GAUGE INVARIANCE FOR GENERALLY COVARIANT SYSTEMS

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Abstract

Previous analyses on the gauge invariance of the action for a generally covariant system are generalized. It is shown that if the action principle is properly improved, there is as much gauge freedom at the endpoints for an arbitrary gauge system as there is for a system with “internal” gauge symmetries. The key point is to correctly identify the boundary conditions for the allowed histories and to include the appropriate end-point contribution in the action. The path integral is then discussed. It is proved that by employing the improved action, one can use time-independent canonical gauges even in the case of generally covariant theories. From the point of view of the action and the path integral, there is thus no conceptual difference between general covariance and “ordinary gauge invariance”. The discussion is illustrated in the case of the point particle, for which various canonical gauges are considered.
1. Introduction

A generally covariant action is one invariant under reparametrization of the manifold on which the fields are defined. Important examples of theories which are generally covariant in their standard formulation are the theory of gravitation on a closed space and string theory. When one passes to the hamiltonian formulation of these theories one finds that there are constraints among the canonical variables and, moreover, the hamiltonian is a linear combination of the constraints. In Yang-Mills theory, on the other hand, one also finds constraints, but besides them there is a non-zero hamiltonian, the total energy. This difference has led people over the years to regard general covariance as a gauge symmetry deserving special consideration, since a straightforward approach would seemingly imply no motion because the hamiltonian vanishes. In the early days one even spoke of the “frozen formalism”.

However the difference hangs from a thin line. A zero-hamiltonian is a consequence of general covariance only when the canonical coordinates are taken to be scalars under reparametrization $\tau \rightarrow f(\tau)$ of the time variable. This happens to be the case in the standard formulation of gravity and string theories, but it is by no means a theoretical necessity. Indeed, it suffices to perform a $\tau$–dependent canonical transformation to obtain new variables that are not scalars and whose hamiltonian does not vanish. Conversely given a system with a non-vanishing hamiltonian one may always perform a time dependent canonical transformation that brings the hamiltonian to zero. In short, the difference between general covariance and “ordinary gauge symmetries” is not invariant under time dependent canonical transformations.
The absence of a clear distinction between general covariance and ordinary gauge invariance is particularly dramatic in 2+1 dimensions, where one can reformulate the vacuum Einstein theory as a Chern-Simons theory with gauge symmetries of the Yang-Mills type [1]. In that case, the changes of coordinates can be expressed as ordinary internal gauge symmetries modulo trivial gauge transformations vanishing on-shell.

The purpose of this paper is to reanalyse two precise issues on which generally covariant systems have been argued to differ from ordinary gauge systems. The first of these issues is the invariance of the action under gauge transformations that do not vanish at the time boundaries. The second -related- issue is whether canonical gauge conditions are permissible in the path integral. On both issues we find that, if the action principle is properly modified by a surface term, one can treat generally covariant systems just as ordinary gauge systems of the Yang-Mills type. Thus one can go beyond the limitations previously found in [2], which were based on the standard form of the action for generally covariant systems. The standard form is too restrictive in this sense.

Although we deliberately restrict the analysis of this paper to precise questions dealing with the transformation properties of the action and do not deal with issues of interpretation, we believe that our results support the view that generally covariant systems can be quantized à la Yang-Mills. This viewpoint agrees with the results found in [3] for 2+1 gravity, as well as with the reduced phase space discussion given in [4,5].
2. Gauge invariance of the action

2.1 Action principle.

Consider a gauge system with canonical coordinates \( q^i, p_i \) \( (i = 1, \ldots, n) \), first class constraints \( G_a(q, p) \approx 0 \) \( (a = 1, \ldots, m) \) and first class Hamiltonian \( H_0(q, p) \),

\[
G_a(q, p) \approx 0 \tag{2.1}
\]

\[
[G_a, G_b] = C_{ab}^c(q, p)G_c \tag{2.2a}
\]

\[
[H_0, G_a] = V^b_a(q, p)G_b \tag{2.2b}
\]

We assume for simplicity that the degrees of freedom and the constraints are bosonic and that there are no second class constraints.

Our starting point is the variational principle in the class of paths \( q^i(\tau), p_i(\tau), \lambda^a(\tau) \) taking prescribed values of a complete set of commuting variables \( Q^i(q, p, \tau) \) at the endpoints \( \tau_1 \) and \( \tau_2 \),

\[
Q^i(q(\tau_1), p(\tau_1), \tau_1) = Q^i_1 \tag{2.3a}
\]

\[
Q^i(q(\tau_2), p(\tau_2), \tau_2) = Q^i_2 \tag{2.3b}
\]

\[
[Q^i, Q^j] = 0 \text{ (at equal times)} \tag{2.3c}
\]

These commuting variables are in equal number as the \( q \)'s ("completeness") and could be for instance the \( q \)'s themselves \( (Q^i = q^i) \) or the \( p \)'s \( (Q^i = p^i) \). The action for the variational principle in which the \( Q \)'s are fixed at the endpoints is
\[ S[q^i(\tau), p_i(\tau), \lambda^a(\tau)] = \int_{\tau_1}^{\tau_2} (p_i \dot{q}^i - H_0 - \lambda^a G_a) d\tau - B(\tau_2) + B(\tau_1) \]  
\quad \text{(for paths obeying (2.3))} 

where the phase space function \( B(q, p, \tau) \) is such that

\[ p_i \delta q^i = P_i \delta Q^i + \delta B \] 
\quad \text{(for fixed \( \tau \)). Here, the \( P \)'s are the momenta conjugate to the \( Q \)'s,}

\[ [P_i, P_j] = 0, \quad [Q^i, P_j] = \delta^i_j \] 

The action (2.4) has an extremum within the class of paths defined by (2.3) since one finds that when the equations of motion and the constraints hold,

\[ \delta S = \int_{\tau_1}^{\tau_2} \frac{d}{d\tau} (p_i \delta q^i) - [\delta B]_{\tau_1}^{\tau_2} = 0 \] 
\quad \text{thanks to (2.5) and } \delta Q^i(\tau_1) = \delta Q^i(\tau_2) = 0.

It should be noted that \( B \) is determined only up to the addition of a function of \( Q^i \). The addition to \( B \) of the function \( V(Q^i) \) amounts to modifying \( P_i \) as \( P_i \rightarrow P_i - \partial V / \partial Q^i \).

2.2 Transversality Conditions

The infinitesimal gauge transformations read [6,7,4]

\[ \delta_e q^i = [q^i, G_a e^a] \] 
\quad \text{(2.8a)}

\[ \delta_e p_i = [p_i, G_a e^a] \] 
\quad \text{(2.8b)}
\[
\delta \epsilon \lambda^a = \frac{\partial \epsilon^a}{\partial \tau} + [\epsilon^a, H_0 + \lambda^b G_b] + \lambda^c \epsilon^b C^a_{bc} - \epsilon^b V^a_b. \tag{2.8c}
\]

Here, the gauge parameters \( \epsilon^a(q, p, \tau) \) are arbitrary functions of \( q^i, p_i \) and \( \tau \). We take \( \epsilon^a \) inside the Poisson bracket in (2.8) so that the gauge transformations are canonical transformations. On the constraint surface \( G_a \approx 0 \), one can pull \( \epsilon^a \) out of the bracket. It follows from (2.8a,b) that the gauge variation of an arbitrary function \( F \) of \( q^i \) and \( p_i \) is given by

\[
\delta \epsilon F = [F, G_a \epsilon^a]. \tag{2.8d}
\]

We shall assume that the functions \( Q^i(q, p, \tau) \) and their conjugates \( P_i(q, p, \tau) \) are such that the “transversality condition”

\[
\frac{\partial G_a}{\partial P_i} \text{ of maximum rank on } G_a \approx 0 \tag{2.9}
\]

holds. This implies that the constraints can be solved for \( m \) of the momenta conjugate to \( Q^i \), say \( P_a \ (a = 1, \ldots, m) \). Consequently, no gauge transformation leaves a given set of \( Q^i \)’s invariant (the variables \( Q^a \) conjugate to \( P_a \) are pure gauge and necessarily transform under gauge transformations), while any set of values for \( Q^i \) is compatible with \( G_a \approx 0 \).

