AMBROSE SINGER THEOREMS AND TOTALLY GEODESIC Riemannian Foliations on Compact Lie Groups

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Abstract. Let \( \mathcal{F} \) be a Riemannian foliation with connected totally geodesic fibers on a connected compact Lie group with bi-invariant metric. We answer a question of A. Ranjan proving that, up to isometry, \( \mathcal{F} \) is homogeneous.

1. Introduction

The present work is devoted to a simple question on geometric spaces: given a space with a geometric structure, how to fill it with a (locally) repetitive pattern? Here we deal with a Riemannian manifold and the pattern is (up to universal cover) a fixed submanifold. Assuming some regularity, the decomposition of a manifold into such immersed submanifolds is called a foliation.

For instance, if we start with a Lie group \( G \) as ambient space, we could try to use its algebraic structure to fill the entire space. A common example is given by cosets of a subgroup: given a Lie subgroup \( H < G \), it induces foliations by both right cosets, \( \mathcal{F}^+_H = \{ gH \mid h \in G \} \), and left cosets, \( \mathcal{F}^-_H = \{ Hg \mid h \in G \} \). We call both as homogeneous foliations.

In general, a foliation on \( M \) is decomposition into the integrable maximal submanifolds of an involutive subbundle \( T \mathcal{F} \subset TM \). Each submanifold is called a leaf. Existence, obstructions and classifications of foliations are deep topological subjects (see Haefliger [10] and Thurston [27, 28, 26] for interesting advancements). Such subject acquires a very geometric flavor by imposing distance rigidity on the fibers: a foliation is called Riemannian if its leaves are locally equidistant.

Riemannian submersions are main examples of Riemannian foliations: a submersion \( \pi \) is Riemannian if the restriction \( d\pi_p \mid (\ker d\pi_p)^\perp \) is an isometry onto \( T_{\pi(p)}M \) for every \( p \in M \) (see [8, 20] for details and results). The classification of Riemannian submersions from compact Lie groups with bi-invariant metrics was asked by Grove [9, Problem 5.4]. His question can be motivated in several ways, for instance, most examples of manifolds with positive sectional curvature are related to Riemannian submersion from Lie groups (see [31] for a more complete account).

On the other hand, given a compact Lie group \( G \) with bi-invariant metric, it is known that homogeneous foliations are Riemannian and have totally geodesic leaves. Therefore, it is natural to ask if every Riemannian foliation with totally geodesic leaves is homogeneous or not ([23]). The following conjecture is commonly called “Grove’s Conjecture” (see also [19]):

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1In this paper, only non singular foliations are considered
Conjecture 1. Let $G$ be a compact simple Lie group with a bi-invariant metric. A Riemannian submersion $\pi: G \to B$ with connected totally geodesic fibers is either by left or right cosets.

Here Conjecture 1 is proved affirmatively in the context of Riemannian foliations on compact (not necessarily simple) groups (as suggested in [19]).

Theorem 1.1. Let $\mathcal{F}$ be a Riemannian foliation with totally geodesic connected leaves on $G$, a compact Lie group with bi-invariant metric. Then $\mathcal{F}$ is locally isometric to an homogeneous foliation. More specifically, the universal cover $\pi: \tilde{G} \to G$ splits metrically, $\tilde{G} = G_1 \times G_2$ and

$$\pi^* \mathcal{F} = \{(g_1H_1, H_2g_2) \mid (g_1, g_2) \in G_1 \times G_2\}$$

where $H_i < G_i$.

One may ask whether the statement of Theorem 1.1 is local in nature or not. The proof works for any foliation in an open subset of $G$, as far as its horizontal connection is complete (see section 2).

We observe that the hypothesis on $\mathcal{F}$ cannot be relaxed: Kerin–Shankar [13] presented infinite families of Riemannian submersions $G \to B$ from compact Lie groups with bi-invariant metrics that are not homogeneous (e.g., the composition of the usual fibration $SO(16) \to S^{15}$ with the Hopf map $S^{15} \to S^8$ cannot be realized by the free action of a Lie group). Conversely, the simple group $SO(8)$ admits a foliation $\mathcal{F}_{SO(8)}$ whose leaves are totally geodesic 7-spheres (recall that the standard bundle $SO(8) \to S^7$ is trivial). Kerin–Shankar examples does not have totally geodesic leaves and the $\mathcal{F}_{SO(8)}$ is not Riemannian.

The general classification of Riemannian foliations is wide open. For instance, it is not known classifications neither for totally geodesic Riemannian foliations on symmetric spaces, nor for generic Riemannian foliations on Lie groups (although, we refer to [29, 16, 17] for important developments in other contexts). The author believes many resources provided here can be applied in the symmetric space case.

The main issue in proving Theorem 1.1 is restricting the leaf type. Since we assume that leaves are totally geodesic, each leaf is a symmetric space. Once proved that the leaf through identity is a subgroup, we follow arguments resembling [6, Lemma 3.3] or [12, Theorem 23] to prove homogeneity. The strategy adopted here is to control local holonomy diffeomorphisms (see section 1.1 for a definition – by holonomy we mean the holonomy defined by the horizontal distribution, as in [8], which is fundamentally different from [21]).

After developing a general theory for infinitesimal holonomy diffeomorphism and applications, we proceed to the case of Riemannian foliation with totally geodesic leaves on bi-invariant metrics. The first step in the proof (section 7) is to relate the root system of $G$ with a new root system related to the foliation (introduced in section 6). Such algebraic step provides control over Grey-O’Neill’s integrability tensor (Proposition 1.9) and reduces the proof to the case of irreducible foliations (Theorem 7.16). The sense of irreducibility we deal here is analogous to the irreducibility of a principal bundle with a connection. In modern terms, we call a foliation irreducible if it has a single dual leaf (see [24, 30]). O’Neill’s integrability tensor measures how much the distribution orthogonal to leaves is not integrable (similar to the Levi tensor in [22]). In the case of Riemannian foliations with totally geodesic fibers, O’Neill integrability tensor gives rise to (local) Killing fields along the leaves.
Proposition 1.9 together with a suitable version of the Ambrose-Singer holonomy theorem [2] shows that leaves have the local Killing property ([3, section 6]): around each point, the tangent space of each leaf has an orthonormal frame of Killing fields. The universal cover of a Riemannian manifold with such property is isometric to a product of an Euclidean space, constant curvature 7-spheres and compact simple Lie groups with bi-invariant metrics [3, Theorem 11].

In the second step of the proof, we prove that 7-spheres cannot appear as factors. By taking advantage of Theorem 1.5, we assume that the leaves are locally isometric to a constant curvature 7-sphere and show that the holonomy group of such foliation acts locally freely and transitive on the leaves, deriving a contradiction.

Knowing that the leaves are locally isometric to a Lie group with bi-invariant metric, the next step in the proof is to construct a special family of Killing fields in $G$ that spans the tangent space of the leaves (as in [12, Theorem 23]). They are constructed in section 4.2, under assumptions in the holonomy group, which are verified in section 9.2.

The results on section 2-6 extrapolates the hypothesis of Theorem 1.1 and might be of independent interest. Below, we give a more detailed account of the paper.

1.1. Main Results. Given a Riemannian foliation $\mathcal{F}$ on $M$, we might think of $\mathcal{F}$ locally as an stripped fabric, by placing leaves in the vertical direction. At each point $x \in M$, we decompose $T_x M$ as the tangent to the leaf $V_{c(x)}$ and its orthogonal complement $H_{c(x)} = (V_{c(x)})^\perp$. We call $V_{c(x)}$ and $H_{c(x)}$ as the vertical space and the horizontal space at $x$. A (local) vector field $X$ is said to be basic horizontal, if it is horizontal and, for every vector field $V$ tangent to the leaves, $[X, V]$ is vertical. The flow of a basic horizontal vector field $X$ induces local diffeomorphisms between leaves (as a direct computation using [11, Proposition 17.6] shows). These diffeomorphisms are called holonomy diffeomorphisms. For instance, it is known that holonomy diffeomorphisms are (local) isometries if and only if leaves are totally geodesic submanifolds.

Given a horizontal curve $c: [0, 1] \to M$, there is a natural holonomy diffeomorphism associated to $c$: any local basic horizontal extension of $\dot{c}$ induces a local diffeomorphism between a neighborhood of $c(0)$ restricted to $L_{c(0)}$ to a neighborhood of $c(1)$ restricted to $L_{c(1)}$. Given two local basic horizontal extensions of $\dot{c}$, the induced diffeomorphisms coincides around $c(0)$ (also see [3] Examples and Remarks 1.3.1) for an equivalent definition using local ‘horizontal lifts’ of $c$). In particular, given $\xi \in V_{c(0)}$ and $\Phi_t^X$, the flow of any extension $X$ of $\dot{c}$, the differential $(d\Phi_t^X)_{c(0)}(\xi)$ is well-defined for all $t$. The vector field $\xi(t) = (d\Phi_t^X)_{c(0)}(\xi)$ is called the holonomy field along $c$ with initial value $\xi(0) = \xi$. We call the linear map $\hat{\mathcal{F}}: V_{c(0)} \to V_{c(1)}$, $\hat{\mathcal{F}}(t) = (d\Phi_t^X)_{c(0)}(\xi)$ as the infinitesimal holonomy diffeomorphism defined by $c$. Here, infinitesimal holonomy diffeomorphisms play the role a connection 1-form does in a principal bundle.

Given a Riemannian foliation $\mathcal{F}$, Gray-O’Neill’s integrability tensor of $\mathcal{F}$, $A: \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ is defined as

$$A_X Y = \frac{1}{2} [\bar{X}, \bar{Y}]^v,$$

where $\bar{X}, \bar{Y}$ are horizontal extensions of $X, Y$ and $v$ (respectively, $h$) stands for orthogonal projection onto $\mathcal{V}$ (respectively, $\mathcal{H}$).

The dual leaf passing through $p \in M$, $L_p^\#$, is the subset of $M$ that can be joined to $p$ by horizontal curves.
When $\mathcal{F}$ is induced by a principal $G$-bundle, $\pi: P \to B$, the integrability tensor, infinitesimal holonomy fields and dual leaves represent classical objects: the curvature 2-form satisfies $\Omega(X,Y) = -2A_XY$; given $p \in M$, for any horizontal curve $c$, $\hat{c}(1)(\xi) = \omega_{c(1)}^{-1}\omega_p(\xi)$ -- for simplicity, we consider the linear isomorphism $\omega_{c(1)}: \mathcal{V}_{c(1)} \to \mathfrak{g}; \mathcal{V}_{c(1)}$ the reduction of $P$ through $p$, is the set of points that are reached by horizontal curves starting at $p$ \cite[section II]{8}, i.e., it coincides with $L_p^\#$ \cite[section 1.8]{8}. Ambrose-singer theorem gives infinitesimal information about the holonomy group of $P$ at $p$, which coincides with $\omega(T\pi P(p))$. We provide an analogous theorem, giving information about the vertical part of the dual leaf:

**Theorem 1.2.** Let $\mathcal{F}$ be a Riemannian foliation with complete connection on a path connected space $M$. Then

$$TL_p^\# \cap \mathcal{V} = \text{span}\{\hat{c}(1)^{-1}(A_XY) \mid X,Y \in \mathcal{H}_{c(1)}, \ c \text{ horizontal}\}.$$  

Following the terminology for principal bundles, we call a foliation $\mathcal{F}$ **irreducible**, if its dual foliation is trivial. That is, if it has only one dual leaf. Let $\mathcal{F}$ be irreducible. We generalize the holonomy bundle of a Riemannian submersion with totally geodesic fibers (the connection reduction of the bundle $P \to B$ in the proof of \cite[Theorem 2.7.2]{8}) using infinitesimal holonomy transformations (as in \cite{24}). Given $p \in M$, we define a principal bundle $\tau_p: \mathcal{E}_p \to M$ with group $H_p(\mathcal{F})$, together with a foliation $\tilde{\mathcal{F}} = \{\tau_p^{-1}(L) \mid L \in \mathcal{F}\}$. In section 3 $\mathcal{E}_p$ and $\tau_p$ are proved to be smooth and $\tilde{\mathcal{F}}$ to be Riemannian (with a suitable metric). The local holonomy diffeomorphisms induced by a horizontal curve $c$ defines (not in a natural way) a diffeomorphism between the universal covers $\hat{\phi}_c: \hat{L}_{c(0)} \to \hat{L}_{c(1)}$ (Lemma 2.2). We call the set of holonomy diffeomorphisms induced by horizontal loops at $p$ as the holonomy group at $p$, $\text{Hol}_p(\mathcal{F})$. $H_p(\mathcal{F})$ is a Lie subgroup of $GL(\mathcal{V}_p)$ that recovers the isotropy representation of the holonomy group at the point $p$.

In section 4 we specialize to the case of bounded $H_p(\mathcal{F})$ (recovering the concept of bounded holonomy in \cite{24}). Such is the case when leaves are totally geodesic leaves or coincides with the orbits of a locally free action, i.e., when the foliation is principal. In \cite[Theorem 23]{12}, a foliation is guaranteed to be principal given the existence of a subalgebra of vector fields fields satisfying special properties. In section 4 such a subalgebra is constructed provided $H_p(\mathcal{F}) = \{\text{id}\}$, characterizing principal foliation. More specifically, we prove:

**Theorem 1.3.** Let $\mathcal{F}$ be an irreducible Riemannian foliation with bounded holonomy on a compact manifolds. Then the group of holonomy diffeomorphisms $\text{Hol}_p(\mathcal{F})$ is a finite dimensional Lie group with compact algebra. Furthermore, $\text{Hol}_p(\mathcal{F})$ acts locally freely on $\mathcal{E}_p$ and the orbits coincide with the leaves of $\tilde{\mathcal{F}}$.

Corollary 1.4 is directly used in the proof of Theorem 1.1.

**Corollary 1.4.** Let $\mathcal{F}$ be as in Theorem 1.3. If $H_p(\mathcal{F}) = \{\text{id}\}$, $\text{Hol}_p(\mathcal{F})$ acts locally freely on $M$, with orbits coinciding with the leaves of $\mathcal{F}$. Furthermore, if the restriction of the action of $\text{Hol}_p(\mathcal{F})$ to each leaf is by isometries, then the action is by isometries on $M$.

In section 4 follows by extending the (local) de Rham decomposition of a leaf to the entire foliation.
Theorem 1.5. Let $F$ be as in Theorem 1.3 and $M$ simply connected. Let $TL_p = \bigoplus_i \Delta_i$ be the de Rham decomposition of $TL_p$. Then there are smooth integrable distributions $\Delta_i$ on $M$ such that:

1. $\Delta_i$ is vertical for every $i$ and $\mathcal{V} = \bigoplus \Delta_i$
2. $\Delta_i \perp \Delta_j$ for $i \neq j$
3. $F_i$, the foliations defined by $\Delta_i$, is Riemannian with totally geodesic leaves

Theorem 1.5 is used in the proof of Theorem 1.1 to rule out 7-sphere factors on the leaves.

In sections 5 and 6 we prove key results to the proof of Theorem 1.1. Their roles are complementary. Theorem 1.6, proved in section 5, greatly refines Theorem 1.2. Theorems 1.8 and 1.7 provide a vertical system of roots, based on the integrability tensor. The comparison between such root system and the Lie algebraic root system of $G$ discloses special properties of the integrability tensor (sections 7 and 8).

Theorem 1.6. Let $F$ be a totally geodesic Riemannian foliation on a manifold $M$ of non-negative sectional curvature. Then

$$TL_p \cap V_p = \text{span}\{A_XY \mid X, Y \in H_p\}.$$ 

Theorem 1.7. Let $F$ be a totally geodesic Riemannian foliation, $\gamma$ a vertical geodesic and $X$ a basic horizontal vector field along $\gamma$. Then $A^\gamma X$ is basic horizontal.

Here $A^\gamma : \mathcal{H} \to \mathcal{H}$ is the negative dual of $A$:

$$\langle A^\gamma X, Y \rangle = - \langle A_X Y, \dot{\gamma} \rangle.$$ 

Theorem 1.8. Let $F$ be a totally geodesic Riemannian foliation and $v \subset V_p$ exponentiate to a maximal totally geodesic flat in $L_p$. If $A$ has bounded norm, then $A^\xi \cdot A^\eta = A^\eta \cdot A^\xi$ for all $\xi, \eta \in v$.

In sections 7-9, we specialize to totally geodesic foliations on bi-invariant metrics. Section 7 is built upon Theorem 1.8 and [19, Theorem 1.5] and provides the main algebraic identities in the proof of Theorem 1.1. In section 8, the algebraic results and Theorem 1.7 are used to prove Proposition 1.9.

Proposition 1.9. Let $F$ be as in Theorem 1.1. If $X, Y, Z, W$ are basic horizontal fields, then $\langle A_X Y, A_Z W \rangle$ is basic, i.e., it is locally constant along leaves.

In particular, if $F$ is irreducible, Theorem 1.6 guarantees that the leaves have the local Killing property. Section 9 deals with the 7-sphere factor in the leaves and the completion of the proof of Theorem 1.1.

The classification of Riemannian foliations on symmetric spaces is greatly explored in literature (see [6, 16, 17, 18, 19, 23]). The arguments in the proof of Theorem 1.1 resemble ideas in [18, 19, 23]. Ranjan [22] used the relation $(A^\xi)^2 = (\frac{1}{2} \text{ad}_\xi)^2$ (induced by O’Neill’s formulas - [8]) to prove Conjecture 1, assuming there is a maximal torus contained in a leaf. Such torus provides a decomposition of the basic horizontal fields into left and right invariant fields. Ranjan then uses the simplicity of the group to prove that either the left or the right invariant basic horizontal fields are zero.

In contrast to the algebraic approach of Ranjan [22], Munteanu and Tapp [19] introduced the geometric concept of good triples: a triple $\{X, V, A\} \subset T_pM$ is good if

2The author was informed that M. Radeschi proved Theorem 1.6 assuming $M$ a compact Lie group with bi-invariant metric.
\[ \exp_p(tV(s)) = \exp_p(sX(t)) \] for all \( s, t \in \mathbb{R} \), where \( V(s), X(t) \) denote the Jacobi fields along \( \exp(sV) \) and \( \exp(tX) \), respectively, that satisfy \( V(0) = V, \ X(0) = X \) and \( V'(0) = A = X'(0) \). Such conditions are achieved in totally geodesic Riemannian foliations by a horizontal \( X \) and a vertical \( V \) (or vice-versa). In this case \( A = A^V X \).

Theorem 1.5 in [19] provides a key identity that is used throughout section 7.

Section 7 provides a suitable splitting of the horizontal space which turns to represent the splitting into left and right invariant horizontal fields. Section 8 explores the concept of good triple to prove Proposition 1.9. Sections 7-9 are the most involved part in this work.

Munteanu [18] deals with not necessarily totally geodesic one-dimensional Riemannian foliation on Lie groups, proving their homogeneity. [18, Proposition 2] has a direct, although not clear, connection with Proposition 1.9.

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### 1.2. Notation

We mostly use the notation of Gromoll and Walschap [8]. We follow the usual nomenclature in Riemannian foliations, calling vectors tangent to leaves verticals and vectors orthogonal to leaves horizontals. They define the vector bundles \( V \) and \( H \), respectively. Horizontal vectors will be denoted by capital Arabic letters: \( X, Y, Z, W \); vertical vectors by Greek lower case letter: \( \xi, \eta \).

Gray-O'Neill’s tensors will be denoted as in [6] or [8]:

\[ A_X Y = \nabla_{\tilde{X}} \tilde{Y} = \frac{1}{2} [\tilde{X}, \tilde{Y}], \quad S_X \xi = -\nabla_\xi \tilde{X} \]

where \( \tilde{X}, \tilde{Y} \) are horizontal extensions of \( X, Y \) and \( \tilde{\xi} \) is a vertical extension of \( \xi \). We recall that a foliation is totally geodesic if and only if \( S \equiv 0 \).

Holonomy fields along a horizontal curve \( c(t) \) are vertical solutions of

\[ \nabla_\xi \xi = A_\xi \dot{c} - S_\xi \dot{\xi}. \]  

(1)

Analogously, basic horizontal fields along a vertical curve \( \gamma \) are horizontal solutions of

\[ \nabla_\bar{X} \bar{X} = A^\bar{X} \dot{\gamma} - S^\bar{X} \dot{\gamma}. \]  

(2)

These are equivalent definitions of holonomy and basic horizontal fields.

Holonomy fields along horizontal geodesics (respectively basic horizontal fields along vertical geodesics, in the totally geodesic case) are the Jacobi fields induced by local horizontal lifting (respectively, by holonomy transport).

Given \( \mathcal{F} \), the dual leaf through \( p \in M \) is the set

\[ L^p_\# = \{ q \in M \mid q \text{ can be joined to } p \text{ via a horizontal curve} \}. \]

\( L^p_\# \) is known to be a submanifold and \( \mathcal{F}^\# = \{ L^p_\# \mid p \in M \} \), to be a foliation (see [5] [30]). We follow the nomenclature in [24] and call a foliation \( \mathcal{F} \) irreducible if it has only one dual leaf.

