Analytical calculations of ground and excited states of unequal mass heavy pseudoscalar and vector mesons mass spectra using Bethe-Salpeter formalism

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Abstract

Considering the fact that, a wide range of variation in masses of various states of heavy pseudoscalar and vector mesons can be seen in different models and the experimental data on masses of many of these states is not yet currently available, in this paper, we study the analytical calculation of heavy pseudoscalar and vector mesons mass spectra (such as $\eta_c$, $B_c$, $\eta_b$, $J/\psi$, $B_c^*$ and $\Upsilon$) and their corresponding wave functions in the framework of Bethe-Salpeter equation under the covariant instantaneous ansatz. The parameters of the framework were determined by a fit to the heavy pseudoscalar and vector mesons mass spectrum of ground states. These input parameters so fixed were found to give good agreements with data on mass spectra of ground and excited states of these mesons mass spectra.

1 Introduction

For a long time, a number of studies dealing with heavy pseudoscalar and vector mesons mass spectra at quark level of compositeness have been carried out to understand the internal structure and dynamics of hadrons. Theoretically, our knowledge of hadron physics is mainly based on phenomenological quark confinement models, as the hadron domain generally falls in the non-perturbative regime of QCD. The analytical calculation of mass spectrum and the corresponding wave functions are very important topic for heavy pseudoscalar and vector mesons. Through these studies, one may have insight of the heavy
mesons and lead to better understanding of QCD. Thus, a realistic description of pseudoscalar and vector mesons mass spectra framework at the quark level of compositeness would be an important element in our understanding of hadron dynamics and internal structures. A number of approaches such as Lattice QCD [1, 2], Chiral perturbation theory [3], heavy quark effective theory [4], QCD sum rules [5, 6], N.R.QCD [7, 8], dynamical-equation based approaches like Schwinger-Dyson equation and Bethe-Salpeter equation (BSE) [9, 10, 11, 12, 13, 14, 15], and potential models [16, 17] have been proposed to deal with the long distance property of QCD. However, since the task of calculating hadron structures from QCD itself is very difficult, as can be seen from various Lattice QCD approaches, one generally relies on specific models to gain some understanding of QCD at low energies.

Thus, the mass spectrum and the corresponding wave functions of the heavy quark bound state system have been studied in the context of many QCD-motivated models. These model predictions give a wide range of variations. Relativistic quark models are important in hadronic physics. A relativistic framework for analyzing mesons as composite objects is provided by Bethe-Salpeter Equation (BSE), which is fully rooted in field theory and is a conventional non-perturbative approach in dealing with relativistic bound state problems in QCD. It provides a realistic description for analyzing inner structure of hadrons, which is also crucial in treating hadronic decays and the decay constants [18, 19, 20] and is a direct source of information on the success of any model.

In this paper we will calculate the mass spectra and the corresponding wave functions of heavy pseudoscalar and vector mesons for ground and excited states, such as 1S, 2S, 3S and 4S for ηc, Bc and ηb mesons and 1S, 2S, 1D, 3S 2D, 4S and 3D for J/ψ, Bc* and Υ mesons which are composites of c and b quarks, in the framework of BSE under Covariant Instantaneous Ansatz (CIA) [21]. This work is a generalization of our previous work [19] (involving equal mass quarkonia) to calculation of mass spectra of unequal mass quarkonia. We have made use of unequal mass kinematics, using the Wightmann Garding definition of masses [10, 21]. This effectively increases the number of number of states of mesons studied. The correctness of our generalization to unequal mass quarks can be gauged by the fact that under equal mass condition (m1 = m2), all our equations reduce to the corresponding equations for equal mass quarkonia such as, ηc, ηb, J/Ψ, and Υ in [19].

To derive the mass spectral equation and their solutions, i.e. the mass spectrum and the wave functions, we have to start with the most general structure of Bethe-Salpeter wave function for pseudoscalar ($J^{PC} = 0^{-+}$ state) and vector ($J^{PC} = 1^{--}$ state) mesons [9]. We then put the formulated BS wave functions into the instantaneous BSE (Salpeter equations) and turn the equation into a set of proper coupled equations for the components which appear in the formulation. These equations are then explicitly shown to decouple in the heavy-quark approximation, and are reduced to a single mass spectral equation for any given state,
whose analytic solutions in an approximate harmonic oscillator basis yield the mass spectrum and wave functions for ground and excited states of pseudoscalar and vector mesons, $\eta_c, B_c, \eta_b, J/\psi, B_c^*,$ and $\Upsilon$. We have explicitly shown that in the heavy quark approximation (valid for heavy quarkonia), these equations can be decoupled, and finally analytical solutions (both mass spectrum and eigenfunctions) of this equation can be obtained using approximate harmonic oscillator basis.

Our main focus in this paper was to show that this problem of $4 \times 4$ BSE under heavy quark approximation can indeed be handled analytically for both masses as well as the wave functions. With the above approximation, that is quite justifiable and well under control, in the context of heavy quark systems, we have been able to give analytical solutions of mass spectral equations for heavy pseudoscalar and vector mesons, giving us a much deeper insight into this problem. Analytical forms of the wave functions for $S$ and $D$ state for $n = 0, 1, 2$ for $c\bar{c}$, $c\bar{b}$ and $b\bar{b}$ systems thus obtained. The analytical forms of eigenfunctions for ground and excited states so obtained will be used to evaluate many transition amplitudes of these mesons. In this paper, we are focusing on the analytical calculation of the ground and excited states of unequal mass heavy pseudoscalar and vector mesons spectroscopy using Bethe-Salpeter equation and our future study will be the calculation of transition amplitudes involving these mesons. We have also plotted the graphs of all the wave functions for the states, $1S, ..., 2D$. The over all features of all these plots show that the states $nS$ have $n - 1$ nodes in their wave functions, and are very similar to the corresponding plots of wave functions in [13], obtained by purely numerical methods, which show the correctness of our approach.

