Ideal quantum gas in expanding cavity:  
nature of non-adiabatic force

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(Dated: January 22, 2013)

We consider a quantum gas of non-interacting particles confined in the expanding cavity, and investigate the nature of the non-adiabatic force which is generated from the gas and acts on the cavity wall. Firstly, with use of the time-dependent canonical transformation which transforms the expanding cavity to the non-expanding one, we can define the force operator. Secondly, applying the perturbative theory which works when the cavity wall begins to move at time origin, we find that the non-adiabatic force is quadratic in the wall velocity and thereby does not break the time-reversal symmetry, in contrast with the general belief. Finally, using an assembly of the transitionless quantum states, we obtain the nonadiabatic force exactly. The exact result justifies the validity of both the definition of force operator and the issue of the perturbative theory. The mysterious mechanism of nonadiabatic transition with use of transitionless quantum states is also explained.

The study is done on both cases of the hard-wall and soft-wall confinement with the time-dependent confining length.

PACS numbers: 05.20.Dd, 51.10.+y

I. INTRODUCTION

The nonadiabatic transition in the quantum gas of non-interacting particles confined in an expanding microscopic cavity is the origin of the nonadiabatic force acting on the cavity wall. Let’s consider non-interacting Fermi particles confined in a billiard with a moving wall. The wall receives the forces from Fermi particles in the billiard. Under the condition that whole system consisting of Fermi particles and a moving wall keeps the energy conservation, the work done on the wall by the force is supplied by the excess energy due to the energy loss of Fermi particles showing the non-adiabatic transition. In this way one can conceive both the adiabatic and nonadiabatic forces. In the adiabatic limit, the adiabatic force due to the quantal gas on the cavity wall is proportional to the derivative of the confining energy with respect to the cavity size. What is a characteristic feature of the nonadiabatic force when the cavity wall is moving? The general belief is that this force should be linear in the wall velocity, breaking the time reversal symmetry. In fact, in compound systems like molecules where two kind of coordinates with different time scales coexist, the Born-Oppenheimer approximation leads to both the adiabatic and nonadiabatic forces acting on the degree of freedom characterized by the slow time scale, and the latter force is linear in the velocity of the slow degree of freedom and is called an irreversible or frictional force \[^{1,3}\]. In the case of the expanding cavity, however, the Hilbert space as well as the domain of Hamiltonian is time-dependent because of the time-dependent length scale of the cavity confining particles, which requires a deeper insight into the nature of the nonadiabatic force.

The investigation of the above subject was started by Hill and Wheeler in 1952 \[^{4}\] in the context of nuclear physics. The nature of the nonadiabatic force on the cavity wall were intensively studied by Blocki et al. \[^{5}\]. Wilkinson developed the extensive study on this subject \[^{6, 7}\] in the context of energy diffusion and of random matrix theory assimilating the chaotic motion of particles inside the cavity, which was followed by other groups \[^{8, 11}\]. Most of these works regarded the force due to the quantal gas as conjugate to a time-dependent wall coordinate. However, the definition of the force operator is not clear at all in the case of the hard wall. In fact, one cannot define the force operator by using a given Hamiltonian for the billiard with a moving boundary. Further, without the valid definition of the force operator, essential results so far would be highly questionable.

In this paper, we introduce the force operator with use of the time-dependent canonical transformation which transforms the expanding cavity to the non-expanding one. Applying the perturbative and exact theories, we
evaluate the non-adiabatic force whose nature thoroughly differs from the conventional frictional force. For comparison we shall also investigate the case of the soft-wall confinement with the time dependent confining length.

In Section III we construct the force operator acting on the moving wall in an unambiguous way. We consider the case that the cavity wall suddenly begins to move at time origin (see Fig.1). In Section III within a framework of von Neumann equation for the density operator, we apply a perturbative theory to obtain the expectation of the nonadiabatic force. In Section IV, with use of the transitionless basis functions [15, 19, 20], we exactly evaluate the energy expectation to see the nature of the nonadiabatic force, and justify the validity of both the definition of force operator and the issue of the perturbative theory. In Section V we investigate the case of soft-wall confinement by treating a tunable harmonic trap. Section VI is devoted to summary and discussions.

In Appendix A we investigate the exactly-solvable case when the expansion rate of the cavity is time-dependent, to see the universality of the assertion of the text. Appendix B treats technical details of the integrals used in Section IV.

II. FORCE OPERATOR

When a given cavity in 1 dimension has a size \( L \) and its wall is fixed, the force on the wall due to the quantal gas inside the cavity is defined by \( F = -\frac{\partial E}{\partial L} \), which, with use of eigenvalues \( E_n = \frac{\hbar^2 n^2}{2mL^2} \), gives rise to \( F = \frac{\hbar^2 n^2}{mL^2} \). And the contribution from all particles is expressed as \( F = \sum_{n=1}^{\infty} \frac{\hbar^2 n^2}{mL^2} f(E_n) \) where \( f(E_n) \) stands for the Fermi distribution function. At zero temperature, \( f(E_n) = 1 \) for \( 1 \leq n \leq N \) and \( f(E_n) = 0 \) otherwise. This force is called the adiabatic force. When the wall will move, the wall receive the extra force depending on its velocity, which comes from the nonadiabatic transition occurring in the quantal gas. However, the definition of the force operator is far from obvious in the case of a moving hard wall. Below we shall define the force operator in two ways.

A. Classical force and quantization

As a first step to provide the force operator, we show a kinetic evaluation of the force due to the classical ideal gas. Then the force is expressed as a dynamical quantity, and is thus quantized straightforwardly.

Suppose that \( N \) mutually noninteracting particles with a common mass \( m \) are confined in a 3-dimensional (3-d) box whose 3 edges have a common length \( L \). Along each of 3 coordinates, one wall is fixed at the origin, for instance, at \( x = 0 \), and another one is initially located, for instance, at \( x = L \) and begins to move with a constant velocity \( (\dot{L}) \). The velocity of the wall \( \dot{L} \) is assumed to be slow compared with the mean square velocity of the particles \( \sqrt{\langle v_i^2 \rangle} \). In the course of time evolution, the particles eventually become uniformly distributed in the box. The time \( t \) for a particle at a position \( x = X_i \) running to right with \( v_i > 0 \) (to left with \( -v_i \)) to reach the moving wall is given by \( t = \frac{L + X_i}{v_i - \dot{L}} \) and its average is

\[
    t = \frac{L}{v_i - \dot{L}}.
\]

The average time for a particle to come back to the initial position is given by 2\( t \). On the other hand, the momentum change at each collision with the moving wall is given by

\[
    \Delta p_i = m(v_i - \dot{L}) - m(-v_i - \dot{L}) = 2mv_i.
\]

Since the collision rate is given as the inverse of 2\( t \), the force acting on the wall is given by

\[
    \sum_{i=1}^{N} \frac{v_i - \dot{L}}{2L} \Delta p_i = \sum_{i=1}^{N} \left( \frac{v_i}{L} m v_i - \frac{\dot{L}}{L} m v_i \right).
\]

The first and second terms on the second line are the adiabatic and non-adiabatic forces, respectively. These forces are rewritten as

\[
    \sum_{i=1}^{N} \frac{mv_i^2}{L} = \langle p^2 / mL \rangle,
\]


where \( p \) is the momentum. For each particle, the bracket evaluates the value of the position \( x \) and momentum \( p \) at the instance of collision.

