HYPERSURFACES OF PRESCRIBED SCALAR CURVATURE IN LORENTZIAN MANIFOLDS

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Dedicated to Robert Finn on the occasion of his eightieth birthday

Abstract. The existence of closed hypersurfaces of prescribed scalar curvature in globally hyperbolic Lorentzian manifolds is proved provided there are barriers.

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0. Introduction

Consider the problem of finding a closed hypersurface of prescribed curvature $F$ in a globally hyperbolic (n+1)-dimensional Lorentzian manifold $N$ having a compact Cauchy hypersurface $S_0$. To be more precise, let $\Omega$ be a connected open subset of $N$, $f \in C^{2,\alpha}(\overline{\Omega})$, $F$ a smooth, symmetric function defined in an open cone $\Gamma \subset \mathbb{R}^n$, then we look for a space-like hypersurface $M \subset \Omega$ such that

\[ F|_M = f(x) \quad \forall x \in M, \]

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where $F_{|M}$ means that $F$ is evaluated at the vector $(\kappa_i(x))$ the components of which are the principal curvatures of $M$. The prescribed function $f$ should satisfy natural structural conditions, e. g. if $\Gamma$ is the positive cone and the hypersurface $M$ is supposed to be convex, then $f$ should be positive, but no further, merely technical, conditions should be imposed.

In [1, 2, 8, 14] the case $F = H$, the mean curvature, has been treated, and in [15] we solved the problem for curvature functions $F$ of class $K^*$ that includes the Gaussian curvature, see [15, Section 1] for the definition, but excludes the symmetric polynomials $H_k$ for $1 < k < n$. Among these, $H_2$, that corresponds to the scalar curvature operator, is of special interest.

However, a solution of equation (0.1) with $F = H_2$ is in general not a hypersurface of prescribed scalar curvature—unless the ambient space has constant curvature—since the scalar curvature of a hypersurface also depends on $\bar{R}_{\alpha\beta\nu}^{\alpha\nu\beta}$. Thus, we have to allow that the right-hand side $f$ also depends on time-like vectors and look for hypersurfaces $M$ satisfying

\begin{equation}
F_{|M} = f(x, \nu) \quad \forall x \in M,
\end{equation}

where $\nu = \nu(x)$ is the past-directed normal of $M$ in the point $x$.

To give a precise statement of the existence result we need a few definitions and assumptions. First, we assume that $\Omega$ is a precompact, connected, open subset of $N$, that is bounded by two achronal, connected, space-like hypersurfaces $M_1$ and $M_2$ of class $C^{4,\alpha}$, where $M_1$ is supposed to lie in the past of $M_2$.

Let $F = H_2$ be the scalar curvature operator defined on the open cone $\Gamma_2 \subset \mathbb{R}^n$, and $f = f(x, \nu)$ be of class $C^{2,\alpha}$ in its arguments such that

\begin{equation}
0 < c_1 \leq f(x, \nu) \quad \text{if} \quad (\nu, \nu) = -1,
\end{equation}

\begin{equation}
\|f_\beta(x, \nu)\| \leq c_2(1 + \|\nu\|^2),
\end{equation}

and

\begin{equation}
\|f_{\nu,\beta}(x, \nu)\| \leq c_3(1 + \|\nu\|),
\end{equation}

for all $x \in \bar{\Omega}$ and all past directed time-like vectors $\nu \in T_\times(\Omega)$, where $\|\cdot\|$ is a Riemannian reference metric that will be detailed in Section 2.

We suppose that the boundary components $M_i$ act as barriers for $(F, f)$.

**Definition 0.1.** $M_2$ is an upper barrier for $(F, f)$, if $M_2$ is admissible, i.e. its principal curvatures $(\kappa_i)$ with respect to the past directed normal belong to $\Gamma_2$, and if

\begin{equation}
F_{|M_2} \geq f(x, \nu) \quad \forall x \in M_2.
\end{equation}
$M_1$ is a lower barrier for $(F,f)$, if at the points $\Sigma \subset M_1$, where $M_1$ is admissible, there holds

\begin{equation}
F|_{\Sigma} \leq f(x,\nu) \quad \forall x \in \Sigma.
\end{equation}

$\Sigma$ may be empty.

**Remark 0.2.** This definition of upper and lower barriers for a pair $(F,f)$ also makes sense for other curvature functions $F$ defined in an open convex cone $\Gamma$, with a corresponding meaning of the notion admissible.

Now, we can state the main theorem.

**Theorem 0.3.** Let $M_1$ be a lower and $M_2$ an upper barrier for $(F,f)$, where $F = H_2$. Then, the problem

\begin{equation}
F|_{M} = f(x,\nu)
\end{equation}

has an admissible solution $M \subset \bar{\Omega}$ of class $C^{4,\alpha}$ that can be written as a graph over $S_0$ provided there exists a strictly convex function $\chi \in C^2(\bar{\Omega}).$

**Remark 0.4.** As we have shown in [15, Lemma 2.7] the existence of a strictly convex function $\chi$ is guaranteed by the assumption that the level hypersurfaces \{x^0 = \text{const}\} are strictly convex in $\bar{\Omega}$, where $(x^\alpha)$ is a Gaussian coordinate system associated with $S_0$.

Looking at Robertson-Walker space-times it seems that the assumption of the existence of a strictly convex function in the neighbourhood of a given compact set is not too restrictive: in Minkowski space e.g. $\chi = -|x^0|^2 + |x|^2$ is a globally defined strictly convex function. The only obstruction we are aware of is the existence of a compact maximal slice. In the neighbourhood of such a slice a strictly convex function cannot exist.

The existence result of our main theorem would also be valid in Riemannian manifolds if one could prove $C^1$- estimates. For the $C^2$- estimates the nature of the ambient space is irrelevant though the proofs are slightly different.

For prescribed curvature problems it seems more natural to assume that the right-hand side $f$ depends on $(x,\nu)$, and we shall prove in a subsequent paper existence results for curvature functions $F \in (K^*)$, where the ambient space can be Riemannian or Lorentzian, cf. [16].

The paper is organized as follows: In Section 1 we take a closer look at curvature functions and define the concept of elliptic regularization for these functions, and analyze some of its properties.
In Section 2 we introduce the notations and common definitions we rely on, and state the equations of Gauß, Codazzi, and Weingarten for space-like hypersurfaces.

In Section 3 we look at the curvature flow associated with our problem, and the corresponding evolution equations for the basic geometrical quantities of the flow hypersurfaces.

In Section 4 we prove lower order estimates for the evolution problem, while a priori estimates in the $C^2$-norm are derived in Section 5.

In Section 6, we demonstrate that the evolutionary solution converges to a stationary approximation of our problem, i.e. to a solution for a curvature problem, where $F$ is replaced by its elliptic regularization $F_\varepsilon$.

The uniform $C^1$-estimates for the stationary approximations are derived in Sections 7 and 8, the $C^2$-estimates are given in Section 9, while the final existence result is contained in Section 10.

1. Curvature Functions

Let $\Gamma \subset \mathbb{R}^n$ be an open cone containing the positive cone $\Gamma_+$, and $F \in C^{2,\alpha}(\Gamma) \cap C^0(\bar{\Gamma})$ a positive symmetric function satisfying the condition

\begin{equation}
F_i = \frac{\partial F}{\partial \kappa_i} > 0; 
\end{equation}

then, $F$ can also be viewed as a function defined on the space of symmetric matrices $\mathcal{C}$, the eigenvalues of which belong to $\Gamma$, namely, let $(h_{ij}) \in \mathcal{C}$ with eigenvalues $\kappa_i$, $1 \leq i \leq n$, then define $F$ on $\mathcal{C}$ by

\begin{equation}
F(h_{ij}) = F(\kappa_i).
\end{equation}

If we define

\begin{equation}
F^{ij} = \frac{\partial F}{\partial h_{ij}}
\end{equation}

and

\begin{equation}
F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}
\end{equation}

then,

\begin{equation}
F^{ij} \xi_i \xi_j = \frac{\partial F}{\partial \kappa_i} |\xi|^2 \quad \forall \xi \in \mathbb{R}^n,
\end{equation}

in an appropriate coordinate system,

\begin{equation}
F^{ij}
\end{equation}

is diagonal if $h_{ij}$ is diagonal,
and
\[(1.7)\quad F^{ij,kl} \eta_{ij} \eta_{kl} = \frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{F_i - F_j}{\kappa_i - \kappa_j} (\eta_{ij})^2,\]
for any \((\eta_{ij}) \in S\), where \(S\) is the space of all symmetric matrices. The second term on the right-hand side of (1.7) is non-positive if \(F\) is concave, and non-negative if \(F\) is convex, and has to be interpreted as a limit if \(\kappa_i = \kappa_j\).

The preceding considerations are also applicable if the \(\kappa_i\) are the principal curvatures of a space-like hypersurface \(M\) with metric \((g_{ij})\). \(F\) can then be looked at as being defined on the space of all symmetric tensors \((h_{ij})\) the eigenvalues of which belong to \(\Gamma\). Such tensors will be called \textit{admissible}; when the second fundamental form of \(M\) is admissible, then, we also call \(M\) admissible.

For an admissible tensor \((h_{ij})\)
\[(1.8)\quad F^{ij} = \frac{\partial F}{\partial h_{ij}}\]
is a contravariant tensor of second order. Sometimes it will be convenient to circumvent the dependence on the metric by considering \(F\) to depend on the mixed tensor
\[(1.9)\quad h^i_j = g^{ik} h_{kj}.
\]
Then,
\[(1.10)\quad F^i_j = \frac{\partial F}{\partial h^i_j}\]
is also a mixed tensor with contravariant index \(j\) and covariant index \(i\).

Such functions \(F\) are called curvature functions. Important examples are the symmetric polynomials of order \(k\), \(H_k\), \(1 \leq k \leq n\),
\[(1.11)\quad H_k(\kappa) = \sum_{i_1 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k}.
\]
They are defined on an open cone \(\Gamma_k\) that can be characterized as the connected component of \(\{H_k > 0\}\) that contains \(\Gamma^+\).

Since we have in mind that the \(\kappa_i\) are the principal curvatures of a hypersurface, we use the standard symbols \(H\) and \(|A|\) for
\[(1.12)\quad H = \sum_i \kappa_i,\]
and
\[(1.13) \quad |A|^2 = \sum \kappa_i^2.\]

The scalar curvature function \(F = H_2\) can then be expressed as
\[(1.14) \quad F = \frac{1}{2}(H^2 - |A|^2),\]

and we deduce that for \((\kappa_i) \in \Gamma_2\)
\[(1.15) \quad |A|^2 \leq H^2,\]
\[(1.16) \quad F_i = H - \kappa_i,\]
and hence,
\[(1.17) \quad H F_i \geq F,\]
for (1.17) is equivalent to
\[(1.18) \quad H \kappa_i \leq \frac{1}{2} H^2 + \frac{1}{2} |A|^2,\]
which is obviously valid.

In important ingredient in our existence proof will be the method of elliptic regularization

**Lemma 1.1.** For each \(\epsilon > 0\), consider the linear isomorphism \(\varphi_\epsilon \) in \(\mathbb{R}^n\) given by
\[(1.19) \quad (\tilde{\kappa}_i) = \varphi_\epsilon(\kappa_i) = (\kappa_i + \epsilon H).\]
Let \(F \in C^2(\Gamma) \cap C^0(\bar{\Gamma})\) be a curvature function such that
\[(1.20) \quad F|_{\partial \Gamma} = 0.\]
Then, \(\Gamma_\epsilon = \varphi_\epsilon^{-1}(\Gamma)\) is an open cone and \(F_\epsilon = F \circ \varphi_\epsilon \in C^2(\Gamma_\epsilon) \cap C^0(\bar{\Gamma}_\epsilon)\) a curvature function satisfying
\[(1.21) \quad F_\epsilon|_{\partial \Gamma_\epsilon} = 0.\]
Assume furthermore, that
\[(1.22) \quad H > 0 \quad \text{in} \quad \Gamma.\]
Then,

\[(1.23) \quad \Gamma \subset \Gamma,\]

and

\[(1.24) \quad H > 0 \quad \text{in} \quad \Gamma.\]

**Proof.** We only prove the assertions (1.23) and (1.24) since the other assertions are obvious. Let \((\kappa_i) \in \Gamma\) be fixed. Then,

\[(1.25) \quad 0 < F(\kappa_i) \leq F(\kappa_i + \epsilon H),\]

because \(F\) is monotone, and we deduce

\[(1.26) \quad (\kappa_i + \epsilon H) \in \Gamma \quad \forall \epsilon > 0,\]

in view of (1.22) and the monotonicity of \(F\), cf. (1.1).

To prove (1.24), we observe that

\[(1.27) \quad \sum_i \kappa_i = (1 + \epsilon \alpha) \sum_i \kappa_i.\]

q.e.d.

**Remark 1.2.** (i) Let \(F\) be as in Lemma 1.1 and assume moreover, that \(F\) is homogeneous of degree 1, and concave, then,

\[(1.28) \quad F(\kappa_i) \leq \frac{1}{n} F(1, \ldots, 1) H \quad \forall (\kappa_i) \in \Gamma,\]

and we conclude that condition (1.22) is satisfied.

(ii) Let \(F\) be as in Lemma 1.1, but suppose that \(F\) is homogeneous of degree \(d_0 > 0\) and \(F^{\frac{1}{d_0}}\) concave, then, the relation (1.22) is also valid.

**Proof.** The inequality (1.28) follows easily from the concavity and homogeneity

\[(1.29) \quad F(\kappa_i) \leq F(1, \ldots, 1) + \sum_i F_i(1, \ldots, 1)(\kappa_i - 1)\]

\[= \frac{1}{n} F(1, \ldots, 1) H,\]

since \(F_i(1, \ldots, 1) = \frac{1}{n} F(1, \ldots, 1)\), while the other assertions are obvious. q.e.d.
For better reference, we use a tensor setting in the next lemma, i.e. the \((\kappa_i) \in \Gamma\) are the eigenvalues of an admissible tensor \((h_{ij})\) with respect to a Riemannian metric \((g_{ij})\). In this setting the elliptic regularization of \(F\) is given by

\[
\tilde{F}(h_{ij}) \equiv F(h_{ij} + \epsilon H g_{ij}).
\]

**Lemma 1.3.** Let \(\tilde{F}\) be the elliptic regularization of a curvature function \(F\) of class \(C^2\), then,

\[
\tilde{F}^{ij} = F^{ij} + \epsilon F^{rs} g_{rs} g^{ij},
\]

and

\[
\tilde{F}^{ij,kl} = F^{ij,kl} + \epsilon F^{ij,ab} g_{ab} g^{kl} + \epsilon F^{rs,kl} g_{rs} g^{ij} + \epsilon^2 F^{rs,ab} g_{rs} g_{ab} g^{ij} g^{kl}.
\]

If \(F\) is concave, then, \(\tilde{F}\) is also concave.

