Gravity Duals of Fractional Branes in Various Dimensions

Christopher P. Herzog and Igor R. Klebanov
Joseph Henry Laboratories, Princeton University,
Princeton, New Jersey 08544, USA

Abstract

We derive type II supergravity solutions corresponding to space-filling regular and fractional Dp branes on (9 − p)-dimensional conical transverse spaces. Fractional Dp-branes are wrapped D(p + 2)-branes; therefore, our solutions exist only if the base of the cone has a non-vanishing Betti number $b_2$. We also consider 11-dimensional SUGRA solutions corresponding to regular and fractional M2 branes on 8-dimensional cones whose base has a non-vanishing $b_3$. (In this case a fractional M2-brane is an M5-brane wrapped over a 3-cycle.) We discuss the gauge theory interpretation of these solutions, as well as of the solutions constructed by Cvetič et al. in hep-th/0011023 and hep-th/0012011.

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1 Introduction

The basic AdS/CFT correspondence [1, 2, 3] is motivated by comparing a stack of elementary branes with the metric it produces (for reviews, see for example [4, 5]). In order to break some of the supersymmetry, we may place the stack at a conical singularity [6, 7, 8, 9]. Consider, for instance, a stack of D3-branes placed at the apex of a Ricci-flat 6-d cone $Y_6$ whose base is a 5-d Einstein manifold $X_5$. Comparing the metric with the D-brane description leads one to conjecture that type IIB string theory on $AdS_5 \times X_5$ is dual to the low-energy limit of the world volume theory on the D3-branes at the singularity.

For certain cones $Y_6$, in addition to the regular D3-branes which may be moved away from the apex, there are also fractional D3-branes which can exist only at the singularity [10, 11, 12, 13]. These fractional branes may be thought of as D5-branes wrapped over 2-cycles of $X_5$, the base of the cone. If we are interested in stacking arbitrary numbers of such branes, we have to require that the second homology group $H_2(X_5)$ is $\mathbb{Z}$ or bigger.

An example of a smooth $X_5$ with Betti number $b_2 = 1$ is the Einstein space $T^{1,1}$ which is topologically $S^2 \times S^3$. The cone over $T^{1,1}$ is a singular Calabi-Yau space known as the conifold [14]; it is described by the following equation in $C^4$:

$$\sum_{n=1}^4 w_n^2 = 0.$$  \hfill (1)

When $N$ regular D3-branes are placed at the singularity, the resulting $\mathcal{N} = 1$ superconformal field theory has gauge group $SU(N) \times SU(N)$. It contains chiral superfields $A_i, A_j$ transforming as $(N, \bar{N})$ and superfields $B_i, B_j$ transforming as $(\bar{N}, N)$, with superpotential $W = \lambda \epsilon^{ij} \epsilon^{kl} \text{Tr} A_i B_k A_j B_l$. The two gauge couplings do not flow and can be varied continuously without ruining conformal invariance [5, 9]. The type IIB background dual to this gauge theory is the near-horizon region of the solution describing $N$ D3-branes at the apex of the conifold, namely, $AdS_5 \times T^{1,1}$ with $N$ units of 5-form flux.

Adding $M$ fractional D3-branes, i.e. $M$ wrapped D5-branes, changes the $\mathcal{N} = 1$ gauge theory to $SU(N + M) \times SU(N)$ coupled to the bifundamental chiral superfields $A_i, B_j$. This theory is no longer conformal: it can be easily seen that the NSVZ beta function for $g_1^{-2} - g_2^{-2}$ does not vanish [12, 13]. Supergravity solutions corresponding to $N$ regular and $M$ fractional branes were considered in [13, 13]. The $M$ wrapped D5-branes introduce $M$ units of 3-form RR flux through the 3-cycle of $T^{1,1}$:

$$F_3 = Q \omega_3,$$  \hfill (2)

where $Q \sim g_s M$ and $\omega_3$ is the harmonic 3-form on $T^{1,1}$. The 10-d metric is [13]

$$ds_{10}^2 = H(r)^{-1/2} \eta_{\alpha\beta} dx^\alpha dx^\beta + H(r)^{1/2} (dr^2 + r^2 ds_{T^{1,1}}^2),$$

where $\alpha, \beta = 0, 1, 2, 3$ and

$$H(r) = \frac{Q^2 \ln(r/r_*)}{4r^4}.$$  \hfill (3)
A surprising feature of the solution found in [13] is that the 5-form flux, which corresponds to the regular D3-branes, is not constant; in fact, it varies logarithmically with $r$. This presence of an extra logarithm in the metric and $\tilde{F}_5$ constitutes a new type of UV behavior. Its gauge theory interpretation in terms of a cascade of Seiberg dualities was given in [16].

The solution of [13] is smooth at large $r$ but possesses a naked singularity in the IR, at $r = r_*$. In [16] this problem was removed by replacing the singular conifold (1) by the deformed conifold

$$\sum_{n=1}^{4} w_n^2 = e^2.$$  \hspace{1cm} (4)

In the deformed conifold the 2-sphere shrinks at the apex $\tau = 0$, but the 3-sphere does not. Hence the conserved 3-form flux does not lead to a singularity of the metric; the warp factor $H(\tau)$ approaches a constant at $\tau = 0$. This implies that the dual gauge theory is confining [16]. The deformation (4) breaks the $U(1)_R$ symmetry $w_n \rightarrow w_n e^{i\alpha}$ down to the $Z_2$ generated by $w_n \rightarrow -w_n$, geometrically realizing the chiral symmetry breaking in the dual gauge theory. Another interesting solution with the same asymptotics as those of [13] was found by Pando Zayas and Tseytlin [17]. This solution is based on the resolved conifold. It is singular in the IR but presumably the singularity may be resolved through the enhançon mechanism of [18].

In this paper we consider more general solutions which describe $D_p$-branes on $R^{p-1} \times Y_{9-p}$ where $Y_{9-p}$ is a Ricci flat $(9 - p)$-dimensional cone. We consider only space-filling branes: those that fill all the dimensions of $R^{p-1}$. The extremal background corresponding to a stack of regular branes at the apex of the cone is well-known [22]:

$$ds^2 = H(r)^{-1/2} \eta_{\alpha\beta} dx^\alpha dx^\beta + H(r)^{1/2} (dr^2 + r^2 h_{ij} dx^i dx^j).$$  \hspace{1cm} (5)

We let $\alpha = 0, 1, \ldots, p$ and $h_{ij} dx^i dx^j$ denotes the metric on $X_{8-p}$, the base of the cone. The dilaton and the $(p + 2)$-form RR field strength are given by

$$e^{4\Phi} = H(r)^{3-p}, \quad F_{\theta dummy} = \partial r H^{-1}.$$  \hspace{1cm} (6)

It is interesting to ask how this background is modified by adding a large number $M$ of $D(p + 2)$-branes wrapped over a 2-cycle of $X_{8-p}$. For this question to make sense, we

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1To be more precise, the $U(1)_R$ is actually first broken to $Z_{2M}$ by instanton effects. For large $M$, however, the $Z_{2M}$ is well approximated by the $U(1)$.

2Gravity duals of fractional branes which are not space-filling were considered for orbifold models in [13, 20, 21].
have to require that $H_2(X_{8-p})$ is at least as big as $\mathbb{Z}$. This is a stringent requirement: for some $p$ we will be unable to find any supersymmetry preserving smooth spaces of this type.

The solutions we construct in this paper are analogues of the Klebanov-Tseytlin (KT) solution for $p = 3$ [13]: the transverse space is taken to be conical. Such solutions are typically smooth in the UV but posses a naked singularity in the IR. We will assume that the naked singularity may be resolved by an appropriate smoothing of the cone, but we leave construction of such solutions for the future.

Our work is closely related to the very nice papers [23, 24] where the effect of adding RR field strength $F_{6-p}$ on various $p$-brane solutions was investigated. In order for this field to be related to $M$ fractional D$p$-branes we require that it carries $M$ units of conserved flux. This is the generalization of the requirement (2) for $p = 3$. In [23, 24], however, the asymptotic form of $F_{6-p}$ (or of the magnetic components of $F_4$ in the case of M2-branes) is typically such that it does not produce any new flux at infinity. We believe that such solutions should not be interpreted as gravity duals of fractional branes. The presence of fractional branes affects the rank of the gauge group itself. In the dual gravity description this shift of the rank manifests itself as a new flux at infinity. Instead, some solutions in [23, 24] have a more conventional interpretation as deformations of the field theory present on $N$ regular branes by certain relevant operators. At large $r$, through the usual AdS/CFT correspondence [2, 3], the power law fall-off of the field strength, and hence also of the gauge potential, determines the dimension of the operator added to the dual gauge theory action. Some specific examples of this interpretation will be presented in section 5.

