Super Hilbert Spaces

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ABSTRACT

The basic mathematical framework for super Hilbert spaces over a Graßmann algebra with a Graßmann number-valued inner product is formulated. Super Hilbert spaces over infinitely generated Graßmann algebras arise in the functional Schrödinger representation of spinor quantum field theory in a natural way.

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1 Introduction

The purpose of this article is to define and study the notion of super Hilbert space in a mathematically proper way and to establish generalizations of some basic results of the theory of Hilbert spaces for super Hilbert spaces. According to our definition a super Hilbert space is a module over a Graßmann algebra endowed with a Graßmann number-valued inner product.

The notion of super Hilbert space has first been considered by DeWitt [1]. Our definition, though being more general than DeWitt’s definition (see Sections 4 and 7 below), is motivated by DeWitt’s work. We shall see that DeWitt’s definition is not general enough for certain physical applications and in particular we shall see that his notion of physical observable on a super Hilbert space is in general not well-defined. Within the framework developed in this paper we shall arrive at a more transparent notion of physical observable for super Hilbert spaces.

In standard complex quantum theory the physical transition amplitudes are given by the complex-valued inner product on the underlying complex Hilbert space. For quantum field theories with fermionic degrees of freedom or supersymmetric quantum theories super Hilbert spaces Graßmann number-valued inner products may be introduced as long as a prescription to compute physical probabilities and transition amplitudes is given alongside.

In the functional Schrödinger representation of spinor quantum field theory super Hilbert spaces with Graßmann number-valued inner products arise naturally, see [4] and below. Therefore super Hilbert spaces provide a means to bring quantum (field) theories with fermionic degrees of freedom and supersymmetric quantum (field) theories into a form resembling standard quantum mechanics which is of potential interest in several branches of quantum field theory and may shed new light on some technical and conceptual issues in quantum field theory. It is the aim of the present investigation to develop and study this notion more thoroughly.

This paper is organized as follows: in Section 2 we review some facts about Graßmann algebras; in Section 3 we define and discuss the notion of a Hilbert module over a Graßmann algebra; in Section 4 we define super Hilbert spaces and in Section 5 we study physical observables on super Hilbert spaces. In Section 6 we review the Schrödinger representation of spinor quantum field theory as a simple example for a situation where a super Hilbert space arises naturally as the state space in a quantum field theory. In Section 7 we briefly review definitions of the notion of super Hilbert space previously given by other authors and discuss the relation of these definitions with our approach.

2 Graßmann algebras

The Graßmann algebra (or exterior algebra) $\Lambda_n$ with $n$ generators is the algebra (over $\mathbb{C}$) generated by a set of $n$ anticommuting generators $\{\xi_i\}_{i=1}^n$ and by $1 \in \mathbb{C}$

$$\xi_i \xi_j = -\xi_j \xi_i, \text{ for all } i, j.$$

The Graßmann algebra generated by a countably infinite set of generators will be denoted by $\Lambda_\infty$. In the sequel we shall write $\Lambda_n$ where $n \in \mathbb{N} \cup \{\infty\}$ is possibly infinite unless indicated otherwise. Let $M_n := \{(m_1, \ldots, m_k) | 1 \leq k \leq n, m_i \in \mathbb{N}, 1 \leq m_1 < \cdots < m_k \leq n\}$. A special basis of $\Lambda_n$ is given by the set of elements of the form $\xi_{m_1} \xi_{m_2} \cdots \xi_{m_k}$, with $(m_1, \ldots, m_k) \in M_n$, together with the unit 1 \in \mathbb{C}$.
Graßmann algebras are more fully discussed in, e.g., [3, 14]. We define an involution * on \( \Lambda_n \), i.e., a map \( * : \Lambda_n \to \Lambda_n \) satisfying \((q^*)^* = q\) and \((pq)^* = p^* q^*\) for \( q, p \in \Lambda_n \). We define an involution \(*\) that the Hodge star operator is independent of the basis used to define it. For \( q \in \Lambda_n \), we have \( (\alpha q)^* = \alpha^* q^* \) for \( q, p \in \Lambda_n \). The Graßmann algebra carries a natural \( \mathbb{Z}_2 \)-grading: \( \Lambda_n = \Lambda_{n,0} \oplus \Lambda_{n,1} \), where \( \Lambda_{n,0} \) consists of the even (commuting) elements in \( \Lambda_n \) and \( \Lambda_{n,1} \) consists of the odd (anticommuting) elements of \( \Lambda_n \). A basis \( \{e_1, \ldots, e_n\} \) of \( \Lambda_n \) may serve as a possible choice of (possibly complex) generators of \( \Lambda_n \). In general there may be no physically preferred choice for the set of generators of a Graßmann algebra.

The Graßmann algebra can also be written as a direct sum of all generators of \( \Lambda_n \) by setting

\[
q = q_B 1 + q_S = q_B 1 + \sum_{(m_1, \ldots, m_k) \in M_n} q_{m_1, \ldots, m_k} \xi_{m_1} \cdots \xi_{m_k},
\]

where \( q_B, q_{m_1, \ldots, m_k} \in \mathbb{C} \). The complex number \( q_B \) is called the body of \( q \) and the Graßmann number \( q_S \) is called the soul of \( q \).

Now expand every \( q \in \Lambda_n \) with respect to the basis of \( \Lambda_n \) as in Equation 1. Then we can define for each \( 1 \leq \kappa < \infty \)

\[
|q|_\kappa := \left( |q_B|^\kappa + \sum_{(m_1, \ldots, m_k) \in M_n} |q_{m_1, \ldots, m_k}|^\kappa \right)^{1/\kappa}.
\]

Moreover, we define \( |q|_\infty := \sup_{(m_1, \ldots, m_k) \in M_n} |q_{m_1, \ldots, m_k}| \). If \( n \) is finite, it is straightforward to verify that each \( |\cdot|_\kappa \) defines a norm on \( \Lambda_n \) and that \( \Lambda_n \) becomes a complex Banach space with each of the norms \( |\cdot|_\kappa \), \( 1 \leq \kappa \leq \infty \), which we denote by \( \Lambda_n(\kappa) \) respectively. In the case of \( \Lambda_\infty \), \( |\cdot|_\kappa \) defines a seminorm on \( \Lambda_\infty \) and we denote the set of all \( q \in \Lambda_\infty \) for which the above expression for \( |q|_\kappa \) satisfies \( |q|_\kappa < \infty \) by \( \Lambda_\infty(\kappa) \). Again it is easy to see that \( \Lambda_\infty(\kappa) \) with the norm \( |\cdot|_\kappa \) is a Banach space for all \( 1 \leq \kappa \leq \infty \). The norm \( |\cdot|_1 \) is sometimes also called the Rogers norm and \( \Lambda_n(1) \) the Rogers algebra, see [10].

The norms \( |\cdot|_\kappa \) in (2) depend implicitly on the choice of the set of generators of the Graßmann algebra and are not invariant under a change of the set of generators of \( \Lambda_n \). Graßmann number-valued variables appearing in physical theories are by their very nature not directly observable and therefore in general there may be no physically preferred choice for the set of generators of a Graßmann algebra.
However, for \( n \) finite, it is well-known that not only all the norms in (2) are equivalent and therefore generate the same topology on \( \Lambda_n \) for all \( 1 \leq \kappa \leq \infty \) but the resulting topology is in fact independent of the choice of generators of the Graßmann algebra (this is an immediate consequence of Proposition 1.2.16 in [5]). It is evident that the Hodge star operator is continuous in this topology.

There is a norm, invariant under change of generators, on \( \Lambda_n \) which can be constructed as follows. Firstly, it is known that there is a norm \( \| \cdot \|_r \) on \( V_r \) given by, \([3, 14]\).

\[
\|q_r\|_r = \inf \left\{ \sum_{(m_1, \ldots, m_r) \in M_n} |q_{m_1, \ldots, m_r}| \right\},
\]

for \( q_r \in V_r \) where the infimum is taken over all possible choices of the set of generators of the Graßmann algebra. The norm \( \| \cdot \|_r \) satisfies \( \|q_r p_s\|_{r+s} \leq \|q_r\|_r \|p_s\|_s \), for all \( q_r \in V_r \) and \( p_s \in V_s \), see \([3, 14]\). Now define a seminorm on \( \Lambda_n \) by

\[
\|q\| := \sum_{r=0}^{n} \|q_r\|_r.
\]

For \( n \) finite it is obvious that \( \| \cdot \| \) is a norm on \( \Lambda_n \). This norm \( \| \cdot \| \) is called the mass (norm) on \( \Lambda_n \) (\( n \) finite). By construction the mass norm is independent of the choice of the set of generators of \( \Lambda_n \).

