Acoustic black holes from supercurrent tunneling

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Abstract

We present a version of acoustic black holes by using the principle of the Josephson effect. We find that in the case two superconductors $A$ and $B$ are separated by an insulating barrier, an acoustic black hole may be created in the middle region between the two superconductors. We discuss in detail how to describe an acoustic black hole in the Josephson junction and write the metric in the language of the superconducting electronics. Our final results infer that for big enough tunneling current and thickness of the junction, experimental verification of the Hawking temperature could be possible.
1 Introduction

Analog models of gravity have recently received great interests since these models may provide possible experimental verifications of the evaporation of black holes. Unruh was the first to propose the idea of using hydrodynamic flows as analogues to mimic some properties of black hole physics [1] (for reviews see [2] and references there in). Any moving fluid with speed exceeding the local sound velocity through a spherical surface could in principle form an acoustic black hole. For acoustic black holes, it is sound waves instead of light waves that cannot escape from the horizon where the horizon locates on the boundary between subsonic and supersonic flow regions. In particular, superfluid helium II [3], atomic Bose-Einstein condensates [4–7], one-dimensional Fermi-degenerate noninteracting gas [8] were proposed to create an acoustic black hole geometry in the laboratory.

Because the Hawking temperature depends on the gradients of the flow speed at the horizon, detecting thermal phonons radiating from the horizons is very difficult. In fact the Hawking temperature calculated from models in Bose-Einstein condensates so far is very low (∼ nano Kelvin). Up to now, only few experimenters have claimed that acoustic black holes are able to be observed. The very recent experimental realization of acoustic black hole reported was conducted in a Bose-Einstein condensate [9]. Considering that the Bose-Einstein condensate systems have a very strict requirement on the environment temperature (for example 170-nano Kelvin for the gas of Rubidium atoms), the authors in [10] proposed that acoustic black holes may be realized in superconducting materials which have much higher critical temperatures (∼ 100 Kelvin) and a relativistic version of acoustic black holes was presented there (see also [11–16] for further reading). It was observed that an acoustic black hole may form near the spiral vortex core in a type II superconductor [10]. But the experimental detecting of
such behavior in superconductors is so far very difficult, because the superconducting phase could be destroyed when the speed of the current carriers exceeds the Fermi-velocity. The purpose of this paper is to investigate the possibility of creating acoustic black holes in the supercurrent tunneling, in particular the Josephson junctions, because there are some very sensitive detectors based on the principle of the Josephson effect, for example, the superconducting quantum interference device (SQUID).

2 Brief reviews on the Josephson effect

In the theory of superconductivity, the Ginzburg-Landau equation is a phenomenologically based theory, which has been proven to be very successful partly because it can describe the mixed states in type II superconductors. It was later shown by Gorkov that this equation can be derived from the Bardeen-Cooper-Schrieffer (BCS) theory in the case of thermodynamic equilibrium and for temperatures close to the transition temperature [17]. The generalization of the Ginzburg-Landau equation, which includes the time dependent term, allows the superconductor to relax in its equilibrium state. Now we consider two superconductors, A and B, separated by an insulating barrier. If the barrier is thick enough so that the superconductors are isolated from each other, the time-dependent Ginzburg-Landau equation for each side is

\[ i\hbar \partial_t \psi_A = H_A \psi_A + \kappa \psi_B, \]  

\[ i\hbar \partial_t \psi_B = H_B \psi_B + \kappa \psi_A, \]

where

\[ H_A = \left[ -\frac{\hbar^2}{2m_A} \left( \nabla + \frac{2\text{i}e}{\hbar} \vec{A} \right)^2 + a_A(T) + b_A(T)|\psi_A|^2 \right], \]  

\[ H_B = \left[ -\frac{\hbar^2}{2m_B} \left( \nabla + \frac{2\text{i}e}{\hbar} \vec{A} \right)^2 + a_B(T) + b_B(T)|\psi_B|^2 \right]. \]