**Proof.** The relations \((1.31)\) and \((1.32)\) are straight-forward calculations.

To prove the concavity of \(\tilde{F}\), let \((\eta_{ij})\) be a symmetric tensor, then,

\[
\tilde{F}^{ij,kl} \eta_{ij} \eta_{kl} = F^{ij,kl} \eta_{ij} \eta_{kl} + 2 \epsilon F^{ij,rs} \eta_{ij} g_{rs} g^{kl} \eta_{kl} + \epsilon^2 F^{rs,ab} g_{rs} g_{ab} (g^{ij} \eta_{ij})^2 \leq 0.
\]

q.e.d.

2. Notations and preliminary results

The main objective of this section is to state the equations of Gauß, Codazzi, and Weingarten for space-like hypersurfaces \(M\) in a \((n+1)\)-dimensional Lorentzian space \(N\). Geometric quantities in \(N\) will be denoted by \((\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta}), \) etc., and those in \(M\) by \((g_{ij}), (R_{ijkl}), \) etc. Greek indices range from 0 to \(n\) and Latin from 1 to \(n\); the summation convention is always used. Generic coordinate systems in \(N\) resp. \(M\) will be denoted by \((x^\alpha)\) resp. \((\xi^i)\). Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e. for a function \(u\) in \(N\), \((u_\alpha)\)
will be the gradient and \((u_{\alpha \beta})\) the Hessian, but e.g., the covariant derivative of the curvature tensor will be abbreviated by \(\bar{R}_{\alpha \beta \gamma \delta ; i} x_i^\tau\). We also point out that

\[
(2.1) \quad \bar{R}_{\alpha \beta \gamma \delta ; i} = \bar{R}_{\alpha \beta \gamma \delta ; x_i^\tau} x_i^\tau
\]

with obvious generalizations to other quantities.

Let \(M\) be a space-like hypersurface, i.e. the induced metric is Riemannian, with a differentiable normal \(\nu\) that is time-like.

In local coordinates, \((x^\alpha)\) and \((\xi^i)\), the geometric quantities of the space-like hypersurface \(M\) are connected through the following equations

\[
(2.2) \quad x_{ij}^\alpha = h_{ij}^\alpha \nu^\alpha
\]

the so-called \(Gauß\) formula. Here, and also in the sequel, a covariant derivative is always a full tensor, i.e.

\[
(2.3) \quad x_{ij}^\alpha = x_{ij}^\alpha - \Gamma_k^{ij} x_k^\alpha + \bar{\Gamma}_\alpha^{\beta \gamma} x_j^\beta x_i^\gamma.
\]

The comma indicates ordinary partial derivatives.

In this implicit definition the second fundamental form \((h_{ij})\) is taken with respect to \(\nu\).

The second equation is the \(Weingarten\) equation

\[
(2.4) \quad \nu_i^\alpha = h_i^k x_k^\alpha,
\]

where we remember that \(\nu_i^\alpha\) is a full tensor.

Finally, we have the \(Codazzi\) equation

\[
(2.5) \quad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha \beta \gamma \delta} \nu_i^\alpha x_j^\beta x_k^\gamma x_i^\delta
\]

and the \(Gauß\) equation

\[
(2.6) \quad R_{ijkl} = -\left\{h_{ik} h_{jl} - h_{il} h_{jk}\right\} + \bar{R}_{\alpha \beta \gamma \delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.
\]

Now, let us assume that \(N\) is a globally hyperbolic Lorentzian manifold with a \(compact\) Cauchy surface. \(N\) is then a topological product \(\mathbb{R} \times S_0\), where \(S_0\) is a compact Riemannian manifold, and there exists a Gaussian coordinate system \((x^\alpha)\), such that \(x^0\) represents the time, the \((x^i)\) \(1 \leq i \leq n\) are local coordinates for \(S_0\), where we may assume that \(S_0\) is equal to the level hypersurface \(\{x^0 = 0\}\)—we don’t distinguish between \(S_0\) and \(\{0\} \times S_0\)—, and such that the Lorentzian metric takes the form

\[
(2.7) \quad ds_N^2 = e^{2\psi} \left\{-dx^0^2 + \sigma_{ij}(x^0, x) dx^i dx^j\right\},
\]
where \( \sigma_{ij} \) is a Riemannian metric, \( \psi \) a function on \( N \), and \( x \) an abbreviation for the space-like components \( (x^i) \), see [17], [19, p. 212], [18, p. 252], and [8, Section 6]. We also assume that the coordinate system is future oriented, i.e. the time coordinate \( x^0 \) increases on future directed curves. Hence, the contravariant time-like vector \( (\xi^\alpha) = (1,0,\ldots,0) \) is future directed as is its covariant version \( (\xi_\alpha) = e^{2\psi}(-1,0,\ldots,0) \).

Let \( M = \text{graph} \ u|_{S_0} \) be a space-like hypersurface

\[
(2.8) \quad M = \{ (x^0, x) : x^0 = u(x), x \in S_0 \},
\]

then the induced metric has the form

\[
(2.9) \quad g_{ij} = e^{2\psi}(-u_iu_j + \sigma_{ij})
\]

where \( \sigma_{ij} \) is evaluated at \( (u, x) \), and its inverse \( (g^{ij}) = (g_{ij})^{-1} \) can be expressed as

\[
(2.10) \quad g^{ij} = e^{-2\psi}\left(\sigma^{ij} + \frac{u^i u^j}{v^2}\right),
\]

where \( (\sigma^{ij}) = (\sigma_{ij})^{-1} \) and

\[
(2.11) \quad u^i = \sigma^{ij} u_j \quad v^2 = 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2.
\]

Hence, graph \( u \) is space-like if and only if \( |Du| < 1 \).

We also note that

\[
(2.12) \quad v^{-2} = 1 + e^{2\psi} g^{ij} u_i u_j \equiv 1 + e^{2\psi} |Du|^2.
\]

The covariant form of a normal vector of a graph looks like

\[
(2.13) \quad (\nu_\alpha) = \pm v^{-1} e^\psi (1, -u_i).
\]

and the contravariant version is

\[
(2.14) \quad (\nu^\alpha) = \mp v^{-1} e^{-\psi} (1, u^i).
\]

Thus, we have
**Remark 2.1.** Let $M$ be a space-like graph in a future oriented coordinate system. Then, the contravariant future directed normal vector has the form

\[(\nu^\alpha) = v^{-1}e^{-\psi}(1, u^i)\]

and the past directed

\[(\nu^\alpha) = -v^{-1}e^{-\psi}(1, u^i).\]

In the Gauß formula (2.2) we are free to choose the future or past directed normal, but we stipulate that we always use the past directed normal for reasons that we have explained in [15].

Look at the component $\alpha = 0$ in (2.2) and obtain in view of (2.16)

\[e^{-\psi}v^{-1}h_{ij} = -u_{ij} - \bar{\Gamma}^0_{00}u_i u_j - \bar{\Gamma}^0_{0j}u_i u_j - \bar{\Gamma}^0_{ij}u_i u_j.\]

Here, the covariant derivatives are taken with respect to the induced metric of $M$, and

\[-\bar{\Gamma}^0_{ij} = e^{-\psi}\bar{h}_{ij},\]

where $(\bar{h}_{ij})$ is the second fundamental form of the hypersurfaces \(\{x^0 = \text{const}\}\).

An easy calculation shows

\[\bar{h}_{ij}e^{-\psi} = -\frac{1}{2}\dot{\sigma}_{ij} - \dot{\psi}\sigma_{ij},\]

where the dot indicates differentiation with respect to $x^0$.

Next, let us analyze under which condition a space-like hypersurface $M$ can be written as a graph over the Cauchy hypersurface $S_0$.

We first need

**Definition 2.2.** Let $M$ be a closed, space-like hypersurface in $N$. Then, $M$ is said to be *achronal*, if no two points in $M$ can be connected by a future directed time-like curve.

In [5] it is proved, see also [15, Proposition 2.5].

**Proposition 2.3.** Let $N$ be connected and globally hyperbolic, $S_0 \subset N$ a compact Cauchy hypersurface, and $M \subset N$ a compact, connected space-like hypersurface of class $C^m$, $m \geq 1$. Then, $M = \text{graph } u_{|S_0}$ with $u \in C^m(S_0)$ iff $M$ is achronal.
Remark 2.4. The $M_i$ are barriers for the pair $(F, f)$. Let us point out that without loss of generality we may assume

\begin{equation}
F|_{M_2} > f(x, \nu) \quad \forall x \in M_2, \tag{2.20}
\end{equation}

and

\begin{equation}
F|_{\Sigma} < f(x, \nu) \quad \forall x \in \Sigma, \tag{2.21}
\end{equation}

for let $\eta \in C^\infty(\bar{\Omega})$ be a function with support in a small neighbourhood of $M_1 \cup M_2$—the dot should indicate that the union is disjoint—such that

\begin{equation}
\eta|_{M_1} > 0 \quad \text{and} \quad \eta|_{M_2} < 0 \tag{2.22}
\end{equation}

and define for $\delta > 0$

\begin{equation}
f_\delta = f + \delta \eta. \tag{2.23}
\end{equation}

Then, if we assume $f$ to be strictly positive with a positive lower bound, we have for small $\delta$

\begin{equation}
f_\delta \geq \frac{1}{2} f, \tag{2.24}
\end{equation}

and the $M_i$ are barriers for $(F, f_\delta)$ satisfying the strict inequalities; since we shall derive $C^{4,\alpha}$ estimates independent of $\delta$, we shall have proved the existence of a solution for $f$ if we can prove it for $f_\delta$.

Lemma 2.5. Let $M_i$ be barriers for $(F, f)$ satisfying the strict inequalities (2.20) and (2.21), where $F$ is supposed to be monotone and concave. Then, they are also barriers for the elliptic regularizations $F_\epsilon$ for small $\epsilon$.

Proof. In view of Lemma 1.1, we know that $\Gamma \subset \Gamma_\epsilon$ and $H$ is positive in $\Gamma_\epsilon$. Hence, $M_2$ is certainly an upper barrier for $(F_\epsilon, f)$ because of the monotonicity of $F$.

Let $\Sigma_\epsilon$ resp. $\Sigma$ be the points in $M_1$ where the principal curvatures belong to $\Gamma_\epsilon$ resp. $\Gamma$ and assume that $\Sigma \neq \emptyset$. Suppose $M_1$ were not a lower barrier for $(F_\epsilon, f)$ for small $\epsilon$, then, there exist a sequence $\epsilon \to 0$ and a corresponding convergent sequence $x_\epsilon \in \Sigma_\epsilon$, $x_\epsilon \to x_0 \in \Sigma$, such that

\begin{equation}
F_\epsilon \geq f(x_\epsilon, \nu), \tag{2.25}
\end{equation}

and hence,

\begin{equation}
F \geq f(x_0, \nu) \tag{2.26}
\end{equation}
contradicting (2.21). q.e.d.

**Remark 2.6.** The condition (0.3) is reasonable as is evident from the Einstein equation

\begin{equation}
\bar{R}_{\alpha\beta} - \frac{1}{3} \bar{R} g_{\alpha\beta} = T_{\alpha\beta},
\end{equation}

where the energy-momentum tensor $T_{\alpha\beta}$ is supposed to be positive semi-definite for time-like vectors (*weak energy condition*, cf. [19, p. 89]), and the relation

\begin{equation}
R = -\left[H^2 - h_{ij} h^{ij}\right] + \bar{R} + 2 \bar{R}_{\alpha\beta} \nu^\alpha \nu^\beta
\end{equation}

for the scalar curvature of a space-like hypersurface; but it would be convenient for the approximations we have in mind, if the estimate in (0.3) would be valid for all time-like vectors.

In fact, we may assume this without loss of generality: Let $\vartheta$ be a smooth real function such that

\begin{equation}
\frac{c_1}{2} \leq \vartheta \quad \text{and} \quad \vartheta(t) = t \quad \forall t \geq c_1,
\end{equation}

then, we can replace $f$ by $\vartheta \circ f$ and the new function satisfies our requirements for all time-like vectors.

We therefore assume in the following that the relation (0.3) holds for all time-like vectors $\nu \in T_\nu(N)$ and all $x \in \Omega$.

Sometimes, we need a Riemannian reference metric, e.g. if we want to estimate tensors. Since the Lorentzian metric can be expressed as

\begin{equation}
\bar{g}_{\alpha\beta} dx^\alpha dx^\beta = e^{2\psi} \{-dx^0 dx^0 + \sigma_{ij} dx^i dx^j\},
\end{equation}

we define a Riemannian reference metric $(\tilde{g}_{\alpha\beta})$ by

\begin{equation}
\tilde{g}_{\alpha\beta} dx^\alpha dx^\beta = e^{2\psi} \{dx^0 dx^0 + \sigma_{ij} dx^i dx^j\}
\end{equation}

and we abbreviate the corresponding norm of a vectorfield $\eta$ by

\begin{equation}
||\eta|| = (\tilde{g}_{\alpha\beta} \eta^\alpha \eta^\beta)^{1/2},
\end{equation}

with similar notations for higher order tensors.
For a space-like hypersurface $M = \text{graph } u$ the induced metrics with respect to $(\tilde{g}_{\alpha\beta})$ resp. $(\check{g}_{\alpha\beta})$ are related as follows

\[
\tilde{g}_{ij} = \tilde{g}_{\alpha\beta} x^\alpha_i x^\beta_j = e^{2\psi} |u_i u_j + \sigma_{ij}|
\]

(2.33) 

Thus, if $(\xi^i) \in T_p(M)$ is a unit vector for $(g_{ij})$, then

\[
\tilde{g}_{ij} \xi^i \xi^j = 1 + 2 e^{2\psi} |u_i \xi|^2,
\]

(2.34) 

and we conclude for future reference

**Lemma 2.7.** Let $M = \text{graph } u$ be a space-like hypersurface in $N$, $p \in M$, and $\xi \in T_p(M)$ a unit vector, then

\[
\|x^\beta_i \xi^i\| \leq c(1 + |u_i \xi^i|) \leq \tilde{c} \tilde{v},
\]

where $\tilde{v} = v^{-1}$.

