\(\mathcal{PT}\)-symmetric mode-locking

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Parity-time (\(\mathcal{PT}\)) symmetry is one of the most important accomplishments in optics over the past decade. Here the concept of \(\mathcal{PT}\) mode-locking of a laser is introduced, in which active phase locking of cavity axial modes is realized by asymmetric mode coupling in a complex time crystal. \(\mathcal{PT}\) mode-locking shows a transition from single to double pulse emission as the \(\mathcal{PT}\) symmetry breaking point is crossed. The transition can show a turbulent behavior, depending on a dimensionless modulation parameter that plays the same role as the Reynolds number in hydrodynamic flows. © 2016 Optical Society of America

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Mode-locking (ML) of a laser is a rather complex phenomenon in which many cavity axial modes lock together to generate ultrashort pulses [1, 2]. ML has been a milestone of laser science, with major applications to such different areas as ultrafast spectroscopy, high-speed optical communications, metrology, attosecond science, etc. Traditionally, ML methods are classified into active and passive methods [1, 2]. In active ML phase locking of cavity axial modes is forced by either intracavity amplitude (AM) or frequency (FM) modulation, which provides a symmetric transfer of the optical power among the longitudinal cavity modes [1]. The hindered complex dynamics of active and passive ML has attracted great interest since the invention of lasers [2], providing an experimentally accessible laboratory tool for the investigation of universal phenomena of dissipative dynamical systems and phase transitions [3–15]. For example, actively ML lasers can show excess noise and turbulent behavior similar to hydrodynamics flows [3–5, 9, 11], or a light-mode transition similar to Bose-Einstein condensation [12]. Phenomena like Anderson localization, Bloch oscillations and metal-insulator phase transitions typical of the solid-state physics can be observed in the spectrum of ML lasers [6, 8, 10]. Even the emergence of the mode-locked state from initial noise is an intrinsically singular transition, which has been measured in a recent experiment [15]. Recently, an interesting link has been established between phase transitions in certain dissipative systems driven out of equilibrium and parity-time (\(\mathcal{PT}\)) symmetric models [16, 17]. \(\mathcal{PT}\)-symmetry, originally introduced in quantum physics as a complex extension of quantum mechanics [18], has provided a fruitful concept in optics in the past few years (see, for instance, [19–26] and references therein). \(\mathcal{PT}\) optical structures show balanced gain and loss distributions, undergoing a symmetry breaking phase transition when the gain/loss contrast is increased. A particular and important class of \(\mathcal{PT}\)-symmetric systems is provided by periodic optical media, so-called complex crystals [25, 27, 28], which are one-way invisible at the symmetry breaking point [25, 28, 29].

In this Letter the concept of \(\mathcal{PT}\)-symmetric ML is introduced, in which the symmetry breaking phase transition can show a turbulent behavior of laser pulse emission. As compared to conventional active ML, in the \(\mathcal{PT}\)-symmetric ML transfer of the optical power between adjacent cavity axial modes is asymmetric. The asymmetry of mode coupling is realized by a suitable combination of intracavity AM and FM, so that the optical pulse circulating in the cavity is repeatedly scattered off by a complex time crystal [30]. Assuming exact synchronism between AM/FM modulation frequency \(\omega_m\) and cavity axial mode separation \(\omega_{ax}\), i.e. \(\omega_m = \omega_{ax}\), the equation of motion for the pulse envelope \(\psi(t, n)\) circulating in the cavity at successive round trips is given by the ML master equation [2, 4, 8, 31, 32]

\[
\frac{\partial \psi}{\partial n} = \left( g - l + D_g \frac{\partial^2 \psi}{\partial n^2} - \Delta_{AM}(1 - \cos(\omega_m t)) + i \Delta_{FM} \sin(\omega_m t) \right) \psi
\]

(1)

where \(t\) is the fast time variable that varies over the cavity round trip interval \((-T_m/2 < t < T_m/2)\), \(n\) is the round-trip number, \(\omega_m = 2\pi/T_m = \omega_{ax}\) is the modulation frequency, \(g = g(n)\) and \(l\) are the saturated gain and loss per transit in the cavity, \(D_g = 1/\omega_{ax}^2\) is the spectral filtering parameter determined by gain bandwidth \(\omega_g\) of the cavity, and \(\Delta_{AM}, \Delta_{FM}\) are the modulation depths of the amplitude and phase modulators, respectively. For a slow-gain medium, saturated gain \(g\) obeys the rate equation

\[
\frac{dg}{dn} = -\gamma (g - g_0 + gP)
\]

