Direct collisional simulation of 10 000 particles past core collapse

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ABSTRACT

A collisional N-body simulation using NBODY5 on a single CRAY YMP processor is followed well into the post-collapse regime. This is presently one of the largest particle numbers of all such models published, but some data for an even larger N produced by using special-purpose computers have recently been presented. In contrast to previous ensemble-averaged N-body simulations the noise here is low enough just to compare this one single run with the expectations from statistical models based on the Fokker-Planck approximation. Agreement is as good as could be expected for the case of the evolution of the Lagrangian radii, radial and tangential velocity dispersions and various core quantities. We discuss briefly approximate models to understand the number and the energy of escapers and the question of gravothermal core oscillations; although the system exhibits post-collapse oscillations they turn out to be directly binary driven and we cannot prove the existence of a gravothermal expansion at this particle number. Finally in a detailed examination of the wandering of the density centre we find, in contrast to some previous studies, a clear long-time period of the order of approximately 14 half-mass crossing times.

Key words: methods: numerical – celestial mechanics, stellar dynamics – galaxies: star clusters.

1 INTRODUCTION

One of the grand challenges of theoretical astrophysics is to understand the dynamics of globular star clusters. Apart from the intrinsic interest of understanding the behaviour of large N-body systems, the importance of the problem stems from its relation with current observational research, and from the aim to study the behaviour of even larger systems such as galactic nuclei. Without understanding globular clusters by means of special tools for investigating their evaluation, the possibility of unravelling the mysterious behaviour of galactic nuclei seems very remote. Unfortunately, the direct simulation of such rich stellar systems as globular clusters with star-by-star modelling is not yet possible. The gap between the largest useful N-body models with N being of the order of $10^7$ particles and the median globular star cluster ($N \sim 5 \times 10^6$) can only be bridged at present by the use of theory. There are two main classes of theory: (i) Fokker-Planck models (henceforth FP model), which are based on the Boltzmann equation of the kinetic theory of gases (Cohn, Hut & Wise 1989; Murphy, Cohn & Hut 1990), and (ii) isotropic (Lynden-Bell & Eggleton 1980; Heggie 1984; Bettwieser & Sugimoto 1984) and anisotropic gaseous models (henceforth AG models, Louis & Spurzem 1991; Spurzem 1992, 1994), which can be thought of as a set of moment equations of the Fokker-Planck equation.

These simplified models are the only detailed models which are directly applicable to large systems such as globular clusters. But their simplicity stems from many approximations and assumptions which are required in their formulation. One example is the assumption of spherical symmetry, which is inconsistent with the asymmetry of the galactic tidal field; another one is the use of statistical estimates of cross-sections for the formation of close binaries by three-body or dissipative (tidal) two-body encounters, and for their subsequent hardening by gravitational interactions with field stars. Such processes play a dominant role in reversing core collapse of globular clusters, which otherwise would inevitably lead to a singular density profile with infinite density at the centre (Lynden-Bell & Wood 1968; Bettwieser & Sugimoto 1984; Elson, Hut & Inagaki 1987; Hut 1993).

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Recent detailed comparisons have established a good relationship between such simplified models and direct N-body simulations, but for the even more idealized case of a single-mass system (identical individual mass in contrast to real star clusters) and rather low particle numbers (Giersz & Spurzem 1994, henceforth GS, for N = 250, 500, 1000 and 2000; this paper already contains some information on the present N = 10 000 direct model, and also Giersz & Heggie 1994a, b, henceforth GHI, GHIII). Only in one case has a two-mass system been compared with direct N-body simulations (Spurzem & Takahashi 1995). Unfortunately larger N post-collapse star clusters cannot be understood easily by extrapolating the results for low N. Whereas two-body relaxation scales with N/log γ N (with γ ≈ 0.11), or in other words, all N-body results before core bounce (the point where binary effects reverse core collapse) match exactly if one scales the time with this factor (GHI, GHIII), the formation of three-body and tidal two-body binaries introduces other time-scales into the problem, which do not scale with the same factor of N.

Therefore it remains very important to get more data from direct N-body simulations with N as high as allowed by the present-day computational technology and software, in order to strengthen confidence in the statistical models based on the Fokker–Planck and other approximations. To follow, for example, an N = 10 000 star cluster to core bounce one needs to calculate some 900 initial half-mass crossing times. To do this accurately a high order integration scheme with high intrinsic accuracy and no artificial softening of the potential is necessary. The latter is very often used in so-called tree codes (Barnes & Hut 1986), for collisionless stellar dynamics. Such models typically need only simulations of some 10 to 20 initial crossing times.

The high-accuracy integration scheme (Aarseth 1985, nbody5) used here includes a fourth-order Adams–Bashforth–Moulton integrator based on force interpolation polynomials at four past-time points. Also, it includes a two- and more body regularization technique (for the two-body technique see Kustaanheimo & Stiefel 1965) to deal efficiently with close encounters and a two-level time-step scheme (Ahmad–Cohen neighbour scheme, Ahmad & Cohen 1973), which updates the irregular force arising from neighbour particles in much shorter time intervals than the regular, less fluctuating force exerted by the more distant particles.

Nbody5 has recently been improved to Nbody6, which uses a Hermite interpolation for the force and its time derivative (based on only two time points) and a hierarchical block time-step scheme (variation of time-steps allowed only by a factor of two to keep as many time-steps commensurate as possible, Makino & Aarseth 1992). The latter is suitable for implementation on parallel machines, and actually has been implemented in a slightly advanced version Nbody6++ on parallel computers (Spurzem 1996b).

