General Tensor Spectral Co-clustering for Higher-Order Data

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ABSTRACT
Spectral clustering and co-clustering are well-known techniques in data analysis, and recent work has extended spectral clustering to square, symmetric tensors and hypermatrices derived from a network. We develop a new tensor spectral co-clustering method that applies to any non-negative tensor of data. The result of applying our method is a simultaneous clustering of the rows, columns, and slices of a three-mode tensor, and the idea generalizes to any number of modes. The algorithm we design works by recursively bisecting the tensor into two pieces. We also design a new measure to understand the role of each cluster in the tensor. Our new algorithm and pipeline are demonstrated in both synthetic and real-world problems. On synthetic problems with a planted higher-order cluster structure, our method is the only one that can reliably identify the planted structure in all cases. On tensors based on n-gram text data, we identify stop-words and semantically independent sets; on tensors from an airline-airport multimodal network, we find worldwide and regional co-clusters of airlines and airports; and on tensors from an email network, we identify daily-spam and focused-topic sets.

Keywords
tensor; spectral clustering; co-clustering

1. INTRODUCTION
Clustering is a fundamental task in data mining that aims to assign closely related entities to the same group. Traditional methods optimize some aggregate measure of the strength of pairwise relationships between items, such as the sum of similarity between all pairs of entities in the same group. Spectral clustering is a particularly powerful technique for computing the clusters when the pairwise similarities are encoded into the adjacency matrix of a graph. However, many graph-like datasets are more naturally described by higher-order connections among several entities. For instance, multiplex networks describe the interactions between several graphs simultaneously with layers of node-node relationships [21].

Tensors are a common representation for many of these higher-order datasets. A tensor, or hypermatrix, is a multidimensional array with an arbitrary number of indices or modes. A tensor with two-modes is equivalent to a matrix, and a tensor with three-modes looks like a three-dimensional brick. We recently proposed a Tensor Spectral Clustering (TSC) framework as a generalization of spectral methods for higher-order graph data [6]. This method was designed for the case when the higher-order tensor recorded the occurrences of small subgraph patterns within the network. For instance, the i, j, kth element of the tensor denoted the presence of a triangle or a directed 3-cycle. The method was particularly successful at identifying clusters that corresponded to different layers of a directed network.

The TSC framework had a number of limitations, however. First, it was primarily designed for the case when the tensor arose based on some underlying graph. The partitioning metric used was designed explicitly for this case. Thus, the applications are limited in scope and cannot model, for example, multiplex networks. Second, the TSC framework involved viewing the tensor as the transitions on a higher-order Markov chain—similar to how spectral clustering views pairwise data as the transitions of a first-order Markov chain—and then using a spacey random walk on this higher-order Markov chain to identify the clusters [5]. When the data are sparse, the spacey random walk required a correction whose magnitude is proportional to the sparsity in the tensor. In many instances, this correction was substantial because the tensor was extremely sparse. This made it difficult to accurately identify clusters.

Here we develop the General Tensor Spectral Co-clustering (GTSC) framework for clustering general sparse tensor data. Our method is based on a new super-spacey random walk model that more accurately models higher-order Markov chains and avoids the correction necessary for the previous spacey random walk model. Furthermore, we show how to use our method on rectangular tensor data through a tensor symmetrization procedure [29]. This allows us to simultaneously cluster the rows, columns, and slices of three-mode tensors. The idea generalizes to any number of modes where we cluster the objects represented in each mode independently. We also introduce a variant on the well-known conductance measure for partitioning graphs [30] that we call biased conductance and describe how this provides a tensor partition quality metric. In particular, biased conductance is approximately the exit probability from a set following our new super-spacey random walk model.

The algorithm underlying our GTSC framework recursively partitions the tensor data. We stop the process when the partitioning metric is bad, which lets us cluster the data without specifying the number of clusters ahead of time. This provides an advantage over existing tensor decomposition methods such as PARAFAC that require the number of clusters as an input to the algorithms. Finally, we show that our method asymptotically scales linearly (up to logarithmic
factors) in the size of data. In Section 6.2, we perform numerical scaling experiments to demonstrate the scalability of our method.

We use extensive experiments on both synthetic and real-world problems to validate the effectiveness of our method. For the synthetic experiments, we devise a “planted cluster” model for tensors and show that GTSC has superior performance compared to other state-of-the-art clustering methods in recovering the planted clusters. In Section 6, we provide an extensive analysis of the performance compared to other state-of-the-art clustering methods in terms of normalized mutual information, F1 score, and adjusted rand index. We find that our GTSC framework identifies stop-words and semantic structures in the data.

In Section 6, we empirically demonstrate that GTSC identifies coherent clusters in real-world datasets (Section 6).

Our contributions are summarized as follows:

- We create a new super-spacey random walk model for tensor data and use it to create a partitioning quality measure based on a notion of biased conductance (Section 4).
- We develop GTSC, a new framework for clustering rectangular tensor data (Section 3.4). The framework simultaneously clusters the rows, columns, and slices of the tensor.
- We show that GTSC outperforms other state-of-the-art clustering methods in identifying the planted cluster structure in synthetic tensor data, in terms of normalized mutual information, F1 score, and adjusted rand index (Section 5).
- We empirically demonstrate that GTSC identifies coherent clusters in real-world datasets (Section 6).

2. FIRST-ORDER SPECTRAL METHODS

We first review graph clustering methods from the view of first-order data in order to develop our GTSC framework.

2.1 Preliminaries and Notation

Let \( A \in \mathbb{R}^{n \times n} \) be the adjacency matrix of an undirected graph \( G = (V, E) \) and let \( n = |V| \) be the number of nodes in the graph. Define \( D = \text{diag}(Ae) \) to be the diagonal matrix of degrees of vertices in \( V \). The graph Laplacian is \( L = D - A \) and the transition matrix is \( P = A^T D^{-1} \). The transition matrix represents the transition probabilities of a random walk on the graph. If a walker is at node \( j \), it transitions to node \( i \) with probability \( P_{ij} = A_{ij} / D_{jj} \).

