Radiation emission due to fluxon scattering on inhomogeneity in large two-dimensional Josephson junction

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Interaction of a fluxon in the two-dimensional large Josephson junction with the finite area inhomogeneity is studied within the sine-Gordon theory. The spectral density of the emitted plane waves is computed exactly for the rectangular and rhombic inhomogeneities. Total emitted energy as a function of the fluxon velocity exhibits at least one local maximum. Connection to the previously studied limiting cases including the point impurity and the one-dimensional limit has been performed.

I. INTRODUCTION

Studies of the fluxon (Josephson vortex) dynamics in large Josephson junctions (LJJ) is an important problem in the modern superconductivity. The LJJ can be spatially inhomogeneous either due to the production defects or can be manufactured in such a way on purpose. Thus, the problem of the fluxon interaction with the spatial inhomogeneity (microshort, microresistor, Abrikosov vortex etc.) is of remarkable importance. As a result of the fluxon-impurity interaction radiation of the small-amplitude linear waves (Josephson plasmons) occurs. The issue of the linear wave radiation due to the fluxon-collision with the spatial inhomogeneity has been studied in detail for the one-dimensional case (1D). Most of these (both theoretical and experimental) studies have focused on the scattering on the point-like inhomogeneity being either a microshort or a microresistor or a magnetic impurity. An extended inhomogeneity has been investigated as well as the interface separating two different junctions.

An important thing to note is that a 1D Josephson junction is only an 1D approximation of the two-dimensional (2D) LJJ of the final width. Thus, a natural question is to take the transverse direction into account and to study the fluxon scattering on an impurity in this situation. Moreover, fluxon dynamics in the large area JJ is an interesting and important problem in its own right. It has been studied in different contexts such as dynamical properties, pinning on impurities, and applications. However, up to now the radiation emission due to the 2D fluxon scattering on the impurity has been studied in detail only for the special case of the point-like impurity described by the δ-function. Thus, the aim of this paper is to study the properties of the small-amplitude wave radiation that appears as a result of fluxon transmission on the inhomogeneity of the general shape.

The paper is organized as follows. In the next section, the model is described. Section III is devoted to the studies of the radiation emitted due to the fluxon-impurity interaction. In the last section, the discussion and conclusions are presented.

II. THE MODEL

We consider fluxon dynamics in the LJJ with spatial inhomogeneities. The main dynamical variable is the difference between the phases \( \theta_2(x, y; t) - \theta_1(x, y; t) \) of the macroscopic wave functions of the superconducting layers of the junction, also known as the Josephson phase. In the bulk of the junction this variable satisfies the following equation

\[
\partial_t H_y - \partial_x H_x = j_c [1 + f_1(x, y)] \sin \phi + \frac{C_h}{2e} \partial_x^2 \phi ,
\]

(1)

where the function \( f_1(x, y) \) describes the critical current change on the spatial inhomogeneity and the magnetic field components \( H_{x,y} \) are related to the Josephson phase as

\[
H_x = \frac{\hbar}{2e\mu_0 l(x, y)} \partial_y \phi , \quad H_y = \frac{\hbar}{2e\mu_0 l(x, y)} \partial_x \phi .
\]

(2)

Among other parameters \( C \) is the junction capacitance, \( j_c \) is the critical current density away from the impurity, \( e \) is the electron charge, \( \mu_0 \) is the vacuum permeability and \( \hbar \) is the Planck’s constant. The value \( l(x, y) \) describes the thickness of the layer that allows magnetic field penetration. It varies in space due to the presence of the impurity and can be written as \( l(x, y) = 2\lambda_L + d_i(x, y) \), where \( \lambda_L \) is superconductors London penetration depth and \( d_i(x, y) \) is the insulating layer thickness. Away from the impurity \( d_i(x, y) = d_0 = \text{const} \) while \( d_i(x, y) = d_0 + d_1 = \text{const} \) inside the impurity. For the impurity of the general shape that covers a certain segment \( \Omega \subset \mathbb{R}^2 \) of the junction one can write

\[
f_1(x, y) = \begin{cases} 
\mu_1 & \text{if } (x, y) \in \Omega , \\
0 & \text{if } (x, y) \notin \Omega ,
\end{cases}
\]

(3)
and
\[ l(x, y) = \begin{cases} d_0 + 2\lambda_L + d_1 & \text{if } (x, y) \in \Omega, \\ l_0 = d_0 + 2\lambda_L & \text{if } (x, y) \notin \Omega. \end{cases} \tag{4} \]

The impurity is a microshort if \( \mu_1 > 0, d_1 < 0 \) and a microresistor if \( \mu_1 < 0, d_1 > 0 \). If the dimensionless variables are introduced, the equation \( (1) \) becomes the two-dimensional perturbed sine-Gordon (SG) equation
\[ \{ \partial_x^2 - \partial_y^2 \} \phi + \{ 1 + f_1(x, y) \} \sin \phi = 0, \tag{5} \]
where
\[ l_0 \frac{l(x, y)}{l_0 + d_1} = 1 + f_2(x, y) = 1 + \begin{cases} 0 & \text{if } (x, y) \notin \Omega, \\ \mu_2 & \text{if } (x, y) \in \Omega. \end{cases} \tag{6} \]

The ratio \( \mu_1/\mu_2 \) is positive both for microshorts and microresistors. In the dimensionless equation \( (5) \) spatial variables \( x \) and \( y \) are normalised to the Josephson penetration depth \( \lambda_J \), the temporal variable \( t \) is normalised to the inverse Josephson plasma frequency \( \omega_J^{-1} \). For details one might consult the textbook\(^{1,2} \).

### III. RADIATION EMISSION

Fluxon interaction with the spatial inhomogeneity is normally accompanied with the radiation of the small-amplitude electromagnetic waves\(^{2} \) (Josephson plasmons). Below we present the general scheme for the calculation of the radiation created by the fluxon-impurity interaction which is based on the method developed for the delta-like impurity\(^{20} \) or for the respective 1D problem\(^{23,24} \). Only the main points of the derivation procedure are presented. For the details the interested reader can consult the abovementioned papers.