On the other hand special-purpose computers have been built to calculate forces for gravitational N-body systems effectively (Sugimoto et al. 1990). There are also so-called field programmable arrays, hardware devices which can be programmed in a similar way by hardware switches (Brown et al. 1992), which may be used for direct N-body simulations in the near future (D. Merritt, private communication). Such computers became increasingly available in the past (for example the HARP-2 and HARP-3 computers operational in Kiel and Cambridge, respectively) and in the autumn of 1995 a teraflop machine consisting of many such HARP boards has been presented during the IAU symposium 174 in Tokyo (Hut & Makino 1996), which would be able to follow core collapse of a reasonably realistic globular cluster. By using existing pieces of this machine it already became possible to integrate systems larger than N = 10^4 past core bounce (N = 3.2 × 10^4, Makino 1996) and observe the onset of gravothermal oscillations.

In this paper we report on one of the largest published collisional N-body simulations so far, which was performed during the last two years on a single CPU of a CRAY YMP computer by using nbody5. Although it was not possible to detect gravothermal oscillations without any doubt in that run, we demonstrate that 10^4 is a large enough particle number to exhibit a predictable evolution of the system as a whole. Predictable means here, that there is already reasonable agreement of this one simulation with the expectations from statistical models based on the Fokker–Planck approximation. This increases confidence that it will be possible to detect gravothermal oscillations in even larger simulations, which are also predicted by the FP models. On the contrary, for lower N, to check compatibility with theoretical models it is necessary to average a sufficient number of statistically-independent N-body models (GHI, GHII, GS). Single systems of N = 1000, for example, behave practically unpredictably because of the inherent chaotic nature of the orbits in an encounter-dominated star cluster. For large N, however, such chaotic behaviour of the individual orbits is balanced by the decrease of the fluctuations, because the relative energies of individual stellar encounters decrease compared to the total energy of the system, as N increases.

After some technical remarks about the model simulation (Section 2) we report in the following on the time evolution of some relevant observables in the N-body simulation of an isolated, single-point mass, N = 10^4 star cluster. These are mainly Lagrangian radii, velocity dispersions and anisotropy, core parameters (Section 3.1), and the results are compared with anisotropic gaseous models. Escaper properties, the individual evolution of hard binaries and their role for the generation of post-collapse oscillations are briefly examined (Section 3.2), and the movement of the density centre is extracted from the data and discussed in some detail (Section 3.3).

2 THE SIMULATION

A 10^4-body realization (equal masses) of Plummer's model was integrated by using the standard nbody5 scheme on one processor of a CRAY YMP machine for 2760 N-body time units. In the following we use standard N-body time units (Heggie & Mathieu 1986), in which the total energy of Plummer's sphere is E = -0.25, G = 1 and M = 1 (G, M, gravitational constant, total mass of the system, respectively). In such a system the individual mass of the body is m = 1/N for an equal-mass cluster, where N is the total particle number. The initial half-mass crossing time is t_{oc} = 2/\sqrt{2} ≈ 2.828. Hence we have integrated the system over a little less than 1000 t_{oc}, which took approximately 1500

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CPU hours on the CRAY, equivalent to approximately two months continuous running time. The accumulated error of the entire simulation in total energy was \(\Delta E = 8.08 \times 10^{-5}\). The accuracy of the integration was checked by the criterion that the relative change in energy within a given time interval (0.5 \(t_{\text{eq}}\)) should be less than \(10^{-3}\). Since NBODY5 contains the Ahmad–Cohen neighbour scheme each particle has two time-steps, a short one, after which the irregular force from the neighbours (typically less than \(\sqrt{N}\) particles) is updated, and another regular time-step (which is usually about 10 times larger than the irregular timestep), after which the total force exerted by all particles is updated, and changes in the neighbour list are maintained. Such a neighbour scheme saves a considerable amount of CPU time as compared with a scheme which at any step always calculates the full force, at least if one considers a non-parallel general purpose computer. Both the regular and irregular time-step are fully adaptive according to the individual force variations on the particle (for details see Aarseth 1985). Our model performed in total approximately 13.5 billion irregular steps, one billion regular steps and 500 million regularized steps for closely bound pairs, the internal motion of which is integrated separately in regularized coordinates. This means on average each particle was moved 1.35 million times (its regular force updated 10 thousand times). Information which is more quantitative can be found in Fig. 1 which shows the CPU time used and the number of regular, irregular and regularized (Kustaanheimo & Stiefel 1965, KS) steps per \(N\)-body time unit as a function of time (Fig. 1a) and the same data per time unit per particle (Fig. 1b). The fluctuations after \(t = 2300\) are owing to the onset of the activity of extremely hard binaries in the core (core bounce is at \(t = 2380\), see below). These binaries suffer from strong superelastic scatterings with field stars in the core, so the time-step distribution of the field stars is affected itself. It is remarkable, however, that from the beginning until the core bounce, when there is a strong core–halo structure, the CPU time rates do not increase by more than half an order of magnitude, which is a proof that the two-level neighbour scheme of NBODY5 is indeed very well tailored to simulate centrally concentrated systems.

3 RESULTS

3.1 Spatial evolution

In the following we compare our \(N\)-body results with those of a standard AG model (Louis & Spurzem 1991; Spurzem 1992, 1994, 1996a), using for the latter those parameters which gave best agreement between all the different kinds of star cluster models in previous work, namely \(\lambda = 0.4977\), \(\lambda_c = 0.1\) and \(C = 90\). Here \(\lambda\) is the numerical parameter of order unity scaling the heat flux, related to the standard \(C\) constant in isotropic gaseous models (see e.g., Heggie & Stephenson 1988) by

\[
\lambda = \frac{27}{10} \sqrt{\pi C}.
\]

