ASSUMED: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary $\partial \Omega$ in an $n$-dimensional Euclidean space $\mathbb{R}^n$. Consider the following Dirichlet eigenvalue problem of poly-Laplacian with arbitrary order:

$$
\begin{cases}
(-\Delta)^l u = \Lambda u, & \text{in } \Omega, \\
u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0, & \text{on } \partial \Omega,
\end{cases}
$$

where $\Delta$ is the Laplacian in $\mathbb{R}^n$ and $\nu$ denotes the outward unit normal to the boundary $\partial \Omega$ of $\Omega$. In this paper, we obtain a sharp upper bound for the sum of the first $k$-th eigenvalues for this Dirichlet problem, which is viewed as an extension of the result due to Cheng and Wei (Journal of Differential Equations, 255 (2013), 220-233). In particular, if $l = 2$ and $k$ is large enough, we give an important improvement of their result.

**Keywords:** Dirichlet problem, eigenvalues, Poly-Laplacian, Sharp upper bound.

**2010 Mathematics Subject Classification:** 35P15, 35G15.
When \( l = 1 \), the eigenvalue problem (1.1) is also called a fixed membrane problem. For this case, we know that, according to Weyl’s law, the asymptotic formula with respect to the \( k \)-th eigenvalue \( \Lambda_k \) is given by (cf. [4, 25])

\[
\Lambda_k \sim \frac{4\pi^2}{(B_n V(\Omega))^\frac{2}{n}} k^\frac{2}{n}, \quad \text{as } k \to +\infty.
\]

From the above asymptotic formula, one can derive

\[
\frac{1}{k} \sum_{i=1}^{k} \Lambda_i \sim \frac{n}{n + 2} \frac{4\pi^2}{(B_n V(\Omega))^\frac{2}{n}} k^\frac{2}{n}, \quad \text{as } k \to +\infty.
\]

In 1961, Pólya [23] proved that the following inequality:

\[
\Lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^\frac{2}{n}} k^\frac{2}{n}, \quad \text{for } k = 1, 2, \cdots,
\]

holds when \( \Omega \) is a tiling domain in \( \mathbb{R}^n \). However, for a general bounded domain, he proposed a famous conjecture as follows:

**Conjecture of Pólya.** If \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), then the \( k \)-th eigenvalue \( \lambda_k \) of the fixed membrane problem satisfies

\[
\Lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^\frac{2}{n}} k^\frac{2}{n}, \quad \text{for } k = 1, 2, \cdots.
\]

Attacking Pólya’s conjecture, Berezin [5] and Lieb [21] gave a partial solution. In particular, by making use of the fact that all of the eigenfunctions of fixed membrane problem form an orthonormal basis of the Sobolev Space \( W^{2,2}_0(\Omega) \), Li and Yau [20] obtained, by means of Fourier transform, a lower bound for eigenvalues as follows:

\[
\frac{1}{k} \sum_{i=1}^{k} \Lambda_i \geq \frac{n}{n + 2} \frac{4\pi^2}{(B_n V(\Omega))^\frac{2}{n}} k^\frac{2}{n}, \quad \text{for } k = 1, 2, \cdots,
\]

which is sharp in the sense of average according to (1.2). From this formula (1.3), one can infer

\[
\Lambda_k \geq \frac{n}{n + 2} \frac{4\pi^2}{(B_n V(\Omega))^\frac{2}{n}} k^\frac{2}{n}, \quad \text{for } k = 1, 2, \cdots,
\]

which gives a partial solution for the conjecture of Pólya with a factor \( \frac{n}{n + 2} \). A similar fact for the Neumann problem of the Laplacian is also used by Kröger in [18] to obtain an upper bound of the eigenvalues for the Neumann problem on a bounded domain in Euclidean space. In addition, many eigenvalue inequalities in various settings are established by mathematicians, e.g. in [2, 3, 14, 15, 24].

When \( l = 2 \), the eigenvalue problem (1.1) is called a clamped plate problem. For the eigenvalues of the clamped plate problem, Agmon [1] and Pleijel [22] obtained

\[
\Lambda_k \sim \frac{16\pi^4}{(B_n V(\Omega))^\frac{4}{n}} k^\frac{4}{n}, \quad \text{as } k \to +\infty.
\]
From the above formula (1.5), one can obtain

\[ \frac{1}{k} \sum_{i=1}^{k} \Lambda_i \sim \frac{n}{n + 4} \frac{16\pi^4}{(B_n V(\Omega))^\frac{4}{n}} k^{\frac{4}{n}}, \text{ as } k \to +\infty. \]

By the mid-1980's, Levine and Protter [19] proved that eigenvalues of the clamped plate problem satisfy the following inequality:

\[ \frac{1}{k} \sum_{i=1}^{k} \Lambda_i \geq \frac{n}{n + 4} \frac{16\pi^4}{(B_n V(\Omega))^\frac{4}{n}} k^{\frac{4}{n}} \text{ for } k = 1, 2, \cdots. \]

Assume that \( \kappa_1(y), \kappa_2(y), \cdots, \kappa_{n-1}(y) \) are the principal curvatures of \( \partial\Omega \) at the point \( y \) and \( |\kappa_j(y)| \leq \kappa_0 \), where \( 1 \leq j \leq n - 1 \). Recently, Cheng and Wei [12] gave an estimate for upper bound of the eigenvalues of the Clamped plate problem as follows:

\[ \frac{1}{k + 1} \sum_{j=1}^{k+1} \Lambda_j \leq \frac{n}{n + 4} \frac{(2\pi)^4}{(1 - \frac{V(\Omega_0)}{V(\Omega)})^\frac{3n+4}{2n}} (1 + k)^\frac{4}{n} \sum_{j=1}^{k+1} \Lambda_j \leq \frac{n}{n + 4} \frac{(2\pi)^4}{(1 - \frac{V(\Omega_0)}{V(\Omega)})^\frac{3n+4}{2n}} (1 + k)^\frac{4}{n} \]

\[ + 24n \frac{(2\pi)^4}{(1 - \frac{V(\Omega_0)}{V(\Omega)})^\frac{3n+4}{2n}} (1 + k)^\frac{4}{n} \]

\[ + 4n^2 \frac{(2\pi)^4}{(1 - \frac{V(\Omega_0)}{V(\Omega)})^\frac{3n+4}{2n}} (1 + k)^\frac{4}{n}. \]

for \( k \geq V(\Omega)\sigma_0^n > V(\Omega)(n\kappa_0)^n \), where \( \sigma_0 \) is a constant and \( \Omega_\sigma \) is defined by

\[ \Omega_\sigma = \{ x \in \Omega \mid r(x) < \frac{1}{\sigma} \}. \]

Here \( r(x) = \text{dist}(x, \partial\Omega) \) denotes the distance function from the point \( x \) to the boundary \( \partial\Omega \) of \( \Omega \). We shall remind the readers that the formula (1.6) implies that the coefficient of \( k^{\frac{4}{n}} \) in both (1.7) and (1.8) are also the best possible constant in the sense of the asymptotic formula (1.6).

When \( l \) is arbitrary, Birman and Solomyak obtained the following asymptotic formula (cf. [6,7]):

\[ \Lambda_k \sim \frac{(2\pi)^{2l}}{(B_n V(\Omega))^\frac{2l}{n}} k^{\frac{2l}{n}}, \text{ as } k \to +\infty. \]

From (1.9), we have

\[ \frac{1}{k} \sum_{j=1}^{k} \Lambda_j \sim \frac{n}{n + 2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^\frac{2l}{n}} k^{\frac{2l}{n}}, \text{ as } k \to +\infty. \]
Furthermore, Levine-Protter [19] obtained a lower bound estimate for the eigenvalues:

$$
\frac{1}{k} \sum_{i=1}^{k} \Lambda_i \geq \frac{n}{n + 2l} \left( \frac{\pi}{B_n V(\Omega)} \right)^{2l} k^{2l/n} \quad \text{for } k = 1, 2, \cdots.
$$

By adding \( l \) terms of lower order of \( k^{2l/n} \) to its right hand side, Cheng, Qi and Wei [11] obtained more sharper result than (1.11):

$$
\frac{1}{k} \sum_{i=1}^{k} \Lambda_i \geq \frac{n}{n + 2l} \left( \frac{2\pi}{B_n V(\Omega)} \right)^{2l} k^{2l/n} + \frac{n}{n + 2l} \left( \frac{l + 1 - p}{(24)^p n \cdots (n + 2p - 2)} \right) \left( \frac{V(\Omega)}{I(\Omega)} \right)^{2l + 1} \frac{n}{n^{2l/n}} k^{2l/n},
$$

where

$$
I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 \, dx
$$

is called the moment of inertia of \( \Omega \).

