A new approach to euclidean plane geometry based on projective geometric algebra

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Abstract
The article presents a new approach to euclidean plane geometry based on projective geometric algebra (PGA). After introducing the algebra, it presents the first detailed study of the geometric product of basic elements: pairs of lines, pairs of points, a point-line pair, 3 lines, and 3 points, with particular attention to the seamless integration of euclidean and ideal aspects. This yields a compact, powerful geometric toolkit which the article then applies to a variety of topics in plane euclidean geometry: distance formulae, sums and differences of points and of lines, isometries via sandwiches, the join operator, orthogonal projection, and a step-by-step solution of a sample geometric construction. In conclusion, the article compares the PGA approach to the analytic geometric approach and also alternative geometric algebra approaches to plane geometry. Numerous figures accompany the text. For readers with the requisite mathematical background, a self-contained coordinate-free introduction to the algebra is provided in an appendix.

1 Introduction
The 19th century witnessed an unprecedented development of geometry and algebra. These impact of these advances on the teaching and practice of euclidean geometry in the 20th century was limited to the introduction of vector and linear algebra to supplement the standard tools of analytic geometry. In recent years, however, there has been a dramatic increase in the
application of geometric algebra to euclidean geometry. A geometric algebra is an encompassing algebraic structure that models both incidence relations and metric relations in a concise and easy-to-implement form.

Those seeking geometric algebra toolkits for doing n-dimensional euclidean geometry find two popular solutions in the contemporary literature: the vector space algebra \( \mathbb{R}^n, 0, 0 \) (VGA), using \( n \)-dimensional coordinates ([DFM07], Ch. 10); and conformal geometric algebra (CGA), which uses \((n + 2)\)-dimensional coordinates ([DFM07], Ch. 13). [Gun11b] and [Gun11c] feature a third model, less well known than these two, which fits naturally between them: projective geometric algebra (or PGA for short), which uses \((n + 1)\)-dimensional coordinates to model \( \mathbb{E}^n \). This article provides an introduction to PGA, by applying it to the euclidean plane \( \mathbb{E}^2 \).

1.1 Structure of the article

This article begins by introducing dual projective geometric algebra \( P(\mathbb{R}^*_2, 0, 1) \) as a tool for euclidean plane geometry. It discusses the basis elements in different grades and how they can be normalized, paying special attention to the distinction between euclidean and ideal elements. It then surveys the geometric product of 2 or 3 blades of various grades and types. There follows a sequence of applications of this product to plane geometry: distance formulae; isometries as sandwiches; the join operator; sums and differences of \( k \)-vectors; orthogonal projections; and a step-by-step solution to a classical geometric construction problem. Finally, the article compares \( P(\mathbb{R}^*_2, 0, 1) \) to alternative approaches to doing plane geometry.

2 The algebra \( P(\mathbb{R}^*_2, 0, 1) \)

We assume the reader is familiar with the geometric algebra \( \mathbb{R}^*_3, 0, 0 \) as a model for \( \mathbb{R}^3 \), and knows that angles between vectors or between planes can be measured via the inner product of the corresponding \( k \)-vectors. If \( \mathbf{u} \) and \( \mathbf{v} \) are two unit-length 1-vectors, then the inner product \( \mathbf{u} \cdot \mathbf{v} \) is well known to be the cosine of their angle. This behaviour is typical of any geometric algebra with non-degenerate metric: the inner product provides the necessary information to calculate the angle or distance between the two elements. What is the situation in the euclidean plane \( \mathbb{E}^2 \)?

Let

\[
\begin{align*}
a_0 x + b_0 y + c_0 &= 0, \\
a_1 x + b_1 y + c_1 &= 0
\end{align*}
\]

be two oriented lines which intersect at an angle \( \alpha \). We can assume WLOG
that the coefficients satisfy \( a_i^2 + b_i^2 = 1 \). Then it is not difficult to show that
\[
a_0a_1 + b_0b_1 = \cos \alpha
\]

Unlike the inner product for the case of \( \mathbb{R}^{3,0,0} \), here the third coordinate makes no difference in the angle calculation: one can translate either line as one pleases without changing the angle between the lines. Refer to Fig. 1 which shows an example involving a general line and a pair of horizontal lines. Hence the proper signature for measuring angles in \( \mathbb{E}^2 \) is \( (2,0,1) \). This is a so-called degenerate inner product since the last entry in the signature is non-zero.

Notice that to model lines and points in a symmetric way we adopt homogeneous coordinates so line equations appear as \( ax + by + cz = 0 \). That is, we work in projective space \( \mathbb{R}P^n \). Hence, to produce a geometric algebra for the euclidean plane we must attach the signature \( (2,0,1) \) to a projectivized Grassmann algebra. As the above discussion yields a way to measure the angle between lines rather than the distance between points, we choose the dual projectivized Grassmann algebra \( \mathbb{P}(\bigwedge \mathbb{R}^3) \) for this purpose. This leads to the geometric algebra \( \mathbb{P}(\mathbb{R}^{2,0,1}_e) \) as the correct one for plane euclidean geometry. We call it projective geometric algebra (PGA) due to its close connections to projective geometry. The standard Grassmann algebra leads to \( \mathbb{P}(\mathbb{R}^{2,0,1}) \), which models dual euclidean space, a different metric space.

PGA for euclidean geometry first appeared in the modern literature in \[Sel00\] and \[Sel05\] and was extended and developed in \[Gun11a\], \[Gun11b\], \[Gun11c\], and \[Gun15\]. Readers interested in a fuller, more rigorous treatment should consult the latter references. The even subalgebra, also known
as the *planar quaternions*, has a long history as a tool for kinematics in the plane ([Bla38], [McC90]).

### 2.1 Meet and join

The wedge operator $\wedge$ in $\mathbf{P}(\mathbb{R}^2_0, 0, 1)$ is the *meet* operator. It is important to have access to the *join* operator also. Since the typical solution to this challenge assumes a non-degenerate metric, we sketch the non-metric approach used here. The Poincaré isomorphism $\mathbf{J} : G \leftrightarrow G^*$ between the Grassmann algebra $G$ and the dual Grassmann algebra $G^*$ can be used to define the *join* operator $\vee$ in $\mathbf{P}(\mathbb{R}^2_0, 0, 1)$ ([Gun11a]):

$$A \vee B := \mathbf{J}(\mathbf{J}(A) \wedge \mathbf{J}(B))$$

One can also implement the join operator using the so-called shuffle operator within $\mathbf{P}(\mathbb{R}^2_0, 0, 1)$ ([Sel05], Ch. 10). We describe below in Sect. 7 a closely related method to calculate the join in $\mathbf{P}(\mathbb{R}^2_0, 0, 1)$.

### 2.2 Basis vectors of the algebra

We provide here a treatment of the algebra based on a choice of basis elements; a coordinate-free treatment can be found in Appendix 1.