\( m_{A,B} \) is the mass of each cooper pair, \( a_{A,B}(T) \) and \( b_{A,B}(T) \) are two parameters that depend on the temperature, \( \kappa \) is the coupling constant for the wave-functions across the barrier. It is worth noting that \( a_{A,B}(T) \) and \( b_{A,B}(T) \) are phenomenological parameters that can be fixed in experiments. Without the coupling \( \kappa \), no mass term is generated in our paper. We will set \( m_A = m_B \) in the following. Actually, equations (2.1) and (2.2) are used to describe the Josephson effect when a voltage \( V \) is applied between the two superconductors and then one can replace the Hamiltonian with \( H_A = eV \) and \( H_B = -eV \). We use the coupled time-dependent Ginzburg-Landau equations in order to obtain the effective acoustic metric. Note that the similar computation was done for a two-species Bose-Einstein condensate by
using two-component time-dependent Gross-Pitaevskii equations \[7\]. One may regard \( b_A \) and \( b_B \) as self-interactions \( U_{AA} \) and \( U_{BB} \) in \[7\]. But different from the two component Bose-Einstein condensate, here the two superconductors \( A \) and \( B \) are separated by a thin film and the interactions between the two superconductors are very weak. So the interactions \( b_{AB} \) and \( b_{BA} \) can be neglected. After obtaining the metric, we will ask what we can learn from the Josephson effect for our understanding on the acoustic black hole physics.

The two macroscopic wave function can be written in the form

\[
\psi_A = \sqrt{\rho_A} e^{i\theta_A},
\]

\[
\psi_B = \sqrt{\rho_B} e^{i\theta_B}.
\]

The equations of motion then becomes

\[
\partial_t \rho_i + \frac{\hbar}{m_i} \nabla \cdot (\rho_i\vec{v}_i) + \frac{\kappa}{\hbar} \sqrt{\rho_i\rho_j} \sin(\theta_i - \theta_j) = 0,
\]

\[
\hbar \partial_t \theta_i = \frac{\hbar^2}{2m_i} \nabla^2 \frac{\sqrt{\rho_i}}{\sqrt{\rho_i}} - \frac{m_i}{2} \vec{v}_i^2 - a_i(T) - b_i(T)\rho_i - \kappa \sqrt{\rho_i \rho_j} \cos(\theta_j - \theta_i),
\]

where \( \vec{v}_i = \frac{\hbar \nabla \theta_i}{m_i} - \frac{2e}{m_i} \vec{A} \) and \( i, j = A, B \) \((i \neq j)\). The first term in the right hand of (2.8) corresponds to the quantum potential. The above two equations are completely equivalent to the hydrodynamic equations for irrotational and inviscid fluid apart from the quantum potential and the \( \kappa \) term. In the long-wavelength approximation, the contribution coming from the linearization of the quantum potential can be neglected. Since the current densities \( \rho_i \) in the superconductor \( A \) is not much different from that in the superconductor \( B \), the quantum potential can be neglected in our derivation of the acoustic metric. The physics of the Josephson current is mainly described by the \( \kappa \) term in equations (2.7) and (2.8) \[19\]. The Josephson relations for the pair density can be obtained from (2.7)

\[
\partial_t (\rho_A - \rho_B) + \frac{\hbar}{m_A} \nabla \cdot (\rho_A \vec{v}_A - \rho_B \vec{v}_B) + \frac{2\kappa}{\hbar} \sqrt{\rho_A \rho_B} \sin(\theta_A - \theta_B) = 0.
\]

It can be reduced to the standard Josephson relation when the second term is dropped out, that is to say,

\[
j = 2e \partial_t (\rho_A - \rho_B) = \frac{4e\kappa}{\hbar} \sqrt{\rho_A \rho_B} \sin(\theta_B - \theta_A) = j_c \sin \theta,
\]

where

\[
j_c = \frac{4e\kappa}{\hbar} \sqrt{\rho_A \rho_B}, \quad \theta = \theta_B - \theta_A.
\]

Actually, only in the Josephson junctions, the speed of electrons cannot be neglected. The kinetic energy of one cooper pair tunneled from one side to the other side changes little. That is why one can
replace the Hamiltonian with $H_A = eV$ and $H_B = -eV$ and ignore the kinetic terms of the Hamiltonian in (2.8). Following this procedure, we obtain the second Josephson equation

$$\frac{\partial (\theta_B - \theta_A)}{\partial t} = \frac{2eV}{\hbar}. \quad (2.12)$$