2.2 Conductance and Markov Chains

One of the most widely-used quality metrics for partitioning a graph is vertices into two sets \( S \) and \( \bar{S} \) is conductance 30. Intuitively, conductance measures the ratio of the number of edges in the graph that go between \( S \) and \( \bar{S} \) to the number of edges in \( S \) or \( \bar{S} \). Formally, we define conductance as:

\[
\phi(S) = \frac{\text{cut}(S)}{\min(\text{vol}(S), \text{vol}(\bar{S}))},
\]

where

\[
\text{cut}(S) = \sum_{i \in S, j \not\in S} A_{ij} \quad \text{and} \quad \text{vol}(S) = \sum_{i \in S, j \in V} A_{ij}.
\]

A set \( S \) with small conductance is a good partition \((S, \bar{S})\). The numerator minimizes the edges going between the sets, and the denominator encourages either \( S \) or \( \bar{S} \) to be large in volume and that both are balanced.

The following well-known proposition relates conductance to random walks on the graph.

Observation 1 (23). Let \( G \) be undirected, connected, and not bipartite. Start a random walk \((X_t)_{t \in \mathbb{N}}\) where the initial state \( X_0 \) is randomly chosen following the stationary distribution of the random walk. Then for any set \( S \subseteq V \),

\[
\phi(S) = \max \{ \Pr(X_1 \in \bar{S} | X_0 \in S), \Pr(X_1 \in S | X_0 \in \bar{S}) \}.
\]

This provides an alternative view of conductance—it measures the probability that one step of a random walk will traverse between \( S \) and \( \bar{S} \). Again, small conductance is indicative of a good partition: a small probability means that the sets have more internal connections than external connections. This random walk view, in concert with the super-spacey random walk, will serve as the basis for our biased conductance idea to partition tensors in Section 3.4.

2.3 Spectral Partitioning with Sweep Cuts

Finding the set of minimum conductance is an NP-hard combinatorial optimization problem 31. However, there are real-valued relaxations of the problem that are tractable to solve and provide a guaranteed approximation 24 12. The most well known computes an eigenvector called the Fiedler vector and then uses a sweep cut to identify a partition based on this eigenvector.

The Fiedler eigenvector \( z \) solves \( Lz = \lambda Dz \) where \( \lambda \) is the second smallest generalized eigenvalue. This can be equivalently formulated in terms of the random walk transition matrix \( P \). Specifically,

\[
Lz = \lambda Dz \iff (I - D^{-1}A)z = \lambda z \iff z^T P = (1 - \lambda)z.
\]

In other words, the Fiedler vector is the eigenvector with the second smallest generalized eigenvalue of the Laplacian and degree and the left eigenvector of \( P \) with the second largest eigenvalue. This equivalence is important for our generalizations to higher-order data in Section 3.

The sweep cut procedure to identify a low-conductance set \( S \) from \( z \) is as follows:

1. Sort the vertices by \( z \) as \( z_1 \leq z_2 \leq \cdots \leq z_n \).
2. Consider the \( n-1 \) candidate sets \( S_k = \{ \sigma_1, \sigma_2, \ldots, \sigma_k \} \) for \( 1 \leq k \leq n-1 \)
3. Choose \( S = \arg \min_{S_k} \phi(S_k) \) as the solution set.

The solution set \( S \) from this algorithm satisfies the celebrated Cheeger inequality 24 11: \( \phi(S) \leq 4 \sqrt{\phi_{opt}} \), where \( \phi_{opt} = \min_{S \subseteq V} \phi(S) \) is the minimum conductance over all sets of nodes. This procedure is extremely efficient since \( S_k \) and \( \sigma_k \) differ only in the vertex \( \sigma_{k+1} \). The conductance value of the set can be updated in time proportional to the degree of vertex \( \sigma_k \) and thus the sweep cut procedure only takes linear time in the number of edges in the graph.

To summarize, the spectral method requires two components: the second left eigenvector of \( P \) and the conductance criterion.

3. HIGHER-ORDER SPECTRAL METHOD

We now generalize the ideas from spectral graph partitioning to nonnegative tensor data. We first review our notation

\[\text{https://github.com/wutaob27/GtensorSC}\]
for tensors and then review how tensor data can be interpreted as a higher-order Markov chain. We briefly review our prior work on Tensor Spectral Clustering before introducing the new super-spacey random walk that we use here. This super-spacey random walk will allow us to compute a vector akin to the Fiedler vector for a tensor and to generalize conductance to tensors. Furthermore, we generalize the ideas from co-clustering in bipartite graph data \cite{13} to rectangular tensors.

### 3.1 Preliminaries and Tensor Notation

We use $\mathbf{T}$ to denote a tensor. As a generalization of a matrix, it may have up to $m$ indices—called an $m$-mode tensor—and so an individual element is $T_{i_1, i_2, \ldots, i_m}$. We will work with non-negative tensors where $T_{i_1, i_2, \ldots, i_m} \geq 0$. We call a subset of the tensor entries with all but the first element fixed a *column* of the tensor. For instance, the $i,j,k$ column is $T_{i,j,k}$. A tensor is square if the dimension of all the modes is equal and rectangular if not; a square tensor is symmetric if it is equal for any permutation of the indices. For simplicity in the remainder of our exposition, we will focus on the three-mode case but everything we talk about generalizes to an arbitrary number of modes. (See \cite{17, 5} for representative examples of how the generalizations look.) We use two operations between a tensor and a vector. First, a tensor-vector product with a three-mode tensor can output a vector, which we denote by:

$$\mathbf{y} = \mathbf{T}\mathbf{x}^2 \iff y_i = \sum_{j,k} T_{i,j,k} x_j x_k.$$  

Second, a tensor-vector product can also produce a matrix, which we denote by:

$$\mathbf{A} = \mathbf{T}[\mathbf{x}] \iff A_{i,j} = \sum_k T_{i,j,k} x_k.$$  

### 3.2 Higher-order Markov Chains

Recall from Section 2 that we could form the transition matrix for a Markov chain from a square non-negative matrix $\mathbf{A}$ by normalizing the columns of the matrix $\mathbf{A}^2$. We can generalize this idea to define a higher-order Markov chain by normalizing a square tensor. This leads to a probability transition tensor $\mathbf{P}$:

$$P_{i,j,k} = \frac{T_{i,j,k}}{\sum_i T_{i,j,k}}$$  

where we assume $\sum_i T_{i,j,k} > 0$. (In Section 3.3 we will discuss the sparse case where the column $T_{i,j,k}$ may not have any non-zero entries.)

