A description of entanglement in terms of quantum phase

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We explore the role played by the phase in an accurate description of the entanglement of bipartite systems. We first present an appropriate polar decomposition that leads to a truly Hermitian operator for the phase of a single qubit. We also examine the positive operator-valued measures that can describe the qubit phase properties. When dealing with two qubits, the relative phase seems to be a natural variable to understand entanglement. In this spirit, we propose a measure of entanglement based on this variable.

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I. INTRODUCTION

The ability to create and manipulate quantum states is a topic of increasing interest, with strong implications in areas of futuristic technology such as quantum computing, quantum cryptography, and quantum communications. Entanglement, or nonlocal quantum correlations between spacelike separated systems, constitutes the essential resource for most, if not all, the applications in quantum information [1, 2]. In fact, the idea of entanglement has proved to be one of the most fertile and thought-generating properties of quantum mechanics and much work has been devoted to understanding how it can be quantified and manipulated [3, 4, 5, 6, 7, 8, 9].

It is a general belief nowadays that bipartite pure-state entanglement is almost completely understood [10] (although some questions are still open for the mixed state case). This entanglement is usually introduced through quantum states that violate the classical locality requirement (i.e., violate the Clauser-Horne-Shimony-Holt inequality). In fact, the mathematical tool underlying this kind of treatments is the Schmidt bi-orthogonal expansion [11, 12], since it allows us to write any pure state shared by two parties in a canonical form, where all the information about the nonlocal properties is contained in the Schmidt coefficients. However, in physical terms, entanglement is deeply linked to the superposition principle, which also plays a key role in the puzzling world of quantum interference. Thus, from this perspective, it is hardly surprising that nonlocal correlations can be interpreted in terms of well-defined phase relations between parties [13], a point that goes almost unnoticed in the vast literature on the subject, in spite of the fact that phase is essential in understanding classical correlation phenomena [14].

Perhaps the most spectacular application of entanglement is the quantum computer, which would allow an exponential increase of computational speed for certain problems, once it is realized. The core of quantum computing are the logic gates [15] (it is known that any quantum computation can be reduced to a sequence of universal two qubit logic gates and one qubit local operation [16]). The realization of these logic gates in trapped ions [17], cavity QED [18, 19] and NMR [20, 21] have already been possible, showing the practical realization of two equivalent kinds of universal two qubit logic gates: a quantum controlled not gate and a quantum phase gate that differ from each other only by local operations [22]. The main problem these systems face is decoherence: it is essential to keep coherence of the qubits themselves and among them.

The moral we wish to extract from the previous discussion is that the concepts of phase and coherence for qubits are ubiquitous in the modern parlance of quantum information. However, the notions of phase for a qubit and of relative phase between qubits is loosely used in the literature, mainly because there is a lack of a clear prescription for dealing with such variables in the quantum world. Even worse, sometimes the concepts employed in this field are misleading. Therefore, a thorough explanation of the techniques needed to characterize the phase of qubit states will be of relevance to workers in the various diverse experimental fields currently under consideration for quantum computing technology. This is the main goal of this paper.
II. PHASE FOR A SINGLE QUBIT

The system we wish to study lives in a two-dimensional Hilbert space spanned by two states, we shall denote by $|0\rangle$ and $|1\rangle$. In this Hilbert space, the operators

\[ \hat{S}_+ = |1\rangle\langle 0|, \quad \hat{S}_- = |0\rangle\langle 1|, \]
\[ \hat{S}_z = \frac{1}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|), \]

(1)

together with the identity $\hat{I}$, form a complete set of linearly independent observables. They satisfy the commutation relations

\[ [\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm, \]
\[ [\hat{S}_+, \hat{S}_-] = 2\hat{S}_z, \]

(2)

which are distinctive from the $su(2)$ algebra that describes angular momentum in quantum mechanics.