If \( n = \infty \), then every finite subset \( \{\xi_{i_1}, \ldots, \xi_{i_m}\} \cup \{1\} \) of the set of all generators \( \{\xi_i\}_i \) of \( \Lambda_\infty \) generates an \( m \)-dimensional Graßmann subalgebra of \( \Lambda_\infty \) denoted by \( \Lambda_{i_1, \ldots, i_m} \). The collection of all such Graßmann subalgebras of \( \Lambda_\infty \) forms a directed set and the canonical imbedding morphisms obviously preserve the mass norm. We consider the algebraic direct limit \( \Delta_\infty \) of this directed set. The mass norm on the finite dimensional Graßmann subalgebras induces a mass norm \( \| \cdot \| \) on \( \Delta_\infty \). We denote the completion of \( \Delta_\infty \) with respect to the mass norm by \( \Lambda^m_\infty \). Obviously, \( \Lambda^m_\infty \) consists of all \( q \in \Lambda_\infty \) with \( \|q\| = \sum_{r=0}^{\infty} \|q_r\|_r < \infty \). The norm on \( \Lambda^m_\infty \) is again called the mass norm.

It is easy to see that the mass norm is submultiplicative \( \|pq\| = \sum_r \|(pq)_r\|_r \leq \sum_r \sum_{k \leq r} \|p_{r-k} q_k\|_r \leq \sum_r \sum_{k \leq r} \|p_{r-k}\|_r \|q_k\|_r \leq \sum_r \sum_{k \leq r} \|p_{r-k}\|_r \|q_k\|_k \leq \sum_r \sum_{k \leq r} \sum_{k \leq r} \|p_{r-k}\|_r \|q_k\|_k = \|p\| \|q\| \).

Notice that there is a seminorm on \( \Lambda_n \) which is trivially independent of the choice of the set of generators, namely

\[
\|q\|_B := |q_B|.
\]

We have \( \|q\|_B \leq |q|_\kappa \) for all \( q \in \Lambda_n(\kappa) \) and \( \|q\|_B \leq \|q\| \) for all \( q \in \Lambda^m_\infty \).

In the sequel it is understood that the symbol \( \Lambda \) stands for either (a) \( \Lambda_n \) with \( n \) finite, (b) for \( \Lambda^m_\infty \), or (c) for \( \Lambda_\infty(\kappa) \) with \( 1 \leq \kappa \leq \infty \).

### 3 Hilbert \( \Lambda \) modules

**Definition 3.1** A pre-Hilbert \( \Lambda \) module is a \( \mathbb{Z}_2 \)-graded right \( \Lambda \) module \( E = E_0 \oplus E_1 \) equipped with a \( \Lambda \)-valued inner product \( \langle \cdot , \cdot \rangle : E \times E \to \Lambda \) that is sesquilinear, definite, and whose body is Hermitian and positive. In other words:
1. \( \langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle \), and \( \langle y_1 + y_2, x \rangle = \langle y_1, x \rangle + \langle y_2, x \rangle \) for \( x, y_1, y_2 \in E \);

2. \( \langle x, \alpha y \rangle = \alpha \langle x, y \rangle = \langle \alpha^* x, y \rangle \), for \( x, y \in E, \alpha \in \mathbb{C} \);

3. \( \langle y_1 + y_2, x \rangle = \langle y_1, x \rangle + \langle y_2, x \rangle \) for \( x, y_1, y_2 \in E \);

4. \( \langle x, x \rangle_B \geq 0 \) for \( x \in E \) and \( \langle x, x \rangle = 0 \) if and only if \( x = 0 \).

An element \( x \) of a pre-Hilbert \( \Lambda \) module \( E = E_0 \oplus E_1 \) is called even if \( x \in E_0 \) and odd if \( x \in E_1 \), respectively.

Immediate consequences of Definition 3.1 are that every pre-Hilbert \( \Lambda \) module is a complex vector space and that every element \( x \) of a pre-Hilbert \( \Lambda \) module \( E \) can be uniquely written as a sum of an even and an odd element of \( E \), i.e., \( x = x_0 + x_1 \), where \( x_0 \in E_0 \) and \( x_1 \in E_1 \).

**Remark 3.2** The rationale behind the positivity requirement 4 in Definition 3.1 is to interpret the body of the inner product \( \langle \cdot, \cdot \rangle \) as physical transition amplitude.

**Remark 3.3** DeWitt [1] requires the inner product on a super Hilbert space to be sesquilinear with respect to Graßmann numbers, i.e., \( \langle x, y \rangle q = \langle x, yq \rangle \), for all \( x, y \) in the super Hilbert space and \( q \in \Lambda_n \). We shall see below, however, that the inner product on the super Hilbert space arising in the functional Schrödinger representation of spinor quantum field theory does not satisfy this condition. Accordingly we have allowed for a more general notion of pre-Hilbert \( \Lambda \) module.

We may now use a norm \( \| \cdot \|_\Lambda \) defined on \( \Lambda \) to define a norm \( \| \cdot \|_E \) on a pre-Hilbert \( \Lambda \) module \( E \) by

\[
\|x\|_E^2 = \|\langle x, x \rangle\|_\Lambda.
\]  

(6)

For instance, if \( \Lambda \) equals \( \Lambda_n \) or \( \Lambda_\infty(\kappa) \) endowed with the norm \( | \cdot |_\kappa \), then the norm on \( E \) is given by

\[
\|x\|_\kappa^2 := |\langle x, x \rangle|_\kappa,
\]  

(7)

for \( x \in E \) and \( 1 \leq \kappa \leq \infty \). The norm

\[
\|x\|^2 := \|\langle x, x \rangle\|
\]  

(8)

corresponding to the mass norm on \( \Lambda = \Lambda_n \) or \( \Lambda = \Lambda_\infty^\kappa \) in Equation 4 is called the **mass norm** on the Hilbert \( \Lambda \) module \( E \).

In the sequel it is understood that we consider only norms on a Hilbert \( \Lambda \) module arising from a norm on \( \Lambda \) as in Equation 5 unless explicitly stated otherwise.

**Lemma 3.4 (Cauchy-Schwarz inequality)** If \( E \) is a pre-Hilbert \( \Lambda \) module and \( x, y \in E \), then

\[
|\langle x, y \rangle_B|^2 \leq \langle x, x \rangle_B \langle y, y \rangle_B.
\]
Proof: Let $p_x := \langle x, x \rangle_B, p_y := \langle y, y \rangle_B, q := \langle x, y \rangle_B$ and $\lambda \in \mathbb{R}$, then

$$0 \leq \langle x - y\lambda q^*, x - y\lambda q^* \rangle_B = p_x - 2\lambda qq^* + \lambda^2 q^2 q^*.$$  

Adding $2\lambda qq^*$ on both sides and taking norms yield

$$2\lambda|q|^2 \leq 2\lambda|q|^2 \leq |p_x| + \lambda^2 |q|^2 |p_y|. \quad (9)$$

This is equivalent to

$$(\lambda|q||p_y| - |q|)^2 \geq (|q|)^2 - |p_x||p_y|.$$  

If $|p_y| \neq 0$, then setting $\lambda := \frac{1}{|p_y|}$ yields the required inequality. Moreover, we find that $|q| = 0$ and $|p_x| \neq 0$ implies $|q| = 0$ (let $\lambda = 1$). From symmetry considerations (or from Equation 9) we also get that $|p_y| = 0$ and $|p_x| \neq 0$ implies $|q| = 0$. In the case that $|p_x| = |p_y| = 0$ we infer from Equation 9 by taking $\lambda$ to be positive that $|q| = 0$. □

On any pre-Hilbert $\Lambda$ module $E$ there is a body operation, i.e., a linear map $B : E \to E_0, x \mapsto x_B$ such that $(\lambda x)_B = x_B \lambda_B$ for all $\lambda \in \Lambda$. First define the soul $s(E)$ and the body $b(E)$ of $E$ by

$s(E) := \{ x \in E | x\lambda = 0 \text{ for some } \lambda \in \Lambda, \lambda \neq 0 \},$

$b(E) := E / s(E).$

The body operation $B : E \to E_0$ is the canonical surjection from $E$ to $b(E)$.

If the inner product of $E$ satisfies $\langle x_B, y_B \rangle = \langle x, y \rangle_B$, then the body of $E$ endowed with the induced inner product is a pre-Hilbert space whose completion is a Hilbert space (by virtue of the Cauchy-Schwarz inequality). But even if the inner product does not respect the body operation, we can prove

**Proposition 3.5** Let $E$ be a pre-Hilbert $\Lambda$ module. Then there exists a map $x \mapsto [x]$ from $E$ into a dense subspace of a Hilbert space $H$ such that

$$\langle [x], [y] \rangle_H = \langle x, y \rangle_B,$$

for all $x, y \in E$, where $\langle \cdot, \cdot \rangle_H$ denotes the inner product on $H$.

