Non-linear operations in quantum information theory

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Abstract

Quantum information theory is used to analyze various non-linear operations on quantum states. The universal disentanglement machine is shown to be impossible, and partial (negative) results are obtained in the state-dependent case. The efficiency of the transformation of non-orthogonal states into orthogonal ones is discussed.

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The rules of a quantum mechanics make certain processes impossible. Non-orthogonal quantum states cannot be cloned [1]. This is one of the fundamental theorems of the quantum information theory. On other hand, there are explicit constructions of quantum circuits that would perform many interesting transformations of quantum states [2, 3], provided that certain devices, like quantum XOR gates [4], can be actually built. Between these two extremes there is a gray area of operations that are not obviously forbidden by the elementary laws of quantum mechanics, but may be ruled out by more careful considerations. The most interesting of them are non-linear operations, and their analysis from the point of view of quantum information theory [5, 6] is the subject of this paper.

The basic procedures of quantum mechanics — unitary transformations and projections — are linear. However, non-linear operations with quantum states are common, even if they are not always regarded as such. Non-linearity naturally enters via selective operations, namely those that involve a filtering at one of their steps. These operations have a finite probability to fail and may look quite exotic, e.g., the transformation [2]

$$\rho_{\text{in}} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \rightarrow \rho_{\text{out}} = \begin{pmatrix} \rho_{11}^2 & \rho_{12}^2 \\ \rho_{21}^2 & \rho_{22}^2 \end{pmatrix}.$$  \hfill (1)

Nevertheless, the possibility of failure is not a necessary condition of non-linearity. An operation may be non-selective, i.e. it is always be completed successfully, but there may be a demand for certain inputs to be transformed into certain outputs, while the processing of other input states remains unspecified. For example, if we want to process the output of a two-state cloning machine [3], the transformation of the two resulting pure entangled states (labelled 1 or 2) into the direct product of their reduced density matrices,

$$\rho_{1,2} \rightarrow \text{Tr}_A \rho_{1,2} \otimes \text{Tr}_B \rho_{1,2}$$  \hfill (2)

would be useful. What happens to all other states is irrelevant and unspecified; everything that makes this process work will be accepted. If we are interested only in these two states, the transformation looks ‘non-linear’.

The most general physical operation

$$\rho \rightarrow \rho' = \frac{T \rho}{\text{Tr} T \rho}$$  \hfill (3)

is represented by a superoperator $T : \mathcal{B}(\mathcal{H})_1 \rightarrow \mathcal{B}(\mathcal{H})_1$, which is (i) positive, (ii) satisfies $T^\dagger T \leq 1$, and (iii) is completely positive [7, 8]. The validity of last condition [9] has
recently become a hot issue in numerous discussions, but it certainly holds for systems which are initially unentangled with their environment.

According to the ‘first representation theorem’ [8]

\[ T \rho = \sum_k A_k \rho A_k^\dagger, \]  

(4)

where the set of $A_k$’s satisfies

\[ \sum_k A_k^\dagger A_k \leq 1. \]  

(5)

Since the probability to get a certain outcome is given by $p = \text{Tr} T \rho$, the transformation in Eq. (2) is trace preserving, so that $\sum_k A_k^\dagger A_k = 1$.

When a ‘quantum black box’ [10] is specified, the superoperator $T$ and the corresponding matrices $A_k$ can be inferred from the complete set of input and output data. Similarly, if we extend Eq. (2) to a larger set of states and then use the approach of Chuang and Nielsen [10], we can confirm or refute different realizations of the desired transformation. If the answer is positive the problem is solved. On the contrary, a negative answer tells us nothing. Another realization may still work.

Quantum information theory can discern absolutely impossible processes from tentatively possible ones. We look at the desired operation as a part of some hypothetical decision scheme, which aims at distinguishing between different input states [6, 11, 12], or as a communication channel [13] with input and output alphabets given by the left and right hand sides of an expression like Eq. (2).

Using Eqs. (3, 4) it is easy to show that any chain of the measurements and/or transformations of the system, with or without ancilla, can be described as a single measurement [14] represented by a positive operator-valued measure (POVM). Thus if the proposed procedure allows to improve any of the distinguishability criteria of the input states beyond the optimal value, it contains some flaw and therefore is not physical.

One of these distinguishability criteria is the probability of error (PE), defined as follows [6, 11]: an observer is given one of two states $\rho_1$ and $\rho_2$. That state is secretly chosen, and since we are interested in the intrinsic difference between the states, the probabilities to pick up either of the states are equal. The task of the observer is to decide, after performing anything that is allowed by quantum mechanics, which state was given to him. The probability that the observer makes a wrong guess with the best possible
decision scheme gives a measure of the distinguishability. Two orthogonal states can be distinguished perfectly, thus giving \( PE = 0 \). When the states are identical \( PE = \frac{1}{2} \). In general the optimal result is

\[
PE(\rho_1, \rho_2) = \frac{1}{2} - \frac{1}{4} \text{Tr} |\rho_1 - \rho_2|.
\] (6)

Any transformation that would give a lower \( PE \) certainly violates the laws of quantum mechanics. Unfortunately, the usefulness of this criterion is limited to the case of two states, which is the only one for which a closed solution is known.