$\mathbf{P}(\mathbb{R}^2_0, 0, 1)$ has an orthogonal basis of 1-vectors $\{e_0, e_1, e_2\}$ satisfying

$$e_0^2 = 0, \quad e_1^2 = e_2^2 = 1, \quad e_i \cdot e_j = 0 \quad \text{for} \quad i \neq j$$

$e_0$ is the *ideal* line of the plane (sometimes called the “line at infinity”) which we write – following the coordinate-free discussion above – as $\omega$, $e_1$ is the $x = 0$ line (vertical!) and $e_2$ is the $y = 0$ line (horizontal). All lines except $\omega$ belong to the euclidean plane and are called *euclidean* lines. We choose the basis 2-vectors

$$E_0 := e_1e_2, \quad E_1 := e_2e_0, \quad E_2 := e_0e_1$$

for the points of the plane. It is easy to check that these satisfy

$$E_0^2 = -1, \quad E_1^2 = E_2^2 = 0, \quad E_i \cdot E_j = 0 \quad \text{for} \quad i \neq j$$

Hence the induced inner product on points is $(1,0,2)$, more degenerate than that for lines. As a result, the distance function between points cannot be obtained from the inner product – but *can* be obtained via the geometric product; see Sect. 3.3 below for details. $E_0$ is the origin of the coordinate
Figure 2: Perspective view of basis 1- and 2-vectors

system, \( \mathbf{E}_1 \) is the \textit{ideal} point in the x-direction and \( \mathbf{E}_2 \) is the ideal point in the y-direction. Ideal points are points incident with the ideal line. In general, ideal elements can be characterised as nilpotent elements, satisfying \( x^2 = 0 \). See Fig. 2 for a perspective view of the fundamental triangle determined by these elements.

The pseudoscalar generates the grade-3 vectors: \( \mathbf{I} := e_0 e_1 e_2 \). It satisfies \( \mathbf{I}^2 = 0 \). This is, the inner product, or metric, is degenerate. We could also say, in analogy with 1- and 2-vectors, the metric (as 3-vector) is \textit{ideal}. A 3-vector \( \mathbf{p} \) has the form \( a \mathbf{I} \) for \( a \in \mathbb{R} \). We define the signed magnitude \( S(\mathbf{p}) := a \). Note that \( \mathbf{E}_i \) was chosen so that \( e_i \mathbf{E}_i = \mathbf{I} \).

### 2.3 Normalized points and lines

A \( k \)-vector whose square is \( \pm 1 \) is said to be \textit{normalized}. Since normalization simplifies the subsequent discussion, we introduce it here, although logically speaking the justification for all the steps in the normalization process will only later be established. The square of any \( k \)-vector in the algebra is a scalar, since all \( k \)-vectors are simple. For a euclidean line \( \mathbf{m} = ce_0 + ae_1 + be_2 \), define the norm

\[
\| \mathbf{m} \| := \sqrt{m^2} = \sqrt{\mathbf{m} \cdot \mathbf{m}} \quad (= \sqrt{a^2 + b^2})
\]

Then \( \mathbf{m}_n := \| \mathbf{m} \|^{-1} \mathbf{m} \) satisfies \( \mathbf{m}_n^2 = 1 \); such a 1-vector is said to be \textit{normalized}. For a euclidean point \( \mathbf{P} = z \mathbf{E}_0 + x \mathbf{E}_1 + y \mathbf{E}_2 \), \( \mathbf{P}^2 = -z^2 \). Define \( \| \mathbf{P} \| := |z| \). Then \( \mathbf{P}_n := z^{-1} \mathbf{P} \) satisfies \( \| \mathbf{P}_n \| = 1 \). Such a point is also called \textit{dehomogenized} since its \( \mathbf{E}_0 \) coordinate is 1. In the following discussions we often assume that euclidean lines and points are normalized.
Ideal elements and free vectors. Ideal points correspond to euclidean “free vectors” (a fact already recognised in [Cli73]). Let $P = aE_1 + bE_2$ be an ideal point. Then, as noted above, $\|P\| = 0$. This leads us to introduce a second norm for ideal points which mirrors their function as free vectors.

Define the ideal norm $\|P\|_\infty := \|P \lor Q\|$ where $Q$ is any normalized euclidean point. Then $\|P\|_\infty = \sqrt{a^2 + b^2}$, as desired. Thus, the points of the ideal line can be treated as free vectors with the positive definite inner product of $\mathbb{R}^2$. We write the corresponding inner product between two ideal points as $\langle U, V \rangle_\infty$. Every euclidean line $m$ has an ideal point $m_\infty$, normalized so that $\|m_\infty\|_\infty = 1$. The ideal norm allows us to represent ideal points in the accompanying figures as familiar free vectors (arrows labeled with capital letters), see Fig. 6 (right). We will have occasion more than once in what follows to confirm that the standard and ideal norms form an organic whole. For a fuller discussion of the ideal norm see §4.4.4 of [Gun11a].

Weight and norm. If one has chosen a standard representative for a projective $k$-vector $X$, and $Y = \lambda X$, we say that $Y$ has weight $\lambda$. Typically the standard element will have norm $\pm 1$. Elements of weight $\pm 1$ are then normalized. The freedom to choose the weight is a consequence of working in projective space, since non-zero multiples of an element are all projectively equivalent. Sometimes the weight is irrelevant, sometimes crucial. When multiplying elements together, one gets the same projective result regardless of the weights; while adding elements, different weights gives different projective results.

3 The geometric product in detail

The general element in a geometric algebra is called a multivector. For a multivector $M$, the grade-$k$ part is written $\langle M \rangle_k$, hence $M = \sum_k \langle M \rangle_k$. For a $k$-vector $A$ and an $m$-vector $B$, the dot product $A \cdot B := \langle AB \rangle_{|k-m|}$ is the lowest grade component of $AB$. The wedge $A \wedge B = \langle AB \rangle_{k+m}$ is the highest grade component. All combinations of $(k, m)$ in $P(\mathbb{R}^{*}_{2,0,1})$ except $(2, 2)$ can then be written as

$$AB = A \cdot B + A \wedge B$$

For $(k, m) = (2, 2)$, $\langle AB \rangle_2 := A \times B \ (= AB - BA)$. The full multiplication table of the 8 basis vectors can be found in Table 1. $\bar{X}$, the reversal of a
multi-vector $X$ is obtained by reversing the order of all products involving 1-vectors.

In the following discussion, $P$ and $Q$ are normalized points (either euclidean or ideal, as indicated), and $m$ and $n$ are normalized lines. We analyze the geometric meaning of products of pairs and triples of $k$-vectors of various grades, paying particular attention to the distinction of euclidean and ideal elements. A selection of these products are illustrated in Fig. 3.

