Nice triples and a moving lemma for motivic spaces

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Abstract

It is proved that for any cohomology theory $A : \text{SmOp}/k \to \text{Gr} - \text{Ab}$ in the sense of [PS] and any essentially $k$-smooth semi-local $X$ the Cousin complex is exact. As a consequence we prove that for any integer $n$ the Nisnevich sheaf $A^n$, associated with the presheaf $U \mapsto A^n(U)$, is strictly homotopy invariant.

Particularly, for any presheaf of $S^1$-spectra $E$ on the category of $k$-smooth schemes its Nisnevich sheaf of stable $\mathbb{A}^1$-homotopy groups is strictly homotopy invariant.

The ground field $k$ is arbitrary. We do not use Gabber’s presentation lemma. Instead, we use the machinery of nice triples as invented in [PSV] and developed further in [P3]. This recovers a known inaccuracy in Morel’s arguments in [M].

The following moving lemma is proved. Let $k$ be a field and $X$ be a quasi-projective variety. Let $Z$ be a closed subset in $X$ and let $U$ be the semi-local scheme of finitely many closed points on $X$. Then the natural morphism $U \to X/(X - Z)$ of Nisnevich sheaves is naively $\mathbb{A}^1$-homotopic to the constant morphism of $U \to X/(X - Z)$ sending $U$ to the distinguished point of $X/(X - Z)$.

A refined version of that moving lemma is proved and is used as well. Moreover, all the above mentioned results are direct consequences of the latter moving lemma.

1 Main results

One of the main aim of the paper is to prove the following result.

**Theorem 1.1.** Let $k$ be a field and let $A : \text{SmOp}/k \to \text{Gr} - \text{Ab}$ be a cohomology theory on the category $\text{SmOp}/k$ in the sense of [PS Sect.1]. Let $A^n_{\text{Nis}}$ be the Nisnevich sheaf associated with the presheaf $W \mapsto A^n(W)$. Then $A^n_{\text{Nis}}$ is homotopy invariant and even it is strictly homotopy invariant on $(\text{Sm}/k)_{\text{Nis}}$.

This theorem is derived in Section 10 in the standard manner from the exactness of the Cousin complex associated with the theory $A$. The exactness of the Cousin complex for the semi-local essentially $k$-smooth schemes is proved in Section 9. The exactness of

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the Cousin complex is derived from a moving lemma (Theorem 8.2) stated and proved in Section 8. That moving lemma is easily derived from Theorem 7.4. Here is a weaker version of the Theorem 7.4, which shows the shape of Theorem 7.4 (we do not claim that the moving lemma can be derived from this weaker version).

**Theorem 1.2** (Geometric). Let $X$ be an affine $k$-smooth irreducible $k$-variety, and let $x_1, x_2, \ldots, x_n$ be closed points in $X$. Let $U = \text{Spec}(\mathcal{O}_X, \{x_1, x_2, \ldots, x_n\})$ and $f \in k[X]$ be a non-zero function vanishing at each point $x_i$. Then there is a monic polynomial $h \in \mathcal{O}_X, \{x_1, x_2, \ldots, x_n\}[t]$, a commutative diagram of schemes with the irreducible affine $U$-smooth $Y$

$$
\begin{array}{ccc}
(A^1 \times U)_h & \xrightarrow{\tau_h} & Y_h := Y_{\tau(h)}(px)|_{Y_h} \\
\downarrow \text{inc} & & \downarrow \text{inc} \\
(A^1 \times U) & \xrightarrow{\tau} & Y \\
\downarrow \text{inc} & & \downarrow \text{inc} \\
 & & X
\end{array}
$$

and a morphism $\delta : U \to Y$ subjecting to the following conditions:

(i) the left hand side square is an elementary distinguished square in the category of affine $U$-smooth schemes in the sense of [MV, Defn.3.1.3];

(ii) $p_X \circ \delta = \text{can} : U \to X$, where can is the canonical morphism;

(iii) $\tau \circ \delta = i_0 : U \to A^1 \times U$ is the zero section of the projection $\text{pr}_U : A^1 \times U \to U$;

(iv) $h(1) \in \mathcal{O}[t]$ is a unit.

Theorem 7.4 is derived from Theorem 7.1. Theorem 7.1 is purely geometric one. Its proof is based on the theory of nice triples developed in Sections 2–6. We especially stress the significance of Section 4, which contains a construction of new nice triples and morphisms between them out of certain simple data. That allows to construct often nice triples with predicted properties.

The article is organized as follows. Section 2 contains a recollection about elementary fibrations. In Section 3 we recall definition of nice triples from [PSV] and inspired by Voevodsky notion of perfect triples. We state in that section two theorems: 3.8 and 3.9. In Section 4 we recall a construction from [OP2]. It turns out that the construction plays a crucial role in our analyses. That construction together with Proposition 4.3 gives a tool to obtain nice triple with prescribed properties. In Section 5 Theorem 3.8 is proved. In Section 6 Theorem 3.9 is proved. In Section 7 a geometric presentation theorem (Theorem 7.1) is proved and a stronger version of Theorem 1.2 is proved too (see Theorem 7.4).

The Appendix A contains the proof of Lemma 6.1. The Appendix B contains the proof of the proposition 2.3.

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2 Elementary fibrations

In this Section we extend a result of M. Artin from [A] concerning existence of nice neighborhoods. The following notion is introduced by Artin in [A, Exp. XI, Déf. 3.1].

**Definition 2.1.** An elementary fibration is a morphism of schemes $p : X \to S$ which can be included in a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{j} & \overline{X} & \xrightarrow{i} & Y \\
\downarrow{p} & & \downarrow{\pi} & & \downarrow{q} \\
S & & & & S
\end{array}
$$

of morphisms satisfying the following conditions:

(i) $j$ is an open immersion dense at each fibre of $\overline{p}$, and $X = \overline{X} - Y$;

(ii) $\overline{p}$ is smooth projective all of whose fibres are geometrically irreducible of dimension one;

(iii) $q$ is finite étale all of whose fibres are non-empty.

**Remark 2.2.** Clearly, an elementary fibration is an almost elementary fibration in the sense of [PSV, Defn.2.1].

We need in the following result, which is a slight extension of Artin’s result [A, Exp. XI,Prop. 3.3]. It is proved in the Appendix B.

**Proposition 2.3.** Let $k$ be a finite field, $X$ be a smooth geometrically irreducible affine variety over $k$, $x_1, x_2, \ldots, x_n \in X$ be a family of closed points. Then there exists a Zariski open neighborhood $X^0$ of the family $\{x_1, x_2, \ldots, x_n\}$ and an elementary fibration $p : X^0 \to S$, where $S$ is an open sub-scheme of the projective space $\mathbb{P}^{\dim X - 1}$.

If, moreover, $Z$ is a closed co-dimension one subvariety in $X$, then one can choose $X^0$ and $p$ in such a way that $p|_{Z \cap X^0} : Z \cap X^0 \to S$ is finite surjective.

The following result is proved in [PSV, Prop.2.4].

**Proposition 2.4.** Let $p : X \to S$ be an elementary fibration. If $S$ is a regular semi-local irreducible scheme, then there exists a commutative diagram of $S$-schemes

$$
\begin{array}{ccc}
X & \xrightarrow{j} & \overline{X} & \xrightarrow{i} & Y \\
\downarrow{\pi} & & \downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{A}^1 \times S & \xrightarrow{\text{in}} & \mathbb{P}^1 \times S & \xrightarrow{i} & \{\infty\} \times S
\end{array}
$$
such that the left hand side square is Cartesian. Here \( j \) and \( i \) are the same as in Definition 2.1, while \( \text{pr}_S \circ \pi = p \), where \( \text{pr}_S \) is the projection \( \mathbb{A}^1 \times S \to S \).

In particular, \( \pi : X \to \mathbb{A}^1 \times S \) is a finite surjective morphism of \( S \)-schemes, where \( X \) and \( \mathbb{A}^1 \times S \) are regarded as \( S \)-schemes via the morphism \( p \) and the projection \( \text{pr}_S \), respectively.

### 3 Nice triples

In the present section we recall and study certain collections of geometric data and their morphisms. The concept of a nice triple was introduced in [PSV, Defn. 3.1] and is very similar to that of a standard triple introduced by Voevodsky [Voe, Defn. 4.1], and was in fact inspired by the latter notion. Let \( k \) be a field, let \( X \) be a \( k \)-smooth irreducible affine \( k \)-variety, and let \( x_1, x_2, \ldots, x_n \in X \) be a family of closed points. Further, let \( \mathcal{O} = \mathcal{O}_{X, \{x_1, x_2, \ldots, x_n\}} \) be the corresponding geometric semi-local ring.

After substituting \( k \) by its algebraic closure \( \bar{k} \) in \( k[X] \), we can assume that \( X \) is a \( \bar{k} \)-smooth geometrically irreducible affine \( \bar{k} \)-scheme. The geometric irreducibility of \( X \) is required in the proposition 2.3 to construct an open neighborhood \( X^0 \) of the family \( \{x_1, x_2, \ldots, x_n\} \) and an elementary fibration \( p : X^0 \to S \), where \( S \) is an open sub-scheme of the projective space \( \mathbb{P}^{\dim X - 1}_k \). The proposition 2.3 is used in turn to prove the proposition 3.6. To simplify the notation, we will continue to denote this new \( \bar{k} \) by \( k \).

**Definition 3.1.** Let \( U := \text{Spec}(\mathcal{O}_{X, \{x_1, x_2, \ldots, x_n\}}) \). A nice triple over \( U \) consists of the following data:

(i) a smooth morphism \( q_U : X \to U \), where \( X \) is an irreducible scheme,

(ii) an element \( f \in \Gamma(X, \mathcal{O}_X) \),

(iii) a section \( \Delta \) of the morphism \( q_U \),

subject to the following conditions:

(a) each irreducible component of each fibre of the morphism \( q_U \) has dimension one,

(b) the module \( \Gamma(X, \mathcal{O}_X)/f \cdot \Gamma(X, \mathcal{O}_X) \) is finite as a \( \Gamma(U, \mathcal{O}_U) = \mathcal{O} \)-module,

(c) there exists a finite surjective \( U \)-morphism \( \Pi : X \to \mathbb{A}^1 \times U \);

(d) \( \Delta^*(f) \neq 0 \in \Gamma(U, \mathcal{O}_U) \).

There are many choices of the morphism \( \Pi \). Any of them is regarded as assigned to the nice triple.
Remark 3.2. Since \( \Pi \) is a finite morphism, the scheme \( \mathcal{X} \) is affine. We will write often below \( k[\mathcal{X}] \) for \( \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) \). The only requirement on the morphism \( \Delta \) is this: \( \Delta \) is a section of \( \eta_U \). Hence \( \Delta \) is a closed embedding. We write \( \Delta(U) \) for the image of this closed embedding. The composite map \( \Delta^* \circ \eta_U : k[\mathcal{X}] \to \mathcal{O} \) is the identity. If \( \text{Ker} = \text{Ker}(\Delta^*) \), then \( \text{Ker} \) is the ideal defining the closed subscheme \( \Delta(U) \) in \( \mathcal{X} \).

Definition 3.3. A morphism between two nice triples over \( U \)

\[ (q' : \mathcal{X}' \to U, f', \Delta') \to (q : \mathcal{X} \to U, f, \Delta) \]

is an \( \acute{e} \)tale morphism of \( U \)-schemes \( \theta : \mathcal{X}' \to \mathcal{X} \) such that

1. \( q'_U = q_U \circ \theta \),
2. \( f' = \theta^*(f) \cdot h' \) for an element \( h' \in \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \),
3. \( \Delta = \theta \circ \Delta' \).

Two observations are in order here.

- Item (2) implies in particular that \( \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/\theta^*(f) \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \) is a finite \( \mathcal{O} \)-module.
- It should be emphasized that no conditions are imposed on the interrelation of \( \Pi' \) and \( \Pi \).

Let \( U \) be as in Definition 3.1 and can : \( U \to X \) be the canonical inclusion of schemes.

Definition 3.4. A nice triple \( (q_U : \mathcal{X} \to U, \Delta, f) \) over \( U \) is called special if the set of closed points of \( \Delta(U) \) is contained in the set of closed points of \( \{ f = 0 \} \).

Remark 3.5. Clearly the following holds: let \( (\mathcal{X}, f, \Delta) \) be a special nice triple over \( U \) and let \( \theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta) \) be a morphism between nice triples over \( U \). Then the triple \( (\mathcal{X}', f', \Delta') \) is a special nice triple over \( U \).

Proposition 3.6. One can shrink \( X \) such that \( x_1, x_2, \ldots, x_n \) are still in \( X \) and \( X \) is affine, and then construct a special nice triple \( (q_U : \mathcal{X} \to U, \Delta, f) \) over \( U \) and an essentially smooth morphism \( q_X : \mathcal{X} \to X \) such that \( q_X \circ \Delta = \text{can}, f = q_X^*(f) \).

Proof of Proposition 3.6. If the field \( k \) is infinite, then this proposition is proved in [PSV, Prop. 6.1]. So, we may and will assume that \( k \) is finite. To prove the proposition repeat literally the proof of [PSV, Prop. 6.1]. One has to replace the references to [PSV, Prop. 2.3] and [PSV, Prop.2.4] with the reference to Propositions 2.3 and 2.4 respectively.

