The generalized KP hierarchy

C.S.Xiong

International School for Advanced Studies (SISSA/ISAS)
Via Beirut 2, 34014 Trieste, Italy

Abstract

We propose one possible generalization of the KP hierarchy, which possesses multi bi–hamiltonian structures, and can be viewed as several KP hierarchies coupled together.
1 Introduction

Non–linear integrable differential systems have been subject to intensive investigations since last century. In the last few years, one of the most exciting developments in this field is the revelation of its close relationship with 2–dimensional exactly solvable field theories, in particular 2–dimensional quantum gravity and string theory. This relation enables us to extract non–perturbative properties of non–critical string. More precisely, the discretization of 2–dimensional quantum gravity can be reformulated as matrix models. Through the double scaling limit, one can prove that 1–matrix model with even potential is described by KdV hierarchy and string equation. Recently it has been shown that the double scaling limit is not an inevitable step. In other words, the KdV hierarchy is not merely an occasional effect of double scaling limit but intrinsic in matrix model formulation. The idea is as follows: we represent matrix models as certain discrete linear system(s), from which we can extract lattice integrable hierarchies, finally we can directly extract differential hierarchies from these lattice hierarchies. Using this approach, one can prove that the 1–matrix model with general potential is characterized by non–linear Schrödinger hierarchy (NLS), which is the two bosonic field representation of KP hierarchy. In other words, the 1–matrix model gives a new solution for the τ–function of KP hierarchy. In the same way, we can apply this procedure to multi–matrix models to obtain their full differential integrable hierarchies. In this process, we are led to consider a possible generalization of KP hierarchy. The study of this generalization is the main purpose of this letter. Actually another enlargement of KP hierarchy was considered some years ago by adding additional flows. However, although each of the additional flows commutes with all the old KP flows, they don’t commute among themselves. Thus this generalization is not consistent. Fortunately, we will see that properly introducing new flows we can obtain an integrable hierarchy, in which all the flows are commutative. We will explain the realization of such generalized KP hierarchy in multi–matrix models elsewhere.

2 KP hierarchy

Let us begin with a pseudo–differential operator(PDO) of arbitrary order

\[ A = \sum_{-\infty}^{n} a_i(x) \partial_i. \] (2.1)

where \(x\) is the space coordinate, while \(\partial^{-1}\) is formal integral operation over \(x\). All the pseudo–differential operators form an algebra \(\wp\) under the generalized Leibnitz rules

\[
\partial a(x) = a(x) \partial + a'(x), \quad [\partial, x] = 1,
\]

\[
\partial^{-1} \partial = \partial \partial^{-1} = 1,
\]

\[
\partial^{-j-1} a(x) = \sum_{l=0}^{\infty} (-1)^l \binom{j+l}{l} a^{(l)}(x) \partial^{-j-l-1}, \]

where \(a^{(l)}(x)\) denotes \(\frac{\partial^{l} a(x)}{\partial x^{l}}\). The algebra \(\wp\) has two sub–algebras:

\[ \wp = \wp_+ \oplus \wp_- . \]
where \( \wp_+ \) denotes the algebra of pure differential operators, while \( \wp_- \) means the algebra of pure integration operations.

For any given pseudo–differential operator \( A \) of type (2.1), we call \( a_{-1}(x) \) its residue, denoted by

\[
\text{res}_\partial A = a_{-1}(x) \quad \text{or} \quad A_{(-1)}
\]

and we define the following functional

\[
< A > = \int a_{-1}(x) dx.
\]

(2.3)

which naturally gives an inner scalar product on the algebra \( \wp_- \).

