The Fidelity of an Encoded [7,1,3] Logical Zero

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Abstract

I calculate the fidelity of a [7,1,3] CSS quantum error correction code logical zero state constructed in a non-equiprobable Pauli operator error environment for two methods of encoding. The first method is to apply fault tolerant error correction to an arbitrary state of 7 qubits utilizing Shor states for syndrome measurement. The Shor states are themselves constructed in the non-equiprobable Pauli operator error environment and their fidelity depends on the number of verifications done to ensure multiple errors will not propagate into the encoded quantum information. Surprisingly, performing these verifications may lower the fidelity of the constructed Shor states. The second encoding method is to simply implement the [7,1,3] encoding gate sequence also in the non-equiprobable Pauli operator error environment. Perfect error correction is applied after both methods to determine the correctability of the implemented errors. I find that which method attains a higher fidelity depends on the which of Pauli operator errors is dominant. Nevertheless, perfect error correction suppresses errors to at least first order for both methods.

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I. INTRODUCTION

Quantum error correction (QEC) will be vital for the proper working of any hoped-for quantum computer [1]. QEC codes must be more elaborate than classical codes due to the need to correct both bit-flip and phase-flip errors [2]. While such codes exist [2] their existence alone is not sufficient to guarantee the theoretical possibility of constructing a working quantum computer. Rather a framework is required to ensure accurate manipulation and recovery of quantum information despite the inevitable presence of errors. This framework has been painstakingly built over the last 15 years and is called quantum fault tolerance [4–7].

Here, we carefully examine a basic component of this fault tolerant framework for the [7,1,3] QEC code: constructing an encoded logical zero. The [7,1,3] code, or Steane code [8], fully protects one qubit of quantum information from bit-flip and phase-flip errors by embedding it into a system of seven qubits. Error correction is implemented by performing controlled-NOT (CNOT) gates between the 7 data qubits and ancilla qubits which are measured to determine the error syndrome. Based on the outcome of the syndrome measurements the appropriate recovery operation is applied. To assure fault tolerance, gates must be applied in such a way as to ensure errors will not propagate in an uncontrolled fashion during algorithm implementation. In addition, the code can be concatenated [9] to decrease the tolerable error probability threshold.

A method of encoding the state \(|0\rangle\) into the Steane code in a fault tolerant manner is to apply fault tolerant error correction to any initial state of 7 qubits. This requires construction of proper ancilla syndrome qubits such that each ancilla interacts with no more than one data qubit, thereby stemming uncontrolled propagation of possible ancilla qubit errors. For the Steane code appropriate ancilla qubits are four qubit Shor states [8]. Shor states are simply GHZ states with Hadamard gates applied to each qubit. A second encoding method is to utilize Steane’s encoding gate sequence [8]. This method is not fault tolerant because an error in the encoding sequence can propagate to multiple qubits.

In this paper I compare the accuracy with which the two methods encode logical zero states using a reasonable, but general, error model: a non-equiprobable Pauli operator error environment [10]. As in [11], this model is a stochastic version of a biased noise model that can be formulated in terms of Hamiltonians coupling the system to an environment. In the model used here, however, the probabilities with which the different error types will take place is left arbitrary: the environment causes qubits to undergo a \(\sigma_x\) error with probability \(p_x\), a \(\sigma_y\) error with probability \(p_y\), and a \(\sigma_z\) error with probability \(p_z\), where \(\sigma_i\), \(i = x, y, z\) are the Pauli spin operators on qubit \(i\). I assume that only qubits taking part in a gate will undergo error and the error is modeled to occur after (perfect) gate implementation. Qubits not involved in a gate are assumed to be perfectly stored. While this represents an idealization, it is justifiable as all accuracy measures are calculated only to second order in the error probabilities \(p_i\). I also make the simplification that possible errors in preparation and measurement can be assumed to have occurred in subsequent or immediately preceding gates.

After (noisy) construction of the logical zero state I apply noiseless quantum error correction. The accuracy of the error corrected state tells us whether or not the errors that had occurred during the encoding are, in principle, correctable. An encoding method that produces uncorrectable errors with probabilities first order in the \(p_i\) would presumably be unusable for practical, fault tolerant, quantum computation.