Loosely speaking, a two-state system is called a quantum bit or qubit [23], in direct analogy with the classical bit of information (which consists in two distinguishable states of some system). Unlike the classical bit, a qubit can be in a superposition of the form

\[ |\Psi\rangle = \sin(\vartheta/2) |0\rangle + e^{i\phi} \cos(\vartheta/2) |1\rangle. \]

(3)

This corresponds to a $1/2$ angular-momentum system, and we know [24] that for this particular case $\hat{S} = \sigma/2$, $\sigma$ being the Pauli matrices. Then it is easy to work out that the mean values $s_j = \langle \Psi | \hat{S}_j | \Psi \rangle$ are given by

\[ s_x = \sin \vartheta \cos \varphi, \]
\[ s_y = \sin \vartheta \sin \varphi, \]
\[ s_z = \cos \vartheta, \]

(4)

where $\hat{S}_\pm = (\hat{S}_x \pm i\hat{S}_y)$. This would support the naive belief that, when viewed in the Bloch sphere $s_x^2 + s_y^2 + s_z^2 = 1$, the parameter $\varphi$ appears as the phase angle associated with the qubit and that it is canonically conjugate to $s_z$ [25]. Therefore, there is a widespread usage of dealing with this qubit phase $\varphi$ as a state parameter instead of a quantum variable, as one could expect from the very basic principles of quantum mechanics [11].

To gain further insight in this point, let us note that

\[ s_- = \langle \Psi | \hat{S}_- | \Psi \rangle = \sin \vartheta e^{i\varphi}, \]

(5)

so it is clear that the parameter $\varphi$ can be obtained through the decomposition of $s_-$ in terms of modulus and phase. Obviously, it is tempting to pursue this simple picture by taking into account that, at the operator level, the equivalent to the decomposition in terms of modulus and phase is a polar decomposition. Thus, it seems appropriate to define the operator counterpart of Eq. (5) by

\[ \hat{S}_- = \sqrt{\hat{S}_- \hat{S}_+} \hat{E}. \]

(6)

Here the unitary operator $\hat{E}$ represents the complex exponential of the qubit phase; i.e., $\hat{E} = e^{i\Phi}$, where $\Phi$ is the Hermitian operator for the phase. The use of the polar decomposition of $su(2)$ was pioneered by Lévy-Leblond [26], and worked out from the mathematical viewpoint in Refs. [27] and [28]. Its application to the proper description of the atomic-dipole phase in quantum optics was fully developed in Ref. [29].

It is not difficult to work out that the unitary solution of Eq. (6) is

\[ \hat{E} = |0\rangle\langle 1| + e^{i\varphi_0} |1\rangle\langle 0|, \]

(7)

where $\varphi_0$ is some arbitrary phase that corresponds to a matrix element undefined by Eq. (6) and that appears due to the unitarity requirement. The main features of this operator are largely independent of $\varphi_0$, however for the sake of concreteness we can make a definite choice. For instance, according to Eq. (6), the complex conjugation of the qubit wave function $|\Psi\rangle$ should reverse the sign of $\hat{\Phi}$ [30]. This leads to the condition $e^{i\varphi_0} = -1$, and therefore

\[ \hat{E} = |0\rangle\langle 1| - |1\rangle\langle 0|, \]

(8)

whose eigenvectors are

\[ |\varphi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle). \]

(9)
It is worth mentioning that these states have been exploited in recent proposals of covariant cloning \cite{31, 32, 33}.

To any smooth function $F(\varphi)$ we can associate the operator

$$F(\hat{\Phi}) = \sum_{\pm} |\varphi_{\pm}\rangle F(\varphi_{\pm}) \langle \varphi_{\pm}|,$$

where the sum runs over the two possible eigenvalues. The mean value of this operator function can be computed as

$$\langle F(\hat{\Phi}) \rangle = \sum_{\pm} F(\varphi_{\pm}) P(\varphi_{\pm}),$$

where $P(\varphi_{\pm})$ is the probability distribution

$$P(\varphi_{\pm}) = \text{Tr} \left[ \hat{\varrho} |\varphi_{\pm}\rangle\langle \varphi_{\pm}| \right],$$

for any state described by the density matrix $\hat{\varrho}$. Two important remarks seem pertinent. First, note that $\hat{E}$ is proportional to $\hat{S}_y$. Also, because $\hat{E}^\dagger = -\hat{E}$, we have $\cos \hat{\Phi} = 0$, which is, certainly, a rather pathological behavior. This is caused by the small dimension of the system space as such strong relations no longer hold for dimensions greater than two. In other words, the two-dimensional Hilbert space where the qubit lives is not large enough to distinctly accommodate all different variables.

Second, and perhaps more striking, is that the qubit phase can take only two values, $\pm \pi/2$, due also to the dimension of the Hilbert space. While this statement seems rather reasonable when dealing with spin systems, it is scarcely recognized when dealing with qubits. Note the clear distinction between the state parameter $\varphi$ in Eq. (3) and the outcome of a measurement of the qubit phase: the former is continuous, while the later is discrete and binary. We shall elaborate on this point in next Section.