The transversality condition (2.9) is by no means necessary for dealing with the path integral in non canonical gauges. However, it becomes mandatory if one wants to write the path integral as a sum over phase space paths obeying gauge conditions restricting directly the canonical variables \( q^i, p_i \),

\[
\chi_a(q, p, \tau) = 0 \tag{2.10}
\]

(“canonical gauges”). Indeed, in order to reach (2.10), one must replace the original boundary conditions (2.3) by gauge related ones that fulfill (2.10)
at the endpoints. In operator language, this means that one must replace the eigenstates \(|Q^i\rangle\) of the operators \(\hat{Q}^i\) selected by the boundary conditions (2.3), by gauge related ones that are annihilated by \(\hat{\chi}_a\),

\[
|Q^i\rangle \rightarrow \exp(i\hat{\mu}^a\hat{G}_a)|Q^i\rangle,
\]

(2.11a)

\[
\hat{\chi}_a \exp(i\hat{\mu}^aG_a)|Q^i\rangle \geq 0.
\]

(2.11b)

This can be achieved only when the states \(|Q^i\rangle\) do transform under the gauge transformations. For instance, if the states \(|Q^i\rangle\) were annihilated by the constraints, one would have \(\exp(i\hat{\mu}^a\hat{G}_a)|Q^i\rangle = |Q^i\rangle\) and it would be impossible to reach (2.11b). [We assume in (2.11) that \([\chi_a,\chi_b] = 0\). This is permissible because any set of canonical gauge conditions \(\bar{\chi}_a = 0\) can be replaced by an equivalent one \(\chi_a = 0\) on \(G_a = 0\) \((\bar{\chi}_a = 0, G_a = 0 \Leftrightarrow \chi_a = 0, G_a = 0\) such that \([\chi_a,\chi_b] = 0\).]

Furthermore, as we shall see, the path integral in a canonical gauge involves only paths that fulfill the constraints everywhere, including the endpoints. The transversality condition enables one to assume that \(G_a \approx 0\) holds at \(\tau_1\) and \(\tau_2\): without (2.9), the conditions \(Q^i = Q^i_1\) or \(Q^i = Q^i_2\) could conflict with \(G_a \approx 0\). Because our ultimate goal is to investigate how to impose canonical gauges in the path integral, we shall assume from now on that the transversality condition holds.
2.3 Invariance of the action under gauge transformations vanishing at the endpoints.

Since the action (2.4) has been defined only for paths obeying the boundary conditions (2.3), it is only meaningful, at this stage, to compute its variation for gauge transformations preserving (2.3). This forces the gauge parameter $\epsilon^a$ in (2.8) to vanish at the endpoints,

$$\epsilon^a(q(\tau_1), p(\tau_1), \tau_1) = 0 \quad (2.12a)$$

$$\epsilon^a(q(\tau_2), p(\tau_2), \tau_2) = 0 \quad (2.12b)$$

The variation of the action (2.4) under the transformation (2.8) is equal to the endpoint term

$$\left[ P_i \frac{\partial (\epsilon^a G_a)}{\partial p_i} - \epsilon^a G_a - [B, \epsilon^a G_a] \right]_{\tau_2}^{\tau_1} \quad (2.13)$$

and hence is zero thanks to (2.12). Therefore, the action (2.4) is invariant under arbitrary gauge transformations vanishing at the endpoints.

For later use, it is convenient to rewrite (2.13) as

$$\left[ P_i \frac{\partial G}{\partial P_i} - G \right]_{\tau_1}^{\tau_2} \quad (2.14a)$$

where

$$G \equiv \epsilon^a G_a \quad (2.14b)$$

is expressed in terms of $Q^i$ and $P_i$. The equation (2.14a) follows from

$$\delta \epsilon B \equiv [B, G] = p_i \delta \epsilon q^i - P_i \delta \epsilon Q^i \quad (2.14c)$$

(see (2.5)).
2.4 Invariance of the action under gauge transformations not vanishing at the endpoints.

Because of the transversality condition, the boundary conditions are not invariant under gauge transformations. Thus, if the gauge parameters $\epsilon^a$ do not vanish at the endpoints, the boundary conditions are modified and the class of paths under consideration is changed.

Now, the action (2.4) has been defined only for those paths that fulfill (2.3). Therefore, in order to study its transformation properties, it is first necessary to extend it off that class of paths. That is, one must adjust the surface term in (2.4) to the new boundary conditions. The observation made in [2], that without an appropriate boundary term the original action (2.4) is not gauge invariant at the endpoints is correct, but it is just a reflection of the fact that if one insists in fixing the $Q^i$’s at the endpoints, then there is less gauge freedom. The new development reported here is that, while this may be convenient in practice, it is not a necessity of principle.

To analyse the issue of how to extend the action, let us first focus on a single gauge transformation with a definite choice of $\epsilon^a$. The boundary conditions obeyed by the new paths read

$$\bar{Q}^i(\tau_1) \equiv (Q^i - [Q^i, G])(\tau_1) = Q^i_1 \quad (2.15a)$$

$$\bar{Q}^i(\tau_2) \equiv (Q^i - [Q^i, G])(\tau_2) = Q^i_2 \quad (2.15b)$$

since $\delta_\epsilon Q^i = [Q^i, G]$. Thus, in terms of the new paths, it is $\bar{Q}^i(q, p, \tau)$ that is kept fixed at the endpoints. According to the discussion of §2.1 above, the action adopted to the new boundary conditions must differ from the action
(2.4) by the endpoint term \(-D(\tau_2) + D(\tau_1)\), where \(D\) is such that

\[ P_i \delta Q^i = \bar{P}_i \delta \bar{Q}^i + \delta (D + M) \quad (2.16a) \]

with

\[ \bar{P}_i = P_i - [P_i, G] \quad (2.16b) \]

In (2.16a), \(M\) is an infinitesimal function of \(\bar{Q}^i\) reflecting the ambiguity in the surface term.

A direct calculation yields

\[ D + M = P_i \frac{\partial G}{\partial P_i} - G \quad (2.17) \]

The action for the new paths obeying the boundary conditions (2.15) is thus

\[ S[q^i(\tau), p_i(\tau), \lambda^a(\tau)] = \int_{\tau_1}^{\tau_2} (p_i \dot{q}^i - H_0 - \lambda^a G_a) d\tau - \left[ B + P_i \frac{\partial G}{\partial P_i} - G \right]_{\tau_1}^{\tau_2} \quad (2.18) \]

(for paths fulfilling (2.15)). We have taken \(M = 0\). This is the only choice that makes the action invariant as we now show.

Under the gauge transformation generated by \(G\), the class of paths (2.3) is mapped on the class of paths (2.15). The variation of the action is equal to

\[ \delta \epsilon S \equiv S[q'(\tau), p'(\tau), \lambda'(\tau)] - S[q(\tau), p(\tau), \lambda(\tau)] \]

\[ = \int_{\tau_1}^{\tau_2} (p_i \dot{q'}^i - H_0' - \lambda'^a G'_a) d\tau - \int_{\tau_1}^{\tau_2} (p_i \dot{q}^i - H_0 - \lambda^a G_a) d\tau - \left[ B' \right]_{\tau_1}^{\tau_2} + \left[ B \right]_{\tau_1}^{\tau_2} - \left[ P_i \frac{\partial G}{\partial P_i} - G \right]_{\tau_1}^{\tau_2} \]

\[ = 0 \quad (2.19) \]

as it follows from (2.14). In (2.19), we have used the notations \(H_0' = H_0(q', p'), G'_a = G_a(q', p')\) and \(B' = B(q', p', \tau)\). Hence, the extension (2.18)
of $S$ possesses the essential property of making the action invariant under the gauge transformation generated by $G$, even though $\epsilon^a$ does not vanish at the endpoints.

What we have done so far concerns a single gauge transformation. However, one can repeat the analysis in exactly the same way for all gauge transformations. One extends in that manner the action to the entire class of paths obeying the original boundary conditions or any set of gauge related ones. While the original class of paths was determined by $n$ boundary conditions at $\tau_1$ and $\tau_2$, the new class of paths is determined by fewer independent boundary conditions, namely $n - m$. The extended action is gauge invariant and reduces to (2.4) when one freezes the gauge freedom at the endpoints by fixing the non gauge invariant coordinates $Q^a$ to $Q^a_1$ or $Q^a_2$, in which case, the boundary conditions reduce to (2.3).

