UNIQUE ERGODICITY OF HARMONIC CURRENTS ON
SINGULAR FOLIATIONS OF $\mathbb{P}^2$

Abstract. Let $\mathcal{F}$ be a holomorphic foliation of $\mathbb{P}^2$ by Riemann surfaces. Assume all the singular points of $\mathcal{F}$ are hyperbolic. If $\mathcal{F}$ has no algebraic leaf, then there is a unique positive harmonic $(1,1)$ current $T$ of mass one, directed by $\mathcal{F}$. This implies strong ergodic properties for the foliation $\mathcal{F}$. We also study the harmonic flow associated to the current $T$.

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1. Introduction

Let $\mathcal{F}$ be a holomorphic foliation of the complex projective space $\mathbb{P}^2$. Our purpose is to study the ergodic properties of $\mathcal{F}$, using the theory of harmonic currents as developed by the authors in [8].

A holomorphic foliation can be seen as a rational vector field in $\mathbb{C}^2$. Our goal is to develop an ergodic theory for the dynamics of such vector fields. The two main difficulties are: the presence of singularities (they always exist) and the absence (generically) of algebraic leaves. And hence it is not clear where to start the analysis. Our method is geometric but requires difficult estimates. To our knowledge, this is the first paper where global dynamical results for rational vector fields are obtained. The subject is classical and related to polynomial vector fields in $\mathbb{R}^2$.

We first recall a few facts. Let $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ denote the canonical projection. The foliation $\pi^* \mathcal{F}$ can be defined in $\mathbb{C}^3$ by a global 1-form $\omega_0 = a_1(x) dx_1 + a_2(x) dx_2 + a_3(x) dx_3$ where the $a_j(x)$ are homogeneous polynomials of the same degree $\delta \geq 1$ without common factors. Moreover since every line through the origin is in the kernel of $\omega_0$, they satisfy the condition $\sum x_i a_i(x) = 0$.

The degree of $\mathcal{F}$ is by definition $\deg \mathcal{F} = d := \deg \delta - 1$. It represents the number of tangencies of a generic line $L$, with $\mathcal{F}$. Let $\text{Fol}(d)$ denote the space of foliations of degree $d$. The space of coefficients of 1 forms of degree $\delta$ is a projective space. The subspace given by $\sum x_i a_i = 0$ is a linear subspace, so also a projective space. The subspace of 1 forms of degree $\delta$ of the form $H \lambda$ where $H$ is a homogenous polynomial of degree $0 < \delta' < \delta$ and $\lambda$ is a 1-form of degree $\delta - \delta'$ is an algebraic subvariety. So together this gives that $\text{Fol}(d)$ is the complement of an algebraic subvariety of some $\mathbb{P}^N$. It follows from the Bézout theorem that the foliation $\mathcal{F}$ has a finite number of singularities bounded uniformly by some function of the degree. If in a coordinate chart $U$, $\mathcal{F}$ is defined by $\omega_1 = \alpha(z,w) dz + \beta(z,w) dw$, then $\text{sing}(\mathcal{F}) \cap U = \{ \alpha = \beta = 0 \}$. We can assume that all the singular points are in the same $\mathbb{C}^2$, $\{ p_j = (\alpha_j, \beta_j) \}_{j \leq N}$.

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Definition 1. Suppose there is a change of coordinates around \( p_j \) sending \( p_j \) to 0 and such that \( \omega_0(z,w) = zdw - \lambda wdz + O(z,w)^2 \) where \( \lambda = a + ib \) and \( b \) is a nonzero number. We say in this case that the singularity is hyperbolic and that we are in the Poincaré domain.

The following is a classical fact due to Poincaré, see [5].

Theorem 1. Suppose that the singular point is hyperbolic. Then there is a local biholomorphic change of coordinates so that the form \( \omega_0 \) in these coordinates can be written \( \omega_0 = zdw - \lambda wdz \) (with the same \( \lambda \)).

We remark that the form \( \omega_0 \) is invariant under scaling except for multiplication by a constant which of course does not affect the zero set. Hence we can assume that the linearization is valid in a fixed large ball, in particular in a neighborhood of the unit bidisc.

The following result is due to Lins Neto, Soares [12] (we give only the two dimensional version, their result is also valid in \( \mathbb{P}^k \)). The result uses Jouanolou’s example of a foliation in \( \mathbb{P}^2 \) without algebraic leaves.

Theorem 2. There exists a real Zariski dense open subset \( \mathcal{H}(d) \subset \text{Fol}(d) \) such that any \( F \subset \mathcal{H}(d) \) satisfies:

i) \( F \) has only hyperbolic singularities and no other singular points.

ii) \( F \) has no invariant algebraic curve.

The global behavior of foliations is not well understood. It is unknown whether every leaf of a given foliation \( F \), clusters at a singular point. This problem, known as the problem of existence of a minimal exceptional set is discussed in [6] and [2] for example. It is conjectured in [11] that a generic holomorphic foliation by Riemann surfaces in \( \mathbb{P}^k \) has dense leaves. Recently Loray and Rebelo [13] have constructed non empty open sets of holomorphic foliations by Riemann surfaces in \( \mathbb{P}^k \) such that every leaf is dense.

L. Garnett [10] has introduced the notion of harmonic measure for smooth foliations (without singularities) of a compact Riemannian manifold. She studied their ergodic properties. The article by Candel [4] contains a recent approach to that theory. In [8] the authors have shown that a \( C^1 \) laminated set in \( \mathbb{P}^2 \), without singularities carries a unique harmonic current of mass 1 directed by the lamination. Very recently Deroin and Klepsyn [7] developed the theory of diffusion on transversally conformal foliations and they showed that there are only finitely many harmonic measures.

For holomorphic foliations (with singularities) of \( \mathbb{P}^2 \) the following analogue was proved in [1]. It is valid for laminations by Riemann surfaces with a small set of singularities, see [1] and [8].

Theorem 3. Let \( F \) be a holomorphic foliation of \( \mathbb{P}^2 \). There exists a positive current \( T \) on \( \mathbb{P}^2 \), of bidimension \((1,1)\) and mass 1 which is harmonic, i.e. \( i\partial \bar{\partial} T = 0 \). Moreover in any flow box \( B \), (without singular points) the current can be expressed as

\[
T = \int h_{\alpha}[V_{\alpha}]d\mu(\alpha).
\]

The functions \( h_{\alpha} \) are positive harmonic on the local leaves \( V_{\alpha} \) and \( \mu \) is a Borel measure on the transversal. The function \( H : B \to \mathbb{R}^+ \), \( H|_{V_{\alpha}} = h_{\alpha} \) is Borel measurable.
Observe that if $F$ is defined in $B$ by a smooth form $\omega_0$, then $T \wedge \omega_0 = 0$. We will say that the current is directed by $F$.

A theory of intersection of positive harmonic currents of bidegree $(1, 1)$ is developed in \cite{8}. The main purpose of the present article is, using that intersection theory, to prove:

**Theorem 4.** Let $F$ be a holomorphic foliation in $\mathbb{P}^2$ without algebraic leaves. Assume that all singular points of $F$ are hyperbolic. Then there is a unique positive harmonic current $T$ of mass one, directed by $F$.

A consequence of Theorem 4 and of results from \cite{8} is that the foliations $F$ with only hyperbolic singular points are uniquely ergodic in a very strong sense, see Corollary 2, i.e. the current $T$ can be obtained by an averaging process on the leaves, whose limit is independent of the leaf. We will show, in Remark 2 section 26, a similar uniqueness result for some classes of foliations with non hyperbolic singularities.

Observe that under the assumption of Theorem 4 there is no non zero positive closed current directed by $F$, see \cite{8} and Brunella \cite{3} for a general discussion of closed cycles on foliations by Riemann surfaces.

The intersection theory of positive harmonic currents in \cite{8} is valid on compact Kähler manifolds. We just recall a few facts restricting to $\mathbb{P}^2$.

Let $T$ be a positive harmonic current of bidegree $(1, 1)$ in $\mathbb{P}^2$, i.e. $i \partial \bar{\partial} T = 0$. Let $\omega$ denote the standard Kähler form on $\mathbb{P}^2$. Then $T$ can be written as

$$T = c \omega + \partial S + \bar{\partial} S + i \partial \bar{\partial} u$$

with $c \geq 0$ and $S$ is a $(0, 1)$ form such that $S, \partial S, \bar{\partial} S$ are in $L^2$ and $u \in L^1$. The current $\partial S$ depends only on $T$ and is zero only if $T$ is closed. So the quantity $\int \partial S \wedge \bar{\partial} S$ which we called energy, measures how far $T$ is from being closed. The expression

$$\int T \wedge T := \int (c \omega + \partial S + \bar{\partial} S) \wedge (c \omega + \partial S + \bar{\partial} S)$$

makes sense and is finite. It is independent on the choice of $S$. Moreover if $T_1$ and $T_2$ are 2 positive harmonic currents such that $\int T_1 \wedge T_2 = 0$, then $T_1$ and $T_2$ are proportional mod $(\partial \bar{\partial} u)$. For currents directed by foliations and whose support does carry a positive closed current, then $\int T_1 \wedge T_2 = 0$ implies that $T_1, T_2$ are proportional, see \cite{8} Lemma 3.10. On the other hand the currents directed by holomorphic foliations can be expressed in a flow box $B$ as

$$T = \int h_\alpha [V_\alpha] d\mu(\alpha)$$

as described in Theorem 3. It is hence possible to consider the geometric self intersection of such currents. More precisely consider suitable automorphisms $\Phi_\epsilon$ of $\mathbb{P}^2$ which are close to the identity. For a current $T$ directed by a foliation $F$, it is possible to define the geometric intersection $T \wedge \Phi_\epsilon(T)$ as the measure on the complement of the singular points given locally by the expression
\[ \int \left[ \sum_{p \in J_{\epsilon, \alpha, \beta}} h_{\alpha}(p) h_{\beta}^*(p) \delta_p \right] d\mu(\alpha) d\mu(\beta) \quad (1) \]

where \( J_{\epsilon, \alpha, \beta} \) denotes the points of intersection of the plaque \( L_\alpha \) and the plaque \((\Phi_\epsilon)_* L_\beta \) and \( \delta_p \) denotes the Dirac mass at \( p \). It is shown in [8] that \( \int T_1 \wedge T_2 = \lim_{\epsilon \to 0} \int T_1 \wedge g T_{2, \epsilon} \) ([8], Lemma 19). To show that \( \int T_1 \wedge T_2 = 0 \) it is enough to count the number of points of intersection of a given plaque with perturbed plaques and estimate the harmonic functions. This is done in [8] (Theorem 6.2) when we assume that the currents \( T_1, T_2 \) are supported on a minimal laminated compact set, which is transversely of class \( C^1 \).

Indeed the minimality hypothesis is not used and the argument there gives the following stronger result.

**Theorem 5.** Let \( \mathcal{F} \) be a \( C^1 \) lamination with singularities by Riemann surfaces in \( \mathbb{P}^2 \). Assume that there is a laminated compact set \( X \) without singularities. Then there is a unique positive harmonic current \( T \), of mass 1, directed by \( \mathcal{F} \).

**Proof.** We know there is a harmonic current \( T_1 \) of mass 1, supported on \( X \). Let \( T_2 \) be another such current directed by \( \mathcal{F} \), but not necessarily supported by \( X \). The argument in [8] Theorem 6.2 shows that \( \lim_{\epsilon \to 0} \int T_1 \wedge g T_{2, \epsilon} = 0 \). Hence \( \int T_1 \wedge T_2 = 0 \). Therefore \( T_1 \) and \( T_2 \) are proportional. \( \square \)

We now deal with the case where the foliation is holomorphic and the current \( T \) contains in its support singular points (which are all hyperbolic).

We will prove the following more general result than Theorem 4.

**Theorem 6.** (MAIN THEOREM) Let \( \mathcal{F} \) be a holomorphic foliation of \( \mathbb{P}^2 \) without algebraic leaves. Let \( X \) be a closed invariant set for \( \mathcal{F} \). Assume that all singular points of \( X \) are hyperbolic. Then there is a unique positive harmonic current \( T \) of mass 1, directed by \( X \).

The result is valid for a laminated set \( (X, \mathcal{L}, E) \) where \( X \setminus E \) is a \( C^1 \) lamination by Riemann surfaces. The set \( E = \{p_1, \ldots, p_\ell\} \) is a finite set and in a neighborhood \( U_j \) of every singular point \( p_j \) we assume that \( X \cap U_j \) is holomorphically equivalent to a lamination contained in \( z = C w^{\lambda_j}, \lambda_j = a_j + ib_j, b_j \neq 0 \). One of the consequences of the main theorem is Corollary 2 (section 26) which says that appropriate weighted averages of the leaves always converge to the current \( T \). This is a strong ergodic theorem. The uniqueness of \( T \) also permits to show that \( \lambda \to T_\lambda \) is continuous when \( \lambda \) varies in a holomorphic family of foliations as considered in the main theorem.

It is easy to see that \( \overline{\partial} T = \varpi \wedge T, \tau \) is a \( (1, 0) \) form along leaves. We consider in Section 27 a metric \( g_T := \frac{1}{2} T \wedge \varpi \) and we show that the curvature \( \kappa \) of that metric satisfies \( \kappa (g_T) = -1 \). We also define a finite measure \( \mu_T := iT \wedge \varpi \wedge T \). We have that the measures vary continuously with the foliation. The metric \( g_T \) and the measure \( \mu_T \) were introduced by S. Frankel [9] in the nonsingular case.

2. **Proof of the Main Theorem**

Let \( T \) be a harmonic current of mass 1 supported on \( X \) and directed by \( \mathcal{F} \). In a flow box
\[ T = \int h_{\alpha}[V_{\alpha}]d\mu(\alpha). \]  

(2)

We have to estimate the number of intersection points of a plaque with perturbed plaques near a singularity and also to study the behaviour of the harmonic continuation \( \tilde{h}_{\alpha} \) of \( h_{\alpha} \) along a leaf near a hyperbolic singularity.

This will give us that the geometric intersection is zero and hence \( \int T \land T = 0 \). Since \( T \) is arbitrary, the intersection theory of positive harmonic currents implies that \( T \) is unique.

After a change of coordinates we do the analysis for the form \( \omega_0 = zdw - \lambda wd\bar{z} \), \( \lambda = a + ib, b \neq 0 \), near \((0,0)\).

In order to study positive harmonic currents near \( 0 \), we cover a deleted neighborhood of \( 0 \) by finitely many "flow boxes" \((B_i)_{1 \leq i \leq N}\), with \( 0 \in B_i \) for every \( i \). Each \( B_i = S_i \times \Delta \), where \( S_i \) is a sector in \( \mathbb{C} \) such that the map \( \zeta \rightarrow e^{\zeta} \) is injective in a strip in the \( \zeta \)-plane \( \gamma_1 < \Im \zeta < \gamma_2 \), with values in \( S_i \), \( \Delta \) is a disc in \( \mathbb{C} \), centered at \( 0 \). So the leaves in \( B_i \) are graphs over all or part of \( S_i \). We will consider them as the local plaques. For the sake of argument we will use the sector \( S \) given by \( 0 < u < 2\pi \).

The strategy for the proof is to choose a family of automorphisms \((\Phi_\epsilon)\) of \( \mathbb{P}^2 \), close to the identity and to estimate the integral (1) in the flow boxes \((B_i)_{i \leq N}\). For that purpose we need to estimate the growth of the harmonic continuation of \( h_{\alpha} \) along the leaves and also the number of intersection points of a plaque \( L_\alpha \), with perturbed plaques \( L_\beta \).

Away from singularities this is just the proof given in [8] for a lamination. In the present case we have to divide the phase space in many regions where the estimates are technically different. The estimates are different close to separatrices and in other regions. This requires a precise subdivision of a polydisc near a singular point. We will describe the subdivision in more details after stating Theorem 7.

Consider again the foliation \( zd\bar{z} - \lambda wdz = 0, \lambda = a + ib, b \neq 0 \). Notice that if we flip \( z \) and \( w \), we replace \( \lambda \) by \( 1/\lambda = \overline{\lambda}/|\lambda|^2 = a/(a^2 + b^2) - ib/(a^2 + b^2) \). We will assume below that the axes are chosen so that \( b > 0 \). However, it is important to note that the estimates are also valid if \( b < 0 \). The point is that it will be seen that the case \( a = 1 \) is a degeneracy that complicates the estimates. However if we flip coordinates, the constant \( a = 1 \) becomes \( a/(a^2 + b^2) = 1/(1 + b^2) < 1 \). We now describe a general leaf.

There are two separatrices, \((w = 0), (z = 0)\). Other than that a leaf \( L_\alpha \) can be parametrized by

\[
\begin{align*}
(z, w) &= \psi_\alpha(\zeta) \\
z &= e^{i(\zeta + (\log |\alpha|)/b)}, \zeta = u + iv \\
w &= \alpha e^{i\lambda(\zeta + (\log |\alpha|)/b)} \\
|z| &= e^{-v} \\
|w| &= e^{-bu-av}
\end{align*}
\]

Notice that as we follow \( z \) once counterclockwise around the origin, \( u \) increases by \( 2\pi \), so the absolute value of \(|w|\) decreases by the multiplicative factor of \( e^{-2\pi b} \).
Proposition 1. The harmonic function $e^{-2\pi b} \leq |x| < 1$. We observe that with the above parametrization, the intersection with the unit bidisc of the leaf is given by $v > 0, u > -av/b$ independently of $\alpha$. In the $(u,v)$-plane this corresponds to a sector $S = S_\alpha$ with corner at $0$ and given by $0 < \theta < \arctan(-b/a)$ where the arctan is chosen to have values in $(0, \pi)$. Let $\gamma := \frac{\pi}{\arctan(-b/a)}$. Then the map $\phi : \tau \to \tau^\gamma$ maps this sector to the upper half plane with coordinates $(U,V)$. The fact that $\gamma > 1$ will be crucial, this is where the hyperbolicity of singularities is used.

Let $h_\alpha$ denote the harmonic function associated to the current $T$ on the leaf $L_\alpha$. The local leaf clusters on both separatrices. To investigate the clustering on the $z-$ axis, we use a transversal $D_{z_0} := \{(z_0, w); |w| < 1\}$ for some $|z_0| = 1$. We can normalize so that $h_\alpha(z_0, w) = 1$ where $(z_0, w)$ is the point on the local leaf with $e^{-2\pi b} \leq |w| < 1$. So $(z_0, w) = \psi_\alpha(z_0) = \psi_\alpha(u_0 + iv_0)$ with $v_0 = 0$ and $0 < u_0 \leq 2\pi$ determined by the equations $|z_0| = e^{-v_0} = 1$ and $e^{-2\pi b} \leq |w| = e^{-bu_0 - av_0} < 1$. Let $\tilde{h}_\alpha$ denote the harmonic continuation along $L_\alpha$. Define $H_\alpha(\zeta) := \tilde{h}_\alpha(e^{i(c+ (\log |x|)/b)}e^{\lambda (\log |z|)/b})$ on $S_\alpha$.

**Proposition 1.** The harmonic function $\tilde{H}_\alpha := H_\alpha \circ \phi^{-1}$ is the Poisson integral of its boundary values. So in the upper half plane $\{U + iv; V > 0\}$,

$$\tilde{H}_\alpha(U + iv) = \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{H}_\alpha(x) \frac{V}{V^2 + (x - U)^2} dx$$

[for a.e. $\alpha, d\mu$]. Moreover,

$$\int_{e^{-2\pi b} \leq |x| < 1} \int_{-\infty}^{\infty} \tilde{H}_\alpha(x)|x|^\frac{1}{2} - 1 dx d\mu(\alpha) < \infty.$$  

**Proof.** Let $A_n := \{(z_0, w); e^{-2\pi b(n+1)} \leq |w| < e^{-2\pi bn}, n = 0, 1, \ldots\}$. The holonomy map around $(z = 0)$ as described above gives a map

$$A_n \to A_{n+1}.$$  

The transverse masses of these sets are $\int_{A_0} H_\alpha(\zeta_0 + 2\pi n) d\mu(\alpha) = B_n(\zeta_0)$. The functions $B_n(\zeta)$ are harmonic on $\{v > 0, u > -av/b - 2\pi n\}$. Since the transverse mass is finite on $(z = z_0)$ and since the annuli $A_n$ are disjoint we get,

$$\sum_{n=0}^{\infty} B_n(\zeta_0) < \infty. \quad (*)$$

We get a similar estimate along the other separatrix. It follows that

$$\int_{A_0} \left( \int_{S_\alpha} H_\alpha \right) d\mu(\alpha) < \infty. \quad (3)$$

We show now that for almost every $\alpha$, $\tilde{H}_\alpha(x, y)$ is equal to the Poisson integral of its restriction to $y = 0$. Every positive harmonic function on the upper half plane can be written as a sum of a Poisson integral and $c y, c \geq 0$. The problem is to show that $c = 0$.

We consider the restriction $L'_\alpha$ of $L_\alpha$ to the bidisc $\Delta^2(0, e^{-1})$. The leaf $L'_\alpha$ equals $\psi_\alpha(S'_\alpha)$ where $S'_\alpha := \{v > 1, u > -av/b + 1/b\}$. The image of this sector under $\phi$ is a domain of the form $\Delta'_{\lambda, \alpha} = \{x + iy; y > \gamma_\alpha(x)\}$ where $\gamma_\alpha$ is a continuous strictly positive function so that $\gamma_\alpha \to +\infty$ when $|x| \to \infty$. The function $B_1$ is bounded on
the edges of $S'$. So $\tilde{B}_1 := B_1 \circ \phi^{-1}$ is bounded on the graph of $\gamma_\alpha$ and hence there is no term $cy, c > 0$ in the canonical representation of $\tilde{B}_1$. The same argument is valid for the functions $H_\alpha$ at least for $\mu$ almost every $\alpha$.

It follows that the representation as a Poisson integral is valid. On the other hand estimate (3) can be read as

$$\int \frac{e^{-2\pi b}}{\alpha} \leq |\alpha| < 1 \int_0^\infty H_\alpha(u) \, du \, d\mu(\alpha) < \infty$$

and

$$\int_0^\infty H_\alpha(u e^{i \arctan(b/a)}) \, du \, d\mu(\alpha) < \infty,$$

which gives after a change of variables the estimate on the growth of $\tilde{H}_\alpha$.

\[ \square \]

**Corollary 1.** Let $F$ be a foliation as in Theorem 6. Then for any positive harmonic current $T$, directed by $F$, the transverse measure $\mu$ is diffuse.

**Proof.** Assume $\mu$ has an atomic part i.e. a Dirac mass at $p$. Let $L$ be the leaf through $p$. The restriction $T$ to $L$ is a non zero positive harmonic current. We can normalize so that the transverse measure is one. Then we have a positive harmonic function $h$ defined on $L$.

If there is a flow box $B$, away from the singularities, such that $L$ crosses $B$ on infinitely many plaques on which $h$ is bounded below by a strictly positive constant, then we get a contradiction because the mass of $T$ should be finite. In any flow box the leaf must intersect in infinitely many plaques $P_j$ and the harmonic functions $h_j = h|_{P_j}$ must go uniformly to zero as $j \to \infty$.

