General dispersion equation for oscillations and waves in non-collisional Maxwellian plasmas

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We propose a new and effective method to find plasma oscillatory and wave modes. It implies searching a pair of poles of two-dimensional (in coordinate $x$ and time $t$) Laplace transform of self-consistent plasma electric field $E(x,t) \rightarrow E_{p_1p_2}$, where $p_1 \equiv -i\omega$, $p_2 \equiv ik$ are Laplace transform parameters, that is determining a pair of zeros of the following equation

$$\frac{1}{E_{p_1p_2}} = 0.$$ 

This kind of conditional equation for searching double poles of $E_{p_1p_2}$ (which correspond to terms of the type $\frac{1}{(p_1-p_1^{(n)})(p_2-p_2^{(n)})}$ in double Laurent expansion of $E_{p_1p_2}$) we call “general dispersion equation”, so far as it is used to find the pair values $(\omega^{(n)} k^{(n)})$, $n = 1, 2, \ldots$. It differs basically from the classic dispersion equation $\epsilon_l(\omega, k) = 0$ (and is not its generalization), where $\epsilon_l$ is longitudinal dielectric susceptibility, its analytical formula being derived according to Landau analytical continuation. In distinction to $\epsilon_l$, which is completely plasma characteristic, the function $E_{p_1p_2}$ is defined by initial and boundary conditions and allows one to find all the variety of asymptotical plasma modes for each concrete plasma problem. In this paper we demonstrate some possibilities of applying this method to the simplest cases of collisionless ion-electron plasma and to electron plasma with collisions described by a collision-relaxation term $-\nu f^{(1)}$.

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I. INTRODUCTION

Up to date the textbook approach to determining dispersion dependence of oscillation frequency on wave number $\omega(k)$ implies solving some dispersion equation, the latter being found using Landau rule of bypassing poles in his theory of collisionless damping (“Landau damping”, see[1]). This theory is based on analytical continuation of complex function defined as Cauchy integral

$$F(\omega) = \int_{-\infty}^{\infty} \frac{\phi(z)dz}{z-\omega},$$

where the integration is performed primarily along the real $z$-axis. This integral is well defined for $\text{Im} \ (\omega) > 0$ and $\text{Im} \ (\omega) < 0$. Landau considers as physically defined just the function $F^+(\omega)$ in the upper half-plane of $\omega$ due to adiabatic in time switching on. Then he tries to determine analytical continuation of this function to the region $\text{Im} \ (\omega) < 0$ through the point $\text{Im} \ (\omega) = 0$ on the real axis, with deformation of the integration contour to the lower half-plane.

One could say that analytical continuation through integration contour $A$ (i.e. the real axis $R$) does not exist by its very definition as a regular function, since $F^+_A(\omega) = F^+_A(\omega)$ is indefinite and not regular for $\omega \in A$, i.e. for $\omega$ on the real axis, discriminated by physics of the problem. Suppose the fixed integration contour $A'$ is parallel to and lies above the contour $A$. When $\omega$ lies above $A'$ according to Cauchy theorem we have $F^+_A(\omega) = F_{A'}(\omega)$. But $F_{A'}(\omega)$ is indefinite for $\omega \in A'$ and therefore both functions $F^+_A$ and $F_{A'}$ for the first sight cannot be regarded as analytical continuation of one to another.

Nevertheless one can resolve above-mentioned paradox and construct the analytical continuation by the following way. Let contour $A'$ to lie over $A$ by $ib$, $b > 0$. According to Cauchy theorem for $\text{Im} \ (\omega) > b$ we have $F_{A'}(\omega) = F^+_A(\omega)$, so

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\[ \int_{-\infty}^{\infty} \frac{\phi(z)dz}{z - \omega} = \int_{-\infty}^{\infty} \frac{\phi(z + ib)dz}{z + ib - \omega}, \]  

(2)

where \( \phi(z) \) is supposed to be analytical in the stripe \( 0 \leq \text{Im} (z) < b \). Analytical continuation of \( F_\alpha' (\omega) \) into the region \( 0 < \text{Im} (\omega) < b \) can be obtained by Eq. (2) with some arbitrary \( b < \text{Im} (\omega) \). In the same way if \( \text{Im} (\omega) < 0 \) one should use Eq. (2) with some arbitrary \( b < \text{Im} (\omega) \). In all cases we obtain the single analytic function (due to singleness theorem). This function is obtained by deforming integration contour in such a way that point \( \omega \) lies always above the contour (this is due to the \( + \) type of considered analytical function). For Maxwellian plasma \( \phi(z) \sim z \exp (-\alpha z^2) \) with \( |\phi(z)| \rightarrow \infty \) at \( \text{Im} (z) \rightarrow \pm \infty \) and it has no poles in the complex plane \( z \in C \). Therefore the corresponding functions \( F^\pm (\omega) \) are analytic functions in the whole complex plane.

