New Discretization of Complex Analysis: The Euclidean and Hyperbolic Planes

Abstract. Discretization of Complex Analysis on the Plane based on the standard square lattice was started in 1940s. It was developed by many people and also extended to the surfaces subdivided by the squares. In our opinion, this standard discretization does not preserve well-known remarkable features of the Completely Integrable System. These features certainly characterize the standard Cauchy Continuous Complex Analysis. They played a key role in the great success of Complex Analysis in Mathematics and Applications. Few years ago we developed jointly with I.Dynnikov a New Discretization of Complex Analysis (DCA) based on the two-dimensional manifolds with colored black/white triangulation (see [9]). Especially deep results were obtained for the Euclidean plane with equilateral triangle lattice in the works [9, 10]. Our approach preserves a lot of features of Completely Integrable Systems. In the present work we develop a DCA theory for the analogs of equilateral triangle lattice in the Hyperbolic plane. This case is much more difficult than Euclidean. Many problems (easily solved for the Euclidean Plane) are not solved here yet. Some specific very interesting ”dynamical phenomena” appear in this case: for example, description of boundaries of the most fundamental geometric objects (like the round ball) leads to dynamical problems. Mike Boyle from the University of Maryland helped me to use here the methods of symbolic dynamics.

Introduction. History. We do not discuss here ”geometric” discretizations of conformal mappings started in early XX century-see survey and history in [5]. By the way, specific topological properties of surfaces subdivided by the squares, was started by the combinatorial geometry/topology group

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in the Steklov Institute in 1980s by the suggestion of the present author (see [6]). This group called this object ”The Quadrillage”. I formulated these problems under the influence of the Statistical Physics of Lattice Models, after discussions with A.Polyakov. It is mentioned also in my Encyclopedia volume number 13, Topology-I. see [7]. In the works of the group [5] made in last decade such surfaces subdivided into squares are called ”quad-graphs”.

Our goal is to discretize Cauchy-Riemann operator \( \bar{\partial} \) as a Linear Difference First Order Operator on Triangulated Surfaces. Let us say at the very beginning of this article that Neither in the Standard Approach nor in our New Discretization The Discrete Analogs of Holomorphic Functions form a Commutative Associative Ring (at least naturally).

The Discretization of Complex Analysis as a Linear Difference Operator was done for the square lattice in \( \mathbb{R}^2 \) by Lelong-Ferrand. Her work was published in 1944 (see [2]). In the work [3] Duffin developed or rediscovered this approach. He also extended it to the rombic lattices later.

Duffin claimed that another mathematician already discovered this idea before and published his work in 1941 in the notes of some small provincial South American University. Our attempts to find this obscure edition failed, so we continue to quote Lelong-Ferrand as a first author who invented this idea.

In the standard approach Discrete Analog of the Cauchy-Riemann Operator \( \bar{\partial} \) acts on the \( C \)-valued functions \( \psi \) of vertices in the square lattice on the plane:

\[
Q_{\text{square}} \psi(m, n) = \psi(m, n) + i\psi(m + 1, n) - \psi(m + 1, n + 1) - i\psi(m, n + 1)
\]

By definition, every solution to the equation \( Q_{\text{square}} \psi = 0 \) is a d-holomorphic function.

A lot of people work here now developing this approach (see, for example, in [5]).

Let us present a hint of the idea: Why we are not satisfied by the standard discretization? Of course, every continuous model admits many difference approximations which converge to it in the continuum limit.

For the continuous systems with huge hidden algebraic symmetry we wish to find optimal discretization preserving as much symmetry as possible. As we already know, there exists no natural discretization preserving multiplication of holomorphic functions. All attempts to invent multiplicative structure with good properties failed. It became clear many years ago.
Let us point out following:
1. In the standard approach discrete analog of the Cauchy-Riemann operator is in fact a second order difference operator: two lengths are involved in the sum (the lengths of sides and diagonals).

**We wish to have a first order difference operator as a natural discrete analog of \( \partial \). Is it possible?** The answer is YES for the Equilateral Triangle Lattice. Corresponding theory of the difference first order ”Triangle Operators” on the simplicial complexes is developed in the Part 1 in more general form. We define also non-standard discrete analogs of \( GL_n \)-Connections for the triangulated n-manifolds.

In the Part 2 we develop a general theory of discrete analogs of holomorphic functions (the d-holomorphic functions) on the 2-manifolds with discrete analog of conformal structure (DCS). **We define DCS as a colored black/white triangulation.**

The d-analogs of Liouville Principle and Maximum Principle will be clarified here for the general 2-surfaces with DCS.

Part 3 is dedicated to the theory of d-holomorphic functions on the Equilateral Triangle Lattice.

