), \partial_x(\partial_y P_{-1}(\vec{R} \cdot \vec{b})b)\|_{L^1_t(L^2_x)}d\tau \\
+ \|t - \tau\|^{-\frac{3}{2}}\|\partial_y P_{-1}(\vec{R} \cdot \vec{b})b\|_{L^1_t(L^2_x)}d\tau \\
\lesssim \int_0^t (t - \tau)^{-1}(t^{-1.01} + t - \tau)^{-\frac{3}{2}}d\tau \|V\|^2_3 \\
\lesssim (t)^{-1}\|V\|^2_3.
$$
where we have used $\|\langle \nabla \rangle f\|_{F^1L^1} \lesssim \|f\|_{F^1L^1} + \|\langle \nabla \rangle f\|_{F^1L^1}$ for the second inequality. We infer from (4.12), (4.13) for $t = 1$, and (4.20), (4.18) that

$$\|R_1(\nabla)L_3\|_{F^1L^1(D_4)} \lesssim \int_0^t \|\partial_x R_1(\nabla)M_2(\partial, t - \tau)(P_{\omega}(\vec{R}^t \cdot \vec{b})\partial_y u)\|_{F^1L^1(D_4)} d\tau$$

$$+ \int_0^t \|R_1(\nabla)M_2(\partial, t - \tau)(\partial_x P_{\omega}(\vec{R}^t \cdot \vec{b})\partial_y u)\|_{F^1L^1(D_4)} d\tau$$

$$\lesssim \int_0^t (t - \tau)^{-\frac{1}{2}}\|\langle \nabla \rangle^2(P_{\omega}(\vec{R}^t \cdot \vec{b})\partial_y u)\|_{F^1L} d\tau$$

$$+ \int_0^t (t - \tau)^{-\frac{1}{2}}\|\langle \nabla \rangle^2 (\partial_x P_{\omega}(\vec{R}^t \cdot \vec{b})\partial_y u)\|_{L^1(L^2)} d\tau$$

$$\lesssim \int_0^t (t - \tau)^{-\frac{1}{2}}\|\langle \nabla \rangle^2\|_{F^1L}^2 d\tau \lesssim (t)^{-1}\|V\|_{3,2}^2 .$$

We can bound “other good parts” by $C(t)^{-1}(\|V\|_3^3 + \|V\|_3^3)$. Thus

$$\|R_1(\nabla)b\|_{F^1L^1(D_4)} \lesssim (t)^{-1}(\|V_0\|_3 + \|V_0\|_3^2 + \|V\|_3^3 + \|V\|_3^3),$$

which, together with (8.6) leads to (8.14).

9. The estimate on $B$ and proof of (1.9)

In this section, we will prove

$$\left\{ \begin{array}{l}
\|B(t)\|_{L^2} \lesssim (t)^{-\frac{1}{2}}(\|V_0\|_3 + \|V\|_3^2); \\
\|\partial_x B(t)\|_{L^2} \lesssim (t)^{-1}(\|V_0\|_3 + \|V\|_3^3); \\
\|B(t)\|_{F^1L^1} \lesssim (t)^{-1}(\|V_0\|_3 + \|V_0\|_3^2 + \|V\|_3^3 + \|V\|_3^3). 
\end{array} \right. \quad (9.1)$$

9.1. The expression of $B$. Thanks to (3.5), we can get

$$B = M_1(\partial, t)v_0 + M_2(\partial, t)B_0 + \int_0^t M_1(\partial, t - \tau)F^2 d\tau + \int_0^t M_2(\partial, t - \tau)G^2 d\tau . \quad (9.2)$$

9.2. The estimate of (9.1). Like the estimate of $\|\hat{v}\|_{L^2(D_1)}$, we can also get

$$\|\hat{B}\|_{L^2(D_1)} \lesssim (t)^{-\frac{1}{2}}(\|V_0\|_3 + \|V\|_3^3). \quad (9.3)$$

Using (3.13), (4.11), (4.3) for $k = 1$, and $B_0 = -R_{11}B_0 + R_{12}b_0$, we have

$$\|L_B\|_{F^1L^2(D_4)} \lesssim \|M_1(\partial, t)v_0\|_{F^1L^2(D_4)} + \|R_1M_2(\partial, t)(B_0, b_0)\|_{F^1L^2(D_4)}$$

$$\lesssim (t)^{-\frac{1}{2}}\|v_0\|_{L^1L^2} + \|G_{1,1}e^{-G_{2,2}t}\{\vec{b}_0\}\|_{L^2(D_4)}$$

$$\lesssim (t)^{-\frac{1}{2}}\|V_0\|_3.$$

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Using (6.3), (4.3) for $k = 2$, and (4.19), one can obtain
\[
\|NL_{B1}\|_{L^2(D_4)} \lesssim \int_0^t \|\nabla |R_1 M_1(\partial, t - \tau)(\vec{b} \otimes \vec{b}, \vec{u} \otimes \vec{u})|_{L^2(D_4)}d\tau
\]
\[
\lesssim \int_0^t \|G_{2,2} e^{-G_{2,2}(t-\tau)} \mathcal{F} \{\vec{b} \otimes \vec{b}, \vec{u} \otimes \vec{u}\}|_{L^2(D_4)}d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-1}(\|\vec{b} \otimes \vec{b}\|_{L^2} + \|\vec{u} \otimes \vec{u}\|_{L^2})d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \langle \tau \rangle^{-\frac{3}{4}}d\tau \|V\|^2_3
\]
\[
\lesssim \langle t \rangle^{-\frac{1}{2}} \|V\|^2_3.
\]
We can deduce from (6.4), (4.12) for $r = 2$, (6.4) and (4.19) that
\[
\|NL_{B2}\|_{L^2(D_4)} \lesssim \int_0^t \|\partial_x M_2(\partial, t - \tau)(uB - bv)\|_{L^2(D_4)}d\tau
\]
\[
\lesssim \int_0^t \|\nabla (uB)\|_{L^2} + \|\nabla (bv)\|_{L^2}d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-\frac{3}{4}} \langle \tau \rangle^{-1,01}d\tau \|V\|^2_3
\]
\[
\lesssim \langle t \rangle^{-\frac{1}{2}} \|V\|^2_3.
\]
As a result, there holds
\[
\|\vec{B}\|_{L^2(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}}(\|V_0\|_3 + \|V\|^2_3). \tag{9.3}
\]
It follows (9.1) by using (9.2) and (9.3).