Strictly speaking, the construction applies only to paths fulfilling the constraints at the endpoints since the gauge transformations may not be integrable off the constraint surface. This, however, is all right because in a canonical gauge, only the paths fulfilling $G_a = 0$ contribute to the path integral (see below), so that only the action for paths lying on the constraint surface is needed.

The action (2.18) depends on the gauge parameters $\epsilon^a(\tau_1)$ and $\epsilon^a(\tau_2)$ through $G(\tau_1)$ and $G(\tau_2)$. By the transversality condition, these can be expressed in terms of the endpoint values of the canonical variables, since the canonical transformation connecting two admissible sets of boundary conditions is unique. The resulting action no longer involves $\epsilon^a$. 

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2.5 Reduced phase space action

One may describe the action obtained in the previous section in terms of the reduced phase space. The reduced phase space is by definition the quotient of the constraint surface by the gauge orbits (see for example [4]). Let us thus consider paths \( q^i(\tau), p_i(\tau), \lambda^a(\tau) \) that project to the same path \( q^{*\alpha}(\tau), p^*_a(\tau) \) in the reduced phase space,

\[
S = S[q^{*\alpha}(\tau), p^*_a(\tau)], \quad \alpha = 1, \ldots, n - m \quad (2.21)
\]

We define \( Q^{*i}(q,p,\tau_k) \) \((k = 1, 2)\) by the following conditions

\[
\begin{align*}
&{[Q^{*i}, G_a]} \approx 0 \quad (2.22a) \\
&Q^{*i} = Q^i + M^i_j(Q^j - Q^j_k) \quad (2.22b) \\
&\frac{\partial (Q^{*i})}{\partial (q,p)} \text{ of rank } n - m \quad (2.22c)
\end{align*}
\]

and we take \( Q^{*i}(q,p,\tau) \) to be solutions of \([Q^{*i}, G_a] \approx 0\) that interpolate between \( Q^{*i}(q,p,\tau_1) \) and \( Q^{*i}(q,p,\tau_2) \). The \( Q^{*i} \) are gauge invariant functions that coincide with \( Q^i \) when \( Q^j = Q^j_1(\tau = \tau_1) \) or \( Q^j_2(\tau = \tau_2) \). The \( Q^{*i} \) are thus reduced phase space functions. They commute because the \( Q^i \) commute,

\[
{[Q^{*i}, Q^{*j}]} \approx 0 \quad (2.23)
\]

On the constraint surface, there are \( n - m \) independent functions among the \( Q^{*i} \), which we denote by \( Q^{*\alpha} \). One may say that the conditions \( Q^i(\tau_k) = Q^i_k \) (on \( G_a \approx 0 \)) (i) contain \( m \) gauge conditions; and(ii) fix \( n - m \) independent
gauge invariant functions that commute (i.e., a “complete set of commuting observables”), namely, the $Q^\alpha_i$. These $Q^\alpha_i$ define “the gauge invariant content” of the $Q^i$.

It is clear that the projected reduced phase space paths that occur in the variational principle diagonalize $Q^\alpha_i(q^*, p^*, \tau_1)$ and $Q^\alpha_i(q^*, p^*, \tau_2)$ at the endpoints. Because the action is invariant under gauge transformations that do not vanish at the endpoints, one concludes that, what is really kept fixed at the endpoints in the new variational principle is the gauge invariant content of the $Q^i$ [8].

Note that $Q^\alpha_i(q^*, p^*, \tau_1)$ and $Q^\alpha_i(q^*, p^*, \tau_2)$ may be different gauge invariant functions even if $Q^i$ does not depend on $\tau$, $Q^i = Q^i(q, p)$. This is because the equations defining $Q^\alpha_i$ involve not only $Q^i$, but also the endpoint values $Q^i_1$ (or $Q^i_2$) of $Q^i$.

One can write explicitly the action $S$ in terms of the reduced phase space coordinates. Because the $Q^*$ are kept fixed at the endpoints, $S$ must be of the form

$$S = \int_{\tau_1}^{\tau_2} \left[ p^*_\alpha \dot{q}^*\alpha - H_0(q^*, p^*) \right] d\tau - \left[ B^* \right]_{\tau_1}^{\tau_2} + \left[ V^*(Q^*) \right]_{\tau_1}^{\tau_2} \quad (2.24)$$

for some function $V$ of the $Q^\alpha$. In (2.24), $B^*$ is the reduced phase space function such that

$$p^*_\alpha \delta q^*\alpha = P^*_\alpha \delta Q^\alpha + \delta B^* \quad (2.25)$$

While the first two terms in (2.24) can be defined within the reduced phase space without reference to the original variational principle (2.3)-(2.4), the term $V^*(Q^*)$ may involve explicitly $Q^i_1$ and $Q^i_2$.

Example: consider a single conjugate pair with action
\[ S[q(\tau), p(\tau), \lambda(\tau)] = \int_{\tau_1}^{\tau_2} (p\dot{q} - \lambda G) \, d\tau \]  
(2.26a)

\[ q(\tau_1) = q_1, \quad q(\tau_2) = q_2 \]  
(2.26b)

\[ G = p - \frac{dV}{dq} \]  
(2.26c)

The system is pure gauge so that the reduced phase space reduces to a single point. The only functional of the reduced phase space paths are thus the constants.

One may extend the action to arbitrary paths \( q(\tau), p(\tau), \lambda(\tau) \) not fulfilling (2.26b) along the lines of the previous sections. One gets

\[ S[q(\tau), p(\tau), \lambda(\tau)] = \int_{\tau_1}^{\tau_2} (p\dot{q} - \lambda G) \, d\tau + V(q_2) - V(q(\tau_2)) + V(q_1) - V(q(\tau_1)) \]  
(2.27)

On the constraint surface, the action reduces to the constant

\[ S = V(q_2) - V(q_1) \]  
(2.28)

If we had started with the variational principle

\[ \delta S = 0, \quad S[q(\tau), p(\tau), \lambda(\tau)] = \int_{\tau_1}^{\tau_2} (p\dot{q} - \lambda G) \, d\tau \]  
(2.29a)

\[ q(\tau_1) = q_1', \quad q(\tau_2) = q_2' \]  
(2.29b)

in which the \( q \)'s are fixed to different values at the endpoints, we would have obtained a different functional of the reduced phase space paths, namely

\[ S' = V(q_2') - V(q_1') \]  
(2.30)
Thus, the action is a functional of the reduced phase space paths that involves also the boundary data $q_1$ and $q_2$.

2.6 Conclusions

By appropriately extending the action for the gauge related paths not fulfilling the original boundary conditions, one can arrange so that it is invariant under arbitrary gauge transformations, and not just those that vanish at the endpoints. The extended action contains a surface term which is, in general, different from zero.

3. Examples

3.1. “Internal gauge symmetries” in the coordinate representation

It has become customary to call a system with constraints that are linear and homogeneous in the momenta “system with internal gauge symmetries”. Yang-Mills systems are of this type. For a system with internal gauge symmetries, the coordinates $q^i$ transform among themselves.

Because the constraints are linear, homogeneous in the momenta, one has

$$p_i \frac{\partial G_a}{\partial p_i} - G_a = 0 \quad (3.1)$$

Hence, if one specifies the $q$’s at the endpoints ($Q^i = q^i$, “coordinate representation”), one finds that the $B$ and $D$-terms introduced above vanish. The action

$$S[q^i(\tau), p_i(\tau), \lambda^a(\tau)] = \int_{\tau_1}^{\tau_2} (p_i \dot{q}^i - H_0 - \lambda^a G_a) d\tau \quad (3.2)$$
without surface term, is gauge invariant. The absence of the surface term, due to the existence of a choice of canonical variables that makes the constraints linear and homogeneous in the momenta, is of great practical value. But – as the analysis of this paper shows –, it is not to be taken as signaling a basic conceptual difference between the systems that obey (3.1) and those that do not.

[Although (3.1) holds for the Yang-Mills field, it does not hold for the Freedman-Townsend model [9]. That model is a theory of a two-form gauge field in Minkowski space. The constraints are of the form \( A_{ij} q^i q^j + B_i q^i \). Thus there are systems which do not obey (3.1) but whose gauge symmetries are “internal” in the sense of not coupling neighbouring spacetime points.]

