A UNIFIED THEORY FOR INERTIAL MANIFOLDS, SADDLE POINT PROPERTY AND EXPONENTIAL DICHOTOMY

ALEXANDRE N. CARVALHO, PHILLIPO LAPPICY, ESTEFANI M. MOREIRA, AND ALEXANDRE N. OLIVEIRA SOUSA

Abstract. Inertial manifold theory, saddle point property and exponential dichotomy have been treated as different topics in the literature with different proofs. As a common feature, they all have the purpose of ‘splitting’ the space to understand the dynamics. We present a unified proof for the inertial manifold theorem, which as a local consequence yields the saddle-point property with a fine structure of invariant manifolds and the roughness of exponential dichotomy. In particular, we use these tools in order to establish the hyperbolicity of certain global solutions for non-autonomous parabolic partial differential equations.

1. Introduction

Inertial manifold theory, saddle point property and exponential dichotomy have been, so far, treated as different topics in the literature with distinct proofs. However, as a common feature, they all have the purpose of ‘splitting’ the space in order to understand the dynamics. The main goal of this paper is to provide a unified treatment to all these phenomena, including a broad range of applications in the autonomous and non-autonomous framework.

The inertial manifold theory has the purpose of reducing the relevant dynamics to an invertible dynamical system (often finite dimensional) and space splitting is implicitly established by the exponential attraction of the reduced dynamics. The proof presented here is inspired in the work of Henry [20] which in turn draws its inspiration from the book of Hale [18]. The idea goes back to the work of Lyapunov [26] and Pliss [33] for the reduction principle when the splitting occurred

2020 Mathematics Subject Classification. 35B42, 37L45, 37D10, 37L25.
Key words and phrases. Non-autonomous equations; Inertial manifolds; Hyperbolicity.
Partially supported by FAPESP grants 2020/14075-6 and CNPq 306213/2019-2.
Partially supported by grants FAPESP 2017/07882-0 and 2018/18703-1.
Supported by CAPES grant PROEX 7547361/D and by FAPESP grants 2018/00065-9 and 2020/00104-4.
Supported by FAPESP grants 2017/21729-0 and 2018/10633-4 and by CAPES grant PROEX-9430931/D.
at zero. The terminology of inertial manifold was introduced in the monumental work of Foiaș, Sell and Temam in [17], and there has been a tremendous effort in understanding these objects ever since. See [42] and references therein for a recent account. In particular, see [22] for inertial manifolds in the non-autonomous context. A central idea in the present paper is the introduction of the idea of an stable manifold of an inertial manifold, characterised by a graph, in contrast with the known results that prove only that the inertial manifold is exponentially attracting.

The saddle-point property has a local nature, and it arises as a nonlinear version of its linear exponential dichotomy counterpart, which splits the space into invariant linear subspaces that contain either exponential expansion or attraction. Its study goes back to Perron, Massera and Schäffer [32, 28, 29, 27] and many developments have been achieved ever since. See [16, 15, 14, 37, 39, 24, 34, 35, 36, 2, 20]. In this direction, our results include: simple proofs of the saddle-point property and its robustness under perturbations; a fine description within the stable and unstable manifolds that allows us identify the possible and preferred directions of approximation (forwards and backwards in time) through an analysis of the growth/decay rates of solutions (new in the non-autonomous context); and an application to obtain that certain global solutions of non-autonomous parabolic partial differential equations are hyperbolic.

In order to properly introduce our results, some terminology is needed. Consider the following semilinear differential initial value problem in a Banach space $(X, \| \cdot \|)$,

\begin{align}
\dot{u} &= A(t)u + f(t, u), \quad t > \tau, \\
u(\tau) &= u_0 \in X,
\end{align}

where the map $f : \mathbb{R} \times X \to X$ is continuous, $f(t, 0) = 0$, for all $t \in \mathbb{R}$ and uniformly Lipschitz in the second variable with Lipschitz constant $\ell > 0$, i.e., $\| f(t, u) - f(t, \tilde{u}) \| \leq \ell \| u - \tilde{u} \|$ for any $(t, u), (t, \tilde{u}) \in \mathbb{R} \times X$. Assume that the family of linear operators $\{ A(t) : t \in \mathbb{R} \}$ (not necessarily bounded) defines a linear evolution process $\{ L(t, \tau) : t \geq \tau \} \subset \mathcal{L}(X)$, i.e., for each $(\tau, u_0) \in \mathbb{R} \times X$, the ‘solution’ of the following linear problem,

\begin{align}
\dot{u} &= A(t)u, \quad t \geq \tau, \\
u(\tau) &= u_0 \in X,
\end{align}

is given by $u(t, \tau, u_0) = L(t, \tau)u_0$, for $t \geq \tau$, $L(t, t) = \text{Id}_X$, $L(t, s)L(s, \tau) = L(t, \tau)$, $t \geq s \geq \tau$ and $[\tau, \infty) \ni t \mapsto L(t, \tau)u_0 \in X$ is continuous, for all $(\tau, u_0) \in \mathbb{R} \times X$.

With this, solutions of (1) define a nonlinear evolution process $\{ T(t, \tau) : t \geq \tau \} \subset \mathcal{C}(X)$ given by the variation of constants formula, that is,

\begin{align}
T(t, \tau)u &= L(t, \tau)u + \int_{\tau}^{t} L(t, s)f(s, T(s, \tau)u) ds, \quad t \geq \tau, u \in X.
\end{align}

Note that the hypothesis that the nonlinearity $f(t, \cdot)$ is globally Lipschitz and satisfy $f(t, 0) = 0$ are not restrictive for the intended analysis and are chosen in order to simplify the calculations, as we discuss in the upcoming remarks.