On the other hand, as the same as the case of the Clamped plate problem [12], from our knowledge, there is no any result on upper bounds for eigenvalue \( \Lambda_k \) with optimal order of \( k \), either. If one can get a sharper universal inequality for eigenvalues of the Dirichlet problem (1.1) of poly-Laplacian with arbitrary order \( l \geq 2 \), we can derive an upper bound for eigenvalue \( \Lambda_k \) by making use of the recursion formula established by Cheng and Yang in [13]. In addition, we recall that, Chen and Qian [9] and Hook [16], independently, proved the following Payne-Pólya-Weinberg type inequality:

$$
\frac{1}{k} \sum_{j=1}^{k} \Lambda_j \leq \Lambda_k + \frac{4l(n + 2l - 2)}{n^2 k^2} \left( \sum_{j=1}^{k} \Lambda_j^{1/l} \right) \left( \sum_{j=1}^{k} \Lambda_j^{(l-1)/l} \right).
$$

In 2009, Cheng Ichikawa and Mametsuka [10] established a Yang type inequality:

$$
\sum_{j=1}^{k} (\Lambda_{k+1} - \Lambda_j)^2 \leq \frac{4l(n + 2l - 2)}{n^2} \sum_{j=1}^{k} (\Lambda_{k+1} - \Lambda_j) \Lambda_j.
$$

Recently, J. Jost, X. Jost, Q. Wang and C. Xia [17] obtained some analagous inequalities. However, when \( l \geq 2 \), all the inequalities are failed to achieve the sharp estimate for the upper bound of eigenvalues of the Dirichlet problem (1.1). Therefore, it is very urgent for us to give a sharp upper bound for the eigenvalues. For this purpose, we investigate eigenvalues of the Dirichlet eigenvalue problem (1.1) of poly-Laplacian with arbitrary order and obtain a sharp upper bound of eigenvalues in the sense of the asymptotic formula in this paper. In more detail, we prove the following:
Theorem 1.1. Let $\Omega$ be a bounded domain with a smooth boundary $\partial \Omega$ in $\mathbb{R}^n$. Then there exists a constant $\sigma_0 > 0$ such that eigenvalues of the Dirichlet problem (1.1) satisfy

$$\frac{1}{k} \sum_{j=1}^{k} \lambda_j \leq \frac{n}{n + 2l} \left( 1 - \frac{V(\Omega_{\sigma_0})}{V(\Omega)} \right)^{\frac{2l+2}{n}} \left( B_n V(\Omega) \right)^{\frac{2l}{n}}$$

$$+ A_1(n, l) \left( \frac{2\pi}{n} \right)^{2l-4} \left( 1 - \frac{V(\Omega_{\sigma_0})}{V(\Omega)} \right)^{\frac{n+2l-4}{n}} \left( B_n V(\Omega) \right)^{\frac{2l-4}{n}}$$

$$+ A_2(n, l) \left( \frac{2\pi}{n} \right)^{2l-8} \left( 1 - \frac{V(\Omega_{\sigma_0})}{V(\Omega)} \right)^{\frac{n+2l-8}{n}} \left( B_n V(\Omega) \right)^{\frac{2l-8}{n}},$$

(1.14)

for $k \geq V(\Omega)\sigma_0^n$, where those coefficients $A_i(n, l), i = 1, 2$, are given by

$$A_1(n, l) = \begin{cases} \frac{n(2l^2 + 4 - 2n)l + 2n - 2}{n + 2l - 2}, & \text{if } l = 1 \text{ or if } l(\neq 1) \text{ is odd and } l \geq n - 3 - \frac{2}{l - 1}; \\ 0, & \text{if } l(\neq 1) \text{ is odd and } l < n - 3 - \frac{2}{l - 1}; \\ \frac{n(2l^2 - 2nl + 4l)}{n + 2l - 2}, & \text{if } l \text{ is even and } l \geq n - 2; \\ 0, & \text{if } l \text{ is even and } l < n - 2, \end{cases}$$

and

$$A_2(n, l) = \begin{cases} \frac{n((l - 1)^2 + n(l - 1))^2}{n + 2l - 4}, & \text{if } l \text{ is odd}; \\ \frac{n(l(l - 2) + nl)^2}{n + 2l - 4}, & \text{if } l \text{ is even}, \end{cases}$$

respectively.

Remark 1.1. Seeing from the fact that $V(\Omega_{\sigma_0}) \to 0$ when $\sigma_0 \to \infty$, we know that the upper bound for the eigenvalue inequality in the theorem 1.1 is sharp in the sense of the asymptotic formula (1.10) due to Birman and Solomyak.

Remark 1.2. When $l = 1$, we give a sharp upper bound of Cheng-Wei type in the sense of the asymptotic formula given by Agmon.

Remark 1.3. When $l = 2$ and $k$ is large enough, combining with (3.12) and by an elementary computation, we know that $A_1(n, 2) < \frac{2m}{n+2}$ and $A_2(n, 2) = 4n^2$ since $\max_{x \in \Omega} |x|^2$ and $\kappa_j(y), y \in \partial \Omega$, where $j = 1, 2, \ldots, n - 1$, are bounded, which means that the inequality (1.14) is sharper than the inequality (1.8). Indeed, according to (3.12), we know that there exist two positive constants $\sigma_1$ and $\sigma_2$ such that $\sigma_1 = (nk_0)^2$ and
\[ \sigma_2 = 2\pi \left( \frac{1 + k}{B_n(V(\Omega) - V(\Omega_{\sigma_0}))} \right)^{\frac{1}{n}}. \]

Let \( \sigma_3 = \max\{\sigma_0, \sigma_1, \sigma_2\} \), then, for any \( k \geq V(\Omega)\sigma_3 \), it is not difficult to prove that, the second term in inequality (1.14) is less than the one in (1.8) and the third term in inequality (1.14) agrees with the one in (1.8). It is equivalent to say that we give an important improvement of the result (1.8) due to Cheng and Wei [12] when \( k \) is a sufficiently large positive integer.

**Remark 1.4.** Let \( l \) be an odd number and
\[ \Theta = \frac{n(2l^2 + (4 - 2n)l + 2n - 2)}{n + 2l - 2}. \]
Assume that \( l \leq 5 \), then all values of \( A_1(n,l) \) can be listed by the following Table I:

| \( n \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) | \( 9 \) | \( 10 \) | \( 11 \) | \( 12 \) | \( \geq 12 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( l \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) |
| 1 | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) |
| 3 | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| 5 | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( \Theta \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |

Table 1. Values of \( A_1(n,l) \) for the case: \( l = 1, 3, 5 \)

However, for the case: \( l \geq 5 \), we have \( A_1(n,l) = \Theta \) when \( l \geq n - 3 \); and \( A_1(n,l) = 0 \) when \( l < n - 3 \).

