Minimal Reachability is Hard To Approximate

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Abstract—In this note, we consider the problem of choosing which nodes of a linear dynamical system should be actuated so that the state transfer from the system’s initial condition to a given final state is possible. Assuming a standard complexity hypothesis, we show that this problem cannot be efficiently solved or approximated in polynomial, or even quasi-polynomial, time.

I. INTRODUCTION

During the last decade, researchers in systems, optimization, and control have focused on questions such as:

- (Actuator Selection) How many nodes do we need to actuate in a gene regulatory network to control its dynamics? [1, 2]
- (Input Selection) How many inputs are needed to drive the nodes of a power system to fully control its dynamics? [3]
- (Leader Selection) Which UAVs do we need to choose in a multi-UAV system as leaders for the system to complete a surveillance task despite communication noise? [4, 5]

The effort to answer such questions has resulted in numerous papers on topics such as actuator placement for controllability [6, 7]; actuator selection and scheduling for bounded control effort [8–11]; resilient actuator placement against failures and attacks [12, 13]; and sensor selection for target tracking and optimal Kalman filtering [14–17]. In all these papers the underlying optimization problems have been proven (i) either polynomially-time solvable [1–3] (ii) or NP-hard, in which case polynomial-time algorithms have been proposed for their approximate solution [4–17].

But in several applications in systems, optimization, and control, such as in power systems [18, 19], transportation networks [20], and neural circuits [21, 22], the following problem also arises:

Minimal Reachability Problem. Given times $t_0$ and $t_1$ such that $t_1 > t_0$, vectors $x_0$ and $x_1$, and a linear dynamical system with state vector $x(t)$ such that $x(t_0) = x_0$, find the minimal number of system nodes we need to actuate so that the state transfer from $x(t_0) = x_0$ to $x(t_1) = x_1$ is feasible.

For example, the stability of power systems is ensured by placing a few generators such that the state transfers from a set of possible initial conditions to the zero state are feasible [19].

The minimal reachability problem relaxes the objectives of the applications in [11–17]. For example, in comparison to the actuator placement problem for controllability [6], the minimal reachability problem aims to place a few actuators only to make a single transfer between two states feasible, whereas the minimal controllability problem aims to place a few actuators to make the transfer among any two states feasible [6, 17].

The fact that the minimal reachability problem relaxes the objectives of the papers [1–17] is an important distinction whenever we are interested in the feasibility of only a few state transfers by a small number of placed actuators. The reason is that under the objective of minimal reachability the number of placed actuators can be much smaller in comparison to the number of placed actuators under the objective of controllability. For example, in the system of Fig. 1 the number of placed actuators under the objective of minimal reachability from $(0, \ldots, 0)$ to $(1, \ldots, 0)$ is one, whereas the number of placed actuators under the objective of controllability grows linearly with the system’s size.

The minimal reachability problem was introduced in [23], where it was found to be NP-hard. Similar versions of the reachability problem were studied in the context of power systems in [19] and [24]. For the polynomial-time solution of the reachability problems in [19, 23, 24], greedy approximation algorithms were proposed therein. The approximation performance of these algorithms was claimed by relying on the modularity result [25, Lemma 8.1], which states that the distance from a point to a subspace created by the span of a set of vectors is supermodular in the choice of the vectors.

In this note, we first show that the modularity result [25, Lemma 8.1] is incorrect. In particular, we show this via a counterexample to [25, Lemma 8.1], and as a result, we prove that the distance from a point to a subspace created by the span of a set of vectors is non-supermodular in the choice of the vectors. Then, we also prove the following strong intractability result for the minimal reachability problem, which is our main contribution in this paper:
**Contribution 1.** Assuming \( \text{NP} \not\equiv \text{BPTIME}(n^{\text{poly-log } n}) \), we show that for each \( \delta > 0 \), there is no polynomial-time algorithm that can distinguish between the two cases where:
- the reachability problem has a solution with cardinality \( k \); 
- the reachability problem has no solution with cardinality \( k2^{\Omega((\log 1 - \varepsilon n)} \), where \( n \) is the dimension of the system.