**Remark 1.1.** The hypothesis that $f(t, 0) = 0$ can always be achieved, as one can translate any global solution to zero as follows. In fact, assuming that $f(t, \cdot) : X \to X$ is Fréchet differentiable and that if $u_* : \mathbb{R} \to X$ is a global solution of (1) and $\mathbb{R} \ni t \mapsto B(t) := f_u(t, u_*(t)) \in \mathcal{L}(X)$ is strongly continuous, for a solution $u$ of (1), we note that $v(t) := u(t) - u_*(t)$ satisfies

\begin{align}
\dot{v} &= A(t)v + B(t)v + g(t, v), \quad t > \tau, \\
v(\tau) &= v_0 \in X,
\end{align}
where \( g(t, v) := f(t, u_\ast(t) + v) - f(t, u_\ast(t)) - f_v(t, u_\ast(t))v \) and \( v_0 := u_0 - u_\ast(\tau) \). We observe that \( g(t, 0) = 0 \in X \) and \( g_\nu(t, 0) = 0 \in \mathcal{L}(X) \). In particular, when we wish to consider the local behaviour in a small tubular neighborhood of \( u_\ast(\cdot) \) in \( X \), it will be often possible to say that \( g \) is Lipschitz with small constant \( \ell > 0 \) in a small neighborhood of \( 0 \in X \). In this case, it is possible to extend \( g \) outside such small neighborhood of \( 0 \in X \) such that it is globally Lipschitz with the same Lipschitz constant \( \ell > 0 \), see for instance [8, page 631].

**Remark 1.2.** The hypothesis that \( f(t, \cdot) \) is globally uniformly Lipschitz in the second variable is actually achieved after we cut-off a nonlinearity outside a region where it is Lipschitz, in order to describe the behaviour of specific global solutions in that region. Indeed, after the procedure that shifts global solutions to zero in Remark 1.1, one can often obtain a cut-off nonlinearity with sufficiently small \( \ell > 0 \) on some neighborhood of zero, since \( g(t, 0) = 0 \) and \( g_\nu(t, 0) = 0 \). See [8, Theorem 6.1] for more details. This cut-off operation will also be applied to bounded neighborhoods of invariant sets, and in particular, bounded neighborhoods of the global attractor.

Let us define an inertial manifold for a nonlinear evolution process.

**Definition 1.1.** A family \( \{\mathcal{M}(t) : t \in \mathbb{R}\} \subseteq X \) is called an **invariant manifold** for the evolution process \( \{T(t, \tau) : t \geq \tau\} \), if

1. \( \mathcal{M}(t) \) is a Lipschitz manifold, for each \( t \in \mathbb{R} \).
2. \( \{\mathcal{M}(t) : t \in \mathbb{R}\} \) is **invariant**, i.e., \( T(t, \tau)\mathcal{M}(\tau) = \mathcal{M}(t) \), for \( t \geq \tau \).
3. \( \{\mathcal{M}(t) : t \in \mathbb{R}\} \) is **exponentially dominated**, i.e.,
   - **Forward exponentially dominated**: for any \( \tau \in \mathbb{R} \) and bounded set \( U \subseteq X \), there are \( t_\ast > \tau, \delta_1 \in \mathbb{R}, K > 0 \) such that \( \text{dist}_H(T(t, \tau)U, \mathcal{M}(t)) \leq Ke^{-\delta_1(t - \tau)} \), for \( t > t_\ast \).
   - **Pullback exponentially dominated**: for any \( t \in \mathbb{R} \) and bounded set \( U \subseteq X \), there are \( \tau_\ast \leq t, \delta_2 \in \mathbb{R}, K > 0 \) such that \( \text{dist}_H(T(t, \tau)U, \mathcal{M}(t)) \leq Ke^{-\delta_2(\tau - t)} \), for \( \tau \leq \tau_\ast \).

If \( \delta_1 > 0 \) and \( \delta_2 > 0 \), \( \{\mathcal{M}(t) : t \in \mathbb{R}\} \subseteq X \) is **exponentially attracting** (pullback and forward) and it is called an **inertial manifold**.

We prove in Theorem 2.1 the existence of an invariant manifold and its exponential attraction for the evolution process given by (3). We also provide the characterization of the invariant manifold as a graph over a linear manifold. Moreover, in Theorem 2.2, we prove that the existence of a stable manifold of an invariant manifold in a general abstract setting, which again is given by a graph. Note that our hypothesis rely on the magnitude of the ratio between the exponential gap of the linear evolution process and the Lipschitz constant of the non-linearity, in contrast with the abstract conditions in [22]. The main tool of our proof is an idea extracted from the existence of an invariant manifold contained in [20, Chapter 6]. This leads to a reduction of (1) to an invertible flow, often on a finite dimensional subspace. As a consequence, we also obtain that the invariant manifold possess the asymptotic phase property in Corollary 2.1.

In particular, we use a cut-off procedure in a neighborhood of the uniform attractor associated with (1), which yield in Theorem 2.3 a localized version of the inertial manifold. In order to achieve such ‘localization’, we use the skew-product semiflow associated with (1).

As particular cases of our study of the invariant manifold theory, we obtain the following:

1. The saddle point property (i.e. the characterization of the local stable and unstable manifolds as graphs of Lipschitz maps) in Corollary 3.1. Furthermore, from Theorem 2.1, we obtain that the unstable manifold is exponentially attracting and we obtain explicit exponential growth/decay rates within the unstable/stable manifolds.
(ii) The roughness of exponential dichotomy for (possibly non-invertible) evolution processes in Corollary 3.4. Our new proof is simpler than the monumental work of Henry in [20] because it does not require a prior proof of admissibility in the discrete case. See also [3, 4] for a different treatment of the robustness without passing through discrete case. However, our proof has the advantage of characterizing the linear stable and unstable manifolds of the perturbed problem as graphs of linear maps.