### Table 1: Geometric product in $\mathbb{P}(\mathbb{R}^*_2,0,1)$

|     | 1   | $e_0$ | $e_1$ | $e_2$ | $E_0$ | $E_1$ | $E_2$ | $I$   |
|-----|-----|-------|-------|-------|-------|-------|-------|------|
| 1   | 1   | $e_0$ | $e_1$ | $e_2$ | $E_0$ | $E_1$ | $E_2$ | $I$   |
| $e_0$ | 0   | 0     | $E_2$ | $-E_1$ | I     | 0     | 0     | 0    |
| $e_1$ | $e_1$ | $-E_2$ | 1     | $E_0$ | $e_2$ | I     | $-e_0$ | $E_1$ |
| $e_2$ | $e_2$ | $E_1$ | $-E_0$ | 1     | $-e_1$ | $e_0$ | I     | $E_2$ |
| $E_0$ | $E_0$ | I     | $-e_2$ | $e_1$ | $-1$ | $-E_2$ | $E_1$ | $-e_0$ |
| $E_1$ | $E_1$ | 0     | I     | $-e_0$ | $E_2$ | 0     | 0     | 0    |
| $E_2$ | $E_2$ | 0     | $e_0$ | I     | $-E_1$ | 0     | 0     | 0    |
| $I$  | $I$  | 0     | $E_1$ | $E_2$ | $-e_0$ | 0     | 0     | 0    |

3.1 **Product with pseudoscalar**

First notice that the pseudoscalar $I$ commutes with everything in the algebra. For a euclidean line, $a$, the polar point $a^\perp := aI = Ia$ is the ideal point perpendicular to the line $a$. We can use the polar point to define a consistent orientation on euclidean lines; we draw the arrow on an oriented line $m$ so that rotating it by $90^\circ$ in the CCW direction produces $m^\perp$. See Fig. 2 right, which shows the resulting orientations on the basis 1-vectors. When $a$ is normalized, so is $a^\perp$, another confirmation that the two norms (euclidean and ideal) have been harmoniously chosen. For a normalized euclidean point $P$, $P^\perp := PI = IP = -e_0$, the ideal line (with CW orientation). The polar of an ideal point or line is 0.

We noted above in Sect. 2.2 that the condition $I^2 = 0$ means the metric is degenerate, or, what is the same, multiplication by $I$ (the so-called metric polarity) is not an algebra isomorphism. Although some researchers see this as a flaw in the algebra (for example, [Li08], p. 11), our experience leads to view it as an advantage, since it accurately mirrors the metric relationships in the euclidean plane. For example, when $m$ and $n$ are parallel, $m^\perp =$
3.2 Product of two lines

In general we have $\bf{mn} = \langle mn \rangle_0 + \langle mn \rangle_2 = \bf{m} \cdot \bf{n} + \bf{m} \wedge \bf{n}$. We say two lines are perpendicular if $\bf{m} \cdot \bf{n} = 0$ – even when one of the lines is ideal. The meaning of the two terms on the RHS depends on the configuration of $\bf{m}$ and $\bf{n}$ as follows.

**Intersecting euclidean lines.** We say that two intersecting euclidean lines meet at an angle $\alpha$ when a rotation of $\alpha$ around their common point brings the first oriented line onto the second, respecting the orientation. Then $\bf{m} \cdot \bf{n} = \| \bf{m} \| \| \bf{n} \| \cos \alpha$ and $\bf{m} \wedge \bf{n} = \| \bf{m} \| \| \bf{n} \|(\sin \alpha) \mathbf{P}$ where $\mathbf{P}$ is their normalized intersection point. Note that the angle $\alpha$ can be deduced from the wedge only because we have used the inner product to normalize the lines in advance. Without normalizing $\bf{m}$ and $\bf{n}$, the formulae are

$$\bf{m} \cdot \bf{n} = \| \bf{m} \| \| \bf{n} \| \cos \alpha \quad \text{and} \quad \bf{m} \wedge \bf{n} = \| \bf{m} \| \| \bf{n} \|(\sin \alpha) \mathbf{P}$$

Similar extensions involving non-normalized arguments could be made for the subsequent formulae given below, but in the interests of space we omit them. **Exercise:** $(\bf{mn})^2 = \cos n\alpha + (\sin n\alpha) \mathbf{P}$. Show that the vector subspace generated by $1$ and $\mathbf{P}$ is isomorphic to the complex plane $\mathbb{C}$. 

\[\text{Figure 3: Selected geometric products of blades.}\]
Parallel euclidean lines. \( m \cdot n = \pm 1 \). We say the lines are *parallel* when this equals 1, otherwise we say they are *anti-parallel*. In the latter case, replace \( n \) by \(-n\) to obtain parallel lines. Then \( m \cdot n = 1 \) and \( m \wedge n = d_{mn}m_\infty \), where \( d_{mn} \) is the oriented euclidean distance between the two oriented lines. The simplicity of this formula confirms the choice of the norm \( \| \cdot \|_\infty \) on ideal points. Note that the geometric product automatically finds the correct form of measuring the distance between the two lines, shifting from the angle measure \((\sin \alpha)\) for intersecting lines to the euclidean distance measure \( d_{mn} \) for two parallel lines.

**Exercise:** \((mn)^n = 1 + nd_{mn}m_\infty\).

Product of a euclidean line with the ideal line. Let \( n = e_0 \) be the ideal line. Then \( m \cdot n = 0 \) and \( m \wedge n = m_\infty \) is the ideal point of \( m \). Note that since \( m \cdot n = 0 \), the ideal line is perpendicular to every euclidean line; since it shares an ideal point with each such line, it is parallel to every euclidean line!

3.3 Product of two points

Here the general formula is \( PQ = (PQ)_0 + (PQ)_2 = P \cdot Q + P \times Q \). The resulting behavior is characterized by the fact that the inner product for points is more degenerate than that for lines.

**Two euclidean points.** \( P \cdot Q = -1 \) and \( P \times Q \) is an ideal point perpendicular to \( P \vee Q \). To be exact \( P \times Q = -(P \vee Q)I \) (notice the negative sign). We also write this as \( (P - Q)^\perp \) since the ideal point \( P - Q \), rotated in the CCW direction by \( 90^\circ \), yields \( P \times Q \). See Fig. 6 right. **Exercise:** The distance \( d_{PQ} \) between two euclidean points satisfies \( d_{PQ} = \|P \times Q\| = \|P \vee Q\| \).

**Euclidean point and ideal point.** If \( Q \) is ideal, then \( P \cdot Q = 0 \) and \( P \times Q \) is the ideal point obtained by rotating \( Q \) \( 90^\circ \) in the CW direction. \( Q \times P \) rotates in the CCW direction. Thus, multiplication of an ideal point by any finite point rotates the ideal point by \( 90^\circ \); the specific location of the euclidean point plays no role. **Exercise:** The product of two ideal points is zero.

3.4 Product of a line and a point

The general formula is \( mP = (mP)_1 + (mP)_3 = m \cdot P + m \wedge P \). The wedge vanishes if and only if \( P \) and \( m \) are incident. As before, we assume that both \( m \) and \( P \) are normalized.

**Euclidean line and euclidean point.** \( m \cdot P \) is the line passing through
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Figure 4: Products of 3 euclidean points

Why? This can be visualised as starting with all the lines through \( P \) and removing all traces of the line parallel to \( m \), leaving the line perpendicular to \( m \). It has the same norm as \( m \), and its orientation is obtained from that of \( m \) by CCW rotation of \( 90^\circ \). This is reversed in the product \( P \cdot m \). This sub-product is important enough to deserve its own symbol. We define \( m_\perp P := m \cdot P = -P \cdot m \). The wedge product satisfies \( m \wedge P = d_{mP}I \), where \( d_{mP} \) is the directed distance between \( m \) and \( P \).

