ON THE ANTI-RAMSEY THRESHOLD FOR NON-BALANCED GRAPHS

PEDRO ARAÚJO, TAÍSA MARTINS, LETÍCIA MATTOS, WALNER MENDONÇA, LUIZ MOREIRA, AND GUILHERME O. MOTA

Abstract. For graphs $G, H$, we write $G \xrightarrow{\text{rb}} H$ if any proper edge-coloring of $G$ contains a rainbow copy of $H$, i.e., a copy where no color appears more than once. Kohayakawa, Konstadinidis and the last author proved that the threshold for $G(n, p) \xrightarrow{\text{rb}} H$ is at most $n^{-1/m_2(H)}$. Previous results have matched the lower bound for this anti-Ramsey threshold for cycles and complete graphs with at least 5 vertices. Kohayakawa, Konstadinidis and the last author also presented an infinite family of graphs $H$ for which the anti-Ramsey threshold is asymptotically smaller than $n^{-1/m_2(H)}$. In this paper, we devise a framework that provides a richer and more complex family of such graphs that includes all the previously known examples.

1. Introduction

We say that a graph $G$ has the anti-Ramsey property $G \xrightarrow{\text{rb}} H$ if every proper edge-coloring of $G$ contains a rainbow copy of $H$. The study of anti-Ramsey properties can be traced back to a question of Spencer, mentioned by Erdős in [4]: Does there exist a graph with arbitrarily large girth and such that every proper edge-coloring contains a rainbow cycle? Rödl and Tuza answered this question affirmatively [14] by showing that $G(n, n^{e^{-1}}) \xrightarrow{\text{rb}} C_k$ for large $k$ and by deleting one edge from each small cycle.

As $G \xrightarrow{\text{rb}} H$ is an increasing property, it admits a threshold function $p_H^{\text{rb}} = p_H^{\text{rb}}(n)$, which is the focus of this work. Kohayakawa, Konstadinidis and the last author [7] proved that, for any fixed graph $H$, we have $p_H^{\text{rb}} \leq n^{-1/m_2(H)}$, where

$$m_2(H) := \max \left\{ \frac{e(J) - 1}{v(J) - 2} : J \subseteq H, v(J) \geq 3 \right\}$$

is the $m_2$-density of the graph $H$. Nenadov, Person, Škorić and Steger [12] showed that this upper bound is sharp for cycles with at least 7 vertices and complete graphs with at least
Figure 1. An example of amalgamation.

19 vertices. This result was extended for cycles and cliques with at least 5 vertices, in [2] and [10], respectively.

Apart from cliques and cycles, not much is known about \( p_{rb}^H \). One might feel compelled to conjecture that indeed this threshold is determined by the \( m_2 \)-density for all graphs, specially because of ‘standard’ Ramsey threshold results such as the one from Rödl and Ruciński [13]. However, the authors of [8] proved that this is not the case for a fairly large class of graphs, which is an evidence of the inherent difference between anti-Ramsey and standard Ramsey properties. To illustrate this difference, we consider the case of \( H = K_3 \). Note that every proper-coloring of a triangle is rainbow and therefore the threshold for the event \( G(n, p) \xrightarrow{rb} K_3 \) is the same as the threshold for the appearance of triangles, which is a local property. For some other graphs \( H \), such as cliques and cycles with more than 3 vertices, it turns out that the property \( G(n, p) \xrightarrow{rb} H \) seems to be related with more global aspects of the host graph. In this paper we explore the interplay of these two cases.

The class of graphs in [8] consists of graphs obtained by ‘attaching’ a triangle to an edge of a graph \( H \) with \( m_2(H) < 2 \), a result that we extend by allowing different graphs to be attached to \( H \). Before we state our results, we formally define an amalgamation of graphs, which we denote by \( \oplus \). A 2-labeled graph is a graph in which exactly one vertex is labeled \( 1 \), exactly one vertex is labeled \( 2 \) and they form an edge. For any 2-labeled graphs \( F \) and \( H \), we define \( F \oplus H \) as the graph obtained by taking the disjoint union of \( F \) and \( H \) and identifying the vertices with label 1 and the vertices with label 2 together with the respective edge. An illustration of this definition is depicted on Figure 1.

We say that a graph \( S \) is 2-balanced if the maximum in (1) is attained by \( S \) itself. We are now ready to state our main theorem.

**Theorem 1.1.** Let \( H, F \) be 2-labeled graphs with \( 1 < m_2(H) < m_2(F) \). For any 2-balanced graph \( S \) such that \( S \xrightarrow{rb} F \), there exists \( C > 0 \) such that

\[
\lim_{n \to \infty} \mathbb{P} \left[ G(n, p) \xrightarrow{rb} F \oplus H \right] = 1, \quad \text{if } p \geq Cn^{-\beta(H,S)},
\]

where

\[
\beta(H, S) = \frac{1}{e(S)} \left( v(S) - 2 + \frac{1}{m_2(H)} \right).
\]
We remark that the definition of $\beta$ comes from a more general definition from [9]. This parameter is defined to ensure that the number of copies of $S$ in $G(n, p)$ is of order $\Omega(n^{2-1/m_2(H)})$, which is essential for our proof (see Section 2).

As an application of Theorem 1.1 we present an infinite family of graphs $H$ such that $n^{-1/m_2(H)}$ is not a threshold for $G(n, p) \xrightarrow{\text{rb}} H$. For any $t \in \mathbb{N}$, let $B_t$ the $t$-book graph, consisted of $t$ triangles sharing a common edge. We prove that $B_{32-2} \xrightarrow{\text{rb}} B_t$ and, since $m_2(B_t) = 2$, that $B_2 \oplus H$ satisfies the hypothesis of Theorem 1.1 for every positive integer $t$ and every graph $H$ with $m_2(H) \in (1, 2)$.

**Corollary 1.2.** For every $t \in \mathbb{N}$ and every graph $H$ with $m_2(H) \in (1, 2)$, we have

$$p_{B_t \oplus H}^{\text{rb}} = o(n^{-1/m_2(B_t \oplus H)})$$

for any 2-labeled copies of $B_t$ and $H$.

The paper is organized as follows: in Section 2 we give an overview of the proof of Theorem 1.1; in Section 3 we recall some definitions and results on the Regularity Method; In Section 4 we explore properties of proper colorings of $G(n, p)$; in Section 5 we prove Theorem 1.1 and in Section 6 we prove Corollary 1.2.

**2. OVERVIEW OF THE PROOF**

We aim to show that, under the hypothesis of Theorem 1.1 there exists $C > 0$ such that, for $p \geq Cn^{-\beta(H,S)}$, every proper edge-coloring of $G(n, p)$ contains a rainbow copy of $F \oplus H$ with high probability. A natural strategy is to partition the colors into $c(F \oplus H)$ classes and find a copy of $F \oplus H$ so that each of its edges belongs to a different class and, consequently, is rainbow. One good attempt would be to make a random partition of the colors, while trying to prove that the graphs induced by each class of colors is pseudo-random. This is indeed the approach used in [7] to provide an upper bound on the threshold of this anti-Ramsey property for any graph.

However, such a general strategy cannot work since we are dealing with lower densities than the one determined by $m_2(F \oplus H)$, which is a barrier to applying an embedding lemma for random graphs such as Theorem 4.1, formerly known as the KLR Conjecture. Moreover, by the results of [2,10,12] we know that the $m_2$-density in fact determines the threshold for cliques and cycles, so if we work with lower densities we must show that the imbalance of $F \oplus H$ plays a role in our proof. Recall that we consider a graph $S$ such that $S \xrightarrow{\text{rb}} F$. By finding many copies of $S$, we will get many rainbow copies of $F$, which we aim to extend to a rainbow copy of $F \oplus H$ using the Theorem 4.1.

Given a realization $G$ of $G(n, p)$ and a proper edge-coloring $c : E(G) \rightarrow \mathbb{N}$, for each color $i \in e(E(G))$ we assign independently and uniformly at random an edge $\sigma(i) \in E(H)$. Let $V(H) = \{u_1, u_2, \ldots, u_h\}$ be a fixed labeling of the vertices of $H$. For each $u_iu_j \in E(H)$, we denote by $G_{u_i, u_j}$ the graph with the edge set $E(G_{u_i, u_j}) = \{e \in E(G) : \sigma(c(e)) = u_iu_j\}$. That is, $E(G_{u_i, u_j})$ are those edges of $E(G)$ for which their color was assigned to $u_iu_j$. 
Let \( V(S) = \{v_1, v_2, \ldots, v_s\} \) be a fixed labeling of the vertices of \( S \) in which \( \{v_1, \ldots, v_f\} \) form a copy of \( F \). Consider \( F \oplus H \) and \( S \oplus H \) to be the amalgamations obtained by identifying \( u_1 \) with \( v_1 \) and \( u_2 \) with \( v_2 \). We will look for a copy \( S' \oplus H' \) of \( S \oplus H \) in \( G \) such that the following holds. If \( e \in E(H') \) is identified with \( u_i u_j \), then \( c(e) \in \sigma^{-1}(u_i u_j) \); if \( e \in E(S') \), then \( c(e) \in \sigma^{-1}(u_i u_j) \). As \( S \rightarrow F \) and the sets \( \sigma^{-1}(u_i u_j) \) are disjoint, such colored copy of \( S \oplus H \) will give us a rainbow copy of \( F \oplus H \).