### 3. An auxiliary curvature problem

Solving the problem (0.2) involves two steps: first, proving a priori estimates, and secondly, applying a method to show the existence of a solution. In a general Lorentzian manifold the evolution method is the method of choice, but unfortunately, one cannot prove the necessary a priori estimates during the evolution when $F$ is the scalar curvature operator. Both the $C^1$ and $C^2$-estimates fail for general $f = f(x,\nu)$.

Therefore, we use the *elliptic regularization* and consider the existence problem for the operators

\[
F_\epsilon(\kappa_i) = F(\kappa_i + \epsilon H), \quad \epsilon > 0,
\]

i.e. we solve

\[
F_\epsilon|_M = f(x,\nu).
\]

Then, we prove uniform $C^{2,\alpha}$-estimates for the approximating solutions $M_\epsilon$, and finally, let $\epsilon$ tend to zero.

The $F_\epsilon$—or some positive power of it—belong to a class of curvature functions $F$ that satisfy the following condition $(H)$: $F \in C^{2,\alpha}(\Gamma) \cap C^0(\bar{\Gamma})$, where $\Gamma \subset \mathbb{R}^n$ is an open cone containing $I_+$, $F$ is symmetric, monotone,
i.e. \( F_i > 0 \), homogeneous of degree 1, concave, vanishes on \( \partial \Gamma \), and there exists \( \epsilon_0 = \epsilon_0(F) > 0 \) such that

\[
F_i \geq \epsilon_0 \sum_k F_k \quad \forall 1 \leq i \leq n.
\]

Furthermore, the set

\[
A_{\delta, \kappa} = \{ (\kappa_i) \in \Gamma : 0 < \delta \leq F(\kappa_i), \kappa_i \leq \kappa \forall 1 \leq i \leq n \}
\]

is compact.

**Remark 3.1.** If the original curvature function \( F \in C^{2,\alpha}(\Gamma) \cap C^0(\bar{\Gamma}) \) is concave, homogeneous of degree 1, and vanishes on \( \partial \Gamma \), then, the \( F_\epsilon \) are of class \((H)\) in the cone \( \Gamma_\epsilon \), and satisfy (3.3) with \( \epsilon_0 = \epsilon \). The set

\[
A_{\delta, \kappa} = \{ (\kappa_i) \in \Gamma_\epsilon : 0 < \delta \leq F_\epsilon(\kappa_i), \kappa_i \leq \kappa \forall 1 \leq i \leq n \}
\]

is compact for fixed \( \epsilon \).

If the parameters \( \kappa \) and \( \delta \) are independent of \( \epsilon \), then the \( A_{\delta, \kappa} \) are contained in a compact subset of \( \Gamma \) uniformly in \( \epsilon \), for small \( \epsilon \), \( 0 \leq \epsilon \leq \epsilon_1(\delta, \kappa, F) \).

**Proof.** In view of the results in Lemma 1.3 we only have to prove the compactness of \( A_{\delta, \kappa} \). We shall also only consider the case when the estimates hold uniformly in \( \epsilon \).

Due to the concavity and homogeneity of \( F_\epsilon \) we conclude from (1.28) that

\[
F_\epsilon(\kappa_i) \leq \frac{1}{n} F(1, \ldots, 1)(1 + n\epsilon)H.
\]

For \((\kappa_i) \in A_{\delta, \kappa}\) we, therefore, infer

\[
\delta \leq F_\epsilon(\kappa_i) \leq \frac{1 + n\epsilon}{n} F(1, \ldots, 1)H \leq (1 + n\epsilon)F(1, \ldots, 1)\kappa,
\]

and thus,

\[
\lim_{\epsilon \to 0} \epsilon H = 0,
\]

uniformly in \( A_{\delta, \kappa} \).

Suppose \( A_{\delta, \kappa} \) would not stay in a compact subset of \( \Gamma \) for small \( \epsilon \), \( 0 < \epsilon \leq \epsilon_1(\delta, \kappa, F) \). Then, there would exist a sequence \( \epsilon \to 0 \) and a corresponding sequence \((\kappa'_i) \in A_{\delta, \kappa} \) converging to a point \((\kappa_i) \in \partial \Gamma \), which is impossible in view of (3.7), (3.8), and the continuity of \( F \) in \( \bar{\Gamma} \). q.e.d.
To prove the existence of hypersurfaces of prescribed curvature \( F \) for \( F \in (H) \) we look at the evolution problem

\[
\dot{x} = (F - f)\nu, \\
x(0) = x_0,
\]

(3.9)

where \( \nu \) is the past-directed normal of the flow hypersurfaces \( M(t) \), \( F \) the curvature evaluated at \( M(t) \), \( x = x(t) \) an embedding and \( x_0 \) an embedding of an initial hypersurface \( M_0 \), which we choose to be the upper barrier \( M_2 \).

Since \( F \) is an elliptic operator, short-time existence, and hence, existence in a maximal time interval \([0, T^\ast]\) is guaranteed. If we are able to prove uniform a priori estimates in \( C^{2,\alpha} \), long-time existence and convergence to a stationary solution will follow immediately.

But before we prove the a priori estimates, we want to show how the metric, the second fundamental form, and the normal vector of the hypersurfaces \( M(t) \) evolve. All time derivatives are total derivatives. The proofs are identical to those of the corresponding results in a Riemannian setting, cf. [9, Section 3] and [15, Section 4], and will be omitted.

**Lemma 3.2** (Evolution of the metric). The metric \( g_{ij} \) of \( M(t) \) satisfies the evolution equation

\[
\dot{g}_{ij} = 2(F - f)h_{ij},
\]

(3.10)

**Lemma 3.3** (Evolution of the normal). The normal vector evolves according to

\[
\dot{\nu} = \nabla_M(F - f) = g^{ij}(F - f)x_j,
\]

(3.11)

**Lemma 3.4** (Evolution of the second fundamental form). The second fundamental form evolves according to

\[
\dot{h}^i \equiv (F - f)_i^i - (F - f)h^i_k h^k_i - (F - f)\tilde{R}_\alpha\gamma\delta\nu^\alpha x^\gamma x^\delta g^{ij},
\]

(3.12)

and

\[
\dot{h}_{ij} = (F - f)_{ij} + (F - f)h^k_i h^k_j - (F - f)\tilde{R}_\alpha\gamma\delta\nu^\alpha x^\gamma x^\delta.
\]

(3.13)

**Lemma 3.5** (Evolution of \( (F - f) \)). The term \( (F - f) \) evolves according to the equation

\[
(F - f) - F^{ij}(F - f)_{ij} = -F^{ij}h_{ik}h^k_j(F - f) - f_\alpha\nu^\alpha(F - f)
\]

\[
- f_\nu x^\alpha_\nu(F - f)_j g^{ij} - F^{ij}\tilde{R}_\alpha\gamma\delta\nu^\alpha x^\gamma x^\delta(F - f),
\]

(3.14)
From (3.9) we deduce with the help of the Ricci identities a parabolic equation for the second fundamental form

**Lemma 3.6.** The mixed tensor $h^i_j$ satisfies the parabolic equation

$$
\begin{align*}
\dot{h}^i_j - F^{kl}h^i_j_{;kl} &= -F^{kl}h^i_{;kl}h^j_k + fh^i_j \\
&= -F^{kl}h^i_{;kl}h^j_k + f h^i_j \\
&= -f_{\alpha\beta}x^\alpha_i x^\beta_k h^i_j - f_{\alpha\nu}x^\alpha_i h^i_k h^j_j - f_{\alpha\nu}x^\alpha_i h^i_k h^j_j - f_{\nu\alpha}h^i_k h^j_j \\
&+ \frac{F}{r_{i,j}}h_{kl;i}h_{rs;j} + 2F^{kl}r_{\alpha\beta\gamma\delta}x^\alpha_m x^\beta_k x^\gamma_i x^\delta_r h^i_j h^m_j \\
&+ F^{kl}r_{\alpha\beta\gamma\delta}x^\alpha_m x^\beta_k x^\gamma_i x^\delta_r h^i_j h^m_j \\
&+ F^{kl}r_{\alpha\beta\gamma\delta\epsilon} \{ h^\alpha_i x^\beta_k x^\gamma_j x^\delta_r h^i_j h^m_j + h^\alpha_i x^\beta_k x^\gamma_j x^\delta_r h^i_j h^m_j \}.
\end{align*}
$$

The proof is identical to that of the corresponding result in the Riemannian case, cf. [9, Lemma 7.1 and Lemma 7.2]; the only difference is that $f$ now also depends on $\nu$.

**Remark 3.7.** In view of the maximum principle, we immediately deduce from (3.14) that the term $(F - f)$ has a sign during the evolution if it has one at the beginning, i.e., if the starting hypersurface $M_0$ is the upper barrier $M_2$, then $(F - f)$ is non-negative

$$
F \geq f.
$$

4. Lower order estimates for the auxiliary solutions

Since the two boundary components $M_1, M_2$ of $\partial Q$ are space-like, achronal hypersurfaces, they can be written as graphs over the Cauchy hypersurface $S_0$, $M_i = \text{graph } u_i$, $i = 1, 2$, and we have

$$
u_1 \leq u_2,
$$

for $M_1$ should lie in the past of $M_2$, and the enclosed domain is supposed to be connected.

Let us look at the evolution equation (3.9) with initial hypersurface $M_0$ equal to $M_2$ defined on a maximal time interval $I = [0, T^*)$, $T^* \leq \infty$. Since the initial hypersurface is a graph over $S_0$, we can write

$$
M(t) = \text{graph } u(t)|_{S_0}, \quad \forall t \in I,
$$
where \( u \) is defined in the cylinder \( Q_T^* = I \times S_0 \). We then deduce from (3.9), looking at the component \( \alpha = 0 \), that \( u \) satisfies a parabolic equation of the form

\[
\dot{u} = -e^{-\psi}v^{-1}(F - f),
\]

where we use the notations in Section 2, and where we emphasize that the time derivative is a total derivative, i.e.

\[
\dot{u} = \frac{\partial u}{\partial t} + u_i \dot{x}^i.
\]

Since the past directed normal can be expressed as

\[
(\nu^\alpha) = -e^{-\psi}v^{-1}(1, u^i),
\]

we conclude from (3.9), (4.3), and (4.4)

\[
\frac{\partial u}{\partial t} = -e^{-\psi}v(F - f).
\]

Thus, \( \frac{\partial u}{\partial t} \) is non-positive in view of Remark 3.7.

Next, let us state our first a priori estimate

**Lemma 4.1.** Suppose that the boundary components act as barriers for \((F, f)\), then the flow hypersurfaces stay in \( \bar{\Omega} \) during the evolution.

The proof is identical to that of the corresponding result in [15, Lemma 4.1].

For the \( C^1 \)- estimate the term \( \tilde{v} = v^{-1} \) is of great importance. It satisfies the following evolution equation

**Lemma 4.2** (Evolution of \( \tilde{v} \)). Consider the flow (3.9) in the distinguished coordinate system associated with \( S_0 \). Then, \( \tilde{v} \) satisfies the evolution equation

\[
\dot{\tilde{v}} - F^{ij} \tilde{v}_{ij} = - F^{ij} h_{ik} h^{kj} \tilde{v} - f \eta_{\alpha \beta} v^\alpha v^\beta \\
- 2F^{ij} h_{ik} x^\alpha_i x^\beta_k \eta_{\alpha \beta} - F^{ij} \eta_{\alpha \gamma \beta} x^\beta_i x^\gamma_j v^\alpha \\
- F^{ij} \bar{R}_{\alpha \beta \gamma \delta} v^\alpha x^\beta_i x^\gamma_j \eta_{\delta} g^{kl} \\
- f_{\beta} x^\beta_i x^\alpha_k \eta_{\alpha} g^{ik} - f_{\gamma} x^\gamma_i x^\delta_k \eta_{\delta} \eta_{\alpha},
\]

where \( \eta \) is the covariant vector field \( (\eta_{\alpha}) = e^\psi(-1, 0, \ldots, 0) \).
The proof uses the relation

\[ \tilde{v} = \eta_\alpha \nu^\alpha \]  

and is identical to that of [15, Lemma 4.4] having in mind that presently \( f \) also depends on \( \nu \).

**Lemma 4.3.** Let \( M(t) = \text{graph} \ u(t) \) be the flow hypersurfaces, then, we have

\[ \dot{u} - F^{ij} u_{ij} = e^{-\psi} \tilde{v} f + \tilde{f}_{0i} F^{ij} u_j + 2 F^{ij} \tilde{f}_{ij} + F^{ij} \tilde{\Gamma}_{ij}, \]

where all covariant derivatives are taken with respect to the induced metric of the flow hypersurfaces, and the time derivative \( \dot{u} \) is the total time derivative, i.e. it is given by (4.4).

**Proof.** We use the relation (4.3) together with (2.17). q.e.d.

As an immediate consequence we obtain

**Lemma 4.4.** The composite function

\[ \varphi = e^{\mu e^{\lambda u}} \]

where \( \mu, \lambda \) are constants, satisfies the equation

\[ \dot{\varphi} - F^{ij} \varphi_{ij} = fe^{-\psi} \mu \lambda e^{\lambda u} \varphi + F^{ij} u_i u_j \tilde{f}_{00} \mu \lambda e^{\lambda u} \varphi \]

\[ + 2 F^{ij} \tilde{f}_{ij} \mu \lambda e^{\lambda u} \varphi + F^{ij} \tilde{\Gamma}_{ij} \mu \lambda e^{\lambda u} \varphi \]

\[ - [1 + \mu e^{\lambda u}] F^{ij} u_i u_j \mu \lambda^2 e^{\lambda u} \varphi. \]

Before we can prove the \( C^1 \)- estimates we need two more lemmata.

**Lemma 4.5.** There is a constant \( c = c(\Omega) \) such that for any positive function \( 0 < \epsilon = \epsilon(x) \) on \( S_0 \) and any hypersurface \( M(t) \) of the flow we have

\[ ||\nu|| \leq c \tilde{v}, \]

\[ g^{ij} \leq c \tilde{v}^2 \sigma^{ij}, \]

\[ F^{ij} \leq F^{kl} g_{kl} g^{ij}, \]
(4.15) \[ |F^{ij}h^{k}_{j}x^{\alpha}_{i}x^{\beta}_{k}\eta_{\alpha\beta}| \leq \frac{\varepsilon}{2}F^{ij}h^{k}_{i}h_{kj}\tilde{v} + \frac{c}{2\varepsilon}F^{ij}g_{ij}\tilde{v}^{3}, \]

(4.16) \[ |F^{ij}x^{\alpha}_{i}x^{\beta}_{j}\nu^{\alpha}| \leq c\tilde{v}^{3}F^{ij}g_{ij}, \]

and

(4.17) \[ |F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x^{\alpha}_{i}x^{\beta}_{j}x^{\gamma}_{k}x^{\delta}_{l}\eta_{\nu}\xi_{\mu}g^{kl}| \leq c\tilde{v}^{3}F^{ij}g_{ij}. \]

**Proof.** (i) The first three inequalities are obvious.

(ii) (4.15) follows from the generalized Schwarz inequality combined with (4.13) and (4.14).

