AdS$_2$ D-branes in AdS$_3$ spacetime

Sylvain Ribault*

Centre de Physique Théorique, Ecole Polytechnique, 91128 Palaiseau, FRANCE

Based on a talk at the 2001 Corfu Summer Institute on Elementary Particle Physics

Abstract

I review some recent progress in understanding the properties of AdS$_2$ branes in AdS$_3$. Different methods – classical string motion, Born-Infeld dynamics, boundary states – are evoked and compared.

1 Introduction

A prominent motivation for the study of strings in AdS$_3$ is the AdS/CFT correspondence [1, 2]. This is a holographic relationship between string theory in an Anti-de Sitter spacetime and field theory on its conformal boundary. The relationship also extends to D-branes in AdS$_3$. This has been made recently quite precise in the case of the AdS$_2$ D-brane [3, 4]. The AdS$_2$ brane extends to the boundary of AdS$_3$ where it may be interpreted as a one-dimensional conformal wall.

I will not focus on this holographic interpretation here, but rather on the properties of the AdS$_2$ branes themselves: their geometry, open-string spectrum, and interactions with closed strings. I do not present any new material, but summarize and relate recent results.

Let me briefly describe the geometry of AdS$_3$. Define it as the following hypersurface of $\mathbb{R}^{2,2}$:

\[-X_0^2 - X_1^2 + X_2^2 + X_3^2 = -L^2.\]

This is a solution to the Einstein equations with negative cosmological constant. The link with the group $SL_2(\mathbb{R})$ is made through

\[g = \frac{1}{L} \left( \begin{array}{cc} X_0 + X_1 & X_2 + X_3 \\ X_2 - X_3 & X_0 - X_1 \end{array} \right).\]

(1)

There is another useful parametrization, the global coordinates :

\[\begin{align*}
X_2 + iX_3 &= L \sinh \rho \ e^{i\theta} \\
X_0 + iX_1 &= L \cosh \rho \ e^{i\tau}
\end{align*}\]

(2)

In these coordinates AdS$_3$ is a solid cylinder ($\tau$ takes all real values) and $SL_2(\mathbb{R})$ is a solid torus ($\tau$ is compactified). The metric is

\[ds^2 = L^2(- \cosh^2 \rho \ dr^2 + d\rho^2 + \sinh^2 \rho \ d\theta^2) = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2.\]

(3)

*ribault@cpht.polytechnique.fr
AdS$_3$ is also part of a string background. This can be seen at the level of supergravity: a solution can be written using the metric of AdS$_3$ and an NS-NS three-form $H$ (the volume form of AdS$_3$). In the equation of motion for the metric, there is no more cosmological constant, but instead the square of $H$.

There are branes with induced AdS$_2$ geometry in the AdS$_3$ background [5, 6]. They have the following equations in global coordinates:

$$\sinh \rho \cos \theta = \sinh \psi$$

The constant $\psi$ is in fact quantized, as we will see later. Now a D-brane is really defined by the geometry, plus the value of the worldvolume two-form $F$. This depends on a choice of gauge for the $B$-field such that $H = dB$. There is a gauge in which we have on each brane:

$$B + 2\pi \alpha' F = \frac{\tanh \psi}{L^2} \text{vol}, \quad 2\pi \alpha' F = -\frac{\psi}{L^2 \cosh^2 \psi} \text{vol},$$

where vol is the volume form induced on the brane by the bulk metric.

We have precisely defined the branes we are interested in. The main tool for studying these branes will be the theory of strings and branes in group manifolds. The main difficulties in this case come from the non-compactness of AdS$_3$ (this results in volume divergences in some physical quantities) and its Minkowskian signature.

