Acceleration of numerical solution of elastic contact problems by a dual-grid approach

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Abstract. Finer meshes employed in the numerical simulation of contact problems provide better accuracy, assuring that the effect of specific features of the initial contact geometry on the stress and displacement field are faithfully captured. Efficient implementation of finer meshes may benefit from a dual-grid scheme, involving a lower level and a desired level. The main idea is to use the lower level for the preparation of a better initial guess for the desired level, obtained by interpolation of results derived on the lower level. A dual-grid scheme involves three steps: (a) solution on the lower level (the coarser grid) with a standard initial guess, (b) interpolation of results into the desired level mesh, providing a more accurate initial approximation for the subsequent step, and (c) the solution of the desired level (the finer mesh) with the improved initial guess. As demonstrated by the conducted numerical simulations, this approach provides important computational advantages, as the solution of the lower mesh is obtained more rapidly than that of the finer mesh, whereas the number of iterations in the upper level is reduced due to the quality of the improved initial guess. The optimal ratio between the lower and the upper levels is investigated through numerical examples and an optimal value is proposed. The proposed dual-grid method should promote faster solutions of the contact problems, and therefore can be implemented in a variety of contact scenarios that rely on the considered algorithm. The improvements should be more beneficial for the contact processes involving the reproduction of the loading path, such as the elastic-plastic or the viscoelastic contact.

1. Introduction
The service life of various machine elements can be greatly affected by high stress gradients arising in concentrated contacts. Apart from the notorious Hertz framework, the contact problem lacks analytical solution and therefore many research efforts were concerned with the numerical analysis of the contact processes. The contact numerical study has gained traction with the application of the Conjugate Gradient Method and the fast Fourier transform to contact mechanics, providing the necessary computational efficiency to address contact scenarios relevant for practical applications.

The numerical treatment of the three-dimensional rough contact problem became feasible with the method proposed by Polonsky and Keer [1], in which the real contact area and the pressure distribution are assessed in a single-loop iteration scheme build around the conjugate gradient method for the resolution of linear systems of equations with symmetric and positive definite matrices. The
method was proven to converge for arbitrary rough surfaces, even when the contact area is highly disconnected, such as in the case of roughness samples of practical sizes.

The method was further improved by substituting the MLMS-based scheme for the displacement computation with the spectral methods that calculate the convolution products in the Fourier transform domain. Moreover, application of the spectral methods allows considering more complicated problems lacking analytical solution in the time/space domain, such as the contact of coated materials. The use of the fast Fourier transform in contact problems pioneered with the works of Yu and Faris [2] and Nogi and Kato [3], whereas full method development including the study of the periodicity error was later achieved by Liu et al. [4] and by Liu and Wang [5].

The improved contact solver was further used in the numerical study of various types of contact processes, the main generalisation coming from the constitutive law of the contacting materials. By coupling the elastic contact solver with the Betti’s reciprocal theorem, a numerical model for the elastic-plastic contact was achieved [6-9]. Friction was later considered and the solution of the partial slip contact problem was achieved [10, 11] based on the same algorithm, leading to the study of fretting wear [12-15]. A contact solution in the presence of inhomogeneities [16] was also advanced. Contact simulations involving linear viscoelastic materials were also conducted [17, 18].

This paper improves the contact solver by proposing a dual-grid computational scheme that uses two mesh levels with increasing grid density. The function of the coarser grid is to prepare a competent initial guess for the finer mesh, thus reducing the number of iterations on the aforementioned mesh. Numerical experimentations show promising results, with time savings reaching 50%.

