BOUNDENDNESS OF LOG PLURICANONICAL REPRESENTATIONS OF LOG CALABI–YAU PAIRS IN DIMENSION 2

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Abstract. We show the boundedness of B-pluricanonical representations of lc log Calabi–Yau pairs in dimension 2. As applications, we prove the boundedness of indices of slc log Calabi–Yau pairs up to dimension 3 and that of non-klt lc log Calabi–Yau pairs in dimension 4.

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1. Introduction

In the framework of Fujino [9], the finiteness of B-pluricanonical representations (or equally, log pluricanonical representations, see Definition 2.2) plays an important role in the study of the abundance conjecture in the minimal model program. The finiteness of B-pluricanonical representations was investigated by Fujino [9] and Gongyo [16] after the work of Nakamura–Ueno [27] and Deligne [30, Section 14], and proved in its full generality by Fujino–Gongyo [15].

In this paper, we are interested in the B-pluricanonical representations of log Calabi–Yau pairs. Log Calabi–Yau pairs form an important class in the minimal model program. It is expected that log Calabi–Yau pairs should satisfy certain boundedness properties, see [1, 2, 7, 6, 4, 21, 32] for related works. Therefore, it is natural to consider the following conjecture on boundedness of B-pluricanonical representations of log Calabi–Yau pairs (cf. [10, Conjecture 3.2], [32, Conjecture 1.9], [8, Conjecture 8.3]).

Date: 2020/2/21, version 0.04.

2010 Mathematics Subject Classification. Primary 14E30; Secondary 14E07, 14J32.

Key words and phrases. log pluricanonical representation, boundedness, index conjecture.
Conjecture 1.1. Let $m, d$ be two positive integers. Then there exists a positive integer $N$ depending only on $m, d$ satisfying the following property: if $(X, \Delta)$ is a projective connected lc pair of dimension $d$ such that $m(K_X + \Delta) \sim 0$, then $|\rho_{km}(\text{Bir}(X, \Delta))| \leq N$ for any positive integer $k$.

Remark 1.2. If $(X, \Delta)$ is a projective connected lc pair of dimension $d$ such that $m(K_X + \Delta) \sim 0$, then $H^0(X, km(K_X + \Delta)) \simeq \mathbb{C}$ for any positive integer $k$. Hence $\rho_{km}(g) \in \mathbb{C}^*$ and $\rho_{km}(g) = \rho_m(g)^k$ for any $g \in \text{Bir}(X, \Delta)$. So

$$|\rho_{km}(\text{Bir}(X, \Delta))| \leq |\rho_m(\text{Bir}(X, \Delta))| \leq k|\rho_{km}(\text{Bir}(X, \Delta))|.$$ 

Therefore, in Conjecture 1.1, it suffices to consider the case $k = 1$.

Remark 1.3. In Conjecture 1.1 the assumptions that $(X, \Delta)$ is connected and lc are necessary. In fact, it is easy to see that $|\rho_m(\text{Bir}(X, \Delta))|$ could not be bounded uniformly unless the number of irreducible components of $X$ is bounded. For example, if $X$ is a cycle of smooth rational curves or a disjoint union of several copies of an elliptic curve, then $K_X \sim 0$ but $|\rho_1(\text{Bir}(X, 0))|$ depends on the number of irreducible components of $X$ (a rotation of irreducible components of $X$ gives a B-pluricanonical representation).

Conjecture 1.1 can be easily proved in dimension 1, see [9, Theorem 3.3], [31, Page 18], or [3] Proposition 8.4. But it was still open even in dimension 2. As the main result of this paper, we give an affirmative answer to Conjecture 1.1 in dimension 2.

Theorem 1.4 (=Theorem 3.1 + Theorem 3.6). Let $m$ be a positive integer. Then there exists a positive integer $N$ depending only on $m$ satisfying the following property: if $(S, B)$ is a projective connected lc pair of dimension 2 such that $m(K_S + B) \sim 0$, then $|\rho_{km}(\text{Bir}(S, B))| \leq N$ for any positive integer $k$.

In the framework of Fujino [9], it is known that Conjecture 1.1 is closely related to the following index conjecture for log Calabi–Yau pairs (cf. [32, Conjecture 1.3]):

Conjecture 1.5 (Index conjecture). Let $I$ be a finite set in $[0, 1] \cap \mathbb{Q}$ and $d$ a positive integer. Then there exists a positive integer $m$ depending only on $I, d$ satisfying the following property: if $(X, \Delta)$ is a projective slc pair of dimension $d$ such that the coefficients of $\Delta$ are in $I$ and $K_X + \Delta \sim_\mathbb{Q} 0$, then $m(K_X + \Delta) \sim 0$.

Recently this conjecture was studied by the first author [21] and Xu [31, 32]. It was proved in dimension 2 [31], and in dimension 3 for lc pairs [21, 32]. As applications of Theorem 1.4 we prove the boundedness of indices of slc log Calabi–Yau pairs up to dimension 3.

Corollary 1.6. Conjecture 1.5 holds in dimension $\leq 3$. To be more precise, let $I$ be a finite set in $[0, 1] \cap \mathbb{Q}$. Then there exists a positive integer $m$ depending only on $I$ satisfying the following property: if $(X, \Delta)$ is a projective slc pair of dimension at most 3 such that the coefficients of $\Delta$ are in $I$ and $K_X + \Delta \sim_\mathbb{Q} 0$, then $m(K_X + \Delta) \sim 0$.

Corollary 1.7. Conjecture 1.5 holds for connected non-klt lc pairs in dimension 4. To be more precise, let $I$ be a finite set in $[0, 1] \cap \mathbb{Q}$. Then there exists a positive integer $m$ depending only on $I$ satisfying the following property: if $(X, \Delta)$ is a projective connected non-klt lc pair of dimension 4 such that the coefficients of $\Delta$ are in $I$ and $K_X + \Delta \sim_\mathbb{Q} 0$, then $m(K_X + \Delta) \sim 0$.

Acknowledgments. The authors would like to thank Osamu Fujino and Jingjun Han for helpful discussions. The idea of this work was carried out during the conference “2019
Suzhou AG Young Forum” at Soochow University and the authors are grateful to Yi Gu and Cheng Gong for hospitality and support. The first author was supported by Start-up Grant No. SXH1414010.

2. Preliminaries

2.1. Notation and conventions. We work over the complex number field \( \mathbb{C} \) throughout this paper. We freely use the basic notation of the minimal model program in [26]. In this paper, we consider only \( \mathbb{Q} \)-divisors instead of \( \mathbb{R} \)-divisors. A scheme is always assumed to be separated and of finite type over \( \mathbb{C} \). The dimension of a scheme is the pure dimension of that scheme, that is, when we consider the dimension of a scheme \( X \), \( X \) is always assumed to be of pure dimension. A variety is a reduced and irreducible scheme. A curve (resp. surface) is a variety of dimension 1 (resp. 2). A normal scheme consists of the disjoint union of irreducible normal schemes.

Let \( D \) be a \( \mathbb{Q} \)-divisor on a normal scheme \( X \), that is, \( D \) is a finite formal sum \( \sum_i d_i D_i \) where \( d_i \in \mathbb{Q} \) and \( \{D_i\}_i \) are distinct prime divisors on \( X \). We put

\[
\begin{align*}
D^{< 1} &= \sum_{d_i < 1} d_i D_i, \\
D^{\leq 1} &= \sum_{d_i \leq 1} d_i D_i, \\
D^{= 1} &= \sum_{d_i = 1} D_i.
\end{align*}
\]

We also put \( |D| = \sum_i |d_i| D_i \) where \( |d_i| \) is the integer defined by \( d_i - 1 < |d_i| \leq d_i \), and put \( \{D\} = D - |D| \).

2.2. Singularities of pairs. A sub-pair \((X, \Delta)\) consists of a normal scheme \( X \) and a \( \mathbb{Q} \)-divisor \( \Delta \) on \( X \) such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. Let \( f : Y \to X \) be a projective birational morphism from a normal scheme \( Y \). Then we can write

\[
K_Y = f^*(K_X + \Delta) + \sum_{E} a(E, X, \Delta) E
\]

with

\[
f_* \left( \sum_{E} a(E, X, \Delta) E \right) = -\Delta,
\]

where \( E \) runs over prime divisors on \( Y \). We call \( a(E, X, \Delta) \) the discrepancy of \( E \) with respect to \((X, \Delta)\). Note that we can define the discrepancy \( a(E, X, \Delta) \) for any prime divisor \( E \) over \( X \) by taking a suitable resolution of singularities of \( X \). If \( a(E, X, \Delta) \geq -1 \) (resp. \( a(E, X, \Delta) > -1 \)) for every prime divisor \( E \) over \( X \), then \((X, \Delta)\) is called sub log canonical (sub-lc for short) or sub Kawamata log terminal (sub-klt for short) respectively.

If \( \Delta \) is effective, then a sub-pair \((X, \Delta)\) is called a pair, and \((X, \Delta)\) is called lc (resp. klt) if it is sub-lc (resp. sub-klt). A divisorial log terminal (dlt for short) pair is a limit of klt pairs in the sense of [26] Proposition 2.43 (see [26] Definition 2.37 and Proposition 2.40 for precise definitions).

Let \((X, \Delta)\) be a sub-lc pair. If there exist a projective birational morphism \( f : Y \to X \) from a normal scheme \( Y \) and a prime divisor \( E \) on \( Y \) with \( a(E, X, \Delta) = -1 \), then \( f(E) \) is called an lc center of \((X, \Delta)\).