In the insulating barrier, the density of the current carriers $\rho_j$ is much lower than the density of current carriers in the superconductors $\rho_s$. For those insulating films who share the same cross-sectional area with the superconductors on each side, they share the same current. That is to say

$$\rho_j v_j eS_1 = \rho_s v_s eS_2, \quad (2.13)$$

where $S_1 = S_2 = S$ is the cross-sectional area, $v_j$ denotes the speed of the current carriers in the film (junction), and $v_s$ the speed in the superconductors. Therefore, the speed of the current carriers in the film (junction) must be much bigger than their speed in the bulk:

$$v_j \gg v_s. \quad (2.14)$$

On the other hand, if the two superconductors, $A$ and $B$, are not separated by an insulating barrier, but by the same superconducting material with a cross-sectional area much smaller than the cross-sectional area of each side (i.e. $S_1 \ll S_2$), this is another kind of Josephson junctions. In this case, the density of current is same everywhere in the around circle (i.e. $\rho_j = \rho_s$). Thus, we still have $v_j \gg v_s$. These can justify why we can drop the second term in (2.9) and kinetic energy terms in (2.8) when we derive the Josephson relations and how it is possible for the creation of an acoustic black hole in the Josephson junction because the speed of the current carriers can be so fast that it may exceed the local speed of sound. In this paper, we will construct an acoustic black hole from the supercurrent tunneling by considering the linearized perturbations of equations (2.7) and (2.8).

### 3 The acoustic black hole metric

Now let us consider a fixed background $(\rho_{i0}, \theta_{i0})$ with small perturbations $\rho_i = \rho_{i0} + \rho_{i1}$ and $\theta = \theta_{i0} + \theta_{i1}$ ($i = A, B$). The leading order equations for $(\rho_{i0}, \theta_{i0})$ can be written as

$$\partial_t \rho_{A0} + \frac{\hbar}{m_A} \nabla \cdot (\rho_{A0} \vec{v}_{A0}) + \frac{\kappa}{\hbar} \sqrt{\rho_{A0} \rho_{B0}} \sin(\theta_{A0} - \theta_{B0}) = 0, \quad (3.1)$$

$$\hbar \partial_t \theta_{A0} = -\frac{m_A}{2} \vec{v}^2_{A0} - a_A(T) - b_A(T) \rho_{A0} - \kappa \sqrt{\rho_{B0} \rho_{A0}} \cos(\theta_{B0} - \theta_{A0}). \quad (3.2)$$

$$\partial_t \rho_{B0} + \frac{\hbar}{m_B} \nabla \cdot (\rho_{B0} \vec{v}_{B0}) + \frac{\kappa}{\hbar} \sqrt{\rho_{B0} \rho_{A0}} \sin(\theta_{B0} - \theta_{A0}) = 0, \quad (3.3)$$

$$\hbar \partial_t \theta_{B0} = -\frac{m_B}{2} \vec{v}^2_{B0} - a_B(T) - b_B(T) \rho_{B0} - \kappa \sqrt{\rho_{A0} \rho_{B0}} \cos(\theta_{B0} - \theta_{A0}). \quad (3.4)$$
Linearizing the equations (2.1) and (2.2), we obtain the two coupled equations for the perturbation of the phases

\[
\frac{\partial \theta_{A1}}{\partial t} + \vec{v}_{A0} \cdot \nabla \theta_{A1} = - \frac{b_A \rho_{A1}}{\hbar} + \frac{\kappa}{2 \hbar} \frac{\rho_{A1} \sqrt{\rho_{B0}}}{\rho_{A0}^{3/2}} - \frac{\kappa}{2 \hbar} \frac{\rho_{B1}}{\sqrt{\rho_{A0} \rho_{B0}}},
\]

(3.5)

\[
\frac{\partial \theta_{B1}}{\partial t} + \vec{v}_{B0} \cdot \nabla \theta_{B1} = - \frac{b_B \rho_{B1}}{\hbar} + \frac{\kappa}{2 \hbar} \frac{\rho_{B1} \sqrt{\rho_{A0}}}{\rho_{B0}^{3/2}} - \frac{\kappa}{2 \hbar} \frac{\rho_{A1}}{\sqrt{\rho_{A0} \rho_{B0}}},
\]

(3.6)

The coupled equations for the density perturbations are given by

\[
\partial_t \rho_{A1} + \nabla \cdot \left( \frac{\hbar}{m_A} \rho_{A0} \nabla \theta_{A1} + \rho_{A1} \vec{v}_{A0} \right) = \frac{2\kappa}{\hbar} \sqrt{\rho_{A0} \rho_{B0}} (\theta_{B1} - \theta_{A1}),
\]

