OVERDETERMINED PROBLEMS FOR FULLY NONLINEAR EQUATIONS IN SPACE FORMS

SHANZE GAO, HUI MA, AND MINGXUAN YANG

Abstract. We consider overdetermined problems for a class of fully nonlinear equations with constant boundary conditions in a bounded domain in space forms. If the domain is star-shaped, the solution to the Hessian quotient overdetermined problem is radial symmetric. By establishing a Rellich-Pohožaev type identity for $k$-Hessian equations with constant boundary conditions, we also show the symmetry of solutions to $k$-Hessian equations without assuming the star-shapedness of the domain.

1. Introduction

Let $(M^n(K), g)$ be a space form of constant sectional curvature $K$ with metric $g$. In this paper, we consider the following class of overdetermined problems:

$$\begin{cases}
\frac{\sigma_k(\nabla^2 u + Kg)}{\sigma_l(\nabla^2 u + Kg)} = \frac{\binom{n}{k}}{\binom{n}{l}} \quad &\text{in } \Omega, \\
u = Kc_1 \quad &\text{on } \partial \Omega, \\
u = c_2 \quad &\text{on } \partial \Omega, \\
K u \geq 0 \quad &\text{in } \Omega,
\end{cases} \quad (1)$$

where $0 \leq l < k \leq n$, $\Omega$ is a bounded, open, connected domain in $M^n(K)$, $c_1$ and $c_2 > 0$ are constants, $\nu$ denotes the outward unit normal to $\partial \Omega$ and $\sigma_k(\nabla^2 u + Kg)$ is the $k$-th elementary symmetric function of the eigenvalues of $\nabla^2 u + Kg$.

Serrin [16] did the pioneer work of overdetermined problem for Poisson equation

$$\Delta u = -1 \quad \text{in } \Omega$$

with boundary conditions

$$u = 0, \quad \nu = \text{constant} \quad \text{on } \partial \Omega$$

2020 Mathematics Subject Classification. 35N25, 58J32, 53C40.

Key words and phrases. Overdetermined problem, Hessian equation, Space form, Rellich-Pohožaev type identity.

After delivering a talk by the third named author, we were recently informed that Zhenghuan Gao, Xiaohan Jia and Dekai Zhang also considered overdetermined problems for Hessian quotients independently (arxiv: 2209.06268v1). Their results are on zero boundary condition in $\mathbb{R}^n$ and $\mathbb{H}^n$. The results and approach are different.
in $\mathbb{R}^n$. Via the moving plane method, he proved the solution is symmetric and the domain is a ball. In a subsequent paper \[18\], Weinberger proved the same result by using the maximum principle to an auxiliary function, which is now called $P$-function. In space forms, Molzon \[11\] used a $P$-function to prove symmetry results for equation $\Delta u = V$ (see the definition of $V$ in Section 2) in $S^n$ and used the moving plane method to obtain the symmetry results for equation $\Delta u = -1$ in space forms; Kumaresan and Prajapat \[9\] used the moving plane method to prove the results for equation $\Delta u + g(u) = 0$, where $g$ is a $C^1$ function, under the condition $u > 0$. For more papers about space forms, the interested readers may refer to \[13, 4, 5, 6\] and references therein.

The overdetermined problems for $k$-Hessian in $\mathbb{R}^n$ is considered by Brandolini, Nitsch, Salani and Trombetti in \[2\]. They used the Pohožaev type identity, Minkowski formula and Newton-MacLaurin inequalities to obtain the symmetry of the solution. For $k$-Hessian equation in hyperbolic space $\mathbb{H}^n$, Gao, Jia and Yan used the maximum principle for the $P$-function to prove the symmetry results in \[7\].

The problems for the other operators, such as $p$-Laplacian and anisotropic $p$-Laplacian in $\mathbb{R}^n$, the interested readers may refer to \[12, 1, 19, 3\].

In this paper, we suppose that $\Omega$ is a bounded, open, connected domain in $M^n(K)$ with boundary $\partial \Omega$ of class $C^2$. For the case $K > 0$, we additionally suppose that $M^n(K)$ is the hemisphere $S^n_+(\frac{1}{\sqrt{K}})$ with radius $\frac{1}{\sqrt{K}}$.

We consider the overdetermined problem for Hessian quotient equations.

**Theorem 1.1.** For given $0 \leq l < k \leq n$, if $\Omega$ is star-shaped and there exists a solution $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ to the problem \[1\], where $c_2 > 0$ is a constant, and $\nu$ denotes the outward unit normal to $\partial \Omega$, then $u$ is radially symmetric and $\Omega$ is a geodesic ball in $M^n(K)$.

**Remark 1.2.** For the case $K = 0$, $Ku \geq 0$ holds automatically and $u|_{\partial \Omega} = 0$ is the same as the conditions in \[2\]. Moreover, the Dirichlet boundary condition $c$ is arbitrary. In fact, since the equation now becomes

$$\frac{\sigma_k(\nabla^2 u)}{\sigma_l(\nabla^2 u)} = \frac{\binom{n}{k}}{\binom{n}{l}},$$

we can consider $\bar{u} = u - c$, if $u|_{\partial \Omega} = c \neq 0$, which solves the same equation with zero boundary condition.

However, in the following cases the conditions of problem \[1\] actually imply a restriction on the constant $c_1$.

(i) For the case $K > 0$, the boundary condition $u_\nu|_{\partial \Omega} = c_2 > 0$ and $Ku \geq 0$ imply $u|_{\partial \Omega} = Kc_1 > 0$.

(ii) For the case $K < 0$, $Ku \geq 0$ implies the constant $c_1 \geq 0$.

**Remark 1.3.** The radially symmetric solution $u$ in Theorem 1.1 can be written explicitly as following:

1. For $K = 0$, $u = \frac{|x|^2 - c_2^2}{2}$;
For $K > 0$, $u = \frac{1}{K} - \frac{c_2}{\sqrt{K} \sin(\sqrt{K} R)} \cos(\sqrt{K} r)$, where

$$R = \frac{1}{\sqrt{K}} \arctan \left( \frac{c_2 \sqrt{K}}{1 - K^2 c_1} \right);$$

for $K < 0$, $u = \frac{1}{K} + \frac{c_2}{\sqrt{-K} \sinh(\sqrt{-K} R)} \cosh(\sqrt{-K} r)$, where

$$R = \frac{1}{\sqrt{-K}} \text{arctanh} \left( \frac{c_2 \sqrt{-K}}{1 - K^2 c_1} \right).$$

In [13] Qiu and Xia introduced another auxiliary function $\tilde{P}$ (see the definition in Section 4) to obtain the symmetry of solutions for Poisson equations in the sphere $S^n$. In the proof of Theorem 1.1, we generalize their approach to Hessian quotient equations in space forms. Due to the nonlinearity of Hessian quotients, we need additional conditions that $K u \geq 0$ and $\Omega$ lies in the hemisphere for the case $K > 0$.