Let $f$ denote the lifting of the harmonic function to the unit disc, so $f = h \circ \phi$ where $\phi : \Delta \to L$ is a universal covering map. Since $f > 0$ there is a set $S \subset \partial \Delta$ of full measure on which $f$ has non-tangential limits $f(e^{i\theta})$.

**Lemma 1.** The function $f(e^{i\theta}) = 0$ a.e. on $S$.

**Proof.** Suppose that $f(e^{i\theta_0}) > 0, \theta_0 \in S$. We consider the curve $\phi(re^{i\theta_0})$. By the above argument, it follows that this curve can only intersect finitely many plaques in any flow box away from the singular points. But if some plaque is visited infinitely many times as $r \to 1$, we see that $h$ must be constant on this plaque, hence constant on the leaf, a contradiction. It follows that the curve converges to a singular point. Then it follows from [8], p 991 that this only happens on a set of measure 0 because almost every radius leaves some ball around the singularity.

\[ \square \]

A consequence of the Lemma is that the function $f$ is given by the convolution of the Poisson kernel with a singular measure. This implies that the function $f$ is unbounded. Outside any given neighborhood of the singular set the function must be uniformly bounded. But then the Poisson integrals of Proposition 1 are also uniformly bounded. Hence by Proposition 1 the function is uniformly bounded everywhere, a contradiction.

\[ \square \]

**Remark 1.** It is convenient in some later calculations to replace $|x|^{1/\gamma - 1}$ by $(|x| + 1)^{1/\gamma - 1}$ in the integral of Proposition 1. By Harnack, this doesn’t effect the order of magnitude of the integral.
We decompose a leaf $L_\alpha$ into plaques $L_{\alpha,n}$ where $2n\pi < u < 2(n+1)\pi$. Here $n$ is an integer. [Note that if $a \leq 0$, these $n$ must be positive to have a nonempty intersection with the bidisc.] In this way $L_{\alpha,n}$ is a graph over some part of the $z-$axis.

We let $(z,w)$ be a point in $L_\alpha$ parametrized by a point $(u,v)$. We write in polar coordinates, $u + iv = \rho e^{i\theta}$ with $\rho = \sqrt{u^2 + v^2}$, $\theta = \arctan(v/u)$. Then in the $(U,V)$ plane this point corresponds to $U + iV = \phi(u + iv) = (u + iv)^\gamma$,

$$U + iV = \rho^\gamma e^{i\gamma\theta} = \rho^\gamma \cos(\gamma\theta) + i\rho^\gamma \sin(\gamma\theta).$$

We hence get the following formula for the function $H_\alpha(u + iv)$:

**Lemma 2.**

$$H_\alpha(u + iv) = \frac{1}{\pi} \int \tilde{H}_\alpha(x) \frac{\rho^\gamma \sin(\gamma\theta)}{(\rho^\gamma \sin(\gamma\theta))^2 + (x - \rho^\gamma \cos(\gamma\theta))^2} dx$$

Now we write the formula for the perturbed foliation $\mathcal{F}_\epsilon = (\Phi_\epsilon)_*\mathcal{F}$ where $\Phi_\epsilon$ is a family of automorphisms of $\mathbb{P}^2$. We will need as in [8] that all our estimates stay valid when composing $\Phi_\epsilon$ with $\Psi$ in a neighborhood of the identity in $U(3)$ (depending on $\epsilon$). We will need that $\Phi_\epsilon$ moves the singular point in a direction away from the separatrices near all the hyperbolic points. We also need the $\Phi_\epsilon$ to have a common fixed point $p$ in the support of $T$ and that the tangent space of the leaf through $p$ moves to first order with $\epsilon$. So we write in $\mathbb{C}^2$

$$\Phi_\epsilon(z,w) = (\alpha(\epsilon), \beta(\epsilon)) + (z,w) + \epsilon \mathcal{O}(z,w)$$

with $\alpha'(0), \beta'(0) \neq 0$. We will also need that $\lambda \neq \beta'(0)/\alpha'(0)$.

Suppose that $(z,w)$ is a point in the perturbed bidisc $\Phi_\epsilon(\Delta^2)$, not on an indicatrix of $\mathcal{F}_\epsilon$. Then $\Phi_\epsilon^{-1}(z,w)$ is on some plaque $L_{\beta,m}$ with parameters $(u',v')$. We ignore the problem that we need $u' \neq 2\pi$ because we can also use other flow boxes. The original point $(z,w)$ is on a plaque $L_{\beta,m}$ and we get:

**Lemma 3.**

$$H_{\beta}(u' + iv') = \frac{1}{\pi} \int \tilde{H}_{\beta}(y) \frac{(v')^\gamma \sin(\gamma\theta')}{((v')^\gamma \sin(\gamma\theta'))^2 + (y - (v')^\gamma \cos(\gamma\theta'))^2} dy$$

Next, for each $(\alpha, \beta, m, n, \epsilon)$, let $L_{\alpha,\beta,m,n,\epsilon}$ denote the set of points $p$ in a slightly smaller bidisc which belong to $L_{\alpha,n} \cap L_{\beta,m}$. Our main technical result is the following theorem, which says that the geometrical intersection is zero, so that the current $T$ is unique, see section 26.

**Theorem 7.**

$$\lim_{\epsilon \to 0} \int \sum_{m,n} \hat{\mu}_{\alpha,n}(p) \hat{\mu}_{\beta,m}(p) d\mu(\alpha) d\mu(\beta) = 0.$$ 

**Proof.** During the proof it will be convenient to divide up the region of integration into several pieces. For constants $0 < c < C$ and $\delta > 0$, we consider the regions around one of the finitely many singular points. The regions are defined as follows:

$$D_1 = \{(z,w); |z| \leq c\epsilon, |w| \leq c\epsilon\}$$

$$D_2 = \{(z,w); |z| \leq C\epsilon, |w| \leq C\epsilon\} \setminus D_1$$

$$D_3 = \{(z,w); |z| \leq \delta, |w| \leq \delta\} \setminus D_1 \cup D_2.$$
By [8], for any given $\delta > 0$, the contribution to the integral from outside these regions goes to zero when $\epsilon \to 0$, this uses that the measure is diffuse. We will subdivide the regions $D_1, D_2, D_3$ further. For most of these new subregions the contributions go to zero with $\epsilon$. But for some of the subregions, we need $\delta$ to go to zero for the contribution to go to zero. Hence in the following arguments, $\delta$ will be an unspecified small number which will later go to zero. So the way the argument works is, in order to show that the integral becomes smaller than some given $\tau > 0$ when $\epsilon \to 0$, we first fix a very small $\delta$ and then let $\epsilon \to 0$. This is the case at the end of Sections 8, 9. We will constantly use the finiteness of the integral in Proposition 1, in order to show that the limits are zero.

The constants $c$ and $C$ are determined by the geometry of the leaves near the singularity. We choose $c > 0$ small enough that the region $D_1$ does not contain the singular point of the perturbed foliation. In fact we will make $c > 0$ so small that the slopes of the leaves of the perturbed foliation are almost constant on $D_1$. The precise estimate is done in Lemma 3.

For the constant $C$ we want to make sure that the singular point of the perturbed foliation is inside $\Delta^2(0, C|\epsilon|/2)$. So for example the choice $C = 3 \max\{|\alpha'(0)|, |\beta'(0)|\}$ will work.

3. Proof of theorem 7 for the intersection points in $D_1$ (close to the singularity)

**Lemma 4.** Let $\delta > 0$. Then for all small enough $c$, $|\epsilon|$, the slopes of the leaves of $F_\epsilon$, $dw/dz \in \Delta(\lambda \beta'(0)/\alpha'(0); \delta)$ at all points in $D_1$.

**Proof.** Recall that 
\[ \Phi_\epsilon(z, w) = (\alpha(\epsilon), \beta(\epsilon)) + (z, w) + \epsilon O(z, w). \]

We estimate $\omega_\epsilon$ in $D_1$.

\[
\omega_\epsilon = (\Phi_\epsilon)_*(\omega_0) = O(\epsilon^2) + [(z - \alpha(\epsilon))(1 + A \epsilon) + B \epsilon (w - \beta(\epsilon))] dw + [(z - \beta(\epsilon))(-\lambda + C \epsilon) + D \epsilon (z - \alpha(\epsilon))] dz = O(\epsilon^2) + (z - \alpha(\epsilon))dw + (z - \beta(\epsilon))(-\lambda)dz = (z - \alpha'(0)\epsilon + O(\epsilon^2))dw - \lambda(w - \beta'(0)\epsilon + O(\epsilon^2))dz.
\]

So,
\[
dw/dz = \frac{\lambda(w - \beta'(0)\epsilon + O(\epsilon^2))}{z - \alpha'(0)\epsilon + O(\epsilon^2)} = \frac{\lambda \beta'(0)\epsilon + \cdots}{-\alpha'(0)\epsilon + \cdots} = \frac{\lambda \beta'(0)}{\alpha'(0)} + \cdots
\]

The Lemma follows immediately. 
\[ \square \]
The following lemma describes the lamination associated to \( \omega_c \) near \( D_1 \) after possibly shrinking \( c \) further and is an immediate consequence of Lemma 3.

**Lemma 5.** The plaques of \( \mathcal{F}_c \) near \( D_1 \) are of the form \( w = f_\eta(z) \) where \( f_\eta(\eta) = 0 \) and \( f_\eta' \in \Delta(u_0, \beta(0), \delta). \)

To estimate the geometric wedge product we will consider three types of points in a plaque \( L^g_{\alpha,m} \), namely if they are close to where the plaque crosses the \(-\) axis (Case 1) or \( w- \) axis or otherwise (Case 2). The estimates for \( h_\alpha \) are fairly independent of which case we are in because of the choice of \( c \), but \( h_\alpha \) is very sensitive to the cases.

We estimate the function \( \tilde{h}_\alpha \) on these plaques. First observe that the points in \( B_2 := \Delta^2((-a'(0)e, -b'(0)e; 2c|e|) \) are mapped by \( \Phi \) to a region covering \( D_1 \).

**Lemma 6.** There is a constant \( C > 0 \) so that if some leaf \( L^g_\alpha \) intersects \( D_1 \) for a parameter value \( u + iv \) then

\[
\frac{1-a}{b} \log(1/|e|) - C < u < \frac{1-a}{b} \log(1/|e|) + C, \log(1/|e|) - C < v < \log(1/|e|) + C.
\]

**Proof.** First recall that \( z = e^{(u + iv + (\log |\beta|/b))}. \) Hence \( |z| = e^{-v}. \) But \( (z, w) \in B_2. \)

Hence

\[
(\alpha'(0)| - 2c)|e| < |z| = e^{-v} < (|\alpha'(0)| + 2c)|e|.
\]

So

\[
\log |e| - C < -v < \log |e| + C
\]

which gives the estimate on \( v. \) To get the estimate on \( u, \) we consider:

\[
|w| = e^{-bu-av}
\]

\[
(\beta'(0)| - 2c)|e| < |w| = e^{-bu-av} < (\beta'(0)| + 2c)|e|
\]

\[
\log |e| - C' < -bu - av < \log |e| + C'
\]

\[
\log(1/|e|) - C' < bu + av < \log(1/|e|) + C'
\]

\[
\frac{1}{b} \log(1/|e|) - C'' < u + av/b < \frac{1}{b} \log(1/|e|) + C''
\]

So

\[
\frac{1}{b} \log(1/|e|) - av/b - C'' < u < \frac{1}{b} \log(1/|e|) - av/b + C'',
\]

hence

\[
u < \frac{1}{b} \log(1/|e|) + \frac{a}{b} (- \log(1/|e|) + C] + C'' [a > 0]
\]

\[
u < \frac{1-a}{b} \log(1/|e|) + C''
\]

\[
u < \frac{1}{b} \log(1/|e|) - \frac{a}{b} (\log(1/|e|) + C] + C'' [a \leq 0]
\]

\[
u < \frac{1-a}{b} \log(1/|e|) + C''
\]

For the other estimate
\[ u > \frac{1}{b} \log(1/|\epsilon|) + \frac{a}{b} [-\log(1/|\epsilon|) - C] - C'' \quad [a > 0] \]
\[ > \frac{1-a}{b} \log(1/|\epsilon|) - C''' \]
\[ u > \frac{1}{b} \log(1/|\epsilon|) - \frac{a}{b} [\log(1/|\epsilon|) - C] - C'' \quad [a \leq 0] \]
\[ > \frac{1-a}{b} \log(1/|\epsilon|) - C''' \]

□

In what follows we use the notation \( A \sim B \) to mean that there is a constant \( L \) so that \( \frac{1}{L} A \leq B \leq L A \) and \( L \) is chosen independent of the other parameters. Also \( A \lessapprox B \) means similarly that there is a constant \( L \) so that \( A \leq LB \).

Next we estimate the value of \( \tilde{h}_\beta \) for a point \((u, v)\) as in the previous Lemma. Let \( \theta, \tan \theta = v/u \) be the argument. By Lemma 5, it follows that for all small \( \epsilon \), \( \tan \theta \sim b/(1-a) \neq b/(-a) \) so that the angle \( \theta \) is uniformly inside the sector \( S_\lambda \) for all small \( \epsilon \). It follows that \( \gamma \theta \) is strictly inside a sector \( 0 < s < \gamma \theta < \pi < s < \pi \) for some fixed \( s > 0 \) which only depends on \( \lambda \) and is independent of all other choices, and for all small enough \( \epsilon \). This implies that \( \sin \gamma \theta > k > 0 \) uniformly. As in Lemma 2, for a point in \( D_1 \), this allows us to estimate the kernel for \( H_\beta \): 

**Lemma 7.** Suppose \((u + iv)\) is such that the corresponding point on the leaf \( L_\beta' \) is in \( D_1 \), then if \(|y| < 2(\log(1/|\epsilon|))^\gamma \),
\[
\frac{(r)^\gamma \sin(\gamma \theta)}{(r)^\gamma \sin(\gamma \theta))^2 + (y - (r)^\gamma \cos(\gamma \theta))^2} \sim \frac{1}{(\log(1/|\epsilon|))^\gamma}
\]
On the other hand if \(|y| \geq 2(\log(1/|\epsilon|))^\gamma \) then
\[
\frac{(r)^\gamma \sin(\gamma \theta)}{(r)^\gamma \sin(\gamma \theta))^2 + (y - (r)^\gamma \cos(\gamma \theta))^2} \sim \frac{(\log(1/|\epsilon|))^\gamma}{y^2}
\]
Hence we get using Lemma 2:

**Lemma 8.** We have the following estimate of \( H_\beta \) for points in \( D_1 \):
\[
H_\beta \sim \frac{1}{(\log(1/|\epsilon|))^\gamma} \int_{|y| < 2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y) \, dy + (\log(1/|\epsilon|))^\gamma \int_{|y| \geq 2(\log(1/|\epsilon|))^\gamma} \frac{\tilde{H}_\beta(y)}{y^2} \, dy
\]

Next we fix \( \alpha, \beta \) and plaques \( L_{\alpha,n}, L_{\beta,m} \) and assume they intersect in \( D_1 \). By Lemma 5, there are conditions on \( m \) for this to happen, namely:
\[
2m\pi < u' < 2(m + 1)\pi
\]
\[
\frac{1-a}{b} \log(1/|\epsilon|) - C < u' < \frac{1-a}{b} \log(1/|\epsilon|) + C
\]
So
\[
\frac{1-a}{b} \log(1/|\epsilon|) - C < 2(m+1)\pi < \frac{1-a}{b} \log(1/|\epsilon|) + C + 2\pi
\]
\[
\frac{1-a}{b} \log(1/|\epsilon|) - C - 2\pi < 2m\pi < \frac{1-a}{b} \log(1/|\epsilon|) + C.
\]

We pick a plaque \( L_{\beta,m}^\epsilon \) with an intersection point in \( D_1 \). Then this plaque is of the form
\[
z = e^{i(u+(\log |\alpha|)/b)-v}
\]
\[
w = \alpha e^{i\lambda (\log |\alpha|)/b}
\]
\[
2n\pi < u < 2(n+1)\pi.
\]
\[
|w| = e^{-bu-av}.
\]

We estimate the location of the intersection points.

Case 1: \(|z - \eta| < d|\eta|, 0 < d << 1 \). The constant \( d \) will be chosen small enough, in order to satisfy an inequality at the end of the proof of Lemma 8.

We estimate the parameter values \((u, v)\) for \( L_{\alpha,n} \).

Since \(|\eta|(1-d) < |z| = e^{-v} < |\eta|(1+d), \log(1/|\eta|) - 2d < v < \log(1/|\eta|) + 2d \).

Note that also, for the point \((z, w)\) to be on \( L_{\beta,m}^\epsilon \) with \(|z - \eta| < d|\eta|\) we must have that \(|w| < 2\lambda |\beta'(0)/|\alpha'(0)|d|\eta|\).

**Lemma 9.** For \((z, w)\) to be an intersection point between \( L_{\alpha,n} \) and \( L_{\beta}^\epsilon \) in \( D_1 \) with \(|z - \eta| < d|\eta|\), we must have

(i) \(2n\pi < u < 2(n+1)\pi\)

(ii) \(2n\pi > \frac{1-a}{b} \log(1/|\eta|) - C\)

(iii) \(\log(1/|\eta|) - 2d < v < \log(1/|\eta|) + 2d\).

Moreover there is at most one such intersection point.

**Proof.** We have already proved (iii) and (i) is given. To prove (ii):

\[
|w| < 2\lambda |\beta'(0)/|\alpha'(0)|d|\eta|.
\]

So,

\[-bu - av < \log |\eta| + C.\]

Using the estimate on \( v \) we get

\[
u > (-a/b)v - (\log |\eta|)/b - C/b\]
\[
u > (-a/b)\log(1/|\eta|) - (\log |\eta|)/b - C'/b\]
\[
u > ((1-a)/b)\log(1/|\eta|) - C''\]
where $C', C''$ are absolute constants.

To prove the last part, notice that the slope of $L^\epsilon_\beta$ is about $\lambda$ while the slope of $L_\alpha$ is $\lambda w/z$ so is at most $|\lambda| (2|\lambda| |\beta'(0)| d||\eta||/||\eta||(1 - d)) << |\lambda|$ if we just make $d$ small enough.

\[ \square \]

Lemma 10. We estimate the value of $H_\alpha$ at intersection points between $L_{\alpha,n}$ and $L^\epsilon_\beta$ in $D_1$ with $|z - \eta| < d||\eta||$. We have two cases:

(i) $\frac{1}{n} \log(1/||\eta||) - C < 2\pi n < C \log(1/||\eta||)$. Then we have

$$H_\alpha(u + iv) \sim \int_{|x| < 2\log(1/||\eta||)} \tilde{H}_\alpha(x) \frac{\log(1/||\eta||)^\gamma}{x^2} + \int_{|x| > 2\log(1/||\eta||)} \tilde{H}_\alpha(x) (\log(1/||\eta||))^\gamma$$

In the next case, (ii) $2\pi n \geq C \log(1/||\eta||)$, we then have: $U + iV \sim n^\gamma + in^{\gamma - 1} \log(1/||\eta||)$ and

$$H_\alpha(u + iv) \sim \int_{|x - U| < n^{\gamma - 1} \log(1/||\eta||)} \tilde{H}_\alpha(x) \frac{n^{\gamma - 1} \log(1/||\eta||)}{(x - U)^2} dx$$

$$+ \int_{|x - U| > n^{\gamma - 1} \log(1/||\eta||)} \tilde{H}_\alpha(x) n^{\gamma - 1} \log(1/||\eta||) (x - U)^2 dx$$

Proof. Case (i): We use that $\sin(\gamma \theta)$ is bounded below by a strictly positive constant. Case (ii) is clear.

\[ \square \]

Case 2: Our next step is to discuss intersection points of $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$ in $D_1$ for which $|z - \eta| > d||\eta||$. Note that $L^\epsilon_{\beta,m}$ intersects the $w-$ axis close to $(0, -\lambda \frac{\beta'(0)}{\alpha'(0)} \eta)$ and the above argument applies as well to the region $|w + \lambda \frac{\beta'(0)}{\alpha'(0)} \eta| < d||\eta||$. Hence we only need to consider intersections of $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$ when $|w + \lambda \frac{\beta'(0)}{\alpha'(0)} \eta| > d||\eta||$ and also $|z - \eta| > d||\eta||$, call this set $S'$.

Note: This is the place in the argument where we will assume that $a \neq 1$.

Since we are excluding the points near where $L^\epsilon_{\beta,m}$ crosses the two axes, we have the following estimate on points in $L^\epsilon_{\beta,m}$: For some fixed constant $R > 1$ we have that

$$\frac{1}{R} |w| < |z| < R|w|$$

for points in $S'$.

Hence
\[
\frac{1}{R} e^{-av - bu} < e^{-v} < Re^{-av - bu}
\]
\[-av - bu - \log R < -v < -av - bu + \log R
\]
\[bu - \log R < (1 - a)v < bu + \log R
\]
\[2n\pi b - \log R < (1 - a)v < 2(n + 1)\pi b + \log R,
\]
\[2n\pi - C < \frac{1 - a}{b} v < 2n\pi + C,
\]
\[2nb\pi/(1 - a) - C' < v < 2nb\pi/(1 - a) + C'.
\]

\[\square\]

**Lemma 11.** (Intersection Lemma) There is a constant \(N > 1\) so that if we cover the rectangle \(2n\pi < u < (2n + 1)\pi, 2nb\pi/(1 - a) - C' < v < 2nb\pi/(1 - a) + C'\) with \(N\) equal squares, then there are at most two intersection points in each square.

**Proof.** We choose \(N\) so that in each square, the slope of \(L_{\alpha,n}\) is almost constant and will produce at most one intersection point. The exception is when the slope is close to \(\lambda \frac{\beta'(0)}{\alpha'(0)}\). Then there might be a tangency between \(L_{\alpha,n}\) and \(L_{\beta}\). Hence there might be two or more intersection points counted with multiplicity. We will show there are at most 2. Note that the slope \(S\) of \(L_{\alpha,n}\) is given by the quotient \(\lambda w/z\).

\[
dw/dz = \lambda w/z
\]
\[= \lambda e^{\lambda(z + (\log |\alpha|/b)) - v}
\]
\[= \lambda e^{((\log |\alpha|)/b)(-b+ia)} e^{i\lambda \zeta} e^{i\zeta/b}
\]
\[= Ce^{i(\lambda - 1)\zeta}
\]

So,
\[S = Ce^{i(\lambda - 1)\zeta}
\]

and
\[
\frac{\partial S}{\partial \zeta} = i(\lambda - 1)S
\]
\[\sim i(\lambda - 1)\lambda \frac{\beta'(0)}{\alpha'(0)}
\]
\[\sim 1.
\]

This says that the slope of \(L_{\alpha,n}\) near intersection points vary very rapidly, while we also see from Lemma 4 that the slope of \(L_{\beta,m}\) varies slowly. This implies that near tangential intersection points there are at most two of them.

\[\square\]

We estimate the value of \(H_\alpha\) at points \(p\) where \(L_{\alpha,n}\) and \(L_{\beta,m}'\) intersect in \(D_1\) away from the axes \((|z - \eta| > d|\eta|, |w + \lambda \alpha'(0)/\alpha(0)\eta| > d|\eta|)\).
Lemma 12. For the intersection point to be in $D_1$ we need $|n| > \frac{|1-a|\log(1/|\epsilon|)}{2\pi b} - C_1$. Then

$$H_\alpha(p) \sim \int_{|x|<C_2|n|^{\gamma}} \frac{\tilde{H}_\alpha(x)dx}{|n|^{\gamma}} + \int_{|x|>C_2|n|^{\gamma}} \frac{\tilde{H}_\alpha(x)|n|^{\gamma}dx}{x^2}.$$

Proof. For the first estimate, recall that $|z| = e^{-v} < c|\epsilon|$ and that $2nb\pi/(1-a) - C' < v < 2nb\pi/(1-a) + C'$. For the integral estimate we see that $(u + iv)^{\gamma} = U + iV$ with $V \sim |n|^{\gamma}$ and $|U| \sim |n|^{\gamma}$. Then the estimate is immediate from the Poisson kernel. \[\square\]

We finish the estimate for $D_1$.