We note that the complexity hypothesis \( \text{NP} \not\equiv \text{BPTIME}(n^{\text{poly-log } n}) \) means there is no randomized algorithm which, after running for \( O(n^{\text{log } n}) \) time for some constant \( c \), outputs correct solutions to problems in NP with probability \( 2/3 \); see [26] for more details.

Notably, Contribution 1 remains true even if we allow the algorithm to search for an approximate solution that is relaxed as follows: instead of choosing the actuators to make the state transfer from the initial state \( x_0 \) to a given final state \( x_1 \) possible, some other state \( \hat{x}_I \) that satisfies \( \|x_I - \hat{x}_I\|_2^2 \leq \varepsilon \) should be reachable from \( x_0 \). This is a substantial relaxation of the reachability problem’s objective, and yet, we show that the intractability result of Contribution 1 still holds.

The rest of this note is organized as follows. In Section II we introduce formally the minimal reachability problem. In Section III, we provide a counterexample to [25, Lemma 8.1]. In Section IV we present Contribution 1; in Section V we prove it. Section VI concludes the paper.

**II. MINIMAL REACHABILITY PROBLEM**

In this section we formalize the minimal reachability problem. We start by introducing the systems considered in this paper and the notions of system node and of actuated node set.

**System.** We consider continuous-time linear systems of the form
\[
\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq t_0, \tag{1}
\]
where \( t_0 \) is a given starting time, \( x(t) \in \mathbb{R}^n \) is the system’s state at time \( t \), and \( u(t) \in \mathbb{R}^m \) is the system’s input vector.

In this paper we want to actuate the minimal number of the system’s nodes in eq. (1) to make a desired state-transfer feasible (and not to achieve necessarily the system’s controllability). We formalize this control objective using the following two definitions.

**Definition 1 (System node).** Given a system as in eq. (1), where \( x(t) \in \mathbb{R}^n \), let \( x_1(t), x_2(t), \ldots, x_n(t) \in \mathbb{R}^n \) such that \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \). We refer to each \( x_i(t) \) as a system node.

**Definition 2 (Actuated node set).** Given a system as in eq. (1), where \( x(t) \in \mathbb{R}^n \), we say that the set \( S \subseteq \{1, 2, \ldots, n\} \) is an actuated node set if for all times \( t \) the input \( u(t) \) affects only the system nodes \( x_i(t) \) where \( i \in S \). Formally, the set \( S \subseteq \{1, 2, \ldots, n\} \) is an actuated node set if the system dynamics are given by
\[
\dot{x}(t) = Ax(t) + I(S)Bu(t), \quad t \geq t_0, \tag{2}
\]
where \( I(S) \) is a \( n \times n \) diagonal matrix such that if \( i \in S \), the \( i \)-th entry of \( I(S) \)’s diagonal is 1, otherwise it is 0.

The definition of \( I(S) \) in eq. (2) implies that the input \( u(t) \) affects only those system nodes \( x_i(t) \) where \( i \in S \).

In more detail,
- if \( i \in S \), the system node \( x_i(t) \) is affected by \( u(t) \), since for \( i \in S \) the \( i \)-th row of \( I(S)B \) is the \( i \)-th row of \( B \);
- if \( i \notin S \), the system node \( x_i(t) \) cannot be affected by \( u(t) \), since for \( i \notin S \) the \( i \)-th row of \( I(S)B \) is zero.

Overall, the set \( S \) determines via the matrix \( I(S)B \) which rows of \( B \) will be set to zero and which will remain the same.

**Problem 1 (Minimal Reachability).** Given
- times \( t_0 \) and \( t_1 \) such that \( t_1 > t_0 \),
- vectors \( x_0, x_1 \in \mathbb{R}^n \), and
- a system \( \dot{x}(t) = Ax(t) + Bu(t), \quad t \geq t_0 \), as in eq. (1),
- with initial condition \( x(t_0) = x_0 \),
find an actuated node set with minimal cardinality such that there exists an input \( u(t) \) defined over the time interval \( (t_0, t_1) \) that achieves \( x(t_1) = x_1 \). Formally, using the notation \( |S| \) to denote the cardinality of a set \( S \):
\[
\begin{align*}
\text{minimize} & \quad |S| \\
\text{subject to} & \quad \text{such that there exist } u : (t_0, t_1) \rightarrow \mathbb{R}^m, x : (t_0, t_1) \rightarrow \mathbb{R}^n \text{ with } \\
& \quad \dot{x}(t) = Ax(t) + I(S)Bu(t), \quad t \geq t_0, \\
& \quad x(t_0) = x_0, \quad x(t_1) = x_1.
\end{align*}
\]

A special case of particular interest is when \( B \) is the identity matrix. Then, minimal reachability asks for the fewest system nodes that need to be directly actuated by an input \( u(t) \) so that at time \( t_1 \), the state \( x_1 \) is reachable from the system’s initial condition \( x(t_0) = x_0 \).

