ON THE DENOMINATOR OF THE POINCARÉ SERIES FOR MONOMIAL QUOTIENT RINGS

HARA CHARALAMBOUS

Abstract. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ and $I$ a monomial ideal of $S$. It is well known that the Poincaré series of $k$ over $S/I$ is rational. We describe the coefficients of the denominator of the series and study the multigraded homotopy Lie algebra of $S/I$.

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1. Introduction

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$, $I$ a monomial ideal of $S$ and $R = S/I$. If $J$ is a subset of the minimal monomial generating set $I$ of $I$, we let $m_J$ denote the least common multiple of the monomials in $J$ or the corresponding monomials in $k[y_1, \ldots, y_n]$ as appropriate. The multigraded Poincaré series of $R$, $P_R(y, t)$ is rational, [Ba82]:

$$P_R(y, t) = \frac{\prod_{i=1}^{m_1}(1 + ty_i)}{Q_R(y, t)}$$

where $Q_R(y, t) \in Z[y_1, \ldots, y_n][t]$.

The few facts that are known in general about the multigraded expansion $Q_R(y, t)$ also date back to [Ba82]: the degree of $Q_R(y, t)$ in $t$ is bounded above by the total degree of $m_1$, and the monomial coefficients of $t$ divide $m_1$. In this paper we discuss the monomial coefficients of $Q_R(y, t)$.

In the first section of this paper we show that the monomial coefficients of $t$ in $Q_R(y, t)$ are least common multiples of the monomial generators of $I$. We discuss the Koszul homology of $R$ when $R$ is Golod. We also present a free resolution of $R$ when $I$ is generic. In the second section we discuss the multigraded acyclic closure of $R$ as well as the multigraded deviations in terms of the multigraded homotopy Lie algebra of $R$, and show that the LCM lattice determines the Poincaré series of $R$. This is the multigraded version of Theorem 1 of [Av02].

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2. The monomials of $Q_R(y, t)$

The argument used in the proof of the next proposition has the flavor of [Ba82] and uses Golod theory. It uses Lescot’s result, [Le86], that the Poincaré series $M$ over $R$

$$P^M_R(y, t) = \sum_{i=0}^{\infty} \sum_{j \geq 0} y^i \dim_k (\text{Tor}^R_i(M, k)) j^t = \frac{f^M_R(y, t)}{Q_R(y, t)}$$

where $f^M_R(y, t)) \in \mathbb{Z}[y_1, \ldots, y_n][t]$.

**Proposition 2.1.** Let $Q_R(y, t) = 1 + \sum (\sum c_j y^j) t^j$. Then $y^j$ is equal to a least common multiple of a subset of the minimal monomial generating set of $I$.

**Proof.** By a standard argument, [Ba82], it is enough to prove Proposition 2.1 when $I$ is generated by squarefree monomials. In this case the $y$ monomials of $Q_R(y, t)$ are squarefree, [Ba82]. We do induction on the number of variables $n$ that divide the monomials of the minimal generating set of $I$, the case $n = 1$ being trivial. Consider a term of $Q_R(y, t)$ whose coefficient in $y$ does not involve $y_i$. We separate the generators of $I$ that involve $x_i$. Let $I = L_i + x_i J_i$ where $L_i, J_i \subset k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ and let $A = k[x_1, \ldots, x_i, \ldots, x_n]/L_i$ and $A' = k[x_1, \ldots, x_i, \ldots, x_n]/(L_i + J_i)$. Therefore $R \cong A[x_i]/x_i J_i$ and $A[x_i] \to R$ is Golod, which implies that

$$P_R(y, t) = \frac{P_A[x_i]}{1 - t(P_A^R[x_i] - 1)} = \frac{(1 + t y_i) P_A}{(1 - t^2 P_A^R[x_i])} = \frac{(1 + t y_i) P_A}{(1 - t^2 y_i P_A^R[x_i])} = \frac{(1 + t y_i) P_A}{(1 - t y_i P_A^R[x_i] - 1)} = \prod (1 + t y_i) = \frac{Q_A + t y_i Q_A - t y_i f_A}{Q_A + t y_i Q_A - t y_i f_A}.$$

The $y$ monomials that do not involve $y_i$ are terms of $Q_A$, so by induction they have the desired form. Suppose now that a monomial of $Q_R(y, t)$ involves all $y_i$: it is the product of all the variables $y_i$ and is equal to the least common multiple of all the generators in $I$.