A previous comparison of these two construction methods using the less general equiprobable error model was done in Ref. [12]. There are other significant distinctions between that work and what is considered here. In the current work Shor states are explicitly constructed and verified and their accuracy is determined. These ( nec-
essarily) noisy Shor states are then used for syndrome measurement in the fault tolerant encoding method. In addition, the fidelity used in this work as an accuracy measure is the fidelity with respect only to the desired state (logical zero). Due to these differences, the results I present here do not reduce to those of Ref. [12].

II. SHOR STATE CONSTRUCTION

To implement fault tolerant encoding of a logical zero state, it is first necessary to construct Shor states for the purpose of syndrome measurement. Perfect Shor states (without the final Hadamard gates as explained below) are given by $|\psi_{\text{Shor}}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$. For our simulations we must determine what the Shor state will look like when constructed in the non-equiprobable error environment. These states can then be used for our simulations of syndrome measurements of the 7 data qubits which are to make up the logical zero. To construct the Shor state we start with four qubits assumed to be perfectly initialized in the state $\rho_i = |\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|000\rangle$. As each gate is applied, the non-equiprobable error environment causes the qubits taking part in the gate to probabilistically undergo errors as described above. Any attempted performance of a unitary CNOT gate with control qubit $j$ and target qubit $k$, $C_J NOT_k$, on any state $\rho$ actually implements:

$$\rho_{\text{Shor-err}} = \sum_{a,b} A_{a,b} A_{a,b}^\dagger \rho (A_{a,b}^\dagger)^\dagger (A_{a,b})^\dagger.$$

To quantify the accuracy of the constructed Shor state we use the fidelity given by $F = \langle |\psi_{\text{Shor}}| \psi_{\text{Shor-err}}\rangle/\langle |\psi_{\text{Shor}}| |\psi_{\text{Shor}}\rangle$. For the non-equiprobable error environment we find that the fidelity of the constructed Shor state is:

$$F_{\text{noAnc}} = 1 - 6p_x - 6p_y + 30p_x^2 - 6p_y^2.$$  

Since we have ignored higher order terms, this expression is accurate only for small values of $p_i$. Were we to simply apply the above described gate sequence we would have not built the Shor states in a fault tolerant fashion. This is because multiple errors in Shor state construction can propagate into the data qubits when the Shor states are used for syndrome measurement. We need to test the Shor states to ensure that multiple errors have not taken place. This is done utilizing an ancilla qubit, initially in the state $|\psi_i\rangle$. The Shor state measurement yields $|1\rangle$, the Shor state is immediately discarded. Of course, the CNOT gates used in this parity measure are themselves performed in the non-equiprobable error environment and their dynamics follows that of Eq. [1].

I utilize an initial ancilla qubit to measure the parity of qubits 1 and 4. After application of the CNOT gates the ancilla will measure $|0\rangle$ with probability (to second order) of $1 - 6p_x - 6p_y + 30p_x^2 + 30p_y^2 + 60p_xp_y$. The resulting Shor state fidelity is now

$$F_{1,\text{Anc}} = 1 - 4p_x - 4p_y - 9p_x^2 - 7p_y^2 - 17p_xp_y + 27p_xp_z - 8p_y^2 + 27p_yp_z + 72p_z^2.$$  

While the verification should work to ensure that errors cannot propagate into the data qubits, it increases the fidelity of the constructed Shor state only for certain values of the $p_i$. However, when $\sigma_z$ is the dominant error
the fidelity with respect to \( \sigma_x \) errors as shown in Fig. 2. This is likely due to the fact that the verification is a parity check and does not check on the phase between the qubits.

Having constructed the noisy Shor states I now use them to perform fault syndrome measurements so as to project an initial state of seven qubits into an encoded logical zero of the Steane code. I choose to utilize Shor states with the two verification steps used above, the first comparing qubits 1 and 4, and the second comparing qubits 1 and 2. A full comparison of the final encoded zero fidelity dependent on the choice of Shor state verification steps will be presented elsewhere.

### III. FAULT TOLERANT ENCODING

To implement error correction on qubits encoded in a Steane code requires 6 syndrome measurements. The first three of the syndrome measurements are used to check for bit-flip errors, using the arrangement of CNOT gates shown in Fig. 1. Then, a Hadamard gate is applied to each qubit, rotating phase-flips to bit-flips, and the final three syndrome measurements are done in the same way to check for phase-flip errors. This is followed by a final set of Hadamards to return the data qubits to their original basis. For a fault tolerant implementation of QEC each syndrome measurement uses four ancilla qubits in a Shor state such that one CNOT only is applied between the appropriate data qubit and a syndrome qubit. In addition, each syndrome measurement should be done twice to account for possible errors in the syndrome measurement itself.

