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A Pseudopolynomial Algorithm
for Alexandrov’s Theorem

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Abstract. Alexandrov’s Theorem states that every metric with the
global topology and local geometry required of a convex polyhedron is
in fact the intrinsic metric of some convex polyhedron. Recent work by
Bobenko and Izmestiev describes a differential equation whose solution
is the polyhedron corresponding to a given metric. We describe an al-
gorithm based on this differential equation to compute the polyhedron
to arbitrary precision given the metric, and prove a pseudopolynomial
bound on its running time.

1 Introduction

Consider the intrinsic metric induced on the surface \( M \) of a convex body in \( \mathbb{R}^3 \).
Clearly \( M \) under this metric is homeomorphic to a sphere, and locally convex in
the sense that a circle of radius \( r \) has circumference at most \( 2\pi r \).

In 1949, Alexandrov and Pogorelov [1] proved that these two necessary con-
ditions are actually sufficient: every metric space \( M \) that is homeomorphic to a
2-sphere and locally convex can be embedded as the surface of a convex body
in \( \mathbb{R}^3 \). Because Alexandrov and Pogorelov’s proof is not constructive, their work
opened the question of how to produce the embedding given a concrete \( M \).

To enable computation we require that \( M \) be a polyhedral metric space,
locally isometric to \( \mathbb{R}^2 \) at all but \( n \) points (vertices). Now the theorem is that
every polyhedral metric, a complex of triangles with the topology of a sphere
and positive curvature at each vertex, can be embedded as an actual convex
polyhedron in \( \mathbb{R}^3 \). This case of the Alexandrov-Pogorelov theorem was proven
by Alexandrov in 1941 [1], also nonconstructively. Further, Cauchy showed in
1813 [3] that such an embedding must be unique. All the essential geometry
of the general case is preserved in the polyhedral case, because every metric
satisfying the general hypothesis can be polyhedrally approximated.

Algorithms for Alexandrov’s Theorem are motivated by the problem of fold-
ing a polygon of paper into precisely the surface of a convex polyhedron. There

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are efficient algorithms to find one or all gluings of a given polygon’s boundary to itself so that the resulting metric satisfies Alexandrov’s conditions [4, 9]. But this work leaves open how to find the actual 3D polyhedra that can be folded from the polygon of paper.

In 1996, Sabitov [12, 11, 13, 5] showed how to enumerate all the isometric maps $M \to \mathbb{R}^3$ for a polyhedral metric $M$, so that one could carry out this enumeration and identify the one map that gives a convex polyhedron. In 2005, Fedorchuk and Pak [6] showed an exponential upper bound on the number of such maps. An exponential lower bound is easy to find, so this algorithm takes time exponential in $n$ and is therefore unsatisfactory.

Recent work by Bobenko and Izmestiev [2] produced a new proof of Alexandrov’s Theorem, describing a certain ordinary differential equation (ODE) and initial conditions whose solution contains sufficient information to construct the embedding by elementary geometry. This work was accompanied by a computer implementation of the ODE [14], which empirically produces accurate approximations of embeddings of metrics on which it is tested.

In this work, we describe an algorithm based on the Bobenko-Izmestiev ODE, and prove a pseudopolynomial bound on its running time. Specifically, call an embedding of $M$ $\varepsilon$-accurate if the metric is distorted by at most a factor $1 + \varepsilon$, and $\varepsilon$-convex if each dihedral angle is at most $\pi + \varepsilon$. For concreteness, $M$ may be represented by a list of triangles with side lengths and the names of adjacent triangles. Then we show the following theorem:

**Theorem 1.** Given a polyhedral metric $M$ with $n$ vertices, ratio $S$ between the largest and smallest distance between vertices, and defect (discrete Gaussian curvature) between $\varepsilon_1$ and $2\pi - \varepsilon_8$ at each vertex, an $\varepsilon_6$-accurate $\varepsilon_9$-convex embedding of $M$ can be found in time $O\left(n^{9/3} 2 S^{831} / (\varepsilon_1^{121} \varepsilon_2^{445} \varepsilon_3^{616})\right)$ where $\varepsilon = \min(\varepsilon_6/nS, \varepsilon_9 \varepsilon_2^2/nS^6)$.

The exponents in the time bound of Theorem 1 are remarkably large. Thankfully, no evidence suggests our algorithm actually takes as long to run as the bound allows. On the contrary, our analysis relies on bounding approximately a dozen geometric quantities, and to keep the analysis tractable we use the simplest bound whenever available. The algorithm’s actual performance is governed by the actual values of these quantities, and therefore by whatever sharper bounds could be proven by a stinger analysis.

