D-branes in Orbifold Singularities and Equivariant K-Theory

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The study of brane-antibrane configurations in string theory leads to the understanding of supersymmetric Dp-branes as the bound states of higher dimensional branes. Configurations of pairs brane-antibrane do admit in a natural way their description in terms of K-theory. We analyze configurations of brane-antibrane at fixed point orbifold singularities in terms of equivariant K-theory as recently suggested by Witten. Type I and IIB fivebranes and small instantons on ALE singularities are described in K-theoretic terms and their relation to Kronheimer-Nakajima construction of instantons is also provided. Finally the D-brane charge formula is reexamined in this context.

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D-branes are extremely interesting objects in string theory and thus their study is still matter of vigorous research (for a review see [1]). Many surprises have been found in the study of various physical situations of D-branes and surely many other surprises remain to be discovered in the near future. Among their multiple applications D-branes have been used to study the strong coupling dynamics of supersymmetric field theories in various dimensions through the construction of configurations of intersecting branes (for a review see [2]). Gauge theories in \( p + 1 \) dimensions with sixteen supercharges can be obtained as the world-volume theories of flat infinite D\( p \)-branes. Intersecting brane configurations are described by BPS bound states of the corresponding superstring theory where the D-brane is living. Intersection of D-branes generically has zero fermionic modes which are chiral and therefore anomalous on the branes. D-branes carries RR charges and the definition of this charge, may be given through the implementation of this anomaly cancellation via the inflow mechanism from the bulk where the D-branes are living [3]. The consideration of these facts in topological terms and a further generalization to non-trivial normal bundle to the cycles where the D-branes are wrapped lead to a formula for the brane charge which widely suggest that it takes values not in the homology of spacetime but in the K-theory of the spacetime [4,5].

Recently some works have been made in the context of stable non-supersymmetric states (non-BPS states) in field theory [6] and string theory [7,8,10,11,12]. In particular, in string theory context, Sen have shed light on the structure of stable non-BPS states in type I and II superstring theories by constructing non-BPS states through the consideration of brane-antibrane pairs and their dynamical properties [8]. Boundary state analysis of these configurations leads to a world-volume field theory, on the brane pair, without the usual super Maxwell multiplet but with a tachyon field on the world-volume which survives the GSO projection. Potential of the tachyon field of this system is shown to be of the same form as the potential for a kink-anti-kink topological defect which interpolates between the two minima of the tachyonic potential. This system is already not supersymmetric but it has an stable vacuum. The vacuum configuration is reached through the mechanism of tachyon condensation. Generalization to other higher dimensional topological defects was considered as well in [8].

Following Sen’s results very recently Witten [13], classified the non-supersymmetric brane configurations in terms of the mathematical structure known as topological K-theory
(for some reviews see [14,15,16,17]). Some earlier evidence of the possible relation between D-branes and K-theory can be found in [18]. At the same paper Witten found further evidence that D-brane charge takes values in the K-theory of the spacetime manifold as before suspected in [3]. Non-BPS states of some brane configurations are classified according to the Type of superstring theory we are dealing. For instance, in Type IIB theory, non-BPS states and their lower dimensional BPS-branes are classified by the complex K-theory group $K(X)$ with $X$ the spacetime manifold. In Type I theory non-BPS objects are classified by the real K-theory group $KO(X)$. Non-BPS states in Type IIA (and their possible lifting to M-theory) are not well understood, however these states are classify by $K(X \times S^1)$. Among other further natural generalizations, Witten argued in [13] that the correct description of non-BPS states in orbifold singularities is given in terms of the equivariant K-theory (for a review of equivariant K-theory see [19,20,21]).

In this paper we continue this analysis of configurations of pairs of branes-antibranes in string theories described by K-theory. In particular we give the first steps to study configurations of D-brane pairs in orbifold singularities. We will see that equivariant K-theory is the natural language to describe these configurations, just as it was argued in [13]. We find that many results, including the global construction of [13], hold also in the equivariant case with minor changes. However, we shall stress the differences and explain some subtleties.

After the previous discussion we apply this formalism to the specific case of six-dimensional gauge theories on the world volume of Type IIB and I BPS-fivebranes (constructed from nine-brane pairs) on a general orbifold singularities. Although the construction is valid for general orbifolds, in order to connect with some well known results of branes in orbifolds, we will restrict ourselves to the case of branes on ALE singularities [22,23,24,25], leaving the theories of D3 branes on $C^3/\Gamma_G$ [26] and D1 branes on $C^4/\Gamma_G$ [27] for further investigation. We have placed coincident fivebranes at at fixed point of the ALE singularity parametrized by the transverse coordinates to the fivebrane. Following Witten [13], the fivebrane charge is now determined by the equivariant K-group of the compact space of the transverse directions to the fivebranes or (in the non-compact case) by the corresponding K-group with compact support. The transverse space is given by a ALE space which is seen as the minimal resolution of the orbifold $C^2/\Gamma_G$ with $\Gamma_G$ a discrete subgroup of SU(2). We will make contact with the Kronheimer-Nakajima construction of Yang-Mills instantons on ALE spaces [28,29]. We show that all information concerning the description of non-trivial RG fixed points of six-dimensional theories in ALE singularities
in Type I and IIB theories, is contained in the K-theoretic description and thus it admits a derivation in pure K-theoretic grounds.

The structure of this paper is as follows, in Section 2 we briefly review Witten description of non-BPS states in terms of K-theory. Section 3 is devoted to study non-BPS states in orbifold singularities and their classification in terms of equivariant K-theory. In Section 4 we discuss in detail the problem of BPS-fivebranes in ALE singularities in Type I and IIB superstring theories. We find a nice relation with Kronheimer-Nakajima construction of instantons and the description of nontrivial RG fixed points of theories in six dimensions is derived from the mathematical formalism of equivariant K-theory. In the process we discuss the origin of the index theorem for ALE manifolds from the equivariant K-theory. In Section 5 we will provide the formula for the charge of the brane in orbifold singularities derived from K-theoretical considerations. Finally is Section 6 our concluding remarks are given.

2. K-theory Description of Pairs D-brane-anti-D-brane

In this section we briefly review the Witten’s construction of D-brane pairs and their relation to topological K-theory following [13]. Our aim is not provide an extensive review, but only recall the relevant structure which will be needed in the following sections. Throughout this paper we will follow the notation introduced by Witten in [13].

2.1. Review of Witten’s Construction

K-Theory Structure

In order to fix some notation let $X$ be the ten-dimensional spacetime manifold and let $W$ be a $(p + 1)$-dimensional submanifold of $X$. Branes or antibranes or both together can be wrapped on $W$. When configurations of $N$ coincident branes or antibranes only are wrapped on $W$, the world-volume spectra on $W$ consists of a vector multiplet and scalars in some representation of the gauge group. These configurations can be described through Chan-Paton bundles which are $U(N)$ gauge bundles $E$ over $W$ for Type II superstring theory and by $SO(N)$ or $Sp(N)$ bundles in Type I theory. Gauge fields from the vector multiplet define a $U(N)$ gauge connection for Type II theory (or $SO(N)$ or $Sp(N)$ gauge connection for Type I theory) on the (corresponding) Chan-Paton bundle. GSO projection
cancels the usual tachyonic degrees of freedom. Something similar occurs for the anti-brane sector.

The description of coincident $N_1$ $p$-branes and $N_2$ $p$-antibranes wrapped on $W$ leads to the consideration pairs of gauge bundles $(E, F)$ (over $W$) with their respective gauge connections $A$ and $A'$. In the mixed configurations GSO projection fails to cancel the tachyon. Thus the system is unstable and may flow toward the annihilation of the brane-antibrane pairs with RR charge for these brane configurations being conserved in the process $[7,8,10,13]$.

On the open string sector Chan-Paton factors are $2 \times 2$ matrices constructed from the possible open strings stretched among the different types of branes. Brane-brane and antibranе-antibrane sectors correspond to the diagonal elements of this matrix. Off-diagonal elements correspond with the Chan-Paton labels of an oriented open string starting at a brane and ending at an antibrane and the other one to be the open string with opposite orientation.

The physical mechanism of brane-antibrane creation or annihilation without violation of conservation of the total RR charge, leads to consider physically equivalent configurations of $N_1$ branes and $N_2$ antibranes and the same configuration but with additional created or annihilated brane-antibrane pairs.

The relevant mathematical structure describing the brane-antibrane pairs in general type I and II superstring theories is as follows:

i) $G_1$ and $G_2$ gauge connections $A$ and $A'$ on the Chan-Paton bundles $E$ and $F$ over $W$, respectively. Bundles $E$ and $F$ corresponding to branes and antibranes are topologically equivalent. The groups $G_1$ and $G_2$ are restricted to be unitary groups for Type II theories and symplectic or orthogonal groups for Type I theories.

ii) Tachyon field $T$ can be seen as a section of the tensor product of bundles $E \otimes F^*$ and its conjugate $\overline{T}$ as a section of $E^* \otimes F$ (where $*$ denotes the dual of the corresponding bundle.)

iii) Brane-antibrane configurations are described by pairs of gauge bundles $(E, F)$.

iv) The physical mechanism of brane-antibrane creation or annihilation of a set of $m$ 9-branes and 9-antibranes is described by the same $U(m)$ (for Type II theories) or $SO(m)$ (for type I theories) gauge bundle $H$. This mechanism is described by the identification of pairs of gauge bundles $(E, F)$ and $(E \oplus H, F \oplus H)$. Actually instead of pairs of gauge bundles one should consider classes of pairs of gauge bundles $[(E, F)] = [E] - [F]$ identified as above. Thus the brane-antibrane pairs really determine an element of
the K-theory group $K(X)$ of gauge bundles over $X$ and the brane-antibrane creation or annihilation of pairs is underlying the $K$-theory concept of *stable equivalence* of bundles$^2$.

Consistency conditions for 9-branes ($p = 9$) in Type IIB superstring theory such as tadpole cancellation implies the equality of the ranks of the structure groups of the bundles $E$ and $F$. Thus $rk(G_1) = rk(G_2)$. The ‘virtual dimension’ $D$ of an element $(E, F)$ is defined by $D = rk(G_1) - rk(G_2)$. Thus tadpole cancellation leads to a description of the theory in terms of pairs of bundles with virtual dimension vanishing, $D = 0$. This is precisely the definition of reduced K-theory, $\tilde{K}(X)$. Thus consistency conditions implies to project the description to reduced K-theory.

In Type I string theory $9 - \overline{9}$ pairs are described by a class of pairs $(E, F)$ of $SO(N_1)$ and $SO(N_2)$ gauge bundles over $X$. Creation-annihilation is now described through the $SO(k)$ bundle $H$ over $X$. In Type I theories tadpole cancellation condition is $N_1 - N_2 = 32$. In this case equivalence class of pair bundles $(E, F)$ determines an element in the real $K$-theory group $KO(X)$. Tadpole cancellation $N_1 - N_2 = 32$, newly turns out into reduced real K-theory group $\tilde{KO}(X)$.

