Abstract

In the present paper, I have re-examined the weak basis invariants at low energies, proposed by C. Jarlskog and Branco et. al, respectively, in their earlier analyses, after confronting them with the assumptions of two zeros and an equality between arbitrary non-zero elements in the Majorana neutrino mass matrix in the flavoured basis. This particular conjecture is found to be experimentally feasible as shown by S. Dev and D. Raj in their recent work. The present analysis attempts to find the necessary and sufficient condition for CP invariance for each experimentally viable ansatz, pertaining to the model along with some important implications.

1 Introduction

In the Standard model (SM), CP violation is considered as a potent tool to explore the flavor sector [1-5], and to probe the signals of New Physics (NP). After the resounding success obtained for K meson decay system, CP violation is well established in the B meson system as well, owing to the efforts made at $e^+e^-$ B factories with their detectors BaBar (SLAC) and Belle(KEK). At present, Cabibbo-Kobayashi-Maskawa (CKM) [6] mechanism appears to be a only way to understand CP violation in the SM. This CKM picture of CP violation is well supported by the precision measurements of $\sin2\beta$ from the CP asymmetry in the decay $B \rightarrow \psi K$, as well as by the reconstruction of unitarity triangle through global fits by various well known groups like Particle Data Group (PDG) [7], CKMfitter [8], UTfit [9] and HFAG [10]. The CP violation in the quark sector stems from the irremovable phase $\delta$, which appears in Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix [6]. On the contrary, SM does not allow the CP violation in the lepton sector. However, in the spirit of quark-lepton symmetry, it is natural to expect an entirely analogous mechanism to arise in the lepton sector, leading to leptonic CP violation. The observation of neutrino oscillation, [11-14], in this regard, provided the first evidence
that CP may be violated in the lepton sector apart from the fact that neutrinos are massive as well as prefer mixing. This further lead to a sudden spurt of activity on the theoretical as well as experimental front to identify the possibilities beyond the standard model, where leptonic CP violation would be observed. The fact that all three neutrino mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$) have been measured to a reasonably good degree of precision [15], only empower our prospects of finding CP violation in the future experiments.

In contrast to the quark case, the parameterization of CP violation in the lepton sector is complicated by several factors. In particular, if the neutrinos are Majorana particles then in comparison to the case of CKM matrix, the corresponding leptonic Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix has two additional phases apart from an analogous CKM phase $\delta$. The leptonic mixing matrix $V$ is given by [16, 17]

$$V = \begin{pmatrix}
    c_{12}c_{13} & \frac{s_{12}s_{23}s_{13} + c_{12}c_{23}e^{-i\delta}}{s_{23}c_{13}} & \frac{s_{13}}{s_{23}c_{13}} \\
    -c_{12}s_{23}s_{13} - s_{12}c_{23}e^{-i\delta} & -s_{12}s_{23}s_{13} + c_{12}c_{23}e^{-i\delta} & -s_{13} \\
    -c_{12}c_{23}s_{13} + s_{12}s_{23}e^{-i\delta} & -s_{12}c_{23}s_{13} - c_{12}s_{23}e^{-i\delta} & c_{23}c_{13}
\end{pmatrix} \times \begin{pmatrix}
    e^{i\rho} & 0 & 0 \\
    0 & e^{i\sigma} & 0 \\
    0 & 0 & 1
\end{pmatrix}.$$  

Here, $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$ for $i, j = 1, 2, 3$, and $\delta$ denotes a Dirac type phase, while $\rho, \sigma$ denote the Dirac and Majorana type CP phases, respectively. The Dirac type CP phase is expected to be measured in the experiments with superbeams and neutrino beams from neutrino factories or indirectly through the area of the unitarity triangles defined for the leptonic sector, while the Majorana type CP phases will contribute to lepton number violating (LNV) processes like neutrinoless double beta decay. Therefore, it is expected that neutrino physics could be useful for investigating the leptonic CP violation at low energies apart from having profound implications for the physics of the early universe.

