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VOLUME ENTROPY RIGIDITY OF NON-POSITIVELY CURVED SYMMETRIC SPACES

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To Werner Ballmann for his 60th birthday

Abstract. We characterize symmetric spaces of non-positive curvature by the equality case of general inequalities between geometric quantities.

1. Introduction

Let \((M, g)\) be a closed connected Riemannian manifold, and \(\pi : (\tilde{M}, \tilde{g}) \to (M, g)\) its universal cover endowed with the lifted Riemannian metric. We denote \(p(t, x, y), t \in \mathbb{R}_+, x, y \in \tilde{M}\) the heat kernel on \(\tilde{M}\), the fundamental solution of the heat equation \(\frac{\partial u}{\partial t} = \text{Div} \nabla u\) on \(\tilde{M}\). Since we have a compact quotient, all the following limits exist as \(t \to \infty\) and are independent of \(x \in \tilde{M}\):

\[
\begin{align*}
\lambda_0 & = \inf_{f \in C_c^2(\tilde{M})} \frac{\int |\nabla f|^2}{\int |f|^2} = \lim_{t \to \infty} \frac{1}{t} \ln p(t, x, x) \\
\ell & = \lim_{t \to \infty} \frac{1}{t} \int d(x, y)p(t, x, y)d\text{Vol}(y) \\
h & = \lim_{t \to \infty} \frac{1}{t} \int p(t, x, y) \ln p(t, x, y)d\text{Vol}(y) \\
v & = \lim_{t \to \infty} \frac{1}{t} \ln \text{Vol}B_{\tilde{M}}(x, t),
\end{align*}
\]

where \(B_{\tilde{M}}(x, t)\) is the ball of radius \(t\) centered at \(x\) in \(\tilde{M}\) and \(\text{Vol}\) is the Riemannian volume on \(\tilde{M}\).

All these numbers are nonnegative. Recall \(\lambda_0\) is the Rayleigh quotient of \(\tilde{M}\), \(\ell\) the linear drift, \(h\) the stochastic entropy and \(v\) the volume entropy. There is the following relation:

\[
(1) \quad 4\lambda_0 \overset{(a)}{\leq} h \overset{(b)}{\leq} \ell v \overset{(c)}{\leq} v^2.
\]

See [L1] for (a), [Gu] for (b). Inequality (c) is shown in [L3] as a corollary of (b) and (2):

\[
(2) \quad \ell^2 \leq h
\]
If \((\tilde{M}, g)\) is a locally symmetric space of nonpositive curvature, all five numbers \(4\lambda_0, \ell^2, h, \ell v\) and \(v^2\) coincide and are positive unless \((\tilde{M}, g)\) is \((\mathbb{R}^n, \text{Eucl.})\). Our result is a partial converse:

**Theorem 1.1.** Assume \((M, g)\) has nonpositive curvature. With the above notation, any of the equalities

\[
\ell = v, \quad h = v^2 \quad \text{and} \quad 4\lambda_0 = v^2
\]

hold if, and only if, \((\tilde{M}, \tilde{g})\) is a symmetric space.

As recalled in [L3], Theorem 1.1 is known in negative curvature and follows from [K], [BFL], [FL], [BCG] and [L1]. The other possible converses are delicate: even for negatively curved manifolds, in dimension greater than two, it is not known that \(h = \ell v\) holds only for locally symmetric spaces. This is equivalent to a conjecture of Sullivan (see [L2] for a discussion). Sullivan conjecture holds for surfaces of negative curvature ([L1], [Ka]). It is not known either whether \(4\lambda_0 = h\) holds only for locally symmetric spaces. This would follow from the hypothetical \(4\lambda_0 \leq \ell^2\) by the arguments of this note.