### 2.1 The integrable structure

Now let \( L \) be a pseudo–differential operator of the first order

\[
L = \partial + \sum_{i=0}^{\infty} u_i(x) \partial^{-i}
\]

(2.4)

which we will call KP(or Lax) operator. We call \( u_i \)'s KP coordinates. \((L - \partial) \in \wp_-\), so we can represent a functional of KP coordinates as

\[
f_X(L) = < LX >, \quad X \in \wp_+
\]

which span a functional space \( \mathcal{F}(\wp_-) \). The remarkable fact is that \( \mathcal{F}(\wp_-) \) is invariant under the co–adjoint action of \( \wp_+ \), consequently the algebraic structure on \( \wp_+ \) determines the Poisson structure on \( \mathcal{F}(\wp_-) \)

\[
\{ f_X, f_Y \}_1(L) = L([X, Y])
\]

(2.5)

The infinite many conserved quantities (or Hamiltonians) are

\[
H_r = \frac{1}{r} < L^r > \quad \forall r \geq 1
\]

(2.6)

They generate infinite many flows,

\[
\frac{\partial}{\partial t_r} L = [L^r_+, L]
\]

(2.7)

where the subindex “+” indicates choosing the non–negative powers of \( \partial \). Since

\[
[L^r_+, L] = [L, L^r_-] \in \wp_-
\]

we see that all the flows preserve the form of KP operator “L”, and they all commute with each other. This commutativity implies the “zero curvature representation”

\[
\frac{\partial}{\partial t_m} L^n_+ - \frac{\partial}{\partial t_n} L^m_+ = [L^m_+, L^n_+], \quad \forall n, m
\]

(2.8)
By KP hierarchy we mean the set of differential equations (2.7) or (2.8). In fact the KP hierarchy possesses another Poisson structure\footnote{5}:

\[
\{f_X, f_Y\}_2(L) = < (XL)_+ YL > - < (LY)_+ LX > + \int [L, Y]_{(-1)} \left( \partial^{-1}[L, X]_{(-1)} \right)
\]  \hspace{1cm} (2.9)

With respect to these Poisson brackets, the KP coordinates \(u_i\) form \(W\)-infinity algebras. The important point is that these two Poisson structures are compatible in the sense

\[
\{f, H_{r+1}\}_1 = \{f, H_r\}_2 \quad \forall \quad \text{function } f
\]  \hspace{1cm} (2.10)

This compatibility ensures the integrability of KP hierarchy. Generally speaking, for a system of infinite many degrees of freedom, for example, the KP hierarchy, we may find various definitions of integrability\footnote{4}. The essential point is that there must exist infinite many conserved quantities in involution. Therefore we can list some of the definitions below

1. There exist two compatible Poisson brackets (or bi-hamiltonian structure).
2. The flows are all commutative.
3. There exists the zero curvature representation (2.8).

For different purposes we may use different definitions. For example, bi-Hamiltonian structure can exhibit the Poisson algebraic structure of the system. But in the next two sections we will mainly use the second definition to prove the integrability of the generalized KP hierarchy due to its simplicity.

### 2.2 The associated linear system

The KP operator \(L\) can be expressed in terms of the “dressing” operator

\[
L = K \partial K^{-1} \quad K = 1 + \sum_{i=1}^{\infty} w_i \partial^{-i}
\]

After defining

\[
\xi(t, \lambda) = \sum_{r=1}^{\infty} t_r \lambda^r
\]  \hspace{1cm} (2.11)

and introducing the Baker–Akhiezer function

\[
\Psi(t, \lambda) = K e^{\xi(t, \lambda)}
\]  \hspace{1cm} (2.12)

we can associate to the KP hierarchy (2.7) a linear system

\[
\begin{cases}
L \Psi = \lambda \Psi, \\
\frac{\partial}{\partial t_r} \Psi = L_r \Psi.
\end{cases}
\]  \hspace{1cm} (2.13)
2.3 The $\tau$–function

One of the important ingredients of the KP system is its $\tau$–function, which can be introduced through Baker–Akhiezer function

$$\Psi(t, \lambda) = \frac{\tau(t_1 - \frac{1}{2}, t_2 - \frac{1}{2^2}, \ldots)}{\tau(t)} e^{\xi(t, \lambda)}$$  \hspace{1cm} (2.14)$$

One can prove that

$$\frac{\partial^2}{\partial t_1 \partial t_r} \ln \tau = \text{res} \frac{\partial L^r}{\partial r}, \quad \forall r \geq 1$$ \hspace{1cm} (2.15)$$

If we define a set of new functionals

$$J_r = \int \text{res} \frac{\partial L^r}{\partial r} dx, \quad \forall r \geq 1$$ \hspace{1cm} (2.16)$$

then

$$\frac{\partial}{\partial t_s} J_r = 0, \quad \forall r, s \geq 1$$ \hspace{1cm} (2.17)$$

So $J_r$’s are the conservation laws of the KP hierarchy.

