Robust Federated Learning Using ADMM in the Presence of Data Falsifying Byzantines

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Abstract

In this paper, we consider the problem of federated (or decentralized) learning using ADMM with multiple agents. We consider a scenario where a certain fraction of agents (referred to as Byzantines) provide falsified data to the system. In this context, we study the convergence behavior of the decentralized ADMM algorithm. We show that ADMM converges linearly to a neighborhood of the solution to the problem under certain conditions. We next provide guidelines for network structure design to achieve faster convergence. Next, we provide necessary conditions on the falsified updates for exact convergence to the true solution. To tackle the data falsification problem, we propose a robust variant of ADMM. We also provide simulation results to validate the analysis and show the resilience of the proposed algorithm to Byzantines.

1 Introduction

Many machine learning and statistics problems fit into the general framework where a finite-sum structure of functions is to be optimized. In general, the problem is formulated as

\[
\min_{x \in \mathbb{R}^N} f(x), \quad f(x) = \sum_{i=1}^{L} f_i(x). \tag{1}
\]

The problem structure in (1) covers collaborative autonomous inference in statistics and linear/logistic regression, support vector machines, and deep neural networks in machine learning. Due to the emergence of the big data era and associated sizes of datasets, solving problem (1) on a single node (or agent) is often impossible, as storing the entire dataset on a single node becomes infeasible. This gives rise to the federated optimization setting [9], in which the training data for the problem is stored in a distributed fashion across a number of interconnected nodes and the optimization problem is solved collectively by the cluster of nodes. However, distributing computation over several nodes induces a higher risk of failures, including communication noise, crashes and computation errors. Furthermore, some nodes, often referred to as Byzantine nodes, may intentionally inject false data to gain unfair advantage or degrade the system performance. While Byzantines (originally proposed in [10]) may, in general, refer to many types of unwanted behavior, our focus in this paper is on data-falsification. Data falsifying Byzantines can easily prevent the convergence of the federated learning algorithm [8, 9].

There exist several decentralized optimization methods for solving (1), including belief propagation [15], distributed subgradient descent algorithms [13], dual averaging methods [4], and the alternating direction method of multipliers (ADMM) [2]. Among these, ADMM has drawn significant attention, as it is well suited for distributed optimization and demonstrates fast convergence in many applications [19, 17]. More specifically, ADMM was found to converge linearly for a large class of problems [7]. In [16], linear convergence rate has also been established for decentralized ADMM. Recently, the performance analysis of ADMM in the presence of inexactness in the updates has received some attention [11, 20, 18, 14, 6, 5]. Most relevant to our work, [3] studies the inexact ADMM algorithm for the decentralized consensus problem. The authors in [11] tried to study the scenario where the error \(e_k\) occurs in the ADMM update \(x_k\). They considered the occurrence of the error in the \(x\)-update step in [7], but failed to consider it in the \(\alpha\)-update step. However, most of the aforementioned papers consider that the inexactness occurs in an intermediate step of proximal mapping in one ADMM iteration, which is
very limited and different from what we have studied
in our paper. The focus of our work is on Byzan-
tine falsification where Byzantines have a large degree
of freedom and can falsify any algorithm parameters
without abiding to an error model. Thus, we consider
a general falsification model where the inexactness \( e^k \)
ocurs in the update \( x^k \) after one ADMM iteration.
Also, our model can be seen as an extension of the
aforementioned works.

Our main contributions can be summarized as follows.

- First, we analyze the convergence behavior of
ADMM for the decentralized consensus optimization
problem with data-falsification errors in up-
dates. A general performance guarantee is estab-
lished, with respect to the distance between the
update \( x^k \) and the solution of the problem \( x^* \).

- Second, we show that ADMM converges linearly
to a neighborhood of the solution if certain condi-
tions involving the network topology, the prop-
erties of the objective function, and the algorithm
parameter, are satisfied. Guidelines are developed
for network structure design to achieve faster con-
vergence.

- Third, we give several conditions on the data-
falsification errors to obtain exact convergence to
the solution.

- Finally, to tackle the data-falsification errors, a ro-
bust variant of ADMM is proposed. This scheme
relies upon a node identity profiling approach to
locate the malicious Byzantine nodes.

1.1 Notations

Non-bold characters will be used for scalars, e.g.,
the algorithm parameter \( c \); bold lower-case characters
will be used for column vectors, e.g., the update \( x^k \); bold,
upper-case characters will be used for matrices, e.g.,
the extended signless Laplacian matrix \( L_+ \); and cal-
ligraphic upper-case characters will be used for sets,
e.g., the set of arcs \( \mathcal{A} \).

For a positive semidefinite matrix \( X \), define that
\( \sigma_{\min}(X) \) is the nonzero smallest eigenvalue of matrix
\( X \) and that \( \sigma_{\max}(X) \) is the nonzero largest eigenvalue.

In particular, \( \mathbb{R} \) denotes the set of real numbers;
\( \mathbb{R}^m \) denotes the \( m \)-dimensional Euclidean space; and
\( \mathbb{R}^{n \times m} \) denotes the set of all \( n \times m \) real matrices.

2 Problem Formulation

2.1 Federated Learning with ADMM

Consider a network consisting of \( D \) agents bidirection-
ally connected by \( E \) edges (and thus \( 2E \) arcs). We can
describe the network as a symmetric directed graph
\( G_d = \{ V, \mathcal{A} \} \) or an undirected graph \( G_u = \{ V, E \} \),
where \( V \) is the set of vertices with cardinality \( |V| = D \),
\( \mathcal{A} \) is the set of arcs with \( |\mathcal{A}| = 2E \), and \( E \) is the set
of edges with \( |E| = E \). In a federated setup, a local
agent generates updates individually (by solving a lo-
cal optimization problem) and communicates with its
neighbors to reach a network-wide common minimizer.

More specifically, the federated learning problem, can
be formulated as follows

\[
\min_{(x_i),(y_j)} \sum_{i=1}^{D} f_i(x_i),
\]

\[\text{s.t. } x_i = y_{ij}, \quad x_j = y_{ij}, \quad \forall (i,j) \in \mathcal{A}\] (2)

Here \( x_i \in \mathbb{R}^N \) is the local copy of the common opti-

mization variable \( \hat{x} \) at agent \( i \) and \( y_{ij} \in \mathbb{R}^N \) is an aux-
illary variable imposing the consensus constraint on
neighboring agents \( i \) and \( j \). In the constraints, \( \{ x_i \} \)
are separable when \( \{ y_{ij} \} \) are fixed, and vice versa.
Obviously (3) is equivalent to (1) when the network is
connected.

Defining \( x \in \mathbb{R}^{DN} \) as a vector concatenating all \( x_i \),
\( y \in \mathbb{R}^{2EN} \) as a vector concatenating all \( y_{ij} \), and \( f(x) = \sum_{i=1}^{D} f_i(x_i) \), (3) can be written in a matrix form as

\[
\min_{x,y} f(x) + g(y)
\]

\[\text{s.t. } Ax + By = 0\] (3)

where \( g(y) = 0 \), which fits the form of (2), and is
amenable to be solved by ADMM. Here \( A = [A_1; A_2]; A_1,A_2 \in \mathbb{R}^{2EN \times LN} \); are both composed of
\( 2E \times D \) blocks of \( N \times N \) matrices. If \( (i,j) \in \mathcal{A} \) and \( y_{ij} \)
is the \( q \)th block of \( y \), then the \( (q,i) \)th block of \( A_1 \) and
the \( (q,j) \)th block of \( A_2 \) are \( N \times N \) identity matrices
\( I_N \); otherwise the corresponding blocks are \( N \times N \) zero
matrices \( 0_N \). Also, we have \( B = [-I_{2EN}; -I_{2EN}] \) with
\( I_{2EN} \) being a \( 2EN \times 2EN \) identity matrix.

We define the following matrices: \( M_+ = A_1^T + A_2^T \)
and \( M_- = A_1^T - A_2^T \). Let \( W \in \mathbb{R}^{DN \times DN} \) be a block
diagonal matrix with its \( (i,i) \)th block being the de-
gree of agent \( i \) multiplying \( I_N \) and other blocks being
\( 0_N \), \( L_+ = \frac{1}{2}M_+M_+^T \), \( L_- = \frac{1}{2}M_-M_-^T \), and we know
\( W = \frac{1}{2}(L_+ + L_-) \). These matrices are related to the
underlying network topology. With regard to the unidi-
rected graph \( G_u \), \( M_+ \) and \( M_- \) are the extended un-
oriented and oriented incidence matrices, respectively;
\( \mathbf{L}_+ \) and \( \mathbf{L}_- \) are the extended signless and signed Laplacian matrices, respectively; and \( \mathbf{W} \) is the extended degree matrix. By extended, we mean replacing every 1 by \( \mathbf{I}_N \), -1 by \(-\mathbf{I}_N \), and 0 by \( \mathbf{0}_N \) in the original definitions of these matrices.

