ASYMPTOTICS OF UNITARY AND ORTHOGONAL MATRIX INTEGRALS

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Abstract. In this paper, we prove that in small parameter regions, arbitrary unitary matrix integrals converge in the large \( N \) limit and match their formal expansion. Secondly we give a combinatorial model for our matrix integral asymptotics and investigate examples related to free probability and the HCIZ integral. Our convergence result also leads us to new results of smoothness of microstates. We finally generalize our approach to integrals over the orthogonal group.

Introduction

Matrix integrals provide models for physical systems (2D quantum gravitation, gauge theory, renormalization, etc...), and generating series for a wide family of combinatorial objects (see e.g \cite{20, 28}).

Gaussian integrals are the most studied. It was shown by Brézin, Itzykson, Parisi and Zuber \cite{7} that perturbations of Gaussian integrals expand formally as a generating function of maps, sorted by their genus when the dimension \( N \) of the matrices is regarded as a parameter. Such ‘topological’ expansions were shown also to hold in the large \( N \) limit, and then to match with the formal expansion on a mathematical level of rigor by two authors \cite{16, 17, 23} and previously in the one matrix case in \cite{1} \cite{2} and \cite{12}. The relation of Gaussian matrices with the enumeration of maps is an easy consequence of Wick calculus -or equivalently Feynman diagrams- see \cite{28} for a good introduction. According to ‘t Hooft \cite{20}, such topological expansion should hold in the more general context of models invariant under unitary conjugation. This leads us to concentrate in this article on matrix integrals given by

\[
I_N(V, A^N_i) := \int_{U_N^N} e^{N \text{Tr}(V(U_i, U^*_i, A^N_i, 1 \leq i \leq m))} dU_1 \cdots dU_m
\]

where \((A^N_i, 1 \leq i \leq m)\) are \( N \times N \) deterministic uniformly bounded matrices, \( dU \) denotes the Haar measure on the unitary group \( U_N \) (normalized so that \( \int_{U_N} dU = 1 \)) and \( V \) is a polynomial function in the non-commutative variables \((U_i, U^*_i, A^N_i, 1 \leq i \leq m)\). \( \text{Tr} \) denotes the usual trace on \( N \times N \) matrices given by \( \text{Tr}(A) = \sum_{i=1}^{N} A_{ii} \).

We will study in this article the first order asymptotics of matrix integrals given by (1) when the joint distribution of the \((A^N_i, 1 \leq i \leq m)\) converges;
namely for all polynomial function $P$ in $m$ non-commutative indeterminates

$$\lim_{N \to \infty} \frac{1}{N} \text{Tr}(P(A_i^N, 1 \leq i \leq m)) = \tau(P)$$

for some linear functional $\tau$ on the set of polynomials. Without loss of
generality, we will assume that $(A_i^N, 1 \leq i \leq m)$ are Hermitian matrices.

For technical reasons, it is convenient to assume that the polynomial $V$
is such that $\text{Tr}(V(U_i, U_i^*, A_i^N, 1 \leq i \leq m))$ is real for all $U_i \in U_N$, all Hermitian
matrices $A_i^N$, for all $i \in \{1, \ldots, m\}$ and $N \in \mathbb{N}$.

Under those very general assumptions, the only result proved so far is the
formal convergence of these matrix integrals. Namely, it was proved in [8]
by one author that for each $k$, the quantity

$$\frac{\partial^k}{\partial z^k} N^{-2} \log \int_{U_N^m} e^{zN\text{Tr}(V(U_i, U_i^*, A_i^N, 1 \leq i \leq m))} dU_1 \cdots dU_m |_{z=0}$$

converges towards a constant $f_k(V, \tau)$ depending only on the limiting distri-
bution of the $A_i^N$’s and $V$. Besides, if $V$ is polynomial with integer coeffi-
cients, $f_k(V, \tau)$ is a polynomial function with integer coefficients of the limit
moments of the $A_i^N$’s.

In this paper we will answer affirmatively to the following, previously
open questions:

1. Does the limit of the matrix integrals exist for small parameters $z$?
2. Does the power series $\sum_k \frac{z^k}{k!} f_k(V, \tau)$ have a strictly positive radius
   of convergence?
3. Is the limit of the matrix integral equal to the power series?

The following Theorem is a precise description of our results:

**Theorem 0.1.** Under the above hypotheses and if we further assume that
the spectral radius of the matrices $(A_i^N, 1 \leq i \leq m, N \in \mathbb{N})$ is uniformly
bounded (by say $M$), there exists $\varepsilon = \varepsilon(M, V) > 0$ so that for $z \in [-\varepsilon, \varepsilon]$,
the limit

$$F_{V, \tau}(z) := \lim_{N \to \infty} \frac{1}{N^2} \log \int_{U_N^m} e^{zN\text{Tr}(V(U_i, U_i^*, A_i^N, 1 \leq i \leq m))} dU_1 \cdots dU_m$$

exists. Moreover, $F_{V, \tau}(z)$ is an analytic function of $z \in \mathbb{C} \cap B(0, \varepsilon) = \{z \in \mathbb{C} : |z| \leq \varepsilon\}$ and for all $k \in \mathbb{N},$

$$\frac{\partial^k}{\partial z^k} F_{V, \tau}(z) \bigg|_{z=0} = f_k(V, \tau).$$

This also implies that the series $F_{V, \tau}(z)$ has a positive radius of con-
vergence, a result which had not been proved by the techniques of [8] based on
Weingarten functions.

Our approach is based on non-commutative differential calculus (in par-
ticular on the resulting Schwinger-Dyson or Master loop equations) and
perturbation analysis as developed in the context of Gaussian matrices in
Another possibility to prove the equality between real and formal limits would have been to show convergence of the integrals for complex parameters \( z \). We have not yet been able to follow this line successfully, and this remains an open question.

An important example of unitary matrix integral is the so-called spherical integral, studied by Harish-Chandra and by Itzykson and Zuber,

\[
HCIZ(A, B) := \int_{U \in U_N} e^{N \text{Tr}(U^* A U B)} dU.
\]

This integral is of fundamental importance in analytic Lie theory and was computed for the first time by Harish-Chandra in [19]. In the last two decades it has also become an issue to study its large dimension asymptotics [18, 36, 11, 15].

Theorem 0.1 holds true for the HCIZ integral. It thus relates the results of [8] (which computed the formal limit of the HCIZ integral) and those of [18] (where the limit of \( HCIZ(A, B) \) was obtained (regardless of any small parameters assumptions) by using large deviations techniques). Let us recall the limit found in [18]. Let us define

\[
I(\mu) = \frac{1}{2} \mu(x^2) + \frac{1}{2} \int \int \log |x - y| d\mu(x) d\mu(y).
\]

If \( \mu_A \) (resp. \( \mu_B \)) denote the limiting spectral measure of \( A \) (resp. \( B \)), assume that \( I(\mu_A) \) and \( I(\mu_B) \) are finite. Then, the limit of \( N^{-2} \log HCIZ(A, B) \) is given, according to [18], by

\[
(3) \quad I(\mu_A, \mu_B) := \lim_{N \to \infty} \frac{1}{N^2} N^2 \log HCIZ(A, B)
= -I(\mu_A) - I(\mu_B) - \frac{1}{2} \inf_{\rho, m} \left\{ \int_0^1 \int \left( \frac{m_t(x)^2}{\rho_t(x)} + \frac{\pi^2}{3} \rho_t(x)^3 \right) dx dt \right\}
\]

where the infimum is taken over \( \rho, m \) so that the measure-valued process \( \mu_t(dx) = \rho_t(x) dx \) is a continuous process, \( \mu_0 = \mu_A, \mu_1 = \mu_B \) and

\[
\partial_t \rho_t(x) + \partial_x m_t(x) = 0.
\]

The inf over \( (\rho_t, m_t) \) is taken (see [14]) at the solution of an Euler equation for isentropic flow with negative pressure \(-\frac{\pi^2}{3} \rho^3\).

Theorem 0.1 shows that \( I(\mu_{\sqrt{\beta} A}, \mu_{\sqrt{\beta} B}) \) depends analytically on \( \beta \) in a real neighbourhood of 0, a result which is not obvious from formula (3). Moreover, the coefficients of this expansion count certain planar graphs (see section 5), as summarized in the following theorem.

**Theorem 0.2.** Denote \( \sqrt{\beta} \mu \) the probability measure

\[
\sqrt{\beta} \mu(f) = \int f(\sqrt{\beta} x) d\mu(x).
\]
Assume that \( \mu_A \) and \( \mu_B \) are two compactly supported probability measures. Then, there exists \( \beta_0 > 0 \) such that for all \( \beta \in [0, \beta_0] \),
\[
I(\sqrt{\beta} \mu_A, \sqrt{\beta} \mu_B) = \sum_{n \geq 0} \beta^n M_n(\mu_A, \mu_B)
\]
converges absolutely. Moreover, we have
\[
M_n(\mu_A, \mu_B) = \sum_{m \text{ admissible maps of } \Sigma_n} M_m(\mu_A, \mu_B).
\]
\( \Sigma_n \) is the set of planar maps drawn above \( n \) vertices defined as stars of type \( U^* A U B \) by gluing pairwise oriented arrows and possibly rings and \( M_m(\mu_A, \mu_B) \) is the weight of the map.

We refer the reader to section 5 for the definitions of stars, admissible maps and weights. Our definition of planar maps is more complicated than those arising in the topological expansion of Gaussian matrix models (and which are directly related with Wick Gaussian calculus and Feynman diagrams): the sums are signed and we have a notion of admissibility. However it was an open question in mathematical physics to have a graphical model for unitary integrals (see \[36\]). Moreover, this graphical interpretation gives a new understanding of cumulants formulae (see section 6.2).

The convergence of other integrals was still unknown and it is one of the points of this paper to show their convergence. We use it to study Voiculescu’s microstates entropy evaluated at a set of laws which are small perturbations of the law of free variables, and prove regularity of microstates.

**Theorem 0.3.** For tracial states \( \mu \) satisfying suitable assumptions described in Theorem 8.1,
\[
\chi(\mu) := \lim_{\epsilon \downarrow 0} \liminf_{k \uparrow \infty} \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log \mu_N^{\otimes m} (\Gamma_R(\mu, \epsilon, k))
\]
\[
= \lim_{\epsilon \downarrow 0} \limsup_{k \uparrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log \mu_N^{\otimes m} (\Gamma_R(\mu, \epsilon, k))
\]
and a formula for \( \chi(\mu) \) can be given.

This result generalizes section 4 in \[16\].

The paper is organized as follows: after setting our working framework (section 1), we study the action of perturbations upon the integral \( I_N(V, A^N) \) and deduce some properties of the related Gibbs measure; namely that the so-called empirical distribution of the matrices under this Gibbs measure satisfies asymptotically an equation called the Schwinger-Dyson equation (section 2). Then, we study this equation and obtain uniqueness for parameters of the potential \( V \) small enough (section 3) and analyticity (section 4).

We also describe a (new) combinatorial solution of Schwinger-Dyson equation (section 5) and therefore of the first order of unitary matrix integrals.
We deduce applications of these results to free probability (section 6) and to the convergence of matrix integrals $I_N(V, A_i^N)$ (section 7). Finally, we point out some consequence of our result for free entropy (section 8). At last, we consider the case where integration holds over the orthogonal group instead of the unitary group and show that the first order of such integrals is the same.

1. Notations

Let $U_N$ be the set of $N \times N$ unitary matrices, $M_N$ the set of $N \times N$ matrices with complex entries, $H_N$ the subset of Hermitian matrices of $M_N$ and $A_N$ the subset of antihermitian matrices of $M_N$. We let $m$ be a fixed integer number throughout this article. We denote by $(A_i^N)_{1 \leq i \leq m}$ a $m$-tuple of $N \times N$ Hermitian matrices. We shall assume that the sequence $(A_i^N)_{1 \leq i \leq m}$ is uniformly bounded for the operator norm, and without loss of generality that they are bounded by one,

$$\sup_{N,i} \|A_i^N\|_\infty = \sup_{N,i} \lim_{p \to \infty} \frac{1}{\sqrt{p}} \leq 1.$$ 

1.1. Free $*$-algebra. Let $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$ denotes the set of polynomial functions in the non-commutative indeterminates $(U_i, U_i^*, A_i)_{1 \leq i \leq m}$ with the relation

$$U_i U_i^* = U_i^* U_i = 1.$$ 

Note that in general we may want to consider models with a number of “deterministic” indeterminates $A_i$ different from the number of “random unitary” indeterminates $U_i$, but this general case can be obtained from the previous one by looking only at a sub-algebra and our convention shortens a little the notations. The algebra $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$ is equipped with the involution $*$ so that $A_i^* = A_i$, $(U_i)^* = U_i^*$; $(U_i^*)^* = U_i$ and for any $X_1, \cdots, X_n \in (U_i, U_i^*, A_i)_{1 \leq i \leq m}$, any $z \in \mathbb{C}$,

$$(zX_1 X_2 \cdots X_{n-1} X_n)^* = z X_n^* X_{n-1}^* \cdots X_2^* X_1^*.$$ 

Note that for any $U_i \in U_N$, $A_i \in H_N$, and $P \in \mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$,

$$(P(U_i, U_i^*, A_i, 1 \leq i \leq m))^* = P^*(U_i, U_i^*, A_i, 1 \leq i \leq m)$$

where in the left hand side $*$ denotes the standard involution on $M_N$. We denote $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})_a$ the set of self-adjoint polynomials; $P = P^*$, and $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})_a$ the set of anti-self-adjoint polynomials ; $P^* = -P$. In the sequel, except when something different is explicitly assumed, we shall make the hypothesis that the potential $V$ belongs to $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})_a$, which insures that $\text{Tr}(V((U_i, U_i^*, A_i^N)_{1 \leq i \leq m}))$ is real-valued for all $U_i \in U_N$ and $A_i^N \in H_N$. Conversely, any potential $V$ such that $\text{Tr}(V((U_i, U_i^*, A_i^N)_{1 \leq i \leq m}))$ is real-valued for all $U_i \in U_N$ and $A_i^N \in H_N$ is self-adjoint up to the addition of some commutators (which does not change the trace). Indeed, this implies that $\text{Tr}((V - V^*)((U_i, U_i^*, A_i^N)_{1 \leq i \leq m}))$ vanishes, which insures that $V - V^* = \sum_i P_i Q_i - Q_i P_i$ for some polynomials $P_i, Q_i$, cf. 9 Lemma 2.9.
for a probabilistic proof or [22], Proposition 2.3 for a direct proof (in the real symmetric case, but directly adaptable to the Hermitian case). Then, $W := V + \sum_i(Q_i P_i - P_i Q_i)/2$ is self-adjoint.

1.2. Non-commutative derivatives. On $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$, we define the non-commutative derivatives $\partial_i$, $1 \leq i \leq m$, given by the linear form such that

$$\partial_i A_j = 0, \quad \partial_i U_j = 1_{i=j} U_j \otimes 1 \quad \partial_i U_j^* = -1_{i=j} 1 \otimes U_j^*, \quad \forall j,$$

and satisfying the Leibnitz rule, that for $P, Q \in \mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$,

$$\partial_i (PQ) = \partial_i P \times (1 \otimes Q) + (P \otimes 1) \times \partial_i Q.$$

Here, $\times$ denotes the product $P_1 \otimes Q_1 \times P_2 \otimes Q_2 = P_1 P_2 \otimes Q_1 Q_2$. We also let $D_i$ be the corresponding cyclic derivatives such that if $m(A \otimes B) = BA$, $D_i = m \circ \partial_i$.