Theorem 8. The contribution to the geometric wedge product of $T$ and $T_\epsilon$ from intersection points in $D_1$ goes to zero when $\epsilon \to 0$.

Proof. Let $I = I_\epsilon$ consist of all intersection points $p$ in $D_1$. They are labeled $p = p_{\alpha,\beta,n,m,\ell}$ if they belong to the plaques $L_{\alpha,n}, L_{\beta,m}$ and $\ell$ lists them (with multiplicity) if there are more than one. By Lemma 5,

$$\frac{(1-a)\log(1/|\epsilon|)}{2\pi b} - C < m < \frac{(1-a)\log(1/|\epsilon|)}{2\pi b} + C$$

so in particular there are at most finitely many values of $m$ and there is a uniform upper bound on the number of them. We can hence restrict to one fixed value of $m$. Next recall that from Lemma 7 we have the estimate on the value of $H_\beta$ at each intersection point:

$$H_\beta(p) \sim \frac{1}{(\log(1/|\epsilon|))^{\gamma}} \int_{|y|<2(\log(1/|\epsilon|))^{\gamma}} \frac{\tilde{H}_\beta(y)dy}{y^{\gamma}}$$

By Lemmas 8 and 10 there is at most a uniformly bounded number of intersection points with $L_{\alpha,n}$ and $L_{\beta,m}$ in $D_1$. Hence when we estimate the geometric wedge product we can factor out the contribution from $\beta$ and we get an upper bound of

$$\int \left(\sum_p H_\beta(p)\right) d\mu(\beta) \lesssim \frac{1}{(\log(1/|\epsilon|))^{\gamma}} \int_{|y|<2(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\beta(y) dy d\mu(\beta)$$

We collect a few equalities that will be used repeatedly in Lebesgue dominated convergence Theorem.
Lemma 13. We have the following integral estimates.

(I) \[
\frac{1}{(\log(1/|\epsilon|))^{\gamma}} \int_{|y|<2(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\beta(y) dy = \frac{1}{(\log(1/|\epsilon|))^{\gamma}} \int_{|y|<2(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\beta(y)|y|^{1/\gamma - 1} |y|^{1-1/\gamma} dy
\]

\[
\sim \int_{|y|<2(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\beta(y)|y|^{1/\gamma - 1} \frac{|y|}{(\log(1/|\epsilon|))^{\gamma}} \cdot \frac{1}{(|y|+1)^{1/\gamma}} dy
\]

(II) \[
(\log(1/|\epsilon|))^{\gamma} \int_{|y|>2(\log(1/|\epsilon|))^{\gamma}} \frac{\tilde{H}_\beta(y)}{y^2} dy = \int_{|y|>2(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\beta(y)|y|^{1/\gamma - 1} \frac{1}{|y|} |y|^{-1/\gamma} dy
\]

(III) If \( U \sim n^\gamma \)

\[
\int_{|x-U|<n^{\gamma-1} \log(1/|\eta|)} H_\alpha(x) \frac{1}{n^{\gamma-1} \log(1/|\eta|)} dx \sim \int_{|x-U|<n^{\gamma-1} \log(1/|\eta|)} H_\alpha(x)|x|^{1/\gamma - 1} \frac{dx}{\log(1/|\eta|)}
\]

We want to show that

\[
\int \sum_p H_\alpha^p d\mu(\beta) \to 0.
\]

We use the a priori bound that we found and estimates (I) and (II) in the previous Lemma.

The Lebesgue dominated convergence theorem and Proposition 1 gives the result.

End of the proof of Theorem 8: After integrating with respect to \( \mu \) and using (I) and (II) of Lemma 12 we can use the dominated convergence theorem. We estimate the value of \( H_\alpha \) at one of the intersection points \( p \in D_1 \). From Lemma 1 we have:

\[
H_\alpha(p) = \frac{1}{\pi} \int \tilde{H}_\alpha(x) \frac{\rho^7 \sin(\gamma \theta)}{\rho^7 \sin(\gamma \theta)^2 + (x - \rho^7 \cos(\gamma \theta))^2} dx
\]

Case (i): \(|z - \eta| < d|\eta|, |n| < C \log(1/|\eta|)\). By Lemma 8 it follows that \( V = \rho^7 \sin(\gamma \theta) \sim (\log(1/|\eta|))^{\gamma} \) and \( |U| \sim (\log(1/|\eta|))^{\gamma} \).

\[
H_\alpha(p) \sim \int_{|x|<C(\log(1/|\eta|))^{\gamma}} \frac{\tilde{H}_\alpha(x)}{(\log(1/|\eta|))^{\gamma}} dx + \int_{|x|>C(\log(1/|\eta|))^{\gamma}} \tilde{H}_\alpha(x) \frac{(\log(1/|\eta|))^{\gamma}}{x^2} dx.
\]

So adding up we get
\[
\sum_{|n|<\log(1/|\eta|)} h_{a,n}(p_n) \sim \int_{|x|<C(\log(1/|\eta|))} \frac{\tilde{H}_\alpha(x) dx}{(\log(1/|\eta|))^{\gamma-1}} \\
+ \int_{|x|>C(\log(1/|\eta|))} \tilde{H}_\alpha(x)\left(\frac{|x|}{(\log(1/|\eta|))^{\gamma}}\right)^{1-1/\gamma} dx \\
\sim \int_{|x|<C(\log(1/|\eta|))} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} \left(\frac{|x|}{(\log(1/|\eta|))^{\gamma}}\right)^{1-1/\gamma} dx \\
+ \int_{|x|>C(\log(1/|\eta|))} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} \left(\frac{|x|}{(\log(1/|\eta|))^{\gamma}}\right)^{1+1/\gamma} dx \\
\int_{|n|<C(\log(1/|\eta|))} H_{a,n}(p_n) d\mu(a) dx.
\]

Integrating with respect to \(\mu\) we get that \(\sum_{|n|<\log(1/|\eta|)} h_{a,n}(p_n) \to 0\) as \(\epsilon \to 0\) since \(|\eta| < \epsilon\). We use again the estimates in Lemma 12 and Proposition 1.

Case (ii): \(|z-\eta| < d|\eta|, |n| > C \log(1/|\eta|)\). Then by Lemma 8, \(n > 0\) and we have \(U_n \sim n^\gamma, V \sim n^{\gamma-1} \log(1/|\eta|)\). From Lemma 9 we have:

\[
H_\alpha(p) \sim \int_{|x-U|<n^{\gamma-1} \log(1/|\eta|)} \frac{\tilde{H}_\alpha(x) dx}{n^{\gamma-1} \log(1/|\eta|)} \\
+ \int_{|x-U|>n^{\gamma-1} \log(1/|\eta|)} \tilde{H}_\alpha(x) \frac{n^{\gamma-1} \log(1/|\eta|)}{|x-U|} dx.
\]

So,

\[
H_\alpha(p) \sim \int_{|x-U|<n^{\gamma-1} \log(1/|\eta|)} \tilde{H}_\alpha(x) \frac{x^{1/\gamma-1} dx}{\log(1/|\eta|)} \\
+ \int_{|x-U|>n^{\gamma-1} \log(1/|\eta|)} \tilde{H}_\alpha(x) \frac{n^{\gamma-1} \log(1/|\eta|)}{|x-U|} dx.
\]

Therefore,

\[
\sum_{n>C \log(1/|\eta|)} H_{a,n}(p) \sim \sum_{n>C \log(1/|\eta|)} \int_{|x-U_n|<n^{\gamma-1} \log(1/|\eta|)} \frac{\tilde{H}_\alpha(x) dx}{(\log(1/|\eta|))^{\gamma}} \frac{n^{\gamma-1} \log(1/|\eta|)}{|x-U_n|} dx \\
+ \sum_{n>C \log(1/|\eta|)} \int_{|x-U_n|>n^{\gamma-1} \log(1/|\eta|)} \tilde{H}_\alpha(x) \frac{n^{\gamma-1} \log(1/|\eta|)}{|x-U_n|} dx
\]

\[
= I + II
\]

We are going to estimate \(I\) and \(II\) separately. Note that for a given \(x\), the number of integers \(n\) for which \(U_n - n^{\gamma-1} \log(1/|\eta|) < x < U_n + n^{\gamma-1} \log(1/|\eta|)\) is bounded above by a multiple of \(\log(1/|\eta|)\). It follows that \(I \sim \int_{\log(1/|\eta|) \log(1/\eta)/C} \tilde{H}_\alpha(x)x^{1/\gamma-1} dx\). This contribution goes to zero as \(|\epsilon| \to 0\) since \(|\eta| < |\epsilon|\).
To study \( \II \) we estimate \( U_n \) more precisely. We have
\[
2n\pi < u_n < 2(n+1)\pi \\
\log(1/|\eta|) - 2d < v_n < \log(1/|\eta|) + 2d,
\]
and
\[
(u_n + iv_n)^\gamma = u_n^\gamma(1 + iv_n/u_n)^\gamma \\
= u_n^\gamma + \gamma u_n^{\gamma-1}iv_n + E_n,
\]
with
\[
|E_n| \lesssim u_n^\gamma(v_n/u_n)^2 \\
\sim n^\gamma - 2(\log(1/|\eta|))^2.
\]
Hence \( |U_n - (2n\pi)^\gamma| << n^\gamma - 1 \log(1/|\eta|) \). We can hence replace \( U_n \) by \((2n\pi)^\gamma\) in \( \II \) without changing the order of magnitude of the expression. We divide \( \II \) into pieces \( \II_A, \II_B, \II_C \). In \( \II_A \), \( x \) is such that \( n > C\log(1/|\eta|) \). In \( \II_B \), \( n \) has a range of the form \( n > x^{1/\gamma} + r(x)\log(1/|\eta|), r(x) \sim 1 \) and in \( \II_C \), \( C\log(1/|\eta|) < n < x^{1/\gamma} - s(x)\log(1/|\eta|), s(x) \sim 1 \). So

\[
\II = \II_A + \II_B + \II_C \\
\II_A = \int_{x = -\infty}^{C_1(\log(1/|\eta|))\gamma} \sum_{n > C\log(1/|\eta|)} \tilde{H}_\alpha(x) \frac{n^{\gamma-1}\log(1/|\eta|)}{|x - n\gamma|^2} \, dx \\
\sim \int_{|x| < C_1(\log(1/|\eta|))\gamma} \tilde{H}_\alpha(x) \frac{\log(1/|\eta|)}{[10\log(1/|\eta|)]^{\gamma} - x} \, dx \\
+ \int_{x = -\infty}^{C_1(\log(1/|\eta|))\gamma} \tilde{H}_\alpha(x) \frac{\log(1/|\eta|)}{[10\log(1/|\eta|)]^{\gamma} - x} \, dx \\
\sim \int_{|x| < C_1(\log(1/|\eta|))\gamma} \tilde{H}_\alpha(x) \frac{1}{[\log(1/|\eta|)]^{\gamma-1}} \, dx \\
+ \int_{x = -\infty}^{C_1(\log(1/|\eta|))\gamma} \tilde{H}_\alpha(x) \frac{\log(1/|\eta|)}{|x|} \, dx \\
\sim \int_{|x| < C_1(\log(1/|\eta|))\gamma} \tilde{H}_\alpha(x) |x|^{1/\gamma - 1} \left( \frac{|x|}{[\log(1/|\eta|)]^{\gamma}} \right)^{1-1/\gamma} \, dx \\
+ \int_{x = -\infty}^{C_1(\log(1/|\eta|))\gamma} \tilde{H}_\alpha(x) |x|^{1/\gamma - 1} \left( \frac{\log(1/|\eta|)}{|x|} \right)^{1/\gamma} \, dx.
\]
For
\[
\II_B \sim \int_{x = C_1(\log(1/|\eta|))\gamma}^{\infty} \sum_{n > x^{1/\gamma} + r(x)\log(1/|\eta|)} \tilde{H}_\alpha(x) \frac{n^{\gamma-1}\log(1/|\eta|)}{|x - n\gamma|^2} \, dx \\
\sim \int_{x = C_1(\log(1/|\eta|))\gamma}^{\infty} \tilde{H}_\alpha(x) |x|^{1/\gamma - 1} \, dx,
\]
and
\[ II_C \sim \int_{x = C_2(\log(1/\eta))}^\infty \sum_{0 < n < x^{1/\gamma} - s(x) \log(1/\eta)} \tilde{H}_\alpha(x) \frac{n^{\gamma - 1} \log(1/\eta)}{|x - n|^2} dx \]
\[ II_C \sim \int_{x = C_2(\log(1/\eta))}^\infty \tilde{H}_\alpha(x) x^{1/\gamma - 1} dx. \]

Integrating with respect to \( \mu \) we get
\[
\int II d\mu(\alpha) \\
\sim \int \int II_A d\mu(\alpha) + \int II_B d\mu(\alpha) + \int II_C d\mu(\alpha) \\
\lesssim \int \int \tilde{H}_\alpha(x)|x|^{1/\gamma - 1} dxd\mu(\alpha) \\
+ \int |x| < C_1(\log(1/\eta)) \tilde{H}_\alpha(x)|x|^{1/\gamma - 1} \left( \frac{|x|}{(\log(1/\eta))^{\gamma}} \right)^{1-1/\gamma} dxd\mu(\alpha),
\]
which tends to 0 as \( \eta \to 0 \). (Recall that \( |\eta| < \epsilon \).)

Case (iii): \( |w + \lambda\eta| < |d|/|\eta| \). This case is symmetric to cases (i) and (ii), so done.

Case (iv): \( |z|, |w| < c|\epsilon|; |z - \eta|, |w + \lambda\eta| > |d|/|\eta| \). We recall the estimate of \( H_{\alpha,n}(p) \) at intersection points from Lemma 11. The contribution \( W \) to the geometric wedge product is:
\[
\int |n| > |1 - \alpha| \log(1/|\epsilon|)/|2\pi b - C| \int |x| < 2|n|^\gamma \frac{\tilde{H}_\alpha(x)dx}{|n|^{\gamma}} + \int |x| > 2|n|^\gamma \frac{\tilde{H}_\alpha(x)dx}{x^2} d\mu(\alpha).
\]

We divide the first integral into two pieces, so \( W = W_A + W_B + W_C \). We get
\[
W_A \\
\sim \int \int |x| < 2|1 - \alpha| \log(1/|\epsilon|)/|2\pi b - C| \frac{\tilde{H}_\alpha(x)dx}{|n|^{\gamma}} d\mu(\alpha) \\
\sim \int \int \frac{\tilde{H}_\alpha(x)dx}{|n|^{\gamma}} d\mu(\alpha) \\
\sim \int \int \frac{\tilde{H}_\alpha(x)dx}{(\log(1/|\epsilon|))^{\gamma - 1}} d\mu(\alpha) \\
\rightarrow 0 \text{ as } \epsilon \to 0.
\]
For $W_B$ we have:

$$W_B \sim \int_\alpha \left[ \int_{|x|>2||1-a\log(1/|x|)||/2\pi b-C|\gamma}^{\infty} \frac{\tilde{H}_\alpha(x)dx}{|n|^\gamma} \right] d\mu(\alpha)$$

$$W_B \sim \int_\alpha \left[ \int_{|x|>2||1-a\log(1/|x|)||/2\pi b-C|\gamma}^{\infty} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} dx \right] d\mu(\alpha)$$

$$\to 0,$$

again by Proposition 1.

$$W_C$$

$$\sim \int_\alpha \left[ \int_{|x|>2||1-a\log(1/|x|)||/2\pi b-C|\gamma}^{\infty} \frac{\tilde{H}_\alpha(x)|n|^\gamma dx}{x^2} \right] d\mu(\alpha)$$

$$\sim \int_\alpha \left[ \int_{|x|>2||1-a\log(1/|x|)||/2\pi b-C|\gamma}^{\infty} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} dx \right] d\mu(\alpha),$$

and hence $W_C \to 0$.

Now we have finished the part of the proof of Theorem 8 where we consider intersection points in $D_1 = \{|z|, |w| < c|\epsilon|\}$.

4. PROOF OF THEOREM 7 FOR INTERSECTION POINTS IN $D_2 \subset \Delta^2(0, C|\epsilon|)$ CLOSE TO THE SEPARATRICS.

We split $D_2$ into regions $A'$ and $B$ where $A'$ denotes points close to the separatrices and $B$ denotes the rest. Then $A'$ has 2 pieces. It suffices to consider one, $A = \{(z, w); c|\epsilon| < |z| < C|\epsilon|, |w| < r|\epsilon|\}$ where $0 < r < c$ depends on the choice of $C$.

We consider intersection points of $L_{\alpha, n}$ and $L_{\beta, m}^\prime$ in $A$. We parametrize $L_\alpha$ with $(u + iv)$ and $L_{\beta, m}^\prime$ with $u' + iv'$. In $A$ we have for $L_{\alpha, n}$:

$$\log(1/|\epsilon|) - C < v < \log(1/|\epsilon|) + C$$

$$|w| = e^{-bu-av} < r|\epsilon|$$

$$bu + av > \log(1/r) + \log(1/|\epsilon|)$$

$$u > \frac{1-a}{b} \log(1/|\epsilon|) - C$$

$$n > \frac{1-a}{2\pi b} \log(1/|\epsilon|) - C.$$
Lemma 14. If intersection points between any pair $L_{\omega, \omega}$, there is at most $N$ intersection points between any pair $L_{\alpha,n}$ and $L_{\beta,m}$.

Proof. This follows from considering the slopes of the plaques, given by the forms $\omega, \omega$. Namely see Case (ii), integrals $I+II$.

Next we estimate $h_{\alpha,n}$ at an intersection point.

Case (i): $n < \log(1/|\epsilon|)$:
In $U, V$ coordinates we have $V \sim (\log(1/|\epsilon|))^\gamma, |U| \lesssim (\log(1/|\epsilon|))^\gamma.$

Using the expression as a Poisson integral we get:

$$h_{\alpha,n}(p) \sim \int_{|x| < C(\log(1/|\epsilon|))^{\gamma}} H_\alpha(x) \frac{1}{(\log(1/|\epsilon|))^{\gamma}} \, dx$$
$$+ \int_{|x| > C(\log(1/|\epsilon|))^{\gamma}} H_\alpha(x) \frac{1}{x^2} \, dx$$
$$\sim \int_{|x| < C(\log(1/|\epsilon|))^{\gamma}} H_\alpha(x) |x|^{1/\gamma - 1} \frac{|x|}{(\log(1/|\epsilon|))^{\gamma}} \, dx$$
$$+ \int_{|x| > C(\log(1/|\epsilon|))^{\gamma}} H_\alpha(x) |x|^{1/\gamma - 1} \frac{1}{|x|} \, dx.$$

Case (ii): $n > \log(1/|\epsilon|)$
Then $U \sim n^{\gamma}, V \sim n^{\gamma - 1} \log(1/|\epsilon|)$. Hence

$$h_{\alpha,n}(p) \sim \int H_\alpha(x) \frac{n^{\gamma - 1} \log(1/|\epsilon|)}{(n^{\gamma - 1} \log(1/|\epsilon|))^2 + |x - n^{\gamma}|^2} \, dx.$$

We observe that this integral has already been estimated above. Namely see Case (ii), integrals $I+II$.

So we get

$$\sum_{n > 10 \log(1/|\epsilon|)} h_{\alpha,n}(p) \lesssim \int_{|x| > (\log(1/|\epsilon|))^{\gamma}} H_\alpha(x) |x|^{1/\gamma - 1} \, dx$$
$$+ \int_{|x| < C(\log(1/|\epsilon|))^{\gamma}} H_\alpha(x) |x|^{1/\gamma - 1} \left(\frac{|x|}{\log(1/|\epsilon|)}\right)^\gamma \, dx.$$
We estimate next \( h_{\beta,m}^\gamma(p) \). From the above estimates for \( u', v' \) we see that \(|u'| \lesssim v'\) and hence \( V' \sim (v')^{\gamma} \), \(|U'| \lesssim (v')^{\gamma} \). We then have:

\[
\begin{align*}
  h_{\beta,m}^\gamma(p) & \sim \int_{|y| < C(v')^{\gamma}} H_{\beta}(y) \frac{1}{(v')^{\gamma}} dy \\
  & \quad + \int_{|y| > C(v')^{\gamma}} H_{\beta}(y) \frac{(v')^{\gamma}}{y^{2}} dy. \\
  h_{\beta,m}^\gamma(p) & \sim \int_{|y| < C(v')^{\gamma}} H_{\beta}(y)|y|^{1/\gamma - 1} \left( \frac{|y|}{(v')^{\gamma}} \right)^{1 - 1/\gamma} \frac{1}{v'} dy \\
  & \quad + \int_{|y| > C(v')^{\gamma}} H_{\beta}(y)|y|^{1/\gamma - 1} \left( \frac{(v')^{\gamma}}{|y|} \right)^{1 + 1/\gamma} \frac{1}{v'} dy.
\end{align*}
\]

Note that for \( a \neq 0 \), we have that

\[
\frac{\log(1/|\epsilon|)/a - bu'/a - 2c/|a|}{b^2m \pi/a} < v' < \log(1/|\epsilon|)/a - bu'/a + 2c/|b| \quad \text{and} \quad \frac{\log(1/|\epsilon|)/a - 2m \pi b/a - C}{b^2m \pi/a} < v' < \log(1/|\epsilon|)/a - 2m \pi b/a + C,
\]

so

\[ v' > \log(1/|\epsilon|) \]

and

\[ m/a < \frac{1}{2\pi b} \log(1/|\epsilon|)(1/a - 1) + C. \]

Define

\[ \Sigma := \sum_{m/a < \frac{1}{2\pi b} \log(1/|\epsilon|)(1/a - 1) + C} h_{\beta,m}^\gamma. \]

Then using the above estimates,

\[
\begin{align*}
  \Sigma & \lesssim \sum_{m} \int_{|y| < C(\log(1/|\epsilon|)/a - b2m \pi/a)^{\gamma}} H_{\beta}(y)|y|^{1/\gamma - 1} \\
  & \quad \times \left( \frac{|y|}{(\log(1/|\epsilon|)/a - b2m \pi/a)^{\gamma}} \right)^{1 - 1/\gamma} \frac{1}{|\log(1/|\epsilon|)/a - b2m \pi/a|} dy \\
  & \quad + \sum_{m} \int_{|y| > C(\log(1/|\epsilon|)/a - b2m \pi/a)^{\gamma}} H_{\beta}(y)|y|^{1/\gamma - 1} \\
  & \quad \times \left( \frac{(\log(1/|\epsilon|)/a - b2m \pi/a)^{\gamma}}{|y|} \right)^{1 + 1/\gamma} \frac{1}{|\log(1/|\epsilon|)/a - b2m \pi/a|} dy \\
  & = I + II.
\end{align*}
\]
We study separately I and II.

\[ I = I_A + I_B \]

\[ I_A = \int_{|y| < C(\log(1/|\epsilon|))} H_\beta(y) |y|^{1/\gamma - 1} \]

\[ \times \left( \frac{|y|}{(\log(1/|\epsilon|)/a - b2m\pi/a)^\gamma} \right)^{1-1/\gamma} \]

\[ \times \frac{1}{|\log(1/|\epsilon|)/a - b2m\pi/a|} dy \]

\[ \sim \int_{|y| < C(\log(1/|\epsilon|))} H_\beta(y) |y|^{1/\gamma - 1} \left( \frac{|y|}{(\log(1/|\epsilon|)/a - b2m\pi/a)^\gamma} \right)^{1-1/\gamma} dy. \]

We estimate

\[ I_B = \int_{|y| = C(\log(1/|\epsilon|))} H_\beta(y) |y|^{1/\gamma - 1} \]

\[ \times \left( \frac{|y|}{(\log(1/|\epsilon|)/a - b2m\pi/a)^\gamma} \right)^{1-1/\gamma} \]

\[ \times \frac{1}{|\log(1/|\epsilon|)/a - b2m\pi/a|} dy \]

\[ \sim \int_{|y| = C(\log(1/|\epsilon|))} H_\beta(y) |y|^{1/\gamma - 1} dy. \]

For II we have

\[ II \sim \sum_{\frac{1}{2\pi}(\log(1/|\epsilon|)(1/a - 1)/m/a > \log(1/|\epsilon|)/C) \gamma} \int_{|y| = C(\log(1/|\epsilon|))} H_\beta(y) |y|^{1/\gamma - 1} \]

\[ \times \left( \frac{|y|}{(\log(1/|\epsilon|)/a - b2m\pi/a)^\gamma} \right)^{1+1/\gamma} \]

\[ \times \frac{1}{\log(1/|\epsilon|)/a - b2m\pi/a} dy \]

\[ \lesssim \int_{|y| = C(\log(1/|\epsilon|))} H_\beta(y) |y|^{1/\gamma - 1} dy. \]

With these estimates it follows that Theorem 7 is proved for the region \( A \) close to the separatrices, in the ball \( D_2 \) provided that \( a \neq 0 \).

