COHOMOLOGY OF TRUNCATED POLYNOMIAL $\lambda$-RINGS

DONALD YAU

Abstract. The $\lambda$-ring cohomology, $H^0_\lambda$ and $H^1_\lambda$, of certain truncated polynomial filtered $\lambda$-rings are computed.

1. Introduction

Lambda-rings were introduced by Grothendieck [3] to describe algebraic objects equipped with operations $\lambda^i$ that behave like exterior powers. Since their introduction by Grothendieck, $\lambda$-rings have been shown to play important roles in several areas of mathematics, including Topology, Representation Theory, and Algebra. Indeed, examples of $\lambda$-rings include the complex $K$-theory of a topological space, the representation ring of a group, and the universal Witt ring of a commutative ring. See the references [1, 3, 4, 5, 6]. Note that what we call a $\lambda$-ring here is sometimes referred to as a special $\lambda$-ring in the literature.

Any reasonable class of algebras (associative, Lie, Poisson, Jordan, von Neumann, etc.) should come with (i) a cohomology theory $H^*$ and (ii) an algebraic deformation theory which is describable in terms of $H^*$. The original theory of algebraic deformations for associative algebras was worked out by Gerstenhaber [2], and the relevant cohomology theory is Hochschild cohomology. Deformations and the associated cohomology for many other types of algebras have since been worked out. Since $\lambda$-rings form such a useful class of algebras, there should likewise be deformations and cohomology for $\lambda$-rings. Indeed, the author defined in [9] $\lambda$-ring cohomology $H_\lambda^*(R)$ for a $\lambda$-ring $R$ and used it to study $\lambda$-ring deformations along the path initiated by Gerstenhaber [2]. There is even a “composition” product on the cochain complex which gives rise to $\lambda$-ring cohomology, and this product descends to $H_\lambda^*(R)$, giving it the structure of a graded, associative, unital algebra.

The current paper can be considered as a sequel to [9]. The main purpose here is to show how $\lambda$-ring cohomology, at least in dimensions 0 and 1, can be computed for some filtered $\lambda$-ring structures on truncated polynomial algebras. Particular attention is paid to those filtered $\lambda$-ring structures that are candidates for topological realizations (i.e. those that can possibly be the $K$-theory of a torsionfree space).

Low-dimensional $\lambda$-ring cohomology is intrinsically interesting. In fact, under the composition product, $H^0_\lambda(R)$ is a non-trivial subalgebra of the
algebra of $\mathbb{Z}$-linear self maps of $R$ that commute with all the Adams operations. Each Adams operation of $R$ does lie in $H^0_\lambda(R)$. The group $H^1_\lambda(R)$ is important because it is the natural habitat for the “infinitesimal” (sort of like the “initial velocity”) of a $\lambda$-ring deformation of $R$. Indeed, $H^1_\lambda(R)$ classifies the order 1 $\lambda$-ring deformations of $R$ up to equivalence.

A description of the rest of the paper follows.

Section 2 gives a brief account of the basics of (filtered) $\lambda$-rings and Adams operations. It also reviews $\lambda$-ring cohomology and the composition product.

The main result in Section 3 is Theorem 3.2, which shows that $H^1_\lambda$ is isomorphic to a smaller group $\bar{H}^1_\lambda$ that involves only the Adams operations $\psi^p$ for $p$ primes (as opposed to all the $\psi^k$). This will allow a more efficient computation of the group $H^1_\lambda$ in later sections. This section ends with a description of the graded algebra $H^{\leq 1}_\lambda(\mathbb{Z}) = H^*_\lambda(\mathbb{Z})/(\oplus_{i \geq 2} H^i_\lambda(\mathbb{Z}))$ for the $\lambda$-ring $\mathbb{Z}$ of integers.

Section 4 gives a description of the graded algebra $H^{\leq 1}_\lambda(R)$ for each of the uncountably many isomorphism classes of filtered $\lambda$-ring structures $R$ on the dual number ring $\mathbb{Z}[x]/(x^2)$ (Theorem 4.1).

Section 5 begins by recalling the uncountably many isomorphism classes of filtered $\lambda$-ring structures on the truncated polynomial ring $\mathbb{Z}[x]/(x^3)$. The main result of this section is Theorem 5.3, which computes the algebra $H^0_\lambda(R)$ for each one of these isomorphism classes of filtered $\lambda$-rings. It turns out that each such $H^0_\lambda(R)$ is a 3-dimensional commutative subalgebra of the algebra $M(3, \mathbb{Z})$ of 3-by-3 integer matrices.

Among the uncountably many isomorphism classes of filtered $\lambda$-ring structures on the truncated polynomial ring $\mathbb{Z}[x]/(x^3)$, at most 64 of them can be realized as the $K$-theory of torsionfree spaces (see [10]). The main result of section 6 is Theorem 6.1, which describes the groups $H^1_\lambda$ and determines (non-)commutativity of the graded algebras $H^{\leq 1}_\lambda$ for these 64 isomorphism classes of filtered $\lambda$-rings. Interestingly enough, exactly 35 of those 64 graded algebras $H^{\leq 1}_\lambda(R)$ are commutative.

It is also shown in [10] that the truncated polynomial ring $\mathbb{Z}[x]/(x^4)$ admits uncountably many isomorphism classes of filtered $\lambda$-ring structures and that, among these classes, at most 61 of them can be realized by the $K$-theory of torsionfree spaces. The main results of section 7, Theorem 7.1 and Theorem 7.2, compute the algebras $H^0_\lambda$ for these 61 isomorphism classes of filtered $\lambda$-rings. Each such $H^0_\lambda$ turns out to be a 4-dimensional commutative subalgebra of the algebra $M(4, \mathbb{Z})$ of 4-by-4 matrices with integer entries.
2. $\lambda$-RINGS AND COHOMOLOGY

The purpose of this section is to review some basic definitions about (filtered) $\lambda$-rings, Adams operations, and $\lambda$-ring cohomology. For more discussions about basic properties of $\lambda$-rings, consult the references [1, 5, 6]. The reference for $\lambda$-ring cohomology is the author’s paper [9].

2.1. $\lambda$-rings.

By a $\lambda$-ring we mean a unital, commutative ring $R$ equipped with functions

\[ \lambda^i : R \to R \quad (i \geq 0), \]

called $\lambda$-operations, which satisfy the following conditions. For any integers $i, j \geq 0$ and elements $r$ and $s$ in $R$:

- $\lambda^0(r) = 1$.
- $\lambda^1(r) = r$.
- $\lambda^i(1) = 0$ for $i > 1$.
- $\lambda^i(r + s) = \sum_{k=0}^{i} \lambda^k(r) \lambda^{i-k}(s)$.
- $\lambda^i(rs) = P_i(\lambda^1(r), \ldots, \lambda^i(r); \lambda^1(s), \ldots, \lambda^i(s))$.
- $\lambda^i(\lambda^j(r)) = P_{i,j}(\lambda^1(r), \ldots, \lambda^j(r))$.

The $P_i$ and $P_{i,j}$ are some universal polynomials with integer coefficients, defined as follows. Consider the variables $\xi_1, \ldots, \xi_i$ and $\eta_1, \ldots, \eta_i$. Denote by $s_1, \ldots, s_i$ and $\sigma_1, \ldots, \sigma_i$, respectively, the elementary symmetric functions of the $\xi$’s and the $\eta$’s. The polynomial $P_i$ in $2i$ variables is defined by insisting that the expression $P_i(s_1, \ldots, s_i; \sigma_1, \ldots, \sigma_i)$ be the coefficient of $t^i$ in the finite product

\[ \prod_{m,n=1}^{i} (1 + \xi_m \eta_n t). \]

Similarly, if $s_1, \ldots, s_{ij}$ are the elementary symmetric functions of $\xi_1, \ldots, \xi_{ij}$, then the polynomial $P_{i,j}$ in $ij$ variables is defined by insisting that the expression $P_{i,j}(s_1, \ldots, s_{ij})$ be the coefficient of $t^i$ in the finite product

\[ \prod_{l_1 < \cdots < l_j} (1 + \xi_{l_1} \cdots \xi_{l_j} t). \]

Note that what we call a $\lambda$-ring here is referred to as a special $\lambda$-ring in [1, 2].

A map of $\lambda$-rings is a ring map that commutes with the $\lambda$-operations.

2.1.1. Adams operations. In a $\lambda$-ring $R$, there are the so-called Adams operations

\[ \psi^n : R \to R \quad (n \geq 1), \]

satisfying the following properties:

- All the $\psi^n$ are ring maps.
- $\psi^1 = \text{Id}$.
- $\psi^m \psi^n = \psi^{mn} = \psi^n \psi^m$. 
\begin{itemize}
  \item $\psi^p(r) \equiv r^p \pmod{pR}$ for each prime $p$ and element $r$ in $R$.
\end{itemize}

The Adams operations can be defined inductively by the Newton formula:

$$
\psi^n(r) - \lambda^1(r)\psi^{n-1}(r) + \cdots + (-1)^{n-1}\lambda^{n-1}(r)\psi^1(r) + (-1)^n n\lambda^n(r) = 0.
$$

Suppose given a unital, commutative, $\mathbb{Z}$-torsionfree ring $R$ with self ring maps $\psi^n : R \rightarrow R$ satisfying the above four properties of Adams operations. Then a result of Wilkerson [7] says that there exists a unique $\lambda$-ring structure on $R$ whose Adams operations are exactly the given $\psi^n$.

2.1.2. Filtered $\lambda$-rings. By a filtered ring we mean a (unital, commutative) ring $R$ equipped with a decreasing sequence of ideals $I^n$. A map of filtered rings is a ring map that preserves the filtration ideals.

A filtered $\lambda$-ring is a filtered ring $R$ that is also a $\lambda$-ring in which the filtration ideals are all closed under the $\lambda$-operations, i.e. $\lambda^i(I^n) \subseteq I^n$ for all $n$ and all $i > 0$. A map of filtered $\lambda$-rings is a $\lambda$-ring map that is also a filtered ring map.

For example, the $K$-theory of a topological space $X$ with the homotopy type of a CW complex is a filtered $\lambda$-ring. Here the filtration ideals are given by the kernels of the restriction maps to the skeletons, i.e.

$$
I^n = \ker(K(X) \rightarrow K(X_{n-1})),
$$

where $X_{n-1}$ is the $(n-1)$-skeleton of $X$ and the map is induced by the inclusion $X_{n-1} \subset X$. The Cellular Approximation Theorem assures that any two CW structures on $X$ give rise to isomorphic filtered $\lambda$-ring structures on $K(X)$.

The obvious analogue of Wilkerson’s Theorem discussed in section 2.1.1 holds for the truncated polynomial algebras $\mathbb{Z}[x]/(x^n)$ with $x$ in some fixed positive filtration.

2.2. $\lambda$-ring cohomology. Let $R$ be a $\lambda$-ring with $\lambda$-operations $\lambda^i$ ($i \geq 0$) and Adams operations $\psi^n$ ($n \geq 1$).

2.2.1. The complex $\mathcal{F}^*$. The $\lambda$-ring cohomology groups $H^*_\lambda(R)$ of $R$ are defined to be the cohomology groups of a certain cochain complex $\mathcal{F}^*(R)$, which is defined as follows.

Denote by $\text{End}(R)$ the algebra of $\mathbb{Z}$-linear endomorphisms of $R$. Let $\overline{\text{End}}(R)$ be the subalgebra of $\text{End}(R)$ consisting of those linear endomorphisms $f$ of $R$ that satisfy the condition,

$$
f(r)^p \equiv f(r^p) \pmod{pR},
$$

for every prime $p$ and each element $r \in R$. Note that every self ring map of $R$ lies in $\overline{\text{End}}(R)$. In particular, $\overline{\text{End}}(R)$ contains all the Adams operations of $R$. 