To describe our approach, consider an embedding of the metric $M$ as a convex polyhedron in $\mathbb{R}^3$, and choose an arbitrary origin $O$ in the surface’s interior. Then it is not hard to see that the $n$ distances $r_i = Ov_i$ from the origin to the vertices $v_i$, together with $M$ and the combinatorial data describing which polygons on $M$ are faces of the polyhedron, suffice to reconstruct the embedding: the tetrahedron formed by $O$ and each triangle is rigid in $\mathbb{R}^3$, and we have no choice in how to glue them to each other. In Lemma 1 below, we show that in fact the radii alone suffice to reconstruct the embedding, to do so efficiently, and to do so even with radii of finite precision.
Therefore in order to compute the unique embedding of $M$ that Alexandrov’s Theorem guarantees exists, we compute a set of radii $r = \{r_i\}$, and derive a triangulation $T$. The exact radii satisfy three conditions:

1. the radii $r$ determine nondegenerate tetrahedra from $O$ to each face of $T$;
2. with these tetrahedra, the dihedral angles at each exterior edge total at most $\pi$; and
3. with these tetrahedra, the dihedral angles about each radius sum to $2\pi$.

In our computation, we begin with a set of large initial radii $r_i = R$ satisfying Conditions 1 and 2, and write $\kappa = \{\kappa_i\}$ for the differences by which Condition 3 fails about each radius. We then iteratively adjust the radii to bring $\kappa$ near zero and satisfy Condition 3 approximately, maintaining Conditions 1 and 2 throughout.

The computation takes the following form. We describe the Jacobian $\left(\frac{\partial \kappa_i}{\partial r_j}\right)_{ij}$, showing that it can be efficiently computed and that its inverse is pseudopolynomially bounded. We show further that the Hessian $\left(\frac{\partial^2 \kappa_i}{\partial r_j \partial r_k}\right)_{ijk}$ is also pseudopolynomially bounded. It follows that a change in $r$ in the direction of smaller $\kappa$ as described by the Jacobian, with some step size only pseudopolynomially small, makes progress in reducing $|\kappa|$. The step size can be chosen online by doubling and halving, so it follows that we can take steps of the appropriate size, pseudopolynomial in number, and obtain an $r$ that zeroes $\kappa$ to the desired precision in pseudopolynomial total time. Theorem 1 follows.

The construction of [2] is an ODE in the same $n$ variables $r_i$, with a similar starting point and with the derivative of $r$ driven similarly by a desired path for $\kappa$. Their proof differs in that it need only show existence, not a bound, for the Jacobian’s inverse, in order to invoke the inverse function theorem. Similarly, while we must show a pseudopolynomial lower bound (Lemma 11) on the altitudes of the tetrahedra during our computation, the prior work shows only that these altitudes remain positive. In general our computation requires that the known open conditions—this quantity is positive, that map is nondegenerate—be replaced by stronger compact conditions—this quantity is lower-bounded, that map’s inverse is bounded. We model our proofs of these strengthenings on the proofs in [2] of the simpler open conditions, and we directly employ several other results from that paper where possible.

The remainder of this paper supplies the details of the proof of Theorem 1. We give background in Section 2, and detail the main argument in Section 3. We bound the Jacobian in Section 4 and the Hessian in Section 5. Finally, some lemmas are deferred to Section 6 for clarity.

## 2 Background and Notation

In this section we define our major geometric objects and give the basic facts about them. We also define some parameters describing our central object that we will need to keep bounded throughout the computation.
2.1 Geometric notions

Central to our argument are two dual classes of geometric structures introduced by Bobenko and Izmestiev in [2] under the names of “generalized convex polytope” and “generalized convex polyhedron”. Because in other usages the distinction between “polyhedron” and “polytope” is that a polyhedron is a three-dimensional polytope, and because both of these objects are three-dimensional, we will refer to these objects as “generalized convex polyhedra” and “generalized convex dual polyhedra” respectively to avoid confusion.

First, we define the objects that our main theorem is about.

**Definition 1.** A metric $M$ homeomorphic to the sphere is a polyhedral metric if each $x \in M$ has an open neighborhood isometric either to a subset of $\mathbb{R}^2$ or to a cone of angle less than $2\pi$ with $x$ mapped to the apex, and if only finitely many $x$, called the vertices $V(M) = \{v_i\}$ of $M$, fall into the latter case.

The defect $\delta_i$ at a vertex $v_i \in V(M)$ is the difference between $2\pi$ and the total angle at the vertex, which is positive by the definition of a vertex.

An embedding of $M$ is a piecewise linear map $f : M \to \mathbb{R}^3$. An embedding $f$ is $\varepsilon$-accurate if it distorts the metric $M$ by at most $1 + \varepsilon$, and $\varepsilon$-convex if $f(M)$ is a polyhedron and each dihedral angle in $f(M)$ is at most $\pi + \varepsilon$.

