Features of the effect of the Earth's ionosphere on the field of an ordinary wave in the vicinity of the caustic

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Abstract. The structure of the electromagnetic wave field in the vicinity of the caustic in the case of propagation in magnetically active ionospheric plasma near the surface of the earth is considered. A comparison of the uniform (caustic) approximation and the non-uniform (ray) one is performed. It is shown that the maximum value of the field amplitude in the vicinity of the caustic can be estimated from the geometrical-optical (ray) approximation. The calculations were performed both taking into account and without taking into account absorption, as well as taking into account the divergence of the radio signal in an anisotropic plasma for the model of night electron concentration and collision frequency of the high-latitudinal ionosphere. An estimate of the field on the caustic based on the ray approximation is proposed.

1. Introduction
When studying the propagation of decimetre-band radio waves by ray methods, the problem of describing fields on caustics arises. The relevance of the study of caustic structures is determined by their special role with respect to ray structures, since caustics, which are the envelopes of ray families, divide the physical space into regions with different propagation patterns [1,2]. We also note that the field in the vicinity of the caustic increases substantially.

In this work, the structure of the wave field in the vicinity of the caustic near the Earth’s surface is simulated without taking into account the radio waves reflected from the surface, but taking into account the absorption and divergence of the radio signal in the ionospheric anisotropic plasma.

2. Mathematical model
If, within the framework of our approximation, the deflecting influence of absorption is neglected, which is acceptable for the ionospheric plasma, then the expression for the effective permittivity has the form [3,4]:

$$\varepsilon_s = 1 - \frac{2\nu (1 - \nu)}{2(1 - \nu) - u \sin^2 \alpha \pm \sqrt{u^2 \sin^4 \alpha + 4u(1-\nu)^2 \cos^2 \alpha}}$$  \hspace{1cm} (1)

The following notations are introduced in expression (1):

$$\nu = \left( \frac{\omega_p}{\omega} \right)^2 = \frac{4\pi e^2 N(\rho)}{m_e \omega^2}$$  \hspace{1cm} (2)

– ratio of the square of the plasma frequency to the square of the working frequency;
\[ u = \left( \frac{\omega_H}{\omega} \right)^2 = \frac{e^2 H_0^2}{m_e^2 c^2 \omega^2} \] (3)

- ratio of the square of the gyrofrequency to the square of the operating frequency.

In formulas (2) and (3), \( m_e \) is the mass of the electron, \( e \) is the charge of the electron, \( c \) is the speed of light, \( H_0 \) is the value of the magnetic field of the Earth, and the function is the electron concentration at a fixed point in space.

In addition to the functions \( u \) and \( v \), the angle \( \alpha \) between the Earth's magnetic field strength \( H_0 = (H_{0x}, H_{0y}, H_{0z}) \) and wave vector \( \mathbf{k} \) enters into formula (1) too.

\[ H_{0x} = H_0 \cos \gamma \cos \varphi, \quad H_{0y} = H_0 \cos \gamma \sin \varphi, \quad H_{0z} = H_0 \sin \gamma \] (4)

The orientation of the magnetic field is given by the angles \( \gamma \) and \( \varphi \). When calculating, you only need to know \( \cos^2 \alpha \):

\[ \cos^2 \alpha = \frac{(H_{0x}k_x + H_{0y}k_y + H_{0z}k_z)^2}{H_0^2 |\mathbf{k}|^2} \] (5)

The sign "+" in formula (1) corresponds to an ordinary wave, and the sign "−" corresponds to an unusual wave.

To construct ray paths, we use the Hamilton – Lukin bi-characteristic system method (see, for example, [5–7]):

\[
\frac{d\mathbf{r}}{dt} = -\frac{\partial \Gamma}{\partial \mathbf{k}}, \quad \frac{d\mathbf{k}}{dt} = -\frac{\partial \Gamma}{\partial \mathbf{r}}, \quad \frac{dt}{dt} = -\frac{\partial \Gamma}{\partial \mathbf{r}}, \quad \frac{d\omega}{dt} = \frac{\partial \Gamma}{\partial \omega} \tag{6}
\]

In formulas (6)

\[ \Gamma = k_x^2 + k_y^2 + k_z^2 - \frac{\omega^2}{c^2} \varepsilon(\mathbf{r}, \mathbf{k}, \omega) \] (7)

- Hamiltonian, \( t \) – a group time, \( \tau \) – a parameter along the trajectory [7,8].

Figure 1 shows the dependence of the electron concentration on height, and figure 2 shows the dependence of the electron collision frequency (see [9]). It is assumed that a monochromatic signal is emitted with an operating frequency \( f = 3.3 \) MHz.