Let \( X \) be a reduced scheme of pure dimension which satisfies Serre’s \( S_2 \) condition and is normal crossing in codimension one. Let \( \Delta \) be an effective \( \mathbb{Q} \)-divisor on \( X \) such that no irreducible component of \( \text{Supp} \Delta \) is contained in the singular locus of \( X \) and \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. We say that \((X, \Delta)\) is a semi log canonical (slc for short) pair if \((X^\nu, \Delta_{X^\nu})\) is log canonical, where \( \nu : X^\nu \to X \) is the normalization of \( X \) and \( K_{X^\nu} + \Delta_{X^\nu} = \nu^*(K_X + \Delta) \), that is, \( \Delta_{X^\nu} \) is the sum of the inverse image of \( \Delta \) and the conductor of \( X \). We say that \((X, \Delta)\) is a semi divisorial log terminal (sdlt for short) pair if \((X^\nu, \Delta_{X^\nu})\) is dlt, and every
irreducible component of $X$ is normal. Note that an sdlt pair is naturally an slc pair. For more details of slc and sdlt pairs, see [9, 11, 23, 24].

2.3. Log Calabi–Yau pairs. A pair $(X, \Delta)$ is called a log Calabi–Yau pair if $X$ is projective and $K_X + B \sim_Q 0$, and the (global) index of $(X, \Delta)$ is the minimal positive integer $m$ such that $m(K_X + \Delta) \sim 0$.

2.4. B-birational maps. We recall basic knowledge on B-birational maps and B-pluricanonical representations introduced by [9]. For more details, see [9, 15, 16] and the references therein.

**Definition 2.1** ([9, Definition 1.5] or [15, Definition 2.11]). Let $(X, \Delta)$ and $(Y, \Delta_Y)$ be two sub-pairs. A proper birational map $f : (X, \Delta) \dashrightarrow (Y, \Delta_Y)$ is called B-birational if there exists a common resolution

$$
\xymatrix{ 
W \\
(\alpha, f, \beta) \\
(X, \Delta) \ar[r] & (Y, \Delta_Y)
}
$$

such that $\alpha^*(K_X + \Delta) = \beta^*(K_Y + \Delta_Y)$. In this case, we say that $(Y, \Delta_Y)$ is a B-birational model or a crepant model of $(X, \Delta)$. Let

$$
\text{Bir}(X, \Delta) = \{f \mid f : (X, \Delta) \dashrightarrow (X, \Delta) \text{ is B-birational}\}.
$$

Then Bir$(X, \Delta)$ has a natural group structure under compositions of maps.

**Definition 2.2** ([9, Definition 3.1] or [15, Definition 2.14]). Let $(X, \Delta)$ be a sub-pair. Fix a positive integer $m$ such that $m(K_X + \Delta)$ is Cartier. Then a B-birational map $f : (X, \Delta) \dashrightarrow (X, \Delta)$ naturally induces a linear automorphism of $H^0(X, m(K_X + \Delta))$. This gives a group homomorphism

$$
\rho_m : \text{Bir}(X, \Delta) \to \text{Aut}_C(H^0(X, m(K_X + \Delta))).
$$

The homomorphism $\rho_m$ is called a B-pluricanonical representation or log pluricanonical representation for $(X, \Delta)$. We sometimes denote $\rho_m(g)$ by $g^*$ for $g \in \text{Bir}(X, \Delta)$ if there is no danger of confusion.

**Remark 2.3.** As explained in [15, Remark 2.12], if $f : (X, \Delta) \dashrightarrow (Y, \Delta_Y)$ is a B-birational map, then there is a group isomorphism $\text{Bir}(X, \Delta) \simeq \text{Bir}(Y, \Delta_Y)$ given by $g \mapsto f \circ g \circ f^{-1}$.

For B-pluricanonical representation, we have the following finiteness theorem proved by Fujino–Gongyo [15] (later Hacon–Xu [20] gave a different proof for a weaker statement). It is a log version of Nakamura–Ueno [27] and Deligne–Ueno’s finiteness theorem of pluricanonical representations [30, Theorem 14.10].

**Theorem 2.4** ([15, Theorem 1.1]). Let $(X, \Delta)$ be a projective lc pair. Assume that $m(K_X + \Delta)$ is Cartier and $K_X + \Delta$ is semi-ample. Then $\rho_m(\text{Bir}(X, \Delta))$ is a finite group.

2.5. Cyclic coverings. We recall the construction of $m$-fold cyclic coverings.

For simplicity, we assume that $X$ is a smooth variety, $\Delta$ is a $\mathbb{Q}$-divisor on $X$ with simple normal crossing support such that $K_X + \Delta \sim_{\mathbb{Q}} 0$. Take $m \in \mathbb{Z}_{\geq 0}$ to be the minimal one such that $m(K_X + \Delta) \sim 0$. Then there is an $m$-fold cyclic covering corresponding to the effective divisor $m\{\Delta\} \sim m(-K_X - [\Delta])$ given by

$$
\mu : \tilde{X} = \text{Spec} \left( \bigoplus_{i=0}^{m-1} \mathcal{L}^{-i}(i\{\Delta\}) \right) \to X.
$$
where \( \mathcal{L} = \mathcal{O}_X(-K_X - [\Delta]) \) (cf. [23, 2.3]). There is also an alternative description of above \( m \)-fold cyclic covering as

\[
\mu : \tilde{X} = \text{Spec} \left( \bigoplus_{i=0}^{m-1} \mathcal{O}_X([i(K_X + \Delta)]) \right) \rightarrow X
\]
as in [13, Section 6] by \( m(K_X + \Delta) \sim 0 \). More precisely, fix a non-zero section \( \omega \in H^0(X, m(K_X + \Delta)) \), the \( \mathcal{O}_X \)-algebra structure of \( \bigoplus_{i=0}^{m-1} \mathcal{O}_X([i(K_X + \Delta)]) \) is given by the natural multiplication

\[
\mathcal{O}_X([i(K_X + \Delta)]) \otimes \mathcal{O}_X([j(K_X + \Delta)]) \rightarrow \mathcal{O}_X([(i + j)(K_X + \Delta)])
\]
if \( i + j \leq m \), and by the multiplication

\[
\mathcal{O}_X([i(K_X + \Delta)]) \otimes \mathcal{O}_X([j(K_X + \Delta)]) \rightarrow \mathcal{O}_X([(i + j)(K_X + \Delta)])
\]
if \( i + j \geq m \). Since we have

\[
\mathcal{L}^{-i}([i\{\Delta\}]) = \mathcal{O}_X(iK_X + i[\Delta] + [i\{\Delta\}]) = \mathcal{O}_X([i(K_X + \Delta)]),
\]
these two descriptions are indeed isomorphic. Note that \( \tilde{X} \) is not necessarily smooth, but it is normal and irreducible by the minimality of \( m \). Also note that \( \mu \) is étale outside \( \text{Supp}\{\Delta\} \). Since the construction of \( \tilde{X} \) depends on the choice of \( \omega \), usually we denote this covering by \( \mu : \tilde{X}_\omega \rightarrow X \). By the construction, there exists a \( \mathbb{Q} \)-divisor \( \Delta_{\tilde{X}_\omega} \) on \( \tilde{X}_\omega \) such that \( K_{\tilde{X}_\omega} + \Delta_{\tilde{X}_\omega} = \mu^*(K_X + \Delta) \) and \( K_{\tilde{X}_\omega} + \Delta_{\tilde{X}_\omega} \sim 0 \).

In order to consider the lifting of B-birational maps after cyclic coverings, we have the following lemma.

**Lemma 2.5.** Keep the above setting. Let \( \alpha : (W, \Delta_W) \rightarrow (X, \Delta) \) be a log resolution such that \( m(K_W + \Delta_W) = m\alpha^*(K_X + \Delta) \sim 0 \). Fix a non-zero section \( \tilde{\omega} \in H^0(W, m(K_W + \Delta_W)) \). Let \( \mu : \tilde{X}_\omega \rightarrow X \) (resp. \( \nu : \tilde{W}_\omega \rightarrow W \)) be the \( m \)-fold cyclic covering given by the section \( \omega \) (resp. \( \tilde{\omega} \)). Then there exists a birational morphism \( \tilde{\alpha}_{\omega,\tilde{\omega}} : \tilde{W}_\omega \rightarrow \tilde{X}_\omega \) making the following diagram commute:

\[
\begin{array}{ccc}
\tilde{W}_\omega & \xrightarrow{\tilde{\alpha}_{\omega,\tilde{\omega}}} & \tilde{X}_\omega \\
\downarrow & & \downarrow \mu \\
W & \xrightarrow{\alpha} & X.
\end{array}
\]

**Proof.** Since \( H^0(W, m(K_W + \Delta_W)) \simeq \mathbb{C} \) by assumption, we can take a \( t \in \mathbb{C}^* \) such that \( \tilde{\omega} = t\alpha^*\omega \). Fix a primitive \( m \)-th root \( \sqrt[\alpha]{t} \) of \( t \). We construct a morphism \( \tilde{\alpha}_{\omega,\tilde{\omega}} : \tilde{W}_\omega \rightarrow \tilde{X}_\omega \) following the construction of the \( m \)-fold cyclic covering. Note that there exists a natural isomorphism (see [28, II.2.11])

\[
\mathcal{O}_X([i(K_X + \Delta)]) \simeq \alpha_* \mathcal{O}_W([i(K_W + \Delta_W)])
\]
for every \( i \geq 0 \). We can consider the following “twisted” isomorphism

\[
\mathcal{O}_X([i(K_X + \Delta)]) \simeq \alpha_* \mathcal{O}_W([i(K_W + \Delta_W)]) \xrightarrow{\sqrt[\alpha]{t}} \alpha_* \mathcal{O}_W([i(K_W + \Delta_W)]),
\]
then it is easy to check that the induced isomorphism

\[
\bigoplus_{i=0}^{m-1} \mathcal{O}_X([i(K_X + \Delta)]) \rightarrow \bigoplus_{i=0}^{m-1} \alpha_* \mathcal{O}_W([i(K_W + \Delta_W)])
\]
is compatible with $O_X$-algebra structures. So this isomorphism gives a birational morphism between coverings $\tilde{\alpha}_{\omega,\omega}: \tilde{W}_\omega \to \tilde{X}_\omega$. \hfill \Box

Then we can show that $B$-birational maps lift to cyclic coverings.