The case \( a = 0 \):

We fix \((\alpha, n)\) and \((\beta, m)\) and investigate intersection points. Note that since \( a = 0 \), we need \((\log(1/|\epsilon|)/b - 2c/b < u' < (\log(1/|\epsilon|)/b + 2c/b). Hence there are at most finitely many possible values for \( m \sim (\log(1/|\epsilon|))/(2\pi b) \). We proceed as if there is at most one. This will suffice. Also note that since we assume that \( |z'| < C|\epsilon| \) we also need \( u' > \log(1/|\epsilon|) - C' \). For every integer \( k > 0 \) we might have an intersection point \( p_{n,k} \) between \( L_{\alpha,n} \) and \( L'_{\beta,m} \) with \( \log(1/|\epsilon|) - C' + k\pi < u' \leq \log(1/|\epsilon|) + (k + 1)\pi \).
We estimate $h_{\beta,m}^\epsilon (p_{n,k})$.

**Lemma 15.** When $a = 0$, then $\gamma = 2$.

**Proof.** The inequalities $|z| < 1, |w| < 1$ lead to $u, v > 0$. 

We have $u' \sim \log(1/|\epsilon|), v' \sim \log(1/|\epsilon|) + k$ so $U' + iV' = (u')^2 - (v')^2 + 2iu'v'$ Hence if $0 < k < C'' \log(1/|\epsilon|)$ we have the estimate $|U'| \lesssim (\log(1/|\epsilon|))^2 \sim V'$. If $k > C'' \log(1/|\epsilon|)$ we have $U' \sim -k^2, V' \sim k \log(1/|\epsilon|)$. We consider various cases:

(i) $0 < k < C'' \log(1/|\epsilon|)$:

$$h_{\beta,m}^\epsilon (p_{n,k}) \sim \int_{|y| < C(\log(1/|\epsilon|))^2} H_\beta (y) \frac{1}{(\log(1/|\epsilon|))^2} dy$$

$$\quad + \int_{|y| > C(\log(1/|\epsilon|))^2} H_\beta (y) \frac{(\log(1/|\epsilon|))^2}{y^2} dy$$

Hence

$$\sum_{k=0}^{C'' \log(1/|\epsilon|)} h_{\beta,m}^\epsilon (p_{n,k}) \sim \int_{|y| < C(\log(1/|\epsilon|))^2} H_\beta (y) \frac{1}{(\log(1/|\epsilon|))^2} dy$$

Adding up we get

$$+ \sum_{k=0}^{C'' \log(1/|\epsilon|)} \int_{|y| > C(\log(1/|\epsilon|))^2} H_\beta (y) \frac{(\log(1/|\epsilon|))^3}{y^2} dy$$

$$\sim \int_{|y| < C(\log(1/|\epsilon|))^2} H_\beta (y) |y|^{-1/2} \left( \frac{|y|}{(\log(1/|\epsilon|))^2} \right)^{1/2} dy$$

$$+ \sum_{k=0}^{C'' \log(1/|\epsilon|)} \int_{|y| > C(\log(1/|\epsilon|))^2} H_\beta (y) |y|^{-1/2} \left( \frac{(\log(1/|\epsilon|))^2}{|y|} \right)^{3/2} dy.$$ 

Next we have case (ii).

(ii) $k > C'' \log(1/|\epsilon|)$:

Using the location of $p_{n,k}$ we have the estimates

$$h_{\beta,m}^\epsilon (p_{n,k}) \sim \int H_\beta (y) \frac{k \log(1/|\epsilon|)}{(k \log(1/|\epsilon|))^2 + (y + k^2)^2} dy$$

$$\sim \int_{|y| < k \log(1/|\epsilon|)} H_\beta (y) \frac{1}{k \log(1/|\epsilon|)} dy$$

$$+ \int_{|y| > k \log(1/|\epsilon|)} H_\beta (y) \frac{k \log(1/|\epsilon|)}{(y + k^2)^2} dy.$$ 

We define $\Sigma$

$$\Sigma := \sum_{k > C'' \log(1/|\epsilon|)} h_{\beta,m}^\epsilon (p_{n,k})$$

$$= I_A + I_B + II_A + II_B + II_C.$$
We have
\[ I_A \sim \int_{y=-(C''+C''') \log(1/|\varepsilon|)}^{(C''^2+C''') \log(1/|\varepsilon|)} \sqrt{y+r(y) \log(1/|\varepsilon|)} \sum_{k=C'' \log(1/|\varepsilon|)} H_\beta(y) \frac{1}{k \log(1/|\varepsilon|)} \, dy \]
where \( r(y) \sim \sqrt{|y|} \).

Hence
\[ I_A \sim \int_{y=-(C''+C''') \log(1/|\varepsilon|)}^{(C''^2+C''') \log(1/|\varepsilon|)} H_\beta(y) \frac{1}{\log(1/|\varepsilon|)} \, dy \]
\[ I_A \sim \int_{y=-(C''^2+C''') \log(1/|\varepsilon|)} H_\beta(y) |y|^{-1/2} \, dy. \]

For \( I_B \) we have
\[ I_B \sim \int_{y=-\infty}^{-(C''^2-C''') \log(1/|\varepsilon|)} \sqrt{y-s(y) \log(1/|\varepsilon|)} \sum_{k=C'' \log(1/|\varepsilon|)} H_\beta(y) \frac{1}{k \log(1/|\varepsilon|)} \, dy \]
with
\[ s(y) \sim \sqrt{|y|}. \]

Hence
\[ I_B \sim \int_{y=-\infty}^{-(C''^2-C''') \log(1/|\varepsilon|)} H_\beta(y) \frac{1}{\log(1/|\varepsilon|)} \, dy \]
\[ I_B \sim \int_{y=-\infty}^{-(C''^2-C''') \log(1/|\varepsilon|)} H_\beta(y) |y|^{-1/2} \left( \frac{|y|}{(\log(1/|\varepsilon|))^2} \right)^{1/2} \, dy. \]

Next we have for \( II_A \)
\[ II_A \sim \int_{y=-\infty}^{-(C''^2-C''') \log(1/|\varepsilon|)} \sqrt{y-s(y) \log(1/|\varepsilon|)} \sum_{k=C'' \log(1/|\varepsilon|)} H_\beta(y) \frac{k \log(1/|\varepsilon|)}{(y+k^2)^2} \, dy \]
\[ II_A \sim \int_{y=-\infty}^{-(C''^2-C''') \log(1/|\varepsilon|)} H_\beta(y) |y|^{-1/2} \, dy \]
and
\[ II_B \sim \int_{y=-\infty}^{-(C''^2+C''') \log(1/|\varepsilon|)} \sqrt{y+r(y) \log(1/|\varepsilon|)} \sum_{k=-C'' \log(1/|\varepsilon|)} H_\beta(y) \frac{k \log(1/|\varepsilon|)}{(y+k^2)^2} \, dy \]
\[ II_B \sim \int_{y=-\infty}^{-(C''^2+C''') \log(1/|\varepsilon|)} H_\beta(y) |y|^{-1/2} \, dy. \]
The last term is
\[
II_C \sim \int_{(-(C''r)^2+C'')}^\infty (y + k^2) dy + \sum_{k=C''}^{\infty} H_\beta(y) dy.
\]
Similarly
\[
II_C \sim \int_{(-(C''r)^2+C'')}^\infty (y + C') dy + \sum_{k=C''}^{\infty} H_\beta(y) dy.
\]

Next we estimate \(h_{a,n}(p_{n,k})\). Note however, that the estimates for the case \(a \neq 0\) still applies to \(h_{a,n}\). This condition was not used to estimate \(h_a\). Hence we are done with the proof of Theorem 7 for the case of intersection points in \(A\).

We next consider the set \(B\) of points in \(\Delta^2(0,C|\epsilon|)\) defined above as consisting of points which are at distance at least \(r|\epsilon|\) from all separatrices.

5. PROOF OF THEOREM 7 FOR POINTS IN \(B\), I.E. POINTS IN \(D_2\) WHICH ARE AT DISTANCE AT LEAST \(r|\epsilon|\) FROM THE SEPARATRICES.

We estimate \(H_a\) on \(L_{a,n} \cap B\). We can assume \(a \neq 1\), otherwise flip the axes. So,

\[
\begin{align*}
    r|\epsilon| &\quad < |z| < C|\epsilon| \\
    r|z| &\quad < e^{-v} < C|\epsilon| \\
    \log(1/|\epsilon|) - C' &\quad < v < \log(1/|\epsilon|) + C'.
\end{align*}
\]

Similarly
\[
\begin{align*}
    r|\epsilon| &\quad < |w| < C|\epsilon| \\
    r|\epsilon| &\quad < e^{-bu-a|\epsilon|} < C|\epsilon| \\
    \log |\epsilon| - C'' &\quad < -bu - av < \log |\epsilon| + C'' \\
    \log(1/|\epsilon|) - C'' - av &\quad < bu < -av + \log(1/|\epsilon|) + C'' \\
    (1 - a) \log(1/|\epsilon|) - C &\quad < bu < (1 - a) \log(1/|\epsilon|) + C \\
    \frac{1-a}{b} \log(1/|\epsilon|) - C &\quad < u < \frac{1-a}{b} \log(1/|\epsilon|) + C.
\end{align*}
\]

Using these estimates on \((u, v)\) and similarly for \((u', v')\), Lemma 10 shows that there is an integer \(N\) independent of \(\epsilon\) so that if we take any two plaques of two leaves \(L_a, L_{\beta}\), then they intersect in \(B\) in at most \(N\) points. In \(U, V\) coordinates,
\[(u + iv)\gamma = U + iV,\]
hence
\[V \sim (\log(1/|\varepsilon|)\gamma, \quad |U| \prec \sim (\log(1/|\varepsilon|)\gamma).

This gives
\[h_{\alpha,n} \sim \int H_\alpha(x) \frac{(\log(1/|\varepsilon|))\gamma}{(\log(1/|\varepsilon|))^{2\gamma} + (x - U)^2} dx \]
\[\sim \int_{|x| < C(\log(1/|\varepsilon|))\gamma} H_\alpha(x) \frac{1}{(\log(1/|\varepsilon|))\gamma} dx + \int_{|x| > (\log(1/|\varepsilon|))\gamma} H_\alpha(x) \frac{\log(1/|\varepsilon|)\gamma}{x^2} dx \]
\[\sim \int_{|x| < C(\log(1/|\varepsilon|))\gamma} H_\alpha(x)(|x| + 1)^{1/\gamma - 1} \frac{|x| + 1}{(\log(1/|\varepsilon|))\gamma} (|x| + 1)^{-1/\gamma} dx + \int_{|x| > (\log(1/|\varepsilon|))\gamma} H_\alpha(x)(|x| + 1)^{1/\gamma - 1} \frac{\log(1/|\varepsilon|)\gamma}{|x| + 1} (|x| + 1)^{-1/\gamma} dx.\]

It follows from these estimates applied to \(H_\beta\) as well, that Theorem 7 is valid for intersection points in \(B\).

6. **Theorem 7 for \(D_3 = \Delta^2(0, \delta) \setminus \Delta^2(0, C|\varepsilon|)\)**

There are 3 regions to consider:

\[D_3 = R_1 \cup R_2 \cup R_3\]
\[R_1 = \{C|\varepsilon| < |z| < \delta, C|\varepsilon| < |w| < \delta\}\]
\[R_2 = \{C|\varepsilon| < |z| < \delta, |w| < C|\varepsilon|\}\]
\[R_3 = \{|z| < C|\varepsilon|, C|\varepsilon| < |w| < \delta\}.\]

Note that since we have assumed \(a \neq 1\), the cases of \(R_2\) and \(R_3\) are not completely symmetric. We will leave it to the reader to verify that the estimates we do later for \(R_2\) nevertheless hold for \(R_3\).

7. **Theorem 7 for \(R_1\), the diagonal part of \(D_3\)**

We first outline our approach. Fix parameters \(\alpha, \beta\) and corresponding plaques \(L_{\alpha,n}, L_{\beta,m}\). Next we divide \(R_1\) into dyadic components, rings, \(\{R(p)\}\) in the z-direction, \(e^{-p-1} < |z| < e^{-p}, C|\varepsilon| < |w| < \delta\). Then we estimate \(h_\alpha\) and \(h_\beta\) on \(L_{\alpha,n} \cap R(p)\) and \(L_{\beta,m} \cap R(p)\) respectively. Next, for fixed \(\alpha, \beta, n, m\) we estimate the values of \(p\) where the leaves \(L_{\alpha,n}, L_{\beta,m}\) might intersect, and the number of intersection points...
for each such $p$. Putting this information together we can estimate the contribution from $R_1$ to the geometric wedge product.

Pick a plaque $L_{\alpha, n}$ and a point $(z, w)$ in $L_{\alpha, n} \cap R(p)$ parametrized by $(u, v)$. Then

$$e^{-p-1} < |z| = e^{-v} < e^{-p}$$
$$\log(1/\delta) < v < \log(1/|\epsilon|) - C$$
$$\log(1/\delta) < p < \log(1/|\epsilon|) - C,$$

and

$$2n\pi < u < 2(n+1)\pi.$$

For $w$ we have

$$C|\epsilon| < |w| < \delta$$
$$\log(1/\delta) < bu + av < \log(1/|\epsilon|) - \log C$$
$$\frac{\log(1/\delta)}{b} - av/b < u < \frac{\log(1/|\epsilon|) - \log C}{b} - av/b.$$

We divide into cases depending on whether $a \neq 0$ or $a = 0$.

First, assume $a \neq 0$. We choose a constant $0 < s < 1$ so that $\frac{1}{2} < 1 + \frac{2ab\pi}{a} < \frac{3}{2}$.

Case (i): $a \neq 0, n < sp$, then

$$(u + iv)^\gamma = U + iV$$
$$\sim U + ip^\gamma, |U| \sim p^\gamma.$$

So we have

$$H_{\alpha, n} \sim \int_{|x| < Cp^{\gamma}} \tilde{H}_\alpha(x)/p^{\gamma} \, dx + \int_{|x| > Cp^{\gamma}} \tilde{H}_\alpha(x)p^{\gamma}/x^2 \, dx$$
$$\sim \int_{|x| < Cp^{\gamma}} \tilde{H}_\alpha(x)|x|^{1/\gamma - 1} \left( \frac{|x|}{p^{\gamma}} \right)^{1-1/\gamma} \frac{1}{p} \, dx$$
$$+ \int_{|x| > Cp^{\gamma}} \tilde{H}_\alpha(x)|x|^{1/\gamma - 1} \left( \frac{p}{|x|} \right)^{1+1/\gamma} \frac{1}{p} \, dx.$$

Case (ii) $a \neq 0, n > sp$

$$(u + iv)^\gamma = U + iV$$
$$\sim n^\gamma + ipn^{\gamma - 1}.$$
Then
\[
H_{\alpha,n} \sim \int_{|x-n\gamma| \leq pn\gamma^{-1}} \tilde{H}_{\alpha}(x) \frac{1}{pn\gamma^{-1}} dx \\
+ \int_{n\gamma/2 < |x-n\gamma| > pn\gamma^{-1}} \tilde{H}_{\alpha}(x) \frac{pn\gamma^{-1}}{|x-n\gamma|^2} dx \\
+ \int_{n\gamma/2 < |x-n\gamma| < 2n\gamma} \tilde{H}_{\alpha}(x) \frac{pn\gamma^{-1}}{n^2} dx \\
+ \int_{|x-n\gamma| > 2n\gamma} \tilde{H}_{\alpha}(x) \frac{pn\gamma^{-1}}{x^2} dx \\
= I + II + III + IV
\]

We will usually leave the case \( a = 0 \) to the reader.

Case (iii) \( a = 0 \)

\[
\gamma = 2 \\
(u + iv)^2 = u^2 - v^2 + 2iuv.
\]

So
\[
h_{\alpha,n} = \int H_{\alpha,n}(x) \frac{uv}{(uv)^2 + (x - (u^2 - v^2))^2} dx \\
\sim \int H_{\alpha,n}(x) \frac{uv}{(uv)^2 + (x + v^2 - u^2)^2} dx \\
\sim \int_{|x + v^2 - u^2| < uv} H_{\alpha,n}(x) \frac{uv}{(uv)^2 + (x + v^2 - u^2)^2} dx \\
+ \int_{|x + v^2 - u^2| > uv} H_{\alpha,n}(x) \frac{uv}{(uv)^2 + (x + v^2 - u^2)^2} dx \\
\sim \int_{|x + v^2 - u^2| < uv} H_{\alpha,n}(x) \frac{uv}{x^2} dx \\
+ \int_{|x + v^2 - u^2| > uv} H_{\alpha,n}(x) \frac{uv}{(x + v^2 - u^2)^2} dx
\]

To estimate the integrals we divide into cases.

\[
u/10 < v < 10u:
\]
\[
\sim \int_{|x| < u^2} H_{\alpha,n}(x) \frac{1}{u^2} dx + \int_{|x| > u^2} H_{\alpha,n}(x) \frac{u^2}{x^2} dx \\
\sim \int_{|x| < u^2} H_{\alpha,n}(x) |x|^{-1/2} \left( \frac{|x|}{u^2} \right)^{1/2} \frac{1}{u} dx \\
+ \int_{|x| > u^2} H_{\alpha,n}(x) |x|^{-1/2} u^2 |x|^{1/2} dx.
\]
Suppose now
\[ u < v/10 \]
\[ h_{\alpha,n} \sim \int_{|x+v^2|<\sim uv} H_{\alpha,n}(x) \frac{1}{uv} dx + \int_{|x+v^2|>\sim uv} H_{\alpha,n}(x) \frac{uv}{(x+v^2-u^2)^2} dx \]
\[ \sim \int_{|x+v^2|<\sim uv} H_{\alpha,n}(x) |x|^{-1/2} \frac{1}{u} dx \]
\[ + \int_{|x+v^2|>\sim uv} H_{\alpha,n}(x) \frac{uv}{(x+v^2)^2} dx \]
\[ \sim \int_{|x+v^2|<\sim uv} H_{\alpha,n}(x) |x|^{-1/2} \frac{1}{u} dx \]
\[ + \int_{|x+v^2|>\sim uv} H_{\alpha,n}(x) |x|^{-1/2} \frac{uv|x|^{1/2}}{(x+v^2)^2} dx. \]

The last case is
\[ u > 10v. \]
\[ h_{\alpha,n} \sim \int_{|x-u^2|<\sim uv} H_{\alpha,n}(x) |x|^{-1/2} \frac{1}{v} dx \]
\[ + \int_{|x-u^2|>\sim uv} H_{\alpha,n}(x) |x|^{-1/2} \frac{uv|x|^{1/2}}{(x-u^2)^2} dx. \]

So now we have estimated \( h_{\alpha} \) on \( L_{\alpha,n} \cap R(p) \). The analogue estimates are valid for \( H_\beta \) on \( L_{\beta,m} \). The reason is that \( e^{-p} >> |\epsilon| \).

Our next step is to locate for which \( R(p) \) there is an intersection between \( L_{\alpha,n} \) and \( L_{\beta,m} \).

Fix \( L_{\alpha,n} \) and \( L_{\beta,m} \) and assume \((z, w) \in L_{\alpha,n} \cap L_{\beta,m} \). We can write
\[ z = e^{i(\zeta + (\log |\alpha|)/b} \]
\[ \zeta = u + iv \]
\[ 2n\pi < u < 2(n+1)\pi \]
\[ |z| = e^{-v}. \]

Also \((z', w') = \Phi_\epsilon(z', w'), (z', w') \in L_{\beta,m} \).
\[ z' = e^{i(\zeta' + (\log |\alpha|)/b} \]
\[ \zeta' = u' + iv' \]
\[ 2m\pi < u' < 2(m+1)\pi \]
\[ |z'| = e^{-v'}. \]
So

\[ \begin{align*}
    z &= \alpha(\epsilon) + e^{i(\zeta + (\log |\beta|)/b)} + \epsilon \mathcal{O}(z', w') \\
    w &= \beta(\epsilon) + \beta e^{i\lambda(\zeta' + (\log |\beta|)/b)} + \epsilon \mathcal{O}(z', w').
\end{align*} \]

Our goal is to locate which \( R(p) \) the point \((z, w)\) can belong to. So we need to find \( p \) so that \( e^{-p} - 1 < |z| = e^{-v} < e^{-p} \), i.e. we need to get a good estimate for \( v \) in terms of \( \alpha, \beta, n, m \).

There are 4 unknowns, \( u, v, u', v' \). However, \( u \sim 2n\pi, u' \sim 2m\pi \), so we only have \( v, v' \) left. Also we have two equations for the \( z \) and \( w \) coordinates respectively. (In fact, since these are complex equations, we have 4 real equations for the two real unknowns \( v, v' \).)

Before we proceed we show at first that for there to be an intersection, we actually must require that \( n \) and \( m \) are very close.