If $q$ is a monomial in $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$, we more specifically have

$$\partial_i q = \sum_{q=1} q_1 U_i \otimes q_2 - \sum_{q=1} q_1 \otimes U_i^* q_2$$

$$D_i q = \sum_{q=1} q_2 q_1 U_i - \sum_{q=1} U_i^* q_2 q_1.$$

1.3. Bounded tracial states. Let $\mathcal{T}$ be the set of tracial states on the algebra generated by the variables $(U_i, U_i^*, A_i)_{1 \leq i \leq m}$, i.e. the set of linear forms on $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$ such that for all $P, Q \in \mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$,

$$\mu(PP^*) \geq 0, \quad \mu(PQ) = \mu(QP), \quad \mu(1) = 1.$$

Throughout this article, we restrict ourselves to tracial states $\mu \in \mathcal{T}$ such that

$$\mu((A_i^* A_i)^n) \leq 1 \quad \forall n \in \mathbb{N}, \forall i \in \{1, \cdots, m\}.$$

We denote $\mathcal{M}$ this subset of $\mathcal{T}$.

Note that for any monomial $q \in \mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$, the Hölder’s inequality implies that for any $\mu \in \mathcal{M}$,

$$\mu(qq^*) \leq 1.$$

We endow $\mathcal{M}$ with its weak topology: $\mu_n$ converges to $\mu$ if and only if for all $P \in \mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$,

$$\lim_{n \rightarrow \infty} \mu_n(P) = \mu(P).$$

By equation (7) and since the above topology is the product topology, $\mathcal{M}$ is a compact metric space by Banach Alaoglu’s theorem.

We denote $\hat{\mu}^N$ the empirical distribution of matrices $\hat{A}_i^N \in \mathcal{H}_N$ and $U_i \in \mathcal{U}_N$ which is given for all $P \in \mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$ by

$$\hat{\mu}^N(P) = \frac{1}{N} \text{Tr} \left( P(U_i, U_i^*, A_i^N, 1 \leq i \leq m) \right).$$

This object will be of crucial interest for us.
We define, for all $P$ distribution on $\mathbb{C}$, a $a, b$ with for all $\tau$ ing distribution $\mathbb{C}$ C we also define the cumulants 1.5. $M$ state of formal generating function: $\log tX = \sum_{k \geq 0} t^k C_k(X, \ldots, X)/k!$

This equality holds also in a complex neighborhood of 0 if $X$ is bounded. We also define the cumulants $C_k$ for $k$ in $\mathbb{N}$:

$$\log E(e^{t_1 X_1 + \cdots + t_n X_n}) = \sum_{k \in \mathbb{N}^n} t^k C_k(X_1, \ldots, X_n)/k!$$

where $k = (k_1, \ldots, k_n)$, $|k| = \sum_i k_i$ and $t^k = \prod_i t_i^{k_i}$. Note that:

$$C_k(X_1, \ldots, X_n) = C_{|k|}(X_1, \ldots, X_1, \ldots, X_n, \ldots, X_n)$$

where in the previous list the variable $X_i$ appears $k_i$ times.

Let us recall some properties of these cumulants.

**Proposition 1.1.** The following two statements hold true:

$$\frac{E(Y e^{t_1 X_1 + \cdots + t_n X_n})}{E(e^{t_1 X_1 + \cdots + t_n X_n})} = \sum_{k \in \mathbb{N}^n} t^k C_{1,k}(Y, X_1, \ldots, X_n)/k!$$

The notation $\mathcal{M}|(A_i)_{1 \leq i \leq m}$ stands for the set of tracial states of $\mathcal{M}$ restricted to the algebra generated by the $(A_i)_{1 \leq i \leq m}$. In particular, the limiting distribution $\tau$ given by (2) belongs to $\mathcal{M}|(A_i)_{1 \leq i \leq m}$.

1.4. **Tracial power states.** Let $V \in \mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})_{sa}$ and $\mu^N_V$ be the distribution on $U_N^m$ given by

$$\mu^N_V(dU_1, \cdots, dU_m) = I_N(V, A_i^N)^{-1} \exp(N\text{Tr}(V))dU_1 \cdots dU_m.$$ 

We define, for all $P \in \mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$, $\mu^N_V(P) := E_{\mu^N_V}[\hat{\mu}^N(V)] := \int \frac{1}{N} \text{Tr} P e^{N\text{Tr} V} dU_1 \cdots dU_n. \int e^{N\text{Tr} V} dU_1 \cdots dU_n.$

In the following, an $n$-tuple of monomials $(q_i)_{1 \leq i \leq n}$ in $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$ will be fixed and we shall take $V = V_i = \sum_{i=1}^n t_i q_i$. Then, $\hat{\mu}^N_V(P)$ can be seen as a power series in the $t_i$'s;

$$\hat{\mu}^N_{V_1}(P) := \sum_{k \in \mathbb{N}^n} \frac{t^k}{k!} \frac{1}{\prod_i \partial t_i^{k_i}} \frac{E[\hat{\mu}^N(P)e^{N^2 \hat{\mu}^N(V_1)}]}{E[e^{N^2 \hat{\mu}^N(V_1)}]}.$$ 

We will call $\mu$ a 'tracial power state' of $\mathcal{M}$ if and only if it is a map $\mu : \mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m}) \rightarrow \mathbb{C}[\mathbb{C}[[t]]$ with for all $a, b, \mu(ab) = \mu(ba)$. Here $\mathbb{C}[\mathbb{C}[[t]]$ is the algebra of power series in the variables $t_1, \cdots, t_n$. In particular, we may view $\mu^N_V$ as a tracial power state of $\mathcal{M}$.

1.5. **Cumulants.** The classical cumulants $\{C_k\}_{k \geq 0}$ are defined via their formal generating function:

$$\log E(e^{tX}) = \sum_{k \geq 0} t^k C_k(X, \ldots, X)/k!$$

The notation $\mathcal{M}|(A_i)_{1 \leq i \leq m}$ stands for the set of tracial states of $\mathcal{M}$ restricted to the algebra generated by the $(A_i)_{1 \leq i \leq m}$. In particular, the limiting distribution $\tau$ given by (2) belongs to $\mathcal{M}|(A_i)_{1 \leq i \leq m}$. 

$$\frac{E(Y e^{t_1 X_1 + \cdots + t_n X_n})}{E(e^{t_1 X_1 + \cdots + t_n X_n})} = \sum_{k \in \mathbb{N}^n} t^k C_{1,k}(Y, X_1, \ldots, X_n)/k!$$
parameters of the potential at a polynomial and study its convergence as a formal power series in the

Assume that

Theorem 2.1.

2.1. Behavior of same type of equations called Schwinger-Dyson (or Master loop) equations.

\[
P \in \mathbb{M}
\]

In particular, any limit point \( \mu \) of variables becomes very close to the identity as \( P \) where the

\[
\mu \otimes (\partial_i P) + \mu(D_i V P) = 0
\]

for all \( P \in \mathbb{C}(\langle U_i, U_i^*, A_i \rangle_{1 \leq i \leq m}) \) and \( \mu|_{(A_i)_{1 \leq i \leq m}} = \tau \).

The idea of the proof, rather common in quantum field theory and successfully used in [16], [17, 23], is to obtain equations on \( \hat{\mu}^N \) by performing an infinitesimal change of variables in \( I_N(V, A_j^N) \). More precisely we make the change of variables \( U = (U_1, \cdots, U_m) \in U_N^m \rightarrow \Psi(U) = (\Psi_1(U), \cdots, \Psi_m(U)) \in U_N^m \) with

\[
\Psi_j(U) = U_j e^{P_j(U)}
\]

where the \( P_j \) are antisymmetric polynomials (i.e. \( P_j^* = -P_j \)). This change of variables becomes very close to the identity as \( N \) goes to infinity, reason why it is called “infinitesimal”.

Lemma 2.1. The function \( \Psi \) is a local diffeomorphism and its Jacobian \( J_\Psi \) has the following expansion when \( N \) goes to infinity

\[
|\det J_\Psi(U)| = e^{N \sum \text{TrTr} \otimes (\partial_i P_i(U_i, U_i^*, A_i, 1 \leq i \leq m)) + O(1)}
\]
where $O(1)$ is uniform on the unitary group (but may depend on $P$).

**Proof.**

Let us first recall the following two elementary results of differential geometry:

1. The map $\exp: \mathcal{M}_N \to \mathcal{M}_N$ is differentiable and:
   \[
   \text{Diff}_M \exp \cdot H := \lim_{\varepsilon \to 0} \varepsilon^{-1}(e^{M + \varepsilon H} - e^M) = \left( \sum_{k=0}^{+\infty} \frac{(Ad_M)^k}{(k+1)!} H \right) e^M
   \]
   where $Ad_M$ is the operator defined by $Ad_M H = MH - HM$.

2. If $P \in \mathbb{C}([U_i, U_i^*, A_i]_{1 \leq i \leq m})$ is considered as a function of the $U_i$'s, then it is differentiable and its differential with respect to the $i$-th variable in the direction $A$, for $A$ in $\mathcal{A}_N$, is
   \[
   \text{Diff}_i P \cdot A := \lim_{\varepsilon \to 0} \varepsilon^{-1}(P(U_1, \cdots, U_{i-1}, U_i e^{\varepsilon A}, U_{i+1}, \cdots) - P(U)) = \partial_i P \cdot A.
   \]

As a consequence, if we fix $A$ in $\mathcal{A}_N$ and $i \in \{1, \cdots, m\}$, one has

\[
\text{Diff}_i \Psi_j(U) \cdot A = 1_{i=j} U_j A + U_j \text{Diff}_i \Psi_j(U) \cdot \exp \left( \frac{\lambda}{N} \partial_i P_j \cdot A \right)
\]

\[
= 1_{i=j} U_j A + \frac{\lambda}{N} \sum_{k=0}^{+\infty} U_j \frac{(Ad_{\frac{\lambda}{N} P_j(U)})^k}{(k+1)!} (\partial_i P_j \cdot A) e^{\frac{\lambda}{N} P_j(U)}
\]

\[
= 1_{i=j} U_j A + U_j \frac{\lambda}{N} \Phi_{ij}(U) A.
\]

with $\Phi_{ij}(U)$ the linear map from $\mathcal{A}_N$ into $\mathcal{M}_N$ given by

\[
\Phi_{ij}(U) A := \sum_{k=0}^{+\infty} \frac{(Ad_{\frac{\lambda}{N} P_j(U)})^k}{(k+1)!} (\partial_i P_j \cdot A) e^{\frac{\lambda}{N} P_j(U)}.
\]

We can factorize the term $U_j$ to obtain

\[
\text{Diff}_i \Psi(U) = U \circ (\text{Id}_{\mathcal{A}_N} + \frac{\lambda}{N} \Phi(U))
\]

with $U \circ (M_1, \cdots, M_m) = (U_1 M_1, \cdots, U_m M_m)$ and $\Phi$ the linear operator from $\mathcal{A}_N$ to $\mathcal{M}_N$ whose blocks are the $\Phi_{ij}(U)$.

Since the operator norms of the $A_i$'s and the $U_i$'s are uniformly bounded in $N$, the operator norm of $Ad_{\frac{\lambda}{N} P_j(U)}$ as an operator on $(\mathcal{M}_N, \|\cdot\|_{\infty})$ is also bounded. Thus, $\Phi_{ij}(U)$ is a uniformly bounded operator from $\mathcal{A}_N$ to $\mathcal{M}_N$. Thus, the norm of $\Phi_{ij}(U)$ is less than 1/2 for $N$ large enough. For those $N$, $\Psi$ is a local diffeomorphism with positive eigenvalues.

We can now compute the Jacobian of $\Psi$

\[
|\det J_\Psi(U)| := |\det \text{Diff}_i \Psi(U)| = |\det U||\det(I + \frac{\lambda}{N} \Phi(U))|.
\]

It can be easily checked that $|\det U| = 1$. 

Besides, the positivity of the eigenvalues of $I + \lambda \Phi(U)/N$ allows us to replace the determinant by the exponential of a trace:

$$|\det J_\Psi(U)| = \exp(\text{Tr} \log(I + \frac{\lambda}{N} \Phi(U))) = \exp \left( - \sum_{p \geq 1} \frac{(-\lambda)^p}{pN^p} \text{Tr}(\Phi(U)^p) \right).$$

Note that since $\Phi$ is a bounded operator on $A_N$, which is a space of dimension $N^2$, the $p$-th term in the previous sum is at most of order $N^{2-p}$. We only look at the terms up to the order $O(N)$. A quick computation shows that if $\varphi : A_N \to A_N$, $X \to \sum_l A_l X B_l$

is considered as a real endomorphism, $\text{Tr} \varphi = \sum_l \text{Tr} A_l \text{Tr} B_l$. Indeed, if we consider $E(kl)$, $1 \leq k, l \leq N$ the canonical basis of $A_N$,

$$E(kl)_{rj} := \begin{cases} \sqrt{-1}^r 1_{r=k, j=l} + 1_{r=l, j=k} & \text{for } k \leq l \\ \sqrt{2(1 + 1/k)} & \text{for } k \geq l \end{cases}$$

for $k \leq l$ and

$$E(kl)_{rj} := \frac{1_{r=k, j=l} - 1_{r=l, j=k}}{\sqrt{2}}$$

for $k \geq l$, $\text{Tr} \varphi = \sum_{k,l} \text{Tr}(E(kl)^* \varphi(E(kl))) = \sum_l \text{Tr}A_l \text{Tr} B_l$. This is sufficient to obtain the first term of the Jacobian:

$$\frac{\lambda}{N} \text{Tr}(\Phi(U)) = \frac{\lambda}{N} \sum_i \text{Tr}(\Phi_{ii}(U)) = \frac{\lambda}{N} \sum_i \text{Tr} \otimes \text{Tr}(\partial_i P(U_i, U_j^*, A_j)) + O(1)$$

with $O(1)$ is uniformly bounded on $U^m_N$. Here we used that the operator norm of $\text{Ad}_{\frac{\lambda}{N} P_j(U)}$ is uniformly small.

Before making the change of variables we show that $\Psi$ is a bijection.

**Lemma 2.2.** For $N$ large enough, $\Psi$ is a diffeomorphism of $U^m_N$.

**Proof.**

First observe that since $\Psi$ is a local diffeomorphism, its image is open in $U^m_N$. Besides, since $U^m_N$ is compact and $\Psi$ is continuous, the image is compact and therefore closed. Thus by connectedness of $U^m_N$, and since $\Psi(U^m_N)$ is closed, open and non-empty, $\Psi$ is surjective.

The only property we still need to prove is the injectivity of $\Psi$. If $\Psi(U) = \Psi(V)$ then for all $j \in \{1, \cdots , m\}$,

$$U_j^* V_j - I = e^{\frac{\lambda}{N} P_j(U)} e^{-\frac{\lambda}{N} P_j(V)} - I.$$
Thus, if $N$ is sufficiently large so that $\frac{1}{N} P_j(U)$ is in a domain where the function exp is 2-Lipschitz, we obtain
\[
\|U_j - V_j\|_\infty = \|U_j V_j^* - 1\|_\infty = \|e^{\frac{1}{N} P_j(U)} e^{\frac{1}{N} P_j(V)} - 1\|_\infty
\]
\[
= \|e^{\frac{1}{N} P_j(U)} - e^{\frac{1}{N} P_j(V)}\|_\infty \leq \frac{2|\lambda|}{N} \|P_j(U) - P_j(V)\|_\infty
\]
with $\|\cdot\|_\infty$ the operator norm. Since $(P_j, 1 \leq j \leq m)$ are uniformly Lipschitz on $U_N$, we conclude that $\sum_{j=1}^m \|U_j - V_j\|_\infty$ vanishes for sufficiently large $N$.