In the flavour basis, where charged lepton mass matrix is diagonal, Majorana neutrino matrix contains all the information regarding the leptonic CP violation at low energy. To facilitate any specific predictions in this regard, it is convenient to restrict the number of free parameters of mass matrix. As an example some elements of neutrino mass matrix can be considered either zero or equal [18, 21] or some co-factors of neutrino mass matrix to be either zero or equal [19, 22, 24]. Among these possibilities, simultaneous existence of texture zeros and an equal elements (or cofactors), also known as hybrid textures is one of the most notable and rigorously studied in the literature. Not only they reduce the number of free parameters to an appreciable extent compare with texture zero, but also can be naturally realized through flavour symmetry. Recently, S. Dev and D. Raj [25], have suggested the new possibility pertaining to the hybrid texture comprising two zero elements, and an equality between the non-zero elements, which reduces the number of free parameters to six. In contrast to the model with one texture zero and an equality, suggested in Ref. [20], the number of experimentally viable possibilities for model with two zeros and an equality, are found to be very less (only eight out of total forty two), and hence relatively more predictive.
However, the hybrid textures like any other texture zero model, are not invariant under weak basis transformation, implying that a given set of texture which exist in a certain weak basis (WB) may not be present or may appear in different entries in another WB, while leading to the same physics. Therefore, it is always instructive to examine the any specific flavor model in a basis independent manner. For that purpose, CP invariants are considered to be an important tool to investigate the CP properties both in the quark and the leptonic sector. The invariants have particularly garnered a special attention, owing to their suitability for the analysis of specific ansätze for neutrino mass matrix without even need to diagonalise it. In this regard, S. Dev et al, have studied the WB invariants for texture two zero mass matrices \cite{26}, as well as for texture one zero and vanishing minor \cite{27}, and derived the CP invariance conditions.

In the present paper, I shall derive the WB basis invariants for hybrid textures with two zero elements and an equality between the non-zero elements. The purpose of such an analysis is to find the leptonic CP violation for a chosen lepton flavour model at low energies. To this end, I shall be finding relationship between the WB invariants in terms of the Majorana neutrino mass matrix for each ansatz. The relations can thus be used to find necessary and sufficient CP invariance condition.

2 CP invariants in lepton sector

Before proceeding further, I would like to briefly define the WB invariants at low energy \cite{28-30}, which must be zero for CP invariance in leptonic sector. The non-zero value signals towards the CP violation. The relevance of CP odd WB invariants in the analysis of the hybrid textures is due to the fact that assumption of two zeros and an equality between the arbitrary elements lead to a decrease in the number of the independent CP violating phases. The number of CP odd invariants coincides with the number of CP-violating phases which arise in the lepton mixing of charged weak current, after all lepton masses have been diagonalised. B. Yu and S. Zhou \cite{31} carried out a numerical analysis to show the minimal set of CP-odd invariants, which lead to the CP conservation, are three, and hence the number is not accidental. The invariant, which is sensitive to Dirac phase only, is given as

\[
I_1 = \text{Tr}[m_\nu m_\nu^\dagger, m_l m_l^\dagger]^3,
\]

where, \(m_\nu\) and \(m_l\) are Majorana neutrino mass matrix and charged lepton mass matrix, respectively. The invariant \(I_1\) is analogous to the \(\text{Tr}[m_d m_d^\dagger, m_u m_u^\dagger]^3\) \cite{28} in the quark sector with three generations. The computation of CP violation through \(I_1\) is possible only in the ”flavoured basis”, \(m_l\) to be real and diagonal. In the flavoured basis, Eq. (1) can be deduced as

\[
I_1 = -6i\Delta_{e\mu}\Delta_{\mu\tau}\Delta_{\tau e}\text{Im}(h_{12}h_{23}h_{31}),
\]

where, \(\Delta_{ij} = m_j^2 - m_i^2\), \(ij\) run over the pairs, \(e\mu, \mu\tau, \tau e\). Also \(\text{arg}(h_{12}h_{23}h_{31})\) is a complex phase responsible for CP violation, and \(h \equiv m_\nu m_\nu^\dagger\), is a hermitian matrix,
and, its nine elements are given by

\[ h_{11} = |m_{ee}|^2 + |m_{e\mu}|^2 + |m_{e\tau}|^2, \]
\[ h_{12} = m_{ee}m_{e\mu}^* + m_{e\mu}m_{ee}^* + m_{e\tau}m_{e\mu}^*, \]
\[ h_{13} = m_{ee}m_{e\tau}^* + m_{e\mu}m_{e\tau}^* + m_{e\tau}m_{e\tau}^*, \]
\[ h_{21} = m_{e\mu}m_{e\mu}^* + m_{\mu\mu}m_{e\mu}^* + m_{\mu\tau}m_{e\tau}^*, \]
\[ h_{22} = |m_{e\mu}|^2 + |m_{\mu\mu}|^2 + |m_{\mu\tau}|^2, \]
\[ h_{23} = m_{e\mu}m_{e\tau}^* + m_{\mu\mu}m_{e\tau}^* + m_{\mu\tau}m_{e\tau}^*, \]
\[ h_{31} = m_{e\tau}m_{e\mu}^* + m_{\mu\tau}m_{e\mu}^* + m_{\tau\tau}m_{e\tau}^*, \]
\[ h_{32} = m_{e\tau}m_{e\tau}^* + m_{\mu\tau}m_{e\tau}^* + m_{\tau\tau}m_{e\tau}^*, \]
\[ h_{33} = |m_{e\tau}|^2 + |m_{\mu\tau}|^2 + |m_{\tau\tau}|^2. \]

The invariant is sensitive to the Dirac type CP phase \( \delta \) and vanishes for \( \delta = 0 \). This can be shown through the relationship between \( I_1 \) and Jarlskog rephasing invariant, \( J_{CP} \) for lepton sector.