**Lemma 16.** If \( L_{\alpha,n} \) and \( L_{\beta,m} \) intersect in \( R_1 \), it follows that \( |m - n| \leq 1 \).

**Proof.** Recall that 

\[ \Phi_\epsilon(z, w) = (\alpha(\epsilon), \beta(\epsilon)) + (z, w) + \epsilon \mathcal{O}(z, w). \]

If \( \delta \) is chosen small enough, this implies that \( |\epsilon \mathcal{O}(z, w)| \leq \sigma |\epsilon| \) for any given \( 0 < \sigma << 1 \).

We pick two plaques, \( L_{\alpha,n}, L_{\beta,m} \) and consider intersection points in \( R_1 \). Let \( S > 0 \) be such that \( |\epsilon|/S < |\alpha(\epsilon)| - |\beta(\epsilon)| - \sigma |\epsilon| < |\alpha(\epsilon)| + |\beta(\epsilon)| + |\sigma| < S \). Note that if we increase the constant \( C \) used in the definition of \( D_4 \), we can still use the same \( S \). When the point is in \( L_{\alpha,n} \) we have

\[ \begin{align*}
    z &= e^{i(\zeta + (\log |\alpha|)/b)} \\
    |z| &= e^{-v} \\
    \log(1/|\delta|) < v < \log(1/|\epsilon|) - C \\
    w &= \alpha e^{i\lambda(\zeta + (\log |\alpha|)/b)} \\
    |w| &= e^{-bu - av}.
\end{align*} \]

If it is also in \( L_{\beta,m} \), then

\[ \begin{align*}
    z' &= e^{i(\zeta' + (\log |\beta|)/b)} \\
    |z'| &= e^{-v'} \\
    \log(1/|\delta|) < v' < \log(1/|\epsilon|) - C \\
    w' &= \beta e^{i\lambda(\zeta' + (\log |\beta|)/b)} \\
    |w'| &= e^{-bu' - av'}.
\end{align*} \]

Since

\[ L_{\beta,n}^\epsilon = \Phi_\epsilon(L_{\beta,m}), \]
the image point can be written
\[ Z = \alpha(\epsilon) + e^{i(\zeta' + (\log |\beta|)/b)} + \epsilon O(z', w') \]
\[ W = \beta(\epsilon) + \beta e^{i\lambda(\zeta' + (\log |\beta|)/b)} + \epsilon O(z', w') \]

Consider an intersection point in \( R_1 \) and set \( \zeta' = \zeta + c + id \). Then
\[ z = Z \]
\[ e^{-v-d} - S|\epsilon| < e^{-v} < e^{-v-d} + S|\epsilon| \]
\[ e^{-d} - Se^v|\epsilon| < 1 < e^{-d} + Se^v|\epsilon| \]

So
\[ Se^v|\epsilon| < S(1/(C|\epsilon|)|\epsilon|) = S/C << 1. \]
\[ |d| < 2Se^v|\epsilon| < 2S/C. \]

For the other coordinate,
\[ w = W \]
\[ e^{-bu-bc-ad} - S|\epsilon| < e^{-bu-av} < e^{-bu-bc-ad} + S|\epsilon| \]
\[ e^{-bc-ad} - Se^{bu+av}|\epsilon| < 1 < e^{-bc-ad} + Se^{bu+av}|\epsilon| \]
\[ Se^{bu+av}|\epsilon| < S/C << 1. \]
\[ |bc + ad| < 2Se^{bu+av}|\epsilon| < 2S/C \]
\[ |bc| < |bc + ad| + |a||d| < 2Se^{bu+av}|\epsilon| + |a|2Se^v|\epsilon|. \]

So
\[ |c| < \frac{1}{|b|} \left( 2Se^{bu+av}|\epsilon| + |a|2Se^v|\epsilon| \right) \]
\[ < 2S \left( 1 + \frac{|a|}{C|b|} \right) \]
\[ |c + id| < \frac{2S}{C} \left( 1 + \frac{1 + |a|}{|b|} \right) << 1. \]

It is also convenient to show that \( \alpha \) and \( \beta \) must be very close if there is an intersection. We estimate first the modulus and next the angle and finally we combine them.

**Lemma 17.** Suppose \( L_{\alpha,n} \) intersects \( L'_{\beta,m} \) in \( R_1 \). Then
\[ \left| \log(|\beta|/|\alpha|) \right| \leq 2S|\epsilon| \left[ e^v (b + |a|) + e^{bu+av} \right]. \]
Proof. We have
\[
z = Z e^{i(\zeta + (\log |\alpha|)/b)} = \alpha(\epsilon) + e^{i(\zeta + c + id + (\log |\beta|)/b)} + \epsilon O(z', w').
\]
So
\[
e^{i(\zeta + (\log |\alpha|)/b)} \left[ 1 - e^{ic - d + i(\log |\beta|)/b} \right] = \alpha(\epsilon) + \epsilon O(z', w').
\]
Taking the modulus
\[
e^{-\nu} \left| 1 - e^{ic - d + i(\log |\beta|)/b} \right| \leq S|\epsilon|
\]
\[
1 - e^{ic - d + i(\log |\beta|)/b} \leq S e^{\nu}|\epsilon| << 1.
\]
This gives
\[
|\theta| \leq 2 Se^{bu + av}|\epsilon| \left[ |a|/b + |a|/b + 1 \right] + 2S|\epsilon| e^{\nu} \left[ |a|^2/b + b + (|a| + |a|^2/b) \right].
\]

The Lemma follows.

We remark that the lemma as stated is slightly inaccurate. We only can conclude the estimate modulo $2\pi$. However, the parameters $e^{-2\pi b} \leq |\alpha|, |\beta| < 1$ so this problem arises when say $|\alpha|$ is close to 1 and $|\beta|$ is close to $e^{-2\pi b}$. We ignore this technicality which just means that $|\alpha|$ and $|\beta|$ get close after we follow the leaf $L_\alpha$ once around 0 counterclockwise.

Lemma 18. Write $\beta/\alpha = |\beta/\alpha|e^{i\theta}$. If there are intersection points in $R_1$, $\theta$ is close to 0 mod $2\pi$. More precisely:
\[
|\theta| \leq 2 Se^{bu + av}|\epsilon| \left[ |a|/b + |a|/b + 1 \right] + 2S|\epsilon| e^{\nu} \left[ |a|^2/b + b + (|a| + |a|^2/b) \right].
\]

Proof. We again use the parametrization.
\[
\alpha e^{i\lambda(\zeta + (\log |\alpha|)/b)} = \beta(\epsilon) + \beta e^{i\lambda(\zeta + c + id + (\log |\beta|)/b)} + \epsilon O(z', w').
\]
So
\[
\beta(\epsilon) + \epsilon O(z', w') = \alpha e^{i\lambda(\zeta + (\log |\alpha|)/b)} \left[ 1 - \frac{\beta}{\alpha} e^{i\lambda(c + id + (\log |\beta|)/b)} \right]
\]
\[
Se^{bu + av}|\epsilon| \geq \left| 1 - \frac{\beta}{\alpha} e^{i\lambda(c + id + (\log |\beta|)/b)} \right|.
\]
Hence
\[
S e^{bu+av} |\epsilon| \geq \left| 1 - \frac{\beta}{\alpha} e^{-bc-ad-(\log |\alpha|/b)+i(ac-bd+a(\log(|\beta|/|\alpha|))/b)} \right|
\]
\[
1 \gg S e^{bu+av} |\epsilon| \geq \left| 1 - \epsilon^{-bc-ad+i(\theta+b\log(|\beta|/|\alpha|))/b} \right|
\]
\[
2Se^{bu+av} |\epsilon| \geq |\theta + ac - bd + a(\log(|\beta|/|\alpha|))/b|.
\]

Therefore
\[
|\theta| \leq |ac| + |bd| + |a|| \log(|\beta|/|\alpha|)/b| + 2Se^{bu+av} |\epsilon|
\]
\[
\leq Se^{bu+av} |\epsilon| \left[ 2|a|/b + 2|a|/b + 2 \right] + S|e^{v} \left[ 2|a|^2/b + 2b + 2(|a| + |a|^2/b) \right].
\]

Which gives the estimate.

Next we locate more precisely the intersections of \( L_{\alpha,n} \) and \( L_{\beta,m} \) in \( R_1 \). Let
\[
z = Z.
\]

Then
\[
e^{i\zeta+i(\log |\alpha|)/b} = \alpha(\epsilon) + e^{i\zeta'+i(\log |\beta|)/b} + \epsilon O(z', w').
\]

We define \( \Delta \) by
\[
\zeta' = \zeta + \Delta.
\]

Then
\[
e^{i\zeta+i(\log |\alpha|)/b} - e^{i\zeta+i\Delta+i(\log |\beta|)/b} = \alpha(\epsilon) + \epsilon O
\]
\[
e^{i\zeta+i(\log |\alpha|)/b} \left[ 1 - e^{i\Delta+i(\log(|\beta|/|\alpha|))/b} \right] = \alpha(\epsilon) + \epsilon O
\]
\[
1 - e^{i\Delta+i(\log(|\beta|/|\alpha|))/b} = e^{-i\zeta-i(\log |\alpha|)/b} \left[ \alpha(\epsilon) + \epsilon O \right].
\]

This gives
\[
2k\pi + \Delta + (\log(|\beta|/|\alpha|))/b = ie^{-i\zeta-i(\log |\alpha|)/b} \left[ \alpha(\epsilon) + \epsilon O \right]
\]
\[
+ O(\epsilon e^{-i\zeta})^2.
\]

Using
\[
w = W,
\]
we have

\[ e^{i\lambda c} \left( \alpha e^{i\lambda (c + (\log |\alpha|)/b)} - \beta e^{i\lambda (c + (\log |\beta|)/b)} \right) = \beta(\epsilon) + \beta e^{i\lambda (c + (\log |\beta|)/b)} + \epsilon O \]

So

\[
1 - \frac{\beta}{\alpha} e^{i\lambda (c + (\log |\beta|)/|\alpha|)/b} - \log \left( \frac{\beta}{\alpha} \right) + i\lambda (c + (\log |\beta|)/|\alpha|)/b \sim e^{-i\lambda c} \frac{e^{-i\lambda (c + (\log |\alpha|)/b)}}{\alpha} [\beta(\epsilon) + \epsilon O].
\]

We get:

\[
\Delta + \log(|\beta|/|\alpha|)/b - \frac{1}{i\lambda} \log \left( \frac{\beta}{\alpha} \right) = \frac{1}{i\lambda} e^{-i\lambda c} \frac{e^{-i\lambda (c + (\log |\alpha|)/b)}}{\alpha} [\beta(\epsilon) + \epsilon O] + O(e^{-i\lambda c})^2.
\]

Adding the two expressions with \(\Delta\):

**Lemma 19.** Suppose that \(L_{\alpha,n} \cap L'_{\beta,m} \cap R_1 \neq \emptyset\). Then:

\[
- \frac{1}{i\lambda} \log \left( \frac{\beta}{\alpha} \right) = ie^{-i\zeta - i(\log |\alpha|)/b} [\alpha(\epsilon) + \epsilon O] + \frac{1}{i\lambda} e^{-i\lambda c} e^{-i\lambda (c + (\log |\alpha|)/b)} [\beta(\epsilon) + \epsilon O] + O(e^{-i\lambda c})^2 + O(e^{-i\lambda c})^2.
\]

To continue the search for intersection points of \(L_{\alpha,n}, L'_{\beta,m}\) in \(R_1\), we divide \(R_1\) into 3 pieces. We let \(C_1 > 1\) be a large constant.

\[
R_{1A} = \{ C|\epsilon| < |z|, |w| < \delta, C_1 |w| \leq |z| \}
\]

\[
R_{1B} = \{ C|\epsilon| < |z|, |w| < \delta, C_1 |z| \leq |w| \}
\]

\[
R_{1C} = \{ C|\epsilon| < |z|, |w| < \delta, |z| \leq C_1 |w| \leq C_1^2 |z| \}
\]

Here the constant \(C_1\) is chosen to work in the slope estimates before Lemma 21. Observe that \(R_{1A}\) and \(R_{1B}\) are similar. We will leave it up to the reader to verify the estimates for \(R_{1B}\).

**8. Theorem 7 for \(R_{1A}\), the part of \(R_3\) close to the \(z\)-axis**

We will assume that \(a \neq 0\) and leave the verification of the case \(a = 0\) to the reader. If \(|w| << |z|\), then the second term in the expression for \(\log(\beta/\alpha)\) in Lemma
18 on the right dominates and we get

\[ e^{av+bu}|\epsilon| \sim |(\beta/\alpha) - 1| \]

\[ 2n\pi < u < 2(n+1)\pi \]

\[ av \sim \log |(\beta/\alpha) - 1| + \log(1/|\epsilon|) - 2nb\pi. \]

So

\[ |v - \frac{\log |(\beta/\alpha) - 1| + \log(1/|\epsilon|) - 2nb\pi}{a}| < C \]

\[ C|\epsilon| < e^{-v} < \delta \]

\[ \log 1/\delta < v < \log 1/|\epsilon| - C \]

\[ p < v < p + 1, \]

see the beginning of Section 7.

**Lemma 20.** For intersection points in \( R_{1A} \), there is a constant \( C' \) such that

\[ \frac{C'|\epsilon|}{\delta} < |\beta - \alpha| < \frac{1}{C'}. \]

**Proof.** Since \( e^{av+bu}|\epsilon| \sim |\beta/\alpha - 1| \sim |\beta - \alpha| \) and \( e^{av+bu} = 1/|w| \) we have \( |\beta - \alpha| \sim |\epsilon|/|w| \). But \( C|\epsilon| < |w| < |z|/C < \delta/C \). The lemma follows. \( \square \)

**Lemma 21.** Suppose that \( L_{\alpha,n} \) intersects \( L_{\beta,m}^\epsilon \) in \( R_{1A} \). Then the intersection points must be in \( R(p) \) for some

\[ |p - \frac{\log |(\beta/\alpha) - 1| + \log(1/|\epsilon|) - 2nb\pi}{a}| < C. \]

For the plaque to enter \( R_1 \) we further need \( n \) to satisfy

\[ \log |(\beta/\alpha) - 1| + \log(1/|\epsilon|) - 2nb\pi \in I \]

where \( I \) is the interval with endpoints \( a\log 1/\delta, a\log(1/|\epsilon|) - aC \)

Our next step is to verify that there is a uniform bound on the number of intersection points of \( L_{\alpha,n}, L_{\beta,m}^\epsilon \) in \( R_{1A} \).

In order to study the number of intersections between plaques, we compare their slopes:

Suppose \( (z,w) = (Z,W) := \Phi_\epsilon(z',w') \) is an intersection point of \( L_{\alpha,n} \) and \( L_{\beta,m}^\epsilon \) in \( R_1 \). The slope \( S_1 \) of \( L_{\alpha,n} \) is \( \lambda w/z \). The slope of the perturbed leaf is \( S_2 \). We choose the constant \( C_1 \) used in the definition of \( R_{1A}, R_{1B}, R_{1C} \) in the following estimates.
\[ \Phi'(z', w')(z', \lambda w') = (z' + \epsilon O(z', w'), \lambda w' + \epsilon O(z', w')) \]
\[ S_2 = \frac{\lambda w' + \epsilon O(z', w')}{z' + \epsilon O(z', w')} \]
\[ = \frac{\lambda W - \lambda \beta(\epsilon) + \epsilon O(Z, W)}{Z - \alpha(\epsilon) + \epsilon O(Z, W)} \]
\[ S_2 = \frac{\lambda w - \lambda \beta(\epsilon) + \epsilon O(z, w)}{z - \alpha(\epsilon) + \epsilon O(z, w)} \]
\[ S_2 - S_1 = \frac{\lambda w - \lambda \beta(\epsilon) + \epsilon O(z, w)}{z - \alpha(\epsilon) + \epsilon O(z, w)} - \frac{\lambda w/z}{z(z - \alpha(\epsilon) + \epsilon O(z, w))} \]
\[ \frac{1}{C_1} |z| \leq |w| \leq C_1 |z| : S_2 - S_1 \sim \frac{\lambda}{z^2} (w \alpha(\epsilon) - z \beta(\epsilon)) \]
\[ |w| > C_1 |z| : S_2 - S_1 \sim \frac{\epsilon w}{z^2} \]
\[ |w| < \frac{1}{C_1} |z| : S_2 - S_1 \sim \frac{\epsilon}{z} \]

**Lemma 22.** There is at most a uniformly bounded number of intersection points in \( R_{1A} \).

**Proof.** The case of \( R_{1A}, R_{1B} \) follows from slope estimates. For the case \( R_{1C} \), note that leaves might be tangent when \( w/z \) is close to \( \beta(\epsilon)/\alpha(\epsilon) \). They both have slope about \( \lambda \). But since we assume that \( \lambda \neq \beta'(0)/\alpha'(0) \), this tangency is at most of order 2. \( \square \)

We estimate the contribution to \( T \wedge g T^\epsilon \) from \( R_{1A} \). We assume again that \( a \neq 0 \). By Lemma 18, the parameters \( \alpha, \beta \) are restricted to the values: \( e^{-2\pi b} < |\alpha|, |\beta| < 1, 1/C > |\beta - \alpha| > C|\epsilon|/\delta \). So fix \( \alpha, \beta \). Next, by Lemma 15, we can set \( n = m \) to be some integer in the interval given by Lemma 19. The case \( n = m \pm 1 \) is similar. Because of the finiteness of the number of intersection points, see Lemma 20, we can set
\[ p = p(n) = \frac{\log |(\beta/\alpha) - 1| + \log(1/|\epsilon|) - 2nb\pi}{a} \]
and consider only one intersection point. Then we multiply the values of \( H_{\alpha,n} \) and \( H_{\beta,n} \) using the formulas in Case (i) or (ii) depending on whether \( n < sp \) or \( n > sp \). We then add these products over \( n \) and integrate the result over \( d\mu(\alpha)d\mu(\beta) \).

**Case (i),** \( n < sp \):
\[ n < \frac{s}{a} \log |(\beta/\alpha - 1) + \log 1/|\epsilon| - 2nb\pi \]
\[ n(1 + \frac{2sb\pi}{a}) < \frac{s}{a} \log |(\beta/\alpha - 1) + \log 1/|\epsilon| |. \]

Recall that \(1/2 < 1 + 2sb\pi a < 3/2\). We get
\[ n < \frac{s}{1 + \frac{2sb\pi}{a}} \frac{\log |(\beta/\alpha - 1) + \log 1/|\epsilon| |}{a} =: n(\alpha, \beta, \epsilon). \]

In this case we have the following estimates at intersection points.

\[ h_{\alpha,n} \sim \int_{|x| < Cv^\gamma} H_\alpha(x)|x|^{1/\gamma - 1} \left( \frac{|x|}{v^\gamma} \right)^{1-1/\gamma} \frac{1}{v} dx + \int_{|x| > Cv^\gamma} H_\alpha(x)|x|^{1/\gamma - 1} \left( \frac{v^\gamma}{|x|} \right)^{1+1/\gamma} \frac{1}{v} dx \]
\[ h_{\beta,m} \sim \int_{|y| < C(v')^\gamma} H_\beta(y)|y|^{1/\gamma - 1} \left( \frac{|y|}{(v')^\gamma} \right)^{1-1/\gamma} \frac{1}{v'} dy + \int_{|y| > C(v')^\gamma} H_\beta(y)|y|^{1/\gamma - 1} \left( \frac{(v')^\gamma}{|y|} \right)^{1+1/\gamma} \frac{1}{v'} dy \]

Here we have used that \(v\) and \(\rho\) are comparable. In fact from the estimate in the beginning of the section, we see that \(u \sim n, n < sp\) so \(u \sim v\), hence \(\rho \sim v\).

Here \(v, v' \sim \log (|\beta/\alpha| - 1) + \log 1/|v| - 2nb\pi\). This allows us to sum over \(v\) instead of over \(n\), \(\log 1/\delta < v < \log 1/|\epsilon| - C\).

We need to estimate \(\sum_v h_{\alpha,n} h_{\beta,m}^v\) and then integrate the answer over the measure \(\mu(\alpha)\mu(\beta)\).

Note we will majorize the sum by the product \(\sum_v h_{\alpha,n} \sum_m h_{\beta,m}^v\). Then we use the dominated convergence theorem.

We finally have
\[ \sim \log 1/|\epsilon| - C \int_{|x| < Cv^\gamma} H_\alpha(x) \frac{dx}{v^\gamma} + \int_{|x| > Cv^\gamma} H_\alpha(x) \frac{|x|^\gamma}{|x|^2} dx \]
We split the integral
\[ \sum_v h_{\alpha,n} \sim \int_{|x|<(\log 1/\delta)\gamma} \tilde{H}_\alpha(x) \frac{\log |x|}{x^\gamma} \, dx + \int_{(\log 1/\delta)\gamma<|x|<\log 1/|\epsilon|\gamma} \tilde{H}_\alpha(x) \frac{\log 1/|\epsilon|}{x^\gamma} \, dx + \int_{|x|>(\log 1/|\epsilon|)\gamma} \tilde{H}_\alpha(x) \frac{1}{x^2} \, dx. \]

We estimate the quantities under $\Sigma$ and we get:
\[ \sim \int_{|x|<(\log 1/\delta)\gamma} \tilde{H}_\alpha(x) \frac{1}{x^\gamma} \, dx + \int_{(\log 1/\delta)\gamma<|x|<\log 1/|\epsilon|\gamma} \tilde{H}_\alpha(x) \frac{1}{|x|^{1-1/\gamma}} \, dx + \int_{|x|>(\log 1/|\epsilon|)\gamma} \tilde{H}_\alpha(x) \frac{1}{x^2} \, dx. \]

Using Lemma 12 this gives:
\[ \sum_v h_{\alpha,n} \sim \int_{|x|<(\log 1/\delta)\gamma} \tilde{H}_\alpha(x) |x|^{1/\gamma-1} \left( \frac{|x|}{(\log 1/\delta)^\gamma} \right)^{1-1/\gamma} \, dx + \int_{(\log 1/\delta)\gamma<|x|<\log 1/|\epsilon|\gamma} \tilde{H}_\alpha(x) |x|^{1/\gamma-1} \, dx + \int_{|x|>(\log 1/|\epsilon|)\gamma} \tilde{H}_\alpha(x) |x|^{1/\gamma-1} \left( \frac{(\log 1/|\epsilon|)^\gamma}{|x|} \right)^{1+1/\gamma} \, dx \to 0, \delta \to 0. \]

Observe that we had to take $\delta$ small.
This finishes the case (i), $n < \text{sp}$. So we have proved:

**Lemma 23.** The contribution to the geometric wedge product from $R_{1A}$ in case (i), $a \neq 0, n < \text{sp}$ goes to zero when $\delta \to 0$.

We next deal with the case $n > \text{sp}$. Recall that:

**Case (ii)** $a \neq 0, n > \text{sp}$
We then have:
(u + iv)^\gamma = U + iV
\sim n^\gamma + ip(n)n^{\gamma-1}.

Then

H_{\alpha,n} \sim \int_{|x-n^\gamma| \leq p(n)n^{\gamma-1}} \tilde{H}_\alpha(x) \frac{1}{p(n)n^{\gamma-1}} dx
+ \int_{n^\gamma/2 > |x-n^\gamma| > p(n)n^{\gamma-1}} \tilde{H}_\alpha(x) \frac{p(n)n^{\gamma-1}}{|x-n^\gamma|^2} dx
+ \int_{n^\gamma/2 < |x-n^\gamma| < 2n^\gamma} \tilde{H}_\alpha(x) \frac{p(n)n^{\gamma-1}}{n^{2\gamma}} dx
+ \int_{|x-n^\gamma| > 2n^\gamma} \tilde{H}_\alpha(x) \frac{p(n)n^{\gamma-1}}{x^2} dx
= I + II + III + IV
= I_n + II_n + III_n + IV_n.

