Multi-bridge graphs are anti-magic

Yu Bin Tai, Gek L. Chia∗, Poh-Hwa Ong

Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science, Universiti Tunku Abdul Rahman, Sungai Long Campus, Malaysia

shotztai@1utar.my, chiagl@utar.edu.my, ongph@utar.edu.my

*Corresponding author

Abstract

An anti-magic graph is a graph whose \(|E|\) edges can be labeled with the first \(|E|\) natural numbers such that each edge receives a distinct number and each vertex receives a distinct vertex sum which is obtained by taking the sum of the labels of all the edges incident to it. We prove that the multi-bridge graph is anti-magic.

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1. Introduction

Let \(G = (V, E)\) be a graph with neither loop nor multiple edges. An anti-magic labeling of \(G\) is a bijection \(\varphi\) from \(E\) to \(\{1, 2, \ldots, |E|\}\) such that the sum of the labels on the edges incident to a vertex, called the vertex sum, is distinct for each vertex. A graph is anti-magic if it admits an anti-magic labeling.

The concept of anti-magic graphs has its origin from the book [7] where Hartsfield and Ringel conjectured that all connected graphs but the single edge \(K_2\) are anti-magic. Since then, the problem of deciding which graphs are anti-magic has attracted much attention. Nevertheless the conjecture remains unsettled despite concerted efforts by mathematicians in graph theory.

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In the same book, Hartsfield and Ringel remarked that even when the conjecture is restricted to trees, no complete affirmative answer has been offered. Some results concerning the anti-magicness of trees are given in [8] and [9].

On the other hand, by confining the attention on regular graphs, the situation turns out to be a lot more delightful. In [4], Cranston showed that every regular bipartite graph with degree at least 2 is anti-magic. In [5], Cranston et al. proved that Hartsfield and Ringel’s conjecture is true for all odd regular graphs. Shortly afterwards, in [3], Chang et al. proved that all even regular graphs are anti-magic. By modifying the argument used in [5], Bérczi et al. in [2] also proved that even regular graphs are anti-magic. For more details on anti-magic graphs, we refer the reader to [6]. For some recent results on anti-magic graphs, we refer the reader to [10].

In view of this, we turn our attention to graphs which are close to being regular. Consider a graph with only two vertices and having \( r \) multiple edges joining them, \( r \geq 3 \). Subdivide the edges of this graph arbitrarily so that at most one edge is not subdivided. Call the result graph an \( r \)-bridge graph and denote it by \( \theta(m_1, m_2, \ldots, m_r) \) if the lengths of the paths are \( m_1, m_2, \ldots, m_r \) respectively.

The purpose of this paper is to prove the following result.

**Theorem 1.1.** Every \( r \)-bridge graph is anti-magic.

In a forth-coming paper, we shall make use of the above result to prove the anti-magicness of a class of not quite regular graphs. Hence it is an appetizer result for a more general result which is to appear later.

We note in passing that in [1], Alon et al. proved that all dense graphs are anti-magic while in [11], Wang initiated the investigation on the anti-magicness of sparse graphs. Incidentally, the graphs in this paper and those in our forth-coming papers are sparse graphs.

2. The proof of Theorem 1.1

Throughout this section, we shall assume that in the graph \( \theta(m_1, m_2, \ldots, m_r) \), the path lengths satisfy the condition \( m_1 \geq m_2 \geq \cdots \geq m_r \). Also, we shall call the paths in \( \theta(m_1, m_2, \ldots, m_r) \) the \( m_i \)-path, \( i = 1, 2, \ldots, r \).

Let \( x \) and \( y \) denote the two vertices of degree \( r \) in \( \theta(m_1, m_2, \ldots, m_r) \) and let \( w(x), w(y) \) denote the vertex sums of \( x, y \) respectively.

The proof is divided into three cases.

**Case 1.** \( r = 3k \).

Suppose \( k = 1 \).

The labelings depicted in Figure 1 show that if \( m_1 \leq 2 \), the 3-bridge graph is anti-magic. Hence we assume that \( m_1 \geq 3 \).

**Subcase 1.1.** \( m_1 + m_2 + m_3 \) is odd.

Let \( \varphi_0 \) denote the following edge labeling on the 3-bridge graph.

