Lattice polytopes with distinct pair-sums

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Let $\mathcal{P}$ be a lattice polytope in $\mathbb{R}^n$, the convex hull of a finite set in $\mathbb{Z}^n$, and let

$$\mathcal{L}(\mathcal{P}) := \mathcal{P} \cap \mathbb{Z}^n = \{v_1, \ldots, v_N\},$$

where $N = N(\mathcal{P}) := |\mathcal{L}(\mathcal{P})|$. Suppose the $N + \binom{N}{2}$ points in $\mathcal{L}(\mathcal{P}) + \mathcal{L}(\mathcal{P})$,

$$2v_1, \ldots, 2v_N; v_1 + v_2, v_1 + v_3, \ldots, v_{N-1} + v_N$$

are distinct. In this case, we say that $\mathcal{P}$ is a distinct pair-sum or dps polytope. Our interest in dps polytopes comes from the study of the representation of polynomials as a sum of squares of polynomials.

The following lemma offers two other geometrical characterizations of dps polytopes.

**Lemma 1.** Let $\mathcal{P}$ be a lattice polytope. Then the following are equivalent:

1. $\mathcal{L}(\mathcal{P})$ is a dps polytope.
2. $\mathcal{L}(\mathcal{P})$ does not contain the vertices of a (nondegenerate) parallelogram, and does not contain three collinear points.
3. Suppose $v \neq v'$ and $w \neq w'$ are in $\mathcal{L}(\mathcal{P})$. Then $v' - v$ and $w' - w$ are parallel only if $\{v, v'\} = \{w, w'\}$.

**Proof.**

1. $\Rightarrow$ (2). Suppose $v_1, v_2, v_3, v_4 \in \mathcal{L}(\mathcal{P})$ are the vertices of a parallelogram. Then $v_1 - v_2 = v_3 - v_4$ implies $v_1 + v_4 = v_2 + v_3$, so that $\mathcal{P}$ is not dps. Now suppose $v_1, v_2, v_3 \in \mathcal{L}(\mathcal{P})$, and $v_2$ is interior to the line segment $v_1v_3$. If $v_2$ is the midpoint...
of the segment, then \( v_2 + v_2 = v_1 + v_3 \), so \( P \) is not dps. Otherwise, we may assume that \( v_2 \) is closer to \( v_1 \) than to \( v_3 \). Then \( v_4 = v_2 + (v_2 - v_1) \) will also be a lattice point on the line segment \( \overline{v_1v_3} \), and \( v_2 \) is the midpoint of \( \overline{v_1v_3} \); again, \( P \) is not dps.

(2) \( \Rightarrow \) (3). For \( u \in \mathbb{Z}^n \), let \( g(u) = \gcd(u_1, \ldots, u_n) \). Suppose \( g(u' - u) = d > 1 \). Then \( u' - u = du'' \) for \( u'' \in \mathbb{Z}^n \), and the line segment \( uu' \) contains the lattice points \( u, u + u'', \ldots, u + du'' = u' \). Thus, if (2) holds and \( u, u' \in L(P) \), \( u \neq u' \), we have \( g(u' - u) = 1 \). Suppose \( w' - w = \alpha \cdot (v' - v) \). Then \( \alpha = p/q \) for nonzero integers \( p, q \), and \( g(w' - w) = p/(v' - v) \). Hence \( |q| = g(p(w' - w)) = g(p(v' - v)) = |p| \), so \( \alpha = \pm 1 \). Now the parallelogram condition in (2) implies that \( \{v, v'\} = \{w, w'\} \).

(3) \( \Rightarrow \) (1). If (3) holds for \( P \), and \( v_i, v_j, v_k, v_\ell \in L(P) \) with \( i \notin \{k, \ell\} \), then \( v_i - v_k \neq v_\ell - v_j \), and so \( v_i + v_j \neq v_k + v_\ell \). This proves (1).

Our main results are these: if \( P \) in \( \mathbb{R}^n \) is a dps polytope, then \( N(P) \leq 2^n \), and, for every \( n \), we construct dps polytopes in \( \mathbb{R}^n \) for which \( N(P) = 2^n \).

**Example 1.** Let \( P \subset \mathbb{R}^2 \) be the triangle with vertices \( \{(0, 1), (1, 2), (2, 0)\} \). Then \( P \) is a dps polytope, because

\[
L(P) = \{(0, 1), (1, 2), (2, 0), (1, 1)\},
\]

and

\[
L(P) + L(P) = \{(0, 2), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (4, 0)\}.
\]

We can view \( P \) as the projection onto the first two coordinates of the triangle with vertices \( \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\} \), which lies in the hyperplane \( x_1 + x_2 + x_3 = 3 \). (In this example, we could have just as well taken the triangle with vertices \( \{(0, 0), (1, 2), (2, 1)\} \); again, \( L(P) \) will consist of the vertices of \( P \) and \( (1, 1) \).)

