Navier–Stokes Flow Past a Rigid Body That Moves by Time-Periodic Motion

Giovanni P. Galdi

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To Professor Yoshihiro Shibata, with friendship and admiration.

Abstract. We study existence, uniqueness and asymptotic spatial behavior of time-periodic strong solutions to the Navier–Stokes equations in the exterior of a rigid body, $\mathcal{B}$, moving by time-periodic motion of given period $T$, when the data are sufficiently regular and small. Our contribution improves all previous ones in several directions. For example, we allow both translational, $\xi$, and angular, $\omega$, velocities of $\mathcal{B}$ to depend on time, and do not impose any restriction on the period $T$ nor on the averaged velocity, $\bar{\xi}$, of $\mathcal{B}$. If $\xi \neq 0$ we assume that $\xi$ and $\omega$ are both parallel to a constant direction, while no further assumption is needed if $\xi \equiv 0$. We also furnish the spatial asymptotic behavior of the velocity field, $u$, associated to such solutions. In particular, if $\mathcal{B}$ has a net motion characterized by $\xi \neq 0$, we then show that, at large distances from $\mathcal{B}$, $u$ manifests a wake-like behavior in the direction $-\bar{\xi}$, entirely similar to that of the velocity field of the steady-state flow occurring when $\mathcal{B}$ moves with velocity $\bar{\xi}$.

1. Introduction

The mathematical analysis of time-periodic viscous flow around a moving body is a relatively new area of research.\(^1\) The basic problem at the foundation of this study can be described as follows \([12]\). A rigid body, $\mathcal{B}$, moves in an otherwise stagnant Navier–Stokes liquid, $\mathcal{L}$, that occupies the whole space outside $\mathcal{B}$. When referred to a frame, $\mathcal{F}$, attached to $\mathcal{B}$, the motion of $\mathcal{B}$ is time-periodic of period $T$, and a given body force $b$ of the same period $T$ may be acting on $\mathcal{L}$. Then, the question to address is whether the corresponding flow of $\mathcal{L}$, in the frame $\mathcal{F}$, will also be time-periodic and of period $T$ (\emph{T-periodic}, in short). From the mathematical viewpoint, this means to find $T$-periodic solutions $(u, p)$ to the following set of equations:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - V \cdot \nabla u + \omega \times u + u \cdot \nabla u &= \Delta u - \nabla p + b \\
\text{div } u &= 0 \\
\end{aligned}
\tag{1.1}
\]

in $\Omega \times (-\infty, \infty)$, $u = V$, at $\partial \Omega \times (-\infty, \infty)$, $\lim_{|x|\rightarrow \infty} u(x, t) = 0$, $t \in (-\infty, \infty)$.

Here, $u$ and $p$ represent velocity and pressure fields of $\mathcal{L}$, and $\Omega$ is the spatial region outside $\mathcal{B}$ and entirely occupied by $\mathcal{L}$. Furthermore, $V = V(t) := \xi(t) + \omega(t) \times x$, where $\xi$ and $\omega$ are prescribed $T$-periodic functions denoting velocity of the center of mass and angular velocity of $\mathcal{B}$, respectively. Finally, $b = b(x, t)$ indicates a $T$-periodic body force acting on $\mathcal{L}$.

In spite of the conspicuous literature dedicated to the resolution of this problem, there is still a number of basic questions that, to date, are still unresolved, as we are going to present next. Well-posedness of (1.1) in the class of $T$-periodic solutions was first investigated in \([15]\) where it was shown that, if the data possess only a mild degree of regularity, there is at least one corresponding weak solution in the sense of Leray-Hopf. For more regular data, but of restricted “size,” in \([15]\) it is also constructed a solution that

\(^1\)For the case when the body is at rest, we refer the reader to \([17,25,29,32,33,36]\).
is strong in the sense of Ladyzhenskaya. The noteworthy aspect of these results is that, besides some regularity (and “smallness” wherever it applies), no further assumption is made on the characteristic vectors $\xi$ and $\omega$. However, the unpleasant drawback is that, even in the class of strong solutions, the uniqueness property is not assured. This is due to the circumstance that solutions constructed in [15] carry very little information about their behavior as $|x| \to \infty$. In fact $(1.1)_4$ is satisfied only in a generalized sense, namely:

$$\sup_{[0,T]} \|u(t)\|_6 < \infty.$$  

(1.2)

The principal, but not only, difficulty in obtaining more relevant information for large $|x|$ is due to the presence in $(1.1)_1$ of the term $\omega(t) \times x \cdot \nabla u$ whose coefficient becomes unbounded as $|x| \to \infty$. For this reason, it is expected that the case $\omega(t) \not= 0$ will be the more challenging one.

More recently, the general problem of existence and uniqueness was investigated and successfully settled by several authors [3–6,8,12,13,18,20,21,24,26,34,35], with approaches and in classes of solutions different from those of [15]. However, unlike [15], all results there established hold under hypotheses on $\xi$ and $\omega$ that are rather more restrictive than just their regularity and smallness. Precisely, in [4,5,8,12,13,20,21,26] one assumes $\omega \equiv 0$, $\xi \not= 0$ (where the bar denotes average over a period), and $\sup_t |\xi(t)|$, $\sup_t |\xi(t) - \bar{\xi}|$ “small” enough. Still keeping $\omega \equiv 0$, in [9,10] the assumption on $\xi$ was relaxed by requiring $\xi$ to be only regular enough and “small.” In [3,6] the request $\omega \equiv 0$ is weakened to $\omega = \text{const.}$, but on condition that $\bar{\xi} \not= 0$ and the period of $\xi$ and $b$ is equal to $2\pi \kappa/|\omega|$, for some $\kappa \in \mathbb{Q} \setminus \{0\}$. Moreover, at all times, $\xi(t)$ must be parallel to the (constant) direction of $\omega$, with $\sup_t |\xi(t)|$, $\sup_t |\xi(t) - \bar{\xi}|$ and $|\omega|$ “small” enough. The method there followed is based on maximal regularity properties for time-periodic problems with zero average, an approach introduced in [8] and successively fully developed and generalized in [27,28]. In [18,20,21,24,34,35], a different line of investigation was undertaken that relies upon sharp “$L^p$ – $L^q$” estimates of the relevant evolution operator in suitable spaces. However, this approach requires $\xi$ and $\omega$ to be constant in time, parallel and “small.” Nevertheless, unlike [3], no restriction is imposed on the period $T$.

The main goal of this paper is to prove existence, uniqueness and asymptotic spatial behavior of $T$-periodic solutions to $(1.1)$, under assumptions on $\xi$ and $\omega$ that are much more general than those requested in all papers cited above. Precisely, $\xi$ and $\omega$ are given $T$-periodic functions that, besides some regularity and “smallness,” satisfy the following condition:

$$\text{If } \xi(t) \not= 0, \text{ then both } \xi(t) \text{ and } \omega(t) \text{ are parallel to the same constant direction} \quad (H)$$

It is worth emphasizing that if, instead, $\xi(t) \equiv 0$, no further assumption is imposed on $\omega(t)$.

The significant feature of these conditions is that, unlike all previous contributions, both $\xi$ and $\omega$ are allowed to depend on time, and no restriction is imposed on the period $T$ nor on $\bar{\xi}$ being not zero. In particular, if the motion of $B$ is only rotatory, its angular velocity can be an arbitrary (sufficiently regular and “small”) periodic function of time. Under the above assumptions, we prove existence and uniqueness of $T$-periodic solutions in a class, $C$, of functions belonging to suitable homogeneous Sobolev spaces (as in [15]), with the further, critical property that the velocity field $u(x,t)$ decays as $|x|^{-1}$, uniformly in time and even faster, outside a paraboloidal region (the “wake”), if $\bar{\xi} \not= 0$. More precisely, in this case, $u$ exhibits a wake-like behavior in the direction $-\bar{\xi}$, entirely analogous to that of the velocity field of the steady-state flow that takes place when $B$ moves with velocity $\bar{\xi}$.

Our method of proof stems from [9], and does not require advanced tools like maximal regularity or $L^p$ – $L^q$ estimates. Instead, it relies just on a skillful combination of the classical Galerkin method with uniform (spatial) estimates for solutions to a suitable Oseen-like Cauchy problem, and can be summarized as follows. The basic idea is to use a standard perturbative argument around solutions to the linear problem, $\mathcal{L} \mathcal{P}$ (say), obtained from $(1.1)$ by suppressing the term $u \cdot \nabla u$ in $(1.1)_1$. The success

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2For notation, see the beginning of next section.

3We are only interested to the case $\omega(t) \not= 0$ because, otherwise, the problem has already been solved in [9,10].

4An early version of this result with $\omega = \text{const.}$ was given in the preprint [11].
of this approach is ensured provided we show existence, uniqueness and continuous data-dependence for solutions to $\mathcal{L} \mathcal{P}$ in the class $\mathcal{C}$. The proof of these properties is obtained in two steps. In the first one, under suitable summability requirements on the data, we show the existence of a corresponding solution in homogeneous Sobolev spaces. Successively, assuming in addition that the data decay pointwise in space at a suitable rate and uniformly in time, we prove that the above solution must enjoy a similar property as well.

To carry out the first step we employ the well-known approach that combines Galerkin method with the “invading domains” technique; see, e.g., [15]. Though this strategy is rather classical for the study of problems in unbounded domains, [30, §17] [31, p. 180], [14,19], in the case at hand its implementation presents a basic difficulty. Actually, in order to prove the desired asymptotic spatial behavior (objective of the second step) we need to construct a solution satisfying, in particular, the crucial requirement

$$\text{ess sup} \int_{\Omega_R} |\frac{\partial u}{\partial t}(t)|^q \leq C, \quad \text{for some } q \geq 2,$$

where $\Omega_R = \Omega \cap \{|x| < R\}$, for some sufficiently large $R > 0$, and $C$ is a positive constant depending on the data. When $\omega \equiv 0$, the proof of (1.3) with $q = 2$ is rather standard by the above method. However, already when $\omega = \text{const.} \neq 0$, to show (1.3) is not trivial [11], due to the presence of the term $\omega \times x \cdot \nabla u$, whose coefficient grows unbounded at large spatial distances. If $\omega$ is a generic (sufficiently smooth) $T$-periodic function, the situation is further complicated by the fact that the coefficient is also time dependent. Thus, in such a case, by no means can the validity of (1.3) be taken for granted, let alone considered an easy generalization of previous results. Nevertheless, by introducing new and non-trivial appropriate estimates we will be able to show (1.3) with $q = 6$; see Proposition 3.1.

The solutions constructed in the first step do not possess much information as $|x| \to \infty$, other than that given in (1.2). As expected, this is not enough to ensure the success of the perturbative argument. Therefore, in the second step, we investigate their asymptotic spatial behavior under the further condition that the data decay “sufficiently fast” at large distances. This is done by reducing $\mathcal{L} \mathcal{P}$ to an analogous problem, $\mathcal{L} \mathcal{P}_0$ (say) with $\Omega \equiv \mathbb{R}^3$ via a standard “cut-off” procedure; see (4.5). Successively, the treacherous term $\omega(t) \times x \cdot \nabla u$ is eliminated by means of a suitable time-dependent change of coordinates (see (4.9)) that brings $\mathcal{L} \mathcal{P}_0$ into a Cauchy problem for an Oseen-like system of equations; see (4.17). This is exactly the point (and the only one) where we need the assumption (H) on $\xi$ and $\omega$. Classical results [7, Theorem VIII.4.4] then ensure existence of a solution, $(v, p)$, to the Cauchy problem where $v$ decays (at least) like $|x|^{-1}$ uniformly in time, on the assumption that the data decay “sufficiently fast” and uniformly in time as well. The property (1.3) plays a fundamental role to guarantee the validity of such an assumption. Then, by uniqueness, also $u$ must decay like $|x|^{-1}$, uniformly in time, and this completes the second step; see Proposition 4.1.

Combining the findings of both steps, we then secure the desired results of existence, uniqueness and continuous data-dependence of solutions to $\mathcal{L} \mathcal{P}$ in the class $\mathcal{C}$; see Theorem 4.1. Thanks to this result and the functional properties of $\mathcal{C}$, we may finally use a standard contraction mapping argument that assures the existence and uniqueness of a solution to the full nonlinear problem (1.1) in $\mathcal{C}$, at least for sufficiently smooth and “small” data, and under the assumption (H) on $\xi$ and $\omega$; see Theorem 5.1.

Before concluding this introductory section, some comments are in order as whether (H) is indeed necessary to prove well-posedness of problem (1.1) in the class of $T$-periodic solutions. I believe that if, as in the present paper, the class includes the pointwise asymptotic spatial behavior and the wake-like property of the velocity field $u$—which, on physical ground, is a rather significant feature—then that hypothesis cannot be removed. However, if the class is enlarged to replace such a pointwise behavior (that, uniformly, is $O(|x|^{-1})$) with the less stringent requirement that $u$ belongs to the weak-$L^3$ space (as in the pioneering work [36]), then there could be a chance that (H) might not be needed. This guess is also supported by the “$L^p - L^q$ estimates” in weak spaces of the evolution operator associated to (1.1), recently proved by T. HISHIDA in [22,23], a fundamental tool for the success of the approach introduced in [36].
The plan of the paper is as follows. After collecting some preliminary results in Sects. 2, 3 we show the existence of a solution to \( \mathcal{L} \mathcal{P} \) in homogeneous Sobolev spaces. In the subsequent Sect. 4, we show the asymptotic spatial properties of these solutions and also their uniqueness and continuous data-dependence. In the final Sect. 5, we use the combined results of Sects. 3 and 4 to show existence, uniqueness and asymptotic spatial behavior of solutions to (1.1) under the assumption of data of restricted magnitude and with \( \xi \) and \( \omega \) satisfying the request (H).