3.2 Parametrized free particle in one dimension

The parametrized free non-relativistic particle in one dimension is the generally covariant system obtained by including the non-relativistic time \( t \) among the dynamical variables. In the Hamiltonian formalism, the parametrized particle is thus described by the original canonical variables \( q \) and \( p \), as well as by \( t \) and its conjugate momentum \( p_t \).

The action for paths obeying the boundary conditions

\[
\begin{align*}
q(\tau_1) &= q_1, & t(\tau_1) &= t_1 & (3.3a) \\
q(\tau_2) &= q_2, & t(\tau_2) &= t_2 & (3.3b)
\end{align*}
\]

\((Q^i \equiv (q, t))\) is given by

\[
S[q(\tau), p(\tau), t(\tau), p_t(\tau)] = \int_{\tau_1}^{\tau_2} (p \dot{q} + p_t \dot{t} - N\mathcal{H}) d\tau \tag{3.4a}
\]
with
\[ \mathcal{H} = p_t + \frac{p^2}{2m} \] (3.4b)

The gauge transformations read
\[ \delta q = [q, \epsilon \mathcal{H}] \approx \frac{p}{m} \epsilon \] (3.5a)
\[ \delta t = [t, \epsilon \mathcal{H}] \approx \epsilon \] (3.5b)
\[ \delta p_t \approx 0 \quad \delta p \approx 0 \] (3.5c)

and modify the boundary conditions. The gauge invariant extension of the action off the class of paths (3.3) is given by
\[ S[q(\tau), p(\tau), t(\tau), p_t(\tau), N(\tau)] = \int_{\tau_1}^{\tau_2} (p\dot{q} + p_t \dot{t} - N \mathcal{H}) d\tau - \frac{1}{2}[t(\tau_2) - t_2] \frac{p^2(\tau_2)}{m} + \frac{1}{2}[t(\tau_1) - t_1] \frac{p^2(\tau_1)}{m} \] (3.6)
in which one fixes “the gauge invariant content of q and t” at \( \tau_1 \) and \( \tau_2 \),
\[ q^*_1(\tau_1) \equiv \left( q - \frac{p}{m} t + \frac{p}{m} t_1 \right) (\tau_1) = q_1 \] (3.7a)
\[ q^*_2(\tau_2) \equiv \left( q - \frac{p}{m} t + \frac{p}{m} t_2 \right) (\tau_2) = q_2 \] (3.7b)

The action (3.6) and the boundary conditions (3.7) reduce respectively to the action (3.4) and the boundary conditions (3.3) if one imposes the gauge conditions \( t(\tau_1) = t_1 \) and \( t(\tau_2) = t_2 \) at the endpoints.

[To derive (3.6) and (3.7), we have integrated the gauge transformations in the explicit case when \( \epsilon \) involves only \( \tau \). If \( \epsilon \) depends also on \( q, t, p \) and \( p_t \), there are corrections proportional to the constraint to both (3.6) and (3.7). We shall not write explicitly these corrections here as they are not needed in what follows (as we have already mentioned, only the paths fulfilling \( \mathcal{H} = 0 \) contribute to the path integral in a canonical gauge)].
3.3 Relativistic free particle

The constraint for the relativistic particle—which is a generally covariant system—reads

\[ \mathcal{H} = p^2 + m^2 \approx 0 \] (3.8a)

with

\[ p^2 \equiv -(p^0)^2 + \sum_{k=1,2,3} (p^k)^2 \] (3.8b)

The gauge invariant action for paths fulfilling

\[ X_{x_1}^{k*}(\tau_1) \equiv \left( x^k - \frac{p^k x^0}{p^0} + \frac{p^k}{p^0} x_1^0 \right)(\tau_1) = x_1^k \] (3.9a)

\[ X_{x_2}^{k*}(\tau_2) \equiv \left( x^k - \frac{p^k x^0}{p^0} + \frac{p^k}{p^0} x_2^0 \right)(\tau_2) = x_2^k \] (3.9b)

at the endpoints is given by

\[
S[x^\mu(\tau), p_\mu(\tau), N(\tau)] = \int_{\tau_1}^{\tau_2} \left( p_\mu \dot{x}^\mu - N\mathcal{H} \right) d\tau \\
- \left( \frac{p^2 - m^2}{2p^0} \right)(\tau_2)[x^0(\tau_2) - x_2^0] \]
\[
+ \left( \frac{p^2 - m^2}{2p^0} \right)(\tau_1)[x^0(\tau_1) - x_1^0]
\] (3.10)

It reduces to

\[
S[x^\mu(\tau), p_\mu(\tau), N(\tau)] = \int_{\tau_1}^{\tau_2} \left( p_\mu \dot{x}^\mu - N\mathcal{H} \right) d\tau
\] (3.11)

for paths which fulfill, in addition to (3.9), the extra condition

\[ x^0(\tau_1) = x_1^0 \quad x^0(\tau_2) = x_2^0 \] (3.12)

(and thus \( x^\mu(\tau_1) = x_1^\mu \), \( x^\mu(\tau_2) = x_2^\mu \)).
There is a factor of $p^0$ in the denominator of (3.9) and (3.10). On the mass-shell (3.8), these denominators do not vanish because $p^0 = \pm \sqrt{\vec{p}^2 + m^2} \neq 0$.

4. Path integral and canonical gauges

General considerations

4.1 Summing over equivalence classes of paths

The path integral for gauge systems is most accurately described within the BRST formalism. However, in order to understand the conceptual points that follow, it is not necessary to resort to that powerful tool. For this reason, we will adopt here the older point of view in which the path integral is defined to be the sum over equivalence classes of gauge related histories of $\exp iS$, with the appropriate Faddeev-Popov measure [10]. This point of view is correct when the gauge transformations “close off shell”, i.e., when the constraints $G_a$ are chosen in such a way that they form a Lie algebra, $[G_a, G_b] = C_{ab}^c G_c$ with $C_{ab}^c = const$. The BRST treatment, valid in the general case, is given in the Appendix A.

In order to select a representative in each equivalence class of gauge related paths, one imposes gauge conditions. If the paths are required to fulfill the original boundary conditions (2.3) at the endpoints, the gauge freedom is fixed at $\tau_1$ and $\tau_2$. The residual gauge freedom is then given by

$$
\delta q^i = [q^i, \epsilon^a G_a], \quad \delta p_i = [p_i, \epsilon^a G_a] \\
\delta \lambda^a = \frac{\partial \epsilon^a}{\partial \tau} + \ldots
$$

(4.1a)  (4.1b)
with
\[ \epsilon^a(\tau_1) = \epsilon^a(\tau_2) = 0, \quad (4.1c) \]
and is the only freedom to be fixed by the gauge conditions.

The path integral reads [10]
\[ P.I. = \int Dq \, Dp \, D\lambda \, DC \, D\bar{C} \prod_{\tau} \delta(\chi) \exp iS^{\text{eff}} \quad (4.2) \]
with
\[ S^{\text{eff}} = \int_{\tau_1}^{\tau_2} (p_i \dot{q}^i - H_0 - \lambda^a G_a) d\tau - [B]_{\tau_1}^{\tau_2} + S^{\text{gh}} \quad (4.3a) \]
\[ S^{\text{gh}} = \int_{\tau_1}^{\tau_2} \bar{C} \delta_C \chi d\tau \quad (4.3b) \]
In (4.2) the functions \( \chi^a \) of the variables \( q^i, p_i, \lambda^a \) and their time derivatives are such that the conditions
\[ \chi^a = 0 \quad (4.4) \]
freeze the gauge freedom (4.1). The paths are subject to the boundary conditions
\[ Q^i(q(\tau_1), p(\tau_1), \tau_1) = Q_1^i, \quad (4.5a) \]
\[ Q^i(q(\tau_2), p(\tau_2), \tau_2) = Q_2^i, \quad (4.5b) \]
and
\[ C(\tau_1) = C(\tau_2) = 0, \quad (4.5c) \]
\[ \bar{C}(\tau_1) = \bar{C}(\tau_2) = 0, \quad (4.5d) \]
The multipliers \( \lambda^a \) are not fixed at the endpoints, that is, they are summed over. The integration range for \( q, p \) and \( \lambda \) is the whole real line. The interpretation of (4.2) in terms of the operator formalism may be found in [4].
The simplest way to freeze the gauge freedom (4.1) is to take the $\chi$’s to depend on the first time derivatives of the Lagrange multipliers $\lambda^a$ [2],

$$\chi^a \equiv \dot{\lambda}^a - \psi^a(q,p) = 0 \quad (4.6)$$

(“derivative gauges”). Indeed, the gauge transformations that preserve (4.6) are characterized by gauge parameters $\epsilon^a$ that obey second order differential equations. This implies, together with (4.1c) that they vanish (except for “unfortunate” choices of $\psi^a$). The Lorentz gauge in electromagnetism is of the derivative type.