Let us state two crucial results which are used in Section 7 to prove Theorem 7.1. Their proofs are given in Sections 5 and 6 respectively. If \( U \) as in Definition 6.1 then for any \( U \)-scheme \( V \) and any closed point \( u \in U \) set

\[ V_u = u \times_U V. \]

For a finite set \( A \) denote \( \sharp A \) the cardinality of \( A \).
Definition 3.7. Let \((X, f, \Delta)\) be a special nice triple over \(U\). We say that the triple \((X, f, \Delta)\) satisfies conditions 1* and 2* if either the field \(k\) is infinite or (if \(k\) is finite) the following holds

1* for \(Z = \{f = 0\} \subset X\) and for any closed point \(u \in U\), any integer \(d \geq 1\) one has

\[ \sharp\{ z \in Z_u | \text{deg}(k(z) : k(u)) = d \} \leq \sharp\{ x \in \mathbb{A}^1_u | \text{deg}(k(x) : k(u)) = d \} \]

(2*) for the vanishing locus \(Z\) of \(f\) and for any closed point \(u \in U\) the point \(\Delta(u) \in Z_u\) is the only \(k(u)\)-rational point of \(Z_u = u \times_U Z\).

Theorem 3.8. Let \(U\) be as in Definition 3.7. Let \((q_U : X' \to U, f', \Delta')\) be a special nice triple over \(U\) subject to the conditions (1*) and (2*) from Definition 3.7. Then there exists a morphism \(\theta = \Pi : Y \to X\) over \(U\), if \(\Pi : U \to X\) between nice triples over \(U\) has, particularly, the following property:

(a) the morphism \(\mathbb{A}^1 \times U \xrightarrow{\sigma} Z'\) is a closed embedding;

(b) \(\sigma\) is étale in a neighborhood of \(Z' \cup \Delta'(U)\);

(c) \(\sigma^{-1}(\sigma(Z')) = Z' \coprod Z''\) scheme theoretically and \(Z'' \cap \Delta'(U) = \emptyset\);

(d) \(\sigma^{-1}(\{0\} \times U) = \Delta'(U) \coprod D\) scheme theoretically and \(D \cap Z' = \emptyset\);

(e) for \(D_1 := \sigma^{-1}(\{1\} \times U)\) one has \(D_1 \cap Z' = \emptyset\).

(f) there is a monic polynomial \(h \in \mathcal{O}[t]\) such that \((h) = \text{Ker}[\mathcal{O}[t] \xrightarrow{-\sigma^*} \Gamma(X, \mathcal{O}_X)/(f')]\)

Theorem 3.9. Let \(U\) be as in Definition 3.7. Let \((X, f, \Delta)\) be a special nice triple over \(U\). Then there exists a morphism \(\theta : (X', f', \Delta') \to (X, f, \Delta)\) of nice triples over \(U\) such that \((X', f', \Delta')\) is a special nice triple satisfying the conditions (1*) and (2*) from Definition 3.7.

4 One of the main construction

In this section beginning with a nice triple we construct a new one. Namely, beginning with a special nice triple \((X, f, \Delta)\) over \(U\) we construct a special nice triple \((X', f', \Delta')\) over \(U\) and a morphism

\[ \theta : (X', f', \Delta') \to (X, f, \Delta) \]

between nice triples over \(U\), which has, particularly, the following property:

if \(\Pi : X \to \mathbb{A}^1 \times U\) is a finite surjective morphism assigned to the nice triple, and \(Y = \Pi^{-1}(\Pi(Z \cup \Delta(U)))\) is the closed subset in \(X\), and \(y_1, \ldots, y_m\) are all its closed points, and \(S = \text{Spec}(\mathcal{O}_{X, y_1, \ldots, y_m})\), and \(S' = \theta^{-1}(S)\), then \(S'\) is étale and finite over \(S\), irreducible.
and the set of closed points of the closed subset \( \{ f' = 0 \} \) in \( X \) is contained in the set of closed points of \( S' \).

We begin with recalling the following geometric lemma [OP1, Lemma 8.2]. For reader’s convenience we state that Lemma adapting notation to the ones of the present section.

The following lemma is equivalent to the one [OP2, Lemma 8.2].

**Lemma 4.1.** Let \( U \) be as in Definition 3.1 and let \((X, f, \Delta)\) be a nice triple over \( U \). Since \((X, f, \Delta)\) is a nice triple over \( U \) there is a finite surjective morphism \( \Pi : X \to \mathbb{A}^1 \times U \) of \( U \)-schemes. Let \( Y \) be a closed nonempty subset of \( X \), finite over \( U \). Let \( V \) be an open subset of \( X \) containing \( \Pi^{-1}(\Pi(Y)) \). Then there exists an open subset \( W \subseteq V \) still containing \( \Pi^{-1}(\Pi(Y)) \) and such that

- the data \((W, f|_W, \Delta)\) is a nice triple over \( U \);
- the open embedding \( i : W \hookrightarrow X \) is a morphism \((W, f|_W, \Delta) \to (X, f, \Delta)\) between the nice triples over \( U \).

Let \( \Pi : X \to \mathbb{A}^1 \times U \) be the finite surjective \( U \)-morphism as in the lemma 4.1. Consider the following diagram

\[
\begin{array}{ccc}
Z & \hookrightarrow & X \\
\downarrow & & \downarrow \Pi \\
\mathbb{A}^1 \times U & \downarrow q_U & \\
& \downarrow \Delta & \\
& U & \\
\end{array}
\]

Here and in the construction 4.2 below \( Z \) is the closed subset defined by the equation \( f = 0 \). By the assumption, \( Z \) is finite over \( U \).

**Construction 4.2.** (compare with [OP2] the proof of lemma 8.1) Let \( U \) be as in Definition 3.1 and let \((X, f, \Delta)\) be a nice triple over \( U \). Since \( \Delta \) is a section of \( q_U \), hence \( \Delta(U) \) is a closed subset in \( X \). Let \( \Pi : X \to \mathbb{A}^1 \times U \) a finite surjective morphism of \( U \)-schemes, which exists, since \((X, f, \Delta)\) be a nice triple over \( U \). Let \( Y = \Pi^{-1}(\Pi(Z \cup \Delta(U))) \) be the closed subset in \( X \). Since \( Z \) and \( \Delta(U) \) are both finite over \( U \) and since \( \Pi \) is a finite morphism of \( U \)-schemes, \( Y \) is also finite over \( U \). Denote by \( y_1, \ldots, y_m \) its closed points and let \( S = \text{Spec}(\mathcal{O}_{X,y_1,\ldots,y_m}) \). Since \( Y \) is also finite over \( U \) it is closed in \( X \). Since \( Y \) is contained in \( S \) it is also closed in \( S \). Set \( T = \Delta(U) \subseteq S \).

So, given a nice triple \((X, f, \Delta)\) and the morphism \( \Pi \) we get certain data \((Z, Y, S, T)\), where \( Z, Y, T \) are closed subsets as in \( X \), so in \( S \). And \( S \) is a semi-local subscheme in \( X \). Particularly, the set of all closed points of \( Z \) is contained in the set of all closed points of \( S \).

Let \( \theta_0 : S' \to S \) be a finite étale morphism with irreducible \( S' \) and let \( \delta : T \to S' \) be a section of \( \theta_0 \) over \( T \). We can extend these data to a neighborhood \( V \).
of \( \{y_1, \ldots, y_n\} \) in \( \mathcal{X} \) and get the diagram

\[
\begin{array}{ccc}
S' & \longrightarrow & \mathcal{V}' \\
\downarrow \delta & & \downarrow \theta \\
T' & \longrightarrow & \mathcal{V} \\
\end{array}
\]

where \( \theta : \mathcal{V}' \to \mathcal{V} \) finite étale (and the square is cartesian). Since \( T \) isomorphically projects onto \( U \), it is still closed viewed as a sub-scheme of \( \mathcal{V} \). Note that since \( Y \) is semi-local and \( \mathcal{V} \) contains all of its closed points, \( \mathcal{V} \) contains \( \Pi^{-1}(\Pi(Y)) = Y \). By Lemma 4.1 there exists an open subset \( W \subseteq \mathcal{V} \) containing \( Y \) (and hence containing \( S \)) and endowed with a finite surjective \( U \)-morphism \( \Pi^* : W \to \mathbb{A}^1 \times U \). Let \( \mathcal{X}' = \theta^{-1}(W) \), \( f' = \theta^*(f) \), \( q_U = q_U \circ \theta \), and let \( \Delta' : U \to \mathcal{X}' \) be the section of \( q_U \) obtained as the composition of \( \delta \) with \( \Delta \). This way we get

1) firstly, a triple \((\mathcal{X}', f', \Delta')\);
2) secondly, the étale morphism of \( U \)-schemes \( \theta : \mathcal{X}' \to \mathcal{X} \) is a morphism between the nice triples

\[ \theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta) \] with \( h' = 1 \);
3) thirdly, inclusions of \( U \)-schemes \( S \subset W \) and \( S' \subset \mathcal{X}' \) and the equality \( S' = \theta^{-1}(S) \).

**Proposition 4.3.** Under the hypotheses and notation of the construction 4.2 the following is true:

(i) the triple \((\mathcal{X}', f', \Delta')\) is a nice triple over \( U \);

(ii) the étale morphism \( \theta : \mathcal{X}' \to \mathcal{X} \) is a morphism between the nice triples

\[ \theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta) \] with \( h' = 1 \);

(iii) if the triple \((\mathcal{X}, f, \Delta)\) is a special nice triple over \( U \), then the triple \((\mathcal{X}', f', \Delta')\) is a special nice triple over \( U \);

(iv) let \( \mathcal{Z}' \) be the vanishing locus of \( f' \) in \( \mathcal{X}' \), then \( \mathcal{Z}' \) is contained in \( S' \) as a closed subset and, particularly, the set of closed points of \( \mathcal{Z}' \) is contained in the set of closed points of the subscheme \( S' \).

**Proof.** Firstly, the structure morphism \( q_U' : \mathcal{X}' \to U \) coincides with the composition

\[ \mathcal{X}' \overset{\theta}{\to} W \hookrightarrow \mathcal{X} \overset{q_U}{\to} U. \]

Thus, it is smooth. The element \( f' \) belongs to the ring \( \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \), the morphism \( \Delta' \) is a section of \( q_U' \). Each component of each fibre of the morphism \( q_U' \) has dimension one, the morphism \( \mathcal{X}' \overset{\theta}{\to} W \hookrightarrow \mathcal{X} \) is étale. Thus, each component of each fibre of the morphism \( q_U' \) is also of dimension one. Since \( \{f = 0\} \subset W \) and \( \theta : \mathcal{X}' \to W \) is finite, \( \{f' = 0\} \) is finite over \( \{f = 0\} \) and hence also over \( U \). In other words, the \( O \)-module \( \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})/f' \cdot \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \) is finite. The morphism \( \theta : \mathcal{X}' \to W \) is finite and surjective. By the first item of Lemma 4.1 there is a finite surjective morphism \( \Pi^* : W \to \mathbb{A}^1 \times U \). It follows that \( \Pi^* \circ \theta : \mathcal{X}' \to \mathbb{A}^1 \times U \) is finite and surjective. Hence, the triple \((\mathcal{X}', f', \Delta')\) is...
a nice triple over $U$. Clearly, the étale morphism $\theta : \mathcal{X} \to \mathcal{X}$ is a morphism between the nice triples

$$\theta : (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta) \quad \text{with} \quad h' = 1$$

If the triple $(\mathcal{X}, f, \Delta)$ is a special nice triple over $U$, then by the remark 3.5, the triple $(\mathcal{X}', f', \Delta')$ is a special nice triple over $U$. The assertions (i) to (iii) are proved.

To prove the assertion (iv) recall that $\mathcal{Z}$ is a semi-local closed subset as in $X$, so in $S$. The equality $f' = \theta^*(f)$ shows that $\mathcal{Z}' = \theta^{-1}(\mathcal{Z})$. Since $S' = \theta^{-1}(S)$, hence $\mathcal{Z}' \subset S'$. Since $(\mathcal{X}', f', \Delta')$ is a nice triple, hence $\mathcal{Z}'$ is finite over $U$. Thus $\mathcal{Z}'$ is closed in $S'$ and semi-local. Hence the set of closed points of $\mathcal{Z}'$ is contained in the set of closed points of the scheme $S'$.

\[ \square \]

5 Proof of Theorem 3.8

Proof of Theorem 3.8 Let $u_1, \ldots, u_n$ be all the closed points of $U$. Let $k(u_i)$ be the residue field of $u_i$. Consider the reduced closed subscheme $u$ of $U$, whose points are $u_1, \ldots, u_n$. Thus

$$u \cong \prod_i \text{Spec } k(u_i).$$

For any closed point $u \in U$ and any $U$-scheme $V$ let $V_u = u \times_U V$ be the scheme fibre of the scheme $V$ over the point $u$. Similarly, set $V_u = u \times_U V$.

Step (i). For any closed point $u \in U$ and any point $z \in \mathcal{Z}_u'$ there is a closed embedding $z^{(2)} \hookrightarrow A^1_{\mathcal{X}_u}$, where $z^{(2)} := \text{Spec } (\Gamma(\mathcal{X}_u, \mathcal{O}_{\mathcal{X}_u})/m_z^2)$ for the maximal ideal $m_z \subset \Gamma(\mathcal{X}_u, \mathcal{O}_{\mathcal{X}_u})$ of the point $z$ regarded as a point of $\mathcal{X}_u'$. This holds, since the $k(u)$-scheme $\mathcal{X}_u$ is equidimensional of dimension one, affine and $k(u)$-smooth.

Step (ii). For any closed point $u \in U$ there is a closed embedding $i_u : \prod_{z \in \mathcal{Z}_u} z^{(2)} \hookrightarrow A^1_{\mathcal{X}_u}$ of the $k(u)$-schemes. To see this apply Step (i) and use that the triple $(\mathcal{X}, f, \Delta)$ satisfies the condition $1^*_u$ from Definition 3.7. Set $i_u = \bigsqcup i_u : \bigsqcup_{u \in U} \prod_{z \in \mathcal{Z}_u} z^{(2)} \hookrightarrow A^1_{\mathcal{X}_u}$.

