WEYL CURVATURE AND THE EULER CHARACTERISTIC IN DIMENSION FOUR

HARISH SESHADRI

Abstract. We give lower bounds, in terms of the Euler characteristic, for the $L^2$-norm of the Weyl curvature of closed Riemannian 4-manifolds. The same bounds were obtained by Gursky, in the case of positive scalar curvature metrics.

1. Introduction

Let $M$ be a smooth closed oriented 4-manifold and let $C = [g] := \{ fg : f \in C^\infty(M) \text{ and } f > 0 \}$ be a conformal class of metrics on $M$. An important numerical invariant associated to $C$ is the Weyl constant $W(C)$. The Weyl constant is defined by

$$W(C) = \int_M |W_g|^2 dV_g,$$

where $g$ is any metric in $C$ and $W$ is the Weyl tensor of $g$. Since the vanishing of the Weyl tensor is equivalent to the conformal flatness of $g$, one can regard $W(C)$ as a quantitative measure of the lack of conformal flatness.

As the existence of a conformal class with prescribed value of $W$ is a diffeomorphism invariant, one can try to relate $W$ to standard topological invariants. In fact, in dimension 4 one has

Theorem 1.1. (Gursky [7]) Let $(M, g)$ be a closed oriented Riemannian 4-manifold. If $g$ has positive scalar curvature, then

$$\int_M |W|^2 \geq 8\pi^2 (\chi(M) - 2).$$

Equality holds if and only if $g$ is conformal to an Einstein metric $h$ with $s_h \Vol_h^\frac{4}{n} = 8\pi \sqrt{6}$, where “$s$” denotes scalar curvature.

Note that $8\pi \sqrt{6}$ is the Yamabe constant of the standard metric on $S^4$. Hence the results of Schoen [12] imply that $(M, g)$ is conformally equivalent to $S^4$ in the case of equality above.

As a corollary of Theorem 1.1, one obtains

Theorem 1.2. (Gursky [7]) Let $(M, g)$ be a closed oriented Riemannian 4-manifold. If $g$ is conformally flat and has positive scalar curvature, then $\chi(M) \leq 0$ unless $(M, g)$ is conformally equivalent to the round 4-sphere.

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Theorems 1.1 and 1.2 were proved by Gursky in [7] (Theorem 1.1 is not stated as such but is contained in the proofs). In the first part of this paper (Section 2) we give a simple, geometric proof of these results using “stereographic” projection. As noted by Gursky, the proofs of these results would be relatively straightforward if one were to assume the existence of a Yamabe metric in every conformal class. However, the known proof of existence of a Yamabe metric in dimension 4 uses the hard and deep Positive Mass Theorem of Schoen and Yau. Hence, in order to make the proofs “elementary”, we try to avoid the use of a Yamabe metric and use it only for the case of equality in Theorem 1.1.

In the second part (Section 3) of the paper, we prove a version of Theorem 1.1 for nonpositive scalar curvature metrics.

**Theorem 1.3.** Let \((M, g)\) be a closed oriented Riemannian 4-manifold. If \(s + c |W| \geq 0\) for some \(c > 0\) and there is a metric \(h\) conformal to \(g\) with \(\int_M sh \leq 0\), then

\[
\int_M |W|^2 \geq \frac{8\pi^2}{1 + \frac{c^2}{24}} \chi(M).
\]

Equality holds if and only if \(g\) is an Einstein metric with \(s + c |W| \equiv 0\).

Let us note that the hypotheses (and the conclusion) in the above theorem are dependent only on the conformal class of the metric \(g\).

It should be mentioned that different (and far subtler) sharp lower bounds for \(\int_M |W|^2\) were obtained by Gursky in [8] (for positive scalar curvature metrics, under the assumption of non-zero first or second Betti number) and in [9] (for negative scalar curvature metrics, under the assumption of the existence of a conformal vector field).

Our strategy for proving Theorems 1.1 and 1.2 is to use the “stereographic projection” of \((M, C)\). This gives us a complete noncompact asymptotically flat scalar-flat 4-manifold \((\hat{M}, \hat{g})\). The two main points for us are: First, under this passage, the Weyl invariant does not change. Second, the scalar-flatness and asymptotic flatness simplify the Chern-Gauss-Bonnet formula for balls in \((\hat{M}, \hat{g})\) considerably. Unfortunately, it is not clear how to extend this method to dimensions beyond 4 since we crucially use the specific form that Chern-Gauss-Bonnet takes in this dimension.

In Section 3 we prove Theorem 1.3. We use Yamabe metrics in this case. It should be possible, with some extra effort, to give a proof using stereographic projections but we do not pursue this approach here.