2. Model outline

The solution of the contact problem under normal and tangential loading is of chief importance for the optimal design of machine elements. Experimental and theoretical studies in the field of contact mechanics are conducted for the concentrated contact, in which case the contact area prior to load application vanish to a point or a line. This type of nonconformal contact is preferred due to its repeatability in controlled conditions, and because of the theoretical development of the problem solution based on elasticity results obtained for the half-space. The concentrated contact exhibits high gradients of stresses in the close vicinity of the initial point or line of contact, which vanish before reaching the other boundaries of the contacting bodies. This allows for bodies of arbitrary initial geometry, nominal or rough, to be assimilated with elastic half-spaces in the calculation of their displacements or stresses. A direct consequence is that the displacements induced by the normal traction (i.e., by contact pressure), are always aligned with the same direction perpendicular to the half-space boundary. By considering a Cartesian coordinate system with its origin in the initial point of contact, the $x_1$ and $x_2$ - axes may be conveniently chosen in the plane that separates best the limiting surfaces of the contacting bodies, whereas $x_3$ becomes the normal contact direction. Although friction and tangential effects can be equally considered, without losing generality, the case investigated in this work is restricted for brevity to the normal frictionless contact under normal load, in which case all considered model parameters are aligned with the normal direction. A schematic of the considered contact scenario in the plane $x_2 = 0$ is depicted in figure 1. A rigid sphere is pressed with a force $W$ into an elastic half-space, which deforms elastically so that the initial point of contact evolves into a circular contact region. The state of the contacting bodies is depicted with dashed line before load application, and with continuous line for the deformed state. The initial point of contact is the origin of the considered coordinate system. The rigid sphere translates normally with a quantity $\delta_1$ referred to as the rigid-body approach (i.e., the approach of points in the contacting bodies distant to the contact region). The gap between the limiting surfaces, measured along the normal direction, are denoted by $h_{\text{init}}$ before deformation and by $h$ in deformed state. Due to the contact pressure $p(x_1, x_2)$ that arises to counterbalance the applied normal force, the elastic half-space boundary undergo elastic displacements $u_3$ that are also aligned with the normal direction.
By equating the model parameters, the equation of the surface of separation yields:

\[ h(x_1, x_2) = h_{in} (x_1, x_2) + u_3(x_1, x_2) - \delta_3, \]  

which, together with the static force equilibrium, builds the model for the contact problem:

\[ W = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x_1, x_2) dx_1 dx_2. \]  

The problem unknowns are not only the pressure distribution, which is related to the displacement by the Boussinesq solution \( G \) for the elastic half-space under point normal load:

\[ u_3(x_1, x_2) = \iint_{A_c} G(x_1 - x'_1, x_2 - x'_2)p(x'_1, x'_2) dx'_1 dx'_2, \]  

but also the shape and size of the contact area \( A_c \). This duality makes the problem solution difficult to pursue analytically, because the continuous integral in equation (3) cannot be generally performed. The closed-form solution of this model cannot be obtained for bodies of arbitrary surfaces, the Hertz contact being one notable exception. A numerical solution was sought that can remove the limiting assumptions of the Hertz model. A discrete model may be constructed by imposing a two-dimensional rectangular mesh of sides lengths \( L_1 \times L_2 \), with \( N_1 \times N_2 \), in the plane \( x_1 x_2 \), as shown in figure 2.

**Figure 1.** Schematic of the spherical frictionless contact under normal load.

**Figure 2.** Meshing of the contact region.
Each rectangular patch has a designated control point in which the problem parameters are computed, and the obtained value is assumed for the whole patch. Any continuous distribution is thus replaced by a piecewise constant function that presumably replicates well the continuous distribution if the number of grids is high enough. Spatial localization in the discrete model can be achieved by using integers indexing the control points, thus building a matrix of elements for each 2D continuous distribution. This discretization allows for the numerical integration of equation (3) for arbitrary contact area and/or pressure distribution. The continuous pressure is substituted by a set of pressure elements acting on a set of rectangular patches constructing the contact area. The displacement induced by a pressure element \( K(i-k,j-\ell) \) in the patch \((k,\ell)\) is known in the literature as the influence coefficient \( K(i-k,j-\ell) \), which is in fact the solution derived by Love [19] for the effect of a uniform pressure on a rectangular patch:

\[
K(i-k,j-\ell) = \int_{x_1(i-k)}^{x_1(i+k/2)} \int_{x_2(j-\ell)}^{x_2(j+\ell/2)} G(x_i-x'_i,x_j-x'_j) \, dx'_i \, dx'_j.
\] (4)

The discrete counterpart of equation (3) becomes:

\[
u_j(i,j) = \sum_{(k,\ell)=k_0}^{N_1} K(i-k,j-\ell) p(k,\ell) = \sum_{k=1}^{N_1} \sum_{\ell=1}^{N_2} K(i-k,j-\ell) p(k,\ell),\]

(5)

which, unlike its continuous version, can be calculated for arbitrary input, thus allowing for a trial-and-error iterative search of the problem unknowns. The derivation of an iterative process that rapidly converges to the contact problem solution has been a topic considered in many research efforts that are reviewed in [20]. In the discrete model, equation (1) generates a linear system of equations with the pressures in the control points as unknowns. Although one equation is constructed for each elementary patch, only equations corresponding to the elementary patches contained in the contact area should be considered. This is an additional difficulty as the contact area is also unknown.

