Composite Adaptive Control for Anti-Unwinding Attitude Maneuvers: An Exponential Stability Result Without Persistent Excitation

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This article provides an exponential stability result for the adaptive anti-unwinding attitude tracking problem of a rigid body with uncertain inertia parameters, without the need for a persistent excitation (PE) condition. Specifically, a composite adaptive control scheme with guaranteed parameter convergence is proposed by integrating the dynamic regressor extension and mixing (DREM) technique into the dynamically scaled immersion and invariance adaptive control framework, wherein we modify the scaling factor so that the algorithm design does not involve any dynamic gains. To avoid the unwinding problem, a barrier function is introduced as the attitude error function, along with the establishment of two key algebraic properties for exponential stability analysis. Aiding by a linear time-varying filter, the scalar regressor of DREM is extended to generate an exciting counterpart. In this manner, the derived controller is shown to permit closed-loop exponential stability under a strictly weaker interval excitation condition than PE, in the sense that both the output-tracking and parameter estimation errors exponentially converge to zero. Furthermore, the composite adaptive law is also augmented to achieve finite/fixed-time parameter convergence in a time-synchronized manner. Simulation results are presented to verify our theoretical findings.

1. INTRODUCTION

Rigid-body attitude control has attracted ever-increasing research attention over the past decades. The interest is not only motivated by aerial and space flight applications, but also arises from applications ranging from underwater/ground vehicles to rigid robotic systems [1]. The rigid-body attitude dynamics for many of the aforementioned mechanical systems usually tend to be nonlinear and uncertain, making the design of attitude controllers that can achieve satisfactory closed-loop performance a challenging task. Several elegant solutions to the attitude control problem have been reported since the early 1990s, such as proportional-derivative (PD) plus feed-forward control (the so-called “PD+” control) [2], [3], adaptive control [4], model predictive control [5], sliding mode control [6], etc., as well as a sophisticated combination of these methods. This article considers the attitude tracking control of a fully actuated rigid body with full-state feedback, and the unit quaternion is chosen to parameterize the rigid body attitude.

Due to the redundancy of the unit-quaternion representation, its state space $\mathbb{S}^3$ (the set of all unit-magnitude vectors in $\mathbb{R}^4$), is a double cover of the special orthogonal group $SO(3)$, and consequently, every physical orientation corresponds to two (antipodal) unit quaternions. It implies that the unit-quaternion tracking error accounts for two equilibria, only one of which is considered a priori, while the other is left unstable in most of the existing works. This fact may cause the rigid body to rotate unnecessarily a large angle or even perform almost a full revolution to reach the desired reference trajectory, even if the initial configuration is very close to it, giving rise to the so-called unwinding problem. The unwinding phenomenon was discussed at length in [7] from the perspective of lifts of paths and vector fields from $SO(3)$ to $\mathbb{S}^3$. In practice, such unwinding would lead to inefficient usage of momentum-management devices or fuel. Kristiansen et al. [8] presented a discontinuous
backstepping controller to eliminate the unwinding behaviors. Later, Mayhew et al. [9] developed a hybrid control scheme with hysteresis-based switching to mitigate the sensitivity of discontinuous feedbacks (e.g., see [8]) to measurement noises. By introducing a class of modified attitude error functions (AEFs), smooth control laws were proposed in [10] for anti-unwinding attitude maneuvers. Costic et al. [11] designed adaptive anti-unwinding controllers based on a barrier function. An alternative barrier function for unwinding avoidance was provided in [12]. Recently, Dong et al. [13] derived a sliding mode controller, in which a especially constructed sliding surface was designed to avoid unwinding. Although the foregoing works can indeed avoid unwinding, most of them have only been shown to deliver asymptotic stability for the resulted closed-loop system, and require exact knowledge of the inertia parameters of the rigid body. However, in practice, the body inertia properties may be uncertain due to, for example, fuel consumption, payload variation, and appendage deployment. This limits the opportunities for practical adoption of the existing anti-unwinding attitude controllers. Adaptive control has been extensively studied as an effective tool to deal with parameter uncertainties. In particular, the successful application of adaptive control theory to the rigid body attitude maneuvers has been enabled by the crucial fact that the governing attitude dynamics permits affine regression of the inertia-related terms [11], [14], [15]. A typical feature of these solutions is that they are based on the classical certainty-equivalence (CE) principle. In general, the CE-based adaptive controllers can recover the deterministic-case of closed-loop performance, only when the persistent excitation (PE) hypothesis holds such that the parameter estimates rapidly converge to their true values. Unfortunately, the PE condition is rarely satisfied in practical applications like setpoint regulation. As a result, the performance resulted by the CE-based adaptive controllers is often seen to be poor relative to the uncertainty-free case. Periodic reference signals can be injected to induce PE [16], but at the cost of unnecessary rotations and energy consumption. To surmount this drawback, Astolfi and Ortega [17] significantly departed from the CE principle and instead proposed a new paradigm, known as immersion and invariance (I&I) adaptive control, yielding a non-CE adaptive controller with better transient performance. In the I&I adaptive control framework, a function satisfying a certain partial differential equation (PDE) is introduced, in combination with the learning term from the adaption law, in order to generate the parameter estimates. But, for general multiinput systems, there usually exist no solutions to the involved PDEs. This is the so-called “integrability obstacle.” The main approaches for overcoming this problem can be classified into two categories: regressor filtering method [18] and dynamic scaling method [19]–[25]. It is noted that these CE and non-CE adaptive control methods can only achieve asymptotic convergence of the output-tracking errors, rather than closed-loop asymptotic stability, making them fragile in the absence of sufficient excitation; moreover, they require a restrictive PE condition for parameter convergence. Indeed, it is sometimes necessary to identify the inertia parameters of the rigid body, e.g., after docking/capture operations, after the deployment of payloads, etc.

Theoretically, a stronger stability result—exponential stability—is beneficial for enhancing the robustness of the closed-loop system and for achieving fast and high-precision attitude tracking maneuvers. However, to the authors’ knowledge, there is no previous work that achieves adaptive attitude tracking of a rigid body with anti-unwinding and exponential stability guarantees, using the unit-quaternion parameterization. The technical obstacles mainly lie in two aspects, which are as follows:

1) The commonly used quaternion-based antiunwinding AEFs can hardly exhibit the algebraic properties necessary for proving the closed-loop exponential stability, unless additional restrictions are imposed upon the attitude convergence behaviors and control gains, as discussed in Remark 1.

2) The rank deficiency of the information matrix necessitates the stringent PE condition for parameter convergence.

Overcoming these two obstacles is the main impetus of this work.

To this end, a logarithmic barrier function is chosen as the AEF for unwinding avoidance, and a composite adaptive control strategy is proposed by incorporating the dynamic regressor extension and mixing (DREM) technique developed in [26] into the dynamically scaled I&I adaptive control framework. Following the method of [24], an essentially bounded scaling factor is constructed such that no dynamic gains are needed for the controller implementation, thus dramatically reducing the computational complexity. The derived controller is shown to deliver the exponential stability for the closed-loop system under a strictly weaker interval excitation (IE) assumption than PE, in the sense that both the output-tracking and estimation errors exponentially converge to zero without causing the unwinding phenomenon. The main theoretical contributions of this work are threefold:

1) Two key algebraic properties of the logarithmic AEF are established for exponential stability analysis (see Lemmas 1 and 2 in Section II-D), without additional restrictions on the attitude convergence behaviors or control gains.

2) An linear time-varying (LTV) filter introduced in [27] is incorporated into the DREM to generate a persistently exciting regressor, whereby the derived adaptive controller can guarantee exponential parameter convergence under a strictly weaker IE assumption than PE. This, together with Lemmas 1 and 2, permits the exponential stability of the closed-loop system. Moreover, this controller inherits all the key features (i.e., performance recovery and parameter locking) of the I&I adaptive control methodology.
regardless of excitation conditions. Thus, if the posteriori IE assumption does not hold or the degree of excitation is low during the entire mission, it can still retain a form of the I&I adaptive controller, and therefore, can achieve a better performance than the DREM-based one that is, in essence, based on the CE principle.

3) By adding a norm-normalized sign-function-related power term, the composite adaptive law is further extended to achieve finite-/fixed-time parameter convergence in a time-synchronized manner.

The rest of this article is organized as follows. Section II introduces the problem formulation and preliminaries. The composite adaptive control algorithm and the finite-/fixed-time-synchronized parameter learning laws are derived in Section III, along with rigorous theoretical analyses. Simulation results are given in Section IV. Finally, Section V concludes this article.

II. PROBLEM FORMULATION AND PRELIMINARIES

Throughout this article, \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space, and \( \mathbb{R}^{m \times n} \) denotes the vector space of \( m \times n \) real matrices. \( \mathbf{I}_n \) is the \( n \times n \) identity matrix. For a matrix or vector \( A, A_{ij} \) (respectively, \( A_i \)) denotes its \( (i, j) \)th entry, while \( \|A\| \) denotes either the Euclidean vector norm or the induced matrix norm. We further write \( | \cdot | \) for the absolute value of a scalar and \( \text{sign}(\cdot) \) for the standard sign function. The notation \( S(\cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3} \) is a cross product operator such that \( S(x) \mathbf{y} = x \times y \) for any vectors \( x, y \in \mathbb{R}^3 \). In addition, the set of unit quaternions is given by \( \mathbb{Q}_u = \{ q = [q_v, q_4]^\top \in \mathbb{R}^4 | q_v^\top q_v + q_4^2 = 1 \} \).

A. Rigid-Body Attitude Dynamics

Denote by \( q = [q_v, q_4]^\top \in \mathbb{Q}_u \) and \( \omega \in \mathbb{R}^3 \), respectively, the attitude and angular velocity of the rigid body relative to the inertial frame \( \mathcal{F}_R \), expressed in the body frame \( \mathcal{F}_B \). The dynamic equations of the rigid body are expressed as [18]

\[
\begin{align}
\dot{q}_v &= \frac{1}{2} (S(q_v^\top) + q_4 \mathbf{I}_3) \omega, \\
\dot{q}_4 &= -\frac{1}{2} q_v^\top \omega \\
J \dot{\omega} &= -S(\omega^\top) \dot{\omega} + u
\end{align}
\]

where \( J = J^\top \in \mathbb{R}^{3 \times 3} \) is the positive definite inertia matrix of the rigid body, and \( u \in \mathbb{R}^3 \) is the control torque. We further define \( q_e = [q_e^\top, q_4]^\top \in \mathbb{Q}_u \) as the reference quaternion that specifies the rotation of the desired reference frame \( \mathcal{F}_R' \) from \( \mathcal{F}_R \), and let \( \omega_e \in \mathbb{R}^3 \) be the reference angular velocity expressed in \( \mathcal{F}_R \).