**Definition 2.** In a tetrahedron $ABCD$, write $\angle CABD$ for the dihedral angle along edge $AB$.

**Definition 3.** A triangulation of a polyhedral metric $M$ is a decomposition into Euclidean triangles whose vertex set is $V(M)$. Its vertices are denoted by $V(T) = V(M)$, its edges by $E(T)$, and its faces by $F(T)$.

A radius assignment on a polyhedral metric $M$ is a map $r : V(M) \to \mathbb{R}_+$. For brevity we write $r_i$ for $r(v_i)$.

A generalized convex polyhedron is a gluing of metric tetrahedra with a common apex $O$. The generalized convex polyhedron $P = (M, T, r)$ is determined by the polyhedral metric $M$ and triangulation $T$ giving its bases and the radius assignment $r$ for the side lengths.

Write $\kappa_i \triangleq 2\pi - \sum_{jk} \angle v_j O v_i v_k$ for the curvature about $O v_i$, and $\phi_{ij} \triangleq \angle v_i O v_j$ for the angle between vertices $v_i, v_j$ seen from the apex.

Our algorithm, following the construction in [2], will choose a radius assignment for the $M$ in question and iteratively adjust it until the associated generalized convex polyhedron $P$ fits nearly isometrically in $\mathbb{R}^3$. The resulting radii will give an $\varepsilon$-accurate $\varepsilon$-convex embedding of $M$ into $\mathbb{R}^3$.

In the argument we will require several geometric objects related to generalized convex polyhedra.

**Definition 4.** A Euclidean simplicial complex is a metric space on a simplicial complex where the metric restricted to each cell is Euclidean.

A generalized convex polygon is a Euclidean simplicial 2-complex homeomorphic to a disk, where all triangles have a common vertex $V$, the total angle at $V$ is no more than $2\pi$, and the total angle at each other vertex is no more than $\pi$. 
Given a generalized convex polyhedron \( P = (M, T, r) \), the corresponding generalized convex dual polyhedron \( D(P) \) is a certain Euclidean simplicial 3-complex. Let \( O \) be a vertex called the apex, \( A_i \) a vertex with \( OA_i = h_i \triangleq 1/r_i \) for each \( i \).

For each edge \( v_i v_j \in E(T) \) bounding triangles \( v_i v_j v_k \) and \( v_j v_i v_l \), construct two simplices \( OA_i A_{jil}, OA_j A_{ijl} \) in \( D(P) \) as follows. Embed the two tetrahedra \( Ov_i v_j v_k, Ov_j v_i v_l \) in \( \mathbb{R}^3 \). For each \( i' \in \{i, j, k, l\} \), place \( A_{i'} \) along ray \( Ov_{i'} \) at distance \( h_{i'} \), and draw a perpendicular plane \( P_{i'} \) through the ray at \( A_{i'} \). Let \( A_{i'j}, A_{ijl} \) be the intersection of the planes \( P_i, P_j, P_k \) and \( P_j, P_l, P_k \) respectively.

Now identify the vertices \( A_{i'j}, A_{ijk}, A_{kij} \) for each triangle \( v_i v_j v_k \in F(T) \) to produce the Euclidean simplicial 3-complex \( D(P) \). Since the six simplices produced about each of these vertices \( A_{i'j} \) are all defined by the same three planes \( P_i, P_j, P_k \) with the same relative configuration in \( \mathbb{R}^3 \), the total dihedral angle about each \( OA_{i'j} \) is \( 2\pi \). On the other hand, the total dihedral angle about \( OA_i \) is \( 2\pi - \kappa_i \), and the face about \( A_i \) is a generalized convex polygon of defect \( \kappa_i \).

The Jacobian bound in Section 4 makes use of certain multilinear forms described in [2] and in the full paper [8].

**Definition 5.** The dual volume \( \text{vol}(h) \) is the volume of the generalized convex dual polyhedron \( D(P) \), a cubic form in the dual altitudes \( h \). The mixed volume \( \text{vol}(\cdot, \cdot, \cdot) \) is the associated symmetric trilinear form.

Let \( E_i \) be the area of the face around \( A_i \) in \( D(P) \), a quadratic form in the altitudes within this face. The \( i \)th mixed area \( E_i(\cdot, \cdot) \) is the associated symmetric bilinear form.

Let \( \pi_i \) be the linear map \( \pi_i(h)_j \triangleq \frac{h_i - h_j \cos \phi_{ij}}{\sin \phi_{ij}} \), so that \( \pi_i(h) = g(i) \). Then define \( F_i(a, b) \triangleq E_i(\pi_i(a), \pi_i(b)) \) so that \( F_i(h, h) \) is the area of face \( i \).