Both (4, 0) and (3, 1) Riemannian curvature tensors are denoted by \( R \). In sections 7 and 8, we work with the complexification of the Lie algebra, among other spaces, throughout the paper. The complexification of a space or an operator will be denoted by a supindex \( ^C \). We recall that the (4, 0) Riemannian curvature in a bi-invariant metric is given by

\[ R(X, Y, Z, W) = -\frac{1}{4} \langle [X, Y], [Z, W] \rangle. \]  

(3)
2. Holonomy transformations in Riemannian Foliations

Let \( \pi: M \to B \) be a Riemannian submersion and \( \bar{c}: [0, 1] \to B \) a curve. The horizontal connection on \( M \) is called complete if for every point \( p \in \pi^{-1}(\bar{c}(0)) \), there is a horizontal curve \( \xi_p: [0, 1] \to M \) such that \( \pi \circ \xi_p = \bar{c} \). If \( \mathcal{H} \) is complete, by lifting horizontally \( \bar{c} \), one gets a holonomy diffeomorphism \( \phi_\xi: \pi^{-1}(\bar{c}(0)) \to \pi^{-1}(\bar{c}(1)). \)

For a foliation, neither \( \bar{c} \) nor its lifts are naturally defined. Here we observe how monodromy arguments gives a diffeomorphism between the universal covers of the leaves \( \phi_\xi: \bar{L}(c(0)) \to \bar{L}(c(1)) \) by patching local ‘lifts’ of a horizontal curve \( c \).

Let \( \mathcal{F} \) be a Riemannian foliation on a complete manifold \( M \). We recall that a Riemannian foliation is locally given by a trivial submersion (\([8, \text{Examples and Remarks 1.2.1, item (ii)}\]) – each point \( p \in M \) has an open neighborhood \( U \) where the restricted foliation \( \mathcal{F}|_U = \{ \text{connected components of } L \cap U \mid L \in \mathcal{F} \} \) is given by a Riemannian submersion \( \pi_U: U \to V \) (the metric on \( V \) is uniquely determined by \( \pi \) \([8, \text{Theorem 1.2.1}]\)); furthermore, one can choose \( U \) diffeomorphic to \( \pi_U^{-1}(\pi_U(p)) \times V \) (take \( V \) a small geodesic ball around \( \pi_U(p) \) and use holonomy translation along radial geodesics). We call such \( U \) a submersive neighborhood. A compactness argument produces a submersion from a tube along a horizontal curve. For this section we assume that all curves have non-zero speed.

**Lemma 2.1.** Let \( c: [0, 1] \to M \) be a horizontal curve. Then \( c \) has a \( \delta \)-neighborhood \( U \) where \( \mathcal{F}|_U \) is given by the fibers of a Riemannian submersion \( \pi: U \to V \).

Given a leaf \( L \in \mathcal{F} \), basic horizontal fields define a parallel frame for the Bott connection in the restriction \( \mathcal{H}|_L \) (\([8, \text{Examples and Remarks 1.3.1 (i)}]\)). Given a horizontal curve \( c: [0, 1] \to M \) and a vertical curve \( \gamma: [0, 1] \to M, c(0) = \gamma(0) \), the horizontal connection is called complete if there are unique curves \( c_{s(t)} \) coinciding with local lifts of \( c \). For completeness sake, we present such property when the geometry of the foliation is bounded (we refer to \([5]\) for more information about complete connections).

**Lemma 2.2.** Suppose the tensors \( A \) and \( S \) are bounded and \( M \) is complete. Then, for each horizontal curve \( c \), there is a unique class \( \Phi_c \) of diffeomorphisms that locally coincide with the local holonomy transformation defined on the submersions on a \( \delta \)-neighborhood of \( c \). Furthermore, every pair of elements in \( \Phi_c \) differs by a deck transformation.

**Proof.** Let \( c \) be a horizontal curve and \( \gamma: [0, 1] \to L_{c(0)} \) be a curve connecting \( p = c(0) \) to some \( q \in L_p \). We prove the existence of a map \( F: [0, 1] \times [0, 1] \to M \) such that,

1. \( \frac{\partial F}{\partial s} \in \mathcal{H} \)
2. for each submersive neighborhood \( U \), \( \pi_U(F(t, s)) \) does not depend on \( s \)
3. \( F(t, 0) = c(t), F(0, s) = \gamma(s) \)

Assuming (1), (2) implies the curves \( s \mapsto F(t, s) \) are obtained from each other by holonomy translations. In fact,

**Claim 2.3.** \( \xi = \frac{\partial F}{\partial s} \) is a holonomy field along \( c_s(t) = F(t, s) \) and \( X = \frac{\partial F}{\partial t} \) is basic horizontal.

**Proof.** It is sufficient to compute \([1]\) and \([2]\) for \( X \) and \( \xi \). But \( \nabla_{\xi}^X \xi = -S_{\xi}X \) and \( \nabla_{\xi}^Y X = A_{\xi}X \) are tensorial and \([X, \xi] = \nabla_{\xi}X - \nabla_{\xi}X = 0. \) \( \square \)
Uniqueness of integral manifolds for Frobenius theorem guarantees that the integral curves of $X \pi_U$ projects to the same curve, characterizing a holonomy diffeomorphism.

We prove the existence of $F$ by an extension argument. From Lemma 2.1 we know there is such an $F$ defined on a small square $[0, 1] \times [0, \varepsilon]$. Let $O \subset [0, 1]$ be the maximal interval containing 0 such that, if $s \in O$, then $F$ is defined on $[0, 1] \times \{s\}$ satisfying (1)-(3). Lemma 2.1 guarantees that $O$ is relatively open.

To prove that $O$ is also closed, suppose (by contradiction) that $O = [0, \beta)$.

Claim 2.4 together with standard ODE arguments, uniformly bounds $X, \xi$ satisfying (1)-(3). Lemma 2.1 guarantees that $O$ is relatively open.

Let $H: [0, 1]^2 \to L_p$ be a homotopy between $\gamma_1, \gamma_2 : [0, 1] \to L_p$ with fixed endpoints. Then $H^* \mathcal{H}$ defines a trivial bundle along $[0, 1]^2$. A translation of this construction through $c(t)$ shows that the ‘horizontal lifts’ of $c$ defined by $\gamma_1$ and $\gamma_2$ coincide.

The Holonomy group of $\mathcal{F}$ at $p$, Hol$_p$, as the set of all holonomy transformations defined by horizontal curves with extremes in $L_p$.

In contrast with holonomy transformation, the infinitesimal holonomy transformation of $c$ can be defined directly with holonomy fields. Let $\xi_0 \in \mathcal{V}_{c(0)}$, we set $\tilde{c}(t)\xi_0 = \xi(t)$, where $\xi(t)$ is the holonomy field defined by $\xi_0$ along $c$. We see in Claim 2.4 that (when restricted to a dual leaf) the set of all such transformations forms a smooth manifold.

**2.1. Proof of Theorem 1.2**

**Proof.** For convenience, write

$$a_p = \text{span}\{\tilde{c}(1)^{-1}(a_{c(1)}) \mid c \text{ horizontal}\}$$

and let $\tilde{a} = \cup_{q \in M} a_q \subset \mathcal{V}$. By construction, $\tilde{a}$ is closed under infinitesimal holonomy transformations, in particular it has constant rank along any dual leaf. Given $p \in M$, in Claim 2.6 we restrict the problem to $L_p^\#$ by proving that $\tilde{a} \subset TL_p^\#$. In Claim 2.6 we observe that $\tilde{a}|_{L_p^\#}$, and then, in Claim 2.6 we show that $\tilde{a} \oplus \mathcal{H}$ is involutive. These three claims complete the proof of Theorem 1.2 since $\mathcal{H} \subset \tilde{a} \oplus \mathcal{H}$, therefore all horizontal curves starting at $p$ must lie in the integral manifold of $\tilde{a} \oplus \mathcal{H}$ through $p$, concluding that this integral manifold must coincide with $L_p^\#$.

**Claim 2.4.** $\tilde{a}_p \subseteq TL_p^\#$.

**Proof.** The claim follows by an usual construction of the A-tensor, that directly relates it to the integrability of $\mathcal{H}$. Consider a small neighborhood $U$ of $p \in M$ such that $\mathcal{F}|_U$ is induced by a submersion $\pi: U \to V$, where $V$ is some open set of an euclidean space. Given $X_0, Y_0 \in \mathcal{H}_p$, let $X, Y$ be basic horizontal extensions of $X_0, Y_0$ such that $[d\pi(X), d\pi(Y)] = d\pi[X, Y]^h = 0$ are commuting vector fields on $V$.

Denote by $\Phi_t^Z$ the flow of a vector-field $Z$. As in the proof of Lemma 2.6, the flow lines of $X, Y$ are the horizontal lifts of flow lines of $d\pi(X), d\pi(Y)$. Since $d\pi(X), d\pi(Y)$ commutes, we conclude that, for $t \geq 0$,

$$\gamma(t) = \Phi_{\sqrt{t}}^X \Phi_{\sqrt{t}}^Y \Phi_{-\sqrt{t}}^X \Phi_{-\sqrt{t}}^Y(p)$$

is a curve in $L_p$ with $\gamma(0) = p$. Furthermore, it is known that $\gamma'(0) = 2A_{X_0, Y_0}$. 

In particular, if \( c \) is a horizontal curve starting at \( p \) and \( X_0, Y_0 \in \mathcal{H}_{c(1)} \), by considering a neighborhood \( U \) where both \( F \) is a submersion and Lemma 2.4 is satisfied, we can again consider suitable extensions of \( X_0, Y_0 \) (as in the first paragraph of the proof) and define the curve

\[
\gamma(t) = \phi_c^{-1} \Phi_v \phi^v \phi^{-1}(p).
\]

\( \gamma \) again lies in \( L_p \) with \( \gamma(0) = p \) and \( \gamma'(0) = 2\hat{c}^{-1}(A_{X_0} Y_0) \). In particular, \( \hat{c}^{-1} A_{X_0} Y_0 \) is tangent to \( L_p \) for every \( c, X_0, Y_0 \), as desired. \( \square \)

**Claim 2.5.** \( \bar{a}|_{L_p^\#} \) is smooth.

**Proof.** Let \( c_i \) be a collection of horizontal curves and \( X_i, Y_i \in \mathcal{H}_{c_i(1)} \) be horizontal vectors such that \( \{ \epsilon_i^{-1}(A_{X_i} Y_i) \}_{1 \leq i \leq k} \) forms a basis for \( \bar{a}_p \). Being careful enough with the neighborhoods in Lemma 2.4, we can extend \( \epsilon_i^{-1}(A_{X_i} Y_i) \) to \( d\phi_c^{-1}(A_{X_i} Y_i) \) in a small neighborhood of \( p \) in \( L_p \). Since \( \bar{a} \) is closed under holonomy diffeomorphisms, this shows that \( \bar{a} \) is spanned by a set of smooth vector fields, concluding it is smooth. \( \square \)

**Claim 2.6.** \( (\bar{a} \oplus \mathcal{H})|_{L_p^\#} \) is integrable.

**Proof.** Let \( X, Y, Z, W \) be horizontal fields and \( \xi, \eta \) sections of \( \bar{a}|_{L_p^\#} \). We prove the claim, by dividing the problem according to the linearity and skew-symmetry of the brackets. We follow the following order:

(i) \( [X, Y] \) is a section of \( \mathcal{H} \oplus \bar{a} \),

(ii) \( [X, \xi] \) is a section of \( \mathcal{H} \oplus \bar{a} \),

(iii) \( [\xi, \eta] \) is a section of \( \mathcal{H} \oplus \bar{a} \).

Since \( [X, Y]^o = 2A_X Y, [X, Y] \in \mathcal{H} \oplus \bar{a} \) by the definition of \( \bar{a} \). For item (ii), it is sufficient to assume \( X \) basic horizontal and \( \xi \) holonomy along the integral curves of \( X \), since basic horizontals and holonomy fields spans the horizontal and the vertical space, respectively. But then \( [X, \xi] = 0 \), since the map \( F \) in the proof of 2.2 integrates the distribution defined by \( X \) and \( \xi \).

As observed in the proofs of Claims 2.6 and 2.5, for small \( U \), \( \bar{a}|_U \) is generated by a finite collection of vectors of the form \( d\phi_c^{-1}(A_X Y) \), \( X, Y \) basic horizontal, satisfying \( [X, Y]^b = 0 \). On the other hand, given \( d\phi_c(A_X Y), d\phi_c(A_Z W) \), we have

\[
[d\phi_c(A_X Y), d\phi_c(A_Z W)] = d\phi_c[A_X Y, (d\phi_c)^{-1} d\phi_c(A_Z W)] = d\phi_c[A_X Y, d\phi_{cc'}(A_Z W)].
\]

Therefore, since \( \bar{a} \) is closed under infinitesimal holonomy transformatoins, it is sufficient to prove that, given \( X, Y, Z, W \) and a horizontal curve \( c, [A_X Y, d\phi_c^{-1}(A_Z W)] \in \bar{a} \).

Writing \( \xi = d\phi_c^{-1}(A_Z W) \), we have

\[
2[A_X Y, \xi] = [[X, Y], \xi] = [[X, \xi], Y] + [X, [Y, \xi]],
\]

which lies in \( \bar{a} \oplus \mathcal{H} \) since, putting items (i) and (ii) together, we have that the bracket of a horizontal fields with any section in \( \bar{a} \oplus \mathcal{H}|_{L_p^\#} \) is again a section in \( \bar{a} \oplus \mathcal{H}|_{L_p^\#} \).

\( \square \)
3. The Infinitesimal Holonomy Bundle

Given \( p, q \in M \), consider the set of linear isomorphisms between \( \mathcal{V}_p \) and \( \mathcal{V}_q \), \( \text{Iso}(\mathcal{V}_p, \mathcal{V}_q) \). The union of such sets for all pairs \( p, q \in M \) defines a Lie groupoid

\[
\text{Aut}(\mathcal{V}) = \{ h : \mathcal{V}_p \to \mathcal{V}_q \mid p, q \in M, \ h \in \text{Iso}(\mathcal{V}_p, \mathcal{V}_q) \}.
\]

with its two natural submersions, the source map and the target map, given for \( h : \mathcal{V}_p \to \mathcal{V}_q \), by

\[
\sigma(h) = p, \quad \tau(h) = q.
\]

Multiplication and inversion of elements given by composition and the usual inverse. Its core \( \mathcal{H} \) and source \( \mathcal{V} \) for which we denote \( \mathcal{V} \) for the linear group of \( \text{Iso}(\mathcal{V}) \), although possibly not smooth. Our first aim is to prove that the intersection of \( \text{Aut}(\mathcal{V}) \) and \( \tau_p \) at \( p \) is a smooth principal bundle over \( L_p^\# \). This result is used in an essential way in sections 3 and 9.1.

Given \( p \in M \), we define the infinitesimal holonomy bundle at \( p, \tau_p : \mathcal{E}_p \to L_p^\# \), as the restriction of \( \tau \) to the set

\[
\mathcal{E}_p = \{ \xi(1) \in \text{Aut}(\mathcal{V}) \mid \xi \text{ horizontal}, \ c(0) = p \}.
\]

That is, \( \tau_p = \tau|_{\mathcal{E}_p} \).

**Theorem 3.1.** \( \mathcal{E}_p \subset \text{Aut}(\mathcal{V}) \) is an immersed submanifold. Furthermore,

\[
H_p(\mathcal{F}) = \mathcal{E}_p \cap \text{GL}(\mathcal{V}_p).
\]

is a Lie subgroup of \( \text{GL}(\mathcal{V}_p) \) and \( \tau_p : \mathcal{E}_p \to L_p^\# \) is a smooth \( H_p(\mathcal{F}) \)-principal bundle.

The idea of the proof is to realize \( \mathcal{E}_p \) as a dual leaf to the following foliation in \( \text{Aut}_p(\mathcal{V}) \):

\[
\mathcal{F} = \{ \mathcal{L}_q = \tau^{-1}(L_q) \mid q \in M \}.
\]

We first see that infinitesimal holonomy transformations naturally induces a horizontal distribution for \( \mathcal{F} \). Our first observation is elementary but used throughout the text.

**Lemma 3.2.** \( \mathcal{L}_q = \tau^{-1}(L_q) \) is a smooth embedded submanifold of \( \text{Aut}_p(\mathcal{V}) \). Furthermore, the restriction \( \tau|_{\mathcal{L}_q} : \mathcal{L}_q \to L_q \) is smoothly isomorphic to the frame bundle of \( L_q \).

The first assertion follows from the smoothness of \( \tau : \text{Aut}_p(\mathcal{V}) \to M \). For the second, recall that the frame bundle can be defined as the collection of isomorphisms \( F(L) = \cup_{x \in L} \text{Iso}(\mathbb{R}^k, T_x L) \). Since \( T_x L = \mathcal{V}_x \), a bundle isomorphism \( \mathcal{L}_q \to F(L_q) \) is induced by right composing with a fixed linear isomorphism \( T : \mathbb{R}^k \to \mathcal{V}_p \). From now on, we stick to \( L_q \) and completely forget any need for identifying \( \mathcal{V}_q \) and \( \mathbb{R}^k \).

The vertical space of \( \mathcal{F} \) is \( \mathcal{V} = d\tau^{-1}(\mathcal{V}) \). Therefore, a distribution \( \mathcal{H} \) defines a horizontal connection for \( \mathcal{F} \) if and only if \( d\tau(\mathcal{H}) \) defines a horizontal distribution to
F. Here, we just lift \( H \). Such lift is achieved in [24] section 3.2 using infinitesimal holonomy transformations or holonomy fields. Given a horizontal curve \( c: [0, 1] \rightarrow M \), we define its \( \tau \)-horizontal lift at \( h \in \tau^{-1}(c(0)) \) as the curve \( \hat{c}_h: [0, 1] \rightarrow \text{Aut}(V) \) given by

\[
\hat{c}_h(t) \xi_0 = \xi_h(t),
\]

where \( \xi_h(t) \) is the holonomy field along \( c \) with initial condition \( \xi_h(0) = h \xi_0 \in V_p(0) \).

Although a horizontal connection in a bundle being completely determined by its horizontal lifts, we remind that not every set of ‘candidates’ for horizontal lifts determine a linear distribution (see [1] Page 2410). We use the identification in Lemma 3.3 to verify such linear condition on the set of velocities of \( \tau \)-horizontal lifts.

**Lemma 3.3.** For each \( h \in \text{Aut}_p(V) \), let \( \hat{H}_h \) be the set of velocities of \( \tau \)-horizontal lifts at \( h \). Then \( \hat{H} = \bigcup_{h \in E_p} \hat{H}_h \) is a smooth distribution satisfying \( d\tau|_{\hat{H}} = \mathcal{H} \).

**Proof.** Identifying \( \text{Aut}_p(V) \) as the frame bundle of \( V \) (much like the identification in Lemma 3.2), a linear connection \( \nabla^V \) on \( V \) induces a horizontal lift \( \zeta: TM \rightarrow T\text{Aut}_p(V) \), which is fiberwise linear: a curve \( \gamma \) on \( \text{Aut}_p(V) \) is \( \zeta(TM) \)-horizontal (i.e., it is always tangent to \( \zeta(TM) \)) if and only if the vector field \( \xi(t) = \gamma(t) \xi_0 \) along \( \tau \circ \gamma \) is parallel with respect to \( \nabla^V \) – here \( \xi_0 \in V_p \) and \( \gamma(t) \) is viewed as an isomorphism \( \gamma(t) \in \text{Iso}(V_p, V_{\tau \gamma(t)}) \). Let \( \nabla^V \) be defined as

\[
\nabla^V_X \xi = (\nabla_X \xi)^v + S_X \xi.
\]

Then, if \( \pi \circ \gamma \) is a horizontal curve, \( \xi(t) \), as defined above, is \( \nabla^V \)-parallel if and only if it is a holonomy fields. Thus, we get \( \hat{H} = \zeta(H) \), therefore \( \hat{H} \) is a linear distribution. \( \square \)

We use \( \hat{H} \) to define a metric \( \langle \cdot, \cdot \rangle_\tau \) on \( \text{Aut}_p(V) \) that makes \( \tilde{F} \) a Riemannian foliation. We declare \( \hat{H} \) orthogonal to \( \hat{V} = TL \) and take \( \langle \cdot, \cdot \rangle_\tau|_{\hat{H} \times \hat{H}} \) such that \( d\tau|_{\hat{H}} \) is an isometry. \( \hat{V}|_{L_q} \), on its turn, admits a natural metric as the frame bundle of \( L_q \).

Consider the induced metric on \( L_q \) and the connection \( \zeta(TM) \cap \hat{V} \) on the bundle \( \tau|_{L_q}: L_q \rightarrow L_q \) (\( \zeta \) as in the proof of Lemma 3.3 – see section 3.1 for more details). For \( \ker d(\tau|_{L_q}) \), we fix an inner product \( Q \) on \( \text{gl}(V_p) \) and use the \( H_p(F) \)-action to identify \( \ker d(\tau|_{L_q}) \) with \( \text{gl}(V_q) \).

**Proposition 3.4.** With respect to \( \langle \cdot, \cdot \rangle_\tau \), \( \tilde{F} \) is a Riemannian foliation with horizontal distribution \( \hat{H} \). Furthermore, a curve \( \alpha: [0, 1] \rightarrow \text{Aut}(V) \) is \( \hat{H} \)-horizontal if and only if \( \alpha = \hat{c}_h \) for \( h = \alpha(0) \) and \( c: [0, 1] \rightarrow M \) is a a horizontal curve. In particular if \( H \) is complete, so it is \( \hat{H} \).