Using analytical continuation of \( F^+ (\omega) \) in the lower half-plane (instead of \( F^- (\omega) \)) is closely related with the absence of any solutions of dispersion equation for any \( \delta \) in \( \omega = \omega_0 + i\delta, \omega_0 \neq 0 \), when dispersion equation is derived by substitution of plane wave solution \( \exp(-i\omega t + ikx) \) into Vlasov equations for electrons (that is linearized kinetic equation + Poisson equation) in the case of background Maxwellian plasma. The theory of Landau damping can also be considered as an equivalent to solving the analytically continued to the region \( \text{Im} (\omega) < 0 \) dispersion equation, that is equation \( F^+ (\omega, \vec{k}) = 1 \) with analytically continued left-hand-side function \( F^+ (\omega, \vec{k}) \).

The mathematical part of Landau theory (see f.e. [1]) contrary to the said in [2] does not give rise to any objections. However there are still the following logical objections:

1. Landau solution in the form of a damping wave must satisfy primary Vlasov equations, but it does not.

2. In the original Landau contour integral \( F(\omega) \) there is the physically discriminated real axis of integration over velocity component \( v_x \). One solves some initial problem in the time interval \( 0 \leq t < \infty \), and coordinate dependence is introduced through the common factor \( \exp(i k x) \) for all times \( t \). Here initial conditions really are in no connection with unknown conditions at \( t < 0 \). Writing solution in the form of inverse Laplace transform ("original") with parameter \( p = -i\omega \) physically leads one to search in the first place some finite (damping or oscillatory non-damping) solutions with \( \omega = \omega_0 - i\delta, \omega_0 > 0 \) and \( \delta > 0 \). These solutions correspond to the lower half-plane of complex frequency \( \omega \).

The function \( F^- (\omega) \) is analytical in the lower half-plane, and there is no logical necessity to use analytical continued from the upper half-plane function \( F^+ (\omega) \), or consider some analytical continuation into upper half-plane, where oscillatory solutions are exponentially divergent. As it is also well-known, the integrand of inverse Laplace transform \( L(F) \) in this case has no poles and, correspondingly, no solutions in the form of damping wave (that means the absence of solutions of equivalent dispersion equation).

3. The application of two-dimensional Laplace transformation (see also this paper) shows that the asymptotic solution of Vlasov equations with Maxwell distribution function exists only in the form of a system of coupled

\[ F^+(\omega) = PV \int_{-\infty}^{\infty} \frac{\phi(z)dz}{z - \omega} + i\pi \phi(\omega) \quad \text{(at Im} (\omega) = 0); \]

\[ F^+(\omega) = PV \int_{-\infty}^{\infty} \frac{\phi(z)dz}{z - \omega} + 2i\pi \phi(\omega) \quad \text{(at Im} (\omega) < 0); \]

\[ F^-(\omega) = PV \int_{-\infty}^{\infty} \frac{\phi(z)dz}{z - \omega} \quad \text{(at Im} (\omega) < 0); \]

\[ F^+(\omega) = F^-(\omega) + 2i\pi \phi(\omega) \quad \text{(for all} \omega). \]

Here \( F^+(\omega) \) is regular at all \( \omega \) with \( -b_0 < \text{Im} (\omega) \leq 0 \), where \( b_0 \) is defined by the analyticity properties of \( \phi(z) \). Cf. also the analogous expressions in [2]. These relations and all features can easily be demonstrated with a simple example of the function \( \phi(z) = 1/ (z^2 + \omega^2) \) (noted by A. P. Bakulev)