By definition, \( J_{CP} \) can be expressed, in terms of \( \delta \) as

\[ J_{CP} = \frac{1}{8} S_{2(12)} S_{2(23)} S_{2(13)} c_{13} \sin \delta, \tag{2} \]

where, \( S_{2(ij)} \equiv \sin 2\theta_{ij}, \ i, j = 1, 2, 3 \).

Also \( J_{CP} \) is an 'invariant function' of mass matrices, and is related to the mass matrices as

\[ \det C = -2 J_{CP} \Delta_{e\mu} \Delta_{\mu\tau} \Delta_{\tau e} \Delta_{12} \Delta_{23} \Delta_{31}, \tag{3} \]

where, \( C \equiv i[m_\nu m_\tau^\dagger, m_\mu m_\mu^\dagger, m_\mu m_\nu^\dagger], \) and, \( \Delta_{12}, etc \) are analogue to the \( \Delta_{ij} \) as defined earlier. The commutator \( C \) is by definition, hermitian and traceless. Thus eigen values are real. In fact they are measurable, even though \( C \) itself is not a measurable. The determination of any traceless \( 3 \times 3 \) may be computed from trace of the third power of the matrix, i.e., \( \det C = \frac{1}{3} \text{Tr}(C^3), \) therefore I have \( \det C = -\frac{1}{3} \text{Tr}[m_\nu m_\tau^\dagger, m_\mu m_\mu^\dagger]^3, \) which is valid for any traceless \( 3 \times 3 \) matrix. Therefore from Eqs.(3), I get

\[ I_1 = -6i J_{CP} \Delta_{e\mu} \Delta_{\mu\tau} \Delta_{\tau e} \Delta_{12} \Delta_{23} \Delta_{31}. \tag{4} \]

The above relation shows that \( I_1 \) is directly proportional to \( \delta \).

Using Eqs. (2), (4) I arrive at

\[ J_{CP} = \frac{\text{Im}(h_{12} h_{23} h_{31})}{\Delta_{\text{sol}}^2 \Delta_{\text{atm}}}, \tag{5} \]

where, \( \Delta_{\text{sol}} \equiv \Delta_{12} \) and \( \Delta_{\text{atm}} \equiv \Delta_{23} \simeq \Delta_{31}, \) are solar and atmospheric neutrino mass squared differences.

The other two invariants, \( I_2 \) and \( I_3, \) are sensitive to both Dirac as well as Majorana type CP phases. The invariant \( I_2 \) is given as

\[ I_2 = \text{Im}(\text{Tr}[m_\nu m_\nu^\dagger m_\mu m_\mu^\dagger m_\mu^\dagger m_\nu^\dagger]). \tag{6} \]
The above invariant was computed, for the first time, to derive the necessary and sufficient condition for CP invariance in the framework of two neutrinos [29]. In this framework, CP violation can occur only due to Majorana type phase. In the flavoured basis, one can re-write $I_2$ as a function of the elements of $m_\nu$,

$$I_2 = m_{e}^4 m_{\nu} H_{11} + m_{\mu}^4 m_{\nu} H_{22} + m_{\tau}^4 m_{\nu} H_{33}$$
$$+ m_{e}^2 m_{\mu}^2 m_{\nu} (H_{12} + H_{21}) + m_{\mu}^2 m_{\tau}^2 m_{\nu} (H_{23} + H_{32})$$
$$+ m_{e}^2 m_{\tau}^2 m_{\nu} (H_{13} + H_{31}),$$

where, $H$ is a complex matrix, and its elements are given as

$$H_{11} = m_{e}^* h_{11} + m_{\mu}^* h_{21} + m_{\tau}^* h_{31},$$
$$H_{12} = m_{e}^* h_{12} + m_{\mu}^* h_{22} + m_{\tau}^* h_{32},$$
$$H_{13} = m_{e}^* h_{13} + m_{\mu}^* h_{23} + m_{\tau}^* h_{33},$$
$$H_{21} = m_{e}^* h_{11} + m_{\mu}^* h_{21} + m_{\tau}^* h_{31},$$
$$H_{22} = m_{e}^* h_{12} + m_{\mu}^* h_{22} + m_{\tau}^* h_{32},$$
$$H_{23} = m_{e}^* h_{13} + m_{\mu}^* h_{23} + m_{\tau}^* h_{33},$$
$$H_{31} = m_{e}^* h_{11} + m_{\mu}^* h_{21} + m_{\tau}^* h_{31},$$
$$H_{32} = m_{e}^* h_{12} + m_{\mu}^* h_{22} + m_{\tau}^* h_{32},$$
$$H_{33} = m_{e}^* h_{13} + m_{\mu}^* h_{23} + m_{\tau}^* h_{33}.$$

From the above, it is apparent that all the elements are complex. Therefore $H$ depends on both Dirac as well as Majorana type CP phases.