**Corollary 1.1.** Let \( \Omega \) be a bounded domain with a smooth boundary \( \partial \Omega \) in \( \mathbb{R}^n \). If there exists a constant \( \delta_0 \) such that
\[ V(\Omega_\sigma) \leq \delta_0 V(\Omega)^{\frac{n-\tau}{n}} \frac{1}{\sigma^\tau} \]
for any \( \sigma > V(\Omega)^{-\frac{1}{n}} \) and \( \tau \geq 1 \). Then, there exists a constant \( \sigma_0 \) such that, for \( k = V(\Omega)\sigma_0^n > \delta_0^n \), eigenvalues of the Dirichlet problem (1.1) satisfy
\[ \frac{1}{k} \sum_{j=1}^{k} \Lambda_j \leq \frac{n}{n + 2l} \left( \frac{2\pi}{B_n V(\Omega)} \right)^{\frac{2l}{n}} k^{\frac{2l}{n}} \]
\[ + \delta_0 \left\{ \alpha_1(n,l) \left( \frac{(2\pi)^{2l}}{B_n V(\Omega)} \right)^{\frac{2l}{n}} + \alpha_2(n,l) \left( \frac{(2\pi)^{2l-4}}{B_n V(\Omega)} \right)^{\frac{2l}{n}} \right\} k^{\frac{2l-\tau}{n}} \]
\[ + \delta_0 \alpha_3(n,l) \left( \frac{(2\pi)^{2l-8}}{B_n V(\Omega)} \right)^{\frac{2l}{n}} k^{\frac{2l-4-\tau}{n}}, \]
where \( \alpha_i(n,l), i = 1, 2, 3 \) are three constants only depending on \( n \) and \( l \), respectively.
2. Functional Space and the Proofs of Lemmas

In this section, we use \( W^{l,2}(\Omega) \) to denote the Sobolev space of all functions in \( L^2(\Omega) \). Furthermore, by \( W^{l,2}_0(\Omega) \) we denote the closure in \( W^{l,2}(\Omega) \) of the space of \( C^\infty \)-functions with compact support in \( \Omega \). For points \( x, y \in \mathbb{R}^n \), we denote by \(|x|, |y|\) their Euclidean norm and by \( \langle x, y \rangle \) their scalar product. Taking an arbitrary fixed point \( \xi \in \mathbb{R}^n \) and \( \sigma > 0 \), we define a function

\[
(2.1) \quad w_{\sigma,\xi}(x) = e^{i\langle \xi, x \rangle} \rho_{\sigma}(x),
\]

with \( i = \sqrt{-1} \), and then, this function belongs to the Sobolev Space \( W^{l,2}_0(\Omega) \). For the purpose of conciseness, we put

\[
\nabla^m = \begin{cases} 
\Delta^{\frac{m}{2}}, & m \text{ is even,} \\
\nabla \Delta^{\frac{m-1}{2}}, & m \text{ is odd,}
\end{cases}
\]

with \( \nabla = \nabla^1 = \nabla \Delta^0 \). Given any positive integer \( p \), let \( f \in W^{2,p}_0(\Omega) \) be a function on \( \Omega \). Define \( [\nabla^p f, \nabla^p f] \) as follows:

\[
[\nabla^p f, \nabla^p f] := \begin{cases} 
\langle \nabla \Delta^{\frac{p-1}{2}} f, \nabla \Delta^{\frac{p-1}{2}} f \rangle, & \text{when } p \text{ is odd;} \\
(\Delta^\frac{p}{2} f)(\Delta^\frac{p}{2} f), & \text{when } p \text{ is even.}
\end{cases}
\]

Then, utilizing Stokes’ formula, we derive

\[
(2.2) \quad \int_{\Omega} f(-\Delta)^p f \, dx = \int_{\Omega} [\nabla^p f, \nabla^p f] \, dx.
\]

Next, we state two lemmas which play significant roles in the proof of Theorem 1.1 and prove them in this section.

**Lemma 2.1.** Let \( r(x) = \text{dist}(x, \partial \Omega) \) denote the distance function from the point \( x \) to the boundary \( \partial \Omega \) of \( \Omega \). Define

\[
\Omega_\sigma = \left\{ x \in \Omega \mid r(x) < \frac{1}{\sigma} \right\}.
\]

Assume that \( \sigma \geq \sigma_0 > \sqrt{\sup_{x \in \Omega} |x|^2} \) and \( w_{\sigma,\xi}(x) = e^{i\langle \xi, x \rangle} \rho_{\sigma}(x) \), where

\[
\rho_{\sigma}(x) = \begin{cases} 
1, & x \in \Omega, r(x) \geq \frac{1}{\sigma}, \\
|\frac{x}{\sigma^2}|, & x \in \Omega, r(x) < \frac{1}{\sigma}, \\
0, & \text{the other},
\end{cases}
\]

and \( i = \sqrt{-1} \). Then,

\[
(2.3) \quad \int_{\Omega_\sigma} [\nabla^l w_{\sigma,\xi}(x), \nabla^l w_{\sigma,\xi}(x)] \leq C_0 V(\Omega_\sigma),
\]
where

\[
C_0 = \begin{cases} \\
|\xi|^{2l-4} \left[ |\xi|^4 + \frac{(2l^2 + (4-2n)l + 2n - 2) |\xi|^2}{\sigma^2} + \frac{((l-1)^2 + n(l-1))^2}{\sigma^4} \right], \\
\quad \text{if } l = 1 \text{ or if } l \neq 1 \text{ is odd and } l \geq n - 3 - \frac{2}{l-1}; \\
|\xi|^{2l-4} \left[ |\xi|^4 + \frac{(l-1)^2 + n(l-1))^2}{\sigma^4} \right], \\
\quad \text{if } l \neq 1 \text{ is odd and } l < n - 3 - \frac{2}{l-1}; \\
|\xi|^{2l-4} \left[ |\xi|^4 + \frac{2l^2 - 2nl + 4l) |\xi|^2}{\sigma^2} + \frac{(l(l-2) + nl)^2}{\sigma^4} \right], \\
\quad \text{if } l \text{ is even and } l \geq n - 2; \\
|\xi|^{2l-4} \left[ |\xi|^4 + \frac{(l(l-2) + nl)^2}{\sigma^4} \right], \\
\quad \text{if } l \text{ is even and } l < n - 2.
\end{cases}
\]

**Remark 2.1.** Recall that a critical function

\[
\tilde{\rho}_\sigma(x) = \begin{cases} \\
1, & x \in \Omega, r(x) \geq \frac{1}{\sigma}, \\
\sigma^2 r^2(x), & x \in \Omega, r(x) < \frac{1}{\sigma}, \\
0, & \text{the other,}
\end{cases}
\]

is constructed by the authors in [12], which plays an important role in the estimate for the upper bound of the eigenvalues of the clamped plate problem. However, when \( l \geq 3 \), it seems to be much more difficult to calculate the left hand of the inequality \( (2.3) \) and further obtain the desired estimate than the case of \( l = 1, 2 \). Fortunately, after making great effort, the author successfully construct a suitable function to take the place of function \( (2.4) \), this is,

\[
\rho_\sigma(x) = \begin{cases} \\
1, & x \in \Omega, r(x) \geq \frac{1}{\sigma}, \\
\frac{|x|^2}{\sigma^2}, & x \in \Omega, r(x) < \frac{1}{\sigma}, \\
0, & \text{the other.}
\end{cases}
\]

Indeed, it is easy to compute all the derivatives with order \( m, m = 1, 2, \cdots, l \), of the function \( (2.5) \), see the equations \((2.6)-\(2.8)\) in the proof of this lemma.

**Proof.** By the definition of \( \rho_\sigma(x) \), we have
\( \nabla \rho_r(x) = \begin{cases} 0, & x \in \Omega, r(x) \geq \frac{1}{\sigma}, \\ \frac{2x}{\sigma^2}, & x \in \Omega, r(x) < \frac{1}{\sigma}, \\ 0, & \text{the other}, \end{cases} \)  

\( \Delta \rho_r(x) = \begin{cases} 0, & x \in \Omega, r(x) \geq \frac{1}{\sigma}, \\ \frac{2n}{\sigma^2}, & x \in \Omega, r(x) < \frac{1}{\sigma}, \\ 0, & \text{the other}, \end{cases} \)  

\( \nabla \Delta^m \rho_r(x) = 0, \) and \( \Delta^{m+1} \rho_r(x) = 0, \)

where 0 denotes zero vector field and \( m \) is a positive integer.