**Lemma 2.6.** Let $X$ be a smooth variety, $\Delta$ a $\mathbb{Q}$-divisor on $X$ with simple normal crossing support such that $K_X + \Delta \sim_\mathbb{Q} 0$. Take $m \in \mathbb{Z}_{>0}$ to be the minimal one such that $m(K_X + \Delta) \sim 0$. Let $\mu: \tilde{X}_\omega \to X$ be the $m$-fold cyclic covering given by a non-zero section $\omega \in H^0(X, m(K_X + \Delta))$. Then a $B$-birational map $g: (X, \Delta) \dashrightarrow (X, \Delta)$ can be lifted to a $B$-birational map $g': (\tilde{X}_\omega, \Delta_{\tilde{X}_\omega}) \dashrightarrow (\tilde{X}_\omega, \Delta_{\tilde{X}_\omega})$ commuting with $\mu$.

**Proof.** Consider a common log resolution

\[
\begin{array}{ccc}
(W, \Delta_W) & & (X, \Delta) \\
\alpha & \beta & \\
\downarrow & \downarrow & g \downarrow \\
(X, \Delta) & \rightarrow & (X, \Delta).
\end{array}
\]

Fix a non-zero section $\tilde{\omega} \in H^0(W, m(K_W + \Delta_W))$. Then by Lemma 2.5, we have the following commutative diagram

\[
\begin{array}{ccc}
\tilde{X}_\omega & & \tilde{X}_\omega \\
\alpha & \beta & \\
\downarrow & \downarrow & \downarrow \\
X & \rightarrow & X
\end{array}
\]

Note that

$\tilde{\alpha}_{\omega,\omega}^*(K_{\tilde{X}_\omega} + \Delta_{\tilde{X}_\omega}) = \nu^*\alpha^*(K_X + \Delta) = \nu^*\beta^*(K_X + \Delta) = \tilde{\beta}_{\omega,\omega}^*(K_{\tilde{X}_\omega} + \Delta_{\tilde{X}_\omega})$.

Hence $\tilde{\beta}_{\omega,\omega} \circ \tilde{\alpha}_{\omega,\omega}^{-1}: (\tilde{X}_\omega, \Delta_{\tilde{X}_\omega}) \dashrightarrow (\tilde{X}_\omega, \Delta_{\tilde{X}_\omega})$ is the required $B$-birational map. \hfill \Box

The following lemma is a special case of [14, Proposition 4.9] (see also [30, Proposition 14.4]). The proof is essentially the same as that in [16, Proposition 4.9]. Note that in [16, Proposition 4.9], cyclic coverings and liftings of $B$-birational maps are constructed locally and analytically, so here we modify the proof by the algebraic construction of coverings and liftings (Lemma 2.6).

**Lemma 2.7** (cf. [16, Proposition 4.9], [15, Remark 3.6]). Let $(X, \Delta)$ be a projective sub-klt pair of dimension $d$ such that $X$ is smooth connected, $\Delta$ is with simple normal crossing support, and $K_X + \Delta \sim_\mathbb{Q} 0$. Take $m \in \mathbb{Z}_{>0}$ to be the minimal one such that $m(K_X + \Delta) \sim 0$. Fix a non-zero section $\omega \in H^0(X, m(K_X + \Delta))$. Let $\mu: \tilde{X}_\omega \to X$ be the $m$-fold cyclic covering given by the section $\omega$. Take $\phi: V \to (\tilde{X}_\omega, \Delta_{\tilde{X}_\omega})$ to be any log resolution. Take $N_V$ to be the least common multiple of all positive integers $k$ such that $\varphi(k) \leq b_d(V)$ where $b_d(V)$ is the $d$-th Betti number of $V$ and $\varphi$ is the Euler function. Then for any $B$-birational map $g \in \text{Bir}(X, \Delta)$, $(g^*)^{N_V}$ is the identity map on $H^0(X, m(K_X + \Delta)) \simeq \mathbb{C}$. In particular, $|\rho_m(\text{Bir}(X, \Delta))| \leq N_V$. 

Proof. Fix any B-birational map $g \in \text{Bir}(X, \Delta)$. Suppose that $g^* \omega = \lambda \omega$ for some $\lambda \in \mathbb{C}$. It suffices to show that $\lambda^{N_V} = 1$.

We can view $\omega \in H^0(X, m(K_X + \Delta))$ as a non-zero meromorphic $m$-ple $d$-form. Then by the cyclic covering construction, there is a non-zero meromorphic $d$-form $\omega_V \in H^0(V, K_V + \Delta_V)$ on $V$ such that

$$(\omega_V)^m = \phi^* \mu^* \omega,$$

where $K_V + \Delta_V = \phi^*(K_{\tilde{X}} + \Delta_{\tilde{X}})$. As $(X, \Delta)$ is sub-klt, $(V, \Delta_V)$ and $(\tilde{X}_\omega, \Delta_{\tilde{X}_\omega})$ are sub-klt by [26 Proposition 5.20]. By [15] Lemmas 3.4, $\omega$ is $L^2$-integrable. Hence $\omega_{V|V\backslash \text{Supp}(\Delta_V)}$ is $L^2$-integrable and $\omega_V$ is a holomorphic $d$-form by [22 Proposition 16] or [15] Lemma 3.3. That is, $\omega_V \in H^0(V, K_V) \subset H^d(V, \mathbb{Z}) \otimes \mathbb{C}$. By Lemma 2.6 there is a lifting $g' \in \text{Bir}(\tilde{X}_\omega, \Delta_{\tilde{X}_\omega})$ which naturally lifts to $g_V \in \text{Bir}(V, \Delta_V)$ such that

$$g_V^*(\omega_V)^m = g_V^* \phi^* \mu^* (\omega) = \phi^* \mu^* g^* \omega = \phi^* \mu^* (\lambda \omega) = \lambda (\omega_V)^m.$$

As $H^0(V, K_V + \Delta_V) \simeq \mathbb{C}$, we can write $g_V^*(\omega_V) = \lambda' \omega_V$ for some $\lambda' \in \mathbb{C}^*$ with $(\lambda')^m = \lambda$. By [30] Proposition 14.4 and Theorem 14.10, we immediately see that $(\lambda)^{N_V} = 1$. More precisely, by [30] Theorem 14.10, $\lambda'$ is a root of unity; by [30] Proposition 14.4, $\lambda'$ is an algebraic integer and the degree of the minimal polynomial of $\lambda'$ with coefficients in $\mathbb{Q}$ is bounded from above by $b_d(V)$, hence $(\lambda)^{N_V} = 1$ by the definition of $N_V$. Therefore, $(\lambda)^{N_V} = (\lambda')^{mN_V} = 1$. 

2.6. Bounded pairs. A collection of projective varieties $\mathcal{D}$ is said to be bounded if there exists a projective morphism $h: \mathcal{X} \to T$ between schemes of finite type such that each $X \in \mathcal{D}$ is isomorphic to $\mathcal{X}_t$ for some closed point $t \in T$ where $\mathcal{X}_t = h^{-1}(t)$.

We say that a collection of projective connected log pairs $\mathcal{D}$ is log bounded if there is a scheme $\mathcal{X}$, a reduced divisor $\mathcal{B}$ on $\mathcal{X}$, and a projective morphism $h: \mathcal{X} \to T$, where $T$ is of finite type and $\mathcal{B}$ does not contain any fiber, such that for every $(X, B) \in \mathcal{D}$, there is a closed point $t \in T$ and an isomorphism $f: \mathcal{X}_t \to X$ such that $\mathcal{B}_t := \mathcal{B}|_{\mathcal{X}_t}$ coincides with the support of $f_*^{-1} B$.

Moreover, if $\mathcal{D}$ is a set of connected log Calabi–Yau pairs, then it is said to be log bounded modulo B-birational contractions if there exists another set $\mathcal{D}'$ of connected log Calabi–Yau pairs which is log bounded, and for each $(X, B) \in \mathcal{D}$, there exists $(X', B') \in \mathcal{D}'$ and a B-birational map $g : (X, B) \dasharrow (X', B')$ which is a contraction, that is, $g$ does not extract any divisor. Here we remark that the concept of log boundedness modulo B-birational contractions is a weaker version of “log boundedness modulo flips” introduced in [6] [21], in which $g$ is assumed to be isomorphic in codimension 1.

3. Proof of the main theorem

In this section, we prove Theorem 1.4. The proof splits into two parts: the klt case and the non-klt case. We will use different methods to treat them.

3.1. Klt case. In this subsection, we deal with the klt case.

**Theorem 3.1.** Let $m$ be a positive integer. Then there exists a positive integer $N$ depending only on $m$ satisfying the following property: if $(S, B)$ is a projective connected klt pair of dimension 2 such that $m(K_S + B) \sim 0$, then $|\rho_m(\text{Bir}(S, B))| \leq N$ for any positive integer $k$.

**Proof.** First we consider the case that $B = 0$ and $S$ has at worst du Val singularities. In this case, the boundedness of $|\rho_m(\text{Bir}(S, B))|$ is well-known to experts (cf. [11] Proposition 3.6]). We briefly recall the proof here for the reader’s convenience. By Remark 2.3, we may
assume that $S$ is smooth after taking the minimal resolution. Take $r$ to be the minimal positive integer such that $rK_S \sim 0$. Take $\tilde{S} \to S$ to be the index 1 cover of $K_S$, that is, the $r$-fold cyclic covering given by a non-zero section in $H^0(S, rK_S)$, then $\tilde{S}$ is a projective smooth surface with $K_{\tilde{S}} \sim 0$. By the classification theory of surfaces (for example [3 Chapter VIII]), $b_2(\tilde{S}) \leq 22$. Hence by Lemma 2.7, there exists a constant $N_1$ independent of $S$ such that $|\rho_r(\text{Bir}(S, B))| \leq N_1$. In particular, $|\rho_m(\text{Bir}(S, B))| \leq |\rho_r(\text{Bir}(S, B))| \leq N_1$ as $r$ divides $m$.

