About Gravitomagnetism

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Abstract

The gravitomagnetic field is the force exerted by a moving body on
the basis of the intriguing interplay between geometry and dynamics
which is the analog to the magnetic field of a moving charged body in
electromagnetism. The existence of such a field has been demonstrated
based on special relativity approach and also by special relativity plus
the gravitational time dilation for two different cases, a moving infinite
line and a uniformly moving point mass, respectively. We treat these
two approaches when the applied cases are switched while appropriate
key points are employed. Thus, we demonstrate that the strength of
the resulted gravitomagnetic field in the latter approach is twice the
former. Then, we also discuss the full linearized general relativity and
show that it should give the same strength for gravitomagnetic field as
the latter approach. Hence, through an exact analogy with the electro-
dynamic equations, we present an argument in order to indicate the
best definition amongst those considered in this issue in the literature.
Finally, we investigate the gravitomagnetic effects and consequences
of different definitions on the geodesic equation including the second
order approximation terms.

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1 Introduction

The analogous idea of the electric theory and the Newtonian gravita-
tional theory inspiring a Maxwell–type gravitational theory is dated back
to the second half of the nineteenth century [1–4]. This idea which was
also explored by Einstein [5] was revived and extended by Sciama [6–7].
Though, his theory was unfruitful because, unlike the electric charge, the mass charges are not invariant nor additive [8]. The properties which are actually due to the linearity of Maxwell–type theories, as, the linearized weak field approximation of the Einstein gravitational theory presented to second order approximation does [9]–[12]. This issue has been demonstrated in several books as well, see, e.g., Refs. [7] [8] [13] and references therein.

Indeed, the introduction of a gravitomagnetic (GM) field is unavoidable when one brings the Newtonian gravitational theory and the Lorentz invariance together in a consistent framework. This effect is the usual Maxwellian feature which has Machian root, see, e.g., Ref. [14]. Dynamical equations for a weak gravitational field similar to the Maxwell equations has been deduced [15] by the parameterized post–Newtonian formalism. Also, more attention has been made [16] in the analogy between general relativity and electromagnetism for slowly motion in a weak gravitational field.

However, there are other dissimilarities. For example, not only negative mass charge has not been detected yet, the like mass charges attract rather than repelling each others. Issues which, partially, have been stated as the weak version of the principle of equivalence, that is, the gravitational field couples to everything, and or, all forms of energy acts as sources of gravitational field.

Nevertheless, in the last two decades, extensive attention, has been taken on this issue, see, e.g., Refs. [17]–[23] and references therein. Especially, and contrary to assertion made in Ref. [24], the existence of GM interaction has been claimed in 1988 [25, 26], and evidence for the GM field has been suggested, see Refs. [13, 17, 27, 28, 29] and references therein.

The introduction of GM field as an analogy to the magnetic field comes about from the need to know the force exerted by a moving body on the basis of the intriguing interplay between geometry and dynamics, as emphasized by Sciama [7]. A fundamental idea that perhaps motivated the two, almost recent, articles, Refs. [30, 31], to present the existence of gravitomagnetism. Indeed, in Ref. [30], based on special relativity (SR), the existence of GM field has been shown for a moving infinite line\(^2\) (MIL) of constant mass charge density, analogous to the magnetic field from a straight current. Whereas, in Ref. [31], again based on SR plus the aid of the gravitational time dilation, the existence of GM field has been demonstrated for a uniformly moving point mass (MPM) by considering its line element and comparing the resulted Lagrangian with the corresponding non–relativistic electromagnetic (EM) case.

The purpose of this article is to study the correspondence of the two approaches employed in Refs. [30, 31] and compare them with the results of the full linearized general relativity (LGR). Hence, in the next section, we briefly iterate these two approaches in a more elegant and consistent manner.

\(^1\)In practice a very long moving line.
while we switch their applied cases and employ appropriate key points in each condition. Implicitly, we refer to these two approaches as SR and semi SR (SSR) approaches, respectively. In Section 3, we apply the LGR to general cases, then we compare this approach with the previous results and give a short discussion and suggestion in order to indicate the best definition amongst those used for this issue in the literature. Then, in Section 4, we investigate the GM effects and consequences of different definitions on the geodesic equation including the second–order approximation terms. Finally, a brief conclusion is given in the last section.