We also establish a Rellich-Pohožaev type identity for $k$-Hessian equations with $u|_{\partial \Omega} = K c_1$ and $u_{\nu}|_{\partial \Omega} = c_2$ in space forms. By this identity, we prove the following theorem.

**Theorem 1.4.** For given $1 \leq k \leq n$, if there exists a solution $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ to the problem

$$\begin{align*}
\sigma_k (\nabla^2 u + K u g) &= \binom{n}{k} \quad \text{in } \Omega, \\
u = K c_1 \quad &\text{on } \partial \Omega, \\
u_{\nu} = c_2 \quad &\text{on } \partial \Omega, \\
K u &\geq 0 \quad \text{in } \Omega,
\end{align*}\tag{2}$$

where $c_2 > 0$ is a constant and constant $c_1$ satisfies $K^2 c_1 \leq 1$, and $\nu$ denotes the outward unit normal to $\partial \Omega$, then $u$ is radially symmetric and $\Omega$ is a geodesic ball in $M^n(K)$.

**Remark 1.5.** Instead of the assumption that $\Omega$ is star-shaped in Theorem 1.1, we assume $K^2 c_1 \leq 1$. For the case $K = 0$, $K^2 c_1 \leq 1$ holds automatically. For the case $K < 0$ and $c_1 = 0$, $K^2 c_1 \leq 1$ also holds automatically and the condition $K u \geq 0$ in $\Omega$ is redundant from the maximum principle, which thus reduces to the result of [7].

Although Theorem 1.1 ($l = 0$) and Theorem 1.4 differ only in $c_1$ and $\Omega$ in terms of conditions, the methods we use are completely different.

In the second part of results in this paper, due to $u|_{\partial \Omega} = K c_1$, we cannot use the Rellich-Pohožaev type identity in [7]. By dealing the boundary terms properly, we obtain a new Rellich-Pohožaev type identity. For the purpose to derive that the $P$-function must be constant by the identity, we also introduce an auxiliary function $w$ to get $u < K c_1$ in $\Omega$ which cannot be obtain by maximum principle directly.
The paper is organized as follows. In Section 2, we recall some notations and facts of space forms, elementary symmetric functions and Hessian operators. In Section 3, we show the convexity of \( u \) to the problem \((\text{1})\). In Section 4, we consider \( P \)-function and \( \tilde{P} \)-function to the Hessian quotient equations. Then we prove Theorem 1.1 in Section 5. In Section 6, we establish a Rellich-Pohožaev type identity for \( k \)-Hessian equations with constant boundary conditions in space forms. In the last section, we prove Theorem 1.4.

Acknowledgments. The authors would like to thank Chao Qian, Chao Xia and Jiabin Yin for their helpful conversations on this work.

The first named author is partially supported by Natural Science Basic Research Program of Shaanxi (Program No.2022JQ-065). The second and the third named authors are partially supported by NSFC grants No. 11831005 and No. 12061131014.

2. Notation and preliminaries

2.1. Space forms. Let \( M^n(K) \) be a complete simply connected manifold with constant sectional curvature \( K \). It is well-known that \( M^n(K) \) is isometric to the Euclidean space if \( K = 0 \), the hyperbolic space if \( K < 0 \) and the sphere if \( K > 0 \). These models can be described as the warped product manifold \( M^n(K) = [0, \bar{r}) \times S^{n-1} \) with metric

\[
g = dr \otimes dr + f(r)^2 g_{S^{n-1}},
\]

where \( r(x) \) is the geodesic distance from \( x \) to a given point \( x_0 \in M^n(K) \), \( g_{S^{n-1}} \) is the metric of the \((n-1)\) dimensional standard unit sphere. The warping function \( f(r) \) is given by

\[
f(r) = \begin{cases} 
  r(x), & \text{for } K = 0, \\
  \frac{\sinh(\sqrt{-K}r)}{\sqrt{-K}}, & \text{for } K < 0, \\
  \frac{\sin(\sqrt{K}r)}{\sqrt{K}}, & \text{for } K > 0.
\end{cases}
\]

In addition, we require that \( \bar{r} = \infty \) for \( K \leq 0 \) and \( \bar{r} = \frac{\pi}{2\sqrt{K}} \) for \( K > 0 \). Hence our restriction on \( \bar{r} \) implies we only focus on a hemisphere \( S^n_+(\frac{1}{\sqrt{K}}) \) with radius \( \frac{1}{\sqrt{K}} \) in the latter case.

We know that space forms \( M^n(K) \) endow a nature conformal Killing vector field, \( f(r) \frac{\partial}{\partial r} \), in terms of the above warped product model. It is the gradient of the potential function \( \Phi \) defined by

\[
\Phi(x) := \begin{cases} 
  \frac{1}{2}r(x)^2, & \text{for } K = 0, \\
  \frac{-K}{\cosh(\sqrt{-K}r(x))}, & \text{for } K < 0, \\
  \frac{-\cos(\sqrt{K}r(x))}{K}, & \text{for } K > 0,
\end{cases}
\]
for \( x \in M^n(K) \).

Denote
\[
V(x) := (f'(r))(x) = \begin{cases} 
1, & \text{for } K = 0, \\
\cosh(\sqrt{-K}r(x)), & \text{for } K < 0, \\
\cos(\sqrt{K}r(x)), & \text{for } K > 0.
\end{cases}
\]

Thus for any vector field \( \xi \) on \( M^n(K) \),
\[
\nabla_{\xi} \left( f(r) \frac{\partial}{\partial r} \right) = V \xi.
\]

Then by direct calculations we have the following nice properties.

**Proposition 2.1.** \(-K \nabla \Phi = \nabla V, \nabla^2 \Phi = V g \) and \( \nabla^2 V = -KV g \).

Let \( \Omega \) be a bounded, open, connected domain in \( M^n(K) \). As a result, \( V(x) > 0 \) for all \( x \in \Omega \).

Let \( \nu \) denotes the outward unit normal of \( \partial \Omega \). We say \( \Omega \) is star-shaped, if \( g(\nu, \frac{\partial}{\partial r})(y) > 0 \) for all \( y \in \partial \Omega \).

2.2. **Elementary symmetric functions.** We recall some properties of elementary symmetric polynomials which will be used later.

For \( k \in \{1, \ldots, n\} \), the \( k \)-th elementary symmetric function of \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) is defined by
\[
\sigma_k(\lambda) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.
\]

Given a real symmetric \( n \times n \) matrix \( A = (a_{ij}) \) with eigenvalues \( \lambda(A) \), we can define the \( k \)-th elementary symmetric polynomial by \( \sigma_k(A) := \sigma_k(\lambda(A)) \).

Thus
\[
\sigma_k(A) = \frac{1}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq n} \delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} a_{i_1 j_1} \cdots a_{i_k j_k},
\]
where \( \delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} \) is the generalized Kronecker symbol defined by
\[
\delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} = \begin{cases} 
1, & \text{if } (i_1 \cdots i_k) \text{ is an even permutation of } (j_1 \cdots j_k), \\
-1, & \text{if } (i_1 \cdots i_k) \text{ is an odd permutation of } (j_1 \cdots j_k), \\
0, & \text{otherwise}.
\end{cases}
\]

We also set \( \sigma_0(A) = 1 \) and \( \sigma_k(A) = 0 \) for \( k < 0 \) and \( k > n \).