(iii) A fine structure and linear approximation within invariant manifolds in Corollary 3.2, which extends the results [7, Lemma 2.2] and [1, Lemma 6] to certain non-autonomous equations. In particular, we are able to compare different asymptotic growth/decay rates within the unstable/stable manifolds of equation (1). This enables us to identify the possible and the preferable directions of approximation (forwards and backwards) of the global hyperbolic solutions or hyperbolic equilibria.

(iv) If the problem under consideration has an invariant manifold, we may first project it into a finite dimensional subspace, to reduce the study of hyperbolicity or normal-hyperbolicity of a global solution to a finite dimensional problem, and then we use our specific knowledge of the finite dimensional problem to achieve it. The hyperbolicity of certain global solutions, which are not obtained by means of perturbations hyperbolic equilibria, for a non-autonomous one-dimensional scalar parabolic problem with localized large diffusion diffusion and reaction term given by $f(t, u) = u - \beta(t)u^3$ is considered in Corollary 3.5.

Remark 1.3. As hyperbolicity is characterized by exponential dichotomy in the non-autonomous context and also for some autonomous infinite dimensional problems for which spectral separation does not necessarily induce a dichotomic behavior, see [40, page 150].

We believe that the results in this paper can be extended to the quasilinear or fully nonlinear case (given that some results on the saddle point property are already available in that case (see [25, Section 9.1.2],[41, 30]). We also believe that the available results on the saddle point property for quasilinear and fully nonlinear problems can be improved requiring less regularity of the nonlinearity $f$ through the ideas introduced in this paper. This will be the subject of a future work.

A crucial notion used throughout the paper is the one of exponential splitting, that allows one to split a linear evolution process in two linear invariant subspaces related to growth/decay.

Definition 1.2. A linear evolution process \( \{L(t, \tau) : t \geq \tau\} \subset \mathcal{L}(X) \) has exponential splitting, with constant \( M \geq 1 \), exponents \( \gamma, \rho \in \mathbb{R} \), with \( \gamma > \rho \), and a family of projections \( \{Q(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X) \), if

i) \( Q(t)L(t, \tau) = L(t, \tau)Q(\tau) \), for all \( t \geq \tau \),

ii) \( L(t, \tau) : \text{Im}(Q(\tau)) \to \text{Im}(Q(t)) \) is an isomorphism, with inverse denoted by \( L(\tau, t) \),

iii) the following estimates hold

\[
\|L(t, \tau)Q(\tau)\|_{\mathcal{L}(X)} \leq M e^{-\rho(t-\tau)}, \quad t \leq \tau,
\]

\[
\|L(t, \tau)(I - Q(\tau))\|_{\mathcal{L}(X)} \leq M e^{-\gamma(t-\tau)}, \quad t \geq \tau.
\]

In particular, if \( \gamma = -\rho \), then we say that \( \{L(t, \tau) : t \geq \tau\} \) has exponential dichotomy with constant \( M \geq 1 \), exponent \( \gamma > 0 \) and a family of projections \( \{Q(t) : t \in \mathbb{R}\} \).

For \( f(t, \cdot) \in C^1(X) \), a global solution \( u_* : \mathbb{R} \to X \) of (1) is called hyperbolic if the linear evolution process given by \( L_*(t, \tau) := L(t, \tau) + \int_\tau^t L(t, s)D_u f(s, u_*(s))L_*(s, \tau) \, ds \) has exponential dichotomy.
Remark 1.4. Note that the linear process \( \{L(t, \tau) : t \geq \tau\} \subset \mathcal{L}(X) \) has exponential splitting, with constant \( M \geq 1 \), exponents \( \gamma, \rho \in \mathbb{R} \), with \( \gamma > \rho \), \( \gamma > 0 \), and family of projections \( \{Q(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X) \) if, and only if, \( \{e^{-\frac{\gamma}{2}(t-\tau)}L(t, \tau) : t \geq \tau\} \subset \mathcal{L}(X) \) has exponential dichotomy, with constant \( M \geq 1 \), exponent \( \frac{\gamma - \rho}{\ell} > 0 \) and family of projections \( \{Q(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X) \).

Our aim with this paper is to prove the robustness of exponential splitting under linear and nonlinear perturbations and to show that this encompasses several topics that so far have been considered as different subjects in the literature, such as roughness of exponential dichotomy, the saddle point property, existence of invariant manifolds and the fine structure of solutions within the stable/unstable manifolds. All of this is achieved through the introduction of the stable manifold of an exponentially attracting invariant manifold.

The remaining of the paper is organized as follows. In Section 2, we present the main result that proves the existence of an invariant manifold in Theorem 2.1 and its stable manifold in Theorem 2.2, in case that \( \gamma, \rho \) are real numbers in Definition 1.2. In Section 3 we prove four consequences of the invariant manifold theorem: (i) the saddle point property in Corollary 3.1, (ii) a fine structure within stable and unstable manifolds in Corollary 3.2, (iii) the roughness of exponential dichotomies in Corollary 3.4, and (iv) the hyperbolicity of certain global solutions of a non-autonomous parabolic differential equation in Corollary 3.5. The first three consequences occur when the Lipschitz constant \( \ell > 0 \) is sufficiently small, and the last occurs when \( \gamma - \rho \) is sufficiently large.

2. Invariant Manifolds and their Stable Manifolds

2.1. Invariant Manifolds. The main result of this section gives sufficient conditions to ensure the existence of an invariant manifold. This condition basically states that the exponential gap \( \gamma - \rho \) must be large when compared with the lipschitz constant \( \ell \) of \( f \). If \( \gamma > 0 \) the invariant manifold will be an exponentially attracting inertial manifold.

