Fractional Spin through Quantum Affine Algebra \( \hat{A}(n) \) and quantum affine superalgebra \( \hat{A}(n, m) \)

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Abstract

Using the splitting of a \( Q \)-deformed boson, in the \( Q \rightarrow q = e^{\frac{2\pi i}{k}} \) limit, the fractional decomposition of the quantum affine algebra \( \hat{A}(n) \) and the quantum affine superalgebra \( \hat{A}(n, m) \) are found. This decomposition is based on the oscillator representation and can be related to the fractional supersymmetry and \( k \)-fermionic spin. We establish also the equivalence between the quantum affine algebra \( \hat{A}(n) \) and the classical one in the fermionic realization.

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1 Introduction

The concept of quantum group and algebra \([1, 2]\), has extensively entered mathematical and physical literatures. In theoretical physics, quantum groups have received considerable interest due to their connection with quantum Yang-Baxter equation \([3]\) and the quantum inverse scattering method \([4]\). From a mathematical point of view, quantum groups and algebras can be regarded as deformations of the universal enveloping algebras of semi-simple Lie algebras. The quantum analogues of Lie superalgebras has been constructed in \([5, 6]\). Quantized enveloping algebras associated to affine algebras and superalgebras are given in \([1, 7]\). Many properties of quantum groups and algebras are quit similar to or richer than ones of the usual Lie groups and algebras in connection with the representations theory. It is well known that boson realization method is a very powerful and elegant method for the study of quantum algebras representations. Based on this method, the representation theory of quantum affine algebras has been an object of intensive studies; Available are the results for the oscillator representations of affine algebra. There are obtained \([8, 9, 10]\) through consistent realizations involving deformed Bose and Fermi operators \([11, 12]\).

Recently, in connection with quantum group theory, a new geometric interpretation of fractional supersymmetry has been developed in \([13, 14, 15, 16, 17]\). In these works, the authors show that the one-dimensional superspace is isomorphic to the braided line when the deformation parameter goes to a root of unity. Similar techniques are used, in the reference \([18]\), to show how internal spin arises naturally in certain limit of the \(Q\)-deformed angular momentum algebra \(U_Q(sl(2))\). Indeed, using \(Q\)-Schwinger realization, it is shown that the decomposition of the \(U_Q(sl(2))\) into a direct product of not deformed \(U(sl(2))\) and \(U_q(sl(2))\) which is the same version of \(U_Q(sl(2))\) at \(Q = q\). The property of splitting of quantum algebras \(A_n, B_n, C_n\) and \(D_n\) and quantum superalgebra \(A(m, n), B(m, n), C(n + 1)\) and \(D(n, m)\) in the \(Q \rightarrow q\) limit is investigated in \([19]\). The case of deformed Virasoro algebra and some other particular quantum (Super)-algebra is given in \([20]\).

The aim of this paper is to investigate the property of decomposition of the quantum affine algebra \(\hat{A}(n)\) and the quantum affine superalgebra \(\hat{A}(n, m)\) in the \(Q \rightarrow q\) limit. As a first step we wish to present in the next section (section 2) a some results concerning k-fermions. In section 3, we discuss the property of \(Q\)-boson decomposition in the \(Q \rightarrow q\) limit.
We introduce the way in which one obtains two independent objects (an ordinary boson and a $k$-fermion) from one $Q$-deformed boson when $Q$ goes to a root of unity. We establish also the equivalence between a $Q$-deformed fermion and a conventional (ordinary) one. Using these results, we analyse the $Q \to q$ limit of the quantum affine algebra $U_Q(\hat{A}(n))$ (section 4) and the quantum affine superalgebra $\hat{A}(m,n)$ (section 5). We note that Q-oscillator realization is crucial in this paper. Therefore, the results obtained in this work are valid for the oscillator representations. In a last section (section 6) we shall give some concluding remarks.