Denote \( \sigma^d_k(A) := \frac{\partial \sigma_k(A)}{\partial a_{ij}} \). The following properties (see [13] for reference) are useful in later calculations.
Proposition 2.2.

(i) \( \sum_{i,j=1}^{n} \sigma_k^{ij}(A)a_{ij} = k\sigma_k(A) \),

(ii) \( \sum_{i=1}^{n} \sigma_k^{ii}(A) = (n - k + 1)\sigma_{k-1}(A) \),

(iii) \( \sigma_k^{ij}(A) = \sigma_{k-1}(A)\delta_{ij} - \sum_{l=1}^{n} \sigma_l^{il}(A)a_{jl} \),

(iv) \( \sum_{i,j,l=1}^{n} \sigma_l^{il}(A)a_{jl}a_{ij} = \sigma_1(A)\sigma_k(A) - (k + 1)\sigma_{k+1}(A) \).

Now we collect some inequalities related the elementary symmetric functions, which are needed in our discussion. Readers can refer to [8, 17, 10].

For \( 1 \leq k \leq n \), we recall that the Gårding’s cone is defined by

\[ \Gamma_k := \{ \lambda \in \mathbb{R}^n| \sigma_i(\lambda) > 0, \text{ for all } 1 \leq i \leq k \} \]

Say a real symmetric matrix \( A \) lies in \( \Gamma_k \) if its eigenvalues \( \lambda(A) \in \Gamma_k \).

Proposition 2.3. For \( A = (a_{ij}) \in \Gamma_k \) and \( 0 \leq l < k \leq n \), the matrix

\[ \left( \frac{\partial}{\partial a_{ij}} \left( \frac{\sigma_k(A)}{\sigma_l(A)} \right) \right) \]

is positive definite.

We recall the Newton-MacLaurin inequalities.

Proposition 2.4. For \( 1 \leq k \leq n - 1 \) and \( A = (a_{ij}) \in \Gamma_k \),

\[ \frac{\sigma_{k+1}(A)}{\binom{n}{k+1}} \leq \frac{\sigma_k(A)}{\binom{n}{k}} \leq \frac{\sigma_{k-1}(A)}{\binom{n}{k-1}} \]

(3)

and

\[ \frac{\sigma_{k+1}(A)}{\binom{n}{k+1}} \leq \left( \frac{\sigma_k(A)}{\binom{n}{k}} \right)^{\frac{k+1}{k}} \]

For both inequalities, the equality occurs if and only if \( A = cI \) for some \( c > 0 \), where \( I \) is the identity matrix.

Since we are considering the Hessain quotient equations, we need some deformation of the above inequality.

Lemma 2.5. If

\[ \frac{\sigma_k(A)}{\sigma_l(A)} = \binom{n}{k} \binom{l}{k} \]

(4)

and \( A \in \Gamma_k \) for \( 0 \leq l < k \leq n \), then

\[ \frac{\sigma_{k-1}(A)}{\sigma_k(A)} \geq \frac{k}{n-k+1} \quad \text{and} \quad \frac{\sigma_{l+1}(A)}{\sigma_l(A)} \geq \frac{n-l}{l+1} \].
Each equality occurs if and only if $A = cI$ for some $c > 0$, where $I$ is the identity matrix. Moreover, we also have

$$\frac{\sigma_{k+1}(A)}{\sigma_k(A)} \leq \frac{n-k}{k+1} \quad \text{and} \quad \frac{\sigma_{l-1}(A)}{\sigma_l(A)} \leq \frac{l}{n-l+1}.$$  

**Proof.** Notice

$$\frac{\sigma_k(A)}{(\begin{smallmatrix} k \end{smallmatrix})} = \frac{\sigma_k(A)}{(\begin{smallmatrix} k-1 \end{smallmatrix})} \frac{\sigma_{k-1}(A)}{(\begin{smallmatrix} k-2 \end{smallmatrix})} \cdots \frac{\sigma_l(A)}{(\begin{smallmatrix} l \end{smallmatrix})}.$$  

It follows from the Newton-MacLaurin inequality [3] that

$$\left( \frac{\sigma_k(A)}{(\begin{smallmatrix} k \end{smallmatrix})} \right)^{k-l} \leq \frac{\sigma_k(A)}{(\begin{smallmatrix} k-1 \end{smallmatrix})} \leq \left( \frac{\sigma_l(A)}{(\begin{smallmatrix} l \end{smallmatrix})} \right)^{l-k}.$$  

Then, the condition [11] implies

$$\frac{\sigma_k(A)}{(\begin{smallmatrix} k \end{smallmatrix})} \leq 1 \leq \frac{\sigma_l(A)}{(\begin{smallmatrix} l \end{smallmatrix})}.$$  

Thus it leads to

$$\frac{\sigma_k(A)}{\sigma_{k-1}(A)} \leq \frac{n-k+1}{k}, \quad \text{and} \quad \frac{\sigma_{l+1}(A)}{\sigma_l(A)} \geq \frac{n-l}{l+1}.$$  

Furthermore, if $\sigma_{k+1}(A) \leq 0$, then we get the following inequality. If $\sigma_{k+1}(A) > 0$, then the Newton-MacLaurin inequality implies

$$\frac{\sigma_{k+1}(A)}{\sigma_k(A)} \leq \frac{n-k}{k+1}$$  

and

$$\frac{\sigma_{l-1}(A)}{\sigma_l(A)} \leq \frac{l}{n-l+1}.$$  

2.3. **Hessian operators.** Let $\Omega$ be an open domain in $M^n(K)$ and $u \in C^2(\Omega)$. The $k$-Hessian operator $\sigma_k(\nabla^2 u + Kg)$ is defined as the $k$-th elementary symmetric function of $\nabla^2 u + Kg$, where $\nabla^2 u$ denotes the Hessian of $u$ and $g$ is the Riemannian metric of $M^n(K)$.

For $2 \leq k \leq n$, the $k$-th Hessian operator is fully nonlinear. We are also interested in the Hessian quotient operator $\sigma_k(\nabla^2 u + Kg)$, for $0 \leq l < k \leq n$, which is a more general class of fully nonlinear operator. When $l = 0$, it is a $k$-Hessian operator.

From now on, we calculate under a local orthonormal frame $\{e_1, \cdots, e_n\}$ on $M^n(K)$ and denote a symmetric 2-form $b := \nabla^2 u + Kg$ and $\nabla_k b_{ij} := b_{ijk}$. Einstein’s summation convention is used for repeated indexes unless otherwise stated.
Proposition 2.6. The 2-form $b$ is a Codazzi tensor and the $k$-Hessian operator is divergence-free. In another words, $b_{ijk} = b_{ikj}$ and for any $u \in C^3(\Omega)$, we have
\[
\sum_i \nabla_i (\sigma_k^j (\nabla^2 u + K u g)) = 0,
\]
with respect to any local orthornormal frame $\{e_1, \ldots, e_n\}$.