When \( l = 1 \), from (2.6), we obtain

\[
\nabla w_{\sigma, \xi}(x) = \nabla (e^{i\langle \xi, x \rangle} \rho_\sigma(x)) = i\rho_\sigma(x)e^{i\langle \xi, x \rangle} \nabla \langle \xi, x \rangle + e^{i\langle \xi, x \rangle} \nabla \rho_\sigma(x) = e^{i\langle \xi, x \rangle} \left( \frac{|x|^2 \xi}{\sigma^2} + \frac{2x}{\sigma^2} \right),
\]

and thus

\[
\nabla w_{\sigma, \xi}(x) = e^{i\langle \xi, x \rangle} \left( -i \frac{|x|^2 \xi}{\sigma^2} + \frac{2x}{\sigma^2} \right),
\]

where \( x \in \Omega \setminus \Omega_\sigma \). Therefore, we infer that

\[
\langle \nabla w_{\sigma, \xi}(x), \nabla w_{\sigma, \xi}(x) \rangle = \frac{|\xi|^2 |x|^4}{\sigma^4} + \frac{4|x|^2}{\sigma^4},
\]

and

\[
\int_{\Omega_\sigma} \left[ \nabla w_{\sigma, \xi}(x), \nabla w_{\sigma, \xi}(x) \right] \, dx = \int_{\Omega_\sigma} \langle \nabla w_{\sigma, \xi}(x), \nabla w_{\sigma, \xi}(x) \rangle \, dx = \int_{\Omega_\sigma} \left( \frac{|\xi|^2 |x|^4}{\sigma^4} + \frac{4|x|^2}{\sigma^4} \right) \, dx,
\]

since \( \sigma \geq \sigma_0 > \sqrt{\sup_{x \in \Omega} |x|^2} \).

When \( l = 2 \), by making use of (2.6) and (2.7), we get
\[ \Delta w_{\sigma, \xi}(x) = \Delta \left( e^{i\langle \xi, x \rangle} \rho_{\sigma}(x) \right) = \Delta \left( e^{i\langle \xi, x \rangle} \frac{|x|^2}{\sigma^2} \right) = \frac{|x|^2}{\sigma^2} \Delta e^{i\langle \xi, x \rangle} + 2 \langle \nabla e^{i\langle \xi, x \rangle}, \nabla \frac{|x|^2}{\sigma^2} \rangle + e^{i\langle \xi, x \rangle} \Delta \frac{|x|^2}{\sigma^2} = e^{i\langle \xi, x \rangle} \left( -\frac{\xi^2 |x|^2}{\sigma^2} + \frac{4i \langle \xi, x \rangle}{\sigma^2} + \frac{2n}{\sigma^2} \right). \]

and thus

\[ \overline{\Delta w_{\sigma, \xi}(x)} = e^{i\langle \xi, x \rangle} \left( -\frac{\xi^2 |x|^2}{\sigma^2} - \frac{4i \langle \xi, x \rangle}{\sigma^2} + \frac{2n}{\sigma^2} \right). \]

In fact, for any \( p, p = 1, 2, \cdots \), we can prove that

\[ \Delta^p w_{\sigma, \xi}(x) = e^{i\langle \xi, x \rangle} \left( (\xi^2 |x|^2)^p - \frac{4i \langle \xi, x \rangle}{\sigma^2} + \frac{2n}{\sigma^2} \right). \]

(2.10)

At first, for any positive integer \( q \), we assume that the following equation

\[ \Delta^q w_{\sigma, \xi}(x) = e^{i\langle \xi, x \rangle} \left( (\xi^2 |x|^2)^q - \frac{4i \langle \xi, x \rangle}{\sigma^2} + \frac{2n}{\sigma^2} \right). \]

(2.11)

holds. Then, according to (2.11), we deduce that
\[ \Delta^{q+1} w_{\sigma, \xi}(x) \]
\[ = \Delta \left\{ e^{i(\xi, x)} \left( (-1)^q \frac{|x|^2}{\sigma^2} |\xi|^2 + (-1)^{q-1} \frac{i4q(\xi, x)}{\sigma^2} |\xi|^{2q-1} \right) \right\} \]
\[ + (-1)^q \frac{4q(q - 1) + 2nq}{\sigma^2} |\xi|^{2q-1} \] 
\[ = e^{i(\xi, x)} \left( (-1)^q \frac{|x|^2}{\sigma^2} |\xi|^2 + (-1)^q \frac{i4q(\xi, x)}{\sigma^2} |\xi|^{2q-1} \right) \]
\[ + (-1)^q \frac{4q(q - 1) + 2nq}{\sigma^2} |\xi|^{2q-1} \] 
\[ + e^{i(\xi, x)} \left( (-1)^q \frac{i4q(\xi, x)}{\sigma^2} |\xi|^{2q-1} \right) \] 
\[ + e^{i(\xi, x)} \left( (-1)^q \frac{i4q(\xi, x)}{\sigma^2} |\xi|^{2q-1} \right) \] 
\[ = e^{i(\xi, x)} \left( (-1)^q \frac{|x|^2}{\sigma^2} |\xi|^2 + (-1)^q \frac{i4q(q + 1)(\xi, x)}{\sigma^2} |\xi|^{2(q+1)-1} \right) \]
\[ + (-1)^{q+1} \frac{4(q + 1)((q + 1) - 1) + 2n(q + 1)}{\sigma^2} |\xi|^{2(q+1)-1} \].

Therefore, according to the principle of induction, we conclude that, for any positive integer \( p \), equation (2.10) holds. By making use of (2.10), we further infer 

\[ (\nabla \Delta^p)(w_{\sigma, \xi}(x)) = \nabla (\Delta^p w_{\sigma, \xi}(x)) \]
\[ = \nabla \left\{ e^{i(\xi, x)} \left( (-1)^p \frac{|x|^2}{\sigma^2} |\xi|^2 + (-1)^{p-1} \frac{i4p(\xi, x)}{\sigma^2} |\xi|^{2(p-1)} \right) \right\} \]
\[ + (-1)^{p-1} \frac{4p(p - 1) + 2np}{\sigma^2} |\xi|^{2(p-1)} \] 
\[ = e^{i(\xi, x)} \left\{ i \left( (-1)^p \frac{|x|^2}{\sigma^2} |\xi|^2 + (-1)^{p-1} \frac{i4p(\xi, x)}{\sigma^2} |\xi|^{2(p-1)} \right) \right\} \xi \]
\[ + (-1)^{p-1} \frac{4p(p - 1) + 2np}{\sigma^2} |\xi|^{2(p-1)} \] 
\[ + e^{i(\xi, x)} \left( (-1)^p \frac{2x}{\sigma^2} |\xi|^2 + (-1)^{p-1} \frac{i4p\xi}{\sigma^2} |\xi|^{2(p-1)} \right). \]
where \( p = 1, 2, \cdots \). By making use of (2.12), for any positive integer \( l \geq 3 \), we have

\[
\int_{\Omega_\sigma} \left[ \nabla \Delta^p w_{\sigma, \xi}(x), \nabla \Delta^p w_{\sigma, \xi}(x) \right] dx
\]

\[
= \int_{\Omega_\sigma} \left( \langle \nabla \Delta^p w_{\sigma, \xi}(x), \nabla \Delta^p w_{\sigma, \xi}(x) \rangle \right) dx
\]

\[
= \int_{\Omega_\sigma} \left| \xi \right|^{4(p-1)} \left[ \left( -\frac{\left| \xi \right|^2 |x|^2}{\sigma^2} + \frac{4p(p-1) + 2np}{\sigma^2} \right)^2 \left| \xi \right|^2 \right] dx
\]

\[
+ \int_{\Omega_\sigma} \left| \xi \right|^{4(p-1)} \left[ \frac{4\left| \xi \right|^4 |x|^2}{\sigma^4} + \frac{16p^2\left| \xi \right|^2}{\sigma^4} + \frac{16p\left| \xi \right|^4 |x|^2}{\sigma^4} \right] dx
\]

\[
+ \int_{\Omega_\sigma} \left| \xi \right|^{4(p-1)} \left[ \left( -\frac{\left| \xi \right|^2 |x|^2}{\sigma^2} + \frac{4p(p-1) + 2np}{\sigma^2} \right) 8p\left| \xi \right|^2 \right] dx
\]

\[
+ \int_{\Omega_\sigma} \left| \xi \right|^{4(p-1)} \left[ \frac{\left( 4p\left( \xi, x \right) \right)^2}{\sigma^2} \left| \xi \right|^2 \right] dx.
\]