2 Introducing $k$-fermionic algebra.

The usual starting commutation relations of $q$-deformed bosonic algebra $\Sigma_q$ are

\[
\begin{align*}
    a^- a^+ - qa^+ a^- &= q^{-N} \\
    a^- a^+ - q^{-1} a^+ a^- &= q^{N} \\
    q^N a^+ q^{-N} &= q a^+ \\
    q^N a^- q^{-N} &= q^{-1} a^- \\
    q^N q^{-N} &= q^{-N} q^N = 1
\end{align*}
\]

(1)

where the deformation parameter

\[
    q = exp\left(\frac{2\pi i}{l}\right) \quad l \in N - \{0, 1\}
\]

(2)

is a root of unity. The annihilation operator $a^-$ is hermitic conjugated to creation operator $a^+$. The number operator $N$ is hermitic. From equation (1), it is not difficult to obtain the following relations

\[
\begin{align*}
    a^-(a^+)^n &= [[n]]q^{-N}(a^+)^{n-1} + q^N(a^+)^n a^- \\
    (a^-)^n a^+ &= [[n]](a^+)^{n-1} q^{-N} + q^N a^+(a^-)^n
\end{align*}
\]

(3)

Where the symbol $[[n]]$ is defined by:

\[
    [[n]] = \frac{1 - q^{2n}}{1 - q^2}
\]

(4)
The cases of odd and even values of \( \ell \) have to be treated in slightly different ways. Hence, we introduce a new variable \( k \) defined by

\[
k = \ell \quad \text{for odd values of } \ell \nonumber \\
k = \frac{\ell}{2} \quad \text{for even values of } \ell
\] (5)

such that for odd \( \ell \) (resp. even \( \ell \)), we have \( q^k = 1 \) (resp., \( q^k = -1 \)). In the particular case \( n = k \), equations (3) are amenable to the form

\[
a^-(a^+)^k = \pm (a^+)^ka^- \\
(a^-)^ka^+ = \pm a^+(a^-)^k
\] (6)

In addition, the equation (1) yield

\[
q^N(a^+)^k = (a^+)^kq^N \\
q^N(a^-)^k = (a^-)^kq^N
\] (7)

We point out that the elements \((a^+)^k\) and \((a^-)^k\) are elements of the centre of \( \sum_q \) algebra (odd \( \ell \)). The irreducible representations are \( k \)-dimensional. Due to the fact that the elements \((a^+)^k\) and \((a^-)^k\) are central, if one deals with a \( k \)-dimensional representation, we have

\[
(a^+)^k = \alpha \ I, \quad (a^-)^k = \beta \ I
\] (8)

The extra possibilities parameterised by

\[
(i) \quad \alpha = 0 \quad \beta \neq 0 \\
(ii) \quad \alpha \neq 0 \quad \beta = 0 \\
(iii) \quad \alpha \neq 0 \quad \beta \neq 0
\]

are not relevant for the considerations of this paper. The case (iii) correspond to the periodic representation and in the cases (i) and (ii) we have the so-called semiperiodic (semicyclic) representation. In what follows, we shall deal with a representation of the algebra \( \sum_q \) such that

\[
(a^+)^k = 0 \ , \quad (a^-)^k = 0
\] (9)

are satisfied. We note that the algebra \( \sum_{-1} \) obtained for \( k = 2 \), correspond to ordinary fermion operators with \((a^+)^2 = 0\) and \((a^-)^2 = 0\) which reflects the
Pauli exclusion principle. In the limit case where \( k \rightarrow \infty \), we have the algebra \( \sum_1 \), which correspond to the ordinary boson operators. For \( k \) arbitrary, the algebra \( \sum_q \) correspond to the \( k \)-fermions(or anyons with fractional spin in the sense of Majid [21, 22]) operators that interpolate between fermion and boson operators.

