Non-divergent Fermi velocity for interacting graphene at the Dirac point

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Recent experiments reveal a significant increase in the graphene Fermi velocity close to charge neutrality. This has widely been interpreted as a confirmation of the logarithmic divergence of the graphene Fermi velocity predicted by a perturbative approach. In this work, we reconsider this problem using functional bosonization techniques calculating the effects of electron interactions on the density of states non-perturbatively. We find that the renormalized velocity is finite and independent of the high energy cut-off, and we argue that the experimental observations are better understood in terms of an anomalous dimension. Our results also represent a bosonized solution for interacting Weyl fermions in $(2+1)$ dimensions at half-filling.

It is generally believed that electron-electron interactions play only a small role in the electronic properties of graphene. Since interaction effects are weak, most experimental observations away from charge neutrality can be understood within the quasi-particle framework of Fermi liquid theory (FL). There is a simple argument for why interaction effects are small in graphene. For Dirac systems, the kinetic energy and Coulomb potential energy have the same scaling with carrier density. In other words, the effective strength of the interactions is density independent and depends only on the Fermi velocity $v$.

It is customary $^1$ to define a dimensionless parameter $r_s = e^2/(\hbar v)$ to parametrize the strength of electron interactions, and using the non-interacting value of $v$ calculated from ab initio methods gives $r_s = 2.2$. Several approaches, but most notably those that combine confined RPA and quantum Monte Carlo $^3$ conclude that this value of $r_s$ is much too small to observe signatures of a correlated ground state such as the opening of a Mott gap.

Moreover, graphene is often on a substrate, and this further reduces the effect of interactions $^4$. For the most common situation of graphene on silicon dioxide or boron nitride, we have $r_s = 0.8$.

Still, at charge neutrality there are other reasons to expect that the effect of interactions in graphene will be strong. Calculating the self-energy to leading order in interaction strength, Gonzales et al. found that there is an interaction correction to the Fermi velocity $^5$. In this equation $v_{\text{int}}$ is the “renormalized” Fermi velocity, $\lambda$ is a high-energy cut-off, which for graphene (up to a factor) is typically taken to be the energy where the bands cease to be linear i.e. $\lambda \sim \hbar v/a$ (for lattice spacing $a$) and $\epsilon$ is the energy. Away from charge neutrality, it is also common in the literature to replace (up to finite terms) $\epsilon \sim E_F$ (where $E_F$ is the Fermi energy) even though Eq. (1) is strictly only valid at charge neutrality. The application of this result to finite chemical potential is then justified because the difference between expanding about $\epsilon = 0$ and $\epsilon = E_F$ can be calculated perturbatively and is non-divergent (for details we refer the reader to calculations done within the random phase approximation framework $^7$ or within a renormalization group framework $^9$).

The important observation from Eq. (1) is that at charge neutrality — for any value of the interactions — the renormalized Fermi velocity diverges, the quasi-particle propagator becomes non-analytic and the system has a non-Fermi liquid ground state $^6$. In other words, even for very weak interactions $r_s \ll 1$, correlation physics dominates close to charge neutrality and the ground state is not a Fermi liquid. It should be obvious that Eq. (1) should be understood as a perturbative result (either in small $r_s$ or in a large $N$ expansion). Yet, this result has been used to explain a factor of 3 enhancement of the Fermi velocity observed experimentally $^6$ or to argue theoretically that graphene electrons at charge neutrality approach the speed of light $^6$. With the breakdown of perturbation theory close to the Dirac point, what is required to understand the experimental observations is a non-perturbative solution for interacting Weyl fermions in $(2+1)$ dimensions.

In this Letter, we present a functional bosonization solution for Dirac fermions at charge neutrality interacting through a Coulomb interaction. We explicitly calculate the interacting density of states (DOS) and find that it is no longer linear in energy. Instead, the DOS scales as $\nu(\epsilon) \sim |\epsilon|^{1+\gamma}$, where $\gamma$ is the anomalous dimension that depends only on $r_s$. Acquiring an anomalous dimension is to be expected for critical systems $^12$, and was anticipated for 2d Dirac fermions by several authors $^6101314$. Our non-perturbative calculation for the anomalous dimension for the interacting DOS is directly relevant to the experimental studies on interacting graphene which we now discuss briefly.

The enhancement of the Fermi velocity close to charge neutrality has been observed in infra-red spectroscopy $^15$, scanning tunnelling spectroscopy (STM) $^16$ $^17$, photoemission $^18$, magnetotransport $^9$ and capacitance measurements $^19$. At least for the mag-
netotransport, STM and capacitance measurements, the experimental quantity actually being measured is related to the interacting density of states. In these works, by ignoring the interaction induced anomalous dimension, and by fixing the density of states to be linear in energy, the energy-dependent “effective velocity” (which we call $v_2$ in our analysis below) acquires a logarithmic divergence in both energy and cut-off. However, our theoretical analysis shows that once the anomalous dimension is taken into account, the velocity becomes finite and both energy and cut-off independent. In particular, we predict that measurements done at charge neutrality will not have a divergent Fermi velocity. Moreover, we note that Ref. [17] reported data for an electron puddle and for a hole puddle, but with the same set-up could have measured graphene at charge neutrality, which is the experimental situation that most closely corresponds to the bosonization results presented here.

Bosonization has been successfully used to describe interacting systems in $d = 1$ where the quasi-particle pictures breaks down and a new strongly correlated state of matter emerges: a Luttinger liquid (LL) [20, 21]. In spatial dimensions $d > 1$, systems interacting with singular interactions have been extensively studied in the past [22] and shown to support non Fermi liquid behaviour. As emphasised by many authors (including Ref. [23]), in these systems an approach that does not rely on the intermediate assumption of Fermi liquid behaviour is desirable. Higher dimensional bosonization has been developed in different forms by many authors [24–30], in which work we adopt the formulation of Kopietz et al. [31–33], who generalised to higher dimensions the work of Lee et al. [34]. This approach to bosonization is closer in spirit to the conventional one used in condensed matter systems and it is deeply connected to the field theoretic approach to critical phenomena.

The key idea of bosonization in a condensed matter system (in any dimension) is that it is possible to describe the low energy sector of an interacting fermionic system in terms of an equivalent system of “non-interacting” bosons. Then, by relating the fermionic and the bosonic representations, it is possible to obtain the correlation functions of the original fermionic system non perturbatively. In one spatial dimension, the Fermi surface (FS) is zero dimensional and it is therefore possible to obtain a low energy description by linearising about the two Fermi points. This procedure defines two fermionic fields, often called the right/left chiral fields. The linear theory is successively bosonized in terms of two bosonic fields, one for each of the original fermionic fields. In $d > 1$, due to the finite extent of the FS, it is generally not possible to linearise the spectrum globally, and the curvature of the FS therefore plays an essential role [24].

The extension of the bosonization approach to systems having a finite Fermi surface has been discussed e.g. in Refs. [31–33, 35–37]. This usually involves the subduction of the FS into patches, each patch identifying a subset of $d - 1$ dimensional fermionic field theories that are subsequently bosonized. It is the curvature of the FS that spoils the exact solution of the model [33]. In other words, the problem of bosonizing a theory with a finite FS lies essentially in the lack of a globally defined low energy fermionic field theory.

The central idea of this work is that for graphene at charge neutrality, there are indeed two isolated Dirac points where the Fermi surface is zero dimensional. The low energy fermionic theory is obtained by linearising the original lattice Hamiltonian about the two Dirac points without requiring any patching. In [38] we redefine the low energy theory of graphene including the Coulomb interaction at the lattice level. The theory is described by the imaginary time action $S = S_0 + S_I$

\[
S_0 = \sum_{\eta = \pm} \int_{\tau, \mathbf{x}} \psi_\eta^\dagger(\mathbf{x}, \tau) \left\{ \sigma_0 \partial_\tau - i \eta \mathbf{v} \sigma^I \partial_i \right\} \psi_\eta(\mathbf{x}, \tau) \tag{2}
\]

\[
S_I = \frac{1}{2} \sum_{\eta, \eta', \tau, \mathbf{x}} \int \psi_\eta^\dagger(\mathbf{x}, \tau) \psi_\eta(\mathbf{x}, \tau) V_{\eta, \eta'} \psi_{\eta'}^\dagger(\mathbf{x}', \tau) \psi_{\eta'}(\mathbf{x}', \tau),
\]

where $\tau \in [0, \beta], \beta = 1/T$ is inverse temperature in units of $\hbar = k_B = c = 1$ and $\eta = \pm$ is the chirality index. We have also introduced the set of Pauli matrices $\sigma_i$, the bare Fermi velocity $\mathbf{v}$ and the spinor representation $\psi_\eta^i = (a_\eta^i, b_\eta^i)$, $\psi_\eta^{\dagger i} = (b_\eta^{i\dagger}, a_\eta^{i\dagger})$. (Throughout this work Einstein’s summation convention applies for repeated indices). $V_{\eta, \eta'}(\mathbf{x} - \mathbf{x}')$ are the matrix elements of the Coulomb interaction describing two kinds of forward scattering events: one (called $g_1$) with the same chiral indices, taking place separately in the two Dirac cones, and the other (called $g_2$) with different indices connecting the two cones. Both terms get most of the contribution from transferred momentum $\mathbf{q} \approx 0$ and preserve the chiral symmetry of the action. Equation (2) can also be obtained directly from the continuum model within a gauge theory approach [35]. Next we outline the main steps in the functional bosonization program [38]. The partition function is given by

\[
Z = \int \prod_{\eta = \pm} \mathcal{D}[\psi_\eta^\dagger, \psi_\eta] e^{-S[\psi_\eta^\dagger, \psi_\eta]}, \tag{3}
\]

where $\psi_\eta^\dagger, \psi_\eta$ are independent Grassman fields. In the operator approach to bosonization, one of the central steps of the procedure consists of realizing that the density field $\rho_\eta = \psi_\eta^\dagger \psi_\eta$ has a bosonic character. Being completely general, this observation can be implemented in the functional formalism by introducing a functional Dirac delta function $\delta[\rho_\eta - \psi_\eta^\dagger \psi_\eta] = C \mathcal{D}[\phi_\eta] e^{\exp \{-i \int \phi_\eta \left[ \rho_\eta - \psi_\eta^\dagger \psi_\eta \right] \}}$, where $\phi$ is a Lange-grange multiplier field and $C$ an irrelevant constant [34]. The resulting partition function contains $\phi$ and $\rho$ as dual fields. In $d = 1$ one usually works with the density field
and integrate out the $\phi$ fields; using the conservation of the topological current, the density field is then expressed as $\rho_0 \sim \partial_0 \varphi_0$. Here $\varphi_0$ is an additional phase field satisfying (in the case of abelian bosonization) a chiral $U(1)$ Kac-Moody algebra \cite{20}. Fermionic vertex fields are then obtained in terms of $\varphi_0$ as $\psi_0 \sim e^{i \varphi_0}$. The generalisation of the vertex representation to $d > 1$ turns out to be problematic, and although several proposal exist \cite{39,40}, there is no general agreement on its form. To avoid this problem, here we work in the dual representation $\phi_0$ since it allows to evaluate the fermionic Green’s functions without the explicit knowledge of the vertex representation. The $\phi_0$ field is a two component spinor encoding the collective degrees of freedom of the system and is analogous to the field used to study critical phenomenas \cite{44}. In order to obtain an effective theory solely in terms of the $\phi_0$ field, we integrate out the fermions and arrive to the intermediate result

$$Z = \prod_{\eta = \pm} \int \mathcal{D} \phi_0 e^{-S_2[\phi_0] - \sum_{\eta} S_{1\eta}(\phi_0)}$$

$$S_2[\phi_0] = \frac{1}{2} \sum_{\eta, \eta'} \int_{\tau, \mathbf{x}, \mathbf{x}'} \phi_0[V^1]_{\eta, \eta'} \phi_0'$$

$$S_{1\eta}(\phi_0) = - \text{Tr} \log \left[ 1 - i \sigma_0 \phi_0 G_{00} \right] = \sum_{n=1}^{\infty} \frac{\text{Tr} \left[ i \sigma_0 \phi_0 G_{00} \right]^n}{n},$$

where $G_{00}^{-1} = (\sigma_0 \partial_x - i \eta \nu \sigma^i \partial_i)$ is the non interacting fermionic Green’s function. Notice that the term originally containing the interactions has been mapped into the effectively free term $S_2$. On the other hand, $S_{1\eta}$ formally contains all possible interactions between the fermions and the background field $\phi_0$ in terms of closed loop diagrams. We note that by rescaling $\phi$ by the coupling strength and introducing an $N$ component field, Eq. \cite{11} can also be interpreted as the starting point of a large $N$ expansion \cite{13,42}. However, in our case, the next step consists of showing that only the $n = 2$ loop in $S_{1\eta}$ is non-zero, and that the resulting bosonic effective theory is Gaussian. This large scale cancellation of $n > 2$ loops is at the very heart of the bosonization procedure. In $d = 1$ it was first proven by Dzyaloshinskii and Larkin \cite{43}, who showed the existence of a Ward identity in the charge sector of the theory. Using functional methods, this loop cancellation theorem (LCT) was rederived in Ref. \cite{14} and then extended to higher dimensions in Refs. \cite{31,32}. The key ingredients of the LCT are the existence of isolated Fermi points, the linearity of the spectrum, the presence of infinitely many states (the Schwinger anomaly) and the scalar interaction vertices. We find that the LCT naturally extends to graphene at charge neutrality interacting via the scalar Coulomb interaction and arrive at the effective Gaussian action \cite{38}

$$S_{1\eta}(\phi_0) = \frac{1}{2} \int_{\mathbf{q}, \omega} \phi_0(\mathbf{q}, \omega) \Pi_\eta(\mathbf{q}, \omega) \phi_0(-\mathbf{q}, -\omega)$$

$$\Pi_\eta(\mathbf{q}, \omega) = \frac{1}{16} \frac{\mathbf{q}^2}{\sqrt{\omega^2 + \nu^2 \mathbf{q}^2}} = \Pi(\mathbf{q}, \omega),$$

in frequency-momentum space and $T = 0$. Here $\mathbf{q}$ and $\omega$ are the bosonic momentum and frequencies respectively.

