Self-similar blow-up for a diffusion-attraction problem

Ignacio A. GUERRA†§, Mark A. PELETIER‡
†Centre for Mathematical Modeling, Universidad de Chile, FCFM Casilla 170 Correo 3 Santiago, Chile
‡Department of Mathematics and Computer Science, TU Eindhoven, P.O Box 513 5600 MB Eindhoven, The Netherlands
E-mail: †iguerra@dim.uchile.cl
E-mail: ‡Mark.Peletier@cwi.nl

Abstract. In this paper we consider a system of equations that describes a class of mass-conserving aggregation phenomena, including gravitational collapse and bacterial chemotaxis. In spatial dimensions strictly larger than two, and under the assumptions of radial symmetry, it is known that this system has at least two stable mechanisms of singularity formation (see e.g. M. P. Brenner et al. 1999, Nonlinearity 12, 1071-1098); one type is self-similar, and may be viewed as a trade-off between diffusion and attraction, while in the other type the attraction prevails over the diffusion and a non-self-similar shock wave results. Our main result identifies a class of initial data for which the blow-up behaviour is of the former, self-similar type. The blow-up profile is characterized as belonging to a subset of stationary solutions of the associated ordinary differential equation.

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1. Introduction

We consider the parabolic-elliptic system

\[ \begin{align*}
    \frac{\partial n}{\partial t} &= \text{div}\{\Theta \nabla n + n \nabla \phi\} \quad \text{in} \ \Omega \times \mathbb{R}^+, \\
    \Delta \phi &= n \quad \text{in} \ \Omega \times \mathbb{R}^+, \\
    0 &= (\Theta \nabla n + n \nabla \phi) \cdot \vec{\nu} \quad \text{on} \ \partial\Omega \times \mathbb{R}^+, \\
    \phi &= 0 \quad \text{on} \ \partial\Omega \times \mathbb{R}^+, \\
    n(x, 0) &= n_0(x) \quad \text{in} \ \Omega,
\end{align*} \]

where \( \Omega = B_1(0) = \{x \in \mathbb{R}^d; |x| \leq 1\} \), \( d > 2 \), and \( \vec{\nu} \) is the outer normal vector from the boundary \( \partial\Omega \). Here \( \Theta > 0 \) is a constant parameter. The initial condition \( n_0 \) is chosen in \( L^2(\Omega) \), radially symmetric, and such that

\[ \int_{\Omega} n_0 \, dx = 1, \quad \text{and} \quad n_0(x) \geq 0 \quad \text{in} \ \Omega. \]
Equations (11)–(16) define a problem for the unknown mass density $n$ and potential $\phi$. Mass is conserved by the no-flux condition (3), and therefore (6) implies
\[ \int_{\Omega} n(x,t) \, dx = \int_{\Omega} n_0(x) \, dx = 1. \] (7)

Problem (11)–(15) is a model for the evolution of a cluster of particles under gravitational interaction and Brownian motion (see [5] and the references therein). Here $n$ represents the mass density, $\phi$ the gravitational potential, and $\Theta$ a rescaled temperature characterizing the Brownian motion. This model also appears in the study of evolution of polytropic stars, by considering the evolution of self-interacting clusters of particles under frictional and fluctuating forces [29]. Finally, problem (11)–(16) also arises in the study of the motion of bacteria by chemotaxis as a simplification (see [21]) of the Keller-Segel model [22, 28, 2, 8]. Here the variables $n$ and $\phi$ represent the density of bacteria and the concentration of the chemo-attractant.

We view the problem (11)–(16) as an evolution equation in $n$, since by equations (2–3) the function $\phi$ is readily recovered from the solution $n$. It is known [6] that problem (11)–(16) has a unique local solution if $n_0 \in L^2(\Omega)$, which satisfies $n \in L^\infty(\Omega \times (\epsilon,\tilde{T}))$ for some $\tilde{T} > 0$ and for every $\epsilon > 0$. We restrict ourselves to the analysis of radially symmetric solutions and write $n(r,t) := n(x,t)$ with $r = |x| \in [0,1]$.

Since we are interested in the question when and how (11)–(16) generates singularities, we define:
\[ T^* = \sup \{ \tau > 0 | \text{Problem (11)–(16) has a solution } n \in L^\infty(\Omega \times (\epsilon,\tau)) \}. \]
If $T^* < \infty$, then we say that blow-up occurs for (11)–(16), in which case
\[ \lim_{t \to T^*} \sup_{[0,1]} n(r,t) = \infty. \] (8)

Various sufficient conditions for blow-up are known [3, 4, 7, 6].

For $d = 3$, Herrero et al. [19, 20] were the first to study the behaviour of the solution close to blow-up, using matched asymptotic expansions. Later Brenner et al. [14] studied the problem for $2 < d < 10$. They used a numerical approach to describe solutions and proved existence and linear stability of similarity profiles. Note however that no proof of convergence or characterization of blow-up in terms of initial data were given in these references. The principal types of blow-up described in [19, 20, 14] are:

(a) A solution $n(r,t)$ consists of an imploding smoothed shock wave which moves towards the origin. As $t \to T^*$, the bulk of such a wave is concentrated at distances $O((T^* - t)^{1/d})$ from the origin, has a width $O((T^* - t)^{(d-1)/d})$, and at its peak it reaches a height of order $O((T^* - t)^{-2(d-1)/d})$. This type of blow-up has the property of concentration of mass at the origin at the blow-up time, i.e.
\[ \lim_{r \to 0} \left[ \lim_{t \to T^*} \int_{0}^{r} n(y,t) y^{d-1} \, dy \right] = C > 0. \] (9)
This situation is depicted in Figure 1 (left).

(b) A solution $n(r,t)$ has a self-similar blow-up of the form
\[ (T^* - t)n \left( \eta \sqrt{(T^* - t)\Theta(t)} \right) \sim \Psi(\eta) \quad \text{as} \quad t \to T^*. \] (10)
Note that this implies that $n$ satisfies (9) with $C = 0$. Therefore no concentration of mass at the origin occurs at the blow-up time. This blow-up behaviour is depicted in Figure 1 (right).
The results of this paper are two-fold. First, we demonstrate rigorously that the self-similar blow-up structure (10) is an attractor for the system (1)–(6); secondly, we identify an explicit class of initial data that converges to a self-similar solution of this type. Let us elaborate on this.

Let \( n_0 = n_0(r) \) be such that

\[
\chi_d r^d n_0(r) \leq \|n_0\|_{L^1(B_r(0))} \quad \text{for} \quad r \in (0, 1),
\]
\[
\Theta(n_0)_r + n_0(\phi_0)_r \geq 0, \quad (r^{d}(\phi_0))_r = r^d n_0 \quad \text{in} \quad (0, 1), \quad \text{and} \quad \phi_0(1) = 0,
\]
where \( \chi_d \) is the measure of the unit ball in \( \mathbb{R}^d \). Suppose also that \( \Theta \leq 1/(4d\chi_d) \), implying that the solution \( n = n(r,t) \) of (1)–(6) blows up at finite time \( T^* > 0 \) and at the point \( r = 0 \) [4]. Finally, assume that the two functions

\[
\|n_0\|_{L^1(B_r(0))} \quad \text{and} \quad \frac{4\Theta r^d}{2(d-2)\Theta T^* + r^2}
\]
intersect exactly once in \( [0,1] \).

Our main result (Theorem 2.1) shows that if (11), (12), and (13) hold, then \( n \) satisfies

\[
n(0,t) \leq \frac{2d}{(d-2)}(T^*-t)^{-1} \quad \text{for} \quad t \in (0,T^*),
\]
and moreover has a structure near blow up given by

\[
n_s(r,t) = (T^* - t)^{-1}\Psi \left( \frac{r}{\sqrt{\Theta(T^*-t)}} \right),
\]
where the function \( \Psi \) is one of a class of solutions of a steady-state problem; a class that includes the functions

\[
\Psi_1(\eta) := (d-2)\frac{(2d+\eta^2)}{(d-2+\frac{1}{2}\eta^2)^2} \quad \text{and} \quad \Psi^*(\eta) := 1 \quad \text{for} \quad \eta > 0.
\]

In particular the initial state \( n_0 \equiv 1/\chi_d \) and \( \Theta \leq 1/(4d\chi_d) \) satisfies the conditions above (Corollary 2.2). If we relax assumption (13) but assume instead that \( n \) satisfies the growth condition

\[
n(0,t) \leq M(T^*-t)^{-1} \quad \text{for} \quad t \in (0,T^*),
\]
for some constant \( M > 0 \), then \( n \) has the same structure of blow-up given above (Theorem 2.3). The hypotheses on the initial data (11), (12), and (13) are more
natural in the context of a transformed problem we introduce in the next section. Note however that $(n_0)_r \leq 0$ in $[0,1]$ implies assumption (11).

This paper is organized as follows. In section 2 we put the problem in terms of a new variable, thus transforming the system (1)–(6) into a single PDE, and then state our results in terms of this new formulation. In section 3 we discuss some non-self-similar blow-up patterns related to case (a). Sections 4, 5, and 6 provide the tools for the proofs of Theorems 2.1 and 2.3, and the arguments are wrapped up in Section 7. A rather technical derivation of a Lyapunov function is placed in Appendix A, and in Appendix B we derive some linear stability results.

2. Precise statements of main results

For radial solutions, the average density function $b(r,t)$ [10] is defined by

$$b(r,t) := \frac{d \chi_d}{r^d} \int_0^r n(y,t) y^{d-1} dy,$$

(14)

This variable turns out to be convenient in the analysis of this system. Note that it has the same scale invariance as $n(r,t)$, but that solutions are smoother when expressed in terms of $b$. For example, if for some fixed $t > 0$ the density $n(r,t)$ is a delta function at the origin with unit mass, then $b(r,t) = r - \frac{d}{r}$. Let $D = (0,1)$ and set $D_T = D \times (0,T)$ for some $T > 0$. Transformation (14) puts system (1)–(6) in the form

$$b_t = \chi_d \Theta \left( b_{rr} + \frac{d+1}{r} b_r + \frac{1}{d} r b_r + b^2 \right), \quad \text{in } D_T$$

(15)

$$b_r(0,t) = 0, \quad b(1,t) = 1, \quad \text{for } t \in [0,T),$$

(16)

$$b(0,r) = b_0(r), \quad \text{for } r \in D.$$  

(17)

Here we have redefined $t := \frac{1}{\chi_d} t$. Regarding the initial condition, we assume

$$b_0 \in C^2(\overline{D}), \quad \text{and } \frac{r}{d}(b_0)_r + b_0 \geq 0 \quad \text{for } r \in D,$$

(18)

where the second condition is equivalent to $n_0 \geq 0$ in $D$. Note that the conservation of the mass (7) is represented by $b(1,t) = 1$ for $t \in [0,T)$. As was done for problem (10)–(13) we define $T > 0$ to be the maximal time of existence for the average density $b(r,t)$. If $T^* < \infty$ in (8), then

$$\lim_{t \to T^*} \sup_{[0,1]} b(r,t) = \infty,$$

where $T = T^*/\chi_d$. Using (14), we deduce $b(r,t) \leq 1/r^d$ for $r \in \overline{D}$, $t > 0$; this implies single point blow-up for $b(r,t)$ at the point $r = 0$. To characterize the asymptotic behaviour near blow-up of the solution $b(r,t)$ of problem (13)–(18), we study the solutions of the associated boundary-value problem

$$\begin{cases} 
\varphi_{\eta\eta} + \frac{d+1}{\eta} \varphi_{\eta} + \frac{1}{d} \eta \varphi_{\eta} - \frac{1}{2} \eta^2 \varphi - \varphi = 0, \quad \text{for } \eta > 0, \\
\varphi(0) \geq 1, \quad \varphi(0) = 0.
\end{cases}$$

(19)

If $b$ is a solution of (13)–(18) which blows up at time $T > 0$ and at the point $r = 0$, then we will show that it has the asymptotic form given by

$$b_*(r,t) = (T-t)^{-1} \varphi \left( \frac{r}{\chi_d \Theta(T-t)} \right).$$
Equation (19) has multiple solutions for \(2 < d < 10\). We classify them by counting the number of times they cross the singular solution \(\varphi_S(\eta) := 2d/\eta^2\). For that purpose, we introduce the set

\[ S_k = \{ \varphi : \varphi \text{ is a solution of (19) that has } k \text{ intersections with } \varphi_S \}. \]

We shall see that \(S_1\) is the relevant subset of solutions of (19) for the characterization of the type of blow-up considered in this paper. Numerical evidence (10) suggests that \(S_1\) contains only two elements:

\[ \varphi^*(\eta) = 1 \quad \text{and} \quad \varphi_1(\eta) := \frac{2d}{(d - 2 + 4\eta^2)} \quad \text{for} \quad \eta \geq 0. \quad (20) \]

For the initial condition, we assume

\[ (b_0)_r \leq 0 \quad \text{for} \quad r \in D, \quad (21) \]

and

\[ \chi_d \Theta \left( (b_0)_{rr} + \frac{d + 1}{r} (b_0)_r \right) + \frac{1}{d} r b_0 (b_0)_e + b_0^2 \geq 0 \quad \text{for} \quad r \in D. \quad (22) \]

We will show that this implies \(b_0 \leq 0\) in \(D_T\) and \(b_t \geq 0\) in \(D_T\). In terms of \(n_0\) assumption (21) becomes \(11\) and assumption (22) becomes \(12\).