For simplicity of notation we assume \( a > 0 \). Then we have the following range for \( n \) from Lemma 20. The number \( n \) satisfies:

\[
a \log \frac{1}{\delta} < \log |(\beta/\alpha) - 1| + \log(1/|\epsilon|) - 2nb\pi < a \log 1/|\epsilon| - aC.
\]

This gives

\[
a \log \frac{1}{\delta} - \log |(\beta/\alpha) - 1| - \log(1/|\epsilon|) < -2nb\pi < \neg \log |(\beta/\alpha) - 1| - \log(1/|\epsilon|) + a \log 1/|\epsilon| - aC.
\]

So

\[
-a \log \frac{1}{\delta} + \log |(\beta/\alpha) - 1| + \log(1/|\epsilon|) > 2nb\pi > 
\log |(\beta/\alpha) - 1| + \log(1/|\epsilon|) - a \log 1/|\epsilon| - aC.
\]

Hence

\[
\frac{\log |(\beta/\alpha) - 1| + (1-a) \log(1/|\epsilon|) - aC}{2b\pi} < n < \frac{-a \log \frac{1}{\delta} + \log |(\beta/\alpha) - 1| + \log(1/|\epsilon|)}{2b\pi}.
\]

However, \( n \) is further restricted because \( n > sp \) and \( p > \log 1/\delta \). If we then estimate IV and sum over \( n \), we get
\[
\sum_n IV_n \sim \int_{|x|> (\log 1/\delta)\gamma} \tilde{H}_\alpha(x) \frac{|x|^{1/\gamma}}{x^2} \sum_{n=\log 1/\delta}^{n\gamma} \frac{|n|^{1/\gamma}}{x^2} \frac{\gamma}{\gamma} \\
\sim \int_{|x|> (\log 1/\delta)\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx \\
\to 0
\]

Similarly for \(\sum_n III_n\) we get to estimate \(\sum 1/n\gamma \sim |x|^{1/\gamma-1}\) which again is fine.

Next we handle the terms \(II_n\). For a given \(x\), the range of \(n\) is on the order of \(2/3|x|^{1/\gamma} < n < |x|^{1/\gamma} - p(x^{1/\gamma})\) and similarly for \(n > |x|^{1/\gamma}\). Also note that the terms \(p(n) \lesssim |x|^{1/\gamma}\) since \(n \sim |x|^{1/\gamma}\) and \(p \lesssim n\). So we sum the expressions \(\frac{n^{\gamma-1}}{|x-n\gamma|^2}\) which integrates to \(\frac{1}{|x-n\gamma|^2}\), so inserting the limits of the summation, we get a bound of the same form as for \(III\).

Finally we sum over the \(I_n\). Here we make the rough estimate that \(\log 1/\delta < p(n) < sn\). So we integrate over \(|x - n\gamma| < sn\gamma\) but in the integrand we replace \(p(n)\) by \(\log 1/\delta\). With this estimate we get the integral \(\tilde{H}_\alpha(x) \frac{1}{\log(1/\delta)|x|} \ll \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\). Hence this also goes to zero with \(\delta\).

Hence we have shown the following:

**Lemma 24.** The contribution to the geometric wedge product in the case of \(R_{1A}\), case (ii), \(a \neq 0\), \(n > sp\) goes to zero when \(\delta \to 0\).

9. **Theorem 7 for \(R_{1C}\), the diagonal part of \(R_1\)**

We are in the set \(\{C|\epsilon| < |z|, |w| < \delta, |z| \sim |w|\}\).

On \(L_{\alpha,n}\), we have the following estimate for \(u, v\).

\[
2n\pi < u < 2(n+1)\pi \\
|v - \frac{2n}{1-a}| < C'', \\
\log 1/\delta < v < \log(1/|\epsilon|) - C.
\]

In the \(U, V\) coordinates,

\[
(u+iv)^\gamma = U + iV \\
V \sim |n|^{\gamma} \\
|U| \ll |n|^{\gamma}.
\]

So at intersection points

\[
h_{\alpha,n} \sim \int \tilde{H}_\alpha(x) \frac{n^{\gamma}}{n^{2\gamma} + (x-U)^2}dx \\
\sim \int_{|x|<2n^{\gamma}} \tilde{H}_\alpha(x) \frac{dx}{n^{\gamma}} + \int_{|x|>2n^{\gamma}} \tilde{H}_\alpha(x) \frac{n^{\gamma}dx}{x^2}.
\]
Adding up the contributions

\[
\sum_n h_{\alpha,n} \sim \int_{|x|<(\log(1/\delta)^{\gamma})} \tilde{H}_\alpha(x) \left( \sum_{n=\log 1/\delta}^{\infty} \frac{1}{n^{\gamma}} \right) \, dx \\
+ \int_{|x|>(\log(1/\delta)^{\gamma})} \tilde{H}_\alpha(x) \left( \sum_{n=1/\gamma}^{1} \frac{1}{n^{\gamma}} \right) \, dx \\
+ \int_{|x|>(\log(1/\delta)^{\gamma})} \tilde{H}_\alpha(x) \left( \sum_{n=\log 1/\delta}^{\infty} \frac{n^{\gamma}}{x^{2}} \right) \, dx.
\]

After estimating the sums we get

\[
\sum_n h_{\alpha,n} \sim \int_{|x|<(\log(1/\delta)^{\gamma})} \tilde{H}_\alpha(x) \frac{1}{(\log(1/\delta)^{\gamma})} \, dx \\
+ \int_{|x|>(\log(1/\delta)^{\gamma})} \tilde{H}_\alpha(x) \frac{1}{x^{1/\gamma} \gamma - 1} \, dx \\
+ \int_{|x|>(\log(1/\delta)^{\gamma})} \tilde{H}_\alpha(x) \frac{x^{1/\gamma} \gamma + 1}{x^{2}} \, dx
\]

So,

\[
\sum_n h_{\alpha,n} \sim \int_{|x|>(\log(1/\delta)^{\gamma})} \tilde{H}_\alpha(x) |x|^{1/\gamma - 1} \, dx \\
+ \int_{|x|<(\log(1/\delta)^{\gamma})} \tilde{H}_\alpha(x) |x|^{1/\gamma - 1} \left( \frac{|x|}{(\log(1/\delta)^{\gamma})} \right)^{1-1/\gamma} \, dx
\]

This is arbitrarily small as long as \( \delta \) is chosen small enough.

10. Theorem 7 for \( R_2 \), the part of \( D_3 \) close to the \( z- \) axis

This case is divided in two subcases depending on whether one is close to one of the indicatrices \( (R_{2A}) \) or not \( (R_{2B}) \).

11. Theorem 7 for \( R_{2A} \) close to an indicatrix

Again we assume that \( a \neq 0 \). There are two indicatrices, \( w = 0 \) and \( w \) close to \( \beta(\epsilon) \). By symmetry it suffices to do one of them. We choose to estimate close to the indicatrix \( w = 0 \). So we set \( R_{2A} = \{ C|\epsilon| < |z| < \delta, |w| < s|\epsilon| \} \) for some small constant \( s > 0 \). Let \( L_{\gamma,m} \) and \( L_{\alpha,n} \) be plaques intersecting at \((z, w)\) in \( R_{2A} \) for parameters \((u', v')\), \((u, v)\).

Since the point \((z, w)\) is about distance \(|\beta'(0)||\epsilon|\) away from the indicatrix for the perturbed lamination, we get \((w' = \beta(\epsilon) + \beta e^{i\lambda(u' + (\log |\beta| / b) + iv')} + \cdots)\). This gives:

\[
2m\pi < u' < 2(m+1)\pi \\
C_1 < av' + 2mb\pi + \log |\epsilon| < C_2.
\]
We also have
\[ C|\epsilon| < |z| = e^{-u} = |z'| = |\alpha(\epsilon) + e^{i(u' + \log|\beta|/b)} - v' + \ldots|, \]
hence
\[ C_3 < v - v' < C_4 \]
\[ C_4 < av + 2mb\pi + \log|\epsilon| < C_5. \]
\[ 2n\pi < u < 2(n + 1)\pi. \]

Using
\[ |w| < s|\epsilon| \]
we get
\[ e^{-bu - av} < s|\epsilon| \]
\[ \log(1/s) < av + 2nb\pi + \log|\epsilon| \]
\[ 2(n - m)b\pi = (av + 2nb\pi + \log|\epsilon|) - (av + 2m\pi b + \log|\epsilon|) \]
\[ > \log(1/s) - C_1. \]

These calculations show that for the given plaques, the pairs \((u, v), (u', v')\) belong to rectangles of uniformly bounded size. Hence the number of intersection points can easily be estimated by using slope estimates for the plaques. We get a uniformly bounded number of intersection points.

We divide this into cases \(I, II, III\).

For \(I\), we have \(1/C\log(1/|\epsilon|) < 2mb\pi + \log|\epsilon| < C\log(1/|\epsilon|)\).

For \(II\) we have \(2mb\pi + \log|\epsilon| < 1/C\log(1/|\epsilon|)\).

For \(III\) we have \(2mb\pi + \log|\epsilon| > C\log(1/|\epsilon|)\). We note however, that in case \(III\), \(v'\) must be very large in comparison with \(\log 1/|\epsilon|\). This implies that \(|z'| << |\epsilon|\) hence there are no intersection points in this case. So we are left with the two cases \(R_{2AI}, R_{2AII}\).

12. Theorem 7 for \(R_{2AI}\) close to an indicatrix.

It follows in this case that \(v, v' \sim \log(1/|\epsilon|)\). Hence
\[ u' + iv' \sim 2m\pi + i\log(1/|\epsilon|) \]
and
\[ U' + iV' \sim U' + i(\log(1/|\epsilon|))^\gamma. \]
In particular

$$|U'| \prec (\log(1/\epsilon))^{\gamma}.$$  

Using the Poisson integral we estimate

$$h_{\beta,m}^\epsilon \sim \int \tilde{H}_\beta(y) \frac{\gamma (\log(1/\epsilon))^{\gamma}}{(\log(1/\epsilon))^{2\gamma} + (x - U')^2} \, dy$$

$$\sim \int |y| < 2(\log(1/\epsilon))^{\gamma} \tilde{H}_\beta(y) \frac{1}{|y|} \, dy$$

$$+ \int |y| > 2(\log(1/\epsilon))^{\gamma} \tilde{H}_\beta(y) \frac{\gamma (\log(1/\epsilon))^{\gamma}}{y^2} \, dy.$$  

Adding up

$$\sum_{m \in I} h_{\beta,m}^\epsilon \sim \int |y| < 2(\log(1/\epsilon))^{\gamma} \tilde{H}_\beta(y) |y|^{1/\gamma - 1} \left( \frac{|y|}{(\log(1/\epsilon))^{\gamma}} \right)^{1 - 1/\gamma} \, dy$$

$$+ \int |y| > 2(\log(1/\epsilon))^{\gamma} \tilde{H}_\beta(y) |y|^{1/\gamma - 1} \left( \frac{(\log(1/\epsilon))^{\gamma}}{|y|} \right)^{1/\gamma + 1} \, dy.$$  

Next we estimate $h_{\alpha,n}$. There are two cases to consider:

a): $n < C \log(1/\epsilon)$

b): $n > C \log(1/\epsilon)$

The contribution for case a) is:

**Case $R_{2AI_1}$**: Recall that we have $n > m - C_0$. Hence we have that $|n| < C \log(1/\epsilon)$. This means that we can write $u + iv \sim 2n\pi + i(\log(1/\epsilon))$. Hence the estimates work as for $h_{\beta,m}^\epsilon$.

$$\sum_{|n| < C \log(1/\epsilon)} h_{\alpha,n} \sim \int |x| < 2(\log(1/\epsilon))^{\gamma} \tilde{H}_\alpha(x) |x|^{1/\gamma - 1} \left( \frac{|x|}{(\log(1/\epsilon))^{\gamma}} \right)^{1 - 1/\gamma} \, dx$$

$$+ \int |x| > 2(\log(1/\epsilon))^{\gamma} \tilde{H}_\alpha(x) |x|^{1/\gamma - 1} \left( \frac{(\log(1/\epsilon))^{\gamma}}{|x|} \right)^{1/\gamma + 1} \, dx.$$  

**Case $R_{2AI_1}$**. We have

$$u + iv \sim n + i \log(1/\epsilon)$$

$$U + iV \sim n^{\gamma} + in^{\gamma - 1}\log(1/\epsilon),$$

and

$$h_{\alpha,n} \sim \int \tilde{H}_\alpha(x) \frac{n^{\gamma - 1}\log(1/\epsilon)}{n^{\gamma - 1}\log(1/\epsilon) + (x - n^{\gamma})^2} \, dx.$$
This integral has already been estimated. See the calculations for the set $D_1$ in the region where $|z - \eta| < d|\eta|$, case (ii) where $n > 10 \log(1/|\eta|)$. It follows that the contributions from that region goes to zero with $\epsilon$.

13. Theorem 7 for $R_{2AIi}$ close to an indicatrix.

We restrict for simplicity to the case $a > 0$. We can divide into three cases:

a) $n > m > v, v'$
b) $n > v, v' > m$
c) $v, v' > n > m$

14. Theorem 7 for $R_{2AIi_a}$ close to an indicatrix.

We have

\[
(u + iv)^\gamma = U + iV \\
\sim n^\gamma + ivn^{\gamma-1} \\
(u' + iv')^{\gamma} = U' + iV' \\
\sim m^\gamma + iv'm^{\gamma-1} \\
\sim (\log(1/|\epsilon|) + iv' \log(1/|\epsilon|))^{\gamma-1} \\
\log 1/\delta < v' < \log 1/|\epsilon|.
\]

We now estimate

\[
H_\beta \sim \int \tilde{H}_\beta(y) \frac{v'(\log(1/|\epsilon|))^{\gamma-1}}{[v'(\log(1/|\epsilon|))^{\gamma-1}]^2 + (y - m)^2} dy.
\]

We divide the integral and estimate each term.

\[
H_\beta \sim \int_{y-m^\gamma < cv'(\log 1/|\epsilon|)^{\gamma-1}} \tilde{H}_\beta(y) \frac{1}{v'(\log(1/|\epsilon|))^{\gamma-1}} dy \\
+ \int_{(\log 1/|\epsilon|)^{\gamma/2}}^{y-(\log 1/|\epsilon|)^{\gamma/2}} \tilde{H}_\beta(y) \frac{v'(\log(1/|\epsilon|))^{\gamma-1}}{(y-(\log 1/|\epsilon|)^{\gamma/2})^2} dy \\
+ \int_{y-(\log 1/|\epsilon|)^{\gamma/2}} \tilde{H}_\beta(y) \frac{v'(\log(1/|\epsilon|))^{\gamma-1}}{(y-(\log 1/|\epsilon|)^{\gamma/2})^2} dy.
\]

So

\[
H_\beta \sim \int_{y-m^\gamma < cv'(\log 1/|\epsilon|)^{\gamma-1}} \tilde{H}_\beta(y) \frac{y^{1/\gamma-1}}{v'} dy \\
+ \int_{y-(\log 1/|\epsilon|)^{\gamma/2}}^{y-(\log 1/|\epsilon|)^{\gamma/2}} \tilde{H}_\beta(y) \frac{v'(\log(1/|\epsilon|))^{\gamma-1}}{(y-(\log 1/|\epsilon|)^{\gamma/2})^2} dy \\
= H_{\beta_{1,v'}} + H_{\beta_{2,v'}}.
\]
For $H_\alpha$ we have

$$H_\alpha \sim \int \tilde{H}_\alpha(x) \frac{v n^{\gamma - 1}}{v n^{\gamma - 1} + (x - n^\gamma)^2} dx$$

$$\sim \int_{|x - n^\gamma| < c v n^{\gamma - 1}} \tilde{H}_\alpha(x) \frac{1}{v n^{\gamma - 1}} dx$$

$$+ \int_{|x - n^\gamma| > c v n^{\gamma - 1}} \tilde{H}_\alpha(x) \frac{v n^{\gamma - 1}}{(x - n^\gamma)^2} dx.$$ 

To sum up over the intersection points, we note at first that for a given plaque $L_{\beta,m}$ there is a finite range of $v'$ and $v - v'$ is bounded, so we can assume that there is one intersection point with $L_{\alpha,n}$ for each $n > m$. Hence we sum first over the plaques $L_{\alpha,n}$, $m < n < \infty$.

$$\sum_n \int_{|x - n^\gamma| < c v n^{\gamma - 1}} \tilde{H}_\alpha(x) \frac{1}{v n^{\gamma - 1}} dx \sim \sum_{n = x^{1/\gamma + v}} \sum_{n = x^{1/\gamma - v}} \int_{|x - n^\gamma| < c v n^{\gamma - 1}} \tilde{H}_\alpha(x) dx.$$

The other contribution is

$$\sum_n \int_{|x - n^\gamma| > c v n^{\gamma - 1}} \tilde{H}_\alpha(x) \frac{v n^{\gamma - 1}}{(x - n^\gamma)^2} dx$$

$$\sim \int_{x > m^\gamma} \tilde{H}_\alpha(x) v n^{\gamma - 1}$$

$$+ \int_{x < m^\gamma - c v m^{\gamma - 1}} \tilde{H}_\alpha(x) \frac{v}{|x - m^\gamma|} dx,$$

so we conclude:

$$\sum_{n > m} H_\alpha \sim \int_{x > m^\gamma} \tilde{H}(x) v n^{\gamma - 1} + \int_{x < m^\gamma - c v m^{\gamma - 1}} \tilde{H}_\alpha(x) \frac{v}{|x - m^\gamma|} dx$$

$$< \sim \int_{|x| > m^\gamma / 2} \tilde{H}_\alpha(x) v n^{\gamma - 1} + \int_{|x| < m^\gamma / 2} \tilde{H}_\alpha(x) v n^{\gamma - 1} \left( \frac{|x|}{m^\gamma} \right)^{1-1/\gamma} dx.$$ 

In this case $m$ will have approximately the range $(\log 1/|\epsilon|)/2 < m < \log 1/|\epsilon|$, hence we have

$$\sum_{n > m} H_\alpha \sim \int_{|x| > (\log 1/|\epsilon|)^\gamma} \tilde{H}_\alpha(x) v n^{\gamma - 1}$$

$$+ \int_{|x| < (\log 1/|\epsilon|)^\gamma} \tilde{H}_\alpha(x) v n^{\gamma - 1} \left( \frac{|x|}{(\log 1/|\epsilon|)^\gamma} \right)^{1-1/\gamma} dx.$$
Next we sum $H_\beta$ over $m$ or equivalently over $v', \log 1/\delta < v' < (\log 1/|\epsilon|)/2$. We integrate first over $H_{\beta_1,v'}$. For a given $y$, the range of $v'$ is in the interval with endpoints $(1 \pm c)\frac{\log 1/|\epsilon|}{(\log 1/|\epsilon|)/2}$.

This part is bounded by
\[
\int_{|y-(\log 1/|\epsilon|)/2| < (\log 1/|\epsilon|)/2} \tilde{H}_\beta(y)|y|^{1/\gamma - 1}dy \to 0.
\]

The second part is bounded by
\[
\int_{|y| < 2(\log 1/|\epsilon|)/2} \tilde{H}_\beta(y)|y|^{1/\gamma - 1} \left(\frac{|y|}{(\log 1/|\epsilon|)/2}\right)^{1-1/\gamma} dy
\]
\[+
\int_{|y| > 2(\log 1/|\epsilon|)/2} \tilde{H}_\beta(y)|y|^{1/\gamma - 1} \left(\frac{|y|}{(\log 1/|\epsilon|)/2}\right)^{1+1/\gamma} dy.
\]

Again the contribution goes to zero by Proposition 1 and Lemma 12.

15. **Theorem 7 for $R_{2AIIb}$ close to an indicatrix.**

In this case $n > v, v' > m$. First we recall the estimates for $H_\alpha$ which are the same as in the case $R_{2AIIa}$. We have
\[
(u + iv)^\gamma = U + iV
\]
\[
\sim n^\gamma + ivn^{\gamma - 1} \text{ with }
\log 1/\delta < v, v' < \log 1/|\epsilon|.
\]

So
\[
H_\alpha \sim \int \tilde{H}_\alpha(x) \frac{vn^{\gamma - 1}}{[vn^{\gamma - 1}]^2 + (x - n^\gamma)^2} dx
\]
\[
\sim \int_{|x-n^\gamma| < cnv^{\gamma - 1}} \tilde{H}_\alpha(x) \frac{1}{vn^{\gamma - 1}} dx
\]
\[+
\int_{|x-n^\gamma| > cnv^{\gamma - 1}} \tilde{H}_\alpha(x) \frac{vn^{\gamma - 1}}{(x - n^\gamma)^2} dx.
\]

Next we estimate $H_\beta$. We have
\[
(u' + iv')^\gamma = U' + iV'
\]
\[
\log 1/|\epsilon|/2 < v' < \log 1/|\epsilon|
\]
\[
m + v' = \log 1/|\epsilon|
\]
\[
V' \sim (\log 1/|\epsilon|)^\gamma
\]
\[
|U'| \sim (\log 1/|\epsilon|)^\gamma.
\]
Hence

\[
H_\beta \sim \int_{|y|<2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\beta \frac{1}{(\log 1/|\epsilon|)^\gamma} dy + \int_{|y|>2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\beta \frac{1}{y^2} dy.
\]

Next we estimate the contribution to the geometric wedge product. So fix \( \alpha, \beta \). Next fix a plaque \( L_{\beta,m}, v, v' \sim \log 1/|\epsilon| - m \). Next we consider the contribution from \( H_\alpha \) for all \( n > v \). This is the same estimate as in the previous section, so goes to zero when \( \epsilon \to 0 \). To sum up over \( m \), notice that we have about \( \log 1/|\epsilon| \) terms of the same order of magnitude. From this we get that the contribution goes to zero when \( \epsilon \to 0 \).

To estimate the geometric wedge product, we sum independently over \( n, m \) throwing out the condition that \( n > m \). We get as in the previous section that the contribution goes to zero.

16. Theorem 7 for \( R_{2A11c} \) close to an indicatrix.

Here we deal with the case when \( v, v' > n > m \). In this case the same formula as in the last section applies to both \( H_\alpha \) and \( H_\beta \). We have:

\[
H_\alpha \sim \int_{|x|<2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\alpha \frac{1}{(\log 1/|\epsilon|)^\gamma} dx + \int_{|x|>2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\alpha \frac{1}{x^2} dx, \text{ and}
\]

\[
H_\beta \sim \int_{|y|<2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\beta \frac{1}{(\log 1/|\epsilon|)^\gamma} dy + \int_{|y|>2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\beta \frac{1}{y^2} dy.
\]

So again the contribution goes to zero.