## 2 Fractional Branes in Type IIA and IIB SUGRA

To construct our type IIA and type IIB fractional D$p$-brane solutions, we start with a warped product (5) of $R^{p,1}$ flat space-time directions and a Ricci flat, $(9-p)$-dimensional cone $Y_{9-p}$. Since the branes are space-filling, the warp factor depends only on the radial coordinate of the cone. Because the cone is Ricci flat, the base of the cone is an $(8-p)$-dimensional Einstein manifold $X_{8-p}$ with metric $h_{ij}$ normalized such that $R_{ij} = (7-p)h_{ij}$. We assume that this Einstein manifold has a harmonic 2-form $\omega_2$, so that wrapping a D$(p+2)$-brane around the 2-cycle corresponding to $\omega_2$, and letting the remaining $p + 1$ dimensions fill $R^{p,1}$, creates a fractional D$p$-brane. The $\omega_2$ may be normalized such that $\omega_2 \wedge *_{8-p} \omega_2$ is the volume form on $X_{8-p}$.

An important issue is whether such smooth manifolds $X_{8-p}$ exist and, if so, whether they preserve any supersymmetry. In 5 and 7 dimensions (corresponding to fractional D3- and D1-branes), we know of examples that preserve some supersymmetry. For D3-branes, the manifold $T^{1,1}$ is such an Einstein space [15]. For D1-branes supersymmetric
examples include $N^{0,1,0}$, $Q^{1,1,1}$ and $M^{1,1,1}$ which we discuss in Section 3. In the D0-, D2-,
and D4-brane cases, we know of some Einstein manifolds with the requisite number of
dimensions and $b_2 > 0$: for example, the $S^2 \times S^{6-p}$. However, such a manifold does not
preserve any supersymmetry and the resulting solution may be unstable. It is certainly
unstable in the absence of fractional branes. It would be very interesting if there are
Einstein spaces which do not preserve supersymmetry, but which remain stable because
of the extra flux from the wrapped $D(p+2)$-branes. We leave such issues of stability for
future work.

$$
\begin{array}{|c|c|c|}
\hline
p & p\text{-brane field strength} & (p+2)\text{-brane field strength} \\
\hline
0 & F_2 = dt \wedge dH^{-1} & \tilde{F}_4 = \frac{Q}{r} dt \wedge dr \wedge \omega_2 \\
1 & \tilde{F}_3 = d^2 x \wedge dH^{-1} & \tilde{F}_5 = -Q (\omega_3 + \ast \omega_3) \\
2 & \tilde{F}_4 = d^3 x \wedge dH^{-1} + Q \omega_4 & \\
3 & \tilde{F}_5 = d^4 x \wedge dH^{-1} + \ast d^4 x \wedge dH^{-1} & \tilde{F}_3 = Q \omega_3 \\
4 & \tilde{F}_4 = H^r r^4 \omega_2 \wedge \omega'_2 & F_2 = Q \omega'_2 \\
5 & \tilde{F}_3 = -H^r r^3 \omega_2 \wedge \omega_1 & dC = Q \omega_1 \\
\hline
\end{array}
$$

Surprisingly, introducing $M$ wrapped $D(p+2)$-branes changes the regular $Dp$-brane solution \[\text{(5)}\] in a controlled way. We account for the $D(p+2)$-brane charge by giving a
nonzero value to $F_{6-p}$, or, for $p=0$, a nonzero value to $F_4$. The precise values are given
in the third column of \[\text{(7)}\] In addition, the NS-NS field must be non-zero due to the
effect of the Chern-Simons (CS) terms in the SUGRA action:

$$
B_2 = \frac{Q}{r^{p-3}} \omega_2, \quad H_3 = dB_2 = \frac{Q}{r^{4-p}} dr \wedge \omega_2
$$

where $Q \sim g_s M$. In the usual $Dp$-brane solution, the warp factor $H(r)$ is an eigenfunction
of the Laplacian on the Ricci flat cone with zero eigenvalue. By adding fractional $Dp$-
branes, we introduce a source term to this differential equation:

$$
(r^{8-p} H(r))' = -\frac{Q^2}{r^{4-p}}.
$$

As has become customary in gauge/gravity duality, we integrate this equation with
the boundary condition that $H$ approaches zero as $r \to \infty$. This boundary condition

\[\text{After the original version of this paper was completed, two supersymmetric fractional D2-brane}
\[\text{solutions were found in [25]. In one example, the base of the cone is topologically}
\[\text{$S^2 \times S^4$, while geometrically it is an $S^2$ bundle over $S^4$.}
\[\text{In the table, $\omega_{6-p} = (-1)^p \ast_{8-p} \omega_2$ where the $-1$ has been added to conform with the conventions}
\[\text{of [10]. For $p = 4$, $\omega'_2 = \ast_4 \omega_2$.}
removes the asymptotically flat region so that we “zoom in” on the low-energy dynamics of the dual gauge theory. Thus, we find that for \( p \neq 3, 5 \) the warp factor is

\[
H(r) = \left( \frac{\rho}{r} \right)^{7-p} - \frac{Q^2}{(3-p)(10-2p)r^{10-2p}}
\]  

where as usual \( \rho^{7-p} \sim g_s N \) and \( N \) is the number of ordinary \( Dp \)-branes. In the fractional \( D3 \)-brane case, the warp factor takes the form familiar from [14],

\[
H(r) = \left( \frac{\rho}{r} \right)^4 + Q^2 \left( \frac{\ln(r)}{4r^4} + \frac{1}{16r^4} \right).
\]  

In the fractional \( D5 \)-brane case, the warp factor is

\[
H(r) = \left( \frac{\rho}{r} \right)^2 - \frac{1}{2}Q^2 \ln r.
\]

Clearly, \( p \) cannot exceed 5: for \( p = 6 \), the base of the cone can only be two dimensional, and there is no place to which the flux from the wrapped D8-brane can escape. The case \( p = 5 \) may also be unphysical as we discuss below.

In the far UV, i.e. at large \( r \), the warp factor of the \( D2-, D1-, \) and \( D0- \)-brane solutions approaches that of its \( M = 0 \) counterpart. In the fractional \( D3 \)-brane case, the logarithmic running of the warp factor was related to a renormalization group flow of the gauge theory dual, and in particular to a logarithmic increase in the number of colors in the theory [13, 15, 16]. Curiously, the warp factor for the fractional \( D4 \)-brane solution appears to be indistinguishable from the \( D5 \)-brane warp factor in the far UV.

The case \( p = 5 \) corresponds to the wrapped \( D7 \)-branes. It is well known that in ten flat space-time dimensions the \( D7 \)-brane solution behaves analogously to a cosmic string in four dimensions [26]: the back reaction from the \( D7 \)-brane makes the metric, at least close to the \( D7 \)-brane, quite complicated. However, in our situation, the wrapped \( D7 \)-brane appears no more badly behaved than a \( D5 \)-brane at close distance, while far away in the UV there is a logarithmic divergence which leads to a naked singularity. This UV behavior seems rather pathological. In reality the \( p = 5 \) solution probably does not exist because there are no requisite smooth three dimensional Einstein spaces. There is a theorem due to Hamilton [27] which states that the only three dimensional Einstein space with positive Ricci scalar curvature is either \( S^3 \) or a quotient of \( S^3 \) by a discrete group. The cone over such a space has no harmonic one or three cycles although it may admit vanishing two cycles.

In the infrared, the fractional \( D0- \) through \( D3 \)-brane solutions exhibit naked singularities. In the case of the \( D3 \)-brane, the naked singularity can be resolved in one of two ways: (1) The flux through the 3-cycle corresponding to the harmonic 3-form \( \omega_3 = (-1)^* \omega_2 \) prevents the cycle from collapsing. The cone is “deformed” and the singularity is avoided
This resolution preserves supersymmetry but breaks the chiral symmetry. (2) The singularity may be hidden behind an event horizon \[28\]. The supersymmetry is now broken and the background describes the high-temperature phase of the gauge theory where the chiral symmetry is restored.

We expect similar IR phenomena to take place in the fractional D0-, D1-, and D2-brane cases. At finite temperature, the naked singularities may be cloaked by event horizons. Alternatively, at zero temperature, the flux through the \((6 - p)\)-dimensional cycle corresponding to the harmonic form \(\omega_{6-p} = (-1)^p *_{8-p} \omega_2\) may deform the cone. We will return to these issues in a future publication.