Then the non-adiabatic force is given as

\[
F_{\text{non-ad}} = -\frac{\dot{L}}{2L^2}(xp + px).
\]

As a dynamical quantity, \( F_{\text{non-ad}} \) is invariant under the time reversal operation, since both the expansion rate \( \dot{L} \) and momentum \( p \) change their signs.

Let’s quantize the non-adiabatic force obtained above. The force operator should satisfy the following conditions.

a) In the classical limit, \( \hat{F}_{\text{non-ad}} \) agrees with \(-\frac{\dot{L}}{2L^2}(xp + px)\).

b) \( \hat{F}_{\text{non-ad}} \) should be Hermitian.

c) \( \hat{F}_{\text{non-ad}} \) does not depend on the particle statistics (boson or fermion).

Consequently the force operator should be

\[
\hat{F}_{\text{non-ad}} = -\frac{\dot{L}}{2L^2}(\hat{x}\hat{p} + \hat{p}\hat{x}).
\]

The idea above is based on the phenomenological argument with use of a 3-d box, but suggesting a promising expression of the force operator. Below we shall provide a rigorous definition of the force operator.

**B. Rigorous definition of force operator via time-dependent canonical transformation**

The original Hamiltonian \( H \) for the billiard with a time-dependent cavity size \( L(t) \) is given by (in unit of \( \hbar^2/m = 1 \))

\[
H = -\frac{1}{2}\frac{\partial^2}{\partial x^2}.
\]

We now see the expectation of \( H \) as given by

\[
\langle \psi | H | \psi \rangle,
\]

where \( |\psi\rangle \) is a solution of the time-dependent Schrödinger equation

\[
\frac{i\hbar}{\partial t}\psi(x, t) = H\psi(x, t)
\]

with a moving Dirichlet boundary condition:

\[
\psi(x = 0, t) = \psi(x = L(t), t) = 0.
\]

The expectation of the force acting on the wall is obtained by

\[
\bar{F} = -\frac{\partial}{\partial L(t)}\langle \psi | H | \psi \rangle.
\]

Noting \( \frac{\partial}{\partial L(t)}|\psi\rangle = \frac{1}{L}\frac{\partial}{\partial t}|\psi\rangle = \frac{i\hbar}{\partial L(t)}|\psi\rangle \) and its Hermitian conjugate, Eq. (12) reduces to

\[
\bar{F} = -\langle \psi | \frac{\partial H}{\partial L(t)} | \psi \rangle.
\]

Hence the force operator is defined by

\[
\hat{F} = -\frac{\partial H}{\partial L(t)}.
\]

However, the original Hamiltonian \( H \) for the billiard with its time-dependent size \( L(t) \) does not formally include \( L(t) \) explicitly. Therefore there is no way to define the force operator directly by using Eq. (14).

To overcome this difficulty, we shall make the time-dependent canonical transformation of \( H \) related to the scale transformation of both the coordinate \( x \) and amplitude of the wave function \( \psi \). This transformation, which was originally developed in the heat equation theory \[12\,13\], is defined by \[14\,16\]

\[
H_1 = e^{-iU}(H - i\frac{\partial}{\partial t})e^{iU},
\]

where

\[
U = -\frac{1}{2\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x}) \ln L(t) = i\left(\frac{x}{\partial x} + \frac{1}{2}\right) \ln L(t).
\]

This canonical transformation leads to the scaling of the coordinate \( x \),

\[
e^{-iU}xe^{iU} = XL(t),
\]

where on the right-hand side the new variable \( x = y \) varies in the range \( 0 \leq y \leq 1 \), which is time-independent! Similarly the amplitude of the wave function is scaled as

\[
\tilde{\psi}(x, t) = e^{-iU}\psi(x, t) = \sqrt{L}\psi(xL, t),
\]

so that the normalization factor of \( \tilde{\phi}(x, t) \) becomes \( L \)-independent. Finally the Schrödinger equation is transformed to

\[
\frac{i\hbar}{\partial t}\tilde{\phi} = H_1\tilde{\phi}
\]
with the new Hamiltonian
\[
H_1 = -\frac{1}{2L^2} \frac{\partial^2}{\partial x^2} + i\hbar \frac{\dot{L}}{L} \frac{\partial}{\partial x} + \frac{i\hbar}{2} \frac{\dot{L}}{L}.
\]
(20)

\(\hat{\phi}(x,t)\) now satisfies the fixed Dirichlet boundary condition \(\hat{\phi}(0,t) = \hat{\phi}(1,t) = 0\). Equation (19) with (20) is also available simply by replacing \(\psi\) and \(x\) and by \(\sqrt{L}\) and \(xL\) respectively in Eq. (10) with Eq. (8).

Taking \(L\) derivative of \(H_1\), we can rigorously define the force operator in the transformed space as
\[
\hat{F} = -\frac{\partial H_1}{\partial L} = -\frac{1}{L^2} \frac{\partial^2}{\partial x^2} + i\hbar \left( \frac{\dot{L}}{L^2} x \frac{\partial}{\partial x} + \frac{1}{2} \frac{\dot{L}}{L} \right)
= \frac{1}{L^3} \hat{p}_x^2 - \frac{\dot{L}}{2L^2} (\hat{x} \hat{p}_x + \hat{p}_x \hat{x}).
\]
(21)

Now, carrying out the inverse canonical transformation \((xL \to x, \text{etc.})\), we have the force operator expressed in the original space as
\[
\hat{F} = e^{iU} \hat{F} e^{-iU} = -\frac{1}{L} \frac{\partial^2}{\partial x^2} + i\hbar \left( \frac{\dot{L}}{L} x \frac{\partial}{\partial x} + \frac{1}{2} \right)
= \frac{\hat{p}_x^2}{L} - \frac{\dot{L}}{2L} (\hat{x} \hat{p}_x + \hat{p}_x \hat{x}),
\]
(22)

which certainly satisfies:
\[
\langle \psi | \hat{F} | \psi \rangle = \langle \hat{\phi} | \hat{F} | \hat{\phi} \rangle.
\]
(23)

The non-adiabatic term in Eq. (22) agrees exactly with the phenomenological result in Eq. (7).

III. PERTURBATIVE THEORY OF NONADIABATIC FORCE

In this Section we shall investigate the expectation of the force operator in Section II perturbatively with use of von Neumann equation for the density operator and adiabatic bases. The method is an extension of the Greenwood’s linear response theory [22]. Let’s assume that the cavity wall is fixed with the cavity size \(L_0\) until the time origin \(t = 0\) and that it suddenly begins to move with constant velocity \(\dot{L}\) at \(t > 0\).