To formulate the attitude tracking issue, an error quaternion \( q_e = [q_e^\top, q_4]^\top \in \mathbb{Q}_u \) is introduced to describe the relative attitude of \( \mathcal{F}_B \) with respect to (w.r.t.) \( \mathcal{F}_R \). As per the multiplication rule of quaternions, \( q_e \) is calculated as follows:

\[
q_e = q_r^{-1} \odot q = \begin{bmatrix}
q_3 q_v - q_4 q_{rv} + S(q_v) q_e \\
q_3 q_4 + q_v^\top q_e
\end{bmatrix}
\]

where \( q_r^{-1} \) is the inverse of \( q_r \), and “\( \odot \)” is the quaternion multiplication operator. The corresponding angular velocity error is defined as \( \omega_e = \omega - C \omega_e \), where the rotation matrix \( C \) from \( \mathcal{F}_R \) to \( \mathcal{F}_B \) is given by

\[
C = (q_e^2 - q_e^\top q_v) \mathbf{I}_3 + 2q_v q_e^\top - 2q_e S(q_v).
\]

As is well known, \( C \) satisfies the following two conditions: \( \|C\| = 1 \) and \( C = -S(\omega_e)C \). Now, the open-loop tracking error dynamics are expressed as [11]

\[
\dot{q}_e = \frac{1}{2} (S(q_e^\top) + q_4 \mathbf{I}_3) \omega_e, \\
\dot{q}_4 &= -\frac{1}{2} q_v^\top \omega_e \\
J \dot{\omega_e} &= -S(\omega_e) \dot{\omega_e} + J (S(\omega_e) \Omega - \dot{\Omega} + \Phi) + u
\]

where \( \Omega = C \omega_e \) and \( \dot{\Omega} = \dot{C} \omega_e \) are defined for brevity.

ASSUMPTION 1 The inertia matrix \( J \) is constant, but otherwise unknown.

ASSUMPTION 2 The reference angular velocity \( \omega_e \) is bounded and at least \( \mathcal{C}^2 \) continuous, and its time derivatives up to order two, i.e., \( \omega_e \) and \( \dot{\omega_e} \) are bounded.

B. Affine Regression

A barrier function is introduced to serve as the AEF for anti-unwinding attitude tracking [12]

\[
V_q = -\alpha \ln \| q_e \|^2, \quad \alpha > 0
\]

which equals to zero only when \( q_e = \pm 1 \) and tends to infinity as \( q_e \rightarrow 0 \). In fact, \( V_q \) imposes a permissible set \( Q_q = \{ q = [q_v, q_4]^\top | q_e = 0 \} \) for the attitude tracking error \( q_e \).

To facilitate the controller design, we further define a filtered tracking error, denoted by \( s \in \mathbb{R}^3 \), as follows:

\[
s = \omega_e + \Lambda q_e
\]

where \( \Lambda = \beta / \text{sign}(q_{4e}) \) with \( \beta > 0 \) being a design constant. Although \( \Lambda \) contains \( \text{sign}(q_{4e}) \), it is a constant (i.e., \( \Lambda = \beta \) or \( \Lambda = -\beta \)) and has the same sign as \( q_{4e}(0) \), under the condition \( q_{4e}(0) \neq 0 \) and the rotation angle is less than 180°.

From a control perspective, the target dynamics desired to be recovered from the uncertainty case is set to

\[
\dot{\omega}_e^* = -k_p s - \xi - \Lambda q_e
\]

where \( k_p > 0 \) is a design constant, and \( \xi = q_{ae} / q_{4e} \) (Gibbs vector) is introduced to cancel the cross-coupling term that will appear in \( V_q \). In view of this, (6) is rewritten as

\[
\dot{\omega}_e = -k_p s - \xi - \Lambda q_e + J^{-1} [u - S(\omega) J \omega + J (S(\omega) \Omega - \dot{\Omega} + k_p s + \xi + \Lambda q_e)]
\]

\[
= -k_p s - \xi - \Lambda q_e + J^{-1} (u + \Phi(\cdot) \theta)
\]

with \( \Phi(\cdot) \in \mathbb{R}^{3 \times 6} \) a known regressor given by

\[
\Phi(\cdot) = -S(\omega) L[\omega] + L[S(\omega) \Omega - \dot{\Omega} + k_p s + \xi + \Lambda q_e]
\]

where \( L[\cdot] : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 6} \) is an affine regression operator such that, for any vector \( x \in \mathbb{R}^3 \), there always has \( Jx = L[\cdot] x \theta \) with \( \theta = [J_{11}, J_{22}, J_{33}, \ldots, J_{12}, J_{23}, J_{31}]^\top \).
C. Control Objective

The control objective of this article is to design an adaptive control law \( u \) for the rigid-body attitude dynamics described by (1) and (2), such that the closed-loop system is exponentially stable in the sense that \( \lim_{t \to \infty} \|q_e(t), \omega_e(t), \dot{\theta}(t)\| = 0 \) (\( \dot{\theta} \) refers to the parameter estimation error) at exponential rates without causing unwinding, despite the absence of PE.

D. Definitions and Lemmas

The IE and PE of a signal are defined as follows [28].

**Definition 1 (IE of a Signal):** A bounded signal \( X(t) \in \mathbb{R}^{m \times n} \), where \( t \in [0, \infty) \), is of IE over a finite-time interval \([t_1, t_2 + t_1]\) if there exist \( t_1 \geq 0 \) and \( t_2 \), \( v > 0 \), such that \( \int_{t_1}^{t+ t_2} X(\tau)X^\top(\tau)d\tau \geq vI_m \).

**Definition 2 (PE of a Signal):** A bounded signal \( X(t) \in \mathbb{R}^{m \times n} \), where \( t \in [0, \infty) \), is of PE if there exist \( t_1 \), \( v > 0 \), such that \( \int_{t_1}^{t+ t_2} X(\tau)X^\top(\tau)d\tau \geq vI_m \) for all \( t_2 \geq 0 \).

Furthermore, two lemmas are provided, which play key roles in showing exponential stability of the closed-loop system.

**Lemma 1:** For all \( q_e \in \mathbb{Q}_q \), the function \((−\ln q_e^2)\) satisfies

\[
−\ln q_e^2 \leq \frac{1 − q_e^2}{|q_e^2|}.
\]

**Proof:** See Appendix A.

**Lemma 2:** Given any constants \( \delta \in (0, 1) \) and \( \alpha > 0 \), there always exist positive constants \( \alpha \) (dependent on \( \delta \) and \( \alpha \)) and \( g \) (dependent on \( \alpha \)) satisfying \( \alpha \geq −\ln \frac{\delta}{2} \) and \( \alpha \geq \alpha \), such that the following holds for \( \delta \leq |q_e| \leq 1 \):

\[
\alpha |\|q_e\|^2 \leq V_e \leq \alpha |\|q_e\|^2 |.
\]

**Proof:** See Appendix B.

**Remark 1:** The commonly used anti-unwinding AEFs e.g., \( V_1 = 1 − q_2^2 \) and \( V_2 = 2(1 − |q_{e1}|) \) discussed in [9], can hardly possess the algebraic properties obtained in Lemmas 1 and 2, to support the establishment of closed-loop exponential stability, without additional restrictions on the attitude convergence behaviors or control gains. This is the primary technical barrier for most of the existing anti-unwinding attitude control schemes to obtain an exponential stability result, even if the inertia matrix \( J \) is known. Here, we take \( V_1 \) and \( V_2 \) as examples to shed more light on the aforementioned argument. For \( V_1 = q_2^2 \), there do exist two positive constants \( \alpha \leq 1 \) and \( \alpha \geq 1 \) such that \( q_2^2 \|q_e\|^2 \leq V_1 \leq \alpha |\|q_e\|^2 | \) holds. This choice of the AEF naturally leads to the control law \( \alpha \omega_e = −\beta_q q_2 \|q_e\| \) with \( \beta_1 > 0 \). With this control law, it follows that \( V_1 = −\beta_q q_2^2 \|q_e\| \) holds. Since \( 0 \leq q_2^2 \leq 1 \), there does not exist a constant \( 0 < \alpha \leq \beta_1 \) such that \( V_1 \leq \alpha |\|q_e\|^2 | \) always holds for any \( q_e \in [-1, 1] \). In view of this, a value of \( \alpha_1 \), one has to require \( |q_{e1}| \to 1 \) such that \( q_2^2(t) \geq \alpha_1/\beta_1 \) always holds after a finite time \( T \). To ensure that \( V_2 \leq −\alpha_1 V_1 \) holds for all \( t > T \). While for \( V_2 \), there also exist two positive constants \( \alpha \leq 1 \) and \( \alpha \geq 2 \) such that \( q_2^2 \|q_e\|^2 \leq V_1 \leq \alpha |\|q_e\|^2 | \) holds. Design the control law as \( \omega_e = −\beta_q \text{sgn}(q_2) q_{e2} \), where \( \beta_2 > 0 \) is the control gain, and \( \text{sgn}(q_2) = 1 \) if \( q_2 \geq 0 \), otherwise \( \text{sgn}(q_2) = −1 \). Using the control law gives \( V_2 = −\beta q_2 q_{e2} \). It is easy to check that, given a \( \alpha_2 > 0 \), \( \beta_2 \) needs to satisfy \( \beta_2 \geq \alpha_2 \) to ensure \( V_2 \leq −\alpha_2 \) always holds. Besides, these two AEFs have the disadvantages of slow convergence rate or discontinuity when \( q_{e1} \to 0 \), as discussed in [9].

III. COMPOSITE ADAPTIVE CONTROL

In this section, a composite adaptive controller is designed to achieve the control objective as stated in Section II-C.