The necessary calculations have also been furnished in the Appendix at the end of the article. We use an isotropic space–time of signature $-2$ and set $c = 1$. Also, we employ the convention that lower case Latin indices run from zero to three, whereas the lower case Greek indices run from one to three.

2 Switching Line and Point Mass Cases

In order to study the correspondence of the two approaches employed in Refs. [30, 31], we iterate them consistently while we switch the applied cases and use an appropriate key point in each condition.

Firstly, following the approach of Ref. [30], we consider a MPM with rest mass $M_o$ and constant linear velocity $v$ with respect to a frame of reference, say $S$, for simplicity along the positive $x$–direction. Also, suppose a point test mass\footnote{A test mass is obviously a mass which experiences a gravitational field but does not itself alter the field or contribute to the field.} with rest mass $m_o$ moving under the influence of $M_o$, which, without loss of generality, we assume has the instantaneous 3–velocity $u^\prime = (0, u^\prime_y, 0)$, where $u^\prime_y \ll 1$, in the rest frame of $M_o$, $S'$, in $x'y'$ plane. That is, we consider the orthogonal relative motion of two bodies.

In this rest frame, the 3–force on $m_o$ is $f^\prime = (0, f^\prime_y, 0)$, where $|f^\prime_y| = G M_o m / r'^2$ and $r'$ is the instantaneous proper distance from $m_o$ to $M_o$. Using the standard Lorentz transformation for 4–force, $F^a = \gamma(u \cdot f; f)$, and $m \simeq m_o$ to the first–order approximation, one gets the 3–force on $m_o$ in the frame $S$ to be

$$f = \left( v u^\prime_y f^\prime_y, \frac{f^\prime_y}{\gamma}, 0 \right) = \pm \left( v \gamma u_y \frac{G M_o m_o}{r'^2}, \frac{G M_o m_o}{\gamma r'^2}, 0 \right),$$

(1)

where obviously $\gamma \equiv \gamma_v = 1 / \sqrt{1 - v^2}$, $r' = r$, for a perpendicular direction of motion, and the negative/positive sign is for when $m_o$ is above/under $M_o$, which is assumed to be on the $x'$–axis. Also, note that, obviously $u_x = v$ in this situation.

Now, as mentioned in the introduction, magnetic type forces can be interpreted as relativistic forces. That is, if all one knows in the EM is the
Coulomb law, then, by using the SR and the invariance of charge, one can demonstrate that a magnetic field, which exerts the Lorentz force on a test charge, must exist \[32\]. Indeed, the force \( f_x \), which is needed to keep the velocity of \( m_o \) in this direction as a constant value, is actually the gravitational analog of the magnetic force, that is \( f_x^{(gm)} \). In Ref. \[30\], where the MIL case has been applied, the length contraction has been employed in order to proceed. In our situation, an appropriate key point is the concentration of the gravitational field lines in the transverse direction, see, e.g., Ref. \[32\]. In order to find the gravitational analog of the electric force on \( m_o \) in \( S \), we use the relation derived in various textbooks, e.g. Ref. \[32\], for the electric field of a moving charge, therefore:

\[
f^{(ge)} = m_o E^{(ge)} = \left( 0, \mp \gamma \frac{GM_o m_o}{r^2}, 0 \right).
\]  

Hence,

\[
f_y^{(gm)} = \left[ f_y - f_y^{(ge)} \right] = \pm v^2 \gamma \frac{GM_o m_o}{r^2} = \pm vu_x \gamma \frac{GM_o m_o}{r^2},
\]

and therefore

\[
f^{(gm)} = v \gamma \frac{GM_o m_o}{r^2} (\mp u_y, \pm u_x, 0).
\]  

This could have been resulted from a GM field as \( f^{(gm)} = m_o u \times B^{(gm)} \), where

\[
B^{(gm)} = \mp v \gamma \frac{GM_o}{r^2} \hat{k}
\]

is the GM field of the MPM case, or equivalently, relating the GM field to the relative velocity and the GE field by

\[
B^{(gm)} = v \times E^{(ge)},
\]

as expected for the electric analog, see, e.g., Ref. \[32\]. The same result has been obtained for the MIL case used in Ref. \[30\].