### III. POSITIVE-OPERATOR VALUED MEASURES FOR QUBIT PHASE

Given the singular behavior exhibited by the description of qubit phase in terms of a Hermitian operator, one may think preferable to use a positive-operator valued measure (POVM) taking continuous values in a $2\pi$ interval. Additionally, this formalism can include also fuzzy generalizations of the ideal phase description provided by $|\varphi_{\pm}\rangle\langle \varphi_{\pm}|$, as in Eq. (12). To keep the discussion as self-contained as possible, we briefly recall that a POVM \cite{34, 35} is a set of linear operators $\Delta(\varphi)$ furnishing the correct probabilities in any measurement process through the fundamental postulate that

$$P(\varphi) = \text{Tr}[\hat{\varrho} \Delta(\varphi)].$$

The real valuedness, positiveness, and normalization of $P(\varphi)$ impose

$$\Delta^\dagger(\varphi) = \Delta(\varphi), \quad \Delta(\varphi) \geq 0, \quad \int_{2\pi} d\varphi \Delta(\varphi) = \hat{I},$$

where the integral extends over any $2\pi$ interval of the form $(\varphi_0, \varphi_0 + 2\pi)$, $\varphi_0$ being a fiducial or reference phase. Note that, in general, $\Delta(\varphi)$ are not orthogonal projectors like in the standard von Neumann measurements described by selfadjoint operators.

In addition to these basic statistical conditions, some other requirements must be imposed to ensure that $\Delta(\varphi)$ provides a meaningful description of the phase as a canonically conjugate variable with respect $\hat{S}_z$ (even in the sense of a weak Weyl relation \cite{36}). Then, we require $\Delta(\varphi)$

$$e^{i\varphi'\hat{S}_z} \Delta(\varphi) e^{-i\varphi'\hat{S}_z} = \Delta(\varphi + \varphi'),$$

which reflects nothing but the basic feature that a phase shifter is a phase-distribution shifter.

We must also take into account that a shift in $\hat{S}_z$ should not change the phase distribution. Therefore, we require as well

$$\hat{E} \Delta(\varphi) \hat{E}^\dagger = \Delta(\varphi),$$

which, loosely speaking, is the physical translation of the fact that phase should be complementary to the action variable $\hat{S}_z$. 
If one takes the dispersion \( D \) that Eq. (11). Indeed, one has \( D \) and then, \( \hat{\Delta} \) with \( c \) and \( \gamma \) where \( \gamma \leq 1 \) is a real number (otherwise its argument can be included in the definition of \( \varphi \)) whose physical meaning will be elucidated soon. The phase distribution \( P(\varphi) \) induced by this POVM is of the form

\[
P(\varphi) = \frac{1}{2\pi} (1 + ce^{i\varphi} + c^*e^{-i\varphi}),
\]

with \( c = \langle 0|\hat{\sigma}|1 \rangle \gamma \). This is a remarkably simple result that holds for any mixed state of the system.

This form allows us to examine some very general properties of the qubit phase. First, note that the information \( P(\varphi) \) conveys goes beyond what would strictly be the phase. Indeed, since words, \( \hat{\Delta}(\varphi) \) cannot be linearly independent because the qubit Hilbert space is two dimensional and the algebra of operators acting on that Hilbert space is four dimensional.

Suppose now that we have the qubit described in terms of two different POVMs with \( \gamma_1 \) and \( \gamma_2 \), such that \( \gamma_1 < \gamma_2 \). If one takes the dispersion \( D \) as a measure of the phase uncertainty of the state, we have

\[
D^2 = 1 - \left| \int_{2\pi} d\varphi \ e^{i\varphi} P_1(\varphi) \right|^2 = 1 - |c|^2
\]

and then, \( D_1 \geq D_2 \). This shows that \( P_1(\varphi) \) is always broader than \( P_2(\varphi) \) when \( \gamma_1 < \gamma_2 \). Moreover, one can also check that

\[
\hat{\Delta}_{\gamma_1}(\varphi) = \frac{1}{2\pi} \int_{2\pi} d\varphi' \left[ 1 + \frac{\gamma_1}{\gamma_2} e^{i(\varphi - \varphi')} + \frac{\gamma_2}{\gamma_1} e^{-i(\varphi - \varphi')} \right] \hat{\Delta}_{\gamma_2}(\varphi').
\]

Therefore, both POVMs contain the same information about the qubit: if one of them is known, the other one can be directly obtained.