We start by fixing an equitable partition of the vertices of \( G \) as follows:

\[
V(G) = \bigcup_{i=1}^s V_i \cup \bigcup_{i=3}^h U_i.
\]

For each \( i \), the set \( V_i \) corresponds to the vertex \( v_i \) in \( S \) and the set \( U_i \) corresponds to the vertex \( u_i \) in \( H \). By convenience, we also set \( U_1 = V_1 \) and \( U_2 = V_2 \). Denote by \( G[V_1, \ldots, V_s] \) the graph formed by the edges of \( G \) whose endpoints belong to different sets \( V_i \) and \( V_j \). First, we prove that with high probability there exist edge-disjoint transversal copies of \( S \) in \( G[V_1, \ldots, V_s] \), that is, copies of \( S \) in which each vertex belongs to a different set \( V_i \). Since \( S \) is 2-balanced and \( m_2(S) > m_2(H) \), we have \( p \leq n^{-1/m_2(S)} \) and hence most copies of \( S \) in \( G \) are isolated, that is, most copies do not share edges with any other copies. We guarantee that a positive proportion of these copies are present in \( G_{u_1 u_2} \) by applying Azuma’s inequality (cf. Lemma 3.3). In this argument we control the influence of each color in the number of copies of \( S \) in \( G_{u_1 u_2} \) by recalling that each color forms a matching and using that the copies are isolated.

Let \( G_{u_1 u_2}^S[V_1, \ldots, V_s] \) be the graph induced by the edges of the isolated transversal copies of \( S \) in \( G_{u_1 u_2}[V_1, \ldots, V_s] \). We show that there exists a bipartite subgraph \( B_{12} = B_{12}[W_1, W_2] \) of \( G_{u_1 u_2}^S[V_1, V_2] \) which is \((\varepsilon, q)\)-regular and has \( \lfloor \alpha n^2 q \rfloor \) edges, where \( \alpha, \varepsilon \in (0, 1) \) and \( q = B^{c(S)} n^{-1/m_2(H)} \). The density of \( B_{12} \) comes from (2), while the regularity comes from Lemma 3.3 combined with Lemma 4.10. Relabeling the sets \( V_1, \ldots, V_s \) if necessary, we also show that each \( e \in E(B_{12}) \) is contained in an isolated copy of \( S \) such that \( e \) corresponds to \( u_1 u_2 \). Moreover, the sets \( W_1 \subseteq V_1 \) and \( W_2 \subseteq V_2 \) have equal size and contain a constant proportion of the vertices in \( V_1 \) and \( V_2 \), respectively.

Now, the plan is to apply Theorem 4.1 to find a rainbow copy of \( H \) in \( G[U_1, \ldots, U_h] \) which contains one edge in \( B_{12} \) and, apart from this edge, only uses colors outside \( c^{-1}(u_1 u_2) \). To do so, we first prove that for all \( j > 2 \) and \( i < j \) the following holds. There exists a bipartite subgraph \( B_{ij} = B_{ij}[W_i, W_j] \) of \( G_{u_i u_j}(U_i, U_j) \) which is \((\varepsilon, q)\)-regular and has \( \lfloor \alpha n^2 q \rfloor \) edges with \( |W_i| = |W_j| \) for all \( 1 \leq i \leq h \). The density of \( B_{ij} \) comes from the concentration of the degrees in \( G_{u_i u_j} \) (cf. Lemma 4.2), using that \( m_2(H) > 1 \) and consequently that \( q \gg \log n/n \). To prove \((\varepsilon, q)\)-regularity, we use Azuma’s inequality to control the number of cycles of a
fixed even length and apply a well-known theorem of Chung and Graham (cf. Lemma 4.4). For more details, see Lemmas 3.4 and 4.3.

By choosing $B$ larger enough, we apply Theorem 4.1 to deduce that the probability that $\cup_{ij} B_{ij}$ does not contain a transversal copy of $H$ is at most of order $\exp(-qn^2)$. Since $m_2(H) > 1$ and consequently $qn^2 \gg n$, this bound is enough to account for the number of choices for the sets $W_1, \ldots, W_h$. This implies that the desired rainbow copy of $H$ in $G[U_1, \ldots, U_h]$ exists with high probability. As $S \overset{\text{rb}}{\rightarrow} F$ and each $e \in E(B_{12})$ is contained in a copy of $S$ so that $e$ corresponds to $u_1 u_2$, we conclude that there is a rainbow copy of $F \oplus H$ in $G$.

3. Tool box

For any graph $G$ and disjoint subsets $U, V \subseteq V(G)$, define the density of the pair $(U, V)$ in $G$ to be

$$d_G(U, V) = \frac{e_G(U, V)}{|U||V|},$$

where $e_G(U, V)$ denotes the number of edges across $U$ and $V$. We suppress $G$ from the notation whenever it is clear from context.

For any $\mu, p \in \mathbb{R}$, we say that $G$ is $(\mu, p)$-upper uniform if

$$d_G(U, V) \leq (1 + \mu)p,$$

for every disjoint pair of sets $U, V \subseteq V(G)$ such that $|U|, |V| \geq \mu v(G)$. If $G = G[U, V]$, we say that $G$ is $(\varepsilon, p)$-regular if

$$|d_G(U, V) - d_G(U', V')| \leq \varepsilon p,$$

for all $U' \subseteq U$ and $V' \subseteq V$ with $|U'| \geq \varepsilon |U|$ and $|V'| \geq \varepsilon |V|$.

The next lemma states that large induced graphs of regular subgraphs are still regular. The proof is straightforward by checking the definition.

**Lemma 3.1.** Let $p \in (0, 1]$ and $0 < \varepsilon < \mu < 1/2$. If $G[U, V]$ is a $(\varepsilon, p)$-regular bipartite graph, then for every $W \subseteq U$, with $|W| \geq \mu |U|$, the graph $G[W, V]$ is $(\varepsilon/\mu, p)$-regular. Furthermore, we have $d(W, V) \geq d(U, V) - \varepsilon p$.

The next lemma states that regular bipartite graphs contain regular subgraphs with any given (sufficiently large) number of edges. A proof can be found in [5, Lemma 4.3].

**Lemma 3.2.** For $\varepsilon \in (0, 1/6)$, there exists $C = C(\varepsilon) > 0$ such that the following holds. Let $G = G[V_1, V_2]$ be a $(\varepsilon, d_G)$-regular bipartite graph, where $d_G = d_G(V_1, V_2)$. For all $Cn \leq m \leq e(G)$, there exists a $(2\varepsilon, d_H)$-regular subgraph $H = H[V_1, V_2]$ of $G$ such that $e(H) = m$, where $d_H = e(H)/|V_1||V_2|$.

The following lemma can be found in [8, Lemma 6]. It states that upper uniform bipartite graphs contain a bipartite subgraph which is regular and has the same density.
Lemma 3.3. For \( \varepsilon \in (0, 1/2) \) and \( \alpha \in (0, 1) \), there exists \( \mu > 0 \) such that the following holds for all \( p \in (0, 1] \). Let \( G = G[V_1, V_2] \) be a \((\mu, p)\)-upper uniform bipartite graph, with \( |V_1| = |V_2| \) and \( d(V_1, V_2) \geq \alpha p \). There exist \( V'_1 \subseteq V_1 \) and \( V'_2 \subseteq V_2 \), with \( |V'_1|, |V'_2| \geq \mu |V_1| \), such that \( G[V'_1, V'_2] \) is \((\varepsilon, p)\)-regular with density at least \( \alpha p \).

Let \( \varepsilon \) be any positive real number. We say that a graph \( G \) has the **discrepancy property** \( \text{DISC}(\varepsilon) \) if for any subsets \( U, V \subseteq V(G) \), we have

\[
|e_G(U, V) - \frac{\text{vol}(U) \text{vol}(V)}{\text{vol}(V(G))}| \leq \varepsilon \text{vol}(V(G)),
\]

where \( \text{vol}(X) := \sum_{x \in X} \text{deg}_G(x) \) for any \( X \subseteq V(G) \). Roughly speaking, if a graph \( G \) has the \( \text{DISC}(\varepsilon) \) property, then its edges are almost uniformly distributed. The next lemma builds a bridge between discrepancy and classical regularity (see [8], Lemma 4).

Lemma 3.4. For every \( \varepsilon, \mu > 0 \) there exist \( \delta, \nu > 0 \) such that the following holds. Let \( p \in (0, 1] \) and \( G \) be an \( n \)-vertex graph which satisfies

1. the discrepancy property \( \text{DISC}(\delta) \);
2. \(|\deg(v) - pn| \leq \nu pn \) for \( v \in V(G) \).

Then, for any disjoint subsets \( U, V \subseteq V(G) \) such that \( |U|, |V| \geq \mu n \), the graph \( G[U, V] \) is \((\varepsilon, p)\)-regular.