2. Continuous 2D Laplace operator admits a natural factorization

\[
\Delta = \partial \partial
\]

Nothing like that exists on the square lattice.

*However, Equilateral Triangle Lattice is much better for this—see below.*

**Problems:**

1. How to construct Discrete Analog of Holomorphic Polynomials without multiplication?

People did it with great efforts after many years in the standard approach. Their constructions are non-canonical and non-unique.

It is very easy in our new approach. Our construction is obvious and canonical.

2. How to construct rational functions without multiplication?

In particular, in the standard continuous complex analysis rational functions are especially important because the function \( 1/z \) (the Cauchy Kernel) is also a unique fundamental solution to the equation

\[
\bar{\partial}(\psi(z)) = 2\pi i \delta(z)
\]
decreasing at infinity?

The answer to this question is also positive and canonical in our approach. It was missed in our first work with Dynnikov [9] and made by Grinevich and R. Novikov later in [10].

Part 4 is dedicated to the Hyperbolic (Lobachevski) Plane $H$.

Equilateral Triangular Lattice in $H$ is defined simply as a triangulation of plane such that $p$ triangles hit every vertex and $p \geq 7$. We need $p = 2s$. For all $s \geq 4$ we have a triangulation of Lobachevski Plane $H$ such that it admits a black/white coloring. Let us concentrate our attention on the case $s = 4$.

At the moment we can construct neither $d$-analog of polynomials nor $d$-rational functions.

Even boundary of round ball $D_r$ of radius $r \in Z$ is a complicated object. How to describe it? How many points does it contain? We succeeded in the solution of this problem using methods of symbolic dynamics. Mike Boyle from the University of Maryland helped us. The boundary $\partial D_r$ contains approximately $\lambda^r$ points where $\lambda = 2 + \sqrt{3}$.

It turns out that for the reconstruction of $d$-holomorphic function in the ball $D_{r+1}$ we need to know values of the boundary function in $[n(r+1)/2] + 1$ points. Here $n(r)$ is a number of vertices in the boundary. It exactly corresponds to the continuous case where only set of coefficients at the nonnegative powers of $z$ (i.e. one half plus one Fourier coefficients of function on the boundary circle) determines completely our function inside. The main unsolved problem is:

**Problem. How to find a bounded basis of $d$-holomorphic functions on Hyperbolic Plane?**

**Part I. Definitions. Discrete $GL_n$-Connections. B/W manifolds. The First Order Triangle Operators.**

Take a simplicial complex $M$. We assume that this complex is given with canonical metric where every simplex is a standard unit linear subsimplex in euclidean space with natural euclidean metric. We fix a family of $n$-simplices such that every vertex belongs at least to one simplex of that family. Fix also collection of nonzero numbers $b_{T,P} \neq 0$ for all $T \in X$ and $P \in T$.

**Definition 1** We call following operator the Triangle Operator $Q^X$ associ...
It maps functions of vertices into functions of simplices \( T \in X \). Our coefficients normally belong to \( R \) or \( C \). The operators \( Q^X \) are the first order linear difference operators.

Now we define a Discrete Version of \( GL_n \)-Connections.

**Definition 2** Let \( X \) be a family of all \( n \)-simplices in \( n \)-manifold \( M \). We call the equation \( Q\psi = 0 \) Discrete Differentially-Geometrical \( GL_n \)-Connection. The coefficients \( b_{T;P} \) are defined in every \( n \)-simplex up to nonzero factor. So every DG-Connection is defined by the set of ratios

\[
\mu^T_{PP'} = b_{T;P}/b_{T;P'}
\]

This discretization is different from the standard Wilson Discretization used by physicists studying Yang-Mills fields; There is no natural way to select compact holonomy groups in our approach.

The theory of discrete \( GL_n \) connections was constructed in the works [1, 4], some first ideas appeared in [8, 9] in connection with completely integrable systems. We are working only with scalar-valued functions. At the same time we are going to define a Nonabelian Curvature and Holonomy with values in the group \( GL_n \) for the \( n \)-manifolds.

As we demonstrated in the [4], two different Holonomy Representations can be constructed here: Abelian ("Framed Abelian") and Nonabelian. We need here only Nonabelian Holonomy.

What is a Nonabelian Discrete Curvature? A Nonabelian Holonomy group is defined along the Thick Paths.

**Definition 3** We call by Thick Path any sequence of \( n \)-simplices \(< T_1 T_2 \ldots T_k > \) such that intersection \( T_j \cap T_{j+1} \) is exactly \( n - 1 \)-dimensional face \( \Delta_j \), and \( \Delta_j \neq \Delta_{j+1} \). We say that Thick Path is closed if \( T_k = T_1 \). Its length is equal to \( k - 1 \).

