A Quantum Algorithm for Minimum Steiner Tree Problem

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Minimum Steiner tree problem is a well-known NP-hard problem. For the minimum Steiner tree problem in graphs with \( n \) vertices and \( k \) terminals, there are many classical algorithms that take exponential time in \( k \). In this paper, to the best of our knowledge, we propose the first quantum algorithm for the minimum Steiner tree problem. The complexity of our algorithm is \( O^*(1.812^k) \). A key to realize the proposed method is how to reduce the computational time of dynamic programming by using a quantum algorithm because existing classical (non-quantum) algorithms in the problem rely on dynamic programming. Fortunately, dynamic programming is realized by a quantum algorithm for the travelling salesman problem, in which Grover’s quantum search algorithm is introduced. However, due to difference between their problem and our problem to be solved, recursions are different. Hence, we cannot apply their technique to the minimum Steiner tree problem in that shape. We solve this issue by introducing a decomposition of a graph proposed by Fuchs et al.

I. INTRODUCTION

Given an undirected graph \( G = (V, E) \), a weight \( w : E \to \mathbb{R}^+ \), and a subset of vertices \( K \subseteq V \), usually referred to as terminals, a Steiner tree is a tree that connects all vertices in \( K \). In this paper, let \( n = |V| \) be the size of vertices and \( k = |K| \) be the size of terminals. A Steiner tree \( T \) is the minimum Steiner tree (MST) when the total edge weight \( \sum_{e \in E(T)} w(e) \) is the minimum among all Steiner trees of \( K \). Note that all leaves of a Steiner tree \( T \) are vertices in \( K \). The task that finds a minimum Steiner tree is called minimum Steiner tree problem, and this problem is known as an NP-hard problem \[2\]. Note that for fixed \( k \), this problem can be solved in polynomial time, which means that the minimum Steiner problem is fixed parameter tractable \[2, 3\].

A naive way to solve the minimum Steiner tree problem is to compute all possible trees. However, the number of all trees in the graph \( G = (V, E) \) is \( O(2^{|E|}) \) at worst. However, an exhaustive search is not realistic. The Dreyfus-Wagner algorithm (the D-W algorithm) is a well-known algorithm based on dynamic programming for solving the Steiner problem in time \( O^*(3^k) \) \[4\]. The \( O^* \) notation hides a polynomial factor in \( n \) and \( k \). This algorithm has been the fastest algorithm for decades. In 2007, Fuchs et al. \[5\] have improved this to \( O^*(2.684^k) \) and Mölle et al. \[6\] to \( O((2 + \delta)^k n^{f(\delta^{-1})}) \) for any constant \( \delta > 0 \). For a graph with a restricted weight range, Björklund et al. have proposed an \( O^*(2^k) \) algorithm using subset convolution and Möbius inversion \[7\]. An important thing is that the dynamic programming part of these algorithms \[6, 7\] use the D-W algorithm.

In order to speed up classical algorithms, use of quantum algorithms is an effective technique. In particular, Grover’s quantum search (Grover search) \[8\] and its generalization, quantum amplitude amplification \[9, 10\], are widely applicable. Grover search brings quadratic speed up to an unstructured search problem \[8, 11\]. This is one of the advantages quantum algorithms have over classical algorithms. For NP-hard problems, speeding up using Grover search is one of the ways to make an algorithm faster than the best classical algorithm in many problems. For example, in \[12\], by using quantum computers, the Travelling Salesman Problem (TSP) for a graph which has \( n \) vertices is solved in time \( O^*(\sqrt{n!}) \) which is the square root of the classical complexity \( O^*(n!) \) of an exhaustive search. However, the best classical algorithm for TSP takes only \( O^*(2^n) \) \[13, 14\] which is clearly faster than \( O^*(\sqrt{n!}) \).