#### A. General framework

We seek the solution of the SG equation \( (5) \) as a superposition of the exact soliton solution and the plasmon radiation on its background: \( \phi(x, y, t) = \phi_0(x, t) + \psi(x, y, t) \). The spatial inhomogeneity is considered as a small \( (|\mu_1,2| \ll 1) \) perturbation. Here \( \phi_0(x, t) = 4 \arctan \left( \frac{x-\tau}{\sqrt{1-v^2}} \right) \) is the exact soliton solution of the unperturbed 1D SG equation and \( \psi(x, y, t) \) is the radiative correction, \( ||\psi|| \ll \phi_0 \). It is convenient to work in the reference frame that moves with the fluxon velocity \( v: \xi = \frac{x-\tau}{\sqrt{1-v^2}}, \tau = \frac{x-\tau}{\sqrt{1-v^2}} \). In these new variables we have \( \phi_0(x, t) = \phi_0(\xi) = 4 \arctan (\exp \xi) \).

In the moving reference frame the equation that describes the emitted radiation reads
\[ \{ \partial_\xi^2 - (\partial^2_\xi + \partial^2_y) + \cos[\phi_0(\xi)] \} \psi = R(\xi, y; \tau), \tag{7} \]
where the right-hand side of Eq. \( (7) \) is completely defined by the impurity:
\[ R(\xi, y; \tau) = \frac{2}{(1-v^2)} \left\{ \left[ 1 - v^2 \right] - \frac{\mu_2}{\mu_1} \frac{\tan \xi}{\cosh \xi} \times \right\} \times f_1 \left( \frac{\xi + v\tau}{\sqrt{1-v^2}}, y \right) + \sqrt{1-v^2} \frac{\sinh \xi}{\cosh \xi} \left( \frac{\xi + v\tau}{\sqrt{1-v^2}}, y \right) \right\}, \tag{8} \]
\[ h_2(x, y) = \partial_x f_2(x, y). \]

In this expression it has been taken into account that \( f_2(x, y) = \mu_2 f_1(x, y)/\mu_1 \) and \( |\sin[\phi_0(\xi)]| = |\partial^2_\xi \phi_0(\xi)| = -2 \tan \xi / \cosh \xi \). Here the last term of \( R(\xi, y; \tau) \) in the curly brackets is associated with the fluxon interaction with the borders of the impurity because \( h_2(x, y) \neq 0 \) only there, i.e. if \( (x, y) \notin \partial \Omega \). The first term corresponds to the radiation produced when the fluxon passes the bulk of the impurity.

The solution of Eq. \( (7) \) can be represented as
\[ \psi(\xi, y; \tau) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} a(q_\xi, q_y; \tau) \varphi(\xi, y; q_\xi, q_y) \, dq_\xi \, dq_y, \tag{9} \]
where \( \varphi(\xi, y; q_\xi, q_y) \) is the eigenfunction\(^{23,24} \) of the homogeneous part of this equation:
\[ \varphi(\xi, y; q_\xi, q_y) = \frac{e^{i(q_\xi \xi + q_y y)} q_\xi + i \tan \xi}{(2\pi)^{3/2} (1 + q^2_y)^{1/2}}, \tag{10} \]
\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi^*(\xi, y; q_\xi, q_y) \varphi(\xi, y; q_\xi', q_y') \, dq_\xi \, dq_y = \frac{1}{2\pi} \delta(q_\xi - q_\xi') \delta(q_y - q_y'). \tag{11} \]
Here \( \delta \) is Dirac’s delta-function, \( q_\xi \) and \( q_y \) are the components of the plasmon wave vector in the moving frame and
\[ \tilde{\omega} = \sqrt{1 + q^2_\xi + q^2_y}, \tag{12} \]
is the plasmon dispersion law in that frame. The function \( a(q_\xi, q_y) \) is the radiation amplitude. It is convenient to introduce another function which also describes emitted radiation, namely \( b(q_\xi, q_y; \tau) = (a_\tau - i\tilde{\omega}a) \exp(i\tilde{\omega} \tau) \). As a result, the following equality holds
\[ \partial_\tau b = e^{i\tilde{\omega} \tau} \left( \partial^2_\xi a + \tilde{\omega}^2 a \right). \tag{13} \]

Multiplying both sides of Eq. \( (7) \) by \( \varphi^*(\xi, y; q_\xi, q_y) \) and integrating simultaneously over \( y \in \mathbb{R} \) and \( \xi \in \mathbb{R} \) we obtain \( \delta(q_\xi - q_\xi') \) and \( \delta(q_y - q_y') \) in the left-hand side [the orthogonality condition \( (11) \) is used] of Eq. \( (7) \). After removing the integration over \( q_\xi \) and \( q_y \) one arrives to the following expression:
\[ \partial_\tau b = 2\pi e^{i\tilde{\omega} \tau} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R(\xi, y; \tau) \times \varphi^*(\xi, y; q_\xi, q_y) \, dq_\xi \, dq_y. \tag{14} \]
The total radiation over the whole time is defined by the function

\[ B(q_x, q_y) = \int_{-\infty}^{+\infty} \partial_{\tau} b(q_x, q_y; \tau) \, d\tau. \quad (15) \]

Thus, with the pair of Eqs. (14)-(15) one has the complete formula for the energy calculation. From this point it is possible proceed to the emitted radiation studies for the particular shapes of \( \Omega \). The return to the laboratory frame is performed with the help of the following Lorentz transformation

\[ q_x = \frac{q_\xi + v\bar{\omega}}{\sqrt{1 - v^2}}, \quad \omega = \frac{vq_\xi + \bar{\omega}}{\sqrt{1 - v^2}}, \quad (16) \]
\[ q_\xi = \frac{q_x - v\bar{\omega}}{\sqrt{1 - v^2}}, \quad \bar{\omega} = \omega - vq_x \sqrt{1 - v^2}. \quad (17) \]