## 2 D-branes in the SL$_2(\mathbb{R})$ WZW model

The AdS$_2$ branes have the general properties of twined conjugacy classes in WZW models. These branes have been known for some time [7]. Let me call $g(z, \bar{z})$ the embedding of the string worldsheet into a group manifold. Taking advantage of the group structure, we construct right- and left-moving currents

$$g(z, \bar{z}) \rightarrow J_L = -g^{-1} \partial g; \quad J_R = \overline{\partial} gg^{-1}$$

From the modes of these currents (i.e. the coefficients of their decomposition in powers of $z$ and $\bar{z}$) we construct two copies of the affine Lie algebra of the group, which are symmetries of the closed-string spectrum.

Let $\omega$ be an automorphism of the Lie algebra of our group. We use the same notation for its natural extension to the affine Lie algebra. The boundary conditions: $J_L = \omega(J_R)$, preserve the conformal symmetry. If $\omega$ is an outer automorphism they break the Lie algebra symmetry and the corresponding brane has the geometry of a twined conjugacy class: \{hg$\omega(h)^{-1}$\}, for $g$ fixed and $h$ varying on the group.

The group SL$_2(\mathbb{R})$ has an outer automorphism: $g \rightarrow \Omega g \Omega$, where $\Omega = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. The corresponding twined conjugacy classes have equations of the following form: Trace $\Omega g = \text{cst}$. This amounts to eq. (4) if we write cst $= 2 \sinh \psi$. General expressions are also known for the $B$ and $F$ fields [8], which in our case amount to eq. (5).

In compact groups this comes with a quantization of the allowed positions of the D-branes due to the mechanism of flux stabilization [9, 8]. This is due to the nontrivial topology of the D-brane, more precisely to its non-trivial two-cycles. Our AdS$_2$ branes have a trivial topology but their position is still quantized [10]:
\[
\sinh \psi = qT_F/T_D \quad \text{where} \quad q \in \mathbb{Z} \quad \text{and} \quad T_D, T_F \quad \text{are the D-string and fundamental string tensions.}
\]

Now let us list some interesting physical quantities characterizing D-branes. The exact quantities are defined in the worldsheet approach, i.e. from boundary conformal field theory. An approximate space-time description is given by the Born-Infeld action.

| SPACE – TIME | WORLD – SHEET |
|--------------|---------------|
| \(S^{BI}(p, F) + \text{corrections} \) | \(S^\sigma = \frac{1}{4\pi \alpha'} \int_{\Sigma} g_{\mu \nu} + B_{\mu \nu} + \int_{\partial \Sigma} A \) |
| Born – Infeld solution | boundary state |
| position | v.e.v. of graviton or dilaton |
| energy | quantum dimension (mass) |
| quadratic fluctuations | open – string spectrum |

- \( \sigma \) denotes the sigma-model, \( S^\sigma \) is the Polyakov action. \( \Sigma \) is the worldsheet and its boundary \( \partial \Sigma \) lies in the D-brane.

- \( S^{BI} = \int \sqrt{\det(p^* g_{\mu \nu} + p^* B_{\mu \nu} + 2\pi \alpha' F)} \) is the Born-Infeld action, depending on the embedding \( p \) of the D-brane in space-time and on the two-form field \( F = dA \) on the D-brane. It is the first term in an expansion in powers of \( \alpha' \) (not counting the factor \( \alpha' \) of \( F \)). In the case of group manifolds, the metric and \( B \) field are proportional to \( L^2 = k \alpha' \) where \( k \) is called the level. So the expansion may be written in powers of \( 1/k \).

- A D-brane, solution of the Born-Infeld equations of motion, is described in CFT by its interactions with closed strings, i.e. by a linear form on closed strings states, i.e. by a closed string state called the boundary state. In particular, the interactions with gravitons and dilatons betray the position of the brane. This is explicitly seen in [7].

- The correspondence between the space-time energy and the quantum dimension is explained in [10].

- The spectrum of Born-Infeld fluctuations around a D-brane is the point-particle limit (\( k \to \infty \)) of the open-string spectrum. This can be seen explicitly in the case of compact groups [9, 8].

