We analyse the ultraviolet divergencies in the ground state energy for a penetrable sphere and a dielectric ball. We argue that for massless fields subtraction of the “empty space” or the “unbounded medium” contribution is not enough to make the ground state energy finite whenever the heat kernel coefficient $a_2$ is not zero. It turns out that $a_2 \neq 0$ for a penetrable sphere, a general dielectric background and the dielectric ball. To our surprise, for more singular configurations, as in the presence of sharp boundaries, the heat kernel coefficients behave to some extend better than in the corresponding smooth cases, making, for instance, the dilute dielectric ball a well defined problem.

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I. INTRODUCTION

The renewed interest in the problem of the ground state energy for the electromagnetic field in the presence of a dielectric body is triggered by the Schwinger suggestion [1] that the Casimir effect may serve as an explanation for the sonoluminescence (for a review, see [2]). The results by different groups are controversially [3–5] and although there is a large number of papers on this topic, basic issues are still to be clarified. As it will turn out, the main difficulty is a proper formulation of the problem in the sense of finding the right physical setup.

There are two ways to calculate the Casimir energy for a given configuration. The first way consists of summing up the retarded Van der Waals forces between individual molecules. The second way, which is studied in the present paper, is to employ the technique of quantum field theory under the influence of external conditions. (For a dilute dielectric ball the two methods have been shown to yield the same answer [6] and in addition they are in agreement with a calculation of the Casimir energy treating the dielectric ball as a perturbation [7].) This latter way is based on the evaluation of closed vacuum loops with suitably modified propagators. Ultraviolet divergencies are then removed by means of a renormalization procedure. While in the case of conducting boundaries it is sufficient to subtract the empty space contribution in order to make the vacuum energy finite, this is not the case for penetrable boundary conditions, like the dielectric ball for instance. The main aim of the present paper is to study the ultraviolet divergencies for the electromagnetic field in the latter case.

The problem can be formulated as follows. Consider a macroscopic body whose electromagnetic properties are described by the permittivity $\epsilon(x)$, which may be position dependent, i.e., which shows dispersion. The function $\epsilon(x)$ enters the photon propagator as an external field respectively as a background. Ultraviolet divergencies of the ground state energy of the electromagnetic field become functionals of $\epsilon$. In principle, it should be possible to interpret the ultraviolet divergencies inherent to the ground state energy in a way which yields a well defined, unique result. This means that a classical model for the body must be found which allows to absorb the divergent contributions (counter terms) into a redefinition of some constants or some meaningful renormalisation condition must be found.

In a standard approach one uses the heat kernel expansion in order to determine the counter terms. In $(3+1)$ dimensions it is the coefficient $a_2$ which must be known for a massless field, having in mind the electromagnetic field for instance. By now, it is still unknown for a dielectric ball. The problem is that such a background is not differentiable and the standard recursive formulas cannot be applied. On the other hand, the known results for
manifolds with boundary (e.g. with Dirichlet boundary conditions) do not apply because the dielectric ball results in some penetrable boundary conditions. Without attempting to solve this problem in general, we calculate the relevant heat kernel coefficients in two sufficiently characteristic examples, first a spherical surface carrying a delta function potential (‘delta sphere’, \(V(r) \sim \delta(r-R)\)) which can be reformulated in terms of matching conditions at \(r=R\), and second, the dielectric ball with constant permittivity and permeability inside and outside. In both cases we discuss in parallel the corresponding formulas for a smooth background.

Before actually calculating the heat kernel coefficients we discuss the renormalization scheme in general, especially the problem of finding a normalization condition. We argue that there is a natural condition for the massive field and show the difficulties for the massless case caused by the conformal anomaly.

Because of the contradicting results mentioned above we found it useful to reconsider the quantization procedure for the Maxwell field in a dielectric. In our previous paper [15] the quantization in terms of usual vector potentials \(A_\mu\) was considered. We observed no cancellation between ghosts and “non-physical” modes. Here we re-examine this problem in terms of the so-called dual potentials. We give a rigorous proof that for the spherically symmetric dielectric ghost contribution can be neglected. We also show that the both approaches give identical results for the heat kernel coefficient \(a_2\) in the dilute approximation.

The paper is organized as follows. In the next section we discuss the renormalization scheme, in sections 3 and 4 we calculate the heat kernel coefficients for a delta sphere and for the dielectric ball, the results are discussed in section 5. Some formulas are given in the appendix.

II. RENORMALIZATION AND CONFORMAL ANOMALY

Let us consider the ground state energy of a massive quantum field in the zeta-function regularisation,

\[
\mathcal{E}_0 = \frac{\mu^{2s}}{2} \zeta_P(s - \frac{1}{2}),
\]

(1)

where

\[
\zeta_P(s) = \sum_{(n)} (\lambda(n) + m^2)^{-s}
\]

(2)

is the zeta function of the operator \(P\) corresponding to a given dynamical system,

\[
P\phi_{(n)}(\vec{x}) = \lambda(n)\phi_{(n)}(\vec{x}).
\]

(3)

The index \((n)\) includes discrete and continuous parts in dependence of the problem considered. An arbitrary parameter \(\mu\) has the dimension of a mass. The zeta function is connected with the heat kernel by means of

\[
\zeta(s) = \int_0^\infty dt \frac{t^{s-1}}{\Gamma(s)} K(t).
\]

(4)

The first few terms (up to \(n = 2\) in \((3+1)\) dimensions) of its asymptotic expansion for \(t \to 0\)

\[
K(t) \sim \frac{e^{-tm^2}}{(4\pi t)^{3/2}} \sum_n a_n t^n
\]

(5)

yield the ultraviolet divergencies of the ground state energy \([1]\). From that we define the ’divergent part’ of the ground state energy as \([16]\)

\[
\mathcal{E}_0^{\text{div}} = -\frac{m^4}{64\pi^2} \left( \frac{1}{s} + \ln \frac{4\mu^2}{m^2} - \frac{1}{2} \right) a_0 - \frac{m^3}{24\pi^{3/2}} a_{1/2}
+ \frac{m^2}{32\pi^2} \left( \frac{1}{s} + \ln \frac{4\mu^2}{m^2} - 1 \right) a_1 + \frac{m}{16\pi^{3/2}} a_{3/2}
- \frac{1}{32\pi^2} \left( \frac{1}{s} + \ln \frac{4\mu^2}{m^2} - 2 \right) a_2.
\]

(6)

Now the renormalised ground state energy can be defined as
\[ \mathcal{E}^{\text{ren}}_0 = \mathcal{E}_0 - \mathcal{E}^{\text{div}}_0 \]

in the limit \( s \to 0 \). In general, the renormalisation is not unique as it is obvious already from the presence of the parameter \( \mu \). Therefor some normalisation condition is necessary.

In general, it depends on the considered physical problem how to handle this non uniqueness. The simplest example is the Casimir effect between two separate reflecting bodies with surfaces \( \partial M_1 \) and \( \partial M_2 \). In flat space and without background potential the coefficients \( a_n \) are integrals over the surface and for \( \partial M_1 \cap \partial M_2 = \emptyset \) they are independent of the distance between the bodies. Therefor all divergent and non unique contributions do not depend on the distance and a force as a physical quantity can be defined. Note that this is sufficient for all current experiments on Casimir force measurements. This applies especially to the case where \( \partial M_1 \) and \( \partial M_2 \) are nonintersecting spheres. But it does not apply to the case of a single sphere when asking for the pressure on its surface, for instance.

In the case of a massive field there is a natural normalisation condition. It is the requirement that the renormalised ground state energy must vanish for the mass \( m \) becoming large,

\[ \mathcal{E}^{\text{ren}}_0 \to 0 \quad \text{for} \quad m \to \infty. \]  

(8)

Because the heat kernel expansion is equivalent to the asymptotic expansion of the ground state energy for large \( m \) (provided there is no other dependence on \( m \) than in Eq. (3)), the definition (3) of the divergent part through the heat kernel coefficients is just equivalent to (3). This condition had been used in [16] for the calculation of the ground state energy of a massive scalar field with boundary conditions on a sphere.

