Existence of solution for a nonlocal dispersal model with nonlocal term via bifurcation theory

Claudianor O. Alves\textsuperscript{a*}, Natan de Assis Lima\textsuperscript{b†} and Marco A. S. Souto\textsuperscript{a‡}

\textsuperscript{a}. Universidade Federal de Campina Grande

Unidade Acadêmica de Matemática
CEP: 58429-900, Campina Grande - Pb, Brazil

\textsuperscript{b}. Universidade Estadual da Paraíba

Centro de Ciências Humanas e Exatas
CEP: 58500-000, Monteiro - Pb, Brazil

Abstract

In this paper we study the existence of solution for the following class of nonlocal problems

\[ L_0u = u\left(\lambda - \int_{\Omega} Q(x, y)|u(y)|^p \, dy\right), \text{ in } \Omega, \]

where \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \), is a smooth bounded domain, \( p > 0 \), \( \lambda \) is a real parameter, \( Q : \Omega \times \Omega \rightarrow \mathbb{R} \) is a nonnegative function, and \( L_0 : C(\Omega) \rightarrow C(\Omega) \) is a nonlocal dispersal operator. The existence of solution is obtained via bifurcation theory.

Mathematics Subject Classifications: 47G20, 35J60, 92B05

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1 Introduction

In this work we study the existence of positive solution for the following equation

\[ L_0u = u\left(\lambda - \int_{\Omega} Q(x, y)|u(y)|^p \, dy\right), \text{ in } \Omega, \]  

(P)

\textsuperscript{*}C. O. Alves was partially supported by CNPq/Brazil 304804/2017-7, coalves@mat.ufcg.edu.br

\textsuperscript{†}N. A. Lima, natan.mat@cche.uepb.edu.br

\textsuperscript{‡}M. A. S. Souto was partially supported by CNPq/Brazil 306082/2017-9 and INCT-MAT, marco@mat.ufcg.edu.br
where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a smooth bounded domain, $p > 0$, $\lambda$ is a real parameter, $Q : \Omega \times \Omega \to \mathbb{R}$ is a nonnegative function with $Q \in C(\overline{\Omega} \times \overline{\Omega})$ and verifying some hypotheses that will be detailed below, and $L_0 : C(\overline{\Omega}) \to C(\overline{\Omega})$ is the nonlocal dispersal operator given by

$$L_0 u(x) = \int_{\Omega} K(x, y) u(y) dy,$$

for $u \in C(\overline{\Omega})$, \hspace{1cm} (1)

with a continuous and nonnegative dispersal kernel $K$. The dispersion mechanism is currently a focus of theoretical interest and has received much attention recently. Most of these continuous dispersion models are based on reaction-diffusion equations, which are widely studied see [3], [4], [5], [8], [9], [15], [16], [21], [22], [24], [25]. This type of diffusion process has been widely used to describe the dispersion of a population (of cell or organisms) through the environment, as indicated in [18], [19], [23], if $u(y)$ is thought of as a density at a location $y$, $K(x, y)$ as the probability distribution of jumping from a location $y$ to a location $x$, then the rate at which the individuals from all other places are arriving at location $x$ is

$$\int_{\Omega} K(x, y) u(y) dy.$$

In this context, $\lambda$ is a parameter which represents the intrinsic growth rate of the species, and the nonlocal term

$$\int_{\Omega} Q(x, y)|u(y)|^p dy$$

can be interpreted as a weighted average of $u$ at all the domain. In many problems in biology (and ecology), for example seed dispersal problems, this formulation of the dispersion of individuals finds its justification; see [7], [12], [27], [28], [31]. The presence of nonlocal reaction term in equation (P) means, from the biological point of view, that the crowding effect depends not only on their own point in space but also depends on the entire population in an $N$-dimensional habitat $\Omega$, see [20].

The motivation to study (P) comes from the model to study the behavior of a species inhabiting in a smooth bounded domain $\Omega$, whose the classical logistic equation with laplacian diffusion is given by

$$\begin{cases}
-\Delta u = u(\lambda - b(x)u^p) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $b(x)$ describes the limiting effect of crowding of the population. In (2), we are assuming that $\Omega$ is surrounded by inhospitable areas, due to the homogeneous Dirichlet boundary conditions. Note that the equation in (2) is a local equation, and so the crowding effect of the population $u$ at $x$ only depends on the value of the population in the same point $x$. In [11], Chipot has considered that crowding effect depends also on the value of the population around of $x$, that is, the crowding effect depends on the value of $u$ in a neighborhood of $x$, $B_r(x)$, the centered ball at $x$ of radius $r > 0$. To be more precisely, Chipot considered the nonlocal problem

$$\begin{cases}
-\Delta u = u \left( \lambda - \int_{\Omega \cap B_r(x)} b(y)u^p(y)dy \right) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(3)
where \( b \) is a nonnegative and nontrivial continuous function. After that, a special attention was given to the problem
\[
\begin{align*}
-\Delta u &= u \left( \lambda - \int_{\Omega} Q(x, y) u^p(y) dy \right) \quad \text{in} \quad \Omega, \\
u &= 0, \quad \text{on} \quad \partial \Omega,
\end{align*}
\]
by supposing different conditions on \( Q \), see for example [1], [2], [10], [13], [14], [26] and [32] and their references.

In [10], Chen and J. Shi have considered the case \( p = 1 \) and the kernel function \( Q(x, y) \) being a continuous and nonnegative function on \( \Omega \times \Omega \) with \( \int_{\Omega} Q(x, y) u(y) dy > 0 \) for all positive continuous functions \( u \) on \( \Omega \). In that paper was proved the existence of \( \lambda^* > \lambda_1 \) such that (4) possesses at least a positive solution for \( \lambda \in (\lambda_1, \lambda^*) \).

In [1], Allegretto and P. Nistri have showed that (4) possesses a unique positive solution when \( \lambda > \lambda_1 \) and \( Q(x, y) = Q_\delta(|x - y|) \) is a mollifier in \( \mathbb{R}^N \), i.e., \( Q_\delta(|x - y|) \in C_0^\infty(\Omega) \), \( \int_{\mathbb{R}^N} Q_\delta(|x - y|) dy = 1 \) for any \( x \) with
\[Q_\delta(|x - y|) = 0 \quad |x - y| \geq \delta \]
and
\[Q_\delta(|x - y|) \quad \text{bounded away from zero is} \quad |x - y| < \mu < \delta.\]
Observe that in this case, \( Q \) vanishes away from the diagonal of \( \Omega \times \Omega \).

In [32], Sun, Shi and Wang have investigated the existence of positive solutions for (4) with \( Q(x, y) = Q_1(|x - y|) \) and \( \Omega = (-1, 1) \), where \( Q_1 : [0, 2] \rightarrow (0, \infty) \) is a nondecreasing and piecewise continuous function satisfying
\[
\int_0^2 Q_1(y) dy > 0.
\]
When \( Q(x, y) \) is a separable variable, i.e., \( Q(x, y) = g(x) h(y) \) with \( h \geq 0; h \neq 0 \) and \( g(x) > 0 \), Corrêa, Delgado and Suárez [13] have studied (4) and proved the existence and uniqueness of positive solution. Moreover, in Coville [14] and Leman, Méléard and Mirrahimi [26], by assuming \( g \equiv 1, p > 1 \) and homogeneous Neumann boundary conditions, the authors have proved that positive solution of (4) attracts all the possible solutions of the corresponding parabolic associated with (4). When \( g \geq 0, g \neq 0 \) and \( g \equiv 0 \) in \( \Omega_0 \subset \Omega \), then (4) possesses a unique positive solution for \( \lambda \in (\lambda_1, \lambda_0) \) where \( \lambda_0 \) is the principal eigenvalue of the minus laplacian in \( \Omega_0 \), for more details see [13].

Finally, in [2], Alves, Delgado, Souto and Suárez have considered the existence and nonexistence of solution for (4). In that paper, they have studied a more general problem than the previous ones, more precisely, they have considered that \( Q \) satisfies:
\((Q_1)\) \(Q \in L^\infty(\Omega \times \Omega)\) and \(Q(x, y) \geq 0\) for all \(x, y \in \Omega\);

\((Q_2')\) If \(w\) is measurable and 
\[ \int_{\Omega \times \Omega} Q(x, y)|w(y)|^p|w(x)|^2\,dxdy = 0, \]
then \(w = 0\) a.e. in \(\Omega\).

In the present paper, our main goal is showing the existence of solution for \((P)\) via the global bifurcation theorem result due to Rabinowitz, see Theorem 29.1 in [17]. The difference of the problem with \(L_0\) of the problem with \(-\Delta\) is that, \((-\Delta)^{-1}\) is a compact operator, while \((L_0 + M)^{-1}\) is not a compact for any \(M \geq 0\) large. This difference between the operators brings some difficulties and the bifurcation here is made in another way.

This paper is organized as follows. In Section 2 we show a Krein-Rutman type result for \(L_0\). In Section 3 we study the existence of a positive solution for equation \((P)\) without boundary conditions, by supposing that \(K\) satisfies:

\((K_1)\) \(K(x, y) = K(y, x)\) for all \(x, y \in \overline{\Omega}\);

\((K_2)\) There exists \(\delta > 0\) such that \(K(x, y) > 0\) for all \(x, y \in \overline{\Omega}\) and \(|x - y| \leq \delta\);

and supposing that \(Q\) is a continuous function that satisfies:

\((Q_2)\) There exist \(r, \sigma > 0\) such that \(Q(x, y) \geq \sigma\) for all \(x, y \in \overline{\Omega}\) and \(|x - y| \leq r\).

If the inequality above works for all \(x, y\) in \(\Omega\), we say that \(Q\) satisfies \((Q_2')\), that is,

\((Q_2')\) \(Q(x, y) \geq \sigma\) for all \(x, y \in \overline{\Omega}\).

The conditions \((K_1)\) and \((K_2)\) above are hypotheses usually considered on the operator \(L_0\), as we can see in [5], [21] and [22].