By elementary geometry \( \text{vol}(h, h, h) = \frac{1}{3} \sum_i h_i F_i(h, h) \), so that by a simple computation \( \text{vol}(a, b, c) = \frac{1}{3} \sum_i a_i F_i(b, c) \).

### 2.2 Weighted Delaunay triangulations

The triangulations we require at each step of the computation are the weighted Delaunay triangulations used in the construction of [2]. We give a simpler definition inspired by Definition 14 of [7].

**Definition 6.** The power \( \pi_v(p) \) of a point \( p \) against a vertex \( v \) in a polyhedral metric \( M \) with a radius assignment \( r \) is \( pv^2 - r(v)^2 \).

The center \( C(v_i v_j v_k) \) of a triangle \( v_i v_j v_k \in T(M) \) when embedded in \( \mathbb{R}^2 \) is the unique point \( p \) such that \( \pi_v(p) = \pi_{v_i}(p) = \pi_{v_k}(p) \), which exists by the radical axis theorem from classical geometry. The quantity \( \pi_v(p) = \pi(v_i v_j v_k) \) is the power of the triangle.

A triangulation \( T \) of a polyhedral metric \( M \) with radius assignment \( r \) is locally convex at edge \( v_i v_j \) with neighboring triangles \( v_i v_j v_k, v_j v_i v_l \) if
A weighted Delaunay triangulation for a radius assignment \( r \) on a polyhedral metric \( M \) is a triangulation \( T \) that is locally convex at every edge.

A weighted Delaunay triangulation can be computed in time \( O(n^2 \log n) \) by a simple modification of the continuous Dijkstra algorithm of [10]. The original analysis of this algorithm assumes that each edge of the input triangulation is a shortest path. In the full paper [8] we show that the same algorithm works in time \( O(S\varepsilon^{-1}n^2 \log n) \) in the general case. Therefore we perform the general computation once at the outset, and use the resulting triangulation as the basis for subsequent runs of the continuous Dijkstra algorithm in time \( O(n^2 \log n) \) each.

The radius assignment \( r \) and triangulation \( T \) admit a tetrahedron \( OV_iv_jv_k \) just if the power of \( v_i v_j v_k \) is negative, and the squared altitude of \( O \) in this tetrahedron is \( -\pi(v_i v_j v_k) \). The edge \( v_i v_j \) is convex when the two neighboring tetrahedra are embedded in \( \mathbb{R}^3 \) just if it is locally convex in the triangulation as in Definition 6. A weighted Delaunay triangulation with negative powers therefore gives a valid generalized convex polyhedron if the curvatures \( \kappa_i \) are positive. For each new radius assignment \( r \) in the computation of Section 3 we therefore compute the weighted Delaunay triangulation and proceed with the resulting generalized convex polyhedron, in which Lemma 11 guarantees a positive altitude and the choices in the computation guarantee positive curvatures.

2.3 Notation for bounds

Definition 7. Let the following bounds be observed:

1. \( n \) is the number of vertices on \( M \).
2. \( \varepsilon_1 \triangleq \min_i \delta_i \) is the minimum defect.
3. \( \varepsilon_2 \triangleq \min_i (\delta_i - \kappa_i) \) is the minimum defect-curvature gap.
4. \( \varepsilon_3 \triangleq \min_{ij \in E(T)} \phi_{ij} \) is the minimum angle between radii.
5. \( \varepsilon_4 \triangleq \max_i \kappa_i \) is the maximum curvature.
6. \( \varepsilon_5 \triangleq \min_{v_i v_j v_k} \angle v_i v_j v_k \) is the smallest angle in the triangulation. Observe that obtuse angles are also bounded: \( \angle v_i v_j v_k < \pi - \angle v_j v_i v_k \leq \pi - \varepsilon_5 \).
7. \( \varepsilon_6 \) is used for the desired accuracy in embedding \( M \).
8. \( \varepsilon_7 \triangleq (\max_i \frac{\delta_i}{\delta_i})/(\min_i \frac{\delta_i}{\delta_i}) - 1 \) is the extent to which the ratio among the \( \delta_i \) varies from that among the \( \delta_i \). We will keep \( \varepsilon_7 < \varepsilon_8/4\pi \) throughout.
9. \( \varepsilon_8 \triangleq \min_i (2\pi - \delta_i) \) is the minimum angle around a vertex.
10. \( \varepsilon_9 \) is used for the desired approximation to convexity in embedding \( M \).
11. \( D \) is the diameter of \( M \).
12. \( L \) is the maximum length of any edge in the input triangulation.
13. \( \ell \) is the shortest distance \( v_i v_j \) between vertices.
14. \( S \triangleq \max(D, L)/\ell \) is the maximum ratio of distances.
15. $d_0 \triangleq \min_{p \in M} Op$ is the minimum height of the apex off of any point on $M$.
16. $d_1 \triangleq \min_{e_i, e_j \in E(T)} d(O, v_i v_j)$ is the minimum distance to any edge of $T$.
17. $d_2 \triangleq \min_i r_i$ is the minimum distance from the apex to any vertex of $M$.
18. $H \triangleq 1/d_0$; the name is justified by $h_i = 1/r_i \leq 1/d_0$.
19. $R = \max_i r_i$, so $1/H \leq r_i \leq R$ for all $i$.
20. $T \triangleq HR$ is the maximum ratio of radii.