17. Theorem 7 for \( R_{2B} \) away from the indicatrices.

At an intersection point \( p = (z, w) \) of \( L_{\alpha,n}, L_{\beta,m} \) we have

\[
|s| |\epsilon| < |w| < C|\epsilon| \quad \text{and} \quad |s| |\epsilon| < |w - \beta(\epsilon)| < C|\epsilon|.
\]

So

\[
\log |\epsilon| - C < -av - bu < \log |\epsilon| + C \quad \text{and} \quad \log |\epsilon| - C < -av' - bu' < \log |\epsilon| + C.
\]
This gives:

\[-C < v - v' < C\]
\[-C < n - m < C\]
\[\log(1/\delta) < v, v' < \log(1/|\epsilon|) - C\]
\[-C \log(1/|\epsilon|) < u, u', n, m < C \log(1/|\epsilon|).\]

Given \((\alpha, \beta, n, m)\) we need to estimate the values of \(v, v'\) corresponding to an intersection, as well as the number of intersections. The following is immediate. There is no dependence on \(\alpha, \beta\).

**Lemma 25.** At intersection points of \(L_{\alpha, n}, L'_{\beta, m}\) in \(R_{2B}\) away from the indicatrices, we have

\[-2nb\pi/a + 1/a \log(1/|\epsilon|) - C < v, v' < -2nb\pi/a + 1/a \log(1/|\epsilon|) + C.\]

It follows that intersection points are localized in bounded rectangles. To show finiteness of number of intersection points for given plaques, we use slope estimates. We divide the estimates in two cases, (i) if \(v, v' \sim \log(1/|\epsilon|)\) and (ii) if \(\log(1/\delta) < v, v' < 1/C \log(1/|\epsilon|)\).

18. **Theorem 7 for** \(R_{2B_i}\) **when** \(v \sim \log(1/|\epsilon|)\)

Recall that this means that for a large constant \(A\), \(\frac{1}{A} \log \frac{1}{m} < v < A \log \frac{1}{m}\). The estimates for \(h_{\alpha, n}\) and \(h'_{\beta, m}\) are similar. We have

\[U + iV = (u + iv)^\gamma\]
\[\sim U + i(\log(1/|\epsilon|))^\gamma\]
\[|U| \sim (\log(1/|\epsilon|))^\gamma,\]

and at intersection points

\[h_{\alpha, n} \sim \int \hat{H}_\alpha(x) \frac{(\log(1/|\epsilon|))^\gamma}{(\log(1/|\epsilon|))^{2\gamma} + (x - U)^2} dx\]
\[\sim \int_{|x| < C(\log(1/|\epsilon|))^{\gamma}} \hat{H}_\alpha(x) \frac{1}{(\log(1/|\epsilon|))^{\gamma}} dx + \int_{|x| > C(\log(1/|\epsilon|))^{\gamma}} \hat{H}_\alpha(x) \frac{(\log(1/|\epsilon|))^\gamma}{x^2} dx\]
\[\sim \int_{|x| < C(\log(1/|\epsilon|))^{\gamma}} \hat{H}_\alpha(x) |x|^{1/\gamma - 1} \left(\frac{|x|}{(\log(1/|\epsilon|))^{\gamma}}\right)^{1 - 1/\gamma} \frac{1}{\log(1/|\epsilon|)} dx + \int_{|x| > C(\log(1/|\epsilon|))^{\gamma}} \hat{H}_\alpha(x) |x|^{1/\gamma - 1} \left(\frac{(\log(1/|\epsilon|))^\gamma}{|x|}\right)^{1 + 1/\gamma} \frac{1}{\log(1/|\epsilon|)} dx.\]
We estimate the total contribution,

\[
\sum_n h_{\alpha,n} \sim \int_{|x| < C \log(1/|\epsilon|)} \tilde{H}_\alpha |x|^{1/\gamma - 1} \left( \frac{|x|}{\log(1/|\epsilon|)\gamma} \right)^{1-1/\gamma} dx \\
+ \int_{|x| > C \log(1/|\epsilon|)} \tilde{H}_\alpha |x|^{1/\gamma - 1} \left( \frac{\log(1/|\epsilon|)\gamma}{|x|} \right)^{1+1/\gamma} dx
\]

which will converge to 0 by Proposition 1.

19. Theorem 7 for \( R_{2Bii} \) when \( v < \frac{1}{2} \log(1/|\epsilon|) \)

In this case we have \( u, u', n, m \sim \log(1/|\epsilon|) \). The estimates for \( h_{\alpha,n}, h_{\beta,m}^\gamma \) are similar. In the following \( 0 < d << 1 \). More precisely, \( d \) will be close to \( |a|A \), see the 4th inequality below.

\[
(1 - d) \log(1/|\epsilon|) < 2nb\pi < (1 + d) \log(1/|\epsilon|) \\
\log |\epsilon| - C < -av - bu < \log |\epsilon| + C \\
\log |\epsilon| + 2nb\pi - C < -av < \log |\epsilon| + 2bn\pi + C \\
d \log(1/|\epsilon|) - C < -av < d \log(1/|\epsilon|) + C.
\]

In \( U, V \) coordinates:

\[
U + iV = (u + iv)^\gamma \\
\sim (\log(1/|\epsilon|)^\gamma + i(\log(1/|\epsilon|))^{\gamma-1}v.
\]

This gives

\[
h_{\alpha,n} \sim \int \tilde{H}_\alpha(x) \frac{(\log(1/|\epsilon|)^{\gamma-1}v}{((\log(1/|\epsilon|)^{\gamma-1}v^2 + (x-U)^2} dx.
\]

When we sum up over \( h_{\alpha,n}, h_{\beta,m}^\gamma \) we can take for simplicity \( n = m \) and \( v = v' \) since \( |n - m|, |v - v'| \) are uniformly bounded in \( R_{2B} \) as stated above. The product of contributions is estimated by

\[
h_{\alpha,n} h_{\beta,m}^\gamma \sim \int \tilde{H}_\alpha(x) \frac{(\log(1/|\epsilon|)^{\gamma-1}v}{((\log(1/|\epsilon|)^{\gamma-1}v^2 + (x-U)^2} dx \\
* \int \tilde{H}_\beta(y) \frac{(\log(1/|\epsilon|)^{\gamma-1}v}{((\log(1/|\epsilon|)^{\gamma-1}v^2 + (y-U)^2} dy
\]
So
\[
\sum h_{\alpha,n}h_{\beta,m}^* \sim \left[ \int_{|x-U|<(\log(1/|\epsilon|))^{\gamma-1}|\epsilon|} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\alpha}(x) dx \right] + \left[ \int_{|x-U|>(\log(1/|\epsilon|))^{\gamma-1}|\epsilon|} \frac{1}{(x-U)^2} \delta_{\alpha}(x) dx \right] 
\]
\[
* \left[ \int_{|y-U|<(\log(1/|\epsilon|))^{\gamma-1}|\epsilon|} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\beta}(y) dy \right] + \left[ \int_{|y-U|>(\log(1/|\epsilon|))^{\gamma-1}|\epsilon|} \frac{1}{(y-U)^2} \delta_{\beta}(y) dy \right]
\]
\[
= [I + II][III + IV].
\]

There are 4 cases to sum over: (I, III), (II, III), (II, IV) and (I, IV). The case \((I, IV)\) is similar to \((II, III)\) so we can skip it without any loss.

20. THEOREM 7 FOR \(R_{2B\mu}^{(1,III)}\)

We have
\[
\sum h_{\alpha,n}h_{\beta,m}^* \sim \left[ \int_{|x-U|<(\log(1/|\epsilon|))^{\gamma-1}|\epsilon|} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\alpha}(x) dx \right] 
\]
\[
* \left[ \int_{|y-U|<(1/|\epsilon|)} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\beta}(y) dy \right]
\]
\[
\lesssim \frac{1}{v^2} \left[ \int_{|x-U|<(1/|\epsilon|)} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\alpha}(x) dx \right]
\]
\[
* \left[ \int_{|y-U|<(1/|\epsilon|)} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\beta}(y) dy \right]
\]
\[
\lesssim \frac{1}{v^2} \left[ \int_{|x-U|<(1/|\epsilon|)} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\alpha}(x) dx \right]
\]
\[
* \left[ \int_{|y-U|<(1/|\epsilon|)} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\beta}(y) dy \right].
\]

Since
\[
\log(1/\delta) < v < 1/A \log(1/|\epsilon|)
\]

we get
\[
\sum h_{\alpha,n}h_{\beta,m}^* \sim \int_{\log(1/\delta)}^{\log(1/|\epsilon|)} \frac{1}{v^2} \left[ \int_{|x-U|<(1/|\epsilon|)} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\alpha}(x) dx \right]
\]
\[
* \left[ \int_{|y-U|<(1/|\epsilon|)} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\beta}(y) dy \right], \text{ or}
\]
\[
\sum h_{\alpha,n}h_{\beta,m}^* \sim \int_{\log(1/\delta)}^{\log(1/|\epsilon|)} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}} \left[ \int_{|x-U|<(1/|\epsilon|)} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\alpha}(x) dx \right]
\]
\[
* \left[ \int_{|y-U|<(1/|\epsilon|)} \frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v} \delta_{\beta}(y) dy \right].
\]
Finally

\[ \sum h_{\alpha,n} h_{\beta,m} \lesssim \frac{1}{\log(1/\delta)} \int_{|x-(\log(1/|e|))^{\gamma}| < 1/C(\log(1/|e|))^{\gamma}} \hat{H}_\alpha(x)|x|^{1/\gamma - 1}dx 
\]

* \[ \int_{|y-(\log(1/|e|))^{\gamma}| < 1/C(\log(1/|e|))^{\gamma}} \hat{H}_\beta(y)|y|^{1/\gamma - 1}dy. \]

This contribution goes to zero when \( \epsilon \to 0 \).

21. Theorem 7 for \( R_{2Bu(II,III)} \)

We estimate

\[ h_{\alpha,n} h_{\beta,m} \sim \int_{|x-U| > (\log(1/|e|))^{\gamma-1}|v|} \hat{H}_\alpha(x) \frac{(\log(1/|e|))^{\gamma-1}v}{(x-U)^2}dx 
\]

* \[ \int_{|y-U| < (\log(1/|e|))^{\gamma-1}|v|} \hat{H}_\beta(y) \frac{1}{(\log(1/|e|))^{\gamma-1}v}dy. \]

Here \( \log(1/\delta) < v < d\log(1/|e|) \), \( 0 < d << 1 \) and \( -av = \log |e| + 2n\pi + O(1) \). Also we can take \( n = m \). When we sum over \( n, v \) runs through \( \log(1/\delta) < v < d\log(1/|e|) \). Hence the contribution to the geometric wedge product is

\[ \sum_{n,m} h_{\alpha,n} h_{\beta,m} \sim \sum_{v=\log(1/\delta)}^{d\log(1/|e|)} \int_{|x-(\log(1/|e|))^{\gamma}| > (\log(1/|e|))^{\gamma-1}|v|} \hat{H}_\alpha(x) \frac{1}{(x-(\log(1/|e|))^{\gamma})^2}dx
\]

* \[ \int_{|y-(\log(1/|e|))^{\gamma}| < (\log(1/|e|))^{\gamma-1}|v|} \hat{H}_\beta(y)dy \]

We introduce a counting function, \( N(x, y) \), which tells us for a given \( (x, y) \) for how many terms of the sum \( (x, y) \) is in the domain of integration for the above integrals:

\[ |x - (\log(1/|e|))^{\gamma}| > (\log(1/|e|))^{\gamma-1}|v| \]
\[ |y - (\log(1/|e|))^{\gamma}| < (\log(1/|e|))^{\gamma-1}|v|. \]

We divide the above domain.

\[ P_1 = \{ |x - (\log(1/|e|))^{\gamma}| > d(\log(1/|e|))^{\gamma}, 
\]
\[ |y - (\log(1/|e|))^{\gamma}| < \log(1/\delta)(\log(1/|e|))^{\gamma-1} \} \]
in which case
\[ N_1(x, y) \sim d \log(1/|\epsilon|). \]
\[ P_2 = \{ |x - (\log(1/|\epsilon|))\gamma| > d(\log(1/|\epsilon|))\gamma, \]
\[ \log(1/\delta)(\log(1/|\epsilon|))^{\gamma-1} < |y - (\log(1/|\epsilon|))\gamma| < d(\log(1/|\epsilon|))\gamma \}, \]
for \( P_2 \)
\[ N_2(x, y) \sim \frac{d(\log(1/|\epsilon|))\gamma - |y - (\log(1/|\epsilon|))\gamma|}{(\log(1/|\epsilon|))^{\gamma-1}}. \]
\[ P_3 = \{ \log(1/\delta)(\log(1/|\epsilon|))^{\gamma-1} < |x - (\log(1/|\epsilon|))\gamma| < d(\log(1/|\epsilon|))\gamma, \]
\[ \log(1/\delta)(\log(1/|\epsilon|))^{\gamma-1} < |y - (\log(1/|\epsilon|))\gamma| < d(\log(1/|\epsilon|))\gamma \}, \]
for \( P_3 \)
\[ N_3(x, y) \sim \frac{|x - (\log(1/|\epsilon|))\gamma| - |y - (\log(1/|\epsilon|))\gamma|}{(\log(1/|\epsilon|))^{\gamma-1}} \]
when positive
\[ N_3(x, y) \sim \frac{|x - y|}{(\log(1/|\epsilon|))^{\gamma-1}}. \]

22. **Theorem 7 for \( R_{2\text{Bi}(II,III)}P_1 \)**

This gives the estimate for the product
\[
\sum_{n,m} h_{\alpha,n} h_{\beta,m}^\gamma \sim d \log(1/|\epsilon|) \int_{P_1} \frac{\tilde{H}_\alpha(x)\tilde{H}_\beta(y)}{(x - (\log(1/|\epsilon|))\gamma)^2} dxdy
\]
\[
\sim \int_{P_1} \frac{\tilde{H}_\alpha(x)|x|^{1/\gamma-1}|x|^{1-1/\gamma}\tilde{H}_\beta(y)|y|^{1/\gamma-1}}{(x - (\log(1/|\epsilon|))\gamma)^2((\log(1/|\epsilon|))^{1/\gamma-1} \log(1/|\epsilon|))} 
\]
\[
\sim \int_{P_1} \frac{\tilde{H}_\alpha(x)|x|^{1/\gamma-1}|x|^{1-1/\gamma}\tilde{H}_\beta(y)|y|^{1/\gamma-1}}{|x - (\log(1/|\epsilon|))\gamma|^{1-1/\gamma}(\log(1/|\epsilon|))^{2/\gamma} \log(1/|\epsilon|)}
\]
\[
\leq \int_{P_1} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\tilde{H}_\beta(y)|y|^{1/\gamma-1} \frac{1}{\log(1/|\epsilon|)}
\to 0
\]
when \( \epsilon \to 0 \).

23. **Theorem 7 for \( R_{2\text{Bi}(II,III)}P_2 \)**

We get the estimate
\[
\sum_{n,m} h_{\alpha,n} h_{\beta,m}^* \sim \int_{P_2} \frac{\tilde{H}_\alpha(x)\tilde{H}_\beta(y) \, d\log(1/|\epsilon|)}{(x-(\log(1/|\epsilon|))\gamma)^2} \frac{|x-y|^\gamma}{|y-(\log(1/|\epsilon|))\gamma|^1} \, dx \, dy
\]
\[
\sim \int_{P_2} \frac{\tilde{H}_\alpha(x)|x|^{1/\gamma-1} \tilde{H}_\beta(y)|y|^{1/\gamma-1}}{(x-(\log(1/|\epsilon|))\gamma)^2} \frac{|x|}{|y-(\log(1/|\epsilon|))\gamma|^1} \, dx \, dy
\]
\[
\times \frac{d\log(1/|\epsilon|)}{(\log(1/|\epsilon|))\gamma-1} \sim \int_{P_2} \frac{\tilde{H}_\alpha(x)|x|^{1/\gamma-1} \tilde{H}_\beta(y)|y|^{1/\gamma-1} \, dx \, dy
\]
\[
< \int_{P_2} \frac{\tilde{H}_\alpha(x)|x|^{1/\gamma-1} \tilde{H}_\beta(y)|y|^{1/\gamma-1} \frac{1}{\log(1/|\epsilon|)} \, dx \, dy \to 0
\]
as \epsilon \to 0.

24. **Theorem 7 for R_{2Bi(II,III)}P_3**

We estimate, using the definition of \( P_3 \),

\[
\sum_{n,m} h_{\alpha,n} h_{\beta,m}^* \sim \int_{P_3} \frac{\tilde{H}_\alpha(x)\tilde{H}_\beta(y) \, d\log(1/|\epsilon|)}{(x-(\log(1/|\epsilon|))\gamma)^2 (\log(1/|\epsilon|))\gamma^{-1}} \, dx \, dy
\]
\[
\sim \int_{P_3} \frac{\tilde{H}_\alpha(x)|x|^{1/\gamma-1} \tilde{H}_\beta(y)|y|^{1/\gamma-1}}{(x-(\log(1/|\epsilon|))\gamma)^2 ((\log(1/|\epsilon|))\gamma)^{1/\gamma-1})^2} \frac{|x-y|}{(\log(1/|\epsilon|))\gamma-1} \, dx \, dy
\]
\[
\sim \int_{P_3} \frac{\tilde{H}_\alpha(x)|x|^{1/\gamma-1} \tilde{H}_\beta(y)|y|^{1/\gamma-1}}{(x-(\log(1/|\epsilon|))\gamma)^2 (\log(1/|\epsilon|))\gamma^{-2}} \frac{|x-y|}{(\log(1/|\epsilon|))\gamma-1} \, dx \, dy
\]
\[
\sim \int_{P_3} \frac{\tilde{H}_\alpha(x)|x|^{1/\gamma-1} \tilde{H}_\beta(y)|y|^{1/\gamma-1}}{(x-(\log(1/|\epsilon|))\gamma)^2 (\log(1/|\epsilon|))\gamma^{-1-\gamma}} \frac{|x-y|}{(\log(1/|\epsilon|))\gamma-1} \, dx \, dy.
\]
Hence
\[
\sum_{n,m} h_{\alpha,n} \tilde{h}_{\beta,m} < \sim \int_{P_3} \tilde{H}_{\alpha}(x)|x|^{1/\gamma-1}\tilde{H}_{\beta}(y)|y|^{1/\gamma-1} \\
\times \frac{1}{|x - (\log(1/|\epsilon|))\gamma|((\log(1/|\epsilon|))^{1-\gamma}} \, dx \, dy \\
\sim \frac{1}{\log(1/\delta)(\log(1/|\epsilon|))^{\gamma-1}} \int_{R_3} \tilde{H}_{\alpha}(x)|x|^{1/\gamma-1}\tilde{H}_{\beta}(y)|y|^{1/\gamma-1} \, dx \, dy \\
\rightarrow 0
\]

because we can choose \( \delta \) small enough.

25. **Theorem 7 for \( R_{2Bi(II,IV)} \)**

Recall from Lemma 24 that:

\[-2nb\pi/a + 1/a \log(1/|\epsilon|) - C < v, v' < -2nb\pi/a + 1/a \log(1/|\epsilon|) + C.\]

We estimate the contribution

\[
h_{\alpha,n} \tilde{h}_{\beta,m} \sim \int_{|x-U|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_{\alpha}(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(x-U)^2} \, dx \\
\times \int_{|y-U|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_{\beta}(y)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(y-U)^2} \, dy \\
\sim \int_{|x-(\log(1/|\epsilon|))\gamma|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_{\alpha}(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(x-(\log(1/|\epsilon|))^{\gamma-1}v)^2} \, dx \\
\times \int_{|y-(\log(1/|\epsilon|))\gamma|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_{\beta}(y)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(y-(\log(1/|\epsilon|))^{\gamma-1}v)^2} \, dy
\]

Note that when we sum over \( n, v \) depends linearly on \( n \) and as seen above, ranges from \( \log 1/\delta \) to \( d \log(1/|\epsilon|) \), \( 0 < d << 1 \).

Hence we need to estimate the expression \( I(\alpha, \beta) \) for given \( (\alpha, \beta) \):

\[
I(\alpha, \beta) := \sum_{k=\log 1/\delta}^{d \log(1/|\epsilon|)} \int_{|x-(\log(1/|\epsilon|))\gamma|>(\log(1/|\epsilon|))^{\gamma-1}k} \tilde{H}_{\alpha}(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}k}{(x-(\log(1/|\epsilon|))^{\gamma-1}k)^2} \, dx \\
\times \int_{|y-(\log(1/|\epsilon|))\gamma|>(\log(1/|\epsilon|))^{\gamma-1}k} \tilde{H}_{\beta}(y)\frac{(\log(1/|\epsilon|))^{\gamma-1}k}{(y-(\log(1/|\epsilon|))^{\gamma-1}k)^2} \, dy
\]
We introduce the integrals

\[ I_{j,\alpha} := \int_{(\log(1/|\epsilon|))^{\gamma-1} < |x-\log(1/|\epsilon|)| < (\log(1/|\epsilon|))^{\gamma-1}(j+1)} \tilde{H}_\alpha(x) \frac{(\log(1/|\epsilon|))^{\gamma-1}}{(x-\log(1/|\epsilon|))^{\gamma} - 1} dx \]

\[ \sim \int_{(\log(1/|\epsilon|))^{\gamma-1} < |x-\log(1/|\epsilon|)| < (\log(1/|\epsilon|))^{\gamma-1}(j+1)} \tilde{H}_\alpha(x) \frac{1}{j^2 (\log(1/|\epsilon|))^{\gamma - 1}} dx \]

\[ \sim \frac{1}{j^2} \int_{(\log(1/|\epsilon|))^{\gamma-1} < |x-\log(1/|\epsilon|)| < (\log(1/|\epsilon|))^{\gamma-1}(j+1)} \tilde{H}_\alpha(x) |x|^{1/\gamma - 1} dx \]

\[ = \frac{1}{j^2} \tilde{I}_{j,\alpha}, \]

and

\[ I_{\infty,\alpha} := \int_{|x-\log(1/|\epsilon|)| > d(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\alpha(x) \frac{(\log(1/|\epsilon|))^{\gamma-1}}{(x-\log(1/|\epsilon|))^{\gamma} - 1} dx \]

\[ \sim \int_{d(\log(1/|\epsilon|))^{\gamma} < |x-\log(1/|\epsilon|)| < C d(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\alpha(x) \frac{1}{(\log(1/|\epsilon|))^{\gamma+1}} dx \]

\[ + \int_{|x-\log(1/|\epsilon|)| > C (\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\alpha(x) \frac{(\log(1/|\epsilon|))^{\gamma-1}}{x^2} dx \]

\[ < \frac{1}{(\log(1/|\epsilon|))^{2}} \int \tilde{H}_\alpha(x) |x|^{1/\gamma - 1} dx \]

\[ = I_{\infty,\alpha}^1 \]

and similarly for \( \beta \).