As in the one dimensional case, and contrary to $d = 3$, the polarization function $\Pi_\eta$ is finite and therefore does not renormalize in the RG sense. However, in Eq. \cite{6} $\Pi_\eta$ differs from the analogous $d = 1$ expression in two crucial aspects: the presence of the square root, and the independence from the chiral index \cite{38}. The former is responsible for the additional branch cut structure while the latter changes the role of the $g_2$ interaction, as shown below. Since we are interested in the fermionic sector of the theory, we need to obtain an expression for the interacting fermionic Green’s function. We follow Ref. \cite{33} and employ a method first introduced by Schwinger in the context of one dimensional QED \cite{15}. The interacting Green’s function satisfies the non-homogenous partial differential equation

$$\{ \sigma_0 \partial_\tau - i \eta \nu \sigma^i \partial_i - i \sigma_0 \phi_0(\mathbf{x}, \tau) \} G_\eta(\mathbf{x}, \mathbf{x}'; \tau, \tau') = \delta(\mathbf{x} - \mathbf{x}') \delta^*(\tau - \tau'),$$

where $\delta^*(\tau - \tau')$ is the anti-periodic Dirac delta. Eq. \cite{7} describes the propagation of an electron in a background field defined by the source term $\phi_0$, whose dynamics is described by the effective action \cite{5}. The complete solution of Eq. \cite{7} is obtained by averaging over all possible configurations of $\phi_0$ and it reads \cite{38}

$$G_\eta(\mathbf{x}, \tau) = G_{0\eta}(\mathbf{x}, \tau) e^{Q_\eta(\mathbf{x}, \tau)}$$

$$G_{0\eta}(\mathbf{x}, \tau) = \frac{-i \eta}{4 \pi} \left[ \sigma^i x_i + i \eta \nu \sigma_0 \right] \sqrt{\omega^2 + \nu^2 \mathbf{q}^2}$$

$$Q_\eta(\mathbf{x}, \tau) = \int_{\mathbf{q}, \omega} \left( 1 - \cos \left( \frac{q^0}{2} x_1 - \omega \tau \right) \right) g(\mathbf{q}) (1 + g(\mathbf{q}) 2 \Pi(\mathbf{q}, \omega))^2,$$

where $g(\mathbf{q}) = e^2 / 2\kappa |\mathbf{q}|$, $\kappa$ the dielectric constant of the substrate and the factor of 2 in front of $\Pi$ comes from the $g_2$ processes. The Debye-Waller factor can be split as $Q_\eta(\mathbf{x}, \tau) = R_\eta(0, 0) - S_\eta(\mathbf{x}, \tau)$, where $R_\eta$ and $S_\eta$ are the static and the dynamic structure factors respectively. A detailed analysis of Eq. \cite{8} is reported in the supplemental material \cite{38}, here we discuss the main results.

The quasi particle residue is related to the static structure factor as $Z = e^{R(0, 0)}$ \cite{33}, where $R$ is always negative and presents a divergence both in the UV and in the IR sector, just like in the 1d case. While the UV divergence can be regularised by introducing a (soft) momentum cutoff $\Lambda = 1/\alpha$ on the scale of the lattice spacing, the IR divergence cannot be cured and the quasi particle residue $Z = e^{-\infty} = 0$ signals non FL behaviour. Indeed, the static structure factor is closely related to the electron’s self energy \cite{33,38}, so that the divergence in $R_\eta$ corresponds to the divergence in the self energy that one
would find from a FL approach. However, $S_\eta$ is also IR divergent in such a way that $Q_\eta$ is IR finite, just like in the 1d case \[21\]. The equal space Green’s function for $\eta = \pm$ and $r_s < 4/\pi$ (in cgs units) reads

$$G_r(\tau) = \frac{\alpha^\gamma}{4\pi} \left( \frac{1}{\sigma_0|\tau|} \right)^{2+\gamma} \frac{1}{v^2 + \gamma_1 v^2 v^- \gamma_3}$$  \hspace{1cm} (9)

$$v_p = v \sqrt{1 - \left( \frac{\pi r_s}{4} \right)^2}, \hspace{0.5cm} v_e = \frac{\pi r_s}{4} \frac{\sqrt{v_p/v}}{\arccos(v_p/v)},$$

where $v_p$ and $v_e$ are the velocities of the plasmon and the incoherent excitations respectively. Notice that unlike in 1d, electrons and plasmons move at different velocities here. Moreover, the branch cut in Eq. (8) is responsible for the incoherent excitations of velocity $v_e$ \[46\]. Remarkably, all quantities in Eq. (9) are finite! The anomalous dimension $\gamma = \gamma_1 + \gamma_2 - \gamma_3$ depends only on $r_s$ and it is a consequence of the quantum critical behaviour of the theory, see inset in Fig. (1)

$$\gamma_1 = \frac{v^2}{(v^2 - v_p^2)} \left( r_s - \frac{r_s^2}{2} \right), \hspace{0.5cm} \gamma_3 = \frac{r_s}{2} \frac{v}{v_p} \frac{v^2 + v_p^2}{v^2 - v_p^2}$$ \hspace{1cm} (10)

$$\gamma_2 = \frac{v^2}{4} \frac{v^2 + v_p^2}{v_p} \frac{v^2}{v^2 - v_p^2} \arccos(v_p/v).$$

The effect of the anomalous dimension on the interacting DOS, is obtained from Eq. (9) by Fourier transform (reinstating $\hbar$)

$$\nu_+(\epsilon) = \frac{\alpha^\gamma}{\pi (hv)^{2+\gamma}} (hv_e)^{\gamma_2} (hv_p)^{\gamma_3} \frac{|\epsilon|^{1+\gamma}}{\Gamma[2+\gamma]}.$$ \hspace{1cm} (11)

Equation (11) is plotted in Fig. (2) for different values of $r_s$. In the non-interacting limit ($\gamma = 0)$, we recover the well known result. The slope of the plotted lines defines the effective velocity $v_1 = v^{1+\gamma_1/2} v_e^{\gamma_2/2} v_p^{\gamma_3/2} \Gamma[2+\gamma]^{1/2}$ plotted in Fig. (1). It is important to emphasize that $v_1$ is independent of both energy and cut-off. The value of $r_s = 4/\pi$ defines two different regimes. For $r_s > 4/\pi$ we find that while $\gamma$ changes continuously between the two regimes, the plasmon mode enters the particle hole continuum and decays, meaning that $v_1$ changes behaviour in this “strong coupling” regime. A complete analysis of the strong coupling regime, together with the study of the space-time propagator and the spectral function will be presented elsewhere.

Imagine now performing an experiment to measure the interacting DOS. If ones insists on interpreting the data in terms of an energy dependent velocity $\tilde{\nu}_e(\epsilon) = |\epsilon| / \pi v^2(\epsilon)$ (with no anomalous dimension), by comparison with Eq. (9) we would define $v_2(\epsilon) = v_1/|\epsilon|^{\gamma/2}$. In the limit $r_s \ll 1$, we find to leading order $v_2(\epsilon)/v \simeq 1 + r_s/4 + (r_s^2)/24 \log(\lambda/\epsilon)$, (where $\lambda = v/\alpha$). The presence of the log-divergence in $r_s^2$ is due to a different definition of the perturbative Fermi velocity. This leads us to conclude that the log-divergence of the velocity is an artefact of the FL assumption and that it disappears in the non-perturbative treatment.

To conclude, we have reconsidered the problem of electronic correlations in graphene. The low energy theory has been solved by means of functional bosonization methods without relying on the FL picture at any stage of the calculation. We find a finite, non-perturbative meaning of the Fermi velocity renormalization in terms of plasmons and incoherent excitations processes. We have evaluated the anomalous dimension due to the quantum critical behaviour of the system and showed that considering it is necessary to correctly interpret the experimental data measuring the interacting density of states.
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SUPPLEMENTAL MATERIAL

In the first part of this supplemental material we discuss in depth the derivation of the low energy action presented in the main text. We discuss the various terms arising in the low energy model and their physical meaning. We proceed by considering an alternative derivation of the low energy model based on a gauge field theoretical approach. This offers additional insights on the emerging symmetries of the theory.

In the second part of this supplemental material we give a detailed derivation of the bosonization results presented in the main text. In order to give a better intuition of the material presented in this section, we draw various parallels with the well known one dimensional case of a Luttinger liquid.
LOW ENERGY MODEL

In this section we give a complete derivation of the low energy model of interacting graphene starting from the lattice theory. Although the derivation of the non-interacting model can be easily found in the literature (see ref. [1] for a comprehensive introduction), here we reconsider it for completeness and for highlighting some important steps. The analysis of the interaction term is not usually presented in the literature.

The concept of low energy effective theories is paramount in modern theoretical physics, and in its current form dates back to the introduction by Wilson [2, 3] of renormalization group ideas. The main idea behind the effective field theory approach is that physics at different energy scales is governed by a restricted, relevant number of degrees of freedom. By progressively discarding high energy degrees of freedom, we can arrive at simpler theories capturing the long wavelength and low energy degrees of freedom of otherwise complicated theories. In condensed matter physics, one usually needs to deal with complicated band structures; it is therefore remarkable that such complicated theories can be reduced at low energies to known field theories presenting a certain number of symmetries. These symmetries are often inherited from the underlying lattice or are sometimes due to the continuum approximation. In the latter case, particular care must be taken in order to obtain meaningful results. The same is true for certain kind of anomalies that appear in the continuum model but are absent in the original lattice model, e.g. Schwinger and chiral anomalies in effective theories described by Dirac electrons [4, 5]. A well studied example of theories where these anomalies play a major role are $(1 + 1)D$ theories of Dirac electrons [35].

