The Coupling-Deformed Pointer Observables and Weak Values

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Weak measurements and weak values have recently attracted wide attention because of their novel applications. However, the current theory and experiments are all based on the perturbative approach with risky approximations. The weakness of the system-pointer coupling leads to problems including deviation between data and predictions of weak value, inefficiency or low signal-to-noise ratio, and systematical bias in weak-value state reconstruction. Here we propose a non-perturbative approach to quantum measurements with post-selected, solving the above problems simultaneously by slightly modifying the current weak-measurement scheme, where coupling-deformed pointer observables are measured. Our result paves the way to put the weak-value and weak-measurement related applications on a more precise and solid foundation.

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The concept of quantum measurement with post-selections was first suggested by Aharonov, Albert and Vaidman (AAV) [1]. They built a perturbative theory where a complex variable called weak value arisen as the reading of the pointer employed in the measurement. The bizarre behaviors of weak value, such as large magnitude and the form resembling matrix element, have stimulated wide interest in explaining the phenomenon, and developing novel applications including weak-value amplification and weak-value tomography [2]. The former focuses on estimating small parameter modeled as the coupling strength of the weak measurements. It has enabled the observation of tiny effects like the spin Hall effect of light [3]. The later aims at directly determining an unknown quantum system by measuring the weak values of specific observables. This idea has been applied to both quantum state [4–12] and quantum dynamical processes [13]. It might be the only workable method for reconstructing high-dimensional states, and the current record is a 19200-dimensional wave function [7]. These achievements are practically impossible for standard tomography based on ordinary projective measurements [14].

However, the two lines of applications are both in controversy which situates weak measurements and weak-value-based methods into awkward positions. A common trouble comes from the failure of the weak values and the real pointer readings [15–17], especially in the region most interested by weak-value amplification, whose superiority over ordinary methods now falls into debate [18–22]. This shortcoming makes weak-value tomography be not universal valid [2 23–24]. Moreover, therein the coupling strength is artificially weakened to fulfill the requirements of AAV’s theory. Compared with projective measurements, weak measurement normally has a signal-to-noise ratio that is much lower; to suppress statistical error to the same level, the required sample size for weak measurement should be several orders of magnitude higher [24]. Meanwhile, as the small but finite coupling strength utilized in experiments cannot reach the weak limit desired by AAV’s perturbation approach, a bias in state reconstruction is inevitable [24]. It seems that weak-value-based quantum tomography is disadvantageous other than the relative simplicity of implementation in some complex systems [24].

In this Letter, we propose a method to remove the above-mentioned problems faced by weak-measurement and weak-value-motivated applications, especially for determining unknown weak values. The core idea is to non-perturbatively treat quantum measurements with post-selections and any system-pointer coupling strength. To retain the original weak-value form, we introduce the concept of coupling-deformed pointer observables by slightly modifying the current weak-measurement scheme. We also studied situations when such modification is unnecessary. This is meaningful for weak-value amplification to avoid the divergence between experiments and predictions of AAV’s perturbation treatment.

AAV’s formalism.—To start, let us briefly review the theory and applications of weak measurements, following the standard AAV perturbation approach. Consider a system initialized in state $\rho_{in} = |\psi_{in}\rangle\langle\psi_{in}|$. To measure observable $A$, we couple the system with a pointer initialized in state $|\phi_0\rangle$, via the typical unitary $U = \exp(-ig\hat{A} \otimes \hat{p})$, where $\hat{p}$ is defined on the pointer and $g$ is the dimensionless coupling strength. After that, we post-select the system into $\Pi_f = |\psi_f\rangle\langle\psi_f|$. AAV showed that [1], if the coupling is weak enough, the unnormalized pointer state $|\psi_f\rangle|U|\psi_{in}\rangle|\phi_0\rangle$ will be

$$|\psi_f\rangle|1 - ig\hat{A} \otimes \hat{p}|\psi_{in}\rangle|\phi_0\rangle + O(g^2) 
\approx |\psi_f\rangle|\psi_{in}\rangle \exp(-igA_{w}\hat{p})|\phi_0\rangle, \quad g \to 0.$$ (1)
This rough derivation suggests that, if the pointer is a one-dimensional continuous-variable system and  is the momentum operator, then the the expectation value of the canonical conjugate observable, position, which is often denoted by  and satisfies , will be shifted by . Here

\[
A_w = \frac{\langle \psi_f | \hat{A} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} = \frac{\text{tr}(\Pi_f \hat{A} \rho_{in})}{\text{tr}(\Pi_f \rho_{in})} \tag{2}
\]

is known as the weak value. However,  is generally a complex number and thus cannot be the exact pointer reading. Its real and imaginary parts could be determined separately by measuring and on the pointer, respectively [1, 25].