3 Generalization of KP hierarchy

Now we come to discuss the generalization of the KP hierarchy, which we promised in the introduction.

3.1 The additional flows

Our purpose is to show that we may introduce other series of flows. In order to do so, we define a new operator[6],

$$M \equiv K \left( \sum_{i=1}^{\infty} rt_i \partial^{i-1} \right) K^{-1} = \sum_{i=-\infty}^{\infty} v_i \partial^i.$$ \hspace{1cm} (3.1)$$

which is conjugate to the KP operator $L$ in the sense that

$$[L, M] = K \left[ \partial, \sum_{i=1}^{\infty} rt_i \partial^{i-1} \right] K^{-1} = 1,$$ \hspace{1cm} (3.2)$$

We can derive the equations of motion for $M$,

$$\begin{cases}
\frac{\partial}{\partial r} M = [L^r_+, M], \\
\frac{\partial}{\partial \lambda} \Psi = M \Psi.
\end{cases} \hspace{1cm} (3.3)$$

So we see that $L$ and $M$ are nothing but the operatorial expressions of $\lambda, \frac{\partial}{\partial \lambda}$ (acting on $\Psi(t, \lambda)$).
As we know, the basic requirement for new flows is that they should preserve the form of KP operator $L$. So we can define new flows like
\[
\frac{\partial}{\partial t_{mn}} L = [L, (M^m L^n)_-], \quad \forall m, n. \tag{3.4}
\]

One can show that each flow commutes with KP flows (2.7), but these additional flows do not commute among themselves. Our aim is to show that properly choosing combinations of these additional flows, we can define new flows which commute with the old KP flows and among themselves.

### 3.2 Another series of flows

Our starting remark is that the $t$–series of perturbations is due to the fact that $[L, L_r^-] \in \mathcal{P}_-, \forall r \geq 1$. Now we also have $[L, M_r] \in \mathcal{P}_-, \forall r \geq 1$, so we could introduce a new series of deformation parameters $^*y_1, y_2, y_3, \ldots$, such that

\[
\begin{align*}
\frac{\partial}{\partial y_r^*} L &= [L, M_r], \\
\frac{\partial}{\partial y_r^*} \Psi &= -M_r^* \Psi. \tag{3.5}
\end{align*}
\]

All of these equations together result in the following enlarged KP system

\[
\begin{align*}
\frac{\partial}{\partial t_r} L &= [L_r^*, L], \\
\frac{\partial}{\partial t_r} M &= [L_r^*, M], \\
\frac{\partial}{\partial y_r^*} L &= [L, M_r^-], \\
\frac{\partial}{\partial y_r^*} M &= [M_r^+, M]. \tag{3.6}
\end{align*}
\]

Now we should prove that these new series of perturbations do not destroy consistency, that is to say, we should check the commutativity of all these flows. In the following we only consider an example, i.e.

\[
\frac{\partial}{\partial t_r} \left( \frac{\partial}{\partial y_s} L \right) = \frac{\partial}{\partial y_s} \left( \frac{\partial}{\partial t_r} L \right). \tag{3.7}
\]

Using eqs. (3.6), we see that the left hand side is

\[
\begin{align*}
\frac{\partial}{\partial t_r} \left( \frac{\partial}{\partial y_s} L \right) &= \frac{\partial}{\partial t_r} [L, M_s^-] \\
&= [[L_r^+, L], M_s^-] + [L, [L_r^+, M_s^-]] \\
&= [[L_r^+, L], M_s^+] + [L, [L_r^+, M_s^+]] - [L, [L_r^+, M_s^-]] + [L, [L_r^+, M_s^+]] + [L_r^+, M_s^+] + L \\
&= r.h.s.
\end{align*}
\]

*That is to say, the KP coordinates $u_i$’s and $v_i$’s depend on both $t$ and $y$. 
The other cases can be checked in the similar way. So the perturbations we introduced before indeed give an enlarged KP hierarchy. Its associated linear system is

\[
\begin{align*}
L \Psi &= \lambda \Psi, \\
\frac{\partial}{\partial t} \Psi &= L^r_+ \Psi, \\
\frac{\partial}{\partial y} \Psi &= -M^r_- \Psi, \\
M \Psi &= \frac{\partial}{\partial \lambda} \Psi.
\end{align*}
\] (3.8)

The usual KP hierarchy (2.7) is a particular case of eqs. (3.6) by fixing the \( y \)-series of the perturbations.

