Robust and Heavy-Tailed Mean Estimation Made Simple, via Regret Minimization

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Abstract

We study the problem of estimating the mean of a distribution in high dimensions when either the samples are adversarially corrupted or the distribution is heavy-tailed. Recent developments in robust statistics have established efficient and (near) optimal procedures for both settings. However, the algorithms developed on each side tend to be sophisticated and do not directly transfer to the other, with many of them having ad-hoc or complicated analyses.

In this paper, we provide a meta-problem and a duality theorem that lead to a new unified view on robust and heavy-tailed mean estimation in high dimensions. We show that the meta-problem can be solved either by a variant of the Filter algorithm from the recent literature on robust estimation or by the quantum entropy scoring scheme (QUE), due to Dong, Hopkins and Li (NeurIPS ’19). By leveraging our duality theorem, these results translate into simple and efficient algorithms for both robust and heavy-tailed settings. Furthermore, the QUE-based procedure has run-time that matches the fastest known algorithms on both fronts.

Our analysis of Filter is through the classic regret bound of the multiplicative weights update method. This connection allows us to avoid the technical complications in previous works and improve upon the run-time analysis of a gradient-descent-based algorithm for robust mean estimation by Cheng, Diakonikolas, Ge and Soltanolkotabi (ICML ’20).

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1 Introduction

Learning from high-dimensional data in the presence of outliers is a central task in modern statistics and machine learning. Outliers have many sources. Modern data sets can be exposed to random corruptions or even malicious tampering, as in data poison attacks. Data drawn from heavy-tailed distributions can naturally contain outlying samples—heavy-tailed data are found often in network science, biology, and beyond [FFF99, LKF05, Bar05, Alb05]. Minimizing the effect of outliers on the performance of learning algorithms is therefore a key challenge for statistics and computer science.

Robust statistics—that is, statistics in the presence of outliers—has been studied formally since at least the 1960s, and informally since long before [Hub64, Tuk60]. However, handling outliers in high dimensions presents significant computational challenges. Classical robust estimators (such as the Tukey median) suffer from worst-case computational hardness, while naïve computationally-efficient algorithms (e.g., throwing out atypical-looking samples) have far-from-optimal rates of error. In the last five years, however, numerous works have developed sophisticated, efficient algorithms with optimal error rates for a variety of problems in high-dimensional robust statistics. Despite significant recent progress, many basic algorithmic questions remain unanswered, and many algorithms and rigorous approaches to analyzing them remain complex and ad hoc.

In this work, we revisit the most fundamental high-dimensional estimation problem, estimating the mean of a distribution from samples, in the following two basic and widely-studied robust settings. In each case, $X_1, \ldots, X_n \in \mathbb{R}^d$ are independent samples from an unknown $d$-dimensional distribution $D$ with mean $\mu \in \mathbb{R}^d$ and (finite) covariance $\Sigma \in \mathbb{R}^{d \times d}$.

- **Robust mean estimation:** Given $Y_1, \ldots, Y_n \in \mathbb{R}^d$ such that $Y_i = X_i$ except for $\epsilon n$ choices of $i$, estimate the mean $\mu$. We interpret the $\epsilon n$ contaminated samples $Y_i \neq X_i$ as corruptions introduced by a malicious adversary. Naïve estimators such as the empirical mean can suffer arbitrarily-high inaccuracy as a result of these malicious samples.

- **Heavy-tailed mean estimation:** Given $X_1, \ldots, X_n$, estimate $\mu$ by an estimator $\hat{\mu}$ such that $\|\mu - \hat{\mu}\|$ is small with high probability (or equivalently, estimate $\mu$ with optimal confidence intervals). Since our only assumption about $D$ is that it has finite covariance, $D$ may have heavy tails. Standard estimators such as the empirical mean can therefore be poorly concentrated.

A significant amount of recent work in statistics and computer science has led to an array of algorithms for both problems with provably-optimal rates of error and increasingly-fast running times, both in theory and experiments [LRV16, DKK+19a, DHL19, CDG19, Hop20, CFB19, DL19, LLVZ20]. However, several questions remain, which we address in this work.

First, the relationship between heavy-tailed and robust mean estimation is still murky: while algorithms are known which simultaneously solve both problems to information-theoretic optimality [DL19], we lack general conditions under which algorithms for one problem also solve the other. This suggests:

*Question 1:* Is there a formal connection between robust mean estimation and heavy-tailed mean estimation which can be exploited by efficient algorithms?

Second, iterated sample downweighting (or pruning) is arguably the most natural approach to statistics with outliers—indeed, the filter, one of the first computationally efficient algorithms for optimal robust mean estimation [DKK+19a]) takes this approach—but rigorous analyses of filter-style algorithms remain ad hoc. Other iterative methods, such as gradient descent, suffer the
same fate: they are simple-to-describe algorithms which require significant creativity to analyze [CDGS20]. We ask:

**Question 2:** Is there a simple and principled approach to rigorously analyze iterative algorithms for robust and heavy-tailed mean estimation?

### 1.1 Our Results

Our main contribution in this work is a simple and unified treatment of iterative methods for robust and heavy-tailed mean estimation.

Addressing Question 1, we begin by distilling a simple meta-problem, which we call *spectral sample reweighing*. While several variants of spectral sample reweighing are implicit in recent algorithmic robust statistics literature, our work is the first to separate the problem from the context of robust mean estimation and show the reduction from heavy-tailed mean estimation. The goal in spectral sample reweighing is to take a dataset \( \{x_i\}_{i \in [n]} \subseteq \mathbb{R}^d \), reweigh the vectors \( x_i \) according to some weights \( w_i \in [0, 1] \), and find a center \( \nu \in \mathbb{R}^d \) such that after reweighing the maximum eigenvalue of the covariance \( \sum_{i \leq n} w_i (x_i - \nu)(x_i - \nu)\top \) is as small as possible.

**Definition 1.1** \((\alpha, \epsilon)\) spectral sample reweighing, informal, see Definition 3.1. For \( \epsilon \in (0, 1/2) \), let \( \mathcal{W}_{n, \epsilon} = \{ w \in \Delta_n : \|w\|_{\infty} \leq \frac{1}{(1-\epsilon)n} \} \) be the set of probability distributions on \([n]\) with bounded \( \ell_{\infty} \) norm. Let \( \alpha \geq 1 \). Given \( \{x_i\}_{i=1}^n \) in \( \mathbb{R}^d \), an \( \alpha \)-approximate spectral sample reweighing algorithm returns a probability distribution \( w \in \mathcal{W}_{n,\epsilon} \) and a spectral center \( \nu \in \mathbb{R}^d \) such that

\[
\left\| \sum_{i \leq n} w_i (x_i - \nu)(x_i - \nu)\top \right\| \leq \alpha \cdot \min_{w' \in \mathcal{W}_{n,\epsilon}, \nu' \in \mathbb{R}^d} \left\| \sum_{i \leq n} w'_i (x_i - \nu')(x_i - \nu')\top \right\|,
\]

where \( \| \cdot \| \) denotes the spectral norm, or maximum eigenvalue.

Note that that spectral sample reweighing is a *worst-case* computational problem. The basic optimization task underlying spectral sample reweighing is to find weights \( w \in \mathcal{W}_{n,\epsilon} \) minimizing the spectral norm of the weighted second moment of \( \{x_i - \nu\}_{i \in [n]} \)—an \( \alpha \)-approximation is allowed to output instead \( w \) in the slightly larger set \( \mathcal{W}_{n,3\epsilon} \) and may only minimize the spectral norm up to a multiplicative factor \( \alpha \). The parameter \( \epsilon \) should be interpreted as the degree to which \( w \in \mathcal{W}_{n,\epsilon} \) may deviate from the uniform distribution.

Our first result shows that robust and heavy-tailed mean estimation both reduce to spectral sample reweighing.

**Theorem 1.1** (Informal, see Theorem 4.1, Theorem 6.3). Robust and heavy-tailed mean estimation can both be solved with information-theoretically optimal error rates (up to constant factors) by algorithms which make one call to an oracle providing a constant-factor approximation to spectral sample reweighing (with \( \epsilon = \epsilon_0 \) a small universal constant) and run in additional time \( \tilde{O}(nd) \).

For robust mean estimation this reduction is implicit in [DKK + 19a] and others (see e.g. [DHL19]). For heavy-tailed mean estimation the reduction was not previously known: we analyze it by a simple convex duality argument (borrowing techniques from [CDG19, DL19]). Our argument gives a new equivalence between two notions of a *center* for a set of high-dimensional vectors—the spectral center considered in spectral sample reweighing and a more combinatorial notion developed by Lugosi
and Mendelson in the context of heavy-tailed mean estimation [LM19]. We believe this equivalence is of interest in its own right—see Proposition 5.1 and Proposition 5.2.

We now turn our attention to Question 2. We offer a unified approach to rigorously analyzing several well-studied algorithms by observing that each in fact instantiates a common strategy for online convex optimization, and hence can be analyzed by applying a standard regret bound. This leads to the following three theorems. We first demonstrate that the filter, one of the first algorithms proposed for efficient robust mean estimation [DKK+19a, Li18, DKK+17, DK19], can be analyzed in this framework. Specifically, we show:

**Theorem 1.2** ([DKK+19a], Informal, see Theorem 3.1). There is an algorithm, filter, based on multiplicative weights, which gives a constant-factor approximation to spectral sample reweighing for sufficiently small $\epsilon$, in time $\tilde{O}(nd^2)$.

Previous approaches to analyzing the filter required by-hand construction of potential functions to track the progress of the algorithm. Our novel strategy to prove Theorem 1.2 demystifies the analysis of the filter by applying an out-of-the-box regret bound: the result is a significantly simpler proof than in prior work. It allows us to capture robust mean estimation in both bounded covariance and sub-gaussian setting.

Moving on, we also analyze gradient descent, giving the following new result, which we also prove by applying an out-of-the-box regret bound. Although it gives weaker running-time bound than we prove for FILTER, the advantage is that the algorithm is vanilla gradient descent. (By comparison, the multiplicative weights algorithm of Theorem 1.2 can be viewed as a more exotic mirror-descent method.)

**Theorem 1.3** (Informal, see Theorem D.3). There is a gradient-descent based algorithm for spectral sample reweighing which gives a constant-factor approximation to spectral sample reweighing in $O(nd^2/\epsilon^2)$ iterations and $\tilde{O}(n^2d^3/\epsilon^2)$ time.

Prior work analyzing gradient descent for robust mean estimation required sophisticated tools for studying non-convex iterative methods [CDGS20]. Our regret-bound strategy shows for the first time that gradient descent solves heavy-tailed mean estimation, and that it solves robust mean estimation in significantly fewer iterations than previously known (prior work shows a bound of $\tilde{O}(n^2d^4)$ iterations in the robust mean estimation setting, where our bound gives $O(nd^2)$ iterations [CDGS20]).

Finally, we demonstrate that the nearly-linear time algorithm for robust mean estimation in [DHL19] fits into this framework as well. Thus, this framework captures state-of-the-art algorithms for robust mean estimation.

**Theorem 1.4** ([DHL19], Informal, see Theorem C.1). There is an algorithm based on matrix multiplicative weights which gives a constant-factor approximation to spectral sample reweighing for sufficiently small $\epsilon$, in time $\tilde{O}(nd\log(1/\epsilon))$.

### 1.2 Related work

For robust mean estimation, [DKK+19a, LRV16] give the first polynomial-time algorithm with optimal (dimension-independent) error rates. Their results have been further improved and generalized

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1We use $\tilde{O}, \tilde{\Omega}$ notation to hide polylogarithmic factors. Also, we remark that a variant of our main algorithm achieves the optimal breakdown point of 1/2; see Appendix E.
by a number of works [BDLS17, DKK+17, DKK+18, DKS18, HL18, SCV18, DHL19, DKK+19b, CDGW19, DKS19]. See [DK19] for a complete survey.

The first (computationally inefficient) estimator to obtain optimal confidence intervals for heavy-tailed distributions in high dimensions is given by [LM19]; this construction was first made algorithmic by [Hop20], using the Sum-of-Squares method. Later works [CFB19, DL19, LLV20] significantly improve the run-time, with the fastest known to be $\tilde{O}(n^2d)$.

Analyses of the Filter algorithm are scattered around the literature [DKK+19, Li18, DKK+17, DK19]. The variant of Filter we present here is based on a soft downweighting procedure first proposed by [Ste18]. However, no prior work analyzes Filter through the lens of regret minimization or points out a connection with the heavy-tailed setting.

Prior works [DL19, PBR19, LM20] have proposed certain unified constructions for heavy-tailed and robust mean estimation. In particular, [DL19] observes a robustness guarantee of [LM19], originally designed for the heavy-tailed setting. However, these works do not distill a meta-problem or obtain the analysis via duality. In addition, although it matches the fastest-known running time in theory, the algorithm of [DL19] is based on semidefinite programming, rendering it relatively impractical. Some constructions from [PBR19, LM20] are not known to be computationally tractable.

Finally, in a concurrent and independent work, [ZJS20] also studies the spectral sample reweighing problem (in the context of robust mean estimation), and provides an analysis of filter-type algorithms based on a regret bounds. The argument of [ZJS20] relies on a technical optimization landscape analysis, which our arguments avoid. The framework of [ZJS20] can be extended to robust linear regression and covariance estimation; it is unclear our techniques extend similarly. Their work also proves an optimal breakdown point analysis of filter-type algorithm for robust mean estimation. We obtain the same result (see Appendix E) with an arguably less sophisticated proof. Lastly, [ZJS20] does not discuss the heavy-tailed setting.

### 1.3 Organization
We formally introduce the spectral sample reweighing problem and analyze an algorithm based on the Filter algorithm in Section 3. We show how this primitive can be immediately used to solve the robust mean estimation problem in Section 4. Then in Section 5 we introduce the duality theorem that connect two notions of centrality. The result is used further in Section 6, where we show how to leverage the duality for heavy-tailed mean estimation.

### 2 Preliminaries
For a set of $n$ real values $\alpha_i$, we let $\text{median}(\{\alpha_i\}_{i=1}^n)$ to denote its median. For a matrix $A$, we use $\|A\|_2$ to denote the spectral norm of $A$ and $\text{Tr}(A)$ its trace. For a vector $v$, $\|v\|_p$ denotes the $\ell_p$ norm. We denote the all-one vector of dimension $k$ by $\mathbf{1}_k$. For vectors $u, v$, we denote the entrywise product by $u \odot v$; that is, the vector such that $w_i = u_i \cdot v_i$ for each $i$. For PSD matrices $A, B$, we write $A \preceq B$ if $B - A$ is PSD. Density matrices refer to the set of PSD matrices with unit trace. For any symmetric matrix $A \in \mathbb{R}^{d \times d}$, let $\exp(A)$ denote the matrix exponential of $A$. For a weight vector $w$ such that $0 \leq w_i \leq 1$ and point set $\{x_i\}_{i=1}^n$, we define $\mu(w) = \sum_{i=1}^n w_i x_i$ and $M(w) = \sum_{i=1}^n w_i (x_i - \mu(w))(x - \mu(w))^\top$. 

\[ M(w) = \sum_{i=1}^n w_i (x_i - \mu(w))(x - \mu(w))^\top. \]
Definition 2.1 (approximate top eigenvector). For any PSD matrix $M$ and $c \in (0, 1)$, we say that a unit vector $v$ is a $c$-approximate largest eigenvector of $M$ if $v^T M v \geq c \|M\|_2$.

For a PSD matrix $M$, we let $\text{APPROXTOPEIGENVECTOR}(M,c,\alpha)$ to denote an approximation scheme that outputs a (unit-norm) $c$-approximate largest eigenvector of $M$ with a failure probability of at most $\alpha$. The classic power method achieves such guarantee with a run-time of $O\left(\frac{1}{1-c}nd\log(1/\alpha)\right)$, when $M$ is given in a factored form $M = X^T X$, for $X \in \mathbb{R}^{n \times d}$.

Definition 2.2 (Kullback–Leibler divergence). For probability distributions $p, q$ over $[n]$, the KL divergence from $q$ to $p$ is defined as $\text{KL}(p||q) = \sum_{i=1}^{n} p(i) \log \frac{p(i)}{q(i)}$.

Definition 2.3 (total variation distance). For probability distributions $p, q$, the total variation distance is defined as $\text{TV}(p, q) = \sup_E |p(E) - q(E)| = \frac{1}{2} \|p - q\|_1$, where the supremum is over the set of measurable events.