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\(^1\) For the analytical continuation of \( F^+ (\omega) \) from the upper half-plane (this is marked by ’plus’-sign in contrast to analytical continuation of \( F^- (\omega) \) from the lower half-plane, which is marked by ’minus’-sign) to the lower one we have

\[ F^+ (\omega) = PV \int_{-\infty}^{\infty} \frac{\phi(z)dz}{z - \omega} + i\pi \phi(\omega) \quad \text{(at Im} (\omega) = 0); \]

\[ F^+ (\omega) = PV \int_{-\infty}^{\infty} \frac{\phi(z)dz}{z - \omega} + 2i\pi \phi(\omega) \quad \text{(at Im} (\omega) < 0); \]

\[ F^- (\omega) = PV \int_{-\infty}^{\infty} \frac{\phi(z)dz}{z - \omega} \quad \text{(at Im} (\omega) < 0); \]

\[ F^+ (\omega) = F^- (\omega) + 2i\pi \phi(\omega) \quad \text{(for all} \omega). \]
oscillatory modes. The absence of solutions in the form of a single wave \( \exp \left( -i\omega t - ik\vec{x} \right) \), where \( \omega \) is complex value, can be proved by *reductio ad absurdum*: the substitution of this solution into Vlasov equations leads to a dispersion equation which has no solutions of the type \( \omega = f (k) \). Landau theory for the solution in a more general form \( f(t) \cdot \exp \left( ik\vec{x} \right) \) ought to be considered as part of the classical proof by *reductio ad absurdum*: using one-dimensional Laplace transformation for the half-plane of damping waves \( \text{Im} (\omega) < 0 \) one obtains, contrary to initial supposition, that damping solutions in the form \( f(t) \sim \exp (-i\omega t) \) are absent, and moreover the analytical continuation into the half-plane \( \text{Im}(\omega) > 0 \) leads, also contrary to initial supposition, to exponentially growing solutions. Besides that, the latters satisfy not the Vlasov equations but some other equations with another dispersion relation, that proves in its turn the failure of the initial supposition about the assumed form of solution.

Therefore there is discrepancy between mathematical correctness of Landau theory and unjustifiyness and unnecessity of its real application.

As it has been pointed out in the solution of “Landau problem” is really non-damping standing oscillations (but not a travelling damping wave) as it has been proved in \( \delta > 0 \). And the desired damping solution can be attained through the transition to a non-Maxwellian electron velocity distribution function (“cut-off” velocity distribution). Probably, it is equivalent to solving dispersion equation which can be solved with respect to the transition to a non-Maxwellian electron velocity distribution (“cut-off” velocity distribution). But evidently, such solution is not completely defined due to the arbitrariness of Maxwell distribution “cut-off”. And besides that, under special condition of “cut-off” \( v < c \) where \( c \) is the light velocity, one should in any case solve the relativistic kinetic equation.

A quite different, but general way to consider plasma oscillations in frames of linearized Vlasov equations was proposed in \( \delta > 0 \). It relies on two-dimensional (in coordinate \( x \) and time \( t \)) Laplace transforms \( f_{p_1p_2}(x,t) \), \( E_{p_1p_2} \) for the perturbation \( f^{(1)}(x,t,\vec{v}) \) of distribution function and the field \( E(x,t) \), where \( p_1, p_2 \) are the parameters \( p_1 = -i\omega, \ \ p_2 = ik \) of Laplace transformation. Depending on initial and boundary conditions one obtains different analytical expressions for the function \( E_{p_1p_2} \). The equation for double poles (in \( p_1 \) and \( p_2 \) of this function defines different asymptotic oscillatory modes as the pairs \( (\omega^{(n)}, k^{(n)}) \), \( n = 1, 2, \ldots \). So, it is natural to consider the equation for poles of \( E_{p_1p_2} \) as “general dispersion equation”. The general asymptotic solution is obtained as a sum of exponential modes with coefficients defined by residues, but substitution of this solution \( f^{(1)}(x,t,\vec{v}) \) and \( E(x,t) \) into Vlasov equations does not lead, in general, to any dispersion equation connecting \( \omega \) and \( k \). That is, the dispersion equation, in common sense, does not exist. Indeed, it ought to suppose that, vice versa, for a given concrete problem its asymptotical solution in the form of coupled oscillatory modes must identically satisfy Eqs. (3)-(4), and this can be tested by direct substitution of solution in Eqs. (3)-(4).

This approach appears to be more general, allowing one to find additional oscillatory modes also in the case of non-Maxwellian distribution functions, when the usual dispersion equation could take place and have solutions.