3 Fractional spin through Q-boson.

In the previous section, we have been working with \( q \) a root of unity. When \( q^l = 1 \), quantum oscillator\((k–fermionic)\) algebra exhibit rich representation behaviour with very special properties different from the generic case. In the first case the Hilbert space is finite dimensional, while in the generic case the Fock space is infinite dimensional Hilbert space. Now, let us consider, in order to investigate the decomposition of a \( Q \)-deformed boson \((q \in C)\) in the \( Q \rightarrow exp(\frac{2\pi i}{k}) \), the \( Q \)-deformed algebra \( \Delta_Q \). The algebra \( \Delta_Q \) is generated by an annihilation operator \( B^- \), a creation operator \( B^+ \) and a number operator \( N_B \) with the relations

\[
\begin{align*}
B^- B^+ - Q B^+ B^- &= Q^{-N_B} \\
B^- B^+ - Q^{-1} B^+ B^- &= Q^{N_B} \\
Q^{N_B} B^+ Q^{-N_B} &= Q B^+ \\
Q^{-N_B} B^- Q^{N_B} &= Q^{-1} B^- \\
Q^{N_B} Q^{-N_B} &= Q^{-N_B} Q^{N_B} = 1
\end{align*}
\]

From equation (10), we obtain

\[
\begin{align*}
[Q^{-N_B} B^-, [Q^{-N_B} B^-, \ldots [Q^{-N_B} B^-, (B^+)^k Q^{2k} \ldots ]Q^4]Q^2] &= Q^{k(k+1)/2} [k]! \tag{11}
\end{align*}
\]

Where the \( Q \)-deformed factorial is given by

\[
[k]! = [k][k-1][k-2] \ldots [1]
\]

\[
[0]! = 1
\]

with

\[
[k] = \frac{Q^k - Q^{-k}}{Q^1 - Q^{-1}}
\]
The $Q$–commutator, in (11), of two operators $A$ and $B$ is defined by

$$[A, B]_Q = AB - QBA$$

The aim of this section is to determine the limit of the $\Delta_Q$ algebra when $Q$ goes to the root of unity $q$ see (2). The starting point is the limit $Q \rightarrow q$ of equation (11)

$$\lim_{Q \rightarrow q} k\left[Q - N B, [Q - NB, B^{-}, [\ldots [Q - NB, B^{+} k_{Q^{2k} \ldots} Q^{4}], B^{+}]]\right] = \lim_{Q \rightarrow q} q^{k(k-1)/2} (k/2)!\left[Q^{-NB}(B^{-})^{k}, (B^{+})^{k}\right] = q^{k(k-1)/2}. \quad (12)$$

The equation (12) can be reduced to:

$$\lim_{Q \rightarrow q} \frac{Q^{kNB} (B^{-})^{k}}{([k]!)^{1/2}}, \frac{(B^{+})^{k} Q^{kNB}}{([k]!)^{1/2}} = 1 \quad (13)$$

We note that since $q$ is a root of unity, it is possible to change the sign on the exponent of $q^{kNB}$ terms in the above and in the following definitions when $Q \rightarrow q$.

Following the work[18], we define the operators

$$b^{-} = \lim_{Q \rightarrow q} \frac{Q^{\pm kNB}}{([k]!)^{1/2}} (B^{-})^{k} \quad (14)$$

then we obtain

$$[b^{-}, b^{+}] = 1 \quad (15)$$

Which are nothing but the commutation relation of an ordinary boson.

The number operator of this new bosonic oscillator is defined, in the usual ways, as $N_{b} = b^{+} b^{-}$.

This type of reasoning, concerning the $Q$–limit of $Q$–boson, has been invoked for the first time in the references [13-16,18] in order to investigate the fractional supersymmetry. (In these references, the authors show that there is an isomorphism between the braided line and one dimensional superspace.)
Now, we are in a position to discuss the splitting of $Q-$deformed boson in the $Q \rightarrow q$ limit. Let us introduce the new set of generators given by:

\begin{align}
A^- &= Bq^{-\frac{kN_b}{2}} \\
A^+ &= B^q q^{-\frac{kN_b}{2}} \\
N_A &= N_B - kN_b
\end{align}

(16)

which satisfy the following commutation relations

\begin{align}
[A^-, A^+]_q^{-1} &= q^{N_A} \\
[A^-, A^+]_q &= q^{-N_A} \\
[N_A, A^\pm] &= \pm A^\pm 
\end{align}

(17)

and then define a $k-$fermion. The two algebras generated by the set of operators $\{b^+, b^-, N_b\}$ and $\{A^+, A^-, N_A\}$ are mutually commutative. We thus conclude that in the $Q$ $toq$ limit, the $Q-$deformed bosonic algebra oscillator decomposes into two independents oscillators, an ordinary boson and $k-$fermion.