Type IIA theory involves more subtle considerations worked out in [13]. It was argued by Witten in [13] that configurations of brane-antibrane pairs are classified by the K-theory group of spacetime with an additional circle space $S^1 \times X$. K-theory group for type IIA configurations is $K(S^1 \times X)$.

Finally, it is also shown in [13] that for both Type I and II theories, the consideration of non-compact usual spacetime $X = \mathbb{R}^4 \times Q$ (where $Q$ some compact space) with appropriate boundary conditions at infinity leads to vacuum configurations with branes. In the language of K-theory this means that a non-zero K-theory class $[(E, F)]$ is equivalent to the trivial class at infinity. Non-compact spacetimes $X$ require K-theory with compact support, but with the boundary conditions at infinity the difference between ordinary and reduced K-theory is irrelevant for the most of physical applications.

$^2$ For 9-branes, the embedded submanifold $W$ coincides with $X$ and the thus brane charges take values in K-theory group of $X$. 

5
Lower Dimensional Branes

We now review Witten’s global construction of lower dimensional $p$-branes, with $p < 9$ from higher-dimensional branes. The flat case will be recovered when it is required only one global coordinate system. The key argument is that it is possible to construct $p$-branes as the bound state of $2^{n-1}$ pairs of $(p+2n)$-branes and antibranes [8,13].

Let $Z$ be a $(p+1)$-dimensional closed orientable submanifold of $Y$. This latter is also a $(p + 1 + 2n)$-dimensional orientable submanifold of spacetime manifold $X$ of the Type II superstring theory. $Z$ is a codimension $2n$ submanifold of $Y$. $Y$ could possess a gauge bundle $\mathcal{L}$ with a section vanishing along $Z$. Now we consider a system of $N_1 (p+2n)$-branes and $N_2 (p+2n)$-antibranes wrapped on $Y$. Tachyon field $T$ of this system transforms in the bifundamental representation $(\square_1, \square_2)$ of the group $U(N_1) \times U(N_2)$. Tadpole cancellation condition $N_1 = N_2 = N$ implies that $T$ actually transforms in the adjoint representation $(\square, \square)$ of $U(N) \times U(N)$. In vacuum tachyon field breaks the group $U(N) \times U(N)$ to the diagonal $U(N)$. Tachyon is a section of the bundle $\mathcal{L}$ and it is vanishing in a codimension $2n$ submanifold $Z$. Tachyon condensation flows the system to a configuration with $N$ charges transforming in the diagonal group $U(N)$, representing the remaining $p$-branes.

The description of lower-dimensional branes wrapped on $Z$ can be nicely described in terms of the pairs of brane-antibrane wrapped in $Y$ as follows. The full formalism requires of the introduction of a $U(N)$ bundle $\mathcal{W}$ over $Z$. In order to extend the definition of this bundle from $Z$ to $Y$ several obstructions can arise. For instance, if $\mathcal{W}$ has trivial normal bundle $\mathcal{W}$ it can extends over $Y$ getting an element of $K(Y)$ given by the class of pairs $[(\mathcal{L} \otimes \mathcal{W}, \mathcal{W})]$. However if $\mathcal{W}$ does not extend over $Y$ a more involved definition of the element of $K(Y)$ has to be given using the standard tools of K-theory [4,13,16,17].

Consider a tubular neighborhood $Z'$ of $Z$ and its closure $\overline{Z}$. Bundles $(\mathcal{L} \otimes \mathcal{W}, \mathcal{W})$ over $Z$ can be easily pulled back to $\overline{Z}$ determining so an element of $K(\overline{Z})$. The corresponding tachyon field $T : \mathcal{W} \rightarrow \mathcal{L} \otimes \mathcal{W}$ defines an isomorphism when it is restricted to the boundary of $Z'$. Thus one can extend $\mathcal{W}$ over $Y' = Y - Z'$ via the map $T$ by declaring that this isomorphism is extendible over $Y'$. If $\mathcal{W}$ does not extend over $Y'$ one can look for a gauge bundle $H$ over $Z$ in such a way that $\mathcal{W} \oplus H$ be trivial over $Z$ and trivial when pulled back to $\overline{Z}$. Thus one can extend $\mathcal{W} \oplus H$ over $Y$ (as the trivial bundle) and extend $\mathcal{L} \otimes \mathcal{W} \oplus H$ by setting it equal to $\mathcal{W} \oplus H$ over $Y'$. The element of $K(X)$ is thus $(\mathcal{L} \otimes \mathcal{W} \oplus H, \mathcal{W} \oplus H)$ and the tachyon field is now $T \oplus 1$. 

6
*The Spinor Description*

Global construction outlined in the above subsection in general might not exist. However, it is possible to implement the global description in terms spin bundles associated to the normal bundle to $Z$. Consider the normal bundle $N$ to $Z$ in $X$. Assume that the codimension of $Z$ is $2n$. Normal bundle has structure group $SO(2n)$. If $N$ is a spin bundle it has associated two spinor bundles $S_+, S_-$ of positive and negative chirality spin representations of $SO(2n)$ respectively. If we suppose that the spinor bundles near $Z$ extend over $X$, thus there are $2^{n-1}$ pairs of $9 - \overline{9}$ branes with gauge bundles $S_+$ and $S_-$ over $X$.

Tachyon field $T$ is a map given by $T : S_- \rightarrow S_+$ and it is naturally given by $T = \vec{\Gamma} \cdot \vec{x}$ where $\vec{x} = (x_1, \ldots, x_{2n})$, where $\vec{x}$ is an element of the tubular neighborhood $Z'$ of $Z$ in $X$. This tachyon map gives a unitary isomorphism in the boundary of $Z'$. This isomorphism is determined by the fact that $\vec{\Gamma} \cdot \vec{x}$ is unitary if $\vec{x}$ is a unitary vector. One can use this fact to extend this configuration over $X$ and the results of the above subsection hold. Thus more generally one can look for a suitable bundle $H$ such that $S_- \oplus H$ is trivial over $Z$ and extends over $X$ with the final configuration given by $(S_+ \oplus H, S_- \oplus H)$ and the tachyon $T \oplus 1$.

### 3. Equivariant K-theory Description of Branes in Orbifolds

In the last section we gave the basic facts of the interplaying between D-branes configurations and K-theory. We consider now brane-antibrane configurations in a generic orbifold and its description in terms of equivariant K-theory. We will show that equivariant K-theory is the natural language to describe supersymmetric and non-supersymmetric brane configurations in orbifolds.

#### 3.1. Non-BPS States From Brane-antibrane Pairs

Before discussing BPS and non-BPS branes in orbifolds, we briefly review the necessary requirements about non-BPS states string theory on compact manifolds. We mainly follow the results \[4,8,9\]. The basic idea is the construction of stable non-BPS states through the construction of brane-antibrane configurations. More precisely, Sen has shown that wrapping branes and antibranes in cycles of some compact manifold in some Type I
and II superstring compactifications, world-volume theory has a tachyon mode expansion which survives GSO projection. Tachyon field is a scalar which has associated a negative potential energy which cancel the tension between the brane-antibrane pair\[7,8\]. It was conjectured than unstabilities associated with the tachyon mode flows the system to the annihilation of RR charge and to an stable state admitting exact boundary conformal field theory description. To be more precise, for instance, in Type I theory pairs of D1-brane-anti D1-brane have a tachyonic kink potential and this system turns out to be stable and coincides with the SO(32) spinor state of the SO(32) heterotic theory\[8\].

Pairs of D1-strings and anti-D1-strings compactified on the circle \(S^1\) of radius \(R\) have a tachyon field of the form \(T(x) = \sum_{n=-\infty}^{+\infty} T_n e^{i \frac{n}{R} x}\). At the self-dual radius tachyon components \(T_{\pm}\) becomes massless and (in the \(-1\) and zero picture) the vertex operators of the boundary CFT are respectively given by

\[
V_{\pm}^{(-1)} = -e^{-\Phi_B} e^{\pm \frac{i}{\sqrt{2}} X_B} \otimes \sigma_1
\]

and

\[
V_{\pm}^{(0)} = \mp i \psi_B e^{\pm \frac{i}{\sqrt{2}} X_B} \otimes \sigma_1,
\]

where \(X_B\) is the bosonic field, \(\Phi_B\) is the bosonized ghost and \(\psi_B\) is the world-sheet fermion. All of them taking its boundary value. \(\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) is the Chan-Paton factor of the brane-antibrane pair open string sector. Brane-antibrane configuration survives \(\Omega\) projection and it does exist in Type IIB theory\[8\].

### 3.2. Branes-antibranes in Orbifolds

Here we describe briefly the dynamics of pairs of branes-antibranes in orbifold singularities. We present the case of a pair of D-string-anti-D-string of the type IIB superstring theory in a \(\mathbb{Z}_2\)-orbifold\[11\]. For definiteness take the pair with world-volume along coordinates \((x^0, x^9)\) leaving transverse coordinates to be \((x^1, \ldots, x^8)\). \(\mathbb{Z}_2\) group act only reversing the coordinates \((x^6, x^7, x^8, x^9)\) and leave fixed the rest of the coordinates i.e. \((x^0, \ldots, x^5)\). Thus action of \(\mathbb{Z}_2\) behaves as an \(A_1\) ALE singularity in the coordinates \((x^6, x^7, x^8, x^9)\). One can redefine complex coordinates \(z_1 = x^6 + i x^7\) and \(z_2 = x^8 + i x^9\) and write down this singularity as \(\mathbb{C}^2/\mathbb{Z}_2\). This is, of course, the most simple non-trivial example of the most simple class of singularities of the \(A\)-\(D\)-\(E\)-type. In order to see how the brane pairs are behaved at orbifold singularities we need to know how the orbifold projection is realized.
on the Chan-Paton sector. As was shown in [10], the Chan-Paton $I$ is even with respect to $Z_2$, while the Chan-Paton factor $\sigma_1$ is odd under $Z_2$-orbifold projection. This is essentially as transforms the tachyon field under the generator $(-1)^F$ in [13]. Thus the tachyon is odd under the orbifold projection. We can immediately generalize the $Z_2$ group to $A_{N-1}$ singularities generated by the group $Z_N$ or for a generic group $\Gamma_G$ of the A-D-E singularities.