These statements which were developed in \( \delta > 0 \) appear to form a new and unexpected, though very simple, approach to the problem of plasma oscillations. In this paper we demonstrate the very possibility of applying the two-dimensional Laplace transformation for finding ion-electron oscillations and waves in collisionless Maxwellian plasma and electron oscillations in low-collisional Maxwellian plasma in the case of electron-neutral collisions described by the collision-relaxation term \( -\nu f^{(1)} \), where \( \nu \approx \text{const} \) is collision frequency. (But, if Landau theory is wrong, then it is necessary also to revise collision corrections for accurate taking into account the Coulomb collisions including one obtained in \( \delta > 0 \) with the method of expansion into asymptotically divergent series over small parameter \( \frac{v}{c} \).

To demonstrate our approach we consider here the simplest cases of one-dimensional (in \( x \)) plane longitudinal plasma waves based on kinetic equations for electrons, correspondingly, ions

\[
\frac{\partial f^{(1e)}}{\partial t} + v_x \frac{\partial f^{(1e)}}{\partial x} - \frac{|e|E_x}{m} \frac{\partial f_0}{\partial v_x} = 0 \tag{3}
\]

\[
\frac{\partial f^{(1i)}}{\partial t} + v_x \frac{\partial f^{(1i)}}{\partial x} + \frac{|e|E_x}{m} \frac{\partial f_0}{\partial v_x} = 0 \tag{4}
\]

and Poisson equation

\[
\frac{\partial E_x}{\partial x} = -4\pi |e| n_e \int_{-\infty}^{\infty} f^{(1e)}(1) d^3\vec{v} + 4\pi |e| n_i \int_{-\infty}^{\infty} f^{(1i)}(1) d^3\vec{v}; \ \ n_e \approx n_i. \tag{5}
\]
II. THE BOUNDARY PROBLEM OF EXCITING ION-ELECTRON OSCILLATIONS AND WAVES

The plasma is assumed Maxwellian, homogeneous and infinite in space and time. The boundary condition is assumed to be given in the plane \( x = 0 \) (the plane geometry). The one-sided Laplace transformation allows one to obtain absolutely definitive solution in the region \( 0 \leq x < \infty \). But for the same boundary condition one can also obtain one-sided solution with the help of analogous Laplace transformation in the region \( -\infty < x \leq 0 \) (after substitution \( x' = -x \)). So one obtains the unified solution in the whole interval \( -\infty < x < \infty \). The solution will be continuous at \( x = 0 \). The derivative \( \partial f^{(1)} / \partial x \) at \( x = 0 \) can be either continuous or discontinuous. The same is true also for the one-sided Laplace time transformation at \( -\infty < t < \infty \) for some given initial condition at \( t = 0 \). The unified solution will not be specially analyzed here.

Applying two-dimensional Laplace transformation to perturbations of the electron and ion distribution functions \( f^{(1 e)}, f^{(1 i)} \)

\[
f^{(1)}(x, t, \vec{v}) = \frac{1}{(2\pi i)^2} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \int_{\sigma_2-i\infty}^{\sigma_2+i\infty} f_{p_1, p_2}^{(1)} e^{p_1 t + p_2 x} dp_1 dp_2,
\]

\[
\frac{\partial f^{(1)}(x, t, \vec{v})}{\partial x} = \frac{1}{(2\pi i)^2} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \int_{\sigma_2-i\infty}^{\sigma_2+i\infty} p_2 f_{p_1, p_2}^{(1)} e^{p_1 t + p_2 x} dp_1 dp_2 - \frac{1}{2\pi i} f^{(1)}(0, t, \vec{v}) \int_{\sigma_2-i\infty}^{\sigma_2+i\infty} e^{p_2 x} dp_2,
\]

and analogous expressions for \( \frac{\partial f^{(1)}(x, t, \vec{v})}{\partial x}, E(x, t), \frac{\partial E(x, t)}{\partial x} \) in eqns. (3)–(7); assuming for instance oscillatory boundary conditions

\[
E(0, t) = E_0 e^{-i\beta t} = \frac{E_0}{2\pi i} \int_{\sigma_1+i\infty}^{\sigma_1+i\infty} e^{p_1 t} dp_1,
\]

\[
f^{(1)}(0, t, \vec{v}) = \alpha e^{-i\beta t} = \frac{\alpha}{2\pi i} \int_{\sigma_1+i\infty}^{\sigma_1+i\infty} e^{p_1 t} dp_1,
\]

one obtains for electrons/ions:

\[
f^{(1e)}_{p_1, p_2} = \frac{1}{p_1 + v_x p_2} \left[ -\frac{v_x |e|}{kT_e} \left( \frac{m}{2\pi kT_e} \right)^{3/2} e^{-\frac{m v_x^2}{2kT_e}} E_{p_1, p_2} + \frac{v_x \alpha_e (\vec{v})}{p_1 + i\beta} + f^{(1e)}_{p_2} (\vec{v}) \right]
\]