We would like point out that assumption \((Q_2)\) implies in \((Q_2')\).

**Definition 1.1.** For each \(Q : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}\) we define the oscillation of \(Q\) in \(x\), uniformly in \(y\), by

\[ [Q] = \sup_{x,y,z \in \Omega} |Q(x, y) - Q(z, y)|. \]

With the above hypotheses we have our first result that establishes the existence of local bifurcation.

**Theorem 1.1.** Suppose that \(p > 0, [Q] > 0, (K_1) - (K_2)\) and \((Q_2')\) hold. Then the problem \((P)\) has a positive solution for all \(\lambda \in (\lambda_1, \lambda_1 + \frac{\lambda_1 \sigma}{[Q]}),\) where \(\lambda_1\) is the principal eigenvalue of \(L_0\).
It is a corollary of the proof of the Theorem 1.1 that if $Q = 0$ we have solution for all $\lambda > \lambda_1$.

In order to obtain a global bifurcation result we will assume the following assumption on $Q$:

$$(Q_3)$$ There are $x_0 \in \overline{\Omega}$ and a nonnegative function $a : \overline{\Omega} \rightarrow \mathbb{R}$, such that $a^{-1} \in L^q(\Omega)$ where $q = \max\{1, p\}$ and $Q(x_0, y) \geq Q(x, y) + a(x)$ for all $x, y \in \Omega$.

Note that $(Q_3)$ implies that $Q(x_0, y) \geq Q(x, y)$, for all $x, y \in \Omega$ and $a(x_0) = 0$. Here is an example of a function $Q$ that satisfies $(Q_3)$:

$$Q(x, y) = h(y)[M - |x - x_1|^{q_1}|x - x_2|^{q_2}...|x - x_k|^{q_k}] + g(y)$$

where $x_i \in \overline{\Omega}$, $q_i < \frac{N}{p}$, $M > 0$ is large enough, $h$ and $g$ are positive continuous functions on $\overline{\Omega}$. It is easy to see that $(Q_3)$ works for $x_0$ being any $x_i$ and $a(x) = m|x - x_1|^{q_1}|x - x_2|^{q_2}...|x - x_k|^{q_k}$, for $m > 0$ small enough.

The next result guarantees the existence of a connected component of solutions which contains solution for our problem $(P)$ for any $\lambda > \lambda_1$.

**Theorem 1.2.** Assume $p > 0$, $(K_1) - (K_2)$ and $(Q_2) - (Q_4)$ hold. Then, there is a connected component of positive solutions of $(P)$ coming out from $\lambda_1 > 0$, with the property that includes solutions of the form $(\lambda, u)$ for all $\lambda > \lambda_1$.

In Section 4, under a weaker condition on $Q$, we can solve the problem for all $\lambda > \lambda_1$, but we can not guarantee the existence of a connected component of positives solutions for $(P)$. Here we consider that $Q$ satisfies:

$$(Q_4)$$ There are $x_0 \in \overline{\Omega}$ such that $Q(x_0, y) \geq Q(x, y)$ for all $x, y \in \Omega$.

The main result this paper is the following:

**Theorem 1.3.** Assume $p > 0$, $(K_1), (K_2), (Q_2)$ and $(Q_4)$ hold. Then, problem $(P)$ has a positive solution for all $\lambda > \lambda_1$.

In fact, we comment in the final of the Section 4 that we have the same result of Theorem 1.3 if $Q$ satisfies a more general condition than $(Q_4)$:

$$(Q_4')$$ There are a decomposition $\Omega = \bigcup_{j=1}^{m} E_j$, $x_1 \in E_1$, $x_2 \in E_2$, ..., $x_m \in E_m$ such that $Q(x_j, y) \geq Q(x, y)$ for all $x \in E_j$, $y \in \Omega$.
2 On the principal eigenvalue of $L_0$

In this section, we consider some preliminary facts related to the principal eigenvalue of $L_0$. Let us denote $C(\overline{\Omega})$ by $X$ and consider the dispersal operator $L_0 : X \to X$, where kernel $K$ is a nonnegative continuous function verifying $(K_1)$. It is easy to see that $L_0$ is a compact operator and $L_0(X_+) \subset X_+$, where $X_+$ is the positive cone in $X$, that is,

$$X_+ = \{ u \in X; u(x) \geq 0, \forall x \in \overline{\Omega} \}.$$

Furthermore, we can also consider $L_0 : L^2(\Omega) \to L^2(\Omega)$, which is well defined, compact and symmetric with $L_0(L^2(\Omega)) \subset X$. It is well known that the resolvent of $L_0$ is defined by

$$\rho(L_0) = \{ \lambda \in \mathbb{R}; L_0 - \lambda I \text{ is bijective} \}$$

and its spectrum is $\sigma(L_0) = \mathbb{R}\setminus \rho(L_0)$. By spectral theory of compact operators, we have for $\lambda \neq 0$

$$\lambda \in \sigma(L_0) \iff N(L_0 - \lambda I) \neq \{0\} \text{ or } R(L_0 - \lambda I) = (L_0 - \lambda I)(X) \neq X.$$  

Here, $EV(L_0)$ denotes the eigenvalues set of $L_0$ given by

$$EV(L_0) = \{ \lambda \in \mathbb{R} : N(L_0 - \lambda I) \neq \{0\} \}.$$ 

To avoid some confusion let us denote by $\tilde{\sigma}(L_0)$ the spectrum of $L_0 : L^2(\Omega) \to L^2(\Omega)$, and $\tilde{EV}(L_0)$ its eigenvalues set. Note that, $\sigma(L_0) = \tilde{\sigma}(L_0)$, that is

$$\lambda \in \tilde{EV}(L_0) \iff \lambda \in EV(L_0).$$

Indeed, from $X \subset L^2(\Omega)$ we have that $EV(L_0) \subset \tilde{EV}(L_0)$, and the sufficient condition is done. Now suppose that $\lambda$ is an eigenvalue of $L_0 : L^2(\Omega) \to L^2(\Omega)$, then there is $w \in L^2(\Omega) \setminus \{0\}$ such that

$$\lambda w = L_0 w \in L_0(L^2(\Omega)) \subset X \Rightarrow \tilde{EV}(L_0) \subset EV(L_0).$$ 

Since $L_0$ is a symmetric operator in $L^2(\Omega)$, that is,

$$\langle L_0 u, v \rangle = \langle u, L_0 v \rangle, \quad \forall u, v \in L^2(\Omega),$$

where $\langle u, v \rangle = \int_{\Omega} u v dx$ is the inner product of $L^2(\Omega)$, we have $\tilde{\sigma}(L_0) \subset [m_o, m]$, where

$$m_o = \inf_{u \in L^2(\Omega) \setminus \{0\}} \frac{\langle L_0 u, u \rangle}{\int_{\Omega} |u|^2 dx} \quad \text{and} \quad m = \sup_{u \in L^2(\Omega) \setminus \{0\}} \frac{\langle L_0 u, u \rangle}{\int_{\Omega} |u|^2 dx}.$$  

Moreover $m_o, m \in \tilde{\sigma}(L_0)$ (see Brézis [6], Proposition 6.9, pg 165), and so, $m = \sup\tilde{\sigma}(L_0)$. From definition $m$ and positiveness of $K$, it follows that

$$0 < m = \sup_{u \in L^2(\Omega) \setminus \{0\}} \frac{\int_{\Omega \times \Omega} K(x, y) u(x) u(y) dxdy}{\int_{\Omega} |u|^2 dx}.$$
and there is \( w \in L^2(\Omega) \setminus \{0\} \) such that
\[
m = \frac{\int_{\Omega \times \Omega} K(x, y)w(x)w(y)dxdy}{\int_{\Omega} |w|^2dx}.
\]
Thus, \( m \in \tilde{EV}(L_0) \) and \( w \) is an eigenfunction of \( L_0 \) associated with \( m \).

The next result establishes that if \( w \in L^2(\Omega) \) is an eigenfunction associated with the eigenvalue \( m \), then \( w \) does not change sign.

**Lemma 2.1.** Suppose that \( w \in L^2(\Omega) \) is such that
\[
m = \frac{\int_{\Omega \times \Omega} K(x, y)w(x)w(y)dxdy}{\int_{\Omega} |w|^2dx},
\]
Then \( w \) is continuous and \( L_0w = mw \), that is, \( m \) is the maximum eigenvalue of \( L_0 \). Moreover, since \( \Omega \) is a connected set, we must have \( w > 0 \) in \( \Omega \) or \( w < 0 \) in \( \Omega \) (\( w \) does not change signal).

**Proof.** To begin with, we show that \( L_0w = mw \). In fact, by definition of \( m \), for any \( t \in \mathbb{R} \) and \( v \in L^2(\Omega) \), we must have
\[
\langle L_0(w + tv), w + tv \rangle \leq m \int_{\Omega} (w + tv)^2dx
\]
that is,
\[
\langle L_0w, w \rangle + 2t \langle L_0w, v \rangle + t^2 \langle L_0v, v \rangle \leq m \int_{\Omega} w^2dx + 2mt \int_{\Omega} wvdx + mt^2 \int_{\Omega} v^2dx,
\]
which yields
\[
\langle L_0w, v \rangle + \frac{t}{2} \langle L_0v, v \rangle \leq m \int_{\Omega} wvdx + \frac{mt}{2} \int_{\Omega} v^2dx, \text{ if } t > 0
\]
and
\[
\langle L_0w, v \rangle + \frac{t}{2} \langle L_0v, v \rangle \geq m \int_{\Omega} wvdx + \frac{mt}{2} \int_{\Omega} v^2dx, \text{ if } t < 0.
\]
Taking the limit as \( t \to 0 \), we get \( \langle L_0w, v \rangle = m \int_{\Omega} wvdx \), for all \( v \in L^2(\Omega) \), from where it follows that, \( L_0w = mw \). Since \( L_0w \) is a continuous function, the equality \( L_0w = mw \) implies that \( w \) is a continuous function on \( \Omega \).

