Classification and Properties of Symmetry Enriched Topological Phases: A Chern-Simons approach with applications to $Z_2$ spin liquids

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We study 2+1 dimensional phases with topological order, such as fractional quantum Hall states and gapped spin liquids, in the presence of global symmetries. Phases that share the same topological order can then differ depending on the action of symmetry, leading to symmetry enriched topological (SET) phases. Here we present a K-matrix Chern-Simons approach to identify all distinct phases with Abelian topological order, in the presence of unitary or anti-unitary global symmetries. A key step is the identification of an edge sewing condition that is used to check if two putative phases are indeed distinct. We illustrate this method for the case of $Z_2$ topological order ($Z_2$ spin liquids), in the presence of an internal $Z_2$ global symmetry. We find 6 distinct phases. The well known quantum number fractionalization patterns account for half of these states. Phases also differ due to the addition of a symmetry protected topological (SPT) phase. Also, we allow for the unconventional possibility that anyons are exchanged by the symmetry. This leads to 4 additional phases with symmetry protected Majorana edge modes. Other routes to realizing protected edge states in SET phases are identified. Symmetry enriched Laughlin states and double semion theories are also discussed. Two surprising lessons that emerge are: (i) gauging the global symmetry of distinct SET phases can lead to states with the same topological order (ii) gauge theories with distinct Dijkgraaf-Witten topological terms may have the same topological order.

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References
quantum spin liquid. Recently, several examples of gapped spin liquids have appeared in numerical calculations of fairly natural spin 1/2 Heisenberg models, on the Kagome and square lattice (with nearest and next neighbor exchange). Calculations of entanglement entropy point to $Z_2$ topological order. However, the precise identification of these phases require understanding the interplay between topological order and symmetry in these systems. The symmetries include both on-site global spin rotation and time reversal symmetries, as well as the space group symmetries of the lattice. Kagome lattice antiferromagnets, such as herbertsmithite, may provide experimental realization of this physics, although experimental challenges arising from disorder and residual interactions continue to be actively studied. This motivates the study of distinct topologically ordered phases that may arise in the presence of symmetry.

In the presence of symmetry, the structure of topological orders is even richer. The microscopic degrees of freedom in the system are either bosons or fermions, and they must form a linear representation of the symmetry group $G_s$. The emergent anyons, however, doesn’t need to form a linear representation of $G_s$. Instead they could transform projectively under symmetry operation, i.e. each of them can carry a fractional quantum number of symmetry. For example, the Laughlin FQH states at filling fraction $\nu = 1/m$, where each elementary quasiparticle carries a fractional ($1/m$) of the electron charge. This phenomena is widely known as fractionalization, although a more appropriate name is perhaps symmetry fractionalization. And the associated symmetry in Laughlin states is the $U(1)$ charge conservation of electrons. While the emergent quasiparticles transform projectively (instead of linearly), the microscopic degrees of freedom always transform linearly under symmetry, simply because each microscopic degrees of freedom can be regarded as a conglomerate of multiple emergent quasiparticles.

Even in the absence of topological order, when symmetry $G_s$ is preserved, different symmetry protected topological (SPT) phases emerge which are separated from each other through phase transitions. These SPT phases feature symmetry protected boundary states which will be gapless, unless symmetry $G_s$ is (spontaneously or explicitly) broken on the boundary. Well known examples of SPT phases are topological insulators and superconductors. In 2+1-D all SPT phases have symmetry protected non-chiral edge modes.

The existence of SPT phases further enrich the structure of topological orders in the presence of symmetry. In other words, topologically ordered phase is not fully determined by how its (anyon) quasiparticles transform (projectively or not) under symmetry: its microscopic degrees of freedom could form a SPT state in parallel with the topological order. The formation of SPT state will e.g. bring in new structures to the edge states of the topologically ordered system, and lead to a distinct symmetry enriched topological (SET) order. Therefore, two different SET phases sharing the same topological order can differ by the symmetry transformation on their anyon quasiparticles, or by their distinct boundary excitations. Clearly we have a question here: given two states sharing the same topological order while preserving symmetry $G_s$, can they be continuously connected to each other without a phase transition, if symmetry $G_s$ is preserved?

We address this issue for 2+1-D (Abelian) topological orders. Focusing on on-site (instead of spatial) symmetries, we present a universal criterion (Criterion I in section IIE) related to the edge states of these 2+1-D SET phases, which works for both unitary and anti-unitary on-site symmetries. The physical picture behind this criterion is demonstrated in FIG. 1. Two SET phases #1 and #2 living on the two cylinders are considered the same if they can be smoothly connected together via tunneling of microscopic degrees of freedom between the two edges. Distinct SET phases on the other hand, present an obstruction to such a smooth sewing.

The above criterion allows us to clarify the structure of symmetry enriched topological (SET) orders in 2+1-D. The method we follow is the Chern-Simons approach, which provides a unified description for low-energy bulk and edge properties of a generic Abelian topological order, in 2+1-D. In particular the bulk-edge correspondence in Chern-Simons approach enables us identify all edge excitations with their bulk counterparts, such as the microscopic degrees of freedom (bosons/fermions) and anyons. Therefore the above criterion of smooth sewing boundary conditions for two dif-

![Figure 1](image)

**Figure 1:** (color online) Edge Sewing Criterion to distinguish symmetry enriched topological (SET) phases. Only the microscopic degrees of freedom i.e. "electrons" (and not gauge charged objects such as anyons/fractionalized quasiparticles) can tunnel between the two edges of a pair of semi-infinite cylinders. If two SET phases can be continuously tuned into one another without a phase transition (while preserving symmetry), there is a "smooth" sewing between the two cylinders of SET phases #1 and #2. This implies that all edge excitations are gapped by a few symmetry-allowed terms that tunnel "electrons" between the two edges. In the thermodynamic limit these tunneling terms lead to $M$ degenerate ground states, corresponding exactly to the $M$-fold torus degeneracy of the topological order. On the other hand, if the two SET phases are different, there is no such "smooth" boundary condition to sew the two edges. A precise version of this statement is formulated in Criterion I in Section IIE.
ferent SET phases can be made precise within the Chern-Simons approach (see section IIE).

More concretely, a 2+1-D Abelian topological phase is fully characterized by a symmetric integer matrix $K$ in the Chern-Simons approach. When symmetry $G_s$ is preserved in the system, the anyons could carry a fractional symmetry quantum number (or transform projectively under the symmetry $G_s = \{ g \}$), while the microscopic degrees of freedom (bosons/fermions) must form linear representations of the symmetry group $G_s = \{ g \}$. The relation between microscopic degrees of freedom and fractionalized anyon excitations is especially clear in the Chern-Simons approach.

Based on the mentioned Criterion I to differentiate distinct SET phases, we can classify all different SET phases with the same topological order $\{ K \}$ and symmetry $G_s$. In this work, we studied various examples: $Z_2$ spin liquids\cite{29}, double semion theory\cite{31,32} and bosonic/fermionic Laughlin states\cite{33} at filling fraction $\nu = 1/m$. We consider both anti-unitary time reversal symmetry considered here. These “unconventional” SET phases. For example, under on-site unitary $Z_2$ symmetry, the two anyons i.e. the electric charge $e$ and magnetic charge $m$ of a $Z_2$ spin liquids are exchanged (see TABLE III). Previously, such a transformation law was considered in the Wen plaquette model\cite{32,34} for translation symmetry, in contrast to the internal $Z_2$ symmetry considered here. These “unconventional” SET phases have some striking properties. First, the edge features gapless Majorana edge modes that are protected by symmetry. Next, if $Z_2$ symmetry is broken at the edge, then a Majorana fermion is trapped at the edge domain wall. Finally, as illustrated in FIG. 2 when a pair of electric charge ($e$) is created at opposite sides of a sphere, we can divide the system into two subsystems $A$ and $B$, so that there is one electric charge $e$ localized in each subsystem. Now if we perform the $Z_2$ symmetry operation only in subsystem $A$ (flip all the spins), the electric charge $e$ therein will become a magnetic charge $m$. Since an electric charge $e$ and a magnetic charge $m$ differs by a fermion $f (e \times f = m$ or $m \times f = e)$ in the $Z_2$ spin liquid, this means a fermion mode $f$ must simultaneously appear at the boundary separating subsystem $A$ and $B$, as the Ising symmetry is acted on $A$. This is discussed in Section IIIB.2 and Section IIIB.3.

Symmetry Protected Edge States: In general, the non-chiral topological orders, like $Z_2$ topological order and double semion models, do not have gapless excitations at the edge. However, these may appear with additional symmetry. Indeed, the ‘unconventional’ $Z_2$ SET phases have Majorana edge states. Since they are protected by an onsite $Z_2$ symmetry, they are stable even in the presence of disorder that breaks translation symmetry along the edge. Two further mechanisms for gapless edge modes in ‘conventional’ SET phases may be identified. The first is the trivial observation that adding an SPT phase leads to a corresponding protected edge state. The second mechanism operates when both the electric and magnetic particle of the $Z_2$ gauge theory transforms projectively under symmetry. Then, one cannot condense neither of them at the edge - implying a protected edge. Details appear in Sec IIIF.

Gauging Symmetry: A powerful tool in studying the effect of an onsite unitary symmetry $G_s$ is the consequence of gauging it\cite{17,25}. This means the global $G_s$ symmetry is promoted to a local “gauge symmetry”, which leads to new topological orders. Distinct topological orders can help distinguish different actions of the symmetry in the ungauged theory. By this procedure in 2+1-D, nonlinear sigma models with topological terms, which describe SPT phases\cite{20} can be mapped to gauge theories with a topological term\cite{17,25}, discussed by Dijkgraaf and Witten\cite{25}. In this work we systematically study the consequences of gauging unitary on-site symmetry in Abelian SET phases. It turns out for “conventional” (type-I) SET phases, the new topological order obtained by gauging symmetry is always Abelian and Chern-Simons theory is a natural framework to derive it. In contrast to the belief that different ‘conventional’ SET phases always lead to different topological orders after gauging its symmetry, we found that many different SET phases (with a common topological order and symmetry group $G_s$) can lead to the same topological order once the symmetry is gauged\cite{20}, as shown in TABLE II and IV. Somewhat surprisingly, this furnishes examples where a gauge theory with two distinct Dijkgraaf-Witten\cite{25} topological terms, correspond to the same topological order. Here the topological terms arising for the gauge group $Z_2 \times Z_2$ are obtained by gauging SPT phases and correspond to elements of $H^3(Z_2 \times Z_2, U(1))$. Theories for distinct elements are shown to be equivalent on relabeling quasiparticles (an $SL(4, Z)$ transformation). Therefore the distinction between these theories requires additional information such as specification of electric vs. magnetic charges (Section IIIB.3).

For “unconventional” (type-II) SET phases, however, gauging the symmetry leads to non-Abelian topological orders. For example the unconventional Ising-symmetry-enriched $Z_2$ spin liquids, after gauging the Ising ($G_s = Z_2$) symmetry, lead to non-Abelian topological orders with 9-fold GSD on a torus. Interestingly, they can be naturally embedded within Kitaev’s 16-fold way classification\cite{20} of 2+1-D $Z_2$ gauge theories (see TABLE III and VI). In this case a vertex algebra approach\cite{32} can be introduced to extract all information of the non-
Abelian topological order. In particular after gauging the on-site Ising ($Z_2$) symmetry, a new quasiparticle $q_a$ (called $Z_2$ vortex) emerges as deconfined excitations. It is a non-Abelian anyon in the unconventional (type-II) SET case, which corresponds to the edge domain wall bound state in FIG. 3.

Spin 1/2 From K-Matrix CS Theory: We demonstrate how an emergent “spin 1/2” excitation can be realized in the Chern Simons formalism, by studying $Z_2$ gauge theories with $Z_2 \times Z_2$ symmetry. The latter has a projective representation that can protect a two fold degenerate state, analogous to spin 1/2. This is accomplished by expanding the $2 \times 2$ K-matrix of a $Z_2$ gauge theory to a $4 \times 4$ matrix by adding a trivial insulator. Symmetry transformations implemented in this expanded space have the desired properties (in Section IIIE).

Connection to Other Work: A symmetry based approach was used to classify $Z_2$ spin liquids in Ref.[13]. An advantage of that approach is that it treated both internal and space group symmetries. However, topological distinctions and the appearance of edge states are not captured. Also, the ‘unconventional’ approach was used to classify both internal and space group symmetries. However, this approach was used to classify Abelian topological orders. In particular after gauging 2D, but only as the surface state of a 3D SPT phase (iii) double semion theory with unitary Ising ($Z_2 \times Z_2$) symmetry, a new quasiparticle can be realized in 2D, but only as the surface state of a 3D SPT phase. Our K-matrix approach does not produce such states.

This paper is organized as follows. In Section II, we introduce the Chern-Simons K-matrix approach to (Abelian) symmetry enriched topological (SET) phases in 2+1-D. Rules for implementing on-site symmetry in a topologically ordered phase are discussed in Section III with criteria to differentiate distinct SET phases in Section II. Next, in Section III, we demonstrate our approach by classifying SET phases in a few examples. They include: (i) $Z_2$ spin liquid with (anti-unitary) time reversal symmetry ($G_a = Z_{2}^{T}$) symmetry (section IIIA) TABLE 1, (ii) $Z_2$ spin liquid with unitary Ising ($G_a = Z_2$) symmetry (section IIIA) TABLE 1 and II, (iii) double semion theory with unitary Ising ($G_a = Z_2$) symmetry (Appendix C TABLE IV) and (iv) even-denominator bosonic Laughlin state with unitary Ising ($G_a = Z_2$) symmetry (section IIIA) TABLE 1 and IIIA. Appendix III and IV provide a detailed instruction on how to gauge unitary symmetries in 2+1-D SET phase.

II. CHERN-SIMONS APPROACH TO SYMMETRY ENRICHED ABELIAN TOPOLOGICAL ORDERS IN 2+1-D

A. Chern-Simons theory description of 2+1-D Abelian topological orders

In two spatial dimensions, a generic gapped phase of matter is believed to be described by a low-energy effective Chern-Simons theory in the long-wavelength limit. Examples include integer and fractional quantum Hall state,38–42 gapped quantum spin liquid,43–45 and topological insulators/superconductors. When we restrict ourselves to the case of gapped Abelian phases where all the elementary excitations in the bulk obey Abelian statistics,46–50 a complete description is given in terms of Abelian $U(1)^N$ Chern-Simons theory.51 To be specific, the low-energy effective Lagrangian of $U(1)^N$ Chern-Simons theory has the following generic form

$$\mathcal{L}_{CS} = \frac{\epsilon_{\mu\nu\lambda}}{4\pi} \sum_{I,J=1}^{N} a_{\mu}^{I} K_{I,J} \partial_{\nu} a_{\lambda}^{J} - \sum_{I=1}^{N} a_{\mu}^{I} j_{I}^{\mu} + \cdots$$

where $\mu, \nu, \lambda = 0, 1, 2$ in 2+1-D and summation over repeated indices are always assumed. Here $\cdots$ represents higher-order terms, such as Maxwell terms $\sim (\epsilon_{\mu\nu\lambda} \partial_{\lambda} a_{\mu})^2$. $K$ is a symmetric $N \times N$ matrix with integer entries. Notice that the $U(1)$ gauge fields $a_{\mu}^{I}$ are all compact in the sense that they are coupled to quantized gauge charges with currents $j_{I}^{\mu}$. In the first quantized language the quantized quasiparticle currents $j_{I}^{\mu}$ are written as

$$\forall \ I = 1, \cdots, N : \ j_{I}^{\mu}(r) = \sum_{n} j_{I}^{\alpha}(n) \delta(r - r^{(n)}),$$

$$j_{I}^{\alpha}(n) = \sum_{\alpha} j_{I}^{\alpha}(n) \delta(r - r^{(n)}), \quad \alpha = 1, 2,$$

where $r^{(n)} = (r^{(n)}_1, r^{(n)}_2)$ denotes the position of the n-th quasiparticle, and gauge charges $j_{I}^{\alpha}(n)$ are all quantized as integers. We can simply label the n-th quasiparticle by its gauge charge vector $I^{(n)} = (I^{(n)}_1, \cdots, I^{(n)}_N)^T$. The self(exchange) statistics of a quasiparticle $I$ is given by its statistical angle

$$\theta_{I} = \pi I^T K^{-1} I, \quad I \in \mathbb{Z}^N.$$ (2)

while the mutual(braiding) statistics of a quasiparticle $I$ and $I'$ is characterized by

$$\tilde{\theta}_{I,I'} = 2\pi I^T K^{-1} I', \quad I, I' \in \mathbb{Z}^N.$$ (3)

The above statistics comes from the nonlocal Hopf Lagrangian of currents $j_{I}^{\mu}$ obtained by integrating out the gauge fields $a_{\mu}^{I}$ in (1). A simple observation from is that for a quasiparticle excitation with gauge charge

$$\tilde{I} = K I, \quad I \in \mathbb{Z}^N.$$ (4)
its mutual statistical with any other quasiparticle \( y \) is a multiple of \( 2\pi \). In other words, the quasiparticles \( I = K l \) are local[12] with respect to any other quasiparticles \( y \). Therefore they are interpreted as the “gauge-invariant” microscopic degrees of freedom in the physical system: such as electrons[13] in a fractional quantum Hall state, and spin-1 magnons in a spin-1/2 \( Z_2 \) spin liquid[12]. Another direct observation is that when all diagonal elements of matrix \( K \) are even integers, the microscopic degree of freedom have bosonic statistics \( \theta = 0 \mod 2\pi \), and \( \{ 1 \} \) describes a bosonic system. When at least one diagonal elements of \( K \) are odd integers, there are fermionic microscopic degrees of freedom in the system.