Since the signal, , could be much larger than , weak measurement can be applied to the estimation of tiny . The procedure is to choose proper and , get the signal by measuring or on the pointer, and then use the known to get the final estimation. Meanwhile, the idea of weak-value tomography is just opposite: the signal is divided by the known to obtained the desired weak values, which is directly related to wave function [3, 26] or the Dirac distribution of general states [8]. Particularly, in the beautiful experiment of Lundeen et al. [3], denoting , i.e., (the projector at position ) and being the zero-momentum eigenstate , the weak value will be , where is an irrelevant global factor.

Thus, to obtain the weak value of is to obtain the wave function , i.e., (the photonics initialized in ) or the projector at position . In the experiment the pointer is played by the photonics initialized in (0) and the photonics are substituted with the Pauli operators, , and . Explicitly, the formula for wave function reconstruction will be

\[
-2g \frac{\tilde{\psi}_i(0)}{\tilde{\psi}_i(0)} \approx \langle \tilde{q} \rangle_f - i\langle \tilde{p} \rangle_f. \tag{3}
\]

Here,  stands for the expectation value conditioned on successful post-selection.

However, the signal cannot be safely described simply by . The behavior for the case where is vitally complicated [13, 14]. That is, the larger the magnitude of the weak value is, the further the signal will deviate from the weak value [13, 16]. Applications of weak-value amplification have to avoid such regions, which are incorrectly hoped to reach the most salient amplification [2]. Particularly, for weak-value tomography,  is unknown and thus hard to judge the validity of Eq. [3], which makes the method not universal [2, 23, 24]. Compared with its counterpart with the projective measurements, the weak-value tomography has two more shortcomings, bias and inefficiency, as we mentioned above. To resolve these existing problems, one has to go beyond the weak-coupling regime.

Our non-perturbative formalism.—AAV’s approach focuses on pointer readings conditioned on successful post-selection, which are actually the projective measurement of . Hence we can focus on the joint measurement performed on the system and the pointer. The expectation value equals to the conditioned pointer reading multiplied by the probability of successful selections.

Suppose the decomposition of  is , where the component of  in subspace orthogonal to any  is known as the weak-coupling regime. However, if , then .

After coupling to the pointer via , the resulting state will be

\[
c_0 |\psi^\perp \rangle |\phi_0 \rangle + \sum_{i=1}^{k} c_{i} |a_i \rangle |\phi_i (g) \rangle, \tag{4}
\]

where  as . We denote as for convenience. Suppose , momentum and position, or Pauli operators, or others accordingly. Then the expectation value of  given coupling strength  can be written in a compact form as

\[
\langle \Pi_f \otimes \hat{s} \rangle_g = \sum_{i,j=0}^{k} (\rho_{in})_{ij} (\Pi_f)_{ji} Q_{ji}(g, \hat{s}). \tag{5}
\]

where , , i.e., . Assuming that , then in the expansion of the matrix element , terms of the first order of can be expressed as

\[
\lim_{g \to 0} Q_{ji}(g, \hat{s}) = ig_{a} \langle \phi_0 | a \hat{s} | \phi_0 \rangle - ig_{i} \langle \phi_0 | \hat{s} a | \phi_0 \rangle. \tag{6}
\]

Here, the limitation is not rigorous in mathematical sense. We use this notation to keep only the first order terms of . As shown on the right hand side of Eq. [6], each of them depends on a single subscript, or . Since , the numerator of weak value, , could be determined via the linear combination of pointer’s shifts for the pair of observables . However, if , the first-order term vanishes while the second-order term dominates. In this case AAV’s first-order perturbation formula still has risks in obtaining the correct expectation value, .
is simple to be measured. One may wonder why not to
directly measure two Hermitian observables, \( \Pi_f \hat{A} + \hat{A} \Pi_f \)
and \( i \Pi_f \hat{A} − i \hat{A} \Pi_f \), to obtain the numerator and thus the
desired information. Actually, while the direct measurement is possible in principle, it is quite hard to implement
in practice. For example, within the experiments of Lun-
deen et al., \( \Pi_f \hat{A} + \hat{A} \Pi_f \) looks like \(|x⟩⟨0|_p⟩ + |0⟩_p⟨x|⟩\), which is
much more difficult to measure than \(|x⟩⟨x|⟩\).