![Figure 1. Dependence of the electron concentration on height.](image1)

![Figure 2. Dependence of the electron frequency of collisions on height.](image2)

To calculate the field in the vicinity of the caustic, a high-latitude night ionospheric plasma model corresponding to March, 80° North latitude and 30° Eastern longitude was used. The angle \( \gamma = -83° \), the angle \( \varphi = 90° \), \( H_0 = 0.551 \) Oe.
3. Ray trajectory calculation

We assume that the radiation source is point-wise and is located on the surface of the Earth at the coordinate origin. Radiation of an electromagnetic wave occurs in the plane \((x, z)\).

Figures 3 and 4 show the ray structure of the radio signal in the \((x, z)\) plane for an o-wave and an x-wave, respectively. The angle of the ray exit varies from 0° to 90°. Each ray in these and subsequent figures is colored in accordance with the rainbow spectrum. As a background, as in figure 1 and 2, the electron concentration of the ionosphere is shown. At an altitude of about 300 km, the maximum of the \(F\) layer is clearly visible, and at an altitude of 115 km the maximum of the \(E\) layer too. Rays corresponding to the propagation of an o-wave are reflected from the \(E\) and \(F\) layers with a small exit angle and return to the ground, and rays with large exit angles pass through the ionosphere.

![Figure 3. Ray structures in the \((x, z)\) plane, o-wave](image)

![Figure 4. Ray structures in the \((x, z)\) plane, x-wave](image)

Figure 3 shows that the family of rays forms a complex caustic structure containing three caustic points. In accordance with the classification of wave catastrophe, this is a catastrophe of \(A_3\) [7, 10]. The lower caustic cusp is caused by layer \(E\), and the two upper caustic cusps are determined by ionospheric layers forming the main maximum.

In the case of an x-wave, all rays are reflected from the ionosphere and returned to the Earth. Part of the rays forms the upper caustic, and part of the rays with small exit angles form a caustic cusp, due to layer \(E\).

Figures 5 and 6 show the ray structure in the lateral plane. It can be seen that the rays leave the original propagation plane \((x, 0, z)\), and the rays that have passed through the ionospheric layer ultimately propagate parallel to this plane, and the reflected rays with this geometry of the problem return to the plane of initial propagation when leaving the ionosphere.

If we turn to figure 3, it can be seen that the three lower branches of caustics, forming the \(A_3\) singularities, descend to the earth at distances of 333.755 km, \(\sim 575\) km and \(\sim 750\) km. The paper considers the structure of the wave field in the vicinity of the first caustic (yellow rays) without taking into account the surface wave, the effect of which at a chosen transmitter frequency at such a distance from the emitter is negligible. According to figure 4 there is an x-wave field that modulates the caustic field of an ordinary wave in this region too.
4. Wave field calculation

Let an isotropic radiation source create an electric field \( E_0 \) at a distance \( r_0 \). Then [11]:

\[
E_0 = \frac{\sqrt{30W}}{r_0} \text{ (V/m)}
\]  

(8)

In formula (8), \( W \) is the radiation power, and \( r_0 \) is the distance to the emitter. In this work, it was assumed that \( W = 1 \text{ kW} \) and \( r_0 = 1 \text{ m} \). The choice of a distance of 1 m is conditional and is determined by the convenience of the calculation. Choosing a different value (100 m, 1 km, 10 km) will not affect the final results in any way until the wave enters the ionosphere.

The field of an o-wave to the right of the caustic is defined as the sum of the contributions of two rays:

\[
u_o^* \approx b_1^* \cdot \exp\left(i\left(\Phi_1^* - \pi/2\right)\right) + b_2^* \cdot \exp\left(i\Phi_2^*\right).
\]  

(9)

The first ray has already touched the caustic, and the second is not yet. The radiation field of an x-wave in this region is single-ray:

\[
u_x^* \approx b_1^* \cdot \exp\left(i\left(\Phi_1^* - \pi/2\right)\right).
\]  

(10)

This ray has already touched the upper caustic. In formulas (9) – (10), the amplitude coefficients can be represented as:

\[
b_j = E_0 \exp[-\psi_j] \left[\frac{J_0}{J_j}\right].
\]  

(11)

In expression (11), \( J_j \) is the Jacobian of the divergence calculated at the observation point using the Lukin extended bi-characteristic system [5, 9], \( J_0 \) is the Jacobian of the divergence calculated at a distance \( r_0 \) from the source, \( \psi_j \) is the absorption determined by the electron collision frequency, and \( \Phi_j \) is the phase calculated like absorption, along the ray path [9]. In our notation, the first ray is the ray that has already touched the caustic.