This paper is organized as follows: In section 2, we give the details of our framework, and the derivation of mass spectral equations using the BS wave functions of heavy pseudoscalar and vector mesons. Finally, discussions and conclusion are given in section 3.

2 Formulation of the mass spectral equation under BSE

We start from the derivation of Salpeter equations in this section, giving only the main steps. The 4D BSE for $q\bar{q}$ comprising of quarks-antiquark of momenta $p_{1,2}$, and masses $m_1$ and $m_2$ respectively is written in $4 \times 4$ representation as:

\[
(2\pi)^4 \left( \frac{p_1}{m_1} - m_1 \right) \Psi(P, q) \left( \frac{p_2}{m_2} + m_2 \right) = i \int d^4q' K(q, q') \Psi(P, q')
\]

where $\Psi(P, q)$ is the 4D BS wave function, $K(q, q')$ is the interaction kernel between the quark and antiquark, and $p_{1,2}$ are the momenta of the quark and anti-quark, which are related to the internal 4-momentum.
\( q \) and total momentum \( P \) of hadron of mass \( M \) as,

\[
p_{1,2\mu} = \hat{m}_{1,2} P_\mu \pm q_\mu
\]  
\[(2)\]

where

\[
\hat{m}_{1,2} = \frac{1}{2} \left( 1 \pm \frac{m_1^2 - m_2^2}{M^2} \right)
\]  
\[(3)\]

are the Wightman-Garding (WG) \[10, 21\] definitions of masses of individual quarks which ensure that \( P \cdot q = 0 \) on the mass shells of either quarks, even when \( m_1 \neq m_2 \). These WG definitions of masses also ensure an effective partitioning of internal momentum for unequal mass quarks in the hadron. We decompose the internal momentum, \( q_\mu \) as the sum of its transverse component, \( \hat{q}_\mu = q_\mu - (q \cdot P) P_\mu / P^2 \) (which is orthogonal to total hadron momentum \( P_\mu \)), and the longitudinal component, \( \sigma P_\mu = (q \cdot P) P_\mu / P^2 \), (which is parallel to \( P_\mu \)). Thus, \( q_\mu = (M\sigma, \hat{q}) \), where the transverse component, \( \hat{q} \) is an effective 3D vector, while the longitudinal component, \( M\sigma \) plays the role of the time component. The 4-D volume element in this decomposition is, \( d^4q = d^3\hat{q}M d\sigma \). To obtain the 3D BSE and the hadron-quark vertex, use an Ansatz on the BS kernel \( K \) in Eq. (1) which is assumed to depend on the 3D variables \( \hat{q}_\mu, \hat{q}'_\mu \) as,

\[
K(q, q') = K(\hat{q}, \hat{q}')
\]  
\[(4)\]

Hence, the longitudinal component, \( M\sigma \) of \( q_\mu \), does not appear in the form \( K(\hat{q}, \hat{q}') \) of the kernel and we define 3D wave function \( \psi(\hat{q}) \) as:

\[
\psi(\hat{q}) = \frac{i}{2\pi} \int M d\sigma \Psi(P, q)
\]  
\[(5)\]

Substituting Eq.(5) in Eq.(1), with definition of kernel in Eq.(4), we get a covariant version of Salpeter equation,

\[
(2\pi)^3 \left( \hat{\phi}_1 - m_1 \right) \Psi(P, q) \left( \hat{\phi}_2 + m_2 \right) = \int d^3\hat{q}' K(\hat{q}, \hat{q}') \psi(\hat{q}')
\]  
\[(6)\]

and the 4D BS wave function can be written as as,

\[
\Psi(P, q) = S_F(p_1)\Gamma(\hat{q}) S_F(-p_2)
\]  
\[(7)\]

where

\[
\Gamma(\hat{q}) = \int \frac{d^3\hat{q}'}{(2\pi)^3} K(\hat{q}, \hat{q}') \psi(\hat{q}')
\]  
\[(8)\]

plays the role of hadron-quark vertex function. Following a sequence of steps given in \[18, 19\], we obtain four Salpeter equations:

\[
(M - \omega_1 - \omega_2)\psi^{++}(\hat{q}) = -\Lambda_{1+}^{+}(\hat{q}) \Gamma(\hat{q}) \Lambda_{2+}^{+}(\hat{q})
\]

\[
(M + 2\omega_1 + \omega_2)\psi^{--}(\hat{q}) = \Lambda_{1-}^{-}(\hat{q}) \Gamma(\hat{q}) \Lambda_{2-}^{-}(\hat{q})
\]

\[
\psi^{+-}(\hat{q}) = \psi^{-+}(\hat{q}) = 0
\]  
\[(9)\]
with the energy projection operators, \( \Lambda_j^\pm (\hat{q}) = \frac{1}{2\omega_j} \left[ \frac{P_j}{M} \pm I(j)(m_j + \hat{q}) \right] \), \( \omega_j^2 = m_j^2 + \hat{q}^2 \), and \( I(j) = (-1)^{j+1} \) where \( j = 1, 2 \) for quarks and anti-quarks respectively. The projected wave functions, \( \psi^{\pm \pm} (\hat{q}) \) in Salpeter equations are obtained by the operation of the above projection operators on \( \psi (\hat{q}) \) (for details see \([18, 19]\) as,

\[
\psi^{\pm \pm} (\hat{q}) = \Lambda_1^\pm (\hat{q}) \frac{P}{M} \psi (\hat{q}) \frac{P}{M} \Lambda_2^\pm (\hat{q})
\]

To obtain the mass spectral equation, we have to start with the above four Salpeter equations and solve the instantaneous Bethe Salpeter equation. However, the last two equations do not contain eigenvalue \( M \), and are thus employed to obtain constraint conditions on the Bethe - Salpeter amplitudes associated with various Dirac structures in \( \psi (\hat{q}) \), as shown in details in \([19]\). The framework is quite general so far. In fact the above four equations constitute an eigenvalue problem that should lead to evaluation of heavy pseudoscalar and vector mesons mass spectrum, such as \( \eta_c, B_c, \eta_b, J/\psi, B^*_c \) and \( \Upsilon \).