From now on, we consider the case that $B \neq 0$ or $S$ has worst than du Val singularities. In this case, $S$ belongs to a bounded family by [1 Theorem 6.9]. Moreover, as $m(K_S + B) \sim 0$, $(S, B)$ belongs to a log bounded family by standard arguments (see, for example, [4 Lemma 2.20]). Then by Theorem 3.2 below, there exist two positive integers $k_2$ and $N_2$ independent of $(S, B)$ such that $k_2(K_S + B) \sim 0$ and $|\rho_{k_2}(\text{Bir}(S, B))| \leq N_2$. By Remark 1.2, $|\rho_m(\text{Bir}(X, \Delta))| \leq k_2|\rho_{k_2m}(\text{Bir}(X, \Delta))| \leq k_2N_2$.

So summarizing two cases, we can just take $N = \max\{N_1, k_2N_2\}$. 

We show the following more general result on boundedness of B-pluricanonical representations in a family which is log bounded modulo B-birational contractions.

**Theorem 3.2.** Let $m, d$ be two positive integers. Let $D$ be a set of connected klt log Calabi–Yau pairs of dimension $d$ which is log bounded modulo $B$-birational contractions, such that for each $(X, B) \in D$, $mB$ is an integral divisor. Then there exist two positive integers $k$ and $N$ depending only on $D$ such that, if $(X, B) \in D$, then $k(K_X + B) \sim 0$ and $|\rho_k(\text{Bir}(X, B))| \leq N$.

**Proof.** Note that if $(X, B) \dashrightarrow (X', B')$ is a B-birational contraction and $(X, B)$ is a connected klt log Calabi–Yau pair, then $(X', B')$ is automatically a connected klt log Calabi–Yau pair and $mB'$ is integral. Moreover, $k(K_X + B) \sim 0$ if and only if $k(K_{X'} + B') \sim 0$, and in this case $|\rho_k(\text{Bir}(X, B))| = |\rho_k(\text{Bir}(X', B'))|$ by Remark 2.3. So after replacing $D$ by its as in the definition of log boundedness modulo B-birational contractions, we may further assume that $D$ is a log bounded family of connected klt log Calabi–Yau pairs of dimension $d$. Note that by Global ACC [17, Theorem 1.5] (see [19 Proof of Proposition 3.1]), there exists a constant $\epsilon \in (0, 1)$ such that $(X, B)$ is $\epsilon$-lc for any $(X, B) \in D$.

By the definition of log boundedness, there is a scheme $\mathcal{X}$ and a projective morphism $h : \mathcal{X} \to T$, a reduced divisor $\mathcal{B}$ on $\mathcal{X}$, where $T$ is of finite type and $\mathcal{B}$ does not contain any fiber, such that for every $(X, B) \in D$, there is a closed point $t \in T$ and an isomorphism $f : \mathcal{X}_t \to X$ such that $\mathcal{B}_t := B|_{\mathcal{X}_t}$ coincides with the support of $f^{-1}_*B$. As the coefficients of $B$ are in a fixed finite set, after replacing $T$ by disjoint union of locally closed subsets, we may assume that there exists a $\mathbb{Q}$-divisor $\mathcal{B}$ on $\mathcal{X}$ such that for every $(X, B) \in D$, $(X, B)|_{\mathcal{X}_t}$ for some fiber $\mathcal{X}_t$. Moreover, by applying [19 Proposition 2.4] and the Noetherian induction, we may further assume that $K_X + \mathcal{B}$ is $\mathbb{Q}$-Cartier, $(\mathcal{X}, \mathcal{B})$ is klt, and the set of points $t$ corresponding to $(X, B) \in D$ is dense in $T$. Replacing $T$ by disjoint union of locally closed subsets $\cup_i T_i$ while taking log resolutions of $\mathcal{X}$, we may assume that there are finitely many smooth varieties $T_i$ and projective morphisms $(\mathcal{W}_i, \mathcal{E}_i) \to (\mathcal{X}_i, \mathcal{B}_i) \to T_i$ such that $(\mathcal{W}_i, \mathcal{E}_i)$ is log smooth over $T_i$ and for every $t \in T_i$, the fiber $\mathcal{X}_{i,t}$ is a normal projective variety of dimension $d$, $(\mathcal{W}_i, \mathcal{E}_i)$ is a log resolution of $(\mathcal{X}_i, \mathcal{B}_i, \mathcal{E}_{i,t})$ with $\mathcal{E}_{i,t}$ the sum of strict transform of $\mathcal{B}_{i,t}$ and the reduced exceptional divisor, and the set of points $t$ corresponding to $(X, B) \in D$ is dense in each $T_i$.

Note that if $(X, B) \in D$ is isomorphic to a fiber $(\mathcal{X}_{i,0}, \mathcal{B}_{i,0})$ of $(\mathcal{X}_i, \mathcal{B}_i) \to T_i$, then it is a good minimal model of $(\mathcal{W}_i, \mathcal{E}_{i,0})$. Hence by [18 Corollary 1.4], for each positive integer
such that $lB_{i,t}$ is integral,
\[
h^0(\mathcal{X}_{i,t}, l(K_{\mathcal{X}_{i,t}} + B_{i,t})) = h^0(\mathcal{W}_{i,t}, l(K_{\mathcal{W}_{i,t}} + \mathcal{E}_{i,t}))
\]
is constant for $t \in T_i$. Since the set of points $t$ corresponding to $(X, B) \in D$ is dense in each $T_i$, over the generic point $\eta_i \in T_i$, $K_{\mathcal{X}_{i,\eta_i}} + B_{i,\eta_i} \sim_{\mathbb{Q}} 0$. So by the Noetherian induction, further replacing $T$ by disjoint union of locally closed affine subsets (still denoted by $\cup_i T_i$), we may assume that $K_X + B \sim_{\mathbb{Q}} 0$.

As there are only finitely many $T_i$, we can reduce to the case that $T = T_i$. Note that by the construction, every fiber $(\mathcal{X}_{t}, \mathcal{B}_{t})$ is a connected klt log Calabi–Yau pair. Recall that by [18, Corollary 1.4], for each positive integer $l$ such that $lB$ is integral (this condition is independent of $t$), $h^0(\mathcal{X}_{t}, l(K_{\mathcal{X}_{t}} + \mathcal{B}_{t}))$ is constant for $t \in T$, which implies that the index of $(\mathcal{X}_{t}, \mathcal{B}_{t})$ is a constant positive integer, denoted by $k$. In particular, $k$ is also the minimal positive integer such that $k(K_X + B) \sim 0$, as $T$ is affine.

Consider the log resolution $\psi : (\mathcal{W}, \Delta) \to (\mathcal{X}, \mathcal{B})$ which is a log resolution on each fiber, where $K_{\mathcal{W}} + \Delta = \psi^*(K_X + B)$. Then for a non-zero section $\omega \in H^0(\mathcal{W}, k(K_{\mathcal{W}} + \Delta))$, we can consider $\mu : \mathcal{W}_\omega \to \mathcal{W}$ to be the $k$-fold cyclic covering given by the section $\omega$ and take $\phi : \mathcal{V} \to (\mathcal{W}_\omega, \Delta_{\mathcal{W}_\omega})$ to be a log resolution. Here we may assume that on each fiber of $t \in T$, $\phi_t : \mathcal{V}_t \to (\mathcal{W}_{i,t}, \Delta_{\mathcal{W}_{i,t}})$ is a log resolution by the Noetherian induction after replacing $T$ by disjoint union of locally closed affine subsets. Since $\mathcal{V}_t$ is in a bounded family, $b_d(\mathcal{V}_t)$ has a uniform upper bound. Hence by Lemma 2.7 there exists a constant $N$ such that for each $t \in T$, $|\rho_k(\text{Bir}(\mathcal{X}_t, \mathcal{B}_t))| = |\rho_k(\text{Bir}(\mathcal{W}_t, \Delta_t))| \leq N$.

Besides the proof of Theorem 3.1, Theorem 3.2 has also the following applications to other known bounded families of log Calabi–Yau pairs.

**Corollary 3.3.** Let $d$ be a positive integer and $I$ a finite set in $[0, 1] \cap \mathbb{Q}$. Then there exist two positive integers $k$ and $N$ depending only on $d$, $I$ satisfying the following property: if $(X, B)$ is a connected klt log Calabi–Yau pair of dimension $d$, $B$ is big and the coefficients of $B$ are in $I$, then $k(K_X + B) \sim 0$ and $|\rho_k(\text{Bir}(X, B))| \leq N$.

**Proof.** It follows directly from [19, Theorem 1.3] and Theorem 3.2.

**Corollary 3.4.** Let $I$ be a finite set in $[0, 1] \cap \mathbb{Q}$. Then there exist two positive integers $k$ and $N$ depending only on $I$ satisfying the following property: if $(X, B)$ is a connected klt log Calabi–Yau pair of dimension $3$ such that $X$ is rationally connected and the coefficients of $B$ are in $I$, then $k(K_X + B) \sim 0$ and $|\rho_k(\text{Bir}(X, B))| \leq N$.

**Proof.** By Theorem 3.2, it suffices to show that such $(X, B)$ belongs to a log bounded family modulo B-birational contractions. In fact, such $(X, B)$ belongs to a log bounded family modulo flops, which is proved by [6, Theorem 4.1] if $B \neq 0$ and by [6, Corollary 5.5] if $B = 0$.

**Corollary 3.5.** There exist two positive integers $k$ and $N$ satisfying the following property: if $X$ is a non-canonical klt Calabi–Yau variety of dimension $3$, then $kK_X \sim 0$ and $|\rho_k(\text{Bir}(X, 0))| \leq N$.