Euclidean line and ideal point. Let \( \alpha \) be the angle between the direction of \( m \) and \( P \): \( \cos \alpha = \langle m, P \rangle \). Then \( m \cdot P = (\cos \alpha)e_0 \) and \( m \wedge P = (\sin \alpha)I \). Notice that \( mP \) is the sum of an ideal line and a pseudoscalar: no euclidean point or line plays any role in the product. The first term, involving the ideal line, is non-zero when the ideal line is the only line through \( P \) perpendicular to \( m \). When \( \alpha = \frac{\pi}{2} \), every line through \( P \) is perpendicular to \( m \), and \( m \cdot P = 0 \).

3.5 3-Way Products

Products of more than 2 \( k \)-vectors can be understood by multiplying the factors out, one pair at a time. The product of 3 euclidean points (or lines) is important enough in its own right to merit a separate discussion. The results provide a promising basis for investigation of euclidean triangles.

Product of 3 euclidean points. Let the 3 points be \( A \), \( B \), and \( C \). See

...
The geometric product in detail

Figure 5: Product of 3 euclidean lines.

Then using the results obtained above for products of two points:

\[ ABC = (AB)C \]
\[ = (-1 + (A - B)\perp)C \]
\[ = -C - (A - B) \]
\[ = A - B + C \]

The first and second steps follow from the results from Sect. 3.3. The final equation indicates the projective equivalence of the two expressions. The result is somewhat surprising, since the scalar part vanishes. **Exercise:**

The product of an odd number of euclidean points is a euclidean point that is the alternating sum of the arguments. Hence, if one begins with the triangle \( ABC \) and generates a lattice of congruent triangles by translating the triangle along its sides, then the vertices of this lattice can be labeled by products of odd numbers of the vertices \( A, B, \) and \( C \) (Fig. 4).

**Product of 3 euclidean lines.** Let the 3 (normalized) lines be \( a, b, \) and \( c \) oriented cyclically. See Fig. 5. These three lines determine a triangle. Then \( a \wedge b = \sin(\pi - \gamma)C, \) etc., produces the interior angle \( \gamma \) and the (normalized) vertex \( C \) of the triangle. Using the results obtained above for products of
two lines:

\[ abc = (ab)c \]

\[ = (- \cos \gamma)c + (\sin \gamma)(Cc) \]

\[ = (- \cos \gamma)c + (\sin \gamma)(C \cdot c + C \wedge c) \]

\[ = -((\cos \gamma)c + (\sin \gamma)c^\perp) + \sin \gamma d_{Cc}I \]

The first two steps follow from the results from Sect. 3.2, the third from Sect. 3.4. Let \( \overline{C} \) be the intersection of \( c \) and \( C \cdot c \). Then the last equation shows that \( \overline{b} := (abc)_1 \) (in parentheses) is the result of rotating \( c \) around \( \overline{C} \) by \( \gamma \). Parenthesizing in a different order yields:

\[ abc = a(bc) = -((\cos \alpha)a - (\sin \alpha)a^\perp_A) + \sin \alpha d_{Aa}I \]

In this form, \( \overline{b} \) is the result of rotating \( a \) around \( \overline{A} \) by \( -\alpha \). Hence \( \overline{b} \) is the joining line of \( \overline{A} \) and \( \overline{C} \).

Since the grade-3 parts are equal, one obtains:

\[ (\sin \gamma)d_{Cc} = (\sin \alpha)d_{Aa} \]

This illustrates an important technique for generating formulas in geometric algebra. By applying the associative principle one can insert parentheses at different positions:

\[ (ab)c = abc = a(bc) \]

The LHS and RHS represent different paths in the algebra to the same result, and these often produce non-trivial identities as this one.

**Exercises.** 1) \( (abc)_1 = \frac{1}{2}(abc + cba) \). 2) \( \frac{1}{2}(abc + bac) = (a \cdot b)c. \)

### 4 Distance and angle formulae

We collect here the various distance formulae encountered in the process of discussing the 2-way vector products above. \( P \) and \( Q \) are normalized euclidean points, \( U \) and \( V \) are normalized ideal points, and \( m \) and \( n \) are normalized euclidean lines. Space limitations prevent further differentiation with respect to signed versus unsigned distances.

- **Intersecting lines.** \( \angle(m, n) = \cos^{-1}(m \cdot n) = \sin^{-1}(\|m \wedge n\|) \)
- **Parallel lines.** \( d(m, n) = \|m \wedge n\| \)
- **Euclidean points.** \( d(P, Q) = \|P \vee Q\| = \|P \times Q\| \)
Figure 6: Left: Sums and differences of euclidean lines. Right: Sums and differences of euclidean and ideal points.

- Ideal points. \[ \angle(U, V) = \cos^{-1}(\langle U, V \rangle_\infty) \]
- Euclidean line, euclidean point. \[ d(m, P) = -d(P, m) = S(m \wedge P) \]
- Euclidean line, ideal point. \[ \angle(m, U) = \cos^{-1}(\|m \cdot U\|_\infty) \]

5 Sums and differences of points, and of lines

Based on the discussion of the geometric product above, it is instructive to examine sums and differences of points, resp. lines. This deceptively simple theme reveals important distinctions between euclidean and ideal points and lines that play a central role throughout this algebra. Consult Fig. 6.

Sums and differences of lines. When \( m \) and \( n \) are both euclidean, and intersect, then \( m + n \) is their mid-line, the line through their common point \( m \wedge n \) that bisects the angle between \( m \) and \( n \). \( m - n \) also passes through their common point, but bisects the supplementary angle between the two lines. (To establish the claim, consider the inner product of \( m \pm n \) with each line separately.) If the two lines are parallel, then \( m + n \) is their mid-line: the line parallel to both, halfway in between them. \( m - n \) is the ideal line, weighted by the signed distance between the lines. If \( m \) is euclidean and \( n = \lambda e_0 \) is a weighted ideal line, then \( m + n \) is a (normalized) euclidean line representing the translation of the line \( m \) a signed distance \( \lambda \) in the direction perpendicular to its own direction, (to be exact, in the direction opposite its polar point \( m^\perp \)).

Sums and differences of points. When \( P \) and \( Q \) are both euclidean, \( P + Q \) is their mid-point. \( \frac{P + Q}{2} \) is the normalized mid-point.) \( P - Q \) is an
ideal point representing their vector difference. If $P$ is euclidean and $V$ is ideal (not necessarily normalized), then $P \pm V$ is a (normalized) euclidean point representing the translation of the point $P$ by the free vector $\pm Q$. If both $U$ and $V$ are ideal (again, not necessarily normalized), then $U \pm V$ is the ideal point representing their vector sum (difference). Here we once again meet the $\mathbb{R}^2$ vector space structure on the ideal line induced by the ideal norm.

6 Isometries

Equipped with our detailed knowledge of 2-way products we now turn to discuss how to implement euclidean isometries in the algebra. The group of isometries of $\mathbb{E}^2$ is generated by reflections in euclidean lines. The product of an even number of reflections yields a direct (orientation-preserving) isometry, while an odd number produces an indirect isometry. Exercise: For the euclidean plane, every isometry can be written using 1, 2, or 3 reflections. We now show how to implement reflections using the geometric product, then extend this result to products of 2 and 3 reflections.

6.1 Reflections

Suppose $a$ and $b$ are two normalized euclidean lines, and let $R_a(b)$ represent the reflection of $b$ in $a$. Purely geometric considerations imply that $R_a(b)$ is a line $x$ satisfying $a \cdot x = a \cdot b$ and $a \wedge x = b \wedge a$. Exercise: Show that $x := aba$ fulfils both conditions, satisfies $x \neq b$ and hence is the desired reflection.