Since the Jacobian \( J_j \) on the caustic is equal to zero due to ray focusing, solution (11) becomes infinite. Therefore, the field on the caustic and in its vicinity is determined using uniform asymptotics through the Airy function and its derivative (see, for example, [7,12]):
\[ u_i \equiv \exp(i\theta) \left( l_1 \cdot Ai(\lambda) - il_2 \frac{dAi(\lambda)}{d\lambda} \right) \]  

In expression (12)

\[ Ai(\lambda) = \int_{\infty}^{\lambda} \exp(i(\xi^3 + \lambda \xi))d\xi \]  

is the Airy function, \( \theta \) is the phase of the traveling wave, and \( \lambda \) is the argument of the Airy function, which in the light region (where two rays intersect) are defined as:

\[ \theta = \frac{1}{2}(\Phi_1 + \Phi_2) \quad \lambda = -\frac{3}{2^{2/3}}|\Phi_1 - \Phi_2|^{2/3} \]  

The coefficients of the asymptotic expression (12) \( l_1 \) and \( l_2 \) in a first approximation have the form:

\[ l_1 \equiv \frac{1}{2\sqrt{\pi}} \left( b_1 + b_2 \right) \sqrt[3]{-3\lambda} \quad l_2 \equiv \frac{3}{2\sqrt{\pi}} \left( b_1 - b_2 \right) \frac{1}{\sqrt[3]{-3\lambda}}. \]  

The problem of the field determination on the caustic and in its vicinity is due to the need to solve the “shooting” problem, that is, calculating with high accuracy at one point the phases and amplitudes of two rays that came along different but near trajectories. For the lower branches of the caustic, this task is especially time-consuming, since the caustic and the rays in its vicinity are quasi-parallel.

In the present work, another algorithm was implemented. This algorithm is inherently close to the method of interpolating local asymptotics [13] and is alternative to the local approach [14]. The angle of exit of the ray forming the caustic was initially calculated. Knowing this angle, the ray family in the vicinity of the caustic was divided into two subfamilies: a subfamily of rays that touched the caustic and did not touch it. Relative to each ray, the point of its intersection with the earth's surface and all the necessary radiation parameters at this point were determined. After that, for each ray subfamily, the interpolation formulas for the phases and amplitude coefficients were constructed using the least squares method. Then the parameters of two intersecting rays were found at each point, and the radiation and caustic fields were calculated using the formulas given above.

Figure 7 shows the ray amplitudes in the vicinity of the caustic without taking into account absorption (green color) and taking into account absorption (purple color).

\[ \text{Figure 7. The amplitudes of the rays in the vicinity of the caustic without taking into account absorption (green color) and taking into account absorption (purple color).} \]

In figure 8 the amplitude of the field of an o-wave in the vicinity of the caustic is shown. Uniform asymptotics calculated by formulas using the Airy function and its derivative (12) – (13) are shown in black and blue colors. The black color shows the calculations made taking into account the absorption, and the blue line shows the calculations without taking into account the absorption. The red dot on the horizontal axis shows the position of the caustic. The maximum value is shifted to the region of light
relative to the position of the caustic as expected. On the caustic, the field amplitude is close in value to the average values of the field amplitude in the light region.

Also the figure shows a comparison of the field amplitude on the caustic calculated using the uniform asymptotics (12) and the ray approximation (9). The green color indicates the geometrical-optical (GO) approximation without taking into account absorption, and the purple color shows the approximation taking into account absorption. It should be noted that the GO approximation in the region of light very well coincides with uniform asymptotics up to the slopes of the main maximum. It follows from the figure that in order to estimate the maximum value of the field amplitude in the vicinity of the caustic using the ray approximation, it is enough to determine where the curve forms the inflection point before going to infinity.

Figures 9 and 10 show the amplitudes of the field of an o-wave, taking into account the contribution of the x-wave in the vicinity of the caustic in the caustic (figure 9) and GO (figure 10) approximations, but excluding the absorption.

It can be seen that the x-wave makes a significant contribution and strongly modulates the caustic structure of the o-wave.
In figure 11 the amplitude of the electric field modulus of an o- and x-wave as a function of distance along the horizontal x axis, taking into account absorption is shown. It can be seen that in this case the x-wave weakly modulates the o-wave, causing only small oscillations.

![Figure 11](image)

**Figure 11.** The amplitude of the electric field strength modulus versus the distance along the horizontal axis; o- and x-waves. Comparison of the uniform asymptotics and the GO approximation.

Also figure 11 compares the uniform asymptotic behavior (black color) and the GO approximation (purple color). The comparison results confirm the above conclusions.

Figure 11 shows that the field amplitude oscillations have a period of 1.5–3 km with a tendency to decrease with distance from the caustic.

5. **Conclusion**

Thus, the structure of the electromagnetic wave field in the vicinity of the caustic in the case of propagation in an anisotropic ionospheric plasma is considered. A comparison of the uniform (caustic) approximation and the non-uniform (ray) one is performed. It is shown that the maximum value of the field amplitude in the vicinity of the caustic can be estimated from the GO (ray) approximation, without resorting to uniform asymptotics, by determining the position of the inflection point on the graph of the amplitude of the radiation solution, immediately before the transition to unlimited growth.

This work was supported by the Russian Science Foundation (grant No. 20-12-00299).

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