A relevant feature of this approach is that it provides a qubit phase where any value of \( \varphi \) is allowed. However, we wish to emphasize that this continuous range of variation is not effective in the sense that the values of \( P(\varphi) \) at every point \( \varphi \) cannot be independent, and we can find relations between them irrespective of the qubit state. In other words, \( \hat{\Delta}(\varphi) \) cannot be linearly independent because the qubit Hilbert space is two dimensional and the algebra of operators acting on that Hilbert space is four dimensional.

This can be stated in a slightly different way: given the general form of \( P(\varphi) \) in Eq. (13), the complex parameter \( c \) can be determined by the value of \( P(\varphi) \) at two \( \varphi \) points. Nevertheless, more manageable expressions appear if we use three points instead of two, such as \( \varphi_r = 2\pi r/3 \) (\( r = -1, 0, 1 \)). After some calculations we get

\[
c = \frac{2\pi}{3} \sum_{r=0,\pm1} P(\varphi_r)e^{-i\varphi_r},
\]

which allows us to express \( P(\varphi) \) as

\[
P(\varphi) = \frac{1}{3} \sum_{r,s=0,\pm1} P(\varphi_r)e^{is(\varphi - \varphi_r)},
\]

and so the knowledge of the three values \( P(\varphi_r) \) gives \( P(\varphi) \) at any other point \( \varphi \).

This effective discreteness allows us to compute the mean values of any function \( F(\varphi) \) in a way very similar to Eq. (11). Indeed, one has

\[
\langle F(\varphi) \rangle = \frac{2\pi}{3} \sum_{r=0,\pm1} \tilde{F}(\varphi_r)P(\varphi_r),
\]
where \( \tilde{F} \) is related to \( F \) by
\[
\int_{2\pi} d\varphi \ e^{ik\varphi} \tilde{F}(\varphi) = \int_{2\pi} d\varphi \ e^{ik\varphi} F(\varphi), \quad k = 0, \pm 1,
\]
\[
\int_{2\pi} d\varphi \ e^{ik\varphi} \tilde{F}(\varphi) = 0, \quad |k| = \pm 2, \pm 3, \ldots
\]
(25)

This last equation guarantees that for the function \( e^{ik\varphi} \) we have \( \langle e^{ik\varphi} \rangle = 0 \), as it should be. Moreover, Eqs. (25) imply that
\[
\int_{2\pi} d\varphi \ P(\varphi) \tilde{F}(\varphi) = \int_{2\pi} d\varphi \ P(\varphi) F(\varphi),
\]
for any \( P(\varphi) \). We conclude then that discreteness is inevitably at the heart of the qubit phase.

Finally, we shall consider two particular examples of POVMs that are admissible to describe the qubit phase. The first one consists in
\[
\hat{\Delta}_{SG}(\varphi) = |\varphi\rangle\langle\varphi|,
\]
(27)
where
\[
|\varphi\rangle = \frac{1}{\sqrt{2\pi}}(|0\rangle + e^{i\varphi}|1\rangle).
\]
(28)

This is a finite-dimensional translation of the POVM generated by the Susskind-Glogower phase states in the Fock space of a harmonic oscillator \([35, 39]\). One can check that it is of the general form (17) with \( \gamma = 1 \). We think that the definition of this POVM is rather reasonable: while the operator \( \hat{E} \) selects an orthogonal basis from the set \( |\varphi\rangle \), this POVM, on the contrary, does not privilege any \( |\varphi\rangle \) and all of them play the same role.

In the case of a single-mode quantum field, a widely used conception of phase is based on examining quasiprobability distributions in phase space \([40, 41]\). Among them, one of the most interesting and studied comes from the \( Q \) function. A POVM for the field phase can be defined then in terms of radial integration, much in the spirit of the classical conception. The natural translation of this procedure to our qubit problem involves the use of SU(2) coherent states for a 1/2 angular momentum \([42]\)
\[
|\vartheta, \varphi\rangle = \sin(\vartheta/2)|0\rangle + e^{i\varphi}\cos(\vartheta/2)|1\rangle.
\]
(29)
and the \( Q \)-function they define as
\[
Q(\vartheta, \varphi) = \frac{1}{2\pi} \text{Tr} [\hat{q} |\vartheta, \varphi\rangle\langle\vartheta, \varphi|].
\]
(30)

This \( Q \) function can be regarded as a natural probability distribution in the associated qubit phase space, which is the manifold of SU(2). Now, we can define a POVM for \( \varphi \) by a marginal integration over \( \vartheta \); that is,
\[
\hat{\Delta}_{Q}(\varphi) = \frac{1}{2\pi} \int_{0}^{\pi} d\vartheta \ \sin(\vartheta/2) |\vartheta, \varphi\rangle\langle\vartheta, \varphi|,
\]
(31)
which corresponds to Eq. (17) with \( \gamma = \pi/4 \).