In the above considerations, in order to be consistent with the approach of Ref. \[31\], one should assume that the speed \( v \) to be much smaller than the speed of light. This assumption just emphasizes that the corresponding results are unobservable. Incidentally, considering the test mass \( m_o \) as a positive mass charge, the direction of \( B^{(gm)} \) found are as if the moving mass charges are negative in analogy with the electric charge case.

Secondly, we follow the approach of Ref. \[31\] and consider a point test mass \( m_o \) moving perpendicularly toward a MIL of constant rest mass charge density \( \lambda_o \), under the influence of its gravitational field in its rest frame, \( S' \). The MIL is assumed to move along its line of direction, as the \( x' \)-axis, with a

\footnote{The gravitoelectric (GE) notation \( E^{(ge)} \), as the gravity analog of the electric field, has been used instead of the usual \( g \) field.}
constant speed \( v \ll 1 \) with respect to a frame of reference, say \( S \), along the positive \( x \)-direction. Again, the orthogonal relative motion of two bodies is considered.

The motion of \( m_o \) is determined by \( \delta \int (-m_o)ds = 0 \). As in Ref. [31], one can use the fact that in a weak gravitational potential, \( \phi \), the line element, in the SSR approach, i.e. SR plus the aid of the gravitational time dilation, is given by \( ds^2 = (1 + 2\phi') dt'^2 - (dx'^2 + dy'^2 + dz'^2) \). Using the standard Lorentz transformation for very small \( v \), one obtains

\[
-m_o ds \simeq -m_o \sqrt{(1 + 2\phi) dt^2 - 4\phi v dt dx - (dx^2 + dy^2 + dz^2)},
\]

where \( \phi \simeq \phi' \) to the first–order approximation, that is \( m_o \) would follow the same geodesic in either frame, \( S \) or \( S' \). As \( L dt = -m_o ds \), one gets

\[
L \simeq -m_o \sqrt{1 + 2\phi - 4\phi vu_x - u^2}.
\]

Concentrating on the non–relativistic case where \( u \ll 1 \) and \( \phi \sim u^2 \), Eq. (8), neglecting a constant term \( -m_o \), reads

\[
L \simeq \frac{1}{2} m_o u^2 - m_o \phi + 2m_o \phi vu_x.
\]

Comparing this result with the EM analog, one can deduce

\[
B^{(gm)} = \nabla \times (2\phi \mathbf{v}) = 2 \mathbf{v} \times (-\nabla \phi) = 2 \mathbf{v} \times \mathbf{E}^{(ge)}.
\]

In Ref. [31], the case MPM has been applied, where the used weak field potential is obviously asymptotically free, as required. In our applied case, an appropriate key point is to maintain the asymptotically free assumption radially for the gravitational field of a long line. This task has been accomplished in the Appendix. Therefore, using weak field potential for the MIL case, derived in the Appendix, Eq. (37), one gets

\[
B^{(gm)} = -2\nu \gamma \frac{2G\lambda_o}{y} \mathbf{k}.
\]

That is, despite a factor of two, Eqs. (10) and (11) are equivalent with Eqs. (6) and (5), respectively.

In conclusion, comparing Eqs. (6) and (10) with the corresponding results of Refs. [31] and [30], respectively, for the same applied cases, it shows that the strength of the GM field obtained in the SSR approach is twice the SR approach, as we expected. However, one may still expect to get the correct qualitative behavior and numerical factor when the full LGR is applied. We examine this approach in the next section.

\[^4\text{Here, the vector potential does not explicitly depend on time.}\]
3 Linearized General Relativity and Discussions

In this section, we consider the full LGR of the weak field limit in the following two parts.