In the sequel, we prove that if \( w \neq 0 \) and \( w \geq 0 \) in \( \Omega \), then \( w > 0 \) in \( \Omega \). Indeed, fixing \( C = w^{-1}(\{0\}) = \{x \in \Omega : w(x) = 0\} \), we have that \( C \) is an open set in \( \Omega \), because if \( z \in C \), we have that
\[
0 \leq \int_{\Omega \cap B_\delta(z)} K(z, y)w(y)dy \leq \int_{\Omega} K(z, y)w(y)dy = mw(z) = 0,
\]
implying that \( w(x) = 0 \) for all \( x \in \Omega \cap B_\delta(z) \), that is, \( \Omega \cap B_\delta(z) \subset C \). This proves that \( C \) is an open set in \( \Omega \). Moreover, from continuity of \( w \), it is easy to see that \( C \) is also a closed set in \( \Omega \). Recalling that \( \Omega \) is connected and \( \Omega \setminus C \) is non-empty, we conclude that \( w > 0 \) in \( \Omega \).
To finish the proof, we will show that $w$ does not change signal. To this end, we fix the sets

$$A = \{ x \in \Omega : w(x) > 0 \} \text{ and } B = \{ x \in \Omega : w(x) < 0 \}.$$ 

If $w$ changes signal, both subsets $A$ and $B$ have positive Lebesgue measure. By a simple computation, we see that

$$\int_{\Omega \times \Omega} K(x, y)w(x)w(y)dx dy = \int_{A \times A} K(x, y)w(x)w(y)dx dy + \int_{B \times B} K(x, y)w(x)w(y)dx dy + \int_{A \times B} K(x, y)w(x)w(y)dx dy + \int_{B \times A} K(x, y)w(x)w(y)dx dy.$$

If $K(x, y)$ is not zero in $A \times B$, we derive that

$$\int_{A \times A} K(x, y)w(x)w(y)dx dy = \int_{A \times A} K(x, y)|w(x)||w(y)|dx dy,$$

$$\int_{B \times B} K(x, y)w(x)w(y)dx dy = \int_{B \times B} K(x, y)|w(x)||w(y)|dx dy,$$

$$\int_{A \times B} K(x, y)w(x)w(y)dx dy < 0 < \int_{A \times B} K(x, y)|w(x)||w(y)|dx dy$$

and

$$\int_{B \times A} K(x, y)w(x)w(y)dx dy < 0 < \int_{B \times A} K(x, y)|w(x)||w(y)|dx dy.$$

These informations lead to

$$m = \frac{\int_{\Omega \times \Omega} K(x, y)w(x)w(y)dx dy}{\int_{\Omega} w^2 dx} < \frac{\int_{\Omega \times \Omega} K(x, y)|w(x)||w(y)|dx dy}{\int_{\Omega} |w|^2 dx} \leq m,$$

which is impossible, then $w > 0$ in $\Omega$ or $w < 0$ in $\Omega$.

Suppose now that $K(x, y) \equiv 0$ in $(A \times B) \cup (B \times A)$. Then,

$$a_1 = \int_{A \times A} K(x, y)w(x)w(y)dx dy,$$

$$a_2 = \int_{B \times B} K(x, y)w(x)w(y)dx dy,$$

and

$$b_1 = \int_{A} w^2 dx \text{ and } b_2 = \int_{B} w^2 dx.$$

Consequently

$$m = \frac{a_1 + a_2}{b_1 + b_2}, \quad \frac{a_1}{b_1} \leq m \text{ and } \frac{a_2}{b_2} \leq m.$$
A simply computation shows that \( a_1 = mb_1 \), and so, the function \( w^+ = \max\{w, 0\} \) satisfies
\[
m = \frac{\int_{\Omega \times \Omega} K(x, y)w^+(x)w^+(y)dx dy}{\int_{\Omega} (w^+)^2 dx}.
\]
By using the first part of this proof, we must have \( w^+ > 0 \) in \( \Omega \), or equivalently, \( w(x) > 0 \) in \( \Omega \), contradicting the fact that \( |B| > 0 \), finishing the proof.\( \square \)

**Corollary 2.1.** If \( w \) is an eigenfunction of \( L_0 \) associated to \( m \) then \( w \) must be a positive (or negative) eigenfunction. Besides, \( \dim N(L_0 - mI) = 1 \).

**Proof.** If \( L_0w = mw \), then \( \langle L_0w, w \rangle = m \int_\Omega w^2 \) and \( w \neq 0 \). From Lemma 2.1, \( w \) must be a positive (or negative) eigenfunction. For the second part of the corollary, suppose that there are two eigenfunctions \( w, \phi \) associated with \( m \) that are linearly independent. Then, without loss of generality we can suppose that \( \int_\Omega w\phi dx = 0 \). However, it is impossible, because \( w \) and \( \phi \) has defined signal in \( \Omega \), then \( \int_\Omega w\phi dx \neq 0 \).\( \square \)

**Corollary 2.2.** Under the conditions \((K_1) - (K_2)\), if \( w \) is an eigenfunction of \( L_0 \) associated to \( m \), then \( w \) is positive in \( \overline{\Omega} \) (or negative in \( \overline{\Omega} \)). Hence, \( w \) is discontinuous on \( \partial \Omega \) if we define \( w = 0 \) in \( \mathbb{R}^N \setminus \Omega \).

**Proof.** Indeed, if \( x_0 \in \partial \Omega \) is such that \( w(x_0) = 0 \), then
\[
0 < \int_{\Omega \cap |x-x_0-y| \leq \delta} K(x_0, y)w(y)dy \leq \int_\Omega K(x_0, y)w(y)dy = \lambda_1 w(x_0) = 0
\]
which is absurd, showing the desired result.\( \square \)

As a byproduct of the study made until moment, we have the following result

**Proposition 2.1.** The eigenvalue problem
\[
L_0u = \lambda u, \quad \text{in} \quad \Omega,
\]
has an unique eigenvalue \( \lambda_1 > 0 \) whose the eigenfunction are continuous on \( \overline{\Omega} \) with defined signal and \( \dim N(L - \lambda_1 I) = 1 \). Moreover, \( \lambda_1 = m = \sup \sigma(L_0) \).

Now, before proving our next result, it is necessary to recall that assumptions on \( K \) implies that for each \( u \in C(\overline{\Omega}) \), with \( u \geq 0 \) in \( \Omega \) only one of the possibilities below holds:
\[
L_0u > 0 \text{ in } \overline{\Omega} \text{ or } u \equiv 0 \text{ in } \overline{\Omega}.
\]
Hence, we have the following lemma

**Lemma 2.2.** Suppose that \( u \in C(\overline{\Omega}) \), \( u \geq 0 \), \( u \neq 0 \) in \( \Omega \) and \( c(x) \) given by \( L_0u = c(x)u \), then \( \|c\|_{\infty} \geq \lambda_1 \).
This inequality becomes equality only when \( c(x) \equiv \lambda_1 \).
Proof. Consider $\varphi_1$ a positive eigenfunction associated to $\lambda_1$. Therefore, $L_0\varphi_1 = \lambda_1 \varphi_1$ and

$$
\lambda_1 \int_\Omega u \varphi_1 dx = \int_\Omega u L_0 \varphi_1 dx = \int_\Omega \varphi_1 L_0 u dx = \int_\Omega c(x) u \varphi_1 dx \leq \|c\|_\infty \int_\Omega u \varphi_1 dx.
$$

Since $\int_\Omega u \varphi_1 dx > 0$ the above inequality leads to $\|c\|_\infty \geq \lambda_1$. \qed

Lemma 2.3. Let $u \in L^1(\Omega)$ be a nonnegative function and $c \in C(\overline{\Omega})$ satisfying

$$
L_0 u(x) = c(x) u(x) \quad a.e. \ in \ \Omega.
$$

Then, if $u \neq 0$ we have that $u$ is continuous and positive in $\overline{\Omega}$. Furthermore, $c$ is positive in $\overline{\Omega}$.

Proof. Clearly $L_0 u \in C(\overline{\Omega})$, and so, $c(\cdot) u \in C(\overline{\Omega})$. Consider the following sets,

$$
V = \{ x \in \overline{\Omega}; \text{ there exists a ball } B \text{ centered at point } x \text{ such that } u \equiv 0 \text{ a.e. in } B \cap \Omega \}
$$

and $W = \{ x \in \overline{\Omega}; L_0 u(x) > 0 \}$. Both subsets $V$ and $W$ are open in $\overline{\Omega}$. Now, we are showing that $W \cap V = \emptyset$ and $V = \overline{\Omega} \setminus W$.

Indeed, if $z \notin W$ we have $L_0 u(z) = 0$. Thus,

$$
0 = L_0 u(z) = \int_\Omega K(z, y) u(y) dy \geq \int_{B_\delta(z) \cap \Omega} K(z, y) u(y) dy,
$$

since $K(z, y) > 0$ for all $|z - y| \leq \delta$, we get $u(y) = 0$ a.e. in $B_\delta(z) \cap \Omega$, that is $z \in V$. Since $\Omega$ is connected, we must have $V = \emptyset$ and $W = \overline{\Omega}$. Moreover, $c$ is positive and $u(x) = \frac{L_0 u(x)}{c(x)}$ is continuous and positive on $\overline{\Omega}$.

Lemma 2.4. If $g(x) > \lambda_1$ in $\Omega$, then $L_0 u = g(x) u$ does not admit a positive solution.