We end this section with a probabilistic tool. We state Azuma’s inequality as it was stated by McDiarmid [11]. Below, the notation \([M]\) stands for \( \{1, 2, \ldots, M\} \).

Lemma 3.5. Let \( X_1, \ldots, X_M \) be independent random variables, with \( X_i \) taking values on a finite set \( A_i \) for each \( i \in [M] \). Suppose that \( f : \prod_{i=1}^M A_i \to \mathbb{R} \) satisfies \(|f(x) - f(x')| \leq c_i \) whenever the vectors \( x \) and \( x' \) differ only in the \( i \)-th coordinate. If \( Y \) is the random variable given by \( Y = f(X_1, \ldots, X_M) \), then, for any \( a > 0 \),

\[
\mathbb{P}(|Y - \mathbb{E}(Y)| > a) \leq 2 \exp \left\{-\frac{2a^2}{\sum_{i=1}^M c_i^2} \right\}.
\]

4. Pseudo-randomness and isolated copies

Recall from Section 2 that we will consider a random partition of the colors of a proper coloring of \( G(n, p) \). In this section we focus on exploring properties of the graphs induced by each color class of this partition. In particular, the proof of Theorem [14] converges to an application of an embedding lemma in a sparse setting: the formerly-called KLR conjecture, proved by Balogh, Morris and Samotij [11]. Therefore, the goal is to guarantee that the graphs that we consider fit in the requirements of that result. In order to state it, we need a little explanation.
Let $m$ and $n$ be positive integers with $m \leq n^2$ and let $\varepsilon > 0$ and $p \in [0, 1]$. For a graph $H$ with $V(H) = [h]$, we denote by $\mathcal{G}(H, n, m, p, \varepsilon)$ the family of graphs obtained in the following way. Consider $h$ disjoint sets $V_1, \ldots, V_h$, each of size $n$, and for each $ij \in E(H)$, add $m$ edges between the pair $(V_i, V_j)$ such that the resulting bipartite graph is $(\varepsilon, p)$-regular. We denote by $\mathcal{G}(H, n, m, p, \varepsilon)$ the collection of all graphs obtained in this way. We say that a copy of $H$ in $G \in \mathcal{G}(H, n, m, p, \varepsilon)$ is transversal if the vertex $i$ is mapped to $V_i$ (we omit the dependence on the order of the $V_i$'s in the definition of $\mathcal{G}$). We denote by $\mathcal{G}^*(H, n, m, p, \varepsilon)$ the set of all graphs $G \in \mathcal{G}(H, n, m, p, \varepsilon)$ that do not contain a transversal copy of $H$.

Now we are ready to state the KŁR conjecture.

**Theorem 4.1.** For every graph $H$ and every positive $\gamma$, there exist positive constants $B$, $n_0$ and $\varepsilon$ such that the following holds. For every $n \in \mathbb{N}$ with $n \geq n_0$ and $m \in \mathbb{N}$ with $m \geq Bn^{2-1/m^2(H)}$,\[
|\mathcal{G}^*(H, n, m, m/n^2, \varepsilon)| \leq \gamma^m \binom{n^2}{m}^{e(H)}.
\]

4.1. **Random partition of the colors.**

In this subsection, we prove that, in a typical outcome of $G(n, p)$, the hypothesis of Lemma 3.4 is met by the graphs induced by the colors assigned to each edge of $H$. Let $G$ be a graph, let $c : E(G) \to \mathbb{N}$ be a proper edge-coloring of $G$ and let $T$ be a positive integer. To each color $i \in \mathbb{N}$, we assign to $i$ an element $\sigma(i) \in [T]$ chosen uniformly at random from $[T]$ and denote by $\mathbb{P}_T$ this distribution. For each $t \in [T]$, let $G_t$ be the spanning subgraph of $G$ with edge set
\[
E(G_t) = \{e \in E(G) : \sigma(c(e)) = t\}.
\]
That is, $E(G_t)$ corresponds to the edges of $G$ for which their color was assigned to $t$.

In our problem, the graph $G$ is an outcome of $G(n, p)$. This makes us consider two probability spaces when dealing with $G_t$: the one which defines $G(n, p)$ and the one which defines the random assignment of colors. We observe that if $A$ is an event depending on $G_t$, for some $t$, then $\mathbb{P}_T(A)$ is a random variable in $G(n, p)$. In the lemmas, $o(1)$ denotes a function tending to 0 as $n \to +\infty$. Whenever we say that an event occurs ‘with high probability’, we mean with probability $1 - o(1)$.

Our aim in this subsection is to show that $G_t$ satisfies the two requirements of Lemma 3.4 which are the concentration of degrees and DISC($\varepsilon$). We omit the proof of the first, since it follows by a straightforward Chernoff’s bound argument, together with the fact that edges
touching each vertex have distinct colors. The proof that \( G_t \) satisfies \( \text{DISC}(\varepsilon) \) is more intricate and relies on results of Chung and Graham [3] on pseudo-random graphs.

**Lemma 4.2.** Let \( \delta > 0 \) and \( T \) be a positive integer. If \( p \gg \log n/n \), then the following holds for \( G = G(n, p) \) with high probability. For any proper edge-coloring \( c : E(G) \to \mathbb{N} \) of \( G \) we have

\[
P_T \left( d_{G_t}(v) = (1 \pm \delta) \frac{pn}{T} \right) = 1 - o(1),
\]

for every \( t \in [T] \) and \( v \in V(G) \).

**Lemma 4.3.** Let \( \varepsilon, \beta \in (0, 1) \) and \( T \) be a positive integer. If \( p \gg n^{-\beta} \), then the following holds for \( G = G(n, p) \) with high probability. For any proper edge-coloring \( c : E(G) \to \mathbb{N} \) of \( G \) we have

\[
P_T \left( G_t \text{ satisfies } \text{DISC}(\varepsilon) \right) = 1 - o(1),
\]

for every \( t \in [T] \).

A straightforward proof that a random graph satisfies \( \text{DISC}(\varepsilon) \) can be tricky, since any concentration inequality we obtain has to be stronger than the choices of subsets of the vertex set. Luckily for us, the works of Chung and Graham [3] relate this property with the distribution of circuits of even length. Given a graph \( G \), we say that a sequence \( C = (v_1, \ldots, v_\ell) \) of vertices of \( G \) is an \( \ell \)-circuit if \( v_i v_{i+1} \in E(G) \), for every \( i \in [\ell - 1] \), and \( v_1 v_\ell \in E(G) \). The weight of an \( \ell \)-circuit \( C = (v_1, \ldots, v_\ell) \) is given by

\[
w(C) = \prod_{i=1}^{\ell} \frac{1}{d_G(v_i)}.
\]

We denote by \( \mathcal{C}_\ell(G) \) the collection of all \( \ell \)-circuits of \( G \). We say that \( G \) has the \( \text{CIRCUIT}_\ell(\varepsilon) \) property if

\[
\sum_{C \in \mathcal{C}_\ell(G)} w(C) = 1 \pm \varepsilon.
\]

The following lemma from [3] shows that \( \text{CIRCUIT} \) essentially implies \( \text{DISC} \).

**Lemma 4.4.** For every \( \varepsilon > 0 \) and positive integer \( \ell \), if \( G \) has the \( \text{CIRCUIT}_{2\ell}(\varepsilon) \) property, then \( G \) has the \( \text{DISC}(\varepsilon^{1/2\ell}) \) property.

Since the degrees in \( G_t \) are concentrated around \( pn/T \) for every \( t \in [T] \), we basically have to show that the number of circuit of some even length \( \ell \) is close to \( (pn/T)^\ell \) in \( G_t \). In principle, it is not clear even how to compute the expectation of this value, since the edges are not selected independently. We simplify this problem in two steps. First we call upon the result stated below and proved in [7, Corollary 4.9], which shows that for certain values
of \( p \), almost all \( \ell \)-circuits are actually cycles. For that, we denote by \( C'_{\ell}(G) \) the number of \( \ell \)-cycles in a graph \( G \).

**Lemma 4.5.** Let \( \ell \geq 2 \) and \( \delta > 0 \). If \( p \gg n^{-1+1/\ell} \), then with high probability \( G = G(n, p) \) satisfies

\[
|C_{2\ell}(G)| \leq (1 + \delta)|C'_{2\ell}(G)|.
\]

Now our aim is to show that almost all cycles in a proper edge-coloring of \( G(n, p) \) are in fact rainbow. If we assume that to be true, it is easy to see that the expected number of \( \ell \)-cycles in each \( G_t \) is roughly \( (pn/T)\ell \), since each color is independently assigned to a class. In order to prove such statement we count the number of non-rainbow cycles basically by counting the shortest path whose first and last edges have the same color and then by completing them into cycles. Therefore, the following special case of a classical result of Spencer [15, Theorem 2] is fairly convenient.

**Lemma 4.6.** Let \( \ell \geq 2 \). If \( p^\ell n^{\ell - 1} \gg \log n \), then with high probability \( G = G(n, p) \) satisfies the following. For every pair of vertices \( u, v \in V(G) \), the number of \( \ell \)-vertex paths connecting \( u \) to \( v \) is \( \Theta(p^{\ell - 1}n^{\ell - 2}) \).