### 2.2 Decentralized ADMM with Byzantines

The iterative updates of ADMM algorithm are given by \[ \] \[ 4 \]

The updates in \[ 4 \] are distributed at the agents. Note that \( \mathbf{x} = [\mathbf{x}_1; \ldots; \mathbf{x}_D] \) where \( \mathbf{x}_i \) is the local solution of agent \( i \) and \( \alpha = [\alpha_1; \ldots; \alpha_D] \) where \( \alpha_i \in \mathbb{R}^{N_i} \) is the local Lagrange multiplier of agent \( i \). Recalling the definitions of \( \mathbf{W}, \mathbf{L}_+ \) and \( \mathbf{L}_- \), \[ 4 \] results in the update of agent \( i \) by

\[
\begin{align*}
\nabla f_i(\mathbf{x}_i^{k+1}) + \alpha_i^k + 2c|\mathcal{N}_i|\mathbf{x}_i^{k+1} &= c|\mathcal{N}_i|\mathbf{x}_i^k + c \sum_{j \in \mathcal{N}_i} \mathbf{x}_j^k, \\
\alpha_i^{k+1} &= \alpha_i^k + c|\mathcal{N}_i|\mathbf{x}_i^{k+1} - c \sum_{j \in \mathcal{N}_i} \mathbf{x}_j^{k+1},
\end{align*}
\]

where \( \mathcal{N}_i \) denotes the set of neighbors of agent \( i \). The algorithm is fully decentralized since the updates of \( \mathbf{x}_i \) and \( \alpha_i \) only rely on local and neighboring information.

Consider the case where a fraction of the nodes generate erroneous updates, and the corresponding nodes are termed as Byzantines. Assume that the true update \( \mathbf{x}^k \) goes through data falsifying manipulations at the Byzantines, and the outcome is modeled as \( \mathbf{x}^k + \mathbf{e}_k \), which is denoted as \( \mathbf{z}^k = \mathbf{x}^k + \mathbf{e}_k \). The corresponding algorithm for the data-falsification Byzantine case is

\[
\begin{align*}
\nabla f_i(\mathbf{x}_i^{k+1}) + \alpha_i^k + 2c|\mathcal{N}_i|\mathbf{x}_i^{k+1} &= c|\mathcal{N}_i|\mathbf{z}_i^k + c \sum_{j \in \mathcal{N}_i} \mathbf{z}_j^k, \\
\alpha_i^{k+1} &= \alpha_i^k + c|\mathcal{N}_i|\mathbf{z}_i^{k+1} - c \sum_{j \in \mathcal{N}_i} \mathbf{z}_j^{k+1},
\end{align*}
\]

For a clearer presentation, we will use the following form of iteration for our analysis

\[
\begin{align*}
\nabla f(\mathbf{x}^{k+1}) + \alpha^k + 2c\mathbf{W}\mathbf{x}^{k+1} - c\mathbf{L}_+\mathbf{z}^k &= 0, \\
\alpha^{k+1} - \alpha^k - c\mathbf{L}_-\mathbf{z}^{k+1} &= 0.
\end{align*}
\]

Compared with the update steps in \[ 4 \], \( \mathbf{x}^k \) is replaced by the erroneous update \( \mathbf{z}^k \) in the first step, and \( \mathbf{x}^{k+1} \) is replaced by \( \mathbf{z}^{k+1} \) in the second step.

### 2.3 Problem Assumptions and Notations

### 2.4 Assumptions

We give two definitions that will be used for the cost function.

**Definition 1** (L-smoothness). A function \( f \) is L-smooth if there is a constant \( L \) such that

\[
\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^{L_N}.
\]

Note that such an assumption is very common in the analysis of first-order optimization methods. From the definition, we can see that a function \( f \) being L-smooth also means that the gradient \( \nabla f \) is L-Lipschitz continuous.

**Definition 2** (v-strongly convex). A function \( f \) is v-strongly convex if there is a constant \( v > 0 \), such that

\[
f(\mathbf{x}) \geq f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + v\|\mathbf{x} - \mathbf{y}\|^2, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^{L_N}.
\]

The constant \( v \) measures how convex a function \( f \) is. In particular, the larger that value of \( v \), the more convex \( f \) is.

We adopt the following assumptions for the problem throughout our analysis.

**Assumption 1.**

1. Function \( f \) is proper and \( \inf_{\mathbf{x} \in \mathbb{R}^{L_N}} f(\mathbf{x}) > -\infty \).
2. Function \( f \) is continuously differentiable and L-smooth.
3. Function \( f \) is v-strongly convex.

### 3 Convergence Analysis

To effectively present the convergence results\(^1\) we first give a few notations and definitions. Let \( \mathbf{Q} = \left( \mathbf{L}_- \right)^{\frac{1}{2}} \). Specifically, let \( \mathbf{Q} = \mathbf{V} \Sigma \mathbf{V}^T \), where \( \frac{\mathbf{L}_-}{2} = \mathbf{V} \Sigma \mathbf{V}^T \) is the singular value decomposition of the positive semidefinite matrix \( \frac{\mathbf{L}_-}{2} \). We also construct a new auxiliary sequence \( \mathbf{r}^k = \sum_{s=0}^{k} \mathbf{Q}(\mathbf{x}^s + \mathbf{e}^s) \). This sequence is not physically generated, but is used only for our analysis. Let \( \mathbf{z}^* = \mathbf{x}^* \), where \( \mathbf{x}^* \) denotes an optimal solution to the problem.

Define the auxiliary vector \( \mathbf{q}^k \) and matrix \( \mathbf{G} \) as

\[
\mathbf{q}^k = \begin{bmatrix} \mathbf{r}^k \\ \mathbf{z}^k \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} c \mathbf{I} & 0 \\ 0 & c \mathbf{L}_+/2 \end{bmatrix}.
\]

\(^1\)Proofs of the theoretical analysis are provided in the supplementary material.
Theorem 1. There exists $q^*$ such that

$$\|q^{k+1} - q^*\|_2^2 \leq \frac{\|q^k - q^*\|_2^2}{1 + \delta} + \frac{P}{1 + \delta} \|e^{k+1}\|_2^2 + \frac{1}{1 + \delta} (e^{k+1}, S)$$

(10)

with

$$S = cLz^{k+1} - z^k + 2cQx^{k+1} - x^* + 2cW(x^{k+1} - x^*)$$

(11)

where

$$P = \frac{\sigma^2_2 L^2(W)}{\sigma^2_{\min}(Q)} + \frac{\sigma^2_3 L^2(L_+)}{\sigma^2_{\min}(Q)}$$

(12)

and

$$\delta = \min \left\{ \frac{(\lambda_1 - 1)(\lambda_2 - 1)L^2_{\min}(Q)}{\lambda_3 \sigma^2_2 L^2(L_+)}, \frac{4\sigma^2_3 L^2(L_+) - \lambda_1 \lambda_3 \sigma^2_{\min}(Q)}{\lambda_1 \lambda_2 (\lambda_3 - 1)L^2 + c^2 \lambda_2 (\lambda_2 - 1)\sigma^2_3 L^2(L_+) \sigma^2_{\min}(Q)} \right\}$$

(13)

with quantities $\lambda_1$, $\lambda_2$, and $\lambda_3$ being greater than 1.

Theorem 1 shows that the sequence $\|q^{k+1} - q^*\|_2^2$ converges linearly with a rate of $\frac{1}{1 + \delta}$ if after a certain number of iterations, there are no data-falsification errors in the updates. Then, it can be easily shown that the sequence $z^{k+1}$ or $x^{k+1}$ converges to the minimizer, since the last two terms in (10) are removed. However, if the errors persist in the updates, this theorem shows how the errors are accumulated after each iteration. As a general result, one can further optimize over $\lambda_1$, $\lambda_2$, and $\lambda_3$ to obtain maximal $\delta$ and minimal $P$ to achieve fastest convergence and least impact from the errors.

