ON THE SHAPE OF THE GROUND STATE EIGENFUNCTION FOR STABLE PROCESSES

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Abstract. We prove that the ground state eigenfunction for symmetric stable processes of order \( \alpha \in (0, 2) \) killed upon leaving the interval \((-1, 1)\) is concave on \((-\frac{1}{2}, \frac{1}{2})\). We call this property “mid-concavity.” A similar statement holds for rectangles in \(\mathbb{R}^d, d > 1\). These results follow from similar results for finite dimensional distributions of Brownian motion and subordination.

1. Introduction

Let \( D \) be a bounded convex domain in \(\mathbb{R}^d, d \geq 1\), and let \( \varphi_1 \) be the first eigenfunction for the Dirichlet Laplacian in \( D \). In their seminal paper [13], Brascamp and Lieb proved that \( \varphi_1 \) is log–concave in \( D \). That is, \( \log(\varphi_1) \) is concave on any segment contained in the domain. This result has led to many interesting applications in analysis, geometry, pde, mathematical physics and probability. For some of these applications, see Borell [10], [11], [12] and the many references therein. In particular, the log–concavity of \( \varphi_1 \) leads to estimates of the spectral gap \( \lambda_2 - \lambda_1 \) which in turn describe the rate to equilibrium of the Brownian motion conditioned to remain forever in the domain \( D \). We refer the reader to [3], [18], [20] and [21] for some of these applications and additional references.

In [4], the first two authors initiated the study of what may be called the “fine spectral theoretic properties” of symmetric stable processes. Unfortunately, given the “nonlocality” of the generator of these processes, even the most basic questions seem to be very difficult. It was proved in [4] (Theorem 5.1) that the ground state eigenfunction for the
Cauchy process in the interval $(-1, 1)$ is concave. We, of course, expect this to be the case for any symmetric stable process. The purpose of this paper is to prove that for any symmetric stable processes, the ground state eigenfunction is concave in $(-\frac{1}{2}, \frac{1}{2})$. We call this property “mid–concavity”. This will follow from a more general result on “mid–concavity” of the finite dimensional distributions of these processes. This “mid–concavity” result is new even for Brownian motion.

We first recall some basic definitions. Let $X^\alpha_t$ be a d-dimensional symmetric stable process of index $0 < \alpha \leq 2$. The process $X^\alpha_t$ has stationary independent increments and its transition density $p^\alpha_t(x, y) = p^\alpha_t(x - y)$, $t > 0, x, y \in \mathbb{R}^d$, is determined by its Fourier transform

$$\exp(-t|\xi|^\alpha) = \int_{\mathbb{R}^d} e^{i\xi \cdot y} p^\alpha_t(y) dy.$$ 

These are Lévy processes with right continuous sample paths. The transition densities satisfy the scaling property

$$p^\alpha_t(x, y) = t^{-d/\alpha} p^\alpha_1(t^{-1/\alpha} x, t^{-1/\alpha} y),$$

hence the process has the scaling property of index $\alpha$. When $\alpha = 2$, $X^2_t$ is just Brownian motion $B_t$ running at twice the speed and when $\alpha = 1$, $X^1_t$ is the Cauchy process. In the first case, $p^2_t(x, y)$ is the usual Gaussian distribution (heat kernel) and in the second, $p^1_t(x, y)$ is the Cauchy distribution (Poisson kernel).

Our interest here is on symmetric stable processes of index $0 < \alpha < 2$ killed upon leaving a domain $D$. That is, let $D \subset \mathbb{R}^d, d \geq 1$, be a nonempty bounded connected open set and let

$$\tau^\alpha_D = \inf\{t \geq 0 : X^\alpha_t \notin D\}$$

be the first exit time of $X^\alpha_t$ from $D$. Let

$$T^D_t f(x) = E_x(f(X^\alpha_t), \tau^\alpha_D > t),$$

for $x \in D, t > 0$ and $f \in L^2(D)$, be the semigroup of the killed process. The killed process has transition densities $p^\alpha_D(t, x, y)$ and

$$T^D_t f(x) = \int_D p^\alpha_D(t, x, y) f(y) dy.$$ (1.1)

As with Brownian motion,

$$p^\alpha_D(t, x, y) = p^\alpha(t, x, y) - r_D(t, x, y),$$ (1.2)

where

$$r_D(t, x, y) = E_x(p^\alpha_{t - \tau^\alpha_D/2}(X^\alpha_{\tau^\alpha_D/2}, y), \tau^\alpha_D < t).$$ (1.3)
From this it follows that the transition function $p_D^\alpha(t, x, y)$ is nonnegative, symmetric, jointly continuous in $x$ and $y$, and that for all $x, y \in D$ and $t > 0$,

$$p_D^\alpha(t, x, y) \leq p_1^\alpha(x, y) = t^{-d/\alpha}p_1^\alpha(t^{-1/\alpha}x, t^{-1/\alpha}y) \leq Ct^{-d/\alpha},$$

where $C = (2\pi)^{-d}\omega_d\Gamma(d/\alpha)/\alpha$ and $\omega_d$ is the surface measure of the unit sphere in $\mathbb{R}^d$. In fact, $p_D^\alpha(t, x, y)$ is strictly positive for $x, y \in D$. These properties and the general theory of heat semigroups (as in [15]) gives an orthonormal basis of eigenfunctions $\{\varphi_n^\alpha\}$ on $L^2(D)$ with eigenvalues $\{\lambda_n^\alpha\}$ satisfying $0 < \lambda_1^\alpha < \lambda_2^\alpha \leq \lambda_3^\alpha \leq \ldots$, and $\lambda_n^\alpha \to \infty$, as $n \to \infty$. That is,

$$T_t^D \varphi_n^\alpha(x) = e^{-\lambda_n^\alpha t} \varphi_n^\alpha(x), \quad x \in D.$$

In addition, the first eigenvalue $\lambda_1^\alpha$ is simple and its corresponding eigenfunction $\varphi_1^\alpha$, which we will refer to as the ground state eigenfunction, is an analytic strictly positive function on $D$. The infinitesimal generator of the semigroup is $-(\Delta)^{\alpha/2}$. We can think of the eigenfunction and eigenvalues as solutions to the eigenvalue problem

$$(-\Delta)^{\alpha/2} \varphi_n^\alpha(x) = \lambda_n^\alpha \varphi_n^\alpha(x), \quad x \in D$$

and $\varphi_n^\alpha(x) = 0$ for $x \in D^c$; the Dirichlet problem for stable processes. We refer the reader to [4], [7], [9], [14] and [16] where many of the general properties of the $\alpha$–stable semigroup and its generator are established.

