Comments on the Gribov Ambiguity

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ABSTRACT

We discuss the existence of Gribov ambiguities in $SU(m) \times U(1)$ gauge theories over the $n$—spheres. We achieve our goal by showing that there is exactly one conjugacy class of groups of gauge transformations for the theories given above. This implies that these transformation groups are conjugate to the ones of the trivial $SU(m) \times U(1)$ fiber bundles over the $n$—spheres. By using properties of the space of maps $Map_*(S^n, G)$ where $G$ is one of $U(1)$, $SU(m)$ we are able to determine the homotopy type of the groups of gauge transformations in terms of the homotopy groups of $G$. The non-triviality of these homotopy groups gives the desired result.

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1. Introduction

Despite their striking successes, non-Abelian gauge theories are far from being well-understood. Most of the progress that has been done is within the region of validity of perturbation theory. Very few things are known about non-perturbative effects, by comparison. Although this lack of understanding is also a problem for QED, it becomes a much greater subject of concern for non-Abelian gauge theories, in particular for the ones that describe such striking physical effects as confinement. The confinement problem is one of enormous importance in QCD. It has become obvious, through extensive studies over a number of years, that this is a genuinely non-perturbative effect. Therefore, its solution requires us to go beyond perturbation theory, in a regime which is virtually unexplored both physically and mathematically.

One of the many characteristics of the non-Abelian gauge theories is the existence of Gribov ambiguities [1]. This is the fact that the Coulomb or Lorentz gauges (or more generally any covariant gauge) fails to globally eliminate the spurious degrees of freedom of a theory defined over a $3$– or $4$– sphere [2]. This effect does not appear when we use the axial gauge (or more generally, algebraic gauges) but in that case we cannot compactify our space or spacetime to a sphere. The physical implications of the existence of the Gribov ambiguities are not known. Despite the early claims, that this phenomenon could provide a solution to the confinement problem [1], [12], no definitive proof has ever been presented, and it is not unreasonable to say that the problem remains still open. It is worthwhile, therefore, to further explore the meaning of the Gribov ambiguity both physically and mathematically since it is one of the very few known non-perturbative effects of the non-Abelian gauge theories.

In this paper we present some mathematical arguments for the existence of Gribov copies of an $SU(m) \times U(1)$ gauge theories over $n$–spheres. In section 2 we present the statement of the Gribov problem. In section 3 we discuss the group of automorphisms (group of gauge transformations) of some principal fiber bundles and the associated conjugacy classes. In section 4 we apply these methods to explicitly determine the homotopy type of the automorphism groups of $SU(m) \times U(1)$ principal fiber bundles over $n$–spheres in terms of the homotopy groups of $SU(m)$ and $U(1)$ respectively. In section 5 we present our conclusions.
2. The Gribov Problem

In classical electrodynamics on $R^4$, the condition $\partial^\mu A_\mu = 0$ fixes the gauge completely. The Lorenz gauge completely eliminates all the redundant degrees of freedom. Under a gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$$

$A'_\mu$ satisfies the Lorenz condition provided that $\partial^\mu \partial_\mu \Lambda = 0$. In order to have a finite energy configuration we impose the following boundary condition at infinity $A_\mu \to \infty \frac{1}{r}$ (we also assume that $A_\mu$ is regular everywhere). This allows us to treat infinity as a point. So, from the viewpoint of dynamics the space on which our theory is defined is $S^4$ (one-point compactification of $R^4$). More generally, to assure finiteness of the energy of the system we confine ourselves to a compact submanifold of $R^4$ with boundary and we are imposing vanishing boundary conditions for the fields. In that case due to the ellipticity of $\partial^\mu \partial_\mu \Lambda = 0$ the only solution is $\Lambda = 0$. We see that by imposing the Lorenz gauge on a compact space with boundary we have completely eliminated the spurious degrees of freedom.