Theorem 2.1. Suppose that the linear evolution process \( \{L(t, \tau) : t \geq \tau\} \) has exponential splitting, with constant \( M \geq 1 \), exponents \( \gamma > \rho \) and a family of projections \( \{Q(t) : t \in \mathbb{R}\} \). If \( f : \mathbb{R} \times X \to X \) is continuous, \( f(t,0) = 0 \), \( f(t,\cdot) : X \to X \) is Lipschitz continuous with Lipschitz constant \( \ell > 0 \), for all \( t \in \mathbb{R} \), and

\[
\frac{\gamma - \rho}{\ell} > \max\{M^2 + 2M + \sqrt{8M^3}, 3M^2 + 2M\},
\]

then there is a continuous function

\[
\Sigma^* : \mathbb{R} \times X \to X
\]

\[
(t, u) \mapsto \Sigma^*(t, u)
\]
such that \( \Sigma^*(t, u) = \Sigma^*(t, Q(t)u) = (I - Q(t))\Sigma^*(t, u) \) and \( \Sigma^*(t, 0) = 0 \), for all \( t \in \mathbb{R} \). In addition \( \Sigma^*(t, \cdot) : X \to X \) Lipschitz continuous with Lipschitz constant \( \kappa = \kappa(\gamma, \rho, \ell, M) > 0 \), for all \( t \in \mathbb{R} \), that is, \( \|\Sigma^*(t, u) - \Sigma^*(t, \bar{u})\| \leq \kappa\|u - \bar{u}\| \), for all \( (t, u), (t, \bar{u}) \in \mathbb{R} \times X \).

Moreover, the graph of \( \Sigma^*(t, \cdot) \), for each \( t \in \mathbb{R} \), given by

\[
M(t) := \{u \in X : u = q + \Sigma^*(t, q), q \in \text{Im}(Q(t))\},
\]
yields an invariant manifold \( \{M(t) : t \in \mathbb{R}\} \) for the evolution process \( \{T(t, \tau) : t \geq \tau\} \) given by (3). In other words, it is invariant and satisfies the following properties, according to the nonlinear projection \( P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u) \) onto \( M(t) \), for \( (t, u) \in \mathbb{R} \times X \),
(i) \( \{ M(t) : t \in \mathbb{R} \} \) has controlled growth: for any \((\tau, u) \in \mathbb{R} \times X\),
(9) \[ \| T(t, \tau)P_{\Sigma^*}(\tau)u \| \leq M(1 + \kappa)e^{-p(t-M(1+\kappa))(t-\tau)}\| P_{\Sigma^*}(\tau)u \|, \quad t \leq \tau. \]
(ii) \( \{ M(t) : t \in \mathbb{R} \} \) satisfies: for any \((\tau, u) \in \mathbb{R} \times X\),
(10) \[ \| T(t, \tau)u - P_{\Sigma^*}(t)T(t, \tau)u \| \leq M\| (I - P_{\Sigma^*}(\tau))u \| e^{-\delta(t-\tau)}, \quad t \geq \tau, \]
where \( \delta := \gamma - M\| \Sigma(\tau, u) - \tilde{\Sigma}(\tau, u) \| \).
If \( \delta > 0 \), \( \{ M(t) : t \in \mathbb{R} \} \) is an inertial manifold.

**Remark 2.1.** When \( \gamma \) is a sufficiently large positive number or the Lipschitz constant \( \ell \) is sufficiently small, the condition that \( \delta > 0 \) is satisfied. Note that the existence of the invariant manifold is independent of the sign of \( \gamma \) and it only depends on the ratio \( \frac{\gamma - \rho}{\ell} \) between the exponential separation \( \gamma - \rho \) and the Lipschitz constant \( \ell \). In particular, \( \kappa \to 0 \) if \( \frac{\gamma - \rho}{\ell} \to +\infty \).

**Proof.** The proof is divided into two parts. First, we show that there is a function \( \Sigma^* \) yielding the graph of the invariant manifold, as desired. Second, we show that this graph is exponentially dominated.

For the first part, given \( \kappa > 0 \), consider the following complete metric space,
(11) \[ \mathcal{L}\mathcal{B}_\Sigma(\kappa) := \left\{ \Sigma \in C(\mathbb{R} \times X, X) : \sup_{t \in \mathbb{R}} \| \Sigma(t, u) - \Sigma(t, \tilde{u}) \| \leq \kappa \| u - \tilde{u} \|, \quad \Sigma(t, 0) = 0, \quad \Sigma(t, u) = \Sigma(t, Q(t)u) = (I - Q(t))\Sigma(t, u), \quad \forall t \in \mathbb{R} \right\}. \]
with the metric \( \| \Sigma - \tilde{\Sigma} \| := \sup_{t \in \mathbb{R}} \sup_{u \neq 0} \frac{\| \Sigma(t, u) - \tilde{\Sigma}(t, u) \|}{\| u \|} \).

We are looking for \( \Sigma \in \mathcal{L}\mathcal{B}_\Sigma(\kappa) \) with the property that, if \((\tau, \eta) \in \mathbb{R} \times X\), then a solution \( u \) of (1) with initial data \( u(\tau) = Q(\tau)\eta + \Sigma(\tau, Q(\tau)\eta) \in X \) can be decomposed as \( u(t) = q(t) + p(t) \), where \( p(t) = \Sigma(t, q(t)) \) for all \( t \in \mathbb{R} \). Thus, \( q \) and \( p \) must satisfy
(12a) \[ q(t) = L(t, \tau)Q(\tau)\eta + \int_{\tau}^{t} L(t, s)Q(s)f(s, q(s) + \Sigma(s, q(s)))ds, \quad t \leq \tau, \]
(12b) \[ p(\tau) = L(\tau, t)(I - Q(t))p(t) + \int_{t}^{\tau} L(\tau, s)(I - Q(s))f(s, q(s) + \Sigma(s, q(s)))ds, \quad t \leq \tau. \]