Our \(\lambda\) corresponds to the standard \(C = 0.104\) of Heggie & Stephenson (1988), which gives the best agreement between FP and AG models. \(\lambda_c\) is a numerical factor scaling the anisotropy decay time \(\tau_a\), defined such that the decay of anisotropy by two-body relaxation is

\[
(\delta A/\delta t)_a = -\frac{A}{\lambda_c \tau_a},
\]

where \(\tau_a\) is the self-consistent anisotropy decay time for a Larson-type anisotropic distribution function \((\tau_a = 107/9)\), with \(T\) the standard two-body relaxation time). Our value of \(\lambda_c = 0.1\) was fitted to averaged \(N\)-body simulations of \(N \leq 2000\). Finally \(C_b\) is the numerical parameter scaling the energy generation by three-body binaries, the standard value of which (Goodman 1987) is \(C_b = 90\), which we adopt here. Goodman & Hut (1993) have recently argued that the formation rate of three-body binaries had been overestimated in previous work, so \(C_b = 75\) would ensue for their new results. However, the still adopted value of \(C_b = 90\) is within the expected error range, regarding their derivations as well as the accuracy with which we can fix \(C_b\) by comparison with the \(N\)-body simulation. The reader more interested in the details of how to define parameters like \(\lambda, \lambda_c\) and \(C_b\) and how to select their values is referred to GS and the other cited papers above.

Fig. 2(a) compares the evolution of Lagrangian radii containing 1–90 per cent of the total mass between the AG and our \(N\)-body model. Whereas the pre-collapse evolution agrees extremely well, and also the slope of the post-collapse expansion agrees fairly well, there is a difference in the collapse time itself, which is at \(t = 2080\) for the gaseous model and \(t = 2380\) in the \(N\)-body system. So is there something wrong? One has to recall that the conductivity parameter \(\lambda\) in the gaseous model was adjusted to match simultaneously the collapse time and slope of the Lagrangian radii evolution in pre-collapse for \(N\)-body simulations \((N = 250, 500\) and 1000) and isotropic FP models (GS). In these averaged models there is always a considerable spread in the collapse times of individual \(N\)-body models, as can be seen from Table 1. First we show in this table results obtained from various independent \(N\)-body calculations \((N \leq 2000)\), as e.g. the variance of Lagrangian radii for the initial model \(\sigma_r\), minimum and maximum collapse time \(t_{\text{min}}\) and \(t_{\text{max}}\) as measured in individual simulations, the average collapse time \(t_c\) for all simulations of a given particle number, the variance of the core collapse time \(\sigma_c\) and its relation to the collapse time itself. For more details about the averaging process and the individual simulations see GHIII and GHIII, from whose models these data were taken. For all quantities except \(t_{\text{min}}\) and \(t_{\text{max}}\), there is an expected scaling with \(N\) denoted in Table 1; we found it to be in general agreement with the measured values, although there is some tendency for mismatch with such scaling for low \(N\) \((N \leq 500)\). In particular the scaling law for the collapse time stems from the assumption that it is two-body relaxation (by taking a Coulomb logarithm factor of \(\gamma = 0.11\), see GHII, GHIII) which dominates the evolution to core collapse. Here and in the following our operational definition of core-collapse time is the time elapsed from the beginning of the simulation (initial model Plummer’s sphere) until core bounce, i.e. the radius containing 1 per cent of the total mass reaches its minimum value. It was checked that this is consistent with other possible criteria
We have used these scaling laws to extrapolate quantities for \( N = 10000 \) and \( N = 20000 \), which cannot be measured yet, because there are not enough individual N-body simulations (values for \( N = 1000 \) were used to extrapolate for \( N = 10000 \), and those of \( N = 2000 \) to extrapolate to \( N = 20000 \)). We find for 10 000 particles an extrapolated average collapse time of \( t_{ce} = 2215 \). Note that even this value is not very accurate, because the measured average collapse times do not exactly obey the assumed relation. However, if we take this value for the time being as a fiducial value it
follows, that our individual N-body simulation has a collapse time 165 time units (1.14σ) larger than the average, and gaseous and FP models have a collapse time 135 time units (0.93σ) smaller than the expected average collapse time.

We stress these numbers here in detail, because recently it has been claimed that the collapse time in anisotropic systems could be somewhat larger than in the isotropic case, as suggested by the new 2D FP models of Takahashi (1995, 1996), and previously by higher order moment models of Louis (1990). However, at this present particle number of \( N = 10000 \) it seems very difficult to distinguish such a variation from the intrinsic uncertainties in determining the

![Figure 2](https://example.com/figure2.png)

**Figure 2.** (a) Lagrangian radii containing the indicated fraction of total mass as a function of time for the \( 10^4 \)-body simulation (fluctuating curves) and the standard anisotropic gaseous model (smooth curves). (b) Lagrangian radii as in (a), but here our \( 10^4 \)-body simulation (time rescaled according to main text) compared to an averaged \( N = 1000 \) simulation of GHI and GHII.
core-collapse time. There are several factors which influence measured collapse times, as e.g. in N-body models the number of independent simulations, and in gaseous and FP models variation of quantities like $t_{\text{bo}}$ and $C_b$ (starting time and strength of binary energy generation, see GS for their definition, and their figs. 4 and 5 for their influence on collapse times) and the factor $\gamma$ used in the Coulomb logarithm could always cause a change of a few per cent in the collapse time. For example, $\gamma = 0.11$ has been adopted as the best value from GHI and GHII, who considered only $N \leq 2000$; a smaller $\gamma$ could always cause a larger collapse time. Hence we conclude here that from the present data there is some suspicion that the collapse time in N-body simulations might be longer than expected from anisotropic gaseous and isotropic FP models, however, we consider the difference of just one $\sigma$ as not yet very convincing. Note that the free parameter in the conductivity of the anisotropic gaseous model was scaled to the same collapse time as the isotropic FP model (GS). Thus if one came to the conclusion that anisotropic 2D FP models and naturally anisotropic N-body systems have a slightly longer collapse time, this just would mean that the anisotropic gaseous model had to be rescaled in its conductivity parameter $\lambda$, now with respect to the 2D FP model. To judge about these questions has to be postponed to the future, when N-body simulations of large particle numbers ($N \geq 10^4$) become available in a large number.