In order to speed up algorithms for the minimum Steiner tree problem, it is thought that use of Grover search is also an effective technique. Combining classical algorithms with Grover search is one of the ways to make an algorithm faster than the best classical algorithm. For example, Ambainis et al. \[15\] have combined Grover search with algorithms for TSP, Minimum Set Cover Problem and so on that use dynamic programming. A naive way is replacing the dynamic programming part of the algorithm of Ambainis et al. by D-W algorithm. However, we cannot use the method of Ambainis et al. in the same way because the characteristic of minimum Steiner tree problem differs from that of TSP. Hence, we adapt this method to a method proposed by Fuchs et al. \[5\] for applying Grover search. The decomposition method of Fuchs et al. is optimized for a classical
computer. We optimize the decomposition for a quantum computer. Our algorithm achieved the complexity $O^{*}(1.812^k)$. Table 3 shows the complexity of classical algorithms for minimum Steiner tree problem and our proposed algorithm.

II. PRELIMINARIES

a. Graph Contraction  For a graph $G = (V, E)$ and a subset of vertices $A \subseteq V$, a graph contraction $G/A$ is a graph which is obtained by removing all edges between two vertices in $A$, replacing all vertices in $A$ with one new vertex $v_A$, and rejoining all the edges joined to vertices in $A$ to $v_A$.

b. Dürr-Høyer algorithm  Dürr-Høyer quantum algorithm (D-H algorithm) [10] is an algorithm for finding the minimum in an unsorted database based on Grover’s quantum search algorithm [8]. Let $a_1, ..., a_n$ be unsorted elements from an ordered set, and consider the task that finds the minimum $\min_i [a_i | i = 1, ..., n]$. We need $O(n)$ operations to solve this task on a classical computer. Dürr-Høyer quantum algorithm solves this task with $O(\sqrt{n})$ operations.

III. PRIOR WORK

In this section we explain three algorithms as prior work. We briefly explain two classical algorithms for the minimum Steiner tree problem in Secs. III A and III B and show a quantum algorithm for TSP proposed by Ambainis et al. in Sec. III C. For explaining the algorithms in Secs. III A and III B we use a graph in Figure 1(a) as an example.

A. The Dreyfus-Wagner algorithm

For a given subset of vertices $K \subseteq V$, the task is to find the minimum Steiner tree $T$ for $K$. The D-W algorithm computes all minimum Steiner trees for $X \cup \{p\}$ where $X \subseteq K, p \in V$. The minimum Steiner tree $T$ is always decomposed into three subtrees $T_1, T_2$ and $T_3$ where $T = T_1 \cup T_2 \cup T_3$. The decomposition is realized as follows.

1. Consider a tree that connects $X \cup \{q\}$ and look at this tree from $q$, where $q$ is a root of this tree.
2. Starting from $q$, go down the tree until reaching either a vertex in $X$ or a vertex of degree at least 3. We call the vertex $p$. Note that it is possible that $p = q$.
3. If $p = q$, we define $D = \{p\}$ and if not, let $D$ be one connected component of $X$ when we remove $p$ from the tree.

Regardless of how to define $D$, $T$ is decomposed into three parts:

- the path from $p$ to $q$
- the tree that connects $p$ and $D$
- the tree that connects $p$ and $X \setminus D$

For example, Figure 2 shows the tree connecting $K = \{q\} \cup X, X = \{d, e, f, g, h, i, j, k, l, m\}$ that are drawn in circled nodes. In the figure, the tree is decomposed into the black path from $p$ to $q$, the red tree that connects $p$ and $D = \{d, e, f, g, h, i\}$, and the blue tree that connects $p$ and $X \setminus D = \{j, k, l, m\}$.

Let $W(Y)$ be the weight of a tree that connects $Y \subseteq V$. By using this decomposition property, we obtain the following recursion.

$$W(\{q\} \cup X) = \min_{p \in V} \{ W(\{p, q\}) + g_p(X) \}$$

$$g_p(X) = \min_{D \subseteq X} \{ W(\{p\} \cup D) + W(\{p\} \cup (X \setminus D)) \}$$

If we compute $W(K)$ by this recursion, then we can easily construct the MST for $K$.

