Non-Fermi liquid behavior and quantum criticality in the multipolar Bose-Fermi Kondo model for cubic heavy fermion systems

SangEun Han,* Daniel J. Schultz,* and Yong Baek Kim

Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada
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In the presence of strong spin-orbit coupling and a crystal electric field, local moments in rare-earth metallic systems can develop higher-rank electric and magnetic multipolar moments. When these moments hybridize with conduction electrons, highly anisotropic Kondo couplings and subsequent RKKY interactions between local moments emerge. Inspired by recent experiments on the Pr(Ti, V)$_2$Al$_{20}$ compounds, we study possible non-Fermi liquids and quantum critical behaviors in the multipolar Bose-Fermi Kondo model, which can be regarded as a local version of the multipolar Kondo lattice model. Here, the multipolar local moments are coupled to fermionic and bosonic bath degrees of freedom representing the multipolar Kondo effect and RKKY interactions. Using a perturbative renormalization group (RG) study up to two loop order, we find critical points between non-Fermi liquid Kondo fixed points and a quadrupolar ordered fixed point. The critical points describe quantum critical behaviors at the corresponding phase transitions and can be distinguished by higher order corrections in the octupolar susceptibility that can be measured by ultrasound experiments. Our results imply the existence of a rich expansion of the phases and quantum critical behaviors in multipolar heavy fermion systems.

I. INTRODUCTION

The interplay between local and itinerant degrees of freedom in a metal often leads to a competition between two or more phases. Such a competition provides a setting for quantum critical behavior. The heavy fermion materials are a central player in this game[1–3], with the Kondo coupling and RKKY interaction competing between forming a paramagnetic phase with large Fermi surface and a magnetically ordered phase, respectively [4–11]. Recently, a number of strongly correlated materials exhibiting unconventional superconductivity, non-Fermi liquid behavior, and higher order electric and magnetic multipoles have emerged and go beyond the picture of the classic heavy fermion materials [12–22]. Due to crystal electric field and strong spin-orbit coupling effects, these multipolar Kondo systems experience anisotropic Kondo couplings between conduction electrons and local multipolar moments, which subsequently induces highly anisotropic RKKY interactions between these moments. Departing from the classic heavy fermion phase diagram, [23, 24], the typical Fermi liquid phase may be replaced by a non-Fermi liquid phase distinct from the quantum critical fan, and the dipolar ordering is replaced by ordering of higher rank moments.

One useful approach to the usual dipolar Kondo lattice problem is inspired by dynamical mean field theory, wherein a lattice problem is mapped to a self-consistent impurity problem in the limit of infinite dimensions [25–28]. Relaxing the self-consistency condition allows one to formulate the Bose-Fermi Kondo model [29–33], whereby a dipolar local moment is coupled to both a fermionic bath representing the conduction electrons, and a bosonic bath representing a local effective magnetic field with the density of states $\sim |\omega|^{1+\epsilon}$ [30, 34–38]; the $\epsilon$ factor is used in a controlled $\epsilon$ expansion procedure. This local effective magnetic field can, roughly speaking, be thought of as a Weiss mean field from the RKKY interaction with nearby local moments.

In this work, motivated by experiments on Pr-based heavy fermion systems, Pr(Ti, V)$_2$Al$_{20}$, we consider the multipolar Bose-Fermi Kondo model in cubic systems as a simplified model for the multipolar Kondo lattice. Here, the Pr$^{3+}$ ions provide a non-Kramers doublet supporting quadrupolar and octupolar moments, which couple to conduction electrons. As shown below, the multipolar Kondo lattice model permits a mapping to a multipolar Bose-Fermi Kondo model, which can be constructed based on local point group symmetry. The fermionic Kondo problem without the bosonic bath or RKKY interaction was theoretically studied earlier and various non-Fermi liquid ground states were identified. These results may have some relevance to experiments in the dilute limit [39–43], where non-Fermi liquid behaviors were observed. It is then natural to study the phase transition between such non-Fermi liquids and a multipolar ordered phase.

In the multipolar Bose-Fermi Kondo model, we set the density of states of the quadrupolar ($Q$) and octupolar ($O$) bosonic baths to be $|\omega|^{1-\epsilon}$ ($i = Q, O$) and perform a perturbative renormalization group (RG) analysis based on an $\epsilon$-expansion to order $\epsilon^2$ to study the zero temperature phase diagram of the model. Similar to previous work, we find that there are two non-Fermi liquid phases [22, 42] in the fermion Kondo part of the model: one is a two-channel Kondo non-Fermi liquid, and the other is a novel non-Fermi liquid phase, not simply classifiable into any multichannel-type model. These phases, upon tuning the Kondo and bosonic bath couplings, can pass through quantum critical points to both arrive at a

* These authors contributed equally to this work.
The remainder of the paper is organized as follows. In Section II, we describe modelling the multipolar Kondo lattice in terms of a multipolar Bose-Fermi Kondo model. In Section III, we perform a renormalization group analysis of our multipolar Bose-Fermi Kondo model to identify the phases and phase transitions in the model. In Section IV, we comment on how these phases can be distinguished experimentally, and in Section V we discuss the implications and possible extensions of our work.

II. MODELS

We start by describing the microscopic origin of the multipolar moment, and then describe how to construct conduction orbitals. Then, we couple the multipolar impurity to the conduction orbitals which constitutes the (Fermi) Kondo coupling. The Bose-Kondo coupling can then be derived from the Fermi-Kondo coupling by pretending as though there are other local moments to interact with, but then replacing these other moments with a Weiss mean field; this Weiss mean field becomes the bosonic bath. Since the setting of interest is the praseodymium cubic compounds Pr(Ti, V)$_2$Al$_{20}$, we need to consider the local symmetry of the Pr$^{3+}$ moment. Here, a Pr$^{3+}$ ion rests at the centre of a Frank-Kasper cage, which is composed of (Ti, V) and Al. Despite the complicated nature of the cage, its point group symmetry is simply the tetrahedral group $T_d$. This means that we can classify the wave function of an electron hopping on the cage according to the irreducible representations of $T_d$. We allow the most general interactions according to the local $T_d$ symmetry and time-reversal; the details of of the symmetry group are listed in Appendix B.