(i) Label the edges of the \( m_1 \)-path with \( 1, 2, \ldots, m_1 \) successively starting from the vertex \( x \).

(ii) Label the edges of the \( m_3 \)-path with \( m_1 + 1, m_1 + 2, \ldots, m_1 + m_3 \) successively starting from the vertex \( y \).
Multi-bridge graphs are anti-magic  |  Yu Bin Tai et al.

(iii) Label the edges of the \( m_2 \)-path with \( m_1 + m_3 + 1, m_1 + m_3 + 2, \ldots, m_1 + m_3 + m_2 \) successively starting from the vertex \( x \).

Figure 2(i) illustrates the case \((m_1, m_2, m_3) = (5, 4, 2)\).

Note that the vertex sums of the degree-2 vertices consist of distinct odd natural numbers and that the vertex sums of \( x \) and \( y \) are both even and are given by \( w(x) = 2(m_1 + m_3 + 1) \) and \( w(y) = 2m_1 + m_1 + m_2 + m_3 + 1 \) respectively.

This shows that \( \varphi_0 \) is an anti-magic labeling of the 3-bridge graph.

Figure 2. Two anti-magic labelings on 3-bridges.

**Subcase 1.2.** \( m_1 + m_2 + m_3 \) is even.

In this case, an anti-magic labeling is obtained by swapping the labels \( m_1 - 1, m_1 \) (on the last two edges of the \( m_1 \)-path) from the anti-magic labeling \( \varphi_0 \) given in Subcase 1.1. Note that there are only three vertices whose vertex-sums are even, namely \( x, y \) and the second last vertex on the \( m_1 \)-path. Since the vertex-sums are \( 2(m_1 + m_3 + 1), 2m_1 + m_1 + m_2 + m_3 \) and \( 2m_1 - 2 \) respectively, they are distinct natural numbers.

The vertex-sums of the rest of the vertices are distinct odd natural numbers.

Figure 2(ii) illustrates the case \((m_1, m_2, m_3) = (5, 4, 3)\).

Now suppose \( k \geq 2 \).

For each \( i = 1, 2, \ldots, k \), let \( H_i \) denote the 3-bridge subgraph induced by the \( m_{3i-2} \)-path, \( m_{3i-1} \)-path and the \( m_{3i} \)-path.

Define \( p_0 = 0 \) and \( p_i = p_{i-1} + m_{3i-2} + m_{3i-1} + m_{3i} \) for \( i \geq 1 \).

For each \( i = 1, 2, \ldots, k \), label the edges of \( H_i \) so that

(i) the edges of the \( m_{3i-2} \)-path receive the labels \( p_{i-1} + 1, p_{i-1} + 2, \ldots, p_{i-1} + m_{3i-2} \) successively starting from the vertex \( x \),

(ii) and then label the edges of the \( m_{3i} \)-path with \( p_{i-1} + m_{3i-2} + 1, p_{i-1} + m_{3i-2} + 2, \ldots, p_{i-1} + m_{3i-2} + m_{3i} \) successively starting from the vertex \( y \).
Finally, label the edges of the \(m_{3i-1}\)-path with \(p_{i-1} + m_{3i-2} + m_{3i} + 1, p_{i-1} + m_{3i-2} + m_{3i} + 2, \ldots, p_{i-1} + m_{3i-2} + m_{3i} + m_{3i-1}\) starting from the vertex \(x\).

Figure 3 illustrates the cases \((m_1, m_2, \ldots, m_6) = (6, 6, 5, 4, 3, 2)\) and \((m_1, m_2, \ldots, m_6) = (2, 2, \ldots, 2)\).

It is routine to check that the vertex sums of \(x\) and \(y\) are given by
\[
w(x) = 2k + 2p_k - 2 \sum_{i=1}^{k} m_{3i-1} + 3 \sum_{i=1}^{k-1} p_i
\]
and
\[
w(y) = k + p_k + 2 \sum_{i=1}^{k} m_{3i-2} + 3 \sum_{i=1}^{k-1} p_i.
\]
respectively.

Also, note that the vertex sums of the degree-2 vertices consist of odd distinct natural numbers and are less than either of \(w(x)\) and \(w(y)\).

This completes the proof for Case 1.

**Case 2.** \(r = 3k + 1\).

Suppose \(k = 1\).