The situation is entirely different in non-Abelian gauge theories. In that case a gauge transformation is

$$A^a_\mu \to A^a_\mu + D^a_{\mu} \Lambda^c,$$  
$$D^a_{\mu} = \partial_\mu \delta^{ac} + g f^{abc} A^c_\mu$$

When we impose the Lorenz gauge condition on the new and the transformed gauge potential we get

$$\partial^\mu \partial_\mu \Lambda^a + g f^{abc} A^b_\mu \partial^\mu \Lambda^c = 0$$

Gribov [1] proved that the above equation admits non-trivial regular solutions $\Lambda^c$ for large values of $A^a_\mu$. This means that the Lorenz condition does not eliminate all the spurious degrees of freedom in a non-Abelian gauge theory defined over a compact space. There is no trouble with this effect in case we are interested solely in perturbative effects. In a non-perturbative treatment however this phenomenon becomes important.

To describe precisely, in a geometric way, the Gribov ambiguity let’s consider a Riemannian manifold $M$ with Euclidean signature metric $g_{ab}$ and let $G$ be a gauge group. Let $P$ be a principal fiber bundle with typical fiber $G$ over $M$ and let $\pi : P \to M$ be the canonical projection of $P$ onto $M$. Following [3],[4] we introduce the associated to $P$ bundle of groups $AdP = P \times_G G$ where $G$ acts on $G$ through the adjoint action and the associated bundle of
Lie algebras $adP = P \times_G LieG$ where $LieG$ is the Lie algebra of $G$ and the action of $G$ on $LieG$ is still the adjoint action. By a connection (gauge potential) we mean a $LieG$ algebra valued one-form on $P$. (Since we are not primarily interested in the functional analytic aspects of the problem we assume from now on that all fields and maps are $C^\infty$. A careful treatment when this is not the case is presented in [4],[5]). For any two such gauge potentials $A_1$ and $A_2$ their difference $A_1 - A_2$ is pulled back on $M$ using $\pi$. This means that

$$A_1 - A_2 \in \Omega^1(M; adP) = \Gamma(T^*M \otimes adP)$$

Let us denote by $A$ the space of gauge potentials. We see that it is an affine space modelled over the vector space $\Omega^1(M; adP)$. By an automorphism of $P$ we mean a diffeomorphism $f : P \to P$ which preserves the right action of $G$ on $P$ i.e. $f(pg) = f(p)g$, $p \in P$, $g \in G$. The space of gauge transformations is the space of sections (with pointwise multiplication) of $AdP$ and it has been proved to form a Lie group i.e.

$$\mathcal{G} = \Gamma(AdP)$$

Let $f \in \mathcal{G}$ and $A \in A$. Then we have the following action

$$A \ni A \mapsto f^*A \in A$$

The problem is that this action is generally not free. This means that there are connections $A$ such that $fA = A$ with $f \neq id$. Non-free actions are the origin of orbifold-type singularities which are quite complicated to handle. In order to simplify technically our treatment we do not consider the space of connections but instead the space of framed connections. A framed connection [6] in $P$ is a pair $(A, \phi)$ where $A$ is a connection on $P$ and $\phi$ is an isomorphism of $G-$spaces $\phi : G \to P_{x_0}$. The group of gauge transformations acts naturally on the framed connections giving the moduli space

$$\tilde{A} = (A \times Hom(G, P_{x_0}))/\mathcal{G}$$

An alternative way of thinking about $\tilde{A}$ is to fix the framing $\phi$ and define $\mathcal{G}_* \subset \mathcal{G}$ to be the isotropy group at point $p_0$ i.e.

$$\mathcal{G}_* = \{f \in \mathcal{G} : f(p_0) = 1, \quad p_0 \in P\} \quad (2)$$

Then $\tilde{A} = A/\mathcal{G}_*$. The isotropy group at $A \in A$ is $\Gamma_A = \{f \in \mathcal{G} : fA = A\}$. Since $fA = A - (D_Af)f^{-1}$ we see that $\Gamma_A = \{f \in \mathcal{G} : D_Af = 0\}$. Because the isotropy group at
A consists of covariantly constant gauge transformations the subgroup $G_*$ acts freely on $\mathcal{A}$. Actually it has been proved that $\tilde{\mathcal{A}}$ is a Hilbert manifold [4],[5]. Physically the point $x_0$ can be chosen to be infinity. Then $G_*$ is the group of gauge transformations that are identity at infinity. Finally, we note that $\tilde{\mathcal{A}}$ is a principal $G_*$ bundle over $\tilde{\mathcal{A}}/G_*$. 