There is also a natural question which emerges: is it possible to find similar splitting property for $Q-$deformed fermionic operators when the deformation parameter $Q$ reduce to a root of unity $q$ ? To answer to this question, we consider the $Q-$deformed fermionic algebra generated by the operators $F^-, F^+$ and $N_F$ satisfying the following relations

\begin{align}
F^-F^+ + QF^+F^- &= Q^N \\
F^-F^+ + Q^{-1}F^+F^- &= Q^{-N} \\
Q^{N_F}F^+Q^{-N_F} &= Q^{F^+} \\
Q^{N_F}F^-Q^{-N_F} &= Q^{-1}F^- \\
Q^{N_F}Q^{-N_F} &= Q^{-N_F}Q^{N_F} = 1 \\
(F^+)^2 &= 0, (F^-)^2 = 0
\end{align}

(18)

We define the new operators

\begin{align}
F^- &= Q^{-\frac{N_F}{2}}F^- \\
F^+ &= F^+Q^{-\frac{N_F}{2}}
\end{align}

(19)

We obtain by a direct calculation the following anti-commutation relation

\begin{align}
\{f^-, f^+\} &= 1
\end{align}

(20)
Moreover, we have the nilpotency conditions
\[
(f^-)^2 = 0 \\
(f^+)^2 = 0
\]  
(21)

Thus, we see that the $Q$–deformed fermion reproduces the conventional (ordinary) fermion.

4 **Quantum affine algebra** $U_Q(\hat{A}(n))$ **at** $Q$ **a root of unity**

We use now the above results to derive the property of decomposition of quantum affine algebras $U_Q(\hat{A}(n))$ in the $Q \to q$ limit. Recall that the $U_Q(\hat{A}(n))$ algebra is generated by the set of generators \( \{ e_i, f_i, k_i^\pm = Q_i^{\pm h_i}, 0 \leq i \leq n \} \) satisfying the following relations:

\[
[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{Q_i - Q_i^{-1}} \\
k_i e_j k_i^{-1} = Q_i^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = Q_i^{-a_{ij}} f_j \\
k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i
\]  
(22)

and the quantum Serre relations described by the expressions:

\[
\sum_{0 \leq p \leq 1 - a_{ij}} (-1)^p \left[ \begin{array}{c} 1 - a_{ij} \\ p \end{array} \right] Q_i^{1 - a_{ij} - p} e_i^{1 - a_{ij} - p} e_j^p = 0 \\
\sum_{0 \leq p \leq 1 - a_{ij}} (-1)^p \left[ \begin{array}{c} 1 - a_{ij} \\ p \end{array} \right] Q_i f_i^{1 - a_{ij} - p} f_j^p = 0
\]  
(23)

In equations (22), (23) $a_{ij}$ is the ij-element of $n \times n$ generalised Cartan matrix:

\[
\hat{A}_n = \begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & -1 \\
-1 & 2 & -1 & 0 & & \\
0 & -1 & \ddots & \ddots & \ddots & \\
& \vdots & 0 & \ddots & \ddots & 0 \\
& & \ddots & \ddots & \ddots & \ddots \\
-1 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}
\]
and \((d_i)\) are the non-zero integers such that \(d_ia_{ij} = a_{ij}d_i\). The quantity \(\left[ \begin{array}{c} m \\ n \end{array} \right]_{Q_i}\) in equation (22) is defined by:

\[
\left[ \begin{array}{c} m \\ n \end{array} \right]_{Q_i} = \frac{[m]_{Q_i}!}{[m-n]_{Q_i}! [n]_{Q_i}!}
\]

with

\[
[H_i]_{Q_i} = \frac{Q_{i1} - Q_{i1}^{-1}}{Q_i - Q_i^{-1}}
\]

