A. Hyperspherical Harmonics

The hyperspherical harmonics are defined by the eigenfunction equation Eq. (8), which can be written explicitly as

$$\Delta_{S^D-1} Y(\hat{r}) \equiv \sin^{D-2} \theta_1 \frac{\partial}{\partial \theta_1} \left[ \sin^{D-2} \theta_1 \frac{\partial}{\partial \theta_1} Y(\hat{r}) \right] + \cdots + \sin^{-2} \theta_1 \cdots \sin^{-2} \theta_{D-2} \frac{\partial^2}{\partial \theta_{D-2}^2} Y(\hat{r}) = -\lambda_{D-1} Y(\hat{r})$$ \hspace{1cm} [S1]

in hyperspherical coordinates. Importantly, the Laplace-Beltrami operator on the $K$-sphere may be written in terms of that on the $(K-1)$-sphere, allowing us to compute the angular basis functions iteratively, first solving for the 1-sphere basis, which can then be used to find the solution on the 2-sphere, et cetera. This leads to the hierarchy $Y^{(K)}(\theta_1, \ldots, \theta_K) = \Theta_K(\theta_K) Y^{(K-1)}(\theta_1, \ldots, \theta_{K-1})$, where

$$\sin^{1-K} \theta_K \frac{\partial}{\partial \theta_K} \left[ \sin^{K-1} \theta_K \frac{\partial \Theta_K(\theta_K)}{\partial \theta_K} \right] - \lambda_{K-1} \sin^{-2} \theta_K \Theta_K(\theta_K) = -\lambda_K \Theta_K(\theta_K)$$ \hspace{1cm} [S2]

(23), writing the eigenvalue for the $K$-sphere as $\lambda_K$. Starting from the 1-sphere basis functions $\Theta_1(\theta_1) = (2\pi)^{-1/2} \exp(-i\ell_1 \theta_1)$, we can form the full hyperspherical harmonics by solving Eq. (S2), giving $Y_{\ell_1 \cdots \ell_{D-1}}(\theta_1, \ldots, \theta_{D-1}) \equiv \Theta_1(\theta_1) \Theta_2(\theta_2) \cdots \Theta_{D-1}(\theta_{D-1})$, with

$$Y_{\ell_1 \cdots \ell_{D-1}}(\theta_1, \ldots, \theta_{D-1}) \equiv \Theta_1(\theta_1) \Theta_2(\theta_2) \cdots \Theta_{D-1}(\theta_{D-1}), \hspace{1cm} [S3]$$

$$\Theta_K(\theta_K) = \sqrt{\frac{2\ell_K + K - 1}{2} \frac{(\ell_K + \ell_{K-1} + K - 2)!}{(\ell_K - \ell_{K-1})!}} \sin^{1-K/2} \theta_K P_{\ell_K + K/2-1}^{(\ell_K - K/2-1)}(\cos \theta_K), \hspace{1cm} (2 \leq K \leq D - 1),$$

where $P_{\ell K}^{(\ell K)}(x)$ is an associated Legendre polynomial (e.g., 34, §14.3). The indices $\ell_i$ are related to the Laplace-Beltrami eigenvalues of Eq. (7) via $\lambda_K = \ell_K(\ell_K + K - 1)$ and must satisfy the selection rules of Eq. (9).

Under complex conjugation and parity transformations, the hyperspherical harmonics transform as

$$Y_{\ell_1 \ell_2 \cdots \ell_{D-1}}(\hat{r}) = (-1)^{\ell_1} Y_{-\ell_1 \ell_2 \cdots \ell_{D-1}}(\hat{r}), \hspace{1cm} \mathcal{P} \left[ Y_{\ell_1 \cdots \ell_{D-1}}(\hat{r}) \right] = (-1)^{D-1} Y_{\ell_1 \cdots \ell_{D-1}}(\hat{r}),$$ \hspace{1cm} [S4]

where the parity operator, $\mathcal{P}$, sends $r \rightarrow -r$ and thus $\theta_1 \rightarrow \pi + \theta_1$, $\theta_k \rightarrow \pi - \theta_k$ for $k > 1$. The latter equation can be verified by noting that the action of the parity operator on the basis components of Eq. (S3) are $\mathcal{P}[\Theta_1(\theta_1)] = (-1)^{\ell_1} \Theta_1(\theta_1)$ and $\mathcal{P}[\Theta_K(\theta_K)] = (-1)^{K-1} \Theta_K(\theta_K)$ for $K > 1$. 