On $r$-Stirling Type Numbers of the First Kind

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Received March 25, 2019; Revised April 28, 2019; Accepted May 16, 2019

Abstract  Some combinatorial properties of $r$-Stirling numbers are proved. Moreover, two asymptotic formulas for $r$-Stirling numbers of the first kind derived using different methods are discussed and corresponding asymptotic formulas for the $r$-Stirling type numbers of the first kind are obtained as corollaries.

Mathematics Subject Classification (2010). 11B73, 41A60.

Keywords: asymptotic analysis, asymptotic formula, Stirling numbers, generalized Stirling numbers

Cite This Article: Cristina B. Corcino, and Roberto B. Corcino, “On $r$-Stirling Type Numbers of the First Kind.”  
Turkish Journal of Analysis and Number Theory, vol. 7, no. 3 (2019): 65-69. doi: 10.12691/tjant-7-3-2.

1. Introduction

The $r$-Stirling numbers and $r$-Stirling type numbers are generalizations of the classical Stirling Numbers of the first kind. Introduced first by Andrei Broder [1], the $r$-Stirling numbers of the first kind count the number of permutations of the set $\{1,2,\ldots,n\}$ with $m$ cycles such that the first $r$ elements are in distinct cycles. Broder denoted these numbers by $\text{n m}_r$. Since $\text{n m}_r=0$ for $m<r$, this study considers the $r$-Stirling numbers of the first kind $\text{n m+r}_r$, where $n,m,r$ are positive integers. These numbers satisfy the relation

\[ z(z+1)(z+2)\ldots(z+(n-1))=\sum_{m=0}^{n} \text{n m+r}_r (z-r)^m. \]  

where $\alpha,\gamma$ are complex numbers. Taking $\alpha = -1$ and $\gamma = r$, (3) becomes

\[ z(z+1)(z+2)\ldots(z+(n-1))=\sum_{m=0}^{n} S^{-1,r}_{n,m} (z-r)^m, \]  

which is exactly (1). Thus,

\[ \text{n m+r}_r = S^{-1,r}_{n,m}. \]  

Taking $\alpha = -\lambda$ and $\gamma = r$, (3) becomes

\[ z(z+\lambda)(z+2\lambda)\ldots(z+(n-1)\lambda)=\sum_{m=0}^{n} S^{-\lambda,r}_{n,m} (z-r)^m \]  

which is exactly (2). Thus,

\[ \text{n m+r}_r = S^{-\lambda,r}_{n,m}. \]  

In this paper, some combinatorial formulas for $r$-Stirling numbers are obtained. Moreover, two asymptotic formulas for these numbers derived using two different methods are mentioned and corresponding asymptotic formulas for the $r$-Stirling type numbers of the first kind are obtained as corollaries. These formulas may be used to compute values of these numbers when the parameters $m$ and $n$ are large within a certain range of $m$.

2. Some Combinatorial Properties

The $r$-Stirling numbers of the second kind, denoted by $\{n\}_k$, are defined by A.Z. Broder as the number of ways to partition the set $\{1,2,\ldots,n\}$ into $k$ nonempty subsets such that the first $r$ elements in $S$ must be in different
subsets. The total number of partitions is defined to be the \( r \)-Bell numbers \([3]\) denoted by \( B_{n,r} \). That is,

\[
B_{n,r} = \sum_{k=1}^{n} (-1)^{n-k} \left\{ \sum_{j=0}^{k} j^{k} \right\} \binom{n}{k}.
\] (7)

If the linear order of the elements in each subset of the partition counts, then the number of ways to partition \( S \) into \( k \) nonempty subsets such that the first \( r \) elements in \( S \) must be in different subsets is equal to the \( r \)-Bell numbers, denoted by \( \left[ \begin{array}{c} n \\ k \end{array} \right] _{r} \). Motivated by the work of Feng Qi \([4]\) the \( r \)-Bell numbers can also be expressed in terms of \( r \)-Lah numbers \([5]\) and \( r \)-Stirling numbers of the second kind as follows,

\[
B_{n,r} = \sum_{k=1}^{n} (-1)^{n-k} \left\{ \sum_{j=0}^{k} j^{k} \right\} \binom{n}{k}.
\] (8)

The proof makes use of the following identity in \([5]\)

\[
\binom{n}{k} = \sum_{j=k}^{n} (-1)^{n-j} \left\{ j \right\} \binom{j}{k},
\] (9)

and inverse relation

\[
b_{n} = \sum_{j=0}^{n} a_{j} \Leftrightarrow a_{n} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} b_{j}.
\]

An identity that involves \( r \)-Stirling numbers of the first kind parallel to (8) is given in the following theorem.