Another criterion is the accessible information \( I(\rho_1, \rho_2) \), which is defined as a maximal mutual information over all possible decision schemes. For a set of states \( \rho_i \) with fixed \textit{a priori} probabilities \( \pi_i \), each possible measurement scheme \( X \) gives a probability distribution of the results. We calculate the mutual information between the probability distributions corresponding to different states. Finally the maximum is taken over all possible measurements. In the case of two inputs the accessible information is given by [5, 6, 12, 13]

\[
I(\rho_1, \rho_2) := \max_X [I(p(\rho_1, X), p(\rho_2, X))].
\] (7)

This definition is naturally extended to more than two states. Accessible information cannot increase, but there are only a handful of cases where it is explicitly known, in particular two pure states and two spin-\( \frac{1}{2} \) states [15].

When \( I \) is unknown we can use different inequalities that relate the accessible information to other distinguishability criteria [6, 12]. The most useful of them involves a new notion, which is called entropy defect or relative entropy [5] and is given by

\[
\Delta S(\rho_1, \ldots, \rho_n) = S(\bar{\rho}) - \sum_i \pi_i S(\rho_i),
\] (8)

where

\[
S(\rho) = -\text{Tr} \rho \log \rho,
\] (9)

is the von Neumann entropy, \( \bar{\rho} = \sum_i \pi_i \rho_i \), and \( \pi_i \) is the \textit{a priori} probability distribution.

In order to measure the intrinsic difference between two of the states, we set \( \pi_1 = \pi_2 = \frac{1}{2} \), as in the definition of \( PE \). The Levitin-Holevo inequality states that the entropy defect is an upper bound of the accessible information [5]

\[
I(\rho_1, \rho_2) \leq \Delta S(\rho_1, \rho_2).
\] (10)
Moreover, the entropy defect does not increase under trace preserving completely positive maps [13],
\[ \Delta S(T\rho_1, T\rho_2) \leq \Delta S(\rho_1, \rho_2). \] (11)

It is obvious that the compliance with known information bounds is only a necessary condition that the proposed transformation must satisfy. However, it is a powerful tool, as the following examples illustrate.

Since the exact cloning of quantum states, \( \rho \rightarrow \rho \otimes \rho \) is impossible [1], let us consider the best approximate cloners. The optimal two-state cloning machine [3] is a device which has one of the two possible pure spin-\( \frac{1}{2} \) states as the input and produces an entangled pair on the output. Two identical reduced density matrices are close to the cloned state with exceptionally high fidelity, \( F = \text{Tr}[(\text{Tr}_A\rho_1^{\text{out}})|\psi_i^{\text{in}}\rangle\langle\psi_i^{\text{in}}|] > 0.985 \). Although the fidelity of the reduced density matrices is very high, it will be shown that it is impossible to separate the output according to the prescription of Eq. (2). If the two input states are parametrized as
\begin{align*}
|u\rangle &= \cos \theta |0\rangle + \sin \theta |1\rangle, \\
|v\rangle &= \sin \theta |0\rangle + \cos \theta |1\rangle,
\end{align*}
the output states are
\begin{align*}
|u'\rangle &= (a \cos \theta + c \sin \theta)|00\rangle + b(\cos \theta + \sin \theta)(|01\rangle + |10\rangle) + (c \cos \theta + a \sin \theta)|11\rangle, \\
|v'\rangle &= (c \cos \theta + a \sin \theta)|00\rangle + b(\cos \theta + \sin \theta)(|01\rangle + |10\rangle) + (a \cos \theta + c \sin \theta)|11\rangle,
\end{align*}
where \( a(\theta), b(\theta) \) and \( c(\theta) \) are given in the Appendix.

Let us suppose that there is a disentanglement procedure which is described by Eq. (2). Then we can include the cloning machine and the ‘disentangler’ into Helstrom’s decision scheme. Instead of performing the optimal measurement on the output states \( \rho_i^{\text{out}} \) (which gives the same PE as the measurement on \( \rho_i^{\text{in}} \), since they are unitarily related) we perform it on the direct product of its disentangled copies, \( \text{Tr}_A\rho_i^{\text{out}} \otimes \text{Tr}_B\rho_i^{\text{out}} \). In both cases the analytical expressions for PE can be found explicitly. They are given in the Appendix and their graphs are plotted on Fig. 1. We see that the disentanglement of the copies decreases PE and, as a result, this process is impossible. Closer look on the construction of the optimal measurement reveals that the proposed transformation is realized by an operator which is not \textit{positive} and, consequently, can represent no physical process.
Before looking at the disentanglement of more general states, it is easy to see why the universal disentanglement machine is impossible. A transformation

\[ \rho \rightarrow \text{Tr}_A \rho \otimes \text{Tr}_B \rho, \quad (14) \]

for all input states is essentially non-linear. Thus it cannot correspond to any physical process.