The characterization of \( \hat{H} \)-horizontal curves given in Proposition 3.4 is a consequence of the definition of \( \hat{H} \) since a curve \( \alpha \) in \( \text{Aut}_p(V) \) is \( \hat{H} \)-horizontal if and only if \( \tau \circ \alpha \) is horizontal (since \( d\tau(H) = \mathcal{H} \)) and \( \alpha(t) \xi_0 \) is a holonomy field for every \( \xi_0 \in V_p \) (recall the proof of Lemma 3.3). In particular, \( E_p \) is readily identified as the dual leaf associated to \( \tilde{F} \) passing through \( \text{id}_{V_p} \in \text{Aut}_p(V) \). We conclude:

**Corollary 3.5.** \( E_p \) is a smooth immersed submanifold of \( \text{Aut}_p(V) \).

We further observe that \( \langle \cdot, \cdot \rangle_\tau \) makes \( \tau: \text{Aut}_p(V) \rightarrow M \) a \( \text{GL}(V_p) \)-principal bundle with principal connection. More specifically, the horizontal distribution \( \mathcal{H}' = \zeta(TM) \) defined in the proof of Lemma 3.3 is \( \text{GL}(V_p) \)-invariant.
Proposition 3.6. There is a principal connection $\mathcal{H}'$ on $\tau: \text{Aut}_p(V) \to M$ that extend $\mathcal{H}$. In particular $H_p(\mathcal{F}) = E_p \cap GL(V_p)$ is a group and $\tau_p: E_p \to L^\#_p$ is a $H_p(\mathcal{F})$-principal bundle.

We now complete the proof of Theorem 3.1

Lemma 3.7. For every $p \in M$, $\tau_p: E_p \to L^\#_p$ is a submersion. In particular, $H_p(\mathcal{F})$ is a Lie group.

Proof. $\tau_p$ is smooth since it is the restriction of a smooth map, $\tau: \text{Aut}_p(V) \to M$, to an immersed submanifold. To show that $d\tau_p$ is surjective over $TL^\#_p$, we use Theorem 1.2. First of all, Theorem 1.2 implies that the rank of $TL^\#_p \cap V$ is constant and spanned by velocities of curves of the form (4). However such a curve $\gamma$ as in (4) is defined by the concatenation of 4 flows of horizontal vector-fields. Therefore, by lifting such vector-fields using $\mathcal{H}$, we get a $\mathcal{H}$-horizontal curve $\tilde{\gamma}$ (thus in $E_p$, if $\tilde{\gamma}(0) \in E_p$, satisfying $\tau_p \circ \tilde{\gamma} = \gamma$. Furthermore, $d\tau_p(\tilde{\gamma}) = \tilde{\gamma}$.

Before concluding the section, we present a direct consequence of Proposition 3.6 and Lemma 3.7

Corollary 3.8. If $L^\#_p$ is simply connected, then $H_p(\mathcal{F})$ is connected.

In fact, since $\mathcal{H}' \supset \mathcal{H}$, every point in $E_p$ can be joined to $\text{id}_V$, with a horizontal curve, in particular, $E_p$ is irreducible as a principal bundle, so $H_p(\mathcal{F})$ coincides with the holonomy group of $\tau_p$, which is connected, if the base of $\tau_p$ is simply connected.

3.1 A further remark on the geometry of $\tilde{\mathcal{F}}$. Given $q \in M$, consider the induced metric on $L_q \subset M$. Consider the connection 1-form $\omega: TL \to gl(V_q)$ defined by the connection $\zeta(TM) \cap V$ on $\tau|_{L_q}: L_q \to L_q$. $\omega$ has the following geometric interpretation: a curve $\alpha$ in $L_q$ is $\omega$-horizontal (i.e., $\omega(\dot{\alpha}) = 0$) if and only if $\alpha(0) \in T_{\tau(\alpha(0))}L_q$ is a parallel field along $\tau \circ \alpha$, for every $\xi \in V_q$. That is, $\omega$ is the standard connection 1-form induced by parallel transport. In this section we compute some important quantities for section 9

Lemma 3.9. Let $\alpha: [0,1] \to L_p$ be a curve with $\alpha(0) = \text{id}$. Then $\omega(\dot{\alpha}(0)): V_p \to V_p$ is the morphism defined by $\xi_0 \mapsto \nabla_{\dot{\alpha}(0)}(\alpha(t))\xi$.

Proof. Denote $\gamma(t) = \tau(\alpha(t))$ and its $\omega$-horizontal lift at $\text{id}$ as $\tilde{\gamma}(t)$. For each $t$, there is a unique $g(t) \in GL(V_p)$ such that $\alpha(t) = \tilde{\gamma}(t)g(t)$. In particular,

$$\omega(\dot{\alpha}(0)) = \omega(\dot{\gamma}(0) + g'(0)) = g'(0).$$

On the other hand, for every vector-field $\xi$ along $\gamma$, $\frac{d}{dt}|_{t=0}(\gamma(t))^{-1}\xi = \nabla_{\gamma(0)}\xi$ (observe that $(\gamma(t))^{-1}\xi$ is a curve in $V_p$). Therefore $g'(0)\xi = \nabla_{\gamma(0)}(\alpha(t))\xi$. $\square$

Lemma 3.10. Denote by $\mathcal{A}$ the integrability tensor of $\tilde{\mathcal{H}}$ on $\tilde{\mathcal{F}}$. Let $X, Y \in \tilde{\mathcal{H}}_{\text{id}}$. Then $\omega(\mathcal{A}_X Y): \xi \mapsto \nabla_{\xi}(\mathcal{A}_{\mathcal{d}\tau X} d\tau Y)$.

Proof. For the proof, we identify $\tilde{\mathcal{H}}$ with $\mathcal{H}$ via $d\tau$. As in the proof of Proposition 2.4, consider $U, \tilde{U}$ neighborhoods of $p$ and id where $F$, $\tilde{F}$ are given by Riemannian submersions $\pi: U \to V$, $\tilde{\pi}: \tilde{U} \to V$ (where $\tau(\tilde{U}) \subset U$). Given $X_0, Y_0 \in \mathcal{H}_{\text{id}}$, let $X, Y$ be a basic horizontal extension of $X_0, Y_0$ such that $d\tau(X), d\tau(Y)$ are commuting vector fields on $V$. We can describe the flow of $X, Y$, the horizontal lift of $d\tau(X), d\tau(Y)$ to $\tilde{U}$, through the flows of $X, Y$: denote by $\Phi^X_t$ the flow of a
vector-field $Z$. Given $\xi_0 \in V_p$, recalling the concept of holonomy transportation in $M$ and horizontal curves in $\mathcal{E}_p$, we get

$$\Phi_t^Z(\xi_0) = (d\Phi_t^Z)_p(\xi_0) = \left. \frac{d}{ds} \Phi_t^Z(\gamma(s)) \right|_{s=0},$$

where $\gamma(s)$ is a curve on $L_p$ with $\gamma(0) = p$ and $\gamma'(0) = \xi_0$. On the other hand, as observed in Proposition 2.4,

$$A_X Y = \left. \frac{d}{dt} \Phi_{\sqrt{t}}^X \Phi_{\sqrt{t}}^Y \Phi_{\sqrt{t}}^X \Phi_{\sqrt{t}}^Y(p) \right|_{t=0}.$$

Therefore, by (6), chain rule and the analogous of (7),

$$\omega(A_{\tilde{X}} \tilde{Y})_{\xi_0} = \nabla_{\xi_0} \left( d\Phi_{\sqrt{t}}^X \Phi_{\sqrt{t}}^Y \Phi_{\sqrt{t}}^X \Phi_{\sqrt{t}}^Y(\xi_0) \right)$$

$$= \nabla_{\xi_0} \left( d\Phi_{\sqrt{t}}^X \Phi_{\sqrt{t}}^Y \Phi_{\sqrt{t}}^X \Phi_{\sqrt{t}}^Y(\gamma(s)) \right)$$

$$= \nabla_{\xi_0} \left( d\Phi_{\sqrt{t}}^X \Phi_{\sqrt{t}}^Y \Phi_{\sqrt{t}}^X \Phi_{\sqrt{t}}^Y(\gamma(s)) \right)$$

$$= \nabla_{\xi_0} (A_X Y).$$

4. Foliations with bounded holonomy

Let $\pi: E \to M$ be a Riemannian submersion whose holonomy group is a finite dimensional compact Lie group. Such submersion enjoy special properties, mainly with respect to the growth of the $S$-tensor (see [25] and references therein).

In [24], the author attempt to emulate a compact holonomy group on foliations through a condition on holonomy fields. Here we explore a slightly weaker condition, which we call by the same name: throughout the paper, we say that a foliation $\mathcal{F}$ has bounded holonomy if $H_p(\mathcal{F})$ is a bounded subgroup of $GL(V_p)$. As in [24], bounded holonomy is readily verified whenever the leaves are totally geodesic or when the foliation is principal (see Proposition [24] for the principal case).

Here we prove Theorem 1.3. Although Corollary 1.4 seems to be a direct consequence of Theorem 1.3, our proof goes in the other way around: we first prove Corollary 1.4, then Theorem 1.3. Theorem 1.5 is proved in section 4.4.

In view of Lemma 2.2, whenever $M$ is not compact, we assume that $A$ and $S$ are bounded. Throughout the section, $\mathcal{F}$ is assumed irreducible (although the author believes $\mathcal{E}_p$ is leafwise diffeomorphic to $\mathcal{E}_p \times \text{Hol}_p(\mathcal{F})$).

4.1. Bounded holonomy and totally geodesic leaves. As Proposition 4.1 suggests, foliations with bounded holonomy are closely related to foliations with totally geodesic leaves. To state this relation precisely, we define below a special deformation of the metric. Proposition 4.1 plays a key role in the proof of Theorem 1.3.

Let $\mathcal{F}$ be a Riemannian foliation on the Riemannian manifold $(M, g)$. A metric $g'$ is called a vertical variation of $g$ if the horizontal distributions of $g$ and $g'$ coincides, together with their values on horizontal vectors, i.e., for every $X \in \mathcal{H}$ and $\xi \in \mathcal{V}$,

$$g'(X + \xi, X + \xi) = g_0(X, X) + g'(\xi, \xi).$$

Proposition 4.1. [24] Theorem 6.5] If the holonomy of $\mathcal{F}$ is bounded, there is a vertical variation $g'$ of $g$ where the leaves of $\mathcal{F}$ are totally geodesic.
Proof. Suppose that $F$ has bounded holonomy. Then, the closure of $H$ on $GL(V_p)$ is compact thus $V_p$ can be endowed with a $H$-invariant inner product $\langle , \rangle$. Given $h \in E_p$, observe that the metric defined by $\langle \xi, \eta \rangle_p = \langle h^{-1}\xi, h^{-1}\eta \rangle$ does not depend on the choice of $h \in \tau_p^{-1}(q)$. In fact, if $h, k \in \tau_p^{-1}(q)$,
$$\langle h^{-1}\xi, h^{-1}\eta \rangle = \langle h^{-1}kk^{-1}\xi, h^{-1}k^{-1}\eta \rangle = \langle k^{-1}\xi, k^{-1}\eta \rangle,$$

since $h^{-1}k \in H$. This metric is smooth since it is the restriction of groupoid action of $E$ on the bundle of symmetric bilinear forms of $V$.

On the other hand, a Riemannian foliation is totally geodesic if and only if holonomy fields have constant length (as (1) indicates). Given a holonomy field $\xi, \eta$ of isometries of $\tilde{\xi}$ such that $\pi \circ h \in \tilde{\xi}$, then $d\pi \circ h \in \tilde{\xi}$. We denote by $\pi_p : \tilde{\xi} \rightarrow L_p$ the map induced by $d\pi$. From now on, we assume that $F$ satisfies all hypothesis in Theorem 1.3. The following claim follows by the definition of $\text{Hol}_p(F)$ and the fact isometries are completely defined by its value and its differential in a single point.

Claim 4.2. Let $z \in \pi^{-1}_s(id_{V_p})$ and denote by $\cdot$ the action of $\text{Hol}_p(F)$ on $\tilde{\xi}_p$ induced by its differential. Then $\text{Hol}_p(F)$ acts freely and transitively on $\pi^{-1}_s(\tilde{\xi}_p \cap E_p)$.

In particular, $\text{Hol}_p(F)$ inherits a differential structure by the map $h \mapsto h \cdot z \pi^{-1}_s(\tilde{\xi}_p \cap E_p)$. Multiplication and inversion are differentiable since the $\text{Hol}_p(F)$-action is induced from $\text{Iso}(\tilde{\xi}_p)$. In view of Claim 4.2, Theorem 1.3 applied to the foliation $F$ on $\tilde{\xi}_p$, characterizes the Lie algebra of $\text{Hol}_p(F)$. Corollary 4.3 is used in section 9.

Corollary 4.3. Let $F$ be as in Theorem 1.3 and denote by $\text{hol}_p(F)$ the set of right invariant vector fields on $\text{Hol}_p(F)$. Then
$$\text{hol}_p(F) = \text{span}\{((d\phi, d\pi)^{-1}(A\xi Y) \mid X, Y \in H_c(1), \ c(0) = p, \ c \text{ horizontal}\},$$

4.2. Proof of Corollary 1.4. Let $G$ be a Lie group. We call a foliation $F$ as $G$-principal if the leaves of $F$ coincide with the orbits of a locally free $G$-action. Here we show that a foliation $F$ satisfying the hypothesis of Theorem 1.3 is principal if and only if $H_p(F) = \{\text{id}\}$.

We begin by observing that, assuming $H_p(F) = \{\text{id}\}$, $F$ automatically has bounded holonomy. Therefore $\text{Hol}_p(F)$ is a finite dimensional Lie group. We proceed with few additional observations.

Lemma 4.4. Let $F$ be a Riemannian foliation satisfying the hypothesis of Theorem 1.3 such that $H_p(F) = \{\text{id}\}$. Then,
(1) the action of $\text{Hol}_p(F)$ on $\pi^{-1}_s(\tilde{\xi}_p \cap E_p)$ is free and transitive;
(2) $\tau_p : E_p \rightarrow M$ is a diffeomorphism;
(3) \( \mathcal{V} \to M \) is a trivial vector bundle.

Item (1) follows from Claim 4.2, (2) from Lemma 3.7. For (3), we observe the more general fact that \( \tau_p^* \mathcal{V} \to \mathcal{E}_p \) is trivial. The trivialization map \( \tilde{\chi} : \mathcal{E}_p \times V \to \tau_p^* \mathcal{V} \) is defined as

\[ \tilde{\chi}(h, \xi_0) = h\xi_0. \]

Since \( \tau_p \) is a diffeomorphism throughout this section, we consider the trivialization \( \chi : M \times V \to \mathcal{V}, \chi(q, \xi_0) = \tau_p^{-1}(q)\xi_0 \). Fixed \( \xi_0 \in V_p \), our first result provides a property of the vector field \( \xi(q) = \chi(q, \xi_0) \).

**Lemma 4.5.** Let \( c \) be a horizontal curve, then \( \hat{c}(1)\xi(c(0)) = \xi(c(1)) \).

**Proof.** Let \( \alpha : I \to M \) be a horizontal curve joining \( p \) to \( c(0) \) and denote by \( c\alpha \) the concatenated curve. Then, on one hand \( c\alpha(1) = \hat{c}(1)\alpha(1) \), on the other hand, since \( \tau_p \) is a diffeomorphism, \( c\alpha(1) = \tau_p^{-1}(c\alpha(1)) = \tau_p^{-1}(c(1)) \). Thus

\[ \hat{c}(1)\xi(c(0)) = \hat{c}(1)\tau_p^{-1}(c(0))\xi_0 = \hat{c}(1)\hat{\alpha}(1)\xi_0 = \hat{c}\alpha(1)\xi_0 = \tau_p^{-1}(c(1))\xi_0 = \xi(c(1)). \]

According to (1) in Lemma 4.5, the action \( \cdot \) of Claim 4.2 induces an embedding \( \mu_z(h) = \tau_p(h \cdot z) \) of \( \text{Hol}_p(\mathcal{F}) \) into \( L \), which is a diffeomorphism since we assume \( \mathcal{F} \) irreducible. This embedding is equivariant with respect to \( \cdot \) and left multiplication in \( \text{Hol}_p(\mathcal{F}) \), i.e., \( \mu_z(gh) = g\mu_z(h) \). In particular, \( \mu_z \) sends right-invariant fields to \( \bullet \)-action fields.

Given \( \xi(q) = \chi(q, \xi_0) \), consider \( \tilde{\xi} \) as the only vector field on \( \text{Hol}_p(\mathcal{F}) \) which is \( \pi \circ \mu_z \)-related to \( \xi|_{L_p} \).

**Claim 4.6.** \( \tilde{\xi} \) is left invariant.

**Proof.** Since \( \text{Hol}_p(\mathcal{F}) \) is a finite-dimensional Lie group, it is sufficient to show that \( \tilde{\xi} \) commutes with a set of generating vectors in \( \text{hol}_p(\mathcal{F}) \), thus, it is sufficient that \( \tilde{\xi} \) commutes with \((d\phi_c d\pi)^{-1}(A_X Y)\) for any basic \( X, Y \). On the other hand, since both \((d\phi_c d\pi)^{-1}(A_X Y)\) and \( \tilde{\xi} \) are \( \pi \)-related, it is sufficient to compute the brackets locally using \( \xi \) and \((d\phi_c)^{-1}(A_X Y)\). Furthermore, since \( d\phi_c(\xi) = \hat{c}(1)\xi = \xi \), we can only consider \([A_X Y, \xi]\), where \( X, Y \) are locally defined basic horizontal fields. Taking \( X, Y \) such that \([X, Y]^h = 0\) as in Claim 2.4 we have

\[ 2[A_X Y, \xi] = [[X, Y], \xi] = [[X, \xi], Y] + [X, [Y, \xi]]. \]

Every term in the right-hand-side is zero since \( \xi \) is invariant by holonomy transformation, as in the proof of Claim 2.6 item (ii).

We are ready to prove the main result of this section:

**Proposition 4.7.** Let \( \mathcal{X} \) be the set of vertical fields \( \xi \) on \( M \) such that \( \xi(q) = \chi(q, \xi_0) \) for some \( \xi_0 \). Then \( \mathcal{X} \) is a subalgebra of vector-fields isomorphic to \( \text{hol}_p(\mathcal{F}) \).

**Proof.** Let \( \xi, \eta \in \mathcal{X}, \xi(p) = \xi_0, \eta(p) = \eta_0 \). Using the notation in Claim 4.6 let us show that \([\xi, \eta]|(q) = \chi(q, d\pi[\xi, \eta]) = \chi(q, d\pi[\hat{\xi}, \hat{\eta}])\) - observe that \( \chi(q, d\pi[\hat{\xi}, \hat{\eta}]) \) is a well-defined vector field since both \( \hat{\xi}, \hat{\eta} \) are \( \pi \)-related. Since both fields are vertical, their flows preserve leaves. Thus \([\xi, \eta] \) is vertical and \([\xi, \eta]|_{L_q} = [\xi|_{L_q}, \eta|_{L_q}] \) for every \( q \in M \). On the other hand, given a horizontal curve \( c \) joining \( p \) and \( q \), we have (locally):

\[ [\xi, \eta] = d\phi_c d\phi_c^{-1}[\xi, \eta] = d\phi_c[d\phi_c^{-1}\xi, d\phi_c^{-1}\eta] = d\phi_c[d\pi\hat{\xi}, d\pi\hat{\eta}] = d\phi_c d\pi[\hat{\xi}, \hat{\eta}]. \]

The proof follows since, at the point \( q \), \( d\phi_c d\pi[\hat{\xi}, \hat{\eta}] = \hat{c}(1)d\pi([\hat{\xi}, \hat{\eta}]) \). \( \square \)
From standard theory, $\mathfrak{X}$ integrates to a smooth locally free $G$-action, where $G$ is the universal cover of $\text{Hol}_p(F)$. We conclude the proof of Corollary 1.4 by recalling that the restriction of the $G$-action to a leaf is locally equivalent to the action in Claim 4.2. In particular, it is locally free.

In the general case, the new $\text{Hol}_p(F)$-action is not by isometries. But it certainly preserves basic horizontal fields: following the proof of Claim 2.6 item (ii), we conclude $[\xi, X] = 0$ whenever $\xi \in \mathfrak{X}$ and $X$ is basic horizontal. The following proposition is a direct consequence.

**Proposition 4.8.** The principal $G$-action is by isometries if and only if its restriction to each leaf is by isometries.

If $F$ has totally geodesic leaves it is sufficient to verify the hypothesis in Proposition 4.9 for a single leaf, since holonomy diffeomorphisms are isometries.

4.3. **Proof of Theorem 1.3** In view of Corollary 1.4 we complete the proof of Theorem 1.3 by showing that $H_\text{id}(\tilde{F}) = \{\text{id}\}$ (Proposition 4.9). In this case, $H_\text{id}(\tilde{F})$ is isomorphic to $\text{Hol}_p(F)$ since the elements of $\text{Hol}_p(F)$ are determined by its first 0- and 1-jet (i.e., the action defined in Claim 1.2 is free).