The following question is motivated from the result of Brascamp and Lieb [13] mentioned above for Brownian motion and by its many applications.

**Question 1.1.** Let $D \subset \mathbb{R}^d$, $d \geq 1$, be a bounded convex domain and $0 < \alpha < 2$. Is $\varphi_1^\alpha$ log–concave? In other words, is $\log(\varphi_1^\alpha)$ concave on any segment contained in $D$?

The only known case is when $D = (-1, 1)$ and $\alpha = 1$, where the question is answered in the affirmative in [4]. In fact, it is shown in [4] that the ground state eigenfunction for the Cauchy process in $(-1, 1)$ is concave. Because of this case we believe this result should hold for all $\alpha$–stable processes. More precisely, we have

**Conjecture 1.1.** Let $\varphi_1^\alpha$ be the ground state eigenfunction for the symmetric stable processes of index $0 < \alpha < 2$ killed upon leaving the interval $I = (-1, 1)$. Then $\varphi_1^\alpha$ is concave on $I$.

There are by now many proofs of the log–concavity result for Brownian motion. None of them, as far as we can see, adapt to the case of general symmetric stable processes. However, Brascamp–Lieb’s proof
does suggest some related questions which may provide some insight. We briefly recall here their argument based on multiple integrals. Let $B_t$ be Brownian motion and let $\tau_D$ be its first exit time from $D$. Then one can show, see [1], that
\[ \varphi_{21}(x) = \lim_{t \to \infty} e^{\lambda_{21} t} P_x\{\tau_D > t\}, \]
uniformly in $x \in D$. From this it is enough to prove that $P_x\{\tau_D > t\}$ is log–
concave in $x$ for every fixed $t > 0$. The latter can be written as the limit as $n$ and $k$ tend to infinity of
\[ P_x\{B_{jt/n} \in D_k; j = 1, 2, \ldots, n\} \]
where $D_k$ is a sequence of convex domains strictly increasing ($D_k \subset D_{k+1}$) up to $D$. We then reduce the problem to prove that for any convex domain $D$, $P_x\{B_{jt/n} \in D; j = 1, 2, \ldots, n\}$ is log–
concave on $D$ as a function of $x$, for all $t > 0$ and all $n$. This, however, is a multiple convolution of
Gaussians with the indicator function of the set $D$. Since the Gaussian $p_t^2(x)$ is log–
concave for all $t > 0$ and the indicator function of a convex
domain is log–concave, the result follows from the fact that convolu-
tions of log–concave functions are log–concave. Using right continuity
of paths, we can try to repeat this argument for $\alpha$–stables processes.
However, this time the argument breaks down right at the end. For ex-
ample, if $\alpha = 1$ the density for the Cauchy process, $p_t^1(x, y) = p_t^1(x - y)$,
is not log–concave for all $t$. The obvious variation of this argument using the fact that $X_t^\alpha = B_{2\sigma_t}$, where $\sigma_t$ is a stable subordinator of index $\alpha/2$ independent of $B_t$, also fails basically due to the fact that the sum
of log–concave functions is not necessarily log–concave.

There is however, a substitute for log–concavity which gives some
insight into the shape of the ground state eigenfunction. We call this property “mid–concavity”.

**Definition 1.1.** Let $D \subset \mathbb{R}^d$ be a convex domain which is symmetric relative to each coordinate axes. Let $J$ be a line segment in $D$ parallel to the $x_1$–axis which intersects the boundary $\partial D$ only at the two points $(-a_1, a_2, \ldots, a_d)$, $(a_1, a_2, \ldots, a_d)$, $a_1 > 0$. We will say that the function $F : D \to \mathbb{R}$, is mid–concave on $J$ if it is concave on the segment (half of $J$) from the point $(-a_1/2, a_2, \ldots, a_d)$ to $(a_1/2, a_2, \ldots, a_d)$. The function is mid–concave along the $x_1$–axis if it is mid–concave on every such segment contained in $D$ which is parallel to the $x_1$–axis. A similar definition applies for mid–concavity along the $x_2$–axis, · · ·, $x_d$–
axis. The function is mid–concave on $D$ if it is mid–concave along each
coordinate axes.

Our main result in this paper is the following

**Theorem 1.1.** Let $Q = (-a_1, a_1) \times (-a_2, a_2) \times \cdots \times (-a_d, a_d)$, $0 < a_i < \infty$ for all $i = 1, 2, \ldots, d$, be a rectangle in $\mathbb{R}^d$. The ground state
eigenfunction $\varphi_{11}^\alpha$ for the symmetric stable process of index $0 < \alpha < 2$
is mid–concave on $Q$. In addition, if $x = (x_1, \ldots, x_n) \in Q$, then

$$\frac{\partial}{\partial x_i} \varphi_1^\alpha(x) \geq 0, \text{ if } x_i < 0, \text{ and } \frac{\partial}{\partial x_i} \varphi_1^\alpha(x) \leq 0, \text{ if } x_i > 0.$$  

(1.4)

Using arguments of multiple integrals as described above, we will show that Theorem 1.1 follows from

Theorem 1.2. Let $Q$ be a rectangle in $\mathbb{R}^d$. Let $0 < t_1 < t_2 < \cdots < t_n < \infty$. The function

$$F(x) = P_x \{ X_1^\alpha \in Q, \ldots, X_n^\alpha \in Q \}$$

is mid–concave in $Q$ for any $0 < \alpha \leq 2$. In addition, if $x = (x_1, \ldots, x_n) \in Q$, then

$$\frac{\partial}{\partial x_i} F(x) \geq 0, \text{ if } x_i < 0, \text{ and } \frac{\partial}{\partial x_i} F(x) \leq 0, \text{ if } x_i > 0.$$  

(1.6)

Remark 1.1. It is important to note here that Theorem 1.2 is new even in the Brownian motion case ($\alpha = 2$). Indeed, as we shall see, the case $\alpha = 2$ implies the general case by subordination.