Now, let us again consider the states

\[ |u\rangle = a|00\rangle + b(|01\rangle + |10\rangle) + c|11\rangle, \quad (15) \]
\[ |v\rangle = c|00\rangle + b(|01\rangle + |10\rangle) + a|11\rangle, \]

where the coefficients \(a, b\) and \(c\) are arbitrary real numbers subject only to the normalization, \(a^2 + 2b^2 + c^2 = 1\). They can be parametrized by spherical coordinates as

\[ a = \sin \vartheta \cos \varphi \quad b = \sin \vartheta \sin \varphi / \sqrt{2} \quad c = \cos \vartheta. \quad (16) \]

It is possible to derive analytical expressions for PE and PE\(_d\), which are given in the Appendix. PE\((\vartheta, \varphi)\) is the actual optimal result, while the decision process which leads to PE\(_d\)\((\vartheta, \varphi)\) includes the hypothetical disentanglement procedure. Obviously, the regions of the \((\vartheta, \varphi)\) plane where PE < PE\(_d\) are forbidden, i.e., the disentanglement procedure cannot be realized. Fig. 2 shows these areas together with the line that corresponds to the disentanglement of the output of the optimal cloner, which lies in one of the forbidden domains.

Another bound can be obtained by using the entropy defect. The explicit expressions for \(\Delta S\) and \(\Delta S_d\) will not be given here, because they are too cumbersome. Proceeding exactly as in the previous case, we look for the regions where \(\Delta S \leq \Delta S_d\), i.e. where the disentanglement is impossible. These domains are presented on Fig. 3. It is instructive to compare it with Fig. 2. The boundary of the regions forbidden by PE coincides with some parts of the boundary obtained by \(\Delta S\). However, the differences are clear. This partial agreement requires further investigation and may bring some new insights on the relationship between different distinguishability criteria.

In the examples that we considered above, these criteria give identical predictions for the states with equal degree of entanglement. For pure states the degree of entanglement is measured by von Neumann entropy of either subsystem [16],

\[ E(\rho) = S(\text{Tr}_A \rho) = S(\text{Tr}_B \rho). \quad (17) \]
The eigenvalues of either of $\text{Tr}_A |u\rangle \langle u|$ or $\text{Tr}_A |v\rangle \langle v|$ are

$$\lambda_{\pm} = \frac{1}{2} \left( 1 \pm (a + c) \sqrt{1 + 2b^2 - 2ac} \right),$$

(18)

These eigenvalues (and, as a result, the degrees of the entanglement of the corresponding states) are unchanged under the transformation $\vartheta \to \pi - \vartheta, \varphi \to \pi - \varphi$. It is also a symmetry of Eqs. (A.6-A.9) and the expressions for $\Delta S$.

The existence of a general ‘no-disentanglement theorem’ remains an open issue. However, it seems that the disentanglement is impossible in general and the way to prove it is similar to the proof of the ‘no-broadcasting theorem’ [6, 17].

A more intricate question [17] is whether there is a transformation that takes an entangled state into a separable one [18] and preserves the reduced density operators,

$$\rho \rightarrow \tilde{\rho} = \sum_i w_i \rho_i^A \otimes \rho_i^B, \quad \text{Tr}_A \tilde{\rho} = \text{Tr}_A \rho, \quad \text{Tr}_B \tilde{\rho} = \text{Tr}_B \rho.$$  

(19)

This is still to be answered.

As another example, we analyze the information approach to the transformation of pure non-orthogonal states into orthogonal ones. Recently [2] a quantum circuit that does this operation for spin-$\frac{1}{2}$ states was proposed. Since non-orthogonal quantum states cannot be distinguished with certainty, such a transformation has only a limited probability of success. It consists in applying a XOR gate [4] to the pair of identically prepared particles in either of the states and measuring the spin of the second particle in the $z$ direction. If the spin is ‘down’ the transformation succeeds. In this case the outputs are $|\phi_{\text{out}}^1\rangle \otimes |\downarrow\rangle$ or $|\phi_{\text{out}}^2\rangle \otimes |\downarrow\rangle$; and we have $\langle \phi_{\text{out}}^2 | \phi_{\text{out}}^1 \rangle = 0$ even if initially there was an overlap between the states.