**III. NON-SUPERMODULARITY OF DISTANCE FROM POINT TO SUBSPACE**

In this section, we provide a counterexample to the supermodularity result [25, Lemma 8.1]. We begin with some notation. In particular, given a matrix \( M \in \mathbb{R}^{n \times n} \), a vector \( v \in \mathbb{R}^n \), and a set \( S \subseteq \{1, \ldots, n\} \), let \( M(S) \) denote the matrix by throwing away columns of \( M \) not in \( S \). In addition, for any set \( S \subseteq \{1, \ldots, n\} \), let the set function
\[
f(S) = \text{dist}^2(v, \text{Range}(M(S))),
\]
where \( \text{dist}(y, X) \) is the distance from a point to a subspace; formally,
\[
\text{dist}(y, X) = \min_{x \in X} \|y - x\|_2.
\]

We show that there exist \( v \) and \( M \) such that the function \( f : \{1, 2, \ldots, n\} \rightarrow \text{dist}^2(v, \text{Range}(M(S))) \) is non-supermodular. We start with the definitions of monotone and supermodular set functions.

**Definition 3 (Monotonicity).** Consider any finite set \( V \). The set function \( f : 2^V \rightarrow \mathbb{R} \) is non-decreasing if and only if for any \( A \subseteq A' \subseteq V \), we have \( f(A) \leq f(A') \).

In words, a set function \( f : 2^V \rightarrow \mathbb{R} \) is non-decreasing if and only if adding elements in any set \( A \subseteq V \) cannot decrease the value of \( f(A) \).
Problem 1, and let the set $S$ in complexity theory, there is no efficient algorithm that runs in polynomial running time. In other words, improving the guarantee of the strategy that achieves the final state is not possible—subject to the complexity-theoretic hypothesis that $\mathsf{NP} \notin \mathsf{BPTIME}(\text{poly log } n)$. Therefore, unless $\mathsf{NP} \notin \mathsf{BPTIME}(\text{poly log } n)$, there exists no quasi-polynomial algorithm for which Problem 7 is $(\Delta(n), 2^{\Omega(\log^{-\epsilon} n)})$-approximable.

Theorem 7 says that if $\mathsf{NP} \notin \mathsf{BPTIME}(\text{poly log } n)$ there is no polynomial time algorithm (or quasi-polynomial time algorithm) that can choose which entries of the system's state to actuate so that $x(t_1)$ is even approximately close to a desired state $x_1 = [1, 1, \ldots, 1, 0, 0, \ldots, 0]^T$ at time $t_1$.

To make sense of Theorem 7 first observe that we can always actuate every entry of the system's state, i.e., we can choose $S = \{1, 2, \ldots, n\}$. This means every system is $(0, n)$-approximable; let us rephrase this by saying that every system is $(0, 2^{\log n})$ approximate. Theorem 7 tells us that we cannot achieve $(0, 2^{\Omega(\log^{-\epsilon} n)})$-approximability for any $\delta > 0$. In other words, improving the guarantee of the strategy that actuates every state by just a little bit, in the sense of replacing $\delta = 0$ with some $\delta > 0$, is not possible—subject to the complexity-theoretic hypothesis $\mathsf{NP} \notin \mathsf{BPTIME}(\text{poly log } n)$. Furthermore, the theorem tells us it remains impossible even if we allow ourselves some error $\Delta(n)$ in the target state, i.e., even $(\Delta(n), 2^{\Omega(\log^{-\epsilon} n)})$-approximability is ruled out.