We know from the basic theory of D-branes in orbifold singularities [22,23] that the action of $\Gamma_G$ on the Chan-Paton factors is given by $g(x(i)) = x(\gamma(g)(i))$ where $g$ is an element of $G$, $\gamma(g)$ of $\Gamma_G$ and $i$ is the index on the Chan-Paton factors, which for the tachyon is $\sigma_1$. The tachyon field transforms under $\Gamma_G$-orbifold projection as

$$g : T(x) \rightarrow \gamma(g)T(x')\gamma^{-1}(g).$$

Thus similar than the vector potential $A_\mu(x)$, the tachyon $T$ transforms in the adjoint representation of $\Gamma_G$ under orbifold projection.

### 3.3. Equivariant K-Theory Structure

In this subsection we shall show in detail that the natural language to deal with stable (non-)BPS branes in Type I and II superstring theory is, as suggested by Witten [13], the equivariant K-theory. Throughout the rest of the paper we limit ourselves to work with $\Gamma_G$ an abelian and discrete subgroup of some A-D-E group.

To be the most self-contained possible we first give some basic definitions of the theory of equivariant K-theory. For details of proofs we encourage the reader to consult [19,20]. Let $G$ be a Lie group, in general $G$ may be a topological group. Let $X$ be a differentiable manifold of finite dimension. $X$ is said to be $G$-manifold if there exist an smooth action of $G$ on $X$. A $G$-map between two $G$-manifolds is a smooth map which commutes with the action of the group $G$. Now consider a principal bundle $E$ over $X$ with canonical projection $\pi$. A $G$-principal bundle $E_G$ over the $G$-manifold $X$, is a $G$-map $\pi$ which carries fibres to fibres linearly and projects to the action over $X$. A $G$-homomorphism $E_G \rightarrow F_G$ between two $G$-bundles over $X$ is a map which is both a bundle homomorphism and a $G$-map. The $G$-isomorphism between two bundles $E_G \rightarrow F_G$ is a $G$-homomorphism with inverse and

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3 The group $G$ does not be confused with the generic A-D-E group.
the inverse is also a $G$-map. Classes of pairs of $G$-bundles have a ring structure known as the Grothendieck ring $K_G(X)$ or equivariant K-theory of $X$ \cite{19,20,21}.

When the group $G$ acts trivially on $X$, $K_G(X)$ factorizes just as the product $K_G(X) = K(X) \otimes R(G)$ where $K(X)$ is the ordinary K-theory ring and $R(G)$ is the ring of irreducible representation spaces of $G$. Let $V_i$ be the $i$-th component of the irreducible vector space representation. For trivial actions $E_G$ can be written as $E_G = \bigoplus_i \text{Hom}_G(E_i, E) \otimes E_i$ where $E_i$ is the trivial bundle $E_i = X \times V_i$.

One of the properties of $K_G(X)$ we will use in this paper is the concept of space of sections of a $G$-bundle or $G$-section space. Let $C(E)$ be the space of sections of an ordinary bundle $E$ over $X$, $C(E)$ has the structure of a vector space and if $X$ is compact $C(E)$ has in addition the structure of a Banach space. Now in the case of we have a $G$-bundle $E_G$, in addition we have an induced action of $G$ on its space of sections $C(E_G)$. This action is continuous an it is given by $(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x)$ with $g \in G$ and $s \in C(E_G)$. To see that this action is continuous consider the sequence of maps: $G \times C(E_G) \times X \to G \times C(E_G) \times X \to G \times E_G \to E_G$ given by $(g, s, x) \mapsto (g, s, g^{-1}x) \mapsto (g, s(g^{-1}x)) \mapsto (g \cdot s)(x)$. Each map is continuous and as composition of continuous mappings is also continuous therefore this action is continuous. Action of $G$ over $C(E_G)$ induces the group homomorphism $J: G \to \text{Aut}(C(E_G))$.

Let $X/\Gamma_G$ be the ten-dimensional spacetime orbifold and let $W/\Gamma_G$ be a $\Gamma_G$-submanifold (of dimension $p + 1$) of $X/\Gamma_G$. Let $j: W/\Gamma_G \hookrightarrow X/\Gamma_G$ be an embedding of $W/\Gamma_G$ into $X/\Gamma_G$ which preserves the action of $\Gamma_G$. Branes or antibranes can be now wrapped on $W/\Gamma_G$. Thus the world-volume spectra on $W/\Gamma_G$ come from the dimensional reduction of the corresponding string theory in ten dimensions with an additional orbifold projection. The spectra can thus be encoded in a quiver diagram as usual in brane theory in orbifolds \cite{22,23,26,27}. The theory on the world-volume $W$ of $N$ branes (or antibranes) wrapped on $W$ is described through Chan-Paton bundles which are given by $\prod_{\mu} U(Nn_{\mu})$ bundles $E$ over $W$, in the case of Type II superstring theory and by $\prod_{\mu} \text{SO}(Nn_{\mu}) \times \prod_{\mu} \text{SU}(Nn_{\mu})$ for the Type I theory, here $n_{\mu}$ is the dimension of the $R_{\mu}$ regular representation of $\Gamma_G$. Gauge fields from the vector multiplet defines a connection $A$ on the corresponding Chan-Paton bundle. This connection satisfies, as usual, the orbifolding condition $A(x) = \gamma(g)A(x')\gamma^{-1}(g)$, with $\gamma(g) \in \Gamma_G$. Just as in the smooth case, GSO projection cancels tachyonic degrees of freedom leaving only the quiver structure of the vector multiplet and hypermultiplets.
When we consider both coincident branes and anti-branes wrapping on $W/\Gamma_G$ tachyon is still preserved by GSO projection (as we have seen in the above subsection) and the vector multiplet is also projected out as the smooth case. In other words, in non-pathological cases, $\Gamma_G$-action commutes with GSO projection. As we saw in the above subsection tachyon field $T$ is required to satisfy the condition $T(x) = \gamma(g)T(x')\gamma^{-1}(g)$ with $\gamma(g)$ being an element of $\Gamma_G$.

In terminology of equivariant K-theory, orbifold spacetime $X/\Gamma_G$ are seen as a $\Gamma_G$-space i.e. spacetime $X$ with the action of $\Gamma_G$ over $X$. Chan-Paton bundles are $\Gamma_G$-bundles $E_{\Gamma_G}$ over $X$ where the projection map preserves the action of group $\Gamma_G$. The tachyon field $T$ can be seen as a $\Gamma_G$-map bundle $T: F_{\Gamma_G} \to E_{\Gamma_G}$. Equivalently, tachyon can be seen now as a $\Gamma_G$-section (up to a constant) of the $\Gamma_G$-bundle $(E \otimes F^*)_{\Gamma_G}$. The way that tachyon field transforms under $\Gamma_G$ given in (3.3), is nicely explained by a consequence of $\Gamma_G$-section definition as follows.

Let $\mathcal{C}((E \otimes F^*)_{\Gamma_G})$ be the space of $\Gamma_G$-sections defined as the space of sections of the $\Gamma_G$-bundle $(E \otimes F^*)_{\Gamma_G}$. The group $\Gamma_G$ acts on $\mathcal{C}((E \otimes F^*)_{\Gamma_G})$ given by

$$(\gamma(g) \cdot T)(x) \equiv \gamma(g) \cdot T(\gamma^{-1}(g) \cdot x) = \gamma(g) \cdot T \cdot \gamma^{-1}(g)(x) \tag{3.4}$$

with $\gamma(g)$ and $T$ being elements of $\Gamma_G$ and $\mathcal{C}((E^* \otimes F)_{\Gamma_G})$ respectively and $x \in X$. The action of $\Gamma_G$ on sections $\mathcal{C}((E \otimes F^*)_{\Gamma_G})$ induces also a group homomorphism between $\Gamma_G$ and $\text{Aut}(\mathcal{C}((E \otimes F^*)_{\Gamma_G}))$.

When one is considering pairs of $9 - \overline{9}$ branes on orbifold singularities similar statements of the smooth case [13] are still valid. Creation and annihilation of $9$-branes in orbifolds is described by a $\Gamma_G$-bundle $H_{\Gamma_G}$ over $X$. So, still conservation of total charge, now including mirror images under $\Gamma_G$, leads to make the identification of pairs of bundles $(E_{\Gamma_G}, E_{\Gamma_G})$ with $(E_{\Gamma_G} \oplus H_{\Gamma_G}, E_{\Gamma_G} \oplus H_{\Gamma_G})$, as two equivalent descriptions. These conditions determine precisely an element of the $\Gamma_G$-equivariant K-group $K_{\Gamma_G}(X)$ of spacetime $X$. Thus brane-antibrane configurations in spacetime orbifold $X/\Gamma_G$ can be described in terms of equivariant K-theory methods. Thus D-brane charges on an orbifold singularity takes values in the equivariant K-theory group of the spacetime $X$ as suggested in [13].

In the specific case of Type IIB theories on orbifold singularities, tadpole cancellation leads, just as the smooth case [13], to the same number $N_1$ of $9$-branes and $N_2$ of $\overline{9}$-branes. The reason of this can be seen from the condition of equality of ranks of the gauge groups.
it is immediate to get $N_1 = N_2 = N$. Thus Chan-Paton bundles $E_{\Gamma_G}$ and $F_{\Gamma_G}$ are \( \prod_{\mu} U(Nn_{\mu}) \) gauge bundles and tadpole cancellation condition implies that virtual dimension is zero and it lies in equivariant reduced K-theory group \( \tilde{K}_{\Gamma_G}(X) \).

In Type I string theory 9−9 pairs on orbifolds are described by a class of pairs \((E_{\Gamma_G}, F_{\Gamma_G})\) of \( \prod_{\mu} \text{SO}(N_1 n_{\mu}) \) and \( \prod_{\mu} \text{SO}(N_2 n_{\mu}) \) gauge bundles over \( X \). Creation-annihilation is now described through the \( \prod_{\mu} \text{SO}(kn_{\mu}) \) bundle \( H \) over \( X \) for some \( k \). In Type I theories tadpole cancellation condition is given by \( \sum_{\mu} N_1 n_{\mu} - \sum_{\mu} N_1 n_{\mu} = \sum_{\mu} 32n_{\mu} \). Thus condition \( N_1 - N_2 = 32 \) holds again in this situation. Equivalence class of pairs of bundles \((E_{\Gamma_G}, F_{\Gamma_G})\) determines an element in the real equivariant K-theory group \( K_{\text{cpt}} \Gamma_G(X) \). Tadpole cancellation \( N_1 - N_2 = 32 \), newly turns out into equivariant reduced real K-theory group \( \tilde{K}_{\text{cpt}} \Gamma_G(X) \).