(2)

where \(g_0\) is the small-signal gain from pumping, \(\gamma = T_m/\tau\) is the ratio between cavity transit time \(T_m\) and upper laser level lifetime \(\tau\) (\(\tau \gg T_m\)), and \(P = (1/T_m) \int_{-T_m/2}^{T_m/2} dt |\psi(t, n)|^2\) is the average laser power normalized to the saturation power. As compared to the conventional AM ML regime, which is attained from Eq.(1) by assuming \(\Delta_{FM} = 0\), the addition of the quarter-phase-shifted FM modulation yields asymmetric power transfer between adjacent axial modes of
the cavity. This can be seen by writing Eq.(1) in the frequency (spectral) domain. After setting \( \psi(t,n) = \sum_n \phi_n(n) \exp(i\omega_m t) \), the following coupled equations for the spectral mode amplitudes \( \phi_n(n) \) are obtained

\[
\frac{d\phi_n}{dn} = (g - l - \Delta_{AM} - D_\sigma \omega_m^2 n^2) \phi_n + \Delta_- \phi_{n+1} + \Delta_+ \phi_{n-1},
\]

where \( q = 0, \pm 1, \pm 2, ... \) is the axial mode number and

\[\Delta = \Delta_+ - \Delta_- = (\Delta_{FM} \pm \Delta_{AM})/2.\]

As is well known, assuming \( t \) to be small, one can make use of the Wick rotation of time, in the physical problem to Wick rotate pairs for \( \Delta = 0 \) and \( \hat{\Delta} \), which represents parametric on the saturated gain \( g \). As is well known, assuming \( g = l + \Delta_{AM} \) the energy spectrum of \( \hat{H} \) is entirely real for \( \Delta_{FM} < \Delta_{AM} \) (unbroken \( PT \) phase), whereas it is formed by complex-conjugate pairs for \( \Delta_{FM} > \Delta_{AM} \) (broken \( PT \) phase) [21, 22]. Owing to Wick rotation of time, in the physical problem Eq.(1) \( PT \) symmetry breaking of \( H \) corresponds to a transition from a regime of a simple lowest-threshold ML state, i.e. a single ML pulse, to a regime of doubly-degenerate lowest-threshold ML state, i.e. to ML pulse doubling [17, 32]. Such a transition has a simple physical explanation in terms of ordinary Krizenga-Siegman theory of AM and FM laser ML [1, 34]: in the limit \( \Delta_{FM} \ll \Delta_{AM} \) the FM signal can be regarded as a small perturbation, and thus the laser operates in the AM regime, which is known to sustain two threshold-degenerate ML pulses centered at \( t = \pm T_m/4 \) [34]. Pulse doubling transition was previously investigated in Ref. [32], however it was not related to a \( PT \) symmetry breaking phase transition. Here we wish to show that, owing to the non-normal nature of \( H \) [4, 5, 16], the \( PT \) phase transition can show a turbulent behavior. To this aim, let us assume \( \Delta_{FM} \leq \Delta_{AM} \), and that there is a single ML pulse centered at \( t \approx 0 \). In such a regime, the pulse dynamics can be captured within a parabolic approximation of the complex sinusoidal potential near the minimum \( t = 0 \) of AM loss, i.e. by letting \( \cos(\omega_m t) \approx 1 - \omega_m^2 t^2/2 \) and \( \sin(\omega_m t) \approx \omega_m t \) in Eqs.(1) and (4). This yields the effective \( PT \)-symmetric Hamiltonian

\[
\hat{H}_{eff} = -D_\sigma \frac{\partial^2}{\partial t^2} + \frac{1}{2} \Delta_{AM} \omega_m^2 (t - i\delta)^2 + \sigma \]

where we have set \( \sigma \equiv l - g + \Delta_{AM}^2 / (2 \Delta_{AM}) \) and \( \delta \equiv \Delta_{FM} / (\omega_m \Delta_{AM}) \).