**Proof:** Let $\mathcal{N} := \{ x \in E | \langle x, x \rangle_B = 0 \}$. Let $[x] := x + \mathcal{N}$. Then $\langle \cdot, \cdot \rangle_B$ induces a well-defined inner product on $E / \mathcal{N}$ by virtue of Lemma 3.4. Therefore $E / \mathcal{N}$ with this inner product is a pre-Hilbert space. □

**Definition 3.6** Let $E$ be a pre-Hilbert $\Lambda$ module and $\| \cdot \|$ a norm on $E$, then $E$ is said to be a Hilbert $\Lambda$ module if $E$ is complete with respect to its norm. A Hilbert submodule of a Hilbert module $E$ is a closed submodule of $E$. 
Definition 3.7 Let $E$ and $F$ be Hilbert $\Lambda$ modules. A $\mathbb{C}$-linear map $O : E \to E$ is called an operator on $E$. We denote the set of all bounded operators on $E$ by $\mathcal{L}(E)$. An operator $T : E \to E$ is called unitary if $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in E$. An operator $S$ is called weakly unitary if $\langle S(x), S(y) \rangle_B = \langle x, y \rangle_B$ for all $x, y \in E$. A (Hilbert) module map is a linear map $T : E \to F$ which respects the module action: $T(xq) = T(x)q$, for $x \in E, q \in \Lambda$.

Definition 3.8 A Hilbert $\Lambda$ module $E$ is said to satisfy the strong definiteness condition if $\langle x, x \rangle_B = 0$ implies $x = 0$ for all $x \in E$.

Every Hilbert $\Lambda$ module $E$ satisfying the strong definiteness condition becomes a pre-Hilbert space with respect to the norm $\| \cdot \|_B^2 := \langle \cdot, \cdot \rangle_B$.

Every Hilbert $\Lambda$ module $E$ is endowed with a $\mathbb{Z}_2$-grading on $\mathcal{L}(E)$: every operator $T : E \to E$ can be written as sum of an even map $T_0 : E_i \to E_i$ and an odd map $T_1 : E_i \to E_{i+1 \text{(mod 2)}}$, i.e. $T = T_0 + T_1$ where $T_0$ and $T_1$ are defined by $T_0u := (Tu_0)_0 + (Tu_1)_1$ and $T_1u := (Tu_0)_1 + (Tu_1)_0$ respectively where $u = u_0 + u_1$.

Definition 3.9 Let $E$ be a Hilbert $\Lambda$ module. An operator $T : E \to E$ is said to be adjointable if there exists an operator $T^* : E \to E$ satisfying $\langle x, Ty \rangle = \langle T^*x, y \rangle$ for all $x, y \in E$. Such an operator $T^*$ is called an adjoint of $T$. We denote the set of all adjointable operators on $E$ by $\mathcal{B}(E)$. An adjointable operator $T \in \mathcal{B}(E)$ is called self-adjoint if $T^* = T$.

An operator $T : E \to E$ is said to be weakly adjointable if there exists an operator $T^\dagger : E \to E$ satisfying $\langle x, Ty \rangle_B = \langle T^\dagger x, y \rangle_B$ for all $x, y \in E$. Such an operator $T^\dagger$ is called a weak adjoint of $T$. We denote the set of all weakly adjointable operators on $E$ by $\mathcal{B}_w(E)$. A weakly adjointable operator $T \in \mathcal{B}_w(E)$ is called weakly self-adjoint if $T^\dagger = T$.

Remark 3.10 Obviously, any adjointable operator is also weakly adjointable. Thus, $\mathcal{B}(E) \subset \mathcal{B}_w(E)$. We have noticed above in Remark 3.2 that the body of the inner product on a Hilbert $\Lambda$ module is interpreted as the physical transition amplitude. Accordingly we also expect that the set $\mathcal{B}_w(E)$ plays a distinguished role and that the operators representing physical observables or physical operations will be elements of $\mathcal{B}_w(E)$.

The following proposition can be proven in analogy to the corresponding result for Hilbert $C^*$-modules, see [13].

Lemma 3.11 (a) Let $E$ be a Hilbert $\Lambda$ module and $T : E \to E$ be an adjointable operator. The adjoint $T^*$ of $T$ is unique. If both $T : E \to E$ and $S : E \to E$ are adjointable operators, then $ST$ is adjointable and $(ST)^* = T^*S^*$.

(b) Let $E$ be a Hilbert $\Lambda$ module satisfying the strong definiteness condition and $T_w : E \to E$ be a weakly adjointable operator. Then the weak adjoint $T_w^\dagger$ of $T_w$ is unique. If both $T_w : E \to E$ and $S_w : E \to E$ are adjointable operators, then $S_wT_w$ is adjointable and $(S_wT_w)^\dagger = T_w^\dagger S_w^\dagger$.

Proof: (a) Assume that $\overline{T}$ and $T^*$ are adjoints of $T$, then

$$0 = \langle \overline{T}x, y \rangle - \langle T^*x, y \rangle = \langle (\overline{T} - T^*)x, y \rangle,$$

for all $x, y \in E$. Let $y = (\overline{T} - T^*)x$. This implies $\overline{T} = T^*$. A similar argument proves (b). □
4 Super Hilbert spaces

The Definitions 3.7 and 3.9 are analogous to parallel definitions in the theory of Hilbert $C^*$-modules [13, 8]. However, the positivity requirement in the definition of a Hilbert $\Lambda$ module is weaker than the positivity requirement for Hilbert $C^*$-modules and all results for Hilbert $C^*$-modules depending on the positivity of the inner product may in general not be valid for a Hilbert $\Lambda$ module. The Cauchy-Schwarz inequality in Lemma 3.4 is a first example. As a consequence of the failure of the general Cauchy-Schwarz inequality the inner product on a pre-Hilbert $\Lambda$ module may in general not be continuous in each argument and therefore in general an inner product on a pre-Hilbert $\Lambda$ module does not extend to an inner product on its completion. In the sequel we shall be mainly interested in inner products which are continuous.

Definition 4.1 We shall call a (pre-) Hilbert $\Lambda$ module $\mathcal{H}$ a super (pre-) Hilbert space if the inner product on $\mathcal{H}$ is continuous, i.e., if there exists a constant $C > 0$ such that $\|\langle x, y \rangle\| \leq C\|x\|\|y\|$.

Remark 4.2 The completion of a super pre-Hilbert space is a super Hilbert space.

Remark 4.3 All concrete examples for super Hilbert spaces we shall discuss below will satisfy the strong definiteness condition. An example for a situation where a super Hilbert space without the strong definiteness condition arises is the Becchi-Rouet-Stora-Tyutin (BRST) formulation of gauge theories. The natural choice of the state space arising the BRST formulation of gauge theories is a Hilbert $\Lambda$ module (as there is always a representation of the Grassmann algebra acting on the state space) endowed with an indefinite inner product. The physical states are annihilated by the BRST operator $\Omega$, i.e., satisfy the condition $\Omega \psi_{\text{phys}} = 0$. The inner product induced on the set $\mathcal{V}_{\text{phys}}$ of all physical states can be shown to be positive but not definite. The states in $\mathcal{V}_{\text{phys}}$ with probability zero are called ghost states and are unobservable. Therefore in the BRST formulation of gauge theories we naturally arrive at a physical state space which carries the structure of Hilbert $\Lambda$ module or a super Hilbert space not satisfying the strong definiteness condition. A good introduction into the BRST formalism can be found, e.g., in [7].

We have already noticed above that the physical transition amplitudes are given by the body of the inner product of a Hilbert $\Lambda$ module. This gives rise to the following definition.

Definition 4.4 Let $\mathcal{H}$ be a super Hilbert space. An element $x \in \mathcal{H}$ is called physical if $\langle x, x \rangle_B \neq 0$. An element $g \in \mathcal{H}$ with $g \neq 0$ and $\langle g, g \rangle_B = 0$ is called a ghost.

Example 4.5 Let $n$ be finite. The Grassmann algebra $\Lambda_n$ endowed with the mass norm $\| \cdot \|$ becomes a super Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle p, q \rangle := \star [p \star [q]] \quad (10)$$
for all \( p, q \in \Lambda_n \), where \(*\) denotes the Hodge star operator. The submultiplicativity of the mass norm implies \( \| \langle p, q \rangle \| \leq \| p \| \| q \| \) for all \( p, q \in \Lambda_n \). Recalling Equation \( \| \cdot \|_{\text{H}} \) we see that

\[
q = qB1 + \sum_{(m_1, \ldots, m_k) \in M_n} q_{m_1, \ldots, m_k} \xi_{m_1} \cdots \xi_{m_k},
\]

we see that

\[
\langle q, q \rangle_B = |qB|^2 + \sum_{(m_1, \ldots, m_k) \in M_n} |q_{m_1, \ldots, m_k}|^2.
\]

Therefore \( \Lambda_n \) with the inner product \( \langle \cdot, \cdot \rangle_B \) satisfies the strong definiteness condition.

More general super Hilbert spaces can be constructed by building the tensor product \( \Lambda_n \otimes \mathcal{H} \) of \( \Lambda_n \) with a complex Hilbert space \( \mathcal{H} \). The inner product of \( \Lambda_n \otimes \mathcal{H} \) is given on simple tensors by \( \langle p \otimes \varphi, q \otimes \psi \rangle = \langle p, q \rangle \langle \varphi, \psi \rangle \), for \( p, q \in \Lambda_n \) and \( \varphi, \psi \in \mathcal{H} \), and extended to arbitrary elements of \( \Lambda_n \otimes \mathcal{H} \) by linearity and continuity. We omit the details of the construction as a more general example will be given below in Example 4.9.