The D-W algorithm uses Eq. (1) recursively for $p \in V$ and $X$ of size $|X| = i, i = 1, ..., k$. If we have already obtained all trees for $p \in V$ and $X$ of size $|X| = i - 1$, then

$$W(\{q\} \cup X)$$

for a given $X$ of size $|X| = i$ can be computed by Eq. (1) in time $O^*(2^i)$ since the number of subsets of $X$ of size $|X| = i$ is $2^i$. Hence, the total complexity of the D-W algorithm is

$$O^*(\frac{k}{i}) = O^*(3^k).$$

B. The algorithm proposed by Fuchs et al.

In the D-W algorithm, we have to search all subsets $D \subseteq X$. Fuchs et al. have improved the D-W algorithm by limiting the sizes of subset of $K$ and dividing the algorithm into two parts: a dynamic programming part and a part which merges three subtrees. We need to define the decomposition that is introduced in [5].

a. Definition  For a tree $T$, an $r$-split of $T$ is a partition

$$T = T_1 \cup \cdots \cup T_r$$

such that each $T_i \cup \cdots \cup T_j, i = 1, ..., r$ is connected.

We also use the following notation, which is illustrated in Fig. 2.

$$A_i := V(T_i) \cap \left[ V(T_1) \cup \cdots \cup V(T_{i-1}) \right]$$

$$A_i := V(T_i) \cap \left[ V(T_1) \cup \cdots \cup V(T_r) \right] \setminus A_i^-$$

$$A_i := A_i^+ \cup A_i^-$$

$$K_i := K \cap V(T_i) \setminus A_i$$

Note that $V(T_i)$ shows the set of vertices of the tree $T_i$. We call $A = \bigcup_i A_i$ the set of the split nodes. According
to this notation, the decomposition in the D-W algorithm has one split node. Fuchs et al. proposed the decomposition that has $\log(1/\varepsilon)$ split nodes where $\varepsilon$ is a small positive number. This allows us to decide the size of $K_i$ in error less than $\varepsilon$. Furthermore, each $T_i$ is a minimum Steiner tree for $K_i \cup A_i^+ \cup v_A^-$ in the graph contraction $G/A_i^-$. We prove this property for 2-partition in Sec. IV.

In Figure 4 the tree that connects $K = \{a\} \cup \{f,g,h,i,j,k,l,m,n,o\}$ is decomposed into 2 parts. The red tree connects nodes $a, f, g, h, i, j$ and the blue tree connects nodes $k, l, m, n, o$. In this situation, the split nodes are nodes $b$ and $d$. In the graph $G$, the red tree and the blue tree have the same edge connecting $b$ and $d$. Hence, if we simply merge them in $G$, nodes $b$ and $d$ are connected by two edges. However, considering the red tree in $G$ and the blue tree in the graph contraction $G/\{b,d\}$, we can count edges without duplication. As shown in Figure 3 the red tree is an MST for $K_1 \cup A = \{a, f, g, h, i, j\} \cup \{b, d\}$ in $G$, and the blue tree is an MST for $K_2 \cup \{v_{(b,d)}\} = \{k, l, m, n, o\} \cup \{v_{(b,d)}\}$ in $G/\{b,d\}$.

The algorithm proposed by Fuchs et al. computes MSTs for all subsets of $K$ whose sizes are less than a certain constant $\alpha k$, $\alpha \in (0, \frac{1}{2}]$ using dynamic programming. Then, apply an exhaustive search for the 3-split $T = T_1 \cup T_2 \cup T_3$. We obtain the complexity $O^*(2.684^k)$ by optimizing the parameter $\alpha$ ($\alpha \approx 0.4361$).