Theorem 2. Let $\beta$ be chosen such that

$$0 < \beta \leq \frac{b(1 + \delta)}{4b\sigma^2_{\min}(L_+)} \left( 1 - \frac{1}{\lambda_4} \right) \left( 1 - \frac{1}{\lambda_3} \right) + 16\sigma^2_{\max}(W)$$

(14)

where $b > 0$ and $\lambda_4 > 1$, then

$$\|z^{k+1} - z^*\|_2^2 \leq B^{k+1} \left( A + \sum_{s=1}^{k+1} L^{-s}C\|e^s\|_2^2 \right)$$

(15)

where

$$A = \|z^0 - z^*\|_2^2 + A_2\|y^0 - r^*\|_2^2$$

(16)

with

$$A_2 = \frac{4}{(1 + 4\beta)\sigma^2_{\max}(L_+)}$$

(17)

and

$$B = \frac{(1 + 4\beta)\sigma^2_{\max}(L_+)}{(1 - b)(1 + \delta - 4\beta)\sigma^2_{\min}(L_+)}$$

(18)

$$C = \frac{4P + 2/\beta}{c^2(1 + \delta - 4\beta)\sigma^2_{\min}(L_+) + b(\lambda_4 - 1)}.$$  

(19)

Theorem 2 presents a general convergence result of ADMM for decentralized consensus optimization with errors, and indicates that the erroneous update $z^{k+1}$ approaches the neighborhood of the minimizer in a linear fashion. The radius of the neighborhood is given as $B^{k+1} \sum_{s=1}^{k+1} B^{-s}C\|e^s\|_2^2$. Note that $B$ is not guaranteed to be less than 1. This is very different from the convergence result of ADMM for decentralized consensus optimization [10], which can guarantee that the update converge to the minimizer linearly fast and the corresponding rate is less than 1. If $\sigma^2_{\max}(L_+) > \sigma^2_{\min}(L_+)$, and it ends up with $B$ being greater than 1, then the algorithm will not converge at all.

In particular, we define $C$ as the radius of the neighborhood in this paper. Thus, we divide the problem into two different parts. The first one is to guarantee that $B$ is within the range $(0, 1)$, and the second one is to minimize the radius of the neighborhood $C$.

Accordingly, we optimize over the variables that appeared in the above theorems and the algorithm parameter $c$, and give the convergence result with $B \in (0, 1)$.

Theorem 3. If $b$ and $\lambda_2$ can be chosen, such that

$$0 < \beta \leq \frac{b(1 + \delta)}{4b\sigma^2_{\min}(L_+)} \left( 1 - \frac{1}{\lambda_4} \right) \left( 1 - \frac{1}{\lambda_3} \right) + 16\sigma^2_{\max}(W)$$

then the ADMM algorithm with a parameter $c = \sqrt{\frac{\lambda_1 \lambda_2 (\lambda_3 - 1)\sigma^2_{\max}(L_+) \sigma^2_{\min}(Q)}{\lambda_3 (\lambda_2 - 1)\sigma^2_{\max}(L_+) \sigma^2_{\min}(Q)}}$ converges linearly with a rate of $B \in (0, 1)$, to the neighborhood of the minimizer with a radius of $C$, which is

$$C = \frac{4\delta \lambda_3 \sigma^2_{\max}(W) + \sigma^2_{\max}(L_+) \sqrt{\lambda_3 (\lambda_2 - 1)\sigma^2_{\max}(L_+) \sigma^2_{\min}(Q)}}{(1 - b)(1 + \delta - 4\beta)\sigma^2_{\min}(L_+) + b(\lambda_4 - 1)}.$$  

(20)

where

$$\lambda_1 = 1 + 2\sigma^2_{\max}(L_+)/(L^2 \sigma^2_{\min}(L_+))$$

(22)
The value of $\sigma^2_{\max}(L_+)$ in the theorem into the expression (25), we have

\[
0 < \frac{\sigma^2_{\max}(L_+)}{(1 + \delta)\sigma^2_{\min}(L_+)} < 1.
\]

One intuition is that we should design a network such that $\frac{\sigma^2_{\max}(L_+)}{\sigma^2_{\min}(L_+)}$ is the smallest possible. Substituting $\delta$ in the theorem into the expression (25), we have

\[
0 < \frac{b(1 + \delta)\sigma^2_{\min}(L_+)^2}{4b\sigma^2_{\min}(L_+)^2(1 - \frac{1}{L_+}) + 16\sigma^2_{\max}(W)}.
\]

Remark 1. The value of $\frac{\sigma^2_{\min}(L_+)}{\sigma^2_{\max}(L_+)}$, which corresponds to the network structure, has to be greater than a certain threshold such that a linear convergence rate of $B \in (0, 1)$ can be achieved. This shows that a decentralized network with a random structure may not converge at all to the neighborhood of the minimizer, when the ADMM algorithm is implemented with erroneous updates.

Remark 2. It is easy to show that the right hand side of the inequality is strictly less than 1. Considering $\frac{\Delta_1 - 1}{\lambda_1}\sigma^2_{\min}(Q)$ as the only variable in the expression on the right hand side, it is upper bounded by $\frac{4\epsilon}{(\lambda_1 + 1)^2 + 16\epsilon^2\lambda_1 - 1\sigma^2_{\min}(Q) - L^2 + 2v}$.

Thus, if we can design a network such that its corresponding value of $\frac{\sigma^2_{\min}(L_+)}{\sigma^2_{\max}(L_+)}$ is greater than this bound, we can ensure that the decentralized ADMM algorithm can converge to the neighborhood of the minimizer.

Remark 3. The right hand side of the above expression depends on the geometric property of the cost function. There exists a certain class of cost functions such that the value of the right hand side can be lowered, compared with other cost functions. Thus, it allows for a more flexible network structure design such that a linear convergence rate can be achieved.

**Corollary 1.** If $\|\epsilon^k\|_2^2$ decreases linearly at a rate $R$ such that $0 < R < B$, and the constraints in Theorem 3 are satisfied, the algorithm converges to the minimizer linearly with a rate of $B$.

This result simply states that if the error in the update decays faster than the distance between the update and the minimizer $\|z^k - z^*\|_2^2$, then the algorithm will reach the minimizer at a linear rate.

**Corollary 2.** If $\|\epsilon^k\|_2^2 \leq e$, and the constraints in Theorem 3 are satisfied, then an upper bound on the error $\|z^k - z^*\|_2^2$ for $k \to \infty$ is $\frac{C\epsilon}{1 - B}$.

This result shows that if the error at every iteration is bounded, then the algorithm will approach the bounded neighborhood of the minimizer.

**Theorem 4.** If for each iteration,

\[
C\|\epsilon^{k+1}\|_2^2 \leq B(A_1 - A_2)\|z^k - z^*\|_2^2
\]

with

\[
A_1 = \frac{4}{(1 - b)\sigma^2_{\min}(L_+)}
\]

and the constraints in Theorem 3 are satisfied, the algorithm converges to the minimizer linearly with a rate of $B \in (0, 1)$.

Recall the result in Corollary 1. Theorem 4 essentially states that when the error decays faster than the distance between the update and the minimizer $\|z^k - z^*\|_2^2$, the algorithm can approach the minimizer, which is intuitively true. However, Theorem 4 gives a much more general condition for convergence to the minimizer. Note that $C$, $B$, $A_1$, and $A_2$ are fixed, and the condition (27) relates the current error to all the previously accumulated errors. The error impacts the performance of decentralized ADMM in two different ways. First, the error at an individual local agent makes the local update deviate from the true update. Second, the local error can propagate over the network through update exchange between neighboring nodes, thus impacting the update precision of the nodes later in the network. Hence, the errors that occurred before the current iteration can diffuse and get accumulated over the network. At this point, Theorem 4 gives an upper bound for the current error based on the past errors, such that the network can tolerate the accumulated errors and the convergence to the minimizer can still be guaranteed.
4 Robust ADMM

Based upon insights provided by our theoretical results in Section 3, we investigate the design of the robust ADMM algorithm which can tolerate the errors in the ADMM updates. We focus on the scenario where a fraction of the nodes generate erroneous updates. The other nodes in the network generate true updates (or their errors can be neglected), which are called honest nodes in this paper.

4.1 Byzantine Profiling

In this section, we describe a scheme for Byzantine profiling. Note that for a decentralized consensus optimization problem, the honest nodes iteratively approach a consensus. Thus, after a certain number of iterations, the Byzantine nodes behave quite differently from others. In particular, if the Byzantine nodes are removed from the neighborhood, the variance of the values of the updates would be significantly reduced.