\[
f^{(1i)}_{p_1, p_2} = \frac{1}{p_1 + v_x p_2} \left[ \frac{v_x |e|}{kT_i} \left( \frac{M}{2\pi kT_i} \right)^{3/2} e^{-\frac{M v_x^2}{2kT_i}} E_{p_1, p_2} + \frac{v_x \alpha_i (\vec{v})}{p_1 + i\beta} + f^{(1i)}_{p_2} (\vec{v}) \right]
\]

where Maxwell functions have been used:

\[
f^{(c)}_{0} (\vec{v}) = \left( \frac{m}{2\pi kT_e} \right)^{3/2} e^{-\frac{m v_x^2}{2kT_e}}; \quad f^{(i)}_{0} (\vec{v}) = \left( \frac{M}{2\pi kT_i} \right)^{3/2} e^{-\frac{M v_x^2}{2kT_i}},
\]

and \( f^{(c)}_{p_1} (\vec{v}), f^{(i)}_{p_1} (\vec{v}) \) are corresponding one-dimensional (either in \( t \) or \( x \)) Laplace images of boundary and initial conditions (see (7)). Then equation for finding \( E_{p_1, p_2} \) can be written in the following form:

\[
E_{p_1, p_2} = \frac{4\pi e^2 n_e}{kT_e} \left( \frac{m}{2\pi kT_e} \right)^{3/2} \int_{-\infty}^{\infty} e^{-\frac{m v_x^2}{2kT_e}} \frac{v_x d^3 \vec{v}}{p_1 + v_x p_2} - \frac{4\pi e^2 n_e}{kT_e} \left( \frac{M}{2\pi kT_i} \right)^{3/2} \int_{-\infty}^{\infty} e^{-\frac{M v_x^2}{2kT_i}} \frac{v_x d^3 \vec{v}}{p_1 + v_x p_2}
\]

\[
= \frac{E_0}{p_1 + i\beta} - 4\pi |e| n_e \left[ \int_{-\infty}^{\infty} \frac{(\alpha_e - \alpha_i) (\vec{v}) v_x d^3 \vec{v}}{(p_1 + v_x p_2) (p_1 + i\beta)} + \int_{-\infty}^{\infty} \frac{f^{(1e)}_{p_2} - f^{(1i)}_{p_2} d^3 \vec{v}}{(p_1 + v_x p_2)} \right].
\]

We can transform integrals in the l.h.s. of eq. (13):
\[
\int_{-\infty}^{\infty} \frac{e^{-\frac{\overline{v}_x^2}{2p_1 + v_x p_2}}}{p_1 + v_x p_2} \, dv_x = -2p_2 \int_{0}^{\infty} \frac{e^{-\frac{\overline{v}_x^2}{2p_1 - v_x^2 p_2}}}{p_1 - v_x^2 p_2} \, dv_x \sim -\left( \frac{2\pi k T_e}{m} \right)^{3/2} \frac{p_2}{2} \frac{\overline{v}_x^2}{p_1^2 - v_x^2 p_2^2},
\]

(14)

where \( \overline{v}_x^2 \) is some characteristic value of squared velocity defined by Maxwellian exponent and \( \gamma \approx 1 \) (see also a discussion on principal value sense of such integrals in [1]). Note that the points \( p_1 = \mp v_x p_2 \) here are not really poles!

The infinities are cancelled after account for the analogous terms with \( p_1 \pm v_x p_2 \) in denominator from \( f_p^{(1)}(\overline{v}) \), with latter being proportional to \( E_{p_1 p_2} \) (see [2]). Let us for simplicity omit off the consideration for a time the additive terms proportional to \( f^{(1e)}(0, t, \overline{v}) \), \( f^{(1i)}(0, t, \overline{v}) \), \( f^{(1e)}(x, 0, \overline{v}) \) and \( f^{(1i)}(x, 0, \overline{v}) \) and obtain

\[
E_{p_1 p_2} = \frac{-\overline{E}_0}{1 + \frac{\omega^2}{p_1^2 - \alpha p_2^2} + \frac{\omega^2}{p_1^2 - \beta p_2^2}},
\]

(15)

where \( \omega_e, \omega_i \) are correspondingly Langmuir electron and ion frequencies; \( \sqrt{\alpha} = \sqrt{2\gamma k T_e/m}, \sqrt{\beta} = \sqrt{2\gamma k T_i/M} \) are nearby thermal mean velocities of one-dimensional thermal movement of electrons and ions. The value \( \gamma \) can be approached in concrete calculations by means of an iteration process with more and more precise values of the poles \((p_1, p_2)\) of expression (15) with their subsequent substitution in eqn. (14). The equation for finding a pair of poles of the function \( E_{p_1 p_2} \) (15) we call “general dispersion equation” for given boundary and initial problem.