Coming back to graphene, we start defining the lattice model in terms of a tight binding Hamiltonian. Carbon atoms in graphene arrange themselves in a honeycomb structure comprising of two triangular Bravais sublattices, identifying two sets $A$ and $B$ of inequivalent points, see Fig. (3). The lattice structure is defined by specifying the lattice basis vectors $r_{n_1 n_2} = n_1 c_1 + n_2 c_2$, where $n_i \in \mathbb{Z}$ and $l$ is the lattice spacing. We define interacting electrons on the honeycomb structure with a Hubbard like Hamiltonian with nearest neighbour (NN) hopping and a long range Coulomb interaction. Physically this corresponds to considering the electrons to be constrained on the lattice while the photons can move in the continuum. Note that this is different from a true lattice gauge theory, where the photons are also constrained on the lattice. The total Hamiltonian $H = H_0 + H_I$, where

$$H_0 = -t \sum_{r_n \in A} \sum_{i=1}^{3} \{ a^\dagger(r_n) b(r_n + \delta_i) + h.c. \}$$

$$H_I = \sum_{r_n, r_m} \frac{1}{2} \left\{ V(r_{nm}) [ \rho_{A,n} \rho_{A,m} + \rho_{B,n} \rho_{B,m} ] + V(r_{nm} - \delta_1) [ \rho_{A,n} \rho_{B,m} + \rho_{B,n} \rho_{A,m} ] \right\} .$$

Here $t$ is the hopping parameter and $\delta_1 = l (0, 1)$, $\delta_2 = l(-\sqrt{3}/2, -1/2)$ and $\delta_3 = l(\sqrt{3}/2, -1/2)$ are the NN vectors. The two ladder operators $a(r_n)$ and $b(r_n)$ operates respectively on sublattice $A$ and $B$ and $\rho_{A,n} = a^\dagger(r_n) a(r_n)$, $\rho_{B,m} \delta_1 = b^\dagger(r_m + \delta_1) b(r_m + \delta_1)$ are the density operators respectively. We have also defined $r_{nm} = r_n - r_m$ and $V(r_{nm})$ as the interaction potential. In $H_I$ the first two terms describe interactions between electrons belonging to the same sublattice, while the last term describes interactions between electrons sitting on

FIG. 3: Lattice structure. The honeycomb structure is defined together with the two sets of inequivalent points $A$ and $B$. The two triangular sub lattices are shown for the $A$ (dashed ) and the $B$ (dashed dotted) lattice sites. Here $l$ is the lattice (constant) spacing and $c_1 = \frac{l}{2} (3, \sqrt{3})$, $c_2 = \frac{l}{2} (3, -\sqrt{3})$ are the two lattice vectors.
two different sublattice sites. We consider first the non interacting part of the Hamiltonian defined in Eq. (1). The momentum space representation of the ladder operators reads

$$a_p = \frac{1}{\sqrt{N}} \sum_p a_p(r_n) e^{i p r_n},$$

(2)

where $p \equiv p_n = 2\pi/r_n$ in a system with periodic boundary conditions. Using the above Fourier decomposition, $H_0$ reads

$$H_0 = \frac{1}{N} \sum_p \left\{ \phi_p a_p^\dagger b_p + \phi_p^* a_p^\dagger b_p^\dagger \right\}, \quad \phi_p = -t \sum_{i=1}^3 e^{i p \delta_i}.$$  

(3)

The three link vectors $\delta_1 = l(0,1)$ $\delta_2 = l(-\sqrt{3}/2,-1/2)$ and $\delta_3 = l(\sqrt{3}/2,-1/2)$ are shown in Fig.(1) of the main text. The energy dispersion of the non interacting system is easily found as:

$$E_0 = \sqrt{\phi_p \phi_p^*} = \pm t \sqrt{3 + 2 \cos[|p|(\delta_1 - \delta_2)] + 2 \cos[|p|(\delta_1 - \delta_3)] + 2 \cos[|p|(\delta_2 - \delta_3)]}$$

$$= \pm t \sqrt{3 + 4 \cos \left( p_x \frac{\sqrt{3}}{2} l \right) \cos \left( p_y \frac{3}{2} l \right) + 2 \cos \left( p_x \sqrt{3} l \right)}. \quad \text{(4)}$$

The energy spectrum vanishes linearly at the corner of the Brillouin zone in three pairs of points called the Dirac points. Choosing for example $p_y = 0$, the energy vanishes for

$$2 \cos \left( p_x \frac{\sqrt{3}}{2} l \right) + \cos \left( p_x \sqrt{3} l \right) = -3/2. \quad \text{(5)}$$

The first couple of zeros is then $K_{\pm} = (\pm 4\pi/3\sqrt{3}l,0)$. We can focus just on the couple of points above, since the description of the remaining ones is completely equivalent. The spectrum of the theory contains therefore two isolated zeros. Since the Fermi surface shrinks to zero at the two Dirac points, it is possible to globally linearise the Hamiltonian in the neighbourhood of $K_+$ and $K_-$. The linearisation procedure affects both the spectrum and the operators of the Hamiltonian, and should always be accompanied by the definition of a (high energy) cutoff $\Lambda$ ($\Lambda$) specifying the energy (momentum) window where the effective theory faithfully describes the original system. Looking at the energy spectrum of $H_0$ close to one of the Dirac points, one notices that the spectrum is actually asymmetric in the $p_x/p_y$ direction. For example, in the $p_x$ direction the band curvature is more pronounced than in the $p_y$ direction, therefore two different cutoffs should be chosen. To avoid this complication, we consider low enough energies for which a linear spectrum indeed constitutes a good approximation (this also corresponds to the energy regime often probed in experiments). Let us consider Eq. (1) first; the wave vectors $p_x$ and $p_y$ can be expanded for small deviations around the two Dirac points: $p_x = K_{\pm x} + k_x$, $p_y = K_{\pm y} + k_y$. From dimensional analysis, being $[l] = \text{length}$ and $[t] = \text{energy}$, multiplying and dividing by $\hbar$, $\hbar[3/2lt/\hbar] = \hbar[\text{velocity}]$. So we can identify $v = 3/2lt/\hbar$ as the bare (i.e. related to the non-interacting system) velocity of the propagating modes. Moreover, by noticing that $\hbar v |K_{\pm}| = \pm \epsilon_0$ defines the ground state energy, one arrives to

$$E_0 = \pm \hbar v |k|, \quad \text{(6)}$$

that is a linear dispersion relation of slope $v$ with a true zero energy vacuum state (the $\epsilon_0$ drops out of the expression). The low energy theory is therefore a well defined quantum field theory.

We proceed next by deriving the low energy expression for $H_1$. First we need to define new operators creating electronic excitations close to the two Dirac points. It is useful to introduce the notation $K_{\pm} = \pm K_D$. In Eq. (2) we have defined the Fourier decomposition of an electron ladder operator acting on sub lattice A (B) by creating or destroying a real electron. The first step in defining the low energy theory is to restrict the momentum sum in the momentum window $k \in [-\Lambda, \Lambda]$. Then we use the decomposition $p = k \pm K_D$ to obtain

$$ a(r_n) \simeq \frac{1}{\sqrt{N}} \sum_{-\Lambda}^{\Lambda} k \left\{ e^{i (k+K_D) r_n} a(k+K_D) + e^{i (k-K_D) r_n} a(k-K_D) \right\} = e^{i K_D r_n} a_+ (r_n) + e^{-i K_D r_n} a_- (r_n), \quad \text{(7)}$$

where we have defined the operators destroying electronic excitations close to the $\pm K_D$ points as $a(\pm K_D) \equiv a_{\pm}(k)$, being $N$ the number of sites in the system. In some graphene literature, the $\pm$ index is called the “valley” index,
However throughout this work we will adopt a different view and call ± the chirality index for reasons that will be evident soon. Let us pause for a moment and think about the meaning of Eq. (7): while $a_\pm(r_n)$ is an operator creating an electron at point $r_n$ in the original lattice theory, $a_\pm(r_n)$ creates excitations of momentum $k \pm K_D$; as long as we are interested in the large scale behaviour of the system, this picture is well defined because the relevant physics takes place in the vicinity of the two Dirac points. Mathematically, the new ladder operators $a_\pm(b_\pm)$ may be considered as smooth functions on the scale $K_D^{-1}$ [9]. However, we need to keep in mind that a real electron is always made of both types of excitations. We first use the decomposition of Eq. (7) in $H_0$

$$H_0 \simeq \frac{1}{N} \sum_{k=-\Lambda}^{\Lambda} \left\{ \phi_{k+K_D} a^\dagger_+ (k) b_+(k) + \phi_{k-K_D} a^-_+(k) b_-(k) + \phi_{k+K_D} b^\dagger_+ (k) a_+(k) + \phi_{k-K_D} b^-_+ (k) a_-(k) \right\}. \quad (8)$$

Note that in principle there are terms of the type $a_\pm^\dagger b_-$, but they come with a pre factor $e^{\pm iK_D \cdot r_n}$ that is suppressed when taking the sum over all space. Expanding the $\phi$-functions for $k \ll K_D$

$$\phi_k = -i \sum_{i=1}^{3} e^{i(k+K_D) \cdot \delta_i} \simeq -i \sum_{i=1}^{3} e^{\pm i2K_D \cdot \delta_i} \left\{ 1 + i \delta_i(k \mp K_D) + \mathcal{O}([k \mp K_D]^2) \right\} = \phi_0(k) + \phi(k). \quad (9)$$

The terms $\phi(0)_\pm$ correspond to the vacuum energy and, as previously discussed, are exactly zero. As for the second term, we obtain for the + chirality

$$\phi_+(k) = -ut \left\{ (k_y - K_{D,y}) + \left[ -\frac{\sqrt{3}}{2} (k_x - K_{D,x}) - \frac{1}{2} (k_y - K_{D,y}) \right] e^{-i\sqrt{3}tK_{D,x}} \right\}$$

$$+ \left[ \frac{\sqrt{3}}{2} (k_x - K_{D,x}) - \frac{1}{2} (k_y - K_{D,y}) \right] e^{i\sqrt{3}tK_{D,x}} \right\} = -v(k_x - K_{D,x}) - i v(k_y - K_{D,y}),$$

and similarly for the − chirality. Using the above expression in Eq. (8) we arrive at

$$H_0 = \frac{1}{N} \sum_{k=-\Lambda}^{\Lambda} \left\{ \left( k_x + ik_y \right) a^\dagger_+ (k) b_+(k) + \left( k_x - ik_y \right) b^\dagger_+ (k) a_+(k) - \left( k_x + ik_y \right) a^\dagger_- (k) b_-(k) - \left( k_x - ik_y \right) a_+^\dagger (k) b^\dagger_-(k) \right\}, \quad (11)$$

where now momentum is measured with respect to $K_D$. It is convenient at this point to take the continuum limit by exchanging the sum over discrete momenta for an integral over two dimensional momentum

$$\frac{1}{N} \sum_{k=-\Lambda}^{\Lambda} \to \int_{-\Lambda}^{\Lambda} \frac{d^2k}{(2\pi)^2}. \quad (12)$$

The last step consists in defining the chiral spinor fields as

$$\psi_+ = \begin{pmatrix} a_+ \\ b_+ \end{pmatrix}, \quad \psi_- = \begin{pmatrix} b_- \\ a_- \end{pmatrix}, \quad (13)$$

together with the three Pauli matrices $\sigma_0 = 1$, $\sigma_1 = \sigma_x$ and $\sigma_2 = \sigma_y$. The sub lattice operators enter in Eq. (13) as spinor degrees of freedom. If we interpret the sub lattice degrees of freedom as a pseudo spin, we see that the two chiral fields are rotated one with respect to the other. In this representation the free piece of the Hamiltonian reads:

$$H_0 = \int_{-\Lambda}^{\Lambda} \frac{d^2k}{(2\pi)^2} \left\{ \psi^\dagger_+ (k) \sigma \cdot k \psi_+ (k) - \psi^\dagger_- (k) \sigma \cdot k \psi_- (k) \right\}. \quad (14)$$

This is the momentum space representation of a two dimensional Dirac Hamiltonian in the chiral basis. We will analyse Eq. (14) in detail in the next section, however here we would like to remark that the chiral form of $H_0$ is due to the decomposition of the ladder operators performed in Eq. (7). Even though in three dimensional (Minkowski) space there should be no real chiral decomposition [9], here this decomposition originates at the level of the effective theory.
Next we proceed as in the non interacting case using the expression in Eq. (7) for

\[ H_1 = \sum_{r_n, r_m} \frac{1}{2} \left\{ V(r_n - r_m) \left[ \rho_A(r_n) \rho_A(r_m) + \rho_B(r_n) \rho_B(r_m) \right] + V(r_n - r_m - \delta_1) \left[ \rho_A(r_n) \rho_B(r_m + \delta_1) + \rho_A(r_n + \delta_1) \rho_B(r_m) \right] \right\} = H_{I1} + H_{I2}. \]

The Fourier transform of the density fields reads:

\[
\rho_A(q) = \frac{1}{\sqrt{N}} \sum_p a_p^\dagger a_p, \quad \rho_A(-q) = \frac{1}{\sqrt{N}} \sum_{p'} a_{p'}^\dagger a_{p'},
\]

where \( q = p' - p \) is the transferred momentum. It is now convenient to define the momentum space density operators as

\[
\rho_A(q) = \frac{1}{\sqrt{N}} \sum_p a_p^\dagger a_p, \quad \rho_A(-q) = \frac{1}{\sqrt{N}} \sum_{p'} a_{p'}^\dagger a_{p'},
\]

and similarly for \( \rho_B(r_n) \). In this way \( H_{I1} \) reads:

\[
H_{I1} = \frac{1}{2N} \sum_q V_1(q) \left\{ \rho_A(q) \rho_B(-q) + \rho_B(q) \rho_A(-q) \right\} = V_1(q) = \sum_{n,m} e^{-i\mathbf{q} \cdot (\mathbf{r}_n - \mathbf{r}_m)} V(\mathbf{r}_n - \mathbf{r}_m).
\]

Similarly we obtain for \( H_{I2} \)

\[
H_{I2} = \frac{1}{2N} \sum_q V_2(q) \left\{ \rho_B(r_n) \rho_B(-q) + \rho_B(q) \rho_B(-q) \right\} = V_2(q) = \sum_{n,m} e^{-i\mathbf{q} \cdot (\mathbf{r}_n - \mathbf{r}_m - \delta_1)} V(\mathbf{r}_n - \mathbf{r}_m - \delta_1).
\]