**Theorem 2.1** Let \(d > 2\) and \(b_0\) satisfy (21) and (22). Let \(b(\cdot, t)\) be the corresponding solution of problem (19)–(15) that blows up at \(r = 0\) and at \(t = T\). If

\[ \Theta \leq \Theta_1 := 1/(4d \chi_d) \quad \text{and} \quad b_0(r) \text{ intersects } T^{-1} \varphi_1(r/\sqrt{\chi_d \Theta T}) \text{ once} \quad (23) \]

then

\[ b(0, t) \leq M_1 (T - t)^{-1} \quad \text{for} \quad t \in (0, T) \quad (24) \]

with \(M_1 := 2d/(d - 2)\). Moreover, \(T < M_1/b_0(0)\), and there exists \(\varphi \in S_1\) such that

\[ \lim_{t \to T} (T - t) b \left( \eta \sqrt{\chi_d \Theta (T - t)} \right) = \varphi(\eta) \quad (25) \]

uniformly on compact sets \(|\eta| \leq C\) for every \(C > 0\).

We remark that there exists a family of \(b_0\) satisfying the conditions (18), (21), and (22), given by \(b_0(r) = K_1 + K_2/(r^d + K_3)\) with positive constants \(K_i\) that satisfy \(K_1 + K_2/(1 + K_3) = 1\) and \(\Theta < K_2/2d^2 \chi_d\). Conditions (18), (21), and (22) are also satisfied for \(b_0 \equiv 1\). Note that condition (23) of Theorem 2.1 can be generalized by changing \(\varphi_1\) for other solution \(\varphi\) of (19). Since these solutions are only known numerically, the counterpart of \(M_1\) and \(\Theta_1\) cannot be given explicitly. The next corollary applies this result to \(b_0 \equiv 1\).

**Corollary 2.2** Let \(d > 2\), \(b_0 \equiv 1\), and \(\Theta < \Theta_1\). Then \(b(\cdot, t)\), the corresponding solution of problem (19)–(15) that blows up at \(r = 0\) and at some time \(t = T < M_1\); moreover, (24) holds and there exists \(\varphi \in S_1\) satisfying (22).

Numerical simulations (10) suggest that for an open set of initial data the convergence in (24) holds for \(\varphi = \varphi_1\). This self-similar behaviour may be seen roughly in Figure 1 (right), by imagining \(n(r, t)\) replaced by \(b(r, t)\) (since \(n\) and \(b\) scale similarly). In (13) we show that \(\varphi_1\) is linearly stable (using the result in (10)) and also that \(\varphi^*\) is linearly unstable.

For more general initial data we have the following result.
Theorem 2.3 Let $d > 2$ and let $b_0$ satisfy (21) and (22). Assume that $b(r, t)$, the corresponding solution of problem (15)–(18), blows up at $r = 0$ and at $t = T$. If $b$ satisfies the growth condition

$$b(0, t) \leq M(T - t)^{-1} \quad \text{for} \quad t \in (0, T)$$

with $M > 0$, then there exists $\varphi \in S_1$ such that the convergence (25) holds.

We now briefly discuss the structure of the proofs of these theorems. Following the scale invariance, we set

$$\tau = \log \left( \frac{T}{T - t} \right), \quad \eta = \frac{r}{(\chi d \Theta(T - t))^{1/2}}, \quad \text{and} \quad B(\eta, \tau) = (T - t)b(r, t).$$

The rectangle $D_T$ transforms into

$$\Pi = \{ (\eta, \tau) \mid \tau > 0, \ 0 < \eta < \ell(\tau) \},$$

where

$$\ell(\tau) := (\chi d \Theta T)^{-1/2} e^{-\tau/2}.$$

The initial-boundary problem (15)–(18) now becomes

$$B_\tau + B + \frac{1}{2} \eta B_\eta = B_\eta + \frac{d + 1}{\eta} B_\eta + \frac{1}{d} \eta BB_\eta + B^2 \quad \text{in} \ \Pi,$$

(27)

$$B_\eta(0, \tau) = 0, \quad B(\ell(\tau), \tau) = e^{-\tau} T \quad \text{for} \ \tau \in \mathbb{R}^+,$$

(28)

$$B(\eta, 0) = B_0(\eta) := T b_0 \left( \chi d \Theta T \right)^{1/2} \quad \text{for} \ \eta \in \Pi(0),$$

(29)

where $\Pi(0) = (0, \ell(0))$. Note that a solution of (19) is a time-independent solution of (27)–(29). Therefore the study of the blow-up behaviour of $b(r, t)$ is reduced to the analysis of the large time behaviour of solutions $B(\eta, \tau)$ of (27)–(29), and in particular stabilization towards solutions $\varphi$ of (19). The proof of Theorem 2.3 consists of two parts. In Section 5, we first prove that $\omega \subset S_1$, where

$$\omega = \{ \phi \in L^\infty(\mathbb{R}^+) : \exists \tau_j \to \infty \quad \text{such that} \quad B(\cdot, \tau_j) \to \phi(\cdot) \quad \text{as} \quad \tau_j \to \infty \quad \text{uniformly on compact subsets of} \ \mathbb{R}^+ \}$$

(30)

is the $\omega$-limit set we introduce for (27)–(29). The proof uses the observation that equation (27), without the convection term $\frac{1}{d} \eta BB_\eta$, is the backward self-similar equation for the parabolic semilinear equation

$$\bar{b}_t = \Delta_N \bar{b} + \bar{b}^2,$$

(31)

where $\Delta_N$ denotes the Laplacian in $\mathbb{R}^N$ and $N = d + 2$ [15, 16]. We use the methods for the analysis of this self-similar equation to prove Theorem 2.3. However, due to the presence of the convection term, a different Lyapunov functional is necessary. This functional is constructed using the method of Zelenyak [30], which yields a Lyapunov functional in implicit form. In section 6, we use intersection comparison arguments based on the ideas of Matano [23] to prove that the $\omega$-limit set (30) is a singleton.

With a result on intersection with $\varphi_S$ this completes the proof of Theorem 2.3.

Note that Theorem 2.3 is similar to a result for the supercritical case ($N > 6$) for equation (31), where two different kinds of self-similar blow-up behaviour may coexist [24].

Finally to obtain Theorem 2.1 and Corollary 2.2, we use Theorem 2.3 and comparison ideas from Samarskii et al. [26, Chapter IV].
3. Discussion on non self-similar blow-up patterns

In this section we discuss a family of blow-up patterns which appears when we refine the asymptotic expansion for the profile \( \varphi = \varphi^* \equiv 1 \). This situation is closely related to the blow-up behaviour of (31) with \( N < 6 \). If a solution \( \bar{b} \) of (31) with \( N < 6 \) blows up at \( x = 0 \) and \( t = T \), then

\[
\lim_{t \to T} (T - t) \bar{b}(\eta \sqrt{T - t}, t) = 1
\]

uniformly on compact sets \( |\eta| < C \) for arbitrary \( C > 0 \) [14, 16]. Moreover it has been shown (see for instance [25, 27]) that a refined description of blow-up gives the existence of two possible types of behaviour: either

\[
\lim_{t \to T} (T - t) \bar{b} \left( \eta \sqrt{(T - t)} |\log(T - t)|^{(d-2)/d}, t \right) = \bar{\varphi}_1(\eta)
\]

(32)

uniformly on compact sets \( |\eta| < C \), with \( C > 0 \) arbitrary; or

\[
\lim_{t \to T} (T - t) \bar{b} \left( \eta(T - t)^{1/2m}, t \right) = \bar{\varphi}_m(\eta)
\]

for some \( m \geq 2 \),

(33)

uniformly on compact sets \( |\eta| < C \), with \( C > 0 \) arbitrary. Here the family \( \{\bar{\varphi}_i\}_{i \geq 1} \) is known explicitly. For problem (15–18), it was shown [20] for \( d = 3 \) that there exists a refined asymptotics for \( \varphi^* \equiv 1 \). Extending the argument to all \( d > 2 \), these asymptotics suggest a convergence given by either

\[
\lim_{t \to T} (T - t) \bar{b} \left( \eta \sqrt{(T - t)} |\log(T - t)|(d-2)/d, t \right) = \tilde{\varphi}_1(\eta)
\]

(34)

or

\[
\lim_{t \to T} (T - t) \bar{b} \left( \eta(T - t)^{1 + \frac{d-2}{2(m+d-1)}}, t \right) = \tilde{\varphi}_m(\eta)
\]

(35)

for some \( m \geq 2 \). An implicit formula for the family \( \{\tilde{\varphi}_m\}_{m \geq 1} \) is given in [10, equation (43)]. The type of convergence in \( \eta \) towards these profiles is an open problem.

In (35), we can take formally the limit \( m \to \infty \) and find a non-trivial scaling,

\[
\lim_{t \to T} (T - t) \bar{b}(\eta(T - t)^{1/d}, t) = \tilde{\varphi}_\infty(\eta).
\]

(36)

Note that this limit cannot be taken for the semilinear equation where (33) holds. The convergence (36) represents the convection-dominant behaviour of (15–18), which in terms of the density \( n = n(r, t) \) describes an imploding wave moving towards the origin, as shown in Figure 1 (left). The function \( \tilde{\varphi}_\infty \) is discontinuous (cf. [19, (3.16)])

\[
\tilde{\varphi}_\infty(\eta) = \begin{cases} 2C^d \eta^d & \text{for } \eta > C \\ 0 & \text{for } \eta < C \end{cases}
\]

where \( 2C^d \) is the mass accumulated in the origin, which can be chosen arbitrarily.

In [19] this type of blow-up was studied using matched asymptotic expansions. There it was suggested that this behaviour is stable and moreover it was expected that there exist initial data such that (36) holds uniformly in \( \eta \) on compact subsets away from the shock. A result of this type was proved in [12, Theorem 3] for a related equation.
4. Preliminaries

4.1. Estimates

In this section we develop some estimates for problem (15)–(17), which in turn will imply bounds for the self-similar problem (27)–(29).

**Lemma 4.1** If \( b_0 \) satisfies (18) then
\[
\frac{r}{d} b_r + b \geq 0 \quad \text{in} \quad D_T.
\]

**Proof.** The solution \( n \) of problem (1)–(6) satisfies the relation
\[
n = \frac{1}{\chi_d} \left( \frac{r}{d} b_r + b \right) \quad \text{in} \quad D_T^*.
\]

Since \( n_0 \geq 0 \) in \( D \), an application of the maximum principle to problem (1)–(6) shows that \( n \geq 0 \) in \( D_T^* \). Using this and (38) the result follows.

To prove the following results, we proceed as in [13] where similar estimates were found for the semilinear parabolic equation (31).

**Lemma 4.2** If \( b_0 \) satisfies (21) then
\[
b_r(r, t) < 0 \quad \text{in} \quad D_T.
\]

**Proof.** Set \( w(r, t) := \frac{r}{d} b_r + b \). Differentiating (15), we find
\[
w_t - \chi_d \Theta \left( w_{rr} - \frac{d + 1}{r} w_r \right) - \frac{1}{d} r b w_r = \left( b + \frac{1}{d} r b_r \right) w.
\]

Assume for the moment a stronger assumption on the initial data
\[
(b_0)_r(r) < 0 \quad \text{for} \quad r \in (0, 1) \quad \text{and} \quad (b_0)_{rr}(0) < 0.
\]

This gives \( w(0, t) = r^{d+1} b_r(0, t) \). Under (41) the function \( b \equiv 1 \) is a sub-solution for (15)–(18), but not a solution; by Hopf’s Lemma, \( w(1, t) = b_r(t, 1) < 0 \) for all \( t > 0 \), so that \( w < 0 \) in \( D_T \), hence \( b_r < 0 \) on \( D_T \). To finish the proof, we note that by the strong maximum principle, if \( b_0 \) satisfies (22), then for each \( t_1 \in (0, T) \) condition (41) holds for the function \( b(r, t_1) \). This proves the result.