We remark that the values of \( p \) needed to apply Lemma 4.6 are lower for longer paths. This fact plays an important role in the proof of Lemma 4.7, which we are now ready to state.

**Lemma 4.7.** For \( \beta \in (0, 1) \), let \( p \geq n^{-\beta} \) and let \( \ell \) be an integer satisfying \( p^{\lceil \ell/2 \rceil}n^{\lceil \ell/2 \rceil - 1} \gg \log n \). With high probability, in every proper edge-coloring of \( G = G(n, p) \) there are \( O(p^{\ell - 1}n^{\ell - 1}) \) non-rainbow \( \ell \)-cycles.

**Proof.** Let \( G = G(n, p) \) be as in the statement. Notice that as a straightforward application of Chernoff’s inequality (which we will omit the details here), it follows that with high probability we have \( d(v) \leq 2pn \), for every \( v \in V(G) \). Fix now a proper edge-coloring of \( G \).

We say that a path in \( G \) is **color-tied** if the first and last edges have the same color. Note that every non-rainbow \( \ell \)-cycle must contain a color-tied path with at most \( \lceil \ell/2 \rceil + 2 \) vertices, by considering the shortest path between edges with the same color. We then count the number of non-rainbow \( \ell \)-cycles by counting the number of such paths and then by counting in how many ways these paths can be extended into an \( \ell \)-cycle in \( G \).

To count the number of color-tied paths with \( k \) vertices, for a fixed \( k \in \lceil \ell/2 \rceil + 2 \), we first choose an ordered pair \( (v_1, v_2) \) such that \( \{v_1, v_2\} \in E(G) \). We then count the number of paths with \( k - 1 \) vertices such that \( (v_1, v_2) \) are respectively the first and second vertices by inductively extending the path choosing vertices from the neighborhood of the last vertex of the current path. Finally, there is at most one way of extending the current path with \( k - 1 \) vertices to a color-tied path with \( k \) vertices. This is because the last edge must have the same color as \( \{v_1, v_2\} \) and since the coloring is proper, there must be at most one neighbor
of the last vertex of the current path with that color. Therefore, the number of color-tied $k$-paths is at most

$$4pn^2 \cdot (2pn)^{k-3} \cdot 1 = O(p^{k-2}n^{k-1}),$$

which is smaller than the number of $k$-paths by a factor of $\Omega(pn)$.

Now let $v_1v_2 \ldots v_k$ be a color-tied path, with $k \in [[\ell/2] + 2]$. Since

$$p^{\ell-k+2}n^{\ell-k+1} \gg p^{[\ell/2]}n^{[\ell/2]-1} \gg \log n,$$

then with high probability the number of $(\ell - k + 2)$-paths connecting $v_1$ to $v_k$ is of order $p^{\ell-k+1}n^{\ell-k}$, by Lemma 4.6. Therefore, the number of non-rainbow cycles is at most

$$\sum_{k=1}^{[\ell/2]+2} O \left( p^{k-2}n^{k-1} \cdot p^{\ell-k+1}n^{\ell-k} \right) = O(p^{\ell-1}n^{\ell-1}).$$

Proof of Lemma 4.3. Let $\varepsilon, \beta \in (0, 1)$ and let $G = G(n, p)$ with $p = n^{-\beta}$ and $c : E(G) \to \mathbb{N}$ a proper edge-coloring. We choose an integer $\ell$ such that $\beta < 1 - 1/(2\ell - 1)$ and we set $\varepsilon' = \varepsilon^{2\ell}$. Our goal is to show that with high probability we have for $G = G(n, p)$ that

$$\mathbb{P}_T \left( G_t \text{ satisfies DISC}(\varepsilon) \right) = 1 - o(1),$$

for every $t \in [T]$, in a random assignment of colors of $c$ into $T$ classes. For any $\delta > 0$ with high probability we have that $\mathbb{P}_T (d_{G_t}(v) = (1 \pm 3\delta)pn/T) = 1 - o(1)$ for all $v \in V(G)$ and all $t \in [T]$, by Lemma 4.2. Therefore, by choosing $\delta$ small enough as a function of $\varepsilon'$, we have

$$(3) \quad \sum_{C \in \mathcal{C}_{2\ell}(G_t)} w(C) = \sum_{C \in \mathcal{C}_{2\ell}(G_t)} \prod_{v \in V(C)} \frac{1}{d_{G_t}(v)} = |\mathcal{C}_{2\ell}(G_t)| \left( 1 \pm \frac{\varepsilon'}{3} \right) \left( \frac{T}{pn} \right)^{2\ell},$$

where $\mathcal{C}_{2\ell}(G_t)$ denotes the set of $2\ell$-circuits in $G_t$.

If we show that, under the distribution $\mathbb{P}_T$, $G_t$ has the CIRCUIT$_{2\ell}(\varepsilon')$ property with probability $1 - o(1)$, then Lemma 4.4 implies that the same holds for DISC$(\varepsilon)$. By the definition of the CIRCUIT$_{2\ell}(\varepsilon')$ property, we have to show that $\sum w(C) = 1 \pm \varepsilon'$. By (3), it is sufficient to prove that

$$(4) \quad \mathbb{P}_T \left( |\mathcal{C}_{2\ell}(G_t)| = \left( 1 \pm \frac{\varepsilon'}{3} \right) \left( \frac{pn}{T} \right)^{2\ell} \right) = 1 - o(1).$$

Recall that $\mathcal{C}_{2\ell}(G_t) \subset \mathcal{C}_{2\ell}(G_t)$ is the collection of $2\ell$-cycles in $G_t$. Since $p \gg n^{-1+1/2\ell}$, the numbers of $2\ell$-circuits and of $2\ell$-cycles in $G(n, p)$ are with high probability asymptotically equal. More precisely, by choosing some small $\delta$ in Lemma 4.5 we infer that with high probability $G$ satisfies
Therefore, we can remove the $2\ell$-circuits that are not $2\ell$-cycles from the computation. In doing that, we reduce the proof of (4) to proving that

\[ P_T(\frac{\|C'_G\|}{2\ell(G_t)} = \left(1 \pm \frac{\varepsilon'}{6}\right)\left(\frac{pn}{2\ell}\right)^{2\ell}) = 1 - o(1), \]

for every $t \in [T]$.

In order to prove (6), fix $t \in [T]$ and for each $i \in c(E(G))$, let $A_i = \{0, 1\}$ and let $X_i$ be the indicator function for the event $\{\sigma(i) = t\}$ and set $Y = |C'_G(\ell_G)|$. Note that $Y = f(X_1, \ldots, X_r)$, for some $f : \prod_{i=1}^r A_i \to \mathbb{R}$. Now, let $c_i$ be the smallest real number for which $|f(x) - f(x')| \leq c_i$, whenever $x, x' \in \prod_{i=1}^r A_i$ differ only on the $i$th coordinate. By double counting the pairs $(i, e)$ such that $i \in c(E(G))$ and $e \in E(G)$ has color $i$ and is contained in a $2\ell$-cycle, we obtain

\[ \sum_{i=1}^r c_i \leq 2\ell|C'_G|, \]

Moreover, since $pn^{1-2/(2\ell-1)} \to \infty$, Lemma 4.6 implies that the number of of $2\ell$-cycles in $G$ containing a given edge $e \in G$ is at most $Dp^{2\ell-1}n^{2\ell-2}$, for some large constant $D > 0$. Since each color $i \in [r]$ induces a matching in $G$, it follows that $c_i = Dp^{2\ell-1}n^{2\ell-1}$. Together with (7), we obtain that

\[ \sum_{i=1}^r c_i^2 \leq Dp^{2\ell-1}n^{2\ell-1}\sum_{i=1}^r c_i \leq 2\ell Dp^{2\ell-1}n^{2\ell-1}|C'_G|. \]

To finish the proof of (6) we have to calculate the expectation of $Y$. Since each edge is in $G_t$ with probability $1/T$, then for each $C \in C'_G$, we have that $P_T(\ell_G) \geq 1/T^{2\ell}$, with equality being attained if $C$ is rainbow. Since $(pn)^{2\ell} \gg 1$, the number of $2\ell$-cycles in $G(n, p)$ is with high probability $1 - o(1)(pn)^{2\ell}$. By Lemma 4.7, since $p^3n^{-4} \gg \log n$, almost all of those $2\ell$-cycles are rainbow. Therefore, with high probability (with respect to $G(n, p)$) we have that

\[ \mathbb{E}_T(Y) = \left(1 \pm \frac{\varepsilon'}{12}\right)\left(\frac{pn}{T}\right)^{2\ell}. \]

Finally, by Lemma 3.5.
\[\mathbb{P}_T \left( |Y - \mathbb{E}[Y]| > \varepsilon' \left( \frac{pn}{T} \right)^{2t} \right) \leq 2 \exp \left\{ - \frac{\Omega((pn)^{4t})}{\sum_i c_i^2} \right\} = 2 \exp -\Omega(pn) = o(1).\]

Therefore, with probability tending to 1 (under \( \mathbb{P}_T \)) equation (6) holds, which, together with (5), implies (4) and finishes the proof.

\[\square\]

4.2. Isolated copies and their distribution.

Given a graph \( G \) on \( n \) vertices, we call a copy of a graph \( S \) in \( G \) isolated if it does not share an edge with any other copy of \( S \) in \( G \). Let \( G^S \) be the spanning subgraph of \( G \) induced by all the edges that belong to isolated copies of \( S \) in \( G \). Since \( G^S \) will play a role in the application of Theorem 1.1 the following issue arises. Theorem 1.1 is a counting result that states that there exists only a very small number of graphs that possess some pseudo-random properties and do not copies of a graph \( H \). However, since \( G^S \) is not \( G(n, p) \) then it is not clear how to bound the probability of appearance of such graphs, which motivates the lemma below. For \( E \subseteq E(K_n) \), we write \( E \subseteq G^S \) if \( E \subseteq G^S \) and if additionally no two edges in \( E \) belong to the same isolated copy of \( S \).