Therefore in every simplex \( T_j \) of the thick path exactly two different \( n - 1 \)-dimensional faces are fixed: \( \Delta_j \subset T_{j+1} \) is called an "In-face" in \( T_{j+1} \), and \( \Delta_j \subset T_j \) is called an "Out-face" in \( T_j \).
Let us define a **Parallel Transport** of the $n$-vector $\psi$ consisting of values in all vertices of the in-face $\Delta_1 \subset T_1$, along the thick path $T_1...T_k$. Solving equation $Q = 0$, we define this function in all vertices of $T_1$. So we know values of $\psi$ in all vertices of the out-face in $T_1$ equal to the in-face in $T_2$. After that we are doing the same procedure for $T_2$ and so on. Finally we define a "Transported value" of $n$-vector $\psi$ in the out-face of the simplex $T_k$. This map is linear. For the closed paths we are coming to the **Nonabelian Holonomy Representation of the Semigroup of the closed thick paths into the Group $GL_n$.**

$$\Omega_{thick}(M, \Delta) :\rightarrow GL_n$$

**Definition 4** We call by the **Nonabelian Discrete Curvature in the vertex $P$** a Holonomy Linear Map along the Thick Closed Paths in the Simplicial Star $St(P)$. It is enough to know holonomy for all thick closed paths around all $n − 2 = $simplices—see Fig 1 for $n = 2$.

![Fig 1](image)

However, the natural gauge group

$$\psi \rightarrow f\psi, Q \rightarrow g(T)Qf^{-1}(P)$$

is abelian, $f \neq 0, g \neq 0$. There are very interesting specific features here realizing non-abelian $GL_n$ Connections using only the spaces of scalar functions of vertices. Some sort of mixing abelian and non-abelian properties appears. Details of classification can be found in the work [4]. No theory of Characteristic Classes is constructed yet.

**Definition 5** We call special discrete $GL_n$-Connection with $b_{T:P} = 1$, a **Canonical Connection**
Consider now any \( n \)-manifold. Is it possible to color its \( n \)-simplices by the black and white colors such that for every pair of \( n \)-simplices attached to each other along \( n-1 \)-face, the colors are opposite?

**Definition 6** We call such colored manifolds a B/W manifolds.

Such coloring exists if and only if every closed thick path consists of even number of \( n \)-simplices. More general, fix any \( n-1 \)-simplex \( \Delta \). We have a natural homomorphism

\[ \Phi_2 : \Omega_{thick}(M, \Delta) \to \mathbb{Z}/2\mathbb{Z} \]

defined as a number of \( n \)-simplices in any thick path modulo 2. This map also looks like simplest Connection whose Holonomy Group is \( \mathbb{Z}/2\mathbb{Z} \), but we do not realize it by the linear operator.

**Definition 7** Let the map \( \Phi_2 \) is trivial for all thick closed paths in the star of every vertex. We call such triangulation "Locally B/W".

In this case following simple lemmas are true:

**Lemma 1** Under this assumptions the map \( \Phi_2 \) defines a map \( \phi_2 : \pi_1(M) \to \mathbb{Z}/2\mathbb{Z} \).

**Lemma 2** Curvature of the Canonical Connection is trivial for every locally B/W manifold.

Proof. Fix any \( n-2 \)-simplex \( S \) in the star \( St(P) \). Let exactly \( s(S) \) \( n \)-simplices contain \( S \). The Nonabelian Holonomy with initial values \( a_0, ..., a_n \) in the vertices of \( n \)-simplex \( T_n = < S, P_{n-1}, P_n > \) containing \( S \) (and around it), is determined by permutation \( A^{s(P)} \). Here \( A(a_0, ..., a_{n-2}, a_{n-1}, a_n) = (a_0, ..., a_{n-2}, a_n, a_{n-1}) \). So \( A^2 = 1 \). We see that Nonabelian Holonomy (Curvature) is equal to 1 if and only if \( s(S) \) is even. Lemma is proved.

Every locally flat Connection (i.e. Nonabelian Curvature is trivial for every vertex) defines a homomorphism of topological holonomy \( \pi_1(M) \to GL_n \).

**Lemma 3** Every flat Canonical Connection defines a Topological Holonomy Homomorphism. Its image belongs to the subgroup \( S_{n+1} \subset GL_n \).
Proof. The imbedding is following: Take arbitrary distinct \( n + 1 \) numbers \( a_0, \ldots, a_n \) such that \( \sum a_j = 0 \). Put this set of numbers in the vertices of initial \( n \)-simplex \( T_1 \). For every thick closed path \( T_1, \ldots, T_k \) the values of Parallel Transport along this path in every simplex \( T_j \) consists of the very same numbers \( a_q \) because \( \sum a_j = 0 \), and Connection is Canonical. In the final simplex \( T_k = T_1 \) we obtain permutation belonging to \( S_{n+1} \) which does not depend on the choice of initial numbers \( a_j \). Lemma is proved.