Vacuum at infinity is \( \Gamma_G \)-invariant, that means that it is reached by tachyon condensation preserving such a property. The reason of this is that vacuum space configuration of \( T' \)'s is always written in an \( U(N) \times U(N) \) invariant way \[8\], thus \( U(N) \times U(N) \) is broken down to the product group \( \prod_{\mu} U(Nn_{\mu}) \times \prod_{\mu} U(Nn_{\mu}) \). The vacuum is reexpressed now in an invariant way in terms or the trace in the regular representation of \( \Gamma_G \). Equivalence to vacuum at infinity means that near infinity bundles \( E_{\Gamma_G} \) and \( F_{\Gamma_G} \) are \( \Gamma_G \)-isomorphic. As the smooth case, the vacuum might have some branes and therefore a non-zero class of \( K_{\text{cpt}} \Gamma_G(X) \) where our spacetime \( X \) is a non-compact orbifold with the generic form \( \mathbb{R}^k \times Q^{10-k}/\Gamma_G \). For Type I theory with \( \Gamma_G = \mathbb{Z}_N \) an abelian and discrete subgroup of \( \text{SU}(2) \) and \( Q = \mathbb{C}^2 \), tadpole cancellation is still 32 for the 9-brane charge. For fivebranes, tadpole constraints can be written in terms of \( \Gamma_G \)-invariant geometric terms. For instance, in the unbroken \( \text{SO}(32) \) theory, anomaly cancellation implies the existence of a certain number of fivebranes \( F \), which is \( F = 24 - N \) for \( N \) even, and \( F = 24 - N + \frac{1}{2} \) for \( N \) odd \[31\]. In general there is not a choice of \( F \) to cancel anomalies and the anomaly inflow mechanism from the bulk should be extended to theories of branes in orbifolds \[31\].

In physical applications descriptions of brane configurations in non-compact space-time, ordinary and reduced equivariant K-groups \( K_{\text{cpt}} \Gamma_G(X) \) and \( \tilde{K}_{\text{cpt}} \Gamma_G(X) \) are equivalent. And the reason of this is that the \( \Gamma_G \)-invariance of the vacuum at infinity is described by a non-zero class of pairs of \( \Gamma_G \)-bundles which is equivalent to the trivial class of pairs \( \sum_{\mu} N_1 n_{\mu} = \sum_{\mu} N_2 n_{\mu} \) \( \text{(3.5)} \).
of bundles which are $\Gamma_G$-isomorphic at infinity. This only is possible if the pair of $\Gamma_G$-bundles have equal rank. Thus virtual dimension is zero and ordinary and equivariant reduced K-groups describe the same physical situation.

Type IIA theory involves more subtle considerations worked out in [13]. It was argued by Witten in [13] that configurations of brane-antibrane pairs are classified by the K-theory group of spacetime with an additional circle space $S^1 \times X$. Equivariant K-theory of Type IIA theory not will be discussed in the paper, however we make only some few comments. K-theory group for type IIA configurations is $K_{\Gamma_G}(S^1 \times X)$. If $\Gamma_G = \mathbb{Z}_2$ and this group acts trivially on space $X$ and non-trivially on $S^1$ then $K_{\mathbb{Z}_2}(X)$ is isomorphic to $K(X \times S^1 / \mathbb{Z}_2)$. It is tantalizing to speculate that equivariant K-theory might be a relevant tool to generalize the K-theoretical description of non-BPS branes to $E_8 \times E_8$ heterotic superstring theory.

3.4. Lower Dimensional Branes on Orbifolds. Global Construction

Up to here we have described $9 - \mathfrak{g}$ pairs of branes in generic orbifold singularities. However to see that the equivariant K-theory classifies also lower dimensional branes than 9, we will attempt to implement to orbifolds the Witten’s construction [13].

For branes of lower dimensions in orbifold singularities in type II superstring theory the description is similar to that given in [13]. However for completeness we include here this discussion. The key argument here is that it is possible to construct $|\Gamma_G| \ p$-branes as the bound state of $2^{n-1} |\Gamma_G|$ pairs of $(p + 2n)$-branes and antibranes.

First we overview some notation. Let $Z_{\Gamma_G}$ be a $(p+1)$-dimensional (and codimension $2n$) closed orientable $\Gamma_G$-submanifold of $Y_{\Gamma_G}$. This latter is a $(p + 1 + 2n)$-dimensional $\Gamma_G$-submanifold of $X$. Let $\mathcal{L}_{\Gamma_G}$ be a $\Gamma_G$-bundle over $Y_{\Gamma_G}$ with $\Gamma_G$-section vanishing along $Z_{\Gamma_G}$. One can consider a set of $N_1$ branes and $N_2$ antibranes wrapped on $Z_{\Gamma_G}$ such that tachyon field $T$ is a $\Gamma_G$-section which transforms in the bifundamental representation $(\mathfrak{i}, \mathfrak{g})$ of the gauge group $\prod \mu U(N_1 n_\mu) \times \prod \mu U(N_2 n_\mu)$. Tadpole cancellation condition implies that tachyon field transforms in the adjoint representation $(\mathfrak{g}, \mathfrak{g})$ of the group $\prod \mu U(N n_\mu) \times \prod \mu U(N n_\mu)$.

Now introduce a $\Gamma_G$-bundle $\mathcal{W}_{\Gamma_G}$ over $Z_{\Gamma_G}$ with structure group $\prod \mu U(N n_\mu)$. As in the smooth case, bundles $\mathcal{L}_{\Gamma_G} \otimes \mathcal{W}_{\Gamma_G}$ and $\mathcal{W}_{\Gamma_G}$ determine an element of the equivariant K-group of $Z_{\Gamma_G}$. If $Z_{\Gamma_G}$ has trivial normal bundle, then the group $K_{\Gamma_G}(Z)$ admits an extension to $Y$ and we get $K_{\Gamma_G}(Y)$. If it does not occurs, then we use the $\Gamma_G$-invariance of the vacuum at infinity $T : \mathcal{W}_{\Gamma_G} \to \mathcal{L}_{\Gamma_G} \otimes \mathcal{W}_{\Gamma_G}$ described by a $\Gamma_G$-isomorphism of bundles.
on the boundary of a tubular neighborhood \( Z'_G \) of \( Z_G \) in \( Y_G \). Thus \( K_{G}(Z) \) extends to \( K_{G}(Y) \) by declaring that this \( \Gamma_G \)-isomorphism extends throughout \( Y_{G}' = Y_G - Z'_G \). If it does not occurs, then we apply the \( \Gamma_G \)-equivariant version [19,20] of the argument given in [3]. One extends rather the trivial bundle \( L_{G} \otimes \mathcal{W}_{G} \oplus H_{G} \) because it is \( \Gamma_G \)-isomorphic (with the isomorphism given by \( T_G \oplus 1 \)) to \( \mathcal{W}_{G} \oplus H_{G} \) over \( Y'_{G} \).

3.5. Examples

Type I zero and -1 Brane

We consider the zero brane of Type I superstring theory. This is described, according to subsection 3.4 by the group \( KO_{G} (S^9) \) or the equivariant compact support group \( KO_{cpt} (R^9) \). The transverse space to the zero brane is the spacetime orbifold \( R^9 / \Gamma_G \). Assume that the group \( \Gamma_G \) acts on the spatial coordinates only, \( R^9 \) leaving fixed the \( x^9 \) coordinate. Thus symbolically, \( \gamma (g) \cdot (x^9, \vec{x}) = (x^9, \gamma(g) \vec{x} \gamma^{-1}(g)) \), where \( \vec{x} = (x^1, \ldots, x^8) \). \( \vec{x} \) vector of the transverse space can be rewritten in terms of complex coordinates \( \vec{z} = (z_1, z_2, z_3, z_4) \in C^4 \) and to be more specific let \( \Gamma_G \) be the cyclic group \( Z_N \). Then choose an action of \( Z_N \) over \( C^4 \) to be \( (z_1, z_2, z_3, z_4) \to (\gamma(g) \vec{z} \gamma^{-1}(g)) = (e^{2\pi i / N} z_1, e^{-2\pi i / N} z_2, e^{2\pi i / N} z_3, e^{-2\pi i / N} z_4) \).

Thus actually we have the zero brane living at an orbifold singularity \( C^4 / \Gamma_G \). An element of \( KO_{G} (R^9) \) is given by a pair of trivial gauge bundles \((E_{\Gamma_G}, F_{\Gamma_G})\) over \( R^{10} \). The tachyon field \( T \) is a \( \Gamma_G \)-isomorphism near infinity and it is given by \( T(x) = \sum_{i=1}^{9} \Gamma_i x^i \).

Separation of coordinates into \((x^9, \vec{x})\) breaks explicitly the symmetry \( SO(9) \) down to \( SO(8) \). Under this decomposition tachyon field reads

\[
T = \begin{pmatrix}
\vec{\Gamma} \cdot \vec{x} & x^9 \\
-x^9 & \vec{\Gamma}^T \cdot \vec{x}
\end{pmatrix}
\]

(3.6)

where \( \vec{\Gamma} \) are the \( SO(8) \) gamma matrices satisfying \( \vec{\Gamma} : S^+_{\Gamma_G} \to S^+_{\Gamma_G} \), \( \vec{\Gamma}^T : S^+_{\Gamma_G} \to S^-_{\Gamma_G} \). \( \vec{\Gamma}^T \) is transpose to \( \vec{\Gamma} \) and \( S^\pm_{\Gamma_G} \) are the positive and negative \( SO(8) \) chirality \( \Gamma_G \)-spinor bundles over the unidimensional space. Each \( S^\pm_{\Gamma_G} \) bundle can be trivially extended over \( R^9 \). The system splits in two components: \( T_1 = \vec{\Gamma} \cdot \vec{x} \) and \( T_2 = \vec{\Gamma}^T \cdot \vec{x} \). Recall that \( \vec{x} \) are the coordinates of the orbifold and the action of \( \Gamma_G \) on \( \vec{x} \) will depends on the explicit group \( \Gamma_G \). \( S^\pm_{\Gamma_G} \) are \( \Gamma_G \)-bundles and the \( SO(8) \) Dirac matrices \( \vec{\Gamma} \) can be seen as \( \Gamma_G \)-sections of the \( \Gamma_G \)-bundle \((S^- \otimes S^+_{\Gamma_G})_{\Gamma_G} \) and according to the transformation of \( \Gamma_G \)-sections, they transform as \( \gamma(g) \vec{\Gamma} \gamma^{-1}(g) \) with \( \vec{\Gamma}_1 = \vec{\Gamma} \) and \( \vec{\Gamma}_2 = \vec{\Gamma}^T \). Under the combination \( T_A = \vec{\Gamma}^A \cdot \vec{x} \), the
transformation of \( \vec{x} \) and \( \Gamma^T_G \) implies that the tachyon field transforms correctly as given from Eq. (3.4)

\[
T_A \to \gamma(g)T_A\gamma^{-1}(g), \quad A = 1, 2.
\] (3.7)

Similar to the smooth case \( T_1 \) describes a set of \( |\Gamma_G| \) D1-branes located at an orbifold singularity at the origin of \( \vec{x} \) and \( T_2 \) describes the other set of \( |\Gamma_G| \) D1-antibranes located at the origin of the orbifold singularity. This orbifold is, as we have seen, an orbifold of fourfolds given by \( \mathbb{C}^4/\Gamma_G \).