**Subcase 2.1.** Not all paths have the same length.

Let \(\varphi_1\) denote the following edge labeling on the 4-bridge graph.

(i) Label the edges of the \(m_1\)-path with \(1, 2, \ldots, m_1\) successively starting from the vertex \(x\).

(ii) Label the edges of the \(m_2\)-path with \(m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\) successively starting from the vertex \(x\).

(iii) Label the edges of the \(m_3\)-path with \(m_1 + m_2 + 1, m_1 + m_2 + 2, \ldots, m_1 + m_2 + m_3\) successively starting from the vertex \(y\).

(iv) Label the edges of the \(m_4\)-path with \(m_1 + m_2 + m_3 + 1, m_1 + m_2 + m_3 + 2, \ldots, m_1 + m_2 + m_3 + m_4\) successively starting from the vertex \(y\).

Figure 4(i) illustrates the case \((m_1, m_2, m_3, m_4) = (5, 4, 3, 2)\).

Note that the vertex sums \(w(x)\) and \(w(y)\) of \(x\) and \(y\) are given by \(3m_1 + 2m_2 + 2m_3 + m_4 + 2\) and \(4m_1 + 3m_2 + m_3 + 2\) respectively. Note that the vertex sums of the degree-2 vertices consist of distinct natural odd numbers and they are all less than either of \(w(x)\) and \(w(y)\).

This means that \(\varphi_1\) is an anti-magic labeling of the 4-bridge.
Subcase 2.2. All paths have the same length \( m \).
In this case, an anti-magic labeling is obtained by labeling the edges of the \( i \)-th path with the labels \((i - 1)m + 1, (i - 1)m + 2, \ldots, im\) successively all starting from \( x \) to \( y \). In this case \( w(x) = 6m + 4 \) and \( w(y) = 10m \). The rest of the vertex sums consist of distinct odd natural numbers.

Figure 4(ii) illustrates the case \( m = 3 \).

Now suppose \( k \geq 2 \).
Let \( H_1 \) denote the 4-bridge subgraph induced by the \( m_j \)-path, \( j = 1, 2, 3, 4 \). Also, for each \( i = 2, \ldots, k \), let \( H_i \) denote the 3-bridge subgraph induced by the \( m_{3i-1} \)-path, \( m_{3i} \)-path and the \( m_{3i+1} \)-path.
Define \( p_0 = 0, p_1 = m_1 + m_2 + m_3 + m_4 \) and \( p_i = p_{i-1} + m_{3i-1} + m_{3i} + m_{3i+1} \) for \( i \geq 2 \).
Label \( H_1 \) using \( \phi_1 \) first. Then for each \( i = 2, \ldots, k \), label the edges of \( H_i \) so that
(i) the edges of the \( m_{3i-1} \)-path receive the labels \( p_{i-1} + 1, p_{i-1} + 2, \ldots, p_{i-1} + m_{3i-1} \) successively starting from the vertex \( x \), and
(ii) label the edges of the \( m_{3i+1} \)-path with \( p_{i-1} + m_{3i-1} + 1, p_{i-1} + m_{3i-1} + 2, \ldots, p_{i-1} + m_{3i-1} + m_{3i+1} \) successively starting from the vertex \( y \).
(iii) Finally, label the edges of the \( m_{3i} \)-path with \( p_{i-1} + m_{3i-1} + m_{3i+1} + 1, p_{i-1} + m_{3i-1} + m_{3i+1} + 2, \ldots, p_{i-1} + m_{3i-1} + m_{3i+1} + m_{3i} \) starting from the vertex \( x \).
Figure 5 illustrates the cases \((m_1, m_2, \ldots, m_7) = (6, 5, 4, 3, 3, 3, 2)\) and \((m_1, m_2, \ldots, m_7) = (2, 2, \ldots, 2)\).

It is routine to check that the vertex sums of \(x\) and \(y\) are given by
\[
w(x) = 2p_k + 2k + m_1 - m_4 + \sum_{i=2}^{k} (3p_{i-1} - 2m_{3i})
\]
and
\[
w(y) = k + 1 + 4m_1 + 3m_2 + m_3 + 2(p_1 - p_k) + \sum_{i=2}^{k} (3p_i + 2m_{3i-1})
\]
respectively.