We use $\Delta_n$ to denote the set of probability distributions over $[n]$ and write $U_n$ for the uniform distribution over $[n]$. We use $i \sim I$ to denote $i$ drawn uniformly from an index set $I \subseteq [n]$. Throughout, we define $\mathcal{W}_{n,\epsilon} = \{w \in \Delta_n : w_i \leq \frac{1}{(1-\epsilon)n}\}$ to be the discrete distributions over $[n]$ with bounded $\ell_\infty$ norm; we call the set good weights.

3 The Meta-Problem and a Meta-Algorithm

We now define the meta-problem, which we call spectral sample reweighing, that underlies both the adversarial and heavy-tailed models. We put it as a promise problem.

Definition 3.1 ($\alpha, \epsilon$-spectral sample reweighing). Let $\epsilon \in (0,1/2)$. The spectral sample reweighing problem is specified by the following.

- **Input**: $n$ points $\{x_i\}_{i=1}^n$ in $\mathbb{R}^d$ and $\lambda \in \mathbb{R}$.
- **Promise**: There exists a $\nu \in \mathbb{R}^d$ and a set of good weights $w \in \mathcal{W}_{n,\epsilon}$ such that
  \[ \sum_{i=1}^{n} w_i (x_i - \nu) (x_i - \nu)^T \preceq \lambda I. \] (†)

- **Output**: A set of good weights $w^\prime \in \mathcal{W}_{n,3\epsilon}$ and $\nu^\prime \in \mathbb{R}^d$ that satisfies the condition above, up to the factor of $\alpha \geq 1$:
  \[ \sum_{i=1}^{n} w^\prime_i (x_i - \nu^\prime) (x_i - \nu^\prime)^T \preceq \alpha \lambda I. \] (3.1)

To provide some intuition, the goal here is to find a set of weights $\{w_i\}_{i=1}^n$, close to the uniform distribution on $[n]$, and a center $\nu$ such that by weighting by $w$ and centering by $\nu$, the covariance is bounded, under the promise that such a set of weights exists. We will refer to our promise (†) as a spectral centrality assumption.
Solving spectral sample reweighing  The main result of this section is an efficient algorithm that achieves a constant factor approximation for the spectral sample reweighing problem.

**Theorem 3.1** ([DKK+19a] spectral sample reweighing via filter). Let \( \{x_i\}_{i=1}^n \) be \( n \) points in \( \mathbb{R}^d \) and \( \epsilon \in (0, 1/10] \). Suppose there exists \( \nu \in \mathbb{R}^d \) and \( w \in W_{n,\epsilon} \) such that

\[
\sum_{i=1}^n w_i (x_i - \nu) (x_i - \nu)^\top \preceq \lambda I
\]

for some \( \lambda > 0 \). Then, given \( \{x_i\}_{i=1}^n, \lambda, \epsilon \) and a failure rate \( \delta \), there is an algorithm that finds \( w' \in W_{n,3\epsilon} \) and \( \nu' \in \mathbb{R}^d \) such that

\[
\sum_{i=1}^n w'_i (x_i - \nu') (x_i - \nu')^\top \preceq 60\lambda I,
\]

with probability at least \( 1 - \delta \).

The algorithm runs in \( O(d) \) iterations and \( \tilde{O} \left( nd^2 \log(1/\delta) \right) \) time in total.

Our algorithm is a multiplicative weights-style procedure. In particular, the output center \( \nu' \) will be a weighted average of the points \( \{x_i\}_{i=1}^n \). The algorithm starts with the uniform weighting and iteratively downweights points which are causing the empirical covariance to have a large eigenvalue. To ensure that we always maintain a set of good weights, we project the weights onto the set \( W_{n,\epsilon} \) at the end of each iteration, according to KL divergence. For technical reason, the algorithm also requires a width parameter \( \rho \). It suffices to set it as the squared diameter of the input points \( \{x_i\}_{i=1}^n \), and it can be bounded by \( O(d\lambda/\epsilon) \) by a simple pruning argument (Lemma 3.3 and Lemma 3.4).

The algorithm should be seen as a variant of the Filter algorithm, due to Diakonikolas, Kamath, Kane, Li, Moitra, and Stewart [DKK+19a]. The procedure we present here most resembles a more streamlined version later by Steinhart [Ste18]. However, neither formulated the problem quite this way or gave this analysis. Instead, we will re-analyze the algorithm for the purpose of spectral sample reweighing and in a different manner than previously done in the literature.

**Lemma 3.2** (analysis of filter). Let \( \epsilon \in (0, 1/10] \) and \( \{x_i\}_{i=1}^n \) be \( n \) points in \( \mathbb{R}^d \). Suppose there exists \( \nu \in \mathbb{R}^d \) and \( w \in W_{n,\epsilon} \) such that

\[
\sum_{i=1}^n w_i (x_i - \nu) (x_i - \nu)^\top \preceq \lambda I
\]

for some \( \lambda > 0 \). Then, given \( \{x_i\}_{i=1}^n \), a failure rate \( \delta \) and \( \rho \) such that \( \rho \geq \tau_i^{(t)} \) for all \( i \) and \( t \), Algorithm 1 finds \( w' \in W_{n,\epsilon} \) and \( \nu' \in \mathbb{R}^d \) such that

\[
\sum_{i=1}^n w'_i (x_i - \nu') (x_i - \nu')^\top \preceq 60\lambda I,
\]

with probability at least \( 1 - \delta \).

The algorithm terminates in \( T = O(\rho \epsilon / \lambda) \) iterations. Further, if \( T = O(\text{poly}(n,d)) \), then each iteration takes \( \tilde{O}(nd \log(1/\delta)) \) time.
Algorithm 1: Multiplicative weights for spectral sample reweighing (Definition 3.1)

**Input:** A set of points \( \{x_i\}_{i=1}^n \), an iteration count \( T \), and parameter \( \rho, \delta \)

**Output:** A point \( \nu \in \mathbb{R}^d \) and weights \( w \in W_{n,\epsilon} \).

1. Let \( w^{(1)} = \frac{1}{n} 1_n \) and \( \eta = 1/2 \).
2. For \( t \) from 1 to \( T \)
   3. Let \( \nu^{(t)} = \sum_i w_i^{(t)} x_i \), \( M^{(t)} = \sum_i w_i^{(t)} (x_i - \nu^{(t)}) (x_i - \nu^{(t)})^T \).
   4. Compute \( \nu^{(t)} = \text{APPROXTOPEIGENVECTOR}(M^{(t)}, t/8, \delta/T) \).
   5. Compute \( \tau_i^{(t)} = \langle \nu^{(t)}, x_i - \nu^{(t)} \rangle^2 \) for each \( i \).
   6. Set \( w_i^{(t+1)} \leftarrow w_i^{(t)} \left( 1 - \eta \tau_i^{(t)}/\rho \right) \) for each \( i \).
   7. Project \( w^{(t+1)} \) onto the set of good weights \( W_{n,\epsilon} \) (under KL divergence):
      \[
      w^{(t+1)} \leftarrow \text{arg min}_{w \in W_{n,\epsilon}} \text{KL}(w || w^{(t)}) .
      \]
3. Return \( \nu^{(t*)}, w^{(t*)} \), where \( t* = \text{arg min}_t \| M^{(t)} \| \).

We first see how to prove Theorem 3.1 via Lemma 3.2. Note that it requires to bound the width parameter \( \rho \). To ensure the condition \( \rho \geq \tau_i^{(t)} \) for all \( i \) and \( t \), observe that as \( \| \nu^{(t)} \| = 1 \), we have
\[
\tau_i^{(t)} = \langle \nu^{(t)}, x_i - \nu^{(t)} \rangle^2 \leq \| x_i - \nu^{(t)} \|^2 .
\]

Also, since \( \nu^{(t)} \) is a convex combination of \( \{x_i\}_{i=1}^n \), we can set \( \rho \) to be the squared diameter of the input data \( \{x_i\}_{i=1}^n \). As the first step, we show that a \( (1 - 2\epsilon) \) fraction of the points lie within a ball of radius \( \sqrt{d\lambda/\epsilon} \) under the spectral centrality condition. Then a (folklore) pruning procedure can be used to extract such set.

Lemma 3.3 (diameter bound). Let \( \{x_i\}_{i=1}^n \) be \( n \) points in \( \mathbb{R}^d \). Suppose there exists \( \nu \in \mathbb{R}^d \) and \( w \in W_{n,\epsilon} \) such that \( \sum_{i=1}^n w_i (x_i - \nu) (x_i - \nu)^T \preceq \lambda I \) for some \( \lambda > 0 \) and \( \epsilon \in (0, 1/2) \). Then there exists a ball of radius \( \sqrt{d\lambda/\epsilon} \) that contains at least \( r = (1 - 2\epsilon)n \) points of \( \{x_i\}_{i=1}^n \).

The proof of the lemma can be found in Appendix A.3

Lemma 3.4 (folklore; see [DHL19]). Let \( \epsilon < 1/2 \) and \( \delta > 0 \). Let \( S \subset \mathbb{R}^d \) be a set of \( n \) points. Assume there exists a ball \( B \) of radius \( r \) and a subset \( S' \subseteq S \) such that \( |S'| \geq (1 - \epsilon)n \) and \( S' \subset B \). Then there is an algorithm \( \text{PRUNE}(S, r, \delta) \) that runs in time \( O(nd \log 1/\delta) \) and with probability \( 1 - \delta \) outputs a set \( R \subseteq S \) so that \( S' \subseteq R \), and \( R \) is contained in a ball of radius \( 4r \).

Using the lemmas above, we can immediately prove the main theorem.

Proof of Theorem 3.1. Given \( S = \{x_i\}_{i=1}^n \), \( \lambda \) and \( \epsilon \), we first run the \( \text{PRUNE}(S, r, \delta/2) \) algorithm, with \( r = \sqrt{d\lambda/\epsilon} \). By Lemma 3.3, the spectral centrality condition (1) implies there exists a ball of radius \( r \) containing at least \( (1 - 2\epsilon)n \) points of \( S \). Therefore, Lemma 3.4 guarantees that it will
Lemma 3.2

Algorithm 1 finds \( w' \in \mathcal{W}_{\mathcal{R}[\epsilon]} \) and \( \nu' \in \mathbb{R}^d \) such that

\[
\sum_{i \in R} w'_i (x_i - \nu') (x_i - \nu')^\top \leq 60\lambda I,
\]

with probability at least \( 1 - \delta/2 \). Let \( w''_i = w'_i \) if \( i \in R \) and \( w''_i = 0 \) otherwise. Since \( \frac{1}{1 - \epsilon (1 - 2\epsilon)} \leq \frac{1}{1 - 3\epsilon} \) for \( \epsilon < 1/3 \), we have \( w''_i \in \mathcal{W}_{n,3\epsilon} \). Moreover, \( \sum_{i=1}^n w''_i (x_i - \nu') (x_i - \nu')^\top \leq 60\lambda I \), as desired.

The overall procedure succeeds with probability at least \( 1 - \delta \) by a union bound, since Algorithm 1 and \textsc{Prune} are both set up to have a failure rate at most \( \delta/2 \). Now for the run-time, \textsc{Prune}(\( S, r, \delta \)) takes \( O(nd\log(1/\delta)) \) by Lemma 3.4. Moreover, by Lemma 3.2, Algorithm 1 runs in time \( \tilde{O}(nd\log(1/\delta) \cdot T) \) time, with \( T = O(\rho e/\lambda) \) being the iteration count. Since \( \rho = 16d\lambda/\epsilon \), we have \( T = O(d) \), and this immediately yields the desired runtime.

\[ \square \]

\textbf{Analysis via regret minimization}  \textup{  } Now it remains to analyze Algorithm 1, proving Lemma 3.2. We will cast the algorithm under the framework of regret minimization using multiplicative weights update (MWU). To see that, we consider \( \{x_i\}_{i=1}^n \) as the set of actions, \( w(t) \) as our probability distribution over the actions at time \( t \), and we receive a loss vector \( \tau(t) \) each round. The weights are updated in a standard fashion. Further, the weights must lie in the constraint set \( \mathcal{W}_{n,\epsilon} \) and thus the projection step. (Note that the algorithm is implementing both the player and the adversary.) The following is a classic regret bound of MWU for the online linear optimization problem.

\textbf{Lemma 3.5} \textup{ (regret bound [AHK12]).} Suppose \( \rho \geq \tau_i^{(t)} \) for every \( t \) and \( i \). Then for any weight \( w \in \mathcal{W}_{n,\epsilon} \), Algorithm 1 satisfies that

\[
\frac{1}{T} \sum_{t=1}^T \langle w(t), \tau(t) \rangle \leq \frac{1}{T} (1 + \eta) \sum_{t=1}^T \langle w, \tau(t) \rangle + \frac{\rho \cdot KL(w||w(t))}{T\eta}, \tag{3.3}
\]

for any choice of step size \( \eta \leq 1/2 \).

In addition, we claim the following lemma and delay its proof to the appendix (Lemma A.2).

\textbf{Lemma 3.6.} \textup{ Under the centrality promise \( \dagger \), for any \( w' \in \mathcal{W}_{n,\epsilon} \),}

\[
\|\nu - \nu(w')\| \leq \frac{1}{1 - 2\epsilon} \left( \sqrt{2\lambda + \sqrt{\epsilon\|M(w')\|}} \right), \tag{3.4}
\]

where \( \nu(w') = \sum_i w'_i x_i \) and \( M(w') = \sum_i w'_i (x_i - \nu(w')) (x_i - \nu(w'))^\top \).

This type of inequality is generally known as the \textit{spectral signature} lemmas from the recent algorithmic robust statistics literature; see [Li18, DK19].

With these technical ingredients, we are now ready to analyze the algorithm.

\textbf{Proof of Lemma 3.2.} Notice first that since \( \nu(t) \) is a \( 7/8 \)-approximate largest eigenvector of \( M(t) = \sum_i w_i^{(t)} (x_i - \nu(t)) (x_i - \nu(t))^\top \), then for all \( t \),

\[
\sum_i w_i^{(t)} \tau_i^{(t)} = \sum_i w_i \langle \nu(t), x_i - \nu(t) \rangle^2 = \nu(t)^\top M(t) \nu(t) \geq \frac{7}{8} \|M(t)\|_2. \tag{3.5}
\]
Let \( w \) be the good weights that satisfies our centrality promise (\( \dagger \)). Summing over the \( T \) rounds and applying the the regret bound (Lemma 3.5), we obtain that

\[
\frac{7}{8T} \sum_{t=1}^{T} \left\| M^{(t)} \right\|_2 \leq \frac{1}{T} \sum_{t=1}^{T} \left\langle w^{(t)}, \tau^{(t)} \right\rangle \leq (1 + \eta) \frac{1}{T} \sum_{t=1}^{T} \left\langle w, \tau^{(t)} \right\rangle + \frac{\rho \cdot \text{KL}(w||w^{(1)})}{T \eta}.
\]

The KL term can be bounded because \( w \) and \( w^{(1)} \) are both close to uniform. Indeed, it is a simple calculation to verify that \( \text{KL}(w||w^{(1)}) \leq 5\epsilon \), using the fact \( w_i \leq 1/(1 - \epsilon)n \) (Lemma A.4). Plugging in \( \eta = 1/2 \), we get

\[
\frac{7}{8T} \sum_{t=1}^{T} \left\| M^{(t)} \right\|_2 \leq \frac{3}{2T} \sum_{t=1}^{T} \left\langle w, \tau^{(t)} \right\rangle + \frac{10\epsilon \rho}{T}.
\]

Our eventual goal is to bound this by \( O(\lambda) \). Note that the second term is easy to control—just set \( T = \Omega(\rho \epsilon / \lambda) \), and this will determine the iteration count and thus the runtime.