In order to obtain the low energy theory, we proceed as in the non interacting case using the expression in Eq. (7) for the ladder operators. Note that since \( q = p - p' \) and \( p \in [-\Lambda, \Lambda] \), the transferred momentum can only describe low energy scattering events. Like in the 1d case, the construction described here leads to a theory valid for small momentum transfer; however, this does not mean that the theory is only valid for weak interactions, as pointed out by Haldane in the 1d case. We proceed by decomposing the density operators into \( \pm \) states

\[
\rho_A(q) \simeq \frac{1}{\sqrt{N}} \sum_k \left\{ a_+^\dagger (\mathbf{k} + q) a_+ (\mathbf{k}) + a_-^\dagger (\mathbf{k} + q) a_- (\mathbf{k}) + e^{i2\mathbf{K}_D \cdot \mathbf{r}_n} a_+^\dagger (\mathbf{k} + q) a_+ (\mathbf{k}) \right\}
\]

and similarly for the other fields. Defining new chiral density operators as

\[
\rho_{A,\pm}(q) = \frac{1}{\sqrt{N}} \sum_k a_+^\dagger (\mathbf{k} + q) a_\pm (\mathbf{k}) \quad \rho_{A,\pm}(-q) = \frac{1}{\sqrt{N}} \sum_{k'} a_\pm^\dagger (\mathbf{k}' - q) a_\pm (\mathbf{k}'),
\]

\[
\rho_{B,\pm}(q) = \frac{1}{\sqrt{N}} \sum_k b_\pm^\dagger (\mathbf{k} + q) b_\pm (\mathbf{k}) \quad \rho_{B,\pm}(-q) = \frac{1}{\sqrt{N}} \sum_{k'} b_\pm^\dagger (\mathbf{k}' - q) b_\pm (\mathbf{k}'),
\]

and substituting into Eq. (18), the low energy form of \( H_{I1} \) reads:

\[
H_{I1} \simeq \frac{1}{2N} \sum_q V_1(q) \left\{ \rho_{A,\pm}(q) \rho_{A,\pm}(-q) + \rho_{A,\pm}(-q) \rho_{A,\pm}(-q) + \rho_{B,\pm}(q) \rho_{B,\pm}(-q) + \rho_{B,\pm}(-q) \rho_{B,\pm}(-q) \right\}
\]

\[
+ \rho_{A,\pm}(q) \rho_{A,\pm}(-q) + \rho_{A,\pm}(-q) \rho_{A,\pm}(-q) + \rho_{B,\pm}(q) \rho_{B,\pm}(-q) + \rho_{B,\pm}(-q) \rho_{B,\pm}(-q) \right\}
\]

\[
+ \frac{1}{2N^2} \sum_{k,k',q} \left\{ a_+^\dagger (k + q) a_- (k) V_1(q + 2K_D) a_+ (p' - q) a_+ (p') + a_-^\dagger (k + q) a_+ (k) V_1(q - 2K_D) a_- (p' - q) a_- (p') + a_-^\dagger (k + q) a_+ (k) V_1(q - 2K_D) a_- (p' - q) a_- (p') \right\}
\]

+ Umklapp terms.
The above interaction Hamiltonian contains two kind of terms: those that can be written in terms of density fields and those that can not. Terms belonging to the first kind correspond to forward scattering events while terms belonging to the second kind correspond to backward and Umklapp scattering. If we assume a Coulomb scattering potential, forward scattering events get most of the contribution from the $q \simeq 0$ sector, while backscattering terms from $q \simeq 2K_D$ (backward) and $q \simeq 4K_D$ (Umklapp) \[37\]. It is clear that for the Coulomb case, the second kind of terms are sub leading at low energies. Among the forward scattering terms we can distinguish two types of processes: the one described by the first line of Eq. \[23\] take place separately in the the two Dirac cones, while those in the second line take place simultaneously in the two cones, see Fig. 4. Unlike the backscattering terms however, both forward scattering processes preserve chirality; in the 1d literature these terms are known as $g_4$ (same Fermi point) and $g_2$ (different Fermi points) processes. It is interesting to note that in the graphene literature the $g_2$-like processes are never explicitly considered. A similar expression is obtained for $H_{12}$, with $V_1 \rightarrow V_2$. Before writing down the complete interaction Hamiltonian, we evaluate the momentum space interaction potential explicitly. Due to translational invariance, in Eq. \[18\] we can use $R = r_n - r_m$, and take the naive continuum limit in real space by exchanging the sum over discrete coordinates for an integral over the continuum space vector $R$. Performing the Fourier transform

$$V(q) = w \int_{-\infty}^{\infty} d^2R \frac{e^{iq \cdot R}}{|R|} = w \int_0^\infty dR 2\pi J_0(|R|) = w \frac{2\pi}{|q|},$$  

where the electromagnetic coupling in a medium is $w = e^2/(4\pi \kappa)$ \[35\], being $\kappa$ the medium dielectric constant. Finally, $J_0(r)$ is the Bessel function of the first kind. In the naive continuum limit it is clear that $V_1 = V_2 \equiv V(q)$. Using the spinor representation introduced in Eq. \[13\] we can write the total interacting Hamiltonian as:

$$H_1 = \frac{1}{2} \int d^2q \, d^2p \, d^2p' \, V(q) \{ \psi^\dagger_+(p+q)\psi_+(p)\psi^\dagger_+(p'-q)\psi_+(p') + \psi^\dagger_-(p+q)\psi^-_+(p)\psi^\dagger_-(p'-q)\psi_-(p') \\
+ \psi^\dagger_+(p+q)\psi^-_+(p)\psi^\dagger_-(p'-q)\psi_-(p') + \psi^\dagger_-(p+q)\psi^-_+(p)\psi^\dagger_+(p'-q)\psi_+(p') \}.$$  

To make connection with the gauge theoretical approach of the next section, we finally write the real space action corresponding to the total low energy effective Hamiltonian:

$$S_{\text{eff}} = \int dt \, dx \left\{ \psi^\dagger_+(x,t) i(\sigma^0 \partial_t - v \sigma^i \partial_i)\psi_+(x,t) + \psi^\dagger_-(x,t) i(\sigma^0 \partial_t + v \sigma^i \partial_i)\psi_-(x,t) \right\} + \frac{1}{2} \int dt \, dx \, dx' \left\{ \psi^\dagger_+(x,t)\psi_+(x,t)V(x,x')\psi^\dagger_+(x',t)\psi_+(x',t) + \psi^\dagger_-(x,t)\psi_-(x,t)V(x,x')\psi^\dagger_-(x',t)\psi_-(x',t) \\
+ \psi^\dagger_+(x,t)\psi_-(x,t)V(x,x')\psi^\dagger_+(x',t)\psi_+(x',t) + \psi^\dagger_-(x,t)\psi_+(x,t)V(x,x')\psi^\dagger_-(x',t)\psi_-(x',t) \right\}.$$
Symmetries and the gauge theory approach

In this section we show that the off diagonal density interaction terms found in the previous section naturally appear from the continuum, non interacting model using the gauge principle. Indeed this was one of the original approaches to the study of electron correlations in graphene and was first considered by González et al. in Ref. [8]. In their approach, the authors obtain a chiral theory of interacting graphene, i.e. without the off diagonal interaction terms. In order to show the source of the discordance, we start from the non-interacting, continuum model described by Eq. (14) and highlights its symmetries. Building on ref. [9, 10], we show how the gauge principle can be used together with dimensional reduction to give the correct result. We will finally support our result with some general considerations on the difference between chiral and non chiral fermionic models.

First of all, it is convenient to rewrite Eq. (14) in real space. The associated Hamiltonian density reads:

\[ H_0 = -i v \left\{ \bar{\psi}^+ (x) \sigma^i \partial_i \psi_+ (x) - \psi^+ (x) \sigma^i \partial_i \psi_- (x) \right\}, \quad (28) \]

where the fermionic fields have been defined in Eq. (13) and we use Einstein’s summation convention with latin indices \( i = 1, 2 \) for spatial coordinates and \( \partial_i \equiv \partial / \partial x_i \). In order to highlights the symmetries of the above Hamiltonian, we introduce the non-chiral spinor representation \([11]\):

\[ \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \quad (29) \]

Using the above representation, the Hamiltonian reads

\[ H_0 = -w \bar{\psi} (x) \left( \begin{array}{cc} \sigma^i & 0 \\ 0 & -\sigma^i \end{array} \right) \partial_i \psi (x) = -w \bar{\psi} (x) \left( \begin{array}{cc} \sigma^i & 0 \\ 0 & \sigma^i \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \partial_i \psi (x), \quad (30) \]

proceeding as in the one dimensional case \([4]\), we recognise the appearance of the chiral gamma matrix of the \( 4 \times 4 \) Clifford algebra

\[ \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (31) \]

Note that \( \gamma_5 \) as found in Eq. (31) is in the Weyl basis representation \([3]\). Using \( \gamma_5 \) we can define the projectors \( P_\pm = (1 \pm \gamma_5) / 2 \) such that \( P_\pm \psi = \psi_\pm \). Since in the Weyl basis we also have

\[ \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (32) \]

we can rewrite \( H_0 \) in covariant form as follow:

\[ H_0 = -i w \bar{\psi} (x) \gamma_0 \gamma_\mu \gamma_\nu \partial_\mu \psi (x) = -i w \bar{\psi} (x) \left( \begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} \sigma^i & 0 \\ 0 & -\sigma^i \end{array} \right) \partial_i \psi (x) \]

\[ = -i w \bar{\psi} (x) \gamma^i \partial_i \psi (x) \quad (33) \]

where we have defined \( \bar{\psi} = \psi^\dagger \gamma_0 \). The Lagrangian density associated with Eq. (33) has the familiar form

\[ \mathcal{L}_0 = i \bar{\psi} (x) \left\{ \gamma^0 \partial_0 + v \gamma^{\dagger} \partial_1 \right\} \psi (x) = i \bar{\psi} (x) \gamma^\mu \partial_\mu \psi (x), \quad (34) \]

where \( x = t, \boldsymbol{x} \) is a three vector and Greek indices run over the three space time components. It is easy to check that \( \mathcal{L}_0 \) is invariant under a global \( U(1) \) transformation \( \psi' = e^{i \alpha} \psi \), with \( \alpha \) an angular parameter. The Lagrangian is also invariant with respect to a chiral transformation, i.e. a rotation of the spinor \( \psi \) in the \( +/- \) space implemented by:

\[ \psi' = e^{i \gamma_5 \theta} \psi. \]

To see this, one has to make use of the anti-commutation property of the gamma matrices \( \{ \gamma_\mu, \gamma_5 \} = 0 \), from which it also follows that \( \bar{\psi}' = \bar{\psi} e^{i \gamma_5 \theta} \). According to Noether’s theorem, every continuum symmetry of \( \mathcal{L}_0 \) corresponds to a conserved current. In the case of the \( U(1) \) gauge symmetry, the electromagnetic current \( J_\mu = \bar{\psi} \gamma_\mu \psi \) is conserved, while in the case of the chiral symmetry, the chiral current \( J_5 = \bar{\psi} \gamma_5 \gamma_\mu \psi \) is conserved. Physically, the global chiral invariance discussed above corresponds to the arbitrariness in labelling the \( +/\) states, so that exchanging this labelling will not change the physics. As it is well known, a mass term breaks the global chiral invariance since \( m \bar{\psi} \gamma_5 \psi = m e^{2i \gamma_5 \theta} \bar{\psi} \psi \). If \( \langle \bar{\psi} \psi \rangle \neq 0 \) in a specific phase, i.e. if it acquires a vacuum expectation value, then we say
that the chiral symmetry is spontaneously broken. This mechanism, familiar in Quantum Chromodynamics (QCD) where it leads to the breakdown of the $SU(2)$ chiral symmetry between left and right quarks, has also been proposed in graphene \[12\,13\]. Note that this way of breaking the chiral symmetry, and therefore introducing a mass gap, is essentially classical. There is in fact another way of introducing a violation of the chiral symmetry that does not involve symmetry breaking nor the appearance of a gap in the spectrum and it is the quantum anomaly \[5\].

We would like to use the gauge principle to determine the interactions in the effective theory of graphene directly from $\mathcal{Z}_0$. However, if we use the gauge principle as it is in 3-D space time, we will obtain electrons interacting through a log-type interaction. The point is that the system we are working with is not a truly 3-D one. As emphasised in the main text, we are considering a system of electrons confined in the two spatial dimensions of the graphene layer but interacting through photons that are free to propagate in the third space dimension, see Fig. \[5\]. However, since the photons themselves originate from the electrons, their dynamics is constrained to the electron’s dynamics.