A. Regressor Reconfiguration

For notational brevity, let

\[
y = −\Omega k_q + k_p \Delta_q + \xi − \Lambda Q(q_e)\Omega
\]

with \( Q(q_e) \equiv 0.5(S(q_e^2) + q_{e2}I_1) \). Decompose \( \Phi(\cdot) \) in (11) into two parts

\[
\Phi(\cdot) = \Phi_1(\omega, y) + \Phi_2(\omega, \Omega, q_e)
\]

with

\[
\Phi_1(\omega, y) = k_q L[\omega] + L[y]
\]

\[
\Phi_2(\omega, \Omega, q_e) = −S(\omega)L[\omega] + L[S(\omega)\Omega] + \Lambda L[Q(q_e)\omega].
\]

It is easy to check that \( \Phi_1^T(\omega, y) \) is a Jacobian matrix. Thus, there exists \( \mu_1 \in \mathbb{R}^6 \) such that the following PDE holds:

\[
\frac{\partial \mu_1}{\partial \omega} = \Phi_1^T(\omega, y).
\]

As \( y \) is independent of \( \omega \), one solution to (16) is

\[
\mu_1 = L^T[y] \omega + k_q \hat{\omega}
\]

where \( \hat{\omega} = [0.5 \omega_1^2, 0.5 \omega_2^2, 0.5 \omega_3^2, \omega_2 \omega_3, \omega_1 \omega_3, \omega_1 \omega_2]^T \). Unlike \( \Phi_1^T(\omega, y) \), \( \Phi_2^T(\omega, \cdot) \) (for brevity, the symbol “·” is used to capture all the other arguments of \( \Phi_2 \) except for \( \omega \)) is not a Jacobian matrix, indicating that there is no \( \mu_2 \in \mathbb{R}^6 \) satisfying the PDE \( \partial \mu_2/\partial \omega = \Phi_2^T(\omega, \cdot) \). This is commonly known as the “integrability obstacle,” which prevents the I&I adaptive control methodology in [17] from being applied to solve the problem under study. To overcome this obstacle, we reconfigure \( \Phi_2(\omega, \cdot) \) in the following way:

\[
\frac{\partial \mu_2}{\partial \omega} = \Phi_2^T(\omega, \omega, \cdot)
\]

with \( \Phi_2^T(\omega, \omega, \cdot) \) being \( \{\phi_1(\omega_1, \hat{\omega}_2, \hat{\omega}_3, \cdot), \Phi_2(\hat{\omega}_1, \omega_2, \omega_3, \cdot), \phi_3(\hat{\omega}_1, \hat{\omega}_2, \omega_3, \cdot), \} \), and \( \hat{\omega} \) a filter state determined by

\[
\hat{\omega} = −\hat{\omega} − k_f \hat{\omega}, \hat{\omega}(0) = \omega(0)
\]

where \( \hat{\omega} = y + k_p \omega + S(\omega)\Omega + \Lambda Q(q_e)\omega \) and \( \hat{\omega} = \omega − \omega \). Noting the fact that \( −S(\omega)J\omega + J\hat{\omega} = \Phi(\cdot) \) and from (2) and (19), it follows that

\[
\hat{\omega} = −k_f \hat{\omega} − J^{-1}(\Phi(\cdot) + u).
\]
A direct solution to (20) is
\[ \mu_2 = \int_0^{\omega_1} \Phi(t) \hat{\omega}_2(\dot{\omega}_2, \dot{\omega}_3) \, dt + \int_0^{\omega_2} \Phi(\dot{\omega}_1, \tau, \dot{\omega}_3) \, d\tau \]
\[ + \int_0^{\omega_1} \Phi(\hat{\omega}_1, \dot{\omega}_2, \tau, \cdot) \, d\tau. \]  
(21)

Let \( \mu = \mu_1 + \mu_2. \) From (15), (16), and (18), we get
\[ \frac{\partial \mu}{\partial \omega} = \Phi^T(\cdot) + \Psi^T(\cdot) \]
where the matrix \( \Psi(\cdot) \in \mathbb{R}^{3 \times 6} \) is defined by
\[ \Psi(\cdot) = \Phi(\omega, \dot{\omega}, \cdot) - \Phi(\omega, \cdot). \]  
(22)

B. Prediction Error Construction via DREM

The regressor filtering method is employed to construct the prediction error using only easily obtainable signals. To this end, we rewrite (2) as \( J = W \theta + u \) with \( W = -S(\omega) L[a(\omega)] \), and introduce a linear time-invariant (LTI) filter \( H \) of which the transfer function is \( H(s) = \frac{1}{s + \tau} \), where \( \tau > 0 \) is the filter time constant. Then, the filtered signals \( \omega_f, W_f, \) and \( u_f \in \mathbb{R}^3 \) are generated by passing \( \omega, W, \) and \( u, \) respectively, through \( H \) as follows:
\[ \omega_f = H[\omega], \quad W_f = H[W], \quad u_f = H[u] \]  
(24)

with \( \omega_f(0) = \omega(0)/a, \) \( W_f(0) = 0, \) and \( u_f(0) = 0. \) After a simple calculation, we get
\[ \omega_f = J^{-1}(W_f \theta + u_f). \]  
(25)

Reorganizing (25) gives a linear regressor equation (LRE)
\[ u_f = (I[\omega_f] - W_f) \theta = W_a \theta. \]  
(26)

Since \( u_f, \omega_f, \) and \( W_f \) are computable from the LTI filters in (24), \( \theta \) can be extracted from the LRE (26) without involving any unmeasurable signals.

A gradient-descent estimator is generally chosen to estimate \( \theta \) and has the form of \( \dot{\theta} = -\Gamma W_a^T (W_a \theta - u_f) = -\Gamma N \theta, \) where \( \Gamma > 0 \) and \( N = W_a^T W_a \) is the information matrix. Notably, \( N \) is at most rank 3, and thus, only positive semidefinite. Under this situation, the parameter convergence can be achieved only when \( W_a^T \) is PE, a condition that is rarely met in practice. To relax the excitation requirement for parameter convergence, \( N \) is designed via DREM [26] to possess full rank under the following rather weak IE assumption, which is almost always satisfied in practice, due to initial transient, system noise, etc. [29].

ASSUMPTION 3 There exist \( t_s \geq 0 \) and \( t_c > 0 \) such that the filtered regressor \( W_a \) in (26) is of IE (as per Definition 1) over \([t_s, t_c + t_s]\), with degree of excitation \( \nu > 0, \) that is,
\[ \int_{t_s}^{t_c + t_s} W_a^T (\tau) W_a(\tau) d\tau \geq \nu I_6. \]

REMARK 2: It is noteworthy that the filtered regressor \( W_a \) is a function of both time and the system state \( \omega \), thus taking the form \( W_a = W_a(t; \psi_\omega(t; \omega_0, u)) \), which is dependent on the initial condition \( \omega_0 = \omega(0) \) and the input \( u, \) with \( \psi_\omega(t; \omega_0, u) \) the flow of the dynamics (2) from \( \omega_0 \) at \( t = 0. \)

Indeed, in the closed loop \( u \) is also a function of the states \( \omega \) and \( q \), and the reference signal \( q_r \). This fact makes \( t_s, t_c, \) and the degree of excitation \( \nu \) in the IE condition dependent on the initial conditions, as well as the reference signal. As such, this work only seeks to achieve exponential stability (ES), rather than uniform ES (UES), for the closed-loop system. For this reason, in Assumption 3, we consider \( W_a^T \) as an IE signal, and omit its argument w.r.t. states for ease of presentation. When studying UES of the closed-loop system, it is necessary to take the dependence of states into consideration by defining the uniform IE of the function \( W_a^T (\cdot; \psi \omega) \) as
\[ \int_{t_s}^{t_c + t_c} W_a^T (\tau; \psi \omega(\tau; \omega_0, u)) W_a(\tau; \psi \omega(\tau; \omega_0, u)) d\tau \geq \nu I_6 \]
for all \( \omega_0 \) and \( u \). The interested reader is referred to [29]–[31] for more details on the uniform IE (u-IE) and uniform PE (u-PE) of a function.

Following the line of [25], we first construct an extended LRE (e-LRE) via the Kreisselmeier’s regressor extension proposed in [32], recently revisited in [33] and [34]. Premultiplying both sides of (26) by \( W_a^T \) gives
\[ W_a^T u_f = W_a^T W_a \theta \]  
(27)

to which we apply an LTI filter \( K \) with transfer function \( K(s) = \frac{1}{s + b} \) (\( b > 0 \) is the filter time constant). To be specific, the state-space realization of \( K \) is as follows:
\[ \dot{M} = -b M + W_a^T u_f, \quad M(0) = 0 \]  
(28)
\[ \dot{N} = -b N + W_a^T W_a, \quad N(0) = 0. \]  
(29)

Solving (28) and (29) readily arrives at \( M + N \theta = -b(M - N \theta), \) from which it is easy to obtain the e-LRE
\[ M = N \theta. \]  
(30)

Next, the regressor mixing step is performed to obtain a set of scalar LREs that share the same scalar regressor. We premultiply the adjunct matrix, denoted by \( \text{adj} \{ \cdot \} \), of \( N \) to both sides of the e-LRE (30) to get
\[ Y_i = \Delta \theta, \quad i \in \{1, \ldots, 6\} \]  
(31)

with definitions in the following compact form:
\[ Y \doteq k_1 \text{adj} \{ N \} M^T, \quad \Delta \doteq k_1 \text{det} \{ N \} \]  
(32)

where \( k_1 > 1 \) is introduced to enhance the regressor strength in case of a low level of excitation.

From Assumption 3, one can deduce that
\[ N(t_s + t_c) \geq \int_{t_s}^{t_c + t_c} e^{-b(t_c + \tau)} W_a^T (\tau) W_a(\tau) d\tau \]
\[ \geq e^{-b t_c} \int_{t_s}^{t_c + t_c} W_a^T (\tau) W_a(\tau) d\tau \]
\[ \geq \nu e^{-b t_s} I_6 > 0 \]  
(33)

\footnote{Based on the Cramer’s rule, \( Y_i \) can be calculated by \( Y_i = k_i \text{det} \{ N_{M_i} \}, \) where \( N_{M_i} \) is the matrix \( N \) with its \( i \)-th column replaced by \( M_i \).}
indicating that $N(t)$ becomes full rank at $t = t_r + t_c$ and accordingly $\Delta(t_r + t_c) \geq (\psi - \kappa)$, as shown in (29). However, if $W^*_I$ is only of IE, the value of $\Delta$ will decay to zero with time after the end of IE, due to the exponential forgetting design in (29). In view of this, the direct use of $\Delta$ in designing the parameter estimator $\hat{\theta} = -\Gamma \Delta (\Delta \hat{\theta} - Y)$ with $\Gamma > 0$, as done in [26], may lead to a remarkable decrease in parameter convergence rate over time. In the sequel, we seek to augment the LREs in (31), with the aim of generating a set of new scalar LREs that share a nondegenerate regressor, under Assumption 3.