4.2 Operator insertions- Proper time

One can also consider insertions of gauge invariant operators $A[q(\tau), p(\tau), \lambda(\tau)]$,

$$\delta_\epsilon A \approx 0 \quad (4.7)$$

The corresponding path integral reads

$$\langle A \rangle = \int Dq \, Dp \, D\lambda \, DC \, D\bar{C} \prod_\tau \delta(\chi) \, A \exp iS_{\text{eff}} \quad (4.8)$$

and does not depend on the gauge fixing conditions.

A particular operator insertion that has attracted considerable attention in the cases of the parametrized non relativistic particle and the relativistic particle is

$$A \equiv \theta(T) \quad (4.9)$$

where $T$ is the proper time associated with the history and $\theta$ is the Heaviside step function. The insertion of $\theta(T)$ in the path integral restricts the paths contributing to the sum to paths having positive proper time. This implements causality and can be generalized to string theory or gravity [2].
The proper time is defined by

\[ T = \int_{\tau_1}^{\tau_2} N \, d\tau \] (4.10)

and is invariant under gauge transformations vanishing at the endpoints. Since the gauge freedom is frozen at \( \tau_1 \) and \( \tau_2 \) in the path integral (4.8), \( \theta(T) \) is a gauge invariant operator and its insertion is permissible.

### 4.3 Canonical gauge conditions

Instead of derivatives gauge conditions, one may implement canonical gauge conditions of the type

\[ \chi^a(q,p) = 0 \] (4.11)

in the path integral. For this to be possible, one needs to perform a gauge transformation at the endpoints in order to enforce (4.11) at \( \tau_1 \) and \( \tau_2 \). That is, one must replace the original boundary conditions by gauge related ones that fulfill (4.11).

In the canonical gauge \( \chi^a = 0 \), the sum (4.8) over equivalence classes of paths becomes

\[ \langle A \rangle = \int Dq \, Dp \, D\lambda \, DC \, D\bar{C} \prod_{\tau} \delta(\chi^a) \, A \exp iS_{\text{eff}} \] (4.12)

where the paths obey instead of (4.5a,b) the boundary conditions

\[ \bar{Q}^i(q,p,\tau_1) = Q_1^i \] (4.13a)

\[ \bar{Q}^i(q,p,\tau_2) = Q_2^i \] (4.13b)

Here, the \( \bar{Q}^i \) are the variables into which the \( Q^i \) transform under the gauge transformation relating the canonical gauges defined by \( Q^i = Q_1^i \) (\( Q^i = Q_2^i \))
and $\chi^a = 0$. The action in (4.12) contain the boundary term worked out in §2.4 and generated by the gauge transformation at the endpoints that implements (4.11).

If the gauge invariant function $A$ does not involve the Lagrange multipliers as one can assume, by using the equations of motion if necessary, the integral over $\lambda^a$ in (4.12) is straightforward and yields the delta function $\delta(G_a)$. The ghost integral gives the Faddeev-Popov determinant $\det[\chi^a, G_b]$. Thus (4.12) becomes

$$\langle A \rangle = \int Dq \, Dp \, \prod_\tau (\delta(G_a)\delta(\chi^b) \det[\chi^a, G_b]) \exp iS$$  \hspace{1cm} (4.14)

with $S$ given by (2.18). In the product over $\tau$ in (4.14), there is one more delta function of the constraints than there are delta functions of the gauge conditions and Faddeev-Popov determinant factors. This is because the $\lambda^a$ are integrated over at the endpoints, while the ghost are kept fixed and the $\chi^a$ are fixing the gauge only “inside” (the gauge freedom at the endpoints being frozen by the boundary conditions)*. Comparison of the path integral (4.14) with the reduced phase space path integral is discussed in the appendix B.

The expression (4.10) as such for the proper time is not invariant under gauge transformations not vanishing at the endpoints. However, one may extend it off the gauge $t(\tau_1) = t_1, t(\tau_2) = t_2$ (non relativistic particle) or $x^0(\tau_1) = x^0_1, x^0(\tau_2) = x^0_2$ (relativistic particle) in such a way that it is

* Since the equations $\chi^a = 0$ define a good gauge slice, the determinant of $[\chi^a, G_b]$ does not vanish on $G_a \approx 0$. It possesses accordingly a definite sign on each connected component of the constraint surface. We choose $\chi^a$ so that the determinant is positive, because the infinite product over $\tau$ in (4.14) is otherwise ill-defined.
invariant. The relevant expressions are

\[ T = \int_{\tau_1}^{\tau_2} N d\tau + t_2 - t(\tau_2) - [t_1 - t(\tau_1)] \quad \text{(non relativistic particle)} \quad (4.15) \]

and

\[ T = \int_{\tau_1}^{\tau_2} N d\tau + \frac{x_2^0 - x^0(\tau_2)}{p^0(\tau_2)} - \frac{x_1^0 - x^0(\tau_1)}{p^0(\tau_1)} \quad \text{(relativistic particle)} \quad (4.16) \]

These are the expressions that should be used in the path integral in an arbitrary gauge.

In the "proper time gauge" \( \dot{N} = 0 \), supplemented by the conditions \( t(\tau_2) = t_2, t(\tau_1) = t_1 \) (or \( x^0(\tau_2) = x^0_2, x^0(\tau_1) = x^0_1 \)) at the endpoints, the proper time becomes

\[ T = N(\tau_2 - \tau_1) \quad (4.17) \]

and the insertion of \( \theta(T) \) in the path integral amounts to integrating over a positive range for the single integration constant \( N \) \( (\dot{N} = 0) [2]\). In a \( \tau \)-independent canonical gauge, like \( t = 0 \) (non-relativistic case) or \( x^0 = 0 \) (relativistic case), one gets \( N = 0, t^0(\tau_2) = t^0(\tau_1) = 0, x^0(\tau_2) = x^0(\tau_1) = 0 \) and thus \( T \) becomes

\[ T = t_2 - t_1 \quad \text{(non relativistic case),} \quad (4.18) \]

\[ T = \frac{x_2^0}{p^0(\tau_2)} - \frac{x_1^0}{p^0(\tau_1)} \quad \text{(relativistic case),} \quad (4.19) \]
5. The free non-relativistic particle

5.1 Gauge $t = 0$

If one sets $t(\tau_1) = 0$ and $t(\tau_2) = 0$ in the action (3.6) for the parametrized non-relativistic particle, one gets the action

$$S[q(\tau), p(\tau)] = \int_{\tau_1}^{\tau_2} d\tau (p\dot{q}) + \frac{1}{2} t_2 \frac{p^2(\tau_2)}{m} - \frac{1}{2} t_1 \frac{p^2(\tau_1)}{m}$$  \hspace{1cm} (5.1)

on the surface $t = 0$, $p_t + \frac{p^2}{2m} = 0$. The path integral is thus

$$P.I. = \int Dq \, Dp \, Dt \, Dp_t \prod_\tau \delta(t) \delta(p_t) \exp iS$$  \hspace{1cm} (5.2)

(the Faddeev-Popov determinant is unity), with $S$ given by (5.1). The paths in (5.2) are subject to the boundary conditions (3.7), which read here ($t = 0$)

$$\left(q + \frac{p}{m} t_1\right)(\tau_1) = q_1$$ \hspace{1cm} (5.3a)

$$\left(q + \frac{p}{m} t_2\right)(\tau_2) = q_2$$ \hspace{1cm} (5.3b)

The integration over $t$ and $p_t$ is direct and yields 1. Hence, (5.2) becomes

$$P.I. = \int Dq \, Dp \exp i \left\{ \int_{\tau_1}^{\tau_2} d\tau (p\dot{q}) + \frac{1}{2m} \left[t_2 p^2(\tau_2) - t_1 p^2(\tau_1)\right] \right\}$$  \hspace{1cm} (5.4)