**Proposition 4.9.** Let $F$ be as in Theorem 1.3. Then $H_\text{id}(\tilde{F}) = \{\text{id}\}$.

**Proof.** We describe $H_\text{id}(\tilde{F})$ explicitly. Recall $H'$, the $H_p(F)$-principal extension of $\tilde{H}$ in $E_p \to M$ of $\tilde{H}$ (Proposition 3.6). $H'$ decomposes $\tilde{V}$ in two factors: the tangent to the fibers of $\tau_p$, which is recognized by the 1-form $\omega$ (section 3.1), and the $H'$-factor, $H' \cap \tilde{H}^\perp$. Given a horizontal loop $\alpha$ at $id \in E_p$, its infinitesimal holonomy is decomposed as

$$\hat{\alpha} = \begin{pmatrix} \hat{\alpha}_{11} & \hat{\alpha}_{12} \\ \hat{\alpha}_{21} & \hat{\alpha}_{22} \end{pmatrix} : (H'_p \cap \tilde{H}^\perp_{id}) \oplus h_p(F) \to (H'_p \cap \tilde{H}^\perp_{id}) \oplus h_p(F).$$

We show that $\hat{\alpha}_{12} = 0$ and both $\hat{\alpha}_{11}, \hat{\alpha}_{22}$ are identity maps.

The elements $\hat{\alpha}_{12}, \hat{\alpha}_{22}$ are easy to understand since, if $\zeta$ is an $H_p(F)$-action field, the restriction $\zeta|_\alpha$ is a holonomy field for $\tilde{F}$. The argument in [24, Lemma 3.5] shows that $\hat{\alpha}_{12} = 0$ and $\hat{\alpha}_{22} = \text{id}_{h_p(F)}$

The remaining terms $\hat{\alpha}_{11}, \hat{\alpha}_{21}$ are related to variations through $\tilde{H}$-horizontal curves. Let $\zeta(t)$ be a holonomy field with $\zeta(0) = \zeta_0 \in H'_p \cap \tilde{H}^\perp_{id}$. Its projection $d\tau_p(\zeta(t)) = \xi(t)$ defines a holonomy field on $F$. Denote $c = \tau \circ \alpha$. We remark two points: (i) $\xi(t)$ is defined by a variation $c_\epsilon$ of $c$ through local ‘horizontal lifts’ of $\pi_U \circ c$, where $U$ is a submersive neighborhood of $p$; (ii) $d\tau|_{H'}$ is an isomorphism; (iii) $\alpha = \hat{c}$. Using (ii) one concludes that $\zeta$ is the variational field induced by $H'$-horizontal lifts of $(\tau \circ \alpha)_*$. In particular, $d\tau_p(\zeta(1)) = \xi(1) = \hat{c}(1)\xi(0) = \alpha(1)\xi(0) = \xi(0)$, since $\alpha(1) = \text{id}$. Therefore $\alpha_{11}$ is the identity.

To conclude that $\hat{\alpha}_{21} = 0$, one can either observe that $\tilde{F}$ has totally geodesic leaves in the metric constructed in Proposition 4.2, therefore $H_p(\tilde{F})$ must be bounded. However, if $\hat{\alpha}_{21} \neq 0$: or observe that, if $\hat{\alpha}_{21} \neq 0$, then

$$\hat{\alpha}^k = \begin{pmatrix} \text{id} & 0 \\ \hat{\alpha}_{21} & \text{id} \end{pmatrix}^k = \begin{pmatrix} \text{id} & 0 \\ k\hat{\alpha}_{21} & \text{id} \end{pmatrix},$$

inducing an unbounded representation of $H_p(\tilde{F})$, a contradiction. $\square$
4.4. Splitting of totally geodesic foliations. We proceed to the proof of Theorem \[1.5\]. Let \( \mathcal{F} \) be as in the hypothesis and consider \( p \in M \). Since \( \mathcal{F} \) has totally geodesic leaves, the arguments in the proof of [8] Theorem 1.4.1] shows that \( \hat{c}(1) \) is an isometry for every horizontal curve \( c \). Moreover:

Claim 4.10. If \( c \) is a horizontal loop at \( p \), then \( \hat{\Delta}_i \) is \( \hat{c}(1) \)-invariant for every \( i \).

Proof. Following the proof of Lemma \[2.2\] we observe that \( H_p(\mathcal{F}) \) is the \( d\pi \)-image of the natural linear representation of the isotropy group of \( \text{Hol}_p(\mathcal{F}) \) at \( z \in \pi^{-1}(p) \). In particular, \( H_p(\mathcal{F}) \) consists of differentials of local isometries fixing \( p \). On the other hand, Corollary \[3.8\] guarantees that \( H_p(\mathcal{F}) \) is a connected subgroup. The claim follows from uniqueness of the de Rham decomposition since it implies that \( \text{Iso}_0(\hat{L}_p) = \Pi \text{Iso}_0(\hat{L}_i) \), where \( \text{Iso}_0 \) stands for the connected component of the isometry group of that contains the identity and \( \hat{L}_p = \Pi \hat{L}_i \) is the de Rham decomposition of \( \hat{L}_p \).

Claim \[4.10\] allows us to extend the distribution \( \hat{\Delta}_i \) via holonomy transportation: let \( c \) be a horizontal curve connecting \( p \) to \( q \). Set \( \Delta_i(q) = \hat{c}(1)(\hat{\Delta}_i(p)) \). We claim that this distribution is well-defined.

In fact, if \( c_0, c_1 \) are horizontal curves joining \( p \) to \( q \), \( \hat{c}_0(1)\hat{c}_1^{-1}(1) \in H_p(\mathcal{F}) \). Thus

\[ \hat{c}_1(1)(\hat{\Delta}_i) = \hat{c}_0(1)(\hat{c}_0(1)^{-1}\hat{c}_1(1)\hat{\Delta}_i) = \hat{c}_0(1)(\hat{\Delta}_i) \]

Claim 4.11. For every \( i \), \( \Delta_i \) is integrable and integrates a Riemannian foliation with totally geodesic leaves.

Proof. Since \( \Delta_i \subset \mathcal{V} \), it is sufficient to analyze its integrability leafwise. However, \( \Delta_i|_{L_q} \) is an isometric translation of a de Rham factor of \( \hat{L}_p \), therefore, \( \Delta_i|_{L_q} \) is both integrable and integrates a Riemannian totally geodesic foliation on \( L_q \). Since \( L_q \) is totally geodesic on \( M \), so it is the integral submanifolds of \( \Delta_i \).

According to [8] Theorem 1.2.1], \( \Delta_i \) integrates a Riemannian foliation if and only if \( \mathcal{L}_{U}g\Delta_i^\perp = 0 \) for every \( U \in \Delta_i \). But \( \Delta_i^\perp = \mathcal{H} \oplus (\Delta_i^\perp \cap \mathcal{V}) \) and: \( g(\mathcal{H}, \mathcal{V}) = 0 \) by definition; \( \mathcal{L}_{U}g(\mathcal{H}, \mathcal{H}) = 0 \), since \( \mathcal{F} \) is Riemannian; \( \mathcal{L}_{U}g(\Delta_i^\perp \cap \mathcal{V}, \Delta_i^\perp \cap \mathcal{V}) = 0 \) since the restriction of \( \Delta_i \) to each leaf is Riemannian.

5. An Ambrose-Singer theorem for totally geodesic foliations on non-negatively curved manifolds

Here we use Theorem \[2\] to prove Theorem \[1.6\]. Theorem \[1.6\] is a direct consequence of a broader rigidity of vertizontal planes with zero sectional curvature in non-negatively curved foliations with totally geodesic fibers (see Proposition \[5.3\] for example). The starting point for Theorem \[1.6\] is the following inequality.

Lemma 5.1. Let \( \mathcal{F} \) and \( M \) be as in Theorem \[1.6\] Then, for every \( x \in M \), there is a neighborhood of \( x \) and a \( \tau > 0 \) such that

\[ \tau ||X||||Z||||A^\xi X|| \geq ||(\nabla_X A^\xi)X, Z|| \]

for all horizontal \( X, Z \) and vertical \( \xi \).

Proof. Given \( X, Z \in \mathcal{H} \) and \( \xi \in \mathcal{V} \), O’Neill’s equations ([8] page 44]) states that the unreduced sectional curvature \( K(X, \xi + tZ) = R(X, \xi + tZ, \xi + tZ, X) \) is given by

\[ K(X, \xi + tZ) = t^2 K(X, Z) + 2t \langle (\nabla_X A)X Z, \xi \rangle + ||A^\xi X||^2. \]
Since, by hypothesis, $K(X, \xi + tZ) \geq 0$, the discriminant of expression (2) (seem as a polynomial on $t$) must be non-negative. That is

$$0 \leq K(X, Z)||A^\xi X||^2 - \langle (\nabla_X A)X Z, \xi \rangle^2.$$

On small neighborhoods, continuity of $K$ guarantees some $\tau > 0$ such that $K(X, Z) \leq \tau||X||^2||Z||^2$. Using a holonomy field, one conclude that

$$\langle (\nabla_X A)Z Y, \xi \rangle = -\langle (\nabla_X A^\xi)Z Y, \xi \rangle$$

for all horizontal $X, Y, Z$.

**Proposition 5.2.** Let $F$ be as in Theorem 1.6. Let $X_0 \in H_p$ and $\xi_0 \in Y_p$ be such that $A^{\xi_0}X_0 = 0$. Then, for $\xi(t)$, the holonomy field along $c(t) = \exp(tX_0)$, $A^{\xi(t)}c(t) = 0$ for all $t$.

**Proof.** Taking $||X_0|| = 1$ and $Z = A^\xi c$ in (5), we get:

$$\tau||A^\xi c||^2 \geq \langle (\nabla_X A^\xi)c, A^\xi c \rangle = \langle \nabla_X (A^\xi c), A^\xi c \rangle = \frac{1}{2} \frac{d}{dt}||A^\xi c||^2.$$  

Inequality (10) is Gronwall’s inequality for $u(t) = ||A_{\xi(t)}(t)||^2$ and implies that

$$||A^{\xi(t)}c(t)||^2 \leq ||A^{\xi(0)}c(0)||^2 e^{2\tau t}$$

for all $t > 0$. In particular, if $A^{\xi(0)}X(0) = 0$, $A^{\xi(t)}c(t) = 0$ for all $t > 0$. The same argument works for $t < 0$, by replacing $X_0$ by $-X_0$.

Fixed a holonomy field $\xi(t)$, our next task is to understand the distribution $D(t) = \ker(A^\xi: H_{\xi(t)} \rightarrow H_{\xi(t)})$. The main result of this section is the constancy of its rank (Proposition 5.3). We prove two technical lemmas first.

**Lemma 5.3.** Let $X, Y \in H$ be unitary orthogonal such that $A^\xi X = 0$. Then,

$$2\tau||A^\xi Y||^2 \geq \langle (\nabla_X A^\xi)Y + (\nabla_Y A^\xi)X, A^\xi Y \rangle.$$

**Proof.** We use (5) to get:

$$2\tau||A^\xi(Y + X)|| \geq \langle (\nabla_X A^\xi)Y + (\nabla_X A^\xi)X + (\nabla_Y A^\xi)X, Z \rangle \geq 2\tau||A^\xi(Y - X)|| \geq -\langle (\nabla_X A^\xi)Y + (\nabla_Y A^\xi)X, Z \rangle$$

We complete the proof by summing up both inequalities and observing that $A^\xi(X + Y) = A^\xi(Y - X) = A^\xi Y$ since $A^\xi X = 0$.

Consider the non-negative symmetric operator $D = -A^\xi A^\xi$. We recall that $\ker A^\xi = \ker D$. Furthermore, if $DY = \lambda^2 Y$ for $\lambda > 0$, we can set $Y = \lambda^{-1}A^\xi Y$, so that $||Y|| = ||Y||$ and $A^\xi Y = -\lambda Y$. In particular, if $||Y|| = 1$, $||DY|| = \lambda^2$ and $||A^\xi Y|| = ||A^\xi Y|| = \lambda$.

**Lemma 5.4.** Let $X, Y$ be unitary horizontals satisfying $A^\xi X = 0$ and $DY = \lambda^2 Y \neq 0$. If $\xi$ is holonomy field along $X$,

$$\langle (\nabla_Y A^\xi)X, A^\xi Y \rangle + \langle (\nabla_Y A^\xi)X, A^\xi Y \rangle = \langle (\nabla_X A^\xi)Y, A^\xi Y \rangle$$

**Proof.** Lemma 1.5.1 in [5] page 26] gives,

$$\langle (\nabla_Y A^\xi)X, Y \rangle = -\langle (\nabla_X A^\xi)Y, Y \rangle - \langle (\nabla_Y A^\xi)Y, X \rangle.$$
Observing that $A^\xi Y = \lambda \bar{Y}$ and $A^\xi \bar{Y} = -\lambda Y$, we have
\[
\langle (\nabla_Y A^\xi) X, A^\xi Y \rangle = \lambda \langle (\nabla_Y A^\xi) \bar{Y}, \bar{Y} \rangle = -\lambda \left( \langle (\nabla_X A^\xi) \bar{Y}, Y \rangle + \langle (\nabla_Y A^\xi) Y, X \rangle \right)
= \langle (\nabla_X A^\xi) \bar{Y}, A^\xi Y \rangle - \langle (\nabla_Y A^\xi) X, A^\xi Y \rangle.
\]

Let $X_0$, $\xi_0$ be as in Proposition 5.2. From the semi-continuity of the rank of symmetric operators (in particular, of the multiplicity of its eigenvalues), there exists an $l > 0$ such that $D$ has a smooth frame of eigenvectors along $c([0, l))$, $c(t) = \exp(t X_0)$.

**Proposition 5.5.** Let $X_0, \xi_0$ satisfy $A^{\xi_0} X_0 = 0$. If $\lambda^2$ is a continuous eigenvalue of $D$ along $c(t) = \exp(t X_0)$, then either $\lambda$ vanishes identically, or $\lambda$ never vanishes.

**Proof.** We argue by contradiction, assuming that $\lambda$ vanishes at $t = 0$ but there is $l > 0$ such that $\lambda(t) > 0$ for all $t \in (0, l)$. We further assume that $D$ admits a smooth frame of eigenvectors along $c([0, l))$. We now prove that
\[
\frac{8}{3} \tau \lambda^2 \geq \frac{d}{dt} \lambda^2.
\]
In particular, $\lambda(t)^2 \leq \lambda(\epsilon)^2 e^{16\tau t}$ for all $\epsilon \in (0, l), t \in (\epsilon, l)$. Thus, $\lambda$ must vanish on $(0, l)$, a contradiction.

Inequality (12) follows from Lemmas 5.3, 5.4. Let $Y$ be a smooth eigenvector field satisfying $D Y = \lambda^2 Y$. Since $||A^\xi Y||^2 = \lambda^2$, (11) gives
\[
2 \tau \lambda^2 \geq \langle (\nabla_X A^\xi) Y + (\nabla_Y A^\xi) X, A^\xi Y \rangle.
\]
Taking $\bar{Y} = A^\xi Y / ||A^\xi Y||$, we have $D \bar{Y} = \lambda^2 \bar{Y}$ and $||A^\xi \bar{Y}||^2 = \lambda^2$. Therefore, replacing $Y$ by $\bar{Y}$ in (13) gives
\[
2 \tau \lambda^2 \geq \langle (\nabla_X A^\xi) \bar{Y}, \bar{Y} \rangle + \langle (\nabla_Y A^\xi) X, A^\xi \bar{Y} \rangle.
\]
Summing up (13) and (14):
\[
4 \tau \lambda^2 \geq \langle (\nabla_X A^\xi) Y + (\nabla_Y A^\xi) X, A^\xi Y \rangle + \langle (\nabla_X A^\xi) \bar{Y}, \bar{Y} \rangle + \langle (\nabla_Y A^\xi) X, A^\xi \bar{Y} \rangle.
\]
According to Lemma 5.4, $(\nabla_Y A^\xi) X, A^\xi \bar{Y}) + (\nabla_Y A^\xi) X, A^\xi \bar{Y}) = \langle (\nabla_X A^\xi) \bar{Y}, A^\xi \bar{Y} \rangle$. On the other hand,
\[
\langle (\nabla_X A^\xi) \bar{Y}, A^\xi Y \rangle = \langle \nabla_X (A^\xi \bar{Y}), A^\xi Y \rangle - \langle A^\xi (\nabla_X \bar{Y}), A^\xi Y \rangle
= \frac{1}{2} \frac{d}{dt} \lambda^2 - \lambda^2 \langle \nabla_X \bar{Y}, \bar{Y} \rangle = \frac{1}{2} \frac{d}{dt} \lambda^2,
\]
and the same equality holds by replacing $\bar{Y}$ by $Y$, concluding the proof. \[\square\]

**Theorem 1.6** follows from Proposition 5.5 and Theorem 1.2.

**Proof of Theorem 1.6.** Let $p \in M$. Observe that
\[
a_p^+ = \{ \xi \in V_p \mid A^\xi X = 0 \ \forall X \in \mathcal{H}_p \}.
\]

**Claim 5.6.** If $c$ is horizontal curve, then $\dot{c}(1)(a_p^+) = a_{c(1)}^+.$

**Proof.** It is sufficient to prove the claim for horizontal geodesics, since $c$ can be smoothly approximated by piece-wise horizontal geodesics. Let $c$ be a horizontal geodesic, $c(0) = p$, and $\xi(t)$ be a holonomy field with $\xi(0) \in a_p^+$, then $\ker A^{\xi(0)} = \mathcal{H}_p$ and $\dim \ker A^{\xi(t)}$ is constant with respect to $t$ (Proposition 5.5). Thus $A^{\xi(t)} = \mathcal{H}_{c(t)}$ for all $t$. \[\square\]
Since \( \hat{c}(1) \) is an isometry, Claim 5.6 implies \( \hat{c}(1)(a_p) = a_{c(1)} \). Theorem 1.2 completes the proof. \( \square \)

6. A Root Decomposition for Basic Horizontal Fields

The usual setting for a root system consists of an abelian Lie algebra (here we assume over \( \mathbb{R} \)) \( \mathfrak{t} \) acting in some linear space \( V \) through a Lie algebra morphism \( \rho: \mathfrak{t} \to \text{End}(V) \). For instance, one may endow \( V \) with an inner product and take \( \mathfrak{t} \) as a subspace of commuting skew-adjoint linear endomorphisms of \( V \). In this case the complexification of \( \mathfrak{a} \) (which we also denote by \( \mathfrak{a} \)) acts in the complexification of \( V, V^\mathbb{C} \), via operators with pure imaginary eigenvalues. The root decomposition induced by \( \rho \) is

\[
V^\mathbb{C} = \sum_{\alpha \in \Pi} V_\alpha,
\]

where \( \Pi \), the root system of \( \mathfrak{t} \), is the set of all linear functions \( \alpha : \mathfrak{t} \to i\mathbb{R} \) such that \( \rho(A)X = \alpha(A)X \) for all \( A \in \mathfrak{t} \) and \( X \in V_\alpha \).

Let \( F \) be a Riemannian foliation and \( i: \mathfrak{t} \subset \hat{\mathfrak{t}} \) be a totally geodesic immersed euclidean space passing through \( z \in \hat{\mathfrak{t}} \) (for instance, if \( \hat{\mathfrak{t}} \) is a compact Lie group, \( T \) can be take as the maximal torus). Here we prove Theorems 1.7 and 1.8.

6.1. Proof of Theorem 1.7

Given \( \xi \in \mathcal{V}_p \), let \( \gamma(s) = \exp(s\xi) \) be a vertical geodesic. Observe that a vector field \( X \) along \( \gamma \) is a basic if and only if \( \nabla_\xi X = A^\xi X \) (20). We verify this equality for the vector \( A^\xi X \).

On one hand, since fibers are totally geodesic, \( \nabla_\xi \) preserves both \( \mathcal{H} \) and \( \mathcal{V} \). On the other hand, if \( X \) and \( Y \) are basic horizontal fields, \( A^\xi X \) is horizontal and

\[
\langle \nabla_\xi (A^\xi X), Y \rangle = \xi \langle A^\xi X, Y \rangle - \langle A^\xi X, \nabla_\xi Y \rangle = - \langle \nabla_\xi (A_X Y), \xi \rangle - \langle A^\xi X, A^\xi Y \rangle = \langle A^\xi A^\xi X, Y \rangle,
\]

since \( A_X Y \) is a Killing field whether \( X, Y \) are basic.