In Fig. 2(a) one can see at a time much later than core bounce ($t \approx 4000$) that the gaseous model becomes unstable to gravothermal oscillations at very late times (compare with gravothermal oscillations, Bettwieser & Sugimoto 1984, Goodman 1987, Breeden, Cohn & Hut 1994). Since the instability at $N = 10000$ is not very strong (particle number not much above the critical number 7000 for gravothermal oscillations in the equal-mass system) the oscillations start late, at a time, which our N-body simulation did not yet reach. It exhibits, however, some kind of oscillations directly after core bounce. However, as will become clear during the discussion of the binary effects, we cannot establish firmly that these oscillations are gravothermal, in contrast to just directly driven by binary energy. Gravothermal oscillations (or expansion) should continue even after binary energy production has ceased. One can deduce that this is not the case, when looking in Fig. 8(c) (time larger than 2620 time units), where there is no significant binary

![Figure 3](https://example.com/figure3.png)

**Figure 3.** Lagrangian radii as in Fig. 2(a), but now an enlarged time segment near core bounce. Gaseous model rescaled according to text in order to compare the post-collapse slopes of both models. The times of prominent hard binary events are marked, as there are binary escapers (dots) and strong hardening events of binaries by three-body encounters (stars).
To relate our single $N=10^{10}$-body model with the lower $N$ cases of GHI and GHII we have scaled its time coordinate by $C(N_0)/C(N)$, where $C(x) = x \log(x)$ ($y = 0.11$) for $N = 10^9$ and $N_0 = 1000$, and plotted the data for the Lagrangian radii together with those of an averaged $N=1000$ body calculation, kindly made available by M. Giersz, in Fig. 2(b). Such scaling would occur if standard two-body relaxation is the dominant force of evolution, which in fact can be clearly seen in the plot. The separation of the curves for different $N$ at core bounce is a result of the three-body activity which sets in at different times and scales with a different power of $N$.

In order to compare the post-collapse evolution to the gaseous model it is better to scale the times of the gaseous model to the exact collapse time ($t = 2380$) of the $N$-body simulation. The result on an enlarged time-scale can be seen in Fig. 3. To get Fig. 3 we have multiplied the time and all radii with a free factor, so as to match the collapse time and the vertical position of the gaseous model and $N$-body curves. Consistently with the previous results of GHI, GHII and GS we find that core bounce in the $N$-body model is deeper in the inner shells and that the intermediate zones of the gaseous model expand faster, whereas the outermost shells lag behind as compared with the $N$-body model. We think this is because of a non-local energy transport by high-velocity stars on radial orbits, which are reaction products from superelastic binary encounters in the core, as also elaborated in the previously cited papers. In the gaseous model energy can only be transported locally by heat conduction with the temperature gradient; thus the energy created by binaries, which is of the correct magnitude (see below) causes more expansion in the adjacent, intermediate shells relative to the $N$-body model. The importance of high-velocity stars carrying energy from superelastic encounters quickly out of the core was first realized by Aarseth (1974) and further elaborated on by Makino & Sugimoto (1987), who denoted them as suprathermal particles.

Figs 4(a–d) show the radial and tangential velocity dispersions, each for the entire evolutionary time (4a and 4c) and for an enlarged time segment around core bounce (4b and 4d). In most cases the general agreement between the gaseous model and our single $N$-body simulation is striking. However, there are some features near and after core bounce worth discussing in more detail. First, the effect that intermediate shells (e.g. 10–20 per cent of total mass) expand a little faster than the $N$-body system, as was detected for the Lagrangian radii, corresponds to a smaller 'temperature' (velocity dispersion) there. On the other hand, further outside, where the $N$-body system expands more rapidly, it is somewhat cooler than the $N$-body model. In general one could say that the large-scale Figs 4(a) and (c) support our claim of the generally fair agreement between $N$-body and AG models, but Figs 4(b) and (d), presenting a zoom in on core bounce and post-collapse, give a clearer picture of what the remaining differences are. How about the anisotropy $A = 2 - 2 \sigma_t^2/\sigma_r^2$ itself? In principle it is redundant, all information is already contained in the previous

Figure 4. (a) Radial 1D velocity dispersion averaged between the indicated Lagrangian radii as a function of time. Comparison between $N$-body (fluctuating curves) and gaseous model (smooth curves). (b) As (a), but for an enlarged time section near core bounce. (c) As (a), but for the tangential 1D velocity dispersion. (d) As (b), but for the tangential 1D velocity dispersion. (e) As (a), but for the anisotropy defined as $A = 2 - 2 \sigma_t^2/\sigma_r^2$. © 1996 RAS, MNRAS 282, 19–39
plots. However, the analysis of \( A \) enhances the remaining differences between the models, so we will discuss it here as well with the help of Fig. 4(e). For the innermost Lagrangian radii (say up to 30 per cent) it is not useful to discuss \( A \), because it is fluctuating strongly; comparisons with the AG models have to rely on the velocity dispersions as above. Note, however, the good agreement between both models for the anisotropy averaged between 30 and 40 per cent of total mass (lowermost curves of Fig. 4e), especially in the reduction of anisotropy after core bounce. Surprisingly, there is not such agreement for the larger radii. For 40–50 per cent the decrease of \( A \) in post-collapse is only visible in the AG model, the anisotropy in the N-body model increases further. Further outside, the disagreement
between both models starts even earlier. This is consistent with the idea of suprathermal particles, moving in radially elongated orbits towards the halo, not being accounted for in the gaseous model. As for the outermost shells of the system one has also to take into account a difference in the boundary conditions (the N-body model is open and may expand freely into space, the AG model is confined by an adiabatic wall at 100 per cent of total mass, so the 100 per cent Lagrangian radius cannot expand, which it does in the N-body model).