First, let us control the growth of \( q(t) \). Since \( \{ L(t, s) : t \geq s \} \) has exponential splitting, \( f(t, 0) = 0 \) and \( f \) and \( \Sigma \) are Lipschitz with respective constants \( \ell \) and \( \kappa \), we obtain
(13) \[ \| q(t) \| \leq Me^{-\rho(t-\tau)}\| \eta \| + \int_{t}^{\tau} \ell Me^{-\rho(t-s)}(1 + \kappa)\| q(s) \| ds, \quad t \leq \tau. \]
Then, by Grönwall’s Lemma,
(14) \[ \| q(t) \| \leq Me^{(\rho + M(1+\kappa))(\tau - t)}\| \eta \|, \quad t \leq \tau. \]

Heuristically, since we wish that \( p(t) = \Sigma(t, q(t)) \) for \( \Sigma \in \mathcal{L}\mathcal{B}_\Sigma(\kappa) \), the growth in equation (14) implies that the limit \( e^{-\gamma(t-\tau)}\| p(t) \| \to 0 \), as \( t \to -\infty \). Thus, due to the exponential splitting of \( \{ L(t, \tau) : t \geq \tau \} \), the first term in (12b) goes to zero as \( t \to -\infty \), yielding
(15) \[ p(\tau) = \int_{-\infty}^{\tau} L(\tau, s)(I - Q(s))f(s, q(s) + \Sigma(s, q(s)))ds. \]
Hence, to rigorously obtain $\Sigma \in \mathcal{LB}_\Sigma(\kappa)$ that satisfies $p(\tau) = \Sigma(\tau, Q(\tau)\eta)$, it is equivalent to find a fixed point of the following map,

$$G(\Sigma)(\tau, \eta) := \int_{-\infty}^{\tau} L(t, s)(I - Q(s))f(s, q(s) + \Sigma(s, q(s)))ds.$$  

Next, we show that $G : \mathcal{LB}_\Sigma(\kappa) \to \mathcal{LB}_\Sigma(\kappa)$ is a well defined contraction in the complete metric space $\mathcal{LB}_\Sigma(\kappa)$.

Let $\eta, \bar{\eta} \in X, \Sigma, \bar{\Sigma} \in \mathcal{LB}_\Sigma(\kappa)$ with corresponding solutions $q(t), \bar{q}(t)$ of (12a). Thus, for $t \leq \tau$,

$$\|q(t) - \bar{q}(t)\| \leq M e^{\rho(\tau-t)}\|\eta - \bar{\eta}\|$$

$$+ M \int_t^\tau e^{\rho(s-t)}\|f(s, q(s) + \Sigma(s, q(s))) - f(s, \bar{q}(s) + \bar{\Sigma}(s, \bar{q}(s)))\|ds$$

$$\leq M e^{\rho(\tau-t)}\|\eta - \bar{\eta}\|$$

$$+ \ell M \int_t^\tau e^{-\rho(t-s)}\left(\|q(s) - \bar{q}(s)\| + \|\Sigma(s, q(s)) - \bar{\Sigma}(s, \bar{q}(s))\|\right)ds$$

$$\leq M e^{\rho(\tau-t)}\|\eta - \bar{\eta}\|$$

$$+ \ell M \int_t^\tau e^{\rho(s-t)}\left(\|\Sigma(s, q(s)) - \bar{\Sigma}(s, \bar{q}(s))\| + (1 + \kappa)\|q(s) - \bar{q}(s)\|\right)ds$$

$$\leq M e^{\rho(\tau-t)}\|\eta - \bar{\eta}\|$$

$$+ \ell M \int_t^\tau e^{\rho(s-t)}\left((1 + \kappa)\|q(s) - \bar{q}(s)\| + \|\Sigma - \bar{\Sigma}\|\|q(s)\|\right)ds.$$

Then, due to (14),

$$\|q(t) - \bar{q}(t)\| \leq M e^{\rho(\tau-t)}\|\eta - \bar{\eta}\| + \ell M (1 + \kappa) \int_t^\tau e^{\rho(s-t)}\|q(s) - \bar{q}(s)\|ds$$

$$+ \ell M^2 \|\eta\|\|\Sigma - \bar{\Sigma}\| e^{(\rho + M(1+\kappa))(\tau-s)} e^{\rho(s-t)}ds$$

$$\leq M e^{\rho(\tau-t)}\|\eta - \bar{\eta}\| + \ell M (1 + \kappa) \int_t^\tau e^{\rho(s-t)}\|q(s) - \bar{q}(s)\|ds$$

$$+ \frac{M \|\eta\|}{(1 + \kappa)} \|\Sigma - \bar{\Sigma}\| e^{(\rho + M(1+\kappa))(\tau-t)},$$

and, by Grönwall’s Lemma

$$\|q(t) - \bar{q}(t)\| \leq M \left[\|\eta - \bar{\eta}\| + \frac{\|\eta\|}{1 + \kappa} \|\Sigma - \bar{\Sigma}\|\right] e^{(\rho + M(1+\kappa))(\tau-t)}, \; t \leq \tau.$$
Finally, we now discuss bounds of the function $G$. Indeed, equations (14) and (18) imply

$$
\left\| G(\Sigma)(\tau, \eta) - G(\tilde{\Sigma})(\tau, \tilde{\eta}) \right\|
\leq M \int_{-\infty}^{T} e^{-\gamma(\tau-s)} \left\| f(s, q(s) + \Sigma(s, q(s))) - f(s, \tilde{q}(s) + \tilde{\Sigma}(s, \tilde{q}(s))) \right\| \chi ds
\leq \ell M \int_{-\infty}^{T} e^{-\gamma(\tau-s)} \left( (1 + \kappa) \left\| q(s) - \tilde{q}(s) \right\| + \left\| \Sigma - \tilde{\Sigma} \right\| \left\| q(s) \right\| \right) ds
\leq \ell M^2 (1 + \kappa) \left[ \left\| \eta - \tilde{\eta} \right\| + \frac{\left\| \eta \right\|}{1 + \kappa} \left\| \Sigma - \tilde{\Sigma} \right\| \right] \int_{-\infty}^{T} e^{-\gamma - \tau} ds
+ \ell M^2 \left\| \eta \right\| \left\| \Sigma - \tilde{\Sigma} \right\| \int_{-\infty}^{T} e^{-\gamma - \tau} ds
$$