In the equilibrium statistical mechanics, the expectation of a given observable \(\hat{O}\) is defined in energy-diagonal representation, as
\[
\langle \hat{O} \rangle = \text{Tr} \left( \hat{O} \frac{1}{e^{\beta(H - \mu)} + 1} \right) = \sum_n O_{nn} \frac{1}{e^{\beta(E_n - \mu)} + 1}.
\]
(24)

In the near-equilibrium, the expectation value is evaluated in terms of the density operator \(\rho\)
\[
\rho = \sum_{\alpha} |\alpha\rangle \omega_{\alpha} \langle \alpha|,
\]
(25)

with \(\sum_{\alpha} \omega_{\alpha} = 1\), as
\[
\langle \hat{O} \rangle = \text{Tr}(\rho \hat{O}) = \sum_{\alpha} \omega_{\alpha} O_{\alpha \alpha}.
\]
(26)

We shall employ the original Hamiltonian \(H\) and coordinate \(x\). The density operator \(\rho\) for the Fermi gas obeys von Neumann equation
\[
i\hbar \frac{\partial \rho}{\partial t} = [H, \rho],
\]
(27)

With use of adiabatic basis \(|n\rangle\), the matrix elements of \(\rho\) satisfies
\[
\dot{\rho}_{nm} = \langle n | \frac{\partial}{\partial t} | m \rangle + \langle n | \rho | m \rangle + \langle n | \rho \delta m \rangle,
\]
(28)

with
\[
\langle n | \rho | m \rangle + \langle n | \rho \delta m \rangle = \sum_\ell \langle \ell | \rho_\ell m \rangle + \rho_{n \ell} \langle \ell | \rho_\ell \delta m \rangle.
\]
(29)

For the system under consideration, the instantaneous (adiabatic) eigenvalue problem is given by
\[
H(t) \psi_n \equiv -\frac{1}{L(t)} \frac{\partial^2}{\partial x^2} \psi_n(t) = E_n \psi_n(t),
\]
(30)

with adiabatic eigenstates and eigenvalues
\[
\psi_n \equiv \langle x | n \rangle = \sqrt{\frac{2}{L(t)}} \sin \left( \frac{n\pi x}{L(t)} \right),
\]
(31a)
\[
E_n = \frac{n^2 \pi^2}{2L^2(t)},
\]
(31b)

where we prescribed \(\hbar^2/m = 1\).

Using Eq. (31), we can obtain the following formulas
\[
\langle \ell | \delta m \rangle = -\frac{\dot{L}}{L} \gamma_{\ell m},
\]
(32a)
\[
\langle \ell | \dot{\delta m} \rangle = \langle \ell | \delta m \rangle^* = -\frac{\dot{L}}{L} \gamma_{\ell m},
\]
(32b)

where
\[
\gamma_{\ell m} \equiv (-1)^{\ell + m + 1} \frac{2\ell\ell_m}{L^2 - m^2} (1 - \delta_{\ell m}).
\]
(33)

Noting the pure-real nature of the adiabatic states in Eq. (31), we see \(\langle \ell | \ell \rangle = \langle \ell | \ell \rangle = 0\). Hence we can put the diagonal element \(\gamma_{\ell \ell} = 0\) in Eq. (33). The von Neumann equation now becomes
\[
\dot{\rho}_{nm} = \frac{1}{i\hbar} (E_n - E_m) \rho_{nm} - \frac{\dot{L}}{L} \left( \sum_{\ell \neq n} \gamma_{\ell n} \rho_{\ell m} + \sum_{\ell \neq m} \gamma_{\ell m} \rho_{n \ell} \right).
\]
(34)
Eq. (34) can be solved perturbatively. Let’s assume the solution to be expanded in $O(\dot{L}/L)$ as,

$$\rho = f(H) + g_1 \frac{\dot{L}(0)}{L(0)} + g_2 \left( \frac{\dot{L}(0)}{L(0)} \right)^2 + \ldots \quad (35)$$

with $f(H) = e^{\beta(E_n(0) - \mu)}$. 

One sees, for $O(1)$,

$$f_{nm} = 0 \quad (n = m). \quad (36)$$

Therefore

$$f_{nm} = \frac{1}{e^{\beta(E_n(0) - \mu)}} \delta_{nm} = f_n \delta_{nm}. \quad (37)$$

Then, for $O(\dot{L}/L)$, one sees

$$\dot{g}_{nm} = \frac{E_n - E_m}{i \hbar} g_{1nm} - (\gamma_{nm} f_m + \gamma_{nm} f_n), \quad (38)$$

where the result in Eq. (37) was used. The solution of Eq. (38) is given by

$$g_{1nm} = \frac{i \hbar \gamma_{nm}}{E_n - E_m} \left( 1 - e^{\frac{E_n - E_m}{\hbar} t} \right) (f_n - f_m). \quad (39)$$

For a correction of $O((\dot{L}/L)^2)$, the dominant contribution comes from the diagonal term satisfying

$$\dot{g}_{2nn} = -\sum_\ell \gamma_{\ell n} (g_{1\ell n} + g_{1n\ell}). \quad (40)$$

With use of Eq. (39),

$$g_{1\ell n} + g_{1n\ell} = -2\gamma_{\ell n} (f_n - f_\ell) \frac{\hbar}{E_n - E_\ell} \sin \frac{E_n - E_\ell}{\hbar} t. \quad (41)$$

So, using Eqs. (40) and (41), we obtain

$$g_{2nn} = -2 \sum_\ell \gamma_{\ell n} (f_n - f_\ell) \left( \frac{\hbar}{E_n - E_\ell} \right)^2 \sin \frac{E_n - E_\ell}{\hbar} t. \quad (42)$$

Now let’s calculate the matrix elements of the force operator in Eq. (22). Using the adiabatic bases, we find

$$F_{mn} = \frac{2}{L(t)} \int_0^{L(t)} \sin \left( \frac{m \pi x}{L(t)} \right) \dot{F} \sin \left( \frac{n \pi x}{L(t)} \right) dx$$

$$= \frac{(n \pi)^2}{L^3(t)} \delta_{mn} + \frac{i \dot{L}(t)}{L^2(t)} \gamma_{mn}. \quad (43)$$

Combining Eq. (43) with Eqs. (37), (39) and (42), the expectation value of the force operator becomes

$$\bar{F} = \langle \bar{F} \rangle = \text{Tr}(\rho \bar{F}) = \sum_{m,n} \rho_{nm} F_{mn} = S_1 + S_2 + S_3, \quad (44)$$

where

$$S_1 = \sum_n f_n F_{nn} = \sum_n \frac{(n \pi)^2}{L^3(t)} f_n, \quad (45)$$

$$S_2 = (\dot{L}/L) \sum_{m \neq n} g_{1mn} F_{mn} = \frac{\dot{L}(t) \dot{L}(0)}{L^2(t) L(0)} \times$$

$$\sum_{m \neq n} \gamma_{nm} \gamma_{mn} \frac{f_n - f_m}{E_n - E_m} \left( 1 - e^{\frac{E_n - E_m}{\hbar} t} \right), \quad (46)$$

$$S_3 = (\dot{L}/L)^2 \sum_n g_{2nn} F_{nn} = -2 \frac{\pi^2 (\dot{L}(0))^2}{(L(t))^2 L^3} \times$$

$$\sum_n \sum_{\ell} \gamma_{\ell n}^2 (f_n - f_\ell) \left( 1 - \cos \frac{E_n - E_\ell}{\hbar} t \right) \left( \frac{\hbar}{E_n - E_\ell} \right)^2. \quad (47)$$

$S_1$ in Eq. (45) gives rise to the expression for the adiabatic force at finite temperature

$$\bar{F}_{ad} = \sum_n \frac{(n \pi)^2}{L^3(0)} \frac{1}{e^{\beta(E_n(0) - \mu)} + 1}. \quad (48)$$

while $S_2$ in Eq. (46) and $S_3$ in Eq. (47) contribute to the nonadiabatic force. In Eq. (48), we have assumed the time range lies in

$$\frac{\hbar}{\Delta E} \ll t \ll \frac{L}{\dot{L}}, \quad (49)$$

where the lower and upper limits of the inequality in Eq. (49) imply the minimum resolution of time and the time necessary for the wall to move by order of $L$, respectively. This is a physically imposed assumption, which will also be employed below.