Proof. In space form $M^n(K)$, the Riemannian curvature tensor $R_{ijkl} = K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$. For $u \in C^3(\Omega)$, the Ricci identity shows
\[
b_{ijk} = u_{ijk} + K u_k \delta_{ij} \\
= u_{ikj} + u_{mj}(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) + K u_k \delta_{ij} \\
= u_{ikj} + K u_j \delta_{ik} = b_{ikj}.
\]
This means $b_{ijk}$ is fully symmetric, so $b$ is a Codazzi tensor. For such $b = \nabla^2 u + K u g$, the $k$-Hessian operator is divergence-free (see [14] etc.). □

3. Convexity of $u$

Now, we show the solution $u$ to the overdetermined problem is naturally $k$-admissible, i.e., $\nabla^2 u + K u g \in \Gamma_k$. This property is used to ensure that we can use the maximum principle and the Newton-Maclaurin inequality later.

Lemma 3.1. For $2 \leq k \leq n$ and $0 \leq l < k$, if $\Omega$ is a $C^2$ bounded domain and $u \in C^2(\bar{\Omega})$ is a solution of (1), then $(\nabla^2 u + K u g) \in \Gamma_k$ for all $x \in \bar{\Omega}$.

Proof. The proof for the case $l = 0$ can be found in [2] and [7]. Hence we assume $1 \leq l < k$. From $u_\nu = c_2 > 0$ on $\partial \Omega$, we know there is a point $x_0 \in \Omega$ such that $u(x_0) = \min \Omega u$. Then $\nabla^2 u(x_0) \geq 0$. From assumption $K u \geq 0$, we know $b(x_0) = (\nabla^2 u + K u g)(x_0) \geq 0$. Moreover,
\[
\frac{\sigma_k(\nabla^2 u + K u g)}{\sigma_l(\nabla^2 u + K u g)} = \binom{n}{k} > 0
\]
shows $\sigma_k(\nabla^2 u + K u g)(x_0) > 0$.

Next, we show $\sigma_k(\nabla^2 u + K u g) > 0$ in $\bar{\Omega}$. If there exists $y_0 \in \bar{\Omega}$ such that $\sigma_k(\nabla^2 u + K u g)(y_0) < 0$, from the smoothness of $u$, we know there exists $z_0 \in \bar{\Omega}$ such that $\sigma_k(\nabla^2 u + K u g)(z_0) = 0$. It is impossible since
\[
\frac{\sigma_k(\nabla^2 u + K u g)}{\sigma_l(\nabla^2 u + K u g)} > 0.
\]
Hence $\sigma_k(\nabla^2 u + K u g)(x) > 0$ for all $x \in \bar{\Omega}$.

The boundary condition implies $|\nabla u|_{\partial \Omega} = c_2 > 0$ which means $\nabla u|_{\partial \Omega} \neq 0$. By the implicit function theorem, we know $\partial \Omega$ is a hypersurface in $M^n(K)$.
and $\nu = \frac{\nabla u}{|\nabla u|}$. By a suitable choice of frame such that $e_n = \nu$, Hessian of $u$ on $\partial \Omega$ has the following form:

$$
\nabla^2 u = \begin{pmatrix}
c_2 \kappa_1 & 0 & 0 & \cdots & 0 & u_{1n} \\
0 & c_2 \kappa_2 & 0 & \cdots & 0 & u_{2n} \\
0 & 0 & c_2 \kappa_3 & \cdots & 0 & u_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & c_2 \kappa_{n-1} & u_{n-1n} \\
u_{1n} & u_{n2} & u_{n3} & \cdots & u_{n_{n-1}} & u_{nn}
\end{pmatrix},
$$

where $\kappa_1, \ldots, \kappa_{n-1}$ are principal curvatures of $\partial \Omega$. On $\partial \Omega$, $u_n = c_2$ implies $u_{1n} = \cdots = u_{n-1n} = 0$. Thus,

$$
\nabla^2 u + KuI = \begin{pmatrix}
c_2 \kappa_1 + Ku & & & & \\
& \ddots & & & \\
& & c_2 \kappa_{n-1} + Ku & & \\
& & & u_{nn} + Ku & \\
& & & & c_2 \kappa_{n} + Ku
\end{pmatrix}.
$$

Let

$$
c_2 \kappa + KuI := \begin{pmatrix}
c_2 \kappa_1 + Ku & & & \\
& \ddots & & \\
& & c_2 \kappa_{n-1} + Ku & \\
& & & c_2 \kappa_{n} + Ku
\end{pmatrix}.
$$

Then

$$
0 < \sigma_k(\nabla^2 u + KuI)(y) = (u_{nn} + Ku)\sigma_{k-1}(c_2 \kappa + KuI) + \sigma_k(c_2 \kappa + KuI). \quad (5)
$$

It is known that boundary of $\Omega$ has at least one elliptic point, i.e., there exists at least one point $y \in \partial \Omega$ such that all principal curvatures $\kappa_1(y), \ldots, \kappa_{n-1}(y)$ are nonnegative. Combining with $c_2 > 0$, $Ku \geq 0$ and [5], we know $\sigma_{k-1}(c_2 \kappa + KuI)(y) > 0$. Then from the Newton-MacLaurin inequality, we have

$$
\sigma_{k-1}(\nabla^2 u + KuI)(y) = (u_{nn} + Ku)\sigma_{k-2}(c_2 \kappa + KuI) + \sigma_{k-1}(c_2 \kappa + KuI)
$$

$$
\quad > -\frac{\sigma_{k-2}(c_2 \kappa + KuI)\sigma_k(c_2 \kappa + KuI)}{\sigma_{k-1}(c_2 \kappa + KuI)} + \sigma_{k-1}(c_2 \kappa + KuI)
$$

$$
\quad > 0.
$$

Similarly, we can get $\sigma_j(\nabla^2 u + KuI)(y) > 0$ for all $1 \leq j \leq k$. Therefore, $\nabla^2 u + KuI \in \Gamma_k$ at $y$.

Now, we consider the set $S := \{x \in \overline{\Omega} | \nabla^2 u + KuI \in \Gamma_k \}$. The above shows $S$ is nonempty. From $u \in C^2(\overline{\Omega})$, we know $S$ is an open set. If a sequence $\{x_i\}_{i=1}^\infty \subset S$ and $\lim_{i \to \infty} x_i = \bar{x} \in \overline{\Omega}$, then $(\nabla^2 u + KuI)(\bar{x}) \in \Gamma_k$. From the Newton-MacLaurin inequality and $\sigma_k(\nabla^2 u + KuI)(\bar{x}) > 0$, we know $(\nabla^2 u + KuI)(\bar{x}) \in \Gamma_k$. This means $S$ is also closed. From the connectness of $\overline{\Omega}$, we know $S = \overline{\Omega}$. Thus, $(\nabla^2 u + KuI) \in \Gamma_k$ for all $x \in \overline{\Omega}$.

\[ \square \]
4. P-FUNCTION AND $\tilde{P}$-FUNCTION

Let $F(b) := \frac{\sigma_k^{ij}(b)}{\sigma_l^{ij}(b)}$ and $F^{ij} := \frac{\partial F(b)}{\partial b_{ij}}$ for $b = \nabla^2 u + K u g$. Proposition 2.3 and Lemma 3.1 show that the matrix $(F^{ij})$ is positive definite. Hence the operator $F^{ij} \nabla^2_{ij}$ is elliptic.