We assume henceforth that \((M, g)\) has nonpositive sectional curvature. Given a geodesic \(\gamma\) in \(M\), Jacobi fields along \(\gamma\) are vector fields \(t \mapsto J(t) \in T_{\gamma(t)} M\) which describe infinitesimal variation of geodesics around \(\gamma\). By nonpositive curvature, the function \(t \mapsto \|J(t)\|\) is convex. Jacobi fields along \(\gamma\) form a vector space of dimension \(2 \dim M\). The rank of the geodesic \(\gamma\) is the dimension of the space of Jacobi fields such that \(t \mapsto \|J(t)\|\) is a constant function on \(\mathbb{R}\). The rank of a geodesic \(\gamma\) is at least one because of the trivial \(t \mapsto \dot{\gamma}(t)\) which describes the variation by sliding the geodesic along itself. The rank of the manifold \(M\) is the smallest rank of geodesics in \(M\). Using rank rigidity theorem ([B1], [BS]), we reduce in section 2 the proof of Theorem 1.1 to proving that if \((M, g)\) is rank one, equality in (2) implies that \((\tilde{M}, \tilde{g})\) is a symmetric space. For this, we show in section 3 that equality in (2) implies that \((M, g)\) is asymptotically harmonic (see the definition below). This uses the Dirichlet property at infinity (Ballmann [B2]). Finally, it was recently observed by A. Zimmer ([Z]) that asymptotically harmonic universal covers of rank one manifolds are indeed symmetric spaces.

2. Generalities and reduction of Theorem 1.1

We recall the notations and results from Ballmann’s monograph [B3] about the Hadamard manifold \((\tilde{M}, \tilde{g})\) that we use. The space \(\tilde{M}\) is homeomorphic to a ball. The covering group \(G := \pi_1(M)\) satisfies the duality condition ([B3] page 45).

2.1. **Boundary at infinity.** Two geodesic rays \(\gamma, \gamma'\) in \(\tilde{M}\) are said to be asymptotic if \(\sup_{t \geq 0} d(\gamma(t), \gamma'(t)) < \infty\). The set of classes of asymptotic unit speed geodesic rays is called the boundary at infinity \(\tilde{M}(\infty)\). \(\tilde{M} \cup \tilde{M}(\infty)\) is endowed with the topology of a compact space where \(\tilde{M}(\infty)\) is a sphere and where, for each unit speed geodesic ray \(\gamma, \gamma(t) \rightarrow [\gamma]\)
as $t \to \infty$. The action of the group $G$ on $\tilde{M}(\infty)$ is the continuous extension of its action on $\tilde{M}$. For any $x, \xi \in \tilde{M} \times \tilde{M}(\infty)$, there is a unique unit speed geodesic $\gamma_{x, \xi}$ such that $\tilde{\gamma}_{x, \xi}(0) = x$ and $[\gamma_{x, \xi}] = \xi$. The mapping $\xi \mapsto \tilde{\gamma}_{x, \xi}(0)$ is a homeomorphism $\pi_x^{-1}$ between $\tilde{M}(\infty)$ and the unit sphere $S_x \tilde{M}$ in the tangent space at $x$ to $\tilde{M}$. We will identify $SM$ with $\tilde{M} \times \tilde{M}(\infty)$ by $(x, v) \mapsto (x, \pi_x v)$. Then the quotient $SM$ is identified with the quotient of $\tilde{M} \times \tilde{M}(\infty)$ under the diagonal action of $G$.

Fix $x_0 \in \tilde{M}$ and $\xi \in \tilde{M}(\infty)$. The Busemann function $b_\xi$ is the function on $\tilde{M}$ given by:

$$b_\xi(x) = \lim_{y \to \xi} d(y, x) - d(y, x_0).$$

Clearly, $b_\xi(gx) = b_\xi(x) + b_\xi(gx_0)$. Moreover, the function $x \mapsto b_\xi(x)$ is of class $C^2$ ([HI]). It follows that the function $\Delta_x b_\xi$ satisfies $\Delta_x b_\xi = \Delta_x b_\xi$ and therefore defines a function $B$ on $G \setminus (\tilde{M} \times \tilde{M}(\infty)) = SM$. It follows from the argument of [HI] that the function $B$ is continuous on $SM$ (see [B3], Proposition 2.8, page 69).

### 2.2. Jacobi fields.

Let $(x, v)$ be a point in $T\tilde{M}$. Tangent vectors in $T_{x,v}T\tilde{M}$ correspond to variations of geodesics and can be represented by Jacobi fields along the unique geodesic $\gamma_{x, v}$ with initial value $\gamma(0) = x, \dot{\gamma}(0) = v$. A Jacobi field $J(t), t \in \mathbb{R}$ along $\gamma_{x, v}$ is uniquely determined by the values of $J(0)$ and $\dot{J}(0)$. We describe tangent vectors in $T_{x,v}T\tilde{M}$ by the associated pair $(J(0), \dot{J}(0))$ of vectors in $T_x \tilde{M}$. The metric on $T_{x,v}T\tilde{M}$ is given by $\|(J_0, J_0')\|^2 = \|J_0\|^2 + \|J_0'\|^2$. Assume $(x, v) \in SM$. A vertical vector in $T_{x,v}SM$ is a vector tangent to $S_x \tilde{M}$. It corresponds to a pair $(0, J(0'))$, with $J(0')$ orthogonal to $v$. Horizontal vectors correspond to pairs $(J(0), 0)$. In particular, let $X$ be the vector field on $SM$ such that the integral flow of $X$ is the geodesic flow. The geodesic spray $X_{x,v}$ is the horizontal vector associated to $(v, 0)$. The orthogonal space to $X$ is preserved by the differential $Dg_t$ of the geodesic flow. More generally, the Jacobi fields representation of $TT\tilde{M}$ satisfies $D_{x,v}g_t(J(0), J'(0)) = (J(t), J'(t))$.