In [ChRe95] we noticed that the terms of $Q_R(y, t)$ when $I$ is a complete or almost complete intersection come from the set $\{-1\}^{l_1} t (\{1\}^{l_1} + \{1\}^{l_2}) M_j \}$. Here $J$ ranges over the subsets of the minimal monomial generating set of $I$ and $l_j$ is the number of connected components of the graph we get when we connect the elements of $J$ by an edge if they have a variable in common. In [ChRe95] we also confirmed that this is the case when the Taylor resolution [Ta60], [El97] of $S/I$ over $S$ is minimal.

In Remark 2.3 we confirm that the same holds for the monomial ideals $I$ such that $R/I$ is Golod. First we recall and expand a comment by Fröberg, [Fr79].

**Remark 2.2.** Let $R = S/I$ be Golod and $K_*$ be the Koszul complex on the variables $\bar{x}_i$ of $R$. Then $H(K_*)$ is isomorphic to $k[X_{A_1}, \ldots, X_{A_r}]$ where $A_1$ is a subset of the minimal monomial generating set of $I$ and the degree of $X_{A_r}$ equals the cardinality of $A_r$. Indeed since $K_*$ admits trivial Massey operations the product of two elements of $H_{\geq 1}(K_*)$ is zero, [GuLe69], [Av98]. Moreover $H(K_*)$ is always of the form $k[X_1, \ldots, X_n]/L$: $L$ is generated by quadratics, the $X_i$’s form a minimal multiplicative generating set for $H(K_*)$, and $X_i X_j = (-1)^{\deg X_i} X_j X_i$ where $\deg X_i$ is the degree of $X_i$. Let $F_*$ be the Taylor complex on the minimal monomial generators of $I$ over $S$, [Ta60] and let $E_*$ be the minimal multigraded resolution of $S/I$ over
Q. $E_\bullet$ is a direct summand of $F_\bullet$. It follows that the multidegrees of the generators of $E_i$ are among the multidegrees of the generators of $F_i$. Therefore they are the multidegrees of least common multiples of subsets $A_j$ of $I$. It follows that $H(K\bullet) = H(k \otimes E_\bullet) = k[X_{A_1}, \ldots, X_{A_r}] / L$. Since $X_{A_j}X_{A_{i}} = 0$, $X_{A_j}X_{A_{i}} \in L$.

**Remark 2.3.** Thus monomial Golod rings are special Golod rings. Let $m_{A} \neq m_{A_{1}} \cdots m_{A_{r}}$, for any partition of $A$ where $X_{A}$ and $X_{A_{j}}$ are among the generators of $H(K\bullet)$ we see that $A$ is connected and $l_{A} = 1$. Let $H_{i}(K)_{j}$ denote the subspace of $H_{i}(K)$ of multidegree $j$. Thus the terms of the denominator of the Poincaré series of $R$, $Q_{R}(y, t) = 1 - \sum_{i} \sum_{j} \dim_{k} H_{i}(K)_{j} y^{i} t^{j+1}$ of the Golod ring $R$ come from the set $\{(−1)^{d_{i}+1}m_{j}\}$ where $J$ ranges over the subsets of the minimal monomial generating set of $I$.