First, we consider $P := |\nabla u|^2 + Ku^2 - 2u$.

**Lemma 4.1.** If $u \in C^3(\Omega)$ be a solution to the problem (1) in Theorem 1.1, then

$$F^{ij} \nabla^2_{ij} P \geq 0.$$ 

Moreover, either

$$P = K^3 c_1^2 + c_2^2 - 2Kc_1 \quad \text{in} \quad \Omega$$

or

$$P < K^3 c_1^2 + c_2^2 - 2Kc_1 \quad \text{in} \quad \Omega.$$

**Proof.** By direct calculation, we have

$$\nabla_i(|\nabla u|^2 + Ku^2) = 2u_{mi}u_m + 2Kuu_i = 2b_{mi}u_m,$n

and

$$\nabla^2_{ij} P = 2b_{mij}u_m + 2b_{mi}u_{mj} - 2u_{ij}$$

$$= 2b_{mij}u_m + 2b_{mi}b_{mj} - 2Ku_{ij} - 2b_{ij} + 2Ku\delta_{ij}.$$ 

From Proposition 2.2 we know

$$F^{ij} b_{ij} = F \left( \frac{\sigma_k^{ij} b_{ij}}{\sigma_k} - \frac{\sigma_l^{ij} b_{ij}}{\sigma_l} \right) = (k - l)F,$n

$$F^{ij} \delta_{ij} = F \left( \frac{\sigma_k^{ij} \delta_{ij}}{\sigma_k} - \frac{\sigma_l^{ij} \delta_{ij}}{\sigma_l} \right) = F \left( (n - k + 1)\frac{\sigma_{k-1}}{\sigma_k} - (n - l + 1)\frac{\sigma_{l-1}}{\sigma_l} \right),$$

and

$$F^{ij} b_{mi}b_{mj} = F \left( \frac{\sigma_k^{ij}b_{mi}b_{mj}}{\sigma_k} - \frac{\sigma_l^{ij}b_{mi}b_{mj}}{\sigma_l} \right)$$

$$= F \left( \frac{\sigma_k - (k + 1)\sigma_{k+1}}{\sigma_k} - \sigma_1 + \frac{(l + 1)\sigma_{l+1}}{\sigma_l} \right)$$

$$= F \left( -(k + 1)\frac{\sigma_{k+1}}{\sigma_k} + \frac{(l + 1)\sigma_{l+1}}{\sigma_l} \right).$$
We also know $\nabla_{m} F = 0$ since $F$ is constant. Then
\[
\frac{1}{2} F^{ij} \nabla_{ij}^2 P = u_{m} \nabla_{m} F + F^{ij} b_{mi} b_{mj} - K u F^{ij} b_{ij} - F^{ij} b_{ij} + K u F^{ij} \delta_{ij}
\]
\[
= F \left( - (k + 1) \frac{\sigma_{k+1}}{\sigma_{k}} + \frac{(l + 1) \sigma_{l+1}}{\sigma_{l}} - (k - l) K u - (k - l) \right)
\]
\[
+ (n - k + 1) K u \frac{\sigma_{k-1}}{\sigma_{k}} - (n - l + 1) K u \frac{\sigma_{l-1}}{\sigma_{l}} \right)
\]
\[
= F \left( - (k + 1) \frac{\sigma_{k+1}}{\sigma_{k}} + \frac{(l + 1) \sigma_{l+1}}{\sigma_{l}} - (k - l) \right)
\]
\[
+ K u F \left( - (k - l) + (n - k + 1) \frac{\sigma_{k-1}}{\sigma_{k}} - (n - l + 1) \frac{\sigma_{l-1}}{\sigma_{l}} \right).
\]

From Lemma 2.5 and $K u \geq 0$, we obtain $F^{ij} \nabla_{ij}^2 P \geq 0$.

Since $F^{ij} \nabla_{ij}^2$ is elliptic, by strong maximum principle, if there exists $x_0 \in \Omega$ such that $P(x_0) = \max_{\Omega} P$, then $P$ is constant. From boundary conditions, we know
\[
P = K^3 c_1^2 + c_2^2 - 2 K c_1 \quad \text{in } \Omega.
\]

If $P(x) < P|_{\partial \Omega}$ for all $x \in \Omega$, then
\[
P < K^3 c_1^2 + c_2^2 - 2 K c_1 \quad \text{in } \Omega.
\]

Now we introduce another auxiliary function
\[
\tilde{P} := -g(\nabla u, \nabla \Phi) + u V + \Phi.
\]

In the case $K = 1$ the function $\tilde{P}$ can be written as
\[
\tilde{P} = g(\nabla u, \nabla V) + u V - V,
\]

which is same as the second auxiliary function in [13]. And they show that $\Delta \tilde{P} = 0$ for the equation $\Delta u + n K u = n$. Due to the nonlinearity of the equation here, we obtain the following nice property of $\tilde{P}$ in the following lemma.

**Lemma 4.2.** Let $u \in C^3(\Omega)$ be a solution to the problem (11) in Theorem 1.1, then
\[
F^{ij} \nabla_{ij}^2 \tilde{P} \geq 0.
\]

**Proof.** Using Proposition 2.1 by direct calculation, we have
\[
\nabla_i \tilde{P} = -u_m \Phi_m - u_m \Phi_{mi} + V_i u + V u_i + \Phi_i
\]
\[
= -u_m \Phi_m + V_i u + \Phi_i
\]
\[ \nabla^2_{ij} \tilde{P} = -u_{mi} \Phi_m - u_{mj} \Phi_m + V_{ij} u + V_i u_j + \Phi_{ij} \]

\[= -u_{mi} \Phi_m - u_{mj} \Phi_m - KV \delta_{ij} + V \delta_{ij} \]

\[= -b_{mi} \Phi_m - V b_{ij} + V \delta_{ij}. \]

We also know \( \nabla m F = 0 \) since \( F \) is constant. Combining Proposition 2.2, we obtain

\[ F_{ij} \nabla^2_{ij} \tilde{P} = -\Phi_m \nabla m F - (k - l)VF + VF \left( (n - k + 1) \frac{\sigma_{k-1}}{\sigma_k} - (n - l + 1) \frac{\sigma_{l-1}}{\sigma_l} \right) = VF \left( (n - k + 1) \frac{\sigma_{k-1}}{\sigma_k} - (n - l + 1) \frac{\sigma_{l-1}}{\sigma_l} - (k - l) \right). \]

