SCATTERING FOR TIME SERIES WITH AN APPLICATION TO THE ZETA FUNCTION OF AN ALGEBRAIC CURVE

Jean-François Burnol

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I explain how the Lax-Phillips theory can be applied to a purely innovating time series and compute the corresponding scattering function. I then associate such a time series to an algebraic curve (of genus at least 1) over a finite field and show that the Riemann Hypothesis (proven long ago) holds if and only if the scattering is causal (this causality is not independently established, though).
1 Lax–Phillips scattering for innovating time series

Let us first briefly review some concepts related to time series. See for example [GreRo57] for this as well as [DymKe72] and [Ho62] for the underlying key results of classical harmonic analysis (Herglotz’s theorem, Hardy spaces, Szegö’s theorem, inner and outer factors, the Beurling–Lax description of invariant subspaces, etc...) which we will use below.

Let \( X = (X_n)_{n \in \mathbb{Z}} \) be a sequence of square-integrable random variables on some probability space, with zero mean, and such that the covariance \( \mathbb{E}(X_nX_m) \) depends only on the difference \( n - m \) (and is denoted \( \gamma_{n-m} \)). We also assume that \( \gamma_0 > 0 \).

The \( \gamma_j \)'s satisfy the positivity condition of Herglotz and are thus the Fourier coefficients of a uniquely determined positive measure \( \mu \) on the circle \( S^1 \):
\[
\gamma_j = \int z^{-j} d\mu \quad (0 < \mu(S^1) < \infty),
\]
the so-called “spectral measure” of the (weakly) stationary time series \( X \).

Let \( H(X) \) (the history of \( X \)) be the Hilbert space spanned by the \( X_n \)'s. There is a unique isometry \( L^2(S^1, d\mu) \to H(X) \) which sends the function \( z^n \) to the random variable \( X_n \). Multiplication by \( z \) then provides a (unique) isometry \( U \) of \( H(X) \) such that \( U(X_n) = X_{n+1} \). We have a chain of increasing subspaces \( H_n \subset H_{n+1} \) with \( H_n \) spanned by the \( X_m \)'s, \( m \leq n \). Let \( H_{-\infty} \) be the intersection of all the \( H_n \)'s (the distant past of the process \( X \)). Then either \( H_{-\infty} = H(X) \), or all inclusions \( H_n \subset H_{n+1} \) are strict. If \( H_{-\infty} = \{0\} \) we are necessarily in this second case and we will say that the time series \( X \) is “purely innovating” (the traditional terminology is “purely non-deterministic”). It then has no distant past, and no distant future either as
\( \alpha(z) \mapsto \overline{\alpha(z)} \) on \( L^2(S^1, d\mu) \) gives an anti-unitary which exchanges \( X_n \) with \( X_{-n} \).

As an example of a stationary time series let us take a (variance 1) white noise process \( Y = (Y_n)_{n \in \mathbb{Z}} \) (by this it is just meant \( \gamma_j(Y) = 0 \) for \( j \neq 0 \), \( \gamma_0 = 1 \), or equivalently that the spectral measure is normalized Lebesgue measure), and also a sequence \((c_n)_{n \in \mathbb{Z}}\) in \( l^2(\mathbb{Z}) \) and define \( X_n \) as the (bi-sided) “moving average” \( \sum_k c_k Y_{n-k} \). The spectral measure of \( X \) is then \( |\psi(\frac{1}{2})|^2 \frac{d\theta}{2\pi} \) (where \( \psi(z) = \sum_n c_n z^n \) in \( L^2(S^1, \frac{d\theta}{2\pi}), z = e^{i\theta} \)). Any absolutely continuous measure can be written in this form, so any process with an absolutely continuous measure can be written as a (bi-sided) moving average. If the coefficients \( c_n \) are such that \( c_n = 0 \) for \( n >> 0 \) or for \( n << 0 \), the moving average is said to be one-sided.

**Theorem 1.1 (see [GreRo57, chap.2])** The following conditions on \( X \) are equivalent:

1. \( X \) can be represented as a one-sided moving average (with respect to a white noise process).
2. \( X \) is purely innovating.
3. The spectral measure of \( X \) is absolutely continuous: \( d\mu = f(\theta) \frac{d\theta}{2\pi} \) and furthermore \( \int \log f(\theta) \frac{d\theta}{2\pi} > -\infty \).
4. The spectral measure of \( X \) is absolutely continuous: \( d\mu = f(\theta) \frac{d\theta}{2\pi} \) and furthermore there exists \( \psi(z) = \sum_{n\geq0} c_n z^n \) in the Hardy Space \( \mathbb{H}^2 \) such that \( f(\theta) \) is almost everywhere equal to \( |\psi(z)|^2 \).