Step (iii) is to introduce some notation. Since $(\mathcal{X}', f', \Delta')$ is a nice triple over $U$ there is a finite surjective morphism $\mathcal{X}' \xrightarrow{\Pi} A^1 \times U$ of the $U$-schemes. Take the composite $\mathcal{X}' \xrightarrow{\Pi} A^1 \times U \xrightarrow{\text{ProjP1}} U$ morphism and denote by $\bar{\mathcal{X}}'$ the normalization of $\text{ProjP1}$ in $U$ in the fraction field $k(\mathcal{X}')$ of the ring $\Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. The normalization of $A^1 \times U$ in $k(\mathcal{X}')$ coincides with the scheme $\mathcal{X}'$, since $\mathcal{X}'$ is a regular scheme. So, we have a Cartesian diagram of $U$-schemes

$$\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\text{inc}} & \bar{\mathcal{X}}' \\
\Pi & & \Pi \\
A^1 \times U & \xrightarrow{\text{inc}} & \text{ProjP1} \times U,
\end{array}$$

in which the horizontal arrows are open embedding.

Let $\mathcal{X}_\infty'$ be the Cartier divisor $(\Pi)^{-1}(\infty \times U)$ in $\bar{\mathcal{X}}'$. Let $\mathcal{L} := \mathcal{O}_{\bar{\mathcal{X}}'}(\mathcal{X}_\infty')$ be the corresponding invertible sheaf and let $s_0 \in \Gamma(\bar{\mathcal{X}}, \mathcal{L})$ be its section vanishing exactly on $\mathcal{X}_\infty'$.  

9
One has a Cartesian square of $U$-schemes

$$
\begin{array}{ccc}
\mathcal{X}_{\infty,u} & \xrightarrow{J_{\infty}} & \mathcal{X}_{\infty} \\
\downarrow \text{in}_u & & \downarrow \text{in} \\
\mathcal{X}_{u,u} & \xrightarrow{j} & \mathcal{X}
\end{array}
$$

which shows that the closed embedding $\text{in}_u : \mathcal{X}_{\infty,u} \hookrightarrow \mathcal{X}_u$ is a Cartier divisor on $\mathcal{X}_u$. Set $\mathcal{L}_u = j^*(\mathcal{L})$ and $s_0,u,u, = s_0|_{\mathcal{X}_u} \in \Gamma(\mathcal{X}_u, \mathcal{L}_u)$.

**Step (iii').** The morphism $\bar{\Pi}$ is finite surjective. Hence the morphism $\bar{\Pi}_u : \mathcal{X}_u \to \mathbb{P}_u^1$ is finite surjective. Hence there is a closed reduced subscheme $W \subset \mathcal{X}_u$ of dimension zero such that $W \cap \mathcal{X}_{\infty} = \emptyset = W \cap \mathcal{Z}'$ and $W$ has exactly one point on each irreducible component of the reduced closed subscheme $(\mathcal{X}_u)_{\text{red}} \subset \mathcal{X}_u$. Let $s_{\infty} : \mathcal{O}_{\mathcal{X}_u} \to \mathcal{L}|_{\mathcal{X}_u}$ be a trivialization of $\mathcal{L}|_{\mathcal{X}_u}$. Let $t$ be the coordinate function on $\mathbb{A}^1_u$ and $i_u$ be the closed embedding from the Step (ii). Let $J \subset \mathcal{O}_{\mathcal{X}}$ be a sheaf of $\mathcal{O}_{\mathcal{X}}$ ideals such that

$$
\mathcal{O}_{\mathcal{X}}/J = \mathcal{O}_{\mathcal{X}_{\infty}} \times \mathcal{O}_W \times \prod_{u \in U} \prod_{z \in \mathcal{Z}_u} \mathcal{O}_{\mathcal{X}_u}/m_z^2.
$$

**Claim.** There exists an integer $n \geq 0$ and a section $s_1 \in H^0(\mathcal{X}', \mathcal{L}^n)$ such that

$$
s_1 \mod J = (s_{\infty}^n, 0, i_u^*(t) \cdot s_0^n).
$$

Prove the Claim. For the coherent sheaf $J$ on $\mathcal{X}$ there is an integer $n(J) \geq 0$ such that for any integer $n \geq n(J)$ one has

$$
H^1(\mathcal{X}, J \otimes \mathcal{L}^n) = H^1(\mathbb{P}_U, (\bar{\Pi})_*(J) \otimes \mathcal{O}(n)) = 0.
$$

Thus the map $H^0(\mathcal{X}, \mathcal{L}^n) = H^0(\mathcal{X}', \mathcal{L}^n \otimes \mathcal{O}_{\mathcal{X}}) \to H^0(\mathcal{X}', \mathcal{L}^n \otimes (\mathcal{O}_{\mathcal{X}}/J)$ is surjective. Whence the Claim.

**Step (iv).** Set $s_{1,u} = s_1|_{\mathcal{X}_u}$. Then $s_{1,u} \in \Gamma(\mathcal{X}_u, \mathcal{L}^n)$ has no zeros on $\mathcal{X}_{\infty,u}$ and the morphism

$$
[s_{0,u}^n : s_{1,u}] : \mathcal{X}_u \to \mathbb{P}_u^1
$$

is finite surjective and it is such that: there is an equality $[s_{0,u}^n : s_{1,u}]^{-1}(\mathbb{A}^1_u) = \mathcal{X}_u$, and

(a) the morphism $\sigma_u = s_{1,u}/s_{0,u}^n : \mathcal{X}_u \to \mathbb{A}^1_u$ is finite surjective,

(b) $\sigma_u|_{\bigcup_{u \in U} \prod_{z \in \mathcal{Z}_u} z^{(2)}} = i_u : \bigcup_{u \in U} \prod_{z \in \mathcal{Z}_u} z^{(2)} \hookrightarrow \mathbb{A}^1_u$, where $i_u$ is from the step (ii); in particular, $\sigma_u$ is étale at every point $z \in \mathcal{Z}_u$.

**Step (v).** For any closed point $u \in U$ one has $s_1|_{\mathcal{X}_u} = s_{1,u}$.

**Step (vi).** If $s_1$ is such as in the step (v), then the morphism

$$
\sigma = (s_1/s_0^n, pr_U) : \mathcal{X} \to \mathbb{A}^1 \times U
$$

is finite and surjective. To see this it suffices to check that the morphism $([s_0 : s_1], pr_U) : \mathcal{X} \to \mathbb{P}_U^1$ is finite surjective and to note that $[s_0 : s_1]^{-1}(\mathbb{A}^1) = \mathcal{X}$. As we already know
the morphism \([s_{0,u} : s_{1,u}] : \mathcal{X}_u \rightarrow \mathbb{P}^1_u\) is finite. Thus the morphism \([s_0 : s_1], pr_U\) is quasi-finite over a neighborhood of \(\mathbb{P}^1_u\). Since \(U\) is semi-local, hence any neighborhood of \(\mathbb{P}^1_u\) coincides with \(\mathbb{P}^1_U\). Since the morphism is projective, hence \([s_0 : s_1], pr_U\) is finite and surjective. Hence, so is the morphism \(\sigma\).

**We are ready now to check step by step all the statements of the Theorem.**

**The assertion (a).** Since the schemes \(\mathcal{X}'\) and \(\mathbb{A}^1 \times U\) are regular and the morphism \(\sigma\) is finite and surjective, the morphism \(\sigma\) is flat by a theorem of Grothendieck [2, Thm. 18.17].

So, to check that \(\sigma\) is étale at a closed point \(z \in \mathcal{Z}'\) it suffices to check that for the point \(u = q'_U(z)\) the morphism \(\sigma_u : \mathcal{X}_u' \rightarrow \mathbb{A}^1_u\) is étale at the point \(z\). The latter does hold by the step (iv), item (b). Whence \(\sigma\) is étale also at all the closed points of \(\mathcal{Z}'\). By the hypotheses of the Theorem the set of closed points of \(\Delta'(U)\) is contained in the set of the closed points of \(\mathcal{Z}'\). Whence \(\sigma\) is étale also at all the closed points of \(\Delta'(U)\). The schemes \(\Delta'(U)\) and \(\mathcal{Z}'\) are both semi-local. Thus, \(\sigma\) is étale in a neighborhood of \(\mathcal{Z}' \cup \Delta'(U)\).

**The assertion (b).** For any closed point \(u \in U\) and any point \(z \in \mathcal{Z}_u'\) the \((k(u))-\)algebra homomorphism \(\sigma_u^\ast : k(u)[t] \rightarrow k(u)[\mathcal{X}_u']\) is étale at the maximal ideal \(m_z\) of the \((k(u))-\)algebra \(k(u)[\mathcal{X}_u']\) and the composite map \(\sigma_u^\ast \circ k(u)[\mathcal{X}_u'] \rightarrow k(u)[\mathcal{X}_u']/m_z\) is an epimorphism. Thus, for any integer \(r > 0\) the \((k(u))-\)algebra homomorphism \(k(u)[t] \rightarrow k(u)[\mathcal{X}_u']/m_z^r\) is an epimorphism. The local ring \(\mathcal{O}_{\mathcal{Z}'_u,z}\) of the scheme \(\mathcal{Z}'_u\) at the point \(z\) of is of the form \(k(u)[\mathcal{X}_u']/m_z^s\) for an integer \(s\). Thus, the \((k(u))-\)algebra homomorphism

\[
k(u)[t] \xrightarrow{\sigma_u^\ast} k(u)[\mathcal{X}_u'] \rightarrow \mathcal{O}_{\mathcal{Z}'_u,z}
\]

is surjective. Since \(\sigma_u|_{\bigsqcup_{z \in \mathcal{Z}_u'}} = i_u\) and \(i_u\) is a closed embedding one concludes that the \((k(u))-\)algebra homomorphism

\[
k(u)[t] \rightarrow \prod_{z \in \mathcal{Z}_u'} \mathcal{O}_{\mathcal{Z}'_u,z} = \Gamma(\mathcal{Z}_u', \mathcal{O}_{\mathcal{Z}_u'})
\]

is surjective. Let \(u_1 = \bigsqcup \Spec(k(u))\) regarded as the closed sub-scheme of \(U\), where \(u_1\) runs over all closed points of \(U\). Then, for the scheme \(\mathcal{Z}_u' = u_1 \times_U \mathcal{Z}'\) the \((k[u])-\)algebra homomorphism

\[
k[u][t] \rightarrow \Gamma(\mathcal{Z}_u', \mathcal{O}_{\mathcal{Z}_u'})
\]

is surjective.

Since \((\mathcal{X}', f', \Delta')\) is a nice triple over \(U\), the \(\mathcal{O}\)-module \(\Gamma(\mathcal{Z}', \mathcal{O}_{\mathcal{Z}'})\) is finite. Thus, the \((k[u])-\)module \(\Gamma(\mathcal{Z}_u', \mathcal{O}_{\mathcal{Z}_u'})\) is finite. Now the surjectivity of the \((k[u])-\)algebra homomorphism \(\sigma_u|_{\mathcal{Z}_u'} : \mathcal{Z}_u' \rightarrow \mathbb{A}^1 \times U\) and the Nakayama lemma show that the \(\mathcal{O}\)-algebra homomorphism \(\sigma_u|_{\mathcal{Z}_u'} : \mathcal{Z}_u' \rightarrow \mathbb{A}^1 \times U\) is surjective. Thus, \(\sigma|_{\mathcal{Z}_u'} : \mathcal{Z}_u' \rightarrow \mathbb{A}^1 \times U\) is a closed embedding.

**The assertion (c).** The morphism \(\Delta'\) is a section of the structure morphism \(q'_U : \mathcal{X}' \rightarrow U\) and the morphism \(\sigma\) is a morphism of the \(U\)-schemes. Hence the composite morphism \(t_0 := \sigma \circ \Delta'\) is a section of the projection \(pr_U : \mathbb{A}^1 \times U \rightarrow U\). This section is defined by an element \(a \in \mathcal{O}\). There is another section \(t_1\) of the projection \(pr_U\) defined by the element \(1 - a \in \mathcal{O}\). Making an affine change of coordinates on \(\mathbb{A}^1_U\) we may and will
assume that \( t_0(U) = \{0\} \times U \) and \( t_1(U) = \{1\} \times U \). If the field \( k \) is infinite we can choose a non-zero \( \lambda \in k \) such that for \( t_1 := \lambda t_1 \) one has: \( \mathcal{D}_1^{new} \cap Z' = \emptyset \) and \( t_0^{new} := \lambda t_0 = t_0 \).

Since \( \mathcal{D}_1 \) and \( Z' \) are semi-local, to prove the assertion (e) it suffices to check that \( \mathcal{D}_1 \) and \( Z' \) have no common closed points. In the infinite field case this is checked just above. It remains to check the finite field case. Let \( z \in \mathcal{D}_1 \cap Z' \) be a common closed point. Then \( \sigma(z) \in \{1\} \times U \). Let \( u = q'_U(z) \). We already know that \( \sigma(z) \) is a closed embedding. Thus \( \deg[z : u] = \deg[\sigma(z) : u] = 1 \). The triple \( (X, f, \Delta) \) satisfies the condition \( 2_U \) from Definition \( \ref{def:goldmann} \). Thus, \( z = \Delta(u) \). In this case \( \sigma(z) \in \{0\} \times U \). But \( \sigma(z) \in \{1\} \times U \). This is a contradiction. Whence \( \mathcal{D}_1 \cap Z' = \emptyset \).