$S_2$ can be rewritten, using $\gamma_{nm} \gamma_{mn} = -\frac{(4m^2)}{(m^2 - n^2)}$, and $L^2(t)(E_n - E_m) = \frac{2}{\pi^2} (n^2 - m^2)$ and noting the fact that, for a symmetric function $s(n, m)$, $\sum_{m \neq n} s(n, m)(1 - e^{\frac{E_n - E_m}{\hbar} t}) = 2 \sum_{m \neq n} s(n, m) \sin^2 \left( \frac{E_n - E_m}{2 \hbar} t \right) \rightarrow \sum_{m \neq n} s(n, m)$, where the final reduction is possible under the assumption in Eq. (49). As a result, $S_2$ becomes

$$S_2 = \frac{16 \hbar^2 \dot{L}(0)^2}{\pi^2 L(0)} \sum_{n > m} \left[ \frac{m^2 n^2}{(n^2 - m^2)^3} (f_n - f_m) \right]. \quad (50)$$

The factor $(f(E_n) - f(E_m))$ in Eq. (50) gives a constraint under which the summation over $m$ and $n$ should be taken. Similarly $S_3$ becomes

$$S_3 = \frac{32 \hbar^2 (\dot{L}(0))^2}{\pi^2 L(0)} \sum_{n > m} \left[ \frac{m^2 n^2}{(n^2 - m^2)^3} (f_n - f_m) \right]. \quad (51)$$
To summarize, the nonadiabatic force is given by

\[ \vec{F}_{\text{non-adv}} = S_2 + S_3 = C \frac{(\dot{L}(0))^2}{(L(0))} \]  

(52)

with

\[ C = \frac{48\hbar^2}{\pi^2} \sum_{n \neq m} \left[ \frac{m^2n^2}{(n^2 - m^2)^2} (f_n - f_m) \right], \]  

(53)

where \( C < 0 \) since Fermi distribution function monotonically decreases with energy and \( f_n - f_m = f(E_n) - f(E_m) < 0 \) for \( n > m \). \( \vec{F}_{\text{non-adv}} \) in Eq. (52) is proportional to \( L^2 \) and does not break the time-reversal symmetry, in marked contrast with the general conjecture so far. The result in Eqs. (48) and (52) will be confirmed by the exact analytical result in the next Section.

Here we should give two comments:

i) The first one is concerned with the level crossing. From an experimental viewpoint, we are considering a quasi-one-dimensional (1-d) hard- or soft-walled rectangular parallelepiped. In this case, the energy gaps between sub-bands are large enough not to meet crossings among sub-bands. Therefore the dynamics within each sub-bands (e.g., the lowest sub-band) used in our scheme is guaranteed. As a more general case, one might consider a 3-d rectangular parallelepiped. In this case, the energy gaps between sub-bands (e.g., the lowest sub-band) used in our scheme is guaranteed. As a more general case, one might consider a 3-d rectangular parallelepiped with the size \( L_x \times L_y \times L_z \), one of whose walls is moving in \( x \)-direction. Then each adiabatic state is characterized by a set of quantum numbers \( (n_x, n_y, n_z) \) and the energy spectra as a function of \( L_x \) might show level crossings among manifolds with different \( n_x, n_y, \) and \( n_z \). If a confined particle is initially in a manifold with the fixed \( n_y \) and \( n_z \) and the cavity expands only in \( x \)-direction, however, there occurs no transition among manifolds with different \( n_y \) and \( n_z \) and thereby energy crossings do not affect the present dynamics at all. Finally, one can conceive an expanding 3-d spherical billiard, which has level crossings among manifolds with different angular momenta. Since there is no transition matrix element among different angular momentum states in the symmetry-keeping dynamics, however, the dynamics is free from the problem of level crossings if a zero-angular momentum state will be chosen as an initial state, which again guarantees our scheme.

ii) The second one is whether or not the expression for the force operator and the expectation of the nonadiabatic force quadratic in the rate of dilation under the "time-independent" Dirichlet boundary condition (TDD), to study the density of diabolical points and both shifts and splitting of level degeneracies. They had recourse to a degenerate perturbation theory with use of diabatic eigenstates at the degenerating point, indicating (in Appendix of their paper) that the off-diagonal energy matrix elements are zero for dilations. It is not easy to interpret the present dynamical result under TDD only in terms of the static one of BW under TDD. The present work is concerned with a one-dimensional billiard with a time-dependent walls, where the adiabatic eigenvalues have no degeneracy and only the level shifts occur against the adiabatic dilation. In the context of the adiabatic (AF) and non-adiabatic forces (NAF) which are given on the second line in Eq. (43), the non-diagonal matrix elements of AF certainly vanish, consistent with BW, but matrix elements of NAF are new, whose counterpart cannot be found in the treatment of BW under TDD. The force operator of BW is only concerned with AF. Non-vanishing matrix elements of NAF in Eq. (43) are due to a dynamical contribution in Eq. (22) coming from TDD. A mechanism of the absence of a term linear in the rate of dilation in the expectation value of the force operator in Eq. (44) is not due to vanishing non-diagonal matrix elements of AF and can not be explained directly within the static framework under TDD. It is caused by a subtle cancellation of the linear cross-coupling terms among the matrix elements of the force operator expressed as a series expansion w.r.t the rate of dilation and the density matrix expressed in the similar expansions in the framework of the extended Kubo-Greenwood formula. On the other hand, Berry and Klein [18] were once involved in the similar subject as the present one, but they showed neither the definition of NAF operator nor the expectation value of the force as a power series in the rate of change of the scale size of the container.

IV. EXACT ANALYSIS

Exact solution of the Schrödinger equation with a moving Dirichlet boundary condition due to the motion of a wall was found by Makowski et.al [19, 20]. The greatness of their work lies in that they discovered the transitionless basis functions where the adiabatic states are also the solution of the time-dependent Schrödinger equation, which recently received a great attention in the context of the shortcut to the adiabatic dynamics [21]. With
use of their basis functions we can proceed to evaluate the nonadiabatic force exactly. After a brief summary of their results, we shall carry out this procedure.

The system we are going to explore is described by the Schrödinger equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

where the wave function satisfies the moving Dirichlet boundary condition in Eq. \(11\).

After the scaling of space coordinate \(x\) and wave function \(\psi\) by \(L(t)\) and \(\sqrt{L(t)}\), respectively as

\[
x \to y = \frac{x}{L(t)} \quad \text{and} \quad \psi \to \tilde{\psi} = \sqrt{L(t)}\psi,
\]

Schrödinger equation with the moving boundary becomes

$$i\hbar \frac{\partial \tilde{\psi}}{\partial t} = -\frac{1}{2L^2} \frac{\partial^2 \tilde{\psi}}{\partial y^2} + i\hbar \frac{\dot{L}}{L} \frac{\partial \tilde{\psi}}{\partial y} + \frac{\hbar^2}{2L} \tilde{\psi},$$

with a static Dirichlet boundary condition, i.e.,

$$\tilde{\psi}(0) = \tilde{\psi}(1) = 0.$$ 

The transformation above is nothing but the time-dependent canonical transformation described in Section 4.