We are now ready to define the cochain complex $F^* = F^*(R)$. Let $T$ be the set of positive integers. Define $F^0$ to be the underlying additive group of $\text{End}(R)$ and $F^1$ to be the set of functions $f : T \to \text{End}(R)$ satisfying the condition $f(p)(R) \subseteq pR$ for every prime $p$. For $n \geq 2$, $F^n$ is defined to be the set of functions $f : T^n \to \text{End}(R)$.

If $f$ and $g$ are elements of $F^n$, then their sum is defined by

$$(f + g)(m_1, \ldots, m_n)(r) = f(m_1, \ldots, m_n)(r) + g(m_1, \ldots, m_n)(r)$$

for $(m_1, \ldots, m_n) \in T^n$ and $r \in R$. This gives $F^n$ ($n \geq 0$) a natural additive group structure.

The differential $d^n : F^n \to F^{n+1}$ is defined by the formula

$$(d^n f)(m_0, \ldots, m_n) = \psi^{m_0} f(m_1, \ldots, m_n) + \sum_{i=1}^{n} (-1)^i f(m_0, \ldots, m_{i-1}, m_i, \ldots, m_n) + (-1)^{n+1} f(m_0, \ldots, m_{n-1}) \psi^{m_n}.$$ 

The $d^n$ are linear maps and satisfy $d^{n+1}d^n = 0$ for each $n \geq 0$. This makes $F^* = (F^*(R), d^*)$ into a cochain complex. We define the $n$th $\lambda$-ring cohomology group of $R$, denoted by $H^n_\lambda(R)$, to be the $n$th cohomology group of the cochain complex $F^* = (F^*(R), d^*)$.

2.2.2. Composition product. Given a $\lambda$-ring $R$, there is an associative, bilinear pairing

$$- \circ - : F^n \otimes F^k \to F^{n+k} \quad (n, k \geq 0)$$

on the complex $F^* = (F^*(R), d^*)$ defined by

$$(f \circ g)(m_1, \ldots, m_{n+k}) = f(m_1, \ldots, m_n) \circ g(m_{n+1}, \ldots, m_{n+k}),$$

for $f \in F^n$ and $g \in F^k$. The element $\text{Id}_R \in F^0$ acts as a two-sided identity for this pairing. Moreover, this pairing satisfies the Leibnitz identity,

$$d(f \circ g) = (df) \circ g + (-1)^{|f|} f \circ (dg),$$

where $|f|$ is the dimension of $f$. We call this pairing the composition product. The complex $F^*$ with the composition product is a differential graded algebra.

The Leibnitz identity implies that the composition product descends to $H^n_\lambda(R) = \oplus_i H^n_\lambda(R)$ with

$$[f] \circ [g] = [f \circ g].$$
where \([f]\) denotes the cohomology class of a cocycle. With this product, the graded group \(H^*_\lambda(R)\) becomes a graded, associative, unital algebra.

3. A reinterpretation of \(H^1_\lambda\)

Since \(\lambda\)-ring cohomology are defined using the Adams operations, which are determined by \(\psi^p\) for \(p\) primes, it should be the case that \(\lambda\)-ring cohomology can be reinterpreted in terms of only the \(\psi^p\). This main point of this section is to do exactly that for \(H^1_\lambda\).

Let \(\mathcal{P}\) denote the set of all primes. For a \(\lambda\)-ring \(R\), define the following:

- **\(\text{Der}_\lambda(R)\)** = The set of \(f \in \mathcal{F}^1(R)\) satisfying the condition
  \[
  f(mn) = \psi^mf(n) + f(m)\psi^n
  \]
  for all positive integers \(m\) and \(n\). This is called the group of \(\lambda\)-derivations in \(R\). Note that \(f(1)\) must be 0.

- **\(\overline{\text{Der}}_\lambda(R)\)** = The set of sequences \(\{f(p)\}_{p \in \mathcal{P}}\) indexed by the primes with each \(f(p) \in \text{End}(R)\), such that \(f(p)(R) \subseteq pR\) and that
  \[
  \psi^p f(q) + f(p)\psi^q = \psi^q f(p) + f(q)\psi^p
  \]
  for all primes \(p\) and \(q\). This forms a group under coordinatewise addition.

- **\(\text{Inn}_\lambda(R)\)** = The set of elements in \(\mathcal{F}^0(R)\) of the form \([\psi^*, g]\), where \(g \in \mathcal{F}^0(R)\) and \([\psi^*, g](n) = \psi^ng - g\psi^n\). This is called the group of \(\lambda\)-inner derivations in \(R\).

- **\(\overline{\text{Inn}}_\lambda(R)\)** = The set of sequences \(\{[\psi^p, g]\}_{p \in \mathcal{P}}\) indexed by the primes in which \(g \in \mathcal{F}^0(R)\). This is clearly a subgroup of \(\overline{\text{Der}}_\lambda(R)\).

- **\(\overline{H}^1_\lambda(R)\)** = The quotient group \(\overline{\text{Der}}_\lambda(R)/\overline{\text{Inn}}_\lambda(R)\).

It follows directly from the definition that \(\text{Der}_\lambda(R)\) is the kernel of \(d^1\) and that \(\text{Inn}_\lambda(R)\) is the image of \(d^0\). Therefore, we have

**Proposition 3.1** (= Proposition 9 in [9]). As an additive group, \(H^1_\lambda(R)\) is the quotient \(\text{Der}_\lambda(R)/\text{Inn}_\lambda(R)\).

Notice that (3.0.1) implies (3.0.2), and so there is a linear map

\[
\pi: \text{Der}_\lambda(R) \rightarrow \overline{\text{Der}}_\lambda(R),
\]

where

\[
\pi(f) = \{f(p)\}_{p \in \mathcal{P}}.
\]

The image of \(\text{Inn}_\lambda(R)\) under \(\pi\) is exactly \(\overline{\text{Inn}}_\lambda(R)\). It follows that \(\pi\) induces a linear map

\[
\pi: H^1_\lambda(R) \rightarrow \overline{H}^1_\lambda(R)
\]

which sends \([f]\) to \(\{f(p)\}_{p \in \mathcal{P}}\).

**Theorem 3.2.** The map \(\pi\) in (3.1.1) is a group isomorphism.
From now on, using the isomorphism $\pi$, we will consistently identify $H_1^\lambda(R)$ with $\bar{H}_1^\lambda(R)$.

**Proof.** Theorem 3.2 will follow from Lemma 3.3 and Lemma 3.4 below and the fact that $\pi(\text{Inn}_\lambda(R)) = \text{Inn}_\lambda(R)$. 

**Lemma 3.3.** Let $f$ be an element of $\text{Der}_\lambda(R)$ and let $n = p_1 \cdots p_r$ be a positive integer, in which the $p_i$ are primes, not necessarily distinct. Then we have that

$$f(n) = \sum_{i=1}^{r} \psi^{p_1 \cdots p_i - 1} f(p_i) \psi^{p_{i+1} \cdots p_r}.$$  

(3.3.1)

In particular, the map $\pi: \text{Der}_\lambda(R) \to \overline{\text{Der}}_\lambda(R)$ is injective.

**Proof.** We proceed by induction on the number $r$ of prime factors. There is nothing to prove when $r = 1$. Suppose that (3.3.1) has been proved for positive integers $< r$. Write $m$ for $n/p_r = p_1 \cdots p_r - 1$. Then we have

$$f(n) = f(mp_r) = \psi^m f(p_r) + f(m) \psi^{p_r} = \left( \sum_{i=1}^{r-1} \psi^{p_1 \cdots p_i - 1} f(p_i) \psi^{p_{i+1} \cdots p_r - 1} \right) \psi^{p_r} + \psi^m f(p_r) = \sum_{i=1}^{r} \psi^{p_1 \cdots p_i - 1} f(p_i) \psi^{p_{i+1} \cdots p_r}.$$  

This finishes the induction and proves the Lemma. 

**Lemma 3.4.** Given an element $\{g(p)\}_p \in \overline{\text{Der}}_\lambda(R)$, there exists an element $f \in \text{Der}_\lambda(R)$ such that $f(p) = g(p)$ for each prime $p$. In other words, the map $\pi: \text{Der}_\lambda(R) \to \overline{\text{Der}}_\lambda(R)$ is surjective.

From now on, using the isomorphism $\pi$ and Lemma 3.3 and Lemma 3.4 we will consistently identify $\text{Der}_\lambda(R)$ (respectively $\text{Inn}_\lambda(R)$) with $\overline{\text{Der}}_\lambda(R)$ (respectively $\overline{\text{Inn}}_\lambda(R)$).

**Proof.** Set $f(1) = 0$. Given an integer $n = p_1 \cdots p_r > 1$, where each $p_i$ is a prime, define $f(n)$ by

$$f(n) \overset{\text{def}}{=} \sum_{i=1}^{r} \psi^{p_1 \cdots p_{i-1}} g(p_i) \psi^{p_{i+1} \cdots p_r}.$$  

(3.4.1)

We first have to make sure that this is well-defined. In other words, we have to show that the right-hand side of (3.4.1) is independent of the order of the $p_i$ appearing in the prime factorization of $n$. This is clearly true when $r = 1$; the case $r = 2$ follows from (3.0.2).

Suppose that $r \geq 2$. Pick any $j$ with $1 \leq j \leq n - 1$ and write

$$n = p_1 \cdots p_{j-1} p_j + 1 p_j p_{j+2} \cdots p_r,$$
i.e., transposes $p_j$ and $p_{j+1}$. With this way of writing $n$, we have

$$f(n) = \left( \sum_{i=1}^{j-1} \psi^{p_1 \cdots p_{i-1}} f(p_i) \psi^{p_{i+1} \cdots p_r} \right) + \psi^{p_1 \cdots p_{j-1}} f(p_{j+1}) \psi^{p_j p_{j+2} \cdots p_r} + \left( \sum_{r=1}^{j+1} \psi^{p_1 \cdots p_{r-1} \cdots p_{j+1}} f(p_j) \psi^{p_{j+2} \cdots p_r} \right) + \left( \sum_{l=j+2}^{r} \psi^{p_1 \cdots p_{l-1}} f(p_l) \psi^{p_l+1 \cdots p_r} \right).$$

(3.4.2)

The sum of the two terms in the middle (those that are not surrounded by parentheses) is

$$\psi^{p_1 \cdots p_{j-1}} f(p_{j+1}) \psi^{p_j p_{j+2} \cdots p_r} + \psi^{p_1 \cdots p_{j-1} p_{j+1}} f(p_j) \psi^{p_j+1 \cdots p_r}$$

$$= \psi^{p_1 \cdots p_{j-1}} (\psi^{p_{j+1}} f(p_j) + f(p_{j+1}) \psi^{p_j}) \psi^{p_j+2 \cdots p_r}$$

$$= \psi^{p_1 \cdots p_{j-1}} (\psi^{p_{j+1}} g(p_j) + g(p_{j+1}) \psi^{p_j}) \psi^{p_j+2 \cdots p_r}$$

$$= \psi^{p_1 \cdots p_{j-1}} (\psi^{p_j} g(p_{j+1}) + g(p_j) \psi^{p_{j+1}}) \psi^{p_j+2 \cdots p_r}$$

$$= \psi^{p_1 \cdots p_{j-1}} (\psi^{p_j} f(p_{j+1}) + f(p_j) \psi^{p_{j+1}}) \psi^{p_j+2 \cdots p_r}.$$

Therefore, the right-hand sides of (3.4.1) and (3.4.2) agree. Since every permutation on the set \{1, \ldots, r\} can be written as a product of transpositions of the form $(j, j+1)$ for $1 \leq j \leq r-1$, the argument above shows that (3.4.1) is indeed well-defined.