(iii) (4.16) is a direct consequence of (4.13) and (4.14).

(iv) The proof of (4.17) is a bit more complicated and uses the symmetry properties of the Riemann curvature tensor.

Let

(4.18) \[ a_{ij} = \bar{R}_{\alpha\beta\gamma\delta}x^{\alpha}_{i}x^{\beta}_{j}x^{\gamma}_{k}x^{\delta}_{l}\eta_{\nu}\xi_{\mu}g^{kl}. \]

We shall show that the symmetrization of \( a_{ij} \) satisfies

(4.19) \[ -c\tilde{v}^{3}g_{ij} \leq \frac{1}{2}(a_{ij} + a_{ji}) \leq c\tilde{v}^{3}g_{ij} \]

with a uniform constant \( c = c(\Omega) \), which in turn yields (4.17).

Let \( p \in M(t) \) be arbitrary, \( (x^{\alpha}) \) be the special Gaussian coordinate of \( N \), and \( (\xi^{i}) \) local coordinates around \( p \) such that

(4.20) \[ x^{\alpha}_{i} = \begin{cases} u_{i}, & \alpha = 0, \\ \delta_{i}^{k}, & \alpha = k. \end{cases} \]

We also note that all indices are raised with respect to \( g^{ij} \) with the exception of the contravariant vector

(4.21) \[ \check{u}^{i} = \sigma^{ij}u_{j}. \]

We point out that

(4.22) \[ \|Du\|^{2} = g^{ij}u_{i}u_{j} = e^{-2\psi}\check{v}^{2}\sigma^{ij}u_{i}u_{j}, \]

(4.23) \[ \check{v}^{2} = 1 + e^{2\psi}\|Du\|^{2}, \]

(4.24) \[ (\nu^{\alpha}) = -\check{v}(1, \check{u}^{i})e^{-\psi}, \]
and
\[(4.25)\]
\[\eta_{\epsilon} x^i g^{kl} = -e^\nu u^k.\]

First, let us observe that in view of \((4.25)\) and the symmetry properties of the Riemann curvature tensor we have
\[(4.26)\]
\[a_{ij} u^j = 0.\]

Next, we shall expand the right-hand side of \((4.18)\) explicitly.
\[(4.27)\]
\[a_{ij} = \bar{R}_{0i0j} \bar{v} \| Du \|^2 + \bar{R}_{0i0k} \bar{v} u_j u^k + \bar{R}_{0ikj} \bar{v} u^k + \bar{R}_{i0k0} \bar{v} u^i u_j + \bar{R}_{i0j0} \bar{v} u_l \| Du \|^2 + \bar{R}_{ik} \bar{v} u^k u_j + \bar{R}_{ikj} \bar{v} u^k \bar{u}^l\]

To prove the estimate \((4.19)\), we may assume that \(Du \neq 0\). Let \(e_i, 1 \leq i \leq n\) be an orthonormal base of \(T_p(M(t))\) such that
\[(4.28)\]
\[e_1 = \frac{Du}{\| Du \|}.\]

Then, for \(2 \leq k \leq n\), the \(e_k\) are also orthonormal with respect to the metric \(e^{2\psi} \sigma_{ij}\), and it is also valid that
\[(4.29)\]
\[\sigma_{ij} \bar{u}^i e^j_k = 0 \quad \forall \ 2 \leq k \leq n,\]

where \(e_k = (e^1_k)\).

For \(2 \leq r, s \leq n\) we deduce from \((4.27)\)
\[(4.30)\]
\[a_{ij} e^r_i e^s_j = \bar{R}_{0i0j} \bar{v} \| Du \|^2 e^r_i e^s_j + \bar{R}_{0i0k} \bar{v} u_j u^k e^r_i e^s_j + \bar{R}_{i0k0} \bar{v} u^k u_j + \bar{R}_{i0j0} \bar{v} u_l \| Du \|^2 + \bar{R}_{ik} \bar{v} u^k u_j + \bar{R}_{ikj} \bar{v} u^k \bar{u}^l\]

and hence,
\[(4.31)\]
\[|a_{ij} e^r_i e^s_j| \leq c \bar{v}^3 \quad \forall \ 2 \leq r, s \leq n.\]

It remains to estimate \(a_{ij} e^1_i e^j_r\) for \(2 \leq r \leq n\), because of \((4.26)\).

We deduce from \((4.27)\)
\[(4.32)\]
\[a_{ij} e^1_i e^j_r = \bar{R}_{0i0j} \bar{v} \| Du \|^2 \bar{v}^{-2} e^1_i e^j_r + \bar{R}_{0ikj} \bar{v}^{-1} u^k e^1_i e^j_r,\]
where we used the symmetry properties of the Riemann curvature tensor.

Hence, we conclude

\[ |a_{ij} e_i^r e_j^r| \leq c \tilde{v}^2 \quad \forall \ 2 \leq r \leq n, \]

and the relation (4.19) is proved. q.e.d.

**Lemma 4.6.** Let \( M \subset \tilde{\Omega} \) be a graph over \( S_0 \), \( M = \text{graph } u \), and \( \epsilon = \epsilon(x) \) a function given in \( S_0 \), \( 0 < \epsilon < \frac{1}{2} \). Let \( \varphi \) be defined through

\[ \varphi = e^{\mu \epsilon \lambda u}, \]

where \( 0 < \mu \) and \( \lambda < 0 \). Then, there exists \( c = c(\Omega) \) such that

\[ 2|F^{ij} \tilde{v}_i \varphi_j| \leq c F^{ij} \eta_{ij} \tilde{v}^3 |\lambda| \mu e^\lambda u \varphi + (1 - 2\epsilon) F^{ij} h_k^i k_j \tilde{v} \varphi \]

\[ + \frac{1}{1 - 2\epsilon} F^{ij} h_i^k u_j \mu^2 \lambda^2 e^{2\lambda u} \tilde{v} \varphi. \]

**Proof.** Since \( \tilde{v} = \eta_\alpha \nu^\alpha \), we have

\[ \tilde{v}_i = \eta_\alpha \nu^\alpha x_i^\alpha + \eta_\alpha h_i^k x_k^\alpha \]

\[ = \eta_\alpha \nu^\alpha x_i^\alpha - e^\psi h_i^k u_k. \]

Thus, we derive

\[ 2|F^{ij} \tilde{v}_i \varphi_j| = 2|F^{ij} \tilde{v}_i u_j| ||\lambda| \mu e^\lambda u \varphi \]

\[ \leq c F^{ij} g_{ij} \tilde{v}^3 |\lambda| \mu e^\lambda u \varphi + 2e^\psi |F^{ij} h_k^i k_j u_j| ||\lambda| \mu e^\lambda u \varphi. \]

The last term of the preceding inequality can be estimated by

\[ (1 - 2\epsilon) F^{ij} h_k^i k_j \tilde{v} \varphi + \frac{1}{1 - 2\epsilon} \tilde{v}^{-1} e^{2\psi} ||Du||^2 F^{ij} h_i^k u_j \mu^2 \lambda^2 e^{2\lambda u} \varphi \]

and we obtain the desired estimate in view of (4.23). q.e.d.

Applying Lemma 4.5 to the evolution equation for \( \tilde{v} \) we conclude
Lemma 4.7. There exists a constant $c = c(\Omega)$ such that for any function $\epsilon$, $0 < \epsilon = \epsilon(x) < 1$, defined on $S_0$ the term $\hat{v}$ satisfies an evolution inequality of the form

$$\dot{\hat{v}} - F^{ij} \hat{v}_{ij} \leq -(1 - \epsilon)F^{ij} h^k_i h_{kj} \hat{v} - f_{\alpha\beta} \nu^\alpha \nu^\beta$$

$$+ \frac{c}{\epsilon} F^{ij} g_{ij} \hat{v}^3 + c \|f_{\beta}(x, \nu)\| \hat{v}^2 + f_{\nu^\beta} x^\beta_i h^j_k u^j_k \epsilon^\psi. \tag{4.39}$$

We are now ready to prove the uniform boundedness of $\hat{v}$.

Proposition 4.8. Assume that there are positive constants $c_i$, $1 \leq i \leq 3$, such that for any $x \in \Omega$ and any past directed time-like vector $\nu$ there holds

$$-c_1 \leq f(x, \nu), \tag{4.40}$$

$$\|f_{\beta}(x, \nu)\| \leq c_2 (1 + \|\nu\|), \tag{4.41}$$

and

$$\|f_{\nu^\beta}(x, \nu)\| \leq c_3. \tag{4.42}$$

Then, the term $\hat{v}$ remains uniformly bounded during the evolution

$$\hat{v} \leq c = c(\Omega, c_1, c_2, c_3, \epsilon_0), \tag{4.43}$$

where $c_0$ is the constant in (3.3). Here, and in the following, the reference that a constant depends on $\Omega$ also means that it depends on the barriers and geometric quantities of the ambient space restricted to $\Omega$.

Proof. We proceed similar as in [14, Proposition 3.7] and show that the function

$$w = \hat{v} \varphi, \tag{4.44}$$

$\varphi$ as in (4.34), is uniformly bounded, if we choose

$$0 < \mu < 1 \quad \text{and} \quad \lambda \ll -1, \tag{4.45}$$

appropriately, and assume furthermore, without loss of generality, that $u \leq -1$, for otherwise replace $u$ by $(u - c)$, $c$ large, in the definition of $\varphi$.

With the help of the lemmata 4.4, 4.6, and 4.7 we derive from the relation

$$\dot{\hat{v}} - F^{ij} \hat{v}_{ij} = [\dot{\hat{v}} - F^{ij} \hat{v}_{ij}] \varphi + [\dot{\varphi} - F^{ij} \varphi_{ij}] \hat{v} - 2 F^{ij} \hat{v}_i \varphi_j \tag{4.46}$$
the parabolic inequality

\[ \dot{w} - F^{ij} w_{ij} \leq -\epsilon F^{ij} h^k_i h_{kj} \tilde{v} \varphi + c[\epsilon^{-1} + |\lambda|e^{\lambda u}] F^{ij} g_{ij} \tilde{v}^3 \varphi \]

\[ + \left[ \frac{1}{1-2\epsilon} - 1 \right] F^{ij} u_i u_j \mu^2 \lambda^2 e^{2\lambda u} \tilde{v} \varphi \]

\[ - F^{ij} u_i u_j \mu \lambda^2 e^{\lambda u} \tilde{v} \varphi \]

\[ + f[\eta_{\alpha\beta} \nu^{\alpha} \nu^{\beta} + e^{-\psi} \mu \lambda e^{\lambda u} \tilde{v}^2] \varphi \]

\[ + c \|f\| \tilde{v}^2 \varphi + f_{\nu^{\beta} x^i_k} h^{kl} u_k e^{\psi} \varphi, \]

where we have chosen the same function \( \epsilon = \epsilon(x) \) in Lemma 4.6 resp. Lemma 4.7.

Setting \( \epsilon = e^{-\lambda u} \) and using Lemma 2.7, the assumption (3.3), which can be rewritten as

\[ F^{ij} \geq \epsilon_0 F^{kl} g_{kl} g^{ij}, \]

as well as the assumptions (4.40), (4.41), and (4.42), and observing, furthermore, that in view of the concavity and homogeneity of \( F \)

\[ F^{ij} g_{ij} \geq F(1, \ldots, 1) > 0, \]

we conclude

\[ \dot{w} - F^{ij} w_{ij} \leq -\frac{1}{2} F^{ij} h^k_i h_{kj} e^{-\lambda u} \tilde{v} \varphi + c |\lambda| \mu e^{\lambda u} F^{ij} g_{ij} \tilde{v}^3 \varphi \]

\[ + \frac{2}{1-2\epsilon} F^{ij} u_i u_j \mu^2 \lambda^2 e^{\lambda u} \tilde{v} \varphi - F^{ij} u_i u_j \mu \lambda^2 e^{\lambda u} \tilde{v} \varphi \]

\[ + cc_1 \mu |\lambda| e^{\lambda u} \tilde{v}^2 \varphi + cc_2 \tilde{v}^3 \varphi + cc_3 e^{-\lambda u} \tilde{v}^3 \varphi, \]

where \( |\lambda| \) is chosen so large that

\[ e^{-\lambda u} \leq \frac{1}{4}. \]

Choosing, furthermore,

\[ \mu = \frac{1}{8}. \]
we see that the terms involving $F^{ij}u_iu_j$ add up to a dominating negative quantity that can be estimated from above by

$$(4.53) \quad -\frac{1}{16}F^{ij}u_iu_j\lambda^2 e^{\lambda u}\tilde{v}\varphi \leq -\frac{c_0}{16} g_{kl}\|Du\|^2\lambda^2 e^{\lambda u}\tilde{v}\varphi,$$

in view of $(4.48)$. 

$\|Du\|^2$ is of the order $\tilde{v}^2$ for large $\tilde{v}$, hence, the parabolic maximum principle yields a uniform estimate for $w$ if $|\lambda|$ is chosen large enough. q.e.d.

5. $C^2$- estimates for the auxiliary solutions

We want to prove that the principal curvatures of the flow hypersurfaces are uniformly bounded.

**Proposition 5.1.** Let $M(t)$, $0 \leq t < T^*$, be solutions of the evolution problem \((3.9)\) with $M(0) = M_2$, $F \in (H)$, and $f \in C^{2,\alpha}$ strictly positive,

\begin{equation}
0 < c_0 \leq f.
\end{equation}

Then, the principal curvatures of the flow hypersurfaces are uniformly bounded provided the $M(t)$ are uniformly space-like, i.e. uniform $C^1$- estimates are valid.

**Proof.** As we have already mentioned in Remark 3.7, we know that

\begin{equation}
0 < c_0 \leq f \leq F
\end{equation}

during the evolution, thus, it is sufficient to estimate the principal curvatures from above.