On the other hand the study of the -1 brane in an orbifold singularity is very similar to the analysis for the zero brane. In this case the system is classified by \( K_{\Gamma_G}(\mathbb{R}^{10}) \) with compact support. Now, there are two \( \Gamma_G \)-spinor bundles \( S^\pm_{\Gamma_G} \) coming from the real spinor representations of \( \text{SO}(10) \). Action of \( \Gamma_G \) on \( \vec{x} = (x^1, \ldots, x^{10}) \) determines a singularity of the type \( \mathbb{C}^5/\Gamma_G \) and thus there is only one tachyon field \( T = \vec{\Gamma} \cdot \vec{x} \).

3.6. **Equivariant Bott Periodicity**

Equivariant Bott periodicity for \( KO_{\Gamma_G} \)-theory with compact support is given by

\[
KO_{\Gamma_G}(\mathbb{R}^n) = KO_{\Gamma_G}(\mathbb{R}^{n+8}).
\] (3.8)

This formula tell us that \( |\Gamma_G| \) 1-branes and \( |\Gamma_G| \) 7-branes or \( |\Gamma_G| \) 0-branes and \( |\Gamma_G| \) 8-branes in the orbifold singularity, are related. In order to see that consider a \((n-1)\)-brane wrapped on \( \mathbb{R}^n \) and let \( j : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+8} \) be an embedding. \( n-1 \) brane is living in an orbifold singularity of the type \( \mathbb{R}^8/\Gamma_G \), being the coordinates \( \vec{x} \) of \( \mathbb{R}^8 \) given by the last eight coordinates of \( \mathbb{R}^{n+8} \). Let \( (E^{(0)}_{\Gamma_G}, F^{(0)}_{\Gamma_G}) \) be an element of \( KO_{\Gamma_G}(\mathbb{R}^n) \). Tachyon field is a \( \Gamma_G \)-map of bundles \( T_0 : F^{(0)}_{\Gamma_G} \to E^{(0)}_{\Gamma_G} \). The embedding \( j \) has an associated normal bundle with structure group \( \text{SO}(8) \). This have two \( \Gamma_G \)-spinor bundles \( S^\pm_{\Gamma_G} \) of positive and negative chiralities associated with the normal bundle. Using the methods of subsection 3.4 we can extend the above pair of bundles over \( \mathbb{R}^8/\Gamma_G \) and thus to get a class of pairs of bundles of \( KO_{\Gamma_G}(\mathbb{R}^{n+8}) \) given by \( (E^{(0)}_{\Gamma_G} \otimes (S^+_{\Gamma_G} \otimes S^-_{\Gamma_G}), F^{(0)}_{\Gamma_G} \otimes (S^+_{\Gamma_G} \otimes S^-_{\Gamma_G})) \) over \( \mathbb{R}^{n+8} \). The associated tachyon field is given by

\[
T = \begin{pmatrix}
\vec{\Gamma} \cdot \vec{x} \\
-T_0 \\
\vec{\Gamma}^T \cdot \vec{x}
\end{pmatrix}.
\] (3.9)
As the above subsection, the system splits in two tachyonic components transforming under the regular representation of $\Gamma_G$ and the full tachyonic field will transform under the 2-dimensional fundamental representation of $\Gamma_G$. Thus we have found that Bott periodicity relating branes in string theory is still valid when we sit them on a generic orbifold singularity relating now not only these branes but also their $|\Gamma_G|$ mirror images under $\Gamma_G$.

Bott periodicity in its equivariant version for complex K-theory is given by the natural isomorphism \[ K^{-q}_G(X) \cong K^{-q-2}_G(X) \] (3.10)

where $K^{-q}_G(X)$ is defined by $K_G(\Sigma^q X)$ with $\Sigma^q X$ the $q$-th reduced suspension of $X$. Thus equivariant Bott periodicity for complex K-theory has, as the ordinary Bott periodicity, periodicity equal to 2. Thus for Type IIB superstring theories there are lower dimensional BPS-branes each two dimensions and Bott periodicity identifies the branes with dimension $q$ with branes with dimension $q + 2$. With $G = \Gamma_G$ essentially the same thing occurs when Type IIB branes are placed in orbifold singularities of the type $\mathbb{C}^k/\Gamma_G$ with $k = 2, 3, 4$. As we have seen, the interpretation of equivariant Bott periodicity theorem is the identification of sets of $|\Gamma_G|$ $q$-branes with $|\Gamma_G|$ $(q + 2)$-branes with $|\Gamma_G|$ the number of mirror images in the orbifold. Later at the end of the Section 4 we will come back to this point.

4. Fivebranes, Small Instantons and Theories in Six Dimensions

As we have reviewed in Type I and IIB is possible to construct some stable D-branes as a bound state of certain number of $9 - \bar{5}$ pairs. In this section we study the description of stable supersymmetric Type I and IIB fivebranes in a background of nine-brane pairs as worked out in [13]. Now we sit the fivebrane in an ALE orbifold singularity $\mathbb{C}^2/\Gamma_G$ in the transverse space of the fivebrane. We have choice an ALE singularity in order to connect with some previous results well known from the literature but in principle its generalization to other orbifold singularities abelian or non-abelian is immediate. Our main claim is that the description of fivebrane from $9 - \bar{5}$ brane pairs in orbifold singularities are classified by equivariant K-theory group $K_G(X)$ with $G = \Gamma_G$. We present evidence of this by deriving the relevant information to describe non-trivial RG fixed points on the fivebrane world-volume theory, through the equivariant K-theory formalism.
4.1. Fivebrane From Nine-brane Pairs

As was shown in [13], fivebranes can be interpreted in terms of nine-brane pairs. In Type I theory, instantons are precisely constructed from bound states of fivebrane and ninebranes [33,34]. We consider four pairs of $9 - \bar{9}$ branes and one fivebrane in Type I theory. This configuration describes a small instanton of charge one and gauge group $SU(2) = Sp(1)$. Instanton charge takes values in the group $K(S^4)$ or $K(R^4)$ with compact support with $R^4$ parametrizing the transverse directions to the fivebrane. This system is equivalently described by the global description of $Z = S^4$. It is easy to see that the normal bundle $N$ to $S^4$ in $R^{10}$ is a bundle with structure group $SO(2n)$ with $n = 3$. Thus following the global constructions reviewed in Section 2, the fivebrane can be constructed from $2^{n-1}$ pairs of 9-branes i.e. four pairs of 9-branes. Normal bundle is a spin bundle and there is a pair of spinor bundles $S_{\pm}$ associated to the positive and negative chiralities of $SO(6)$. Tachyon field is given by $T = \vec{\Gamma} \cdot \vec{x}$ with $\vec{x} = (x_1, \ldots, x_6)$ are a point of the fivebrane world-volume. Tachyon field breaks the group $SO(4) \times SO(4)$ to the diagonal $SO(4)$, which is isomorphic (locally) to $SU(2) \times SU(2)$, this group is broken by the instantons to $SU(2)$.

One can attempt an immediate generalization of the last paragraph. Consider a set of $k$ fivebranes constructed from $N = 2^{n-1}k$ pairs of 9–$\bar{9}$ branes. Following [33], one can construct easily the configuration of $k$ instantons with gauge group $SO(N)$ with $N = 4k$. Later in subsection 4.5 we will discuss with detail the K-theoretic description of the small instanton in an ALE singularity.

4.2. Type IIB Instanton in an ALE Orbifold Singularity

We now consider fivebranes located at general A-D-E type orbifold singularities of Type IIB theory. Orbifold singularity we deal is of the form $R^4/\Gamma_G$ where $\Gamma_G$ is a discrete and abelian subgroup of $SU(2)$. Consider a configuration consisting of $k$ fivebranes whose world-volume is parametrized by the coordinates $(x^0, \ldots, x^5)$ located at a fixed point of the orbifold singularity $C^2/\Gamma_G$, parametrizing the normal directions to the fivebrane $(x^6, x^7, x^8, x^9)$. Fivebrane charge takes values in the equivariant group $K_{\Gamma_G}(S^4)$ or equivalently in $K_{\Gamma_G}(R^4)$ with compact support. Moreover, the relevant group to compute it is rather the K-theory group $K(Z_\zeta)$ of the ALE space $Z_\zeta$, with $Z_\zeta$ being isomorphic to the minimal resolution of $R^4/\Gamma_G$ [28].
4.3. Relation to Kronheimer-Nakajima Construction

In Type IIB theories we have seen that an element of the equivariant K-theory group is given by a class of pairs of $\Gamma_G$-gauge bundles $(E_{\Gamma_G}, F_{\Gamma_G})$ both are $G'$-bundles over $Z_\zeta$, with $G' = \prod_{\mu=0}^r \mathbb{U}(N_n)/\mathbb{U}(1)$ where $r = \text{rank}(G)$ and $G$ is some A-D-E group: $A_r$, $D_r$, $E_6$, $E_7$ and $E_8$. Group $\Gamma_G$ has the regular irreducible representation $R$. With the action of $\prod_{\mu=0}^r \mathbb{U}(N_n)$ over $R$, one can construct the associated $\Gamma_G$-vector bundle given by $R_i - E_{\Gamma_G} \times_{G'} R \to Z_\zeta$. Similarly for $F_{\Gamma_G}$ we can construct a $\Gamma_G$-vector bundle $R_i - F_{\Gamma_G} \times_{G'} R \to Z_\zeta$. One can immediately see that vector bundle $\mathcal{E}_{\Gamma_G} = E_{\Gamma_G} \times_{G'} \mathcal{R}$ it is very well known from the literature [29] and it is known as the tautological bundle. Thus we have found that equivariant K-group determining the charge of $k$ fivebranes in an ALE orbifold singularity is equivalently described as a class of pairs of tautological bundles arising in Kronheimer-Nakajima construction of instantons [29] i.e. $K_{\Gamma_G}(Z_\zeta) = (\mathcal{E}_{\Gamma_G}, \mathcal{F}_{\Gamma_G})$. These bundles are $\Gamma_G$-modules given by $\mathcal{E}_{\Gamma_G} = \oplus_{\mu=0}^r E_{\Gamma_G} \times_{G'} R_\mu$ where $R_\mu = (R_0, \ldots, R_r)$ are the irreducible regular representations of $\Gamma_G$, $\mathcal{E}_{\Gamma_G} \times_{G'} R_\mu$ and $R_\mu$ is the trivial bundle $R_\mu = Z_\zeta \times R_\mu$ over $Z_\zeta$ with fiber $R_\mu$.