We can now prove Theorem 2.1.

**Proof.**

Let us define
\[
Y^N(P) = \sum_i \left( \frac{1}{N} \text{Tr}(D_i V P_i) + \frac{1}{N} \text{Tr} \otimes \frac{1}{N} \text{Tr}(\partial_i P_i) \right).
\]

We expand $\text{Tr} V^i(\Psi(U)_i, \Psi(U)^*_i, A_i, 1 \leq i \leq m)$ as
\[
\text{Tr}(V(\Psi(U)_i, \Psi(U)^*_i, A_i, 1 \leq i \leq m)) - \text{Tr}(V(U_i, U^*_i, A_i, 1 \leq i \leq m))
\]
\[
= \frac{\lambda}{N} \sum_j \text{Tr}(D_j V P_j(U_i, U^*_i, A_i, 1 \leq i \leq m)) + O(N^{-1})
\]

and perform the change of variables $U \to \Psi(U)$ in $I_N(V, A_i^N)$;
\[
I_N(V, A_i^N) := \int e^{N \text{Tr}(V(U_i, U^*_i, A_i, 1 \leq i \leq m))} dU_1 \cdots dU_m
\]
\[
= \int e^{N \text{Tr}(V(\Psi(U)_i, \Psi(U)^*_i, A_i, 1 \leq i \leq m))} |\det J_{\Psi}(U)| dU_1 \cdots dU_m
\]
\[
= \int e^{NY^N(P) + O(1)} e^{N \text{Tr}(V(U_i, U^*_i, A_i, 1 \leq i \leq m))} dU_1 \cdots dU_m
\]

where we used (11) and Lemma 2.1. $O(1)$ is of order one independently of $N$ and uniformly on the unitary matrices $(U_1, \cdots, U_m)$. Thus we have proved that
\[
\int e^{NY^N(P)} d\mu^N(V,U) = O(1).
\]

Borel-Cantelli’s lemma thus insures that
\[
\limsup_{N \to \infty} Y^N(P) \leq 0 \quad a.s.
\]
and the converse inequality holds by changing $P$ into $-P$ since $Y^N$ is linear in $P$. This proves the first statement of Theorem 2.1. The last result is simply based on the compactness of $\mathcal{M}$ and the fact that any limit point must then satisfy the same asymptotic equations than $\hat{\mu}^N$.  

\[ \square \]
Another consequence of this convergence is the existence of solutions to (9) for any self-adjoint potential $V$ (since any limit point of $\hat{\mu}^N$ in the compact metric space $\mathcal{M}$ will satisfy it) a fact already proved in [6]. Moreover, since these solutions are limit points of $\hat{\mu}^N$, they belong to $\mathcal{M}$ and in particular $|\mu(q)| \leq 1$ for any monomial $q$.

2.2. Moments of $\hat{\mu}^N$. In the sequel, we denote by $E$ the expectation with respect to the Haar measure on the unitary group. The goal of this section is to show (see Proposition 2.1) that cumulants also satisfy a formal version of Schwinger-Dyson equation. We start with the following lemma:

Lemma 2.3. One has, for all $i$, all $N$, all monomials $q_1, \cdots, q_n$ and all $k = (k_1, \cdots, k_n)$ in $\mathbb{N}^n$,

$$N^2 E \left( \hat{\mu}^N \otimes \hat{\mu}^N (\partial_i P) \cdot (\hat{\mu}^N(q_1))^{k_1} \cdots (\hat{\mu}^N(q_n))^{k_n} \right)$$

$$+ \sum_j k_j E \left( (\hat{\mu}^N(q_j))^{k_j-1} \cdots (\hat{\mu}^N(q_n))^{k_n} \hat{\mu}^N(D_i q_j \cdot P) \right) = 0$$

Proof.
Following Lemma 2.1, we write down the change of variable

$$\Psi_i : U \rightarrow (U_1, \cdots, U_{i-1}, U_i e^{\lambda P_i(U)}, U_{i+1}, \cdots, U_m)$$

in the integral $\int ((\hat{\mu}^N(q_1))^{k_1} \cdots (\hat{\mu}^N(q_n))^{k_n}) dU_1 \cdots dU_m$, where the integration is taken over the unitary Haar measure. Its Jacobian satisfies

$$| \det J_{\Psi}(U) | = 1 + \frac{\lambda}{N} \text{Tr} \otimes \text{Tr}(\partial_i P) + o(\lambda).$$

and we have the expansion

$$\text{Tr}(q_j(\Psi(U)_i, \Psi(U)_i^*, A_i, 1 \leq i \leq m)) = \text{Tr}(q_j(U_i, U_i^*, A_i, 1 \leq i \leq m))$$

$$+ \lambda \text{Tr}(D_i q_j \cdot P(U_i, U_i^*, A_i, 1 \leq i \leq m)) + \lambda^2 o(\lambda)$$

where the $o(\lambda)$’s are for a given $P$ uniform bounds in $N$. The first order of the Taylor expansion of this change of variables around $\lambda = 0$ proves the claim.

Proposition 2.1. As a formal series equality, one has, for all $i$, for all $t$,

$$E[\hat{\mu}^N \otimes \hat{\mu}^N (\partial_i P) e^{N^2 \hat{\mu}^N(V_t)}] + E[\hat{\mu}^N (D_i V_t \cdot P) e^{N^2 \hat{\mu}^N(V_t)}] = 0.$$  

Proof.
Multiplying the equality of Lemma 2.3 by $t^k N^{|k|}/k!$ and summing over $k$ in $\mathbb{N}^n$ gives the desired identity.

Finally we study the large $N$ limit $\mu^f$ of these formal states (the index $f$ stands for “formal”).
Theorem 2.2. Let $V_t$ be the polynomial $\sum_{j=1}^{n} t_j q_j$. For all $P$, the sequence $\bar{\mu}^N_{V_t}(P)$ converges as a formal series (i.e. coefficientwise) when $N$ goes to infinity to some $\mu^f(P)$. Besides, $\mu^f$ satisfies the family of equations, for all $i$, for all $P$,

$$\mu^f \otimes \mu^f(\partial_i P) + \mu^f(D_i V_t \cdot P) = 0.$$ 

Proof.

First, we prove the existence of a limit. By the first item of Proposition 1.1, we can express $\bar{\mu}^N_{V_t}(P)$ as a sum over cumulants,

$$\bar{\mu}^N_{V_t}(P) = \sum_{k \in \mathbb{N}^n} t^{k} C_{1,k}(\frac{1}{N} \text{Tr} P, N \text{Tr} q_1, \ldots, N \text{Tr} q_n)/k!.$$ 

The limit in $N$, of the $C_{1,k}(\frac{1}{N} \text{Tr} P, N \text{Tr} q_1, \ldots, N \text{Tr} q_n)$ was proved to exists in [8] so that $\mu^f$ is well defined.

Item (2) from Proposition 1.1 implies

$$E(\frac{1}{N} \text{Tr} P_1 \frac{1}{N} \text{Tr} P_2 e^{N \text{Tr} V}) E(e^{N \text{Tr} V}) - E(\frac{1}{N} \text{Tr} P_1 e^{N \text{Tr} V}) E(\frac{1}{N} \text{Tr} P_2 e^{N \text{Tr} V}) E(e^{N \text{Tr} V})$$ 

$$= \sum_{k \geq 0} \frac{t^{k}}{k!} C_{1,1,k}(\frac{1}{N} \text{Tr} P_1, \frac{1}{N} \text{Tr} P_2, N \text{Tr} q_1, \ldots, N \text{Tr} q_n).$$ 

Now, it follows from [8] that elements on the right hand side have decay $N^{-2}$ so that the coefficientwise limit is zero. This can be interpreted as a formal convergence of measure result for the states $\bar{\mu}^N$.

The proof of the Theorem follows from this observation and from Proposition 2.1.

3. Study of the Schwinger-Dyson equation

We have shown that the limit points of the matrix model satisfy the Schwinger-Dyson equation [9]. The aim of this section is to study this equation and show that it has a unique solution.

Definition 3.1. Let $\tau \in \mathcal{M}_{\{A_i\}_{1 \leq i \leq m}}$. A tracial state $\mu \in \mathcal{M}$ is said to satisfy Schwinger-Dyson equation $SD[V, \tau]$ if and only if for all $P \in \mathbb{C}((A_i)_{1 \leq i \leq m})$,

$$\mu(P) = \tau(P)$$

and for all $P \in \mathbb{C}((U_i, U^*_i, A_i)_{1 \leq i \leq m})$, all $i \in \{1, \ldots, m\}$,

$$\mu \otimes \mu(\partial_i P) + \mu(D_i V P) = 0.$$ 

Let $V \in \mathbb{C}((U_i, U^*_i, A_i)_{1 \leq i \leq m})$. One can decompose $V$ in a sum

$$V = \sum_{i=1}^{n} t_i q_i(U_j, U^*_j, A_j, 1 \leq j \leq m)$$
with monomial functions \( q_i \) and complex numbers \( t_i \). The monomials \((q_i, 1 \leq i \leq n)\) will be fixed hereafter. We let \( D \) be the maximal degree of the monomials \( q_i \).

Here we prove that \( \tau \) is uniquely defined provided that the parameters \((t_i, 1 \leq i \leq m)\) are small enough.

**Theorem 3.1.** Let \( D \) an integer and \( \tau \) a tracial state in \( \mathcal{M}|(A_i)_{1 \leq i \leq m} \) be given. There exists \( \varepsilon = \varepsilon(D, m) > 0 \) such that if \(|t_i| \leq \varepsilon\), there exists at most one solution \( \mu \) to \( \text{SD}[V, \tau] \).

From this and Theorem 2.1 we deduce the following

**Corollary 3.1.** Assume that \( V \) is self-adjoint. Let \( D \) an integer and \( \tau \) a tracial state in \( \mathcal{M}|(A_i)_{1 \leq i \leq m} \) be given. There exists \( \varepsilon = \varepsilon(D, m) > 0 \) such that if \(|t_i| \leq \varepsilon\), \( \hat{\mu}^N \) converges almost surely to the unique solution \( \mu \) of the Schwinger-Dyson equation. Moreover, \( \bar{\mu}_V^N = \mu_V^N(\hat{\mu}^N) \) converges as well to this solution as \( N \) goes to infinity.

This result is obvious since Theorems 2.1 and 3.1 show that \( \hat{\mu}^N \) has a unique limit point, and thus converges almost surely. The convergence of \( \bar{\mu}_V^N \) is then a direct consequence of bounded convergence theorem since \( \hat{\mu}^N \in \mathcal{M} \).

Actually Theorem 2.1 and Corollary 3.1 do not use the assumption that the matrices \((A_i^N, 1 \leq i \leq m)\) are deterministic, but only that they are bounded and have a converging joint distribution. Therefore these two results extend to the case where these matrices are random, independent of the \((U_i, 1 \leq i \leq m)\), and satisfy the above two conditions almost surely. This observation implies that our result can also encompass the case of the truncated GUE or other classical bounded matrix models.

We prove now Theorem 3.1

**Proof.**

Let \( \mu \) be a solution to \( \text{SD}[V, \tau] \). Note that if we take \( q \) a monomial in \( \mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m}) \), either \( q \) does not depend on \((U_j, U_j^*, 1 \leq j \leq m)\) and then \( \mu(q) = \tau(q) \) is uniquely defined, or \( q \) can be written as \( q = q_1 U_i^a q_2 \) for some \( i \in \{1, \ldots, m\}, a \in \{-1,+1\} \) and monomials \( q_1, q_2 \) (Here, note that \( U_i^{-1} = U_i^* \)). Then, by the traciality assumption, \( \mu(q) = \mu(q_2 q_1 U_i^a) = \mu(U_i^a q') \) with \( q' = q_2 q_1 \). Remark that we can assume without loss of generality that the last letter of \( q' \) is not \( U_i^{-a} \). We next use \( \text{SD}[V, \tau] \) to compute \( \mu(U_i^a q) \) for some monomial \( q \). We assume first that \( a = -1 \). Then, by (1),

\[
\partial_i (U_i^a q) = -1 \otimes (U_i^* q) + U_i^* \otimes 1 \times \partial_i q.
\]
where the supremum holds over monomials of $C \mathcal{D}_n$

Taking the expectation, we thus find by (5), since $\mu(1) = 1$, that

$$
\mu(U_i^* q) = \mu \otimes \mu(U_i^* \otimes 1 \times \partial_i q) + \mu(D_i V q)
$$

$$
= \sum_{q=q_i U_{ij} q_2} \mu(q_1)\mu(q_2) - \sum_{q=q_i U_{ij} q_2} \mu(U_i^* q_1)\mu(U_i^* q_2)
$$

(12)

$$
+ \sum_{j} t_{ij} \mu(q_{ij} q)
$$

where $(t_{ij}, q_{ij})$ are such that $D_i V = \sum_j t_{ij} q_{ij}$. Note that the sum runs at most on $Dn$ terms and that all the $t_{ij}$ are bounded by $\max|t_i|$. A similar formula is found when $a = +1$ by differentiating $qU_i$ (or by using $\mu(qU_i) = \mu((qU_i)^*) = \mu(U_i^* q^*)$).

We next show that (12) and its equivalent for $a = -1$ characterize uniquely $\mu \in \mathcal{M}$ when the $t_{ij}$ are small enough. It will be crucial here that $\mu(q)$ is bounded independently of the $t_i$’s (here by the constant 1).

Now, let $\mu, \mu' \in \mathcal{M}$ be two solutions to $\mathsf{SD}[V, \tau]$ and set

$$
\Delta(\ell) = \sup_{\deg(q) \leq \ell} |\mu(q) - \mu'(q)|
$$

where the supremum holds over monomials of $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$ with total degree in the $U_j$ and $U_j^*$ less than $\ell$. Namely, if the monomial (or word) $q$ contains $a_j^+$ times $U_j$ and $a_j^-$ times $U_j^*$, we assume $\sum_{j=1}^m (a_j^+ + a_j^-) \leq \ell$.