**Lemma 4.3** If \( b_0 \) satisfies (21) and assuming that blow up occurs at time \( T > 0 \), then
\[
b_0(0, t) \geq (T - t)^{-1} \quad \text{for} \quad t \in [0, T),
\]

**Proof.** Since the maximum of \( b \) in \( D \) is attained at \( r = 0 \) (by Lemma 4.2), we have \( b_r(0, t) \leq 0 \). It follows from (15) that \( b_0(0, t) \leq b^2(0, t) \). Integrating this inequality on \( (0, T) \) gives the result.

**Lemma 4.4** If \( b_0 \) satisfies (22) then \( b_t \geq 0 \) for all \( t \in (0, T) \).

**Proof.** Condition (22) implies that \( b_0 \) is a subsolution for (15)–(18); therefore \( b(r, \epsilon) \geq b(r, 0) \) for small \( \epsilon \geq 0 \). By the comparison principle we find \( b(r, t + \epsilon) \geq b(r, t) \) for \( t \in (0, T - \epsilon) \). It follows that \( b_1 \geq 0 \) on \( D_T \).
The next lemma gives a bound on $|b_r|$ in $D_T$.

**Lemma 4.5** Let $b_0$ satisfy (21) and (22). Then

$$\chi_d \Theta b_r^2(r, t) \leq \frac{2}{3} b(0, t)^3 \quad \text{for} \quad (r, t) \in D_T. \quad (43)$$

**Proof.** Since $b_t \geq 0$ and $b_r \leq 0$ in $D_T$, we multiply equation (15) by $b_r$ and obtain

$$0 \geq \chi_d \Theta \int_0^r b_r b_{rr} \, ds + \frac{1}{3} b^3(r, t) - \frac{1}{3} b^3(0, t)$$

$$= \frac{1}{2} \chi_d \Theta [b_r^2(r, t) - b_r(0, t)] + \frac{1}{3} b^3(r, t) - \frac{1}{3} b^3(0, t).$$

Since $b_r^2(0, t) = 0$ we obtain the desired inequality. \[\square\]

To conclude this section we translate the properties of solutions derived above into estimates for problem (27)–(29). From hypothesis (26) and noting that $b \geq 1$ and $b_r \leq 0$ in $D_T$, we have the a priori bound

$$0 \leq B(\eta, \tau) \leq M \quad \text{for} \quad (\eta, \tau) \in \Pi. \quad (44)$$

Combining this with (43) and (39), we obtain

$$0 \leq -B_{\eta}(\eta, \tau) \leq \bar{M} \quad \text{for} \quad (\eta, \tau) \in \Pi, \quad (45)$$

where $\bar{M}$ depends on $M$. Finally from (42), we get

$$1 \leq B(0, \tau) \quad \text{for} \quad \tau \in (0, \ell(\tau)). \quad (46)$$

### 4.2. The steady state equation (19)

We begin by recalling problem (19):

$$\varphi_{\eta\eta} + \frac{d + 1}{\eta} \varphi_\eta + \frac{1}{d} \eta \varphi_\eta - \frac{1}{2} \eta^2 \varphi_{\eta} + \varphi^2 - \varphi = 0 \quad \text{for} \quad \eta > 0, \quad (47)$$

$$\varphi(0) \geq 1, \quad \varphi_\eta(0) = 0. \quad (48)$$

Condition (48) is required, since $B(0, \tau) \geq 1$ for all $\tau \geq 0$. Equation (47) has three special solutions:

$$\varphi_S(\eta) = \frac{2d}{\eta^2}, \quad \varphi^*(\eta) = 1, \quad \text{and} \quad \varphi_*(\eta) = 0 \quad \text{for} \quad \eta > 0.$$

Note that $\varphi_S$ satisfies

$$\varphi_S + \frac{1}{2} \eta (\varphi_S)_\eta = 0 \quad \text{and} \quad 0 = (\varphi_S)_{\eta\eta} + \frac{d + 1}{\eta} (\varphi_S)_\eta + \frac{1}{d} (\varphi_S)_\eta^2 + (\varphi_S)^2. \quad (49)$$

For bounded non-constant solutions we have the following theorem [10, 20].

**Theorem 4.6** Let $2 < d < 10$. There exists a countable set of solutions $\{\varphi_k\}_{k \in \mathbb{N}}$ of (47)–(48) such that $\varphi_k(0) > 1$ and $\varphi_k(0) \to \infty$ as $k \to \infty$. Moreover $\varphi_k$ intersects the singular solution $\varphi_S$ $k$ times and has the asymptotic behaviour $\varphi_k(\eta)\eta^2 = \text{Const}(k) > 0$. 
The proof is based on the equation for $G(\eta) := \eta^2 \varphi(\eta)$,

$$G_{\eta\eta} + \left(\frac{d-3}{\eta} + \frac{1}{2} \eta \right) G_{\eta} + \frac{2(d-2)G}{2d} \left( \frac{G}{2d} - 1 \right) = 0,$$  

(50)

$$\lim_{\eta \to 0} \frac{G(\eta)}{\eta^2} < \infty, \quad \lim_{\eta \to \infty} \eta G_{\eta}(\eta) = 0.$$  

(51)

Note that $\varphi_S$ corresponds to $G(\eta) = 2d$.

It was formally argued in [10] that for each integer $k \geq 2$ and $2 < d < 10$ the set

$$S_k = \{ \varphi: \varphi \text{ solution of } (11) - (18) \text{ with } k \text{ intersections with } \varphi_S \}$$

is a singleton and that for $d > 2$ the set $S_1$ contains only two elements. More precisely, $S_1$ consists of the functions $\varphi^*$ and $\varphi_1$ given in [20]. If we relax condition (18) to $\varphi(0) > 0$, we conjecture that there is at least one other solution in $S_1$. For $d = 3$ this was shown numerically by Brenner et al., who found a solution $\varphi^*_1$ of (11) such that $\varphi^*_1(0) < 1$ and $(\varphi^*_1)_\eta(0) = 0$, which intersects $\varphi_S$ once [10, Figure 14].

5. Convergence

In this section we prove the following convergence theorem.

**Theorem 5.1** Let conditions (21) and (22) hold. Let $B(\eta, \tau)$ be a uniformly bounded global solution of (24)–(26). Then for every sequence $\tau_n \to \infty$ there exists a subsequence $\tau'_n$ such that $B(\eta_n, \tau'_n)$ converges to a solution $\varphi$ of (47)–(48). The convergence is uniform on every compact subset of $[0, \infty)$.

**Proof.** Define $B^\sigma(\eta, \tau) := B(\eta, \sigma + \tau)$. We will first show that for any unbounded sequence $\{n_j\}$ there exists a subsequence (renamed $\{n_j\}$) such that $B^{n_j}$ converges to a solution $\varphi$ of (11)–(18) uniformly in compact subsets of $\mathbb{R}^+ \times \mathbb{R}$. Without loss of generality we assume that the sequence $\{n_j\}$ is increasing.

Let $N \in \mathbb{N}$. We take $i$ large enough such that the rectangle $Q_{2N} = \{(\eta, \tau) \in \mathbb{R}^2: 0 \leq \eta \leq 2N, |\tau| \leq 2N\}$ lies in the domain of $B^{n_i}$. The function $B(\xi, \tau) = B^{n_i}(\xi, \tau)$ is a solution of

$$\tilde{B}_\tau = \Delta_{d+2} \tilde{B} - \frac{1}{2} \xi \cdot \nabla \tilde{B} + \frac{1}{d} (\xi \cdot \nabla \tilde{B}) \tilde{B} + \tilde{B}^2 - \tilde{B}$$

on the cylinder given by

$$\Gamma_{2N} = \{ (\xi, \tau) : \mathbb{R}^{d+2} \times \mathbb{R} : |\xi| \leq 2N, |\tau| \leq 2N \},$$

and $|\tilde{B}(\xi, \tau)|$ is uniformly bounded in $\Gamma_{2N}$ by (14).

By Schauder’s interior estimates all partial derivatives of $\tilde{B}$ can be uniformly bounded on the subcylinder $\Gamma_N \subset \Gamma_{2N}$. Consequently $B^{n_i}$, $B^{n_i}_\tau$, $B^{n_i}_\eta$, and $B^{n_i}_{\eta\eta}$ are uniformly Lipschitz on $Q_N \subset Q_{2N}$. By Arzela-Ascoli, there is a subsequence $\{n_j\}$ and a function $\tilde{B}$ such that $B^{n_j}$, $B^{n_j}_\tau$, $B^{n_j}_\eta$, and $B^{n_j}_{\eta\eta}$ converge to $\tilde{B}$, $\tilde{B}_\tau$, $\tilde{B}_\eta$, and $\tilde{B}_{\eta\eta}$, uniformly on $Q_N$.

Repeating the construction for all $N$ and taking a diagonal subsequence, we can conclude that

$$B^{n_j} \to \tilde{B}, \quad B^{n_j}_\tau \to \tilde{B}_\tau, \quad B^{n_j}_\eta \to \tilde{B}_\eta, \quad \text{and} \quad B^{n_j}_{\eta\eta} \to \tilde{B}_{\eta\eta},$$

(52)
uniformly in every compact subset in $\mathbb{R}^+ \times \mathbb{R}$. Clearly $\bar{B}$ satisfies (44) and estimates (45) and (46). Finally, it remains to prove that $\bar{B}$ is independent of $\tau$. This implies that $\bar{B}$ is a solution of (19), since $B(0, \tau) \geq 1$ for all $\tau > 0$, and the result follows.

**Claim.** The function $\bar{B}$ is independent of $\tau$.

To prove this we construct a *non-explicit* Lyapunov functional in the spirit of Galaktionov [14] and Zelenyak [30].

1. **Non-explicit Lyapunov functional.** We seek a Lyapunov function of the form

$$E(\tau) = \int_0^{\ell(\tau)} \Phi(\eta, B(\eta, \tau), B_\eta(\eta, \tau)) \, d\eta,$$

where $\ell(\tau) = (\chi_a T)^{-1/2} e^{\tau/2}$ and $\Phi = \Phi(\eta, v, w)$ is a function to be determined. In Appendix A we show that such a Lyapunov function exists; more precisely, we show that a function $\rho = \rho(\eta, v, w)$ exists such that

$$\frac{d}{d\tau} E(\tau) = - \int_0^{\ell(\tau)} \rho(\eta, B(\eta, \tau), B_\eta(\eta, \tau))(B_\tau)^2(\eta, \tau) \, d\eta$$

$$+ \Phi_w(B) \Phi(\ell(\tau), B(\ell(\tau), \tau), B_\eta(\ell(\tau), \tau)).$$

To identify the relevant domain of the functions $\Phi$ and $\rho$, we note that by estimates (44) and (45) the solution $\bar{B}$ satisfies $(\eta, B(\eta, \tau), B_\eta(\eta, \tau)) \in \bar{R}$, with

$$\bar{R} = \mathcal{R} \cap \{0 \leq v \leq M, 0 \leq -w \leq M\},$$

where $\mathcal{R} = \{\eta > 0, v \geq 0, w \leq 0\} \cup \{\eta = 0, v \geq 0, w = 0\}$.

The functions $\rho$ and $\Phi$ are continuous in $\mathcal{R} \setminus \{\eta = \bar{\eta}, v > 1\}$ with $\bar{\eta} > 0$ defined later and they satisfy

$$\frac{1}{C_0} \eta^{d+1} e^{-C_0 \eta^2} \leq \rho(\eta, v, w) \leq \eta^{d+1} e^{-(d-2)\eta^2/4d} \quad \text{for} \quad (\eta, v, w) \in \bar{R},$$

with $C_0 = C_0(M) > 0$ (Lemma A.5), and

$$|\Phi(\eta, v, w)| \leq C_1 \eta^{d+1} e^{-(d-2)\eta^2/4d} \quad \text{for} \quad (\eta, v, w) \in \bar{R}$$

for some positive constants $C_1(M) > 0$ (Lemma A.6).