If the terms \( \alpha_e(\overline{v}), \alpha_i(\overline{v}) \) and \( f_p^{(1e)}, f_p^{(1i)} \) are non-zero, they can be presented as sums of symmetrical and antisymmetrical (with respect to \( v_x \to -v_x \)) parts and be reduced by analogous way to integrals over interval \( 0 \leq v_x < \infty \) with the integrand denominator \((p_1^2 - v_x^2 p_2^2)\). This might lead also to an appearance of new plasma modes. The plasma boundary oscillations apparently can be realized by supplying periodic electric potential to wide flat electrode grid immersed into homogeneous plasma volume.

Asymptotical both in \( x \) and \( t \) plasma modes \((\omega(n), k(n))\) are defined by a pair of residues \((p_1, p_2)\) at poles of function (15). The case \( p_2 = 0 \) (that is, relative displacements of electron and ion parts as a whole) leads to the pole \( p_1 = \pm i \sqrt{\omega_e^2 + \omega_i^2} \). It corresponds to the eigen-mode of non-damping plasma oscillations with the frequency (but not residue) independent on \( \beta \). To excite this mode the boundary field should be increased fast enough up to the value \( E_0 \).

Besides that there is a mode \( p_2 = 0 \), \( p_1 = -i \beta \), which corresponds to forced plasma oscillations of electrons and ions as a whole with the frequency of excitation source.

The more interesting case is \( p_1 = -i \beta \) and \( p_2 \) being the root of equation

\[
1 - \frac{\omega_e^2}{\beta^2 + \alpha p_2^2} - \frac{\omega_i^2}{\beta^2 + \beta p_2^2} = 0.
\]

(16)

Its solution can be found analytically in general form:

\[
p_2^2 = -\frac{1}{2ab} \left[ a(\beta^2 - \omega_e^2) + b(\beta^2 - \omega_i^2) \right] \pm \frac{1}{2ab} \sqrt{a(\beta^2 - \omega_e^2) + b(\beta^2 - \omega_i^2) - 4ab(\beta^2 - \omega_e^2 - \omega_i^2)}.
\]

(17)

We do not consider here all particular cases, but instead note that at common conditions \( M \gg m, T_e \gg T_i \) with \( \beta \gg \omega_i \) one obtains the following roots:

\[
p_2^2 \approx -\frac{\beta^2 - \omega_i^2}{b}; \quad p_2^{(1,2)} \approx \pm i\beta \sqrt{1 - \frac{\omega_i^2/\beta^2}{b}},
\]

(18)

so the wave speeds (if there are no standing waves) are

\[
v^{(1,2)} \approx \pm \sqrt{\frac{b}{1 - \omega_i^2/\beta^2}} = \pm \sqrt{\frac{2\gamma k T_i}{M}} \sqrt{\frac{1}{1 - \omega_i^2/\beta^2}},
\]

(19)

– that is the modes are the Langmuir ion waves.

\[\text{It is necessary to correct in [1] the term } E_0/p_2^2 \text{ by the term } E_0/p_2. \text{ In Eq.(28) the second pair of poles must be } (p_1 = \pm i \xi \sqrt{2A e^{-\xi^2}} \Delta \xi, p_2 = 0)\]
If $\beta \ll \omega_i$, the expression under the root sign can be expanded in the small parameter

$$\frac{4ab\beta^2 \left( \beta^2 - \omega_i^2 - \omega_e^2 \right)}{a \left( \beta^2 - \omega_i^2 \right) + b \left( \beta^2 - \omega_e^2 \right)}.$$ 