For each \(p\), we will now describe the way in which the SUGRA solution satisfies the requisite equations of motion. We start with the type IIA solutions. Readers not interested in the details can skip ahead to the next section. It is a straightforward although tedious matter to check that the field strengths and metrics given above satisfy the Bianchi identities, field strength equations, and the trace of Einstein’s equation. In checking the trace, a useful formula is the equation for the Ricci scalar in Einstein frame:

\[
R = H^{-(1+p)/8} \left( -\frac{p+1}{8} \frac{(r^{8-p}H')'}{r^{8-p}H} + \frac{(p+1)(p-3)}{32} \left( \frac{H'}{H} \right)^2 \right). \tag{11}
\]

We have also partially checked that the field strengths satisfy each component of Einstein’s equation independently although without detailed knowledge of \(\omega_2\), it is difficult to do a complete check. We believe that for a suitably symmetric \(\omega_2\), Einstein’s equation will be fully satisfied.

### 2.1 Type IIA Fractional Branes

#### 2.1.1 Fractional D0-branes

Fractional D0-branes involve a 9-dimensional Ricci flat cone over an 8-dimensional Einstein space \(X_8\). The dilaton is determined by \(e^\Phi = H(r)^{3/4}\), and hence the metric in Einstein frame can be written

\[
ds^2_E = g_s^{1/2} \left[ -H^{-7/8} dt^2 + H^{1/8} (dr^2 + r^2 h_{ij} dx^i dx^j) \right].
\]

The nonzero field strengths are

\[
F_2 = dt \wedge dH^{-1}, \\
\tilde{F}_4 = \frac{Q}{r^4} H^{-1} dt \wedge dr \wedge \omega_2, \\
H_3 = \frac{Q}{r^4} dr \wedge \omega_2.
\]
where $\omega_2$ is a harmonic 2-form on $X_8$. The factor of $H^{-1}$ in $\tilde{F}_4$ is included to satisfy the equation of motion for the $\tilde{F}_4$ field strength

$$d(e^{\Phi/2}*\tilde{F}_4) = -g_s^{1/2}F_4 \wedge H_3.$$  \hfill (12)

For this background $F_4 \wedge H_3 = 0$, and the factor of $H^{-1}$ in $\tilde{F}_4$ guarantees that $e^{\Phi/2}*\tilde{F}_4$ is independent of $r$. The other nontrivial field equation is

$$d(e^{3\Phi/2}*F_2) = g_s e^{\Phi/2} H_3 \wedge *\tilde{F}_4.$$  \hfill (13)

Each side of this equation is proportional to the volume form on the 9-d cone, and it is straightforward to check that the prefactors agree too provided $(r^8 H')' = -Q^2/r^4$. The equation for the NS-NS field $H_3$ is

$$\frac{g_s}{2} F_4 \wedge F_4 = d(e^{-\Phi} H_3 + g_s^{1/2} e^{\Phi/2} C_1 \wedge *\tilde{F}_4).$$  \hfill (14)

It is satisfied because of two facts: (1) $F_4 \wedge F_4 = 0$ and (2) $\tilde{F}_4$ has the factor of $H^{-1}$, making the right side of the NS-NS three form equation of motion homogenous in $H$.

Next, we check the dilaton equation of motion

$$d*d\Phi = -\frac{g_s e^{-\Phi}}{2} H_3 \wedge *H_3 + \frac{3g_s^{1/2} e^{3\Phi/2}}{4} F_2 \wedge *F_2 + \frac{g_s^{3/2} e^{\Phi/2}}{4} F_4 \wedge *\tilde{F}_4.$$  \hfill (15)

and find again that the equation is satisfied provided $(r^8 H')' = -Q^2/r^4$.

Finally, we check the trace of Einstein’s equation. The Ricci scalar on this 10-dimensional space is given by (11) where $p = 0$. The trace of Einstein’s equation is

$$R \text{Vol} = \frac{1}{2} d\Phi \wedge *d\Phi + \frac{g_s e^{-\Phi}}{4} H_3 \wedge *H_3 + \frac{3g_s^{1/2} e^{3\Phi/2}}{8} F_2 \wedge *F_2 + \frac{g_s^{3/2} e^{\Phi/2}}{8} \tilde{F}_4 \wedge *\tilde{F}_4.$$  \hfill (16)

where $\text{Vol}$ is the ten dimensional volume form on the space, and the trace equation is satisfied.

### 2.1.2 Fractional D2-branes

The dilaton is given by $e^\Phi = H^{1/4}$, and the Einstein frame metric is

$$ds_E^2 = g_s^{1/2} \left[ H^{-5/8} \eta_{\alpha\beta} dx^\alpha dx^\beta + H^{3/8} (dr^2 + r^2 h_{ij} dx^i dx^j) \right]$$

where $\alpha, \beta = 0, 1, 2$ and $h_{ij}$ is the metric on the 6-d Einstein space $X_6$. The nonzero field strengths are

$$\tilde{F}_4 = \frac{d^3x \wedge dH^{-1} + Q \ast_6 \omega_2}{r^2 dr} \wedge \omega_2,$$

$$H_3 = \frac{Q}{r^2} dr \wedge \omega_2.$$
Only the RR 4-form field strength is needed because under $\tilde{F}_4$ the D2-branes are charged electrically while the wrapped D4-branes are charged magnetically.

To verify the solution we consider first the equation of motion for the field strength $\tilde{F}_4$ (12). Both sides of this equation are proportional to the volume form on the seven dimensional cone, and the equation is satisfied provided

$$ (r^6 H')' = -Q^2/r^2. $$

(17)

The equation of motion for $H_3$ (14) is satisfied essentially because $\tilde{F}_4$ includes a factor of $dH^{-1}$. The dilaton equation of motion (15) is satisfied because of (17). Last we check the trace of Einstein’s equation (16) using (11).

### 2.1.3 Fractional D4-branes

The dilaton is given by $e^\Phi = H^{-1/4}$, and in Einstein frame the metric is

$$ ds_E^2 = g_s^{1/2} \left[ H^{-3/8} \eta_{\alpha\beta}dx^\alpha dx^\beta + H^{5/8}(dr^2 + r^2 h_{ij}dx^i dx^j) \right] $$

where $\alpha, \beta = 0, 1, \ldots, 4$ and $h_{ij}$ is the metric on the 4-d Einstein manifold $X_4$. The field strengths are

$$ \tilde{F}_4 = H' r^4 \omega_2 \wedge \omega'_2 $$

$$ F_2 = Q \omega'_2 $$

$$ H_3 = Q dr \wedge \omega_2 $$

where $\omega_2$ is the usual harmonic 2-form and $\omega'_2 = *_4 \omega_2$ is its dual. To understand the structure of $\tilde{F}_4$, consider the more familiar dual $*\tilde{F}_4 = -H^{1/8}dx^5 \wedge dH^{-1}$. There remains seemingly an extra factor of $H^{1/8}$. To satisfy the equation of motion for the NS-NS 3-form (13), the $H^{1/8}$ is necessary because it cancels the $e^{\Phi/2}$ that multiplies $*\tilde{F}_4$. The field strength equations (13) and (12) are trivial for this solution. However, the Bianchi identity for $\tilde{F}_4$, which was trivially satisfied in the previous two examples, is nontrivial here:

$$ d\tilde{F}_4 = -F_2 \wedge H_3. $$

(18)

The Bianchi identity leads to the differential equation for the warp factor $(r^4 H')' = -Q^2$. The dilaton equation of motion (15) is also satisfied provided the warp factor obeys this equation. Finally, the trace of Einstein’s equation, which we can check provided we know the Ricci scalar curvature (11), is also satisfied.

### 2.2 Fractional Type IIB Branes
2.2.1 Fractional D1-branes

In this case the transverse space is 8-dimensional, so it can be a Calabi-Yau 4-fold. Such cases will be dual to (1 + 1)-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry, and we will discuss some specific examples in section 4. In general, the dilaton is given by $e^{\Phi} = H^{1/2}$, and the metric in Einstein frame is

$$ds^2_E = g^{1/2}_s \left[ H^{-3/4} \eta_{\alpha\beta} dx^\alpha dx^\beta + H^{1/4} (dr^2 + r^2 h_{ij} dx^i dx^j) \right]$$

where $\alpha, \beta = 0, 1$ and $h_{ij}$ is the metric on the 7-d Einstein space $X_7$. The nonzero field strengths are

$$\tilde{F}_3 = d^2 x \wedge dH^{-1}$$
$$\tilde{F}_5 = -Q (\omega_5 + *\omega_5)$$
$$= -Q \omega_5 - \frac{Q}{r^3} H^{-1} d^2 x \wedge dr \wedge \omega_2$$
$$H_3 = \frac{Q}{r^3} dr \wedge \omega_2$$

where $\omega_5 = (-1)^* \omega_2$. By construction, the five form field strength obeys the self duality constraint $\tilde{F}_5 = *\tilde{F}_5$. Next we check the equation of motion of the RR 3-form field strength

$$d(e^{\Phi} * \tilde{F}_3) = g_s F_5 \wedge H_3 . \quad (19)$$

Both sides are proportional to the volume form on the eight dimensional cone, and the equation is satisfied provided $(r^7 H')' = -Q^2 / r^3$. The NS-NS 3-form field strength is

$$d*(e^{-\Phi} H_3 - C e^\Phi \tilde{F}_3) = -g_s F_5 \wedge F_3 . \quad (20)$$

Note first that the axion $C = 0$ for this solution. Each side of the equation is proportional to the eight form $d^2 x \wedge dr \wedge \omega_5$, and the coefficients match as well.