A. Multipolar moments

Generally speaking, on the site of a local moment, the wave functions of a particular ionic configuration are constrained to an effective ground state by Hund’s rules. These ground states are then split by the local crystalline electric field. The consequence of these restrictions is the formation of localized anisotropic charge and magnetization densities, leading to multipolar moments. In the case of a rare-earth Pr$^{3+}$ ion subjected to a tetrahedral ($T_d$) crystal field, the spin-orbit coupled $J = 4$ multiplet of the 4$f^2$ electrons is split to give rise to a low-lying and energetically well-isolated $\Gamma_3$ non-Kramers doublet [13]; the doublet states are listed in Appendix A. This $\Gamma_3$ doublet supports both time-reversal even quadrupolar moments ($O_{15} = \sqrt{5}/6J_z^2 - J_y^2$, $O_{20} = \sqrt{5}/6J_z^2 - J_y^2$) as well as a time-reversal odd octupolar moment ($T_{xyz} = \sqrt{15}/6J_x^2 J_y J_z$); we use the Stevens operators to describe the multipolar moments and the overline indicates a full symmetrization. These moments can be compactly represented by the pseudospin-1/2 op-
rewrite the local moment

and satisfy a canonically normalized \( \mathfrak{su}(2) \) algebra \( [S^x, S^y] = i \epsilon_{ijk} S^k \). Further details of this pseudospin-1/2 object are described in Appendix A. Note that, although the multipolar moments are written in terms of pseudospin-1/2 operators, their transformations under rotations in \( T_d \) and time reversal reflect the underlying multipolar attributes.

### B. Fermi-Kondo Models

Electrons hopping on Frank-Kasper cage can be thought of as molecular orbitals centred at the Pr ion. It is these molecular orbitals which we couple to the local multipolar moments respecting the local \( T_d \) symmetry and time-reversal are enumerated in Eqs. (3)-(5):

\[
H_0^F = \sum_{k,\alpha,\sigma} E_k \hat{c}_{k,\alpha,\sigma}^\dagger \hat{c}_{k,\alpha,\sigma}, \quad (2)
\]

\[
H_{Q1} = K_{Q1} \sum_{\alpha,\beta} \left( \sigma^0_{\alpha,\beta} \lambda^3_{\alpha,\beta} S^x - \sigma^0_{\alpha,\beta} \lambda^8_{\alpha,\beta} S^y \right) \hat{c}_{\alpha,\beta}, \quad (3)
\]

\[
H_{Q2} = K_{Q2} \sum_{\alpha,\beta} \left( 2\sigma^x_{\alpha,\beta} \lambda^2_{\alpha,\beta} S^y + \sigma^y_{\alpha,\beta} \lambda^5_{\alpha,\beta} \left( \sqrt{3} S^x + S^y \right) \right) \hat{c}_{\alpha,\beta} \quad (4)
\]

\[
H_{O} = K_{O} \sum_{\alpha,\beta} \left( \sigma^x_{\alpha,\beta} \lambda^6_{ab} + \sigma^y_{\alpha,\beta} \lambda^4_{ab} + \sigma^z_{\alpha,\beta} \lambda^1_{ab} \right) S^z \hat{c}_{\alpha,\beta} \quad (5)
\]

The subscript 0 on the conduction electron operators indicates that this interaction occurs only on the impurity site, which is taken to be the origin. The Latin indices sum over orbitals \( a, b = x, y, z \), and the Greek indices sum over spins \( \alpha, \beta = \uparrow, \downarrow \). \( \sigma^x \) are the standard Pauli matrices, and \( \lambda^j \) are the \( 3 \times 3 \) Gell-Mann matrices, listed in Appendix C. For the conduction electrons, we assume a constant density of states near the Fermi surface, \( \sum_k \delta(\omega - E_k) = N_0 \) between \(-D < \omega < D\).

The pseudospin \( \mathbf{S} \) represents the multipolar moments, with \( S^{x,y} \) and \( S^z \) standing for the quadrupolar and octupolar moments respectively. In order to perform the many-body perturbation theory later in this work, we rewrite the local moment \( \mathbf{S} \) in terms of Abrikosov pseudfermions:

\[
\mathbf{S} = \sum_{\alpha,\beta} f_\alpha^{\dagger} \tilde{\sigma}_{\alpha,\beta} f_\beta \quad (6)
\]

where we constrain the occupation of the impurity to be \( \sum_\alpha f_\alpha^{\dagger} f_\alpha = 1 \). In order to impose this physical constraint, we introduce a chemical potential for the pseudofermion by adding \( \lambda \sum_\alpha f_\alpha^{\dagger} f_\alpha \) to the Hamiltonian, and take the limit \( \lambda \to \infty \) at the end of the calculation [30, 44].

### C. Bose-Kondo Models

In the full Kondo lattice, the local Kondo Hamiltonian of Eqs. (3)-(5) appears at each lattice site. Through this Kondo interaction, an effective interaction between local moments is generated, known as the RKKY interaction [4–6]. In the Bose-Fermi Kondo model, this RKKY interaction is represented by the coupling of the local moment to a bosonic bath. The procedure to generate the most general symmetry allowed RKKY-type interaction is described in Appendix D. The resulting kinetic term for bosons and the Bose-Kondo coupling are given in Eqs. (7),(8) respectively,

\[
H_0^B = \sum_k \left[ \Omega_{Qk}(\hat{c}_{k,\alpha,\sigma}^\dagger \hat{\phi}_{k,\alpha,\sigma}^\dagger + \phi_{k,\alpha,\sigma}^\dagger \hat{\phi}_{k,\alpha,\sigma}^\dagger) + \Omega_{Ok}(\hat{c}_{k,\alpha,\sigma}^\dagger \hat{\phi}_{k,\alpha,\sigma}^\dagger \hat{\phi}_{k,\alpha,\sigma}^\dagger) \right], \quad (7)
\]

\[
H_g = g_{Q}(S^x \phi_0^\dagger + S^y \phi_0^\dagger) + g_{O} S^z \phi_0. \quad (8)
\]

Here, \( \Omega_{Qk} \) and \( \Omega_{Ok} \) are the dispersions of the bosonic baths coupled to the quadrupole and octupole moments, respectively. To set up the controlled RG calculation, we introduce an \( \epsilon \) expansion with dimensional regularization in the density of states of the bosonic bath,

\[
\sum_k [\delta(\omega - \Omega_{i,k}) - \delta(\omega + \Omega_{i,k})] = \frac{N_0^2}{2} |\epsilon|^\epsilon \omega^{1-\epsilon} \text{sgn}(\omega). \quad (9)
\]

To consider the most general situation, we introduce \( \epsilon_Q \) and \( \epsilon_O \) for the quadrupolar and octupolar bosonic baths because the density of states power law of the quadrupolar and octupolar bosonic baths are generically different. The multipolar moments localized at \( r = 0 \) couple to the bosonic bath fields \( \phi_0 = \sum_k (\phi_k + \phi_k^\dagger) \).