2. Preliminaries

We begin to recall some notation. \( \Omega \) will always denote the complement of the closure of a bounded domain \( \Omega_0 \subset \mathbb{R}^3 \), of class \( C^2 \). With the origin of the coordinate system in the interior of \( \Omega_0 \), we set \( \mathcal{B}_R = \{ x \in \mathbb{R}^3 : |x| < R \} \), and if \( R \geq R_0 := 2 \text{diam} (\Omega_0) \), \( \Omega_R = \Omega \cap \overline{\mathcal{B}_R} \), \( \Omega^R = \Omega \setminus \mathcal{B}_R \). For \( \Omega \subset \mathbb{R}^3 \) a domain, by \( L^q(A) \), \( 1 \leq q \leq \infty \), \( W^{m,q}(A) \), \( m \geq 0 \), \( (W_0^{0,q} \equiv W_0^0 \equiv L^q) \), we indicate usual Lebesgue and Sobolev spaces, with norms \( \| \cdot \|_{q,A} \) and \( \| \cdot \|_{m,q,A} \), respectively. By \( P \) we indicate the (Helmholtz–Weyl) projector from \( L^2(A) \) onto its subspace, \( H(A) \), of solenoidal (vector) function with vanishing normal component (in the sense of trace) at \( \partial A \). We define \( (u, v)_A := \int_A u \cdot v \). With \( D^{m,2}(A) \) we indicate the homogeneous Sobolev space of (equivalence classes of) functions \( u \) with seminorm \( \sum_{|k|=m} \| D^k u \|_{2,A} < \infty \). In all the above notation, the subscript “A” will be omitted, unless otherwise specified. We also set \( u_t := \partial u/\partial t, \partial_x u := \partial u/\partial x_k, \) and \( D^2 u = \{ \partial_k \partial_j u \} \), the matrix of the second derivatives. A function \( u : A \times \mathbb{R} \to \mathbb{R}^3 \) is \( T \)-periodic, \( T > 0 \), if \( u(\cdot, t + T) = u(\cdot, t) \), for a.a. \( t \in \mathbb{R} \), and we set \( \overline{u} := \frac{1}{T} \int_0^T u(t)dt \). Let \( B \) be a function space endowed with seminorm \( \| \cdot \|_B, \; r = [1, \infty], \) and \( T > 0 \). \( L^r(0,T; B) \) is the class of functions \( u : (0, T) \to B \) such that

\[
\| u \|_{L^r(B)} \equiv \begin{cases} \left( \int_0^T \| u(t) \|_B^r \right)^{\frac{1}{r}} < \infty, & \text{if } r \in [1, \infty) ; \\
\text{ess sup}_{t \in [0,T]} \| u(t) \|_B < \infty, & \text{if } r = \infty. \end{cases}
\]

We also define

\[
W^{m,r}(0,T; B) = \left\{ u \in L^r(0,T; B) : \partial_t^k u \in L^r(0,T; B), \; k = 1, \ldots, m \right\}.
\]

We shall simply write \( L^r(B) \) for \( L^r(0,T; B) \), etc. unless otherwise stated. Moreover, if \( B \equiv \mathbb{R}^d \), \( d \geq 1 \), we set \( L^r(0,T; B) = L^r(0,T) \), etc.

For \( A := \Omega, \mathbb{R}^3, m = 1, 2 \), and \( \lambda > 0 \) we set

\[
\| f \|_{m,\lambda, A} := \sup_{(x,t) \in A \times (0,\infty)} |(1 + |x|)^m (1 + 2\lambda s(x))^m f(x,t)|,
\]

where

\[
s(x) = |x| + x_1, \; x \in \mathbb{R}^3,
\]

and the subscript \( A \) will be omitted, unless necessary.

Finally, we denote by \( c \) or \( C \) a generic positive constant whose specific value is irrelevant and may change even in the same line. When we want to emphasize the dependence of \( c \) on a quantity \( \rho \), we shall write \( c_\rho \) or \( c(\rho) \), and similarly for \( C \).

We now collect some preliminary results whose proof can be found in the literature. We begin with the following one [7, Theorem III.3.1 and Exercise III.3.7].

**Lemma 2.1.** Let \( A \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \), and let \( f \in L^2(A) \) with \( \int_A f = 0 \). Then the problem

\[
\begin{align*}
\text{div } z &= f \quad \text{in } A, \\
z &\in W^{1,2}_0(A), \quad \| z \|_{1,2} \leq C_0 \| f \|_2,
\end{align*}
\]

\[^{5}\text{We shall use the same font style to denote scalar, vector and tensor function spaces.}\]
for some \( C_0 = C_0(A) > 0 \) has at least one solution. Moreover, if also \( f \in W^{1,2}_0(A) \) and \( \|z\|_{2,2} \leq C_0 \|f\|_{1,2} \). Finally, if \( f = f(t) \) with \( f_t \in L^\infty(L^2(A)) \), then we have in addition \( z_t \in L^\infty(W^{1,2}_0(A)) \) and

\[
\|z_t\|_{L^\infty(W^{1,2})} \leq C_0 \|f_t\|_{L^\infty(L^2)}.
\]

The next result is proved in \([15, \text{Lemma } 2.2]\).

**Lemma 2.2.** Let \( \xi, \omega \in W^{2,2}(0, T) \) be \( T \)-periodic. There exists a solenoidal, \( T \)-periodic function \( \tilde{u} \in W^{1,2}(W^{m,q}) \), \( m \in \mathbb{N}, q \in [1, \infty) \), such that

\[
\tilde{u}(x, t) = \xi(t) + \omega \times x, \quad (x, t) \in [0, T] \times \partial \Omega,
\]

\[
\tilde{u}(x, t) = 0, \text{ for all } (x, t) \in \overline{\Omega} \times [0, T],
\]

\[
\|\tilde{u}\|_{W^{2,2}(W^{m,q})} \leq C \left( \|\xi\|_{W^{2,2}(0, T)} + \|\omega\|_{W^{2,2}(0, T)} \right),
\]

where \( C = C(\Omega, m, q) \). Moreover, for any \( \varepsilon > 0 \), the field \( \tilde{u} \) can be chosen in such a way that

\[
\int_{\Omega_R} v \cdot \nabla \tilde{u}(t) \cdot v \leq \varepsilon \|\nabla v\|^2_2, \quad \text{for all } v \in H(\Omega_R) \cap W^{1,2}_0(\Omega_R).
\]

We also have the following.

**Lemma 2.3.** Let \( u \in H(\Omega_R) \cap W^{1,2}_0(\Omega_R) \cap W^{2,2}(\Omega_R) \), and \( a, b \in \mathbb{R}^3 \). Then, the following properties hold

(i) \( a \times x \cdot \nabla u = 0 \) at \( \partial B_R \);

(ii) \( (a \times u - (b + a \times x) \cdot \nabla u) \in H(\Omega_R) \).

**Proof.** The proof of (i) is given in \([14, \text{Lemma } 3]\). There, it was also shown that \( (a \times u - a \times x \cdot \nabla u) \in H(\Omega_R) \), so that, in order to prove (ii), we only have to prove

\[
b \cdot \nabla u \in H(\Omega_R).
\]

By \([7, \text{Lemma III.2.1}]\) this is equivalent to showing that

\[
(b \cdot \nabla u, \nabla \psi) = 0, \quad \text{for all } \psi \in D^{1,2}(\Omega_R).
\]

Since \( u \in H(\Omega_R) \cap W^{1,2}_0(\Omega_R) \), by \([7, \text{Theorem III.4.1}]\) there is a sequence \( \{u_n\} \) of solenoidal functions from \( C_0^\infty(\Omega_R) \) such that \( \nabla u_n \to \nabla u \) in \( L^2(\Omega_R) \). Clearly, as shown by a simple integration by parts,

\[
(b \cdot \nabla u_n, \nabla \psi) = 0,
\]

so that, by Schwarz inequality,

\[
| (b \cdot \nabla u, \nabla \psi) | = | (b \cdot \nabla (u - u_n), \nabla \psi) | \leq |b| \|\nabla (u - u_n)\|_2 \|\nabla \psi\|_2,
\]

which, in turn, by letting \( n \to \infty \), entails (2.4).

We conclude by recalling the next lemma that ensures suitable existence and uniqueness properties for a linear Cauchy problem \([7, \text{Theorem VIII.4.4}]\)

**Lemma 2.4.** Let \( \mathcal{G} = \mathcal{G}(x, t) \) be a second-order tensor field defined in \( \mathbb{R}^3 \times (0, \infty) \) such that

\[
\text{div } \mathcal{G} \in L^2(0, T; L^2),
\]

for all \( T > 0 \), let \( h \in L^\infty(0, \infty; L^q) \), \( q \in (3, \infty) \), with spatial support contained in \( B_r \), some \( r > 0 \), and let \( \lambda \geq 0 \). Then, the problem

\[
\begin{align*}
\mathbf{w}_t &= \Delta \mathbf{w} + \lambda \partial_1 \mathbf{w} - \nabla \mathbf{q} + \text{div} \mathcal{G} + \mathbf{h} \\
\nabla \cdot \mathbf{w} &= 0 \\
\mathbf{w}(x, 0) &= \mathbf{0},
\end{align*}
\]

in \( \mathbb{R}^3 \times (0, \infty) \)

has one and only one solution such that

\[
\mathbf{w} \in L^2(0, T; W^{2,2} \mathbb{R}^3), \quad \mathbf{w}_t \in L^2(0, T; L^2); \quad \nabla \mathbf{q} \in L^2(0, T; L^2).
\]
If, in addition, $|\mathcal{G}|_{\infty,2,\lambda} < \infty$, then

$$|\mathbf{w}|_{\infty,1,\lambda} + \operatorname{ess sup}_{t \geq 0} \|q(t)\|_r < \infty,$$

for arbitrary $r \in (\frac{3}{2}, \infty)$, and, setting

$$D := |\mathcal{G}|_{\infty,2,\lambda} + \operatorname{ess sup}_{t \geq 0} \|h(t)\|_q,$$

the following inequalities hold:

$$|\mathbf{w}|_{\infty,1,\lambda} \leq CD, \quad \operatorname{ess sup}_{t \geq 0} \|q(t)\|_r \leq C_1 D,$$

with $C = C(q,r,\lambda_0), C_1(q,r,r,\lambda_0)$ whenever $\lambda \in [0, \lambda_0]$, for some $\lambda_0 > 0$.

### 3. Linear Problem: Existence

The main objective of this section and the following one is to prove existence, uniqueness and corresponding spatial asymptotic behavior of $T$-periodic solutions, in appropriate function classes, to the following linear problem

$$\begin{aligned}
\mathbf{u}_t - \mathbf{V}(t) \cdot \nabla \mathbf{u} + \mathbf{\omega}(t) \times \mathbf{u} &= \Delta \mathbf{u} - \nabla p + f \quad \text{in } \Omega \times (0,T) \\
\text{div } \mathbf{u} &= 0
\end{aligned}$$

(3.1)

where $\mathbf{V}(t) := \xi(t) + \mathbf{\omega}(t) \times \mathbf{x}$, and $f = f(x,t), \xi = \xi(t)$ and $\mathbf{\omega} = \mathbf{\omega}(t)$, suitably prescribed $T$-periodic functions.

Throughout, we shall consider only the case $\mathbf{\omega}(t) \neq 0$, since, otherwise, the problem has already been solved, even at the full nonlinear level, in [9, 10].

Our plan develops into two steps and goes as follows. In the first one, considered in this section, we show, under suitable summability requirements on $f$, the existence of a corresponding solution in Lebesgue and homogeneous Sobolev spaces. Successively, in the next section, assuming in addition that $f$ decays pointwise in space at a suitable rate uniformly in time, we shall prove that the above solution must enjoy a similar property as well. We shall also show that, in this class, solutions are unique and depend continuously upon the data.