In our case of the AdS_2 brane, the space-time quantities are easily computed [5, 11]. We already described the position and F-field of the brane. Its energy is proportional to \( \cosh \psi \) (for a \( \psi \)-independent regularization of the volume divergence). The spectrum of fluctuations is basically the space of functions on the AdS_2; the Born-Infeld equations of motion should really be solved only in a complete (ten-dimensional) string background, and they roughly correspond to the Virasoro conditions.

To describe this spectrum we organize it in representations of \( SL_2(\mathbb{R}) \) (which acts on the brane by conjugation). It is time to describe those representations as well as the unitary representations of the corresponding affine Lie algebra \( \hat{sl}(2) \).

3
3 Closed and open strings in $AdS_3$

Let me first review some material from [12]. We describe states in representations of $SL_3(\mathbb{R})$ by the spin $j$ (corresponding to a Casimir eigenvalue $-j(j+1)$) and the eigenvalue $m$ of some generator $J^3$ of the Lie algebra. We will use discrete representations $D^\pm_j$ with $j > 0 \in \mathbb{R}$ and $m \in \pm j + \mathbb{N}$; and continuous representations $C_{\alpha}^m$ with $j \in \frac{1}{2} + i\mathbb{R}$ and $0 \leq \alpha < 1$ and $m \in \alpha + \mathbb{Z}$. Then one can build representations of $sl(2)$ as follows: $sl(2)$ can be defined by its generators $J^a_n$ (where $n \in \mathbb{Z}$ and $a$ is an $sl(2, \mathbb{R})$ label), and the relations

$$[J^a_n, J^b_m] = f^{abc}_{\,\,\,\,d} J^d_{m+n} + \frac{n}{2} \kappa^{ab}_{\,\,\,\,c} \delta_m, -n,$$

where $f^{abc}_{\,\,\,\,d}$ and $\kappa^{ab}_{\,\,\,\,c}$ are the $sl(2, \mathbb{R})$ structure constants and Killing form. Unflowed representations of $\hat{sl}(2)$ are defined by starting with an $sl(2, \mathbb{R})$ representation $C_{\alpha}^m$ (or $D^\pm_j$) and acting on it with the operators $J^a_n$ (with $n > 0$; the action of operators $J^a_n$ is assumed to give zero) to get the affine $\hat{C}_{\alpha}^m$ (or $\hat{D}^\pm_j$) representation. Flowed representation can then be built using the spectral flow automorphisms of $sl(2)$, defined for any $w \in \mathbb{Z}$:

$$J^a_n, J^a_{n+w} \rightarrow J^a_n - \frac{w}{2} \delta_{n,0}, J^a_{n+w}.$$

(we use labels $a = 3, +, -$ corresponding to the $sl(2, \mathbb{R})$ relations $[J^3, J^\pm] = \pm J^\pm$, $[J^+, J^-] = 2 J^3$). They are called $C_{\alpha,w}^m$, $D^\pm_{j,w}$.

This is enough to build the spectrum of closed strings in $AdS_3$: it is made of the above-mentioned representations with the constraint $\frac{1}{2} < j < \frac{\alpha - 1}{2}$ on the discrete representations, each tensored with a right-moving copy of itself. This spectrum can be checked by exact partition function computations in $H_3^+$ [13, 14] (the Euclidean version of $AdS_3$, obtained by $\tau \rightarrow i\tau$). One can also build consistent three- and four-point correlators in $H_3^+$ [15, 16].