**Coupling-deformed pointer observables.**—Compared
with AAV’s formula, Eq. (3) successfully separates
pointer’s information from that of the system. Note that
\( g \) and the operator \( \hat{s} \) enter only the \( Q \)-matrix. To see
more clearly the physical meaning of Eq. (3), it is ad-
vantageous to regard \( \{Q_{ji}(g)\} \) as coefficients of the linear
combination of \( \langle \rho_m \rangle_{ij}(\Pi_f)_{ji} \). Importantly, Eq. (4) and
its follow-up discussion show that the value of Eq. (5)
will be formally identical to the numerator of the weak
value, \( \text{tr}(\Pi_f \hat{A} \rho_m) \), if \( Q_{ji} \) equals to the right hand side of
Eq. (6) up to a common factor. Meanwhile, in AAV’s for-
malism, to realize such a special form of \( Q_{ji} \) is to weaken
the coupling and read observables \( \hat{p} \) and \( \hat{q} \) on the pointer
after post-selection. Now an intriguing problem arises as
to whether or not one could have an alternative method
to get the desired form of the \( Q \)-matrix. Surprisingly,
we give an affirmative answer to this question. In this
way, the existing problems of applying AAV’s formalism
to the above-mentioned applications can be removed sim-
ultaneously.

Actually, to solve the demerits brought by the weak-
ness of measurements, we have no way but to strengthen
the interactions. For couplings of finite strength, say
\( g' \), the pointer states become \( |\phi_i(g')⟩ \). Then the ma-
trix \( Q(g', \hat{s}) \) is hardly to be proportional to \( Q(g, \hat{s}) \), to
be clear soon. To have the desired form of the \( Q \)-matrix,
we can modify the observable read on the pointer, i.e.,
replace \( \hat{s} \) with \( \hat{s}(g') \), which may depend on \( g' \) and will
be called the **coupling-deformed** (CD) pointer observable
hereafter. Now the problem reduces to the determination
of this new observable such that

\[
Q^\rightarrow (g, \hat{s}) \propto Q[g', \hat{s}(g')],
\]

where the \( Q \)-matrix at right hand side is defined via
\( \langle \phi_i(g')|\hat{s}(g')|\phi_i(g')⟩ \). Firstly, an orthonormal basis \( \{|^k⟩\} \)
of the space spanned by \( \{|\phi_i(g')⟩\} \) can be fixed so that

\[
|\phi_i(g')⟩ = \sum_k \alpha_{ik} |^k⟩.
\]

Define a matrix \( S(g') \) with elements \( S_{ki} = \langle ^k|\phi_i(g')⟩ \). Then, the matrix \( Q[g', \hat{s}(g')] \)
with entries \( \langle \phi_i(g')|\hat{s}(g')|\phi_i(g')⟩ \) can be linked to the
matrix \( \hat{Q}(g') \) defined by \( \langle \hat{m}|\hat{s}(g')|\hat{m}⟩ \) via \( Q[g', \hat{s}(g')] = S^\dagger(g') \hat{Q}(g') S(g') \).
Actually, \( Q(g') \) is an effective matrix representation of the CD observable \( \hat{s}(g') \). Although the
dimension of pointer’s Hilbert space could be much larger
than \( k \) and even uncountable, parts of \( \hat{s}(g') \) living in
the orthocomplement space are irrelevant to the rea-
ning of the pointer. Assuming that the states \( \{|\phi_i(g')⟩\} \)
are linearly independent (the strongest measurements are
achieved when \( \langle \phi_i(g')|\phi_j(g')⟩ = \delta_{ij}. \)), the matrix \( S(g') \)
has a well-defined inversion. Then, the desired \( \hat{s}(g') \),
which obeys Eq. (7), can be simply determined as

\[
\hat{Q}(g') \propto (S^\dagger)^{-1}(g') Q^\rightarrow(g, \hat{s}) S^{-1}(g').
\]

Finally, it is straightforward to translate \( \hat{Q}(g') \) into
an observable of the pointer according to the basis \( \{|^k⟩\} \).
The proportion coefficient in Eq. (8) is irrelevant to the
signal-to-noise ratio of reading \( \hat{s}(g') \) and can be fixed by
preference. More details are left in the Supplemental
Material. Note that the determination of \( \hat{s}(g') \) is inde-
pendent to the unknown system and thus can be accom-
plished beforehand in the step of pointer calibrating.