Note that we have focused on the case of a single qubit. The generalization of this formalism to a collection of \( N \) identical qubits is straightforward: the operators \( \hat{S}_+ \) and \( \hat{S}_\pm \) constitute then a \( (N+1) \)-dimensional representation of \( \text{su}(2) \) \([11]\) and, in consequence, the qubit phase takes \( N+1 \) distinct values. In terms of POVMs, the discreteness is also present but now we need to know the value of \( P(\varphi) \) at \( N+1 \) independent points. In other words, the general form of \( P(\varphi) \) involves only \( N+1 \) different frequencies, and the Fourier series can be inverted from the knowledge of \( P(\varphi) \) in \( N+1 \) points.

IV. RELATIVE PHASE FOR TWO QUBITS

Let us assume now that we have two such qubits, which we shall label by the subscripts A and B (in the language of quantum information, the first qubit belongs to Alice and the second one belongs to Bob). It seems natural to introduce the exponentials of the phase sum \( \hat{E}_{(+)} \) and phase difference \( \hat{E}_{(-)} \) by the unitary operators
\[
\hat{E}_{(+)} = \hat{E}_A \hat{E}_B, \quad \hat{E}_{(-)} = \hat{E}_A \hat{E}_B^\dagger.
\]
(32)
We have used the subscripts (+) and (−) for sum and difference to distinguish clearly from the symbols of ladder operators. We also introduce the operators (for simplicity we drop henceforth the subscript $z$ from $\hat{S}_+(\pi)$ and $\hat{S}_-(\pi)$, since there is no risk of confusion)

$$
\hat{S}_+ = \hat{S}_{A,+} + \hat{S}_{B,+}, \quad \hat{S} = \hat{S}_{A,+} - \hat{S}_{B,+},
$$

that satisfy the commutation relations

$$
[\hat{E}_+, \hat{S}_+] = \hat{E}_+, \quad [\hat{E}_-, \hat{S}_-] = \hat{E}_-,
$$

$$
[\hat{E}_+, \hat{S}_-] = 0, \quad [\hat{E}_-, \hat{S}_+] = 0.
$$

so $\hat{E}_+$ ($\hat{E}_-$) is canonically conjugate to $\hat{S}_+$ ($\hat{S}_-$). Note also that the vectors $|\varphi_A, \varphi_B\rangle = |\varphi_A\rangle \otimes |\varphi_B\rangle$ are simultaneous eigenvectors of $\hat{E}_+$ and $\hat{E}_-$, with eigenvalues $e^{i\varphi} = e^{i(\varphi_A + \varphi_B)}$ and $e^{i\varphi} = e^{i(\varphi_A - \varphi_B)}$, respectively.

The previous definition of phase sum and difference seems appropriate because it is in accordance with the algebra of complex numbers. However, due to its periodic character, adding and subtracting phases must be done with some care [43, 44]. Since each individual phase is expressed in a $2\pi$ range, the eigenvalue spectra of the sum and difference operators have widths of $4\pi$, and this is not compatible with the idea that a phase variable (even if it is a phase sum or difference) must be $2\pi$ periodic. Thus, we must devise a way to cast the phase sum and difference into the $2\pi$ range.

This problem can be traced back to the fact that while $(\hat{E}_A, \hat{E}_B)$, $(\hat{S}_{A,+}, \hat{S}_{B,+})$ or $(\hat{S}_+, \hat{S}_-)$, are complete sets of commuting operators, this is not true for $(\hat{E}_+, \hat{E}_-)$, since the vectors $|\varphi_A, \varphi_B\rangle$ and $|\varphi_A + \pi, \varphi_B + \pi\rangle$ have the same phase sum and difference. In consequence, another commuting operator must be considered to describe the system. In Ref. [45], dealing with the problem of angle sum and difference, it has been proposed using the operator

$$
\hat{V} = e^{2i\pi\hat{S}_+},
$$

which commutes with $\hat{E}_+$ and $\hat{E}_-$:

$$
[\hat{E}_+, \hat{V}] = [\hat{E}_-, \hat{V}] = 0.
$$

Therefore, $(\hat{E}_+, \hat{E}_-, \hat{V})$, is a complete set of commuting operators, whose associated basis is

$$
|\varphi_+, \varphi_-, v\rangle = \frac{e^{iv\varphi}}{2} [\varphi_A, \varphi_B] + (-1)^v |\varphi_A + \pi, \varphi_B + \pi\rangle,
$$

with $v = 0, 1$ and

$$
\varphi_A = \frac{1}{2}(\varphi_+ + \varphi_-), \quad \varphi_B = \frac{1}{2}(\varphi_+ - \varphi_-).
$$

The complex exponential in the definition (37) has been introduced for convenience, in order to get the same expression $|\varphi_+, \varphi_-, v\rangle$ when $\varphi_A$ and $\varphi_B$ are replaced by $\varphi_A + \pi$ and $\varphi_B + \pi$. Then, the action of $\hat{V}$ in this basis is

$$
\hat{V}|\varphi_+, \varphi_-, v\rangle = (-1)^v |\varphi_+, \varphi_-, v\rangle,
$$

and we have the resolution of the identity

$$
\hat{I} = \sum_{v=0,1} \int_{2\pi} \int_{2\pi} d\varphi_+ d\varphi_- \langle \varphi_+, \varphi_-, v\rangle \langle \varphi_+, \varphi_-, v\rangle.
$$

The proper joint probability distribution $P$ (cast into a $2\pi$ range) for the phase sum and difference associated with a system state $\hat{\phi}$ is

$$
P(\varphi_+, \varphi_-) = \sum_{v=0,1} \langle \varphi_+, \varphi_-, v|\hat{\phi}|\varphi_+, \varphi_-, v\rangle,
$$

which is nothing but the sum of the contributions from each value of $v$. 
We can express $\mathcal{P}(\varphi_+, \varphi_-)$ in terms of the probability distribution for the individual phases $P(\varphi_A, \varphi_B) = \langle \varphi_A, \varphi_B | \hat{\varphi} | \varphi_A, \varphi_B \rangle$ in the form

$$
\mathcal{P}(\varphi_+, \varphi_-) = \frac{1}{2} \{ P[(\varphi_+ + \varphi_-)/2, (\varphi_+ - \varphi_-)/2] \\
+ P[(\varphi_+ + \varphi_-)/2 + \pi, (\varphi_+ - \varphi_-)/2 + \pi] \}.
$$

(42)

Another equivalent procedure is to note that we must get the same mean values for any periodic function of the phase sum and difference whether we use the variables $(\varphi_+, \varphi_-)$ or $(\varphi_A, \varphi_B)$, which translates into

$$
\int_{2\pi} d\varphi_+ d\varphi_- e^{i\varphi_+} e^{i\varphi_-} \mathcal{P}(\varphi_+, \varphi_-) \\
= \int_{2\pi} d\varphi_A d\varphi_B e^{i\varphi_+} e^{i\varphi_-} P(\varphi_A, \varphi_B).
$$

(43)

Since $\mathcal{P}(\varphi_+, \varphi_-)$ and $P(\varphi_A, \varphi_B)$ are $2\pi$-periodic functions, these equalities determine $\mathcal{P}(\varphi_+, \varphi_-)$ completely, as can be shown by using a simple Fourier analysis [45].

We can now generalize the transformation law (42) to any POVM. It is clear that the joint probability distribution function $P(\varphi_A, \varphi_B)$ arises from $\hat{\Delta}(\varphi_A, \varphi_B)$ defined by

$$
\hat{\Delta}(\varphi_A, \varphi_B) = \hat{\Delta}_{\gamma_A}(\varphi_A) \otimes \hat{\Delta}_{\gamma_B}(\varphi_B),
$$

(44)

and then the use of (42) leads to the following POVM for the phase sum and difference cast into a $2\pi$ range

$$
\hat{\Lambda}(\varphi_+, \varphi_-) = \frac{1}{2} \{ \hat{\Delta}[(\varphi_+ + \varphi_-)/2, (\varphi_+ - \varphi_-)/2] \\
+ \hat{\Delta}[(\varphi_+ + \varphi_-)/2 + \pi, (\varphi_+ - \varphi_-)/2 + \pi] \}.
$$

(45)