Figs 5(a)–(c) depict the time development of the central potential, Cohn's $x$ value ($x = \Phi_p/3\sigma_v^2$, where $\sigma_v$ is the core velocity dispersion), and the number of core particles. All of the data in Fig. 5 are scaled in time as in Fig. 2(b), in order...
Figure 5. (a) Central potential as a function of scaled N-body time, see text. Compared with a smoothed curve obtained from average \(N = 1000\) data. (b) Cohn's \(x\) as a function of scaled \(N\)-body time. Smoothed in time, compared with smoothed data from average \(N = 1000\) simulation. (c) Core particle number as a function of scaled N-body time. Compared with data of average lower \(N\) simulations. Note the minimum \(N_c = 13\) for 10,000 particles at core bounce.

5(a) and (c) once more a very good agreement of a single \(N = 10^4\) model with the average \(N = 1000\) data is visible for the pre-collapse phase. During core bounce the single \(N = 10^4\) simulation exhibits strong fluctuations (very deep central potential, correlated with an extremely small core particle number), which presumably would be smoothed out to compare them with the data of GHI and GHII. In some cases the figure caption notes that the data were smoothed, which means the high frequency fluctuations have been cut out by the 'Numerical Recipes' routine smooft, which uses Fourier transforms to do that. The window for smoothing was about 30 data points, which are 45 time units. In Figs
if we would be able to average the results for different independent models, as in the $N=1000$ case.

It is very instructive to look at a graph just showing as tiny dots values of the potential at the individual particle locations as a function of radius for different times in pre- and post-collapse (Fig. 6). This gives a good description of how smooth the particle distribution already is for a total particle number of 10,000, and moreover, it unambiguously clarifies
Figure 7. (a) Cumulative particle loss because of escapers as a function of scaled N-body time (scaled to the collapse time of an isotropic FP model). Curves labelled N and N-BIN denote the N-body data with and without hard escapers (see main text); the data labelled FP-I show the increase of Hénon's (1965) standard escape rate (at $t=0$) due to the evolutionary effect; the curves labelled A-90 and A-75 show the multiplicative combination of the evolutionary effect and another correction for anisotropy evaluated by using Dejonghe's (1987) anisotropic Plummer model with a $q$-value derived from the anisotropy at the 75 or 90 per cent Lagrangian radius, respectively. (b) As (a), but for the energy of the escapers. The final value of the cumulative escaper energy, including the hard escapers, is 0.0557. It cannot be seen in the plot, since its axes have been scaled in order to resolve sufficiently the other curves.
that core bounce, the time with the deepest central potential, occurred at $t = 2380$ (labelled 886 in the plot), which can also be seen in Fig. 5(a) (but note the rescaled time).

### 3.2 Escapers and binaries

At the present endpoint of our simulation a total of 360 particles had escaped (they were removed from the calculation when moving with more than the escape velocity from the entire cluster and being at a radius larger than ten times the initial scaling radius of Plummer's model). Fig. 7 shows the cumulative particle and energy loss. Similarly to GHI we tried to approximate the N-body results by rescaling Hénon's (1965) escape rate (isotropic Plummer sphere) multiplicatively for the effects of evolution and anisotropy (see figs 9 and 11 of GHI). The effect of evolu-

![Figure 8. (a) Time at which stars escape as a function of their energy in $kT$, where $1kT = \sigma_c^2$ with the 1D core velocity dispersion $\sigma_c$.](image)

Figure 8. (a) Time at which stars escape as a function of their energy in $kT$, where $1kT = \sigma_c^2$ with the 1D core velocity dispersion $\sigma_c$.

![Figure 8. (b) As Fig. 8(a), but with an enlarged time-axis to see the decrease in binary activity after core bounce for $t > 2600$.](image)

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Figure 8. (c) Individual fate of binaries close to core bounce; binding energy as a function of time. At some times labels are plotted, which are the names of the stars in the binary and can be used to identify them in the following plot. Three thick vertical line segments show the magnitude of an increase in binding by energy by 40 per cent from energy 20, 100 and 300, respectively, to be compared with the actual energy changes. Binary escapers and exchange reactions of the binaries are indicated.

Figure 8. (d) Radial position of the hard binaries of the previous figure over the same time span. Note that the core radius is much smaller than one during this entire period. In some cases by comparison with Fig. 8(c) the event can be identified which kicked a binary out of the core or even out of the cluster (lines escaping to the top belonging to escaping binaries).
tion was determined by directly calculating the escape rate of an isotropic FP model with $N=10^4$ particles and standard parameters ($\gamma=0.11$ for the Coulomb logarithm, $C_b=90$ for the energy generation), see for reference equation (2) of GHI. As a model for the effect of anisotropy Dejonghe's (1987) anisotropic Plummer models were used, which have a dimensionless parameter $q$. Here $q$ can be determined by taking the anisotropy $A$ at a certain radius $r$ of the N-body model by $q=A/(1+1/r^2)$. GHI adopted the anisotropy at the Lagrangian radius containing 75 per cent of total mass, here we try to bracket the N-body results by computing $q$ and subsequently scaled escape rates for both the 75 and 90 per cent Lagrangian radii (see plots). Since we were able to distinguish every single escaping particle in our N-body calculation, it was possible to remove those escaping particle and energy loss is labelled by 'N-BIN' in the plots. Since we were able to distinguish every single escaping particle in our N-body calculation, it was possible to remove those escaping in the N-body rates of escape is small for the particle number and in pre-collapse, but dominant for the energy of escapers in core bounce and beyond. A fair approximation of the particle and energy loss is labelled by 'N-BIN' in the plots.