**Example 2.** Let

\[
A = \{(4, 1, 0, 0), (0, 4, 1, 0), (0, 0, 4, 1), (1, 0, 0, 4)\},
\]

\[
B = \{(2, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1), (1, 1, 1, 2)\};
\]

and let \( P = \text{cvx}(A \cup B) \subset \mathbb{R}^4 \) be the convex hull of \( A \cup B \). By construction, \( P \) is cyclically symmetric with respect to its coordinates. It is not hard to show that \( L(P) = A \cup B \). Suppose \( w = (w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)}) \in L(P) \). Since \( w \) is a convex combination of \( A \cup B \), we have \( w^{(i)} \geq 0 \) and \( \sum_i w^{(i)} = 5 \). If \( w^{(i)} \geq 1 \) for all \( i \), then \( w \) must be a permutation of \( (2, 1, 1, 1) \) and so lies in \( B \). Otherwise, \( w^{(i)} = 0 \) for some \( i \), and by cycling the coordinates, we may assume that \( w^{(1)} = 0 \). But then \( w \) must be a convex combination of \( (0, 4, 1, 0) \) and \( (0, 0, 4, 1) \) and so \( w \in A \). A routine check, which we omit, shows that the \( 8 + \binom{8}{2} = 36 \) sums in \( L(P) \) are distinct. By projecting \( P \) onto its first three coordinates, we obtain a dps polytope in \( \mathbb{R}^3 \) with \( N(P) = 8 \).
Theorem 2. Suppose $\mathcal{P}$ is a dps polytope in $\mathbb{R}^n$. Then $N(\mathcal{P}) \leq 2^n$.

Proof. If $N(\mathcal{P}) > 2^n$, then by the Pigeonhole Principle, there exist $v_i \neq v_j$ so that $v_k = \frac{1}{2}(v_i + v_j) = v_i + \frac{1}{2}(v_j - v_i)$ is also a lattice point, and it follows from Lemma 1 that $\mathcal{P}$ is not a dps polytope.

This argument is essentially the same one used to solve Putnam Problem 1971-A1 (see [1]): “Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.” The proof of Theorem 2 also applies to the less restrictive class of convex polytopes which do not contain three lattice points on a line. One such polytope is the $n$-cube $C_n = \{0, 1\}^n$, which has many lattice parallelograms.

We shall say that a dps polytope $\mathcal{P} \subset \mathbb{R}^n$ for which $N(\mathcal{P}) = 2^n$ is maximal. The proof of Theorem 2 implies that no two points in a dps polytope are component-wise congruent modulo 2; hence a maximal dps polytope contains one representative from every congruence class modulo 2 (and at most one representative from every congruence class modulo $m, m \geq 3$).

Suppose $M$ is an $n \times n$ unimodular matrix with integer entries. Then $M$ defines a linear mapping on $\mathbb{R}^n$ (viewed as column vectors) by matrix multiplication. Since linear mappings preserve inclusions and both $M$ and $M^{-1}$ have integer entries, it is easy to see that $\mathcal{L}(M(\mathcal{P})) = M(\mathcal{L}(\mathcal{P}))$ for any lattice polytope $\mathcal{P}$, and since linear mappings preserve sums, it is then clear that $\mathcal{P}$ is dps if and only if $M(\mathcal{P})$ is dps.

Theorem 3. There exist maximal dps polytopes in $\mathbb{R}^n$ for every $n$.

Proof. For $n = 1$, let $\mathcal{P} = [0, 1]$; for $n = 2, 3$, consider Examples 1 and 2. Suppose now that $\mathcal{P}$ is a maximal dps polytope in $\mathbb{R}^n, n \geq 3$. Write $\mathcal{L} = \mathcal{L}(\mathcal{P})$ and define the (finite) set of differences

$$\mathcal{D} = (\mathcal{L} - \mathcal{L})^* := \{v - v' : v, v' \in \mathcal{L}, v \neq v'\}.$$

Let $M$ be a unimodular integer matrix such that if $u \in \mathcal{D}$, then $M(u) \notin \mathcal{D}$. (We shall construct such an $M$ below.)