6.2. Proof of Theorem 1.8

Consider \( X, Y \), basic horizontal fields on \( i^*\mathcal{H} \), and \( \xi, \eta \in TT \) commuting parallel vector fields. Since \( T \) is an immersed flat, \( \xi, \eta \) satisfy \( R(\xi, \eta) = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi = 0 \). In particular

\[
\nabla_\eta (A^\xi X) = \nabla_\eta \nabla_\xi X = \nabla_\xi \nabla_\eta X = \nabla_\xi (A^\eta X).
\]

On the other hand,

\[
\langle \nabla_\eta (A^\xi X), Y \rangle = \eta \langle A^\xi X, Y \rangle + \langle A^\eta A^\xi X, Y \rangle = - \langle \nabla_\eta (A_X Y), \xi \rangle + \langle A^\eta A^\xi X, Y \rangle.
\]

Therefore,

\[
0 = \langle \nabla_\eta (A^\xi X) - \nabla_\xi (A^\eta X), Y \rangle = \langle (A^\eta A^\xi - A^\xi A^\eta) X, Y \rangle - \langle \nabla_\eta (A_X Y), \xi \rangle + \langle \nabla_\xi (A_X Y), \eta \rangle.
\]

To prove that \( \langle \nabla_\xi (A_X Y), \eta \rangle = \langle \nabla_\eta \nabla_\xi (A_X Y), \eta \rangle = - \langle R(A_X Y, \xi)\eta, \eta \rangle = 0 \), we first observe that

\[
(15) \quad \xi \eta \langle A_X Y, \eta \rangle = \langle \nabla_\xi \nabla_\eta (A_X Y), \eta \rangle = - \langle R(A_X Y, \xi)\eta, \eta \rangle = 0,
\]

where the second equality follows since \( A_X Y \) is a Killing field and the last since \( R(\eta, \xi) = 0 \). Let \( \gamma(t) = \exp_t(t\xi) \). Equation (15) together with the bound on \( A \) implies that \( \varphi(t) = \langle A_X(\gamma(t)), Y(\gamma(t)), \eta(\gamma(t)) \rangle \) is linear and bounded (note that
\(X, Y, \eta\) have fixed norms). Therefore \(\varphi(t)\) must be constant, thus \(\langle \nabla_\xi(A_XY), \eta \rangle = \xi \langle A_XY, \eta \rangle\) vanishes.

7. Good triples and Totally geodesic foliations on Lie groups

From now on, we specialize to totally geodesic Riemannian foliations on compact Lie groups with bi-invariant metrics. This section has a technical aim: to split \(\mathcal{H}_{\text{id}}\) into two commuting subspaces \(\mathcal{H}_\pm(\mathcal{F})\) which behave as spaces of left and right invariant horizontal fields, respectively (Theorem 7.7). Such aim culminates into the proof of Proposition 1.9 in section 8.

Theorem 1.5 of [19] (Theorem 7.1 below) lay the foundation for Proposition 7.4, the main algebraic identity used in Theorem 7.7. [19, Theorem 1.5] provides a fundamental bracket relation between vertical and horizontal vectors, which is further explored using the root system provided in section 6.

**Theorem 7.1 (Theorem 1.5, [19]).** Let \(G\) be a compact Lie group with a bi-invariant metric and denote its Lie algebra by \(\mathfrak{g}\). The triple \(\{J, V, A\} \subset \mathfrak{g}\) is good if and only if, for all integers \(n, m \geq 0\),

\[
[\text{ad}_J^n B, \text{ad}_V^m \bar{B}] = 0,
\]

where \(B = \frac{1}{2} \text{ad}_V J - A\) and \(\bar{B} = -\frac{1}{2} \text{ad}_V J - A\).

7.1. Decomposition of the Horizontal space at the identity. Theorem 7.1 motivates an interaction between the usual root decomposition in a Lie algebra and the one induced by section 8 which lay the foundation of this section. Here we consider a maximal abelian vertical subalgebra \(t^v \subset \mathcal{V}_{\text{id}}\) completed to a maximal abelian subalgebra \(t = t^v \oplus t^h\). We consider the usual action of \(t\) and an extra action of \(t^h\), together with their root decompositions.

Given a linear map \(\alpha: t^v \to \mathbb{R}\), we call \(X \in g^C, \text{ for all } \xi, \frac{1}{2} \text{ad}_\xi(X) = \alpha(\xi)X\) (respectively if \(A^2(X) = \alpha(\xi)X\)). If \(\alpha\) has a non-trivial weight (respectively, \(A\)-weight) we call it a vertical root (respectively an \(A\)-root), denoting the set of vertical roots as \(\Pi^v(t^v)\) (respectively, the set of \(A\)-roots as \(\Pi^A(t^v)\)). We complete a vertical root to a root \((\alpha, \beta)\) by summing it with a linear function \(\beta: t^v \to \mathbb{R}\). The set of roots of the form \((\alpha, \beta)\) is denoted \(\Pi(t)\). We consider the weight spaces:

\[
\mathfrak{g}_{(\alpha, \beta)}(t) = \{X \in g^C \mid \frac{1}{2} \text{ad}_\xi + \xi^' \text{ad}_\xi, X = (\alpha(\xi) + \beta(\xi')X, \text{ for all } \xi \in t^v, \xi' \in t^h\}
\]

\[
\mathfrak{g}_\alpha(t^v) = \{X \in g^C, \frac{1}{2} \text{ad}_\xi X = \alpha(\xi)X, \text{ for all } \xi \in t^v\}
\]

\[
\mathcal{H}_\alpha(t^v) = \{X \in \mathcal{H}_{\text{id}}^\varepsilon \mid \frac{1}{2} \text{ad}_\xi X = \alpha(\xi)X, \text{ for all } \xi \in t^v\}
\]

taking advantage of a two levels decomposition:

\[
\mathfrak{g} = t^v + \sum_\alpha \mathfrak{g}_\alpha(t^v) = t + \sum_{(\alpha, \beta)} \mathfrak{g}_{(\alpha, \beta)}(t).
\]

Theorem 1.8 guarantees that \(\mathcal{H}_{\text{id}}^\varepsilon = \bigoplus \mathcal{H}_\alpha(t^v)\). The main result is Proposition 7.3 which relates the decompositions in \(A\)-weights with the decomposition in vertical weights. The next Lemma settles the connection between such weights.

**Lemma 7.2.** For any given maximal vertical abelian subalgebra \(t^v\), \(\Pi^v(t^v) = \Pi^A(t^v)\). Moreover, if \(X \in \mathcal{H}_{\text{id}}^\varepsilon\) is an \(\alpha\)-\(A\)-weight then

\[
X = X_\alpha + X_{-\alpha}.
\]
That is $X \in \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$.

**Proof.** Since we are dealing with a totally geodesic Riemannian submersion in a bi-invariant metric, for every $\xi \in \mathcal{V}_{id}$, we have the string of identities (compare \[8\]):

\begin{equation}
- A^\xi A^\xi X = R(X, \xi)\xi = -\frac{1}{4} \text{ad}_{\xi}^2 X
\end{equation}

In particular, if $X$ is an $\alpha$-$A$-weight,

\begin{equation}
(A^\xi)^2 X = \alpha(\xi)^2 X = \frac{1}{4} \text{ad}_{\xi}^2 X.
\end{equation}

for all $\xi \in \mathfrak{t}$. On the other hand, if $\alpha, \alpha' : \mathcal{V}_{id} \to \mathbb{R}$ are linear functionals on $\mathcal{V}_{id}$ such that $\alpha(\xi)^2 = \alpha'(\xi)^2$ for every $\xi$, then $\alpha = \pm \alpha'$. In fact, if we suppose that there exist $\xi_0, \xi_1$ such that $\alpha(\xi_0) = \alpha'(\xi_0) \neq 0$ and $\alpha(\xi_1) = -\alpha'(\xi_1) \neq 0$ then the identity $\alpha(\xi)^2 = \alpha'(\xi)^2$ gives

\begin{align*}
\alpha(\xi_0 + t\xi_1)^2 &= \alpha(\xi_0)^2 + 2t\alpha(\xi_0)\alpha(\xi_1) + t^2\alpha(\xi_1)^2 \\
= \alpha'(\xi_0 + t\xi_1)^2 &= \alpha'(\xi_0)^2 - 2t\alpha(\xi_0)\alpha(\xi_1) + t^2\alpha(\xi_1)^2,
\end{align*}

for all $t$. Which is a contradiction. The statement follows since $\cap_{\xi \in \mathfrak{v}^\perp} \ker(a_{\xi}^2 - \alpha(\xi)^2) = \mathfrak{g}_{\alpha} + \mathfrak{g}_{-\alpha}$. \(\square\)

Given a maximal vertical abelian subalgebra $\mathfrak{t}^v$, Lemma \[7.2\] provides well-defined projection $\pi_{\pm} : \mathcal{H}_{id} \to \mathfrak{g}^c$ defined by sending $X \in \mathcal{H}_\alpha$ to its $\mathfrak{g}_{\pm\alpha}$ component. Let:

\begin{align*}
\mathcal{H}_0(\mathfrak{t}^v) &= \ker \pi_+ \cap \ker \pi_- \\
\mathcal{H}_\pm(\mathfrak{t}^v) &= \pi_{\pm}(\mathcal{H}_{id})
\end{align*}

In particular, $\mathcal{H}_0(\mathfrak{t}^v) \subset \mathcal{H}_0^{c\perp}$, although $\mathcal{H}_\pm(\mathfrak{t}^v)$ are not horizontal. The spaces $\mathcal{H}_\pm(\mathfrak{t}^v)$ correspond to the decomposition of Jacobi fields as left and right invariant fields (see \[19\] Lemma 5.1 and section \[8\]).

We further observe that $\mathcal{H}_0(\mathfrak{t}^v) = \cap_{\xi \in \mathfrak{v}^\perp} \ker \text{ad}_\xi = \cap_{\xi \in \mathfrak{v}^\perp} \ker A^\xi$ since

\begin{align*}
\ker \text{ad}_\xi &= \ker \text{ad}_{\xi}^2 = \ker(\mathcal{A}^\xi)^2 = \ker A^\xi.
\end{align*}

Moreover, since both $A^\xi$ and $\text{ad}_\xi$ are real linear maps (i.e., commute with complex conjugation), $\mathcal{H}_0(\mathfrak{t}^v)$, $\mathcal{H}_+(\mathfrak{t}^v)$, $\mathcal{H}_-(\mathfrak{t}^v)$ are real subspaces, i.e., are complexification of subspaces $\mathcal{H}_0(\mathfrak{t}^v) \subset \mathfrak{g}$, $\epsilon = 0, +, -$. Furthermore, $\mathcal{H}_0(\mathfrak{t}^v) \cap \mathcal{H}_+(\mathfrak{t}^v) + \mathcal{H}_-(\mathfrak{t}^v) = \{0\}$.

We start by characterizing $\mathfrak{t}^v$ as a subset of $\mathcal{H}_0(\mathfrak{t}^v)$.

**Lemma 7.3.** Let $\mathfrak{t}^v$ be a maximal vertical subalgebra and $\mathfrak{t} \supset \mathfrak{t}^v$ a maximal torus. Then, $\mathfrak{t}$ decomposes orthogonally as $\mathfrak{t} = \mathfrak{t}^v \oplus \mathfrak{t}^h$, with $\mathfrak{t}^h \subset \mathcal{H}_0$.

**Proof.** Let $l, t \in \mathfrak{t}$, and decompose $t$ in its vertical and horizontal components $t = t^v + t^h$. On one hand, \[8\] gives $R(t, l) = 0$. On the other hand, \[8\] page 44] gives $R^v(\xi, l)\eta \in \mathcal{V}_{id}$ for all $\xi, \eta, l \in \mathcal{V}_{id}$, thus

\begin{align*}
0 &= R(t, l, \xi, \eta) = R(t^h, l, \xi, \eta) + R(t^v, l, \xi, \eta) = R(t^v, l, \xi, \eta).
\end{align*}

In particular, $\langle R(t^v, l, t^v)^v, l, t^v \rangle = \frac{1}{4}||t^v||^2 = 0$, thus $t^v \in \mathfrak{t}$ and $t^h \in \mathfrak{t}$. Since $t^v$ is maximal and $l$ is arbitrary, $t^v \in \mathfrak{t}^v$. Since $t^v, t \in \mathfrak{t}$, $[t^v, t] = 0$, therefore $t^h \in \mathcal{H}_0$. \(\square\)

Conversely, if $\mathfrak{t}^v$ is a maximal vertical subalgebra, then any maximal abelian subalgebra $\mathfrak{t}' \subset \mathcal{H}_0(\mathfrak{t}^v)$ gives a maximal abelian $\mathfrak{t} = \mathfrak{t}^v + \mathfrak{t}' \subset \mathfrak{g}$. Lemma \[7.3\] is used in the proof of Lemma \[7.0\].
7.2. Bracket identities. We proceed to the main technical result in the section.

**Proposition 7.4.** Let \( t = t^v + t' \) be a maximal torus and \( X \in \mathcal{H}^+(t^v) \), \( Y \in \mathcal{H}^-(t^v) \). Then, for every pair of roots \((\alpha, \beta), (\alpha', \beta')\),

\[
[X_{(\alpha, \beta)}, Y_{(\alpha', \beta')}] = 0.
\]

The first step is to translate Theorem 7.4 to the current language. For the rest of the section, we fix a maximal vertical abelian subalgebra \( t' \) and a complement \( t' \subset H_0(t^v) \).

**Lemma 7.5.** Let \( \xi \in t^v \), \( X \in \mathcal{H}^C_{id} \) and denote by \( X_{e}(t^v) \), \( \epsilon = 0, +, - \), the \( \mathcal{H}_{\epsilon}(t^v) \) component of \( X \). Then, for all \( n, m \geq 0 \),

\[
[\text{ad}^n_{\xi} X_-, \text{ad}^m_{\xi} X_+] = 0.
\]

**Proof.** Let \( \xi \in t^v \) and \( X \in \mathcal{H}^C_{id} \) be formed of non-zero \( \alpha \)-\( \Lambda \) weights, i.e., \( X \in \oplus_{\alpha \neq 0} \mathcal{H}_\alpha(t^v) \). We denote the decomposition of \( X \) into \( \alpha \)-\( \Lambda \) weights as \( X = \sum X_\alpha \), and the projection of each \( \Lambda \) weight into \( \mathcal{H}_{\epsilon}(t^v) \) as \( X_\alpha^\epsilon \). Observe that \( \frac{1}{2} \text{ad}_\xi X_\alpha^\epsilon = \pm \alpha(\xi) X_\alpha^\epsilon \) and \( \mathcal{A}^\epsilon (X_\alpha^- + X_\alpha^+ \) \( = \alpha(\xi) X_\alpha \). Thus, for the good triple \( \{X, \xi, A\} = A^\epsilon X \),

\[
B = \frac{1}{2} \text{ad}_\xi X - A^\epsilon X = \sum_{\alpha \neq 0} \left( \frac{1}{2} \text{ad}_\xi (X_\alpha^- + X_\alpha^+) - A^\epsilon (X_\alpha^- + X_\alpha^+) \right) \sum_{\alpha \neq 0} \alpha(\xi) ((X_\alpha^- - X_\alpha^+) - (X_\alpha^+ + X_\alpha^-)) = -2 \sum_{\alpha \neq 0} \alpha(\xi) X_\alpha = -\text{ad}_\xi X_-
\]

Analogously, \( B = -\text{ad}_\xi X_+ \). \( \square \)

The proof of Proposition 7.4 relies on a computation similar to (19), keeping in mind that \( t' \subset H_0(t^v) \). This computation is made possible by the next Lemma.

**Lemma 7.6.** Given \( X \in \mathcal{H}^C_{id} \), for all integers \( m, n \geq 0 \) and all \( \xi \in t^v \),

\[
[\text{ad}_\xi X_-, \text{ad}^n_{\xi} X_0 X_+] = 0.
\]

**Proof.** The proof is through induction on the index \( s \) on the following identity:

\[
[\text{ad}_\xi X_-, \text{ad}^s_{\xi} X_0 X_+] = 0,
\]

where \( X'_\epsilon = \text{ad}_\xi X_- \) and \( X'_0 = \text{ad}^{s+1}_{\xi} X_+ \). Observe that (18) holds for \( s = 0 \) and \( r \geq 0 \) (Lemma 7.5). As induction hypothesis we assume that (18) holds for \( s \leq k \) and \( r \geq 0 \). We compute \( [\text{ad}_\xi^{k+1} X_-, \text{ad}^s_{\xi} X_0 X_+] \) backwards:

\[
0 = [\text{ad}_\xi^{k+1} X_-, \text{ad}^s_{\xi} X_0 X_+] =
[[X_0, \text{ad}_\xi^{k} X_0 X_+], \text{ad}^s_{\xi} X_0 X_+] + [[X_0, \text{ad}_\xi^{k} X_0 X_+], \text{ad}^s_{\xi} X_0 X_+]
= \text{ad}_\xi^{k} X_0 [\text{ad}_\xi^{k+1} X_-, \text{ad}^s_{\xi} X_0 X_+] + [\text{ad}_\xi^{k+1} X_-, \text{ad}^s_{\xi} X_0 X_+]
+ \text{ad}_\xi^{k} [\text{ad}_\xi^{k} X_0 X_+, \text{ad}^s_{\xi} X_0 X_+] - \text{ad}_\xi^{k} X_0 [\text{ad}_\xi^{k+1} X_-, \text{ad}^s_{\xi} X_0 X_+]
+ [[X_0, \text{ad}_\xi^{k+1} X_0 X_+], \text{ad}^s_{\xi} X_0 X_+] = -[\text{ad}_\xi^{k+1} X_-, \text{ad}^s_{\xi} X_0 X_+]
\]

since, by the induction hypothesis, \( [\text{ad}_\xi^{k} X_-, \text{ad}^s_{\xi} X_0 X_+] = [\text{ad}_\xi^{k} X_-, \text{ad}^s_{\xi} X_0 X_+] = [X_-, \text{ad}^s_{\xi} X_0 X_+] = [X_-, \text{ad}^s_{\xi} X_0 X_+] = 0. \) \( \square \)
Proof of Proposition 7.4. Let $X \in \mathcal{H}^C_{\text{id}}$ be a horizontal field without $\mathcal{H}_0(t^v)$ component and let $t \in t^v < G$. Consider $Z \in \mathcal{H}^C_{\text{id}}$ as $Z = l + X$. Let $X_{\pm}^{\alpha,\beta}$ be the $\mathfrak{g}_{(\alpha,\beta)}$-component of $X_{\pm}$. Replacing $X$ by $Z$ in Lemma 7.3 we have for all $n \geq 0$, $m \geq 1$, $\xi \in t^v$, $t \in t'$:

$$0 = [\text{ad} X_{-}, \text{ad}_{\xi}^{m} X_{+}] = \sum_{(\gamma,\delta) \in \Pi} \gamma(\xi)^m \delta(t)^n \sum_{(\alpha,\beta) \in \Pi} \alpha(\xi)[X_{\alpha,\beta}^{\gamma,\delta}, X_{+}^{\alpha,\beta}].$$

Let $\{\xi_i\}, \{l_i\}$ be bases for $t^v$ and $t'$ where $\alpha(\xi_i) \neq 0$, $\beta(l_i) \neq 0$ whenever $\alpha \beta \neq 0$. Replacing $\xi, l$ by $\xi_i, l_i$ and taking values enough of $m, n$, we conclude that

$$\sum_{(\alpha,\beta) \in \Pi} \alpha(\xi)[X_{\alpha,\beta}^{\gamma,\delta}, X_{+}^{\alpha,\beta}] = 0$$

for every $(\gamma, \delta) \in \Pi$. On the other hand $[X_{\alpha,\beta}^{\gamma,\delta}, X_{+}^{\alpha,\beta}] \in \mathfrak{g}_{(\alpha+\gamma,\beta+\delta)}$, thus each term in the sum (20) lies in a different weight space concluding that $[X_{\alpha,\beta}^{\gamma,\delta}, X_{+}^{\alpha,\beta}] = 0$ for all $(\alpha, \beta), (\gamma, \delta) \in \Pi$.

From now on, we observe that we can assume $G$ is a simple group. If not, we consider the projection of each element $X_{\alpha,\beta}^{\gamma,\delta}$ into simple components of $G$. In any case, we have that the brackets $[., .] : \mathfrak{g}_{(\alpha,\beta)} \times \mathfrak{g}_{(\alpha',\beta')} \rightarrow \mathfrak{g}_{(\alpha+\alpha',\beta+\beta')}$ is either zero, when $\mathfrak{g}_{(\alpha+\alpha',\beta+\beta')} = \{0\}$, or induces a non-degenerate bi-linear pairing, when $\mathfrak{g}_{(\alpha+\alpha',\beta+\beta')} \neq \{0\}$.

Let $\pi_{(\alpha,\beta)} : \mathcal{H}^C_{\text{id}} \rightarrow \mathfrak{g}_{(\alpha,\beta)}$ be the linear projection into $\mathfrak{g}_{(\alpha,\beta)}$ and let $\pi_{\pm}(\alpha,\beta) = \pi(\alpha,\beta) \circ \pi_{\pm}$.

We prove that whenever $(\alpha + \alpha', \beta + \beta')$ is a root, one of the two projections $\pi_{(\alpha,\beta)}, \pi_{(\alpha',\beta')}$ is zero. Thus $[\pi_{(\alpha,\beta)}(\mathcal{H}_+(t^v)), \pi_{(\alpha',\beta')}(\mathcal{H}_-(t^v))] = \{0\}$ for every pair $(\alpha, \beta), (\gamma, \delta)$.