C. A Quantum Algorithm for Travelling Salesman Problem

In [15], Ambainis et al. have proposed quantum algorithms for several problems which are solved using dynamic programming on a classical computer. In particular, they have proposed a quantum algorithm for TSP in time $O^*(1.728^n)$. This algorithm combines D-H algorithm with Bellman-Held-Karp algorithm [13, 14] which solves TSP classically in time $O(n^{2.575})$. For a graph $G = (V,E)$ with $|V| = n$, and a weight $w : E \to \mathbb{R}^+$, TSP is a problem that finds the shortest simple cycle that visits each vertex. For $S \subseteq V$ and $u, v \in S$, let $f(S, u, v)$ be the length of the shortest path which starts in $u$, visits all vertices in $S$ at once, and ends in $v$. The Bellman-Held-Karp algorithm uses the following recursion:

$$f(S, u, v) = \min_{t \in X \setminus \{u,v\}} \{w(u,t) + f(S \setminus \{u\}, t, v)\}. \quad (4)$$

$f(S, u, v)$ can also be calculated recursively by splitting $S$ into two sets. Let $k$ be some fixed number in the range of $[2, |S| - 1]$. Then

$$f(S, u, v) = \min_{X \subseteq S, |X| = k} \min_{u \in X, v \in X^c} f(X, u, t) + f((S \setminus X) \cup \{t\}, t, v). \quad (5)$$

In their algorithm, $f(X, u, t)$ and $f((S \setminus X) \cup \{t\}, t, v)$ are computed classically and then for $|S| = \frac{1}{2}, \frac{3}{2}, k$, $f(S, u, v)$ are computed by applying D-H algorithm to Eq. (5).
In TSP, the two routes that are obtained by dividing the optimal route into two parts are also the optimal routes in the parts. That is, $f(S, u, v)$ is computed by splitting $S$ into two parts of a fixed size and merging the optimums of them. However, in the case of Steiner tree problem in the same way as in [15] since the property shows the set of terminals (the circled nodes) in $T$.

IV. THE PROPOSED ALGORITHM

In TSP, the two routes that are obtained by dividing the optimal route into two parts are also the optimal routes in the parts. That is, $f(S, u, v)$ is computed by splitting $S$ into two parts of a fixed size and merging the optimums of them. However, in the case of $W(K)$ of Eq. (1) in minimum Steiner tree problem, the property like this is not true. In other words,

$$W(K) \neq \min_{K_1 \subseteq K, |K_1| = |T|} \{ W(K_1) + W(K \setminus K_1) \}. \quad (6)$$

Hence, we cannot apply D-H algorithm to minimum Steiner tree problem in the same way as in [15] since the property of minimum Steiner tree problem differs from that of TSP. We propose an alternative recursion based on the method by Fuchs et al. [5]. We regard the split nodes as tentative terminals and then the two subtrees that are obtained by dividing the MST into two parts are also the MSTs for two parts of the terminals.

Here we show the proofs of these things. Let $T$ be a minimum Steiner tree for $K$. $T$ is computed by the following procedure. Using Lemma 1 in [5], we obtain Theorem 1.

**Theorem 1.** For any 2-partition $T = T_1 \cup T_2$,

1. $T_1$ is an MST for $K_1 \cup A$ on $G$.
2. $T_2$ is an MST for $K_2 \cup v_A$ on $G/A$.

**Proof.** 1. Let $T_1$ be any Steiner tree for $K_1 \cup A$ on $G$. $T_1 \cup T_2$ is a Steiner tree for $K \cup A$. Since $T_1 \cup T_2$ is an MST for $K \cup A$,

$$W(T_1 \cup T_2) \leq W(T_1' \cup T_2).$$

Hence, $W(T_1) \leq W(T_1')$, proving 1.

2. Obviously, $T_2$ is a Steiner tree for $K_2 \cup v_A$ in $G/A$. Let $T_2'$ be a Steiner tree for $K_2 \cup v_A$ in $G/A$. $T_1 \cup T_2$ is connected and spans $K \cup A$. Since $T_1 \cup T_2$ is an MST for $K \cup A$,

$$W(T_1 \cup T_2) \leq W(T_1 \cup T_2').$$

Hence $W(T_2) \leq W(T_2')$. \hfill \Box

Furthermore, we use Lemma 2 in [5] as Theorem 2.

**Theorem 2.** There exists a 2-partition $T = T_1 \cup T_2$ which satisfies $|K_1| = |V(T_1) \cap K| = (\alpha + \varepsilon)k$ for any $\varepsilon > 0$ and any $0 < \alpha \leq \frac{1}{2}$. We can choose the corresponding set of split nodes $A$ of size at most $|A| = \lceil \log \frac{1}{\varepsilon} \rceil$.