The remaining is mostly tedious calculations to bound the first term. The reader can simply skip forward to (3.12). For those interested: we proceed by expanding the first term on the right-hand side,

\[
\frac{3}{2T} \sum_{t=1}^{T} \left\langle w, \tau^{(t)} \right\rangle = \frac{3}{2T} \sum_{t=1}^{T} \sum_{i=1}^{n} w_i \left( \left\langle x_i - \nu^{(t)}, \nu^{(t)} \right\rangle \right)^2
\]

(3.7)

\[
= \frac{3}{2T} \sum_{t=1}^{T} \sum_{i=1}^{n} w_i \left( \left\langle x_i - \nu, \nu^{(t)} \right\rangle \right)^2 + \left\langle \nu - \nu^{(t)}, \nu^{(t)} \right\rangle^2
\]

(3.8)

\[
\leq \frac{3}{2} \lambda + \frac{3}{2T} \sum_{t=1}^{T} \left\langle \nu - \nu^{(t)}, \nu^{(t)} \right\rangle^2
\]

(3.9)

\[
\leq \frac{3}{2} \lambda + \frac{3}{2T} \sum_{t=1}^{T} \left\| \nu - \nu^{(t)} \right\|_2^2,
\]

(3.10)

where (3.7) is by the definition that \( \tau^{(t)}_i = \left\langle \nu^{(t)}, x_i - \nu^{(t)} \right\rangle^2 \), (3.8) uses the definition of \( \nu^{(t)} \), (3.9) follows from the spectral centrality assumption (\( \dagger \)), and (3.10) is by the fact that \( \left\| \nu^{(t)} \right\| = 1 \). Since \( \nu^{(t)} = \sum_{i=1}^{n} w^{(t)}_i x_i \), we can apply Lemma 3.6 to bound \( \left\| \nu - \nu^{(t)} \right\| \) and it follows that

\[
\frac{3}{2T} \sum_{t=1}^{T} \left\| \nu - \nu^{(t)} \right\|_2^2 \leq \frac{3}{2T} \left( \sum_{t=1}^{T} \frac{25}{2} \lambda + \frac{1}{3} \left\| M^{(t)} \right\|_2 \right),
\]

for \( \epsilon \leq 1/10 \). Plugging the bound into (3.10), we obtain

\[
\frac{3}{2T} \sum_{t=1}^{T} \left\langle w, \tau^{(t)} \right\rangle \leq \frac{3}{2} \lambda + \frac{3}{2T} \left( \sum_{t=1}^{T} \frac{25}{2} \lambda + \frac{1}{3} \left\| M^{(t)} \right\|_2 \right) = \frac{81}{4} \lambda + \frac{1}{2T} \sum_{t=1}^{T} \left\| M^{(t)} \right\|_2.
\]

(3.11)

Finally, substituting this back into (3.6), we see that

\[
\frac{7}{8T} \sum_{t=1}^{T} \left\| M^{(t)} \right\|_2 \leq \frac{81}{4} \lambda + \frac{1}{2T} \sum_{t=1}^{T} \left\| M^{(t)} \right\|_2 + \frac{10\epsilon \rho}{T}.
\]

(3.12)
Now if we set \( T = 10\rho \epsilon / \lambda \), then the last term is \( \lambda \). Rearranging yields that \( \frac{1}{T} \sum_{t=1}^{T} \| M(t) \|_2 \leq 60\lambda \). This shows that within \( T = O(\rho \epsilon / \lambda) \) iterations we have achieved our goal (3.2).

Now it remain to argue the cost of each iteration. For approximating the largest eigenvector, the well-known power method computes a constant-approximation in \( O(nd \log(1/\alpha)) \) time with a failure probability at most \( \alpha \) [KW92]. We set \( \alpha = \delta / T \), and an application of union bound implies that all the \( T \) calls to the power method jointly succeed with probability at least \( 1 - \delta \). This gives a total run-time of \( O(nd \log(1/\delta)) \), since \( T = O(\text{poly}(n, d)) \), and bounds the overall failure probability of the algorithm by \( \delta \). Finally, we remark that the KL projection onto \( W_{n, \epsilon} \) can be computed exactly in \( O(n) \) time, by the deterministic procedures in [HW01, WK08]. This completes the run-time analysis.

**Faster algorithm** Under the same assumptions, the spectral sample reweighing problem can be solved in \( O(nd \log(1/\delta)) \) time, by adapting a matrix multiplicative weight scheme, due to Dong, Hopkins and Li [DHL19]. The algorithm and its analysis generally follow from the proofs therein. The details can be found in Appendix C.

As we will see soon, applying this procedure directly match the fastest known algorithms for both robust and heavy-tailed settings.

**Gradient descent analysis** As we argued, Algorithm 1 is essentially an online linear optimization scheme, with the objective of minimizing \( \sum_{t=1}^{T} \langle w(t), \tau(t) \rangle \). It is known that the multiplicative weights rule employed here can be seen an entropic mirror descent update [SL14]. Therefore, it is natural to ask whether an additive update/gradient descent procedure would solve the problem as well. In Appendix D, we provide such an analysis (Theorem D.3). More importantly, the resulting scheme is equivalent of the gradient descent algorithm analyzed by [CDGS20]. Our analysis improves upon the iteration complexity from their work (in the concrete settings of robust mean estimation, under bounded second moment and sub-gaussian distributions).

### 4 Estimation under Corruptions

We now apply Algorithm 1 for the robust mean estimation problem. We focus on the bounded second moment distributions, where Algorithm 1 can be invoked in a black-box fashion. A slight variant of it can be used for the sub-gaussian setting, where we achieve a more refined analysis; see Appendix B.

The problem is formally defined below.

**Definition 4.1** (robust mean estimation). Given a distribution \( D \) over \( \mathbb{R}^d \) with bounded covariance and a parameter \( 0 \leq \epsilon < 1/2 \), the adversary draws \( n \) i.i.d. samples \( D \), inspects the samples, then removes at most \( \epsilon n \) points and replaces them with arbitrary points. We call the resulting dataset \( \epsilon \)-corrupted (by an adaptive adversary).

The goal is to estimate the mean of \( D \) only given the \( \epsilon \)-corrupted set of samples.

Using a meta-algorithm for approximating the spectral sample reweighing problem, we will show the following. In particular, using Algorithm 1 matches the run-time and statistical guarantee of the original FILTER algorithm.
Theorem 4.1 (robust mean estimation via sample reweighing). Let $D$ be a distribution over $\mathbb{R}^d$ with mean $\mu$ and covariance $\Sigma \preceq \sigma^2 I$ and $\epsilon \leq 1/10$. Given an $\epsilon$-corrupted set of $n = \Omega(d \log d/\epsilon)$ samples, there is an algorithm that runs in time $\tilde{O}(nd^2)$ that with constant probability outputs an estimate $\hat{\mu}$ such that $\|\hat{\mu} - \mu\| \leq O(\sigma \sqrt{\epsilon})$.

Further, the algorithm is via a black-box application of Algorithm 1, which can be replaced by any constant approximation algorithm for the spectral sample reweighing problem (Definition 3.1).

Information-theoretically, Theorem 4.1 is near optimal. It is known that the sample complexity $\tilde{O}(\log d/\epsilon^2)$ is tight, only up to the log factor. The estimation error $O(\sqrt{\epsilon})$ is tight up to constant factor.

Our analysis requires a set of deterministic conditions to hold for the input, which follow from Lemma A.18 of [DKK+17]. This is meant to obtain the desired spectral centrality condition and to bound the final estimation error.

Lemma 4.2 (deterministic conditions [DKK+17]). Let $S$ be an $\epsilon$-corrupted set of $\Omega(d \log d/\epsilon)$ samples from $D$ with mean $\mu$ and covariance $\Sigma \preceq I$. With high constant probability, $S$ contains a subset $G$ of size at least $(1 - \epsilon)n$ such that

\begin{align*}
\|\mu - \mu_G\| &\leq O(\sqrt{\epsilon}) \\
\left\| \frac{1}{|G|} \sum_{i \in G} (x_i - \mu_G)(x_i - \mu_G)^\top \right\|_2 &\leq O(1),
\end{align*}

where $\mu_G = \frac{1}{|G|} \sum_{i \in G} x_i$.

We now prove the main result of this section—using the meta-algorithm to solve the robust mean estimation problem. Observe that it suffices to prove the theorem with $\sigma^2 = 1$. Without loss of generality, we can first divide every input sample by $\sigma$, execute the algorithm and then multiply the output by $\sigma$.

Proof of Theorem 4.1. First, we check that the centrality promise (†) is satisfied. This would ensure that we are in the setting of the spectral sample reweighing problem so that the meta-algorithm applies. Assume the conditions from Lemma 4.2. Then suppose we let $w_i = 1/|G|$ if $x_i \in G$ and $w_i = 0$ otherwise, so we have that $w \in W_{n, \epsilon}$ and let $\nu = \mu_G$ and $\lambda = O(1)$. Observe that (4.2) is exactly the spectral centrality condition (†). Then we can apply Theorem 3.1 and obtain that the algorithm will find $\nu' \in \mathbb{R}^d$ and $w' \in W_{n, 3\epsilon}$ such that

$$M(w') := \sum_{i=1}^n w'_i (x_i - \nu') (x_i - \nu')^\top \leq O(1) \cdot I$$

Furthermore, by definition of the algorithm, $\nu'$ is a weighted average of the points $\{x_i\}_{i=1}^n$; that is, $\nu' = \nu(w') = \sum_{i=1}^n w'_i x_i$. This allows us again to apply the spectral signature lemma. In particular, Lemma A.3 implies

$$\|\mu_G - \nu'\| \leq \frac{1}{1 - 6\epsilon} \left( \sqrt{6\epsilon \lambda} + \sqrt{3\epsilon \|M(w')\|} \right) = O(\sqrt{\epsilon})$$

since $\lambda = O(1)$ and $\|M(w')\| = O(1)$. Finally, by triangle inequality and (4.1),

$$\|\mu - \nu'\| \leq \|\mu_G - \nu'\| + \|\mu - \mu_G\| \leq O(\sqrt{\epsilon})$$.
Therefore, the output $\nu'$ estimates the true mean up to an error of $O(\sqrt{\epsilon})$, as desired.

Finally, the run-time guarantee follows directly from the statement of Theorem 3.1, since we apply the meta-algorithm in a black-box fashion. This completes the proof.

**Optimal breakdown point**  In Appendix E, we show that a variant of the filter algorithm can be used to achieve the optimal breakdown point of $1/2$. The result also appeared in a concurrent work [ZJS20], with an arguably more sophisticated proof.

**Other algorithms**  To improve the computational efficiency, applying the same argument and using the matrix multiplicative weight algorithm (Theorem C.1), we can obtain a near linear time algorithm, which matches the fastest known algorithm for robust mean estimation [DHL19, CDGW19].

**Corollary 4.3** (faster robust mean estimation [DHL19]). Let $D$ be a distribution over $\mathbb{R}^d$ with mean $\mu$ and covariance $\Sigma \preceq \sigma^2 I$ and $\epsilon$ be a sufficiently small constant. Given an $\epsilon$-corrupted set of $n = \Omega(d \log d/\epsilon)$ samples, there is a matrix multiplicative update algorithm that runs in time $\tilde{O}(nd)$ and with constant probability computes an estimate of error $O(\sigma \sqrt{\epsilon})$.

Since $\lambda = O(1)$ in the robust mean estimation problem under bounded covariance (Lemma 4.2), our analysis of the gradient descent algorithm (Theorem D.3) implies the following.

**Corollary 4.4** (robust mean estimation via gradient descent). Let $D$ be a distribution over $\mathbb{R}^d$ with mean $\mu$ and covariance $\Sigma \preceq \sigma^2 I$ and $\epsilon$ be a sufficiently small constant. Given an $\epsilon$-corrupted set of $n = \Omega(d \log d/\epsilon)$ samples, there is a gradient-descent based algorithm that computes an estimate of error $O(\sigma \sqrt{\epsilon})$ with constant probability in $\tilde{O}(nd^2/\epsilon^2)$ iterations.$^2$

A variant of the gradient descent-based algorithm can be used for robust mean estimation in the sub-gaussian setting as well; see Appendix D.2.

## 5 Equivalent Notions of Centrality

In this section, we prove a duality statement that connects the setting of heavy-tailed and robust estimation. In particular, we will show that the following two (deterministic) notions of a center $\nu$ for points $\{x_i\}_{i=1}^k$ are essentially equivalent. We call them spectral and combinatorial center. The former is the requirement that showed up first in the original formulation of the spectral sample reweighing problem (Definition 3.1) and then in dealing with adversarial corruptions. The latter will yield the right notion of high-dimensional median for estimating the mean of heavy tailed data, now known as the *Lugosi-Mendelson estimator*, due to [LM19].

In the following, let $\{x_i\}_{i=1}^k$ be a set of $k$ points in $\mathbb{R}^d$.

**Spectral center**  Recall that our meta-problem of spectral sample reweighing (Definition 3.1) requires the assumption:

$$\min_{w \in \mathcal{W}_{k,\epsilon}} \left\| \sum_{i=1}^k w_i (x_i - \nu) (x_i - \nu)^\top \right\| \leq \lambda. \quad (5.1)$$

$^2$The $1/\epsilon$ dependence in the run-time can be removed by a simple bucketing trick due to [DL19]; also see Lemma B.1 of [DHL19].
Intuitively, this says that the data are roughly clustered around $\nu$ and no bad point significantly corrupts its shape. Note that by linearity, the objective can be rewritten as a minimax one, and this leads to the following definition

**Definition 5.1** ($(\epsilon, \lambda)$-spectral center). A point $\nu \in \mathbb{R}^d$ is a $(\epsilon, \lambda)$-spectral center of $\{x_i\}_{i=1}^k$ if

$$\min_{w \in \mathcal{W}_{k, \epsilon}} \max_{M \succeq 0, \text{Tr}(M) = 1} \sum_{i=1}^k w_i \left< (x_i - \nu)(x_i - \nu)^\top, M \right> \leq \lambda.$$  

(spectral center)

In the robust mean estimation setting, the deterministic conditions (Lemma 4.2) imply that the true mean is a $(\epsilon, O(1))$-spectral center.

**Combinatorial center** On the other hand, there is another natural way of saying that the data are centered around $\nu$, which proves to be more useful in the heavy-tailed setting. We call it combinatorial centrality condition. It roughly says that when we project the data onto any one-dimensional direction, a majority of them will be close to $\nu$.

**Definition 5.2** ($(\epsilon, \lambda)$-combinatorial center). A point $\nu$ is a $(\epsilon, \lambda)$-combinatorial center of $\{x_i\}_{i=1}^k$ if for all unit $v \in \mathbb{R}^d$,

$$\sum_{i=1}^k 1 \left\{ |\langle x_i - \nu, v \rangle| \geq \sqrt{2\lambda} \right\} \leq \epsilon k,$$

(combinatorial center)

In the heavy-tailed model, the condition directly implies that $\nu$ is close to the true mean by $\lambda$, as shown by Lugosi-Mendelson [LM19], and thus it suffices for the algorithm to find a combinatorial center (with the smallest possible $\lambda$ and a constant $\epsilon < 1/2$). This will be elaborated in the next section.

**Duality** It turns out that for constant $\epsilon$ these two conditions are equivalent (up to some minor gaps in constants). To pave way for the proofs, a key observation, first made by [CDG19], is that the left-side of (spectral center) is an SDP objective. (This is because it is simply minimizing the maximum eigenvalue of $\sum_i w_i (x_i - \nu)(x_i - \nu)^\top$.) And strong duality allows us to swap the min and max, so

$$\min_{w \in \mathcal{W}_{k, \epsilon}} \max_M \sum_{i=1}^k w_i \left< (x_i - \nu)(x_i - \nu)^\top, M \right> = \max_M \min_{w \in \mathcal{W}_{k, \epsilon}} \sum_{i=1}^k w_i \left< (x_i - \nu)(x_i - \nu)^\top, M \right>, \tag{5.2}$$

where the maximization is over the set of density matrices. Using this, we prove the following two propositions, showing (by contrapositives) that the two notions of centrality are equivalent. The constants in the statements are chosen only to serve the purpose of heavy-tailed mean estimation, and they can be tweaked easily by the same arguments.

We consider the easy direction first.

**Proposition 5.1** (spectral center $\implies$ combinatorial center). If for some unit $v \in \mathbb{R}^d$

$$\sum_{i=1}^k 1 \left\{ |\langle x_i - \nu, v \rangle| \geq 10\sqrt{\lambda} \right\} \geq 0.4k,$$

(5.3)
then we have that for $\epsilon = 0.3$,

$$\min_{w \in \mathcal{W}_{k, \epsilon}} \max_{M \succeq 0, \text{Tr}(M) = 1} \sum_{i=1}^{k} w_i \left\langle (x_i - \nu)(x_i - \nu)^T, M \right\rangle \geq \lambda.$$  

**Proof.** The assumption (5.3) immediately implies that

$$\sum_{i=1}^{k} \mathbb{1}\{\langle x_i - \nu, v \rangle^2 \geq 100\lambda\} \geq 0.4k$$

This means that there are (at least) $0.4k$ points in $\{x_i\}_{i=1}^{k}$ such that $t_i := \left\langle (x_i - \nu)(x_i - \nu)^T, M \right\rangle \geq 100\lambda$, where $M = vv^T$. We call them outliers.