Then, applying the gauge transformation

$$\phi(y,t) = G\tilde{\psi}(y,t) = \exp \left( -\frac{i\hbar}{2} \frac{\dot{L}}{L} L(t)y^2 \right) \tilde{\psi}(y,t),$$

Eq. (56) can be reduced to the Schrödinger equation for the time-dependent harmonic oscillator:

$$i\hbar L^2 \frac{\partial \phi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \phi}{\partial y^2} + \frac{\hbar^2}{2} L^3 \dot{L} y^2 \phi.$$ 

Equation (58) can be solved exactly if the time-dependence of the boundary satisfies the following equation [14]:

$$L^3 \dot{L} = \text{const} = -\frac{1}{4} B^2.$$ 

For a linearly expanding or contracting billiard with the constant wall velocity \(\dot{L} = \dot{L}(0)\), i.e.,

$$L(t) = L_0 + \dot{L} t,$$ 

the condition (59) is satisfied, \(B = 0\). (A general case of \(B \neq 0\) will be investigated in Appendix A)

The solution in this case is

$$\psi_n(x,t) = \frac{2}{L(t)} \exp \left( \frac{i\hbar \dot{L}}{2L} x^2 - \frac{in\pi x}{2\hbar} \tau(t) \right) \times \sin \left( \frac{n\pi x}{L(t)} \right),$$ 

where \(\tau(t)\) is a new time variable defined by

$$\tau(t) = \int_0^t \frac{ds}{L^2(s)}.$$ 

The solution (61) is the transitionless state where the adiabatic state also serves as the solution of the time-dependent Schrödinger equation, which recently received a renewed attention [21]. An assembly of states in Eq. (61) constitute the complete ortho-normal set.

Let’s obtain the adiabatic and nonadiabatic forces acting on the moving wall which is confining the Fermi gas into the cavity, by evaluating the expectation of Hamiltonian. Statistical weight factors (Fermi distribution) will be incorporated a posteriori. In the case of a linearly expanding cavity described by Eq. (60), the initial state of a particle is given by \(\sqrt{\frac{L}{L_0}} \sin \left( \frac{\pi x}{L_0} \right)\) with the eigenvalue \(E_0(0) = \frac{\beta^2}{2L_0}\) and the wall suddenly begins to move with constant velocity \(\dot{L}\). The solution of Eq. (61) can be expressed in terms of the transitionless states in Eq. (61) as

$$\psi = \frac{1}{\sqrt{L}} \exp \left( \frac{i\hbar}{2} \dot{L} L y^2 \right) \phi,$$

$$\phi = \sum_n c_n(t) \varphi_n(y), \quad \varphi_n(y) = \sqrt{2} \sin(n\pi y),$$ 

where expansion coefficients are given by

$$c_n^{(t)}(t) = c_n^{(0)}(0) \exp \left( -\frac{in^2\pi^2}{2\hbar} \tau(t) \right)$$ 

with

$$c_n^{(0)}(0) = 2 \int_0^1 \sin(l\pi y) \sin(n\pi y) \exp \left( -\frac{i\hbar}{2} \dot{L}(0) L(0) y^2 \right) dy.$$ 

The average energy can be represented as

$$\langle E^{(t)}(t) \rangle = \int_0^L \psi^*(x,t) \psi(x,t) dx$$

$$= \frac{1}{L} \int_0^1 \psi^*(y,t) \left( -\frac{\partial^2}{\partial y^2} \right) \psi(y,t) dy$$

$$= \frac{1}{2L^2} I_0 + \frac{\hbar \dot{L}}{L} \text{Im}(I_1) + \frac{\hbar^2 \dot{L}^2}{2} I_2,$$ 

where \(I_0, I_1\) and \(I_2\) are respectively defined by

$$I_0 = \int_0^1 |\psi_y|^2 dy$$

$$= 2\pi^2 \sum_n c_n^{*}(t) \int_0^1 \cos(n'\pi y) \cos(n\pi y) dy$$

$$= \pi^2 \sum_n |c_n^{(0)}(0)|^2 n^2,$$ 

(67)
\begin{align}
I_1(t) &= \int_0^1 y \phi^* \phi \, dy = \sum_n c_n^{(t)*}(t) \sum_n c_n^{(t)}(t) n \pi J_1(n, n'), \\
I_2(t) &= \int_0^1 y^2 | \phi |^2 \, dy = \sum_{n'} c_{n'}^{(t)*}(t) \sum_n c_n^{(t)}(t) J_2(n, n'),
\end{align}

(68)

with \( J_1(n, n') \) and \( J_2(n, n') \) given in Appendix B.

Since the work done by the force \( F^{(t)}(t) \) (on the moving wall) from a Fermi particle is supplied by the excess energy induced by its energy loss, we find

\[
F^{(t)}(t) = -\frac{\partial (E^{(t)}(t))}{\partial L} = \frac{1}{L^3} I_0 + \frac{\hbar \tilde{L}}{I^2} \text{Im}(I_1).
\]

(70)

The force can also be calculated by taking the expectation of the force operator \( \tilde{F} \) in Eq. (22) defined in the original space:

\[
\tilde{F}(t) = \int_0^{L(t)} \psi^*(x, t) \tilde{F} \psi(x, t) \, dx = \frac{1}{L^3} I_0 + \frac{\hbar \tilde{L}}{I^2} \text{Im}(I_1)
\equiv \tilde{F}_{ad} + \tilde{F}_{non-ad}.
\]

(71)

Eqs. (70) and (71) exactly agree mutually, which guarantees the validity of the force operator defined in Section 4.

To investigate the nature of the nonadiabatic force more carefully, we must estimate the integrals \( I_0 \) and \( I_1 \). By expanding the exponential in Eq. (69) as

\[
\exp \left( -\frac{i \hbar}{2} \hat{L}(0) L(0) y^2 \right) = 1 - \frac{i \hbar}{2} \hat{L}(0) L(0) y^2 - \frac{\hbar^2}{8} (\hat{L}(0) L(0))^2 y^4 + \ldots
\]

(72)

we find (see Appendix B)

\[
I_0 = \pi^2 \ell^2 + \frac{\pi^2 \hbar^2 (\hat{L}(0) L(0))^2}{4} \left( \sum_n n^2 (J_2(n, \ell))^2 - \ell^2 J_3(\ell, \ell) \right)
\]

(73)

and

\[
\text{Im}(I_1) = -\frac{\pi \hbar}{2} \hat{L}(0) L(0) \sum_n 2n J_1(n, \ell) J_2(n, \ell)
= -\hbar \hat{L}(0) L(0) \sum_{n \neq \ell} \left( \frac{16}{\pi^2} \frac{(n \ell)^2}{(n^2 - \ell^2)^3} \right)
\times \cos \left( \frac{\pi}{2} (\ell^2 - n^2) \tau(t) \right).
\]

(74)

In Eq. (74) the last factor can be taken as \( \cos(\cdot \cdot \cdot) \sim 1 \) in the time range in Eq. (19). We find that the \( \hat{L}(0) \)-dependent terms are included not only in \( I_1 \) but also in \( I_0 \). Substituting Eqs. (73) and (74) into Eq. (71), we see: (i) the \( \hat{L}(0) \)-independent term in Eq. (71) gives rise to the adiabatic force \( \langle \tilde{F}_{ad} \rangle \); (ii) the remaining terms give the nonadiabatic force \( \langle \tilde{F}_{non-ad} \rangle \).

Picking up the first term on r.h.s of Eq. (63), multiplying statistical weight \( f_n \) and summing up over all initial eigenstates, we find

\[
\tilde{F}_{ad} = \sum_n n^2 \frac{\pi^2}{L^3(0)} f_n,
\]

(75)

which justifies Eq. (18).