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**2. Stereographic projection and the Weyl constant**

For rest of this section we assume that \((M, g)\) is a closed oriented Riemannian 4-manifold with positive scalar curvature. Fix \(p \in M\) and let \(G\) denotes the Green’s function of the conformal Laplacian \(L = 6\Delta - s\) at \(p\). Since \(s > 0\), \(G\) exists and is positive. Also \(\hat{g} = G^2 g\) is a complete, scalar-flat, asymptotically flat metric on \(\hat{M} := M - \{p\}\) (cf. [11]). \((\hat{M}, \hat{g})\) is sometimes referred to as the ”stereographic projection” of \((M, [g])\). Let \(S_r\) and \(B_r\) denote the sphere and closed ball of radius \(r\) at \(p\) in \((M, g)\).
Lemmas 2.1 and 2.2 will imply that the boundary integral in Chern-Gauss-Bonnet applied to certain large domains in \((\hat{M}, \hat{g})\) will give the same value as for balls in flat \(\mathbb{R}^4\). The domains we consider are the complements of \(B_r\) in \(M\). Forms of the Gauss-Bonnet theorem for asymptotically flat manifolds have been described in [1] and [5]. For the specific result that we need and for the sake of completeness, we give the computation of the boundary integrals in detail.

In the next lemma the principal curvatures are with respect to the \(\text{inward}\) pointing normal of \(S_r \subset M\).

**Lemma 2.1.** If \(\hat{\lambda}_r\) is a principal curvature of \(S_r\) with respect to \(\hat{g}\), then \(\hat{\lambda}(r) = -r + O(r^2)\) as \(r \to 0\).

**Proof.** In what follows, hats will denote quantities defined with respect to \(\hat{g}\). The second fundamental form \(\hat{B}\) of \(S_r\) is related to \(B\) by

\[
\hat{B} = G B + \frac{\partial G}{\partial r} g,
\]

where we have used standard formulas for conformal changes. Hence we have the following equations for the shape operator \(S\), which is given by \(B(X, Y) = g(S(X), Y)\), and the principal curvatures, which are the eigenvalues of \(S\):

\[
\hat{S} = G^{-1} S + G^{-2} \frac{\partial G}{\partial r} I, \quad \hat{\lambda}_r = G^{-1} \lambda_r + G^{-2} \frac{\partial G}{\partial r}.
\]

Now let \(\{x^i\}\) denote \(\text{conformal normal coordinates}\) at \(p\), as defined in [11]. If \(r = d(x, p)\), then we have

\[
(2.1) \quad G(x) = r^{-2} + A + O''(r) \quad \text{as} \quad r \to 0,
\]

where \(f = O''(r^k)\) means \(f = O(r^k), \nabla f = O(r^{k-1})\) and \(\nabla^2(f) = O(r^{k-2})\).

We do not use this information but we note that \(A\) is a multiple of the \(\text{mass}\) of the asymptotically flat manifold \((M, \hat{g})\). From the above expression we get \(\frac{\partial G}{\partial r} = -2r^{-3} + O(1)\). Finally

\[
\hat{\lambda}_r = G^{-1} \lambda_r + G^{-2} \frac{\partial G}{\partial r} = \frac{r^{-1} + O(r)}{r^{-2} + A + O(r)} + \frac{-2r^{-3} + O(1)}{r^{-4} + O(r^{-2})} = -r + O(r^2)
\]

In the second equality we have used the well-known (see [11], for instance) and easily verified fact that \(\lambda_r = r^{-1} + O(r)\) on any Riemannian manifold. \(\square\)

The Chern-Gauss-Bonnet formula for a manifold with boundary \(N\) states (see [4] and also [2]) that

\[
(2.2) \quad 8\pi^2 \chi(N) = \int_N ([|W|^2 - \frac{1}{2} |z|^2 + \frac{1}{24} s^2] - 4 \int_{\partial N} \prod_{i=1}^{3} \lambda_i - \int_{\partial N} \sum_{\sigma \in S_3} K_{\sigma_1, \sigma_2, \sigma_3} \lambda_{\sigma_1} \lambda_{\sigma_2} \lambda_{\sigma_3}).
\]

Here \(W, z = \text{ric} - \frac{1}{2} g\) and \(s\) are the Weyl, trace-free Ricci and scalar curvature, respectively, \(K\) denotes sectional curvature and \(\lambda_i\) the principal curvatures of \(\partial N\).

Let us denote by \(I_r^1 = \int_{\hat{S}} \prod_{i=1}^{3} \lambda_i\) and \(I_r^2 = \int_{\hat{S}} \sum_{\sigma \in S_3} K_{\sigma_1, \sigma_2, \sigma_3} \lambda_{\sigma_1} \lambda_{\sigma_2} \lambda_{\sigma_3}\) the two boundary integrals in the above formula applied to \((M, G^2 g) := (M - Int B_r, G^2 g))\).