The assertion (c). The finite morphism \( \sigma \) is étale in a neighborhood of \( Z' \) by the item (b) of the Theorem. By the item (a) of the Theorem \( \sigma|_{Z'} \) is a closed embedding. Thus, the morphism \( \sigma^{-1}(\sigma(Z')) = \sigma(Z') \) of affine schemes is finite and there is an affine open sub-scheme \( V \) of the scheme \( \sigma^{-1}(\sigma(Z')) \) such that the morphism \( V \to \sigma(Z') \) is étale. Since \( \sigma|_{Z'} \) is a closed embedding there is a unique section \( \sigma \) of this morphism \( \sigma^{-1}(\sigma(Z')) \) with the image \( Z' \) and this image is contained in \( V \). By \( \ref{lem:goldmann} \) the scheme \( \sigma^{-1}(\sigma(Z')) = Z' \) is an isomorphism. By a similar reasoning the scheme \( \sigma^{-1}(\{0\} \times U) \) has the form \( \Delta'(U) \amalg D \). The triple \( (X, f, \Delta) \) is a special nice triple. Thus all the closed points of \( \Delta'(U) \) are closed points of \( Z' \) and \( Z' \cap Z'' = \emptyset \). Thus, \( \Delta'(U) \cap Z'' = \emptyset \).

The assertion (d). It remains to show that \( \mathcal{D} \cap Z' = \emptyset \). Recall that \( \sigma \) is étale in a neighborhood of \( Z' \cup \Delta'(U) \). It’s easy to check that \( \sigma|_{Z' \cup \Delta'(U)} \) is a closed embedding. Thus assuming as above one gets a disjoint union decomposition

\[
\sigma^{-1}(\sigma(Z' \cup \Delta'(U))) = (Z' \cup \Delta'(U)) \cup E
\]

for a closed subscheme \( E \) in \( X' \). It’s checked already that \( \Delta'(U) \amalg D = \emptyset \). Thus, \( \mathcal{D} \cap Z' = \emptyset \).

The assertion (f). Recall that \( X' \) is affine irreducible and regular. Thus by Auslander-Goldmann theorem every height one prime ideal is locally principal and every locally principal ideal \( I \subset \Gamma(X', \mathcal{O}_{X'}) \) is of the form \( I = q_1 a_1 q_2 a_2 \cdots q_m a_m \) with integers \( a_i \geq 0 \). Moreover, such a presentation of the ideal \( I \) is unique. Hence the principal ideal \( (f') \) has the form \( p_1 r_1 q_2 r_2 \cdots p_m r_m \) with integers \( r_i \geq 1 \), where \( p_i \)’s are height one prime ideals in \( k[X] := \Gamma(X', \mathcal{O}_{X'}) \). Let \( Z'_i \) be the closed subscheme in \( X' \) defined by the ideal \( p_i \). Since \( I \subset p_i \), hence \( Z'_i \) is a closed subscheme in \( Z' \).

Let \( q_i = \mathcal{O}[t] \cap p_i \). The morphism \( \sigma|_{Z'} : Z' \to A^1 \times U \) is a closed embedding by the item (a) of Theorem \( \ref{thm:auslander-goldmann} \). This yields that \( \sigma|_{Z'_i} : Z'_i \to A^1 \times U \) is a closed embedding too. Thus \( q_i \) is a height one prime ideal in \( \mathcal{O}[t] \). So, it is a principal prime ideal. Since \( Z' \) is finite over \( U \) the scheme \( Z'_i \) is finite over \( U \) too. Hence the principal prime ideal \( q_i \) is of the form \( (h_i) \) for a unique monic polinomial \( h_i \in \mathcal{O}[t] \).

Set \( h = h_1^{r_1} h_2^{r_2} \cdots h_m^{r_m} \). Clearly, \( h \in \text{Ker}[\mathcal{O}[t] \to k[X] \to k[x]/(f')] \). Since the map \( \mathcal{O}[t] \to k[x]/(f') \) is surjective, to prove the assertion (f) it suffices to check that the surjective \( \mathcal{O} \)-module homomorphism \( - \otimes \sigma^* : \mathcal{O}[t]/(h) \to k[X]/(f') \) is an isomorphism.

12
Consider a decreasing filtration of principal $\mathcal{O}[t]$-ideals

\[ \mathcal{O}[t] \supset (h_1) \supset ... \supset (h_i) \supset (h_i^2) \supset ... \supset (h_i^m) \supset ... \supset (h). \]

Since $\mathcal{X}$ is an affine scheme there is an element $g \in k[\mathcal{X}]$ such that $g|_{\mathcal{Z}} = 0$ and $g|_{\mathcal{Z}} = 1$. The $\mathcal{O}[t]$-algebra $k[\mathcal{X}]$ is flat. Hence $\mathcal{O}[t]$-algebra $k[\mathcal{X}]$ is flat. Thus taking the tensor product of the latter filtration with the $\mathcal{O}[t]$-algebra $k[\mathcal{X}]$ we get a decreasing filtration by principal $k[\mathcal{X}]$-ideals

\[ k[\mathcal{X}] \supset h_1 \cdot k[\mathcal{X}] \supset ... \supset h \cdot k[\mathcal{X}] . \]

The $\mathcal{O}[t]$-algebra map $\mathcal{O}[t] \to k[\mathcal{X}]$ induces the map of the filtered $\mathcal{O}[t]$-modules

\[ \mathcal{O}[t]/(h) \to k[\mathcal{X}] / h \cdot k[\mathcal{X}] . \]

It sufficient to check that the induced map on the adjoint graded $\mathcal{O}[t]$-modules is an isomorphism. Graded summands for the first filtration are isomorphic to one of the $\mathcal{O}[t]$-module $\mathcal{O}[t]/(h_i)$. Graded summands for the second filtration are isomorphic to one of the $\mathcal{O}[t]$-module $k[\mathcal{X}] / (p_i)$. Moreover the map between the corresponding graded summands is the map of the $\mathcal{O}[t]$-modules $\mathcal{O}[t](h_i) \to k[\mathcal{X}] / (p_i)$ induced by the scalar extension from $\mathcal{O}[t]$ to $k[\mathcal{X}]$. It remains to show that for any $i$ the map of the $\mathcal{O}$-modules $\mathcal{O}[t](h_i) \to k[\mathcal{X}] / (p_i)$ is an isomorphism. To show this note that the composite map

\[ \mathcal{O}[t] \xrightarrow{\sigma^*} k[\mathcal{X}] / (f') \to k[\mathcal{X}] / p_i \]

is an $\mathcal{O}[t]$-algebra epimorphism (already $\sigma^*$ is an epimorphism). The kernel of the epimorphism $\sigma^* \circ \sigma^*$ is the ideal $a_i = (h_i)$. Thus $\mathcal{O}[t](h_i) = k[\mathcal{X}] / p_i$.

Since $g \equiv 1$ mod($f'$) one has equalities $k[\mathcal{X}] / (f') = k[\mathcal{X}] / (f) \cdot k[\mathcal{X}]$, $k[\mathcal{X}] / p_i = k[\mathcal{X}] / (p_i)$. Hence $\mathcal{O}[t](h_i) = k[\mathcal{X}] / p_i = k[\mathcal{X}] / (p_i)$.

The assertion (f) is proved. Whence the Theorem. \hfill $\square$

6 Proof of Theorems 3.9

Let $k$ be a field. Let $U$ be as in Definition 3.1. Let $S'$ be an irreducible regular semi-local scheme over $k$ and $p : S' \to U$ be a $k$-morphism. Let $T' \hookrightarrow S'$ be a closed sub-scheme of $S'$ such that the restriction $p|_{T'} : T' \to U$ is an isomorphism. Let $\delta : U \to T'$ be the inverse to $p|_{T'}$. We will assume below that $\dim(T') < \dim(S')$, where $\dim$ is the Krull dimension. For any closed point $u \in U$ and any $U$-scheme $V$ let $V_u = u \times_U V$ be the fibre of the scheme $V$ over the point $u$. For a finite set $A$ denote $\#A$ the cardinality of $A$.

Lemma 6.1. If the field $k$ is finite and all the closed points of $S'$ have finite residue fields. Suppose that for any closed point $u \in U$ the scheme $S'_u$ is a semi-local Dedekind scheme. Then there exists a finite étale morphism $\rho : S'' \to S'$ (with an irreducible scheme $S''$) and a section $\delta' : T' \to S''$ of $\rho$ over $T'$ such that the following holds.
(1) for any closed point \( u \in U \) let \( u' \in T' \) be a unique point such that \( p(u') = u \), then the point \( \delta'(u') \in S''_u \) is the only \( k(u) \)-rational point of \( S''_u \).

(2) for any closed point \( u \in U \) and any integer \( d \geq 1 \) one has
\[
\sharp\{z \in S''_u | [k(z) : k(u)] = d\} \leq \sharp\{x \in \mathbb{A}^1_u | [k(x) : k(u)] = d\}
\]

If the field \( k \) is infinite, then set \( S'' = S' \), \( \rho = \text{id} \), and \( \delta' = i \).

Proof. The proof is given at the very end of the Appendix A.

\[ \square \]

Proof of Theorems 3.3. Since \((X, f, \Delta)\) is a special nice triple over \( U \), there is a finite surjective \( U \)-morphism \( \Pi : X \to \mathbb{A}^1_U \). Applying to the this nice triple and to the morphism \( \Pi \) the first part of the construction 4.2 we get the data \((Z, \gamma, S, T)\).

Let \( p = q_u|_S : S \to U \) and \( \delta = \Delta : U \to S \). Applying the lemma 6.1 to \( S' = S \), \( T' = T \) and \( \delta \) we get \( S'', \rho : S'' \to S \), and \( \delta' : T \to S'' \) subjecting to the conditions (1) and (2) from the lemma 6.1. Recall that \( \rho : S'' \to S \) is a finite étale morphism (with an irreducible scheme \( S'' \)) and \( \delta' \) is a section of \( \rho \) over \( T \subseteq S \).

Applying the second part of the construction 4.2 and also the proposition 4.3 to the special nice triple \((X, f, \Delta)\), the finite surjective morphism \( \Pi \) and to the finite étale morphism \( \rho \) and to its section \( \delta \) over \( T \) we get

(i) the nice triple \((X', f', \Delta')\) over \( U \);
(ii) the morphism \( \theta : (X', f', \Delta') \to (X, f, \Delta) \) between the special nice triples;
(iii) the equality \( f' = (\theta)^*(f) \);
(iv) the vanishing locus \( Z' \) of \( f' \) on \( X' \) such that its set of closed points is contained in the set of closed points of the subscheme \( S' \).

The properties (1) and (2) of the \( U \)-scheme \( S'' \) show that the conditions \((1^*)\) and \((2^*)\) from the second assertion of the theorem 3.3 are full filled for the closed sub-scheme \( Z'' \) of \( X'' \) defined by \( \{f'' = 0\} \). That follows from the property (iv) mentioned just above.

The property (ii) shows that the triple \((X'', f'', \Delta'') \to (X', f', \Delta') \) is a special nice triple. This completes the proof of the theorem.

\[ \square \]

7 Theorems 7.1 and proof of Theorem 1.2

Theorem 7.1 is a purely geometric one. If the base field \( k \) is finite, say, of two elements, if the closed point of \( U \) is \( k \)-rational and the scheme \( Z' \) below is such that its closed fibre \( Z'_u \) contains three \( k \)-rational points, then there are no closed embedding \( Z' \) into \( \mathbb{A}^1 \times U \). So, one of the main problem in the proof of the next theorem is to find such an \( X \), a morphism \( q_X \) and a function \( f' \) to overcome the mentioned difficulties.

**Theorem 7.1.** Let \( X \) be an affine \( k \)-smooth irreducible \( k \)-variety, and let \( x_1, x_2, \ldots, x_n \) be closed points in \( X \). Let \( U = \text{Spec}(\mathcal{O}_X(x_1, x_2, \ldots, x_n)) \). Given a non-zero function \( f \in k[X] \)
vanishing at each point \( x_i \), there is a diagram of the form

\[
\begin{array}{c}
\mathbb{A}^1 \times U \\
\downarrow^{\sigma} \quad \Delta \quad \downarrow^{q_X} \\
\Delta \quad \downarrow^{\text{can}} \\
U
\end{array}
\]

with an irreducible affine scheme \( \Delta \), a smooth morphism \( q_U \), a finite surjective \( U \)-morphism \( \sigma \) and an essentially smooth morphism \( q_X \), and a function \( f' \in q_X^*(f)k[\Delta] \), which enjoys the following properties:

(a) if \( \Delta' \) is the closed subscheme of \( \Delta \) defined by the principal ideal \( (f') \), the morphism \( \sigma|_{\Delta'} : \Delta' \to \mathbb{A}^1 \times U \) is a closed embedding and the morphism \( q_U|_{\Delta'} : \Delta' \to U \) is finite;

(a') \( q_U \circ \Delta = \text{id}_U \) and \( q_X \circ \Delta = \text{can and} \sigma \circ \Delta = i_0 \)

(the first equality shows that \( \Delta(U) \) is a closed subscheme in \( \Delta \));

(b) \( \sigma \) is \( \text{étale} \) in a neighborhood of \( \Delta' \cup \Delta(U) \);

(c) \( \sigma^{-1}(\sigma(\Delta')) = \Delta' \coprod \Delta'' \) scheme theoretically for some closed subscheme \( \Delta'' \) and \( \Delta'' \cap \Delta(U) = \emptyset \);

(d) \( \mathcal{D}_0 := \sigma^{-1}(\emptyset) \times U = \Delta(U) \coprod \mathcal{D}_0' \) scheme theoretically for some closed subscheme \( \mathcal{D}_0' \) and \( \mathcal{D}_0' \cap \Delta' = \emptyset \);

(e) for \( \mathcal{D}_1 := \sigma^{-1}(\{1\} \times U) \) one has \( \mathcal{D}_1 \cap \Delta' = \emptyset \).