2. **Proof of the claim.** An integration over the interval $(a, b)$ of (53) gives

$$\int_a^b \rho(\eta, B(\eta, \tau), B_\eta(\eta, \tau))B_\tau^2(\eta, \tau) \, d\eta d\tau = E(a) - E(b) + \psi(a, b)$$

where

$$\psi(a, b) := \int_a^b \frac{1}{2} \Phi(\ell(\tau), B(\ell(\tau), \tau), B_\eta(\ell(\tau), \tau)) \, d\tau +$$

$$+ \int_a^b B_\tau(\ell(\tau), \tau) \left[ \int_0^{B_\eta(\ell(\tau), \tau)} \rho(\ell(s), B(\ell(s), \tau), s) \, ds \right] \, d\tau.$$  

Since $B_\tau(\ell(\tau), \tau) = -B(\ell(\tau), \tau) - \frac{1}{2} \ell(\tau) B_\eta(\ell(\tau), \tau)$,

$$B_\tau(\ell(\tau), \tau) = -Te^{-\tau} - \frac{1}{2} b(1, T(1 - e^\tau)).$$
Applying (57) at \( r = 1 \) gives \(|b_r(1, T(1 - e^r))| \leq d\) and consequently \(B_\tau\) is uniformly bounded as \(\tau \to \infty\). Employing this bound on \(B_\tau\) and the estimates (55) and (56) we find

\[
\lim_{n \to \infty} \{\sup_{a > b} \psi(a, b)\} = 0. \quad (59)
\]

By (52) we have that there exists a sequence \(n_j \to \infty\) such that \(B^{n_j}(\eta, \tau)\) converges to \(B\) uniformly in compact subsets of (\(\mathbb{R}^+\))^2. For any fixed \(N\) we will prove for a subsequence satisfying \(\lim_{j \to \infty} (n_{j+1} - n_j) = \infty\) that

\[
\lim_{n_j \to \infty} \int_{Q_N} \rho(\eta, \{B^{n_j}(\eta, \tau), B^{n_j}_\eta(\eta, \tau)\})(B^{n_j}_\tau)^2(\eta, \tau) \, d\eta d\tau = 0, \quad (60)
\]

where we recall that \(Q_N = \{(\eta, \tau) \in \mathbb{R}^2 : 0 \leq \eta \leq N, |\tau| \leq N\}\). Since \(\rho\) is bounded from below on bounded subsets of \(\mathcal{R}\), it then follows that

\[
\int_{Q_N} B^2 \, d\eta d\tau = \lim_{n_j \to \infty} \int_{Q_N} (B^{n_j}_\tau)^2(\eta, \tau) \, d\eta d\tau = 0,
\]

proving the claim. For all \(j\) sufficiently large,

\[
N \leq (\chi_d \Theta T)^{-1/2} e^{\frac{1}{2} (n_j - N)} \quad \text{and} \quad n_{j+1} - n_j \geq 2N.
\]

Consequently using (52), we find

\[
\int_{-N}^{N} \int_{0}^{\infty} \rho(\eta, \{B^{n_j}(\eta, \tau), B^{n_j}_\eta(\eta, \tau)\})(B^{n_j}_\tau)^2(\eta, \tau) \, d\eta d\tau
\]

\[
\leq \int_{-N}^{n_{j+1} - n_j} \int_{0}^{\pi} \rho(\eta, \{B^{n_j}(\eta, \tau), B^{n_j}_\eta(\eta, \tau)\})(B^{n_j}_\tau)^2(\eta, \tau) \, d\eta d\tau
\]

\[
\leq \int_{n_j - N}^{n_j + 1 - N} \int_{0}^{\pi} \rho(\eta, \{B^{n_j}(\eta, \tau), B^{n_j}_\eta(\eta, \tau)\})(B^{n_j}_\tau)^2(\eta, \tau) \, d\eta d\tau
\]

\[
\leq E(n_j - N) - E(n_{j+1} - N) + \psi(n_j - N, n_{j+1} - N).
\]

Hence applying (52), we discover

\[
\int_{Q_N} \rho(\eta, \{B^{n_j}(\eta, \tau), B^{n_j}_\eta(\eta, \tau)\})(B^{n_j}_\tau)^2(\eta, \tau) \, d\eta d\tau \leq \limsup_{j \to \infty} |E(n_j - N) - E(n_{j+1} - N)|.
\]

Next we divide the expression \(E(n_j - N) - E(n_{j+1} - N)\) into three integrals, choosing \(K\) arbitrarily large:

\[
E(n_j - N) - E(n_{j+1} - N) =
\]

\[
= \int_{0}^{\pi} \Phi(\eta, \{B^{n_j}(\eta, -N), B^{n_j}_\eta(\eta, -N)\}) - \Phi(\eta, \{B^{n_j}(\eta, -N), B^{n_j}_\eta(\eta, -N)\}) \, d\eta \quad (61)
\]

\[
+ \int_{\pi}^{\pi + \frac{1}{2} \frac{n_j - N}{2}} \Phi(\eta, \{B^{n_j+1}(\eta, -N), B^{n_j+1}_\eta(\eta, -N)\}) \, d\eta \quad (62)
\]
Lemma 6.1
Under the assumptions (21) and (22), there exists a continuously differentiable function $\eta(\tau)$ with domain $[0, \infty)$ such that $\eta(0) = \eta_1$ and $B(\eta_1, \tau) = \varphi_S(\eta_1(\tau))$ for all $\tau \geq 0$.

Proof. Define $H(\eta, \tau) := B(\eta, \tau) - \varphi_S(\eta)$. We first claim that $H, H_\eta$, and $H_\tau$ do not vanish simultaneously. Using Lemma 4.4 and the strong maximum principle we find

$$b_1 = (T - t)^{-2} \left( B_\tau + B + \frac{1}{2} \eta B_\eta \right) > 0 \quad \text{in} \quad D_T.$$  

6. Comparison results

6.1. Comparison with the singular solution $\varphi_S$

This section closely follows [1]. From section 4.2, we recall that solutions $\varphi$ of (19)–(22) are classified by their intersections with $\varphi_S$. In this section we study the intersections of solutions $B$ of (27)–(29) with $\varphi_S$. Our results are closely related to the ones found in [1], where equation (61) was studied.

We first see that for $\Theta < 1/(2d\chi_d)$ a solution $B$ of (27)–(29) intersects the singular solution $\varphi_S$ at least once in $\Pi(0)$ since

$$\varphi_S(0) = \infty > B(0,0), \quad \text{and} \quad \varphi_S \left( (\chi_d \Theta T)^{-1/2} \right) < B \left( (\chi_d \Theta T)^{-1/2}, 0 \right) = T.$$  

On the other hand, for $\Theta \geq 1/(2d\chi_d)$ it can also be shown that $B$ intersects $\varphi_S$ at least once in $\Pi(0)$. Assuming the contrary, suppose that $B(\cdot, 0) < \varphi_S(\cdot)$ in $\Pi(0)$. By the maximum principle, we obtain $B < \varphi_S$ in $\Pi$. Therefore in the limit $\tau \to \infty$, thanks to the Theorem 5.1 and since $B(0, \tau) \geq 1$ for all $\tau > 0$, we find a solution $\varphi$ of (10) such that $\varphi < \varphi_S$. However we can show that every bounded non zero solution $\varphi$ of (10) has to cross $\varphi_S$. This is equivalent to proving that there exists no solution $G$ of (60)–(63) such that $G(\eta) < 2d$ for $\eta \geq 0$. To check this, we assume that such a solution exists; we examine two cases. Suppose that for some $\eta^*$, we have $G_\eta(\eta^*) = 0$ and $G(\eta^*) < 2d$. By (60), $G$ has a strict minimum at $\eta^*$, which contradicts the boundary condition (61). On the other hand if, $G(\eta)$ is increasing for all $\eta > 0$, then for large $\eta$, equation (60) implies that $G_\eta > 0$, which also contradicts (61).

We conclude that there exists $\eta_1 \in \Pi(0)$ such that $B(\eta_1, 0) = \varphi_S(\eta_1)$ and $B(\eta, 0) < \varphi_S(\eta)$ for $\eta < \eta_1$.

Lemma 6.1 Under the assumptions (27) and (28), there exists a continuously differentiable function $\eta_1(\tau)$ with domain $[0, \infty)$ such that $\eta_1(0) = \eta_1$ and $B(\eta_1(\tau), \tau) = \varphi_S(\eta_1(\tau))$ for all $\tau \geq 0$.

Proof. Define $H(\eta, \tau) := B(\eta, \tau) - \varphi_S(\eta)$. We first claim that $H, H_\eta$, and $H_\tau$ do not vanish simultaneously. Using Lemma 4.4 and the strong maximum principle we find

$$b_1 = (T - t)^{-2} \left( B_\tau + B + \frac{1}{2} \eta B_\eta \right) > 0 \quad \text{in} \quad D_T.$$  

The integral of $\Phi$ tends to zero as $j \to \infty$. In fact by the continuity of $\Phi$ in the second and third argument we obtain pointwise convergence and by the bounds on $\Phi$, we apply the Dominated Convergence Theorem to conclude. Expressions 12 and 13 can be made arbitrarily small since they can be bounded by

$$C \int_{\mathbb{R}} \eta^{d+1} e^{-(d-2)\eta^2/4d} d\eta,$$

where $C$ is a positive constant, and $K$ can be chosen arbitrary large. Thus we have proved (60), concluding the proof of the Theorem. \hfill \blacksquare
Suppose there exists a point in $\Pi$ where $H_\eta = H_r = H = 0$. Then $H_\eta = 0$ implies $B_\eta = 0$, and condition $H_\eta = 0$ combined with $H = 0$ gives

$$B + \frac{1}{2}\eta B_\eta = 0 \text{ in } \Pi,$$

using (54). This implies that $b_1 = 0$ at some point of $D_T$, a contradiction with (53). Secondly, we claim that $H_\eta \neq 0$ at any point $(\bar{\eta}, \bar{\tau}) \in \Pi$ where $H(\bar{\eta}, \bar{\tau}) = 0$ and moreover $H(\eta, \tau) < 0$ in a left neighborhood of $\bar{\eta}$. A proof of this can be done as in [1]. Moreover, from the proof, we find $H_\eta(\bar{\eta}, \bar{\tau}) > 0$.

Now we prove that $H_\eta(\eta_1, 0) > 0$. This follows from the equation satisfied by $H(\eta, 0)$. To the left of $\eta_1$, we find

$$H_\eta(\eta_1, 0) + \frac{d + 1}{\eta}H_\eta(\eta_1, 0) + \frac{1}{2d}\eta H_\eta(\eta_1, 0)(B(\eta_1, 0) + \varphi_S) + \frac{1}{2d}\eta H_\eta(\eta_1, 0)(B(\eta_1, 0) + \varphi_S)_\eta \geq 0. \quad (65)$$

Since $(B(\eta_1, 0) + \varphi_S)_\eta \leq 0$ and $H(\eta_1, 0) = 0$, we can apply Hopf’s Lemma to obtain that $H_\eta(\eta_1, 0) > 0$. Finally, to conclude the proof of the lemma, we use the implicit function theorem as in [1].

Define the set $\Pi_1 = \{(\eta, \tau) \mid 0 < \eta < \eta_1(\tau)\}$ and the function

$$\eta_2(\tau) = e^{\tau/2} \cdot \sup \{\eta \in (\eta_1, (\chi_\omega \Theta T)^{-1/2}] : H(s, 0) \geq 0 \text{ for } s \in [\eta_1, \eta]\}.$$

Since $H(\eta_1, 0) = 0$ and $H_\eta(\eta_1, 0) > 0$, the above supremum is finite. Define the set

$$\Pi_2 = \{(\eta, \tau) \mid \eta_1(\tau) < \eta < \eta_2(\tau)\}.$$

Let $F(\tau) = H(\eta_2(\tau), \tau)$. By definition of $\eta_2$, $F(0) \geq 0$. Also,

$$\frac{d}{d\tau} F(\tau) = H_\tau(\eta_2(\tau), \tau) + \frac{1}{2} \eta_2(\tau) H_\eta(\eta_2(\tau), \tau).$$

Using (64), we have $d[e^{\tau} F(\tau)]/d\tau \geq 0$. An integration yields $F(\tau) \geq 0$ for $\tau \geq 0$.

As was done in [1], applying the maximum principle, using Lemma 6.1 and noting that $H(\eta_2(\tau), \tau) \geq 0$ for $\tau \geq 0$, we can prove the following lemma and its corollary.

**Lemma 6.2** The function $H(\eta, \tau) = B(\eta, \tau) - \varphi_S(\eta)$ satisfies $H < 0$ in $\Pi_1$ and $H > 0$ in $\Pi_2$.

**Corollary 6.3** Assume the conditions in Lemma 6.2. For each $N > 0$ there is $\tau_N > 0$ such that for $\tau > \tau_N$, $B(\eta, \tau)$ intersect $\varphi_S(\eta)$ at most once in $\eta \in (0, N)$.

### 6.2. Intersection comparison

In this section we derive comparison results, which will be used to prove that $\omega$, the limit set of $\mathcal{X}$, is a singleton.