If we consider the eigenfunction for the Laplacian in the unit disk $D$ in the plane, one can show, by analysis of the Bessel function, that such a function is not concave in $D$ but it is mid–concave. Also, it may be tempting to conjecture that for any symmetric domain in the plane the eigenfunction is mid–concave. This, however, is not the case, even for the Brownian motion, as we will show at the end of the paper.

The paper is organized as follows. In §2, we prove that the multiple convolutions of Gaussians in the interval $(-1, 1)$ is mid–concave. In §3, we show how this and subordination implies Theorem 1.2. Here we also show that full concavity fails for general multiple integrals and that mid–concavity fails in general symmetric domains in the plane.

2. Mid–concavity for Brownian motion

Let

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

be the Gaussian density in one dimension. With the notation of the introduction, we have $p_t^2(x, y) = p_{2t}(x - y)$.

Proposition 2.1. Let $n = 1, 2, \ldots$ and let $t_1, t_2, \ldots, t_n$ be real numbers in $(0, \infty)$. For $x \in (-1, 1)$ define

$$\Phi_n(x) = \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{i=1}^{n} p_{t_i}(x_i - x) \, dx_1 \ldots dx_n,$$  

(2.1)
where \( x_0 = x \). The function \( \Phi_n(x) \) is mid-concave on \((-1, 1)\). That is, \( \Phi_n(x) \) is concave on \((-\frac{1}{2}, \frac{1}{2})\).

Clearly \( \Phi_n(x) \) is a positive even function on \([-1, 1]\). Integrating by parts we obtain

\[
\frac{\partial}{\partial x} \Phi_n(x) = \frac{1}{\sqrt{2\pi t_n}} \int_{-1}^{1} \left( \frac{\partial}{\partial y} e^{-\frac{(y-x)^2}{2t_n}} \right) \Phi_{n-1}(y) \, dy
\]

\begin{equation}
= \frac{\Phi_{n-1}(1)}{\sqrt{2\pi t_n}} \left( e^{-\frac{(1-x)^2}{2t_n}} - e^{-\frac{(1+x)^2}{2t_n}} \right)
- \frac{1}{\sqrt{2\pi t_n}} \int_{-1}^{1} e^{-\frac{(y-x)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy.
\end{equation}

Notice that for all \( t > 0 \),

\begin{equation}
\left( e^{-\frac{(1-x)^2}{2t}} - e^{-\frac{(1+x)^2}{2t}} \right) = e^{-\frac{(1-x)^2}{2t}} \left( 1 - e^{-\frac{2x}{t}} \right),
\end{equation}

is a positive increasing function on \([0, 1]\).

**Lemma 2.1.** The function \( \Phi_n(x) \) is decreasing on \((0, 1)\) for all \( n \geq 1 \).

**Proof.** We argue by induction. If \( n = 1 \), then

\begin{equation}
\frac{\partial}{\partial x} \Phi_1(x) = \frac{1}{\sqrt{2\pi t_1}} \int_{-1}^{1} \frac{\partial}{\partial y} e^{-\frac{(y-x)^2}{2t_1}} \, dy
\end{equation}

\[
= \frac{1}{\sqrt{2\pi t_1}} \left( -e^{-\frac{(1-x)^2}{2t_1}} + e^{-\frac{(1+x)^2}{2t_1}} \right)
< 0,
\]

for all \( x \in (0, 1) \). Thus \( \Phi_1(x) \) is decreasing on \((0, 1)\).

Let us assume that \( \Phi_{n-1}(x) \) is decreasing on \((0, 1)\). That is, suppose that

\[
\frac{\partial}{\partial x} \Phi_{n-1}(x) \leq 0,
\]

for all \( x \in (0, 1) \). Because of (2.3), it is enough to prove that

\begin{equation}
\frac{-1}{\sqrt{2\pi t_n}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy \geq 0.
\end{equation}

By symmetry

\[
\frac{\partial}{\partial y} \Phi_{n-1}(y) = -\frac{\partial}{\partial y} \Phi_{n-1}(-y).
\]

On the other hand, if \( x > 0 \) then

\[
e^{-\frac{(x-y)^2}{2t}} \geq e^{-\frac{(x+y)^2}{2t}},
\]
for all \( t, y > 0 \). Hence for all \( y > 0 \),

\[
- e^{-\frac{(x-y)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) - e^{-\frac{(x+y)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(-y) =
\]

\[
- e^{-\frac{(x-y)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) + e^{-\frac{(x+y)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) =
\]

\[
\left( -e^{-\frac{(x-y)^2}{2tn}} + e^{-\frac{(x+y)^2}{2tn}} \right) \frac{\partial}{\partial y} \Phi_{n-1}(y) \geq 0.
\]

Integrating this inequality on \([0,1]\) we obtain (2.5). \( \square \)

Notice that

\[
\frac{\partial^2}{\partial x^2} \Phi_1(x) = \frac{1}{\sqrt{2\pi t_1}} \int_{-1}^{1} \frac{\partial^2}{\partial y^2} e^{-\frac{(y-x)^2}{2t_1}} dy
\]

(2.6)

\[
= \frac{1}{t_1 \sqrt{2\pi t_1}} \left( -(1-x)e^{-\frac{(1-x)^2}{2t_1}} - (1+x)e^{-\frac{(1+x)^2}{2t_1}} \right)
\]

\[
< 0,
\]

for all \( x \in (-1,1) \). Thus \( \Phi_1(x) \) is concave in \((-1,1)\). We will now prove that \( \Phi_n(x) \) is concave in \((-\frac{1}{2}, \frac{1}{2})\).