As an important example, let us revisit the exper-
diment of Lundeen et al. [2], where \( |\phi_0⟩ = |0⟩ \) and
\( |\phi_1⟩ = \cos(g)|0⟩ - i \sin(g)|1⟩ \). For the real part of the weak value, we can measure \( g \to \sigma_y \) such that

\[
Q^\rightarrow(g, \hat{q}) = -g \left( \begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array} \right).
\]

With the method proposed above, we can identify the
CD observable \( \hat{q}(g') \) as

\[
\hat{q}(g') = \sigma_y - \tan(g') |I - \sigma_z|, \quad g' \in [0, \frac{\pi}{2}].
\]

The strong limit is when \( g' = \frac{\pi}{2} \) so that \( |\phi_1⟩ = -i|1⟩ \).
Straightforward calculation shows that in the measure-
ment with coupling constant \( g' \) and pointer reading ob-
servable \( \hat{q}(g') \), the corresponding \( Q \)-matrix has the same
form as Eq. (9) up to a global factor and reads

\[
Q(g', \hat{q}(g')) = -\sin(g') \left( \begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array} \right),
\]

which matches Eq. (7) exactly. Explicitly, we have

\[
\langle \Pi_f \otimes \hat{q}(g') | g' \rangle = -2 \sin(g') \Re [\text{tr}(\Pi_f \hat{A} \rho_m)].
\]

Here \( \Re \) means the real part.

With this example, the usual weak-measurement
scheme calls for a slight modifications made by the idea of
CD observables as following. In the modified scheme, one
keeps most settings unchanged and needs to strengthen
the coupling and then measure the CD observables on
the pointer. The weak-value information, previously gained
at the weak limit in AAV’s formalism, can now be ex-
tracted exactly with the modified measurements.

We will elaborate in the Supplemental Material how
such a modified measurement scheme solves the prob-
lems mentioned above simultaneously. Concisely speak-
ing, (1) exactness of our formalism solves the conflicts
between theory and experiments for the previous weak-
measurement scheme, and implies the elimination of bias;
(2) by contrast to AAV’s formula, the validity of ours is
universal regardless of the relation between input states and post-selections, or others; (3) by going beyond the weak-coupling regime to even the strong limit, the signal-to-noise ratio and thus the efficiency of the resource utilization, will be significantly enhanced.

Additionally, we emphasize that the post-selections do not waste resource, at least in principle. For mixed states, the post-selected states compose an orthogonal basis thus can be implemented together \[11\]. For pure states, data belongs to different posterior states can be processed separately with different equations to reconstruct the unknown state \([11]\).

The \(g\)-Invariant pointer observables.—Generally speaking, \(\hat{s}(g')\) depends on \(g'\). Can we find particular situations where the CD observable \(\hat{s}(g')\) is, instead, independent on the coupling? Such \(g\)-Invariant pointer observables are important for applications (e.g., the weak-value amplification) in which the coupling constant is either unknown or uncertain with inevitable errors. Thus if the CD observable read on the pointer is \(g\)-invariant, then the above merit of our formalism can be also shared by, e.g., weak-value amplification. The existence of such observables seems quite seldom unless in situations where \(Q\)-matrix is 2-dimensional (common in the literature). Below we find several \(g\)-invariant observables that are easy to measure.

Consider again the experiment of Lundeen et al. \([4]\) where the imaginary part of the weak value is determined by measuring \(\hat{p} \to \sigma_x\). We have

\[
Q(g', \sigma_x) = \sin(g')(0 \quad -i) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

Thus it is straightforward to see that Eq. \((17)\) is fulfilled and the pointer observable is \(\sigma_x\), certainly \(g\)-invariant. If the pointer is replaced by a continuous-variable system and initialized in a Gaussian state with standard deviation \(\Delta\),

\[
\langle q|\psi_0 \rangle = \frac{1}{(2\pi\Delta)^{1/4}} \exp(-\frac{q^2}{4\Delta^2}),
\]

which is very common in reported experiments on weak measurements \([2]\), we will have \((\tilde{p} = -i \frac{\partial}{\partial q})\)

\[
Q(g', \tilde{p}) = \frac{g'}{4\Delta^2} \exp(-\frac{g'^2}{8\Delta^2}) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

The CD observable is found to be the original observable \(\hat{p}\), which is of course \(g\)-invariant. Interestingly, here the strong limit will not bring the largest magnitude of the \(Q\)-matrix. The optimal selection of coupling strength is \(g' = 2\Delta\). This result can be applied to state reconstruction. It is useful to notice that there exist more examples of \(g\)-invariant observables. Other than projectors, if \(\hat{A}\) is formed like Pauli matrices, then both \(\hat{p}(g')\) and \(\hat{q}(g')\) could be \(g\)-invariant, for pointers played by both qubit and Gaussian wave pockets \([13]\).