Now by the SDP duality (5.2), we only need to show that for any feasible $w$ the objective is at least $\lambda$. Observe first that for a fixed $M$, the optimal $w^*$ for the max-min objective is to put weight $1 / (1 - \epsilon)k$ on the $(1 - \epsilon)k$ points with the smallest $t_i$. Recall we set $\epsilon = 0.3$. Hence, by pigeonhole principle, the support of $w^*$ must have an overlap of size $0.1k$ with the outliers. It follows that

$$\sum_{i=1}^{k} w_i^* \left\langle (x_i - \nu)(x_i - \nu)^T, M \right\rangle \geq 0.1k \cdot \frac{1}{(1 - 0.3)k} \cdot 100\lambda \geq 10\lambda.$$

Since $w^*$ is the optimal choice, this completes the proof.

The other direction is a bit more involved. The key idea is to round the maximizing PSD matrix $M$ into a single vector $v$, via gaussian sampling, and this part of the argument is due to [DL19].

**Proposition 5.2** (combinatorial center $\Rightarrow$ spectral center). Let $\epsilon = 0.1$. If for some $\nu \in \mathbb{R}^d$

$$\min_{w \in \mathcal{W}_{k, \epsilon}} \max_{M \succeq 0, \text{Tr}(M) = 1} \sum_{i=1}^{k} w_i \left\langle (x_i - \nu)(x_i - \nu)^T, M \right\rangle \geq \lambda$$

then we have for some unit $v$,

$$\sum_{i=1}^{k} \mathbb{1}\{|\langle x_i - \nu, v \rangle| \geq 0.1\sqrt{\lambda}\} \geq 0.01k.$$  

**Proof.** Strong duality (5.2) implies that there exists PSD $M$ of unit trace such that

$$\sum_{i=1}^{k} w_i \left\langle (x_i - \nu)(x_i - \nu)^T, M \right\rangle \geq \lambda$$

for all $w \in \mathcal{W}_{k, \epsilon}$. As we observed, the optimal $w^*$ for a fixed $M$ would put weights on the points with smallest value of $t_i = \left\langle (x_i - \nu)(x_i - \nu)^T, M \right\rangle$. The fact that the objective is large implies that there must be more than $\epsilon k = 0.1k$ points with $t_i \geq \lambda$. Let $B$ be this set of points such that $t_i \geq \lambda$.

It remains to demonstrate a vector $v$ such that

$$\sum_{i=1}^{k} \mathbb{1}\{|\langle x_i - \nu, v \rangle| \geq 0.1\sqrt{\lambda}\} \geq 0.01k. \quad (5.4)$$

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The idea is to round the PSD matrix $M$ to a single vector $v$ that achieves this inequality. The right rounding method is simply \textit{gaussian sampling}. Namely, if we draw $v_M \sim \mathcal{N}(0, M)$, then it can be shown that with constant probability $v = v_M / \|v_M\|$ satisfies the property above.

For that, we apply the argument from [DL19]. First let $g_i = \langle x_i - \nu, v_M \rangle$ for each $i \in [k]$. Note that $g_i$ is a mean-zero Gaussian random variable with variance $\sigma_i^2 = t_i$. A standard anti-concentration calculation shows that for any $i \in B$, $\Pr(|g_i| \geq 0.5\sqrt{\lambda}) \geq 1/2$. Therefore, if we define

$$Y = \sum_{i=1}^{k} 1 \left\{ |\langle x_i - \nu, v \rangle| \geq 0.5\sqrt{\lambda} \right\},$$

then by linearity of expectations we have $\mathbb{E} Y \geq 0.05k$. It follows from the Payley-Zigmund inequality that $\Pr(Y \geq 0.01k) \geq 0.0018$. Moreover, by Borell-TIS inequality (Theorem 7.1 of [Led01]), we can bound that with probability at least 0.999,

$$\|v_M\| \leq \mathbb{E} \|v_M\| + 4\sqrt{\|M\|} \leq \sqrt{\text{Tr}(M)} + 4\sqrt{\text{Tr}(M)} \leq 5,$$

since $\text{Tr}(M) = 1$. Combining these facts immediately proves (5.4). \hfill \Box

### 6 Estimation under Heavy-Tails

We now come to the heavy-tailed mean estimation problem and show how to solve it using the machinery developed in the last sections. The setting is very simple.

**Definition 6.1** (heavy-tailed mean estimation with optimal rates). Given $n$ random vectors $\{X_i\}_{i=1}^n$ drawn i.i.d. from a distribution $D$ over $\mathbb{R}^d$ with mean $\mu$ and (finite) covariance $\Sigma$ and a desired confidence $2^{-O(n)} \leq \delta < 1$, compute an estimate $\hat{\mu}$ such that with probability at least $1 - \delta$,

$$\|\hat{\mu} - \mu\| \lesssim r_\delta \overset{\text{def}}{=} \sqrt{\frac{\text{Tr}(\Sigma)}{n}} + \sqrt{\frac{\|\Sigma\| \log(1/\delta)}{n}}. \quad (6.1)$$

We note that the error rate (6.1) is information-theoretically optimal, up to a constant. The bound is known as \textit{sub-gaussian} error, since when $D$ is sub-gaussian, the empirical average obtains the guarantee. Moreover, in general, the estimator needs to depend on the parameter $\delta$, and the requirement that $\delta \geq 2^{-O(n)}$ is necessary [Cat12, DLL016]. In the following, we will aim only at a computationally efficient, $\delta$-\textit{dependent} construction that attains the optimal error $r_\delta$.

**Lugosi-Mendelson Estimator.** In one dimension, the well-known \textit{median-of-means} construction, due to [NY83, JVV86, AMS99], provides such strong guarantee:

(i) Bucket the data into $k = \lceil 8\log(1/\delta) \rceil$ disjoint groups and compute their means $Z_1, Z_2, \ldots, Z_k$.

(ii) Output the median $\hat{\mu}$ of $\{Z_1, Z_2, \ldots, Z_k\}$.

In high dimensions, however, the question is a lot more subtle, with the correct notion of median being elusive. A long line of work culminated in the celebrated work of Lugosi and Mendelson [LM19]. The estimator follows the median-of-means paradigm by first bucketing the data into $k$ groups and taking the means $\{Z_i\}_{i=1}^k$. The key structural lemma of their work is that the true mean is a $(0.01, O(r_\delta^2))$-combinatorial center of the bucket means, where $r_\delta$ is the sub-gaussian error rate (6.1). Recall that it means that if we consider projecting the bucket means to a one-dimensional direction, a majority of them are close to the true mean.
**Lemma 6.1** (Lugosi-Mendelson structural lemma [LM19]). Consider the setting of heavy-tailed mean estimation (Definition 6.1). Let \( \{Z_i\}_{i=1}^k \) be the \( k \) bucket means with \( k = \lceil 800 \log(1/\delta) \rceil \). Then with probability at least \( 1 - \delta \), for all unit \( v \in \mathbb{R}^d \),

\[
|\langle Z_i - \mu, v \rangle| \leq 3000 \left( \sqrt{\frac{\text{Tr}(\Sigma)}{n}} + \sqrt{\frac{\|\Sigma\| \log(1/\delta)}{n}} \right),
\]

(E\(_v\))

for 0.99\( k \) of the bucket means \( \{Z_i\}_{i=1}^k \).

This is exactly the combinatorial centrality condition (Definition 5.2) we introduced in Section 5. To build more intuition, we should visualize it as a clustering property. That is, under any one-dimensional projection, the bucket means are clustered around the true mean, and the width of the cluster is precisely the optimal sub-gaussian error \( O(r_\delta) \).

This enables a natural estimator/algorithm—we can search for a point \( \hat{\mu} \) that is a \((0.01, r_\delta^2)\)-combinatorial center for \( \{Z_i\}_{i=1}^k \). Of course, such \( \hat{\mu} \) exists, since Lugosi-Mendelson (Lemma 6.1) showed that \( \mu \) itself satisfies the condition (with probability at least \( 1 - \delta \)). Furthermore, one can check any valid \((\epsilon, O(r_\delta^2))\)-combinatorial center (Definition 5.2) \( \hat{\mu} \) with \( \epsilon < 1/2 \) is indeed an estimator with sub-gaussian error rate \( O(r_\delta) \), by a simple “pigeonhole + triangle inequality” argument.

**Lemma 6.2** (combinatorial center has sub-gaussian rate). Let \( \{Z_i\}_{i=1}^k \) be defined as above and \( \epsilon < 1/2 \). Suppose that the condition in the Lugosi-Mendelson structural lemma (Lemma 6.1) holds. Then any \((\epsilon, O(r_\delta^2))\)-combinatorial center \( \hat{\mu} \) of \( \{Z_i\}_{i=1}^k \) attains the sub-gaussian error (6.1) (up to constant).

**Proof.** Let \( v \) be the unit vector in the direction of \( \mu - \hat{\mu} \). Then since \( \hat{\mu} \) is an \((\epsilon, O(r_\delta^2))\)-combinatorial center with \( \epsilon < 1/2 \), we have \( |\langle Z_i - \hat{\mu}, v \rangle| \leq r_\delta \) for most \( Z_i \). Also, \( |\langle Z_i - \mu, v \rangle| \leq O(r_\delta) \) for most \( \{Z_i\}_{i=1}^k \) by our assumption from Lugosi-Mendelson lemma. By the pigeonhole principle, there must be a \( Z_j \) such that \( |\langle Z_j - \hat{\mu}, v \rangle| \leq O(r_\delta) \) and \( |\langle Z_j - \mu, v \rangle| \leq O(r_\delta) \). By triangle inequality,

\[
\|\hat{\mu} - \mu\| = |\langle \mu - \hat{\mu}, v \rangle| \leq |\langle Z_i - \mu, v \rangle| + |\langle Z_i - \hat{\mu}, v \rangle| \leq O(r_\delta).
\]

as desired, and this completes the proof. \( \square \)

However, the problem of efficiently finding a combinatorial center appears difficult. If one sticks to its definition, it is required to ensure that for all unit vector \( v \), the clustering property (E\(_v\)) holds. It seems that even just certifying this condition would naïvely take exponential time (say, by enumerating a 1/2-net of unit sphere). Yet, we can actually resort to duality, to avoid the pain of designing a new algorithm from scratch. As we showed, a combinatorial center is just a spectral center, which our meta-algorithm can find for us.

**Theorem 6.3** (heavy-tailed mean estimation via spectral sample reweighing). Given \( \{X_i\}_{i=1}^n \) and \( \delta \), any constant-factor approximation algorithm for the spectral sample reweighing problem (Definition 3.1) can be used to compute an estimate \( \hat{\mu} \) that obtains the sub-gaussian error rate for heavy-tailed mean estimation (Definition 6.1), with probability at least \( 1 - \delta \).

**Proof.** Let \( \{Z_i\}_{i=1}^k \) be the bucket means with \( k = \lceil 800 \log(1/\delta) \rceil \) and let \( \lambda = 3000r_\delta \). We assume that the true mean \( \mu \) is a \((0.01, \lambda^2)\)-combinatorial center of \( \{Z_i\}_{i=1}^k \). Suppose that we can obtain an \( \alpha \)-factor approximation the spectral sample reweighing, with the input being \( \{Z_i\}_{i=1}^k \).
\textbf{Promise:} First let’s check the spectral centrality condition holds. Since, by assumption, $\mu$ is a $(0.01, \lambda^2)$-combinatorial center of $\{Z_i\}_{i=1}^k$, we have that for all unit $v$

$$\sum_{i=1}^k 1 \{|\langle x_i - \mu, v \rangle| \geq \lambda\} \leq 0.01k.$$ 

Thus, Proposition 5.2 (with $\nu = \mu$) implies that

$$\min_{w \in \mathcal{W}_{k, \epsilon}} \max_{M \succeq 0, \text{Tr}(M) = 1} \sum_{i=1}^k w_i \langle (x_i - \mu)(x_i - \mu)^T, M \rangle \leq 100\lambda^2,$$

where $\epsilon = 0.1$. This means that there exists $w \in \mathcal{W}_{k, \epsilon}$ such that

$$\left\| \sum_{i=1}^n w_i (x_i - \mu)(x_i - \mu)^T \right\| \leq 100\lambda^2.$$

\textbf{Output:} Now the guarantee of an $\alpha$-factor approximation for spectral sample reweighing (Definition 3.1) is that we have $\hat{\mu} \in \mathbb{R}^d$ and $w' \in \mathcal{W}_{k, 3\epsilon}$ such that

$$\left\| \sum_{i=1}^n w'_i (x_i - \hat{\mu})(x_i - \hat{\mu})^T \right\| \leq 100\alpha\lambda^2.$$

It immediately follows that

$$\min_{w \in \mathcal{W}_{k, 3\epsilon}} \max_{M \succeq 0, \text{Tr}(M) = 1} \sum_{i=1}^k w_i \langle (x_i - \hat{\mu})(x_i - \hat{\mu})^T, M \rangle \leq 100\alpha\lambda^2.$$

Now we can apply Proposition 5.1. Since $\alpha$ is a constant by assumption, we obtain that for all unit $v$,

$$\sum_{i=1}^k 1 \{|\langle x_i - \hat{\mu}, v \rangle| \geq C(\alpha) \cdot \lambda\} \leq 0.4k, \quad (6.2)$$

for some constant $C(\alpha) = O(1)$ that depends on $\alpha$. Therefore, we get that a majority of the points cluster around $\hat{\mu}$, along any direction $v$, so it is a $(0.4, O(\lambda))$-combinatorial center. It follows from Lemma 6.2 that $\|\hat{\mu} - \mu\| \leq O(r_\delta)$, as $\lambda = O(r_\delta^2)$.

Finally, note that the only condition of the argument is that the true mean is a combinatorial center, which occurs with probability at least $1 - \delta$, by Lemma 6.1.

We remark that the exact constants we choose in the proof are immaterial, and no efforts have been given in optimizing them.

The theorem implies that the filter algorithm (Algorithm 1) combined with a simple pruning step from [LLVZ20]) can be used for heavy-tailed mean estimation as well.

\textbf{Corollary 6.4} (filter for heavy-tailed mean estimation). Given $\{X_i\}_{i=1}^n$ drawn i.i.d. from a distribution with mean $\mu$ and covariance $\Sigma$ and a failure probability $2^{-O(\alpha)} \leq \delta < 1$, there is an efficient algorithm that outputs $\hat{\mu}$ such that with probability at least $1 - \delta$, $\|\hat{\mu} - \mu\| \leq O(r_\delta)$.

Further, the algorithm is a black-box application of Algorithm 1 and runs in time $O(k^2d^2 + nd)$. 

Proof. Given the input, we first compute the bucket means \( \{Z_i\}_{i=1}^{2k} \), which takes \( O(nd) \) time. Assume that the condition of the Lugosi-Mendelson structural lemma (Lemma 6.1) holds; that is, \( \mu \) is a \( (0.01, \lambda^2) \)-combinatorial center of \( \{Z_i\}_{i=1}^{k} \), where \( \lambda = 3000r_\delta \). We use the filter algorithm (Algorithm 1) with the input being a pruned subset of \( \{Z_i\}_{i=1}^{k} \) and apply its guarantees.

Here, we will not use the pruning step (Lemma 3.4), since it requires the knowledge of \( \lambda \). Instead, we first compute the coordinate-wise median-of-means \( \tilde{\mu}_0 \) of \( \{Z_i\}_{i=1}^{2k} \) and the distances \( d_i = \|Z_i - \tilde{\mu}_0\| \) for each \( i \in [k] \). We then sort the points by \( d_i \) (in descending order) and remove the top 0.01\( k \) points in \( \{Z_i\}_{i=1}^{k} \) with large \( d_i \). It can be shown that the remaining points has diameter at most \( O(\sqrt{\delta}r_\delta) \); see Lemma E.1 of [LLVZ20]. Let \( S \) the remaining points in \( \{Z_i\}_{i=1}^{k} \).

For the run-time, we can apply the guarantee of the filter algorithm (Lemma 3.2), given the input \( S \) and a failure probability \( \delta/3 \). Since the squared diameter is \( \rho = O(\delta r_\delta^2) \) and \( \lambda = O(r_\delta^2) \), this gives a run-time of \( \tilde{O}(k^2d^2) \), since \( k = O(\log(1/\delta)) \).

We now have a constant-factor approximation for the spectral sample reweighing problem. By Theorem 6.3, this gives an estimate with the sub-gaussian error (6.1). Finally, the procedure's success depends on the condition of Lugosi-Mendelson (Theorem 6.3), success of the pruning procedure, and the guarantees of constant-approximation of spectral sample reweighing (Theorem 3.1). The failure probability of each event can be bounded by \( \delta/3 \). Applying union bound completes the proof. \( \square \)

Other algorithms for heavy-tailed mean estimation This argument also enables us to solve the heavy-tailed mean estimation problem using other approximation algorithms for the spectral sample reweighing problem. Let \( \lambda = 3000r_\delta \). Recall that the argument for Theorem 6.3 shows that there is a \( (0.1, O(\lambda^2)) \)-spectral center (which is the true mean \( \mu \)). Moreover, the pruning step in the proof of Corollary 6.4 allows us to bound the squared diameter of a large subset of \( \{Z_i\}_{i=1}^{k} \) by \( \rho = O(d\lambda^2) \).