Then we know \( F_{ij} \nabla^2_{ij} \tilde{P} \geq 0 \) from Lemma 2.5 and \( V > 0 \).

\[ \square \]

5. Proof of Theorem 1.1

From Lemma 4.1, we know either

\[ P = K^3 c_1^2 + c_2^2 - 2Kc_1 \quad \text{in} \quad \Omega \]

or

\[ P < K^3 c_1^2 + c_2^2 - 2Kc_1 \quad \text{in} \quad \Omega. \]

If the former occurs, namely \( P \) is constant, calculations in the proof of Lemma 4.1 and the Newton-MacLaurin inequality imply \( \nabla^2 u + Ku = a(x)g \) for some function \( a \). From the equation

\[ \frac{\sigma_k (\nabla^2 u + Ku)}{\sigma_l (\nabla^2 u + Ku)} = \left( \begin{array}{c} k \\ k \end{array} \right) \left( \begin{array}{c} n \\ l \end{array} \right), \]

we know \( a(x) \equiv 1 \). Hence \( \nabla^2 u + Ku = g \).

If the latter occurs, the Hopf Lemma implies

\[ 0 < \nabla_P P(y) = 2u_{\nu} u_{\nu\nu} + 2K u_{\nu\nu} - 2u_{\nu} = 2c_2 (u_{\nu\nu} + K^2 c_1 - 1) \quad (6) \]

for any \( y \in \partial \Omega \).

From Lemma 4.2, we know \( F_{ij} \nabla^2_{ij} \tilde{P} \geq 0 \). We show \( \tilde{P} \) must be constant. If \( \tilde{P} \) is not constant, by the strong maximum principle, there exists \( y_0 \in \partial \Omega \) such that \( P(x) < \tilde{P}(y_0) \) for any \( x \in \Omega \). From Hopf Lemma, we know

\[ \nabla_P \tilde{P}(y_0) > 0. \]
Since
\[ \nabla_\nu \bar{P} = -u_\nu \Phi_\nu - u_\nu \Phi_{\nu\nu} + u_\nu V + u V_\nu + \Phi_\nu \]
\[ = -u_\nu \Phi_\nu - u_\nu V + u V - Ku_\nu + \Phi_\nu \]
\[ = -\Phi_\nu (u_{\nu\nu} + Ku - 1), \]
we obtain
\[ 0 < \nabla_\nu \bar{P}(y_0) = -\Phi_\nu (u_{\nu\nu} + K^2 c_1 - 1). \tag{7} \]
Moreover,
\[ \Phi_\nu = g(\nabla \Phi, \nu) = f(r)g(\frac{\partial}{\partial r}, \nu) > 0, \]
where the last inequality is from \( f(r) > 0 \) and \( \Omega \) is star-shaped. This implies
\[ u_{\nu\nu} + K^2 c_1 - 1 < 0. \]
It contradicts with \eqref{8}.

Now, since \( \bar{P} \) is constant, calculations in the proof of Lemma 4.2 and the Newton-MacLaurin inequality imply \( \nabla^2 u + Ku = a(x)g \) for some function \( a \). As before, we obtain \( \nabla^2 u + Ku = g \).

At last, we show \( \nabla^2 u + Ku = g \) implies \( u \) is radial symmetric. The boundary condition \( u|_{\partial \Omega} = Kc_1 \) makes the discussion different from \[15, \, 5\].

Since \( u_\nu = c_2 > 0 \) on \( \partial \Omega \), we know there exists \( p \in \Omega \) such that \( u(p) = \min_{\Omega} u \). Let \( \gamma : I \to \Omega \) be a maximal geodesic of arc-length parameter satisfies \( \gamma(0) = p \). We consider \( v(s) := u(\gamma(s)) \), then
\[
\begin{cases}
  v'' + Ku = 1, \\
  v'(0) = 0, \\
  v(0) = u(p). 
\end{cases}
\tag{8}
\]
Given two different such geodesics \( \gamma_1 \) and \( \gamma_2 \), by the existence and uniqueness of solution to ODE, we have \( u(\gamma_1(s)) = u(\gamma_2(s)) \), hence \( u \) is a radial symmetric function in the geodesic ball \( B_R(p) \) where \( R \) is chosen such that \( \partial B_R(p) \) touch the \( \partial \Omega \), so \( u(\partial B_R(p)) = Kc_1 \).

From the boundary condition of the problem \eqref{11}, we know \( \partial \Omega \subset \{ u = Kc_1 \} \). Now we consider \( \bar{u} := u - Kc_1 \), then \( \nabla^2 u + Ku = g \) implies
\[
\Delta \bar{u} + nK \bar{u} = \Delta u + nKu - nK^2 c_1 = n - nK^2 c_1.
\]
Since \( V > 0 \) in \( \Omega \subset M^n(K) \), we can set \( w := \frac{\bar{u}}{V} \), by direct computation,
\[
w_i = \frac{\bar{u}_i V - V_i \bar{u}}{V^2},
\]
\[
w_{ij} = \frac{1}{V^2} ((\bar{u}_{ij} V - V_{ij} \bar{u}) + \bar{u}_i V_j - \bar{u}_j V_i) V^2 - 2VV_j (\bar{u}_i V - V_i \bar{u})),
\]
\[
\Delta w = \frac{1}{V} (\Delta \bar{u} + nK \bar{u}) - \frac{2}{V} \langle \nabla \bar{V}, \nabla \bar{w} \rangle.
\]
Thus,
\[
\Delta w + \frac{2}{V} \langle \nabla \bar{V}, \nabla \bar{w} \rangle = \frac{n(1 - K^2 c_1)}{V}.
\]
If $K^2c_1 \geq 1$, then

$$
\Delta w + \frac{2}{V} (\nabla V, \nabla w) \leq 0,
$$

by the strong maximum principle and $w|_{\partial \Omega} = 0$, the auxiliary function $w > 0$ in $\Omega$. Otherwise, $w = 0$ in $\Omega$ implies $u = Kc_1$ in $\Omega$ which contradicts with $u_\nu = c_2 > 0$ in $\partial \Omega$. So the Hopf lemma and the boundary conditions of the problem (1) imply

$$
0 > \frac{\partial w}{\partial \nu} = \bar{u}_\nu > 0.
$$

Therefore, $K^2c_1 < 1$.

Now the equation $\Delta w + \frac{2}{V} (\nabla V, \nabla w) > 0$, in the same way, the auxiliary function $w < 0$ in $\Omega$. Hence, $u < Kc_1$ in $\Omega$.

Hence, $\partial \Omega = \partial B_R(p)$. Thus, we finish the proof.

6. Rellich-Pohožaev type identity

In the following, $\sigma_k = \sigma_k(b)$ for convenience.