We get:

\[ I(\alpha, \beta) = \sum_{k=\log 1/\delta}^{d \log(1/|\epsilon|)} \left[ k \left( \sum_{j=k}^{d \log(1/|\epsilon|)} \tilde{I}_{j,\alpha} + I_{\infty,\alpha} \right) \right] \left[ k \left( \sum_{i=k}^{d \log(1/|\epsilon|)} \tilde{I}_{i,\beta} + I_{\infty,\beta} \right) \right] \]

\[ \sim \sum_{k=\log 1/\delta}^{d \log(1/|\epsilon|)} k^2 \left[ \left( \sum_{j=k}^{d \log(1/|\epsilon|)} \tilde{I}_{j,\alpha} \right) + I_{\infty,\alpha} \right] \left[ \left( \sum_{i=k}^{d \log(1/|\epsilon|)} \tilde{I}_{i,\beta} \right) + I_{\infty,\beta} \right] \]

\[ = \sum_{k=\log 1/\delta}^{d \log(1/|\epsilon|)} k^2 \left[ \sum_{j=k}^{d \log(1/|\epsilon|)} \tilde{I}_{j,\alpha} \right] \left[ \sum_{i=k}^{d \log(1/|\epsilon|)} \tilde{I}_{i,\beta} \right] \]

\[ + \sum_{k=\log 1/\delta}^{d \log(1/|\epsilon|)} k^2 I_{\infty,\alpha} \left[ \sum_{i=k}^{d \log(1/|\epsilon|)} \tilde{I}_{i,\beta} \right] + \sum_{k=\log 1/\delta}^{d \log(1/|\epsilon|)} k^2 \left[ \sum_{j=k}^{d \log(1/|\epsilon|)} \tilde{I}_{j,\alpha} \right] I_{\infty,\beta} \]

\[ + \sum_{k=\log 1/\delta}^{d \log(1/|\epsilon|)} k^2 I_{\infty,\alpha} I_{\infty,\beta} \]

\[ = I + II + III + IV \]

Here II and III are symmetric. It suffices to estimate II.
We estimate IV first. Since $\sum k^2 \sim (\log(1/\epsilon))^3$, this is immediately small when multiplied with $I_{\infty,\alpha}, I_{\infty,\beta}$. For II, we get:

\[
II = \sum_{k=\text{log } 1/\delta} k^2 I_{\infty,\alpha} \left[ \sum_{i=k}^{d \log(1/\epsilon)} \frac{I_{i,\alpha}}{i^2} \right] < I_{\infty,\alpha} \sum_{k=\text{log } 1/\delta}^{d \log(1/\epsilon)} \left[ \sum_{i=k}^{d \log(1/\epsilon)} \frac{I_{i,\beta}}{i^2} \right] < \frac{1}{(\log(1/\epsilon))} \int \overline{H}_\alpha(x)|x|^{1/\gamma-1}dx\int \overline{H}_\beta(y)|y|^{1/\gamma-1}dy \rightarrow 0
\]

Finally we estimate I.

\[
I = \sum_{k=\text{log } 1/\delta} k^2 \left[ \sum_{j=k}^{d \log(1/\epsilon)} \frac{I_{j,\alpha}}{j^2} \right] \left[ \sum_{i=k}^{d \log(1/\epsilon)} \frac{I_{i,\beta}}{i^2} \right] < \sum_{k=\text{log } 1/\delta} \frac{1}{k^2} \left[ \sum_{j=k}^{d \log(1/\epsilon)} \frac{I_{j,\alpha}}{j^2} \right] \left[ \sum_{i=k}^{d \log(1/\epsilon)} \frac{I_{i,\beta}}{i^2} \right]
\]

We can make this as small as we wish by choosing $\delta$ small.

## 26. Proof of Theorem 4

**Proof.** We use the approach in [8].

Let $T$ be a positive harmonic current directed by $F$. We want to show that $\int T \wedge T = 0$. Let $T_\epsilon = (\Phi_\epsilon)_* T$ and define $T_\delta$ as the average of $T_\epsilon$ using a small neighborhood of identity in $U(3)$. Then since $T_\epsilon \rightarrow T$, we have $\int T \wedge T = \lim_{\epsilon \rightarrow 0} \int T \wedge T_\epsilon$. On the other hand $T_\delta = \omega + \partial S_\delta + \overline{\partial S_\delta} + i\partial \overline{\partial S_\delta}$ and $S_\delta \rightarrow S_\epsilon$ in $L^2$. So $\int T \wedge T_\epsilon = \lim_{|\delta|,|\delta'| \rightarrow 0,|\delta|,|\delta'| \ll \epsilon} \int T_\epsilon \wedge T_\delta$. Hence as in [8] it is enough to show that

\[
\lim_{\delta,\delta',\epsilon \rightarrow 0,|\delta|,|\delta'| \ll \epsilon} \int T_\epsilon \wedge T_\delta = 0.
\]

We can compute the geometric intersection $T_\epsilon \wedge T_\delta'$ and it is enough to estimate $T_\epsilon \wedge g T$. Recall that if $\phi$ is a test function supported in $B$, then we define

\[
<T_\epsilon \wedge g T, \phi> = \int J_{\alpha,\beta} \phi(p) H_\alpha(p) H_\beta(p) d\mu(\alpha) d\mu(\beta).
\]

where $J_{\alpha,\beta}$ consists of intersection points of $\Delta_\alpha$ and $\Delta_\beta$. The following Lemma is proved in [8].

**Lemma 26.** We have that $\int T \wedge T_\epsilon = \int T \wedge g T_\epsilon$. The same holds for $T_\epsilon, T_\epsilon'$. Then

\[
<T_\epsilon \wedge g T, \phi> \leq C\|\phi\|_\infty \int J_{\alpha,\beta} H_\alpha(p) H_\beta(p) d\mu(\alpha) d\mu(\beta),
\]
We know that the number of points in $J_{\alpha,\beta}$ is bounded by a fixed constant independent of $\epsilon$. For $p$ out of a fixed neighborhood of the singularities the integral converges to zero. This is the case considered in [8]. So it is enough to show that for $\delta > 0$ small enough

$$J_\epsilon(\delta) := \int \sum_{J_{\alpha,\beta}} H_\alpha(p) H^{\epsilon}_\beta(p) d\mu(\alpha) d\mu(\beta)$$

is arbitrarily small. This is precisely the content of Theorem 7, since all estimates are valid after composition by automorphisms in a small neighborhood of $U(3)$.

Consequently if $T_1, T_2$ are two such currents then $\int \frac{T_1 + T_2}{2} \wedge \frac{T_1 - T_2}{2} = 0$. Hence $\int T_1 \wedge T_2 = 0$, therefore $T_1, T_2$ are proportional.

We give a dynamical consequence of the uniqueness of the harmonic current for $F \in \mathcal{H}(d)$, here $\mathcal{H}(d)$ is the Zariski open set of foliations of degree $d$, introduced in Theorem 2.

**Corollary 2.** Let $F \in \mathcal{H}(d)$. Let $\phi : \Delta \to L$ be the universal covering of a leaf $L$. Let $\tau_r := \phi_* [\log^+ \frac{r}{|z|} \Delta_r]$. Then $\lim_{r \to 1} \tau_r = T$, where $T$ is the unique harmonic current directed by $F$.

Here $\Delta_r$ denotes the disc of center 0 and radius $r$. The Corollary which is a consequence of paragraph 5 in [8] says that the normalized images of $[\log^+ \frac{r}{|z|} \Delta_r]$ converge to $T$. This is similar to the pointwise ergodic theorem, since we are averaging on an orbit.

Recall that the limit set of a leaf $L$ is defined as $\lim(L) = \cap_n L \setminus K_n$, where $K_n \subset K_{n+1}$ is an exhaustion of $L$ by compact sets. One of the main questions in foliation theory is to describe the limit set of a foliation $F$: $\lim(F) := \cup_{L \in F} \lim(L)$. Corollary 2 implies in particular that for $F \in \mathcal{H}(d)$, for every leaf $L \in F$, $\lim(L)$ contains $\supp(T)$. Indeed as shown in [8],

$$\| \phi_* \left[ \log^+ \frac{r}{|z|} \Delta_r \right] \| \to \infty$$

as $r \to 1$. Hence $\supp(T) \subset \overline{L \setminus K_n}$ for every $n$.

**Corollary 3.** The map $\lambda \to T_\lambda$ is continuous from $\mathcal{H}(d)$ with values in the positive harmonic currents of mass one. Let $F_\lambda$ be a holomorphic family of foliations in $\mathcal{H}(d)$. Let $(T_\lambda)$ be the associated currents. If a hyperbolic point $p_0 \in \supp(T_{\lambda_0})$, then the perturbed hyperbolic point $p_\lambda$ belongs to $\supp(T_\lambda)$.

**Proof.** Assume $F_{\lambda_n} \to F_{\lambda_0}$ in $\mathcal{H}(d)$. Let $(T_{\lambda_n})$ be the normalized positive harmonic currents associated to $F_{\lambda_n}$. Since $\|T_{\lambda_n}\| = 1$, the sequence $(T_{\lambda_n})$ has cluster points. It is clear that any cluster point $S$ is positive harmonic and directed by $F_{\lambda_0}$. So $S = T_{\lambda_0}$ by uniqueness. Assume the support of $T_{\lambda_0}$ intersects a ball $B(p_0, r)$ where $p_0$ is a hyperbolic singular point of $F_{\lambda_0}$ and the ball is contained in the common domain of linearization of $p_\lambda \in \Sigma(F_\lambda), p_\lambda \to p_0, p_\lambda$ hyperbolic.

From our local study of positive harmonic currents near a hyperbolic singular point $p_0 \in \supp(T_{\lambda_0})$. Since $T_\lambda \to T_{\lambda_0}$, $T_{\lambda}$ gives mass to $B(p_0, r)$, applying again the local study for $T_{\lambda}$ we get that $p_\lambda \in \supp(T_\lambda)$. □
Remark 2. Let \( f \) be a holomorphic endomorphism of \( \mathbb{P}^2 \). Let \( \mathcal{F} \) be a foliation with only hyperbolic singularities. Then \( f^* \mathcal{F} \) is a foliaton and its singularities are not necessarily hyperbolic. However there is only one positive harmonic current of mass 1, directed by \( f^* \mathcal{F} \). Indeed let \( T \) be any such current. We will show that \( \int T \wedge T = 0 \) which implies the uniqueness. Observe that \( f_* T \) is a current directed by \( \mathcal{F} \). Hence \( \int f_* T \wedge f_* T = 0 \). Since \( f^* \) is a finite covering of degree \( d^2 \) we have

\[
\int T \wedge T \leq \int f^* [f_* T \wedge f_* T] = d^2 \int f_* T \wedge f_* T = 0.
\]

27. Measure associated to a harmonic current

Let \( \mathcal{F} \in \mathcal{H}(d) \) be a holomorphic foliation as in Theorem 2. We know that there is a unique positive harmonic current \( T \) of mass one directed by \( \mathcal{F} \).

We are going to associate to \( T \) a conformal, measurable metric along leaves that we will denote by \( g_T \) and also a positive finite measure \( \mu_T \) which is invariant under the harmonic flow associated also to \( T \). The metric \( g_T \) and the measure \( \mu_T \) where first considered by S. Frankel, in the non singular case [9] he proved in that case a version of Proposition 2 and Proposition 3.

On a flow box \( B \) disjoint from \( E = \text{Sing}(\mathcal{F}) \), the current \( T \) can be written

\[
T = \int h_\alpha [V_\alpha] d\mu(\alpha)
\]

where \( h_\alpha \) are positive harmonic functions and \( \mu \) is a positive measure on a transversal \( A \). The \( [V_\alpha] \) are the currents of integration on plaques. On \( B \), \( \partial T = \tau \wedge T \) with \( \tau = \frac{\partial h_\alpha}{h_\alpha} \), \( \mu \) almost everywhere. Observe that \( \tau \) is independent of the choice of \( h_\alpha \): if we replace \( h_\alpha \) by \( c_\alpha h_\alpha, c_\alpha \in \mathbb{R}^+ \) then \( \tau \) is unchanged.

We define the metric \( g_T \) on leaves by \( g_T = \frac{1}{2} \tau \wedge \tau \). Along the plaque \( V_\alpha \) with a choice of coordinate \( (z_\alpha) \) we have

\[
g_T = \frac{i}{2} \left| \frac{\partial h_\alpha}{\partial z_\alpha} \right|^2 \frac{1}{h_\alpha^3} dz_\alpha \wedge d\bar{z}_\alpha \quad (1)
\]

Define \( \mathcal{C}_T = \{(\alpha, z); \frac{\partial h_\alpha}{\partial z_\alpha}(\alpha, z) = 0\} \) it’s the critical set of the "metric" \( g_T \). We also define the current of bidegree \((2, 2), \mu_T \), which we identify with a measure

\[
\mu_T := i \tau \wedge \tau \wedge T.
\]

In local coordinates in a flow box \( B \), we have:

\[
\mu_T = \int d\nu(\alpha) \int_{[V_\alpha]} \frac{1}{h_\alpha} \left| \frac{\partial h_\alpha}{\partial z_\alpha} \right|^2 (idz_\alpha \wedge d\bar{z}_\alpha). \quad (2)
\]

Proposition 2. Let \( \mathcal{F} \in \mathcal{H}(d) \). The metric \( g_T \) has constant negative curvature out of the set \( \mathcal{C}_T \) where the metric vanishes.

Proof. Since the current \( T \) is unique, every measurable set of leaves \( A \) has zero or full measure with respect to \( \|T\| \). Define \( \mathcal{N}_g := \{\text{leaves on which } g_T \text{ vanishes identically}\} \).

Since \( h_\alpha \) is measurable, then \( \mathcal{N}_g \) is measurable. So \( \mathcal{N}_g \) is of zero or full measure. But if \( \mathcal{N}_g \) is of full measure, \( \partial T = 0 \) and by conjugation \( \overline{\partial T} = 0 \), hence \( T \) is closed. A foliation \( \mathcal{F} \) in \( \mathcal{H}(d) \) admits no positive closed current directed by \( \mathcal{F} \) since all singularities are hyperbolic. So \( \mathcal{N}_g \) is of zero \( \|T\| \) measure.
From (1) it is clear that the metric is conformal. On a flow box $B$, the curvature $\kappa(g)$ has the following expression out of $C_T$. The curvature is given by

$$\kappa(g) = \frac{-1}{4} \frac{\Delta \log g}{g} = \frac{1}{2} \left( \frac{\partial h}{\partial z} \right)^2 \frac{1}{h_\alpha}.$$ 

So

$$\kappa(g_T) = \frac{h_\alpha^2}{|h_{\alpha, z}|^2} \left( \frac{\partial}{\partial z} \left( \frac{h_{\alpha, z}}{h_\alpha} \right) \right).$$

Since $h_\alpha$ is harmonic we get $\kappa(g_T) = -1$.

Because of the nature of the singularities, the leaves are uniformized by the unit disc $\Delta$. Let $g$ denote the Poincaré metric on leaves. We choose a normalization so that the curvature $\kappa(g)$ of $g$ on leaves is $-1$.

**Proposition 3.** Let $T$ be the harmonic current associated to $F \in \mathcal{H}(d)$. If $g_T$ is the associated metric on leaves, then $g_T \leq g$.

**Proof.** We have normalized the metric $g_T$ so that on each leaf $L_\alpha$, $g_T$ has curvature $-1$ on $L_\alpha \setminus C_T$. The Ahlfors’ Schwarz lemma, applied to the abstract Riemann surface $L_\alpha \setminus C_T$ implies that $g_T \leq g$. □

We will denote by $\Phi_\alpha : \Delta \rightarrow L_\alpha$, the uniformizing map from $\Delta$ to $L_\alpha$. When we fix a transversal $A$ in a flow box we can choose for each $\alpha \in A$ a uniformizing map $\Phi_\alpha(0) = \alpha$, then $\Phi_\alpha$ vary measurably. We will denote by $\Gamma_\alpha$ the group of deck transformations for the map $\Phi_\alpha$.

We want to define a vector field $\chi$ on $F$ associated to the current $T$. The vector field will be defined as the metric $g_T$ only $\|T\|$ a.e. On $L_\alpha$, $\chi_\alpha$ is collinear with the gradient field of $h_\alpha$. We define $\chi_\alpha$ on a flow box with local coordinates $z_\alpha = x_\alpha + iy_\alpha$ by

$$\chi_\alpha := c \frac{h_\alpha}{|h_z|^2} (h_{x_\alpha}, h_{y_\alpha}).$$

We choose the constant $c$ so that $g_T(\chi_\alpha, \chi_\alpha) = 1$. The vector field $\chi_\alpha$ is independent of the choice of $h$. It blows up at every point of $C_T$. Which means that the integral curves of $\chi_\alpha$ approach these points at infinite speed. So we have to take out these trajectories in order to have a well defined flow. Observe that the set of these trajectories is of $\mu_T$ measure zero. It is clear that the integral curves of $\chi_\alpha$ are along the level sets of the harmonic conjugates of $h_\alpha$ such that $f_\alpha = h_\alpha + iv_\alpha$ is holomorphic.

**Theorem 9.** Let $T$ be the positive harmonic current associated to $F \in \mathcal{H}(d)$. Then the measure $\mu_T$ is finite. Moreover, if $\mathcal{F}_\lambda$ is a holomorphic family of foliations in $\mathcal{H}(d)$, $\lambda \in \Delta(\lambda_0, r)$, then the mass of $\mu_{T_\lambda}$ near hyperbolic singularities is uniformly small in a fixed neighborhood of the singularities.

**Proof.** For a flow box $B$ away from the singularities, it is clear that $\mu_T$ has finite mass. Indeed the functions $h_\alpha$ are positive harmonic, and by Harnack $\frac{h_\alpha}{|h_z|} \leq c$, hence $\mu_T$ has finite mass in $B$. It is enough to show that $\mu_T$ has finite mass in a flow
bounded. It follows that $\lambda$ is continuous for every $z$. We use the parametrization 

$$
\psi_\alpha(\zeta) = (e^{i(\zeta+\log|\alpha|)/b}, \alpha e^{i\lambda(\zeta+\log|\alpha|)/b})
$$

by a sector near the hyperbolic singularity. Since $\psi_\alpha^* h_\alpha = H_\alpha$ is a positive harmonic function and $\mu$ a.e. $H_\alpha(\zeta) \to 0$ when $3\zeta \to +\infty$, then again by Harnack $\psi_\alpha^*(\tau)$ is bounded. The total mass of $\mu_T$ in $B_\iota$ satisfies

$$
\int_{B_\iota} \mu_T \leq \int_{D(w_0, \iota) \times S_\lambda} i\psi_\alpha^*(\tau) \land \psi_\alpha^*(\tau) \land \psi_\alpha^*[V_\alpha] H_\alpha d\mu(\alpha)
$$

$\psi_\alpha^*[V_\alpha]$ is a graph in the flow box. It is of bounded area and $\int_{D(w_0, \iota)} H_\alpha d\mu(\alpha)$ defines a bounded harmonic function. So the mass $\mu_T$ is bounded near the origin.

Basic ally the slicing of $\mu_T$ along the leaves gives the area measure on leaves associated to the metric $g_T$. Let $T_\lambda$ be the current associated to $F_\lambda$, and let $\mu^\lambda$ denote the corresponding measure on a transversal. The linearizations associated to a holomorphically varying hyperbolic singularity vary holomorphically. Then $\int H_\alpha^\lambda d\mu^\lambda(\alpha) \to 0$ when $3\zeta \to +\infty$, uniformly when $\lambda$ is near $\lambda_0$. (We don’t say that $H_\alpha^\lambda$ vary holomorphically.) So the mass of $\mu_{T_\lambda}$ is uniformly small in a fixed neighborhood of the singularities if $\lambda$ is close enough to $\lambda_0$.

\begin{proof}

Let $\lambda \to F_\lambda$ be a holomorphic family of foliations in $\mathcal{H}(d)$, parametrized by a disc $\Delta$. Then $\lambda \to \mu_\lambda$ is a continuous family of measures.

Recall that $\mu_{T_\lambda} = i\tau_\lambda \land \tau_\lambda \land T_\lambda$.

Fix a flow box $B$ for $F_{\lambda_0}$ away from the singularities. We can consider $(\phi_\lambda)_\iota$ local biholomorphisms straightening $F_\lambda$ in $B$, when $\lambda \to \lambda_0$. We know that the currents $S_\lambda := (\phi_\lambda)_\iota T_\lambda$ depend continuously on $\lambda$. We can write in $B$,

$$
S_\lambda = \int [\omega = \alpha] h_\alpha^\lambda(z) d\mu_\lambda(\alpha)
$$

where $\mu_\lambda$ is the measure on a fixed transversal ($z = z_0$). We can assume that $h_{\alpha_0}^\lambda(z_0) = 1$ for all $\alpha, \lambda$.

Since $S_\lambda \to S_{\lambda_0}$ then for every $z$ we have $h_\alpha^\lambda(z) \mu_\lambda(\alpha) \to h_{\alpha_0}^{\lambda_0} \mu_{\lambda_0}(\alpha)$ weakly when $\lambda \to \lambda_0$.

The $(h_\alpha^\lambda)^2$ also vary slowly, by Harnack, so we also get that $\lambda \to (h_\alpha^\lambda(z))^2 \mu_\lambda(\alpha)$ is continuous for every $z$. Define

$$
U_\lambda := \int [\omega = \alpha] (h_\alpha^\lambda)^2(z) d\mu_\lambda(\alpha).
$$

The family of positive currents $U_\lambda$ is also continuous because $(h_\alpha^\lambda)^2$ is uniformly bounded. It follows that $\lambda \to i\partial\bar{\partial}U_\lambda$ is continuous i.e.

$$
\lambda \to \int |h_{\alpha,z}^\lambda|^2 [\omega = \alpha] d\mu_\lambda(\alpha).
$$

Using again Harnack inequalities for $\frac{1}{h_\alpha^\lambda}$, we find that $\lambda \to \mu_{T_\lambda}$ is continuous in $B$.

\end{proof}
We have seen in Theorem 9 that $\mu_T\lambda$ has uniformly small mass near the singularities. Hence $\lambda \rightarrow \mu_T\lambda$ is continuous.

Let $|g_T^2|$ denote the measure induced by the metric $g_T$ on the leaf $L_\alpha$. We will omit $\alpha$, most frequently.

We will say that a set $E$ is invariant if up to a set of $\mu_T$ measure zero, it is a union of orbits of $\chi$. For a measurable set $E$ we denote by $E_\alpha$ the intersection $E \cap L_\alpha$.

**Theorem 11.** Either there is an invariant set $E$ for $\chi$ such that for $\|T\|$ almost every leaf $L_\alpha$, $|g_T|(E_\alpha) > 0$ and $|g_T|(E^c_\alpha) > 0$ or the measure $\mu_T$ is ergodic.

**Proof.** Fix a countable family $(B_i)$ of flow boxes such that $\bigcup_i B_i = \mathbb{P}^2 \setminus \text{Sing}(F)$. Let $E$ be an invariant set for $\chi$ such that $\mu_T(E) > 0$. Define $E_i = \{\alpha; |g_T|(L_\alpha \cap E \cap B_i) = 0\}$. $E := \bigcap_i E_i$ is measurable. It is a union of leaves. Since the current $T$ is unique and $\mu_T(E) = 0$, then $\|T\|$ almost every leaf is in $E^c$.

For $L_\alpha \in E^c$, $|g_T|(E_\alpha) > 0$. We can do a similar construction for $E^c$ if $\mu_T(E^c) > 0$. We then get a set of $\|T\|$ full measure of leaves such that $|g_T|(E_\alpha) > 0$ and $|g_T|(E^c_\alpha) > 0$.

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