This implies that the gradient descent-based algorithm that we analyze in Appendix D solves the heavy-tailed setting in \( O(kd^2) \) iterations.

Corollary 6.5 (heavy-tailed mean estimation via gradient descent). Assume the setting of Corollary 6.4. A black-box application of the gradient descent-based algorithm (Algorithm 4, Appendix D) solves the heavy-tailed mean estimation problem with optimal error rate within \( O(nkd^2) \) iterations and \( \tilde{O}(n^2\lambda) \) time.

The quantum entropy scoring scheme (Appendix C), however, runs in \( \tilde{O}(\log(\rho/\lambda)) \) number of iterations. Setting its failure probability to be \( \delta/3 \), we obtain the following, which matches the fastest-known algorithm for the problem [DL19, LLVZ20].

Corollary 6.6 (heavy-tailed mean estimation via quantum entropy scoring). Assume the setting of Corollary 6.4. A black-box application of the matrix multiplicative update algorithm (Algorithm 3, Appendix C) solves the heavy-tailed mean estimation problem with optimal error rate, in \( \tilde{O}(1) \) iterations and \( \tilde{O}(k^2d) \) total run-time.

7 Discussion
Estimating the mean of a distribution is arguably the most fundamental problem in statistics. We showed that in robust and heavy-tailed settings, the problem can be approached by techniques from
regret minimization and online learning. We believe the ideas we present here may be more broadly applicable to other problems in high-dimensional robust statistics, such regression and covariance estimation.

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A Technical Lemmas and Proofs

A.1 Spectral Signatures

Lemma A.1. Let \( \{x_i\}_{i=1}^n \) be \( n \) points in \( \mathbb{R}^d \). Suppose there exists \( \nu \in \mathbb{R}^d \) and a set of good weights \( w \in W_{n,\epsilon} \) such that
\[
\sum_{i=1}^n w_i (x_i - \nu) (x_i - \nu)^\top \preceq \lambda I.
\] (A.1)
Then there exists a set \( G \subseteq [n] \) of size \((1 - \epsilon) n\) such that
\[
\frac{1}{(1 - \epsilon)n} \sum_{i \in G} (x_i - \nu) (x_i - \nu)^\top \preceq \lambda I.
\] (A.2)

Proof. Define \( w(T) \) to be the uniform distribution over a subset \( T \subseteq [n] \) of data. Let \( \mathcal{T} \) denote the collection of all subsets of size \((1 - \epsilon) n\). Observe that the set of good weights \( W_{n,\epsilon} \) is simply the convex hull of \( \{w(T) : T \in \mathcal{T}\} \). Thus, for each \( i \), we can rewrite \( w_i = \sum_{T \in \mathcal{T}} \alpha_T w(T)_i \) for some distribution \( \alpha \) over \( \mathcal{T} \). Then we get that
\[
\sum_{i=1}^n w_i (x_i - \nu) (x_i - \nu)^\top = \sum_{i=1}^n \sum_{T \in \mathcal{T}} \alpha_T w(T)_i (x_i - \nu) (x_i - \nu)^\top
\]
\[
= \sum_{T \in \mathcal{T}} \alpha_T \sum_{i=1}^n w(T)_i (x_i - \nu) (x_i - \nu)^\top
\]
\[
= \sum_{T \in \mathcal{T}} \alpha_T A_T,
\]
where \( A_T = \frac{1}{(1-\epsilon)n} \sum_{i \in T} (x_i - \nu) (x_i - \nu)^\top \). It follows that the spectral centrality condition (A.1) is equivalent of
\[
\sum_{T \in \mathcal{T}} \alpha_T A_T \preceq \lambda I.
\]
Therefore, there must exist a \( G \in \mathcal{T} \) such that \( A_G \preceq \lambda I \), as we desired in (A.2).

Lemma A.2. Let \( \{x_i\}_{i=1}^n \) be \( n \) points in \( \mathbb{R}^d \). Suppose there exists \( \nu \in \mathbb{R}^d \) and a set of good weights \( w \in W_{n,\epsilon} \) such that
\[
\sum_{i=1}^n w_i (x_i - \nu) (x_i - \nu)^\top \preceq \lambda I.
\] (A.3)
for some \( \lambda > 0 \). Then for any \( w' \in W_{n,\epsilon} \),
\[
\|\nu - \nu(w')\| \leq \frac{1}{1 - 2\epsilon} \left( \sqrt{\lambda} + \sqrt{2\epsilon \lambda} + \sqrt{\epsilon \|M(w')\|} \right),
\] (A.4)
where \( \nu(w') = \sum_i w_i' x_i \) and \( M(w') = \sum_i w_i'(x_i - \nu(w'))(x_i - \nu(w'))^\top \).

The lemma and its proof strategy is similar to the spectral signature lemma in robust statistics and is now somewhat standard in the literature; see, e.g., [DHL19, Li18].
Proof. First, by Lemma A.1, there exists a set \( G \) of data of size \((1 - \epsilon)n\) such that
\[
\frac{1}{(1 - \epsilon)n} \sum_{i \in G} (x_i - \nu) (x_i - \nu)^\top \leq \lambda I. \tag{A.5}
\]
Let \( \mu_G = \frac{1}{(1 - \epsilon)n} \sum_{i \in G} x_i \) and \( B = [n] \setminus G \).

Next, to bound \( \|\nu - \nu(w')\|\), we note
\[
\|\nu(w') - \nu\|_2^2 = \sum_i w'_i \langle \nu(w') - \nu, x_i - \nu \rangle
\]
\[
= \sum_{i \in G} \frac{1}{(1 - \epsilon)n} \langle \nu(w') - \nu, x_i - \nu \rangle + \sum_{i \in G} \left( w'_i - \frac{1}{(1 - \epsilon)n} \right) \langle \nu(w') - \nu, x_i - \nu \rangle + \sum_{i \in B} w'_i \langle \nu(w') - \nu, x_i - \nu \rangle \tag{A.6}
\]
We bound the three terms respectively as follows.

(i) For the first term, by Cauchy-Schwarz,
\[
\sum_{i \in G} \frac{1}{(1 - \epsilon)n} \langle \nu(w') - \nu, x_i - \nu \rangle = \langle \nu(w') - \nu, \mu_G - \nu \rangle \leq \|\nu(w') - \nu\| \cdot \|\mu_G - \nu\|.
\]
By Jensen’s inequality and since \( A_G \preceq \lambda I \), we have for all unit \( u \),
\[
\langle \mu_G - \nu, u \rangle^2 = \left( \frac{1}{(1 - \epsilon)n} \sum_{i \in G} x_i - \nu, u \right)^2 \leq \frac{1}{(1 - \epsilon)n} \sum_{i \in G} \langle x_i - \nu, u \rangle^2 \leq \lambda.
\]
Thus, \( \|\mu_G - \nu\| \leq \sqrt{\lambda} \).

(ii) For the second term, let \( \alpha_i = w'_i - 1/(1 - \epsilon)n \). Then
\[
\left( \sum_{i \in G} \alpha_i \langle \nu(w') - \nu, x_i - \nu \rangle \right)^2 \leq \left( \sum_{i \in G} (1 - \epsilon)n\alpha_i^2 \right) \cdot \sum_{i \in G} \frac{1}{(1 - \epsilon)n} \langle \nu(w') - \nu, x_i - \nu \rangle^2
\]
\[
\leq \left( \sum_{i \in G} (1 - \epsilon)n\alpha_i^2 \right) \cdot \lambda \cdot \|\nu(w') - \nu\|_2^2 \tag{A.7}
\]
\[
\leq \left( \sum_{i \in G} |\alpha_i| \right) \cdot \lambda \cdot \|\nu(w') - \nu\|_2^2 \tag{A.8}
\]
\[
\leq 2\lambda \cdot \|\nu(w') - \nu\|_2^2, \tag{A.9}
\]
where (A.7) is by the covariance bound that \( A_G \preceq \lambda I \), (A.8) follows since \( |(1 - \epsilon)n\alpha_i| \leq 1 \), and (A.9) since \( \sum_{i=1}^n |\alpha_i| \leq \epsilon/(1 - \epsilon) \leq 2\epsilon \).

(iii) For the third term, we have
\[
\sum_{i \in B} w'_i \langle \nu(w') - \nu, x_i - \nu \rangle = \sum_{i \in B} w'_i \langle \nu(w') - \nu, x_i - \nu(w') \rangle + \left( \sum_{i \in B} w_i \right) \|\nu(w') - \nu\|_2^2
\]

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\[
\leq \sum_{i \in B} w_i' \langle \nu(w') - \nu, x_i - \nu(w') \rangle + \epsilon \cdot \|\nu(w') - \nu\|^2
\]

By Cauchy-Schwarz,

\[
\left( \sum_{i \in B} w_i' \langle \nu(w') - \nu, x_i - \nu(w') \rangle \right)^2 \leq \left( \sum_{i \in B} w_i' \right) \left( \sum_{i \in B} w_i' \cdot \langle \nu(w') - \nu, x_i - \nu(w') \rangle \right)^2 \\
\leq \epsilon \cdot \sum_{i=1}^n w_i' \langle \nu(w') - \nu, x_i - \nu(w') \rangle^2 \\
= \epsilon \cdot \langle \nu(w') - \nu \rangle^\top M(w') (\nu(w') - \nu) \\
\leq \epsilon \cdot \|M(w')\|_2 \cdot \|\nu(w') - \nu\|^2
\]

Substituting the three bounds back into (A.6) immediately yields the result (A.4).

**Lemma A.3.** Let \( \{x_i\}_{i=1}^n \) be \( n \) points. Suppose there exists a subset \( G \subset [n] \) of size \((1 - \epsilon)\) such that \( \frac{1}{|G|} \sum_{i \in G} (x_i - \mu_G) (x_i - \mu_G)^\top \leq \lambda I \) for some \( \lambda > 0 \), where \( \mu_G = \frac{1}{|G|} \sum_{i \in G} x_i \). Then for any \( w \in W_{n,\epsilon} \),

\[
\|\mu_G - \mu(w)\| \leq \frac{1}{1 - 2\epsilon} \left( \sqrt{2\epsilon\lambda} + \sqrt{\epsilon \|M(w')\|} \right). \tag{A.10}
\]

**Proof.** The proof follows from the same argument of Lemma A.2, with \( \nu = \mu_G \). Observe that the first term in (A.6) becomes \( \frac{1}{|G|} \sum_{i \in G} (\mu(w) - \nu, x_i - \nu) \), which equals 0 when \( \nu = \mu_G \), shaving the \( \sqrt{\lambda} \) term in the final bound.

**A.2 A KL Divergence Bound**

**Lemma A.4.** Let \( p \in W_{n,\epsilon} \) and \( q \) be the uniform distribution over \( n \) points. Then \( KL(p||q) \leq 5\epsilon \).

**Proof.** The lemma follows from direct calculations. By definition of KL divergence,

\[
KL(p||q) = \sum_i p_i \log \frac{p_i}{q_i} \\
= \sum_i p_i \log(np_i) \\
\leq \sum_i \frac{1}{(1 - \epsilon)n} \log \frac{1}{1 - \epsilon} \\
= \frac{1 - \epsilon}{1 - \epsilon} \log \frac{1}{1 - \epsilon} \\
\leq 5\epsilon.
\]

where the last inequality holds when \( 0 < \epsilon \leq 1/2 \).
A.3 Proof of Lemma 3.3

Proof of Lemma 3.3. We show that there exists a ball of radius $\sqrt{d\lambda/\epsilon}$ that contains at least $(1 - 3\epsilon)n$ points. Note that the spectral centrality condition $\sum_{i=1}^{n} w_{i}(x_{i} - \nu)(x_{i} - \nu)^{\top} \preceq \lambda I$ implies that

$$\sum_{i=1}^{n} \text{Tr} \left( w_{i}(x_{i} - \nu)(x_{i} - \nu)^{\top} \right) \leq d\lambda.$$ 

By the cyclic property of trace, we get

$$\sum_{i=1}^{n} w_{i} \|x_{i} - \nu\|^2 \leq d\lambda.$$ 

Therefore, by Markov’s inequality,

$$\Pr_{i \sim w} \left( \|x_{i} - \nu\|^2 \geq d\lambda/\epsilon \right) \leq \epsilon, \quad \text{(A.11)}$$

where $i \sim w$ denotes $i$ drawn from the discrete distribution defined by $w$. Observe that since $\mathcal{W}_{n,\epsilon}$ is the convex hull of all uniform distributions over a subset of size $(1 - \epsilon)n$, we have $\|w - U_{n}\|_{1} \leq 2\epsilon$. Thus, $\text{TV}(w, U_{n}) \leq \epsilon$. Hence, using the definition of total variation distance, (A.11) implies that

$$\Pr_{i \sim U_{n}} \left( \|x_{i} - \nu\|^2 \geq d\lambda/\epsilon \right) \leq 2\epsilon, \quad \text{(A.12)}$$

as desired. \qed

B Extension to sub-gaussian distributions

We now consider a variant of the filter algorithm (Algorithm 1) analyzed in Section 3. The difference is that instead of fixing the step size to be $\eta = 1/2$, we set it as $\epsilon$. That is, we will perform the multiplicative update less aggressively when there are few bad points. In addition, we require a stronger approximation for the largest eigenvector computation. This increases the run-time by an $O(\text{poly}(1/\epsilon))$ factor. For technical reasons, we also ask the algorithm to stop early if the weighted covariance has been reduced to a desired value. Formally, the algorithm is described by the pseudo-code below (Algorithm 2).

First, we need a stronger spectral signature lemma.

Lemma B.1 ([DHL19]). Let $S = \{x_{i}\}_{i=1}^{n}$ be an $\epsilon$-corrupted set of $n$ samples from a sub-gaussian distribution over $\mathbb{R}^{d}$, with mean $\mu$ and identity covariance. Suppose $n \geq \tilde{\Omega}(d/\epsilon^2)$. If $\|M(w)\| \leq 1 + \lambda$, for some $\lambda \geq 0$, then for any $w \in \mathcal{W}_{n,2\epsilon}$,

$$\|\mu - \mu(w)\| \leq \frac{1}{1 - \epsilon} \left( \sqrt{\epsilon \lambda} + C\epsilon \sqrt{\log(1/\epsilon)} \right),$$

for some universal constant $C > 0$.

Moreover, we assume that for all $w \in \mathcal{W}_{n,2\epsilon}$ we have

$$\left\| \sum_{i \in G} w_{i}(x_{i} - \mu)(x_{i} - \mu)^{\top} - I \right\| \leq \lambda = O(\epsilon \log(1/\epsilon)). \quad \text{(B.1)}$$

This condition holds with high probability over the draws of samples [DKK+19a].
Algorithm 2: Multiplicative weights for sub-gaussian robust mean estimation

**Input:** A set of points \( \{x_i\}_{i=1}^n \), an iteration count \( T \), and parameter \( \rho, \delta \)

**Output:** A set of weights \( w \in W_{n, \epsilon} \).

1. Let \( w^{(1)} = \frac{1}{n} 1_n \).
2. For \( t \) from 1 to \( T \)
   3. Let \( \nu^{(t)} = \sum_i w_i^{(t)} x_i, M^{(t)} = \sum_i w_i^{(t)} (x_i - \nu^{(t)})(x_i - \nu^{(t)})^T \).
   4. Compute \( \nu^{(t)} = \text{APPROXTOPEIGENVECTOR}(M^{(t)}, 1 - \epsilon^2, \delta/T) \).
   5. If \( \lambda^{(t)} = \nu^{(t)}^T M^{(t)} \nu^{(t)} \leq 1 \), return \( w^{(t)} \).
   6. Compute \( \tau_i^{(t)} = \langle \nu^{(t)}, x_i - \nu^{(t)} \rangle^2 \).
   7. Set \( w_i^{(t+1)} \leftarrow w_i^{(t)} \left(1 - \epsilon \tau_i^{(t)}/\rho \right) \) for each \( i \).
   8. Project \( w^{(t+1)} \) onto the set of good weights \( W_{n, \epsilon} \) (under KL divergence).
9. Return \( w^{(t^*)} \), where \( t^* = \arg \min_i \|M^{(t)}\| \).