**Proof.** By Theorem 3.2, it suffices to show that such $(X, 0)$ belongs to a log bounded family modulo B-birational contractions. In fact, such $(X, 0)$ belongs to a log bounded family modulo flops, which is proved by [21, Theorem 1.4].

### 3.2. Non-klt case.

In this subsection, we deal with the non-klt case.
Theorem 3.6. Let \( m \) be a positive integer. Then there exists a positive integer \( N \) depending only on \( m \) satisfying the following property: if \((S, B)\) is a projective connected non-klt lc pair of dimension 2 such that \( m(K_S + B) \sim 0 \), then \(|\rho_m(\text{Bir}(S, B))| \leq N\) for any positive integer \( k \).

Remark 3.7. In the proof of Theorem 3.6, we follow the idea in [15, Proof of Theorem 3.15, Case 1]. But as we are interested in the boundedness of B-pluricanonical representations, we need to put extra effort. For example, we need to show the lc centers of \((S, B)\) satisfy certain boundedness. To this end, we show in Lemma 3.13 that certain special lc centers of \((S, B)\) are bounded after replacing \((S, B)\) by a B-birational model. Then we can modify the argument in [15, Proof of Theorem 3.15, Case 1] to prove Theorem 3.6. On the other hand, in Section 4, following the idea in [9, Theorem 3.5] and [16, Page 560, Step 2], we can give an alternative proof of Theorem 3.6 which involves more technical inductive arguments by using (pre-)admissible sections, see Remark 4.6.

Firstly we recall the following well-known connectedness lemma (cf. [24, 12.3.2], [9, Proposition 2.1], [10, Proposition 2.4], [25, Proposition 5.1], [16, Claim 5.3]).

Lemma 3.8 (Connectedness lemma, [16, Claim 5.3]). Let \((X, \Delta)\) be a projective connected lc pair with \( K_X + \Delta \sim_\mathbb{Q} 0 \). Then \( \lfloor \Delta \rfloor \) has at most 2 connected components.

Recall that a set of smooth curves is a chain (resp. cycle) if their dual graph is a chain (resp. cycle), and end curves of a chain are those curves corresponding to end points of the dual graph. Here we allow two curves intersecting at two distinct points to be a cycle.

Lemma 3.9. Let \((S, B)\) be a projective connected dlt pair of dimension 2 such that \( K_S + B \sim_\mathbb{Q} 0 \). Then \( |B| \) has at most 2 connected components, and each connected component is either a chain or a cycle. In this setting, we usually write \( |B| = T_0 + T_1 \), where \( T_0 \) consists of connected components which are cycles, and \( T_1 \) consists of connected components which are chains, and denote \( T_e \) to be the sum of all end curves in \( T_1 \).

Proof. As \((S, B)\) is dlt, all irreducible components of \(|B|\) are normal curves and any two curves intersect transversally at a smooth point of \( S \). Moreover, for any irreducible component \( D \) of \(|B|\), we have \( K_D + B_D = (K_S + B)|_D \sim_\mathbb{Q} 0 \) by the adjunction formula. Hence \( D \) is either a rational curve or an elliptic curve, and \( \deg B_D \leq 2 \), which implies that each irreducible component of \(|B|\) intersects at most two other irreducible components of \(|B|\). \( \square \)

Lemma 3.10. Let \((S, B)\) be a projective connected dlt pair of dimension 2 such that \( K_S + B \sim_\mathbb{Q} 0 \). Let \( \pi : S' \to S \) be any log resolution of \((S, B)\), write \( \pi^*(K_S + B) = K_{S'} + B' \). Then the image of \( \pi \) gives a natural 1-1 correspondence between connected components of \( B_{i=1}' \) and those of \(|B|\), which preserves the type of each connected component. Moreover, if one connected component of \( B_{i=1}' \) is a chain, then \( \pi \) maps the end curves of this connected component to those of the corresponding connected component of \(|B|\).

Proof. After replacing \( S \) by its minimal resolution, we may assume that \( S \) is smooth. Note that this does not change \(|B|\). We can factor \( \pi : S' \to S \) into blowups at points

\[ S' = S_k \to S_{k-1} \to \cdots S_1 \to S_0 = S, \]

where \( \pi_i : S_i \to S_{i-1} \) is a blowup at a smooth point, and write \( \pi_i^*(K_{S_{i-1}} + B_{i-1}) = K_{S_i} + B_i \). Then we can consider \( B_{i=1}' \) in each step. If the blowed up point is not a 0-dimensional stratum of \( B_{i-1}' \), then \( B_{i=1}' \) is isomorphic to \( B_{i-1}' \); if the blowed up point is a 0-dimensional stratum of \( B_{i-1}' \), then \( B_{i=1}' \) is obtained by replacing this point by a smooth rational curve,
and in this case, this blowup preserves types of irreducible components and end curves of chains. So we can conclude the lemma inductively.

The following lemma is well-known to experts (see, for example, [2, Lemma 1.4]).

**Lemma 3.11.** Let \( S \) be a smooth projective minimal surface and \( B = \sum_i b_i B_i \) an effective \( \mathbb{Q} \)-divisor on \( S \) such that \( b_i \leq 1 \) for all \( i \). Assume that \( K_S + B \sim_{\mathbb{Q}} \mathbb{Q} \) 0. Then one of the following is true:

1. \( B = 0 \) and \( K_S \sim_{\mathbb{Q}} 0 \);
2. \( S \sim_{\mathbb{Q}} \mathbb{P}^2 \) with \( \sum_i b_i \leq 3 \);
3. \( S \sim_{\mathbb{Q}} \mathbb{Q} \) for some integer \( n \geq 2 \) with \( \sum_i b_i \leq 4 \);
4. \( S \) is a \( \mathbb{P}^1 \)-bundle over an elliptic curve with \( \sum_i b_i \leq 2 \).

In particular, \( [B] \) has at most 4 irreducible components.

**Proof.** If \( B = 0 \), then \( K_S \sim_{\mathbb{Q}} 0 \), which gives (1). From now on, we assume that \( B \neq 0 \). Then \( K_S \sim_{\mathbb{Q}} -B \neq 0 \) is not pseudo-effective. By the minimality of \( S \), \( S \) is either \( \mathbb{P}^2 \) or a \( \mathbb{P}^1 \)-bundle over a smooth curve \( C \).

Suppose that \( S \sim_{\mathbb{Q}} \mathbb{P}^2 \). Then \( \sum_i b_i \leq \sum_i b_i (B_i \cdot \ell) = (-K_S \cdot \ell) = 3 \) by \( K_S + B \equiv 0 \), where \( \ell \) is a line on \( \mathbb{P}^2 \). This is (2).

Suppose that \( f : S \to C \) is a \( \mathbb{P}^1 \)-bundle over a smooth curve \( C \). Since \( K_S + B \sim_{\mathbb{Q}} 0 \), by the canonical bundle formula (see [14, Theorem 3.1]), \( -K_C \) is pseudo-effective, which implies that \( C \) is either a rational curve or an elliptic curve. If \( C \) is a rational curve, then \( S \sim_{\mathbb{Q}} \mathbb{Q} \) and \( \sum_i b_i \leq 4 \) by [2, Lemma 1.3]. This is (3). If \( C \) is an elliptic curve, then we know that \( K_S + B \sim_{\mathbb{Q}} f^* K_C \). Again from the canonical bundle formula, \( B \) does not contain any fiber of \( f \). Hence \( \sum_i b_i \leq \sum_i b_i (B_i \cdot F) = (-K_S \cdot F) = 2 \), where \( F \) is a fiber of \( S \to C \).

**Lemma 3.12.** Let \( (S, B) \) be a projective connected lc pair of dimension 2 such that \( K_S + B \sim_{\mathbb{Q}} 0 \). Suppose that \( (S, [B]) \) is log smooth. Then there exists a projective birational morphism \( \pi : (S', B') \to (S, B) \), where \( K_{S'} + B' = \pi^*(K_S + B) \), such that \( S' \) is smooth, \( (S', B') \) is dlt, and if we write \( [B'] = T_0 + T_1 \) as in Lemma 3.9, then \( T_0 \leq \pi_*^{-1} [B] \).

**Proof.** As \( (S, [B]) \) is log smooth, we can construct a dlt model of \( (S, B) \) by a sequence of blowups along 0-dimensional lc centers of \( (S, B) \) avoiding blowing up 0-dimensional strata of \([B] \), say \( \pi : (S', B') \to (S, B) \), where \( K_{S'} + B' = \pi^*(K_S + B) \), such that \( S' \) is smooth and \( (S', B') \) is dlt. Note that irreducible components of \([B']\) consist of irreducible components of \( \pi_*^{-1} [B] \) and some exceptional divisors of \( \pi \). By the construction of \( S' \), there is no exceptional divisor appearing in \( T_0 \), so \( T_0 \leq \pi_*^{-1} [B] \).

The following lemma is the key lemma in this subsection, which tells that for any connected lc log Calabi–Yau pair of dimension 2, there is a “nice” B-birational model.

**Lemma 3.13.** Let \( (S, B) \) be a projective connected lc pair of dimension 2 such that \( K_S + B \sim_{\mathbb{Q}} 0 \). Then there exists a dlt pair \((S', B')\) B-birational to \((S, B)\) such that \( S' \) is smooth, and if we write \([B'] = T_0 + T_1\) as in Lemma 3.9, then \( T_0 \) has at most 6 irreducible components.