### III. RENORMALIZATION GROUP ANALYSIS

#### A. \( \epsilon \)-Expansion and Dimensional Regularization

We perform the renormalization group analysis by using dimensional regularization with minimal subtraction [30]. The bosonic bath already has an \( \epsilon \) factor modifying its density of states which can be used in the minimal subtraction procedure, but the conduction electron bath
does not. We therefore introduce $\epsilon'$ for the conduction electron density of states to enable the minimal subtraction of poles:

$$\sum_k \delta(\omega - E_k) = N_0 |\omega|^{-\epsilon'}.$$  \hfill (10)

Note that $\epsilon'$ will set to zero at the end of the calculation. Consequently, we define a renormalized field $f$ and dimensionless coupling constants $g_i$ and $K_i$,

$$f^B = Z_f^{1/2} f, \quad g_i^B = g_i Z_f^{-1} Z_g i^{\epsilon_i/2}, \quad K_j^B = K_j Z_f^{-1} Z_K j^{\epsilon_j},$$  \hfill (11)

(12)

(13)

where $\mu$ is the renormalization energy scale, and $Z_f$, $Z_g$, and $Z_K$ are the renormalization constants for the pseudofermion $f$, bosonic couplings $g_i$ (here $i = Q, O$), and fermionic couplings $K_j$ (here $j = Q, Q, Q, 2, O$). The superscript $B$ stands for the bare value which does not evolve under the RG flow. In addition, we absorb the density of states $N_i$ into the dimensionless couplings as $N_0 K_j \rightarrow K_j$ and $N_i^2 g_i \rightarrow g_i$, respectively, in the following section. The details of the RG analysis and corresponding Feynman diagrams are enumerated in Appendix E. Note that we ignore the self-energies of the conduction electrons and bosonic baths because they vanish in the thermodynamic limit \([35]\).

### B. Beta Functions for the Fermionic Kondo Model

The beta functions with the multipolar moment couplings up to cubic order in $K_i$ are given by \([22, 42, 43]\)

$$\frac{dK_{Q1}}{d\ln \mu} = 6K_{Q}K_O + 2K_{Q1}(K_{Q1}^2 + 6K_{Q2}^2 + 3K_{Q}^2),$$  \hfill (14)

$$\frac{dK_{Q2}}{d\ln \mu} = K'_O (K_{Q1} - \sqrt{3}K_{Q2})$$

$$+ 2K_{Q2}(K_{Q1}^2 + 6K_{Q2}^2 + 3K_{Q}^2),$$  \hfill (15)

$$\frac{dK_{Q}}{d\ln \mu} = 2K_{Q2}(2K_{Q1} - \sqrt{3}K_{Q2})$$

$$+ 4K_{O}(K_{Q2}^2 + 6K_{Q2}^2).$$  \hfill (16)

This RG flow has two distinct stable fixed points. The two types of stable fixed points are the multipolar fixed points, $F_{1\pm} = (K_{Q1}^*, K_{Q2}^*, K_{Q}^*) = (\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{2\sqrt{2}}, \pm \frac{1}{2\sqrt{3}})$, and two-channel Kondo fixed points, $F_{2\pm} = (\pm \frac{1}{2\sqrt{2}}, \pm \frac{1}{\sqrt{3}})$. The stable fixed points have perturbative scaling dimensions $\Delta = 1/4$ and $\Delta = 1$, respectively, which are the slope of the beta function at the respective fixed points; both fixed points are non-Fermi liquid phases. $\Delta$ is also related to the scaling dimension $(1 + \Delta)$ of the leading irrelevant operator at the fixed point. The physical observables such as resistivity $\rho$ and heat capacity $C_V$ at the fixed points are obtained by using the scaling dimension $\Delta$; at low temperatures we have $\rho \sim T^\Delta$ and $C_V \sim T^{2\Delta}$.

### C. Beta Functions for the Bosonic Kondo Model

The beta functions for the coupling of the local moment to the bosonic bath up to $g_i^0$ order are given by

$$\frac{d\lambda_Q}{d\ln \mu} = -\lambda_Q (\epsilon_Q - \lambda_Q - \lambda_Q + \lambda_Q^2 + \lambda_Q^2),$$  \hfill (17)

$$\frac{d\lambda_O}{d\ln \mu} = -\lambda_O (\epsilon_O - 2\lambda_Q + 2\lambda_Q^2),$$  \hfill (18)

where $\lambda_Q, O \sim g_Q^2$. Eqs. (17) and (18) have two stable fixed points, a quadrupolar ordered fixed point, $B_Q = (\lambda_Q^2, \lambda_Q^2) = (\epsilon_Q - \epsilon_Q, 0)$, and an octupolar ordered fixed point, $B_O = (0, \infty)$. The quadrupolar and octupolar fixed points can be identified with an $XY$ fixed point and Ising fixed point in the ordinary Fermi-Bose Kondo model \([30, 37]\). The octupolar fixed point is, strictly speaking, outside of the regime of our perturbative calculation. The beta functions also have another fixed point, $C_{QO} = (\lambda_Q^2 + \omega Q - \omega O, \lambda_Q^2 Q - \omega O + \frac{\pi^2}{4})$, which is a critical point between the quadrupolar and octupolar fixed points, and corresponds to the XXZ fixed point in the ordinary Fermi-Bose Kondo model \([30, 37]\). All the fixed point values are calculated up to $g_i^1$ order. In the limit $\epsilon_Q = \epsilon_O = \epsilon, C_{QO} = (\frac{\pi^2}{4} + \frac{\pi^2}{4} + \frac{\pi^2}{4} + \frac{\pi^2}{4})$ becomes isotropic \([30, 37]\).