Remark 1. In [23, Theorem 3] it is claimed that for any $\epsilon > 0$ the minimal reachability Problem 7 is $(\epsilon, O(\log^{1/2} n))$-approximable, which contradicts Theorem 7. Moreover, the proof of this claim was based on [25, Lemma 8.1], which we proved incorrect in Section III.

Remark 2. The minimal controllability problem [6] seeks to place the fewest number of actuators to make the system...
controllable. Theorem 1 is arguably surprising, as it was shown in [6] that the sparsest set of actuators for controllability can be approximated to a multiplicative factor of $O(\log n)$ in polynomial time. By contrast, we showed in this note that an almost exponentially worse approximation ratio cannot be achieved for minimum reachability.

V. PROOF OF INAPPROXIMABILITY OF MINIMAL REACHABILITY

In this section, we provide a proof of our main result, namely Theorem 1. We use some standard notation throughout.

A. Reachability Space for continuous-time linear systems

**Definition 7** (Reachability space). Consider a system $\dot{x}(t) = Ax(t) + Bu(t)$ as in eq. (1) whose size is $n$. The Range($[B, AB, A^2B, \ldots, A^{n-1}B]$) is called the reachability space of $\dot{x}(t) = Ax(t) + Bu(t)$.

The reason why Definition 7 is called the reachability space is explained in the following proposition.

**Proposition 1** ([29] Proof of Theorem 6.1]). Consider a system as in eq. (1), with initial condition $x_0$. There exists a real input $u(t)$ defined over the time interval $(t_0, t_1)$ such that the solution of $\dot{x} = Ax + Bu$, $x(t_0) = x_0$ satisfies $x(t_1) = x_1$ if and only if

$$x_1 - e^{A(t_1-t_0)}x_0 \in \text{Range}([B, AB, A^2B, \ldots, A^{n-1}B]).$$

The notion of reachability space allows us to redefine the minimal reachability Problem 1 as follows.

**Corollary 1.** The minimal reachability Problem 1 is equivalent to

$$\min_{S \subseteq \{1, 2, \ldots, n\}} |S|$$

such that $x_1 - e^{A(t_1-t_0)}x_0 \in \text{Range}([\mathbb{1}(S)B, A\mathbb{1}(S)B, \ldots, A^{n-1}\mathbb{1}(S)B]).$

Overall, Problem 1 is equivalent to picking the fewest rows of the input matrix $B$ such that $x_1 - e^{A(t_1-t_0)}x_0$ is in the linear span of the columns of $[\mathbb{1}(S)B, A\mathbb{1}(S)B, A^2\mathbb{1}(S)B, \ldots, A^{n-1}\mathbb{1}(S)B]$.

B. Variable Selection Problem

We show the intractability of the minimum reachability by reducing it to the variable selection problem, defined next.

**Problem 2** (Variable Selection). Let $U \in \mathbb{R}^{m \times l}$, $z \in \mathbb{R}^m$, and let $\Delta$ be a positive number. The variable selection problem is to pick $y \in \mathbb{R}^l$ that is an optimal solution to the following optimization problem.

$$\min_{y \in \mathbb{R}^l} \|y\|_0$$

such that $\|Uy - z\|_2 \leq \Delta,$

where $\|y\|_0$ refers to the number of non-zero entries of $y$.

The variable selection Problem 2 is found in [30] to be inapproximable:

**Theorem 2** ([30] Proposition 6]). Unless $\text{NP} \subseteq \text{BPTIME}(n^{\text{poly log }n})$, we have that for each $\delta \in (0, 1)$ there exist

- a function $\Delta(l) : \mathbb{N} \to \mathbb{N}$ which is $2^{\Omega((\log \log l) \log \log \log l)}$;
- a function $q_1(l) : \mathbb{N} \to \mathbb{N}$ which is in $2^{\Omega((\log \log l) \log \log \log l)}$ and $O(l)$;
- a polynomial $p_1(l)$ which is $O(l)$;
- a polynomial $m(l)$,

such that, given an $m(l) \times l$ matrix $U$, no quasi-polynomial algorithm can distinguish between the following two cases:

1. There exists $y \in \{0, 1\}^l$ such that $Uy = 1_{m(l)}$ and $\|y\|_0 \leq p_1(l)$.
2. For any $y \in \mathbb{R}^l$ such that $\|Uy - 1_{m(l)}\|_2 \leq \Delta(l)$, we have $\|y\|_0 \geq p_1(l)q_1(l)$.