9.3. The estimate of (9.1)2. As the estimate of $\|\partial_x v\|_{L^2}$, it is easy to obtain
\[
\|\partial_x B\|_{L^2(D_4)} \lesssim \langle t \rangle^{-1}(\|V_0\|_3 + \|V\|^2_3). \tag{9.4}
\]
Similar to the estimate of $\|L_B\|_{L^2(D_4)}$, by (5.13), (1.11) for $r = 2$, and (4.3) for $k = 2$, we infer
\[
\|\partial_x L_B\|_{L^2(D_4)} \lesssim \|\partial_x M_1(\partial, t)v_0\|_{L^2(D_4)} + \|\partial_x M_2(\partial, t)B_0\|_{L^2(D_4)}
\]
\[
\lesssim \langle t \rangle^{-1}\|v_0\|_{L^2} + \|\nabla M_2(\partial, t)\vec{b}_0\|_{L^2(D_4)}
\]
\[
\lesssim \langle t \rangle^{-1}\|v_0\|_{L^2} + \|G_{2,2} e^{-G_{2,2}t} \mathcal{F} \{\nabla \vec{b}_0\}|_{L^2(D_4)}
\]
\[
\lesssim \langle t \rangle^{-1}\|v_0\|_{L^2}.
\]
It follows by (3.13), (4.11) for $r = 2$, and (4.19) that
\[
\|\partial_x NL_{B1}\|_{L^2(D_4)} \lesssim \int_0^t \|\partial_x M_1(\partial, t - \tau)(\vec{b} \cdot \nabla \vec{b}, \vec{u} \cdot \nabla \vec{u})|_{L^2(D_4)}d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-1}(\|\vec{b} \cdot \nabla \vec{b}\|_{L^2} + \|\vec{u} \cdot \nabla \vec{u}\|_{L^2})d\tau
\]
\[
\lesssim \int_0^t \langle t - \tau \rangle^{-1} \langle \tau \rangle^{-1,1}d\tau \|V\|^2_3
\]
\[
\lesssim \langle t \rangle^{-1} \|V\|^2_3.
\]
Using (6.4), (3.13), (4.12), and (4.19), one has
\[ \| \partial_NL_{B2} \|_{FL^2(D_4)} \lesssim \int_0^t \| \partial_x^2 M_2(\partial, t - \tau)(uB - bv) \|_{FL^2(D_4)} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-1}(\| \nabla^2(uB) \|_{L^2} + \| \nabla^2(bv) \|_{L^2}) d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-1} d\tau \| V \|^2_3 \]
\[ \lesssim \langle t \rangle^{-1} \| V \|^3_3. \]
Hence, we have
\[ \| \partial_NB \|_{FL^2(D_4)} \lesssim \langle t \rangle^{-1}(\| V_0 \|_3 + \| V \|^2_3). \] (9.5)
Combining with (9.4) and (9.5) implies (9.1).

9.4. The estimate of (9.1)\textsubscript{3}. Like the previous way dealing with \( \| \hat{B} \|_{L^1(D_1)} \), one can get
\[ \| B \|_{FL^1(D_1)} \lesssim \langle t \rangle^{-1}(\| V_0 \|_3 + \| V \|^2_3). \] (9.6)
For the estimate on \( D_4 \), by \( B = -R_{11}B + R_{12}b \), and
\[ \| R_{12}b \|_{FL^1(D_4)} \lesssim \| R_1(\nabla)b \|_{FL^1(D_4)} \lesssim \langle t \rangle^{-1}(\| V_0 \|_3 + \| V \|^2_3 + \| V \|_3), \] (9.7)
it suffices to bound \( \| R_{11}B \|_{FL^1(D_4)} \). By (3.13), (4.11), (4.12), (4.15) for \( l = 2 \) and (4.20), we deduce
\[ \| R_{11}L_B \|_{FL^1(D_4)} \lesssim \| R_{11}M_1(\partial, t)v_0 \|_{FL^1(D_4)} + \| R_{11}M_2(\partial, t)b_0 \|_{FL^1(D_4)} \lesssim \langle t \rangle^{-1}(\| v_0 \|_3 + \| b_0 \|_3), \]
\[ \| R_{11}NL_1 \|_{FL^1(D_4)} \lesssim \int_0^t \| R_{11}M_1(\partial, t - \tau)(\bar{u} \cdot \nabla \bar{u}, \bar{b} \cdot \nabla \bar{b}) \|_{FL^1(D_4)} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-1}(\| \nabla \bar{u}, \nabla \bar{b} \|_{FL^1(D_4)} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-1} d\tau \| V \|^3_3 \lesssim \langle t \rangle^{-1} \| V \|^3_3, \]
and
\[ \| R_{11}NL_2 \|_{FL^1(D_4)} \lesssim \int_0^t \| R_{11}M_2(\partial, t - \tau)(\bar{u} \cdot \nabla B, \bar{b} \cdot \nabla v) \|_{FL^1(D_4)} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-1}(\| \nabla \bar{u}, \nabla B \|_{FL^1(D_4)} d\tau \]
\[ \lesssim \int_0^t \langle t - \tau \rangle^{-1} d\tau \| V \|^3_3 \lesssim \langle t \rangle^{-1} \| V \|^3_3. \]
So we achieve
\[ \| R_{11}B \|_{FL^1(D_4)} \lesssim \langle t \rangle^{-1}(\| V_0 \|_3 + \| V \|^2_3). \] (9.8)
Finally, it follows (9.1)\textsubscript{3} by using (9.6), (9.8) and (9.7).

9.5. Proof of (1.9). (1.9) is a direct consequence by adding (5.1), (6.1), (7.1), (8.1), (9.1) and \( \| V \|^2_3 \lesssim \| V \|^3_3 + \| V \|_3^3. \)
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Appendix A.