**Lemma 6.1.** Let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution to the problem in Theorem 1.4. Then

$$
k \binom{n}{k} \int_\Omega uVd\mu = c_1K \int_\Omega \sigma^i_j u_i \Phi^j = \frac{c_2^2}{2}K \int_\Omega \sigma^i_j \Phi^j dS + \frac{n^2 - k + 1}{2}K \int_\partial \Omega \sigma_{k-1} u^2 Vd\mu
$$

$$
- \frac{c_2^2}{2} \int_\partial \Omega \sigma^i_j \Phi^j dS + \frac{n - k + 1}{2} \int_\Omega \sigma_{k-1} |\nabla u|^2 Vd\mu.
$$

**Proof.** From

$$
k \binom{n}{k} uV = \sigma^i_j b_{ij}uV.
$$

By $b_{ij} = u_{ij} + Ku_{ij}$ and $\sigma^i_j \delta_{ij} = (n - k + 1)\sigma_{k-1}$, we know

$$
k \binom{n}{k} uV = \sigma^i_j (u_{ij} + Ku_{ij}) uV
$$

$$
= \sigma^i_j u_{ij} uV + (n - k + 1)K \sigma_{k-1} u^2 V.
$$

(9)

Using $\Phi_{ij} = V \delta_{ij}$ and $\nabla_j \sigma^i_j = 0$, we notice

$$
\sigma^i_j u_{ij} uV = \sigma^i_j u_i \Phi^j
$$

$$
= (\sigma^i_j u_i \Phi^j)_{ij} - \sigma^i_j u_{il} \Phi^j - \sigma^i_j u_{ij} \Phi^l.
$$

(10)
Since $b_{ijk} = b_{ikj}$, $b_{ijk} = u_{ijk} + Ku_k\delta_{ij}$ and $\sigma_k = \binom{n}{k}$,

$$\sigma_k^{ij}u_{ij}^k\Phi_l = \sigma_k^{ij}b_{ij}^k\Phi_l - K\sigma_k^{ij}u_j^i\Phi_l$$

$$= \sigma_k^{ij}b_{ij}^k\Phi_l - K\sigma_k^{ij}u_j^i\Phi_l$$

$$= \nabla_i\sigma_k^i\Phi_l - K\sigma_k^{ij}u_j^i\Phi_l$$

$$= -K\sigma_k^{ij}u_j^i\Phi_l.$$

Thus,

$$\sigma_k^{ij}u_{ij}^k\Phi_l = -K\sigma_k^{ij}u_j^i\Phi_l$$

$$= -\frac{1}{2}K\sigma_k^{ij}\Phi_l(u^2)_j$$

$$= -\frac{1}{2}K(\sigma_k^{ij}\Phi_l u^2)_j + \frac{1}{2}K\sigma_k^{ij}\Phi_l u^2.$$

By

$$\sigma_k^{ij}\Phi_l = V\sigma_k^{ij}\delta_{ij} = (n - k + 1)\sigma_{k-1}V,$$

we obtain

$$\sigma_k^{ij}u_{ij}^k\Phi_l = -\frac{1}{2}K(\sigma_k^{ij}\Phi_l u^2)_j + \frac{n - k + 1}{2}K\sigma_{k-1}u^2V. \quad (11)$$

Since $u_{ij}$ and $b_{ij}$ can be diagonalized at the same time,

$$\sigma_k^{ij}u_{ij}^k\Phi_l = \sigma_k^{ji}_{kj}\Phi_l.$$

From

$$2\sigma_k^{hi}u_{ij}^k\Phi_l = (\sigma_k^{hi}|\nabla u|^2\Phi_l)_i - \sigma_k^{ji}|\nabla u|^2\Phi_l,$$

we obtain

$$\sigma_k^{ij}u_{ij}^k\Phi_l = \frac{1}{2}(\sigma_k^{ji}|\nabla u|^2\Phi_l)_i - \frac{n - k + 1}{2}\sigma_{k-1}|\nabla u|^2V. \quad (12)$$

Substitute (11) and (12) into (10), we have

$$\sigma_k^{ij}u_{ij}^kV = (\sigma_k^{ij}u_i^j\Phi_l)_j + \frac{1}{2}K(\sigma_k^{ij}\Phi_l u^2)_j - \frac{n - k + 1}{2}K\sigma_{k-1}u^2V$$

$$- \frac{1}{2}(\sigma_k^{ji}|\nabla u|^2\Phi_l)_i + \frac{n - k + 1}{2}\sigma_{k-1}|\nabla u|^2V.$$

Furthermore, from (9), we obtain

$$k\binom{n}{k}uV = (\sigma_k^{ij}u_i^j\Phi_l)_j + \frac{1}{2}K(\sigma_k^{ij}\Phi_l u^2)_j + \frac{n - k + 1}{2}K\sigma_{k-1}u^2V$$

$$- \frac{1}{2}(\sigma_k^{ji}|\nabla u|^2\Phi_l)_i + \frac{n - k + 1}{2}\sigma_{k-1}|\nabla u|^2V.$$
Integrating and using divergence theorem,
\[
k(n) \int_{\Omega} uV \, d\mu = \int_{\partial \Omega} \sigma_{ij} u_i \Phi_{ij} \, dS + \frac{1}{2} K \int_{\partial \Omega} \sigma_k \Phi_k u^2 \, dS \\
+ \frac{n-k+1}{2} K \int_{\Omega} \sigma_{k-1} u^2 V \, d\mu \\
- \frac{1}{2} \int_{\partial \Omega} \sigma_{ij}\nabla u^2 \Phi_{ij} \, dS + \frac{n-k+1}{2} \int_{\Omega} \sigma_{k-1} \nabla u^2 V \, d\mu.
\]
Since \( u \vert_{\partial \Omega} = Kc_1 \) and \( \vert \nabla u \vert \vert_{\partial \Omega} = c_2^2 \), it is
\[
k(n) \int_{\Omega} uV \, d\mu = c_1 K \int_{\partial \Omega} \sigma_k \Phi_{ji} u_i \, dS + \frac{c_2^2 K^3}{2} \int_{\partial \Omega} \sigma_k \Phi_k u^2 \, dS \\
+ \frac{n-k+1}{2} K \int_{\Omega} \sigma_{k-1} u^2 V \, d\mu \\
- \frac{c_2^2}{2} \int_{\partial \Omega} \sigma_{ij} \delta_{ij} \nabla u^2 \, dS + \frac{n-k+1}{2} \int_{\Omega} \sigma_{k-1} \nabla u^2 V \, d\mu.
\]

**Lemma 6.2.** Under the same assumption of Lemma 6.1, the following identities hold

i) \( \int_{\partial \Omega} \sigma_{ij} \Phi_{ij} \, dS = (n-k+1) \int_{\Omega} \sigma_{k-1} V \, d\mu, \)

ii) \( \int_{\partial \Omega} \sigma_k u_i \Phi_{ij} \, dS = \int_{\Omega} \left( k \binom{n}{k} - K^2 c_1 (n-k+1) \right) \sigma_{k-1} \, d\mu. \)