Taking together Eq. (74) and the second term on r.h.s. of Eq. (69), multiplying statistical weight \( f_n \) and summing up over all initial eigenstates, we find

\[
\tilde{F}_{non-ad} = C' \frac{\langle \hat{L}(0) \rangle^2}{L(0)}
\]

(76)

with

\[
C' = \sum_n \left[ \sum_{m \neq n} \frac{16 \hbar^2}{\pi^2} \left( \frac{m^2 n^2}{(m^2 - n^2)^3} + \frac{m^4 n^2}{(m^2 - n^2)^3} \right) - \frac{\hbar^2}{4} \frac{\pi^2}{2} \left( \frac{n^2}{5} - \frac{1}{\pi^2} + \frac{3}{2 n^2 \pi^4} \right) \right] f_n,
\]

(77)

which is again negative due to the dominant term proportional to \( n^2 \). Irrespective of the direction of the moving wall, the non-adiabatic force always acts inwards and is proportional to the square of the wall velocity, which is in marked contrast with the general belief that the non-adiabatic force should be linear in the wall velocity and mimic the irreversible or frictional force. There is a minor discrepancy between the absolute values of \( C \) and \( C' \), which is due to the difference in the way of solving the problem (23).

In closing this Section, we should note the following two remarks:

(i) Firstly there is a mystery in obtaining the nonadiabatic force in the exact analysis above. In this Section we had recourse to the transitionless states as basis functions. We can see neither nonadiabatic transition nor nonadiabatic force so long as tracking individual transitionless states. In fact, if we shall evaluate the expectation value \( \bar{F} = \langle \tilde{F} \rangle \) using only a single transitionless state in Eq. (61), we will obtain formally the same result as in Eq. (71) but with \( I_0 = \pi^2 \ell^2 \) and \( \text{Im}(I_1) = 0 \) and can see no nonadiabatic force. Throughout this paper, we are
considering the case that the wall is fixed up to the initial time $t = 0$ and suddenly moves at $t > 0$. Therefore the eigenstate under the fixed boundary generates at $t = 0$ a mixture of the transitionless states that are eigenstates of the moving boundary, giving rise to nonvanishing coefficients $\{c_n\}$. Exploring Eqs. (67) and (65), we can understand that the correlation among non-zero coefficients $\{c_n\}$ resulted in the nonvanishing nonadiabatic force in Eq. (76). By contrast, it is quite easy to see the mechanism for nonadiabatic force in the perturbative theory of Section III, where the energy diffusion among standard adiabatic states can explain the nonadiabatic force.

ii) Secondly the exact analysis can also reveal the nature of the non-adiabatic force in the case when the expanding rate of the cavity is not constant, so long as $L(t)$ obeys Eq. (53), namely when $L(t) = \sqrt{at^2 + bt + c}$. Under the initial condition with $L(0) = L_0$ and $\dot{L}(0) = \frac{b}{2\sqrt{a}} \neq 0$, $\dot{F}(t)$ can be calculated, leading to the identical result $F_{\text{non-ad}} = \tilde{C}(\frac{L(t)}{L(0)})^2$ with a negative constant $\tilde{C}$. The details are given in Appendix A. Thus our assertion that the non-adiabatic force is quadratic in the wall velocity and thereby does not break the time-reversal symmetry does hold also for the hard-wall cavity with the time-dependent wall velocity.

V. CASE OF SOFT-WALLED CONFINEMENT

To see the universality of our argument so far, we proceed to investigate the case of the soft-wall confinement, and consider the force acting on the soft wall. Here the Fermi gas is assumed to be confined in a harmonic trap with the confining length $L$ changing linearly in time (see Fig. 2).

To evaluate the expectation of energy and force, we shall solve the Schrödinger equation for a particle under the harmonic trap with the time-dependent trapping frequency,

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = H \psi(x,t) = -\frac{1}{2} \frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{1}{2} m \omega^2(t) x^2 \psi(x,t),$$

$$-\infty < x < +\infty,$$  

(78)

where, due to the prescription $\hbar^2/m = 1$, we see $m = \hbar^2$. In Eq. (78) $\omega(t)$ is expressed in terms of the time-dependent confining length: $\hbar \omega(t) = \frac{1}{L(t)}$ where $L(t) = L_0 + \dot{L} t$ with constant $\dot{L}$.

After scale and gauge transformations like Eqs. (55) and (57), Eq. (68) is reduced to

$$i\hbar L(t)^2 \frac{\partial \phi(y,t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \phi(y,t)}{\partial y^2} + \frac{1}{2} \hbar^2 y^2 \phi(y,t),$$

(79)

where $\psi(x,t) = \frac{1}{\sqrt{L(t)}} \exp(\frac{i}{\hbar} \tilde{L}(t)L(t)y^2) \phi(y,t)$ and $x = L(t)y$. The solution of Eq. (79) can be written as

$$\phi_n(y,t) = \exp \left( -i \int_0^t \frac{n + \frac{1}{2}}{\hbar L^2(s)} ds \right) Y_n(y),$$

(80)

where

$$Y_n(y) = \sqrt{\frac{\sqrt{\hbar}}{2^n n! \sqrt{\pi}}} \exp \left( -\frac{\hbar y^2}{2} \right) H_n(\sqrt{\hbar}y)$$

(81)

with

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}.$$  

(82)

An assembly of solutions in Eq. (81), which are transitionless states, constitute the complete ortho-normal set.

As in the case of the hard-wall cavity, let's require that the initial state is an eigenstate under the fixed harmonic trap:

$$\psi(x,t)|_{t=0} = \psi_0(x) = \frac{1}{\sqrt{L(0)}} Y_0 \left( \frac{x}{L(0)} \right),$$

(83)

which is equivalent to

$$\phi(y,0) = Y_0(y) \exp \left( -i \frac{\hbar L(0) L(0) y^2}{2} \right).$$

(84)

The time-dependent solution satisfying this initial condition is expressed as

$$\phi(y,t) = \sum_{n=0}^{+\infty} c_n \phi_n(y,t),$$

(85)
Expansion coefficients $c_n$ are determined by

$$c_n = \int_{-\infty}^{+\infty} \phi(y,0)Y_n^*(y)dy$$

$$= \int_{-\infty}^{+\infty} Y_n^*Y_n(y)exp(-\frac{i}{2}\hbar\hat{L}(0)L(0)y^2)dy. \quad (86)$$

The average energy $\langle E(t) \rangle$ is calculated by

$$\langle E(t) \rangle = \int_{-\infty}^{+\infty} \psi^*(x,t)H\psi(x,t)dx$$

$$= \frac{1}{2L^2}K_0 + \frac{\hbar L}{L^2}\,{\rm Im}(K_1) + \frac{\hbar^2 L^2}{2}K_2,$$ \quad (87)

where $K_0 = \int_{-\infty}^{+\infty} (|\phi_y|^2 + y^2|\phi|^2)$, $K_1 = \int_{-\infty}^{+\infty} y\phi^*\phi_y dy$ and $K_2 = \int_{-\infty}^{+\infty} y^2|\phi|^2 dy$.