**Example 4.6** Consider a measure space \((X, \Omega)\), where \( \Omega \) is a set and \( \Omega \) a \( \sigma \)-algebra of subsets of \( X \), endowed with a \( \sigma \)-finite measure \( \mu \). Every function \( f : X \to \Lambda_n \) can be expanded as

\[
f(x) = f_B(x) + \sum_{(m_1, \ldots, m_k) \in M_n} f_{m_1, \ldots, m_k}(x) \xi_{m_1} \cdots \xi_{m_k},
\]

with complex-valued functions \( f_B : X \to \mathbb{C} \) and \( f_{m_1, \ldots, m_k} : X \to \mathbb{C} \). We restrict ourselves here to the case that \( n \) is finite. Now consider the set \( E \) of all functions \( f : X \to \Lambda_n \) such that \( f_B \) and all \( f_{m_1, \ldots, m_k} \) are square integrable with respect to \( \mu \). This requirement is independent of the basis chosen. We define a \( \Lambda_n \)-valued inner product on \( E \) by

\[
\langle f, g \rangle = \int f(x)^* g(x) d\mu(x),
\]

for all \( f, g \in E \). If \( \Lambda_n \) is furnished with the Rogers norm \( | \cdot |_1 \), then define

\[
\| f \| := \sum_{(m_1, \ldots, m_r) \in M_n^0} \sqrt{\int |f_{m_1, \ldots, m_r}(x)|^2 d\mu(x)},
\]

where we introduced the notation \( M_n^0 := \{(m_1, \ldots, m_k) | 0 \leq k \leq n, m_i \in \mathbb{N}, 1 \leq m_1 < \cdots < m_k \leq n \} \) and \( f_B := f_B \). Further let \( \mathcal{N} := \{ f \in E | \| f \| = 0 \} \). It is easy to see that Equation \( \| \cdot \| \) defines a norm on \( E/\mathcal{N} \) and that \( E/\mathcal{N} \) equipped with the norm \( \| \cdot \| \) becomes a super Hilbert space. Indeed, let \( f, g \in E \), then

\[
|\langle f, g \rangle|_1 = \sum_{(m_1, \ldots, m_r) \in M_n^0} \left| \sum_{k=0}^r \sum_{\sigma} (-1)^{\text{sgn}(\sigma)} \int f_{\sigma(m_1), \ldots, \sigma(m_k)}^*(x) g_{\sigma(m_{k+1}), \ldots, \sigma(m_r)} d\mu(x) \right|
\]

\[
\leq \sum_{(m_1, \ldots, m_r) \in M_n^0} \sum_{k=0}^r \sum_{\sigma} \left| \int f_{\sigma(m_1), \ldots, \sigma(m_k)} d\mu(x) \right| \left| g_{\sigma(m_{k+1}), \ldots, \sigma(m_r)} \right| d\mu
\]

\[
\leq \sum_{(m_1, \ldots, m_r) \in M_n^0} \sum_{k=0}^r \sum_{\sigma} \left[ \int |f_{\sigma(m_1), \ldots, \sigma(m_k)}|^2 d\mu(x) \right]^{1/2} \left[ \int |g_{\sigma(m_{k+1}), \ldots, \sigma(m_r)}|^2 d\mu(x) \right]^{1/2}
\]

\[
\leq \| f \| \| g \|,
\]
where the sum $\sum'_{\sigma}$ in the first three lines runs over all permutations $\sigma$ of $(m_1, \cdots, m_r)$ such that $(\sigma(m_1), \cdots, \sigma(m_k)) \in M^0_n$ and $(\sigma(m_{k+1}), \cdots, \sigma(m_r)) \in M^0_n$. If we replace (I4) by $\langle f, g \rangle = \int [f(x)]g(x) d\mu(x)$, a similar argument holds.

Example 4.7 For $n$ infinite we also can make $\Lambda^m_n$ a super Hilbert space by defining an appropriate inner product. For simplicity we assume that the set of all generators is countable $\{\xi_i\}_{i \in \mathbb{N}}$. The generalization of the following to the situation where the set of generators is uncountable is obvious. First of all we observe that the inner product (10) is not well-defined as the Hodge star operator is not defined on $\Lambda^m_n$. This difficulty can be overcome by suitably imbedding $\Lambda^m_n$ into the direct sum $\Lambda^m_\infty \oplus \Lambda^m_\infty$ of two copies of $\Lambda^m_\infty$. The basic idea is to introduce the formal infinite product of all generators $\xi_\infty \equiv \prod_i \xi_i$. We do not make any attempt to give a precise meaning to this infinite product of Graßmann numbers and just introduce $\xi_\infty$ as an auxiliary object which has certain properties we would expect from the product of all generators of the Graßmann algebra. Namely, we require that $q\xi_\infty = q\xi_\infty$ for all $q \in \Lambda^m_\infty$. Analogously we define cofinite products of the generators of the Graßmann algebra, i.e., infinite products obtained from $\xi_\infty$ by removing at most finitely many terms in the product. E.g., the infinite product $\prod_{i \neq 1} \xi_i$ of all generators except $\xi_1$ is denoted by $\hat{\xi}_1 \equiv \frac{\partial}{\partial \xi_1} \xi_\infty$. We require

$$\frac{\partial}{\partial \xi_j} \frac{\partial}{\partial \xi_i} = - \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_j}$$

and $\xi_i \hat{\xi}_j = \xi_\infty$ and $\xi_i \frac{\partial}{\partial \xi_j} = - \frac{\partial}{\partial \xi_j} \xi_i$, for all $i \neq j$. Moreover we require $\xi_\infty$ to be even. Therefore the algebra $\star \Lambda^m_\infty$ generated by the $\frac{\partial}{\partial \xi_i}$ and 1 is isomorphic to $\Lambda^m_\infty$.

Now we are able to define the action of the Hodge star operator on $\Lambda^m_\infty$ by setting

$$\star[q] \equiv q_B^* \xi_\infty + \sum_{(m_1, \cdots, m_k) \in M_\infty} q_{m_1}^* \cdots q_{m_k}^* \frac{\partial}{\partial \xi_{m_1}} \cdots \frac{\partial}{\partial \xi_{m_k}} \xi_\infty,$$

(13)

for all $q \in \Lambda^m_\infty$. Moreover, we require $\star[\star[q]] = q$, for all $q$. The algebra generated by the $\frac{\partial}{\partial \xi_i}$ is isomorphic to $\Lambda^m_\infty$ with the isomorphism given by the Hodge star operator (I4).

The inner product $\langle p, q \rangle = \star[p \star [q]]$, for all $p, q \in \Lambda^m_\infty$ is now well-defined. Notice that although $\star[q] \notin \Lambda^m_\infty$ for all $q \in \Lambda^m_\infty$, the inner product satisfies $\langle p, q \rangle \in \Lambda^m_\infty$ if $p, q \in \Lambda^m_\infty$. Since, by virtue of the properties of the mass norm, we also have $\|\langle p, q \rangle\| \leq \|p\| \|q\|$ for all $p, q \in \Lambda^m_\infty$ and since

$$\langle q, q \rangle_B = |q_B|^2 + \sum_{(m_1, \cdots, m_r) \in M_n} |q_{m_1} \cdots q_{m_r}|^2$$

we see that $\Lambda^m_\infty$ with the inner product (I4) is a super Hilbert space satisfying the strong definiteness condition.

Example 4.8 $\star \Lambda^m_\infty$ can be made a super Hilbert space (over $\Lambda^m_\infty$) by setting

$$\langle p, q \rangle = \star[p] q,$$

for all $p, q \in \star \Lambda^m_\infty$ (when we identify $\xi_\infty$ formally with 1 $\in \mathbb{C}$). Obviously $\star \Lambda^m_\infty$ satisfies the strong definiteness condition.
Example 4.9 We are now going to construct the tensor product of two super Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We denote the inner products on $\mathcal{H}_1$ and $\mathcal{H}_2$ by $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively, and the norms on $\mathcal{H}_1$ and $\mathcal{H}_2$ are denoted by $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively.