Proof of Theorem \( \text{[7, I]} \) By Proposition \( \text{[3.6]} \) one can shrink \( \Delta \) such that \( x_1, x_2, \ldots, x_n \) are still in \( \Delta \) and \( \Delta \) is affine, and then to construct a special nice triple \( (q_U : \Delta \to U, \Delta, f) \) over \( U \) and an essentially smooth morphism \( q_X : \Delta \to \Delta \) such that \( q_X \circ \Delta = \text{can,} f = q_X^*(f) \) and the set of closed points of \( \Delta(U) \) is contained in the set of closed points of \( \{ f = 0 \} \).

By Theorem \( \text{[3.7]} \) there exists a morphism \( \theta : (\Delta', f', \Delta') \to (\Delta, f, \Delta) \) such that the triple \( (\Delta', f', \Delta') \) is a special nice triple over \( U \) subject to the conditions (1*) and (2*) from Definition \( \text{[3.7]} \).

The triple \( (\Delta', f', \Delta') \) is a special nice triple over \( U \) subject to the conditions (1*) and (2*) from Definition \( \text{[3.7]} \). Thus by Theorem \( \text{[3.8]} \) there is a finite surjective morphism \( \mathbb{A}^1 \times U \to \Delta' \) of the \( U \)-schemes satisfying the conditions (a) to (f) from that Theorem. Hence one has a diagram of the form

\[
\begin{array}{c}
\mathbb{A}^1 \times U \\
\downarrow^{\sigma} \quad \Delta' \quad \downarrow^{\text{can}} \\
\Delta' \quad \downarrow^{\text{can}} \\
U
\end{array}
\]
with the irreducible scheme $X'$, the smooth morphism $q_U \circ \theta$, the finite surjective morphism $\sigma$ and the essentially smooth morphism $q_X \circ \theta$ and with the function $f' \in (q_X \circ \theta)^*(f) k[X']$, which after identifying notation enjoy the properties (a) to (f) from Theorem 7.1. Whence the Theorem 7.1. 

To formulate a first consequence of the theorem 7.1 (see Corollary 7.2), note that using the items (b) and (c) of Theorem 7.1 one can find an element $g \in I(X')$ such that

1. $(f') + (g) = \Gamma(X, \mathcal{O}_X)$,
2. $Ker((\Delta)^*) + (g) = \Gamma(X, \mathcal{O}_X)$,
3. $\sigma_g = \sigma|_{X_g} : X_g \to \mathbb{A}^1_U$ is étale.

Corollary 7.2 (Corollary of Theorem 7.1). The function $f'$ from Theorem 7.1, the polynomial $h$ from the item (f) of that Theorem, the morphism $\sigma : X \to \mathbb{A}^1_U$, and the function $g \in \Gamma(X, \mathcal{O}_X)$ defined just above enjoy the following properties:

(i) the morphism $\sigma_g = \sigma|_{X_g} : X_g \to \mathbb{A}^1 \times U$ is étale,

(ii) data $(\mathcal{O}[t], \sigma'_{\mathcal{O}} : \mathcal{O}[t] \to \Gamma(X, \mathcal{O}_X)_g, h)$ satisfies the hypotheses of $\text{C-T/Q. Prop.2.6}$, i.e. $\Gamma(X, \mathcal{O}_X)_g$ is a finitely generated $\mathcal{O}[t]$-algebra, the element $(\sigma_g)^*(h)$ is not a zero divisor in $\Gamma(X, \mathcal{O}_X)_g$ and $\mathcal{O}[t]/(h) = \Gamma(X, \mathcal{O}_X)_g/h\Gamma(X, \mathcal{O}_X)_g$,

(iii) $(\Delta(U) \cup Z') \subset X_g$ and $\sigma_g \circ \Delta = i_0 : U \to \mathbb{A}^1 \times U$,

(iv) $X_{gh} \subset X_g f' \subset X_f' \subset X_{\Delta}(t)$

(v) $\mathcal{O}[t]/(h) = \Gamma(X, \mathcal{O}_X)/(f')$ and $h\Gamma(X, \mathcal{O}_X) = (f') \cap I(X')$ and $(f') + I(X') = \Gamma(X, \mathcal{O}_X)$.

Proof of Corollary 7.2. We will use notation from Theorem 7.1. Since $X$ is a regular affine irreducible scheme and $\sigma : X \to \mathbb{A}^1_U$ is finite surjective the induced $\mathcal{O}$-algebra homomorphism $\sigma^* : \mathcal{O}[t] \to \Gamma(X, \mathcal{O}_X)$ is a monomorphism. We will regard below the $\mathcal{O}$-algebra $\mathcal{O}[t]$ as a subalgebra via $\sigma^*$.

The assertions (i) and (iii) of the Corollary hold by our choice of $g$. The assertion (iv) holds, since $\sigma^*(h)$ is in the principal ideal $(f')$ (use the properties (a) and (f) from Theorem 7.1). It remains to prove the assertion (ii). The morphism $\sigma$ is finite. Hence the $\mathcal{O}[t]$-algebra $\Gamma(X, \mathcal{O}_X)_g$ is finitely generated. The scheme $X$ is regular and irreducible. Thus, the ring $\Gamma(X, \mathcal{O}_X)$ is a domain. The homomorphism $\sigma^*$ is injective. Hence, the element $h$ is not zero and is a non zero divisor in $\Gamma(X, \mathcal{O}_X)_g$.

It remains to check that $\mathcal{O}[t]/(h) = \Gamma(X, \mathcal{O}_X)_g/h\Gamma(X, \mathcal{O}_X)_g$. Firstly, by the choice of $h$ and by the item (a) of Theorem 7.1 one has $\mathcal{O}[t]/(h) = \Gamma(X, \mathcal{O}_X)/(f')$. Secondly, by the property (1) of the element $g$ one has $\Gamma(X, \mathcal{O}_X)/(f') = \Gamma(X, \mathcal{O}_X)_g/f'\Gamma(X, \mathcal{O}_X)_g$. Finally, by the items (c) and (a) of Theorem 7.1 one has

$$\Gamma(X, \mathcal{O}_X)/(f') \times \Gamma(X, \mathcal{O}_X)/I(Z') = \Gamma(X, \mathcal{O}_X)/(h).$$

(10)

Localizing both sides of (10) in $g$ one gets an equality

$$\Gamma(X, \mathcal{O}_X)_g/f'\Gamma(X, \mathcal{O}_X)_g = \Gamma(X, \mathcal{O}_X)_g/h\Gamma(X, \mathcal{O}_X)_g,$$

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hence \( \mathcal{O}[t]/(h) = \Gamma(X, \mathcal{O}_X)/(f') = \Gamma(X, \mathcal{O}_X^G/f') \Gamma(X, \mathcal{O}_X)_g = \Gamma(X, \mathcal{O}_X^G/h \Gamma(X, \mathcal{O}_X)_g) \). Whence the Corollary.

**Remark 7.3.** The item (ii) of this corollary shows that the cartesian square

\[
\begin{array}{ccc}
\mathcal{X}_{gh} & \xrightarrow{\text{inc}} & \mathcal{X}_g \\
\downarrow{\sigma_{gh}} & & \downarrow{\sigma_g} \\
(A^1 \times U)_h & \xrightarrow{\text{inc}} & A^1 \times U
\end{array}
\]  

(11)
can be used to glue principal \( G \)-bundles for a reductive \( U \)-group scheme \( G \).

Set \( Y := \mathcal{X}_g \), \( p_X = q_X : Y \to X \), \( p_U = q_U : Y \to U \), \( \tau = \sigma_g \), \( \tau_h = \sigma_{gh} \), \( \delta = \Delta \) and note that \( pr_U \circ \tau = p_U \). Take the monic polinomial \( h \in \mathcal{O}[t] \) from the item (f) of Theorem 7.1.

With this replacement of notation and with the element \( h \) we arrive to the following

**Theorem 7.4** (stronger than Theorem 7.2). *Let the field \( k \), the variety \( X \), its closed points \( x_1, x_2, \ldots, x_n \), the semi-local ring \( \mathcal{O} = \mathcal{O}_{X, \{x_1, x_2, \ldots, x_n\}} \) the semi-local scheme \( U = \text{Spec}(\mathcal{O}) \), the function \( f \in k[X] \) be the same as in Theorem 7.1. Then the monic polinomial \( h \in \mathcal{O}[t] \), the commutative diagram of schemes with the irreducible affine \( U \)-smooth \( \mathcal{Y} \)

\[
\begin{array}{ccc}
(A^1 \times U)_h & \xrightarrow{\tau_h} & Y_h \ := \ Y_{\tau^*(h)} \mid_{Y_h} \\
\downarrow{\text{inc}} & & \downarrow{\text{inc}} \\
(A^1 \times U) & \xrightarrow{\tau} & Y \end{array}
\]  

(12)
the morphism \( \delta : U \to Y \) subject to the following conditions:

(i) the left hand side square is an elementary distinguished square in the category of affine \( U \)-smooth schemes in the sense of \([\text{MV}, \text{Defn.3.1.3}]\);

(ii) \( p_X \circ \delta = \text{can} : U \to X \), where \( \text{can} \) is the canonical morphism;

(iii) \( \tau \circ \delta = i_0 : U \to A^1 \times U \) is the zero section of the projection \( pr_U : A^1 \times U \to U \);

(iv) \( h(1) \in \mathcal{O}[t] \) is a unit.

**Proof.** The items (i) and (iv) of the Corollary 7.2 show that the morphisms \( \delta : U \to Y \) and \( (p_X)\mid_{Y_h} : Y_h \to X_f \) are well defined. The items (i), (ii) of that Corollary show that the left hand side square in the diagram (12) is an elementary distinguished square in the category of smooth \( U \)-schemes in the sense of \([\text{MV}, \text{Defn.3.1.3}]\). The equalities \( p_X \circ \delta = \text{can} \) and \( \tau \circ \delta = i_0 \) are clear. The property (iv) follows from the items (e), (f) and (a) of Theorem 7.1.

**Remark 7.5.** The latter commutative diagram gives rise to a pointed motivic space morphism

\[
\alpha : \mathbb{P}^1 \times U/\infty \times U \to A^1 \times U/(A^1 \times U)_h \xrightarrow{(p_X)\mid_{Y_h}} X/X_f
\]
such that \( \alpha|_{0 \times U} : 0 \times U \to X/X_f \) coincides with the morphism \( U \xrightarrow{\text{can}} X \to X/X_f \).
8 A moving lemma

Let $k$ be a field. Particularly, $k$ can be a finite field. We prove the following useful geometric theorem

**Theorem 8.1** (A moving lemma). Let $X$ be a $k$-smooth quasi-projective irreducible $k$-variety, and let $x_1, x_2, \ldots, x_n$ be closed points in $X$. Let $U = \text{Spec}(O_{X,(x_1,x_2,\ldots,x_n)})$. Let $Z$ be a closed subset in $X$. Let $U \xrightarrow{\text{can}} X$ be the inclusion and $X \xrightarrow{p} X/(X-Z)$ be the factor morphism. Let $* = (X-Z)/(X-Z) \in X/(X-Z)$ be the distinguished point of $X/(X-Z)$. Given a closed subset $Z \subset X$ there is a Nisnevich sheaf morphism

$$\Phi_t : \mathbb{A}^1 \times U \to X/(X-Z)$$

such that $\Phi_0 : U \to X/(X-Z)$ is the composite morphism $U \xrightarrow{\text{can}} X \xrightarrow{p} X/(X-Z)$ and $\Phi_1 : U \to X/(X-Z)$ takes $U$ to the distinguished point $*$ in $X/(X-Z)$.

**Proof.** Take a function $f \in k[X]$ such that $f$ vanishes as on $Z$, so at all the points $x_1, x_2, \ldots, x_n$. Consider the commutative diagram (12). Since the left hand side square is a distinguished elementary square, hence the morphism $\sigma : Y/Y_h \to \mathbb{A}^1_U/(\mathbb{A}^1 \times U)_h$ of Nisnevich sheaves is an isomorphism. Thus there is a composite morphism of motivic spaces of the form

$$\Phi_t : \mathbb{A}^1_U \to \mathbb{A}^1_U/(\mathbb{A}^1 \times U)_h \xrightarrow{\sigma^{-1}} Y/Y_h \xrightarrow{p_X} X/X_f,$$

Let $i_0 : 0 \times U \to \mathbb{A}^1_U$ be the natural morphism. By the properties (a') and (d) from Theorem 7.1 the morphism $\Phi_0 := \Phi \circ i_0$ equals to the one

$$U \xrightarrow{\text{can}} X \xrightarrow{p} X/X_f,$$

where $p : X \to X/X_f$ is the canonical morphisms. By the item (e) of Theorem 7.1 the morphism $\Phi_1 := \Phi \circ i_1 : U \to X/X_f$ is the constant morphism to the distinguished point $*$ of $X/(X-Z)$. \qed

**Theorem 8.2** (An extended moving lemma). Let $X$ be a $k$-smooth quasi-projective irreducible $k$-variety, and let $x_1, x_2, \ldots, x_n$ be closed points in $X$. Let $U = \text{Spec}(O_{X,(x_1,x_2,\ldots,x_n)})$. Let $c > 0$ be an integer. Let $Z$ be a closed subset in $X$ of pure codimension $c$ in $X$. Then there is a closed subset $Z^{new}$ in $X$ containing $Z$ and of pure codimension $c-1$ and a morphism of pointed Nisnevich sheaves

$$\Phi_t : \mathbb{A}^1 \times (U/(U-Z^{new})) \to X/(X-Z)$$

such that $\Phi_0 : U/(U-Z^{new}) \to X/(X-Z)$ is the composite morphism $U/(U-Z^{new}) \xrightarrow{\text{can}} X/(X-Z^{new}) \xrightarrow{p} X/(X-Z)$; and

$\Phi_1 : U/(U-Z^{new}) \to X/(X-Z)$

takes $U/(U-Z^{new})$ to the distinguished point $*$ in $X/(X-Z)$. 