Specifically, after $T$ iterations, agent $i$ starts to check the update quality of its neighbors. Say at $k$-th iteration, agent $i$ constructs a new vector $X_i^k \in \mathbb{R}^{DN}$, using the updates from its neighbors $x_{ij}^k, j \in M_i$, where $M_i = \{i\} \cup N_i$. While comparing with $x_i$, $X_i^k$ substitutes the elements indicating updates from agent $j \notin M_i$ with the mean of updates from agent $i$ and its neighbors, which is denoted by $m_i^k \in \mathbb{R}^N$, i.e.,

$$m_i^k = \frac{1}{|M_i|-1} \sum_{j \in M_i} x_{ij}^k.$$  

Note that for the decentralized ADMM algorithm, the constraint $Qx = 0$ guarantees a minimizer of consensus, and the norm $\|QX_i^k\|_2^2$ is shown to be a good metric of the deviation from a consensus for the current update [12]. Then, if $\|QX_i^k\|_2^2$ is greater than a predefined threshold $\tau$, i.e., $\|QX_i^k\|_2^2 > \tau$, it triggers an alarm and the profiling process starts.

First, for every agent $j \in M_i$, a unique vector $X_{ij}^k$ is assigned, which is constructed in a similar way as $X_i^k$, except that the elements of $x_{ij}^k$ and $m_{ij}^k$ are replaced with $m_{ij}^k$, the mean of the rest of the updates in $N_i$, i.e.,

$$m_{ij}^k = \frac{1}{|M_i|-1} \sum_{h \in M_i \setminus \{j\}} x_{ih}^k, \quad j \in M_i. \quad (30)$$

Then, a score $S_{ij}^k = \|QX_{ij}^k\|_2^2$ is calculated for agent $j \in M_i$. Node $h$ is profiled by node $i$ as a Byzantine node with the following criterion

$$h = \arg \min_{j \in N_i} S_{ij}^k. \quad (31)$$

4.2 Robust Algorithm

According to Theorem 1, if the Byzantine nodes are correctly profiled and their updates are not used for iteration, the ADMM algorithm can still converge to the true minimizer. Thus, if agent $i$ has profiled its neighbor $h$ as a Byzantine, the update from $h$ will no longer be used for iteration in $i$.

Even though it looks like every node has to maintain the knowledge of the whole network structure $Q$, it can be shown after simple manipulations that $\|QX_i^k\|_2^2 = \sum_{j \in M_i} \|x_{ij}^k - x_i^k\|_2^2$, which does not require the knowledge of the network structure and ensures the applicability of the scheme to networks with high mobility.

We give the corresponding pseudocode for ease of presentation.

**Algorithm 1** Robust Decentralized ADMM

1: function $f = \sum_{i=1}^D f_i(x_i)$
2: Initialization: $x^0, c, a^0, K$
3: for $k = 1$ to $K$
4: \quad Iteratively update using (7)
5: if $k \geq T$ then
6: \quad Start Byzantine Profiling
7: \quad Discard neighbor $h$ from future updates
8: \quad end if
9: end for
10: end function

5 Experiments

In this section, we use ADMM to solve the decentralized consensus optimization problem

$$\min_{x=\mathbb{R}} \sum_{i=1}^D \frac{1}{2} \|y_i - B_ix_i\|_2^2.$$  

A decentralized network with $D = 10$ nodes is employed to perform the optimization task. We assume that there are 3 Byzantine nodes in the network and the update is falsified by Gaussian random variables with mean $\mu_b$ and variance 1. We initialize $y_i$ and $B_i$ randomly with elements generated by standard Gaussian distribution $(0, 1)$.

The algorithm stops when the number of iterations reaches 80, and we record the average error $\|z^k - z^*\|_2$ (where average is over all the agents).

Figure 1 shows the distance of the current update to the minimizer $\|z^k - z^*\|_2$ versus the number of iterations. We can see that if there are no Byzantine
Figure 1: Performance comparison with different Byzantine attack intensity.

Figure 2: Performance comparison with different choices of algorithm parameter.

nodes in the network, the conventional ADMM converges quickly to the minimizer. However, in the presence of Byzantines, with $\mu_b = 0.5$ and $\mu_b = 1$, it can be seen that the performance of the conventional ADMM degrades significantly. We can observe that the algorithm approaches a neighborhood of the minimizer and cannot converge to the minimizer. The radius of the neighborhood depends on the strength ($\mu_b$) of the Byzantine attacks. With the parameters $\tau = 1$ and $T = 5$, our proposed robust ADMM algorithm can perfectly identify the Byzantine nodes and achieve a comparable convergence speed as the case where there are no Byzantines. We found that, within 10 iterations, our Byzantine profiling scheme can identify all the Byzantine nodes, and the algorithm starts to converge.

Next, we employ the derived optimal choice of the algorithm parameter $c$ and show the performance comparison. The optimal $c$, which is termed as $c_{opt}$, is given in Theorem 3. We compare the performance of the robust algorithm in the cases where $c = 0.9$ and $c = c_{opt}$. We can see clearly from Figure 2 that with the optimal $c$, the robust algorithm achieves a much faster convergence speed. Even though the optimal algorithm parameter is derived for the situation where there are Byzantine nodes, the conventional ADMM can also obtain an acceleration with the optimal $c$.

6 Conclusion

We considered the problem of federated learning using ADMM in the presence of data falsification. We studied the convergence behavior of the decentralized ADMM algorithm and showed that the ADMM converges linearly to a neighborhood of the solution under certain conditions. We suggested a guideline for network structure design to achieve faster convergence. We also gave several conditions on the errors to obtain exact convergence to the solution. We proposed a robust ADMM scheme to enable federated learning in the presence of data falsifying Byzantines. We also gave simulation results to validate the analysis and showed the effectiveness of the proposed robust scheme. We assumed the strong convexity of the cost function, and one might follow our lines of analysis for general convex functions.

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**Supplementary Materials**

**Lemma 1.** The update of the the algorithm can be written as

\[ x^{k+1} = -\frac{1}{2c} \mathbf{W}^{-1} \nabla f(x^{k+1}) + \frac{W_+^{-1} L_+}{2}(x^k + e^k) - \frac{W_-^{-1} L_-}{2} \left( \sum_{s=0}^{k} x^s + e^s \right). \]  

**Proof.** Using the second step of the algorithm, we can write

\[ \alpha^{k+1} = \alpha^k + cL_-(x^{k+1} + e^{k+1}) \]  

and

\[ \alpha^k = \alpha^{k-1} + cL_-(x^k + e^k). \]  

Sum and telescope from iteration 0 to k using (34), and we can get the following by assuming \( \alpha^0 = 0 \)

\[ \alpha^k = cL_- \sum_{s=0}^{k} (x^s + e^s). \]  

Substitute the above result to the first step in the algorithm and it yields

\[ 2cWx^{k+1} = -\nabla f(x^{k+1}) + cL_+(x^k + e^k) - cL_- \sum_{s=0}^{k} (x^s + e^s), \]  

which completes the proof.

**Lemma 2.** The sequences satisfy

\[ \frac{L_+}{2}(z^{k+1} - e^k) - We^{k+1} = -Qr^{k+1} - \frac{1}{2c} \nabla f(x^{k+1}) \]  

**Proof.** Based on Lemma 1 and the fact \( \mathbf{W} = \frac{1}{2} (\mathbf{L}_- + \mathbf{L}_+) \), we can write

\[ \mathbf{W}(x^{k+1} - x^k - e^k) + \mathbf{W}(x^k + e^k) - \frac{L_+}{2}(x^k + e^k) = -Qr^k - \frac{1}{2c} \nabla f(x^{k+1}). \]  

Subtracting \( \frac{L_+}{2}(x^{k+1} + e^{k+1}) \) from both sides of the above equation provides

\[ \mathbf{W}(x^{k+1} - x^k - e^k) + \frac{L_-}{2} (x^k + e^k) - \frac{L_-}{2} (x^{k+1} + e^{k+1}) = -Qr^{k+1} - \frac{1}{2c} \nabla f(x^{k+1}). \]  

Rearrange and we have the desired result.

**Lemma 3.** The null space of \( Q \) null(\( Q \)) is span\{1\}.

**Proof.** Note that the null space of \( Q \) and \( \mathbf{L}_- \) are the same. By definition, \( \mathbf{L}_- = \frac{1}{2} \mathbf{M}_- \mathbf{M}_T \) and \( \mathbf{L}_- = \mathbf{A}_1^T - \mathbf{A}_2^T \). Recall that if \((i, j) \in \mathcal{A}\) and \( y_{ij} \) is the \( q \)th block of \( y \), then the \((q, i)\)th block of \( \mathbf{A}_1 \) and the \((q, j)\)th block of \( \mathbf{A}_2 \) are \( N \times N \) identity matrices \( \mathbf{I}_N \); otherwise the corresponding blocks are \( N \times N \) zero matrices \( 0_N \). Therefore, \( \mathbf{M}_T = \mathbf{A}_1 - \mathbf{A}_2 \) is a matrix that each row has one “1”, one “-1”, and all zeros otherwise, which means \( \mathbf{M}_T \mathbf{1} = 0 \), i.e., null(\( \mathbf{M}_T \))=span\{1\}.