We write the reflection operator $b \rightarrow aba$ as $\overline{a}(b)$. We sometimes refer to this as a sandwich operator since the $a$ “sandwiches” the operand $b$ on both sides. Exercise: Show that $\overline{a}(P)$ is also a reflection applied to a euclidean point $P$. [Hint: Write $P = mn$ for orthogonal $m$ and $n$.]

6.2 Product of two reflections

Before we discuss the product of several reflections, we introduce some terminology. The product of any number of euclidean lines is called a versor; the product of an even number is called a rotor. Versors and rotors are important since sandwich operators based on them yield euclidean isometries.

The concatenation of two reflections in lines $a$ and $b$ can be written $\overline{b}(\overline{a}(x)) = b(ax)a$. Moving the parentheses yields $(ba)x(ab)$. Define $r := ba$, and an operator $r(x) := rx\overline{r}$ which represents the composition
The reflection in the line $a$ is implemented by the sandwich $aXa$; the product of the reflection in line $a$ followed by reflection in (non-parallel) line $b$ is a rotation around their common point $a \wedge b$ through $2 \cos^{-1}(a \cdot b)$.

of these two reflections using $r$. Such a composition can take two forms, depending on the position of the lines.

When the lines intersect in a euclidean point, then $r$ is a rotation around that point by twice the angle between the lines. See Fig. 7. When the lines are parallel, $r$ is a translation by twice the distance between the lines in the direction perpendicular to the direction of the lines. The details can be confirmed by applying the results above involving products of two lines in Sect. 3.2 to write out $r$ for these two cases and then by multiplying out the resulting sandwich operators. If the rotor has a euclidean fixed point, the rotor is called a rotator; if not, a translator.

**Exercise:** Show that for a translator $t$, $tx = \tilde{x}t$ represents half the translation of the sandwich $\tilde{t}(x)$. That is, translators also make good “open-faced” sandwiches :-) **Exercise:** Discuss the rotator $\cos \alpha + \sin \alpha \mathbf{P}$ when $\alpha = \frac{\pi}{2}$.

**6.3 Product of 3 reflections**

We first show that the sandwich operator formed by the sum of a line and a pseudoscalar implements a glide reflection along the line. Let $r = (r)_1 +$
Figure 8: Glide reflection generated by \( r = m + \lambda I \) applied to line \( x \).

\[
\langle r \rangle_3 = m + \lambda I
\]

where \( m \) is normalized. Then for an arbitrary line \( x \):

\[
\tilde{r}(x) = r \bar{x} = \langle m + \lambda I \rangle x(m - \lambda I) = mxm + \lambda \lambda x - \lambda x m - \lambda^2 I^2
\]

\[
= mxm + \lambda \lambda x - \lambda x m
\]

\[
= m(x) + \lambda (mx - x m)
\]

\[
= m(x) + 2\lambda mx^\perp
\]

\[
= m(x) + 2\lambda (m \cdot x^\perp)
\]

\[
= m(x) + 2\lambda (\cos \alpha) e_0
\]

The first term is the reflection in the line \( m \); by Sect. 3 above, the second term represents the translation of the reflected line perpendicular to its own direction by the distance \( 2\lambda \cos(\alpha) \). The translation component reveals itself more clearly by considering \( \tilde{r}(X) \) for an arbitrary point \( X \). A calculation similar to the above yields:

\[
\tilde{r}(X) = ... = \tilde{m}(X) + 2\lambda (m \wedge X^\perp)
\]

\[
= \tilde{m}(X) + 2\lambda (m \wedge e_0)
\]

\[
= \tilde{m}(X) + 2\lambda (m_\infty)
\]

In this form it is clear that the translation component is \( 2\lambda m_\infty \): a translation in the direction of the line \( m \) through a distance \( 2\lambda \). Consult Fig. 8.

Applying this to the situation of 3 reflections: By Sect. 3.5 above, the product of three lines has the form \( r = abc = \tilde{B} + \sin(\alpha)d_{aA}I \), hence the above results can be applied. Recall that \( \tilde{B} \) is the joining line of \( \tilde{A} \) and \( \tilde{C} \),
the feet of the altitudes from $A$ and $C$, resp. Refer to Fig. 5. The geometric meaning of the translation distance $2\sin(\alpha)d_{Aa}$ has still to be clarified.

### 6.4 Exponential form for direct isometries

It’s not necessary to write a rotator as the product of two lines. If one knows the desired angle of rotation, one can generate the rotor directly from the fixed point $P$ of the rotation. We know that it is normalized so that $P^2 = -1$. Then, using a well-known technique of geometric algebra, one looks at the exponential power series $e^{tP}$ and shows, in analogy to the case of complex number $i^2 = -1$, that $e^{tP} = \cos t + (\sin t)P$. The RHS we already met above as the product of two euclidean lines meeting in the point $P$ at the angle $t$. Setting $t = \alpha$ one obtains the rotor $r$ from the previous paragraphs. What’s more, letting $t$ take values from 0 to $\alpha$ one obtains a smooth interpolation between the identity map and the desired rotation. Note that this sandwich operator rotates through the angle $2\alpha$; to obtain a rotation of $\alpha$ around $P$, set $r = e^{\alpha P}$.

**Exercise:** Carry out the same analysis for an ideal point $V$ to obtain an exponential form for a translator that moves a distance $d$ in the direction perpendicular (CCW) to $V$. (Answer: $e^{dV} = 1 + \frac{d}{2}V$.)

![Figure 9: Calculating the join of two points.](image)

### 7 Calculating the join operator

As promised above we show now how to calculate the join of two points without leaving the algebra $P(\mathbb{R}_{2,0,1}^*)$. This is the only join operation required since $m \vee P = S(m \wedge P)$. Consult Fig. 9. Let the two normalized
points be \( P \) and \( Q \) and the joining line be \( n \). Let \( m \) be any normalized euclidean line. Then
\[
d_m P := S(m \wedge P) \quad \text{and} \quad d_m Q := S(m \wedge Q)
\]
are the signed distances from \( P \), resp. \( Q \), to \( m \) (see Sect. 3.4). If both are zero, \( n := m \). Otherwise define
\[
M := d_m Q - d_m P
\]
Verify that \( M \wedge m = M \wedge n = 0 \). \( m_\perp := m M \) is the perpendicular to \( m \) at \( M \). For numerical efficiency, assume \( d_m P > d_m Q \), if not relabel \( P \) and \( Q \). \( d_m P := S(m_\perp \wedge P) \) is the signed distance between \( m_\perp \) and \( P \). Then \( n := d_m P m - d_m P m_\perp \) is the desired line. This calculation is closely related to the shuffle product of Cayley-Grassmann algebras (Sel05, Ch. 10).

8 Orthogonal projections

When one has two geometric entities it is often useful to be able to express one in terms of the other. Orthogonal projection is one method to obtain such a decomposition. The algebra \( P(\mathbb{R}_2,0,1) \) offers a variety of such projections which we now discuss, both for their utility as well as to gain practice in using the geometric product introduced above. We can project a line onto a line or a point; and a point onto a line or a point. As before all points and lines are assumed to be normalized. Consult Fig. 10.