### 3.3 The new basic derivative and the new bi–hamiltonian structure

As we remarked a moment ago, when we discard the \( y \)-series of flows, we recover the usual KP hierarchy, whose Hamiltonians are

\[H_{r(L)} = \frac{1}{r} < L^r>,\]

here we use the subindex \( (L) \) to indicate that the Hamiltonians are constructed from the KP operator \( L \). We may also use the same symbol to denote the Poisson brackets, \( \{,\}_{(L)}. \)

Now if we fix all the \( t \)-series of parameters, then we get another subset of the enlarged hierarchy (3.6), that is

\[
\begin{align*}
\frac{\partial}{\partial y} L &= [L, M^r_-], \\
\frac{\partial}{\partial y} M &= [M^r_-, M].
\end{align*}
\] (3.9)

The second equation is in fact a KP hierarchy with KP operator \( M \) of the form (3.1). Since all these flows commute, this is an integrable system, and there should exist two compatible Poisson brackets written in terms of coordinates \( v_i \)'s. However, this bi–hamiltonian structure is unknown due to the fact that the positive powers of \( \partial \) in \( M \) go to infinity.

Fortunately, we may overcome the difficulty by introducing a new basic derivative. To this end, we recall that in our previous analysis we treated \( t_1 \) as the space coordinate. For later convenience, we denote \( \frac{\partial}{\partial y_1} \) by \( \tilde{\partial} \). From the \( y_1 \)-flows of \( \Psi \), we may extract an operator identity

\[
\tilde{\partial} = -M_ - = \sum_{i=1}^{\infty} \Gamma_i \partial^{-i}.
\] (3.10)

Since any positive powers of \( \tilde{\partial} \) belongs to \( \mathcal{P}_-(\partial) \), so \( \{\tilde{\partial}^i; i \geq 1\} \) forms a basis of \( \mathcal{P}_-(\partial) \).
We may invert the relation (3.10) to express \( \partial \) in terms of the new derivative \( \tilde{\partial} \)

\[
\partial = \sum_{i=1}^{\infty} \tilde{\gamma}_i \tilde{\partial}^{-i}.
\] (3.11)

Using this fact, we get

\[
M = -\tilde{\partial} + \sum_{i=1}^{\infty} \tilde{v}_i \tilde{\partial}^{-i} = -\tilde{K} \tilde{\partial} \tilde{K}^{-1}.
\] (3.12)

with new dressing operator \( \tilde{K} \) and new KP coordinates \( \tilde{v}_i \)'s. Obviously

\[
M(r - (\partial)) = M(r + (\tilde{\partial})), \quad \forall r \geq 1,
\] (3.13)

where LHS is expanded in powers of \( \partial \), while the RHS is expanded in powers of \( \tilde{\partial} \).

Using eq.(3.11), we can reexpress all the formulas (3.6) in terms of this new derivative \( \tilde{\partial} \), i.e.

\[
\begin{align*}
\frac{\partial}{\partial y_r}(-M(\tilde{\partial})) &= (-1)^{r+1} [(-M)^{+}_r(\tilde{\partial}), (-M)(\tilde{\partial})], \\
\frac{\partial}{\partial y_r} L(\tilde{\partial}) &= (-1)^{r+1} [(-M)^{+}_r(\tilde{\partial}), L(\tilde{\partial})], \\
\frac{\partial}{\partial y_r}(-M)(\tilde{\partial}) &= -[(-M)(\tilde{\partial}), L^{+}_r(\tilde{\partial})], \\
\frac{\partial}{\partial y_r} L(\tilde{\partial}) &= L^{+}_r(\tilde{\partial}), (\tilde{\partial}).
\end{align*}
\] (3.14)

Apart from some additional signs, these equations are isomorphic to eqs.(3.6). This reminds us that we can even consider \((-M)\) as a KP operator, and alternatively interpret \(y_1\) as space coordinate, all the other parameters as time parameters. Therefore we can define two compatible Poisson brackets for KP operator \((-M)\) by simply replacing \(L\) in (2.5) and (2.9) by \((-M)\), which shows that on the space \(y_1\), the fields \(\tilde{v}_i\)'s form \(W_\infty\) algebras too.