To accomplish the first step, let $\mathcal{S} = \{R_m, m \in \mathbb{N}\}$ be an increasing, unbounded sequence of positive numbers with $R_1 > \rho$ (see Lemma 2.2), and let $\{\Omega_R, R \in \mathcal{S}\}$ be the sequence of bounded domains such that $\bigcup_{R \in \mathcal{S}} \Omega_R = \Omega$. For each $R \in \mathcal{S}$, we look for a $T$-periodic solution $(\mathbf{u}_R, p_R)$ to the problem

$$\begin{aligned}
(\mathbf{u}_R)_t - \mathbf{V}(t) \cdot \nabla \mathbf{u}_R + \mathbf{\omega}(t) \times \mathbf{u}_R &= \Delta \mathbf{u}_R - \nabla p_R + f \\
\text{div } \mathbf{u}_R &= 0 \\
\mathbf{u}_R(x,t) &= \mathbf{V}(t), \quad (x,t) \in \partial \Omega \times [0,T]; \quad \mathbf{u}_R(x,t) = 0, \quad (x,t) \in \partial B_R \times [0,T].
\end{aligned}$$

(3.2)

Setting $\mathbf{v}_R := \mathbf{u}_R - \bar{\mathbf{u}}$ with $\bar{\mathbf{u}}$ given by Lemma 2.2, we can equivalently rewrite (3.2) in the following form

$$\begin{aligned}
(\mathbf{v}_R)_t - \mathbf{V}(t) \cdot \nabla \mathbf{v}_R + \mathbf{\omega}(t) \times \mathbf{v}_R &= \Delta \mathbf{v}_R - \nabla p_R \\
-\bar{\mathbf{u}} \cdot \nabla \mathbf{v}_R - \mathbf{v}_R \cdot \nabla \bar{\mathbf{u}} + \tilde{f} &= \mathbf{f} \\
\text{div } \mathbf{v}_R &= 0 \\
\mathbf{v}_R(x,t) &= 0, \quad (x,t) \in \partial \Omega_R \times [0,T],
\end{aligned}$$

(3.3)

where

$$\tilde{f} = f + \Delta \bar{\mathbf{u}} - \bar{\mathbf{u}}_t + \mathbf{V} \cdot \nabla \bar{\mathbf{u}} - \mathbf{\omega} \times \bar{\mathbf{u}} := f + f_c.$$

(3.4)

To show the existence of solutions to (3.3)–(3.4), we shall employ Galerkin method with the orthonormal base of $H(\Omega_R)$, $\{\mathbf{w}_{Rj}\}_{j \in \mathbb{N}}$, constituted by the eigenfunctions of the Stokes operator:

$$P \Delta \mathbf{w}_{Rj} = -\lambda_{Rj} \mathbf{w}_{Rj}, \quad \mathbf{w}_{Rj} \in H(\Omega_R) \cap W^{1,2}_0(\Omega_R) \cap W^{2,2}(\Omega_R).$$

(3.5)
We thus search for an “approximating” solution to (3.3) of the form
\[
\mathbf{v}_{Rk}(x, t) = \sum_{i=1}^{k} c_{Rk_i}(t) \mathbf{w}_{R_i}(x),
\] (3.6)
where the coefficients \( c_{Rk} = \{c_{Rk1}, \cdots, c_{Rkk}\} \) solve the following system of equations
\[
\begin{aligned}
(\mathbf{v}_{Rk})_t + \nabla \mathbf{v}_{Rk} &= \mathbf{f} + \mathbf{w}_R + (\mathbf{v} \cdot \nabla \mathbf{v}_{Rk}) - (\nabla \mathbf{v}_{Rk} \cdot \nabla) \mathbf{v}_{Rk} + (\omega \times \mathbf{v}_{Rk}) - (\mathbf{v} \cdot \nabla) \mathbf{w}_R + (\mathbf{v} \cdot \nabla) \mathbf{w}_R + \mathbf{v} \cdot \nabla \mathbf{v}_R - \mathbf{v} \cdot \nabla \mathbf{v}_R + (\mathbf{V} \cdot \nabla) \mathbf{v}_{Rk} + \mathbf{w}_R \Omega_R, \\
&= (\mathbf{f}, \mathbf{w}_R)_{\Omega_R}, \quad j = 1, \ldots, k.
\end{aligned}
\] (3.7)

Our next goal is to establish a number of estimates for \( \mathbf{v}_{Rk} \) [respectively, \( \mathbf{v}_R \)] with bounds that are independent of \( k \) [respectively, \( R \)]. In what follows, we will denote by \( V_0 \) a fixed positive number such that
\[
\| \mathbf{v} \|_{W^{2,2}(0, T)} + \| \mathbf{w} \|_{W^{2,2}(0, T)} \leq V_0.
\] (3.8)

We begin to recall the following result whose proof is given in [15, Lemmas 3.1, 3.2, 4.1 and 4.3].

**Lemma 3.1.** Let \( \mathbf{f} = \text{div } \mathbf{F} \in L^2(L^2) \) with \( \mathbf{F} \in L^2(L^2) \), \( \xi, \omega \in W^{1,2}(0, T) \) be \( T \)-periodic. Then, for each \( k \in \mathbb{N} \) problem (3.7) has at least one \( T \)-periodic solution \( c_{Rk} = c_{Rk}(t) \). Moreover, the approximating solution \( \mathbf{v}_{Rk} \) satisfies the following uniform estimates
\[
\sup_{t \in [0, T]} (\| \mathbf{v}_{Rk}(t) \|_{L^{6}(\Omega_R)} + \| \nabla \mathbf{v}_{Rk}(t) \|_{L^{2}(\Omega_R)} + \| D^2 \mathbf{v}_{Rk}(t) \|_{L^2(\Omega_R)}).
\] (3.9)

with \( C = C(\Omega, V_0, T) \).

We shall next prove additional uniform estimates (in \( k \)) for the approximating solution (3.6).

**Lemma 3.2.** Let the assumptions of Lemma 3.1 hold and suppose, in addition, \( \mathbf{f}, \mathbf{F} \in W^{1,2}(L^2) \) and \( \xi, \omega \in W^{2,2}(0, T) \). Then \( \mathbf{v}_{Rk} \) obeys the following bound
\[
\sup_{t \in [0, T]} \| (\mathbf{v}_{Rk})_t(t) \|_{L^2} + \| (\mathbf{v}_{Rk})_{tt}(t) \|_{L^2} + \| D^2 \mathbf{v}_{Rk}(t) \|_{L^2} \leq C \left( \| \mathbf{f} \|_{W^{1,2}(L^2)} + \| \mathbf{F} \|_{W^{1,2}(L^2)} + V_0 \right),
\] (3.10)

where \( C = C(R, T, V_0) \).

**Proof.** In what follows, all norms and scalar products are taken in \( \Omega_R \) which, therefore, will be omitted as a subscript. Moreover, we set \( \mathbf{v} \equiv \mathbf{v}_{Rk} \). By taking the time-derivative of both sides of (3.7), dot-multiplying the resulting equations by \( \dot{c}_{Rk_j} \), summing over \( j \) from 1 to \( k \), and then integrating by parts over \( \Omega_R \) we get:
\[
\frac{1}{2} \frac{d}{dt} \| \mathbf{v}_t \|^2_2 = -\| \nabla \mathbf{v}_t(t) \|^2_2 + (\nabla \cdot \mathbf{v} - \omega \cdot \nabla \mathbf{v} - \nabla \cdot \mathbf{v} - \mathbf{v} - \mathbf{v} - \mathbf{v} - \mathbf{v} - \mathbf{v} - \mathbf{v} - \mathbf{v} + f_c, \mathbf{v}_t) - (\mathbf{F}_t, \nabla \mathbf{v}_t).
\] (3.11)

By (2.3) we may choose \( \mathbf{u} \) such that
\[
\| (\mathbf{v}_t \cdot \nabla \mathbf{u}, \mathbf{v}_t) \| \leq \frac{1}{2} \| \nabla \mathbf{v}_t \|^2_2.
\] (3.12)

Also, from classical embedding theorems and Lemma 2.2, we have
\[
\| \mathbf{u} \|_{L^\infty(W^{1,2})} + \| \mathbf{u}_t \|_{L^\infty(W^{1,2})} \leq c \| \mathbf{u} \|_{W^{2,2}(W^{3,2})} \leq c_1 V_0,
\] (3.13)

and so it follows that
\[
\| (\mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v}_t \cdot \nabla \mathbf{v} + \mathbf{v}_t \cdot \nabla \mathbf{v} - f_c, \mathbf{v}_t) \| \leq c \| \mathbf{u} \|^2_2 \| \nabla \mathbf{v} \|^2 + \| \mathbf{v} \|^2_{\Omega_R} \| \mathbf{v}_t \|^2_{\Omega_R},
\]
where, we recall, $\Omega_R$ contains the bounded (spatial) support of $\tilde{u}$. Thus, with the help of Lemma 2.2 and Poincaré inequality, we infer

$$
|\langle \dot{V} \cdot \nabla v - \omega \times v - \tilde{u}_t \cdot \nabla v - v \cdot \nabla \tilde{u}_t + f_c, v_t \rangle| \leq c \|\nabla v\|_2 \|\nabla v_t\|_2, 
$$

with $c = c(R, V_0, T)$. Finally, by Schwarz inequality,

$$
|\langle F_t, \nabla v_t \rangle| \leq \|F_t\|_2 \|\nabla v_t\|_2. 
$$

Replacing in (3.11) the estimates (3.12)–(3.15), we find

$$
\frac{d}{dt} \|v_t\|_2^2 + \|\nabla v_t\|_2^2 \leq c (V_0 + \|\nabla v\|_2 + \|F_t\|_2) \|\nabla v_t\|_2, 
$$

which, in view of Lemma 3.1, implies

$$
\frac{d}{dt} \|v_t\|_2^2 + \|\nabla v_t\|_2^2 \leq c (V_0 + \|f\|_{L^2(L^2)} + \|F\|_{L^2(L^2)} + \|F_t\|_2) \|\nabla v_t\|_2, 
$$

Integrating both sides of the latter from 0 to $T$ and using the $T$-periodicity of $v$ we readily conclude

$$
\|\nabla v_t\|_{L^2(0,T)} \leq c (V_0 + \|f\|_{L^2(L^2)} + \|F\|_{W^{1,2}(L^2)}), 
$$

where $c = c(R, V_0, T)$ is independent of $v$. We next take the time derivative of both sides of (3.7), dot-multiply the resulting equations by $-\lambda R_j \dot{c}_R k_j$ and use (3.5). We then sum over $j$ from 1 to $k$, and integrate by parts over $\Omega_R$ to get

$$
\frac{d}{dt} \|v_t\|_2^2 - \|P \Delta v_t\|_2^2 = ( V \cdot \nabla v_t + \dot{V} \cdot \nabla v - \omega \times v_t, P \Delta v_t ) 

-( \tilde{u}_t \cdot \nabla v + \tilde{v}_t \cdot \nabla v_t + v_t \cdot \nabla \tilde{u} - v \cdot \nabla \tilde{u}_t + \tilde{f}_t, P \Delta v_t ). 
$$

By arguing as before, and using Cauchy-Schwarz inequality, we can show

$$
\|\langle \tilde{u}_t \cdot \nabla v + \tilde{v}_t \cdot \nabla v_t + v_t \cdot \nabla \tilde{u} - v \cdot \nabla \tilde{u}_t + \tilde{f}_t, P \Delta v_t \rangle \| 

\leq c \left( \|V_0 + \|\nabla v\|_2 + \|\nabla v_t\|_2 + \|f_t\|_2 \right)^2 + \frac{1}{4} \|P \Delta v_t\|_2^2, 
$$

with $c = c(V_0, T)$. Also, by Poincaré and Cauchy-Schwarz inequalities, we get

$$
\|\langle V \cdot \nabla v_t + \dot{V} \cdot \nabla v - \omega \times v_t, P \Delta v_t \rangle \| \leq c (\|\nabla v\|_2^2 + \|\nabla v_t\|_2^2 + \frac{1}{4} \|P \Delta v\|_2^2, 
$$

with $c = c(V_0, R, T)$. If we replace (3.18) and (3.19) into (3.17) we deduce

$$
\frac{d}{dt} \|v_t\|_2^2 + \frac{1}{4} \|P \Delta v_t\|_2^2 \leq c (V_0 + \|\nabla v\|_2 + \|\nabla v_t\|_2 + \|f_t\|_2)^2, 
$$

which, in turn, with the help of (3.9) and (3.16) furnishes

$$
\frac{d}{dt} \|v_t\|_2^2 + \frac{1}{4} \|P \Delta v_t\|_2^2 \leq c (V_0 + \|f\|_{W^{1,2}(L^2)} + \|F\|_{W^{1,2}(L^2)})^2, 
$$

with $c = c(V_0, T)$. We now observe that, by (3.16), there is at least one $\tilde{t} \in (0, T)$ such that

$$
\|\nabla v_t(\tilde{t})\|_{L^2} \leq c \left( V_0 + \|f\|_{L^2(L^2)} + \|F\|_{W^{1,2}(L^2)} \right), 
$$

and so, integrating (3.20) between $\tilde{t}$ and arbitrary $t > \tilde{t}$, and exploiting the $T$-periodicity of $v$, we readily get

$$
\sup_{t \in [0,T]} \|v_t(t)\|_2 + \int_0^T \|P \Delta v_t\|_2^2 \leq c \left( V_0 + \|f\|_{W^{1,2}(L^2)} + \|F\|_{W^{1,2}(L^2)} \right). 
$$