The quantum affine algebra \(U_Q(A_n)\) admits two \(Q\)-oscillator representations: bosonic and fermionic. In the bosonic realization, the generators of \(U_Q(A_n)\) algebra can be constructed, by introducing \((n+1)\) \(Q\)-deformed bosons as follows.

\[
\begin{align*}
e_i &= B_i^-B_{i+1}^+, \quad 1 \leq i \leq n \\
f_i &= B_i^+B_{i+1}^-, \quad 1 \leq i \leq n \\
k_i &= Q^{-N_i+N_{i+1}}, \quad 1 \leq i \leq n \\
e_0 &= B_{n+1}^-B_1^+, \\
f_0 &= B_1^-B_{n+1}^+, \\
k_0 &= Q_{n1-N_{n+1}}
\end{align*}
\]

Equation (25)

The fermionic realization is given by:

\[
\begin{align*}
e_i &= F_i^+F_{i+1}^-, \quad 1 \leq i \leq n \\
f_i &= F_i^-F_{i+1}^+, \quad 1 \leq i \leq n \\
k_i &= Q_{N_i-N_{i+1}}, \quad 1 \leq i \leq n \\
e_0 &= F_{n+1}^+F_1^-, \\
f_0 &= F_1^+F_{n+1}^-, \\
k_0 &= Q^{-N_i+N_{n+1}}
\end{align*}
\]

Equation (26)

At this stage we investigate the limit \(Q \rightarrow q\) of the quantum affine algebra \(U_Q(A_n)\). As already mentioned in the introduction, our analysis is based on the \(Q\)-oscillator representation. Therefore all results obtained are specific to the use of \(Q\)-Schwinger realization. In the \(Q \rightarrow q\), the splitting of \(Q\)-deformed bosons leads to classical bosons \(\{b_i^+, b_i^-, N_{b_i} \quad (1 \leq i \leq n)\}\) given by the equation (24) and \(k\)-fermionic operators \(\{A_i^+, A_i^-, N_{A_i} \quad (1 \leq i \leq n)\}\)
defined by equations (16). From the classical bosons, we define the operators
\[
e_i = b_i^+ b_{i+1}^-, \quad f_i = b_i^- b_{i+1}^+, \quad h_i = -N_{b_i} + N_{b_{i+1}},
\]
for \(i = 1, \ldots, n\) and
\[
e_0 = b_1^- b_{n+1}^+, \quad f_0 = b_1^+ b_{n+1}^-, \quad h_0 = N_{b_1} - N_{b_{n+1}},
\]
(27)

The set \(\{e_i, f_i, k_i, \; 0 \leq i \leq n\}\) generate the classical algebra \(U(\hat{A}(n))\). From the remaining operators \(\{A_i^+, A_i^-, N_{A_i}, \; (1 \leq i \leq n+1)\}\), one can realize the \(U_q(\hat{A}(n))\) algebra. Indeed, the generators defined by
\[
E_i = A_i^- A_{i+1}^+, \quad (1 \leq i \leq n) \quad F_i = A_i^+ A_{i+1}^-, \quad (1 \leq i \leq n) \quad K_i = q^{-N_{A_i}+N_{A_{i+1}}}, \quad (1 \leq i \leq n) \\
E_0 = A_1^- A_{n+1}^+, \quad F_0 = A_1^+ A_{n+1}^-, \quad K_0 = q^{N_{A_1}-N_{A_{n+1}}},
\]
(28)
generate the \(U_q(\hat{A}(n))\) algebra which is the same version of \(U_Q(\hat{A}_n)\) obtained by simply setting \(Q = q\), rather than by taking the limit as above. Due to the commutativity of elements of \(U_q(\hat{A}_n)\) and \(U(\hat{A}_n)\), we obtain the following decomposition of the quantum affine algebra \(U_q(\hat{A}_n)\):
\[
\lim_{Q \to q} U_Q(\hat{A}_n) = U_q(\hat{A}_n) \otimes U(\hat{A}_n).
\]
in the bosonic realization.