The ground state degeneracy (GSD), as an important characteristic for the topologically ordered phase described by effective theory \( \{ 1 \} \),

\[
\text{GSD} = |\det K|^g.
\]

on a Riemann surface of genus \( g \). On the torus with \( g = 1 \), the corresponding GSD= \( |\det K| \) also equals the number of different anyon types (or the number of superselection sectors) in a two-dimensional topological ordered system. A simple picture is the following: any two anyons differing by a (local) microscopic excitations are the same (or more precisely, belong to the same superselection sector) in the sense that they share the same braiding properties:

\[
I' \simeq I'' \Leftrightarrow I - 1'' = K I, \quad I', I'' \in \mathbb{Z}^N.
\]

Therefore different quasiparticle types correspond to inequivalent integer vectors \( I \in \mathbb{Z}^N \) in a \( N \)-dimensional lattice, where the Bravais lattice primitive vectors are nothing but the \( N \) column vectors of matrix \( K \). As a result \( |\det K| \), the volume of the primitive cell in \( 1 \)-space, counts the number of different quasiparticle types (or superselection sectors) in a topologically ordered system described by effective theory \( \{ 1 \} \).

B. Edge excitations of an Abelian topological order

There is a bulk-edge correspondence[11,12] for effective theory \( \{ 1 \} \). When put on an open manifold \( M \) with a boundary \( \partial M \), the gauge invariance of effective Lagrangian \( \{ 1 \} \) implies the existence of edge states on the boundary \( \partial M \). The \( N \) chiral boson fields \( \phi_I = \phi_I + 2\pi |I|, 1 \leq I \leq N \) capture the edge excitations. To be specific, assuming the manifold \( M \) covers the lower half-plane \( r_2 < 0 \), then edge excitations localized on the boundary \( \partial M = \{(r_1, r_2)| r_2 = 0 \} \) has the following effective Lagrangian

\[
\mathcal{L}_{\text{edge}} = \frac{1}{4\pi} \sum_{I, J} \left( K_{I, J} \partial_0 \phi_I \partial_t \phi_J - V_{I, J} \partial_1 \phi_I \partial_t \phi_J \right).
\]

where \( rE \) stands for the right edge. On the other hand, if the manifold \( M \) instead covers the upper half-plane \( r_2 > 0 \), the corresponding edge theory becomes

\[
\mathcal{L}_{\text{edge}} = -\frac{1}{4\pi} \sum_{I, J} \left( K_{I, J} \partial_0 \phi_I \partial_t \phi_J + V_{I, J} \partial_1 \phi_I \partial_t \phi_J \right).
\]

where \( lE \) means left edge here. \( V \) is a positive-definite real symmetric matrix, determined by microscopic details of the system. The edge effective theories \( \{ 1 \} \) imply the following Kac-Moody algebra[14] of chiral boson fields:

\[
[\phi_I(x), \partial_y \phi_J(y)] = \pm 2\pi l_{I, J} \delta(x - y).
\]

where \( +(-) \) sign corresponds to the right(left) edge. The signature \( (n_+, n_-) \) of matrix \( K \) now has a clear physical meaning from \( \{ 5 \} \) to \( \{ 6 \} \): each positive(negative) eigenvalue of \( K \) corresponds to a right-mover (left-mover) on the right edge \( \{ 5 \} \) and a left-mover (right-mover) on the left edge \( \{ 6 \} \).

Similar to the quasiparticle excitations in the bulk labeled by their gauge charge \( I \), associated quasiparticles on the edge \( V_I = \exp(i \sum_{I} l_{I} \phi_I) \) are also labeled by an integer vector \( I = (I_1, \ldots, I_N)^T \). This identification between bulk quasiparticle \( I \) and edge excitations \( V_I \) indicates that each (local) microscopic degree of freedom \( \{ 1 \} \) in the bulk also has a correspondent local excitation on the edge: \( V_I = V_{KI} \). For a \( N \times N \) matrix \( K \), all these local excitations on the edge are composed of the following \( N \) independent local excitations (microscopic degrees of freedom on the edge):

\[
e^{i \sum_{I_J} K_{I, J} \phi_J(x, t)}, \quad 1 \leq I \leq N.
\]

In the context of fractional quantum Hall states, these local operators on the edge are called[11,12] “electron operators”.

Here let’s go over the simplest case with no symmetry, by symmetry group \( G_s = \{ e \} \) and \( e \) denotes the identity element of a group. In this case all the (local) microscopic boson degrees of freedom can condense in the bulk, and accordingly on the edge the following Higgs terms can be added to Lagrangian \( \{ 5 \} \) to \( \{ 6 \} \):

\[
\mathcal{L}_{\text{Higgs}} = \sum_I C_I \left( e^{i \chi_I} M_I + \text{h.c.} \right) = \sum_{I=1}^N C_I \cos \left( p_I \sum_{J} K_{I, J} \phi_J(x, t) + \chi_I \right).
\]

where \( C_I \) and \( \chi_I \) are all real parameters. Notice that constant factor

\[
p_I \equiv (3 - (-1)^{K_{I, I}})/2, \quad 1 \leq I \leq N.
\]

guarantees the self statistics[2] of local quasiparticle

\[
\tilde{M}(x, t) \equiv e^{i p_I \sum_J K_{I, J} \phi_J(x, t),} \quad 1 \leq I \leq N.
\]

is bosonic, since if \( \bar{M} \) is fermionic the Higgs term \( \{ 8 \} \) will violate locality. The Abelian topological order (featured by GSD on genus-\( g \) Riemann surfaces and anyon statistics) will not be affected by these Higgs terms[20,21,22,23].
since all anyon excitations are local with respect to the microscopic bosonic degrees of freedom. As a result the condensation of local bosonic degrees of freedom \{\mathcal{M}_t\} will not trigger a phase transition, when there is no symmetry in the Abelian topological order. Hence in a general ground these Higgs terms \([9]\) should be included in the low-energy effective theory \([1,5,8]\) of an Abelian topological order, in the absence of any symmetry.

Since Higgs terms \([8]\) are generally present in the edge effective theory (when there is no symmetry), they will introduce backscattering processes on the edge. A natural question is the stability of gapless edge excitations\([22]\). When \(n_+ \neq n_-\) for the signature \((n_+, n_-)\) of matrix \(K\), there is a net chirality for edge states \([6,0]\) and they cannot be fully gapped out by the Higgs terms \([8]\). A physical consequence is a nonzero thermal Hall conductance in the system\([33]\). If \(n_+ = n_-\) on the other hand, there is no net chirality on the edge. But this doesn’t mean the edge states can be gapped out by Higgs term \([8]\): the simplest counterexample is \(K = \begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}\), whose edge cannot be gapped out even in the absence of any symmetry. When the system preserves symmetry \(G_s\), the structure of edge states is richer. Typically some Higgs terms in \([8]\) will be forbidden by symmetry, and there will be symmetry-protected edge excitations\([20,21,22]\) in the Abelian topological order. In other words certain branches of edge excitations will either remain gapless when symmetry \(G_s\) is preserved, or become gapped out when symmetry \(G_s\) is spontaneously broken on the edge. For a general discussion on the stability of edge modes in an Abelian topological order we refer the readers to section III of Ref.\([26]\). For the SET phases studied in this work, their edge stabilities are briefly discussed in section IIIIF.

C. Different Chern-Simons theories can describe the same topological order

For symmetric unimodular \(K\) matrix with \(\det K = \pm 1\), the ground state of system \([1]\) is unique on any closed manifold. Consistent with the nondegenerate ground state on torus, any quasiparticle \(I\) is either bosonic or fermionic with trivial mutual statistics with each other. Hence there is no topological order in the system\([11,26]\) when \(\det K = \pm 1\). However the corresponding gapped phase can still have gapless chiral edge modes on its boundary, which are stable against any perturbations. Well-known examples are the integer quantum Hall effects where \(K\) is an \(N \times N\) identity matrix. On the other hand, if

Trivial phase: \(\det K = (-1)^N/2\), \(N = \dim K = \text{even}(10)\)

the edge excitations will be non-chiral (the same number of right- and left-movers) and are generally gapped in the absence of symmetry\([26]\). In these cases we call the corresponding phase a trivial phase in 2+1-D, since it’s featureless both in the bulk and on the edge and can be continuously connected to a trivial product state without any phase transition\([20]\).

One key point we want to emphasize is that the Chern-Simons theory description for a certain topologically ordered phase is not unique. In other words, two different \(K\) matrices for effective theory \([1]\) can correspond to the same topological phase, with the same set of quasiparticle (anyon) excitations. The two features described below are crucial for the classification of symmetry enriched topological orders.

First of all, the following \(GL(N,\mathbb{Z})\) transformation on the \(K\) matrix yields an equivalent description for the same phase

\[
K \simeq \tilde{K} = X^T K X, \quad X \in GL(N,\mathbb{Z}).
\]  

(11)

where \(GL(N,\mathbb{Z})\) represents the group of \(N \times N\) unimodular matrices. This \(GL(N,\mathbb{Z})\) transformation \(X\) merely relabels the quasiparticle (anyon) excitations so that \(1 \rightarrow \tilde{1} = X^{-1} 1\). It’s straightforward to see that all the topological properties, such as quasiparticle statistics and GSD are invariant under such a \(GL(N,\mathbb{Z})\) transformation. A brief introduction to \(GL(N,\mathbb{Z})\) group is given in Appendix A.

Secondly, notice that a trivial phase satisfying \([10]\) can always be added to a topologically ordered phase without changing any topological properties (such as quasiparticle statistics, GSD and chiral central charge of edge excitations\([39]\)). One just needs to enlarge the Hilbert space to include some new microscopic degrees of freedom, which form a trivial phase. Mathematically addition of a topologically ordered phase with matrix \(K\) and a trivial phase with matrix \(K_t\) satisfying \([10]\) is carried out by the matrix direct sum\([26]\)

\[
\det K_t = (-1)^{N_t/2}, \quad N_t = \dim K_t = \text{even}. 
\]  

(12)

Therefore two \(K\) matrices of different dimensions can describe the same topologically ordered phase. Typically in a bosonic system (where the microscopic degrees of freedom are all bosons) the generic trivial phase is represented by\([11,26]\)

\[
K_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. 
\]  

(13)

Meanwhile in a fermionic system, both \([13]\) and

\[
K_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. 
\]  

(14)

together represent a generic trivial phase.

D. Implementing symmetries in Abelian topological orders

Our discussions in the previous section didn’t assume any symmetry\([26]\) in the topologically ordered phase.
Without any symmetry, an Abelian topological order is fully characterized by its \( K \) matrix. In the presence of symmetry, however, \( K \) matrix alone is not enough to describe a symmetry enriched topological (SET) phase: e.g. distinct SET phases that are separated from each other by phase transitions can share the same \( K \) matrix. The missing information is how the bulk quasiparticles (with currents \( j^I \)) in effective theory (1) transform under the symmetry. The corresponding information in the edge states (5)-(6) is how the chiral boson fields \( \{ \phi_I, 1 \leq I \leq N \} \) transform under symmetry.

We will restrict to unitary and anti-unitary onsite symmetries in this work. By onsite symmetries we mean the local Hilbert space is mapped to itself\(^{24}\) under the symmetry transformation, so that the symmetries act in a “on-site” fashion. In this case studying the symmetry transformations of bulk quasiparticles (with currents \( j^I \)) is equivalent to studying the symmetry transformations of edge chiral bosons \( \{ \phi_I, 1 \leq I \leq N \} \). Henceforth we’ll focus on the chiral boson variables on the edge to study their transformation rules under symmetry operations, in the presence of a symmetry group \( G_s \).

Most generally, under the operation of symmetry group element \( g \in G_s \), the chiral boson fields \( \{ \phi_I \} \) transform in the following way\(^{26}\):

\[
\begin{align*}
\phi_I(x, t) &\to \sum_J \eta^g W^g_{IJ} \phi_J(x, t) + \delta \phi^g, \\
\eta^g K &= (W^g)^T K W^g, \quad W^g \in GL(N, \mathbb{Z}).
\end{align*}
\]  

where \( \eta^g = +1(-1) \) for a unitary (anti-unitary) onsite symmetry. This is simply because under an anti-unitary symmetry operation (such as time reversal \( t \to -t \)) the Chern-Simons term \( e^{i\phi_I(x)} \) changes sign, and in order to keep the Lagrangian (1) in the bulk or (5),(6) on the edge invariant, \( K \) must change sign under the \( GL(N, \mathbb{Z}) \) rotation \( W^g \).

Notice that the above symmetry transformations \( \{ W^g, \delta \phi^g \} \) must be compatible with group structure of symmetry group \( G_s \). This provides a strong constraint on the allowed choices of \( GL(N, \mathbb{Z}) \) rotations \( \{ W^g \} \) and \( U(1) \) phase shifts \( \delta \phi^g \simeq \delta \phi^1 + 2\pi \). To be precise, the consistent conditions for symmetry transformations \( \{ W^g, \delta \phi^g \} \) on an Abelian topological order characterized by matrix \( K \) is summarized in the following statement:

The (nonlocal) quasiparticle excitations \( \{ \hat{Q}_I(x, t) \} \) transform projectively under symmetry group \( G_s \), while the (local) microscopic boson degrees of freedom \( \{ \tilde{M}_I(x, t) \} \) in (9) must form a linear representation of symmetry group \( G_s \). Here constant factor \( p_I = 1 \) if \( K_{I,I} = \) even, or \( p_I = 2 \) if \( K_{I,I} = \) odd.

In the following we’ll discuss why (local) microscopic boson degrees of freedom must form a linear representation of symmetry group \( G_s \). Imagine an Abelian topological ordered phase preserves symmetry \( G_s \). For simplicity let’s consider \( G_s = Z_2 = \{ g, e \} \) for an illustration. We denote the generator of the \( Z_2 \) group as \( g \). It satisfies the following \( Z_2 \) multiplication rule:

\[
 g \cdot g \equiv g^2 = e. 
\]  

And under this \( Z_2 \) symmetry operation \( g \) the edge chiral bosons transform as \( [5] \). Consider we weakly break the \( Z_2 \) symmetry without closing the bulk energy gap (no phase transition). Now \( Z_2 \) operation \( g \) is not a symmetry anymore and there is no symmetry in the system. Therefore all the local bosonic degrees of freedom \( \{ \tilde{M}_I(x, t) \} \) can condense and Higgs terms \([3]\) should be allowed. At the same time, notice that \( g^2 = e \) is still a “symmetry” of the system. When symmetry operation \( g \) act twice, its transformations on chiral bosons \( \tilde{\phi}(x, t) \) becomes  

\[
\begin{align*}
\tilde{\phi}(x, t) &\to g \eta^g W^g \tilde{\phi}(x, t) + \delta \tilde{\phi}^g \to \\
(\eta^g)^2 &\tilde{\phi}(x, t) + (1_N + \eta^g W^g) \delta \tilde{\phi}^g.
\end{align*}
\]  

where \( 1_N \in \mathbb{N} \) denotes an \( N \times N \) identity matrix. And we must require all Higgs terms \([5]\) with arbitrary parameters \( \{ C_I, \chi_I \} \) are allowed by “symmetry” \( g^2 = e \) in \( [17] \). In other words all the Higgs terms in \([5]\) must remain invariant when \( Z_2 \) operation \( g \) act twice as in \( [17] \! \). This means the argument of any cosine (Higgs) terms in (8) should remain invariant up to a \( 2\pi \) phase, leading to the following relation:

\[
PK(W^g)^2 = PK(1_N + \eta^g W^g) \delta \phi^g = 2\pi n.
\]  

where we defined \( N \times N \) diagonal matrix \( P_I = \delta_{I,J} \) and \( n = (n_1, \cdots, n_N)^T \in \mathbb{Z}^N \) is an integer vector. The above relation can be rewritten as

\[
\begin{align*}
(W^g)^2 &= 1_N + \eta^g K = (W^g)^T K W^g, \\
(1_N + \eta^g W^g) \delta \phi^g &= 2\pi (PK)^{-1} n, \quad n \in \mathbb{Z}^N.
\end{align*}
\]  

These are the group compatibility conditions on the symmetry transformation (15) for a \( Z_2 \) symmetry group \( G_s = \{ g, e \} \). These conditions will be applied in the examples later.