The \( q_y \) component remains unchanged. Taking into account that the emitted energy density equals

\[ E(x, y) \simeq \frac{|B(q_x, q_y)|^2}{4\pi}, \]

the total energy equals

\[ E = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(q_x, q_y) \, dq_x dq_y. \quad (18) \]

A following simplification can be achieved if \( \Omega \) has the properties, defined below. Suppose the impurity covers the area that is limited by the lines \( x = x_1 \) and \( x = x_2 \) along the \( y \) axis and by the continuous and single-valued functions \( y = g_{\pm}(x) \) along the \( x \) axis, as is shown in Fig. 1. In this case

\[ f_{1,2}(x, y) = \mu_{1,2} \left[ \theta \left( x + \frac{d_x}{2} \right) - \theta \left( x - \frac{d_x}{2} \right) \right] \times \]
\[ \times \left[ \theta \left( y + \frac{d_y}{2} \right) - \theta \left( y - \frac{d_y}{2} \right) \right], \quad (20) \]
\[ h_{2}(x, y) = \mu_{2} \left[ \delta \left( x + \frac{d_x}{2} \right) - \delta \left( x - \frac{d_x}{2} \right) \right] \times \]
\[ \times \left[ \theta \left( y + \frac{d_y}{2} \right) - \theta \left( y - \frac{d_y}{2} \right) \right], \quad (21) \]

is considered. The parameters \( d_x \) and \( d_y \) are the impurity length and width, respectively.

1. Spectral density of the emitted waves

At this point we can substitute the actual expressions (20)-(21) that corresponds to the rectangular impurity into Eqs. (14)-(15). Then the radiation function (15) in the moving frame is obtained after the consecutive integration over the \( y, \tau \) and \( \xi \) variables:
\[ B(q_x, q_y) = i \frac{2\sqrt{2\pi} \mu_1}{\omega q_y \sqrt{1 + q^2 (1 - v^2)v^2}} \left\{ \left( 1 - v^2 - \frac{\mu_2}{\mu_1} \right) \left[ \tilde{\omega}^2 - \left( 1 + q^2 \right) v^2 \right] + 2 \frac{\mu_2}{\mu_1} \left( 1 - v^2 \right) \tilde{\omega}^2 \right\} \sin \left( \frac{\tilde{\omega} v dq_y}{2} \right) \times \sin \left( \frac{\tilde{\omega} \sqrt{1 - v^2}}{2 v} dx \right) \text{sech} \left[ \frac{\pi}{2 v} (q_x + \tilde{\omega}) \right]. \] (22)

The first term in the curly brackets in Eq. (22) appears due to the first term in \( R \) [see Eq. (8)] and can be treated as a result of the fluxon interaction with the bulk of the impurity. The second term in the curly brackets appears due to the second term (associated with the function \( h_2 \)) in Eq. (8) and can be considered as the radiation that appears due to the fluxon interaction with the border of the impurity. After returning to the laboratory frame of reference with the help of Eqs. (16)-(17) the final formula for the spectral density reads:

\[ \mathcal{E}(q_x, q_y) = 2\mu_1^2 \frac{(1 - v^2 + \mu_2/\mu_1)^2}{v^4} \left[ \sin(q_d dq_y/2) \right]^2 \left[ \sin(dx(q - q_x v)/2v) \right]^2 \frac{(\omega - q_x v)^2 + \frac{1 - v^2}{v^2} \mu_2/\mu_1 q^2 v^2}{(\omega - q_x v)^2 - q^2_0 (1 - v^2)} \times \text{sech}^2 \left[ \frac{\pi \omega}{2v} \sqrt{1 - v^2} \right], \]

\[ \omega = \sqrt{1 + q^2 + q^2_0}. \] (23) (24)

This function is symmetric with respect to the mirror symmetry \( q_y \rightarrow -q_y \) and to the transform \( q_x \rightarrow -q_x, v \rightarrow -v \). Therefore, it is sufficient to restrict the plots of \( E(v) \) to the interval \( 0 \leq v \leq 1 \). In order to compute the total emitted energy \( E(v) \) [see Eq. (18)] it is necessary to use numerical methods because it is not possible to take the respective double integral explicitly.

2. 1D limit

Before embarking on the investigation of the full 2D problem it is instructive to recall the corresponding one-dimensional (1D) case of the fluxon scattering on the impurity with the length \( d_x \). Formally this limit can be achieved if \( d_y \rightarrow \infty \). The energy density in this case is already known13

\[ \mathcal{E}(q) = \frac{\pi}{v^4} \mu_1^2 \left[ \frac{\mu_2}{\mu_1} + (1 - v^2) \right]^2 \times \sin^2 \left[ \frac{dx}{2v} \sqrt{1 + q^2 - qv} \right] \times \text{sech}^2 \left[ \frac{\pi \sqrt{1 - v^2}}{2v} \sqrt{1 + q^2} \right]. \] (25)

We note that this expression can be obtained in the limit \( q_y \rightarrow 0 \) from Eq. (23) (\( \mu_1 \) should be renormalized as \( \mu_1 d_y \rightarrow \mu_1 \)). This means that the impurity width \( d_y \) tends to infinity, and, as a result, the scattering does not create any radiation in the \( y \) direction, leaving the problem completely invariant in that direction, i.e. one-dimensional.