In turn, the latter, combined with the Poincaré inequality and the well-known inequality $\|D^2 w_{Ri}\|_2 \leq c \|P \Delta w_{Ri}\|_2$, with $c = c(\Omega, R)$ [7, Lemma IV.6.1], furnishes

$$
\sup_{t \in [0,T]} \|v_t(t)\|_{1,2} + \int_0^T \|D^2 v_t\|_2^2 \leq c \left( V_0 + \|f\|_{W^{1,2}(L^2)} + \|F\|_{W^{1,2}(L^2)} \right). 
$$
Finally, we take the time-derivative of both sides of (3.7), dot-multiply the resulting equations by \( \tilde{c}_{Rkj} \) and sum over \( j \) from 1 to \( k \) to get
\[
\|v_{t}\|^2 = \left( \dot{V} \cdot \nabla v + V \cdot \nabla v_t - \omega \times v - \omega \times v_t - \tilde{u}_t \cdot \nabla v \\
- v_t \cdot \nabla \tilde{u} - \tilde{u} \cdot \nabla v_t - v \cdot \nabla \tilde{u}_t + P \Delta v_t, v_{tt} \right) + (f_t, v_{tt}).
\]
Using Schwarz inequality on the right-hand side of this relation along with (3.13), and taking into account (3.4), we show
\[
\|v_{tt}\| \leq c (\|v_t\|_{2,2} + \|v\|_{1,2} + \|f_t\|_{2}), \tag{3.22}
\]
with \( c = c(R, V_0, T) \). The lemma then follows from (3.21), (3.22) and Lemma 3.1. \( \square \)

**Lemma 3.3.** Suppose \( f, \xi \) and \( \omega \) satisfy the assumptions of Lemma 3.2. Then, for any \( R \in S \), problem (3.3) has one \( T \)-periodic solution \( (v, p_R) \) such that
\[
v_R \in W^{1,2}(W^{2,2}(\Omega)) \cap W^{2,2}(L^2(\Omega)), \quad p_R \in W^{1,2}(W^{1,2}(\Omega)).
\]
Moreover, there is a constant \( C = C(V_0, \Omega, T) \) independent of \( R \), such that
\[
\|v_R\|_{L^\infty(L^6(\Omega))} + \|\nabla v_R\|_{L^\infty(L^2(\Omega))} + \|D^2v_R\|_{L^2(L^2(\Omega))} + \|\nabla p_R\|_{L^2(L^2(\Omega))} \leq C \left( \|f\|_{L^2(L^2)} + \|\xi\|_{W^{2,2}(0,T)} + \|\omega\|_{W^{2,2}(0,T)} \right) \tag{3.23}
\]

**Proof.** In view of Lemmas 3.1 and 3.2, the sequence of “approximating solutions,” \( \{v_{Rk}\} \), is bounded in the class \( W^{1,2}(W^{2,2}) \cap W^{2,2}(L^2) \), uniformly in \( k \). Therefore, one may find a subsequence \( \{v_{Rk'}\} \) and a function
\[
v_R \in W^{1,2}(W^{2,2}(\Omega)) \cap W^{2,2}(L^2(\Omega)) \tag{3.24}
\]
such that \( v_{Rk'} \to v_R \) as \( k' \to \infty \) in appropriate topology. By a classical argument [15], we then show that there is \( p_R \in L^2(W^{1,2}(\Omega)) \) such that \( (v_R, p_R) \) is a \( T \)-periodic solution to (3.3). However, from (3.3)_1 and (3.24) it follows that \( p_R \in W^{1,2}(W^{1,2}(\Omega)) \). Clearly, by Lemma 3.1, \( v \) satisfies (3.23). As a result, in order to complete the proof of the lemma, it remains to show the estimate for \( p_R \) in (3.23). By the Helmholtz decomposition, it follows that
\[
\tilde{f} = P\tilde{f} + \nabla \tilde{p}, \quad \tilde{p} \in L^2(D^{1,2}), \tag{3.25}
\]
where
\[
\|\nabla \tilde{p}\|_{L^2(L^2)} \leq c \|\tilde{f}\|_{L^2(L^2)},
\]
and \( c = c(\Omega) \). From (3.4) and the latter, we deduce
\[
\|\nabla \tilde{p}\|_{L^2(L^2)} \leq c (\|f\|_{L^2(L^2)} + \|\xi\|_{W^{2,2}(0,T)} + \|\omega\|_{W^{2,2}(0,T)}), \tag{3.26}
\]
where \( c = c(\Omega, V_0) \). Recalling that, by Lemma 2.3,
\[
((\xi + \omega \times x) \cdot \nabla v_R - \omega \times v_R) \in H(\Omega_R),
\]
from (3.3)_1 we show that \( p_R := p_R - \tilde{p}_{\Omega_R} \) satisfies the following Neumann problem (in the sense of distributions)
\[
\Delta p_R = \text{div } F, \quad \text{in } \Omega_R, \tag{3.27}
\]
\[
\frac{\partial p_R}{\partial n} = F \cdot n, \quad \text{at } \partial \Omega_R,
\]
\[Notice that, since \( p_R \) is \( T \)-periodic, from (3.3)_1 it follows that \( v_t \) is also \( T \)-periodic.
where $F := -v_R \cdot \nabla \tilde{u} - \tilde{u} \cdot \nabla v_R + \Delta v_R$. Formally multiplying both sides of (3.27) by $p_R$, integrating by parts over $\Omega_R$ and using (3.27)$_2$ we deduce
\[
\|\nabla p_R\|_{L^2(\Omega_R)}^2 = (F, \nabla p_R)_{\Omega_R},
\]
which, in turn, by Schwarz inequality, entails
\[
\|\nabla p_R\|_{L^2(\Omega_R)}^2 \leq \|F\|_2^2.
\] (3.28)

From (3.13) we get
\[
\|F\|_{L^2(L^2)} \leq c \left( \|v_R\|_{L^2(W^{1,2}(K))} + \|P\Delta v_R\|_{L^2(L^2(\Omega_R))} \right),
\] (3.29)
with $c = c(V_0, T)$ and where $K$ is the bounded (spatial) support of $\tilde{u}$. Collecting (3.25), (3.26), (3.28), (3.29) and (3.23) for $v_R$ we show
\[
\|\nabla p_R\|_{L^2(L^2(\Omega_R))} \leq C \left( \|f\|_{L^2(L^2)} + \|\mathcal{F}\|_{L^2(L^2)} + \|\xi\|_{W^{1,2}(0, T)} + \|\omega\|_{W^{2,2}(0, T)} \right).
\] (3.30)

Finally, since $F$ is $T$-periodic, by the (well known) uniqueness property for the problem (3.27) we infer that $P_R$, and hence $p_R$, is $T$-periodic as well. The proof of the lemma is thus completed. \hfill \Box

Our next objective is to show that the solution obtained in Lemma 3.3, in addition to (3.23), satisfies some further estimates, uniformly with respect to $R$. In this regard, we need some preparatory results.

**Lemma 3.4.** Let $f$, $\xi$, $\omega$ and $(v_R, p_R)$ be as in Lemma 3.3. Then
\[
\left\|\left( v_R \right)_t - V(t) \cdot \nabla v_R + \omega(t) \times v_R \right\|_{L^2(L^2(\Omega_R))} \leq c_1 \left( \|f\|_{L^2(L^2)} + \|\mathcal{F}\|_{L^2(L^2)} + \|\xi\|_{W^{1,2}(0, T)} + \|\omega\|_{W^{2,2}(0, T)} \right),
\] (3.31)
where $c_1 = c_1(\Omega, V_0, T)$. Moreover,
\[
\left\| \tilde{u}_t \cdot \nabla v_R + \tilde{u} \cdot \nabla (v_R)_t + (v_R)_t \cdot \nabla \tilde{u} + v_R \cdot \nabla \tilde{u}_t \right\|_2 \leq c_2 \left( \|\nabla v_R\|_2 + \|\nabla (v_R)_t\|_2 \right).
\] (3.32)

\textbf{Proof.} From (3.3) we have$^8$
\[
\left\|\left( v_R \right)_t - V(t) \cdot \nabla v_R + \omega(t) \times v_R \right\|_{L^2(L^2)} = \|\Delta v_R - \nabla p_R - \tilde{u} \cdot \nabla v_R - v_R \cdot \nabla \tilde{u} + \tilde{f}\|_{L^2(L^2)}.
\] (3.33)

Clearly, by (3.13), (3.4), and Lemma 2.2, we infer
\[
\|\Delta v_R - \nabla p_R - \tilde{u} \cdot \nabla v_R + \tilde{f}\|_{L^2(L^2)} \leq c \left( \|\nabla v_R\|_{L^2(W^{1,2})} + \|\nabla p_R\|_{L^2(L^2)} + \|f\|_{L^2(L^2)} + \|\xi\|_{W^{2,2}(0, T)} + \|\omega\|_{W^{2,2}(0, T)} \right),
\] (3.34)
with $c = c(V_0, \Omega, T)$. Moreover, recalling that the support of $\tilde{u}$ is contained in $\Omega$, and taking into account (3.13), we obtain
\[
\|v_R \cdot \nabla \tilde{u}\|_{L^2(L^2)} \leq c_0 \|v_R\|_{L^2(L^2(\Omega_R))} \leq c_1 \|v_R\|_{L^2(L^2)}
\] (3.35)
with $c_1$ independent of $R$. The inequality (3.31) is then a consequence of combining (3.33)–(3.35) with (3.23). Finally, (3.32) readily follows from (3.13) and, since $v_R \equiv 0$ at $\partial \Omega$, the Poincaré inequality applied to $v_R$ and $(v_R)_t$ on the domain $\Omega_R$. \hfill \Box

**Lemma 3.5.** Let $\xi$, $\omega$ and $v_R$ be as in Lemma 3.4. The following inequality holds$^9$
\[
\left( \|b + a \cdot x\| \cdot \nabla (v_R)_t - \omega \times (v_R)_t, P\Delta (v_R)_t \right) \leq c \|\nabla (v_R)_t\|_2^2 + \eta \|P\Delta (v_R)_t\|_2^2.
\] (3.37)

---

$^8$All spatial norms of $v_R$ and $p_R$ are meant to be taken in $\Omega_R$.

$^9$All scalar products and norms are taken on the domain $\Omega_R$. 

Proof. For simplicity, we set $v \equiv v_R$. Clearly, by Schwarz inequality,
\[
|\langle b \cdot \nabla v, P \Delta v \rangle| \leq V_0 \|\nabla v\|_2 \|P \Delta v\|_2.
\] (3.38)
Furthermore, by Lemma 2.3(ii),
\[
\mathcal{I} := (a \times x \cdot \nabla v - a \times v, P \Delta v) = (a \times x \cdot \nabla v - a \times v, \Delta v),
\]
so that, integrating by parts over $\Omega_R$ and taking into account Lemma 2.3(i) and that $v$ vanishes at $\partial \Omega_R$, we show
\[
\mathcal{I} = -\int_{\partial \Omega} n \cdot \nabla v_t : (a \times x \cdot \nabla v) + \int_{\Omega_R} \nabla(a \times x \cdot \nabla v - a \times v) : \nabla v_t.
\] (3.39)
Also with the help of classical trace theorems, from this relation we infer
\[
|\mathcal{I}| \leq c V_0 \|\nabla v\|_{1,2}(\|\nabla v_t\|_2 + \|D^2 v\|_2) + \int_{\Omega_R} \nabla(a \times x \cdot \nabla v) : \nabla v_t,
\] (3.40)
with $c = c(\Omega)$. We now have
\[
\int_{\Omega_R} \nabla(a \times x \cdot \nabla v) : \nabla v_t = \int_{\Omega_R} \varepsilon_{pqk} a_q \partial_p v_t \partial_k v_{ti} + \int_{\Omega_R} \varepsilon_{pqr} a_r \partial_p \partial_k v_t \partial_k v_{ti} := \mathcal{I}_1 + \mathcal{I}_2,
\] (3.41)
where $\varepsilon_{pqr}$ is the alternating symbol. Obviously,
\[
|\mathcal{I}_1| \leq c V_0 \|\nabla v\|_2 \|\nabla v_t\|_2,
\] (3.42)
with $c$ numerical constant. Moreover,
\[
\mathcal{I}_2 = \frac{1}{2} \int_{\partial \Omega_R} a \times x \cdot \nabla(\partial_t |\nabla v|^2) = \frac{1}{2} \int_{\partial \Omega_R} a \times x \cdot n \partial_t |\nabla v|^2 + \frac{1}{2} \int_{\partial B_R} a \times x \cdot n \partial_t |\nabla v|^2
\]
\[
= \int_{\partial \Omega_R} a \times x \cdot n(\nabla v : \nabla v_t).
\]
Therefore, again by trace theorems
\[
|\mathcal{I}_2| \leq c V_0 \|\nabla v\|_{1,2}(\|\nabla v_t\|_2 + \|D^2 v\|_2).
\] (3.43)
We now recall Heywood inequality [19, Lemma 1]
\[
\|D^2 w\|_2 \leq c(\|P \Delta w\|_2 + \|\nabla w\|_2), \quad w \in H(\Omega_R) \cap W^{1,2}_0(\Omega_R) \cap W^{2,2}(\Omega_R),
\] (3.44)
where the constant $c$ is independent of $R$. Thus, the proof of (3.36) becomes a consequence of (3.38), (3.40)–(3.44). To prove (3.35) we observe that by Cauchy-Schwarz inequality
\[
|\langle \xi \cdot \nabla v_t, P \Delta v_t \rangle| \leq c V_0 \|\nabla v_t\|_2^2 + \eta \|P \Delta v_t\|_2^2,
\] (3.45)
with $c = c(V_0)$, and that by Lemma 2.3(ii),
\[
\mathcal{T} := (\omega \times x \cdot \nabla v_t - \omega \times v_t, P \Delta v_t) = (\omega \times x \cdot \nabla v - \omega \times v, \Delta v_t)
\] (3.46)
As a result, integrating by parts over $\Omega_R$ and proceeding exactly as in (3.39), we show
\[
\mathcal{T} = -\int_{\partial \Omega} n \cdot \nabla v_t : (\omega \times x \cdot \nabla v) + \int_{\Omega_R} \nabla(\omega \times x \cdot \nabla v_t - \omega \times v_t) : \nabla v_t.
\] (3.47)
By using an argument entirely analogous to that employed in the proof of (3.36), we may prove
\[
\left| \int_{\Omega_R} \nabla(\omega \times x \cdot \nabla v_t - \omega \times v_t) : \nabla v_t \right| \leq \frac{1}{2} \int_{\partial \Omega} \omega \times x \cdot n |\nabla v_t|^2 + c V_0 \|\nabla v_t\|_2^2,
\] (3.48)
where $c$ is a numerical constant. Therefore, from (3.47) and (3.48), we deduce
\[
|\mathcal{T}| \leq c V_0(\|\nabla v_t\|_2^2 + \|\nabla v_t\|_2^2),
\] (3.49)
with $c = c(\partial \Omega)$. Again by trace theorems, we know that for any $\eta > 0$ there exists $c = c(\eta, \Omega)$ such that [7, Exercise II.4.1]
\[
\|\nabla v_t\|_{2,0, \Omega} \leq c \|\nabla v_t\|_2^2 + \eta \|D^2 v_t\|_2^2
\]
Lemma 3.6. Let \( f, \xi, \omega \) and \((v_R, p_R)\) be as in Lemma 3.3. Then, the following estimate holds