(3.7)

\[
\partial_t \rho_{B1} + \nabla \cdot \left( \frac{\hbar}{m_B} \rho_{B0} \nabla \theta_{B1} + \rho_{B1} \vec{v}_{B0} \right) = \frac{2\kappa}{\hbar} \sqrt{\rho_{A0} \rho_{B0}} (\theta_{A1} - \theta_{B1}).
\]

(3.8)

The above equations governing the perturbation of phases and density are very hard to decouple. We need impose some constraints on the background parameters for the purpose of deriving the acoustic metric. The superconductors $A$ and $B$ can be the same so that $\rho_{A0} = \rho_{B0} = \rho_0$, $b_A = b_B = b$, $m_A = m_B$ and the background phases also can be set to be equal ($\theta_{A0} = \theta_{B0}$). This implies that without perturbations there are no currents crossing the junction and background velocities $\vec{v}_{A0} = \vec{v}_{B0} = \vec{v}_0$. But when the phase $\theta$ is fluctuated, the supercurrent tunneling happens in the junction and the background velocity $v_0$ can be regarded as the function of the space variables $x_i$. In the region where $v_0$ exceeds the “sound velocity” $c_s$, an acoustic black hole forms. In the following, we will see how this can happen.

The coupled equations for phase and density perturbation can be written as

\[
\partial_t (\theta_{A1} - \theta_{B1}) + \vec{v}_0 \cdot \nabla (\theta_{A1} - \theta_{B1}) = - \frac{b}{\hbar} (\rho_{A1} - \rho_{B1}) + \frac{\kappa}{\hbar} \left( \frac{\rho_{A1} - \rho_{B1}}{\rho_0} \right),
\]

(3.9)

\[
\partial_t (\rho_{A1} - \rho_{B1}) + \nabla \left( \frac{\hbar}{m} \rho_0 \nabla (\theta_{A1} - \theta_{B1}) + (\rho_{A1} - \rho_{B1}) \vec{v}_0 \right) = - \frac{4\kappa \rho_0}{\hbar} (\theta_{A1} - \theta_{B1}).
\]

(3.10)

It is convenient to introduce the notations

\[
\theta_1 = \theta_{A1} - \theta_{B1}, \quad \rho_1 = \rho_{A1} - \rho_{B1},
\]

(3.11)

and

\[
\chi = \frac{b \rho_0 - \kappa}{\hbar \rho_0}.
\]

(3.12)

After combining (3.9) and (3.10) as a single equation, we have the wave equation for $\theta_1$

\[
- \partial_t \left[ \frac{1}{\chi} (\partial_t \theta_1 + \vec{v}_0 \cdot \nabla \theta_1) \right] + \nabla \cdot \left[ \frac{\hbar \rho_0}{m} \nabla \theta_1 - (\partial_t \theta_1 + \vec{v}_0 \cdot \nabla \theta_1) \frac{\vec{v}_0}{\chi} \right] = - \frac{4\kappa \rho_0}{\hbar} \theta_1.
\]

(3.13)
The above equation is comparable with a massive Klein-Gordon equation in curved space-time
\[
\frac{1}{\sqrt{-g}} \partial \mu (\sqrt{-g} g^{\mu \nu} \partial \nu \theta_1) - \tilde{m}^2 \theta_1 = 0.
\] (3.14)

We can therefore read off the inverse acoustic metric
\[
g^{\mu \nu} \equiv \frac{\sqrt{m^3}}{\sqrt{\hbar^2 \rho_0^3 \chi}} \left[ \begin{array}{cc} -1 & -v_0^j \\
\cdots & \cdots \\
v_i^i & \left( c_s^2 \delta^{ij} - v_0^i v_0^j \right) \end{array} \right],
\] (3.15)

and
\[
\tilde{m}^2 = -\frac{4\kappa \rho_0}{\hbar} \sqrt{\frac{m^3 \chi}{\hbar^2 \rho_0^3}},
\] (3.16)

where the local speed of sound is defined as
\[
c_s^2 = \frac{\hbar \rho_0}{m \chi} = \frac{b \rho_0 - \kappa}{m}.
\] (3.17)

Note that in absence of the coupling constant \(\kappa\), the local speed of sound has the form
\[
c_s = \frac{\hbar}{\sqrt{2m \xi(T)}},
\] (3.18)

where \(\xi(T)\) is the Ginzburg-Landau coherence length \(\xi(T) = \frac{\hbar}{\sqrt{2m |a(T)|}}\).