Since $f(p) = g(p)$ for all primes $p$, to show that $f = \{f(n)\}$ lies in $\text{Der}_\chi(R)$, it remains to show that $f$ satisfies (3.0.1). Given $m = p_1 \cdots p_r$ and $n = q_1 \cdots q_s$, where the $p_i$ and $q_j$ are primes, we have

$$f(mn) = f(p_1 \cdots p_r q_1 \cdots q_s)$$

$$= \sum_{i=1}^{r} \psi^{p_1 \cdots p_{i-1}} f(p_i) \psi^{p_{i+1} \cdots p_r q_1 \cdots q_s} + \sum_{j=1}^{s} \psi^{p_1 \cdots p_r q_1 \cdots q_{j-1}} f(q_j) \psi^{q_j+1 \cdots q_s}$$

$$= \left( \sum_{i=1}^{r} \psi^{p_1 \cdots p_{i-1}} f(p_i) \psi^{p_{i+1} \cdots p_r} \right) \psi^n + \psi^m \left( \sum_{j=1}^{s} \psi^{q_1 \cdots q_{j-1}} f(q_j) \psi^{q_j+1 \cdots q_s} \right)$$

$$= \psi^n f(m) + f(m) \psi^m.$$

This finishes the proof of the Lemma. □

Consider the quotient

$$H^1_{\leq 1}(R) = H^{0}_{\leq 1}(R) \oplus H^1_{\leq 1}(R) \cong \frac{H^0(R) \oplus H^1(R)}{\oplus_{i=2}^{\infty} H^i(R)}$$

of the graded algebra $H^\chi(R)$ (under the composition product) by the homogeneous ideal of elements of degree at least 2. We still consider $H^\leq_{\leq 1}(R)$ a graded algebra, with $H^0(R)$ and $H^1(R)$ in degrees 0 and 1, respectively, and 0 in degrees $\neq 0, 1$. Similarly, define the graded group

$$\tilde{H}^\leq_{\leq 1}(R) \overset{\text{def}}{=} \frac{H^{0}_\leq(R) \oplus \tilde{H}^1_{\leq 1}(R),}$$

in which $H^0(R)$ and $\tilde{H}^1_{\leq 1}(R)$ are in degrees 0 and 1, respectively.
With these notations, an immediate consequence of Theorem 3.2 is

**Corollary 3.5.** The group isomorphism

\[(\text{Id}, \pi): H^{\leq 1}_\lambda(R) \xrightarrow{\cong} \tilde{H}^{\leq 1}_\lambda(R)\] (3.5.1)

induces a (unital, associative) graded algebra structure on \(\tilde{H}^{\leq 1}_\lambda(R)\) that is isomorphic to \(H^{\leq 1}_\lambda(R)\). Under this graded algebra structure of \(\tilde{H}^{\leq 1}_\lambda(R)\), one has that, for \(g \in H^0_\lambda(R)\) and \(\{f(p)\}_p \in \text{Der}_\lambda(R)\),

\[g \circ \{f(p)\}_p \in \mathbb{P} = \{g f(p)\}_p \in \mathbb{P}\]

and

\[\{f(p)\}_p \circ g = \{f(p)g\}_p \in \mathbb{P} \].

Here \(\circ\) denotes the algebra product in \(\tilde{H}^{\leq 1}_\lambda(R)\), the composition product, and \(gf(p)\) is just composition of \(\mathbb{Z}\)-linear self maps of \(R\).

Using Corollary 3.5, we will consistently identify the graded algebras \(H^{\leq 1}_\lambda(R)\) and \(\tilde{H}^{\leq 1}_\lambda(R)\). Recall that the ring \(\mathbb{Z}\) of integers has a unique \(\lambda\)-ring structure with \(\lambda^i(n) = \binom{n}{i}\) and \(\psi^k = \text{Id}\) for all \(k\). Using Corollary 3.5, the result in \([9]\) describing the groups \(H^0_\lambda(\mathbb{Z})\) and \(H^1_\lambda(\mathbb{Z})\) can be restated as follows.

**Theorem 3.6** (= Corollary 8 and Corollary 10 in \([9]\)). There is a graded algebra isomorphism

\[H^{\leq 1}_\lambda(\mathbb{Z}) \cong \mathbb{Z} \oplus \prod_{p \in \mathbb{P}} p\mathbb{Z},\]

in which \(\mathbb{Z}\) is the degree 0 part and the other summand is the degree 1 part. Given \(m, n \in \mathbb{Z}\) and \((n_p) \in \prod_{p \in \mathbb{P}} p\mathbb{Z}\), we have

\[m \circ n = mn\]

and

\[m \circ (n_p) = (mn_p) = (n_p) \circ m.\]

In particular, \(H^{\leq 1}_\lambda(\mathbb{Z})\) is a commutative graded algebra.

### 4. The Dual Number Ring

Recall from \([8]\) Corollary 4.1.2] that the dual number ring \(\mathbb{Z}[x]/(x^2)\), with \(x\) in some fixed positive filtration \(d\), admits uncountably many isomorphism classes of filtered \(\lambda\)-ring structures. In fact, there is a bijection between this set of isomorphism classes and the set of sequences \((b_p)\) indexed by the primes in which \(b_p \in p\mathbb{Z}\). The (isomorphism class of) filtered \(\lambda\)-ring \(R\) corresponding to a sequence \((b_p)\) has Adams operations

\[\psi^p(x) = b_p x\]

for each prime \(p\).

We will continue to denote by \(\mathbb{P}\) the set of all primes.
Theorem 4.1. Let $R$ be a representative of any one of the uncountably many isomorphism classes of filtered $\lambda$-ring structures on the dual number ring $\mathbb{Z}[x]/(x^2)$. Then there is a graded algebra isomorphism

$$H_\lambda^{\leq 1}(R) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \prod_{p \in \mathcal{P}} (p\mathbb{Z} \oplus p\mathbb{Z}),$$

in which $\mathbb{Z} \oplus \mathbb{Z}$ is the degree 0 part and the infinite product is the degree 1 part. Given $(a, b), (c, d) \in \mathbb{Z} \oplus \mathbb{Z}$ and $\{(x_p, y_p)\} \in \prod_{p \in \mathcal{P}} p\mathbb{Z} \oplus p\mathbb{Z}$, we have

$$(a, b) \circ (c, d) = (ac, bd)$$

and

$$(a, b) \circ \{(x_p, y_p)\} = \{(ax_p, by_p)\} = \{(x_p, y_p)\} \circ (a, b).$$

In particular, $H_\lambda^{\leq 1}(R)$ is a commutative graded algebra.

Proof. Step 1: $H_\lambda^0(R)$.

We begin by computing $\underline{\text{End}}(R)$, which, by definition, consists of the $\mathbb{Z}$-linear maps $g: R \to R$ for which

$$g(r^p) \equiv g(r^p) \pmod{pR} \quad (4.1.1)$$

for all primes $p$ and elements $r \in R$. This last condition is clearly equivalent to

$$g(x^p) \equiv g(x ip) \pmod{pR} \quad (4.1.2)$$

for $i = 0, 1$ and all primes $p$. Consider first the case $i = 0$. Writing $g(1) = a + bx$, the condition then says

$$a + bx \equiv (a + bx)^p \pmod{p}$$

$$\equiv a + bx^p \pmod{p}$$

$$= a,$$

since $x^2 = 0$ in $R$. In other words, $b = 0$ and $g(1) = a \in \mathbb{Z}$. Now write $g(x) = c + dx$. Since $x^p = 0$ for any prime $p$ and since $g(0) = 0$, the condition above for $i = 1$ says

$$g(x^p) = 0$$

$$\equiv (c + dx)^p \pmod{p}$$

$$\equiv c + dx^p \pmod{p}$$

$$= c.$$

Since this is true for all primes $p$, we infer that $c = 0$ and $g(x) = dx$.

Summarizing this discussion, we have shown that $\underline{\text{End}}(R)$ consists of precisely the $\mathbb{Z}$-linear self-maps $g$ of $R$ for which $g(1) \in \mathbb{Z}$ and $g(x) \in \mathbb{Z}x$. It is clear that any such map commutes with $\psi^k$ for all $k$. Therefore, $H_\lambda^0(R) = \underline{\text{End}}(R)$ is isomorphic to the product ring $\mathbb{Z} \oplus \mathbb{Z}$, as stated in the statement of the Theorem.

Step 2. $H_\lambda^1(R)$. 


Observe that, since each \( g \in \mathcal{F}^0(R) = \overline{\text{End}}(R) \) commutes with all the \( \psi^k \) in \( R \), the image of the differential
\[
d^0 = [\psi^k, -] : \mathcal{F}^0(R) \to \mathcal{F}^1(R)
\]
is trivial. Therefore, \( H^1_\lambda(R) = \text{Der}_\lambda(R) \), and so it suffices to compute \( \text{Der}_\lambda(R) \).

To do this, let \( (f(p))_p \) be a sequence of \( \mathbb{Z} \)-linear self maps of \( R \) indexed by the primes with \( f(p)(R) \subseteq pR \). Thus, using the \( \mathbb{Z} \)-basis \( \{1, x\} \) of \( \mathbb{Z}[x]/(x^2) \), we can write each \( f(p) \) as a 2-by-2 matrix with entries in \( p\mathbb{Z} \), say, \( f(p) = p(a(p)_{ij}) \). If \( \psi^p(x) = b_p x \) in \( R \), then we can similarly represent \( \psi^p \) as a 2-by-2 diagonal matrix with entries 1 and \( b_p \) along the diagonal. In this context, we have
\[
\psi^p f(q) + f(p) \psi^q = q \begin{bmatrix} 1 & a(q)_{11} + a(q)_{12} \\ b_p & a(q)_{21} + a(q)_{22} \end{bmatrix} + p \begin{bmatrix} a(p)_{11} & a(p)_{12} \\ a(p)_{21} & a(p)_{22} \end{bmatrix} \begin{bmatrix} 1 \\ b_q \end{bmatrix}
\]
As usual, the empty entries denote 0. One obtains the matrix representation of \( \psi^q f(p) + f(q) \psi^p \) by interchanging \( p \) and \( q \) in the above matrix. Using this, the condition \( \text{(3.0.2)} \) can now be seen to be equivalent to the equalities
\[
pa(p)_{ij}(b_q - 1) = qa(q)_{ij}(b_p - 1)
\]
for all primes \( p \) and \( q \) and \( (i, j) = (1, 2) \) and \( (2, 1) \). Since each \( b_p \) is divisible by \( p \), it follows that \( a(q)_{ij} \) is divisible by \( p \) for all \( p \neq q \), i.e. \( a(q)_{ij} = 0 \). It follows that \( a(p)_{ij} = 0 \) for all primes \( p \) and \( (i, j) = (1, 2) \) and \( (2, 1) \). In other words, each \( f(p) \) is a diagonal matrix with entries in \( p\mathbb{Z} \). Therefore, we have the group isomorphisms
\[
H^1_\lambda(R) \cong \text{Der}_\lambda(R) \cong \prod_{p \in \mathcal{P}} (p\mathbb{Z} \oplus p\mathbb{Z}).
\]

**Step 3:** Ring structure.

To finish the proof, we only need to observe that if \( (a, b) \in H^0_\lambda(R) \) and \( \{(x_p, y_p)\} \in H^1_\lambda(R) \), then, by Corollary \( \text{3.5} \) we have
\[
(a, b) \circ \{(x_p, y_p)\} = \left\{ \begin{bmatrix} a x_p \\ b y_p \end{bmatrix} \right\} = \left\{ \begin{bmatrix} ax_p \\ by_p \end{bmatrix} \right\}
\]
\[
= \{(ax_p, by_p)\} = \{(x_p, y_p)\} \circ (a, b),
\]
as desired. \( \Box \)

5. \( H^0_\lambda \) of filtered \( \lambda \)-ring structures on \( \mathbb{Z}[x]/(x^3) \)

In this section, we will compute the algebra \( H^0_\lambda(R) \) for each of the uncountably many isomorphism classes of filtered \( \lambda \)-ring structures on \( \mathbb{Z}[x]/(x^3) \), with \( x \) in some fixed positive filtration. (See \( \text{[10]} \) for a proof of this uncountability statement.)
5.1. **Filtered \(\lambda\)-ring structures on \(\mathbb{Z}[x]/(x^3)\).** Let us begin by recalling the classification of filtered \(\lambda\)-ring structures on \(\mathbb{Z}[x]/(x^3)\). See [10] for more details.