**Proof.** We may assume that \( S \) is smooth after taking the minimal resolution. After running a \( K_S \)-MMP, we get a birational morphism \( \pi : S \to S_0 \) to a minimal surface \( S_0 \). Then \( K_{S_0} + \pi_* B \sim_{\mathbb{Q}} 0 \) and \([\pi_* B]\) has at most 4 irreducible components by Lemma 3.11. Note that \((S_0, \pi_* B)\) is lc but not necessarily dlt. \((S_0, \pi_* B)\) being lc implies that any two curves in \([\pi_* B]\) intersect transversally.
For a component $D$ of $[\pi_*B]$, we know that 
$$2p_a(D) - 2 = (K_{S_0} + D) \cdot D \leq (K_{S_0} + \pi_*B) \cdot D = 0.$$ 
Hence $p_a(D) = 0$ or 1, moreover, if $p_a(D) = 1$, then $D$ is disjoint from $\text{Supp}(\pi_*B - D)$. As $[\pi_*B]$ has at most 2 connected components, it has at most 2 singular irreducible components. Moreover, a singular irreducible component can be resolved by the blowup at its singular point. So after at most two blowups, we get a B-birational model $(S_1, B_1) \to (S_0, \pi_*B)$ such that $(S_1, B_1)$ is log smooth and $[B_1]$ has at most 6 irreducible components. Then we can apply Lemma 3.12 to $(S_1, B_1)$. 

**Remark 3.14.** The bound on the number of irreducible components of $T_0$ could be much sharper by carefully discussions case by case. Indeed, it can be shown that we can construct a sharper bound in this paper, we are satisfied with the bound in Lemma 3.13 and left the details to those interested readers.

The following lemma is a modification of [15, Remark 2.15] in our situation.

**Lemma 3.15.** Let $(S, B)$ be a projective connected dlt pair of dimension 2 such that $m(K_S + B) \sim 0$ for a positive integer $m$. Consider $[B] = T_0 + T_1$ and $T_e$ as in Lemma 3.9. Assume that irreducible components of $T_e$ are disjoint from each other. Then a B-birational map $g: (S, B) \dashrightarrow (S, B)$ induces natural automorphisms

$$g^*: H^0(T_0, m(K_{T_0} + B_{T_0})) \to H^0(T_0, m(K_{T_0} + B_{T_0})), $$

$$g^*: H^0(T_e, m(K_{T_e} + B_{T_e})) \to H^0(T_e, m(K_{T_e} + B_{T_e}))$$

where $K_{T_0} + B_{T_0} = (K_S + B)|_{T_0}$, $K_{T_e} + B_{T_e} = (K_S + B)|_{T_e}$.

**Proof.** Consider a common log resolution

$$(S', B') \xrightarrow{\alpha} (S, B) \xrightarrow{g} (S, B),$$

where $K_{S'} + B' = \alpha^*(K_S + B) = \beta^*(K_S + B)$. By Lemma 3.10, we can consider $T_e' = T_0' + T_1'$ and $T'_e$ accordingly to $[B] = T_0 + T_1$ and $T_e$. Note that this expression is independent of $\alpha$ and $\beta$. Since irreducible components of $T_e$ are disjoint from each other, it is clear that $\alpha^*\mathcal{O}_{T'_e} = T_e$ and $\beta^*\mathcal{O}_{T'_e} = T'_e$. By [15, Remark 2.15], we have $\alpha^*\mathcal{O}_{T_0'} = T_0$ and $\beta^*\mathcal{O}_{T_0'} = T_0$. Hence we get natural automorphisms

$$g^*: H^0(T_0, m(K_{T_0} + B_{T_0})) \xrightarrow{\beta^*} H^0(T_0', m(K_{T_0'} + B_{T_0}'))$$

$$\xrightarrow{(\alpha^*)^{-1}} H^0(T_0, m(K_{T_0} + B_{T_0})), $$

$$g^*: H^0(T_e, m(K_{T_e} + B_{T_e})) \xrightarrow{\beta^*} H^0(T_e', m(K_{T_e'} + B_{T_e}'))$$

$$\xrightarrow{(\alpha^*)^{-1}} H^0(T_e, m(K_{T_e} + B_{T_e})), $$

where $K_{T_0'} + B_{T_0'} = (K_{S'} + B')|_{T_0'}$ and $K_{T_e'} + B_{T_e'} = (K_{S'} + B')|_{T_e'}$. 

The following lemma is a modification of [15, Lemma 2.16] in our situation.
Lemma 3.16. Let \((S, B)\) be a projective connected dlt pair of dimension 2 such that \(K_S + B \sim_{\mathbb{Q}} 0\). Consider \([B] = T_0 + T_1\) and \(T_e\) as in Lemma 3.9. Let \(g : (S, B) \rightarrow (S, B)\) be a \(B\)-birational map. Take a common log resolution \(T\) as in Lemma 3.13. In particular, we can assume that \((S, B)\) is in \((S, B)\) contained in \(P\), an lc center \(P\) of \((S', B')\), and an lc center \(Q'\) of \((S, B)\) contained in \(T_0\) such that \(\alpha|_R\) and \(\beta|_R\) are \(B\)-birational morphisms

\[
\begin{align*}
\xymatrix{(Q, B_Q) & (Q', B_{Q'}) \\
(R, B'_R) & (Q', B_{Q'})}
\end{align*}
\]

where \(K_Q + B_Q = (K_S + B)|_Q\), \(K_{Q'} + B_{Q'} = (K_S + B)|_{Q'}\), and \(K_R + B'_R = (K_S' + B')|_R\). Therefore, \(\beta|_R \circ (\alpha|_R)^{-1} : (Q, B_Q) \rightarrow (Q', B_{Q'})\) is a \(B\)-birational map. Moreover, by the natural restriction map, \(H^0(P, m(K_P + B_P)) \simeq H^0(Q, m(K_Q + B_Q))\) where \(K_P + B_P = (K_S + B)|_P\).

Proof. (1) is a direct consequence of Lemma 3.10. (2) is just \([15\text{, Lemma }2.16]\), the fact that \(Q'\) is in \(T_0\) follows from Lemma 3.10.

Now we are ready to prove Theorem 3.6.

Proof of Theorem 3.6. By Remark 2.3, we can replace \((S, B)\) by a \(B\)-birational dlt model as in Lemma 3.13. In particular, we can assume that \((S, B)\) is dlt, \(S\) is smooth, and \(T_0\) has at most 6 irreducible components. Here we consider \([B] = T_0 + T_1\) and \(T_e\) as in Lemma 3.9. Note that \([B] \neq 0\) as \((S, B)\) is not klt. Fix any \(g \in \text{Bir}(S, B)\).

Suppose that \(T_1 \neq 0\). We can consider \(T_e\). As \(T_1\) has at most 2 connected components, \(T_e\) has at most 4 irreducible components. After further blowups, we may assume that irreducible components of \(T_e\) are disjoint from each other. Since \(m(K_S + B) \sim 0\),

\[
H^0(S, m(K_S + B) - T_e) = H^0(S, -T_e) = 0.
\]

Therefore, the restriction map

\[
H^0(S, m(K_S + B)) \rightarrow H^0(T_e, m(K_{T_e} + B_{T_e}))
\]
is injective, where $K_{T_e} + B_{T_e} = (K_S + B)|_{T_e}$. Write $T_e = \bigoplus_{i=1}^k C_i$ with $k \leq 4$. Write $K_{C_i} + B_{C_i} = (K_S + B)|_{C_i}$. Applying Lemma 3.16(1), for each index $i$, $g$ induces a B-birational map $(C_i, B_{C_i}) \to (C_i', B_{C_i'})$ for some index $i'$. Therefore, $g^{k!}$ induces a B-birational map $(C_i, B_{C_i}) \to (C_i, B_{C_i})$ for each $i$. Note that $m(K_{C_i} + B_{C_i}) \sim 0$ and $C_i$ is of dimension 1, hence by [8, Proposition 8.4], there exists an integer $N_1$ depending only on $m$ such that

$$|\rho_m(\text{Bir}(C_i, B_{C_i}))| \leq N_1.$$ Let $h := 24 \cdot (N_1)!$. Then $(g^*)^h$ induces the identity map on $H^0(C_i, m(K_{C_i} + B_{C_i}))$ for each $i$. By Lemma 3.15, there is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & H^0(S, m(K_S + B)) \\
\downarrow{(g^*)^h} & & \downarrow{(g^*)^h = \text{id}} \\
0 & \longrightarrow & H^0(T_e, m(K_{T_e} + B_{T_e})).
\end{array}
$$

It follows that $(g^*)^h = \text{id}$ on $H^0(S, m(K_S + B))$. In particular, $|\rho_m(\text{Bir}(S, B))| \leq h$.