To end this section, we discuss the equivalence between \(U_Q(\hat{A}(n))\) and \(U(\hat{A}(n))\) algebras in the fermionic construction. Indeed, We have discussed in the second section how one can identify the conventional fermions with \(Q\)-deformed fermions. There have an equivalence between these two objects. Consequently, due to this equivalence, it is possible to construct \(Q\)-deformed affine algebras \(U_Q(\hat{A}_n)\) using ordinary fermions. It is also possible to construct the affine algebra \(\hat{A}_n\) by considering \(Q\)-deformed fermions. So, in the fermionic realization we have equivalence between \(U(\hat{A}_n)\) and \(U_Q(\hat{A}_n)\). To be more
clear, we consider the $U_Q(\hat{A}_n)$ in the Q-fermionic representation. The generators are given by:

$$
e_i = F^-_i F^+_i, \quad 1 \leq i \leq n$$

$$f_i = F^+_i F^-_{i+1}, \quad 1 \leq i \leq n$$

$$k_i = Q^{N_{F_i}-N_{F_{i+1}}}, \quad 1 \leq i \leq n$$

$$e_0 = F^+_n F^-_1,$$

$$f_0 = F^+_1 F^-_{n+1},$$

$$k_0 = Q^{-N_{F_1}+N_{F_{n+1}}},$$

(30)

due to equivalence fermion - Q-fermion, the operators $f^-_i, f^-_i$ are defined as a constant multiple of conventional fermion operators, i.e,

$$f^-_i = Q^{-N_{F_i}} F^-_i$$

$$f^+_i = F^+_i Q^{-N_{F_i}}$$

(31)

from which we can realize the generators:

$$E_i = f^-_i f^+_i, \quad 1 \leq i \leq n$$

$$F_i = f^+_i f^-_{i+1}, \quad 1 \leq i \leq n$$

$$H_i = N_{f_i} - N_{f_{i+1}}, \quad 1 \leq i \leq n$$

$$E_0 = f^+_n f^-_1,$$

$$F_0 = f^+_1 f^-_{n+1},$$

$$H_0 = - N_{f_1} + N_{f_{n+1}},$$

(32)

The set $\{E_i, F_i, H_i \, , 0 \leq i \leq n\}$ generate the classical affine algebra $U(\hat{A}_n)$.

5 Quantum affine Superalgebra $U_Q(\hat{A}(m, n))$

at $Q$ a root of unity

Let $Q \in C - \{0\}$ be the deformation parameter. we shall use also $Q_i = Q^{d_i}$ with $d_i$ are numbers, that symmetries the Cartan matrix $(a_{ij})$. The quantum affine superalgebra $U_Q(\hat{A}(m, n))$ is described in the Serre-Chevalley basis in terms of the simple root $e_i, f_i$ and Cartan generators $h_i$ where $i = 1, \ldots, m + n - 1$ which satisfy the following super-commutations relations
\[ [e_i, f_j] = \delta_{ij} \frac{Q^{d_i h_i} - Q^{-d_i h_i}}{Q_i - Q_i^{-1}} \]
\[ [h_i, h_j] = 0 \]
\[ [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j \]
\[ [e_i, e_i] = [f_i, f_i] = 0, \quad m_{a_{ii}} = 0 \]

with the bracket \([,]\) is the \(Z_2\)-graded one
\[ [X, Y] = XY - (-1)^{deg(X)deg(Y)} YX. \]

In the equation (33), \((a_{ij})\) is the element of the following Cartan matrix:

\[
\hat{A}(m, n) = \begin{pmatrix}
0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 \\
-1 & 2 & -1 & 0 & & & & & & \\
0 & -1 & \ddots & \ddots & \ddots & & & & & \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & & & & \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
\vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
0 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & -1
\end{pmatrix}
\]