Due to (6) and upcoming choice of $\kappa$, we obtain that $\gamma - \rho - 2\ell M(1 + \kappa) \geq 0$ and the above integrals are convergent. Thus,

$$
\left\| G(\Sigma)(\tau, \eta) - G(\tilde{\Sigma})(\tau, \tilde{\eta}) \right\|
\leq \frac{\ell M^2 (1 + \kappa)}{\gamma - \rho - 2\ell M(1 + \kappa)} \left[ \left\| \eta - \tilde{\eta} \right\| + \frac{\left\| \eta \right\|}{1 + \kappa} \left\| \Sigma - \tilde{\Sigma} \right\| \right]
+ \frac{\ell M^2}{\gamma - \rho - 2\ell M(1 + \kappa)} \left\| \Sigma - \tilde{\Sigma} \right\| \left\| \eta \right\|,
$$

where the denominators are positive, due to (6). Consequently,

$$
\left\| G(\Sigma)(\tau, \eta) - G(\tilde{\Sigma})(\tau, \tilde{\eta}) \right\|
\leq \kappa \left\| \eta - \tilde{\eta} \right\| + \nu \left\| \Sigma - \tilde{\Sigma} \right\| \left\| \eta \right\|,
$$

in case that

$$
\begin{align*}
(20a) \quad \frac{\ell M^2 (1 + \kappa)}{\gamma - \rho - 2\ell M(1 + \kappa)} & \leq \kappa, \\
(20b) \quad \frac{2\ell M^2}{\gamma - \rho - 2\ell M(1 + \kappa)} & < 1.
\end{align*}
$$

Equation (20a) can be rewritten as $2\ell M^2 + (\ell^2 + M - (\gamma - \rho)/\ell) \kappa + M^2 \leq 0$. This can be seen as a quadratic polynomial (in $\kappa$), which admits two real roots due to (6), given by

$$
\kappa_{\pm} := \frac{-(\gamma - \rho)/\ell - M^2 - 2M \pm \sqrt{(\gamma - \rho)/\ell - M^2 - 2M}^2 - 8M^3}{4M}.
$$

Moreover, the condition $(\gamma - \rho)/\ell > M^2 + 2M + \sqrt{8M^3}$ in (6) implies that $(\gamma - \rho)/\ell > M^2 + 2M$ and thus $\kappa_{+} > \kappa_{-} > 0$. Thus, (20a) is satisfied for any $\kappa \in [\kappa_{-}, \kappa_{+}]$. Equation (20b) holds true for $\kappa_{-}$, due to $(\gamma - \rho)/\ell > 3M^2 + 2M$ in (6). Moreover, we can isolate $\kappa$ in (20b), and thus this inequality is satisfied for any $\kappa < \kappa_{+} := (\gamma - \rho)/(2M\ell) - M - 1$. Therefore, both conditions (20) are satisfied for any $\kappa \in [\kappa_{-}, \min\{\kappa_{+}, \kappa_{+}\}]$. Note that $\kappa_{-} < \kappa_{+}$, due to the hypothesis $(\gamma - \rho)/\ell > 3M^2 + 2M$ in (6) and hence the interval $[\kappa_{-}, \min\{\kappa_{+}, \kappa_{+}\})$ is not empty.

1Note that $\kappa_{\pm}$ given by (21) satisfy $\kappa_{-} \rightarrow 0$ and $\kappa_{+} \rightarrow +\infty$, if either $(\gamma - \rho) \rightarrow +\infty$ or $\ell \rightarrow 0$. 
Therefore, the map $G$ is invariant manifold and its invariance. Furthermore, to bound the variable $\tau$, the property that any solution satisfies $t, \Sigma(t, t) = 0$, together with (13), implies the growth estimate (9) within the invariant manifold. This completes the first part of the proof.

We now embark in the second part of the proof. For $(t, u) \in \mathbb{R} \times X$, define the nonlinear projection $P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, u)$. We show that $M(t) = \{\text{Im}(P_{\Sigma^*}(t)) : t \in \mathbb{R}\}$ is has the property that any solution satisfies $t, \Sigma = 0$ (exponential attraction if $\delta > 0$), and thus we wish to bound the variable $\xi(t) := T(t, \tau)u - P_{\Sigma^*}(t)T(t, \tau)u$ for any $\eta \in X$ and $t \geq \tau$. Note that $\xi(t) = p(t) - \Sigma^*(t, q(t))$ due to the definitions in (12).