Suppose that $T_0 \neq 0$. Then by construction, $T_0$ has at most 6 irreducible components and at most 6 0-dimensional strata. In particular, $T_0$ contains at most 12 lc centers of $(S, B)$, denoted by $P_1, \ldots, P_l$ for $l \leq 12$. Since $m(K_S + B) \sim 0$,

$$H^0(S, m(K_S + B) - T_0) = H^0(S, -T_0) = 0.$$ Therefore, the restriction map

$$H^0(S, m(K_S + B)) \to H^0(T_0, m(K_{T_0} + B_{T_0}))$$

is injective, where $K_{T_0} + B_{T_0} = (K_S + B)|_{T_0}$. By Lemma 3.16(2), for every $P_i$, we can find lc centers $Q_i, Q'_i$ of $(S, B)$ in $T_0$ such that $(Q_i, B_{Q_i}) \to (Q'_i, B_{Q'_i})$ is a B-birational map and

$$H^0(P_i, m(K_{P_i} + B_{P_i})) \simeq H^0(Q_i, m(K_{Q_i} + B_{Q_i})).$$

by the natural restriction map, where $K_{P_i} + B_{P_i} = (K_S + B)|_{P_i}$ and $K_{Q_i} + B_{Q_i} = (K_S + B)|_{Q_i}$. Here $Q_i, Q'_i$ also belong to the set $\{P_1, \ldots, P_l\}$. Therefore, $g^{k!}$ induces a natural B-birational map $g^{k!} : (Q_i, B_{Q_i}) \to (Q_i, B_{Q_i})$ for each $i$. Note that $m(K_{Q_i} + B_{Q_i}) \sim 0$ and $Q_i$ is of dimension at most 1, hence by [8, Proposition 8.4], there exists an integer $N_1$ depending only on $m$ such that

$$|\rho_m(\text{Bir}(Q_i, B_{Q_i}))| \leq N_1.$$ Let $h' := 12! \cdot (N_1)!$. Then $(g^*)^{h'}$ induces the identity map on $H^0(Q_i, m(K_{Q_i} + B_{Q_i}))$ for each $i$. By the natural embedding

$$H^0(T_0, m(K_{T_0} + B_{T_0})) \subseteq \bigoplus_i H^0(P_i, m(K_{P_i} + B_{P_i})) \simeq \bigoplus_i H^0(Q_i, m(K_{Q_i} + B_{Q_i})),
$$

$(g^*)^{h'}$ induces the identity map on $H^0(T_0, m(K_{T_0} + B_{T_0}))$. By Lemma 3.15, there is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & H^0(S, m(K_S + B)) \\
\downarrow{(g^*)^{h'}} & & \downarrow{(g^*)^{h'} = \text{id}} \\
0 & \longrightarrow & H^0(T_0, m(K_{T_0} + B_{T_0})).
\end{array}
$$

It follows that $(g^*)^{h'} = \text{id}$ on $H^0(S, m(K_S + B))$. In particular, $|\rho_m(\text{Bir}(S, B))| \leq h'$. □
4. Applications

In this section, we discuss applications of Theorem 1.4 to Conjecture 1.5. Note that Conjecture 1.5 can be viewed as the effective version of the abundance conjecture for log Calabi–Yau pairs, so the framework of Fujino [9] provides an inductive argument between Conjecture 1.1 and Conjecture 1.5.

4.1. Fujino’s work on (pre-)admissible sections. In this subsection, we recall some key ideas of [9] (see also [16, 32]). Pre-admissible sections and admissible sections are used in the inductive proof of the index conjecture.

**Definition 4.1 ([9], Definition 4.1).** Let $(X, \Delta)$ be a projective sldt pair of dimension $d$. Let $m$ be a positive integer such that $m(K_X + \Delta)$ is Cartier. Let $\nu : X' = \coprod_i X'_i \to X = \coprod_i X_i$ be the normalization. Let $\Delta'$ be the $\mathbb{Q}$-divisor such that $K_{X'} + \Delta' = \nu^*(K_X + \Delta)$ and $\Delta'_i = \Delta'|_{X'_i}$. We define **pre-admissible section** and **admissible section** inductively on dimension:

1. $s \in H^0(X, m(K_X + \Delta))$ is pre-admissible if the restriction
   $$\nu^*s|_{(\coprod_i \Delta'_i)} \in H^0((\coprod_i \Delta'_i), m(K_{X'} + \Delta'))$$
   is admissible. Denote the set of pre-admissible sections by $\text{PA}(X, m(K_X + \Delta))$.

2. $s \in H^0(X, m(K_X + \Delta))$ is admissible if $s$ is pre-admissible and $g^*(s|_{X_j}) = s|_{X_i}$ for every $B$-birational map
   $$g : (X_i, \Delta_i) \to (X_j, \Delta_j)$$
   for every $i, j$. Denote the set of admissible sections by $A(X, m(K_X + \Delta))$.

   Note that if $s \in A(X, m(K_X + \Delta))$, then $s|_{X_i}$ is $\text{Bir}(X_i, \Delta_i)$-invariant for every $i$.

We can run the same inductive argument as in [9, Section 4] (see also [16, Section 5] and [31, Section 5]). In the following we briefly recall the key results with proofs following their ideas. Taking boundedness into account, Theorems A, B, C in [16] can be formulated into the following conjectures:

**Conjecture A.** Let $m, d$ be two positive integers. Then there exists a positive integer $M$ depending only on $m, d$ satisfying the following property: if $(X, \Delta)$ is a projective (not necessarily connected) dlt pair of dimension $d$ such that $m(K_X + \Delta) \sim 0$, then $\text{PA}(X, Mm(K_X + \Delta)) \neq 0$.

**Conjecture B (≡Conjecture 1.1).** Let $m, d$ be two positive integers. Then there exists a positive integer $N$ depending only on $m, d$ satisfying the following property: if $(X, \Delta)$ is a projective connected dlt pair of dimension $d$ such that $m(K_X + \Delta) \sim 0$, then $|\rho_m(\text{Bir}(X, \Delta))| \leq N$.

**Conjecture B’.** Let $m, d$ be two positive integers. Then there exists a positive integer $N$ depending only on $m, d$ satisfying the following property: if $(X, \Delta)$ is a projective connected klt pair of dimension $d$ such that $m(K_X + \Delta) \sim 0$, then $|\rho_m(\text{Bir}(X, \Delta))| \leq N$.

**Conjecture C.** Let $m, d$ be two positive integers. Then there exists a positive integer $M$ depending only on $m, d$ satisfying the following property: if $(X, \Delta)$ is a projective (not necessarily connected) dlt pair of dimension $d$ such that $m(K_X + \Delta) \sim 0$, then $A(X, Mm(K_X + \Delta)) \neq 0$.

In the following, Conjecture $\bullet_d$ (resp. Conjecture $\bullet_{≤d}$) stands for Conjecture $\bullet$ with $\dim X = d$ (resp. $\dim X ≤ d$). Note that for the above conjectures, Conjecture $\bullet_d$ naturally implies Conjecture $\bullet_{d−1}$ by considering fiber products with an elliptic curve.
The following propositions are taken out from [16] Section 5 and [9] Section 4 by minor modifications of proofs.

**Proposition 4.2** (cf. [9] Proposition 4.5, [10] Proposition 4.15, [16] Claim 5.4, [31] Lemma 5.11). If Conjecture $C_{d-1}$ holds true, then Conjecture $A_d$ holds true.

**Proof.** Let $(X, \Delta)$ be a projective dlt pair of dimension $d$ such that $m(K_X + \Delta) \sim 0$. Write $(X, \Delta) = \Pi_i (X_i, \Delta_i)$ where $(X_i, \Delta_i)$ is a projective connected dlt pair of dimension $d$ for each $i$.

If $|\Delta| = 0$, then this is trivial because any section in $H^0(X, m(K_X + \Delta))$ is pre-admissible.

If $|\Delta| \neq 0$, then by Conjecture $C_{d-1}$, there exists a positive integer $M$ depending only on $m, d$ such that $A(|\Delta|, Mm(K_X + \Delta)) \neq 0$. We may assume that $M$ is even. Then by [9] Proposition 4.5, for each $i$,

$$PA(X_i, Mm(K_{X_i} + \Delta_i)) \rightarrow A(|\Delta_i|, Mm(K_{X_i} + \Delta_i))$$

is surjective. Here we remark that [9] Proposition 4.5 requires that $\dim X_i \leq 3$ and $Mm$ is sufficiently divisible, where the former condition is for applying [9] Proposition 2.1 which can be removed by using [16] Claim 5.3, and the latter condition can be replace by $Mm(K_{X_i} + \Delta_i) \sim 0$ and $Mm$ is even, hence we can apply [9] Proposition 4.5 in our situation, see also [31] Lemma 5.11 for detailed discussions. As

$$A(|\Delta|, Mm(K_X + \Delta)) \subset \bigoplus_i A(|\Delta_i|, Mm(K_{X_i} + \Delta_i)),$$

there exists a non-zero section $t \in H^0(X, Mm(K_X + \Delta))$ such that $t|_{\Delta_i} \in A(|\Delta_i|, Mm(K_{X_i} + \Delta))$. By definition, $t$ is pre-admissible, and hence $PA(X, Mm(K_X + \Delta)) \neq 0$. \hfill $\square$

**Proposition 4.3** (cf. [9] Theorem 3.5, [16] Page 560, Step 2). If Conjecture $A_d$ and Conjecture $B'_d$ hold true, then Conjecture $B_d$ holds true.

**Proof.** Let $(X, \Delta)$ be a projective connected dlt pair of dimension $d$ such that $m(K_X + \Delta) \sim 0$.

If $(X, \Delta)$ is klt, then by Conjecture $B'_d$, there exists a positive integer $N$ depending only on $m, d$ such that $|\rho_m(Bir(X, \Delta))| \leq N$.

If $(X, \Delta)$ is not klt, then there exists a non-zero section $t \in PA(X, m(K_X + \Delta))$ by Conjecture $A_d$. Note that in this case $PA(X, m(K_X + \Delta)) = H^0(X, m(K_X + \Delta)) \simeq \mathbb{C}$. So $\rho_m(g) \in \mathbb{C}^*$ for any $g \in Bir(X, \Delta)$. On the other hand, by [9] Proposition 4.9, for any $g \in Bir(X, \Delta)$, $g^*t|_{\Delta_i} = t|_{\Delta_i}$. So $\rho_m(g) = 1$ for any $g \in Bir(X, \Delta)$, that is, $|\rho_m(Bir(X, \Delta))| = 1$. \hfill $\square$

**Proposition 4.4** (cf. [9] Lemma 4.9, [31] Proposition 5.10). If Conjecture $A_d$ and Conjecture $B'_d$ hold true, then Conjecture $C_d$ holds true.

**Proof.** Let $(X, \Delta)$ be a projective (not necessarily connected) dlt pair of dimension $d$ such that $m(K_X + \Delta) \sim 0$. Write $(X, \Delta) = \Pi_i (X_i, \Delta_i)$ where $(X_i, \Delta_i)$ is a projective connected dlt pair of dimension $d$ for each $i$. Our goal is to construct a non-zero section in $A(X, km(K_X + \Delta))$ for some positive integer $k$ independent of $(X, \Delta)$.