According to the construction in [27, Proposition 1], an LTV filter is introduced

$$\begin{align*}
\dot{\hat{a}} &= \Delta(Y - \hat{a}) \\
\hat{b} &= -\Delta \hat{a} Z, \quad Z(0) = 1.
\end{align*}$$

(34)

From (31) and (34), it is straightforward to get

$$\hat{a}(t) - \theta = \Xi(t)(\hat{a}(0) - \theta).$$

(35)

By inserting (35) into the original LREs (31), we obtain a set of new scalar LREs as follows:

$$Y = \Delta \hat{a} \theta$$

(36)

where $Y$ and $\Delta$ are defined, respectively, by

$$Y(t) = Y(t) + k_0(\hat{a}(t) - \Xi(t)(\hat{a}(0) - \theta))$$

(37)

and

$$\Delta(t) = \Delta(t) + k_0(1 - \Xi(t))$$

(38)

with $k_0 > 0$ being a design constant.

**Lemma 3:** The LRE extension based on the LTV filter (34) guarantees the following:

1. the LTV filter (34) is internally stable;
2. the newly obtained scalar regressor $\Delta$ satisfies $\Delta(t) \geq 0$ on $t \in [0, \infty)$;
3. if Assumption 3 holds, then there exists a constant $h > 0$ such that $\Delta(t) > h$ on $t \in [t_r + t_c, \infty)$.

**Proof:** From (35) and the fact that $\Delta'(t) \geq 0$ for $t \geq 0$, it can be claimed that $\|\hat{a}(t) - \theta\|$ is not increasing, showing the internal stability of the LTV filter (34). Recalling (29), we find that the information matrix $N$ is a symmetric positive semidefinite, which indicates that $\Delta(t) \geq 0$ for all $t \geq 0$. Besides, one can easily infer that $(1 - \Xi(t)) \geq 0$. It is then evident from (38) that $\Delta(t) \geq 0$ on $t \in [0, \infty)$.

Furthermore, we consider Assumption 3 holds, that is, $W^*_I$ is of IE. Given this, (33) and its accompanying result $\Delta(t_r + t_c) \geq (\psi - \kappa)$, there exist $t_\Delta > 0$ and $\alpha > 0$ such that

$$\int_{t_r + t_c}^{t} \Delta(t) dt \geq \alpha \Rightarrow \int_{0}^{t} \Delta^2(t) dt \geq \alpha$$

(39)

for all $t \geq t_r + t_c$. Then, we have the following:

$$W^*_I \in \text{IE} \Rightarrow \Delta \in \text{IE}, \quad \forall t \geq t_r + t_c.$$ (40)

C. Composite Adaptive Controller

A composite adaptive controller is designed by applying a dynamically scaled $I^2L$ adaptive control method with parallel combination of a prediction-error-driven (also called DREM-based) learning law. Design the control law and the accompanying adaptive laws as

$$u = -\Phi(\hat{\theta} + \xi)$$

(41)

$$\dot{\hat{a}} = -\gamma [\hat{\mu} - (\Phi + \Psi) \hat{y}] - \gamma \lambda \epsilon$$

(42)

$I^2L$-based learning law: Part I DREM-based learning law

$$\xi = \frac{\gamma \mu}{\epsilon}$$

(43)

$I^2L$-based learning law: Part II

where $\gamma, \lambda > 0$ are constant gains, $\hat{\mu} = \hat{\mu} - (\partial \mu / \partial \omega) \omega$, and $\epsilon \in \mathbb{R}^5$ is the prediction error vector given by

$$\epsilon = \Delta(\hat{\theta} + \xi) - Y.$$ (44)

Actually, the composite term $\dot{\theta} + \xi$ acts as the estimate of $\theta$, thus the parameter estimation error is defined as $\dot{\theta} = \hat{\theta} + \xi - \theta$. From (36) and (44), it follows that $\epsilon = \Delta(\hat{\theta})$.

Inserting (41) into (10) yields

$$\dot{\mu} = J^{-1} \dot{\Phi} - \gamma \lambda \epsilon.$$ (45)

Bearing (2), (22), (42)–(44) in mind and recalling the definitions of $\Phi$ and $\bar{y}$, we have

$$\dot{\theta} = \gamma \frac{\partial \mu}{\partial \omega} \omega + \gamma \hat{\mu} - \gamma \left[\hat{\mu} - (\Phi + \Psi) \bar{y}\right] - \gamma \lambda \epsilon$$

(46)

In what follows, the dynamic scaling technique is applied to obviate the effect of perturbation $\Psi$ on parameter estimation. Inspired by [24], a naturally bounded scaling factor $R(t)$ that satisfies $R(t) > R_0 \geq 0$ for a constant $R_0 > 0$ is presented to form the scaled estimation error

$$z = \frac{\hat{\theta}}{R}$$

(47)

with $R$ having the following form

$$R = \frac{\sqrt{J_m}}{\epsilon^0 \epsilon^m} e^{-\nu m}$$

(48)

where $J_m$ denotes the minimum eigenvalue of $J$, and $f(r)$ is defined as

$$f(r) = f_m \tanh(r) + 1.$$ (49)

Note that $f_m > 0$ can be freely chosen to adjust the maximum value of $f(r)$ (noting that $1 < f(r) \leq f_m + 1$), while $r$ is a time-varying scalar satisfying $\dot{r}(t) > 0 \forall t \geq 0$ and is determined by

$$\dot{r} = \gamma \frac{f(r) \sqrt{\ln f(r)}}{\frac{\partial f(r)}{\partial r}} \|\Psi\|^2, \quad r(0) > 0.$$ (50)

Taking the time derivative of $z$ and noting (46)–(50) lead to

$$\dot{z} = -\gamma (\Phi + \Psi) J^{-1} \Phi z - \gamma \lambda \Delta \xi z - \frac{\gamma}{2 \nu m} \|\Psi\|^2 z.$$ (51)
Now, consider a Lyapunov-like function
\[ V_c = \frac{1}{2\gamma'} z^T z. \]  \hfill (52)

By Young’s inequality, we deduce that
\[ \dot{V_c} \leq -\frac{J_m}{2} \| J^{-1} \Phi z \|^2 - \lambda \Delta_0 \| z \|^2 \leq 0 \]  \hfill (53)

where the fact that \( \Delta_0(t) \geq 0 \forall t \geq 0 \) has been used. As a consequence, the equilibrium \( z = 0 \) of the scaled estimation error dynamics (51) is uniformly globally exponentially stable and accordingly uniformly bounded.

**THEOREM 1:** Consider the rigid-body attitude dynamics described by (1) and (2) under Assumptions 1 and 2. Given the initial conditions \( q(0), \omega(0) \) and the reference trajectory \( q_r(t), \omega_r(t) \) satisfying \( q_r(0) \in Q_a \), if \( k_p \) and \( k_f \) are chosen as \( k_p = k_f = \kappa (f_m + 1) \) with \( \kappa > 0 \) a design constant, then the implementation of the control law (41) in conjunction with the parameter estimator (42) and (43) leads to the following:

1. All closed-loop trajectories converge asymptotically to an invariant attracting manifold \( M \) given by
   \[ M = \{ \tilde{\theta} \in \mathbb{R}^6 | \Phi \tilde{\theta} = 0 \}. \]  \hfill (54)

As a result, the uncertain plant dynamics are ultimately immersed into the target dynamics (9);

2. The output-tracking errors \( q_{ev}(t) \) and \( \omega_{et}(t) \) asymptotically converge to zero on \( t \in [0, \infty) \), and the unwinding phenomenon is strictly avoided.

3. If Assumption 3 holds, the origin of the closed-loop system is exponentially stable on \( t \in [t_r + t_e, \infty) \), in the sense that \( q_{ev}(t), \omega_{et}(t), \) and \( \tilde{\theta}(t) \) converge to zero exponentially fast on \( t \in [t_r + t_e, \infty) \).

**PROOF:** Consider the overall Lyapunov-like function
\[ V = V_q + \frac{1}{2} s^T s + \frac{1}{2} \omega^T \omega + \eta V_c \]  \hfill (55)

where \( \eta = 2(1/\kappa + \varrho) \) with \( \varrho > 0 \) is introduced just for stability analysis. Taking the time derivative of \( V \) and noting (5), (8), (20), (45), and (53), we have
\[ \dot{V} \leq \frac{q_{ev}^T}{|q_{ev}|} \omega_{et} + s^T s + \omega^T \omega - \eta \frac{J_m}{2} \| J^{-1} \Phi z \|^2 - \eta \lambda \Delta_0 \| z \|^2 \leq -\beta \frac{|q_{ev}|}{|q_{ev}|} q_{ev} - k_p s^T s - R s^T J^{-1} \Phi z - k_f \omega^T \omega \]
\[ + R \omega^T J^{-1} \Phi z - \eta \frac{J_m}{2} \| J^{-1} \Phi z \|^2 - \eta \lambda \Delta_0 \| z \|^2. \]  \hfill (56)

By Young’s inequality, we have
\[ R \leq \frac{\sqrt{J_m}}{e^{1/2L_2}} \cdot \frac{e^{1/2L_2}}{\sqrt{J_m} e^{3/2L_2}} \leq \sqrt{J_m} e^{3/2L_2} \]  \hfill (56a)
\[ - R s^T J^{-1} \Phi z \leq \kappa f(r) \| s \|^2 + \frac{J_m}{2\kappa} \| J^{-1} \Phi z \|^2 \]  \hfill (56b)
\[ R \omega^T J^{-1} \Phi z \leq \kappa f(r) \| \omega \|^2 + \frac{J_m}{2\kappa} \| J^{-1} \Phi z \|^2. \]  \hfill (56c)

Using (57a)–(57c) in (56) and further from the fact that \( \eta = 2(1/\kappa + \varrho) \) and \( 1 < f(r) \leq f_m + 1 \), it follows that
\[ \dot{V} \leq -\frac{\beta}{|q_{ev}|} \| q_{ev} \|^2 - \frac{\kappa (f_m + 1)}{2} \| s \|^2 - \frac{\kappa (f_m + 1)}{2} \| \omega \|^2 - \eta \lambda \Delta_0 \| z \|^2. \]  \hfill (58)