Therefore, it follows with the Cauchy-Schwarz inequality and (2.13) that,

\[
\int_{\Omega_\sigma} \left[ \nabla \Delta^p w_{\sigma, \xi}(x), \nabla \Delta^p w_{\sigma, \xi}(x) \right] dx
\]

\[
\leq \int_{\Omega_\sigma} \left| \xi \right|^{4(p-1)} \left[ \left( -\frac{\left| \xi \right|^2 |x|^2}{\sigma^2} + \frac{4p(p-1) + 2np}{\sigma^2} \right)^2 \left| \xi \right|^2 \right] dx
\]

\[
+ \int_{\Omega_\sigma} \left| \xi \right|^{4(p-1)} \left[ \frac{4\left| \xi \right|^4 |x|^2}{\sigma^4} + \frac{16p^2\left| \xi \right|^2}{\sigma^4} + \frac{16p\left| \xi \right|^4 |x|^2}{\sigma^4} \right] dx
\]

\[
+ \int_{\Omega_\sigma} \left| \xi \right|^{4(p-1)} \left[ \left( -\frac{\left| \xi \right|^2 |x|^2}{\sigma^2} + \frac{4p(p-1) + 2np}{\sigma^2} \right) 8p\left| \xi \right|^2 \right] dx
\]

\[
+ \int_{\Omega_\sigma} \left| \xi \right|^{4(p-1)} \left[ \frac{\left( 4p\left( \xi, x \right) \right)^2}{\sigma^2} \left| \xi \right|^2 \right] dx
\]

\[
= \int_{\Omega_\sigma} \left| \xi \right|^{4p-2} \left[ \frac{\left| \xi \right|^4 |x|^4}{\sigma^4} + \frac{4(2p^2 + (4-n)p + 1)\left| \xi \right|^2 |x|^2}{\sigma^4} + \frac{(4p^2 + 2np)^2}{\sigma^4} \right] dx.
\]

In fact, according to (2.3), we know (2.14) also holds for the case of \( l = 1 \). Furthermore, by substituting \( p = \frac{l-1}{2} \) into (2.14), we yield

\[
\int_{\Omega_\sigma} \left[ \nabla \Delta^{\frac{l-1}{2}} w_{\sigma, \xi}(x), \nabla \Delta^{\frac{l-1}{2}} w_{\sigma, \xi}(x) \right] dx
\]

\[
\leq \int_{\Omega_\sigma} \left| \xi \right|^{2l-4} \left[ \left| \xi \right|^4 + \frac{(2l^2 + (4-2n)l + 2n - 2)\left| \xi \right|^2}{\sigma^2} + \frac{((l-1)^2 + n(l-1))^2}{\sigma^4} \right] dx
\]

\[
= \left| \xi \right|^{2l-4} \left[ \left| \xi \right|^4 + \frac{(2l^2 + (4-2n)l + 2n - 2)\left| \xi \right|^2}{\sigma^2} + \frac{((l-1)^2 + n(l-1))^2}{\sigma^4} \right] V(\Omega_\sigma),
\]
Therefore, from (2.15), we obtain

\[
\int_{\Omega_\sigma} \left[ \nabla \Delta^{\frac{l-1}{2}} w_{\sigma,\xi}(x), \nabla \Delta^{\frac{l-1}{2}} w_{\sigma,\xi}(x) \right] dx \\
\leq \int_{\Omega_\sigma} |\xi|^{2l-4} \left[ |\xi|^4 + \frac{(l-1)^2 + n(l-1)^2}{\sigma^4} \right] dx \\
= |\xi|^{2l-4} \left[ |\xi|^4 + \frac{(l-1)^2 + n(l-1)^2}{\sigma^4} \right] V(\Omega_\sigma),
\]

when \(l(\neq 1)\) is an odd number and \(l < n - 3 - \frac{2}{l-1}\); and

\[
\int_{\Omega_\sigma} \left[ \nabla \Delta^{\frac{l}{2}} w_{\sigma,\xi}(x), \nabla \Delta^{\frac{l}{2}} w_{\sigma,\xi}(x) \right] dx \\
= \int_{\Omega_\sigma} |\Delta^{\frac{l}{2}} w_{\sigma,\xi}(x)|^2 dx \\
\leq \int_{\Omega_\sigma} |\xi|^{2l-4} \left[ \left( -\frac{|\xi|^2|x|^2}{\sigma^2} + \frac{l(l-2) + nl}{\sigma^2} \right)^2 + \left( \frac{2l\langle \xi, x \rangle}{\sigma^2} \right)^2 \right] dx \\
\leq \int_{\Omega_\sigma} |\xi|^{2l-4} \left[ \left( -\frac{|\xi|^2|x|^2}{\sigma^2} + \frac{l(l-2) + nl}{\sigma^2} \right)^2 + \frac{4l^2|\xi|^2|x|^2}{\sigma^4} \right] dx \\
= \int_{\Omega_\sigma} |\xi|^{2l-4} \left[ \frac{|\xi|^4|x|^4}{\sigma^4} + \frac{(2l^2 - 2nl + 4l)|\xi|^2|x|^2}{\sigma^4} + \frac{(l(l-2) + nl)^2}{\sigma^4} \right] dx.
\]

Therefore, from (2.15), we obtain

\[
\int_{\Omega_\sigma} \left[ \Delta^{\frac{l-1}{2}} w_{\sigma,\xi}(x), \Delta^{\frac{l-1}{2}} w_{\sigma,\xi}(x) \right] dx \\
= \int_{\Omega_\sigma} |\Delta^{\frac{l-1}{2}} w_{\sigma,\xi}(x)|^2 dx \\
\leq \int_{\Omega_\sigma} |\xi|^{2l-4} \left[ |\xi|^4 + \frac{(2l^2 - 2nl + 4l)|\xi|^2}{\sigma^2} + \frac{(l(l-2) + nl)^2}{\sigma^4} \right] dx \\
= |\xi|^{2l-4} \left[ |\xi|^4 + \frac{(2l^2 - 2nl + 4l)|\xi|^2}{\sigma^2} + \frac{(l(l-2) + nl)^2}{\sigma^4} \right] V(\Omega_\sigma),
\]

when \(l\) is an even number and \(l \geq n - 2\); and

\[
\int_{\Omega_\sigma} \left[ \Delta^{\frac{l}{2}} w_{\sigma,\xi}(x), \Delta^{\frac{l}{2}} w_{\sigma,\xi}(x) \right] dx = \int_{\Omega_\sigma} |\Delta^{\frac{l}{2}} w_{\sigma,\xi}(x)|^2 dx \\
\leq \int_{\Omega_\sigma} |\xi|^{2l-4} \left[ |\xi|^4 + \frac{(l(l-2) + nl)^2}{\sigma^4} \right] dx \\
= |\xi|^{2l-4} \left[ |\xi|^4 + \frac{(l(l-2) + nl)^2}{\sigma^4} \right] V(\Omega_\sigma),
\]

when \(l\) is an even number and \(l < n - 2\). This finishes the proof of the lemma. \(\square\)
Lemma 2.2. Under the same assumption as in Lemma 2.1, the following equation

\[(2.16) \quad \int_{\Omega \setminus \Omega_\sigma} \left[ \nabla^l \phi(x), \nabla^l \phi(x) \right] dx = |\xi|^{2l} (V(\Omega) - V(\Omega_\sigma))\]

holds.