Firstly, by using the length contraction as well as the gravitational time dilation, we employ the metric \(ds^2 = (1 + 2\phi')dt'^2 - (1 - 2\phi')(dx'^2 + dy'^2 + dz'^2)\) in the rest frame. The static nature of this metric, same as the Schwarzschild metric, implies that it can only produce GE fields. However, the existence of cross terms with \(dt\) can show that a stationary space also has a GM field, as in the case of NUT metric \([34]\), as the generalized Schwarzschild metric, it has been shown \([35]\) with the existence of the cross term \(d\phi dt\). Therefore, in order to extract the GM field from the Lagrangian of the system for small quantities analogous to the EM case, similar to the SSR approach, for general cases, including both examples of the MIL and the MPM, the metric, where the source is moving with a constant speed \(v \ll 1\) along the positive \(x\)-direction, can be written as

\[
ds^2 \simeq (1 + 2\phi)dt^2 - (1 - 2\phi)(dx^2 + dy^2 + dz^2) - 8\phi v dx dt. \tag{12}\]

With a similar calculation, one gets

\[
\mathcal{L} \simeq \left(\frac{1}{2} - \phi\right)m_o u^2 - m_o \phi + 4m_o \phi v u_x. \tag{13}\]

Ignoring the second term, \(-m_o \phi u^2\), which is a fourth–order term in \(u\), one obtains

\[
B^{(gm)} = 4 v \times E^{(ge)}, \tag{14}\]

see, e.g., Refs. \([8, 16]\).

Interpreting this result in comparison with the light deflection by a mass where the SR explains only one–half of the deflection, it has been concluded \([31]\) that the effect of space curvature to the GM field is equal to the effect of the gravitational time dilation. On the other hand, the analogy of the Lagrangian \([13]\) with the corresponding classical EM case, which is not already affected by space curvature, is questionable and doubtful. Indeed, instead of the classical EM case, one should apply the Einstein–Maxwell equations which, since the EM energy–momentum tensor, as it is well known, is trace–free, does exert a sort of constraint on geometry/space curvature \([37]\). The above idea has almost been studied \([16]\) by taking into account the effect of the EM field tensor on the geodesic of a particle of mass \(m_o\) and charge \(e\) up to the first–order in \(v/c\). However, the EM effect has not practically been considered into the resulting deduction, instead the analogy with the electromagnetism has just been used.

\[5\] See, for example, Ref. \([33]\).

\[6\] For a discussion on NUT space and GM monopoles see, e.g., Ref. \([30]\).
Implicitly, the existence of the term $-m_0 \phi u^2$ in Eq. (13) fades a complete analogy between the Lagrangian (13) and the EM case. Besides, it has been claimed [38] that Eq. (14) fails to correctly describe the free–fall problem in general relativity, for it apparently does not produce all terms of $v^2/c^2$ order. However, in this work, we have taken all of these terms into account and we speculate that the correct result out of the full linearized equation should be as Eq. (10). Though, the equality of the effect of space curvature and the gravitational time dilation looks to be legitimate.

In the following method, we emphasize on this noticeable point in order to indicate the best definition amongst those considered in the literature. Incidentally, there has recently been some debates in the literature about which one of these definitions leads to physically correct predictions. Indeed, it has been claimed [39] that Eq. (14) has been verified as a source of perturbing acceleration of the lunar orbit. But, it has been pointed out [40] that this effect depends merely on a coordinate frame. Though, it has also been asserted [41] that these type of effects are unobservable, hence the confirmation of Eq. (14) is unwarranted [38].

Secondly, we apply the weak field approximation, $g_{ab} = \eta_{ab} + h_{ab}$ with $|h_{ab}| \ll 1$, and the definition $\bar{h}_{ab} = h_{ab} - \frac{1}{2} \eta_{ab} h$ with the gauge condition $\bar{h}_{ab,b} = 0$, as derived in several textbooks, in order to get the full LGR as

$$\square \bar{h}_{ab} = 16\pi GT_{ab},$$

(15)

where $\square \equiv \eta^{ab} \partial_a \partial_b$. This equation is the analogous of the EM potential equation, that is $\square A^a = 4\pi j^a$.