Note that by traciality of $\mu$,

$$
\Delta(\ell) = \max_{1 \leq \ell \leq m} \sup_{a \in \{+1, -1\}} \sup_{\deg(q) \leq \ell-1} |\mu(U_i^* q) - \mu'(U_i^* q)|
$$

and that by (12), we find that, for $q$ with degree less than $\ell - 1$,

$$
|\mu(U_i^* q) - \mu'(U_i^* q)| \leq \sum_{q=q_i U_{ij} q_2} |(\mu - \mu')(q_1)| + \sum_{q=q_i U_{ij} q_2} |(\mu - \mu')(q_2)|
$$

$$
+ \sum_{q=q_i U_{ij} q_2} |(\mu - \mu')(U_i^* q_1)| + \sum_{q=q_i U_{ij} q_2} |(\mu - \mu')(U_i^* q_2)|
$$

$$
+ \sum_{j} t_{ij} |(\mu - \mu')(q_{ij} q)|.
$$

A similar formula holds for $|\mu(U_i q) - \mu'(U_i q)|$ by conjugation, and therefore

$$
\Delta(\ell) \leq 2 \sum_{p=1}^{\ell-2} \Delta(p) + 2 \sum_{p=1}^{\ell-1} \Delta(p) + nD\varepsilon\Delta(\ell + D - 1)
$$

where we used that $\deg(q_1) \in \{0, \cdots, \ell - 2\}$, $\deg(q_2) \in \{0, \cdots, \ell - 2\}$ (but $\Delta(0) = 0$) and $\deg(q_{ij}) \leq D$ and assumed $|t_i| \leq \varepsilon$. Hence, we have proved that
\[
\Delta(\ell) \leq 4 \sum_{p=1}^{\ell-1} \Delta(p) + nD\varepsilon\Delta(\ell + D).
\]

Multiplying these inequalities by \(\gamma^\ell\) we get, since \(H(\gamma) := \sum_{\ell \geq 1} \gamma^\ell \Delta(\ell)\) is finite for \(\gamma < 1\),
\[
H(\gamma) \leq \frac{\gamma}{1-\gamma} H(\gamma) + \gamma \frac{nD\varepsilon}{1-\gamma} H(\gamma)
\]
resulting with \(H(\gamma) = 0\) for \(\gamma\) so that \(1 > \frac{\gamma}{1-\gamma} + \frac{nD\varepsilon}{\gamma}\). Such a \(\gamma > 0\) exists when \(\varepsilon\) is small enough. This proves the uniqueness.

\[\Box\]

As a corollary, we characterize asymptotic freeness by a Schwinger-Dyson equation, a result which was already obtained in [32], Proposition 5.17.

**Corollary 3.2.** A tracial state \(\mu\) satisfies SD\([0,\tau]\) if and only if, under \(\mu\), the algebra generated by \((A_i, 1 \leq i \leq m)\) and \((U_i, U_i^*, 1 \leq i \leq m)\) are free and the \(U_i\)'s are two by two free and satisfy
\[
\mu(U_i^a) = 0 \quad \forall a \in \mathbb{Z}\backslash\{0\}.
\]

**Proof.**

By the previous theorem, it is enough to verify that the law \(\mu\) of free variables \((A_i, U_i, U_i^*)\) with some \(B_k\)’s in the algebra generated by \((A_i, 1 \leq i \leq m)\). We wish to show that for all \(i \in \{1, \ldots, m\}\),
\[
\mu \otimes \mu(\partial_i P) = 0.
\]

Note that by linearity, it is enough to prove this equality when \(\mu(B_j) = 0\) for all \(j\). Now, by definition, we have
\[
\partial_i P = \sum_{k; i_k = i, \alpha_k > 0} \sum_{l=1}^{\alpha_k} U_{i_1}^{a_1} B_{i_1} \cdots B_{i_{l-1}} U_i \otimes U_{i_l}^{a_l} B_k \cdots U_{i_p}^{a_p} B_p
\]

\[- \sum_{k; i_k = i, \alpha_k < 0} \sum_{l=0}^{\alpha_k-1} U_{i_1}^{a_1} B_{i_1} \cdots B_{i_{l-1}} U_i \otimes U_{i_l}^{a_l+1} B_k \cdots U_{i_p}^{a_p} B_p.
\]

Taking the expectation on both sides, since \(\mu(U_i^l) = 0\) and \(\mu(B_j) = 0\) for all \(i \neq 0\) and \(j\), we see that freeness implies that the right hand side is null (recall here that in the definition of freeness, two consecutive elements have to be in free algebras but the first and the last element can be in the same algebra). Thus, \(\mu \otimes \mu(\partial_i P) = 0\) which proves the claim.

\[\Box\]
4. Formal solution and analyticity

We have shown in Theorem 2.2 that the limit points of the formal model also satisfy an equation similar to Schwinger-Dyson’s equation. The only difference is that one of these equations is on the space of tracial states while the other one is on the space of tracial power states. In order to prove that the formal model matches the matrix model we need to study this formal equation and show that the series have a positive radius of convergence, hence providing a solution to $\text{SD}[V, \tau]$ as defined in Definition 3.1.

**Definition 4.1.** Let $V_i = \sum_i t_i q_i$ be a polynomial. Let $\tau$ be a tracial power state in $\mathcal{M}[A_i]_{1 \leq i \leq m}$. A tracial power state $\mu \in \mathcal{M}$ is said to satisfy Schwinger-Dyson equation $\text{SD}^f[V_i, \tau]$ if and only if for all $P \in \mathbb{C}[\langle A_i \rangle_{1 \leq i \leq m}]$, $\mu(P) = \tau(P)$ and for all $P \in \mathbb{C}[\langle (U_i, U_i^*, A_i) \rangle_{1 \leq i \leq m}]$, all $i \in \{1, \cdots, m\}$, $\mu \otimes \mu(\partial P) + \mu(D_i V_i P) = 0$.

(Here, both terms of the above equality are elements of $\mathbb{C}[t]$ and the equality is formal.)

We already know, due to Theorem 2.2, that there exists a solution to this equation. We now prove that this solution is unique.

**Theorem 4.1.** There exists a unique tracial power state $t \to \mu_t$ which satisfies Schwinger-Dyson equation $\text{SD}^f[V_i, \tau]$.

**Proof.**

Let $\mu_t$ be a tracial power state solution of $\text{SD}^f[V_i, \tau]$. There exists a family $\mu_t^k, k = (k_1, \cdots, k_n) \in \mathbb{N}^n$ in the algebraic dual of $\mathbb{C}[\langle (U_i, U_i^*, A_i) \rangle_{1 \leq i \leq m}]$ such that for all $P$,

$$\mu_t(P) = \sum_{k \in \mathbb{N}^n} \prod_{i=1}^n t_{k_i} \mu_t^k(P).$$

We will now show that the $\mu_t^k$ are uniquely inductively defined by the relation given by $\text{SD}^f[V_i, \tau]$. Let us define $1_j$, the vector in $\mathbb{N}^n$ which vanishes on every coordinate except the $j$-th which is 1. We get the following equalities, for all $k$,

1. If $P$ is in $\mathbb{C}[\langle A_i \rangle_{1 \leq i \leq m}]$, $\mu_t^k(P) = \tau(P) 1_{k=0}$,
2. If $P = RU_i S$ with $S$ in $\mathbb{C}[\langle A_i \rangle_{1 \leq i \leq m}]$, $\mu_t^k(P) = \mu_t^k(SRU_i)$,
3. If $P = RU_i^* S$ with $R$ in $\mathbb{C}[\langle A_i \rangle_{1 \leq i \leq m}]$ and $S$ does not contain any $U_j$ (but may contain the $U_j^*$), $\mu_t^k(P) = \mu_t^k(U_j^* SR)$,
4. If $q$ does not contain any $U_j$,

$$\mu_t^k(U_j^* q) = - \sum_{q=q_1 U_j^* q_2} \binom{k}{k'} \sum_{k''} \mu_t^{k'}(U_j^* q_1) \mu_t^{k''}(U_j^* q_2) + \sum_j k_j \mu_t^{k-1_j}(U_j q D_i q_j).$$
(5) And for all $q$,

$$
\mu^k(qU_i) = - \sum_{q=q_1U_{k'}q_2} \sum_{k''=k} \binom{k}{k'} \mu^{k'}(q_1U_i) \mu^{k''}(q_2U_i)
$$

$$
+ \sum_{q=q_1U_{k'}q_2} \sum_{k''=k} \binom{k}{k'} \mu^{k'}(q_1U_i) - \sum_{j} k_j \mu^{k-1,j}(D_jq_jqU_i).
$$

One can see that this allows to compute uniquely any $\mu^k(P)$. The first relation takes care of the non random case, the relations 2 and 3 use the traciality to place a variable $U$ in a convenient place. Finally relations 4 and 5 allow to compute $\mu^k(P)$ as a function which depends on the $\mu^{k'}(Q)$ with $\deg Q < \deg P$ and $k' \leq k$ (first terms) or on the $\mu^{k''}(Q)$ with $k'' < k$ (last term). This is a well founded induction. Thus the $\mu^k$ are uniquely defined.

\[ \square \]

We next show that this solution is not only formal but that it gives a family of solutions $\mu_t$ of the non-formal equation $SD[V_t, \tau]$, which depends analytically on the parameters $t_i$.

**Theorem 4.2.** There exists $\varepsilon > 0$ such that for $t \in \mathbb{C}^n, \max_{1 \leq i \leq n} |t_i| \leq \varepsilon$, the formal solution $\mu_t$ of $SD^f[V_t, \tau]$ is indeed a convergent series. For all polynomials $P$, $t \in B(0, \varepsilon) = \{t \in \mathbb{C}^n : \max_{1 \leq i \leq n} |t_i| \leq \varepsilon\} \rightarrow \mu_t(P)$ is analytic.

In other words, there exists a family $({\mu^k, k = (k_1, \ldots, k_n) \in \mathbb{N}^n})$ in the algebraic dual of $\mathbb{C}((U_i, U_i^*, A_i)_{1 \leq i \leq m})$ such that for all $P$,

$$
\mu_t(P) = \sum_{k \in \mathbb{N}^n} \prod_{i=1}^{n} \frac{t_i}{k_i!} \mu^k(P)
$$

converges absolutely for $\max_{1 \leq i \leq n} |t_i| \leq \varepsilon$.

An immediate consequence of this result is to deduce that the formal solution is a real solution of $SD[V_t, \tau]$ in a small parameters region, and therefore by Theorem 3.1 equals the real solution. This will be a key to prove Theorem 0.1 (see section 7).

**Corollary 4.1.** For small $t$, the formal solution of Schwinger-Dyson equation $SD^f[V_t, \tau]$ converges as a series. In addition, it matches the real solution of $SD[V_t, \tau]$ which thus depends analytically in the parameters $t$ of the potential in a neighborhood of the origin.

Let us now prove Theorem 4.2.

**Proof.**

According to the proof of Theorem 1.1 the $\mu^k$ are uniquely defined by the family of relations (1)-(5). We only need to control the growth of the coefficients $\mu^k(P)$ to show that $\mu_t(P)$ is indeed convergent for small enough parameters.
To bound these quantities, we use the Catalan numbers
\[ C_0 = 1, C_{k+1} = \sum_{0 \leq p \leq k} C_p C_{k-p} \]
and the fact that they do not explode too fast; \( C_{k+1} \leq 4C_k \). We denote \( C_k := \prod_i C_{k_i} \) and \( D_k := A^{k-1}C_{k-1} \) for \( k \geq 1 \), \( D_0 := 0 \). The two key properties of this sequence is first that it is sub-geometric (\( D_{k+1} \leq 4AD_k \)) and secondly it satisfies \( D_k = A\sum_{0<p<k} D_p D_{k-p} \). Now our induction hypothesis is that there exists \( A, B > 0 \) such that for all \( k \), for all monomial \( P \) of degree \( p \),
\begin{equation}
|\mu^k(P)| \leq C_k B^k D_p.
\end{equation}
We prove this bound by induction, and the relations (1)-(5) which define the \( \mu^k \). For \( k = (0, \cdots, 0) \) this bound is satisfied since \( D_p \geq 1 \). We will check the induction for a polynomial of the form \( qU_i \) since it is the most complicated case.
\begin{align*}
&\frac{|\mu^k(qU_i)|}{k!} \leq \sum_{q=q_1, q_2} \frac{|\mu^{k'}(q_1 U_i)|}{k'!} \frac{|\mu^{k''}(q_2 U_i)|}{k''!} \\
&+ \sum_{q=q_1, q_2} \frac{|\mu^{k'}(q_1)|}{k'!} \frac{|\mu^{k''}(q_2)|}{k''!} + \sum_{k_j \neq 0} \frac{|\mu^{k-1}(D_i q_j q)|}{(k-1)!}.
\end{align*}
Now we use the induction hypothesis. If \( q \) is of degree \( p - 1 \),
\begin{align*}
&\frac{|\mu^k(qU_i)|}{k!C_k B^k D_p} \leq 2 \sum_{0<q<p} \frac{C_k C_k' B^k D_q C_k'' B^k D_{p-q}}{C_k B^k D_p} \\
&+ nD \sum_j \frac{C_{k-1-j} B^{k-1} D_{p+j}}{C_k B^k D_p} \\
&\leq 2 \prod_i \frac{C_{k_i+1}}{C_{k_i}} \frac{1}{A} + nD \frac{(4A)^D}{B}.
\end{align*}
The point is that we can choose \( A, B > 0 \) such that this last quantity is lesser than 1. For example take \( A > 4_{n+1}^2 \) and then \( B > 2nD(4A)^D \).

Thus, for \( \|t\| := \max_i |t_i| < 1/4A \), for all \( P \) in \( C \langle (U_i, U_i^*, A_i) \rangle \), the series \( \sum_k \prod_{i \leq m} \mu^k(P) \) is absolutely convergent.

\[ \square \]

5. Combinatorics.

The purpose of this section is to provide a graphical approach to the solution of the Schwinger-Dyson equation, and therefore to the computation of unitary matrix integrals and free entropy (see sections 6, 7 and 8). Actually,
the proof of Theorem 4.1 gives a recursive way of computing formal solutions to the Schwinger-Dyson equation, and therefore numerical solutions with arbitrary precision.

Before giving a detailed description of our combinatorial model, we start with an overview. We need the notions of a star, which is a pictorial encoding of a monomial of $\mathbb{C} \langle (U_i, U_i^*, A_i)_{1 \leq i \leq m} \rangle$, of root star, which is a distinguished star, and of a map, which is a specific planar decoration over a set of stars and one root star.

The goal of this section is to show that the limits of integrals on the space of unitary matrices are generating function of the number of some maps as described above. However we are not interested in all maps, but rather on some that arise from an admissible construction, which leads us to the concept of admissible maps. Last, we need the notion of weight of a map, and our result will be in terms of sum over admissible maps of weights.

Let us point out that for the sake of clarity, although our natural playground is the algebra $\mathbb{C} \langle (U_i, U_i^*, A_i)_{1 \leq i \leq m} \rangle$ and our definitions work in full generality, we restrict ourselves in the examples to the case of one single unitary matrix $U$ and two variables $A_1 = A$ and $A_2 = B$. We first start with the definition of stars and root stars, in the spirit of [16, 17].

**Definition 5.1.**

1. A **star** is a circle endowed with the clockwise orientation, decorated with elements such as colored incoming or outgoing arrows, and colored diamonds. One of the element is marked.

2. To each letter $X_i$ in the alphabet $(A_i, U_i, U_i^*)_{1 \leq i \leq m}$, we associate (bijectively) an element as follows; a diamond of color $i$ if $X_i = A_i$ and a ring of color $i$ if $X_i = U_i$ or $U_i^*$; in the case of $U_i$ (resp. $U_i^*$) we attach before the ring an outgoing arrow of color $i$ (resp. we attach after the ring an incoming arrow of color $i$) outside of the circle.

3. To a monomial $q \in \mathbb{C} \langle (U_i, U_i^*, A_i)_{1 \leq i \leq m} \rangle$, we associate in a canonical way a **star of type** $q$ by drawing on the clockwise oriented circle the elements associated to the successive letters of $q$, while the element corresponding to the first letter of $q$ is marked (or distinguished).