For any vector $Y \in T_x \tilde{M}$, there is a unique vector $Z = S_{x,v}Y$ such that the Jacobi field $J$ with $J(0) = Y, J'(0) = Z$ satisfies $\|J(t)\| \leq C$ for $t \geq 0$ ([B3], Proposition 2.8 (i)). The mapping $S_{x,v} : T_x \tilde{M} \to T_x \tilde{M}$ is linear and selfadjoint. The vectors $(Y, SY)$ describe variations of asymptotic geodesics and the subspace $E^*_s \subset T_{x,v} \tilde{M}$ they generate corresponds to $TW^s_{x,v}$, where $W^s_{x,v}$, the set of initial vectors of geodesics asymptotic to $\gamma_{x,v}$, is identified with $\tilde{M} \times \pi_x(v)$ in $\tilde{M} \times \tilde{M}(\infty)$. Observe that $S_{x,v} \gamma_{x,v}(0) = 0$ and that the operator $S_{x,v}$ preserves $(\gamma_{x,v}(0))$. Recall from [B3], Proposition 3.2 page 71, that, for $Y \in (\gamma_{x,v}(0))$, with $\pi_x v = \xi$, 

$$D_Y (\nabla b_\xi) = -S_{x,v}Y,$$

and therefore $\Delta_x b_\xi = -\mathrm{Tr} S_{x,v}$ with $\pi_x(v) = \xi$.

Similarly, there is a selfadjoint linear operator $U_{x,v} : T_x \tilde{M} \to T_x \tilde{M}$ such that the Jacobi field $J$ with $J(0) = Y, J'(0) = UY$ satisfies $\|J(t)\| \leq C$ for $t \leq 0$. The subspace $E^*_v \subset$
they generate corresponds to $TW^u$, where $W^u_{x,v}$ is the set of opposite vectors to vectors in $W^s_{x,v}$. By definition, $S_{x,v}(0) = -U_{x,v}(0)$, so that we also have:

$$B(x,v) := -\text{Tr} \ S_{x,v} = \text{Tr} \ U_{x,v}.$$  

We have $\text{Ker} \ S = \text{Ker} \ U$ and $Y \in \text{Ker} \ S$ if, and only if, the Jacobi field $J(t)$ with $J(0) = Y, J'(0) = 0$ is bounded for all $t \in \mathbb{R}$. The rank of the geodesic $\gamma_{x,v}$ therefore is $\kappa = \text{Dim} \ \text{Ker} \ S$ and the geodesic $\gamma_{x,v}$ is of rank one only if $\text{Det}((U - S)|_{(\gamma_{x,v}(0))_{\perp}}) = 0$.

Recall that $SM$ is identified with the quotient of $\tilde{M} \times \tilde{M}(\infty)$ under the diagonal action of $G$. Clearly, for $g \in G$, $g(W^s_{x,v}) = W^s_{Dg(x,v)}$ so that the $W^s$ define a foliation $W^s$ on $SM$. The leaves of the foliation $W^s$ are quotient of $\tilde{M}$, they are naturally endowed with the Riemannian metric induced from $\tilde{g}$.