Inspecting (58), it is found that \( \dot{V}(t) \leq 0 \) for all \( t \geq 0 \), from which we establish the boundedness of \( V \), and hence, \( V_{q} \). The latter, together with the fact that \( q_{ev}(0) \in Q_a \), implies that there exists a positive constant \( \delta \in (0, 1) \) such that \( q_r \) remains in a compact subset \( \mathbb{B} \subset Q_a \), defined by \( \mathbb{B} = \{ |q_r| \in Q_a | \delta \leq |q_{ev}| \leq 1 \} \) for all \( r \geq 0 \); in other words, the set \( \mathbb{Q}_a \) is forward invariant. Thus, the unwinding phenomenon is strictly avoided during the entire mission. It is then clear that \( \tilde{\theta} \in (\mathbb{S}_\infty) \mathbb{S}_\infty \). By integrating both sides of (58), we know that \( \int_0^\infty \dot{V}(t) \) exists and is finite, which then implies that \( q_{ev}, s, \omega, \) and \( J^{-1} \Phi z \) are bounded on \( \mathbb{L}_2 \cap \mathbb{L}_\infty \). The boundedness of \( q_{ev}, \) and \( s, \) we have \( \omega \in \mathbb{L}_\infty \), and \( \mathbb{N} \). Furthermore, invoking Assumption 2 and the fact that \( \omega, \tilde{\omega} \in \mathbb{L}_\infty \), one can easily deduce \( \omega \in \mathbb{L}_\infty \), so does \( \tilde{\omega} \). Based on the aforementioned argument, it is evident from (11) that \( \Phi \in \mathbb{L}_\infty \), and from (18) and (23) that \( \Psi \in \mathbb{L}_\infty \). As the scaling factor \( R \) is naturally bounded [see (48)], \( J^{-1} \Phi \) is \( \mathbb{L}_2 \cap \mathbb{L}_\infty \) and is actually equivalent to \( J^{-1} \Phi \) \( \in \mathbb{L}_2 \cap \mathbb{L}_\infty \). In addition, the state and regressor filtering operations \( \mathcal{H}(\omega) \) and \( \mathcal{H}(W) \) described by (24) with bounded inputs generate bounded signals \( \omega \), and \( W \), respectively, which leads to the boundedness of \( \mathbb{W} \) and \( \mathbb{N} \). From \( \mathbb{N} \), we show that \( \Delta \), and hence, \( \Delta_0 \in \mathbb{L}_\infty \). It is now straightforward to check from (46) that \( \tilde{\theta} \in \mathbb{L}_\infty \). Noting (10) and (11), we have \( \omega, \Phi \in \mathbb{L}_\infty \). A direct deduction gives the conclusion that \( q_{ev}, s, \omega, \) and \( J^{-1} \Phi \) are bounded, indicating the uniform continuity of \( q_{ev}, s, \omega, \) and \( J^{-1} \Phi \). Then, by applying Barbalat’s lemma, we guarantee that
\[ \lim_{t \to \infty} q_{ev}(t), s(t), \omega(t), J^{-1}(\Phi(t) \tilde{\theta}(t)) = 0. \]  \hfill (59)

From (8) and (59), it follows that \( \lim_{t \to \infty} \omega_{et}(t) = 0 \). In fact, the convergence condition \( \lim_{t \to \infty} \mathcal{J}(\Phi(t) \tilde{\theta}(t)) = 0 \) directly contributes to the establishment of an invariant attracting manifold \( M \) as defined in (54). In view of this, all the closed-loop trajectories (in the domain of interest) asymptotically converge to \( M \), showing that the uncertain plant dynamics will be immersed into the target dynamics (9). Consequently, the ideal closed-loop performance obtained in the deterministic case can be ultimately recovered without requiring convergence of the parameter estimates to the corresponding true values.

Next, we consider Assumption 3 holds such that \( \Delta_0(t) > h > 0 \) for all \( t \geq t_r + t_e \), as stated in Lemma 3. Under this condition, let \( \mathcal{Z} = [q_{ev}^T, s^T, \omega^T, z^T]^T \). Then, recalling the fact that \( |q_{ev}| \geq \delta (\delta \in (0, 1)) \) and Lemma 2, it can be easily shown that \( V \) in (55) is bounded by the following:
\[ \xi \| Z \|^2 \leq V(q_{ev}, s, \omega, z) \leq \xi \| Z \|^2 \]  \hfill (60)

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where \( \zeta \leq \min\{\alpha, 1/2, \eta/2\gamma \} \) and \( \zeta \leq \max\{\alpha, 1/2, \eta/2\gamma \} \). Furthermore, by invoking Lemma 1, (58) becomes

\[
V \leq \beta \ln(q_{d2}) - \frac{\kappa (f_m + 1)}{2} \|s\|^2 - \frac{\kappa (f_m + 1)}{2} \|\omega\|^2 \\
- \eta \lambda h \|z\|^2 \leq -\zeta V,
\]

for all \( t \geq t_1 + t_c \), where \( \zeta \leq \min\{\beta, \kappa (f_m + 1), 2\gamma \lambda \} \). By the comparison lemma, we get

\[
V(t) \leq V(T_r) \exp(-\zeta (t - T_r)) \quad \forall t \geq T_r
\]

where \( T_r = t_1 + t_c \) is defined hereafter for notational brevity. This implies that \( V(t) \to 0 \) uniformly exponentially fast on \( t \in [T_r, \infty) \), which together with (60) allows us to conclude the exponential stability of the equilibrium point \( Z = 0 \) on \( t \in [T_r, \infty) \) by [35, Th. 4.10]. Furthermore, from (8) and the exponential convergence of \( q_{r1}(t) \) and \( s(t) \) on \( t \in [T_r, \infty) \), we conclude that \( \omega_a(t) \) converges to zero exponentially fast on \( t \in [T_r, \infty) \), thus completing the proof.

**Remark 3:** The key steps that permit achieving exponential stability for the closed-loop dynamics on \( t \in [T_r, \infty) \) are to deduce (60)–(62) under the strictly weak Assumption 3, through the introduction of a DREM-based learning term \(-\gamma \lambda e \) into the adaptive law (42) and our establishment of Lemmas 1 and 2 for \( V_q \). Strictly speaking, the stability condition can be considered as a local result, since the proposed adaptive controller itself provides exponential stability on \( t \in [T_r, \infty) \) for the initial conditions satisfying \( q_r(0) \in \mathbb{D} \subseteq \mathbb{B} \), a subset of \( \mathbb{Q}_a \). Although it is difficult to quantify the interior bound of \( \mathbb{D} \), \( q_{r1}(0) \) can be chosen as close as possible to zero until the actuators under magnitude limits cannot provide sufficient torques to ensure system stability. Thus, the local result has a quite large domain of attraction.

**Remark 4:** A bounded scaling factor \( R \) in (48) is introduced, instead of a single term \( \Gamma r \) determined generally by [20]

\[
\dot{r} = \gamma \Gamma r \|\Psi\|^2 r, \quad \text{any } \Gamma > 1/(2J_m)
\]

to construct the scaled estimation error \( z \). This modification offers the following two prominent advantages for control design and analysis:

1) it eliminates the need for the minimum eigenvalue of \( J \) in designing the scaling factor dynamics (50), as discussed in [21], [24], and [25];

2) only constant gains \( k_r \) and \( k_r \) are chosen for the target dynamics (9) and the state filter (20), respectively, rather than dynamic ones involving \( r \) that sustains unlimited growth under perturbed and noisy conditions due to the lack of damping.

As stated in [23] and [24], this avoids causing high-gain control, and hence, undesirable transient behavior of the closed-loop system.

**Remark 5:** Concerning the DREM-based gradient descent estimator \( \hat{\theta} = -\Gamma (\Delta \hat{\theta} - Y) \) derived based upon the LRE (31), it has been claimed in [26] and [33] that nonsquare integrability of the scalar regressor \( \Delta \) is required to ensure asymptotic parameter convergence, which further becomes exponential by imposing the PE condition. However, the conditions \( \Delta \notin \mathbb{L}_2 \) and \( \Delta \in \mathbb{P} \) can hardly be fulfilled in practice. In many cases, \( W_q \) has only a weak IE, so does \( \Delta \). As a result, the gradient estimator usually fails to achieve consistent parameter convergence. For the scenarios satisfying IE, a determinant detection and updating freeze mechanism [25], [36], [37] can be introduced to ensure nondegradation of \( \Delta \) after the end of excitation. But, such a manner would make the estimator lose alertness to parameter variations. To circumvent the aforementioned problems, an LTV filter (34) borrowed from [27] is applied to extend the scalar LREs in (31) to (36), yielding a new scalar regressor that satisfies \( \Delta(t) \in \mathbb{P} \) on \( [t_r + t_c, \infty) \) under the IE Assumption, as shown in Lemma 3. Thus, the excitation requirement for parameter convergence is greatly relaxed.

**Remark 6:** Although this work shares some similarities with [23], it remains substantial differences and improvements, which lie in the following two aspects.

1) In [23], the sign function of the initial attitude quaternion is involved in the AEF to avoid the unwinding problem. However, this AEF does not possess algebraic properties that can support the establishment of closed-loop exponential stability. Different from [23], a logarithmic barrier function \( V_q \) given in (7) is chosen in this work as the AEF, whereby the unwinding phenomenon is strictly avoided. Moreover, it is shown that \( V_q \) satisfies Lemmas 1 and 2, which pave the way for exponential stability analysis.

2) The work of [23] only achieves asymptotic convergence of the output errors, while failing to guarantee parameter convergence in the absence of PE. On the contrary, the proposed adaptive control scheme, as an extension of the I&I adaptive control methodology, can not only preserve all the key features of [23], but also ensure exponential parameter convergence under a strictly weak IE condition, instead of PE. The latter, together with Lemmas 1 and 2, directly contributes to exponential stability of the closed-loop system under IE.