Gauge fixing means picking one connection from each orbit of $\tilde{\mathcal{A}}/G_*$. This corresponds to a section $s : \tilde{\mathcal{A}}/G_* \rightarrow \tilde{\mathcal{A}}$ such that $\pi \circ s = id$. If gauge fixing is possible, we will be able to define a global section $s$ satisfying these properties i.e. the corresponding fiber bundle $\tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}/G_*$ will be trivial. Then $\tilde{\mathcal{A}} \approx \tilde{\mathcal{A}}/G_* \times G_*$. For some positive or zero $q$

$$\pi_q(\tilde{\mathcal{A}}) \approx \pi_q(\tilde{\mathcal{A}}/G_*) \otimes \pi_q(G_*) \quad (3)$$

The space $\tilde{\mathcal{A}}$ is an affine space and as such it is contractible. Therefore $\pi_q(\tilde{\mathcal{A}}) = 0$. So, if we are able to fix a gauge we must have

$$0 \approx \pi_q(\tilde{\mathcal{A}}/G_*) \otimes \pi_q(G_*) \quad (4)$$

In order for this condition to hold for any $q$, the homotopy groups of the right-hand side should be vanishing. By examining in detail the homotopy groups of $G_*$ we will prove that this is not the case thus establishing the impossibility of gauge fixing.

3. AUTOMORPHISM GROUPS OF PRINCIPAL FIBER BUNDLES

In this section we present the general formalism for determining the conjugacy classes of the group of gauge transformations [4]. Assume that $M$ is a manifold, $P$ a principal $G$–bundle over $M$ with the projection $\pi : P \rightarrow M$. We denote the space of gauge transformations of this bundle by $G_*(\pi)$. This notation indicates that we consider the base manifold $M$ as well as the typical fiber $G$ fixed and we only vary the projection $\pi$. In this way we are able to obtain all the possible $G$– bundles over $M$. As we mentioned before, $G_*(\pi)$ is a topological group with topology induced from the compact-open topology of the space $Map_*(P, P)$. Let $\{U_\alpha, \alpha \in I\}$ be a covering of $M$ and $\{(U_\alpha, \phi_\alpha), \alpha \in I\}$ a chart. For every $x \in U_\alpha$ let $\phi_{\alpha x} : G \rightarrow \pi^{-1}(x) = P_x$ be the map $\phi_{\alpha x}(g) = \phi_\alpha(x, g)$. Define $\theta_{U_\alpha} : G(\pi_\alpha) \rightarrow Map_*(U_\alpha, G)$ by the condition

$$\theta_{U_\alpha} = \phi_{\alpha x}^{-1} \circ f_x \circ \phi_{\alpha x} \quad (5)$$
where $f$ denotes the right action of $G$ on $P$, $f_x$ the restriction of $f$ to the fiber $P_x$ and $\pi_\alpha$ the restriction of $\pi$ to the chart $U_\alpha \times G$. Locally every fiber bundle is trivial, so the map $\theta_{U_\alpha}$ is an isomorphism of topological groups. Let’s consider the group

$$\prod_{\alpha \in I} \text{Map}_*(U_\alpha, G)$$

and for every $\alpha \in I$ the restriction map $r_\alpha : \mathcal{G}(\pi) \to \mathcal{G}(\pi_\alpha)$. We define the function

$$\theta : \mathcal{G}(\pi) \to \prod_{\alpha \in I} \text{Map}_*(U_\alpha, G)$$

with $\theta = \{\theta_{U_\alpha} \cdot r_\alpha\}_{\alpha \in I}$. $\theta$ is an embedding of topological groups which shows that the group $\mathcal{G}(\pi)$ corresponding to the fibration $P \xrightarrow{\pi} M$ can be considered as a subgroup of