In what follows, we will describe a filtered \(\lambda\)-ring structure on \(\mathbb{Z}[x]/(x^3)\) in terms of its Adams operations \(\psi^p\) for \(p\) primes. This is sufficient to determine the filtered \(\lambda\)-ring structure by a result of Wilkerson [7], as discussed in section 2.1.1.

Let \(R\) be a filtered \(\lambda\)-ring structure on \(\mathbb{Z}[x]/(x^3)\) with Adams operations
\[
\psi^p_R(x) = \psi^p(x) = b_p x + c_p x^2. \tag{5.1.1}
\]
It is shown in [10] that if \(b_2 = 0\), then

- \(b_p = 0\) for all primes \(p\),
- \(c_2\) is an odd integer, and
- \(c_p \equiv 0 \pmod{p}\) for all odd primes \(p\).

In this case, we write \(S((c_p))\) for \(R\), since its filtered \(\lambda\)-ring structure is completely determined by the \(c_p\) for \(p\) primes. Conversely, any such sequence \((c_p)\) gives rise to a filtered \(\lambda\)-ring structure on \(\mathbb{Z}[x]/(x^3)\). Two such filtered \(\lambda\)-ring structures, \(S((c_p))\) and \(S((c'_p))\), are isomorphic if and only if \((c_p) = \pm(c'_p)\). In particular, there are uncountably many isomorphism classes of filtered \(\lambda\)-ring structures on \(\mathbb{Z}[x]/(x^3)\) of the form \(S((c_p))\).

On the other hand, if \(b_2 \neq 0\), then there exists an odd integer \(h\) such that
\[
c_p = h \frac{b_p(b_p - 1)}{G} \tag{5.1.2}
\]
for all primes \(p\), where
\[
G = \gcd\{b_q(b_q - 1) : \text{all primes } q\}.
\]

Moreover, the following conditions have to hold:

- \(b_p \equiv 0 \pmod{p}\) for all primes \(p\).
- \(b_p(b_p - 1) \equiv 0 \pmod{2^{\nu_2(b_2)}}\), where \(\nu_2\) denotes the 2-adic evaluation of an integer (i.e. the exponent of the prime factor 2 in the integer).
- \(h\) is in the range \(1 \leq h \leq G/2\).
- Suppose that there exists an odd prime \(p\) for which \(b_p \neq 0\) and
  \[
  \nu_p(b_p) = \min\{\nu_p(b_q(b_q - 1)) : b_q \neq 0\}. \tag{5.1.3}
  \]
  (There are at most finitely many such primes, since each such \(p\) divides \(b_2(b_2 - 1) \neq 0\).) Then any such prime \(p\) divides \(h\).

In this case, we denote \(R\) by \(S((b_p), h)\), since the entire filtered \(\lambda\)-ring structure is determined by the \(b_p\) for \(p \in \mathcal{P}\) and \(h\). Conversely, given integers \(b_p\) (\(p \in \mathcal{P}\)) and \(h\) satisfying the above properties, there is a filtered \(\lambda\)-ring structure on \(\mathbb{Z}[x]/(x^3)\) whose \(\psi^p\) is given by (5.1.1), in which \(c_p\) is determined by \(b_p\) and \(h\) via (5.1.2). Two such filtered \(\lambda\)-rings, \(S((b_p), h)\) and \(S((b'_p), h')\), are isomorphic if and only if \(b_p = b'_p\) for all primes \(p\) and \(h = h'\). In particular,
there are uncountably many isomorphism classes of filtered $\lambda$-ring structures on $\mathbb{Z}[x]/(x^3)$ of the form $S((b_p), h)$.

For example, let $F$ denote the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, or the Cayley octonions $\mathbb{O}$, and let $FP^2$ be the corresponding projective 2-space. Then the $K$-theory filtered $\lambda$-ring of $FP^2$ is

$$K(FP^2) = \begin{cases} S((p), 1) & \text{with } G = 2(2 - 1) = 2 \quad \text{if } F = \mathbb{C} \\ S((p^2), 1) & \text{with } G = 2^2(2^2 - 1) = 12 \quad \text{if } F = \mathbb{H} \\ S((p^4), 1) & \text{with } G = 2^4(2^4 - 1) = 240 \quad \text{if } F = \mathbb{O}. \end{cases}$$

(5.1.3)

5.2. The algebras $H_0^*(R)$. Let $R$ be any filtered $\lambda$-ring structure on $\mathbb{Z}[x]/(x^3)$. Using $\{1, x, x^2\}$ as a $\mathbb{Z}$-basis for $\mathbb{Z}[x]/(x^3)$, we can write each $\mathbb{Z}$-linear self map of $R$ as an element of $M(3, \mathbb{Z})$, the algebra of 3-by-3 matrices with integer entries. We will continue to omit entries that are 0 in a matrix.

Here is the main result of this section. Recall that $P$ denotes the set of all primes.

**Theorem 5.3.** Let $R$ be a filtered $\lambda$-ring structure on $\mathbb{Z}[x]/(x^3)$.

1. If $R = S((c_p))$ for some $c_p, p \in P$, then

   $$H_0^\lambda(R) = \left\{ \begin{bmatrix} a & j & k \\ j & k & j \end{bmatrix} : a, j, k \in \mathbb{Z} \right\}$$

   as a subalgebra of $M(3, \mathbb{Z})$.

2. If $R = S((b_p), h)$ for some $b_p, p \in P$, and $h$, then

   $$H_0^\lambda(R) = \left\{ \begin{bmatrix} a & j & t \\ j & k & t \end{bmatrix} : h(t - j) = kG \right\}$$

   as a subalgebra of $M(3, \mathbb{Z})$, where $G = \gcd(b_q(b_q - 1))_{q \in P}$.

In each case, $H_0^\lambda(R)$ is a commutative algebra and has rank 3 over $\mathbb{Z}$ as an additive group.

To prove this Theorem, we will first compute $\mathfrak{F}^0(R) = \overline{\text{End}}(R)$, still using the $\mathbb{Z}$-basis $\{1, x, x^2\}$.

**Lemma 5.4.** Let $R$ be any filtered $\lambda$-ring structure on $\mathbb{Z}[x]/(x^3)$ and let $g$ be a $\mathbb{Z}$-linear self map of $R$. Then $g \in \overline{\text{End}}(R) = \mathfrak{F}^0(R)$ if and only if

$$g = \begin{bmatrix} a \\ j \\ k \\ s \\ t \end{bmatrix}$$

with

$$s \equiv 0 \equiv t - j \pmod{2}.$$
Proof. By definition, the \( \mathbb{Z} \)-linear map \( g \) lies in \( \text{End}(R) \) if and only if (4.1.1) holds for all primes \( p \) and elements \( r \in R \). This condition only needs to be checked for \( r = 1, x, \) and \( x^2 \). If \( g(1) = a + bx + cx^2 \), then (4.1.1), when applied to \( r = 1 \), becomes
\[
 g(1^p) = g(1) = a + bx + cx^2 \\
\equiv (a + bx + cx^2)^p \quad (\text{mod } p) \\
\equiv a + bx^p + cx^{2p} \quad (\text{mod } p) \\
= a \quad (\text{mod } p) \text{ if } p > 2.
\]
It follows that \( b = c = 0 \). The case \( p = 2 \) gives no additional information.

Write \( g(x) = i + jx + kx^2 \). Then (4.1.1), when applied to \( r = x \), becomes
\[
 g(x^p) = 0 \text{ if } p > 2 \\
\equiv (i + jx + kx^2)^p \quad (\text{mod } p) \\
\equiv i \quad (\text{mod } p).
\]
Since this is true for all odd primes \( p \), we have that \( i = 0 \). The case \( p = 2 \) has not been used yet; we will come back to this below.

Write \( g(x^2) = r + sx + tx^2 \). Then (4.1.1) becomes
\[
 0 = g(x^{2p}) \\
\equiv (r + sx + tx^2)^p \quad (\text{mod } p) \\
\equiv r + sx^p \quad (\text{mod } p).
\]
Therefore, \( r = 0 \) and \( s \equiv 0 \pmod{2} \). Moreover, the condition
\[
 g(x)^2 \equiv g(x^2) \pmod{2}
\]
is equivalent to
\[
 jx^2 \equiv tx^2 \pmod{2},
\]
i.e. \( j \equiv t \pmod{2} \).

This finishes the proof of the Lemma. \( \square \)

Proof of Theorem 5.3. It is immediate from the definition that \( H^0_\chi(R) \) is the subalgebra of \( \text{End}(R) \) consisting of those \( \mathbb{Z} \)-linear self maps \( g \) of \( R \) for which \( g\psi^p = \psi^p g \) for all primes \( p \). Write \( \psi^p(x) = b_p x + c_p x^2 \) and let \( g \in \text{End}(R) \) be as in the statement of Lemma 5.4. Then the equation \( g\psi^p = \psi^p g \) can be written in matrix form as
\[
 g\psi^p = \begin{bmatrix} a & jb_p + sc_p & sb_p^p \\ kb_p + tc_p & \end{bmatrix} \begin{bmatrix} a & jb_p \\ kb_p + tc_p & jc_p + kb_p^2 \\ \end{bmatrix} = \begin{bmatrix} a & jb_p + sc_p + tb_p^2 \\ jc_p + kb_p^2 & sc_p + tb_p^2 \\ \end{bmatrix} = \psi^p g.
\]
In the case that \( R = S((c_p)) \), \( b_p = 0 \) for all \( p \). Therefore, (5.4.1) is equivalent to the conditions
\[
 sc_p = 0 = (t - j)c_p.
\]
Since $c_2$ is an odd integer, it is non-zero in any case. It follows that
\[ s = 0 = t - j, \]
and the first part of Theorem 5.3 is proved.

Now suppose that $R = S((b_p), h)$ for some $b_p$ and $h$. By comparing the $(2, 3)$ entries in (5.4.1), using the fact that $b_2 \neq 0$ is an even integer, one infers that $s = 0$. The only condition left in (5.4.1) now is
\[ kb_p + tc_p = jc_p + kb_p^2, \]
or equivalently,
\[ h \frac{b_p(b_p - 1)}{G}(t - j) = kb_p(b_p - 1). \tag{5.4.2} \]
This condition is trivially true if $b_p = 0$. If $b_p \neq 0$, then, since $b_p \neq 1$ in any case, this condition is equivalent to
\[ h(t - j) = kG. \tag{5.4.3} \]
Conversely, it is easy to see that the conditions, $s = 0$ and (5.4.3), imply (5.4.1) for all primes $p$.