**Proof.** There exists $v \in V(T)$ such that all components $T_1, T_2, \ldots$ of $T \setminus v$ have sizes $k'_i := |V(T'_i) \cap K| \leq \frac{1}{2}$. For these components $T'_1, T'_2, \ldots$, we let $T'' = T'_1 \cup \cdots \cup T'_r$ where $r$ is the minimum number that satisfies $\sum_{i=1}^{r} k'_i \geq \alpha k$. $T = T'' \cup T_2$ meets all conditions for $|A| = 1$. For the case of $|A| \geq 2$, also we consider $T'' = T'_1 \cup \cdots \cup T'_r$. If $k'' > (\alpha + \varepsilon)k$, then by induction on $|A|$, we can construct a 2-partition $T'' = T''_1 \cup \cdots \cup T''_{r-1} \cup T''_r$ meets all conditions. For more information, you can see the proof of Lemma 1 in [5]. \hfill \Box
FIG. 4. A decomposition proposed by Fuchs et al. (a) A tree in $G$ which has two edges connecting $b$ and $d$. (b) A tree in $G/\{b, d\}$. (c) Graph $G$ containing the tree of (a). (d) Graph $G/\{b, d\}$ containing the tree of (b). (e) The tree with red dotted edges in graph $G$. (f) The tree with blue dashed edges in graph $G/\{b, d\}$. (g) The minimum Steiner tree. This is obtained by merging the tree with red dotted edges of (e) extracted from $G$ and the tree with blue dashed edges of (f) extracted from $G/\{b, d\}$. 
Algorithm 1 Quantum algorithm for Minimum Steiner Tree

MinimumSteinerTree (graph $G = (V, E)$, edge weights $w$, a subset of vertices $K \subseteq V$): a minimum Steiner tree for $K$.

1. Classically compute the values of $W(X)$ for all $X \subseteq K_1, |X| \leq (1-\beta)4+\varepsilon)k$ using dynamic programming.
2. (a) To calculate $W(K')$ for $K' \subseteq K, |K'| = (1\pm \varepsilon)k$, apply D-H algorithm to Eq. (7) with $|K_1| = (\frac{3}{4} \pm \varepsilon)k$.
   (b) To calculate $W(K')$ for $K' \subseteq K, |K'| = (\frac{1}{2} \pm \varepsilon)k$, apply D-H algorithm to Eq. (7) with $|K_1| = (\frac{1}{4} \pm \varepsilon)k$.
   (c) Apply D-H algorithm to Eq. (7) with $|K_1| = (\frac{1}{2} \pm \varepsilon)k$ to find the solution.

By Theorems 1 and 2, we obtain the following recursion.

$$W(K) = \min_{|K_1| = |V(T_1) \cap K| = (\frac{3}{4} \pm \varepsilon)k} \min_{A \subseteq V \setminus |A| = |\log \frac{1}{k}|} \{W(K_1 \cup A) + W(K_2 \cup v_A)_{G/A}\} \quad (7)$$

where $W(K_2 \cup v_A)_{G/A}$ is a weight of an MST for $K_2 \cup v_A$ on $G/A$. As shown in Eq. (6), the tree that is obtained by dividing an MST is not an MST. However, a tree becomes an MST by regarding split nodes $A$ and the node $v_A$ as tentative terminals.

Algorithm 1 shows the proposed algorithm. Our algorithm is constructed by the classical part and the quantum part. First, the algorithm computes minimum Steiner trees for all subsets of sizes at most $\frac{1}{4} - \beta k$ classically. Then, we apply Dürr-Hoyer algorithm to combine these subtrees in three steps. Splitting the tree more than 3 parts makes the complexity of the quantum part worse and using D-H algorithm more than 3 levels also makes the complexity worse. We will discuss these things in Appendix A.