By inverting (3.15), we determine the metric
\[
g_{\mu \nu} \equiv \left( \frac{\hbar \rho_0}{m c_s} \right) \left[ \begin{array}{cc} -(c_s^2 - v_0^2) & -v_0^j \\
\cdots & \cdots \\
v_i^i & \delta^{ij} \end{array} \right].
\] (3.19)

In the presence of an external magnetic field, we will show in the following that the structure of the acoustic black hole may have “draining bathtub” form. The general acoustic metric is given by
\[
ds^2 = \left( \frac{\hbar \rho_0}{m c_s} \right) \left[ -(c_s^2 - v_0^2) dt^2 - 2v_0^j \cdot d\vec{r} dt + d\vec{r} \cdot d\vec{r} \right] + \left( v_0^i \cdot d\vec{r} \right)^2 + \left( v_0^0 \cdot d\vec{r} \right)^2,
\] (3.20)

where the corresponding horizon locates at \(c_s = v_0\).

In the cylindrical coordinate \((r, \theta, z)\), suppose that the superconducting current is along the \(z\)-direction and the magnetic field \(A\) is along the \(\theta\)-direction. In this case, the background fluid flow will be bended by the magnetic field. It is convenient to set \(v_z(r, \theta, z) \neq 0\) and \(v_\theta(r, \theta, z) \neq 0\) and let \(v_r = 0\). The metric (3.20) becomes
\[
ds^2 = \left( \frac{\hbar \rho_0}{m c_s} \right) \left[ -(c_s^2 - v_0^2) dt^2 - 2v_0^j \cdot d\vec{r} dt + d\vec{r} \cdot d\vec{r} \right] + \left( v_0^i \cdot d\vec{r} \right)^2 + \left( v_0^0 \cdot d\vec{r} \right)^2
\] (3.21)
where \( v_0^2 = v_z^2 + v_\theta^2 \). If we make the coordinate transformations

\[
dt = d\tau - \frac{v_z}{c^2_s - v_z^2} dz, \tag{3.22}
\]
\[
d\theta = d\vartheta - \frac{v_\theta v_z}{r(c^2_s - v_z^2)} dz, \tag{3.23}
\]

then the line-elements of the metric can be written as

\[
ds^2 = \left( \frac{\hbar \rho_0}{mc_s} \right) \left\{- [c_s^2 - (v_z^2 + v_\theta^2)] d\tau^2 - 2rv_\theta d\tau d\vartheta + \frac{c_s^2}{c_s^2 - v_z^2} dz^2 + r^2 d\vartheta^2 + dr^2 \right\}. \tag{3.24}
\]

The formation of an acoustic black hole requires \( v_i^0 c_s > 1 \) in some regions. In [10], the authors pointed out that it would be very difficult to realize acoustic black holes by using type I and type II superconductors. Especially, for type I superconductors, when the speed of superconducting electrons is equivalent to the “sound velocity”, say \( v_0 = c_s \), the superconducting phase is broken and return to the normal state. It was expected to form an acoustic black hole in the region of the spiral vortex core in a type II superconductor. The calculation shows that \( \xi < r < \sqrt{2}\xi \), electron velocity may exceed the sound velocity [10], where \( r \) denotes the distance from the vortex core and \( \xi \) the coherence length. But experimental verification of superconducting electron speed in the bulk of a superconductor is not easy. From (2.11) (i.e. \( j_c = 2e v_i^0 \rho_0 \sim 4e\kappa \rho_0 / \hbar \)) and (3.17), we know that \( \frac{v_i^0}{c_s} > 1 \) means

\[
\frac{v_i^0}{c_s} = \frac{2\kappa}{\hbar} \sqrt{\frac{m}{\hbar \rho_0 - \kappa}} > 1. \tag{3.25}
\]

As an example, let us consider the material PbCu at \( T_c = 7.2K \) with coherence length \( \xi = 80nm \) [19]. Then (3.25) requires the coupling constant \( \kappa > 1.788 \times 10^{-18} J \), which is possible in experiments. The Josephson junctions and related instruments, such as superconducting quantum interference device (SQUID), may open a door to build an acoustic black hole directly.