In order to extract physical information it is needed to describe the K-group $K_{\Gamma_G}(Z_\zeta)$ in terms of more familiar grounds. K-theoretical description can be usually translated in terms of cohomological language through the famous index theorem of elliptic operators [35]. To do this Atiyah and Singer used an alternative definition of equivariant K-theory in terms of complexes of $G$-bundles over $G$-spaces [13, 20]. To proceed further we need a tubular neighborhood of $Z_\zeta$ and the Thom isomorphism. Tubular neighborhood $Z'_\zeta$ of $Z_\zeta$ in $\mathbb{R}^{10}$ is familiar for us from the above sections so we will focus on the Thom isomorphism.

One can construct the Thom isomorphism as follows: Start from the cotangent bundle $T^*Z_\zeta$, to $Z_\zeta$ with canonical projection $\pi$. The embedding of $Z_\zeta$ into regular representation vector space $\mathcal{R}$ induces an embedding of $T^*Z_\zeta$ into $T^*\mathcal{R}$ and so the Thom isomorphism is given by $\psi : K_{\Gamma_G}(T^*Z_\zeta) \to K_{\Gamma_G}(\pi^*(Z'_\zeta \otimes_{\mathbb{R}} \mathbb{C}))$. Also there is an embedding $i : \pi^*(Z'_\zeta \otimes_{\mathbb{R}} \mathbb{C}) \hookrightarrow T^*\mathcal{R}$ and its corresponding inducing map $i_* : K_{\Gamma_G}(\pi^*(Z'_\zeta \otimes_{\mathbb{R}} \mathbb{C})) \to K_{\Gamma_G}(T^*\mathcal{R})$. Thus one can construct the Gysin map $i! : K_{\Gamma_G}(T^*Z_\zeta) \to K_{\Gamma_G}(T^*\mathcal{R})$ as $i! = i_* \circ \psi$. The inclusion of a point $P$ into $\mathcal{R}$ induces similarly the map $j! : K_{\Gamma_G}(T^*P) \to K_{\Gamma_G}(T^*\mathcal{R})$. Thus $\Gamma_G$-equivariant topological index $\text{ind}_t : K_{\Gamma_G}(T^*Z_\zeta) \to R(\Gamma_G)$ is defined as $\text{ind}_t = ((j)^{-1} \circ i!)([\mathcal{O}_{\Gamma_G}])$ where $\mathcal{O}_{\Gamma_G}$ is a $\Gamma_G$-vector bundle over $T^*Z_\zeta$. This is precisely the well known Atiyah-Singer $\Gamma_G$-index formula [35]. Equivariant Chern character $ch_{\Gamma_G} : K_{\Gamma_G}(Z_\zeta) \to H_{\Gamma_G}(Z_\zeta; \mathbb{Q})$ is an useful tool to translate the index formula in cohomological
G has to be modified to \( \Gamma \rightarrow W \) operators \( \eta \) boundary term given by the classification coming from the limiting flat connection on the end of \( Z \).

As \( Z_\xi \) is a ALE manifold with non-trivial boundary \( S^3/\Gamma_G \), the \( \Gamma_G \)-index theorem has to be modified to \( \Gamma_G \)-index for manifolds with boundary [36,37]. For twisted Dirac operators \( D^\pm : \mathcal{C}(S^\pm \otimes \mathcal{E}_{\Gamma_G}) \rightarrow \mathcal{C}(S^\mp \otimes \mathcal{E}_{\Gamma_G}) \) the above equation has to be corrected by a boundary term given by the \( \eta \) invariant [29]

\[
\text{Ind}(D) = - \int_{Z_\xi} \text{ch}_{\Gamma_G}(\mathcal{E}_{\Gamma_G}) \hat{A}(Z) + \frac{1}{|\Gamma_G|} \sum_{\gamma \neq 1} \chi_{\mathcal{W}(\gamma)} \frac{\chi_{\mathcal{Q}(\gamma)}}{2 - \chi_{\mathcal{Q}(\gamma)}}.
\]

This formula determines the dimension of the moduli space of instantons in ALE gravitational spaces [29]. In cohomological terms the use of characteristic classes gives relevant information about the non-trivial submanifolds of the ALE space. For instance the first Chern class \( c_1(\mathcal{E}_{\Gamma_G}) \) of \( \mathcal{E}_{\Gamma_G} \) with \( \mu = 1, \ldots, r \) form a basis of \( H^2(Z_\xi) \) [29]. Poincaré duality determines \( r \) non-trivial homology cycles \( \Sigma_\mu = (\Sigma_1, \ldots, \Sigma_r) \). The representation \( R_\mu \) is strongly associated to the monodromy of \( \mathcal{E}_{\Gamma_{G}} \) at the end of \( Z_\xi, Z_\infty \). In other words \( \pi_1(Z_\infty) = \Gamma_G \) and consequently there are non-trivial Wilson loops at infinity providing a representation of \( \Gamma_G \) into the gauge group.

Kronheimer-Nakajima construction of instantons on ALE spaces [23], requires the construction of \( \Gamma_G \)-vector bundle \( \mathcal{J}_{\Gamma_G} \) over \( Z_\xi \) with anti-self-dual connection, from some ADHM data. This bundle is of course a generalization of our friend \( \mathcal{E}_{\Gamma_{G}} \). Among these data there are two complex vector spaces \( V \) and \( W \) which are \( \Gamma_G \)-modules and can be written as \( V = \bigoplus_{\mu=0}^r V_\mu \otimes R_\mu \) and \( W = \bigoplus_{\mu=0}^r W_\mu \otimes R_\mu \), of dimensions \( k = \dim(V) = \sum_\mu n_\mu v_\mu \) and \( N = \dim(W) = \sum_\mu n_\mu w_\mu \) respectively. The mentioned bundle is given by \( \mathcal{J}_{\Gamma_G} = \text{Ker}(D^{\dagger}_{\Gamma_G}) = \text{Coker}(D_{\Gamma_G}) \) where \( D^{\dagger}_{\Gamma_G} : \mathcal{V} \oplus \mathcal{W} \rightarrow \mathcal{U} \) where \( \mathcal{V} = (\mathcal{Q} \otimes \mathcal{V} \otimes E)_{\Gamma_G} \), \( \mathcal{W} = (\mathcal{W} \otimes E)_{\Gamma_G} \) and \( \mathcal{U} = S^+ \otimes (\mathcal{V} \otimes E)_{\Gamma_G} \).

The vector \( \vec{w} = (w_0, \ldots, w_r) \) contains information about the monodromy representation coming from the limiting flat connection on the end of \( Z_\xi \). In this limit the bundle \( \mathcal{J} \) is given by \( \mathcal{J}_\infty = \bigoplus_{\mu=0}^r w_\mu R_\mu \).

The first Chern class of \( \mathcal{J}_{\Gamma_G} \) is given by \( c_1(\mathcal{J}_{\Gamma_G}) = c_1(\mathcal{V}) + c_1(\mathcal{W}) - c_1(\mathcal{U}) = \sum_{\mu=0}^r u_\mu c_1(\mathcal{E}_{\Gamma_{G}}) \) with \( u_\mu = w_\mu - \sum_\nu \widetilde{C}_{\mu\nu} v_\nu \) where \( \widetilde{C}_{\mu\nu} \) is the extended Cartan matrix \( \widetilde{C}_{\mu\nu} = 2\delta_{\mu\nu} - a_{\mu\nu} \) and \( a_{\mu\nu} \) is the matrix constructed from the extended Dynkin
diagram of the corresponding A-D-E group. $a_{\mu\nu}$ enters in the representation formula
\[ R_Q \otimes R_\mu = \bigoplus_{\mu=0}^r a_{\mu\nu} R_\nu \]
with $R_Q$ the fundamental two-dimensional representation of $\Gamma_G$.

It is convenient rewrite the resulting formula from first Chern class for future reference as
\[ \tilde{C}_{\mu\nu} v_\nu = w_\mu - u_\mu. \] (4.3)

While the second Chern class has two contributions: the ordinary contribution due to the non-trivial topology of $Z_\zeta$ and that of the asymptotic behavior. Thus second Chern class is given by
\[ c_2(J_{\Gamma_G}) = \sum_\mu u_\mu c_2(\mathcal{E}_{\Gamma_G} \mu) + \frac{k}{|\Gamma_G|}. \]
The integrals of $c_2(J_{\Gamma_G})$ determines the instanton number $I = k(J_{\Gamma_G}) = \sum_\mu u_\mu k(\mathcal{E}_{\Gamma_G} \mu) + (\text{dim}V)/|\Gamma_G|$. 

Finally, the moduli space $\mathcal{M}_{k,N}$ of $k U(N)$ instantons is the space of anti-self-dual Yang-Mills instantons on an ALE space. Construction of this moduli space and uniqueness was proved using the ADHM data by Kronheimer and Nakajima [29]. The dimension of $\mathcal{M}_{k,N}$ can be determined by the Atiyah-Patodi-Singer for manifolds with boundary [30,37]. The result is $\text{dim}(\mathcal{M}_{k,N}) = \frac{1}{2} v_\mu (w_\mu + u_\mu)$.

4.4. Type IIB Six-dimensional Gauge Theories From Fivebranes in ALE Orbifolds

Six-dimensional theories on the world-volume of a type IIB $k$ coincident fivebranes in a general A-D-E ALE singularity and in a background of $N$ 9-branes, have many interesting properties [24,25]. One of the most interesting is the existence of non-trivial RG fixed points. Some results are known but it remains to explore other descriptions of these points using new techniques. Important dynamical information is extracted from these points, which are described by a supersymmetric gauge theories with eight supercharges and gauge group $\prod_{\mu=0}^r U(kn_\mu)$. The theory without 9-branes has $\mathcal{N} = (1,0)$ supersymmetry and their matter multiplets transforms as $\frac{1}{2} a_{\mu\nu}(\square_\mu, \square_\nu)$. In addition there are $r$ $\mathcal{N} = (1,0)$ hypermultiplets and $r$ $\mathcal{N} = (1,0)$ tensor multiplets. These come from the six-dimensional reduction of the ten-dimensional two-form and four-form potentials on the $r$ two-dimensional cycles $\Sigma_\mu = (\Sigma_1, \ldots, \Sigma_r)$ of $Z_\zeta$.