By making the canonical change of integration variables

$$Q = q + \frac{p}{m} t_1 + \frac{p}{m} \frac{t_2 - t_1}{\tau_2 - \tau_1} (\tau - \tau_1)$$ \hspace{1cm} (5.5a)

$$P = p$$ \hspace{1cm} (5.5b)

and the time rescaling $\tilde{\tau} = \frac{t_2 - t_1}{\tau_2 - \tau_1} (\tau - \tau_1) + t_1$, one can recast (5.4) in the form

$$P.I. = \int DQ \, DP \exp i \left[ \int_{\tau_1}^{\tau_2} d\tau \left( P \frac{dQ}{d\tau} - \frac{P^2}{2m} \right) \right]$$  \hspace{1cm} (5.6a)
with

\[ Q(\bar{\tau} = \tau_1) = q_1 \quad Q(\bar{\tau} = \tau_2) = q_2 \quad (5.6b) \]

This is just the standard path integral for the non-parametrized particle, known to be equal to

\[ P.I. = [2 \pi i(t_2 - t_1)]^{-1/2} \exp \frac{i(q_2 - q_1)^2}{2(t_2 - t_1)}, \quad (5.7) \]

as it should. Note that (5.7) is a solution of the constraint equations at the endpoints [11, 4].

One can also compute the quantum average of \( \theta(T) \) in the canonical gauge \( t = 0 \). Since the proper time does not depend on the paths in the gauge \( t = 0 \) (see (4.18)), one can pull \( \theta(T) \) out of the path integral. One gets the causal Green function

\[ \langle \theta(T) \rangle = \theta(t_2 - t_1)[2 \pi i(t_2 - t_1)]^{1/2} \exp \frac{i(q_2 - q_1)^2}{2(t_2 - t_1)}, \quad (5.8) \]

This is not a solution of the constraint equation at the endpoints (even though (5.7) is) because the operator being inserted depends on the endpoints data.

5.2 Gauge \( t = p\tau \) – Time flowing back and forth

The path integral may be written in any other canonical gauge by the same method. The boundary term in the action will take a different form and a Jacobian different from unity will generically arise. For example, the gauge

\[ t = p\tau \quad (5.9) \]
can be reached by a transformation (3.5) with

\[ \epsilon(\tau_2, p(\tau_2)) = -t_2 + p(\tau_2)\tau_2 \]  
(5.10a)

\[ \epsilon(\tau_1, p(\tau_1)) = -t_1 + p(\tau_1)\tau_1 \]  
(5.10b)

The Faddeev-Popov determinant is unity and the path integral becomes

\[ [2\pi i(t_2 - t_1)]^{-1/2} \exp \frac{i(q_2 - q_1)^2}{2(t_2 - t_1)} = \int Dq \ Dp \ Dt \ Dp_t \prod_\tau \delta(t - p\tau)\delta \left(p_t + \frac{p^2}{2m}\right) \]
\[ \times \exp \left\{ \int_{\tau_1}^{\tau_2} \left(p\dot{q} + p_t\dot{t}\right)d\tau + \frac{1}{2m}\left[p^2(\tau_2)(t_2 - p(\tau_2)\tau_2) - p^2(\tau_1)(t_1 - p(\tau_1)\tau_1)\right] \right\} \]

(5.11)

The boundary conditions are

\[ q(\tau_1) - \frac{p^2(\tau_1)}{m}\tau_1 + \frac{p(\tau_1)}{m}t_1 = q_1 \]  
(5.12a)

\[ q(\tau_2) - \frac{p^2(\tau_2)}{m}\tau_2 + \frac{p(\tau_2)}{m}t_2 = q_2 \]  
(5.12b)

\[ t(\tau_1) = p(\tau_1)\tau_1 \quad t(\tau_2) = p(\tau_2)\tau_2 \]  
(5.12c)

In the gauge (5.9), the histories with constant negative spatial momentum go back in coordinate time \( t \). Thus we see that even for a non relativistic system time can go backwards.
6. The free relativistic particle

6.1 Gauge $x^0 = 0$ - Schwinger $\Delta_1$ function [12]

The same analysis can be carried out in the case of the relativistic particle (3.8). In the gauge $x^0 = 0$, the path integral reduces to *

$$\text{P.I.} = \int Dx^\mu DP_\mu \prod_\tau (\delta(p^2 + m^2)\delta(x^0)2|p^0|) \exp iS$$ (6.1)

$$= \int Dx^k DP_k \frac{1}{2\omega(\tau_2)} \exp iS^+ + \int Dx^k DP_k \frac{1}{2\omega(\tau_2)} \exp iS^-$$ (6.2)

where $S^+$ (respectively $S^-$) is obtained from (3.10) by setting $x^0 = 0$ and $p^0 = +\omega$ (respectively $-\omega$), with

$$\omega = \sqrt{\vec{p}^2 + m^2}$$ (6.3)

That is,

$$S^+ = \int_{\tau_1}^{\tau_2} p_k \dot{x}^k d\tau - \frac{m^2}{\omega(\tau_2)} x_2^0 + \frac{m^2}{\omega(\tau_1)} x_1^0$$ (6.4a)

$$S^- = \int_{\tau_1}^{\tau_2} p_k \dot{x}^k d\tau + \frac{m^2}{\omega(\tau_2)} x_2^0 - \frac{m^2}{\omega(\tau_1)} x_1^0$$ (6.4b)

The paths in (6.2) are subject to (3.9), i.e.,

$$\left(x^k + p^k \omega x_1^0\right)(\tau_1) = x_1^k$$ (6.5a)

$$\left(x^k + p^k \omega x_2^0\right)(\tau_2) = x_2^k$$ (6.5b)

(for $S^+$) and

$$\left(x^k - p^k \omega x_1^0\right)(\tau_1) = x_1^k$$ (6.5c)

* To achieve $[\chi, \mathcal{H}] = 2|p^0|$ (rather than $2p^0$), one should really take $\chi = +x^0$ on the sheet $p^0 > 0$ and $\chi = -x^0$ on the sheet $p^0 < 0$, e.g. $\chi = \epsilon(p^0)x^0$. 

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\[(x^k - p^k \omega x_0^0) (\tau_2) = x_2^k \quad (6.5d)\]

(for \(S^\pm\)).

To carry out the integral, we make the canonical change of integration variables

\[y^k = x^k + \frac{p^k}{\pm \omega} x_0^0 + \frac{p^k}{\pm \omega} \frac{x_2^0 - x_1^0}{\tau_2 - \tau_1} (\tau - \tau_1) \quad (6.6a)\]

\[P^k = p^k \quad (6.6b)\]

togther with the time rescaling \(\bar{\tau} = \frac{x_2^0 - x_1^0}{\tau_2 - \tau_1} (\tau - \tau_1) + x_1^0\). This brings the path integral to the form

\[P.I. = \int DyDP \frac{1}{2\omega(\tau_2)} \exp i \left\{ \int_{x_1^0}^{x_2^0} \left[ P_k \frac{dy^k}{d\bar{\tau}} - \omega \right] d\bar{\tau} \right\} \]

\[+ \int DyDP \frac{1}{2\omega(\tau_2)} \exp i \left\{ \int_{x_1^0}^{x_2^0} \left[ P_k \frac{dy^k}{d\bar{\tau}} + \omega \right] d\bar{\tau} \right\} \quad (6.7a)\]

where the paths \(y^k(\bar{\tau})\) and \(P_k(\bar{\tau})\) are subject to

\[y^k(x_1^0) = x_1^k, \quad y^k(x_2^0) = x_2^k \quad (6.7b)\]

By descretizing (6.7) into \(N\) time intervals, one finds that the integral over the intermediate \(y\)'s yields \(N - 1\) \(\delta\)-functions \(\delta(P_k^i - P_k^{i-1})\), leaving one with a single integral over \(P_k\),

\[P.I. = \int \frac{dP_k}{2\omega} \left\{ \exp i \left[ P_k(x_2^k - x_1^k) - \omega(x_2^0 - x_1^0) \right] \right\} \]

\[+ \exp i \left[ P_k(x_2^k - x_1^k) + \omega(x_2^0 - x_1^0) \right] \quad (6.8)\]

\[= \int d^4 p \exp[ip(x_2^\mu - x_1^\mu)] \delta(p^2 + m^2)\]

This is just Schwinger’s \(\Delta_1\)-function,

\[P.I. = \Delta_1(x_2, x_1), \quad (6.9)\]
which is a solution of the homogeneous Klein-Gordon equation.