Now we are going to consider only Orientable Manifolds \( M \).

**Lemma 4** Let Curvature and Topological Holonomy are trivial for the Canonical Connection (i.e. it is flat). Then B/W coloring exists globally.

Proof. Let a closed thick path \( < T_1, \ldots, T_k > \) defines a trivial permutation corresponding to the Canonical Connection. We claim that this path consists of the even number of \( n \)-simplices. Indeed, it follows from the fact that our Nonabelian Holonomy is equal to the product of elementary transpositions. Every one of them changes orientation. The whole path preserves orientation. So our lemma follows.

**Lemma 5** For every manifold \( M \) with flat Canonical Connection the equation \( Q\psi = 0 \) defines \( n \)-dimensional linear space of Covariant Constants.

Proof obviously follows from the fact that every initial values in the \( n-1 \)-simplex \( \Delta \) defines a covariant constant correctly because our connection is flat.

So we fix a manifold \( M \) with B/W structure. We always assume that our Canonical Connection is flat. It is always true for simply connected manifolds. It is locally flat because B/W structure is given. For non-simply-connected case B/W structure implies that Nonabelian Holonomy maps \( \pi_1(M) \) into \( S_{n+1} \). So Canonical Connection became globally flat on some finite covering with number of sheets no more than \( n! \). We define \( d \)-analog of holomorphic functions in every B/W manifolds (see below).

**How to construct Continuous Limit for our Discretization of Complex Analysis?**

For that we are doing the following procedure: Let \( n = 2 \). Construct a special Covariant Constant \( f_0 \) whose values in every triangle are \( 1, \zeta, \zeta^2 \) where \( \zeta^3 = 1 \). It is unique up to the group \( S_3 \) permuting vertices of the initial triangle. Apply following gauge transformation to the Canonical Connection:

\[
Q \rightarrow f_0^{-1}Qf_0, \quad \psi \rightarrow f_0^{-1}\psi
\]
We still have two-dimensional space of covariant constants for this connection but in this gauge one of covariant constants became an ordinary constant. We work over the extended field containing $\zeta$. For the case $R$ and extended field $C$ we used this construction to get Continuum Limit of our theory to the ordinary complex analysis. For $M = R^2$ exactly one half of our theory converges to the ordinary complex analysis. The second half diverges for small triangulations.

Now we start to construct New Discretization of Complex Analysis.

Let $M$ be any B/W-manifold. Following two families play fundamental role in our theory: $X_1 = b$ (all black simplices) and $X_2 = w$ (all white simplices)—see Fig 2.

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**Definition 8** The operators $Q^b, Q^w$ are called Black and White Triangle Operators. The union of this families $b \cup w$ is complete. Using the Canonical Connection, we define an operator $Q = Q^b \oplus Q^w$.

Following lemma is very simple:

**Lemma 6** The operators $Q^b, Q^w, Q$ have following properties:

$$L = Q^*Q = 2Q^{bw}Q^b = 2Q^{bw}Q^w$$
Let \( \Delta = dd^* \) be an ordinary scalar simplicial Laplace-Beltrami Operator, \( m_P \) is the number of edges entering vertex \( P \). For 2-manifolds we have \( Q^*Q = -2\Delta + 3m_P \).

Proof follows from the elementary calculation.

**Definition 9** Consider any function \( \psi \) such that \( Q^b \psi = 0 \). For \( n = 2 \) we call such functions \( d \)-holomorphic. We call solutions to the equation \( Q^w \psi = 0 \) \( d \)-antiholomorphic.

Now we assume that the Canonical Connection is flat. For every black triangle \( T \) we define a unique covariant constant \( E_\psi(T) \) whose values in \( T \) coincide with \( \psi \).

**Definition 10** We call \( E_\psi \) an Evaluation of function \( \psi \) in the black triangle \( T \).

As a conclusion, we can view every \( d \)-holomorphic function either as a real-valued function \( \psi \) on the vertices or as a \( R^2 \)-valued function \( E_\psi(T) \) of the black triangle \( T \).

**Part II. D-holomorphic functions on 2-manifolds. Liouville Principle and Maximum Principle**

As above, we consider triangulated orientable B/W manifolds \( M \) with flat Canonical Connection. Our field here is \( R \). Following general properties of \( d \)-holomorphic functions were found in [1]:

**Theorem 1** (The Liouville Principle). For the compact closed 2-manifold every \( d \)-holomorphic function is a covariant constant.