**Lemma 4.8.** Let \( F \) be a graph and \( G = G(n, p), \) with \( p = p(n) \in (0, 1] \). If \( q = n^{v(F)-2}p^{e(F)} \), then for any \( E \subseteq E(K_n) \) we have that

\[\mathbb{P}[E \subseteq E(G^S)] \leq q^{|E|}.\]

We move to other properties of \( G^S \), namely the density and upper-uniformity, to be able to apply Lemma 3.3 and hence the Theorem 1.1. For disjoint sets \( V_1, \ldots, V_{v(S)} \subseteq V(G) \), we denote by \( Z_G(V_1, \ldots, V_{v(S)}) \) the number of transversal copies of \( S \) in \( G[V_1, \ldots, V_{v(S)}] \), i.e., copies of \( S \) in \( G \) with one vertex in each \( V_i \) with \( i \in [v(S)] \) and by \( Y_G(V_1, \ldots, V_{v(S)}) \) the number of such copies that are also isolated. We may omit the sets \( V_1, \ldots, V_{v(S)} \) from the notation if they are clear from the context. For \( G = G(n, p) \) and for disjoint linear-sized sets \( V_1, \ldots, V_{v(S)} \subseteq V(G) \), we have that

\[\mathbb{E}[Z(V_1, \ldots, V_{v(S)})] = \Theta \left( n^{v(S)}p^{e(S)} \right).\]

Our analysis of \( G_i^S \) starts by proving that in a typical outcome of \( G = G(n, p) \) and in a typical assignment of colors we have \( Y_{G_t}(V_1, \ldots, V_{v(S)}) = \Omega(n^{v(S)}p^{e(S)}) \). In words, we mean that the number of isolated copies of \( S \) in each \( G_t \), with \( t \in [T] \), is a considerable proportion of all copies of \( S \) in \( G \). Therefore, the bipartite graph \( G_i^S[V_i, V_j] \), for \( ij \in E(S) \), has \( \Omega(n^{v(S)}p^{e(S)}) \) many edges because the copies are isolated. Under the hypothesis of Theorem 1.1, this lower
bound is larger than $\Omega(n^{2-1/m_2(H)})$, which is one of the requirements for applying Theorem 4.1 successfully.

**Lemma 4.9.** Let $S$ be a graph. For every $\mu > 0$ and integer $T > 0$ there exists $\alpha > 0$ such that for $p = n^{-\beta}$ and $1/m_2(S) < \beta \leq (v(S) - 1)/e(S)$ the following holds. With high probability for every proper edge-coloring of $G = G(n, p)$ and for a fixed family of disjoint sets $V_1, \ldots, V_{v(S)} \subseteq V(G)$ with $|V_i| \geq \mu n$, for $i \in [v(S)]$, we have

$$\mathbb{P}_T \left( Y_{G_t} \geq \alpha n^{v(S)} p^{e(S)} \right) = 1 - o(1),$$

for every $t \in [T]$.

Another requirement for our proof is that $G^S$, and hence $G^S_t$, is $(\mu, q)$-upper uniform for $q = O(n^{v(S)} p^{e(S)})$, which is stated in the next lemma. The proof follows by an application of Lemma 4.8 and it was originally stated in \cite{9, Lemma 14}.

**Lemma 4.10.** Let $S$ be a graph and $\beta > 0$ be such that $\beta < (v(S) - 1)/e(S)$. Let $p = Bn^{-\beta}$, with $B > 0$, and $G = G(n, p)$. Then for every $\mu > 0$ with high probability we have that $G^S$ is $(\mu, q)$-upper-uniform, where $q = 6e(S)n^{v(S)}p^{e(S)}$.

The rest of this section is dedicated to the proof of Lemma 4.9. First proving that for some regimes of $p$ and for a fixed family of pairwise disjoint sets $V_1, \ldots, V_{v(S)}$ with high probability we have that a big proportion of the transversal copies are actually isolated.

**Proposition 4.11.** Let $S$ be a 2-balanced graph. If $p = n^{-\beta}$ and $1/m_2(S) < \beta \leq (v(S) - 1)/e(S)$, then with high probability $G = G(n, p)$ satisfies the following. For every $\mu > 0$ and every family of pairwise disjoint sets $V_1, \ldots, V_{v(S)} \subseteq V(G)$, with $|V_i| \geq \mu n$, we have

$$\mathbb{E}[Z(V_1, \ldots, V_{v(S)})] = \Omega \left( n^{v(S)} p^{e(S)} \right).$$

**Proof.** Let $X$ be the number of copies of any graph that is a union of two copies of $S$ intersecting in at least one edge and which is not $S$. Let $V_1, \ldots, V_{v(S)} \subset V(G)$ be a fixed family of pairwise disjoint sets. The proof lies basically on showing that

$$\Delta := \sum_{K_2 \subset J \subset S} O(n^{2v(S) - v(J)} p^{2e(S) - e(J)}) = o(\mathbb{E}(Z)). \tag{8}$$

Indeed, since $S$ is 2-balanced we have that

$$\frac{v(S) - v(J)}{e(S) - e(J)} \leq \frac{1}{m_2(S)} \implies n^{v(S) - v(J)} p^{e(S) - e(J)} = o(1),$$

for every $K_2 \subset J \subset S$, which proves (8) since $\mathbb{E}Z = \Omega(n^{v(S)} p^{e(S)})$. By Janson’s inequality \cite{0} and by the fact that $\Delta < \mathbb{E}Z$, we have that

$$\mathbb{P}(Z \leq 3\mathbb{E}Z/4) = \exp(-\Omega(\mathbb{E}Z)).$$
Since $\mathbb{E}Z = \Omega((n^{v(S)}p^{e(S)})) \gg n$, by taking union bound $Z \geq 3Z/4$ for every choice of $V_1, \ldots, V_{v(S)}$.

Moreover, since $\mathbb{E}X = O(\Delta)$, then by Markov’s inequality we have that with high probability $X = o(\mathbb{E}Z)$ and, therefore, we can discount every copy of $S$ which shares at least an edge with other copy, to finish the proof. □

We use Proposition 4.11 together with Azuma’s inequality to prove Lemma 4.9 and finish this section.

**Proof of Lemma 4.9.** Let $G = G(n, p)$, with $p = n^{-\beta}$ and $1/m_2(S) < \beta \leq (v(S) - 1)/e(S))$, and let $c : E(G) \rightarrow [r]$ be a proper edge-coloring of $G$, for some $r \in \mathbb{N}$. For an integer $T > 0$ consider a random partition of the colors into $T$ classes. For each $i \in [r]$, let $X_i$ be the indicator function for the event $\sigma(i) = t$ and observe that $Y_{G_t}$ is a function of $X_1, \ldots, X_r$. For each $i \in [r]$, let $c_i = c_i(G)$ be the smallest real number such that if we change the value of $X_i$ only, then the value of $Y_{G_t}$ will be altered by at most $c_i$. Since the coloring of $G$ is proper, by altering the value of $X_i$ we add or remove at most a perfect matching from $G_t$, which implies that it will affect at most $n$ isolated copies of $S$. Therefore, we have $c_i \leq n$. Furthermore, since a transversal isolated copy of $S$ in $G$ can be affected by at most $e(S)$ changes in the value of $X_1, \ldots, X_r$, we also have

$$\sum_{i=1}^{r} c_i \leq e(S)Y_G.$$ 

Hence,

$$\sum_{i=1}^{r} c_i^2 \leq n \sum_{i=1}^{r} c_i \leq e(S)nY_G.$$ 

Furthermore, notice that each copy of $S$ in $G$ belongs to $G_t$ with probability $(1/T)^k$, where $k$ is the number of colors that appears in such copy of $S$. In particular, such copy of $S$ is in $G_t$ with probability at least $(1/T)^{e(S)}$. Therefore,

$$\mathbb{E}_T[Y_{G_t}] \geq \frac{Y_G}{T^{e(S)}}.$$ 

Note that, by Proposition 4.11 we have that $Y_G = \Omega((n^{v(S)}p^{e(S)}))$. Therefore, Lemma 3.5 yields that
\[ \mathbb{P}_T \left[ |Y_{G_t} - \mathbb{E}[Y_{G_t}]| \geq \frac{1}{2} \mathbb{E}[Y_{G_t}] \right] \leq 2 \exp \left\{ -\frac{\mathbb{E}[Y_{G_t}]^2}{2 \sum_{i \in [q]} c_i^2} \right\} \leq 2 \exp \left\{ -\frac{Y_{G_t}^2}{T^{2e(S)} e(S)n Y_G} \right\} = \exp \left\{ -\Omega(n^{v(S) - 1} p e(S)) \right\} = o(1). \]