### 4 A microscopic picture

From (3.17), we know that the local speed of sound in the junction could be smaller than that in the bulk of the superconductor for a positive-valued coupling constant \( \kappa \). In the above derivation, we have used the Ginzburg-Landau theory, which is very useful in describing qualitative and macroscopic behaviors. In order to have a clear picture for the formation of acoustic black holes, now we present a microscopic description by using the BCS theory. In fact, Gorkov in 1959 proved that the Ginzburg-Landau theory can be derived from full the BCS theory in a suitable limit [17]. The microscopic model
can be constructed by using the tunneling Hamiltonian
\begin{equation}
H = H_R + H_L + H_T,
\end{equation}
\begin{equation}
H_T = \sum_{k\rho\sigma} \left( T_{k\rho} C_{k\rho\sigma}^\dagger C_{k\rho\sigma} + \text{h.c.} \right).
\end{equation}

The Hamiltonian is identical with equations (2.1) and (2.2). $H_R$ is the Hamiltonian for particles on the right side of the tunneling junction. Similarly, $H_L$ has all the physics for particles on the left side of the junction. The tunneling is caused by the term $H_T$ and $T_{\rho\sigma}$ denotes the tunneling matrix that can transfer particles through an insulating junction. The derivation of the tunneling current led to two terms: the single-particle terms and the Josephson term. We only consider the Josephson term here.

\begin{equation}
\text{i} \hbar \partial_t \psi_i = H_i \psi_i + \kappa \psi_j,
\end{equation}
where
\begin{equation}
H_i = -\frac{\hbar^2}{2m_i} \left( \nabla + \frac{2ie}{\hbar} \vec{A} \right)^2 + \frac{1}{\eta} \left[ \frac{T_c - T}{T_c} - \frac{1}{\rho_0} |\psi_i|^2 \right],
\end{equation}
\begin{equation}
\eta = \frac{7\zeta(3)}{6(\pi T_c)^2} \varepsilon_F,
\end{equation}
where $\varepsilon_F$ is the Fermi energy. In this sense, the formation of an acoustic black hole requires
\begin{equation}
\frac{v_0}{c_s} = \frac{2\kappa}{\hbar} \sqrt{\frac{m\eta}{1 - \eta\kappa}} > 1.
\end{equation}

As pointed out in section 2, in the insulating barrier, the density of the current carriers is much lower than that in the superconductors, but the speed of the current carriers in the film should be much bigger than their speed in the bulk. Therefore, there may exist a region in where $v_0 < c_s$ that continuously connected to the region $v_0 > c_s$.

5 An acoustic black hole in the Josephson junction

Let us consider a weak link tunnel junction with a magnetic field $B_x(y)\vec{i}$ applied along the x-direction, as shown in Fig.1. The junction is of thickness $2a$ normal to the z-axis with cross-sectional dimensions $d$ and $w$ along $y$ and $x$, respectively. We assume that the external magnetic field is larger than the field produced by the currents. The applied field is derived from the vector potential $\vec{A} = B_x(y)\vec{k}$. In the barrier film the material is normal and the magnetic field is constant-valued $B_x(y) = B_0$, but the magnetic field decays exponentially into the superconductors on either side of the junction.
From (3.21), we know that the magnetic field can change the direction of the fluid flow. The formation of an acoustic black hole should satisfy the condition \( v_i^0 > c_s \). Note that when we derive the acoustic metric, we consider the perturbations around a fixed background \((\rho_{0i}, \theta_{0i})\) without the fluctuations of the magnetic field. The vector potential \( \vec{A} \) is regarded as an external source. We know that the phase \( \theta_0^i \) is determined by the magnetic field and the supercurrent. Let us first review the derivation of the Josephson junction diffraction equation (see [18, 19] for more details). Consider a rectangle circle

\[
PC_1QC_2P \text{ in the Josephson junction, where } P \text{ and } Q \text{ locate at the middle of the junction. We can neglect the thickness of the film and assume } \Delta y \text{ deep inside the superconductor where the induced current decays away (i.e. the magnetic field is vanishing). The integration of } v_z \cdot d\vec{l} \text{ along the rectangle circle } PC_1QC_2P \text{ is zero. Therefore, only the magnetic field contributes to the change of the phase }
\]

\[
\nabla \theta_0^i = \frac{2e}{\hbar} \vec{A}. \tag{5.1}
\]