### D. Beta Functions for the Bose-Fermi Kondo Model

In order to study the destruction of the Kondo effect to magnetic ordering, we consider the full model of coupling the local moment to both the fermionic conduction electron bath and the bosonic bath. In this case, the beta functions are as follows:

$$\frac{dK_{Q1}}{d\ln \mu} = K_{Q1} \left( \frac{\lambda_Q + \lambda_O}{4} - \frac{\lambda_Q (\lambda_Q + \lambda_O)}{2} \right)$$

$$+ 6K_{Q2}K_O + 2K_{Q1}(K_{Q1}^2 + 6K_{Q2}^2 + 3K_{Q}^2),$$  \hfill (19)

$$\frac{dK_{Q2}}{d\ln \mu} = K_{Q2} \left( \frac{\lambda_Q + \lambda_O}{2} - \frac{\lambda_Q (\lambda_Q + \lambda_O)}{2} \right)$$

$$+ K_O (K_{Q1} - \sqrt{3}K_{Q2})$$

$$+ 2K_{Q1}(K_{Q1}^2 + 6K_{Q2}^2 + 3K_{Q}^2),$$  \hfill (20)

$$\frac{dK_{Q}}{d\ln \mu} = K_O (\lambda_Q - \lambda_Q \lambda_O)$$

$$+ 2K_{Q2}(2K_{Q1} - \sqrt{3}K_{Q2}) + 4K_{O}(K_{Q1}^2 + 6K_{Q2}^2),$$  \hfill (21)

$$\frac{d\lambda_Q}{d\ln \mu} = -\lambda_Q (\epsilon_Q - \lambda_Q + \lambda_Q) + \lambda_Q (\lambda_Q + \lambda_O)$$  \hfill (22)
Under this full renormalization group flow, all of the previously found stable fixed points in the fermionic Kondo $F_{1,\pm}$ and bosonic Kondo $B_Q$, and $B_O$ cases remain stable. Further, new fixed points emerge, which describe critical points between the phases described in Secs. III B-III C. For the case $\lambda_Q \neq 0, \lambda_O = 0$, we find two pairs of critical points. The first critical point is given by $C_{1,\pm}^Q = (\pm (\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}), \pm (\frac{\sqrt{Q}}{4\sqrt{2}}, \frac{\sqrt{Q}}{4\sqrt{2}}, \epsilon_Q - 2\epsilon_Q^2, 0))$ which is a critical point between $F_{1,\pm}$ and $B_Q$. The flow diagram corresponding to this transition is given in Fig. 2. The second critical point is $C_{1,\pm}^Q = (\pm (\frac{Q}{2\sqrt{2}}, \frac{Q}{2\sqrt{2}}), \pm (\frac{\sqrt{Q}}{6\sqrt{2}}, \frac{\sqrt{Q}}{6\sqrt{2}}, \epsilon_Q + \epsilon_Q^2, 0))$. This is a critical point between $F_{1,\pm}$ and $B_Q$, and its flow diagram is Fig. 3. In the case $\lambda_Q = 0, \lambda_O \neq 0$, we find the critical points, $C_{1,2,\pm}^O$, between $F_{1,2,\pm}$ and $B_O$. However, the fixed point values of $\lambda_O$ in $B_Q$ and $C_{1,2,\pm}^O$ are order one numbers, so they are outside of the perturbative regime. Despite this, we believe the the existence of the fixed points to be maintained under specialized non-perturbative methods. For example, the octupolar-type critical points may be accessible via the Coulomb gas representation [30, 45].

\[
\frac{d\lambda_Q}{d\ln \mu} = -\lambda_O[\epsilon_O - 2\lambda_Q + 2\lambda_Q \lambda_O - 8(\lambda_Q^2 + 6\lambda_Q^2)\].
\]

IV. PHYSICAL OBSERVABLES

A. Zero-Temperature Multipolar Susceptibility

In order to compare our results with experiment, we consider the local multipolar moment susceptibility exponent. The local quadrupolar and octupolar moment susceptibilities, $\chi_Q$ and $\chi_O$, are defined as

\[
\chi_Q\tau = \langle T, S^{x,y}(\tau) S^{x,y}(0) \rangle \propto \left( \frac{\tau_0}{|\tau|} \right)^{\gamma_Q},
\]

\[
\chi_O\tau = \langle T, S^{z}(\tau) S^{z}(0) \rangle \propto \left( \frac{\tau_0}{|\tau|} \right)^{\gamma_O},
\]

where $\gamma_i (i = Q, O)$ is the multipolar susceptibility exponent and $\tau \gg \tau_0$ with the cutoff $\tau_0 = 1/\Lambda$. We emphasize that the multipolar susceptibility exponents describe how the susceptibility scales as imaginary time evolves, but do not directly yield the temperature scaling. When the fixed point value of $\lambda_i (i = Q, O)$ is non-zero, the corresponding susceptibility exponent is given by [30, 37]

\[
\gamma_i = \epsilon_i + \left[ \frac{1}{\chi_i} \frac{d\lambda_i}{d\ln \mu} \right]_{f.p.},
\]

where f.p. stands for value at the fixed point. By definition, $\frac{d\lambda_i}{d\ln \mu} = 0$ at the fixed point, so $\gamma_i = \epsilon_i$, which is exact to all orders of $\epsilon$ [30, 37]. Since our critical points $C_{1,2,\pm}^Q$ all have a non-zero fixed point value for $\lambda_Q$, the quadrupolar susceptibility exponent of the critical points
When the fixed point value of \( \lambda_1 \) is zero, as is the case for \( \lambda_O \) at both \( C_{1,2}^Q \), then the corresponding susceptibility exponent is given by \([37]\)

\[
\gamma_i = \epsilon_i + \left[ \lim_{\lambda_i \to 0} \frac{1}{\lambda_i} \frac{d \lambda_i}{d \ln \mu} \right]_{\text{f.p.}}
= \epsilon_i + \left[ \frac{\partial}{\partial \lambda_i} \frac{d \lambda_i}{d \ln \mu} \right]_{\text{f.p.}}. \tag{27}
\]

In contrast to the previous case, Eq. (27) includes higher order corrections in \( \epsilon_{Q \gamma} \). Since our calculation applies to order \( \epsilon^2 \), we can use \( \epsilon^2 \) corrections to the octupolar susceptibility to distinguish between different fixed points. The resulting susceptibility exponents for the two critical points are \( \gamma_O = 2 \epsilon_Q \) and \( 2 \epsilon_Q + 3 \epsilon_O^2 / 2 \), for \( C_{1,2}^O \), respectively. The results for the susceptibility exponents at different fixed points are summarized at Table I. The full expression of the multipolar susceptibility exponent \( \gamma_i \) for \( \lambda_i^* = 0 \) is presented in Appendix F. An additional point is that we may also distinguish between these critical points and the non-Fermi liquid phases \( F_{1,2} \) using this octupolar susceptibility. This is useful because we can then distinguish non-Fermi liquid behavior due to a quantum critical regime from non-Fermi liquid behavior in a phase (\( F_{1,2} \) or \( F_{2,2} \) in our case).