Proof. Suppose that $L_0 u = g(x) u$ admit a positive solution $u$ and consider $\varphi_1$ a positive eigenfunction associated to $\lambda_1$. Then,

$$
\lambda_1 \int_\Omega u \varphi_1 dx = \int_\Omega u L_0 \varphi_1 dx = \int_\Omega \varphi_1 L_0 u dx = \int_\Omega g(x) u \varphi_1 dx
$$

that is,

$$
\int_\Omega (g(x) - \lambda_1) u \varphi_1 dx = 0,
$$

since $u \varphi_1 > 0$ we have $g(x) - \lambda_1 = 0$, which is absurd. \qed

Here, we would like to point out that in [21, 22], García-Melián and Rossi also have considered an nonlocal eigenvalue problem of the type

$$
\begin{cases}
  \int_{\mathbb{R}^N} K(x - y) u(y) dy - u = -\lambda u(x) & \text{ in } \Omega, \\
  u = 0, & \text{ in } \mathbb{R}^N \setminus \Omega
\end{cases}
$$

(5)
with a kernel $K \in C^1(\mathbb{R}^N)$, $K > 0$ in $B_1$ (the unit ball), $K = 0$ in $\mathbb{R}^N \setminus B_1$, $K(-z) = K(z)$ and $
int_{B_1} K(x) dx = 1$. They proved that the problem admits a unique principal eigenvalue $\lambda_1$, that is, an eigenvalue with an associated positive eigenfunction $\phi_1 \in C(\overline{\Omega})$, it is simple and verifies $0 < \lambda_1(\Omega) < 1$. In this way, the first eigenfunction $\phi_1$ of the problem is strictly positive in $\Omega$ (with a positive continuous extension to $\Omega$) and vanishes outside $\Omega$. Therefore, a discontinuity may occur in $\partial \Omega$ and the boundary value is not taken in the usual sense, for more details see [30, Chapter 2]. From these comments, we see that Proposition 2.1 continues the study made in the above papers for another class of nonlocal problems.

3 Framework

In whole this section, we are assuming that $K$ is a nonnegative continuous function that verifies $(K_1)$ and $(K_2)$. Moreover, for each $w \in C(\overline{\Omega})$, we set the function $\Phi_w : \overline{\Omega} \to \mathbb{R}$ by

$$\Phi_w(x) = \int_{\Omega} Q(x, y)|w(y)|^p dy,$$

where $p > 0$ and $Q$ is a continuous function satisfying $(Q_2)$.

Since $Q$ and $w$ are bounded, we have that $\Phi_w$ is well defined. Furthermore, the ensuing properties will be useful later on:

(Φ1) $t^p\Phi_w = \Phi_{tw}$, for all $w \in C(\overline{\Omega})$;

(Φ2) $\|\Phi_w\|_\infty \leq \|Q\|_\infty \|w\|_p^p |\Omega|$, for all $w \in C(\overline{\Omega})$;

(Φ3) $\|\Phi_w - \Phi_v\|_\infty \leq \|Q\|_\infty \|w^p - |v|^p\|_\infty |\Omega|$, for all $w \in C(\overline{\Omega})$;

(Φ4) $\Phi : C(\overline{\Omega}) \to C(\overline{\Omega})$, given by $\Phi(w) = \Phi_w$, is uniformly continuous in $C(\overline{\Omega})$.

Hereafter, we intend to prove the existence of positive solution for $(P)$ by using the Global Bifurcation Theorem. Having this in mind, it is very important to observe that if $(\lambda, u)$ is a solution of $(P)$, then from Lemma 2.3 $\lambda > \Phi_u(x)$ for all $x \in \overline{\Omega}$ (we will see that it is a necessary condition to obtain positive solution) and so

$$L_0 u = (\lambda - \Phi_u(x))u \iff u = \frac{L_0 u}{\lambda - \Phi_u(x)} \iff u = \lambda^{-1} L_0 u + \frac{\Phi_u(x)L_0 u}{\lambda(\lambda - \Phi_u(x))}.$$

For $\gamma = \lambda^{-1}$, we have

$$u = \gamma L_0 u + \frac{\gamma^2 \Phi_u(x)L_0 u}{1 - \gamma \Phi_u(x)}.$$
or equivalently

\[ u = \gamma L_0 u + G(\gamma, u), \]

where

\[ G(\gamma, u) = \frac{\gamma^2 \Phi_u(x) L_0 u}{1 - \gamma \Phi_u(x)}. \]

Furthermore, for each \(0 < a < b\),

\[ \lim_{v \to 0} \frac{G(\gamma, v)}{\|v\|}\infty = 0, \quad \text{uniformly in } \gamma \in [a, b]. \quad (G) \]

Next, we recall the definition of compact operator when the domain is not a closed set. This type of operator applies an important role in our approach.

**Definition 3.1.** Let \( A \) be an open set in \((0, +\infty) \times C(\Omega)\). A nonlinear operator \( G : A \to C(\Omega) \) is said to be compact if \( G \) is continuous, and for each \( B \subset A \) such that \( B \) is bounded and \( \text{dist}(B, \partial A) \) is positive, then \( G(B) \) is relatively compact in \( C(\Omega) \).

**Remark 3.1.** The operator \( G \) is very well defined in

\[ A = \{(\gamma, v) \in (0, +\infty) \times C(\Omega); \gamma \left\| \Phi_v \right\|\infty < 1\}. \]

Moreover, \( A \) is an open set which contains \((\lambda_1^{-1}, 0)\). It is easy to see that \( G \) is compact in each \( U_{\lambda, \rho, M} \), where

\[ U_{\lambda, \rho, M} = \{(\gamma, v) \in (0, +\infty) \times C(\Omega); \|v\|\infty < M, \lambda^{-1} < \gamma \text{ and } 1 - \gamma \left\| \Phi_v \right\|\infty > \rho\} \]

and \( U_{\lambda, \rho, M} \subset A \).

### 3.1 Proof of Theorems 1.1 and 1.2

Using the above notations, we see that \((\lambda, u)\) solves \((P)\) if, and only if,

\[ L_0 u + \Phi_u(x) u = \lambda u, \]

or equivalently, \( u = F(\gamma, u) := \gamma L_0 u + G(\gamma, u) \), where \( \gamma = \lambda^{-1} \).

In the sequel, we will apply a Global Bifurcation Theorem found in [17, Theorema 29.1], which improves a well known Global Bifurcation Theorem found in [29].

**Theorem 3.1. (Global bifurcation)** Let \( X \) be a Banach space, \( U \subset \mathbb{R} \times X \) a neighbourhood of \((\gamma_0, 0)\), \( G : \overline{U} \to X \) completely continuous and \( G(\gamma, u) = o(\|u\|_X) \) as \( u \to 0 \), uniformly in \( \gamma \), in compacts of \( \mathbb{R} \). Let \( T \in L(X) \) be compact and \( \gamma_0 \) a characteristic value of odd algebraic multiplicity, \( F(\gamma, u) = u - \gamma T + G(\gamma, u) \) and

\[ \Sigma = \{(\gamma, u) \in U; F(\gamma, u) = 0, u \neq 0\}. \]

Then the component \( \mathcal{C} = \mathcal{C}_\gamma \) of \( \Sigma \), containing \((\gamma_0, 0)\), has at least one of the following properties:
(i) \( \mathcal{C} \cap \partial U \neq \emptyset \)

(ii) \( \mathcal{C} \) contains an odd number of trivial zeros \((\gamma_i, 0) \neq (\gamma_0, 0)\), where \( \gamma_i \) is a characteristic value of \( T \) of odd algebraic multiplicity.

By the previous section, we know that there is a first positive eigenfunction \( \varphi_1 \) associated to \( \lambda_1 \). Moreover, \( \lambda_1 \) is an eigenvalue of \( L_0 \) with multiplicity equal to 1. From global bifurcation theorem, there exists a closed connected component \( \mathcal{C} = \mathcal{C}_{\lambda_1^{-1}} \) of solutions for \((P)\) that satisfies (i) or (ii). We claim that (ii) does not occur. In order to show this claim, we need of the lemma below

**Lemma 3.1.** There exists \( \delta > 0 \) such that, if \((\gamma, u) \in \mathcal{C} \) with \(|\gamma - \lambda_1^{-1}| + \|u\|_{\infty} < \delta \) and \( u \neq 0 \), then \( u \) has defined sign, that is,

\[
 u(x) > 0, \quad \forall x \in \overline{\Omega} \quad \text{or} \quad u(x) < 0, \quad \forall x \in \overline{\Omega}.
\]

**Proof.** It is enough to prove that for any two sequences \( (u_n) \subset C(\overline{\Omega}) \) and \( \gamma_n \to \lambda_1^{-1} \) with \( u_n \neq 0, \quad \|u_n\|_{\infty} \to 0 \) and \( u_n = F(\gamma_n, u_n) = \gamma_n L_0 u_n + G(\gamma_n, u_n) \), \( u_n \) has defined signal for \( n \) large enough.

Setting \( w_n = u_n/\|u_n\|_{\infty} \), we have that \( (w_n) \subset C(\overline{\Omega}) \) and

\[
 w_n = \gamma_n L_0 (w_n) + \frac{G(\gamma_n, u_n)}{\|u_n\|_{\infty}} = \gamma_n L_0 (w_n) + o_n(1),
\]

where we have used \((G)\) in the last equality. From compactness of operator \( L_0 \), we can assume that \( (L_0(w_n)) \) is convergent for some subsequence. Then,

\[
 w_n \to w \text{ in } C(\overline{\Omega}),
\]

for some \( w \in C(\overline{\Omega}) \) with \( \|w\|_{\infty} = 1 \). Thereby,

\[
 w = \lambda_1^{-1} L_0 (w)
\]

or equivalently,

\[
 L_0 w = \lambda_1 w \text{ in } \Omega.
\]

Thereby, \( w \neq 0 \) is an eigenfunction associated with \( \lambda_1 \), and by Proposition 2.1 and Corollary 2.2

\[
 w(x) > 0, \quad \forall x \in \overline{\Omega} \quad \text{or} \quad w(x) < 0, \quad \forall x \in \overline{\Omega}.
\]

In the sequel, without loss of generality we assume that \( w \) is positive in \( \overline{\Omega} \). As \( w \) is the uniform limit of \( (w_n) \) in \( C(\overline{\Omega}) \), we must have \( w_n > 0 \) for all \( x \in \overline{\Omega} \) and \( n \) large enough. As \( u_n \) and \( w_n \) has the same signal, \( u_n \) is also positive, which is the desired conclusion.
Remark 3.2. It is easy to check that if \((\gamma, u) \in \Sigma\) if, and only if, \((\gamma, -u)\) is also in \(\Sigma\). Thus, by considering the sets

\[
C^+ = \{(\gamma, u) \in C : u(x) > 0, \ \forall x \in \overline{\Omega}\} \cup \{(\lambda_1^{-1}, 0)\}
\]

and

\[
C^- = \{(\gamma, u) \in C : u(x) < 0, \ \forall x \in \overline{\Omega}\} \cup \{(\lambda_1^{-1}, 0)\},
\]

we have

\[
C = C^+ \cup C^-.
\]

Moreover, \(C^- = \{(\gamma, u) \in C : (\gamma, -u) \in C^+\}\) and \(C^+ \cap C^- = \{(\lambda_1^{-1}, 0)\}\).