We define the polytope $\mathcal{P}'$ in $\mathbb{R}^{n+1}$ as follows. Let

$$\mathcal{A} = \{(v, 0) \in \mathbb{R}^{n+1} : v \in \mathcal{L}(\mathcal{P})\}, \quad \mathcal{B} = \{(M(v), 1) \in \mathbb{R}^{n+1} : v \in \mathcal{L}(\mathcal{P})\},$$

and let $\mathcal{P}' = \text{cx}(\mathcal{A} \cup \mathcal{B})$. If $w = (w^{(1)}, \ldots, w^{(n+1)}) \in \mathcal{L}(\mathcal{P}')$, then $0 \leq w^{(n+1)} \leq 1$, hence $w^{(n+1)}$ equals 0 or 1. Thus, $w$ lies either on the face determined by $\mathcal{A}$, in which case $w = (v, 0)$, or on the face determined by $\mathcal{B}$, in which case $w = (M(v), 1)$.

It follows that $\mathcal{L}(\mathcal{P}') = \mathcal{A} \cup \mathcal{B}$, so $N(\mathcal{P}') = 2^{n+1}$.

Now consider $\mathcal{L}(\mathcal{P}') + \mathcal{L}(\mathcal{P})$; this consists of three disjoint sets of points:\nn\{(v_i, 0) + (v_j, 0)\}, \quad \{(v_i, 0) + (M(v_j), 1)\}, \quad \{(M(v_i), 1) + (M(v_j), 1)\},
where \( v_i, v_j \in L(\mathcal{P}) \). Since both \( \mathcal{P} \) and \( M(\mathcal{P}) \) are dps, the sums in the first and the third set are distinct. For the second set, we suppose that

\[(v_i, 0) + (M(v_j), 1) = (v_k, 0) + (M(v_{\ell}), 1), \tag{1}\]

or equivalently,

\[v_i - v_k = M(v_{\ell}) - M(v_j) = M(v_{\ell} - v_j).\]

If \( j = \ell \), then \( v_i - v_k = 0 \), so \( i = k \), which is the only possible way for (1) to hold in a dps polytope. Otherwise, \( j \neq \ell \), so \( M(v_{\ell} - v_j) = v_i - v_k \in \mathcal{D} \), a contradiction to the choice of \( M \). Thus, \( \mathcal{P}' \) is a maximal dps polytope in \( \mathbb{R}^{n+1} \).

We now construct a matrix \( M \) with the desired properties. First, let

\[R = \max \{|u_j^{(k)}| : u_j \in \mathcal{D}, 1 \leq k \leq n\}.\]

and let \( M \) be the \( n \times n \) matrix given below:

\[
M = \begin{pmatrix}
1 + (R + 1)^2 & R + 1 & 0 & 0 & \ldots & 0 & 0 \\
R + 1 & 1 & R + 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & R + 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & R + 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}.
\]

(In words, the only non-zero entries in \( M \) are the diagonal, the superdiagonal, and the first entry in the second row.) It is easy to see that \( M \) is unimodular.

We show now that for every \( u \in \mathcal{D} \), at least one entry of \( w = M(u) \) has absolute value greater than \( R \). This implies that \( M(u) \notin \mathcal{D} \), and will complete the proof. Write \( u = (u^{(1)}, \ldots, u^{(n)}) \) and suppose that \( k \) is the smallest index such that \( u^{(k)} \neq 0 \). (Such an index exists because \( 0 \notin \mathcal{D} \).)

If \( k = 1 \), then \( w^{(1)} = (1 + (R + 1)^2)u^{(1)} + (R + 1)u^{(2)} \), and hence

\[|w^{(1)}| \geq |(1 + (R + 1)^2)u^{(1)}| - (R + 1)|u^{(2)}| \geq 1 + (R + 1)^2 - R(R + 1) = R + 2.\]

If \( k \geq 2 \), then \( u^{(1)} = \cdots = u^{(k-1)} = 0 \), so \( w^{(k-1)} = (R + 1)u^{(k)} \) and \( |w^{(k-1)}| \geq R + 1 \).