Entries of $\mathbf{P}$ can be interpreted as the transition probabilities of a higher-order Markov chain $Z_t$:

$$P_{i,j,k} = \Pr(Z_t = i \mid Z_{t-1} = j, Z_{t-2} = k).$$

In terms of random walks, if the last two states were $j$ and $k$, then the next state is $i$ with probability $P_{i,j,k}$.

It is possible to turn any higher-order Markov chain into a first-order Markov chain on the product state space of all ordered pairs $(i,j)$. The new Markov chain moves to the state-pair $(i,j)$ from $(j,k)$ with probability $P_{i,j,k}$. Computing the Fiedler vector associated with this chain would be one possible strategy. However, this approach has two immediate problems. First, the eigenvector is of size $n^2$, which quickly becomes too large. Second, the eigenvector gives information about the product space—not the original data. In other words, it is unclear how to use this eigenvector even if we could afford to compute and store it.

In our prior work, we used the *spacey random walk* and *spacey random surfer* stochastic processes for a scalable random walk process \cite{6, 3}. This stochastic process is non-Markovian and generates a sequence of states $X_t$ as follows. After arriving at state $X_t$, the walker promptly “spaces out” and forgets the state $X_{t-1}$, yet it still wants to transition according to the higher-order transitions $\mathbf{P}$. So it invents a state $Y_t$ by drawing a random state from its history and then transition to state $X_{t+1}$ with probability $P_{X_{t+1},X_t,Y_t}$. If $H_t$ denotes the history of the process up to time $t$ then

$$\Pr(Y_t = j \mid H_t) = \frac{1}{\sum_{r=1}^t \Ind\{X_r = j\}}.$$  

In this case, we assume that the process has a non-zero probability of picking any state by inflating its history count by 1 visit. The spacey random surfer is a generalization where the walk follows the above process with probability $\alpha$ and teleports at random following a stochastic vector $\mathbf{v}$ with probability $1 - \alpha$. This is akin to how the PageRank random walk includes teleportation.

These spacey random walk processes are instances of vertex reinforced random walks \cite{4, 28}. Limiting stationary distributions are solutions to the multilinear PageRank problem \cite{17}:

$$\alpha \mathbf{P}\mathbf{x}^2 + (1 - \alpha)\mathbf{v} = \mathbf{x}.$$  

As shown in \cite{5}, the limiting distribution $\mathbf{x}$ represents the stationary distribution of the transition matrix $\mathbf{P}[\mathbf{x}]$. The transition matrix $\mathbf{P}[\mathbf{x}]$ asymptotically approximates the spacey walk or spacey random surfer. Thus, it is feasible to compute an eigenvector of this matrix and use it with the sweep cut procedure on a generalized notion of conductance. However, we had to assume that all the columns of $\mathbf{T}$ were non-zero, which does not occur in real-world datasets. In other words, all of the $n^2$ possible transitions from pairs of states were well defined. In our prior work, we adjusted the tensor $\mathbf{T}$ and replaced any columns of all zeros with the uniform distribution vector. (This is easy to do in a way that does not require $n^2$ storage.) Because the number of zero-columns could be extremely large, this was not ideal—although it gave promising results when the tensors arose from graphs for a notion of conductance that was specific to these graphs. We deal with this issue more generally in the following section, and note that our new solution outperforms the old one as described in the experiments.

### 3.3 A Stochastic Process for Sparse Tensors

Here we consider another model of the random surfer that avoids the issue of undefined transitions—which correspond to columns of $\mathbf{T}$ that are all zero—entirely. If the surfer attempts to use an undefined transition, then the surfer moves to a random state drawn from history. Formally, define the set of feasible states by

$$\mathcal{F} = \{(j,k) \mid \sum_i T_{i,j,k} > 0\}.$$  

Here, the set $\mathcal{F}$ denotes all the columns in $\mathbf{T}$ that are non-zero. The transition probabilities of our proposed stochastic

\footnotesize{Formally, this is the $\sigma$-algebra generated by the states $X_1, \ldots, X_t$.}
process are given by
\[
\Pr(X_{t+1} = i \mid X_t = j, H_t) = (1 - \alpha)v_i + \alpha Pr(X_{t+1} = i \mid X_t = j, Y_t = k, H_t)Pr(Y_t = k \mid H_t)
\]
where \(v_i\) is the teleportation probability. Again \(Y_t\) is chosen according to Equation (4). We call this process the super-spacey random surfer because when the transitions are not defined it picks a random state.

This process is also a vertex-reinforced random walk. Let \(P\) be the normalized tensor \(P_{i,j,k} = T_{i,j,k}/\sum T_{i,j,k}\) only for the columns in \(F\) and where all other entries are zero. Stationary distributions of the stochastic process must satisfy the equation:
\[
\alpha P\mathbf{x}^2 + \alpha(1 - \|P\mathbf{x}\|_{1})\mathbf{x} + (1 - \alpha)\mathbf{v} = \mathbf{x},
\]
where \(\mathbf{x}\) is a probability distribution vector. (See Appendix A for the derivation.) At least one solution must exist, which follows directly from the Brouwer fixed-point theorem. Here we give a sufficient condition for it to be unique.

**Theorem 3.1.** If \(\alpha < 1/(2m - 1)\) then there is a unique solution \(\mathbf{x}\) to (9) for the general \(m\)-mode tensor. Furthermore, the iterative fixed point algorithm
\[
\mathbf{x}_{k+1} = \alpha P\mathbf{x}_k^2 + \alpha(1 - \|P\mathbf{x}_k\|_1)\mathbf{x}_k + (1 - \alpha)\mathbf{v}_k
\]
will converge to this solution.

(Theorem 3.1 and this proof are in Section 2.2.3 in the Appendix.)

In practice we found high values (e.g., 0.95) of \(\alpha\) did not impede convergence. We use \(\alpha = 0.8, v_i = 1/n\) for all of our experiments in this paper.

### 3.4 Biased Conductance for Tensor Partitions

From Observation [1] in Section 2.2.2 we know that conductance may be interpreted as the exit probability between two sets that form a partition of the nodes in the graph. In this section, we derive an equivalent first-order Markov chain from the stationary distribution of the super-spacey random surfer. If this Markov chain was guaranteed to be reversible, then we could apply the standard definitions of conductance and the Fiedler vector. This will not generally be the case, and so we introduce a biased conductance measure to partition this non-reversible Markov chain with respect to starting in the stationary distribution of the super-spacey random walk. We use the second largest, real-valued eigenvector of the Markov chain as an approximate Fiedler vector. Thus, we can use the sweep cut procedure described in Section 2.3 to identify the partition.