### B. Finite Temperature Scaling and Elastic Constants

The results in the previous section only apply at zero temperature, and do not directly correspond to a measurable quantity. In order to obtain the temperature dependence of the susceptibility, we assume that we have conformal invariance at the critical point, and that the multipolar moment is a primary operator with conformal dimension \( \gamma_i/2 \) \([37, 46]\). The results for the real part \( \chi' \) and imaginary part \( \chi'' \) are given by Eqs. (29)-(28); see Appendix G for details.

\[
\chi'(T) = \begin{cases} T^{2 \gamma_i - 2}, & |\omega/T| \ll 1, \\ T^{-\gamma_i - 3}, & |\omega/T| \gg 1, \end{cases} \tag{28}
\]

\[
\chi''(T) = \begin{cases} T^{2 \gamma_i - 2}, & |\omega/T| \ll 1, \\ T^{-\gamma_i - 2}, & |\omega/T| \gg 1. \end{cases} \tag{29}
\]

We expect that the multipolar susceptibility exponent in Eq. (28) can be observed by measuring the temperature dependence of elastic constants \([47, 48]\). The elastic free energy including the symmetry-allowed coupling between the multipolar moments and strains is given by \([48, 49]\)

\[
F = \frac{C_{11}^0}{2} - \frac{C_{12}^0}{2} (\epsilon_{\mu}^2 + \epsilon_{\nu}^2) + \frac{C_{44}^0}{2} (\epsilon_{xy}^2 + \epsilon_{yz}^2 + \epsilon_{xz}^2)
- s_Q (\epsilon_{\mu} O_{122} + \epsilon_{\nu} O_{212})
- s_O T_{yz} [h_x \epsilon_{xy} + h_y \epsilon_{xz} + h_z \epsilon_{yz}], \tag{30}
\]

where \( \epsilon_{ij} \) is the strain tensor, \( \epsilon_{\mu} = (2 \epsilon_{xx} - \epsilon_{yy}) / \sqrt{3} \) and \( \epsilon_{\nu} = (\epsilon_{xx} - \epsilon_{yy}) \), \( C_{11}, C_{12}, C_{44} \) are the elastic constants which are coefficients of \( \epsilon_{11}^2, \epsilon_{ij} \epsilon_{ij}, \epsilon_{ij}^2 (i \neq j) \) for

| Label | \((K_{Q1}, K_{Q2}, K_O, \lambda_Q, \lambda_O)\) | Type | \(\gamma_Q\) | \(\gamma_O\) |
|-------|---------------------------------|------|----------|----------|
| \(F_{1\pm}\) | \(\pm \frac{1}{2}\sqrt{6}, \pm \frac{1}{12}\sqrt{2}, -\frac{1}{4}\sqrt{3}, 0, 0\) | Multipolar | 1/2 | 1/2 |
| \(F_{2\pm}\) | \(\pm \frac{1}{2}\sqrt{3}, \pm \frac{1}{2}\sqrt{3}, 0, 0\) | 2-channel Kondo | 2 | 2 |
| \(B_Q\) | \((0, 0, 0, \epsilon_Q + \epsilon_O^2, 0)\) | Quadrupolar | \(\epsilon_Q\) | \(2 \epsilon_Q + 2 \epsilon_O^2\) |
| \(B_O\) | \((0, 0, 0, 0, 0)\) | Octupolar | \(\epsilon_O\) | \(-\epsilon_O\) |
| \(C_{Q\gamma}\) | \(0, 0, 0, \frac{\epsilon_O^2}{4} + \frac{\epsilon_{Q\gamma}(2 \epsilon_Q - \epsilon_O)}{4} + \frac{\epsilon_Q^2}{4}\) | Critical | \(\epsilon_Q\) | \(-\epsilon_Q\) |

Table I. Table of the fixed points and their multipolar susceptibility exponents. \(F_{1,2}\) are the multipolar and 2-channel Kondo fixed points, respectively, and \(B_Q\) and \(B_O\) are the quadrupolar fixed point and octupolar fixed line, respectively (all four of these are stable). \(C_{Q\gamma}\) is the critical point between \(F_{1,2}\) and \(B_Q\), and \(C_{Q\gamma}\) is the critical point between \(F_{1,2}\) and \(B_O\). C_{Q\gamma} is the critical point between \(B_Q\) and \(B_O\). \(\gamma_Q\) and \(\gamma_O\) stand for the quadrupolar and octupolar susceptibility exponents defined in Eq. (24) and (25), respectively. The schematic diagram for their quantum phase transitions is presented in Fig. 1. Note that \(B_O\) and \(C_{Q\gamma}\) are outside of the perturbative regime, so their \(\gamma_Q\) values are omitted.
the deformation free energy in the cubic lattice, respectively. \( h_i \) is the magnetic field \((i = x, y, z)\), the superscript \( 0 \) stands for the bare value of the elastic constants, and \( s_Q \) and \( s_O \) are the couplings between the multipolar moments and lattice strain tensors. From second-order perturbation theory, we can get the following corrections to the bare elastic constants,

\[
(C_{11} - C_{12}) = (C_{11}^0 - C_{12}^0) - (s_Q^2) \chi_Q, \quad (31)
\]

\[
C_{44} = C_{44}^0 - (s_Q h^2) \chi_O, \quad (32)
\]

where \( h \) is the magnetic field. The octupolar susceptibility is therefore only detectable when measured in the presence of both strain and magnetic field simultaneously. As a result, the multipolar susceptibility can be observed by measuring the temperature dependence of the elastic constants \( (C_{11} - C_{12}) \) and \( C_{44} \) via ultrasonic measurements.

V. CONCLUSIONS

Inspired by experiments on \( \text{Pr(Ti, V)}_2\text{Al}_{20} \) [39–41], we have studied the multipolar Bose-Fermi Kondo model in this setting of a non-Kramers doublet carrying quadrupolar and octupolar moments coupled to \( p \)-orbital electrons in the presence of a tetrahedral crystal field. By using an RG analysis on our model, we find not only two non-Fermi liquid phases and a quadrupolar ordered phase, but also two quantum critical points between the non-Fermi liquid phases and quadrupolar ordered phase. To distinguish between each of these non-Fermi liquid phases and quantum critical points, we compute the multipolar susceptibility exponents at zero temperature and show that the octupolar susceptibility exponent is different at second order in \( \epsilon \) at all of these fixed points. Furthermore, we obtain the temperature scaling behavior of the multipolar susceptibility, and explain how the quadrupolar and octupolar susceptibilities are related to the elastic constants \( (C_{11} - C_{12}) \) and \( C_{44} \), respectively. We propose that measurement of the temperature dependence of the elastic constants \( (C_{11} - C_{12}) \) and \( C_{44} \) using through an ultrasonic measurement in the presence of a magnetic field can be used to distinguish the non-Fermi liquid phases and quantum critical points experimentally.