The algebraic tensor product $\mathcal{H}_1 \otimes_{\mathrm{alg}} \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ is defined as usual as the set of all finite sums of the form $\sum_i p_i \otimes q_i$ with $p_i \in \mathcal{H}_1$ and $q_i \in \mathcal{H}_2$. We define a function $\mu$ on $\mathcal{H}_1 \otimes_{\mathrm{alg}} \mathcal{H}_2$ by

$$
\mu(t) := \inf \left\{ \sum_i \|p_i\|_1 \|q_i\|_2 \bigg| t = \sum_i p_i \otimes q_i \right\}. \quad (14)
$$

$\mu$ is a cross norm on $\mathcal{H}_1 \otimes_{\mathrm{alg}} \mathcal{H}_2$ and the completion of $\mathcal{H}_1 \otimes_{\mathrm{alg}} \mathcal{H}_2$ with respect to $\mu$ is a Banach algebra which we denote by $\mathcal{H}_1 \otimes_{\mu} \mathcal{H}_2$ (for a proof, see, e.g., Proposition T.3.6 in [13]). The inner products on $\mathcal{H}_1$ and $\mathcal{H}_2$ induce an inner product on $\mathcal{H}_1 \otimes_{\mathrm{alg}} \mathcal{H}_2$ given by

$$
\langle a, b \rangle = \sum_{i,j} \langle p_i, t_j \rangle_1 \otimes \langle q_i, s_j \rangle_2
$$

if $a = \sum_i p_i \otimes q_i$ and $b = \sum_j t_j \otimes s_j$. As

$$
\mu(\langle a, b \rangle) = \inf \left\{ \sum_i \|c_i\|_1 \|d_i\|_2 \bigg| \langle a, b \rangle = \sum_i c_i \otimes d_i \right\}
$$

$$
\leq \text{restr. inf} \sum_{i,j} \|\langle p_i, t_j \rangle_1\|_1 \|\langle q_i, s_j \rangle_2\|_2
$$

$$
\leq \text{restr. inf} \sum_{i,j} \|p_i\|_1 \|q_i\|_2 \|t_j\|_1 \|s_j\|_2
$$

$$
= \text{restr. inf} \left( \sum_i \|p_i\|_1 \|q_i\|_2 \right) \left( \sum_j \|t_j\|_1 \|s_j\|_2 \right)
$$

$$
= \mu(a)\mu(b),
$$

where the infimum in the first line runs over all possible decompositions of $\langle a, b \rangle$ as sums over elementary tensors, whereas the 'restricted infima' in the following three lines run over all decompositions of $a$ and $b$ into sums of elementary tensors. Consequently the inner product $\mu$ on $\mathcal{H}_1 \otimes_{\mathrm{alg}} \mathcal{H}_2$ is continuous and can be extended to the completion $\mathcal{H}_1 \otimes_{\mu} \mathcal{H}_2$ of $\mathcal{H}_1 \otimes_{\mathrm{alg}} \mathcal{H}_2$. We denote this extension also by $\mu$. Therefore $\mathcal{H}_1 \otimes_{\mu} \mathcal{H}_2$ is a super Hilbert space when endowed with the norm $\mu$.

When both $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy the strong definiteness condition, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are pre-Hilbert spaces with respect to the body of their inner products. Therefore also the body $\mu_B$ of $\mu$ is a complex-valued scalar product on $\mathcal{H}_1 \otimes_{\mathrm{alg}} \mathcal{H}_2$ and, by virtue of the Cauchy-Schwarz inequality, $\mu_B$ can be extended to a complex-valued scalar product $\tilde{\mu}_B$ on $\mathcal{H}_1 \otimes_{\mu} \mathcal{H}_2$. $\tilde{\mu}_B$ obviously coincides with the body of the extension of $\mu$ to $\mathcal{H}_1 \otimes_{\mu} \mathcal{H}_2$. Therefore we conclude that $\mathcal{H}_1 \otimes_{\mu} \mathcal{H}_2$ is a super Hilbert space satisfying the strong definiteness condition.

In Section 6 we shall be interested in the case $\mathcal{H}_l = \Lambda^m_{\infty}$ and $\mathcal{H}_r = \ast [\Lambda^m_{\infty}]$. The norm $\mu_m$ arising from the mass norms on $\Lambda^m_{\infty}$ and $\ast [\Lambda^m_{\infty}]$ via Equation [4] is called the mass norm on $\Lambda^m_{\infty} \otimes_{\mu_m} [\Lambda^m_{\infty}]$. It follows from our discussion above that $\Lambda^m_{\infty} \otimes_{\mu_m} [\Lambda^m_{\infty}]$ is a super Hilbert space satisfying the strong definiteness condition. We shall see in Section 6 that in the functional Schrödinger representation of
spinor quantum field theory the super Hilbert space \( \Lambda_m^\infty \otimes \mu_m \star [\Lambda_m^\infty] \) arises naturally as the quantum theoretical state space.

**Proposition 4.10** Let \( \mathcal{H} \) be a super Hilbert space and \( T : \mathcal{H} \to \mathcal{H} \) be an adjointable operator. Then \( T \) and \( T^* \) are bounded with respect to the operator norm

\[
\|T\| := \sup \{ \|Tx\| : \|x\| \leq 1 \}.
\]

If Equation (15) holds, then \( \|T\| = \|T^*\| \).

*Proof:* Let \( x_\lambda, x, y \in \mathcal{H} \), such that \( x_\lambda \to x \) and \( Tx_\lambda \to y \). The inner product of a super Hilbert space is separately continuous in each variable. Thus

\[
0 = \langle T^*e, x_\lambda \rangle - \langle T^*e, x_\lambda \rangle = \langle e, Tx_\lambda \rangle - \langle e, y \rangle - \langle T^*e, x \rangle = \langle e, y - Tx \rangle,
\]

for all \( e \in \mathcal{H} \). Putting \( e = y - Tx \) implies \( y = Tx \). The boundedness of \( T \) and \( T^* \) follows now from the closed graph theorem. As \( \|Tx\|^2 = \|\langle T^*Tx, x \rangle\| \leq \|T^*\| \|T\| \|x\|^2 \), we find \( \|T\| \leq \|T^*\| \). But then also \( \|T^*\| \leq \|T^{**}\| = \|T\| \). \( \square \)

A similar argument proves

**Proposition 4.11** Let \( \mathcal{H} \) be a super Hilbert space satisfying the strong definiteness condition and \( T : \mathcal{H} \to \mathcal{H} \) be a weakly adjointable operator. Then \( T \) and \( T^\dagger \) are bounded with respect to the operator norm in Equation (15) and with respect to the norm

\[
\|T\|_w := \sup \{ \|\langle Tx, Tx \rangle_B\|^{1/2} : \|x\| \leq 1 \}
\]

and \( \|T\|_w = \|T^\dagger\|_w \).

*Proof:* The boundedness of \( T \) and \( T^\dagger \) with respect to the norm in Equation (15) follows as in the proof of Proposition 4.10. The boundedness with respect to \( \|\cdot\|_w \) follows from \( \|q\|_B \leq \|q\| \) for all \( q \in \Lambda \). \( \square \)

**Proposition 4.12** Let \( \mathcal{H} \) be a super Hilbert space. When equipped with the operator norm (15), \( \mathfrak{B}(\mathcal{H}) \) is an involutive Banach algebra with continuous involution.

*Proof:* It is easy to see that (15) defines a norm on \( \mathfrak{B}(\mathcal{H}) \). The operator norm is clearly submultiplicative. It remains to show that \( \mathfrak{B}(\mathcal{H}) \) is norm complete. If \( (T_n)_{n \in \mathbb{N}} \) is a Cauchy sequence of adjointable operators, then \( (T_nx)_{n \in \mathbb{N}} \) and \( (T^*_nx)_{n \in \mathbb{N}} \) are Cauchy sequences in \( \mathcal{H} \) for every \( x \in \mathcal{H} \). We call the limits \( Tx \) and \( T^*x \) respectively. Since \( \langle y, Tx \rangle = \lim \langle y, T_nx \rangle = \lim \langle T^*_nx, y \rangle = \langle T^*_y, x \rangle \), we see that \( T \) is adjointable and \( T^* = T^\dagger \). This shows that \( \mathfrak{B}(\mathcal{H}) \) is norm complete. From \( \|T_n - T\| = \|T^*_n - T^*\| \) it is easy to see that the involution is continuous. \( \square \)
5  Physical observables

**Definition 5.1** Let $E$ be a Hilbert $\Lambda$ module and $T \in \mathcal{B}_w(E)$. Then we say that a Grassmann number $\lambda$ is a spectral value for $T$ when $T - \lambda I$ does not have a two-sided inverse in $\mathcal{B}_w(E)$. The set of spectral values for $T$ is called the spectrum of $T$ and is denoted by $\text{sp}(T)$. The subset $\text{sp}_c(T) := \text{sp}(T) \cap \mathbb{C}$ is called the complex spectrum of $T$.

It is well-known that a Grassmann number $q \in \Lambda_n$, $n$ finite, has an inverse if and only if its body $q_B$ is nonvanishing [1]. Therefore the following proposition that the spectrum of a bounded module map $T$ on a Hilbert $\Lambda_n$ module, $n$ finite, is fully determined by the complex spectrum of $T$ is not surprising.

**Proposition 5.2** Let $E$ be a Hilbert $\Lambda_n$ module, $n$ finite, and $T \in \mathcal{B}_w(E)$ be a Hilbert module map. Then $\lambda \in \text{sp}(T)$ if and only if $\lambda_B \in \text{sp}_c(T)$.