In the following derivation, we use the property of the two tensor-vector products:
\[
P[T] = Px^2.
\]
The stationary distribution \(\mathbf{x}\) for this super-spacey random surfer (described in Section 3.3) is equivalently the stationary distribution of the Markov chain with transition matrix
\[
\alpha(P[\mathbf{x}] + \mathbf{x}(e^T - e^TP[\mathbf{x}])) + (1 - \alpha)v e^T.
\]
(To understand this expression, note that \(x \geq 0\) and \(e^T x = 1\).)

This matrix represents a useful (but crude) approximation of the higher-order structure in the data. First, we determine how often we visit states using the super-spacey random surfer to get a vector \(\mathbf{x}\). Then the Markov chain \(P\) will tend to have a large probability of spending time in states where the higher-order information concentrates. This matrix represents a first-order Markov chain on which we can compute an eigenvector and run a sweep cut.

We now define a biased conductance to partition it.

**Biased Conductance.** Consider a random walk \((X_t)_{t \in \mathbb{N}}\). The biased conductance \(\phi_p(S)\) of a set \(S \subseteq \{1, \ldots, n\}\) is
\[
\phi_p(S) = \max \{Pr(X_1 \in S \mid X_0 \in S), Pr(X_1 \in S \mid X_0 \in \bar{S})\}
\]
where \(X_0\) is chosen according to a fixed distribution \(p\).

This definition is general and has a few subtle aspects that we wish to acknowledge. First, the initial state \(X_0\) is not chosen following the stationary distribution (as in the standard definition on a reversible chain) but following \(p\) instead. This is why we call it biased conductance. We apply this measure to \(P\) using \(p = x\) (the stationary distribution of the super-spacey walk). This choice emphasizes the higher-order information.

We use the eigenvector of \(P\) with the second-largest real eigenvalue as an analogue of the Fiedler vector. If the chain were reversible, this would be exactly the Fiedler vector. When it is not, then it encodes largely the same type of information and is a practical heuristic. It is important to note that although \(P\) is a dense matrix, we can implement the two operations we need with \(P\) in time and space that depends only on the number of non-zeros of the sparse tensor \(P\) using standard iterative methods for eigenvalues of matrices.

### 3.5 Rectangular Tensor and Co-clustering

So far, we have only considered square, symmetric tensor data. However, tensor data are often rectangular. This is usually the case when the different modes represent different types of data. For example, in Section 3.3 we examine a tensor \(T \in \mathbb{R}^{p \times n \times n}\) of airline flight data, where \(T_{i,j,k}\) represents that there is a flight from airport \(i\) to airport \(k\) on airline \(j\). Our approach is to embed the rectangular tensor into a
The tensor $C$ and if

1: $C = \{1, \ldots, n\}$
2: IF $n \leq \min$-size: RETURN
3: Generate transition tensor $P$ by
   $P_{ijk} = \begin{cases} T_{ijk}/\sum_{j=1}^{n} T_{ijk} & \text{if } \sum_{j=1}^{n} T_{ijk} > 0 \\ 0 & \text{otherwise} \end{cases}$
4: Compute super-spacey stationary vector $\lambda$ (Equation 9) and form $P[\lambda]$.
5: Compute second largest left, real-valued eigenvector $z$ of $P = P[\lambda] + \kappa (\kappa - \lambda I P[\lambda])$ (that is, $z^T \lambda = \lambda z^T$).
6: $\sigma \leftarrow$ Sort eigenvector $z$.
7: $(S, \phi_p) \leftarrow$ Biased Conductance Sweep Cut($\sigma, P[\lambda]$) with bias $p = \sigma$.
8: IF $n \geq \max$-size or $\delta_p \leq \delta^*$ then
9: $C_S = \text{Algorithm 1}$ on sub-tensor $T_{S,S,S}$.
10: $C_S = \text{Algorithm 1}$ on sub-tensor $T_{S,S,S}$.
11: $C = C_S \cup C_S$.
12: end if
13: RETURN $C$

larger square tensor and then symmetrize this tensor, using approaches developed by Ragnarsson and Van Loan [29]. After the embedding, we can run our algorithm to simultaneously cluster rows, columns, and slices of the tensor. This approach is similar in style to the symmetrization of bipartite graphs for co-clustering proposed by Dhillon [13].

Let $U$ be an $n$-by-$m$-by-$\ell$ rectangular tensor. Then we embed $U$ into a square three-mode tensor $T$ with $n + m + \ell$ dimensions and where $U_{ijk} = T_{i,j+n,k+m}$. This is illustrated in Figure 1 (left). Then we symmetrize the tensor by using all permutations of the indices Figure 1 (right). The result is, when viewed as a 3-by-3-by-3 block tensor,

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U_{(2,3,1)} & 0 \\ 0 & 0 & U_{(1,3,2)} \end{bmatrix} \begin{bmatrix} 0 & 0 & U_{(2,3,1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & U_{(1,3,2)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The tensor $U_{(1,3,2)}$ is just a generalized transpose of $U$ with the dimensions permuted.

Finally, we note that “rectangular” tensors are not determined by sizes of modes only. We may consider a tensor $T$ to be rectangular if each mode represents a different type of object. The important idea is that if a tensor is declared to be rectangular, then the result of clustering is a subset of each mode.

4. THE ALGORITHM

In this section, we put together the pieces from Section 3 to build the GTSC framework and analyze its computational complexity. In Section 4.3, we also derive a “popularity” metric for clusters that will be useful for discussing clustering results on real-world data in Section 5.

4.1 Recursive Two-way Cuts

Our GTSC algorithm works by recursively applying the sweep cut procedure, similar to the recursive bisection procedures for clustering matrix-based data [8]. We continue partitioning as long as the clusters are large enough or we can get good enough splits. Specifically, if a cluster has dimension less than a specified size $\min$-size, we do not consider it for splitting. Otherwise, the algorithm recursively splits the cluster if either (1) its dimension is above some threshold $\max$-size or (2) the biased conductance of a new split is less than a target value $\phi^*$. The overall algorithm is summarized in Algorithm 1.

We also have a couple of pre-processing steps. First, we have to symmetrize the data if the tensor is rectangular. Second, we look for “empty” indices that do not participate in the tensor structure. Formally, index $i$ is empty if $T_{ijk} = 0$ for all $j$ and $k$.