The state-of-the-art is clearly indicated [20] as the algorithm proposed by Polonsky and Keer [1], which iterates simultaneously the contact area and the pressure distribution in a iteration scheme constructed around the Conjugate Gradient method for the resolution of linear systems with symmetric and positive definite matrices. An initial guess is adopted: the entire computational domain is in contact, i.e., the contact area and the imposed mesh coincide, and the pressure is distributed uniform on the contact area. This allows for the calculation of displacement according to equation (5), which in its turn allows computing a new, more accurate, pressure distribution. However, the updated pressure matrix will contain negative values corresponding to the patches that are not actually in the contact area. Those patches should be eliminated from the system in the next iteration and their attached pressure set to zero. Thus, the size of the system to be solved varies as the contact area is also sought iteratively. There exist an opposing process, which reintegrates in the contact area the cells for which the gap \( h \) results negative. The size of the contact area and the associated pressure distribution are thus found simultaneously, and the convergence is judged based on the condition that:

\[
\frac{\Delta_1 \Delta_2 \sum_{(i,j) \in k} \left| p^{(k+1)}(i,j) - p^{(k)}(i,j) \right|}{w} \leq \varepsilon_0,
\] (6)

where the optimal value for the imposed precision was proposed based on numerical experimentations:

\[
\varepsilon_0 = (\min(N_1,N_2))^{-3/2}.
\] (7)

A complete algorithm description is beyond the point of this paper, an interested reader being redirected to [1]. The before mentioned algorithm was further improved by substituting the MLMS
sequence [1] by the DCFFT [4, 5], capitalising on the fact that the multi-summation in equation (5) is in fact a discrete cyclic convolution that can be evaluated efficiently in the frequency domain.

The algorithm was further employed beyond the elastic domain, in the simulation of dissipative processes that require the reproduction of the loading history. The latter replication is mainly achieved by solving a series of subsequent contact states, together with load and/or time incrementation. Enhancing the computational efficiency of the contact algorithm without sacrificing accuracy may lead to improved convergence in the simulation of history-dependent contact processes. It may also lead to increased resolution in facing the complexity of real microtopography as resulting from different manufacturing processes.

3. Dual-grid computational scheme
Capitalising on the fact that a more qualitative initial guess is expected to promote a faster solution, a dual-grid scheme is proposed in this paper to improve the computational efficiency of the original algorithm. Two (or more) mesh levels are involved: a lower mesh level with a standard initial guess, and a desired mesh level, which benefits from the improved guess resulting from calculations performed on the coarser mesh. As depicted in figure 3, the lower mesh covers the same computational domain and consequently has the same side lengths, but larger grid steps. An element size ratio \( r = N_{\text{fine}} / N_{\text{coarse}} \), identical for the both directions \( x_1 \) and \( x_2 \), may be introduced for convenience, resulting in the relation between the grid steps depicted in figure 3. The total number of control points is \( N_1 \times N_2 \) for the desired mesh level and \( N_1 \times N_2 / r^2 \) for the lower mesh level. The lower mesh level prepares a more accurate initial guess for the desired mesh level, thus reducing the number of iterations on the finer mesh. The computational advantage stems from the fact that the solution obtained on the lower mesh level is obtained faster than on the desired mesh level, and the loss of accuracy due to the lower resolution can be rapidly corrected on the desired mesh level.

The dual-grid scheme thus consists in three steps:
(a) Solve the contact problem on the coarser mesh with a standard initial guess, i.e. a contact pressure uniform on the entire computational domain. Obtain the pressure distribution that verifies equation (6) on the lower level mesh.
(b) Interpolate and extrapolate the latter solution to obtain a pressure distribution matrix corresponding to the control points of the desired level mesh.
(c) Solve the contact problem on the finer mesh having as initial guess the pressure distribution prepared in the previous step. Stop iterations when condition (6) is verified for the finer mesh.

It should be noted that, according to equation(7), the precision goal on the desired mesh is higher than that on the lower mesh level.

![Figure 3. Lower-level mesh (right) and desired mesh level (left).](image-url)
4. Results and discussions
The proposed scheme for improvement of computational efficiency of the contact solver is benchmarked by computing the solution of the Hertz contact with different element size ratios $r$ between the lower level and the desired mesh level. The computing times are compared with those needed for the solution of the desired mesh level as a fixed level, i.e. with the original algorithm applied to the finer mesh. Computations are performed for different resolutions, which are all powers of 2. This choice is reasonable as the fast Fourier transform needed in the displacement computation by the DCFFT scheme is computed by zero-padding the input series to the next power of 2.