In this section, we give the proof of some lemmas.

Proof of Lemma 4.3. Thanks to (3.13), (4.3), for $k = 0$ and (4.3) for $k = 1$ and $r = 2$, we get
\[
\|M_1(\partial, t) f\|_{L^2(D_4)} \lesssim \min\{\|G_{1,2} e^{-G_{2,1} t} \hat{f}\|_{L^2(D_4)}, \|G_{1,1} e^{-G_{2,1} t} \nabla^{-1} f\|_{L^2(D_4)}\} \lesssim \langle t \rangle^{-\frac{1}{2}} \min\{\|f\|_{L^1 \cap L^2}, \|\nabla^{-1} f\|_{L^2}\},
\]
which yields (4.11)1. It follows (4.11)2 by using (3.13),
\[
\|\nabla M_1(\partial, t) f\|_{L^r(D_4)} \lesssim \|G_{1,2} e^{-G_{2,1} t} \hat{f}\|_{L^r(D_4)}
\]
and (4.3) for $k = 1$. (4.11)3 can be obtained by using (3.13),
\[
\|\partial_x M_1(\partial, t) f\|_{L^r(D_4)} \lesssim \|G_{2,2} e^{-G_{2,1} t} \hat{f}\|_{L^r(D_4)}
\]
and (4.3) for $k = 2$. Combining with (3.13),
\[
\|M_1(\partial, t) f\|_{L^r(D_4)} \lesssim \|G_{1,2} e^{-G_{2,1} t} \hat{f}\|_{L^r(D_4)};
\]
(4.3)2 for $k = 1$ and $\delta = 0.01$ leads (4.11)4. Combining with (3.13),
\[
\|\nabla^{-1} M_1(\partial, t) f\|_{L^1(D_4)} \lesssim \|G_{1,3} e^{-G_{2,1} t} \hat{f}\|_{L^1(D_4)}
\]
and (4.3) for $k = 0, q = p = 4/3$, we can get (4.11)5. It follows (4.11)6 by using (3.13),
\[
\|R_1 M_1(\partial, t) f\|_{L^1(D_4)} \lesssim \|G_{2,3} e^{-G_{2,1} t} \hat{f}\|_{L^1(D_4)}
\]
and (4.3) for $k = 1$. It follows (4.11)7 by using (3.13),
\[
\|R_1 M_1(\partial, t) f\|_{L^1(D_4)} \lesssim \|G_{2,3} e^{-G_{2,1} t} \hat{f}\|_{L^1(D_4)}
\]
and (4.3) for $k = 1$. Applying
\[
\|\nabla R_1 M_1(\partial, t) f\|_{L^1(D_4)} + \|R_{11} M_1(\partial, t) f\|_{L^1(D_4)} \lesssim \|G_{2,3} e^{-G_{2,1} t} \hat{f}\|_{L^1(D_4)}
\]
and (4.3) for $k = 2$ and $r = 1$ can lead (4.11)8. Similarly, using
\[
\|\partial_x R_1 M_1(\partial, t) f\|_{L^1(D_4)} + \|\nabla R_{11} M_1(\partial, t) f\|_{L^1(D_4)} \lesssim \|G_{2,3} e^{-G_{2,1} t} \hat{f}\|_{L^1(D_4)}
\]
and (4.3) for $k = 3$ and $r = 1$ can lead (4.11)9. \qed

Proof of Lemma 4.4. Using (3.13) and (4.3) for $k = 0$, one can easily get (4.12)1. By (3.13),
\[
\|\partial_x M_2(\partial, t) f\|_{L^r(D_4)} \lesssim \|\nabla G_{1,1} e^{-G_{2,1} t} \hat{f}\|_{L^r(D_4)}
\]
and (4.3) for $k = 1$, we can obtain (4.12)2. Similarly, one can also get (4.12)3. Using a similar way leading (4.5) for $k = 0$, we can get (4.12)4. Thanks to
\[
\|R_1^t M_2(\partial, t) f\|_{L^r(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}} \|M_2(\partial, t) f\|_{L^r(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}} \|G_{0,0} e^{-G_{2,1} t} \hat{f}\|_{L^r(D_4)}
\]
it follows (4.12)5 by using (4.3) for $k = 0$ and $r = 1$. Due to
\[
\|\partial_x R_1^t M_2(\partial, t) f\|_{L^r(D_4)} \lesssim \||\nabla R_1^t \hat{f} M_2(\partial, t) f\|_{L^r(D_4)}
\]
it follows (4.12)6 by using (4.12)5. \qed
Proof of Lemma 4.3. Using (3.13), (4.1) for $k = 1$, (4.3) for $k = 1$ and $r = 2$, we have

$$\|M_3(\partial, t)f\|_{L^2(D_4)} \lesssim \|e^{\frac{1}{2}t \Delta} f\|_{L^2(D_4)} + \|\xi\|^{-1} \langle \xi \rangle^{-1} \xi^{-1} \xi_0^{-1} \frac{\xi}{\xi} f \|_{L^2(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}} \|\nabla f\|_{L^1}.$$  

Using (3.13), (4.1) for $k = 0$ and (4.3) for $k = 0$, we have

$$\|M_3(\partial, t)f\|_{L^2(D_4)} \lesssim \|e^{\frac{1}{2}t \Delta} f\|_{L^2(D_4)} + \|G_1 e^{-G_2 t} \|_{L^2(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}} \|f\|_{L^1 \cap L^2}.$$  