In a generic case, symmetry group \( G_s \) (and its multiplication table) is fully determined by a set of algebraic relations

\[
A_{m_1, \cdots, m_{N_g}} = g_{m_1} m_2 \cdots g_{N_g} = e.
\]  

where \( \{ g_1, \cdots, g_{N_g} \} \) is a set of generators in group \( G_s \). Each algebraic relation \( A_{m_1, \cdots, m_{N_g}} \) gives rise to a consistent condition with an integer vector \( n_{m_1, \cdots, m_{N_g}} \), just like (18) in the \( G_s = Z_2 \) case. When all these group compatibility conditions are satisfied, any local bosonic degrees of freedom

\[
\hat{B}_I(x, t) \equiv \exp \left( iLT PK \tilde{\phi}(x, t) \right), \quad 1 \in \mathbb{Z}^N.
\]
is invariant under symmetry operation $A_{m_1,\ldots,m_{N_g}}$. By definition they form a linear representation of the symmetry group $G_s$. On the other hand, a generic quasiparticle excitation

$$\tilde{V}_I(x,t) \equiv \exp \left(i T \tilde{\phi}(x,t)\right), \quad I \in \mathbb{Z}^N.$$ could still transform nontrivially under consecutive symmetry operations $A_{m_1,\ldots,m_{N_g}}$ (which equals identity in symmetry group $G_s$). Therefore these (fermionic or anyonic) excitations transform projectively under symmetry group $G_s$.

E. Criteria for different symmetry enriched topological orders

In the previous section we discussed the consistent conditions on the symmetry transformations on the quasiparticle excitations in an Abelian topological order. In terms of chiral boson fields $\tilde{\phi}(x,t)$ which captures the quasiparticle contents in an Abelian topological order, under symmetry transformations (15) (labeled by $\{W^g, \delta \tilde{\phi}^g | g \in G_s\}$ for symmetry group $G_s$), the (local) bosonic degrees of freedom (9) transform linearly while (nonlocal) anyonic degrees of freedom $\{e^{i \tilde{\phi}^s}\}$ can transform projectively. Together with matrix $K$ which contains all the topological properties, the following set of data

$$\{K, \{\eta^g, W^g, \delta \tilde{\phi}^g | g \in G_s\}\} \quad (19)$$

fully characterizes a symmetry enriched topological (SET) phase in the presence of symmetry group $G_s$.

A natural question is: is such a data a unique fingerprint for a SET phase? Can two different sets of data describe the same SET phase? Not surprisingly the answer is yes. A trivial example is discussed earlier when symmetry group is trivial $G_s = \{e\}$ and two different $K$ matrices corresponds to the same Abelian topological order. So how can we tell whether two sets of data (19) describe the same SET phase or not? In the following we’ll propose a few criteria, which thoroughly address this question.

The first criterion comes from the physical picture that there is no “smooth” boundary condition under which we can sew two different SET phases with the same topological order and symmetry group $G_s$. This is rooted in the fact that two different SET phases cannot be continuously (no phase transitions in between) connected to each other without breaking the symmetry. The above physical picture can be made more precise mathematically in the following way. First we require $K^L \simeq K^R$ describe the same Abelian topological order in the absence of symmetry, i.e., they have the same topological properties such as GSD ($|\det K^L| = |\det K^R|$) and quasiparticle statistics. This is because two SET phases are certainly different if they correspond to different topological orders when symmetry is broken.

Consider a left edge $[0]$ of SET phase $[K^L, \{\eta^g, W^g_L, \delta \tilde{\phi}^g | g \in G_s\}]$ and the right edge $[5]$ of SET phase $[K^R, \{\eta^g, W^g_R, \delta \tilde{\phi}^g | g \in G_s\}]$ are sewed together by introducing tunneling terms between the two edges (see FIG. 1). We denote the chiral boson fields as $\{\phi^I_L\}$ on the left edge and $\{\phi^I_R\}$ on the right edge. Notice that only microscopic degrees of freedom whose mutual statistics (3) with any quasiparticle are multiples of $2\pi$, can appear in the tunneling term between the right and left edges (20) as shown in FIG. 1. Therefore a general tunneling term has the following Lagrangian density

$$\mathcal{H}_{tunnel} = \sum_{\alpha} T_\alpha \cos \left( (I_0^L)^T K^L \tilde{\phi}^L - (I_0^R)^T K^R \tilde{\phi}^R + \varphi_\alpha \right),$$

where $T_\alpha, \varphi_\alpha$ are real parameters. According to Kac-Moody algebra (7) for the chiral bosons, the condition on the 2nd line means the variables in each cosine term of (20) commute with itself and can be localized at a classical value $\langle (I_0^L)^T K^L \tilde{\phi}^L - (I_0^R)^T K^R \tilde{\phi}^R \rangle$. Of course every tunneling term in (20) must be allowed by symmetry, i.e., they remain invariant under symmetry transformation (15). The edge states is fully gapped (20) if each chiral boson field $\phi^I_L/R$ is either pinned at a classical value or doesn’t commute with at least one variable of the cosine terms in (20). Notice that each cosine term in (20) must contain local operators from both edges.

Let’s take a look at the simplest case, when the two SET states share exactly the same set of data (19). In this case the tunneling term (20) essentially sews the left and right edge of the same SET phase. The smooth sewing between left and right edge basically tunnels the same local microscopic degrees of freedom $V_{K^L/R}$ of one edge with its counterpart $V_{K^R/I}$ on the other edge. The following tunneling term

$$\mathcal{H}_{tunnel}^0 = \sum_{I=1}^{N} T_I \cos \left( \sum_{J \neq I} (K_{IJ} \phi^I_J - \phi^J_I) + \varphi_I \right), \quad (21)$$

is allowed by symmetry $G_s$ and will gap out the edge states. Notice that all cosine terms commute with each other, so they can be minimized simultaneously. One important feature of the above tunneling terms is that there are $|\det K|$ inequivalent classical minima (20) for the $\{\phi^I_J - \phi^J_I\}$ variables of the cosine terms. In other words, the chiral bosons will be pinned at one of the $|\det K|$ classical values by the above tunneling terms. These $|\det K|$ have a one-to-one correspondence to the $|\det K|$ degenerate ground states on torus of the Abelian topological order here.

When the two sets of data, $[K^L, \{\eta^g, W^g_L, \delta \tilde{\phi}^g | g \in G_s\}]$ for the left edge and $[K^R, \{\eta^g, W^g_R, \delta \tilde{\phi}^g | g \in G_s\}]$ for the right edge correspond to two different SET phases, on the other hand, there is no way to smoothly sew the left and right edges together. In this case when tunneling term (20) allowed by symmetry $G_s$
is added, either certain chiral boson modes remain gapless or the number of classical minima is more than \(|\det K^L| = |\det K^R|\). And our first criterion is

**Criterion I:** Two sets of data \([K^L, \{\theta^g, W^g, \delta \vec{\phi}_L^g | g \in G_s}\}] \text{ (for the left edge)}\) and \([K^R, \{\theta^g, W^g, \delta \vec{\phi}_R^g | g \in G_s}\}] \text{ (for the right edge)}\) belong to the same SET phase if and only if there exists a tunneling term \([20]\) connecting the two edges, which gaps out all chiral boson fields and has \(|\det K^{L/R}|\)-fold degenerate classical minima.

This criterion applies universally to both unitary and anti-unitary symmetries (such as time reversal symmetry). When \(\det K^{L/R} = \pm 1\) it automatically reduces to the criterion for different symmetry protected topological (SPT) phases in the Chern-Simons approach. A direct consequence of Criterion I are the following two corollaries

**Corollary I:**

If the two sets of data share the same matrix \(K^L = K^R\) and all their local microscopic degrees of freedom \([2]\) transform in the same way under symmetry \(G_s\), then they belong to the same SET phase since their edges can be sewed together smoothly by term \([27]\).

Next, notice that a \(GL(N,\mathbb{Z})\) transformation \([11]\) can always be performed on a \(K\) matrix without changing the topological order. It simply relabels different quasiparticles. Besides, \(U(1)\) gauge transformations can always be performed on gauge fields \(a_i^J\) and chiral bosons \(\{\phi_I\}\). The most general gauge transformations that relabel quasiparticles have the following form

\[
\phi_I(x,t) \rightarrow \sum_j X_{I,J} \phi_J(x,t) + \Delta \phi_I, \quad X \in GL(N,\mathbb{Z}) \tag{22}
\]

where \(\Delta \phi_I \in [0,2\pi]\) are constants. We denote such a gauge transformation as \([X, \Delta \vec{\phi}]\). Under such a gauge transformation, the set of data \([19]\) changes as

\[
K \xrightarrow{[X, \Delta \vec{\phi}]} X^T K X, \tag{23}
\]

\[
\forall g \in G_s, \quad W^g \xrightarrow{[X, \Delta \vec{\phi}]} X^{-1} W^g X,
\]

\[
\delta \vec{\phi}^g \xrightarrow{[X, \Delta \vec{\phi}]} X^{-1} (\delta \vec{\phi}^g + (\eta^g W^g - 1_{N \times N}) \Delta \vec{\phi}),
\]

and \(\eta^g\) remains invariant. Here comes the second corollary

**Corollary II:** any two sets of data \([19]\) that can be related to each other by a gauge transformation \([23]\) correspond to the same SET phase.

Last but not least, an important lesson from studies of SPT phases is that there is a duality between SPT phases and gauge theories (or intrinsic topological orders). This duality is established by gauging the (unitary) symmetry \(G_s\) in the SPT phase, \(i.e.\) coupling the physical degrees of freedom (which transform under symmetry \(G_s\)) to a gauge field \([23]\) (with gauge group \(G_s\)). One conjecture is that different SPT phases with \(G_s\) symmetry always leads to distinct \(G_s\) gauge theories. Naively one would expect this also holds for SET phases \([11]\), two different \(G_s\)-symmetry-enriched \(G_g\) topological orders (gauge theories) will lead to different \(G\) topological orders (gauge theories) where \(G/G_g = G_s\), when the symmetry \(G_s\) is gauged. However as will be shown by the examples studied in this work, this is not true. In spite of that, the converse (inverse) statement is still true:

**Criterion II:** After gauging the unitary symmetry \(G_s\), if two SET phases (with symmetry \(G_s\)) lead to two different topological orders, they must belong to two distinct SET phases.

Criterion II only applies to unitary symmetries. As will become clear in the examples, in certain cases (which we call “unconventional” SET phases), gauging an Abelian symmetry in an Abelian topological order will lead to non-Abelian topological orders \([55]\).

In the following sections we will demonstrate these criteria by classifying different SET phases with (anti-)unitary \(Z_2\) symmetries. The Abelian topological orders that will be studied include \(Z_2\) spin liquid \([33,34]\) with \(K \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\), double semion theory \([33,34]\) with \(K \simeq \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\) and Laughlin states \([11]\) \((K \simeq m)\) at different filling fractions \(\nu = 1/m\). Among them \(Z_2\) spin liquid and double semion theory are non-chiral Abelian phases, in the sense that their edge excitations have no net chirality. And in the absence of symmetry their edge excitations will generically be gapped. On the other hand Laughlin states are chiral Abelian phases with quantized thermal Hall conductance.

**III. EXAMPLES**

In this section we’ll apply the Chern-Simons approach discussed in previous sections to various Abelian topological orders. We start by classifying \(Z_2\) spin liquid with time reversal symmetry \(G_s = Z_2^T\) and with a unitary \(Z_2\) gauge \(G_s = Z_2\). Usually by \(Z_2\) spin liquids people assume that there are certain fractionalized quasiparticles carrying spin quantum numbers, coined “spinons” and other fractionalized quasiparticles carrying no spin quantum numbers, coined “visons”. The mutual (braiding) statistics of a spinon and a vison is semionic \((\theta_{s,v} = \pi)\), while the self statistics of a vison is bosonic. A \(Z_2\) spin liquid has 4-fold GSD on a torus. All these topological properties are captured by the Chern-Simons theory \([1]\) with

\[
K \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{24}
\]
In the context of this work, we don’t assume spin rotational symmetry and vision/spinon generically cannot be distinguished from their quantum numbers. Despite this fact we still use the name “$Z_2$ spin liquid” to label this Abelian topological order. The 4 degenerate ground states on a torus correspond to the 4 superselection sectors, which are associated with the 4 inequivalent quasiparticles:

$$1 \simeq \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$e \simeq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad m \simeq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$  

where both $e$ and $m$ have bosonic self statistics and they denote electric and magnetic charge in a $Z_2$ gauge theory. $f$ is the bound state of an electric charge and a magnetic charge, with fermionic self statistic. $0$ corresponds to any local microscopic degrees of freedom, belonging to the vacuum sector. In the folklore of $Z_2$ spin liquid, a viron is $e$ (or $m$), and accordingly a bosonic spinon is $m$ (or $e$).

A. Classifying $Z_2$ spin liquids with time reversal symmetry

As a warmup we consider Abelian topological order [24] with symmetry group $Z_2^T = \{g, e = g^2\}$ with algebra [16]. Notice that the generator of $Z_2^T$ group, $g$ is an anti-unitary operation with $g^2 = -1$ in (15). In this case we rely on Criterion I and its corollaries to differentiate different $Z_2^T$-SET phases.

The associated group compatibility condition (18) for $G_s = Z_2^T$ in Abelian topological order (24) is

$$\begin{pmatrix} \eta^g & W^g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad n \in Z^2.$$  

The $W^g \in GL(2, Z)$ solution to the above conditions is $W^g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. However notice that the following $GL(2, Z)$ gauge transformations keep the $K$ matrix invariant:

$$\begin{pmatrix} \eta^g & W^g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Therefore $W^g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $W^g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ are equivalent, related by gauge transformation $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in (23). And we can fix the gauge by choosing $W^g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with symmetry $G_s = Z_2^T = \{g, e = g^2\}$

| Label | #1 | #2 |
|-------|----|----|
| $W^g$ | (1 0), (0 -1) | (1 0), (0 -1) |
| $\delta \phi^g$ | (1 0), (0, $\pi/2$) | (0, $\pi$) |
| Proj. Sym. | (m = (0,1)$^T$) | No | Yes |
| Gapless edges | No | No |

Table I: Classification of different $Z_2$ spin liquids with (anti-unitary) time reversal symmetry. There are two different SET phases: in SET phase #1 all quasiparticles in (25) transform linearly under $Z_2^T$ symmetry, while in SET phase #2 quasiparticle $m$ transforms projectively under $Z_2^T$ symmetry. The data set in the 2nd line completely characterize these SET phases. "Proj. Sym." is short for “projective realization of symmetry” in the table.

for time reversal operation $g$. Then solving the 2nd line of conditions (26) we obtain

$$n_2 = 0, \quad \delta \phi = \left( \frac{\phi_1}{2} \right) \mod 2\pi.$$  

We can always choose a gauge transformation $\{X = \begin{pmatrix} 1 & 2x \end{pmatrix}, \Delta \phi \}$ in (23) so that $\delta \phi_1 = 0$. Meanwhile since $1 = (2,0)^T$ and $\mathbf{1} = (0,2)^T$ are the local excitations in the system, according to Corollary II, $n_1 =$ all corresponds to the same SET phase. Meanwhile $n_1 =$ odd leads to another SET phase, which is distinctive from the $n_1 =$ even SET phase. This is because the magnetic charge $m \simeq (0,1)^T$ transforms projectively in $n_1 =$ odd phase but transforms linearly in the $n_2 =$ even phase under time reversal symmetry. It’s straightforward to check that there is no way to smoothly sew the two edges of $n_1 =$ even and $n_1 =$ odd SET phases by a time-reversal-invariant tunneling term (20), which has 4-fold degenerate classical minima. Therefore according to Criterion I they belong to two different SET phases. These are the only two different classes of $Z_2^T$-symmetry-enriched $Z_2$ spin liquids, as summarized in Table I.

Since the previous calculations are based on 2 $K$ matrix (24), it is natural to ask: what if we enlarge the dimension of $K$ matrix (12) by introducing the trivial part with (13)? In this new representation of the same Abelian topological order, will we get more SET phases or not? Notice that the trivial part (13) is nothing but the $K$ matrix for a bosonic SPT phase $2$ in 2+1-D. For an anti-unitary $Z_2^T$ symmetry, there is no nontrivial bosonic SPT phase in 2+1-D. This means the edge chiral bosons for the trivial parts can always be gapped out by introducing symmetry-allowed backscattering cosine (Higgs) terms, whose classical minima is pinned at a unique classical value since $|\det K_i| = 1$. According to Criterion I, in the presence of $Z_2^T$ symmetry, when the dimension
of $K$ is enlarged by adding the trivial parts, it will not introduce any new SET phases.