The Higgs branch of this theory is shown to be isomorphic to the hyper-Kähler quotient construction of Yang-Mills instantons in ALE gravitational instantons [22,24,25]. First Chern class is realized by the integral of NS-NS $B$ in the non-trivial cycles $\Sigma_\mu$ as $\int_{\Sigma_\mu} B = n_\mu/|\Gamma_G|$ with $(\mu = 1, \ldots, r)$. In the first reference of [25], it was shown that for Type I and II theories, worldsheet tadpole cancellation condition for instanton configurations in ALE
spaces implies the cancellation of gauge anomalies of the six-dimensional theory on the world-volume of the fivebranes. Tadpole cancellation condition fivebranes in ALE spaces is shown to be contained in the topological and group information of the ALE singularity through the formula (4.3).

The six-dimensional theory on the fivebranes world-volume is anomaly free requiring exactly the \( r \) hypermultiplets and the \( r \) tensor multiplets to cancel anomalies. The anomaly coming from the \( r \) U(1) factors of \( \prod_{\mu=0}^{r} U(kn_{\mu}) \) are canceled by the \( r \) mentioned hypermultiplets. The other contribution to the anomaly is canceled by the coupling of the \( r \) tensor multiplets to the gauge fields. Thus a sensible description of these theories requires the full topological items coming from Kronheimer-Nakajima construction. From the point of view of equivariant K-theory, fivebranes configurations are completely classified by the group \( K_{\Gamma_G}(Z_\zeta) \) and as we have seen in this Section, an element of this group can be expressed in terms of a pair of tautological bundles providing thus a connection to Kronheimer-Nakajima construction. Thus all relevant topological information to describe anomaly free gauge theories in six-dimensions is contained as we have shown, in equivariant K-theory group \( K_{\Gamma_G}(X) \) as suggested by Witten in [13].

4.5. Type I Small Instantons and Six-dimensional Gauge Theories

Now we consider type I theory of \( k \) coincident fivebranes located at general ALE singularity \( \mathbb{R}^4/\Gamma_G \) and in a background of \( N \) pairs of \( 9 - \bar{9} \) branes. These configurations describe \( k \) SO(N) small instantons on the ALE space. The description of these configurations is in much as the above subsection. We only will comments the differences and make some remarks. In these type of theories there is two possible cases: the case with vector structure and the case without vector structure. We submit the reader to [23] for details. Here we only consider the case with vector structure.

In Type I theories we have seen that an element of the equivariant KO-theory group is given by a class of pairs of real \( \Gamma_G \)-gauge bundles \( (E_{\Gamma_G}, F_{\Gamma_G}) \) both are \( G' \)-bundles over \( Z_\zeta \), with gauge group \( G' = \prod_{\mu \in \mathcal{R}} \text{Sp}(v_{\mu}) \times \prod_{\mu \in \mathcal{P}} \text{SO}(v_{\mu}) \times \prod_{\mu \in \mathcal{C}} U(v_{\mu}) \) were \( \mathcal{R} \) denotes real, \( \mathcal{P} \) pseudoreal, \( \mathcal{C} \) complex and \( \overline{\mathcal{C}} \) complex conjugate, representations of \( \Gamma_G \). With the action of \( G' \) over \( R \), one can construct the associated vector bundle given by \( R \rightarrow E_{\Gamma_G} \times_{G'} R \rightarrow Z_\zeta \) and similar for \( F_{\Gamma_G} \). Once again the vector bundle \( E = E_{\Gamma_G} \times_{G'} R \) is the tautological bundle. Thus the equivariant K-group determining the charge of \( k \) fivebranes in an ALE
orbifold singularity is equivalently described as a class of pairs of tautological bundles from the Kronheimer-Nakajima construction of instantons \[29\].

KO-theoretical description can be translated in cohomological terms. Thus the basic formulas are still valid. First Chern class leads to \( \bar{C}_{\mu
u} V_{\nu} = w_{\mu} - D_{\mu} \) where \( V_{\mu} = \dim(\square_{\mu}) \) i.e. \( V_{\mu} = 2v_{\mu} \) for \( \text{Sp}(v_{\mu}) \) and \( V_{\mu} = v_{\mu} \) for \( \text{SO}(v_{\mu}) \) and \( \text{U}(v_{\mu}) \). Instantons on ALE space are characterized by the instanton number \( k \) which come from the Chern second class and by the vector \( \vec{w} = (w_0, \ldots, w_r) \) which characterize the Wilson loops at the end of \( Z_\zeta \) and satisfying the condition \( \sum_{\mu} n_{\mu} w_{\mu} = 32 \). The theory also possesses matter as hypermultiplets and tensor multiplets which are associated by phase transitions of the six-dimensional gauge theory \[25\].

Equivariant \( \Gamma_G \) K-theory groups provide once again all topological and group theoretical information to describe the Higgs and Coulomb branches of type I six-dimensional theory associated to small instantons. Actually this is not surprising, but what is interesting is that all relevant information is encoded from the beginning in the equivariant KO-theory groups. Thus we claim that K-theoretical tools enter deeply in the problem and may provide of a very useful tool to study and classify non-trivial RG fixed points of six-dimensional theories.

4.6. AdS/CFT Correspondence and Orbifold Singularities

In Type IIB string theory D-brane theories on orbifolds also can be realized in the context of the AdS/CFT correspondence \[38\]. It is well known the equivalence of superconformal field theories with various degrees of supersymmetry on the D-brane world-volume is equivalently described by Type IIB \( D(p-2) \)-branes in orbifolds \( \text{AdS}_p \times S^{10-p}/\Gamma_G \). The field theory on the world-volume of a D brane probes on singularities can be determined using open string techniques introduced in \[22,26\]. In particular for \( p = 5 \) one have a theory of Type IIB D3-branes with supersymmetry on the world-volume \( \mathcal{N} = 2, 1, 0 \) corresponding to \( \Gamma_G \) being a subgroup of \( \text{SU}(2), \text{SU}(3), \text{SU}(4) \) respectively \[39,40\]. Admissible values of \( p \) are given by the practice to be \( p = 3, 5, 7 \).

We focus for the moment on the case \( p = 5 \), this corresponds to \( \text{AdS}_5 \times S^5/\Gamma_G \). When \( \Gamma_G \) is a subgroup of \( \text{SU}(2) \) of the \( A_{N-1} \) type i.e. \( \Gamma_G = \mathbb{Z}_N \) then, the field theory on the world-volume of \( n_1 \) D3 branes corresponds to a superconformal field theory in four dimensions with \( \mathcal{N} = 2 \) supersymmetry and gauge group \( \text{U}(n_1)|\Gamma_G| \). For \( \Gamma_G \) a subgroup of \( \text{SU}(3) \) the supersymmetry on the world-volume theory of D3-branes is \( \mathcal{N} = 1 \) and
thus finite superconformal chiral gauge theories in four dimensions can be constructed \[39,40,41\]. Threebrane charge on the orbifold singularity will be enhanced by the orbifold projection by a factor of \(|\Gamma_G|\) and it takes values at \(K(\mathbb{R}^6/\Gamma_G)\) with compact support or \(K(\mathbb{S}^6/\Gamma_G)\). Gauge and chiral multiplets are determined by the orbifold projection of the spectrum into \(\Gamma_G\) invariant states. Thus the supersymmetry, spectra and interactions of the world-volume theory depends on the choice of the group \(\Gamma_G\). Similar descriptions are applied to theories in two-dimensions on the worldsheet of Type IIB D1-branes in orbifold singularities of the type \(\mathbb{C}^4/\Gamma_G\) \[27,42\]. Here the choice of the \(\Gamma_G\) corresponds with different enhanced worldsheet supersymmetries.

In general the brane charge of these systems take values at \(K(\mathbb{R}^{p+1}/\Gamma_G)\) with compact support or \(K(\mathbb{S}^{p+1}/\Gamma_G)\). Using equivariant Bott periodicity for complex groups one immediately find that these groups are non-zero for odd values of \(p\) and more precisely for \(p = 3, 5, 7\). Moreover just as in the previous case for the fivebrane, the additional structure of the \(\Gamma_G\) action of the equivariant K-theory provides the necessary tools to construct sensible gauge theories on the world-volume of D-branes.

D-brane theory in orbifold singularities might be generalized to more general singularities such as conifolds. In the context of AdS/CFT correspondence the field theory of branes in conifolds was originally discussed in \[43\] (and further studied for singular spaces at \[44\]), where the \(S^5/\Gamma_G\) is substituted by \(SU(2) \times SU(2)/U(1)\) and in general by an Einstein manifold \(X\). It would be interesting to pursue an appropriate description of the brane charges and their classification of supersymmetric D3-branes (coming from \(9 - \bar{9}\) pairs) in conifold singularities in terms of an suitable generalization of the equivariant K-theory.

5. Brane Charge and Equivariant K-theory

In this section we will discuss equivariant K-theory description of configurations of Type II D-branes in abelian ALE singularities. We give a formula which generalizes the formula to compute the RR charge found in \[4,5\]. First of all let us recall the Minasian and Moore result \[5\] concerning the basic formula for obtaining the RR charges is given by

\[ Q = ch(f_1E) \sqrt{A(TX)} \]  

(5.1)
where $E$ is a $U(N)$ Chan-Paton bundle over a $(p+1)$-dimensional submanifold $W$ of the spacetime $X$, $f_1$ is a K-theoretic Gysin map: $f_1 : K(W) \to K(X)$, $\hat{A}(TX)$ is the Dirac genus of $X$ and $ch : K(X) \to H^*(X)$ is the usual Chern character. Formula (5.1) was obtained from a matching between the anomaly contribution from the chiral fermions on the world-volume $W$ given by

$$I = \int_W c \wedge Y$$

(5.2)

and the inflow anomaly from the bulk in such a way that the ten-dimensional theory is anomaly free [3]. With the definition $ch(E) = \text{tr}_N(e^{F/2\pi})$ it was shown in [3] that $Y$ at (5.2) is given by $Y = ch(E)j^*\sqrt{\hat{A}(TX)}$ where $j : W \hookrightarrow X$ is an embedding.

At the presence of orbifold singularities $\mathbb{R}^{10-p}/\Gamma_G$, with $\Gamma_G$ an discrete and abelian subgroup of some Lie group, the above action is modified as [22]

$$I_W = \int_W c \wedge Y$$

$$= \int_W c \wedge \text{Tr}(\gamma(g)e^{F/2\pi})$$

(5.3)

where $\gamma(g)$ is an element of the discrete group $\Gamma_G$ and Tr is given by a product of traces of $\text{tr}_R(\gamma(g))$ and $\text{tr}_N(e^{F/2\pi})$.

This action is valid for the case of flat spaces but is has to be modified for submanifolds $W$ with non-trivial normal bundle [3]. The purpose of this section is to get such a modification. This formula as we shall see generalizes (5.1) and it is described appropriately by equivariant K-theory.

To be specific consider $W$ to be a six-dimensional cycle where $N$ fivebranes can be wrapped. Assume that these fivebranes are placed in an ALE singularity of the A-type and thus $\Gamma_G = \mathbb{Z}_N$.