Suppose that $(\alpha + \alpha', \beta + \beta')$ is a root. Then the pairing $[., .] : \mathfrak{g}_{(\alpha,\beta)} \times \mathfrak{g}_{(\alpha',\beta')} \rightarrow \mathfrak{g}_{(\alpha+\alpha',\beta+\beta')}$ is non-degenerate. Since $[\pi_{(\alpha,\beta)} (X), \pi_{(\alpha,\beta)} (X)] = 0$ for every $X \in \mathcal{H}$, $\mathcal{H} = \ker \pi_{(\alpha,\beta)} \cup \ker \pi_{(\alpha',\beta')}$, which is only possible if one of the kernels is $\mathcal{H}$. □

7.3. The left-right horizontal splitting. In this section we refine the $\mathcal{H}$-splitting to be independent of $t^v$. By taking advantage of Theorem 1.5 we assume that $G$ is simply connected and each leaf is an irreducible symmetric space.

Let $H < G$ be the maximal connected subgroup of $G$ whose adjoint representation leaves $V$ invariant, i.e., $h \in H$ if and only if $\text{Ad}_h (V) = V$. Here we fix an initial maximal abelian subalgebra $t$ (containing a maximal vertical abelian subalgebra $t^v$) and consider the family of maximal (respectively, maximal vertical) abelian subalgebras $\text{Ad}_h t$ ($\text{Ad}_h t^v$). We recall a few points:

1. Every maximal vertical subalgebra is of the form $\text{Ad}_h t^v$ (see [4])
2. By denoting $(\alpha \circ \text{Ad}_h^{-1}, \beta \circ \text{Ad}_h^{-1}) = h^*(\alpha, \beta)$,
   $$\Pi(\text{Ad}_h t) = \{ h^*(\alpha, \beta) \mid (\alpha, \beta) \in \Pi(t) \}$$
   which we denote by $h^* \Pi(t)$
3. $\mathfrak{g}_{h^*(\alpha,\beta)}(\text{Ad}_h t) = \text{Ad}_h (\mathfrak{g}_{(\alpha,\beta)}(t))$

Define the vector spaces $\mathcal{H}_{\pm}(F) = \sum_{h \in H} \mathcal{H}_{\pm}(\text{Ad}_h t^v)$ and $\mathcal{H}_0(F) = \mathcal{H}^C_{\text{id}} \cap (\mathcal{H}_+(F) + \mathcal{H}_-(F))^\perp$. The main result of section 7 is the next theorem.

Theorem 7.7. Suppose the leaves of $F$ are locally isometric to a Lie group or to an irreducible symmetric space. Then $\mathcal{H}_+(F) \perp \mathcal{H}_-(F)$ and $[\mathcal{H}_+(F), \mathcal{H}_-(F)] = 0$. 

Theorem 7.7 follows from Proposition 7.4 and Proposition 7.8 below.

**Proposition 7.8.** Let \( t^v \) be a maximal vertical torus and \( h \in H \). Then \( \text{Ad}_h t^v = \text{Ad}_h (t^v) \).

The proof of Proposition 7.8 takes advantage of Proposition 7.4 to control the set of \( \text{Ad}_h \)-t-roots. We proceed with three Lemmas.

Denote by \( \Upsilon_\pm(t) \) the set of roots that appear as components of elements in \( \mathcal{H}_\pm(t^v) \). Since \( \mathcal{H}_{id} \) is isomorphic to \( \mathcal{H}_{id} \), it possesses a natural complex conjugation. \( \mathcal{H}_\pm(t^v) \) is closed by such conjugation, i.e., if \( Z \in \mathcal{H}_\pm(t^v) \), then \( \bar{Z} \in \mathcal{H}_\pm(t^v) \).

In particular, if \( (\alpha, \beta) \in \Upsilon_\pm(t) \) then \( (-\alpha, -\beta) \in \Upsilon_\pm(t) \).

**Lemma 7.9.** \( \Upsilon_+(t) \cap \Upsilon_-(t) = \emptyset \).

**Proof.** Recall that, given a set of positive roots \( \Pi^+(t) \), there is a basis \( \{H_{(\alpha, \beta)}\} \) of \( \mathfrak{t} \), parametrized by \( (\alpha, \beta) \in \Pi^+(t) \), satisfying

\[
[X_{(\alpha, \beta)}, Y_{(-\alpha, \beta)}] = \langle X_{(\alpha, \beta)}, Y_{(-\alpha, \beta)} \rangle H_{(\alpha, \beta)}.
\]

According to Proposition 7.4 \( [X_+, Y_-] = 0 \) for all pairs \( X_+ \in \mathcal{H}_+(t^v), Y_- \in \mathcal{H}_-(t^v) \).

Suppose that \( (\alpha, \beta) \) appears as a component of \( X_+ \). The \( t \)-component of \( [X_+, Y_-] \) satisfies

\[
0 = [X_+, Y_-]^t = \sum_{(\gamma, \delta) \in \Pi^+(t)} \langle X_{(\gamma, \delta)}, Y_{(-\gamma, -\delta)} \rangle H_{(\gamma, \delta)}.
\]

Since the set \( \{H_{(\gamma, \delta)}\} \) is linearly independent and \( \langle , \rangle : \mathfrak{g}(\gamma, \delta) \times \mathfrak{g}(-\gamma, -\delta) \to \mathbb{C} \) is non-degenerate, \( Y_{(-\alpha, -\beta)} = 0 \).

**Lemma 7.10.** For every \( g \in H \), \( \mathcal{H}_0(\text{Ad}_g t^v) = \text{Ad}_g \mathcal{H}_0(t^v) \).

**Proof.** \( \mathcal{H}_0(t^v) = \mathcal{H} \cap \xi \in t^v \ker \text{ad}_\xi \). Therefore,

\[
\text{Ad}_g \mathcal{H}_0(t^v) = (\text{Ad}_g \mathcal{H}) \cap (\text{Ad}_g \ker \text{ad}_\xi) = \mathcal{H} \cap \ker \text{ad}_{\text{Ad}_g \xi} = \mathcal{H} \cap \ker \text{ad}_\xi \cap \xi \in \text{Ad}_g t^v.
\]

Let \( \Upsilon(t) \) be the set of roots that appears as components of elements in \( \mathcal{H} \). Since \( \text{Ad}_g \) fixes \( \mathcal{V}_{id} \), \( \text{Ad}_g \mathcal{H}_{id} = \mathcal{H}_{id} \) and \( g^* \Upsilon(t) = \Upsilon(\text{Ad}_g t) \). Furthermore, since \( \mathcal{H}_0(\text{Ad}_g t^v) = \text{Ad}_g \mathcal{H}_0(t^v) \), for any \( X \in \oplus_{\alpha \neq \beta} \mathcal{H}_{(\alpha, \beta)}(t^v) \), \( \text{Ad}_g X \in \oplus_{\alpha \neq \beta} \mathcal{H}_{(\alpha, \beta)}(\text{Ad}_g t^v) \).

In particular, if \( (\alpha, \beta) \in \Upsilon_+(t) \cup \Upsilon_-(t) \), then \( g^*(\alpha, \beta) \in \Upsilon_+(\text{Ad}_g t) \cup \Upsilon_-(\text{Ad}_g t) \). That is \( \Upsilon_+(\text{Ad}_g t) \cup \Upsilon_-(\text{Ad}_g t) = g^*(\Upsilon_+(t) \cup \Upsilon_-(t)) \). We refine this identity in the next Lemma.

**Lemma 7.11.** For every \( g \in H \), \( \Upsilon_\pm(\text{Ad}_g t) = g^*(\Upsilon_\pm(t)) \).

**Proof.** Given \( (\alpha, \beta) \in \Upsilon_+(t) \cup \Upsilon_-(t) \), consider \( H^\pm_{(\alpha, \beta)} = \{g \in H \mid g^*(\alpha, \beta) \in \Upsilon_\pm(\text{Ad}_g t)\} \). \( H^\pm_{(\alpha, \beta)} \cap H^\mp_{(\alpha, \beta)} = \emptyset \) since \( \Upsilon_+(\text{Ad}_g t) \cap \Upsilon_-(\text{Ad}_g t) = \emptyset \) (Lemma 7.9).

Moreover, \( H^+_+(\alpha, \beta) \cup H^-_{(\alpha, \beta)} = H \) since \( g^* \Upsilon(t) \subset \Upsilon(\text{Ad}_g t) \) for all \( g \in H \). Since \( H \) is connected, the proof is completed by showing that \( H^+_+(\alpha, \beta) \) and \( H^-_{(\alpha, \beta)} \) are open subsets.

**Claim 7.12.** The rank of \( \mathcal{H}_\pm(\text{Ad}_g t^v) \) does not depend on \( g \).

**Proof.** For the set \( \mathcal{H}_\alpha(t^v) \), of \( \alpha \)-\( \mathcal{A} \)-weights, observe that

\[
\bigcap_{\xi \in t^v} \ker((A^t)^2 - \alpha(\xi)^2) = \mathcal{H}_\alpha(t^v) \oplus \mathcal{H}_{-\alpha}(t^v).
\]
On the other hand, \((A^d)\xi)^2 = \frac{1}{4} \text{ad}_\xi^2 = \frac{1}{4} \text{Ad}_\xi^2 = \text{Ad}_\xi(A\xi)^2\). Thus
\[
(\text{Ad}_\xi t)(A\xi(t)) = \frac{1}{4} \text{ad}_\xi(\text{Ad}_\xi t)(A\xi(t)) = \frac{1}{4} \text{Ad}_\xi(\text{Ad}_\xi t)(A\xi(t))^2.
\]
Furthermore, since \(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t)^0\) only involves components with vertical roots \(\alpha\) and \(-\alpha\), for any \(\xi \in t^0\) such that \(\alpha(\xi) \neq 0\),
\[
(\alpha(\xi) + \frac{1}{2} \text{ad}_\xi)(\mathcal{H}_\alpha(t)) = \pi^+(\mathcal{H}_\alpha(t))^0 \oplus \pi^-(\mathcal{H}_{-\alpha}(t)),
\]
(21) \[
(\alpha(\xi) - \frac{1}{2} \text{ad}_\xi)(\mathcal{H}_\alpha(t)) = \pi^-(\mathcal{H}_\alpha(t))^0 \oplus \pi^+(\mathcal{H}_{-\alpha}(t)),
\]
(22) \[
(A^d\xi \pm \frac{1}{2} \text{ad}_\xi)(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t)) = \pi^+(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t)) \oplus \pi^-(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t)).
\]
(23) \[
(A^d\xi \mp \frac{1}{2} \text{ad}_\xi)(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t)) = \pi^-\pi^+(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t)) \oplus \pi^-(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t)).
\]
(24) On one hand, the left hand sides of (21) and (22) are \(\text{Ad}_g\)-equivariant, therefore their ranks are constant with respect to \(g\). On the other hand, the sum of the LHS in (21) with the LHS in (22) coincides with the sum of LHS in (23) with the LHS in (24), i.e., with \(\pi^+(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t)) \oplus \pi^-(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t))\), thus such sums have constant rank in \(g\).
To prove constancy of the rank of \(\pi^\pm(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t))\), observe that the RSHs in (23) and in (24) are images of analytic families of operators with respect to the variable \(g\) — recall that the adjoint representation of a Lie group \(\text{Ad} : g \to GL(g)\) is an analytic map. Therefore, the functions \(k^\pm(g) = \text{rank } \pi^\pm(\mathcal{H}_\alpha(t) \oplus \mathcal{H}_{-\alpha}(t))\) are lower semi-continuous, i.e., if \(k^\pm|_A\) is constant equal to \(c^\pm\), then \(k^\pm|_{A^\perp} \geq c^\pm\). On the other hand, \(k^+(g) + k^-(g) = k\) is constant. Let \(A \subset H\) be the set where \(k^+\) has its minimum value. It is open by lower semi-continuity and it is closed since it coincides with the set where \(k^-\) admits its maximum value. Therefore \(C = H\) and \(k^\pm\) are constant functions.

We now prove that \(H^\pm_{(\alpha,\beta)}\) are open sets for every \((\alpha,\beta)\). Taking advantage of Claim 7.12 we define the following smooth subbundles of \(H \times g\):
\[
\pi^\pm_{(\alpha,\beta)} : \pi^\pm_{(\alpha,\beta)}(t^0) = \{(g,X) \in H \times g \mid X \in \mathcal{H}_{\pm(\text{Ad}_g t)}\} \to H.
\]
Given \((\alpha,\beta) \in \Upsilon_+(t) \cup \Upsilon_-(t)\), define the continuous maps
\[
\pi^\pm_{(\alpha,\beta)} : \pi^\pm_{(\alpha,\beta)}(t^0) \to g
\]
\[
(g,X) \mapsto \pi^\pm_{(\alpha,\beta)}(X).
\]
Observe that \(g^\ast(\alpha,\beta) \in \Upsilon_{\pm(\text{Ad}_g t)}\) if and only if \(\pi^\pm_{(\alpha,\beta)}((g,\mathcal{H}_{\pm}(\text{Ad}_g t)))\) is a non-zero element. That is, \(H^\pm_{(\alpha,\beta)} = \pi^\pm(\pi^\pm_{(\alpha,\beta)})^{-1}(\mathcal{H}_{\pm}(\text{Ad}_g t))\). The proof is concluded observing that \(\pi^\pm_{(\alpha,\beta)}\) are continuous and \(\pi^\pm\) are open maps.

For the proofs of Proposition 7.8 and Theorem 7.7 let \(\mathfrak{h}_{\pm}(t)\) be the subalgebra generated by \((\otimes_{(\alpha,\beta) \in \Upsilon_+(t)}\mathfrak{h}_{(\alpha,\beta)}(t))\). Proposition 7.10 Lemma 7.9 and invariance by complex conjugation guarantees that \([\mathfrak{h}_{\pm}(t),\mathfrak{h}_{\mp}(t)] = 0\), \(\mathfrak{h}_{\pm}(t)\perp \mathfrak{h}_{\mp}(t)\) and \(\mathfrak{h}_{\pm}(t) \cap \mathfrak{h}_{\mp}(t) = \{0\}\). Moreover, Lemma 7.11 implies \(\mathfrak{h}_{\pm}(\text{Ad}_g t) = \text{Ad}_g \mathfrak{h}_{\pm}(t)\).

**Proof of Proposition 7.5**

Let \(\pi_{\pm}(t) : \mathcal{H}_{\text{id}} \to \mathfrak{h}_{\pm}(t)\) be the projections defined by the decomposition \(g = \mathfrak{h}_{\pm}(t) + \mathfrak{h}_{\mp}(t) + \mathfrak{h}_0(t)\), where \(\mathfrak{h}_0(t) = (\mathfrak{h}_{\pm}(t) + \mathfrak{h}_{\mp}(t))^\perp\). Then, for
We have reached a new characterization of the spaces $\mathcal{H}_{\pm}(F)$: given any $t^v$, $\mathcal{H}_{\pm}(t^v)$ is the smallest $\text{Ad}_{t^v}$-invariant subset containing $\mathcal{H}_{\pm}(t^v)$.

### 7.4. Proof of Theorem 7.7

We divide the proof in two cases: whether $\mathcal{V}_p$ is a Lie subalgebra or not. In both cases we reduce the proof to the identity $[\text{ad}_g^k \mathcal{H}_+(t^v), \mathcal{H}_-(t^v)] = \{0\}$ for every $k \in \mathbb{N}, Z \in \mathfrak{h}$. To achieve such reduction, observe that Proposition 7.8 implies that $[\mathcal{H}_+(F), \mathcal{H}_-(F)] = \{0\}$ if and only if $[\text{Ad}_{t^v} \mathcal{H}_+(t^v), \mathcal{H}_-(t^v)] = \{0\}$ for some fixed $t^v$. However, since $\mathfrak{h}$ is compact and $H$ connected, every element in $H$ can be written as $e^Z$ for some $Z \in \mathfrak{h}$. Therefore, $[\text{Ad}_{t^v} \mathcal{H}_+(t^v), \mathcal{H}_-(t^v)] = \{0\}$ if and only if $[\text{ad}_g^k \mathcal{H}_+(t^v), \mathcal{H}_-(t^v)] = \{0\}$.

#### 7.4.1. The subalgebra case.

Assuming that $\mathcal{V}_p$ is a subalgebra, we set $H = \exp(\mathcal{V}_p)$ instead of the original $H$, since, in this case, conjugation by $\exp(\mathcal{V}_p)$ is transitive in the set of maximal vertical tori. In particular,

$$\mathcal{H}_{\pm}(F) = \sum_{g \in H} \mathcal{H}_{\pm}(\text{Ad}_g t^v) = \sum_{V \in \mathcal{V}_p} \mathcal{H}_{\pm}(\text{Ad}_V t^v).$$

Given $\xi \in \mathcal{V}_p$, since $\mathcal{V}_p$ is a subalgebra, $\text{ad}_{\xi}(\mathcal{V}_p) \subset \mathcal{V}_p$. Being $\text{ad}_{\xi}$ a skew-symmetric operator, we conclude that $\text{ad}_{\xi}(\mathcal{H}_p) \subset \mathcal{H}_p$.

**Claim 7.13.** For every $\xi \in \mathcal{V}_p$ and maximal vertical torus $t^v$, $\mathcal{H}_{\pm}(t^v)$ is horizontal and $\text{ad}_{\xi}(\mathcal{H}_{\pm}(t^v)) \subset \mathcal{H}_{\mp}(t^v) \cap \mathcal{H}_p$.

**Proof.** $\mathcal{H}_{\pm}(t^v)$ is horizontal since we can express such subspaces as the image of the sum of two operators that preserves $\mathcal{H}_p$ (equation (23), taking $g = \text{id}$). $\text{ad}_{\xi}$ preserves $\mathcal{H}_{\pm}(t^v)$ since, given $X_+ \in \mathcal{H}_+(t^v)$ and $Y_- \in \mathcal{H}_-(t^v)$, $[\text{ad}_{\xi} X_+, Y_-] = \langle \xi, [X_+, Y_-] \rangle = 0$.

**Claim 7.14.** For any $\xi \in \mathcal{V}_p$, $[\text{ad}_{\xi}^k \mathcal{H}_+(t^v), \mathcal{H}_-(t^v)] = 0$.

**Proof.** We proceed by induction on $k$. From Claim 7.13, we see that $\text{ad}_{\xi} X_+ \in \mathcal{H}_+(t^v) + \mathcal{H}_0(t^v)$. We choose $t'$ to contain $[\text{ad}_{\xi} X_+]_0$, thus, for any $Y_- \in \mathcal{H}_-(t^v)$, $[\text{ad}_{\xi} X_+, Y_-] = ([\text{ad}_{\xi} X_+]_0, Y_-) \in \mathfrak{h}_-(t')$. On the other hand, Jacobi identity gives $[\text{ad}_{\xi} X_+, Y_-] = [\text{ad}_{\xi} Y_-, X_+]$. However, $[\text{ad}_{\xi} Y_-, X_+] = [\text{ad}_{\xi} Y_-, [W_-, X_+]] = 0$ for all $W_- \in \mathfrak{h}_-(t)$. Thus $[\text{ad}_{\xi} X_+, Y_-] = 0$.

For the inductive step, we assume $[\text{ad}_{\xi}^k \mathcal{H}_+(t^v), \mathcal{H}_-(t^v)] = 0$ (in particular, $[\text{ad}_{\xi}^k \mathcal{H}_+(t^v), \mathcal{H}_-(t^v)]$, and take $[\text{ad}_{\xi}^{k+1} X_+]_0 \in t'$. Proceeding as in the last paragraph, we conclude that $[\text{ad}_{\xi}^k \mathcal{H}_+(t^v), \mathcal{H}_-(t^v)] = 0$.

#### 7.4.2. The non-subalgebra case.

According to [3] Lemma 7, if $\mathcal{V}_p$ is not a subalgebra, $\mathfrak{h} \oplus \mathcal{V}_p, \mathfrak{h}$) is a symmetric pair, therefore $\mathfrak{h}$ is horizontal.

Let $Z$ be an element in the Lie algebra of $H$. When $\mathcal{V}_p$ is not a subalgebra, observing that $\mathcal{H}_{\pm}(t^v)$ does not depend on the completion $t'$, we take $Z_0 \in t'$.

Let us conclude that $\mathfrak{h}_+(t) = \text{ad}_Z$-invariant. Decomposing $Z = Z_0 + Z_+ + Z_-$ we conclude that $\mathfrak{h}_+(t)$ is $\text{ad}_{Z_+}$-invariant since $Z_0 \in t$, $\text{ad}_{Z_+}$-invariant since $\mathfrak{h}_+(t)$ is a subalgebra and $Z_+ \in \mathfrak{h}_+(t)$; $\text{ad}_{Z_-}$-invariant since $Z_- \in \mathfrak{h}_-(t)$ and $[\mathfrak{h}_-(t), \mathfrak{h}_+(t)] = \{0\} \subset \mathfrak{h}_+(t)$.
7.5. **Splitting of the dual foliation.** Given $\mathcal{F}$ with irreducible leaves, the decomposition of horizontal vectors with respect to $\mathcal{H}_0(\mathcal{F}) + \mathcal{H}_+(\mathcal{F}) + \mathcal{H}_-(\mathcal{F})$ gives an explicit $A$:

**Corollary 7.15.** Suppose $\mathcal{F}$ has irreducible leaves. Then $A^\xi X = \frac{1}{2} \text{ad}_\xi(X_+ - X_-)$.

**Proof.** Let $t^\prime$ be a maximal abelian vertical subalgebra such that $\xi \in t^\prime$ and consider the decomposition $X = X_0 + \sum_{\alpha \in \Pi^+(t^\prime)} X_\alpha$. Since $\text{ad}_\xi X_0 = 0$, it is sufficient to prove the Corollary when $X$ is an $\alpha$-$A$-weight. Momentarily denote the $\mathcal{H}_\alpha(t^\prime)$ decomposition of $X$ as $X = X'_+ + X'_+ + X'_-$, so that $A^\xi X = \alpha(\xi)X$, $\frac{1}{2} \text{ad}_\xi X'_\pm = \pm \alpha(\xi)X'_\pm$. But then,

$$A^\xi X = \alpha(\xi)(X'_+ + X'_-) = \frac{1}{2} \text{ad}_\xi X'_+ - \frac{1}{2} \text{ad}_\xi X'_-.$$  

The importance of Theorem 7.7 can be summarized in two results: Corollary 7.15 and Theorem 1.9. Corollary 7.15 is used throughout.