We now show the method for calculation of heavy pseudoscalar and vector mesons mass spectrum. We first write down the most general formulation of the relativistic BS wave functions, according to the total angular momentum (J), parity (P) and charge conjugation(C) of the concerned bound state. We then put this BS wave function in Eq.(9) to derive the mass spectral equations, which are a set of coupled equations.

1. For pseudoscalar mesons, the complete decomposition of 4D BS wave function in terms of scalar functions \( \phi_i (P, q) \) is \([13, 19]\):

\[
\Psi_P (P, q) \approx \left\{ \phi_1 (P, q) + \bar{P} \phi_2 (P, q) + \bar{q} \phi_3 (P, q) + [\bar{P}, \bar{q}] \phi_4 (P, q) \right\} \gamma_5
\]

Following \([13, 18, 19]\), we write the general decomposition of the instantaneous BS wave function for pseudoscalar mesons \( (J^{PC} = 0^{-+}) \), in the center of mass frame as:

\[
\psi_P (\hat{q}) \approx \left\{ M \phi_1 (\hat{q}) + \bar{P} \phi_2 (\hat{q}) + \bar{q} \phi_3 (\hat{q}) + \frac{\bar{P} \hat{q}}{M} \phi_4 (\hat{q}) \right\} \gamma_5
\]

where \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \) are even functions of \( \hat{q} \) and \( M \) is the mass of the bound state. Now lets come to derive the mass spectral coupled equations from Eq.(9). Putting Eq.(12) into the last two equations of Eq.(9):

\[
\psi^{+-} (\hat{q}) = \psi^{-+} (\hat{q}) = 0
\]

One can obtain:

\[
\phi_3 = \frac{\phi_1 M (-\omega_1 + \omega_2)}{\omega_1 m_2 + \omega_2 m_1} ; \phi_4 = -\frac{\phi_2 M (\omega_1 + \omega_2)}{\omega_1 m_2 + \omega_2 m_1}
\]

So we can apply Eq.(14) into Eq.(12) and rewrite the relativistic wave function of \( 0^{-+} \) as:

\[
\psi_P (\hat{q}) \approx \left\{ \left( M + \frac{\hat{q} M (\omega_2 - \omega_1)}{m_2 \omega_1 + m_1 \omega_2} \right) \phi_1 (\hat{q}) + \left( \bar{P} \phi_2 (\hat{q}) + \frac{\bar{q} \bar{P} (\omega_2 + \omega_1)}{m_2 \omega_1 + m_1 \omega_2} \right) \phi_2 (\hat{q}) \right\} \gamma_5
\]
where, we have been able to express the instantaneous wave function $\psi_P(\hat{q})$ in terms of only the leading Dirac structures associated with amplitudes $\phi_1$ and $\phi_2$. Put the wave function Eq.(15) into the first two Salpeter equations of Eq.(9) and by evaluating traces of $\gamma$-matrices on both sides, we obtain the independent BS coupled integral equations:

\[
(M - \omega_1 - \omega_2)[H_1\phi_1 + H_2\phi_2] = \int \frac{d^3\hat{q}'}{(2\pi)^3} K(\hat{q}, \hat{q}') [H'_1\phi'_1 + H'_2\phi'_2]
\]
\[
(M + \omega_1 + \omega_2)[H_1\phi_1 - H_2\phi_2] = -\int \frac{d^3\hat{q}'}{(2\pi)^3} K(\hat{q}, \hat{q}') [H'_1\phi'_1 - H'_2\phi'_2]
\]

where

\[
H_1 = M(\omega_1\omega_2 + m_1m_2 - \hat{q}^2) + A(m_1 - m_2)\hat{q}^2
\]
\[
H_2 = M \left[\omega_1m_2 + m_1\omega_2 - B(\omega_1 + \omega_2)\hat{q}^2\right]
\]
\[
H'_1 = M(\omega_1\omega_2 + m_1m_2 - \hat{q}^2) - A'(m_1 - m_2)\hat{q}\hat{q}'
\]
\[
H'_2 = M \left[\omega_1m_2 + m_1\omega_2 - B'(\omega_1 + \omega_2)\hat{q}\hat{q}'\right]
\]
\[
A = \frac{M(\omega_2 - \omega_1)}{\omega_1m_2 + \omega_2m_1}
\]
\[
B = \frac{\omega_1 + \omega_2}{\omega_1m_2 + \omega_2m_1}
\]
\[
A' = \frac{M(\omega'_2 - \omega'_1)}{\omega'_1m_2 + \omega'_2m_1}
\]
\[
B' = \frac{\omega'_1 + \omega'_2}{\omega'_1m_2 + \omega'_2m_1}
\]
\[
\omega_{1,2}^2 = m_{1,2}^2 + \hat{q}^2
\]
\[
(17)
\]

The correctness of these equations can be gauged by the fact that under equal mass condition ($m_1 = m_2$), these equations reduce to the corresponding equal mass equations, Eq.(21) for equal mass pseudoscalar quarkonia $\eta_c$, and $\eta_b$ in [19].