At the end we discuss the stability of edge excitations in the two SET phases. Notice that the chiral bosons $\{\phi_{1,2}\}$ transform as

$$\begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{pmatrix} \xrightarrow{g} \begin{pmatrix} -\phi_1(x,t) \\ \phi_2(x,t) + \frac{\alpha_1}{T} \pi \end{pmatrix}.$$\textsuperscript{15}

under time reversal operation $g$. As a result the edges can be completely gapped by introducing Higgs terms

$$\mathcal{H}_{\text{higgs}} = C \cos \left( 2\phi_1(x,t) \right).$$

which pins chiral boson field $\phi_1(x,t)$ to a classical value $\langle \phi_1(x,t) \rangle = 0$ or $\pi$, without breaking the time reversal symmetry. Therefore in general there are no gapless edge states for the two SET phases with $G_s = Z_2^T$.

Potentially, one could conceive of a phase where both electric and magnetic charges transform projectively under time reversal symmetry. However, such a phase is only possible as the surface state of a 3D SPT phase with time reversal symmetry.\textsuperscript{21} The K-matrix classification correctly reproduces the fact that this phase is forbidden.

### B. Classifying $Z_2$ spin liquids with onsite $Z_2$ symmetry

As discussed earlier, the reason why $2 \times 2$ matrix $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ is enough to describe all $Z_2^T$-symmetric $Z_2$ spin liquids is that there is no nontrivial $Z_2^T$-SPT phases of bosons in 2+1-D. In other words the possible trivial part\textsuperscript{13} that can be added to $K$ in \textsuperscript{12} doesn’t contribute to new structure to SET phases. However, for a unitary $G_s = Z_2$ symmetry, as will become clear later, there is a nontrivial bosonic SPT phase\textsuperscript{20-22} whose edge cannot be gapped without breaking the $Z_2$ symmetry. This $Z_2$-SPT phase can be understood in Chern-Simons approach with $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore we need to consider $4 \times 4$ matrix

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

as a generic (unitary) $Z_2$-symmetry-enriched $Z_2$ spin liquid. The group compatibility condition\textsuperscript{18} for the unitary $Z_2$ symmetry ($\eta^g = 1$) transformation

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

becomes

$$\begin{pmatrix} W^g \end{pmatrix}^2 = 1_{4 \times 4},$$

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} W^g \end{pmatrix}^T \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W^g,$$

$$(1_{4 \times 4} + W^g) \delta \phi^g = \pi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} n, \quad n \in \mathbb{Z}^4.$$\textsuperscript{23}

The gauge inequivalent solutions to $W^g$ is the following\textsuperscript{23}

$$W^g = 1_{4 \times 4}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1_{2 \times 2}.$$\textsuperscript{24}

We will discuss these two cases separately in the following.

#### 1. “Conventional” $Z_2$-symmetry-enriched $Z_2$ spin liquids

First we discuss the solution $W^g = 1_{4 \times 4}$. In this case the anyon quasiparticles\textsuperscript{25} in the $Z_2$ spin liquids merely obtains a $U(1)$ phase factor under the $Z_2$ symmetry operation $g$, and we call them “conventional” SET phases. Due to the gauge transformations $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1_{2 \times 2}, 1_{2 \times 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which leave the $K$ matrix invariant, we know that the integer vector $n$ in \textsuperscript{27} has the following equivalence relation

$$n \sim \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} \approx \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} \sim \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} \sim \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}.$$\textsuperscript{25}

Moreover Corollary II tells us

$$n \sim \begin{pmatrix} n_1 + 2 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} \sim \begin{pmatrix} n_1 + 2 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} \sim \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 + 2 \end{pmatrix} \sim \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 + 2 \end{pmatrix}.$$\textsuperscript{25}

and

$$n \sim \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} \sim \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} \sim \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}.$$\textsuperscript{25}

As a result we obtain 6 inequivalent SET phases with $W^g = 1_{4 \times 4}$ under $Z_2$ symmetry, with their symmetry transformations $\delta \phi^g$ summarized in TABLE II. We require $\delta \phi \neq 0$ so that the local excitations\textsuperscript{4} form a faithful representation\textsuperscript{26} of symmetry group $G_s$. In the following we briefly discuss the consequence of gauging the
unitary $Z_2$ symmetry. A detailed prescription of gauging a unitary symmetry in the Chern-Simons approach is given in Appendix B where we’ve shown that gauging $Z_2$ symmetry in the Chern-Simons approach yields an Abelian topological order described by matrix $K_g$ in (B2). Take #6 for an example, after gauging $Z_2$ symmetry we have

$$K_g \simeq \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ 2 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

where the first equivalency $\simeq$ is realized by gauge transformation (23) with

$$X = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 3 & 1 & 1 \\ 2 & -2 & 0 & -1 \end{pmatrix}, \quad \det X = 1.$$

From TABLE I one can see that two different SET phases can lead to the same (intrinsic) topological order by gauging their $Z_2$ symmetry, such as #3 and #4, or #5 and #6.

The stability of edge excitations is also summarized in TABLE I. For SET phases #2 and #4 the gapless edge excitations come from the trivial part (lower $2 \times 2$ part) of $K$ matrix, which corresponds to the symmetry protected edge modes of bosonic $Z_2$-SPT phases. However for #5 and #6, the topologically ordered part (upper $2 \times 2$) of $K$ matrix also contribute to $c = 1$ gapless edge states. In other words in a $Z_2$ spin liquids, if both $e$ and $m$ transform projectively under $Z_2$ symmetry, the edge excitations is protected to be gapless unless symmetry is broken. The edge chiral bosons $\{\phi_{1,2}\}$ for $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ can be refermionized as right-moving branch $\psi_R \sim \exp [i(\phi_1 + \phi_2)]$ and left-moving branch $\psi_L \sim \exp [i(\phi_1 - \phi_2)]$. The edge effective theory [5] can be rewritten as

$$\mathcal{L}_{rE} = i\psi_R^\dagger (\partial_t - v_+ \partial_x)\psi_R - i\psi_L^\dagger (\partial_t - v_- \partial_x)\psi_L.$$ (28)

where $v_\pm = (V_{1,2} \pm \sqrt{V_{1,2}^2 + V_{1,1} V_{2,2}})/2$ are the velocities of edge modes. It’s easy to see that under $Z_2$
Table III: Classification of “unconventional” $Z_2$ spin liquids enriched by onsite (unitary) $G_s = Z_2$ symmetry. There are 4 different “conventional” SET phases, where under $Z_2$ symmetry quasiparticles $c$ and $m$ will exchange. The data set in the 2nd line completely characterizes these SET phases. $K_g$ denotes the topological order, which is obtained by gauging the unitary $G_s = Z_2$ symmetry in the $Z_2$ spin liquid. All these SET phases have $Z_2$ symmetry protected edge states, which will be gapless unless $Z_2$ symmetry is spontaneously broken. However, the central charge $c$ of (symmetry protected) gapless edge states are different. $q_g$ is the new quasiparticle excitation (the $Z_2$ vortex) obtained by gauging the “unconventional” $Z_2$ symmetry in the system, as described in Appendix D. Gauging this “unconventional” $Z_2$ symmetry leads to new non-Abelian quasiparticles (blue entries), which has quantum dimension $d_{q_g} = \sqrt{2}$ and topological spin $\exp(2\pi i h_{q_g})$. The quasiparticle contents of the non-Abelian topological orders obtained by gauging $Z_2$ symmetry is summarized in TABLE IV. The “gauged” non-Abelian topological orders for all these 4 SET phases have 9-fold GSD on a torus, corresponding to 9 different superselection sectors.

2. “Unconventional” $Z_2$-symmetry-enriched $Z_2$ spin liquids

Here we focus on solutions to (27) with $W^g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1_{2 \times 2}$. Notice that one can always choose a proper gauge $\Delta \Phi$ in (23) so that $\delta \Phi^g_1 = \delta \Phi^g_2$, and hence $n_1 = n_2$ in (27). There are 4 inequivalent $\delta \Phi^g$ solutions of this type to (27), as summarized in TABLE III. We can see that the electric charge $e$ and magnetic charge $m$ form a two-dimensional representation of $Z_2$ symmetry group. They transform projectively in SET phases #3 and #4, or transform linearly in SET phases #1 and #2. All these phases host symmetry protected edge excitations on the boundary, but the structure of these gapless edge states (when symmetry is preserved) are different.

For SET phase #1 and #3, the edge chiral bosons $\phi_{3,4}$ for the trivial $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ part of $K$ matrix can be fully gapped by a cos $\phi_4$ term. However, the chiral bosons $\phi_{1,2}$ for topologically ordered $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ part of $K$ are protected by $Z_2$ symmetry. To be precise, in the refermionized description (28) for edge states, the chiral fermions transform as

$$\psi_R \xrightarrow{g} (-1)^{n_1} \psi_R, \quad \psi_L \xrightarrow{g} \psi_L^\dagger.$$ (29)

for $\delta \Phi^g = \pi(n_1/2, n_1/2, 1, n_3)$ with $n_{1,3} = 0, 1$ in TABLE III. We can rewrite each chiral fermion in terms of two Majorana fermion $\xi_{R/L}$ and $\eta_{R/L}$

$$\psi_{R/L} \equiv \xi_{R/L} \pm i \eta_{R/L}.$$
backscattering term is allowed by $Z_2$ symmetry
\[ \mathcal{H}_{bs} \propto \psi_R (\psi_L + \psi^\dagger_L) + h.c. = 4\xi_R \xi_L. \]

Therefore the $\xi_{R/L}$ branch of Majorana fermions are gapped, while the $\eta_{R/L}$ branch is protected by $Z_2$ symmetry. For phases #2 and #4 with $n_1 = 1$, similarly the following backscattering term
\[ \mathcal{H}_{bs} \propto \psi_R (\psi_L - \psi^\dagger_L) + h.c. = 4\eta_R \eta_L. \]
is allowed by $Z_2$ symmetry. It will gap out $\eta_{R/L}$ modes and leave Majorana modes $\xi_{R/L}$ gapless. As a consequence a $c = 1/2$ branch of Majorana fermions will remain gapless, unless $Z_2$ symmetry is spontaneously broken on the edge. Together with the $c = 1$ $Z_2$-symmetry-protected chiral boson edge\[^{20}\] from
\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]
part for phases #2 and #4, we obtain the total central charge $c$ for all 4 “unconventional” SET phases as summarized in TABLE III.

Aside from gapless edge states, another important feature for these “unconventional” SET phases is that they lead to non-Abelian topological orders once the unitary $Z_2$ symmetry is gauged. For these unconventional SET phases, a vertex algebra approach is introduced in Appendix D to gauge the unitary symmetry. The quasiparticle contents of the “gauged” non-Abelian topological orders for these 4 SET phases are summarized in TABLE VI. The “gauged” topological orders are related to the “unconventional” $Z_2$ gauge theories describing fermions with odd Chern number $\nu$ coupled to a $Z_2$ gauge field, as Kitaev described in his 16-fold way classification of 2+1-D $Z_2$ gauge theories\[^{21}\]. More specifically, these non-Abelian topological orders are $Z_2 \times Z_2$ gauge theories, the direct product of $\nu =$odd $Z_2$ gauge theory and its time reversal counterpart $\bar{\nu} = 16 - \nu$, as summarized in TABLE III. Hence these “gauged” topological orders all have non-chiral edge states (chiral central charge $c_c = 0$), which will generally be gapped in the absence of extra symmetry.

After gauging the symmetry, new quasiparticles with quantum dimension $d_{qu} = \sqrt{2}$ emerge as deconfined excitations, called $Z_2$ vortices. When any quasiparticle $q$ in the original SET phase is moved adiabatically around a $Z_2$ vortex once, it becomes its image $gq$ under $Z_2$ symmetry operation. These $Z_2$ vortices are similar to a Majorana bound state in the vortex core of a spinless $p + i q$ superconductor\[^{25}\] in 2+1-D. However, they have different topological spin than those in $p + i q$ superconductors. To be specific, there are 4 inequivalent $Z_2$ vortices with topological spin $\exp(\pm \pi (2\nu_i))$ and $\exp(\pm \pi (2\nu_i - 2\pi))$, as shown in TABLE III. All these non-Abelian topological orders have 9-fold GSD on a torus, corresponding to 9 different superselction sectors shown in TABLE VI. It’s not hard to see that SET phases #1 and #3 lead to the same non-Abelian theory by gauging the $Z_2$ symmetry, and so do SET phases #2 and #4. However, SET phases #1 and #2 do lead to different non-Abelian topological orders after $Z_2$ symmetry is gauged. In particular, their $9 \times 9$ modular $S$ and $T$ matrices in the basis of TABLE VI are shown in the end of Appendix D. Combining the edge states and “gauged” topological orders summarized in TABLE III, indeed there are 4 distinct “unconventional” $Z_2$-symmetry-enriched $Z_2$ spin liquids.

3. Discussions on $Z_2$-symmetry-enriched $Z_2$ gauge theories

First we discuss the results for “conventional” SET phases with onsite $Z_2$ symmetry. For $Z_2$ spin liquids $K \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, with on-site $Z_2$ symmetry we obtain 6 different conventional $Z_2$-SET phases as summarized in TABLE II. For double semion theory $K \simeq \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, with on-site $Z_2$ symmetry we obtain 8 different conventional $Z_2$-SET phases as summarized in TABLE IV. Here we make some connection between these SET phases and $Z_2$-symmetry-enriched $Z_2$ gauge theories obtained in previous studies\[^{16,17}\].

In Ref. 16 the exact solvable lattice models for 8 different $G_s = Z_2$-symmetry-enriched $G_s = Z_2$ gauge theories are obtained. They correspond to the following group cohomology $\mathcal{H}^3(G_s \times G_s, U(1)) = \mathcal{H}^3(Z_2 \times Z_2, U(1)) = \mathbb{Z}_2^3$.

Among these 8 different SET phases, 4 come from $Z_2$ spin liquids with onsite $G_s = Z_2$ symmetry, and the other 4 from double semion theory with onsite $G_s = Z_2$ symmetry. They are nothing but #1, #2, #3, #4 in TABLE I together with #1, #2, #7, #8 in TABLE IV.

In Ref. 17 twelve different $Z_2$-symmetry-enriched $Z_2$ topological orders are proposed, among which six are $Z_2$ spin liquids and the others are double semion theories. It was conjectured that different $G_s$-symmetry-enriched $G_s$-gauge theory (with gauge group $G_g$) are classified by group cohomology $\mathcal{H}^{d+1}(G, U(1))$ in d-dimensional, where $G$ is an extension of symmetry group $G_s$ by gauge group $G_g$. When $G_s = G_g = Z_2$ we have $G/G_s = G_g$ and hence $G = Z_2 \times Z_2$ or $G = Z_4$. The number $12 = 2^4 + 4$ is associated with $\mathcal{H}^3(Z_2 \times Z_2, U(1))$ and $\mathcal{H}^3(Z_4, U(1)) = \mathbb{Z}_2^3 \oplus \mathbb{Z}_4$. The 8 different SET phases from $\mathcal{H}^3(Z_2 \times Z_2, U(1))$ are the same as in Ref. 16 which are discussed earlier. After gauging the $G_s = Z_2$ symmetry, these 8 different SET phases lead to Abelian topological orders described by a $4 \times 4$ matrix $K$.

\[
K(n_1 n_2 n_3) = \begin{pmatrix}
2n_1 & 2 & n_2 & 0 \\
2 & 0 & 0 & 0 \\
n_2 & 0 & 2n_3 & 2 \\
0 & 0 & 2 & 0
\end{pmatrix}
\]

where $n_1, n_2, n_3 = 0, 1$. It’s not difficult to check that these 8 different SET phases labeled by $(n_1 n_2 n_3)$ have
\[
K \simeq \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}
\]
with unitary symmetry \(G_s = Z_2 = \{g, e = g^2\}\)

Data set in (19): \([K = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \{\eta^9 = +1, W^9 = 1_{4\times4}, \delta \vec{\phi}\}]\)

| Label | #1 | #2 | #3 | #4 | #5 | #6 | #7 | #8 |
|-------|----|----|----|----|----|----|----|----|
| \(\delta \vec{\phi}\) | 0 0 | 0 0 | \(\pi/2\) | \(\pi/2\) | 0 0 | \(\pi/2\) | \(\pi/2\) | \(\pi/2\) |
| Proj.Sym.(s) | No | No | Yes | Yes | No | No | Yes | Yes |
| Proj.Sym.(\(s^\ast\)) | No | No | No | Yes | Yes | Yes | Yes | Yes |
| Proj.Sym.(b) | No | No | Yes | Yes | Yes | Yes | No | No |
| SP edge | No | Yes | Yes | Yes | Yes | Yes | No | Yes |

On gauging the symmetry

\[
K_g \simeq \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}
\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}
\begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix}
\begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix}
\begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix}
\begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix}
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}
\]