Thus, we complete the proof of (4.13) for $k = 0$. Using (3.13), one can get the first estimate of (4.13) by applying

$$\|M_3(\partial, t)f\|_{L^1(D_4)} \lesssim \|e^{\frac{1}{2}t \Delta} f\|_{L^1(D_4)} + \|G_2 e^{-G_2 t} \|_{L^1(D_4)};$$

for $k = 0$, and (4.3) for $k = 0$. Using (3.13), one can get the second estimate of (4.13) by applying

$$\|M_3(\partial, t)f\|_{L^1(D_4)} \lesssim \|e^{\frac{1}{2}t \Delta} f\|_{L^1(D_4)} + \|G_2 e^{-G_2 t} \|_{L^1(D_4)};$$

for $k = 1$, and (4.3) for $k = 1$. Using (3.13), one can get the first estimate of (4.13) by applying

$$\|\partial_y M_3(\partial, t)f\|_{L^1(D_4)} \lesssim \|e^{\frac{1}{2}t \Delta} f\|_{L^1(D_4)} + \|G_2 e^{-G_2 t} \|_{L^1(D_4)};$$

for $k = 0$, and (4.3) for $k = 0$. Using (3.13), one can get the second estimate of (4.13) by applying

$$\|\partial_y M_3(\partial, t)f\|_{L^1(D_4)} \lesssim \|e^{\frac{1}{2}t \Delta} f\|_{L^1(D_4)} + \|G_2 e^{-G_2 t} \|_{L^1(D_4)};$$

for $k = 2$, and (4.3) for $k = 1$. Using (3.13), one can get the third estimate of (4.13) by applying

$$\|\partial_y M_3(\partial, t)f\|_{L^1(D_4)} \lesssim \|e^{\frac{1}{2}t \Delta} f\|_{L^1(D_4)} + \|G_2 e^{-G_2 t} \|_{L^1(D_4)};$$

for $k = 1$, and (4.3) for $k = 1$. So we conclude the proof of (4.13). For (4.13), we only show the case $r = 2$, and other cases can be bounded similarly. Using $|\xi| \lesssim \|\xi\|^2$, one can get (4.13) for $r = 2$ by (4.1) for $k = 2$ and (4.3) for $k = 2$. Using $|\xi| \lesssim \|\xi\|^2$, one can get the first bound of (4.13) by (4.1) for $k = 2$ and (4.3) for $k = 0$. Using $|\xi| \lesssim \|\xi\|^2$, one can get the second bound of (4.13) by (4.1) for $k = 2$ and (4.3) for $k = 2$. Using $e^{ct \Delta} f \|_{L^1} \lesssim \langle t \rangle^{-\frac{1}{2}} (\|f\|_{L^1} + \|\xi f\|_{L^1})$, which can be proved by using the similar arguments yielding (4.2), and

$$\|G_2 e^{-G_2 t} \|_{L^1(D_4)} \lesssim \|G_2 e^{-G_2 t} \|_{L^1(D_4)} + \|G_2 e^{-G_2 t} \|_{L^1(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}} (\|f\|_{L^1} + \|\xi f\|_{L^1})$$

which can be obtained by using (4.3) for $k = 2$ and (4.3) for $k = 1$, $p = \frac{1}{2}$ and $q = 2$, we have

$$\|\partial_y M_3(\partial, t)f\|_{L^1(D_4)} \lesssim \|e^{\frac{1}{2}t \Delta} f\|_{L^1(D_4)} + \|G_2 e^{-G_2 t} \|_{L^1(D_4)} \lesssim \langle t \rangle^{-\frac{1}{2}} (\|\xi f\|_{L^1} + \|\partial_x f\|_{L^1})$$

which completes the proof of the third bound of (4.13). Using (4.1) for $k = 2$, (4.3) for $k = 2$ and

$$\|\Delta M_3(\partial, t)f\|_{L^2(D_4)} \lesssim \|\Delta e^{\frac{1}{2}t \Delta} f\|_{L^2(D_4)} + \|G_2 e^{-G_2 t} \|_{L^2(D_4)}$$

for $k = 2$.
we can get (4.13)6. Using $|\xi| \lesssim |\tilde{\xi}|^2$, (4.1)4 for $k = 5$, (4.3)1 for $k = 5$ and 
\[ \| \partial_2^2 R_1 M_5(\partial, t)f \|_{F^1_{L^1}} \lesssim \| \partial_2^2 R_1 e^{\xi^2 t}f \|_{F^1_{L^1}} + \| G_{5,5} e^{-G_{2,2} t} \tilde{f} \|_{F^1_{L^1}}, \]
we can get (4.13)7. Using $|\xi| \lesssim |\tilde{\xi}|^2$, (4.1)4 for $k = 3$, (4.3)1 for $k = 3$ and 
\[ \| \partial_x |\nabla| M_5(\partial, t)f \|_{F^1_{L^1}} \lesssim \| \partial_x |\nabla| e^{\xi^2 t}f \|_{F^1_{L^1}} + \| G_{3,3} e^{-G_{2,2} t} \tilde{f} \|_{F^1_{L^1}}, \]
we can get (4.13)8 for $r = 1$. Other cases $1 < r \leq 2$ can be bounded similarly. Using $|\xi| \lesssim |\tilde{\xi}|^2$, (4.1)4 for $k = 4$, (4.3)1 for $k = 4$ and 
\[ \| \partial_2^2 M_5(\partial, t)f \|_{F^1_{L^1}} \lesssim \| \partial_2^2 e^{\xi^2 t}f \|_{F^1_{L^1}} + \| G_{4,4} e^{-G_{2,2} t} \tilde{f} \|_{F^1_{L^1}}, \]
we can get (4.13)9. Using $|\xi| \lesssim |\tilde{\xi}|^2$, (4.1)4 for $k = 2$, (4.3)1 for $k = 2$ and 
\[ \| R_1 M_3(\partial, t)f \|_{F^1_{L^1}} \lesssim \| \partial_2 e^{\xi^2 t} |\nabla|^{-1} f \|_{F^1_{L^1}} + \| G_{2,2} e^{-G_{2,2} t} \tilde{f} \|_{F^1_{L^1}}, \]
we can get (4.13)10. Using $|\xi| \lesssim |\tilde{\xi}|^2$, (4.1)4 for $k = 3$, (4.3)1 for $k = 3$ and 
\[ \| \partial_x R_1 M_3(\partial, t)f \|_{F^1_{L^1}} \lesssim \| \partial_x R_1 e^{\xi^2 t} f \|_{F^1_{L^1}} + \| G_{3,3} e^{-G_{2,2} t} \tilde{f} \|_{F^1_{L^1}}, \]
we can get (4.13)11. 