2. **For vector mesons**, the complete decomposition of 4D BS wave function in terms of various Dirac structures is [9, 19]:

\[
\Psi_V(P, q) \approx \ell X_1 + \ell P X_2 + [q \cdot \epsilon - \ell \hat{q}]X_3 + [2q \cdot \epsilon P + \ell (P\hat{q} - \hat{q}P)]X_4
\]
\[
(q \cdot \epsilon)X_5 + (q \cdot \epsilon)\hat{P} X_6 + (q \cdot \epsilon)\hat{q} X_7 + (q \cdot \epsilon)(\hat{P}\hat{q} - \hat{q}\hat{P})X_8
\]

\[
(18)
\]

where $\chi_\alpha = \chi_\alpha(q^2, q.P, P^2); (\alpha = 1, 2, 3, ..., 8)$ are the Lorentz scalar amplitudes multiplying the various Dirac structures in the BS wave function, $\Psi(P, q)$. We can write the instantaneous BS wave
function $\psi^V(\hat{q})$ with dimensionality $M$ for vector quarkonia, ($J^{PC} = 1^{--}$) up to sub-leading order as per the power counting scheme [21] as in case of pseudoscalar mesons, in the center of mass frame as [13, 19]:

$$\psi^V(\hat{q}) \approx M\hat{\epsilon}\chi_1 + \epsilon\hat{P}\chi_2 + [\epsilon\hat{\epsilon} - \hat{q}\epsilon]\chi_3 + [\hat{P}\epsilon\hat{\epsilon} - (\hat{q}\epsilon)\hat{P}]\frac{1}{M}\chi_4 + (\hat{q}\epsilon)\chi_5$$

$$+ (\hat{q}\epsilon)\hat{P}\chi_6$$

(19)

where $\chi_1, ..., \chi_6$ are even functions of $\hat{q}$ and $M$ is the mass of the bound state (of the corresponding meson). We now derive the mass spectral coupled equations from Eq.(9). Putting Eq.(19) into the last two equations of Eq.(9), and we obtain the independent constraints on the components for the Instantaneous BS wave function:

$$\chi_5 = \frac{M(\omega_1 + \omega_2)\chi_1}{(m_2\omega_1 + m_1\omega_2)}; \chi_4 = -\frac{M(\omega_1 + \omega_2)\chi_2}{(m_2\omega_1 + m_1\omega_2)}$$

(20)

with the heavy quark approximation, $\hat{q} \ll P$, the terms $\chi_6$ and $\chi_3$ have minimum contributions. Then, applying the constraints in Eq.(20) to Eq.(19), we can rewrite the relativistic wave function of state ($J^{PC} = 1^{--}$) as:

$$\psi^V(\hat{q}) \approx M\left[\hat{\epsilon} + \frac{(\omega_1 + \omega_2)}{(m_2\omega_1 + m_1\omega_2)}\hat{q}\epsilon\right]\chi_1(\hat{q}) + \left[\hat{P}\epsilon - (\hat{P}\epsilon\hat{\epsilon} + \hat{P}\hat{q}\epsilon)\frac{(\omega_1 + \omega_2)}{(m_2\omega_1 + m_1\omega_2)}\right]\chi_2(\hat{q})$$

(21)

where, we have been able to express the instantaneous wave function $\psi^V(\hat{q})$ in terms of only the leading Dirac structures associated with amplitudes $\chi_1$ and $\chi_2$. Putting the wave function Eq.(21) into the first two Salpeter equations of Eq.(9) and by evaluating trace over the $\gamma$-matrices on both sides, we can obtain two independent BS coupled integral equations:

$$(M - \omega_1 - \omega_2)(N_1\chi_1 + N_2\chi_2) = -\int \frac{d^3\hat{q}}{(2\pi)^3}K(\hat{q}, \hat{q}')[N_1'\chi_1' + N_2'\chi_2']$$

$$(M + \omega_1 + \omega_2)(N_1\chi_1 - N_2\chi_2) = \int \frac{d^3\hat{q}}{(2\pi)^3}K(\hat{q}, \hat{q}')[N_1'\chi_1' - N_2'\chi_2']$$

(22)

where

$$N_1 = \left[\omega_1\omega_2A + m_1 + m_2 - A\hat{q}^2\right]\hat{q}\epsilon$$

$$N_2 = \left[2A\omega_1m_2 - \omega_1 - 2A\omega_2m_1 - m_1m_2A - \omega_2\right]\hat{q}\epsilon$$

$$N_1' = (\omega_1\omega_2A' - m_1m_2A' - A'\hat{q}^2)\hat{q}\epsilon - (m_1 + m_2)\hat{q}\epsilon$$

$$N_2' = (2A'\omega_1m_2 - 2A'\omega_2m_1)\hat{q}\epsilon - (\omega_1 - \omega_2)\hat{q}\epsilon$$

$$A = \frac{\omega_1 + \omega_2}{\omega_1m_2 + \omega_2m_1}$$

$$A' = \frac{\omega_1' + \omega_2'}{\omega_1'm_2 + \omega_2'm_1}$$

$$\omega_{1,2}^2 = m_{1,2}^2 + \hat{q}^2$$

$$\omega_{1,2}'^2 = m_{1,2}'^2 + \hat{q}^2$$

(23)
These above two equations also reduce to the corresponding coupled equations for vector quarkonia in \cite{19} under equal mass condition. For the solution of Eq.(16) and Eq.(22), we briefly mention some features of the BS formulation employed. We now introduce the BS kernel, $K(q,q')$ \cite{10, 21}, which is taken to be one-gluon-exchange like as regards the spin dependence ($\gamma_\mu \otimes \gamma_\mu$), and color dependence, $(\frac{1}{2} \vec{\lambda}_1, \frac{1}{2} \vec{\lambda}_2)$, and has a scalar part $V$:

$$K(q,q') = (\frac{1}{2} \vec{\lambda}_1, \frac{1}{2} \vec{\lambda}_2)(\gamma_\mu \otimes \gamma_\mu)V(q-q')$$

$$V(q,q') = \frac{3}{4} \omega^2 \int d^3 \vec{r} \left[ r^2 (1 + 4 \hat{m}_1 \hat{m}_2 A_0 M^2 r^2)^{-\frac{1}{2}} - \frac{C_0}{\omega_0^2} \right] e^{i(q-q') \cdot \vec{r}}$$

$$\omega^2 = 4M \hat{m}_1 \hat{m}_2 \omega_0^2 \alpha_s(M^2)$$

$$\alpha_s(M^2) = \frac{12\pi}{33 - 2n_f} \left[ \log \left( \frac{M^2}{\Lambda^2} \right) \right]^{-1}$$

$$\hat{m}_{1,2} = \frac{1}{2} \left[ 1 \pm \left( \frac{m_1^2 - m_2^2}{M^2} \right) \right]$$

$$\kappa = (1 + 4 \hat{m}_1 \hat{m}_2 A_0 M^2 r^2)^{-\frac{1}{2}}$$

$$M_\gamma = \text{Max}(M, m_1 + m_2).$$

(24)