We start by considering the following linear equation with inhomogeneous boundary conditions:

$$
\begin{cases}
  v_t = v_{rr} + \frac{d + 1}{r} v_r + a(r, t)v, & \text{for } 0 < r < 1, T_1 < t < T_2, \\
  v_r(0, t) = 0 & \text{for } T_1 < t < T_2, \\
  v(1, t) = h(t) & \text{for } T_1 < t < T_2;
\end{cases}
$$

(67)
where $T_1, T_2$ are positive constants and
\[ a \in L^\infty([0, 1] \times (T_1, T_2)), \quad h \in C^1((T_1, T_2)), \]
are given functions. Moreover we assume
\[ h(t) > 0 \quad \text{for} \quad T_1 < t < T_2. \] (69)

The zero number functional of (67) is defined by
\[ z[v(\cdot, t)] = \# \{ r \in [0, 1] : v(r, t) = 0 \}, \]
and the following lemma provides some properties of this zero number functional.

**Lemma 6.4** \[24\] Let $v = v(r, t)$ be a nontrivial classical solution of (67) and assume that (68) and (69) hold. Then the following properties hold true:

(i) $z[v(\cdot, t)] < \infty$ for any $T_1 < t < T_2$;
(ii) $z[v(\cdot, t)]$ is nonincreasing in time;
(iii) if $v(r_0, t_0) = v(r, t_0) = 0$ for some $r_0 \in [0, 1]$ and $t_0 > T_1$, then $z[v(\cdot, t)]$ drops strictly at $t = t_0$, that is, $z[v(\cdot, t_1)] > z[v(\cdot, t_2)]$ for any $T_1 < t_1 < t_0 < t_2 < T_2$.

From this lemma we deduce a property of intersection between a solution $\varphi$ of (11) and a solution $B$ of (27)–(29).

**Lemma 6.5** Let $B$ be a bounded solution of (27)–(29) and let $\varphi$ be a solution of (47). Denote $Z(\tau) = \# \{ r \in [0, \ell(\tau)] : B(\eta, \tau) = \varphi(\eta) \}$. Then the following properties hold true:

(i) $Z(\tau) < \infty$ for any $\tau > \tau^*$;
(ii) $Z(\tau)$ is nonincreasing in time;
(iii) if $B(\eta_0, \tau_0) = \varphi(\eta_0)$ and $B(\eta, \tau_0) = \varphi(\eta_0)$ for $\tau_0 > \tau_1$, and $\eta_0 \leq \ell(\tau)$ then $Z(\tau_1) > Z(\tau_2)$ for any $\tau_1 < \tau_0 < \tau_2$.

**Proof.** Writing $\hat{V} = U - b$, where $U(r, t) = (T - t)^{-1} \varphi(r / (\chi_d \Theta(T - t)))^{1/2}$, we have
\[
\begin{cases}
\hat{V}_t = \hat{V}_{rr} + \left( \frac{d + 1}{r} + \frac{r}{d} U \right) \hat{V}_r + \left( \frac{r}{d} b_r + b + U \right) \hat{V} & \text{for } 0 < r < 1, 0 < t < T, \\
\hat{V}_t(0, t) = 0, & \hat{V}(1, t) = U(1, t) - b(1, t) & \text{for } 0 < t < T.
\end{cases}
\]

Let $T_1 < T_2 < T$. For the variable $V(r, t) = \exp \left( \frac{1}{2d} \int_0^r yU(y, t) \, dy \right) \hat{V}(r, t)$, we find
\[
\begin{cases}
V_t = V_{rr} + \frac{d + 1}{r} V_r + A(r, t)V & \text{for } 0 < r < 1, T_1 < t < T_2, \\
V_r(0, t) = 0, & \text{for } T_1 < t < T_2, \\
V(1, t) = (U(1, t) - 1) \exp \left( \frac{1}{2d} \int_0^1 yU(y, t) \, dy \right) & \text{for } T_1 < t < T_2,
\end{cases}
\]
where
\[ A(r, t) = \frac{r}{d} b_r + b + U + \frac{1}{2d} \int_0^r yU(y, t) \, dy - \frac{1}{4d^2} y^2 U^2 - \frac{1}{2d}(U + rU_r) - \frac{d + 1}{2d} U. \]
Note that $A \in L^\infty([0,1] \times (T_1,T_2))$ since $b,b_r,U,U_r \in L^\infty([0,1] \times (T_1,T_2))$. If we show that $V(1,t)$ does not change sign for $t > t_0$, then setting $T_1 = t_0$ and using Lemma 6.3 we have proved the lemma.

We claim that there exists $\bar{t}_0$ such that $U_t(1,t)$ does not change sign for $t > \bar{t}_0$. By definition of $V$, this implies that there exists $t_0 \geq \bar{t}_0$ such that $V(1,t)$ does not change sign for $t > t_0$.

Since $U_{t}(r,t) = (T-t)^{-1/2} \eta^2 \varphi(t)/\eta$, if $r = 1$ and $t > t^*$, then
\[ U_t(1,t) = (T-t)^{-1/2} \eta^2 \varphi(t) \text{ for } t > t^*, \quad \text{and} \quad \eta > \eta^*(t^*), \tag{72} \]
where $\eta^*(t^*) := (\chi d \Theta(T-t^*))^{-1/2}$. From [9, Lemma A.1], we know that for a given $a \in (0,4d)$, any solution $\varphi$ of (17) satisfying
\[ \eta^2 \varphi(\eta) \rightarrow a \quad \text{as} \quad \eta \rightarrow \infty, \tag{73} \]
is such that there exists $\bar{t}_0 = \bar{t}_0(a)$ so that the sign of $(\eta^2 \varphi(\eta))$ does not change on $[\bar{t}_0, \infty)$. Using (72), this implies that there exists $\bar{t}_0 = \bar{t}_0(\bar{t}_0)$ such that the claim holds.

7. Proofs of main results

We start by proving that the $\omega$-limit set of problem (27)–(29) is a singleton.

**Theorem 7.1** Assume the hypotheses of Theorem 2.3. Then the set $\omega$ defined in (30) is a singleton.

**Proof.** For this proof we extend a solution $B$ of (27)–(29) to all $(\mathbb{R}^+)^2$ by setting $B(\eta,\tau) = e^{-\tau} T$ for $(\eta,\tau) \in (\mathbb{R}^+)^2 \setminus \Pi$. We also define the weight function $\rho(\eta) = e^{-\eta^2/4}$ for $\eta > 0$.

The hypothesis (29) implies that $B$ is uniformly bounded; Theorem 3.1 therefore states that $\omega$ is non-empty, and that each $\varphi \in \omega$ is a solution of (17)–(18).

We claim that for each $\varphi \in \omega$ there exists $\tau^* > 0$ such that $B(0,\tau) - \varphi(0)$ never changes sign in $[\tau^*, \infty)$. By contradiction, we assume that there exists a sequence $\tau_k$, such that $\tau_k \rightarrow \infty$, and $B(0,\tau_k) = \varphi(0)$. Since $B_\eta(0,\tau_k) = \varphi_{\eta}(0) = 0$, by Lemma 6.5 the function $Z(\tau)$ has to decrease at least by one. However this cannot happen an infinite number of times. This proves the claim.

Suppose now that $\omega$ is not a singleton. Since the $\omega$-limit set is connected, closed, and non empty, it contains an infinite number of elements. We select three different elements $\varphi_1, \varphi_2, \varphi_3$ in the $\omega$-limit set. Since these functions are different and each solves (18), we may assume that $\varphi_1(0) < \varphi_2(0) < \varphi_3(0)$. By the claim above, $B(0,\tau) - \varphi_2(0)$ never changes sign in $[\tau^*, \infty)$. This contradicts the fact that $\varphi_1$ and $\varphi_3$ are elements of $\omega$; it follows that $\omega$ is a singleton.

We now conclude the proof of Theorems 2.3 and 2.4 and Corollary 2.2.

**Proof of Theorem 2.3**. By the previous theorem $\omega$ is a singleton, say $\{\bar{B}\}$. From Corollary 6.3 we find that for every $N > 0$ there exists a $\tau_N > 0$ such that the solution $B(\eta,\tau)$ intersects $\varphi_S(\eta)$ at most once in $\eta \in [0,\bar{N}]$ for each $\tau > \tau_N$. This implies that in the limit $\tau \rightarrow \infty$, $\bar{B}$ intersects $\varphi_S$ at most once, concluding the proof.
Theorem 2.1. Since $b$ and $U_1(r, t) = (T - t)^{-1} \varphi_1(r/(\chi_d \Theta(T - t)))^{1/2}$ are solutions of (4.3) with the same blow up time, $\bar{V} = b - U_1$ satisfies equation (41). Using that $U_1(r, t) = 2d/(d - 2)(T - t) + r^2/(2\chi_d \Theta)$, we find
\[
\bar{V}(1, t) = (1 - U_1(1, t)) > 0 \quad \text{if} \quad \Theta \leq 1/(4d\chi_d) \quad \text{for any} \quad t < T.
\]
The functions $U_1$ with $b$ necessarily intersect exactly once for all $t$, since non-intersection implies that the solutions must have different times of blow-up [26, p. 271]. It follows that $b(0, 0) < U_1(0)$, and one finds $(T - t)b(0, t) \leq 2d/(d - 2)$. An application of Theorem 2.3 proves the theorem.

Proof of Corollary 2.2. If $b_0 \equiv 1$ and $\Theta < 1/(2(d + 2)\chi_d)$, we know from [2, Theorem 2] that the corresponding solution $b$ blows up. Now assuming $\Theta \leq 1/(4d\chi_d) < 1/(2(d + 2)\chi_d)$, we can apply Theorem 2.1 to conclude.

A. Appendix: The Lyapunov functional

In this appendix we construct the Lyapunov functional $E$ satisfying (58) with the suitable properties of $\rho$ and $\Phi$ to prove Theorem 5.1. We start with a formal construction of the functional. This requires solving a first-order equation for $\rho$ after which $\Phi$ can be expressed in terms of $\rho$. Finally, we explain how to use smooth approximations of $\Phi$ to obtain a rigorous derivation of (58).

A.1. Formal derivation of a Lyapunov functional

Assume that $\Phi$ and $\rho$ are regular. To find such functions satisfying (58), we compute
\[
\frac{d}{d\tau} E(\tau) = \int_0^{\ell(\tau)} \Phi_v B_{\tau} \, d\eta + \int_0^{\ell(\tau)} \Phi_w B_{\tau \eta} \, d\eta + \frac{\ell(\tau)}{2} \Phi(\ell(\tau), B(\ell(\tau), \tau), B_\eta(\ell(\tau), \tau)). \tag{A.1}
\]
Wherever possible we omit the arguments of $\Phi$ and $\rho$ for clarity. Integrating by parts the second integral in (A.1) becomes
\[
\int_0^{\ell(\tau)} \Phi_w B_{\tau \eta} \, d\eta = - \int_0^{\ell(\tau)} \left[ \Phi_{\eta w} + \Phi_v \Phi_{\tau w} B_{\eta} + \Phi_w B_{\eta} \Phi_{\tau w} \right] B_{\tau} \, d\eta + \Phi_w B_{\tau}^{\ell(\tau)} \tag{A.1.1}
\]
Defining
\[
f(\eta, v, w) = \frac{d + 1}{\eta} w - \frac{\eta}{2} w + \frac{1}{d} \eta v w + v^2 - v,
\]
equation (27) takes the form $B_{\tau} = B_{\eta \eta} + f(\eta, B, B_\eta)$, by which equation (A.1) becomes
\[
\frac{d}{d\tau} E(\tau) = \int_0^{\ell(\tau)} \left\{ \left[ \Phi_v - \Phi_{\eta w} - \Phi_v \Phi_{\tau w} B_\eta + \Phi_w \Phi_{\tau w} \right] B_{\tau} - \Phi_w B_{\tau}^2 \right\} \, d\eta
\]
\[+ \Phi_w B_{\tau}^{\ell(\tau)} + \frac{\ell(\tau)}{2} \Phi(\ell(\tau), B(\ell(\tau), \tau), B_\eta(\ell(\tau), \tau)).
\]
Now if functions $\rho = \rho(\eta, v, w) > 0$ and $\Phi = \Phi(\eta, v, w)$ exist that satisfy the system of equations
\[- \Phi_v + \Phi_{\eta w} + w \Phi_{v w} = \rho f \quad \text{and} \quad \Phi_{w w} = \rho,
\tag{A.2}
\]
then $E$ has the form of a Lyapunov functional with a contribution on the boundary, i.e.

$$
\frac{d}{d\tau} E(\tau) = - \int_{0}^{\ell(\tau)} \rho(\eta, B, B_{\eta}) (B_{\eta})^2 d\eta + \Phi_{w} B_{\eta}[\ell(\tau)] + \frac{\ell(\tau)}{2} \Phi(\ell(\tau), B(\ell(\tau), \tau), B_{\eta}(\ell(\tau), \tau)).
$$

(A.3)

Therefore we may obtain this formula by solving system (A.2), which we do by transforming it to a first-order equation for $\rho$,

$$
w \rho_{v} + \rho_{\eta} - f \rho_{w} = f_{w} \rho.
$$

(A.4)

If we supplement a given solution $\rho$ of this equation with the function $\Phi$ given by

$$\Phi(\eta, v, w) = \int_{0}^{\eta} (w - s) \rho(\eta, v, s) ds - \int_{0}^{\eta} \rho(\eta, \mu, 0) f(\eta, \mu, 0) d\mu,
$$

(A.5)

then the pair $(\rho, \Phi)$ solves (A.2). In order to find the pair $(\rho, \Phi)$ we therefore only need to solve equation (A.4).