As a last step in this section, we reduce Theorem 1.1 to the irreducible case. In section 5, we identify $TL^\#_p$ with $A(\Lambda^2 H_p)$. Here we show that the orthogonal to this image is an ideal (therefore the dual foliation splits whenever $G$ is simply connected).

**Theorem 7.16.** Let $\mathfrak{s} = \{ \xi \in \mathfrak{v}_{id} \mid A^\xi = 0 \}$. Then $\mathfrak{s}$ is an ideal. In particular, if $G$ is simply connected, $G$ is isometric to the metric product $L^\#_{id} \times \exp(\mathfrak{s})$.

**Proof.** Since $\mathfrak{g} = \mathcal{H}_{id} + \mathfrak{v}_{id}$, we prove that bracketing with horizontals and verticals stabilizes $\mathfrak{s}$. Fix a maximal vertical abelian subalgebra $t^\prime$ and observe that $\mathfrak{s} \subset t^\prime$.

Let $\xi \in \mathfrak{s}$ and $X \in \mathcal{H}_{id}$, then $[\xi, X] = 0$ since $\mathcal{H} = \ker(\mathcal{A}) = \ker a^2_\mathcal{H} \cap \mathcal{A}' = \ker a_\mathcal{H} \cap \mathcal{A}'$. In particular, if $\eta \in \mathfrak{v}_{id}$, then $[\xi, \eta] \in \mathfrak{v}_{id}$ $\langle [\xi, \eta], [\xi, \eta] \rangle = 0$. Furthermore, $A^\xi \eta X = 0$ for all $X \in \mathcal{H}_{id}$; let $X = X_0 + X_+ + X_-$ be the $\mathcal{H}(\mathcal{F})$-decomposition. For all $Y \in \mathcal{H}_{id}$,

$$2\langle A^{\xi, \eta} X, Y \rangle = \langle \text{ad}_\xi \text{ad}_\eta - \text{ad}_\eta \text{ad}_\xi(X_+ - X_-), Y \rangle = \langle \text{ad}_\xi \text{ad}_\eta(X_+ - X_-), Y \rangle = -\langle A^\xi X, \text{ad}_\xi Y \rangle = 0.$$  

8. **Totally geodesic foliations on Lie groups**

Here we combine Theorem 7.7 with ideas of [19] section 5 to prove Proposition 1.14. Theorem 1.14 is proved in section 6.

8.1. **The algebra of bounded Jacobi fields.** Let $G$ be a Compact Lie group with bi-invariant metric and $\gamma$ a geodesic with $\gamma(0) = \text{id}$. There are three (usually intersecting) families of Jacobi fields along $\gamma$: the parallel fields, the restriction of left invariant fields and restrictions of right invariant fields. In [18] [19] it is shown that every bounded Jacobi field along $\gamma$ is uniquely expressed as the sum of one element in each family. The aim of this section is to present an analogous decomposition for basic horizontal fields using Theorem 7.8 (Theorem 8.1 and Lemma 8.3). We start recalling the construction in [19], then generalize the decomposition in [19] and extend the bracket identity in Theorem 7.7.

Given $\gamma$, one decomposes $\mathfrak{g}$ as the sum eigenspaces $V_i = \ker(\mathcal{R}_\gamma(0) - k_i)$, where $\mathcal{R}_\gamma(0)X = R(X, \gamma(0))\dot{\gamma}(0)$ and $0 = k_0 < k_1 < ... < k_s$ are the eigenvalues of $\mathcal{R}_\gamma(0)$. 


Then every Jacobi field $J$ can be expressed as

\begin{equation}
J(t) = E_0 + tF_0 + \sum_{i=1}^{s} \cos(t\sqrt{k_i})E_i + \sin(t\sqrt{k_i})F_i,
\end{equation}

where $E_i, F_i$ are parallel fields satisfying $E_i(0), F_i(0) \in V_i$. $J$ is completely defined by its initial conditions $J(0) = \sum_{i=0}^{s} E_i, J'(0) = F_0 + \sum_{i=1}^{s} \sqrt{k_i} F_i$.

A Jacobi field as in (25) has bounded norm if and only if $F_0 = 0$. In this case, $J'(0) \in \oplus_{i=1}^{s} V_i$, thus $\text{ad}_{J(0)}^{-1}(J'(0))$ is well defined. So does

\begin{equation}
J_+ = \frac{1}{2} \left( J(0) - E_0 + \text{ad}_{J(0)}^{-1}(J'(0)) \right), \quad J_- = \frac{1}{2} \left( J(0) - E_0 - \text{ad}_{J(0)}^{-1}(J'(0)) \right).
\end{equation}

Thus the decomposition of $J$ in the three families is given by: $J_0 = E_0$, the parallel field; $J_L$ (respectively $J_R$), the left (respectively right) invariant field with $J_L(0) = J_+$ (respectively $J_R(0) = J_-$). It is straightforward to see that $J(0) = E_0 + J_+ + J_-$ and $J'(0) = J'_L(0) + J'_R(0)$.

When we deal with totally geodesic foliations, basic horizontal fields restrict to bounded (constant norm, actually) Jacobi fields along vertical geodesics, so they can be decomposed accordingly to [19]. The aim of this section is to use the decomposition $\mathcal{H}_{id} = \mathcal{H}_{0}(\mathcal{F}) + \mathcal{H}_{+}(\mathcal{F}) + \mathcal{H}_{-}(\mathcal{F})$ to induce a decomposition on the basic horizontal fields along $\tilde{L}_{id}$.

**Theorem 8.1.** Let $X$ be a basic horizontal field along $\tilde{L}_{id}$ and consider the decomposition $X(\text{id}) = X_0 + X_+ + X_-$, with $X_0 \in \mathcal{H}_e(\mathcal{F})$, $\epsilon = 0, +, -. \quad$ Then $X = X_B + X_L + X_R$, where

1. $X_L, X_R$ are the restrictions of a left invariant field with $X_L(\text{id}) = X_+$ and a right invariant field with $X_R(\text{id}) = X_-$, respectively;

2. $X_B$ is the parallel translation of $X_0$ and can be realized by the restriction of a left invariant field as well as a right invariant field.

**Proof.** Given $X(\text{id}) = X_0 + X_+ + X_-$, the conditions $X_B(\text{id}) = X_0, X_L(\text{id}) = X_+, X_R(\text{id}) = X_-$ completely determine fields $\tilde{X}_B, \tilde{X}_L, \tilde{X}_R$, where $\tilde{X}_B$ is parallel and $\tilde{X}_L, \tilde{X}_R$ are invariant. To show that $X = \tilde{X}_B + \tilde{X}_L + \tilde{X}_R$, it is sufficient to show that $X(e^{\xi}) = \tilde{X}_B(e^{\xi}) + \tilde{X}_L(e^{\xi}) + \tilde{X}_R(e^{\xi})$ for the dense set of $\xi \in \mathcal{V}_{id}$ where $\text{ad}_{\xi} X_\pm \neq 0$. On the other hand, Corollary 7.15 gives

\[ \nabla_{\xi} X(\text{id}) = \mathcal{A}_{\xi} X(\text{id}) = \frac{1}{2} \text{ad}_{\xi} X_+ - \frac{1}{2} \text{ad}_{\xi} X_-.
\]

The result follows from equation (26), suitably changing $E_0$. \hfill \Box

In order to compute the $A$-tensor of $\mathcal{F}$ (taking advantage of Theorem 7.7) we consider the left translation of the Lie bracket as a $(2,1)$-tensor on $G$. To avoid ambiguity, we denote the Lie bracket of $\mathfrak{g}$ as $[,]_{\mathfrak{g}}$ and the usual Lie bracket of vector-fields as $[,].$

Let $X, Y$ be vector fields on $G$. We define $[X, Y]$ to be the vector-field

\[ [X, Y](g) = d_l g[dl^{-1}_g X(g), dl^{-1}_g Y(g)]_g. \]

**Proposition 8.2.** $[,]$ is the only $(2,1)$ tensor that satisfies

\begin{equation}
[X_L, Y_L] = [X_L, Y_L]_X
\end{equation}
for any pair $X_L, Y_L$ of left invariant fields. In particular $[,]$ is parallel, satisfies
Jacobi identity and $\nabla_{X_L} Y_L = \frac{1}{2}[X_L, Y_L]$. Moreover, for any pair $X_R, Y_R$ of right
invariant fields, $[X_R, Y_R] = [X_R, Y_R]_X$ and $\nabla_{X_R} Y_R = -\frac{1}{2}[X_R, Y_R]$.

Proof. $[,]$ is a tensor since it is defined as a fiber-wise bilinear map in a trivialization
$TM \cong M \times \mathfrak{g}$. Equation (27) follows from the well-known identity
$$[X_L, Y_L] = dl_g(dl^{-1}_g X_L(g), dl^{-1}_g Y_L(g)) = dl_g[X_L(id), Y_L(id)]_{\mathfrak{g}}.$$ All other properties follows by computing $[,]$ on left invariant fields. To observe that $[X_R, Y_R] = [X_R, Y_R]_X$, first recall that a vector field $Z$ is right invariant if and
only if $Z(g) = dl_g Ad^{-1}_g Z(id)$. Therefore,
$$[X_R, Y_R]_X = dl_g Ad^{-1}_g[X_R(id), Y_R(id)]_{\mathfrak{g}} = dl_g[Ad^{-1}_g X_R(id), Ad^{-1}_g Y_R(id)]_{\mathfrak{g}}$$
$$= dl_g[dl^{-1}_g X_R(g), dl^{-1}_g Y_R(g)]_{\mathfrak{g}} = [X_R, Y_R].$$

The main result of this section (item (2) of Proposition 8.3) is a direct application of
Theorem 7.7.

Proposition 8.3. Let $X, Y$ be basic horizontal fields with decomposition $X = X_B + X_L + X_R, Y = Y_B + Y_L + Y_R$. Then:
(1) $[X_L, Y_B]$ and $[X_R, Y_B]$ are restrictions of left and right invariant fields, respectively
(2) $[X_R, Y_L] = 0$

Proof. (1) follows from Proposition 8.2 and Theorem 8.1 item (2). For (2), we use
geometric arguments to improve Proposition 8.7.

Lemma 8.4. Let $p^{-1} \in L_{id}$. Then $[\mathcal{H}(F), Ad_p \mathcal{H}(F)]_{\mathfrak{g}} = 0$.

Proof. Given $p \in F$, we consider the translated foliation $F^p = \{lp(L) \mid L \in F\}$. Since $l_p : G \rightarrow G$ is an isometry, we conclude that $F^p$ is a Riemannian foliation
with totally geodesic leaves. Furthermore, its vertical and horizontal spaces at
$pq \in M$ are given by $V^p_{pq} = dl_p(V_q), H^p_{pq} = dl_p(\mathcal{H}_q)$. Therefore, if $X$ is a basic $F$-
horizontal field along $L_{p^{-1}} = L_{id}, dl_p X$ is a basic $F^p$-horizontal field along $l_p(L_{p^{-1}})$. Moreover, if $X = X_B + X_L + X_R$, then $dl_p X = dl_p(X_B) + dl_p(X_L) + dl_p(X_R)$ is one, therefore the only, decomposition of $dl_p(X)$ as a parallel basic horizontal, a left invariant and a right invariant field. Since $dl_p(X_\epsilon(p^{-1}))(0) = X_\epsilon(id)$ for $\epsilon = B, L$ and $dl_p(X_R(p^{-1}))(0) = Ad_p X_-$, we conclude that
$$\mathcal{H}_0(F^p) = \mathcal{H}_0(F), \quad \mathcal{H}_+(F^p) = \mathcal{H}_+(F), \quad \mathcal{H}_-^p = Ad_p \mathcal{H}_-(F).$$ Since $F^p$ satisfies the hypothesis in Proposition 7.7 $[\mathcal{H}_+(F^p), \mathcal{H}_-(F^p)] = 0$. $\square$

With Lemma 8.4, we compute $[X_R, Y_L]_{(p^{-1})}$ directly:
$$[X_R, Y_L](p^{-1}) = dl_{p^{-1}}[dl_p X_R, dl_p Y_L] = dl_{p^{-1}}[Ad_p X_R(id), Y_L(id)]_{\mathfrak{g}}$$
$$= dl_{p^{-1}}[Ad_p X_-, Y_+]_{\mathfrak{g}} = 0,$$
which vanishes by Lemma 8.3. $\square$

Proposition 8.5 is the main step into the proof of Proposition 8.9. Proposition 8.6 is used in section 9.4.

Proposition 8.5. Let $X, Y$ be basic horizontal fields along $L_{id}$. Then $A_X Y = \frac{1}{2}([X_L, Y_L] - [X_R, Y_R])^v$. 
Proof. Let $\xi \in \mathcal{V}_d$ and consider the restriction of $X$ to the geodesic $e^{t\xi}$, so that we can think of $\xi$ as a left invariant field along $e^{t\xi}$. Observe that $\langle A_X Y, \xi \rangle = \langle A^\xi X, Y \rangle = \langle \nabla^\xi Y, X \rangle = \langle \nabla^\xi (X_L + X_R), Y \rangle$, since $X_B$ is parallel. Thus, according to Proposition 8.2

$$\langle A_X Y, \xi \rangle = \frac{1}{2} \langle [\xi, X_L] - [\xi, X_R], Y \rangle = \frac{1}{2} \langle \xi, [X_L, Y] - [X_R, Y] \rangle.$$ 

On the other hand, Proposition 8.3 implies that

$$\frac{1}{3} \langle \xi, [X_L, Y] - [X_R, Y] \rangle = \frac{1}{2} \langle \xi, [X_L, Y_B + Y_L] - [X_R, Y_B + Y_R] \rangle.$$ 

The proposition follows since $X_B, Y_B$ are horizontals and $A$ is skew-symmetric. □

Propositions 8.3 and 8.6 has two main consequences: Propositions 1.9 and 8.6

Proof of Proposition 1.9. We define the auxiliary tensor $\hat{A}_X Y = \frac{1}{2}([X_L, Y_L] - [X_R, Y_R])$. On one hand, $\langle \hat{A}_X Y, \hat{A}_Z W \rangle$ is basic since the mixed terms in

$$2\langle \hat{A}_X Y, \hat{A}_Z W \rangle = \langle ([X_L, Y_L] - [X_R, Y_R]), [Z_L, Z_R] - [W_R, W_R] \rangle$$ 

vanishes by the Jacobi identity of $[,]$ and Proposition 8.3. The $([X_L, Y_L], [Z_L, W_L]), ([X_R, Y_R], [Z_R, W_R])$ are basic since they are the inner product of either right invariant or left invariant fields.

On the other hand, according to Proposition 8.3, the vertical part of $\hat{A}$ coincides with $A$. We conclude the proof by observing that $\hat{A}$ differs from $A$ by a basic horizontal field: recall that a horizontal field is basic if and only if its inner product with basic horizontal fields is basic. Let $Z$ be a basic horizontal field. Then

$$2\langle \hat{A}_X Y, Z \rangle = \langle ([X_L, Y_L] - [X_R, Y_R], Z) \rangle = \langle [X_R, Y_R], Z_B + Z_R - [X_L, Y_L], Z_B + Z_L \rangle,$$

where the second equality follows from Proposition 8.3. The terms $([X_R, Y_R], Z_B + Z_R)$ and $([X_L, Y_L], Z_B + Z_L)$ are constant since they can be realized as the inner products of right, respectively left, invariant fields. Therefore

$$\langle A_X Y, A_Z W \rangle = \langle \hat{A}_X Y, \hat{A}_Z W \rangle - \langle (\hat{A}_X Y)^h, (\hat{A}_Z W)^h \rangle$$

is the difference of two basic functions, so it is basic. □

Proposition 8.6. If $\xi \in \mathcal{V}$ and $X, Y \in \mathcal{H}$, then $\nabla^\xi A = 0$.

Proof. For the computation, (possibly using a translated foliation as in Lemma 8.4) we assume that $\xi, X, Y \in \mathfrak{g}$ and take $t^v$ a maximal vertical subalgebra that contains $\xi$. We compute $\nabla^\xi (A_X Y)$ directly by taking $X$ an $\alpha$-$A$-weight and $Y$ a $\beta$-$A$-weight. According to Propositions 8.3 and 8.2, we have

$$\nabla^\xi (A_X Y) = \frac{1}{2} \nabla^\xi ([X_L, Y_L] - [X_R, Y_R])^v$$

$$= \frac{1}{2} \left( \langle \nabla^\xi X_L, Y_L \rangle + [X_L, \nabla^\xi Y_L] - \langle \nabla^\xi X_R, Y_R \rangle - [X_R, \nabla^\xi Y_R] \right)^v$$

$$= (\alpha(\xi) + \beta(\xi)) \frac{1}{2} ([X_L, Y_L] - [X_R, Y_R])^v.$$ □
9. Proof of Theorem 1.1

Conjecture 1 can be divided in two problems:

**Question 1.** Prove that leaves are (locally isometric to) subgroups.

Once settled Question 1, it is still left to prove that the foliation is homogeneous (in the sense of Section 1).

**Question 2.** Suppose that the leaves of $\mathcal{F}$ are locally isometric to a subgroup. Prove that $\mathcal{F}$ is homogeneous.

We observe that Question 2 is not straightforward. Assuming that leaves are subgroups, it was settled in Jimenez [12, Corollary 24] for the submersion case. To settle Question 2, Jimenez uses [12, Theorem 23] which requires the existence of a special algebra of vector fields. The strategy in section 4.2 was to produce such an algebra using the triviality of $H_{id}(\mathcal{F})$. However, the triviality of $H_{id}(\mathcal{F})$ cannot be sufficient: the Gromoll–Meyer fibration $Sp(2) \to \Sigma^7$, [7], is a principal bundle which is not isometric to an homogeneous foliation (although it posses a totally geodesic fiber $SU(2)$ passing through the identity).

This section combine the algebraic and geometric results so far to show that: (1) $H_{id}(\mathcal{F}) = \{id\}$; (2) the field $\chi(q, \xi_0)$ in section 4.2 is a constant length Killing field for every $\xi_0 \in V_{id}$. Item (1) shows that $S^7$ can not be a factor in the decomposition of the leaf (answering question one) and proves that $\mathcal{F}$ is principal. Knowing that $\mathcal{F}$ is principal, (2) guarantees that the group action is either by right or left invariant fields on each simple component of $G$, completing the proof (in the simply connected case. When $G$ is not simply connected, it is sufficient to work on the universal cover – section 9.3).

9.1. Ruling out the 7-sphere. The main difficult to prove Conjecture 1 is to control the leaf type (the arguments in [12] could be adapted to foliations). We know that the leaves of $\mathcal{F}$ must be locally symmetric spaces, since they are (immersed) totally geodesic submanifolds of a symmetric space. Theorem 1.2 and Proposition 1.9 show that the typical leaf has the local Killing property. Here we prove $S^7$ can not appear as a factor in the leaf. For this aim, we show that $H_{id}(\mathcal{F}) = \{id\}$. In particular, $Hol_{id}(\mathcal{F})$ is a transitive group of isometries acting locally free on $L_{id}$.