Finally, we remark that the same proof applies in the case \( n = 2 \), if we take as our matrix the \( 2 \times 2 \) submatrix at the upper left of \( M \).

\[\Box\]

**Example 3.** We illustrate the last construction by applying it to the polytope in Example 1, for which

\[\mathcal{D} = \{\pm(0, 1), \pm(1, -2), \pm(1, -1), \pm(1, 0), \pm(1, 1), \pm(2, -1)\},\]

so \( R = 2 \) and

\[M = \begin{pmatrix}
10 & 3 \\
3 & 1
\end{pmatrix}.
\]
Thus, \(\text{cvx}(A \cup B)\) is a maximal dps polytope in \(\mathbb{R}^3\), where
\[
A = \{(0, 1, 0), (1, 1, 0), (1, 2, 0), (2, 0, 0)\}, \\
B = \{(3, 1, 1), (13, 4, 1), (16, 5, 1), (20, 6, 1)\}.
\]
We could now apply the shear \((x_1, x_2, x_3) \mapsto (x_1 - 3x_2 - 5x_3 + 5, x_2 - x_3, x_3)\), which maps \(A\) and \(B\) to
\[
A' := \{(2, 1, 0), (3, 1, 0), (0, 2, 0), (7, 0, 0)\}
\]
and
\[
B' := \{(0, 0, 1), (1, 3, 1), (1, 4, 1), (2, 5, 1)\},
\]
respectively, in order to reduce the magnitude of the coordinates in the example.

Since any translate of a dps polytope is also dps, we may always assume, as we have done in the examples, that \(P\) lies in the non-negative orthant of \(\mathbb{R}^n\). In this case, we define \(s(P)\), the size of \(P\):
\[
s(P) = \max\{v_j^{(1)} + \cdots + v_j^{(n)} : v_j \in \mathcal{L}(P)\}.
\]
If \(s = s(P)\), then \(P\) can be viewed as a projection onto the first \(n\) coordinates of a polytope in \(\mathbb{R}^{n+1}\) which lies in the simplex
\[
\Delta_{n+1}(s) := \{u = (u^{(1)}, \ldots, u^{(n+1)}) : u^{(i)} \geq 0, \sum_{i=1}^{n+1} u^{(i)} = s\}.
\]

Let \(s_n\) denote the minimum size of any maximal dps polytope in \(\mathbb{R}^n\). Examples 1 and 2 show that \(s_2 \leq 3\) and \(s_3 \leq 5\). It is not difficult to show that these estimates are sharp. The first case can be done by hand: if \(P\) is a maximal dps polytope with size 2 in \(\mathbb{R}^2\), then \(\mathcal{L}(P)\) must consist of four points chosen from
\[
\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0)\}.
\]
Since each congruence class is represented in \(\mathcal{L}(P)\), it must contain \((0, 1), (1, 0)\) and \((1, 1)\). These three points form a parallelogram with each of the points \((0, 0), (0, 2)\) and \((2, 0)\). Hence no fourth point can exist in \(\mathcal{L}(P)\) while preserving the dps property. The second case is similar, but much more complicated. Computer-aided calculations can be used to conclude that no dps polytope in \(\mathbb{R}^3\) has size 4 or less. (We thank Dr. Bruce Carpenter for doing the Mathematica coding.)

It can also be shown, using the style of argument of [4, Ch. 3], that every maximal dps polytope in \(\mathbb{R}^2\) is the image of the triangle in Example 1 under an affine unimodular linear mapping, and consists of a triangle with area 3/2, and a single lattice point inside, which will always be the centroid of the triangle. The tetrahedron determined by \(B\) in Example 2 lies within the tetrahedron determined by \(A\), whereas in Example 3, each point in \(\mathcal{L}\) is on the boundary of the polytope.
Thus there are at least two distinct combinatorial types of maximal dps polytopes in $\mathbb{R}^3$.

We make no serious conjecture about the growth of $s_n$. On the one hand, any maximal dps polytope must contain a lattice point with odd coordinates, so $s_n \geq n$. In the other direction, it is not difficult to use the proof of Theorem 3 to obtain a doubly-exponential bound for $s_n$. Since this bound is likely to be very crude, we do not present it explicitly. Another open question is to determine the minimum volume of a maximal dps polytope in $\mathbb{R}^n$ for $n \geq 3$. We also do not know the answer to the following question: is every dps polytope a subset of a maximal dps polytope?

We now discuss our original interest in this subject. Given $u \in \mathbb{Z}_n^+$, define the monomial $x^u \in \mathbb{R}[x_1, \ldots, x_n]$ by

$$x^u = x_1^{u_1(1)} \cdots x_n^{u(n)}.$$

Suppose $\mathcal{U} \subseteq \mathbb{Z}_+^n$ and consider the polynomial

$$p(x_1, \ldots, x_n) = \sum_{u \in \mathcal{U}} b_u x^u.$$

In [4], the present authors developed an algorithm for determining whether $p$ can be written as a sum of squares of polynomials. A necessary condition is that $p$ is psd; that is, $p(x_1, \ldots, x_n) \geq 0$ for all $x \in \mathbb{R}^n$. Suppose $p$ is psd and let