$$\prod_{\alpha \in I} \text{Map}_*(U_\alpha, G)$$

$\mathcal{G}(\pi)$ divides $\prod_{\alpha \in I} \text{Map}_*(U_\alpha, G)$ into conjugacy classes each of which is denoted by $\overline{\mathcal{G}(\pi)}$ and $C(M, G)$ denotes the set of all conjugacy classes of the groups $\mathcal{G}(\pi)$. Let $k : M \to BG$ be a classifying map for $(P, \pi, M)$, $i$ be the isomorphism $i : G/ZG \simeq \text{In}(G)$ where $\text{In}(G)$ is the group of inner automorphisms of $G$ and $l$ the quotient homomorphism $l : G \to G/ZG$ (where $ZG$ denotes the center of $G$). Consider the map $\eta : G \to \text{In}(G)$ defined by $\eta = i \circ l$ and let $B\eta : BG \to B\text{In}(G)$ be the induced map at the level of the classifying spaces $BG$ and $B\text{In}(G)$. Assume that $G$ has the form $A \times K$ where $A$ is a path-connected Abelian group ($U(1)$ in the finite dimensional case) and $K$ any path-connected group with trivial center. Let $(B\eta)_*$ be the map

$$(B\eta)_* : [M, BG]_* \to [M, B\text{In}(G)]_*$$

induced by $B\eta$ on the homotopy classes of based maps $\text{Map}_*(M, BG)$ and $\text{Map}_*(M, B\text{In}(G))$. If moreover $\text{card}[M, B\text{In}(G)]_* = 1$ then the map $\Xi : C(M, G) \to [M, B\text{In}(G)]_*$ defined by

$$\Xi(\overline{\mathcal{G}(\pi)}) = (B\eta) \circ k$$

is a bijection. The previous theorem establishes a very interesting correspondence. Generally, it is very hard to compute $C(M, G)$ directly. Instead, when the conditions of the theorem hold, we have to compute the homotopy classes of maps $[M, B\text{In}(G)]_*$ which is by comparison a much easier space to handle. One very interesting consequence of the above theorem is in the case in which $[M, B\text{In}(G)]_*$ is trivial. Then $C(M, G)$ has only one element.
In that case the “twisted” fiber bundle with projection map \( P \xrightarrow{\pi} M \) has an automorphism group which is conjugate to that of the trivial fiber bundle \( P \xrightarrow{\pi_0} M \). But it is known that in the latter case the group of gauge transformations is the group \( \text{Map}_*(M, G) \). By that we have succeeded in reducing the calculation of the group of gauge transformations of the fiber bundle \( \pi : P \to M \) to the computation of the space \( \text{Map}_*(M, G) \) which is much more manageable. This is the strategy that we follow in the next section to compute the homotopy type of the automorphism group of \( SU(m) \times U(1) \).

4. **Application to \( SU(m) \times U(1) \) over \( S^n \)**

Let \( G = SU(m) \times U(1) \) and \( M = S^n \). We have the isomorphisms

\[
[S^n, B(SU(m) \times U(1))]_* \simeq [S^{n-1}, SU(m) \times U(1)]_*
\]

By definition

\[
[S^{n-1}, SU(m) \times U(1)]_* \equiv \pi_{n-1}(SU(m) \times U(1)) \simeq \pi_{n-1}(SU(m)) \otimes \pi_{n-1}(U(1))
\]

For \( n = 3 \) we get \( \pi_2(SU(m)) \simeq 0 \)

For \( n = 4 \) we get \( \pi_3(SU(m)) \simeq \mathbb{Z} \)

Generally we can prove that \( \pi_n(U(1)) = 0, \quad n > 1 \). Taking this into account

\[
[S^{n-1}, SU(m) \times U(1)]_* \simeq \pi_{n-1}(SU(m))
\]

We also know that

\[
[S^n, BIn(SU(m) \times U(1))]_* \simeq [S^{n-1}, In(SU(m) \times U(1))]_*
\]

and this means that

\[
[S^{n-1}, In(SU(m) \times U(1))]_* \simeq \pi_{n-1}(In(SU(m) \times U(1))]
\]

But

\[
Z(SU(m) \times U(1)) \simeq \mathbb{Z}_m \times U(1)
\]