**Lemma B.2** (analysis of sub-gaussian filter). Let \( \epsilon \) be a sufficiently small constant and \( \{x_i\}_{i=1}^n \) be \( n \) points in \( \mathbb{R}^d \). Assume the following (deterministic) conditions hold.

(i) There exists \( \nu \in \mathbb{R}^d \) and \( w \in W_{n, \epsilon} \) such that

\[
\left\| \sum_{i=1}^n w_i (x_i - \nu) (x_i - \nu)^T \right\| \leq 1 + O(\epsilon \log (1/\epsilon)). \tag{B.2}
\]

(ii) If \( \|M(w)\| \leq 1 + \lambda \), for some \( \lambda \geq 0 \), then for any \( w \in W_{n, \epsilon} \),

\[
\|\nu - \mu(w)\| \leq \frac{1}{1 - \epsilon} \left( \sqrt{\epsilon \lambda} + C \epsilon \sqrt{\log(1/\epsilon)} \right), \tag{B.3}
\]

Then, given \( \{x_i\}_{i=1}^n \), a failure rate \( \delta \) and \( \rho \) such that \( \rho \geq \tau_i^{(t)} \) for all \( i \) and \( t \), Algorithm 2 finds \( w' \in W_{n, \epsilon} \) such that

\[
\|M(w')\| \leq 1 + O(\epsilon \log (1/\epsilon)), \tag{B.4}
\]

with probability at least \( 1 - \delta \).

The algorithm terminates in \( T = O(\rho/\epsilon) \) iterations. Further, if \( T = O(\text{poly}(n, d)) \), then each iteration takes \( \tilde{O}(nd \log (1/\delta)/\epsilon^2) \) time.

**Proof of Lemma B.2.** If the algorithm gets stopped early (at Line 5), then it means that

\[
\|M^{(t)}\| \leq \lambda^{(t)}/(1 - \epsilon^2) \leq 1/(1 - \epsilon^2) \leq 1 + O(\epsilon^2),
\]

since \( \nu^{(t)} \) is a \((1 - \epsilon^2)\) approximate largest eigenvector of \( M^{(t)} \). Hence, in this case, we immediately achieves the goal \( \text{(B.4)} \).

Now assume the algorithm did not stop early and so \( \|M^{(t)}\| > 1 \) for all \( t \). Then we have

\[
\sum_i w_i^{(t)} \tau_i^{(t)} = \sum_i w_i^{(t)} \left\langle \nu^{(t)}, x_i - \nu^{(t)} \right\rangle^2 = \nu^{(t)^T} M^{(t)} \nu^{(t)} \geq (1 - \epsilon^2) \|M^{(t)}\|_2, \tag{B.5}
\]
for all $t$. Since the step size $\epsilon < 1/2$ and $\rho \geq \tau_i^{(t)}$ for all $i, t$ by assumption, we can apply the regret bound of MWU (Lemma 3.5) and conclude that for $w$ that satisfies assumption (B.2),

$$
\frac{1 - \epsilon^2}{T} \sum_{t=1}^{T} \| M^{(t)} \|_2^2 \leq \frac{1}{T} \sum_{t=1}^{T} \langle w^{(t)}, \tau^{(t)} \rangle \leq (1 + \epsilon) \frac{1}{T} \sum_{t=1}^{T} \langle w, \tau^{(t)} \rangle + \frac{\rho \cdot \text{KL}(w||w^{(1)})}{T \epsilon}.
$$

We now focus on bounding $\frac{1}{T} \sum_{t=1}^{T} \langle w, \tau^{(t)} \rangle$.

**Claim B.3.** In the setting above, we have

$$
\frac{1}{T} \sum_{t=1}^{T} \langle w, \tau^{(t)} \rangle \leq 1 + O \left( \epsilon \log(1/\epsilon) \right) + \frac{2\epsilon}{(1 - \epsilon)^2} \frac{1}{T} \sum_{t=1}^{T} \| M^{(t)} \| - \frac{2\epsilon}{(1 - \epsilon)^2}
$$

**Proof.** Note that

$$
\frac{1}{T} \sum_{t=1}^{T} \langle w, \tau^{(t)} \rangle = \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} w_i \langle x_i - \nu^{(t)}, v^{(t)} \rangle
$$

$$
= \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} w_i \left( \langle x_i - \nu, v^{(t)} \rangle^2 + \langle \nu - \nu^{(t)}, v^{(t)} \rangle^2 \right)
$$

$$
\leq 1 + O \left( \epsilon \log(1/\epsilon) \right) + \frac{1}{T} \sum_{t=1}^{T} \langle \nu - \nu^{(t)}, v^{(t)} \rangle^2
$$

$$
\leq 1 + O \left( \epsilon \log(1/\epsilon) \right) + \frac{1}{T} \sum_{t=1}^{T} \| \nu - \nu^{(t)} \|_2^2,
$$

where (B.7) follows from the assumption (B.4). Now we apply assumption (B.3) to bound $\| \nu - \nu^{(t)} \|_2^2$. Since we may assume $\| M^{(t)} \| \geq 1$ by the early stopping of Line 5, we have

$$
\| \nu - \nu^{(t)} \|_2^2 \leq \frac{2}{(1 - \epsilon)^2} \left( \epsilon \left( \| M^{(t)} \| - 1 \right) + C^2 \epsilon^2 \log(1/\epsilon) \right)
$$

$$
= \frac{2\epsilon}{(1 - \epsilon)^2} \| M^{(t)} \| - \frac{2\epsilon}{(1 - \epsilon)^2} + O(\epsilon^2 \log(1/\epsilon)).
$$

Substituting the bound back into (B.8) completes the proof. $\square$

Using Claim B.3, the KL bound (Lemma A.4) and (B.6), we have

$$
\frac{1 - \epsilon^2}{T} \sum_{t=1}^{T} \| M^{(t)} \|_2^2 \leq \frac{2(1 + \epsilon)\epsilon}{(1 - \epsilon)^2} \frac{1}{T} \sum_{t=1}^{T} \| M^{(t)} \| + 1 - \frac{2(1 + \epsilon)\epsilon}{(1 - \epsilon)^2} + O(\epsilon \log(1/\epsilon)) + \frac{5\rho}{T}.
$$

For sufficiently small $\epsilon$, rearrange and divide through to obtain

$$
\frac{1}{T} \sum_{t=1}^{T} \| M^{(t)} \|_2^2 \leq 1 + O(\epsilon \log(1/\epsilon)) + O(\epsilon) + \frac{O(\rho)}{T}.
$$

Setting $T = O(\rho/\epsilon)$ completes the correctness proof. Finally, the per-iteration cost follows from the run-time of using power method to approximate the largest eigenvector. $\square$

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Using the lemma we can prove our main theorem.

**Theorem B.4** (sub-gaussian robust mean estimation, [DKK+19a]). Let $S = \{x_i\}_{i=1}^n$ be an $\epsilon$-corrupted set of $n$ samples from a sub-gaussian distribution over $\mathbb{R}^d$, with mean $\mu$ and identity covariance. Suppose $n \geq \tilde{\Omega}(d/\epsilon^2)$. Given $S$, there is an algorithm that outputs $\hat{\mu}$ such that $\|\hat{\mu} - \mu\| \leq O(\epsilon \log(1/\epsilon))$ with high constant probability. The algorithm runs in time $\tilde{O}(nd^2/\epsilon^3)$.

**Proof.** Let $\delta = 0.01$. We apply Algorithm 2 with a simple pruning procedure as a preprocessing. By standard concentration of sub-gaussian random vectors, with high constant probability, $\|x_i - \mu\| \leq r = O(\sqrt{d \log n})$ for all $i \in G$. Hence, we apply PRUNE($S, r, \delta$), and by Lemma 3.4 it guarantees to terminate in $O(nd)$ time and removes at most $en$ (bad) points.

We feed the remaining (at least) $(1 - \epsilon)n$ points $R \supseteq G$ into Algorithm 2 with $\rho = r^2$. Notice that $\frac{1}{(1-\epsilon)[1-\epsilon]} \leq \frac{1}{1-2\epsilon}$ for $\epsilon \leq 1/2$, so assumptions (i)-(ii) of Lemma B.2 are satisfied by the claim of (B.1) and Lemma B.1, respectively.

It then follows from Lemma B.2 that Algorithm 2 outputs $w' \in \mathcal{W}_{R, \epsilon}$ such that

$$\left\| \sum_{i \in R} (x_i - \mu(w')) (x_i - \mu(w'))^\top \right\| \leq 1 + O(\epsilon \log(1/\epsilon)),$$

where $\mu(w') = \sum_{i \in R} w'_i x_i$. Let $w''_i = w'_i$ if $i \in R$ and $w''_i = 0$ otherwise. We obtain $w'' \in \mathcal{W}_{n, \epsilon}$ such that $\|M(w'')\| \leq 1 + O(\epsilon \log(1/\epsilon))$. Applying the spectral signature (Lemma B.1) proves that $\mu(w'')$ attains the desired estimation error. Moreover, the run-time simply follows from Lemma B.2.

---

### C Sampling reweighing via Matrix Multiplicative Update

We now show that the spectral sample reweighing problem (Definition 3.1) can be solved in near linear time via a matrix multiplicative update scheme from the recent work of [DHL19]. Our analysis will closely resemble the arguments therein.

**Theorem C.1.** Let $\{x_i\}_{i=1}^n$ be $n$ points in $\mathbb{R}^d$. Suppose there exists $\nu \in \mathbb{R}^d$ and $w \in \mathcal{W}_{n, \epsilon}$ such that $\sum_{i=1}^n w_i (x_i - \nu) (x_i - \nu)^\top \preceq \lambda I$ for some $\lambda > 0$ and a sufficiently small $\epsilon$. Then, given $\{x_i\}_{i=1}^n, \lambda$, the squared diameter $\rho$ of the points and a failure rate $\delta$, there is a matrix multiplicative weights-based algorithm (Algorithm 3) that, with probability at least $1 - \delta$, finds $w' \in \mathcal{W}_{n, \epsilon}$ and $\nu' \in \mathbb{R}^d$ such that

$$\sum_{i=1}^n w'_i (x_i - \nu') (x_i - \nu')^\top \preceq O(\lambda) I.$$

Further, the algorithm terminates in $O(\log(\rho/\lambda))$ iterations, where $\rho$ is the squared diameter of the input points $\{x_i\}_{i=1}^n$, and each iteration can be implemented in $\tilde{O}(nd \log(1/\delta))$ time.

**Remark C.1.** In the following, we will consider an idealized version of the algorithm and omit the detail of implementing the numerical linear algebra primitives in $\tilde{O}(nd \log(1/\delta))$ time each iteration. The exact details can be found in [DHL19].

The algorithm is based on the matrix multiplicative weights update. For a sequence of PSD matrices $M_1 \succeq M_2 \succeq \cdots \succeq M_{t-1}$, we will apply the matrix multiplicative weight (MMW) update,
given by

$$\text{MMW}(M_0, M_1, \cdots, M_{t-1}) = \exp \left( \frac{1}{\|M_0\|_2} \sum_{k=1}^{t-1} M_k \right) / \text{tr} \exp \left( \frac{1}{\|M_0\|_2} \sum_{k=1}^{t-1} M_k \right).$$  \hspace{1cm} (C.1)

For technical reasons, we will not maintain a set of weights that is a probability distribution throughout. Instead, recall by Lemma A.1 that there exists a subset $G$ of size $(1 - \epsilon)n$ such that $A_G \preceq \lambda I$, where $A_G = \frac{1}{(1 - \epsilon)n} \sum_{i \in G} (x_i - \nu)(x_i - \nu)^\top$. Thus, our new notion of a good set of weights is that starting from the uniform distribution over $n$ points, more weights are removed from $G$ than from $G$. Let $w_G, w_B$ denote the restriction of $G$ to the indices of vector $w$ and $B = [n] \setminus G$.

**Definition C.1** (mostly-good weight vector). Given $\{x_i\}_{i=1}^n$ that satisfy the spectral centrality condition (\dagger), let $G$ be a subset of size $(1 - \epsilon)n$ such that $A_G \preceq \lambda I$. The set of mostly-good weight vectors is

$$\mathcal{C}_{n, \epsilon} = \left\{ w \in \mathbb{R}^n : 0 \leq w_i \leq \frac{1}{n} \text{ and } \frac{1}{n} \|1_G \| - w_G \leq \frac{1}{n} \|1_B \| - w_B \right\}.$$

A crucial subroutine we use is a deterministic down-weighting scheme, directly from [DHL19], that maintains the mostly-good property of the input weights.

**Lemma C.2** (1D Filter [DHL19]). Let $\eta \in (0, 1/2)$, let $b \geq 2\eta$, and let $w_1, \ldots, w_m$ and $\tau_1, \ldots, \tau_m$ be non-negative numbers so that $\sum_{i=1}^m \tau_i \leq 1$. Let $\tau_{\text{max}} = \max_{i \in [m]} \tau_i$. Suppose there exist two disjoint sets $G, B$ so that $G \cup B = [m]$, and moreover,

$$\sum_{i \in G} w_i \tau_i \leq \eta \sigma \text{, where } \sigma = \sum_{i=1}^n w_i \tau_i.$$

Then $1D\text{Filter}(w, \tau, b)$ runs in time $O((1 + \log \tau_{\text{max}})m)$ and outputs $0 \leq w' \leq w$ so that:

- more weight is removed from $B$ than from $G$: $\sum_{i \in G} w_i - w_i' \leq \sum_{i \in B} w_i - w_i'$, and
- the weighted sum of $\tau_i$ is decreased: $\sum_{i=1}^m w_i' \tau_i \leq b \sigma$.

The algorithm is formally described in Algorithm 3. Throughout let $M^{(s)} = M(w^{(s)})$ and $M_t^{(s)} = M(w_t^{(s)})$, where $M(w) = \sum_{i=1}^n w_i (x_i - \mu(w))(x_i - \mu(w))^\top$. The procedure runs by epochs, where each epoch $s$ reduces the largest eigenvalue of $M^{(s)}$ by a constant factor. We will show that the inner loop achieves the reduction within $O(\log d)$ iterations while maintaining the invariant that the weights are mostly-good (Definition C.1).

To start with the analysis, we first establish certain invariants of the algorithm. This requires the following lemma. The proof follows from exactly the same argument for Lemma A.2, which we omit for the sake of brevity.

**Lemma C.3** (spectral signature for mostly-good weights). Let $\{x_i\}_{i=1}^n$ be $n$ points in $\mathbb{R}^d$. Suppose there exists $\nu \in \mathbb{R}^d$ and a mostly-good weight vector $w \in \mathcal{C}_{n, \epsilon}$ such that $\sum_{i=1}^n w_i (x_i - \nu)(x_i - \nu)^\top \preceq \lambda I$, for some $\lambda > 0$. Then for any $w' \in \mathcal{C}_{n, \epsilon}$,

$$\|\nu - \nu(w')\| \leq \frac{1}{1 - 2\epsilon} \left( 2\sqrt{\lambda} + \sqrt{\epsilon \|M(w')\|} \right),$$  \hspace{1cm} (C.3)

where $\nu(w') = \sum_i w_i' x_i$ and $M(w') = \sum_i w_i' (x_i - \nu(w'))(x_i - \nu(w'))^\top$. 

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Algorithm 3: Matrix multiplicative update for spectral sample reweighting (Definition 3.1)

**Input:** A set of points \( x_1, \ldots, x_n \), \( \lambda, \rho \) and a failure rate \( \delta \)

**Output:** A point \( v' \in \mathbb{R}^d \) and weights \( w' \in \mathcal{W}_{n,\epsilon} \) that satisfy (3.1) up to a constant factor.

1. Let \( w^{(0)} = \frac{1}{n}(1, 1, \ldots, 1) \).
2. For \( s \) from 0 to \( O(\log \rho) \)
   
   - Compute \( \lambda^{(s)} = \|M^{(s)}\| \).
   
   - If \( \lambda^{(s)} \leq 300\lambda \)
     
     Return \( w^{(s)}/\|w^{(s)}\|_1, \mu(w^{(s)}) \).

3. For \( t \) from 0 to \( O(\log d) \)
   
   - Compute \( \lambda^{(s)}_t = \|M^{(s)}_t\| \) and terminate epoch if \( \lambda^{(s)}_t \leq \frac{1}{3}\lambda^{(s)}_0 \).
   
   - Compute \( U^{(s)}_t = \text{MMW}(M^{(s)}_1, M^{(s)}_2, \ldots, M^{(s)}_{t-1}) \).

