On the Consensus Threshold for the Opinion Dynamics of Krause-Hegselmann

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Abstract

In the consensus model of Krause-Hegselmann, opinions are real numbers between 0 and 1 and two agents are compatible if the difference of their opinions is smaller than the confidence bound parameter $\epsilon$. A randomly chosen agent takes the average of the opinions of all neighbouring agents which are compatible with it. We propose a conjecture, based on numerical evidence, on the value of the consensus threshold $\epsilon_c$ of this model. We claim that $\epsilon_c$ can take only two possible values, depending on the behaviour of the average degree $d$ of the graph representing the social relationships, when the population $N$ goes to infinity: if $d$ diverges when $N \rightarrow \infty$, $\epsilon_c$ equals the consensus threshold $\epsilon_i \sim 0.2$ on the complete graph; if instead $d$ stays finite when $N \rightarrow \infty$, $\epsilon_c = 1/2$ as for the model of Deffuant et al.

Keywords: Sociophysics, Monte Carlo simulations.

In recent years several models of opinion formation have been proposed \[1, 2, 3, 4, 5, 6\]. In general they deal with simple cellular automata, where people become the vertices of a graph and neighbouring vertices represent agents which have a personal relationship (acquaintance). A simple rule determines how the opinion of an agent is influenced from (or can influence) that of its neighbours. The aim is to understand how it happens that large groups of people ultimately share the same opinion, starting from a situation in which everybody has its own ideas independently of those of the people with whom they interact. In particular, for large-scale phenomena it is possible to predict and/or reproduce general features like statistical distributions: with a voter model based on the Sznajd \[4\] dynamics one was able to reproduce the final distribution of votes among candidates in Brazilian and Indian elections \[7, 8\].
In this paper we focus on a special model, that of Krause-Hegselmann (KH) \cite{KrauseHegselmann}. It is a compromise model based on the principle of bounded confidence, which characterizes as well the opinion dynamics of Deffuant et al. \cite{Deffuant}. Bounded confidence is nothing but the reasonable consideration that a discussion between two individuals is constructive, i.e. it may lead to a change of opinion, only if the initial positions of the two persons are close enough, otherwise everybody retains its own opinion. In the model of KH, the opinion \( s \) is a real number in \([0:1]\), and two opinions are 'close' to each other if the absolute value of their difference is smaller than a positive real parameter \( \epsilon \), called confidence bound.

One starts from a graph \( G \) and assigns to each of its vertices a real number between 0 and 1, with uniform probability. Next, we perform ordered sweeps through the whole system and iteratively update the opinion of each agent. Suppose we want to update the opinion of an agent \( i \): one has to check which agents among the neighbours of \( i \) have opinions which are 'close' to that of \( i \), in the sense explained above. Only such neighbours, that span some set \( V \), can influence the opinion of \( i \). The new opinion \( s_i \) of \( i \) is given by the average of the opinions of the agents in \( V \). By repeating the procedure over and over, the system will reach a configuration which is a fixed point for the dynamics, so it is stable. Such configuration is characterized by just a few surviving opinions, with many agents sharing the same opinion. Strictly speaking, a stable configuration must be a superposition of Dirac \( \delta \)'s, located in such a way that the distance between two consecutive spikes is larger than \( \epsilon \).

The number of opinion clusters in the final configuration depends on the value of \( \epsilon \). In particular, above some value \( \epsilon_c \), all agents are bound to share the same opinion at the end of the process (consensus). The location of this threshold for complete synchronization is very important, because it provides useful information on the dynamics of the model. For the model of Deffuant et al., for instance, we have recently discovered that the threshold for complete consensus is \( \epsilon_c = 1/2 \), independently of the particular graph used to modelize society \cite{Deffuant}. In the model of Deffuant et al. \cite{Deffuant} the interactions between the agents are binary processes: an individual chooses at random one

\footnote{We remark that one could put the agent \( i \) itself in the set \( V \), so that the opinion \( s_i \) would contribute to the average too. This fact would be reasonable because the initial opinion should somehow be taken into account in the decision process of the individual, but it would have no influence whatsoever on the final configurations attained by the system.}
of its neighbours and discusses with it. If the two neighbours are compatible, i.e. if their opinions differ from each other (in absolute value) by less than the confidence bound $\epsilon$, the opinions of the agents move towards each other by a relative amount $\mu$, where $\mu$ is real in $[0 : 1/2]$.