The average force is obtained by taking the derivative of $\langle E(t) \rangle$ with respect to $L$ as

$$\bar{F} = \frac{\partial E}{\partial L(t)} = \frac{1}{L^3}K_0 + \frac{\hbar L}{L^2}\,{\rm Im}(K_1). \quad (88)$$

On the other hand, one should evaluate the expectation value of the force operator to reproduce the above result. Although the range of $x$ is not limited in the case of soft-wall confinement, one can define the force operator using the time-dependent canonical transformations related to scaling of both coordinates and wave functions and its inverse transformations in Section II. The force operators in the original space is given by

$$\hat{F} = -\frac{1}{L} \partial^2 \frac{\partial}{\partial y^2} + \frac{x^2}{L^3} + \frac{\hbar L}{L^2} \left( x \frac{\partial}{\partial x} + \frac{1}{2} \right), \quad (89)$$

which includes the second term missing in Eq. (22). Therefore its expectation is

$$\bar{F} = \langle \hat{F} \rangle = \frac{1}{L^3}K_0 + \frac{\hbar L}{L^2}\,{\rm Im}(K_1), \quad (90)$$

which accords with Eq. (88) and guarantees the validity the definition of the force operator in Eq. (89).

With use of asymptotic expressions for $K_0$ and $K_1$, we again find $F_{\text{non-adi}} = C_{\text{soft}}L^2$ with a coefficient $C_{\text{soft}} < 0$, namely, the nonadiabatic force is proportional to the square of velocity of the soft wall, and never breaks the time-reversal symmetry.

VI. CONCLUSIONS

We investigated the nature of the non-adiabatic force acting on the cavity wall, which is generated from the non-interacting quantal gas confined in the expanding cavity. Firstly, with use of the time-dependent canonical transformations by which we can move to the non-expanding cavity, the force operator is defined. Secondly, we analyzed the expectation of the force operator perturbatively with use of von Neumann equation for the density operator, which works when the cavity wall suddenly begins to move at time origin. We found that the non-adiabatic force is quadratic in the wall velocity and thereby does not break the time-reversal symmetry, in marked contrast with the existing conjecture. Finally, using an assembly of the transitionless quantum states, we obtain the nonadiabatic force exactly. The exact result justifies the validity of both the definition of force operator and the issue of the perturbative theory, and guarantees the present findings in the general case when the expansion rate of the cavity is time-dependent. The mysterious mechanism of nonadiabatic transition with use of transitionless quantum states is also explained. The study is done on both cases of the hard-wall and soft-wall confinement with the time-dependent confining length. Quantum fluctuation theorem, deviation from the standard Fermi-Dirac distribution and equation of states, etc. in the expanding cavity where Hilbert space is time-dependent also constitute interesting subjects, which will be investigated in due course.

Acknowledgments. We are grateful to S. Tanimura, A. Sugita and A. Terai for useful comments. K.N. expresses special thanks to B. Mehlig for enlightening discussions in the early stage of the present work and to M.V. Berry for kindly informing us of his old paper with Klein touching on the similar subject as the present one. T.M. acknowledges a support under the JSPS program (Grant in Aid 22·7744). The work is also supported through a project of the Uzbek Academy of Sciences (FA-F2-084).

Appendix A: The case $L(t) = \sqrt{at^2 + bt + c}$

We consider the expanding cavity of the size $L(t)$ governed by Eq. (58), which has the general solution $L(t) = \sqrt{at^2 + bt + c}$ with $B^2 = b^2 - 4ac$ and the initial velocity $\dot{L}(0) = \frac{b}{2\sqrt{c}} \neq 0$. In this case the reduced Schrödinger equation in Eq. (58) takes the following form:

$$i\hbar L^2 \frac{\partial \phi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \phi}{\partial y^2} - \frac{\hbar^2 B^2 y^2 \phi}{8}, \quad (A1)$$

where $y = x/L(t)$ and

$$\phi(y,t) = \sqrt{L} \exp \left( -\frac{i\hbar}{2} L \dot{y} y^2 \right) \psi(yL(t),t). \quad (A2)$$
Changing the time variable from \( t \) to \( \tau \) defined by Eq. \([A2]\), Eq. \([A1]\) can be solved as

\[
\phi_n(y, t) = \varphi_n(y) \exp \left( \frac{i}{\hbar} K_n \tau(t) \right) \tag{A3}
\]

with

\[
\varphi_n(y) = A_n \exp \left( -\frac{B y^2}{2} \right) y \text{ Re } \left[ M \left( \frac{3}{4} + \frac{i K_n}{\hbar B} \cdot \frac{3}{2} i \hbar B - y^2 \right) \right] \tag{A4}
\]

where \( M(a, b, y) \) is Kummer function (i.e., the confluent hypergeometric function) and \( K_n \) are the solutions of the equation

\[
M \left( \frac{3}{4} + \frac{K_n}{\hbar B} \cdot \frac{3}{2} i \hbar B \right) = 0. \tag{A5}
\]

In the semiclassical region where \( \hbar B \sim 0 \), one can see \( K_n \sim \frac{n^2 \pi^2}{2} \).

Now we shall solve the time-dependent problem under the initial state

\[
\phi(y, 0) = \sqrt{2} \sin(\pi \ell y), \tag{A6}
\]

which corresponds to \( \psi(x, 0) = \sqrt{\frac{2}{L(0)}} \sin(\pi \ell x) \) with the eigen-energy \( E_\ell(0) = \frac{\ell^2 \pi^2}{2L^2} \). Expanding \( \phi(y, t) \) in terms of the complete set of functions \( \varphi_n(y) \), we have

\[
\phi(y, t) = \sum_n c_n(t) \varphi_n(y), \tag{A7}
\]

where

\[
c_n(t) = c_n(0) \exp \left( -\frac{i}{\hbar} K_n \tau(t) \right) \tag{A8}
\]

with \( c_n(0) \) given by

\[
c_n(0) = \sqrt{2} \int_0^1 \varphi_n^*(y) \sin \ell \pi y \exp \left( -\frac{i \hbar}{2} L(0) \hat{L}(0) y^2 \right) dy
\]

\[= J_0(n, \ell) - \frac{i \hbar}{2} L(0) \hat{L}(0) J_2(n, \ell)
- \frac{\hbar^2}{8} (L(0) \hat{L}(0))^2 J_3(n, \ell) + \cdots. \tag{A9}\]

\( J_0, J_2 \) and \( J_3 \) are defined respectively as

\[
J_0(n, \ell) = \sqrt{2} \int_0^1 \varphi_n^*(y) \sin(\pi \ell y) dy, \tag{A10}
\]

\[
J_2(n, \ell) = \sqrt{2} \int_0^1 y^2 \varphi_n^*(y) \sin(\pi \ell y) dy, \tag{A11}
\]

and

\[
J_3(n, \ell) = \frac{1}{\sqrt{2}} \int_0^1 y^4 \varphi_n^*(y) \sin(\pi \ell y) dy. \tag{A12}
\]

Pair products of \( c_n \)s are

\[
c_{n'}(0)c_n(0) \sim \tilde{J}_0(n, \ell) \tilde{J}_0(n, \ell)
- \frac{i \hbar}{2} \hat{L}(0) \hat{L}(0) \left[ \tilde{J}_0(n', \ell) \tilde{J}_2(n, \ell) - \tilde{J}_0(n, \ell) \tilde{J}_2(n', \ell) \right]
\]

\[\frac{\hbar}{8} (\hat{L}(0) \hat{L}(0))^2 \left[ \tilde{J}_0(n', \ell) \tilde{J}_3(n, \ell) + \tilde{J}_0(n, \ell) \tilde{J}_3(n', \ell) + 2 \tilde{J}_2(n', \ell) \tilde{J}_2(n, \ell) \right] \tag{A13}\]

and

\[
c_{n'}(t)c_n^*(t) = c_n(0)c_n^*(0) \exp \left( -i \int_0^t \frac{K_{n'} - K_n}{\hbar L^2(s)} ds \right). \tag{A14}\]