Let us suppose from now on that \( X \) is such a purely innovating process. Let:

\[
D_0^0 = \text{Span}\{X_n, n < 0\}
\]
$D_0^+ = \text{Span}\{X_n, n \geq 0\}$

Then $D_0^-$, $D_0^+$, and the unitary shift operator $U$ satisfy the Lax-Phillips axioms for scattering ([LaxPh89]):

$$j \leq 0 \Rightarrow U^j(D^-_0) \subset D^-_0$$

$$\bigwedge U^jD^-_0 = 0 \quad \bigvee U^jD^-_0 = H(X)$$

$$j \geq 0 \Rightarrow U^j(D^+_0) \subset D^+_0$$

$$\bigwedge U^jD^+_0 = 0 \quad \bigvee U^jD^+_0 = H(X)$$

except for the orthogonality axiom $D_0^- \perp D_0^+$ which would be satisfied only for a white noise process. The orthogonality axiom guarantees the causality of the scattering matrix (more on this later). For this reason we will say that the choice $\{D_0^-, D_0^+\}$ defines the naive scattering associated with $X$.

**Definition 1.2** The (dual) scattering associated with the purely innovating process $X$ is given by the following choice of “incoming” and “outgoing” spaces:

$$D_- = \text{Span}\{X_n, n \geq 0\}^\perp = (D_0^+)^\perp$$

$$D_+ = \text{Span}\{X_n, n < 0\}^\perp = (D_0^-)^\perp$$

We note that the orthogonality condition for $\{D_-, D_+\}$ boils down to $(D_0^-)^\perp \subset D_0^+$ (rather than $D_0^\perp \subset (D_0^-)^\perp$).

As the codimension of $H_n(X)$ in $H_{n+1}(X)$ is 1, the Lax–Phillips theory teaches us that there exist (unique up to constants) isometric intertwiners

$$\phi_- : H(X) \to L^2(S^1; \frac{d\theta}{2\pi})$$

$$\phi_+ : H(X) \to L^2(S^1; \frac{d\theta}{2\pi})$$
between the action of $U$ and multiplication by $z$ and such that

$$\phi_-(D_-) = (\mathbb{H}^2)^\perp$$

$$\phi_+(D_+) = \mathbb{H}^2$$

The unitary $S = \phi_+ \cdot \phi_-^{-1}$ from $L^2(S^1, \frac{d\theta}{2\pi})$ to itself is called the scattering matrix (in its spectral representation). As $S$ commutes with multiplication by $z$ it is multiplication with a measurable function of unit modulus $s(\theta)$, which we will call the scattering function.

**Theorem 1.3** Let $X$ be a purely innovating process, and $\psi(z)$ an element of the Hardy space $\mathbb{H}^2$ such that the spectral measure of $X$ is normalized Lebesgue measure multiplied by $|\psi(z)|^2$. Let $\psi_{\text{out}}$ be the outer part $\psi$ (see [DymKe72], [Ho62]). The dual scattering function associated with the process $X$ is given as

$$s(\theta) = \frac{\overline{\psi_{\text{out}}(e^{i\theta})}}{\psi_{\text{out}}(e^{i\theta})}$$

**Proof 1.4** We first determine the *naive* scattering function associated with the pair $\{D_0^0, D_0^+\}$. Let $\phi_0^+$ be the corresponding intertwiner to the outgoing spectral representation. The map $L^2(d\mu) \rightarrow L^2(\frac{d\theta}{2\pi})$ given by $\alpha(z) \mapsto \psi(z)\alpha(z)$ is a unitary embedding (commuting with multiplication by $z$). It is in fact onto, as by a well-known result the zero set of $\psi(z)$ has Lebesgue measure 0. It sends $D_0^0$ to $\mathbb{H}^2$ which by a theorem of Beurling is also equal to $\psi_{\text{inn}}\mathbb{H}^2$, with $\psi_{\text{inn}}$ the inner part of $\psi$ (an inner function on the circle is a measurable function of modulus 1 which is almost everywhere the non-tangential boundary value of an analytic function in the interior of the unit disc, itself bounded in modulus by 1). Division by $\psi_{\text{inn}}(z)$ is a unitary, so the operator $\phi_0^+$ is given as $\alpha(z) \mapsto \psi_{\text{out}}(z)\alpha(z)$ from $L^2(d\mu)$ to $L^2(\frac{d\theta}{2\pi})$. 