Possible directions for future work could include applying the work to a variety of other heavy fermion systems. For example, several Yb and Ce compounds exhibit local moments with very high degeneracies, which enables the formation of a large number of multipolar moments [50–54]. Another direction could be to verify our results from the (extended) dynamical mean field theory perspective. One subtlety is that the \( \epsilon_Q, \epsilon_O \) parameters in the bosonic bath density of states should be determined self-consistently. We did not address this detail in our work, but in a full dynamical mean field treatment, this would be taken into account explicitly. Furthermore, in the extended scheme we may even be able to predict order-wave vectors [55].

More generally, our results are indicative of the large variety of multipolar ordered phases and exotic electronic states found in rare-earth metallic systems. The root of the multipolar moments, unusual Kondo couplings, and anisotropic RKKY interactions is the strong spin-orbit coupling and crystal electric field effects, which, as we have shown, can lead to a myriad of quantum critical behaviors beyond the Landau paradigm of symmetry breaking. This suggests there may be new classes of quantum critical points relating Kondo destruction, multipolar ordering, and non-Fermi liquids in multipolar Kondo lattice systems, and that they are experimentally accessible. This opens new doors for exploring the landscape of multipolar quantum matter.

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Appendix A: Non Kramers doublet microscopic environment

In a vacuum, a \( \text{Pr}^3+ \) ion forms a spin \( J = 4 \) system by Hund’s rules. In the presence of a tetrahedral crystal field, these 9 degenerate states are split, and the resulting ground state in the \( \text{Pr(V, Ti)}_2\text{Al}_{20} \) compounds is a non-Kramers \( \Gamma_3 \) doublet spanned by the following two states:

\[
|\Gamma_3^{(1)}\rangle = \frac{1}{\sqrt{2}} |2\rangle + \frac{1}{\sqrt{2}} |−2\rangle, \quad (A1)
\]

\[
|\Gamma_3^{(2)}\rangle = \frac{1}{\sqrt{2}} |2\rangle - \frac{1}{\sqrt{2}} |−2\rangle. \quad (A2)
\]

To determine which multipolar moments are supported by these wave functions, we can compute the matrix elements of Stevens operators in the doublet \( \{ |\Gamma_3^{(1)}\rangle, |\Gamma_3^{(2)}\rangle \} \). In this doublet, we find that, defining a different basis

\[
|\uparrow\rangle = \frac{1}{\sqrt{2}} (|\Gamma_3^{(1)}\rangle + i |\Gamma_3^{(2)}\rangle), \quad (A3)
\]

\[
|\downarrow\rangle = \frac{1}{\sqrt{2}} (i |\Gamma_3^{(1)}\rangle + |\Gamma_3^{(2)}\rangle), \quad (A4)
\]

we find that

\[
\langle \alpha | (\frac{1}{4} O_{22}) | \beta \rangle = \frac{1}{2} \sigma^\alpha_{\alpha\beta}, \quad (A5)
\]

\[
\langle \alpha | (\frac{1}{4} O_{20}) | \beta \rangle = \frac{1}{2} \sigma^\beta_{\alpha\beta}, \quad (A6)
\]

\[
\langle \alpha | (\frac{1}{3\sqrt{6}} T_{yz}) | \beta \rangle = \frac{1}{2} \sigma^\alpha_{\alpha\beta}, \quad (A7)
\]
where $\alpha, \beta$ take the values $\uparrow, \downarrow$ (where these $\uparrow, \downarrow$ are the ones listed in Eqs. (A3)-(A4)), and $\sigma^i$ are the standard Pauli matrices. We emphasize that these $\sigma^i$ matrices and indices $\alpha, \beta$ relate to matrix elements of operators in local moment states, and have nothing to do with the Pauli matrices and $\alpha, \beta$ indices for the conduction electrons in Eqs. (3)-(5).

**Appendix B: Action of Tetrahedral Group**

In order to test which terms in the Hamiltonian are allowed, we need to know how candidate terms transform under action of the tetrahedral group $T_d$, and under time-reversal $\mathcal{T}$. The most economical way to check all transformations is pick two generators of $T_d$, which are $C_{31}$ and $S_{42}$. $C_{31}$ is the rotation by $2\pi/3$ about the $(1,1,1)$ axis, and $S_{42}$ is a rotation by $\pi/2$ about the $z$-axis followed by a mirror reflection across the $xy$ plane. Both of these transformations leave a tetrahedron invariant. Checking all possible Kondo terms respecting the symmetry yields Eqs. (3)-(5). The table of all symmetry transformation is given by Table II.

| Object | $S_{42}$ | $C_{31}$ | $\mathcal{T}$ |
|--------|----------|----------|---------------|
| $x$    | $-y$     | $y$      | $x$           |
| $y$    | $x$      | $z$      | $y$           |
| $z$    | $-z$     | $x$      | $z$           |
| $\sigma^x$ | $\sigma^y$ | $\sigma^z$ | $\sigma^y$ |
| $\sigma^y$ | $-\sigma^z$ | $\sigma^x$ | $-\sigma^z$ |
| $\sigma^z$ | $\sigma^y$ | $\sigma^x$ | $-\sigma^y$ |

TABLE II. Symmetry transformations of various objects under two generators of the tetrahedral group as well as time-reversal $\mathcal{T}$.

**Appendix C: SU(3) Gell-Mann Matrices**

In our multipolar Kondo models, we have three orbitals. To account for all possible traceless hermitian matrices which describe possible fermionic bilinears, we use the generators of SU(3), normalized to $\text{tr}(\lambda^i \lambda^j) = 2\delta_{ij}$. We enumerate these $3 \times 3$ Gell-Mann matrices that appear in the Fermi-Kondo Hamiltonians here:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & l \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad \text{(C3)}$$

**Appendix D: Bose Kondo Coupling**

In order to construct the coupling of the local moment to the bosonic bath while respecting the local symmetry, we construct the effective interaction between spins in the corresponding Kondo lattice. Starting with the Fermi-Kondo Hamiltonian in Eqs. (3)-(5), we can compute the effective interaction between two spins by computing the diagram in Fig. 4. We then replace one of the spin operators in this resulting RKKY interaction with the bosonic field and thereby find the symmetry-allowed coupling of the local moment to the bosonic bath. We emphasize that this is not an actual RKKY interaction between local moments on different sites, and should be conceptually likened to a Weiss mean field coupled to the impurity.

$$S_i \quad \cdots \cdots \quad S_j$$

FIG. 4. Effective Kondo Lattice RKKY Interaction; dotted lines refer to the pseudospin operators (not to be confused with dashed lines in other diagrams referring to pseudofermion propagators) and the solid lines are fermion propagators.