The two remaining equations to check are the dilaton equation of motion and the trace of Einstein’s equation

$$d* d\Phi = e^{2\Phi} dC \wedge *dC - \frac{g_s e^{-\Phi}}{2} H_3 \wedge *H_3 + \frac{g_s e^{\Phi}}{2} \tilde{F}_3 \wedge *\tilde{F}_3 \quad (21)$$
$$R \text{ Vol} = \frac{1}{2} d\Phi \wedge *d\Phi + \frac{e^{2\Phi}}{2} dC \wedge *dC + \frac{g_s e^{-\Phi}}{4} H_3 \wedge *H_3 + \frac{g_s e^{\Phi}}{4} \tilde{F}_3 \wedge *\tilde{F}_3 . \quad (22)$$

The Ricci scalar is (11), which ensures that (22) is satisfied.

2.2.2 Fractional D3-branes

The dilaton is a constant which we will set to zero. The metric is

$$ds^2 = g^{1/2}_s \left[ H^{-1/2} \eta_{\alpha\beta} + H^{1/2} (dr^2 + r^2 h_{ij} dx^i dx^j) \right]$$
where $\alpha, \beta = 0, 1, 2, 3$ and $h_{ij}$ is the metric on the 5-d Einstein space $X_5$. The nonzero field strengths are

\[
\tilde{F}_5 = d^4x \wedge dH^{-1} + *d^4x \wedge dH^{-1} = d^4x \wedge dH^{-1} - H'r^5\omega_2 \wedge \omega_3
\]

\[
\tilde{F}_3 = Q\omega_3
\]

\[
H_3 = \frac{Q}{r} dr \wedge \omega_2.
\]

This type of solution was worked out in detail in [15] for the supersymmetric example where the 6-d cone is the conifold. The warp factor is given in (10). Even though this solution is singular in the IR, it may be thought of as the asymptotic UV part of the non-singular deformed solution presented in [16]. In particular, it incorporates the cascade of Seiberg dualities which takes place in the dual $\mathcal{N} = 1$ supersymmetric SU($N + M$) $\times$ SU($N$) gauge theory.

The crucial feature of the conifold which enables us to write down the fractional brane solutions is that its base $T^{1,1}$ has $b_2 = b_3 = 1$. Other supersymmetry preserving cones with non-trivial 2-cycles are certain generalized conifolds [29, 30, 31]. For example, there is an $A_k$ series described by

\[
w^k_1 + w^2_2 + w^2_3 + w^2_4 = 0.
\]

Finding an explicit form of the metric on the base and of the forms $\omega_2, \omega_3$ is an interesting challenge.

One could also consider a variety of 6-d cones which do not preserve any supersymmetry. The simplest example is the space $T^{1,0}$ which is a product of $S^2$ and $S^3$ geometrically:

\[
h_{ij}dx^i dx^j = \frac{1}{4} ds_2^2 + \frac{1}{2} ds_3^2,
\]

where $ds^2_n$ is the metric on a unit $n$-sphere. In this case $\omega_2 \sim vol(S^2)$ and $\omega_3 \sim vol(S^3)$ and the warp factor still has the form (10) [32]. The main concern about this explicit solution is whether it is stable. In the absence of the $F_3$ flux we would find a supersymmetry breaking $AdS_5 \times T^{1,0}$ background which is unstable. For example, there is a scalar mode with $m^2 = -8$ in $AdS_5$ which is below the Breitenlohner-Freedman stability bound $m^2 = -4$. This particular mode corresponds to changing the relative volumes of the $S^2$ and $S^3$ while keeping the overall volume of $T^{1,0}$ unchanged. It is clear, however, that adding the $F_3$ flux has a stabilizing effect on this particular mode since it adds a term to the potential which is sensitive to the volume of $S^3$. It would be remarkable if the fractional D3-brane solution based on $T^{1,0}$ turns out to be stable. We postpone investigation of the stability issue to a future publication.

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5This result was found in collaboration with A. Tseytlin.
2.2.3 Fractional D5-branes

The fractional D5-brane solution depends on a 3-d Einstein manifold which might not exist. We will thus be brief in our description of this formal solution to the type IIB SUGRA equations of motion. The dilaton is given by $e^\Phi = H(r)^{-1/2}$, and the Einstein frame metric is

$$ds_E^2 = g_s^{1/2} \left[ H^{-1/4} \eta_{\alpha\beta} dx^\alpha dx^\beta + H^{3/4}(dr^2 + r^2 h_{ij} dx^i dx^j) \right].$$

The nonzero field strengths are

$$dC = Q_1 \omega_1$$
$$\tilde{F}_3 = H^{1/2} \ast d^6 x \wedge dH^{-1}$$
$$= -H' \gamma^3 \omega_2 \wedge \omega_1$$
$$H_3 = Q r dr \wedge \omega_2$$

where $\omega_1 = (-1)^* \omega_2$. The factor of $H^{1/2}$ in $\tilde{F}_3$ is necessary in order to satisfy the NS-NS 3-form field strength equation of motion (20): the $H^{1/2}$ cancels with the $e^\Phi$.

The RR field strength equations of motion are trivial for this solution. On the other hand, the Bianchi identity $d\tilde{F}_3 = -dC \wedge H_3$ results in the relation $(r^3 H')' = -Q^2 r$ which ensures that the dilaton equation of motion (21) and the trace of Einstein’s equation (22) are also satisfied.

3 An eleven dimensional SUGRA solution

Recall that the SUGRA solution describing regular M2-branes at the apex of a cone $Y_8$ is given by the metric

$$ds^2 = H(r)^{-2/3} \eta_{\mu\nu} dx^\mu dx^\nu + H(r)^{1/3}(dr^2 + r^2 h_{ij} dx^i dx^j),$$

where $\mu, \nu = 0, 1, 2$ and $h_{ij}$ is the Einstein metric on the base of the cone $X_7$. The 4-form field strength is $F_4 = d^3 x \wedge dH^{-1}$. The warp factor $H(r)$ satisfies the differential equation $\Box_8 H(r) = 0$ where $\Box_8$ is the Laplacian on $Y_8$.

As the only other branes in eleven dimensional supergravity are 5-dimensional, in order to get fractional M2 branes, we need a 3-cycle corresponding to a harmonic 3-form in the 7-dimensional base of the cone. It is on this three cycle that we wrap our M5-branes. Wrapping M5 branes on such a harmonic three form $\omega_3$ produces the following modification to the M2-brane solution. We set

$$F_4 = d^3 x \wedge dH^{-1} + M*7 \omega_3 - \frac{M}{r} dr \wedge \omega_3.$$
It is straightforward to verify that $F_4$ satisfies the Bianchi identity $d*F_4 = F_4 \wedge F_4/2$. The Ricci scalar on this space is

$$R = -H^{-1/3} \left( \frac{1}{3} \frac{(r^7 H')'}{r^7 H} + \frac{1}{6} \left( \frac{H'}{H} \right)^2 \right).$$

From the Ricci scalar and the trace of Einstein’s equation, we see that the additions to $F_4$ produce a source term in the equation for $H(r)$:

$$(r^7 H(r))' = -\frac{M^2}{r}.$$ 

Thus

$$H(r) = \left( \frac{\rho}{r} \right)^6 + M^2 \left( \frac{\ln(r)}{6r^6} + \frac{1}{36r^6} \right).$$

We conclude that adding wrapped M5-branes produces a logarithmic renormalization group flow in what would otherwise be a 3-dimensional SCFT. This result is surprising in that dimensional arguments suggest that the RG flow in 3-d field theories should be polynomial. We should keep in mind, however, that we have not been able to find supersymmetry-preserving 8-d cones with a harmonic 3-form. Non-supersymmetric such cones certainly exist – for example, a cone over $S^3 \times S^4$ – but it is unclear whether they give stable solutions.

### 4 Fractional D-strings

In order to produce fractional D-strings in our framework, we need a 7-d Einstein space with nonzero Betti numbers $b_2$ and $b_5$, and this space should preferably preserve some supersymmetry. Fortunately, several examples are known to exist, and some have been well studied. For example, the manifold $N^{0,1,0}$ has $b_2 = 1$ [33], admits an Einstein metric, and can be described by the coset SU(3)/U(1). Moreover, this manifold, as it has three Killing spinors, is expected to lead to a 2-d gauge theory on the regular and fractional D-strings with $\mathcal{N} = 3$ supersymmetry [34].