Typical dependencies of the spectral density \( \mathcal{E} = \mathcal{E}(q) \) for the different values of the fluxon velocity are given in Fig. 2. It is easy to see that the energy density \( \mathcal{E}(q) \) [Eq. (25)] has an infinite countable set of global minima for which \( \mathcal{E}(q_{min}) = 0 \). They are the roots of the equation

\[ \frac{dx(\sqrt{1 + q^2_{min} - qmin v})}{2v} = \pi n, \quad n = n_0, n_0 + 1, \ldots, \]

\[ n_0 = \left[ dx(1 - v^2)^{1/2}/(2v\pi) \right] > 0, \] (26)

where \( \lceil x \rceil \) is the ceiling function of \( x \). Similarly, there are maxima that are placed between those minima at the values of \( q \) that are the roots of the equations

\[ \frac{dx(\sqrt{1 + q^2_{max} - qmax v})}{2v} \approx \frac{\pi(2n - 1)}{2}, \] (27)

The minima and maxima are associated with the constructive and destructive interference of the plasmons, emitted when the fluxon enters and exits the impurity. Depending on the length of the impurity and the fluxon velocity, the radiated plasmons can either cancel each other if their phases differ by \( \pm \pi \) or can enhance each other if their phases coincide. The radiation consist of
the forward \((q > 0)\) and backward \((q < 0)\) emitted plasmons, and the energy of these plasmons is distributed non-homogeneously with respect to \(q\). First of all, the most of the energy is concentrated in the long-wavelength modes due to the presence of the \(\text{sech}^2(\cdot \cdot \cdot)\) term in Eq. \((25)\). Secondly, as can be seen from Fig. 2 the distribution of the backward radiation is defined by the extrema \((26)-\(27)\) that lie on the negative half-axis \((q < 0)\). These extrema are distributed almost in an equidistant way with the step \(2\pi v/d_x(1 + v)\), therefore, the small change of \(v\) will lead to the small change in the area under the \(E(q)\) curve. On the contrary, the forward radiation depends strongly on \(v\), especially if \(v\) is not small \((v < 1)\) but not \(v \ll 1\). Only for large \(q\)’s the extrema are distributed with the almost fixed step \(2\pi v/d_x(1-v)\].

The minima of \(E(q)\) given by Eq. \((26)\) come in pairs, numbered by the index \(n\). These pairs are placed on the different sides from the value \(q = v/\sqrt{1-v^2}\), which is the minimum of the l.h.s of Eqs. \((26)-(27)\). The pair with \(n = n_0\) is the pair of the minima, that are the closest to each other. There always should be a maximum between these minima. If the abovementioned minima are very close to each other \((2\pi n_0 v/d_x \gg \sqrt{1-v^2})\), the maximum between them cannot be associated with Eq. \((27)\), as seen in Figs. 2(a, c), thus, the respective value of \(E\) lies not on the \(\text{sech}^2(\cdot \cdot \cdot)\) envelope function, but significantly below it. As a result, for these values of \(v\) the forward radiation can be insignificant, as can be observed from the area below the curve \(E(q)\) at \(q > 0\). In another case, the pair of minima that correspond to \(n_0\) are significantly separated, and the maximum between them belongs to the set \((27)\). It is again the first maximum at the positive axis, and it attains the value of \(E\) which is quite large comparing to the previous case, as can be seen in Figs. 2b, d.

The dashed lines 4 and 5 in Fig. 3 show the dependence of the total emitted energy on the fluxon velocity (the solid lines correspond to the 2D case which will be discussed later). The values of \(v\) which correspond to the minima of the \(E(v)\) in curve 5 in Fig. 3 have the minimal forward emission, and the respective spectral energy distributions are shown in Fig. 2(a, c). The values of \(v\) that are the maxima of \(E(v)\) correspond to the maximal forward emission and the respective spectral distributions are given in Figs. 2b, d. Thus, the maxima of the total energy coincide with the maximal forward emission while the minima of \(E(v)\) correspond to the minimal forward emission. It should be noted that the minima [Eq. \((26)\)] and maxima [Eq. \((27)\)] of the energy density are distributed approximately equidistantly for the short-wavelength \((|q| > 1)\) modes but with the different step for \(q > 0\) and \(q < 0\). In the limit \(|v| \ll 1\) this step becomes approximately the same, it equals \(2\pi v/d_x\). Consequently, in the limit \(|v| \to 0\) one cannot expect sharp pronounced extrema of the \(E(v)\) dependence, and this can be noticed from the inset.

![FIG. 2:](image)

**FIG. 2:** (Color online). Energy density [see Eq. \((26)\)] for the 1D junction with \(d_x = 8\), \(\mu_2 = 0\) at the fluxon velocity \(v = 0.398\) (a), \(v = 0.488\) (b), \(v = 0.552\) (c) and \(v = 0.676\) (d). The red dashed line depicts the \(\text{sech}^2(\cdot \cdot \cdot)\) “envelope” term in Eq. \((26)\).
3. Total emitted energy in the 2D case

First of all, we discuss the dependence of the total emitted energy \( E(v) \) on the impurity parameters \( \mu_1 \) and \( \mu_2 \). We remind that \( \mu_1 \) is associated with the change of the critical current and \( \mu_2 \) is associated with the narrowing or distension of the insulating area. If \( \mu_2 = 0 \) the impurity corresponds only to the local change of the critical current without any changes in the insulating layer thickness. The total emitted energy for different values of the \( \mu_2/\mu_1 \) ratio is given in Fig. 3. We note the principal difference in the behaviour of the \( E(v) \) function in the limit \( v \to 1 \) if \( \mu_2 \neq 0 \) comparing to the case \( \mu_2 = 0 \).

In the latter case \( E(v) \) tends to zero while in the former case it diverges: \( E(v)_{v \to 1} \to +\infty \). The same is observed in the 1D case (shown by the dashed lines).

The explanation of the divergence can be given by the following arguments. In both the 1D and 2D cases at \( v = 1 \) the soliton width becomes infinitely small, i.e. zero. In the 1D case everywhere except the entrance \( (x = -d_x/2) \) and exit \( (x = d_x/2) \) points of the impurity the fluxon propagates in the homogeneous media, hence the plasmons are radiated only at the time instants when the fluxon enters and exits the impurity. The \( \mu_2 \) term contributes to the sharp growth of the perturbative term \( \delta \) due to the \( \delta \)-function in the function \( h_2 \), while the term with \( \mu_1 \) does not produce such growth. In the 2D case the situation is similar, however the radiation process goes on all the way during the fluxon propagation through the inhomogeneity due to the fluxon interaction with its borders at \( y = \pm d_y/2 \).

The divergence at \( v \to 1 \) seems to be non-physical. Within the current model the dissipative effects have been neglected. If they are taken into account, the radiated energy will always be finite. Also, the infinite growth of the emitted radiation was obtained in the first order of the perturbation theory. This might signal the breaking of the first order approximation and the necessity of the higher order approximation that should yield a more realistic picture.