Figure 4(a) compares the results obtained by solving the contact problem on a fixed mesh but with different grids densities. The pressure profiles in the plane $x_2 = 0$ follow closely the semi-elliptic Hertz distribution of central pressure $P_H$ and contact radius $a_H$. The original algorithm seems to provide accurate results even with a limited number of elementary patches, thus supporting the dual-grid formulation by assuring the quality of the initial guess for the desired level mesh. This result suggests that with this method, qualitative predictions are prepared on the coarser meshes. A typical 3D pressure distribution obtained on a $32 \times 32$ grid is depicted in figure 4(b).

The computational times for a fixed mesh contact simulation are presented in table 1 for different grid densities. The CPU time increases rapidly with number of grids, especially due to the convolution products in equation (5) that require important computational resources. It should be noted that the complexity of computing the convolution product by conventional multi-summation is $O(N^2)$, where $N$ is the data size. By using the DCFFT algorithm [4,5], this complexity is reduced to $O(N \log N)$.

The computing times for the fixed and the dual-grid scheme with $r = 2$ and $r = 4$ are compared in figure 5. The individual times for the lower level and the desired final grid are compared with the times presented in table 1, which stand for 100% in each mesh density case. Two types of lower level meshes are considered for each desired mesh density, namely a coarser mesh with half (on each direction) the points of the desired mesh, i.e. $r = 2$, and a second one with a quarter (on each direction) of the points, i.e. $r = 4$. The computing times for each $r$ are represented as stacked columns to allow easy comparison with the fixed mesh case.

| Mesh density $N_1 \times N_2$ | 256×256 | 512×512 | 1024×1024 | 2048×2048 | 4096×4096 |
|-------------------------------|---------|---------|-----------|-----------|-----------|
| Running time, [sec]           | 0.41    | 2.62    | 14.01     | 80.47     | 513.94    |

Figure 4. (a) Radial pressure distributions obtained with coarser meshes; (b) 3D pressure distribution.
The comparison is extended to the number of iterations performed to achieve the imposed precision (7). It should be noted that higher resolutions lead to smaller precision goals that require an increased number of iterations. The results depicted in figures 5 and 6 suggest that, although the number of iterations for the dual-grid scheme is higher than the one for the fixed mesh approach, important computing time savings can be achieved. These savings, approaching 50%, appear to favour the dual-grid scheme with $r = 2$, although in one case, i.e. $2048 \times 2048$, the opposite is true. It appears that, in the $r = 2$ case, more CPU time is spent on the coarser mesh, leading to a higher quality initial guess for the upper mesh level. In the case $r = 4$, the solution of the coarser mesh is achieved, as expected, very rapidly, but more time is spent on the finer mesh to achieve the precision goals. The goal of the dual-grid approach is, however, to move a consistent part of the computational burden to the coarser mesh. Providing a better initial guess for the desired level seems to take precedence in achieving better CPU total times. Whichever the case, the overall computing time is consistently smaller than for the fixed mesh classical algorithm. Considering these results, it is tantalising to investigate the efficiency of a multi-grid scheme with a fixed element ratio of 2 between subsequent grids.
5. Conclusions
Finding the contact pressure is a fundamental problem in machine elements design, and the complexity of real materials and contact conditions can only be accommodated by numerical analysis. The efficiency of a state-of-the-art algorithm for the contact of arbitrarily-shaped bodies whose materials can be described by complex constitutive equations, has been enhanced in this paper by proposing a dual-grid scheme that reduces the computing time with up to 50%.

The contact problem is first solved on a coarser mesh, having as initial guess the medium pressure over the considered computational domain, as in the original formulation. The obtained solution is then transferred to a finer mesh by interpolation and extrapolation, and then inputted as initial guess to a second instance of the contact solver, but this time applied to the finer mesh. Due to the quality of the initial guess prepared on the coarser mesh, the convergent distribution on the finer mesh is calculated in fewer iterations than with the standard initial guess. The overall number of iterations increases, but giving that the computing time per iteration is much smaller for the coarser mesh, the total simulation computing time is reduced by near a half.

The numerical experimentations suggest that an element size ratio of 2 works best, resulting in a more evenly distributed computing time between the lower and the desired mesh levels, and in a smaller total. Future research work may demonstrate even more computing time savings by means of a multi-grid approach.

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