**D. Finite-/Fixed-Time Parameter Convergence**

In this subsection, we show that the composite adaptive law (42) can be extended to achieve finite-/fixed-time-synchronized parameter convergence. Before proceeding, a continuous vector function of the following form is introduced:

\[
|x| = \max\{\|x\|, |\text{Sgn}_n(x)|\}
\]

where \( \tau > 0 \) and \( \text{Sgn}_n(x) \) is the so-called norm-normalized sign function [38]

\[
\text{Sgn}_n(x) = \begin{cases} 
\frac{x}{\|x\|}, & \text{if } x \neq 0 \\
0, & \text{if } x = 0
\end{cases}
\]
Then, the adaptive law (42) is modified as
\[ \dot{\theta} = -\gamma (\tilde{\theta} - (\Phi + \Psi)^\top y) - \gamma (\lambda \epsilon + \Theta) \] (66)
with \( \Theta \) being a power term defined by
\[ \Theta = \lambda_1 |\epsilon|^{\alpha_1} + \lambda_2 |\epsilon|^{\alpha_2} \] (67)
where \( \lambda_1, \lambda_2 > 0 \) are adaption gains, \( t_1 \in (0, 1) \), and \( t_2 > 1 \).
This results in slight modifications of the estimation error dynamics (46) and the scaled error dynamics (51) as follows:
\[ \dot{\hat{\theta}} = -\gamma (\Phi + \Psi)^\top J^{-1} \Phi \hat{\theta} - \gamma (\lambda \epsilon + \Theta) \] (68)
\[ \dot{\tilde{z}} = -\gamma (\Phi + \Psi)^\top J^{-1} \Phi \tilde{z} - \gamma \lambda \Delta_1 \tilde{z} - \gamma \frac{\Theta}{R} - \frac{\gamma}{2m} \|\tilde{z}\|^2 \] (69)
As per the definitions of \( \cdot^\top \) in (64) and \( \Theta \) in (67), it is easy to check that \( z^\top \Theta/R \geq 0 \) always holds for all \( t \geq 0 \).
Thus, the aforementioned modifications will not affect the results presented in Theorem 1. As for all \( t \in [T_s, \infty) \), it has \( \Delta_1(t) > \bar{h} \), whereby we further get
\[ z^\top (\Theta/R) = \left\{ \begin{array}{ll}
\lambda_1 \Delta_1^u R^{-1} ||z||^{\alpha_1 + 1} + \lambda_2 \Delta_1^u R^{-1} ||z||^{\alpha_2 + 1}, & \text{if } \dot{\hat{\theta}} \neq 0 \\
0, & \text{if } \dot{\theta} = 0
\end{array} \right. \]
Consider again the Lyapunov-like function \( V_z \) in (52). Then, using the aforementioned equation in \( V_z \) yields
\[ V_z \leq -2\lambda \gamma h V_z - c_1 V_z^{\alpha_1} - c_2 V_z^{\alpha_2} \] (70)
for all \( t \geq T_s \), where \( c_1 \equiv (2\gamma)^{\alpha_1} \lambda_1 h R_1^{-1} \) and \( c_2 \equiv (2\gamma)^{\alpha_2} \lambda_2 h R_1^{-1} \), and \( R_m \equiv \inf_{t \in [T_s, \infty)} R(t) > 0 \). From (70) and the finite-/fixed-time stability theorem [39], it is shown that the equilibrium point \( z = 0 \) of the scaled estimation error dynamics (69) is fixed-time (resp., finite-time) stable on \( t \in [T_s, \infty) \), if \( \lambda_1, \lambda_2 \neq 0 \) (resp., \( \lambda_1 = 0 \) and \( \lambda_2 \neq 0 \)). As \( R > 0 \), \( \dot{\theta} \) also converges to zero in finite/fixed time.

Apart from the finite-/fixed-time convergence, an interesting phenomenon called “time-synchronized convergence” is also observed for \( \theta \). By this concept, we mean that all the elements of \( \dot{\theta} \) converge to zero almost at the same time. Consider that \( \lim_{t \to \infty} J^{-1}(t) \dot{\theta}(t) = 0 \) and the convergence can be made arbitrarily fast by tuning the learning gain \( \gamma \), the first term on the right-hand side of (68) can be guaranteed to decay much faster than \( \dot{\theta} \). Ignoring this term in (68) yields
\[ \dot{\hat{\theta}} = -\gamma (\lambda \epsilon + \Theta). \] (71)
From (67) and (71), it is easy to verify that, for any \( \tilde{\theta}_i, \tilde{\theta}_j \neq 0, i \neq j \), \( \delta (\tilde{\theta}_i \tilde{\theta}_j) = 0 \) always holds, whereby we conclude that \( \tilde{\theta}_i / \tilde{\theta}_j \) (with \( l \neq 0 \) being a constant) for any nonzero \( \tilde{\theta}_i \) and \( \tilde{\theta}_j, i, j \in \{1, 2, \ldots, 6\}, i \neq j \). Hence, \( \tilde{\theta}_i \) and \( \tilde{\theta}_j \) converge to zero almost at the same time, indicating the time-synchronized convergence. We should emphasize that the aforementioned analysis is established by ignoring the term \( -\gamma (\Phi + \Psi)^\top J^{-1} \Phi \dot{\theta} \) in (68), which makes the formula (71) approximately hold. This is the reason why we claim that all the elements of \( \dot{\theta} \) converge to zero almost at the same time.

IV. NUMERICAL SIMULATIONS

In this section, numerical simulations are carried out to show the effectiveness and key features of the composite adaptive control scheme developed in this article. Consider the rigid-body attitude dynamics (1) and (2), where the inertia matrix is given by \( \Theta = [20, 17, 15, 1.4, 0.9, 1.2]^\top \) kg · m². The reference attitude is set to (20): \( q_r(0) = [0, 0, 0, 1]^\top \) and the velocity profile \( \omega = \omega_1 = 1 \) rad/s, where \( \omega_1 = [1, 1, 1]^\top \) and \( \omega_2 \) is of the following form:
\[ \omega_2 = 0.3(1 - e^{-0.01t}) \cos t + e^{-0.01t} (0.08\pi + 0.006 \sin t). \]
This setting evidently generates a non-PE reference trajectory. To verify the anti-unwinding capability of the proposed adaptive control algorithm, two sets of initial body attitudes that correspond to the same physical orientation but render opposite signs of \( q_{ad}(0) \) are considered in the following simulations:

Case 1: \( q_{ad}(0) = q_0 \) and \( q_{ad}(0) = \sqrt{1 - \|q_{ad}(0)q_{ad}(0)\|}; \)
Case 2: \( q_{ad}(0) = -q_0 \) and \( q_{ad}(0) = -\sqrt{1 - \|q_{ad}(0)q_{ad}(0)\|}; \)
with zero rate, where \( q_0 = [0.33, -0.3, -0.62]^\top \). It is shown that \( q_{ad}(0) > 0 \) for Case 1 and \( q_{ad}(0) < 0 \) for Case 2.

A. Nominal Performance

Here, the composite adaptive controller is simulated for Case 1 under a nominal (i.e., perturbation free) scenario, in order to support the theoretical findings. The control parameters are selected as \( \alpha = 0.5, \beta = 0.1, a = 5, b = 0.5, k_{f1} = 8, \gamma = 25, \lambda = 0.01, \kappa = 0.5, \) and \( f_m = 2 \).
As the reference trajectory is weakly exciting, \( \dot{\kappa}_c = 1 \times 10^6 \) is introduced into (32) to enhance the regressor signal strength. In addition, the initial conditions of the LTV filter (34) and the parameter estimator (42) are set as \( \chi(0) = 0 \) and \( \hat{\theta}(0) + \xi(0) = [10, 30, 8, 0, 0, 0]^\top \) (in fact, \( \xi(0) = 0 \)), respectively.

From Fig. 1(a) and (b), it is shown that the attitude and angular velocity tracking errors converge asymptotically to zero. Note that an IE condition is satisfied during the initial phase of the mission such that \( \Delta_n(t) \in PE \) after \( T_e = 4 \) s, as clearly seen in Fig. 1(d). Thus, more precisely speaking, \( q_{ad} \) and \( \omega_r \) are exponentially convergent on \( t \in [T_e, \infty) \).

The control torques are plotted in Fig. 1(c), where we observe that the torque demands are smooth and remain time varying at the steady state to ensure tracking of the assigned reference trajectory. The time responses of \( \Phi \hat{\theta}, \Delta_n, \) and \( \hat{\theta} \) are depicted in Fig. 1(d), from the left subplot of which it is clear that \( \Phi \hat{\theta} \) converges asymptotically to zero, indicating that the closed-loop system trajectory is indeed attracted to the invariant manifold \( \mathcal{M} \). Thus, the uncertain attitude dynamics will be ultimately immersed into the target dynamics (9). As can be seen in the top
right subplot of Fig. 1(d), the extended scalar regressor \( \Delta_3(t) \) turns to be strictly positive \( (\Rightarrow \Delta_3(t) \in \text{PE}) \) after \( T_s = 4 \) s, only under an extremely weak IE condition, which is consistent with Lemma 3. The parameter convergence with good transient behaviors is observed in the bottom right subplot of Fig. 1(d).

To show the key role played by the DREM-based learning law, respectively, at regular intervals. Although \( T_s = 4 \) s, \( \Delta_3 \) is very small at \( T_s \) and gradually increases until to \( t \approx 12 \) s. Bearing this in mind, the initial point in Fig. 2 is taken at \( t = 5 \) s for more clear illustration. By observing the arrows in Fig. 2, we intuitively see that the two learning laws synchronously drive the estimates of the principle inertia parameters in two linearly independently directions to their true values. This shows the importance of the DREM-based learning law in ensuring parameter convergence, in the absence of PE. To further justify the finite-/fixed-time convergence property of the generalized parameter estimator (66), we simulate the finite- and fixed-time adaption cases in which the gains in \( \Theta \) are set to \( \lambda_1 = 0.01, \lambda_2 = 0 \), and \( \lambda_1 = \lambda_2 = 0.01 \), respectively, and the power of numbers are identically chosen as \( \iota_1 = 0.85 \) and \( \iota_2 = 1.1 \). The time responses of \( |\tilde{\theta}_i|, i = 1, 2, \ldots, 6 \) under finite- and fixed-time adaptation extensions are depicted in Fig. 3. The estimation results from the original estimator is also provided (see the left subplot of Fig. 3) to serve as baseline for convergence time comparisons. As seen in Fig. 3, the inclusion of \( \Theta \) in the adaptive law helps achieve finite-time-synchronized parameter convergence, and the fixed-time adaptation extension delivers faster convergence rate than the finite-time one.