This proves the second part of Theorem 5.3.

It remains to establish the last statement in the Theorem. When $R = S((c_p))$, it is easy to see that $H^0_\rho(R)$ is free of rank 3 over $\mathbb{Z}$ as a group. Indeed, the following elements form a $\mathbb{Z}$-basis for $H^0_\rho(R)$:
\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]
Moreover, any two elements in $H^0_\rho(S((c_p)))$ commute, since
\[
\begin{bmatrix}
a \\
j \\
k
\end{bmatrix}
\begin{bmatrix}
a' \\
j' \\
k'
\end{bmatrix}
= \begin{bmatrix}
aa' \\
jj' \\
kj' + jk'
\end{bmatrix}.
\]
This last matrix remains the same if $a$ (respectively $j$ and $k$) and $a'$ (respectively $j'$ and $k'$) are interchanged. This shows that the algebra $H^0_\rho(S((c_p)))$ is commutative.

For the second case, $R = S((b_p), h)$, we have
\[
\begin{bmatrix}
a \\
j \\
k
\end{bmatrix}
\begin{bmatrix}
a' \\
j' \\
k'
\end{bmatrix}
= \begin{bmatrix}
aa' \\
jj' \\
kj' + tk'
\end{bmatrix}.
\]
To see that this is equal to the product with the reserve order, observe that we can write $kk'G/h$ in two ways:
\[
\frac{kk'G}{h} = k'(t - j) = k(t' - j').
\]
Therefore, we have that
\[ kj' + tk' = k'j + t'k, \]
which shows that $H^0_\lambda(R)$ is a commutative algebra. To see that $H^0_\lambda(R)$ is free of rank 3 over $\mathbb{Z}$, observe that
\[(j, k, t) \in \mathbb{Z}^3 : h(t - j) = kG\] is the kernel of the surjective $\mathbb{Z}$-linear map
\[\varphi : \mathbb{Z}^3 \to d\mathbb{Z} \cong \mathbb{Z},\]
where $d = \gcd(h, G)$ and
\[\varphi((j, k, t)) = h(t - j) - kG.\]
So the group in (5.4.4) is free of rank 2. It follows that $H^0_\lambda(R)$ is free of rank 3, as the $(1, 1)$ entries of elements in it are arbitrary.

This finishes the proof of Theorem 5.3. □

6. $H^1_\lambda$ of the 64 $S((p^r), h)$

The main purpose of this section is to compute the groups $H^1_\lambda(R)$ and to determine the commutativity of the graded algebras $H^\leq1_\lambda(R)$ for the following 64 filtered $\lambda$-ring structures $R$ over $\mathbb{Z}[x]/(x^3)$:
\[S((p^r), h) = \left\{ \psi^p(x) = p^r x + h \frac{p^r (p^r - 1)}{2^r (2^r - 1)} x^2 \right\}.\] (6.0.5)
Here $r \in \{1, 2, 4\}$ and
\[h \in \left\{ \begin{array}{ll}
\{1\} & \text{if } r = 1, \\
\{1, 3, 5\} & \text{if } r = 2, \\
\{1, 3, \ldots, 119\} & \text{if } r = 4.
\end{array} \right.\]
In the notation of section 5.1, these are the only filtered $\lambda$-ring structures on $\mathbb{Z}[x]/(x^3)$ (up to isomorphism) of the form $S((p^r), h)$ for $r = 1, 2,$ and 4.

Before we proceed, we should explain the significance of these 64 filtered $\lambda$-rings. The following statement is shown in [10]: If $X$ is a torsionfree topological space (i.e. its integral cohomology is $\mathbb{Z}$-torsionfree) whose unitary $K$-theory $K(X)$, as a filtered ring, is the ring $\mathbb{Z}[x]/(x^3)$, then $K(X)$ is isomorphic as a filtered $\lambda$-ring to one of the 64 $S((p^r), h)$ in (5.0.5). In other words, among the uncountably many isomorphism classes of filtered $\lambda$-ring structures on $\mathbb{Z}[x]/(x^3)$, only these 64 isomorphism classes can possibly be topologically realized by torsionfree spaces.

Given one of these 64 $\lambda$-rings $S((p^r), h)$, define
\[D = D(r, h) \overset{\text{def}}{=} \gcd(h, 2^r (2^r - 1)).\] (6.0.6)
For example, $D = 1$ if $h = 1$. These are the three cases for the $K$-theory of the projective 2-spaces $\mathbb{F}P^2$ with $\mathbb{F} = \mathbb{C}$ ($r = 1$), $\mathbb{H}$ ($r = 2$), and $\mathbb{O}$ ($r = 4$) (see (5.1.3)). Exactly 35 of these $D(r, h)$ are equal to 1, namely, $D(1, 1)$, $D(2, 1)$, $D(2, 5)$, and 32 of the $D(4, h)$. Also, let $\mathbb{Z}^N$ denote a countably infinite product of copies of the additive group of integers $\mathbb{Z}$. 


With these notations, we can now state the main result of this section.

**Theorem 6.1.** Let $R = S((p^r), h)$ be any one of the 64 filtered $\lambda$-rings in \textbf{[6.0.5]}. Then there is a group isomorphism

$$H^1_\lambda(R) \cong \frac{\mathbb{Z}}{h\mathbb{Z}} \times \frac{\mathbb{Z}}{D\mathbb{Z}} \times \mathbb{Z}^N.$$  

The graded algebra $H^1_\lambda(R)$ is commutative if and only if $D = 1$.

We will need an explicit description of $\text{Der}_\lambda(R)$. Using the $\mathbb{Z}$-basis $\{1, x, x^2\}$, if $f$ is a $\mathbb{Z}$-linear self map of $R$ satisfying $f(R) \subseteq pR$ for some $p$, then it can be represented by a 3-by-3 matrix $(pa_{ij})$ with entries in $p\mathbb{Z}$.

**Lemma 6.2.** Let

$$R = S((b_p), h) = \{ \psi^p(x) = b_p x + c_p x^2 \}$$

be any filtered $\lambda$-ring structure on $\mathbb{Z}[x]/(x^3)$ with $b_2 \neq 0$. Let $\{ f(p) \}_{p \in \mathbb{P}} = \{ (pa(p)_{ij}) \}_{p \in \mathbb{P}}$ be a sequence of $\mathbb{Z}$-linear self maps of $R$ with $f(p)(R) \subseteq pR$ for each $p$. Then $\{ f(p) \}_{p \in \mathbb{P}}$ is an element of $\text{Der}_\lambda(R)$ if and only if the following three conditions hold for all primes $p$ and $q$:

\begin{align*}
\text{(6.2.1)} & \quad a(p)_{12} = a(p)_{13} = a(p)_{21} = a(p)_{31} = 0, \\
\text{(6.2.2)} & \quad pb_q(b_q - 1)a(p)_{23} = qb_p(b_p - 1)a(q)_{23}, \\
\text{(6.2.3)} & \quad qc_p(a(q)_{22} - a(q)_{33}) + qb_p(b_p - 1)a(q)_{32} = pc_q(a(p)_{22} - a(p)_{33}) + pb_q(b_q - 1)a(p)_{32}.
\end{align*}

**Proof.** A little bit of matrix computation shows that for any primes $p$ and $q$, one has

$$\psi^p f(q) + f(p)\psi^q = 
\begin{bmatrix}
pa(p)_{11} + qa(q)_{11} & qa(q)_{12} + pb_qa(p)_{12} + pc_qa(p)_{13} & qa(q)_{13} + pb_q^2a(p)_{13} \\
qb_p a(q)_{21} + qa(p)_{21} & qb_p a(q)_{22} + pb_q a(p)_{22} + pc_q a(p)_{23} & qb_p a(q)_{23} + pb_q^2 a(p)_{23} \\
qc_p a(q)_{21} + qa(p)_{31} & qc_p a(q)_{22} + pb_q^2 a(q)_{32} & qc_p a(q)_{23} + pb_q^2 a(q)_{33}
\end{bmatrix}$$

The matrix for the map $(\psi^q f(p) + f(q)\psi^p)$ is obtained from this matrix by interchanging $p$ and $q$. The sequence $\{ f(p) \}$ is an element of $\text{Der}_\lambda(R)$ if and only if \textbf{[3.0.2]} holds, which is equivalent to saying that the 3-by-3 matrices represented by the two sides are equal. We will now think of \textbf{[3.0.2]} as a matrix equation. It is clear that the $(1, 1)$ entries give no information, and the $a(p)_{11}$ can be any integers.

The equality of the $(1, 3)$ entries in \textbf{[3.0.2]} is equivalent to

$$qa(q)_{13} + pb_q^2 a(p)_{13} = pa(p)_{13} + qb_p^2 a(q)_{13},$$
which can be rewritten as
\[ p(b_q^2 - 1)a(p)_{13} = q(b_p^2 - 1)a(q)_{13}. \]
Since each \( b_p \) is a multiple of \( p \), this equation forces \( a(p)_{13} = 0 \) for all \( p \). Applying this to the \((1, 2)\) entries in \((3.0.2)\), one obtains
\[ q(b_p - 1)a(q)_{12} = p(b_q - 1)a(p)_{12}. \]
Just as above, this forces \( a(p)_{12} = 0 \) for all \( p \). A similar argument applies to the \((2, 1)\) and the \((3, 1)\) entries in \((3.0.2)\) and gives rise to \( a(p)_{21} = a(p)_{31} = 0 \) for all primes \( p \). In other words, the condition imposed by the first rows and the first columns in \((3.0.2)\) is exactly \((6.2.1)\).

From the matrix above, it is immediate that the equality of the \((2, 3)\) (respectively \((3, 2)\)) entries in \((3.0.2)\) is exactly \((6.2.2)\) (respectively \((6.2.3)\)).

It remains to show that the \((2, 2)\) and the \((3, 3)\) entries in \((3.0.2)\) give rise to redundant conditions. In each case, the condition is
\[ pc_qa(p)_{23} = qc_pa(q)_{23}. \]
This can be obtained from \((6.2.2)\) by multiplying both sides of that equation by \( h/G \) (see \((5.1.2)\)), as claimed. (It is here that we are using the condition \( b_2 \neq 0 \).)

This finishes the proof of the Lemma. \(\square\)

We also need an explicit description of \( \text{Inn}_\lambda(R) \).

**Lemma 6.3.** Let \( R \) be any filtered \( \lambda \)-ring structure on \( \mathbb{Z}[x]/(x^3) \) with \( \psi^p(x) = b_px + c_px^2 \). Then, using the \( \mathbb{Z} \)-basis \( \{1, x, x^2\} \), \( \text{Inn}_\lambda(R) \) consists of the following sequences (indexed by the primes) of matrices,
\[
\begin{cases}
0 & 0 \\
0 & -sc_p \\
0 & kbp(b_p - 1) + (j - t)c_p \\
0 & -sb_p(b_p - 1) & sc_p
\end{cases}
\]
\[ p \in \mathbb{P}, \quad (6.3.1) \]
such that \( s \equiv 0 \equiv t - j \pmod{2} \).

Notice that any such sequence of matrices is completely determined by the four parameters \( j, k, s, \) and \( t \).