Interestingly, \( \hat{H}_{eff} \) describes the Hamiltonian of a \( PT \)-symmetric quantum harmonic oscillator [35], which is obtained from the ordinary (hermitian) quantum oscillator Hamiltonian \( H_{OQ} = -D_\sigma \frac{\partial^2}{\partial t^2} + 1/2 \Delta_{AM} \omega_m^2 x^2 + \sigma \) after complexification of the spatial variables \( x = t - i\delta \). \( \hat{H}_{eff} \) and \( H_{OQ} \) thus shear the same eigenvalues, whereas the eigenmodes of \( \hat{H}_{eff} \) are obtained from the Gauss-Hermite modes of \( H_{OQ} \) after the substitution \( x = t - i\delta \). From the eigenvalues of \( H_{OQ} \) one can then readily obtain the gain thresholds \( g_{0\text{th}} \) of the various Gauss-Hermite modes. In particular, the lowest-threshold mode is the fundamental Gaussian state, given by

\[
\psi_0(t) = \exp[-\rho(t - i\delta)^2]
\]

with corresponding gain threshold

\[
g_{0\text{th}} = l + \sqrt{\Delta_{AM} D_\sigma \omega_m^2 / 2} + \Delta_{AM}^2 / 2 \Delta_{AM}.
\]

where we have set \( \rho \equiv \omega_m / (2 \Delta_{AM} / 2 D_\sigma)^{1/2} \). Note that, like in the ordinary AM ML, the fundamental Gaussian state is centered at \( t = 0 \), i.e. at the minimum of AM modulation loss, however the effect of a non-vanishing \( \delta \) is to spectrally shift the ML pulse away from the center of the gainline by the amount

\[
\Delta_{\text{shift}} = 2 \rho \delta = \Delta_{FM} (2 \Delta_{AM} D_\sigma)^{-1/2},
\]

as one can see from Eq.(6). Moreover, the fundamental Gaussian state turns out to be linearly stable: it saturates the gain \( g \) so as all other higher-order modes experience net loss. However, owing to the complex displacement \( \delta \neq 0 \), \( \hat{H}_{eff} \) is a non-normal operator, i.e. \( \hat{H}_{eff} \) does not commute with its Hermitian adjoint, its eigenmodes are not orthogonal, and transient amplification of small perturbations can be thus observed.
of the FM parameter $\Delta FM$, trips, starting form a small random field, for increasing values and peak pulse intensity (bottom panel) at successive round figures show the numerically-computed evolution of normal-

![Fig. 2. (Color online) Transient formation of laser ML. The figures show the numerically-computed evolution of normalized pulse intensity (left panel), pulse spectrum (right panel) and peak pulse intensity (bottom panel) at successive round trips, starting form a small random field, for increasing values of the FM parameter $\Delta FM$: (a) $\Delta FM = 0.005$ (unbroken $\PT$ phase), (b) $\Delta FM = 0.01$ (symmetry breaking point), and (c) $\Delta FM = 0.02$ (broken $\PT$ phase). Other parameter values are: $\Delta AM = 0.01$, $\omega g/\omega m = 50$, $\gamma = 1 \times 10^{-3}$, $l = 0.04$, and $g_0 = 0.15$. The inset in the bottom panel of (c) shows an enlargement of the peak pulse intensity evolution after transient relaxation oscillations, showing small undamped oscillations.](image)

![Fig. 3. (Color online) Same as Fig. 2, except for $\omega g/\omega m = 400$.](image)

$4,5,11,36$. The maximum energy amplification $G$ of perturbations is given by the Petermann excess noise factor [11], which is given by $G = \langle \psi_0, \psi_0 \rangle \langle \psi_0^\dagger, \psi_0^\dagger \rangle / |\langle \psi_0, \psi_0^\dagger \rangle|^2$, where $\langle f, g \rangle = \int_{-\infty}^{\infty} dt f^*(t) g(t)$ denotes the usual (Hermitean) scalar product and $\psi_0^\dagger(t)$ is the adjoint mode of $\psi_0(t)$, which is simply obtained from $\psi_0(t)$ after the change $\delta \rightarrow -\delta$ on the right hand side of Eq.(6). One obtains $G = \exp(2R)$, where we have set

$$R \equiv 2\rho \delta^2 = (\omega g/\omega m) \Delta FM^2 (2\Delta AM)^{-1/2}$$