The model of KH was originally introduced for a community where everybody talks to everybody else [3], and in this special case the algorithm runs very slowly as compared to that of Deffuant et al., due to the large averages needed to update the opinions of the agents. In this way, at present one can simulate at most systems with populations of the order of $10^5$ agents, whereas with the Deffuant dynamics systems as large as the population of the European Union can be simulated [10]. That is the main reason why much less is known [11, 12] on the model of KH as compared to that of Deffuant et al.

Nevertheless, we believe that the model of KH deserves more attention from the sociophysics community. In a sense, it is similar to the model of Deffuant et al., as it follows the criterion of bounded confidence and the opinion of an agent is affected by that of its neighbours in an "average" way: the crucial difference is that in KH the agent feels in one shot the influence of all its (compatible) neighbours, in Deffuant this happens after some time because each interaction of the agent involves only one of its (compatible) acquaintances. This argument suggests that there may be a considerable difference between the two models when the number of neighbours, or degree, of each agent is large, but that this difference should reduce when the degree is small. Moreover, in the latter case it takes just a short time to calculate the average opinion of the compatible neighbours, because there are only a few, and the algorithm can compete in speed with that of Deffuant. Indeed, for special graphs like regular lattices or random graphs with low average degree, which are much more appropriate for realistic applications, the KH algorithm is faster than the Deffuant algorithm, except perhaps in the very narrow bands of $\epsilon$ corresponding to the transition from a stable final opinion configuration to the next one, where the dynamics slows down.

Here we will investigate the consensus threshold of the KH model, by performing a similar analysis as in [9]. Basically, we carried out simulations of the model on different graph topologies and determined in each case the value of the consensus threshold. It turns out that the scenario is more complex than for the model of Deffuant et al., as we expected, but that the consensus threshold keeps its character of universality, labeling large classes of graphs. We analyzed five different types of graphs.
• a complete graph, where everybody talks to everybody else [3];
• a square lattice;
• a scale free graph á la Barabási-Albert [13];
• a random graph á la Erdös and Rényi [15];
• a star-like graph where a vertex is connected to all the others and no further connections exist.

![Figure 1: Fraction of samples with a single opinion cluster in the final configuration, for a society where everybody talks to everybody. The two data sets refer to a population of 5000 and 10000 agents.](image)

Before discussing the single cases, we give some details on the Monte Carlo simulations. We chose to update the opinions of the agents in ordered sweeps over the population. The equally legitimate choice of random updating would not have influence on the final number of opinion clusters\(^2\). The program

\(^2\)Nevertheless for special types of graphs this choice can influence the probability for complete consensus (see Figs. 6 and 7).
stops if no agent changed opinion after an iteration; since opinions are 64-bit real numbers, our criterion is to check whether any opinion varied by less than $10^{-9}$ after a sweep. We proceeded as follows: for a given population $N$ and confidence bound $\epsilon$ we produced 1000 configurations. After that we analyzed the final configurations, by checking whether all agents are labeled by the same opinion variable or not ("the same" means still within $10^{-9}$). The fraction of samples with all agents sharing the same opinion is the probability $P_c$ to have complete consensus, that we study as a function of $\epsilon$. For each social topology we repeated the procedure for several values of the population size, because the scaling of the curves with $N$ is useful to better identify the position of the consensus threshold.

Figure 2: As Fig. 1 but for agents sitting on the sites of a square lattice with periodic boundary conditions. The two data sets refer to a population of 1600 and 6400 agents.

We present our results starting from the case of the complete graph. Fig. 1 shows how the consensus probability $P_c$ varies as a function of $\epsilon$. The two curves correspond to a population of 5000 and 10000 agents, respectively. For the reasons we explained above, it is virtually impossible to go to much larger values of $N$, as the algorithm would become terribly slow. Nevertheless, from
Fig. 1 we observe that $P_c$ rapidly rises in a rather narrow interval of the opinion space. From the figure it seems that the curve will approach a step function, as already observed in [9]. We cannot determine with precision the position of the step in the limit where the population $N$ goes to infinity, but it will almost surely lie within the observed variation range $[0.195, 0.202]$. We set the upper limit to 0.202 because for $\epsilon > 0.202$ the curve corresponding to $N = 10000$ is definitely above the curve relative to $N = 5000$, which suggests that when $N$ diverges $P_c$ will probably attain the value 1 in that region. So, we conclude that the consensus threshold for KH on the complete graph, that we indicate with $\epsilon_i$, is in the interval $[0.195, 0.202]$.