**Lemma 2.2.** \(\lim_{r \to 0} I_r^1 = -2\pi^2\) and \(\lim_{r \to 0} I_r^2 = 0\).
Proof. If $dA_r$ denotes the volume form of $S_r$ in $(M_r, g)$, then $\hat{d}A_r = G^3 dA_r = (r^{-6} + O(r^{-4})) dA_r$. Now

$$I_r^1 = \int_{S_r} \prod_{i=1}^3 \lambda_i dA_r = \int_{S_r} (-r^3 + O(r^4))(r^{-6} + O(r^{-4}) dA_r,$$

where we have used (2.2) in the last equation. Since $\text{Vol}(S_r) = O(r^3)$,

$$\lim_{r \to 0} I_r^1 = \lim_{r \to 0} -r^{-3} \text{Vol}(S_r) = -2\pi^2.$$ 

The last equation above can be easily seen by using normal coordinates. As for $I_r^2$, it is clear from (2.1) that for $r$ small enough, $|K| \leq 1$ on $S_r$. Hence the integrand (with respect to $dA_r$) in $I_r^2$ is of $O(r^{-2})$ and $I_r^2 \to 0$, as above. □

Now we come to the proof of Theorem 1.1.

Proof. Applying the Chern-Gauss-Bonnet theorem to $(M_r, G^2 g)$, setting $\hat{s} = 0$ and getting rid of the $|\hat{z}|^2$ term, we get

$$8\pi^2 \chi(M_r) \leq \int_{M_r} |\hat{W}|^2 d\hat{V} + I_r^1 + I_r^2.$$ 

From the conformal invariance of $W$, we have $\int_{M_r} |\hat{W}|^2 d\hat{V} = \int_{M_r} |W|^2 dV \to W(M)$ as $r \to 0$. By Lemma 2.2 and (2.2) we have

$$8\pi^2 \chi(M - \{p\}) = 8\pi^2 \chi(M_r) \leq W(M) + 8\pi^2.$$ 

Since $\chi(M) = \chi(M - \{p\}) + 1$, we finally get $W(M) \geq 8\pi^2 (\chi(M) - 2)$.

Now suppose that (2.3) $W(M) = 8\pi^2 (\chi(M) - 2)$.

Let $C$ denote the conformal class of $g$. Let $h \in C$ be a Yamabe metric, i.e, a metric minimizing the total scalar curvature functional $E$

$$\tilde{g} \to E(\tilde{g}) = \frac{\int_M s_{\tilde{g}} d\tilde{g}}{\text{Vol}(\tilde{g})^{\frac{2}{n}}}, \quad \tilde{g} \in C.$$ 

The existence of $h$ is guaranteed by [12]. $h$ has constant scalar curvature, which implies that

$$\int_M s_h^2 = \left(\frac{\int_M s_h}{\text{Vol}(h)}\right)^2$$

Moreover, by Aubin, the infimum of $E$ cannot be greater than the value of $E$ on the round sphere:

$$\frac{\int_M s_h}{\text{Vol}(h)^{\frac{2}{n}}} \leq 8\pi \sqrt{n}.$$ 

Combining (2.2) and the Chern-Gauss-Bonnet formula for $M$, we get

$$-\frac{1}{2} |z_h|^2 + \frac{1}{24} \int_M s_h^2 - 16\pi^2 = 0,$$

By (2.4) and (2.5) we see that the sum of the last two terms above is nonpositive. Hence we must have $z_h = 0$, i.e., $h$ is Einstein, and also $\frac{1}{24} \int_M s_h^2 = 16\pi^2$. □

Now we come to the proof of Theorem 1.2.
Proof. We assume that \((M, g)\) is conformally flat, i.e. \(W = 0\). If \(g\) has positive scalar curvature and \(\chi(M) > 0\), then we claim that \(\chi(M) = 2\). This is because \(\chi(M) = 2 - 2\beta_1 + \beta_2\), by Poincare Duality. However, \(\beta_2 = 0\) by the Bochner formula for harmonic 2-forms on \((M, g)\).

Since \(\chi(M) = 2\) and \(g\) is conformally flat, we can appeal to the theorem above and conclude that a Yamabe metric \(h\) in \([g]\) is Einstein. Since \(W_h = 0\) it would follow that \(h\) is of constant (positive) sectional curvature and by orientability, it would follow that \((M, h)\) is isometric to \((S^4, g_0)\) and we would be done. However, we give a different proof which avoids the existence of a Yamabe metric: First, let \(\hat{\chi}\) be the modified scalar curvature for harmonic 2-forms on \((M, g)\) defined by \(\hat{\chi} = \chi + \sigma\), where \(\sigma\) is the curvature at the north pole) for the conformal Laplacian on \((S^4, g_0)\).