The last line is due to the fact that \( \beta < (v(S) - 1)/e(S) \). Consequently, with probability \( 1 - o(1) \) under \( \mathbb{P}_T \), we have that

\[ Y_{G_t} \geq \frac{1}{2} \mathbb{E}[Y_{G_t}] \geq \frac{Y_G}{2T^{v(S)}} \geq \alpha n^{v(S)} p e(S), \]

for some \( \alpha > 0 \) that only depends on the graph \( S \) and the values of \( T \) and \( \mu \).

\[ \square \]

5. **Proof of Theorem 1.1**

Let \( H, F \) and \( S \) be as in the statement of Theorem 1.1. Let us say that \( V(H) = \{u_1, \ldots, u_h\} \), \( V(S) = \{v_1, \ldots, v_s\} \) and \( V(F) = \{w_1, \ldots, w_f\} \), where \( u_1 = v_1 \) and \( u_2 = v_2 \). We consider an equi-partition of \([n]\) into \( s + h - 2 \) sets

\[ V = \left( \bigcup_{i=1}^s V_i \right) \cup \left( \bigcup_{i=3}^h U_i \right). \]

For technical reasons related to the Theorem 4.11, instead of working in \( G(n, p) \), we work on the random graph \( G \) obtained in the following way. Each pair of vertices in \( V \) contained in \( \bigcup_{i=1}^s V_i \) forms an edge with probability \( p \) and every pair of vertices in \( V \) forms an edge with probability \( q' = e(H) q \), where \( q := 6 e(S) n^{v(S) - 2} p e(S) \), all independently from each other. Notice that \( q' < p \), and therefore, since we are dealing with a monotone property, in order to prove Theorem 1.1 it suffices to prove that with high probability the random graph \( G \) satisfies \( G \xrightarrow{\text{th}} F \oplus H \).

Let \( c : E(G) \to \mathbb{N} \) be a proper coloring of \( E(G) \). For each color \( i \in e(E(G)) \), we assign independently and uniformly at random an edge \( \sigma(i) \in E(H) \). For each \( u_i u_j \in E(H) \), let \( G_{u_i u_j} \) be the spanning subgraph of \( G \) with edge set

\[ E(G_{u_i u_j}) = \{ e \in E(G) : \sigma(c(e)) = u_i u_j \}. \]

That is, \( E(G_{u_i u_j}) \) are those edges of \( E(G) \) for which their color was assigned to \( u_i u_j \).
Recall that \( Y_{G_{u_1u_2}}(V_1, \ldots, V_s) \) denotes the number of transversal isolated copies of \( S \) in \( G_{u_1u_2}[V_1, \ldots, V_s] \) (c.f. Section 4.2). By Lemma 4.9 there exists \( \alpha > 0 \) such that the following holds with high probability. For every proper edge-coloring \( c \) of \( G \) we have

\[
P_{e(H)}(Y_{G_{u_1u_2}} \geq \alpha n^{v(S)} p^{e(S)}) = 1 - o(1).
\]

From now on, we assume that \( \alpha \in (0, s^{-2f}) \). For an outcome \( G \) and a proper edge-coloring \( c : E(G) \to \mathbb{N} \), we denote by \( \mathcal{E}_1 = \mathcal{E}_1(G, c) \) the event in which \( Y_{G_{u_1u_2}} \geq \alpha n^{v(S)} p^{e(S)} \). Note that \( \mathcal{E}_1 \) is an event in the probability space given by \( \sigma \).

Let \( G^S(V[s]) \) be the spanning subgraph of \( G \) induced by all the edges that belong to isolated copies of \( S \) in \( V[s] := V_1 \cup \cdots \cup V_s \). By Lemma 4.10 \( G^S(V[s]) \) is \((\mu, q)\)-upper uniform with high probability for any constant \( \mu > 0 \). As we need regularity to apply Theorem 4.1 from now on we fix \( \varepsilon \in (0, 1/8) \) (to be chosen later) and we let \( \mu = \mu(\varepsilon, \alpha) > 0 \) be given by Lemma 3.3.

Let \( \mathcal{P} \) be the set of graphs \( J \) for which \( J^S(V[s]) \) is \((\mu, q)\)-upper uniform. Our next claim states that, if \( G \in \mathcal{P} \) and the event \( \mathcal{E}_1 \) occurs, then the following holds. There is a large and dense regular bipartite graph subgraph of \( G_{u_1u_2}[V_1, V_2] \) whose edges are contained in distinct isolated rainbow copies of \( F \).

**Claim 5.1.** Suppose that \( G \in \mathcal{P} \) and let \( c : E(G) \to \mathbb{N} \) be a proper edge-coloring. If the event \( \mathcal{E}_1 \) occurs, then the following holds. For some \( i_1, i_2 \in [s] \) there exist \( W_{i_1} \subseteq V_{i_1}, W_{i_2} \subseteq V_{i_2} \) with \( |W_{i_1}| = |W_{i_2}| \geq \mu|V_{i_1}| \) and a bipartite graph \( B_{i_2} \subset G_{u_1u_2}[W_{i_1}, W_{i_2}] \) such that

1. For every \( ab \in E(B_{i_2}) \), with \( a \in W_{i_1} \) and \( b \in W_{i_2} \), there is an isolated rainbow copy of \( F \) in \( V[s] \) containing \( ab \) whose colors are assigned to \( u_1u_2 \). Moreover, in these copies the vertices \( a \) and \( b \) correspond to the vertices \( w_1 \) and \( w_2 \), respectively.
2. The graph \( B_{i_2} \) is \((\varepsilon, q)\)-regular with density at least \( \alpha^2 q \).

**Proof.** Since \( S \rightharpoonup F \), in each transversal isolated copy of \( S \) in \( G_{u_1u_2}[V_1, \ldots, V_s] \) we can find an transversal isolated rainbow copy of \( F \). Note that there are at most \( s^f \) different ways for a copy of \( F \) to be transversal in \( G_{u_1u_2}[V_1, \ldots, V_s] \). As \( \mathcal{E}_1 \) holds, by the pigeon-hole principle we have for some \( i_1, \ldots, i_f \in [s] \) at least

\[
\frac{\alpha}{s^f} n^{v(S)} p^{e(S)}
\]

transversal isolated rainbow copies of \( F \) in \( G_{u_1u_2}[V_{i_1}, \ldots, V_{i_f}] \) with the corresponding copy of \( w_t \) belonging to \( V_{i_t} \), for each \( t \in [f] \). We turn our attention to the bipartite graph \( B_{i_2} = (V_{i_1} \cup V_{i_2}; E') \) induced by the pairs contained in those copies of \( F \). Observe that \( B_{i_2} \) already satisfies property (1). Note that each edge of \( B_{i_2} \) is in exactly one of the previously considered copies of \( F \). Therefore, for \( \alpha < s^{-2f} \) we have
\[
E(B_{12}) \geq \frac{\alpha}{s^2} n^{v(S)} p^{e(S)} = \frac{\alpha}{6s^3 e(S)} q n^2 \geq \alpha^2 q |V_1| |V_2|.
\]

As \( G \in P \), the graph \( G^S(V_s) \) is \((\mu, q)\)-upper uniform and so it is \( B_{12} \). Moreover, as \( d_{B_{12}}(V_1, V_2) \geq \alpha^2 q \), we can apply Lemma 3.3 to show that the following holds. There exists \( W_{i_1} \subseteq V_{i_1}, W_{i_2} \subseteq V_{i_2} \) with \( |W_{i_1}| = |W_{i_2}| \geq \mu |V_{i_2}| \), such that the bipartite graph \( B_{12}[W_{i_1}, W_{i_2}] \) is \((\varepsilon, q)\)-regular with density at least \( \alpha^2 q \). This shows property (2).

From now on we assume that, if the event \( \mathcal{E}_1 \) occurs, then in Claim 5.1 we have \( i_1 = 1 \) and \( i_2 = 2 \). We shall denote by \( W_1 \) and \( W_2 \) the sets obtained from this claim.

For a pair \( ij \) such that \( u_i u_j \in E(H) \setminus \{u_1 u_2\} \), let \( G_{ij} \) be the graph induced by \( G \) on \( U_i \cup U_j \). Our next goal is, roughly speaking, to show that there exists a large and dense regular bipartite graph in \( G_{u_i u_j}[U_i, U_j] \). The density will be given by the concentration on the degrees (c.f. Lemma 4.2). The regularity will be given by the degrees and the DISC property (c.f. Lemma 4.3) combined with Lemma 3.4.