Note that \( \mathbf{L}_- = \frac{1}{2} \mathbf{M}_- \mathbf{M}_T \) and \( \mathbf{Q} = \left( \frac{L_-}{2} \right)^2 \), thus null(\( \mathbf{Q} \))=null(\( \mathbf{M}_T \)), completing the proof.

**Lemma 4.** For some \( r^* \) that satisfies \( Qr^* + \frac{1}{2c} \nabla f(x^*) = 0 \) and \( r^* \) belongs to the column space of \( Q \), the sequences satisfy

\[ \frac{L_+}{2}(z^{k+1} - e^k) - We^{k+1} = -Q(r^{k+1} - r^*) - \frac{1}{2c}(\nabla f(x^{k+1}) - \nabla f(x^*)) \]  

(40)
Proof. Using Lemma 2, we have

\[ \frac{L_+}{2} (z^{k+1} - z^k) - W e^{k+1} = -Q r^{k+1} - \frac{1}{2c} \nabla f(x^{k+1}). \] (41)

According to Lemma 3, \( \text{null}(Q) \) is span\{1\}. Since \( 1^T \nabla f(x^*) = 0, \nabla f(x^*) \) can be written as a linear combination of column vectors of \( Q \). Therefore, there exists \( r \) such that \( \frac{1}{2c} \nabla f(x^*) = -Q r \). Let \( r^* \) be the projection of \( r \) onto \( Q \) to obtain \( Q r = Q r^* \) where \( r^* \) lies in the column space of \( Q \).

Hence, we can write

\[ \frac{L_+}{2} (z^{k+1} - z^k) - W e^{k+1} = -Q (r^{k+1} - r^*) - \frac{1}{2c} (\nabla f(x^{k+1}) - \nabla f(x^*)) \] (42)

Lemma 5. \( \langle x^*, Q \rangle = 0. \)

Proof. Since the optimal consensus solution \( x^* \) has an identical value for all its entries, \( x^* \) lies in the space spanned by 1. Thus, according to Lemma 3, we have the desired result, and also \( \langle x^*, L_- \rangle = 0. \)

7 Proof of Theorem 1

Proof.

\[ v \|x^{k+1} - x^*\|^2 \leq \langle x^{k+1} - x^*, \nabla f(x^{k+1}) - \nabla f(x^*) \rangle \] (43)

\[ = \langle x^{k+1} - x^*, cL_+(z^k - z^{k+1}) + 2cW e^{k+1} - 2cQ (r^{k+1} - r^*) \rangle \] (44)

\[ = \langle x^{k+1} - x^*, cL_+(z^k - z^{k+1}) \rangle + \langle x^{k+1} - x^*, 2cW e^{k+1} \rangle \] (45)

\[ + \langle x^{k+1} - x^*, -2cQ (r^{k+1} - r^*) \rangle \] (46)

\[ = \langle z^{k+1} - z^*, cL_+(z^k - z^{k+1}) \rangle - \langle e^{k+1}, cL_+(z^k - z^{k+1}) \rangle \] (47)

\[ + \langle x^{k+1} + e^{k+1} - x^*, -2cQ (r^{k+1} - r^*) \rangle \] (48)

\[ - \langle e^{k+1}, -2cQ (r^{k+1} - r^*) \rangle + \langle e^{k+1}, 2cW (x^{k+1} - x^*) \rangle \] (49)

\[ = 2 \langle z^{k+1} - z^*, cL_+(z^k - z^{k+1}) \rangle + 2 \langle r^k - r^{k+1}, c(r^{k+1} - r^*) \rangle \] (50)

\[ + \langle e^{k+1}, cL_+(z^k - z^{k+1}) + 2cQ (r^{k+1} - r^*) + 2cW (x^{k+1} - x^*) \rangle \] (51)

\[ = \|q^k - q^*\|^2_G - \|q^{k+1} - q^*\|^2_G - \|q^k - q^{k+1}\|^2_G \] (52)

\[ + \langle e^{k+1}, cL_+(z^{k+1} - z^k) + 2cQ (r^{k+1} - r^*) + 2cW (x^{k+1} - x^*) \rangle \] (53)

For any \( \lambda > 0 \), using the basic inequality

\[ \|a + b\|^2 \leq (\lambda - 1)\|a\|^2 \geq (1 - \frac{1}{\lambda})\|b\|^2 \] (54)

we can write for \( \lambda_1 > 1 \) and \( \lambda_2 > 1 \)

\[ \frac{\sigma_{\max}^2(L_+)}{4} \|z^{k+1} - z^k\|^2 + \frac{(\lambda_1 - 1)L_0^2\|x^{k+1} - x^*\|^2}{4c^2} \] (55)

\[ \geq \| \frac{L_+}{2} (z^{k+1} - z^k) \|_G^2 + (\lambda_1 - 1)\| \frac{1}{2c} (\nabla f(x^{k+1}) - \nabla f(x^*)) \|_G^2 \] (56)

\[ \geq \left( 1 - \frac{1}{\lambda_1} \right) \| We^{k+1} - Q(r^{k+1} - r^*) \|_G^2 \] (57)

\[ \geq \left( 1 - \frac{1}{\lambda_1} \right) \| Q(r^{k+1} - r^*) \|_G^2 - \left( 1 - \frac{1}{\lambda_1} \right) (\lambda_2 - 1)\| We^{k+1} \|_G^2 \] (58)

\[ \geq \left( 1 - \frac{1}{\lambda_1} \right) (1 - \frac{1}{\lambda_2})\sigma_{\min}^2(Q)\|r^{k+1} - r^*\|_G^2 - \left( 1 - \frac{1}{\lambda_1} \right) (\lambda_2 - 1)\sigma_{\max}^2(W)\| e^{k+1} \|_G^2. \] (59)
Thus, for a positive quantity $\delta$,

$$\frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_1 \lambda_2}{4\sigma_{\text{min}}^2(Q)(\lambda_1 - 1)(\lambda_2 - 1)} \|z^{k+1} - z^k\|_2^2 + \frac{\delta \lambda_1 \lambda_2 L^2}{4\sigma_{\text{min}}^2(Q)(\lambda_2 - 1)} \|x^{k+1} - x^*\|_2^2 + \frac{\delta \lambda_2 \sigma_{\text{max}}^2(W)}{4\sigma_{\text{min}}^2(Q)} \|e^{k+1}\|_2^2.$$  (60)

$$\geq \delta \|r^{k+1} - r^*\|_2^2 - \frac{\delta \lambda_2 \sigma_{\text{max}}^2(W)}{4\sigma_{\text{min}}^2(Q)} \|e^{k+1}\|_2^2.$$  (61)

Since $x^{k+1} - x^* = z^{k+1} - z^* - e^{k+1}$, for any $\lambda_3 > 1$, we can get

$$\|x^{k+1} - x^*\|_2^2 \geq \left(1 - \frac{1}{\lambda_3}\right) \|z^{k+1} - z^*\|_2^2 - (\lambda_3 - 1) \|e^{k+1}\|_2^2.$$  (62)

Therefore, the addition of $\text{(60)} \times c^2$ and $\text{(62)} \times \frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_3}{4(\lambda_3 - 1)}$ yields

$$\frac{c^2 \delta \sigma_{\text{max}}^2(L_+)\lambda_1 \lambda_2}{4\sigma_{\text{min}}^2(Q)(\lambda_1 - 1)(\lambda_2 - 1)} \|z^{k+1} - z^k\|_2^2 + \left(\frac{\delta \lambda_1 \lambda_2 L^2}{4\sigma_{\text{min}}^2(Q)(\lambda_2 - 1)} + \frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_3}{4(\lambda_3 - 1)}\right) \|x^{k+1} - x^*\|_2^2$$

$$\geq \delta \|r^{k+1} - r^*\|_2^2 + \delta \|z^{k+1} - z^k\|_2^2 - \frac{c^2 \delta^2 \lambda_2 \sigma_{\text{max}}^2(W)}{4\sigma_{\text{min}}^2(Q)} - \frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_3}{4} \|e^{k+1}\|_2^2.$$  (63)

$$\geq \delta \|r^{k+1} - r^*\|_2^2 + \delta \|z^{k+1} - z^k\|_2^2 - \frac{c^2 \delta^2 \lambda_2 \sigma_{\text{max}}^2(W)}{4\sigma_{\text{min}}^2(Q)} - \frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_3}{4} \|e^{k+1}\|_2^2.$$  (64)

$$= \delta \|q^{k+1} - q^*\|_G^2 - \left(\frac{c^2 \delta \lambda_2 \sigma_{\text{max}}^2(W)}{4\sigma_{\text{min}}^2(Q)} + \frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_3}{4}\right) \|e^{k+1}\|_2^2.$$  (65)

