Odd harmonious labeling of $S_n(m,r)$ graph

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Abstract. A graph labeling is an assignment of integers to vertices or edges of a graph subject to certain conditions. There are various kinds of graph labeling, one of them is an odd harmonious labeling. An odd harmonious labeling $\ell$ of a graph $G$ on $q$ edges is an injective function $\ell$ from the set of vertices of $G$ to the set $\{0,1,2,\ldots,2q-1\}$ such that the induced function $\ell^*$, where $\ell^*(uv) = \ell(u) + \ell(v)$ for every edge $uv$ of $G$, is a bijection from the set of edges of $G$ to $\{1,3,5,\ldots,2q-1\}$. A graph is said to be odd harmonious if it admits an odd harmonious labeling. A graph $S_n(m,r)$ is a graph formed from $r$ stars, each of which has $n + 1$ vertices, and every center of the star is joined to one new vertex $v_0$ by a path of length $m$. In this paper we show that the graph $S_n(m,r)$, $m \geq 2$, $1 \leq r \leq 3$, is odd harmonious.

1. Introduction

Graph theory is a branch of mathematics that many people work on it. In this paper, we only consider finite and simple graph. We follow that of Diestel [1] for most part of notation and terminology. Here, $G$ denotes a graph, $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. When $A$ is a set, then cardinality of $A$ is denoted by $|A|$. Hence, $|V(G)|$ and $|E(G)|$ are the number of vertices and the number of edges of $G$, respectively.

One of the topics in graph theory is graph labeling. Graph labeling was first introduced in the mid 1960’s. The definition of graph labeling can be found in Gallian [3]; it is an assignment of integers to the vertices or edges subject to certain conditions. More than 2800 papers on graph labelings have been published [3].

There are many types of graph labelling. In this paper we discuss about odd harmonious labeling. The definition of odd harmonious labeling can be found in [8]. A graph $G$ with $|V(G)| = p$ and $|E(G)| = q$ is said to be odd harmonious if there is an injective function $\ell: V(G) \to \{0,1,2,\ldots,2q-1\}$ such that induce a function $\ell^*: E(G) \to \{1,3,5,\ldots,2q-1\}$, defined by $\ell^*(uv) = \ell(u) + \ell(v)$, is bijective. The function $\ell$ is called an odd harmonious labelling of $G$.

Jeyanthi and Philo [4, 5] studied odd harmonious labeling of certain graphs, including subdivided shell graphs. Further, Jeyanthi, et al. [6] studied odd harmonious labeling of super subdivision graphs, and Jeyanti, et al. [7] studied odd harmonious labeling of grid graphs. Liang and Bai [8] have studied some classes of graphs, including path, cycles, complete graphs, complete $k$-partite graphs, and windmill graphs. Selvaraju, et al. [9] studied odd harmonious labeling of some path related graphs. Vaidya and Shah [10] studied odd harmonious labeling of some graphs including shadow graph and splitting graph. Recently, Febriana and Sugeng [2] studied odd harmonious labeling on squid graph and double squid graph.
Let \( n, m \) and \( r \) be positive integers. A graph \( S_n(m,r) \) is a graph formed from \( r \) stars, each of which has \( n + 1 \) vertices, and every center of the star is joined to one new vertex \( v_0 \) by a path of length \( m \). From the above discussion, we can see that the odd harmonious labeling of \( S_n(m,r) \) has not been studied. In this paper we study the odd harmonious labeling of \( S_n(m,r) \) for the case \( m \geq 2, 1 \leq r \leq 3 \), and show that \( S_n(m,r) \) is odd harmonious. A special case, when \( n = r = 1 \), graph \( S_n(m,r) \) is a path. Liang and Bai [8] say that paths are odd harmonious.