4. Compute
   
   \[
   \tau^{(s)}_{t,i} = \left( x_i - \mu \left( w^{(s)}_i \right) \right)^\top U^{(s)}_t \left( x_i - \mu \left( w^{(s)}_i \right) \right) \tag{C.2}
   
   \]

5. Let \( w^{(s+1)}_t = w^{(s)}_t \) if \( \sum_i w^{(s)}_{t,i} \tau^{(s)}_{t,i} \leq \frac{1}{3}\lambda^{(s)}_1 \); otherwise \( w^{(s+1)}_t = 1\text{DFILTER} \left(w^{(s)}_t, \tau^{(s)}_t \right) \).

6. Let \( w^{(s+1)} = w^{(s)} \).

Using this, we establish a key lemma of the inner loop of the algorithm.

**Lemma C.4.** Let \( w \in \mathcal{C}_{n,\epsilon} \) be such that \( \beta = \|M(w)\|_2 \geq 300\lambda \) and \( U \) be a density matrix. Let \( \tau_i = (x_i - \mu(w))^\top U (x_i - \mu(w)) \). If \( \beta \geq \frac{1}{4}\beta \) and \( w' = 1\text{DFILTER}(w, \tau, 1/4) \), then we have \( w' \in \mathcal{C}_{n,\epsilon} \) and \( \langle M(w'), U \rangle \leq \frac{1}{4}(M(w), U) \).

**Proof.** Let \( \bar{w}_i = 1/n \) if \( i \in G \) and \( \bar{w}_i = 0 \) otherwise. Let \( \mu(\bar{w}) = \sum_i \bar{w}_i x_i \). Then for any unit vector \( u \), we have

\[
\langle \mu(\bar{w}) - \nu, u \rangle^2 \leq \left( \frac{1}{(1-\epsilon)n} \sum_{i \in G} x_i - \nu, u \right)^2 \leq \frac{1}{(1-\epsilon)n} \sum_{i \in G} (x_i - \nu, u)^2 \leq \lambda.
\]

Thus, \( \|\mu(\bar{w}) - \nu\|_2^2 \leq \lambda \). Expanding the definition of \( \tau_i \), we get

\[
\sum_{i \in G} w_i \tau_i = \left\langle \sum_{i \in G} w_i \left( X_i - \mu(w) \right) \left( X_i - \mu(w) \right)^\top, U \right\rangle \\
\leq \left\langle \sum_{i=1}^n \bar{w}_i \left( X_i - \mu(w) \right) \left( X_i - \mu(w) \right)^\top, U \right\rangle \\
= \left\langle \sum_{i=1}^n \bar{w}_i \left( X_i - \mu(\bar{w}) \right) \left( X_i - \mu(\bar{w}) \right)^\top, U \right\rangle + \|\bar{w}\|_1 \cdot (\mu(\bar{w}) - \mu(w))^\top U (\mu(\bar{w}) - \mu(w)) \\
\leq \langle M(\bar{w}), U \rangle + (1-\epsilon)\|\mu(\bar{w}) - \mu(w)\|_2^2 \\
\leq \lambda + 2\|\mu(\bar{w}) - \nu\|_2^2 + 2\|\mu(w) - \nu\|_2^2 \\
\leq 3\lambda + (5\lambda + 2\epsilon\|M(w)\|) \tag{C.4}
\]

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\[ \leq \frac{1}{30} \| M(w) \| = \frac{1}{30} \sum_i w_i \tau_i, \]  
(C.6)

where (C.4) follows since \( M(\tilde{w}) \leq A_G \leq M \), (C.5) follows from Lemma C.4, and (C.6) uses our assumption that \( \| M(w) \| \geq 300\lambda \) and the definition of \( \tau_i \). This allows us to apply the guarantee of the 1D filter procedure (Lemma C.2) and get that

\[ \langle M(w'), U \rangle = \left( \sum_{i=1}^n w'_i (X_i - \mu(w)) (X_i - \mu(w)) \right), U \right \rangle = \sum_{i=1}^n w'_i \tau_i \leq \frac{1}{4} \sum_{i=1}^n w_i \tau_i = \frac{1}{4} \langle M(w), U \rangle. \]

Furthermore, \( w' \in C_{n, \epsilon} \). This completes the proof.

We are now ready to prove the main theorem of this section.

**Proof of Theorem C.1.** Consider a fixed epoch and drop the super script for simplicity of notation. It is not hard to observe that \( M(w_{t+1}) \leq M(w_t) \) (see Lemma 3.4 [DHL19]). Let \( \alpha = 1/\| M(w_0) \| \). A regret bound of matrix multiplicative weights [AZLO15] implies that

\[ \left\| \sum_{t=0}^{T-1} M(w_{t+1}) \right\|_2 \leq \sum_{t=0}^{T-1} \langle M(w_{t+1}), U_t \rangle + \alpha \sum_{t=0}^{T-1} \langle U_t, M(w_{t+1}) \rangle \| M(w_{t+1}) \|_2 + \frac{\log d}{\alpha} \]

\[ \leq 2 \sum_{t=0}^{T-1} \langle M(w_{t+1}), U_t \rangle + \| M(w_0) \|_2 \cdot \log d. \]

Now by definition of Line 10, we have \( \langle M(w_{t+1}), U_t \rangle \leq \frac{1}{4} \| M(w_0) \|_2 \). Hence,

\[ T \| M(w_T) \|_2 \leq \left\| \sum_{t=0}^{T-1} M(w_t) \right\|_2 \leq T \cdot \frac{1}{2} \| M(w_0) \|_2 + \| M(w_0) \|_2 \cdot \log d. \]

Setting \( T \gg \log d \) shows that the inner loop terminates in \( O(\log d) \) iterations and reduces the largest eigenvalue of the covariance by, say, \( 4/5 \).

Finally, to bound the number of epochs, we simply note that \( \| M^{(0)} \| \leq \rho \). Therefore, \( O(\log(\rho/\lambda)) \) epochs suffice drive the largest eigenvalue of \( \| M^{(t)} \| \) down to \( O(\lambda) \), since it is reduced geometrically each epoch.

**D Sample reweighing via Online Gradient Descent**

**D.1 Regret analysis of gradient descent**

We now consider a gradient updated-based algorithm for solving the spectral sample reweighing problem (Definition 3.1). The analysis will be through the classic regret guarantee of online gradient descent for convex optimization [Zin03]. Though the resulting run-time is higher than the MWU scheme we analyzed in Section 3, it nonetheless betters the recent work of [CDGS20], where essentially the same gradient descent-based algorithm is studied.

We will leverage the following regret guarantee of online gradient descent; the definition of the algorithm in the general setting can be found in standard text [Haz16].
Lemma D.1 (Theorem 3.1 [Haz16], originally due to [Zin03]). Let $f_t : \mathcal{K} \to \mathbb{R}$ be the convex cost function revealed at iteration $t$, where $\mathcal{K}$ is a convex feasible set. Suppose $f_t$ is $L$-Lipschitz (in $\ell_2$ norm) and $\|x_0 - x^*\|_2 \leq R$ for some $x^* \in \arg\min_{x \in \mathcal{K}} \sum_t f_t(x)$. The online gradient descent algorithm with step sizes $\eta_t = \frac{R}{L\sqrt{t}}$ achieves

$$\sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{K}} \sum_{t=1}^{T} f_t(x) \leq \frac{3}{2} L R \sqrt{T}. \quad \text{(D.1)}$$

Our algorithm implicitly defines the cost functions $f_t(w) = \langle w, \tau^{(t)} \rangle$, where the feasible set is $\mathcal{W}_{n,\epsilon}$, and implements the online gradient descent algorithm for the linear objective. Note that $\nabla f_t(w) = \tau^{(t)}$, and the main difference of this algorithm from the MWU scheme (Algorithm 1) is that we use an additive/gradient-descent update, in lieu of the multiplicative update.

**Algorithm 4:** Gradient descent for spectral sample reweighing (Definition 3.1)

**Input:** A set of points $\{x_i\}_{i=1}^n$, an iteration count $T$, and step sizes $\eta_t$

**Output:** A point $\nu \in \mathbb{R}^d$ and weights $w \in \mathcal{W}_{n,\epsilon}$

1. Let $w^{(1)} = \frac{1}{n}(1, 1, \ldots, 1)$.
2. For $t$ from 1 to $T$
   3. Let $\nu^{(t)} = \sum_i w_i^{(t)} x_i$, $M^{(t)} = \sum_i w_i^{(t)} (x_i - \nu^{(t)}) (x_i - \nu^{(t)})^T$.
   4. Let $v^{(t)}$ be the top eigenvector of $M^{(t)}$ (with $\|v^{(t)}\| = 1$).
   5. Compute $\tau_i^{(t)} = \langle v^{(t)}, x_i - \nu^{(t)} \rangle^2$.
   6. Set $w_{i} \leftarrow w_i - \eta_t \tau_i^{(t)}$.
   7. Project $w^{(t+1)}$ onto the set of good weights $\mathcal{W}_{n,\epsilon}$ (under $\ell_2$ distance).
8. Return $\nu^{(t^*)}, w^{(t^*)}$, where $t^* = \arg\min_t \|M^{(t)}\|$.

Lemma D.2. Let $\rho$ be the squared diameter of the inputs points $\{x_i\}_{i=1}^n$. The cost function $f_t(\cdot)$ is $\sqrt{n} \rho$-Lipschitz (in $\ell_2$ norm), for all $t$.

**Proof.** Since $f_t$ is differentiable, we only need the bound $\|\nabla f_t\|$. We have that for all $t$ and $i$,

$$\tau_i^{(t)} = \langle v^{(t)}, x_i - \nu^{(t)} \rangle^2 \leq \|x_i - \nu^{(t)}\|_2^2 \leq \rho.$$ 

Therefore, $\|\nabla f_t\| = \|\tau^{(t)}\| \leq \sqrt{n} \rho$. \hfill $\square$

Theorem D.3. Given $\{x_i\}_{i=1}^n$ and $\eta_t = R/L\sqrt{t}$ with $L = \sqrt{n} \rho, R = \sqrt{2}$, the online gradient descent algorithm (based on Algorithm 4) yields a constant-factor approximation for the spectral sample reweighing problem (Definition 3.1) in $O(nd^2/e^2)$ iterations and $O(n^2d^2/e^2)$ total run-time.

**Proof.** We first apply the PRUNE procedure of Lemma 3.4 to bound the diameter. By Lemma 3.3 and the guarantee of PRUNE, we can have $\rho = 16d / \epsilon$. Then we apply Algorithm 4.

We will use Lemma D.1 to analyze Algorithm 4. First, by Lemma D.2, we have $L = \sqrt{n} \rho$, and further, since the $\ell_2$ diameter of the probability simplex can be (trivially) bounded by $\sqrt{2}, R = \sqrt{2}$. Moreover, observe for any $t$,

$$f_t (w^{(t)}) = \langle w^{(t)}, \tau^{(t)} \rangle = \sum_i w_i^{(t)} \langle v^{(t)}, x_i - \nu^{(t)} \rangle^2 = v^{(t)}^T M^{(t)} v^{(t)} = \|M^{(t)}\|_2.$$  

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Let \( w \in W_{n, \varepsilon} \) be a weight that satisfies the spectral centrality condition. Then, from the regret guarantee (D.1),

\[
\frac{1}{T} \sum_{t=1}^{T} \left\| M^{(t)} \right\|_2 \leq \frac{1}{T} \sum_{t=1}^{T} \langle w, \tau^{(t)} \rangle + \frac{3LR}{2\sqrt{T}} \tag{D.2}
\]

We bound the two terms on the right side individually.

(i) A bound on the first term follows exactly from the calculations we did in the analysis of MWU algorithm (Algorithm 1). In particular, from (3.11) we have

\[
\frac{1}{T} \sum_{t=1}^{T} \langle w, \tau^{(t)} \rangle \leq 15\lambda + \frac{1}{3T} \sum_{t=1}^{T} \left\| M^{(t)} \right\|_2 .
\]

(ii) Observe that it suffices to set \( T = 3L^2R^2/\lambda^2 \) to bound the second term by \( \lambda \).

Substituting the two bounds back into (D.2),

\[
\frac{1}{T} \sum_{t=1}^{T} \left\| M^{(t)} \right\|_2 \leq 16\lambda + \frac{1}{3T} \sum_{t=1}^{T} \left\| M^{(t)} \right\|_2 . \tag{D.3}
\]

Rearranging and dividing through immediately yields the desired guarantee.

Given that \( L = \sqrt{n\rho}, R = \sqrt{2} \), we have that the iteration count \( T = 6n\rho^2/\lambda^2 \). Since \( \rho = 16d\lambda/\varepsilon \), \( T = O(nd^2/\varepsilon^2) \). For the run-time, note that instead of computing the exact largest eigenvector, we can use power method to find an \( 7/8 \)-approximate one. Observe that this suffices for our analysis of the method above. Finally, the Euclidean projection onto \( W_{n, \varepsilon} \) can be computed in \( O(n \log n) \) time [WL15]. This yields the desired run-time.

\[\square\]

### D.2 Extension to sub-gaussian setting

Theorem D.3 implies that a gradient descent-based algorithm (Algorithm 4) can be used for robust mean estimation under bounded covariance. We now extend the result to the sub-gaussian setting, showing that the same iteration and run-time complexity holds. The optimal estimation error we will aim for is \( O(\varepsilon \sqrt{\log(1/\varepsilon)}) \). We assume the spectral signature Lemma B.1 and the deterministic condition (B.1).

In particular, we will analyze Algorithm 5 and prove the following set of guarantees.

**Lemma D.4.** Let \( \varepsilon \) be a sufficiently small constant and \( \{x_i\}_{i=1}^{n} \) be \( n \) points in \( \mathbb{R}^d \). Assume the following (deterministic) conditions hold.

(i) There exists \( \nu \in \mathbb{R}^d \) and \( w \in W_{n, \varepsilon} \) such that

\[
\left\| \sum_{i=1}^{n} w_i (x_i - \nu) (x_i - \nu)^\top \right\| \leq 1 + O(\varepsilon \log(1/\varepsilon)). \tag{D.4}
\]

(ii) If \( \| M(w) \| \leq 1 + \lambda \), for some \( \lambda \geq 0 \), then for any \( w \in W_{n, \varepsilon} \),

\[
\| \nu - \mu(w) \| \leq \frac{1}{1-\varepsilon} \left( \sqrt{\varepsilon\lambda} + C\varepsilon \sqrt{\log(1/\varepsilon)} \right), \tag{D.5}
\]
Algorithm 5: Gradient descent for sub-gaussian robust mean estimation

\textbf{Input}: A set of points \(\{x_i\}_{i=1}^n\), step sizes \(\eta_t\), an iteration count \(T\), and parameter \(\rho\)

\textbf{Output}: A set of weights \(w \in W_{n,\epsilon}\).

1. Let \(w^{(1)} = \frac{1}{n} \mathbf{1}_n\).
2. For \(t\) from 1 to \(T\)
   3. Let \(\nu^{(t)} = \sum_i w_i^{(t)} x_i\), \(M^{(t)} = \sum_i w_i^{(t)} (x_i - \nu^{(t)})(x_i - \nu^{(t)})^T\).
   4. Compute \(\nu^{(t)} = \text{APPROXTOPEIGENVECTOR}(M^{(t)}, 1 - \epsilon^2, \delta/T)\).
   5. If \(\lambda^{(t)} = \nu^{(t)\top} M^{(t)} \nu^{(t)} \leq 1\), return \(w^{(t)}\).
   6. Compute \(\tau_i^{(t)} = \langle v^{(t)}, x_i - \nu^{(t)} \rangle^2\).
   7. Set \(w_t \leftarrow w_t - \eta_t \tau_i^{(t)}\).
   8. Project \(w^{(t+1)}\) onto the set of good weights \(W_{n,\epsilon}\) (under \(\ell_2\) distance).
9. Return \(w^{(t^*)}\), where \(t^* = \arg\min_t \|M^{(t)}\|\).

Then, given \(\{x_i\}_{i=1}^n\), a failure rate \(\delta\) and \(\rho\) such that \(\rho \geq \tau_i^{(t)}\) for all \(i\) and \(t\), Algorithm 5 finds \(w' \in W_{n,\epsilon}\) such that

\[
\|M(w')\| \leq 1 + O(\epsilon \log (1/\epsilon)),
\] (D.6)

with probability at least \(1 - \delta\).

The algorithm terminates in \(T = O(n\rho^2/\epsilon^2)\) iterations. Further, if \(T = O(\text{poly}(n,d))\), then each iteration takes \(O(nd\log (1/\delta)/\epsilon^3)\) time.