**Proof.** This is immediate from Lemma 5.4 and \((5.4.1)\), noting that the matrix displayed above is exactly \([\psi^p, g]\) with \( g \) as in the statement of Lemma 5.4. \(\square\)

**6.4. Proof of Theorem 6.1 when \( r = 1 \).** Using Lemma 6.2 we infer that \( \text{Der}_\lambda(R) = \text{Der}_\lambda(S((p), 1) \text{ consists of the sequences } \{f(p) = p(a_{ij})\}_{p \in \mathbb{P}} \text{ of } \mathbb{Z} \text{-linear self maps of } R \text{ satisfying } (6.2.1), \]
\[ a(p)_{23} = (p - 1)a(2)_{23}, \quad (6.4.1) \]
and
\[ a(p)_{22} - a(p)_{33} + 2a(p)_{32} = (p-1)(a(2)_{22} - a(2)_{33} + 2a(2)_{32}) \tag{6.4.2} \]
for all primes \( p \). In other words, to obtain an element in \( \text{Der}_\lambda(R) \), one chooses five arbitrary elements in \( 2\mathbb{Z} \) for the \((1, 1), (2, 2), (2, 3), (3, 2), \) and \((3, 3)\) entries in \( f(2) \). Then for each odd prime \( p \), one chooses three arbitrary elements in \( p\mathbb{Z} \) for the \((1, 1), (2, 2), \) and \((3, 2)\) entries in \( f(p) \). The \((2, 3)\) and \((3, 3)\) entries are then determined by \( \text{(6.4.1)} \) and \( \text{(6.4.2)} \), respectively. Therefore, we can make the identification
\[ \text{Der}_\lambda(R) = (2\mathbb{Z})^5 \times \prod_{p>2} (p\mathbb{Z})^3. \tag{6.4.3} \]

As for \( \text{Inn}_\lambda(R) \), note that the element displayed in \( \text{(6.3.1)} \) in Lemma 6.3 now becomes
\[ \left\{ \begin{array}{c} (p) \\ 2 \end{array} \right\} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -s & -2s \\ 0 & j-t+2k & s \end{bmatrix} \text{ for } p \in P. \tag{6.3.1} \]

In particular, the \( p = 2 \) component of this element is the matrix
\[ \begin{bmatrix} -s & -2s \\ j-t+2k & s \end{bmatrix}. \]

As these elements vary through \( \text{Inn}_\lambda(R) \), \( s \) and \((j-t)\) run through all even integers. Therefore, using the identification \( \text{(6.4.3)} \), we conclude that
\[ H^1_\lambda(R) \cong \text{Der}_\lambda(R) \cong \text{Inn}_\lambda(R) \cong (2\mathbb{Z})^3 \times \prod_{p>2} (p\mathbb{Z})^3 \cong \mathbb{Z}^N \]
as groups. This is exactly the assertion of Theorem 6.1 in the case \( r = 1 \), since \( h = D = 1 \) here.

The commutativity assertion is dealt with below, after the \( r = 4 \) case.

6.5. **Proof of Theorem 6.1 when \( r = 2 \).** In this case, \( \text{(6.2.2)} \) and \( \text{(6.2.3)} \) become, respectively,
\[ a(p)_{23} = \frac{p(p^2-1)}{6} a(2)_{23} \tag{6.5.1} \]
and
\[ h(a(p)_{22} - a(p)_{33}) + 12a(p)_{32} = \frac{p(p^2-1)}{6} (h(a(2)_{22} - a(2)_{33}) + 12a(2)_{32}), \tag{6.5.2} \]
where \( R = S((p^2), h) \) with \( h \in \{1, 3, 5\} \). If \( h = 1 \) or \( 3 \), then the same argument as in the \( r = 1 \) case allows us to once again make the identification \( \text{(6.4.3)} \) for \( \text{Der}_\lambda(R) \).

If \( h = 5 \), then, since 5 and 12 are relatively prime, the left-hand side of \( \text{(6.5.2)} \) can still attain any integer by choosing \( a(p)_{22} \), \( a(p)_{33} \), and \( a(p)_{32} \).
appropriately. In particular, to choose an element \( \{ f(p) \} \) of \( \text{Der}_\lambda(R) \), one can choose five arbitrary elements in \( 2\mathbb{Z} \) for the \((1,1), (2,2), (2,3), (3,2), \) and \((3,3) \) entries in \( f(2) \). Furthermore, for each odd prime \( p \), one chooses an element in \( p\mathbb{Z} \) for the \((1,1) \) entry in \( f(p) \) and two more arbitrary integers to form \( f(p) \). Indeed, denoting the value in the right-hand side of (6.5.2) by \( N_p \), one has

\[
a(p)_{32} = -2N_p + 5r_p
\]

(6.5.3)

for some integer \( r_p \). Therefore, there are two degrees of freedom left, namely, \( r_p \) and, say, \( a(p)_{22} \). This allows us to make the identification

\[
\text{Der}_\lambda(S((p^2), 5) = (2\mathbb{Z})^5 \times \prod_{p>2} (p\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})
\]

(6.5.4)

\[
\cong (2\mathbb{Z})^5 \times \mathbb{Z}^N.
\]

As for \( \text{Inn}_\lambda(R) \), note once again that each element in \( \text{Inn}_\lambda(R) \) is still determined by the four parameters \( s, j, k, \) and \( t \) with \( s \) and \( (j-t) \) even. The \( p = 2 \) component of a typical element in \( \text{Inn}_\lambda(R) \) (see (6.3.1)) now takes the form

\[
\begin{bmatrix}
-s h & -12 s \\
  h(j-t) + 12 k & s h
\end{bmatrix}.
\]

To finish the proof, observe that

\[
\{ s h : s \in 2\mathbb{Z} \} = 2h\mathbb{Z}
\]

and

\[
\{ h(j-t) + 12 k : (j-t) \in 2\mathbb{Z}, k \in \mathbb{Z} \} = \begin{cases} 
2\mathbb{Z} & \text{if } h = 1, 5, \\
6\mathbb{Z} & \text{if } h = 3.
\end{cases}
\]

Summarizing this discussion, we have shown that

\[
H^1_\lambda(S((p^2), h)) \cong \begin{cases} 
(2\mathbb{Z})^3 \times \mathbb{Z}^N & \text{if } h = 1 \\
(2\mathbb{Z}/6\mathbb{Z})^2 \times (2\mathbb{Z})^3 \times \mathbb{Z}^N & \text{if } h = 3 \\
(2\mathbb{Z}/10\mathbb{Z}) \times (2\mathbb{Z})^3 \times \mathbb{Z}^N & \text{if } h = 5
\end{cases}
\]

This is exactly the claim in Theorem 6.1 when \( r = 2 \), since \( D = 1 \) (respectively 3) when \( h = 1 \) or 5 (respectively 3).

The commutativity assertion is dealt with below, after the \( r = 4 \) case.

6.6. **Proof of Theorem 6.1 when \( r = 4 \).** When \( r = 4 \), (6.2.2) and (6.2.3) can be rewritten as, respectively,

\[
a(p)_{23} = \frac{p^3(p^4-1)}{120} a(2)_{23}
\]

(6.6.1)
and
\[ \frac{h}{D}(a(p)_{22} - a(p)_{33}) + \frac{240}{D}a(p)_{32} \]
\[ = \frac{h}{D} \cdot \frac{p^3(p^4 - 1)}{120} (a(2)_{22} - a(2)_{33}) + 2 \cdot \frac{p^3(p^4 - 1)}{D}a(2)_{32}, \]
(6.6.2)

where \( D = D(4, h) = \gcd(h, 240). \) Since \( h/D \) and \( 240/D \) are relatively prime, we can make the same argument as in the case \( r = 2, h = 5. \) More precisely, the two conditions, (6.6.1) and (6.6.2), above tell us that to choose an element of \( \text{Der}_\lambda(R), \) one first chooses five arbitrary elements in \( 2\mathbb{Z} \) for the \((1, 1), (2, 2), (2, 3), (3, 2), \) and \((3, 3)\) entries in \( f(2). \) Then for each odd prime \( p, \) one chooses an element in \( p\mathbb{Z} \) for the \((1, 1)\) entry in \( f(p) \) and two more arbitrary integers to form \( f(p). \) These last two degrees of freedom come from (6.6.2) and are completely analogous to (6.5.3). This allows us to make the identification (6.5.4) for \( \text{Der}_\lambda(R) = \text{Der}_\lambda(S((p^4), h)). \)

To complete the proof, observe that the \( p = 2 \) component of a typical element in \( \text{Inn}_\lambda(R) \) (see (6.3.1)) has the form
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & -sh & -240s \\
0 & h(j - t) + 240k & sh
\end{bmatrix}
\]

Moreover, we have that
\[ \{sh: s \in 2\mathbb{Z}\} = 2h\mathbb{Z} \]

and
\[ \{h(j - t) + 240k: j \equiv t \pmod{2}\} = 2D\mathbb{Z}. \]

Therefore, it follows as in the case when \( r = 2 \) and \( h = 5 \) that there is a group isomorphism
\[ H^1_\lambda(S((p^4), h)) \cong \frac{2\mathbb{Z}}{2h\mathbb{Z}} \times \frac{2\mathbb{Z}}{2D\mathbb{Z}} \times \mathbb{Z}^N, \]
as desired.

This proves Theorem 6.1 when \( r = 4, \) except for the commutativity assertion, to which we now turn.

6.7. Proof of the commutativity assertion in Theorem 6.1. We can now show that the graded algebra \( H^{\leq 1}_\lambda(R) = H^{\leq 1}_\lambda(S((p^r), h)) \) is commutative if and only if \( D = 1, \) i.e. \( h \) and \( 2^r(2^r - 1) \) are relatively prime. Since Theorem 5.3 already established the commutativity of \( H^0_\lambda(R), \) commutativity of \( H^{\leq 1}_\lambda(R) \) here means that elements in \( H^0_\lambda(R) \) commute with those in \( H^1_\lambda(R). \)

To do this, let \( g \in H^0_\lambda(R) \subseteq \bar{\text{End}}(R) = \mathfrak{g}^0(R) \) be a 0-cocycle, as in Theorem 5.3 (2), and let \( f = \{f(p) = (f(p)_{ij})\}_{p \in \mathbb{P}} \) be a 1-cocycle, i.e. an element of \( \text{Der}_\lambda(R). \) Then the map
\[ (g \circ f - f \circ g)(p) = gf(p) - f(p)g \]
is represented by the matrix
\[
\begin{bmatrix}
0 & 0 & 0 \\
-kf(p)_{23} & (j-t)f(p)_{23} & 0 \\
0 & kf(p)_{22} - kf(p)_{33} + (t-j)f(p)_{32} & kf(p)_{23}
\end{bmatrix}
\] (6.7.1)

Notice that in all 64 cases under consideration, we have
\[
G = \gcd(b_p(b_p - 1))_{p \in \mathcal{P}} = \gcd(p^r(p^r - 1))_{p \in \mathcal{P}}
\]
\[
= \begin{cases} 
2^r(2^r - 1) & \text{if } r = 1 \\
12 & \text{if } r = 2 \\
240 & \text{if } r = 4.
\end{cases}
\] (6.7.2)

In particular, \(D = D(r, h) = 1\) if and only if \(h\) and \(G\) are relatively prime.