\(h_{\eta^g} \text{ mod } 1\)
\(\theta_{\eta^g,s} \text{ mod } 2\pi\)
\(\theta_{\eta^g,b} \text{ mod } 2\pi\)

Notation in Ref. [17] (001) (101) \(m_1 = 3\) \(m_1 = 1\) (011) (111)

Table IV: Classification of double semion theory ([17] enriched by onsite (unitary) \(G_s = Z_2\) symmetry, see Appendix C for details. There are 8 different ‘conventional’ SET phases, where under \(Z_2\) symmetry all quasiparticles \((s, \bar{s}, b)\) merely obtain a \(U(1)\) phase factor. The data set in the 2nd line completely characterizes these SET phases. \(K_g\) denotes the topological order, which is obtained by gauging the unitary \(G_s = Z_2\) symmetry in the double semion theory. Some of these SET phases have \(Z_2\) symmetry protected (SP) edge states, which will be gapless unless \(Z_2\) symmetry is spontaneously broken. On gauging the \(Z_2\) symmetry (blue entries) a new quasiparticle \(\eta^g\) is obtained, as described in Appendix B. Its statistics ([3]–[6]) are also summarized in the table: its self statistics \(\theta_{\eta^g} = 2\pi h_{\eta^g}\) has a one-to-one correspondence with its topological spin \(\exp(2\pi i h_{\eta^g})\). Note, there are no ‘unconventional’ symmetry realizations with this topological order.

the following correspondence with our results: and

K(001) \(\simeq \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \)
\(\Leftrightarrow \#1\) in TABLE IV

K(101) \(\simeq \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{pmatrix} \)
\(\Leftrightarrow \#2\) in TABLE IV

K(011) \(\simeq \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \end{pmatrix} \)
\(\Leftrightarrow \#7\) in TABLE IV

K(111) \(\simeq \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \)
\(\Leftrightarrow \#8\) in TABLE IV

We want to emphasize that when the on-site unitary \(Z_2\)
symmetry is gauged, different SET phases could result in the same topological order. Therefore the structure of the topological order obtained by gauging the symmetry doesn’t fully characterize one SET phase! For example with $X_{1,2} \in GL(4,\mathbb{Z})$

$$X_1^T K(01) X_1 = X_2^T K(011) X_2 = \begin{pmatrix}
2 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & -2 & 1 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} ,$$

$$X_1 = \begin{pmatrix}
2 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & -2 & 1 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} , X_2 = \begin{pmatrix}
2 & 2 & -1 & 1 \\
0 & 1 & 0 & 0 \\
0 & -2 & 1 & 0 \\
-1 & 1 & 0 & 0
\end{pmatrix} .$$

and it’s straightforward to see $K(100) \simeq K(001)$ and $K(110) \simeq K(011)$. In fact the 8 different matrices $K(n_1 n_2 n_3)$ describe 5 different Abelian topological orders. Since these 8 theories correspond to different Dijkgraaf-Witten theories, gauge theories with topological terms specified by $\mathcal{H}^3(Z_2 \times Z_2, U(1)) = \mathbb{Z}_2^3$, this also implies that different Dijkgraaf-Witten theories based on a particular gauge group share the same topological order. Perhaps further information regarding which particles comprise electric and magnetic charges is required to uniquely define those phases.

The other 4 SET phases proposed in Ref. [17] are associated to group cohomology $\mathcal{H}^3(Z_4, U(1)) = \mathbb{Z}_4$. Ref. [17] asserted that after gauging the $Z_2$ symmetry they lead to Abelian $Z_4$ topological orders described by

$$K(m_1) = \begin{pmatrix}
2 m_1 & 4 \\
4 & 0
\end{pmatrix} , \quad m = 0, 1, 2, 3.$$ 

We found that these 4 different SET phases have overlap with those 8 SET phases associated to $\mathcal{H}^3(Z_2 \times Z_2, U(1)) = \mathbb{Z}_2^3$: they are nothing but

$$K(m_1 = 0) = \begin{pmatrix}
0 & 4 \\
4 & 0
\end{pmatrix} \Leftrightarrow \#3 \text{ in TABLE [II]}$$

$$K(m_1 = 2) = \begin{pmatrix}
4 & 0 \\
0 & -4
\end{pmatrix} \Leftrightarrow \#5 \text{ in TABLE [II]}$$

$$K(m_1 = 1) = \begin{pmatrix}
2 & 0 \\
0 & -8
\end{pmatrix} \Leftrightarrow \#5 \text{ in TABLE [IV]}$$

$$K(m_1 = 3) = \begin{pmatrix}
8 & 0 \\
0 & -2
\end{pmatrix} \Leftrightarrow \#3 \text{ in TABLE [IV]}$$

“Conventional” SET phases #6 in TABLE [II] ($Z_2$ spin liquids) and #4, #6 in TABLE [IV] however, doesn’t fall into these 12 classes. It’s not clear at this point whether these SET phases can be described by group cohomology theory.

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**Figure 2:** (color online) A fermion mode ($f$) localized at the boundary between two subsystem $A$ and $B$ which from a bipartition of the on a sphere, where the “unconventional” (type-II) Ising-symmetry-enriched $Z_2$ spin liquid resides. Under the Ising symmetry operation, an electric charge $e$ will transform into a magnetic charge $m$. Consider one electric charge is created in each subsystem. If we perform Ising ($Z_2$) symmetry only on subsystem $A$, a fermion mode will emerge on the boundary, as the electric $e$ charge turns into a magnetic charge $m$ in $A$.

4. **Measurable effects of “unconventional” $Z_2$ symmetry realizations**

Suppose there is a $Z_2$ spin liquid which preserves $Z_2$ spin rotational symmetry (for integer spins), are there measurable effects for these SET phases? More specifically, what are the distinctive measurable features of the “unconventional” $Z_2$-SET phases? In this section we’ll try to answer these questions in two aspects, i.e. measurements in the bulk and on the edge. We’ll focus on unconventional SET phases in this section.

First of all, an important ingredient of SET phases is how their quasiparticles transform under symmetry $G_s$. This gives us one way to measure an SET phase: to create quasiparticle excitations and apply symmetry operation on them. For example for a $Z_2$ spin liquid on a closed manifold (a sphere, say), a pair of electric charges $e \simeq (1,0,0,0)^T$ (or magnetic charges $m \simeq (0,1,0,0)^T$) can be created on top of the groundstate. For an onsite unitary symmetry (such as $Z_2$ spin-flip symmetry $g$), one can choose to perform the symmetry operation only on a part of the whole system. For example, FIG. 2 shows such a striking measurable effects on the unconventional $Z_2$-symmetry-enriched $Z_2$ spin liquids. Assume in the SET phase a pair of electric charges $e$ are created, one in subsystem $A$ and the other in subsystem $B$. The whole system $A \cup B$ lives on a closed manifold, say a sphere on which the groundstate is unique. If we only perform the “unconventional” $Z_2$ symmetry operation $g$ in subsystem $A$ (but not in $B$), the electric charge $e \simeq (1,0,0,0)^T$ in subsystem $A$ will become a magnetic charge $m \simeq (0,1,0,0)^T$. However one electric charge and one magnetic charge cannot be created simultaneously.
linear combinations under the "unconventional" (type-II) Ising symmetry $Z_2$ for a set of basis called the minimal entropy states (MESs) for a subsystem $A$ living on the boundary (dashed line in FIG. 2) between subsystems $A$ and $B$, as shown by the wavy line in FIG. 2. This effect happens in all 4 SET phases in TABLE III.

Secondly, there are degenerate ground states once we put the SET phases on a closed manifold with nontrivial topology (with nonzero genus). For example, they have 4-fold GSD on a torus (or infinite cylinder). How these ground states transform under symmetry is a reflection of how anyon quasiparticles transform under symmetry. Specifically, one can always choose a set of basis called the minimal entropy states (MESs) for a flux eigenstates $|e\rangle$ of the infinite cylinder as shown in FIG. 3. The MESs cut is chosen: e.g. the degenerate ground states which minimizes the bipartite entanglement entropy, once a certain entanglement cut is chosen: e.g. along the y-direction in the middle of the infinite cylinder as shown in FIG. 3. The MESs are flux eigenstates, which keeps maximum knowledge of the states after the entanglement cut. Specifically for a $Z_2$ spin liquid, we can label the 4 MESs as $|1\rangle$, $|e\rangle$, $|m\rangle$, $|f\rangle$ on an infinite cylinder. Remarkably under the “unconventional” (type-II) Ising symmetry operation, two MESs ($|e\rangle$ and $|m\rangle$ exchanges) and their linear combinations $|\pm\rangle = (|e\rangle \pm |m\rangle)/\sqrt{2}$ are the Ising symmetry eigenstates. Therefore the MESs necessarily breaks Ising symmetry in such an unconventional SET phase! This phenomena can be actually measured in numerical studies.

Thirdly, the edge state structure encodes many information of a SET phase, when it supports symmetry protected edge modes. For unconventional SET phases, there are always gapless edge excitations protected by symmetry, unless the symmetry is spontaneously broken on the boundary. In the specific case of unconventional $Z_2$-symmetry-enriched $Z_2$ spin liquids summarized in TABLE III, one important feature is that SET phases #1 and #2 supports gapless (non-chiral) Majorana fermion excitations on the boundary, with central charge $c = 1/2$ mod 1. However this is not universal for all unconventional SET phases. A more interesting effect comes from the bound state localized at a $Z_2$ domain wall on the edge. Take SET phase #1 for example, a perturbation on the edge

$$H_1 = A_1 \cos(2\phi_1) + A_2 \cos(2\phi_2) + A_4 \cos \phi_4. \quad (30)$$

can fully gap out the edge excitations in (35)-[39] with $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, if $A_{1,4} \neq 0$ (or $A_{2,4} \neq 0$). On one side of the $Z_2$ domain wall on the edge of SET phase #1, we break $Z_2$ symmetry with $A_2 = 0$ and $A_{1,4} \neq 0$. On the other side of the $Z_2$ domain wall, $Z_2$ symmetry is broken in the opposite way so that $A_1 = 0$ and $A_{2,4} \neq 0$. Physically the electric charges are condensed on one side of the domain wall, while magnetic charges condense on the other side. At the domain wall a non-Abelian (Majorana) bound state is localized, which has quantum dimension $\sqrt{2}$, as illustrated in FIG. 4. Such a domain-wall-bound state is similar to those localized at the (ferromagnetism/superconductivity) mass domain walls of a quantum spin Hall insulator.

In the vertex algebra context, these domain-wall-bound states correspond to quasiparticle $\tilde{q}_6$ (in the 8th role) in TABLE VI. In SET phase #1 e.g. it has topological spin

$$\sqrt{2}|\pm\rangle = |e\rangle \pm |m\rangle =$$

![Figure 3](color online) The Ising symmetry eigenstates are linear combinations of minimal entropy states (MESs) for a “unconventional” Ising-symmetry-enriched $Z_2$ spin liquid, since one MES $|e\rangle$ transforms into another MES $|m\rangle$ under Ising symmetry operation.

![Figure 4](color online) Domain wall bound state on the edge of “unconventional” Ising-symmetry-enriched $Z_2$ spin liquids (see TABLE III). In these SET phases, under $Z_2$ symmetry operation, one electric charge will transform into a magnetic charge and vice versa. The on-site unitary $Z_2$ (Ising) symmetry can be, e.g. a spin-flip symmetry. On the two sides of the Ising-symmetry domain wall, two different backscattering “mass” terms related by spin-flip Ising symmetry are added to gap out the edge states. These two mass terms break $Z_2$ symmetry in opposite ways. A non-Abelian bound state with quantum dimension $d_{sg} = \sqrt{2}$ is localized at each Ising domain wall. For a “conventional” $Z_2$-SET phase, such a Ising mass domain wall will trap an Abelian bound state with quantum dimension 1.
\[ \exp(-i\pi/8). \] In the bulk-edge correspondence of SET phases, such a bound state on the edge is related to the \( Z_2 \) vortex \( q_g \) in the bulk.

In previous sections we use the Chern-Simons approach to study SET phase which are non-chiral, i.e. there are no gapless edge excitations in the absence symmetry. Chern-Simons approach also applies to chiral topological phases, which has gapless edge modes even in the absence of symmetry. These chiral phases have a nonzero chiral central charge \( c_- \) and quantized thermal Hall effect, which necessarily breaks time reversal symmetry. In the following we’ll use Laughlin states as illustrative examples of chiral SET phases.

### C. Classifying bosonic Laughlin state at \( \nu = \frac{1}{N\pi}, (k \in \mathbb{Z}) \) with onsite \( Z_2 \) symmetry

A Laughlin state at filling fraction \( \nu = 1/m \) is described by \( K \simeq m \) in effective theory \( [1] \). When \( m \) is even it describes a bosonic topological order, while \( m \) is odd corresponds to a fermionic state. Such an effective theory also describes chiral spin liquid. Here we start from the simplest case \( i.e. m = 2 \). It has 2-fold GSD on torus, corresponding to two different types of quasiparticles (or two superselection sectors): boson 1 and semion \( s \). Under a unitary \( Z_2 \) symmetry a semion always transforms into a semion, hence we don’t expect any unconventional SET phases where two inequivalent quasiparticles exchange under \( Z_2 \) operation. Again due to the existence of nontrivial \( Z_2 \)-SPT phase of bosons in 2+1-D, we use the following 3 \( \times \) 3 matrix

\[
K = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

in [1] to represent a generic \( \nu = 1/2 \) Laughlin state with \( Z_2 \) symmetry. The group compatibility conditions \( [18] \) for symmetry transformations \( [15] \) are \( \eta^g = 1 \) for unitary \( Z_2 \) symmetry

\[
(W^g)^2 = 1_{3 \times 3}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = (W^g)^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} W^g,
\]

\[
(1_{3 \times 3} + W^g)\delta\phi^g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} n\pi, \quad n \in \mathbb{Z}^3.
\]

The inequivalent solutions to the above conditions are \( W^g = 1_{3 \times 3} \) and

\[
\delta\phi^g = \left( \frac{i1\pi}{2}, \pi, i3\pi \right)^T.
\]

They correspond to 4 different SET phases as summarized in TABLE [V] with \( k = 1 \).

Accordingly the new quasiparticle \( i.e. Z_2 \) vortex \( q_g \) emerging after we gauge the unitary \( Z_2 \) symmetry is \( q_g = (i_1, i_3, 1)^T/2 \). Its topological spin is given by \( \exp(2\pi i h_{g_\nu}) \), where

\[
h_{g_\nu} = \frac{\theta_{g_\nu}}{2\pi} = \frac{1}{2} \left( \frac{\delta\phi^g}{2\pi} \right)^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} W^g = \frac{i_1^2 + 4i_3}{8} \mod 1.
\]

Its mutual statistics with semion \( s = (1, 0, 0)^T \) is

\[
\hat{\theta}_{g_\nu, s} = \frac{i_1}{2} \pi \mod 2\pi.
\]

Following the Chern-Simons approach to gauge the unitary \( Z_2 \) symmetry described in Appendix [13] we obtain the following topological order

\[
K_{g_\nu}^{-1} = M^T K^{-1} M, \quad M = \begin{pmatrix} 1 & 0 & i_2/2 \\ 0 & 1 & i_3/2 \\ 0 & 0 & 1/2 \end{pmatrix}.
\]

and hence

\[
K_g = \begin{pmatrix} 2 & -i_1 & 0 \\ -i_1 & -2i_2 & 2 \\ 0 & 2 & 0 \end{pmatrix}.
\]

Specifically for SET phase \#4 in TABLE [V] its “gauged” theory has the following quasiparticle contents in [36]:

\[
\gamma \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \theta_\gamma = \frac{5\gamma^2}{8}\pi, \quad \gamma = 0, 1, \cdots, 7.
\]

where \( (1, 0, 0)^T \approx (0, 0, 2)^T \) and \( (0, 1, 0)^T \approx (0, 0, 4)^T \).

In a complete parallel fashion we can study generic “conventional” \( Z_2 \)-symmetry-enriched even-denominator Laughlin state at \( \nu = 1/(2k), k \in \mathbb{Z} \). Without loss of generality, a \( \nu = 1/(2k) \) Laughlin state with unitary \( Z_2 \) symmetry is represented by

\[
K = 2k \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2k & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

It has 12 different quasiparticles (or superselection sectors) labeled as

\[
q_\gamma = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, \quad h_{q_\gamma} = \frac{\theta_{q_\gamma}}{2\pi} = \frac{\gamma^2}{2k}, \quad \gamma = 0, 1, \cdots, 2k - 1.
\]

The group compatibility conditions [18] for symmetry transformations [15] have the following inequivalent solutions

\[
W^g = 1_{3 \times 3}, \quad \delta\phi^g = \left( \frac{i1\pi}{2k}, \pi, i3\pi \right)^T.
\]
where $i_{1,3} = 0, 1$. The solution $i_1 = 2$ represents the same SET phase as $i_1 = 0$, according to Corollary II in the criterions. Comparing with the $\nu = 1/2$ bosonic Laughlin state case, we can see there is a universal structure for all bosonic Laughlin state with $K \simeq 2k, k \in \mathbb{Z}$. To be specific, for a $\nu = 1/2k$ bosonic Laughlin state, there are 4 different $Z_2$-SET phases as summarized in Table V.