Indeed, in what follows, we fix

\[
C^\pm = \{(\gamma, u) \in C : u^\pm \neq 0\}
\]

that is, \(C^\pm\) is the subset of \(C\) of the functions that change signal. Since

\[
C = C^+ \cup C^- \cup C^\pm,
\]

we deduce that to prove (6), it is enough to show that \(C^\pm = \emptyset\). Supposing by contradiction that \(C^\pm \neq \emptyset\), as \(C\) is a connected set in \((0, +\infty) \times C(\overline{\Omega})\) and \(C^+ \cup C^-\) is closed nonempty set with \((C^+ \cup C^-) \cap C^\pm = \emptyset\), we must have

\[
(C^+ \cup C^-) \cap C^\pm \neq \emptyset.
\]

Therefore, there is a solution \((\gamma, u) \in C\) and sequences \((\gamma_n, u_n) \subset C^+ \cup C^-\) and \((s_n, w_n) \subset C^\pm\) such that

\[
\gamma_n, s_n \to \gamma \ \text{in} \ R, \ u_n \to u \ \text{in} \ C(\overline{\Omega}) \ \text{and} \ w_n \to u \ \text{in} \ C(\overline{\Omega}).
\]

Consequently \(u \geq 0\) in \(\overline{\Omega}\) or \(u \leq 0\) in \(\overline{\Omega}\) and \(u \neq 0\). Without loss of generality, suppose that \(u \geq 0\) in \(\overline{\Omega}\). Since \((\gamma, u)\) verifies \(u = \gamma L_0 u + G(\gamma, u), L_0 u > 0\) and \(G(\gamma, u) \geq 0\), it follows that \(u > 0\) in \(\overline{\Omega}\). Hence, \(w_n\) is positive for \(n\) large enough, obtaining a contradiction. Thereby, \(C^\pm = \emptyset\), finishing the proof of (6).

Remark 3.3. Lemma 2.4 shows that the connected component that leaving \((\lambda_1, 0)\), has no accumulation points of the form \((\lambda, 0)\) with \(\lambda > \lambda_1\).

Indeed, if \(u > 0\) and \(\|u\|_\infty\) is small enough that \(\lambda - \Phi_u(x) > \lambda_1\), from Lemma 2.4, \((\lambda, u)\) can not belong to this component.

Now, consider \(U \subset A\) as in Remark 3.1 that is, \(U := U_{\lambda, \rho, M}\). Then,

Lemma 3.2. \(C^+ \cap \partial U \neq \emptyset\).
Proof. Suppose by contradiction that \( C^+ \cap \partial U = \emptyset \). Then, from global bifurcation theorem, there exists \((\hat{\gamma}, 0)\) \(\in C\), where \(\hat{\gamma} \neq \lambda_1^{-1}\) and \(\hat{\gamma}\) is a characteristic value of \(L_0\) with odd algebraic multiplicity. Hence, there exists \((u_n) \subset C(\overline{\Omega})\) and \(\gamma_n \to \hat{\gamma}\), such that

\[
\gamma_n \neq 0, \quad \|u_n\|_\infty \to 0 \quad \text{and} \quad u_n = F(\gamma_n, u_n).
\]

Thus, \(L_0u_n = (\lambda_n - \Phi_{u_n}(x))u_n\) and \(\Phi_{u_n}(x) \to 0\) in \(\Omega\), where \(\lambda_n = \gamma_n^{-1}\). Then from Lemma 2.4, we have that

\[
\lambda_n - \Phi_{u_n}(x) > \lambda_1 \quad \text{for all } n \text{ sufficiently large},
\]

that is, \((\hat{\gamma}, 0)\) cannot belong to this component, which is an absurd.

The next result establishes more some properties of the positive solutions of \((P)\).

Lemma 3.3. If \((\gamma, u)\) is a solution of \(u = F(\gamma, u) := \gamma L_0u + G(\gamma, u)\) with \(u > 0\) in \(\Omega\), then we have

\[
\gamma \Phi_u(x) < 1 \quad \text{for all } x \in \overline{\Omega},
\]

that is, \((\gamma, u) \in A = \{(\gamma, v) \in (0, +\infty) \times C(\overline{\Omega}); \gamma \|\Phi_v\|_\infty < 1\}\). This also states that \(C^+ \subset A\).

Proof. Note that, \(u = F(\gamma, u) := \gamma L_0u + G(\gamma, u)\) is equivalent to \(L_0u + \Phi_u(x) = \lambda u\), where \(\gamma = \lambda^{-1}\) and as \(u > 0\) in \(\Omega\), then

\[
0 < L_0u = (\lambda - \Phi_u(x))u
\]

this implies

\[
\lambda - \Phi_u(x) > 0, \quad \text{for all } x \in \overline{\Omega}.
\]

Therefore, \(\lambda^{-1}\Phi_u(x) < 1\) for all \(x \in \overline{\Omega}\) and so,

\[
\gamma \Phi_u(x) < 1, \quad \text{for all } x \in \overline{\Omega},
\]

that is \((\gamma, u) \in A\).

Since \(\overline{\Omega}\) is a compact, we can cover it with a finite number of balls centered at some points of \(\overline{\Omega}\) and radius \(r > 0\), that is, there are \(x_1, \ldots, x_m \in \Omega\) and \(m \in \mathbb{N}\) such that

\[
\overline{\Omega} = \bigcup_{j=1}^{m} (B_{r/2}(x_j) \cap \overline{\Omega}),
\]

where \(r > 0\) was given in assumption \((Q_2)\). The integer \(m\) will appear in the next lemma.

Lemma 3.4. Let \(\Lambda > 0\) and suppose that \((\lambda, u)\) is such that \(L_0u + \Phi_u(x)u = \lambda u\) for some \(\lambda \in (0, \Lambda]\). Then \(\|u\|_p \leq \left(\frac{m\lambda}{\sigma}\right)^{\frac{1}{p}}\), that is, \(u\) is uniformly bounded in \(L^p(\Omega)\).
Proof. By the above coverage consider \( E_j = \Omega \cap B_r(x_j) \), thus

\[
\sigma \int_{E_j} |u(y)|^p \, dy \leq \int_{E_j} Q(x_j, y) |u(y)|^p \, dy \leq \Phi_u(x_j),
\]

from Lemma 3.3

\[
\sigma \int_{E_j} |u(y)|^p \, dy \leq \lambda.
\]

Then,

\[
\sigma \int_{\Omega} |u(y)|^p \, dy \leq m\lambda
\]

that is, \( \|u\|_p \leq \left( \frac{m\lambda}{\sigma} \right)^{\frac{1}{p}} \).

As a consequence of this last proof, we have:

**Corollary 3.1.** Under condition \((Q''_2)\), \( \Phi_v(x) \geq \sigma \|v\|_p^p \), for all \( v \in X, x \in \overline{\Omega} \).

**Proof of Theorem 1.1**

If \( u \) is a positive solution of \( L_0 u + \Phi_u(x)u = \lambda u \) for some \( \lambda \), then \( \lambda - \Phi_u(x) \geq \lambda_1 \) for all \( x \in \Omega \). Indeed, let \( x_* \in \Omega \) be such that \( \Phi_u(x_*) \leq \Phi_u(x) \) for all \( x \in \Omega \). From Lemma 2.2 we have \( \|\lambda - \Phi_u\|_\infty > \lambda_1 \), hence

\[
\lambda - \Phi_u(x_*) > \lambda_1,
\]

or equivalently, \( \lambda - \lambda_1 > \Phi_u(x_*) \). Moreover, from Corollary 3.1

\[
\lambda - \lambda_1 > \Phi_u(x_*) \geq \sigma \|u\|_p^p, \tag{7}
\]

On the other hand,

\[
\lambda - \Phi_u(x) = \lambda - \Phi_u(x) + \Phi_u(x_*) - \Phi_u(x_*)
\]

or

\[
\lambda - \Phi_u(x) > \lambda_1 - |\Phi_u(x) - \Phi_u(x_*)|. \tag{8}
\]

Furthermore, from (7),

\[
|\Phi_u(x) - \Phi_u(x_*)| \leq \int_{\Omega} |Q(x, y) - Q(x_*, y)||u(y)|^p \, dy \leq |Q| \|u\|_p^p \leq |Q| \left( \frac{\lambda - \lambda_1}{\sigma} \right). \tag{9}
\]

Note that, if \( \lambda_1 < \lambda < \lambda_1 + \frac{\lambda_1 \sigma}{|Q|} - \epsilon \), for a small fixed \( \epsilon > 0 \), we have \( 0 < \lambda - \lambda_1 < \frac{\lambda_1 \sigma}{|Q|} - \epsilon \). Thereby, from (3.1) and (9),

\[
\lambda - \Phi_u(x) > \lambda_1 - |Q| \left( \frac{\lambda - \lambda_1}{\sigma} \right)
\]

and then,

\[
\lambda - \Phi_u(x) > \frac{|Q|x}{\sigma} > 0 \text{ for all } x \in \Omega.
\]
Therefore, with study above we obtain that for each $\epsilon > 0$, there exists $\rho > 0$ ($\rho = |Q|\varepsilon\sigma^{-1}$) such that $\lambda - \|\Phi_u\|_\infty \geq \rho$, for all $\lambda \in (\lambda_1, \lambda_1 + \frac{\lambda_1 \sigma}{2} - \epsilon)$. Moreover,

$$|u(x)| \leq \frac{|L_0u(x)|}{\lambda - \Phi_u(x)} \leq \frac{|L_0u(x)|}{\lambda - \Phi_u(x)} \leq \frac{\|K\|_{\rho'}}{\rho} \|u\|_p, \quad \text{for all } x \in \overline{\Omega},$$

where we have used the Hölder Inequality in the last inequality and $\frac{1}{p'} + \frac{1}{p} = 1$. From Lemma 3.4, $\|u\|_p$ is bounded, then there exists $M > 0$ such that $\|u\|_\infty \leq M$.