The expectation for the force operator in Eq. \([22]\) is obtained as

\[
\langle F(t) \rangle = \int_0^L \psi^*(x, t) \hat{F} \psi(x, t) dx
= \frac{1}{L^3} \tilde{I}_0 + \frac{\hbar}{L^2} \text{Im}(\tilde{I}_1), \tag{A15}\]

with \( \tilde{I}_0 \) and \( \tilde{I}_1 \) being expressed as

\[
\tilde{I}_0 = \int_0^1 |\psi_0|^2 dy = 2 \sum_n |c_n(0)|^2 K_n
+ \frac{1}{4} \hbar^2 B^2 \sum_n \sum_{n'} \text{Re} \left[ c_{n'}^*(t)c_n(t) \right] \int_0^1 \varphi_{n'} \varphi_n y^2 dy \tag{A16}\]

and

\[
\tilde{I}_1 = \int_0^1 y \psi^* \dot{\psi} dy
= \sum_n \sum_{n'} c_{n'}^*(t)c_n(t) \int_0^1 y \varphi_{n'} \frac{\partial \varphi_n}{\partial y} dy, \tag{A17}\]

respectively. Using the asymptotic forms for the pair products in Eq. \([A13]\) together with Eq. \([A14]\) and taking the short-time region employed in the main text, we reach the result:

\[
F_{\text{non-ad}} = \frac{C}{L(0)} \left( \hat{L}(0) \right)^2, \tag{A18}\]

which is again proportional to the square of the wall velocity.
Appendix B: Calculation of $I$ and $J$

Coefficients $c_n^{(\ell)}(0)$ are defined by

$$c_n^{(\ell)}(0) = 2\int_0^1 \sin(l\pi y) \sin(n\pi y) \exp\left(-\frac{i\hbar}{2} \hat{L}(0) L(0) y^2\right) dy = \delta_{n\ell} - \frac{i\hbar}{2} \hat{L}(0) L(0) J_2(n, \ell) - \frac{\hbar^2}{8} (\hat{L}(0) L(0))^2 J_3(n, \ell) + \cdots. \quad (B1)$$

Their products are

$$c_{n'}^{(\ell')^*}(0) c_n^{(\ell)}(0) \sim \delta_{n'\ell} \delta_{n\ell} - \frac{i\hbar}{2} \hat{L}(0) L(0) \left[ \delta_{n'\ell} J_2(n, \ell) - \delta_{n\ell} J_2(n', \ell) \right] - \frac{\hbar^2}{8} (\hat{L}(0) L(0))^2 \left[ \delta_{n'\ell} J_3(n, \ell) + \delta_{n\ell} J_3(n', \ell) \right] + 2 J_2(n', \ell) J_2(n, \ell). \quad (B2)$$

Therefore $I_1$ is expressed as

$$I_1 \sim -\frac{1}{2} - \frac{i\hbar}{2} \hat{L}(0) L(0) \times \left[ \sum_n n J_1(n, \ell) J_2(n, \ell) - \ell \sum n J_1(\ell, n') J_2(n', \ell) \right]. \quad (B3)$$

$J_1$, $J_2$ and $J_3$ used above are given respectively by

$$J_1(n, \ell) = 2 \int_0^1 y \sin(\ell \pi y) \cos(n \pi y) dy = \begin{cases} (-1)^{n+\ell+1} \frac{1}{2(n+\ell+1)^2} & \text{for } n = \ell \\ (-1)^n (\frac{1}{2n^2} - \frac{1}{2(n+\ell)^2}) & \text{otherwise} \end{cases} \quad (B4)$$

and

$$J_2(n, \ell) = 2 \int_0^1 y^2 \sin(\ell \pi y) \sin(n \pi y) dy = \begin{cases} \frac{1}{4} - \frac{1}{8(n+\ell)^2} & \text{for } n = \ell \\ (-1)^n (\frac{1}{2n^2} - \frac{3}{2(n+\ell)^2}) & \text{otherwise} \end{cases} \quad (B5)$$

and

$$J_3(n, \ell) = 2 \int_0^1 y^4 \sin(\ell \pi y) \sin(n \pi y) dy = \begin{cases} \frac{1}{4} - \frac{1}{16(n+\ell)^2} + \frac{3}{8(n+\ell)^2} & \text{for } n = \ell \\ (-1)^n (\frac{16n^2}{(n^2+\ell^2)^2} - \frac{192n^2(\ell^2+1)}{(n^2+\ell^2)^4}) & \text{otherwise} \end{cases}. \quad (B6)$$

[1] M.V. Berry and J.M. Robbins, Proc. R. Soc. London. A. 442 659 (1993).
[2] M.V. Berry and E.C. Sinclair, J. Phys. A. 30 2853 (1997).
[3] C. Jarzynski, Phys. Rev. Lett. 74 2937 (1995).
[4] D.A. Hill and J.A. Wheeler, Phys. Rev. 89 1102 (1952).
[5] J. Blocki et al., Ann. Phys. 113 330 (1978).
[6] M. Wilkinson, J. Phys. A. 21 4021; J. Phys. A. 23 3603 (1990).
[7] M. Wilkinson and E. J. Austin, J. Phys. A. 28 2277 (1995).
[8] S. Mizutori and S. Aberg, Phys. Rev. E. 56 6311 (1997).
[9] A. Bulgac, G.D. Dang and D. Kusnezov, Chaos, Solitons and Fractals 8 1149 (1997); Phys. Rev. E. 58 196 (1998).
[10] D. Cohen, Phys Rev Lett. 82 4951 (1999).
[11] O.M. Auslaender and S. Fishman, Phys Rev Lett. 84 1886 (2000).
[12] R.J. Tait, Q. Appl. Math.37 313 (1979).
[13] A. Munier et al., J. Math. Phys. 22 1211 (1981).
[14] M. Razavy, Lett. Nuovo Cimento 37 2384 (1983).
[15] M. Razavy, Phys. Rev. A 44 2384 (1991).
[16] J.M. Cervero and J.D. Lejarreta, Europhys. Lett. 45 6 (1999).
[17] M. V. Berry and M. Wilkinson, Proc. R. Soc. Lond. A 392 15 (1984).
[18] M.V. Berry and G. Klein, J. Phys. A 17 1805 (1984).
[19] A.J. Makowski and S.T. Dombinski, Phys. Lett. A 154 217 (1991).
[20] A.J. Makowski and P. Peplowski, Phys. Lett. A 163 142 (1992).
[21] Xi Che et al., Phys. Rev. Lett. 104 063002 (2010).
[22] D.A. Greenwood, Proc. Phys. Soc. A 71 585 (1958).
[23] The wave function in Section 11V describes a statistical pure state whereas the density operator in Section III is concerned with a mixed state causing the correlation among initial states, which resulted in a discrepancy in the treatment of statistical weights. At zero temperature and for the particle number $N \gg 1$, for example, this discrepancy amounts to $|C| \sim N$ and $|C'| \sim N^3$, according to our numerical calculations.