Poisson structure on a space with linear SU(2) fuzziness

Mohammad Khorrami
Amir H. Fatollahi
Ahmad Shariati

Department of Physics, Alzahra University, Tehran 1993891167, Iran.

Abstract
The Poisson structure is constructed for a model in which spatial coordinates of configuration space are noncommutative and satisfy the commutation relations of a Lie algebra. The case is specialized to that of the group SU(2), for which the counterpart of the angular momentum, as well as the Euler parameterization of the phase space are introduced. SU(2)-invariant classical systems are discussed, and it is observed that the path of particle can be obtained by the solution of a first-order equation, as the case with such models on commutative spaces. The examples of free particle, rotationally-invariant potentials, and specially the isotropic harmonic oscillator are investigated in more detail.
1 Introduction

There have been arguments supporting the idea that the ordinary picture of spacetime breaks down at spacetime intervals in which the quantum gravity effects would be important, the so-called Planck length and time [1]. The reasonings lie on the expectation that gravity, as the theory responsible for partial properties of spacetime, would present an undetermined character at scales where quantum effects are dominant. As a consequence, one is inclined to believe in some kinds of space-space and space-time uncertainty relations [1], whose appearances usually point to noncommutative objects. In fact, by a reverse way of reasoning, it has been argued that one might had to expect that a proper description of quantum gravity should be possible based on noncommutative theories of spacetime [2–5]. In the simplest case of canonical noncommutative space the coordinates satisfy

$$[\hat{x}_\mu, \hat{x}_\nu] = i \theta_{\mu\nu} \mathbf{1},$$  \hspace{1cm} (1)

in which $\theta$ is an antisymmetric constant tensor and $\mathbf{1}$ represents the unit operator. It has been understood that the longitudinal directions of D-branes in the presence of a constant B-field background appear to be noncommutative, as seen by the ends of open strings [6–9]. The theoretical and phenomenological implications of such noncommutative coordinates have been extensively studied [10]. In particular, it is argued that the spacetime background that emerges from the gravity theory based on canonical noncommutativity corresponds to a flat one [2–5].

One direction to extend studies on noncommutative spaces is to consider spaces where the commutators of the coordinates are not constants. Examples of this kind are the noncommutative cylinder and the $q$-deformed plane [11], the so-called $\kappa$-Poincaré algebra [12–15], and linear noncommutativity of the Lie algebra type [16,17]. In the latter the dimensionless spatial positions operators satisfy the commutation relations of a Lie algebra:

$$[\hat{x}_a, \hat{x}_b] = f_{a b}^{\ c} \hat{x}_c,$$  \hspace{1cm} (2)

where $f_{a b}^{\ c}$’s are structure constants of a Lie algebra. One example of this kind is the algebra SO(3), or SU(2). A special case of this is the so called fuzzy sphere [18,19], where an irreducible representation of the position operators is used which makes the Casimir of the algebra, $(\hat{x}_1)^2 + (\hat{x}_2)^2 + (\hat{x}_3)^2$, a multiple of the identity operator (a constant, hence the name sphere). One can consider the square root of this Casimir as the radius of the fuzzy sphere. This is, however, a noncommutative version of a two-dimensional space (sphere).

In [20–22] a model was introduced in which the representation was not restricted to an irreducible one, instead the whole group was employed. In particular the regular representation of the group was considered, which contains all representations. As a consequence in such models one is dealing with the whole space, rather than a sub-space, like the case of fuzzy sphere as a 2-dimensional surface. In [20] basic ingredients for calculus on a linear fuzzy space, as well as
basic notions for a field theory on such a space, were introduced. In [21] basic elements for calculating the matrix elements corresponding to transition between initial and final states were discussed. Models based on the regular representation of SU(2) were treated in more detail, giving explicit forms of the tools and notions introduced in their general forms [20, 21]. In [21] and [22] the tree and 1-loop diagrams for a self-interacting scalar field theory were discussed, respectively. It is observed that models based on Lie algebra type noncommutativity enjoy three features:

- They are free from any ultraviolet divergences if the group is compact.
- There is no momentum conservation in such theories.
- In the transition amplitudes only the so-called planar graphs contribute.

The reason for latter is that the non-planar graphs are proportional to $\delta$-distributions whose dimensions are less than their analogues coming from the planar sector, and so their contributions vanish in the infinite-volume limit usually taken in transition amplitudes [22].