2.3. **Proof of Theorem 1.1.** We continue assuming that $(\tilde{M}, \tilde{g})$ has nonpositive curvature. By the Rank Rigidity Theorem (see [B3]), $(\tilde{M}, \tilde{g})$ is of the form

$$(\tilde{M}_0 \times \tilde{M}_1 \times \cdots \times \tilde{M}_j \times \tilde{M}_{j+1} \times \cdots \times \tilde{M}_k, \tilde{g})^1,$$

where $\tilde{g}$ is the product metric $\tilde{g}^2 = (\tilde{g}_0)^2 + (\tilde{g}_1)^2 + \cdots + (\tilde{g}_j)^2 + (\tilde{g}_{j+1})^2 + \cdots + (\tilde{g}_k)^2$, $(\tilde{M}_0, \tilde{g}_0)$ is Euclidean, $(\tilde{M}_i, \tilde{g}_i)$ is an irreducible symmetric space of rank at least two for $i = 1, \cdots, j$ and a rank-one manifold for $i = j + 1, \cdots, k$. If the $(\tilde{M}_i, \tilde{g}_i), i = j + 1, \cdots, k$, are all symmetric spaces of rank one, then $(\tilde{M}, \tilde{g})$ is a symmetric space. Moreover in that case, all inequalities in (1) are equalities: this is the case for irreducible symmetric spaces (all numbers are 0 for Euclidean space; for the other spaces, $4\lambda_0$ and $v^2$ are classically known to coincide ([O]) and we have:

$$4\lambda_0(\tilde{M}) = \sum_i 4\lambda_0(\tilde{M}_i), \quad v^2(\tilde{M}) = \sum_i v^2(\tilde{M}_i).$$

To prove Theorem 1.1, it suffices to prove that if $\ell^2 = h$, all $\tilde{M}_i$ in the decomposition are symmetric spaces. This is already true for $i = 0, 1, \cdots, j$. It remains to show that $(\tilde{M}_i, \tilde{g}_i)$ are symmetric spaces for $i = j + 1, \cdots, k$. Eberlein showed that each one of the spaces $(\tilde{M}_i, \tilde{g}_i)$ admits a cocompact discrete group of isometries (see [Kn], Theorem 3.3). This shows that the linear drifts $\ell_i$ and the stochastic entropies $h_i$ exist for each one of the spaces $(\tilde{M}_i, \tilde{g}_i)$. Moreover, we clearly have

$$\ell^2 = \sum_i \ell_i^2, \quad h = \sum_i h_i.$$  

Therefore Theorem 1.1 follows from

**Theorem 2.1.** Assume $(M, g)$ is a closed connected rank one manifold of nonpositive curvature and that $\ell^2 = h$. Then $(\tilde{M}, \tilde{g})$ is a symmetric space.\footnote{With a clear convention for the cases when Dim $\tilde{M}_0 = 0, j = 0$ or $k = j$.}
A Hadamard manifold \( \tilde{M} \) is called asymptotically harmonic if the function \( B(=\Delta x b) \) is constant on \( SM \). Theorem 2.1 directly follows from two propositions:

**Proposition 2.2.** Assume \((M, g)\) is a closed connected rank one manifold of nonpositive curvature and that \( \ell^2 = h \). Then \((\tilde{M}, \tilde{g})\) is asymptotically harmonic.

**Proposition 2.3.** [Z], Theorem 1.1] Assume \((M, g)\) is a closed connected rank one manifold of nonpositive curvature such that \((\tilde{M}, \tilde{g})\) is asymptotically harmonic. Then, \((\tilde{M}, \tilde{g})\) is a symmetric space.

### 3. Proof of Proposition 2.2

We consider the foliation \( W \) of subsection 2.2. Recall that the leaves are endowed with a natural Riemannian metric. We write \( \Delta^W \) for the associated Laplace operator on functions which are of class \( C^2 \) along the leaves of \( W \). A probability measure \( m \) on \( SM \) is called harmonic if it satisfies, for any \( C^2 \) function \( f \), we have:

\[
\int_{SM} \Delta^W f dm = 0.
\]

Let \( M \) be a closed connected manifold such that \( \ell^2 = h \). In [L3] it is shown that then, there exists a harmonic probability measure \( m \) on \( SM \) such that, at \( m \)-a.e. \((x, v), B(x, v) = \ell \). Since \( B \) is a continuous function, Proposition 2.2 follows from

**Theorem 3.1.** Let \((M, g)\) be a closed connected rank one manifold of nonpositive curvature, \( W \) the stable foliation on \( SM \) endowed with the natural metric as above. Then, there is only one harmonic probability measure \( m \) and the support of \( m \) is the whole space \( SM \).

