Optimal antipodal spherical codes in the space of spherical harmonics

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Abstract
In a previous study, we presented a construction of spherical 3-designs. In the current study, using this construction, we present new optimal antipodal spherical codes in the space of spherical harmonics. Our construction is a generalization of Bondarenko’s work.

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1 Introduction
This study is a sequel to the previous study [5], which is inspired by [1]. [1] gives an optimal antipodal spherical $(35,240,1/7)$ code whose vectors form a spherical 3-design. Generalizing the study [1], [5] gives a construction of spherical 3-designs.

In the current study, using this construction, we present optimal antipodal spherical codes in the space of spherical harmonics. To explain our results, we review [1] and the concept of spherical codes.

First, we quote some results from [1]. Let

$$\Delta = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial x_j^2}.$$ 

We say that a polynomial $P$ in $\mathbb{R}^{d+1}$ is harmonic if $\Delta P = 0$. For integer $k \geq 1$, the restriction of a homogeneous harmonic polynomial of degree $k$ to $S^d$ is called

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a spherical harmonic of degree \( k \). We denote by \( \text{Harm}_k(S^d) \) the vector space of the spherical harmonics of degree \( k \). Note that, (see for example [7])

\[
\dim \text{Harm}_k(S^d) = \frac{2k + d - 1}{k + d - 1} \binom{d + k - 1}{k}.
\]

For \( P, Q \in \text{Harm}_k(S^d) \), we denote by \( \langle P, Q \rangle \) the usual inner product

\[
\langle P, Q \rangle := \int_{S^d} P(x)Q(x)d\sigma(x),
\]

where \( d\sigma(x) \) is a normalized Lebesgue measure on the unit sphere \( S^d \). For \( x \in S^d \), there exists \( P_x \in \text{Harm}_k(S^d) \) such that

\[
\langle P_x, Q \rangle = Q(x) \quad \text{for all } Q \in \text{Harm}_k(S^d).
\]

It is known that

\[
P_x(y) = g_{k,d}((x, y)),
\]

where \( g_{k,d} \) is a Gegenbauer polynomial. Let

\[
G_x = \frac{P_x}{g_{k,d}(1)^{1/2}}.
\]

It should be noted that

\[
\langle G_x, G_y \rangle = \frac{g_{k,d}((x, y))}{g_{k,d}(1)}.
\]

(For a detailed explanation of Gegenbauer polynomials, see [7].) Therefore, if we have a set \( X = \{x_1, \ldots, x_N\} \) in \( S^d \), then we obtain the set \( G_X = \{G_{x_1}, \ldots, G_{x_N}\} \) in \( S^{\dim \text{Harm}_k(S^d) - 1} \).

Thereafter, we recall the concept of an optimal antipodal \((d + 1, N, a)\) code.

**Definition 1.1** ([3]). An antipodal set \( X = \{x_1, \ldots, x_N\} \) in \( S^d \) (i.e. \( X = -X \)) is called an antipodal spherical \((d + 1, N, a)\) code if \(|(x_i, x_j)| \leq a\) for some \( a > 0 \) and all \( x_i, x_j \in X, i \neq j \), are not antipodal.

This code is called optimal if, for any antipodal set \( Y = \{y_1, \ldots, y_N\} \) on \( S^d \), there exists \( y_i, y_j \in Y, i \neq j \), that are not antipodal and are such that \(|(y_i, y_j)| \geq a\).

Let \( X = \{x_1, \ldots, x_{120}\} \) be an arbitrary subset of 240 normalized minimum vectors of the \( E_8 \) lattice such that no pair of antipodal vectors is present in \( X \). Set \( P_x(y) = g_{2,7}((x, y)) \). [1] showed that \( G_X \cup -G_X \) is an optimal antipodal spherical \((35, 240, 1/7)\) code whose vectors form a spherical 3-design, where

\[-G_X := \{-G_x \mid G_x \in G_X\}.\]

However, this fact is an example that extends to a more general setting as follows: The spherical 3-design obtained by Bondarenko in [1] is a special case of our main result, which is presented as the following theorem:
Theorem 1.1. Let $X$ be an $(d+1, N, a)$ code and

$$IP := \{(x_i, x_j) \in X^2 \mid x_j \neq -x_i, i \neq j\}$$

We assume that for all $a \in IP$,

$$g_{2,d}(a) = \ell \text{ (constant)}$$

and $X$ satisfies the condition

$$\frac{(2N)^2}{d(d+3)/2} - 4N = \ell^2((2N)^2 - 4N). \tag{1}$$

Then, $G_X \cup -G_X$ is an optimal antipodal $(d(d+3)/2, 2N, \ell)$ code and spherical $3$-design in $S^{\dim \text{Harm}_2(s^d)} - 1$.

In the following, we give some examples satisfying the condition of Theorem 1.1. We quote a table of the known sharp configurations which form spherical $t$-design with $t \geq 4$, together with the 600-cell. (For a detailed explanation of sharp configurations, see \[2\].)