Suppose that \(D = 1\). We need to show that the cohomology classes represented by the 1-cocycles \(g \circ f\) and \(f \circ g\) are the same, i.e. the 1-cocycle \((g \circ f - f \circ g)\) is actually a 1-coboundary. Since the entries in \(g\) satisfies
\[
h(t - j) = kG,
\] (6.7.3)
it follows that \(k \equiv 0 \pmod{h}\). Using Lemma 5.4 one observes that the matrix
\[
\beta = \frac{k}{h} \begin{bmatrix}
0 & 0 & 0 \\
0 & f(2)_{22} & f(2)_{23} \\
0 & f(2)_{32} & f(2)_{33}
\end{bmatrix}
\]
represents a 0-cochain in \(\mathcal{F}^*(R)\). We claim that
\[
d^0 \beta = g \circ f - f \circ g.
\] (6.7.4)

To see this, first note that the matrix in (6.3.1) is the \(p\)th component of a 1-coboundary in \(\mathcal{F}^*(R)\). Applying this to the 0-cochain \(\beta\), we see that the first column and first row of \((d^0 \beta)(p)\) are both 0 and that its lower-right 2-by-2 submatrix is
\[
\frac{k}{h} \begin{bmatrix}
-f(2)_{23}c_p & -f(2)_{23}p^r(p^r - 1) \\
f(2)_{32}p^r(p^r - 1) + f(2)_{22} - f(2)_{33}c_p & f(2)_{23}c_p
\end{bmatrix}.
\] (6.7.5)

To prove (6.7.4), we only need to show that the matrix in (6.7.5) coincides with the lower-right 2-by-2 block of the matrix (6.7.1). Now (6.7.2) and (6.7.3) imply that
\[
f(p)_{23} = \frac{p^r(p^r - 1)}{G} f(2)_{23}.
\] (6.7.6)

One infers from (6.7.6) and (6.7.3) that
\[
-\frac{k}{h} f(2)_{23}p^r(p^r - 1) = -\frac{k}{h} Gf(p)_{23} = (j-t)f(p)_{23}.
\]

This shows that the \((2, 3)\) entries in \(d^0 \beta\) and \((g \circ f - f \circ g)\) coincide.
Similarly, we have
\[ \frac{k}{h} f(2) c_{p} = \frac{k}{h} f(2) \frac{hp^r(p^r - 1)}{G} = kf(p) \]
This shows that the (3, 3) (and hence also the (2, 2)) entries in \( d^0 \beta \) and \((g \circ f - f \circ g)\) coincide.

Finally, we have
\[ \frac{k}{h} \left( c_{p}(f(2)_{22} - f(2)_{33}) + p^r(p^r - 1)f(2)_{32} \right) \]
\[ = \frac{k}{h} \left( c_{2} f(p)_{22} - f(p)_{33} + 2^r(2^r - 1)f(p)_{32} \right) \text{ by (6.2.3)} \]
\[ = \frac{k}{h} \left( h(f(p)_{22} - f(p)_{33}) + G f(p)_{32} \right) \]
\[ = \frac{k}{h} (f(p)_{22} - f(p)_{33}) + (t - j)f(p)_{32}. \]
The second equality follows from the fact that \( c_{2} = h \) and \( G = 2^r(2^r - 1) \). This shows that the (3, 2) entries in \( d^0 \beta \) and \((g \circ f - f \circ g)\) coincide, and thus the claim (6.7.4) is proved. Of course, this implies that the cohomology classes represented by the 1-cocycles \( g \circ f \) and \( f \circ g \) are equal.

Now suppose that \( D = \gcd(h, 2^r(2^r - 1)) > 1 \). (This can only happen when \( r \neq 1 \).) We must show that the graded algebra \( H_{1}^{\leq 1}(R) \) is not commutative. Consider the 0-cocycle
\[ g = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{h}{D} & \frac{G}{D} \end{array} \right]. \]
That this is indeed a 0-cocycle follows from Theorem 5.3 (2). Also, by Lemma 6.2, the sequence (indexed by the primes) of linear self maps on \( R \) represented by the matrices
\[ f = \left\{ \begin{array}{c} \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \frac{2p^r(p^r - 1)}{G} \\ 0 & 0 & 0 \end{array} \right] \end{array} \right\}_{p \in P} \]
is an element of \( \text{Der}^1_{\lambda}(R) \), i.e. a 1-cocycle in \( \mathcal{F}^1(R) \). We claim that the 1-cocycle \((g \circ f - f \circ g)\) is not a 1-coboundary, which would imply that the graded algebra \( H_{1}^{\leq 1}(R) \) is not commutative. To see this, observe from (6.7.1) that the (3, 3) entry in the matrix of \((gf(2) - f(2)g)\) is
\[ \frac{h}{D} \cdot 2 \cdot \frac{2^r(2^r - 1)}{G} = \frac{2h}{D}, \]
From Lemma 6.3 the (3, 3) entry of the component for \( p = 2 \) of an element in \( \text{Inn}^1_{\lambda}(R) \) is of the form \( sc_{2} = sh \) for some even integer \( s \). But then the equality
\[ sh = \frac{2h}{D} \]
would imply
\[ Ds = 2, \]
which is absurd, since $D > 1$. In other words, we have shown that the cohomology classes $g \in H^0_\lambda(R)$ and $[f] \in H^1_\lambda(R)$ do not commute with each other, as desired.

This finishes the proof of the commutativity assertion in Theorem 6.1. The proof of Theorem 6.1 is now complete.

7. $H^0_\lambda$ OF CERTAIN FILTERED $\lambda$-RING STRUCTURES ON $\mathbb{Z}[x]/(x^4)$

The purpose of this section is to compute the algebras $H^0_\lambda(R)$ for the following 61 filtered $\lambda$-ring structures $R$ on the truncated polynomial algebra $\mathbb{Z}[x]/(x^4)$, with $x$ in a fixed positive filtration:

1. $R = K(\mathbb{C}P^3)$ with Adams operations
   \[ \psi^p(x) = (1 + x)^p - 1 \]
   for $p$ primes.

2. $R = S(h, d_2)$ with
   \[ S(h, d_2) = \left\{ \psi^p(x) = p^2 x + h\frac{p^2(p^2 - 1)}{12}x^2 + d_p x^3 \right\}_{p \in \mathbb{P}}, \]
   where $h \in \{1, 5\}$, $d_2 \in \{0, 2, 4, \ldots, 58\}$, and \[ d_p = \frac{p^2(p^4 - 1)}{60}d_2 + \frac{p^2(p^2 - 1)(p^2 - 4)h^2}{360} \] for odd primes $p$.

In this notation, the $K$-theory filtered $\lambda$-ring of $\mathbb{H}P^3$, the quaternionic projective 3-space, is $S(1, 0)$.

It is shown in [10] that the truncated polynomial algebra $\mathbb{Z}[x]/(x^4)$ admits uncountably many isomorphism classes of filtered $\lambda$-ring structures. Among those classes are the 61 filtered $\lambda$-rings above. Moreover, if $X$ is a torsionfree space (i.e. its integral cohomology is $\mathbb{Z}$-torsionfree) whose $K$-theory filtered ring is the truncated polynomial ring $\mathbb{Z}[x]/(x^4)$, then $K(X)$ is isomorphic as a filtered $\lambda$-ring to one of the above 61 filtered $\lambda$-rings. In other words, among the uncountably many isomorphism classes of filtered $\lambda$-ring structures on $\mathbb{Z}[x]/(x^4)$, at most 61 of them, those listed above, can possibly be topologically realized by torsionfree spaces.

As in previous sections, we will use the standard $\mathbb{Z}$-basis $\{1, x, x^2, x^3\}$ for the ring $\mathbb{Z}[x]/(x^4)$, and, using this basis, each $\mathbb{Z}$-linear self map is represented by an element in $M(4, \mathbb{Z})$, the algebra of 4-by-4 matrices with integer entries. The algebra $H^0_\lambda(R)$ then becomes a subalgebra of the matrix algebra $M(4, \mathbb{Z})$.

Here are the main results of this section.

**Theorem 7.1.** With the standard basis $\{1, x, x^2, x^3\}$ of $\mathbb{Z}[x]/(x^4)$, $H^0_\lambda(K(\mathbb{C}P^3))$ is the subalgebra of $M(4, \mathbb{Z})$ consisting of matrices of the form
This algebra is commutative and, as an additive group, is free of rank 4 over \( \mathbb{Z} \).

**Theorem 7.2.** Consider any one of the 60 filtered \( \lambda \)-rings \( S(h,d_2) \). Then \( H_\lambda^0(S(h,d_2)) \) is the subalgebra of \( M(4,\mathbb{Z}) \) consisting of matrices of the form

\[
\begin{bmatrix}
a & j \\
j & k + 2k \\
k & j + 6k + 6l \\
l & 4k + 6l & j + 6k + 6l
\end{bmatrix},
\]

(7.1.1)

such that the following is true:

(1) If \( h = 1 \), then

\[
\begin{align*}
r &= j + 12k, \\
(6d_2 + 1)s &= (8 - 12d_2)k + 60l, \text{ and} \\
w &= 6s + j + 12k.
\end{align*}
\]

(7.2.2)

(2) If \( h = 5 \), then

\[
\begin{align*}
k &\equiv 0 \pmod{5}, \\
r &= j + \frac{12k}{5}, \\
(6d_2 + 25)s &= (200 - 12d_2)k + 300l, \text{ and} \\
(6d_2 + 25)w &= (6d_2 + 25)j + 300k + 360l.
\end{align*}
\]

In each of the 60 cases, the algebra \( H_\lambda^0(S(h,d_2)) \) is commutative and, as an additive group, is free of rank 4 over \( \mathbb{Z} \).

To prove these results, we begin by computing the group \( F^0(R) \) of 0-cochains.

**Lemma 7.3.** Let \( R \) be any filtered \( \lambda \)-ring structure on \( \mathbb{Z}[x]/(x^4) \). Then the group \( \text{End}(R) = \mathcal{F}^0(R) \) consists of the matrices

\[
\begin{bmatrix}
a & j & n & u \\
j & k & r & v \\
k & r & v \\
l & s & w
\end{bmatrix}
\]

(7.3.1)

such that

(1) \( n \equiv u \equiv 0 \pmod{6} \),

(2) \( r - j \equiv s \equiv 0 \pmod{2} \),

(3) \( w - j \equiv v \equiv 0 \pmod{3} \).
Proof. Let $g$ be a $\mathbb{Z}$-linear self map on $R$ represented by the matrix

$$
g = \begin{bmatrix}
a & i & m & t \\
b & j & n & u \\
c & k & r & v \\
d & l & s & w
\end{bmatrix}.
$$

By definition, $g \in \text{End}(R)$ if and only if it satisfies (4.1.1), which is equivalent to (4.1.2) for $0 \leq i \leq 3$. For $i = 0$, (4.1.2) says

$$
g(1^p) = g(1) = a + bx + cx^2 + dx^3 \\equiv (a + bx + cx^2 + dx^3)^p \pmod{p} \\
\equiv a + bx^p \pmod{p} \\
= a \pmod{p} \quad \text{if } p \geq 5.
$$

This implies that $b = c = d = 0$; that is, $g(1) = a \in \mathbb{Z}$. The cases $p = 2$ and 3 give no additional information.

When $i = 1$, the condition (4.1.2) says

$$
g(x^p) = (i + jx + kx^2 + lx^3)^p \\equiv i \pmod{p} \quad \text{if } p \geq 5 \\
\equiv g(x^p) \pmod{p} \\
= 0 \pmod{p} \quad \text{if } p \geq 5.
$$

This implies that $i = 0$. We have not used the cases $p = 2$ and 3 yet. If $p = 2$, then

$$
g(x)^2 \equiv jx^2 \pmod{2} \\
\equiv g(x^2) \pmod{2} \\
= m + nx + rx^2 + sx^3.
$$

This implies that

$$
m \equiv n \equiv s \equiv j - r \equiv 0 \pmod{2}.
$$

If $p = 3$, then

$$
g(x)^3 \equiv jx^3 \pmod{3} \\
\equiv g(x^3) \pmod{3} \\
= t + ux + vx^2 + wx^3.
$$

This implies that

$$
t \equiv u \equiv v \equiv j - w \equiv 0 \pmod{3}.
$$
For \( i = 2 \), the condition (4.1.2) says
\[
\begin{align*}
g(x^2)^p &= (m + nx + rx^2 + sx^3)^p \\
&\equiv m + nx^p \pmod{p} \\
&\equiv g(x^{2p}) \pmod{p} \\
&= 0.
\end{align*}
\]
This implies that \( m = 0 \) and \( n \equiv 0 \pmod{6} \).