2. Main results
Our result is on odd harmonious labeling of \( S_n(m,r) \), where \( m \geq 2 \) and \( 1 \leq r \leq 3 \). We show that the graph \( S_n(m,r) \) is odd harmonious. First, we define a vertex labeling \( f \), a function from \( V(S_n(m,r)) \) to \( \{0,1,2,...,2q-1\} \), where \( q \) is the number of edges. Then, we show that this function is injective. Further, we show the induced function function \( f^*:E(G)\to\{1,3,5,...,2q-1\} \), defined by \( f^*(uv) = f(u) + f(v) \), is bijective. Since \( |E(G)| = q = \{1,3,5,...,2q-1\} \), to show that \( f^* \) is bijective, it is sufficient to show that \( f^* \) is injective. We divide it on three cases, when \( r = 1 \), \( r = 2 \), and \( r = 3 \).

Our first result is for the case \( r = 1 \).

**Theorem 1.** Let \( m, n \) be positive integer and \( r = 1 \). Graph \( S_n(m,1) \) is odd harmonious.

**Proof:**
Let graph \( S_n(m,1) \) have vertex set
\[
V(S_n(m,1)) = \{v_i^1|1 \leq i \leq n\} \cup \{v_i^1|1 \leq i \leq m\} \cup \{v_0\}
\]
and edge set
\[
E(S_n(m,1)) = \{v_i^1v_j^1|1 \leq i \leq n\} \cup \{v_i^1v_i^{1+1}|1 \leq i \leq m - 1\} \cup \{v_m^1v_0\}
\]
As in Figure 1.

![Figure 1. S_n(m,1) graph.](image)

We have, the number of vertices is
\[
p = |V(S_n(m,1))| = m + n + 1
\]
and the number of edges is
\[
q = |E(S_n(m,1))| = m + n.
\]
Define \( f \), a vertex labelling, \( f:V(S_n(m,1))\to\{0,1,2,...,2q-1\} \) as follow:
\[
f(v_i^1) = 2(i-1), \quad 1 \leq i \leq n
\]
\[
f(v_i^1) = \begin{cases} 
  i, & \text{when } i \text{ is odd, } 1 \leq i \leq m \\
  2(n - 1) + i, & \text{when } i \text{ is even, } 1 \leq i \leq m \\
  m + 2n - 1, & \text{when } m \text{ is odd}
\end{cases}
\]
\[
f(v_0) = \begin{cases} 
  m + 1, & \text{when } m \text{ is even}
\end{cases}
\]
We show will that all the values of \( f \) are different. It is easy to see that the odd values of \( f \) are different, \( f(v^1_i) = i \) when \( i \) is odd, and \( 1 \leq i \leq m \); and \( f(v_0) = m + 1 \) when \( m \) is even. The even values of \( f \) are \( f(v^1_i) = 2(i - 1) \leq 2n - 2 \) when \( 1 \leq i \leq n \); \( f(v^2_i) \), \( 2n \leq f(v^2_i) \leq 2n + m - 2 \), when \( 1 \leq i \leq m \); and \( f(v_0) = 2n + m - 1 \) when \( m \) is odd. The even values of \( f \) are different; and so the values of \( f \) are all different, \( f \) is injective.

Now we will show that the induced function \( f^* \) where \( f^*(uv) = f(u) + f(v) \) for every edge \( uv \) of \( G \), is a bijection from the set of edges \( E(G) \) to \( \{1, 3, 5, \ldots, 2q - 1\} \). Since \( |E(G)| = q \) and \( \{|1, 3, 5, \ldots, 2q - 1\} = q \), it is sufficient to show that \( f^* \) is injective. By the definition of vertex labeling, the induced labeling of edges is as follows.

\[
\begin{align*}
f^*(v^1_i v^1_j) &= f(v^1_i) + f(v^1_j) = 2i - 1, \quad 1 \leq i \leq n, \\
f^*(v^1_i v^1_{i+1}) &= f(v^1_i) + f(v^1_{i+1}) = 2n + 2i - 1, \quad 1 \leq i \leq m - 1, \\
f^*(v^1_i v_0) &= f(v^1_i) + f(v_0) = 2m + 2n - 1.
\end{align*}
\]

All values of \( f^* (uv) \), \( uv \in E(G) \), are odd. For \( 1 \leq i \leq n \), all values of \( f^*(v^1_i v^1_j) \) are different and \( 1 \leq f^*(v^1_i v^1_j) \leq 2n - 1 \); for \( 1 \leq i \leq m - 1 \), all values of \( f^*(v^1_i v^1_{i+1}) \) are different and \( 2n + 1 \leq f^*(v^1_i v^1_{i+1}) \leq 2m + 2n - 3 \); and \( f^*(v^1_i v_0) = 2m + 2n - 1 = 2q - 1 \). All values of \( f^* (uv) \) are odd and different, \( f^* \) is injective. This completes the proof that graph \( S_n(m, 1) \) is odd harmonious.