On substituting $m_{\nu} m_{\nu}^\dagger$ with $m_{\nu}(m_{\nu} m_{\nu}^\dagger + m_{\nu}^\dagger m_{\nu})$ in Eq. (1), $I_3$ can be trivially expressed as,

$$I_3 \equiv \text{Tr}[m_{\nu} m_{\nu}^\dagger m_{\nu}^* m_{\nu}^\dagger m_{\nu} m_{\nu}^\dagger]$$

The computation of above invariant predicts CP violation for three or more generation of Majorana neutrinos even in the limit of complete neutrino mass degeneracy [29,30]. This is contrary to the case of Dirac neutrinos, where in the limit of exact degeneracy it is well known that there is no CP violation or physical lepton mixing, for an arbitrary number of generations.

In the chosen basis, invariant $I_3$ can be written as

$$I_3 = -6i \Delta_{\nu} \Delta_{\mu} \Delta_{\tau} \Delta_{\tau} \text{Im}(h_{12}^* h_{23}^* h_{31}^*)$$

where, $\text{arg}(h_{12}^* h_{23}^* h_{31}^*)$ is complex phase responsible for CP violation.

Similar to $h$, $h'$ is also hermitian matrix, and its elements are given as

$$h_{11}' = m_{e}^2 |m_{ee}|^2 + m_{\mu}^2 |m_{e\mu}|^2 + m_{\tau}^2 |m_{e\tau}|^2,$$
$$h_{12}' = m_{e}^2 m_{ee} m_{e\mu}^* + m_{\mu}^2 m_{e\mu} m_{e\mu}^* + m_{\tau}^2 m_{e\tau} m_{e\tau}^*,$$
$$h_{13}' = m_{e}^2 m_{ee} m_{e\tau}^* + m_{\mu}^2 m_{e\mu} m_{e\tau}^* + m_{\tau}^2 m_{e\tau} m_{e\tau}^*.$$
\[ h'_{21} = m_e^2 m_{e\mu} m_{e\tau}^* + m_\mu^2 m_{\mu\mu} m_{e\mu}^* + m_\tau^2 m_{\mu\tau} m_{e\tau}^*, \]
\[ h'_{22} = m_e^2 |m_{e\mu}|^2 + m_\mu^2 |m_{\mu\mu}|^2 + m_\tau^2 |m_{\mu\tau}|^2, \]
\[ h'_{23} = m_e^2 m_{e\mu} m_{e\tau}^* + m_\mu^2 m_{\mu\mu} m_{e\mu}^* + m_\tau^2 m_{\mu\tau} m_{e\tau}^*, \]
\[ h'_{31} = m_e^2 m_{e\tau} m_{e\mu}^* + m_\mu^2 m_{\mu\tau} m_{e\mu}^* + m_\tau^2 m_{\mu\tau} m_{e\tau}^*, \]
\[ h'_{32} = m_e^2 m_{e\tau} m_{e\mu}^* + m_\mu^2 m_{\mu\tau} m_{e\mu}^* + m_\tau^2 m_{\mu\tau} m_{e\tau}^*, \]
\[ h'_{33} = m_e^2 |m_{e\tau}|^2 + m_\mu^2 |m_{\mu\tau}|^2 + m_\tau^2 |m_{\tau\tau}|^2, \]

where, \( m_e, m_\mu \) and \( m_\tau \) denote the electron, muon and tau neutrino, respectively.

In Table 1 I have encapsulated the eight viable ansätze belonging to hybrid texture with two texture zeros and one equality condition, as shown by S. Dev and D. Raj [25]. Among the eight ansätze, there are certain pairs \((A1^{IV}, A2^{IV}), (A1^{V}, A2^{V})\) and \((A1^{VI}, A2^{VI})\), which exhibit the similar phenomenological implications and are related via permutation symmetry. This corresponds to the permutation of 2-3 rows and 2-3 columns of \( m_\nu \). The corresponding permutation matrix can be given by

\[
P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{9}\]

With the help of permutation symmetry, one obtains the following relations among the neutrino oscillation parameters,

\[
\theta_{12}^X = \theta_{12}^Y, \quad \theta_{23}^X = 90^\circ - \theta_{23}^Y, \quad \theta_{13}^X = \theta_{13}^Y, \quad \delta^X = \delta^Y - 180^\circ, \tag{10}\]

where X and Y denote the cases related by 2-3 permutation. On the other hand, ansätze C\( ^{II} \) and C\( ^{IV} \) transform onto themselves independently.