Choose $\delta$ to be such that

$$\frac{c^2 \delta \sigma_{\text{max}}^2(L_+)\lambda_1 \lambda_2}{4\sigma_{\text{min}}^2(Q)(\lambda_1 - 1)(\lambda_2 - 1)} \leq \frac{\delta c^2 \sigma_{\text{min}}^2(L_+)}{4}.$$  (67)

$$\left(\frac{\delta \lambda_1 \lambda_2 L^2}{4\sigma_{\text{min}}^2(Q)(\lambda_2 - 1)} + \frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_3}{4(\lambda_3 - 1)}\right) \leq v,$$  (68)

and we can have

$$\frac{c^2 \sigma_{\text{min}}^2(L_+)}{4} \|z^{k+1} - z^k\|_2^2 + v \|x^{k+1} - x^k\|_2^2 \geq$$

$$\delta \|q^{k+1} - q^*\|_G^2 - \left(\frac{c^2 \delta \lambda_2 \sigma_{\text{max}}^2(W)}{4\sigma_{\text{min}}^2(Q)} + \frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_3}{4}\right) \|e^{k+1}\|_2^2.$$  (69)

Thus, it is straightforward to write

$$\|q^{k+1} - q^*\|_G^2 + v \|x^{k+1} - x^k\|_2^2 \geq$$

$$\geq \|c(r^{k+1} - r^k)\|_2^2 + \frac{cL_+}{2} \|z^{k+1} - z^k\|_2^2 + v \|x^{k+1} - x^k\|_2^2.$$  (71)

$$\geq \|c(r^{k+1} - r^k)\|_2^2 + \frac{c \sigma_{\text{min}}^2(L_+)}{4} \|z^{k+1} - z^k\|_2^2 + v \|x^{k+1} - x^k\|_2^2.$$  (72)

$$\geq \|c(r^{k+1} - r^k)\|_2^2 + \delta \|q^{k+1} - q^*\|_G^2 - \left(\frac{c^2 \delta \lambda_2 \sigma_{\text{max}}^2(W)}{4\sigma_{\text{min}}^2(Q)} + \frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_3}{4}\right) \|e^{k+1}\|_2^2.$$  (73)

$$\geq \|c(r^{k+1} - r^k)\|_2^2 + \delta \|q^{k+1} - q^*\|_G^2 - \left(\frac{c^2 \delta \lambda_2 \sigma_{\text{max}}^2(W)}{4\sigma_{\text{min}}^2(Q)} + \frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_3}{4}\right) \|e^{k+1}\|_2^2.$$  (74)

Recall the result in [43] regarding the bound to $v \|x^{k+1} - x^k\|_2^2$, and we can further write

$$\|q^k - q^*\|_G^2 + \|q^{k+1} - q^*\|_G^2 + (e^{k+1}, cL_+(z^{k+1} - z^k) + 2cQ(r^{k+1} - r^k) + 2cW(x^{k+1} - x^*)) \geq$$

$$\geq \|q^{k+1} - q^*\|_G^2 - \left(\frac{c^2 \delta \lambda_2 \sigma_{\text{max}}^2(W)}{4\sigma_{\text{min}}^2(Q)} + \frac{\delta c^2 \sigma_{\text{max}}^2(L_+)\lambda_3}{4}\right) \|e^{k+1}\|_2^2.$$  (75)
Let \( P = \frac{\sigma_2^2(\mathbf{L})}{\sigma_2^2(Q)} + \frac{\delta^2 \sigma_2^2(\mathbf{L}_+)^2 \lambda_3}{4} \). Rearrange the expression and we get

\[
\|q^{k+1} - q^*\|_2^2 \leq \frac{\|q^k - q^*\|_2^2}{1 + \delta} + \frac{P}{1 + \delta} \|e^{k+1}\|_2^2 + \frac{1}{1 + \delta} (e^{k+1}, c\mathbf{L}_+(z^{k+1} - z^k) + 2cQ(r^{k+1} - r^*) + 2c\mathbf{W}(x^{k+1} - x^*))
\]

(78)

\[
(1 - b) \left( \frac{1}{4} - \frac{\beta}{1 + \delta} \right) \sigma_{\min}^2(\mathbf{L}_+) \|z^{k+1} - z^*\|_2^2 + \left( 1 - \frac{4\beta}{1 + \delta} \right) \|r^{k+1} - r^*\|_2^2
\]

(80)

\[
+ b \left( \frac{1}{4} - \frac{\beta}{1 + \delta} \right) \sigma_{\min}^2(\mathbf{L}_+) \left( 1 - \frac{1}{\lambda_4} \right) \|x^{k+1} - x^*\|_2^2
\]

(81)

\[
\leq \frac{1}{4 + \beta \sigma_{\max}^2(\mathbf{L}_+)} \|z^k - z^*\|_2^2 + \frac{1}{1 + \delta} \|r^k - r^*\|_2^2
\]

(82)

\[
+ \left( \frac{P + 1/2\beta}{1 + \delta} \right) + b \left( \frac{1}{4} - \frac{\beta}{1 + \delta} \right) \sigma_{\min}^2(\mathbf{L}_+) (\lambda_4 - 1) \|e^{k+1}\|_2^2
\]

(83)

\[
+ \frac{4\beta \sigma_{\max}^2(\mathbf{W})}{1 + \delta} \|x^{k+1} - x^*\|_2^2.
\]

(84)

**Lemma 6.** Let \( \beta \in (0, \frac{1 + \delta}{4}) \), \( b \in (0, 1) \), \( \lambda_4 > 1 \), and then we have

\[
(1 - b) \left( \frac{1}{4} - \frac{\beta}{1 + \delta} \right) \sigma_{\min}^2(\mathbf{L}_+) \|z^{k+1} - z^*\|_2^2 + \left( 1 - \frac{4\beta}{1 + \delta} \right) \|r^{k+1} - r^*\|_2^2
\]

(85)

\[
+ b \left( \frac{1}{4} - \frac{\beta}{1 + \delta} \right) \sigma_{\min}^2(\mathbf{L}_+) \left( 1 - \frac{1}{\lambda_4} \right) \|x^{k+1} - x^*\|_2^2
\]

(86)

\[
\leq \frac{1}{4 + \beta \sigma_{\max}^2(\mathbf{L}_+)} \|z^k - z^*\|_2^2 + \frac{1}{1 + \delta} \|r^k - r^*\|_2^2
\]

(87)

\[
+ \frac{1/\beta}{1 + \delta} \|c\mathbf{L}_+^2(z^{k+1} - z^*) + c\mathbf{Q}(r^{k+1} - r^*) + c\mathbf{W}(x^{k+1} - x^*)\|_2^2
\]

(88)

\[
\leq \frac{1}{1 + \delta} \left( \|c\mathbf{L}_+^2(z^{k+1} - z^*)\|_2^2 + 4\alpha \|c\mathbf{L}_+^2(z^{k+1} - z^*)\|_2^2 + \frac{c\mathbf{L}_+^2(z^{k+1} - z^*)}{1 + \delta} \|c\mathbf{W}(x^{k+1} - x^*)\|_2^2
\]

(89)

\[
\leq \frac{1}{1 + \delta} \left( \|c\mathbf{L}_+^2(z^{k+1} - z^*)\|_2^2 + 4\alpha \|c\mathbf{L}_+^2(z^{k+1} - z^*)\|_2^2 + \left( \frac{P}{1 + \delta} + \frac{1/\beta}{1 + \delta} \right) \|e^{k+1}\|_2^2
\]

(90)

\[
+ \frac{4\beta \sigma_{\max}^2(\mathbf{L})}{1 + \delta} \|e^{k+1}\|_2^2 + \frac{4\beta \sigma_{\max}^2(\mathbf{L})}{1 + \delta} \|e^{k+1}\|_2^2
\]

(91)

where \( \beta > 0 \).