**Lemma 2.2.** If \( 0 \leq x \leq \frac{1}{2} \), then for all \( n \geq 1 \),

\[
\frac{\partial}{\partial x} \Phi_n(x) \geq \frac{\partial}{\partial x} \Phi_n(1-x).
\]

**Proof.** By (2.6) the result is true for \( n = 1 \). Let us assume that the result is true for \( n-1 \). Let

\[
\psi_n(x) = \frac{-1}{\sqrt{2\pi t_n}} \int_{-1}^{1} e^{-\frac{(y-x)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy.
\]

Because of (2.3), it is enough to prove that

(2.7) \( \psi_n(1-x) \geq \psi_n(x) \).

Let \( y \in (-1,0) \), then

\[
1 - x - y \geq x - y \geq 0.
\]

Thus

\[
e^{-\frac{(1-x-y)^2}{2tn}} \leq e^{-\frac{(x-y)^2}{2tn}}.
\]

Since

\[
- \frac{\partial}{\partial y} \Phi_{n-1}(y) \leq 0,
\]
for all $y < 0$, it follows that

$$- \int_{-1}^{0} e^{-\frac{(x-y)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy \leq - \int_{-1}^{0} e^{-\frac{(1-x-y)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy.$$ 

To simplify notation let

$$\phi(y) = - \frac{\partial}{\partial y} \Phi_{n-1}(y).$$

Let $y \in (0, \frac{1}{2})$, and consider $\hat{y} = 1 - y$. Notice that $\hat{y} \in (\frac{1}{2}, 1)$ and

$$\frac{1}{2} - y = \hat{y} - \frac{1}{2}.$$

By induction,

$$0 \leq \phi(y) \leq \phi(\hat{y}).$$

On the other hand,

$$e^{-\frac{(x-y)^2}{2tn}} = e^{-\frac{(x-\hat{y})^2}{2tn}},$$

$$e^{-\frac{(x-\hat{y})^2}{2tn}} = e^{-\frac{(x-y)^2}{2tn}},$$

$$e^{-\frac{(x-y)^2}{2tn}} \geq e^{-\frac{(x-\hat{y})^2}{2tn}}.$$

Thus

$$\left( e^{-\frac{(x-y)^2}{2tn}} - e^{-\frac{(x-\hat{y})^2}{2tn}} \right) \phi(y) \leq \left( e^{-\frac{(x-\hat{y})^2}{2tn}} - e^{-\frac{(x-y)^2}{2tn}} \right) \phi(\hat{y}),$$

and we conclude that

$$e^{-\frac{(x-y)^2}{2tn}} \phi(y) + e^{-\frac{(x-\hat{y})^2}{2tn}} \phi(\hat{y}) \leq e^{-\frac{(x-y)^2}{2tn}} \phi(y) + e^{-\frac{(x-\hat{y})^2}{2tn}} \phi(\hat{y}).$$

Integrating over $(0, \frac{1}{2})$ we obtained that

$$- \int_{0}^{1} e^{-\frac{(x-y)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy \leq - \int_{0}^{1} e^{-\frac{(1-x-y)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy,$$

the desired result immediately follows. \qed

**Lemma 2.3.** If $0 \leq x < u \leq \frac{1}{2}$, then for all $n \geq 1$,

$$\frac{\partial}{\partial x} \Phi_n(x) \geq \frac{\partial}{\partial x} \Phi_n(u).$$

**Proof.** By (2.6) the result is true for $n = 1$. Let us assume that the result is true for $n - 1$. As in Lemma 2.1, we let

$$\psi_n(x) = -\frac{1}{\sqrt{2\pi tn}} \int_{-1}^{1} e^{-\frac{(y-x)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy$$
and
\[ \phi(y) = -\frac{\partial}{\partial y} \Phi_{n-1}(y). \]

By (2.3), it is enough to prove that
\[ \psi_n(u) \geq \psi_n(x). \]

Let \( y \in (-1, 0) \), then \( |u - y| \geq |x - y| \). Thus
\[ e^{-\frac{(u-y)^2}{2tn}} \leq e^{-\frac{(x-y)^2}{2tn}}. \]

Lemma 2.1 implies that
\[ -\int_{-1}^{0} e^{-\frac{(x-y)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy \leq -\int_{-1}^{0} e^{-\frac{(u-y)^2}{2tn}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy. \]

Let \( m = \frac{x+u}{2} \). For all \( y \in (0, m) \), define \( \tilde{y} = x + u - y \). Notice that \( \tilde{y} \in (m, x+u) \) and that
\[ |x - y| = |u - \tilde{y}|. \]

We can easily check that
\[ e^{-\frac{(x-y)^2}{2tn}} = e^{-\frac{(u-y)^2}{2tn}}, \]
\[ e^{-\frac{(x-y)^2}{2tn}} = e^{-\frac{(u-y)^2}{2tn}}, \]
\[ e^{-\frac{(x-y)^2}{2tn}} \geq e^{-\frac{(x-y)^2}{2tn}}, \]

for all \( y \in (0, m) \). We claim that
\[ \phi(y) \leq \phi(\tilde{y}). \]

This follows immediately from the induction hypothesis if \( \tilde{y} \leq \frac{1}{2} \). On the other hand, if \( \tilde{y} = (x+u) - y \geq \frac{1}{2} \), then
\[ 1 - (x + u) + y \leq \frac{1}{2}, \text{ and } y \leq 1 - (x + u) + y. \]

Lemma 2.2 and the induction hypothesis imply that
\[ 0 \leq \phi(y) \leq \phi(1 - (x + u) + y) \leq \phi((x + u) - y) = \phi(\tilde{y}). \]

Thus
\[ e^{-\frac{(x-y)^2}{2tn}} \phi(y) + e^{-\frac{(x-y)^2}{2tn}} \phi(\tilde{y}) \leq e^{-\frac{(u-y)^2}{2tn}} \phi(y) + e^{-\frac{(u-y)^2}{2tn}} \phi(\tilde{y}). \]

Integrating over \((0, m)\) we obtained that
\[
- \int_0^{x+u} e^{-\frac{(y-x)^2}{2\alpha}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy \leq - \int_0^{x+u} e^{-\frac{(y+u)^2}{2\alpha}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy.
\]

Finally if \( y \in [x + u, 1] \) then
\[
e^{-\frac{(x-y)^2}{2\alpha}} \leq e^{-\frac{(u-y)^2}{2\alpha}}.
\]

Therefore
\[
- \int_{x+u}^1 e^{-\frac{(y-x)^2}{2\alpha}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy \leq - \int_{x+u}^1 e^{-\frac{(y+u)^2}{2\alpha}} \frac{\partial}{\partial y} \Phi_{n-1}(y) \, dy.
\]

By symmetry, Proposition 2.1 follows from Lemma 2.3. The following is an immediate corollary of Proposition 2.1.