Informally, for the variable selection Problem 2 in Theorem 2 unless $\text{NP} \subseteq \text{BPTIME}(n^{\text{poly log }n})$, there is no quasi-polynomial algorithm that can distinguish between the case where there exists a solution to Problem 2 with a few non-zero entries, and the case where every approximate solution has almost every entry nonzero.

C. Sketch of Proof of Theorem 1

We begin by sketching the intuition behind the proof of Theorem 1. Our general approach is to find instances of Problem 1 that are as hard as inapproximable instances of the variable selection Problem 2. We begin by discussing a construction that is not necessary, and then explain how to fix it. Given the matrix $U$ coming from a variable selection Problem 2 we first attempt to construct an instance of the minimal reachability Problem 1 where

- the system’s initial condition is $x(t_0) = 0$,
- the destination state $x_1$ at time $t_1$ is of the form $[1, 0]^T$ (the exact dimensions of 1 and 0 are to be determined);
- the system’s input matrix is $B = I$;
- the system’s matrix $A$ is

$$A = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix},$$

where the number of zeros is large so that $A^2 = 0$.

Whereas the variable selection problem involves finding the smallest set of columns of $U$ so that a certain vector is in their span, for the minimum reachability problem, every time we add the $k$-th state to the set of actuated variables $S$, the reachability space expands by adding the span of the set of columns of the controllability matrix that correspond to the vector $e_k$ being added in $\mathbb{1}(S)$. In particular, for the above construction, because $A^2 = 0$, when the $k$-th state is added to the set of actuated variables, the span of the two columns $e_k$ and $Ue_k$ is added to the reachability space.

1In this context, a function with a fractional exponent is considered to be a polynomial, e.g., $t^{1/5}$ is considered to be a polynomial in $t$. 

In other words, with the above construction we are basically constrained to make “moves” which add columns in pairs, and we are looking for the smallest number of such “moves” making a certain vector lie in the span of the columns.

It should be clear that there is a strong parallel between this and variable selection (where the columns are added one at a time). However, because the columns are being added in pairs, this attempt to connect minimum reachability with variable selection does not work quite well. To fix this idea, we want only the columns of $U$ to contribute meaningfully to the addition of the span, with any vectors $e_k$ we add along the way being redundant; this would reduce minimal reachability to exactly variable selection. We accomplish this by further defining,

$$U' = \begin{pmatrix} U & U & \vdots & U \\ \end{pmatrix},$$

where we stack $U$ some large number of times (to be determined in the main proof of Theorem 1 at Section V-D). We then set

$$A = \begin{pmatrix} 0 & U' & 0 \\ \end{pmatrix}.$$

The idea is that because $U$ is “stacked” many times, adding a column of $U$ to a set of vectors expands the span much more than adding any vector $e_k$, so there is never an “incentive” to even consider the contributions of the vectors $e_k$ to the reachability space.

We next make this argument precise. First, given a matrix $M \in \mathbb{R}^{l \times l}$, for $n \geq kp$ we define $\phi_{n,d}(M)$ to be the $n \times n$ matrix which stacks $U$ in the top-right hand corner $d$ times. For example,

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \phi_{5,2}(M) = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

i.e., $\phi_{5,2}(M)$ stacks $M$ twice, and then pads it with enough zeros to make the resulting matrix $5 \times 5$. Observe that if $n \geq 2dl$, then $\phi_{n,d}(M)^2 = 0$. We adopt the notation that the last $l$ columns of $\phi_{n,d}(M)$ are called the non-identity columns, while the first $n - l$ columns are called the identity columns.

**D. Proof of Theorem 1**

We turn to the proof of Theorem 1. We adopt the definitions in the previous sections.

Proof of Theorem 1 Let $U$ be an $l \times l$ matrix and consider solving the minimum variable selection problem with $y = 1$; by Theorem 2 this cannot be computed in quasi-polynomial time unless $\text{NP} \neq \text{BPTIME}(1^{\text{poly log } n})$. Adopting the notation of Theorem 2 we set:

- $d = m(l)[p_1(l)q_1(l)];$
- $n = 2 \max(d,l);$
- for simplicity, we use $m$ and $m(l)$ interchangeably.