Other features of the \( E(v) \) dependence like the multiple extrema will be discussed below. At this point we only note that as \( \mu_2 \) decreases, the positions of the extrema do not shift significantly, but the absolute values of \( E \) at the extrema decrease. This happens because the contribution to the emitted radiation due to the narrowing/expansion of the insulating layer, decreases. The limit \( v \to 0 \) is given in the inset of Fig. 3. One can notice that the extrema of the total energy persist in this limit both in the 1D and 2D cases, although they can be spotted only on the logarithmic scale.

In Fig. 3 the total emitted energy is plotted for the fixed value of the impurity length \( d_x = 8 \) while its width \( d_y \) is varied. The 1D result for the same length is plotted with the dashed line as a reference. Naturally, the value of the emitted energy decreases as \( d_y \) decreases. More interestingly, the extrema become less pronounced, and, finally only the main maximum remains for \( d_y = 2 \) (see the curve 4 in Fig. 3). The following interpretation of the obtained results can be made. The shape of the energy density distribution is given in Fig. 5. The absolute minima of the energy density satisfy \( \mathcal{E}(q_x, q_y) = 0 \) and these minimal values are attained at the following set on the \((q_x, q_y)\) plane:

\[
q_y = \frac{2\pi n}{d_y}, \quad n = \pm 1, \pm 2, \ldots \quad \text{for any } q_x, \quad (28)
\]

\[
(1-v^2)q_x^2 + q_y^2 = \left( \frac{2\pi n v}{d_x} \right)^2 - 1 + \frac{4\pi mv^2}{d_x} q_x, \quad (29)
\]

where \( n_0 \) is given by Eq. (28). Thus, the minima are located on the set of parallel lines (28) as well as on the set of embedded ellipses given by Eq. (29). The ridges of the maximal \( \mathcal{E} \) lie between the curves, defined by the roots of Eq. (28). For large \( d_y \) these ridges are strongly localized in the \( q_y \) direction [see Figs. 5(a,b)], while the decreasing of \( d_y \) makes them concentric and crescent-like as shown in Figs. 5(c,d).

For the large values of \( d_y \) the problem can be treated as an almost 1D, so that most of the emitted radiation travels in the \( x \) direction while the \( y \)-component of the radiation remains insignificant. This can be clearly ob-
the minimum of the total emitted energy corresponds to the minimal forward emission. This can be easily observed in Fig. 5(a) that the global maximum is placed on the $q_x$ axis at $q_x < 0$ while the first local maximum at $q_x > 0$ is rather small. In Fig. 5(b) it can be seen that the global maximum is placed on the positive half-axis of the $q_x$ axis, and this happens at $v = 0.7$ which is quite close to the maximum of the $E(v)$ function (curve 1) in Fig. 4. The further decreasing of $d_y$ smears out maxima up to the point when only one local maximum can be observed in Fig. 5(c-d). The maxima of a quasi-1D any more. The radiation distribution becomes spotted. The scattering problem cannot be considered as quasi-1D any more. The radiation distribution becomes rather different as shown in Figs. 4(c-d). The maxima of $E(q_x, q_y)$ still lie on the $q_x$ axis, but the curves (29) that define the minimal values become distinctly arc-shaped. The $y$-component of the radiation becomes more delocalized and the analogy with the 1D picture breaks down.

It is interesting to note how the shape of the energy density function varies in the extreme limits of the velocity value: $v \to 0$ and $v \to 1$. In the small velocity limit $|v| \ll 1$ the ellipses Eq. (29) that correspond to the minima of $E$ are almost circles and the density function is close to being radially symmetric, see Fig. 5(e). The increasing of $v$ makes the ellipses Eq. (29) more elongated in the $x$ direction, as has been demonstrated previously [see Fig. 5(a)-(d)]. An interesting situation emerges in the opposite limit, namely if $1 - |v| \ll 1$. The global maximum that was positioned on the $q_x$ axis splits up into two maxima that are now located off the $q_x$ axis symmetrically with respect to each other, as shown in Fig. 5(f). Physically this means the following. The slow fluxon “feels” the impurity as a wall and the emergent radiation moves mostly along the fluxon propagation direction. The fast (relativistic) fluxon interacts with the impurity in such a way that the impurity acts like a groin (a wave-breaker) and the emitted radiation is splitted by the impurity into two halves that have both $x$ and $y$ components. At the same time the $x$ component of the radiation becomes insignificant.

The number of local extrema of $E(v)$ depends on the impurity length $d_x$. This is easily demonstrated by Fig. 6, where the number of the maxima decreases with the decreasing of $d_x$. This result is similar to the same situation in the 1D model.

**FIG. 5:** (Color online). Emitted energy density $E$ for the data in Fig. 4 curve 1 at $v = 0.56$ (a) and $v = 0.7$ (b); curve 4 at $v = 0.5$ (c), $v = 0.75$ (d), $v = 0.1$ (e) and $v = 0.99$ (f).

**FIG. 6:** Total emitted energy (normalized to $\mu^2_1$) as a function of the fluxon velocity for $\mu_2/\mu_1 = 0.1$, $d_y = 8$ and $d_x = 2$ (curve 1), $d_x = 4$ (curve 2), $d_y = 8$, $d_x = 6$ (curve 3).