A.2. The first-order equation for $\rho$

We solve equation (A.4) by the method of characteristics. Characteristic curves of equation (A.4) are curves $x = (\eta, v, w)$ in $\mathbb{R}^{3}$, which we consider parametrised by $\eta$, along which

$$
d\eta v = w \quad \text{and} \quad d\eta w = -f.
$$

(A.6)

If a curve $x(\eta) = (\eta, v^{1}(\eta), w^{1}(\eta))$ satisfies these equations, then equation (A.4) reduces to

$$
\frac{d}{d\eta} \rho(x(\eta)) = f_{w}(x(\eta)) \rho(x(\eta)).
$$

(A.7)

In order to solve the system of ODE’s (A.6) and (A.7), we select a vector $(\eta_{0}, v_{0}, w_{0}) \in \mathbb{R}^{+} \times \mathbb{R}^{2}$ and define $\phi(\xi) = \phi(\xi; \eta_{0}, v_{0}, w_{0})$ to be the solution of the initial value problem

$$
\phi'' + f(\xi, \phi, \phi') = 0, \quad \text{with} \quad \phi|_{\xi = \eta_{0}} = v_{0} \quad \text{and} \quad \phi'|_{\xi = \eta_{0}} = w_{0},
$$

(A.8)

where $' = \frac{\partial}{\partial \xi}$. If the curve $x$ passes through $(\eta_{0}, v_{0}, w_{0})$, i.e. if $x(\eta_{0}) = (\eta_{0}, v_{0}, w_{0})$, then this curve can be identified with $\phi(\cdot; \eta_{0}, v_{0}, w_{0})$, since $x(\eta) = (\eta, v^{1}(\eta), w^{1}(\eta))$ where

$$
v^{1}(\eta) = \phi(\eta; \eta_{0}, v_{0}, w_{0}) \quad \text{and} \quad w^{1}(\eta) = \phi'(\eta; \eta_{0}, v_{0}, w_{0}).
$$

(A.9)

Since $f_{w} = \frac{d+1}{2} - \frac{1}{2} + \frac{1}{2} \eta v$, we may integrate (A.7) to find

$$
\rho(\eta, v, w) = \rho(\eta_{0}, v_{0}, w_{0}) \exp \left\{ \int_{\eta_{0}}^{\eta} \left[ \frac{d+1}{\xi} - \frac{\xi}{2} + \frac{1}{d} \xi v^{1}(\xi) \right] d\xi \right\}
$$

$$= \rho(\eta_{0}, v_{0}, w_{0}) \eta_{0}^{d+1} \eta_{0}^{-d+1} e^{-\eta^{2}/4+\eta_{0}^{2}/4} \exp \left\{ 1 \int_{\eta_{0}}^{\eta} \xi v^{1}(\xi) d\xi \right\}.
$$

(A.10)
To prove Theorem 5.1 we need to define \( \rho \) in the set \( \tilde{\mathcal{R}} \subset \mathcal{R} \) given by (54).

\[
\mathcal{R} = \{ \eta > 0, v \geq 0, w \leq 0 \} \cup \{ \eta = 0, v \geq 0, w = 0 \}
\]

\[
\tilde{\mathcal{R}} = \mathcal{R} \cap \{ 0 \leq v \leq M, 0 \leq -w \leq M \}.
\]

We do so in the following way: for each \((\eta, v, w) \in \mathcal{R}\), we define \(\rho(\eta, v, w)\) by following the characteristic curve through \((\eta, v, w)\) to a reference point \((\eta_0, v_0, w_0)\) for which \(\rho(\eta_0, v_0, w_0)\) is fixed by choice; the value of \(\rho(\eta, v, w)\) is then given by (A.10). To select an appropriate set of reference points, we study some of the properties of solutions \(\phi\) of (A.8), since they define the characteristic curves.

It follows from standard ODE theory that solutions of (A.8) are locally smooth and continuous under changes of \((\eta_0, v_0, w_0)\). In general, however, we cannot extend these solutions to the whole of \(\mathbb{R}^+ \); in fact, for each \((\eta, v, w) \in \mathcal{R}\), there may exist \(0 \leq \xi_1 < \eta\) and/or \(\xi_2 > \eta\) such that

\[
\phi(\xi_1; \eta, v, w) = \infty \quad \text{and/or} \quad \phi(\xi_2; \eta, v, w) = -\infty.
\]

Partly because of this difficulty, we choose to only use forward solutions of (A.8) to define the characteristic curves. The next result details the behaviour of a forward solution \(\phi\) of (A.8).

**Lemma A.1** Let \((\eta, v, w) \in \mathcal{R}\), and let \(\phi(\xi) = \phi(\xi; \eta, v, w)\) be the solution of (A.8).

For \(\xi \geq \eta\), exactly one of the following three alternatives holds:

(i) \(\phi \equiv 1\) or \(\phi \equiv 0\);

(ii) there exists \(\eta^* > \eta\) such that \(\phi(\eta^*) = 0\) and \(\phi(\xi) < 0\) for \(\xi > \eta^*\);

(iii) \(\phi(\xi) \to 0\) as \(\xi \to \infty\) and there exists a constant \(C > 0\) such that \(\phi(\xi)\xi^2 \to C\) as \(\xi \to \infty\).

**Proof.** See [17, p. 95]. The proof is based on results from [20]. \(\blacksquare\)

Since we need to define \(\rho\) with the appropriate estimates, we introduce a parameter \(\tilde{\eta}\) in the following lemma.

**Lemma A.2** There exists \(\tilde{\eta} > 0\) such that for every \(\eta_1 \geq \tilde{\eta}\) any solution \(\phi\) of (A.8) with \(\phi(\eta_1) = 1\) and \(\phi'(\eta_1) \leq 0\) satisfies

\[
\phi'(\eta_2) < -1 \quad \text{for all } \eta_2 > \eta_1 \text{ with } \phi(\eta_2) \in [0, 1/2].
\]

(11)

**Corollary A.3** For every \(\eta_2 \geq \tilde{\eta}\), we have

\[
\phi(\xi; \eta_2, \epsilon, -\epsilon) < 1 \quad \text{for } \xi \in [\tilde{\eta}, \eta_2]
\]

(12)

for all \(0 \leq \epsilon \leq 1/2\), and \(0 < \epsilon \leq 1\).

**Proof of Corollary A.3** A violation of (A.12) implies the existence of \(\eta_1 \in [\tilde{\eta}, \eta_2]\) with \(\phi(\eta_1) = 1\) and \(\phi'(\eta_1) \leq 0\); then (A.11) contradicts the condition \(\phi'(\eta_2; \eta_2, \epsilon, -\epsilon) = -\epsilon \geq -1\).

**Proof of Lemma A.2** We fix \(\eta_1 \gg 1\) and define the variable \(y = \xi/\eta_1\). Changing variables, equation (A.8) transforms into

\[
0 = \frac{1}{\eta_1^2} \left( \phi + \frac{d + 1}{y} \phi \right) - \frac{y}{2} \phi + \frac{1}{d} y \phi' \phi + \phi^2 - \phi, \quad \text{for } y > 1
\]

(13)

\[
\dot{\phi}(1) = -D\eta_1, \quad \phi(1) = 1,
\]

(14)
where \( \gamma = \frac{d}{dy} \). Define
\[
y_0 = \sup\{ y > 1 : \dot{\phi}(y) < 0 \text{ and } \phi(y) > 0 \}
\]
and note that \( y_0 > 1 \) if \( \eta_1 \) is large. On \([1, y_0]\), \( \dot{\phi} \leq -D\eta_1 \); therefore \( y_0 \leq 1 + 1/(D\eta_1) \), and consequently, on \([1, y_0]\),
\[
-\frac{1}{d}(y-1)\phi \leq \frac{1}{dD\eta_1} |\dot{\phi}| \leq \frac{1}{4} \left( \frac{1}{2} - \frac{1}{d} \right) |\dot{\phi}| \quad \text{if} \quad \eta_1 \geq \frac{8d}{d-2} \frac{1}{dD}.
\]
Similarly,
\[
-\frac{d+1}{y\eta_1^2} \phi \leq \frac{1}{4} \left( \frac{1}{2} - \frac{1}{d} \right) |\dot{\phi}| \quad \text{if} \quad \eta_1^2 \geq \frac{8d}{d-2} (d+1).
\]
Therefore,
\[
\frac{1}{\eta_1^2} \dot{\phi} \leq \frac{1}{2} \left( \frac{1}{2} - \frac{1}{d} \right) \phi - \phi^2 + \phi \quad \text{on} \ [1, y_0],
\]
for large \( \eta_1 \). Estimating \( |\dot{\phi}| \) by \( D\eta_1 \), we find
\[
\frac{1}{\eta_1^2} \phi \leq -\frac{1}{2} \left( \frac{1}{2} - \frac{1}{d} \right) D\eta_1 + \frac{1}{4} \quad \text{on} \ [1, y_0],
\]
and since the right-hand side of this expression is negative for large \( \eta_1 \) it follows that \( \dot{\phi} < 0 \) on \([1, y_0]\); therefore \( y_0 \) may be redefined as
\[
y_0 = \sup\{ y > 1 : \phi(y) > 0 \}.
\]
It follows that on \([1, y_0]\),
\[
\dot{\phi}(y) \leq -D\eta_1 - \eta_1^2 d \left[ \frac{1}{4} \left( \frac{1}{2} - \frac{1}{d} \right)^2 - \left( \frac{d-2}{2d} \right)^2 \right] + \int_1^y (\phi - \phi^2).
\]
When \( 0 \leq \phi(y) \leq 1/2 \), this expression is bounded from above by \(-\eta_1^2/64\) for large \( \eta_1 \). In terms of the original variable \( \xi \) we obtain \( \phi'(\xi) \leq -\eta_1/64 \), thus proving the lemma.

**A.3. Definition of \( \rho \) in \( \mathcal{R} \)**

The general idea is to use \( \eta_0 = \bar{\eta} \) as a reference point. In this way, owing to Corollary A.1, we can obtain the required estimates for \( \rho \). It can happen, however, that the function \( \phi(\xi ; \eta, v, w) \) is not defined at \( \xi = \bar{\eta} \). In such a situation, to define \( \rho \), we introduce functions representing the intersection of \( \phi(\cdot ; \eta, v, w) \) with the lines \( \phi = 0 \) for \( \eta < \bar{\eta} \) and \( \phi = 1 \) for \( \eta > \bar{\eta} \). Thus it is useful to define the following subsets of \( \mathcal{R} : \)
\[
\mathcal{R}_1 = \{(\eta, v, w) \in \mathcal{R} : \phi(\xi; \eta, v, w) \text{ satisfies (i) in Lemma A.1} \};
\]
\[
\mathcal{R}_2 = \{(\eta, v, w) \in \mathcal{R} : \phi(\xi; \eta, v, w) \text{ satisfies (ii) in Lemma A.1} \}, \text{ with }
\]
\[
\mathcal{R}_{2a} = \mathcal{R}_2 \cap \{ \eta \leq \bar{\eta} \} \text{ and } \mathcal{R}_{2b} = \mathcal{R}_2 \cap \{ \eta > \bar{\eta} \};
\]
\[
\mathcal{R}_3 = \{(\eta, v, w) \in \mathcal{R} : \phi(\xi; \eta, v, w) \text{ satisfies (iii) in Lemma A.1} \}.\]
We treat the cases in turn.