Our next result is for the case \( r = 2 \).

**Theorem 2.** Let \( m, n \) be positive integer and \( r = 2 \). Graph \( S_n(m, 2) \) is odd harmonious.

**Proof:**

Let graph \( S_n(m, 2) \) have vertex set

\[
V(S_n(m, 2)) = \{v^1_i | 1 \leq i \leq n\} \cup \{v^l_i | 1 \leq i \leq m, l = 1, 2\} \cup \{v_0\} \cup \{v^2_m, v^2_i | 1 \leq i \leq n\}
\]

and edge set

\[
E(S_n(m, 2)) = \{v^1_i v^1_j | 1 \leq i \leq n\} \cup \{v^1_i v^1_{i+1} | 1 \leq i \leq m - 1, l = 1, 2\} \cup \{v^2_m v^2_i | 1 \leq i \leq n\}
\]

as in Figure 2.

![Figure 2. S_n(m, 2) graph.](image)

We have, the number of vertices is

\[
p = |V(S_n(m, 2))| = 2m + 2n + 1
\]

and the number of edges is

\[
q = |E(S_n(m, 2))| = 2m + 2n.
\]

Define \( f \), a vertex labelling, \( f : V(S_n(m, 2)) \rightarrow \{0, 1, 2, \ldots, 2q - 1\} \) as follow:
Let graph $G = (V, E)$ be a graph with $V = \{v_0\} \cup \{v_i | 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E = E(G)$. Let $m, n \in \mathbb{Z}^+$ with $m \geq 2, n \geq 2, r \in \mathbb{Z}$. 

Theorem 3. Let $m, n$ be positive integer and $m \geq 2, n \geq 2, r = 3$. Graph $S_n(m, 3)$ is odd harmonious.

Proof:
Let graph $S_n(m, 3)$ have a vertex set 

$V(S_n(m, 3)) = \{v_{i,j}^1 | 1 \leq i \leq n, j = 1, 3\} \cup \{v_{i,j}^2 | 1 \leq i \leq m, 1 \leq j \leq 3\} \cup \{v_0\} \cup \{v_{m,i}^2 | 1 \leq i \leq n\}$

and edge set 

$E(S_n(m, 3)) = \{v_{i,j}^1 v_{i,j+1}^1 | 1 \leq i \leq n, j = 1, 3\} \cup \{v_{i,j}^2 v_{i,j+1}^2 | 1 \leq j \leq 3, 1 \leq i \leq m - 1\} \cup \{v_{0} v_{0}^2 | j = 1, 3\} \cup \{v_{0} v_{i}^1 | 1 \leq i \leq n\}$

as in Figure 3.
We have, the number of vertices is 
\[ p = |V(S_n(m, 3))| = 3m + 3n + 1 \]
and the number of edges is 
\[ q = |E(S_n(m, 3))| = 3m + 3n. \]

Define \( f \), a vertex labelling, \( f : V(S_n(m, 2)) \to \{0, 1, 2, \ldots, 2q – 1\} \) as follow:

\[
\begin{align*}
  f(v^1_0) &= 2(m + 1) + i, \quad \text{when } i \text{ is odd, } 1 \leq i \leq m - 1 \\
  f(v^2_0) &= 3m + 2(n + 1), \quad \text{when } i = m \text{ is odd} \\
  f(v^3_0) &= 4m + 4n + 2i - 6, \quad \text{when } i \text{ is odd, } 1 \leq i \leq n, \\
  f(v^1_i, j) &= \begin{cases} 
  2m + 4n + 2i - 4, & \text{when } m \text{ is even, } 2 \leq i \leq n \\
  2m + 4i + 2n + 2i - 6, & \text{when } m \text{ is odd, } 1 \leq i \leq n \\
  4m + 4n + 2i - 6, & \text{when } i \text{ is even, } 1 \leq i \leq m, \\
  2m + 4n + i - 2, & \text{when } i \text{ is even, } m \text{ is odd, } 1 \leq i \leq m \\
  2m + 4i + 2, & \text{when } i \text{ is odd, } 1 \leq i \leq m \\
  2m + 4n + i + 1, & \text{when } i \text{ is odd, } 1 \leq i \leq m - 1 \\
  m + 1 + i, & \text{when } i \text{ is even, } 1 \leq i \leq m \\
  m + 2n - 1, & \text{when } m \text{ is odd} \\
  m + i + 1, & \text{when } i \equiv m \pmod{2}, 1 \leq i \leq m - 1 \\
  2(n - 1) + i, & \text{when } i \text{ is even, } 1 \leq i \leq m \\
  i, & \text{when } i \text{ is odd, } 1 \leq i \leq m \\
\end{cases}
\end{align*}
\]

We will show that all the values of \( f \) are different. The odd values of \( f \) are:

\[
\begin{align*}
  f(v^1_0) &= i \leq m, \quad \text{when } i \text{ is odd, } 1 \leq i \leq m; \\
  f(v^2_0) &= m + 1, \quad \text{when } m \text{ is even;} \\
  f(v^3_0), m + 2 \leq f(v^2_i) = m + i + 1 \leq 2m + 1, & \text{when } i \equiv m \pmod{2} \text{ and } 1 \leq i \leq m. \\
  f(v^3_i), 2m + 3 \leq f(v^3_i) = \begin{cases} 
  2(m + 1) + i, & \text{when } i \text{ is odd, } 1 \leq i \leq m - 1 \\
  3m + 2(n + 1), & \text{when } i = m \text{ is odd} \\
\end{cases} \\
\end{align*}
\]
We can see that the odd values of \( f \) are different. The even values of \( f \) are:

\[
f(v_i^1) = 2(n - 1) + i \leq m + 2n - 2, \text{ when } i \text{ is even, } 1 \leq i \leq m;
\]

\[
f(v_0^1) = m + 2n - 1, \text{ when } m \text{ is odd};
\]

\[
f(v_i^2), \text{ when } i \equiv m \text{ (mod2) and } 1 \leq i \leq m;
\]

\[
f(v_{m+1}^2). \text{ when } m + 2n \leq f(v_{m+1}^2) = 2(m + n + i - 1) \leq 2m + 4n - 2, \ 1 \leq i \leq n.
\]

We can see, by the definition, all even values of \( f(v_i^1) \) and \( f(v_{m+1}^2) \) are different. When \( m \) is even,

\[
f(v_i^2), \text{ when } i \equiv m \leq 3m + 4n, \text{ when } i \text{ is even, } 1 \leq i \leq m,
\]

\[
f(v_{m+1}^2), \text{ when } m \text{ is even, } i = 1,
\]

\[
f(v_{m+1}^2), \text{ when } m \text{ is odd,}
\]

\[
f(v_i^2), \text{ when } i \equiv m \leq 3m + 4n - 3, \ 1 \leq i \leq m - 1, m \geq 3,
\]

\[
f(v_{m+1}^2), \text{ when } 4m + 4n - 4 \leq f(v_{m+1}^2) = 4m + 4n + 2i - 6 \leq 4m + 6n, \ 1 \leq i \leq n.
\]

Thus we can find that all even values of \( f \) are different. Hence, the values of \( f \) are all different, \( f \) is injective.

It is remain to show that the induced function \( f^* \) where \( f^*(uv) = f(u) + f(v) \) for every \( uv \in E(G) \), is a bijection from \( E(G) \) to \( \{1, 3, 5, \ldots, 2q - 1\} \). Again, since \( |E(G)| = q \) and \( |\{1, 3, 5, \ldots, 2q - 1\}| = q \), it is sufficient to show that \( f^* \) is injective.