4. Limiting cases

It is of interest to check the limiting cases when in one of the direction ($x$ or $y$) the impurity becomes infinitely narrow. In the first case the limit $d_x \to 0$, while $\mu_x = \mu_1 d_x$ remains constant, corresponds to the situation when the impurity becomes infinitely thin in the $x$ direction. In Eqs. (20)-(21) the difference of the $\theta$-functions that form the first factor in the function $f_1(x,y)$ (20) becomes the Dirac’s $\delta$-function while $b_2 = 0$. This case with $\mu_2 = 0$ has been studied previously. Yet another interesting limit can be considered if $d_y \to 0$, $\mu_y = \mu_1 d_y$. In other words, the impurity remains elongated in the $x$ direction, but becomes infinitely thin in the $y$ direction. The spectral density of the emitted plasmons in the cases mentioned above, reads...
In these limits the modulation in the $q$-space, caused by the interference, disappears along the $q_x$ direction in the first formula because the impurity length becomes infinitely small. For the same reason there is no interference along the $q_y$ component when $d_y \to 0$ in the second formula of Eq. (30). When any of these limits are approached, the multiple maxima of the $E(v)$ dependence disappear leaving only one local maximum. The limit of the point $f_1(x,y) = \mu_1 \delta(x) \delta(y)$ impurity\textsuperscript{20} can be achieved easily from the stripe impurity by taking Eqs. (30) the limits $\mu_2 \to 0$ and $d_x \to 0$ or $d_y \to 0$ where appropriate. When the impurity shrinks into a point the local change of the insulating layer thickness is ignored, thus $\mu_2 = 0$. The obtained formula coincides with the previous result\textsuperscript{20}.

C. Rhombic impurity

Now we consider the rhombus(diamond)-shaped impurity with $d_x$ and $d_y$ being its length and width respectively:

$$\Omega: \ |x| \leq d_x/2 \bigcap |y| \leq g(x) = d_y(1/2 - |x|/d_x) \ . \quad (31)$$

The tip of the rhombus is perpendicular to the fluxon line. Then

$$h_2(x,y) = \mu_2 \{[\delta(x + d_x/2) - \delta(x - d_x/2)] \times$$
$$\times [\theta(y + g(x)) - \theta(y - g(x))] +$$
$$\times \{\delta(y - g(x)) + \delta(y + g(x))\} g'(x)\} \ , \quad (32)$$

$$g'(x) = -\frac{d_y}{d_x} \text{sign}(x) - \frac{d_y}{d_x} \text{sign}(x + v \tau) \ . \quad (33)$$

Substituting the formulae (32)-(33) into Eqs. (12)-(15) we obtain the radiation function in the moving frame

$$B(q_x, q_y) = i \frac{2 \sqrt{2 \pi \mu_1}}{q_y^2 \sqrt{1 + q_x^2(1 - v^2)^{1/2} d_y}} \frac{d_x}{d_y} \left\{1 - v^2 - \frac{\mu_2}{\mu_1}\right\} \left[\bar{\omega}^2 - (1 + q_x^2)\omega^2 + 2\frac{\mu_2}{\mu_1} \omega^2\right] \times$$
$$\times \left[\frac{\cos(q_y d_y/2) - \cos(q_x(\omega - q_x v)/2v)}{(d_x/d_y)^2(\omega - q_x v)^2 - v^2 q_y^2}\right] \sech\left[\frac{\pi}{2v}(q_x \bar{\omega} + \bar{\omega})\right] \ , \quad (34)$$

where the dispersion law $\bar{\omega} = \bar{\omega}(q_x, q_y)$ in the moving frame is given by Eq. (12). The transition to the laboratory frame is performed in the standard way, and, as a result, the spectral energy density in the laboratory frame is expressed by the following formula:

$$E(q_x, q_y) = 2\mu_1^2 \left(1 - v^2 + \frac{\mu_2}{\mu_1}\right)^2 \frac{d_x^2}{d_y^2} \left[\frac{\cos(q_y d_y/2) - \cos(q_x(\omega - q_x v)/2v)}{(d_x/d_y)^2(\omega - q_x v)^2 - v^2 q_y^2}\right] \times$$

$$\times \left\{\cos(q_y d_y/2) - \cos(q_x(\omega - q_x v)/2v)\right\}^2 \ .$$
where the dispersion law \( \omega = \omega(q_x, q_y) \) in the laboratory frame is given by Eq. (23). It may seem that this dependence has a singularity where the equation \( \left( \frac{d^2}{d \omega^2} \right) (\omega - q_x v)^2 = v^2 q_y^2 \) is satisfied. However, with the help of the trigonometric formula \( \cos a - \cos b = 2 \sin((a + b)/2) \sin((b - a)/2) \) it is straightforward to show that the respective divergences cancel out.

The total emitted energy as a function of the fluxon velocity \( v \) is shown in Figs. 1-3. The first figure (Fig. 1) focuses on the situation when the ratio \( d_y/d_x \) is fixed while the area covered by the impurity is varied. In general, the dependence \( E(v) \) grows with \( v \) in the limit \( v \ll 1 \) and diverges at \( v \to 1 \) due to the presence of the \( \mu_x \) term [otherwise, if \( \mu_2 = 0 \), we have \( E(v) \to 0 \)]. This behavior is quite similar to the case of the rectangular impurity studied in the subsection III.B.

![Figure 7: Total emitted energy (normalized to \( \mu_2^2 \)) as a function of the fluxon velocity for the rhombic impurity \( \mu_2/\mu_1 = 0.1 \) with the fixed ratio \( d_y/d_x = 4 \).](image)

First we consider the rhombus, elongated towards the fluxon propagation direction (main part of Fig. 7). We observe that there is one broad local maximum if the area of the inhomogeneity is small, as shown by the curve 1 where \( d_y/d_x = 2 \). The increasing of the impurity area causes emergence the new local maxima (curves 2-3). Interestingly, the value of the local maximum decreases while the impurity area is approximately doubled: compare the curve 2 (\( d_y/d_x = 50 \)) and the curve 3 (\( d_y/d_x = 112.5 \)). Here one can clearly observe the situation, analogous to the rectangle impurity studied in the previous subsection. The number of local maxima increases with the growth of the impurity length.

The inset of Fig. 7 corresponds to the situation when the impurity is elongated in the \( y \)-direction with the ratio \( d_x/d_y = 1/4 \) being fixed. In this case there is only one local maximum that decreases while the adjacent local minimum becomes more shallow as the area \( d_x d_y/2 \) decreases. This limit is qualitatively close to the limit of the strip impurity \( \mu_s \), but the limit \( \mu_2 \) is not restored.