Regrettably, this normalisation condition cannot be extended to the case of a massless field. In general, for \( m = 0 \), all dangerous contributions are proportional to \( a_2 \). In case \( a_2 = 0 \) there is no divergence and no arbitrariness at all and, hence, no need for a normalisation condition. But for \( a_2 \neq 0 \) the limit \( m \to 0 \) cannot be performed in \( \mathcal{E}^{\text{ren}}_0 \) as can be seen quite easily. For this we observe that the ground state energy (11) can be expanded into a power series for small \( m \) provided there are no zero modes (we assume all \( \lambda^{(n)} > 0 \))

\[ \mathcal{E}_0 = \frac{1}{2} \left( \sum_{(n)} \lambda^{1/2, n} + m^2 \sum_{(n)} \lambda^{(n)} - \frac{1}{2} \right. \left. + \ldots \right). \]  

(9)

Now, by means of (7) and (6) we subtract a contribution containing \( (a_2/16\pi^2) \log m \). Therefor, the behaviour of the renormalised ground state energy \( \mathcal{E}^{\text{ren}}_0 \), eq. (3), for small \( m \) is

\[ \mathcal{E}^{\text{ren}}_0 \sim - \frac{a_2}{16\pi^2} \log m. \]  

(10)

Consequently, there is no transition from the properly normalised ground state energy of a massive field to the corresponding massless case as long as \( a_2 \neq 0 \).

A more general procedure for handling the divergencies is the following. One considers the quantum field in the background of some classical system. One may think of quantised matter fields in the background of the classical gravitational field or of quantum fluctuations around a classical solution. The minimal structure of the background is determined by the heat kernel coefficient \( a_n \) with \( n = 0, \ldots, 2 \) which are certain functions respectively functionals of the background. In fact, the structure of the energy of the classical system is determined by \( \mathcal{E}^{\text{div}}_0 \), eq. (3). In writing

\[ \mathcal{E}^{\text{class}} = pa_0 + \sigma a_{1/2} + Fa_1 + ka_{3/2} + ha_2, \]  

(11)

where \( p, \sigma, F, k, h \) are the parameters of the classical energy, the subtraction in (3) can be interpreted as a redefinition of these parameters. They remain undetermined and, hence, cannot be predicted from the calculation of the ground state energy. In fact, by means of this, the heat kernel coefficients give the physical prediction which contributions are of quantum origin and which are of classical nature.

Again, a remark must be added for the massless case. Consider, for example, a scalar field with boundary conditions on a sphere. The dependence of the eigenvalues on the radius is \( \lambda^{(n)} = \lambda^{(n)}(R)/R^2 \) where \( \lambda^{(n)} \) are numbers (depending on the kind of boundary conditions). Hence the ground state energy can be written as

\[ \mathcal{E}_0 = \frac{(R^\mu)^{2s}}{2R} \zeta_p(s - 1/2), \]  

(12)

where \( \zeta_p \) is the zeta function for \( m = 0 \) and \( R = 1 \). In case \( a_2 \neq 0 \), one has the expansion

\[ \zeta(s - 1/2) = -\frac{a_2}{16\pi^2} s + \bar{h} + O(s), \]
where $\tilde{a}_2 = a_2(R = 1)$ and $\tilde{h}$ is some number which cannot be calculated from the asymptotic expansion. The ground state energy becomes

$$
E_0 = \frac{-\tilde{a}_2}{32\pi^2 R} \left( \frac{1}{s} + \log(R\mu)^2 \right) + \frac{\tilde{h}}{R}.
$$

(13)

The appearance of $\log R$ is just a result of the conformal anomaly. This was clearly stated in [7], where the scaling behavior of the Casimir energy was investigated (see also [9]). In order to setup the renormalisation scheme one has to introduce the classical energy

$$
\mathcal{E}_{\text{class}} = \frac{h}{R},
$$

(14)

which allows to absorb the pole term into a redefinition of the parameter $h$. However, the contribution $\tilde{h}/R$ in $E_0$ which could be viewed as the genuine result of the calculation of the ground state energy (sometime it is called the nonlocal contribution) cannot be distinguished from the classical part and its calculation does not have a predictive power. Thus the only outcome from the calculation of the ground state energy is the contribution containing $\log R$. This consideration applies in the case $a_2 \neq 0$. No renormalization procedure and no classical system are required in case $a_2 = 0$. The ground state energy can be calculated directly and becomes

$$
E_0 = \frac{\tilde{h}}{R}.
$$

(15)

We remind that for the scalar field with Dirichlet or Robin boundary conditions and for the electromagnetic field with conductor boundary conditions, the coefficient $a_2$ is the same except for the sign for the interior and the exterior regions (the reason being that the extrinsic curvature has opposite signs on both sides). Therefore, when considering the ground state energy in the whole space, $a_2$ is zero and a result like (15) can be obtained. An example for this is just the well known Casimir effect for a conducting sphere, where 'delicate' cancellations have been observed in early papers already. In contrary, when considering only the interior (or the exterior or an odd dimensional spacetime), one has $a_2 \neq 0$ and only the logarithmic contribution in (13) is the part which it is meaningful to calculate, i.e., it is sufficient to calculate $a_2$.

From this point of view, let us consider a conducting sphere of finite thickness. Let a massless scalar field be given in the regions $0 \leq r \leq R_1$ $\cup$ $(R_2 \leq r < \infty)$, $R_1 < R_2$, obeying Dirichlet boundary conditions at $r = R_1$ and $r = R_2$. Let the field be zero in between the two spherical shells having in mind an ideally conducting medium there. The heat kernel coefficient is $\tilde{a}_2 = -16\pi/315$. In zeta functional regularization, the ground state energy for a single sphere is given by eq. (13). From that we obtain for two concentric spheres

$$
E_0 = \frac{1}{630\pi} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \left( \frac{1}{s} + \log(R\mu)^2 \right) + \frac{\tilde{h}_1}{R_1} + \frac{\tilde{h}_2}{R_2}.
$$

(16)

where $\tilde{h}_1 \neq \tilde{h}_2$ are the 'genuine', nonlocal contributions of the ground state energy inside respectively outside a sphere of unit radius.

Now, by writing

$$
\frac{\tilde{h}_1}{R_1} + \frac{\tilde{h}_2}{R_2} = \tilde{h}_1 \left( \frac{1}{R_1} - \frac{1}{R_2} \right) + \frac{\tilde{h}_1 + \tilde{h}_2}{R_2}
$$

we can define a classical energy

$$
\mathcal{E}_{\text{class}} = h \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
$$

(17)

where $h$ is some free parameter, and the renormalised energy by (11)

$$
\mathcal{E}_0^{\text{ren}} = \frac{\tilde{h}_1 + \tilde{h}_2}{R_2},
$$

which delivers a meaningful expression for a pressure on the surface at $r = R_2$.

By writing
\[
\frac{\tilde{h}_1}{R_1} + \frac{\tilde{h}_2}{R_2} = -\tilde{h}_2 \left( \frac{1}{R_1} - \frac{1}{R_2} \right) + \frac{\tilde{h}_1 + \tilde{h}_2}{R_1}
\]

we get a similar result on the inner surface at \( r = R_1 \). Thus, for the given configuration, one is forced to introduce a classical model with the energy (17) in order to obtain a renormalised Casimir energy. However, although this procedure might look natural, it isn’t. So in the first case, the renormalized ground state energy is independent on the inner radius which seems unlikely to be reasonable.

In summary, whereas for the massive field a unique normalized Casimir energy may be defined by considering the \( m \to \infty \) behaviour, for the massless case, as seen for different situations, finite renormalizations remain undetermined as soon as the heat kernel coefficient \( a_2 \neq 0 \).

### III. PENETRABLE SPHERICAL SHELL

In the preceding section we discussed the role of the coefficient \( a_2 \) for the ground state energy, for instance the well known fact that there is a cancellation between the two sides of a surface dividing the quantisation volume. Here we show that this is a rather special case and that generally the renormalization ambiguity remains. We start from the known example of a smooth background field \( V(\vec{x}) \). The operator \( P \) in (3) is

\[
P = -\Delta + V(\vec{x})
\]

and the corresponding heat kernel coefficients are

\[
a_1 = -\int d\vec{x} \ V(\vec{x}) , \quad a_2 = \frac{1}{2} \int d\vec{x} \ V(\vec{x})^2 .
\]

The coefficient \( a_2 \) can vanish only in the trivial case of a vanishing background potential. Consequently, the presence of \( a_2 \), i.e., of the conformal anomaly, is a rather general case. The question arises how this is connected with the cancellation of \( a_2 \) in the presence of boundaries. As is known, there is no transition in the coefficients from a smooth background to boundary conditions. For instance, boundary conditions lead to coefficients with half integer numbers which are not present in (19). In the following we consider a singular background potential

\[
V(r) = \frac{\alpha}{R} \delta(r - R) ,
\]

which can be viewed as standing in the gap between a smooth background and boundary conditions. The delta function potential (20) in the operator \( P \), eq. (18), can be replaced by penetrable boundary conditions, namely by the matching conditions

\[
\phi_{r=R} = 0 , \quad \phi'_{r=R} = 0 \quad \text{for Dirichlet boundary conditions}.
\]

For \( \alpha > 0 \) this potential is repulsive and in the formal limit \( \alpha \to \infty \) Dirichlet boundary conditions at \( r = R \) are restored. The Casimir force between two planes carrying delta potentials had been calculated in [19,20] and the similar problem for dissipative mirrors in [21]. In these cases one had been interested in the distance dependent contributions only.