| IV | V | VI |
|----|---|----|
| Class A1 | \[\begin{pmatrix} 0 & 0 & e \\ b & b & c \end{pmatrix}\] | \[\begin{pmatrix} 0 & 0 & e \\ b & f & \end{pmatrix}\] | \[\begin{pmatrix} 0 & 0 & e \\ b & c & \end{pmatrix}\] |
| | | | |
| IV | V | VI |
| Class A2 | \[\begin{pmatrix} 0 & d & 0 \\ b & b & c \end{pmatrix}\] | \[\begin{pmatrix} 0 & d & 0 \\ b & f & \end{pmatrix}\] | \[\begin{pmatrix} 0 & d & 0 \\ b & c & \end{pmatrix}\] |
| | | | |
| II | IV |
| Class C | \[\begin{pmatrix} a & d & a \\ 0 & c & 0 \end{pmatrix}\] | \[\begin{pmatrix} a & d & e \\ 0 & e & 0 \end{pmatrix}\] |

Table 1: The eight viable ansätze of Majorana neutrino mass matrices having two texture zeros and one equality.
3 CP invariants for eight experimentally viable hybrid textures

In this section, I compute the WB invariants in terms of mass matrix elements for all the eight experimentally viable hybrid textures.

**Ansatz (A1)\textsuperscript{IV}:** Using Eq. (1), I obtain,

\[ I_1 = -6i\Delta_{e\mu}\Delta_{\mu\tau}\Delta_{e\tau}|m_{e\tau}|^2|m_{\mu\mu}|^2 \operatorname{Im} Q(A1)_{\text{IV}}, \tag{11} \]

where, \( Q(A1)_{\text{IV}} = m_{\tau\tau}m_{\mu\mu}^* \).

The Jarlskog rephasing invariant, which is a measurable neutrino oscillation parameter, can be defined as

\[ J_{\text{CP}} = \frac{|m_{e\tau}|^2|m_{\mu\mu}|^2 \operatorname{Im} Q(A1)_{\text{IV}}}{\Delta_{\text{sol}}\Delta_{\text{atm}}^2}. \tag{16} \]

The main advantage of above equation is to find \( J_{\text{CP}} \) directly without even need to diagonalize the \( m_\nu \). Interestingly, for \( m_{\mu\mu} = m_{\tau\tau} \), I have \( J_{\text{CP}} = 0 \) implying that \( \sin\delta = 0 \). Hence \( m_{\mu\mu} = m_{\tau\tau} \) is essential to find the Dirac type CP invariance.

Using Eqs. (7) and (8), the expressions for invariants \( I_2 \) and \( I_3 \), sensitive to both Dirac and Majorana phases, are found as

\[ I_2 = \Delta_{\mu\tau}^2|m_{\mu\mu}|^2 \operatorname{Im} Q(A1)_{\text{IV}}, \tag{12} \]

and,

\[ I_3 = -6i\Delta_{e\mu}\Delta_{\mu\tau}\Delta_{e\tau}m_{\mu\tau}^2|m_{e\tau}|^2|m_{\mu\mu}|^2 \operatorname{Im} Q(A1)_{\text{IV}}. \tag{13} \]

respectively. From the above discussion, it is clear that CP violation is possible if the phases associated with the \( m_{\mu\mu} \) and \( m_{\tau\tau} \) are not finely tuned.

More apparently, CP invariance condition is allowed for **ansatz** (A1)\textsuperscript{IV} if and only if

\[ \arg(m_{\mu\mu}) = \arg(m_{\tau\tau}). \tag{14} \]

**Ansatz (A1)\textsuperscript{V}:** Similarly, using Eq. (1), one can find invariant \( I_1 \) as

\[ I_1 = -6i\Delta_{e\mu}\Delta_{\mu\tau}\Delta_{e\tau}|m_{e\tau}|^2 \operatorname{Im} Q(A1)_{\text{V}}, \tag{15} \]

where, \( Q(A1)_{\text{V}} = m_{\mu\mu}^2(m_{\mu\tau}^*)^2 \).

The Jarlskog rephasing invariant parameter, can be found in terms of mass matrix elements as

\[ J_{\text{CP}} = \frac{|m_{e\tau}|^2|m_{\mu\mu}|^2 \operatorname{Im} Q(A1)_{\text{V}}}{\Delta_{\text{sol}}\Delta_{\text{atm}}^2}. \tag{16} \]

Therefore, for \( m_{\mu\mu} = m_{\mu\tau} \), I have \( J_{\text{CP}} = 0 \) implies \( \sin\delta = 0 \).