At $\left[ a \left( \beta^2 - \omega_i^2 \right) + b \left( \beta^2 - \omega_e^2 \right) \right] < 0$ one of solutions will take the form

$$p_2^2 \simeq \frac{-\beta^2\omega_e^2}{a\omega_i^2 + b\omega_e^2}; \quad p_2^{(3,4)} \simeq \pm \frac{i\beta\omega_e}{\sqrt{a\omega_i} \sqrt{1 + b\omega_e^2/a\omega_i^2}},$$ (20)

so the travel speeds are

$$v^{(3,4)} \simeq \pm \frac{\sqrt{a\omega_i}}{\omega_e} \sqrt{1 + b\omega_e^2/a\omega_i^2} = \pm \sqrt{\frac{2\gamma k T_e}{M} \left( 1 + \frac{T}{T_e} \right)},$$ (21)

that is these solutions correspond to the modes of non-damping ion-acoustic waves. The other pair of values $p_2$ are

$$p_2^2 \simeq \frac{1}{ab} \left| a \left( \beta^2 - \omega_i^2 \right) + b \left( \beta^2 - \omega_e^2 \right) \right| \simeq \frac{\omega_i^2 - \beta^2}{b};$$

$$p_2^{(1,2)} \simeq \pm \omega_i \sqrt{\frac{1 - \beta^2/\omega_i^2}{b}}$$ (22)

and correspond to exponential damping and exponential growing modes.

Besides that in all the cases the damping solutions are not of the specific Landau damping type.

As it has been noted in the previous section, the presence of growing solutions is connected with the inconsistency of the initial and boundary conditions on the field $E(0, t)$ and functions $f^{(1)}(0, t; \vec{v})$ and $f^{(1)}(x, 0; \vec{v})$, which are not independent.

The initial and boundary conditions must be given in such a way, that the residues sums in the coefficients of exponentially growing modes be cancelled. Such consistency condition leads to linear equations connecting pre-

consistent plasma electric field $E$ and potential according to $\odot$ boundary condition $\odot$ external field (its inconsistency means that it is defined by external source, not by plasma). In practice one usually transits from field to potential according to $E_{ext} = -\nabla \phi_{ext}/\nabla x$, and $E_{con}$ must satisfy Poisson equation $\odot$ and is defined as self-consistent field with boundary condition $E_{con}(0, t)$.

### III. PROPER ELECTRON WAVES IN THE LOW-COLLISION PLASMA

Adding into the r.h.s. of eqn.(3) the collision term $-\nu (\vec{v}) f^{(1)}$ in close analogy with the previous section we obtain

$$E_{p_1 p_2} = \frac{E_e}{p_2 (p_1 + i\beta)} \left( \frac{\omega_i^2}{\omega_e^2} \right)^{p_1 + i\beta},$$ (23)

where $\nu (\vec{v})$ is an effective collision frequency between electrons and neutral particles. At $p_2 = 0$ one obtains the pole $p_1 = \pm i\omega_e - \nu$ - that means the damping in time Langmuir oscillations. Besides that there are non-damping oscillations with the boundary field frequency $(p_2 = 0, p_1 = -i\beta)$.

At $p_1 = -i\beta$ one obtains also according to (23) the equation for determining $p_2$:
\[ p_2^2 = \frac{1}{a} \left[ (\nu^2 - \beta^2 + \omega_c^2) - 2i\beta\nu \right]. \]  

(24)

Omitting the elementary algebraic procedures of extracting the root of a complex variable, we give the final result:

\[ p_2 = \pm \beta\nu \sqrt{\frac{\sqrt{2}}{\left(\sqrt{F^2 + 4\beta^2\nu^2} - F\right)^{1/2}}} + \frac{i}{\sqrt{2a}} \left( \sqrt{F^2 + 4\beta^2\nu^2} - F \right)^{1/2}, \]  

(25)

where

\[ F \equiv \nu^2 - \beta^2 + \omega_c^2. \]

At \( F < 0 \) and small \( \nu \ll \sqrt{\beta^2 - \omega^2_c}, \beta > \omega_c \) one obtains

\[ p_2 \simeq \pm \beta\nu \sqrt{\frac{\sqrt{a}}{\beta^2 - \omega_c^2}} \pm \frac{i}{\sqrt{a}} \sqrt{\beta^2 - \omega_c^2}. \]  

(26)

At \( F > 0 \) it is more convenient to rewrite eqn. (25) in the form

\[ p_2 = \pm \frac{\beta\nu \sqrt{\frac{\sqrt{2}}{\left(\sqrt{F^2 + 4\beta^2\nu^2} + F\right)^{1/2}}}}{\sqrt{2a}} \mp \frac{i}{\sqrt{a}} \left( \sqrt{F^2 + 4\beta^2\nu^2} + F \right)^{1/2}, \]  