**B. Comparison Results**

A nominal scenario is considered, whereas the initial body attitude is chosen as Case 2 to further show the
efficiency of the proposed composite adaptive controller (denoted as CI&IAC) in unwinding avoidance. For comparison purposes, two classical adaptive controllers are also simulated.

1) **Non-CE adaptive controller in [20] (denoted as NCEAC):** This controller is derived using the I&I adaptive control method with dynamic scaling. The reader is referred to [20] for the design details of NCEAC, and the design parameters are chosen as $k_p = 0.48$, $k_v = 1$, $k_r = 0.2$, $k = 0.01$, $k_H = 0.1$, $k_1 = k_2 = k_3 = 1$, $\Gamma = 501\lambda$, $v = 0.5$, and $\epsilon = 0.00001$. We note that although the proposed CI&IAC is partially inspired by the NCEAC, it gives substantial improvements. As dictated by Theorem 1, the CI&IAC preserves the key features of the NCEAC, while achieving exponential parameter and tracking error convergence without causing unwinding.

2) **CE-based adaptive controller in [11] (denoted as CEAC):** The CEAC not only achieves asymptotic attitude tracking, but is capable of avoiding the unwinding phenomenon by introducing a potential function $\frac{1}{2}q_e^Tq_e(1 - q_e^Tq_e)$ to ensure that $q(t) \neq 0$ for all $t \geq 0$. The structure of the CEAC is detailed in [11] with its design parameters given as $\alpha = 10$, $K = 10$, and $\Gamma = 0.02$.

The initial values of the parameter estimates for the aforementioned two controllers are chosen the same as those of the proposed CI&IAC. In addition, to permit a fair comparison, the design parameters of the NCEAC and the CEAC are judiciously tuned by trial and error to obtain similar tracking error convergence rates as the CI&IAC. As can be seen in Fig. 4(a) and (b), all the three controllers achieve asymptotic convergence of the tracking error vectors $q_e$ and $\omega_e$, but quantitatively speaking, the proposed CI&IAC delivers the best transient performance, as it ensures UES of the closed-loop system after $T_i = 4$ s, which is not the case for the NCEAC and the CEAC. Note that $q_e$ and $\omega_e$ under the CEAC exhibit very slow convergence trends, and a long simulation time is needed to see their asymptotically convergent behaviors. This is because the CE-based estimator cannot guarantee parameter convergence due to nonsatisfaction of the PE condition, which in turn renders the control performance of the CEAC to be arbitrarily poor. In contrast, the CI&IAC and the NCEAC deviate significantly from the CE principle and effectively overcome the aforementioned deficient inherent in the CEAC, through introducing an invariant attracting manifold. In addition, from the bottom subplot of Fig. 4(d), we find that both the CI&IAC and the CEAC drive $q_e$ to the nearest equilibrium $[0, -1]^T$, while the NCEAC steers $q_e$ to $[0, 1]^T$, leading to the unwinding phenomenon (as will soon be witnessed in Fig. 6).

The norms of the control torques commanded by the three controllers are plotted on a semilogarithmic scale as shown in Fig. 4(c), from which it is observed that the torque demands due to the CI&IAC and especially the CEAC are higher than that of the NCEAC during the initial transient. This may be caused by the introduction of potential functions in the CI&IAC and the CEAC to achieve unwinding avoidance. The comparison results in terms of parameter estimation error are depicted in Fig. 4(d), in which we recognize that the CI&IAC achieves exponential parameter convergence after $T_i = 4$ s, without PE, while the other two controllers fails to obtain such a result.

The 3-D motion trajectories of the body frame $\mathcal{F}_B$ w.r.t. the reference frame $\mathcal{F}_R$ observed in $\mathcal{F}_R$ are provided in Figs. 5 and 6 to intuitively show the attitude tracking processes of the three controllers under Cases 1 and 2. The mutually perpendicular solid lines (red, green, and blue) in Figs. 5 and 6 denote the axes of $\mathcal{F}_R$, while the dashed counterparts denotes the initial axes of $\mathcal{F}_R$. Inspecting Fig. 5 reveals that, for Case 1, all three controllers can track the desired reference trajectory via a rotation less than $180^\circ$, but intuitively, the proposed CI&IAC renders a smoother trajectory with less fluctuation than the other two controllers. For Case 2, it is shown from Fig. 6 that the tracking trajectories from the CI&IAC and the CEAC remain the same as in Fig. 5, without suffering from the unwinding phenomenon. However, the NCEAC exhibits an unnecessarily long rotation path due to unwinding.

Fig. 3. Time responses of $|\dot{\theta}_i|$, $i = 1, 2, \ldots, 6$ under finite- and fixed-time adaptation extensions.
Fig. 4. Control performance comparisons of different controllers under the nominal scenario. (a) Attitude error norm. (b) Angular velocity error norm. (c) Control torque norm. (d) Parameter estimation error norm.

Fig. 5. 3-D attitude tracking trajectories observed in $\mathcal{F}_R$ for Case 1. The initial and desired orientations are marked by “solid dot” and “asterisk.”. (a) CI&IAC. (b) NCEAC. (c) CEAC.

Fig. 6. 3-D attitude tracking trajectories observed in $\mathcal{F}_R$ for Case 2. The initial and desired orientations are marked by “solid dot” and “asterisk.”. (a) CI&IAC. (b) NCEAC. (c) CEAC.
C. Robustness Validations

At this point, we examine the robustness of the simulated controllers against external disturbances and measurement noises. The disturbance of the following form [25]:

\[ u_d = 10^{-4} \times \begin{bmatrix} 3 \cos(0.2t) + 4 \sin(0.06t) - 10 \\ -1.5 \sin(0.04t) + 3 \cos(0.1t) + 15 \\ 3 \sin(2.2t) - 8 \sin(0.08t) + 5 \end{bmatrix} \text{Nm} \]

is added to (2). The attitude measurement noises are modeled following the method in [40]. Toward this end, we rewrite the attitude quaternion as \( \hat{q} = [\hat{n}^T \sin(\psi/2), \cos(\psi/2)]^T \), where \( \hat{n} \) and \( \psi \) are known as the Euler eigenaxis and eigenangle, respectively. Within this setting, the noisy measurements of \( \hat{q} \) are generated by randomly perturbing the true \( \hat{n} \) with uniform distribution in a spherical cone centered around it. The cone half-angle is set here to 0.1 deg. In addition, the measurement noises with mean zero and standard deviation \( 10^{-3} \text{ rad/s} \) are added to the feedback of \( \omega \). The simulation scenario in Section IV-B is repeated under consideration of external disturbances and measurement noises described previously. To clearly illustrate the comparison results in terms of steady-state performance, the simulation duration is prolonged to 100 s.

The performance comparisons under the perturbed scenario are presented in Fig. 7. By comparing Fig. 7(a) and (b) with Fig. 4(a) and (b), we find that all the controllers suffer in performance degradation, especially the steady-state accuracy. The attitude and angular velocity tracking errors only converge to small residual sets around the origin, rather than zero. From a practical viewpoint, the I&I-based adaptive controllers—CI&IAC and NCEAC—still exhibit acceptable steady-state performance and outperform the CEAC. We further underscore that the proposed CI&IAC preserves the transient-state behaviors, that is, it can steer the tracking error to converge exponentially fast to the steady-state values, and moreover, it performs slightly better than the NCEAC. The control torque norms of all three controllers are depicted in Fig. 7(c), in which some burrs are observed due to the noisy feedback signals. From Fig. 7(d), it is clear that the CI&IAC can still ensure parameter convergence with an acceptable accuracy in the perturbed scenario. Furthermore, to quantitatively compare the steady-state performance, the root mean square (rms) values, denoted as \( \text{rms}(\cdot) \), of the tracking and estimation errors at the steady state (40–100 s) under different controllers are summarized in Table I. As can be seen, the CI&IAC shows smallest rms values of tracking and estimation errors among all three controllers. In summary, the proposed CI&IAC has an inherent strong robustness against external disturbances and measurement noises.

| Method   | \( \text{rms}(q_{\theta}) \) | \( \text{rms}(\omega_z) \), rad/sec | \( \text{rms}(\theta) \), kg \cdot m^2 |
|----------|-----------------------------|-----------------------------------|-----------------------------------|
| CI&IAC   | \( 4.803 \times 10^{-4} \)  | \( 9.234 \times 10^{-4} \)       | 0.1433                            |
| NCEAC    | \( 6.165 \times 10^{-4} \)  | 0.0012                            | 5.2812                            |
| CEAC     | 0.0094                      | 0.0181                            | 12.7127                           |

*It is taken as the maximum rms value across all vector components.*
V. CONCLUSION

A composite adaptive control scheme is proposed to address the antiunwinding attitude tracking problem of an uncertain rigid body, which ensures exponential stability of the closed-loop system under a strictly weaker IE assumption than PE, and consequently, guarantees exponential convergence of both the output-tracking and parameter estimation errors to zero without causing unwinding. The key ideas behind permitting exponential stability without PE are twofold:

1) a logarithmic barrier function is introduced as the AEF for unwinding avoidance, along with the establishment of two crucial algebraic properties for exponential stability analysis;

2) an LRE extension procedure based on an LTV filter is proposed for DREM to generate a persistently exciting regressor, whereby a prediction-error-driven learning law is presented for relaxing the dependence of parameter convergence on the PE condition.

Saliently, the control algorithm developed preserves all the key beneficial features of the I&I adaptive control methodology and does not involve any dynamic gains. Another contribution of this article is to achieve the finite/fixed-time-synchronized parameter convergence, by simply augmenting the composite learning law with a power term.

Appendix A
Proof of Lemma 1

Two cases are considered to complete the proof.

1) Case 1: \( q_{c4} \in (0, 1) \). For analysis, we introduce an auxiliary variable defined by \( h(q_{c4}) = (1 - q_{c4}^2)/q_{c4} + \ln q_{c4}^2 \). Via simple algebraic manipulations, it is shown that

\[
\frac{\partial h(q_{c4})}{\partial q_{c4}} = -\frac{(1 - q_{c4}^2)^2}{q_{c4}^2} \leq 0
\]

in the set \( q_{c4} \in (0, 1) \), indicating that \( h(q_{c4}) \) is a non-increasing function of \( q_{c4} \). Thus, \( h(q_{c4}) \geq h(1) = 0 \) holds in this case, which coincides with the result in (12).