(9)

From a physical viewpoint, the transient amplification of perturbations can be explained by the appearance of a net spectral gain window, as schematically shown in Fig. 1. In fact, within the parabolic approximation of the complex sinusoidal $\PT$ potential, the dynamics of the spectral modes, as given by Eq.(3), can be cast in the form

$$\frac{\partial \phi}{\partial n} = (g-l)\phi + \frac{1}{2} \Delta AM \frac{\partial^2 \phi}{\partial q^2} - D \omega_m q^2 \phi - \Delta FM \frac{\partial \phi}{\partial q},$$

(10)

where $\phi(q,n)$ is the spectrum (Fourier transform) of
the pulse envelope at the n-th round trip in the cavity. Equation (10) is readily obtained from Eq.(3) by considering the mode index \( q \) as a continuous variable, 
\[ \phi_q(n) \rightarrow \phi(q, n), \]
and after setting \( \phi_{q+1} \simeq \phi(q) \pm (\partial \phi / \partial q) + (1/2)(\partial^2 \phi / \partial q^2)^2 \). When \( \Delta_{FM} = 0 \), the pulse spectrum is Gaussian and centered at \( q = 0 \), i.e. at the center of the gainline. However, for asymmetric mode coupling \( \Delta_{FM} \neq 0 \) a drift term arises in Eq.(10) (the last term on the right hand side), which shifts the pulse spectrum away from the center of the gainline by the amount \( \Delta \omega_{\text{shift}} \) [Eq.(8)]. Such a spectral shift leads to an increase of the laser threshold by the excess gain amount \( \Delta g \). \[ \Delta g = \Delta_{FM}(2\Delta_{AM}) \] [see Eq.(7)]. In this way, a spectral window with net gain arises: cavity axial modes excited by noise at the center of the gainline can be transiently amplified, and then convected away from the center of the gainline because of frequency drift introduced by the FM modulation; Fig.1. Such a scenario of transient amplification of perturbations is analogous to the one predicted by Kärntner et al. in detuned AM ML [4], however in the \( \mathcal{PT} \)-symmetric ML, the drift dynamics occurs in the frequency (rather than time) domain and originates from asymmetric mode coupling. Like in hydrodynamics models [36], the dimensionless parameter \( \mathcal{R} \) that determines the amount of transient energy amplification plays the role of the Reynolds number [4].

Depending on the level of noise in the system, a sufficiently large value of \( \mathcal{R} \) can bring the system to a turbulent regime [4]. Like in Ref. [4], in our model we did not consider spontaneous emission noise in the ML master equation (1), however just the noise introduced in the numerical solution to Eq.(1) can induce a turbulence behavior for \( \mathcal{R} \) larger than \( \sim 27.6 \), corresponding to a transient growth \( \mathcal{G} \sim 10^{24} \) [4]; turbulence can be observed at lower values of \( \mathcal{R} \), down to \( \mathcal{R} \sim 8 \) [4], if spontaneous emission noise is included in Eq.(1). In the turbulent regime, the system does not reach a steady state, because it is non-periodically interrupted by a new spectrally-shifted pulse created out of the net gain spectral window that destroys the previous almost stationary pulse; see Fig.3(b) to be discussed below. While in the detuned AM ML the turbulent regime is always attained by increasing the detuning between modulation period and cavity round-trip time, in our \( \mathcal{PT} \)-symmetric ML turbulence can be prevented by the onset of \( \mathcal{PT} \) symmetry breaking. In fact, the effective description of the \( \mathcal{PT} \)-symmetric ML in terms of the \( \mathcal{PT} \)-symmetric quantum oscillator Hamiltonian \( \hat{H}_{\text{eff}} \) is accurate provided that \( \Delta_{FM} \) remains smaller than \( \Delta_{AM} \), i.e. in the unbroken \( \mathcal{PT} \) phase. Therefore, the maximum energy transient growth that can be attained in the \( \mathcal{PT} \)-symmetric ML can be estimated as \( \mathcal{G}_{\text{max}} \approx \exp(2\mathcal{R}_{\text{max}}) \), where \( \mathcal{R}_{\text{max}} \approx (\omega_g/\omega_m) \sqrt{\Delta_{AM}/2} \) is obtained from Eq.(9) by assuming \( \Delta_{FM} = \Delta_{AM} \) as an upper limit. In a typical ML laser \( \Delta_{AM} \) is generally small (\( \Delta_{AM} \sim 0.01 - 0.1 \)), therefore turbulence is observed when a sufficiently large number of cavity axial modes \( N = \omega_g/\omega_m \) falls within the laser gainline. In previous analysis, we neglected group velocity dispersion (GVD) effects, which is a reasonable assumption for ML pulses with duration down to a few ps, such as in ML Nd:YAG lasers. GVD would make \( \mathcal{D}_{\text{g}} \) complex, thus breaking exact \( \mathcal{PT} \) invariance of \( \hat{H} \). However, even in the presence of small-to-moderate GVD, i.e. for \( |\text{Im}(\mathcal{D}_{\text{g}})| \ll |\text{Re}(\mathcal{D}_{\text{g}})| \), pulse splitting and turbulence discussed above can be still observed.