Therefore

\[
In(SU(m) \times U(1)) \simeq (SU(m) \times U(1))/(\mathbb{Z}_m \times U(1))
\]
To compute
\[ \pi_n((SU(m) \times U(1))/(Z_m \times U(1))) \]
we observe that \( SU(m) \times U(1) \) is a covering space for \((SU(m) \times U(1))/(Z_m \times U(1))\). According to a theorem [8] when this is the case we get
\[ \pi_{n-1}((SU(m) \times U(1))/(Z_m \times U(1))) \otimes \pi_{n-1}(Z_m \times U(1)) \simeq \pi_{n-1}(SU(m) \times U(1)), \quad n > 3 \]
With the compact-open topology \( \pi_{n-1}(Z_m) \simeq 0 \) and \( \pi_{n-1}(U(1)) \simeq 0 \) for these values of \( n \). Therefore
\[ \pi_{n-1}((SU(m) \times U(1))/(Z_m \times U(1))) \simeq \pi_{n-1}((SU(m) \times U(1))) \simeq \pi_{n-1}(SU(m)) \]
Then we observe that the map
\[ (B\eta)_* : [S^n, B(SU(m) \times U(1))]_{*} \rightarrow [S^n, BIn(SU(m) \times U(1))]_{*} \]
is the trivial map so \( cardC(S^n, SU(m) \times U(1)) = 1 \). This implies, according to the theorem stated above, that \( G(\pi) \) is conjugate to \( G(\pi_0) \) or that
\[ G(\pi) = Map_{*}(S^n, SU(m) \times U(1)) \] (8)
Next we prove that the space \( Map_{*}(S^n, SU(m) \times U(1)) \) is not contractible. We begin by noticing that in the compact-open topology
\[ Map_{*}(S^n, SU(m) \times U(1)) \simeq Map_{*}(S^n, SU(m)) \times Map_{*}(S^n, U(1)) \]
where “\( \simeq \)” denotes homeomorphism. To reach our goal, it suffices to prove that at least one homotopy group of \( Map_{*}(S^n, SU(m) \times U(1)) \) is non-trivial. By applying the homotopy functor \( \pi_k \) to the previous equation we get
\[ \pi_k(Map_{*}(SU(m) \times U(1))) \simeq \pi_k(Map_{*}(S^n, SU(m)) \times Map_{*}(S^n, U(1))) \]
From a known property of the homotopy functor for the Cartesian product of topological spaces, this is equal to
\[ \pi_k(Map_{*}(SU(m) \times U(1))) \simeq \pi_k(Map_{*}(S^n, SU(m))) \otimes \pi_k(Map_{*}(S^n, U(1))) \] (9)
By definition
\[ \pi_k(Map_{*}(S^n, U(1))) \equiv [S^k, Map_{*}(S^n, U(1))]_{*} \]
As a topological space $U(1)$ is $S^1$, therefore

$$\pi_k(Map_*(S^n, U(1))) \equiv [S^k, Map_*(S^n, S^1)]_*$$

The topological spaces $S^n$, with the compact-open topology, are compact so they are compactly generated. This does not guarantee that the spaces $Map_*(S^n, S^1)$ equipped with the compact-open topology are also compactly generated. In order to make them so, we retopologize $Map_*(S^n, S^1)$ by applying to it the functor $k : Top_* \to Comp$ where $Top_*$ is the category of pointed topological spaces and $Comp$ is the category of compactly generated spaces. The functor $k$ is defined as follows [8]: let $X$ be a topological space. $k(X)$ and $X$ have the same underlying set. Let $A$ be a subset of $X$. Then $A$ is closed in $k(X)$ if and only if $A \cap C$ is closed in $X$ for every compact subset $C$ of $X$. By $Map_*(\ , \ )$ in the sequel we mean the element $kMap_*(\ , \ )$ in $Comp$. We also have by definition that

$$[S^k, Map_*(S^n, S^1)] \approx \pi_0(Map_*(S^k, Map_*(S^n, S^1)))$$

In the category $Comp$ we know that

$$Map_*(S^n, Map_*(S^n, S^1)) \approx Map_*(S^k \wedge S^n, S^1)$$

where "\~" is a natural homeomorphism. The symbol "\^" denotes the smash product of the topological spaces $S^k$ and $S^n$. From the definition of the smash product we see that