For the calculation of the ground state energy and the heat kernel coefficients we adopt the technique developed in [22] for a smooth background potential. The starting point is the expression of the ground state energy in zeta functional regularisation

\[
E_0 = -\cos \frac{\pi s}{2} \sum_{l=0}^{\infty} \nu \int_0^\infty dk \ (k^2 - m^2)^{1/2-s} \frac{\partial}{\partial k} \log f_l(i k)
\]

with \( \nu = l + 1/2 \). Here, \( f_l(i k) \) is the Jost function of the scattering problem associated with the operator \( P \), eq. (20), on the imaginary axis. It can be found by standard methods and reads

\[
f_l(i k) = 1 + \alpha I_\nu(kR)K_\nu(kR) .
\]
where $I_\nu$ and $K_\nu$ are the modified Bessel functions. In order to calculate the heat kernel coefficients it is sufficient to calculate the pole part of $E_0$ in $s = 0$. For this reason we use the uniform asymptotic expansion of the Jost function for $k \to \infty, \nu \to \infty$. From the corresponding asymptotic expansions of the Bessel functions [22] we obtain

$$\log f^{as}_l(ik) = \sum_{n=1}^{3} \sum_i X_{n,i} f^i_n$$

with $t = 1/\sqrt{1 + (kR/\nu)^2}$ and $X_{1,1} = \alpha/2, X_{2,2} = \alpha^2/8, X_{3,3} = \alpha/16 + \alpha^3/24, X_{3,5} = -3\alpha/8, X_{3,7} = 5\alpha/16$. Higher terms of the expansion do not contribute to the pole. Inserting $\log f^{as}_l$ into the representation (22) we denote the corresponding part of the ground state energy by $E^{as}_l$. There the integration over $k$ can be carried out explicitly (e.g., using formula (31) in [22]). After that, the sum over $l$ will be converted into two integrals by means of

$$\sum_{l=0}^{\infty} f(l + \frac{1}{2}) = \int_0^\infty d\nu \ f(\nu) - i \int_0^\infty \frac{d\nu}{1 + e^{2\pi \nu}} (f(i\nu) - f(-i\nu))$$

and $E^{as}$ splitted into two parts $E^{as} = E^{as,1} + E^{as,2}$ accordingly. In the first part the integral over $\nu$ can be carried out explicitly and we obtain

$$E^{as,1} = \frac{m^2}{32\pi^2} \left( \frac{1}{s} + \frac{4\mu^2}{m^2} - 1 \right) (-4\pi \alpha R) + \frac{m}{16\pi^{3/2}} (\pi^{3/2} \alpha^2)$$

$$- \frac{1}{32\pi^2} \left( \frac{1}{s} + \frac{4\mu^2}{m^2} - 2 \right) \left( \frac{\pi \alpha}{3R} (1 - 2\alpha^2) \right) .$$

In $E^{as,2}$, from $X_{1,1}$ after some calculation a further pole contribution results

$$E^{as,2} = - \frac{1}{32\pi^2} \left( \frac{1}{s} + \frac{4\mu^2}{m^2} - 2 \right) \left( \frac{-\pi \alpha}{3R} \right) + O\left( \frac{1}{m} \right) ,$$

where all other contributions vanish for $m \to \infty$ and hence do not contribute to the considered heat kernel coefficients. Comparing these expressions with (19) we read off the coefficients

$$a_{1/2} = 0, \quad a_1 = -4\pi \alpha R, \quad a_{3/2} = \pi^{3/2} \alpha^2, \quad a_2 = -\frac{2\pi \alpha^3}{3R}$$

(25)

(a0 does not depend on the potential $V(r)$).

From these formulas we see that the coefficients $a_{1/2}$ and $a_1$ are the same as for a smooth background. In fact, in [19] $a_{1/2} = 0$ and $a_1$ equals (22) when inserting $V(r)$, eq. (24). In contrast, $a_{3/2}$ in (23) has no counterpart in the smooth case, which is not surprising as in the case of boundary conditions coefficients with half integer numbers are present. Finally, the coefficient $a_2$ in (24) cannot vanish except for the trivial case $\alpha = 0$ and it cannot be obtained from $a_2$, eq. (15), because of a squared delta function which would appear. It is interesting to note that contributions linear in $\alpha$, which are present in intermediate expressions $E^{as,1}$ and $E^{as,2}$, cancelled. We will see below that for the dielectric ball there is a similar cancellation. In order to renormalize the groundstate energy one needs to introduce the classical part $E^{class} = \hbar a_2$, see eq. (11), of the energy and there is no condition at hand to fix the free parameter $\hbar$.

IV. DIELECTRIC BACKGROUND

There is a general interest in a careful consideration of the quantisation procedure in a dielectric background. The ground state energy is the sum over all states of the considered Hilbert space, physical as well as unphysical ones. This is in opposite to the usual situation where one has to consider expectation values in states with particles, i.e., in states of the physical subspace. Therefore the unphysical states, including ghost states, must be considered carefully. A general investigation whether the contributions of the ghosts and of the unphysical photons cancel each other is still to be done. So a reconsideration of the in general well known quantisation procedure of QED in a dielectric background is in order. Thereby we consider two quantisation procedures, first in terms of the electromagnetic potentials and, secondly, in terms of the dual potentials. In fact, the first case was already done in [15]. We reconsider it here in order to derive specific expressions in the spherically symmetric case.
The action for the electromagnetic field in a medium with permittivity $\epsilon(x)$ and permeability $\mu = 1$ is

$$S = \frac{1}{2} \int d^4x (\epsilon(x) E^2 - B^2).$$  \hfill (26)

First we consider the case of $\epsilon(x)$ being a sufficiently smooth function so that the necessary number of derivatives exists. Furthermore, we assume that $\epsilon(x)$ goes sufficiently fast to a constant value at spatial infinity such that integrals appearing later on are well defined. Volume divergencies still present have to be subtracted by hand. These terms are associated with the contributions of an unbounded medium, where $\epsilon$ everywhere takes its value at infinity. As is well known, for the calculation of the ground state energy in gauge field theories care must be taken in order not to loose modes, ghost contributions for instance. This had been done in [15]. There, after introducing standard vector potentials, a gauge fixing term, ghost fields, and taking into account the proper functional measure, the path integral takes the form

$$Z = \int D\tilde{A}_i DA_0 Dc D\bar{c} \exp \left\{ i \int d^4x \left[ \frac{1}{4} (2\epsilon(x) (\partial_0 \epsilon)^{-1/2} \tilde{A}_i - \partial_i A_0)^2 
- (\partial_0 \epsilon^{-1/2} \tilde{A}_k - \partial_k \epsilon^{-1/2} \tilde{A}_i)^2 + \mathcal{L}_{gf} + \mathcal{L}_{ghost} \right] \right\}$$  \hfill (27)

with $\tilde{A}_i = \sqrt{\epsilon} A_i$ and

$$\mathcal{L}_{gf} = -\frac{1}{2} (\epsilon^{-1} \partial_i \epsilon^{1/2} \tilde{A}_i - \epsilon \partial_0 A_0)^2,$$

$$\mathcal{L}_{ghost} = -\bar{c} (\epsilon^{-1} \partial_i \epsilon \partial_i + \epsilon \partial_0^2) c.$$

The action for the electromagnetic field $A$ is then found to be

$$\frac{1}{2} \int d^4x \left[ \epsilon_0 (\partial_0 A_0)^2 - \epsilon^2 (\partial_0 \tilde{A}_i)^2 + (\partial_0 A_0)^2 
+ \tilde{A}_i \epsilon^{-1/2} (\partial_k^2 \delta_{ij} - e_i \partial_k + e_j \partial_k - e_i e_k) \epsilon^{-1/2} \tilde{A}_k] \right],$$

with the notation $e_i = \partial_i \ln \epsilon$, and later also $e_{ij} = \partial_i \partial_j \ln \epsilon$ and so on.