Rearranging the inequality provides

\[
\left( 1 - \frac{4\beta}{1 + \delta} \right) \|c\mathbf{L}_+^2(z^{k+1} - z^*)\|_2^2 + \left( 1 - \frac{4\beta}{1 + \delta} \right) \|c\mathbf{r}^{k+1} - r^*\|_2^2
\]

(94)

\[
\leq \left( 1 + \frac{4\beta}{1 + \delta} \right) \frac{\|c\mathbf{L}_+^2(z^{k+1} - z^*)\|_2^2}{1 + \delta} + \frac{1}{1 + \delta} \|c\mathbf{r}^{k+1} - r^*\|_2^2
\]

(95)

\[
+ \left( P + 1/\beta \right) \|e^{k+1}\|_2^2 + \frac{4\beta \sigma_{\max}^2(\mathbf{L})}{1 + \delta} \|c\mathbf{W}(x^{k+1} - x^*)\|_2^2.
\]

(96)

Note that the parameters should be chosen such that \( 1 - \frac{4\beta}{1 + \delta} > 0 \).
Thus, we can write
\[\left(\frac{1}{4} - \frac{\beta}{1 + \delta}\right) \sigma_{\min}(L_+) \|z^{k+1} - z^*\|^2 + \left(1 - \frac{4\beta}{1 + \delta}\right) \|r^{k+1} - r^*\|^2 \leq \left(\frac{1}{4(1 + \delta)} + \frac{\beta}{1 + \delta}\right) \sigma_{\max}(L_+) \|z^k - z^*\|^2 + \frac{1}{1 + \delta} \|r^k - r^*\|^2 \]
\[+ \frac{P + 1/2\beta}{(1 + \delta)c^2} \|e^{k+1}\|^2 + \frac{4\beta \sigma_{\max}(W)}{1 + \delta} \|x^{k+1} - x^*\|^2.\]

Since we have the inequality \(\|z^{k+1} - z^*\|^2 \geq \left(1 - \frac{1}{\lambda_4}\right) \|x^{k+1} - x^*\|^2 - (\lambda_4 - 1) \|e^{k+1}\|^2,\) for \(b \in (0, 1),\) we can get
\[b \left(\frac{1}{4} - \frac{\beta}{1 + \delta}\right) \sigma_{\min}(L_+) \|z^{k+1} - z^*\|^2 \geq b \left(\frac{1}{4} - \frac{\beta}{1 + \delta}\right) \sigma_{\min}(L_+) \left(1 - \frac{1}{\lambda_4}\right) \|x^{k+1} - x^*\|^2\]
\[+ b \left(\frac{1}{4} - \frac{\beta}{1 + \delta}\right) \sigma_{\min}(L_+) \left(1 - \frac{1}{\lambda_4}\right) \|x^{k+1} - x^*\|^2\]
\[\leq \frac{1/4 + \beta}{1 + \delta} \sigma_{\max}(L_+) \|z^k - z^*\|^2 + \frac{1}{1 + \delta} \|r^k - r^*\|^2\]
\[+ \frac{P + 1/2\beta}{(1 + \delta)c^2} + b \left(\frac{1}{4} - \frac{\beta}{1 + \delta}\right) \sigma_{\min}(L_+) (\lambda_4 - 1) \|e^{k+1}\|^2\]
\[+ \frac{4\beta \sigma_{\max}(W)}{1 + \delta} \|x^{k+1} - x^*\|^2.\]

Thus,
\[\left(1 - b\right) \left(\frac{1}{4} - \frac{\beta}{1 + \delta}\right) \sigma_{\min}(L_+) \|z^{k+1} - z^*\|^2 + \left(1 - \frac{4\beta}{1 + \delta}\right) \|r^{k+1} - r^*\|^2\]
\[\leq \frac{1}{4 + \beta} \sigma_{\max}(L_+) \|z^k - z^*\|^2 + \frac{1}{1 + \delta} \|r^k - r^*\|^2\]
\[+ \frac{P + 1/2\beta}{(1 + \delta)c^2} + b \left(\frac{1}{4} - \frac{\beta}{1 + \delta}\right) \sigma_{\min}(L_+) (\lambda_4 - 1) \|e^{k+1}\|^2\]
\[+ \frac{4\beta \sigma_{\max}(W)}{1 + \delta} \|x^{k+1} - x^*\|^2.\]

Define
\[A_1 = \frac{4}{(1 - b) \sigma_{\min}(L_+)},\]
and
\[A_2 = \frac{4}{(1 + 4\beta) \sigma_{\max}(L_+)}.
\]

8 Proof of Theorem 2

8.1 Eliminate \(\|x^{k+1} - x^*\|^2\)

First, we want to eliminate the term \(\|x^{k+1} - x^*\|^2\) in Lemma 6 which requires
\[b \left(\frac{1}{4} - \frac{\beta}{1 + \delta}\right) \sigma_{\min}(L_+) \left(1 - \frac{1}{\lambda_4}\right) \geq \frac{4\beta \sigma_{\max}(W)}{1 + \delta}\]
and it is equivalent to that
\[\beta \leq \frac{b(1 + \delta) \sigma_{\min}(L_+) \left(1 - \frac{1}{\lambda_4}\right)}{4b \sigma_{\min}(L_+) \left(1 - \frac{1}{\lambda_4}\right) + 16\sigma_{\max}(W)}\]
\[= \frac{b(1 + \delta) \sigma_{\min}(L_+) \left(1 - \frac{1}{\lambda_4}\right)}{4b \sigma_{\min}(L_+) \left(1 - \frac{1}{\lambda_4}\right) + 16\sigma_{\max}(W)}\]
Then we can write

\[
(1 - b)\left( \frac{1}{4} - \frac{\beta}{1 + \delta} \right) \sigma_{\min}^2(L_+) \|z^{k+1} - z^*\|^2_2 + \left( 1 - \frac{4\beta}{1 + \delta} \right) \|r^{k+1} - r^*\|^2_2 
\]

\[
\leq \frac{1}{1 + \delta} - \sigma_{\max}^2(L_+) \|z^k - z^*\|^2_2 + \frac{1}{1 + \delta} \|r^k - r^*\|^2_2 
\]

\[
+ \frac{P + 1/2\beta}{(1 + \delta)c^2} + b \left( \frac{1}{4} - \frac{\beta}{1 + \delta} \right) \sigma_{\min}^2(L_+(\lambda_4 - 1)) \|e^{k+1}\|^2_2 
\]

which can be further simplified

\[
\|z^{k+1} - z^*\|^2_2 + A_1 \|r^{k+1} - r^*\|^2_2 \leq B(\|z^k - z^*\|^2_2 + A_2 \|r^k - r^*\|^2_2) + C\|e^{k+1}\|^2_2 
\]

We require the following for convergence analysis

\[
A_1 \geq A_2 
\]

which leads to the requirement

\[
(1 - b)\sigma_{\min}^2(L_+) \leq (1 + \beta)\sigma_{\max}^2(L_+). 
\]

Note that this requirement is satisfied intrinsically.

Therefore, we get

\[
\|z^{k+1} - z^*\|^2_2 + A_1 \|r^{k+1} - r^*\|^2_2 \leq B^{k+1}\left( \|z^0 - z^*\|^2_2 + A_2 \|r^0 - r^*\|^2_2 + \sum_{s=1}^{k+1} B^{-s} C\|e^s\|^2_2 \right) 
\]

and we have the desired result since \(A_1 \|r^{k+1} - r^*\|^2_2 > 0\).