The other two invariants \( I_2 \) and \( I_3 \) can be derived using Eqs. (7), (8)

\[ I_2 = \Delta_{\mu\tau}^2|m_{e\tau}|^2 \operatorname{Im} Q(A1)_{\text{V}}, \tag{17} \]
and,

\[ I_3 = -6i\Delta_{\mu\tau}\delta_{\mu\tau}\Delta_{\tau e}m_{\mu}^2m_{\tau}^4|m_{e\tau}|^2\text{Im}Q_{(A1)^{VI}}. \]  

(18)

The CP invariance condition can be found as

\[ \text{arg}(m_{\mu\mu}) = \text{arg}(m_{\mu\tau}). \]  

(19)

The mismatch between the elements \(m_{\mu\mu}\) and \(m_{\mu\tau}\) can lead to CP violation. **Ansatz (A1)^{VI}**: Using Eqs.\((\text{I})\), it is found that

\[ I_1 = -6i\Delta_{\mu\tau}\delta_{\mu\tau}\Delta_{\tau e}m_{e\tau}^2|m_{e\tau}|^2\text{Im}Q_{(A1)^{VI}}, \]  

(20)

where, \(Q_{(A1)^{VI}} = m_{\mu\mu}^*m_{\tau\tau}^*\).

Using Eq.\((\text{II})\), \(J_{CP}\) can be written as

\[ J_{CP} = \frac{|m_{e\tau}|^2|m_{\tau\tau}|^2\text{Im}Q_{(A1)^{IV}}}{\Delta_{sol}\Delta_{atm}^2}. \]  

(21)

As found in ansatz \(A1^{IV}\), Eq. \((\text{II})\) provides \(J_{CP} = 0\) for \(m_{\mu\mu} = m_{\tau\tau}\). In addition phase relation for CP invariance is similar to ansatz \(A1^{IV}\).

The other invariants \(I_2\) and \(I_3\) can be derived using Eqs.\((\text{II})\) and \((\text{III})\),

\[ I_2 = \Delta_{\mu\tau}^2|m_{e\tau}|^2\text{Im}Q_{(A1)^{VI}}, \]  

(22)

and,

\[ I_3 = -6i\Delta_{\mu\tau}\delta_{\mu\tau}\Delta_{\tau e}m_{\mu}^2m_{\tau}^4|m_{e\tau}|^2|m_{\tau\tau}|^2\text{Im}Q_{(A1)^{VI}}. \]  

(23)

Similarly, one can find the expressions for WB invariants for ansätze \(A2^{IV}, A2^{V}\) and \(A2^{VI}\) from \(A1^{IV}, A1^{V}\) and \(A1^{VI}\), respectively, by using the \(\mu - \tau\) exchange permutation symmetry, as mentioned earlier.

The condition for CP invariance is similar for \(A2^{IV}\) (\(A2^{VI}\)) as found for \(A1^{IV}\) (\(A1^{VI}\)).

**Ansatz C^{II}**: The invariant \(I_1\) can be derived by using Eq.\((\text{I})\)

\[ I_1 = -6i\Delta_{\mu\tau}\delta_{\mu\tau}\Delta_{\tau e}(|m_{e\mu}|^2 - |m_{e\tau}|^2)|m_{e\tau}|^2\text{Im}Q_{C^{II}}, \]  

(24)

where, \(Q_{C^{II}} = m_{\mu\tau}m_{\epsilon\epsilon}^*\).

The Jarlskog rephasing invariant parameter \(J_{CP}\) can be written as

\[ J_{CP} = \frac{(|m_{e\mu}|^2 - |m_{e\tau}|^2)|m_{e\epsilon}|^2\text{Im}Q_{C^{II}}}{\Delta_{sol}\Delta_{atm}^2}. \]  

(25)

From the above equation, it is explicit that \(J_{CP} = 0\) might be due to \(|m_{e\mu}| = |m_{e\tau}|\).

Keeping in mind the texture zero condition for ansatz \(C^{II}\), it can be inferred that Dirac type CP symmetry is related to \(\mu - \tau\) reflection symmetry \((32,33)\). The other possibility for CP invariance arises if \(m_{\mu\tau} = m_{\epsilon\mu}\) is considered.

Interestingly, it is found from the above equation that \(J_{CP}\) and \(|m_{e\epsilon}|\) are strongly related to each other, i.e. \(J_{CP} \propto |m_{e\epsilon}|^2\). This implies the parabolic relation between these experimentally measurable parameters, and further point out that the absence
of neutrinoless double beta decay in future experiments implies $\sin \delta = 0$. The remaining invariants $I_2$ and $I_3$ can be written as
\begin{equation}
I_2 = 2\Delta_{ee}\Delta_{\tau e}|m_{ee}|^2\text{Im}Q_{CI}, \tag{26}
\end{equation}
and,
\begin{equation}
I_3 = -6i\Delta_{e\mu}\Delta_{\mu\tau}\Delta_{\tau e}m_\beta^2(m_\mu^2|m_{ee}|^2 - m_\tau^2|m_{ee}|^2)|m_{ee}|^2\text{Im}Q_{CI}, \tag{27}
\end{equation}
The CP invariance condition can be given as
\begin{equation}
\arg(m_{\mu\tau}) = \arg(m_{e\mu}). \tag{28}
\end{equation}

**Ansatz C^{IV}:** Using Eq.(11), invariant $I_1$ can be written as
\begin{equation}
I_1 = -6i\Delta_{e\mu}\Delta_{\mu\tau}\Delta_{\tau e}m_\beta^4(m_\mu^2|m_{e\mu}|^2 - m_\tau^2|m_{e\tau}|^2)|m_{ee}|^2\text{Im}Q_{C^{IV}}, \tag{29}
\end{equation}
where, $Q_{C^{IV}} \equiv m_{ee}m_{e\mu}^\ast$.