Let $\varphi$ be defined by

\begin{equation}
\varphi = \sup \{ h_{ij}\eta^i\eta^j : \|\eta\| = 1 \}.
\end{equation}

We claim that $\varphi$ is uniformly bounded.

Let $0 < T < T^*$, and $x_0 = x_0(t_0)$, with $0 < t_0 \leq T$, be a point in $M(t_0)$ such that

\begin{equation}
\sup_{M_0} \varphi < \sup_{M(t)} \{ \sup_{M(t)} \varphi : 0 < t \leq T \} = \varphi(x_0).
\end{equation}
We then introduce a Riemannian normal coordinate system \((\xi^i)\) at \(x_0 \in M(t_0)\) such that at \(x_0 = x(t_0, \xi_0)\) we have
\[
g_{ij} = \delta_{ij} \quad \text{and} \quad \varphi = h_n^i.
\]

Let \(\tilde{\eta} = (\tilde{\eta}^i)\) be the contravariant vector field defined by
\[
\tilde{\eta} = (0, \ldots, 0, 1),
\]
and set
\[
\tilde{\varphi} = \frac{h_{ij}\tilde{\eta}^i\tilde{\eta}^j}{g_{ij}\tilde{\eta}^i\tilde{\eta}^j}.
\]
\(\tilde{\varphi}\) is well defined in neighbourhood of \((t_0, \xi_0)\), and \(\tilde{\varphi}\) assumes its maximum at \((t_0, \xi_0)\). Moreover, at \((t_0, \xi_0)\) we have
\[
\tilde{\varphi} = h_n^i,
\]
and the spatial derivatives do also coincide; in short, at \((t_0, \xi_0)\) \(\tilde{\varphi}\) satisfies the same differential equation (3.15) as \(h_n^i\). For the sake of greater clarity, let us therefore treat \(h_n^i\) like a scalar and pretend that \(\varphi = h_n^i\).

At \((t_0, \xi_0)\) we have \(\tilde{\varphi} \geq 0\), and, in view of the maximum principle, we deduce from Lemma 3.6
\[
0 \leq -\epsilon_0 F^{ij}g_{ij}|A|^2h_n^i + f|h_n^i|^2 + cF^{ij}g_{ij}(h_n^i + 1)
\]
\[
+ c(1 + |A|^2)(1 + f + ||Df|| + ||D^2f||),
\]
where we used the concavity of \(F\), the Codazzi equations, (4.48), and where
\[
|A|^2 = g^{ij}h_i^kh_kj.
\]

Thus, \(\varphi\) is uniformly bounded in view of (4.49). q.e.d.

6. Convergence to a stationary solution

We shall show that the solution of the evolution problem (3.9) exists for all time, and that it converges to a stationary solution.

**Proposition 6.1.** The solutions \(M(t) = \text{graph}\ u(t)\) of the evolution problem (3.9) with \(F \in (H)\), and \(M(0) = M_2\) exist for all time and converge to a stationary solution provided \(f \in C^{2,\alpha}\) satisfies the conditions (4.41), (4.42), and (5.1).
Proof. Let us look at the scalar version of the flow as in (4.6)

\[
\frac{\partial u}{\partial t} = -e^{-\psi} v(F - f).
\]

This is a scalar parabolic differential equation defined on the cylinder

\[
Q_{T^*} = [0, T^*) \times S_0
\]

with initial value \( u(0) = u_2 \in C^{4,\alpha}(S_0) \). In view of the a priori estimates, which we have established in the preceding sections, we know that

\[
|u|_{2,0, S_0} \leq c
\]

and

\[
F \text{ is uniformly elliptic in } u
\]

independent of \( t \) due to the definition of the class \((H)\). Thus, we can apply the known regularity results, see e.g. [20, Chapter 5.5], where even more general operators are considered, to conclude that uniform \( C^{2,\alpha}\)-estimates are valid, leading further to uniform \( C^{4,\alpha}\)-estimates due to the regularity results for linear operators.

Therefore, the maximal time interval is unbounded, i.e. \( T^* = \infty \).

Now, integrating (6.1) with respect to \( t \), and observing that the right-hand side is non-positive, yields

\[
u(0, x) - u(t, x) = \int_0^t e^{-\psi} v(F - f) \geq c \int_0^t (F - f),
\]

i.e.,

\[
\int_0^\infty |F - f| < \infty \quad \forall x \in S_0.
\]

Hence, for any \( x \in S_0 \) there is a sequence \( t_k \to \infty \) such that \((F - f) \to 0\).

On the other hand, \( u(\cdot, x) \) is monotone decreasing and therefore

\[
\lim_{t \to \infty} u(t, x) = \tilde{u}(x)
\]

exists and is of class \( C^{4,\alpha}(S_0) \) in view of the a priori estimates. We, finally, conclude that \( \tilde{u} \) is a stationary solution, and that

\[
\lim_{t \to \infty} (F - f) = 0.
\]

q.e.d.
An immediate consequence of the results we have proved so far—cf. especially Lemma 2.5 and Remark 3.1—is the following theorem which is of independent interest.

**Theorem 6.2.** Let \( F \in C^{2,\alpha}(\Gamma) \cap C^0(\bar{\Gamma}) \) be a concave curvature function vanishing on \( \partial\Gamma \) and homogeneous of degree 1. Let \( f = f(x,\nu) \) of class \( C^{2,\alpha} \) satisfy the conditions \((4.41), (4.42)\), and \((5.1)\), and suppose that the boundary components \( M_i \) act as barriers for \((F,f)\), then, there exists an admissible hypersurface \( M = \text{graph} \, u, \, u \in C^{4,\alpha}(\bar{S}_0) \), solving

\[
F_{\epsilon|M} = f(x,\nu)
\]

for small \( \epsilon > 0 \).

### 7. Stationary Approximations

We want to solve the equation

\[
H_2|_M = f(x,\nu),
\]

where \( f \) satisfies the conditions of Theorem 0.3. The curvature function \( F = H_2^\gamma \) is concave and the elliptic regularization \( F_\epsilon \) of class \((H)\), cf. \( 3.1 \) and Remark 3.1.

Thus, we would like to apply the preceding existence result to find hypersurfaces \( M_\epsilon \subset \bar{O} \) such that

\[
F_{\epsilon|M_\epsilon} = f^{\frac{1}{2}}.
\]

But, unfortunately, the derivatives \( f_\beta \) grow quadratically in \( \|\nu\| \) contrary to the assumption \((4.41)\) in Proposition 4.8.

Therefore, we define a smooth cut-off function \( \theta \in C^\infty(\mathbb{R}_+) \), \( 0 < \theta \leq 2k \), where \( k \geq k_0 > 1 \) is to be determined later, by

\[
\theta(t) = \begin{cases} 
   t, & 0 \leq t \leq k, \\
   2k, & 2k \leq t,
\end{cases}
\]

such that

\[
0 \leq \dot{\theta} \leq 4,
\]
and consider the problem

\[(7.5) \quad F_\epsilon|_{M_\epsilon} = \tilde{f}(x, \tilde{\nu}),\]

where for a space-like hypersurface \(M = \text{graph } u\) with past directed normal vector \(\nu\) as in (2.16), we set

\[(7.6) \quad \tilde{\nu} = \theta(\tilde{\nu})\tilde{\nu}^{-1}\nu\]

and

\[(7.7) \quad \tilde{f}(x, \tilde{\nu}) = f^\frac{1}{2}(x, \nu).\]

Then,

\[(7.8) \quad ||\tilde{\nu}|| \leq ck,\]

so that the assumptions in Proposition 4.8 are certainly satisfied.

The constant \(k_0\) should be so large that \(\tilde{\nu} = \nu\) in case of the barriers \(M_i, i = 1, 2\).

If we now start with the evolution equation

\[(7.9) \quad \dot{x} = (F_\epsilon - \tilde{f})\nu,\]

then, the \(M_i\) are barriers for \((F_\epsilon, \tilde{f})\) for small \(\epsilon\), cf. Lemma 2.5 and we conclude

**Lemma 7.1.** The flow hypersurfaces \(M_\epsilon(t) = \text{graph } u_\epsilon\) stay in \(\Omega\) during the evolution if \(\epsilon\) is small \(0 < \epsilon \leq \bar{\epsilon}(\Omega)\).

**Remark 7.2.** When we consider the elliptic regularizations \(F_\epsilon\), we would like to generalize the meaning of *admissible* hypersurface by calling a hypersurface admissible if the tensor \(h_{ij} + \epsilon H g_{ij}\) is admissible, i.e. if its eigenvalues belong to \(\Gamma_2\).

Next, let us consider the evolution equations for \(\tilde{\nu}\) and \(h^{ij}\) which look slightly different: In (4.7) the term involving \(f_{\nu \beta}\) has to be replaced by

\[(7.10) \quad -\hat{f}_{\nu \alpha} [\theta(\tilde{\nu})\tilde{\nu}^{-1}\nu^\alpha + \tilde{\theta} \tilde{\nu} \tilde{\nu}^{-1}\nu^\alpha - \theta \tilde{\nu}^{-2} \tilde{\nu} \nu^\alpha] x_k g^{ik} \eta_\alpha.\]

But in view of (4.36), the additional terms do not cause any new problems in the proof of Proposition 4.8, and hence, the uniform \(C^1\)-estimates are still valid for the modified evolution problem, where the estimates depend on \(k\).
The $C^2$-estimates in Section 5 remain valid, too, since the second derivatives of $\tilde{f}$, $\tilde{f}_i^j$, that occur on the right-hand side of (3.15), can be expressed as—we only consider the covariant form $\tilde{f}_i^i$, no summation over $i$—

\begin{equation}
- \tilde{f}_{ii} = -\tilde{f}_{\alpha\beta}x^\alpha_i x^\beta_i - 2\tilde{f}_{\alpha\nu}x^\alpha_i \tilde{v}_i^\nu - \tilde{f}_{\alpha\nu}h_{ii} - \tilde{f}_{\nu\nu}x^\nu_i \tilde{v}_i^\nu - \tilde{f}_{\nu\nu}h_{ii},
\end{equation}

where

\begin{equation}
\tilde{v}_i^\alpha = \theta\tilde{v}^{-1}v_i^\alpha + \dot{\theta}\tilde{v}_i\tilde{v}^{-1}v^\alpha - \theta\tilde{v}^{-2}\tilde{v}_i v^\alpha,
\end{equation}

\begin{equation}
\tilde{v}_{ii}^\alpha = 2\dot{\theta}\tilde{v}_i\tilde{v}^{-1}v_i^\alpha - 2\theta\tilde{v}^{-2}\tilde{v}_i v_i^\alpha + \theta\tilde{v}^{-1}v_{ii}^\alpha + \dot{\theta}\tilde{v}_i\tilde{v}^{-1}v_i^\alpha
\end{equation}

and

\begin{equation}
\tilde{v}_{ii} = \eta_{\alpha\beta\gamma}x^\beta_i x^\gamma_i v_i^\alpha + \eta_{\alpha\beta}v_i^\nu h_{ii} + 2\eta_{\alpha\beta}x^\beta_i v_i^\alpha + \eta_{\alpha}v_{ii}^\alpha.
\end{equation}

Hence, the result of Proposition 5.1 is still valid since no additional bad terms occur in inequality (5.9) as one easily checks, and since, furthermore, we also have

\begin{equation}
\tilde{f} \leq F_\epsilon
\end{equation}

during the evolution, for the modified version of (3.14) now has the form

\begin{equation}
(F_\epsilon - \tilde{f})' - F_{ij}^\nu (F_\epsilon - \tilde{f})_{ij} = -F_{ij}^\nu h_{ii} h_{ij}^\nu (F_\epsilon - \tilde{f}) - \tilde{f}_{\alpha\nu} h_{ii} (F_\epsilon - \tilde{f})
\end{equation}

\begin{equation}
- \tilde{f}_{\nu\nu}h_{ii} (F_\epsilon - \tilde{f}) - [\dot{\theta}\tilde{v}^{-1} - \theta\tilde{v}^{-2}]\eta_{\alpha\beta}v_i^\nu \nu_{ij}^\beta (F_\epsilon - \tilde{f})
\end{equation}

\begin{equation}
- \theta\tilde{v}^{-1}\tilde{f}_{\nu\nu}x_i^\nu (F_\epsilon - \tilde{f})_{ij}^\nu
\end{equation}

\begin{equation}
- F_{ij}^\nu \bar{R}_{\alpha\beta\gamma\delta}v_i^\nu x^\beta_i x^\delta_j (F_\epsilon - \tilde{f}).
\end{equation}

Here, we used the relation

\begin{equation}
\dot{\tilde{v}} = \eta_{\alpha\beta}\tilde{x}_i^\beta v_i^\alpha + \eta_{\alpha}\tilde{v}^\alpha,
\end{equation}
which follows immediately from (4.8), together with (3.9) and (3.11).

The conclusions of Section 6 are therefore applicable leading to a solution of equation (7.5).

8. \(C^1\)-ESTIMATES FOR THE STATIONARY APPROXIMATIONS

Consider the solutions \(M_\epsilon = \text{graph } u_\epsilon\) of equation (7.5), which at the moment not only depend on \(\epsilon\) but also on \(k\), the parameter of the cut-off function \(\theta\), cf. (7.3). We shall prove that the hypersurfaces \(M_\epsilon\) are uniformly space-like independent of \(\epsilon\) and \(k\), or, equivalently, that there exists a constant \(m_1\) such that

\[
\tilde{v} = (1 - |Du_\epsilon|^2)^{-\frac{1}{2}} \leq m_1 \quad \forall \epsilon, k,
\]

where the parameter \(\epsilon\) is supposed to be small and \(k\) to be large, so that the barrier condition is satisfied.

**Lemma 8.1.** Let \(u_\epsilon\) be a solution of (7.5), then, the estimate (8.1) is valid uniformly in \(\epsilon\) and \(k\). Hence, \(M_\epsilon = \text{graph } u_\epsilon\) is a solution of equation (7.2), if we choose \(k \geq 2m_1\).

**Proof.** For arbitrary but fixed values of \(\epsilon\) and \(k\), let us introduce the notation \(\tilde{F}\) for \(F_\epsilon\), where from now on through the rest of the article

\[
(8.2) \quad F = H_2,
\]

and where \(f = f(x, \nu)\) satisfies

\[
(8.3) \quad 0 < c_1 \leq f(x, \nu),
\]

\[
(8.4) \quad \|f_\beta(x, \nu)\| \leq c_2(1 + \|\nu\|^2),
\]

and

\[
(8.5) \quad \|f_{\nu,\nu}(x, \nu)\| \leq c_3(1 + \|\nu\|),
\]

for all \(x \in \bar{\Omega}\) and all past directed time-like vectors \(\nu \in T_x(\Omega)\).