**Appendix E: Details of the renormalization group method**

From the bare Hamiltonian presented in the main text, we can introduce counterterms in order to remove divergences in the loop integrals. When calculating the Fermi-Kondo and Bose-Kondo vertex functions, as well as the pseudofermion self-energy, we can solve for these counterterms order by order and use them to compute the renormalization factors. The corresponding diagrams for the pseudofermion self-energy are given in Fig. 5, the diagrams for the Fermi-Kondo vertex corrections are given in Figs. 6-11, and the diagrams for the Bose-Kondo vertex corrections are given in Figs. 12-13. Details for how to extract renormalization constants from the vertex corrections and self-energy are presented in an excellent reference [30]. In the Feynman diagrams of Figs. 5-13, solid lines refer to conduction electron propagators, dashed lines corresponds to pseudofermion propagators, and the squiggly lines refer to bosonic bath propagators.
FIG. 5. Pseudofermion self-energy, both direct and counterterm contribution.

FIG. 6. Order $K$ corrections to the Fermi-Kondo vertex. There are only direct contributions at this order.

FIG. 7. Order $K^2$ and $g^2$ corrections to the Fermi-Kondo vertex.

FIG. 8. Order $Kg^2$ direct correction to the Fermi-Kondo vertex.

FIG. 9. Order $Kg^2$ counterterm corrections to the Fermi-Kondo vertex.

FIG. 10. Order $g^4$ direct corrections to the Fermi-Kondo vertex.
FIG. 11. Order $g^4$ counterterm corrections to the Fermi-Kondo vertex.

The renormalization constants and wavefunction renormalization up to third order in $K_i$ and fifth order in $g_i$, are given in Eqs. (E1)-(E6).

\[ Z_{KQ_1} = 1 - \frac{\frac{g^2_0}{\epsilon_Q} (g^2_0 - g^2_Q)}{8} + \frac{1}{\epsilon_O} \left[ \frac{g^2_0}{4} + \frac{g^2_0 g^2_Q}{8} \right] + \frac{1}{\epsilon_Q} \left[ \frac{g^2_0}{32} + \frac{3g^2_0 g^2_Q}{8} \right] + \frac{1}{\epsilon_O \epsilon_Q} \left[ \frac{3K^2_{Q2} K_0 g^2_Q}{2K_Q} \right] 
+ \frac{1}{\epsilon'(\epsilon_Q + \epsilon')} \left[ \frac{3K^2_{Q2} K_0 g^2_Q}{2K_Q} \right] 
+ \frac{1}{\epsilon'O} \left[ \frac{K_{Q1} K_{O} g^2_Q}{K_{Q2}} - \frac{3K^2_{Q2}}{2} \right] 

+ \frac{1}{\epsilon'O} \left[ \frac{K_{Q1} K_{Q2} g^2_Q}{K_{Q2}} \right] - \frac{1}{\epsilon_Q} \left( \epsilon_Q + \epsilon' \right) \left[ \frac{g^2_0 g^2_Q}{4} \right] 
+ \frac{1}{\epsilon'(\epsilon_Q + \epsilon') \epsilon'} \left[ \frac{K_{Q1} K_{Q2} g^2_Q}{K_{Q2}} - \frac{3K^2_{Q2}}{2} \right], \quad (E1) \]

\[ Z_{KQ_2} = 1 - \frac{\frac{g^2_0}{\epsilon_Q} (g^2_0 - g^2_Q)}{8} + \frac{1}{\epsilon_O} \left[ \frac{g^2_0}{4} + \frac{g^2_0 g^2_Q}{8} \right] + \frac{1}{\epsilon_Q} \left[ \frac{g^2_0}{32} + \frac{3g^2_0 g^2_Q}{8} \right] + \frac{1}{\epsilon_O \epsilon_Q} \left[ \frac{3K^2_{Q2} K_0 g^2_Q}{2K_Q} \right] 
+ \frac{1}{\epsilon'(\epsilon_Q + \epsilon')} \left[ \frac{3K^2_{Q2} K_0 g^2_Q}{2K_Q} \right] 
+ \frac{1}{\epsilon'O} \left[ \frac{K_{Q1} K_{Q2} g^2_Q}{K_{Q2}} \right] - \frac{1}{\epsilon_Q} \left( \epsilon_Q + \epsilon' \right) \left[ \frac{g^2_0 g^2_Q}{4} \right] 
+ \frac{1}{\epsilon'(\epsilon_Q + \epsilon') \epsilon'} \left[ \frac{K_{Q1} K_{Q2} g^2_Q}{K_{Q2}} - \frac{3K^2_{Q2}}{2} \right], \quad (E2) \]