\textbf{Proof of Lemma 4.6.} (4.17)1. By Hölder’s inequality, product estimate in one dimension, interpolation inequality and 
\[ \| \nabla P_r(\tilde{R} \cdot \tilde{b}) \|_{L^2} \lesssim \| \tilde{b} \|_{L^2} \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3, \]  \hspace{1cm} (A.1)
we have 
\[ \| \nabla P_r(\tilde{R} \cdot \tilde{b})b \|_{L^2} + \| \tilde{b} \tilde{b} \|_{L^2} \lesssim \| \tilde{b} \tilde{b} \|_{L^2} \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2, \]
\[ \| \partial_x \partial_\beta (b \tilde{b}) \|_{L^2} \lesssim \| b \|_{L^2} \| \partial_x \partial_\beta b \|_{L^2} \lesssim \| b \|_{L^2} \| \partial_x \partial_\beta b \|_{L^2} \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2, \]
which yields 
\[ \| \partial_x \partial_\beta (b \tilde{b}) \|_{L^2} \lesssim \| b \|_{L^2} \| \partial_x \partial_\beta b \|_{L^2} \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2. \]  \hspace{1cm} (4.17)2

Using Hölder’s inequality and $\partial_y v = -\partial_x u$, we have 
\[ \| \partial_y v_{(t)-8} \|_{L^2} + \| \partial_y v_{(t)-20.05} \|_{L^2} \lesssim \| \partial_x u \|_{L^2} \| b \|_{L^2} \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2. \]
Thanks to $\| \partial_x P_r(\tilde{R} \cdot \tilde{b}) \|_{L^2} \lesssim \| B \|_{L^2}$, we can get 
\[ \| \partial_x P_r(\tilde{R} \cdot \tilde{b}) \|_{L^2} \lesssim \| B \|_{L^2} \| \partial_x u \|_{L^2} \lesssim \| B \|_{L^2} \| \partial_x u \|_{L^2} \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2. \]  \hspace{1cm} (4.17)3

Using product estimate and interpolation inequality, we have 
\[ \| \nabla \partial_\beta (\tilde{u} \otimes \tilde{u}) \|_{L^2} \lesssim \| \tilde{u} \|_{L^2} \| \nabla \partial_\beta \tilde{u} \|_{L^2} \lesssim \| \tilde{u} \|_{L^2} \| \nabla \tilde{u} \|_{L^2} \lesssim \langle t \rangle^{-\frac{1}{2}} \| V \|_3^2 \]
and 
\[ \| \nabla \partial_\beta (\tilde{b} B) \|_{L^2} \lesssim \| \nabla \partial_\beta (\tilde{b} B) \|_{L^2} \| B \|_{L^2} + \| \nabla \partial_\beta (\tilde{b} B) \|_{L^2} \lesssim \langle t \rangle^{-0.21-0.5} \| V \|_3^2 + \langle t \rangle^{-0.25-0.6} \| V \|_3^2 \]
\[ \lesssim \langle t \rangle^{-0.7} \| V \|_3^2, \]
where we have used 
\[ \| \nabla \partial_\beta (\tilde{b} B) \|_{L^2} \leq \| \nabla \partial_\beta (\tilde{b} B) \|_{L^2} \lesssim \| \nabla \partial_\beta (\tilde{b} B) \|_{L^2} \lesssim \langle t \rangle^{-0.21} \| V \|_3 \]
and 
\[ \| \nabla \partial_\beta (\tilde{b} B) \|_{L^2} \lesssim \| \nabla \partial_\beta (\tilde{b} B) \|_{L^2} \lesssim \langle t \rangle^{-0.6} \| V \|_3. \]
This concludes the proof of Lemma 4.6. \hfill \square
Proof of Lemma (4.18) \(\text{(L.1S)}\) Using interpolation inequality
\[
\|f\|_{L^p_x L^2_y} \lesssim \|f\|_{L^2_x}^{\frac{1}{p}} \|\partial_x f\|_{L^2_x}^{1 - \frac{1}{p}},
\]
we obtain
\[
\|b \partial_x b\|_{L^1_t(L^p_y)} \lesssim \|b\|_{L^2_y}^{\frac{1}{2}} \|\partial_x b\|_{L^2_y}^{\frac{1}{2}} \lesssim \|b\|_{L^2_y}^{\frac{1}{2}} \|\partial_x b\|_{L^2_y}^{\frac{1}{2}} \|\partial_x b\|_{L^2_y}^{2 - \frac{1}{p}} \|b\|_{L^p_y}^{\frac{1}{2} - \frac{1}{p}} \lesssim \|b\|_{L^2_y}^{\frac{1}{2}} \|\partial_x b\|_{L^2_y}^{\frac{2 - \frac{1}{p}}{5}} \lesssim \langle t \rangle^{-\frac{3}{2} + \frac{1}{2p}} \|V\|_{L^2_y}^2.
\]

We only give the estimate of \(\|\langle \nabla \rangle^3 (\vec{b} \cdot \nabla u)\|_{L^1_t(L^p_y)}\), since other terms can be bounded similarly. Using
\[
\|\langle \nabla \rangle^k f\|_{L^1_t(L^p_y)} \lesssim \|f\|_{L^1_t(L^p_y)} + \|\nabla^k f\|_{L^1_t(L^p_y)}, \quad k \geq 0,
\]
we have
\[
\|\langle \nabla \rangle^3 (\vec{b} \cdot \nabla u)\|_{L^1_t(L^p_y)} \lesssim \|\langle \nabla \rangle^3 (B \partial_y u)\|_{L^1_t(L^p_y)} + \|\langle \nabla \rangle^3 (b \partial_x u)\|_{L^1_t(L^p_y)} \lesssim \|B \partial_y u\|_{L^1_t(L^p_y)} + \|B \partial_y \nabla^3 u\|_{L^1_t(L^p_y)} + \text{other similar terms}.
\]