Recall that each edge in \( G_{ij} \) is included independently with probability \( q' = e(H)q \), where \( q = 6e(S)n^{v(S)-2}p^{e(S)} \). For \( \delta > 0 \), an outcome \( G \) and a proper edge-coloring \( c : E(G) \rightarrow \mathbb{N} \), we denote by \( \mathcal{E}_2 = \mathcal{E}_2(G, c, \delta) \) the event in which

\[
d_{G_{u_i u_j}}(v) = (1 \pm \delta)qv(G_{ij})
\]

for all \( v \in U_i \cup U_j \) and pairs \( ij \) such that \( u_i u_j \in E(H) \setminus \{u_1 u_2\} \). Note that \( \mathcal{E}_2 \) is an event in the probability space given by \( \sigma \). By Lemma 4.2, the following holds with high probability for any fixed \( \delta > 0 \). For every proper edge-coloring of \( G_{ij} \) we have

\[
\mathbb{P}_{e(H)}(\mathcal{E}_2) = 1 - o(1).
\]

From now on, we fix \( \delta(\varepsilon, \mu/2) \in (0, 1/4) \) given by Lemma 3.4. Applying Lemma 4.2 with \( \mu/2 \) instead of \( \mu \) it is only a subtle technicality. This choice ensures that we have a regular bipartite graph with classes of size \( |W_i| \), which will be necessary to apply Theorem 1.1.

Let \( \varepsilon' = \varepsilon'(\varepsilon, \mu/2) \in (0, \mu^22^{-5sh}) \) be given by Lemma 3.4. For an outcome \( G \) and a proper edge-coloring \( c : E(G) \rightarrow \mathbb{N} \), define \( \mathcal{E}_3 = \mathcal{E}_3(G, c) \) to be the event in which \( G_{u_i u_j}^{ij} \) satisfies DISC(\( \varepsilon' \)). By Lemma 4.3, with high probability we have that for every proper edge-coloring

\[
\mathbb{P}_{e(H)}(\mathcal{E}_3) = 1 - o(1).
\]

Our next claim states that if \( G \in P \) and the event \( \mathcal{E}_2 \cap \mathcal{E}_3 \) holds, then we can find a large and dense regular bipartite graph in \( G_{u_i u_j}[U_i, U_j] \).
Claim 5.2. For a fixed outcome of $G$ and a proper edge-coloring $c$, suppose that the event $\mathcal{E}_2 \cap \mathcal{E}_3$ occurs. Then, for $i \in [h] \setminus \{1, 2\}$ there exists $W_i \subset U_i$, with $|W_i| = |W_1|$, such that the following holds. For every $u_iu_j \in E(H) \setminus \{u_1u_2\}$ the bipartite graph $G_{u_iu_j}[W_i, W_j]$ is $(\varepsilon, q)$-regular with density at least $\alpha^2 q$.

Proof. Under the event $\mathcal{E}_2 \cap \mathcal{E}_3$, Lemma [3,4] guarantees that the following holds. For $u_iu_j \in E(H) \setminus \{u_1u_2\}$ and any disjoint subsets $W_i \subset U_i$ and $W_j \subset U_j$ such that $|W_i| = |W_j| = |W_1|$, the bipartite graph $G_{u_iu_j}[W_i, W_j]$ is $(\varepsilon, q)$-regular. Fix any choice for those sets $W_i$. Now we are left to show that $G_{u_iu_j}[W_i, W_j]$ has density at least $\alpha^2 q$.

As $G_{u_iu_j}$ satisfies DISC($\varepsilon'$), we have

$$(13) \quad e(G_{u_iu_j}[W_i, W_j]) \geq \frac{\text{vol}(W_i) \text{vol}(W_j)}{\text{vol}(G_{u_iu_j}^{ij})} - \varepsilon' \cdot \text{vol}(G_{u_iu_j}^{ij}),$$

where the volume is over $G_{u_iu_j}^{ij}$. By simplicity, set $k := v(G_{u_iu_j}^{ij})$. By [13], we have $\text{vol}(G_{u_iu_j}^{ij}) < 2qk^2$. Moreover, the volumes $\text{vol}(W_i)$ and $\text{vol}(W_j)$ are both lower bounded by $qk|W_i|/2$. Therefore, it follows from (13) that

$$e(G_{u_iu_j}[W_i, W_j]) \geq \left(\frac{qk^2}{2}\right)^2 \left(\frac{1}{2qk^2}\right)|W_i||W_j| - 2\varepsilon' qk^2 \geq \frac{q|W_i||W_j|}{16}.$$

In the last inequality, we used that $\varepsilon' \leq \mu^2 2^{-5sh}$. As $\alpha \leq 1/8$, it follows that the density in $G_{u_iu_j}[W_i, W_j]$ is at least $\alpha^2 q$. \hfill \Box

By Lemma [4,10] with high probability we have that $G \in \mathcal{P}$. By combining this with Lemmas [4.2, 4.3] with high probability every proper edge-coloring of $G$ satisfies

$$\mathbb{P}_{e(H)}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3) = 1 - o(1).$$

From this we infer that for a typical sample of $G$ and a fixed proper edge-coloring $c$ there must exist an assignment $c'$ of colors of $c$ into $E(H)$ such that the conclusion of Claims [5.1] and [5.2] hold. Therefore, in this setup we find a subgraph $G'[W_1, \ldots, W_h]$ of $G$ with the following properties:

(a) $|W_i| = n_0$ for all $i \in [h]$, for some $n_0 \geq \mu |V_i|$;

(b) $G'[W_i, W_j] \subseteq G_{u_iu_j}[W_i, W_j]$ is $(2\varepsilon/\mu, m/n^2)$-regular and has $m = \alpha^2 qn_0^2 \gg n$ edges for all $u_iu_j \in E(H)$;

(c) For every $ab \in G'[W_1, W_2]$, with $a \in W_1$ and $b \in W_2$, there is a copy $F_{ab}$ of $F$ in $V_{[n]}$ containing $ab$. Moreover, in these copies the vertices $a$ and $b$ correspond to the vertices $w_1$ and $w_2$, respectively.

(d) $F_{ab}$ is a rainbow graph whose colors are assigned to $u_1u_2$. 

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Observe that properties \((a)\) and \((b)\) guarantee that \(G' \in \mathcal{G}(H, n_0, m, m/n_0^2, \varepsilon)\).

Now we claim that if \(G'\) contains a transversal copy of \(H\), then \(G\) contains a rainbow copy of \(F \oplus H\) in the coloring \(c\). In fact, as the edges in \(G'[W_i, W_j]\) only use colors assigned to \(u_iu_j\), any transversal copy of \(H\) in \(G'\) is rainbow. Moreover, by item \((c)\), each transversal copy of \(H\) in \(G'\) can be extended to a copy of \(F \oplus H\) in \(G\), where \(F\) is rainbow and only uses colors assigned to \(u_1u_2\). Therefore, we conclude that the copy of \(F \oplus H\) we found is also rainbow.

Let \(\gamma \in (0, 1)\) be a constant to be chosen later and let \(B(\gamma)\) be given by Theorem 4.1. Let \(p \geq Cn^{-\beta(H,S)}\), where \(C > B(\gamma)\) is another constant to be chosen later. We observe that the constants chosen before do not depend on these parameters. Let \(G^*(n) = G^*\) be the family of graphs \(G'\) such that \(V(G') \subseteq [n]\), the items \((a)-(c)\) are satisfied and no transversal copy of \(H\) is contained in \(G'\). If \(G \nrightarrow F \oplus H\) and \(G\) is a typical graph then, by the discussion above, there is a coloring \(c\) and graph \(G'(c) \subseteq G\) satisfying items \((a)-(d)\) such that it does not contain a transversal copy of \(H\). In particular, we have

\[
\mathbb{P}(G \nrightarrow F \oplus H) \leq \mathbb{P}\left( \bigcup_{G' \in G^*} \{G' \subseteq G\} \right) + o(1).
\]

By Lemma 4.8 and properties \((b)-(c)\), we have

\[
\mathbb{P}(G'[W_1, W_2] \subseteq G^F) \leq q^m \quad \text{and} \quad \mathbb{P}(G'[W_i, W_j] \subset G) \leq q^m
\]

for \(u_iu_j \in E(H) \setminus \{u_1, u_2\}\), and hence

\[(14) \quad \mathbb{P}(G \nrightarrow F \oplus H) \leq |G^*|q^m + o(1).
\]

Now we move to the application of Theorem 4.1. Let \(\varepsilon(\gamma) \in (0, 1/2)\) given by Theorem 4.1. As long as \(m > Bn^{2-1/m_2(H)}\), we have

\[
|G^*| \leq 2^{e(H)n_0^2} \left( \frac{n_0^2}{m} \right)^{e(H)}.
\]

The factor of \(2^{e(H)n}\) accounts for the choices of the subsets \(W_1, \ldots, W_h\). To see if \(m > Bn^{2-1/m_2(H)}\), recall that \(m = \alpha^2q_n^2\) and \(q = 6\varepsilon(S)n^e(S)-2p^e(S)\). It follows that

\[
m > \alpha^2q_n^{e(S)}p^e(S) \geq \alpha^2q_n^{e(S)}n^{-s+2-1/m_2(H)} = \alpha^2Cn^e(S) \left( \frac{\mu}{sh} \right)^s n^{2-1/m_2(H)}.
\]
In the last inequality, we used that $n_0 > \mu |V_1| > \frac{4m}{sh}n$. Therefore, in order to apply Theorem 4.1, we take

$$C(\gamma) := \left( \left( \frac{sh}{\mu} \right)^s B(\gamma) \right)^{1/\alpha(S)}.$$

Observe that $C$ is, in fact, only a function of $\gamma$. Indeed, $\alpha \in (0, s^{-2f})$ is given by Lemma 4.9 and it only depends on the original partition. That is, $\alpha$ is a function of $h$ and $s$. The constant $\mu$ only depends on $\varepsilon$ and $\alpha$, but $\varepsilon$ depends on $\gamma$ and it is given by Theorem 4.1.