**Theorem 2.1.** For \( n \) and \( r \) positive integers, the following explicit formula holds

\[
r + 1_{m} = \sum_{j=0}^{n} (-1)^{j-k} \left\{ \sum_{j=0}^{k} j^{k} \right\} \binom{n}{j} j \binom{n}{j}.
\] (10)

**Proof.** To establish (10), another form of inverse relation will be needed. Using the orthogonality relation

\[
\sum_{j=k}^{n} (-1)^{j-k} \binom{n}{j} j^{k} \binom{j}{k} = \delta_{n,k},
\]

where \( \delta_{n,k} \) is the Kronecker delta, one can easily prove that

\[
b_{k} = \sum_{j=k}^{n} j^{k} \binom{n}{j} j \binom{n}{j}.
\] (11)

Note that (9) can be expressed as

\[
(-1)^{k} \binom{n}{k} = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} (-1)^{j} \binom{n}{j}.
\] (12)

By making use of inverse relation in (11), the identity (12) with

\[
a_{k} = (-1)^{k} \binom{n}{k} \quad \text{and} \quad b_{j} = (-1)^{j} \binom{n}{j},
\]

can be expressed as

\[
(-1)^{k} \binom{n}{k} = \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \binom{n}{j},
\]

which gives

\[
\binom{n}{k} = \sum_{j=0}^{n} (-1)^{j-k} \binom{j}{k} \binom{n}{j}.
\]

Summing up both sides over \( k \) from 0 to \( n \) yields

\[
\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} (-1)^{j-k} \binom{j}{k} \binom{n}{j}.
\]

This is equivalent to

\[
r + 1_{m} = \sum_{j=0}^{n} (-1)^{j-k} \left\{ \sum_{j=0}^{k} j^{k} \right\} \binom{n}{j} j \binom{n}{j},
\] (13)

where \( (x)_n = x(x + 1)(x + 2) \ldots (x + n - 1) \).

The preceding equation counts the total number of permutations of \( S \) such that the first \( r \) elements of \( S \) are in distinct cycles. We observe that the structure of the identity (13) is analogous to (8). Thus, one may try to construct another combinatorial interpretation for \( r \)-Bell numbers using (8) which may be the basis to construct another combinatorial interpretation for \( (r + 1)_n \).

The first values of the classical Stirling numbers of the first kind can be computed using the recurrence relation

\[
s(n,k) = s(n-1,k-1) - s(n-1,k),
\] (14)

and the Schlömilch formula

\[
s(n,k) = \sum_{r=0}^{n-k} (-1)^{i+r} \binom{n}{j} \binom{n-1+r}{j-k} \binom{2n-k}{n-k-r} \frac{(r-j)^{r-k+n}}{r!}.
\] (15)

On the other hand, the first values of \( r \)-Stirling numbers of the first kind can also be computed using the recurrence relation (see \([1]\))

\[
\binom{n}{k} = \binom{n-1}{k-1} + (n-1) \binom{n-1}{k}.
\] (16)

And the Schlömilch-type formula \([6]\)

\[
\binom{n+m}{k+r} = \sum_{m-k}^{n} \sum_{h=0}^{m-k} (-1)^{m+k+h+i} \binom{n}{m} \binom{h}{j} \binom{m-h}{m-k-h} \frac{(h-j)^{m-k+h}}{h!} r_{m}.\]
(17)

This explicit formula is derived in \([6]\) using the following exponential generating function

\[
\sum_{n} \frac{n^{r}}{k+r} z^{n} = \frac{1}{k!} \left[ \frac{1}{1-z} \right] \left[ \ln \left( \frac{1}{1-z} \right) \right]^{k}.
\] (18)

and the facts that
In this paper,\textsuperscript{11} \( ln !! 1 \)

Theorem 3.1. \([\text{C.B. Corcino, L.C. Hsu and E.L. Tan, [8]}]\)

approximation: \( \text{Cauchy-Integral Formula to (1) gives} \)

holds,

and \( \) the desired explicit formula for \( r\)-Stirling numbers of the first kind is easily obtained.