By interpolation inequality, we can get
\[
\|B \partial_y u\|_{L^1_t(L^p_y)} \lesssim \|B\|_{L^2_y}^{\frac{1}{2}} \|\partial_x b\|^{\frac{1}{2}} \|u\|_{L^2_y} \lesssim \|B\|_{L^2_y}^{\frac{1}{2}} \|\partial_x b\|_{L^2_y}^{\frac{1}{2}} \|u\|_{L^2_y} \lesssim \langle t \rangle^{-\frac{3}{2}} \|V\|_{L^2_y}^2.
\]

Using
\[
\|\nabla (\langle \nabla \rangle^3 u)\|_{L^2_y} \leq \|\nabla (\langle \nabla \rangle^3 u)_{\langle t \rangle^4} + \|\nabla (\langle \nabla \rangle^3 u)_{\langle t \rangle^4}\|_{L^2_y} \lesssim \langle t \rangle^{-\frac{3}{2}} \|V\|_{L^2_y}^2,
\]
then
\[
\|B \partial_y \nabla^3 u\|_{L^1_t(L^p_y)} \lesssim \|B\|_{L^2_y}^{\frac{1}{2}} \|\partial_y \nabla^3 u\|_{L^2_y} \lesssim \langle t \rangle^{-\frac{3}{2}} \|V\|_{L^2_y}^2 \lesssim \langle t \rangle^{-1.01} \|V\|_{L^2_y}^2.
\]

Thus
\[
\|\langle \nabla \rangle^3 (\vec{b} \cdot \nabla u)\|_{L^1_t(L^p_y)} \lesssim \langle t \rangle^{-1.01} \|V\|_{L^2_y}^2.
\]

Use (A.2), we only show the estimates of
\[
\|u_{\langle t \rangle^{-s}} \partial_y \nabla^3 b\|_{L^1_t(L^p_y)}, \quad \|u_{\langle t \rangle^{-0.05}} \partial_y \nabla^3 b\|_{L^1_t(L^p_y)},
\]
and other terms can be bounded similarly. Using Hölder’s inequality and interpolation inequality, one can get
\[
\|u_{\langle t \rangle^{-s}} \partial_y \nabla^3 b\|_{L^1_t(L^p_y)} \lesssim \|u_{\langle t \rangle^{-s}} \|_{L^2_y} \|\partial_y \nabla^3 b\|_{L^2_y} \lesssim \|u\|_{L^2_y}^{\frac{1}{2}} \|\partial_y u\|_{L^2_y}^{\frac{1}{2}} \|\partial_y \nabla^3 b\|_{L^2_y} \lesssim \langle t \rangle^{-2} \|V\|_{L^2_y}^2,
\]
\[
\|u_{\langle t \rangle^{-0.05}} \partial_y \nabla^3 b\|_{L^1_t(L^p_y)} \lesssim \|u_{\langle t \rangle^{-0.05}} \|_{L^2_y} \|\partial_y \nabla^3 b\|_{L^2_y} \lesssim \langle t \rangle^{-0.05} \|V\|_{L^2_y}^2.
\]

Using (A.2), we only estimate \(\|B \partial_y \nabla^2 b\|_{L^1_t(L^p_y)}\), and other terms can be controlled similarly. Using interpolation inequality, and
\[
\|\langle \nabla \rangle^3 b\|_{L^2_y} \lesssim \|\langle \nabla \rangle^3 b_{\langle t \rangle^{\frac{1}{2}}}\|_{L^2_y} + \|\langle \nabla \rangle^3 b_{\langle t \rangle^{\frac{1}{2}}}\|_{L^2_y} \lesssim \langle t \rangle^{-\frac{3}{2}} \|V\|_{L^2_y}.
\]
we have
\[ \| B \partial_y \nabla^2 b \|_{L^1_t(L^2_y)} \lesssim \| B \|_{L^2_t(L^\infty_y)} \langle \nabla \rangle^3 b \|_{L^2_t} \lesssim \langle t \rangle^{-\frac{3}{4}} \| \nabla \|_3 \lesssim \langle t \rangle^{-0.75} \| V \|_3. \]

Using (4.19) again, we only bound \( \| \partial_y \langle \nabla \rangle^2 P_y u b \|_{L^1_t(L^2_y)} \), while other terms can be bounded similarly. Using interpolation inequality, and
\[ \| \partial_x \langle \nabla \rangle^3 u \|_{L^2_t} \leq \| \partial_x \langle \nabla \rangle^3 u_{<\langle t \rangle^{0.2}} \|_{L^2_t} + \| \partial_x \langle \nabla \rangle^3 u_{\geq \langle t \rangle^{0.2}} \|_{L^2_t} \lesssim \langle t \rangle^{-0.8} \| V \|_3, \]
we have
\[ \| \partial_x \langle \nabla \rangle^2 P_y u \|_{L^1_t(L^2_y)} \lesssim \| \partial_x \langle \nabla \rangle^2 u \|_{L^2_t} \lesssim \langle t \rangle^{-0.95} \| V \|_3. \]