Proof is based on the ”instanton phenomenon” for the quadratic functional \( (L\psi,\psi) \) if \( L \) is factorizable \( L = Q^*Q \). Let the global minima \( \psi \) in the Hilbert space \( L_2 \) is a zero mode \( L\psi = 0 \). The order of Euler-Lagrange equation drops twice reducing to the ”Self-Duality Equation” which is in
our case simply the d-analog of the Cauchy-Riemann Equation. Let us re-
mind here that This Property is the most fundamental feature of
the standard continuous Complex Analysis unifying it with The
Completely Integrable Systems. The proof of our theorem is following:

\( Q^b \psi = 0 \) implies \( (Q^b)^* Q^b \psi = 0 \) implies \( L \psi = Q^* Q \psi = 0 \) implies \( (L \psi, \psi) = 0 \) implies \( (Q \psi, Q \psi) = 0 \) implies \( Q \psi = 0 \), so our theorem is proved.

In principle, the famous Instanton Phenomena discovered by Polyakov,
Belavin, Schwarz and Tiupkin in 1974 in the Nonlinear Variational Calculus
(Yang-Mills Field Theory), is based on the similar arguments.

Let \( D \) be a bounded domain in \( M \) consisting of black triangles.

**Definition 11** We call \( T \in D \) a boundary triangle if some of its vertices
belong to black triangle not belonging to the domain \( D \).

**Theorem 2** *(The Maximum Principle)* For every d-holomorphic function \( \psi 
in D \) the set of covariant constants \( E \psi(T) \in \mathbb{R}^2 \) for all \( T \in D \) is contained
in the convex hull of the image of boundary triangles.

Proof can be found in [1].

In the next chapters we consider specific results obtained for the Euclidean
(Part III) and Lobachevski or Hyperbolic Planes (Part IV).

**Part III. Equilateral Triangle Lattice in \( \mathbb{R}^2 \).**

The standard regular equilateral lattice in \( \mathbb{R}^2 \) (see Fig 3) admits two basic
(unitary) shift operators \( t_1, t_2 \) acting on vertices

\[
t_1(m, n) = (m + 1, n), \quad t_2(m, n) = (m, n + 1)
\]
All six elementary shifts $t_i^{\pm 1}, (t_1 t_2^{-1})^{\pm 1}$ have equal length. The operators $Q^b, Q^w$ are equal to

$$Q^b = 1 + t_1 + t_2, \quad Q^w = (Q^b)^* = 1 + t_1^{-1} + t_2^{-1}$$

They map the space of functions of vertices into itself and are adjoint to each other. Our field here is $R$. In the works [8, 9] we defined first time black and white triangle operators and invented the idea of new type discrete $GL_2$ connection. We proved also that every difference second order self-adjoint operator $L = a + bt_1 + ct_2 + dt_1 t_2^{-1} + (adjoint)$ admits Laplace-type factorization $L = Q^* Q + W$ where $Q$ is some triangle operator $Q = u + vt_1 + wt_2$. and $W, a, b, c, d, u, v, w$ are some real functions on this lattice. In particular, $-\Delta + 9 = Q^b Q^w$ for the standard Laplace-Beltrami operator. A theory of discrete completely integrable systems based on the discretized second order operators, was started.

**Definition 12** We call $d$-holomorphic function $\psi$ polynomial (i.e. $\psi \in \text{Pol}_k$) of degree $k$ if $Q^b \psi = 0$ and $(Q^w)^{k+1} \psi = 0$.

As it was established in [11], these functions have a $k$-polynomial growth at infinity in $R^2$. They are completely determined by their values in any standard $k$-triangles $T_k$, black from inside, with $2k + 2$ points in each edge (see Fig 4).
So the dimension of the space \( \text{Pol}_k \) of \( k \)-polynomials is \( 2k + 2 \). We can choose the canonical \( k \)-polynomials \( \psi_{T_k, \alpha}, \alpha = 1, 2, 3 \). They are equal to zero in \( T_k \) everywhere except one boundary edge \( \alpha \) where they have values \( \pm 1 \).

Sum of them belongs to the space \( \text{Pol}_{k-1} \):

\[
\sum_{\alpha} \psi_{k, \alpha} \in \text{Pol}_{k-1}
\]

Therefore for the choice of basis we have to select pair of edges in \( T_k \).

Following theorem ([1]) give natural analog of polynomial approximation of holomorphic functions leading to the "d-Taylor Series":

**Theorem 3** For every function \( \psi \), \( d \)-holomorphic in \( \mathbb{R}^2 \), and every canonical triangle \( T_k \), there exists a unique \( k \)-polynomial \( \phi \) such that \( \psi - \phi \) is identically equal to zero in the triangle \( T_k \).

For the Taylor decomposition of \( \psi \) we have to choose an increasing sequence of triangles \( T_k, k = 0, 1, 2, ... \) and pair of boundary edges in each of them. The choice of such sequence is non-canonical.