Let \(\tilde{\chi}\) extend to a smooth metric conformal to \(g\). Since \((M, g)\) is simply-connected and hence isometric to flat \(\mathbb{R}^4\), it follows that \((M, g)\) is conformally equivalent to \((S^4, g_0)\) and \(\tilde{\chi}\) is of constant (positive) sectional curvature and by orientability, it extends to a smooth metric conformal to \(g\) on \(M\).

3. Nonpositive scalar curvature and the Weyl constant

Here we prove Theorem 1.3. So assume that \(s + c|W| \geq 0\) for \(g\).

Lemma 3.1. For any metric \(h\) in \([g]\) we have \(\int_M (s_h + c|W_h|) \geq 0\).

Proof. Let us introduce, following Gursky and LeBrun [10], the modified scalar curvature \(\sigma_g = s_g + c|W_g|\). Under a conformal change \(g \to \tilde{g} = u^2 g\), the modified scalar curvature transforms (with our convention, the Laplacian \(\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \cdot \frac{\partial}{\partial r}\) on \(\mathbb{R}\)) by

\[
\sigma_g \to \tilde{\sigma}_g = u^{-2} \sigma_g - 6u^{-3} \Delta u.
\]

We also have the functional

\[
E_\sigma(g) = \int_M \sigma_g dV_g.
\]

When we restrict \(E_\sigma\) to a conformal class we get an operator \(L\) on \(C^\infty(M)\) defined by \(L(u) = E_\sigma(u^2 g)\) or

\[
L(u) = -6 \Delta u + \sigma_g u.
\]

Let \(\langle , \rangle\) denotes the \(L^2\) inner product on \(C^\infty(M)\). and let

\[
\lambda = \inf_{f \in W^{1,2}} \frac{\langle Lf, f \rangle}{\|f\|_{L^2}^2} < \lambda_f,
\]

and \(u\) be the corresponding eigenfunction. We note that since \(\sigma_g\) is, in general, Lipschitz continuous but not smooth (at the zero locus of \(|W|\)), the best regularity we can obtain for \(u\) is that \(u \in C^{2,\alpha}\) for any \(0 < \alpha < 1\). This is sufficient for our purposes. By the minimum principle \(u > 0\) and by definition, \(u\) satisfies

\[
\sigma_{g_0} = u^{-3} L(u) = \lambda u^{-2}.
\]

Claim: If \(g_0 = u^2 g\), then \(\sigma_{g_0} \geq 0\).

Proof: By (3.1) and (3.3), we see that

\[
\sigma_{g_0} = u^{-3} L(u) = \lambda u^{-2}.
\]
Hence $\sigma_{g_0}$ has a fixed sign. Suppose $\sigma_{g_0} < 0$. Then $\Delta u \leq 0$. Hence, by the minimum principle, $u$ would be constant. But this would contradict (3.1). This proves the Claim.

Suppose that $h = f^2 g$. Let $h_0 = \|f\|^{-2} f^2 g$. Then $E_\sigma(h_0) \geq E_\sigma(g_0) > 0$. Since $E_\sigma(h) = \|f\|^2 E_\sigma(h_0)$, the lemma is proved.

Now let $h$ be a metric of constant scalar curvature in $\{g\}$. This exists by the solution to the Yamabe problem. Note that since we have assumed that $\int_M s_g dV_g \leq 0$, the Yamabe metric has nonpositive scalar curvature. We work with $h$ for rest of the proof. By the Lemma 3.3, we have $\int_M (s + c |W|) \geq 0$. Hence

$$\int_M s^2 = Vol^{-1}(\int_M s)^2 \leq c^2 Vol^{-1}(\int_M |W|)^2 \leq c^2 \int_M |W|^2,$$

where we have used the Cauchy-Schwartz inequality at the last step.

Combining this with the Chern-Gauss-Bonnet formula, we are done. If equality holds in Theorem 1.3, we must have $s_h + c |W_h| \equiv 0$ and $h$ must be Einstein. Again referring to (3.1), we see that $h$ must be a constant multiple of $g$. Hence $\sigma_g \equiv 0$ and $g$ must be Einstein.

For $M$ which do not admit positive scalar curvature metrics, it would be interesting to estimate (in terms of the topology of $M$) the smallest $c$ such that $\sigma \geq 0$ for some $g$.

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department of mathematics, Indian Institute of Science, Bangalore 560012, India
E-mail address: harish@math.iisc.ernet.in