One way to have an intuition of these different kinds of representations is to study classical closed string motion in $SL_3(\mathbb{R})$. The continuous representations correspond to strings reaching the spatial infinity (long strings), whereas discrete representations correspond to strings staying at finite distance (short strings) [12]. We can also do such an analysis for open strings ending on an $AdS_2$ brane [11, 17]. Let me call $x^+ = \sigma^0 \pm \sigma^1$ the world-sheet light-cone coordinates ($\sigma^1 \in [0, \pi]$) and $J_+ = -\partial_+ gg^{-1}$, $J_- = g^{-1} \partial_- g$ the world-sheet currents. The general solution to the bulk equations of motion, plus boundary condition $J_- = \omega(J_+)$ at $\sigma^1 = 0$ is [18, 1]

$$g = a(x^+) \, m \, \omega(a(x^-))^{-1},$$

(8)

where $a$ is a group-valued function which should be $2\pi$-periodic up to left-translation by a constant, and $m$ is a group element (defining the twined conjugacy class to which $\sigma^1 = 0$ is attached).

To have an idea of the different sectors appearing in the open-string spectrum it is enough to first restrict ourselves to functions $a(x) = e^{x C}$, where $C$ is a constant matrix, then apply spectral flow $a \rightarrow \exp x w \left( \begin{array}{cc} 0 & 1/2 \\ -1/2 & 0 \end{array} \right) \times a$.

However, if $w$ is odd, then the image of our open string has no more its two ends
on the same D-brane, but stretches between two opposite branes of parameters \( \psi, -\psi \). If we restrict to the study of a single brane, it is still possible to construct classical solutions which will be interpreted as strings in \( \tilde{D}^{\pm, w} \), but this no longer works for long strings [17]. For those, only half of the spectral flow symmetry is preserved.

In the case of the "straight brane" \( \psi = 0 \), such problems do not arise. Moreover, all classical open string solutions give rise to closed string solutions by reflection with respect to the brane [11]. It is then natural to expect that the open-string spectrum is the holomorphic square-root of the closed-string spectrum (i.e. it consists of the same representations, but without the right-movers), as can indeed be proved [17].

4 Some speculations

I finally comment on the significance of these results for the exact spectrum of open strings living on the \( AdS_2 \) brane in \( AdS_3 \).

Since the rôle of spectral flow seems quite clear, what remains to be determined is the density of unflowed representations in the spectrum. Let me start by coming back to the Born-Infeld analysis. It predicts a constant density of discrete representations, independent of the position \( \psi \) of the brane. The continuous representations are not really seen here because they have positive Casimir, thus they are not able to fulfill the Virasoro conditions in a background whose only time direction is included in \( AdS_3 \). They nevertheless appear in the physical spectrum because of spectral flow (which changes the Virasoro generators), however the Born-Infeld analysis is valid in the point-particle limit in which there is no spectral flow (because we are really dealing with representations of the mere Lie algebra, not \( \hat{sl}(2) \)).

In the case of the \( S^2 \) D-brane in \( SU(2) \), the Born-Infeld analysis is known to give exact results for the open-string density of states, up to a truncation on the allowed spins [3]. This could be due to the vanishing of contributions from corrections to the Born-Infeld action, which would not be surprising given the supersymmetry of the \( AdS_2 \times S^2 \) D-brane in the background \( AdS_3 \times SU(2) \times U(1)^4 \) [3] [4]. If these corrections vanish for the \( S^2 \) brane in \( SU(2) \) they may as well vanish for the \( AdS_2 \) brane in \( AdS_3 \), given the formal similarity between the two cases. This would indicate that the spectrum of discrete representations on \( AdS_2 \) branes is brane-independent.

We can also compare the \( AdS_2 \) brane to its counterpart in \( H_3^+ \), obtained by the analytic continuation \( \tau \to i\tau \). For this Euclidean \( AdS_2 \) brane in \( H_3^+ \), exact expressions for the boundary state and the open-string spectrum are known [20] [21] (however, there are still problems to be solved in the construction of the corresponding boundary CFT, see [22]). Because the \( AdS_2 \) brane in \( AdS_3 \) is \( \tau \)-independent, it is still possible to interpret these expressions for this brane. Only continuous representations appear in the spectrum of \( H_3^+ \), so the boundary state for \( AdS_2 \) in \( H_3^+ \) does not give a complete boundary state in \( AdS_3 \). This is not really a problem since the discrete closed string states are not expected to couple to the \( AdS_2 \) brane because it is static (this can be seen in harmonic analysis: the integral over \( AdS_2 \) of a point-particle wavefunction in a discrete representation vanishes). However, discrete representations do appear in the open-string spectrum of the \( AdS_2 \) brane in \( AdS_3 \), whereas they are absent from
the $H_3^+$ results. Here we should notice that the Cardy computation in \cite{20} takes into account only the difference of open-string densities of states between two D-branes of different positions. This means that this computation is insensitive to $\psi$-independent terms in the density of states. To interpret this in $AdS_3$ is thus possible, if the density of discrete representations is brane-independent. This conclusion agrees with the previous reasoning about effective actions.