Two more examples are the well known manifolds $Q^{1,1,1}$ and $M^{1,1,1}$ [4]. Both of these spaces admit two Killing spinors; hence, we expect the gauge theory dual living on $N$ regular and $M$ fractional D-strings to have $\mathcal{N} = 2$ supersymmetry [34]. In what follows, we will exhibit the harmonic 2-forms on these spaces. We will also review some facts about these spaces which allow us to guess the Lie group structure of the gauge theory.

---

6 From a knowledge of the harmonic two forms described later, we were able to check completely Einstein’s equation for these two examples.
duals. We start with the simpler \(Q^{1,1,1}\) which can be described as the coset manifold

\[
\frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)}
\]

which has \(SU(2)^3 \times U(1)^R\) global symmetry. The metric on \(Q^{1,1,1}\) can be written

\[
ds^2 = \frac{1}{16} (d\psi - \sum_{i=1}^{3} \cos \theta_i \, d\phi_i)^2 + \frac{1}{8} \sum_{i=1}^{3} (d\theta_i^2 + \sin^2 \theta_i \, d\phi_i^2),
\]

which makes it clear that it is a \(U(1)\) bundle over \(S^2 \times S^2 \times S^2\). For \(Q^{1,1,1}\), \(b_2 = 2\). Two linearly independent harmonic two forms are

\[
\omega_2(ij) = \frac{\sqrt{2}}{16} (\sin \theta_i \, d\theta_i \wedge d\phi_i - \sin \theta_j \, d\theta_j \wedge d\phi_j)
\]

where \((ij) = (12)\) or \((13)\). We have normalized \(\omega_2\) such that \(\omega_2 \wedge *\omega_2\) is the volume form on \(Q^{1,1,1}\).

To understand the possible gauge theory dual to this SUGRA solution, we review some possible embeddings of the cone over \(Q^{1,1,1}\). The locus of points describing this cone can be embedded in \(\mathbb{C}^8\)

\[
\begin{align*}
w_1 w_2 - w_3 w_4 &= 0 \\
w_1 w_2 - w_5 w_6 &= 0 \\
w_1 w_2 - w_7 w_8 &= 0 \\
w_1 w_4 - w_5 w_8 &= 0.
\end{align*}
\]

(23)

Similar to what was done with the coset space \(T^{1,1} = SU(2) \times SU(2)/U(1)\), we can reparametrize this system of equations using instead the six complex variables \(A_i, B_i, C_i, \ i = 1, 2\), and set each \(w_i\) to some different combination \(A_i B_j C_k\). In particular

\[
\begin{align*}
w_1 &= A_1 B_1 C_1 & w_2 &= A_2 B_2 C_2 \\
w_3 &= A_1 B_2 C_1 & w_4 &= A_2 B_1 C_2 \\
w_5 &= A_1 B_1 C_2 & w_6 &= A_2 B_2 C_1 \\
w_7 &= A_1 B_2 C_2 & w_8 &= A_2 B_1 C_1.
\end{align*}
\]

However, we get the same \(w_i\) if we act on the \(A, B,\) and \(C\) by

\[
\begin{align*}
A_j &\to \lambda A_j & B_k &\to \lambda^{-1} B_k \\
A_j &\to \mu A_j & C_l &\to \mu^{-1} C_l \\
j, k, l &= 1, 2
\end{align*}
\]

where \(\lambda, \mu \in \mathbb{C}^*\). We fix this freedom by selecting the absolute value of \(\lambda\) and \(\mu\) to guarantee the two D-term equations:

\[
\begin{align*}
|A_1|^2 + |A_2|^2 &= |B_1|^2 + |B_2|^2 \\
|A_1|^2 + |A_2|^2 &= |C_1|^2 + |C_2|^2.
\end{align*}
\]

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Further quotienting by the two U(1)'s corresponding to the phases of $\lambda$ and $\mu$ gives the cone over $Q^{1,1,1}$. If we fix $|A_1|^2 + |A_2|^2 = 1$, we get the space $Q^{1,1,1}$ itself. $A_1$ and $A_2$ form a doublet under the first global SU(2), $B_1$ and $B_2$ under the second, and $C_1$ and $C_2$ under the third.

In [33] (see also [37]) these algebraic considerations were used to argue that M-theory on $\text{AdS}_4 \times Q^{1,1,1}$ is dual to a $\text{SU}(N) \times \text{SU}(N) \times \text{SU}(N)$ superconformal gauge theory in $2 + 1$ dimensions. The $A_i$, $B_j$, and $C_k$ could then be understood as chiral superfields in the $(N, \overline{N}, 1)$, $(1, N, \overline{N})$, and $(\overline{N}, 1, N)$ representations of the gauge group, respectively. Similar considerations lead us to argue that the gauge theory on $N$ regular D-strings placed at the apex of the cone over $Q^{1,1,1}$ has the same Lie group structure and matter content as above, except in $1 + 1$ dimensions. Adding $M$ fractional branes changes the gauge theory to $\text{SU}(N+M) \times \text{SU}(N) \times \text{SU}(N)$ coupled to chiral superfields $A_i$, $B_j$, and $C_k$ in the $(N+M, \overline{N}, 1)$, $(1, N, \overline{N})$, and $(\overline{N}+M, 1, N)$ representations, respectively.

The next example is $M^{1,1,1}$ which has the coset structure $[38, 34]$

\[
\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\text{SU}(2) \times \text{U}(1) \times \text{U}(1)}.
\]

The Einstein metric on this seven dimensional space can be written [38]

\[
d s^2 = \frac{1}{64} \left( (d\tau + 3 \sin^2 \mu \sigma_3 + 2 \cos \theta_2 d\phi_2)^2 \\
+ \frac{3}{4} (d\mu^2 + \frac{1}{4} \sin^2 \mu (\sigma_1^2 + \sigma_2^2 + \cos^2 \mu \sigma_3^2)) \\
+ \frac{1}{8} (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) \right)
\]

where

\[
\begin{align*}
\sigma_1 &= d\theta_1 \\
\sigma_2 &= \sin \theta_1 d\phi_1 \\
\sigma_3 &= (d\psi + \cos \theta_1 d\phi_1).
\end{align*}
\]

The manifold has $b_2 = 1$ and thus admits one harmonic two form which can be written in this basis as

\[
\omega_2 = \frac{1}{\sqrt{6}} \left( \frac{1}{8} \sin \theta_2 d\theta_2 \wedge d\phi_2 - \frac{3}{16} \sin^2 \mu \sigma_1 \wedge \sigma_2 + \frac{3}{8} \sin \mu \cos \mu \, d\mu \wedge \sigma_3 \right).
\]

Again, we have chosen to normalize the harmonic two form such that $\omega_2 \wedge *\omega_2$ is the volume form on $M^{1,1,1}$.

The cone over $M^{1,1,1}$ can be described as the locus of points in $\mathbb{C}^5$ satisfying the D-term equation [33]

\[
2(|U_1|^2 + |U_2|^2 + |U_3|^2) = 3(|V_1|^2 + |V_2|^2)
\]

quotiented by the action of a U(1), acting on $U_i$ with charge +2 and on $V_i$ with charge -3. To recover the manifold $M^{1,1,1}$, we set $|V_1|^2 + |V_2|^2 = 1$. To express $M^{1,1,1}$ as an
embedding in $\mathbb{C}^p$ for some $p$, just as $Q^{1,1,1}$ was embedded in $\mathbb{C}^8$ above, we need $p = 30$. As a result, we will not describe this embedding here.

From this type of algebraic consideration, the authors of [33] are able to deduce that in an M theory context, where the gauge theory dual is three dimensional and conformal, the gauge group is $\text{SU}(N) \times \text{SU}(N)$. Moreover, they assert that $U_i$ and $V_j$ can be understood as chiral superfields corresponding respectively to $\text{Sym}^2N \times \text{Sym}^2\overline{N}$ and $\text{Sym}^3N \times \text{Sym}^3\overline{N}$ representations of the gauge group. Here $N$ and $\overline{N}$ are respectively the fundamental and anti-fundamental representations of $\text{SU}(N)$. Similar considerations lead us to argue that the gauge theory on $N$ regular D-strings at the apex of the cone over $M^{1,1,1}$ has analogous structure, except in $1+1$ dimensions. Addition of $M$ fractional branes again changes the group to $\text{SU}(N + M) \times \text{SU}(N)$ and modifies the matter representation accordingly.