**Proof:** Let $\lambda \not\in \text{sp}(T)$. Then $T - \lambda I$ has a two-sided inverse in $\mathcal{B}_w(E)$, denoted by $T^{-1}_\lambda$. Evidently $T^{-1}_\lambda$ is a module map. Now let $s$ be a Grassmann number with vanishing body. Then $T^{-1}_{\lambda - s,L} := \left(\sum_{n=0}^{\infty} (-sT^{-1}_\lambda)^n \right)^{-1}$ is a left inverse for $T - (\lambda - s)I$ and $T^{-1}_{\lambda - s,R} := T^{-1}_\lambda \left(\sum_{n=0}^{\infty} (-sT^{-1}_\lambda)^n \right)^{-1}$ is a right inverse for $T - (\lambda - s)I$. Both sums are actually finite. This follows from the bodylessness of $s$ and from the fact that $T^{-1}_\lambda$ is decomposable into an even and an odd part: $T^{-1}_\lambda = T^{-1}_{\lambda,0} + T^{-1}_{\lambda,1}$. Therefore the left and right inverse exist for all $s \in \Lambda_n$, $s_B = 0$.

As $T^{-1}_{\lambda - s,L} (T - (\lambda - s)I) T^{-1}_{\lambda - s,R} = T^{-1}_{\lambda - s,L}$ the left and right inverse coincide. This proves that $\lambda \not\in \text{sp}(T)$ implies $\lambda - s \not\in \text{sp}(T)$ for all $s \in \Lambda_n$ with $s_B = 0$. □

**Example 5.3** Consider $\Lambda_n$ endowed with the inner product $\langle \cdot | \cdot \rangle$. Let $\xi_1, \ldots, \xi_n$ denote the set of generators of $\Lambda_n$. Consider the module map $\xi_1 : \Lambda_n \to \Lambda_n, \xi_1 q := \xi_1 q$. Obviously 0 is the only complex spectral value of $\xi_1$ and, as $\xi_1^2 = \xi_1$, it does not have an inverse for all bodyless $s \in \Lambda_n$, all Grassmann numbers with vanishing body are spectral values for $\xi_1$. The element $\xi_1 \xi_2 \cdots \xi_n \in \Lambda_n$ is an “Eigenstate” for $\xi_1$ for any bodyless spectral value: $\xi_1 \xi_2 \cdots \xi_n = s_1 \xi_2 \cdots \xi_n = 0$, for all $s \in \Lambda_n$ with $s_B = 0$.

**Definition 5.4** Let $\mathcal{H}$ be a super Hilbert space. A physical observable on $\mathcal{H}$ is a weakly self-adjoint operator $O : \mathcal{H} \to \mathcal{H}$.

**Proposition 5.5** Let $\mathcal{H}$ be a super Hilbert space and let $H$ be the Hilbert space from Proposition 4.3. Then there exists a * homomorphism $\varphi$ from $\mathcal{B}_w(\mathcal{H}) \cap \mathcal{L}(\mathcal{H})$ (equipped with the norm $\| \cdot \|_w$) into the $C^*$-algebra $\mathcal{B}(H)$ of bounded operators on $H$.

**Proof:** Let $\mathcal{N} := \{ x \in \mathcal{H} | \langle x, x \rangle_B = 0 \}$ and let $T \in \mathcal{B}_w(\mathcal{H})$. For $n \in \mathcal{N}$ we have by virtue of Lemma 3.4: $|\langle Tn, Tn \rangle_B |^2 \leq (T^\dagger Tn, T^\dagger Tn)_B(n,n)_B = 0$. Thus $T(\mathcal{N}) \subset \mathcal{N}$. This shows that every $T \in \mathcal{B}_w(\mathcal{H})$ induces a bounded linear operator on $\mathcal{H}/\mathcal{N}$ which we denote by $\varphi(T)$ via $\varphi(T)(x + \mathcal{N}) := T(x) + \mathcal{N}$. The operator $\varphi(T)$ can be uniquely extended to a bounded linear operator $\varphi(T)$ on $H$ (compare, e.g., Theorem 1.5.7 in [3]). Obviously, the correspondence $\varphi$ is linear, multiplicative and satisfies $\varphi(T^\dagger) = \varphi(T)^*$ and $\varphi(I) = I_H$, i.e., $\varphi$ is a * homomorphism. □
Proposition 5.6 Let $\mathcal{H}$ be a super Hilbert space satisfying the strong definiteness condition. Then the \* homomorphism $\varphi$ from Proposition 5.5 is an isometric isomorphism from $\mathfrak{B}_w(\mathcal{H})$ to the $C^*$-algebra $\mathfrak{B}(H)$. Hence $\mathfrak{B}_w(\mathcal{H})$ is a $C^*$-algebra with norm $\|T\|_w := \sup\{|\langle Tx, Tx \rangle_B|^{1/2}||x|| \leq 1\}$.

Proof: This follows, e.g., from Theorem 1.5.7 in [5]. $\square$

Remark 5.7 Proposition 5.2 and Example 5.3 show that in general it is meaningless to attribute physical relevance to the soul of a spectral value of a physical observable. Instead it becomes clear that in general only the elements of the complex spectrum of a physical observable may admit an interpretation as possible physical values of the observable. This is in accordance with and further substantiated by the fact that Graßmann numbers cannot be measured. Above we have argued that the physical transition amplitudes on a super Hilbert space are given by the body of the inner product. Therefore it seems reasonable to define the physical spectrum of an operator $T$ on a super Hilbert space $\mathcal{H}$ as, loosely speaking, the subset of $\text{sp}_\mathbb{C}(T)$ corresponding to physical elements of $\mathcal{H}$, i.e.,

Definition 5.8 The physical spectrum of a bounded physical observable $O$ on a super Hilbert space $\mathcal{H}$, denoted by $\text{sp}_{\text{ph}}(O)$, is the set of $\lambda \in \mathbb{C}$ such that $\varphi(O - \lambda I)$ has no two-sided inverse in $\mathfrak{B}(H)$.

Proposition 5.9 Let $O$ be a bounded physical observable on a super Hilbert space $\mathcal{H}$. Then $\text{sp}_{\text{ph}}(O) = \text{sp}(\varphi(O)) \subset \text{sp}_\mathbb{C}(O)$. If $\mathcal{H}$ satisfies the strong definiteness condition, then $\text{sp}_{\text{ph}}(O) = \text{sp}_\mathbb{C}(O)$.

Proof: Let $\lambda \notin \text{sp}_\mathbb{C}(O), \lambda \in \mathbb{C}$. Then $O - \lambda I$ has a two-sided inverse in $\mathfrak{B}_w(\mathcal{H})$ and as $\varphi$ is an algebra-homomorphism, also $\varphi(O) - \lambda I$ has a two-sided inverse in $\mathfrak{B}(H)$. This implies $\lambda \notin \text{sp}(\varphi(O))$. The other direction follows analogously if $\varphi$ is an isomorphism. $\square$

Corollary 5.10 The physical spectrum of a bounded physical observable on a super Hilbert space is a compact subset of $\mathbb{R}$.

If $\mathcal{H}$ is a super Hilbert space, $T \in \mathcal{L}(\mathcal{H})$, then we say that an element $\lambda \in \text{sp}(T)$ is a right Eigenvalue for $T$ if there is an $x \in \mathcal{H}$ such that $T(x) = x\lambda$. $x$ is called an Eigenvector corresponding to $\lambda$. Notice that an Eigenvector might correspond to more than one Eigenvalue, see Example 5.3.

Corollary 5.11 Let $O$ be a bounded physical observable $O$ on a super Hilbert space. Let $\lambda, \lambda' \in \text{sp}_{\text{ph}}(O)$ be physical spectral values of $O$ with $\lambda \neq \lambda'$ and let $x, x'$ be Eigenvectors corresponding to $\lambda$ and $\lambda'$ respectively. Then $\langle x, x' \rangle_B = 0$.

Remark 5.12 It is instructive to compare our definition of a physical observable with the definition given by DeWitt in [7]. According to DeWitt’s definition an element of a super Hilbert space is called physical if it has nonvanishing body. A physical observable is then defined as a self-adjoint module map on the super Hilbert space such that

- all Eigenvalues are even Graßmann numbers;
for every Eigenvalue there is a physical Eigenstate;

• the set of physical Eigenvectors that correspond to soulless Eigenvalues contains a complete basis (for a definition of this notion see [7]).

Our argument above shows that DeWitt’s additional assumptions are unnecessary to assure real-valuedness of physical spectral values and weak orthogonality of Eigenvectors of physical observables. Moreover, Proposition 5.4 shows that for super Hilbert spaces over finitely generated Graßmann algebras there are no physical observables in DeWitt’s sense, as there will always be Eigenvalues which are not even. For super Hilbert spaces over an infinitely generated Graßmann algebra it is also easy to see that in general DeWitt’s conditions cannot be satisfied. For let \( T \) be a physical observable in DeWitt’s sense with Eigenvalue \( \lambda \), and let \( x \) denote a physical Eigenstate for \( \lambda \). Then \( \lambda + \xi_1 \) is an Eigenvalue of \( T \) with Eigenstate \( x \xi_1 \neq 0 \). The Eigenvalue \( \lambda + \xi_1 \) is neither an even Graßmann number nor it is guaranteed that there is a physical Eigenstate for this Eigenvalue. This is a contradiction. Therefore we conclude that in general there are no physical observables in DeWitt’s sense on a super Hilbert space.