Let us now see what happens for a society where the agents are on the sites of a square lattice, with periodic boundary conditions. The situation is illustrated in Fig. 2. Again, two population sizes were taken, $N = 1600$ and $N = 6400$, respectively. At variance with the case of the complete graph, we see that the onset is $1/2$, as for the model of Deffuant et al. (see [9]). This is interesting, as it reveals that the dynamics of the model of KH does not
suffice, as in Deffuant, to determine the value of the consensus threshold, but that it is necessary to take into account the interplay between the dynamics and the underlying graph topology. The result also confirms our expectation that KH becomes very similar to Deffuant when the average degree of the graph is low.

Fig. 3 further supports our conjecture. Here the graph is a scale free network á la Barabási-Albert (BA) [13], which has become very popular in the last years [14]. This object can be constructed by means of a simple dynamical procedure. One starts from a complete graph with $m$ vertices. At each iteration a new vertex is added and $m$ new edges are built between the new vertex and the old ones, so that the probability of connection to some vertex $i$ is proportional to the degree of $i$. One repeats the procedure until the network reaches the desired (total) number of vertices $N$. It is known that this growth process leads to a graph characterized by a degree distribution with a power law tail (in the limit $N \to \infty$): the exponent of the power law is 3, independently of the value of the parameter $m$ (or "outdegree"). For our network we took $m = 3$; Fig. 3 shows the same pattern as in Fig. 2 so that the consensus threshold is $1/2$ for the BA network too. In this way, the value $1/2$ is not relative to a special topology, but it labels at least two classes of graphs. What do lattices and BA networks have in common? The average degree $d$ is 4 for the square lattice and $2m$ for the BA network, so, in both cases $d$ remains finite when the graph becomes infinitely large ($N \to \infty$). In the case of the complete graph, instead, $d = N - 1$, so $d \to \infty$ when $N \to \infty$. Based on this fact, we propose the following conjecture:

- There are only two possible values for the consensus threshold $\epsilon_c$ of the model of Krause-Hegselmann: if the average degree $d$ of the graph stays finite when the order $N$ of the graph diverges, then $\epsilon_c = 1/2$; if $d$ diverges when $N \to \infty$, then $\epsilon_c = \epsilon_i \sim 0.2$.

We remark that this conjecture distinguishes between two regimes: a regime where each agent interacts on average with a few agents (microscopic interaction), and a regime where each agent interacts on average with a finite fraction of the whole population (mean field interaction). It is known that the two situations are well separated in statistical mechanics, and that they are characterized by different behaviours\(^3\).

\(^3\)In spin models like Ising, for instance, mean field theory applies for space dimensions $d \geq 4$, and the relative critical exponents differ from those at lower dimensions.
Figure 4: As Fig. 11 but for agents sitting on the sites of a random network à la Erdős-Rényi. Here the average degree $d = p(N - 1) \sim pN$ is kept constant when increasing the population $N$ from 1000 to 10000 agents.

So we claim that there are two different "universality classes of graphs" for the KH model, that we call $G_F$ (finite degree) and $G_I$ (infinite degree), each of them being labeled by a special value of the consensus threshold. The ideal way to test our conjecture would be to pass smoothly from one class of graphs to the other, so that we can see how the consensus threshold varies. If there were only two possible values for $\epsilon_c$, we would expect to observe a discontinuous variation by passing from the one to the other class. There is a special type of graph which allows us to perform this test, the random graph of Erdős and Rényi [15]. It is characterized by a parameter $p$, which is the bond probability of the vertices. One assumes that each of the $N$ vertices of the graph has probability $p$ to be linked to any other vertex. In this way, the total number of edges $m$ is $m = pN(N - 1)/2$ and the average degree is $d = p(N - 1)$ which can be well approximated by $pN$ when $N \to \infty$. This class of graphs is especially interesting for our purposes because it contains
both graphs in $G_F$ and graphs in $G_I$. In fact, suppose that $p$ is fixed to some value $> 0$: then $d = p(N - 1) \rightarrow \infty$ when $N \rightarrow \infty$. On the other hand, if $p \rightarrow 0$ when $N \rightarrow \infty$ in such a way that the product $p(N - 1)$ remains constant, then $d$ would remain finite. The random graph of Erdős and Rényi provides us then a natural way to interpolate between $G_F$ and $G_I$.