Define $q^*(s, t)$, for $s \leq t$, as

$$q^*(s, t) := L(s, t)q(t) + \int_s^t L(s, r)Q(r)f(r, q^*(r, t) + \Sigma^*(r, q^*(r, t)))dr.$$  

Since $f, \Sigma^*$ are Lipschitz with respective constants $\ell, \kappa > 0$, we obtain

$$\|q^*(s, t) - q(s)\| \leq M \int_s^t e^{\rho(r-s)}\|f(r, q^*(r, t) + \Sigma^*(r, q^*(r, t))) - f(r, q(r) + p(r))\|dr$$

$$\leq M\ell \int_s^t e^{\rho(r-s)}(\|\Sigma^*(r, q^*(r, t)) - p(r)\| + \|q^*(r, t) - q(r)\|)dr,$$

$$\leq M\ell \int_s^t e^{\rho(r-s)}(\|\Sigma^*(r, q(r)) - p(r)\| + (1 + \kappa)\|q^*(r, t) - q(r)\|)dr, \quad s \leq t.$$  

Hence, by Gronwall’s Lemma and definition of $\xi$,

$$\|q^*(s, t) - q(s)\| \leq M\ell \int_s^t e^{(\rho + M\ell(1+\kappa))(r-s)}\|\xi(r)\|dr, \quad s \leq t.$$  

Also, for $s \leq \tau \leq t$, we obtain

$$\|q^*(s, t) - q^*(s, \tau)\| \leq \|L(s, \tau)Q(\tau)[q^*(\tau, t) - q(\tau)]\|$$

$$+ \|\int_{\tau}^s L(s, r)Q(r)[f(r, q^*(r, t) + \Sigma^*(r, q^*(r, t))) - f(r, q^*(r, \tau) + \Sigma^*(r, q^*(r, \tau)))\|dr,$$

$$\leq M^2\ell e^{\rho(r-s)} \int_{\tau}^t e^{(\rho + M\ell(1+\kappa))(r-\tau)}\|\xi(\tau)\|dr + M\ell(1 + \kappa) \int_s^\tau e^{\rho(r-s)}\|q^*(r, t) - q^*(r, \tau)\|dr,$$

and by Grönwall’s Lemma

$$\|q^*(s, t) - q^*(s, \tau)\| \leq M^2\ell \int_{\tau}^t e^{(\rho + M\ell(1+\kappa))(r-s)}\|\xi(r)\|dr.$$
Now, we use these inequalities to estimate $\|\xi(t)\|$. Note that
\[
\xi(t) - L(t, \tau)(I - Q(\tau))\xi(\tau) = p(t) - L(t, \tau)p(\tau) - \Sigma^*(t, q(t)) + L(t, \tau)\Sigma^*(\tau, q(\tau))
\]
\[=
\int_{\tau}^{t} L(t, s)(I - Q(s))f(s, q(s) + p(s))ds - \int_{-\infty}^{\tau} L(t, s)(I - Q(s))f(s, \Sigma^*(s, q^*(s, t)) + q^*(s, t))ds
\]
\[+ \int_{-\infty}^{\tau} L(t, s)(I - Q(s))f(s, q^*(s, \tau) + \Sigma^*(s, q^*(s, \tau)))ds
\]
\[\leq M\ell \int_{\tau}^{t} e^{-\gamma(t-s)}\|p(s) - \Sigma^*(s, q^*(s, t))\| + \|q(s) - q^*(s, t)\| ds
\]
\[+ M\ell(1 + \kappa) \int_{-\infty}^{t} e^{-\gamma(t-s)}\|q^*(s, \tau) - q^*(s, t)\| ds
\]
Thus, using (24) and (25), we obtain
\[
\|\xi(t) - L(t, \tau)(I - Q(\tau))\xi(\tau)\|
\leq M\ell \int_{\tau}^{t} e^{-\gamma(t-s)}\|p(s) - \Sigma^*(s, q^*(s, t))\| + \|q(s) - q^*(s, t)\| ds
\]
\[+ M\ell(1 + \kappa) \int_{-\infty}^{t} e^{-\gamma(t-s)}\|q^*(s, \tau) - q^*(s, t)\| ds
\]
\[\leq M\ell \int_{\tau}^{t} e^{-\gamma(t-s)}\|\xi(s)\| ds + M^2\ell^2(1 + \kappa) \int_{\tau}^{t} e^{-\gamma(t-r)}\|\xi(r)\| \int_{\tau}^{t} e^{-(\gamma - \rho - M\ell(1 + \kappa))(r-s)} ds dr
\]
\[+ M^3\ell^2(1 + \kappa) \int_{\tau}^{t} e^{-\gamma(t-r)}e^{-(\gamma - \rho - M\ell(1 + \kappa))(r-r)}\|\xi(r)\| \int_{-\infty}^{\tau} e^{-\gamma(\tau-s)}e^{(\rho + M\ell(1 + \kappa))(\tau-s)} ds dr
\]
\[\leq M\ell \int_{\tau}^{t} e^{-\gamma(t-s)}\|\xi(s)\| ds + \frac{M^2\ell^2(1 + \kappa)}{\gamma - \rho - M\ell(1 + \kappa)} \int_{\tau}^{t} e^{-\gamma(t-r)}\|\xi(r)\| dr
\]
\[+ \frac{M^3\ell^2(1 + \kappa)}{\gamma - \rho - M\ell(1 + \kappa)} \int_{\tau}^{t} \|\xi(r)\| e^{-\gamma(t-r)} dr
\]
and we have that
\[\|\xi(t) - L(t, \tau)(I - Q(\tau))\xi(\tau)\| \leq \left[ M\ell + \frac{M^2\ell^2(1 + \kappa)(1 + M)}{\gamma - \rho - M\ell(1 + \kappa)} \right] \int_{\tau}^{t} e^{-\gamma(t-r)}\|\xi(r)\| dr.
\]
Thus,
\[\|\xi(t)\| \leq Me^{-\gamma(t-\tau)}\|\xi(\tau)\| + \left[ M\ell + \frac{M^2\ell^2(1 + \kappa)(1 + M)}{\gamma - \rho - M\ell(1 + \kappa)} \right] \int_{\tau}^{t} e^{-\gamma(t-r)}\|\xi(r)\| dr
\]
and
\[e^{\tau\gamma}\|\xi(t)\| \leq Me^{\tau\gamma}\|\xi(\tau)\| + \left[ M\ell + \frac{M^2\ell^2(1 + \kappa)(1 + M)}{\gamma - \rho - M\ell(1 + \kappa)} \right] \int_{\tau}^{t} e^{\tau\gamma}\|\xi(r)\| dr.
\]
By Grönwall’s Lemma, we obtain the bound in equation (10).
\[\square\]