\textbf{Proof}. If the algorithm gets early stopped, then \(\|M^{(t)}\| \leq 1 + O(\epsilon^2)\), so assumption (D.4) guarantees that \(\mu(w^{(t)})\) achieves the desired bound (D.6). We now assume that \(\|M^{(t)}\| > 1\) for any \(t\).

By the regret bound (Lemma D.1) and the inequality \(\langle w^{(t)}, \tau^{(t)} \rangle \geq (1 - \epsilon^2) \|M^{(t)}\|_2\), for a \(w\) that satisfies assumption (D.4)

\[
\frac{1 - \epsilon^2}{T} \sum_{t=1}^T \|M^{(t)}\|_2 \leq \frac{1}{T} \sum_{t=1}^T \langle w, \tau^{(t)} \rangle + \frac{3LR}{2\sqrt{T}},
\] (D.7)

where \(L = \sqrt{n\rho}\) and \(R = \sqrt{2}\). For the first term, note that we may apply Claim B.3 and obtain

\[
\frac{1}{T} \sum_{t=1}^T \langle w, \tau^{(t)} \rangle \leq 1 + O(\epsilon \log(1/\epsilon)) + \frac{2\epsilon}{(1 - \epsilon)^2} \frac{1}{T} \sum_{t=1}^T \|M^{(t)}\| - \frac{2\epsilon}{(1 - \epsilon)^2}
\]

By setting \(T = 3L^2 R^2/\epsilon^2 = O(n\rho^2/\epsilon^2)\), we can bound the second term by \(O(\epsilon)\)

Substituting the bounds back into (D.7), we obtain

\[
\frac{1 - \epsilon^2}{T} \sum_{t=1}^T \|M^{(t)}\|_2 \leq 1 - \frac{2\epsilon}{(1 - \epsilon)^2} + O(\epsilon \log(1/\epsilon)) + \frac{1}{T} \sum_{t=1}^T \frac{2\epsilon}{(1 - \epsilon)^2} \|M^{(t)}\|
\]

For sufficiently small \(\epsilon\), we can move the last term to the left side and divide through. This immediately yields that

\[
\frac{1}{T} \sum_{t=1}^T \|M^{(t)}\|_2 \leq 1 + O(\epsilon \log(1/\epsilon)).
\]
The run-time follows from the cost of computing \((1 - \epsilon^2)\)-approximate largest eigenvector via power iteration.

Using the same argument for \textit{Theorem B.4}, \textit{Lemma D.4} implies the following theorem.

**Theorem D.5.** Let \(S = \{x_i\}_{i=1}^n\) be an \(\epsilon\)-corrupted set of \(n\) samples from a sub-gaussian distribution over \(\mathbb{R}^d\), with mean \(\mu\) and identity covariance. Suppose \(n \geq \Omega(d/\epsilon^2)\). Then given \(S\), there is an algorithm (based on Algorithm 5) that finds \(\tilde{\mu}\) such that with high constant probability \(\|\tilde{\mu} - \mu\| \leq O\left(\epsilon \sqrt{\log(1/\epsilon)}\right)\).

The algorithm runs in \(\tilde{O}(nd^2/\epsilon^2)\) iterations and \(\tilde{O}(n^2d^3/\epsilon^2)\) total time.

### D.3 Equivalence with [CDGS20]

The recent work of Cheng, Diakonikolas, Ge and Soltanolkotabi [CDGS20] studies a gradient-descent-based algorithm for solving the following non-convex formulation of robust mean estimation.

\[
\min_{w} \|\Sigma_w\| \quad \text{such that } w \in W_{n,\epsilon},
\]

where \(\Sigma_w = \sum_{i=1}^{n} w_i (x_i - \mu(w)) (x - \mu(w))^\top\). This is equivalent to

\[
\min_w \max_{u \in S^{d-1}} F(w, u) = u^\top \Sigma_w u \quad \text{such that } w \in W_{n,\epsilon}.
\]

The sub-gradient of \(F(w, u)\) with respect to \(w\) (for a fixed \(u\)) is given by

\[
\nabla_w F(w, u) = Xu \odot Xu - 2 \left(w^\top Xu\right) Xu, \tag{D.8}
\]

where \(X \in \mathbb{R}^{n \times d}\) is the data matrix whose the \(i\)th row is \(x_i\).

Based on the observation, they consider and analyze an algorithm that computes a (approximately) maximizing \(u\) and performs a projected gradient descent on \(w\) each iteration.

Since Algorithm 4 can be directly applied to the same robust setting (Corollary 4.4), it is natural to consider the relationships between the two algorithms. Indeed, one can argue that they are essentially the same. First, we unpack our gradient update (\textit{i.e.}, the spectral scores) of iteration \(t\). Note that

\[
\nabla_i f_t(w^{(t)}) = \tau^{(t)}_i = \langle v^{(t)}, x_i - \nu^{(t)} \rangle^2 = \langle v^{(t)}, x_i \rangle^2 + \langle v^{(t)}, \nu^{(t)} \rangle^2 - 2 \langle v^{(t)}, x_i \rangle \langle v^{(t)}, \nu^{(t)} \rangle = \left(X v^{(t)} \odot X v^{(t)}\right)_i + \left(w^{(t)\top} X v^{(t)}\right)^2 - 2 \left(w^{(t)\top} X v^{(t)}\right) \left(X v^{(t)}\right)_i
\]

since \(\nu^{(t)} = \sum_i w_i x_i = X^T w^{(t)}\), where \(\odot\) denotes entrywise product of vectors. Let \(C_t = w^{(t)\top} X v^{(t)}\). Therefore, we can rewrite the gradient as

\[
\nabla f_t(w^{(t)}) = C_t^2 \cdot 1_n + X v^{(t)} \odot X v^{(t)} - 2 C_t \cdot X v^{(t)}
\]

Note that the gradient (D.8) used in [CDGS20] is exactly the same as above, except without the term of all-one vector \(C_t^2 \cdot 1_n\). In the gradient update step, the additional term reduces the weight
of every point uniformly by the same quantity $C_l^2$. However, observe that by Pythagorean theorem, the (Euclidean) projection onto $W_{n,\epsilon}$ can be decomposed into two (sequential) steps: (1) first an orthogonal projection onto the affine subspace containing $W_{n,\epsilon}$, and then (2) a projection onto $W_{n,\epsilon}$ itself. Note that reducing each coordinate by the same quantity or not results in the same vector by the first step. Therefore, the two algorithms yield the same sequence of iterates $(w^{(t)})_t$.

**E Optimal Breakdown Point Analysis**

We now consider a slight variant of the filter algorithm and show that it achieves the optimal breakdown point of $\epsilon = 1/2$ (for the spectral sample reweighting problem). Recall that both the classic filter algorithm and our Algorithm 2 work with the spectral scores defined as $\tau_i = \left(\langle v(t), x_i \rangle - \langle v(t), \nu(t) \rangle\right)^2$, where the second term is the (weighted) average of the first. Instead, the following variant replaces that by the median.

Throughout we let $\nu^{(t)} = \sum_i w_i^{(t)} x_i$, $M^{(t)} = \sum_i w_i^{(t)} (x_i - \nu^{(t)})(x_i - \nu^{(t)})^T$.

**Algorithm 6:** Optimal filter for spectral sample reweighing (Definition 3.1)

- **Input:** A set of points $\{x_i\}_{i=1}^n$, an iteration count $T$, and parameter $\delta$
- **Output:** A point $\nu \in \mathbb{R}^d$ and weights $w \in C_{n,\epsilon}$.

1. Let $w^{(1)} = \frac{1}{n} 1_n$.
2. While $\|M^{(t)}\| \geq \frac{16}{7} \lambda \left(1 + \frac{1}{1/2-\epsilon}\right)$
   3. Compute $\nu^{(t)} = \text{APPROXTOPEIGENVECTOR}(M^{(t)}, 7/8, \delta/T)$.
   4. Compute $\alpha_i^{(t)} = \langle v(t), x_i \rangle$ for each $i$ and let $m^{(t)} = \text{median}(\{\alpha_i^{(t)}\}_{i=1}^n)$
   5. Compute $\tau_i^{(t)} = \left(\alpha_i^{(t)} - m^{(t)}\right)^2$ for each $i$ and $\tau_{\max} = \max_{i:w_i>0} \tau_i^{(t)}$.
   6. Set $w_i^{(t+1)} \leftarrow w_i^{(t)} \left(1 - \tau_i^{(t)}/\tau_{\max}\right)$ for each $i$, and $t \leftarrow t + 1$.
7. Return $\nu^{(t)}, w^{(t)}$.

Our proof follows by tracing the argument of the soft down-weighting filter [Li19b]. First, note Lemma A.1 guarantees that there exists $G \subseteq [n]$ such that $|G| \geq (1 - \epsilon)n$ and

$$\frac{1}{(1 - \epsilon)n} \sum_{i \in G} (x_i - \nu) (x_i - \nu)^T \preceq \lambda I. \quad (E.1)$$

Let $B = [n] \setminus G$, and we first establish a technical condition on $m^{(t)}$.

**Lemma E.1.** Let $\beta(t) = \frac{1}{n} \sum_{i \in G} \alpha_i^{(t)}$. Then we have $|m^{(t)} - \beta(t)|^2 \leq \frac{\lambda}{1/2-\epsilon}$.

**Proof.** We fix one iteration and drop the superscript. Let $\mu_G = \frac{1}{n} \sum_{i \in G} x_i$. First, observe that by (E.1), we have

$$\frac{1}{(1 - \epsilon)n} \sum_{i \in G} (x_i - \mu_G) (x_i - \mu_G)^T \preceq \lambda I. \quad (E.2)$$

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Therefore, $E_{i \sim G}[(\alpha_i - \beta)^2] = E_{i \sim G}(v, \mu_G - x_i)^2 \leq \lambda$. By Chebyshev’s inequality,

$$\Pr_{i \sim G}(|\alpha_i - \beta| > \sqrt{\frac{\lambda}{1/2 - \epsilon}}) \leq \frac{1}{2} - \epsilon.$$  \hspace{1cm} (E.3)

This means that we have $|G| \cdot (1/2 + \epsilon)$ points $i \in [n]$ that satisfy $|\alpha_i - \beta|^2 \leq \frac{\lambda}{1/2 - \epsilon}$. Our claim now follows since $|G| \geq (1 - \epsilon)$ and $(1 - \epsilon)(1/2 + \epsilon) > 1/2$ for any $\epsilon \in (0, 1/2)$.

This allows us to establish the key invariant of the algorithm.

**Lemma E.2.** Suppose at iteration $s$, we have that

$$\left\|M^{(s)}\right\| \geq \frac{16}{7} \frac{\lambda}{1 + \frac{1}{1/2 - \epsilon}}.$$  \hspace{1cm} (E.4)

and for $t = s$

$$\sum_{i \in G} \frac{1}{n} - w_i^{(t)} < \sum_{i \in B} \frac{1}{n} - w_i^{(t)}.$$  \hspace{1cm} (E.5)

Then the condition (E.5) continues to hold for $t = s + 1$.

**Proof.** Observe that to prove the claim inductively, it suffices to show that for any $s$,

$$\sum_{i \in G} w_i^{(s)} - w_i^{(s+1)} < \sum_{i \in B} w_i^{(s)} - w_i^{(s+1)}.$$  \hspace{1cm} (E.6)

We now just focus on these two iterations, drop the superscript and denote $w^{(s+1)}$ by $w'$. By definition of the update step (line 7 of Algorithm 6), we just need to prove that

$$\sum_{i \in G} w_i t_i < \sum_{i \in B} w_i t_i.$$  \hspace{1cm} (E.7)

Now note that since $\nu = \mu(w) = \sum_{i=1}^n w_i x_i$, we have

$$\sum_{i=1}^n w_i t_i = \sum_{i=1}^n w_i ((v, x_i) - m)^2$$

$$= \sum_{i=1}^n w_i ((v, x_i - \nu) + (\nu, v) - m)^2$$

$$= \sum_{i=1}^n w_i ((v, x_i - \nu)^2 + (m - \nu, v)^2)$$

$$\geq \sum_{i=1}^n w_i (v, x_i - \nu)^2$$

$$= v^\top M v \geq \frac{7}{8} \|M\|_2.$$ 

Hence, to establish invariant (E.7), we proceed by showing that

$$\sum_{i \in G} w_i t_i \leq \frac{7}{16} \|M\|_2.$$  \hspace{1cm} (E.8)
Since \( w_i \leq \frac{1}{n} \), we have \( \sum_{i \in G} w_i \tau_i \leq \sum_{i \in G} \frac{1}{n} (\langle v, x_i \rangle - m)^2 \). On the other hand, let \( \mu_G = \frac{1}{n} \sum_{i \in G} x_i \), and so by condition (E.1) and Lemma E.1,

\[
\sum_{i \in G} \frac{1}{n} (\langle v, x_i \rangle - m)^2 = \frac{1}{n} \sum_{i \in G} \langle v, x_i - \mu_G \rangle^2 + |\langle \mu_G, v \rangle - m|^2 \\
\leq \lambda + \frac{\lambda}{1/2 - \epsilon} \\
\leq \frac{7}{16} \|M\|,
\]

by our assumption (E.4). This completes the proof. \( \square \)

**Theorem E.3.** For any \( \epsilon \in (0, 1/2) \), Algorithm 6 gives a constant approximation to the spectral sample reweighting problem (Definition 3.1). The algorithm terminates in \( T = O(n) \) iterations.

**Proof.** The run-time follows from the invariant Lemma E.2, which guarantees weights on bad points are removed more than those on good points. Hence, after \( 2\epsilon n \) iterations, the algorithm must terminate. Moreover, when the algorithm terminates, we have

\[
\|M^{(t)}\| \leq \frac{16}{7} \lambda \left( 1 + \frac{1}{1/2 - \epsilon} \right).
\]

(E.9)

For any constant \( \epsilon \leq 1/2 - O(1) \), the bound is \( O(\lambda) \). \( \square \)

**Robust mean estimation.** Unfortunately, our reduction (Theorem 4.1) is not sufficiently tight for the purpose of attaining the optimal breakdown point of 1/2 in robust mean estimation. Hence, it cannot be directly applied here. Instead, we will appeal to the following more refined spectral signature.

**Claim E.4** (refined spectral signature [Li19a]). Let \( S = S_g \cup S_b \setminus S_r \) be \( n \) points with \( |S_b| = |S_r| = \epsilon n \). Define \( \mu_g = \frac{1}{n} \sum_{i \in S_g} x_i \) and \( \Sigma = \frac{1}{n} \sum_{i \in S_g} (x_i - \mu)(x_i - \mu)^\top \). Let \( w(S) \) be the uniform distribution on \( S \) and \( \mathcal{C}_{n, \epsilon} = \{ w : \|w - w(S)\|_1 \leq \epsilon, 0 \leq w_i \leq 1/n \text{ for } i \in [n] \} \). Then for any \( w \in \mathcal{C}_{n, \epsilon} \),

\[
\left( \sum_{i \in S \cap S_g} w_i \right) \|\mu - \mu(w)\| \leq \sqrt{2\epsilon\|\Sigma\|} + \sqrt{\epsilon \|\Sigma(w)\|}.
\]

**Theorem E.5.** For the problem of robust mean estimation (under bounded second moment), Algorithm 6 attains the optimal estimation error \( O(\sqrt{\epsilon}) \) for any \( \epsilon < 1/2 \).

**Proof.** By Lemma E.2, our algorithm always removes more weights from bad points than from good points. Thus, \( w^{(t)} \in \mathcal{C}_{n, 2\epsilon} \), as there are at most \( \epsilon n \) bad points. Moreover, \( \sum_{i \in S \cap S_g} w_i \geq 1 - 2\epsilon \).

For robust mean estimation, if we have \( n = \Omega(d \log d/\epsilon) \) samples, then \( \|\mu_g - \mu\| \leq O(\sqrt{\epsilon}) \) and \( \lambda = \|\Sigma\| \leq 2 \) [DKK+19a]. Hence, applying Claim E.4 and the guarantee that \( \|M^{(t)}\| \leq \frac{16}{7} \lambda \left( 1 + \frac{1}{1/2 - \epsilon} \right), \)

\[
\|\mu_g - \nu^{(t)}\| \leq \frac{1}{1-2\epsilon} \left( 2\sqrt{\epsilon\|\Sigma\|} + \sqrt{2\epsilon \|M^{(t)}\|} \right) \leq O(\sqrt{\epsilon}),
\]

for any \( \epsilon < 1/2 \). Finally, triangle inequality implies that \( \|\mu - \nu^{(t)}\| = O(\sqrt{\epsilon}) \). \( \square \)