It is possible to consider the limiting cases of the infinitely narrow stripes: \( d_x \to 0 \) and \( d_y \to 0 \). If the impurity amplitude is redefined as \( \mu = \mu_1 d_x \) (or \( \mu = \mu_1 d_y \)), the spectral density in these limits reads

\[
\mathcal{E}(q_x, q_y) \to \begin{cases} 
\frac{8(1-v^2+\mu_2/\mu_1)^2}{v^b} \sin^4 \left( \frac{2d_y q_y}{4} \right) \left[ (\omega - q_x v)^2 + \frac{1-v^2-\mu_2/\mu_1}{1-v^2+\mu_2/\mu_1} q_y^2 \right]^2 \sech^2 \left( \frac{\pi \omega \sqrt{1-v^2}}{2v} \right), \\
\frac{8(1-v^2+\mu_2/\mu_1)^2}{d_x^2 v^2} \left[ \sin(d_x (\omega - q_x v)/4v) \right]^4 \left[ (\omega - q_x v)^2 + (v^2 - 1) q_y^2 \right]^2 \sech^2 \left( \frac{\pi \omega \sqrt{1-v^2}}{2v} \right), & \text{if } d_x \to 0,
\end{cases}
\]

These limiting values of \( \mathcal{E} \) are very similar to the analogous limits for the rectangular impurity (III.B). The only principal difference is the interference terms that are responsible for the oscillations in the \( q_x \) or \( q_y \) direction.
Figure 8: The value of the local maximum $\max_{v \in [0,1]} E(v)$ of the emitted energy (normalized to $\mu_1^2$) as a function of the angle $\arctan(d_y/d_x)$ for the parameters $\mu_2/\mu_1 = 0.1$, $d_y = 5$ (a) and $d_y = 1$ (b). The solid line is used as a guide for an eye. The inset shows the total emitted energy (normalized to $\mu_1^2$) as a function of the fluxon velocity for $d_x = 100$.

The emission depends on the rhombus angle and may equal $\arctan(d_y/d_x) \approx \pi/3$ if $d_y$ is fixed at the value $d_y = 5$ or $\arctan(d_y/d_x) \approx 0.3$ if $d_y = 1$. In the panel (b) the dependence of $\max_{v \in [0,1]} E(v)$ stops at some value close to $\pi/3$. This means that the local maximum disappears and the $E(v)$ dependence becomes strictly monotonic. In the limits $d_x \to 0$ and $d_x \to \infty$ the radiated energy decreases significantly. In the former case one obtains infinitely thin impurity in the $x$ direction. When this limit is approached the local maximum of the $E(v)$ dependence becomes less and less pronounced. The energy density is proportional to $d_x^2$, thus, it is not surprising that the total energy tends to zero in this limit. The renormalization of the impurity amplitude $\mu_\ast = \mu_1 d_x$ and $d_x \to 0$ will lead the first formula of Eq. (36). In the case $d_x \to \infty$ the increasing of the impurity length leads to the increasing of the number of local maxima, as has been demonstrated in Fig. 7. The further increasing of $d_x$ to the values when the impurity becomes extremely narrow, increases significantly the number of local maxima but makes their values much smaller, as shown in the inset of Fig. 8(a).

This limit means that the impurity acts as an extremely narrow groin that does not cause much radiation due to its narrowness. The area of the impurity does not decrease, and the total energy is always finite although is small comparing to the $d_x \sim d_y$ case.

If the rhombus becomes a square ($d_x = d_y$) the local maximum of the radiation becomes more pronounced if the area of the impurity increases, as shown in Fig. 9. Also, the decreasing of the impurity area makes the local maximum less pronounced. For very small impurities the local maximum may disappear completely, as shown in the inset. Another limit of very large impurity produces secondary local maxima, to the left from the main maximum, although they are very small. Reducing the size of

the impurity in both directions ($d_x, d_y \to 0$ and $\mu_2 \to 0$) brings the spectral energy density function $\mathcal{E}(q_x, q_y)$ to the already known limit of the point-like impurity. The same limit can be obtained from any of the Eqs. (39) by setting $d_y \to 0$, $\mu_2 \to 0$ in the first equation or $d_x \to 0$, $\mu_2 \to 0$ in the second equation. The impurity amplitude should be redefined as $\mu_\ast = \mu_1 d_y$ or $\mu_\ast = \mu_1 d_x$, respectively.

The energy density profiles $\mathcal{E}(q_x, q_y)$ that correspond to the rhombic impurity are presented in Fig. 10. As a particular example an impurity that corresponds to the curve 2 from Fig. 7 i.e. for $d_x = 20$, $d_y = 5$. This energy density distribution bears many qualitative similarities with the energy density function for the rectangular impurity shown in Fig. 5. The global minima of the energy density satisfy the condition $\mathcal{E}(q_x, q_y) = 0$ and are given by the set of equations

$$d_x \left( \sqrt{1 + q_x^2 + q_y^2} - q_x \right) \pm q_y d_y = 4\pi n_x \ . \quad (37)$$
FIG. 10: (Color online). Emitted energy density $E$ for the rhombic impurity with $d_x = 20$, $d_y = 5$, $\mu_2/\mu_1 = 0.1$ (curve 2 in Fig. 7) at $v = 0.64$ (a), $v = 0.79$ (b), $v = 0.86$ (c), $v = 0.94$ (d); $v = 0.962$ (e); $v = 0.98$ (f); and $v = 0.99$ (g). The panel (h) corresponds to $v = 0.99$ and $\mu_2 = 0$.