6 The Schrödinger representation of spinor quantum field theory

In the Schrödinger representation of quantum field theory the commutation relations of the field operators are realized by representing the field operators by functionals and representing the conjugate momenta by functional derivatives [9]. This formulation of quantum field theory is equivalent to the standard operator formulation and to the functional-integral representation of quantum field theory. In the Schrödinger representation of spinor quantum field theory super Hilbert spaces naturally arise as the quantum mechanical state space. The material presented in this section is taken from Hatfield [9].

The Hamiltonian for free spinor field theory is given by

\[
H = \int d^3x \Psi^\dagger(x)(-i\alpha \cdot \nabla + \beta m)\Psi(x),
\]

\[
= \int d^3x \Psi(x)(-i\gamma^k \partial_k + m)\Psi(x)
\]

where the matrices \( \alpha_i \) and \( \beta \) satisfy \( \beta^2 = 1, \{\alpha_i, \alpha_j\} = 2\delta_{ij}, \{\alpha_i, \beta\} = 0 \) and the \( \gamma \) are related by \( \gamma^0 = \beta \) and \( \alpha_i = \gamma^0 \gamma_i \).

The canonical anticommutation relations of the fields are given by

\[
\{\Psi_\alpha(\vec{x},t), \Psi^\dagger_\beta(\vec{y},t)\} = \delta_{\alpha\beta}\delta^3(\vec{x} - \vec{y})
\]

\[
\{\Psi_\alpha(\vec{x},t), \Psi_\beta(\vec{y},t)\} = \{\Psi^\dagger_\alpha(\vec{x},t), \Psi^\dagger_\beta(\vec{y},t)\} = 0.
\]

The time evolution is given by the functional Schrödinger equation

\[
dt \frac{\partial}{\partial t} |\Psi\rangle = H |\Psi\rangle.
\]
In the “coordinate” Schrödinger representation the state space at time \( t \) is spanned by the Eigenfunctions \( |\Psi\rangle \) of the field operator \( \Psi(x) \). The corresponding Eigenvalues \( \psi(x) \) are spinors of Grassmann number-valued functions. The conjugate momentum operator \( \Psi^\dagger(x) \) is represented as a functional derivative

\[
\Psi^\dagger_\beta(x) = \frac{\delta}{\delta \Psi_\beta(x)}.
\]

It is well-known that the Eigenfunctions and the functional derivative can be rewritten as a plane wave expansion in terms the creation and annihilation operators

\[
\Psi_\alpha(x) = \sum_{i=1}^{2} \int \frac{d^3 p}{\sqrt{(2\pi)^3}} \int \frac{m}{E} \left[ b_i(\vec{p}) u_\alpha^i(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + d_i^\dagger(\vec{p}) v_\alpha^i(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right],
\]

\[
\frac{\delta}{\delta \Psi_\alpha(x)} = \sum_{i=1}^{2} \int \frac{d^3 p}{\sqrt{(2\pi)^3}} \int \frac{m}{E} \left[ \frac{\delta}{\delta b_i(\vec{p})} (u_\alpha^i(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + \frac{\delta}{\delta d_i^\dagger(\vec{p})} v_\alpha^i(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right],
\]

where

\[
u^i(\vec{p}) = \frac{-\vec{p} + m}{\sqrt{2m(m+E)}} \begin{pmatrix} \delta_{i1} \\ \delta_{i2} \\ 0 \\ 0 \end{pmatrix}, \quad u^i(\vec{p}) = \frac{-\vec{p} + m}{\sqrt{2m(m+E)}} \begin{pmatrix} 0 \\ 0 \\ \delta_{i1} \\ \delta_{i2} \end{pmatrix}.
\]

The operators \( b(\vec{p}), \frac{\delta}{\delta b_i(\vec{p})}, d_i^\dagger(\vec{p}) \) and \( \frac{\delta}{\delta d_i^\dagger(\vec{p})} \) act on the state space and obviously satisfy the equal time anticommutation relations

\[
\left\{ b_i(\vec{p}), \frac{\delta}{\delta b_j(\vec{k})} \right\} = \delta^3(\vec{p} - \vec{k}) \delta_{ij}
\]

\[
\left\{ \frac{\delta}{\delta d_i^\dagger(\vec{p})}, d_j^\dagger(\vec{k}) \right\} = \delta^3(\vec{p} - \vec{k}) \delta_{ij}
\]

with all other anticommutators vanishing. Accordingly, the \( \frac{\delta}{\delta d_i^\dagger(\vec{p})} \) and the \( \frac{\delta}{\delta d_i^\dagger(\vec{p})} \) are interpreted as creation operator of field quanta with positive or negative energy respectively whereas the \( b_i \) and the \( d_i^\dagger \) are interpreted as the corresponding annihilation operators.

At this stage the important observation for our purposes is that the state space can be naturally identified as the tensor product super Hilbert space \( \tilde{\Lambda}^m_{\infty} \equiv \Lambda^m_{\infty,d} \otimes \mu * \Lambda^m_{\infty,b} \) where the uncountable set of generators of the first factor (\( \Lambda^m_{\infty,d} \)) is identified with \( \{d_i^\dagger(\vec{p})\} \) and the set of generators of \( \Lambda^m_{\infty,b} \) is identified with \( \{b_i(\vec{p})\} \), see Example 4.9. As all generators of \( \Lambda^m_{\infty,d} \) commute with all generators of \( \Lambda^m_{\infty,b} \), we omit the tensor symbol in our notation. A factor \( b_i(\vec{p}) \) (or \( d_i^\dagger(\vec{p}) \)) in an element \( x \in \tilde{\Lambda}^m_{\infty} \) corresponds to the situation that the field quantum annihilated by \( b_i(\vec{p}) \) (or \( d_i^\dagger(\vec{p}) \)) is absent. For
example, the vacuum state, where all negative energy states are filled and all positive energy states are empty, is represented by

\[ |0\rangle = \prod_{i=1}^{2} b_i(\vec{p}_i) = \xi_{\infty,b} \]

and the state with one positron of momentum \(\vec{p}_p\) with spin up is given by

\[ |\vec{p}_p, \uparrow\rangle = d_2^\dagger(\vec{p}_p) \prod_{i=1}^{2} b_i(\vec{p}) \].

Physical transition amplitudes are computed by performing a functional integration over all Grassmann degrees of freedom, see [4]. Mathematically this is equivalent to taking the body of the inner product (10) as the physical transition amplitude, compare Proposition 3.5. It is now obvious that – when identifying the Hilbert space \(H\) in Proposition 3.5 with the state space in the operator (Heisenberg) representation of spinor quantum field theory – these physical transition amplitudes in the Schrödinger representation coincide with the physical transition amplitudes in the operator representation.

### 7 Previous definitions revisited

#### 7.1 DeWitt super Hilbert spaces

Super Hilbert spaces were first considered by DeWitt in his book [1]. The basic features of DeWitt’s definition may be summarized as follows: DeWitt defines a super Hilbert space \(\mathcal{H}\) basically as a \(\mathbb{Z}_2\)-graded \(\Lambda_n\) module, where \(n\) is possibly infinite, with a \(\Lambda_n\)-valued inner product \(\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \Lambda_n\) subject to the following conditions

1. \(\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle\), for \(x, y_1, y_2 \in \mathcal{H}\);
2. \(\langle x, \alpha y \rangle = \alpha \langle x, y \rangle = \langle \alpha^* x, y \rangle\), for \(x, y \in \mathcal{H}, \alpha \in \mathbb{C}\);
3. \(\langle x, y q \rangle = \langle x, y \rangle q\) for all \(x, y \in \mathcal{H}, q \in \Lambda_n\).
4. \(\langle x, y \rangle = \langle y, x \rangle^*\), for \(x, y \in \mathcal{H}\);
5. \(\langle x, x \rangle_B \geq 0\) for \(x \in \mathcal{H}\); \(x \in \mathcal{H}\) has nonvanishing body if and only if \(\langle x, x \rangle_B > 0\);
6. \(\langle x_s, x_r \rangle q_t = (-1)^t(s+r) q_t \langle x_s, x_r \rangle\) for all pure \(x_s, y_r \in \mathcal{H}_s, y_r \in \mathcal{H}_r\) and \(q \in \Lambda_n, \deg(q_t) = t\).

DeWitt moreover requires that the body of \(\mathcal{H}\) is an ordinary complex Hilbert space. The central difference between our definition and DeWitt’s is the requirement of sesqui-\(\Lambda\)-linearity of the inner product. It is easy to see that sesqui-\(\Lambda\)-linearity implies \(\langle x_B, y_B \rangle = \langle x, y \rangle_B\) for all \(x, y \in \mathcal{H}\). DeWitt concentrated on algebraic properties of super Hilbert spaces and did not consider the topological or metric structure of \(\Lambda_n\) or \(\mathcal{H}\).
Nagamachi and Kobayashi formalized and refined DeWitt’s definition by taking also into account the topological and norm structure on super Hilbert spaces [9]. It becomes clear from their work that the requirement of sesqui-$\Lambda$-linearity in some sense trivializes the theory, as it is possible to show along their lines that, when $\Lambda_n$ is equipped with the Rogers norm, every super Hilbert space is of the form $\mathcal{H} = H \otimes \Lambda_n$ where $H$ is an ordinary complex Hilbert space.