We may write $(X, \Delta) = (X', \Delta') \amalg (X'', \Delta'')$ into two parts where $(X', \Delta')$ consists of all klt $(X_i, \Delta_i)$ and $(X'', \Delta'')$ consists of all non-klt $(X_i, \Delta_i)$. Then

$$A(X, km(K_X + \Delta)) = A(X', km(K_X' + \Delta')) \oplus A(X'', km(K_{X''} + \Delta''))$$
for all positive integer \(k\) because there is no B-birational map between a klt pair and a non-klt pair. Hence we only need to consider 2 cases: \((X_i, \Delta_i)\) is klt for each \(i\), or \((X_i, \Delta_i)\) is not klt for each \(i\).

Suppose that \((X_i, \Delta_i)\) is not klt for each \(i\). By Conjecture \(A_d\), there exists a positive integer \(M\) depending only on \(m, d\) such that \(\rho_{m}(\text{Bir}(X_i, \Delta_i))\) is not klt for each \(M\) integer. So either \(\rho(\text{Bir}(X_i, \Delta_i))\) induces a B-birational map \(\gamma\) such that \(\rho_{\gamma}(\text{Bir}(X_i, \Delta_i))\) acts trivially on \(\mathbb{C}^*\). On the other hand, \(\gamma\) induces a natural B-birational map \(\bar{\gamma}\) by exchanging \(X_i\) and \(X_j\), hence by [7] Lemma 4.9, \(\gamma^*(s|_{\Delta_i}) = s|_{\Delta_j}\). So either \(\bar{\gamma}\) is klt for each \(i\) is klt for each \(i\), then

\[
\rho_{m}(\text{Bir}(X_i, \Delta_i)) = \bigoplus_{i} \rho_{m}(\text{Bir}(X_i, \Delta_i)) = \bigoplus_{i} \rho_{m}(\text{Bir}(X_i, \Delta_i)).
\]

Fix a section \((s_i)_i \in H^0(X, m(K_X + \Delta))\) where \(s_i \in H^0(X, m(K_X + \Delta))\) is non-zero for each \(i\). By Conjecture \(B'_{\leq 2}\), there exists a positive integer \(N\) depending only on \(m, d\) such that \(|\rho_{m}(\text{Bir}(X_i, \Delta_i))| \leq N\). This implies that \(\rho_{Nm}(\text{Bir}(X_i, \Delta_i))\) acts trivially on \(H^0(X, N!m(K_X + \Delta))\) for each \(i\). Denote \(t = (t_i)_i = (s_i^N)_i \in H^0(X, N!m(K_X + \Delta))\). If \(G_{ij} = (X_i, \Delta_i) \longrightarrow (X_j, \Delta_j)\) is a B-birational map, then we can write \(\gamma_{ij}t_i = \lambda_{ij}t_i\) for some \(\lambda_{ij} \in \mathbb{C}^*\). Note that if \(f_{ij} = (X_i, \Delta_i) \longrightarrow (X_j, \Delta_j)\) is another B-birational map, then \(f_{ij}^{-1} \circ G_{ij} \in \text{Bir}(X_i, \Delta_i)\). As \(\rho_{Nm}(\text{Bir}(X_i, \Delta_i))\) acts trivially on \(H^0(X, N!m(K_X + \Delta))\), this implies that \(\lambda_{ij}\) is independent of the choice of \(G_{ij}\) and \(\lambda_{ij} = 1\) for each \(i\). So we can find \(\lambda_i \in \mathbb{C}^*\) for each \(i\) such that \(\lambda_{ij} = \lambda_i \lambda_j^{-1}\) if there is a B-birational map \((X_i, \Delta_i) \longrightarrow (X_j, \Delta_j)\). Then it is easy to check that the non-zero section \(t' = (\lambda_i t_i)_i \in H^0(X, N!m(K_X + \Delta))\) is actually admissible. □

From the above inductive arguments, it is easy to get the following corollary.

**Corollary 4.5.** Conjecture \(C_{\leq 2}\) and Conjecture \(A_{\leq 3}\) hold true.

**Proof.** This directly follows from Propositions 4.2 and 4.4 as Conjecture \(B'_{\leq 2}\) holds true by Theorem 3.1. □

**Remark 4.6.** By Proposition 4.3 and Corollary 4.5, Conjecture \(B_{\leq 2}\) holds true. Note that here we only use Theorem 3.1 so this indeed gives an alternative proof of Theorem 3.6.

4.2. **Applications to the index conjecture.** In this subsection, we give applications of Theorem 3.4 to the index conjecture.

The following propositions are well-known to experts as inductive steps towards the index conjecture.

**Proposition 4.7 (cf. [16 Theorem 1.5]).** If Conjecture \(L_3\) holds true for dlt pairs of dimension \(d\) and Conjecture \(A_d\) holds, then Conjecture \(L_3\) holds true in dimension \(d\).

**Proof.** The proof is essentially the same as [16 Theorem 1.5]. Let \((X, \Delta)\) be a projective slc pair of dimension \(d\) such that the coefficients of \(\Delta\) are in \(I\) and 

\[K_X + \Delta \sim_{Q} 0.\]

We may assume that \(X\) is connected. Take the normalization \(X' = \Pi_i X'_i \to X = \bigcup_i X_i\) and a dlt blowup \([25\text{ Theorem 3.1}]\) on each \(X'_i\). We get \(\phi : (Y, \Gamma) \to (X, \Delta)\) such that \((Y, \Gamma)\) is a projective (not necessarily connected) dlt pair and \(K_Y + \Gamma = \phi^*(K_X + \Delta)\). Note that \(K_Y + \Gamma \sim_{Q} 0\) and the coefficients of \(\Gamma\) are in \(I \cup \{1\}\), hence by Conjecture \(L_3\) for dlt pairs of dimension \(d\), there exists a positive integer \(m\) depending only on \(d, I\) such
that \( m(K_Y + \Gamma) \sim 0 \). By Conjecture A\(_d\), after replacing \( m \) by a constant multiple, we may assume that \( m(K_Y + \Gamma) \) has a non-zero pre-admissible section, which descends to a non-zero section of \( m(K_X + \Delta) \) by \([9, \text{Lemma 4.2}]\). Hence \( m(K_X + \Delta) \sim 0 \). 

**Proposition 4.8** (cf. \([32, \text{Theorem 1.7}]\)). If Conjecture 1.3 holds true in dimension \( d-1 \), then Conjecture 1.5 holds true for connected non-klt lc pairs in dimension \( d \).

Here we give a simple proof different from \([32, \text{Theorem 1.7}]\), which follows the idea of \([19, 9.9]\).

**Proof.** Let \((X, \Delta)\) be a projective connected non-klt lc pair of dimension \( d \) such that the coefficients of \( \Delta \) are in \( I \) and \( K_X + \Delta \sim _{\mathbb{Q}} 0 \). After taking a dlt blowup (\([23, \text{Theorem 3.1}]\)) and replacing \( I \) by \( I \cup \{1\} \), we may assume that \((X, \Delta)\) is dlt and in particular \( |\Delta| \neq 0 \).

If we write \((K_X + \Delta)|_{|\Delta|} = K_{|\Delta|} + \Theta\), then \(|\Delta|, \Theta\) is an sdlt pair of dimension \( d-1 \) by \([9, \text{Remark 1.2(3)}]\) such that \( K_{|\Delta|} + \Theta \sim _{\mathbb{Q}} 0 \). By the adjunction formula (\([24, \text{Corollary 16.7}]\) or \([17, \text{Lemma 4.1}]\)), the coefficients of \( \Theta \) belong to a set \( D(I) \) which is a DCC (i.e., descending chain condition) set of rational numbers depending only on \( I \). Then by the Global ACC (\([17, \text{Theorem 1.5}]\)), the coefficients of \( \Theta \) belong to a finite set \( I_0 \) of rational numbers depending only on \( I \). Hence by Conjecture 1.5 in dimension \( d-1 \), there exists a positive integer \( m \) depending only on \( d, I_0 \) such that \( m(K_X + \Delta)|_{|\Delta|} \sim 0 \). Note that \( m \) depends only on \( d, I \) as \( I_0 \) depends only on \( I \). As \( I \) is a finite set, after replacing \( m \) by a multiple, we may also assume that \( m(K_X + \Delta) \) is a Weil divisor. Take the minimal positive integer \( r \) such that \( rm(K_X + \Delta) \sim 0 \). Take the index 1 cover of \( m(K_X + \Delta) \), say \( \pi : Y \to X \), then \( K_Y + \Delta_Y = \pi^*(K_X + \Delta) \sim _{\mathbb{Q}} 0 \) where \( \Delta_Y \) is an effective \( \mathbb{Q} \)-divisor since \( \pi \) is étale in codimension one. Then \((Y, \Delta_Y)\) is a connected lc pair of dimension \( d \) by \([26, \text{Proposition 5.20}]\). As \( m(K_X + \Delta)|_{|\Delta|} \sim 0 \), \( |\Delta_Y| = \pi^{-1}(\Delta) \) has at least \( r \) connected components. Hence by the connectedness lemma (\(\text{Lemma 3.8}\), \( r \leq 2 \). In particular, \( 2m(K_X + \Delta) \sim 0 \).

Now we can get some partial results of the index conjecture in low dimensions.

**Proof of Corollary 1.6.** It suffices to proof the corollary for \( \dim X = 3 \), as lower dimensional case can be reduced to higher dimensional case by taking fiber products with an elliptic curve. By Proposition 1.7 and Corollary 1.5, it suffices to prove Conjecture 1.5 for dlt pairs in dimension 3. If \( X \) is klt and \( \Delta = 0 \), then this is proved by \([21, \text{Corollary 1.7}]\). The remaining cases are proved by \([32, \text{Theorem 1.13}]\).

**Proof of Corollary 1.7.** This follows directly from Proposition 4.8 and Corollary 1.6.

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