4. A **root star of type** $q$ is obtained by drawing on the oriented circle the elements associated to the successive letters of $q$ in the counter clockwise order, the arrows being drawn inside the circle. Its first element is distinguished. Although the maps are on the sphere, in the graphical representation of this section we will draw them on the plane and, to highlight the role of the root star we will draw it in this section such that it contains all the other stars. It can be viewed as the star centered in infinity or as the outer face of the dual map. Besides, on a root star we will distinguish a root element. If $q$ contains no $U_i$ nor $U_i^*$, there are no root element. If $q$ contains $U_i$, the ring associated to the last $(U_i, 1 \leq i \leq m)$ is the root element. If $q$ contains no $U_i$ but some $U_i^*$, the ring associated to the first $(U_i^*, 1 \leq i \leq m)$ is called the root element.
A multistar is a set of \( k \) stars inside a root star drawn on the same plane with a coherent orientation.

The figure 1 shows a concrete example of a multistar. In the middle of the picture there is a star of type \( U^*AUB \) and, surrounding it, a root star of type \( U^*A^5UB^2U^*A^3UB \).

**Figure 1.** Star of type \( U^*AUB \) and root star of type \( U^*A^5UB^2U^*A^3UB \).

We are now ready to introduce the main objects in our combinatorial model, namely, maps:

**Definition 5.2.** A map is a decoration of a multistar into a connected graph embedded in the plane by drawing two species of edges between rings:

1. A first category of edges, called “dotted edges”, can be drawn between rings either attached to two outgoing arrows of the same color or to two incoming arrows of the same color. These edges can only have rings as end points, not diamonds or arrows. Rings can have any number of dotted edges going out of them, possibly none.
2. A second category of edges, called “colored oriented edge” arises from the connection of an arrow going out of a star (associated with a variable \( U_i \)) into an incoming arrow (associated to a variable \( U^*_i \)) of the same color. These colored oriented edges is a pairing between the set of \( U_i \)'s and the set of \( U^*_i \)'s: exactly one incoming arrow is glued to each outgoing arrow.

In addition, all the above edges do not cross, all arrows are paired but rings can be attached to any number of dotted edges (including to none).

In the remainder of this section we keep considering pictures drawn on the sphere (and in fact on the plane). They therefore give rise to graphs...
with vertices, edges and faces - together with additional decoration. For our forthcoming definitions, we need to clarify the notion of ‘face’: we consider that faces of a graph are the connected components of the complementary of the graph on the sphere. However, we take the convention that the original stars are ‘fattened vertices’. Therefore the interior of stars will not be considered as faces (neither is the exterior of the root star).

Each ‘face’ component of a map is isomorphic to a disc; thus this is an actual face. This is due to the fact that our map is embedded into a sphere. This condition would not be granted in the case of an embedding into a higher genus oriented 2D compact manifold. In this case it would have to stand in the definition of a map of ‘higher genus’: this will be of use for future work but for the sake of simplicity we do not emphasize this notion in this paper.

Next, we define the weight of a map. The boundary of a face is homeomorphic to a circle, it is given an orientation (the orientation of the sphere) and is decorated with diamonds (note that all arrows have been paired); it thus has the structure of a star except for the distinguished element.

**Definition 5.3.** Assume we are given the tracial state $\tau$ of $\mathcal{M}$.

- First we define the weight of the faces of a map. The boundary of a face have the structure of a star, i.e. it has the topology of a circle with some diamonds on it. We can therefore associated each of these boundaries with a monomial in the $A_i$’s, given up to cyclic permutation (or equivalently up to knowing its first letter). The weight of a face is the trace $\tau(q)$ (which does not depend on cyclic permutations) of the monomial $q$ associated with its boundary.
- The weight of the map $m$, denoted by $M_m(\tau)$, is the product of the weights of its faces times a sign given by $-1$ to the power the number of dotted edges.

As we said before, not all maps will contribute and we need to define now the notion of admissible maps. Admissibility can be checked by an inductive procedure $IP$, which resembles Tutte’s surgery [29] and which amounts to check one after the other whether edges of the map are admissible. Once an edge has been checked, it is frozen and we continue by checking the other edges.

**Inductive Procedure IP :**

a- If the root star has no root element, then it can not be connected to any other star and hence the graph can not be a map unless there is no other star in which case the map is just the trivial graph with no edges.

b- The root star has a root element which is associated to a $U_i$ (resp. a $U^*_i$), for some $i \in \{1, \ldots, m\}$.

1-Then, we first check the admissibility of the dotted edges starting from this root element. We first consider the dotted edge which is the farthest from the arrow and declare it admissible if its other vertex is a ring of an outgoing (resp. ingoing) arrow and that there is no other dotted edge
attached to this ring which is farther (amongst the unfrozen dotted edges) from its arrow. Once this condition is verified, we freeze this dotted edge and the root element remains the root element. We check all dotted edges of the root element inductively. Once a dotted edge has been checked to be admissible, it is frozen and we go on checking the others. Once all the dotted edges of the root element have been checked, they are frozen and may separate the graph into subgraphs. Thus, the map may have been cut into disjoint subgraphs whose boundary (which may contain dotted edges) is homeomorphic to a disc (In the case where it has edges glued with an internal star, we see these other stars as part of the external star by following all the graph connected to the external boundary). In each of these subgraphs, we declare the first (following the orientation of the plane) element (corresponding to a $U_i$ or a $U_i^*$) after the last frozen dotted edge of its boundary as distinguished. We then define the root element of the boundaries of these subgraphs by the same procedure as for the root star. The boundary of each subgraph is then a star and these subgraphs have now the structure of a map; we will call them submaps.

For instance, in figure 2, once the two dotted edges have been checked and frozen, the map is cut into two disjoint submaps, the left one having a fixed frozen dotted edge between the root element and the inner star (thus forbidding other edges to cross it and allowing us to consider the inner star as part of the external star). The boundary of this left subgraph is now seen as a star of type $q = U^*A^5UBU^*AUB$. This left subgraph has the same distinguished element as before but a new root element (here the outgoing arrow on its boundary corresponding to the first $U$ in $q$).

2- When all dotted edges are frozen, we check that the arrow of the root element is paired with an arrow of the opposite direction (note that if the root element comes from a $U_i^*$, it can only be paired with an element of another star since by definition there is no more outgoing arrows on the root star). The oriented edge is seen as a fat edge. In particular, if the oriented edge link the root star with another star, we see this other star as part of the root star for the next step, i.e we identify the root star of type $QU_iP$ glued to the star of type $RU_i^*S$ (by the marked $U_i$’s) with the root star of type $PQRS$ with, by convention, the distinguished element chosen to be the closest element after the glued $U_i^*$. If the oriented edge link two rings of the root star, two disjoint subgraphs are formed and we proceed as in -1-.

c- We continue the inductive procedure on the submaps until all edges have been checked to be admissible.

Now we can define weighted sum of admissible maps.

**Definition 5.4.** Assume we are given the tracial state $\tau$ of $[2]$.

We define the weighted sum of admissible maps constructed above the stars $r_1, \cdots, r_n$ and the root star $P$:

$$M_{r_1, \cdots, r_n}(P) = \sum M_m(\tau)$$
where the sum runs over all admissible maps $m$ constructed above $r_1, \ldots, r_n$ with root star $P$. Assuming that $V_t = t_1 q_1 + \ldots + t_n q_n$ where $q_i$ are monomials, we define the formal series:

$$M_t(P) = \sum_{k \in \mathbb{N}^n} \frac{t^k}{k!} M_k(P)$$

with $M_{k_1, \ldots, k_n}(P) = M_{q_1, \ldots, q_1, \ldots, q_n, \ldots, q_n}(P)$ where the monomial $q_j$ appears in $k_j$ successive position and $t^k = \prod t_i^{k_i}$, $k! = \prod k_i!$.

Remark that we do not count all the maps which contain the stars $r_1, \ldots, r_n$ but only those that are constructed using our inductive rules; they for instance forbid to glue the two same rings more than twice.

However, a given map is counted at most once since there is only one way to decompose it using the procedure $IP$. Indeed, it is easy to check that at each step we have only one possibility for the next step since the dotted edges have to be drawn one after the other following the orientation and no new dotted edge can be drawn after the arrow of the root has been glued.

**Example**

Let us show some examples. We start from one root star and a star on the sphere (see figure 1). We want to construct maps above these stars with our rules, starting with the root element shown by the arrow outside the root star. Figures 2, 3 and 5 are examples of such maps. Note that the weights of the maps of figures 2 and 3 are the same, the only difference is the way the three rings are glued. There is a third way to glue those three rings shown in figure 4 which is a map but can not be obtained by our rule of construction (and thus is not admissible).

![Figure 2](image)

**Figure 2.** A possible map. Its weight is $\tau^{05}(A^6 \otimes B \otimes B^2 \otimes A^3 \otimes B)$

We now come to the main theorem of this section, namely the graphical expansion result for $M_4$: 
Theorem 5.1. Let $V = \sum_{1 \leq i \leq n} t_i q_i$ be a polynomial. Let $\mu_t$ be a solution of $\text{SD}[V_t, \tau]$ and $M_t$ be the formal series defined for monomials $P$ by

$$M_k(P) = \sum_{k \in \mathbb{N}^n} \prod_{i=1}^n \frac{t_i^{k_i}}{k_i!} M_{k}(P)$$
where \( M_k(P) \) is the weighted sum of planar maps with one root star of type \( P \) and \( k \) stars of type \( q_i \). If we extend the definition of \( M_t \) by linearity to any polynomial \( P \) then the series \( M_t(P) \) is absolutely convergent in a neighborhood of the origin and,

\[
M_t(P) = \mu_t(P).
\]

**Proof.**

For the sake of clarity we first prove the case \( V = 0 \) and show that \( M(P) := M_0(P) = \mu_t(P) \) for a monomial \( P \).

We proceed by induction on the total degree in \( U_i, 1 \leq i \leq m \), in \( q \).

Suppose that there is no variable \( U_i \) in \( P \). Then either there is no variable \( U_i^* \) and both sides of the equality are equal to \( \tau(P) \), or there is a \( U_i^* \) and both sides vanish: the left hand side by freeness between \( U_i \) and the \( A_i \)'s and the fact that all non-trivial moments of \( U_i \) is 0 and the right hand side because one can not glue the arrow coming out from this \( U_i^* \) anywhere.

We assume our identification proved when the degree of \( P \) in the \( U_i \)'s is less than \( k \). We next take \( q \) with degree in the \( U_i \)'s equal to \( k + 1 \). Thus we can assume that there is a \( U_i \) in \( P \), and we consider the last one in \( P \) so that \( P = pU_i b \) with \( b \) a polynomial in the \( U_j^* \) and the \( A_j \)'s, \( 1 \leq j \leq m \). By definition, \( M(pU_i b) = M(bpU_i) \) since it depends only on the position of the last \( U_i \). Thus, we may assume that \( P \) is of the form \( QU_i \) with \( Q \) of degree less than \( k \). We apply Schwinger-Dyson equation to this quantity:

\[
\mu(QU_i) = - \sum_{Q=RU_iS} \mu(RU_i) \otimes \mu(SU_i) + \sum_{Q=RU_i^*S} \mu(R) \otimes \mu(S) \tag{15}
\]

Now, we can apply our induction hypothesis since all polynomials appearing in the right hand side have degree strictly smaller than \( k + 1 \).

We need to show that this is exactly the induction relation for maps. To construct a map above a star of type \( QU_i \), we first look at the root element \( U_i \) and we have to decide what to do first with the dotted edges. There are two possibilities:

1. The first possibility is that there is no dotted edge going outside of the ring of the root. In such a case, we can glue the arrow to any other arrow of opposite direction and of the same color (corresponding to a variable \( U_i^* \)). This implies that \( Q \) decomposes into \( RU_i^*S \) and we construct an oriented edge between \( U_i \) and \( U_i^* \). Thus we separate the map into two parts and we have to construct a map above the \( R \) part and another one above the \( S \) part (this is the case 2 of \( \text{IP} \)). This gives

\[
M(R)M(S)
\]

possibilities which is exactly the possibilities counted by the second term in the right hand side of (15).

2. The second possibility is that we glue the root ring to another ring with a dotted edge. Thus \( Q \) must decompose into \( RU_i S \) and the
creation of the dotted edge amounts to decompose the map into $RU_i$ and $SU_i$ and again to continue the construction of the map we will have to construct a map above the $RU_i$ part and another one above the $SU_i$ part (note here that when a dotted edge is attached to a circle of an $U_i$, the arrow and the circle keep their structure and live on the right of the dotted edge). In this procedure, we have fixed one dotted edge and thus multiplied the contribution of the resulting map by $-1$ (this is the case 1 of IP). The resulting contribution to $M$ is therefore $-M(RU_i)M(SU_i)$. Thus, the first term in (15) computes the operation of gluing rings by dotted edges.

Putting these two possibilities together we see that the state $\mu$ and the enumeration of maps $M$ satisfy the same induction so that they are equal; $M(pU_i b) = \mu(pU_i b)$ for any $b$ monomial which does not contain any of the $(U_i, 1 \leq i \leq m)$. Note here that no dotted edges between rings of incoming arrows can be drawn since if there are no outgoing arrows in a map, but some $U_i^*$, there is no contribution. By traciality of $\mu$, we deduce as well that $M_0$ is tracial. Indeed, if we decompose $p, q$ into $p = p_1U_{i_1}p_2U_{i_2}\cdots p_{n-1}U_{i_{n-1}}p_n$ and $q = q_1U_{j_1}q_2U_{j_2}\cdots q_{r-1}U_{j_{r-1}}q_r$ with monomials $p_i, q_i$ which does not contain any of the $(U_i, 1 \leq i \leq m)$,

$$
M(pq) = M((pq_1U_{i_1}q_2U_{j_2}\cdots q_{r-1}U_{j_{r-1}}q_r)U_{i_{r-1}}q_r)
= \mu(pq_1U_{i_1}q_2U_{j_2}\cdots q_{r-1}U_{j_{r-1}}q_r) = \mu(pq)
= \mu(qp) = \mu((qp_1U_{i_1}p_2U_{i_2}\cdots p_{n-1})U_{i_{n-1}}p_n) = M(qp).
$$

Now we turn to the general $V$ case.

We first check the induction relation when the root star $P$ contains a $U_i$ for some $i \in \{1, \ldots, m\}$ so that we can write $P = QU_i$. Let us denote for $n$-tuples $k = (k_1, \ldots, k_n)$ and $l = (l_1, \ldots, l_n)$, $\binom{k}{l} = \prod_{i} \binom{k_i}{l_i}$. We check the formal equality by considering the induction relation, now given by:

$$
\mu^{k+1_j}(QU_i) = - \sum_{\ell \leq k+1_j} \sum_{Q=RU_iS} \binom{k+1_j}{\ell} \mu^\ell(RU_i) \otimes \mu^{k+1_j-\ell}(SU_i)
+ \sum_{\ell \leq k+1_j} \sum_{Q=RU_i^*S} \binom{k+1_j}{\ell} \mu^\ell(R) \otimes \mu^{k+1_j-\ell}(S)
- \sum_{q_j=RU_i^*S} k_j \mu^{k}(QU_iSRU_i) - \sum_{q_j=RU_i^*S} k_j \mu^{k}(QRSR)
$$

We need to show that the enumeration of maps satisfies the same relation.

We start by putting stars of type $(q_j, 1 \leq j \leq n)$ inside a root star of type $QU_i$ and we wonder what happens to the root element $U_i$. We apply one step of IP. Two things can happen. Either we link $U_i$ to another part of $Q$ and in that case we have already shown that the possibilities are enumerated by the first two terms of the induction relation. Here, note that the product of $\binom{k_i}{l_i}$
corresponds to the possible distribution of stars in each part (or submap) of the map, since all the stars are labeled.