When focusing on the phase difference, the associated POVM is defined by

$$
\hat{\Lambda}(\varphi_-) = \int_{2\pi} d\varphi_+ \hat{\Lambda}(\varphi_+, \varphi_-),
$$

(46)

which is equivalent to

$$
\Lambda(\varphi_-) = \int_{2\pi} d\varphi' \hat{\Delta}_{\gamma_A}(\varphi_- + \varphi') \hat{\Delta}_{\gamma_B}(\varphi').
$$

(47)

This equation allows us to provide an alternative approach to the fuzzy description of phase [46]. If we consider that the density operator factorizes, $\hat{\varphi}_A \otimes \hat{\varphi}_B$, the phase difference can be regarded as a measure of the phase $\varphi_A$ relative to a given reference state described by $\hat{\varphi}_B$.

V. DEGREE OF ENTANGLEMENT FOR TWO QUBITS

Various measures of entanglement have been proposed [47, 48, 49, 50, 51, 52] so far, each one having its own merits and emphasizing different aspects of the phenomenon. We stress, however, that a closer examination reveals that much of these seemingly unconnected notions are actually identical [51].

Given the distinguished role assigned in this paper to the relative phase, it seems almost compulsory to discuss a possible measure of entanglement within this framework. For simplicity, we shall restrict our attention to the case of two qubits.

Let us first recall some previous well-established definitions. A bipartite state $|\Psi_{A\times B}\rangle$ is said to be factorizable if it can be factored into a product $|\Psi_{A\times B}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, where $|\Psi_A\rangle \in \mathcal{H}_A$ and $|\Psi_B\rangle \in \mathcal{H}_B$, and $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces of the individual qubits. An entangled state is one for which this is not possible. A maximally entangled bipartite state $|\Psi_{\text{max}}\rangle$ satisfies the conditions

$$
\text{Tr}_A(|\Psi_{\text{max}}\rangle \langle \Psi_{\text{max}}|) = \frac{1}{2} \hat{I}_B, \quad \text{Tr}_B(|\Psi_{\text{max}}\rangle \langle \Psi_{\text{max}}|) = \frac{1}{2} \hat{I}_A.
$$

(48)
where \(\text{Tr}_A\) and \(\text{Tr}_B\) stand for tracing over the subspaces \(\mathcal{H}_A\) and \(\mathcal{H}_B\), respectively. Equation (48) implies that each subsystem, when considered alone, is in a maximally mixed state, although the state of the system as a whole is pure.

The general bipartite state of two qubits may be expanded in the \(\{|0\}, |1\}\) bases of \(\mathcal{H}_A\) and \(\mathcal{H}_B\) in the usual form

\[
|\Psi_{AB}\rangle = \alpha_1|0_A0_B\rangle + \alpha_2|0_A1_B\rangle + \alpha_3|1_A0_B\rangle + \alpha_4|1_A1_B\rangle, \tag{49}
\]

where \(\sum_j |\alpha_j|^2 = 1\). However, for our purposes here it proves more convenient to write this state in terms of a Schmidt decomposition \[1\]

\[
|\Psi_{AB}\rangle = \kappa_1|x_1,y_1\rangle + \kappa_2|x_2,y_2\rangle, \tag{50}
\]

where \(\{|x_1\}, |x_2\}\) and \(\{|y_1\}, |y_2\}\) are orthonormal bases of \(\mathcal{H}_A\) and \(\mathcal{H}_B\), respectively, and \(\kappa_1\) and \(\kappa_2\) are real nonnegative coefficients satisfying \(\kappa_1^2 + \kappa_2^2 = 1\) and \(\kappa_1 \geq \kappa_2\). In particular, we shall choose

\[
|x_k\rangle = a_k|0_A\rangle + b_k|1_A\rangle, \\
|y_k\rangle = \alpha_k|0_B\rangle + \beta_k|1_B\rangle, \tag{51}
\]

for \(k = 1, 2\); i.e., the corresponding bases are related by general local unitary transformations (obviously, the coefficients must fulfill constraints in order to ensure the orthonormality of the transformed bases).