Each projection follows the same pattern: take a product of the form \( XYY \) and apply associativity to obtain \( X(YY) = (XY)Y \). Assuming normalized arguments, \( YY = \pm 1 \), yielding \( \pm X = (XY)Y \). The RHS typically consists of two terms which represent an orthogonal decomposition of the LHS.

8.1 Orthogonal projection of a line onto a line

Assume both lines are euclidean. Multiply the equation \( mn = m \cdot n + m \wedge n \) with \( n \) on the right and use \( n^2 = 1 \) to obtain
\[
m = (m \cdot n)n + (m \wedge n)n = (\cos \alpha)n + (\sin \alpha)Pn = (\cos \alpha)n - (\sin \alpha)n_P
\]
Note that \( Pn = -n_P^\perp \) since \( P \wedge n = 0 \). Thus one obtains a decomposition of \( m \) as the linear combination of \( n \) and the perpendicular line \( n_P^\perp \) through \( P \). Exercise: If the lines are parallel one obtains \( m = n + d_m n_0 \).
8.2 Orthogonal projection of a line onto a point

Assume both point and line are euclidean. Multiply the equation $mP = m \cdot P + m \wedge P$ with $P$ on the right and use $P^2 = -1$ to obtain

\[
m = -(m \cdot P)P - (m \wedge P)P = -(m_P^0)P - (d_{mP_I})P = m_P^\parallel - d_{mP} e_0
\]

In the third equation, $m_P^\parallel$ is the line through $P$ parallel to $m$, with the same orientation. $e_0$ is the ideal line. Thus one obtains a decomposition of $m$ as the sum of a line through $P$ parallel to $m$ and a multiple of the ideal line (adding which, as noted above in Sect. 5, translates euclidean lines parallel to themselves).

8.3 Orthogonal projection of a point onto a line

Assume both point and line are euclidean. Multiply the equation $mP = m \cdot P + m \wedge P$ on the left with $m$ on the right and use $m^2 = 1$ to obtain

\[
m = m(m \cdot P) + (m \wedge P) = m(m_P^0) + m(d_{mP}) = P_m + d_{mP} m^\perp = P_m + (P - P_m)
\]

In the third equation, $P_m$ is the point of $m$ closest to $m$. The second term of the third equation is a vector perpendicular to $m$ whose length is $d_{P_m}$:
exactly the vector $\mathbf{P} - \mathbf{P}_m$. Thus one obtains a decomposition of $\mathbf{P}$ as the point on $\mathbf{m}$ closest to $\mathbf{P}$ plus a vector perpendicular to $\mathbf{m}$.

**Exercise:** Show that the orthogonal projection of a euclidean point $\mathbf{P}$ onto another euclidean point $\mathbf{Q}$ yields $\mathbf{P} = \mathbf{Q} + (\mathbf{P} - \mathbf{Q})$.

### 9 Worked-out example of euclidean plane geometry

We pose a problem in euclidean plane geometry on which to practice the theory developed up to now:

Given a point $\mathbf{A}$ lying on an oriented line $\mathbf{m}$, and a second point $\mathbf{A}'$ lying on a second oriented line $\mathbf{m}'$, construct the unique direct isometry mapping $\mathbf{A}$ to $\mathbf{A}'$ and $\mathbf{m}$ to $\mathbf{m}'$.

The problem is illustrated in Fig. 11 (left), including orientation on the two lines. We assume the points and lines are normalized, and define to begin with the intersection point of the lines and the joining line of the points:

$$M := \mathbf{m} \land \mathbf{m}', \quad a := \mathbf{A} \lor \mathbf{A}'$$

The direct isometry we are seeking is either a rotation or a translation. In the former case, the center of rotation has to be equidistant from $\mathbf{A}$ and $\mathbf{A}'$, that is, it lies on the perpendicular bisector of the segment $\overline{AA'}$. To construct this we first obtain the midpoint, and then, applying Sect. 3.4, construct the perpendicular line through the midpoint:

$$\mathbf{A}_m := \mathbf{A} + \mathbf{A}', \quad r := \mathbf{A}_m \cdot a \quad (= A_m a)$$
The condition that \( m \) maps to \( m' \) implies that the center of rotation is the same distance from \( m \) as from \( m' \), that is, lies on the angle bisector of the two lines. We choose the difference in order to respect the orientations of the lines, as the reader can readily confirm. The desired center is then the intersection \( C \) of \( r \) and \( c \).

\[
c := m - m', \quad C := r \wedge c
\]

The final step is to construct the desired isometry. We can (for a rotation) find two lines through \( C \) that meet at half the desired angle of rotation: the line \( A \lor C \) and the perpendicular bisector \( r \) satisfy this condition. Then form the rotor of their product; the rotation is then the sandwich operator defined by this rotor.

\[
s := A \lor C, \quad g := rs, \quad g(X) := gXg
\]

One can also calculate the angle \( \alpha \) between the two mirror lines from the equation \( \cos \alpha = r \cdot s \), and use this to calculate \( g \) as an exponential: \( g = e^{\alpha C} \).

**Exercise:** Show that the above construction also yields valid results when \( C \) is ideal, and that the resulting isometry is a translation.

### 10 Evaluation and conclusion

We have shown that traditional plane geometry can be formulated in a compact and elegant form using \( P(\mathbb{R}_{2,0,1}^*), \) in which euclidean and ideal elements are tightly interwoven in an organic whole. We have successfully applied the algebra to a variety of practical problems of plane geometry and have encountered no obstacles that would prevent extending these results to all of plane geometry. How do these results compare to existing approaches?

Plane geometry is usually handled with a mixture of analytic geometry, linear algebra, and vector algebra. \( P(\mathbb{R}_{2,0,1}^*) \) offers a variety of desirable “infrastructure” features which this mixed approach does not offer:

1. It is coordinate-free.
2. Points and lines are equal citizens.
3. Join and meet operators are obtained from the Grassmann algebra.
4. Isometries are represented by sandwich operators.
5. The geometric product provides a rich, interrelated family of formulas for distance and angle covering both euclidean and ideal elements.

How does \( P(\mathbb{R}_{2,0,1}^*) \) compare to the other two geometric algebras mentioned at the beginning of the article? [Cal07] is a treatment of plane geometry based on \( \mathbb{R}_{2,0,0} \). While entirely appropriate as an introduction to
GA at the high school level, it makes extensive use of non-GA techniques to overcome the limitations of $\mathbb{R}_{2,0,0}$, which unlike the euclidean plane contains a distinguished point (the origin), and can model neither parallelism nor translations. One of the leitmotifs of this article has been to show how $\mathbb{R}_{2,0,0}$ is embedded organically within $P(\mathbb{R}^*_2,0,1)$ as the ideal line $\omega$, so all the features of $\mathbb{R}_{2,0,0}$ can be accessed easily in the model presented here.

We are not aware of an analogous treatment of plane geometry in CGA to the one presented here; such a contribution would make a concrete comparison of the two approaches easier to carry out. Then one could more confidently answer questions such as: “What is analogous in CGA to the seamless integration of parallelism and free vectors found in $P(\mathbb{R}^*_2,0,1)$?” and “Which approach makes more sense for undergraduate (or high school) instruction?” A broader comparison of PGA and CGA can be found in [Gun15].