**Case $R_3$.** Fix a point $(\eta,v,w) \in R_3$. We choose $\eta_0 = \eta$, $v_0 = \phi(\eta;\eta,v,w)$ and $w_0 = \phi'(\eta;\eta,v,w)$. Note that this choice is well defined: since $\phi(\xi)\xi^2 \to C > 0$ as $\xi \to \infty$, there exists $\eta_c > \eta$ such that $\phi(\eta_c;\eta,v,w) = \epsilon < 1/2$, and $-\phi'(\eta_c;\eta,v,w) = \epsilon < 1$, with $\epsilon \sim \frac{2\eta_c}{\eta}$. Then Corollary A.8 implies that the solution $\phi'(\eta;\eta,v,w)$ can be continued to $\eta$, even if $\eta < \eta_c$. Setting $\rho(\eta_0,v_0,w_0) = \eta_0^{d+1}e^{-\eta_0^2/4}$, we find (cf. (A.10))

$$
\rho(\eta,v,w) = \eta^{d+1}e^{-\eta^2/4} \exp \left\{ \frac{1}{d} \int_{\eta}^{\eta_c} \xi \phi(\xi;\eta,\phi(\eta;\eta,v,w),\phi'(\eta;\eta,v,w)) d\xi \right\}. \quad (A.15)
$$

The choice of $\eta_0 = \eta$ also allows us to estimate the value of $\phi$ for $\xi > \eta$, which in turn permits us to control $\rho$ for large $\eta$, since the bound $\phi(\xi) \leq 1$ for $\xi > \eta$ implies an exponential decay for $\rho$ as $\eta \to \infty$.

**Case $R_1$.** Points in $R_1$ are of the form $(\eta,1,0)$ and $(\eta,0,0)$. We again choose $\eta_0 = \eta$; substituting $\phi \equiv 1$ and $\phi \equiv 0$ into formula (A.15) gives

$$
\rho(\eta,1,0) = \eta^{d+1}e^{-\frac{(d-2)\eta^2}{4d} - \frac{\eta^2}{2d}} \quad \text{and} \quad \rho(\eta,0,0) = \eta^{d+1}e^{-\frac{\eta^2}{2d}}. \quad (A.16)
$$

**Case $R_{2a}$.** Fix a point $(\eta,v,w) \in R_{2a}$. Let $\eta^*$ be given by Lemma A.1 and define the function $L_0: R_{2a} \to \mathbb{R}^+$ such that $L_0(\eta,v,w) = \min\{\eta^*,\eta\}$. Note that the function $L_0$ is continuous and equals either the point $\eta^*$ where $\phi(\eta^*;\eta,v,w)$ vanishes or $\eta$ if $\phi(\eta;\eta,v,w) \geq 0$. To find $\rho$, we choose $(\eta_0,v_0,w_0) = (\eta^*,0,\phi'(\eta^*,\eta,v,w))$, and set $\rho(\eta_0,v_0,w_0) = \eta_0^{d+1}e^{-\eta_0^2/4}$. This gives

$$
\rho(\eta,v,w) = \eta^{d+1}\exp\{-\eta^2/4 + I_0\}, \quad (A.17)
$$

where

$$
I_0 = \int_{L_0(\eta,v,w)}^{\eta} \frac{1}{d} \xi \phi(\xi; L_0(\eta,v,w), \phi(L_0(\eta,v,w);\eta,v,w), \phi'(L_0(\eta,v,w);\eta,v,w)) d\xi.
$$

**Case $R_{2b}$.** Here it is convenient to define for any $(\eta,v,w) \in R_{2b}$ the function $L_1: R_{2b} \to \mathbb{R}^+$, by

$$
L_1(\eta,v,w) = \begin{cases} 
\max\{\eta, \max\{\xi \in (0,\eta) \mid \phi(\xi;\eta,v,w) \geq 1\}\} & \text{if } v < 1, \\
\min\{\xi \in (\eta,\infty) \mid \phi(\xi;\eta,v,w) \leq 1\} & \text{if } v \geq 1. 
\end{cases} \quad (A.18)
$$

The function $L_1$ is well defined for $v < 1$ since if $\phi(\xi;\eta,v,w) = 0$ for some $\xi \in (\eta,\eta_c)$ then $\phi < 1$ in $(\eta_c,\eta)$ by Corollary A.8 and $\phi$ has to attain a local maximum in $(\eta_c,\eta)$, which is a contradiction with equation (A.8). For $v \geq 1$, $L_1$ is well-defined by Lemma A.1.

Note that $\phi(L_1(\eta,v,w);\eta,v,w) \leq 1$. The function $L_1$ is continuous and equals either $\eta$, where $\phi(\eta;\eta,v,w) = 1$ or $\eta$ if $\phi(\eta;\eta,v,w) \in (0,1)$.

Now fix a point $(\eta,v,w) \in R_{2b}$, choose $\eta_0 = L_1(\eta,v,w)$ and set $\rho(\eta_0,v_0,w_0) = \eta_0^{d+1}e^{-(d-2)\eta_0^2/4d} - \eta_0^2/2d$. Using (A.10), we find

$$
\rho(\eta,v,w) = \eta^{d+1}\exp\{-\eta^2/4 + \eta_0^2/2d - \eta^2/2d + I_1\} \quad (A.19)
$$

where

$$
I_1 = \int_{L_1(\eta,v,w)}^{\eta} \frac{1}{d} \xi \phi(\xi; L_1(\eta,v,w), \phi(L_1(\eta,v,w);\eta,v,w), \phi'(L_1(\eta,v,w);\eta,v,w)) d\xi.
$$
and
\[\rho(\eta, v, w) = \eta^{d+1} \exp\{- (d-2)\eta^2 / 4d - \eta^2 / 2d + I'_1\}\] (A.20)
where
\[I'_1 = \int_{L_1(\eta, v, w)}^{\eta} \frac{1}{d} \xi [\phi(\xi, L_1(\eta, v, w), \phi(L_1(\eta, v, w); \eta, v, w), \phi'(L_1(\eta, v, w), \eta, v, w)) - 1] d\xi.\]

A.4. Properties of \(\rho\) and \(\Phi\)

In the previous section, we have found a solution \(\rho\) of (A.4). Here we show that this solution, together with the function \(\Phi\) given by (A.3), satisfies the properties required for the proof of Theorem 5.1. We start by stating a result which provides a lower bound for \(\rho\) in \(\mathbb{R}_+\).

**Lemma A.4** Let \(M\) and \(\bar{M}\) be the constants in estimates (44) and (45), and let \(L_1\) be defined as in (A.10). Then there exists a large constant \(\bar{\eta}_0\) such that the function \(G: [\bar{\eta}_0, \infty) \to \mathbb{R}_+\) given by
\[G(\eta) = \max\{L_1(\eta, a, -\bar{b}) | 1 \leq a \leq M \text{ and } 0 \leq b \leq \bar{M}\} \quad \text{for } \eta \geq \bar{\eta}_0,
\] satisfies \(G(\eta) \leq C\eta\) for some constant \(C = C(M) > 0\).

**Proof.** We take \(\bar{\eta}_0\) large and we fix \(\eta \geq \bar{\eta}_0\). Using the continuity of \(L_1\), we have that \(G(\eta) = L_1(\eta, \bar{a}, -\bar{b})\) for some \(\bar{a} \in [1, M]\), and \(\bar{b} \in [0, \bar{M}]\). Now we define the variable \(y = \xi / \eta \geq 1\); the result is proved if we show that \(\sup\{y \geq 1 : \phi(y) > 1\} \leq C(M)\).

As in the proof of Lemma A.2, equation (A.8) transforms into
\[0 = \frac{1}{\eta^2} \left( \frac{d+1}{y} \phi + \phi - \phi \right) - \frac{y}{2} \phi + \frac{1}{d} y \phi + \phi^2 - \phi \quad \text{for } y > 1,
\] (A.21)
\[\phi(1) = -\bar{b} \eta \quad \text{and} \quad \phi(1) = \bar{a}.
\] (A.22)

Note that for \(\phi > 1\) we have \(\phi(y) < 0\) for all \(y > 1\), since \(\phi(\bar{y}) = 0\) implies that \(\bar{y}\) can only be a maximum, a contradiction with equation (A.8).

We prove the claim in two steps. In the first step we consider the case \(\bar{a} > d/2 - \delta > 1\), where \(\delta = (d+1)d/\eta^2\). Define \(y_1 = \sup\{y > 1 : \phi(y) > d/2 - \delta\}\). We write (A.21) as
\[\frac{1}{\eta^2} \phi = -y A_2(y) \phi - A_1(y) \quad \text{for } y > 1,
\] (A.23)
where
\[A_1(y) = \phi^2 - \phi \quad \text{and} \quad A_2(y) = \left(\frac{1}{\eta^2} \phi - \frac{1}{2} + \frac{d+1}{y^2 \eta^2}\right).
\]
Since \(\phi(\cdot) \in [d/2 - \delta, \bar{a}]\) on \([1, y_1]\), \(A_2\) is non-negative and bounded by \(\bar{A}_2 := \bar{a} / d\). The function \(A_1\) is positive and bounded from below:
\[A_1(y) \geq A_{11} := \left(\frac{d}{2} - \delta\right)^2 - \left(\frac{d}{2} - \delta\right) > 0.
\]
Integrating equation (A.23), we have
\[\phi(y) = -\bar{b} \eta e^{-\eta^2 \int_y^{\infty} A_2(t) dt} - \eta^2 \int_1^y A_1(s) e^{-\eta^2 \int_1^s A_2(t) dt} ds \quad \text{for } y > 1.
\] (A.24)
We observe that

\[ \eta^2 \int_1^y A_1(s) e^{-\eta^2 s t A_2(t)} dt \geq A_1 f(y; \eta) \quad \text{for} \quad 1 \leq y \leq y_1 \quad (A.25) \]

where \( f(y; \eta) = \eta^2 \int_1^y e^{-\eta^2 A_2(y^2 - s^2)/2} ds \) is a positive bounded function satisfying \( yf(y; \eta) \to 1/\bar{A}_2 \) as \( y \to \infty \) (the latter claim follows from considering the integrand close to \( s = y \)), and more precisely,

\[ yf(y; \eta) \geq \frac{1}{2A_2} \quad \text{for} \quad y \geq 2 \quad \text{and for sufficiently large} \quad \eta. \]

Therefore the primitive function \( F(y; \eta) = \int_1^y f(s; \eta) ds \)

satisfies

\[ F(y; \eta) \geq \frac{1}{2A_2} (\log y - \log 2). \quad (A.26) \]

Integrating \( A.24 \) on \([1, y_1]\) and using \( A.25 \), we obtain

\[ \phi(y_1) \leq \bar{a} - \bar{b}n(y_1 - 1) - A_1 F(y_1; \eta). \]

To obtain a bound on \( y_1 \), we use \( \phi(y_1) = d/2 - \delta \) and conclude

\[ A_1 F(y_1; \eta) \leq \bar{a} \leq M, \]

from which it follows that \( y_1 \leq C(M) \) by \( A.26 \).

For the second step, we replace \( \eta \) by \( y_1 \eta \) in the rescaling above, by which we can assume that we are in the same situation: \( \phi(1) = \bar{a}, \phi(1) = \bar{b}n, \) but this time \( 1 \leq \bar{a} \leq d/2 - \delta. \)

Similarly define \( y_2 = \sup\{ y \geq 1 : \phi(y) > 1 \} \). Since \( 1 \leq \phi(\cdot) \leq d/2 - \delta \) on \([1, y_2]\),

the function \( A_2(\cdot) \) in \( A.23 \) is negative, so that \( \phi \) satisfies the differential inequality

\[ \frac{1}{\eta^2} \dot{\phi} \leq -\phi^2 + \phi < -2(\phi - 1). \quad (A.27) \]

Let the function \( \psi \) solve

\[ \frac{1}{\eta^2} \dot{\psi} = -2(\psi - 1), \quad \text{with} \quad \psi(1) = \bar{a} \quad \text{and} \quad \dot{\psi}(1) = 0. \]

The solution of this equation is \( \psi(y) = 1 + (\bar{a} - 1) \cos(\eta \sqrt{2}(y - 1)) \), and note that \( \psi(\tilde{y}_2) = 1 \) for \( \tilde{y}_2 := \pi/(2\eta \sqrt{2}) \). From \( A.27 \), \( \phi(1) < \psi(1) \); if \( \phi(y) = \psi(y) \) for some \( y \in (1, \tilde{y}_2) \), then by the comparison principle (which the operator \( u \leftrightarrow \dot{u}/\eta^2 + 2(u - 1) \) satisfies on intervals of length less than \( \tilde{y}_2 \)) we find \( \phi \geq \psi \) on the interval \([1, y]\), a contradiction with the previous remark.

In conclusion we find that \( y_2 \leq \tilde{y}_2 \), thus proving the lemma.