$$S^k \wedge S^n \approx S^{k+n}$$

Therefore

$$[S^k, Map_*(S^n, S^1)]_* \approx [S^{k+n}, S^1]_* \approx \pi_{k+n}(S^1)$$

Then

$$\pi_k(Map_*(S^n, U(1))) \approx 0, \quad k + n > 1 \quad (10)$$

So, the second factor of the right-hand side of equation is trivial. In order to compute the homotopy type of the first factor of equation we follow an almost identical procedure. In this case we use the fact that for any topological group $G$ the following homeomorhism is true

$$G \approx \Omega BG$$

where $BG$ is the classifying space of $G$ and $\Omega X$ denotes the loop space of a topological space $X$. Remembering that, we have the following equivalence

$$Map_*(S^n, SU(m)) \approx Map_*(S^n, \Omega BSU(m))$$
It is known though, that in the category of compactly generated spaces the functors \( \Sigma \) and \( \Omega \) form an adjoint pair e.g. the following spaces are naturally homeomorphic

\[
\text{Map}_*(S^n, \Omega SU(m)) \simeq \text{Map}_*(\Sigma S^n, BSU(m))
\]

where \( \Sigma \) is the (reduced) suspension functor. From the definition of the (reduced) suspension we can see that \( \Sigma S^n \approx S^{n+1} \). Therefore

\[
\text{Map}_*(S^n, SU(m)) \simeq \text{Map}_*(S^{n+1}, BSU(m))
\]

We want to calculate

\[
\pi_k(\text{Map}_*(S^{n+1}, SU(m))) \equiv [S^k, \text{Map}_*(S^{n+1}, SU(m))]_*
\]

Using the procedure described above for the case of \( \text{Map}_*(S^n, U(1)) \) we get

\[
\pi_k(\text{Map}_*(S^{n+1}, BSU(m))) \simeq [S^{k+n+1}, BSU(m)]_* \simeq \pi_{k+n+1}(BSU(m))
\]

But from the definition of the loop functor \( \Omega \) it follows that

\[
\pi_i(SU(m)) \simeq \pi_i(\Omega SU(m)) \simeq \pi_{i+1}(BSU(m))
\]

So

\[
\pi_k(\text{Map}_*(S^{n+1}, BSU(m))) \simeq \pi_{k+n}(SU(m)) \quad (11)
\]

Combining equations (9), (10), (11) we find

\[
\pi_k(\text{Map}_*(S^n, SU(m) \times U(1))) \simeq \pi_{k+n}(SU(m)) \quad (12)
\]

This means that the space of automorphisms of the principal fiber bundle with typical fiber \( SU(m) \times U(1) \) over \( S^n \) has the same homotopy groups as \( SU(m) \). The only difference is the shift of the index of the homotopy groups of \( SU(m) \) by \( n \).

The physically interesting theory is the \( SU(2) \times U(1) \) (the Glashow-Salam-Weinberg model of the electroweak interactions) and potentially an \( SU(m) \times U(1) \) theory (probably as a byproduct of a grand unified theory). The physically interesting dimensions of spacetimes are \( n = 3 \) (corresponding to a one-point compactified spatial slice of a Lorentz manifold) and \( n = 4 \) (corresponding to the one-point compactification of \( R^4 \)). So for the physically interesting cases we have

\[
\pi_1(\text{Map}_*(S^3, SU(2) \times U(1))) \simeq \pi_4(SU(2)) \simeq Z_2 \quad (13)
\]
\[ \pi_1(\text{Map}_*(S^4, SU(2) \times U(1))) \simeq \pi_5(SU(2)) \simeq \mathbb{Z}_2 \]  

Therefore the spaces \( \text{Map}_*(S^3, SU(2) \times U(1)) \) and \( \text{Map}_*(S^4, SU(2) \times U(1)) \) are not contractible. This means that any \( SU(2) \times U(1) \) theory over \( S^3 \) and \( S^4 \) has Gribov ambiguities.