![Figure 4](image.png)

Figure 4: As Fig. 4 but for a fixed bond probability $p = 0.002$. In this case the average degree $d = p(N - 1) \sim pN$ diverges when $N \rightarrow \infty$, and the consensus threshold jumps from $1/2$ to a smaller value which coincides, within errors, with the threshold obtained for the complete graph.

Fig. 5 shows the probability for complete consensus $P_c$ as a function of $\epsilon$ for two random graphs á la Erdős and Rényi, with 1000 and 10000 vertices, respectively. The bond probability $p$ is varied so to keep the product $pN$ fixed in both cases ($pN = 2$). In the limit $N \rightarrow \infty$ we would then get a graph of the class $G_F$, and indeed we see from the figure that the consensus threshold is $1/2$, as for the square lattice and the BA network. The situation changes abruptly in Fig. 5: here we have three random graphs, with 1000, 10000 and 50000 vertices, respectively, but now the value of the bond probability $p$ is
fixed to 0.002. For $N = 1000$ we see that the consensus threshold is close to $1/2$, but this is clearly a finite size effect, as the system is small and has a few edges, so it is in the same situation as the smaller graph in Fig. [4]. However, when $N$ increases to 10000 we notice that the variation of $P_c$ takes place in a large range of $\epsilon$ values, which indicates the crossover between one regime and the other. Finally, for $N = 50000$, the variation of $P_c$ is again sharp and takes place in a narrow interval of $\epsilon$, which goes from 0.194 to 0.202, in excellent agreement with the range we have found for the complete graph.

The last issue we would like to discuss here concerns the definition of consensus threshold. In all cases we have dealt with so far, the pattern of $P_c$ seems to approach a step function when the graph becomes infinite. In this way, the onset $\epsilon_c$ indicates the value of $\epsilon$ above which the system can reach only a single stable final configuration, i.e. complete consensus. This consideration is true as well for the model of Deffuant et al [9]. For the opinion dynamics of KH, however, this is not the end of the story. As a matter of fact, it can happen that consensus is not the only possible stable state, but that it coexists with other stable states, even when the number of agents goes to infinity. In such cases, how is the consensus threshold $\epsilon_c$ defined? A possible definition would be the value of the confidence bound above which complete consensus is a possible stable state for the system, i.e. the value $\epsilon_c$ such that $P_c(\epsilon) > 0$ (but not necessarily 1) for $\epsilon > \epsilon_c$.

There are indeed some special graphs for which the probability for complete consensus does not converge to a step function. A typical example is shown in Fig. [5] where we plot once again the probability for complete consensus $P_c$ as a function of $\epsilon$. The social topology is now a star, i.e. a graph where one vertex (the ”core” of the star, that we call $C$) is linked to all others (the connections are the ”rays” of the star). We performed ordered updates of the agent opinions starting with the most connected central agent. In this case, it is possible to determine analytically the expression of the consensus probability as a function of $\epsilon$ in the limit $N \to \infty$. The average degree of the graph is finite, as the total number of edges is $N - 1$, so $d = 2(N - 1)/N \to 2$ when $N \to \infty$. If we believe our conjecture, the consensus threshold should be $\epsilon_c = 1/2$. Let us see what happens when $\epsilon > 1/2$. Suppose that the central agent $C$ gets initially the opinion $x \in [0 : 1]$. There are three possibilities:

1. $0 < x < 1 - \epsilon$, all agents with opinions between 0 and $x + \epsilon$ are compatible with $C$, so the new opinion $s_C$ of $C$ is the average of all opinions in the interval $[0, x + \epsilon]$, i.e. $s_C = (x + \epsilon)/2$;
2. $1 - \epsilon \leq x < \epsilon$, all agents are compatible with $C$, so its new opinion is $1/2$;

3. $\epsilon \leq x < 1$, all agents with opinions between $x - \epsilon$ and $1$ are compatible with $C$, so the new opinion $s_C$ of $C$ is the average of all opinions in the interval $[x - \epsilon, 1]$, i.e. $s_C = (1 + x - \epsilon)/2$.