From the figures we conclude the following: the effect of hard escapers in the N-body rates of escape is small for the particle number and in pre-collapse, but dominant for the energy of escapers in core bounce and beyond. A fair approximation of the particle and energy loss in the N-body system is reached by rescaling Hénon's escape rate for the effect of evolution and anisotropy, provided the energy of the hard escapers is subtracted first from the N-body data. The effect of evolution visible in the curves labelled FP-I is much less significant than that of the anisotropy. However, since Dejonghe's models are only a very approximate model it is not a priori clear, how to fit the best $q$ parameter to an individual N-body system. To illustrate this uncertainty we have used two different Lagrangian radii (containing 75 and 90 per cent of total mass, with correspondingly different anisotropies and thus varying $q$ parameter for the same time). The escape rate of particles seems to be best approximated by using a $q$-value derived from the 90 per cent Lagrangian radius (Fig. 7a), whereas the energy-loss rate is bracketed by the two choices (Fig. 7b). It follows that it is not possible to find a simultaneous best fit to both data; we think that this is owing to the fact that Dejonghe's models are not an accurate representation of the real N-body distribution. Note that we have to rescale here once more the N-body time to the same collapse time as in the FP model ($t=2080$, see Fig. 2a above and its discussion) used for comparison, in order to get a useful comparison of both data.

We started initially without any binaries. At $t=1200$ a transient mildly hard binary formed, the effect of which can be seen in a slight dip of the N-body Lagrangian radii against the gaseous model data in Fig. 2(a). Such very early transient binary activity may be a reason why our model system has a collapse time which is at more than $1\sigma$ above the expected average value. The really hard binaries then formed shortly before core bounce. This can be seen in Figs 8(a) and (b) where the high-energy ($E \gg kT$) escapers occur just at core bounce, which is the time of formation of hard binaries, as is shown in Fig. 8(c). Fig. 8(b) is an enlargement from the data of Fig. 8(a); we see that the binary activity reduces to a significantly lower level again at $t=2600$, since we interpret any escaper with more than $1kT$ as arising from an interaction with a hard binary. It is

![Figure 8](https://example.com/figure8.png)

*Figure 8.* (e) Energy balance as a function of time near core bounce. Labelled are the curves for the binding energy of the binaries still inside the cluster ($E_{bin}$), the cumulative translational energy of single escapers ($E_{esc}$), the cumulative binding energy carried away by escaping binaries ($E_{escbin}$), and the cumulative energy due to three-body binaries produced in a standard AG model ($E_{bin}$). The core-collapse time of the gaseous model was scaled to the actual collapse time of the N-body model, see text.
interesting to note, that near the times $t=2400$, 2500 and 2600 there is a crowding of high-energy escapers in Fig. 8(a). We relate this to the spikes in the velocity dispersions of the outer Lagrangian shells occurring in Fig. 4 just near the same time, only for the radial velocity dispersion, not for the tangential one. This supports the idea, that the high-energy reaction products of the strong three-body encounters move radially outwards and generate the spikes in $\sigma_t$. Moreover, these three times can be identified as periods of strong binary activity in Fig. 8(c) (binaries exceeding a binding energy per unit mass of 200 in N-body units). If one then looks at Fig. 3 these times correspond approximately to expansion phases of the Lagrangian radii; after the binary activity has ceased, the expansion does not continue, on the contrary, the system starts to contract again. Note, that after $t=2600$ there is also no strong binary activity, because the hardest binary, although it is still contained in the system (not yet escaped), moves in an elongated orbit after a recoil. Hence it has a small probability for three-body encounters. From all this evidence we conclude that the post-collapse oscillations of the N-body system are binary driven and not gravothermal. If they were gravothermal, expansion should continue after the binary energy generation has ceased. One can deduce that this is not the case, when looking in Fig. 8(c) (time larger than 2620 time units), where there is no significant binary energy change until the end of the run. Shortly afterwards according to Fig. 2(a) a recollapse occurs.

Moreover, as was found already Bettwieser & Sugimoto (1984), there should be an inversion of the temperature (i.e. velocity dispersion) profile in the case of a gravothermal expansion, which we could not observe with a sufficient statistical significance in our N-body data. The system was divided into various amounts (20–100) of radial shells containing nearly equal amounts of particles, and we look at single configurations as well as suitable ensemble averaged configurations (up to 20 models at consecutive times close to each other averaged). Moments of the velocity distribution up to fifth order were calculated; however, neither the radial nor the averaged velocity dispersion showed a persistent temperature inversion. At some times there could be observed a temperature rise from the innermost shell to the next, but the gradient was small and changed its sign quickly with time. Therefore our conclusion is that we do not find a statistically significant temperature inversion.

On the other hand, however, there is a steady expansion consistent with the gaseous model in post-collapse superposed on the binary driven oscillations. So we would not exclude that on a secular time-scale our N-body system will undergo a gravothermal expansion, presumably with a temperature inversion too small to be extracted from the noisy data.