We also see that by using equation (12) we can prove that many more theories with gauge groups \( SU(m) \times U(1) \) over \( S^n \) have Gribov ambiguities. These results also apply for gauge groups of the form \( SU(m) \times [U(1)]^r \) which are quite often encountered as gauge groups of superstring inspired models.

We can repeat the procedure given in this section for the group \( SO(m) \times U(1) \), \( m: \text{odd} \), with similar results. All the arguments presented above for the case of \( SU(m) \times U(1) \) carry through in this case. The end result is that

\[ \pi_k(\text{Map}_*(S^n, SO(m) \times U(1))) \simeq \pi_{k+n}(SO(m)) \]

Theories with gauge groups \( SO(m) \times U(1) \) may conceivably arise as intermediate steps of the symmetry breaking pattern \( GUT \to \text{Standard Model} \) using supersymmetric or non-supersymmetric GUT as the starting point. They may also arise as effective field theories for some systems or even as toy models for testing new ideas. In all these cases we expect the index \( m \) of the gauge group \( SO(m) \) to be relatively small (probably \( m \leq 5 \)).

5. Discussion and Conclusions

In this paper we proved that the Gribov ambiguity exists for \( SU(m) \times U(1) \) over the \( S^n \). This is a result which should not come as a big surprise, in view of the fact that the non-Abelian gauge theories with semi-simple gauge group exhibit this behavior. What is remarkable though, is that although we did not consider a trivial fiber bundle over \( S^n \), it turned out that the groups of gauge transformations were conjugate to the ones of the corresponding trivial fiber bundles. The spirit of this approach is strongly reminiscent of the approaches encountered in Galois theory. There, we reduce a problem involving a non-commutative group to a problem involving the automorphism group of that group which is commutative. This simplifies the algebraic structure considerably and allows us to obtain results that are unreachable through more conventional methods. Similarly, in the present paper, instead of
considering a “twisted” principal fiber bundle, we analyse its automorphism group. It turns out that this automorphism group is conjugate to that of a trivial fiber bundle. Therefore many problems of the “twisted” structure can be addressed in the “untwisted” case. It would be hard to imagine that this would be the case before solving the problem.

In section 2 we defined the moduli space $\tilde{A}$. Using the same ideas of framed connections we see that

$$\mathcal{G}/\mathcal{G}_s \approx \text{Aut}(P_{x_0}) \approx G$$

This statement as well as the lack of orbifold type singularities of $\tilde{A}$ and the existence of singularities for $A$ may tempt us to believe that the Gribov ambiguities have their source in the existence of gauge transformations not continuously connected to the identity, in short in the fact that $\pi_0(\mathcal{G}) \neq 0$. It has been known for some time that this is not the case. Therefore potential connection between global anomalies and the Gribov ambiguity does not exist in that context so far as we know today. Gribov copies exist even within the first Gribov horizon. It is not yet known which is the maximal subspace of the moduli space $\tilde{A}$ that does not have Gribov copies. A first step in the direction of the determination of that fundamental modular domain is the proof that all gauge orbits pass inside the Gribov horizon.

It would be more informative to extend our analysis to a base manifold that does not have the topology of an $n$–sphere or of a product of spheres. This can be, presumably, done by using a combination of the methods used in this paper and a Postnikov system in which we can decompose the more complicated base manifold. However, apart from its mathematical interest, we do not expect this approach to give anything unexpected (i.e. the non-existence of a Gribov ambiguity in the generic case).

The main task is to understand the physical meaning of the Gribov ambiguity and the effects that it has, not only on the formal structure of non-Abelian gauge theories but also on observable quantities. The former case will provide a better understanding of the non-perturbative structure of gauge theories. The latter goal can also be attained, in principle. It is our opinion, however, that the Gribov ambiguity is most likely a gauge artifact, therefore it should not be expected to have any observable consequences. It may present considerable complication in the calculations, in handling global issues, but the lack of its existence in algebraic gauges, seems compelling enough to make us suspect that it is a mathematical
artifact. We believe that the issue can be settled after performing a BRST analysis and see the form that the Gribov ambiguity takes in that formalism.

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