Both for the $Q^{1,1,1}$ and for the $M^{1,1,1}$ example, $\int B_2 \sim M/r^2$. As for the D3-brane solution, it is tempting to interpret this as RG flow of a relative gauge coupling in the dual gauge theory. Another interesting effect is the radial variation of the RR 3-form flux. If we assume that this flux measures the effective number of regular branes, $N$, then

$$N_{\text{eff}}(r) = N - a_0 g_s M^2 r^2,$$

where $a_0$ is a proportionality constant. Unlike in the D3-brane solutions of [15, 16], $N_{\text{eff}}$ does not diverge in the UV. This is in accord with the expectation that (1+1)-dimensional theories should have weak UV but strong IR dynamics and hence should have power law rather than logarithmic flow. It would be very interesting to find detailed explanations of these supergravity effects in the dual gauge theory. It is possible that the reduction in the number of colors is due to some 2-d analogue of Seiberg duality, but a more mundane scenario, which was recently discussed in some 4-dimensional examples [19], is that the gauge symmetry is broken by the Higgs mechanism. In order to understand the IR effects in more detail on the SUGRA side it will be necessary to find resolved solutions without naked singularities.

5 Fractional Branes and Generalizations

Our type II and 11-dimensional supergravity solutions can be understood as special cases of the more general ansatz presented by Cvetic, Lü, Gibbons, and Pope in [23, 24]. In the type II context they consider non-compact Ricci flat manifolds which are asymptotically conical and possess a harmonic 3-form $\Omega_3$. They then set the NS-NS 3-form $H_3 \sim g_s M \Omega_3$. The warp factor must satisfy the differential equation

$$\Box_{9-p} H \sim -(g_s M)^2 |\Omega_3|^2.$$

The symbol $\Box_{9-p}$ is the Laplacian on the $(9-p)$-dimensional Ricci flat space. To translate to our situation, note that $H_3 \sim (g_s M/r^{1-p}) dr \wedge \omega_2$ is indeed harmonic when considered as
a 3-form on the \((9-p)\)-dimensional cone. However, we impose the additional requirement that \(\Omega_3\) originates from the \(\omega_2\) which is harmonic on the base of the cone. Some specific examples worked out in \([23, 24]\) involve \(\Omega_3\) which only exist on the full \((9-p)\)-dimensional space. We believe that such solutions do not correspond to fractional D-branes although they are very interesting in their own right.

Our 11-dimensional SUGRA solution can also be understood as a special case of the general ansatz worked out in \([23, 24]\) (for earlier work, see \([40, 30]\)) where the authors consider asymptotically conical 8-d Ricci flat manifolds with a self-dual or anti-self-dual harmonic 4-form \(\omega_4\). The field strength is then simply

\[
F_4 = d^3x \wedge dH^{-1} + M\omega_4 .
\]

We restrict

\[
\omega_4 = dr \wedge \omega_3/r + *_8 (dr \wedge \omega_3/r) ,
\]

where \(\omega_3\) is a harmonic 3-form on the base of the cone corresponding to the 3-cycle wrapped by the M5-branes. \([10, 31, 23, 24]\) allow for more general kinds of harmonic 4-forms. Of the specific examples worked out in \([23, 24]\), three of the spaces have SU(4) holonomy; hence the corresponding \((2+1)\)-dimensional field theory has \(\mathcal{N} = 2\) supersymmetry. The bases of these cones are the well studied Einstein manifolds \(V_{5,2}, M^{1,1,1},\) and \(Q^{1,1,1}\) \([24]\). The fourth space has Spin(7) holonomy corresponding to \(\mathcal{N} = 1\) supersymmetry; here the base of the asymptotic cone is the squashed \(S^7\). Although the detailed nature of \(\omega_4\) is complicated in each of the examples, a little bit of power counting is useful in understanding what happens. We find that in the case of \(V_{5,2}\), at large \(r\), \(\omega_4\) scales as \(1/r^{4/3}\), for \(M^{1,1,1}\) and \(Q^{1,1,1}\) as \(1/r^2\), and for the squashed \(S^7\) as \(1/r^{2/3}\). We can ask if these asymptotics correspond to our 11-dimensional solution found above. The answer is no, and the reason is simple: all four Einstein manifolds have \(b_3 = b_1 = 0\). Hence, the solutions found in \([23, 24]\) do not correspond to regular and fractional M5-branes. Instead, as we now show, they describe \((2+1)\)-dimensional CFT’s perturbed by relevant operators.

In each case, the 11-dimensional metric of the solution can be written as

\[
ds^2 = H(r)^{-2/3} \eta^{\alpha\beta} dx^\alpha dx^\beta + H(r)^{1/3} ds_8^2
\]

where \(ds_8^2\) is the metric on an 8-dimensional Ricci flat manifold. Just as in the fractional brane cases, \(\omega_4\) introduces a source term into the equation for the warp factor: \(\Box_8 H(r) \sim M^2 |\omega_4|^2\). At small \(r\), \(H(r)\) approaches a constant in all four examples, which means that the solutions are non-singular and free of horizons. For large \(r\), the metric on the 8-dimensional manifold \(ds_8^2 \rightarrow dr^2 + r^2 ds_7^2\), where \(ds_7^2\) is the metric on the Einstein space \(X_7\). Expansions of \(H(r)\) in powers of \(1/r\) have the form

\[
H(r) = \left( \frac{\rho}{r} \right)^6 \left( 1 + \frac{c_1}{r^\gamma} + \ldots \right).
\]

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If $\gamma > 0$, then in the UV we recover the background $AdS_4 \times X_7$ which is dual to a $(2 + 1)$-dimensional CFT. (To make the $AdS_4$ space completely manifest, we also need to make the change of variables $r^2 = 1/z$.) The leading deformation of the $AdS_4 \times X^7$ metric is encoded in the term

$$\sim z^{\gamma/2} ds^2_7.$$  (28)

$\gamma$ is 4 for $M^{1,1,1}$ and $Q^{1,1,1}$, $8/3$ for $V_{5,2}$ and $4/3$ for the squashed $S^7$ [23, 24]. Specific examples of some of these CFT’s were discussed in [33, 41].

A perturbation of an $AdS_4 \times X_7$ background which falls off at large $r$ can correspond either to adding a relevant operator to the CFT action or to giving an expectation value to an operator [42, 43]. In all four cases above, the perturbation that falls off the slowest at large $r$ is contained in the 4-form $\omega_4$. Asymptotically, we may write $\omega_4 = dc_3$, where the $c_3$ is the perturbation of the 3-form potential. Purely angular components of the 3-form potential are dual to pseudoscalar operators in the CFT [44, 45]. Indeed, we will be able to confirm that the rate at which $c_3$ falls off at large $r$ is consistent with the addition of pseudoscalar operators to the action via the usual AdS/CFT relation [2, 3]

$$\phi(\vec{x}, z) \sim z^{3-\Delta} \phi_0(\vec{x}).$$  (29)

where $\Delta$ is the dimension of the operator corresponding to $\phi$ in the conformal gauge theory dual.

Let us recall that an $\mathcal{N} = 1$ chiral superfield of dimension $\Delta$ decomposes into a scalar of dimension $\Delta$, a fermion of dimension $\Delta + 1/2$, and a pseudoscalar of dimension $\Delta + 1$. In searching for a pseudoscalar with the right dimension, we need to find a chiral superfield in the conformal gauge theory dual whose dimension is appropriate and protected. We will present explicit findings for the $V_{5,2}$ and $Q^{1,1,1}$ cases.

For $V_{5,2}$, the cone can be embedded in $\mathbb{C}^5$ via the equation

$$F = \sum_{i=1}^5 w_i^2 = 0.$$

We expect to find protected operators in the conformal gauge theory dual that correspond to monomials in the $w_i$ quotiented by the embedding relation $F = 0$. In the proposed CFT dual, each $w_i$ contributes $2/3$ to the operator dimension [41]. These dimensions can be derived from the AdS/CFT correspondence as follows. From symmetry requirements, the Kaehler potential on $V_{5,2}$ must take the form $K = (\sum_{i=1}^5 |w_i|^2)^{\alpha}$. The Calabi-Yau 4-form

$$\Omega = \frac{dw_1 \wedge dw_2 \wedge dw_3 \wedge dw_4}{w_5}.$$

For the metric to be Ricci flat, the CY 4-form and the Kaehler form $\omega = \partial \bar{\partial} K$ must obey the relation $\wedge^4 \omega \sim \Omega \wedge \bar{\Omega}$. Power counting then shows that $\alpha = 3/4$. It is natural to equate $K = r^2$ because we are looking for a cone, and the metric component $g_{rr}$ should
be of order zero in $r$. We see that the eigenvalue of these monomials under the action of $r \partial_r$ is $4k/3$ where $k$ is the total number of $w_i$ in the monomial. Finally, because we need to make the transformation $r^2 = 1/z$ to make the $AdS_4$ manifest, we see that these monomials will have dimension $2k/3$ in the conformal theory and each $w_i$ has dimension $2/3$.