4.2 Computational Complexity

We now provide an analysis of the running time of our algorithm. Let $N$ be the number of non-zeros in the tensor $T$. First, note that the pre-processing (tensor symmetrization and finding empty nodes) takes $O(N)$ time. Now, we examine the computational complexity of a single partition:

1. Generating the transition tensor $P$ costs $O(N)$.
2. Each step of (10) to find the stationary distribution is $O(N)$.
3. Constructing $P[\lambda]$ costs $O(N)$. (The matrix $P$ is not formed explicitly).
4. Each iteration of the eigenvector computation takes time linear in the number of non-zeros in $P[\lambda]$, which is $O(N)$.
5. Sorting the eigenvector takes $O(n \log n)$ computations, which is negligible considering $N$ is big compared to $n$.
6. The sweep cut takes time $O(n + N)$, which is $O(N)$.

In practice, we find that only a few iterations are needed to compute the stationary distribution, which is consistent with past results [6][17]. For these systems, we do not know how many iterations are needed for the eigenvector computations. However, for the datasets we analyze in this paper, the eigenvector computation is not prohibitive. Thus, we can think of the time of each cut as roughly linear in the number of non-zeros. Provided that the cuts are roughly balanced, the depth of the recursion tree is $O(\log N)$, and the total time is $O(N \log N)$. Again, in our experiments, this is the case.

4.3 Ranking Clusters with Popularity Scores

For our analysis of real-world datasets in Section 6, it will be useful to order clusters based on the volume of their interactions with other clusters. To this end, we compute a “popularity score” for each cluster. Let $k$ be the number of clusters and define a $k \times k$ interaction matrix $M$ by the total tensor weight between the clusters. Formally, for a three-mode tensor, $M_{ij} = \sum_{x \in S_i, j \in S_j} T_{ijk}$. Finally, define the popularity score of cluster $i$ as the PageRank score of node $i$ in the graph induced by $M$. (To handle corner cases, isolated nodes in this graph are given a popularity score of 0). For our experiments, we compute the PageRank score.
with $\alpha = 0.99$ and uniform teleportation vector.

5. SYNTHETIC EXPERIMENTS

We generate tensors with planted cluster structures and try to recover the planted clusters. We compare our GTSC framework with a variety of other methods and find that it recovers the planted structure most often.

5.1 Synthetic Dataset Generation

**Square Tensor Data.** We first generate 20 groups of nodes that will serve as our planted clusters. The number of nodes in each group $n_g$ is drawn from a normal distribution with mean 20 and variance 5, with truncated minimum value so that each group has at least 4 nodes. Our square synthetic data tensor will have mode 3 and dimension $\sum_{g=1}^{20} n_g$. For each group $g$ we also assign a weight $w_g$ where the weight depends on the group number. For group $i$, the weight is $\left(\frac{\sigma}{\sqrt{2\pi}}\right)^{-1} \exp\left(-\frac{(i-10.5)^2}{2\sigma^2}\right)$, where $\sigma$ varies by experiment. With these weights, groups 10 and 11 have the largest weight and groups 1 and 20 have the smallest weight. As described next, these weights will be used to provide a skew in the distribution of the number of interactions for a given dimension.

Non-zeros correspond to interactions between three indices, and we call these assignments triples. We generate $t_w$ triples whose indices are within a group and $t_a$ triples whose indices span across more than one group. The $t_w$ triples are chosen by first uniformly selecting a group $g$ and then uniformly selecting three indices $i$, $j$, and $k$ from group $g$ and finally assigning a weight of $w_g$. Formally, the value of the tensor data tensor $T$ is

$$T_{i,j,k} = w_g.$$

If the same three indices are chosen more than once in the sampling procedure, we simply increment the value in the tensor. For the $t_a$ triples that span multiple groups, the sampling procedure first selects an index $i$ from group $g_i$ proportional to the weights of the group. In other words, indices in group $g$ are chosen proportional to $w_g$. Two indices $j$ and $k$ are then selected uniformly at random from groups $g_j$ and $g_k$ other than $g_i$. Finally, the weight in the tensor is assigned to be the average of the three group weights:

$$T_{i,j,k} = (w_{g_i} + w_{g_j} + w_{g_k})/3.$$

For our experiments, $t_w = 10,000$ and $t_a = 1,000$, and the variance $\sigma$ that controls the group weights is 2 or 4. For each value of $\sigma$, we create 5 sample datasets.

**Rectangular Tensor Data.** For rectangular data, we distinguish between the indices for each mode of our mode-3 tensor. We label the modes as $x$, $y$, and $z$. The groups are generated by mode subgroups of size $n_g^x$, $n_g^y$, and $n_g^z$, $1 \leq g \leq 20$. Each subgroup size is sampled from the same normal distribution as before with the same truncated minimum value. The dimension of the data tensor is then $\sum_{g=1}^{20} n_g^x \times \sum_{g=1}^{20} n_g^y \times \sum_{g=1}^{20} n_g^z$. Each group $g$ is assigned a weight in the same way as the synthetic square data.

Index triples $(i, j, k)$ now correspond to three subgroups $g_i^x$, $g_j^y$, and $g_k^z$, and we generate within-group and across-group triples similarly to the square case. The $t_w$ within-group triples are chosen by first uniformly selecting a group and then uniformly selecting one index from each subgroup. The $t_a$ across-group triples are chosen by (1) selecting a mode $(x, y, or z)$ uniformly at random, (2) selecting the mode index proportional to the weights $w_g$, and (3) selecting the other two indices uniformly at random from the other groups. The value of triple $t_w$ and $t_a$ is assigned in a similar manner to the square tensor case.

We again set $t_w = 10,000$ and $t_a = 3,000$ and generate 5 synthetic datasets for each value of $\sigma \in \{2, 4\}$.

5.2 Clustering Methods and Evaluation

We compared the results of our GTSC framework to several other state-of-the-art methods for clustering tensor data.  

**GTSC.** This is the method presented in this paper (Algorithm 1). We use the parameters max-size = 100, min-size = 5, and $\phi^* = 0.35$.  

**TSC.** This is the original tensor spectral clustering algorithm [6]. We use the algorithm with recursive bisection to find 20 clusters.