**Proof of Lemma 4.8.** By Hölder’s inequality, and (A.1), it is easy to get the estimate of (4.19) and (A.2). Let us begin with the estimate of (4.19)\_3. We only give the estimate of \( \| b \cdot \nabla b \|_{H^2} \), while one can bound other terms by the similar way. We have
\[ \| b \cdot \nabla b \|_{H^2} \leq \| b \partial_x \nabla^2 b \|_{L^2_t} + \text{other similar terms.} \]
Using
\[ \| \partial_x \nabla^2 b \|_{L^2_t} \lesssim \| \partial_x \nabla^2 b_{<\langle t \rangle^{0.2}} \|_{L^2_t} + \| \partial_x \nabla^2 b_{\geq \langle t \rangle^{0.2}} \|_{L^2_t} \lesssim \langle t \rangle^{-\frac{7}{4}} \| V \|_3^2, \]
we have
\[ \| b \partial_x \nabla^2 b \|_{L^2_t} \lesssim \| b \|_{L^\infty_t} \| \partial_x \nabla^2 b \|_{L^2_t} \lesssim \langle t \rangle^{-1} \| V \|_3. \]
Thus
\[ \| b \cdot \nabla b \|_{H^2} \lesssim \langle t \rangle^{-1} \| V \|_3. \]
Using (4.19)\_4
\[ \| \partial_x P_y (\tilde{R} \cdot \tilde{b}) \|_{L^\infty_t} \lesssim \| \tilde{B} \|_{L^1_t} \lesssim \langle t \rangle^{-1} \| V \|_3 \]
and (A.4), one can get
\[ \| \partial_x P_y (\tilde{R} \cdot \tilde{b}) \partial_y b \|_{L^2_t} \lesssim \langle t \rangle^{-1} \| V \|_3. \]
Then we can get the estimates of other terms by the similar way. We have only bound \( \| P_y (\tilde{R} \cdot \tilde{b}) \partial_x \nabla^2 b \|_{L^2_t} \). Using (A.5), and
\[ \| P_y (\tilde{R} \cdot \tilde{b}) \|_{L^1} \lesssim \| \nabla \|^{-1} b \|_{F^1} \sum \| \nabla \|^{-1} P_M b \|_{F^1} + \sum \| \nabla \|^{-1} P_M b \|_{F^1} \]
\[ \lesssim \langle t \rangle^{-0.5} \| V \|_3 + \sum \| P_M b \|_{L^2_t} \lesssim \langle t \rangle^{-0.5} \| V \|_3 + \sum \| P_M b \|_{L^2_t} \]
\[ \lesssim \langle t \rangle^{-0.5} \| V \|_3 + \| B \|_{L^2_t} \sum \| P_M b \|_{L^2_t} \lesssim \langle t \rangle^{-0.5} \| V \|_3 + \langle t \rangle^{-0.5} \| V \|_3 \lesssim \langle t \rangle^{-0.492} \| V \|_3. \]
we have
\[ \| P_y (\tilde{R} \cdot \tilde{b}) \partial_x \nabla^2 b \|_{L^2_t} \lesssim \| P_y (\tilde{R} \cdot \tilde{b}) \|_{L^1} \| \partial_x \nabla^2 b \|_{L^2_t} \lesssim \langle t \rangle^{-1} \| V \|_3. \]
Here we only show the estimates of \( \| b \partial_x \nabla^2 b \|_{L^2_t} \) and \( \| P_y (\tilde{R} \cdot \tilde{b}) \partial_y \nabla^3 u \|_{L^2_t} \). Using
\[ \| \partial_x \langle \nabla \rangle^3 b \|_{L^2_t} \lesssim \| \partial_x \langle \nabla \rangle^3 b_{<\langle t \rangle^{0.2}} \|_{L^2_t} + \| \partial_x \langle \nabla \rangle^3 b_{\geq \langle t \rangle^{0.2}} \|_{L^2_t} \lesssim \langle t \rangle^{-\frac{1}{4}} \| V \|_3^2, \]
we have
\[ \|b\partial_x^2 \nabla^2 b\|_{L^2} \lesssim \|b\|_{L^\infty} \|\partial_x^2 \nabla^2 b\|_{L^2} \lesssim \langle t \rangle^{-1} \|V\|_2. \]
Thanks to (A.6) and (A.3), we can get the estimate of \( \|P\partial_y \nabla^3 u\|_{L^2} \) by using Hölder’s inequality.

By using (A.6) and (A.4), one can easily get this estimate. \( \Box \)

**Proof of Lemma 4.9**

The first three terms can be bounded easily. By
\[ \|\nabla P\partial_y \nabla^3 u\|_{F_{L^1}} \lesssim \|\nabla \bar{b}\|_{F_{L^1}} \lesssim \langle t \rangle^{-\frac{5}{4}} \|V\|_3, \]
we can get
\[ \|\partial_x \langle \partial_y P\partial_y (\hat{R} \cdot \bar{b}) \rangle \nabla \partial_y \langle \partial_y P\partial_y (\hat{R} \cdot \bar{b}) \rangle u\|_{F_{L^1}} \lesssim \|\nabla \bar{b}\|_{F_{L^1}} (\|\partial_x b\|_{F_{L^1}} + \|\nabla \bar{u}\|_{L^\infty}) \lesssim \langle t \rangle^{-\frac{7}{4}} \|V\|_2. \]
Like the previous procedure, it suffices to estimate \( \|b\partial_x u\|_{F_{L^1}} \) and \( \|\partial_x b\partial_x v\|_{F_{L^1}} \). Using
\[ \|\langle \nabla \rangle \partial_x u\|_{F_{L^1}} \lesssim \|\langle \nabla \rangle \partial_x u\|_{F_{L^1}} + \|\langle \nabla \rangle \partial_x u\|_{F_{L^1}} \lesssim \langle t \rangle^{-0.75} \|V\|_3, \]
\[ \|\langle \nabla \rangle b\|_{F_{L^1}} \lesssim \|\langle \nabla \rangle b\|_{F_{L^1}} + \|\langle \nabla \rangle b\|_{F_{L^1}} \lesssim \langle t \rangle^{-0.42} \|V\|_3, \]
\[ \|\partial_x v\|_{F_{L^1}} \lesssim \|\partial_x u\|_{F_{L^1}} + \|\partial_x u\|_{F_{L^1}} \lesssim \langle t \rangle^{-0.85} \|V\|_3, \]
one gets
\[ \|b\partial_x u\|_{F_{L^1}} \lesssim \|\nabla \bar{b}\|_{F_{L^1}} \|\partial_x \nabla^2 b\|_{F_{L^1}} \lesssim \langle t \rangle^{-\frac{5}{4}} \|V\|_2, \]
and
\[ \|\partial_x b\partial_x v\|_{F_{L^1}} \lesssim \|\partial_x b\|_{F_{L^1}} \|\partial_x v\|_{F_{L^1}} \lesssim \langle t \rangle^{-\frac{5}{4}} \|V\|_2. \]
Using (A.6) and the fact that \( P \) can be bounded by the process dealing with \( P \), we can get the desired estimate by Hölder’s inequality.