Now we are able to integrate over $A_0$, $\tilde{A}$ and the ghosts. The resulting path integral reads after Wick rotation to the Euclidean domain

$$Z = Z[A_0] Z[\tilde{A}] Z[\bar{c}, c],$$

where the separate contributions are of the form:

$$Z[A_0] = \det^{-1/2} (-\partial_i \epsilon \partial_i - \epsilon^2 \partial_0^2),$$

$$Z[\tilde{A}] = \det^{-1/2} \left( \frac{1}{\epsilon} \partial^2 \delta_{ij} - \partial_0^2 \delta_{ij} - e_i \epsilon \partial_j + e_j \epsilon \partial_i - \frac{1}{\epsilon} (e_{ij} - e_i e_j) \right),$$

$$Z[\bar{c}, c] = \det(\epsilon^{-1} \partial_i \epsilon \partial_i - \epsilon \partial_0^2).$$

For the functional determinants we use the integral representation

$$\log \det(L) = \int_0^\infty dt \frac{dt}{t} K(L; t),$$

where the heat kernel $K(L; t)$ for a second order elliptic operator $L$ is

$$K(L; t) = \text{Tr} \exp(-tL).$$

In [15] we calculated the coefficients $a_n$ of the asymptotic expansion of the heat kernel

$$K(\exp(-tL)) = (4\pi t)^{-\frac{n}{2}} \sum_{n=0}^\infty t^n a_n(L).$$  \hfill (28)
Note that here $L$ is a four-dimensional operator. In eq. (3) above $P$ is a three-dimensional operator. This explains different powers of $4\pi t$ in (3) and (28). Let us list the individual contributions to the different coefficients (summation over repeated indices is assumed). First

\[
\begin{align*}
a_0^{gh} &= \int d^4xe^{-1/2} \\
a_0^{A_0} &= \int d^4x e^{-5/2} \\
a_0^\mathcal{A} &= 3 \int d^4x e^{3/2}. 
\end{align*}
\]

For $a_1$ we obtain

\[
\begin{align*}
a_1^{gh} &= \int d^4x e^{-1/2} \left( -\frac{7}{12} e_{ii} - \frac{13}{48} e_{ij} e_{ij} \right), \\
a_1^{A_0} &= \int d^4x e^{-3/2} \left( -\frac{7}{12} e_{ii} - \frac{13}{48} e_{ij} e_{ij} \right), \\
a_1^\mathcal{A} &= \int d^4x e^{1/2} \left( \frac{3}{4} e_{ii} - \frac{1}{16} e_{ij} e_{ij} \right), 
\end{align*}
\]

and for the coefficient $a_2$ the result is

\[
\begin{align*}
a_2^{gh} &= \int d^4x e^{-1/2} \frac{1}{360} \left( -33 e_{ii} e_{jj} - 18 e_{ij} e_{ij} - 33 e_{ij} e_{ij} + \frac{237}{4} e_{ii} e_{jj} \\
&\quad+ \frac{531}{8} e_{ij} e_{ij} + \frac{33}{4} e_{ij} e_{ij} e_{ij} + \frac{837}{64} e_{ij} e_{ij} e_{ij} \right), \\
a_2^{A_0} &= \int d^4x e^{-1/2} \frac{1}{360} \left( -27 e_{ii} e_{jj} - 60 e_{ij} e_{ij} - 41 e_{ij} e_{ij} + \frac{119}{4} e_{ii} e_{jj} \\
&\quad- \frac{91}{8} e_{ij} e_{ij} + \frac{415}{8} e_{ij} e_{ij} e_{ij} + \frac{4411}{64} e_{ij} e_{ij} e_{ij} \right) + O(t^3), \\
a_2^\mathcal{A} &= \int d^4x e^{-1/2} \frac{1}{360} \left( 33 e_{ii} e_{jj} - 63 e_{ij} e_{ij} + 231 e_{ij} e_{ij} - \frac{723}{4} e_{ii} e_{jj} \\
&\quad- \frac{1029}{4} e_{ij} e_{ij} + \frac{793}{8} e_{ij} e_{ij} e_{ij} + \frac{4263}{16} e_{ij} e_{ij} e_{ij} \right)
\end{align*}
\]

(in the first line of $a_2^\mathcal{A}$ we corrected some typo's which appeared in [15]). Because the background, which we consider here, is static we can drop the integration over $x_0$. The contributions must be summed up according to

\[
a_n = a_n^{A_0} + a_n^\mathcal{A} - 2a_n^{gh}. \tag{29}
\]

The expression for $a_2$ appearing in this way represents the necessary counterterm which is required to perform the renormalisation of the ground state energy of the electromagnetic field. As it is seen, this is a quite complicated functional of $\epsilon(x)$. It is interesting to consider the case of $\epsilon$ close to unity (dilute dielectric medium). By means of $\epsilon = 1 + \delta$, $e_i = \delta_i/(1 + \delta)$, etc. we obtain

\[
\begin{align*}
a_0 &= \int d\vec{x} \left( 2 + 3\delta + \frac{19}{4}\delta^2 \right) + O(\delta^3) \\
a_1 &= \int d\vec{x} \left( -\frac{11}{24} \right) \delta_i^2 + O(\delta^3)
\end{align*}
\]

and

\[
a_2 = \int d\vec{x} \frac{69}{720} \delta_i^2 + O(\delta^3). \tag{30}
\]

In $a_1$ and $a_2$ there are no terms of the first order, they cannot be present for symmetry and dimensional reasons. (In some detail, the coefficient $a_1$ has the dimension of length, $a_2$ of $1/\text{length}$ and there is no possibility to have
nonvanishing terms linear in $\delta$ which do not integrate away.) But the terms of the second order are present. Moreover, they cannot vanish (unless $\epsilon$ is globally constant).

From this one can conclude, that for a smooth $\epsilon(x)$ the ultraviolet divergencies in the ground state energy of the electromagnetic field are present generically and can cancel at most for very specific choices of $\epsilon$ but not for small $\epsilon - 1$ as shown by (31).

The functional dependence of the needed counterterms is very involved and the classical model necessary for renormalization lacks physical intuition as well as motivation.

So we obtained the relevant heat kernel coefficients for a $\epsilon(x)$ which must be at least for times differentiable in order to deliver a finite result when inserted into the equation for $a_2$. When restricting to a spherically symmetric situation, the quantization approach using dual potentials will provide several advantages. (For the general formalism of dual potentials see [24].) Probably most important is, that the calculation of the relevant determinant can be reduced to the problem of two scalar fields. This procedure can be taken over to the non-smooth dielectric ball treated below and for that reason is described in some detail.

In this approach one starts with the action

$$ S = -\frac{1}{2} \int d^4x \left( \frac{1}{\epsilon} \vec{B}^2 - \vec{B}_0^2 \right), $$

which generates the Maxwell equations in dielectric medium with permittivity $\epsilon(x)$. Here $\vec{D} = \epsilon \vec{E}$. Introducing the dual potentials $C_\mu$: $\vec{D} = \vec{\nabla} \times \vec{C}$, $\vec{B} = \partial_t \vec{C} - \vec{\nabla} C_0$, one gets the canonical Poisson brackets,

$$ \{C_i(x,t), B^j(y,t)\} = \delta_i^j \delta(x - y), $$

thus $\vec{B}$ is the momentum conjugate to $\vec{C}$.

One can rewrite the action (31) in the canonical first-order form,

$$ S = \int d^4x \left( \vec{B} \partial_t \vec{C} + C_0(\vec{\nabla} \vec{B}) - \frac{1}{2} \vec{B}^2 - \frac{1}{2\epsilon} \vec{B}_0^2 \right). $$

(33)

$C_0$ plays the role of the Lagrange multiplier.