**Corollary 2.1.** Let \( B_t \) be one dimensional Brownian motion and set \( I = (-a, a) \). For \( 0 < t_1 < t_2 < \cdots < t_n \), the function
\[
F(x) = P_x \{ B_{t_1} \in I, B_{t_2} \in I, \ldots, B_{t_n} \in I \}
\]
is mid-concave in \( I \). In addition, if \( x \in I \), then
\[
F'(x) \geq 0, \text{ if } x < 0, \text{ and } F'(x) \leq 0, \text{ if } x > 0.
\]

**Proof.** By the Markov property,
\[
F(x) = \int_{-a}^a \cdots \int_{-a}^a \prod_{i=1}^n \rho_{t_i-t_{i-1}}(x_{i-1} - x_i) \, dx_1 \cdots dx_n,
\]
where \( x_0 = x \) and \( t_0 = 0 \). This is exactly the same expression as in Lemma 2.1 and Proposition 2.1 except for the fact that the interval \((-1, 1)\) has been replaced by the interval \((-a, a)\). The proof of the proposition is the same for this case and the corollary follows.

**Corollary 2.2.** Let \( B_t \) be Brownian motion in \( \mathbb{R}^d \) and let \( Q = I_1 \times I_2 \times \cdots \times I_d \) where \( I_i = (-a_i, a_i) \), be a rectangle in \( \mathbb{R}^d \). For \( 0 < t_1 < t_2 < \cdots < t_n \), the function
\[
F(x) = P_x \{ B_{t_1} \in Q, B_{t_2} \in Q, \ldots, B_{t_n} \in Q \}
\]
is mid-concave in \( Q \). In addition, if \( x = (x_1, x_2, \ldots, x_d) \in Q \), then
\[
\frac{\partial}{\partial x_i} F(x) \geq 0, \text{ if } x_i < 0, \text{ and } \frac{\partial}{\partial x_i} F(x) \leq 0, \text{ if } x > 0.
\]
Proof. With \( x = (x_1, x_2, \ldots, x_d) \) and \( B_t = (B^1_t, B^2_t, \ldots, B^d_t) \), it follows by independence that
\[
F(x) = \prod_{i=1}^{d} P_{x_i} \{ B_{t_i}^i \in I_i, B_{t_{i+1}}^i \in I_i, \ldots, B_{t_{n+1}}^i \in I_i \}
\]
and the conclusion of the corollary follows from Corollary 2.1 and our definition of mid–concavity for domains in \( \mathbb{R}^d \).

3. Mid–concavity for stable processes

In this section we prove Theorems 1.1 and 1.2. First, let us recall that for \( 0 < \alpha < 2 \) the symmetric stable process \( X^\alpha_t \) in \( \mathbb{R}^d \) has the representation
\[
X^\alpha_t = B_{2\sigma_t},
\]
where \( \sigma_t \) is a stable subordinator of index \( \alpha/2 \) independent of \( B_t \) (see [6]). Thus
\[
p^\alpha_t(x - y) = \int_0^\infty p^2_s(x - y) g_{\alpha/2}(t, s) ds,
\]
where \( g_{\alpha/2}(t, s) \) is the transition density of \( \sigma_t \) and
\[
p^2_{\alpha/2}(x - y) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x-y|^2}{2t}}.
\]
Now, let \( Q \) and \( t_1, t_2, \ldots, t_n \) be as in the statement of Theorem 1.2. Set \( x_0 = x \) and \( t_0 = 0 \). Using the Markov property of the stable process \( X^\alpha_t \), the subordination formula (3.2), Fubini’s theorem, and the Markov property of the Brownian motion, in this order, we obtain,
\[
F(x) = P_x \{ X^\alpha_{t_1} \in Q, \ldots, X^\alpha_{t_n} \in Q \}
\]
\[
= \int_Q \cdots \int_Q \prod_{i=1}^{n} p^\alpha_{t_i - t_{i-1}}(x_{i-1} - x_i) \, dx_1 \ldots dx_n
\]
\[
= \int_0^\infty \cdots \int_0^\infty \left( \int_Q \cdots \int_Q \prod_{i=1}^{n} p^2_s(x_{i-1} - x_i) \, dx_1 \ldots dx_n \right)
\]
\[
\times \prod_{i=1}^{n} g_{\alpha/2}(t_i - t_{i-1}, s_i) \, ds_1 \ldots ds_n
\]
\[
= \int_0^\infty \cdots \int_0^\infty P_x \{ B_{2s_1} \in Q, B_{2(s_1+s_2)} \in Q, \ldots, B_{2(s_1+s_2+\cdots+s_n)} \in Q \}
\]
\[
\times \prod_{i=1}^{n} g_{\alpha/2}(t_i - t_{i-1}, s_i) \, ds_1 \ldots ds_n.
\]
Since the function
\[ P_x \{ B_{2s_1} \in Q, B_{2(s_1+s_2)} \in Q, \ldots, B_{2(s_1+s_2+\cdots+s_n)} \in Q \} \]
is mid–concave and satisfies the monotonicity property (2.14), by Corollary 2.2, so is the integral against the densities \( g_{\alpha/2}(t_i - t_{i-1}, s_i) \) and this completes the proof of Theorem 1.2.