The facts that in such theories the mass-shell condition is different, and there is no momentum conservation, lead to different consequences (with respect to ordinary theories) in collisions. This was exploited in [23], where it was seen that there may be a new threshold for the collision of two massless particles to produce massive particles.

The purpose of the present work is to examine the classical mechanics defined on space with SU(2) fuzziness. In particular, the Poisson structure induced by noncommutativity of SU(2) type is investigated, as well as the consequences of rotational symmetry in such spaces.

The scheme of the rest of this paper is the following. In section 2, the phase space corresponding to a space with Lie algebra type noncommutativity is studied. Specifically, the corresponding Poisson structure is investigated. In section 3, these are specialized to the group SU(2). In section 4 classical systems are studied which are SU(2)-invariant, and a formulation is presented to obtain the path of the particle. Section 5 is devoted to some examples.

2 The Poisson structure

Consider a Lie group G. Denote the members of a basis for the left-invariant vector fields corresponding to this group by $\hat{x}_a$’s. These fields satisfy (2), with the structure constants of the Lie algebra corresponding to G. The coordinates $\hat{k}^a$ are defined such that

$$U(\hat{k}) := [\exp(\hat{k}^a \hat{x}_a)] U(0),$$

where $U(\hat{k})$ is the group element corresponding to the coordinates $\hat{k}$, $U(0)$ is the identity, and $\exp(\hat{x})$ is the flux corresponding to the vector field $\hat{x}$. The action
of \( L_{\hat{x}_a} \) (the Lie derivative corresponding to the vector field \( \hat{x}_a \)) on an arbitrary scalar function \( F \) can be written like

\[
L_{\hat{x}_a}(F) = \hat{x}_a^b \frac{\partial F}{\partial k^b},
\]  
(4)

where \( \hat{x}_a^b \)'s are scalar functions, and satisfy

\[
\hat{x}_a^b(\mathbf{k} = \mathbf{0}) = \delta_a^b.
\]  
(5)

One can define the vector fields \( \hat{X}_a \) locally through

\[
L_{\hat{X}_a}(F) = \frac{\partial F}{\partial \hat{k}^a},
\]  
(6)

so that

\[
\hat{x}_a = \hat{x}_a^b \hat{X}^b.
\]  
(7)

Then, considering scalar functions as operators acting on scalar functions through simple multiplications, and vector fields as operators acting on scalar functions through Lie derivation, one arrives at the following commutation relations

\[
[\hat{X}_a, \hat{X}_b] = 0,
\]  
(8)

\[
[\hat{X}_a, \hat{k}^b] = \delta_a^b,
\]  
(9)

\[
[\hat{k}^a, \hat{k}^b] = 0.
\]  
(10)

One should, however, remember that the functions \( \hat{k}^a \) and the vector fields \( \hat{X}_a \) are only locally defined. One can write the above commutation relations in terms of \( \hat{x}_a \)'s instead of \( \hat{X}_a \)'s. The equation corresponding to (8) would be (2), while that corresponding to (9) would be

\[
[\hat{x}_a, \hat{k}^b] = \hat{x}_a^b,
\]  
(11)

and as \( \hat{x}_a^b \)'s are scalar functions, they commute with \( \hat{k}^a \)'s.

Next consider the right-invariant vector fields \( \hat{x}_a^R \), so that they coincide with their left-invariant analogues at the identity of the group:

\[
\hat{x}_a^R(\mathbf{k} = \mathbf{0}) = \hat{x}_a(\mathbf{k} = \mathbf{0}).
\]  
(12)

These field satisfy the commutation relations

\[
[\hat{x}_a^R, \hat{x}_b^R] = -f^c_{ab} \hat{x}_c^R,
\]  
(13)

\[
[\hat{x}_a^R, \hat{k}^b] = 0.
\]  
(14)

Using these, one defines the new vector field \( \hat{J}_a \) through

\[
\hat{J}_a := \hat{x}_a - \hat{x}_a^R.
\]  
(15)
These are the generators of the adjoint action, and satisfy the commutation relations

\[ [\hat{J}_a, \hat{J}_b] = f^{c}_{a b} \hat{J}_c, \]  
\[ [\hat{J}_a, \hat{X}_b] = f^{c}_{a b} \hat{X}_c, \]  
\[ [\hat{J}_a, \hat{x}_b] = f^{c}_{a b} \hat{x}_c, \]  
\[ [\hat{k}^c, \hat{J}_a] = f^{c}_{a b} \hat{k}^b. \]  

Equations (16), (17), and (19) show that

\[ \hat{J}_a = -f^{c}_{a b} \hat{k}^b \hat{X}_c. \]  