Proof. When \( x \in \Omega \setminus \Omega_\sigma \), we know that \( \rho_\sigma(x) = 1 \) and thus \( \phi(x) = e^{i\langle \xi, x \rangle} \). If \( l \) is an even number, by a direct calculation, we have

\[\Delta^{\frac{l}{2}} e^{i\langle \xi, x \rangle} = (-1)^l |\xi|^l e^{i\langle \xi, x \rangle}\]

and

\[\Delta^{\frac{l}{2}} f = (-1)^l |\xi|^l f\]

Hence, we obtain

\[\int_{\Omega \setminus \Omega_\sigma} \left[ \nabla^l \phi(x), \nabla^l \phi(x) \right] dx = \int_{\Omega \setminus \Omega_\sigma} \left( \Delta^{\frac{l}{2}} \phi(x) \right) \left( \Delta^{\frac{l}{2}} \phi(x) \right) dx\]

\[= \int_{\Omega \setminus \Omega_\sigma} \left( \Delta^{\frac{l}{2}} e^{i\langle \xi, x \rangle} \right) \left( \Delta^{\frac{l}{2}} e^{i\langle \xi, x \rangle} \right) dx\]

\[= \int_{\Omega \setminus \Omega_\sigma} |\xi|^{2l} dx\]

\[= |\xi|^{2l} (V(\Omega) - V(\Omega_\sigma)).\]

If \( l \) is an even number, by a direct calculation, we derive

\[\nabla \Delta^{\frac{l}{2}} e^{i\langle \xi, x \rangle} = (-1)^{l-1} i |\xi|^{l-1} e^{i\langle \xi, x \rangle} \nabla \langle \xi, x \rangle\]

and

\[\nabla \Delta^{\frac{l}{2}} f = (-1)^l i |\xi|^{l-1} f \nabla \langle \xi, x \rangle\]

Therefore, we infer that

\[\int_{\Omega \setminus \Omega_\sigma} \left[ \nabla \Delta^{\frac{l-1}{2}} \phi(x), \nabla \Delta^{\frac{l-1}{2}} \phi(x) \right] dx\]

\[= \int_{\Omega \setminus \Omega_\sigma} \left( \nabla \Delta^{\frac{l-1}{2}} \phi(x), \nabla \Delta^{\frac{l-1}{2}} \phi(x) \right) dx\]

\[= \int_{\Omega \setminus \Omega_\sigma} \left( \nabla \Delta^{\frac{l-1}{2}} e^{i\langle \xi, x \rangle}, \nabla \Delta^{\frac{l-1}{2}} e^{i\langle \xi, x \rangle} \right) dx\]

\[= |\xi|^{2l} (V(\Omega) - V(\Omega_\sigma)).\]

This finishes the proof of the lemma. \( \square \)
3. Upper Bound for Eigenvalues

In this section, we continue to use those notations given in the previous section and give the proofs of theorem 1.1 and corollary 1.1.

Proof of Theorem 1.1. We let $u_k$ be an orthonormal eigenfunction corresponding to the eigenvalue $\Lambda_k$, which is equivalent to say that $u_k$ satisfies

$$
\begin{align*}
&\left\{ (-\Delta)^l u_j + \Lambda_j u_j = 0, \quad \text{in } \Omega, \\
&u_j = \frac{\partial u_j}{\partial \nu} = \cdots = \frac{\partial^{l-1} u_j}{\partial \nu^{l-1}} = 0, \quad \text{on } \partial \Omega, \\
&\int_\Omega u_j(x) u_s(x) dx = \delta_{js}, \quad \text{for any } j, s.
\end{align*}
$$

(3.1)

It is easy to see that $\{u_j\}_{j=1}^\infty$ forms an orthonormal basis of the Sobolev space $W^{l,2}_0(\Omega)$. Thus, we have

$$
\begin{align*}
w_{\sigma, \xi}(x) &= \sum_{j=1}^\infty a_{\sigma, j}(\xi) u_j(x), \\
a_{\sigma, j}(\xi) &= \int_\Omega w_{\sigma, \xi}(x) u_j(x) dx.
\end{align*}
$$

Defining a function

$$
\varphi_k(x) = w_{\sigma, \xi}(x) - \sum_{j=1}^k a_{\sigma, j}(\xi) u_j(x),
$$

we can verify that $\varphi_k|_{\partial \Omega} = \frac{\partial \varphi_k}{\partial \nu}|_{\partial \Omega} = \cdots = \frac{\partial^{l-1} \varphi_k}{\partial \nu^{l-1}}|_{\partial \Omega} = 0$ and

$$
\int_{\Omega} \varphi_k(x) u_j(x) dx = 0, \quad \text{for } j = 1, 2, \cdots, k.
$$

Hence, $\varphi_k$ is a trial function. By making use of the Rayleigh-Ritz formula (cf. [8]), we know that

$$
\Lambda_{k+1} \int_{\Omega} |\varphi_k(x)|^2 dx \leq \int_{\Omega} \left[ \nabla^l \varphi_k(x), \nabla^l \varphi_k(x) \right] dx.
$$

(3.2)

From the definition of $\varphi_k$ and (2.5), we have

$$
\int_{\Omega} |\varphi_k(x)|^2 dx = \int_{\Omega} \left| w_{\sigma, \xi}(x) - \sum_{j=1}^k a_{\sigma, j}(\xi) u_j(x) \right|^2 dx
$$

$$
= \int_{\Omega} |\rho_\sigma(x)|^2 dx - \sum_{j=1}^k |a_{\sigma, j}(\xi)|^2 dx
$$

$$
\geq \text{vol}(\Omega) - \text{vol}(\Omega_\sigma) - \sum_{j=1}^k |a_{\sigma, j}(\xi)|^2.
$$

(3.3)
Using (3.3) and Stokes’ formula, we deduce
\[
\int_\Omega \left[ \nabla^l \varphi_k(x), \nabla^l \varphi_k(x) \right] dx \\
= \int_\Omega \varphi_k(x)(-\Delta)^l \varphi_k(x) dx \\
= \int_\Omega \left( w_{\sigma, \xi}(x)(-\Delta)^l w_{\sigma, \xi}(x) + \sum_{j=1}^k |a_{\sigma,j}(\xi)|^2 u_j(x)(-\Delta)^l u_j(x) \right) dx
\]
\[
(3.4)
\]
\[
- \int_\Omega \left( w_{\sigma, \xi}(x) \sum_{j=1}^k a_{\sigma,j}(\xi)(-\Delta)^l u_j(x) + (-\Delta)^l w_{\sigma, \xi}(x) \sum_{j=1}^k a_{\sigma,j}(\xi) u_j(x) \right) dx
\]
\[
= \int_\Omega w_{\sigma, \xi}(x)(-\Delta)^l w_{\sigma, \xi}(x) dx + \sum_{j=1}^k \Lambda_j |a_{\sigma,j}(\xi)|^2
\]
\[
- \int_\Omega \left( w_{\sigma, \xi}(x) \sum_{j=1}^k a_{\sigma,j}(\xi)(-\Delta)^l u_j(x) + (-\Delta)^l w_{\sigma, \xi}(x) \sum_{j=1}^k a_{\sigma,j}(\xi) u_j(x) \right) dx
\]

Substituting (2.2) into (3.4), we have
\[
\int_\Omega \left[ \nabla^l \varphi_k(x), \nabla^l \varphi_k(x) \right] dx \\
= \int_\Omega \left[ \nabla^l w_{\sigma, \xi}(x), \nabla^l w_{\sigma, \xi}(x) \right] dx + \sum_{j=1}^k \Lambda_j |a_{\sigma,j}(\xi)|^2 \\
- \int_\Omega \left( w_{\sigma, \xi}(x) \sum_{j=1}^k a_{\sigma,j}(\xi)(-\Delta)^l u_j(x) + (-\Delta)^l w_{\sigma, \xi}(x) \sum_{j=1}^k a_{\sigma,j}(\xi) u_j(x) \right) dx \\
(3.5)
\]