The quasiparticles (or edge chiral bosons) transform as

$$\phi \xrightarrow{g} \phi + \left(\frac{4\pi}{2k}, \pi, \pi\right)^T, \quad i_{1,3} = 0, 1.$$ (40)

under “conventional” $Z_2$ operation. After gauging the $Z_2$ symmetry, one obtains a quasiparticle (the $Z_2$ vertex) $q_g = (i_1, i_3, 1)^T/2$. Its topological spin is given by $\exp(2\pi i h_{q_g})$ where

$$h_{q_g} = \frac{\theta_{q_g}}{2\pi} = \frac{1}{2} \left(\frac{\delta\tilde{\phi}^g}{2\pi}\right)^T \mathbf{K}_g \frac{\delta\tilde{\phi}^g}{2\pi} = \frac{i_1^2 + 4i_1 k}{8k} \mod 1.$$ (41)

Its mutual statistics with anyon $q_1 = (1, 0, 0)^T$ is

$$\tilde{\theta}_{q_g,q_1} = \frac{i_1}{2k}\pi \mod 2\pi.$$ (42)

Again following the Chern-Simons approach to gauge the unitary $Z_2$ symmetry described in Appendix B, we obtain the following topological order

$$\mathbf{K}_g^{-1} = \mathbf{M}^T \mathbf{K}^{-1} \mathbf{M}, \quad \mathbf{M} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \frac{1}{2} \end{pmatrix}.$$ and hence

$$\mathbf{K}_g = \begin{pmatrix} 2k & -i_1 & 0 \\ -i_1 & -2i_2 & 2 \\ 0 & 0 & 2 \end{pmatrix}, \quad i_{1,3} = 0, 1.$$ (43)

All these 4 Abelian topological orders obtained by gauging symmetry has $|\det \mathbf{K}_g| = 8k$ fold GSD on a torus.

### D. Fermionic Laughlin state at $\nu = \frac{1}{2k+1}, (k \in \mathbb{Z})$ with $Z_2^f$ symmetry is unique

Now let’s turn to the simplest fermionic Laughlin state with $K \simeq 3$. It has 3-fold GSD on a torus and anyon excitations with statistics $\theta = \pm \pi/3$. We consider the following matrix in effective theory [1]

$$\mathbf{K} = 3 \oplus \mathbf{K}_f.$$ (44)

where $\mathbf{K}_f$ generically take the form of [13] and [14]. Notice that for a fermion system with only $Z_2^f = \{e, g = (-1)^N\}$ (fermion number parity) symmetry, there is no nontrivial SPT phases [26,27] in 2+1-D, which hosts gapless edge excitations protected by $Z_2^f$ symmetry. This fact suggests that $K = 3$ is enough to describe a $\nu = 1/3$ fermionic Laughlin state with only $Z_2^f$ symmetry. Now for such a $1 \times 1$ matrix $K = 3$, the group compatibility conditions [18] becomes (note that $\mathbf{P}$ is 2 for fermions)

$$\mathbf{W}^g = 1, \quad 2\delta\phi^g = \frac{2\pi}{6} n, \quad n \in \mathbb{Z}.$$ (45)

for a unitary $Z_2^f$ symmetry $g$. The gauge inequivalent solutions are $\delta\phi^g = \frac{\pi}{3} n$ with $n \in \mathbb{Z}$. However notice a fermions in this system have gauge charge 3 in [1], or alternatively it’s represented by $e^{3i\phi}$ on the edge [3-6]. Under the $Z_2^f$ operation $g$ each fermion obtains a $-1$ sign, which means $3\delta\phi^g = \pi \mod 2\pi$ and $n$ must be even in [15]. Therefore the quasiparticle (chiral boson) transform as

$$\phi \xrightarrow{g} (-1)^N \phi, \quad \phi = \frac{2\pi}{3} n + 1, \quad n \in \mathbb{Z}.$$ (46)

According to Corollary II on smooth sewing condition between edges, we know different integer $n \in \mathbb{Z}$ above correspond to the same $Z_2^f$-SET phase. As a result when
only fermion number parity ($Z_2^f$ symmetry) is conserved, the Laughlin $\nu = 1/3$ state of fermions is unique.

It’s straightforward to see that after gauging the $Z_2^f$ symmetry, we obtain an Abelian topological order

$$K_g = 3 \times 4 = 12.$$  \hspace{1cm} (47)

In fact, the above conclusion is true for any fermionic Laughlin state at $\nu = 1/(2k + 1)$, $k \in \mathbb{Z}$ with conserved fermion number parity. In the presence of $Z_2^f$ symmetry, it is unique with $K = 2k + 1$. After gauging the $Z_2^f$ symmetry, we obtain an Abelian topological order $K_g = 4(2k + 1)$.

E. $Z_2$ spin liquids with onsite $Z_2 \times Z_2$ symmetry

In the end we turn to a $Z_2$ spin liquid $K \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ in the presence of $Z_2 \times Z_2$ symmetry. The symmetry group $G_s = Z_2 \times Z_2 = \{ e, g_1, g_2, g_1g_2 \}$ consists of two generators $g_1$ and $g_2$, satisfying the following algebra:

$$g_1^2 = g_2^2 = (g_1g_2)^2 = e.$$  \hspace{1cm} (48)

In an integer-spin$^{20}$ system these two generators $g_{1,2}$ can correspond to e.g. spin rotations along $\hat{x}$ ($g_1$) and $\hat{y}$ ($g_2$) direction by an angle of $\pi$. Naturally the $\pi$-spin-rotation along $\hat{y}$ direction corresponds to group element $g_1g_2$.

Here we’ll not attempt to fully classify all $Z_2 \times Z_2$ symmetry-enriched $Z_2$ spin liquids. Instead, we focus on one nontrivial example, where spinons transforms projectively under $Z_2 \times Z_2$ symmetry, in the sense that under $2\pi$-spin-rotation along any ($\hat{x}, \hat{y}, \hat{z}$) direction the spinon (or electric charge $e$) obtains a Berry phase $-1$, just like a half-integer spin. On the other hand, the visor (or magnetic charge $m$) transforms trivially under the spin rotations. Such a SET phase can be easily realized by e.g. Schwinger boson$^{21}$ representation of $Z_2$ spin liquids, for integer spin-$S$ ($S = 0, 1, 2, \cdots$)

$$S = \frac{1}{2} \begin{pmatrix} b_1^\dagger b_1 \\ b_1 \end{pmatrix}^T \sigma \begin{pmatrix} b_1^\dagger b_1 \\ b_1 \end{pmatrix}$$  \hspace{1cm} (49)

where $\sigma$ are Pauli matrices. The following constraint needs to be enforced for each spin

$$b_1^\dagger b_1 + b_1^\dagger b_1 = 2S$$  \hspace{1cm} (50)

to guarantee $S^2 = S(S+1)$ for a spin-$S$ system. Once the bosons $b_{1/2}$ form a pair superfluid (but not a superfluid) with $\langle bb \rangle \neq 0$ (but $\langle b \rangle = 0$), the resulting spin-$S$ state after projection into the physical Hilbert space$^{20}$ is a $Z_2$ spin liquid$^{15}$. Its spinon excitations $b_{1/2}$ carry half-spin each, hence transforming projectively under $SO(3)$ (and hence $Z_2 \times Z_2$) spin rotations. By transforming projectively" we simply mean that after all three symmetry operations in$^{15}$ which equals identity operation $e$, all spinons obtain $-1$ Berry phase instead of remaining invariant (or transforming linearly). In the following we’ll show such a $Z_2 \times Z_2$ SET phase can be captured in the Chern-Simons approach.

Starting from a $4 \times 4$ matrix $K_0 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to describe $Z_2$ spin liquid, for clarity we first perform a $GL(4, \mathbb{Z})$ gauge transformation$^{23}$ on $K_0$

$$K = X^T K_0 X = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (51)

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \det X = 1.$$

We study $Z_2$ spin liquid with $Z_2 \times Z_2$ spin rotational symmetry in the above representation $K$. Notice that

$$K^{-1} = \frac{1}{2} X K X^T = \begin{pmatrix} 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ -1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

Apparently the first two components ($\phi_{1,2}$ in the edge chiral boson context) can be regarded as spinons, which obey semionic mutual statistics with the last two components ($\phi_{3,4}$) i.e. the visons. Two spinons (visons) are mutually local w.r.t. each other as indicated by$^{3}$.

In a $Z_2 \times Z_2$ symmetry group$^{48}$, the group compatibility conditions for symmetry transformations $\{ W^{g_{1,2}}, \delta \phi^{g_{1,2}} \}$ in$^{15}$ are

$$\begin{pmatrix} W^{g_{1,2}} \end{pmatrix}^2 = \begin{pmatrix} W^{g_1} W^{g_2} \end{pmatrix}^2 = 1_{4 \times 4};$$

$$\begin{pmatrix} (1_{4 \times 4} + W^{g_{1,2}}) \delta \phi^{g_{1,2}} \end{pmatrix} = 2\pi K^{-1} n_{1,2};$$

$$\begin{pmatrix} (1_{4 \times 4} + W^{g_1} W^{g_2}) (\delta \phi^{g_2} + W^{g_2} \delta \phi^{g_1}) \end{pmatrix} = 2\pi K^{-1} n.$$

where $n_{1,2} \in \mathbb{Z}^4$. Among all inequivalent solutions to these group compatibility conditions$^{48}$, the following one

$$W^{g_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \delta \phi^{g_1} = \begin{pmatrix} \pi/2 \\ \pi/2 \\ 0 \\ 0 \end{pmatrix},$$

$$W^{g_2} = 1_{4 \times 4}, \quad \delta \phi^{g_2} = (\pi/2, -\pi/2, 0, 0)^T.$$

$$n_1 = \mathbf{n} = (0, 0, 0, 1)^T, \quad n_2 = (0, 0, -1, 0)^T.$$

corresponds to such a integer-spin $Z_2$ spin liquid where spinons transform projectively under the $Z_2 \times Z_2$ spin
rotations. To be specific, the quasiparticles transform as

\[
\bar{\phi} = \begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{pmatrix} \rightarrow \begin{pmatrix}
\phi_2 + \pi/2 \\
\phi_1 + \pi/2 \\
-\phi_3 \\
-\phi_4
\end{pmatrix},
\]

where \( g_{\pi/2} \) is the \( \pi/2 \) rotation matrix.

Indeed each spinon (\( \phi_{1,2} \)) acquires a Berry phase after every \( 2\pi \)-spin-rotation, while visons (\( \phi_{3,4} \)) transform trivially.

Notice that in such a SET phase there is no symmetry protected gapless edge states, i.e. generically all edge states are gapped in the presence of \( Z_2 \times Z_2 \) symmetry. Specifically the following backscattering terms can be added to the edge action \( \mathcal{L}_{\text{Higgs}} \) without breaking symmetry

\[
\mathcal{L}_{\text{Higgs}} = C_3 \cos(2\phi_3) + C_4 \cos(2\phi_4)
\]

All the chiral boson modes \( \{\phi_i\}_{i = 1, 2, 3, 4} \) on the edge will be gapped out by this term.

\[ \mathcal{L}_{\text{Higgs}} = C_3 \cos(2\phi_3) + C_4 \cos(2\phi_4) \]

\[ \text{All the chiral boson modes } \{\phi_i\}_{i = 1, 2, 3, 4} \text{ on the edge will be gapped out by this term.} \]

\[ F. \text{ Comments on symmetry protected edge states in SET phases} \]

In this section we briefly comments the symmetry protected edge states in all SET phases discussed above. First of all for a chiral topological order, such as Laughlin states at filling fraction \( \nu = 1/m \), its edge excitations have net chirality \( n_+ - n_- \neq 0 \) and hence cannot be destroyed even in the absence of any symmetry. These chiral topological orders are featured by quantized thermal Hall transport\([33]\).

On the other hand, the edge excitations of a non-chiral topological order have both right and left movers and can be fully gapped out in the absence of any symmetry\([29]\) such as in \( Z_2 \) spin liquids \( K \simeq \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \) and double semion theory \( K \simeq \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \). In the presence of global symmetry \( G_x \), they might have symmetry protected gapless edge modes, if the edge backscattering terms are forbidden by symmetry. To be specific, the backscattering terms are typically Higgs terms \([8]\). The edge states will be fully gapped, if and only if each chiral boson fields \( \phi_i \) is either pinned at a classical minimal by the Higgs terms or doesn’t commute with at least one Higgs term\([29]\).

Take \( Z_2 \) spin liquids with \( K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) for example, either \( A_1 \cos(2\phi_1 + \alpha_1) \) or \( A_2 \cos(2\phi_2 + \alpha_2) \) could gap out chiral boson fields \( \phi_{1,2} \), since \( \phi_1(x), \phi_2(y) \neq 0 \). Similarly either \( A_3 \cos(\phi_3 + \alpha_3) \) or \( A_4 \cos(\phi_4 + \alpha_4) \) could gap out chiral bosons \( \phi_{3,4} \). When these terms are not allowed by symmetry, there could be gapless excitations on the edge protected by symmetry. The symmetry protected edge modes in “conventional” (Type-I) Ising-symmetry-enriched \( Z_2 \) spin liquids are summarized in TABLE III. Among the 6 different conventional SET phases, only \#1 and \#3 don’t support symmetry protected edge modes. Phase \#5 provides an interesting example. Here, both electric and magnetic particles transform projectively under the global \( Z_2 \) symmetry. Hence the edge perturbations \( A_1 \cos(2\phi_1 + \alpha_1) \) and \( A_2 \cos(2\phi_2 + \alpha_2) \) which attempt to condense them, are both disallowed, leading to a symmetry protected edge state.

For “unconventional” (Type-II) Ising-symmetry-enriched \( Z_2 \) spin liquids summarized in TABLE III there are always Ising-symmetry protected Majorana edge modes with central charge \( c = 1/2 \mod 1 \).

For double semion theory with \( K = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) on the other hand, either \( A_1 \cos(2\phi_1 + 2\phi_2 + \alpha_+) \) or \( A_- \cos(2\phi_1 - 2\phi_2 + \alpha_-) \) can fully gap out chiral bosons \( \phi_{1,2} \). Meanwhile again either \( A_3 \cos(\phi_3 + \alpha_3) \) or \( A_4 \cos(\phi_4 + \alpha_4) \) could gap out chiral bosons \( \phi_{3,4} \). When symmetry forbids these terms on the edge, there will be gapless edge excitations. The results are summarized in TABLE IV for Ising-symmetry-enriched double semion theories. Among the 8 different SET phases, only \#1 and \#7 don’t host symmetry protected gapless edge states.

IV. CONCLUDING REMARKS

In summary, we have presented a general framework to study 2+1-D symmetry enriched topological phases with Abelian topological order, using the Chern-Simons approach. It allows us to implement generic on-site unitary (or anti-unitary) symmetry in an Abelian topologically ordered phase in 2+1-D, to differentiate whether two states belong to the same SET phase or not, and to gauge a generic unitary symmetry and extract the resultant topological order. Based on this general formulation, we classify all different SET phases in a series of examples, including \( Z_2 \) spin liquids with time reversal symmetry (TABLE I), \( Z_2 \) spin liquids with unitary Ising symmetry (TABLE II and III), double semion theory with unitary Ising symmetry (TABLE IV), bosonic Laughlin states with unitary Ising symmetry (TABLE V) and others. We also show that (odd-denominator) fermionic Laughlin states with only conserved fermion number parity \( (Z_2^f) \) symmetry is unique. Consequences of gauging symmetries and measurable effects, such as gapless edge states, are also discussed for these SET phases.

A number of directions remain. Can the approach applied be extended to spatial symmetries? Can we extend...
this framework to symmetry enriched non-Abelian topological orders in 2+1-D and SET phases in 3+1-D? While SPT phases form an Abelian group, it is presently unclear if the set of SET states have additional structure. We leave these questions to future work.

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Appendix A: Introduction to $GL(N,Z)$

$GL(N,Z)$ is the group of all $N \times N$ unimodular matrices. All $GL(N,Z)$ matrices can be generated by the following basic transformations ($i \neq j$):

\[
T_{a,b}^{(i,j)} = \delta_{a,b} + \delta_{a,i} \delta_{b,j},
\]

\[
S_{a,b}^{(i,j)} = \delta_{a,b}(1 - \delta_{a,i})(1 - \delta_{a,j}) + \delta_{a,j} \delta_{b,i} - \delta_{a,i} \delta_{b,j},
\]

\[
D_{a,b} = \delta_{a,b} - 2 \delta_{a,N} \delta_{b,N}. \quad \text{(A1)}
\]

$T^{(i,j)}K$ will add the $j$-th row of matrix $K$ to the $i$-th row of $K$, while $S^{(i,j)}K$ will exchange the $i$-th and $j$-th row of $K$ by a factor of $-1$ multiplied on the $i$-th row. $DK$ will just multiply the $N$-th row of $K$ by a factor of $-1$. $K^{-1}$, $KS^{(i,j)}$ and $KD$ correspond to similar operations to columns (instead of rows). A subgroup of $GL(N,Z)$ with determinant $+1$ is called $SL(N,Z)$ and it’s generated by \{\(T^{(i,j)}, S^{(i,j)}\}\).