Were we considering a truly 2-d system, under a local gauge transformation the fermionic fields would transform as
\[
\psi'(x) = e^{i\alpha} \psi(x), \quad x \in \mathbb{R}^2.
\]
So there is no way we will obtain a photon propagating in the third spatial dimension in this way. The correct way to analyse this system is within the context of dimensional reduction \[9\] as we are now going to explain. First of all we should start thinking about the fermionic fields as not living strictly in 2-d, but instead as a 3-d field constrained to 2-d. In condensed matter physics there are many examples of systems supporting this picture. In a heterostructure for example, a scalar potential along the (say) $z$ direction confines the electron’s wave function $\psi(x, y, z)$ along $z$. In the absence of other interactions the following ansatz is used \[14\]
\[
\psi(x, y, z) \simeq \sum_n \phi_n(z) \psi(x, y),
\]
where $\psi$ is taken in the plane wave basis. The function $\phi_n(z)$ is the solution of the potential-well problem. At low temperature and for not too high electron densities the Fermi energy can be at the lowest level (i.e. $n = 0$) such that higher energy bands do not play a significant role and the system dynamics can be considered 2-d. When obtaining graphene, we physically decouple the 2-d sheet from the 3-d structure of graphite. So we may think about this situation as that of having an infinitely strong confinement potential along the $z \equiv x_3$ direction. Guided by this picture, we employ the following ansatz of the 4-D electron field:
\[
\psi(x, x_3, t) \simeq \phi_0(x_3, t) \psi(x, t).
\]
Then, under a local gauge transformation
\[
\psi'(x, x_3, t) \simeq e^{i\alpha(x_3, t)} \phi_0(x_3, t) e^{i\alpha(x, t)} \psi(x, t).
\]
It is now convenient to introduce the following notation for the indices: $m = \{3, \mu\}$ and $\mu = \{0, 1, 2\}$. An action equivalent to the one in Eq. \[34\] but in 4-D reads:
\[
S_4 = \int d^4x \bar{\psi}(x) \gamma^m \partial_m \psi(x) \simeq \int dt \int dx_3 \int d^2x \bar{\psi}(x_3, t) \psi(x, t) \gamma^m \partial_m \phi_0(x_3, t) \psi(x, t),
\]
where in the second step we have used the ansatz for the $\psi$-fields. Performing the local gauge transformation as in Eq. \[36\], we need to introduce the covariant derivative $\partial_m \rightarrow D_m = \partial_m - iA_m$ in order to leave the action gauge invariant. Here $A_m(x_3, x)$ is a $U(1)$ gauge field that transforms as $A'_m = A_m + \partial_3 \alpha(x_3, t) + \partial_\mu \alpha(x, t)$. Finally, in order to mimic the sharp confinement along the $x_3$ direction, we take $\phi_0(x_3) = \delta(x_3)$. In this way we arrive at
\[
S_4 = \int dx_3 dx^2 \delta(x_3) \bar{\psi}(x) \gamma^m \{\partial_m - iA_m(x_3, x)\} \delta(x_3) \psi(x) = \int dx_3 dx^2 \delta(x_3) \bar{\psi}(x) \gamma^m \{\delta^\mu_m \partial_m - iA_m(x_3, x)\} \psi(x).
\]

**FIG. 5:** Sketch of the model. Electrons ($e^-$) are confined in a two dimensional plane while the photons ($\gamma$) can escape in the $z$ direction.
where the second term in Eq. (39) is a gauge fixing term and can be obtained using the Fadeev-Popov method \[3\]. Finally, we want to rewrite the gauged fermionic action in terms of the fermionic currents as

\[
j^\mu = \bar{\psi}\gamma^\mu \delta(x_3), \quad j^3 = \bar{\psi}\gamma^3 \delta(x_3).
\]

In this way we obtain an action for fermions moving in a 3-D space time but electromagnetically interacting via photons living in a 4-D space time. Introducing the current \( J^\mu = \delta^{\mu \nu} j_\nu + \delta^{\mu 3} j_3 \), we arrive at the action for QED_{3,4} \[15\]:

\[
S_{3,4} = \int d^3x \bar{\psi} i\gamma^\mu \partial_\mu \psi + \int d^3x d^4k \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right\}.
\]

As we can see, the first term is just the action for the lagrangian density of Eq. (34). The second term however describes the coupling between the fermionic currents and the gauge fields in 4-D and the dynamics of the gauge fields respectively. The next step consists in deriving an effective action for the second term in which the third dimension \( x_3 \) does not appear anymore. This procedure is called dimensional reduction and was first employed in Kaluza-Klein theories; here we use a different way of performing the dimensional reduction, for more details see Ref. \[9\] \[10\] and references therein. We start by rewriting \( S_{j\gamma} \) in terms of the gauge fields only:

\[
S_{j\gamma} = \int d^3x d^3x \left\{ \frac{1}{2} A_m \left( \eta^{mn} \partial^2 - (1 - \xi^{-1}) \partial^m \partial^n \right) A_n + A_m J^m \right\},
\]

where \( \eta^{mn} \) is the flat metric in Minkowsky space \( \mathbb{M}_4 \). Defining the propagator of the gauge fields as \( (Q^{mn})^{-1} = \eta^{mn} \partial^2 - (1 - \xi^{-1}) \partial^m \partial^n \), the path integral over the gauge field’s configurations is

\[
Z = \int \mathcal{D}A e^{i \int d^3x d^3x \left\{ \frac{1}{2} A_m (Q^{mn})^{-1} A_n + A_m J^m \right\}} = e^{i \int d^3x d^3x \left\{ -\frac{1}{2} J_m (x_3) Q^{mn} (x_3,x';x',x') J_n (x',x') \right\}},
\]

where in the second equality we have integrated out the gauge fields. Note that the current \( J_m \) contains also the “out of plane” current \( j_3 \); this means that in order to obtain the correct physical result, Eq. (43) must be supplemented by the boundary condition \( j_3 = 0 \), meaning that electrons cannot escape from the plane. Moving to momentum space and imposing the boundary condition on \( j_3 \) we arrive at

\[
Q^{\mu\nu}(x,x') = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot (x-x')} \left( -\eta^{\mu\nu} + (1 - \xi) \frac{k^\mu k^\nu}{k^2 + k_3^2} \right),
\]

where \( k = \omega, \mathbf{k} \). Integrating out \( k_3 \) we arrive to the desired expression for the effective propagator of the gauge fields:

\[
Q_{eff}^{\mu\nu}(x,x') = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot (x-x')} \left( -\eta^{\mu\nu} + (1 - \tilde{\xi}) \frac{k^\mu k^\nu}{k^2} \right),
\]

where we have defined \( \tilde{\xi} = 2\xi - 1 \) \[10\]. Inverting the Kernel, we can go back to the real space representation of the effective propagator

\[
[Q_{eff}^{\mu\nu}(x,x')]^{-1} = \frac{2}{\sqrt{-\partial^2}} \left( \eta^{\mu\nu} \partial^2 + (1 - \tilde{\xi}^{-1}) \partial^\mu \partial^\nu \right).
\]

In the gauge field strength representation, the effective action now reads:

\[
S_{eff} = \int d^3x \left\{ \bar{\psi} i\gamma^\mu D_\mu \psi - \frac{1}{2g^2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{\xi g^2} \partial_\mu A^\mu - \frac{1}{\sqrt{-\partial^2}} \partial_\mu A^\mu \right\}
\]
In the above action, all fields are now defined in 3-D space time and we have reinstated the gauge coupling $g$. In order to make contact with the result of the previous section, we need to consider the non-relativistic limit of Eq. (47), i.e. $v/c \to 0$, where $c$ is the velocity of light at which the photons propagate. In the covariant derivative, the space component of the gauge field $A^\mu$ has a pre factor of $v/c$ and it is therefore suppressed in the non-relativistic limit \[11\]. Using the Feynman’s gauge $\tilde{\xi} = 1$ and integrating out the gauge fields, we arrive at a new effective action expressed solely in terms of the fermionic fields

$$S'_{\text{eff}} = \int d^3 x \left\{ \bar{\psi} \gamma^\mu \partial_\mu \psi + \frac{1}{4} \epsilon_{\mu \nu \rho \sigma} \left( \eta v q^2 \right) j_\nu \right\} \frac{v/c \to 0}{\int dt d^2 x \bar{\psi}(x,t) i(\gamma^0 \partial_t + v \gamma^i \partial_i) \psi(x,t)}$$

$$+ \frac{1}{2} \int dt d^2 x d^2 x' \bar{\psi}(x,t) \gamma^0 \psi(x,t) \frac{g^2}{2\sqrt{-\partial^2}} \bar{\psi}(x',t) \gamma^0 \psi(x',t).$$

Fourier transforming (FT)

$$\frac{1}{2 \sqrt{-\partial^2}} \xrightarrow{\text{FT}} \frac{1}{2|q|} \xrightarrow{\text{FT}} \frac{1}{4\pi|x - x'|}.$$ $\text{FT}$

Using the value of the electromagnetic gauge coupling (in a medium) $g^2 = e^2/\kappa$, we arrive to Eq. (27) upon identifying $w = g^2/4\pi$. We would like to stress that this derivation does not only serve as a way to confirm our previous result, but it highlights some interesting aspects concerning the symmetries of the problem. The main message of this analysis is that the system is non-chiral and $g_2$-like processes need to be included just like in one dimension. We would like to remind the reader that in a LL the $g_4$ process merely renormalizes the Fermi velocity while the $g_2$ process is responsible for the effective Luttinger parameter, i.e. the anomalous dimension of the theory \[10\]. However, as we will show in the next section, the $g_2$ process in graphene merely accounts for a degeneracy factor.

**BOSONIZATION SOLUTION**

In this section we present details concerning the bosonization solution described in the main text. We briefly recapitulate some of the steps described in the main text and then discuss the loop cancellation theorem (LCT) from which Eq. (5) in the main text is obtained. We proceed by explaining Eq. (7) of the main text and show how to obtain a formal definition of the interacting fermionic Green’s function. Finally, we discuss the non-interacting Green’s function, the Debye-Waller factor and its connection to the equal space Green’s function.

After integrating out the density fields $\rho$ in Eq.(3) of the main text, we arrive at the action

$$S = \int d\tau d^2 x \sum_\eta \bar{\psi}_\eta^n \left( G_{0\eta}^{-1} - i \sigma_0 \phi_\eta \right) \psi_\eta + \frac{1}{2} \int d\tau d^2 x d^2 x' \sum_{\eta, \eta'} \phi_\eta [V^{-1}]_{\eta, \eta'} \phi_{\eta'} = S_1[\psi_\eta, \psi_\eta^\dagger, \phi_\eta] + S_2[\phi_\eta],$$

$$G_{0\eta}^{-1} = \left( \sigma_0 \partial_\tau - i \eta v \sigma^i \partial_i \right),$$

where $G_{0\eta}$ is the non-interacting fermionic Green’s function. Writing $G^{-1}_{\eta}[\phi] = G_{0\eta}^{-1} - i \sigma_0 \phi_\eta$, we can recognise the Dyson equation for an electron moving in the background field $\phi_\eta$. However, the $\phi_\eta$-field is not static, its dynamics being determined by $S_2$. Since in our model the internal degrees of freedom do not play any role, we can work within an abelian bosonization scheme. Integrating out the fermions we arrive at Eq.(4) in the main text. The action of the theory is now expressed solely in terms of the collective field $\phi_\eta$. This action can be divided in two terms: $S_2$, containing the interaction potential and $S_1$ containing the interaction between the $\phi$-field and the electrons. The resulting theory is not a perturbative expansion in a dimensionless coupling parameter since interactions now appear in $S_2$. On the other hand, The $S_1$ term contains an infinite expansion in terms of loop diagrams without any small parameter

$$S_{1\eta}[\phi_\eta] = \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left[ i \sigma_0 \phi_\eta G_{0\eta} \right]^n.$$ $\text{FT}$

The next step in the bosonization program consists in showing that only the $n = 2$ loop in Eq. (52) is different from zero and therefore the resulting effective theory is Gaussian. This large scale cancellation of $n \geq 2$ loops is at the very heart of the bosonization procedure and in 1$d$ it is equivalent to the Dzyaloshinskii-Larkin theorem \[17\]. The LCT we present here was first proven in higher dimensional fermionic systems by Kopietz et. al \[18\][20]. Following
the one dimensional case, we remove the band (hard) cutoff and use a soft momentum cutoff later on. Although one may argue that in this way the problem is not exactly the same, we would like to stress that in a Dirac theory an hard momentum cutoff is never a good choice since it leads to the violation of most of the symmetries of the action. This is a well known problem of QFTs and has led to the development of sophisticated regularisation schemes such as dimensional regularisation, Pauli-Villars, the heat kernel regularisation and so on. The interested reader is advised to consult ref. [5] for an up to date analysis of the problem. In particular, the hard momentum cutoff leads to violation of gauge invariance and therefore charge conservation. In the context of bosonization, this leads to the violation of the Ward identity on which the LCT is based on [20].