Appendices

A Coordinate-free description

We provide here a modern, coordinate-free description of the algebra instead of the more traditional coordinate-based approach used above in Sect. 2.2 and Sect. 2.3.

A.1 Foundations

Let $V$ be a real, 3-dimensional vector space with dual space $V^*$. We construct a geometric algebra $A$ based on $V$ using the signature $(2, 0, 1)$. We describe it here algebraically, and postpone until later the geometric interpretation. We begin by recalling some basic facts and definitions regarding the underlying Grassmann algebra $G$ based on $V$:

- $G$ is a graded algebra consisting of 4 grades:
  - $\bigwedge^0(V)$ is the 1-dimensional subspace of scalars $\mathbb{R}1$.
  - $\bigwedge^1(V)$ can be identified with $V$.
  - $\bigwedge^2(V)$ can be identified with $V^*$.
  - $\bigwedge^3(V)$ is a 1-dimensional vector space of pseudoscalars $\mathbb{R}I$. $I$ is defined more precisely below in Sect. A.2.2.

- Let $a \in \bigwedge^1(V)$ and $A \in \bigwedge^2(V)$. We say $a$ and $A$ are incident $\iff a \wedge A = 0$. 

• For a vector subspace $T \subset \bigwedge^k(V)$ define the outer product null space $T^\perp := \{ x \in \bigwedge^{3-k}(V) | t \wedge x = 0 \ \forall \ t \in T \}$.
• Notation: For a multi-vector $M \in G$, $M = \sum_k \langle M \rangle_k$ where $\langle M \rangle_k$ is the grade-$k$ part of $M$.

A.2 Euclidean and ideal elements

The inner product of the geometric algebra can be represented by a symmetric bilinear form $B : V \otimes V \to \mathbb{R}$. The kernel of $B$ is defined as:

$$N := \{ n \in V | B(n, x) = 0 \ \forall \ x \}$$

The signature of the inner product is $(2, 0, 1)$. The 1 in the third position gives the dimension of $N$. So, $N$ is a 1-dimensional vector sub-space of $V$.

As such, it is generated by an element $\omega$, which we will specify more precisely below in Sect. A.2.2. Elements of $N$ are called ideal vectors. Vectors not in $N$ are called euclidean (or proper). $N^\perp$ consists of bivectors incident with $\omega$, and is a 2-dimensional subspace of $\bigwedge^2(V)$. An element of $N^\perp$ is said to be an ideal bivector; all other bivectors are euclidean (or proper).

A.2.1 The square of a 1-vector; normalized euclidean vectors

In a geometric algebra, the geometric product is defined on 1-vectors by

$$ab = a \cdot b + a \wedge b$$

where $a \cdot b = B(a, b)$ and $a \wedge b$ is the exterior product of the underlying Grassmann algebra. The geometric product $m^2$ for a 1-vector $m$ reduces to $m \cdot m$ since the wedge product is antisymmetric. For $m = \omega$, $\omega^2 = \omega \cdot \omega = 0$ since $\omega \in N$. For any euclidean vector $m$, $m^2 = m \cdot m = k \in \mathbb{R}^+$. We define the norm $\|m\| := \sqrt{m^2}$. Then $m_n := \sqrt{k^{-1}} m$ satisfies $\|m_n\| = 1$; such a vector is said to be normalized.

A.2.2 The square of a 2-vector

From the above, there are two sorts of bivectors, ideal and euclidean. For ideal $U$, $U = \omega \wedge m$ for some euclidean vector $m$. And, since $\omega \in N$, $\omega \wedge m = \omega m$. Then $U^2 = -\omega^2 m^2 = 0$. Exercise: For euclidean $P$, one can find two orthonormal euclidean 1-vectors $m$ and $n$ such that $P = mn$. Then it is easy to calculate that $P^2 = -1$. Hence a bivector is ideal $\iff$ its square is zero.
A.2.3 Normalized euclidean 2-vectors

We could define a normalized euclidean bivector to be a bivector satisfying $P^2 = -1$. But we can do better, as the following discussion shows. Let $P$ be any euclidean 2-vector satisfying $P^2 = -1$. We fix $\omega$ to be the unique element of $N$ satisfying $S(\omega \wedge P) = 1$, and define $I := \omega \wedge P$. We show that these definitions don’t depend on $P$, and that the value of $S(\omega \wedge P)$ can serve as a norm for bivectors.

Lemma 1. For euclidean bivector $P$ and ideal bivector $U$, $\langle PU \rangle_0 = 0$.

Proof. Choose $m \in U_{\land}^\perp \cap P_{\land}^\perp$ with $\|m\| = 1$. Then $U = \lambda m \omega$ for $\lambda \in \mathbb{R}^*$. Write $P = nm$ where $n$ is normalized and orthogonal to $m$. Then

$$PU = (nm)(\lambda m \omega) = \lambda n (m^2) \omega = \lambda n \omega$$

Here we have used associativity of the geometric product, and the fact that $m$ is normalized. Finally, since $\omega \in N$, $\langle n \omega \rangle_0 = n \cdot \omega = 0$. \qed

Lemma 2. Given euclidean bivectors $P$ and $Q$, $Q = \lambda P + U$ for some $\lambda \in \mathbb{R}, \lambda \neq 0$ and $U \in N_{\land}^\perp$. Furthermore, $Q^2 = \lambda^2 P^2$.

Proof. The first part follows by observing that $N_{\land}^\perp \subset \wedge^2(V)$ is a subspace of co-dimension 1 in $\wedge^2(V)$, and $Q, P \notin N_{\land}^\perp$. The second assertion follows by observing:

$$Q^2 = (\lambda P + U)^2 = \lambda^2 P^2 + \lambda (PU + UP) + U^2 = \lambda^2 P^2 + 2 \lambda \langle PU \rangle_0 = \lambda^2 P^2$$

Here we have used the fact that the grade-0 part of the geometric product $PU$ is the symmetric part of the product, that $U^2 = 0$ for ideal $U$, and the previous lemma. \qed

Theorem 1. Given euclidean bivectors $P$ and $Q$ such that $P^2 = Q^2$, and $\omega \in N$. Then $\omega \wedge P = \pm \omega \wedge Q$.

\footnote{Or define $m = U \lor P$ and normalize $m$.}
Proof. By the lemma, \( Q = \lambda P + U \) for \( U \in N_\Lambda^\perp \). Since \( Q^2 = P^2 \), \( \lambda = \pm 1 \). Wedging with \( \omega \) yields \( \omega \wedge Q = \pm \omega \wedge P + \omega \wedge U = \pm \omega \wedge P \).

The preceding theorem allows us to obtain a stronger normalization than the condition \( Q^2 = -1 \). Define the norm of a bivector to be \( \|Q\| := S(\omega \wedge Q) \). We say that a euclidean bivector \( Q \) is normalized when \( \|Q\| = 1 \). In every one-dimensional vector subspace of \( \wedge^2(V) \), there are two solutions \( \{Q, -Q\} \) to \( Q^2 = -1 \). \( \|Q\| = 1 \) picks out exactly one of these solutions. The uniqueness of this result simplifies many calculations.