Thus we need to show that the two other terms take into account the case where $U_i$ is linked to another star of type $q_j$. According to our construction rules we have two possibilities:

1. Starting from $U_i$ we glue the arrow to an arrow of the same color entering a star of type $q$. This rule forbids any other gluing from $U_i$, this is counted by

$$\sum_{q_j=RU^*_iS} k_j \mu(QSR).$$

The coefficient $k_j$ counts the number of choices for the star of type $q_j$ since they are all labelled.

2. The other possibility is to glue the ring to a ring of the same color. This leads to

$$- \sum_{q_j=RU_iS} k_j \mu(QU_iSRU_i)$$

possibilities.

In the case where $P$ does not contain any $U_i$, $1 \leq i \leq m$ but still some $U^*_i$, the root of the root star can only be glued by a dotted edge to any other $U^*_i$, or by a directed edge to a $U_i$ of a star. The resulting induction relation is exactly given by the formula obtained by conjugation of (16), hence again $M_k(P) = \mu^k(P)$. This completes the proof.

This theorem gives a combinatorial interpretation in term of maps to the unitary integrals. The fact that we do not take the sum on all maps but only on admissible ones makes this interpretation less transparent than the one for the gaussian case found in [7]. However, now that we know that the series can be identified to the matrix integral, we obtain some combinatorial identities which show that $IP$ is less rigid than it looks like.

**Corollary 5.1.** Let $V = \sum t_i q_i$.

1. For all $P,Q$,

$$M_t(PQ) = M_t(QP).$$

2. For all monomials $r_1, \ldots, r_n, r_{n+1}$, and all permutation $\sigma$ of $n+1$ elements,

$$M_{r_1, \ldots, r_n}(r_{n+1}) = M_{r_{\sigma(1)}, \ldots, r_{\sigma(n)}}(r_{\sigma(n+1)}).$$

3. Assume that we define another procedure to define the root element of the root star (for example we pick the root element to be the second ring available if possible, or we pick a ring at random, or any other choice which may change during $IP$ for the root stars that are created during the procedure when new faces are added). This will change the notion of admissible maps and we can define a new weighted sum
\( M'_{r_1, \ldots, r_n}(P) \) and a new series \( M'_t(P) \) where the sum occurs on these new maps. For all \( r_1, \ldots, r_n, P, \)

\[
M_{r_1, \ldots, r_n}(P) = M'_{r_1, \ldots, r_n}(P)
\]

\[
M_t(P) = M'_t(P).
\]

Note that due to the definition of admissible maps via the procedure \( \text{IP} \), those properties are far from being obvious from a purely combinatorial point of view. Still they will appear as an easy consequence of the identification with the matrix model.

Obviously different roots lead to a different procedure \( \text{IP} \), and thus potentially to different maps. It is actually possible to see through examples that this phenomenon actually happens.

However, it follows from the second point of the corollary that the choice of the root does not affect the weighted sum. The first and third points show that the choice of the root element and of the root star does not affect the final series. We were not able to give a more direct combinatorial proof of that result.

To be more specific on the impact of the choice of the roots on the maps, let us call clusters the equivalence class of rings for the equivalence relation generated by \( a \sim b \) if the ring \( a \) is glued to the ring \( b \) by a dotted edge. Changing the choices of the roots will lead to different admissible maps since it will allow different positions for the dotted edges. For example, they were three choices for the starting root in figure 1. For each of these choices, two of the three maps represented in figures 2, 3 and 4 would have been reachable by the inductive construction \( \text{IP} \) but not the third one. The one who is not constructible depends on the choice of the first root. It seems that if the maps are different, nevertheless the clusters are the same and in that simple case, knowing this cluster is sufficient to define the faces created by the dotted edges and thus the weight of the maps.

**Proof.**

Changing the root element of a star is the same thing than making a circular permutation of the variable of the associated monomial. The theorem shows that weighted sums are equal to the limit of the empirical measure of the matrix model which are tracial. The first and third items are a direct consequence of this identification.

For the second item, observe that permuting the first \( n \) monomials doesn’t change the sum by its definition. Thus we only need to show that

\[
M'_{r_1, \ldots, r_n}(P) = M_{P, r_2, \ldots, r_n}(r_1).
\]

Let us define \( V = \sum_i u_i r_i + tP \). We will again use the identification with the matrix model but now we will use the formal version. The coefficient \( M_{r_1, \ldots, r_n}(P) \) appears as the coefficient of the limit tracial power state \( \mu' \) by
Corollary 3.1 and Theorem 5.1. More precisely,

\[ M_{r_1, \ldots, r_n}(P) = \lim_{N} \frac{\partial^n}{\partial u_1 \cdots \partial u_n} \mu^f(P) \bigg|_{u_i=0} . \]

We now use the fact that \( \mu^f \) is the limit coefficientwise of the formal model defined in (8). Thus,

\[ M_{r_1, \ldots, r_n}(P) = \lim_{N} \frac{\partial^n}{\partial u_1 \cdots \partial u_n} \exp N^2 \mu^N(V) \bigg|_{u_i=0} . \]

We conclude by noticing that this last expression is symmetric in the monomials \( r_1, \ldots, r_n, P \).

\[ \blacksquare \]

6. Application to free probability

In this section we look at applications of the combinatorial results of section 5 to free probability.

Let us assume that the \( U_i \)'s are chosen independently according to the Haar measure. If we define \( X_i = U_i^* A_i U_i \) then the \( X_i \)'s are asymptotically free (according to a theorem of Voiculescu [31]) and with fixed distribution \( \mu \) uniquely defined by the distribution of the \( A_i \)'s. We are interested in using our setup to compute limits of moments of these variables or in other word to compute the moments of free variables:

\[ \mu(X_{i_1} \ldots X_{i_k}) . \]

According to our interpretation this can be computed by looking at the maps above the star of type \( X_{i_1} \ldots X_{i_k} \) without any other stars, in other words we have to focus on computations of \( \mathbb{M}(q) = \mathbb{M}_0(q) \) which turns out to be equal to \( \mu(q) \) where \( \mu \) is the free state product (see Corollary 3.2).

We are interested in using this method to compute some non-commutative moments of free variables, in relation with Speicher’s non-crossing cumulants theory, cf [27].

6.1. One star maps. For these purposes we need to find a simplified interpretation of \( \mathbb{M}(q) \) in the single star map.

For this case with only one star, the combinatorial interpretation can be slightly modified. First, we do not need to consider dotted edges between incoming arrows since if there is a \( U_i^* \) there must be a \( U_i \) which can be chosen as the root element or we can not build any map. But the main difference is that now each time we glue two rings, the edge newly created separate these two rings into two different faces so that they can no longer be glued together. Thus, we can forget about the restriction of the construction rules and present a simpler description in that case. Instead of gluing the ring two
by two we will now glue them together. We define a new structure which we
will call a node and now rings can only be glued to node and a node can be
 glued to any number of rings. A one star map is a map with one star where
the arrows has been glued two by two while respecting the orientation and
rings may be glued to exactly one node, each node is glued to an arbitrary
number of rings but at least one. Figure 6 shows the new representation
of a one star map. The trick to go from the previous interpretation to this
one is to glue together to a node all the rings that are in the same class
of the equivalence relation generated by being glued. In order to compute
the weight of such a map, observe that several maps give the same one-star
map, but the weight is easy to compute since as we will see we only need to
add a factor $C_{d-1}$ for each node of degree $d$.

![Figure 6. Reduction of a map on one star to a one-star map.](image)

**Definition 6.1.** A one-star map is a connected graph embedded on a sphere
above one star and with some edges such that

1. Edges are drawn only between rings and must not intersect.
2. Arrows must be glued two by two while respecting the orientation and
   the color: an arrow going out of a star (associated with a variable
   $U_i$) is always glued to exactly one other arrow going into a star (as-
   sociated to a variable $U_i^*$) of the same color. This pair of arrows
   creates an oriented edge.
3. Any number of rings may be glued together on a node.

The weight of a one-star map is the product of the weight of its faces
which is defined as before as trace of products of $A_i$'s times the product of
the weight of the nodes. The weight of a node of degree $d$ is $(-1)^{d-1}C_{d-1}$.

We define $\widetilde{M}_0(q)$ the weighted sum of one-star map above a star of type $q$.

**Proposition 6.1.** For all monomial $q$,

$$\mu(q) = \widetilde{M}_0(q).$$
Proof.

We only need to show that $\mathcal{M}(q) = \mathcal{M}_0(q)$. For this we need to compute the number of maps above one star that are reduced to a given one-star map. The reduction goes as follows: two rings are glued to the same node if they are linked by a sequence of dotted-edges. We have to count how many configurations of dotted edges lead to a node of degree $d$. When one of the ring glued to this node becomes the root in the recursive construction, it has to be glued to one of the other ring glued to the node. Thus it separates the set of ring into two subsets, so according to our inductive procedure of section 5, we have to continue to glue this ring to other ones while we continue the construction in the face newly created. This yields a structure of tree on this set of rings. We have as many choices as they are trees with $d-1$ edges (to glue the $d$ ring we need exactly $d-1$ edges). This explains the factor $C_{d-1}$. The factor $(-1)^{d-1}$ simply comes from the factor $-1$ which comes with each edge.

\[\blacksquare\]

6.2. Maps and cumulants. Let $A_1, \ldots, A_n$ be self-adjoint variables and $U$ a unitary matrix, free from the $A_i$'s. Then choosing $k$ indices $i_1, \ldots, i_k$ in \{1, $n$\} one has

$$\mu(A_{i_1} \ldots A_{i_k}) = \mu(U^*UA_{i_1} \ldots U^*UA_{i_k})$$

Let us apply Schwinger-Dyson equation with respect to $U$ to the above equality, and let us rearrange the sum according to the non-crossing partition of $A_i$'s generated by the oriented edges. Obviously one obtains a formula of type

$$\mu(A_{i_1} \ldots A_{i_k}) = \sum_{\pi \in NC(k)} \tilde{K}_\pi(A_{i_1}, \ldots, A_{i_k})$$

where $NC(k)$ is the non-crossing partitions and $\tilde{K}_\pi$ is a $k$-linear form multiplicative along the blocks of $\pi$ in the sense of Speicher: if $\pi = \{V_1, \ldots, V_n\}$ with the block $V_i = \{a^i_1, \ldots, a^i_{r_i}\}$

$$\tilde{K}_\pi(X_{1} \ldots X_k) = \prod_i \tilde{K}_{(r_i)}(X_{a^i_1}, \ldots, X_{a^i_{r_i}})$$

where $(r_i)$ represents the partition on $r_i$ elements with only one block.

The fact that such a formula holds true for any choice of non-commutative laws for $A_i$’s proves via the moment-cumulant formula that $\tilde{K}_\pi$ has to be Speicher’s non-crossing cumulants $K_\pi$. But it is also given as a sum on maps by our graphical model.

Let us recap this in the following proposition:

**Proposition 6.2.** The $n$-th non-crossing cumulant of the variables $A_1, \ldots, A_p$ is the weight of all one-star maps over the star build by putting in the
clockwise order a ring, a diamond of color $i_1$, a ring, a diamond of color $i_2$, . . . , a ring, a diamond of color $i_p$.

Note that we have defined this map above a star which is not of type $q$ for any monomial $q$. This would be a problem for admissible maps since IP requires the presence of oriented edges. But the definition of one-star map is fine in this context.

Actually, Proposition 6.1 gives us a new proof of the following Corollary, due to Speicher and known as non-crossing Moebius formula

**Corollary 6.1.** The following inversion formula holds true:

$$K_n(A_1, \ldots, A_n) = \sum_{\pi \in NC(n)} \mu(\pi)(A_1, \ldots, A_n)(-1)^{n-|\text{blocks}(\pi)|} \prod_{B \text{ block of } \pi} C_{|B|-1},$$

where $\pi^c$ is the Kreweras complement (see [26]) and $C_q$ the Catalan number.

**Proof.**

This is a direct consequence of the previous proposition. Remember that $K_n(A_1, \ldots, A_n)$ is a weighted sum over maps with dotted edges since the star contains some rings and no arrows. These dotted edges form a non-crossing partition of $[1, \ldots, n]$ by saying that two rings are in the same component if they are linked to the same node. The weight associated to this map is a product whose factors are: $(-1)^{d-1}C_{d-1}$ for each node of degree $d$ and the weight of each face. The faces are by definition the component of the Kreweras complement of $\pi'$. Thus we obtain:

$$K_n(A_1, \ldots, A_n) = \sum_{\pi' \in NC(n)} \mu(\pi')(A_1, \ldots, A_n) \prod_{B \text{ block of } \pi'} (-1)^{|B|-1}C_{|B|-1}.$$  

The formula follows after taking $\pi' = \pi^c$.

As a further remark, one can also read graphically the main properties of cumulants, for example, $K_n(X_1, \ldots, X_n) = 0$ as soon as there are occurrence of free elements. More precisely, assume that we can partition the $X_i$’s into two families the $A_j$’s and the $B_k$’s with the algebra generated by the $A_j$’s free from the algebra generated by the $B_k$’s. Then if all the $X_i$’s do not take value in the same algebra, $K_n(X_1, \ldots, X_n) = 0$. Indeed, one can replace all the family of $A_j$’s by the one of $V^*A_jV$ with $V$ unitary and free from the other variables. Now when looking at the combinatorial interpretation of $\mu(X_1, \ldots, X_n)$ we can see that the oriented edges coming from $V$ separate the components containing the $A_j$’s from the others. By following those edges we see that the faces they are defining contain only variable from one of the two algebras (The edges are going in the clockwise order around the faces which contain the $B_k$’s and in the counter-clockwise order around the faces containing the $A_j$’s). Thus, in the decomposition (17), the terms corresponding to partitions with one component containing both some $A_j$’s and some $B_j$’s vanish. By uniqueness of the decomposition into cumulants we deduce that those elements vanish i.e. $K_n(X_1, \ldots, X_n) = 0$. 

These remarks are not new but this shows that our graphical model fully encompasses the theory of non-crossing cumulants and that the Schwinger-Dyson equation can also be read in terms of cumulants.

It is interesting to mention here that papers [24] and [25] have developed a calculus on annuli which seems to be related to our graphical model. However these approaches only deal with the asymptotics of second order cumulants whereas our approach via formal calculus, see section 4, allows us to deal with arbitrary order cumulants.

The actual relation can be found in [10], where convolution on partitioned permutations is introduced and showed to be the relevant algebraic tool to handle higher order freeness, namely, the asymptotic behaviour of cumulants of unitarily invariant random matrices.

But the results in our paper give an explicit algorithmic description of the Moebius inversion formula and therefore of higher order cumulants. As in the one star case, cumulants are also obtained by inserting an outer $U^*U$ between each variable of each star and by looking at generating function where $U$ is linked to its neighboring $U^*$.

It is interesting to see that a direct (yet difficult to describe) graphical reading of the Schwinger-Dyson equation, which is our main tool of investigation of unitarily invariant matrix models, yields non-crossing and could yield higher order moments related series and operations similar to convolution, although these latter results rely on more representation theoretic grounds (Weingarten function theory as developed in [11]).

It is not obvious to us how the Schwinger-Dyson equation can be read off from the results of [10] (without writing a change of variable invariance formula), and it would be interesting to attempt to figure out the meaning of Schwinger-Dyson equation at the representation theoretic level.