The theory (33) is then quantized according to the general method [25]. The path integral measure reads

$$ d\mu = DC_j DB^k DC_0 J_{FP} \delta(\chi), $$

(34)

where $\chi$ is a gauge fixing condition, $J_{FP}$ is the Faddeev–Popov determinant, $J_{FP} = \det \{\chi, (\vec{\nabla} \vec{B})\}$. For $\chi$ we choose the condition

$$ \chi = \nabla_i C^i = 0. $$

(35)

The Faddeev–Popov determinant becomes

$$ J_{FP} = \det(-\nabla^i \nabla_i). $$

(36)

The ghost operator in (36) is non-elliptic, and, therefore, the determinant (36) is ill-defined. Such operators must be treated with great care in order not to break the gauge invariance of the theory (see e.g. [24]). Fortunately, in our case $J_{FP}$ does not depend on $\epsilon$ and can be neglected altogether. Moreover, at the expense of some technical complications we can obtain an equivalent result in the Lorentz type gauge $\nabla^\mu C_\mu = 0$ where no such problems occur.

Now one can perform the intergrations over all fields arriving at the following expression for the path integral,

$$ Z[\epsilon] = \int d\mu \exp(iS) = \int DC_{\perp} \exp(i\overline{S}), $$

(37)

where the subscript $\perp$ denotes transversal 3-vectors, $\nabla_i C^i_{\perp} = 0$.

Two substantial simplifications occur here compared to the quantization in terms of non-dual vector potentials $A_\mu$ described before [27]. First of all, the gauge condition on $A_\perp$ [27], which removes the mixing between $A_0$ and $A_i$, depends on $\epsilon$. As a consequence, the ghost operator depends on $\epsilon$ as well and cannot be neglected. Also, in the case of non-dual potentials the term quadratic in momenta is multiplied by $\epsilon^{-1}$. After momentum integration it leads to a modification of the path integral measure.

Even though the path integral (37) is Gaussian its direct evaluation is not easy. The corresponding determinant is to be computed on the space of transversal vectors. To use standard heat kernel methods one must extend the
functional space to all unconstrained vector fields at the expense of introducing a compensating scalar determinant. This procedure is equivalent to introducing ghost fields. However, if the function $\epsilon(x)$ is spherically symmetric, one can reduce the determinant arising from (37) to two scalar ones by making use of standard TE and TM modes. We turn to this case, now.

The transversality equation (37) can be solved by introducing two independent scalar modes $\phi$ and $\psi$,

$$C_r = r \Delta \psi,$$
$$C_a = -\vec{\nabla}_a \partial_r r \psi + r \epsilon_{ab} \partial_b \phi. \quad (38)$$

Here the coordinates $x^a$ denote angular variables, $\Delta$ and $\vec{\nabla}$ are the Laplace operator and covariant derivative on the two-sphere. $\epsilon_{ab}$ is the two-dimensional Levi–Civita tensor.

It is clear from (38) that the scalar harmonics with zero angular momentum do not give rise to any transversal vector fields. Such harmonics must be excluded from the path integral.

The normalization conditions for the scalar modes one can read off from the integral

$$\int r^2 dr d\Omega C_i C^i = \int r^2 dr d\Omega \left( \psi \Delta (\Delta r^2) \psi + \phi (-\Delta r^2) \phi \right), \quad (39)$$

where $d\Omega$ is the measure of the angular integration.

By substituting (38) in the action (31) we obtain

$$S = \int r^2 dr d\Omega \left[ \psi (\epsilon_0^2 + \frac{1}{\epsilon} \Delta) \psi + \phi (-\epsilon_0^2 + \frac{1}{\epsilon} \Delta + \frac{1}{r}) \phi \right]. \quad (40)$$

We can now evaluate the path integral (37). Taking into account the Jacobian factors appearing due to (39), one arrives at the following expression

$$Z[\epsilon] = \det^{-1/2} (L_\psi) \det^{-1/2} (L_\phi), \quad (41)$$

with

$$L_\psi = -\partial_0^2 + \frac{1}{\epsilon} \Delta,$$
$$L_\phi = -\partial_0^2 + \frac{1}{\epsilon} \Delta + \left( \partial_r + \frac{1}{r} \right) \left( \partial_r + \frac{1}{r} \right). \quad (42)$$

The prime in (41) reminds us to subtract zero angular momentum fields. The operator $L_\psi$ is not hermitian. However, its heat kernel coincides with that of the hermitian operator $\epsilon^{-1/2} L_\psi \epsilon^{1/2}$.

The heat kernel coefficients for the operators $L_\psi$ and $L_\phi$ which include zero angular momentum fields can be easily calculated by using the standard methods [27]. After continuation to the Euclidean domain the operators (42) become elliptic operators of the Laplace type. This means that they can be represented in the form:

$$L = -(g_{\mu\nu} \partial_\mu \partial_\nu + a^\sigma \partial_\sigma + b) \quad (43)$$

where $g_{\mu\nu}$ is a symmetric tensor field which plays the role of the metric. By introducing a connection $\omega_{\mu}$, one can bring $L$ to the form:

$$L = -(g_{\mu\nu} \nabla_\mu \nabla_\nu + E) \quad (44)$$

where $\nabla$ is a sum of the Riemannian covariant derivative with respect to the metric $g$ and the connection $\omega$. The explicit form of $\omega$ and $E$ is

$$\omega_{ij} = \frac{1}{2} g_{ij} (a^\nu + g^{\mu\sigma} \Gamma_{\mu\nu}^\sigma)$$
$$E = b - g_{\mu\nu} (\partial_\mu \omega_{\nu} + \omega_{\mu} \omega_{\nu} - \omega_{\sigma} \Gamma_{\mu\nu}^\sigma) \quad (45)$$

As usual, $\Gamma$ denotes the Christoffel connection. We must stress here that the metric $g_{\mu\nu}$ is an auxilliary object. It does not reflect any real geometry of the problem. The heat kernel coefficients can be expressed in terms of the fields introduced above [27].
Here \( R, \rho \) and \( \tau \) are Riemann tensor, Ricci tensor and scalar curvature of the metric \( g \) respectively. Semicolon denotes covariant differentiation, \( E_{\mu \nu} = \nabla_\mu E_\nu \). All indices are lowered and raised with the metric tensor. \( \Omega \) is the field strength of the connection \( \omega \), which is zero for the operators \((42)\). This is the most effective, though maybe a bit artificial way to calculate the leading terms of the heat kernel expansion on a smooth background.

Metric and curvature tensors coincide for the operators \( L_\phi \) and \( L_\psi \). In the dilute approximation they are:

\[
g_{ij} = \epsilon \delta_{ij},
\]
\[
R^i_{\ jkl} = \frac{1}{2}(e_{ji}\delta_{kl} - e_{jk}\delta_{il} - e_{il}\delta_{jk} + e_{ik}\delta_{lj}) + \ldots
\]
\[
\rho_{jk} = -\frac{1}{2}(e_{jk} + e_{pp}\delta_{jk}) + \ldots
\]
\[
\tau = -\frac{1}{\epsilon}2e_{pp} + \ldots,
\]

where we dropped the terms which do not contribute to the first non-trivial order in the \((\epsilon - 1)\) expansion. The functions \( E \) are different for \( L_\psi \) and \( L_\phi \):

\[
E_\psi = \frac{1}{4}e_{ii} + \ldots, \quad E_\phi = \frac{3}{4}e_{rr} + \frac{e_r}{2r} + \ldots.
\]

From these expressions we obtain the conformal anomaly in the dilute approximation:

\[
a_2'[\phi] = \frac{1}{360} \int r^2 dr d\Omega \left[ \frac{39}{2} \frac{\delta_r^2}{r^2} + \frac{129}{4} \delta_{rr}^2 \right],
\]
\[
a_2'[\psi] = \frac{1}{360} \int r^2 dr d\Omega \left[ \frac{9}{4} \delta_{ii}^2 \right].
\]

Note, that the coefficient \((48)\) can not be represented in the covariant form of the coefficient \((42)\). The prime means that the zero angular momentum modes \((\phi^0, \psi^0)\) are still to be subtracted. Introduce the fields \(\bar{\phi}^0 = r\phi^0\) and \(\bar{\psi}^0 = r\psi^0\) which are square integrable on the half-line with the unit weight. Acting on the fields \(\psi^0\) and \(\phi^0\) the operators \(L_\psi\) and \(L_\phi\) become

\[
L_\psi^0 = -\partial_r^2 + \frac{1}{\epsilon}\delta_r^2
\]
\[
L_\phi^0 = -\partial_r^2 + \partial_r \frac{1}{\epsilon}\partial_r
\]