We assume $|\bar{h}_{00}| \gg |\bar{h}_{\alpha\beta}|$ and $|\bar{h}_{0\alpha}| \gg |\bar{h}_{\alpha\beta}|$, which have been extensively used in the literature, hence neglecting $\bar{h}_{\alpha\beta}$, and defining

$$\bar{h}_{00} := k_1 \phi,$$
$$\bar{h}_{0\alpha} := k_2 A_\alpha,$$

(16)

where $k_1$ and $k_2$ are constants, therefore the gauge condition reads

$$k_1 \frac{\partial \phi}{\partial t} + k_2 \partial_\alpha A^\alpha = 0.$$  

(17)

Also, if the gravitoelectromagnetic fields are defined as

$$E^{(ge)} = -\nabla \phi - \frac{k_2}{k_1} \frac{\partial A}{\partial t} \quad \text{and} \quad B^{(gm)} = \nabla \times A,$$

(18)

the gravitoelectromagnetic equations will be

$$\nabla \cdot E^{(ge)} = \frac{4}{k_1} (4\pi G \rho),$$  

$$\nabla \cdot \left( \frac{k_2}{k_1} B^{(gm)} \right) = 0.$$  

7
\[ \nabla \times \mathbf{E}^{(ge)} = -\frac{\partial}{\partial t} \left( \frac{k_2}{k_1} \mathbf{B}^{(gm)} \right), \]
\[ \nabla \times \left( \frac{k_2}{k_1} \mathbf{B}^{(gm)} \right) = \frac{\partial \mathbf{E}^{(ge)}}{\partial t} + \frac{4}{k_1} (4\pi G j). \] (19)

An exact analogy with the corresponding EM case, that is the Lorentz gauge and the Maxwell equations, leads to \(k_1 = k_2 = 4\). This choice is self-consistent, as it regards the special solution of Eq. (15), that is \(\bar{h}_{ab} = 4G \int T_{ab} \frac{(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'\), (20)

where \(T^{00} = \rho\) and \(T^{0\alpha} = j^\alpha\), namely the matter and current density, as expected.

The weak field approximation, as Eq. (12), but in a general case, yields \(\bar{h}_{0\alpha} = h_{0\alpha} = -4\phi v^\alpha = 4\phi v^\alpha\), hence, one obtains
\[ \mathbf{A} = \phi \mathbf{v}. \] (21)

However, as mentioned before, in order to get
\[ \mathbf{B}^{(gm)} = 2 \mathbf{v} \times \mathbf{E}^{(ge)}, \] (22)

one must replace \(\phi\), as the effective gravitopotential, by \(2\phi\) arising from the EM analogy. This exertion, as discussed before, can again be better justified by the analogy that has been taken between the full linearized equation, which includes the space curvature, with the classical EM situation. That is, one should account for the lack of the effect of space curvature in the EM case \[37\]. Similar replacement has also been performed in the literature, see Refs. \[17, 23, 27, 42\], based on the fact that the linear approximation of general relativity involves a spin–2 field whereas the electrodynamics involves a spin–1 field. Thus, they have taken the GM charge twice that of the GE charge.

Implicitly, Eq. (12) gives \(\bar{h}_{00} = 4\phi\) and \(\bar{h}_{\alpha\beta} = 0\), as we have assumed. Also, the gauge condition (17), in the case of Eq. (21) and when \(\phi\) does not explicitly depend on time, leads to \(\mathbf{v} \cdot \mathbf{E}^{(ge)} = 0\), which is true in the orthogonal relative motion cases that we have employed in this work.

On the other hand, if one sets \(k_1 = 4\) and \(k_2 = 2\), as used in the literature \[17, 23, 42, 43\], the same result of Eq. (22) will be obtained\[7\]. However, with this choice, the compensation is that the appearances of the analog EM equations are altered by a factor of one–half. Though, these factors have also been justified by the interpretation of the effective GM charge, however, the point we have raised after Eq. (14) will not be clarified in this choice.

\[ ^7\text{Note that, the appeared sign differences are due to sign conventions.} \]
The choice $k_1 = 4$ and $k_2 = 1$, actually used in Refs. [8, 16], gives Eq. (14). Despite the interpretation made by Ref. [31], we doubt this choice to be a correct analog, as mentioned before. Besides, one can also rewrite the work of Ref. [16] similar to our work with constants $k_1$ and $k_2$, and, e.g., concludes $k_1 = k_2 = 4$.