\[ Z_{KQ_0} = 1 + \frac{1}{\epsilon_Q} \left[ \frac{g^2_0}{2} + \frac{g^2_0}{8} + \frac{g^2_0 g^2_Q}{4} \right] + \frac{1}{\epsilon_Q} \left[ \frac{3g^2_0}{8} + \frac{g^2_0 + g^2_Q}{4} \right] + \frac{1}{\epsilon_Q} \left[ \frac{3g^2_0 g^2_Q}{4} \right] 
- \frac{1}{\epsilon_Q \epsilon_O} \left[ \frac{K_{Q1} K_{Q2} g^2_Q}{K_O} + \frac{\sqrt{3}K^2_{Q2} g^2_Q}{K_O} \right] + \frac{1}{\epsilon_O} \left[ \frac{2K_{Q1} K_{Q2} g^2_Q}{K_O} - \frac{\sqrt{3}K^2_{Q2} g^2_Q}{K_O} \right] 
+ \frac{1}{\epsilon} \left[ \frac{K^2_{Q2}}{2} - \frac{3K^2_{Q2}}{2} \right] + \frac{1}{\epsilon} \left[ \frac{2K_{Q2} (2K_{Q2} - \sqrt{3}K_{Q2})}{K_O} \right], \quad (E3) \]
\[
2K_{Q1}^2 - 4\sqrt{3}K_{Q1}K_{Q2} + 18K_{Q2}^2 + 3K_{Q2}K_O(2K_{Q1} - \sqrt{2}K_{Q2}) + \frac{2K_{Q2}^2}{K_O}(K_{Q1} + 6K_{Q2})(2K_{Q1} - \sqrt{3}K_{Q2})
\]
\[
- \frac{1}{\epsilon_Q(\epsilon_Q + \epsilon_O)} \left[ \frac{K_{Q1}K_O\gamma^2}{K_{Q2}} - \sqrt{3}K_{O\gamma^2} \right],
\]
\[
Z_{g_{Q1}} = 1 - \frac{2}{\epsilon_Q(\epsilon_Q + \epsilon_O)} \left[ \frac{\gamma^2}{4} + \frac{3\gamma^2}{32} \right] + \frac{1}{\epsilon_Q(\epsilon_Q + \epsilon_O)} \left[ \frac{K_{Q1} + 6K_{Q2} - \frac{3K_{Q2}^2}{2}}{2} \right]
\]
\[
Z_{g_{Q2}} = 1 + \frac{1}{\epsilon_Q(\epsilon_Q + \epsilon_O)} \left[ \frac{\gamma^2}{4} + \frac{\gamma^2}{32} \right] + \frac{1}{\epsilon_Q(\epsilon_Q + \epsilon_O)} \left[ \frac{K_{Q1} + 6K_{Q2} - \frac{3K_{Q2}^2}{2}}{2} \right]
\]
\[
Z_f = 1 + \frac{1}{\epsilon_Q(\epsilon_Q + \epsilon_O)} \left[ \frac{\gamma^2}{2} + \frac{\gamma^2}{4} \right] - \frac{1}{\epsilon_Q(\epsilon_Q + \epsilon_O)} \left[ \frac{\gamma^2}{4} + \frac{\gamma^2}{32} \right] + \frac{1}{\epsilon_Q(\epsilon_Q + \epsilon_O)} \left[ \frac{K_{Q1} + 6K_{Q2} + \frac{3K_{Q2}^2}{2}}{2} \right] - \frac{1}{\epsilon_Q(\epsilon_Q + \epsilon_O)} \left[ \frac{3\gamma^2}{8} \right]
\]

From the renormalization constants, we can compute the beta functions,
\[
\frac{dK_i}{d\ln \mu} = K_i \left[ \sum_{k=Q_1, Q_2, O} K_k \partial_{K_k} G^{(0,0,1)}_{K_i} + \frac{g_{Q2}}{2} \partial_{g_{Q2}} G^{(1,0,0)}_{K_i} + \frac{g_{O}}{2} \partial_{g_{O}} G^{(0,1,0)}_{K_i} \right],
\]
\[
\frac{dg_j}{d\ln \mu} = g_j \left[ -\epsilon_j + \sum_{k=Q_1, Q_2, O} K_k \partial_{K_k} G^{(0,0,1)}_{g_j} + \frac{g_{Q2}}{2} \partial_{g_{Q2}} G^{(1,0,0)}_{g_j} + \frac{g_{O}}{2} \partial_{g_{O}} G^{(0,1,0)}_{g_j} \right],
\]

where we Taylor expand the products \(Z_f^{-1}Z_{K_i}\) and \(Z_f^{-1}Z_{g_j}\), as follows in order to obtain the \(G^{(m,n,\ell)}\) factors which appear in Eqs. (E7),(E8):

\[
G_{K_i} \equiv Z_f^{-1}Z_{K_i} = \sum_{m,n,\ell=0} \frac{G^{(m,n,\ell)}_{K_i}}{\epsilon_Q^n \epsilon_O^\ell},
\]
\[
G_{g_j} \equiv Z_f^{-1}Z_{g_j} = \sum_{m,n,\ell=0} \frac{G^{(m,n,\ell)}_{g_j}}{\epsilon_Q^n \epsilon_O^\ell}.
\]

The first terms of the series are \(G^{(0,0,0)}_{K_i} = G^{(0,0,0)}_{g_j} = 1\), and the indices \(i = Q_1, Q_2, O, j = Q, O\).

**Appendix F: Expression for multipolar susceptibility**

From the Bose-Kondo beta functions Eq. (22)-(23) in Sec. III D, the local multipolar moment susceptibilities for the case of zero fixed point values in our model are given by Eq. (27), and turn out to be

\[
\gamma_Q = \lambda_Q^* + 4((K_Q^*)^2 + 6(K_{Q2})^2 + 3(K_Q^*)^2),
\]
\[
\gamma_O = 2\lambda_Q^* + 8((K_Q^*)^2 + 6(K_{Q2}^*)^2),
\]
if \(\lambda_Q^* = 0\), and

\Appendix G: Scaling behaviors of multipolar susceptibility at finite temperature

The scaling behavior of the multipolar susceptibility as a function of imaginary time in the previous section is for zero temperature. Here, we will discuss how to obtain the scaling behavior for finite temperature. Let us assume that we have conformal invariance at the critical point. Assuming that the multipolar moments are primary operators with conformal dimension \(\gamma_i/2\), the correlation function (susceptibility) of the multipolar moment is [37, 46]

\[
\langle S^i(\tau_1)S^i(\tau_2) \rangle \propto \frac{1}{|\tau_1 - \tau_2|^{\gamma_i}}.
\]

Performing a conformal mapping, \(\tau \rightarrow f(\tau) = \frac{\pi}{\tau} \tan\left(\frac{\pi \tau}{\tau}\right)\),

\[
\langle S^i(\tau_1)S^i(\tau_2) \rangle \rightarrow
\]
The scaling behavior of the real and imaginary parts of $F(\omega, T) \equiv \Gamma(\frac{\gamma_i}{2} - \frac{i\omega}{\pi T})/\Gamma(1 - \frac{\gamma_i}{2} - \frac{i\omega}{\pi T})$ is

$$\text{Im}[F(x)] = \begin{cases} 
C_{\text{Im}, >}[x]^{-1}, & |x| \gg 1, \\
C_{\text{Im}, <}[x], & |x| \ll 1.
\end{cases}$$

$$\text{Re}[F(x)] = \begin{cases} 
C_{\text{Re}, >}[x]^{-1}, & |x| \gg 1, \\
C_{\text{Re}, <}[x]^2, & |x| \ll 1.
\end{cases}$$

Then, the temperature dependence of the imaginary part of the multipolar susceptibility is

$$\chi''(\omega, T) \propto \begin{cases} 
T^\gamma_{-2}, & |\omega| \gg T, \\
T^\gamma_{-3}, & |\omega| \ll T.
\end{cases}$$

where $\chi_i = \chi'_i + i\chi''_i$. This is the same result reported in the main text Eqs. (28)-(29).
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