Since the operators \((50)\) are two-dimensional, the coefficient before the zeroth power of proper time is given by \(a_1\). Again, it can be calculated by the same methods of \((27)\). The result is

\[
a_1^0[\psi] + a_1^0[\phi] = \int_0^\infty dr \left( -\frac{1}{8} \delta_r^2 \right) = \frac{1}{4\pi} \int r^2 dr d\Omega \left( -\frac{1}{8} \delta_r^2 \right).
\]

The total coefficient \(a_2\) after subtracting the zero angular momentum modes reads

\[
a_2 = a_2'[\psi] + a_2'[\phi] - 4\pi(a_1^0[\psi] + a_1^0[\phi])
\]
\[
= \int r^2 dr d\Omega \frac{69}{120} \delta_{ii}^2,
\]

where the factor \(4\pi\) appeared due to different normalization conventions for the heat kernel coefficients in 2 and 4 dimensions. We see, that the covariance of the result is restored. The coefficient \((52)\) coincides with the result of the calculations in ordinary potentials, eq. \((50)\), providing a check of our previous calculation.
Now, because the problem is reduced to two scalar ones we can turn to the case of a dielectric ball of radius \( R \) whose permittivity and permeability are constant inside and outside

\[
\epsilon(r) = \epsilon_1 \Theta(R - r) + \epsilon_2 \Theta(r - R),
\]

(and similar for \( \mu(r) \)). We remind that it is just the step contained in this \( \epsilon(r) \) why the above approach cannot be applied.

The most convenient form of the operators \([42]\) to analyse the present case is

\[
L_\psi = -\partial_\rho^2 + \frac{1}{r} \partial_r r + \frac{1}{\epsilon} \Delta,
\]

\[
L_\phi = -\partial_\rho^2 + \frac{1}{r} \partial_r r + \frac{1}{\epsilon} \Delta.
\]

The corresponding eigenfunctions, which can be divided into two parts according to

\[
\psi(r) = \psi_1(r) \Theta(R - r) + \psi_2(r) \Theta(r - R)
\]

(similarly for \( \phi \)), are determined by \( \psi(r) \) and its first derivative being continuous,

\[
\partial_r \psi_1 = \partial_r \psi_2 = \partial_r \psi |
\]

and for \( \phi \) we have

\[
\partial_r \phi_1 = \partial_r \phi_2 = \partial_r \phi |
\]

i.e., the function and a known combination of the first derivative with \( \epsilon \) must be continuous. It is clear that these conditions are just the same as known from the standard approach, see for example a textbook on classical electrodynamics as \([42]\).

Note that this situation is quite similar to the delta shell \([21]\). Just like in that case, to proceed further one has to define the corresponding scattering problem and the corresponding Jost functions. They turn out to be proportional to the expressions known for example from \([4]\)

\[
\Delta^{TE}_l(kR) = \sqrt{\epsilon_1 \mu_2} s'_l(kR) e_l(kR) - \sqrt{\epsilon_2 \mu_1} s_l(kR) e'_l(kR)
\]

\[
\Delta^{TM}_l(kR) = \sqrt{\epsilon_2 \mu_1} s'_l(kR) e_l(kR) - \sqrt{\epsilon_1 \mu_2} s_l(kR) e'_l(kR)
\]

with the notations

\[
s_l(x) = \sqrt{\frac{\pi x}{2}} L_\nu(x), \quad e_l(x) = \sqrt{\frac{2x}{\pi}} K_\nu(x)
\]

\((\nu = l + 1/2)\). Here, \( \epsilon_{1,2} \) and \( \mu_{1,2} \) are the permittivity and permeability (we keep it in these formulas) inside respectively outside the ball. Accordingly we use the notations \( k_{1,2} = k \sqrt{\epsilon_{1,2} \mu_{1,2}} \). These expressions for \( \Delta^{TE} \) and \( \Delta^{TM} \) can be inserted into formula \([24]\) for the ground state energy whereby any multiplicative factors (even dependent on \( k \) as long as they are analytic near the real axis) by which \( \Delta^{TE} \) and \( \Delta^{TM} \) may differ from the Jost functions, drop out. By means of Eq. \([9]\) the corresponding zeta function then reads

\[
\zeta(s) = \frac{\sin \pi s}{\pi} \sum_{l=1}^{\infty} (2l + 1) \int_0^{2\pi} dk \ k^{-2s} \ |\Delta^{TE}_l(kR)\Delta^{TM}_l(kR)|.
\]

As will be seen clearly afterwards (eq. \([73]\)), the Casimir energy of the dielectric filling all space, and having permittivity \( \epsilon_2 \) and permeability \( \mu_2 \), has already been subtracted. This is the translational invariant so called 'Minkowski space' contribution.

It is useful to introduce the notation

\[
\Delta_{\rho,l}(ka) = \xi^p s'_l(k_1a) e_l(k_2a) - s_l(k_1a) e'_l(k_2a)
\]

with \( \xi = \sqrt{\frac{\epsilon_1 \mu_2}{\epsilon_2 \mu_1}} \) and with the correspondence \( \Delta^{TE} \rightarrow \Delta_{1,l} \) and \( \Delta^{TM} \rightarrow \Delta_{-1,l} \). In passing from Eqs. \([59]\) to \([62]\) only irrelevant factors have been neglected.
indicating to which variable the contribution belongs. Then, after some calculation, we get

\[ \zeta(s) = \frac{\sin \frac{\pi s}{\pi}}{\pi} \sum_{i=0}^{\infty} (2l + 1) \int_{0}^{\infty} dz \left( \frac{z^2}{R} \right)^{-2s} \frac{\partial}{\partial z} \ln |\Delta_{\rho,i}(z\nu)|. \]  

In order to calculate the heat kernel coefficients we need the uniform asymptotic expansion of \( \Delta_{\rho,i} \). It can be obtained just as in the preceding section by using the known uniform asymptotic expansions of the Bessel functions. But because the resulting expressions are more involved now, we introduce some obvious abbreviations involving the Debye polynomials \( u_k(t) \) and \( v_k(t) \)

\[ A = \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}, \quad B = \sum_{k=1}^{\infty} (-1)^k \frac{u_k(t)}{\nu^k}, \]

\[ C = \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k}, \quad D = \sum_{k=1}^{\infty} (-1)^k \frac{v_k(t)}{\nu^k}. \]

In terms of these the asymptotic expansion of \( s_i, \) \( e_i, \) and their derivatives read

\[ s_i(\nu z) = \frac{1}{2} \frac{z^{1/2}}{(1 + z^2)^{1/4}} e^{\nu \eta(z)} (1 + A), \]

\[ e_i(\nu z) = \frac{1}{4} \frac{z^{1/2}}{(1 + z^2)^{1/4}} e^{-\nu \eta(z)} (1 + B), \]

\[ s_i'(\nu z) = \frac{1}{4} \frac{z^{1/2}}{(1 + z^2)^{1/4}} e^{\nu \eta(z)} (1 + A) \]

\[ + \frac{1}{2} \frac{1}{z^{1/2}} e^{\nu \eta(z)} (1 + C), \]

\[ e_i'(\nu z) = \frac{1}{2} \frac{z^{1/2}}{(1 + z^2)^{1/4}} e^{-\nu \eta(z)} (1 + B) \]

\[ - \frac{1}{z^{1/2}} e^{-\nu \eta(z)} (1 + D). \]

Before stating the asymptotic expansion of \( \Delta_{\rho,i} \) some further notation is necessary. First, the velocities of light are defined by \( c_i = 1/\sqrt{\epsilon_i \mu_i} \) \((i = 1, 2)\). Furthermore we use \( t_i = 1/\sqrt{1 + z_i^2}, \) \( z_i = \sqrt{\epsilon_i \mu_i} z, \) and we will write \( A_i, B_i, C_i, D_i \) indicating to which variable the contribution belongs. Then, after some calculation, we get

\[ \Delta_{\rho,i} = \frac{1}{2} e^{\nu(\eta(z_1) - \eta(z_2))} \frac{\xi \xi c_1 t_2 + c_2 t_1}{\xi \xi c_1 t_1 c_2 t_2} \times \]

\[ \left\{ 1 + \frac{\sqrt{c_1 c_2 t_1 t_2}}{\xi \xi c_1 t_2 + c_2 t_1} \left[ \xi \xi \left( \frac{c_1 t_2}{c_2 t_1} \right) \left( C_1 + \frac{t_1}{2\nu} (1 + A_1) \right) + B_2 \right] \right. \]

\[ + \left. \sqrt{\frac{c_2 t_1}{c_1 t_2}} \left( D_2 - \frac{t_2}{2\nu} (1 + B_2) \right) (1 + A_1) + A_1 \right\}. \]