5
In the same manner the “incoming spectral representer” \( \phi^0 \) is just multiplication by \( \psi_{\text{out}}(z) \) from \( L^2(d\mu) \) to \( L^2(\frac{d\theta}{2\pi}) \). The naive scattering function is thus \( \psi_{\text{out}}(e^{i\theta}) \cdot \left( \psi_{\text{out}}(e^{i\theta}) \right)^{-1} \). And the looked-for \( s(\theta) \) is its inverse. The conclusion follows.

**Definition 1.5** The purely innovating process has causal scattering if the associated dual scattering function is a causal function (equivalently if the naive scattering function is an inner function).

**Note 1.6** A causal function is an inner function with respect to the exterior domain \( |z| > 1 \) (including \( \infty \)). Sometimes “causal function” refers to the values taken in that domain (which we still denote by \( s(z) \)), so we should perhaps say “boundary value of a causal function”. This causality condition is equivalent to \( s_{\text{naive}}(e^{i\theta}) \) being an inner function (with respect to \( |z| < 1 \)) as the following relations show:

\[
|s(e^{i\theta})| = 1 \\
s_{\text{naive}}(e^{i\theta}) \cdot s(e^{i\theta}) = 1 \\
(|z| < 1; \text{causal case}) \quad \frac{s_{\text{naive}}(e^{i\theta})}{s_{\text{naive}}(e^{i\theta})} = s\left(\frac{1}{z}\right)
\]

**Note 1.7** The outer part of \( \psi \) can be expressed as (the boundary values of the exponential of) an integral (see [Ho62]) involving only the modulus of \( \psi \), so that it is possible to express \( s(\theta) \) directly in terms of the spectral measure \( \mu \) (up to an arbitrary multiplicative constant of modulus 1). We don’t write up this formula as we will not need it.
**Note 1.8** Perhaps our definition of causal scattering is a little too narrow and we should allow a pole at $z = \infty$ (equivalently at $z = 0$ for the naive scattering function). Replacing either the incoming or the outgoing subspace with a suitable shift would eliminate such a pole. The criterion of the next section would then also apply to genus 0 curves.

## 2 The congruence zeta–function as a time series

Let $C$ be a smooth, geometrically irreducible, complete algebraic curve with field of constants the field with $q$ elements. Let $g$ be its genus, $h(C)$ its class number (the number of distinct divisor classes in each degree $d \in \mathbb{Z}$), and $Z(T)$ ($T = q^{-s}$) its zeta function (for all of this and more, see for example [Mor91]). We will use $\mathcal{D}$ to denote an equivalence class of divisors. Two integers are associated with each such $\mathcal{D}$: its degree $d(\mathcal{D})$ which belongs to $\mathbb{Z}$ and a dimension $l(\mathcal{D})$ which belongs to $\mathbb{N}$.

Most of what follows can be done also for $g = 0$ but behaves in the end slightly differently, so we will assume $g \geq 1$.

Let us now define a set of coefficients $e_m$ indexed by $m \in \mathbb{Z}$:

- \(|m| \geq g\) : \(e_m = -q^{-|m|}\)
- \(|m| < g\) : \(e_m = \frac{1}{h(C)} \left[ \sum_{d(\mathcal{D}) = m+g-1} q^{l(\mathcal{D}) - \frac{m}{2}} \right] - \left[ q^{\frac{m}{2}} + q^{-\frac{m}{2}} \right]\)

and a time series $X = (X_n)_{n \in \mathbb{Z}}$ as the bi-sided moving average

\[X_n = \sum_m e_m Y_{n-m}\]

where $Y$ is a white noise process.
It is apparent that the Fourier series \( \sum_m e_m z^m \) represents a rational function (with poles at \( \frac{1}{\sqrt{q}} \) and \( \sqrt{q} \)), so that its modulus has an integrable logarithm and \( X \) is a purely innovating process to which the theory described before applies.