3. Asymptotic Formulas for \( r\)-Stirling Numbers of the First Kind

Let \( C \) be any closed contour enclosing \( r \). Applying the Cauchy-Integral Formula to (1) gives

\[
\left[ \frac{n+r}{m+r} \right]_{r} = \frac{1}{2\pi i} \int_{C} \frac{z(z+1)(z+2)\ldots(z+n-1)}{(z-r)^{m+1}} dz. \tag{21}
\]

A modified saddle point method used in \([7]\) was applied to the integral above to obtain the following asymptotic approximation:

**Theorem 3.1.** \([\text{C.B. Corcino, L.C. Hsu and E.L. Tan, [8]}]\)

For positive integers \( m, n \) and \( r \), the asymptotic formula holds,

\[
\left[ \frac{n+r}{m+r} \right]_{r} \sim e^{B} g(s_{0}) (n-1)_{m} r^{a-m-1} \frac{R^{a-m-1}}{m!}, \tag{22}
\]

as \( n \to \infty \) valid uniformly with \( m \) in the range \( 0 < m < n \), where

\[
s_{0} = \frac{nr}{n-m}, \tag{21}
\]

\[
B = \phi(z_{0}) - nlog s_{0} + nlog(s_{0} - r), \tag{24}
\]

and

\[
g(s_{0}) = \frac{1}{z_{0} - r} \sqrt{s_{0}(s_{0} - r)(n-m)} \phi''(z_{0}). \tag{25}
\]

The number \( z_{0} \) is the unique positive solution to the equation \( \phi(z) = 0 \), the function \( \phi(z) \) is

\[
\phi(z) = log[z(z+1)(z+2)\ldots(z+n-1)] - mlog(z-r), \tag{26}
\]

and

\[
(n-1)_{m} = (n-1)(n-2)\ldots(n-1-m+1). \tag{26}
\]

Remark: The number \( z_{0} \) may be obtained using mathematica.

Using the method in \([9]\), Vega and Corcino \([10]\) obtained an asymptotic formula for the generalized Stirling numbers of the first kind which is given by

\[
S_{n,m}^{\alpha,r} \sim \frac{(\alpha)^{-m} R^{a} (R-n) + 1 + 3C_{4} H^{2} - 15C_{3} H^{2}}{(2\pi H)^{1/2} R^{m} \Gamma(R) \Gamma(R-n)} \tag{27}
\]

as \( n \to \infty \) valid for \( m \) in the range \( h(n) < m < n - O(n^{1/2}), \) where \( h(n) \) is a function such that \( \lim_{n \to \infty} h(n) = \infty \) and \( 0 < \delta < 1, \) \( \Gamma(x) \) is the gamma function,

\[
\nu = \frac{a}{2} < 1. \tag{27}
\]

In this paper, \( h(n) = log n \) and \( \delta = \frac{1}{2} \). The \( H \) that appears in (27) is

\[
H = \sum_{h=1}^{n-1} \frac{(h-\nu) R}{R + h - \nu}, \tag{28}
\]

and \( R \) is the unique positive solution to the equation

\[
\sum_{h=1}^{n-1} \frac{R}{R + h - \nu} = m - 1. \tag{29}
\]

The constants \( C_{3} \) and \( C_{4} \) are given by

\[
C_{3} = \frac{1}{6} \left( 3H - 2(m-1) + 2 \sum_{h=1}^{n-1} \frac{R^{3}}{R + h - \nu} \right), \tag{30}
\]

and

\[
C_{4} = \frac{1}{24} \left[ 36C_{3} - 11H + 6(m-1) - 6 \sum_{h=1}^{n-1} \frac{R^{4}}{R + h - \nu} \right]. \tag{31}
\]

With a little modification in the computations in \([10]\), the same formula as (27) is obtained when

\[
H = \sum_{h=0}^{n-1} \frac{(h-\nu) R}{R + h - \nu}, \tag{32}
\]

and \( R \) is the unique positive solution to the equation

\[
\sum_{h=0}^{n-1} \frac{R}{R + h - \nu} = m. \tag{33}
\]

Since \( \left[ \frac{n+r}{m+r} \right]_{r} = S_{n,m}^{a,r} \) \([\text{see [6]}]\), taking \( a = -1, \gamma = r \) in (27), the following asymptotic formula for the \( r\)-Stirling numbers of the first kind is obtained:

**Theorem 3.2.** \((\text{Corcino-Corcino, [11]}\) \)

For positive integers \( m, n \) and \( r \), and as \( n \to \infty \), the following asymptotic formula for the \( r\)-Stirling numbers of the first kind holds:

\[
\left[ \frac{n+r}{m+r} \right]_{r} = \frac{\Gamma(R+r+n)}{(2\pi H)^{1/2} R^{m} \Gamma(R)} \left[ 1 + \frac{3C_{4}}{H^{2}} - \frac{15C_{3}^{2}}{H^{2}} \right], \tag{34}
\]

valid for \( m \) in the range \( log n < m < n - O(n^{1/2}) \), where \( R \) is the unique positive solution to the equation

\[
\sum_{h=0}^{n-1} \frac{R}{R + h + r} = m, \tag{35}
\]

and

\[
H = \sum_{h=0}^{n-1} \frac{(h+r) R}{R + h + r}, \tag{36}
\]
The corresponding constants $C_3$ and $C_4$ are as follows,

$$C_3 = \frac{1}{6} \sum_{h=0}^{n-1} \frac{R(h+r)(3R+h+r)}{(R+h+r)^3}, \quad (37)$$

$$C_4 = \frac{1}{24} \sum_{h=0}^{n-1} \frac{R(h+r)[(-3R^2 + 4R(h+r)+(h+r)^2]}{(R+h+r)^4}. \quad (38)$$

The next lemma gives the connection formula for the $z_0$ defined in Theorem 3.1. and the number $R$ defined in Theorem 3.2.

**Lemma 3.3.** (Corcino-Corcino, [11]) The numbers $z_0$ and $R$ satisfy the relation $z_0 = R + r$.

### 4. r-Stirling Type Numbers of the First Kind

Applying the Cauchy Integral Formula to (2) we obtain

$$\left[ \begin{array}{c} n+r \\ m+r \end{array} \right]_{\lambda,r} = \frac{1}{2\pi i} \int_{C} \frac{z^{\lambda} (z+2\lambda) \ldots (z+n-1\lambda)}{(z-r)^{m+1}} \, dz$$

$$= \lambda^{n-m} \frac{1}{2\pi i} \int_{C} \frac{u(u+1)(u+2) \ldots (u+n-1)}{(u-\eta)^{m+1}} \, du$$

$$= \lambda^{n-m} \left[ \begin{array}{c} n+\eta \\ m+\eta \end{array} \right]_{\eta} = \lambda^{n-m} S_{n,m}^{1,\eta}, \quad (41)$$

where $\eta = \frac{r}{\lambda}, u = \frac{z}{\lambda}$.

Following (17) and (41), we have the following corollary.

**Corollary 4.1.** The r-Stirling type numbers satisfy

$$\left[ \begin{array}{c} n+r \\ m+r \end{array} \right]_{\lambda,r} = \lambda^{n-m} \sum_{m=1}^{n} \sum_{h=0}^{n-m} \sum_{j=0}^{m-k} \frac{(-1)^{m-k+h+j}}{m! j!} \binom{n}{m} \binom{h}{j} \binom{m-k}{h} \eta^{h-m}.$$  

where $\eta = \frac{r}{\lambda}$.

The asymptotic formula corresponding to (22) is given in Corollary 4.2.

**Corollary 4.2.** For positive integers $n, m$ and $r$,

$$\left[ \begin{array}{c} n+r \\ m+r \end{array} \right]_{\lambda,r} \sim \lambda^{n-m} e^B g(s_0) \frac{(n-1)_m \eta^{n-m-1}}{m!}, \quad (43)$$

as $r \to \infty$, where

$$s_0 = \frac{m\eta}{n-m}.$$

$$B = \phi(u_0) - m \log s_0 + m \log (s_0 - \eta),$$

$$\phi(u) = \log \left[ u(u+1)(u+2) \ldots (u+n-1) \right] - m \log (u-\eta),$$

$u_0$ is the unique positive solution to the equation

$$\phi(u) = 0,$$

and

$$g(s_0) = \frac{1}{u_0 - \eta} \int_0^{s_0} \frac{1}{\phi''(u_0)} \, du.$$  

**Proof.** That $R = u_0 - \eta$ follows from Lemma 3.3. With (27), where $\nu = -\eta$ and (41), the corollary is then an immediate consequence of Theorem 3.2.

**Remark.** The asymptotic formulas in Theorem 3.1 and Theorem 3.2 can be shown to be asymptotically equivalent in the range of $m$ where both are valid. Proof for the equivalence is done in [11]. This implies the equivalence of the asymptotic formulas in Corollary 4.2 and 4.3.

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