The expansion for \( \nu \to \infty \) of the logarithm entering (63) reads

\[ \ln \Delta_{\rho,i} \sim \sum_{n=-1,0,1,...} \frac{D_{n,\rho}(z)}{\nu^n} \]

with

\[ D_{-1,\rho}(z) = \eta \left( \frac{z}{c_1} \right) - \eta \left( \frac{z}{c_2} \right), \]

\[ D_{0,\rho}(z) = \ln \left( \frac{1}{2} \frac{\xi \xi c_1 t_2 + c_2 t_1}{\sqrt{c_1 c_2 t_1 t_2}} \right), \]

the consecutive expressions are listed in the appendix. We define the corresponding contributions to the zeta function \( \zeta_{\rho}(s) \), Eq. (63) to be

\[ A_{n,\rho}(s) = \frac{\sin \frac{\pi s}{\pi}}{\pi} \sum_{i=1}^{\infty} (2l + 1) \int_{0}^{\infty} dz \left( \frac{z^2}{R} \right)^{-2s} \frac{\partial}{\partial z} D_{n,\rho}(z) \]  

\[ \zeta_{\rho}(s) = \frac{\sin \frac{\pi s}{\pi}}{\pi} \sum_{i=1}^{\infty} (2l + 1) \int_{0}^{\infty} dz \left( \frac{z^2}{R} \right)^{-2s} \frac{\partial}{\partial z} D_{n,\rho}(z) \]  

\[ (63) \]
(n = -1, 0, 1, 2, 3). The contribution from \( n = -1 \) can be obtained doing the same calculations as in [10] for the massive scalar field in a ball with the result

\[
A_{-1, \rho}(s) = \frac{R^{2s} \Gamma(s - \frac{1}{2})}{2\sqrt{\pi} \Gamma(s + 1)} \zeta_H(2s - 2; 3/2) \left[ \left( \frac{1}{c_1} \right)^{2s} - \left( \frac{1}{c_2} \right)^{2s} \right].
\]

The contribution following for the residuum at \( s = -1/2 \) of the zeta function is

\[
\text{Res} A_{-1, \rho}(s = -1/2) = \frac{127}{1920\pi R}(c_1 - c_2).
\]

The \( z \)-integral appearing in \( A_{0, \rho} \) cannot be done by elementary means. However the analytical structure around \( s = -1/2 \) is easily found. First, for \( z \to 0 \) the integrand behaves like \( O(z) \) at \( s = -1/2 \) and the integral exists at the lower bound. Furthermore one can verify, that for \( z \to \infty \) it behaves as \( O(z^{-2}) \) and the integral is well defined also at the upper integration limit. Thus \( A_{0, \rho} \) is analytical around \( s = -1/2 \) and gives no contribution to the pole of the zeta function.

Let us now come to the most delicate part of the calculation. As we will see even the pole contribution will be, in general, a very involved integral which can only be analysed numerically. This will happen for the contribution arising from \( D_{3, \rho}(z) \). But let us continue step by step. First, the angular momentum sum in the \( A_{1, \rho} \) can be performed in terms of the Hurwitz zeta function,

\[
A_{n, \rho}(s) = 2\frac{\sin \pi s}{\pi} R^{2s} \zeta_H(2s + n - 1; 3/2) \int_0^\infty dz z^{-2s} \frac{\partial}{\partial z} D_{n, \rho}(z)
\]

In the limit \( z \to 0 \) all \( D_{n, \rho}(z) \) converge to a constant so that the integrals always exist at the lower bound. At the upper bound however the integral might not be defined at \( s = -1/2 \). This happens for example for \( D_{1, \rho}(z) \). Here one finds

\[
D_{1, \rho}(z) = \frac{1}{8}(c_1 - c_2) \frac{1}{z} + O\left( \frac{1}{z^2} \right)
\]

and the integral as it stands does not exist at \( s = -1/2 \). However, the analytical continuation is obtained by writing for \( A_{1, \rho} \) instead of (67) the following,

\[
A_{1, \rho}(s) = 2\frac{\sin \pi s}{\pi} R^{2s} \zeta_H(2s; 3/2) \int_0^\infty dz z^{-2s} \frac{\partial}{\partial z} D_{1, \rho}(z) + \frac{c_1 - c_2}{8} \frac{1}{(z + 1)^2}
\]

Here, the asymptotics for \( z \to \infty \) has been added and subtracted, the first term is now analytical around \( s = -1/2 \), the second one clearly contains the pole, namely

\[
\text{Res} A_{1, \rho}(s = -1/2) = -\frac{11}{192\pi R}(c_1 - c_2).
\]

For \( A_{2, \rho} \) similar arguments than the ones provided for \( A_{0, \rho} \) show that there is no pole. Finally we are left with

\[
A_{3, \rho}(s) = 2\frac{\sin \pi s}{\pi} R^{2s} \zeta_H(2s + 2; 3/2) \int_0^\infty dz z^{-2s} \frac{\partial}{\partial z} D_{3, \rho}(z).
\]

Here at \( s = -1/2 \) the Hurwitz zeta function has already a pole such that the residue of \( A_{3, \rho} \) is determined by the integral. In detail, after a partial integration one finds

\[
\text{Res} A_{3, \rho}(s = -1/2) = \frac{1}{\pi R} \int_0^\infty dz \, D_{3, \rho}(z).
\]

An integration of \( D_{3, \rho}(z) \) in terms of elementary functions seems impossible. For the case of \( \mu_i = 1 \) simplifications occur, but still a complete analytical treatment seems only possible for \( \rho = 1 \), this is for the TE modes. For \( \rho = -1 \) the integral cannot be expressed in terms of usual special functions. Instead we present the behaviour of the complete pole (that is for the pole of the complete zeta function) as a function of the velocities of light in Figure 1.
FIG. 1. The dependence of the heat kernel coefficient $a_2$ on the speed of light $c_1 = 1/\sqrt{\epsilon_1}$ inside the ball while $c_2 = 1$ outside.

In the limit of small differences of the velocities of light (dilute dielectric ball) one might expand the residue in powers of $c_1 - c_2$. Surprisingly, the leading $O(c_1 - c_2)$ pole coming from $A_1$ and $A_3$, the second order is zero so that we are left with a pole of the order $(c_1 - c_2)^3$. This result is in agreement with [13], where the calculation for a dilute ball was performed up to the order $(c_1 - c_2)^2$ and a finite result was obtained in the zeta function scheme. Here we see that in the next order of the dilute approximation a pole appears. In detail we find

$$\text{Res} \zeta(s = -1/2) = \frac{166}{5005 \pi R} \frac{(c_1 - c_2)^3}{c_2^3},$$

(73)

especially, for equal velocities of light in the interior and exterior region no pole is present. By means of (6), the heat kernel coefficient is

$$a_2 = -16\pi^2 \text{Res} \zeta(s = -1/2) = -\frac{2656\pi}{5005R} \frac{(c_1 - c_2)^3}{c_2^3} + O((c_1 - c_2)^4),$$

(74)

a value quite close to $a_2$, eq. (25) for the delta sphere.

Let us mention that also for the $TE$ and $TM$ modes separately the divergent term is of third order in the difference of the velocities of light.

The peculiarity of this divergent term is, that it has the same $1/R$ dependence as the finite part of the result. This does not happen to quadratic order where independently of the scheme used in [6,7,13], no such divergence appears. Thus in contrast to the second order calculation, in the next order one must raise again the question how to fix the finite part of the Casimir energy. Until this question is not answered, the calculation of finite parts in higher orders of the dilute approximation makes no sense. The situation is even more involved for smooth dielectrics as seen in eq. (30) and as discussed further in the Conclusions.