**PARAFAC.** The PARAFAC method is a widely used tensor decomposition procedure [18] that finds an approximation to the tensor by the sum of outer-products of vectors. We compute a rank-20 decomposition using the Tensor Toolbox [20, 10], and then assign nodes to clusters by taking the index of the vector with highest value in the nodes index. We use the default tolerance of $10^{-4}$ and a maximum of 1000 iterations.

**Spectral Clustering.** Our clustering framework (Algorithm 1) works on mode-2 tensors, i.e., matrices. In this case, with $\alpha = 1$, our algorithm reduces to a standard spectral clustering method. We create a matrix $M$ from the tensor data $T$ by summing along the third mode: $M_{ij} = \sum_{k=1}^{20} T_{i,j,k}$. We then run Algorithm 1 with the same parameters as GTSC.

**Evaluation metrics.** We evaluate the clustering results using the Adjusted Rand Index (ARI) [20], Normalized Mutual Information (NMI) [22], and F1 score. The ground truth labels correspond to the generated groups.

5.3 Experimental Results

Table 1 depicts the performances of the four algorithms. In all cases, GTSC has the best performance. In the square tensor case when $\sigma = 4$, the standard spectral method performs as well as GTSC. However, when $\sigma = 2$, the score drops for all four methods with the least impact on GTSC.

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3 We tested several values $\phi^* \in [0.3, 0.4]$ and obtained roughly the same results.
Table 1: Results of various clustering methods on the synthetically generated data. Bold indicates the best mean performance in terms of ARI, NMI, or F1 score, and ± entries are the standard deviations over 5 trials. Our method (GTSC) has the best performance in all cases.

|                | Square tensor with $\sigma = 4$ | Rectangular tensor with $\sigma = 4$ |
|----------------|----------------------------------|-------------------------------------|
| **GTSC**       | **0.99 ± 0.01** 0.99 ± 0.00      | **0.99 ± 0.01** 0.97 ± 0.06         |
| **TSC**        | 0.42 ± 0.05 0.60 ± 0.04          | 0.45 ± 0.04 0.38 ± 0.17             |
| **PARAFAC**    | 0.82 ± 0.05 0.94 ± 0.02          | 0.83 ± 0.04 0.81 ± 0.04             |
| **Spectral Clustering** | **0.99 ± 0.01** 0.99 ± 0.01 | **0.99 ± 0.01** 0.91 ± 0.06         |

|                | Square tensor with $\sigma = 2$ | Rectangular tensor with $\sigma = 2$ |
|----------------|----------------------------------|-------------------------------------|
| **GTSC**       | **0.78 ± 0.13** 0.89 ± 0.06      | **0.79 ± 0.12** 0.96 ± 0.06         |
| **TSC**        | 0.41 ± 0.11 0.60 ± 0.09          | 0.44 ± 0.10 0.28 ± 0.08             |
| **PARAFAC**    | 0.48 ± 0.08 0.67 ± 0.04          | 0.50 ± 0.07 0.10 ± 0.04             |
| **Spectral Clustering** | 0.43 ± 0.07 0.66 ± 0.04 | 0.47 ± 0.06 0.38 ± 0.07             |

6. REAL-WORLD EXPERIMENTS

We now use our GTSC framework to cluster real-world tensor datasets. Table 2 describes the relevant statistics of these datasets.

| Dataset                        | Size                  | # non-zeros |
|--------------------------------|-----------------------|-------------|
| Airline-airport                | $539 \times 2,939 \times 2,939$ | 51,982      |
| English 3-grams                | square $30,966$       | 1,020,009   |
| Chinese 3-grams                | square $18,387$       | 966,138     |
| English 4-grams                | square $23,734$       | 1,034,307   |
| Chinese 4-grams                | square $14,728$       | 1,002,660   |
| Enron email                    | $185 \times 184 \times 184 \times 34$ | 61,891      |

Airline-airport. This dataset consists of global air flight routes from 539 airlines and 2,939 airports from OpenFlight.\(^6\) The $539 \times 2,939 \times 2,939$ data tensor $T$ summarizes flights connecting airports on several airlines. Formally, $T_{ijkl}$ is 1 if airline $i$ flies between airports $j$ and $k$ and 0 otherwise.

English 3,4-grams. An $n$-gram is a contiguous sequence of $n$ words from a sequence of text. We generated a tensor dataset from the one million most frequent 3- and 4-grams from the Corpus of Contemporary American English.\(^7\) The 3-gram tensor entry $T_{ijkl}$ is given by the number of times that words $i$, $j$, and $k$ co-occur in an $n$-gram in the corpus. The 4-gram tensor dataset is defined analogously. We symmetrize these tensors for our method.

Chinese 3,4-grams. We also constructed $n$-gram tensor datasets from the Chinese language using the Google Books $n$-grams dataset.\(^8\) We constructed 3- and 4-gram tensors in the same way as the English dataset.

Enron email. This dataset is constructed from emails between Enron employees with labeled topics.\(^7\) The tensor data represents the volume of communication between two employees discussing a given topic during a particular week. In total, there are 185 weeks of data, 184 employees, and 34 topics, leading to a $185 \times 184 \times 184 \times 34$ tensor where $T_{ijkl}$ is the number of emails between employee $j$ and $k$ on topic $l$ during week $i$.

In all of our experiments, we use the stopping criterion $\phi^* = 0.4$ for Algorithm 1. For the $n$-gram data and airline-airport data, we use the parameters max-size = 100 and min-size = 5; for the Enron email data, we use max-size = 50 and min-size = 10.

6.1 Airline-airport

Our GTSC framework groups the airports and airlines of $n$-grams is only roughly 1 million—see Table 2.

\(^6\)http://openflights.org/data.html#route
\(^7\)http://www.ngrams.info/intro.asp
\(^8\)The datasets contain ties in frequency, so the actual number of times that words $i$, $j$, and $k$ co-occur in an $n$-gram in the corpus. The 4-gram tensor dataset is defined analogously. We symmetrize these tensors for our method.