As a simple example when $N = 2$, group $GL(2,Z)$ is generated by the following basic transformations:

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \text{(A2)}
\]

The following results will be useful

\[
T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad (-STS)^n = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}, \quad n \in \mathbb{Z}.
\]

Appendix B: Chern-Simons approach to gauge a unitary symmetry

In this section we discuss how to obtain the (intrinsic) topological order by gauging the unitary symmetry in an Abelian SET phase. We’ll restrict ourselves to “conventional” SET phases, characterized by data $[K, \{\eta^g = +1, W^g = 1_{N \times N}, \delta g^g[g \in G_s]\}]$. In these cases the chiral bosons $\phi_I$ only acquire $U(1)$ phase factors $\phi_I \rightarrow \phi_I + \delta \phi^g_I$ after unitary symmetry operation $g \in G_s$. When we couple the quasiparticles to a gauge field (with gauge group $G_s$), the following gauge flux

\[
\epsilon^{0\mu\nu} \partial_\mu a_\nu^I(r, t) = \delta \phi^g_I (r - r^{(0)}),
\]

becomes deconfined excitations in the system. Since in a Chern-Simons theory gauge charges are always combined with gauge fluxes by the following equation of motion

\[
j^\mu_I = \frac{\epsilon^{\mu\nu\lambda}}{2\pi} \sum_j K_{I,j} \partial_\nu a^j_\lambda.
\]

Therefore the new excitation (emerged after gauging the symmetry, called $Z_2$ vortex) carries gauge charge vector $K \delta \phi^g / (2\pi)$. We will denote a unit of this new excitation (the $Z_2$ vortex) as $q_g$. Therefore the new topological order $K_g$ obtained by gauging symmetry must include this excitation in its quasiparticle contents. More precisely, the quasiparticle contents of topological order $K_g$ is expanded by all the integer vectors as well as multiples of vectors \{\(K \delta \phi^g / (2\pi)\}\)

\[
\{\mathbf{l}' = 1 + \sum_n n_g \frac{K \delta \phi^g}{2\pi}, \quad 1 \in \mathbb{Z}^N, \quad n_g \in \mathbb{Z}\}. \quad \text{(B1)}
\]

where \{\(g\}\} are the generators of symmetry group $G_s$ that are gauged. And we can identify the new matrix $K_g$ which contains all these quasiparticles in its spectra.

In the following we work on one example to demonstrate this gauging procedure. We consider SET phase #6 in TABLE II with $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and symmetry transformation $W^g = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ with $i_{1,2,4} = 0,1$. According to (B1) we know here a generic quasiparticle is labeled by gauge charge vector

\[
\{\mathbf{l}' = 1 + \sum_n n_g \frac{K \delta \phi^g}{2\pi}, \quad 1 \in \mathbb{Z}^4\}.
\]

Since this new Abelian topological order is determined by the statistics of its quasiparticles, we immediately obtain

\[
\{\mathbf{l}'_1\}^T K_g^{-1} \{\mathbf{l}'_2\} = \{\mathbf{l}'_1\}^T K_g^{-1} \{\mathbf{l}'_2\}, \quad \mathbf{l}'_\alpha = M \mathbf{l}_\alpha.
\]

and therefore

\[
K_g^{-1} = M^T K^{-1} M = \begin{pmatrix} i_{1,2} + 2i_4 & i_4 & i_4/2 & 1/2 \\ i_4 & i_4/2 & 1/2 & 0 \\ i_4/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\Rightarrow K_g = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & -i_2 \\ 0 & 2 & 0 & -i_1 \\ 2 & -2 & -2 & -2 \end{pmatrix}. \quad \text{(B2)}
\]
Clearly from (2)–(3) we know the self statistics of new quasiparticle \( q_g \) is
\[
\theta_{q_g} = \pi \left( \frac{\delta \vec{\phi}}{2\pi} \right)^T \mathbf{K} \delta \vec{\phi} = \pi \frac{i_1 i_2 + 2 i_4}{4},
\] (B3)
and its mutual statistics with original quasiparticles \( e, m, f \) of \( Z_2 \) spin liquid are
\[
\begin{align*}
\tilde{\theta}_{q_g,e} &= \frac{i_2 \pi}{2}, \\
\tilde{\theta}_{q_g,m} &= \frac{i_2 \pi}{2}, \\
\tilde{\theta}_{q_g,f} &= \tilde{\theta}_{q_g,e} + \tilde{\theta}_{q_g,m} = \frac{(i_1 + i_2) \pi}{2}.
\end{align*}
\] (B4)

Topological spin \(^{10}\) exp(\(2\pi i h_g\)), the Berry phase obtained by adiabatically rotating a quasiparticle \( q \) by \( 2\pi \), is an important character of a 2+1-D topological order. In Abelian topological orders, the topological spin \( \exp(2\pi i h_{q_g}) \) has a one-to-one correspondence to the self-statistics \( ^{[B3]} \) of a quasiparticle in unit of \( 2\pi \):
\[
h_{q_g} = \frac{\theta_{q_g}}{2\pi} = \frac{1}{2} \left( \frac{\delta \vec{\phi}}{2\pi} \right)^T \mathbf{K} \delta \vec{\phi} = \frac{i_1 i_2 + 2 i_4}{8}.
\] (B5)

For the “unconventional” SET phases, e.g. in our case with \( G_s = Z_2 \), the \( Z_2 \) symmetry would exchange quasiparticles that belong to different superselection sectors in Abelian topological order \( \mathbf{K} \). Gauging this kind of \( Z_2 \) symmetry will in general lead to \( U(1)^3 \times Z_2 \) Chern-Simons theory \( ^{23} \), which describes non-Abelian topological orders in relation to \( Z_2 \) orbifold conformal field theory \( ^{07,10} \).

**Appendix C: Classifying double semion theory with onsite \( Z_2 \) symmetry**

Double semion theory \( ^{13,14} \) is a “twisted” \( Z_2 \) gauge theory in 2+1-D, with Abelian topological order described by \( \mathbf{K} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \). Due to the presence of nontrivial bosonic \( Z_2 \)-SPT phase in 2+1-D, again here we believe a 4 \( \times \) 4 matrix
\[
\mathbf{K} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\] (C1)
can capture all the different \( Z_2 \)-symmetry-enriched double semion theory. Such a theory has the following quasiparticle contents in its spectra
\[
s \simeq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{s} \simeq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad b \simeq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\] (C2)
where \( s \) and \( \bar{s} \) represents semion and anti-semion respectively, and \( b \) is the bound state of a semion and an anti-semion. \( b \) has bosonic self statistics \( ^{2} \) but mutual semion(anti-semion) statistics with \( s(\bar{s}) \). Here \( \{1, s, \bar{s}, b\} \) represents the 4 superselection sector of double semion theory. Any two quasiparticles differing by a local excitation \( \simeq 0 \) belong to the same superselection sector.

Now let’s consider the implementation of unitary \( G_s = Z_2 \) symmetry on double semion theory. We have group compatibility condition \( ^{18} \) for symmetry transformation \( ^{15} \) on quasiparticles:
\[
(W^g)^2 = I_{4 \times 4},
\] (C3)
\[
\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (W^g)^T \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} W^g,
\]
\[\text{(1}_{4\times4}+W^g)\delta \vec{\phi} = \pi \begin{pmatrix} 1 & 0 & 0 & 0 \\ -i_2/2 & 0 & 1 & 0 \\ i_4/2 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & 0 \end{pmatrix} \mathbf{n}, \quad \mathbf{n} \in \mathbb{Z}^4. \]

We consider \( ^{10} \) the solutions to \( ^{[C3]} \) with \( W^g = I_{4 \times 4} \). Due to Criterion I, the gauge inequivalent solutions of \( \delta \vec{\phi} \) to \( ^{[C3]} \) are summarized in TABLE IV.

Following Appendix B we briefly discuss consequences of gauging the unitary \( Z_2 \) symmetry in the double semion theory. Since the symmetry transformation is a \( U(1) \) phase shift \( \delta \vec{\phi} = \pi (i_1/2, i_2/2, 1, i_4)^T \) as shown in TABLE IV the quasiparticle content in the new topological order obtained by gauging \( Z_2 \) symmetry is expanded by gauge charge vector:
\[
l' = Ml:
\quad M = \begin{pmatrix} i_1/2 & 0 & 0 \\ -i_2/2 & 0 & 1 \\ i_4/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}, \quad l \in \mathbb{Z}^4,
\]
and therefore
\[
K_9^{-1} = M^T K^{-1} M = \begin{pmatrix} i_1^2 - i_1^2 + i_4^2 & i_1/4 & i_2/4 & 1/2 \\ i_1/4 & 1/2 & 0 & 0 \\ i_2/4 & 0 & -1/2 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix},
\]
\[\Rightarrow K_g = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & -i_1 \\ 0 & 0 & -2 & i_2 \\ 2 & -i_1 & i_2 & -2i_4 \end{pmatrix}.
\] (C4)

Take \#7 as an example with \( \delta \vec{\phi} = \pi (1/2, 1/2, 1, 0)^T \) (or \( i_1 = i_2 = 1, i_4 = 0 \), the Abelian topological order
and for SET phase

\[ \mathbf{K}_g \simeq \mathbf{X}^T \mathbf{K}_g \mathbf{X} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in GL(4, \mathbb{Z}). \]

Obtained by gauging \( Z_2 \) symmetry is

\[ \mathbf{K}_g \simeq \mathbf{X}^T \mathbf{K}_g \mathbf{X} = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}, \]

\[ \mathbf{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2 & 2 & 1 & 1 \end{pmatrix} \in GL(4, \mathbb{Z}). \]

Notice that

\[ \begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix} \simeq \begin{pmatrix} 0 & 4 \\ 4 & 6 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix} \simeq \begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix}. \]

Again from (2) we can obtain the self statistics of new quasiparticle \( g \) as

\[ \tilde{\theta}_{q_g} = \pi \left( \frac{\delta \tilde{\varphi}}{2 \pi} \right)^T \mathbf{K} \delta \tilde{\varphi} = \pi \left( \frac{i^2 - i^2 + 4i^4}{8} \right), \quad (C5) \]

and its mutual statistics with original quasiparticles \( e, m, f \) of \( Z_2 \) spin liquid are

\[ \tilde{\theta}_{q_g} = \tilde{\theta}_{q_g}, \tilde{\theta}_{q_g}, \tilde{\theta}_{q_g} = \frac{i^2}{2}, \quad \tilde{\theta}_{q_g}, \tilde{\theta}_{q_g} = \frac{(i+1)(i+2)}{2}. \quad (C6) \]

The topological spin of this new quasiparticle is given by \( \exp(2\pi i h_{q_g}) \) where

\[ h_{q_g} = \frac{1}{2} \left( \frac{\delta \tilde{\varphi}}{2 \pi} \right)^T \mathbf{K} \delta \tilde{\varphi} = \frac{i^2 - i^2 + 4i^4}{16}. \quad (C7) \]

Unlike others, for SET phases #1, #6, #8 it’s not easy to find a \( GL(4, \mathbb{Z}) \) transformation on \( \mathbf{K}_g \) matrix \((C4)\) to reduce it to a simpler form. e.g. one can only show for SET phase #4

\[ \mathbf{K}_g \simeq \mathbf{X}^T \mathbf{K}_g \mathbf{X} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in GL(4, \mathbb{Z}). \]

And it has 16 different types of quasiparticles:

\[ \tilde{\gamma} \equiv \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ 0 \\ 0 \end{pmatrix} \quad \text{in} \quad (C4), \quad \gamma_{1,2} = 0, 1, 2, 3. \]

Among them four has bosonic self statistics \( (\theta = 0 \mod 2\pi) \), six with semionic statistics \( (\theta = \frac{\pi}{2} \mod 2\pi) \) and the other six with anti-semionic statistics \( (\theta = -\frac{\pi}{2} \mod 2\pi) \). In the above basis, the \( 16 \times 16 \) modular \( \mathcal{S} \)-matrix of this Abelian topological order is given by

\[ \mathcal{S}_{\tilde{\gamma}, \tilde{\gamma}'} = \frac{1}{4} \exp \left[ \frac{\pi i}{2} \left( 2 \sum_{a=1}^{2} \gamma_a \gamma'_a + \gamma_1 \gamma'_2 + \gamma_2 \gamma'_1 \right) \right]. \quad (C9) \]

Appendix D: Vertex algebra approach to gauge a unitary symmetry

The Chern-Simons approach to gauge a unitary symmetry, introduced in Appendix B applies to all cases.
where we obtain an Abelian topological order after gauging the symmetry. Thus for any “conventional” SET phases we can gauge its unitary symmetry and obtain an Abelian topological order in the Chern-Simons approach. For “unconventional” SET phases, as those summarized in TABLE II gauging a unitary (e.g. $Z_2$) symmetry will result in non-Abelian topological orders. In the case $G_e = Z_2$ as discussed in this work, these non-Abelian topological orders are described by $U(1)^N \times Z_2$ Chern-Simons theory. In these “unconventional” cases the Chern-Simons approach introduced previously is not enough. In order to obtain the full structure (such as topological spin exp$(2\pi i h)$ of quasiparticles and modular $\mathcal{S}$ matrix associated with quasiparticle statistics) of these non-Abelian topological orders, here we introduce a vertex algebra approach to gauge the unitary symmetry. It applies to both the “conventional” and “unconventional” SET phases and in the following we’ll demonstrate its power by two examples: “conventional” and “unconventional” $Z_2$-symmetry-enriched $Z_2$ spin liquids.

1. The vertex algebra formalism, and application to “conventional” SET phases

The vertex algebra approach is based on the close connection between the bulk topological order (described by 2+1-D topological field theory) and its boundary excitations (described by 1+1-D conformal field theory) in two spatial dimensions. Let’s take $Z_2$ spin liquid as an example. The edge effective theory contains two branches of chiral bosons $\{\phi_1, \phi_2\}$, which could be reformulated by a $c = 1$ $U(1) \times U(1)$ Gaussian model with a holomorphic and anti-holomorphic part:

$$\varphi(x + i\tau) \equiv \varphi(z) = \phi_1(x, t) + \phi_2(x, t),$$
$$\overline{\varphi}(x - i\tau) \equiv \overline{\varphi}(\bar{z}) = \phi_1(x, t) - \phi_2(x, t).$$

The Gaussian model has Lagrangian density $L_{\text{Gaussian}} = \frac{1}{2\pi} \overline{\varphi}(z) \overline{\partial \varphi}(z) = \frac{1}{8\pi} |\nabla \varphi|^2$, yielding the following correlation function

$$\langle \varphi(z) \varphi(w) \rangle = -\ln(z - w).$$

and

$$\langle \overline{\varphi}(z) \overline{\varphi}(w) \rangle = -\ln(\bar{z} - \bar{w}).$$

The free boson field $\varphi$ has compactification radius $R = 2$ for $Z_2$ spin liquid so that periodicity $\varphi \sim \varphi + 2\pi R$ holds. In general for $K = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$ the associated compactification radius of scalar boson

$$\sqrt{\frac{2}{N}} \varphi(z) = \phi_1(x, t) + \phi_2(x, t),$$
$$\sqrt{\frac{2}{N}} \overline{\varphi}(\bar{z}) = \phi_1(x, t) - \phi_2(x, t).$$

is $R = \sqrt{2N}$. The allowed physical excitations must be compatible with $2\pi R$ periodicity of bosons and they are

$$V_k(z) = e^{i k \varphi(z)/\sqrt{2N}}e^{i \overline{\varphi}(\bar{z})/\sqrt{2N}}, \quad k = 0, 1, \cdots, 2N - 1.$$
It’s straightforward to check that all allowed quasiparticles (local w.r.t. the above electron operator) have the following form
\[ e^{i(l_1 \phi_1 + l_2 \phi_2)} = \exp \left[ i \left( l_1 \psi_1(z) + (l_1 \phi_2(z) + l_2 \bar{\phi}_2(z)) \right) \right]. \]
Lastly, any two primary fields differing by an electron operator are regarded as the same (or belong to the same superselection sector).