We have argued that the density of states for open string living on a single $AdS_2$ brane in $AdS_3$ is brane-independent for discrete representations, and given by \cite{20} for continuous representations.

Acknowledgements

I thank the organizers of the Corfu meeting for the opportunity to talk, Angelos Fotopoulos for carefully reading this manuscript, and Marios Petropoulos for discussions and encouragements.

References

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113] \texttt{arXiv:hep-th/9711200}.

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323 (2000) 183 \texttt{arXiv:hep-th/9905111}.

[3] C. Bachas, J. de Boer, R. Dijkgraaf and H. Ooguri, JHEP 0206 (2002) 027 \texttt{arXiv:hep-th/0111210}.

[4] C. Bachas, \texttt{arXiv:hep-th/0205115}.

[5] C. Bachas and M. Petropoulos, JHEP 0102 (2001) 025 \texttt{arXiv:hep-th/0012234}.

[6] S. Stanciu, JHEP 9909 (1999) 028 \texttt{arXiv:hep-th/9901122}.

[7] G. Felder, J. Frohlich, J. Fuchs and C. Schweigert, J. Geom. Phys. 34 (2000) 162 \texttt{arXiv:hep-th/9909030}.

[8] P. Bordalo, S. Ribault and C. Schweigert, JHEP 0110 (2001) 036 \texttt{arXiv:hep-th/0108201}.

[9] C. Bachas, M. R. Douglas and C. Schweigert, JHEP 0005, 048 (2000) \texttt{arXiv:hep-th/0003037}.

[10] J. A. Harvey, S. Kachru, G. W. Moore and E. Silverstein, JHEP 0003 (2000) 001 \texttt{arXiv:hep-th/9909072}.

[11] P. M. Petropoulos and S. Ribault, JHEP 0107 (2001) 036 \texttt{arXiv:hep-th/0105252}.

[12] J. M. Maldacena and H. Ooguri, J. Math. Phys. 42 (2001) 2929 \texttt{arXiv:hep-th/0001053}.

[13] K. Gawedzki, \texttt{arXiv:hep-th/9110076}.
[14] J. M. Maldacena, H. Ooguri and J. Son, J. Math. Phys. 42 (2001) 2961 [arXiv:hep-th/0005183].

[15] J. Teschner, Phys. Lett. B 521 (2001) 127 [arXiv:hep-th/0108121].

[16] J. M. Maldacena and H. Ooguri, Phys. Rev. D 65 (2002) 106006 [arXiv:hep-th/0111180].

[17] P. Lee, H. Ooguri, J. w. Park and J. Tannenhauser, Nucl. Phys. B 610 (2001) 3 [arXiv:hep-th/0106125].

[18] K. Gawedzki, I. Todorov and P. Tran-Ngoc-Bich, arXiv:hep-th/0101170.

[19] C. Bachas, arXiv:hep-th/0106234.

[20] B. Ponsot, V. Schomerus and J. Teschner, JHEP 0202 (2002) 016 [arXiv:hep-th/0112198].

[21] P. Lee, H. Ooguri and J. w. Park, Nucl. Phys. B 632 (2002) 283 [arXiv:hep-th/0112188].

[22] B. Ponsot, arXiv:hep-th/0204085.