The Jarlskog rephasing invariant can be written as
\begin{equation}
J_{CP} = \frac{(|m_{e\mu}|^2 - |m_{e\tau}|^2)|m_{e\tau}|^2\text{Im}Q_{C^{IV}}}{\Delta_{sol}\Delta_{atm}^2}. \tag{30}
\end{equation}
Again for $m_{ee} = m_{e\mu}$, $J_{CP} = 0$. Similar to ansatz $C^V$, $J_{CP} = 0$ can also hold if $\mu - \tau$ reflection symmetry is assumed.

The other invariants $I_2$ and $I_3$ can be written as
\begin{equation}
I_2 = 2\Delta_{e\mu}\Delta_{\tau e}|m_{e\tau}|^2\text{Im}Q_{C^{IV}}, \tag{31}
\end{equation}
and,
\begin{equation}
I_3 = -6i\Delta_{e\mu}\Delta_{\mu\tau}\Delta_{\tau e}m_\beta^4(m_\mu^2|m_{e\mu}|^2 - m_\tau^2|m_{e\tau}|^2)|m_{e\tau}|^2\text{Im}Q_{C^{IV}}. \tag{32}
\end{equation}

The CP invariance condition is given as
\begin{equation}
\arg(m_{ee}) = \arg(m_{e\mu}). \tag{33}
\end{equation}

From Eqs.(14), (19), (28) and (33) I found that that it is sufficient to assume one of the elements of mass matrix to be real,in order to find CP invariance for all the ansatz corresponding to hybrid texture model. For illustration, if Im($m_{ee}$) = 0 is assumed, one can find, Im($m_{e\mu}$) = 0, Im($m_{\mu\tau}$) = 0, Im($m_{\mu\mu}$) = 0, Im($m_{\tau\tau}$) = 0 coexist from Eqs.(14), (19), (28) and (33).

The Dirac CP phase, which is sensitive to the CP violation in neutrino oscillation is contained in WB invariant $I_1$, while invariants $I_2$ and $I_3$ are measures of Majorana type CP phases, contribute to CP violation in neutrinoless double beta decay. However, it is explicit from the analysis, that all the three CP violating phases are not independent and there is only one physical phase in all the phenomenologically viable ansatze with two texture zeros and an equality between non-zero elements. However Dirac and Majorana CP type phases can not be distinguished in a concerned hybrid texture of Majorana mass matrix.
4 Summary and conclusion

To summarize the analysis, I have computed the three weak basis invariants in terms of neutrino mass matrix elements at low energy for flavored hybrid textures of Majorana neutrinos with two texture zeros and an equality between mass matrix elements. In the analysis, some useful equalities between the different neutrino mass matrix elements are found to be connected with CP invariance corresponding to lepton number conserving process (LNC) for each ansatz. Another interesting result is found, where if one of the element of Majorana neutrino mass matrix is assumed real, it implies CP invariance is automatically favoured by all the viable ansätze. To this end, a necessary and sufficient condition for CP invariance is provided for each ansatz. Similar to the earlier analyses by S. Dev et al [26,27], it is maintained that there is only one physical phase which describes the CP properties in present model. Therefore Dirac and Majorana phases can not be extracted without considering certain assumptions.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] S. L. Glashow, Nucl. Phys. 22, 597 (1961).
[2] S. Weinberg, Phys. Rev. Lett.19, 1264 (1967).
[3] A. Salam, Proc. of the 8th Noble Symposium on Elementary Particle Theory, Relativistic Groups and Analyticity, edited by N.Svartholm (1969).
[4] H. Fritzsch, Gell-Mann and H. Leutwyler, Phys. Lett. B 4, 365 (1973).
[5] For excellent reviews on the Standard Model see, J. F. Dooghue, E. Golowich and B. R.Holstein, Dynamics of the Standard Model, Cambridge University Press, 1992.
[6] N. Cabbibo, Phys. Rev. Lett. 10, 531 (1963); M. Kobayashi and K. Maskawa, Prog. Theor. Phys. B49, 652 (1973).
[7] M. Tanabashi et al. (Particle Data Group) Phys. Rev. D 98 030001 (2018), updated results available at http://pdg.lbl.gov/.

[8] J. Charles et al., CKMfitter Group, updated result available at http://wwwckmfitter.in2p3.fr/.