A. Running Time

The complexity of the classical part of this algorithm is

$$O^* \left( (1 + \varepsilon)k ((1 - \beta)4 + \varepsilon)k \right)^{2((1-\beta)/4+\varepsilon)k} \quad (8)$$

The complexity of the quantum part of this algorithm is

$$O^* \left( \sqrt{k \left( (1 + \varepsilon)/k \right)^{((1 + \varepsilon)/k^4)}} \right) \quad (9)$$

Following [5], by substituting $\varepsilon = 0$ in Eqs. (8) and (9), we respectively obtain the following:

$$O^* \left( \left( (1 - \beta)4 k \right)^{2((1-\beta)k/4)} \right) \quad (10)$$
$$O^* \left( \sqrt{k \left( (1/2) k \right)^{((k/2)/k^4)}} \right) \quad (11)$$

The overall complexity is minimized when both parts are equal. Hence the optimal choice for $\beta \in (0, \frac{1}{2}]$ is approximately 0.28325. We will show this in Appendix A. The running time of the algorithm then is $O^* (2^0.8574...k) \approx O^*(1.812k^k)$. 

V. CONCLUSION

In this paper we presented a quantum algorithm for solving the minimum Steiner tree problem. We explained that the complexity of our algorithm is $O^*(1.812k^k)$. As shown in Table 1, the complexity of this algorithm is smaller than those of any classical ones.

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Appendix A: Optimizing Parameters of the Proposed Method

First, we show that applying D-H algorithm in three steps is optimal by discussing a method that uses D-H algorithm in \( r \in \{1, 2, 3, \ldots \} \) steps. We show that applying D-H algorithm in more than three steps increases the complexity of the quantum part and applying D-H algorithm in less than three steps also increases the complexity of the classical part. If the algorithm uses D-H algorithm in \( r \) steps, the complexity of the classical part of the algorithm is

\[
O^* \left( \left( \frac{k}{k/2^r} \right)^2 \right) = O^* \left( \frac{k}{k/2^r} \right)^{2^{k/2^r}} = O^* \left( 2^{k(1-2^{-r})} \right).
\]  

(A1)

The quantum part of the algorithm is

\[
O^* \left( \sqrt{ \left( \frac{k}{k/2^r} \right)^2 \left( \frac{k/2^r}{k/2^{2r}} \right)^2 \left( \frac{k/2^{2r}}{k/2^{3r}} \right)^2 \cdots \left( \frac{k/2^{(r-1)r}}{k/2^{rr}} \right)^2 } \right) = O^* \left( 2^{k(1-2^{-r})} \right).
\]  

(A2)

Figure 5 shows the relationship between the classical part and the quantum part of the complexity as a function of \( r \). As shown in the figure, the two complexities equal at about \( r = 2.55 \). Hence, as its nearest integer, we should choose \( r = 2 \) or 3. If \( r = 2 \), the classical part is over \( O^* (2^k) \), which is worse than the complexity of the best classical algorithm. On the other hand, if \( r = 3 \), both of the complexities of the classical and quantum parts are better than \( O^* (2^k) \). Hence, \( r = 3 \) is optimal.

In the rest of this section, in order to obtain the best possible complexity of the method with \( r = 3 \), we optimize the parameter \( \beta \). According to Stirling’s Formula, Eqs. (10) and (11) are respectively deformed as

\[
O^* \left( \frac{k}{(1-\beta)k/4} \right)^{2(1-\beta)k/4} = O^* \left( \frac{4}{1-\beta} \right)^{\frac{1-\beta}{4} k} \left( \frac{4}{3+\beta} \right)^{\frac{3+\beta}{4} k} 2^{(1-\beta)k/4} = O^* \left( 2^{(\frac{1-\beta}{4} \log_2 \frac{4}{1-\beta} + \frac{3+\beta}{4} \log_2 \frac{4}{3+\beta} + \frac{1-\beta}{4})k} \right) \]  

\[
O^* \left( \frac{\sqrt{k/2^r}}{k/2^r} \right)^{k/4} \left( \frac{k/2^r}{k/4} \right)^{\beta k/4} = O^* \left( \sqrt{2^{k2^{-r}}} \left( \frac{1}{\beta} \right)^{\frac{1-\beta}{4} k} \right) \left( \frac{1}{1-\beta} \right)^{\frac{1-\beta}{4} k} = O^* \left( 2^{k \left( \frac{1-\beta}{4} \log_2 \frac{4}{1-\beta} + \frac{3+\beta}{4} \log_2 \frac{4}{3+\beta} + \frac{1-\beta}{4} \right) k} \right). \]