This set of equations describes the sequence of pairs of ellipses that are numbered by the integers $n_\pm$,

$$n_\pm = n_0, n_0 + 1, \ldots, n_0 = \left\lfloor \frac{d_x}{4\pi} \sqrt{\frac{1}{v^2} + \frac{d_y^2}{d_x^2}} \right\rfloor. \quad (38)$$

if

$$|v| < \frac{d_x}{\sqrt{d_x^2 + d_y^2}}. \quad (39)$$

Otherwise, the Eqs. (37) yield the set of hyperbolas that are numbered with $n_\pm = \pm 1, \pm 2, \ldots$. The two curves (ellipses or hyperbolas) given by Eq. (37) that correspond to the opposite signs but with $n_+ = n_-$ are mapped into each other with the mirror symmetry with respect to the $q_x$ axis. If we consider the set of curves with the same sign, say $+$, they are embedded into each other and they expand with the growth of the index $n_\pm$. Between these curves lie the ridges of the $E(q_x, q_y)$ function, and the local maxima of the energy density lie on these ridges. Signatures of these curves can be spotted in all panels of Fig. 10. For the small and intermediate values of the fluxon velocity the emitted radiation is localized predominantly in one peak in the $q-$space, as shown in Figs. 10(a-b). This peak lies on the $q_x$ axis, thus, most of the radiation does not propagate in the perpendicular direction. Here the panel (a) corresponds to the local minimum of $E(v)$ [curve 2 of Fig. 7] at $v = 0.64$ while the panel (b) corresponds to the local maximum at $v = 0.79$. The structure of both these functions is similar and the only difference is that on the panel (b) the main peak is more pronounced and is shifted further along the $q_x$ axis. In both cases the maximal emitted energy corresponds to the forward radiation ($q_x > 0$), what is in contrast to the rectangular impurity, where the minimum of $E(v)$ corresponded to the minimal forward radiation. The panel (c) corresponds to the next local minimum of the $E(v)$ curve at $v = 0.86$, and here one observes the increasing of the share of the perpendicular radiation in the total radiated energy. The further increasing of $v$ leads to the appearance of the pair of equivalent global maxima off the $q_y$ axis [see the panel (d) that corresponds to the local maximum of the $E(v)$ dependence at $v = 0.94$]. These maxima are suppressed at $v = 0.962$ as the global maximum lies again on the $q_x$ axis [see the panel (e) that corresponds to the last local minimum of $E(v)$ at $v = 0.962$]. In general, we observe the increasing of the perpendicular radiation that reaches its climax in the relativistic limit $v \to 1$, as demonstrated in the panels (f) and (g). These panels correspond to the diverging part of the $E(v)$ dependence. Here we see that the peaks of the emitted energy density lie off the $q_x$ axis. They appear to be strongly localized in the $q_y$ direction while their localization in the $q_x$ is significantly weaker. In this limit the interaction time is too small to generate significant longitudinal radiation while the shape of the obstacle breaks the incident fluxon as a groin and generates predominantly transverse radiation.

Finally, we mention the dependence of the emitted energy on the parameter $\mu_2$. The panel (h) corresponds to the same parameters of the model as in the panel (g) but with $\mu_2 = 0$. Comparing the panels (g) and (h) we see that the structure of these functions is quite similar while the absolute values of $E$ are significantly smaller in the $\mu_2 = 0$ case. If $\mu_2 = 0$, but for the same value of the fluxon velocity, the values of the maxima actually decrease with $v$. Thus, the total emitted energy tends to zero, in the same way as shown by the dashed line in Fig. 3. This has been confirmed for the values of $v$ even closer to unity as well as for the different values of $d_x, d_y$. The qualitative behaviour of $E(v)$ in the limit $|v| \to 1$ appears to be the same both for the rectangular
and rhombic impurities.

IV. DISCUSSION AND CONCLUSION

The radiation emitted as a result of the fluxon interaction with the impurity of a general geometrical shape in the large two-dimensional Josephson junction has been studied. The emitted energy distribution in the $q$-space has been computed as well as the total emitted energy. This energy distribution can always be represented as a triple integral. In principle, any geometrical shape can be taken into consideration, however, the explicit integration is not always possible, but if the inhomogeneity area can be represented by the piecewise-linear functions, this integration can be done. In this article the rectangular and rhombic impurities have been studied.

The main result of this work has been formulated in the dependence of the total emitted energy as a function of the incident fluxon velocity. It appears that this dependence has local maxima that depend strongly on the geometric properties of the impurity. These local maxima do not exist if the impurity is treated as a point. Controlling the shape of the impurity one can remove the extrema or make them more pronounced. The limit of the 1D problem with the finite-size inhomogeneity can be restored.

In this article the junction thickness change due to the homogeneity is taken into account. Its role is measured by the ratio $\mu_2/\mu_1$ [see Eqs. (3) and (6)]. In some papers it is ignored (especially it is always ignored if the point impurities are considered), and, in general, it is considered to be weak. However, it influences significantly the asymptotic behaviour of the total emitted energy in the “relativistic” (i.e., $v \to 1$) limit of the fluxon velocity. If the thickness change is ignored, the total emitted energy goes to zero, while it exhibits unbounded growth if the thickness change is taken into account. This is true for the both 1D and 2D junctions. The emitted energy has been computed under the assumption that it is a small perturbation on the fluxon background. Consideration of the higher order corrections may block the infinite radiation growth. Also, the dissipative effects, which have been ignored in this work, should contribute to the decreasing of the emitted energy.

Although in the real large area Josephson junctions have finite dimensions, in this article the infinitely-sized junction has been considered. This approximation is sufficient if the physical dimensions of the LJJ exceed by the order of magnitude the Josephson penetration depth, and, consequently, the fluxon length in the $x$ direction. The boundary conditions are also important, however, if the junction width is large enough (exceeds the Josephson length at least by the order of magnitude) the fluxon distortion from the linear shape is insignificant. In any case, before focusing on the more concrete setup an idealised, but more easily solvable model should be studied.

Finally, we discuss the possible application of the obtained results. Recently, a number of papers have focused on the different application of the fluxon dynamics in the 2D LJJ, such as fluxon splitting on the $T$-shaped junctions, excitation of the different modes that move along the fluxon front, and the fluxon logic gates, where the interaction with the spatial inhomogeneity takes place. If the incident fluxon velocity is large enough, the emitted radiation becomes sufficient and it should influence the fluxon motion. In particular, the non-monotonicity of the $E(u)$ dependence may produce the hysteresis-like branches on the current-voltage characteristics (IVC) of the LJJ. The studies of these IVCs for the different shapes of the inhomogeneity in the genuinely 2D case are in progress and will be published elsewhere.

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