Note that \( S_n(m, 3) \) contains \( S_n(m, 2) \), and we label all the vertices of \( S_n(m, 2) \), subgraph of \( S_n(m, 3) \), as when we label them in Theorem 2, and \( S_n(m, 2) \) is odd harmonious. To show that \( S_n(m, 3) \) is odd harmonious, it is sufficient to show that all values of the induced function \( f^*(uv) = f(u) + f(v) \), \( uv \in (E(S_n(m, 3)) - E(S_n(m, 2))) \) are different, odd, and \( 4m + 4n + 1 \leq f^*(uv) \leq 6m + 6n - 1 \). By the definition of vertex labeling, the induced labeling of edges is as follows.

\[
\begin{align*}
 f^*(v_m^3v_0) &= f(v_m^3) + f(v_0) = 4m + 4n + 1, \\
 f^*(v_i^3v_{i+1}^3) &= f(v_i^3) + f(v_{i+1}^3) = \begin{cases} 
 4m + 4n + 2i + 1, & 1 \leq i \leq m - 2, \\
 6m + 6n - 1, & i = m - 1.
\end{cases}
\end{align*}
\]

The value of \( f^*(v_1^3v_{i+1}^3) \):

\[a. \text{ When } m \text{ is odd,}\]

\[
 f^*(v_1^3v_{i+1}^3) = f(v_1^3) + f(v_{i+1}^3) = \begin{cases} 
 4m + 4n + 2i + 1, & 1 \leq i \leq m - 2, \\
 6m + 6n - 1, & i = m - 1.
\end{cases}
\]

\[b. \text{ When } m \text{ is even,}\]

\[
 f^*(v_1^3v_{i+1}^3) = f(v_1^3) + f(v_{i+1}^3) = 4m + 4n + 2i + 3, \ 1 \leq i \leq m - 1.
\]

The value of \( f^*(v_i^3v_{i+1}^3) \):

\[a. \text{ When } m \text{ is odd,}\]

\[
 f^*(v_i^3v_{i+1}^3) = f(v_i^3) + f(v_{i+1}^3) = 6m + 4n + 2i - 3, \ 1 \leq i \leq n.
\]

\[b. \text{ When } m \text{ is even,}\]

\[
 f^*(v_i^3v_{i+1}^3) = f(v_i^3) + f(v_{i+1}^3) = \begin{cases} 
 6m + 4n + 2i - 1, & 2 \leq i \leq n.
\end{cases}
\]

All values of \( f^*(uv) \), \( uv \in (E(S_n(m, 3)) - E(S_n(m, 2))) \), are odd, different, and \( 4m + 4n + 1 \leq f^*(uv) \leq 6m + 6n - 1 \). This completes the proof that all values of \( f^*(uv) \) are odd and different, \( f^* \) is injective, and hence graph \( S_n(m, 3) \) is odd harmonious. Then, the odd harmonious labeling to graph \( S_n(m, 3) \) for \( n \) positive integers applies to \( m \geq 2 \). It is because when \( m = 1 \) or when graph \( S_n(1, 3) \) there is the same vertex label so that is not an injective function. Therefore, the graph \( S_n(1, 3) \) for \( n \) positive integers cannot be labeled with odd harmonious labeling. So it is proven that odd harmonious labeling to graph \( S_n(m, 3) \) for \( n \) positive integers applies when \( m \geq 2 \). This completes the proof that graph \( S_n(m, 3) \) is odd harmonious when \( m \geq 2 \).
For example, graphs $S_6(7,1)$, $S_7(4,2)$, and $S_6(5,3)$, are shown in Figure 4, Figure 5, and Figure 6, respectively.

**Figure 4.** Odd harmonious labeling of $S_6(7,1)$ graph.

**Figure 5.** Odd harmonious labeling to $S_7(4,2)$ graph.
Figure 6. Odd harmonious labeling to $S_6(5,3)$ graph.

3. Conclusion

We have shown that graph $S_n(m,r)$ for $1 \leq r \leq 3$, is harmonious. When $n = r = 1$, $S_n(m,r)$ is a path. So, this result is more general than the result which says that paths are harmonious, that can be found in Liang and Bai [8]. There is still a problem for the general case. For next study, one can prove (or disprove for some cases) that $S_n(m,r)$ is odd harmonious, for $r > 3$.

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