We can write the complete potential as in \cite{19}:

$$K(\hat{q}, \hat{q}') = V(\hat{q}) \delta^3(\hat{q} - \hat{q}')$$

$$V(\hat{q}) = \omega^2 \left[ \kappa \nabla^2 \hat{q} + \frac{C_0}{\omega_0^2} (2\pi)^3 \right]$$

$$\kappa = \left( 1 - 4 \hat{m}_1 \hat{m}_2 A_0 M^2 \nabla^2 \hat{q} \right)^{-1/2}$$

(25)

The potential is purely confining (as in Ref. \cite{14, 19, 22}, and the Martin potential \cite{23} employed for heavy mesons). Here in the expression for $K(\hat{q}, \hat{q}')$, the proportionality of $\omega^2$ is needed to provide a more direct QCD motivation \cite{10} to confinement and the constant term $C_0/\omega_0^2$ is designed to take account of the correct zero point energies. Hence the term $(1 - 4 \hat{m}_1 \hat{m}_2 A_0 M^2 \nabla^2 \hat{q})^{-1/2}$ in the above expression is responsible for effecting a smooth transition from harmonic ($q\bar{q}$) to linear ($QQ$) confinement.

To decouple the mass spectral equations Eq.(16) and Eq.(22), we first add them and subtract the second equations from the first equations. For a kernel expressed as $K(\hat{q} - \hat{q'}) \approx V(\hat{q}) \delta^3(\hat{q} - \hat{q'})$, we get two algebraic equations which are still coupled. Then from one of the two equations so obtained, we eliminate $\phi_1(\hat{q})$ and $\chi_1(\hat{q})$ in terms of $\phi_2(\hat{q})$ and $\chi_2(\hat{q})$, and plug these expressions for $\phi_1(\hat{q})$ and $\chi_1(\hat{q})$ in the second equations of the coupled set so obtained to get a decoupled equations
in $\phi_2(\hat{q})$ and $\chi_2(\hat{q})$ respectively from Eq.(16) and Eq.(22). Similarly, we eliminate $\phi_2(\hat{q})$ and $\chi_2(\hat{q})$ from the second equation of the set of coupled algebraic equations in terms of $\phi_1(\hat{q})$ and $\chi_1(\hat{q})$, and plug these expressions into the first two equations to get a decoupled equations entirely in $\phi_1(\hat{q})$ and $\chi_1(\hat{q})$, respectively from Eq.(16) and Eq.(22). Thus, we get two identical decoupled equations, one entirely in $\phi_1(\hat{q})$ and $\chi_1(\hat{q})$, and the other that is entirely in, $\phi_2(\hat{q})$ and $\chi_2(\hat{q})$ respectively for heavy pseudoscalar and vector mesons. Employing the limit, $\omega_{1,2} \approx m_{1,2}$, the mass spectral equations can be expressed for heavy pseudoscalar mesons as:

$$
\frac{M^2}{4} - \frac{1}{4}(m_1 + m_2)^2 - q^2 \phi_1(\hat{q}) = -2(m_1 + m_2)\omega_{qq}^2 \left[ \kappa \nabla_{q}^2 + \frac{C_0}{\omega_0^2} \right] \phi_1(\hat{q}) - 4\omega_{qq}^4 \left[ \kappa \nabla_{q}^2 + \frac{C_0}{\omega_0^2} \right]^2 \phi_1(\hat{q}) 
$$

and for heavy vector mesons as:

$$
\frac{M^2}{4} - \frac{1}{4}(m_1 + m_2)^2 - q^2 \chi_1(\hat{q}) = -(m_1 + m_2)\omega_{qq}^2 \left[ \kappa \nabla_{q}^2 + \frac{C_0}{\omega_0^2} \right] \chi_1(\hat{q}) + 2\omega_{qq}^4 \left[ \kappa \nabla_{q}^2 + \frac{C_0}{\omega_0^2} \right]^2 \chi_1(\hat{q})
$$

These two decoupled Eq. (26) and Eq. (27), would resemble harmonic oscillator equations, except for second term on the RHS of these equations. It will be shown later from the definition of kernel in Eq.(24), that these second terms are negligible in comparison to the first terms on the RHS, and can be dropped, and these equations would then resemble exact harmonic oscillator equations. However, due to identical nature of these equations, their solutions are written as: $\phi_1(\hat{q}) = \phi_2(\hat{q}) \approx \phi_P(\hat{q})$ for pseudoscalar mesons and $\chi_1(\hat{q}) = \chi_2(\hat{q}) \approx \phi_V(\hat{q})$ for vector mesons, which represent the eigenfunctions of unequal mass heavy pseudoscalar and vector mesons obtained by solving the full Salpeter equation and we can write the wave function for heavy pseudoscalar mesons as:

$$
\psi_p(\hat{q}) \approx \left[ M + \hat{P} + \frac{\hat{q}M(\omega_2 - \omega_1)}{m_2\omega_1 + m_1\omega_2} + \frac{\hat{q}\hat{P}(\omega_2 + \omega_1)}{m_2\omega_1 + m_1\omega_2} \right] \gamma_5 \phi_P(\hat{q})
$$

and the wave function for heavy vector mesons as:

$$
\psi_v(\hat{q}) \approx \left[ M(\hat{\gamma} + \frac{(\omega_1 + \omega_2)}{(m_2\omega_1 + m_1\omega_2)}\hat{q}\cdot\epsilon) + \frac{\hat{q}\hat{P}(\omega_2 + \omega_1)}{m_2\omega_1 + m_1\omega_2} \right] \phi_V(\hat{q})
$$