Thus, \(F\) is homogeneous of degree 2, and we recall that

\[
(8.6) \quad F^{ij} = Hg^{ij} - h^{ij},
\]

and

\[
(8.7) \quad \tilde{F}^{ij} = F^{ij} + \epsilon(n - 1)(1 + \epsilon n)Hg^{ij},
\]
where $\tilde{F}^{ij}$ is evaluated at $h_{ij}$ and $F^{ij}$ at $(h_{ij} + \epsilon H g_{ij})$.

We also drop the index $\epsilon$, writing $u$ for $u_{\epsilon}$ and $M$ for $M_{\epsilon}$, i.e. $M$ solves the equation

$$(8.8) \quad \tilde{F}_{|M} = f(x, \tilde{\nu}).$$

The $C^1$- estimate will follow the arguments in the proof of Proposition 4.8, where at one point we shall introduce an additional observation especially suitable for the curvature function $F = H_2$.

**Remark 8.2.** The former parabolic equations and inequalities, (4.11), (4.39), and (4.47) can now be read as elliptic equations resp. inequalities by simply assuming that the terms involved are time independent. Though, to be absolutely precise, one has to observe that the present curvature function is homogeneous of degree 2, which means that, whenever the term $F$—not derivatives of $F$—occurs explicitly in the equations or inequalities just mentioned, it has to be replaced by $2F$ because it was obtained as a result of Euler’s formula for homogeneous functions of degree $d_0$

$$(8.9) \quad d_0 F = F^{ij} h_{ij}.$$ 

We mention it as a matter of fact only, since it doesn’t affect the estimates at all.

However, we have to be aware that $f$ now depends on $\tilde{\nu}$ instead of $\nu$, i.e. the elliptic version of inequality (4.47) now takes the form

$$-(\tilde{F}^{ij} w_{ij} \leq -\delta \tilde{F}^{ij} h_i^k k_j \tilde{\nu} \varphi + c|\delta^{-1} + |\lambda|\mu e^{\lambda u}\tilde{F}_{ij} y_{ij} \tilde{\nu}^3 \varphi $$

$$+ \left[\frac{1}{1-2\delta} - 1\right] \tilde{F}^{ij} u_i u_j \mu^2 e^{2\lambda u} \tilde{\nu} \varphi$$

$$(8.10) \quad - \tilde{F}^{ij} u_i u_j \mu^2 e^{\lambda u} \tilde{\nu} \varphi + 2f[-\eta_{\alpha\beta} \nu^\alpha \nu^\beta e^{-\psi} \mu e^{\lambda u} \tilde{\nu}^2] \varphi + c \|f\| \|\tilde{\nu}^2 \varphi$$

$$+ f_{\tilde{\nu}^3} [\theta \tilde{\nu}^{-1} \nu^3 + \tilde{\theta} \tilde{\nu}^{-1} \nu^3 - \theta \tilde{\nu}^{-2} \tilde{\nu} \nu^3] u^i e^{\psi} \varphi.$$ 

Here, we used the notation $\delta = \delta(x)$ for the small parameter in the Schwarz inequality instead of $\epsilon$, which has a different meaning in the present context, $w$ is defined as in (4.44), where the parameters $\mu, \lambda$ should satisfy the conditions in (4.45), and $u$ is supposed to be less than $-1.$
We claim that $w$ is uniformly bounded provided $\mu$ and $\lambda$ are chosen appropriately. Following the arguments in Section 4, we shall use the maximum principle and consider a point $x_0 \in M$, where

$$w(x_0) = \sup_M w.$$ \hfill (8.11)

As before, we choose $\delta = e^{-\lambda u}$. But the further conclusions are no longer valid, since we have a really bad term on the right-hand side of (8.10) that is of the order $\tilde{v}^4$ due to the assumption (8.4).

The only possible good term which can balance it, is

$$-\delta \tilde{F}^{ij} h^k_i h^l_j \tilde{v} \varphi.$$ \hfill (8.12)

To exploit this term we use the fact that $Dw(x_0) = 0$, or, equivalently

$$-\tilde{v}_i = \mu \lambda e^{\lambda u} \tilde{v} u_i = e^\psi h^k_i u_k - \eta_{\alpha\beta} \nu^\alpha x^\beta_i,$$ \hfill (8.13)

where the second equation follows from (4.8) and the definition of the covariant vectorfield $\eta = e^\psi(-1,0,\ldots,0)$.

Next, we choose a coordinate system $(\xi^i)$ such that in the critical point

$$g_{ij} = \delta_{ij} \quad \text{and} \quad h^k_i = \kappa_i \delta^k_i,$$ \hfill (8.14)

and the labelling of the principal curvatures corresponds to

$$\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n.$$ \hfill (8.15)

Then, we deduce from (8.13)

$$e^\psi \kappa_i u_i = \mu \lambda e^{\lambda u} \tilde{v} u_i + \eta_{\alpha\beta} \nu^\alpha x^\beta_i.$$ \hfill (8.16)

Assume that $\tilde{v}(x_0) \geq 2$, and let $i = i_0$ be an index such that

$$|u_{i_0}|^2 \geq \frac{1}{n} \|Du\|^2.$$ \hfill (8.17)

Setting $(e^i) = \frac{\partial}{\partial \xi^0}$ and assuming without loss of generality that $0 \leq u_i e^i$ in $x_0$ we infer from Lemma 2.7

$$e^\psi \kappa_{i_0} u_{i_0} e^i = \mu \lambda e^{\lambda u} \tilde{v} u_{i_0} e^i + \eta_{\alpha\beta} \nu^\alpha x^\beta_i e^i \leq \mu \lambda e^{\lambda u} \tilde{v} u_i e^i + c \tilde{v}^2,$$ \hfill (8.18)
and we deduce further in view of (2.12) and (8.17) that

\[(8.19) \quad \kappa_{t_0} \leq [\mu \lambda e^{\lambda u} + c] \bar{v} e^{-\psi} \leq \frac{1}{2} \mu \lambda e^{\lambda u} \bar{v} e^{-\psi},\]

if \(|\lambda|\) is sufficiently large, i.e. \(\kappa_{t_0}\) is negative and of the same order as \(\bar{v}\).

The Weingarten equation and Lemma 2.7 yield

\[(8.20) \quad \|\nu^\beta u^i\| = \|h^k_i u^i x^\beta_k\| \leq c \bar{v}[h^k_i u^i h_{kl} u^l]^{\frac{1}{2}},\]

and therefore, we infer from (8.13)

\[(8.21) \quad |\tilde{v}_1 u^i| + \|\nu^\beta u^i\| \leq c \mu |\lambda| e^{\lambda u} \bar{v}^3\]

in critical points of \(w\), and hence, that in those points, the term involving \(f_{\beta,\beta}\) on the right-hand side of inequality (8.10) can be estimated from above by

\[(8.22) \quad |f_{\beta,\beta}[\theta \bar{v}^{-1} \nu^\beta_i + \theta \tilde{v}_i \bar{v}^{-1} \nu^\beta_i - \theta \bar{v}^{-2} \tilde{v}_i \nu^\beta_i]u^i e^\psi \varphi| \leq c \mu |\lambda| e^{\lambda u} \bar{v}^4 \varphi.\]

Next, let us estimate the crucial term in (8.12). Using (8.7), the particular coordinate system (8.14), as well as the inequalities (8.15), together with the fact that \(\kappa_{t_0}\) is negative, we conclude

\[(8.23) \quad -\hat{F}^i j h^k_i h_{kj} \leq -F^i j h^k_i h_{kj} \leq -\sum_{i=1}^{\kappa_0} F^i_i \kappa^2_i\]

where we recall that the argument of \(F^i_i\) is the \(n\)-tupel with components

\[(8.24) \quad \tilde{\kappa}_j = \kappa_j + \epsilon H\]

and observe that in the present coordinate system

\[(8.25) \quad F^i_i = \frac{\partial F}{\partial \tilde{\kappa}_i}.\]

Let \(\hat{F} = \log F\), then, \(\hat{F}\) is concave, and therefore, we have in view of (8.15)

\[(8.26) \quad \hat{F}^1_i \geq \hat{F}^2_i \geq \cdots \geq \hat{F}^n_i,\]
cf. [6, Lemma 2], or equivalently,

\[(8.27)\quad F_1^1 \geq F_2^2 \geq \cdots \geq F_n^n.\]

Hence, we conclude

\[(8.28)\quad -\sum_{i=1}^{i_0} F_i^i \leq -F_1^1 \leq -\frac{1}{n} \sum_{i=1}^{n} F_i^i = -\frac{1}{n} (n - 1) [H + cnH] \leq -\frac{n - 1}{n} H,
\]

where we also used (8.6) and (8.24).

Combining (8.19), (8.23), (8.28), and the estimate (1.15), we deduce further

\[(8.29)\quad -\tilde{F}^{ij} h^k_i h^l_j \leq -\frac{n - 1}{n} H \kappa^2_{i_0}
\]

\[\leq -\frac{n - 1}{2n} cn |\kappa_{i_0}|^3 \leq -\frac{n - 1}{2n} H \kappa^2_{i_0}
\]

\[\leq -a_0 \mu^3 |\lambda|^3 e^{3\lambda u} \tilde{v}^3 - a_1 H \mu^2 \lambda^2 e^{2\lambda u} \tilde{v}^2
\]

with some positive constants \(a_0 = a_0(n, \Omega)\) and \(a_1 = a_1(n, \Omega)\).

Inserting this estimate, and the estimate in (8.22) in the elliptic inequality (8.10), with \(\delta = e^{-\lambda u}\), we finally obtain

\[(8.30)\quad -\tilde{F}^{ij} w_{ij} \leq -a_0 \mu^3 |\lambda|^3 e^{2\lambda u} \tilde{v}^4 \varphi - a_1 H \mu^2 \lambda^2 e^{\lambda u} \tilde{v}^3 \varphi
\]

\[+ \frac{2}{1 - 2\delta} \tilde{F}^{ij} u_i u_j \mu^2 \lambda^2 e^{\lambda u} \tilde{v} \varphi + c [1 + |\lambda|] H e^{\lambda u} \tilde{v}^3 \varphi
\]

\[- \tilde{F}^{ij} u_i u_j \lambda^2 e^{\lambda u} \tilde{v} \varphi + c [c_2 + c_3 \mu |\lambda| e^{\lambda u}] \tilde{v} \varphi
\]

\[+ 2f [c + e^{-\psi} \mu \lambda e^{\lambda u}] \tilde{v}^2 \varphi.
\]

Choosing, now, \(\mu = \frac{1}{4}\) and \(|\lambda|\) large, the right-hand side of the preceding inequality is negative, contradicting the maximum principle, i.e. the maximum of \(w\) cannot occur at point where \(\tilde{v} \geq 2\). Thus, the desired uniform estimate for \(w\) and, hence, \(\tilde{v}\) is proved. q.e.d.

Let us close this section with an interesting observation that is an immediate consequence of the preceding proof, we have especially (8.23) and the first line of inequality (8.28) in mind,
Lemma 8.3. Let $F \in C^2(\Gamma)$ be a positive symmetric curvature function such that the partial derivatives $F_i$ are positive and $\tilde{F} = \log F$ is concave. Suppose $F$ is evaluated at a point $(\kappa_i)$, and assume that $\kappa_{i_0}$ is a component that is either negative or the smallest component of that particular $n$-tupel, then

$$\sum_{i=1}^{n} F_i \kappa_i^2 \geq \frac{1}{n} \sum_{i=1}^{n} F_i \kappa_{i_0}^2.$$  

9. $C^2$- estimates for the stationary approximations

We want to prove uniform $C^2$- estimates for admissible solutions $M$ of

$$\tilde{F}_{|M} = f(x, \nu),$$

where we use the notations and conventions of the preceding section.

The starting point is an elliptic equation for the second fundamental form.

Lemma 9.1. The tensor $h_i^j$ satisfies the elliptic equation

$$-\tilde{F}^{kl}_{;h_i^j} + F^{kl}_{;h_i^j} - f_{x^\alpha} x_i^\alpha x_k^\beta g^{kj} + f_{x^\alpha} x_i^\alpha h_i^j t^{kj} - f_{x^\alpha} x_i^\alpha h_i^j k^{kj} - f_{x^\alpha} \nu^\alpha h_i^j k^{kj} + f_{x^\alpha} \nu^\alpha h_i^j t^{kj} + 2 f R_{\alpha\beta\gamma\delta} x_m^\alpha x_i^\beta x_k^\gamma x_l^\delta g^{mj}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary indices.

Proof. The elliptic equation can be immediately derived from the corresponding parabolic equation (3.15) by dropping the time-derivative, replacing $F$ by $\tilde{F}$, and observing that the present curvature function is homogeneous of degree 2, cf. Remark 8.2. q.e.d.
Contracting over the indices \((i, j)\) in (9.2) we obtain a differential equation for \(H\)

\[-\tilde{F}^{kl} H_{kl}\]

\[= -\tilde{F}^{kl} h_{rkl} h_{i}^{r} H + 2 f |A|^{2}\]

\[= f_{\alpha\beta} x_{i}^{\alpha} x_{k}^{\beta} g^{ki} - f_{\alpha}^{\nu} x_{i}^{\alpha} |A|^{2}\]

\[- f_{\alpha\nu} x_{i}^{\alpha} x_{k}^{\beta} h_{ki} - f_{\nu}^{\beta} (H^{k} x_{k}^{\beta} + |A|^{2} \nu^{\beta})\]

\[- \tilde{R}_{\alpha\beta} h^{\alpha} x_{i}^{k} x_{j}^{\gamma} f_{\nu}^{\gamma} + \tilde{F}^{kl} r_{rs} h_{kl}^{i} h_{rs}^{i}\]

\[+ 2 \tilde{F}^{kl} \tilde{R}_{\alpha\beta\gamma\delta} x_{i}^{\alpha} x_{m}^{\beta} x_{i}^{\gamma} x_{m}^{\delta} h_{mi}^{i} - 2 \tilde{F}^{kl} \tilde{R}_{\alpha\beta\gamma\delta} x_{i}^{\alpha} x_{m}^{\beta} x_{i}^{\gamma} x_{m}^{\delta} h^{mi}\]

\[- \tilde{F}^{kl} \tilde{R}_{\alpha\beta\gamma\delta} x_{i}^{\alpha} x_{m}^{\beta} x_{i}^{\gamma} x_{m}^{\delta} H + 2 f \tilde{R}_{\alpha\beta} \nu^{\alpha} \nu^{\beta}\]

\[+ \tilde{F}^{kl} \tilde{R}_{\alpha\beta\gamma\delta} \epsilon \{ \nu^{\alpha} x_{i}^{\beta} x_{m}^{\gamma} x_{i}^{\delta} g^{mi} + \nu^{\alpha} x_{i}^{\beta} x_{m}^{\gamma} x_{i}^{\delta} g^{mi}\},\]

where we also used the symmetry properties of the Riemann curvature tensor and the Codazzi equations at one point.