\[
\| (v_R) \|_{L^\infty(L^\infty(\Omega_R))} + \| (v_R) t \|_{L^\infty(L^2(\Omega_R))} + \| \nabla (v_R) t \|_{L^\infty(L^2(\Omega_R))} + \| D^2 (v_R) t \|_{L^2(L^2(\Omega_R))} + \| \nabla (p_R) t \|_{L^2(L^2(\Omega_R))} \leq C \left( \| f \|_{W^{1,2}(L^2)} + \| \mathcal{F} \|_{L^2(L^2)} + \| \xi \|_{W^{2,2}(0,T)} + \| \omega \|_{W^{2,2}(0,T)} \right),
\]

(3.50)

where \( C = C(\Omega, V_0, T) \).

Proof. As usual, we shall omit the subscript \( \Omega_R \) in the various scalar products and norms, and set \( v \equiv v_R \). Moreover, we let

\[
U := v_t - V \cdot \nabla v - \omega \times v, \quad E := \| U \|_2
\]

We take the time derivative of both sides of (3.3) and dot-multiply both sides of the resulting equation, called (3.3)_t, by \( U \). Integrating by parts as necessary and observing that, by Lemma 2.3(ii), \( (V \cdot \nabla v - \omega \times v, \nabla P) = 0 \), we show

\[
\frac{1}{2} \frac{d}{dt} \| \nabla v_t \|_2^2 = (\Delta v_t, V \cdot \nabla v - \omega \times v) - (\tilde{u}_t \cdot \nabla v + \tilde{u} \cdot \nabla v_t + v_t \cdot \nabla u + v \cdot \nabla \tilde{u}_t - \tilde{f}_t, U).
\]

By virtue of Lemma 2.3(ii), Lemma 3.4 and Lemma 3.5 the previous relation furnishes

\[
\frac{1}{2} \frac{d}{dt} \| \nabla v_t \|_2^2 \leq c \left[ \| \nabla v \|_{1,2} (\| \nabla v_t \|_2 + \| P \Delta v_t \|_2) + (\| \nabla v_t \|_2 + \| \nabla v_t \|_2 + \| \tilde{f}_t \|_2) E \right],
\]

where \( c = c(\Omega, V_0, T) \). Using, suitable, Cauchy-Schwarz inequality on the right-hand side of the latter, we get

\[
\frac{d}{dt} \| \nabla v_t \|_2^2 \leq \eta \| P \Delta v_t \|_2^2 + c_1 (\| \nabla v \|_{1,2}^2 + E + \| \tilde{f}_t \|_2^2),
\]

(3.51)

where \( \eta > 0 \) is arbitrary and \( c_1 = c_1(\Omega, V_0, T, \eta) \). Next, we dot-multiply both sides of (3.3) by \( P \Delta v_t \) and integrate by parts as necessary. In this way we show

\[
\frac{1}{2} \frac{d}{dt} \| \nabla v_t \|_2^2 + \| P \Delta v_t \|_2^2 = - (\tilde{V} \cdot \nabla v - \omega \times v, P \Delta v_t) - (V \cdot \nabla v_t - \omega \times v_t, P \Delta v_t)
\]

\[
+ (\tilde{u}_t \cdot \nabla v + \tilde{u} \cdot \nabla v_t + v_t \cdot \nabla u + v \cdot \nabla \tilde{u}_t - \tilde{f}_t, P \Delta v_t).
\]

Again by Lemmas 3.4 and 3.5, from this equation we deduce

\[
\frac{1}{2} \frac{d}{dt} \| \nabla v_t \|_2^2 + \| P \Delta v_t \|_2^2 \leq c \left[ \| \nabla v \|_{1,2} (\| \nabla v_t \|_2 + \| P \Delta v_t \|_2) + (\| \nabla v_t \|_2 + \| \tilde{f}_t \|_2) \| P \Delta v_t \|_2^2 + \frac{1}{4} \| P \Delta v_t \|_2^2 \right],
\]

where \( c = c(\Omega, V_0, T) \). Thus, using Cauchy-Schwarz inequality, we readily obtain

\[
\frac{d}{dt} \| \nabla v_t \|_2^2 + \| P \Delta v_t \|_2^2 \leq c_2 (\| \nabla v \|_{1,2}^2 + \| \nabla v \|_{1,2}^2 + \| \tilde{f} \|_2^2),
\]

(3.52)

with \( c_2 = c_2(\Omega, V_0, T) \). We now multiply both sides of (3.51) by \( 2c_2 \), choose \( \eta = 1/4c_2 \) and sum, side by side, the resulting inequality and (3.52). We thus get

\[
\frac{d}{dt} \left( \| \nabla v_t \|_2^2 + 2c_2 E \right) + c_2 \| \nabla v_t \|_2^2 + \frac{1}{2} \| P \Delta v_t \|_2^2 \leq c_3 (E + \| \nabla v \|_{1,2}^2 + \| \tilde{f} \|_{W^{1,2}(L^2)}).
\]

(3.53)

Next, observe that, by (3.23) and Lemma 3.5 it follows that

\[
\int_0^T (E(t) + \| \nabla v(t) \|_{1,2}^2) dt \leq c \left[ \| f \|_{W^{1,2}(L^2)} + \| \mathcal{F} \|_{L^2(L^2)} + \| \xi \|_{W^{2,2}(0,T)} + \| \omega \|_{W^{2,2}(0,T)} \right],
\]

(3.54)

and that, by Lemma 2.2 and (3.4),

\[
\| \tilde{f} \|_{W^{1,2}(L^2)} \leq c \left( \| f \|_{W^{1,2}(L^2)} + \| \xi \|_{W^{2,2}(0,T)} + \| \omega \|_{W^{2,2}(0,T)} \right)
\]

(3.55)
with \( c = c(\Omega, T, V_0) \). Therefore, Integrating both sides of (3.53) from 0 to \( T \), then using the \( T \)-periodicity of \( E \) and \( \nabla v_t \), along with (3.54), (3.55), and (3.44) we find

\[
\| \nabla v_t \|^2_{L^2(L^2)} + \| D^2 v_t \|^2_{L^2(L^2)} \leq c \left( \| f \|_{W^{1,2}(L^2)}^2 + \| \mathbf{F} \|_{L^2(L^2)}^2 + \| \xi \|_{W^{2,2}(0,T)}^2 + \| \omega \|_{W^{2,2}(0,T)}^2 \right),
\]

(3.56)

Further, from (3.54) and (3.56) it follows that there is \( \tilde{t} \in (0, T) \) such that

\[
E(\tilde{t}) + \| \nabla v_t(\tilde{t}) \|^2_2 \leq c \left[ \| f \|_{W^{1,2}(L^2)}^2 + \| \mathbf{F} \|_{L^2(L^2)}^2 + \| \xi \|_{W^{2,2}(0,T)}^2 + \| \omega \|_{W^{2,2}(0,T)}^2 \right],
\]

(3.57)

so that, integrating (3.53) from \( \tilde{t} \) to \( 2T \) and using the \( T \)-periodicity, (3.54), (3.55), and (3.57) we conclude, in particular,

\[
\sup_{t \in [0,T]} \| \nabla v_t(t) \|^2_2 \leq c \left[ \| f \|_{W^{1,2}(L^2)}^2 + \| \mathbf{F} \|_{L^2(L^2)}^2 + \| \xi \|_{W^{2,2}(0,T)}^2 + \| \omega \|_{W^{2,2}(0,T)}^2 \right].
\]

(3.58)

If we now extend \( v_t \) to 0 outside \( \Omega_R \) and continue to denote by \( v_t \) the extension, we may use the Sobolev inequality to get

\[
\| v_t \|_6 \leq c_\Omega \| \nabla v_t \|_2.
\]

(3.59)

Furthermore, by well–known embedding theorems (e.g. [1, Corollary 5.16]) it follows that

\[
\| v \|_{L^\infty} \leq c (\| v \|_6 + \| \nabla v \|_2 + \| D^2 v \|_2)
\]

with \( c \) depending only on the regularity of \( \Omega \). Thus, since by Lemma 3.3 and (3.56) \( v \in L^\infty(L^6 \cap W^{1,2}(D^2,2)) \cap W^{1,2}(D^2,2) \), and \( W^{1,2}(D^2,2) \subset L^\infty(D^2,2) \), the last displayed inequality furnishes

\[
\| v \|_{L^\infty(L^\infty)} \leq c (\| v \|_{L^\infty(L^6)} + \| \nabla v \|_{L^\infty(L^2)} + \| D^2 v \|_{W^{1,2}(D^2,2)}),
\]

where \( c \) is depending only on the regularity of \( \Omega \). As a result, the desired estimate for \( v \) follows from the latter, Lemma 3.3 and (3.56), (3.58), (3.59). Concerning the stated estimate for \( \nabla p_t \), we see that its proof can be carried out exactly in the same way as (3.30) and will, therefore, omitted.

With the help of Lemmas 3.3 and 3.6 we are now able to prove a general existence result of \( T \)-periodic solutions to (3.1). To this end, define the function spaces

\[
\mathcal{U} := \{ T\text{-periodic } u : u \in W^{1,\infty}(L^6 \cap D^{1,2}) \cap W^{1,2}(D^{2,2}) \cap L^\infty(\Omega) \cap \Omega = 0 \}
\]

(3.60)

\[
\mathcal{P} := \{ T\text{-periodic } p : p \in L^\infty(L^6) \cap W^{1,2}(D^{1,2}) \}.
\]

Clearly, both \( \mathcal{U} \) and \( \mathcal{P} \) become Banach spaces when endowed with the norms

\[
\| u \|_{\mathcal{U}} := \| u \|_{L^\infty(\Omega)} + \| u \|_{W^{1,\infty}(L^6)} + \| \nabla u \|_{W^{1,\infty}(L^2)} + \| D^2 u \|_{W^{1,2}(L^2)};
\]

(3.61)

\[
\| p \|_{\mathcal{P}} := \| p \|_{L^\infty(\Omega)} + \| \nabla p \|_{W^{1,2}(L^2)}.
\]

Our next result represents the main finding of this section, and establishes the existence of \( T \)-periodic solutions to problem (3.1) in suitable Lebesgue and homogeneous Sobolev spaces.

**Proposition 3.1.** Suppose \( f, \xi \) and \( \omega \) satisfy the assumptions of Lemma 3.2. Then, problem (3.1) has at least one solution \((u, p, t) \in \mathcal{U} \times \mathcal{P}\) such that

\[
\| u \|_{\mathcal{U}} + \| p \|_{\mathcal{P}} \leq C \left( \| f \|_{W^{1,2}(L^2)} + \| \mathbf{F} \|_{W^{1,2}(L^2)} + \| \xi \|_{W^{2,2}(0,T)} + \| \omega \|_{W^{2,2}(0,T)} \right),
\]

with \( C = C(\Omega, T, V_0) \).