These different values of $k_1$ and $k_2$ are the best and mostly used ones in the literature. It should be interesting to see whether these different values give any new information on the gravitational interactions. In other words, it is instructive if one makes clear what happens to the equation of motion of a test particle in the different choices one takes into account. For this purpose, in the next section, we investigate any effects on the geodesic equation due to the interplay between the constants $k_1$ and $k_2$ for the GM field.

### 4 Geodesic Equation Including Second–Order Approximation

In order to examine the GM effects and consequences of these different definitions on the geodesic equation, we consider the conditions and the metric (12) with the definitions (16) and Eq. (21), for simplicity, in the $x$–direction. Hence, the weak field potential is

\[
(h_{ab}) = \phi \begin{pmatrix}
\frac{1}{2}k_1 & -k_2 v & 0 & 0 \\
-k_2 v & \frac{1}{2}k_1 & 0 & 0 \\
0 & 0 & \frac{1}{2}k_1 & 0 \\
0 & 0 & 0 & \frac{1}{2}k_1
\end{pmatrix}.
\]

As we know, the effects of the GM field on the geodesic equation can be shown when one maintains the terms of order $\phi^2$, hence, we will write the relations up to the third–order approximation. That is, corresponding to the linear approximation $g_{ab} = \eta_{ab} + h_{ab}$, one gets

\[
g^{ab} = \eta^{ab} - \eta^{ac} \eta^{bd} h_{cd} + \eta^{ac} \eta^{bd} \eta^{ef} h_{ce} h_{df} + \mathcal{O}(\varepsilon^3),
\]

where $\varepsilon \sim \phi \sim v^2$ and, in the rest of this section, the indices of $h_{ab}$ are not raised by the Minkowski metric, $\eta^{ab}$, as they usually do up to the first order approximation.

As before, for a test particle with $u_x' = 0 = u_z'$ and $u_y'^2 \sim \varepsilon$, which gives $u_x = v$, $u_y \simeq u_y'$ and $u_z = 0$, the geodesic equation, $\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$ where $\tau$ is the proper time, in the $y$–direction, yields

\[
\frac{d^2 y}{d\tau^2} = -\left[ \Gamma^2_{00} + 2 \left( v^2 \Gamma^2_{01} + u_y \Gamma^2_{02} \right) \right. \\
\left. + \left( v^2 \Gamma^2_{11} + 2vu_y \Gamma^2_{12} + u_y^2 \Gamma^2_{22} \right) \right] + \mathcal{O}(\varepsilon^3),
\]
where we have neglected the term containing $du_y'/dt'$, as a negligible acceleration using the slow–motion approximation assumption.

Using Eqs. (23) and (24), we get the required Christoffel symbols, up to the necessary orders, as

\[
\Gamma^2_{00} = \frac{k_1}{4} \frac{\partial \phi}{\partial y} + \frac{k_2^2}{8} \frac{\partial \phi}{\partial y} + \mathcal{O}(\varepsilon^3), \\
\Gamma^2_{01} = -\frac{k_2}{2} \frac{v}{\partial y} - \frac{k_1 k_2}{4} v \frac{\partial \phi}{\partial y} + \mathcal{O}(\varepsilon^3), \\
\Gamma^2_{11} = \frac{k_1}{4} \frac{\partial \phi}{\partial y} + \mathcal{O}(\varepsilon^2), \\
\Gamma^2_{12} = -\frac{k_1}{4} \frac{\partial \phi}{\partial x} + \mathcal{O}(\varepsilon^2), \\
\Gamma^2_{22} = -\frac{k_1}{4} \frac{\partial \phi}{\partial y} + \mathcal{O}(\varepsilon^2) 
\]  

(26)

and null result for $\Gamma^2_{02}$ up to the $\mathcal{O}(\varepsilon^3)$. Hence, Eq. (25) reads

\[
\frac{d^2 y}{dt^2} = \left( -\frac{k_1}{4} \frac{\partial \phi}{\partial y} \right) + \left\{ -\frac{k_1^2}{8} \frac{\partial \phi}{\partial y} + \left( k_2 - \frac{k_1}{4} \right) v^2 \right. \\
\left. + \frac{k_1}{4} u_y^2 \frac{\partial \phi}{\partial y} + \frac{k_1}{2} v u_y \frac{\partial \phi}{\partial x} \right\} + \mathcal{O}(\varepsilon^3). 
\]  