Of these bounds, $n, \varepsilon_1, \varepsilon_8$, and $S$ are fundamental to the given metric $M$ or the form in which it is presented as input, and $D, L$, and $\ell$ are dimensionful parameters of the same metric input. The values $\varepsilon_6$ and $\varepsilon_9$ define the objective to be achieved, and our computation will drive $\varepsilon_4$ toward zero while maintaining $\varepsilon_2$ large and $\varepsilon_7$ small. In Section 6 we bound the remaining parameters $\varepsilon_3, \varepsilon_5, R, d_0, d_1$, and $d_2$ in terms of these.

**Definition 8.** Let $J$ denote the Jacobian $(\frac{\partial \kappa_i}{\partial r_j})_{ij}$, and $H$ the Hessian $(\frac{\partial^2 \kappa_i}{\partial r_j \partial r_k})_{ijk}$.

### 3 Main Theorem

In this section, we prove our main theorem using the results proved in the remaining sections. The algorithm of Theorem 1 obtains an approximate embedding of the polyhedral metric $M$ in $\mathbb{R}^3$. Its main subroutine is described by the following theorem:

**Theorem 2.** Given a polyhedral metric $M$ with $n$ vertices, ratio $S$ (the spread) between the diameter and the smallest distance between vertices, and defect at least $\varepsilon_1$ and at most $2\pi - \varepsilon_8$ at each vertex, a radius assignment $r$ for $M$ with maximum curvature at most $\varepsilon$ can be found in time $O(n^{913/2}S^{831}/(\varepsilon^{121} \varepsilon_4^{445} \varepsilon_8^{616}))$.

**Proof.** Let a good assignment be a radius assignment $r$ that satisfies two bounds: $\varepsilon_7 < \varepsilon_8/4\pi$ so that Lemmas 9–11 apply and $r$ therefore by the discussion in Subsection 2.2 produces a valid generalized convex polyhedron for $M$, and $\varepsilon_2 = \Omega(\varepsilon_1 \varepsilon_8^2 / n^2 S^2)$ on which our other bounds rely. By Lemma 6, there exists a good assignment $r^0$. We will iteratively adjust $r^0$ through a sequence $r^i$ of good assignments to arrive at an assignment $r^N$ with maximum curvature $\varepsilon_4^N < \varepsilon$ as required. At each step we recompute $T$ as a weighted Delaunay triangulation according to Subsection 2.2.

Given a good assignment $r = r^0$, we will compute another good assignment $r' = r^{n+1}$ with $\varepsilon_4 - \varepsilon_4' = \Omega(\varepsilon_1^{445} \varepsilon_4^{121} \varepsilon_8^{616}/(n^{907/2} S^{831}))$. It follows that from $r^0$ we can arrive at a satisfactory $r^N$ with $N = O((n^{907/2} S^{831})/(\varepsilon_1^{121} \varepsilon_4^{445} \varepsilon_8^{616}))$.

To do this, let $J$ be the Jacobian $(\frac{\partial \kappa_i}{\partial r_j})_{ij}$ and $H$ the Hessian $(\frac{\partial^2 \kappa_i}{\partial r_j \partial r_k})_{ijk}$, evaluated at $r$. The goodness conditions and the objective are all in terms of $\kappa$, so we choose a desired new curvature vector $\kappa^*$ in $\kappa$-space and apply the inverse Jacobian to get a new radius assignment $r' = r + J^{-1}(\kappa^* - \kappa)$ in $r$-space. The
embed each neighboring tetrahedron in turn. is made up of rigid tetrahedra and we embed one tetrahedron arbitrarily, then exact values for \( \kappa \) and therefore \( P \) polyhedron \( P \) matrix multiplication. Each iteration therefore costs time \( O(3n^2) \). Choose a step size \( \epsilon \) in [2]. Choose a step size \( \epsilon \) p \in \mathbb{R} \). This produces a good radius assignment \( r' \) in which \( \epsilon_4 \) has declined by at least

\[
\frac{pc_4}{2} = \frac{\epsilon_1^2}{128\pi^2 nC} = \Omega \left( \frac{\epsilon_4^{445} \epsilon_4^{121} \epsilon_8^{616}}{n^{907/2} S^{831}} \right)
\]

as required.