Table 1: Table of the known sharp configurations which form are spherical $t$-design with $t \geq 4$, together with the 600-cell. (from \[2\].)

| $n$ | $N$   | $M$ | Inner products                  | Name            |
|-----|-------|-----|---------------------------------|-----------------|
| 3   | 12    | 5   | $-1, \pm 1/\sqrt{5}$           | icosahedron     |
| 4   | 120   | 11  | $-1, \pm 1/2, 0, (\pm 1 \pm \sqrt{5})/4$ | 600-cell        |
| 8   | 240   | 7   | $-1, \pm 1/2, 0$              | $E_8$ roots     |
| 7   | 56    | 5   | $-1, \pm 1/3$                 | kissing         |
| 6   | 27    | 4   | $-1/2, 1/4$                   | kissing/Schl"afli|
| 24  | 196560| 11  | $-1, \pm 1/2, \pm 1/4, 0$     | Leech lattice   |
| 23  | 4600  | 7   | $-1, \pm 1/3, 0$              | kissing         |
| 22  | 891   | 5   | $-1/2, -1/8, 1/4$             | kissing         |
| 23  | 552   | 5   | $-1, \pm 1/5$                 | equiangular lines|
| 22  | 275   | 4   | $-1/4, 1/6$                   | kissing         |

Some of these examples satisfy the condition of Theorem 1.1.

Corollary 1.1. There exists antipodal spherical codes whose vectors form a spherical $3$-design with the following parameters:
(d + 1, N, a) code | \|x_i, x_j\| | \|G_{x_i}, G_{x_j}\|
---|---|---
(5, 12, 1/5) | \{1/\sqrt{5}\} | \{1/5\}
(9, 120, (1 + \sqrt{5})/6) | \{0, (\pm 1 + \sqrt{5})/4, 1/2\} | \{0, (\pm 1 + \sqrt{5})/6, 1/3\}
(35, 240, 1/7) | \{0, 1/2\} | \{1/7\}
(27, 56, 1/27) | \{1/3\} | \{1/27\}
(20, 54, 1/8) | \{1/4, 1/2\} | \{1/10, 1/8\}
(299, 196560, 5/23) | \{0, 1/4, 1/2\} | \{1/46, 1/23, 5/23\}
(275, 4600, 7/99) | \{0, 1/3\} | \{1/22, 7/99\}
(252, 1782, 3/14) | \{1/8, 1/4, 1/2\} | \{1/56, 1/32, 3/14\}
(275, 552, 1/275) | \{1/5\} | \{1/275\}
(252, 275, 1/54) | \{1/6, 1/4\} | \{1/56, 1/54\}

Moreover, (5, 12, 1/5), (27, 56, 1/27), (35, 240, 1/7), and (275, 552, 1/275) codes are optimal antipodal spherical codes whose vectors form a spherical 3-design.

The following corollary gives parameters satisfying the condition of Theorem 1.1.

**Corollary 1.2.** Let X be a \((d + 1, N, a)\) code satisfying the condition of Theorem 1.1. Assume that there exists \(3 \leq d + 1 \leq 100\) and \(1/\sqrt{m} \in \text{IP}\) for some \(1 \leq m \leq 200\) such that \(g_{2,d}(1/\sqrt{m}) = \ell \neq 0\). Then, in \([6]\), we list the possible parameters \(d, N, \) and \(\ell\). If a \((d + 1, N, a)\) code with the parameter in \([6]\) exists, then there exists an optimal antipodal \((d(d + 3)/2, 2N, \ell)\) code and spherical 3-design in \(S_{\dim\text{Harm}_2(S^d)}-1\).

We note that \(\text{IP}\) must be a subset of “Inner Product” in \([6]\).

In section 2, we give a definition of spherical \(t\)-designs and a construction of spherical \(t\)-designs. In section 3, we give proofs of Theorem 1.1, Corollary 1.1, and Corollary 1.2, along with a concluding remark.

All computer calculations in this study were done with the help of Mathematica.

## 2 Preliminary

In this section, we explain the concept of spherical \(t\)-designs and give the construction of spherical 3-designs.

**Definition 2.1** \([3]\). For a positive integer \(t\), a finite non-empty set \(X\) in the unit sphere

\[S^d = \{x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \mid x_1^2 + \cdots + x_{d+1}^2 = 1\}\]
is called a spherical $t$-design in $S^d$ if the following condition is satisfied:

$$
\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^d|} \int_{S^d} f(x) d\sigma(x),
$$

for all polynomials $f(x) = f(x_1, \ldots, x_{d+1})$ of degree not exceeding $t$. Here, the right hand side involves the surface integral over the sphere and $|S^d|$, the volume of sphere $S^d$.

The meaning of spherical $t$-designs is that the average value of the integral of any polynomial of degree up to $t$ on the sphere can be replaced by its average value over a finite set on the sphere.