To fault tolerant encode a logical zero, we start with 7 qubits all in state \( |0\rangle \) assumed to be initialized perfectly. This choice absolves us from performing the first set (bit-flip) of syndrome measurements as they will result in no evolution of the data qubits. Instead we proceed to measuring the phase flip syndrome which we would like to do with the least number of gates. We can eliminate all of the necessary Hadamard gates by utilizing the equality pictured in the box of Fig. 1. In words, this equality states that applying Hadamard gates on the control and target qubits before and after application of a CNOT gate is equivalent to simply reversing the roles of the qubits in the CNOT gate. Thus, we can apply the phase-flip syndrome measurement CNOT gates with the data qubits as the target and the ancilla qubits as the control as long as the proper Hadamard gates are applied. However, Hadamard gates are already necessary on the data qubits before and after the phase-flip syndrome measurement. Thus, reversing the CNOT gates eliminates these Hadamards. In addition, reversing the CNOT gates removes the need to apply the final Hadamard gates in the Shor state construction leaving only the need to perform Hadamards on the Shor state qubits after the CNOT gates. We can obviate even this need by simply measuring the Shor state qubits in the \( x \)-basis rather than the

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**FIG. 2:** Contour plots comparing the fidelity of three methods of constructing Shor states, no ancilla based verifications (white), one verification between qubits 1 and 4 (black), and two verifications, the first between qubits 1 and 4 and the second between qubits 1 and 2 (gray). The contours show where in the error space \((p_x, p_y, p_z)\) the fidelity of the constructed Shor state is equal to .99. Note that the origin (where the three probabilities equal zero) is at the lower back left corner. The once verified Shor state achieves a fidelity of .99 at higher error probabilities than the twice verified Shor state except in the case of very low \( p_z \). For higher values of \( p_z \) the unverified state achieves a fidelity of .99 at the highest error probabilities.
z-basis and in this way eliminate all Hadamard gates.

In the actual performance of the first encoding method we join a (necessarily) noisy Shor state to the 7 qubits all in the state |0⟩ and apply the required CNOT gates as outlined above (with the ancilla qubits as the control). As with the construction of the Shor states themselves, these CNOT gates are performed in a non-equiprobable error environment such that the actual CNOT evolution follows Eq. 4. The four syndrome qubits are measured in the x-basis and the syndrome measurement result is determined from the parity of these measured qubits. The entire syndrome measurement is then repeated such as to get the same result twice in a row. The repetition is necessary to protect against the possibility of errors during the syndrome measurement itself. Since it is the parity of the four measured qubits that is important, there are different possible success outcomes each of which would lead to a slightly different final density matrix. In this paper I choose to analyze the scenario where all four qubits are measured as zero.

Assuming perfect initialization of the 7 data qubits, utilizing the noisy Shor states discussed above, implementing the necessary (error prone) CNOT gates, selecting the case where all syndrome measurements give all zeros, and repeating each syndrome measurement twice, provides a full simulation of fault tolerant encoding of a logical zero for the [7,1,3] QEC code and constructs an imperfect logical zero state ρ_{TL}. To quantify the accuracy of the encoding process we look at two distinct fidelity measures. The first compares the seven qubit state ρ_{TL} to the ideal seven qubit logical zero state |0_L⟩,

\[ F_{0_L} = \langle 0_L | \rho_{TL} | 0_L \rangle. \]

This quantifies how well the encoding process was implemented. For the non-equiprobable error model we find (to second order):

\[ F_{0_L} = 1 - 48p_x - 12p_y - 12p_z + 1236p_x^2 + 504p_xp_y + 528p_xp_z - 596p_y^2 + 1795p_yp_z - 1092p_z^2. \]

Here, the fidelity is most sensitive to σ_x errors than other types of errors.