As a simplification, we need not compute \( p \) exactly according to (1). Rather, we choose the step size \( p' \) at each step, trying first \( p'^{t-1} \) (with \( p^0 \) an arbitrary constant) and computing the actual curvature error \( |\kappa' - \kappa^*| \). If the error exceeds its maximum acceptable value \( pc_4^2 \epsilon_4 / 16\pi^2 \) then we halve \( p' \) and try step \( t \) again, and if it falls below half this value then we double \( p' \) for the next round. Since we double at most once per step and halve at most once per doubling plus a logarithmic number of times to reach an acceptable \( p \), this doubling and halving costs only a constant factor. Even more important than the resulting simplification of the algorithm, this technique holds out the hope of actual performance exceeding the proven bounds.

Now each of the \( N \) iterations of the computation go as follows. Compute the weighted Delaunay triangulation \( T^t \) for \( r^t \) in time \( O(n^2 \log n) \) as described in Subsection 2.2. Compute the Jacobian \( J^t \) in time \( O(n^2) \) using formulas (14, 15) in [2]. Choose a step size \( p' \), possibly adjusting it, as discussed above. Finally, take the resulting \( r' \) as \( r'^{t+1} \) and continue. The computation of \( \kappa^* \) to check \( p' \) runs in linear time, and that of \( r' \) in time \( O(n^2) \) where \( \omega < 3 \) is the time exponent of matrix multiplication. Each iteration therefore costs time \( O(n^3) \), and the whole computation costs time \( O(n^3 N) \) as claimed.

Now with our radius assignment \( r \) for \( M \) and the resulting generalized convex polyhedron \( P \) with curvatures all near zero, it remains to approximately embed \( P \) and therefore \( M \) in \( \mathbb{R}^3 \). To begin, we observe that this is easy to do given exact values for \( r \) and in a model with exact computation: after triangulating, \( P \) is made up of rigid tetrahedra and we embed one tetrahedron arbitrarily, then embed each neighboring tetrahedron in turn.
In a realistic model, we compute only with bounded precision, and in any case Theorem 2 gives us only curvatures near zero, not equal to zero. Lemma 1 produces an embedding in this case, settling for less than exact isometry and exact convexity.

**Lemma 1.** There is an algorithm that, given a radius assignment \( r \) for which the corresponding curvatures \( \kappa \) are all less than \( \varepsilon = O \left( \min \left( \frac{\varepsilon_6}{nS}, \frac{\varepsilon_9 \varepsilon_1^2}{nS^6} \right) \right) \) for some constant factor, produces explicitly by vertex coordinates in time \( O(n^2 \log n) \) an \( \varepsilon_6 \)-accurate \( \varepsilon_9 \)-convex embedding of \( M \).

**Proof (sketch).** As in the exact case, triangulate \( M \), embed one tetrahedron arbitrarily, and then embed its neighbors successively. The positive curvature will force gaps between the tetrahedra. Then replace the several copies of each vertex by their centroid, so that the tetrahedra are distorted but leave no gaps. This is the desired embedding. The proofs of \( \varepsilon_6 \)-accuracy and \( \varepsilon_9 \)-convexity are straightforward and left to the full paper [8].

A weighted Delaunay triangulation takes time \( O(n^2 \log n) \) as discussed in Subsection 2.2, and the remaining steps take time \( O(n) \).

We now have all the pieces to prove our main theorem.

**Proof (Theorem 1).** Let \( \varepsilon = O \left( \min \left( \frac{\varepsilon_6}{nS}, \frac{\varepsilon_9 \varepsilon_1^2}{nS^6} \right) \right) \), and apply the algorithm of Theorem 2 to obtain in time \( O \left( \frac{n^{913/2}S^{831}}{\varepsilon_1^{121} \varepsilon_8^{445} \varepsilon_8^{616}} \right) \) a radius assignment \( r \) for \( M \) with maximum curvature \( \varepsilon_6 \leq \varepsilon \).

Now apply the algorithm of Lemma 1 to obtain in time \( O(n^2 \log n) \) the desired embedding and complete the computation.

\[ \Box \]

### 4 Bounding the Jacobian

**Theorem 3.** The Jacobian \( J = \left( \frac{\partial \kappa}{\partial r^j} \right)_{ij} \) has inverse pseudopolynomially bounded by \( |J^{-1}| = O \left( \frac{n^{7/2}T^2}{\varepsilon_6 \varepsilon_3^2 \varepsilon_4^2} R \right) \).