If the observable measured in a weak-value amplification is \(g\)-invariant, then the product of conditioned pointer readings and post-selection probability would be exactly proportional to (real or imaginary part of) \(\text{tr}(\Pi_f \hat{A} \rho_{in})\), although either behaves weirdly. This removes the errors caused by previous problematic approximation.

Finally, to put the above examples into a wider context, here we show further that every measurement of the weak value can be transformed into equivalent experiments where \(Q\)-matrices are 2-dimensional. Let us define

\[
|\psi_A \rangle = \mathcal{N} \sum_i a_i (|\psi_f \rangle |a_i \rangle),
\]

where \(\mathcal{N}\) is the normalization factor. Then we have

\[
\Pi_f \hat{A} \propto \Pi_f |\psi_A \rangle \langle \psi_A |,
\]

which means that the weak value of \(\hat{A}\) is equivalent to the weak value of the rank-1 projector, \(|\psi_A \rangle \langle \psi_A |\). If by accidence \(\langle \psi_f | \psi_A \rangle = 0\), we can measure the generalized Pauli matrix \(|\psi_A \rangle \langle \psi_f | + |\psi_f \rangle \langle \psi_A |\) instead \([13]\). Strikingly, both cases are the examples we have given for \(\hat{s}(g')\) and \(g\)-invariant observables.

Such a transformation is important if only poor pointers with less dimensions are available while finite-coupling measurements are desired, since the existence of the CD observables requires states \(\{|\phi_i(g')\rangle\}\) to be linearly independent, which can not be fulfilled now. Note that after the transformation, the \(Q\)-matrices become 2-dimensional. Thus poor pointers like single qubits are adequate for the strong measurements of any weak-value information. However, to implement the transformation means to alter the Hamiltonian which couples the system and the pointer. The modified experiments may not be as simple as the original to implement, as the example of measuring \(|x\rangle \langle 0|_p + |0\rangle_0 |x\rangle\) mentioned before.

Conclusions and discussions.—To summarize, we have developed a non-perturbative approach to measurements with post-selections. Here the system-pointer coupling strength can be of any finite value, not necessarily small enough as in AA V’s formalism. To retain the original form of the weak-value, we can slightly modify the current weak-measurement scheme, in which the CD observables are read on the pointers. We also studied situations when such modification is unnecessary, namely, the CD observables is \(g\)-invariant. This is meaningful for weak-value amplification to avoid the mismatch between experiments and theoretical predictions by AA V’s perturbation treatment. Thus, while keeping the advantages of current weak-measurement and weak-value motivated applications, our method eliminates main problems therein, such as inefficiency or low signal-to-noise ratio, bias, and universal validity in the current weak-value tomography scheme, without introducing much complexity in experimental implementations.
Our result also has other implications. For example, in the literature of weak-value direct characterization, the possibility of measuring coherent information about unknown system is often attributed to the negligible disturbance caused by weak measurements. However, our results show that such interpretation is unnecessary. We hope this work could stimulate more sparks on theories and applications of “weak” measurements.

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SUPPLEMENTAL MATERIAL

1 SYSTEMATIC METHODS FOR COUPLING-DEFORMED POINTER OBSERVABLE

Here we give the detailed method for constructing the coupling-deformed (CD) pointer observables.

Suppose the set of relevant pointer states in the form of $|\phi_i(g')\rangle = \exp(-ig'a_i\hat{\pi})|\phi_0\rangle$ span a space. We can find an orthonormal basis of the space. Denote one such basis by $\{|\hat{m}\rangle\}$. Then we can insert the identity operator $\mathcal{I} = \sum_{m=1}^{n} |\hat{m}\rangle\langle\hat{m}|$ into the elements of matrix $\hat{Q}[g',\hat{s}(g')]$, that is,

$$Q_{ij}[g',\hat{s}(g')] = \sum_{m,n} \langle\phi_i(g')|\hat{m}\rangle\langle\hat{m}|\hat{s}(g')|\hat{n}\rangle\langle\hat{n}|\phi_j(g')\rangle.$$  \hspace{1cm} (18)