Figure 6: Fraction of samples with a single opinion cluster in the final configuration, for a special community where one agent is connected with all the others, but the others have no further connections (star-like graph). We performed ordered sweeps over the whole population starting with the center of the star. For the system to always reach complete consensus (i.e. with probability 1), $\epsilon$ must be greater than $2/3$.

The aim of this calculation is to identify those intervals of the opinion space $[0:1]$ such that, if $x$ falls in any of them, agent $C$ will be compatible with all other agents after the first step of the calculation. If this happens, in fact, $C$ will sooner or later "convince" all other people to accept its opinion (through successive shifts), so the system would reach complete consensus.
We indicate the size of the "consensus" intervals for $x$ in the three cases as $p_1$, $p_2$ and $p_3$, respectively. In case 2, for any $x$ in the interval $1 - \epsilon \leq x < \epsilon$, agent $C$ will take opinion 1/2 after the first step and will then be compatible with all agents of the community, as $\epsilon > 1/2$. For this reason, the size $p_2$ of the "consensus" interval for $x$ is just the length of the whole range $[1-\epsilon, \epsilon]$, i.e., $p_2 = 2\epsilon - 1$. It is easy to see that, in the remaining two cases, the sectors of the opinion space in which $x$ has to fall in order to obtain consensus have the same size as in case 2, i.e. $p_1 = p_3 = p_2 = 2\epsilon - 1$. As the opinions are uniformly distributed at the beginning of the process, the probability for $x$ to fall in the "consensus" intervals, which coincides with the probability $P_c$ of having complete consensus, is $p_1 + p_2 + p_3 = 6\epsilon - 3$. This ansatz is represented by the skew straight line in Fig. 6 and it reproduces the data very well. The data sets are actually two, corresponding to $N = 1000$ and $N = 10000$ agents, respectively. Their excellent overlap shows that what we observe is indeed the asymptotic pattern. We remark that $P_c > 0$ for $\epsilon > 1/2$, in agreement with our conjecture, and that $P_c = 1$ only$^4$ for $\epsilon \geq 2/3$.

For the pattern of Fig. 6 it is essential to perform an ordered update of the agent opinions starting with the central agent $C$. If we would perform a random update of the agents, the situation would look quite different. In this case, in fact, before coming to the update of $C$, the opinions of some finite fraction of the whole population have been varied, and that alters the distribution of the opinions that can influence $C$, which are now no longer uniformly distributed. This fact prevents us from repeating the same argument we have presented above, for which the uniformity of the opinion distribution was crucial, and the results are different. Fig. 7 illustrates the new situation. We took two population sizes, $N = 1000$ and $N = 2000$: their remarkable overlap again shows that the observed pattern is the asymptotic one. However, it is no longer a simple straight line, but a curve which attains its limit value 1 when $\epsilon \to 1$. We have as well carried on simulations on the star-like graph for the opinion dynamics of Deffuant et al.; in contrast with the results of Figs. 6 and 7, we found that the probability for complete consensus converges to a step function as for all other graphs (the onset is still 1/2).

We have found that the consensus threshold $\epsilon_c$ of the opinion dynamics

$^4$As a matter of fact, the other stable configurations of the system for $1/2 < \epsilon < 2/3$ are characterized by one large cluster of agents with the same opinion of the core $C$, and by single-agent clusters with opinions close to the extremes 0 and/or 1.
Figure 7: As Fig. 6, but for random updating order of the agent opinions. In this case the pattern of the probability for complete consensus is not a simple straight line, but a curve which reaches the value 1 for $\epsilon \to 1$.

The criterion which distinguishes the two possibilities is the behaviour of the average degree $d$ of the graph when the number of vertices $N$ goes to infinity. If $d$ stays finite, $\epsilon_c = 1/2$, as in the model of Deffuant et al.; if instead $d \to \infty$, then $\epsilon_c = \epsilon_i \sim 0.2$. We have tested our conjecture on different types of graphs: the complete graph, the square lattice, the Barabási-Albert network, the random graphs a lá Erdös and Rényi. Further tests on star-like graphs show that the probability $P_c$ for complete consensus does not always converge to a step function when $N \to \infty$, and that the updating order of the agents may influence the final shape of the curve. We stress that for the result to hold it is necessary that the opinion space be symmetric with respect to the center opinion $1/2$, as in the case of the model of Deffuant et al. For modifications of the model
violating this symmetry we expect to find different values of the consensus threshold, as it was recently found for Deffuant  

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