Generic ideals and their minimal resolutions were first introduced in [BaPeSt98] and generalized in [MiStYa00]. Let $I$ be a generic ideal. If the multidegrees of two minimal monomial generators of $I$ are equal for some variable then there is a third monomial generator of $I$ whose multidegree is strictly smaller than the multidegree of the least common multiple of the other two. The minimal resolution $E_\bullet$ of $S/I$ over $S$ is determined by the Scarf complex of $\Delta_{f}$. The latter consists of the subsets $J$ of the index set of $I$, $\Phi$, such that $m_{J} \neq m_{L}$ for any other subset $L$ of $\Phi$. We note that $E_\bullet$ has the structure of a DG algebra, [BaPeSt98]. The Eagon resolution of $k$ over $R = S/I$ was discussed in [GuLe69]. Below we give the generators for the Eagon resolution when $I$ is generic.

**Theorem 2.4.** Let $R = S/I$ where $I$ is a generic ideal of $S$, let $K_\bullet$ be the Koszul complex on the $x_{i}$ over $R$, $X_{0} = 0$, and $X_{i}$ be the free $R$-module with generators $T_{i}$, where $|L| = i$ and $L$ is in $\Delta_{f}$, the Scarf complex of $I$. Then $(Y_\bullet, d_\bullet)$ is a free resolution of $k$ over $R$, where

$$\delta_{i} = \sum_{j+t_{1}+\cdots+t_{i}+t = i} K_{j} \otimes X_{t_{1}} \otimes \cdots \otimes X_{t_{i}},$$

and
d$(T_{1}) = (m_{J}/x_{J})e_{1},$
d$(T_{1} \otimes \cdots \otimes T_{i}) = d(T_{1} \otimes \cdots \otimes T_{i-1}) \otimes T_{i} + \sum_{i=1}^{I-1} \sum_{i=1}^{(|I_{i}| + 1)T_{i} \otimes \cdots \otimes T_{i-2} \otimes \delta(I_{i-1}, I_{i})(T_{i-1} \wedge T_{i}),$

where $\delta(I_{i-1}, I_{i}) = m_{I_{i-1} \cup I_{i}} \wedge m_{I_{i-1} \cup I_{i}}$ if $I_{i-1} \cup I_{i} \in \Delta_{f}$ and 0 otherwise
d$(e_{L} \otimes T_{1} \otimes \cdots \otimes T_{i})d(e_{L}) \otimes T_{1} \otimes \cdots \otimes T_{i} + (−1)^{|L|}e_{L} \otimes d(T_{1} \otimes \cdots \otimes T_{i}).$

**Proof.** The theorem follows immediately from [Iy97], Theorem 1.2 and the preceding comments. \hfill \Box

As an immediate corollary we immediately get the characterization of Golod rings when $I$ is a generic ideal, which generalizes Proposition 2.4 of [Ga00].

**Proposition 2.5.** Let $I$ be a generic ideal. $S/I$ is Golod if and only if $m_{1}m_{2} \neq m_{1}m_{2}$ whenever $I_{1} \cup I_{2} \in \Delta_{f}$. 
3. Multigraded deviations and acyclic closures

In this section we introduce the multigraded acyclic closure \( R<X> \) of \( k \) over \( R \). For the construction of the usual acyclic closure, we refer to Construction 6.3.1, [Av98]. To this end we choose \( x_j \), the \( \Gamma \)-variables of \( X_n \) so that \( \text{cls}(\theta(x_j)) \) has multidegree \( j \) and \( \{\text{cls}(\theta(x_j))\} \) is a minimal multigraded generating set of \( H_{n-1}(R<X\leq n-1>) \). We define the multidegree of \( x_j \) to be \( j \) and we let \( X_{n,j} \) consist of all \( x \in X_n \) whose multidegree is \( j \). We will show that the cardinality of \( X_{n,j} \) appears as an exponent in a product decomposition of the multigraded Poincaré series.

First we state and prove the multigraded version of Remark 7.1.1, [Av98].