Gathering all the above informations, for fixed $\epsilon > 0$ and considering $\Lambda = \lambda_1 + \frac{\lambda_1 \sigma}{2} - \epsilon$ we find $\rho > 0$ and $M > 0$ such that, for $U = U_{\Lambda, \frac{\rho}{2}2M}$ we have $C^+ \cap \partial U \neq \emptyset$, thus we have $(\lambda, u) \in C^+$ and one of the following conditions occur: $\lambda - \|\Phi_u\|_\infty = \frac{\rho}{2}$ or $\|u\|_\infty = 2M$ or $\lambda = \Lambda$. But we have seen that, under the above conditions, $\lambda - \|\Phi_u\|_\infty > 0$ and $\|u\|_\infty \leq M$, that is, we should have $\lambda = \Lambda$ and the connected component $C^+$ crosses the hyperplane $\{\lambda\} \times C(\overline{\Omega})$, for all $\lambda \in (\lambda_1, \Lambda]$. Completing the proof of the Theorem 1.1.

Now we will study the bifurcation for all $\lambda > \lambda_1$. In this way, as in the last proof, we want to find a positive number $\rho > 0$ such that, whatever the number $\Lambda > \lambda_1$, if $u$ is a positive solution of $L_0u + \Phi_u(x)u = \lambda u$ for some $\lambda \in (\lambda_1, \Lambda)$, then $\lambda - \|\Phi_u\|_\infty \geq \rho$. To obtain this number, we need more information on $Q$ and $p$.

**Lemma 3.5.** Suppose $p > 0$ and $(Q_2) - (Q_3)$ hold. Then there exist $\rho > 0$ and $M > 0$ such that $\lambda - \|\Phi_u\|_\infty \geq \rho$ and $\|u\|_\infty \leq M$ for all $(\lambda, u)$ such that $L_0u + \Phi_u(x)u = \lambda u$, $u > 0$ and $\lambda \in (\lambda_1, \Lambda]$.

**Proof.** To begin with, we will prove first the existence of $\rho$, after we show the existence of $M$. If there is no $\rho$, then we can find a sequence $(\lambda_n, u_n)$ such that $L_0u_n + \Phi_{u_n}(x)u_n = \lambda_n u_n$, $u_n > 0$, $\lambda_n \rightarrow \lambda \in (\lambda_1, \Lambda]$ and $\|\Phi_{u_n}\|_\infty \rightarrow \lambda$. In the sequel we divide into two cases our study, namely $p > 1$ and $p \in (0, 1)$.

**Case 1:** $p > 1$. By Lemma 3.4, $(u_n)$ is a bounded sequence in $L^p(\Omega)$, and as $p > 1$, there is a some subsequence of $(u_n)$, still denoted by itself, such that $u_n \rightarrow u$ in $L^p(\Omega)$. As $(L_0u_n)$ and $(\Phi_{u_n})$ are uniformly convergent in $C(\overline{\Omega})$, we assume that $L_0u_n \rightarrow w$ and $\Phi_{u_n} \rightarrow v$ in $C(\overline{\Omega})$ respectively. As $L_0$ is a linear and compact operator, we have

$$L_0u_n(x) = \int_{\Omega} K(x, y)u_n(y)dy \rightarrow \int_{\Omega} K(x, y)u(y)dy = L_0u(x) \quad \text{in } \Omega,$$

and so, $L_0u_n \rightarrow L_0u$ in $C(\overline{\Omega})$. Next we are going to show that $\Phi_{u_n} \rightarrow \Phi_u$ in $C(\overline{\Omega})$, however as $\Phi$ is not linear the above argument does not work well, and we need to use others arguments. From the limit $\Phi_{u_n} \rightarrow v$ in $C(\overline{\Omega})$, we know that $\Phi_{u_n}(x) \rightarrow v(x)$, $\forall x \in \Omega$. Now, as $\lambda_n - \Phi_{u_n}(x) > 0$, we have $\lambda - v(x) \geq 0$. 


Passing to the weak limit in the $L^p(\Omega)$ sense in $L_0u_n + \Phi_{u_n}(x)u_n = \lambda_n u_n$, we obtain

$$L_0 u = (\lambda - v(x))u \quad \text{a.e. in } \Omega.$$  

In the sequel, we will consider the cases $u \equiv 0$ and $u \neq 0$, and in the both cases, we will arrive to a contradiction. This proves the existence of $\rho$.

**The case $u \equiv 0$:** In this case, $u_n \to 0$ in $L^p(\Omega)$ and as $u_n > 0$, we have that $u_n \to 0$ in $L^1(\Omega)$ and $L_0u_n \to 0$ in $C(\Omega)$. This yields $u_n \to 0$ in $L^p(\Omega)$. In fact, by $(Q_3)$,

$$\lambda_n > \Phi_{u_n}(x_0) \geq \Phi_{u_n}(x) + a(x)\|u_n\|_p^p \quad \text{for } x \in \Omega$$

therefore,

$$\lambda_n - \Phi_{u_n}(x) > a(x)\|u_n\|_p^p \quad \text{for } x \in \Omega. \quad (10)$$

From this,

$$\int_{\Omega} u_n(y)^p dy = \int_{\Omega} \left[ \frac{L_0u_n(y)}{\lambda_n - \Phi_{u_n}(y)} \right]^p dy \leq \frac{\|L_0u_n\|_\infty^p}{(\|u_n\|_p)^p} \int_{\Omega} \frac{1}{a(y)^p} dy$$

which implies

$$\left( \int_{\Omega} u_n(y)^p dy \right)^{p+1} \leq \|L_0u_n\|_\infty^p \int_{\Omega} \frac{1}{a(y)^p} dy < \infty.$$

Passing to the limit, and using the fact that $\|L_0u_n\|_\infty \to 0$, we find $\|u_n\|_p \to 0$. Therefore, $u_n \to 0$ in $C(\Omega)$, which contradicts the limit $\|\Phi_{u_n}\|_\infty \to \lambda > 0$.

**The case $u \neq 0$:** From Lemma 2.5, $u > 0$ and $\lambda - v(x) > 0$ in $\overline{\Omega}$. On the other hand, arguing as above, $u_n \to u$ in $L^p(\Omega)$. Hence, $\Phi_{u_n} \to \Phi_u$ in $C(\Omega)$, $\|\Phi_{u_n}\|_\infty \to \|\Phi_u\|_\infty$ and $\lambda - \Phi_u(x) > 0$ for all $x \in \Omega$. Then, $\lambda - \|\Phi_u\|_\infty > 0$, which contradicts $\|\Phi_{u_n}\|_\infty \to \lambda$.

Now, we are going to prove the existence of $M$. Note that

$$|u(x)| \leq \frac{|L_0u(x)|}{\lambda - \Phi_u(x)} \leq \frac{\|L_0u\|_\infty}{\lambda} \leq \frac{\|K\|_\infty}{\rho} \|u\|_p, \quad \text{for all } x \in \overline{\Omega},$$

where we have used the Hölder inequality in the last inequality with $\frac{1}{p} + \frac{1}{q} = 1$. From Lemma 3.4, $\|u\|_p$ is bounded, then there is a $M > 0$ such that $\|u\|_\infty \leq M$.

**Case 2:** $p \in (0,1]$. As in the first case, we can assume that $u_n^p \not\to 0$ in $L^1(\Omega)$, otherwise we will get $\Phi_{u_n} \to 0$ in $C(\Omega)$, which contradicts the limit $\|\Phi_{u_n}\|_\infty \to \lambda > 0$. In the sequel, as $(u_n^p)$ is bounded in $L^1(\Omega)$, for some subsequence, we can assume that $u_n^p \to \mu$ in $M(\Omega)$ for some $\mu \in M(\Omega)$, where $M(\Omega)$ denotes the space of positive finite measure on $\Omega$. Thereby,

$$\int_{\Omega} \phi u_n^p dx \to \int_{\Omega} \phi d\mu, \quad \forall \phi \in C(\overline{\Omega}),$$

18
and so,
\[ \Phi_{u_n}(x) = \int_{\Omega} Q(x,y)u_n^p \, dx \rightarrow \int_{\Omega} Q(x,y)d\mu = v(x), \quad \forall x \in \overline{\Omega}. \]
As \( \mu \in \mathcal{M}(\Omega) \), a simple computation gives \( v \in C(\overline{\Omega}) \) and \( v(x) \geq 0 \) for all \( x \in \overline{\Omega} \). Using the fact that
\[
\lambda_n - \Phi_{u_n}(x) \geq a(x) \int_{\Omega} u_n^p \, dx,
\]
by taking the limit of \( n \rightarrow +\infty \), we get
\[
\lambda - v(x) \geq a(x)W, \quad \forall x \in \overline{\Omega}
\]
where \( W = \int_{\Omega} d\mu > 0 \). Here we know that \( W > 0 \), because we are supposing that \( u_n^p \not\rightarrow 0 \) in \( L^1(\Omega) \).
Since \( a \) is a nonnegative function and \( a^{-1} \in L^1(\Omega) \), we have that set \( O = \{ x \in \Omega : a(x) = 0 \} \) has null measure, thus
\[
\lambda - v(x) > 0, \quad \text{a.e. in } \overline{\Omega}.
\]