(27)

With the choice of $k_1 = 4$, the first part on the right hand side, that is the first order approximation, just reveals the Newtonian theory, thus, Eq. (27), with $v = u_x$, reads

\[
\frac{d^2 y}{dt^2} = \left( \frac{\partial \phi}{\partial y} \right) + \left\{ -2 \frac{\partial \phi}{\partial y} + \left( k_2 - 1 \right) u_x^2 + u_y^2 \right\} \frac{\partial \phi}{\partial y} + 2 u_x u_y \frac{\partial \phi}{\partial x} \right\} + \mathcal{O}(\varepsilon^3), 
\]  

(28)

where the second part gives the terms of second–order approximation and its first term can be justified as the second–order correction to the Newtonian theory. Its second and third terms are the GM effect on the same and the transverse directions, though, in our situation $\phi$ does not depend on $x$.

It also shows that the resulting GM acceleration of a test particle, i.e. the second term, is proportional to the square of speed, as has been discussed in Ref. [38]. The proportional coefficients of this term for different choices of $k_2$ discussed in the previous section, namely 4, 2 and 1, are $3 u_x^2 + u_y^2$, $u_x^2 + u_y^2 = u^2$ and $u_y^2$, respectively. It will be up to, and may lead easier to, experimental detections to reveal which one of these values, and hence definitions, may lead to physically correct predictions. However, the existence of these coefficients in geodesic equations may also be employed in tuning the numerical results of experiments.
5 Conclusion

We have investigated apparent GM fields, which can be removed with suitable choice of coordinates, or which can be arisen on the basis of the intriguing interplay between geometry and dynamics. We have demonstrated that the strength of the GM field obtained in the SSR approach is twice the SR approach, and we have argued that the results of LGR equations must be the same as SSR approach. Our argument is mainly based on the fact that the corresponding EM case is not affected by space curvature, which is shown to have the same effect on the GM field as the gravitational time dilation. By this, we have justified that one must replace the effective gravitopotential, $\phi$, by $2\phi$, and thus achieve an exact analogy with the corresponding EM equations for the GM fields. Hence, the best definition for the gravitoelectromagnetic equations with coefficients $k_1$ and $k_2$ are obtained when the values for both of them are set equal to four. Also, our stimulated hope in deriving the geodesic equation including second-order approximation is that more theoretical understanding of this procedure may lead to a practical detection of this phenomenon.

Appendix

In this Appendix, we derive an asymptotically flat metric, in the radial direction, for the MIL case, although, a sort of general treatments can be found in Ref. [44].

A most general cylindrically symmetric static metric in four dimensions with signature $-2$ can be written in the canonical form, in a given frame as

$$ds^2 = e^{2F(\rho)}dt^2 - e^{2H(\rho)}d\rho^2 - \rho^2 d\phi^2 - e^{2U(\rho)}dz^2,$$

(29)

where the existence of a function $U(\rho)$ and the factor of two in the exponentials are for later convenience.

The unknown functions $F(\rho)$, $H(\rho)$ and $U(\rho)$ can be determined using the Einstein vacuum equations. After some calculations, one gets the following differential equations

$$F''(\rho) + \frac{F'(\rho)}{\rho} = 0,$$

$$U''(\rho) + \frac{U'(\rho)}{\rho} = 0,$$

$$H'(\rho) = F'(\rho) + U'(\rho)$$

(30)

and

$$F''(\rho) + U''(\rho) - F'(\rho)U'(\rho) - \frac{F'(\rho) + U'(\rho)}{\rho} = 0,$$

(31)

8 The $z$ direction in this Appendix corresponds to the $x$ direction in the text.
where the prime represents derivative with respect to \( \rho \). Eqs. (30) can easily be used to get

\[
F(\rho) = F_o \ln \frac{\rho}{a}, \\
U(\rho) = U_o \ln \frac{\rho}{b}, \\
H(\rho) = H_o + F(\rho) + U(\rho),
\]

(32)

where \( F_o, U_o, H_o, a \) and \( b \) are constants of integration, Eq. (31) implies

\[
2(F_o + U_o) + F_o U_o = 0.
\]

(33)

This is actually due to the contracted Bianchi identities that makes Eqs. (30) and (31) not to be independent.