**A.2.4 Multiplication by the pseudoscalar**

Multiplication by the basis pseudoscalar \( I \) is an important operation, sometimes called the polarity on the metric quadric. It maps an element to its orthogonal complement with respect to the inner product. This multiplication is important enough to merit its own notation \( \Pi(X) := X^\perp := IX \), and the result is called the polar of \( X \). By the previous section, a euclidean bivector \( Q \) is normalized \( \iff \) \( I := \omega Q \). Then the polar of a normalized euclidean point \( Q \) is given by \( P^\perp = IP = -\omega \) since \( P^2 = -1 \).

The situation is a little more complicated for 1-vectors. Let \( m \) be a normalized euclidean 1-vector. Let \( n \) be a 1-vector orthogonal to \( m \). Then the product \( nm \) is a normalized euclidean 2-vector, hence \( I = \omega nm \) and \( m^\perp := Im = \omega n = U \), where \( U \) is an ideal bivector. **Exercise:** The kernel of \( \Pi \), restricted to 1-vectors, is \( N \), while the kernel of \( \Pi \), restricted to 2-vectors, is \( N_\Lambda^\perp \).

**A.3 Ideal inner product on ideal bivectors**

We saw above that euclidean bivectors can be normalized, but an ideal bivector \( U \) satisfies \( U^2 = 0 \) hence cannot be normalized in the same way. However, there is a way to provide a norm, and associated inner product, on the ideal bivectors. In fact, we then show how to derive the complete inner product structure on the euclidean elements from this ideal inner product.

**A.3.1 The quotient space \( V/\omega \)**

Define an equivalence relation on the set of euclidean vectors: \( m \equiv n \iff \) there exists \( c \in \mathbb{R} \) such that \( m - n = c\omega \). Let the equivalence class of \( m \) be denoted by \([m]\). Define a symmetric bilinear form \( \bar{B} \) on the resulting quotient space \( V/\omega \) by \( \bar{B}([m],[n]) := B(m,n) \). This is well-defined. For if
\( \tilde{m} \) and \( \tilde{n} \) are two other representatives, then \( \tilde{m} = m + c\omega \) and \( \tilde{n} = n + d\omega \). 

\[ B(\tilde{m}, \tilde{n}) = B(m + c\omega, n + d\omega) = B(m, n) \] since \( \omega \in N \).

**A.3.2 An “ideal” inner product on \( N^1_\wedge \)**

Furthermore, \( m \equiv n \iff \Pi(m) = \Pi(n) \). (\( \rightarrow \)): If \( m \equiv n \), then \( m = n + c\omega \) and \( \Pi(m) = \Pi(n) + c\Pi(\omega) = \Pi(n) \) since \( \omega \in \ker(\Pi) \). (\( \leftarrow \)): If \( \Pi(m) = \Pi(n) \), then by linearity \( \Pi(m) - \Pi(n) = I(m - n) = 0 \). This means \( m - n \in \ker(\Pi) \).

Hence, by the exercises above, \( m - n = c\omega \) for \( c \in \mathbb{R} \). Thus \( \tilde{\Pi} : V/\omega \rightarrow N^1_\wedge \) defined by \( \tilde{\Pi}([m]) := \Pi(m) \) is well-defined. In fact we have showed that it is a bijection and hence has a well-defined inverse.

Use this inverse to transfer the inner product \( \tilde{B}([m], [n]) \) onto the ideal bivectors via

\[ \langle U, W \rangle_\infty := \tilde{B}(\tilde{\Pi}^{-1}(U), \tilde{\Pi}^{-1}(W)) \]

It’s not hard to show that \( \langle , \rangle_\infty \) is the standard positive definite inner product on \( N^1_\wedge \) (since we began with the signature \( (2, 0, 1) \)). This induces a norm on ideal bivectors by \( \|U\|_\infty := \sqrt{\langle U, U \rangle_\infty} \). It is always possible to choose a representative for \( U \) so that \( \|U\|_\infty = 1 \).

**A.4 Recreating the inner product \( (2, 0, 1) \) inner product from the ideal inner product**

It’s tempting to view the ideal norm \( \langle , \rangle_\infty \) as something \textit{ad hoc} added on to the algebra \( P(\mathbb{R}^*_2, 0, 1) \). However, the above discussion supports the contrary interpretation that the ideal inner product \( \langle , \rangle_\infty \) on ideal bivectors is the \textit{primary} structure from which the inner product \( (2, 0, 1) \) on vectors is derived, rather than vice-versa. For, let \( a \) and \( b \) be two euclidean vectors, and \( A := a \wedge \omega \) and \( B := b \wedge \omega \) be their wedge product with the ideal vector. Then define a symmetric bilinear form \( \hat{B}(a, b) := \langle A, B \rangle_\infty \). From the above discussion it is clear that \( \hat{B} \) is well-defined, and in fact, \( \hat{B} = B \). So one can begin with the ideal bivector subspace \( N^1_\wedge \) equipped with the signature \( (2, 0, 0) \) and “push” it in this straightforward way onto the euclidean 1-vectors to obtain the euclidean plane. Similar constructions work for any dimension.

**A.5 Interpretation with respect to \( P(\mathbb{R}^*_2, 0, 1) \)**

The above treatment has been carried out for an abstract real vector space \( V \) of dimension 3. To arrive at the algebra \( P(\mathbb{R}^*_2, 0, 1) \) one must specify \( V \), as outlined in Sect. 2 above, which leads to the choice \( V := (\mathbb{R}^3)^* \), the
dual space of $\mathbb{R}^3$. In the resulting geometric algebra, 1-vectors represents oriented planes through the origin and 2-vectors represent standard vectors. In the second step, the algebra has to be projectivized. Hence, 1-vectors transform to lines and 2-vectors become points. In particular, $\omega$ represents a plane in $(\mathbb{R}^3)^*$, and when projectivized represents a line, the ideal line of the euclidean plane. The ideal bivectors are ideal points, incident with $\omega$. Interpreting the contents of Sect. A.2.2 in this light: the difference $P - Q$, for normalized euclidean points $P$ and $Q$, is an ideal point. This is reminiscent of how free vectors are defined to be the difference of two euclidean points. In fact, ideal points are equivalent to free vectors, an insight already made by Clifford in [Cli73], so that the vector algebra $\mathbb{R}_{2,0,0}$ is contained here as the ideal line with its ideal inner product.

The equivalence classes of $V/\omega$, in the context of $\mathbb{P}(\mathbb{R}_{2,0,1}^*)$, are families of parallel lines, which share a common ideal point. Such a set of lines is known as a line pencil in classical projective geometry; in this case the pencil is centered on (or carried by) an ideal point. To see this: $m \equiv n \iff m - n = \omega$. The point $U := m \land n$ satisfies $U \land m = U \land n = 0$. Hence $U \land \omega = 0$, which shows that $U$ is ideal, as claimed. The metric polarity $\tilde{\Pi}$, in this context, maps an equivalence class $[m]$ to an ideal point perpendicular to the ideal point $U$. It maps all euclidean points (2-vectors) to the ideal line.

Equipped with this coordinate-free foundation of the algebra, the reader can now “rejoin” the article at Sect. 3.

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