7.2 El Gradechi and Nieto’s super Hilbert space

El Gradechi and Nieto studied in [2] a super extension of the Kirillov-Kostant-Souriau geometric quantization method. They defined a super Hilbert space to be a direct sum $\mathcal{H} = H_0 \oplus H_1$ of two complex Hilbert spaces $(H_1, \langle \cdot, \cdot \rangle_0)$ and $(H_2, \langle \cdot, \cdot \rangle_1)$ equipped with the super Hermitean form $\langle \langle \cdot, \cdot \rangle \rangle = \langle \cdot, \cdot \rangle_0 + i \langle \cdot, \cdot \rangle_1$.

At first sight this definition looks rather different from the approach given in this paper. However, El Gradechi’s and Nieto’s definition is actually abstracted from the concrete example arising in their study of super unitary irreducible representations of $OSp(2/2)$ in super Hilbert spaces of $L^2$ super-holomorphic sections of prequantum bundles of the Kostant type. It is beyond the scope of this paper to review the construction in [2] in detail. For our purposes it is enough to know that the super Hilbert space $L_p^{(1|2)}$ constructed in [2] is a $\mathbb{Z}_2$-graded $\Lambda_4$ module and that its elements are sections of the form $\psi(z, \bar{z}, \theta, \bar{\theta}, \chi, \bar{\chi})$, where $z \in \mathbb{C}$ denotes a complex variable with $|z| \leq 1$ and $\theta, \bar{\theta}, \chi, \bar{\chi}$ denote a complexified set of generators of $\Lambda_4$. Notice that the definition for complex conjugation in $\Lambda_4$ adopted in [2] differs from our definition: $pq = p q$ for all $p, q \in \Lambda_4$. The even super Hermitean form on $L_p^{(1|2)}$ is defined for $\psi = \psi_0 + \psi_1$ and $\psi' = \psi'_0 + \psi'_1$ by (see Equation 5.8 and 5.10 in [2])

$$\langle \langle \psi', \psi \rangle \rangle \equiv \int (\psi', \psi) h dz d\bar{z} d\theta d\bar{\theta} d\chi d\bar{\chi},$$

where $h$ is an integrating constant factor for the super Liouville measure constructed in [2] and where

$$(\psi', \psi) = \overline{\psi'} \psi = \overline{\psi'_0} \psi_0 + \overline{\psi'_1} \psi_1 + \overline{\psi'_0} \psi_1 + \overline{\psi'_1} \psi_0.$$

It is now crucial to realize that it is possible to construct an $\Lambda_4$-valued inner product $\langle \cdot, \cdot \rangle$ on $L_p^{(1|2)}$ in the sense of Definition 3.1 by replacing $(\psi', \psi)$ by

$$\langle \psi', \psi \rangle \sim = \psi'^* \psi$$

(where $*$ denotes the complex conjugation operation introduced in Section 2) and by setting

$$\langle \psi', \psi \rangle \sim \equiv \int (\psi', \psi) \sim h dz d\bar{z}. \quad (18)$$

This is an inner product of the form Equation 11 and therefore the physical super Hilbert space can be identified with the super Hilbert space discussed in Example 4.6. Notice that what is called a super unitary operator with respect to the super Hermitean product in [2] would be called a weakly unitary operator with respect to the inner product (18) in our terminology. El Gradechi and Nieto correctly note that the examples for super Hilbert spaces considered by them are not covered by the definitions of DeWitt, and of Nagamachi and Kobayashi.
7.3 Other definitions

There are some other notions of super Hilbert space in the literature which we briefly mention here. Khrennikov \cite{6} defines a super Hilbert space to be a Banach (commutative) \( \Lambda \) module which is isomorphic to the space \( \ell_2(\Lambda) \) of square-summable sequences in \( \Lambda \) with the inner product \( \langle x, y \rangle := \sum x_n y_n^* \) and norm \( \|x\|^2 := \langle x, x \rangle \).

Schmitt proposed a different notion of super Hilbert space in \cite{12}. According to his definition a super Hilbert space is just a complex \( \mathbb{Z}_2 \)-graded ordinary Hilbert space (consequently the inner product is always complex-valued). It is beyond the scope of the present work to discuss this definition in detail and the reader is referred to \cite{12} and references therein. We only remark that Schmitt’s definition of super Hilbert space is general enough to cover even the functional Schrödinger representation of spinor quantum field theory and contact to our approach can be made for instance by identifying his “super Hilbert space” with the Hilbert space of Proposition 3.5.

7.4 Discussion

The definitions of the notion of super Hilbert space put forward by DeWitt \cite{1}, Nagamachi and Kobayashi \cite{9} and Khrennikov \cite{6} are all special cases of our more general definition. From a mathematical point of view they are viable generalizations of ordinary Hilbert space theory, and indeed, unlike the definition of super Hilbert space introduced in this paper, the definitions by Nagamachi, Kobayashi and Khrennikov are actually designed such that analogies or extensions of most basic structural results of ordinary Hilbert space theory remain valid in the super case. However, these definitions suffer from the problem that they are too narrow to cover the physically important example of the state space arising in the functional Schrödinger representation of spinor quantum field theory discussed in Section 6. As repeatedly stated above, in physical applications of super Hilbert spaces the physical transition amplitudes have to be identified with the body of the inner product (this is in accordance with DeWitt’s remark that “real physics is restricted to the ordinary Hilbert space that sits inside the super Hilbert space”). However, in the theories by DeWitt, Nagamachi, Kobayashi and Khrennikov the inner product respects the body operation \( \langle x_B, y_B \rangle = \langle x, y \rangle_B \). In the functional Schrödinger representation of spinor quantum field theory the physical state space is identified with the tensor product of two isomorphic copies of the underlying Grassmann algebra: the presence of, e.g., a Grassmann number \( d_1^i(\bar{p}) \) indicates the presence of the corresponding positron field quantum. Therefore the body of a DeWitt-type inner product gives essentially only the contribution of the vacuum-to-vacuum transition amplitude to the full physical transition amplitude. To obtain the full physical transition amplitude we need to introduce the more general inner product on the state space, given by Equation \[10\] which is not sesqui-\( \Lambda \)-linear and for which in general \( \langle x_B, y_B \rangle \neq \langle x, y \rangle_B \). The super Hilbert spaces arising in the work of El Gradechi and Nieto \cite{2} and of Samsonov \cite{11} provide further examples for super Hilbert spaces which are not super Hilbert spaces in the sense of DeWitt, Nagamachi and Kobayashi, or Khrennikov but which are super Hilbert spaces in the sense of Definition 4.1. Therefore we conclude that the introduction of the more general notion of super Hilbert space put forward in this paper is physically justified.
Acknowledgements

This research was performed while the author was a Marie Curie Research Fellow in the Theoretical Physics Group at Imperial College, London, UK, under the Training and Mobility of Researchers (TMR) programme of the European Commission.

References

[1] DeWitt, B.: *Supermanifolds*, 2nd Edition. Cambridge: Cambridge University Press, 1992

[2] El Gradechi, A.M., Nieto, L.M.: Supercoherent States, Super Kähler Geometry and Geometric Quantization. Commun. Math. Phys. **175**, 521-564 (1996)

[3] Federer, H.: *Geometric Measure Theory*. Berlin: Springer, 1969

[4] Hatfield, B.: *Quantum Field Theory of Point Particles and Strings*. Redwood City: Addison-Wesley, 1992

[5] Kadison, R.V., Ringrose, J.R.: *Fundamentals of the Theory of Operator Algebras* I. Orlando: Academic, 1983

[6] Khrennikov, A.Yu.: The Hilbert super space. Sov. Phys. Dokl. **36**, 759-760 (1991)

[7] Kugo, T.: *Eichtheorie*. Berlin: Springer, 1997

[8] Lance, E.C.: *Hilbert C*-modules*, London Mathematical Society Lecture Notes Series **210**. Cambridge: Cambridge University Press, 1995

[9] Nagamachi, S., Kobayashi, Y.: Hilbert superspace. J. Math. Phys. **33**, 4274-4282 (1992)

[10] Rogers, A.: A global theory of supermanifolds. J. Math. Phys. **21**, 1352-1365 (1980)

[11] Samsonov, B.F.: Supersymmetry and supercoherent states of a nonrelativistic free particle. J. Math. Phys. **38**, 4492-4503 (1997)

[12] Schmitt, T.: Supergeometry and hermitean conjugation. J. Geom. Phys. **7**, 141-169 (1990)

[13] Wegge-Olsen, N.E.: *K-Theory and C*-algebras*. Oxford: Oxford University Press, 1993

[14] Whitney, H.: *Geometric Integration Theory*. Princeton, New Jersey: Princeton University Press, 1957