By utilizing Stokes’ formula, we have
\[
\int_\Omega \left( w_{\sigma, \xi}(x) \sum_{j=1}^k a_{\sigma,j}(\xi)(-\Delta)^l u_j(x) + (-\Delta)^l w_{\sigma, \xi}(x) \sum_{j=1}^k a_{\sigma,j}(\xi) u_j(x) \right) dx \\
= \int_\Omega \left( w_{\sigma, \xi}(x) \sum_{j=1}^k a_{\sigma,j}(\xi)(-\Delta)^l u_j(x) + w_{\sigma, \xi}(x) \sum_{j=1}^k a_{\sigma,j}(\xi)(-\Delta)^l u_j(x) \right) dx \\
(3.6)
\]
\[
= \int_\Omega \sum_{j=1}^k \left( \frac{w_{\sigma, \xi}(x)}{a_{\sigma,j}(\xi)} a_{\sigma,j}(\xi) + \frac{w_{\sigma, \xi}(x)}{a_{\sigma,j}(\xi)} a_{\sigma,j}(\xi) \right) (-\Delta)^l u_j(x) dx \\
= \int_\Omega \sum_{j=1}^k \Lambda_j \left( \sum_{i=1}^\infty a_{\sigma,i}(\xi) u_i(x) a_{\sigma,j}(\xi) + \sum_{i=1}^\infty a_{\sigma,i}(\xi) u_i(x) a_{r,j}(\xi) \right) u_j(x) dx \\
= 2 \sum_{j=1}^k \Lambda_j |a_{\sigma,j}(\xi)|^2.
\]
Without loss of generality, we only consider the case that \( l \) is an even number and \( l \geq n - 2 \). For the other cases, we can obtain the desired result by making use of the same method as above case. Applying lemma 2.1 and lemma 2.2 we have

\[
\int_{\Omega} \left[ \nabla^l w_{\sigma,\xi}(x), \nabla^l w_{\sigma,\xi}(x) \right] \, dx \\
= \int_{\Omega} \left[ \nabla^l w_{\sigma,\xi}(x), \nabla^l w_{\sigma,\xi}(x) \right] \, dx + \int_{\Omega} \left[ \nabla^l w_{\sigma,\xi}(x), \nabla^l w_{\sigma,\xi}(x) \right] \, dx
\]

(3.7)

\[
\leq |\xi|^{2l}(V(\Omega) - V(\Omega_\sigma)) + |\xi|^{2l-4} \left[ |\xi|^{2} + \frac{(2l^2 - 2nl + 4l)|\xi|^2}{\sigma^2} + \frac{(l(l - 2) + nl)^2}{\sigma^4} \right] V(\Omega_\sigma)
\]

\[
= |\xi|^{2l}V(\Omega) + |\xi|^{2l-4} \left[ \frac{(2l^2 - 2nl + 4l)|\xi|^2}{\sigma^2} + \frac{(l(l - 2) + nl)^2}{\sigma^4} \right] V(\Omega_\sigma)
\]

Uniting (3.2), (3.3), (3.5), (3.6) and (3.7), we infer

\[
\Lambda_{k+1}(V(\Omega) - V(\Omega_\sigma)) \\
\leq |\xi|^{2l}V(\Omega) + |\xi|^{2l-4} \left[ \frac{(2l^2 - 2nl + 4l)|\xi|^2}{\sigma^2} + \frac{(l(l - 2) + nl)^2}{\sigma^4} \right] V(\Omega_\sigma)
\]

\[
+ \sum_{j=1}^{k} (\Lambda_{k+1} - \Lambda_j) |a_{\sigma,j}(\xi)|^2
\]

(3.8)

\[
= |\xi|^{2l}V(\Omega) + |\xi|^{2l-4} \left[ \frac{(2l^2 - 2nl + 4l)}{\sigma^2} V(\Omega_\sigma) + \frac{(l(l - 2) + nl)^2}{\sigma^4} \right] V(\Omega_\sigma)
\]

\[
+ \sum_{j=1}^{k} (\Lambda_{k+1} - \Lambda_j) |a_{\sigma,j}(\xi)|^2;
\]

here \( \sigma > \sigma_0 > \sqrt{\max_{x \in \Omega} |x|^2} \). We use the symbol \( B_n(\sigma) \) and \( O \) to denote the ball on \( \mathbb{R}^n \) with a radius \( \sigma \) and the origin in \( \mathbb{R}^n \), respectively. By integrating the above inequality on the variable \( \xi \) on the ball \( B_n(\sigma) \subset \mathbb{R}^n \), we derive from (3.8)

\[
\sigma^n B_n(V(\Omega) - V(\Omega_\sigma)) \Lambda_{k+1}
\]

\[
\leq B_n \sigma^{n+2l} \left\{ \frac{n}{n+2l} V(\Omega) + \frac{n(2l^2 - 2nl + 4l)}{n+2l - 2} \frac{V(\Omega_\sigma)}{\sigma^4} \right. \\
\left. + \frac{n(l(l - 2) + nl)^2}{n+2l - 4} \frac{V(\Omega_\sigma)}{\sigma^8} \right\} + \sum_{j=1}^{k} (\Lambda_{k+1} - \Lambda_j) \int_{B_n(\sigma)} |a_{\sigma,j}(\xi)|^2 d\xi,
\]

(3.9)

for \( \sigma \geq \sigma_0 > \sqrt{\max_{x \in \Omega} |x|^2} \). Putting

\[
\psi_j(x) = \begin{cases} u_j(x), & x \in \Omega, \\
0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases}
\]

From Parseval’s identity for Fourier transform, we infer
\[ \int_{B_n(\sigma)} |a_{\sigma,j}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} |a_{\sigma,j}(\xi)|^2 d\xi \]
\[ = \int_{\mathbb{R}^n} \left| \int_{\Omega} e^{i(\xi,x)} \rho_\sigma(x) u_j(x) dx \right|^2 d\xi \]
\[ = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{i(\xi,x)} \rho_\sigma(x) \psi_j(x) dx \right|^2 d\xi \]
\[ = (2\pi)^n \int_{\mathbb{R}^n} \left| \hat{\rho}_\sigma \psi_j(\xi) \right|^2 d\xi \]
\[ = (2\pi)^n \int_{\mathbb{R}^n} \left| \rho_\sigma(x) \psi_j(x) \right|^2 dx \]
\[ = (2\pi)^n \int_{\Omega} \left| \rho_\sigma(x) u_j(x) \right|^2 dx \]
\[ \leq (2\pi)^n. \]

Therefore, from (3.10) and (3.9), we obtain

\[ \sigma^n B_n (V(\Omega) - V(\Omega_\sigma)) \Lambda_{k+1} \]
\[ \leq B_n \sigma^{n+2l} \left\{ \frac{n}{n+2l} V(\Omega) + \frac{n(2l^2 - 2nl + 4l)}{n+2l-2} \frac{V(\Omega)}{\sigma^4} \right\} \]
\[ + \frac{n(l(l-2) + nl)}{n+2l-4} \frac{V(\Omega)}{\sigma^8} \]
\[ + (2\pi)^n \sum_{j=1}^k (\Lambda_{k+1} - \Lambda_j), \quad \sigma \geq \sigma_0 > \sqrt{\max_{x \in \Omega} |x|^2}. \]

Taking

\[ \sigma = 2\pi \left( \frac{1 + k}{B_n (V(\Omega) - V(\Omega_\sigma))} \right)^{\frac{1}{n}}, \]

noting \( k \geq V(\Omega)\sigma_0^n \) and \( \frac{2\pi}{(B_n)^{\frac{1}{n}}} > 1 \), we have

\[ \sigma^n = \frac{(2\pi)^n}{B_n} \left( \frac{1 + k}{V(\Omega) - (\Omega_\sigma)} \right) \geq \frac{1 + V(\Omega)\sigma_0^n}{V(\Omega) - V(\Omega_\sigma)} > \frac{\sigma_0^n}{1 - \frac{V(\Omega_\sigma)}{V(\Omega)}} \geq \sigma_0^n \]

and thus \( \sigma > \sigma_0 \). Furthermore, synthesizing (3.11) with (3.12), we deduce
respectively. Putting

\[ \frac{1}{1 + k} \sum_{j=1}^{k+1} \lambda_j \]

\[ \leq \frac{n}{n + 2l} \frac{(2\pi)^{2l} V(\Omega)}{(V(\Omega) - V(\Omega_{\sigma}))} \frac{n + 2l}{n} B^n \frac{(1 + k)^{2l}}{n} \]

\[ + \frac{n(2l^2 - 2nl + 4l)}{n + 2l - 2} \frac{(2\pi)^{2l-4} V(\Omega_\sigma)}{(V(\Omega) - V(\Omega_{\sigma_0}))} \frac{n + 2l - 4}{n} B^n \frac{(1 + k)^{2l-4}}{n} \]

\[ + \frac{n(l(l - 2) + nl)^2}{n + 2l - 4} \frac{(2\pi)^{2l-8} V(\Omega_\sigma)}{(V(\Omega) - V(\Omega_{\sigma_0}))} \frac{n + 2l - 8}{n} B^n \frac{(1 + k)^{2l-8}}{n} \]