With Theorem 1.2 proved, we argue as in the proof of the log–concavity for Brownian motion discussed in the introduction. Recall that \( \varphi^\alpha_1 \) is the ground state eigenfunction for the stable process of index \( \alpha \), \( \alpha \in (0, 2) \), killed upon leaving \( Q \) and \( \lambda^\alpha_1 \) is its eigenvalue. Let \( \tau^\alpha_Q \) be the first exit time of the symmetric stable process from \( Q \). Since \( Q \) is certainly intrinsically ultracontractive, see [17], we have that
\[
\varphi^\alpha_1(x) = \lim_{t \to \infty} e^{\lambda^\alpha_1 t} P_x \{ \tau^\alpha_Q > t \}.
\]
(3.3)

The convergence is uniform for \( x \in Q \). Thus to prove mid–concavity for \( \varphi^\alpha_1(x) \) it is enough to prove mid–concavity for \( P_x \{ \tau^\alpha_Q > t \} \). By the right continuity of the sample paths, we have,

\[
P_x \{ \tau^\alpha_Q > t \} = P_x \{ X^\alpha_s \in Q, 0 \leq s \leq t \}
= \lim_{n \to \infty} P_x \{ X^\alpha_{nt/n} \in Q, i = 1, \ldots, n \}.
\]
(3.4)

Theorem 1.1 now follows from this and Theorem 1.2.

We remark that in the case of Brownian motion, there is an extra approximation by an increasing sequence of domains in passing from the first equality to the second in (3.4). This is not needed for our stable processes since, as explain in [8], Lemma 6, for any domain \( D \subset \mathbb{R}^d \) with Lipschitz boundary,

\[ P_x \{ X^\alpha_{t_D} \in \partial D \} = 0 \text{ for } x \in D. \]

The above argument applies not only to symmetric stable processes but also to any other process which is obtained by subordination of Brownian motion. In particular, the above results hold for the so called “relativistic” Brownian motion and “relativistic” \( \alpha \)-stable processes studied in [19].

It is of course natural to ask if the function of Proposition 2.1 is concave in the whole interval \( (-1, 1) \) for all \( n \) and all \( t_i \). Notice that, thanks to the proof of Lemma 2.2, this is the case for \( n = 1 \). If this were the case, it would show that the same is true for the function \( P_x \{ \tau^\alpha_Q > t \} \) and hence for the function \( \varphi^\alpha_1 \), as desired. Unfortunately, this is not the case.
Proposition 3.1. Let

(3.5) \[ \Phi_n(x) = \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{i=1}^{n} p_t(x_{i-1} - x_i) \, dx_1 \cdots dx_n, \]

where \( x_0 = x \). Then there exist a positive integer \( n \) and real numbers \( t_1, t_2, \ldots, t_n \) in \( (0, \infty) \) such that the function \( \Phi_n(x) \) is not concave on \( (-1, 1) \).

Proof. We may replace, to simplify certain notation below, the interval \((-1, 1)\) by the interval \((0, \pi)\). Fix \( t \) and \( s \) both positive. Let \( t = t_1 \) and \( t_2 = \cdots = t_n = \frac{s}{n-1} \). If the function \( \Phi_n(x) \) is concave on \((0, \pi)\) for all \( n \) with these chosen \( t_1, t_2, \ldots, t_n \), letting \( n \to \infty \) we see that the function

(3.6) \[ \int_0^\pi p_t(x - y) P_y \{ \tau_{(0,\pi)} > s \} \, dy \]

is also concave on \((0, \pi)\). Here we have used \( \tau_{(0,\pi)} \) to denote the first exit time of Brownian motion from the interval. We have

(3.7) \[ \lim_{s \to \infty} e^{\lambda_1 s} P_y \{ \tau_{(0,\pi)} > s \} = c \sin(y), \]

uniformly for \( y \in (0, \pi) \), where \( c > 0 \) and \( \lambda_1 = 1 \) (the first eigenvalue for \((0, \pi)\)). It follows that for each \( t > 0 \), the function

(3.8) \[ F_t(x) = \int_0^\pi p_t(x - y) \sin(y) \, dy \]

must also be concave on \((0, \pi)\).

We will now show that the function \( F_t(x) \) is not concave. Without any difficulty we may differentiate under the integral to obtain that

(3.9) \[ F''_t(x) = \frac{1}{\sqrt{2\pi t}^{5/2}} \int_0^\pi \left[ (x-y)^2 - t \right] e^{-\frac{(x-y)^2}{2t}} \sin(y) \, dy. \]

Taking \( x = 0 \) and using the elementary inequality

\[ y - \frac{y^3}{3!} \leq \sin(y) \leq y \]

uniformly for \( y \in (0, \pi) \), we find that \( F''_t(x) \) is negative for every \( x \neq 0 \). Hence \( F_t(x) \) is not concave on \((-1, 1)\).
valid for all $y > 0$, we see that $F''(0)$ is equal to

$$\frac{1}{\sqrt{2\pi t^{5/2}}} \int_0^\pi \left( y^2 - t \right) e^{-\frac{y^2}{2t}} \sin(y) \, dy$$

$$= \frac{1}{\sqrt{2\pi t^{5/2}}} \left( \int_0^\pi y^2 e^{-\frac{y^2}{2t}} \sin(y) \, dy - t \int_0^\pi e^{-\frac{y^2}{2t}} \sin(y) \, dy \right)$$

$$\geq \frac{1}{\sqrt{2\pi t^{5/2}}} \left( \int_0^\pi y^3 e^{-\frac{y^2}{2t}} \, dy - \frac{1}{3!} \int_0^\pi y^5 e^{-\frac{y^2}{2t}} \, dy - t \int_0^\pi ye^{-\frac{y^2}{2t}} \, dy \right)$$

$$= \frac{1}{\sqrt{2\pi t^{5/2}}} \left( \int_0^\pi y^3 e^{-\frac{y^2}{2t}} \, dy - \frac{t}{3!} \int_0^\pi y^5 e^{-\frac{y^2}{2t}} \, dy - t \int_0^\pi ye^{-\frac{y^2}{2t}} \, dy \right).$$