Let us assume that the phases of this bipartite system are described by a POVM such as (45). For the general state described in (50) one can easily compute the joint probability distribution \(P(\varphi_+, \varphi_-)\) as

\[
P(\varphi_+, \varphi_-) = \frac{1}{(2\pi)^2} \sum_{k,l} \kappa_k \kappa_l [1 + \gamma_A \gamma_B] \\
\times (a_k a_l b_l^* \beta_l^* e^{i\varphi_+} + b_k \beta_k a_k^* \alpha_k^* e^{-i\varphi_+} + a_k \beta_k b_k^* \alpha_k^* e^{i\varphi_-} + \alpha_k b_k a_k^* \beta_l^* e^{-i\varphi_-})]. \tag{52}
\]

Once obtained this joint probability distribution, we can proceed further by calculating the associated dispersions

\[
D^+ = 1 - \left| \int_{2\pi} d\varphi_+ e^{\pm i\varphi_+} P(\varphi_+, \varphi_-) \right|^2 \\
= 1 - \left( \frac{\gamma_A \gamma_B}{4\pi^2} \right)^2 \left[ \sum_k \kappa_k a_k \alpha_k \right]^2 \left[ \sum_l \kappa_l b_l \beta_l \right]^2,
\]

\[
D^- = 1 - \left| \int_{2\pi} d\varphi_- e^{\pm i\varphi_-} P(\varphi_+, \varphi_-) \right|^2 \\
= 1 - \left( \frac{\gamma_A \gamma_B}{4\pi^2} \right)^2 \left[ \sum_k \kappa_k a_k b_k \right]^2 \left[ \sum_l \kappa_l a_l \beta_l \right]^2. \tag{53}
\]

We stress that these are the only relevant phase-related quantities involved in the problem and, in addition, they are accessible to the experiment \[53\].

We propose to define the degree of entanglement by

\[
\mathcal{D} = \frac{|D^+ - D^-|}{\Gamma}, \tag{54}
\]

where the constant

\[
\Gamma = \left( \frac{\gamma_A \gamma_B}{2\pi^2} \right)^2 \tag{55}
\]

has been chosen so as to normalize \(0 \leq \mathcal{D} \leq 1\). Obviously, for any separable state we have \(\kappa_1 = 1, \kappa_2 = 0\) and then \(\mathcal{D} = 0\). On the opposite limit, for the “magic” Bell basis, which is especially germane for displaying correlations between Alice’s and Bob’s qubits \[1\],

\[
|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0_A1_B\rangle \pm |1_A0_B\rangle), \\
|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0_A0_B\rangle \pm |1_A1_B\rangle). \tag{56}
\]
one can immediately check that $\mathbb{D} = 1$.

As a simple, but nontrivial example, let us consider the family of nonmaximally entangled states parametrized by

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}} \left\{ \varepsilon |0_A\rangle + (1 - \varepsilon) |1_A\rangle \right\}/\mathcal{N} \otimes |0_B\rangle \pm |1_A1_B\rangle, \tag{57}$$

where $\mathcal{N}$ is a constant ensuring the proper normalization of the state. Note that when $\varepsilon \to 1$ these states become the (maximally entangled) Bell states $|\Phi^\pm\rangle$, while for $\varepsilon \to 0$ they tend to the (separable) states $|1_A\rangle(0_B\rangle \pm |1_B\rangle)/\sqrt{2}$. For this family one has

$$\mathbb{D} = \frac{\varepsilon^2}{\mathcal{N}^2}. \tag{58}$$

Apart from mathematical subtleties, we think that the appeal of this new measure is that it relies on observable quantities that can be recast in terms of measurements of two-particle visibility [47]. It is a general belief [2] that the information encoded by these bipartite entangled states only lies in the relative qubit properties and not in the local ones. This is exactly the purpose of the parameter $\mathbb{D}$: using phase sum and phase difference this property is fully brought out.

VI. CONCLUSIONS

In this paper we have investigated a description of the phase for a qubit in terms of a proper polar decomposition of its amplitude, much in the spirit of our previous work on the subject. Perhaps, the most striking consequence of this description is that such a phase can take only two values: $\pm \pi/2$.

We have also considered some other generalized descriptions in terms of POVMs. Although these formalisms give different results, they share a lot of properties. In particular, we have shown an effective discreteness even if, in principle, a continuous range of variation is assumed.

We have discussed the subtleties that arise when considering the relative phase for two qubits. We have presented a procedure to cast individual phases to phase sum and phase difference. The relative phase that emerges from this casting procedure is a powerful tool for examining entanglement. In particular, we have proposed a measure of entanglement involving only relative-phase dispersions, which are physically measurable quantities.
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[53] B. Julsgaard, A. Kozhinkin, and E. Polzik, Nature 413, 400 (2001).