The following is an equivalent condition of the antipodal spherical $3$-designs:

**Proposition 2.1 ([7]).** An antipodal set $X = \{x_1, \ldots, x_N\}$ in $S^d$ forms a spherical $3$-design if and only if

$$
\frac{1}{|X|^2} \sum_{x_i, x_j \in X} (x_i, x_j)^2 = \frac{1}{d+1}.
$$

We note that, for any $Y := \{y_1, \ldots, y_N\} \in S^d$, the following inequality holds:

$$
\frac{1}{|Y|^2} \sum_{y_i, y_j \in Y} (y_i, y_j)^2 \geq \frac{1}{d+1}.
$$

The following corollary give a construction of spherical $3$-designs. We denote by $\tilde{G}_X$ the set $G_X \cup -G_X$ defined in Theorem 1.1.

**Corollary 2.1 ([5]).** 1. Let $X$ be a spherical $4$-design in $S^d$. Then, $\tilde{G}_X$ is a spherical $3$-design in $S^{\dim \text{Harm}_2(S^d)} - 1$.

2. Let $X$ be a spherical $4$-design in $S^d$ and an antipodal set. Let $X'$ be an arbitrary subset of $X$ with $|X'| = |X|/2$ such that no pair of antipodal vectors is present in $X'$. Then, $\tilde{G}_{X'}$ is a spherical $3$-design in $S^{\dim \text{Harm}_2(S^d)} - 1$.

3 **Proofs of Main results**

In this section, we give the proofs of Theorem 1.1, Corollary 1.1, and Corollary 1.2.
Proof of Theorem 1.1. Let \(X = \{x_1, \ldots, x_N\}\) be a \((d+1, N, a)\) code. We have the following Gegenbauer polynomial of degree 2 on \(S^d\):

\[
g_{2,d}(x) = \frac{d + 1}{d} x^2 - \frac{1}{d}.
\]

First, we show that \(G_X \cup -G_X\) is a spherical 3-design. By Proposition 2.1, it is enough to show that

\[
\frac{1}{|X|^2} \sum_{x_i, x_j \in X} \langle G_{x_i}, G_{x_j} \rangle^2 = \frac{2}{d(d+3)}
\]

because

\[
dim \text{Harm}_2(S^d) = \frac{d + 3}{d+1} \binom{d+1}{2} = \frac{d(d+3)}{2}
\]

and \(G_X \cup -G_X\) is an antipodal set. In fact, by the equation (1)

\[
\frac{1}{|X|^2} \sum_{x_i, x_j \in X} \langle G_{x_i}, G_{x_j} \rangle^2 = \frac{1}{(2N)^2} \sum_{x_i, x_j \in X} g_{2,d}((x_i, x_j))^2
\]

\[
= \frac{1}{(2N)^2} (4N + \ell^2((2N)^2 - 4N))
\]

\[
= \frac{2}{d(d+3)}.
\]

Therefore, \(G_X \cup -G_X\) is a spherical 3-design.

Thereafter, we prove the optimality. For any antipodal set of points \(Y = \{y_1, \ldots, y_{2N}\}\), by the equation (1), the inequality

\[
\frac{1}{(2N)^2} \sum_{i,j=1}^{2N} (y_i, y_j)^2 \geq \frac{1}{\text{dim Har}_{2}(S^d)}
\]

hold and we have

\[
(y_i, y_j)^2 \geq \ell^2
\]

for some \(y_i, y_j \in Y, i \neq j\) and \(y_j \neq -y_i\).

The proof is completed.

Finally, we give the proofs of Corollary 1.1 and Corollary 1.2.

Proof of Corollary 1.1. The first part follows from Corollary 2.1. We give the proof of optimality for the cases \((5, 12, 1/5)\). The other cases can be proved similarly.
Let $X$ be a $(3, 12, 1/\sqrt{5})$ code. Let $X'$ be an arbitrary subset of $X$ with $|X'| = |X|/2$ such that no pair of antipodal vectors is present in $X'$. Therefore, $X'$ satisfies the condition of Theorem 1.1.

This completes the proof of Corollary 1.1.

\textbf{Proof of Corollary 1.2.} Recall the condition of Theorem 1.1:

\[
\frac{(2N)^2}{d(d+3)/2} - 4N = \ell^2((2N)^2 - 4N),
\]

where $g_{2,d}(a) = \ell$ for all $a \in IP$. Using Mathematica, for $3 \leq d+1 \leq 100$ and for $1 \leq m \leq 200$, we solve the equation:

\[
\frac{(2N)^2}{d(d+3)/2} - 4N = g_{2,d}(1/\sqrt{m})^2((2N)^2 - 4N)
\]

We thereafter list the solutions $d+1, 2N$, and $\ell$ in [6]. We eliminate the parameters, which do not satisfy the Fisher bounds [4].

Moreover, we solve

\[
g_{2,d}(x) = \ell
\]

and we list as "Inner Product" in [6].

This completes the proof of Corollary 1.2.

\textbf{Remark 3.1.} Is there a $(d + 1, N, a)$ code with the parameters in [6] except for the cases in Corollary 1.2?

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