7. Application to the asymptotics of $I_N(V, A_i^N)$

In this section, we investigate the free energy by using the combinatorial interpretation of the previous section.

Let $(q_1, \ldots, q_n)$ be fixed monomials in $\mathbb{C}[(U_i, U_i^*, A_i)_{1 \leq i \leq m}]$, let $V = \sum t_i q_i$ be a self-adjoint polynomial and $I_N(V, A_i)$ be given by (1).

**Theorem 7.1.** There exists $\varepsilon = \varepsilon(q_1, \ldots, q_n)$ so that for any $t \in \mathbb{C}^n \cap B(0, \varepsilon)$ such that $V = V^*$ for any $\alpha \in [-1, 1]$,

$$F_{V, \tau}(\alpha) := \lim_{N \to \infty} \frac{1}{N^2} \log I_N(\alpha V, A_i^N) = \sum_{k \in \mathbb{N}_0 \setminus \{0, \ldots, 0\}} \prod_{i=1}^n \frac{(\alpha t_i)^{k_i}}{k_i!} M_k(q_1, \ldots, q_n, \tau).$$

Moreover,

$$M_k(q_1, \ldots, q_n, \tau) = \sum \text{admissible maps with } k_i \text{ stars } q_i.$$
is the weighted sum of maps constructed above \(k_i\) stars of type \(q_i\) for all \(i\), after choosing one of them as a root star (this is well defined according to Corollary [5.7]).

**Proof.**

Let

\[
F^N_t = \frac{1}{N^2} \log I_N(V_t, A_i^N).
\]

Then, if \(\alpha \in \mathbb{R}\),

\[
\partial_\alpha F^N_\alpha t = \int \hat{\mu}^N(V_t) d\mu^N_\alpha V^\alpha_t.
\]

Assume that \(t\) is small enough so that Corollary [3.1] holds and remark that \(V^\alpha_t\) is self-adjoint and such that \(|\alpha t_i| \leq \varepsilon\) for all \(i\) and all \(0 \leq \alpha \leq 1\). Thus, for \(\alpha \in [0, 1]\),

\[
\lim_{N \to \infty} \partial_\alpha F^N_\alpha = \mu_\alpha(V_t)
\]

with \(\mu_\alpha\) the solution to \(\text{SD}[\alpha V_t, \tau]\). By dominated convergence theorem (since \(\partial_\alpha F^N_\alpha\) is uniformly bounded in \(N\) and \(\alpha \in [0, 1]\)), we deduce that

\[
\lim_{N \to \infty} F^N_\alpha = \int_0^1 \mu_\alpha(V_t) d\alpha
\]

where we used that \(F^N_0 = 0\).

Here also, we obtain the following important corollary, as a consequence of Corollary [4.1].

**Corollary 7.1.** The following holds true:

\[
\lim_{N \to \infty} \frac{\partial^k}{\partial z^k} N^{-2} \log \int_{U_1^N} e^{zNTr(V(U_i, U_i^* A_i^{N,1 \leq i \leq m}))} dU_1 \cdots dU_m|_{z=0} = \frac{\partial^k}{\partial z^k} F_{V, \tau}(z)|_{z=0}
\]

In particular, this result allows us to give an expansion of the Harish-Chandra-Itzykson-Zuber integral as a generating function of the number of some maps. Let us recall the exact expression of this integral:

\[
F^{A, B}_N(z) := \frac{1}{N^2} \log HCIZ(z A, B) = \frac{1}{N^2} \log \int_{U_1^N} e^{zNTr(U^* AUB)} dU.
\]

The maps appearing in the expansion contain only stars of type \(U^* AUB\) (see the star in the middle of figure [1]). Besides we can build these maps without considering the rings attached to variable \(U^*\) since we will always be able to choose the root element to be a \(U\) (a \(U^*\) always comes with a \(U\) for this potential).

Since the number of diagrams is growing quickly we compute only the first term of the expansion. Note that when gluing the arrow of the root of the root star, we must always glue it to another incoming arrow of another
star and hence we shall never see the case of a root star with no $U_i$’s. Again, we therefore do not see dotted edges between incoming arrows.

Besides, we consider only the case where the distribution is centered, that is when $\tau(A) = \tau(B) = 0$. The other cases can be deduced easily from this one since we have the relation

$$F_{N}^{a+A,b+B}(z) = F_{N}^{A,B}(z) + \frac{z}{N}(b\text{Tr}A + a\text{Tr}B) + zab.$$  

In terms of diagrams, this means that we only need to consider diagrams such that no face contains only one diamond.

According to the previous theorem, $\lim_{N \to \infty} F_{N}^{A,B}(z)$ has, for small $z$, an expansion $\sum_n F_n z^n$. We now use this graphical representation to compute the first terms of this integral.

Since the distributions are centered, the first term $F_1$ is zero.

The second term $F_2$ consists of maps constructed with two stars of type $U^*AUB$. There is only one way to add edges between these two stars to construct a connected map without faces which contains only one diamond, this is represented by figure 7. We obtain a map with two faces. One has two diamonds associated to $A$ and the other one two diamonds associated to $B$. Thus the weight of this map is $\tau(A^2)\tau(B^2)$. Since there is no gluing between the rings they are no other signs. They are only one way to distribute the labels on this picture (that is the second distribution leads to the same map) thus to obtain $F_2$ we only need to divide by $2!$,

$$F_2 = \frac{1}{2} \tau(A^2)\tau(B^2).$$

We can continue this for the next terms in the expansion, the third term (see figure 8) is in the same spirit and leads to

$$F^3 = \frac{1}{3} \tau(A^3)\tau(B^3).$$

The fourth term is the first one where gluings between the rings appear. Thus weights with negative coefficients can occur. The sign of a map is easy
to compute, it is $-1$ to the power the number of dotted lines in the map. Equivalently since in the case of HCIZ integral the number of oriented edges is equal to the number of stars, this number is also equal to the number of faces of the map and thus to the number of factor in the product of moments of the weight. In figure 8 we have drawn all unlabelled planar maps one can construct with 4 stars. To compute the exact coefficient of each map one has to multiply it by the number of way to distribute the labels and divide by $4!$.

This leads to,

$$F_4 = \frac{1}{4} \tau(A^4)\tau(B^4) - \frac{1}{2} \tau(A^2)^2 \tau(B^4) - \frac{1}{2} \tau(A^4)\tau(B^2)^2$$

$$+ \frac{1}{2} \tau(A^2)^2 \tau(B^2)^2 + \frac{1}{4} \tau(A^2)^2 \tau(B^2)^2.$$ 

Here the weight are given in the same order than the maps in the figure. Note a new and interesting feature that appears in the third map: two rings are linked by more than one dotted edge.

The other terms can be computed in the same way, for example figure 10 represents the fifth term and gives

$$F_5 = \frac{1}{5} \tau(A^5)\tau(B^5) - \tau(A^2)\tau(A^3)\tau(B^5) - \tau(A^5)\tau(B^2)\tau(B^3)$$

$$+ 4 \tau(A^2)\tau(A^3)\tau(B^2)\tau(B^3).$$

Thus the first terms agree with the expansion given in [36] on page 23, besides this allows us to answer a question raised in this paper. Indeed, the authors ask if there is an explanation to the fact that the coefficient of $F_n$ all seem to be integer multiple of $\frac{1}{n}$. This is easy to prove with this graphical interpretation. To compute the contribution of a given unlabelled map we must distribute the labels $\{1, \ldots, n\}$ on its stars, count the number of different map that we obtain and divide by $n!$. But after choosing the star which received the label 1 we have $(n-1)!$ ways to distribute the remaining labels and they all lead to different maps (note that on the other hand, due to possible symmetry in the unlabelled map, different choices for the star
with the label 1 may lead to the same maps). Thus the coefficient in front of this map is a multiple of \( \frac{(n-1)!}{n!} = 1/n \). More precisely it is \( 1/n \) times the number of choices of the star which carry the label 1 that will lead to different maps, in particular it is always less than 1.

To finish, we wish to point out that we can recover results in [8] and [15] about scalings of HCIZ integral. In these two papers, one considers the scaling where \( A \) has small rank, which amounts to considering only terms \( \tau(A^k) \times P(B) \). Here the transformation depicted in section 6 applies and one sees that \( P(B) \) has to be \( k^{-1}K_k(B) \). In particular this means in the case that \( A \) is a rank 1 projection, that \( N^{-1} \log HCIZ \) tends to the primitive of Voiculescu’s \( R \)-transform.

8. Application to Voiculescu free entropy

Voiculescu’s microstates free entropy is given as the asymptotic the volume of matrices whose empirical distribution approximates sufficiently well a given tracial state. Up to a Gaussian factor, it is given by

\[
\chi(\mu) = \lim_{N \to \infty} \sup_{k \to \infty} \lim_{\epsilon \to 0} \sup_{\Gamma_R(\mu, \epsilon, k)} \frac{1}{N^2} \log \mu_N^{\otimes m}(\Gamma_R(\mu, \epsilon, k))
\]
with $\mu_N$ the Gaussian measure on $\mathcal{H}_N$ and $\Gamma_R(\mu, \varepsilon, k)$ the microstates

$$\Gamma_R(\mu, \varepsilon, k) = \{X_1, \ldots, X_m \in \mathcal{H}_N: \left| \frac{1}{N} \text{Tr}(X_{i_1} \cdots X_{i_p}) - \mu(X_{i_1} \cdots X_{i_p}) \right| < \varepsilon \} \quad p \leq k, i_\ell \in \{1, \ldots, m\}, \|X_i\|_\infty \leq R \}.$$ 

When $m = 1$, it is well known [30] that $\mu \in \mathcal{P}(\mathbb{R})$ and

$$\chi(\mu) = I(\mu) = \int \int \log |x - y| d\mu(x) d\mu(y) - \frac{1}{2} \int x^2 d\mu(x) + \text{const.}$$ 

Moreover, one can replace the lim sup by a lim inf in the definition of $\chi$. Such answers (convergence and formula for $\chi$) are still open in general when $m \geq 2$ (see [5] for bounds). However, if $\mu$ is the law of $m$ free variables with respective laws $\mu_i$, then these questions are settled and

$$\chi(\mu) = \sum_{i=1}^{m} I(\mu_i).$$

We here want to emphasize that our result provides a small step towards dependent variables by showing convergence and giving a formula for the

**Figure 10.** Fifth term in the expansion of the IZ integral.
type of laws $\mu$ solutions of Schwinger-Dyson’s equations $SD[V, \tau]$. Indeed, we shall prove that

**Theorem 8.1.** Let $\mu$ be the law of $m$ self-adjoint variables $X_i$ with marginal distribution $(\mu_1, \ldots, \mu_m)$. Assume that $X_i$ can be decomposed as $X_i = U_iD_iU_i^*$ with $U_i$ unitary matrices in such a way that the joint law $\nu$ of $(D_i, U_i, U_i^*)_{1 \leq i \leq m}$ satisfy $SD[V, \tau]$ with $\tau$ the law of $m$ free variables with marginal distribution $\mu_1, \ldots, \mu_m$ and some potential $V = \sum_{i=1}^n t_i q_i$. Assume that the $t_i$’s are small enough so that Corollary 3.1 holds. Assume also that the hypotheses of Theorem 7.1 hold. Then,

$$
\chi(\mu) = \lim_{\varepsilon \downarrow 0} \lim_{N \to \infty} \frac{1}{N^2} \log \mu_N^{\otimes m} \left( \Gamma_R(\mu, \varepsilon, k) \right)
$$

and a formula of $\chi(\mu)$ can be given in terms of the $\mu^k$’s of Theorem 7.2.

**Proof.**

Indeed, let us consider $V = V(U_iA_iU_i^*, 1 \leq i \leq m)$ with $V$ a self-adjoint polynomial and $\mu$ the unique solution of $SD[V, \tau]$ with $\tau$ the law of the $A_i, 1 \leq i \leq m$ which is now chosen to be the law of $m$ free variables with marginals distribution $\mu_i, 1 \leq i \leq m$. Under the law $\mu_N^{\otimes m}$, we can diagonalize the matrices $X_i = U_iD_iU_i^*$ with $U_i$ following the Haar measure on $U_N$, and $d$ is the Dudley metric, we find that for $N$ sufficiently large

$$
\mathbb{L}_N := \mu_N^{\otimes m} \left( \Gamma_R(\mu, \varepsilon, k) \right)
$$

$$
= \mu_N^{\otimes m} \left( d(\hat{\mu}_{D_i}^N, \mu_i) < \varepsilon; \hat{\mu}_{U_iD_iU_i^*}^N, 1 \leq i \leq m \in \Gamma_R(\mu, \varepsilon, k) \right)
$$

$$
= \int_{d(\hat{\mu}_{D_i}^N, \mu_i) < \varepsilon} \left( \int_{\hat{\mu}_{U_iD_iU_i^*}^N, 1 \leq i \leq m \in \Gamma_R(\mu, \varepsilon, k)} dU_1 \cdots dU_m \right) \prod_{1 \leq i \leq m} d\sigma_N(\lambda_i)
$$

where we denoted $\Delta(\lambda_j) = \prod_{k \neq j} |\lambda_k - \lambda_j|$ and $d\sigma_N$ the probability measure

$$
d\sigma_N(\lambda) := Z_N^{-1} \prod_{k \neq j} |\lambda_k - \lambda_j|^2 e^{-\frac{1}{4} \sum_{\lambda}^2} \prod_{1 \leq j \leq N} d\lambda.
$$

In these notations, $D_i = \text{diag}(\lambda_i^1, \ldots, \lambda_i^N)$ and $\lambda = (\lambda_1, \ldots, \lambda_N)$. Hereafter, $\hat{\mu}_{\{E_i\}_{1 \leq i \leq n}}^N$ denotes the empirical distribution of $\{E_i\}_{1 \leq i \leq n}$: $\hat{\mu}_{\{E_i\}_{1 \leq i \leq n}}^N(P) = N^{-1} \text{Tr}(P(E_i, 1 \leq i \leq n))$. As a consequence, applying the large deviations result of [3] to the diagonal matrices $D_i$, we find that there exists $o(1)$ going to zero with $\varepsilon$ such that

$$
\mathbb{L}_N \leq e^{N^2 \sum_{i=1}^m f(\mu_i) + N^2 o(1)} \sup_{d(\hat{\mu}_{D_i}^N, \mu_i) < \varepsilon} \int_{\hat{\mu}_{U_iD_iU_i^*}^N, 1 \leq i \leq m \in \Gamma_R(\mu, \varepsilon, k)} dU_1 \cdots dU_m
$$

$$
:= e^{N^2 \sum_{i=1}^m f(\mu_i) + N^2 o(1)} \mathbb{L}_N
$$
with for $k$ greater than the degree of $V$,

$$
\mathbb{L}_N^1 = \sup_{d(\hat{\mu}_N D_i, \mu_i) < \varepsilon, \|D_i\|_{\infty} \leq R} e^{N \text{Tr}(V) - \frac{N}{2} \text{Tr}(V)} dU_1 \cdots dU_m
$$

$$
= e^{-N^2 \mu(V) + N^2 \varepsilon} \sup_{d(\hat{\mu}_N D_i, \mu_i) < \varepsilon, \|D_i\|_{\infty} \leq R} e^{N \text{Tr}(V)} dU_1 \cdots dU_m
$$

$$
\leq e^{-N^2 \mu(V) + N^2 \varepsilon} \sup_{d(\hat{\mu}_N D_i, \mu_i) < \varepsilon, \|D_i\|_{\infty} \leq R} e^{N \text{Tr}(V)} dU_1 \cdots dU_m
$$

$$
= e^{-N^2 \mu(V) + N^2 \varepsilon} \sup_{d(\hat{\mu}_N D_i, \mu_i) < \varepsilon, \|D_i\|_{\infty} \leq R} I_N(V, D_i)
$$