We consider an instance of the minimal reachability where:

- the system’s initial condition is $x(t_0) = 0$;
- the destination state $x_1$ at time $t_1$ is $[1_1^T, 0_{n-d}^T]^T$;
- the system’s input matrix is $B = I$, where $I$ is the identity matrix;
- the system’s matrix is $A = \phi_{n,d}(U)$.

Given the above instance for Problem 1, we next prove Theorem 1 in two steps.

First step of proof: Suppose that there exists a vector $y \in \{0,1\}^l$ with $Uy = 1_m$ and $||y||_0 \leq p_1(l)$. In that case, we claim there exists a set $S \subseteq \{1,2,\ldots,n\}$ with $|S| \leq p_1(l)$ such that $[1_1^T, 0_{n-d}^T]_S$ reachable. Indeed, let $S$ be a set of columns of $U$ that have $1_m$ in their span, and set $S = \{k + n - l \mid k \in S\}$. Then $|S| \leq p_1(l)$, and

$$1_m = \sum_{k \in S} U_k,$$

where $U_k$ denotes the $k$’th column of the matrix $U$; hence, we have

$$\begin{pmatrix} 1_1 \\ 0_{n-d} \end{pmatrix} = \sum_{k \in S} \begin{pmatrix} U_k \\ 0_{n-d} \end{pmatrix} = \sum_{k \in S} A_{k + n - l} = \sum_{k \in S} A_{k + n - l},$$

where the final step follows by definition of $\phi_{n,d}(\cdot)$. Now each of the vectors in the last term is a column of $\text{All}(S)$ with this choice of $S$, so $[1_1^T, 0_{n-d}]_S^T$ indeed lies in the range of the controllability matrix.

Second step of proof: Conversely, suppose that any $z$ with $||z||_0 \leq \Delta(l)$ has the property that $||z||_0 \geq p_1(l)q_1(l)$. We refer to this as assumption A1. We claim that in this case there is no $S \subseteq \{1,2,\ldots,n\}$ with cardinality strictly less than $p_1(l)q_1(l)$ that makes any $y$ with $||y - [1_1^T, 0_{n-d}]_S^T||_2 \leq \Delta(l)$ reachable. To prove this, assume the contrary, i.e., assume there exists $S \subseteq \{1,2,\ldots,n\}$ with cardinality strictly less than $p_1(l)q_1(l)$ that makes some $y$ with $||y - [1_1^T, 0_{n-d}]_S^T||_2 \leq \Delta(l)$ reachable. We call this assumption A2. We obtain a contradiction as follows:

- Break up $S$ into identity columns and non-identity columns such that $S = S_{id} \cup S_{non-id}$.
- By the pigeonhole principle, it follows that in the set $\{1,2,\ldots,d\}$ there is some interval $E = \{km + 1, km + 2,\ldots, km + \kappa\}$, where $\kappa$ is a non-negative integer, such that $S \cap V = \emptyset$, because $|S| < p_1(l)q_1(l)$ and $d \geq m[p_1(l)q_1(l)]$.
- In particular, there is no $k \in S_{id}$ such that $k \in E$, since in the previous bullet point we showed $S \cap E = \emptyset$, and therefore $S_{id} \cap E = \emptyset$.
- As a consequence of the assumption that there is $S \subseteq \{1,2,\ldots,n\}$ with cardinality strictly less than $p_1(l)q_1(l)$ that makes any $y$ with $||y - [1_1^T, 0_{n-d}]_S^T||_2 \leq \Delta(l)$ reachable, we have that there is $y \in \text{Range}(\text{All}(S), \text{All}(S), 0, 0, \ldots, 0)$ such that $||y - [1_1^T, 0_{n-d}]_S^T||_2 \leq \Delta(l)$. Define $y_0 \in \mathbb{R}^n$ by taking the rows of $y$ corresponding to indices in $E$. Then, $||y_0 - 1_m||_2^2 \leq \Delta(l)$. Moreover, $y_0$ is in the span
of the vectors obtained by taking the rows \( km + 1, \ldots, km + m \) of the columns of the reachability matrix \([I(S), A|S|, 0, \ldots, 0]\). Since in the previous bullet point we concluded \( S_A \cap \mathcal{E} = \emptyset \), all such columns are either zero or equal to a column of \( U \).