**Claim 3.1.** The sequence \( (u_n) \) is bounded in \( L^1(\Omega) \).

Indeed, assume by contradiction that \( \|u_n\|_1 \rightarrow +\infty \) and set \( w_n = \frac{u_n}{\|u_n\|_1} \). Using the fact that \((\lambda_n, u_n)\) is a solution of \((P)\), we get
\[
L_0 w_n + \Phi_{u_n} w_n = \lambda_n u_n
\]
and so,
\[
w_n = \frac{L_0 w_n}{\lambda_n - \Phi_{u_n}}.
\]
As \( (w_n) \) is bounded in \( L^1(\Omega) \), for some subsequence, we have that \( L_0 w_n \rightarrow w_* \) in \( C(\overline{\Omega}) \), consequently
\[
w_n(x) \rightarrow \frac{w_*(x)}{\lambda - v(x)} = w(x) \quad \text{a.e. in } \overline{\Omega}.
\]
From definition of \( w \), we see that \( w \in C(\Omega \setminus O) \) and \( w(x) \geq 0 \) a.e. in \( \overline{\Omega} \). Moreover, we also have
\[
w_n(x) \leq \frac{2\|L_0 w_n\|_\infty}{a(x)W} \quad \text{a.e. in } \overline{\Omega}.
\]
Since \( a^{-1} \in L^1(\Omega) \), the above informations ensure that
\[
w_n \rightarrow w \quad \text{in } L^1(\Omega),
\]
then \( \|w\|_1 = 1 \). On the other hand, we also have that
\[
L_0 \left( \frac{u_n}{\|u_n\|^{p+1}_1} \right) + \Phi_{u_n}(x)w_n = \lambda_n \frac{u_n}{\|u_n\|^{p+1}_1},
\]
then \( \Phi_{u_n} w_n \rightarrow 0 \) in \( L^1(\Omega) \), and so, by Fatou’s Lemma
\[
\int_{\Omega} \int_{\Omega} Q(x,y)w^p(y)w^2(x) \, dxdy = 0.
\]
From \((Q_2)\) we get \(w = 0\), which is an absurd. This proves that \((u_n)\) is bounded in \(L^1(\Omega)\). Arguing as above, replacing \(w_n\) by \(u_n\), we can prove that \(u_n \rightarrow u\) in \(L^1(\Omega)\), and the lemma follows by repeating the same arguments explored in Case 1.

\begin{proof}

Proof of Theorem 1.2

Gathering all the above informations, for all \(\Lambda > \lambda_1\) we find \(\rho > 0\) and \(M > 0\) such that, for \(U = U_{\Lambda, \frac{\rho}{2}, 2M}\) we have \(C^+ \cap \partial U \neq \emptyset\), thus we have \((\lambda, u) \in C^+\) and one of the following conditions occur: \(\lambda - \|\Phi u\|_{\infty} = \frac{\rho}{2}\) or \(\|u\|_{\infty} = 2M\). But we have seen that, under the above conditions, \(\lambda - \|\Phi u\|_{\infty} > \rho\) and \(\|u\|_{\infty} \leq M\), that is, the connected component \(C^+\) crosses the hyperplane \(\{\Lambda\} \times C(\Omega)\), for all \(\Lambda > \lambda_1\).

To conclude the proof, we will show the nonexistence of solution for \(\lambda \leq \lambda_1\). In fact, suppose that \((\lambda, u)\) satisfies \(u \geq 0\), \(\lambda > 0\) and \(L_0 u + \Phi u u = \lambda u\). Then, for all \(v \in L^2(\Omega)\),

\[\langle L_0 u, v \rangle + \langle \Phi u u, v \rangle = \lambda \langle u, v \rangle.\]

Taking \(v = \varphi_1\), the eigenfunction associated with \(\lambda_1\), we derive

\[\langle L_0 u, \varphi_1 \rangle + \langle \Phi u u, \varphi_1 \rangle = \lambda \langle u, \varphi_1 \rangle.\]

As \(L_0\) is symmetric in \(L^2(\Omega)\), it follows that

\[\lambda_1 \langle \varphi_1, u \rangle + \langle \Phi u u, \varphi_1 \rangle = \lambda \langle u, \varphi_1 \rangle.\]

Using the fact \(\langle \Phi u u, \varphi_1 \rangle > 0\), we have

\[\lambda_1 \langle \varphi_1, u \rangle < \lambda \langle \varphi_1, u \rangle\]

i.e.,

\[(\lambda_1 - \lambda) \int_\Omega \varphi_1 u dx < 0,\]

showing that \(\lambda_1 < \lambda\).

4 Proof of Theorem 1.3

In this section, we will fix \(\lambda > \lambda_1\) and show that Problem \((P)\) has a positive solution that will be the uniform limit of solutions given by Theorem 1.2.

First, consider \(0 < \epsilon \leq \epsilon_0 = \frac{N}{2p}\) and define

\[Q_\epsilon(x, y) = Q(x, y)(2 - a_\epsilon(x))\]

20
Proof. Note that, \(2Q(x,y) \geq Q^e(x,y) \geq Q(x,y) \geq 0\) for all \(x, y \in \Omega\) and \(Q^e(x,y) \geq \sigma\) when \(|x - y| \leq r\), and so, \(Q^e\) verifies \((Q_2)\). Moreover, 
\[
Q^e(x_0, y) - Q^e(x, y) = 2Q(x_0, y) - Q(x, y)(2 - a^e(x)) \geq 2Q(x, y)(2 - a^e(x))
\]
that is, 
\[
Q^e(x_0, y) - Q^e(x, y) \geq Q(x, y)a^e(x) \geq \frac{1}{2}Q^e(x, y)a^e(x), \quad \text{for all } x, y \in \Omega. \tag{12}
\]

Related to \(a^e\) we have: \(a^e(x) \leq 1\) for all \(x \in \Omega\).

We will consider the following family of auxiliary problems:

\[
L_0u + \Phi^e_u(x)u = \lambda u, \quad \text{in } \Omega. \tag{P^e}
\]

**Lemma 4.1.** Suppose that \((Q_2), (Q_4)\) hold and fix \(\epsilon > 0\). Then, if \((\lambda, u)\) is a positive solution of \((P^e)\) with \(\lambda > \lambda_1\) and \(u > 0\), we have 
\[
\lambda - \Phi^e_u(x) \geq \theta a^e(x), \quad \text{for all } x \in \Omega, \tag{13}
\]
where \(\theta = \min\{\lambda_1, \lambda - \lambda_1\}\).

**Proof.** Note that, \((Q_2), (Q_4)\) and \([12]\) imply in \(\Phi^e_u(x_0) \geq \Phi^e_u(x)\), for all \(x \in \Omega\) and,
\[
\lambda - \Phi^e_u(x) \geq \Phi^e_u(x_0) - \Phi^e_u(x) \geq \sigma(x) \int_{\Omega} Q(x, y)u(y)dy \geq \frac{1}{2}a^e(x)\Phi^e_u(x). \tag{14}
\]
Thus, if \(\Phi^e_u(x) \leq \lambda_1\), we have \(\lambda - \Phi^e_u(x) \geq \lambda - \lambda_1 \geq (\lambda - \lambda_1)a^e(x)\). On the other hand, if \(\Phi^e_u(x) > \lambda_1\), from \([14]\) we have \(\lambda - \Phi^e_u(x) \geq \lambda_1 a^e(x)\). Anyway, since \(\theta = \min\{\lambda_1, \lambda - \lambda_1\}\) we have that 
\[
\lambda - \Phi^e_u(x) \geq \theta a^e(x), \quad \text{for all } x \in \Omega.
\]

As a consequence of Lemma [4.1] we see that the same conclusion of Lemma [3.3] for the auxiliary problem \((P^e)\).

**Lemma 4.2.** Suppose that \(0 < \epsilon \leq \epsilon_0\) is fixed, \(p > 0\), \((Q_2)\) and \((Q_4)\) hold. Then there exist \(\rho > 0\) and \(M > 0\) such that \(\lambda - \|\Phi^e_u\|_\infty \geq \rho\) and \(\|u\|_\infty \leq M\) for all \((\lambda, u)\) satisfying \((P^e)\), with \(u > 0\) and \(\lambda \in (\lambda_1, \Lambda]\).
Proof. We proceed exactly as in the proof of Lemma 3.5. Here we replace (10) by (13), in the case $p > 1$, and we replace (11) by (13) for the case $p \leq 1$.

Using Lemma 4.2 and following all the steps in the proof of Theorem 1.2, for any fixed $\epsilon \in (0, \epsilon_0]$, we have a connected component $C_\epsilon^+$, associated to the bifurcation equation $(P_\epsilon)$, which crosses the hyperplans $\{\lambda\} \times C(\Omega)$ for all $\lambda > \lambda_1$.

Remark 4.1. We resume this last remark as follows: for any $\lambda > \lambda_1$ and any $\epsilon \in (0, \epsilon_0]$ we have a positive $u \in C(\Omega)$ satisfying $(P_\epsilon)$.

Now we are ready to prove Theorem 1.3.

Proof. (Theorem 1.3) Fix $\lambda > \lambda_1$ again and consider the functions $g_n : \overline{\Omega} \to \mathbb{R}$ defined by

$$g_n(x) = \int_\Omega Q_\lambda(x, y)|u_n(y)|^p dy,$$

where $u_n$ is given by Remark 4.1 with $\epsilon = \frac{1}{n}$, which verifies $L_0u_n + g_n(x)u_n = \lambda u_n$. The proof consists in proving that the problem $(P)$ has a solution which is a limit of a subsequence of $u_n$ when $n$ goes to infinity.

We know that $\|u_n\|_p$ is bounded and $g_n$ is bounded in $L^\infty(\Omega)$. Moreover, $g_n$ is uniformly convergent in compact parts of $\overline{\Omega} \setminus \{x_0\}$. In fact,

$$|g_n(x) - g_m(x)| \leq 2\|Q\|_\infty \|u_n\|_p^p|x - x_0|^\frac{1}{p} - |x - x_0|^\frac{1}{p}, \quad \text{if } |x - x_0| \leq 1$$

and

$$|g_n(x) - g_m(x)| = 0, \quad \text{if } |x - x_0| > 1.$$

In the sequel we divide into two cases our study, namely $p > 1$ and $p \in (0, 1)$.