Before going into details of the LCT, it is convenient, although not necessary, to rewrite Eq. (52) in frequency momentum space. In order to do that, we start with the frequency momentum space version of Eq. (50):

\[ S_1[\psi_\eta, \psi_\eta^\dagger, \phi_\eta] = \sum_{q} \frac{1}{\beta L^2} \sum_{k,n} \left\{ \psi_\eta^\dagger(k, \nu_n) \left\{ \delta_{k',k} \delta_{\nu_n',n} G_{0\eta}^{-1}(k, \nu_n) \right\} \psi_\eta(k', \nu_n') \right\} + \frac{i}{\beta L^2} \sum_{\mathbf{q},m} \delta_{\mathbf{k}',\mathbf{k}-\mathbf{q}} \delta_{\nu_n',m-n} \psi_\eta^\dagger(k, \nu_n) \phi_\eta(q, \omega_m) \psi_\eta(k - \mathbf{q}, \nu_n - \omega_m) \right\}, \]

where \( \nu_n \) and \( \omega_m \) are Matsubara fermionic and bosonic frequencies respectively. We have also used \( \mathbf{k} \) for the fermionic momentum, \( \mathbf{q} \) for the bosonic one and \( L^2 \) for the volume element. Below we introduce the notation \( k = (\nu_n, \mathbf{k}) \) for the fermionic “3-momentum” and \( q = (\omega_m, \mathbf{q}) \) for the bosonic one, to be understood within an euclidean metric signature. We also use the shorthand notation \( (\beta L^2)^{-1} \sum_{\mathbf{k},n} \rightarrow \sum_k \). Using Eq. (53), the effective action can be written as

\[ S_{1\eta}[\phi_\eta] = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathbf{q},\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_n} \Gamma_{\eta,n}(\mathbf{q}, \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) \phi_\eta(q) \phi_\eta(q_1) \phi_\eta(q_2) \ldots \phi_\eta(q_n), \]

\[ \Gamma_{\eta,n}(\mathbf{q}, \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) = \delta \left( \sum_{i=0}^{n} q_i = 0 \right) \left( \frac{i}{\beta} \right)^n \sum_{P(1, \ldots, n)} \sum_{\mathbf{k}} \text{Tr} \left\{ G_{0\eta}(k) G_{0\eta}(k + q) \right\} \times G_{0\eta}(k + q + q_1) G_{0\eta}(k + q + q_1 + q_2) \ldots G_{0\eta}(k + q + q_1 + q_2 + \ldots + q_n) \right\}, \]

where \( \Gamma_{\eta,n} \) is the full vertex function and \( \phi_\eta \) are the “external” fields. In the expression for the vertex, \( \delta(\sum_{i=0}^{n} q_i = 0) \) is a Kronecker-\( \delta \) enforcing conservation of the internal 3-momentum; as we show in Fig. (6), this means that the diagrammatic representation of Eq. (53) consists of closed (fermionic) loops and \( n \) external \( \phi_\eta \) fields attached to them, represented by wavy lines. The symbol \( P(1, \ldots, n) \) implies a sum over all possible permutations of the \( q \) internal labels (i.e. all possible ways of attaching the external lines to the loop); there are in principle \( n! \) of such permutations, however due to the Kronecker-\( \delta \) restriction one of the external lines is always fixed by the 3-momentum conservation. This means that the sum is extended over \( (n - 1)! \) permutations, such that \( (n - 1)!/n! = 1/n \) gives back the correct pre-factor coming from the expansion of the logarithm. Finally, the trace in the vertex function is taken only over the Pauli matrices.

The term with \( n = 1 \) in Fig. (6) corresponds to the number of occupied states in the non-interacting limit, i.e. the (infinite) ground state energy. Since in our model everything is measured with respect to the filled Dirac sea, this term is zero. Stated otherwise, we are considering normal ordered fields such that the infinite ground state energy is subtracted from any physical quantities [4]. The \( n = 2 \) loop corresponds to particle-hole fluctuations over the ground state and it leads to the RPA term [42]. Finally, \( n > 2 \) loop diagrams describe interactions between particle-hole excitations. Note that the vertex function of Eq. (55) corresponds to the expectation value of density operators, i.e.

\[ \Gamma_{\eta,n}(\mathbf{q}, \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) = \langle T_\tau \rho_\eta(q) \rho_\eta(q_1) \rho_\eta(q_2) \ldots \rho_\eta(q_n) \rangle, \]

where \( T_\tau \) is the imaginary time ordering operator and the expectation value is taken with respect to the canonical ensemble and evaluated by means of Wick’s theorem. For \( n = 2 \) there is only one loop corresponding to the vacuum polarization term \( \Pi_{\eta}(q) \), whose value has been evaluated by different authors [21, 22]. The \( n = 3 \) term is identically zero due to Furry’s theorem [23, 24], stating that expectation values of any odd number of current operators is zero:

\[ \langle T_\tau J^\mu(q) J^\nu(q_1) J^\lambda(q_2) \rangle = 0. \]
FIG. 6: The first three diagrams in the loop expansion. \( n = 1 \) gives the ground state energy, that is zero in our scheme. \( n = 2 \) gives the particle-hole excitation spectrum at the RPA level. \( n = 3 \) is the first term including interactions between particle-hole excitations. The wavy lines on the right are understood as outgoing momentum channels.

expectation values of density operators, corresponding in Eq. (57) to \( \mu = \nu = \lambda = 0 \). In our case, there is an easy way [20] of proving the cancellation of the odd vertices that uses solely the fact that the fermionic Green’s functions are odd functions of their arguments, i.e.

\[
G_{0\eta}(\mathbf{k}, -\nu_n) = \frac{1}{i\sigma_0 \nu_n - \eta \nu \sigma^3 k_i} = -G_{0\eta}(\mathbf{k}, \nu_n) \equiv -G_{0\eta}(k).
\] (58)

Now consider the two \( n = 3 \) loops in Fig. (6); in every loop there is a sum over \( \mathbf{k} \in (-\infty, \infty) \) and one over the periodic frequency \( \nu_n \). This means we can send \( \mathbf{k} \to -\mathbf{k} \) and \( \nu_n \to -\nu_n \) in the second loop without changing the result of the sum. Diagrammatically this corresponds to changing the loop momentum flow in the second diagram from clockwise to counterclockwise; since the flow direction of the loop is arbitrary, it is clear that the clockwise and the anticlockwise loops are topologically equivalent, see Fig. (7). The \( n = 3 \) vertex reads explicitly

\[
\Gamma_{n,3}(q, q_1) \propto \sum_k \left\{ G_{0\eta}(k) G_{0\eta}(k + q) G_{0\eta}(k + q + q_1) + G_{0\eta}(-k) G_{0\eta}(-k - q) G_{0\eta}(-k - q - q_1) \right\} = 0,
\] (59)

where we have used Eq. (58). It is easy to see that the above result extends to any vertex function with odd \( n \). This method however does not work for vertices with even \( n \). As we are going to show, as long as only density correlation

FIG. 7: Cancellation of the \( n = 3 \) vertex. In the right loop we have sent \( k \to -k \) and correspondingly inverted the momentum flow from clockwise to counterclockwise.
functions are considered, also the vertices with even $n > 2$ are identically zero \([20]\). To see this we use another property of the bare Green’s function, the partial fraction decomposition

$$G_{0q}(k) G_{0q}(k + q) = G_{0q}(q) \{G_{0q}(k) - G_{0q}(k + q)\},$$  \hspace{1cm} (60)

that can be explicitly proven by using the definition of the Green’s function, Eq. \([54]\). Using this property for the $n = 3$ case we obtain again the cancellation of the vertex. Here we show the cancellation explicitly for the first non-trivial case $n = 4$, Fig. 8. In this case we have $(4 - 1)! = 6$ loops that can be grouped in the two blocks represented schematically in Fig. (8). Loops in the same block add up to zero, therefore we consider here only the first block $\Gamma_{\eta,4}^1$:

$$\Gamma_{\eta,4}^1(q, q_1, q_2) = \sum_k \left\{ G_{0q}(k) G_{0q}(k + q) G_{0q}(k + q + q_1) G_{0q}(k + q + q_1 + q_2) \right. \right.$$

$$\left. + G_{0q}(k) G_{0q}(k + q) G_{0q}(k + q + q_2) G_{0q}(k + q + q_1 + q_2) + G_{0q}(k) G_{0q}(k + q_1) G_{0q}(k + q + q_1) G_{0q}(k + q + q_1 + q_2) \right\},$$

now the factorisation is chosen in such a way to always have a Green’s function depending on the same $q$ in all the three loops

$$\Gamma_{\eta,4}^1(q, q_1, q_2) = \sum_k \left\{ G_{0q}(k) G_{0q}(q_1) [G_{0q}(k + q) - G_{0q}(k + q + q_1)] G_{0q}(k + q + q_1 + q_2) \right.$$

$$\left. + G_{0q}(k) G_{0q}(q_2) G_{0q}(q_1) [G_{0q}(k + q + q_2) - G_{0q}(k + q + q_1 + q_2)] \right.$$  

$$\left. + G_{0q}(q_1) [G_{0q}(k) - G_{0q}(k + q_1)] G_{0q}(k + q + q_1) G_{0q}(k + q + q_1 + q_2) \right\}$$

$$= \sum_k \left\{ G_{0q}(q_1) G_{0q}(k + q) G_{0q}(k) G_{0q}(k + q + q_2) - G_{0q}(q_1) G_{0q}(k + q + q_1) G_{0q}(k + q + q_1 + q_2) \right\} = 0.$$
To obtain the cancellation, in the last line we have shifted \( k + q_1 \to k \). By iteration, it can be shown that the argument above is valid for every \( n > 2 \) \cite{20}. Since the \( n = 2 \) term is the only one (beside the \( n = 1 \)) to contain only one loop, the arguments used to show the cancellation of higher order loops do not apply in this case and we find that, as in the \((1 + 1)D\) case, the effective theory is Gaussian:

\[
S_{1\eta} = \frac{1}{2} \sum_q \phi_\eta(q) \Pi_\eta(q) \phi_\eta(-q), \quad \Pi_\eta(q) = \sum_k \text{Tr} \{ i\sigma_0 G_{0\eta}(k) i\sigma_0 G_{0\eta}(k + q) \}.
\]  

(62)

The complete effective theory is therefore given by \( S_{\text{eff}} = \sum_\eta S_{1\eta} + S_2 \). We would like to remark that the LCT is valid only for scalar vertices. The \((1 + 1)D\) case is special in this sense since also the “vector” potential is a scalar. For this reason the relativistic version of the LL model, the Thirring model, can be completely solved by means of bosonization methods \cite{32}. Since the effective theory has been obtained mostly using functional manipulations, it does not come as a surprise that it closely resembles the theory describing interacting Fermions in \((1 + 1)D\). Obviously, in the case of graphene, the summations are taken in \((2 + 1)D\) and the polarization function reads

\[
\Pi_\eta(q, \omega_m) = \frac{1}{16} \frac{q^2}{\sqrt{\omega_m^2 + v^2 q^2}} \equiv \Pi(q, \omega_m),
\]

(63)

Independent from the chiral index \( \eta \). It is instructive to compare the polarization function of graphene with the ones of one and three dimensional Dirac electrons

\[
\Pi^{1d}_\eta(q, \omega_m) = \frac{1}{2\pi^2} \frac{\eta q}{\omega_m^2 + \eta v q}, \quad \Pi^{3d}_\eta(q, \omega_m) = \eta \frac{q^2}{24\pi v} \log \frac{\Lambda^2}{\omega_m^2 + v^2 q^2}.
\]

(64)

We see that in one and two dimensions, the polarization function is finite and therefore it does not renormalize in the RG sense. On the other hand, in three dimensions the polarization function is logarithmically divergent and it does renormalize as it is well known from QED \cite{25}. However, Eq. (63) crucially differs from Eq. (64) in two main aspects: the presence of the square root and the dependence on the chiral index \( \eta \). We conclude this section defining the dressed propagator of the \( \phi_\eta \)-fields

\[
D_{\eta\eta'}(q, \omega_m) = \frac{1}{\Pi(q, \omega_m) + [V(q)^{-1}]_{\eta,\eta'}},
\]

(65)

that is the RPA propagator.