- Thus, we have that a vector \( y_\epsilon \in \mathbb{R}^m \) such that \( ||y_\epsilon - 1||_2^2 \leq \Delta(l) \) and \( y_\epsilon \) is in the span of \( [S] \) columns of \( U \).

Moreover, assumption A2 tells us that \( |S| < p_1(q_1(l)) \) while assumption A1 tells us the opposite.

To summarize, we showed the dichotomy of (1a) and (1b):

1a) “There exists a vector \( y \in \{0,1\}^l \) with \( Uy = 1 \) and \( ||y||_0 \leq p_1(l) \).”

1b) “Any \( y \) with \( ||Uy - 1||_2^2 \leq \Delta(l) \) has the property that \( ||y||_0 \geq p_1(q_1(l)) \).”

implies the dichotomy of (i-a) and (i-b):

i-a) “There exists a set \( S \subseteq \{1, 2, \ldots, n\} \) with \( |S| \leq p_1(l) \) such that \( [1^T_l, 0_{n-d}^{1-T}] \) reachable.”

i-b) “There is no \( S \subseteq \{1, 2, \ldots, n\} \) with cardinality strictly less than \( p_1(q_1(l)) \) that makes any \( y \) with \( ||y - [1^T_l, 0_{n-d}^{1-T}]||_2^2 \leq \Delta(l) \) reachable.”

in the sense that (1a) implies (i-a) (first step of the proof) and (1b) implies (i-b) (second step of the proof).

Theorem 2 showed that unless \( \text{NP} \subseteq \text{BPTIME}(\text{polylog} n) \), no quasi-polynomial time algorithm can distinguish between (1a) and (1b). This implies that, under the same assumption, no quasi-polynomial time algorithm can distinguish between (i-a) and (i-b). In particular, since for any \( \delta \in (0,1) \), we can take \( q_1(l) = 2^{\Omega((\log^{-1} \delta) l)} \) in Theorem 2 this implies that the smallest number of inputs rendering \( [1^T_l, 0_{n-d}^{1-T}] \) reachable cannot be approximated withing a multiplicative factor of \( \phi(l) \) which grows slower than \( 2^{\Omega((\log^{-1} \delta) l)} \).

Finally, we note that because the dimension of \( A \) is polynomial in \( l \) (since \( A \) is \( n \times n \), where \( n = 2 \max(d,l) \) with \( d = m(l)[p_1(l)q_1(l)] \)), we have that \( \phi(l) = 2^{\Omega((\log^{-1} \delta) n)} \).

VI. CONCLUDING REMARKS

We focused on the minimal reachability Problem 1 which is a fundamental question in optimization and control with applications such as power systems and neural circuits. By exploiting the connection to the variable selection Problem 2 we proved that Problem 1 is hard to approximate. Future work will focus on properties for the system matrix \( A \) so that Problem 1 is approximable in polynomial time.

We conclude with an open problem. As we have discussed, the minimum reachability problem is \( (0, 2^{\log n}) \)-approximable by the algorithm which actuates every variable; but \( (0, 2^{\Omega((\log^{-1} \delta) n)} \) is impossible for any positive \( \delta \). We wonder, whether the minimum number of actuators can be approximated to within a multiplicative factor of say, \( \sqrt{n} \) in polynomial time, or, more generally, \( n^c \) for some \( c \in (0, 1) \). Indeed, observe that since \( \sqrt{n} = 2^{(1/2) \log n} \), the function \( \sqrt{n} \) does not belong to \( 2^{\Omega((\log^{-1} \delta) n)} \) for any \( \delta > 0 \). Thus, the present paper does not rule out the possibility of approximating the minimum reachability problem up to a factor of \( \sqrt{n} \), or more broadly, \( n^c \) for \( c \in (0, 1) \). We remark that such an approximation guarantee would have considerable repercussions in the context of effective control, as at the moment the best polynomial-time protocol for actuation to meet a reachability goal (in terms of worst-case approximation guarantee) is to actuate every variable.

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