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Proof. Take a function \( f \in k[X] \) such that \( f \) vanishes as on \( Z \), so that at all the points \( x_1, x_2, \ldots, x_n \). Consider the commutative diagram \((12)\). Recall [PSV, Sect.6] that additionally there is a commutative diagram of schemes

\[
\begin{array}{ccc}
Y & \xrightarrow{\rho} & X \\
\downarrow{p_U} & & \downarrow{q_X} \\
X & \xrightarrow{q_V} & X \\
\downarrow{r_U} & & \downarrow{r_X} \\
S & & S
\end{array}
\]

with \( X = U \times_s X \) such that \( r_X \) is an almost elementary fibration such that \( r_X|_{\{f=0\}} : \{f=0\} \to S \) is finite, \( q_X \) and \( q_V \) are the projections, and \( rho \) is étale.

Set \( r_U = r_X \circ \text{can} \), \( p_U = q_U \circ \rho \), \( p_X = q_X \circ \rho \). Then \( p_U \) is a smooth morphism of relative dimension 1 with an irreducible \( Y \). Set \( Z^{\text{new}} := r_X^{-1}(r_X(Z)) \). Since \( r_X|_{\{f=0\}} : \{f=0\} \to S \) is finite and \( Z \) is in \( \{f=0\} \), hence \( Z \) is finite over \( S \) and \( r_X(Z) \) is equi-dimensional. Since \( r_X \) is an almost elementary fibration, hence \( Z^{\text{new}} \) is equi-dimensional of pure codimension \( c - 1 \) in \( X \). Clearly, \( Z \) is in \( Z^{\text{new}} \). Set \( Z = p_X^{-1}(Z) \). Then in the diagram the left hand side square

\[
\begin{array}{ccc}
(A^1 \times U) & \xrightarrow{\tau(Z)} & Y - Z \\
\downarrow{\text{inc}} & & \downarrow{\text{inc}} \\
(A^1 \times U) & \xrightarrow{\tau} & Y \\
\downarrow{p_X} & & \downarrow{p_X} \\
X & & X
\end{array}
\]

is an elementary distinguished square. Hence \( \tau \) induces a Nisnevich sheaf isomorphism \( \tau : Y/(Y - Z) \to (A^1 \times U)/((A^1 \times U) - \tau(Z)) \). Note that for \( Z^{\text{new}}_U := \text{can}^{-1}(Z^{\text{new}}) \) one has \( p_U^{-1}(Z^{\text{new}}_U) = p_X^{-1}(Z^{\text{new}}) \) and \( \tau^{-1}(A^1 \times Z^{\text{new}}_U) = p_U^{-1}(Z^{\text{new}}) \). Let

\[
\Phi_1 : A^1 \times (U/U - Z^{\text{new}}_U) \xrightarrow{\Pi} (A^1 \times U)/((A^1 \times U) - \tau(Z)) \xrightarrow{\tau^{-1}} Y/(Y - Z) \xrightarrow{p_X} X/(X - Z)
\]

be the composite morphism.

It is straightforward to check that \( \Phi_0 : U/(U - Z^{\text{new}}) \to X/(X - Z) \) is the composite morphism \( U/(U - Z^{\text{new}}) \xrightarrow{\text{can}} X/(X - Z^{\text{new}}) \xrightarrow{p} X/(X - Z) \). The property (iv) of the polynomial \( h \) from Theorem 7.4 yields that the morphism \( \Phi_1 : U/(U - Z^{\text{new}}) \to X/(X - Z) \) takes \( U/(U - Z^{\text{new}}) \) to the distinguished point \( * \) in \( X/(X - Z) \). \( \square \)

9 Coisin complex and strict homotopy invariance

Let \( k \) be a field. Let \( A : \text{SmOp}/k \to Gr - Ab \) is a cohomology theory on the category \( \text{SmOp}/k \) in the sense of [PS] Sect. 1]. Let \( \mathcal{O} \) be the semi-local ring of finitely many closed points on a \( k \)-smooth irreducible affine \( k \)-variety \( X \), \( d = \text{dim}X \). Let \( U = \text{Spec}(\mathcal{O}) \). The main result of the present section and of the paper states that the Cousin complex of \( U \) associated with the theory \( A \) is exact. A relative version of that result is proved too.
Theorem 9.1. For any integer \( n \) the Cousin complex

\[
0 \to A^n(U) \to A^n(\eta) \xrightarrow{\partial} \bigoplus_{x \in U(1)} A_x^{n+1}(U) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \bigoplus_{x \in U(0)} A_x^{n+d}(U) \to 0
\]

is exact. If \( d = 1 \) and \( O \) is local, then the sequence \( 0 \to A^n(U) \to A^n(\eta) \xrightarrow{\partial} A_x^{n+1}(U) \to 0 \) is exact.

Proof. It follows in the standard manner from Theorem 5.2 and the definition of a cohomology theory on \( SmOp/k \).

Corollary 9.2. The Zariski sheaf \( A^n_{Zar} \) on \( X \), associated with the presheaf \( W \mapsto A^n(W) \) has a flasque resolution of the form

\[
0 \to A^n_{Zar} \to \eta_*(A^n(\eta)) \xrightarrow{\partial} \bigoplus_{x \in X(1)} (i_x)_*(A_x^{n+1}(X)) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \bigoplus_{x \in X(n)} (i_x)_*(A_x^{n+d}(X)) \to 0.
\]

Lemma 9.3. For any field \( K \) containing \( k \) the sequence

\[
0 \to A^n(K) \xrightarrow{i} A^n(K(t)) \xrightarrow{\partial} \bigoplus_{x \in A_k^n} A_x^{n+1}(A_K^1) \to 0
\]

is exact.

Proof of the lemma. By the definition \( \partial = \Sigma \partial_x \), where the sum runs over all the closed points of \( A_k^n \). Since \( A^n(K) = A^n(A_K^1) \), hence the composite map \( \partial \circ i \) is zero. Particularly, \( \partial_x \circ i = 0 \). Hence \( \text{Im}(i) \) is in \( \text{Ker}(\partial_x) \). By Theorem 9.1 applied to \( \text{Spec}(O_{A_k^n}) \) one has \( \text{Im}(i) \) is in \( A^n(O_{A_k^n}) \). Taking the origin of coordinates as the point \( x \) and consider the composition map \( i^*_0 \circ i : A^n(K) \to A^n(K) \), where \( i^*_0 : A^n(\text{Spec}(O_{A_k^n})) \to A^n(\text{Spec}(K)) \) is the pull-back map. Since \( i^*_0 \circ i = i \circ \partial \), \( i \circ \partial \) is the identity, hence \( i \) is injective (for any integer \( n \)). The sequence

\[
A^n(A_K^1) \xrightarrow{i} A^n(K(t)) \xrightarrow{\partial} \bigoplus_{x \in A_k^n} A_x^{n+1}(A_K^1) \to A^n(A_K^1) \xrightarrow{i} A^{n+1}(K(t))
\]

is a part of the long exact sequence. The maps \( i \) are injective. Thus the maps \( \partial \) are surjective for any integer \( n \).

We need in the following modification of Theorem 9.1.

Theorem 9.4. Under the notation and hypotheses of Theorem 9.1, let \( C \) be a \( k \)-smooth absolutely irreducible curve. Let \( x \in X \) be a point, \( c \in C \times_k k(x) \) be a closed point and \( \eta_x \in C \times_k k(x) \) be the generic point. Then the cousin complex

\[
0 \to A^n(\mathcal{O}_{C \times X,c}) \xrightarrow{i^*} A^n(\mathcal{O}_{C \times X,\eta_x}) \xrightarrow{\partial} A_x^{n+1}(C \times X) \to 0
\]

is exact. Particularly, \( A^n(\mathcal{O}_{C \times X,c}) = \text{Ker}(\partial) \) and there is a well-defined map \( c^* : \text{Ker}(\partial) = A^n(\mathcal{O}_{C \times X,c}) \to A^n(c) \).
Proof. The localization sequence
\[ A^n(\mathcal{O}_{C \times X,c}) \xrightarrow{j^*} A^n(\mathcal{O}_{C \times X,n_x}) \xrightarrow{\partial} A^{n+1}(C \times X) \rightarrow A^{n+1}(\mathcal{O}_{C \times X,c}) \xrightarrow{j^*} A^{n+1}(\mathcal{O}_{C \times X,n_x}) \]
is exact. Both maps \( j^* \) are injective by Theorem 9.1. Thus the map \( \partial \) is surjective. \( \Box \)

**Corollary 9.5.** [of Theorem 9.4] Under the notation and hypotheses of Theorem 9.1 let \( x \in X \) be a point and \( n_x \in \mathbb{A}_k^{1(x)} \) be the generic point. Then the complex
\[ 0 \rightarrow A^n_x(X) \xrightarrow{j^* \circ p^*} A^n(\mathcal{O}_{\mathbb{A}^1 \times X,n_x}) \xrightarrow{\partial} \oplus_{c \in \mathbb{A}_k^{1(x)}} A^{n+1}_x(\mathbb{A}^1 \times X) \rightarrow 0 \]
is exact. Both maps \( j^* \) are injective by Theorem 9.1. Thus the map \( \partial \) is surjective for any integer \( n \). Whence the lemma.

**Proof of the corollary.** By the definition \( \partial = \Sigma \partial_c \), where the sum runs over all the closed points of \( \mathbb{A}_k^{1(x)} \). The composite map \( \partial \circ i \) is zero. Particularly, \( \partial_c \circ i = 0 \). Hence \( \text{Im}(j^* \circ p^*) \) is in \( \text{Ker}(\partial_c) \).

By Theorem 9.3 there is a well-defined pull-back map \( c^* : \text{Ker}(\partial_c) = A^n(\mathcal{O}_{C \times X,c}) \rightarrow A^n(c) \). Taking \( c \) to be the origing of coordinates on \( \mathbb{A}_k^{1(x)} \), we see that \( i_0^* \circ j^* \circ p^* = id^* = id_{A^2}(X) \). Hence the map \( j^* \circ p^* \) is injective. The map \( p^* : A^n_x(X) \rightarrow A^n_{\mathbb{A}_k^{1(x)}}(\mathbb{A}^1 \times X) \) is an isomorphism by the homotopy invariance of the theory \( A \). Thus the map \( j^* \) is injective. The sequence
\[ A^n_{\mathbb{A}_k^{1(x)}}(\mathbb{A}^1 \times X) \xrightarrow{j^*} A^n(\mathcal{O}_{\mathbb{A}^1 \times X,n_x}) \xrightarrow{\partial} \oplus_{c \in \mathbb{A}_k^{1(x)}} A^{n+1}_x(\mathbb{A}^1 \times X) \rightarrow A^{n+1}_{\mathbb{A}_k^{1(x)}}(\mathbb{A}^1 \times X) \xrightarrow{j^*} A^{n+1}(\mathcal{O}_{\mathbb{A}^1 \times X,n_x}) \]
is a part of the long exact sequence. The maps \( j^* \) are injective. Thus the maps \( \partial \) are surjective for any integer \( n \). Whence the lemma. \( \Box \)

**10 Strict homotopy invariance of sheaves \( \mathcal{A}^n_{Nis} \)**

For basic definitions used in this section see [PS, Sect.1]. We will work below with \( \mathbb{Z} \)-graded cohomology theories on the category \( SmOp/k \) and will suppose that the boundary maps \( \partial_{(X,U)} \) have degree +1.

**Theorem 10.1.** Let \( k \) be a field and let \( A : SmOp/k \rightarrow Gr - Ab \) be a cohomology theory on the category \( SmOp/k \) in the sense of [PS, Sect.1]. Let \( \mathcal{A}^n_{zar} \) be the Zariski sheaf associated with the presheaf \( W \mapsto A^n(W) \). Then \( \mathcal{A}^n_{zar} \) is homotopy invariant and even it is strictly homotopy invariant on \( (Sm/k)_{zar} \).

**Proof.** Let \( X \) be a \( k \)-smooth variety. Let \( p : \mathbb{A}^1 \times X \rightarrow X \) be the projection. The morphism \( p \) induces the pull-back morphisms of the Cousin complexes \( p^* : \text{Cous}(X) \rightarrow \text{Cous}(\mathbb{A}^1 \times X) \). By Corollary 9.3 that morphism is a quasi-isomorphism of complexes of abelian groups. By Corollary 9.2 complexes \( \text{Cous}(X) \) and \( \text{Cous}(\mathbb{A}^1 \times X) \) computes the Zarisky cohomology of the sheaf \( \mathcal{A}^n \) on \( X \) and on \( \mathbb{A}^1 \times X \) respectively. And the the pull-back morphisms of the Cousin complexes \( p^* \) induces the pull-back map
\[ p^* : H^n_{zar}(X, \mathcal{A}^n_{zar}) \rightarrow H^n_{zar}(\mathbb{A}^1 \times X, \mathcal{A}^n_{zar}). \]
Hence the latter map is an isomorphism. Whence the theorem. \( \Box \)
Theorem 10.2. Let $k$ be a field and let $A : SmOp/k \to Gr–Ab$ be a cohomology theory on the category $SmOp/k$ in the sense of [PS, Sect.1]. Let $\mathcal{A}^n_{Nis}$ be the Nisnevich sheaf associated with the presheaf $W \mapsto A^n(W)$. Then for any $X \in Sm/k$ one has $\mathcal{A}^n_{Zar}(X) = \mathcal{A}^n_{Nis}(X)$, $H^p_{Zar}(X, \mathcal{A}^n_{Zar}) = H^p_{Nis}(X, \mathcal{A}^n_{Nis})$.