**Proof.** Set \( u_R := u_R + \tilde{u} \), with \((v_R, p_R)\) given in Lemma 3.3. Then, \((u_R, p_R)\) is a solution to (3.2). Because of the properties of the extension \( \tilde{u} \), it is at once recognized that both estimates (3.23) and (3.50) hold with \( v_R \equiv u_R \), namely,

\[
\| u_R \|_{L^\infty(\Omega_R)} + \| u_R \|_{W^{1,\infty}(L^6(\Omega_R))} + \| \nabla u_R \|_{W^{1,\infty}(L^2(\Omega_R))} + \| D^2 u_R \|_{W^{1,2}(L^2(\Omega_R))} \leq c \mathcal{D}.
\]

(3.62)

where \( c = c(\Omega, T, V_0) \) and

\[
\mathcal{D} := \| f \|_{W^{1,2}(L^2)} + \| \mathbf{F} \|_{W^{1,2}(L^2)} + \| \xi \|_{W^{2,2}(0,T)} + \| \omega \|_{W^{2,2}(0,T)}.
\]

\(^{10}\)See Footnote 7.
We want to let \( R \to \infty \) (\( R \in \mathcal{S} \)) and show that \((u_R, p_R)\) tends (in suitable topology) to a solution \((u, p)\) ∈ \( \mathcal{H} \times \hat{\mathcal{P}} \) to (3.1). This can be done by an argument similar to that given in [8, Section 3]. Let \( \chi = \chi(s) \), \( s > 0 \) be a smooth, non-increasing real function such that \( \chi(s) = 1 \) for \( s \leq 1/2 \) and \( \chi(s) = 0 \) for \( s \geq 1 \). For \( R_m \in \mathcal{S} \), set \( \chi_m(x) = \chi(|x|/R_m) \). We thus have, for all \( x \in \Omega \),

\[
\chi_m(x) = \begin{cases} 
1 & \text{if } |x| \leq \frac{1}{2} R_m, \\
0 & \text{if } |x| \geq R_m.
\end{cases}
\]

Notice that

\[
|D^\alpha \chi_m(x)| \leq CR_m^{-|\alpha|}, \quad |\alpha| \geq 0; \quad \text{supp} \chi \subseteq \left\{ \frac{1}{2} R_m \leq |x| \leq R_m \right\},
\]

where \( C \) is independent of \( x \) and \( m \). Let \( u_m := u_{R_m} \). We observe that \( u_m \) satisfies the following inequality

\[
\int_{\frac{1}{2} R_m \leq |x| \leq R_m} |u_m(x)|^2 |x|^{-2} \leq 4 \| \nabla u_m \|^2_{2, \Omega_{R_m}}.
\]  

In fact, let’s extend \( u_m \) to 0 outside the ball of radius \( R_m \) and continue to denote by \( u_m \) such an extension. Then, clearly, \( u_m \in D^{1,2}(\Omega) \) and \( \lim_{|x| \to \infty} u_m(x) = 0 \). Thus, (3.64) follows from [7, Theorem II.6.1(i)].

We now set \( \tilde{u}_m := \chi_m u_m \). From (3.62), (3.63) and (3.64) we readily deduce that \( \tilde{u}_m \in \mathcal{H} \) and

\[
\| \tilde{u}_m \|_\mathcal{H} \leq c \mathcal{D}
\]

where, from now onward, \( c \) denotes a constant independent of \( m \). Therefore, there exists \( u \in \mathcal{H} \) such that (possibly, along a subsequence)

\[
\tilde{u}_m \to u, \quad \text{weakly in } \mathcal{H},
\]

and, in addition,

\[
\| u \|_\mathcal{H} \leq c \mathcal{D}.
\]  

We shall show that

\[
\int_0^\tau \int_\Omega (u_t - V(t) \cdot \nabla u + \omega(t) \times u - \Delta u - f) \cdot \psi = 0,
\]

for all \( \psi \in C_0^\infty(\Omega) \) with \( \text{div } \psi = 0 \), and all \( \tau \in [0, T] \). Actually, let us dot-multiply by \( \psi \) both sides of (3.2)1 where \( R = R_m \) is so large that \( \Omega_{\frac{1}{2} R_m} \) contains the support, \( K \), of \( \psi \). By integrating over \( \Omega_{R_m} \times (0, \tau) \), \( \tau \in [0, T] \) the resulting equation, (also by parts, when necessary), we get

\[
\int_0^\tau \int_K (u_{mt} - V(t) \cdot \nabla u_m + \omega(t) \times u_m - \Delta u_m - f) \cdot \psi = 0.
\]

Since \( u_m(x,t) = \tilde{u}_m(x,t) \) for all \( (x,t) \in K \times [0,T] \), this relation furnishes

\[
\int_0^\tau \int_\Omega (\tilde{u}_{mt} - V(t) \cdot \nabla \tilde{u}_m + \omega(t) \times \tilde{u}_m - \Delta \tilde{u}_m - f) \cdot \psi = 0.
\]  

Thus, passing to the limit \( m \to \infty \) in (3.68) and employing (3.65) we arrive at (3.67). Now, since \( \psi \) is arbitrary in its class, from well-known results and the fact that \( u \in \mathcal{H} \), we deduce that there is \( p \in L^2(W^{1.2}(\Omega_R)) \), arbitrary \( R > R_m \), such that \((u,p)\) satisfy (3.1) a.e. in \( \Omega \times [0,T] \). Moreover, it is not difficult to prove that \( u \) obeys (3.1)3 a.e. in \( \partial \Omega \times [0,T] \). Finally, for any \( \phi \in C_0^\infty(\Omega) \) we have for all sufficiently large \( m \)

\[
\int_0^\tau \int_\Omega \tilde{u}_m \cdot \nabla \phi = \int_0^\tau \int_{\Omega_{R_m}} u_m \cdot \nabla \phi = 0,
\]

which, by (3.65), implies that \( u \) satisfies (3.1)2 a.e. in \( \Omega \times [0,T] \). Thus, in order to complete the proof of the proposition, it remains to show the stated properties for the pressure field \( p \). To this end, we recall the Hardy-type inequality [7, Theorem II.6.1(i)]

\[
\|u \|_{L^6(\Omega)} \cap D^{1/2}(\Omega)} \leq C \| \nabla u \|_2, \quad u \in L^6(\Omega) \cap D^{1.2}(\Omega).
\]  

\[
\|u \|_{L^6(\Omega)} \cap D^{1/2}(\Omega)} \leq C \| \nabla u \|_2, \quad u \in L^6(\Omega) \cap D^{1.2}(\Omega).
\]
As a result, multiplying both sides of (3.1) by $|x|^{-1}$ and bearing in mind that $u \in \mathcal{W}$, we deduce a.e. in $[0, T]$

$$|x|^{-1} \nabla p \in L^2(\Omega).$$  \hspace{1cm} (3.70)

Observing that

$$\text{div} (V \cdot \nabla u + \omega \times u) = \text{div} u_t = \text{div} \Delta u = 0$$
$$V \cdot \nabla u = 0, \quad \text{at } \partial \Omega; \quad \Delta (V + \omega \times V) = 0,$$

from (3.1) we obtain for a.a. $t \in [0, T]$ and in distributional sense

$$\Delta p = \text{div} f \quad \text{in } \Omega; \quad \frac{\partial p}{\partial n} = -n \cdot \Delta u \quad \text{at } \partial \Omega. \hspace{1cm} (3.72)$$

Classical results on the Neumann problem ensure the existence of at least one solution, $\hat{p}$, to (3.72) such that

$$\|\nabla \hat{p}\|_2 + \|\nabla p_t\|_2 \leq c (\|f\|_2 + \|f_t\|_2 + \|\mathcal{F}\|_2 + \|\Delta u\|_2 + \|\Delta u_t\|_2).$$  \hspace{1cm} (3.73)

Setting $p := \hat{p} - \hat{p}$, it follows then, for a.a. $t \in [0, T]$

$$\Delta p = 0 \quad \text{in } \Omega; \quad \frac{\partial p}{\partial n} = 0 \quad \text{at } \partial \Omega. \hspace{1cm} (3.74)$$

Since $\hat{p} \in D^{1,2}(\Omega)$ and $p$ satisfies (3.70), again by classical results on the Neumann problem, we have [7, Exercise V.3.6]

$$D^\alpha p = O(|x|^{-|\alpha|} - 1), \quad |\alpha| \geq 0, \quad \text{as } |x| \to \infty. \hspace{1cm} (3.75)$$

Therefore, multiplying both sides of (3.74) by $p$, integrating by parts over $\Omega_R$, and then letting $R \to \infty$, with the help of (3.76) we deduce $\|\nabla p\|_2 = 0$, namely,

$$\nabla p = \nabla \hat{p} \quad \text{in } \Omega.$$  

From the latter, (3.73) and (3.66) we thus infer

$$\|\nabla p\|_{W^{1,2}(L^2)} \leq c \mathcal{D}. \hspace{1cm} (3.76)$$

Finally, possibly adding to $p$ a function of time, we observe that [7, II.9.1(i)]

$$\|p\|_6 \leq c \|\nabla p\|_2$$

which, in turn, by (3.76) and the embedding $W^{1,2}(D^{1,2}) \subset L^\infty(D^{1,2})$, proves $p \in \mathcal{P}$ and the corresponding estimate in (3.61). The proof of the proposition is completed. \hspace{1cm} \Box

4. Linear Problem: Existence, Uniqueness and Asymptotic Behavior

Our next goal is to show that, if $\mathcal{F}$ also decays “sufficiently fast” at large spatial distances, then a similar property must hold for the solution $u$ given in Proposition 3.1. In this regard, we notice that, with a suitable choice of the axes, the average of $\xi(t)$ over a period can be written as

$$\lambda e_1 := \frac{1}{T} \int_0^T \xi(t)dt, \quad \lambda \geq 0, \hspace{1cm} (4.1)$$

The following result holds.

**Proposition 4.1.** Let $f$, $\xi$ and $\omega$ be as in Proposition 3.1. Suppose in addition that, if $\xi(t) \neq 0$, it is $\xi(t) = \xi(t)e_1$, and $\omega(t) = \omega(t)e_1$, while no further assumption is imposed on $\omega$ if $\xi(t) \equiv 0$. Then, if $\|\mathcal{F}\|_{2,2} < \infty$, it follows $\|u\|_{1,1} < \infty$ and $p \in L^{\infty}(L^r)$ for all $r \in (\frac{2}{3}, 6)$. Moreover, there are $C = C(\Omega, T, V_0)$ and $C_1 = C_1(\Omega, T, V_0, r)$ such that, setting

$$\mathcal{D} := \|\mathcal{F}\|_{2,2} + \|f\|_{W^{1,2}(L^2)} + \|\mathcal{F}\|_{W^{1,2}(L^2)} + \|\xi\|_{W^{2,2}(0, T)} + \|\omega\|_{W^{2,2}(0, T)} \hspace{1cm} (4.2)$$
we have
\[ \|u\|_{\infty,1,\lambda} \leq C_D; \quad \|p\|_{L^\infty(L^r)} \leq C_1 D. \]  
(4.3)

**Proof.** Let \( \chi = \chi(s) \) be the “cut-off” function introduced in Proposition 3.1, and set \( \chi_R(x) = \chi(|x|/R) \), where \( R > 2R_0 \). Further, let \( z \) be a solution to problem (2.2) with \( A \equiv \{ x \in \mathbb{R}^3 : \ |x| < R \} \), and \( f \equiv -\nabla \chi_R \cdot u \). Since \( \int_A f = 0 \), Lemma 2.1 guarantees the existence of such a \( z \) with the properties stated there. Setting
\[ w := \chi_R u + z, \quad p := \chi_R p, \quad H = \chi_R F \]  
(4.4)
from (3.1) we deduce that \((w, p)\) is a \( T \)-periodic solution to the following problem
\[ \begin{align*}
\partial_t w - V \cdot \nabla w + \omega \times w &= \Delta w - \nabla p + \text{div} H + g \\
\text{div} w &= 0
\end{align*} \]  
in \( \mathbb{R}^3 \times (0, T) \),
(4.5)
where
\[ g := -z_t + V \cdot \nabla z - \omega \times z + \Delta z - 2\nabla \chi_R \cdot \nabla u + p \nabla \chi_R - \xi \cdot \nabla \chi_R u - u \Delta \chi_R - F \cdot \nabla \chi_R. \]
Extending \( z \) to 0 outside its (spatial) support, we obtain that \( g \) is of bounded support as well. From Lemma 2.1, Proposition 3.1 and the assumption on \( f \) we readily show
\[ \sup_{t \geq 0} \|g(t)\|_4 \leq c (\|f\|_{W^{1,2}(L^2)} + \|F\|_{W^{1,2}(L^2)} + \|\xi\|_{W^{2,2}(0,T)} + \|\omega\|_{W^{2,2}(0,T)}), \]  
(4.6)
where \( c \) depends, at most, on \( \Omega, T \) and \( V_0 \). Define
\[ A = A(t) := \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\
\omega_3(t) & 0 & -\omega_1(t) \\
-\omega_2(t) & \omega_1(t) & 0 \end{pmatrix}, \]
and let \( Q = Q(t) \) be the family of orthogonal transformations that are solutions to the following initial-value problem
\[ \dot{Q} = Q \cdot A, \quad Q(0) = I, \]
with \( I \) the identity matrix. It is readily checked that, for any \( a \in \mathbb{R}^3 \),
\[ Q^T \cdot Q \cdot a = -Q^T \cdot Q \cdot a = \omega \times a. \]  
(4.7)
Moreover, in the case when \( \omega_2 \equiv \omega_3 \equiv 0 \), we also have
\[ Q(t) \cdot e_1 = e_1, \quad \text{for all } t \geq 0. \]  
(4.8)
We next introduce the new variable \( y \) defined by
\[ y = Q(t) \cdot x + x_0(t) \]  
(4.9)
where
\[ x_0(t) := \int_0^t (\xi(s) - \lambda e_1) \, ds, \]  
(4.10)
Since \((1/T) \int_0^T (\xi(t) - \lambda e_1) \, dt = 0\), by [9, Proposition 1] there is \( M = M(T, V_0) > 0 \) such that
\[ \sup_{t \geq 0} |x_0(t)| \leq M \]  
(4.11)
which, by (4.9) implies, in particular,
\[ |x| - M \leq |y| \leq |x| + M. \]  
(4.12)
Set
\[ v(y, t) = Q(t) \cdot w(Q^\top(t) \cdot (y - x_0(t)), t), \]
\[ p(y, t) = p(Q^\top(t) \cdot (y - x_0(t)), t), \]
\[ h(y, t) = Q(t) \cdot g(Q^\top(t) \cdot (y - x_0(t)), t) \]
\[ G(y, t) = Q(t) \cdot H(Q^\top(t) \cdot (y - x_0(t)), t) \cdot Q^\top(t). \] (4.13)