For weak gravitational fields, one can assume \( F_o \) and \( U_o \) to be small. Hence, neglecting the term \( F_o U_o \) in Eq. (33), one gets \( F_o \approx -U_o \). In this approximation, the line element (29) reads

\[
ds^2 \simeq \left( 1 + 2F_o \ln \frac{\rho}{a} \right) dt^2 - h_o d\rho^2 - \rho^2 d\varphi^2 - \left( 1 - 2F_o \ln \frac{\rho}{b} \right) dz^2,
\]

(34)

where \( H(\rho) = H_o + F_o \ln(b/a) \equiv h_o \) is actually a constant.

In a local frame of reference, using the standard Lorentz transformation between inertial frames, that is \( x \to x, y \to y, z \to \gamma(z - vt) \) and \( t \to \gamma(t - vz) \), one gets a stationary metric, namely

\[
ds^2 \simeq \left( 1 + 2\gamma^2 F_o \ln \frac{\rho}{a} \right) dt^2 - h_o d\rho^2 - \rho^2 d\varphi^2 - \left( 1 - 2\gamma^2 F_o \ln \frac{\rho}{b} \right) dz^2 + 4\gamma^2 vF_o \ln \frac{ab}{\rho^2} dzdt,
\]

(35)

where we have neglected the third–order terms containing \( v^2 F_o \).

Now, let us use the above results for the MIL case. First of all, our \textit{a priori} assumption that the space–time would be static can be justified by the constant linear velocity of the MIL. Besides, it is also clear that a static space–time can be evident only in its adapted coordinate system, and not in a general coordinate, e.g., Eq. (35). However, one should note that, as has been discussed in Ref. [45] and references therein, a rotating line mass and/or mixed time–independent electric and magnetic fields, in general, cause rotational effects in space–time, and hence, the space–time will be stationary but not static.

In order for the metric to be asymptotically flat in the radial direction, we assume the following simple case of \( b = a \) and \( h_o = 1 \), hence the metric (34) reads

\[
ds^2 \simeq \left( 1 + 2\gamma^2 F_o \ln \frac{\rho}{a} \right) dt^2 - d\rho^2 - \rho^2 d\varphi^2 - \left( 1 - 2\gamma^2 F_o \ln \frac{\rho}{a} \right) dz^2 - 8\gamma^2 vF_o \ln \frac{ab}{\rho^2} dzdt.
\]

(36)
The metric (36), for the radial distance $\rho = a$, obviously gives the Minkowski flat metric. Henceforth, to comply with the required assumption, we assume that this should be true when $\rho > a$ as well, i.e. a cutoff has been applied. That is, $a$ must be a sufficiently large perpendicular distance from the line where, on and beyond it, the gravitational field tends to zero, and actually, the metric (36) is valid for $\rho \leq a$.

To determine $F_o$, one should note that the gravitational potential for a long line mass charge density at rest, $\lambda_o$, is $\phi' = \phi'_o + 2G\lambda_o \ln \rho$. Choose, $\phi'_o = -2G\lambda_o \ln a$, and amend equations when the line is moving, that is replace $\lambda_o$ by $\gamma \lambda_o$. For the weak field case, where $g_{00} = 1 + 2\phi$, one obtains

$$F_o = 2G\lambda_o/\gamma.$$ That is, the asymptotically flat metric, in the radial direction for the MIL case in the frame $S$, when $\rho \leq a$, is given by

$$ds^2 \simeq \left( 1 + 4G\gamma \lambda_o \ln \frac{\rho}{a} \right) dt^2 - d\rho^2 - \rho^2 d\varphi^2 - \left( 1 - 4G\gamma \lambda_o \ln \frac{\rho}{a} \right) dz^2 - 16G\lambda_o v \ln \frac{\rho}{a} dz dt.$$ (37)

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9Evidently, $a$ depends on the line mass charge density.

10If one keeps the third order terms containing $v^2 F_o$, one obtains $F_o = 2G\lambda_o/\gamma(1 + v^2)$. 
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