In the superconductor A, the integral around \( C_1 \) path gives

\[
\theta_{0Q_1}(y) - \theta_{0P_1}(y + \Delta y) = \frac{2e}{\hbar} \int_{C_1} \vec{A} \cdot d\vec{l}. \tag{5.2}
\]

For the \( C_2 \) path

\[
\theta_{0P_2}(y) - \theta_{0Q_2}(y + \Delta y) = \frac{2e}{\hbar} \int_{C_2} \vec{A} \cdot d\vec{l}. \tag{5.3}
\]
Then we have
\[ \theta_0(y + \Delta y) - \theta_0(y) = \frac{2e}{\hbar} \int \vec{A} \cdot d\vec{l}. \] (5.4)

By using the Stokes theorem, we find that
\[ \frac{\partial \theta_0}{\partial y} \Delta y = \frac{2e}{\hbar} \int \int \vec{B} \cdot d\vec{s} = \frac{2e}{\hbar} B_x(2\kappa + 2a) \Delta y. \] (5.5)

Now, we have
\[ \nabla_y \theta_0 = \frac{2e\kappa}{\hbar} B_x(y), \] (5.6)
where \( \Lambda = 2\kappa + 2a \) denotes the effective thickness of the junction and \( \kappa_L \) is the penetration depth of the magnetic field. The phase \( \theta_0 \) then depends on the coordinate \( y \)
\[ \theta_0(y) \approx \frac{2e\Lambda B_0}{\hbar} y + c_1. \] (5.7)

It is worth noting that we can replace \( B_x \) with the total flux across the junction \( \Phi_J = \Lambda dB_x \),
\[ \frac{2e\Lambda B_x}{\hbar} = \frac{2\pi}{\hbar} dB_x \approx \frac{2\pi \Phi_J}{d\phi_0}, \] (5.8)
where \( \phi_0 = \frac{\hbar}{2e} \) is the magnetic quantum flux. If this is substituted in Eq. (2.10) and integrated over the area \( S = wd \) of the junction, we have
\[ I_s(B) = j_c S \frac{\sin(\frac{\pi \Phi_J}{\phi_0})}{\frac{\pi \Phi_J}{\phi_0}} \sin c_1. \] (5.9)

We call this the \textit{Josephson junction diffraction equation}. This equation indicates that the \( n \)th maximum of the current \( I_s \) occurs at the flux value \( \Phi_J = (n + \frac{1}{2})\phi_0 \), but cancels for \( \Phi_J = n\phi_0 \), where \( n \) is an integer.

We are interested in (5.6), since this equation gives us \( v_y = \frac{\hbar}{m} \nabla_y \theta_0 = \frac{2e\Lambda}{m} B_x(y) \). Therefore, in the cartesian coordinate system, from (3.19) we have the metric
\[ ds^2 = \left( \frac{\hbar \rho_0}{mc_s} \right) \left[ - (c_s^2 - \left( v_z^2 + v_y^2 \right)) dt^2 - 2v_z dz dt - 2v_y dy dt + d\vec{x}_i \cdot d\vec{x}_i \right]. \] (5.10)

Taking the coordinate transformation
\[ dt = d\tau - \frac{v_y}{c_s^2 - v_y^2 - v_z^2} dy - \frac{v_z}{c_s^2 - v_y^2 - v_z^2} dz, \] (5.11)
we obtain
\[ ds^2 = \left( \frac{\hbar \rho_0}{mc_s} \right) \left[ - [c_s^2 - (v_z^2 + v_y^2)] d\tau^2 + \left( \delta_{ij} + \frac{v_i v_j}{c_s^2 - (v_z^2 + v_y^2)} dx^i dx^j \right) \right]. \] (5.12)
where we should note that \( c_s = \sqrt{\frac{\hbar \omega_0 - \kappa}{m}} \), \( v_z \approx \frac{2 \kappa}{\hbar} \) and \( v_y = \frac{2e \Lambda_B}{m} B_x(y) \). Note that the transformation from the metric (5.10) to (5.12) means that the resulting metric (5.10) is a static one. The variable \( t \) measures the time of the background fluid and \( \tau \) is a redefined time. It would be non-trivial to consider the case in which the total phase change across the junction is \( 2n\pi \), with \( n \) Josephson vortices side by side in the junction, each containing one flux quantum. Then the total current in the junction is vanishing because the supercurrent flows down across the junction on the left and up on the right. The current flows horizontally within a penetration depth \( \lambda \) inside the superconductor to form a closed loops. These current loops encircle flux and the resulting configuration is known as a Josephson vortex. The supercurrent along the \( z \)-direction can somehow be regarded as a constant value and then \( v_z \) does not change in the junction.