**How to get analog of the Cauchy formula?**

We need to construct a "Cauchy Kernel" (d-analog of \( 1/z \)) satisfying to equation:

\[
Q^b G(x, y) = \delta(x - y)
\]

where \( x, y \) are points in the lattice \( x = (m, n), y = (m', n') \), and difference operator \( Q^b \) acts on the variables \( x = (m, n) \). Having any such function, we
consider any bounded domain $D$ and $d$-holomorphic function $\psi$ in $D$. Let us extend this function to the function $\bar{\psi}$ such that

$$\bar{\psi}(x) = \psi(x), x \in D; \bar{\psi}(x) = 0, x \in R^2 \text{minus} D$$

We have

$$\sum_y [Q^b \bar{\psi}(y)] G(x - y) = \psi(x)$$

for all $x \in D$. Let us point out that the function $Q^b \bar{\psi}(x)$ is concentrated along the "boundary strip", so it is really analog of the Cauchy formula. In the work $[1]$ we constructed following "hyperbolic-like" Cauchy Kernel (see Fig 6). We call it Pascal Triangle. It is equal to zero outside of infinite triangle and has exponential growth inside of it.

As it was pointed out by P. Grinevich and R. Novikov in $[10]$, much better Green function can be obtained simply by the Fourier Transform

$$G(x) = (\text{const}) \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 e^{imk_1} e^{ink_2} (1 + \exp\{ik_1\} + \exp\{ik_2\})^{-1}$$

where $x = (m, n)$.

They proved following theorem:

**Theorem 4** This integral converges. This Green Function decays at infinity as $c/d(P)$ where $d(P)$ is equal to the distance of this point to zero. In particular, every $d$-holomorphic function on the whole equilateral triangle lattice whose growth at infinity is no more that polynomial, is a $d$-holomorphic polynomial.
Conclusion. This result gives a unique function decaying like $1/|x|$ at $|x| \to \infty$. All functions $(Q^w)^k G(x)$ has a growth like $1/|x|^{k+1}$. Therefore we have also analogs of rational function without any of multiplication (which does not exist).

Part IV. D-holomorphic functions on The Hyperbolic (Lobachevski) Plane.

Every triangulated plane $H^2$ such that more than six triangles (edges) hit every vertex, can be viewed as a negative curvature plane. An Equilateral Triangle Lattice we get in the case then this number is the same for all vertices (i.e. $m_P \geq 7$). We need also $B/W$ structure so our Equilateral Triangle Lattices $H^2_m$ in $H^2$ with $d$-conformal structure (i.e. with $B/W$ coloring) correspond to $m = 8, 10, 12, \ldots$. We consider here only the case $m = 8$.

Let us point out that the triangles of this lattice cannot be made arbitrary small in the standard Lobachevski metric (their size is fixed in $H^2$ by the number $m$) but our domain $D$ can be made arbitrarily large.

For example, for every finite set of vertices $K \subset H^2_m$ and positive integer $r \in \mathbb{Z}_+$ we define a domain $D_{K,r}$ consisting of vertices $x \in D_{K,r}$ such that distance $d(x,K)$ is no more than $r$. We measure distances between vertices (sets) counting minimal number of edges needed for joining them. The simplest important domains of that kind correspond to the cases: 1. $K$=vertex 0 (we call it ”standard ball” $D_r$); 2. $K$=triangle $T$ i.e. $K$=3 vertices of $T$. We denote it $D_{T,r}$. It is also like a ball with center in the center of triangle. For $K = D_{K',r}$ we have $D_{K,r} = D_{K',r+1}$ if $K$ is connected (i.e. no jumps of the length more than one are needed to reach one point from another).

There is very big automorphism group mapping this Lattice into itself but this group is non-commutative. It does not contain big enough commutative subgroups, so nothing like Fourier transform exists here. We cannot construct good enough Green function. Our operators $Q^b, Q^w$ map space of functions of vertices into the space of functions of black and white triangles correspondingly. Therefore we cannot iterate them. So we cannot construct analogs of polynomials here.

**Problem:** How to construct basis of d-holomorphic functions $\psi_l(x)$ in $H^2_8$ which are globally bounded in all space?

In the continuous case our space is realized as a unit disc $D^2 : |z| < 1$. We have basis of bounded holomorphic functions $z^k, k = 0, 1, 2, \ldots$. Every
rational function with poles outside of unit disc is bounded in it.

Easy to construct some d-holomorphic functions $z_{P,r}(x)$ equal to zero inside of the $r$-ball $D_r$ (i.e. for $x \in D_l, l < r$) and equal to zero along the boundary $\partial D_r$ except of the specific place $P \subset \partial D_r$ looking like in Fig 6, a and b:

![Fig 6](image)

Its extension to the external domain is non-unique. **Can we construct a globally bounded extension?**

Let us introduce class of right-convex paths.