It is convenient to introduce the quantities \(k_i = Q^{d_i h_i}\) in terms of which the defining relations (33) become

\[
[e_i, f_j] = \delta_{ij} \frac{k_i k_j^{-1}}{Q_i - Q_i^{-1}} \]
\[k_i e_j k_i^{-1} = Q^{a_{ij}}_i e_j, \quad k_i f_j k_i^{-1} = Q^{-a_{ij}}_i f_j \]
\[k_i k_j^{-1} = k_j k_i, \quad k_i k_i^{-1} = k_i, \quad k_i k_j = k_j k_i \]
\[ [e_i, e_i] = [f_i, f_i] = 0, \quad \text{if} \quad a_{ii} = 0 \]

Further the quantum affine superalgebra \(U_Q(\hat{A}(m, n))\) generators obey to generalised Serre relations. The latter’s are most simply presented in terms
of the following rescaled generators [23]:

\[ \xi_i = e_i k_i^{-\frac{1}{2}}, \quad \zeta_i = f_i k_i^{-\frac{1}{2}} \]

they then take the form

\[
\begin{align*}
(ad_Q \xi_i)^{1-\tilde{a}_{ij}} \xi_j &= 0, \quad i \neq j, \\
(ad_Q \zeta_i)^{1-\tilde{a}_{ij}} \zeta_j &= 0, \quad i \neq j.
\end{align*}
\]

(35)

where \( \tilde{a}_{ij} \) matrix is obtained from the non-symmetric Cartan matrix \( a_{ij} \) by substituting -1 for the strictly positive elements in the rows with 0 on the diagonal entry. The quantum adjoint action \( ad_Q \) can explicitly written in terms of the coproduct and the antipode as:

\[
(ad_Q X)Y = (-1)^{deg(X(2)).deg(Y)} X(1)Y S(X(2))
\]

with \( \Delta(X) = X(1) \otimes X(2) \). and some supplementary relations for \( i \) such that \( a_{ii} = 0 \).

\[
\begin{align*}
[[e_{i-1}, e_i]_Q, [e_i, e_{i+1}]_Q] &= 0, \\
[[f_{i-1}, f_i]_Q, [f_i, f_{i+1}]_Q] &= 0
\end{align*}
\]

The universal quantum affine Lie superalgebra \( U_Q(\hat{A}(m, n)) \) is endowed with a Hopf superalgebra structure with coproduct:

\[
\begin{align*}
\Delta(k_i) &= k_i \otimes k_i \\
\Delta(e_i) &= e_i \otimes k_i^{-\frac{1}{2}} + k_i^{\frac{1}{2}} \otimes e_i \\
\Delta(f_i) &= f_i \otimes k_i^{-\frac{1}{2}} + k_i^{\frac{1}{2}} \otimes f_i
\end{align*}
\]

counit:

\[
\epsilon(k_i) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0
\]

and antipode :

\[
\begin{align*}
S(k_i) &= -k_i, \quad S(e_i) = -Q^{a_{ii}} e_i, \quad S(f_i) = -Q^{-a_{ii}} f_i
\end{align*}
\]

We shall give now the \( Q \)-oscillator representation of the quantum affine superalgebra \( U_Q(A(m, n)) \). We shall provide explicit expressions for corresponding
generators as linear and bilinear in Q-deformed bosonic and fermionic oscillators. The quantum affine superalgebra $U_Q(\hat{A}(m, n))$ can be realized simply by $(m+1)Q$-deformed fermions and $(n+1)$ Q-bosons. Explicitly the generators of $\hat{A}_Q(m, n)$ are given by:

\[
\begin{align*}
  e_i &= F_i^+ F_{i+1}^-, \quad 1 \leq i \leq m \\
  f_i &= F_i^- F_{i+1}^+, \quad 1 \leq i \leq m \\
  k_i &= Q^{(N_{F_{m+1}} - N_{F_{i+1}})}, \quad 1 \leq i \leq m \\
  e_{m+1} &= F_{m+1}^+ B_1^- \\
  f_{m+1} &= F_{m+1}^- B_1^+ \\
  k_{m+1} &= Q^{(N_{F_{m+1}} + N_{B_1})}
\end{align*}
\]