The interacting fermionic Green’s function

In the previous section we have found the effective field theory of interacting electrons in \((2 + 1)D\) in terms of an action involving only the bosonic degrees of freedom. However, we are interested in the fermionic sector of the theory, i.e. we are interested in obtaining an expression for the interacting fermionic Green’s function. In order to do that we follow again ref. \cite{20} and employ a method first introduced by Schwinger in the context of one dimensional QED \cite{20}. Consider Eq. (50), introducing grassmanian source fields in the partition function and functionally differentiating twice we obtain the formal expression for the interacting, fermionic Green’s function in the main text, Eq. (7):

\[
\{ (\sigma_0 \partial_\tau - i \eta v \sigma^i \partial_i) - i\sigma_0 \phi_\eta(x, \tau) \} G_\eta(x, x'; \tau, \tau') = \delta(x - x') \delta^*(\tau - \tau'),
\]

(66)

where \( \delta^*(\tau - \tau') = \beta^{-1} \sum_n e^{-iu_n(\tau - \tau')} \) is the anti-periodic Dirac delta function. The above equation defines a linear, non-homogeneous partial differential equation. The source term \( \phi_\eta(x, \tau) \) describes a background field in which the electrons propagate. Following Schwinger, the differential equation can be solved using the following ansatz:

\[
G_\eta(x, x'; \tau, \tau') = G_{0\eta}(x - x'; \tau - \tau') e^{\Phi_\eta(x, \tau) - \Phi_\eta(x', \tau')} \]

(67)

\[
(\sigma_0 \partial_\tau - i \eta v \sigma^i \partial_i) G_{0\eta}(x - x'; \tau - \tau') = \delta(x - x') \delta^*(\tau - \tau')
\]

where \( G_{0\eta} \) (the non-interacting Green’s function), is the solution of the homogenous problem. Note that since \( G_\eta \) is an anti-periodic function of (imaginary) time and \( G_{0\eta} \) on the r.h.s. of Eq. (67) is also anti-periodic, it follows that \( \Phi_\eta \)
must be a periodic function of time, i.e. \( \Phi_q(x, \tau + \beta) = \Phi_q(x, \tau) \). Using the Fourier transform technique, the solution of the differential equation reads

\[
G_{0q}(x' - x) = \frac{1}{\beta L^2} \sum_{k,n} e^{-i(k \cdot x - v_n \tau)} \frac{e^{-i(q \cdot x' - \omega_m \tau)}}{-i\sigma_0 \omega_m + \eta \nu \sigma^4 k_i} \tag{68}
\]

\[
\Phi_q(x, \tau) = \frac{1}{\beta L^2} \sum_{q,m} e^{-i(q \cdot x - \omega_m \tau)} \frac{e^{-i(q \cdot x' - \omega_m \tau')}}{-i\sigma_0 \omega_m + \eta \nu \sigma^4 q_i} \tag{69}
\]

Using the above solution in Eq. (67) we arrive at the desired result

\[
G_q(x, x'; \tau, \tau', [\phi_q]) = G_{0q}(x - x'; \tau - \tau') \exp \left\{ \frac{1}{\beta L^2} \sum_{q,m} \frac{e^{-i(q \cdot x - \omega_m \tau)} - e^{-i(q \cdot x' - \omega_m \tau')}}{-i\sigma_0 \omega_m + \eta \nu \sigma^4 q_i} \right\} \tag{70}
\]

N.b. here we have made explicit the dependence on \( \phi_q \) of the interacting Green’s function in order to stress the fact that \( G_q \) is evaluated for a specific, frozen configuration of the background field \( \phi_q \). The solution of the interacting Green’s function is obtained by averaging Eq. (70) over all possible configurations of the \( \phi_q \) field

\[
G_q(x, x'; \tau, \tau', [\phi_q]) = \langle G_q(x, x'; \tau, \tau', [\phi_q]) \rangle_{\text{eff}} = Z_{0, q}^{-1} \int \mathcal{D}\phi \; e^{-S_{\text{eff}}[\phi_q]} G_q(x, x'; \tau, \tau', [\phi_q])
\]

The above equation resembles the standard \((1 + 1)D\) representation of the fermionic Green’s function in terms of vertex operators [4]. It is convenient to rewrite the bosonic exponent as

\[
\Phi_q(x, \tau) - \Phi_q(x', \tau') = \sum_{q,m} J_q(q, \omega_m; x; \tau; x'; \tau') \sigma_0 \phi_q(q, \omega_m) \tag{72}
\]

\[
J_q(q, \omega_m; x; \tau; x'; \tau') = \frac{1}{\beta L^2} \left\{ \frac{e^{-i(q \cdot x - \omega_m \tau)} - e^{-i(q \cdot x' - \omega_m \tau')}}{-i\sigma_0 \omega_m + \eta \nu \sigma^4 q_i} \right\}, \tag{73}
\]

where \( J_q \) could be interpreted as a source field. The last step consists in evaluating the expectation value in Eq. (71)

\[
G_q(x, x'; \tau, \tau') = G_{0q}(x - x'; \tau - \tau') \times Z_{0, q}^{-1} \int \mathcal{D}\phi \; e^{-iS_{\text{eff}}[\phi_q]} \left\{ \sum_{q,m} - \frac{1}{2} \mathcal{D}\phi \; e^{iS_{\text{eff}}[\phi_q]} \right\}
\]

\[
= G_{0q}(x - x'; \tau - \tau') e^{Q_q(x, x'; \tau, \tau')}, \tag{74}
\]

where in the second equality we have performed the integration over the \( \phi \)-fields and defined the Debye-Waller (DW) factor

\[
Q_q(x, \tau; x'; \tau') = \sum_{q,m} \frac{\beta L^2}{2} J_q(q, \omega_m; x; \tau; x'; \tau') D_{\eta \nu} \phi_q(q, \omega_m) \delta_{q, \eta \nu} \tag{75}
\]

In the next sections we are going to evaluate explicitly the two objects appearing in Eq. (74): the non interacting Green’s function and the DW factor. We conclude this section with few comments concerning Eq. (70): the exponential term corresponds diagrammatically to a re-summation of self energy skeleton diagrams plus the vertex corrections. Since the vacuum polarization only contains the Gaussian term due to the LCT, Eq. (70) gives a complete solution of the interacting problem.

The non interacting Green’s function in real space

We start evaluating the real space form of the non interacting Green’s function. This form is particularly useful to understand the difference between the one and the two dimensional problem. We need to evaluate

\[
G_{0q}(x) = \frac{1}{\beta L^2} \sum_{k,n} e^{-i(k \cdot x - \nu_n \tau)} \frac{e^{-i(q \cdot x - \omega_m \tau)}}{-i\sigma_0 \nu_n + \eta \nu \sigma^4 k_i} \tag{77}
\]
FIG. 9: Momentum distribution function. The momentum distribution function of non interacting graphene is plotted for \( \eta = + \). The hole-like (Red) and electron-like (blue) energy states are shown in correspondence with the energy spectrum, represented by the Dirac cone on the right. The result for \( \eta = - \) is obtained by exchanging the two colours in the figures.

either directly or by using the known relation between the Dirac \((G_D)\) and the Klein-Gordon \((G_{KG})\) propagator \[27\]. In imaginary time this relation reads

\[
G_D(x, \tau) = (\partial_\tau + i v \sigma^i \partial_i)G_{KG}(x, \tau).
\]

(78)

The interesting aspect of relation (78) is that it factorises the propagator into a “spinor” and a “bosonic” part. In the rest of this work we will consider zero temperature and take the continuum limit at the beginning of the calculation. This limit is easily accounted for by taking

\[
\frac{1}{L^2} \sum_k \rightarrow \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2}, \quad \frac{1}{\beta} \sum_n \rightarrow \int_{-\infty}^{\infty} \frac{d\nu}{(2\pi)},
\]

(79)

and similarly for the bosonic variables. The imaginary time Klein-Gordon (KG) propagator both for \( \tau > 0 \) and \( \tau < 0 \) is readily evaluated as

\[
G_{0,KG} = \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{e^{-ik \cdot x \nu \tau}}{\nu^2 + v^2k^2} = \frac{1}{4\pi v} \frac{1}{\sqrt{x^2 + v^2 \tau^2}}.
\]

(80)

Using relation (78), we arrive at the imaginary time Dirac propagator

\[
G_{0\eta}(x, \tau) = \frac{i}{4\pi} \sigma^i x_i + i \eta v \sigma_0 |\tau| \frac{1}{\sqrt{x^2 + v^2 \tau^2}}.
\]

(81)

From the above expression we can clearly distinguish the “spinor” propagator (i.e. the term containing the Pauli matrices) from the KG one. We see that the standard LL propagator is, so to speak, proportional only to the KG part, but with a linear dependence in space and time \[16\]. Moreover, in one dimension a connection between the chirality and the sign of \( \tau \) exists. As a check for this expression, and for later reference, we can evaluate the momentum distribution function (at zero temperature) and the spectral function of the system. The former corresponds to the evaluation of the FT of the equal time propagator

\[
n_\eta(k) = \int_{-\infty}^{\infty} d^2x e^{i k \cdot x} \text{Tr} \left[ G_{0\eta}(x, 0) \sigma_1 \right] = \eta \text{sign}(k) \equiv \eta \theta(k) - \eta \theta(-k).
\]

(82)

In the last equality we have adopted a form in terms of the more familiar step function. The above result is shown in Fig. (9), where we make explicit the meaning of Eq. (82) in terms of positive (electron-like) and negative (hole-like) energy eigenstates. The non-interacting density of states (DOS) is obtained in a similar way, this time taking the FT
of the equal space propagator

\[ \nu_\eta(\epsilon) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} dt \epsilon^{ext} \text{Tr} \left[ G_{0,\eta}(0, t + + \text{sign}(t)) \right] = \frac{|e|}{\pi v^2}, \]  

independent from the chirality index. The result of Eq. (83) is well known and its generalisation to arbitrary dimensions is \( \nu(\epsilon) \propto |e|^{d-1} \). Note that the trace is taken in spinor space, i.e. it accounts for the factor of 2 “pseudo spin” degeneracy.

The Debye-Waller factor at equal space

The second, and most important ingredient in Eq. (74) is the DW factor, describing the effect of interactions at a non perturbative level. Without any loss of generality we fix the chiral index \( \eta = + \) and choose \( x' = 0 \) and \( \tau' = 0 \). In graphene, we are interested in the case where the matrix elements of the interaction potential are all equal, i.e. \( g_1 = g_2 \equiv g \). It is worth reminding that in the one dimensional case, due to the chiral nature of the polarization function Eq. (64), this special case is non trivial and different from setting \( g_2 = 0 \). In the latter case, interactions only lead to the renormalization of the Fermi velocity while in the former case an anomalous dimension is also obtained. In our case however, since \( \Pi_{\eta} \propto |q| \), we find

\[ Q_+(x, \tau) = \frac{1}{\beta L^2} \sum_{q, m} \frac{[1 - \cos (q \cdot x - \omega_m \tau)]}{(-i \omega_m + v \sigma \cdot q)^2} \frac{g(|q|)}{1 + g(|q|)^2 \Pi(q, \omega_m)}, \]  

where \( g(|q|) = w/|q| \), and we have redefined \( w = e^2/2\kappa \). It is interesting to note that Eq. 84 is very similar to the expression of the self energy obtained in [30, 31]. In Eq. (74) we can identify a space-time independent part, known as the static structure factor, and a dynamical one. We would like to stress that while the static contribution can be identified with the self energy term in the RG approach, the dynamical part is completely absent in the RG schemes mostly employed in the literature. As we are now going to show, the dynamical part is essential for obtaining a finite value of \( Q_\eta \).

It is convenient to rewrite \( D_{++} \) in a different form, useful for explicit calculations, that also gives an insight into the physics at play. Using the definition of the polarization function Eq. (63) and the interaction term \( g(|q|) \), the \( \phi_+ \)-propagator can be written as

\[ D_{++} = \frac{g(|q|)}{1 + g(|q|)^2 \Pi(q, \omega_m)} \approx \frac{w}{\sqrt{\omega^2 - v^2 q^2 - (w/8)|q|}} \frac{\omega^2 + v^2 q^2 - (w/8)^2 q^2}{|q| \Pi(q, \omega_m)}, \]  

This form suggests that, as in the one dimensional model, we can define the plasmon’s velocity

\[ v_p^2 = v^2 \left[ 1 - \left( \frac{w}{8\kappa} \right)^2 \right] = v^2 \left[ 1 - \left( \pi R_s \right)^2 \right], \]  

where we have introduced the dimensionless coupling \( R_s = e^2/\kappa v \). Since \( R_s \) is dimensionless, Eq. (86) correctly defines an effective velocity. Before proceeding with the evaluation of the DW factor, we would like to make clear the physical identification of \( v_p \) as the plasmon’s velocity. We note that the denominator of the r.h.s of Eq. (86) is essentially the “RPA” dielectric function

\[ \epsilon_{RPA}(q, \omega) = 1 + g(|q|) \Pi(q, \omega), \]  

whose zeros define the excitation spectrum of the system. First of all we need to perform the analytic continuation to real frequencies

\[ \epsilon_{RPA}(q, \omega \rightarrow \omega + \text{i} 0^+) \approx 1 + \frac{w}{8} \frac{q^2}{\sqrt{v^2 q^2 - \omega^2 - \text{i} 0^+}} \approx 1 + \frac{w}{8} \frac{q^2}{\sqrt{v^2 q^2 - \omega^2}} \]  

where the last expression is valid as long as we are only interested in the pole contribution. Note that for \( |v q| < \omega \), the dielectric function has an imaginary part that is responsible for Landau damping. Solving for \( \epsilon_{RPA} = 0 \) we find

\[ \omega_q = q v \sqrt{1 - \left( \frac{\pi R_s}{4} \right)^2} = q v_p. \]
A linear plasmon is present in the excitation spectrum of the theory. This is not surprising since, according to Ref. [28], the spectrum of excited states usually consists of two parts: a continuum resulting from the excitations of individual particles and a discrete one coming from the excitation of a collective mode, i.e., the plasmon. One dimension is special in this sense since its spectrum only contains collective excitations. As a result of the continuum spectrum, the linear plasmons are usually damped in $d > 1$. In this work we restrict ourselves to $r_s < 4/\pi$, the analysis of the opposite regime will be considered elsewhere.