In order to see the relation with the K-theory group $K_{\Gamma_G}(X)$ we consider the action of $\Gamma_G$ on the spacetime $X = W \times \mathbb{R}^4$. For the purpose of studying fivebranes sitting in ALE singularity in the transverse directions of the fivebrane world-volume, the action of $\Gamma_G$ on $W$ is trivial, acting only on $\mathbb{R}^4$ to get $X = W \times \mathbb{R}^4/\Gamma_G$. From the general theory of equivariant K-theory one have that $W$ becomes a $\Gamma_G$-trivial space [19,20,21]. Thus Chan-Paton bundle becomes a $\Gamma_G$-bundle and it is given by

$$E = \bigoplus_{\mu=0}^{r} E_\mu \otimes R_\mu$$

(5.4)

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where $E_\mu = Hom(R_\mu, E)$ and $R_\mu$ is the trivial bundle $W \times R_\mu$ with \{R_\mu\} the set of irreducible regular representations of $\Gamma_G$. In this orbifold singularity the gauge group is broken from $U(N)$ to $\prod_\mu U(Nn_\mu)$. Thus Chan-Paton bundle are obviously modified to be

$$\prod_{\mu=0}^{r} U(Nn_\mu) \to E \to W.$$  \hspace{1cm} (5.5)

Is well known that for the case of trivial actions, the group $K_{\Gamma_G}(W)$ decomposes just as the tensor product $K(W) \otimes R(\Gamma_G)$ where $R(\Gamma_G)$ is the ring of regular irreducible representations of $\Gamma_G$ \[19,20,21\].

On the other hand chiral fermions in intersections of two cycles $Z = W_1 \cap W_2$ leads to the well known world-volume gauge anomaly $A$ \[3,5\]. The only modification is in the Chern character

$$\mathcal{A} = ch_{\Gamma_G}(E_1)ch_{\Gamma_G}(E_2)\hat{A}(TZ)$$  \hspace{1cm} \hspace{1cm} (5.6)

where $E_1$ and $E_2$ are the Chan-Paton bundles over $W_1$ and $W_2$ respectively and $ch_{\Gamma_G}(E)$ is the equivariant Chern character given by

$$ch_{\Gamma_G}(E) = \text{Tr}(\gamma(g)e^{F/2\pi}).$$  \hspace{1cm} \hspace{1cm} (5.7)

If one rewrites the last equation in terms of properties of $W$ and $X$ only as in \[5\] this leads to the modification of $\bar{Y}$ in Eq. (5.3) as

$$\bar{Y} \rightarrow \bar{Y}' = \bar{Y}e^{\frac{2d}{j^*\hat{A}(TX)}}.$$  \hspace{1cm} (5.8)

Thus Eq. (5.3) should to be rewritten as

$$I_W = \int_W c \wedge ch_{\Gamma_G}(E)\hat{A}(TW) \cdot \frac{1}{j^*\hat{A}(TX)}.$$  \hspace{1cm} \hspace{1cm} (5.9)

Following the same procedure as \[5\] we lead finally to the definition of brane charge

$$Q = \int_G j^* \left( ch_{\Gamma_G}(E)\hat{A}(TW) \right) \cdot \frac{1}{j^*\hat{A}(TX)}.$$  \hspace{1cm} (5.10)

In the equivariant case it is still valid Atiyah-Hirzebruch theorem \[19,20\] and we get
\[ j_* \left( ch_{\Gamma G}(E) \hat{A}(TW) \right) = ch_{\Gamma G}(j_! E) \hat{A}(TX) \]  
(5.11)

where \( j_! : K_{\Gamma G}(W) \to K_{\Gamma G}(X) \) is a \( \Gamma_G \)-isomorphism induced by the embedding \( j : W \hookrightarrow X \). Thus the formula for the brane charge reads

\[ Q = ch_{\Gamma G}(j_! E) \sqrt{\hat{A}(TX)} \]  
(5.12)

where \( ch : K_{\Gamma G}(X) \to H^*(X; \mathbb{Q}) \otimes R(\Gamma_G) \). One can further proceed specifying \( \Gamma_G = \mathbb{Z}_N \) and the fact that for each \( \gamma(g) \in \Gamma_G \) we have the regular irreducible representation \( R_\mu \) and for each regular representation we have a character \( \chi : R(\Gamma_G) \to \mathbb{Z} \) given by \( \chi_\mu \equiv tr_R(R_\mu(g)) \). As we have seen \( E \) is of the form given by Eq. (5.4), thus equivariant Chern character is given by

\[ ch_{\Gamma G}(E) = ch_{\Gamma G} \left( \bigoplus_{\mu=0}^{r} E_\mu \otimes R_\mu \right). \]  
(5.13)

Using the fact that \( ch_{\Gamma G} \) is a ring homomorphism between \( K_{\Gamma G}(W) \) and \( H^*(W; \mathbb{Q}) \otimes R(\Gamma_G) \) we get

\[ ch_{\Gamma G}(E) = \sum_{\mu=0}^{r} ch(E_\mu) \chi_\mu \]  
(5.14)

where \( \chi_\mu = \chi(R_\mu) \) and \( ch(E_\mu) = tr_N(e^{F_\mu/2\pi}) \). Finally, action of Gysin map on the bundle \( j_! E \in K_{\Gamma G}(W) \) is

\[ j_! \left( \bigoplus_{\mu=0}^{r} E_\mu \otimes R_\mu \right) = \bigoplus_{\mu=0}^{r} j_!(E_\mu) \otimes R_\mu. \]  
(5.15)

Thus the formula for the brane charge in a \( \mathbb{C}^2/\Gamma_G \) orbifold singularity says that it takes values in the equivariant K-theory group \( K(X) \otimes R(\Gamma_G) \)

\[ Q = \sum_{\mu=0}^{r} ch(j_! E_\mu) \chi_\mu \sqrt{\hat{A}(X)}. \]  
(5.16)

This formula can be generalized for non-trivial action and thus obtaining that the brane charge can take values at \( K_{\Gamma G}(X) \).

Although we have computed the brane charge for ALE singularities, this procedure immediately generalizes to other singularities as \( \mathbb{C}^k/\Gamma_G \) with \( k = 3, 4, \) i.e. for threefolds and fourfolds singularities.
6. Concluding Remarks

In this paper we have further investigated (non-)BPS states on orbifold singularities using for that the language of equivariant K-theory. The idea of describing pairs of D-branes in orbifold singularities in terms of equivariant K-theory was originally argued by Witten in [13]. Mathematically the relation between the ordinary K-theory and the equivariant ones is the additional action of a group on the spacetime $X$ and the relevant Chan-Paton bundles. We have take this group action to be a discrete and abelian group action of $\Gamma_G$ on the spacetime $X$. We have shown that this mathematical framework describes and classifies correctly configurations of (non-)BPS branes in orbifold singularities. We have seen that several results of 9-branes in orbifold singularities are nicely reproduced in the K-theory formalism. Among them of particular interest is the result of [31], where inflow mechanism for branes on singularities is shown to be play an important role. We have also discussed how the incorporation of lower-dimensional branes than 9, follows a parallel treatment given in [13], in the language of equivariant K-theory. In addition in discussing examples we consider Type I zero and -1 branes on specific orbifold singularities. Moreover for Type I theory since Bott periodicity gives a correspondence between 0-branes 8-branes and -1-branes an 7-branes (with period equal to 8), the equivariant Bott periodicity determines a map between the corresponding branes and their associated mirror images produced by the orbifold projection.

We have also discussed the description of fivebranes in orbifold singularities of the ALE type, for Type I and IIB superstring theories. For Type IIB theory we have placed coincident fivebranes at a point of an ALE singularity $\mathbb{C}^2/\Gamma_G$ in the transverse directions of the fivebranes. Following Witten results for the fivebrane [13], the fivebrane charge is determined by the equivariant K-theory group $K_{\Gamma_G}(Z_\zeta)$ of the ALE space $Z_\zeta$ coming from the minimal resolution of the ALE singularity $\mathbb{C}^2/\Gamma_G$. This system is completely determined by a pair of gauge bundles over $Z_\zeta$. Description of these pair in terms of associated pair of vector bundles leads naturally to identify an element of $K_{\Gamma_G}(X)$ with a pair of tautological vector bundles arising in the Kronheimer-Nakajima construction of Yang-Mills instantons on ALE spaces [29]. This relation is not a coincidence as we have shown using equivariant K-theory to get all relevant topological information which enters in the description of non-trivial RG fixed points of six-dimensional theories in ALE singularities [24,25]. Similarly the description of Type I fivebrane or small instantons in ALE singularities [24,25] leads to the description of the charge of the fivebranes in terms of the
equivariant real KO-theory group $\text{KO}_{\Gamma_G}(X)$. The relation between both K-theory groups (complex and real) is a multiplication by $2|\Gamma_G|$. We shown that $\text{KO}_{\Gamma_G}(X)$ determines also all data of the dynamics of the non-trivial RG fixed points of the supersymmetric gauge theories on the world-volume of the Type I fivebranes.

In section 5 we have reexamined the cohomological formula to get the RR charge of D-branes, obtained in [34] in the context of branes in singularities. We have found a formula which generalizes formula (5.1) in the context of cohomological terms of equivariant K-theory.

One can study the theory in four dimensions of Type IIB theory D3-branes on $\mathbb{R}^4 \times \mathbb{C}^3/\Gamma_G$ where $\Gamma_G$ being a discrete abelian or non-abelian subgroup of SU($K$) with $K = 2, 3, 4$. Also one could study the two-dimensional world-sheet theory on coincident D1-branes on orbifolds $\mathbb{C}^4/\Gamma_G$, where $\Gamma_G$ is a discrete and abelian (or non-abelian) subgroup of SU(4) [27]. It would be interesting a better understanding of such theories in the context of equivariant K-theory describing (non-)BPS brane configurations. Finally AdS/CFT correspondence on orbifolds $\text{AdS}_k \times S^{10-k}/\Gamma_G$ would be an interesting arena to study non-BPS brane configurations and it is tantalizing to speculate that equivariant K-theory may be an invaluable tool to study these configurations within AdS/CFT correspondence. The problem of finding an appropriate mathematical framework to generalize to $[H] \neq 0$ argued in [13] remains still open in our present description of branes in orbifolds.

Very recently Horava in [45], has studied non-stable brane-antibrane systems in Type IIA theory and its relation to Matrix theory. Specifically he shown that all supersymmetric Type IIA D-branes can be seen as bound states of non-supersymmetric 9-branes. It would be extremely interesting to apply Horava results and the results of the present paper to study Type IIA D-branes and possibly to M-theory membranes and fivebranes in orbifolds following [46].

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