Finally, for \( i = 3 \), (4.1.2) says
\[
\begin{align*}
g(x^3)^p &= (t + ux + vx^2 + wx^3)^p \\
&\equiv t + ux^p \pmod{p} \\
&\equiv g(x^{3p}) \pmod{p} \\
&= 0.
\end{align*}
\]
So \( t = 0 \) and \( u \equiv 0 \pmod{6} \).

This proves the Lemma. \( \square \)

Proof of Theorem 7.1. Let \( R \) denote \( K(\mathbb{CP}^3) \) and let \( g \in \text{End}(R) \) be as in (7.3.1). Then \( g \in H_0^0(R) \) if and only if \( g\psi^p = \psi^pg \) for all primes \( p \). Since the matrices for both \( g \) and \( \psi^p \) have 0’s in their first rows and first columns, except in the \( (1,1) \) entries, \( g \) commutes with \( \psi^p \) if and only if their lower-right 3-by-3 submatrices commute. Note that the matrix for \( \psi^p \) is
\[
\psi^p = \begin{bmatrix}
1 \\
p \\
p^2 & p^2 & p^2(p - 1) & p^3 \\
\end{bmatrix}.
\]

Denote by \( A \) (respectively \( B_p \)) the lower-right 3-by-3 submatrix of \( g \) (respectively \( \psi^p \)). Then \( A \) commutes with \( B_p \) means that
\[
AB_p = \begin{bmatrix}
pj + \binom{p}{2}n + \binom{p}{3}u & p^2n + p^2(p - 1)u & p^3u \\
jk + \binom{p}{3}(\binom{p}{2}j + p^2k) & p^2r + p^2(p - 1)v & p^3v \\
pl + \binom{p}{3}s + \binom{p}{4}w & p^2r + p^2(p - 1)v & p^3w
\end{bmatrix}
= B_pA.
\]

For any prime \( p \), comparing the \( (1,3) \) entries in (7.3.2) yields \( u = 0 \). Applying this to the \( (1,2) \) and the \( (2,3) \) entries gives \( n = v = 0 \). In particular, if \( g \) commutes with \( \psi^p \) for any \( p \), then \( g \) must be lower-triangular, i.e. it respects the filtration of \( R \). The diagonal entries in (7.3.2) provide no new information about the entries of \( g \). Since \( v = 0 \), the \( (2,1) \) entries in (7.3.2) yields
\[
r = j + 2k,
\]
(7.3.3)
as stated in the Theorem.

Now consider the \((3, 1)\) entries in (7.3.2). If \(p = 2\), then, since \(\binom{2}{3} = 0\), one obtains
\[
\begin{align*}
s &= 4k + 6l. && (7.3.4)
\end{align*}
\]
If \(p = 3\), then one obtains the equation
\[
3l + 3s + w = j + 18k + 27l,
\]
which is equivalent to
\[
w = j + 6k + 6l. && (7.3.5)
\]
The \((3, 2)\) entries in (7.3.2) give no new information, since the equation obtained from them is
\[
w = r + s
\]
for each prime \(p\). It is clear from (7.3.3), (7.3.4), and (7.3.5) that this is true. Conversely, it is straightforward to check that (7.3.3), (7.3.4), and (7.3.5) imply that \(AB_p\) and \(B_pA\) have the same \((3, 1)\) and \((3, 2)\) entries for all primes \(p\).

This proves that \(H^0_\lambda(R)\) is the subalgebra of \(M(4, \mathbb{Z})\) consisting of matrices of the form (7.1.1). It is now also easy to see that, as an additive group, \(H^0_\lambda(R)\) is free of rank 4 over \(\mathbb{Z}\), since \(a, j, k,\) and \(l\) are arbitrary and the other three entries are linear combinations in \(j, k,\) and \(l\).

To show that \(H^0_\lambda(R)\) is a commutative algebra, let \(g'\) be another element in it of the form (7.1.1) with \(a', j', k',\) and \(l'\) in place of \(a, j, k,\) and \(l\), respectively. It is clearly enough to show that the \((i, j)\) entries in \(gg'\) and \(g'g\) coincide for \((i, j) = (3, 2), (4, 2),\) and \((4, 3)\). Denote the \((i, j)\) entry in a matrix \(A\) by \(A_{ij}\). A little bit of matrix computation then shows that \((gg')_{ij}\) is
\[
\begin{align*}
&\bullet kj' + k'j + 2kk' \text{ if } (i, j) = (3, 2), \\
&\bullet lj' + l'j + 4kk' + 6(k' + 6l') \text{ if } (i, j) = (4, 2), \text{ and} \\
&\bullet (4k + 6l)(j' + 2k') + (4k' + 6l')(j + 2k) + (4k + 6l)(4k' + 6l') \text{ if } (i, j) = (4, 3).
\end{align*}
\]
Since each of these entries remains the same if \(j\) (respectively \(k, l\)) and \(j'\) (respectively, \(k'\) and \(l'\)) are interchanged, it follows that \(gg'\) is equal to \(g'g\), as desired.

This finishes the proof of Theorem 7.1. \(\square\)

**Proof of Theorem 7.2.** Let \(R = S(h, d_2)\). This proof is quite similar to the proof of Theorem 7.1.

The Adams operation \(\psi^p\) in \(R\) is given by
\[
\psi^p(x) = p^2x + h\frac{p^2(p^2 - 1)}{12}x^2 + d_px^3
\]
and $\psi^p$ is a ring map on $R$. Therefore, the matrix of $\psi^p$ has 0's in its first row and first column, except for the entry 1 in the (1, 1) spot, and its lower-right 3-by-3 submatrix is

$$B_p = \begin{bmatrix}
p^2 & 0 & 0 \\
hp^2(p^2 - 1)/12 & p^4 & 0 \\
d_p & hp^4(p^2 - 1)/6 & p^6
\end{bmatrix}.$$  

Let $g \in \text{End}(R)$ be a 0-cochain whose matrix is as in (7.3.1), and let $A$ denote its lower-right 3-by-3 submatrix. Then $g$ commutes with $\psi^p$ if and only if $A$ commutes with $B_p$. As in (7.3.2), this means that

$$\begin{bmatrix}
p^2 j + hp^2(p^2 - 1)n/12 + dp u & p^4 n + hp^4(p^2 - 1)u/6 & p^6 u \\
p^2 k + hp^2(p^2 - 1)r/12 + dp v & p^4 r + hp^4(p^2 - 1)v/6 & p^6 v \\
p^2 l + hp^2(p^2 - 1)s/12 + dp w & p^4 s + hp^4(p^2 - 1)w/6 & p^6 w
\end{bmatrix} = \begin{bmatrix}
p^2 j \\
hp^2(p^2 - 1)j/12 + p^4 k & hp^2(p^2 - 1)n/12 + p^4 r & hp^2(p^2 - 1)u/12 + p^4 v \\
d_p j + hp^4(p^2 - 1)k/6 + p^6 l & dp n + hp^4(p^2 - 1)r/6 + p^6 s & dp u + hp^4(p^2 - 1)v/6 + p^6 w
\end{bmatrix}.$$  

(7.3.6)

Again, for any prime $p$, the (1, 3) entries tell us that $u = 0$. Applying this to the (1, 2) and the (2, 3) entries, one obtains $n = v = 0$, i.e. $g$ is lower-triangular. The diagonal entries in (7.3.6) give no new information about the entries in $g$.

The (2, 1) entries in (7.3.6) yield, for any prime $p$, the equation

$$h(r - j) = 12k.$$  

If $h = 1$, this means that

$$r = j + 12k,$$  

as stated in the Theorem. If $h = 5$, then 5 must divide $k$ and

$$r = j + \frac{12k}{5},$$  

as desired.

Setting $p = 2$, the (3, 1) and (3, 2) entries in (7.3.6) yield the simultaneous equations

$$hs + d_2 w = d_2 j + 8hk + 60l$$

$$6s - hw = -hj - 12k$$  

(7.3.7)

in $s$ and $w$. It is an elementary exercise to see that this system of linear equations gives rise to the required conditions, (7.2.2) and (7.2.3), for $s$ and $w$ in the statement of Theorem 7.2. Furthermore, using (7.0.7), which expresses each $d_p$ in terms of $d_2$ and $h$, a slightly tedious but easy calculation shows that (7.2.2) and (7.2.3) make the (3, 1) (respectively (3, 2)) entries in $AB_p$ and $B_p A$ coincide for any prime $p$. This proves that $H^0_\lambda(R)$ consists of exactly the matrices (7.2.1) satisfying (7.2.2) or (7.2.3), depending on whether $h$ is 1 or 5.
To see that $H^0_\lambda(R)$ is a commutative algebra, note that
\[
\begin{align*}
    r &= j + a_1k \\
    s &= a_2k + a_3l \\
    w &= j + a_4k + a_5l
\end{align*}
\]
for some rational numbers $a_1, \ldots, a_5$. Therefore, the proof for the commutativity of $H^0_\lambda(K(\mathbb{C}P^3))$ can be used virtually verbatim here as well.

To see that $H^0_\lambda(R)$ is free of rank 4, first consider the case $h = 1$. We will use the notations in (7.2.2). Since $r$ (respective $w$) are linear combinations in $j$ and $k$ (respectively $s, j,$ and $k$), it suffices to show that
\[
\{(k, l, s) \in \mathbb{Z}^3 : (6d_2 + 1)s = (8 - 12d_2)k + 60l\} \quad (7.3.8)
\]
is free of rank 2. Define $\alpha$ by
\[
\alpha = \gcd(8 - 12d_2, 60, 6d_2 + 1).
\]
Then there is a surjective $\mathbb{Z}$-linear map
\[
\varphi: \mathbb{Z}^3 \to \alpha \mathbb{Z} \cong \mathbb{Z},
\]
where
\[
\varphi((k, l, s)) = (8 - 12d_2)k + 60l - (6d_2 + 1)s.
\]
The kernel of $\varphi$ is exactly the group in (7.3.8), which shows that it has rank 2, as desired.

Now consider the case $h = 5$. Let $H$ be the subgroup of
\[
K = \mathbb{Z} \times (5\mathbb{Z}) \times \mathbb{Z}^3 \cong \mathbb{Z}^5
\]
consisting of elements $(j, k, l, s, w) \in K$ for which the last two equations in (7.2.3) hold. As in the case $h = 1$, it suffices to show that $H$ is free of rank 3. Define $\beta$ and $\gamma$ by
\[
\begin{align*}
    \beta &= \gcd(5(200 - 12d_2), 300, 6d_2 + 25) \\
    \gamma &= \gcd(5(300), 360, 6d_2 + 25).
\end{align*}
\]
Then there is a surjective $\mathbb{Z}$-linear map
\[
\phi: K \to \beta \mathbb{Z} \times \gamma \mathbb{Z} \cong \mathbb{Z}^2,
\]
where
\[
\phi((j, k, l, s, w)) = ((200 - 12d_2)k + 300l - (6d_2 + 25)s, \quad (6d_2 + 25)(j - w) + 300k + 360l).
\]
The kernel of $\phi$ is exactly $H$, which shows that $H$ is free of rank 3, as desired.

This finishes the proof of Theorem (7.2).
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E-mail address: dyau@math.uiuc.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801 USA