**Proof.** By divergence theorem, \( \Phi_{ij} = V \delta_{ij} \) and \( \sigma_k \delta_{ij} = (n-k+1) \sigma_{k-1}, \)
\[
\int_{\partial \Omega} \sigma_k u_i \Phi_{ij} \, dS = \int_{\Omega} (\sigma_k \Phi_k u^2) \, d\mu = (n-k+1) \int_{\Omega} \sigma_{k-1} V \, d\mu. \quad (13)
\]
Similarly, since \( u \vert_{\partial \Omega} = Kc_1 \) and \( \sigma_k = \binom{n}{k}, \)
\[
\int_{\partial \Omega} \sigma_k u_i \Phi_{ij} \, dS = \int_{\partial \Omega} \sigma_k u_i \Phi_{ij} - K \sigma_k u_i \delta_{ij} \Phi_{ij} \, dS \\
= \int_{\Omega} \sigma_k b_{ij} \Phi_{ij} \, d\mu - K^2 c_1 \int_{\partial \Omega} \sigma_k \Phi_k \Phi_{ij} \, dS \\
= \int_{\Omega} \sigma_k b_{ij} V - K^2 c_1 \sigma_k \Phi_{ij} \, d\mu.
\]

Due to \( \sigma_k \Phi_{ij} = k \sigma_k = k \binom{n}{k} \)
and \( \sigma_k \Phi_{ij} = \sigma_k \delta_{ij} V = (n-k+1) \sigma_{k-1} V, \)
\[
\int_{\partial \Omega} \sigma_{ij}^k u_{ij} \Phi \nu_j \, dS = \int_{\Omega} \left( k \binom{n}{k} - K^2 c_1 (n-k+1) \sigma_{k-1} \right) V \, d\mu.
\]

\[\square\]

Combining Lemma 6.1 and Lemma 6.2, we obtain the following lemma.

**Lemma 6.3.** Let \( u \in C^3(\Omega) \cap C^2(\overline{\Omega}) \) be a solution to the problem in Theorem 1.4. Then

\[
\left( \binom{n}{k} - 1 \right) \int_{\Omega} (u - K c_1) V \, d\mu = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + Ku^2 - K^3 c_1^2 - c_2^2) \sigma_{k-1} V \, d\mu. \tag{14}
\]

**Proof.** Substitute the identities in Lemma 6.2 into Lemma 6.1, we have

\[
k \binom{n}{k} \int_{\Omega} u V \, d\mu = k \binom{n}{k} c_1 K \int_{\Omega} V \, d\mu - (n-k+1)c_1^2 K^3 \int_{\Omega} \sigma_{k-1} V \, d\mu
\]

\[
+ \frac{n-k+1}{2} K \int_{\Omega} \sigma_{k-1} V \, d\mu
\]

\[
- \frac{n-k+1}{2} c_2^2 \int_{\Omega} \sigma_{k-1} V \, d\mu
\]

\[
+ \frac{n-k+1}{2} \int_{\Omega} \sigma_{k-1} |\nabla u|^2 V \, d\mu.
\]

The above equality is equivalent to

\[
k \binom{n}{k} \int_{\Omega} (u - K c_1) V \, d\mu = \frac{n-k+1}{2} \int_{\Omega} (|\nabla u|^2 + Ku^2 - K^3 c_1^2 - c_2^2) \sigma_{k-1} V \, d\mu.
\]

Noticing \( \binom{n}{k-1} = \frac{k}{n-k+1} \binom{n}{k} \), we finish the proof. \[\square\]

**7. Proof of Theorem 1.4**

First, we show \( u < K c_1 \) in \( \Omega \). Since \( \sigma_{ij}^k u_{ij} = \sigma_{ij}^k (b_{ij} - Ku \delta_{ij}) = k \sigma_k - (n-k+1) Ku \sigma_{k-1} \), we obtain

\[
\sigma_{ij}^k u_{ij} + (n-k+1) Ku \sigma_{k-1} = k \binom{n}{k} > 0.
\]

For case \( K = 0 \), by the strong maximum principle, the above inequality shows \( u(x) < u|_{\partial \Omega} = K c_1 \) for all \( x \in \Omega \).
For case $K \neq 0$, similar to the proof of Theorem 1.1, we consider $\bar{u} := u - Kc_1$, then
\[
\Delta \bar{u} + nK\bar{u} = \Delta u + nKu - nK^2c_2 = \sigma_1(b) - nK^2c_1.
\]
The Newton-MacLaurin inequality implies
\[
\frac{\sigma_1(b)}{n} \geq \left( \frac{\sigma_k(b)}{\binom{n}{k}} \right)^{\frac{1}{k}} = 1.
\]
Thus, we have
\[
\Delta \bar{u} + nK\bar{u} \geq n(1 - K^2c_2) \geq 0,
\]
where the last inequality is from assumption $K^2c_2 \leq 1$.

Since $V > 0$ in $\Omega \subset M^n(K)$, we set $w := \frac{\bar{u}}{V}$, by direct computation,
\[
\Delta w + \frac{2}{V} \langle \nabla V, \nabla w \rangle = \frac{1}{V}(\Delta \bar{u} + nK\bar{u}).
\]
Thus,
\[
\Delta w + \frac{2}{V} \langle \nabla V, \nabla w \rangle \geq 0.
\]
If $w$ is constant, $\bar{u}|_{\partial \Omega} = 0$ implies $u = Kc_1$ which contradicts with $u_\nu = c_2 > 0$. Therefore, strong maximum principle implies $w < 0$ in $\Omega$. Hence, $u < Kc_1$ in $\Omega$.

Now, as the proof of Theorem 1.1, we know either
\[
P = K^3c_1^2 + c_2^2 - 2Kc_1 \quad \text{in} \quad \overline{\Omega}
\]
or
\[
P < K^3c_1^2 + c_2^2 - 2Kc_1 \quad \text{in} \quad \Omega
\]
from Lemma 4.1. And we only need to show the latter can not occur. If $P = |\nabla u|^2 + Ku^2 - 2u < K^3c_1^2 + c_2^2 - 2Kc_1$ in $\Omega$, using $\sigma_{k-1} > 0$ and $V > 0$ in $\Omega$, we have
\[
\int_{\Omega} (|\nabla u|^2 + Ku^2 - K^3c_1^2 - c_2^2)\sigma_{k-1}V d\mu < 2\int_{\Omega} (u - Kc_1)\sigma_{k-1}V d\mu.
\]
The Newton-MacLaurin inequality implies
\[
\frac{\sigma_{k-1}}{\binom{n}{k-1}} \geq \left( \frac{\sigma_k(b)}{\binom{n}{k}} \right)^{\frac{k-1}{k}} = 1.
\]
Since $u < Kc_1$ in $\Omega$, we obtain
\[
\int_{\Omega} (|\nabla u|^2 + Ku^2 - K^3c_1^2 - c_2^2)\sigma_{k-1}V d\mu < 2\binom{n}{k-1}\int_{\Omega} (u - Kc_1)V d\mu.
\]
This contradicts with (14) in Lemma 6.3. Thus $P$ must be constant in $\overline{\Omega}$. The rest of the proof is the same as in Section 5.
OVERDETERMINED PROBLEMS IN SPACE FORMS

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