(A3)

Since the complexity is minimized when the complexities of the classical and quantum parts equal, we can optimize the parameter \( \beta \) by solving the following equation:

\[
\frac{1-\beta}{4} \log_2 \frac{4}{1-\beta} + \frac{3+\beta}{4} \log_2 \frac{4}{3+\beta} + \frac{1-\beta}{4} = \frac{1}{2} \left( \frac{3+\beta}{4} \log_2 \frac{4}{3+\beta} + \frac{1-\beta}{4} \log_2 \frac{1}{1-\beta} \right).
\]  

(A5)
Figure 6 shows the right side and the left side of this equation. The solution of this equation is $\beta \approx 0.28325$. Hence, we can achieve a total time

$$O^* \left( \left( \frac{k}{1 - 0.28325)k/4} \right) 2^{(1 - 0.28325)k/4} \right) \approx O^* (1.812^k)$$

by an appropriately small choice of $\varepsilon > 0$.

Appendix B: Other approaches

We can consider three different strategies to apply D-H algorithm other than the proposed algorithm.

First, there is a way to simply apply D-H algorithm to the method proposed by Fuchs et al., in which D-H algorithm is applied for merging three subtrees. However, to classical preprocessing, we have to take $O^*(2.684^k)$, which is the complexity of the algorithm proposed by Fuchs et al., because the decomposition ratio of the method of Fuchs et al. is optimized for classical computing. Hence, this approach is inefficient.

Second, as discussed above, the ratio in dividing the tree of the algorithm proposed by Fuchs et al. is optimized for classical computing. Hence, we examine the optimum case for quantum computing that the algorithm divides the tree into $\frac{3}{7} : \frac{2}{7} : \frac{2}{7}$ while the method proposed by Fuchs et al. divides the tree into $\alpha : 1 - 2\alpha$ where $\alpha \approx 0.436$. Compared to our algorithm which applies D-H algorithm after trees are divided into two parts, the algorithm examining here applies D-H algorithm after trees are divided into three parts. If we already compute all $W(D)$, $|D| = k/3^r$, we can obtain $W(K)$ by applying D-H algorithm in $r$ levels. In this case the complexity of the classical part is

$$O^* \left( \left( \frac{k}{k/3} \right)^{2k^3 - r} \right) 
= O^* \left( 2^{\left( \log_2 3 + \frac{\varepsilon}{2} \right) k} \left( \frac{3}{2} \right)^k \right). \quad \text{(B1)}$$

That of the quantum part is

$$O^* \left( \sqrt{ \frac{2k/3}{k/3} \left( \frac{k/3}{k/3^2} \right) \left( \frac{k/3^2}{k/3^3} \right) \cdots \left( \frac{k/3^{r-1}}{k/3^r} \right)} \right) 
= O^* \left( 3^{\frac{k}{2} \left( \frac{1}{3} + \frac{1}{3} + \cdots + \frac{1}{3} \right) k} \left( \frac{3}{2} \right)^k 2^{(1 - \frac{1}{3})^k} \right) 
= O^* \left( 2^{\left( \frac{1}{3} \log_2 3 + \frac{1}{3} \log_2 \left( \frac{3}{2} \right) + (1 - \frac{1}{3})^k \right)} \right). \quad \text{(B3)}$$

Note that the second square root of Eq. (B2) exists because how to select the second subtree is not uniquely determined when the tree is divided into three parts. In Figure 7, the red line shows the quantum part and the blue line shows the classical part. We can minimize the complexity when $r \approx 1.09$. However, even when $r \approx 1.09$, the complexity exceed $2^k$, which is the complexity of the best classical algorithm.