Next, let us improve the estimate in Proposition 5.1.

**Lemma 9.2.** Let \(M = \text{graph } u\) be an admissible solution of (9.1), then the principal curvatures of \(M\) satisfy the estimate

\[(9.4)\]

\[\epsilon |A|^{2} \leq \text{const},\]

where the constant depends on \(\| Df \\|, \| D^{2} f \\|\), the constant \(c_{1}\) in (8.3), and on known estimates of the \(C^{0}\) and \(C^{1}\)-norm of \(u\).

**Proof.** We argue as in the proof of Proposition 5.1 and define

\[(9.5)\]

\[\varphi = \sup \{ h_{ij} \eta^{i} \eta^{j} : \| \eta \| = 1 \} .\]

Let \(x_{0} \in M\) be a point, where \(\varphi\) achieves its maximum, and assume without loss of generality that, after having introduced normal Riemannian coordinates around \(x_{0}\), we may write \(\varphi = h_{n}^{n}\), cf. the corresponding arguments in the proof of Proposition 5.1.

Applying the maximum principle in \(x_{0}\), we deduce from (9.2) the following inequality

\[(9.6)\]

\[0 \leq -\epsilon (n - 1) H |A|^{2} h_{n}^{n} + 2 f |h_{n}^{n}|^{2} + \tilde{F}^{kl} r_{rs} h_{kl}^{i} h_{rs}^{i} \]

\[+ c(1 + \| Df \\| + \| D^{2} f \\|)(1 + |A|^{2}) + c(\tilde{F}^{ij} g_{ij} + f),\]
where we also used (8.7), the Weingarten and Codazzi equations, and the fact that the pair \((x, \nu)\) stays in a compact subset of \(\bar{\Omega} \times C_-(\bar{\Omega})\), where \(C_-(\bar{\Omega})\) stands for the set of past directed time-like vectorfields in \(\bar{\Omega}\).

Furthermore, we know that

\[
\tilde{F}^{ij} g_{ij} \leq cH,
\]

and

\[
\tilde{F}^{kl,rs} h_{kl;n} h_{rs;n} \leq \tilde{F}^{-1} \tilde{F}_n \tilde{F}^n = f^{-1} f_n f^n \leq cc_1^{-1} (1 + |A|^2),
\]

since \(\log \tilde{F}\) is concave, cf. Lemma 1.3.

Thus, we conclude that in \(x_0\) the following inequality is valid

\[
0 \leq -c|A|^4 + c(1 + |A|^2)
\]

with a known constant \(c\), and the lemma is proved. q.e.d.

The estimate (9.4) will play an important role in the final a priori estimate.

**Lemma 9.3.** Let \(F = H_2, M = M^n\) a Riemannian manifold with metric \(g_{ij}\), \(h_{ij}\) a symmetric tensor field on \(M\) the eigenvalues of which belong to \(\Gamma_2\), and \(p \in M\) an arbitrary point. Choose local coordinates around \(p\) such that the relations (8.14) and (8.15) are satisfied. Then, we have for \(1 \leq j \leq n\)

\[
\sum_{i \neq j} \kappa_i^2 + 2F = |F_j|^2 + 2F_j\kappa_j,
\]

\[
\sum_{i \neq n} \kappa_i^2 + 2F \leq cF^n \kappa_n,
\]

and

\[
\sum_{i \neq j} \frac{F_{ij}}{F_j} + \frac{F_j}{H} (1 - \frac{F_i}{F_j})^2 \leq c|F_j|^2 F^{-1},
\]

with \(c = c(n)\), \(F\) is evaluated at \(h_{ij}\), and where we point out that the summation convention is not used.
Proof. Throughout the proof we shall use the ambivalent meaning of $F$ as a function depending on $\kappa_i$ or on $h_{ij}$ switching freely from one viewpoint to the other.

(i) From the definition of $F$

\begin{equation}
F = \frac{1}{2}(H^2 - |A|^2),
\end{equation}

and (8.6) we conclude

\begin{equation}
2F = (F_j^2 + \kappa_j)^2 - \sum_i \kappa_i^2
\end{equation}

\begin{equation}
= |F_j^2|^2 + 2F_j^2 \kappa_j - \sum_{i \neq j} \kappa_i^2,
\end{equation}

which proves (9.10).

(ii) If $j = n$, and thus $\kappa_n$ the largest eigenvalue, then, we derive from (8.6)

\begin{equation}
F_n^2 \leq H \leq n\kappa_n,
\end{equation}

and (9.11) follows at once from (9.10).

(iii) A simple algebraic transformation yields

\begin{equation}
\frac{F_j}{F_j^2} + \frac{F_j}{F_j^2} \frac{1 - \frac{F_j^2}{F_j^2}}{H} = \frac{F_j}{HF_j^3} (H - \kappa_j + \kappa_i)
\end{equation}

\begin{equation}
= \frac{F_j}{HF_j^3} \left( \sum_{k \neq j} \kappa_k + \kappa_i \right),
\end{equation}

and hence,

\begin{equation}
\sum_{i \neq j} \left| \frac{F_j}{F_j^2} + \frac{F_j}{F_j^2} \frac{1 - \frac{F_j^2}{F_j^2}}{H} \right|^2 \leq c \frac{|F_j|^2}{H^2 |F_j|^2 \sum_{i \neq j} \kappa_i^2}.
\end{equation}

We, now, treat the cases $j = n$ and $j \neq n$ separately.

If $j = n$, we apply (9.11) and (1.17) and conclude

\begin{equation}
\sum_{i \neq j} \left| \frac{F_j}{F_j^2} + \frac{F_j}{F_j^2} \frac{1 - \frac{F_j^2}{F_j^2}}{H} \right|^2 \leq c \frac{|F_j|^2}{H^2 |F_j|^2} \leq c \frac{|F_j|^2}{F}.
\end{equation}
If $j \neq n$, we deduce from (9.10)

\[ \kappa^2_n \leq 6|F^j_j|^2, \]

and deduce further

\[ \sum_{i \neq j} \kappa^2_i \leq 8|F^j_j|^2, \]

where apparently we only had to worry about the case $0 \leq \kappa_j$.

Thus, the right-hand side of (9.17) is estimated from above by

\[ \frac{c|F^j_j|^2}{H^2}, \]

which in turn is less than

\[ \frac{c|F^j_j|^2}{F}. \]

q.e.d.

**Corollary 9.4.** Let $M$ be an admissible solution of (9.1) and $p \in M$ arbitrary. Choose local coordinates around $p$ such that the relations (8.14) and (8.15) are valid. Then, for any $1 \leq j \leq n$, the following inequality is valid in $p$

\[ \sum_{i \neq j} \left| \frac{f^j_j + f^j_j}{F^j_j} (1 - \tilde{F}^i_i) \right|^2 \leq c|f^j_j|^2 f^{-1}, \]

where we use the notation $\tilde{H} = (1 + \epsilon n)H$, and do not apply the summation convention.

**Proof.** Let us recall the relation (8.7), which we can also express in the form

\[ \tilde{F}^{ij} = \tilde{H} g^{ij} - \tilde{h}^{ij} + \epsilon F^{rs} g^{rs} g^{ij}, \]

where

\[ \tilde{h}^{ij} = h^{ij} + \epsilon H g^{ij}. \]
Consider each summand in (9.23) separately. We have

\[
\left| \frac{f_j}{F_j} + \frac{\tilde{f}_i}{F_j} \left(1 - \tilde{F}_i \right) \right|^2 = \frac{f_j^2}{H^2 |F_j|^2} |\tilde{\varphi} + \tilde{F}_j - \tilde{F}_i|^2
\]

(9.26)

\[ \leq \frac{f_j^2}{H^2 |F_j|^2} \left| \sum_{k \neq j} \tilde{\kappa}_k + \tilde{\kappa}_i \right|^2, \]

in view of (9.24), i.e. we are exactly in the same situation as in the proof of Lemma 9.3 after the equation (9.16) with the following modifications: we replace \( h_{ij}, \kappa_i \) and \( H \) by \( \tilde{h}_{ij}, \tilde{\kappa}_i \) resp. \( \tilde{H} \) and observe that \( F(\tilde{h}_{ij}) = f. \) q.e.d.

Lemma 9.5. Let \( M \) be an admissible solution of equation (9.1), then, the estimate

\[
\tilde{F}^{ij,kl} h_{ij} h_{kl}^{r} + H^{-1} \tilde{F}^{ij} H_i H_j \leq c f^{-1} |Df|^2 + c \epsilon \|DH\|^2 + c
\]

is valid in every point, where the smallest principal curvature \( \kappa_1 \) satisfies

\[
\max(-\kappa_1, 0) \leq \frac{1}{2(n-1)} H = \epsilon_1 H.
\]

Proof. The proof is a modification of a similar result in [3, Sections 6.1.4 and 6.1.5] or [4, Section 5.1.1].

It follows immediately from the definition of \( F \) that

\[
F^{ij,kl} = g^{ij} g^{kl} - \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk})
\]

and

\[
\tilde{F}^{ij,kl} h_{ij} r h_{kl}^{rs} = F^{ij,kl} h_{ij} r h_{kl}^{rs} + \epsilon [2(n-1) + \epsilon(n-1)n] |DH|^2.
\]

In a fixed point \( p \in M \) introduce normal Riemannian coordinates such that the relations (8.14) and (8.15) are valid, and define the matrix \( (a^{kl}) \) through

\[
a^{kl} = \begin{cases} 1, & k \neq l, \\ 0, & k = l. \end{cases}
\]

We also set

\[
h_{ijk} = h_{ij;k}.
\]
Then, we conclude from (9.29)

\[ F^{ij,kl}h_{ijr}h_{klr}g_{rs} = \|DH\|^2 - h_{ijk}h_{ijk} \]

\[ = \sum_{i,k,l} (h_{kkki}h_{lli} - h_{kli}h_{kli}) \]

\[ = \sum_{i} a^{kl}(h_{kkki}h_{lli} - h_{kli}^2) \]

\[ = \sum_{i} a^{kl}h_{kkki}h_{lli} - \sum_{l} \sum_{k,i} a^{kl}h_{kli}^2. \]

(9.33)

In the last summand let us interchange the roles of \(i\) and \(l\) to obtain

\[ - \sum_{l} \sum_{k,i} a^{kl}h_{kli}^2 = - \sum_{i} \sum_{k,l} a^{ki}h_{kil}^2. \]

(9.34)

The Codazzi equations yield

\[ h_{kil} = h_{kli} + c_{kil}, \]

where \((c_{kil})\) is a uniformly bounded tensor in \(\bar{\Omega}\)

(9.35)

\[ \|c_{kil}\| \leq \text{const}, \]

and hence,

\[ h_{kil}^2 = h_{kli}^2 + 2h_{kli}c_{kil} + c_{kil}^2. \]

(9.36)

and

\[ - \sum_{i} \sum_{k,l} a^{ki}h_{kil}^2 \leq - \sum_{i} \sum_{k,l} a^{ki}h_{kli}^2 - 2 \sum_{i,k,l} a^{ki}h_{kli}c_{kil} \]

\[ \leq -2 \sum_{i} \sum_{k} a^{ki}h_{kki}^2 - \sum_{[i,j,k]} h_{ijk}^2 \]

\[ - 2 \sum_{i,k,l} a^{ki}h_{kli}c_{kil} - 2 \sum_{i,k} a^{ki}h_{iik}c_{iki}, \]

(9.37)

where \(\sum_{[i,j,k]}\) means that the summation is carried out over those triples \((i,j,k)\) where all three indices are different from each other.
The first linear term in the last inequality can be estimated from above by

\[-2 \sum_{i,j,k,l} a^{ki} h_{kii} c_{kij} = -2 \sum_{i} \sum_{k} a^{ki} h_{kki} c_{kik} - 2 \sum_{i} \sum_{k \neq l} a^{ki} h_{kli} c_{kil} \]

\[(9.39)\]

\[\leq \delta \sum_{i} \sum_{k} a^{ki} h_{kki}^2 + \frac{\delta}{2} \sum_{[i,j,k]} h_{ijk}^2 + c\delta^{-1} \]

for any \(\delta > 0\), and the second term similarly. Thus, we deduce from (9.33) and (9.38)

\[F_{ij,kl}^x h_{ijr} h_{klr} g^{rs} \leq -2(1 - \delta^2) \sum_{i} \sum_{k} a^{ki} h_{kki}^2 \]

\[(9.40)\]

\[+ \sum_{i} a^{ki} h_{kki} h_{lil} + c\delta^{-1} \]

for any \(0 < \delta \leq 1\).

Next, let us consider the second term on the left-hand side of (9.27); we have

\[H^{-1} \tilde{F}_{ij} H_i H_j = \tilde{H}^{-1} \tilde{F}_{ij} H_i H_j + \frac{en}{1 + en} H^{-1} \tilde{F}_{ij} H_i H_j \]

\[(9.41)\]

\[\leq \tilde{H}^{-1} \sum_{i} \tilde{F}_{ii} (\sum_{k} h_{kki})^2 + c\|DH\|^2, \]

where \(c = c(n)\), in view of (8.7), (9.25), and (1.15).