**Proof.** Our argument parallels that of Corollary 2 in [2], which concludes that the same Jacobian is nondegenerate. Theorem 4 of [2] shows that this Jacobian equals the Hessian of the volume of the dual \( D(P) \). The meat of the corollary’s proof is in Theorem 5 of [2], which begins by equating this Hessian to the bilinear form \( 6 \operatorname{vol}(h, \cdot, \cdot) \) derived from the mixed volume we defined in Definition 5. So we have to bound the inverse of this bilinear form.

To do this it suffices to show that the form \( \operatorname{vol}(h, x, \cdot) \) has norm at least \( \Omega \left( \frac{\varepsilon_5 \varepsilon_3 \varepsilon_4}{nT^2} |x| \right) \) for all vectors \( x \). Equivalently, suppose some \( x \) has \( |\operatorname{vol}(h, x, z)| \leq |z| \) for all \( z \); we show \( |x| = O \left( \frac{n^{7/2}T^2}{\varepsilon_6 \varepsilon_3^2 \varepsilon_4^2} R \right) \).

To do this we follow the proof in Theorem 5 of [2] that the same form \( \operatorname{vol}(h, x, \cdot) \) is nonzero for \( x \) nonzero. Throughout the argument we work in terms of the dual \( D(P) \).
Recall that for each $i$, $\pi_ix$ is defined as the vector $\{x_ij\}$. It suffices to show that for all $i$

$$|\pi_ix|^2 = O\left(\frac{n^3T^3}{\varepsilon_3^2\varepsilon_4^2}R^2 + \frac{n^2T^2}{\varepsilon_2\varepsilon_4}R|x|_1\right)$$

since then by Lemma 2

$$|x|^2 \leq \frac{4n}{\varepsilon_3^2} \max_i |\pi_ix|^2 = O\left(\frac{n^3T^3}{\varepsilon_3^2\varepsilon_4^2}R^2 + \frac{n^2T^2}{\varepsilon_2\varepsilon_4}R|x|_1\right),$$

and since $|x|_1 \leq \sqrt{n}|x|_2$ and $X^2 \leq a + bX$ implies $X \leq \sqrt{a + b}$, $|x|_2 = O\left(\frac{n^{7/2}T^2}{\varepsilon_2\varepsilon_3\varepsilon_4}R\right)$. Therefore fix an arbitrary $i$, let $g = \pi_i h$ and $y = \pi_i x$, and we proceed to bound $|y|/|z|_1$.

We break the space on which $E_i$ acts into the 1-dimensional positive eigenspace of $E_i$ and its $(k-1)$-dimensional negative eigenspace, since by Lemma 3.4 of [2] the signature of $E_i$ is $(1,k-1)$, where $k$ is the number of neighbors of $v_i$. Write $\lambda_+$ for the positive eigenvalue and $-E_i^-$ for the restriction to the negative eigenspace so that $E_i^-$ is positive definite, and decompose $g = g_+ + g_-$, $y = y_+ + y_-$ by projection into these subspaces. Then we have

$$G \overset{\Delta}{=} E_i(g,g) = \lambda_+g_+^2 - E_i^-(g_-,g_-) \overset{\Delta}{=} \lambda_+g_+^2 - G_-$$
$$E_i(g,y) = \lambda_+g_+y_+ - E_i^-(g_-,y_-)$$
$$Y \overset{\Delta}{=} E_i(y,y) = \lambda_+y_+^2 - E_i^-(y_-,y_-) \overset{\Delta}{=} \lambda_+y_+^2 - Y_-$$

and our task is to obtain an upper bound on $Y_- = E_i^-(y_-,y_-)$, which will translate through our bound on the eigenvalues of $E_i$ away from zero into the desired bound on $|y|$.

We begin by obtaining bounds on $|E_i(g,y)|$, $G_-$, $G$, and $Y$. Since $|z| \geq |\text{vol}(h,x,z)|$ for all $z$ and $|\text{vol}(h,x,z)| = \sum_j z_j F_j(h,x)$, we have $|E_i(g,y)| = |F_i(h,x)| \leq 1$. Further, $\text{det}\left(\begin{smallmatrix} E_i(g,g) & E_i(g,y) \\ E_i(y,g) & E_i(y,y) \end{smallmatrix}\right) < 0$ because $E_i$ has signature $(1,1)$ restricted to the $(y,g)$ plane, so by Lemma 3 $Y = E_i(y,y) < \frac{R^2}{\varepsilon_2}$.