By a straightforward calculation we show
\[ \frac{\partial v}{\partial t} = Q \cdot w + Q \cdot \left\{ (Q^\top \cdot (y - x_0) - Q^\top \cdot \dot{x}_0) \cdot \nabla w + \frac{\partial w}{\partial t} \right\}, \]
\[ = Q \cdot \left\{ Q^\top \cdot Q \cdot w + (Q^\top \cdot (y - x_0) - Q^\top \cdot \dot{x}_0) \cdot \nabla w + \frac{\partial w}{\partial t} \right\}, \]
which, in turn, in view of (4.7), (4.9) and (4.10), provides
\[ \frac{\partial v}{\partial t} = Q \cdot \left\{ \omega \times w - \nabla w + (\xi - \lambda e_1) \cdot \nabla w + \frac{\partial w}{\partial t} \right\}. \] (4.14)

Now, suppose \( \xi \neq 0 \). Then, by assumption and (4.8) we obtain
\[ Q^\top \cdot (\xi - \lambda e_1) = \xi - \lambda e_1. \] (4.15)

If, on the other hand, \( \xi = 0 \), then, obviously, \( Q^\top \cdot (\xi - \lambda e_1) = \xi - \lambda e_1 = 0 \). Thus, in either case, from (4.14) we deduce
\[ \frac{\partial v}{\partial t} = Q \cdot \left\{ \omega \times w - \nabla w + (\xi - \lambda e_1) \cdot \nabla w + \frac{\partial w}{\partial t} \right\}, \]
and so, by (4.5)_1,
\[ \frac{\partial v}{\partial t} = Q \cdot (\lambda \partial_1 w + \Delta w - \nabla p + \text{div} \ H + g). \] (4.16)

By a straightforward calculation, we show
\[ \Delta_y v = Q \cdot \Delta_x w, \quad \nabla_y p = Q \cdot \nabla_x p, \quad \text{div}_y G = Q \cdot \text{div}_x \mathcal{H}, \quad \text{div}_y v = \text{div}_x w, \]
and also, by assumption and (4.8),
\[ \partial_1 v = Q \cdot \partial_1 w. \]

As a result, from (4.16) we infer that \((v, p)\) is a solution to the following Cauchy problem
\[ \begin{align*}
\begin{cases}
\partial_t v - \lambda \partial_1 v = \Delta v - \nabla p + \text{div} G + h, \\
\text{div} v = 0
\end{cases}
\end{align*} \quad \text{in } \mathbb{R}^3 \times (0, \infty), \]
\[ v(x, 0) = w(x, 0). \] (4.17)

In order to obtain the desired spatial decay property, we would like to apply Lemma 2.4 to (4.17). To this end, we begin to observe that, by (2.1) and (4.12) it follows that
\[ (1 + |x|)(1 + 2\lambda s(x)) \leq (1 + |y|)(1 + 2\lambda s(y)) + 2\lambda (M + x_{01}(t)) \]
\[ \leq c (1 + |y|) (1 + 2\lambda s(y)), \] (4.18)
and, likewise,
\[ (1 + |y|)(1 + 2\lambda s(y)) \leq c (1 + |x|)(1 + 2\lambda s(x)). \] (4.19)

We next look for a solution to (4.17) of the form \((v_1 + v_2, p)\) where
\[ \begin{align*}
\begin{cases}
\partial_t v_1 - \lambda \partial_1 v_1 = \Delta v_1 - \nabla p_1 + \text{div} G + h, \\
\text{div} v_1 = 0
\end{cases}
\end{align*} \quad \text{in } \mathbb{R}^3 \times (0, \infty), \]
\[ v_1(x, 0) = 0, \] (4.20)
and
\[
\begin{aligned}
(v_2)_t - \lambda \partial_t v_2 &= \Delta v_2 \\
\text{div } v_2 &= 0 \\
v(x,0) &= w(x,0).
\end{aligned}
\] (4.21)

From (4.13), (4.19) and (4.4) we infer
\[
|\mathcal{G}|_{\infty,2,\lambda} \leq C |\mathcal{F}|_{\infty,2,\lambda}
\] (4.22)

Moreover, by (4.13) and (4.6), it follows that
\[
\sup_{t \geq 0} \|h(t)\|_1 \leq c (\|f\|_{W^{1,2}(L^2)} + \|\mathcal{F}\|_{W^{1,2}(L^2)} + \|\xi\|_{W^{2,2}(0,T)} + \|\omega\|_{W^{2,2}(0,T)}).
\] (4.23)

As a result, from Lemma 2.4 we conclude that (4.21) has one and only one solution such that for all \(T > 0\),
\[
v_1 \in L^2(0; W^{2,2}), \quad (v_1)_t \in L^2(0; L^2); \quad \nabla p \in L^2(0; L^2);
\]
\[
|v_1|_{\infty,1,\lambda} \leq c, \quad p_1 \in L^\infty(L^r), \quad \text{for all } r \in (\frac{3}{2}, \infty)
\]
satisfying, in addition, the inequality
\[
|v_1|_{\infty,1,\lambda} \leq C D, \quad \|p\|_{L^\infty(L^r)} \leq C_1 D.
\] (4.24)

with \(C\) and \(C_1\) as in (4.3). Concerning (4.21), since by Proposition 3.1, (4.4) and (2.2) it is \(w(x,0) \in L^6(\mathbb{R}^3)\) and \(\text{div } w(x,0) = 0\), it follows that there exists a (unique) solution \(v_2\) such that (see, e.g., [7, Theorem VIII.4.3])
\[
v_2, \partial_t v_2 D^2 v_2 \in L^\infty(\mathbb{R}^3), \quad \|v_2(t)\|_{\infty} \leq C_2 t^{-\frac{1}{2}} \|w(0)\|_6, \quad \sup_{t \in (0,\infty)} \|v_2(t)\|_6 \leq C_2 \|w(0)\|_6.
\] (4.25)

In view of the regularity properties of \(u\) (and hence of \(w\)) and those in (2.6), (4.25) for \(v_i, i = 1, 2\), respectively, we may use the results proved in [7, Lemma VIII.4.2] to guarantee, by uniqueness, that \(v_1 + v_2 \equiv v\) and \(p_1 \equiv p\). Thus, in particular, from (4.24) we get
\[
\|p\|_{L^\infty(L^r(\Omega \times \mathbb{R}^3))} \leq c (\|\mathcal{F}\|_{\infty,2,\lambda} + \|f\|_{W^{1,2}(L^2)} + \|\mathcal{F}\|_{W^{1,2}(L^2)}
+ \|\xi\|_{W^{2,2}(0,T)} + \|\omega\|_{W^{2,2}(0,T)}), \quad \text{for all } r \in (\frac{3}{2}, \infty),
\] (4.26)

where \(c = c(\Omega, T, V_0, r)\). Further, due to the \(T\)-periodicity of \(w\) and (4.13), for arbitrary positive integer \(n\) and \(t \in [0, T]\) we obtain
\[
|w(x,t)|((1 + |x|)(1 + 2\lambda s(x)) = |v(y,t + nT)|(1 + |x|)(1 + 2\lambda s(x)) \leq (|v_1(y,t + nT)| + |v_2(y,t + nT)|)(1 + |x|)(1 + 2\lambda s(x)).
\] (4.27)

Employing (4.18), (4.24) and (4.25) in this inequality we get
\[
|w(x,t)|(1 + |x|)(1 + 2\lambda s(x)) \leq c ((1 + |x|)(1 + 2\lambda s(x))(t + nT)^{-\frac{1}{2}} \|w(0)\|_6
+ \|\mathcal{F}\|_{\infty,2,\lambda} + \|f\|_{W^{1,2}(L^2)} + \|\mathcal{F}\|_{W^{1,2}(L^2)} + \|\xi\|_{W^{2,2}(0,T)} + \|\omega\|_{W^{2,2}(0,T)})
\]
so that, by letting \(n \to \infty\) and recalling that, uniformly in \(t \geq 0\), \(u(x,t) \equiv w(x,t)\) for \(|x|\) sufficiently large (\(> \frac{T}{n}\)) we deduce
\[
|u|_{\infty,1,\lambda,\Omega \times \mathbb{R}^3} \leq c (\|\mathcal{F}\|_{\infty,2,\lambda} + \|f\|_{W^{1,2}(L^2)} + \|\mathcal{F}\|_{W^{1,2}(L^2)} + \|\xi\|_{W^{2,2}(0,T)} + \|\omega\|_{W^{2,2}(0,T)}).
\] (4.28)

Moreover, by Proposition 3.1 we have
\[
|u|_{L^\infty(L^\infty)} + \|p\|_{L^\infty(L^6)} \leq c (\|f\|_{W^{1,2}(L^2)} + \|\mathcal{F}\|_{W^{1,2}(L^2)} + \|\xi\|_{W^{2,2}(0,T)} + \|\omega\|_{W^{2,2}(0,T)})
\] (4.29)

and the desired result then follows from (4.26), (4.28) and (4.29).
Combining the results of Proposition 3.1 and Proposition 4.1 we can prove the main achievement of this section. Precisely, let
\[
\mathcal{U}_\lambda := \{ \mathbf{u} \in \mathcal{V} : \| \mathbf{u} \|_{\mathcal{U}_\lambda} := \| \mathbf{u} \|_\mathcal{V} + \| \mathbf{u} \|_{C^{0,1}_\lambda} < \infty \},
\]
and
\[
\mathcal{P} := \{ p \in \mathcal{P} : \| p \|_\mathcal{P} := \| p \|_\mathcal{V} + \| p \|_{L^\infty(L^r)} < \infty, \text{ for all } r \in \left( \frac{3}{2}, 6 \right) \},
\]
with \( \lambda \) defined in (4.1). The following theorem holds.

**Theorem 4.1.** Let \( \mathbf{f} = \text{div} \mathbf{F} \), and \( \mathbf{f}, \mathbf{F} \in W^{1,2}(L^2) \), \( \xi, \omega \in W^{2,2}(0, T) \) be given \( T \)-periodic functions, with \( \| \mathbf{F} \|_{\mathcal{C}^{0,2,\lambda}} < \infty \). Suppose in addition that, if \( \xi(t) \neq 0 \), it is \( \xi(t) = \xi(t)e_1 \), and \( \omega(t) = \omega(t)e_1 \), while no further assumption is imposed on \( \omega \) if \( \xi(t) \equiv 0 \). Then, there exists one and only one \( T \)-periodic solution \( (\mathbf{u}, p) \) to (3.1) with \( (\mathbf{u}, p) \in \mathcal{U}_\lambda \times \mathcal{P} \). Moreover, the following estimate holds
\[
\| \mathbf{u} \|_{\mathcal{U}_\lambda} \leq C D, \quad \| p \|_\mathcal{P} \leq C_1 D
\]
where \( D \) is defined in (4.2), \( C = C(\Omega, T, V_0) \), \( C_1 = C_1(\Omega, T, V_0, r) \), and \( V_0 \) is given in (3.8).