**8.2 \( B \in (0, 1) \)**

The above convergence result requires that \( B \in (0, 1) \). First, having \( \beta \) in Theorem 2 at hand, we can make sure that \( B \) is greater than 0. Then, it requires that \( B < 1 \) and correspondingly

\[
(1 + 4\beta)\sigma_{\max}^2(L_+) \leq (1 - b)(1 + \delta - 4\beta)\sigma_{\min}^2(L_+) 
\]

which is equivalent to that

\[
\beta \leq \frac{(1 - b)(1 + \delta)\sigma_{\min}^2(L_+) - \sigma_{\max}^2(L_+)}{4\sigma_{\max}^2(L_+) + (1 - b)\sigma_{\min}^2(L_+)} 
\]

and

\[
(1 - b)(1 + \delta)\sigma_{\min}^2(L_+) - \sigma_{\max}^2(L_+) > 0. 
\]

Since \( b \) can be arbitrarily chosen from \((0, 1)\), we also need

\[
0 < \frac{\sigma_{\max}^2(L_+)}{(1 + \delta)\sigma_{\min}^2(L_+)} < 1 
\]

One intuition is that we should design a network such that \( \frac{\sigma_{\max}^2(L_+)}{\sigma_{\min}^2(L_+)} \) is the smallest possible. Substituting \( \delta \) in the expression and we have

\[
\frac{\sigma_{\min}^2(L_+)}{\sigma_{\max}^2(L_+)} > \frac{L^2 - 2\nu + \sqrt{(L^2 + 2\nu)^2 + 16\nu^2\lambda_2\lambda_2 - 1}\sigma_{\min}^2(Q)}{4\nu\lambda_2 - 1\sigma_{\min}^2(Q) + 2L^2}. 
\]

(123)
9 Proof of Theorem 3

Note that \( \delta \) is chosen as

\[
\delta = \min \left\{ \frac{(\lambda_1 - 1)(\lambda_2 - 1)\sigma^2_{\min}(Q)\sigma_{\min}(L_+)}{\lambda_1\lambda_2\sigma^2_{\max}(L_+)}, \frac{4v(\lambda_3 - 1)\sigma^2_{\min}(Q)}{\lambda_1\lambda_2(\lambda_3 - 1)L^2 + c^2\lambda_3(\lambda_2 - 1)\sigma^2_{\max}(L_+)} \right\}
\]  

(124)

We choose \( c \) such that

\[
\lambda_1\lambda_2(\lambda_3 - 1)L^2 = c^2\lambda_3(\lambda_2 - 1)\sigma^2_{\max}(L_+)\sigma^2_{\min}(Q),
\]

which yields

\[
c = \sqrt{\frac{\lambda_1\lambda_2(\lambda_3 - 1)L^2}{\lambda_3(\lambda_2 - 1)\sigma^2_{\max}(L_+)\sigma^2_{\min}(Q)}}
\]

(126)

and

\[
\delta = \min \left\{ \frac{(\lambda_1 - 1)(\lambda_2 - 1)\sigma^2_{\min}(Q)\sigma_{\min}(L_+)}{\lambda_1\lambda_2\sigma^2_{\max}(L_+)}, \frac{2v(\lambda_2 - 1)\sigma^2_{\min}(Q)}{\lambda_1\lambda_2L^2} \right\}
\]

(127)

\[
= \frac{(\lambda_2 - 1)\sigma^2_{\min}(Q)}{\lambda_2} \min \left\{ \frac{(\lambda_1 - 1)\sigma^2_{\min}(L_+)}{\lambda_1\sigma^2_{\max}(L_+)}; \left( \frac{2v}{\lambda_1L^2} \right) \right\}
\]

(128)

It is desirable that \( \delta \) can achieve its maximum, which is obtained by

\[
\frac{(\lambda_1 - 1)\sigma^2_{\min}(L_+)}{\lambda_1\sigma^2_{\max}(L_+)} = \frac{2v}{\lambda_1L^2}.
\]

(129)

Therefore, we can set \( \lambda_1 \) as

\[
\lambda_1 = 1 + \frac{2v\sigma^2_{\max}(L_+)}{L^2\sigma^2_{\min}(L_+)},
\]

(130)

and thus, we have \( \delta \) as

\[
\delta = \frac{(\lambda_2 - 1)}{\lambda_2} \frac{2v\sigma^2_{\min}(Q)\sigma^2_{\min}(L_+)}{L^2\sigma^2_{\min}(L_+) + 2v\sigma^2_{\max}(L_+)}
\]

(131)

The constraint on \( \alpha \) in Eq. (14) ensures that \( B > 0 \).

Note that \( \lambda_3 \) only appears in \( C \) and \( P \). It is straightforward to derive the optimal \( \lambda_3 \) to minimize \( C \), and we arrive at

\[
\lambda_3 = \sqrt{\frac{L^2\sigma^2_{\min}(L_+) + 2v\sigma^2_{\max}(L_+)}{\beta\lambda_1L^2\sigma^2_{\min}(L_+)}} + 1
\]

(132)

thus resulting in

\[
C = \frac{4\delta\lambda_2\sigma^2_{\max}(W)}{\sigma^2_{\min}(Q)} + \sigma^2_{\max}(L_+) \left( \sqrt{\delta} + \sqrt{\frac{2(\lambda_2 - 1)\sigma^2_{\min}(Q)}{\beta\lambda_1\lambda_2L^2}} \right)^2 + \frac{b(\lambda_4 - 1)}{(1 - b)(1 + \delta)(1 + \delta - 4\beta)\sigma^2_{\min}(L_+)}
\]

(133)

10 Proof of Corollary 1

Recall the result in (115),

\[
\|z^{k+1} - z^*\|_2^2 \leq B^{k+1}(\|z^0 - z^*\|_2^2 + A_1\|r^0 - r^*\|_2^2) + B^{k+1} \sum_{s=1}^{k+1} B^{-s} C\|e^s\|_2^2
\]

(134)
which then can be written as

\[ \| z^{k+1} - z^* \|_2^2 \leq B^{k+1}(\| z^0 - z^* \|_2^2 + A_1 \| r^0 - r^* \|_2^2) + B^{k+1} C \sum_{s=1}^{k+1} B^{-s} R^s \| e^0 \|_2^2 \]  
\[ \leq B^{k+1}(\| z^0 - z^* \|_2^2 + A_1 \| r^0 - r^* \|_2^2) + B^{k+1} C \| e^0 \|_2^2 \sum_{s=1}^{k+1} \left( \frac{R}{B} \right)^s \]  
\[ \leq B^{k+1}(\| z^0 - z^* \|_2^2 + A_1 \| r^0 - r^* \|_2^2) + B^{k+1} C \| e^0 \|_2^2 \frac{R}{B - R} \]  
\[ = B^{k+1}(\| z^0 - z^* \|_2^2 + A_1 \| r^0 - r^* \|_2^2 + \frac{R C \| e^0 \|_2^2}{B - R}) \]  

completing the proof.

11 Proof of Corollary 2

According to the result in Theorem 3, we have

\[ \| z^{k+1} - z^* \|_2^2 \leq B^{k+1}(\| z^0 - z^* \|_2^2 + A_1 \| r^0 - r^* \|_2^2) + B^{k+1} \sum_{s=1}^{k+1} B^{-s} C \| e^0 \|_2^2 \]  

and then

\[ \| z^{k+1} - z^* \|_2^2 \leq B^{k+1}(\| z^0 - z^* \|_2^2 + A_1 \| r^0 - r^* \|_2^2) + C e B^{k+1} \sum_{s=1}^{k+1} B^{-s} \]  
\[ = B^{k+1}(\| z^0 - z^* \|_2^2 + A_1 \| r^0 - r^* \|_2^2) + C e B^{k+1} \frac{1}{1 - B} \]  
\[ \leq B^{k+1}(\| z^0 - z^* \|_2^2 + A_1 \| r^0 - r^* \|_2^2) + \frac{C e}{1 - B} \]

Since \( B \in (0, 1) \), we have the desired result.

12 Proof of Theorem 4

Recall the result in [115],

\[ \| z^{k+1} - z^* \|_2^2 + A_1 \| r^{k+1} - r^* \|_2^2 \leq B(\| z^k - z^* \|_2^2 + A_2 \| r^k - r^* \|_2^2) + C \| e^{k+1} \|_2^2. \]  

If \( C \| e^{k+1} \|_2^2 \leq B(A_1 - A_2) \| r^k - r^* \|_2^2 \), we can write

\[ \| z^{k+1} - z^* \|_2^2 + A_1 \| r^{k+1} - r^* \|_2^2 \leq B(\| z^k - z^* \|_2^2 + A_2 \| r^k - r^* \|_2^2) + C \| e^{k+1} \|_2^2 \]  
\[ \leq B(\| z^k - z^* \|_2^2 + A_2 \| r^k - r^* \|_2^2) + B(A_1 - A_2) \| r^k - r^* \|_2^2 \]  
\[ \leq B(\| z^k - z^* \|_2^2 + A_1 \| r^k - r^* \|_2^2) \]

Then we have

\[ \| z^{k+1} - z^* \|_2^2 + A_1 \| r^{k+1} - r^* \|_2^2 \leq B^{k+1}(\| z^0 - z^* \|_2^2 + A_1 \| r^0 - r^* \|_2^2), \]

which leads to

\[ \| z^{k+1} - z^* \|_2^2 \leq B^{k+1}(\| z^0 - z^* \|_2^2 + A_1 \| r^0 - r^* \|_2^2), \]

completing the proof as \( B \in (0, 1) \).