As a special condition, let us consider the Hawking temperature contributed by \( v_y \). In the vicinity of the horizon, we can split up the fluid flow into normal and tangential components (i.e. \( \mathbf{v} = v_z + v_y \)) and we choose \( \hat{j} \) as the unit vector field that at the horizon is perpendicular to it [2]. From the definition, we know that the horizon locates at \( c_s = v_y \). In this case, the Hawking temperature at the event horizon is given by

\[
T_H = \frac{\hbar}{2\pi k_B} \left| \partial_y (c_s - v_y) \right|_{\text{horizon}}. \tag{5.13}
\]

More explicitly,

\[
T_H = (1.2 \times 10^{-9} K m) \left( \frac{1}{1000 \text{m/s}} \right) \left| \partial_y (c_s - v_y) \right|_{\text{horizon}}
\]

\[
= (1.2 \times 10^{-9} K m) \left( \frac{1}{1000 \text{m/s}} \right) \left| \partial_y c_s - \frac{2e \Lambda \mu_0 j_z}{m} \right|_{\text{horizon}}, \tag{5.14}
\]

where we have used the Maxwell equation \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \). If the speed of sound is to be a position-independent constant, the resulting Hawking temperature then has the form

\[
T_H = (1.2 \times 10^{-9} K m) \left( \frac{1}{1000 \text{m/s}} \right) \left| \frac{2e \Lambda \mu_0 j_z}{m} \right|_{\text{horizon}}. \tag{5.15}
\]

This is a very interesting result which indicates that for big enough value of the effective thickness of the junction \( \Lambda \) and the tunneling current \( j_z \), the Hawking temperature would be detectable in the future. As an example, let us estimate the Hawking temperature of a particular kind of Josephson junction: given the tunneling current \( j_z = 5 \times 10^7 \text{ A/m}^2 \) and the effective length \( \Lambda = 50 \text{nm} \) (including the penetration depth) [18], the resulting Hawking temperature is about \( T_H \sim 10^{-7} \text{Kelvin} \). This value varies for different tunneling currents \( j_z \). Compared with the Hawking temperature (\( \sim \text{nano Kelvin} \)) of acoustic black holes in Bose-Einstein condensate, the temperature obtained here is two orders of magnitude higher that maybe possible for the future experiments.
6 Conclusion

In summary, we have presented a version of acoustic black holes by using the Josephson effect. We started from two coupled Ginzburg-Landau equations with a coupling constant $\kappa$ for Josephson junctions and reviewed the basic equations for the Josephson effect.

The acoustic black hole metric can be obtained from the perturbation equation for $\theta_1$ and $\rho_1$. The advantage of creating acoustic metric by using the Josephson effect is that the coupling constant $\kappa$ can be tuned. So that the sound velocity $c_s$ can be tuned to be very small and then $v_{0i} > c_s$ would become easier. We discuss in detail how to describe an acoustic black hole in the Josephson junction and write the metric in the language of the superconducting electronics. Finally, we estimate the Hawking temperature of acoustic black hole created in the Josephson junction. Although we have set up a theoretical model for acoustic black holes in Josephson junctions, the experimental detection of the Hawking temperature would be difficult. Our result indicates that the Hawking temperature strongly depends on the tunneling current. The enhancement of the Josephson current is thus crucial for the measurement of the acoustic black hole.

On the other hand, the obstacle of detecting Hawking radiation may come from its instability against other mechanisms. For instance, for type I superconductors, when $v_0 = c_s$, the superconducting phase is broken and return to normal states [10]. In [20], the authors studied acoustic horizons in the quantum de Laval nozzle. They solved the Gross-Pitaevskii equation and found that both in hydrodynamic and non-hydrodynamic regimes there exist dynamically unstable regions associated with the creation of positive and negative energy quasiparticle pairs in analogy with the gravitational Hawking effect. In this paper, we may suffers the same problems since the Ginzburg-Landau equation and the Gross-Pitaevskii equation are very similar. Also, the quasinormal modes analysis of the obtained acoustic black holes may reveal that in the high momentum regime the configuration would be unstable against perturbations. We leave discussion of the quantum instability to a future publication.

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