We checked the predictions of the theoretical analysis by direct numerical simulations of the ML laser equations (1) and (2) using a standard pseudo spectral split-step method and assuming a small random amplitude of the intracavity field \( \psi(t, 0) \) at initial round trip. Parameter values used in the simulations are \( \gamma_0 = 1 \times 10^{-3} \), \( \mathcal{N}_{\text{AM}} = 0.01 \), and \( g_0 = 0.15 \), which are typical of solid-state lasers (e.g. Nd:YAG). Two different values of \( N = \omega_g/\omega_m \) are considered. In an experiment, for a given modulation frequency \( N \) can be controlled by changing the effective gain bandwidth \( \omega_g \) using an intracavity etalon. In the first set of simulations, a relatively small value \( N = 50 \) is considered, and the ML pulse build-up dynamics was numerically simulated for increasing values of the FM amplitude \( \Delta_{FM} \), from below to above the \( \mathcal{PT} \) symmetry breaking transition. The results are shown in Fig.2. According to the theoretical predictions, below the symmetry breaking point a single ML pulse, with a spectrum shifted from the center of the gainline owing to the FM signal, is observed [Figs.2(a) and (b)], whereas pulse splitting is observed in the broken \( \mathcal{PT} \) phase [Fig.2(c)]. In the latter case the amplitudes of the two ML pulses are generally different, depending on the initial random conditions. Note that, after an initial transient pulse built-up interval associated to relaxation oscillations, the highest peak pulse intensity shows a small but visible undamped oscillations in the broken \( \mathcal{PT} \) phase [see the inset in Fig.2(c)]. Such oscillations arise from the interference of the two non-orthogonal and temporally-shifted ML pulses [17, 32]. Note also that the symmetry breaking transition does not show a turbulent behavior: indeed, for such a relatively narrow gain bandwidth the Reynold number at the symmetry breaking point is \( \mathcal{R}_{\text{max}} \sim 3.53 \) according to Eq.(11), which is smaller than the critical Reynolds number that brings the system to turbulence. On the other hand, turbulence can be observed by increasing the gain bandwidth. Figure 3 shows, as an example, numerical results obtained for \( N = 400 \), corresponding to a Reynolds number \( \mathcal{R}_{\text{max}} \sim 28.3 \) at the symmetry breaking point. Note that \( \mathcal{PT} \) symmetry breaking is now associated to a turbulent behavior, with transiently growing spectral modes [indicated by the arrows in Fig.3(b)] that irregularly disrupt the stationary ML state.

In conclusion, the concept of \( \mathcal{PT} \)-symmetric ML has been introduced, in which asymmetric mode coupling leads to a transition from a single to double ML pulse emission. Such a transition is the signature of \( \mathcal{PT} \) symmetry breaking and can show a turbulent behavior. The
present results provide an important link between a fundamental operational regime of a laser, i.e. mode locking, and the emerging field of PT optics, suggesting that laser ML could provide a fertile laboratory tool to investigate the physics of PT symmetry in optics.

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