(27)

and at \( \nu \ll \sqrt{\omega_c^2 - \beta^2}, \omega_c > \beta \) one obtains

\[ p_2 \simeq \pm \frac{\sqrt{\omega_c^2 - \beta^2}}{\sqrt{a}} \mp \frac{i\beta\nu}{\sqrt{a}} \frac{1}{\sqrt{\omega_c^2 - \beta^2}}. \]  

(28)

In both cases one obtains the exponential growing solutions. But substitution of any exponential growing expression of the form \( \exp(i\omega t + ikx) \) with complex values \( \omega, k \) in eqns. (3)–(5) leads to a dispersion equation which has no solutions. It means that at the same extent as taking into account the boundary condition \( E(0, t) \) one must also for the sake of consistency with \( E(0, t) \) take into account initial and boundary values of functions \( f^{(1e)}(x, t, \vec{v}), f^{(1i)}(x, t, \vec{v}), \) in such a way that growing terms be cancelled (as it has been already noted in [4]).

The possibility of such cancellation is provided with the same poles of growing solutions for \( E_{p_1p_2} \) and Laplace images of initial and boundary values of \( f^{(1e)} \) and \( f^{(1i)} \).

IV. CONCLUSIONS

We have considered Vlasov differential equations for collisionless plasma and successfully solved them asymptotically by the Laplace transform method of operational calculus.

We have proposed absolutely new, very simple and effective way of finding plasma oscillation modes: they are defined by the pairs \( (\omega(n), k(n)), n = 1, 2, \ldots, \) which are determined as pairs of roots (“double-zeros”) of the “generalized dispersion equation”

\[ \frac{1}{E_{p_1p_2}} = 0, \]  

(29)

where \( E_{p_1p_2} \) is two-dimensional (in \( x \) and \( t \)) Laplace image of self-consistent plasma electric field \( E(x, t) \). Some additional plasma density oscillation modes appear to be determined from

\[ \frac{1}{f^{(1e)}_{p_1p_2}} = 0; \quad \frac{1}{f^{(1i)}_{p_1p_2}} = 0, \]  

(30)

but really this has no place. According to the required mutual consistency of initial and boundary values of \( f^{(1e)}(x, t, \vec{v}), f^{(1i)}(x, t, \vec{v}) \) with boundary field \( E(0, t) \) (the solutions finiteness condition) these additional oscillatory modes must be connected with the electric field, thus the coefficients in the total sum of the modes have to be proportional to \( E_0 \).
The eqn.(29) is in principle different from equation commonly used in literature (see, for instance, the classic text-book exposition in

\[ \varepsilon_l(\omega, k) = 0, \]  

(31)

where (according to the considered in this paper cases) \( \varepsilon_l \) is longitudinal dielectric susceptibility defined by only the intrinsic parameters of a homogeneous infinite plasma. Contrary to this approach the function \( E_{p_1p_2} \) accounts for initial and boundary conditions concretely. The finding \( E_{p_1p_2} \), contrary to \( \varepsilon_l \), does not require using Landau theory with its doubtful foundations (using analytical continuation, expansions in asymptotically divergent series (see, for instance, 4, 6), problems with the principle of causality and so on). At the same time the equation (29) allows one, contrary to (31), to find all the oscillation modes in a given concrete problem. It reveals immediately the tight interconnection of oscillatory modes with the concrete conditions, that is with methods of excitation, excitation frequency, functional forms of initial and boundary conditions, whereas eqn.(31) connects plasma oscillations and waves only with proper intrinsic parameters of plasma.

Moreover, the required connection of initial and boundary conditions means that it is impossible completely to discriminate between a pure boundary or pure initial problems. So, to find the finite solution of a “boundary” problem it is certainly required to supply also non-zero initial values of distribution functions \( f^{(1)} \) under consideration.

It should be emphasized once again that in general case using the theory of plasma oscillations, based on susceptibility \( \varepsilon \) and plasma modes equation \( \varepsilon = 0 \), either is erroneous (because it is based on the dispersion equation which does not exist) or has rather limited range of applicability.

In this paper we have considered only the simplest examples of application of suggested method. Besides of diverse generalizations one of the main problems here stays the account for Coulomb collisions, which requires cumbersome mathematical computations with revision of results described in literature (see, for instance, 6) and based mainly on expansions in asymptotically divergent series in small parameter \( \delta/\omega_0 \).

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