Now, a monomial quadratic in the $w_i$ is a chiral field with dimension $4/3$. Also in this chiral multiplet is a pseudoscalar of dimension $7/3$. Adding this operator to the action corresponds to $c_3 \sim \frac{z^{2/3}}{z}$, precisely the rate at which $c_3$ falls off at small $z$. This perturbation breaks the conformal invariance of the CFT dual to $AdS_4 \times V_{5,2}$ and, according to the solution found in [24], produces confinement far in the IR.

Our next example, the cone over $Q^{1,1,1}$, can be embedded in $C^8$ via (23). We expect to find protected operators in the gauge theory dual which correspond to monomials in the $w_i$ quotiented by the embedding relations. From [33], we know that each of the $w_i$ contributes 1 to the operator dimension. Thus we expect and find [46] a pseudoscalar with dimension 2 within the chiral multiplet corresponding to the $w_i$ themselves, explaining the falloff $c_3 \sim z$ found in this case.

Thus, both the $Q^{1,1,1}$ and the $V_{5,2}$ cases have a convenient explanation in terms of perturbing the action by relevant operators. We also note that for the squashed $S^7$, adding relevant operators to the action is the only available interpretation. Here $c_3 \sim z^{1/3}$ which cannot be interpreted as due to an expectation value; $\Delta = 1/3$ required for this interpretation violates the unitarity bound in $2 + 1$ dimensions, $\Delta \geq 1/2$. Instead, the fall-off in $c_3$ corresponds to adding an operator of dimension $8/3$ to the action. An analogous explanation of the $M^{1,1,1}$ case requires the existence of a pseudoscalar operator with dimension 2. However, we have not found such an operator in the results of [47]. We hope to return to this issue in the future.

A point of similarity between the fractional brane solutions considered here and the solutions constructed in [23, 24] is that all of them have varying flux corresponding to regular branes. It is natural to interpret the varying flux as a reduction in the number of degrees of freedom as the theory flows towards the IR. In some situations [16] this reduction is due to Seiberg duality while in others it is due to Higgsing [19]. In the case of solutions without extra conserved fluxes, such as those in [23, 24], the Higgsing interpretation is more likely to be applicable. It would be very interesting to understand this in more detail.

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A Supergravity Actions and Equations of Motion

A.1 Type IIA

The bosonic piece of the IIA Einstein frame action is

\[
\frac{1}{2\kappa^2} \int d^{10}x (-G)^{1/2} R - \frac{1}{4\kappa^2} \int \left( d\Phi \wedge *d\Phi + g_s e^{-\Phi} H_3 \wedge *H_3 \\
+ g_s^{1/2} e^{3\Phi/2} F_2 \wedge *F_2 + g_s^{3/2} e^{\Phi/2} \tilde{F}_4 \wedge *\tilde{F}_4 + g_s^2 B_2 \wedge F_4 \wedge \tilde{F}_4 \right),
\]

(30)

where

\[
\tilde{F}_4 = F_4 - C_1 \wedge H_3, \quad F_4 = dC_3, \quad F_2 = dC_1.
\]

We define the Einstein metric by \((G_{\mu\nu})_{\text{Einstein}} = g_s^{1/2} e^{-\Phi/2} (G_{\mu\nu})_{\text{string}}\). As a result \(g_s\) appears in the action, explicitly and also through \(2\kappa^2 = (2\pi)^7 \alpha' g_s^2\).

The field equations are [48]

\[
d^*d\Phi = -g_s e^{-\Phi} H_3 \wedge *H_3 + \frac{3g_s^{1/2} e^{3\Phi/2}}{4} F_2 \wedge *F_2 + \frac{g_s^{3/2} e^{\Phi/2}}{4} \tilde{F}_4 \wedge *\tilde{F}_4
\]

\[
d(e^{3\Phi/2} F_2) = g_s e^{3\Phi/2} H_3 \wedge *\tilde{F}_4
\]

\[
d(e^{\Phi/2} \tilde{F}_4) = -g_s^{1/2} F_4 \wedge H_3
\]

\[
\frac{g_s}{2} F_4 \wedge F_4 = d(e^{-\Phi} H_3 + g_s^{1/2} e^{\Phi/2} C_1 \wedge *\tilde{F}_4)
\]

\[
R_{MN} = \frac{1}{2} \partial_M \Phi \partial_N \Phi + \frac{g_s e^{-\Phi}}{4} (H_M^{\ PQR} H_N^{\ PQ} - \frac{1}{12} G_{MN} H^{PQR} H_{PQR})
\]

\[
+ \frac{g_s^{1/2} e^{3\Phi/2}}{2} (F_M^{\ P} F_N^{\ PQ} - \frac{1}{16} G_{MN} F^{PQ} F_{PQ})
\]

\[
+ \frac{g_s^{3/2} e^{\Phi/2}}{12} (F_M^{\ PQR} \tilde{F}_{NPQR} - \frac{3}{32} G_{MN} \tilde{F}^{PQRS} \tilde{F}_{PQRS}) e^{\Phi/2}.
\]

(32)

We use indices \(M, N, \ldots\) in ten dimensions. The Bianchi identities are

\[
d\tilde{F}_4 = -F_2 \wedge H_3, \quad dF_2 = 0.
\]

A.2 Type IIB

The bosonic piece of the IIB Einstein frame action [49] is

\[
\frac{1}{2\kappa^2} \int d^{10}x (-G)^{1/2} R - \frac{1}{4\kappa^2} \int \left( d\Phi \wedge *d\Phi + e^{2\Phi} dC \wedge *dC +
\]

...
\[ g_s e^{-\Phi} H_3 \wedge * H_3 + g_s e^\Phi \tilde{F}_3 \wedge * \tilde{F}_3 + \frac{g_s^2}{2} \tilde{F}_5 \wedge * \tilde{F}_5 + g_s^2 C_4 \wedge H_3 \wedge F_3 \], \quad (33)

supplemented by the self-duality condition

\[ * \tilde{F}_5 = \tilde{F}_5 . \quad (34) \]

Here

\[ \tilde{F}_3 = F_3 - C H_3 \] \[ F_3 = dC_2 \]
\[ \tilde{F}_5 = F_5 - C_2 \wedge H_3 \] \[ F_5 = dC_4 . \quad (35) \]

The field equations are [50]

\[ ds*d\Phi = e^{2\Phi} dC \wedge * dC - \frac{g_s e^{-\Phi}}{2} H_3 \wedge * H_3 + \frac{g_s e^\Phi}{2} \tilde{F}_3 \wedge * \tilde{F}_3, \]
\[ d(e^{2\Phi}*dC) = -g_s e^\Phi H_3 \wedge * \tilde{F}_3, \]
\[ d*(e^\Phi \tilde{F}_3) = g_s F_5 \wedge H_3 , \]
\[ d*(e^{-\Phi} H_3 - C e^\Phi \tilde{F}_3) = -g_s F_5 \wedge F_3 , \]
\[ d* \tilde{F}_5 = -F_3 \wedge H_3 , \]
\[ R_{MN} = \frac{1}{2} \partial_M \Phi \partial_N \Phi + \frac{e^{2\Phi}}{2} \partial_M C \partial_N C + \frac{g_s^2}{96} \tilde{F}_{MPQRS} \tilde{F}_N^{PQRS} \]
\[ + \frac{g_s}{4} (e^{-\Phi} H_{MPQH} H_N^{PQ} + e^\Phi \tilde{F}_{MPQ} \tilde{F}_N^{PQ}) \]
\[ - \frac{g_s}{48} G_{MN} (e^{-\Phi} H_{PQRH} H^{PQR} + e^\Phi \tilde{F}_{PQR} \tilde{F}_P^{PQR}) . \quad (36) \]

The Bianchi identities are

\[ d\tilde{F}_3 = -dC \wedge H_3 \]
\[ d\tilde{F}_5 = -F_5 \wedge H_3 . \quad (37) \]

A.3 M theory

The eleven dimensional SUGRA action[51] is

\[ \frac{1}{2 \kappa_{11}^2} \int d^{11}x (\text{vol})^{1/2} R - \frac{1}{4 \kappa_{11}^2} \int (F_4 \wedge * F_4 + \frac{1}{3} A_3 \wedge F_4 \wedge F_4) . \quad (38) \]

The field equations are then

\[ ds F_4 = \frac{1}{2} F_4 \wedge F_4, \]
\[ R_{MN} = \frac{1}{12} \left( F_M^{PQR} F_{NPQR} - \frac{1}{12} G_{MN} F^{PQRS} F_{PQRS} \right) \quad (39) \]

supplemented by the Bianchi identity \( dF_4 = 0 \).
References

[1] J. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.

[2] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B428 (1998) 105, hep-th/9802109.

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.

[4] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, “Large N Field Theories, String Theory and Gravity,” Phys. Rept. 323 (2000) 183, hep-th/9905111.