As mentioned above, we have dropped the second terms of Eq. (26) and Eq. (27), in comparison to the first terms on the RHS of Eq. (26) and Eq. (27). As the coefficients, $\rho_2 = 4\omega_{qq}^4$ and $\rho'_2 = 2\omega_{qq}^4$, respectively for heavy pseudoscalar and vector mesons, associated with second term have a very small contribution ($\leq 1.2556\%$) in comparison to the coefficient $\rho_1 = 2(m_1 + m_2)\omega_{qq}^2$ and $\rho'_1 = (m_1 + m_2)\omega_{qq}^2$ respectively for heavy pseudoscalar and vector mesons, associated with first term, due to $\omega_{qq}^4 << \omega_{qq}^2$ for $\eta_c$, $B_c$, $\eta_b$, $J/\psi$,
The numerical values of these coefficients, and their percentage ratio for heavy pseudoscalar and vector mesons are given in Table 1 below, which justifies this term being dropped or negligible. We left the signs and the units with equations of Eq. (26) and Eq. (27), we have taken only the constant coefficients when we calculate the contributions of the first and second terms of these equations.

|       | $\rho_1$ or $\rho'_1$ | $\rho_2$ or $\rho'_2$ | $\frac{m}{\rho_1}$ or $\frac{\rho'_2}{\rho_1}$% |
|-------|----------------------|----------------------|-----------------------------------------------|
| $c\bar{c}$ | 0.1115               | 0.0014               | 1.2556%                                       |
| $c\bar{b}$ | 0.3188               | 0.0024               | 0.7528%                                       |
| $b\bar{b}$ | 0.9653               | 0.0091               | 0.9427%                                       |

Table 1: Numerical values of coefficients for heavy pseudoscalar and vector mesons and their percentage ratio, along with the corresponding values. The input parameters of our model are: $C_0 = 0.21$, $\omega_0 = 0.15\text{GeV}$, QCD length scale $\Lambda = 0.200\text{GeV}$, $A_0 = 0.01$, and the input quark masses, $m_c = 1.49\text{GeV}$, and $m_b = 5.070\text{GeV}$.

The form of $K(\hat{q}, \hat{q}')$ in Eq.(24) suggests that Eqs. (26) and Eqs. (27) have to be solved numerically. However, to solve these equations analytically, we follow an analytical procedure on lines of Ref.[18, 19], where we treat $\kappa$ as a "correction" factor due to small value of parameter, $A_0$ ($A_0 << 1$), while we work in an approximate harmonic oscillator basis, due to its transparency in bringing out the dependence of mass spectral equations on the total quantum number $N$. (In this connection, we wish to mention that recently, harmonic oscillator basis has also been widely employed to study heavy mesons using a Light-front quark model [24].) The latter is achieved through the effective replacement, $\kappa = (1 - 4\hat{m}_1\hat{m}_2A_0M^2\hat{q}^2)^{-\frac{1}{2}} \Rightarrow (1 + 8\hat{m}_1\hat{m}_2A_0(N + \frac{3}{2})^{-\frac{1}{2}}$, (which is quite valid for heavy $c\bar{c}$, $c\bar{b}$ and $b\bar{b}$ systems). With this, we can reduce Eq. (26) and Eq. (27) to equations of a simple quantum mechanical 3D- harmonic oscillator with coefficients depending on the hadron mass $M$ and total quantum number $N$. The wave function satisfies the 3D BSE for heavy pseudoscalar and vector mesons respectively as given below:

$$
\left[\frac{M^2}{4} - \frac{1}{4}(m_1 + m_2)^2 - \hat{q}^2\right] \phi_P(\hat{q}) = -\beta_P^4 \left[\hat{\nabla}_\hat{q}^2 + \frac{C_0}{\omega_0^2}\sqrt{1 + 8\hat{m}_1\hat{m}_2A_0(N + \frac{3}{2})} \right] \phi_P(\hat{q}) \quad (30)
$$

and

$$
\left[\frac{M^2}{4} - \frac{1}{4}(m_1 + m_2)^2 - \hat{q}^2\right] \phi_V(\hat{q}) = -\beta_V^4 \left[\hat{\nabla}_\hat{q}^2 + \frac{C_0}{\omega_0^2}\sqrt{1 + 8\hat{m}_1\hat{m}_2A_0(N + \frac{3}{2})} \right] \phi_V(\hat{q}) \quad (31)
$$

where the inverse range parameter for heavy pseudoscalar and vector mesons respectively is, $\beta_P = \left[2\frac{(m_1 + m_2)\omega_{\text{q}}^2}{\sqrt{1 + 8\hat{m}_1\hat{m}_2A_0(N + \frac{3}{2})}}\right]^\frac{1}{2}$ and $\beta_V = \left[\frac{(m_1 + m_2)\omega_{\text{q}}^2}{\sqrt{1 + 8\hat{m}_1\hat{m}_2A_0(N + \frac{3}{2})}}\right]^\frac{1}{2}$ and is dependent on the input kernel parameters and contains the dynamical information. The numerical values of inverse range parameters $\beta_P$, and $\beta_V$ for various heavy pseudoscalar and vector mesons in the mass spectrum studied in this paper are listed in...
Table 2: $\beta_P$ and $\beta_V$ values for ground state and excited states of $\eta_c$, $B_c$, $\eta_b$, $J/\psi$, $B_c^*$ and $\Upsilon$ (in GeV units) in present calculation (BSE-CIA).

|       | 1S   | 2S   | 1D  | 3S  | 2D  |
|-------|------|------|-----|-----|-----|
| $\beta_{\eta_c}$ | 0.3314 | 0.3282 | 0.3252 |     |     |
| $\beta_{B_c}$    | 0.5588 | 0.5516 | 0.5447 |     |     |
| $\beta_{\eta_b}$ | 0.9753 | 0.9660 | 0.9572 |     |     |
| $\beta_{J/\psi}$ | 0.2343 | 0.2321 | 0.2321 | 0.2300 | 0.2300 |
| $\beta_{B_c^*}$  | 0.3952 | 0.3900 | 0.3900 | 0.3852 | 0.3852 |
| $\beta_{\Upsilon}$ | 0.6896 | 0.6831 | 0.6831 | 0.6768 | 0.6768 |