6.2 Gauge $x^0 = 0$ - Causal propagator

One can similarly compute the functional average of $\theta(T)$. The proper time is given by (4.19) in the gauge $x^0 = 0$. For paths lying on the positive sheet of the mass-shell hyperboloid, $\theta(T)$ is equal to $\theta \left( x^0_2 - x^0_1 \frac{\omega(\tau_2)}{\omega(\tau_1)} \right)$; for paths lying on the negative sheet, $\theta(T)$ is equal to $\theta \left( x^0_1 - x^0_2 \frac{\omega(\tau_1)}{\omega(\tau_2)} \right)$. Hence, the path integral for $\langle \theta(T) \rangle$ reads

$$\langle \theta(T) \rangle = \int Dx^k Dp_k \frac{1}{2\omega(\tau_2)} \times \left[ \theta \left( x^0_2 - x^0_1 \frac{\omega(\tau_2)}{\omega(\tau_1)} \right) \exp iS^+ + \theta \left( x^0_1 - x^0_2 \frac{\omega(\tau_1)}{\omega(\tau_2)} \right) \exp iS^- \right].$$

By making the same changes of variables as in previous section, the expression (6.10) becomes

$$\langle \theta(T) \rangle = \theta(x^0_2 - x^0_1) \int \frac{dP_k}{2\omega} \exp i \left[ P_k(x^k_2 - x^k_1) - \omega(x^0_2 - x^0_1) \right] + \theta(x^0_1 - x^0_2) \int \frac{dP_k}{2\omega} \exp i \left[ P_k(x^k_2 - x^k_1) + \omega(x^0_2 - x^0_1) \right],$$

which is the Feynman propagator for the Klein-Gordon equation (up to the factor $i(2\pi)^{-3}$).

6.3 Light cone gauge $x^+ \sim \tau$

The calculations of $\langle 1 \rangle$ and $\langle \theta(T) \rangle$ in the light cone gauge where $x^+$ is proportional to the time $\tau$ proceed along the same lines. To simplify the derivation, we take as gauge conditions

$$x^+ = \frac{\tau - \tau_1}{\tau_2 - \tau_1} (x^+_2 - x^+_1) + x^+_1$$

(6.12)
so that the boundary conditions

\[ x^\mu(\tau_2) = x_2^\mu, \quad x^\mu(\tau_1) = x_1^\mu \]  \hfill (6.13)

fulfill (6.12). The action is then just \( \int_{\tau_1}^{\tau_2} (p_\mu \dot{x}^\mu - N \mathcal{H}) d\tau \) without boundary term. The path integral for \( \langle 1 \rangle \) is

\[
P.I. = \int Dx^\alpha Dx^- Dp^\alpha Dp^+ Dp^- \prod_\tau \delta(p^2 + m^2) |2p^+| \]
\[
\times \exp i \int_{\tau_1}^{\tau_2} d\tau \left[ p_\alpha \dot{x}^\alpha - p^+ \dot{x}^- - p^- \left( \frac{x_2^+ - x_1^+}{\tau_2 - \tau_1} \right) \right] \]  \hfill (6.14)

with \( \alpha = 1, 2 \). The paths are subject to the boundary conditions

\[
x^\alpha(\tau_1) = x_1^\alpha, \quad x^\alpha(\tau_2) = x_2^\alpha \quad (\alpha = 1, 2) \hfill (6.15a)
\]
\[
x^-(\tau_1) = x_1^-, \quad x^-(\tau_2) = x_2^- \hfill (6.15b)
\]

In the product over \( \tau \) in (6.8), there is, as before, one more \( \delta \)-function of the constraint than there are Faddeev-Popov factors \( |2p^+| \).

One can evaluate (6.14) by descretization of the time interval. The integral over the \( x \)'s is direct and provides \( N - 1 \) \( \delta \)-functions \( \delta(p_{i+1} - p_i) \). This leaves a single integral over the momenta \( p_\alpha \) and \( p^+ \). The integral over \( p^- \) is also direct because of the \( \delta \)-function \( \delta(p^2 + m^2) \). The path integral becomes thus

\[
P.I. = \int dp_\alpha \frac{dp^+}{2|p^+|} \exp i \left[ p_\alpha (x_2^\alpha - x_1^\alpha) - p^+(x_2^- - x_1^-) - p^- (x_2^+ - x_1^+) \right] \]  \hfill (6.16a)

with

\[
p^- = \frac{1}{2p^+} [(p_\alpha)^2 + m^2] \]  \hfill (6.16b)
The expression (6.16) can easily be brought to a more familiar form by making the change of variables

$$p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3), \quad p^0 = \pm \omega(p), \quad \omega(p) = \sqrt{(p_1)^2 + (p_2)^2 + (p_3)^2 + m^2}$$

(6.17)

This change of variables maps the negative half line $p^+ < 0$ on the whole real line for $p^3$, with $p^0 = -\omega(p) < 0$. And similarly, it maps the positive real line $p^+ > 0$ on the whole real line $-\infty < p^3 < +\infty$, but this time with $p^0 = \omega(p) > 0$. Furthermore, $(dp^+)/p^+ = dp^3/p^0$. Hence, by separating the $p^+$-integral over $(-\infty, +\infty)$ into an integral over $(-\infty, 0)$ and an integral over $(0, +\infty)$, one gets again the standard Fourier representation of $\Delta_1$,

$$P.I. = \int \frac{dp_0}{2\omega} \exp i\vec{p} \cdot (\vec{x}_2 - \vec{x}_1) \left[ \exp i\omega(p)(x_2^0 - x_1^0) + \exp -i\omega(p)(x_2^0 - x_1^0) \right]$$

(6.18)

The same steps go through if one inserts $\theta(T)$ with $T$ given by (4.19) in the path integral.
7. Conclusions

We have shown in this paper that the action for any gauge system may be made invariant under gauge transformations not vanishing at the endpoints. The invariance is achieved by (i) specifying properly what is fixed at the endpoints; and (ii) including a corresponding boundary term in the action.

It follows from our analysis that one can use time-independent canonical gauges (analogous to the Coulomb gauge) such as $tr\pi = 0$ for gravity on a compact space, rather than the more familiar $tr\pi = g^{1/2}\tau$, or $x^+ = 0$ for the string, rather than $x^+ = p^+\tau$. The discussion has been illustrated in the case of the point particle (relativistic and non-relativistic), for which solutions of the homogeneous wave equation and propagators (causal Green functions) have been computed in canonical gauges.

Our analysis assumes throughout that the transversality condition holds at the endpoints. Locally in phase space, this is not a restriction. Indeed, one can always replace locally the constraints by equivalent ones that are some of the momenta in a new canonical system of coordinates. The corresponding $Q$’s are clearly not left invariant by the gauge transformations and fulfill accordingly the transversality conditions. However, there may be obstructions to the global existence of such $Q$’s. In that case, the transversality condition would necessarily fail at ”singular” points. The question depends on the form of the constraints. The eventual failure of the transversality condition is by no means characteristic of generally covariant systems, since we have given examples of such systems for which the transversality condition hold globally on the constraint surface (parametrized systems, free relativistic massive particle).

Finally, we emphasize that the issues discussed in this paper are more
conceptual than practical. Even though it is permissible to adopt, in principle, time-independent gauge for generally covariant systems, such gauges require the explicit integration of the gauge transformations generated by the constraints in order to adjust the boundary conditions. This is usually not tractable.