2) Case 2: \( q_{c4} \in [-1, 0) \). In this case, let us define \( h(q_{c4}) = -(1 - q_{c4}^2)/q_{c4} + \ln q_{c4}^2 \). Following a similar reasoning as Case 1, it is not difficult to check that \( \partial h(q_{c4})/\partial q_{c4} \geq 0 \), and hence, \( h(q_{c4}) \geq h(-1) = 0 \) in the set \( q_{c4} \in [-1, 0) \) so that (12) is also obtained.

Appendix B
Proof of Lemma 2

Let us first prove that \( \alpha \|q_{c4}\|^2 = \alpha (1 - q_{c4}^2) \leq V_q \). To facilitate the analysis, similar to the proof of Lemma 1, we here define \( h_1(q_{c4}) = -\alpha \ln q_{c4}^2 - \alpha (1 - q_{c4}^2) \). Taking the partial derivative of \( h_1(q_{c4}) \) w.r.t. \( q_{c4} \) gives

\[
\frac{\partial h_1(q_{c4})}{\partial q_{c4}} = -\frac{2\alpha}{q_{c4}^2} (\alpha - \alpha q_{c4}^2)
\]

(B1)

since \( \alpha \leq \alpha \) and \( q_{c4}^2 \leq 1 \), an intuitive observation reveals that \( (\alpha - \alpha q_{c4}^2) \geq 0 \). With this in mind, one can claim that for \( q_{c4} \in (0, 1) \), \( \partial h_1(q_{c4})/\partial q_{c4} \leq 0 \), while for \( q_{c4} \in [-1, 0) \), \( \partial h_1(q_{c4})/\partial q_{c4} \geq 0 \). From the aforementioned, together with the fact that \( \lim_{q_{c4} \to 0} h_1(q_{c4}) = +\infty \), it can be concluded that \( h_1(q_{c4}) \geq h_1(\pm 1) = 0 \). Thus, \( g(1 - q_{c4}^2) \leq -\alpha \ln q_{c4}^2 \) always holds for any \( \alpha \leq \alpha \), so does the inequality \( \alpha \|q_{c4}\|^2 \leq V_q \).

Next we prove that \( V_q \leq \bar{\alpha} \|q_{c4}\|^2 = \bar{\alpha}(1 - q_{c4}^2) \) holds for any \( |q_{c4}| \in [\delta, 1) \). To this end, define an auxiliary variable in the set \( \delta \leq |q_{c4}| \leq 1 \) as follows:

\[
h_2(q_{c4}) = -\ln q_{c4}^2
\]

(B2)

whose partial derivative w.r.t. \( q_{c4} \) is given by

\[
\frac{\partial h_2(q_{c4})}{\partial q_{c4}} = \frac{-2(1 - q_{c4}^2) - 2q_{c4} \ln q_{c4}^2}{q_{c4}(1 - q_{c4}^2)^2}.
\]

(B3)

For brevity, we denote by \( P(q_{c4}) \) the numerator of (B3). Taking its partial derivative w.r.t. \( q_{c4} \) gives

\[
\frac{\partial P(q_{c4})}{\partial q_{c4}} = -4q_{c4} \ln q_{c4}^2
\]

(B4)

from which it is not difficult to check that \( \partial P(q_{c4})/\partial q_{c4} > 0 \) for \( q_{c4} \in [\delta, 1) \) and \( \partial P(q_{c4})/\partial q_{c4} < 0 \) for \( q_{c4} \in (-1, -\delta) \). Consequently, the maximum value of \( P(q_{c4}) \) takes \( \lim_{q_{c4} \to \pm 1} P(q_{c4}) = 0 \) in the set \( \delta \leq |q_{c4}| < 1 \), indicating that \( P(q_{c4}) < 0 \) for all \( |q_{c4}| \in [\delta, 1) \). In view of this, from (B3), it is clear that \( \partial h_2(q_{c4})/\partial q_{c4} < 0 \) for \( q_{c4} \in [\delta, 1) \) and \( \partial h_2(q_{c4})/\partial q_{c4} > 0 \) for \( q_{c4} \in (-1, -\delta) \), whereby one can observe that \( h_2(q_{c4}) < -\ln \delta^2/(1 - \delta^2) \) for any \( |q_{c4}| \in [\delta, 1) \). By using simple arithmetic operations, we can conclude that \( -\alpha \ln \delta^2/(1 - \delta^2) \) for any \( |q_{c4}| \in [\delta, 1) \). As \( \bar{\alpha} \geq \alpha \ln \delta^2/(1 - \delta^2) \), it can be further claimed that \( -\alpha \ln q_{c4}^2 < \bar{\alpha}(1 - q_{c4}^2) \) holds for any \( |q_{c4}| \in [\delta, 1) \). In addition, it is noted that \( -\alpha \ln q_{c4}^2 = \bar{\alpha}(1 - q_{c4}^2) = 0 \) when \( |q_{c4}| = 1 \). Based on the aforementioned argument, we can draw the conclusion that \( -\alpha \ln q_{c4}^2 \leq \bar{\alpha}(1 - q_{c4}^2) \) holds for any \( |q_{c4}| \in [\delta, 1) \).

Furthermore, it can be readily verified that \( -\ln \delta^2/(1 - \delta^2) > 1 \) strictly holds for \( 0 < \delta < 1 \). This directly contributes to the fact that \( \bar{\alpha} > \alpha \), thus completing the proof.

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REFERENCES

[1] N. A. Chaturvedi, A. K. Sanyal, and N. H. McClamroch, “Rigid-body attitude control,” IEEE Control Syst. Mag., vol. 31, no. 3, pp. 30–51, Jun. 2011.
[2] J.-Y. Wen and K. Kreutz-Delgado, “The attitude control problem,” IEEE Trans. Autom. Control, vol. 36, no. 10, pp. 1148–1162, Oct. 1991.

[3] S. Arjun Ram and M. R. Akella, “Uniform exponential stability result for the rigid-body attitude tracking control problem,” J. Guid., Control, Dyn., vol. 43, no. 1, pp. 39–45, 2020.

[4] L. Zhao, J. Yu, and H. Yu, “Adaptive finite-time attitude tracking control for spacecraft with disturbances,” IEEE Trans. Aerosp. Electron. Syst., vol. 54, no. 3, pp. 1297–1305, Jun. 2018.

[5] H. Li, W. Yan, and Y. Shi, “Continuous-time model predictive control of under-actuated spacecraft with bounded control torques,” Automatica, vol. 75, pp. 144–153, 2017.

[6] B. Li, W. Gong, Y. Yang, B. Xiao, and D. Ran, “Approximated-fixed-time observer based sliding mode control for a quaternion UAV under external disturbances,” IEEE Trans. Aerosp. Electron. Syst., vol. 58, no. 1, pp. 290–303, Feb. 2022, doi: 10.1109/TAES.2021.3101562.

[7] S. P. Bhat and D. S. Bernstein, “A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon,” Syst. Control Lett., vol. 39, no. 1, pp. 63–70, 2000.

[8] R. Kristiansen, P. J. Nicklasson, and J. T. Gravdahl, “Satellite attitude control by quaternion-based backstepping,” IEEE Trans. Control Syst. Technol., vol. 17, no. 1, pp. 227–232, Jan. 2009.

[9] C. G. Mayhew, R. G. Sanfelice, and A. R. Teel, “Quaternion-based adaptive control of spacecraft under velocity and control constraints,” Aerosp. Sci. Technol., vol. 67, pp. 257–264, 2017.

[10] Q. Hu and X. Tan, “Unified attitude control for spacecraft,” J. Guid., Control, Dyn., vol. 36, no. 11, pp. 2555–2566, Nov. 2011.

[11] Q. Hu and X. Tan, “Unified attitude control for spacecraft,” J. Guid., Control, Dyn., vol. 36, no. 1, pp. 63–70, 2000.

[12] R. Dong, A.-G. Wu, and Y. Zhang, “Anti-unwinding attitude control of spacecraft with constrained control inputs,” J. Guid., Control, Dyn., vol. 42, no. 4, pp. 822–835, 2019.

[13] R. Dong, A.-G. Wu, and Y. Zhang, “Anti-unwinding sliding mode attitude maneuver control for rigid spacecraft,” IEEE Trans. Autom. Control, vol. 67, no. 2, pp. 978–985, Feb. 2022, doi: 10.1109/TAC.2021.3079220.

[14] D. Thakur, S. Srivastava, and M. Akella, “Adaptive attitude-tracking control of spacecraft with uncertain time-varying inertia parameters,” J. Guid., Control, Dyn., vol. 38, no. 1, pp. 41–52, 2015.

[15] Y. Xiao, A. De Ruiter, D. Ye, and Z. Sun, “Adaptive fault-tolerant control for spacecraft with guaranteed performance bounds,” IEEE Trans. Aerosp. Electron. Syst., vol. 58, no. 3, pp. 1922–1940, Jun. 2022, doi: 10.1109/TAES.2021.3123295.

[16] J. Ahmed, V. T. Coppola, and D. S. Bernstein, “Adaptive asymptotic tracking of spacecraft attitude motion with inertia matrix identification,” J. Guid., Control, Dyn., vol. 21, no. 5, pp. 684–691, 1998.

[17] A. Astolfi and R. Ortega, “Immersion and invariance: A new tool for stabilization and adaptive control of nonlinear systems,” IEEE Trans. Autom. Control, vol. 48, no. 4, pp. 590–606, Apr. 2003.

[18] D. Seo and M. R. Akella, “High-performance spacecraft adaptive attitude-tracking control through attracting-manifold design,” J. Guid., Control, Dyn., vol. 31, no. 4, pp. 884–891, 2008.

[19] D. Karagiannis, M. Sassano, and A. Astolfi, “Dynamic scaling and observer design with application to adaptive control,” Automatica, vol. 45, no. 12, pp. 2883–2889, 2009.

[20] S. Yang, M. R. Akella, and F. Mazenc, “Dynamically scaled immersion and invariance adaptive control for Euler-Lagrange mechanical systems,” J. Guid., Control, Dyn., vol. 40, no. 11, pp. 2844–2856, 2017.

[21] H. Wen, X. Yue, and J. Yuan, “Dynamic scaling-based noncertainty-equivalent adaptive spacecraft attitude tracking control,” J. Aerosp. Eng., vol. 31, no. 2, 2018, Art. no. 04017098.
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