Now let’s go back to \( Z_2 \) spin liquids with \( K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \) \( \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), which have 4 branches of chiral bosons \( \{ \phi_i, 1 \leq i \leq 4 \} \). We can introduce free bosons \( \varphi_1(z), \bar{\varphi}_1(z) \) for chiral bosons \( \phi_{1,2} \) as in (D2) with \( N = 2 \), and free bosons \( \varphi_2(z), \bar{\varphi}_2(z) \) for chiral bosons \( \phi_{3,4} \) as in (D2) with \( N = 1 \). In other words we have
\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{pmatrix} = \begin{pmatrix}
1/2 & 1/2 & 1/\sqrt{2} & 1/\sqrt{2} \\
1/2 & -1/2 & 1/\sqrt{2} & -1/\sqrt{2} \\
1/\sqrt{2} & 1/\sqrt{2} & 1 & 0 \\
1/\sqrt{2} & -1/\sqrt{2} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi_4
\end{pmatrix}.
\]
Before gauging the unitary \( Z_2 \) symmetry, the four superselection sectors (or 4 types of different quasiparticles) correspond to
\[
1 \sim \bar{j}_1(z) \sim j_1(z) \sim e^{2i \varphi_1(z)} \sim e^{2i \bar{\varphi}_1(z)} \sim e^{i \left[ \varphi_1(z) \pm \bar{\varphi}_1(z) \right]}
\]
\[
\bar{j}_2(z) \sim j_2(z) \sim e^{i \varphi_2(z) + i \bar{\varphi}_2(z)} \sim e^{i \varphi_2(z) - i \bar{\varphi}_2(z)} \sim e^{i \bar{\varphi}_2(z)} \sim e^{i \varphi_2(z)}.
\]
\[
e \sim e^{i \varphi_1(z)/2} \sim m \sim e^{i \varphi_1(z) - i \bar{\varphi}_1(z)} \sim f \sim e^{i \bar{\varphi}_1(z)} \sim e^{i \varphi_1(z)}.
\]
Now after gauging the “conventional” \( Z_2 \) symmetries in Table I as discussed in Appendix I, a new type of quasiparticles \( q_g \) becomes deconfined excitations:
\[
q_g \sim e^{i \sum_{\gamma} \phi_1 \mathcal{K}_{\gamma} / 2\pi} \sim \exp \left[ i \left( \frac{l_1 \psi_1(z) + (l_1 \phi_2(z) + l_2 \bar{\phi}_2(z))}{\sqrt{2} (1 + i \psi_1(z)) + \sqrt{2} (1 - i \psi_1(z))} \right) \right],
\]
where \( l_{1,2,4} = 0, 1 \) in \( \delta \psi_2 \). Notice that when such a \( Z_2 \) vortex \( q_g \) is deconfined, we have to modify the previous definition of electron operators 1 in (D8). The new electron operator is defined as anything that is local w.r.t. quasiparticles \( \{ e, m, f, q_g \} \). With this new definition for electron operators, we can track down all the inequivalent quasiparticles (superselection sectors) and obtain the full structure of the topological order obtained by gauging \( Z_2 \) symmetry. One can easily check this approach indeed reproduces Table I consistent with the result of Chern-Simons approach.

In the vertex algebra context, the scaling dimension \( h \) of a quasiparticle determines its topological spin \( \exp(2\pi i h) \), the Berry phase obtained by self-rotating a quasiparticle adiabatically by \( 2\pi \). On the other hand, the mutual statistics of quasiparticle \( A \) and \( B \) is given by \( \theta_{A,B} = -2\pi \alpha_{A,B} \) in OPE (D6).

2. Application to “unconventional” SET phases

For a “unconventional” SET phase, e.g. where two inequivalent quasiparticles \( e \) and \( m \) in \( Z_2 \) spin liquid) are exchanged under \( Z_2 \) symmetry operation as summarized in Table III, a non-Abelian topological order is obtained by gauging the \( Z_2 \) symmetry. Here we apply the vertex algebra approach to extract the full structure of these non-Abelian topological orders.

First let’s review some known results, discussed in detail in Ref. [55-72]. When the “unconventional” \( Z_2 \) symmetry \( \{ W^g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \delta \psi_2 = 0 \} \) is gauged for Abelian topological order \( K = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix} \), the resultant topological order is described by \( U(1) \times U(1) \times Z_2 \) Chern-Simons theory (coined “twisted” \( Z_N \) gauge theory in Ref. [72], which has GSD = \( N^g / 2 \{ \frac{N^g}{2} + 1 + (2^g - 1) (N^g - 1) \} \) on a genus-\( g \) Riemann surface. It contains 2\( N \) different quasiparticles with quantum dimension \( d = 1 \), another 2\( N \) quasiparticles with \( d = \sqrt{N} \), and \( N(N - 1) / 2 \) quasiparticles with \( d = 2 \). Under the unconventional \( Z_2 \) symmetry operation two superselection sectors \( e \leftrightarrow m \) exchanges and so does chiral bosons \( \phi_1 \leftrightarrow \phi_2 \). Therefore in the context of vertex algebra (D2) the anti-holomorphic free boson \( \bar{\psi} \rightarrow -\bar{\psi} \) under \( Z_2 \) symmetry operation! After this “unconventional” \( Z_2 \) symmetry is gauged for \( Z_2 \) spin liquids (\( N = 2 \)), we obtain an non-Abelian topological order whose quasiparticle content has an antiholomorphic part (from \( \bar{\varphi} \)) given by \( Z_2 \) orbifold CFT (D2) with compactification radius \( R = 2 \). It has been shown that (D8)\( Z_2 \) orbifold CFT is equivalent to Ising × Ising (or Ising\(^2 \)) CFT (D7). In each Ising CFT there are 3 different quasiparticles: vacuum (or boson) 1, fermion \( \psi \) and the “disorder” field \( \sigma \) with the following fusions rules:
\[
\psi \times \psi = 1, \quad \psi \times \sigma = \sigma, \quad \sigma \times \sigma = 1 + \psi.
\]
Both 1 and \( \psi \) have quantum dimension 1 while disorder operator \( \sigma \) has quantum dimension \( \sqrt{2} \). Their scaling dimensions are 0, 1/2 and 1/16. Therefore the \( Z_2 \) orbifold CFT, equivalent to the direct product of two copies of Ising CFTs, contains 9 = 3 × 3 inequivalent quasiparticles (superselection sectors). The quasiparticle contents of the \( Z_2 \) orbifold CFT are summarized in the first 3 columns of Table IV.

Now let’s get back to our cases of \( Z_2 \) spin liquids with unconventional on-site \( Z_2 \) symmetry. There are 4 such SET phases as summarized in Table III. After gauging the unitary \( Z_2 \) topological orders with 9 inequivalent quasiparticles (superselection sectors). In the vertex algebra context, they all share the same antiholomorphic \( \bar{\varphi}_1 \) part which gives rise to the non-Abelian quasiparticles. However, their different holomorphic parts discriminates these 4 SET phases. A key issue in determining the quasiparticle contents is: which quasiparticles are identical (or belong to
the same superselection sector), after the symmetry is gauged.

In the vertex algebra context, once we fix the electron operator $1 \sim ?$ (or the trivial sector) which is local w.r.t. all quasiparticles, the full structure of inequivalent quasiparticles is determined. So the above issue becomes the following question: how to determine the electron operators in the vertex algebra, once we gauge the unitary symmetry? The answer lies in the following physical principle:

*If in the original SET phase, two quasiparticles belong to the same superselection sector (i.e. they are equivalent) and transform in the same way under a unitary symmetry, then they belong to the same superselection sector after the unitary symmetry is gauged.*

To be specific, if two quasiparticles $q_A$ and $q_B$ belong the same superselection sector and transform in the same way under unitary symmetry, then after gauging the symmetry, quasiparticle $q_Aq_B^{-1} \sim 1$ ($q_B^{-1}$ is the anti-particle of $q_B$) belong to the trivial sector. For instance, in SET phase #1 in TABLE [III] and [VI] the following two quasiparticles belong the the trivial sector and are both odd under $Z_2$ symmetry $g$:

\[
\tilde{j}_1 \sim e^{i\phi_3} = e^{i \frac{\pi}{4}}
\]

and they are both their own anti-particles. Besides the following two fermions also belong the same superselection sector and are both even under $Z_2$ symmetry:

\[
\tilde{f}_1 = \cos(\phi_1 - \phi_2) = \cos(\bar{\phi}_1) \sim e^{i(\phi_1 + \phi_2)} = e^{i \phi_1}.
\]

Both of them are also their own anti-particles. Therefore we have the following definitions of electron operators (trivial sector) as shown in TABLE [VI]:

\[
1 \sim \tilde{j}_1 e^{i \frac{\phi_1 + \phi_2}{2}} \sim \tilde{f}_1 e^{i \phi_1}.
\]

This enables us to obtain all the 9 inequivalent quasiparticles (superselection sectors) as summarized in TABLE [VI] for the non-Abelian topological order acquired by gauging $Z_2$ symmetry in these SET phases.

As discussed earlier, in the vertex algebra approach, the mutual statistics of two quasiparticles $A$ and $B$ is
given in their OPE \([D6]\) by \(S_{A,B} = \exp(i\delta_{A,B}) = \exp(-2\pi i\alpha_{A,B})\). If quasiparticles \(A\) and \(B\) leads to more than one fusion channels, the corresponding entry \(S_{A,B} = 0\) vanishes in the modular \(S\) matrix. Besides, scaling dimensions \(\{\nu\}\) of quasiparticles \(\{q\}\) determine their topological spins \(\Delta_{A,B} = \delta_{A,B}\exp(2\pi i\alpha_{A,B})\), which corresponds to the modular \(T\) matrix. So we can extract all the topological properties of the non-Abelian topological orders, obtained by gauging \(Z_2\) symmetry in SET phases.

The modular \(S\) matrix in the basis \(q_a\) \((0 \leq a \leq 8\), see TABLE [VI] of the 9 different quasiparticles (superselection sectors) is \(S_{\#1} =\)

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 2 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
1 & 1 & 1 & 1 & 2 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
1 & 1 & 1 & 1 & 1 & -2 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\
1 & 1 & 1 & 1 & 1 & -2 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\
2 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\end{pmatrix}
\]

for gauged “unconventional” SET phase \#1. Meanwhile the \(T\) matrix is a diagonal unitary matrix \(T_{\#1} = \delta_{a,b}\exp(2\pi i\alpha_{a,b})\), where \(\alpha_{a,b}\) gives the topological spin \(\exp(2\pi i\alpha_{a,b})\) of quasiparticle \(q_a\) shown in TABLE [VI]. To be specific we have

\[
T_{\#1} = \begin{pmatrix}
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 2 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
1 & 1 & 1 & 1 & 2 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
1 & 1 & 1 & 1 & 1 & -2 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\
1 & 1 & 1 & 1 & 1 & -2 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\
2 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\end{pmatrix}
\]

and its \(T\) matrix as

\[
T_{\#2} = \begin{pmatrix}
1 & 1 & 1 & 1 & 2 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
1 & 1 & 1 & 1 & 2 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
1 & 1 & 1 & 1 & 1 & -2 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\
1 & 1 & 1 & 1 & 1 & -2 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\
2 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\end{pmatrix}
\]

Clearly after gauging the unconventional \(Z_2\) symmetry, SET phases \#1 and \#3 lead to the same non-Abelian topological order. They share the same \(S\) and \(T\) matrices, differing by a relabel of quasiparticles in TABLE [VI]. For example quasiparticle \(q_4\) in \#1 phase corresponds to quasiparticle \(q_6\) in \#3 phase. Similarly SET phases \#2 and \#4 lead to the same non-Abelian topological order, by gauging the unconventional \(Z_2\) symmetry.

It’s easy to verify that they satisfy the following consistency condition\[^{19}\] for modular transformations:

\[(ST)^3 = \Theta \cdot S^2, \quad S^4 = 1.\] (D10)

where the \(U(1)\) phase factor \(\Theta\) is defined as

\[\Theta \equiv d_a^2 \cdot e^{2\pi i\alpha_a} / \sqrt{\sum_a d_a^2} = e^{2\pi i c_-/8}.\] (D11)

d\(_a\) and \(\alpha_a\) corresponds to the quantum dimension and topological spin \(\exp(2\pi i\alpha_a)\) of quasiparticle \(q_a\) respectively. \(c_-\) is the chiral central charge of the edge excitations of the topological ordered phase. All the non-Abelian topological orders in TABLE [VI] have \(c_- = 0\) and hence \(\Theta = 1\). In fact for all the 4 non-Abelian topological orders (\#1–\#4) summarized in TABLE [VI] their modular \(S\) and \(T\) matrices satisfy \(S^2 = (ST)^3 = 1_{9\times9}\).

Starting from a \(Z_2\) gauge theory (\(Z_2\) spin liquid or double semion theory) with unitary \(Z_2\) symmetry, once the symmetry is gauged, a resultant \(Z_2 \times Z_2\) gauge theory is expected\[^{10}\]. The above non-Abelian topological orders can be regarded as “unconventional” \(Z_2 \times Z_2\) gauge theories, related to Kitaev’s 16-fold way classification\[^{39}\] of \(Z_2\) gauge theories in \(2+1\)-D. In particular, they are associated with \(Z_2\) gauge theories where fermions having an odd Chern number \((\nu = \text{odd})\) couple to \(Z_2\) gauge fields. Notice that before gauging the symmetry, all 4 SET phases have non-chiral edge excitations with chiral central charge \(c_- = 0\). As a result, we expect that after gauging the \(Z_2\) symmetry their edge states remain non-chiral and should be gapped due to backscattering in a generic situation. Indeed in all the “gauged” non-Abelian topological orders in TABLE [VI] a \(Z_2\) gauge theory with fermion Chern number \(\nu\) is always accompanied by its time-reversal counterpart \(\nu = 16 - \nu\) mod 16 through a direct product.
To be specific, in Ref. [39] Kitaev introduced a 16-fold way classification of 2+1-D $Z_2$ gauge theories, describing fermions coupled to a $Z_2$ gauge field. When the Chern number $\nu$ of fermions changes by 1, one ends up with the same $Z_2$ gauge field. Specifically when $\nu = \text{odd}$, associated $Z_2$ gauge theory contains 3 inequivalent quasiparticles: vacuum (or boson) 1, fermion $\psi$ ($\varepsilon$ in Kitaev’s notation) [39] and vortex $\sigma$. Their fusion rules are the same as [9], i.e. those in Ising anyon theory [39]. Their quantum dimensions are

\[ d_1 = d_\psi = 1, \quad d_\sigma = \sqrt{2}. \]

The topological spin $\exp(2\pi i h)$ of these quasiparticles are given by

\[ h_1 = 0, \quad h_\psi = \frac{1}{2}, \quad h_\sigma = \frac{\nu}{16}. \]

Therefore when $\nu = 1$ this corresponds to the Ising anyon theory. When a direct product of a $Z_2$ gauge theory with Chern number $\nu$ (we denote this $Z_2$ gauge theory by $\nu$) and its time reversal counterpart $\bar{\nu} = 16 - \nu$ is made, one can combine the fermion $\psi$ in $\nu$ and the vortex $\sigma$ in $\bar{\nu}$ to form a new vortex operator, which have scaling dimension $\frac{1}{2} - \frac{\nu}{16} = \frac{8 - \nu}{16}$. Therefore one can clearly see the following two seemingly different direct products

\[ \nu \otimes (16 - \nu) \simeq (8 - \nu) \otimes (8 + \nu). \]  

lead to the same topological order. As a result SET phases #1 and #3 (#2 and #4) in TABLE [I] lead to the same non-Abelian topological order, by gauging the unitary $Z_2$ symmetry.

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76. Recently we noticed that Cheng and Gu have reached the same conclusion.
77. Although the formulation of $U(1)^N$ Chern-Simons theory in \[ \] seems to suggest existence of $N$ conserved $U(1)$-currents, they can actually be explicitly broken by $e.g.$ introducing Higgs terms. These Higgs terms merely condense local bosonic excitations (instead of nonlocal anyonic excitations) and hence don’t change the superselection sectors or any topological properties of the phase.
78. In fact there are other solutions to conditions (27), such as $W^g = -\frac{1}{4} \times 4$, $W^g = -\sigma_z \times 1_2 \times 2$, $W^g = \pm \frac{1}{4} \times 2 \times 2 \times \sigma_x$, and $W^g = \pm \sigma_z \times \sigma_y$, where $\sigma_x, \sigma_y, \sigma_z$ are the three Pauli matrices. Whether they may lead to new SET phases are is clear to us at this moment, and we don’t consider them in this work.
79. In a half-integer spin system, on the other hand, spin rotations by an angle of $2\pi$ will lead to a Berry phase $-1$. Therefore the group structure generated by $\pi$-spin-rotations along $\hat{x}$ and $\hat{z}$ directions is not $Z_2 \times Z_2$ as in [45].
80. We believe there is no “unconventional” implementation of onsite $Z_2$ symmetry in the double semion theory. This is because here the three types of anyons $s, \bar{s}, b$ have different self statistics. As a result exchanging any two of them shouldn’t be a symmetry of the system.