Particularly, the Nisnevich sheaf $\mathcal{A}^n_{Nis}$ is strictly homotopy invariant on $(Sm/k)_{Nis}$.

Proof. It is derived in a standard manner from Corollary 9.2 and Theorem 10.1.

11 Appendix A: proof of Lemma 6.1

Let $k$ be a finite field. Let $\emptyset$ be the semi-local ring of finitely many closed points on a $k$-smooth irreducible affine $k$-variety $X$. Set $U = \text{Spec} \emptyset$. Let $u \subset U$ be the set of all closed points in $U$. For a point $u \in u$ let $k(u)$ be its residue field.

Notation 11.1. Let $k$ be the finite field of characteristic $p$, $k'/k$ be a finite field extensions. Let $c = \sharp(k)$ (the cardinality of $k$). For a positive integer $r$ let $k^{q}(r)$ be a unique field extension of the degree $r$ of the field $k'$. Let $A^1_k(r)$ be the set of all degree $r$ points on the affine line $A^1_k$. Let $\text{Irr}(r)$ be the number of the degree $r$ points on $A^1_k$.

Lemma 11.2. Let $k$ be the finite field of characteristic $p$, $c = \sharp(k)$, $k'/k$ be a finite field extension of degree $d$. Let $q \in \mathbb{N}$ be a prime which is co-prime as to the characteristic $p$ of the field $k$, so to the integer $d$. Then

1. $\text{Irr}(q) = (c^q - c)/q$;
2. the $k$-algebra $k' \otimes_k k(q)$ is the field $k^{q}(q)$;
3. $\text{Irr}(dq) \geq \text{Irr}(q)$;
4. $\text{Irr}(dq) \geq (c^q - c)/q \gg 0$ for $q \gg 0$.

Proof. The assertions (1) and (2) are clear. The assertion (3) it is equivalent to the inequality $\varphi(c^{dq} - 1)/dq \geq \varphi(c^q - 1)/q$. The latter is equivalent to the one $\varphi(c^{dq} - 1)/d \geq \varphi(c^q - 1)$.

The norm map $N : k(dq)^x \to k(q)^x$ is surjective. Both groups are the cyclic groups of orders $c^q - 1$ and $c^d - 1$ respectively. Thus for any $c^q - 1$-th primitive root of unity $\xi \in k(q)^x$ there is a $c^d - 1$-th primitive root of unity $\zeta \in k(dq)^x$ such that $\text{Norm}(\zeta) = \xi$.

On the other hand, if $N(\zeta) = \xi$ and $\sigma \in \text{Gal}(k(dq)/k(q))$, then $N(\zeta^\sigma) = \xi$. Thus $\varphi(c^{dq} - 1)/d \geq \varphi(c^q - 1)$. The assertion (3) is proved.

The assertions (1) and (3) yield the assertion (4).

Notation 11.3. For any étale $k$-scheme $W$ set $d(W) = \max\{d_{q,k}(v) | v \in W\}$.

Lemma 11.4. Let $Y_u$ be an étale $k$-scheme. For any positive integer $d$ let $Y_u(d) \subseteq Y_u$ be the subset consisting of points $v \in Y_u$ such that $d_{q,k}(v) = d$. For any prime $q \gg 0$ the following holds:

1. if $v \in Y_u$, then $k(v) \otimes_k k(q)$ is the field $k(v)(q)$ of the degree $q$ over $k(v)$;
2. $k[Y_u] \otimes_k k(q) = \left( \prod_{v \in Y_u} k(v) \right) \otimes_k k(q) = \prod_{v \in Y_u} k(v)(q)$;
3. there is a surjective $k$-algebra homomorphism

$$\alpha : k[t] \to k[Y_u] \otimes_k k(q)$$

(15)
Proof. The first assertion follows from the lemma \[\text{Lemma 11.2}(2)\]. The second assertion follows from the first one. Prove now the third assertion. The $k$-algebra homomorphism

$$
k[t] \rightarrow \prod_{d=1}^{d(Y_u)} \left( \prod_{x \in A_t^1(qd)} k[t]/(m_x) \right) = \prod_{d=1}^{d(Y_u)} \left( \prod_{x \in A_t^1(qd)} k(dq) \right)
$$
is surjective. The $k$-algebra $k[Y_u]$ is equal to $\prod_{d=1}^{d(Y_u)} \prod_{v \in Y_u(d)} k(v) = \prod_{d=1}^{d(Y_u)} \prod_{v \in Y_u(d)} k(d)$. Thus for any prime $q \gg 0$ one has

$$k[Y_u] \otimes_k k(q) = \prod_{d=1}^{d(Y_u)} \prod_{v \in Y_u(d)} k(dq).$$

Choose a prime $q \gg 0$ such that for any $d = 1, 2, ..., d(Y_u)$ one has $\text{Irr}(dq) \geq \sharp(Y_u(d))$. This is possible by the lemma \[\text{Lemma 11.2}\]. In this case for any $d = 1, 2, ..., d(Y_u)$ there exists a surjective $k$-algebra homomorphism $\prod_{x \in A_t^1(qd)} k(dq) \rightarrow \prod_{d=1}^{d(Y_u)} \prod_{v \in Y_u(d)} k(dq)$. Thus for this specific choice of prime $q$ there exists a surjective $k$-algebra homomorphism $k[t] \rightarrow k[Y_u] \otimes_k k(q)$. The third assertion of the lemma is proved.

Under the notation \[\text{Lemma 11.1}\] and \[\text{Lemma 11.3}\] state one more lemma, which is used to prove the lemma \[\text{Lemma 11.1}\].

**Lemma 11.5.** Let $l/k$ be a finite field extension. Let $V$ be an étale l-scheme containing a $l$-rational point $v_0$. Let $V' := V - \{v_0\}$. Then for any prime $q \gg 0$ the following holds

1. There is a surjective $k$-algebra map

$$\beta : k[t] \rightarrow [l[V'] \otimes_k k(q)] \times [l[v_0] \times l(q - 1)];$$

2. the étale extensions $l[V'] \rightarrow l[V'] \otimes_k k(q)$ and in : $l \rightarrow [l[v_0] \times l(q - 1)]$ are of the form $l[V'][t]/(f(t))$ and $l[t]/(t \cdot g)$ respectively. Here $f(t) \in l[V'][t]$ is a monic degree $q$ polynomial and $g(t) \in l[t]$ a monic degree $(q - 1)$ irreducible polynomial;

3. the projection $p : [l[v_0] \times l(q - 1)] \rightarrow l[v_0] = l$ is such that $p \circ \text{in} = \text{id}_E$.

**Proof.** For any positive integer $d$ let $V'(d) \subseteq V'$ be the subset consisting of points $v \in V'$ such that $\text{deg}(l(v)) = d$. Let $d(V') = \max\{\text{deg}(l(v))|v \in V'\}$. Let $d_0 = \text{deg}_k(l)$.

Choose a prime $q \gg 0$ such that for any $d = 1, 2, ..., d(V')$ one has $\text{Irr}_k(dq) \geq \sharp(V'(d))$. This is possible by the lemma \[\text{Lemma 11.2}\]. Regarding the $l$-scheme $V'$ as the $k$-scheme and applying the lemma \[\text{Lemma 11.4}\] we get a surjective $k$-algebra map $\beta' : k[t] \rightarrow [l[V'] \otimes_k k(q)] = \prod_{v \in V', k(v)(q)}$. Now consider a $k$-algebra map

$$\beta = (\beta', \beta_2, \beta_3) : k[t] \rightarrow [l[V'] \otimes_k k(q)] \times [l[v_0] \times l(q - 1)],$$

where $\beta_2, \beta_3$ are surjective. Since $q \gg 0$ and $q$ is co-prime to $(q - 1)$, hence for any integer $d = 1, 2, ..., d(V')$ one has $d \cdot q \neq d_0$, $dq \neq d_0 \cdot (q - 1)$ and $d_0 \neq d_0 \cdot (q - 1)$. The Chinese remainder theorem yields now the surjectivity of $\beta$.

The second and the third assertions are obvious. □

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Proof of the lemma 6.1. We prove this lemma for the case of local \( U \) and left the general case to the reader. So there is only one closed point \( u \in U \). Let \( k(u) \) be its residue field. It is a finite field extension of the finite field \( k \). Let \( Y \) be the set of closed points of the scheme \( S' \) and \( V \) be the scheme \( \sqcup_{v \in Y} v \). Set \( v_0 = \delta(u) \) and \( V' = V - \{v_0\} \). Then \( V, V' \) and \( v_0 \) are étale \( k(u) \)-schemes and \( v_0 \) is a \( k(u) \)-rational point in \( V \).

Set \( l = k(u) = k(v_0) \). By the lemma 11.5 there is the prime number \( q \gg 0 \) and the surjective \( k \)-algebra map \( \beta : k[t] \rightarrow [l[V'] \otimes_k k(q)] \times [l[v_0] \times l(q-1)] \). We will use \( \beta \) at the end of this proof.

By the lemma 11.5(2) one have equalities of the étale algebras

\[
[l[V'] \otimes_k k(q)] = l[V'][t]/(f) \quad \text{and} \quad l[v_0] \times l(q-1) = l[t]/(t \cdot g).
\]

Here \( f(t) \in l[V'][t] \) is a monic degree \( q \) polynomial and \( g(t) \in l[t] \) a monic degree \( (q-1) \) irreducible polynomial.

The closed embedding \( \delta|_u : u \rightarrow T' \) induces a surjection \((\delta|_u)* : k[T'] \rightarrow k(u) = l \) of the \( k \)-algebras. Choose a monic degree \( (q-1) \) polynomial \( \tilde{g}(t) \in k[T'] \), which is a lift of \( g(t) \in l[t] \). Consider the \( l[V'] \times k[T'] \)-algebra \((l[V'] \times k[T'])(t)/\langle f, t \cdot \tilde{g} \rangle) = \langle l[V'] \times k[T'] \rangle_{(t)} \). The closed embedding \( i_V \sqcup i_T : V' \sqcup T' \hookrightarrow S' \) induces a surjection \((i_V \sqcup i_T)* : k[S'] \rightarrow l[V'] \times k[T'] \). Choose a monic degree \( q \) polynomial \( F(t) \in k[S'] \), which is a lift of \( \langle f, t \cdot \tilde{g} \rangle \in l[V'][t] \times k[T'][t] = \langle l[V'] \times k[T'] \rangle_{(t)} \). Consider the \( k[S'] \)-algebra \( k[S'][t]/(F(t)) \). Set

\[
S'' = \text{Spec}(k[S'][t]/(F(t))).
\]

The inclusion \( k[S'] \rightarrow k[S''] = k[S'][t]/(F(t)) \) is a \( k[U] \)-algebra homomorphism. Let \( \rho : S'' \rightarrow S' \) be the corresponding morphism of the \( U \)-schemes. Nakayama lemma yields that the closed subscheme \( \rho^{-1}(T') \subseteq S'' \) coincides with

\[
\text{Spec}(k[T'][t]/(t \cdot \tilde{g})) = \text{Spec}(k[T'][t]/(t) \sqcup \text{Spec}(k[T'][t]/(\tilde{g}))) = T' \sqcup \text{Spec}(k[T'][t]/(\tilde{g}))
\]

and the morphism \( \rho|_{T'} : T' \rightarrow S' \) coincides with the closed embedding \( i_T : T' \hookrightarrow S' \). Denoting \( \delta' : T'' \rightarrow S'' \) the inclusion we see that \( \delta' \) is a section of \( \rho \) over \( T'' \subset S'' \).

Clearly, the the morphism \( \rho \) is finite flat. The \( l \)-scheme \( \rho^{-1}(V) \) coincides with the one \( \text{Spec}(l[V'] \otimes_k k(q)) \sqcup [\text{Spec}(l[v_0] \times l(q-1))] \). It is étale of degree \( q \) over the \( l \)-scheme

\[
V = \text{Spec}(l[V']) \sqcup \text{Spec}(l).
\]

Hence the morphism \( \rho \) is finite étale. The set of points of the scheme \( \rho^{-1}(V) \) coincides with the set \( \{ z \in S''_\text{nd} : k(u) \} \). The surjectivity of the homomorphism \( \beta \) shows now that the latter set satisfies to the condition (2) of the lemma 6.1.

Clearly, \( v_0 \) is the only \( l \)-rational point of the scheme \( \rho^{-1}(V) \). Thus the morphisms \( \rho \) and \( \delta' \) satisfy to the condition (1) of the lemma. Whence the lemma 6.1.

\[ \Box \]

12 Appendix B: proof of the proposition 2.3

Let \( F_q \) be a finite field of \( q = p^a \) elements. Let \( S = F_q[x_0, ..., x_n] \) be the homogeneous coordinate ring of \( \mathbb{P}^n \), let \( S_d \subset S \) be the \( F_q \)-subspace of homogeneous polynomials of degree
density of $P_m$ of dimension point $x$. The following Bertini type proposition is an extension of Artin’s result \[A\] Exp. XI, Thm. 2.1. It is a straightforward corollary of two theorems \[Poo\ Thm. 1.2], \[ChPoo\ Thm. 1.1].

**Proposition 12.1.** Let $X$ be a projective smooth geometrically irreducible subscheme of $\mathbb{P}^n$ over $\mathbb{F}_q$. Let $Z$ be a finite subset of $X$. Let $m \geq 2$ be the dimension of $X$. Then there is an integer $N_0 > 0$ such that for any integer $d \geq N_0$ there is an integer $x$ such that the scheme $H(f) \cap X$ is smooth geometrically irreducible of dimension $m - 1$.