While the fidelity of the entire seven qubit system is an appropriate accuracy measure for the implementation of a process, it may not be an appropriate measure for the accuracy with which the single qubit of quantum information is stored. Errors accounted for in the seven qubit fidelity may reside in degrees of freedom of the system that do not affect the actual stored quantum information. To check the fidelity of the stored quantum information one may, for example, perform destructive measurement on all of the qubits followed by classical error correction and the determination of the parity of the obtained codeword. The probability of correctly reading out zero (in our case) would then serve as a measure of accuracy. Or, one may perform a parity measure of the seven qubit system using an ancilla qubit. These accuracy measures will be explored elsewhere. Here, I apply a perfect decoding sequence to the seven qubit system and partial trace over qubits 2 through 7. The state of the remaining qubit ρ_1 is then compared to the single qubit state |0⟩, via the fidelity \( F_{0} = \langle 0 | \rho_1 | 0 \rangle \). For fault tolerant encoding this fidelity (up to second order) is given by:

\[ F_{0} = 1 - 16p_x - 4p_y + 226p_x^2 + 102p_xp_y - 188p_y^2. \]

Again we see that the fidelity is most sensitive to σ_x errors.

### IV. GATE SEQUENCE ENCODING

The second logical encoding method is implemented via the gate sequence shown in Fig. 2. As above, we assume perfect initialization of the 7 qubits each in the state |0⟩ and an environment that causes non-equiprobable errors, σ_x, σ_y, and σ_z on each of the qubits involved in a gate. Evolution of each of the 11 CNOT gates follows Eq. 4 and the evolution of a Hadamard gate on a state ρ is given by:

\[ \sum_{a=x,y,z} p_a \sigma_a^\dagger H \rho H^\dagger \sigma_a^a. \]

When the state to be encoded is |0⟩, as explored here, we can skip the first two CNOT gates (put in brackets in the figure) as they will have no effect on the target qubits. The entire encoding thus requires 3 Hadamard and 9 CNOT gates. The fidelity of the seven qubit logical zero state after application of this gate sequence (to second order) is:

\[ F_{0_L} = 1 - 18p_x - 21p_y - 21p_z + 166p_x^2 + 360p_xp_y + 360p_xp_z + 211p_y^2 + 433p_yp_z + 248p_z^2. \]

This fidelity is better than that achieved using the first encoding method when σ_x errors are dominant, despite the fact that this second method is not fault tolerant.

The fidelity of the single qubit of stored quantum information for the encoding gate sequence is given by:

\[ F_{0} = 1 - 8p_x - 8p_y + 56p_z^2 + 112p_xp_y + 56p_y^2. \]

Again, with respect to this measure the gate sequence encoding is more accurate when σ_x errors are dominant.

### V. APPLYING PERFECT ERROR CORRECTION

Applying perfect error correction allows us to test the ‘correctability’ of the types of errors that occur during the encoding. If even perfect error correction cannot (to first order) correct the errors that occur during encoding then the encoding method cannot be usable for practical implementations of quantum computation. I apply perfect error correction to the states constructed using each of the two methods described in the previous two sections.
and calculate both fidelity measures (of the seven qubit system and the one encoded qubit). For the fault tolerant encoding method both of the fidelities are found to be $1 - O(p^3)$. For the gate sequence method, both fidelities are found to be $1 - 9p^2$. Both encoding methods thus produce usable logical zeros for quantum computation and the advantage of utilizing fault tolerant techniques is demonstrated by the suppression of the second order error term. It should be stressed, however, that in realistic systems the error correction itself will be imperfect and thus the error correction may not correct all errors to first order. This will be explored in future work.

VI. CONCLUSION

In conclusion, I have simulated two methods of encoding a logical zero for the [7, 1, 3] CSS quantum error correction code, or Steane code, in a non-equiprobable error environment. For both methods, each of the seven qubits are initially in the state $|0\rangle$. To implement the first, fault tolerant, method requires the use of Shor states for syndrome measurement. Shor state construction involves (error prone) CNOT gates followed by verification of the state via parity measurements between random pairs of data qubits. While fault tolerance demands these verifications I have found that the fidelity of the constructed Shor state may decrease upon applying verification in cases where $\sigma_z$ errors are dominant. The noisy Shor states are then used for error syndrome measurement in the logical zero encoding. The second, non-fault tolerant, encoding method is to implement an error encoding gate sequence. Which method provides more accurate encoded logical zero states depends on the values of the $p_i$ characterizing the non-equiprobable error model. Finally, I apply perfect error correction to the states constructed by the two methods and find that, to second order, all of the errors are corrected up to second order. This implies that either construction method can be used for fault tolerant quantum computation.

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