Here we only show the estimate of \( \|P\partial_y \nabla^2 u\|_{F_{L^1}} \). Using
\[ \|\langle \nabla \rangle^3 u\|_{F_{L^1}} \lesssim \|\langle \nabla \rangle^3 u\|_{F_{L^1}} + \|\langle \nabla \rangle^3 \bar{u}\|_{F_{L^1}} \lesssim \langle t \rangle^{-0.99} \|V\|_3 \]
and (A.6), we have
\[ \|P\partial_y \nabla^2 u\|_{F_{L^1}} \lesssim \|P\partial_y \nabla^2 \bar{u}\|_{F_{L^1}} \|\partial_y \nabla^2 u\|_{F_{L^1}} \lesssim \langle t \rangle^{-1.01} \|V\|_3. \]

We only give the estimate of \( \|\partial_x (b\partial_x b)\|_{F_{L^1}} \). Using
\[ \|\partial_x^2 b\|_{F_{L^1}} \lesssim \|\partial_x^2 b\|_{F_{L^1}} + \|\partial_x^2 b\|_{F_{L^1}} \lesssim \langle t \rangle^{-0.83} \|V\|_3, \]
we have
\[ \|\partial_x (b\partial_x b)\|_{F_{L^1}} \lesssim \|\partial_x b\partial_x b\|_{F_{L^1}} + \|b\partial_x b\|_{F_{L^1}} \lesssim \langle t \rangle^{-1.3} \|V\|_3. \]
Using the same way yielding (A.7)3, we can get
\[ \|\nabla B\|_{F_{L^1}} \lesssim \langle t \rangle^{-0.85} \|V\|_3, \]
which, together with (A.7)2 yields
\[ \|\nabla (B\bar{b})\|_{F_{L^1}} \lesssim \langle t \rangle^{-1.2} \|V\|_3. \]
Other terms can be bounded similarly.

We only estimate \( \|\nabla^2 B\partial_y b\|_{F_{L^1}} \), while other terms can be bounded similarly. Using the same way yielding (A.7)3, one has
\[ \|\nabla^2 B\|_{F_{L^1}} \lesssim \langle t \rangle^{-0.71} \|V\|_3, \]
which, with (A.7)2 leads
\[ \|\nabla^2 B\partial_y b\|_{F_{L^1}} \lesssim \|\nabla^2 B\|_{F_{L^1}} \|\partial_y b\|_{F_{L^1}} \lesssim \langle t \rangle^{-1.1} \|V\|_3. \]
Using (A.7) and (A.6), we have
\[
\|P_\omega(\tilde{R}^\omega \cdot \tilde{b})\partial_y b\|_{F^L_1} \lesssim \|P_\omega(\tilde{R}^\omega \cdot \tilde{b})\|_{F^L_1} \|\partial_y b\|_{F^L_1} \lesssim \langle t \rangle^{-0.9} \|V\|_3^2.
\]

References

[1] H. Abidi and P. Zhang, On the global well-posedness of 3-D MHD system with initial data near the equilibrium state, *Comm. Pure. Appl. Math* **70**, (2017) 1509-1561.

[2] C. Bardos, C. Sulem and P. L. Sulem, Longtime dynamics of a conductive fluid in the presence of a strong magnetic field, *Trans. Amer. Math. Soc.* **305**, (1988) 175-191.

[3] H. Cabannes, Theoretical Magneto-Fluid Dynamics, Academic Press, New York, London, 1970.

[4] F. Califano, C. Chiuderi, Resistivity-independent dissipation of magnetohydrodynamic waves in an inhomogeneous plasma, *Phys. Rev. E* **60**, (Part B) (1999) 4701-4707.

[5] J. Y. Chemin, D. S. McCormick, J. C. Robinson and J. L. Rodrigo, Local existence for the non-resistive MHD equations in Besov spaces, *Adv. Math.* **286** (2016) 1-31.

[6] W. Deng, P. Zhang, Large time behavior of solutions to 3-D MHD system with initial data near equilibrium, *Arch. Ration. Mech. Anal.* online.

[7] G. Duvaut and J. L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, *Arch. Ration. Mech. Anal.* **46** (1972) 241-279.

[8] F. Lin, L. Xu and P. Zhang, Global small solutions to 2-D MHD system with small data, *J. Differential Equations* **259** (2015), 5440-5485.

[9] F. Lin and P. Zhang, Global small solutions to MHD type system (I): 3-D case, *Comm. Pure. Appl. Math* **67** (2014) 531-580.

[10] F. Lin and T. Zhang, Global small solutions to a complex fluid model in three dimensional, *Arch. Ration. Mech. Anal.* **216** (2015) 905-920.

[11] T. Kato, G. Ponce, Commutator estimates and the Euler and Navier- Stokes equations, *Comm. Pure Appl. Math.* **41**, (1988) 891-907.

[12] R. Pan, Y. Zhou, Y. Zhu, Global classical solutions of three dimensional viscous MHD system without magnetic diffusion on periodic boxes, *Arch. Ration. Mech. Anal.* **227**, (2017) 637-662.

[13] X. Ren, J. Wu, Z. Xiang and Z. Zhang, Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion, *J. Funct. Anal.* **267** (2014) 503-541.

[14] M. Sermange and R. Temam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.* **36** (1983) 635-664.

[15] R. Wan, On the uniqueness for the 2D MHD equations without magnetic diffusion, *Nonlinear Analysis Real World Applications* **30**, (2016) 32-40.

[16] J. Wu, Y. Wu, X. Xu, Global small solution to the 2D MHD system with a velocity damping term, *SIAM J. Math. Anal.* **47** (2015), 2630-2656.

[17] J. Wu, Y. Wu, Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion, *Adv. Math.* **310** (2017) 759-888.

[18] L. Xu and P. Zhang, Global small solutions to three-dimensional incompressible MHD system, *SIAM J. Math. Anal.* **47** (2015) 26-65.

[19] T. Zhang, An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system, *arXiv:1404.5681v2 [math.AP]*.

[20] T. Zhang, Global solutions to the 2D viscous, non-resistive MHD system with large background magnetic field, *J. Differential Equations* **260** (2016) 5450-5480.

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