Figure 3: Visualization of the Airline-airport data tensor. The $x$ and $y$ axes index airports and the $z$ axis indexes airlines. A dot represents that an airline flies between those two airports. On the left, indices are sorted randomly. On the right, indices are sorted by the co-clusters found by our GTSC framework, which reveals structure in the tensor.
Table 4: Fraction of words in the top two groups (in terms of popularity, see Section 1.2.3) that are among the top 100 (200) most frequently used words in the English (Chinese) written language corpora.

| Groups | English 3-gram | English 4-gram | Chinese 3-gram | Chinese 4-gram |
|--------|----------------|----------------|----------------|----------------|
| 1st Group | 24/29 | 11/11 | 42/84 | 31/31 |
| 2nd Group | 23/25 | 42/47 | 33/74 | 23/36 |

into 129 co-clusters. Figure 3 illustrates the connectivity of the tensor with a random ordering of the indices (left) and the ordering given by the popularity of co-clusters (right). We can see that after the co-clustering, there is clear structure in the data tensor. Table 3 summarizes some of the larger clusters found by our method, and we discuss these clusters in more detail below.

The co-cluster with the highest popularity score is the one marked as Worldwide Metropolises. This group is composed of large international airports in cities such as Beijing and New York City. The 250 airports in this group are responsible for 59% of the total routes, even though they only account for 8.5% of the total number of airports. Figure 3 illustrates this result—the airports with the highest indices are connected to almost every airport. This cluster is analogous to the “stop word” group we will see in the n-gram experiments.

Groups with medium levels of popularity are organized geographically. Our GTSC framework found one large cluster for Europe, the United States, China/Taiwan, Oceania/SouthEast Asia, and Mexico/Americas. Interestingly, Cancun International Airport is included with the United States cluster, probably due to the large amounts of tourism. In addition to the large groups, we find many mini groups which consist of 5–30 airports. These airports compose the long, concentrated diagonal in Figure 3. The airports and airlines in these co-clusters are also closely connected geographically but they are more isolated than the large, continental co-clusters.

6.2 English n-grams

Our GTSC framework clusters the 3-gram and 4-gram tensors into 486 and 383 groups, respectively. We rank the groups by decreasing order of the popularity score described in Section 1.2.3. We find several conclusions that hold for both tensor datasets. Highly ranked groups are mostly composed of stop words (i.e., common words) such as the, a, we, is, by (we define stop words as those most connected with other words). In fact, the top two groups consist nearly entirely of stop words (see Table 4). In addition, 48% (3-gram) and 64% (4-gram) of words in the first group are prepositions (e.g., in, of, as, to) and link verbs (e.g., is, get, does). In the second group, 64% (3-gram) and 57% (4-gram) of the words are pronouns (e.g., we, you, them) and link verbs. This result matches the structure of English language where link verbs can connect both prepositions and pronouns whereas prepositions and pronouns are unlikely to appear in close vicinity. Since prepositions are more common than pronouns, they rank first.

Groups ranked in the middle mostly consist of semantically related English words. For instance, a few of these groups are \{cheese, cream, sour, low-fat, frosting, nonfat, fat-free\}, \{bag, plastic, garbage, grocery, trash, freezer\}, \{infection, chronic, sexually, transmitted, diseases, hiv\}. The lowest ranked groups are comprised mostly of non-English words that appear in English text. For example, one such group is \{je, ne, sais, quoi\}, which is a French phrase. We may consider these groups as outliers.

The clustering of the 4-gram tensor contains some groups that the 3-gram tensor fails to find. For example, one cluster is \{german, chancellor, angela, merkel, gerhard, Schroeder, helmut, kohl\}. Angela Merkel, Gerhard Schroeder, and Helmut Kohl have all been German chancellors, but it requires a 4-gram to make this connection strong. Likewise, some clusters only appear from clustering the 3-gram tensor. One such cluster is \{church, bishop, catholic, priest, greek, orthodox, methodist, roman, priests, episcopal, churches, bishops\}. In 3-grams, we may see phrases such as “catholic church bishop”, but 4-grams containing these words likely also contain stop words, e.g., “bishop of the church”. However, since stop words are in a highly ranked cluster, the connection is destroyed once the top clusters are formed.

6.3 Chinese n-grams

We find that many of the conclusions from the English n-gram datasets also hold for the Chinese n-gram datasets. For example, many of the words in the top two groups are stop words (see Table 4). Groups scoring in the middle consist of semantically similar words, and low-scoring groups are foreign language phrases that appear in Chinese literature.

However, there are also several differences from the English n-gram data. First, we find that groups in Chinese are usually larger than those in English. The average sizes of small-to-medium groups (3–100 words) is 6.8 for English and 10 for Chinese. We suspect this pattern is due to the fact that Chinese words are more semantically dense than English words. For example, it is common that Chinese words have multiple meanings and can co-occur with a large number of words.

We note that there are several words form the top two groups that are not typically considered as stop words. For example, society, economy, develop, -ism, nation, government are among the top 100 most common words. (These are translations of the Chinese words). This is a consequence of the dataset coming from scanned Chinese-language books and is a known issue with the Google Books corpus [27]. In this case, it is a feature as we are illustrating the efficacy of our tensor clustering framework rather than making any linguistic claims.

6.4 Enron Email Tensor

In total, the algorithm finds 23 co-clusters of topics, people, and time. The most popular group corresponds to three topics, 19 people, and 0 time intervals. Similar to the n-grams and airport-airline data, this cluster corresponds to high-volume entities, in this case common topics and people who send a lot of emails. The three topics are “Daily business”, “too few words”, and “no matching topic”, which account for roughly 90% of the total email volume. (The latter two topics are capturing outliers and emails that do not fall under an obvious category). The 19 employees include 11 managers: the CEO, (vice) presidents, and directors. These employees are involved in 42% of the total emails. There is no time interval in this co-cluster because these high-volume
Table 3: High-level descriptions of the larger co-clusters found by our GTSC framework on the Airline-airport dataset. The algorithm finds one co-cluster of international hubs and large commercial airlines and several geographically coherent groups.

| Name                | # Airports | # Airlines | Airports description                              | Airlines description |
|---------------------|------------|------------|---------------------------------------------------|----------------------|
| Worldwide metropolises | 250        | 77         | Large hubs, e.g., Beijing Capital and JFK in New York | Large commercial airlines, e.g., United, Air Canada, Air China |
| Europe              | 184        | 32         | 177 in Europe, rest in Morocco                     | 29 European airlines  |
| United States       | 137        | 9          | 136 in U.S., Cancín International                   | 29 all U.S. airlines  |
| China/Taiwan        | 170        | 33         | 136 in China or Taiwan,                            | 21 in China/Taiwan 14 in S. Korea and Thailand |
| Oceania/S.E. Asia   | 302        | 52         | 231 in Oceania or S.E. Asia,                       | 41 in East Asia or Canada 66 in China, Japan, or Canada |
| Mexico/Americas     | 399        | 68         | 396 in Mexico or Central and South America         | 43 in Mexico or Central and South America |