We now derive estimates for \( \rho \) and \( \Phi \) in \( \tilde{R} \) and \( R \).
Lemma A.5 The function $\rho$ is continuous in $R \setminus \{\eta = \tilde{\eta}, v > 1\}$; for $(\eta, v, w) \in R$, one finds

$$\rho(\eta, v, w) \leq \eta^{d+1}e^{-(d-2)\eta^2/4d}.$$  \hspace{1cm} (A.28)

In addition, if $(\eta, v, w) \in \tilde{R}$, then

$$\rho(\eta, v, w) \geq \frac{1}{C_0}\eta^{d+1}e^{-C_0\eta^2}.$$  \hspace{1cm} (A.29)

for some constant $C_0 = C_0(M) > 0$.

Proof. We start by proving $\frac{A.28}{A.29}$. Let $\tilde{R}_i = \tilde{R} \cap R_i$ for $i = 1, 2, 3$. If $(\eta, v, w) \in R_i$, then the estimates follow by definition. If $(\eta, v, w) \in R_{2a}$, then as $\phi > 0$ on $(\eta, L_0(\eta, v, w))$ the integral in (A.17) is negative. This gives

$$\rho(\eta, v, w) \leq \eta^{d+1}e^{-\eta^2/4}$$ for $(\eta, v, w) \in R_{2a}$.

Now for $(\eta, v, w) \in \tilde{R}_{2a}$, we have that $[\eta, L_0(\eta, v, w)] \subset [0, \tilde{\eta}], v \in [0, M]$ and $w \in [-M, 0]$. Then the continuity of $\phi$ on $[\eta, L_0(\eta, v, w)]$ implies that

$$\phi(\xi; L_0(\eta, v, w), \phi(L_0(\eta, v, w); \eta, v, w), \phi'(L_0(\eta, v, w); \eta, v, w)) - 1 > 0$$ for $\xi \geq 1$,

$$\phi(\xi; L_1(\eta, v, w), \phi(L_1(\eta, v, w); \eta, v, w), \phi'(L_1(\eta, v, w); \eta, v, w)) - 1 < 0$$ for $\xi < 1$.

Next for $(\eta, v, w) \in \tilde{R}_{2b}$, we find

$$\rho(\eta, v, w) \geq \eta^{d+1}e^{-\eta^2/4}$$

for $v \leq 1$,

$$\rho(\eta, v, w) \geq \eta^{d+1}e^{-(d-2)\eta^2/4d}e^{-\eta^2/2d}e^{-\tilde{C}(M)\eta^2}$$

for $v > 1$,

where $\tilde{C}(M) > 0$. The estimate when $v \leq 1$ follows directly from (A.19). To obtain the estimate for $\rho$ when $v > 1$, we use (A.20). In fact, noting that $\phi$ is non increasing in $[\eta, L_1(\eta, v, w)]$, we have that

$$\phi(\xi; L_1(\eta, v, w), \phi(L_1(\eta, v, w); \eta, v, w), \phi'(L_1(\eta, v, w); \eta, v, w)) \leq M.$$  \hspace{1cm} (A.20)

Using this bound together with the estimate $L_1(\eta, v, w) \leq C(M)\eta$ (see Lemma A.4), we find that $I_1'$ in (A.20) satisfies $-I_1' \leq \tilde{C}(M)\eta^2$, which gives the desired estimate.

To prove (A.28) for $(\eta, v, w) \in R_3$, we examine two cases, if $\eta \leq \tilde{\eta}$ then the estimate for $R_{2a}$ holds and for $\eta > \tilde{\eta}$ the estimate for $R_{2b}$ holds. Finally, to obtain (A.29) for $(\eta, v, w) \in R_3$, we also check two cases, if $\eta \leq \tilde{\eta}$ then the estimate for $R_{2a}$ holds and for $\eta > \tilde{\eta}$ the estimate for $R_{2b}$ with $v \leq 1$ holds.
Claim. \( \rho \) is continuous in \( \mathcal{R} \setminus \{ \eta = \bar{\eta}, \ v > 1 \} \).

Before we prove this, note that \( \mathcal{R}_2 \) is an open set and \( \mathcal{R}_1 \) and \( \mathcal{R}_3 \) are closed.

We first see that \( \rho \) is continuous within \( \mathcal{R}_{2a} \) and \( \mathcal{R}_{2b} \), by continuity of \( L_0 \) and \( L_1 \). For the elements in \( \mathcal{R}_1 \), the definition of \( \rho \) is as for \( \mathcal{R}_2 \), therefore there is continuity of \( \rho \) between \( \mathcal{R}_2 \) and \( \mathcal{R}_1 \).

The delicate part is to proof continuity between \( \mathcal{R}_3 \) and \( \mathcal{R}_2 \). Taking a sequence \( (\eta_n, v_n, w_n) \in \mathcal{R}_2 \), we associate a solution \( \phi_n(\cdot, \eta_n, v_n, w_n) \).

For the elements in \( \mathcal{R}_1 \), the definition of \( \rho \) is as for \( \mathcal{R}_2 \), therefore there is continuity of \( \rho \) between \( \mathcal{R}_2 \) and \( \mathcal{R}_1 \).

Finally if \( v \leq 1 \) and \( \eta = \bar{\eta} \), then \( \rho \) is continuous. If \( \eta \) close enough to \( \bar{\eta} \) then we have that \( \eta_0 = \bar{\eta} \). So the computation of \( \rho \) uses the same formula, independent of the subset of \( \mathcal{R} \) to which \( (\eta, v, w) \in \mathcal{R}_3 \) belongs.

For \( \Phi \) we deduce the following lemma, which implies (56).

**Lemma A.6** The function \( \Phi \) is continuous in \( \mathcal{R} \setminus \{ \eta = \bar{\eta}, \ v > 1 \} \) and if \( (\eta, v, w) \in \mathcal{R} \), then

\[
\Phi(\eta, v, w) \leq \left\{ w^2 + \frac{v^2}{2} \right\} \eta^{d+1} e^{-(d-2)\eta^2/4d}
\]

and

\[
\Phi(\eta, v, w) \geq -\left\{ \frac{v^3}{3} - \frac{v^2}{2} \right\} \eta^{d+1} e^{-(d-2)\eta^2/4d}.
\]

**Proof.** Follows directly from the definition (A.5) of \( \Phi \) and uses the upper bound (A.28) of \( \rho \).

### A.5. Regularizing argument

In the beginning of this appendix, we formally constructed a Lyapunov functional \( E(\tau) \) with \( \Phi \) and \( \rho \) satisfying (A.3). In the previous section, we obtained a solution \( \rho \) of (A.4) and \( \Phi \) given by (A.2). Moreover these functions satisfy the properties found in Lemmas A.5 and A.6. From these results we do not obtain enough regularity to derive (A.3). To do this, we introduce a regularization of \( \Phi \) using standard mollifiers and translation function to avoid the singularity of \( f \) at \( \eta = 0 \). See the details of the proof in [17, p. 102].

### B. Appendix: Linear stability of blow-up profiles

In this appendix, we study the linear stability of the blow-up profiles \( \varphi_1 \) and \( \varphi^* \), see (20).

Let \( B \) be a solution of (27)–(29) and let \( \varphi \) be a solution of (19). The idea is to study the linearized equation for the difference \( \Phi(\eta, \tau) := B(\eta, \tau) - \varphi(\eta) \), i.e.

\[
\Phi_\tau = \Phi_{\eta\eta} + \frac{d+1}{\eta}\Phi_\eta + \left( \frac{1}{d}\phi - \frac{1}{2} \right) \eta\Phi_\eta + \left( \frac{1}{d\eta}\varphi_\eta + 2\phi - 1 \right) \Phi.
\]

Here, we have implicitly assumed that sufficiently close to blow-up only the linear terms play a role in describe the singularity formation.
For the stability analysis, let $\lambda > 0$ and consider a solution of (B.1) of the form $\psi_\lambda(\eta)e^{\lambda \eta}$. By (B.1), $\psi_\lambda(\eta), \lambda)$ satisfies
\[
(\psi_\lambda)_\eta + \frac{d + 1}{\eta}(\psi_\lambda)\eta + \left(\frac{1}{d} \varphi - \frac{1}{2}\right) \eta(\psi_\lambda) + \left(\frac{1}{d} \eta \varphi_\eta + 2 \varphi - 1 - \lambda\right) \psi_\lambda. \tag{B.2}
\]
For the analysis of boundary conditions we consider first $\varphi = \varphi_1$. We note that at $\eta = 0$ we have either $\psi_\lambda \sim 1$ or $\psi_\lambda \sim 1/\eta^d$. To have $\psi_\lambda$ bounded near 0, we impose
\[
(\psi_\lambda)_\eta(\eta) \to 0, \quad \text{as} \quad \eta \to 0. \tag{B.3}
\]
For large $\eta$, we can either have $\psi_\lambda \sim \eta^{-(2\lambda+3)}e^{\eta^2/4}$ or $\psi_\lambda \sim \eta^{2\lambda-2}$. We see that both behaviours diverge with $\eta$, however the second asymptotic is bounded in terms of $r$ and $t$ as $t \to T$. Therefore to have a polynomial behaviour at infinity, we prescribe
\[
\psi_\lambda(\eta)e^{-\eta} \to 0, \quad \text{as} \quad \eta \to \infty. \tag{B.4}
\]
Now solving equation (B.2) together with (B.3) and (B.4), we find a sequence solutions of (B.1) given by $\{e^{\lambda_n \eta} \psi_n(\eta)\}_{n \in \mathbb{N} \cup \{0\}}$, with $\lambda_0 > \lambda_1 > \ldots$, where $\psi_n := \psi_{\lambda_n}$. If the blow-up time $T > 0$ is chosen correctly in the definition of $\eta$ and $\tau$, we can eliminate, see [10], the first mode ($n = 0$) corresponding to change of blow-up and write
\[
B(\eta, \tau) = \varphi(\eta) + \psi_1(\eta)e^{\lambda_1 \tau} + O(e^{\lambda_2 \tau}).
\]
Therefore from the sign of $\lambda_1$ we obtain the linear stability of $\varphi$.

In [10], Brenner et al. proved, using (B.1), the following stability result for various blow-up profiles.

**Theorem B.1** Every solution $\varphi$ of (47) satisfying $\eta \varphi_\eta/\varphi \to 2$ as $\eta \to \infty$ has an unstable mode corresponding to changing the blow-up time. Also, a blow-up profile with $k$ intersections with the singular solution $\varphi_S$ has at least $k - 1$ additional unstable modes.

In addition, the authors in [10] found numerically that $\lambda_1 < 0$ when $\varphi = \varphi_1$ and $d > 2$. In particular, they computed $\lambda_1 = -0.272 \ldots$ for $d = 3$. This implies that $\varphi_1$ is linearly stable for $d > 2$.

For $\varphi = \varphi^*$, we can proceed as above and solve the eigenvalue problem for (B.1). Considering (B.2) with $\varphi = \varphi^*$, we find that $(\psi_\lambda, \lambda)$ satisfies
\[
(\psi_\lambda)_\eta + \left(\frac{d + 1}{\eta} - \frac{d - 2}{2d} \eta\right)(\psi_\lambda)_\eta + (1 - \lambda)\psi_\lambda = 0, \tag{B.5}
\]
with (B.3) and (B.4). These boundary conditions are chosen by the same arguments for $\varphi = \varphi_1$; however in the current case we have either $\psi_\lambda \sim \eta^{\lambda_2(d-1)-d-2}e^{\eta^2/(4\eta^2)}$ or $\psi_\lambda \sim \eta^{\lambda_2(d-1)-\lambda}$ as $\eta \to \infty$. Note that by changing $\eta$ by $(-\eta)$ the equation remains invariant, so only solutions consisting on even powers are allowed. Then we construct a sequence of solutions of the form
\[
\psi_n(\eta) = \sum_{i=0}^{n} A_i \eta^{2i} \quad \text{for any} \quad n = 0, 1, 2, 3 \ldots,
\]
where the coefficients are given by $A_i(2i(2i-1) + (d+1)2i) = A_{i-1}(1-\lambda-2i(d-2)/2d)$ for $i = 1, 2, \ldots$ and $A_0$ an arbitrary constant. This means that when $(1-\lambda-2(n+1)(d-2)/2d) = 0$, we find an explicit polynomial solution of degree $2n$, where $\lambda$ is given by
\[
\lambda_n = \frac{d - n(d - 2)}{d}. \tag{B.6}
\]
Consequently, we have obtained an explicit sequence of solution \( \{\langle \psi_n, \lambda_n \rangle \}_{n \in \mathbb{N} \cup \{0\}} \) for the eigenvalue problem (B.5). The eigenvalue \( \lambda_0 = 1 \) corresponds to the unstable mode of change of blow-up time and since \( \lambda_1 > 0 \) for all \( d > 2 \), by (B.6), this means that \( \varphi^* \) is linearly unstable.

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