Since

$$\frac{t}{3!} \int_0^\pi y^5 e^{-\frac{y^2}{2t}} \, dy \to 0,$$

$$\int_0^\pi y^3 e^{-\frac{y^2}{2t}} \, dy \to 2,$$

and

$$\int_0^\pi ye^{-\frac{y^2}{2t}} \, dy \to 1,$$

as $t \to 0^+$, we see that $F''(0)$ is positive for sufficiently small $t$. By continuity, we have that $F''(x) > 0$ for sufficiently small $x \in (0, \pi)$ and sufficiently small $t$. This, of course, contradicts the concavity of the function and shows that $\Phi_n(x)$ is not concave.

\[\square\]

Of course, it may still be the case that the function $\Phi_n(x)$ is concave on the whole interval when we restrict to a sequence of times satisfying $t_1 = t_2 = \cdots = t_n$ and substitute $p_{t_i}(x)$ by $p_{t_i}^\alpha(x)$, which is what is needed for our applications (Conjecture 1.1). That is, the following conjecture may still be true.

**Conjecture 3.1.** Let $I = (-1,1)$ and let $n$ be a positive integer. If $t_i = \frac{t}{n}$ for $1 \leq i \leq n$, then the function

\[F(x) = P_x \{ X_{t_1}^\alpha \in I, \ldots, X_{t_n}^\alpha \in I \}\]

is concave on $I$.

A natural question is whether $\varphi^\alpha_1$ is mid-concave for any symmetric, convex domain in the plane. We will now show that for a large enough
rhombus and \( \alpha = 2 \) (Brownian motion), this is not the case. Below we use \( \lambda_1(D) \) and \( \varphi_D \) to denote the first eigenvalue for the domain \( D \) and its corresponding eigenfunction, respectively, for the Brownian motion. We also denote the first exit time of the Brownian motion from a domain \( D \) by \( \tau_D \).

Proposition 3.2. For \( n \geq 1 \), set
\[
D(n) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \in (-n, n), \ x_2 \in \left(-1 + \frac{|x_1|}{n}, 1 - \frac{|x_1|}{n}\right) \right\}.
\]
There exists an \( n \) large enough such that \( \varphi_{D(n)} \) is not mid-concave on \( D(n) \).

Proof. The rectangle
\[
R(n) = (-\sqrt{n}, \sqrt{n}) \times \left(-1 + \frac{1}{\sqrt{n}}, 1 - \frac{1}{\sqrt{n}} \right)
\]
is a subset of \( D(n) \) and therefore,
\[
\lambda_1(D(n)) < \lambda_1(R(n)) = \frac{\pi^2}{(2 - 2/\sqrt{n})^2} + \frac{\pi^2}{(2\sqrt{n})^2} \leq \frac{\pi^2}{4} \left(1 + \frac{3}{\sqrt{n}}\right),
\]
for \( n \) large enough. Now, for any \( a \in (0, 1/2) \), consider the subset of \( D(n) \) define by
\[
Q(a, n) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \in (an, n), \ x_2 \in \left(-1 + \frac{|x_1|}{n}, 1 - \frac{|x_1|}{n}\right) \right\}.
\]
Since
\[
\Delta \varphi_{D(n)} + \lambda_1(D(n)) \varphi_{D(n)} = 0
\]
in \( Q(a, n) \) and \( \lambda_1(D(n)) < \lambda_1(Q(a, n)) \). That is, \( \varphi_{D(n)} \) is a \( q \)-harmonic function with \( q = \lambda_1(D(n)) \). The Feynman–Kac formula gives that for any \( x \in Q(a, n) \),
\[
\begin{align*}
\varphi_{D(n)}(x) &= E_x \left[ e^{\lambda_1(D(n)) \tau_{Q(a, n)}} \varphi_{D(n)}(B(\tau_{Q(a, n)})) \right] \\
(3.11) &\leq \varphi_{D(n)}(0) E_x \left[ e^{\lambda_1(D(n)) \tau_{Q(a, n)}} ; B(\tau_{Q(a, n)}) \in D(n) \setminus Q(a, n) \right].
\end{align*}
\]
Of course,
\[
\varphi_{D(n)}(0) = \max \{ \varphi_{D(n)}(x) : x \in D \},
\]
by symmetry. Let \( p(a) = (1 - a/2)^2/(1 - a)^2 \) and \( q(a) \) be such that \( 1/p(a) + 1/q(a) = 1 \). Note that \( p(a) > 1 \) so \( q(a) > 0 \). By Hölder’s inequality the expression in (3.11) is bounded above by
\[
\varphi_{D(n)}(0) \left( E_x \left[ e^{\lambda_1(D(n)) \tau_{Q(a, n)} p(a)} \right] \right)^{1/p(a)} \times \left( P_x \left[ B(\tau_{Q(a, n)}) \in D(n) \setminus Q(a, n) \right] \right)^{1/q(a)}.
\]
Since

\[ Q(a, n) \subset (-\infty, \infty) \times (-1 + a, 1 - a) \]

we have that for any \( x \in Q(a, n) \),

\[
    E_x \left[ e^{\lambda_i(D(n)) \tau_{Q(a,n)}(p(a)} \right] \\
    \leq E_0 \left[ \exp \left( (\pi^2/4) (1 + 3/\sqrt{n} \right) p(a) \tau_{(-1+a,1-a)} \) \left. \right) \right] \\
    = E_0 \left[ \exp \left( (\pi^2/4) (1 + 3/\sqrt{n} \right) p(a) \right. \left. (1 - a)^2 \tau_{(-1,1)} \right) \right] \\
    = E_0 \left[ \exp \left( (\pi^2/4) (1 + 3/\sqrt{n} \right) (1 - a/2)^2 \tau_{(-1,1)} \right) \right].
\]

By a simple calculation we see that \((1 + 3/\sqrt{n})(1 - a/2) \leq 1\) when \( n \geq (6 - 3a)^2/a^2 \). For such \( n \), we have

\[
    \left( E_x \left[ e^{\lambda_i(D(n)) \tau_{Q(a,n)}(p(a)} \right] \right)^{1/p(a)} \\
    \leq \left( E_0 \left[ \exp \left( (\pi^2/4) (1 - a/2) \tau_{(-1,1)} \right) \right] \right)^{1/p(a)} = C_1(a).
\]

Using the fact that \( \pi^2 \) is the eigenvalue for the interval \((-1,1)\), we have that for any \( c \in (0, 1) \), \( E_0[\exp(c \tau_{(-1,1)} \pi^2/4)] < \infty \). Thus for any \( a \in (0, 1/2) \) we have \( C_1(a) < \infty \).