Now, for fixed $R$, any $D_i, D_i'$ in $d(\hat{\mu}_N D_i, \mu_i) < \varepsilon, \|D_i\|_{\infty} \leq R$

$$
\left| \frac{1}{N^2} \log I_N(V, D_i) - \frac{1}{N^2} \log I_N(V, D_i') \right| \leq \eta(\varepsilon, R),
$$

with $\eta(\varepsilon, R)$ going to zero as $\varepsilon$ goes to zero for any fixed $R$. Hence,

$$
\limsup_{N \to \infty} \frac{1}{N^2} \log I_N(V, D_i) \leq F(V, \mu_i) + \eta(\varepsilon, R)
$$

with $F(V, \mu_i)$ the limit of $N^{-2} \log I_N(V, A_i)$ given in Theorem 7.1 when the distribution of the $A_i$ converges to free variables with marginal distribution $\mu_i$. We thus have proved, letting $\varepsilon$ going to zero and then $R, k$ to infinity, that

$$
\chi(\mu) \leq \sum_{i=1}^{m} I(\mu_{A_i}) - \mu(V) + F(V, \mu_i).
$$

Conversely, we have

$$
\mathbb{L}_N \geq e^{N^2 \sum_{i=1}^{m} I(\mu_i) + N^2 o(\varepsilon)} \mathbb{L}_N^2
$$

with

$$
\mathbb{L}_N^2 := \inf_{d(\hat{\mu}_N D_i, \mu_i) < \varepsilon, \|D_i\|_{\infty} \leq R} e^{N \text{Tr}(V) - \frac{N}{2} \text{Tr}(V)} dU_1 \cdots dU_m
$$

$$
= e^{-N^2 \mu(V) + N^2 o(\varepsilon)} \inf_{d(\hat{\mu}_N D_i, \mu_i) < \varepsilon, \|D_i\|_{\infty} \leq R} e^{N \text{Tr}(V)} dU_1 \cdots dU_m
$$

$$
\geq e^{-N^2 \mu(V) + N^2 o(\varepsilon)} \inf_{d(\hat{\mu}_N D_i, \mu_i) < \varepsilon, \|D_i\|_{\infty} \leq R} e^{N \text{Tr}(V)} dU_1 \cdots dU_m
$$
for any $\delta < \varepsilon$. Now, choosing $\delta$ and using the continuity of $\hat{\mu}^N_{U_iD_iU_i^*}$ in the distribution of the uniformly bounded variables $D_i$, we find by Corollary 3.1 and our hypothesis that

$$\liminf_{N \to \infty} \frac{\int_{\Gamma_R(\mu, \varepsilon, k)} \mu^N_{U_iD_iU_i^*}dU_1 \cdots dU_m}{\int e^{N\text{Tr}(V)}dU_1 \cdots dU_m} = 1$$

which insures that

$$\chi(\mu) \geq \sum_{i=1}^m I(\mu_i) - \mu(V) + F(V, \mu_i).$$

Thus we have proved that

$$\chi(\mu) = \sum_{i=1}^m I(\mu_i) - \mu(V) + F(V, \mu_i).$$

Note that $\mu(V)$ and $F(V, \mu_i)$ can be written in terms of the $\mu^k$ of Theorem 4.2 by Theorem 7.1.

\[\Box\]

9. Generalization to integrals over the orthogonal group

In a recent article [35], Zuber shows that the large $N$ asymptotics of two matrix integrals (the integral with external magnetic field and the Harish-Chandra-Itzykson-Zuber integral) enjoy a universality property in the sense that they are the same (up to a proper rescaling) if one integrates over the unitary or the orthogonal group. This property was also obtained (but not explicitly stated) in the case of the Harish-Chandra-Itzykson-Zuber integral in [18] where the rate functions for the large deviation principle for the law of the spectral measure process of the Hermitian and the symmetric Brownian motion were shown to differ only by a factor two. The Harish-Chandra-Itzykson-Zuber integral is rather special in the family of angular integrals and one can compute many interesting related quantities, regardless of the group on which integration is taken (see [4, 13]).

In this section, we generalize this universality property by relating the large $N$ limit of any small parameter integrals over the orthogonal group with its complex analogue.

Let us define

$$I_N^1(V, \mathbf{A}_i^N) := \int_{\mathcal{O}_N^m} e^{N\text{Tr}(V(O_i, O_i^*, \mathbf{A}_i^N, 1 \leq i \leq m))}dO_1 \cdots dO_m$$

where $(\mathbf{A}_i^N, 1 \leq i \leq m)$ are $N \times N$ deterministic symmetric uniformly bounded matrices, $dO$ denotes the Haar measure on the orthogonal group $\mathcal{O}_N$ (normalized so that $\int_{\mathcal{O}_N} dO = 1$). In this section we will assume that $V$ is a non-commutative polynomial in the $O_i, O_i^*, \mathbf{A}_i^N$ with real coefficients. Here, $O^* = O^t$ is the standard involution $O^*_ij = O_{ji}$. Observe that if $P$ is
Moreover, we then claim that we have the following analogue of Theorem 9.1, which shows that the first order of integrals over the orthogonal group is the same as on the unitary group (up to proper renormalizations);

**Theorem 9.1.** There exists \( \varepsilon = \varepsilon(q_1, \ldots, q_n) \) so that for any \( \mathbf{t} \in \mathbb{R}^n \cap B(0, \varepsilon) \) such that \( V = V^* = \sum t_i q_i \), if we define

\[
F_{V, \tau}^1 := \lim_{N \to \infty} \frac{1}{N^2} \log I_N^1(V_t, A_i^N)
\]

then \( F_{V, \tau}^1 \) exists and

\[
F_{V, \tau}^1 = \frac{1}{2} \sum_{k \in \mathbb{N}^n \setminus \{0\}} \prod_{1 \leq i \leq n} \frac{t_{k_i}}{k_i} M_k(q_1, \ldots, q_n, \tau).
\]

Moreover,

\[
M_k(q_1, \ldots, q_n, \tau) = \sum_{\text{admissible maps with } k_i \text{ stars } q_i} M_m(\tau)
\]

is the weighted sum of maps constructed above \( k_i \) stars \( q_i \) for all \( i \), after choosing one of them as a root star.

The proof is based on the fact that if \( \mu_N^{1,1} \) denotes the law on \( O_N^m \) given by

\[
\mu_N^{1,1}(dO_1, \cdots, dO_m) := \frac{1}{I_N^1(V^N, A_i^N)} \prod_{1 \leq i \leq m} \text{Tr}(O_i, O_i^*, A_i^N) dO_1 \cdots dO_m
\]

and \( \hat{\mu}^N \) is the empirical distribution of \( (O_i, O_i^*, A_i, 1 \leq i \leq m) \), then we have the analogue of Corollary 8.1.

**Theorem 9.2.** Assume that \( V = \sum t_i q_i \) is self-adjoint. Let \( D \) an integer and \( \tau \) a tracial state in \( M_1(\mathcal{A}_t)_{1 \leq i \leq m} \) be given. There exists \( \varepsilon = \varepsilon(D, m) > 0 \) such that if \( |t_i| \leq \varepsilon \), \( \hat{\mu}^N \) converges almost surely under \( \mu_N^{1,1} \) to the unique solution \( \mu_t \) of the Schwinger-Dyson equation \( \text{SD}[V, \tau] \). Moreover, \( \hat{\mu}_N^{1,1} = \mu_N^{1,1}(\hat{\mu}^N) \) converges as well to this solution as \( N \) goes to infinity.

In fact, since then we know that \( \mu_t(P) \) expands as a generating function of the \( M_k(q_1, \cdots, q_n, \tau) \)'s, Theorem 9.1 follows readily since for any \( \alpha \in [0, 1] \),

\[
\partial_\alpha \frac{1}{N^2} \log I_N^1(\frac{\alpha}{2} V, A_i^N) = \frac{1}{2} \hat{\mu}_N^{1,1}(V)
\]

converges towards \( \frac{1}{2} \mu_t(V) \).

**Proof of Theorem 9.2** The proof follows the same lines as the proof of Theorem 2.1, we make the change of variables \( \mathbf{O} = (O_1, \cdots, O_m) \in O_N^m \rightarrow \Psi(O) = (\Psi_1(O), \ldots, \Psi_m(O)) \in O_N^m \) with

\[
\Psi_j(O) = O_j e^{\sum F_j(O)}
\]
where the $P_j$ are antisymmetric polynomials (i.e. $P_j^* = -P_j$). The only change is that now $P_j(O)$ are matrices with real coefficients and the differentials hold in the direction of $A_N^1$ which are the antisymmetric matrices with real coefficients. For $N$ large enough, $\Psi$ is a diffeomorphism; it is as in the complex case a local diffeomorphism which is injective. As such, its image is open and compact. $O_N^m$ is not connected but the union of copies of $SO^\varepsilon(N) = \{ O \in O_N; \det(O) = \pm \varepsilon \}$, $\varepsilon = +1$ or $-1$. Since $\det(\Psi_j(O)) = \det(e^{N P_j(O)}) = \det(O_j)$, $\Psi$ maps $SO^\varepsilon_1(N) \times \cdots \times SO^\varepsilon_m(N)$ into itself for each choice of $\varepsilon_j \in \{1, -1\}$. Therefore, by connectedness of this set, $\Psi(SO^\varepsilon_1(N) \times \cdots \times SO^\varepsilon_m(N))$ is open and closed and therefore equals $SO^\varepsilon_1(N) \times \cdots \times SO^\varepsilon_m(N)$. Thus, $\Psi$ is a diffeomorphism of $O_N^m$.

Like in the proof of Lemma 2.1, we need to compute the Jacobian of this change of variable. The same arguments apply to show that

$$|\det J_\Psi(O)| = \exp(\frac{\lambda}{N} \text{Tr} \Phi + O(1))$$

with $\Phi$ the linear operator defined on antisymmetric matrices by

$$\Phi A = \sum_i \partial_i P_{i}^\sharp A.$$ 

A basis of $A_N^1$ is given, for $k < l$, by

$$E^1(kl)_{rj} = \frac{1_{r=k, i=l} - 1_{r=l, j=k}}{\sqrt{2}}.$$

Therefore, the trace of any linear endomorphism $\varphi$ on $A_N^1$, defined by $\varphi(X) = \sum_\ell A_\ell XB_\ell$, for uniformly bounded matrices $A_\ell$, $B_\ell$, is now given by

$$\text{Tr}(\varphi) = \sum_{k<l} \text{Tr}(E^1(kl)^* \varphi(E^1(kl))) = \frac{1}{2} \sum_\ell \left( \sum_{k<l} A_\ell^t B_{lk}^\ell - \sum_{k\neq l} A_{lk}^\ell B_{lk}^\ell \right)$$

$$= \frac{1}{2} \sum_\ell \text{Tr}(A^\ell) \text{Tr}(B^\ell) + \text{Tr}(A_\ell B_\ell^\ell)$$

$$= \frac{1}{2} \sum_\ell \text{Tr}(A^\ell) \text{Tr}(B^\ell) + NO(1)$$

since the operator norm of $A_\ell$ and $B_\ell$ is uniformly bounded, $O(1)$ is uniformly bounded in $N$.

We can apply this bound to our case where $A_\ell$ and $B_\ell$ are given by $\partial_i P_i := \sum_\ell A_\ell \otimes B_\ell$. The $A_\ell$ and $B_\ell$’s are uniformly bounded since the $O_j$’s and the $A_j$’s are and non zero for a finite number of $\ell$’s, thus we deduce that

$$|\det J_\Psi(O)| = \exp(\frac{\lambda}{2N} \sum_{i=1}^m \text{Tr} \otimes \text{Tr}(\partial_i P_i) + O(1))$$

with $O(1)$ bounded uniformly in $N$. Since $O(1)$ is uniformly bounded, we can now proceed exactly as in the proof of Theorem 2.1 to show that for
any \( r \in \{1, \cdots, m\} \),
\[
\lim_{N \to \infty} \left\{ \frac{1}{2} \hat{\mu}^N \otimes \hat{\mu}^N (\partial_r P) + \frac{1}{2N} \hat{\mu}^N(D_r V P) \right\} = 0 \quad \mu^N_{1/2V} \text{ a.s.}
\]
As a consequence, for any limit point \( \tau \) of \( \hat{\mu}^N \), any antisymmetric polynomial \( P \),
\[
\tau \otimes \tau(\partial_r P) + \tau(D_r V P) = 0.
\]
If \( P \) is symmetric, we claim that for any \( r \in \{1, \cdots, m\} \),
\[
\tau \otimes \tau(\partial_r P) = \tau(D_r V P) = 0
\]
so that (19) still holds. Indeed, if \( Q \) is a word in the \((\hat{O}_1, \hat{O}_1^*, A_i, 1 \leq i \leq m)\),
\[
\partial_r Q = \sum_{Q=Q_1O_rQ_2} Q_1O_r \otimes Q_2 - \sum_{Q=Q_1O_r^*Q_2} Q_1 \otimes O_r^* Q_2
\]
\[
\partial_r Q^* = \sum_{Q^*=Q_1O_rQ_2} Q_1O_r \otimes Q_2 - \sum_{Q^*=Q_1O_r^*Q_2} Q_1 \otimes O_r^* Q_2
\]
\[
= \sum_{Q=Q_1O_r^*Q_2} (O_r^* Q_2) \otimes Q_r^* - \sum_{Q=Q_1O_r^*Q_2} Q_r^* \otimes (Q_1O_r)^*.
\]
Since the trace is invariant under transposition, we deduce that for all \( P \),
\[
\hat{\mu}^N(P^*) = \hat{\mu}^N(P)
\]
and thus,
\[
\hat{\mu}^N \otimes \hat{\mu}^N (\partial_r Q + \partial_r Q^*) = 0.
\]
With the same method, we can deal with the cyclic derivative term. Indeed, since \( D_r (Q^*) = - (D_r Q)^* \), if we write \( V = Q + Q^* \), we obtain:
\[
\hat{\mu}^N(D_r V (P + P^*)) = \hat{\mu}^N(D_r (Q + Q^*) (P + P^*))
\]
\[
= \hat{\mu}^N(D_r Q (P + P^*)) - \hat{\mu}^N((D_r Q)^* (P + P^*))
\]
\[
= \hat{\mu}^N(D_r Q (P + P^*)) - \hat{\mu}^N((P + P^*) D_r Q) = 0.
\]
To sum up,
\[
\hat{\mu}^N \otimes \hat{\mu}^N (\partial_r P) = \hat{\mu}^N(D_r V P) = 0
\]
from which we get (20) by going to the limit. Since any polynomial \( P \) can be decomposed as the sum of a symmetric polynomial \((P + P^*/2)\) and an antisymmetric polynomial \((P - P^*/2)\), we conclude by linearity that (19) holds for any polynomial \( P \). By uniqueness of the solutions to this equation for sufficiently small parameters \( t_i \) proved in Theorem 3.1 the proof is complete. \( \square \)

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