The Table 2 below. Solutions of Eq.(30) and Eq.(31), their eigenfunctions and eigenvalues are well known. The analytical solutions of these equations for ground and excited states are given below.

\[
\begin{align*}
\phi_{P(V)}(1S, \hat{q}) &= \frac{1}{\pi^{3/4} \beta^{3/2}} e^{-\frac{\hat{q}^2}{2\beta^2}} \\
\phi_{P(V)}(2S, \hat{q}) &= \left(\frac{3}{2}\right)^{1/2} \frac{1}{\pi^{3/4} \beta^{3/2}} (1 - \frac{2\hat{q}^2}{3\beta^2}) e^{-\frac{\hat{q}^2}{2\beta^2}} \\
\phi_V(1D, \hat{q}) &= \left(\frac{4}{15}\right)^{1/2} \frac{1}{\pi^{3/4} \beta^{3/2}} \hat{q}^2 e^{-\frac{\hat{q}^2}{2\beta^2}} \\
\phi_{P(V)}(3S, \hat{q}) &= \left(\frac{15}{8}\right)^{1/2} \frac{1}{\pi^{3/4} \beta^{3/2}} (1 - \frac{20\hat{q}^2}{15\beta^2} + \frac{4\hat{q}^4}{15\beta^4}) e^{-\frac{\hat{q}^2}{2\beta^2}} \\
\phi_V(2D, \hat{q}) &= \left(\frac{14}{15}\right)^{1/2} \frac{1}{\pi^{3/4} \beta^{3/2}} (1 - \frac{2\hat{q}^2}{7\beta^2}) \hat{q}^2 e^{-\frac{\hat{q}^2}{2\beta^2}}
\end{align*}
\]

(32)

where $\beta = \beta_{P(V)}$. We will use the wave functions for a description of various transition amplitudes of these heavy pseudoscalar and vector mesons in our future work. We now give the plots of these normalized wave functions Vs. $\hat{q}$ (in GeV.) for different states of heavy pseudoscalar and vector mesons (such as composite of $c\bar{c}$, $c\bar{b}$ and $b\bar{b}$) in Fig.1 and in Fig.2, respectively. It can be seen from these plots that the wave functions corresponding to $nS$ and $nD$ states have $n - 1$ nodes.

The mass spectrum equation of ground and excited states for heavy pseudoscalar and vector mesons are written respectively as:

\[
\begin{align*}
\frac{1}{2\beta_P^2} \left[ \frac{M^2}{4} - \frac{1}{4} (m_1 + m_2)^2 + \frac{C_0 \beta_P^4}{\omega_0^2} \sqrt{1 + 8\hat{m}_1 \hat{m}_2 A_0 (N + \frac{3}{2})} \right] &= N + \frac{3}{2}; N = 0, 2, 4, \ldots \quad (33) \\
\frac{1}{2\beta_V^2} \left[ \frac{M^2}{4} - \frac{1}{4} (m_1 + m_2)^2 + \frac{C_0 \beta_V^4}{\omega_0^2} \sqrt{1 + 8\hat{m}_1 \hat{m}_2 A_0 (N + \frac{3}{2})} \right] &= N + \frac{3}{2}; N = 0, 2, 4, \ldots \quad (34)
\end{align*}
\]
where $N = 2n + l = 0, 2, 4, \ldots$ for $n = 0, 1, 2, \ldots$ and the orbital quantum number $l = 0$ for S-state and $l = 2$ for D-state.

**Numerical Results**

The input parameters of our model are: $C_0 = 0.21$, $\omega_0 = 0.15 GeV$, QCD length scale $\Lambda = 0.200 GeV$, $A_0 = 0.01$, and the input quark masses, $m_c = 1.49 GeV$, and $m_b = 5.070 GeV$. The results of mass spectral predictions of heavy pseudoscalar and vector mesons for both ground and excited states with the above set of parameters are given in tables 3 and 4.

## 3 Discussions and Conclusion

In this work, using BSE framework under CIA for heavy quarks, we have done the analytical calculation of the mass spectrum of ground states and excited states of heavy quarks pseudoscalar and vector mesons in approximate harmonic oscillator basis. This work is a generalization of our previous work [19] involving equal mass quarkonia to calculation of mass spectra of unequal mass quarkonia studied in this paper. Further, it can be easily checked that our equations right from the coupled equations for unequal mass pseudoscalar mesons (Eqs.(16-17)) and unequal mass vector mesons (Eqs.(22-23)), all the way up to their mass spectral equations (Eqs.(33-34)) reduce to the corresponding equations in [19] for equal mass...
| State  | BSE-CIA  | Expt. [25]     | PM [28]  | Rel. PM [26] | Rel. QM [27] |
|--------|----------|----------------|----------|--------------|--------------|
| $M_{n_c(1S)}$ | 2.9489  | 2.983±0.0007 | 2.985    | 2.980        |              |
| $M_{n_c(2S)}$ | 3.7296  | 3.639±0.0013 | 3.626    | 3.589        |              |
| $M_{n_c(3S)}$ | 4.3623  | 4.047         |          |              |              |
| $M_{B_c(1S)}$ | 6.1611  | 6.2756±0.0011 | 6.256    | 6.270        | 6.271        |
| $M_{B_c(2S)}$ | 6.8442  | 6.929         | 6.835    | 6.855        |              |
| $M_{B_c(3S)}$ | 7.4501  | 7.308         | 7.193    |              | 7250         |
| $M_{b(1S)}$  | 8.8590  | 9.398 ±0.0032 | 9.425    | 9.314        |              |
| $M_{b(2S)}$  | 9.6866  | 9.999±0.0028 | 10.012   | 9.931        |              |
| $M_{b(3S)}$  | 10.4354 | 10.319        | 10.288   |              |              |

Table 3: Mass spectrum for ground and excited states of heavy pseudoscalar mesons (in GeV units) in present calculation (BSE-CIA).