Case 1: $p > 1$. By Lemma 3.4 $(u_n)$ is a bounded sequence in $L^p(\Omega)$, and as $p > 1$, there is a some subsequence of $(u_n)$, still denoted by itself, such that $u_n \rightharpoonup u$ in $L^p(\Omega)$. As in the proof of Theorem 1.2 $L_0u_n$ converges to $L_0u$ uniformly in $\overline{\Omega}$. Denote by $v$ the uniform limit of $g_n$ in the compact parts of $\overline{\Omega} \setminus \{x_0\}$. Let us show that, $u_n$ converges to $u$ in $L^p(\Omega)$. From Lemma 3.1 we have

$$\lambda - g_n(x) \geq \theta a_{\lambda, \epsilon}(x) \geq \theta R^\frac{1}{p}, \quad \text{for } x \in \overline{\Omega} \setminus B_R(x_0), \quad (R < 1)$$

that is, $\lambda - g_n(x)$ converges uniformly to $\lambda - v(x)$ in $\overline{\Omega} \setminus B_R(x_0)$ which implies $\lambda - v(x) \geq \theta$ in $\overline{\Omega} \setminus B_R(x_0)$, hence

$$\frac{u_n(x) = \frac{L_0u_n(x)}{\lambda - g_n(x)} \to \frac{L_0u(x)}{\lambda - v(x)}}{\text{uniformly in } \overline{\Omega} \setminus B_R(x_0)}.$$

22
Moreover, \( v(x) < \lambda \) for \( x \neq x_0 \). Now, in the neighborhood of \( x_0 \), we have

\[
\int_{B_R(x_0)} u_n(y)^p dy = \int_{B_R(x_0)} \left[ \frac{L_0 u_n(y)}{\lambda - g_n(y)} \right]^p dy \leq \frac{\|L_0 u_n\|_{L_p}^p}{\theta^p} \int_{B_R(x_0)} \frac{1}{u_n^{1/p}(y)} dy \leq \frac{\|L_0 u_n\|_{L_p}^p \omega(N) R^{N-\frac{p}{p}}} {\theta^p (N - \frac{2}{p})}.
\]

The \( L^p \)-convergence of \( (u_n) \) follows from \([19]\) and \([10]\).

Now, we claim that \( g_n \) converges to \( \Phi_u \) in \( \overline{\Omega} \). In fact, since \( (u_n) \) converges in \( L^p(\Omega) \), passing to a subsequence if necessary, \( (u_n) \) is dominated by a function \( h \in L^p(\Omega) \) and

\[
Q_\perp(x, y) u_n(y)^p \leq 2\|Q\|_\infty h(y)^p.
\]

The assertion follows from the dominated convergence. As \( g_n(x) < \lambda \), we have \( \Phi_u(x) \leq \lambda \). Passing to the weak limit in the \( L^p(\Omega) \) sense in \( L_0 u_n + g_n(x) u_n = \lambda u_n \), we obtain

\[
L_0 u = (\lambda - \Phi_u(x)) u \quad \text{a.e. in } \Omega.
\]

In the sequel, we will consider the cases \( u \equiv 0 \) and \( u \neq 0 \).

**The case** \( u \equiv 0 \): We can not have \( u \equiv 0 \), because \( g_n(x) \leq 2\|Q\|_\infty \|u_n\|^p_p \) and thus \( g_n \) converges uniformly to \( 0 \) in \( \overline{\Omega} \), which contradicts the Lemma \([25]\) because we would have \( \lambda - g_n(x) > \lambda_1 \) in \( \overline{\Omega} \) for \( n \) large enough.

**The case** \( u \neq 0 \): From Lemma \([25]\), \( u > 0 \) and \( \lambda - v(x) > 0 \) in \( \overline{\Omega} \). On the other hand, arguing as above, \( u_n \to u \) in \( L^p(\Omega) \). Hence, \( g_n \to \Phi_u \) in \( C(\overline{\Omega}) \), \( \|g_n\|_\infty \to \|\Phi_u\|_\infty \) and \( \lambda - \Phi_u(x) > 0 \) for all \( x \in \overline{\Omega} \). Then \( u \) is a positive solution sought.

**Case 2:** \( p \in (0, 1] \): As in the first case, we can assume that \( u_n^p \not\to 0 \) in \( L^1(\Omega) \). In the sequel, as \( (u_n^p) \) is bounded in \( L^1(\Omega) \), for some subsequence, we can assume that \( u_n^p \to \mu \) in \( M(\Omega) \) for some \( \mu \in M(\Omega) \), where \( M(\Omega) \) denotes the space of positive finite measure on \( \Omega \). Thereby,

\[
\int_{\Omega} \phi u_n^p \, dy \to \int_{\Omega} \phi \, d\mu, \quad \forall \phi \in C(\overline{\Omega}),
\]

and so,

\[
g_n(x) = \int_{\Omega} Q_\perp(x, y) u_n^p \, dy \to \int_{\Omega} Q(x, y) \, d\mu = v(x), \quad \forall x \in \overline{\Omega}.
\]

As \( \mu \in M(\Omega) \), a simple computation gives \( v \in C(\overline{\Omega}) \) and \( v(x) \geq 0 \) for all \( x \in \overline{\Omega} \). Using the fact that

\[
\lambda_n - g_n(x) \geq \theta a_\perp(x), \quad \text{for } x \neq x_0
\]

by taking the limit of \( n \to +\infty \), we get

\[
\lambda - v(x) \geq \theta > 0, \quad \forall x \in \overline{\Omega} \setminus \{x_0\}
\]

Hence, \( \lambda - v(x) \geq \theta > 0, \ a.e. \ in \overline{\Omega} \).
Claim 4.1. The sequence \((u_n)\) is bounded in \(L^1(\Omega)\).

Indeed, assume by contradiction that \(\|u_n\|_1 \to +\infty\) and set \(w_n = \frac{u_n}{\|u_n\|_1}\). Using the fact that \((\lambda, u_n)\) is a solution of \((P)\), we get

\[
L_0 w_n + g_n(x) w_n = \lambda w_n
\]

and so,

\[
w_n = \frac{L_0 w_n}{\lambda - g_n(x)}.
\]

As \((w_n)\) is bounded in \(L^1(\Omega)\), for some subsequence, we have that \(L_0 w_n \to w_*\) in \(C(\overline{\Omega})\), consequently

\[
w_n(x) \to \frac{w_*(x)}{\lambda - v(x)} = w(x) \quad \text{a.e. in } \overline{\Omega}.
\]

From definition of \(w\), we see that \(w \in C(\overline{\Omega})\) and \(w(x) \geq 0\) a.e. in \(\overline{\Omega}\). Moreover, we also have

\[
w_n(x) \leq \frac{2\|L_0 w_n\|_\infty}{\theta} \quad \text{a.e. in } \overline{\Omega}.
\]

The above informations ensure that

\[
w_n \to w \quad \text{in } L^1(\Omega),
\]

then \(\|w\|_1 = 1\). On the other hand, we also have that

\[
L_0 \left( \frac{u_n}{\|u_n\|_1^{p+1}} \right) + \Phi_{\frac{1}{w_n}}(x) w_n = \lambda \frac{u_n}{\|u_n\|_1^{p+1}},
\]

where

\[
\Phi_{\frac{1}{w_n}}(x) = \int_{\Omega} Q_{\frac{1}{w_n}}(x, y) |w_n(y)|^p dy.
\]

Then \(\Phi_{\frac{1}{w_n}} w_n \to 0\) in \(L^1(\Omega)\), and so, by Fatou’s Lemma

\[
\int_{\Omega} \int_{\Omega} Q(x, y) w^p(y) w^2(x) dx dy = 0,
\]

from \((Q_2)\) we get \(w = 0\), which is absurd. This proves that \((u_n)\) is bounded in \(L^1(\Omega)\). Arguing as above, replacing \(w_n\) by \(u_n\), we can prove that \(u_n \to u\) in \(L^1(\Omega)\), and the result follows by repeating the same arguments explored in Case 1.

Final comments: We expect that the bifurcation results in this work remain valid for weaker conditions on \(Q\), without assumptions \((Q_3)\) and \((Q_4)\) for instance.

To finish this work, we would like to remark that the same result of this section holds under the condition \((Q_4')\):
There are a decomposition \( \Omega = \bigcup_{j=1}^{m} E_j \), \( x_1 \in E_1, x_2 \in E_2, \ldots, x_m \in E_m \) such that \( Q(x_j, y) \geq Q(x, y) \) for all \( x \in E_j, y \in \Omega \).

It is easy to check if we consider \( Q_{\varepsilon} \) replacing \( a_{\varepsilon} \) by

\[
a_{\varepsilon}(x) = \begin{cases} 
|x - x_1|^\tau \ldots |x - x_m|^\tau, & \text{if } |x - x_0| \leq 1 \\
1, & \text{if } |x - x_0| \geq 1,
\end{cases}
\]

then the proof works with the following adjustments. The uniform convergence of \( g_n \) is in the compact parts of \( \Omega \setminus \{x_1, \ldots, x_m\} \). In the case 1, \( p > 1 \), \( u_n \) converges uniformly in

\[
\Omega \setminus \bigcup_{j=1}^{m} B_R(x_j), \text{ for small } R > 0
\]

and for any \( j = 1, 2, \ldots, m \),

\[
\int_{B_R(x_j)} u_n(y)^p dy = \int_{B_R(x_j)} \left[ \frac{L_0 u_n(y)}{\lambda - g_n(y)} \right]^p dy \leq \frac{\|L_0 u_n\|_{p, \infty}}{\theta^p} \int_{B_R(x_j)} \frac{1}{a_{N/\theta}^p(y)} dy \leq \frac{\|L_0 u_n\|_{p, \infty}}{\theta^p(N - \frac{\theta}{\omega})}.
\]

The proof follows as in the proof of Theorem 1.3.

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