**Proof.** Let \( m \) be a \( W \) harmonic probability measure on \( SM \). Then, there is a unique \( G \)-invariant measure \( \tilde{m} \) on \( SM \) which coincide with \( m \) locally. Seen as a measure on \( \tilde{M} \times \tilde{M}(\infty) \), we claim that \( \tilde{m} \) is given, for any \( f \) continuous with compact support, by:

\[
\int f(x, \xi) d\tilde{m}(x, \xi) = \frac{1}{\text{Vol} M} \int_{\tilde{M}} \left( \int_{\tilde{M}(\infty)} f(x, \xi) d\nu_x(\xi) \right) dx,
\]

where the family \( x \mapsto \nu_x \) is a family of probability measures on \( \tilde{M}(\infty) \) such that, for all \( \varphi \) continuous on \( \tilde{M}(\infty) \), \( x \mapsto \int \varphi(\xi) d\nu_x(\xi) \) is a harmonic function on \( \tilde{M} \) and the measure \( dx \) is the Riemannian volume on \( \tilde{M} \). The claim follows from [Ga]. For convenience, let us reprove it: on the one hand, the measure \( \tilde{m} \) projects on \( \tilde{M} \) as a \( G \)-invariant measure satisfying \( \int \Delta f dm = 0 \). The projection of \( \tilde{m} \) on \( \tilde{M} \) is proportional to Volume, gives measure 1 to fundamental domains and formula (3) is the desintegration formula. On the other hand, if one projects \( \tilde{m} \) first on \( \tilde{M}(\infty) \), there is a probability measure \( \nu \) on \( \tilde{M}(\infty) \) such that

\[
\int f(x, \xi) d\tilde{m}(x, \xi) = \int_{\tilde{M}(\infty)} \left( \int_{\tilde{M}} f(x, \xi) dm_x(dx) \right) d\nu(\xi).
\]
For $\nu$-a.e. $\xi$, the measure $m_\xi$ is a harmonic measure on $\tilde{M}$; therefore, for $\nu$-a.e. $\xi$, there is a positive harmonic function $k_\xi(x)$ such that $m_\xi = k_\xi(x)\text{Vol}$. Comparing the two expressions for $\int f d\tilde{m}$, we see that the measure $\nu_x$ is given by

$$\nu_x = k_\xi(x)\nu$$

and $x \mapsto \int_{\tilde{M}(\infty)} \varphi(\xi) d\nu_x(\xi)$ is indeed a harmonic function.

The $G$-invariance of $\tilde{m}$ implies that, for all $g \in G$, $g_*\nu_x = \nu_{gx}$. In particular, the support of $\nu$ is $G$-invariant. By [E] (see [B3], page 48), the support of $\nu$ is the whole $\tilde{M}(\infty)$ and therefore the support of $m$ is the whole $SM$. This result would be sufficient for proving Proposition 2.2, but using discretization, we are going to identify the measure $\nu_x$ on $\tilde{M}(\infty)$ as the hitting measure of the Brownian motion on $\tilde{M}$ starting from $x$. This shows Theorem 3.1.

Fix $x_0 \in \tilde{M}$. The discretization procedure of Lyons and Sullivan ([LS]) associates to the Brownian motion on $\tilde{M}$ a probability measure $\mu$ on $G$ such that $\mu(g) > 0$ for all $g$ and that any bounded harmonic function $F$ on $\tilde{M}$ satisfies

$$F(x_0) = \sum_{g \in G} F(gx_0) \mu(g).$$

Recall that for all $\varphi$ continuous on $\tilde{M}(\infty)$, $x \mapsto \nu_x(\varphi)$ is a harmonic function and that $\nu_{gx} = g_*\nu_x$. It follows that the measure $\nu_{x_0}$ is stationary for $\mu$, i.e. it satisfies:

$$\nu_{x_0} = \sum_{g \in G} g_*\nu_{x_0} \mu(g).$$

Since the support of $\mu$ generates $G$ as a semigroup (actually, it is already the whole $G$), there is only one stationary probability measure on $\tilde{M}(\infty)$ (see [B3], Theorem 4.11 page 58). We know one already: the hitting measure $m_{x_0}$ of the Brownian motion on $\tilde{M}$ starting from $x_0$. This shows that $\nu_{x_0} = m_{x_0}$. Since $x_0$ was arbitrary in the above reasoning, we have $\nu_x = m_x$ for all $x \in \tilde{M}$ and the measure $\tilde{m}$ is given by:

$$\int f(x, \xi) d\tilde{m}(x, \xi) = \frac{1}{\text{Vol}M} \int_{\tilde{M}} \left( \int_{\tilde{M}(\infty)} f(x, \xi) d\nu_x(\xi) \right) dx.$$

\[\square\]

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