$$\mathcal{C}(p) = \text{cvx}\{u : b_u \neq 0\}.$$

Then $\mathcal{C}(p)$ is a lattice polytope; in fact it can be shown that the vertices of $\mathcal{C}(p)$ lie in $(2\mathbb{Z})^n$, so that $P := \frac{1}{2}\mathcal{C}(p)$ is a lattice polytope. Let

$$\mathcal{L}(P) = \{v_1, \ldots, v_N\},$$

and for $u \in \mathcal{C}(p)$, let $D(u) = \{(i, j) : v_i + v_j = u\}$. It is proved in [4, Thm. 2.4] that $p$ can be written as a sum of at most $r$ squares of polynomials if and only if there is a real $N \times N$ symmetric psd matrix $A = [a_{ij}]$ of rank at most $r$, so that

$$\sum_{(i, j) \in D(u)} a_{ij} = b_u \quad \text{for all } u \in \mathcal{C}(p).$$

If $P$ is a dps polytope in $\mathbb{R}^n$, then either $|D(u)| \leq 1$ or $D(u) = \{(i, j), (j, i)\}$. In either case, $a_{ij}$ is completely determined by $b_u$. In particular, if

$$h_P(x_1, \ldots, x_n) := \sum_{i=1}^N (x_i^{v_i})^2,$$

then $A$ must equal $I_N$, the $N \times N$ identity matrix, so that $p$ is a sum of $N$ squares, and no fewer.
Finally, we note that the homogenization of polynomials with \( n \) variables into forms with \( n + 1 \) variables is precisely analogous to the embedding of polytopes in \( \mathbb{R}^n_+ \) into the hyperplane \( \Delta_{n+1}(s) \).

**Example 4.** (See [4, Ex. 3.9])

We return to Example 1, in its homogeneous version. Let \( A = [a_{ij}] \) be a real symmetric \( 4 \times 4 \) matrix and let

\[
f(t_1, t_2, t_3, t_4) = \sum_{i=1}^{4} \sum_{j=1}^{4} a_{ij}t_i t_j
\]

be its associated quadratic form. We use the substitution suggested by \( \mathcal{L}(\mathcal{P}) \) and define the ternary sextic form

\[
p(x_1, x_2, x_3) = f(x_2x_3^2, x_1x_2^2, x_1^2x_3, x_1x_2x_3).
\]

Then \( p \) is a sum of squares of polynomials (cubic forms) if and only if \( f \) is a psd quadratic form; that is,

\[
f(t_1, t_2, t_3, t_4) \geq 0 \quad \text{for all } (t_1, t_2, t_3, t_4) \in \mathbb{R}^4.
\]

Since \( t_4^3 = t_1t_2t_3 \), the condition for \( p \) to be a psd form is weaker:

\[
f(t_1, t_2, t_3, (t_1t_2t_3)^{1/3}) \geq 0 \quad \text{for all } (t_1, t_2, t_3) \in \mathbb{R}^3.
\]

If \( f(t_1, t_2, t_3, t_4) = t_1^2 + t_2^2 + t_3^2 - 3t_4^2 \), then \( f \) is not psd, but \( f(t_1, t_2, t_3, (t_1t_2t_3)^{1/3}) \) \( \geq 0 \) by the arithmetic-geometric inequality. It follows that

\[
p(x_1, x_2, x_3) = x_2^2x_3^4 + x_1^2x_2^4 + x_1^4x_3^2 - 3x_1^2x_2^2x_3^2
\]

is a form which is psd, but not a sum of squares of polynomials. This particular example was discussed in [3]. For a history and bibliography of this subject and its relation to Hilbert’s 17th Problem, see [5].

More generally, the *Pythagoras number* of a ring \( A, P(A) \), is the smallest number \( n \leq \infty \) such that any sum of squares in \( A \) can be expressed as a sum of at most \( n \) squares in \( A \). Pfister [5] proved in 1967 that \( P(\mathbb{R}[x_1, \ldots, x_n]) \leq 2^n \). It is easy to see that \( P(\mathbb{R}[x_1]) = 2 \). Since maximal dps polytopes exist in \( \mathbb{R}^n \) for every \( n \), a consideration of \( h_P \) (c.f. (2)) shows that \( P(\mathbb{R}[x_1, \ldots, x_n]) \geq 2^n \). This is not the strongest result possible; in [2, p.60], using other methods, Dai and the present authors have shown that \( P(\mathbb{R}[x_1, \ldots, x_n]) = \infty \) for \( n \geq 2 \).

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