Leaving the explicit calculations aside, let us look at the bounds on this operation. Orthogonal states can be, in principle, distinguished unambiguously. Thus it is possible to use this procedure as part of an error-free scheme of discrimination between two non-orthogonal states [2]. The exact solution of this problem [19] is given by a POVM with two outputs which correspond to the unambiguous results, $A_1$ and $A_2$, and an output which is a failure of the measurement, $A_\tau = 1 - A_2 - A_1$. The optimal POVM is explicitly known and the probability to have a definite answer is

$$P = 1 - |\langle \phi_1 | \phi_2 \rangle|. \quad (20)$$
Since the direct products of two copies of the same states have an overlap equal to 
$|\langle \phi_1 | \phi_2 \rangle|^2$ the probability of success of any transformation of two non-orthogonal states
into orthogonal ones is bounded by

$$P(\text{success}) \leq 1 - |\langle \phi_1^{\text{in}} | \phi_2^{\text{in}} \rangle|^2.$$  \hfill (21)

Moreover, there should be a transformation that achieves this upper bound: if the state
is known, any number of its copies can be produced. As a result, a transformation
which consists of the unambiguous state identification and the corresponding preparation
achieves the bound of Eq. (21). However, the more reasonable (and physically fruitful)
transformation of [2] has this maximal efficiency too. The importance of Eq. (21) is
that generalizations of this result [20, 21] are valid for an arbitrary number of linearly
independent states. For example, if $m$ copies of three pure non-orthogonal states are used
to produce the orthogonal ones, the efficiency of the operation is bounded by [20]

$$P(\text{success}) \leq 1 - \frac{3}{m} \sum_{i=1}^{3} k_i |[\phi_1^{\text{in}} \phi_2^{\text{in}} \phi_3^{\text{in}}]|^2m / 3,$$ \hfill (22)

where coefficients $k_i$ depend on the relative orientation of the state vectors and $[uvw]$ stands for the triple product of the vectors, i.e., the determinant of their components, in
any basis.

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Appendix

Parameters of the cloning machine:

$$a = \frac{1}{\cos 2\theta} [\cos \theta (P + Q \cos 2\theta) - \sin \theta (P - Q \cos 2\theta)],$$ \hfill (A.1)

$$b = \frac{1}{\cos 2\theta} P \sin 2\theta (\cos \theta - \sin \theta),$$ \hfill (A.2)

$$c = \frac{1}{\cos 2\theta} [\cos \theta (P - Q \cos 2\theta) - \sin \theta (P + Q \cos 2\theta)],$$ \hfill (A.3)
\[ P = \frac{1}{2} \sqrt{1 + \sin 2\theta} \frac{1}{2 \sqrt{1 + \sin^2 2\theta}}. \]  
(A.4)

\[ Q = \frac{1}{2} \sqrt{1 - \sin 2\theta}. \]  
(A.5)

Probabilities of error for the output of cloning machine:

\[ PE(\theta) = \frac{1}{2} - \frac{1}{2}|\cos^2 \theta - \sin^2 \theta|, \]  
(A.6)

\[ PE_d(\theta) = \frac{1}{2} - \frac{1}{\sqrt{2}} \frac{\cos^2 2\theta(13 - 10 \cos 4\theta + \cos 8\theta + 6 \sin 2\theta - 2 \sin 6\theta)}{\sqrt{(3 - \cos 4\theta)^3}}. \]  
(A.7)

Probabilities of error in more general case:

\[ PE(\vartheta, \phi) = \frac{1}{2} - \frac{1}{2}|\cos^2 \vartheta - \cos^2 \phi \sin^2 \vartheta| \times \sqrt{\cos^2 \vartheta + \cos^2 \phi \sin^2 \vartheta + 2 \sin^2 \phi \sin^2 \vartheta + \cos \phi \sin 2\vartheta}. \]  
(A.8)

\[ PE_d(\vartheta, \phi) = \frac{1}{2} - \frac{1}{2}|\cos^2 \vartheta - \cos^2 \phi \sin^2 \vartheta| \sqrt{F(\vartheta, \phi)}, \]  
(A.9)

where \( F \) is

\[ F(\vartheta, \phi) = \cos^4 \vartheta + \cos^2 \vartheta \sin^2 \vartheta (3 - \cos 2\phi) \]
\[ + \ 4 \cos \phi \cos \vartheta \sin^2 \phi \sin^2 \vartheta + (5 - \cos 4\phi) \sin^4 \vartheta/4. \]
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CAPTIONS OF THE FIGURES

**Figure 1.** Probabilities of Error:

- PE in the optimal measurement
- PE in the optimal measurement after copying and disentanglement

**Figure 2.** Domains of the parameters for the disentanglement transformation according to the PE criterion

**Figure 3.** Domains of the parameters for the disentanglement transformation according to $\Delta S$ criterion

The forbidden domain is drawn in black.
Fig. 2