Therein, $\langle\hat{m}|\hat{s}(g')|\hat{n}\rangle$ defines a new matrix, namely, $\hat{Q}(g')$. It is an effective matrix representation of $\hat{s}(g')$ in the space spanned by the relevant pointer states. If we have known this $\hat{Q}(g')$ matrix, then the CD observable $\hat{s}(g')$ will be easily determined as

$$\hat{s}(g') \sim \sum_{mn} \hat{Q}_{mn}(g')|\hat{m}\rangle\langle\hat{m}|,$$  \hspace{1cm} (19)

where "$\sim$" means "equal effectively" since $\hat{s}(g')$ could have spectrum outside of the relevant pointer space. In Eq. (18), we could define another matrix $\hat{S}(g')$ with elements in the form of $\langle\hat{n}|\phi_1(g')\rangle$. With such a decomposition, the requirement that $\hat{Q}[g',\hat{s}(g')]$ is proportional to $Q^{g\rightarrow 0}(g,\hat{s})$ can be expressed through a factor $\eta$ as

$$\eta Q^{g\rightarrow 0}(g,\hat{s}) = S^1(g')\hat{Q}(g')\hat{S}(g').$$  \hspace{1cm} (20)

In such a form, it is straightforward to see that

$$\hat{Q}(g') = \eta S^{-1}(S^{-1})^{-1}Q^{g\rightarrow 0}(g,\hat{s})S^{-1}.\hspace{1cm} (21)$$

Together with Eq. (19), the CD observable $\hat{s}(g')$ can be determined up to a global factor, given the existence of the inverse matrix $S^{-1}(g')$. The existence is ensured if the relevant pointer states $\{|\phi_i(g')\rangle\}$ are linearly independent.

The global factor $\eta$ is not relevant to the signal-to-noise ratio. It serves like the unit of pointer readings and are thus not important. A viable paradigm of fixing this factor is to make the eigenvalues of the CD observable be comparable with those of the original observable. Or, one can follow other standards for the sake of convenience.

In the experiment of Lundeen et al., the observable of weak measurement is a rank-1 projector. The pointer’s state (polarization) will not be shifted if the photon is not at the position $x$, and otherwise be shifted. Thus, there are two relevant pointer states,

$$|\phi_0\rangle = |0\rangle,$$

$$|\phi_1(g)\rangle = \cos(g)|0\rangle - i\sin(g)|1\rangle.$$  \hspace{1cm} (22)

For the real part of weak value, the standard method is to measure $\hat{q} = \sigma_y$ on the polarisation. Then the Q-matrix with element $\langle\phi_i|\sigma_y|\phi_j\rangle$ reads as

$$\begin{pmatrix}
0 & -\sin(g) \\
-\sin(g) & -\sin(2g)
\end{pmatrix},$$  \hspace{1cm} (23)

so that the Q-matrix in the weak limit reads as

$$Q^{g\rightarrow 0}(g,\hat{q}) = -g \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (24)

When we increase the coupling constant to some value $g'$, the relevant pointer states are in the same form of Eq. (22). The orthonormal basis $\{|\hat{m}\rangle\}$ can be naturally selected as the eigenstates of $\sigma_z$, i.e., $\{|0\rangle, |1\rangle\}$. Then the matrix $\hat{S}(g')$ defined with elements in the form of $\langle\hat{m}|\phi_i(g')\rangle$ is written as

$$\hat{S}(g') = \begin{pmatrix} 1 & \cos(g') \\ 0 & -i\sin(g') \end{pmatrix}.$$  \hspace{1cm} (25)

According to Eq. (21), we have

$$\hat{Q}(g') = \eta \frac{g}{\sin(g')} \begin{pmatrix} 0 & -i \\ i & -2\tan\frac{g'}{2} \end{pmatrix},$$  \hspace{1cm} (26)

where we have used the relation that $\tan\frac{g'}{2} = \frac{1-\cos(g')}{\sin(g')}$. Then we can fix $\eta$ as $\frac{\sin(g')}{g}$ so that the CD observable is normalized as

$$\hat{q}(g') = \sigma_y - \tan\frac{g'}{2}(I - \sigma_z).$$  \hspace{1cm} (27)

At the strong limit, $g' = \frac{\pi}{2}$, the eigenvalues of $\hat{q}(g')$ are $-1 \pm \sqrt{2}$. Their scale is comparable with the eigenvalues of $\hat{q} = \sigma_y$, i.e., $\pm 1$. Using the notation $\langle\cdots\rangle_g$ to denote the expectation value of the omitted observable with respect to the joint state of system and pointer after their coupling at strength $g'$, we have

$$\langle\Pi_f \otimes \hat{q}(g')\rangle_g = -2\sin(g')\Re \left[ \text{tr}(\Pi_f \hat{A}_{\rho_{in}}) \right].$$  \hspace{1cm} (28)

This equation also covers the weak limit $g' \rightarrow 0$, the range where AAV’s formula applies.