Finally, by using the estimate $|G^*| \leq 2^{e(H)n\gamma^m}(n_0^2/m)^{(e(H)/m)}$, we obtain from (14) that

$$\mathbb{P}(G \nabla F \oplus H) \leq 2^{e(H)n\gamma^m}(n_0^2/m)^{e(H)/m} + o(1)$$

$$\leq \gamma^m \left( \frac{2e(m_0^2)}{m} \right)^{e(H)/m} + o(1)$$

$$\leq \gamma^m \left( \frac{2e}{\alpha^2} \right)^{e(H)/m} + o(1).$$

In the second inequality above, we used that $n \ll m$ and that $(a/b)^{e(H)/m^2} \leq (\varepsilon/\alpha^2)^b$ for $a \geq b \geq 1$. Choose $\gamma := (8\varepsilon/\alpha^2)^{-e(H)/m}$. Thus we conclude that

$$\mathbb{P}(G \nabla F \oplus H) \leq 2^{-m} + o(1),$$

which finishes our proof.

6. Book graphs

Recall that, for a positive integer $t$, the book graph $B_t$ is defined by $t$ triangles sharing one edge. In this section we prove that book graphs fit the framework of Theorem 1.1. We start with the following result.

**Lemma 6.1.** $B_{3t-2} \nabla_{1b} B_{2t}$ for every $t \geq 1$.

**Proof.** The base case $t = 1$ is trivial since every proper coloring of a triangle is rainbow. We assume the lemma to be true for every integer up to $t - 1$ and we move one step in the induction. Let $\Phi$ be a proper-coloring of $B_{3t-2}$ and let $\Phi(B_{3t-2}) = \{u_1, u_2, v_1, \ldots, v_{3t-2}\}$ where $\{u_{i1}, u_{i2}, v_i\}$ is a triangle for every $t \in [3t - 2]$ and $v_1, \ldots, v_{3t-2}$ is an independent set. By induction, we have that $\Phi$ induces a rainbow copy of $B_{(t-1)}$ which, without loss of generality, we assume to be induced by $\{u_{11}, u_{21}, v_1, \ldots, v_{t-1}\}$.

Let $X$ be the set containing any $v_k$, with $t \leq k \leq 3t - 2$, such that $\{u_1, u_2, \ldots, v_{t-1}, v_k\}$ does not induces a rainbow copy $B_{2t}$. Since the coloring is proper, then $\Phi(u_{i1}u_{i2})$ is different of $\Phi(u_{i1}u_{i2})$ for every $i \in \{1, 2\}$ and $v_k \in X$. Therefore, if $v_k$ belongs to $X$ then we must have that $\Phi(u_{i1}v_k) = \Phi(u_{3-i}v_{t})$ for some $i \in \{1, 2\}$ and $\ell \in [t - 1]$. For fixed $i \in \{1, 2\}$ there can be at most $t - 1$ values $k$ such that $\Phi(u_{i1}v_k) = \Phi(u_{3-i}v_{t})$ for some $\ell \in [t - 1]$, since the coloring

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is proper. Therefore we have that $|X| < 2t - 2$ and we conclude that there exists a vertex that yields a rainbow copy of $B_t$. □

Now we are ready to prove Corollary 1.2.

**Proof of Corollary 1.2.** Let $H$ be a graph with $m_2(H) \in (1, 2)$. It is straightforward to check that for any $t \geq 1$ we have that $m_2(B_t) = 2$, which implies that the hypothesis $1 < m_2(H) < m_2(B_t)$ of Theorem 1.1 is satisfied. Since $B_{3t-2}$ is a $2$-balanced graph and $B_{3t-2} \xrightarrow{rb} B_t$, Theorem 1.1 implies that for some $B > 0$ we have that

$$\lim_{n \to \infty} \mathbb{P}[G(n, p) \xrightarrow{rb} B_t \oplus H] = 1$$

whenever $p \geq Bn^{-\beta(H, B_{3t-2})}$. Now we only have to show that $\beta(H, B_{3t-2}) > 1/m_2(B_t \oplus H)$. Note that $v(B_{3t-2}) = 3t$ and $e(B_{3t-2}) = 6t - 3$. Since $m_2(H) < 2$, then we have that

$$\beta(H, B_{3t-2}) = \frac{1}{6t-3} \left(3t - 2 + \frac{1}{m_2(H)}\right) > \frac{1}{2}.$$

On the other hand, since $B_t$ is contained in $B_t \oplus H$, then $1/m_2(B_t \oplus H) \leq 1/2$, which finishes the proof. □

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**References**

[1] J. Balogh, R. Morris, and W. Samotij. Independent sets in hypergraphs. *J. Amer. Math. Soc.*, 28(3):669–709, 2015.

[2] G. F. Barros, B. P. Cavalar, G. O. Mota, and O. Parczyk. Anti-Ramsey threshold of cycles for sparse graphs. *Electron. Notes Theor. Comput. Sci.*, 346:89–98, 2019.

[3] F. Chung and R. Graham. Quasi-random graphs with given degree sequences. *Random Structures Algorithms*, 32(1):1–19, 2008.

[4] P. Erdős. Some old and new problems in various branches of combinatorics. *Proc. 10th southeast. Conf. Combinatorics, graph theory and computing*, 1(23):19–37, 1979.

[5] S. Gerke and A. Steger. The sparse regularity lemma and its applications. In *Surveys in combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 227–258. Cambridge Univ. Press, Cambridge, 2005.

[6] S. Janson. Poisson approximation for large deviations. *Random Structures & Algorithms*, 1(2):221–229, 1990.

[7] Y. Kohayakawa, P. B. Konstadinidis, and G. O. Mota. On an anti-Ramsey threshold for random graphs. *European J. Combin.*, 40:26–41, 2014.
[8] Y. Kohayakawa, P. B. Konstadinidis, and G. O. Mota. On an anti-Ramsey threshold for sparse graphs with one triangle. *J. Graph Theory*, 87(2):176–187, 2018.

[9] Y. Kohayakawa and B. Kreuter. Threshold functions for asymmetric Ramsey properties involving cycles. *Random Structures Algorithms*, 11(3):245–276, 1997.

[10] Y. Kohayakawa, G. O. Mota, O. Parczyk, and J. Schnitzer. The anti-ramsey threshold of complete graphs. 2019.

[11] C. McDiarmid. On the method of bounded differences. In *Surveys in combinatorics, 1989 (Norwich, 1989)*, volume 141 of *London Math. Soc. Lecture Note Ser.*, pages 148–188. Cambridge Univ. Press, Cambridge, 1989.

[12] R. Nenadov, Y. Person, N. Škorić, and A. Steger. An algorithmic framework for obtaining lower bounds for random Ramsey problems. *J. Combin. Theory Ser. B*, 124:1–38, 2017.

[13] V. Rödl and A. Ruciński. Threshold functions for ramsey properties. *J. Amer. Math. Soc.*, 8(4):917–942, 1995.

[14] V. Rödl and Z. Tuza. Rainbow subgraphs in properly edge-colored graphs. *Random Structures Algorithms*, 3(2):175–182, 1992.

[15] J. Spencer. Counting extensions. *Journal of Combinatorial Theory, Series A*, 55(2):247–255, 1990.

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**Institute of Computer Science of the Czech Academy of Sciences, Pod Vodárenskou věží 2, 18207, Prague, Czech Republic (P. Araújo)**

*Email address:* araujo@cs.cas.cz

**Instituto de Matemática, Universidade Federal Fluminense, Niterói, Brazil (T. Martins)**

*Email address:* tlmartins@id.uff.br

**Freie Universität Berlin and Berlin Mathematical School (BMS/MATH+), Arnimallee 3, 14195 Berlin, Germany (L. Mattos)**

*Email address:* lmattos@zedat.fu-berlin.de

**IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria (W. Mendonça)**

*Email address:* w.mendonca@ist.ac.at

**Guaca Macramê, Rua do Z 33, 20251-600, Rio de Janeiro, RJ, Brazil (L. Moreira)**

*Email address:* lzplfm@gmail.com

**Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508-090 São Paulo, Brazil (G. Mota)**

*Email address:* mota@ime.usp.br