| State  | BSE-CIA  | Expt. [25]     | PM [28]  | Rel. PM [26] | Rel. QM [27] |
|--------|----------|----------------|----------|--------------|--------------|
| $M_{J/ψ(1S)}$ | 3.1003  | 3.0969± 0.000011 | 3.096    | 3.097        |              |
| $M_{ψ(2S)}$  | 3.6468  | 3.6861± 0.00034 | 3.690    | 3.686        |              |
| $M_{ψ(1D)}$  | 3.6468  | 3.773± 0.00033  | 3.814    |              |              |
| $M_{ψ(3S)}$  | 4.1134  | 4.03± 0.001     | 4.082    |              |              |
| $M_{ψ(2D)}$  | 3.1134  | 4.191±0.005     |          |              |              |
| $M_{B^*_c(1S)}$ | 6.4718  | 6.314          | 6.332    | 6.338        |              |
| $M_{B^*_c(2S)}$ | 6.9381  | 6.968          | 6.881    | 6.887        |              |
| $M_{B^*_c(1D)}$ | 6.9381  | 7.072          | 7.028    |              |              |
| $M_{B^*_c(3S)}$ | 7.3643  | 7.326          | 7.235    | 7.272        |              |
| $M_{B^*_c(2D)}$ | 7.3643  | 7.365          |          |              |              |
| $M_{Τ(1S)}$  | 9.6476  | 9.4603± 0.00026 | 9.461    | 9.460        |              |
| $M_{Τ(2S)}$  | 10.1944 | 10.0233±0.00031 | 10.027   | 10.023       |              |
| $M_{Τ(1D)}$  | 10.1944 | 10.147         |          |              |              |
| $M_{Τ(3S)}$  | 10.7043 | 10.3552±0.00005 | 10.329   | 10.355       |              |
| $M_{Τ(2D)}$  | 10.7043 | 10.442         |          |              |              |

Table 4: Mass spectrum for ground and excited states of heavy vector mesons (in GeV units) in present calculation (BSE-CIA).
It is to be noted that the coupled integral equations one obtains by solving the 4X4 BSE for unequal mass mesons are quite complicated, which in most works is solved only numerically [13], but in this study, we have attempted to solve the coupled integral equations analytically, and we have explicitly shown that in the heavy quark approximation (valid for quarkonium systems), these equations can be decoupled, and finally analytical solutions (both mass spectrum and eigenfunctions) of these equation can be obtained using approximate harmonic oscillator basis.

All numerical calculations have been done using Math lab. We first fit our parameters to the ground state masses of $\eta_c$, $B_c$, $\eta_b$, $J/\psi$, $B_c^*$ and $\Upsilon$ with data. Using the input parameters, we obtained the best fit to these ground state masses. With the same set of parameters, we calculate the masses of all the other excited states of $\eta_c$, $B_c$, $\eta_b$, $J/\psi$, $B_c^*$ and $\Upsilon$. These input parameters of our model are the same with our recent papers [19, 20]. The calculated mass spectrum in the framework of BSE are listed in Tables 3 and 4. The results obtained for masses of ground and excited states of $\eta_c$, $B_c$, $\eta_b$, $J/\psi$, $B_c^*$ and $\Upsilon$ are in reasonable agreement with experiment. However, a wide range of variation in masses of various states can be seen in Tables 3 and 4 and the experimental data on masses of some of these states is not yet currently available.

From the tables, one can read out for vector mesons, there is a degeneracy in the masses of $S$ and $D$ states with the same principal quantum number $N$ for $J/\Psi$, $B_c^*$ and $\Upsilon$. This is due to the fact that we did not incorporate the one-gluon-exchange (OGE) effects in the kernel, and used only the confining part.
of interaction taking analogy from \cite{14,18,19,22,23} for heavy mesons. However our results also reflect
the fact that the OGE term becomes more and more important for very heavy mesons and the inclusion
of OGE terms in the potential will lift up the degeneracy in these states.

However, our main focus was to show that this problem of $4 \times 4$ BSE under heavy quark approximation
can indeed be handled analytically for both masses, as well as the wave functions in an approximate
harmonic oscillator basis. The analytical forms of wave functions are obtained as solutions of mass spectral
equations (that are derived from 3D BSE). Analytical forms of the wave functions for both $S$ and $D$ states
for $n = 0, 1, 2, ..., \alpha \pi$, $c \overline{c}$ and $b \overline{b}$ systems thus obtained are given in Eq.(32). We have also plotted the
graphs of all these wave functions for the states, $1S, ... , 3S$ for $\eta_c$, $B_c$ and $\eta_b$ mesons, and for the states
$1S, 2S, 1D, 3S$ and $2D$ for $J/\psi$, $B_c^*$ and $\Upsilon$ mesons, in Fig. 1 and Fig. 2.

We are not aware of any other BSE framework, involving $4 \times 4$ BS amplitude, that treats the mass
spectral problem involving heavy mesons analytically. To the best of our knowledge, all the other $4 \times 4$
BSE approaches treat this problem numerically just after they obtain the coupled set of equations (see
Ref.\cite{13}). We further wish to mention that the over all features of our plots of wave functions in Eqs.(32)
derived in our framework are very similar to the corresponding plots of wave functions obtained by purely
numerical methods in \cite{13}, suggesting that our approach is not only in good agreement with the numerical
approaches followed in other works, but also gives a deeper understanding of the problem, by showing
an explicit dependence of the mass spectrum on principal quantum number $N$. In the present paper,
we have restricted our study only to the analytical calculation of heavy pseudoscalar and vector mesons
mass spectrum and their corresponding wave functions. We will extend this study to the calculation of
various transition amplitudes of these heavy pseudoscalar and vector mesons in our framework to do as
our further work. It is seen that in this framework, we were able to obtain fairly good values of masses for
their ground as well as their excited states of $\eta_c$, $B_c$, $\eta_b$, $J/\psi$, $B_c^*$ and $\Upsilon$.

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