By standard results for Brownian motion (or the trivial estimate of the harmonic measure in the strip obtained by conformal mapping to the disk), for any \( b \geq 0 \) and \( x_1 > b \) we have

\[ P_{(x_1,0)} \left[ B(\tau_{b,\infty} \times (-1,1)) \right] \subset (-\infty, b) \times (-1, 1) \] 

\[ C_2 e^{-\frac{n}{2}(x_1-b)}, \]

where \( C_2 > 0 \) is an absolute constant.

Note that \( x = (2an, 0) \in Q(a, n) \). It follows that

\[ P_{(2an,0)} \left[ B(\tau_{Q(a,n)}) \right] \subset D(n) \setminus Q(a, n) \leq C_2 e^{-\frac{n}{2}an}. \]

Now choose \( a = 1/8 \). For such \( a \) we have \((2an, 0) = (n/4, 0)\). For \( n \geq (6 - 3a)^2/a^2 \) we have

\[ \varphi_{D(n)}(n/4, 0) \leq \varphi_{D(n)}(0, 0) C_1(1/8) \left[ C_2 e^{-\frac{n}{2}a n^{1/3}} \right]^{1/2^{(1/3)} \varphi(n)}. \]

(3.12) \[ \varphi_{D(n)}(n/4, 0) \leq \varphi_{D(n)}(0, 0) C_1(1/8) \left[ C_2 e^{-\frac{n}{2}a n^{1/3}} \right]^{1/2^{(1/3)} \varphi(n)}. \]

If \( \varphi_{D(n)} \) were mid-concave, we would have

\[ \varphi_{D(n)}(n/4, 0) \geq \frac{1}{2} \left[ \varphi_{D(n)}(0, 0) + \varphi_{D(n)}(n/2, 0) \right] \geq \frac{1}{2} \varphi_{D(n)}(0, 0). \]

However, by (3.12) for large enough \( n \) we have that \( \varphi_{D(n)}(n/4, 0) \) is smaller than \( \varphi_{D(n)}(0, 0)/2 \). Thus \( \varphi_{D(n)} \) is not mid-concave. Indeed, the same argument shows that for any \( c \in (0, 1) \) there exists an \( n \) large enough such that \( \varphi_{D(n)} \) is not concave on the interval with endpoints \((-cn, 0), (cn, 0)\). \( \square \)
References

[1] R. Bañuelos, Intrinsic ultracontractivity and eigenfunction estimates for Schrödinger operators, J. Funct. Anal. 100 (1991), 181-206.

[2] R. Bañuelos, R. Latala, P. J. Méndez-Hernández, A Brascamp-Lieb-Luttinger-type inequality and applications to symmetric stable processes, Proc. Amer. Math. Soc. 129(10) (2001), 2997–3008.

[3] R. Bañuelos, P. J. Méndez-Hernández, Sharp inequalities for heat kernels of Schrödinger operators and applications to spectral gaps, J. Funct. Anal. 176(2) (2000), 368–399.

[4] R. Bañuelos and T. Kulczycki, The Cauchy process and the Steklov problem, J. Funct. Anal. 211 (2004), 355–423.

[5] R.M. Blumenthal and R.K. Getoor, The asymptotic distribution of the eigenvalues for a class of Markov operators Pacific J. Math. 9 (1959), 399–408.

[6] R.M. Blumenthal and R.K. Getoor, Some Theorems on Symmetric Stable Processes, Trans. Amer. Soc. 95 (1960), 263-273.

[7] R.M. Blumenthal, R.K. Getoor and D.B. Ray, On the distribution of first hits for the symmetric stable process, Trans. Amer. Math. Soc. 99 (1961), 540–554.

[8] K. Bogdan The boundary Harnack principle for the fractional Laplacian, Studia Math. 123(1) (1997), 43–80.

[9] K. Bogdan and T. Byczkowski, Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains, Studia Math. 133(1) (1999), 53-92.

[10] C. Borell Examples of Brunn-Minkowski inequalities in diffusion theory, (preprint).

[11] C. Borell, Diffusion equations and geometric inequalities, Potential Anal. 12 (2000), 49–71.

[12] C. Borell, Geometric inequalities in option pricing, Convex geometric analysis (Berkeley, CA, 1996), 29–51.

[13] H.L. Brascamp and E.H. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, J. Functional Analysis, 22(1976), 366–389

[14] Z.Q. Chen and R. Song Intrinsic ultracontractivity and conditional gauge for symmetric stable processes, J. Funct. Anal. 150(1) (1997), 204-239.

[15] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge University Press, Cambridge, (1989).

[16] R. K. Getoor, Markov operators and their associated semi-groups, Pacific J. Math. 9 (1959) 449–472.

[17] T. Kulczycki, Intrinsic ultracontractivity for symmetric stable processes, Bull. Polish Acad. Sci. Math. 46(3) (1998), 325–334.

[18] J. Ling, A lower bound for the gap between the first two eigenvalues of Schrödinger operators on convex domains in $S^n$ or $R^n$, Michigan Math. J. 40(2) (1993), 259–270.

[19] M. Ryznar, Estimates of Green functions for relativistic α-stable processes, Potential Analysis, 17 (2002), 1–23.

[20] I. M. Singer, B. Wong, S.-T. Yau, S. S.-T. Yau, An estimate of the gap of the first two eigenvalues in the Schrödinger operator, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 12(2) (1985), 319–333.
[21] R. Smits, *Spectral gaps and rates to equilibrium for diffusions in convex domains*, Michigan Math. J., 43 (1996), 141–157.

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