Finally, we consider a possibility that makes our algorithm faster by reducing the complexity of the classical part. In our algorithm, the size of the subtree which is computed classically is $\frac{1 - \frac{1}{3} \varepsilon}{6} k \approx 0.1792k$ and $0.1792 > \frac{1}{6} \approx 0.1666$. Hence, if the size of the subtree which is computed classically can be $\frac{1}{6}$ of the original tree size, the complexity of the algorithm may be better than our algorithm. Then, we think a method that

FIG. 6. Running time of our algorithm.

FIG. 7. Running time in the case of the method proposed by Fuchs et al. into three even parts. In this case, the algorithm divides the tree into $\frac{3}{7} : \frac{2}{7} : \frac{2}{7}$ for $r$ times.
splits the tree into three parts followed by splitting the
tree into two parts. That is, \( W(X), |X| \leq k/6 \approx 0.1666k \)
is classically computed and three of them are merged to
\( W(X), |X| = k/2 \) by using D-H algorithm, and finally
we obtain \( W(K) \) by merging two of them by using D-H
algorithm. In this case the complexity of the classical
part is reduced to \( O^*(1.569^k) \) and that of the quantum
part is increased to
\[
O^* \left( \sqrt{\left( \frac{k}{k/2} \right) \left( \frac{k/2}{k/6} \right) \left( \frac{k/3}{k/6} \right)} \right)
\]
\[
= O^* \left( \sqrt{2^{k/2} 3^{k/6}} \left( \frac{3}{2} \right)^{4k} \right)
\]
\[
= O^* \left( 2^k \left( \frac{1}{2} + \frac{1}{3} \log_2 3 + \frac{1}{4} \log_2 \frac{3}{2} \right) \right)
\]
\[
\approx O^*(2^{0.89624k})
\]
\[
= O^*(1.8612^k). \quad (B4)
\]

We can improve it by classically computing \( W(X), |X| \leq k/6 + \alpha \). Note that \( 0 < \alpha \leq 0.1792 - \frac{1}{6} \approx 0.0125 \). Then
the complexity of quantum part is modified as
\[
O^* \left( \sqrt{\left( \frac{k}{k/2} \right) \left( \frac{k/2}{(1/6 + \alpha)k} \right) \left( \frac{1/3 - \alpha}{(1/6 + \alpha)k} \right)} \right). \quad (B5)
\]

When \( \alpha \approx 0.0125 \), this is minimized to \( O^*(1.859^k) \), which
is slower than our algorithm. We can also consider the
method in which the merging order is reversed. That is,
\( W(X), |X| \leq k/6 \approx 0.1666k \) is classically computed and
two of them are merged to \( W(X), |X| = k/3 \) by using D-H
algorithm, and finally we obtain \( W(K) \) by merging three of them by using D-H
algorithm. In this case the complexity of the quantum part is
\[
O^* \left( \sqrt{\left( \frac{k}{k/3} \right) \left( \frac{2k/3}{k/3} \right) \left( \frac{k/3}{k/6} \right)} \right)
\]
\[
= O^* \left( \sqrt{3^k \left( \frac{2k}{2} \right) \left( \frac{2k}{3} \right)^{2/3} \left( \frac{2k}{6} \right)^{2/3}} \right)
\]
\[
= O^* \left( 2^k (1 + \frac{1}{3} \log_2 3 + \frac{1}{2} \log_2 \frac{3}{2} \right)^k
\]
\[
\approx O^*(2^{0.95915k})
\]
\[
= O^*(1.9441^k). \quad (B6)
\]

We can also improve it by classically computing
\( W(X), |X| \leq k/6 + 0.0125k \approx 0.1792k \). Then, the
complexity of quantum part is
\[
O^* \left( \sqrt{\left( \frac{k}{k/3} \right) \left( \frac{2k/3}{k/3} \right) \left( \frac{k/3}{0.1792k} \right)} \right)
\]
\[
\approx O^* \left( 2^k (1 + \frac{1}{3} \log_2 3 + 0.3320) \right)
\]
\[
\approx O^*(2^{0.9301k})
\]
\[
= O^*(1.905^k). \quad (B7)
\]

Hence, this method is also slower than our algorithm.