However, before we come to the summary of our main results let us comment on other possible regularization schemes in order to make contact with the results of [4,6,7]. It is known, that the zeta function scheme often gives finite answers whereas using other techniques divergences show up. An example is the dilute ball to second order in $(c_1 - c_2)$ where the zeta-function method yields a finite result whereas cutoff functions of different types yield poles proportional to the volume and surface for example. These divergent terms are also easily recovered by our calculation. There are well established connections between zeta function and cutoff regularization of Casimir energies. Using a frequency cutoff $\lambda$ the divergent terms for $\lambda \to 0$ are [25]

$$E^{\text{div}}_C = \frac{3a_0}{2\pi^2 \lambda^4} + \frac{a_{1/2}}{4\pi^{3/2} \lambda^3} + \frac{a_1}{8\pi^2 \lambda^2} + \frac{a_2}{16\pi^2} \ln \lambda$$

(75)
with the heat-kernel coefficients \( a_i \) of the problem at hand. These are immediately determined by the formulas of \( A_{i,\rho} \) already presented. One gets

\[
a_0 = \frac{8}{3} \pi R^3 \left[ \left( \frac{1}{c_1} \right)^3 - \left( \frac{1}{c_2} \right)^3 \right],
\]

\[
a_{1/2} = -2R^2 \pi^{3/2} \frac{(c_1^2 - c_2^2)^2}{c_1^4 c_2^3 (c_1^2 + c_2^2)},
\]

\[
a_1 = -\frac{22}{3} \pi R \left( \frac{1}{c_1} - \frac{1}{c_2} \right) + 8\pi R \int_0^\infty dy \frac{1}{y} \frac{\partial}{\partial y} [D_{1,1}(y) + D_{1,-1}(y)]
\]

\[
a_{3/2} = \pi^{3/2} \frac{(c_1^2 - c_2^2)^2}{(c_1^2 + c_2^2)^2}
\]

and \( a_2 \) is the sum of eqs. (66), (70) and (72). It is clearly seen, that volume, surface and more divergencies are present. Using an expansion in \( \delta = c_1 - c_2 \), one finds explicitly

\[
\mathcal{E}_0 = \mathcal{E}_0^\text{div} + \mathcal{E}_0^\text{ren}
\]

\[
= -\delta \frac{9}{2\pi^2 c_2^4} \frac{V}{\lambda^4} + \delta^2 \left( \frac{18}{\pi^2 c_2^4} \frac{V}{\lambda^4} - \frac{1}{4\pi c_2^3} \frac{S}{\lambda^3} + \frac{23}{1530\pi} \frac{1}{R} \right) + \mathcal{O}(\delta^3),
\]

an expansion in parallel with Barton [7], and where \( \mathcal{E}_0^\text{ren} \) has been taken from that reference. Let us mention that the factors of the divergent pieces can not be compared to [7] because there a wave number cutoff has been used. The fact that there is no \( 1/\lambda \) term is generic in this cutoff definition [29]. In contrast, the reason for \( 1/\lambda^2 \) not being there is a result of cancellations between the TE and TM modes. In the approximation given, volume and surface divergencies are absorbed by counterterms and a unique finite result is obtained. However, let us repeat and emphasize that at third order a term \( (\delta^3/R) \ln \lambda \) appears and renormalization and normalization of this term remains unclear.

V. CONCLUSIONS

We have calculated the lowest heat kernel coefficients for the delta sphere and for the dielectric ball. For a massless field, it is the coefficient \( a_2 \) which determines the interpretation of the ground state energy. In both cases it does not vanish. For a delta sphere it is the simple expression (25), for the dielectric ball it is an involved expression, including the integral (23). Therefore, just subtracting the “empty space” or “unbounded medium” contribution is clearly insufficient to make the ground state energy finite. At this point the conventional field theoretical approach fails to give a unique result. It must be supplemented by an extra normalization condition, which is not at hand by now.

A common feature of the two cases considered here is that for small coupling to the background, i.e., for small \( \alpha \) in (20) or \( \epsilon \rightarrow 1 \) in (3), the lowest two orders vanish ( contrary to the third and higher orders). Thereby nontrivial cancellations appear in intermediate expressions. Therefore, the good news is that calculations in the leading orders of the dilute dielectric ball approximation are physically relevant and give a unique result. It seems to be true for all penetrable boundary conditions. This observation is supported by an example of the paper [4], where Casimir-Polder forces between the particles constituting a dielectric sphere, had been summed up over the volume. In fact, the \( r^{-7} \) potential had been integrated using dimensional regularization. It turned out, that the analytic continuation to the physical dimension yields a unique and finite value. In general, one expects the Casimir-Polder forces to be equivalent to the ground state energy. Because these forces are nonadditive, the result found in [4] should have relevance to a dilute dielectric medium and in fact the identity of the van der Waals force and the Casimir effect has been shown in [5]. Thus to the leading non-vanishing order the situation is quite clear now for the dilute dielectric ball, but, as we have described in detail at the end of section 4 in higher orders the situation remains unclear. Even worse, if we consider a smoothly varying dielectric background, we have seen that already in the second order an ambiguity proportional to \( a_2 \), eq. (20), remained. If one is going to calculate the Casimir energy by summing up the van der Waals energies between the molecules that make up the medium taking into account that \( \epsilon \) and \( \mu \) may vary in space, we expect that in the scheme used in [4] in contrast to the dilute calculation divergencies or at last a logarithmic contribution delivering a nonuniqueness will show up when the regularization is removed.

So for the case of nonconstant dielectricum the way how to fix a unique finite value of the Casimir energy still remains unclear and further investigations will be necessary to shed more light into this very basic question.
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APPENDIX A: ASYMPTOTIC FUNCTIONS $D_N(Z)$

In this appendix we give a list of the functions used in order to get the analytical properties of the relevant zeta function. We have,

$$D_{1,p}(z) = \frac{1}{24 (c_2 t_1 + c_1 t_2 \xi^p)} \times \left\{ -c_2 t_1 \left( -3 t_1 + 5 t_1^3 + 3 t_2 + 7 t_2^3 \right) + c_1 t_2 \left( 3 t_1 + 7 t_1^3 - 3 t_2 + 5 t_2^3 \right) \xi^p \right\}$$  \hspace{1cm} (A1)

$$D_{2,p}(z) = \frac{1}{16 (c_2 t_1 + c_1 t_2 \xi^p)} \times \left\{ c_2^2 t_1^2 \left( t_1^2 - 6 t_1^4 + 5 t_1^6 - t_2^2 + 6 t_2^4 - 7 t_2^6 \right) + 4 c_1 c_2 t_1^4 t_2^4 \xi^p + c_1^2 t_2^2 \left( -t_1^6 + 6 t_1^4 - 7 t_1^2 + t_2^2 - 6 t_2^4 + 5 t_2^6 \right) \xi^{2p} \right\}$$  \hspace{1cm} (A2)

$$D_{3,p}(z) = \frac{1}{5760 (c_2 t_1 + c_1 t_2 \xi^p)^3} \left\{ c_2^3 t_1^3 \left( 375 t_1^3 - 4779 t_1^5 + 9945 t_1^7 - 5525 t_1^9 \right) + c_1^3 \left( 345 - 4941 t_2^6 + 11655 t_1^6 - 7315 t_1^8 \right) \right\}$$

$$\left\{ -3 c_1 c_2^2 t_1^2 t_2 \left( 1659 t_1^2 - 3465 t_1^4 + 1925 t_1^6 + 120 t_1^2 t_2^2 - 720 t_1^4 t_2^2 + 600 t_1^6 t_2^2 \right) - 15 t_1^4 \left( 9 + 8 t_2^2 - 48 t_1^2 + 56 t_2^4 + t_2^6 \right) + t_2^3 \left( -105 + 1581 t_2^2 - 3735 t_2^4 + 2275 t_2^6 \right) \right\} \xi^p$$

$$+ 3 c_1^2 c_2 t_1 t_2^2 \left( 1581 t_1^2 - 3735 t_1^4 + 2275 t_1^6 - 120 t_1^4 t_2^2 + 720 t_1^6 t_2^2 - 840 t_1^6 t_2^2 \right) + 15 t_1^4 \left( -7 + 8 t_2^2 - 48 t_1^2 + 40 t_2^4 + t_2^6 \right) + t_2^3 \left( -135 + 1659 t_2^2 - 3465 t_2^4 + 1925 t_2^6 \right) \right\} \xi^{2p}$$

$$+ c_1^3 t_2^3 \left( -345 t_1^3 + 4941 t_1^5 - 11655 t_1^7 + 7315 t_1^9 \right) + t_2^3 \left( -375 + 4779 t_2^6 - 9945 t_2^8 + 5525 t_2^10 \right) \right\} \xi^{3p},$$  \hspace{1cm} (A3)

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