**Theorem 2.1** The Riemann Hypothesis for the algebraic curve \( C \) holds if and only if the associated process \( X \) has causal scattering.

**Proof 2.2** First of all, let us recall the expression for the zeta function \( Z(T) \) which exhibits it as a rational function (see [Mor91, chap. 3])

\[
(q - 1)Z(T) = \sum_{0 \leq d(D) \leq 2g-2} q^{d(D)T^d(D)} + h(C) \left[ \frac{q^g T^{2g-1}}{1 - qT} - \frac{1}{1 - T} \right]
\]

and a straightforward calculation then shows the following identity in \( L^2(S^1, \frac{d\theta}{2\pi}) \):

\[
z^{-(g-1)} q^{g-1} \frac{q - 1}{h(C)} \cdot Z\left(\frac{z}{\sqrt{q}}\right) = \sum_m e_m z^m
\]

The spectral measure of \( X \) is thus (note that \( Z(T) = \overline{Z(T)} \))

\[
d\mu = q^{g-1} \left( \frac{q - 1}{h(C)} \right)^2 \cdot |Z\left(\frac{z}{\sqrt{q}}\right)|^2 \frac{d\theta}{2\pi}
\]

It can be written as \( |\psi(z)|^2 \frac{d\theta}{2\pi} \) for

\[
\psi(z) = q^{g-1} \frac{q - 1}{h(C)} \cdot \frac{z - \frac{1}{\sqrt{q}}}{1 - \frac{1}{\sqrt{q}}} \cdot Z\left(\frac{z}{\sqrt{q}}\right)
\]

which belongs to \( \mathbb{H}^2 \) as the only pole of \( Z\left(\frac{z}{\sqrt{q}}\right) \) in the open disc is at \( z = \frac{1}{\sqrt{q}} \).

We now compute the naive scattering function as

\[
s_{\text{naive}}(\theta) = \frac{\psi_{\text{out}}(e^{i\theta})}{\psi_{\text{out}}(e^{i\theta})} = \frac{\overline{\psi_{\text{inn}}(e^{i\theta})}}{\psi_{\text{out}}(e^{i\theta})} = \frac{\overline{\psi_{\text{inn}}(e^{i\theta})}}{\psi_{\text{inn}}(e^{i\theta})} = \frac{\psi(e^{i\theta})}{\overline{\psi}(e^{-i\theta})} \cdot \left( \psi_{\text{inn}}(e^{i\theta}) \right)^{-2}
\]
where \( \bar{\psi}(z) = \psi(\overline{z}) \) and \( |\psi_{\text{inn}}(e^{i\theta})| = 1 \) were used. The functional equation reads

\[
Z\left(\frac{1}{qT}\right) = q^{1-g}T^{1-2g}Z(T)
\]

which translates into

\[
\psi\left(\frac{1}{z}\right) = \left(\frac{1 - z \sqrt{q}}{z - \frac{1}{\sqrt{q}}}\right)^2 z^{2g-2} \psi(z)
\]

We obtain, for \( |z| = 1 \):

\[
s_{\text{naive}}(z) = \left(\frac{z - \frac{1}{\sqrt{q}}}{1 - \frac{1}{\sqrt{q}}}\right)^2 z^{2g-2} \psi(z)
\]

Let us assume that \( s_{\text{naive}}(z) \) is in fact an inner function. The rational function on the right-hand side can be inner only if it has no poles in the open disc. But this means that \( \psi_{\text{inn}} \) has no zeroes (as 0 and \( \frac{1}{\sqrt{q}} \) are not acceptable candidates). This is just a way of phrasing the Riemann Hypothesis. And conversely under the Riemann Hypothesis the naive scattering function is

\[
s_{\text{naive}}(z) = \left(\frac{z - \frac{1}{\sqrt{q}}}{1 - \frac{1}{\sqrt{q}}}\right)^2 z^{2g-2}
\]

which is indeed an inner function.

Similar things can be done in the number field case too, but the Tate Gamma functions at infinite places are not of bounded characteristic in the half-plane \( \text{Re}(s) > \frac{1}{2} \) and this is a source of additional hurdles. In [Bu99] a \( p \)-adic scattering problem was studied and the causality was established in that local setting.
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Jean-François Burnol
Université de Nice-Sophia-Antipolis
Laboratoire J.-A. Dieudonné
Parc Valrose
F-06108 Nice Cédex 02
France
burnol@math.unice.fr