Proposition 3.1. For each formal multigraded power series with integer coefficients

\[
P(y, t) = 1 + \sum_{i=1}^{\infty} \left( \sum_{j} a_{i,j} y^j \right) t^i
\]

where for each \( i \), \( a_{i,j} = 0 \) for all but finitely many values of \( j \), there exist uniquely defined integers \( e_{n,j} \in Z \) such that

\[
P(y, t) = \frac{\prod_{i=1}^{\infty} \prod_{j}(1 + y^j t^{(i-1)e_{i-1,j}})}{\prod_{i=1}^{\infty} \prod_{j}(1 - y^j t^{(i-1)e_{i-1,j}})}
\]

and the product converges in the \((t)\)-adic topology of the ring \( Z[y, t] \).

Proof. We set \( P_0(t) = 1 \) and assume by induction that \( P(y, t) \equiv P_{n-1}(y, t) \) modulo \( t^{n+1} \). If \( P(y, t) - P_{n-1}(y, t) = \sum a_{n,j} y^j t^n \) (mod \( t^{n+1} \)), then we set \( P_n(y, t) = P_{n-1}(y, t) \prod_{j}(1 + y^j t^{n})^{e_{n,j}} \) if \( n \) is odd and \( P_n(y, t) = P_{n-1}(y, t) / \prod_{j}(1 - y^j t^{n})^{e_{n,j}} \) if \( n \) is even. Then it is clear that \( P(t) \equiv P_n(t) \) (mod \( t^{n+1} \)) and the other assertions follow as well.

We define the multigraded \((n, j)\) deviation of \( R \), denoted by \( e_{n,j} \), to be the exponent \( e_{n,j} \) in the product decomposition of the Poincaré series of \( R \). As in Theorem 7.1.3, [Av98] it follows that

Proposition 3.2. Let \( I \) be a monomial ideal of \( S \), \( R = S/I \) and \( R<X> \) be the multigraded acyclic closure of \( k \) over \( R \). Then

\[
P_H(y, t) = \frac{\prod_{i=1}^{\infty} \prod_{j}(1 + y^j t^{2(i-1)\text{card}(X_{2i-1,j})})}{\prod_{i=1}^{\infty} \prod_{j}(1 - y^j t^{2(i-1)\text{card}(X_{2i,j})})}
\]

and \( \text{card} X_{n,j} = e_{n,j}(R) \).

Since \( R<X> \) is multigraded, it follows that the homotopy Lie algebra of \( R \), \( \pi^*(R) = \text{HDer}_{\mathbb{Q}}(R<X>, R<X>) \) has a multigraded structure. Moreover \( \pi^n(R) \cong \text{Hom}_k(kX_{n,j}, k) \) (Theorem 10.2.1, [Av98]), and \( \pi^j(R) \cong \text{Hom}_k(kX_{n,-j}, k) \) while \( \text{rank}_k \pi^j(R) = e_{n,-j} \).

Remark 3.3. We let \( K_\bullet \) denote the Koszul complex on \( x_1, \ldots, x_n \) over \( S \) and \( T_\bullet \) the Taylor complex on the minimal generators of \( I \) over \( S \). The complexes \( K_\bullet, T_\bullet \) have the structure of a DG \( \Gamma \) algebra by Lemma 9, [Av02]; they are also multigraded and \( H_\bullet(T \otimes_S K_j) \cong H_\bullet(T \otimes_S k_j) \cong H_\bullet(R \otimes_S K_j) \), where \( j \in L_I \). In Lemma 11 of [Av02], it is shown that \( \pi^{\geq 2}(R) \cong \pi^*(R \otimes_S K) \). We remark that \( R \otimes_S K \) is a multigraded DG \( \Gamma \) algebra, and one can choose a multigraded DG \( \Gamma \)
algebra $U$ so that we have the factorization $R \otimes_S K \longrightarrow U \longrightarrow k$ with the properties of Lemma 11 and Remark 10 of [Av02]: $\pi(R \otimes_S K)$ is the graded $k$ dual of the residue of $H_1(U \otimes_{R \otimes_S K} k)$ modulo multigraded relations and is multigraded. The homomorphisms of Lemma 11, [Av02] are multigraded and $\pi^{1+2}_j(R) \cong \pi^{1}_j(R \otimes_S K)$. Finally we have $\pi^{1}_j(T \otimes_S K) \cong \pi^{0}_j(T \otimes_S k) \cong \pi^{1}_j(R \otimes_S K)$.