(3.13)

This completes the proof of Theorem 1.1.

\[ \square \]

**Proof of the Corollary 1.1.** Without loss of generality, we firstly consider the case of \( l \) is an even number and \( n \leq l + 2 \). Let

\[ B_1 = \frac{n}{n + 2l} \frac{(2\pi)^{2l}}{(V(\Omega_{\sigma_0}))} \frac{n + 2l}{n} (B_n V(\Omega))^\frac{2l}{n} (1 + k)^\frac{2l}{n}, \]

(3.14)

\[ B_2 = \frac{(2\pi)^{2l-4} V(\Omega_{\sigma_0})}{V(\Omega)} \frac{n + 2l - 4}{n} (B_n V(\Omega))^\frac{2l-4}{n} (1 + k)^\frac{2l-4}{n}, \]

(3.15)

and

\[ B_3 = \frac{(2\pi)^{2l-8} V(\Omega_{\sigma_0})}{V(\Omega)} \frac{n + 2l - 8}{n} (B_n V(\Omega))^\frac{2l-8}{n} (1 + k)^\frac{2l-8}{n}, \]

(3.16)

respectively. Putting

\[ \theta = \frac{V(\Omega_{\sigma_0})}{V(\Omega)}. \]

Next, the first step is to estimate the value of \( B_1 \). From (3.14), we deduce that
\[ B_1 = \frac{n}{n + 2l (B_n V(\Omega))^{\frac{2l}{n}}} (1 + k)^{\frac{2l}{n}} \frac{1}{(1 - \theta)^{\frac{n + 2l}{n}}} \]

\[ (3.17) \]

\[ = \frac{n}{n + 2l (B_n V(\Omega))^{\frac{2l}{n}}} (1 + k)^{\frac{2l}{n}} + \frac{n}{n + 2l (B_n V(\Omega))^{\frac{2l}{n}}} (1 + k)^{\frac{2l - 4}{n} C_1(\theta)}, \]

where

\[ C_1(\theta) = (1 + k)^{\frac{4}{n}} \left( \frac{1}{(1 - \theta)^{\frac{n + 2l}{n}}} - 1 \right), \]

with \( C_1(0) = 0 \) and

\[ C_1'(\theta) = (1 + k)^{\frac{4}{n}} \frac{(1 + \frac{2l}{n})}{(1 - \theta)^{\frac{n + 2l}{n}}}. \]

Since

\[ V(\Omega_\sigma) \leq \delta_0 V(\Omega)^{\frac{n - \tau}{n}} \frac{1}{\sigma^\tau} \]

for \( \sigma > V(\Omega)^{-\frac{\tau}{n}} \), there exists a constant \( \theta_0 \) such that

\[ (3.18) \quad 0 < \theta = \frac{V(\Omega_\sigma)}{V(\Omega)} \leq \frac{\delta_0}{(1 + k)^{\frac{4}{n}}} \leq \theta_0 < 1 \]

with \( \sigma_0 = \left( \frac{1 + k}{V(\Omega)} \right)^{\frac{n}{\tau}} \). By Lagrange mean value theorem, \((3.17)\) and \((3.18)\), it is not difficult to see that there exists \( 0 < \epsilon_1 < 1 \) such that

\[ (3.19) \quad C_1(\theta) = C_1(0) + C_1'(\epsilon_1) \theta = \frac{(1 + \frac{2l}{n})(1 + k)^{\frac{4}{n}}}{(1 - \epsilon_1)^{\frac{2n + 2l}{n}}} \theta \leq \delta_0 \frac{(1 + \frac{2l}{n})}{(1 - \epsilon_1)^{\frac{2n + 2l}{n}}} (1 + k)^{\frac{4 - \tau}{n}}. \]

Therefore, there exists a constant \( \alpha_1(n, l) \) such that

\[ B_1 \leq \frac{n}{n + 2l (B_n V(\Omega))^{\frac{2l}{n}}} (1 + k)^{\frac{2l}{n}} + \delta_0 \alpha_1(n, l) \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} (1 + k)^{\frac{2l - \tau}{n}}. \]

Second, we estimate the value of \( B_2 \). From \((3.15)\), we know that

\[ (3.20) \quad B_2 = \mathcal{A}_1(n, l) \frac{(2\pi)^{2l - 4}}{(B_n V(\Omega))^{\frac{2l - 4}{n}}} (1 + k)^{\frac{2l - \tau}{n}} C_2(\theta), \]

where

\[ C_2(\theta) = \frac{\theta}{(1 - \theta)^{\frac{n + 2l - 4}{n}}} (1 + k)^{\frac{\tau - 4}{n}} \]

with \( C_2(0) = 0 \) and

\[ C_2'(\theta) = \frac{1 - \theta + \frac{n + 2l - 4}{n} \theta}{(1 - \theta)^{\frac{2n + 2l - 4}{n}}} (1 + k)^{\frac{\tau - 4}{n}}. \]
Likewise, by means of Lagrange mean value theorem and \((3.19)\), we know that there exists \(0 < \varepsilon_2 < 1\) such that
\[
C_2(\theta) = C_2(0) + C'_2(\varepsilon_2)\theta \\
= \frac{1 - \varepsilon_2 + \frac{n+2l-4}{n}\varepsilon_2}{(1 - \varepsilon_2)^{2n+2l-4}}(1 + k)^{\frac{r-4}{n}} \\
\leq \delta_0 \frac{1 - \varepsilon_2 + \frac{n+2l-4}{n}\varepsilon_2}{(1 - \varepsilon_2)^{2n+2l-4}}.
\]
(3.21)

Therefore, according to \((3.20)\) and \((3.21)\), we know that there exists a constant \(\alpha_2(n, l)\) such that
\[
B_2 \leq \delta_0 \frac{\alpha_2(n, l)(2\pi)^{2l-4}}{(B_nV(\Omega))^{\frac{2l-4}{n}}}(1 + k)^{\frac{2l-4-r}{n}}.
\]
Similarly, there exists a constant \(\alpha_3(n, l)\) such that
\[
B_3 \leq \delta_0 \frac{\alpha_3(n, l)(2\pi)^{2l-8}}{(B_nV(\Omega))^{\frac{2l-8}{n}}}(1 + k)^{\frac{2l-4-r}{n}}.
\]
(3.23)

Therefore, substituting \((3.17)\), \((3.22)\) and \((3.23)\) into \((3.13)\) we finally obtain
\[
\frac{1}{1 + k} \sum_{j=1}^{k+1} \Lambda_j \leq \frac{n}{n + 2l} \frac{(2\pi)^{2l}}{(B_nV(\Omega))^{\frac{2l}{n}}}(1 + k)\frac{2l}{n} \\
+ \delta_0 \left\{ \alpha_1(n, l) \frac{(2\pi)^{2l}}{(B_nV(\Omega))^{\frac{2l}{n}}} + \alpha_2(n, l) \frac{(2\pi)^{2l-4}}{(B_nV(\Omega))^{\frac{2l-4}{n}}} \right\}(1 + k)^{\frac{2l-4-r}{n}} \\
+ \delta_0 \alpha_3(n, l) \frac{(2\pi)^{2l-8}}{(B_nV(\Omega))^{\frac{2l-8}{n}}}(1 + k)^{\frac{2l-4-r}{n}}.
\]

For the other cases (i.e., \(l = 1; l\) is an even number and \(l < n - 2; l(\neq 1)\) is an odd number and \(l \geq n - 3 - \frac{2}{l-1}; \) or \(l(\neq 1)\) is an odd number and \(l < n - 3 - \frac{2}{l-1}\)), we can also obtain the corresponding results by means of the same method. Therefore, we finish the proof of this corollary.

\begin{flushright}
\Box
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