Combining (9.40) and the preceding estimate, we conclude

\[F_{ij,kl}^x h_{ijr} h_{klr} g^{rs} + H^{-1} \tilde{F}_{ij} H_i H_j \]

\[(9.42)\]

\[\leq -2(1 - \frac{\delta}{2}) \sum_{i} \sum_{k} a^{ki} h_{kki}^2 + \sum_{i} a^{ki} h_{kki} h_{lil} + c\delta^{-1} \]

\[+ \tilde{H}^{-1} \sum_{i} \tilde{F}_{ii} (\sum_{k} h_{kki})^2 + c\|DH\|^2. \]

For each index \(i\), let us estimate the corresponding summand separately, i.e. let us look at—no summation over \(i—\)

\[-(1 - \frac{\delta}{2}) \sum_{k} a^{ki} h_{kki}^2 + \frac{1}{2} a^{ki} h_{kki} h_{lil} + \frac{1}{2H} \tilde{F}_{ii} (\sum_{k} h_{kki})^2, \]

\[(9.43)\]

where we have divided the terms by 2.
Denote by $\sum'$ a sum where the index $i$ is omitted during the summation, then, (9.43) can be expressed as

$$-(1 - \frac{\delta}{2}) \sum_{k} a^{ki} h_{kki} + h_{iii} \sum_{k} a^{ki} h_{kki} + \sum_{k < l} h_{kkl} h_{lli}$$

(9.44)

$$+ \frac{1}{2H} \tilde{F}_{ii} (\sum_{k} h_{kki})^2.$$

To replace $h_{iii}$ in the preceding expression we use the chain rule

$$f_i \equiv \tilde{F}_i = \tilde{F}^{kk} h_{kki}$$

to derive

$$h_{iii} = \frac{1}{F_{ii}} (f_i - \sum_{k} h_{kkk} h_{kki}).$$

(9.45)

Inserting (9.46) in (9.44) we obtain, after some simple algebraic manipulations, cf. [3, equ. (36) on p. 78] or [4, equ. (17)],

$$-(1 - \frac{\delta}{2}) \sum_{k} h_{kki}^2 - 
\sum_{k < l} h_{kkk} h_{kki} (1 - \frac{F_i^k}{F_i^l}) \left(1 - \frac{F_i^k}{F_i^l}\right) \tilde{F}_{kki} \tilde{F}_{lli}$$

$$- \sum_{k} h_{kki}^2 \left(1 - \frac{F_i^k}{F_i^l}\right) \left(1 - \frac{F_i^k}{F_i^l}\right) \tilde{F}_{kki} \tilde{F}_{lli}$$

(9.47)

$$+ \sum_{k} h_{kki}^2 \left(1 - \frac{F_i^k}{F_i^l}\right) \left(1 - \frac{F_i^k}{F_i^l}\right) \tilde{F}_{kki} \tilde{F}_{lli}.$$
In view of (9.23) we can estimate $I_1$ from above by

\[
I_1 \leq c \delta^{-1} f^{-1} |f_i|^2.
\]

(9.51)

$I_2$ is estimated by

\[
I_2 = \frac{1}{2H \hat{F}^i_i} |f_i|^2 \leq c f^{-1} |f_i|^2,
\]

because of (1.17).

Finally, we claim that $I_3 \leq 0$ if we choose $\delta = \frac{1}{4}$. To verify this assertion, let us multiply $I_3$ by $2 \hat{H} \tilde{F}^i_i$ to obtain

\[
-2(1 - \delta) \hat{H} \hat{F}^i_i \sum_k h^2_{kki} - \sum_k' [2 \hat{H} \hat{F}^k_k - (\hat{F}^i_i - \hat{F}^k_k)^2] h^2_{kki}
\]

(9.53)

\[
- \sum_{k<l} [2 \hat{H} (\hat{F}^k_k + \hat{F}^l_l) - 2 \hat{H} \hat{F}^i_i - 2(\hat{F}^i_i - \hat{F}^k_k)(\hat{F}^i_i - \hat{F}^l_l)] h_{kki} h_{lli}.
\]

Now, we use (8.7) and replace any $\hat{F}^j_j$, $1 \leq j \leq n$, by

\[
F^j_j + \epsilon(n - 1)(1 + \epsilon n) H g^j_j \equiv F^j_j + \epsilon \gamma \epsilon H.
\]

(9.54)

The expression in (9.53) is then equal to the sum of two terms $I_4 + I_5$, where

\[
I_4 = -2(1 - \delta) \hat{H} F^i_i \sum_k h^2_{kki} - \sum_k' [2 \hat{H} F^k_k - (F^i_i - F^k_k)^2] h^2_{kki}
\]

(9.55)

\[
- \sum_{k<l} [2 \hat{H} (F^k_k + F^l_l) - 2 \hat{H} F^i_i - 2(F^i_i - F^k_k)(F^i_i - F^l_l)] h_{kki} h_{lli},
\]

and

\[
I_5 = -2(1 - \delta) \hat{H} \epsilon \gamma H \sum_k h^2_{kki} - 2 \hat{H} \epsilon \gamma H \sum_k h^2_{kki}
\]

\[
- 2 \hat{H} \epsilon \gamma H \sum_{k<l} h_{kki} h_{lli}.
\]

From the binomial formula

\[
\left( \sum_k h_{kki} \right)^2 = \sum_k h^2_{kki} + 2 \sum_{k<l} h_{kki} h_{lli}.
\]

(9.57)
we infer that $I_5 \leq 0$, while $I_4$ is non-positive provided we choose $\delta = \frac{1}{4}$ and assume

\begin{equation}
\max(-\tilde{\kappa}, 0) \leq \frac{1}{2(n-1)} \tilde{H},
\end{equation}

cf. [3, pp. 81–85] or [4, pp. 23–29].

But the condition (9.58) is certainly satisfied in view of (9.28).

Combining (9.30), (9.42), (9.51), and (9.52) gives (9.27), and thus, the Lemma is proved. q.e.d.

From Lemma 9.2 and Lemma 9.5 we conclude

**Corollary 9.6.** Let $M = \text{graph } u$ be an admissible solution of equation (9.1) in $\Omega$. Then, the estimate

\begin{equation}
\tilde{F}^{ij}h_{ij} r + \tilde{F}^{ij}(\log H)_i (\log H)_j \leq c H^{-1} f^{-1} \|Df\|^2 + c H^{-1} \|D \log H\|^2 + c H^{-1}
\end{equation}

is valid in every point $p \in M$, where (9.28) is satisfied. The constant $c$ depends on $\Omega$, $\|Df\|$, $\|D^2f\|$, the constant $c_1$ in (8.3), and on known estimates of the $C^0$ and $C^1$-norm of $u$.

As we already mentioned we have to assume the existence of a strictly convex function $\chi \in C^2(\bar{\Omega})$, i.e. $\chi$ satisfies

\begin{equation}
\chi_{\alpha\beta} \geq c_0 g_{\alpha\beta}
\end{equation}

with a positive constant $c_0$.

We observe that then the restriction $\chi = \chi|_M$ of $\chi$ to an admissible solution $M \subset \bar{\Omega}$ of (9.1) satisfies the elliptic inequality

\begin{equation}
-\tilde{F}^{ij} \chi_{ij} = -2\tilde{F} \chi_{\alpha} \nu^{\alpha} - \tilde{F}^{ij} \chi_{\alpha\beta} x^\alpha x^\beta \leq -2\tilde{F} \chi \nu^{\alpha} - c_0 \tilde{F}^{ij} g_{ij},
\end{equation}

where we used the homogeneity of $\tilde{F}$.

We can now prove uniform $C^2$- estimates.

**Theorem 9.7.** Let $M = \text{graph } u$ be an admissible solution of equation (9.1) in $\Omega$, where $f$ satisfies the estimates (8.3), (8.4) and (8.5). Then, the principal curvatures of $M$ are uniformly bounded.
Proof. Let $\chi$ be the strictly convex function and $\mu$ a large positive constant. We shall prove that $w = \log H + \mu \chi$ is uniformly bounded from above.

Let $x_0 \in M$ be such that

$$w(x_0) = \sup_M w,$$

and choose in $x_0$ a local coordinate system satisfying (8.14) and (8.15). Applying the maximum principle, we conclude from (9.3) and (9.61)

$$0 \leq \tilde{F}^{kl} h_{kr} h^{r}_{l} + c \tilde{F}^{ij} g_{ij} + c \mu f - \mu c_0 \tilde{F}^{ij} g_{ij}$$

$$+ c(1 + f + \|Df\| + \|D^2 f\|)(1 + H + \|D \log H\|)$$

$$+ \tilde{F}^{ij,kl} h\mbox{,}_{ij} h\mbox{,}_{kl} g^{rs} H^{-1} + \tilde{F}^{ij} (\log H)\mbox{,}_{i} (\log H)\mbox{,}_{j},$$

where we also assumed $H$ to be larger than 1.

We now consider two cases.

Case 1: Suppose that

$$|\kappa_1| \geq \epsilon_1 H = \frac{1}{2(n-1)} H.$$

Then, we infer from Lemma 8.3 and (9.24)

$$-\tilde{F}^{kl} h_{kr} h^{r}_{l} \leq -\frac{n-1}{n} H \kappa_1^2 \leq -\frac{n-1}{n} \epsilon_2^3 H^3 = -\epsilon_2 H^3.$$

Moreover, the concavity of $\log \tilde{F}$ implies

$$\tilde{F}^{ij,kl} h\mbox{,}_{ij} h\mbox{,}_{kl} g^{rs} H^{-1} \leq \tilde{F}^{ij} g^{ij} \tilde{F}^{kl} h\mbox{,}_{kl} \tilde{F}^{rs} h\mbox{,}_{rs} H^{-1}$$

$$= f^{-1}\|Df\|^2 H^{-1}$$

$$\leq c f^{-1}\|Df\|^2 |A|^2 H^{-1}$$

$$\leq c f^{-1}\|Df\|^2 H.$$

Furthermore, $Dw(x_0) = 0$, or,

$$(\log H)\mbox{,}_{i} = -\mu \chi_\mbox{,}_{i}.$$

Inserting the last three relations in (9.63) we obtain

$$0 \leq -\epsilon_2 H^3 + c(1 + H + \mu) + c \mu^2 H,$$
where, now, $c$ depends on $f$ and its derivatives in the ambient space.

Hence, $H$, and therefore $w$, are a priori bounded in $x_0$.

**Case 2:** Suppose that

\[(9.69)\quad |\kappa_1| < \epsilon_1 H.\]

Then, Corollary 9.6 is applicable, and we infer from (9.63) and (9.67)

\[(9.70)\quad 0 \leq c(1 + H + \mu + \mu^2 H^{-1}) + (c - \mu c_0)\bar{F}^{ij} g_{ij}.\]

Choosing now $\mu$ sufficiently large we obtain an a priori bound for $H(x_0)$, since

\[(9.71)\quad \bar{F}^{ij} g_{ij} \geq (n - 1)H.\]

Thus, $w$, or equivalently $H$, are uniformly bounded. q.e.d.

# 10. Existence of a solution

We can now demonstrate the final step in the proof of Theorem 0.3. Let $M_\epsilon = \text{graph } u_\epsilon$ be the stationary approximations. In the preceding sections we have proved uniform estimates for $u_\epsilon$ up to the order two. Since, by assumption, $f$ is strictly positive, the principal curvatures of $M_\epsilon$ stay in a compact subset of the cone $\Gamma_2$ for small $\epsilon$, cf. Remark 3.1, and therefore, the operator $\bar{F}$ is uniformly elliptic for those $\epsilon$. Taking the square root on both sides of equation (9.1) without changing the notation, we also know that $\bar{F}$ is concave. Hence the $C^{2,\alpha}$-estimates of Evans and Krylov are applicable, cf. [7] and [20], and we deduce

\[(10.1)\quad |u_\epsilon|_{2,\alpha,S_0} \leq \text{const}\]

uniformly in $\epsilon$. If $\epsilon$ tends to zero, a subsequence converges to a solution $u \in C^{2,\alpha}(S_0)$ of our problem. From the Schauder estimates we further conclude $u \in C^{4,\alpha}(S_0)$.
References

[1] R. Bartnik, *Existence of maximal surfaces in asymptotically flat spacetimes*, Commun. Math. Phys. 94 (1984) 155–175.

[2] ______, *Remarks on cosmological spacetimes and constant mean curvature surfaces*, Comm. Math. Phys. 117 (1988) 615–624.

[3] P. Bayard, *Problème de Dirichlet pour la courbure d’ordre m lorentzienne*, Thèse de doctorat. Université de Nice-Sophia-Antipolis (2001).

[4] P. Bayard, *Dirichlet problems for space-like hypersurfaces with prescribed scalar curvature in $R^n$, preprint* 41, 44, 46

[5] R. Budic, J. Isenberg, L. Lindblom & P. Yasskin, *On the determination of the Cauchy surfaces from intrinsic properties*, Commun. Math. Phys. 61 (1978) 87–95.

[6] K. Ecker & G. Huisken, *Immersed hypersurfaces with constant Weingarten curvature*, Math. Ann. 283 (1989) 329–332.

[7] L. C. Evans, *Classical solutions of fully nonlinear, second-order elliptic equations*, Comm. Pure Appl. Math. XXV (1982) 333–363.

[8] C. Gerhardt, *H-surfaces in Lorentzian manifolds*, Commun. Math. Phys. 89 (1983) 523–553.

[9] ______, *Closed Weingarten hypersurfaces in Riemannian manifolds*, J. Differential Geom. 43 (1996) 512–641.

[10] ______, *Closed Weingarten hypersurfaces in space forms*, Geometric Analysis and the Calculus of Variations (J. Jost, ed.), International Press, Boston, 1996, pp. 71–98.

[11] ______, *Hypersurfaces of prescribed Weingarten curvature*, Math. Z. 224 (1997) 167–194.

[12] ______, *Closed hypersurfaces of prescribed mean curvature in conformally flat Riemannian manifolds*, J. Differential Geom. 48 (1998) 587–613.

[13] ______, *Erratum “Hypersurfaces of prescribed Weingarten curvature”*, Math. Z. 233 (2000) 829–830.

[14] ______, *Hypersurfaces of prescribed mean curvature in Lorentzian manifolds*, Math. Z. 235 (2000) 83–97.

[15] ______, *Closed Weingarten hypersurfaces in semi-riemannian manifolds*, preprint, Nov. 2001, [http://arXiv.org/pdf/math.DG/0207053](http://arXiv.org/pdf/math.DG/0207053)

[16] R. P. Geroch, *The domain of dependence*, J. Math. Phys. 11 (1970) 437–449.

[17] R. P. Geroch & G. T. Horowitz, *Global structure of space-time*, General Relativity. An Einstein centenary survey (S. W. Hawking & W. Israel, eds.), Cambridge University Press, Cambridge, 1979, pp. 212–293.

[18] S. W. Hawking & G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge, 1973.

[19] N. V. Krylov, *Nonlinear elliptic and parabolic equations of second order*, Reidel, Dordrecht, 1987.

[20] B. O’Neill, *Semi-riemannian geometry*, Academic Press, New-York, 1983.

[21] J. I. E. Urbas, *On the expansion of convex hypersurfaces by symmetric functions of their principal radii of curvature*, J. Differential Geom. 33 (1991) 91–125.