**Definition 13** We call oriented path consisting of edges Right Convex if it bounds two or three triangles only from the right site in every vertex.

We are coding all right convex paths by the words in two symbols $w, b$ assigning to every edge letter $w$ or $b$ depending on which color has triangle from the right site of this path.-see Fig 7

![Fig 7](image)
Theorem 5  Let $\psi$ be a $d$-holomorphic function and $R$ is a full set of zeroes $\psi(x) = 0$. Consider boundary of its complementary domain, i.e. the set $D_{R,1minusR}$, which is a set of points-closest neighbors of zeroes. Every connected component of this boundary set is right convex choosing orientation such that corresponding component of zeroes lies inside.

Proof. In order to have zero value of $\psi$ in the point $x$ we should have following values in the neighboring point outside of zero set (see Fig 8, a and b).

![Fig 8](image)

No other possibilities exist because of the equation $Q^b\psi = 0$. It means in this case that $c \neq 0$ implies either $d \neq 0$ or $d = 0$ in the Fig 8. In the first case our boundary contains only two triangles inside. In the second case ($d = 0$) our boundary contains three triangles inside. Anyway, it is convex. Our theorem is proved.

![Fig 9](image)
A maximal right convex set has a boundary path coded by the infinite sequence \( \ldots bbb \ldots \). For every point \( x \) and direction-edge \( l \) started in \( x \) and having black triangle from the right, we uniquely construct such maximal path \( \ldots bbb \ldots \) and denote it by \( \gamma_{x,l} \). Easy to construct d-holomorphic function \( \psi_{x,l}(y) \) such that \( \psi = 0 \) in the domain to the right of the path \( \gamma_{x,l} \), and \( \psi(x) = \pm 1 \) along the path \( \gamma_{x,l} \)–see Fig 9. Its continuation to the complementary domain is non-unique. We can define it from the requirement that the growth is minimal if distance \( d(y, \gamma_{x,l}) \to \infty \), but this definition is noneffective. **How to find this growth? What is a ”minimal” function \( \psi_{x,l} \)?** This function and its group shifts give basis in the space of all d-holomorphic functions.

Every right-convex path \( \gamma \) can be shifted to the left side by the distance equal to one. We get a new path \( T(\gamma) \). Between these two paths we have a strip containing all triangles \( T \) having at least one common vertex with path \( \gamma \). New path \( T(\gamma) \) consists of edges opposite to vertices belonging to \( \gamma \). Only triangles having exactly one common vertex with \( \gamma \) participate in this construction (see Fig 10).

![Fig 10](image)

**Lemma 7** New path \( T(\gamma) \) is also right convex. Its word can be obtained from the word describing \( \gamma \) by the following procedure: take every pair of neighboring letters in \( \gamma \) and put between them words written below

\[
\begin{align*}
bw & \rightarrow bwbw, wb \rightarrow wbwb \\
bb & \rightarrow bwb, ww \rightarrow wbw
\end{align*}
\]

After that delete old letters. What remains is exactly a word \( T(\gamma) \).
Proof easily follows from the picture (see Fig 11).

\[ \gamma \xrightarrow{T(\gamma)} \]

This map can be studied by the technic of symbolic dynamics. Mike Boyle from the University of Maryland helped me a lot with this business.

We introduce new letters in order to describe this map. Let \( bw \) corresponds to the symbol \( w_b \), \( bb \) corresponds to \( b_b \), \( wb \) corresponds to \( b_w \) and \( ww \) corresponds to \( w_w \). Written in these symbols, our map \( T \) has a form

\[
\begin{align*}
    w_b &\rightarrow w_b b_w w_b w_w, \quad w_w \rightarrow b_w w_b w_w \\
    b_w &\rightarrow b_w w_b w_b b_b, \quad b_b \rightarrow w_b w_b b_b
\end{align*}
\]

After abelianization and replacing product by sum, we are coming to the "Perron matrix" \( A \) whose largest eigenvalue is \( \lambda = 2 + \sqrt{3} \). So we proved following

**Theorem 6** The size of right convex path \( \gamma \) shifted to the left side by the distance one, increases asymptotically by the factor \( \lambda = 2 + \sqrt{3} \), so we have
$|T(\gamma)| \sim (2 + \sqrt{3})|\gamma|$. In particular, this is true for the boundary of $r$-ball $|\gamma| = |\partial D_r| \sim \lambda r^k|\partial D_k|$, $k \geq 1$

**Example 1** We have $|\partial D_r| = 8, 32, 120, 448, 1672, ...$ for $r = 1, 2, 3, 4, 5, ...$, so our asymptotic formula is practically exact for $r \geq 3$.

**How to calculate dimension of the space of d-holomorphic functions in the ball $D_r$?** How many data on the boundary $\partial D_r$ are needed to recover d-holomorphic function in $D_r$?