To construct the phase space, all that is needed is to transform the commutation relations to Poisson brackets. This can be done through the correspondence \([\ldots]/(i\hbar) \rightarrow \{\ldots\}\). However, one should also take care of the dimension of the quantities, and their reality. To do so, let us define the following quantities.

\[ p^a := (\hbar/\ell) \hat{k}^a, \]  
\[ X_a := i \ell \hat{X}_a, \]  
\[ x_a := i \ell \hat{x}_a, \]  
\[ x_a^b(p) := \hat{x}_a^b[(\ell/\hbar)p], \]  
\[ J_a := i \hbar \hat{J}_a, \]

where \(\ell\) is a constant of dimension length. One then arrives at the following Poisson brackets.

\[ \{p^a, p^b\} = 0, \]  
\[ \{X_a, p^b\} = \delta^b_a, \]  
\[ \{X_a, X_b\} = 0, \]  
\[ \{x_a, p^b\} = x_a^b, \]  
\[ \{x_a, x_b\} = \lambda f^{c}_{a b} x_c, \]  
\[ \{J_a, X_b\} = f^{c}_{a b} X_c, \]  
\[ \{J_a, x_b\} = f^{c}_{a b} x_c, \]  
\[ \{p^c, J_a\} = f^{c}_{a b} p^b, \]  
\[ \{J_a, J_b\} = f^{c}_{a b} J_c, \]

where the dimension of \(\lambda\) is that of inverse momentum:

\[ \lambda := \frac{\ell}{\hbar}. \]  

Using (5) it is seen that in the limit \(\lambda \to 0\) (corresponding to \(\ell \to 0\)), the ordinary Poisson brackets are retrieved.
3 The group SU(2), and the Euler parameters

For the group SU(2), one also can define the Euler parameters through

\[ \exp(\phi T_3) \exp(\theta T_2) \exp(\psi T_3) := \exp(k^a T_a), \quad (36) \]

where \( T_a \)’a are the generators of SU(2) satisfying the commutation relation

\[ [T_a, T_b] = \epsilon_{abc} T_c. \quad (37) \]

Using these, one arrives at

\[ L_{\hat{x}_1}(F) = -\frac{\cos \psi}{\sin \theta} \frac{\partial F}{\partial \phi} + \sin \psi \frac{\partial F}{\partial \theta} + \frac{\cos \psi \cos \theta}{\sin \theta} \frac{\partial F}{\partial \psi}, \quad (38) \]
\[ L_{\hat{x}_2}(F) = \frac{\sin \psi}{\sin \theta} \frac{\partial F}{\partial \phi} + \cos \psi \frac{\partial F}{\partial \theta} - \frac{\sin \psi \cos \theta}{\sin \theta} \frac{\partial F}{\partial \psi}, \quad (39) \]
\[ L_{\hat{x}_3}(F) = \frac{\partial F}{\partial \psi}, \quad (40) \]

and

\[ L_{\hat{J}_1}(F) = \frac{\cos \phi}{\sin \theta} \cos \theta \left( -\cos \psi \frac{\partial F}{\partial \phi} + \sin \phi \frac{\partial F}{\partial \theta} \right) \]
\[ + \frac{\cos \phi + \cos \psi \cos \theta}{\sin \theta} \frac{\partial F}{\partial \psi}, \quad (41) \]
\[ L_{\hat{J}_2}(F) = \frac{\sin \phi}{\sin \theta} \cos \theta \left( -\cos \psi \frac{\partial F}{\partial \phi} + \sin \phi \frac{\partial F}{\partial \theta} \right) \]
\[ + \frac{-\sin \phi - \sin \psi \cos \theta}{\sin \theta} \frac{\partial F}{\partial \psi}, \quad (42) \]
\[ L_{\hat{J}_3}(F) = -\frac{\partial F}{\partial \phi} + \frac{\partial F}{\partial \psi}, \quad (43) \]

for an arbitrary scalar field \( F \). Again, using the dimensionalization process of the previous section one arrives at