For the imaginary part of the numerator of the weak value, AAV’s formula requires the measurement of $\hat{p}$, which is $\sigma_x$ here. As shown in the main text, the CD observable is also $\sigma_x$, i.e., it is $g$-invariant. Parallel to Eq. (28), we have

$$\langle\Pi_f \otimes \hat{p}\rangle_{g'} = 2\sin(g')\Im \left[ \text{tr}(\Pi_f \hat{A}_{\rho_{in}}) \right].$$  \hspace{1cm} (29)

With Eqs. (28) and (29), the numerator of the weak value, $\text{tr}(\Pi_f \hat{A}_{\rho_{in}})$ can be directly obtained from experimental data.
II ELIMINATING THE DEFECTS WITH CD POINTER OBSERVABLES

Here we analyse the three defects of weak-value-based methods in the existing schemes. We shall show how the idea of CD pointer observables could solve them simultaneously.

A. Failure of AAV’s formula and the validity of weak-value tomography

As a tradition, in the literature of weak measurements with post-selection, people mostly focus on the conditioned pointer reading, i.e., the data of pointer reading is kept in statistics if and only if the system is successfully selected into the particular state $|\psi_f\rangle$. Explicitly, in AAV’s formula the expectation value of such conditioned pointer reading of observable, say $\hat{s}$, can be expressed as

$$\frac{\langle \Pi_f \otimes \hat{s} \rangle_g}{\langle \Pi_f \otimes I \rangle_g}$$

(30)

where the denominator is actually the probability of successful post-selection, $P_f(g)$. In AAV’s formula, the coupling constant $g$ is required to be very tiny so that the denominator enjoys a zero-order approximation that

$$\langle \Pi_f \otimes I \rangle_g \approx \text{tr}(\Pi_f \rho_{in})$$

(31)

So if $\text{tr}(\Pi_f \rho_{in}) = 0$, the mathematical rigor usually breaks down and the weak value becomes ill-defined. This singularity can be removed by considering the joint expectation value, i.e., $\langle \Pi_f \otimes \hat{s} \rangle_g$. It can be determined from the experiments by multiplying the conditioned pointer reading and post-selection probability. However, even though, AAV’s formula is still imperfect. Within the equality emphasized in the main text,

$$\langle \Pi_f \otimes \hat{s} \rangle_g = \sum_{ij} (\rho_{in})_{ij} \langle \Pi_f \rangle_{ji} Q_{ji}(g, \hat{s})$$

$$\approx g \text{tr}(\Pi_f \hat{A} \rho_{in})$$

(32)

we have shown in the main text that it is valid if we consider only the first order terms in the series expansion of $Q_{ji}(g, \hat{s})$. Meanwhile, let us consider the situation when $\langle \psi_{in} | \psi_f \rangle = O(g)$. Then as the pointer reading predicted by AAV,

$$g \text{tr}(\Pi_f \hat{A} \rho_{in}) = g \langle \psi_{in} | \psi_f \rangle \langle \psi_f | \hat{A} | \psi_{in} \rangle = O(g^2).$$

(33)

This means that, we should take the second-order terms of $Q_{ji}(g, \hat{s})$ into account and the prediction of weak value fails.

Importantly, to make the judgement of the validity of AAV’s formula, we should know the relation between initial state and post-selected state, which is impossible in the task of unknown state tomography. This is why the weak-value tomography is believed to be not a universal valid method.

Comparatively, thanks to the CD pointer observables, our formulae, such as Eqs. (28) and (29), are universally valid. We do not need to worry about the extreme cases as in the applications of AAV’s formula.

B. Exactness removes the bias

If the expectation value of the estimator does not exactly equal to the value of the estimated, the estimation is called biased. In Ref. [1], the authors have shown such systematic error caused by the finite coupling strength, and thus finite higher-order terms, for current weak measurement scheme. The bias was shown to be very robust. To resolve it, the authors of Ref. [1] suggested to implement weak-value tomography with different coupling strengths, and extrapolate to the weak limit where AAV’s formula becomes exact. The method is, however, unpractical.

Here, Eqs. (28) and (29) clearly imply that our scheme with CD observables are unbiased.

C. Lift the efficiency and signal-to-noise ratio

Real experiments cannot be infinitely repeated so that the statistical error must be taken into account. As a good figure of merit of the relative error in one time of measurement, the signal-to-noise ratio (SNR) is defined as the quotient of the expectation value and the standard deviation. If the SNR is too tiny, we should repeat the measurement for more times to reduce the standard deviation, and thus to reduce the random error. As AAV’s formula requires weak measurements, compared with the standard projective measurements, the signal is much weaker. Thus, in experiments of weak measurements, we need more repetitions to reach the comparable level of random noise. That is, we need much larger sample size. For instance, weak-value tomography is inefficiency in physical resources as confirmed rigorously in Ref. [1].