4 Further perturbations and the full generalized KP hierarchy

In the previous section we have shown that the KP hierarchy can be perturbed by the conjugate operator \(M\) of the KP operator \(L\). In fact, the KP system allows further deformations.

\footnote{Rigorously speaking, this is only true when it acts on the function \(\Psi\). But we may think of it in the following way, starting from

\[
\tilde{\partial} \Psi = \sum_{i=1}^{\infty} \Gamma_i \partial^{-i} \Psi.
\]

properly choosing the combinations of \(\tilde{\partial}\) such that we can reexpress the \(\partial^{-1} \Psi\) in terms of new derivatives \(\partial\), we replace all the derivatives \(\partial\) in the linear system (3.8) by \(\tilde{\partial}\). So we may interpret \(y_1\) as another space coordinate.}
4.1 The new series of the flows

In order to explain the further perturbations just mentioned, we change a little bit our notation. Denote \(t_r\)’s and \(y_r\)’s by \(t_1r\) and \(t_2r\) respectively. Furthermore define

\[
L(1) \equiv L, \quad V(1) \equiv \sum_{r=1}^{\infty} rt_1r L^{r-1}(1)
\]

\[
L(2) \equiv -\frac{1}{c_{12}} M, \quad V(2) \equiv \sum_{r=1}^{\infty} rt_2r L^{r-1}(2)
\]

Now let us introduce new operators in the following way

\[
L(\alpha) \equiv -\frac{1}{c_{\alpha-1,\alpha}} \left(c_{\alpha-2,\alpha-1} L(\alpha - 2) + V(\alpha - 1)\right) \quad (4.1a)
\]

\[
V(\alpha) = \sum_{r=1}^{\infty} rt_{\alpha,r} L^{r-1}(\alpha), \quad \alpha = 3, 4, \ldots, n \quad (4.1b)
\]

where \(c_{\alpha,\alpha+1}\)’s are arbitrary constants, which amount to rescaling the space coordinates, and \(n\) is an arbitrary positive integer. Then, in the same way, we can perturb the system further as follows

\[
\frac{\partial}{\partial t_{\beta r}} L(\alpha) = [L^r(\beta), L(\alpha)], \quad 1 \leq \beta < \alpha \quad (4.2a)
\]

\[
\frac{\partial}{\partial t_{\beta r}} L(\alpha) = [L(\alpha), L^r(\beta)], \quad \alpha \leq \beta \leq n \quad (4.2b)
\]

Now in order to justify the consistency of these perturbations, we once again should prove that all the flows commute among themselves. Let us check one example,

\[
\frac{\partial}{\partial t_{\alpha l}} \left( \frac{\partial}{\partial t_{\beta m}} L(\gamma) \right) = \frac{\partial}{\partial t_{\beta m}} \left( \frac{\partial}{\partial t_{\alpha l}} L(\gamma) \right), \quad \alpha < \beta < \gamma.
\]

Using the above hierarchy and Jacobi identities, we see that

l.h.s. = \[\frac{\partial}{\partial t_{\alpha l}} [L^m_+(\beta), L(\gamma)] = [[L^l_+(\alpha), L^m_+(\beta)], L(\gamma)] + [L^m_+(\beta), [L^l_+(\alpha), L(\gamma)]].\]

and

r.h.s. = \[\frac{\partial}{\partial t_{\beta m}} [L^l_+(\alpha), L(\gamma)] = [[L^l_+(\alpha), L^m_+(\beta)], L(\gamma)] + [L^l_+(\alpha), [L^m_+(\beta), L(\gamma)]].\]