In Fig. 8(c) the fate of the hard binaries in the system can be followed individually, e.g. one can estimate by eye that the rate of change of the binding energy of a hard binary is related to its binding energy $x$; in an encounter binaries with large $x$ suffer a change in binding energy of the same order as $x$. This is consistent with the early experimental (Hills 1975) and theoretical (Heggie 1975) predictions, that on average the energy change $\Delta x$ during a superelastic scattering should be $\Delta x = \sqrt{x}$ ($\eta = 0.4$). GHII argue that an experimental value of $\eta$, found by their statistics of a large number of N-body simulations for $N \leq 2000$, is about 0.2, in better agreement with more recent predictions by Spitzer (1987) and scattering experiments by Heggie & Hut (1993). Since, however, in contrast to the averaged N-body simulations of GHII and GHIII, we only have a rather small number of strong encounters, i.e. a poor statistics, we will not elaborate further on this point here. We think our results are at least consistent with a value of $\eta$ between 0.2 and 0.4; a change with $\eta = 0.4$ is indicated in the figure as a thick vertical bar for three energies $x = 20, 100, 300$. One may wonder why in one case at about $t = 2570$ the binding energy of a binary decrease so dramatically. This is because of the formation of a hierarchical triple, which was unfortunately not included in our binary list. Later, its binding energy occurs again in the most strongly bound binary, after dissolution of the triple. Fig. 8(d) gives an optical impression of the orbital motion of the centre of mass of hard binaries, their recoils by superelastic scatterings to very large radii, and subsequent shrinking again by dynamical friction to the core. Especially the motion of the hardest binary consisting of the particles numbered 4805 and 5745 (see Fig. 8c, its binding energy amounts to several hundred $kT$) reaches far into the halo after an energetic recoil. Thus one can understand that exotic products of binary evolution (like soft X-ray binaries or blue stragglers, see for the first e.g. Rappaport et al. 1994) may be found in the outskirts of a cluster.

As a final remark we want to demonstrate with Fig. 8(e) how the cumulative binary energy generation in the N-body system compares to the steady energy input modelling the binary heating in the gaseous model. One can see that the energy liberated in the gaseous model is of the same order as the binding energy of the hard binaries still in the N-body system, and also of the same order as the binding energy carried away by escaping binaries.

### 3.3 Movement of the density centre

A very interesting and not yet completely understood feature of a real N-body system is the oscillations of its density centre and also other quantities, like the Lagrangian radii. To define the density centre we first compute for each particle $i$ a neighbour density $\rho_i$, according to Casertano & Hut (1985) by the mass inside $r_{\text{in}}$ (excluding the test particle itself), where $r_{\text{in}}$ is the distance of the sixth nearest neighbour. Then by

$$r_s = \frac{\sum \rho_i r_i}{\sum \rho_i},$$

we define a density centre for a special subset of particles ($r_i$, $r_s$ are the position vectors of the density centre and the particle $i$, respectively). For the summation only the inner parts of the system should be included, in order to exclude strong biases by asphericities in the outermost parts of the system (large weight due to large $|r_i|$). On the other hand selecting the core particles or some fraction of them (based on the previous core radius, which was also determined in a similar way as the density centre) proved to produce extreme fluctuations of the density centre in the late phases of core collapse. So we chose a procedure suggested by McMillan, Hut & Makino (1990), to take all particles whose $\rho_i$ is some fraction (actually 1/20) of the maximum $\rho$. This
yielded the data for the density centre of Figs 9(a)–(c). In recent years there were detailed examinations of core wandering and oscillations by Heggie, Inagaki & McMillan (1994) and Sweatman (1993). Our data are different from their data in several respects. First, we cover a much larger time, however, we do not resolve the small time-scales as they did. We cover approximately 1200 N-body time units with a constant time interval of roughly $0.5 t_{\text{in}} (\Delta t = 1.4)$. In principle our data set is even larger, but unfortunately we did not always keep a constant time interval for the data.

Figure 9. (a) Modulus of the vector of the density centre and the centre of mass of the entire cluster as a function of time. (b) As (a), with enlarged time-axis, and added thick curve for the smoothed density centre variations, which serves as a reference for the autocorrelation analysis, see text. (c) As (a), but even more zoomed in on the time-axis, to see the limiting resolution of our data at small time-scales. (d) Autocorrelation function explained in the text as a function of displacement (in time) $\tau$ (in N-body time units). (e) Power spectrum of the differences between the density centre data in (a)–(c) and their smoothed counterpart (see b and c), as a function of period in N-body time units.
collection, which turns out to be essential for the subsequent analysis. Secondly, we are not able to resolve the noise at high frequencies, as was the aim of Heggie, Inagaki & McMillan (1994). In fact, our data become significant just at frequencies smaller than 0.1 (see their fig. 7), where their statistics becomes poor. Also Sweatman (1993), who did a very thorough examination, which we closely follow here in our data processing, was limited to periods smaller than 10 (frequencies larger than 0.1) owing to the duration of his simulation.

We first present the original data of the density centre in three different scales in Figs 9(a)–(c), together with the centre-of-mass data for the entire system. As noted already by the cited authors and also by Makino & Sugimoto (1987) there are variations on different time-scales, beginning from very abrupt changes on times comparable to the dynamic
time up to larger changes on time-scales of the order of several crossing times. We will now inspect the time variability of the data in more detail.

In Figs 9(b) and (c) a third (thick) curve is plotted, which is the density centre data \( r_{sd} \) smoothed by the SMOOT procedure of ‘Numerical Recipes’ with a window of 50 data points (we varied the window to 30 and 100 data points without any significant change in the following results). The reason is that for a proper autocorrelation and frequency analysis we need data the average of which is zero. Sweatman (1993) took the centre of mass as a reference point, but as one can see in Fig. 9(a) for a long time simulation, where many, partly high-energy, escapers are removed and the outermost parts of the system are not nicely symmetric anymore, there is a systematic shift between the centre of mass and the density centre. Therefore we need the smoothed data as reference, such that we can compute a quantity \( \delta R = r_{sd} - r_s \), the time average of which is zero. So we cannot detect any periods in the data which are larger than the smoothing window, and we checked that our results are independent of its size. With \( \delta R = |\delta R| \) we performed a autocorrelation analysis identical to that described by Sweatman (1993) in his chapter 3.3. The striking result is in Fig. 9(d), which exhibits a dominant period of the order of 40 time units, which is also about 14 initial half-mass crossing times. The previous studies by Heggie, Inagaki & McMillan (1994) and Sweatman (1993) could not find such periods, because their time series was not long enough. Oscillations of the core with periods large compared to the crossing time have been observed earlier by Makino & Sugimoto (1987) in their N-body simulation.