The DW factor can be split as $Q_\eta(x, \tau) = R_\eta(0,0) - S_\eta(x, \tau)$, where $R$ and $S$ are the static and the dynamic structure factors respectively. Using Eq. (85) and taking $T \to 0$ and the continuum limit, the static structure factor reads

$$R_+(0,0) = \int \frac{d^2q}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \left( \frac{1}{\omega^2 + v^2 q^2} - \frac{1}{\omega^2 + v_p^2 q^2} \right) = R_{+1}(0,0) - R_{+2}(0,0).$$

At this point our choice of writing $D_{++}$ in Eq. (85) should be clear: $R_{+1}$ contains only the pole contribution while $R_{+2}$ contains the branch cut contribution due to the presence of the square root. In the rest of this work we will restrict ourselves to the analysis of the equal space propagator, i.e., $x = 0$. The analysis of the space time propagator will be presented elsewhere. The dynamic structure factor at equal space reads

$$S_{+}(0, \tau) = \int \frac{d^2q}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \cos(\omega \tau) \left( \frac{1}{\omega^2 + v^2 q^2} - \frac{1}{\omega^2 + v_p^2 q^2} \right)$$

$$= S_{+1}(0, \tau) - S_{+2}(0, \tau).$$

We proceed by evaluating the static structure factor. In Eq. (90) we have divided the static factor into two contributions: one having only simple poles and the other having branch cuts in the complex $\omega$ plane. The frequency integral of $R_{+1}(0,0)$ is easily evaluated using the residue theorem, while for $R_{+2}(0,0)$ a direct integration is preferable. Performing the frequency integral we obtain

$$R_{+1}(0,0) = \frac{w}{v_p (v + v_p)} \int_0^\infty dq \frac{1}{2\pi q}$$

$$R_{+2}(0,0) = \frac{w^2}{8 \pi (v^2 - v_p^2)} \int_0^\infty dq \frac{1}{2\pi q} \left[ \frac{2 - 2 v^2 + v_p^2}{v_p \sqrt{v^2 - v_p^2}} \arccos \left( \frac{v_p}{v} \right) \right],$$

where $v > v_p$ in the regime considered here. According to Ref. [20], the quasi particle residue $Z$ is related to the static structure factor as $Z = e^{R_{1}(0,0)}$. It can be checked explicitly that $R_+$ in Eq. (92) is always negative. The remaining momentum integral is divergent both in the UV and in the IR just like in the one dimensional case. Indeed, the integral over momentum $q$ is the same appearing in the problem of one dimensional bosonization! The reason is that in standard one dimensional bosonization, electrons interacting with a contact interaction are usually considered. In our case, even though we are working in two dimensions, the Coulomb interaction provides exactly the factor of $q$ necessary to obtain the same divergence of the one dimensional case. While we can cure the UV divergence by introducing an high energy cutoff on the scale of the inverse lattice spacing, the IR divergence cannot be cured and the quasi particle residue $Z = e^{-\infty} = 0$ signalling non Fermi liquid behaviour. Indeed, the static structure factor is closely related to the electron’s self energy, so that the divergence in $R_\eta(0,0)$ corresponds to the divergence in the self energy that one would find from a FL treatment of the problem. Below we will show that, exactly as in the one dimensional case, the combination $R_\eta(0,0) - S_\eta(0,\tau)$ is always finite. However, the propagator will not describe anymore well defined quasi particles but a truly correlated system. We would like to point out that the above result is in qualitative agreement with the ones found in Ref. [29][31]. We move on the evaluation of the dynamic factors, performing the frequency integral it reads

$$S_{+1}(0, \tau) = \frac{w}{(v^2 - v_p^2)} \int_0^\infty dq \frac{1}{2\pi q} \left[ \frac{v^2 + v_p^2}{v_p} e^{-v_p q |\tau|} - 2 v e^{-v q |\tau|} \right].$$

$$S_{+2}(0, \tau) = \frac{w^2}{8} \int_0^\infty dq \frac{1}{2\pi q} \left[ \frac{v^2 + v_p^2}{v^2 - v_p^2} \arctan \left( \frac{v}{v_p} \right) K_1(v_q |\tau|) - 2 v \sqrt{v^2 - v_p^2} K_1(v_q |\tau|) \right],$$

where $K_1(aq)$ is the special Bessel function of the second kind [34]. N.b. $K_1(aq)$ diverges as $1/q$ for $q \to 0$, i.e. it
Being interested in the low energy behaviour of the system, in the last step we have taken.

The type of integrals can be performed essentially in the same way (here we take identical to those one finds when employing bosonization in one dimension. They give \( i = p, e \) /\( \alpha \) a soft momentum cut off \( \Lambda = 1/\alpha \).

Following the one dimensional case, we regularise the theory introducing

Written in this way, the artificial IR divergence cancels. However, the integrals are still UV divergent as we should expect from an effective low energy theory. Following the one dimensional case, we regularise the theory introducing a soft momentum cut off \( \Lambda = 1/\alpha \) on the scale of the sub lattice constant. The first two integrals in Eq. (95) are identical to those one finds when employing bosonization in one dimension. They give \( i = p, e \).

\[
I_1 = \int_0^\infty dq \frac{e^{-\alpha q}}{q} \left( 1 - e^{-v_i q|\tau|} \right) = \log \left( \frac{\alpha + v_i |\tau|}{\alpha} \right) \simeq \log \left( \frac{v_i |\tau|}{\alpha} \right),
\]

Being interested in the low energy behaviour of the system, in the last step we have taken \( v_i |\tau| \gg \alpha \). The second type of integrals can be performed essentially in the same way (here we take \( v_i |\tau| \gg \alpha \) from the beginning):

\[
I_2 = \int_0^\infty dq \left( \frac{e^{-\alpha q}}{q} - v |\tau| K_1(vq|\tau|) \right) \simeq \log \left( \frac{v |\tau|}{\alpha} \right).
\]

Using the above results, we arrive at the final expression of the DW factor

\[
Q_+(0, \tau) = \gamma_1 \log \left( \frac{\alpha}{v |\tau|} \right) + \gamma_2 \log \left( \frac{\alpha}{v_e |\tau|} \right) - \gamma_3 \log \left( \frac{\alpha}{v_p |\tau|} \right),
\]

FIG. 10: **Excitation’s velocities.** Plot of \( v_p \) and \( v_e \) as a function of the interaction strength. Here \( i = e, p \) denotes one of the two velocities. In the interaction regime considered here, we find that \( v_p \geq v_e \), meaning that the plasmon does not decay into the particle-hole continuum.

shows the same degree of divergence of the static factor in the IR regime. We see that the \( S_{+1} \) term, containing only the pole contribution, is similar to the same factor ones would obtain in one dimension \([20]\), while the \( S_{+2} \) term is completely new and it contains the branch cut contribution. Since the branch cut physically corresponds to single and multi particle excitations in the continuum, we can identify

\[
v_e = v \frac{\pi r_s}{4} \frac{\sqrt{v_p/v}}{\arccos(v_p/v)}
\]

as the velocity of these incoherent excitations. In Fig. 10 we plot \( v_p \) and \( v_e \). As it can be seen, \( v_p \geq v_e \) in the considered interaction regimes. As a consequence, the plasmon mode does not decay in the particle-hole continuum. In the regime \( r_s > 4/\pi \) the situation changes and we find that the linear plasmon disappear from the spectrum of the theory.

The dynamics of the plasmon’s decay for \( r_s > 4/\pi \) will be discussed in details elsewhere. The DW factor reads

\[
Q_+(0, \tau) = \frac{w}{v^2 - v_p^2} \frac{v_p + v^2}{v_p} \int_0^\infty dq \frac{1}{2\pi q} \left( 1 - e^{-v_p q|\tau|} \right) - 2v \frac{w}{v^2 - v_p^2} \int_0^\infty dq \frac{1}{2\pi q} \left( 1 - e^{-v q|\tau|} \right)
\]

\[
+ \frac{w^2}{8} \frac{2}{\pi(v^2 - v_p^2)} \int_0^\infty dq \frac{1}{2\pi} \left( \frac{1}{q} - v |\tau| K_1(vq|\tau|) \right) - \frac{w^2}{8} \frac{v^2 + v_p^2}{\pi v_p(v^2 - v_p^2)} \int_0^\infty dq \frac{1}{2\pi} \frac{\arccos(v_p/v)}{\sqrt{v^2 - v_p^2}} \frac{\arccos(v/v)}{\sqrt{v^2 - v_p^2}} K_1(v_e q|\tau|).
\]
FIG. 11: **Interaction parameters.** Behaviour of the effective interaction parameters as a function of the coupling $r_s$. It can be seen that in the regime considered in this work the effective interaction related to the electron sector ($\gamma_1$) is positive and decreases while the one coming from the excitations is negative and increases.

where the $\gamma_i$s depend only on the dimensionless coupling $r_s$

$$\gamma_1 = \frac{v^2}{v^2 - v_p^2} \left(\frac{r_s}{2} - \frac{r_s^2}{2}\right), \quad \gamma_2 = \frac{v^2}{4} \left(\frac{v^2 + v_p^2}{v^2} - \frac{v^2}{v^2 - v_p^2}\right) \arccos \frac{v_p}{v}, \quad \gamma_3 = \frac{r_s}{2} \frac{v^2 + v_p^2}{v^2 - v_p^2}.$$  \hspace{1cm} (99)

In Fig. 11, we plot the value of $\gamma_1$ and $\gamma_2 - \gamma_3$ as a function of $r_s$. The opposite behaviour of these two functions gives rise to the peculiar dependence of $\gamma = \gamma_1 + \gamma_2 - \gamma_3$ in the main text. Using Eq. (99) in the definition of the interacting Green's function Eq. (74) we finally obtain Eq. (9) of the main text

$$G_{+}(\tau) = -\frac{\alpha^2}{4\pi} \left(\frac{1}{\sigma_0 |\tau|}\right)^{2+\gamma} \frac{1}{v^{2+\gamma} v_p^{2+\gamma}}.$$  \hspace{1cm} (100)

Looking at the scaling behaviour of the Green’s function, we can identify $\gamma$ as the anomalous dimension. Using Eq. (100) in the definition of the DOS Eq. (83) we obtain the non perturbative expression for the interacting DOS discussed in the main text.

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[35] Here we use $D$ for the space time dimensions while $d$ is used for the space dimension only.
[36] For the sake of simplicity we keep the notation $q$ also for the low energy transferred momentum.
[37] To be more precise, Umklapp terms contain a phase factor $e^{2i(r_n + r_m) \cdot q}$, so that momentum is not conserved. However, if the total momentum violation equals a reciprocal lattice vector, these kind of processes are indeed allowed [4].
[38] If not otherwise stated, throughout this work we use natural units $\hbar = k_B = c = 1$. In this units, the vacuum dielectric constant $\epsilon_0 = c = 1$.
[39] Note that whenever we want to highlights the Lorentz invariance of the theory, we should work with $\bar{\psi}$ and not $\psi$. In fact, it is $\bar{\psi}\psi$ that transforms like a scalar under a Lorentz transformation not $\psi^\dagger\psi$ [3].
[40] Note that in Feynman’s gauge $\xi = 1$ corresponds to $\bar{\xi} = 1$.
[41] Remind that $A_i$ corresponds to the transverse field describing electromagnetic effects such as induction.
[42] One of the advantages of the functional integral formulation is that the partition function directly generates dressed propagators, i.e. it automatically sums over one particle reducible diagrams.
[43] Note that here we are using cgs units, i.e. we are taking $4\pi\epsilon_0 = 1$. In this way the DOS will have units of $1/eV m^2$. 