Next we will make use of the LCM lattice $L_I$, [GaPeWe99] and the GCD graph $G_I$, [Av02]. We recall that $L_I$ and $G_I$ have vertices the least common multiples of the monomial generators of $I$ while the edges of $G_I$ join least common multiples that are relatively prime. In [Av02] it is shown that if $I$ and $I'$ are two monomial ideals of $S$ and $S'$ respectively with an isomorphism of lattices $\lambda : L_I \longrightarrow L_{I'}$ which induces an isomorphism of the GCD graphs $\lambda : G_I \longrightarrow G_{I'}$ then the homotopy Lie algebras of $R = S/I$ and $R' = S/I'$ are isomorphic and the Poincaré series of $R$ and $R'$ have the same denominator. We will show that the multigraded version of this result is actually true. We consider the multigraded acyclic closures $R < X >$ and $R' < X' >$ of $k$ over $R$ and $R'$.

Remark 3.4. Let $I \subset S = k[x]$ and $I' \subset S' = k[x']$ be two monomial ideals, $K, K'$ the Koszul complexes on the $x_i$ and $x'_i$ respectively and $\lambda : L_I \longrightarrow L_{I'}$, an isomorphism of lattices. If $\lambda$ induces an isomorphism of the GCD graphs $\lambda : G_I \longrightarrow G_{I'}$ then $H(K \otimes R) \cong H(K' \otimes R')_{\lambda(j)}$ through the isomorphism that sends the algebra generators $X_A$ of $H(K \otimes R) = k[X_A]/L$ to the generators $X'_{\lambda(A)}$ of $H(K' \otimes R') = k[X'_{\lambda(A)}/L']$ (here our notation is as in the remarks preceding Proposition 2.2). Indeed, let $\lambda$ be as above. Since $\lambda$ is an isomorphism of the LCM lattices the minimal resolution of $I$ gives rise to a minimal resolution of $I'$ and the generators $X_A$ of $H(K \otimes R)$ correspond to generators $X'_{\lambda(A)}$ of $H(K' \otimes R')$. Since $\lambda$ is an isomorphism of the GCD graphs the relations among the $X_A$ correspond to relations among the $X'_{\lambda(A)}$. Conversely if the Koszul homology algebras are isomorphic as above then there is an isomorphism of multigraded vector spaces between the minimal resolutions of the two ideals and a map $\lambda$ which is an isomorphism of lattices and GCD graphs for the multidegrees that appear in the resolutions.

Proposition 3.5. Let $I'$ be a monomial ideals of $S' = k[x']$ such that $\lambda : L_I \longrightarrow L_{I'}$ is an isomorphism of lattices that induces an isomorphism of the GCD graphs $\lambda : G_I \longrightarrow G_{I'}$. Then there is a map $\lambda$ from the set of the multidegrees of the variables $X_n$ to the set of the multidegrees of the variables $X'_n$ such that $\text{card } X_{n,j} = \text{card } X'_{n,\lambda(j)}$ and $\epsilon_{n,j} = \epsilon'_{n,\lambda(j)}$.