[5] I.R. Klebanov, “TASI Lectures: Introduction to the AdS/CFT Correspondence,” hep-th/0009139.

[6] S. Kachru and E. Silverstein, “4d conformal field theories and strings on orbifolds,” Phys. Rev. Lett. 80 (1998) 4855, hep-th/9802183; A. Lawrence, N. Nekrasov and C. Vafa, “On conformal field theories in four dimensions,” Nucl. Phys. B533 (1998) 199, hep-th/9803011.

[7] A. Kehagias, “New Type IIB Vacua and Their F-Theory Interpretation,” Phys. Lett. B435 (1998) 337, hep-th/9805131.

[8] I.R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B536 (1998) 199, hep-th/9807080.

[9] D. Morrison and R. Plesser, “Non-Spherical Horizons I,” Adv. Theor. Math. Phys. 3 (1999) 1, hep-th/9810201.

[10] E.G. Gimon and J. Polchinski, “Consistency Conditions for Orientifolds and D Manifolds,” Phys. Rev. D54 (1996) 1667, hep-th/9601038.

[11] M.R. Douglas, “Enhanced Gauge Symmetry in M(atrix) theory,” JHEP 007 (1997) 004, hep-th/9612126.

[12] S.S. Gubser and I.R. Klebanov, “Baryons and Domain Walls in an N=1 Superconformal Gauge Theory,” Phys. Rev. D58 (1998) 125025, hep-th/9808075.

[13] I. R. Klebanov and N. Nekrasov, “Gravity Duals of Fractional Branes and Logarithmic RG Flow,” Nucl. Phys. B574 (2000) 263, hep-th/9911096.

[14] P. Candelas and X. de la Ossa, “Comments on Conifolds,” Nucl. Phys. B342 (1990) 246.
[15] I. R. Klebanov and A. A. Tseytlin, “Gravity Duals of Supersymmetric $SU(N) \times SU(N + M)$ Gauge Theories,” *Nucl. Phys. B578* (2000) 123. [hep-th/0002155]

[16] I. R. Klebanov and M. J. Strassler, “Supergravity and a Confining Gauge Theory: Duality Cascades and $\chi$SB-Resolution of Naked Singularities,” *JHEP 0008* (2000) 052. [hep-th/0007191]

[17] L. Pando Zayas and A. Tseytlin, “3-branes on resolved conifold,” *JHEP 0011* (2000) 028. [hep-th/0010083]

[18] C. Johnson, A. Peet and J. Polchinski, “Gauge Theory and the Excision of Repulson Singularities,” *Phys. Rev. D61* (2000) 086001. [hep-th/9911161]

[19] O. Aharony, “A Note on the Holographic Interpretation of String Theory Backgrounds with Varying Flux,” [hep-th/0101013].

[20] J. Polchinski, “$\mathcal{N} = 2$ gauge-gravity duals,” [hep-th/0011193] based on the talk at Strings 2000, Ann Arbor, Michigan.

[21] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta and I. Pesando, “Fractional D-branes and their gauge duals,” [hep-th/0011077].

[22] G. T. Horowitz and A. Strominger, “Black strings and p-branes,” *Nucl. Phys. B360* (1991) 197.

[23] M. Cvetič, H. Lü, and C. N. Pope, “Brane Resolution Through Transgression,” [hep-th/0011023].

[24] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, “Ricci-flat Metrics, Harmonic Forms and Brane Resolutions,” [hep-th/0012011].

[25] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, “Supersymmetric non-singular fractional D2-branes and NS-NS 2-branes,” [hep-th/0101096].

[26] G. W. Gibbons, M. B. Green, M. J. Perry, “Instantons and Seven-Branes in Type IIB Superstring Theory,” *Phys. Let. B370* (1996) 37. [hep-th/9511080]

[27] R. S. Hamilton, “Three manifolds with positive Ricci curvature,” *J. Diff. Geom. 17* (1982) 255.

[28] A. Buchel, “Finite temperature resolution of the Klebanov-Tseytlin singularity,” [hep-th/0011146]; S. S. Gubser, C. P. Herzog, I. R. Klebanov and A. A. Tseytlin, “Restoration of chiral symmetry: A supergravity perspective,” [hep-th/0102172]; A. Buchel, C. P. Herzog, I. R. Klebanov, L. Pando Zayas and A. A. Tseytlin, “Non-extremal gravity duals for fractional D3-branes on the conifold,” [hep-th/0102105].
[29] S. Gubser, N. Nekrasov and S. Shatashvili, “Generalized conifolds and four dimensional N = 1 superconformal theories,” JHEP 9905 (1999) 003, hep-th/9811230.

[30] S. Gukov, C. Vafa and E. Witten, “CFT’s from Calabi-Yau four-folds,” Nucl. Phys. B584 (2000) 69, hep-th/9906070.

[31] K. Oh and R. Tatar, “Renormalization group flows on D3 branes at an orbifolded conifold,” JHEP 0005 (2000) 030, hep-th/0003183.

[32] I. R. Klebanov and A. A. Tseytlin, unpublished.

[33] D. Fabbri, P. Fré, L. Gualtieri, C. Reina, A. Tomasiello, A. Zaffaroni, and A. Zampa, “3D superconformal theories from Sasakian seven-manifolds: new nontrivial evidences for AdS4/CFT3,” Nucl. Phys. B577 (2000) 547, hep-th/9907219.

[34] L. Castellani, L. J. Romans, and N. P. Warner, “A Classification of Compactifying Solutions for d=11 Supergravity,” Nucl. Phys. B241 (1984) 429.

[35] D. Page and C. N. Pope, “Which Compactifications of D=11 Supergravity are Stable,” Phys. Let. 144B (1984) 346.

[36] K. Oh and R. Tatar, “Three dimensional SCFT from M2 branes at conifold singularities,” JHEP 9902 (1999) 025, hep-th/9810244.

[37] C. Ahn, “N = 2 SCFT and M theory on AdS(4) x Q(1,1,1),” Phys. Lett. B466 (1999) 171, hep-th/9908162.

[38] E. Witten, “Search for a Realistic Kaluza-Klein Theory,” Nucl. Phys. B186 (1981) 412.

[39] D. Page and C. N. Pope, “Stability Analysis of Compactifications of D=11 Supergravity with SU(3) x SU(2) x U(1) Symmetry,” Phys. Let. 145B (1984) 337; L. Castellani, R. D’Auria, and P. Fré, “SU(3) x SU(2) x U(1) from D=11 Supergravity,” Nucl. Phys. B239 (1984) 610.

[40] K. Becker and M. Becker, “M Theory on Eight-Manifolds,” Nucl. Phys. B477 (1996) 155, hep-th/9605053.

[41] A. Ceresole, G. Dall’Agata, R. D’Auria, and S. Ferrara, “M-Theory on the Stiefel Manifolds and 3d Conformal Field Theories,” JHEP 0003 (2000) 011, hep-th/9912107.

[42] V. Balasubramanian, P. Kraus and A. Lawrence, “Bulk vs. Boundary Dynamics in Anti-de Sitter Spacetime,” hep-th/9805171.
[43] I. R. Klebanov and E. Witten, “AdS/CFT Correspondence and Symmetry Breaking,” Nucl. Phys. B556 (1999) 89, hep-th/9905104.

[44] O. Aharony, Y. Oz and Z. Yin, “M-theory on AdS(p) x S(11-p) and superconformal field theories,” Phys. Lett. B430 (1998) 87, hep-th/9803051.

[45] S. Minwalla, “Particles on AdS(4/7) and primary operators on M(2/5) brane worldvolumes,” JHEP 9810 (1998) 002, hep-th/9803053.

[46] P. Merlatti, “M-theory on AdS4 × Q111: the complete Osp(2|4) × SU(2) × SU(2) spectrum from harmonic analysis,” hep-th/0012159.

[47] D. Fabbri, P. Fré, L. Gualtieri, and P. Termonia, “M-theory on AdS4 × M111: the complete Osp(2|4) × SU(3) × SU(2) spectrum from harmonic analysis,” Nucl. Phys. B560 (1999) 617, hep-th/9903036.

[48] L. J. Romans, “Massive N=2a Supergravity in Ten Dimensions,” Phys. Let. 169B (1986) 374.

[49] E. Bergshoeff, H. J. Boonstra, and T. Ortin, “S duality and dyonic p-brane solutions in type II string theory,” Phys. Rev. D53 (1996) 7206, hep-th/9508091.

[50] J. H. Schwarz, “Covariant Field Equations Of Chiral N=2 D = 10 Supergravity,” Nucl. Phys. B226 (1983) 269.

[51] E. Cremmer, B. Julia, and J. Scherk, “Supergravity Theory in Eleven Dimensions,” Phys. Lett. 76B (1978) 409.