Figure 4: Enron email volume on three labeled topics. Our GTSC framework finds a co-cluster consisting of these three topics at the time points labeled in red, which seems to correlate with various events involving the CEO.

dices in the rows and columns of a matrix of interaction clustering. Prior work in this area has focused on finding groups of different entities through their interactions. We studied standard algorithms for the PARAFAC model. We also found several interesting co-clusters. One consists of the topics “California bankruptcy”, “California legislature”, and “India (general)”, during three weeks in December 2000 and January 2001, and 13 employees. These time points correspond to various events involving CEO Skilling (Figure 4). Each of the 13 employees in the co-cluster sent at least one email from at least one of the topics. Another co-cluster consists of the topics “General newsfeed”, “Downfall newsfeed”, and “Federal Energy Regulatory Commission/Department of Energy” and several weeks from March 2001 and December 2001. These time intervals coincide with investor James Chanos finding problems with Enron in early 2001 and the serious financial troubles encountered by the company in late 2001.

7. RELATED WORK

There are a variety of methods for tensor decomposition that could be used to cluster a tensor. For instance, Huang et al. use a higher-order SVD as a basis for clustering [19] and Anandkumar et al. [1] use a latent-variable model. We studied standard algorithms for the PARAFAC decomposition in this light and found that our tensor clustering method outperformed this general strategy.

Our work provides a technique for “co-clustering”, i.e., finding clusters of different entities through their interactions. Prior work in this area has focused on finding groups of indices in the rows and columns of a matrix of interaction data [13] [14]. Our work provides the first generalization of these techniques for the higher-order interactions encountered in tensor data. We note that recent work has looked at co-clustering several entity types simultaneously [18] [3]. However, these methods are still based on first-order information (graph structure).

Our GTSC framework is immediately applicable as a clustering method for multiplex networks [25]. Indeed, we found interesting structure in the airline-airport dataset, which is an example of a multiplex network. Related work in this area includes multi-view clustering [32], ensemble methods [15], and multi-network clustering [26]. However, these methods are specialized for multiplex networks, whereas our method provides a general tensor clustering framework.

8. CONCLUSION

Tensors are increasingly common in modern applications and clustering tensor data is fundamental for discovering patterns and data analysis. However, tensor clustering is a difficult task for two reasons: higher-order structure in tensors is difficult to model and obvious extensions of models to higher-order data are computationally challenging. In this paper we proposed the General Tensor Spectral Co-clustering (GTSC) method. Our method addresses these issues by modeling higher-order data with a new stochastic process, the super-spacey random walk, which is a variant of a higher-order Markov chain. Our iterative solver and sweep cut procedure for biased conductance can achieve near-linear complexity in the number of non-zeros in tensors.

In synthetic experiments our GTSC out-performs state-of-the-art spectral methods and tensor decomposition methods and can efficiently handle skew in the distribution of the indices of the non-zeros in the tensor. Furthermore, our GTSC framework can find clear cluster structures in various tensor datasets, including English and Chinese n-gram text, an airline and airport multiplex network, and Enron email data.

In terms of future work, we’d like to create tensors that bridge information from multiple modes. For instance, the clusters from the 3-gram data were different from the 4-gram data and it would be useful to have a holistic tensor to jointly partition both 3- and 4-gram information. This is important because some of the clusters in the n-gram data correspond to automatically extracted knowledge, such as the cluster with the names of various German chancellors. This aspect of our output also merits further investigation as it would require overlapping clusters to be useful to knowledge extraction efforts.
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APPENDIX

A. STATIONARY DISTRIBUTION
We provide a heuristic proof here and a reference for how to make it formal. Suppose the process has run for a very long time and that $x_t$ is the current empirical distribution. From equation (1), we have

$$\Pr(X_{i+1} = i | X_t = j) = (1-\alpha)v_i + \alpha \sum_{(j,k) \in E} P_{i,j,k} x_t(k) + \alpha \sum_{(j,k) \notin E} x_t(k) x_t(i).$$

Because the process has run for a very long time, the empirical distribution $x_t$ is essentially fixed. Thus, the stochastic process can be approximated by a Markov chain with the transition matrix:

$$\alpha P[x_i] + \alpha x_t (e^T - e^T P[x_i]) + (1-\alpha)e^T.$$

At stationarity of the super-spacey random walk, the stationary distribution of this Markov chain must be equal to $x_t$. A more formal proof is to use the results from [4] to show that (9) is the necessary condition of the stationary distribution in a vertex reinforced random walk.

B. PROOF OF THEOREM 3.1
Let $R$ denote the mode-1 unfolding of $P^1$:

$$R = [P(:,i,:)] P(:,i,:) P(:,i,:); \ldots; P(:,i,:)].$$

Note that $R(x \otimes x) = P^2 x^2$ where $\otimes$ is the Kronecker product. Assume $x$ and $y$ are two solutions of (9). Let $r_x = \|R(x \otimes x)\|_1$ and $r_y = \|R(y \otimes y)\|_1$. Then

$$\|x - y\|_1 \leq \alpha \|R(x \otimes x - y \otimes y)\|_1 + \alpha \|1 - r_x\| \|x\|_1 + (1 - r_y)\|y\|_1.$$  

By Lemma 4.5 of [17], the first term

$$\alpha \|R(x \otimes x - y \otimes y)\|_1 \leq \alpha \|R\|_1 \|x \otimes x - y \otimes y\|_1 \leq 2\alpha \|x - y\|_1.$$

The second term satisfies

$$\alpha \|(1 - r_x)(x - y)\|_1 + \alpha r_y - r_x \leq \alpha \|x - y\|_1 + \alpha \|R(x \otimes x - y \otimes y)\|_1 \leq \alpha \|x - y\|_1.$$

Combining the above two facts, we know when $\alpha < 1/2$ the solution is unique. For an $m$-mode tensor, this idea generalizes to $\alpha < 1/(2m-1)$.

For the convergence of the fixed point algorithm (10), the same analysis shows that $\|x_{k+1} - x^*\|_1 \leq 5\alpha \|x_k - x^*\|_1$, and so the iteration converges when the solution is unique.