Consider strip between $D_r$ and $D_{r+1}$-see Fig 12, $r=1$.

![Fig 12](image)

Every letter $b$ in $\partial D_{r+1}$ defines black triangle in the strip touching $\partial D_r$ in one vertex. Every letter $w$ in $\partial D_r$ defines black triangle in the same strip touching $\partial D_{r+1}$ in one vertex. So total number of equations $Q^b\psi = 0$ in this strip is equal to $B_{r+1} + W_r$, the numbers of black and white letters in the boundaries $\partial D_{r+1}, \partial D_r$. Unifying all strips with $k \leq r + 1$, we get total number of equations $Eq_{r+1} : Q^b\psi = 0$:

$$Eq_{r+1} = (B_{r+1} + W_r) + ... + (B_2 + W_1) + B_1$$

For the number of points in $D_{r+1}$ we have

$$N_{r+1} = 1 + B_1 + W_1 + ... + B_k + W_k + ... + B_{r+1} + W_{r+1}$$

and $|\partial D_k| = B_k + W_k$. So we have $Eq_{r+1} = B_{r+1} + N_r - 1$. For the number of necessary data on the boundary $\partial D_{r+1}$ we obtain (taking into account $W_k = B_k$)

$$N_{r+1} - Eq_{r+1} = W_{r+1} + 1 = |\partial D_{r+1}|/2 + 1$$

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Conclusion: Our result that "The Number of Necessary Data is equal to \([|\partial D_{r+1}|/2] + 1\)" is an exact analog of the Continuous case: Only the set of Fourier coefficients on the boundary circle corresponding to the exponents \(\exp\{in\phi\}\) with nonnegative \(n \geq 0\), is needed for the reconstruction of holomorphic function in the whole disc inside.

References

[1] I.Dynnikov, S.Novikov. Geometry of Triangle Equation, Moscow Math Journal-MMJ (2003) v 3, pp 410-438

[2] J.(Le-long)-Ferrand. Fonctions preharmoniques et fonctions preholomorphes, Bull Sci Math(1944) v 68 second series, pp 152-180

[3] R.Duffin. Basic properties of discrete analytic function, Duke Math Journal 1956 vol 23 pp 335-363

[4] S.Novikov. Discrete \(GL_n\)-Connections. Proceeding of Steklov Math Institute, (2004) v 247, pp 186-201

[5] A.Bobenko, C.Mercat, Yu.Suris. Linear and nonlinear theories of discrete analytic functions. Integrable structure and isomonodromic Green functions, J.Reine Angew. Mathematics(2005) v 583, pp 117-161

[6] N.Dolbilin, M.Shtan’ko, M.Shtogrin. a) Cubic subcomplexes in regular lattices. Dokl. Akad. Nauk SSSR Dokl. 291, 277-279 (1986), English translation: Sov. Math. Dokl. 34, 467-469 (1987); b) (jointly with Sedrakyan, A.G.). A topology for the family of parametrizations of two-dimensional cycles arising in the three-dimensional Ising model. Dokl. Akad. Nauk SSSR 295, 12-23 (1987), English translation: Sov. Math., Dokl. 36, No.1, 11-15 (1987); c) The problem of parametrization of cycles modulo 2 in a three-dimensional cubic lattice. Izv. Akad. Nauk SSSR, Ser. Mat., 52, No. 2, 355-377 (1988); English translation: Math. USSR, Izv. 52, No.2, 359-383 (1989); d) Quadrillages and parametrizations of lattice cycles. Tr. Mat. Inst. Steklova 196, 66-85 (1991); English translation: Proc. Steklov Inst. Math. 196, 73-93 (1992); d) Cubic manifolds in lattices. Izv. Ross. Akad. Nauk, Ser. Mat., 58, No. 2, 93-107 (1994), English translation: Izv., Math. 44, No.2, 301-313 (1995).
[7] S. Novikov. Topology I. Encyclopedia Math Sciences (Editor General-R. Gamkrelidze), vol 13 (S. Novikov, B. Botvinnik, R. Burns–editors), Springer Verlag (in english), page 41-42

[8] S. Novikov. Algebraic properties of 2D difference operators, Russian Math Surveys (1997) v 52, n 1, pp 225-226

[9] S. Novikov, I. Dynnikov. Discrete Spectral Symmetries of differential and difference low dimensional operators, Russian Math Surveys (1997) v 52, n 5, pp 175-234

[10] P. Grinevich, R. Novikov. The Cauchy kernel for Novikov-Dynnikov (DN) discrete complex analysis in triangular lattices, Russian Math Surveys (2007) v 62, n 4, pp 799-801

[11] P. Grinevich, R. Novikov, to appear in 2010