\[ x_1 = \lambda \left( -\frac{\cos \psi}{\sin \theta} X_\phi + \sin \psi X_\theta + \frac{\cos \psi \cos \theta}{\sin \theta} X_\psi \right), \quad (44) \]
\[ x_2 = \lambda \left( \frac{\sin \psi}{\sin \theta} X_\phi + \cos \psi X_\theta - \frac{\sin \psi \cos \theta}{\sin \theta} X_\psi \right), \quad (45) \]
\[ x_3 = \lambda X_\psi, \quad (46) \]
\[ J_1 = \frac{\cos \phi}{\sin \theta} \cos \theta \left( -\cos \psi X_\phi + \sin \phi X_\theta \right) \]
\[ + \frac{-\cos \phi + \cos \psi \cos \theta}{\sin \theta} X_\psi, \quad (47) \]
\[ J_2 = \frac{\sin \phi}{\sin \theta} \cos \theta \left( -\cos \psi X_\phi + \sin \phi X_\theta \right) \]
\[ + \frac{-\sin \phi - \sin \psi \cos \theta}{\sin \theta} X_\psi, \quad (48) \]
\[ J_3 = -X_\phi + X_\psi, \quad (49) \]
where $\phi$, $\theta$, and $\psi$ are the canonical momenta corresponding to the canonical coordinates $X_\phi$, $X_\theta$, and $X_\psi$, respectively. $\phi$, $\theta$, and $\psi$ are dimensionless, while $X_\phi$, $X_\theta$, and $X_\psi$ have the dimension of action.

One also has

$$\cos \frac{\theta}{2} = \cos \frac{\phi + \psi}{2},$$

or in terms of the dimensionful quantities

$$\cos \frac{\lambda p}{2} = \cos \frac{\phi + \psi}{2},$$

where

$$\hat{k} := (\delta_{ab} \hat{k}^a \hat{k}^b)^{1/2},$$

$$p := (\delta_{ab} p^a p^b)^{1/2}.$$

One could obtain $\hat{x}_a$’s in a different way, trying to use only (2) and (5). It turns out that these are not sufficient to determine $\hat{x}_a$’s uniquely [17]. However, defining $\hat{x}_a$’s as the left invariant vector fields of the group manifold determines them uniquely. It can be shown that adding

$$[\hat{x}_a, \hat{k}] = \frac{\hat{k}_a}{\hat{k}},$$

to (2) and (5), completely determines $\hat{x}_a$’s. One then arrives at

$$\hat{x}^a = \frac{\hat{k}}{2} \cot \frac{\hat{k}}{2} \delta^a_b + \left(1 - \frac{\hat{k}}{2} \cot \frac{\hat{k}}{2}\right) \frac{\hat{k}_a \hat{k}_b}{\hat{k}^2} + \frac{1}{2} \epsilon^{abc} \hat{k}_c,$$

which is equivalent to (38)-(40).

### 4 SU(2)-invariant classical systems

Consider a configuration space with linear SU(2)-fuzziness and its corresponding phase space. This is like what introduced in section 2, with $f$ equal to $\epsilon$. A classical system which is characterized by a Hamiltonian $H$, is said to be SU(2)-invariant, if $H$ is SU(2)-invariant, that is if the Poisson brackets of $H$ with $J_a$’s vanish. A Hamiltonian which is a function of only $(p \cdot p)$ and $(x \cdot x)$ is clearly so, where

$$A \cdot B := \delta_{ab} A^a B^b.$$

From now on, assume that the system is so, that is the Hamiltonian is a function of only $(p \cdot p)$ and $(x \cdot x)$. Then $J$ is a constant vector and one can choose the axes so that the third axis is parallel to this vector:

$$J_1 = 0,$$

$$J_2 = 0.$$
Solving these, one arrives at
\[ \phi + \psi = 0, \]
\[ X_\phi + X_\psi = 0. \]  
(57)

Defining
\[ \frac{\phi - \psi}{2} =: \chi, \]
\[ X_\phi - X_\psi = J, \]  
(58)

it is seen by (49) that
\[ \{J, \chi\} = 1, \]  
(59)

and that \( J \) and \( \chi \) are in involution with \( X_\theta \) and \( \theta \), so that \( (X_\theta, J, \theta, \chi) \) are canonical coordinates left after applying (57). Applying (57), one arrives at
\[ x_1 = \lambda \left( -\frac{J}{2} \frac{1 + \cos \theta}{\sin \theta} \cos \chi - X_\theta \sin \chi \right), \]
\[ x_2 = \lambda \left( -\frac{J}{2} \frac{1 + \cos \theta}{\sin \theta} \sin \chi + X_\theta \cos \chi \right), \]
\[ x_3 = \frac{J}{2}, \]
\[ \mathbf{x} \cdot \mathbf{x} = \lambda^2 \left[ X_\theta^2 + \frac{J^2}{4} \left( 1 + \cot^2 \frac{\theta}{2} \right) \right], \]
\[ \cos \frac{k}{2} = \cos \frac{\theta}{2}. \]  
(60)

It is seen that the motion is not in the plane \( x_3 = 0 \), but in a plane parallel to that, as \( x_3 \) does not vanish but is a constant.