Proof. We define $\lambda$ by induction on $n$. Let $T'$ be the Taylor complex of $I'$. Then there is an isomorphism of vector spaces $(T' \otimes_S k)_j \cong (T' \otimes_{S'} k)_{\lambda(j)}$, for $j \in L_I$. Suppose that $\lambda$ is an extension of $\lambda$ and $R[X_{\leq n,j}] \otimes k \cong R'[X'_{\leq n,\lambda(j)}] \otimes k$. Extend $\lambda$ to the multidegrees of the multigraded generators of $H_n(R[X_{\leq n}])$ and $H_n(R'[X'_{\leq n}])$. It follows that there is an isomorphism of homotopy Lie algebras $\pi^{*}_j(T \otimes_S k) \cong \pi^{*}_{\lambda(j)}(T' \otimes_{S'} k)$, and the proposition now follows. \hfill $\square$

Remark 3.6. Let $I$ be a monomial ideal and $I_{pol}$ be the square free monomial ideal in $A = k[z]$ that corresponds to $I$, [Fr82]. The map $\lambda$ that sends a monomial of $I$ to a squarefree monomial in $I_{pol}$ is an isomorphism of lattices and of GCD graphs. Moreover $Q_H(y,t)$ can be obtained from $Q_A(z,t)$ by applying $\lambda^{-1}$ to the monomial coefficients of $t$, [Ba82].
Theorem 3.7 now completes the multigraded version of Theorem 1, \[Av02\].

**Theorem 3.7.** Let \( S = k[x] \) and \( S' = k[x'] \) be polynomial rings over a field \( k \), and \( I \subset S \), \( I' \subset S' \) be ideals generated by monomials of degree at least 2. Let \( \lambda: L_I \rightarrow L_{I'} \) and the induced \( \lambda: G_I \rightarrow G_{I'} \) be isomorphisms. Suppose that \( Q_R(y, t) = 1 + \sum (\sum c_j y^{j}) t^i \) then \( Q_{R'}(y', t) = 1 + \sum (\sum c_j y^{j}) t^i \).

**Proof.** By Proposition 2.1 the multidegrees \( j \) in \( Q_R(y, t) \) are in \( L_I \), so \( \lambda(j) \) makes sense. By the above remark it is enough to examine the case where \( I \) and \( I' \) are both squarefree. The key point is that since \( I \) and \( I' \) are squarefree, \( \lambda \) is additive: \( \lambda(j + i) = \lambda(j) + \lambda(i) \). It follows that its extension \( \hat{\lambda} \) is also additive. Since

\[
Q_R(y, t) = \prod_{i=1}^{\infty} \frac{(1 + ty_i)}{P_R(y, t)} = \prod_{i=1}^{\infty} \frac{\prod_{j=1}^{\infty} (1 - y^2 t^{2i})^{\text{rank} \pi_{j}^{2i}}}{\prod_{i=2}^{\infty} \prod_{j=1}^{\infty} (1 + y^2 t^{2i-1})^{\text{rank} \pi_{j}^{2i-1}}}
\]

the linear coefficient \( c_j \) of the monomial \( y^j t^n \) of \( Q_R(y, t) \), depends on the deviations \( \epsilon_i, f \) when \( i \leq n \). The corresponding combination of the deviations \( \epsilon_i, f \) gives a term of \( Q_{R'}(y', t) \) of \( t \)-degree \( n \). The theorem now follows. \( \square \)

The condition that \( \lambda \) is an isomorphism of GCD graphs is a necessary condition as the example from \[Ga00\] shows. The isomorphism of LCM lattices for \( I = (x_1^2, x_2^2, x_3^3) \) and \( I' = (x_1 x_2^2, x_1 x_3^3) \) is not an isomorphism of GCD graphs and

\[
Q_{S/I}(y, t) = 1 - t^2(y_1^2 - y_2^2 y_3) + t^4(y_1^2 y_2^2 y_3)
\]

while

\[
Q_{S/I'}(y, t) = 1 - t^2(y_1 y_2^2 + y_1 y_3^2) - t^3(y_1 y_2^2 y_3^2).
\]

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Department of Mathematics, University at Albany, SUNY, Albany, NY 12222

E-mail address: hara@math.albany.edu