The first term of “l.h.s” can be written as

the 1st term = \[[L^l_+(\alpha), L^m_+(\beta)], L(\gamma)] + [[L^l_+(\alpha), L^m_+(\beta)], L(\gamma)]

= \[[L^l_+(\alpha), L^m_+(\beta)], L(\gamma)] + [L^l_+(\alpha), [L^m_+(\beta), L(\gamma)]

therefore

l.h.s. = \[[L^l_+(\alpha), L^m_+(\beta)], L(\gamma)] + [[L^l_+(\alpha), L^m_+(\beta)], L(\gamma)] + [L^m_+(\beta), [L^l_+(\alpha), L(\gamma)]

= \[[L^l_+(\alpha), L^m_+(\beta)], L(\gamma)] + [L^l_+(\alpha), [L^m_+(\beta), L(\gamma)]

= r.h.s.
All the other cases can be done in the same way. Therefore eqs. (4.2) really define an integrable system. The associated linear system is

\[
\begin{align*}
L(1)\Psi &= \lambda\Psi, \\
\frac{\partial}{\partial t_{1,r}}\Psi &= L_+^r(1)\Psi, \\
\frac{\partial}{\partial \alpha_{r}}\Psi &= -L_-^r(\alpha)\Psi, \quad \alpha = 2, 3, \ldots, n, \\
M\Psi &= \frac{\partial}{\partial \lambda}\Psi.
\end{align*}
\]

(4.3)

In fact we can rewrite this linear system in a better way by choosing a new function

\[
\Psi(\lambda, t) \mapsto \xi(\lambda, t) = \exp\left(-\sum_{r=1}^{\infty} t_{1,r}^r\lambda^r_1\right)\Psi(\lambda, t),
\]

then all the flows can be summarized by a single equation

\[
\frac{\partial}{\partial t_{\alpha,r}}\xi = -L_-^r(\alpha)\xi.
\]

(4.4)

The consistency conditions of the above linear system exactly give the hierarchy (4.2) We would like to remark that the hierarchy (4.2) have several important sub–hierarchies.

(i). \(\alpha = \beta = 1\), the eqs. (4.2b) are nothing but the usual KP hierarchy (2.7).

(ii). \(2 \leq \alpha = \beta \leq n\), the eqs. (4.2b) give \((n - 1)\) KP hierarchies whose KP operator possess the form (1.1a).

(iii). All the flows commute.

(iv). When \(n \to \infty\), the full hierarchy (4.2) contains all possible combinations of additional flows (3.4).

We may conclude that the hierarchy (4.2) possess \(n\) bi–hamiltonian structures, each of them generates a KP hierarchy, all of these hierarchies couple together. The integrability of the system is guaranteed by the commutativity of the flows. However, we are not sure whether the hamiltonians in different series are commutative.

### 4.2 New bi–hamiltonian structures

In the above analysis, all the operators are expanded in terms of \(\partial\). However, if we use the same trick as the one in previous section, it is not difficult to reexpress them in terms of any one of \(\frac{\partial}{\partial t_{\alpha,1}}\)'s. Let us define

\[
\partial_\alpha \equiv \frac{\partial}{\partial t_{\alpha,1}}
\]

(4.5)

and expand \(L_-^r(\alpha)\) in powers of \(\partial\)

\[
L_-^r(\alpha) = -\sum_{i=1}^{\infty} \Gamma_{i}^{(r)}(\alpha)\partial^{-i}
\]