[9] A. J. Bevan et al. [Utfit Collaboration], JHEP 03, 123 (2014), updated results available at http://www.utfit.org/.

[10] Y. Amhis et al., Heavy Flavor Averaging Group (HFAG), hep-ex/1207.1158 v2(2013), updated results available at http: www.slac.stanford.edu/xorg/hfag/.

[11] R. Davis, Prog. Part. Nucl. Phys. 32, 13 (1994); B.T. Cleveland et al., Astrophys. J. 496, 505 (1998); S. Fukuda et al.,SuperKamiokande Collaboration, Phys. Rev. Lett. B 539, 179 (2002); Q. R. Ahmad et al., SNO Collaboration Phys. Rev. Lett. 89, 011301(2002); S. N Ahmad et al., Phys. Rev. Lett. 92, 181301 (2004).

[12] Y. Fukuda et al. SuperKamiokande Collaboration, Phys. Rev. Lett. 81, 1562 (1998); A. Surdo, MACRO Collaboration, Nucl. Phys. Proc. Suppl. 110, 342 (2002); M. Sanchez, Soudan Collaboration, Phys. Rev D68, 113004 (2003).

[13] K. Eguchi et al., KamLAND Collaboration, Phys. Rev Lett. 90, 021802 (2003); Phys. Rev. Lett. 94, 081801 (2005).

[14] M. H. Ahn et al., K2K Collaboration, Phy. Rev. Lett. 90, 041801 (2003).

[15] I. Esteban et al, JHEP 01 (2019) 106, arXiv: 1811.05487v1 [hep-ph].

[16] H. Fritzsch, Z. Z. Xing, Phys. Lett. B 517 (2001) 363-368, arXiv: hepph/0103242

[17] B. Pontecorvo , Zh. Eksp. Teor. Fiz. (JETP) 33, 549 (1957); ibid. 34, 257 (1958); ibid. 53, 1717(1967); Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. 28, 870(1962).

[18] Paul H. Frampton, Sheldon L. Glashow and Danny Marfatia, Phys. Lett. B 536, 79 (2002), hep-ph/0201008, Zhi-zhong Xing, Phys. Lett. B 530, 159 (2002), hep-ph/0201151; J. Liao, D. Marfatia, K. Whisnant, arXiv:1311.2639 [hep-ph]; D. Meloni, A. Meroni, E. Peinado, Phys. Rev. D 89 (2014) 053009, arXiv:1401.3207 [hep-ph]; P. O. Ludl, W. Grimus, JHEP 07, 090 (2014), arXiv: 1406.3546 [hep-ph].

[19] S. Dev, Radha Raman Gautam and Lal Singh, Phys. Rev. D 87, 073011 (2013), arXiv: 1303.3092 [hep-ph].

[20] Ji-Yuan Liu, Shun Zhou, Phys. Rev. D 87, 093010 (2013), arXiv:1304.2334 [hep-ph].
[21] S. Kaneko, H. Sawanaka and M. Tanimoto, JHEP 0508, 073 (2005), hep-ph/0504074.

[22] L. Lavoura, Phys. Lett. B 609, 317 (2005), hep-ph/0411232; E. I. Lashin and N. Chamoun, Phys. Rev. D 78, 073002 (2008), arXiv:0708.2423 [hep-ph]; E. I. Lashin, N. Chamoun, Phys. Rev. D 80, 093004 (2009), arXiv:0909.2669 [hep-ph].

[23] J. Liao, D. Marfatia and K. Whisnant, JHEP 1409, 013 (2014), arXiv:1311.2639 [hep-ph].

[24] Weijian Wang, Eur. Phys. J. C 73, 2551 (2013), arXiv:1306.3556 [hep-ph]; S. Dev, R. R. Gautam and Lal Singh, Phys. Rev. D 88, 033008 (2013), arXiv:1306.4281 [hep-ph].

[25] S. Dev, D. Raj, To appear in Nuclear physics B, https://doi.org/10.1016/j.nuclphysb.2020.115081.

[26] S. Dev, Sanjeev Kumar, Surender Verma, Phys. Rev. D 79, 033011 (2009).

[27] S. Dev, Shivani Gupta, R. R. Gautam, J. Phys. G 37, 125003 (2010).

[28] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985).

[29] G. C. Branco, L. Lavoura and M. N. Rebelo, Phys. Lett. B 180 (1986) 264.

[30] G. C. Branco, M. N. Rabelo, J. I. Silva-Marcos Phys. Rev. Lett. 82 (1999) 683.

[31] B. Yu, S. Zhou, Phys. Lett. B 800 (2020) 135085, arXiv: 1908.09306 [hep-ph].

[32] P.F. Harrison and W. G. Scott, Phys. Lett. B 547,219 (2002), arXiv: hep-ph/0210197.

[33] W. Grimus and L. Lavoura, Phys. Lett. B 579,113 (2004), arXiv: hep-ph/0305309.