Now by further calculation and the use of Lemma 4, the theorem follows; the details are left to the full paper [8] for brevity.

Three small lemmas used above follow from the geometry of spherical polygons and of generalized convex dual polyhedra. Their proofs are left to the full paper [8] for brevity.

**Lemma 2.** $|x|^2 \leq (4n/\varepsilon_3^2) \max_i |\pi_ix|^2$.

**Lemma 3.** $F_i(h,h) > \varepsilon_2/R^2$.

**Lemma 4.** The inverse of the form $E_i$ is bounded by $|E_i^{-1}| = O(n/\varepsilon_4)$. 

5 Bounding the Hessian

In order to control the error in each step of our computation, we need to keep the Jacobian $J$ along the whole step close to the value it started at, on which the step was based. To do this we bound the Hessian $H$ when the triangulation is fixed, and we show that the Jacobian does not change discontinuously when changing radii force a new triangulation.

Each curvature $\kappa_i$ is of the form $2\pi - \sum_{j,k:v_i,v_j,v_k \in T} \angle v_j O v_i v_k$, so in analyzing its derivatives we focus on the dihedral angles $\angle v_j O v_i v_k$. When the tetrahedron $Ov_i v_j v_k$ is embedded in $\mathbb{R}^3$, the angle $\angle v_j O v_i v_k$ is determined by elementary geometry as a smooth function of the distances among $O, v_i, v_j, v_k$. For a given triangulation $T$ this makes $\kappa$ a smooth function of $r$. Our first lemma shows that no error is introduced at the transitions where the triangulation $T(r)$ changes.

**Lemma 5.** The Jacobian $J = (\frac{\partial \kappa_i}{\partial r_j})_{ij}$ is continuous at the boundary between radii corresponding to one triangulation and to another.

**Proof (sketch).** The proof, which can be found in the full paper [8], uses elementary geometry to compare the figures determined by two triangulations near a radius assignment on their boundary. \(\square\)

It now remains to control the change in $J$ as $r$ changes within any particular triangulation, which we do by bounding the Hessian.

**Theorem 4.** The Hessian $H = (\frac{\partial \kappa_i}{\partial r_j r_k})_{ijk}$ is bounded in norm by $O(n^{5/2}S^{14}R^{23}/(\varepsilon_3^3 n^6 d_1^6 D^{14}))$.

**Proof.** By direct computation and computer algebra. See the full paper [8] for the details. \(\square\)

6 Intermediate Bounds

Here we bound miscellaneous parameters in the computation in terms of the fundamental parameters $n, S, \varepsilon_1, \varepsilon_8$ and the computation-driving parameter $\varepsilon_4$.

**Lemma 6.** Given a polyhedral metric space $M$, there exists a radius assignment $r$ with curvature skew $\varepsilon_7 \leq \varepsilon_8/4\pi$, maximum radius $R = O(n D/\varepsilon_1 \varepsilon_8)$, and minimum defect-curvature gap $\varepsilon_2 = \Omega(\varepsilon_1^2 \varepsilon_8^3/n^2 S^2)$.

**Proof (sketch).** Take $r_i = R$ for all $i$, with $R$ sufficiently large. Then each $\kappa_i$ is nearly equal to $A_i$, so that $\varepsilon_7$ is small. For the quantitative bounds and a complete proof, see the full paper [8]. \(\square\)

Two bounds on angles can be proven by elementary geometry; details are left to the full paper [8] for brevity.

**Lemma 7.** $\varepsilon_3 > \ell d_1/R^2$. 

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Two bounds on angles can be proven by elementary geometry; details are left to the full paper [8] for brevity.

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\textbf{Lemma 8.} \( \varepsilon_5 > \varepsilon_2/6S. \)

Finally we bound \( O \) away from the surface \( M \). The bounds are effective versions of Lemmas 4.8, 4.6, and 4.5 respectively of \cite{2}, and the proofs, left for brevity to the full paper \cite{8}, are similar but more involved.

Recall that \( d_2 \) is the minimum distance from \( O \) to any vertex of \( M \), \( d_1 \) is the minimum distance to any edge of \( T \), and \( d \) is the minimum distance from \( O \) to any point of \( M \).

\textbf{Lemma 9.} \( d_2 = \Omega(D\varepsilon_4^2\varepsilon_5^2\varepsilon_8/(nS^4)). \)

\textbf{Lemma 10.} \( d_1 = \Omega(D\varepsilon_2^2\varepsilon_4^2\varepsilon_5^2\varepsilon_8/(n^2S^{10})). \)

\textbf{Lemma 11.} \( d_0 = \Omega(D\varepsilon_4^2\varepsilon_5^2\varepsilon_8^2/(n^4S^{22})). \)

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