**Theorem 9.1.** $\tilde{L}_{id}$ is a Lie group with a bi-invariant Riemannian metric.

The argument follows by contradiction. Using Theorem 1.5 and 7.16 we assume that $\mathcal{F}$ is irreducible and its leaves are locally isometric to a round 7-sphere. We proceed with the following argument:

We compute $\mathcal{A}$, the O’Neill $A$-tensor of the foliation $\tilde{\mathcal{F}}$ on $Aut(\mathcal{V})$ ($A_{XY}(h)$ can be understood as the 1-jet extension of the vector-field $A_{X}Y$ along $L_{p}$, $p = \tau(h)$). Here we identify $\tilde{\mathcal{H}}$ with $\mathcal{H}$ via $d\tau$ and prove that $\nabla_{\mathcal{A}}X = 0$ for all $X, Y \in \mathcal{H}$. Following arguments in section 5 we show that the association $h \mapsto \mathcal{A}(\Lambda^{2}\tilde{\mathcal{H}}_{h})$ is invariant under infinitesimal holonomy diffeomorphisms. Recalling that $\mathcal{E}_{h}$ is the dual leaf of $\tilde{\mathcal{F}}$, Theorem 1.2 gives $T_{h}\mathcal{E}_{h} = \mathcal{A}(\Lambda^{2}\tilde{\mathcal{H}}_{h})$ for all $h$. On one hand, Theorem 1.3 gives

$$hol_{p}(\mathcal{F}) = \mathcal{A}(\Lambda^{2}\tilde{\mathcal{H}}_{h})$$

(28) $$\cong \text{span}\{A_{XY} \in \mathcal{X}(L_{p}) \mid X, Y \in \mathcal{X}(L_{p}), \ X, Y \text{ basic horizontals}\}.$$
where the Lie algebra of $\text{Hol}_p(F)$ and the last isomorphism follows since Killing fields are completely determined by their 1-jet extension. On the other hand, Proposition 1 implies that evaluation at $h$ induces an isomorphism from \(\mathcal{H}_p\) to $A(\Lambda^2\mathcal{H}_p)$. In particular $\text{Hol}_p(F)$ acts freely and transitively on $\tilde{L}_h$. The main step in the proof is to conclude:

**Proposition 9.2.** Let $p \in G$. Then $T_{\text{id}}\mathcal{E}_p = A(\Lambda^2\mathcal{H}_p)$.

Its proof depends on Lemmas 9.3-9.8. Although we assume $F$ a totally geodesic Riemannian foliation on a bi-invariant metric, Lemma 9.3 applies generally. We consider $\text{Aut}_p(V)$ with the Riemannian metric

$$\langle X + \zeta, Y + \chi \rangle_{\tau} = \langle drX, drY \rangle + \sum_i \langle \omega(\zeta)\xi_i, \omega(\chi)\xi_i \rangle,$$

where $\omega : T\text{Aut}(V) \to \mathfrak{gl}(V)$ is the connection 1-form defined in Lemma 3.9 and $\{\xi_i\}$ is an orthonormal basis for $V$. Given $\zeta \in \hat{V}$, we uniquely decompose $\zeta = \zeta^M + \zeta^\omega$ where $\omega(\zeta^M) = 0$, i.e., $\zeta^M$ is obtained by recalling that $d\tau_{\text{id}}|_{\ker \omega}$ is an isomorphism. We use the principal structure of $\text{Aut}_p(V)$ to identify the set of $\eta^\omega$ components, $\hat{V}^\omega$, as the subset

$$\hat{V}^\omega_h = h \text{End}(V_p) = \{hh' \in \text{End}(V_p, V_{\text{r}(h)}) \mid h' \in \text{End}(V_p)\}.$$

$V^\omega$ is the set spanned by the action field of the principal $GL(V_p)$-action on $\text{Aut}_p(V)$.

**Lemma 9.3.** Suppose that $F$ has totally geodesic leaves and that $\zeta$ is a $\hat{F}$-holonomy field along a $\hat{F}$-horizontal curve $\hat{c}$. Then both $\zeta^M$ and $\zeta^\omega$ are holonomy fields. Furthermore:

(i) $\zeta^M(t)$ is a holonomy field along $c = \tau \circ \hat{c}$

(ii) $\zeta^\omega$ is an $H_p(F)$-action field. In particular, for every $\xi_0 \in V_p$, $\zeta^\omega(t)\xi_0$ is a holonomy field along $c$.

**Proof.** Let $\hat{V}^M$ be the space spanned by the $\zeta^M$ components of $\hat{V}$. Since $V^M = \ker \omega \cap V$ and $V^\omega = \ker dr$, we have $\langle V^\omega, V^M \rangle = 0$. Given $\hat{c}$, using the initial data we conclude that the space of holonomy fields along $\hat{c}$ splits into two subspaces: the restriction of the $GL(V_p)$-action fields and the fields with initial data in $\hat{V}^M$. We use the totally geodesic condition to show that a field with initial data in $\hat{V}^M$ stays in $\hat{V}^M$ along the entire geodesic.

For simplicity, by restricting $F$ to a tube along $c$ (as in Lemma 2.1), we assume that holonomy diffeomorphisms are well defined between leaves. Let $\varphi_{\hat{c}}$ be the $\hat{F}$-holonomy diffeomorphism defined by $\hat{c}$.

**Claim 9.4.** $\varphi_{\hat{c}}(h) = d\phi_{\hat{c}}h$.

**Proof.** Given $q \in L_c(0)$, denote by $c^\theta$ the $H$-horizontal curve induced by $\phi_{\hat{c}}$, i.e., $c^\theta(t) = \phi_{\hat{c}}(q)$. Recall that the holonomy field $\xi$ along $c^\theta$ is given by $(d\phi_{ct})_q(\xi(0))$. Therefore $(\tilde{c}^\theta)_h = d\phi_{\hat{c}}h$. On the other hand, both $t \mapsto \varphi_{\hat{c}}(h)$ and $(\hat{c}^\theta)_h = d\phi_{\hat{c}}h$ must be the only $H$-horizontal curve starting at $h$ that $\tau_{\text{id}}$-projects to $c^\theta(h)$. Thus $\varphi_{\hat{c}}(h) = (\tilde{c}^\theta)_h = d\phi_{\hat{c}}h$.

Since $F$ has totally geodesic leaves, $\phi_{\hat{c}}$, the $\hat{F}$-holonomy diffeomorphism defined by $c$, is an isometry. Let $\gamma$ be a vertical geodesic in $G$. Its $H$-horizontal lift at $h \in \text{Aut}(V)_p$ is given by the curve $\tilde{\gamma}_h(s) = P_{\gamma}(s)h$, where $P_{\gamma}(s) : V_{\gamma(0)} \to V_{\gamma(s)}$ is the parallel transport along $\gamma$. Since $\phi_{\hat{c}}$ is an isometry, it sends parallel transport
to parallel transport, more precisely, $P_{\phi_h}(s)d\phi_h = d\phi_h P_\gamma(s)$. In particular, $\varphi_{\tilde{t}}$ sends the $H$-horizontal lift of $\gamma$ at $h$ to the $H$-horizontal lift of $\phi_h \gamma$ at $d\phi_h h = \varphi_{\tilde{t}}(h)$.

**Lemma 9.5.** Let $X \in \mathcal{H}_h$, $\zeta \in \mathcal{V}_h$ and $\{\xi_i\}$ be an orthonormal basis of $\mathcal{V}_{r(h)}$. Then $A^*_{X}\zeta = A^*_{X}\zeta^M + \sum R(\zeta\xi_i, \xi_i) X$.

**Proof.** Using basic horizontal fields on Lemma 3.10 and adding the equality on Corollary 8.6, we conclude that

$$\omega(A_X Y) : \xi \mapsto A_{A_X Y} + A_X A^i Y.$$ 

By extending the inner-product $\langle , \rangle$ On the other hand, the trace norm in $\text{Aut}(\mathcal{V})_p$ gives

$$\langle A_X Y, \zeta \rangle = \langle A_X Y, d\tau \zeta \rangle + \sum \langle\omega(A_X Y)\xi_i, \zeta \xi_i \rangle$$

$$= \langle A_X \zeta^M, Y \rangle + \sum \langle A_{A^i, X} Y + A_X A^i Y, \zeta \xi_i \rangle$$

$$= \langle A_X \zeta^M, Y \rangle + \sum \langle [A^i \zeta, \zeta^i, X, Y \rangle$$

Corollary 8.6 together with O'Neill formulas (see [8, page 44]) gives $[A^i \zeta, \zeta^i, X, Y \rangle$.

Therefore, $A^i \zeta = 0$ for all $X$ if and only if

$$[A^i \zeta, \zeta^i, X, Y \rangle = 0.$$ 

In the next lemma resides the core of the contradiction argument.

**Lemma 9.6.** Assume that $\mathcal{F}$ is irreducible and that the leaves of $\mathcal{F}$ are locally isometric to round 7-spheres. Suppose $\zeta \in \mathcal{V}_h$ satisfies $A^*_{X}\zeta = 0$ for all $X \in \mathcal{H}_{id}$. Then $A^*_{X}\zeta^M = 0$ for all $X \in \mathcal{H}_{r(p)}$ and $\sum R(\zeta\xi_i, \xi_i) = 0$.

**Proof.** By possibly translating $\zeta\xi_i, \xi_i$ to id, we have $R(\zeta\xi_i, \xi_i) X = -\frac{1}{4} \text{ad}_{[\zeta\xi_i, \xi_i]} X$. Therefore, $A^*_{X}\zeta = 0$ for all $X$ if and only if

$$A^*_{X}\zeta^M = \frac{1}{4} \text{ad}_{[\zeta\xi_i, \xi_i]} X$$

(recall that on totally geodesic foliations, $\langle R(\mathcal{V}, \mathcal{V})\mathcal{H}, \mathcal{V} \rangle = 0$, thus $\text{ad}_{[\zeta\xi_i, \xi_i]} X \in \mathcal{H}$.) We argue that (29) is only possible if $\zeta^M = \sum [\zeta\xi_i, \xi_i]_\theta \neq 0$. According to [3, Lemma 7], $[\mathcal{V}_{id}, \mathcal{V}_{id}]_\theta + \mathcal{V}_{id}, [\mathcal{V}_{id}, \mathcal{V}_{id}]_\theta$ must be isomorphic to the symmetric pair $(\mathfrak{so}(8), \mathfrak{so}(7))$, satisfying the identities

$$[\mathfrak{e}, \mathfrak{e}] \subset \mathfrak{e}, \quad [\mathfrak{e}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] = \mathfrak{e},$$

for $\mathfrak{m} = \mathcal{V}_{id}$. Therefore, there is a Lie algebra isomorphism of the pair $(\mathfrak{e} + \mathfrak{m}, \mathfrak{e})$ to $(\mathfrak{so}(8), \mathfrak{so}(7))$, sending $\mathfrak{e}$ to the set of skew-symmetric matrices with vanishing first column and $\mathfrak{m} = \mathcal{V}_{id}$ to its orthogonal complement

$$\mathfrak{so}(7)^+ = \{e_0 \wedge w \mid w \in e_0^\perp\},$$

where $\{e_0, ..., e_7\}$ is a basis for $\mathbb{R}^8$ and, given $v, w \in \mathbb{R}^8$, $v \wedge w$ is the endomorphism

$$v \wedge w(z) = \langle v, z \rangle w - \langle w, z \rangle v.$$ 

For simplicity we denote $\zeta^M = \xi \in \mathcal{V}_{id}$ and $\sum [\zeta\xi_i, \xi_i]_\theta = A \in \mathfrak{e}$. Putting together (10) and (29), we have $\text{ad}_{\xi}^2 B = \text{ad}_{A}^2 B$ for all $B \in \mathfrak{so}(7)$. We assume the contradicting hypothesis that $\xi \neq 0$ and show that $\text{ad}_{\xi}^2$ can not be realized by an endomorphism of the form $\text{ad}_A^2$. Assuming $\xi \neq 0$, up to isomorphism (and dividing $\xi$ by its
norm), we can assume \( \xi = e_0 \wedge e_1 \). For every \( B \in \mathfrak{so}(7) \), \( Be_0 = 0 \), \( Be_1 \in \{e_0, e_1\} \),

On the other hand \( -\xi^2 \) is the orthogonal projection to \( \text{span}\{e_0, e_1\} \). Thus,

\[
(30) \quad [\xi, [\xi, B]] = \xi^2 B + B\xi^2 = -e_1 \wedge (Be_1),
\]

which is minus the orthogonal projection from \( \mathfrak{so}(7) \) to the space \( V = \text{span}\{e_1 \wedge e_2, ..., e_1 \wedge e_7\} \), when we consider the Cartan-Killing metric on \( \mathfrak{so}(7) \).

For \( A \), we have

\[
[A, [A, v \wedge w]](z) = \langle v, z \rangle A^2 w - \langle w, z \rangle A^2 v + 2(\langle Av, z \rangle Av - \langle Aw, z \rangle Av) + \langle A^2 v, z \rangle w - \langle A^2 w, z \rangle v.
\]

for all \( v, w, z \in \mathfrak{e} \). Assuming \( v \perp w \), for \( z = v \) we have

\[
(31) \quad [A, [A, v \wedge w]](v) = A^2 w - 2 \langle Av, v \rangle Av + \langle A^2 v, v \rangle w - \langle A^2 w, v \rangle v.
\]

From \((30)\) we conclude that \((31)\) must vanish whenever \( v, w \perp e_1 \). Thus \( A^2 w \in \text{span}\{Av, w\} \) for all \( v, w \perp e_0, e_1 \).

Claim 9.7. \( \{e_0, e_1\} \perp \ker A \).

Proof. From \( A^2 w = 2 \langle Av, v \rangle Av - \langle A^2 v, v \rangle w \), we conclude that \( A^2 w = \lambda_w \) \( w \) for all \( w \), where \( \lambda_w \) can be computed as \( \lambda_w = -\langle A^2 v, v \rangle \) for any \( v \perp \{e_0, e_1, Av\} \) (which always exist by the dimension of the space). In particular, every non-zero vector in \( \{e_0, e_1\} \perp \ker A \) is an eigenvector for \( A^2 \), thus \( A^2 \) must be diagonal. On the other hand \( \lambda_w = \langle A^w, w \rangle \leq 0 \) since \( A \) is skew symmetric, but \( \lambda_w = -\langle A^2 v, v \rangle = |Av|^2 \geq 0 \).

Thus \( \lambda_w = 0 \) for all \( w \perp e_0, e_1 \).

We conclude that \( A = 0 \) by recalling that \( A \) is a skew-symmetric operator in \( \mathfrak{e}_0 \perp \ker A \) and must have even co-dimensional kernel. Therefore \((A, \mathfrak{e}_0) = e_0 \perp \ker A \) and \( \lambda_0^2 = \lambda_0^2 = (A^0)^2 = 0 \). Since \( \ker A^0 = \ker (A^0)^2 \) and \( \mathcal{F} \) is irreducible, Theorem 1.6 gives \( \lambda = \gamma \).

Lemma 9.8. If \( \xi, \eta \) are holonomy fields along a horizontal curve \( \gamma \), then \( \sum \xi \eta \) has constant length.

Proof. Note that \( ||\sum \xi \eta||^2 = 4 \sum_{i,j} R(\xi, \eta, \xi, \eta) \). But

\[
\frac{d}{dt} R(\xi, \eta, \xi, \eta) = R(A^0 X, \xi, \eta, \xi) + R(\xi, A^0 X, \eta, \xi) + \ldots
\]

Which vanishes since \( R(\mathcal{V}, \mathcal{Y}) = 0 \) is a Riemannian foliation with totally geodesic leaves (see 4 page 44)).

Proof of Proposition 9.2. Let \( \zeta \in (A(A^0 \mathcal{H})) \perp \). Following the proof of Theorem 1.6 it is sufficient to show that the holonomy field \( \zeta(t) \) with initial condition \( \zeta(0) = \zeta \) satisfies \( \mathcal{A}_\mathcal{Y}(t) \zeta(t) = 0 \), for every horizontal geodesic \( \gamma \) and horizontal field \( Y(t) \).

Let \( \zeta = \gamma \). From Lemma 9.6 we know that \( \gamma \) \( \gamma \). Since \( \mathcal{A}_\mathcal{Y}(t) \) is a holonomy field (Lemma 9.3), \( \gamma \) \( \gamma \) vanishes identically. Lemma 9.5 now gives

\[
\mathcal{A}_\mathcal{Y}(t) \zeta \gamma = \sum_{i,j} R(\xi_i(t), \xi_j(t)) Y(t) = -\frac{1}{4} \text{ad}_{\sum \xi_i(t), \xi_j(t)} Y(t).
\]

Since \( \sum \xi_i(t), \xi_j(t) = 0 \) (Lemma 9.6) and \( \xi_i(t), \xi_j(t) \) are holonomy fields (Lemma 9.5), we have \( \sum \xi_i(t), \xi_j(t) = 0 \) for all \( t \) (Lemma 9.8). Therefore \( \mathcal{A}_\mathcal{Y}(t) \zeta \) vanishes along \( \gamma \).
9.2. **Foliations whose leaves are (locally) isometric to subgroups.** In section 9.1 we prove that the only irreducible factors of $L_{id}$ are abelian or compact simple Lie groups with bi-invariant metrics. Given a symmetric space $L$, a subspace $CK$ Killing fields is called Clifford-Killing if for any two elements $Z, W \in CK$, $\langle Z, W \rangle$ is constant. In particular, the elements of $CK$ are constant length Killing fields whose integral flows are the so-called Clifford-Wolf translations ([3 Proposition 3]). On one hand, Proposition 1.9 shows that $A(\text{L}^2\mathcal{H})$ (see as the space in [28]) is a Clifford-Killing space. On the other hand, constant length Killing-fields on compact simple Lie groups are either left or right invariant fields (never both). To construct an action on $G$ which is equivalent to a left action, we invoke Theorem 1.6 to control the geometry of the fields in section 4.2.

We again assume that $G$ is simply connected and that the foliation is irreducible (in the light of Theorem 7.10). For simplicity, we implicitly identify the space of Killing fields on $L_{id}$ with the germs of Killing fields on $L_{id}$ around id.

Let $L$ be a leaf and let $a_L$ denote the space in (28), i.e., $a_L$ is the space spanned by the (local) fields $A_X Y$, where $X, Y$ are basic horizontal.

**Lemma 9.9.** Let $\tilde{L} = L_0 \times \ldots \times L_s$ be the decomposition of $\tilde{L}$ onto an abelian group $L_0$ and simple compact groups. Then, $a_L = \oplus a_L^i$, where $a_L^i$ is either the sum of right invariant fields or of left invariant fields on $L_i$. In particular, $a_L$ is isomorphic to the Lie algebra of $L_{id}$.

**Proof.** Observe that a Clifford-Killing space on a compact simple Lie group that trivializes its tangent bundle must be either a set of left invariant fields or the set of right invariant fields. Furthermore, using [3 Theorem 4 and Proposition 3] we conclude that every element $\eta \in a_L$ is the sum of constant length Killing fields $\eta_i \in \mathfrak{X}(L_i)$. From [3 Proposition 7] and Theorem 1.6 we conclude that the projection of the elements of $a_L$ to each component $L_i$ must trivialize their tangent bundle, concluding the proof. $\square$

Let $c$ be a horizontal curve, $c(0) = id$. Decompose $L(t) = \tilde{L}_{c(t)} = \Pi L_i(t)$ according to $L_i(t) = \phi_{c_i}(L_i)$. Consider then the decomposition of $a_{L_{c(t)}}$ given in Lemma 9.9 $a_{L_{c(t)}} = \oplus a^i_{L_{c(t)}}$. According to Lemma 9.9 $a^i_{L_{c(t)}}$ must be a smooth bundle with constant rank along $c$. Furthermore, at every $t$, it must be either all left or all right invariant fields, if $i > 0$, therefore the property of being right invariant (respectively, left invariant) must be constant with respect to $t$. Keeping the proof of Theorem 1.6 in mind, we just have proved:

**Lemma 9.10.** $\text{hol}_{id}(\mathcal{F}) = a_{L_{id}}$. In particular, if $G$ is simply connected and $\mathcal{F}$ is irreducible, then $H_{id}(\mathcal{F}) = \{id\}$.

**Proof.** According to Corollary 4.1 it is sufficient to show that $\dot{c}(1)^{-1}(a_{L_{id}}) \subset a_{L_{id}}$ for every horizontal curve $c$, $c(0) = id$. This condition holds from the discussion above. $\square$

We are in position to apply Corollary 4.4. We recall that the fields $p \mapsto \chi(\xi_0, p)$ in section 4.2 were constructed using holonomy transportation. In particular, they have constant length. We now use Proposition 4.5 to prove that they are Killing fields. This completes the proof.

**Lemma 9.11.** Let $\mathcal{F}$ be irreducible and $G$ simply connected. Identify $\mathcal{E}_{id}$ with $G$. The set of vector fields $\{h \mapsto h \xi_0 \mid \xi_0 \in \mathcal{V}_{id}\}$ is a Clifford-Killing space on $G$. 

Proof. We already know that the $\text{Hol}_\text{id}(\mathcal{F})$-action is transitive and commutes leaf-wise with $\alpha_L$. Therefore, its restriction to each leaf must be by components-wise by only right or left invariant fields (opposing $\alpha_L$). Now, Proposition 4.8 guarantees that the $\text{Hol}_\text{id}(\mathcal{F})$-action fields are Killing, and the arguments here guarantee that they are Clifford-Killing, completing the proof of Theorem 1.1. □

9.3. The non-simply connected case. We recall that, if $G$ is connected, $\pi_1(M)$ is naturally a central subgroup of the universal cover $\widetilde{M}$. So far, we have proved that a Riemannian foliation with totally geodesic fibers on a simply connected space is isometric to a homogeneous foliation. Therefore, given $\widetilde{F}$, there is a subgroup $\widetilde{H} < \widetilde{M}$ that realizes (up to isometry) $\pi^*(\mathcal{F})$. But the two actions, by $\widetilde{H}$ and $\pi_1(M)$ commute, since the last is in the center. Therefore $H$ acts locally freely on $M = \widetilde{M}/\pi_1(M)$, since $\pi_1(M) < \widetilde{M}$ is discrete. Since the Lie algebra of $\widetilde{H}$ is a Lie subalgebra of $G$, it integrates as an immersed subgroup $H$ of $G$. It is straightforward that the action of $H$ realizes the action of $\widetilde{H}$.

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