Let us consider the SNR of the projective measurements required in standard tomography. Typical selections of the observables there are rank-1 projectors $\Pi$. The expectation value would be $P_\Pi = \langle \psi_{in} | \Pi | \psi_{in} \rangle$. Meanwhile, since $\Pi^2 = \Pi$, its SNR denote by $R_\Pi$ will be

$$R_\Pi = \frac{\sqrt{P_\Pi}}{\sqrt{1 - P_\Pi}}.$$ 

(34)

With the CD observables, the applicable coupling is extended to the full range of strength, from the weak limit to the strong limit. Thus, we have an opportunity to find the optimal strength. Here, let us calculate the SNR in qubit weak-value tomography. The observable
measured on the system is a projector and the pointer is the polarisation of photons.

Now the observable we measure on the joint system is \( \Pi_f \otimes \hat{s}(g') \), of which the expectation value is given by Eq. \[23\] or Eq. \[20\]. For the noise, we have to obtain the expectation value of \( \Pi_f \otimes \hat{s}^2(g') \). For the real part of the numerator of the weak value, we have

\[
\langle \hat{q}^2(g') \rangle = I + 4 \tan^2\left(\frac{g'}{2}\right)\langle 1 | - 2 \tan\left(\frac{g'}{2}\right)\sigma_y
\]

\[
= I - 2 \tan\left(\frac{g'}{2}\right)\hat{q}(g').
\]  \(35\)

Then, the standard deviation \( \Delta_q \) of this measurement can be written as

\[
\Delta_q^2(g') = \langle \Pi_f \otimes \hat{q}^2(g') \rangle_{g'} - \langle \Pi_f \otimes \hat{q}(g') \rangle_{g'}^2
\]

\[
= P_f(g') + 8 \sin^2\left(\frac{g'}{2}\right) \text{Re} \text{tr}(\Pi_f \hat{A} \rho_{in})
\]

\[
- 4 \sin^2(g') [\text{Re} \text{tr}(\Pi_f \hat{A} \rho_{in})]^2.
\]  \(36\)

As defined above, \( P_f(g') = \langle \Pi_f \otimes I \rangle_{g'} \) is the successful probability of post-selection. In the current case, the Q-matrix is a \( 2 \times 2 \) matrix and for the formula of \( P_f(g') \), the element of its Q-matrix is \( \langle \phi_i(g')|\phi_j(g') \rangle \). Explicitly, in this case we have

\[
P_f(g') = \sum_{ij} (\rho_{in})_{ij} (\Pi_f)_{ji} Q_{ji}(g', I),
\]

\[
Q(g', I) = \begin{pmatrix} 1 & \cos g \\ \cos g & 1 \end{pmatrix}.
\]  \(37\)

For the imaginary part, the associated standard deviation \( \Delta_p(g') \) can be expressed as

\[
\Delta_p^2(g') = P_f(g') - 4 \sin^2(g')[\text{Im} \text{tr}(\Pi_f \hat{A} \rho_{in})]^2.
\]  \(38\)

For a fast and primary comparison, we consider randomly the case when the input state \( |\psi_{in} \rangle = \frac{1}{\sqrt{2}}(|0 \rangle + i|1 \rangle) \), \( \hat{A} = |0 \rangle \langle 0 | \) and \( |\psi_f \rangle = \frac{1}{\sqrt{2}}(|0 \rangle + |1 \rangle) \). For projective measurements, \( P_f = \frac{1}{2} \) and thus \( R_{g1} = 1 \); for weak-value tomography, we have \( \text{tr}(\Pi_f \hat{A} \rho_{in}) = \frac{1}{4} (1 - i) \). The values of the SNRs are plotted in Fig. 1 against the coupling strength, from the weak limit \( g = 0 \), to the strong limit \( g = \Delta \). Clearly, the SNR is significantly enhanced relative to that in the weak-measurement region covered by the shade. In the strong-coupling region, the SNRs for the real and imaginary parts of the weak value for the purpose of tomography are in the same order with the standard tomography. Thus, the problems of inefficiency is no longer a problem.

The comparison is actually not fair. In physical system such as the spatial degree of freedom of photons, the standard tomography is practically impossible, but weak-value tomography really works. \[2\]

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\[\text{Footnotes:}\]

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