Due to the property of Q-boson decomposition in the $Q \rightarrow q$ limit, each Q-boson $\{B_i^-, B_i^+, N_{B_i}\}$ reproduce an ordinary boson $\{b_i^-, b_i^+, N_{b_i}\}$ and a k-fermion operator $\{A_i^-, A_i^+, N_{A_i}\}$. In the limit the Q-fermions become q-fermions which are objects equivalents to conventional fermions $\{f_i^-, f_i^+, N_{f_i}\}$.

From the classical bosons $\{b_i^-, b_i^+, N_{b_i}\}$ and conventional fermions $\{f_i^-, f_i^+, N_{f_i}\}$, one can realize the classical affine algebra $U(\hat{A}(m, n))$:

\[
\begin{align*}
  E_i &= F_i^+ f_{i+1}^- \quad 1 \leq i \leq m \\
  F_i &= f_i^- F_{i+1}^+ \quad 1 \leq i \leq m \\
  H_i &= N_{f_i} - N_{f_{i+1}} \quad 1 \leq i \leq m \\
  E_{m+1} &= f_{m+1}^+ b_1^- \\
  F_{m+1} &= f_{m+1}^- b_1^+ \\
  H_{m+1} &= N_{f_{m+1}} + N_{b_1} \\
  E_{m+j} &= b_{j+1}^- b_j^+ \quad 2 \leq j \leq n \\
  F_{m+j} &= b_{j+1}^+ b_j^- \quad 2 \leq j \leq n \\
  H_{m+j} &= N_{b_{j+1}} - N_{b_j} \quad 2 \leq j \leq n \\
  E_0 &= b_{n+1}^+ f_{1}^- \\
  F_0 &= f_{1}^+ b_{n+1}^- \\
  H_0 &= N_{f_{n+1}} + N_{b_1}
\end{align*}
\]
From the operators \( \{ A_i^-, A_i^+, N_{A_i} \} \) we construct the generators

\[
\begin{align*}
e_i &= A_i^- A_{i+1}^+, \quad 1 \leq i \leq n + 1 \\
f_i &= A_i^+ A_{i+1}^-, \quad 1 \leq i \leq n + 1 \\
k_i &= q^{-N_{A_i} + N_{A_{i+1}}}, \quad 1 \leq i \leq n + 1 \\
e_0 &= A_{n+1}^- A_1^+, \\
f_0 &= A_{n+1}^+ A_1^-, \\
k_0 &= q^{N_{A_{n+1}} + N_{A_1}}
\end{align*}
\]

for \( 1 \leq i \leq n + 1 \), which generates the algebra \( U_q(\hat{A}_n) \). It is easy to verify that \( U_q(\hat{A}(n)) \) and \( \hat{A}(m, n) \) are mutually commutative. As results, we have the following decomposition of quantum superalgebra \( A_Q(m, n) \) in the \( Q \to q \) limit

\[
\lim_{Q \to q} U_Q(\hat{A}(m, n)) = U(\hat{A}(m, n)) \otimes U_q(\hat{A}(n)).
\]

## 6 Conclusion

We have presented the general method leading to the investigation the \( Q \to q = e^{2\pi i} \) limit of the quantum affine algebra \( U_Q(\hat{A}_n) \) and quantum affine superalgebra \( U_Q(\hat{A}(m, n)) \) based on the decomposition of \( Q \)-bosons in this limit. We note that \( Q \)-oscillator realization is crucial in this manner of splitting in this paper. We believe that the techniques and formulae, used here, will be useful foundation to extend this study to all quantum affine algebras, quantum affine superalgebras and \( Q \)-deformed exceptional Lie algebras and superalgebras.
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