(4.6)
then the first flows of the linear system (4.3) suggest
\[ \partial_\alpha = \sum_{i=1}^{\infty} \Gamma_i^{(\alpha)} \partial^{-i} \] (4.7)
similar to the argument in the previous section, we can invert these relations, such that
\[ \partial = \sum_{i=1}^{\infty} \Gamma_i^{(\alpha)} \partial^{-i} \] (4.8)
Substituting them into the formulas (4.1a), we get the expansions of \( L(\alpha) \) in \( \partial \partial_t \beta \), \( r \) (for any \( \alpha, \beta \)). In particular \( L(\alpha) \) expanded in \( \partial \partial_t \alpha, 1 \) is also a KP operator,
\[ L(\alpha) = -(\partial_\alpha + \sum_{i=1}^{\infty} v_i^{(\alpha)} \partial^{-i}) \] (4.9)
and its \( \alpha - th \) series of flows is nothing but the ordinary KP hierarchy
\[ \frac{\partial}{\partial t_{\alpha,r}} L(\alpha) = (-1)^{r+1} [L_r^{\alpha}(\alpha), L(\alpha)] \] (4.10)
where the operators are expanded in powers of \( \partial_\alpha \), and the additional sign indicates rescaling of the parameters. Of course, for this subsystem, we can construct its integrable structure, by replacing \( L \) in (2.5) and (2.9) by \( L(\alpha) \). Therefore, we may say that KP system (4.2) possesses multi bi–Hamiltonian structures, and it contains “n” coupled ordinary KP hierarchies. The coupling comes from the dynamical equations (4.2) with \( \alpha \neq \beta \).

4.3 The \( \tau \)–function of the generalized  KP hierarchy

Using eqs.(4.2), we get
\[ \frac{\partial}{\partial t_{\beta,s}} \text{res} \partial L^r(\alpha) = \frac{\partial}{\partial t_{\alpha,r}} \text{res} \partial L^s(\beta), \quad \forall \alpha, \beta; \quad r, s. \] (4.11)

These equalities imply the existence of \( \tau \)–function
\[ \frac{\partial^2}{\partial t_{1,1} \partial t_{\alpha,r}} \ln \tau = \text{res} \partial L^r(\alpha), \quad \forall \alpha, r. \] (4.12)
Using this \( \tau \)–function, we can introduce a series of the Baker–Akhiezer functions,
\[ \Psi_{\alpha}(t, \lambda_{\alpha}) = \frac{\tau(t_{\alpha,1} - \frac{1}{\lambda_{\alpha}}, t_{\alpha,2} - \frac{1}{2\lambda_{\alpha}}, \ldots)}{\tau(t)} e^{\xi(t, \lambda_{\alpha})} \] (4.13)
where \( \alpha = 1, 2, \ldots, n \). To each \( \Psi_{\alpha} \) we can associate a linear system. Among them, the \( \alpha = 1 \) case was described above. The other cases can be analysed in the similar way.

11


5 Discussion

We have shown that the KP hierarchy can be extended to a much larger hierarchy by introducing additional KP operators. This generalized hierarchy can be considered as several coupled KP hierarchies. For each of the KP operators, we have constructed its bi–hamiltonian structure by introducing new basic derivatives. Although we do not know if all these hamiltonians are in involution, the commutativity of the flows guarantees the integrability of the system.

As we know in the ordinary KP hierarchy case, the series of flows reflects the large symmetry of the system generated by its Hamiltonians. In our case, the multi–series of flows imply that this new hierarchy (4.2) should possess a much larger symmetry. However we are not sure what this large symmetry is.

It is not clear if this new hierarchy relates to the multi–component KP hierarchy. Another interesting problem is to reduce this hierarchy to the known hierarchies like generalized KdV hierarchies. This is under investigation.

Acknowledgement

I would like to thank Prof. L. Bonora for his constant encouragement, valuable suggestions and fruitful discussions.

References

[1] L. Bonora and C.S.Xiong, Phys.Lett.B285(1992)191.
[2] L. Dickey, Soliton equations and Hamiltonian Systems, World Scientific, 1991.
[3] L.Bonora and C.S.Xiong, Multi–matrix models without scaling limit, SISSA preprint SISSA–211/92/EP.
[4] B. Babelon and C. Viallet, Lecture notes in SISSA(1989).
[5] Y. Watanabe, Lett.Math.Phys. 7(1983)99–106.
I.M. Gelfand and L.A. Dikii, Funct. Anal. Appl. 11:2 (1977) 93.
Y.S.Wu, F.Yu, Nucl.Phys.B373(1992)713.
[6] H. Chen, Y. Lee and J. Lin, Physica D9(1983)359–445.
A. Orlov, and E. Shulman, Lett.Math.Phys.12(1986)171–179.
L.Dickey, Oklahoma preprint, it Additional symmetries, Grassmanian and string equation I,II.
[7] C. S. Xiong, Ph.D. thesis, SISSA, 1992.