Defining \( \rho \) and \( \alpha \) (the so called polar coordinates corresponding to \( x_1 \) and \( x_2 \)) as
\[ x_1 =: \rho \cos \alpha, \]
\[ x_2 =: \rho \sin \alpha, \]  
(61)

one has
\[ \mathbf{x} \cdot \mathbf{x} = \rho^2 + \frac{\lambda^2 J^2}{4}, \]
\[ \rho^2 = \lambda^2 (X_\theta^2 + J^2 u^2), \]
\[ \alpha = \chi - \tan^{-1} \frac{X_\theta}{J u}, \]  
(62)

where
\[ u := \frac{1}{2} \cot \frac{\theta}{2}. \]  
(63)
One then has
\[ x \cdot x = \lambda^2 \left[ X_\theta^2 + J^2 \left( \frac{1}{4} + u^2 \right) \right]. \] (64)

The Hamiltonian is a function of only \( u \) and \( (x \cdot x) \), where \( (x \cdot x) \) itself contains \( J \) (a constant of motion), \( u \), and \( X_\theta \). Our aim is now to reduce the problem to that of a system of one degree of freedom. The aim is to write an equation for the path, in terms of \( \rho \) and \( \alpha \). To do so, one notices by (64) that

\[ \frac{du}{dt} = \{ u, H \}, \]
\[ = \lambda^2 \frac{\partial H}{\partial (x \cdot x)} (2 X_\theta) \left( \frac{1}{4} + u^2 \right), \] (65)

and

\[ \frac{d\chi}{dt} = \{ \chi, H \}, \]
\[ = -\lambda^2 \frac{\partial H}{\partial (x \cdot x)} (2 J) \left( \frac{1}{4} + u^2 \right). \] (66)

Using these, one arrives at

\[ X_\theta = -J \frac{du}{d\chi}, \] (67)

so that

\[ \rho^2 = \lambda^2 J^2 \left[ \left( \frac{du}{d\chi} \right)^2 + u^2 \right], \] (68)
\[ \alpha = \chi + \tan^{-1} \left( \frac{1}{u} \frac{du}{d\chi} \right). \] (69)

Using these, one arrives at

\[ \frac{d\rho}{d\chi} = \lambda^2 \frac{J^2}{\rho} \frac{du}{d\chi} \left( u + \frac{d^2 u}{d\chi^2} \right), \]
\[ \frac{d\alpha}{d\chi} = u \left( u + \frac{d^2 u}{d\chi^2} \right) \left[ u^2 + \left( \frac{du}{d\chi} \right)^2 \right]^{-1}, \]
\[ = \lambda^2 \frac{J^2}{\rho^2} u \left( u + \frac{d^2 u}{d\chi^2} \right), \] (70)

from which

\[ \frac{d\rho}{d\alpha} = \frac{\rho}{u} \frac{du}{d\chi}. \] (71)

Using this and (68), one can eliminate \( (du/d\chi) \) and arrive at

\[ \frac{1}{u^2} = \lambda^2 J^2 \left[ \frac{1}{\rho^2} + \frac{1}{\rho^4} \left( \frac{d\rho}{d\alpha} \right)^2 \right]. \] (72)
The Hamiltonian is a function of \( \hat{k} \) and \((x \cdot x)\). One can express it in terms of \( u \) and \((x \cdot x)\), and arrive at

\[
H \left\{ \frac{1}{u^2} = \lambda^2 J^2 \left[ \frac{1}{\rho^2} + \frac{1}{\rho^3} \left( \frac{d\rho}{d\alpha} \right)^2 \right], x \cdot x = \rho^2 + \frac{\lambda^2 J^2}{4} \right\} = E, \quad (73)
\]

where \( E \) is the energy. This is a first order differential equation for the path of the system, which should be compared to the corresponding commutative case \((\lambda \to 0)\):

\[
H \left\{ p \cdot p = J^2 \left[ \frac{1}{\rho^2} + \frac{1}{\rho^3} \left( \frac{d\rho}{d\alpha} \right)^2 \right], x \cdot x = \rho^2 \right\} = E, \quad (74)
\]

where it is understood that in the commutative limit \((\lambda u)\) has been kept constant (equal to the inverse of the momentum).