Also, note that the vertex sums of the degree-2 vertices consist of distinct odd natural numbers each of which is less than either of \(w(x)\) and \(w(y)\).

This completes the proof for Case 2.

**Case 3.** \(r = 3k + 2\).

Suppose \(k = 1\).

Let \(\varphi_2\) denote the following edge labeling on the 5-bridge graph.

(i) Label the edges of the \(m_1\)-path with \(1, 2, \ldots, m_1\) successively starting from the vertex \(x\).

(ii) Label the edges of the \(m_2\)-path with \(m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\) successively starting from the vertex \(y\).

(iii) For each \(i \in \{3, 4, 5\}\), label the edges of the \(m_i\)-path with \(q_i + 1, q_i + 2, \ldots, q_i + m_i\) successively all starting from \(x\) to \(y\). Here \(q_3 = m_1 + m_2\) and \(q_j = q_{j-1} + m_{j-1}\) for \(j \in \{4, 5\}\).

Figure 6 illustrates the case \((m_1, m_2, m_3, m_4, m_5) = (6, 5, 4, 3, 2)\).

![Anti-magic labeling of a 5-bridge](image)

Note that the vertex sums of \(x\) and \(y\) are given by \(w(x) = 4(m_1 + m_2) + 2m_3 + m_4 + 4\) and \(w(y) = 5m_1 + 3(m_2 + m_3) + 2m_4 + m_5 + 1\) respectively.

Clearly the vertex sums of the degree-2 vertices in \(\varphi_2\) consist of odd distinct natural numbers and each is less than either of \(w(x)\) and \(w(y)\).

Hence \(\varphi_2\) is an anti-magic labeling of the 5-bridge.

Now suppose \(k \geq 2\).

Let \(H_1\) denote the 5-bridge induced by the \(m_j\)-path, \(j = 1, 2, \ldots, 5\). Also, for each \(i = 2, \ldots, k\), let \(H_i\) denote the 3-bridge subgraph induced by the \(m_{3i}\)-path, \(m_{3i+1}\)-path and the \(m_{3i+2}\)-path.

Define \(p_0 = 0, p_1 = m_1 + m_2 + \cdots + m_5\) and \(p_i = p_{i-1} + m_{3i} + m_{3i+1} + m_{3i+2}\) for \(i \geq 2\).

Label \(H_1\) using \(\varphi_2\) first. Then for each \(i = 2, \ldots, k\), label the edges of \(H_i\) so that

(i) the edges of the \(m_{3i}\)-path receive the labels \(p_i + 1, p_{i-1} + 2, \ldots, p_{i-1} + m_{3i}\) successively starting from the vertex \(x\), and
(ii) label the edges of the \( m_{3i+2} \)-path with \( p_{i-1} + m_{3i} + 1, p_{i-1} + m_{3i} + 2, \ldots, p_{i-1} + m_{3i} + m_{3i+2} \) successively starting from the vertex \( y \).

(iii) Finally, label the edges of the \( m_{3i+1} \)-path with \( p_{i-1} + m_{3i} + m_{3i+2} + 1, p_{i-1} + m_{3i} + m_{3i+2} + 2, \ldots, p_{i-1} + m_{3i} + m_{3i+2} + m_{3i+1} \) starting from the vertex \( x \).

Figure 7 illustrates the case \((m_1, m_2, \ldots, m_8) = (6, 5, 4, 3, 3, 3, 2, 2)\).

Figure 7. Anti-magic labeling of an 8-bridge.

It is routine to check that the vertex sums of \( x \) and \( y \) are given by

\[
\begin{align*}
 w(x) &= 2(p_k + k + 1 + m_1 + m_2 - m_5) - m_4 + \sum_{i=2}^{k} (3p_{i-1} - 2m_{3i+1}) \\
 w(y) &= 2(2m_1 + m_2 + m_3) + m_4 + k + p_k + \sum_{i=2}^{k} (3p_{i-1} + 2m_{3i})
\end{align*}
\]

respectively.

Also, note that the vertex sums of the degree-2 vertices consist of distinct odd natural numbers each of which is less than either of \( w(x) \) and \( w(y) \).

This completes the proof for Case 3 and so is the proof for Theorem 1.1.

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