**Proof.** We give a proof in the simplest case, when the set $Z$ consists of one point and this point is rational. The general case we left to the reader. So, we may and will assume that $Z = \{z_0 = [1 : 0 : \ldots : 0] \in X\}$. Let $\tau \subseteq \mathbb{P}^n$ be the tangent space to $X$ at the point $x_0$. In this case let $l = l(t_0, t_1, \ldots, t_n)$ be a linear form in $\mathbb{F}_q[t_0, t_1, \ldots, t_n]$ such that $l|_\tau \neq 0$. Set

$$\mathcal{P}_1 := \{f \in S_{\text{homog}} : H(f) \cap (X-x_0) \text{ is smooth of dimension } m-1, f(x) = 0, f_1 = l\}.$$ 

By \[Poo\ Thm. 1.2\] one has $\mu(\mathcal{P}) = \frac{1}{q^{\dim (X-x_0)}}(m + 1)^{-1} > 0$. On the other hand the density of $\mathcal{P}_2 := \{f \in S_{\text{homog}} : H(f) \cap X \text{ is geometrically irreducible}\}$ is 1. These yield that there in an integer $N_0 > 0$ such that for any integer $d \geq N_0$ there is an integer $d \geq N_0$ such that for any integer $d \geq N_0$ there is an $f \in S_d$ with $f(x) = 0$, $f_1 = l$ such that the scheme $H(f) \cap (X-x_0)$ is smooth geometrically irreducible of dimension $m - 1$. Since $f(x) = 0$ and $l|_\tau \neq 0$, hence the scheme $H(f) \cap X$ contains the point $x_0$ and it is smooth at this point too. Whence the proposition.

We next prove the proposition \[2.3\]. Recall that it extends a result of Artin from \[A\] concerning existence of nice neighborhoods. The proof follows the proof of the original Artin’s result. However we do not have in hand at the moment any kind of theorem about existence of d appropriate family of sections of the bundle $\mathcal{O}(d)$ for some $d \gg 0$. This is why there is a slight technical difference between our proof and the original Artin’s proof of his result. The details are given below in the proof. Since the differences are purely technical we will give the proof for the case of a smooth geometrically irreducible surface and left to the reader to recover the general case.

**Proof of the proposition \[2.3\]**. We may and will assume that $X \subseteq \mathbb{A}_k^1$ is closed $k$-smooth geometrically irreducible surface. Set $x := \bigcap_{j=1}^n x_i$. Let $X_0$ be the closure of $X$ in $\mathbb{P}_k^1$. Let $\bar{X}$ is the normalization of $X_0$ and set $Y = \bar{X} - X$ with the induced reduced structure. Let $S \subseteq \bar{X}$ be the closed subset of $\bar{X}$ consisting of all singular points. Then one has

(i) $S \subseteq Y$,

(ii) $\dim \bar{X} = \dim X = 2$,

(iii) $\dim Y = 1$,

(iv) $\dim S \leq 0$ (so, $S$ consists of finitely many closed points in $Y$).
Embed $\tilde{X}$ in a projective space $\mathbb{P}^r$. By the proposition \[12.1\] there is an integer $N_0 > 0$ such that for any integer $d_1 \geq N_0$ there is there is an $f_1 \in S_{d_1}$ such that

(0) for any $j = 1, \ldots, n$ one has $f_1(x_i) \neq 0$,

(1) the scheme $X_1 := H(f_1) \cap X$ is geometrically irreducible of dimension 1 and it is smooth at all the points, where $X$ is smooth,

(2) the scheme $Y_1 := H(f_1) \cap Y$ is of dimension 0 and is smooth at all the points, where $Y$ is smooth;

(3) the scheme $S_1 := H(f_1) \cap S$ is empty (particularly, the scheme $Y_1$ is smooth).

The properties (1) and (3) show that $X_1$ is smooth.

By the proposition \[12.1\] there is an integer $N_1 > 0$ such that for any integer $d_1d_2 \geq N_1$

there is there is an $f_2 \in S_{d_1d_2}$ such that

(0) for any $j = 1, \ldots, n$ one has $f_2(x_i) = 0$,

(1) the scheme $X_2 := H(f_2) \cap X$ is geometrically irreducible of dimension 1 and it is smooth at all the points, where $X$ is smooth,

(2) the scheme $Y_2 := H(f_2) \cap Y$ is of dimension 0 and is smooth at all the points, where $Y$ is smooth;

(3) the scheme $S_2 := H(f_2) \cap S$ is empty (particularly, the scheme $Y_1$ is smooth),

(4) the scheme $X_2 \cap X_1$ is smooth of dimension 0,

(5) the scheme $X_2 \cap X_1 \cap Y$ is empty.

The properties (1) and (3) show that $X_2$ is smooth. The property (5) shows that $X_2 \cap X_1 \subset X$. Let $t_1, t_2$ be the homogeneous coordinates on the projective line $\mathbb{P}^1_{\mathbb{F}_q}$.

Consider the section $t_2 \cdot f_1^{d_2} - t_1f_2$ on $\tilde{X} \times \mathbb{P}^1_{\mathbb{F}_q}$ of the line bundle $\mathcal{O}(d_1d_2)|_{\tilde{X}} \otimes \mathcal{O}_{\mathbb{P}^1(1)}$. Let $\tilde{X}' \subset \tilde{X} \times \mathbb{P}^1_{\mathbb{F}_q}$ be a Cartier divisor defined by the equation $t_2 \cdot f_1^{d_2} - t_1f_2 = 0$. Let

$$\sigma = pr_{\tilde{X}'|_{\tilde{X}'}} : \tilde{X}' \to \tilde{X}.$$ 

It is easy to check that the scheme $\tilde{X}'$ is a reduced and even normal. One has $\sigma^{-1}(X_1 \cap X_2) = \mathbb{P}^1_{X_1 \cap X_2}$.

Moreover, for any smooth open $U$ in $\tilde{X}$ the open subscheme $\sigma^{-1}(U)$ in $\tilde{X}'$ is a smooth and open. Particularly, $\sigma^{-1}(X-S)$ and $\sigma^{-1}(X) = \sigma^{-1}(\tilde{X}-Y)$ in $\tilde{X}'$ are open and smooth.

**Lemma 12.2.** Let $\tilde{X}_{f_1} := \tilde{X} - X_1$ and $\tilde{X}'_{f_1} := \sigma^{-1}(\tilde{X} - X_1)$ and $\sigma_1 := \sigma|_{\tilde{X}'_{f_1}}$. Then $\sigma_1 : \tilde{X}'_{f_1} \to \tilde{X}_{f_1}$ is an affine scheme isomorphism.

Furthermore, if $Y_{f_1} = Y - Y_1$, then $Y_{f_1}$ is closed in $\tilde{X}_{f_1}$ and we will identify $Y_{f_1}$ with the closed subscheme $\sigma_1^{-1}(Y_{f_1})$ in $\tilde{X}'_{f_1}$.

**Proof of the lemma.** In fact, both schemes are affine and the morphism $\sigma_1$ induces a $\mathbb{F}_q[\tilde{X}_{f_1}]$-algebra map

$$\sigma_1^* : \mathbb{F}_q[\tilde{X}_{f_1}] \to \mathbb{F}_q[\tilde{X}'_{f_1}] = \mathbb{F}_q[\tilde{X}_{f_1}]/(t - f_2/f_1^{d_2}),$$

which is an isomorphism. \[\square\]

We left to the reader the following lemma.
Lemma 12.3. The morphism \( p : \bar{X}' \to \mathbb{P}_q^1 \) is flat. Let \( X'_1 \subset \bar{X}' \) be the closure of \( \sigma^{-1}(X_1 - (X_1 \cap X_2)) \) in \( \bar{X}' \) equipped with the reduced scheme structure. Then one has

(i) \( \sigma^{-1}(X_1 \cap X_2) = \mathbb{P}^1_{X_1 \cap X_2} \);

(ii) \( \sigma^{-1}(X_1) = \mathbb{P}^1_{X_1 \cap X_2} \cup X'_1 \);

(iii) \( p^{-1}([0 : 1]) = \mathbb{P}^1_{X_1 \cap X_2} \cup X'_1 = \sigma^{-1}(X_1) \);

(iv) \( X'_1 \cap \mathbb{P}^1_{X_1 \cap X_2} = \{ \infty \} \times (X_1 \cap X_2) \), where \( \infty := [0 : 1] \in \mathbb{P}^1_q \).

These two lemmas yield: the morphism \( \bar{P} = p|_{\bar{X}' - \bar{X}'} : \bar{X}' - \bar{X}' \to \mathbb{A}^1_{\mathbb{P}_q} \) is projective and flat, \( \mathbb{A}^1_{\mathbb{P}_q} \) is a closed subvariety in \( (\bar{X}' - \bar{X}') \), \( \sigma_1 : (\bar{X}' - \bar{X}') - \mathbb{A}^1_{\mathbb{P}_q} = \bar{X}'_f \to \bar{X}_f \) is an affine scheme isomorphism. Set \( J = (\bar{X}'_f \stackrel{\sigma_1^{-1}}{\to} (\bar{X}' - \bar{X}') - \mathbb{A}^1_{\mathbb{P}_q} \stackrel{I}{\to} \bar{X}' - \bar{X}'). \) For \( x \in \bar{X}_f \) set \( P(x) = f_2(x)/f_1^d(x) \). Summarizing we get a commutative diagram of the form

\[ \begin{array}{ccc}
\bar{X}_f & \xrightarrow{J} & \bar{X}' - \bar{X}'_1 \\
\downarrow{P} & & \downarrow{Q} \\
\mathbb{A}^1_{\mathbb{P}_q} & & \mathbb{A}^1_{\mathbb{P}_q} \\
\end{array} \]

of morphisms satisfying the following conditions:

(i) \( J \) is the open immersion dense at each fibre of \( P \), and \( J(\bar{X}_f) = (\bar{X}' - \bar{X}') - \mathbb{A}^1_{\mathbb{P}_q} \);

(ii) \( P \) is flat projective all of whose fibres equi-dimensional of dimension one;

(iii) \( Q = id_{\mathbb{A}^1} \times r \), where \( r : X_1 \cap X_2 \to Spec(\mathbb{P}_q) \) is the structure map.

Recall that by the property (5_1) the scheme \( X_2 \cap X_1 \cap Y \) is empty. Thus \( Y_f = J(Y'_f) \) is closed even in \( \bar{X}' - \bar{X}'_1 \). So, \( Y_f \) as the \( \mathbb{A}^1_{\mathbb{P}_q} \) is projective and affine. Hence it is finite over \( \mathbb{A}^1_{\mathbb{P}_q} \).

The scheme \( \bar{P}^{-1}(\{0\}) \) is isomorphic via the morphism \( \sigma \) to \( X_2 \) and thus it is geometrically irreducible of dimension 1 and smooth. Hence the morphism \( \bar{P} \) is smooth over a neighborhood \( S \) of the point \( 0 \in \mathbb{A}^1_{\mathbb{P}_q} \). By the property (2_1) after shrinking the neighborhood \( S \) of the point \( 0 \in \mathbb{A}^1_{\mathbb{P}_q} \), we may and will assume that \( Y_f \) is finite and étale over \( S \).

Pulling back the diagram (16) via the open embedding \( S \to \mathbb{A}^1_{\mathbb{P}_q} \) we get a commutative diagram of the form

\[ \begin{array}{ccc}
X_{prel} & \xrightarrow{j_{prel}} & \bar{X}_{prel} \\
\downarrow{p_{prel}} & & \downarrow{\bar{p}_{prel}} \\
S & \xleftarrow{q_{prel}} & S_{X_1 \cap X_2} \\
\end{array} \]

of morphisms satisfying the following conditions:

(i) \( j_{prel} \) is the open immersion dense at each fibre of \( \bar{p}_{prel} \), and \( X_{prel} = \bar{X}_{prel} - S_{X_1 \cap X_2} \).
(ii) \( \tilde{p}_{\text{prel}} \) is smooth projective all of whose fibres are geometrically irreducible of dimension one;

(iii) \( q_{\text{prel}} = id_S \times r \), where \( r : X_1 \cap X_2 \to \text{Spec}(\mathbb{F}_q) \) is the structure map.

(iv) if \( Y_S = \tilde{p}_{\text{prel}}^{-1}(S) \cap Y_{f_1} \), then \( Y_S \) is finite étale over \( S \) and \( Y_S \) is in \( X_{\text{prel}} \).

Taking now \( X_{\text{fin}} = X_{\text{prel}} - Y_S, Y_{\text{fin}} = S_{X_1 \cap X_2} \cup Y_S, \tilde{X}_{\text{fin}} = \tilde{X}_{\text{prel}}, p_{\text{fin}} = p_{\text{prel}}|X_{\text{fin}}, \)
\( \tilde{p}_{\text{fin}} = \tilde{p}_{\text{prel}}, q_{\text{fin}} = q_{\text{prel}} \sqcup (p_{\text{prel}}|Y_S), f_{\text{fin}} = f_{\text{prel}}|X_{\text{fin}}, i_{\text{fin}} = i_{\text{prel}}|S_{X_1 \cap X_2} \sqcup j_{\text{prel}}|Y_S \) we get a diagram of the form (2) subject to the conditions 2.1(i) to 2.1(iii). This proves that the morphism \( p_{\text{fin}} : X_{\text{fin}} \to S \) is an elementary fibration.

To prove the last assertion of the proposition 2.3 it sufficient to choose \( f_1 \) and \( f_2 \) such that they additionally satisfy the condition \( H(f_1) \cap H(f_2) \cap Z = \emptyset \). Clearly, this is possible.

The proposition follows. \hfill \Box

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