4 CONCLUSIONS AND DISCUSSION

We have performed a very long (in terms of computer time as well as of physical time) simulation of a collisional \( N = 10^4 \) particle system well into the post core-collapse evolutionary phase by using the standard NBODY5 integrator (Aarseth 1985). As initial model a Plummer sphere with no binaries and equal masses was selected. In contrast to previous averaged N-body simulations (GHI; GHII; GS) this single model already compares well with expectations of gasdynamical models based on the Fokker–Planck approximation. In particular the evolution until core collapse is entirely dominated by standard two-body relaxation, so for example Lagrangian radii, velocity dispersions and anisotropy compare fairly well with the other, statistical models, although there is some disagreement. Moreover, especially in the post-collapse phase and for the outer halo regions there are further stronger differences between the models, which we think are because of the poor representation of suprathermal particles, originating from superelastic scatterings with hard binaries, in the gaseous model. Anisotropy is also the dominant factor determining the time evolution of the escape rate, which in deep core collapse is much larger than Hénon’s (1965) standard estimate. Within some uncertainty the effect of the anisotropy on the escape rate can be modelled by using one of Dejonghe’s (1987) anisotropic Plummer models, provided the energy of escapers
from hard three-body encounters is not taken into account.

Thereafter the core bounces because of the energy generation by superelastic scatterings between single stars and binaries. The energy generation is in fair accord with statistical predictions by Heggie (1975) and Hut (1993). From our simulation we see that individual hard binaries are sometimes ejected by recoil effects very far from the core after encounters. Since in our simulation we are able to follow the individual fate of each binary, we can correlate strong binary activity with expansion phases of the whole system. Since each expansion does not continue after the binary activity ceases, the post-collapse oscillations present in our model are not gravothermal in nature. Although theoretically 10,000 particles should be enough to observe gravothermal oscillations (but they set in long time after core bounce in the gaseous model), the fluctuations induced by individual strong three-body encounters suppress them to a non-observable level. Nevertheless the \( N \)-body system expands on a secular time-scale as prescribed by the gaseous model, so it cannot be excluded that there is a gravothermal expansion with superimposed binary-driven oscillations. In recent simulations (Makino 1996) with even larger particle number (32,000 particles) the onset of gravothermal oscillations including a temperature inversion has been observed, which would mean that for increasing \( N \) the gravothermal evolution as predicted by gaseous and FP models will dominate the evolution as compared to the stochastic variations directly induced by binary activity. For a particular case, namely the fluctuations of core-collapse time in an individual \( N \)-body simulation, we could show that it will decrease relative to the collapse time itself with larger \( N \).

Most features of our single \( N = 10^4 \) model are consistent with expectations from statistical (gaseous or FP models), although our collapse time is 1.14\( \sigma \) larger than an extrapolated average value for \( N \)-body simulations, and the latter is another 0.934\( \sigma \) larger than the collapse time of anisotropic gaseous and isotropic FP models. Since this is rather large, one may speculate whether it reflects a recent result by Takahashi (1995, 1996), who finds that in his new 2D FP models collapse times are some 10 per cent larger than expected from the previous isotropic models. However, our particle number is still not large enough to discriminate with a sufficient statistical significance between such small differences in the collapse time.

We have also examined the wandering of the density centre. By autocorrelation analysis and Fourier transformation we find a dominant period of the order of 40-60 time units, which is about 13-20 initial half-mass crossing times. Since our data base is much longer in time, these oscillations could not have been found in previous studies like Heggie, Inagaki & McMillan (1994) and Sweatman (1993). The existence of long-time-scale oscillations like those we found here has, however, been noted already by Makino & Sugimoto (1987).

From data of our simulation which were not shown in the plots here we conclude that the velocity distribution in the core changes significantly between the pre- and post-collapse state of the cluster. Thus for example the ratio of the fourth order moment of the velocity distribution related to the second order moments is larger after core bounce than before, consistent with the idea that a high velocity tail of stars is generated by superelastic scatterings with the binaries. The presentation of these data as well as possible relations to observable line profiles in star clusters will be the subject of future work. In parallel to this it could be useful to extend the gaseous models to higher order moments of the velocity distribution in order to achieve a better representation of the suprathermal particles in the post-collapse phase. One might argue that 2D FP simulations, if they become more common in the near future, already deliver a complete and unrestricted 2D distribution function. However, for the grid-based solutions of the orbit-averaged Fokker–Planck equation an isotropized distribution function is still used to calculate the diffusion coefficients. This is in contrast to the gaseous models, which use a consistent anisotropic background for the Fokker–Planck collisional terms. If the Fokker–Planck equation is solved by using a variational principle (Inagaki & Lynden-Bell 1990; Takahashi & Inagaki 1992) it is self-consistent (contrary to the discretization method), but other uncertainties occur, as e.g. the proper choice of trial functions. We conclude that none of the presently used methods to simulate the evolution of large-\( N \) star clusters should be discarded; it will remain very important in the future to produce a larger data base, especially for the case of \( N \)-body simulations with large \( N \) and to study thoroughly the good and weak features of each method.

To improve the statistics at particle numbers of \( N \geq 10^4 \) more independent computations are useful and in progress, especially by using the recent fast versions of the HARP special purpose hardware (Taiji 1996). In order to see whether a single-star cluster behaves in accordance with the statistical models and, for example, shows gravothermal oscillations, a still larger \( N \) is required, as in the case of the recent 32,000 particle simulation (Makino 1996).

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