**Proof.** In view of Proposition 3.1 and Proposition 4.1, it remains to show the uniqueness property, namely, that, under the given assumption on \( \mathbf{V} \), the problem
\[
\begin{align*}
\mathbf{u}_t - \mathbf{V}(t) \cdot \nabla \mathbf{u} + \omega(t) \times \mathbf{u} &= \Delta \mathbf{u} - \nabla p, \\
\text{div} \mathbf{u} &= 0, \\
\mathbf{u}(x, t) &= 0, \quad (x, t) \in \partial \Omega \times [0, T],
\end{align*}
\]
has only the null solution in the class \( (\mathbf{u}, p) \in \mathcal{U}_\lambda \times \mathcal{P} \). To this end, we recall [16, Lemma 3] [7, Lemma II.6.4] that there exists a “cut-off” function, \( \psi_R \in C^\infty_0(\mathbb{R}^3) \), \( R \in (0, \infty) \), with the following properties
\begin{enumerate}[(i)]
\item \( \psi_R(x) \in [0, 1] \), for all \( x \in \mathbb{R}^3 \) and \( R > 0 \);
\item \( \lim_{R \to \infty} \psi_R(x) = 1 \) for all \( x \in \mathbb{R} \);
\item \( \nabla \psi_R(x) \cdot (\omega \times x) = 0 \), for all \( x \in \mathbb{R} \) and \( R > 0 \);
\item \( \text{supp} \psi_R \subset \{ x \in \mathbb{R}^3 : 2R < |x| < (2R)^2 \} \);
\item \( \| \mathbf{u} \nabla \psi_R \|_2 \leq c \| \nabla \mathbf{u} \|_2 \frac{R}{\sqrt{2}} \), with \( c \) independent of \( R \);
\item \( |x|^{-2} \partial_1 \psi_R \in L^1(\Omega) \), and \( \lim_{R \to \infty} \int_{\Omega} |x|^{-2} \partial_1 \psi_R = 0 \).
\end{enumerate}

Let us dot-multiply both sides of (4.30) by \( \psi_R \mathbf{u} \) and integrate by parts over \( \Omega \times [0, T] \). Using \( T \)-periodicity, (iii) and (4.30) 2,3, we thus obtain
\[
\int_0^T \nabla \psi_R \nabla \mathbf{u} = \int_0^T \int_\Omega \left( -\nabla \psi_R \cdot \nabla \mathbf{u} + \frac{1}{2} \xi \partial_1 \psi_R \mathbf{u}^2 + p \nabla \psi_R \cdot \mathbf{u} \right)
\]
\[
:= I_1(R) + I_2(R) + I_3(R).
\]
By Schwarz inequality, (iv) and (v), we show
\[
|I_1(R)| \leq \| \mathbf{u} \nabla \psi_R \|_{2, \Omega} \frac{R}{\sqrt{2}} \| \nabla \mathbf{u} \|_{2, \Omega} \frac{R}{\sqrt{2}} \leq c \| \nabla \mathbf{u} \|_{2, \Omega}^2 \frac{R}{\sqrt{2}},
\]
and, likewise,
\[
|I_3(R)| \leq c \| p \|_{2, \Omega} \frac{R}{\sqrt{2}} \| \nabla \mathbf{u} \|_{2, \Omega} \frac{R}{\sqrt{2}}.
\]

Since \( \mathbf{u}(x, t) = O(|x|^{-1}) \) uniformly in time, we also get
\[
|I_2(R)| \leq c \int_0^T \int_{\Omega} \left| \partial_1 \psi_R \right|.
\]

Thus, letting \( R \to \infty \) in (4.31), and taking into account (4.32)–(4.34), (i), (vi) and that \( (\mathbf{u}, p) \in \mathcal{U}_\lambda \times \mathcal{P} \), by the dominated convergence theorem we infer
\[
\int_0^T \| \nabla \mathbf{u}(t) \|_2^2 = 0,
\]
which, in view of (4.30) 3 and the summability properties of \( p \), furnishes \( \mathbf{u} \equiv p \equiv 0 \). \( \square \)
5. On the Unique Solvability of the Nonlinear Problem

In this section we shall prove existence and uniqueness of $T$-periodic solutions to the full nonlinear problem (1.1), provided the magnitude of the data is suitably restricted. This goal will be reached by combining Theorem 4.1 with a contraction mapping argument.

Set

$$\mathcal{F}_\lambda := \{ \mathcal{F} : \Omega \times [0,T] \to \mathbb{R}^{3 \times 3} : \mathcal{F}, \text{div} \mathcal{F} \in W^{1,2}(L^2), \| \mathcal{F} \|_{\infty,2,\lambda} < \infty \},$$

with $\lambda$ defined in (4.1), and define

$$\mathcal{D}_\lambda := \{ (\mathcal{F}, \xi, \omega) \text{ $T$-periodic : } \mathcal{F} \in \mathcal{F}_\lambda; \ \xi, \omega \in W^{2,2}(0,T) \}$$

endowed with the norm

$$\|(\mathcal{F}, \xi, \omega)\|_{\mathcal{D}_\lambda} := \|\text{div} \mathcal{F}\|_{W^{1,2}(\Omega)} + \|\mathcal{F}\|_{W^{1,2}(\Omega)} + \|\mathcal{F}\|_{\infty,2,\lambda} + \|\xi\|_{W^{2,2}(0,T)} + \|\omega\|_{W^{2,2}(0,T)}.$$

We need the following preliminary result.

Lemma 5.1. Let $u, w \in \mathcal{D}_\lambda$. Then $u \otimes w \in \mathcal{D}_\lambda$ and

$$\|\text{div} (u \otimes w)\|_{W^{1,2}(L^2)} + \|u \otimes w\|_{W^{1,2}(L^2)} + \|u \otimes w\|_{\infty,2,\lambda} \leq c\|u\|_{\mathcal{D}_\lambda} \|w\|_{\mathcal{D}_\lambda}. \tag{5.1}$$

Proof. Obviously,

$$\|u \otimes w\|_{\infty,2,\lambda} \leq \|u\|_{\mathcal{D}_\lambda} \|w\|_{\mathcal{D}_\lambda}. \tag{5.2}$$

We next observe that, since $\text{div} u = 0$, it follows

$$\text{div} (u \otimes w) = u \cdot \nabla w. \tag{5.3}$$

Now, clearly,

$$\|u \cdot \nabla w\|_{L^2(L^2)} \leq \|u\|_{\infty,1,\lambda} \|\nabla w\|_{L^\infty(L^2)} \leq \|u\|_{\mathcal{D}_\lambda} \|w\|_{\mathcal{D}_\lambda}, \tag{5.4}$$

and

$$\|u \otimes w\|_{L^2(L^2)} \leq c \|u\|_{\infty,1,\lambda} \|w\|_{\infty,1,\lambda} \leq c \|u\|_{\mathcal{D}_\lambda} \|w\|_{\mathcal{D}_\lambda}. \tag{5.5}$$

Moreover, by using the inequality $\|\nabla w\|_3 \leq c \|\nabla w\|^\frac{1}{2}_2 \|D^2 w\|^\frac{1}{2}_2$ (see [2, Theorem 2.1]) along with Hölder inequality, we get

$$\|u_t \cdot \nabla w\|_{L^2(L^2)} + \|u \cdot \nabla w_t\|_{L^2(L^2)} \leq \|u_t\|_{L^{\infty}(L^6)} \|\nabla w\|_{L^2(L^2)} + \|u\|_{\infty,1,\lambda} \|\nabla w_t\|_{L^2(L^2)}$$

$$\leq c \left( \|u\|_{\mathcal{D}_\lambda} (\|\nabla w\|_{L^{\infty}(L^2)} \|D^2 w\|_{L^2(L^2)}^\frac{1}{2} + \|u\|_{\mathcal{D}_\lambda} \|w\|_{\mathcal{D}_\lambda}) \right)$$

$$\leq c \|u\|_{\mathcal{D}_\lambda} \|w\|_{\mathcal{D}_\lambda}. \tag{5.6}$$

Thus, combining (5.6) and (5.4) and (5.3) we conclude

$$\|\text{div} (u \otimes w)\|_{W^{1,2}(L^2)} \leq c\|u\|_{\mathcal{D}_\lambda} \|w\|_{\mathcal{D}_\lambda}. \tag{5.7}$$

By Hardy inequality [7, Theorem II.6.1], we infer

$$\|u_t \otimes w\|_{L^2(L^2)} + \|u \otimes w_t\|_{L^2(L^2)} \leq \|w\|_{\infty,1,\lambda} \|u_t\|_{L^2(L^2)} + \|u\|_{\infty,1,\lambda} \|w_t\|_{L^2(L^2)}$$

$$\leq c \left( \|w\|_{\infty,1,\lambda} \|u_t\|_{L^2(L^2)} + \|u\|_{\infty,1,\lambda} \|w_t\|_{L^2(L^2)} \right)$$

$$\leq c \|u\|_{\mathcal{D}_\lambda} \|w\|_{\mathcal{D}_\lambda}. \tag{5.8}$$

The latter and (5.5) furnish

$$\|u \otimes w\|_{W^{1,2}(L^2)} \leq c \|u\|_{\mathcal{D}_\lambda} \|w\|_{\mathcal{D}_\lambda}. \tag{5.9}$$

The lemma follows from (5.2), (5.7) and (5.8).

We are now in a position to prove the main result of this paper.
\textbf{Theorem 5.1.} Let \((\mathcal{B}, \xi, \omega) \in \mathcal{D}_\lambda\). Suppose in addition that, if \(\xi(t) \neq 0\), it is \(\xi(t) = \xi(t)e_1\), and \(\omega(t) = \omega(t)e_1\), while no further assumption is imposed on \(\omega\) if \(\xi(t) = 0\). Then, there exists \(\varepsilon_0 = \varepsilon_0(\Omega, T, V_0) > 0^{11}\) such that if

\[
\|([\mathcal{B}, \xi, \omega])\|_{\mathcal{D}_\lambda} < \varepsilon_0,
\]

problem (1.1) has one and only one solution \((u, p) \in \mathcal{V}_\lambda \times \mathcal{P}\) with \(\|u\|_{\mathcal{V}_\lambda} + \|p\|_{\mathcal{P}} \leq c\|([\mathcal{B}, \xi, \omega])\|_{\mathcal{D}_\lambda}\), for some \(c = c(\Omega, T, V_0, r)\).

\textit{Proof.} We want to apply the contraction mapping theorem to the map

\[ M : u \in \mathcal{V}_\lambda \mapsto u \in \mathcal{V}_\lambda, \]

with \(u\) solving the linear problem

\[
\begin{aligned}
\frac{du}{dt} - \nabla \cdot \nu u + \omega \times u &= \Delta u - \nu p + u \cdot \nabla u + \nabla \mathcal{B} \\
\text{div} u &= 0
\end{aligned}
\]

in \(\Omega \times (0, T)\) \hspace{1cm} (5.9)

\[
u u(x, t) = \nu (t), \quad (x, t) \in \partial \Omega \times [0, T],
\]

Set

\[ u \cdot \nabla u = \text{div} (u \otimes u) := \text{div} F, \hspace{1cm} (5.10)\]

where we used the condition \(\text{div} u = 0\). In virtue of Lemma 5.1, by assumption, and by the obvious inequality

\[
|F|_{\infty, 2, \lambda} \leq c_1 |u|_{\infty, 1, \lambda}^2, \quad u \in \mathcal{V}_\lambda,
\]

we infer that \(F, B, \xi\) and \(\omega\) satisfy the assumptions of Theorem 4.1. Therefore, by that theorem we conclude that the map \(M\) is well defined and, in particular, that

\[
\|u\|_{\mathcal{V}_\lambda} \leq c_2 \left(\|u\|^2_{\mathcal{V}_\lambda} + \|([\mathcal{B}, \xi, \omega])\|_{\mathcal{V}_\lambda}\right), \hspace{1cm} (5.11)
\]

with \(c_2 = c_2(\Omega, T, V_0)\). If we now take

\[
\|u\|_{\mathcal{V}_\lambda} < \delta, \quad \delta := 4c_2\|([\mathcal{B}, \xi, \omega])\|_{\mathcal{V}_\lambda}, \quad \|([\mathcal{B}, \xi, \omega])\|_{\mathcal{V}_\lambda} < \frac{1}{16c_2^2},
\]

from (5.11) we deduce \(\|u\|_{\mathcal{V}_\lambda} < \frac{1}{2}\delta\). Let \(u_i \in \mathcal{V}_\lambda\), \(i = 1, 2\), and set

\[
u u_1 \equiv u_2, \quad u := M(u_1) - M(u_2).
\]

From (5.9) we then show

\[
\begin{aligned}
\frac{du}{dt} - \nabla \cdot \nu u + \omega \times u &= \Delta u - \nu p + u_1 \cdot \nabla u + u_2 \cdot \nabla u \\
\text{div} u &= 0
\end{aligned}
\]

in \(\Omega \times (0, T)\) \hspace{1cm} (5.13)

\[
u u(x, t) = 0, \quad (x, t) \in \partial \Omega \times [0, T].
\]

Proceeding as in the proof of (5.11) we can show

\[
\|u\|_{\mathcal{V}_\lambda} \leq c_2 \left(\|u_1\|_{\mathcal{V}_\lambda} + \|u_2\|_{\mathcal{V}_\lambda}\right)\|u\|_{\mathcal{V}_\lambda}.
\]

As a result, since \(\|u_i\|_{\mathcal{V}_\lambda} < \delta, i = 1, 2\), from the previous inequality we infer

\[
\|u\|_{\mathcal{V}_\lambda} < 2c_2\delta \|u\|_{\mathcal{V}_\lambda}.
\]

By (5.12), we have \(2c_2\delta < 1/2\), and so, from the last displayed relation we may conclude that \(M\) is a contraction, which, along with (5.12), completes the proof of the theorem. \hfill \Box

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\textsuperscript{11}V_0 \text{ defined in (3.8).}
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Giovanni P. Galdi
Department of Mechanical Engineering and Materials Science
University of Pittsburgh
Pittsburgh PA15261
USA
e-mail: galdi@pitt.edu

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