A special situation is when the angular momentum vanishes. In this case the differential equation \((73)\) is not a good equation, as \( J \) vanishes and \((d\rho/d\alpha)\) goes to infinity. Using \((66)\) it is seen that \( \chi \) is constant, from which and \((69)\) it turns out that \( \alpha \) is constant as well. So \( \phi \) and \( \psi \) are constants and \( X_\phi \) and \( X_\psi \) vanish. One then arrives at an effectively one-degree-of-freedom Hamiltonian (involving \( \theta \) and \( X_\theta \)), from which a relation between \( \theta \) and \( X_\theta \) is found to be

\[
H(u, x \cdot x = \lambda^2 X_\theta^2) = E. \quad (75)
\]

One can also write a first-order differential equation for \( \theta \), using \((65)\). To do so, one obtains \( X_\theta \) from \((65)\), and inserts it in \((75)\).

### 5 Examples

Here we consider a case where the Hamiltonian is the sum of a Kinetic term, which is a function of only \( p \), and a potential term, which is a function of only \((x \cdot x)\). Following \([20–22]\), the kinetic term is taken to be

\[
K = \frac{4}{\lambda^2 m} \left( 1 - \cos \frac{\lambda p}{2} \right) \quad (76)
\]

for a particle of mass \( m \). This function is increasing in \( p \), as long as \( p \) does not exceed \((2 \pi/\lambda)\), and periodic in \( p \) with the period \((4 \pi/\lambda)\), showing that it is a function of the group manifold \( SU(2) \). It is also reduced to \( p^2/(2 m) \) for small values of \( p \), which shows that for small momenta the commutative results are obtained. One can express this kinetic term in terms of \( u \):

\[
K = \frac{4}{\lambda^2 m} \left( 1 - \frac{2 u}{\sqrt{1 + 4 u^2}} \right). \quad (77)
\]

For a free particle, the Hamiltonian is equal to the kinetic term, so that \( u \) is a constant. Then using \((72)\) one arrives at

\[
(r^{-1})^2 + \left( \frac{d\rho^{-1}}{d\alpha} \right)^2 = C^{-2}, \quad (78)
\]
where $C$ is a constant. The solution to (79) is
\[ \rho \cos(\alpha - \alpha_0) = C, \quad (79) \]
where $\alpha_0$ is another constant. This, combined with the fact that $x_3$ is a non-vanishing constant, describes a line in a plane parallel to the plane $x_3 = 0$. For a commutative space, one would obtain a line in the plane $x_3 = 0$.

If the particle is not free, one can still use (73) to obtain a differential equation for $\rho$, similar to the equation of the commutative case. To do so, one obtains $u$ in terms of $K$:
\[ \frac{1}{u^2} = 4 \left[ \left( 1 - \frac{\lambda^2 m}{4} K \right)^{-2} - 1 \right]. \quad (80) \]
Expressing the kinetic term in terms of the energy and the potential term, and using (73), one arrives at
\[ \frac{1}{\rho^2} + \frac{1}{\rho^4} \left( \frac{d\rho}{d\alpha} \right)^2 = \frac{4}{\lambda^2 J^2} \left\{ \left[ 1 - \frac{\lambda^2 m}{4} (E - V) \right]^{-2} - 1 \right\}, \quad (81) \]
where $V$ is the potential energy which is a function of only $(x \cdot x)$. Also note that $(x \cdot x)$ is not equal to $\rho^2$, but is obtained from (62).

A special case is when the angular momentum vanishes. Then, using (76) one arrives at
\[ H = \frac{4}{\lambda^2 m} \left( 1 - \cos \frac{\lambda p}{2} \right) + V(x \cdot x), \quad (82) \]
where
\[ x := \lambda X_\theta, \quad (83) \]
showing that
\[ \{x, p\} = 1. \quad (84) \]
Equations (82) and (84) describe a one-degree-of-freedom system, the only difference of which with the commutative case is in the kinetic term. Of course it is also known that the coordinate $p$ is a periodic one, with the period $(4 \pi/\lambda)$.

An example is the case of a simple harmonic potential:
\[ V = \frac{1}{2} m \omega^2 x^2, \quad (85) \]
where $\omega$ is a constant. One then has
\[ H = \frac{4}{\lambda^2 m} \left( 1 - \cos \frac{\lambda p}{2} \right) + \frac{1}{2} m \omega^2 x^2, \quad (86) \]
which is like the Hamiltonian of a simple pendulum, with the roles of $x$ and $p$ interchanged.

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