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Appendix A: Reaching a canonical gauge in the BRST path integral

The BRST path integral that replaces (4.2) – (4.3) in the general case of an arbitrary gauge system with closed or open algebra is [4,13,14]

\[
P.I. = \int Dq \ Dp \ DL \ D\eta \ D\beta \ D\theta \ \exp iS_K \tag{A.1}
\]

\[
S_K = \int_{\tau_1}^{\tau_2} d\tau (\dot{q}^i p_i + \dot{\eta}^a \mathcal{P}_a + \dot{\lambda}^a b_a + x\bar{C}^a \rho^a - H - [K, \Omega] - [B]_{\tau_1}^{\tau_2} \tag{A.2}
\]

where the sum is extended over all paths that obey the boundary conditions

\[
Q^i(Q(\tau_2), p(\tau_2), \tau_2) = Q^i_2 \quad (A.3a)
\]
\[
Q^i(Q(\tau_1), p(\tau_1), \tau_1) = Q^i_1 \quad (A.3b)
\]
\[
\eta^a(\tau_1) = \eta^a(\tau_2) = 0 \quad (A.3c)
\]
\[
\bar{C}^a(\tau_1) = \bar{C}^a(\tau_2) = 0 \quad (A.3d)
\]
\[
b_a(\tau_1) = b_a(\tau_2) = 0 \quad (A.3e)
\]

The path integral (A.1)-(A.2) does not depend on the choice of \(K\) [6, 13, 14, 4]. By taking \(K\) to be of the form

\[
K = -\mathcal{P}_a \lambda^a + i\bar{C}^a \psi^a(q, p) \tag{A.4}
\]

one may perform the integral over \(b_a\). This yields the expression (4.2) with, however, higher order ghost vertices in \(S_{gh}\) when the gauge algebra does not close (and \(\eta^a\) is replaced by \(C^a\)).

In order to implement the canonical gauge \(\chi^a(q, p) = 0\) \((\delta Q^i \neq 0)\), we first perform the canonical change of variables
\[ q'^i = q^i + [q^i, C], \quad p'_i = p_i + [p_i, C] \quad (A.4a) \]

\[ \eta'^a = \eta^a + [\eta^a, C], \quad \mathcal{P}'_a = \mathcal{P}_a + [\mathcal{P}_a, C] \quad (A.4b) \]

\[ \lambda'^a = \lambda^a + [\lambda^a, C], \quad b'_a = b_a + [b_a, C] \quad (A.4c) \]

\[ \bar{C}'_a = \bar{C}_a + [\bar{C}_a, C], \quad \rho'_a = \rho_a + [\rho_a, C] \quad (A.4d) \]

where \( C \) is the BRST-exact function

\[ C = [-\epsilon^a(q,p,\tau)\mathcal{P}_a, \Omega] \quad (A.5a) \]

\[ = \epsilon^a G_a + \text{"higher order terms in } \mathcal{P}_a \text{ and } \eta^a" \quad (A.5b) \]

The gauge parameters \( \epsilon^a \) in (4.5) are chosen so that the modification of \( q^i \) and \( p_i \) at the endpoints, which reduces to

\[ q'^i = q^i + [q^i, \epsilon^a G_a] \quad (A.6a) \]

\[ p'_i = p_i + [p_i, \epsilon^a G_a] \quad (A.6b) \]

because \( \eta^a(\tau_1) = \eta^a(\tau_2) = 0 \), makes the canonical gauge condition hold at \( \tau_1 \) and \( \tau_2 \).

The change of variables (A.4) has the following effect: (i) it induces in the action the surface term discussed in the text because the ghosts vanish at \( \tau_1 \) and \( \tau_2 \); and (ii) it replaces \( K \) by \([H + [K, \Omega], \epsilon^a \mathcal{P}_a] - (\partial \epsilon^a / \partial \tau) \mathcal{P}_a\). But this second change can be discarded since the path integral does not depend on \( K \).
The next step in order to reach the canonical gauge $\chi_a = 0$ is to take $K$ to be of the form

$$K_\epsilon = -\mathcal{P}_a \lambda_a^a + \frac{i}{\epsilon} \bar{C}_a \chi^a$$  \hspace{1cm} (A.7)

Making the rescaling of variables [6]

$$b_a = \epsilon B_a, \quad \bar{C}_a = \epsilon \bar{C}_a$$  \hspace{1cm} (A.8)

whose Jacobian is unity, taking the limit $\epsilon \to 0$ (the path integral does not depend on $\epsilon$), and integrating over $\mathcal{P}_a$, $\rho^a$, $\lambda^a$, $B_a$, $\bar{C}_a$, and $\eta^a$, one finally gets the desired path integral expression used in the text

$$P.I. = \int Dq \, Dp \, \prod_\tau (\delta(G_a)\delta(\chi^a) \det[\chi^a, G_b]) \exp iS$$  \hspace{1cm} (A.9)

with $S$ given by (2.18).

The same derivation can be repeated for the quantum averages of BRST invariant operators (which may be assumed to depend only on the variables $q$, $p$, $\eta$ and $\mathcal{P}$ of the “minimal sector”).

Appendix B Comparison with the reduced phase space path integral

The path integral $(A.1) - (A.3)$ is physically equivalent to the reduced phase space path integral, but differs from it by endpoint wave function factors whose explicit form depends on the representation of the constraint surface [4].

The reduced phase space path integral is invariant under the redefinitions $G_a \to a_b^a G_b$ of the constraints, while the path integral (A.9) is not since there is in (A.9) one more $\delta(G_a)$ than there are Faddeev-Popov determinant factors.
The reduced phase space path integral can also be cast in the form (A.9),

\[
\int DqDp \prod_\tau (\delta(G_a)\delta(\chi^a) \det[\chi^a, G_b]) \exp \left[i \int_{\tau_1}^{\tau_2} (p_i \dot{q}_i - H_0) d\tau + \text{bound. term} \right]
\]

but the precise meaning of the “measure” \(\prod_\tau (\delta(G_a)\delta(\chi^a) \det[\chi^a, G_b])\) is now different. In (B.1), this measure stands for \(Dq^* Dp^*\) where \(q^*\alpha\) and \(p^*_\alpha\) form a complete set of reduced phase space canonical coordinates [One has \(dq^*\alpha dp^*_\alpha = dq^i dp^i \delta(\chi^a) \delta(G_a) \det[\chi^a, G_b]\) but in \(Dq^* Dp^*\), there is one “unmatched” \(dp^*_\alpha\) in, say the \(q^*\)-representation. Accordingly, the numbers of \(\delta(\chi^a)\), \(\delta(G_a)\) and \(\det[\chi^a, G_b]\) in (A.9) and (B.1) are different].

If \(Q^*\alpha(\tau_1)\) and \(Q^*\alpha(\tau_2)\) are the reduced phase space variables kept fixed in the reduced phase space variational principle, the reduced phase space path integral is equal to

\[
(Q^*_2 \exp -iH_0(\tau_2 - \tau_1))|Q^*_1\rangle
\]

where \(\hat{Q}^*\alpha(\tau_2)|Q^*_2\rangle = Q^*_2 |Q^*_2\rangle\) and \(\hat{Q}^*\alpha(\tau_1)|Q^*_1\rangle = Q^*_1 |Q^*_1\rangle\) (note that \(\hat{Q}^*\alpha(\tau_1)\) and \(\hat{Q}^*\alpha(\tau_2)\) may be different operators). In (B.2), ( | ) stands for the scalar product of the reduced phase space quantization. The states \(|Q^*_1\rangle\) and \(|Q^*_2\rangle\) are complete sets of physical states. Let \(|Q^*_1\rangle\) and \(|Q^*_2\rangle\) be the corresponding states of the Dirac quantization, solutions of the constraint equations

\[
\hat{G}_a |Q^*_1\rangle = 0, \quad \hat{G}_a |Q^*_2\rangle = 0
\]

(these states depend on the form of \(\hat{G}_a\) since the physical state condition (B.3) does). One may show that the path integral (A.1) – (A.3) is equal to
\[ \int dQ_1^{*\alpha} dQ_2^{*\alpha} \langle Q_2^{i\alpha}|Q_2^{*\alpha}\rangle \langle Q_2^{*\alpha}|Q_1^{i\alpha}\rangle \exp -iH_0(\tau_2 - \tau_1)\langle Q_1^{*\alpha}|Q_1^{i\alpha}\rangle \] (B.4)

where \( |Q_1^{i}\rangle \) and \( |Q_2^{i}\rangle \) are respectively the eigenstates of \( \hat{Q}^i(\tau_1) \) and \( \hat{Q}^i(\tau_2) \) with eigenvalues \( Q_1^{i} \) and \( Q_2^{i} \) [4].

The difference between the reduced phase space path integral and the path integral analyzed in the text is of the same type as–and possesses no more significance than–the difference between the kernels of the same operator in two different representations.
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