**Remark 2.2.** Continuity of the invariant (inertial if $\delta > 0$) manifold with respect to perturbations on $f$, and thus continuity with respect to parameters, follows as usual; see [8] for a detailed proof in the case that $\rho < 0$. In contrast with the usual regularity issues for stable/unstable manifolds, when the $C^k$ regularity of the vector field implies the $C^k$ regularity of these invariant manifolds,
the regularity of invariant (inertial if \( \delta > 0 \)) manifolds is more delicate, see [13, 23]. However, such regularity can be achieved by stronger exponential gap conditions. Moreover, regularity for the invariant (inertial if \( \delta > 0 \)) manifold in the non-autonomous case can be achieved in a similar manner as in [6, Chapter 4]. These considerations are also valid for the upcoming results.

Note that, if \( \delta > 0 \), the exponentially attracting property (10) of the inertial manifold in Theorem 2.1 shows that any solution in phase-space converges to some set within the inertial manifold. If, in addition, we assume that the projections associated with the dichotomy have finite rank, then any solution converges to a solution within the inertial manifold, and thereby the inertial manifold also possess the asymptotic phase property described below in a more general context in which \( \delta \) does not need to be positive.

**Corollary 2.1.** Consider the hypothesis of Theorem 2.1 and that \( \dim \text{Im}(Q(\tau)) < \infty \). Then, for any \( u_0 \in X \), there exists a global solution \( \bar{q}(t) + \Sigma^*(t, \bar{q}(t)) \in \mathcal{M}(t) \) of (1) and a constant \( c > 0 \) such that

\[
\| T(t, \tau)u_0 - \bar{q}(t) - \Sigma^*(t, \bar{q}(t)) \| \leq c e^{-\delta(t-\tau)} \| (I - Q(\tau))u_0 - \Sigma^*(\tau, \bar{q}(\tau)) \|.
\]

**Proof.** Since \( f, \Sigma^* \) are Lipschitz with respective constants \( \ell, \kappa > 0 \), we obtain

\[
\| q^*(s, t) - q(s) \| \leq M \int_s^t e^{\rho(r-s)} \| f(r, q^*(r, t) + \Sigma^*(r, q^*(r, t))) - f(r, q(r) + p(r)) \| dr
\]

\[
\leq M\ell \int_s^t e^{\rho(r-s)} (\| \xi(r) \| + (1 + \kappa) \| q^*(r, t) - q(r) \|) dr,
\]

hence

\[
\| q^*(s, t) - q(s) \| \leq M\ell \int_s^t e^{(\rho + M(1+\kappa))(r-s)} \| \xi(r) \| dr,
\]

and that

\[
\| \xi(r) \| \leq M \| \xi(s) \| e^{-\left(\gamma - \rho - \frac{M^2\ell^2(1+\kappa)}{\gamma - \rho - \ell M (1+\kappa)}\right)(r-s), \ r \geq s.}
\]

and

\[
\| q^*(s, t) - q(s) \| \leq M^2\ell \int_s^t e^{-\left(\gamma - \rho - \frac{M^2\ell^2(1+\kappa)}{\gamma - \rho - \ell M (1+\kappa)}\right)(r-s)} dr \| \xi(s) \|,
\]

Since \( \{q^*(\tau, t) : t \geq \tau \} \) is bounded in \( \text{Im}(Q(\tau)) \), there is a sequence \( t_n \to \infty \) and \( \bar{q}(\tau) \in \text{Im}(Q(\tau)) \) such that \( q^*(\tau, t_n) \xrightarrow{n \to \infty} \bar{q}(\tau) \). Then, for every \( s \geq \tau \), we have that \( q^*(s, t_n) \xrightarrow{n \to \infty} \bar{q}(s) \) and thereby \( \bar{q} \) defines a solution in the invariant manifold, i.e.,

\[
\bar{q}(s) := L(s, \tau)\bar{q}(\tau) + \int_{s}^{\tau} L(s, r)Q(r)f(r, \bar{q}(r) + \Sigma^*(r, \bar{q}(r))) dr, \ s \leq \tau
\]

\[
\Sigma^*(s, \bar{q}(s)) := \int_{-\infty}^{\tau} L(\tau, \theta)(I - Q(\theta))f(\theta, \bar{q}(\theta) + \Sigma(\theta, \bar{q}(\theta)))d\theta, \ s \leq \tau.
\]

From (29), we deduce that

\[
\| \bar{q}(s) - q(s) \| \leq \frac{M^2\ell \| \xi(\tau) \|}{\gamma - \rho - M\ell(2 + \kappa) - \frac{M^2\ell^2(1+\kappa)(1+\kappa)}{\gamma - \rho - \ell M (1+\kappa)}} e^{-\left(\gamma - \rho - \frac{M^2\ell^2(1+\kappa)(1+\kappa)}{\gamma - \rho - \ell M (1+\kappa)}\right)(s-\tau}).
\]

The remaining part of the proof follows directly from Theorem 2.1. 

\( \square \)
2.2. Stable Manifold of an Invariant Manifold.

**Theorem 2.2.** Suppose that the linear evolution process \( \{L(t, \tau) : t \geq \tau\} \) has exponential splitting, with constant \( M \geq 1 \), exponents \( \gamma > \rho \) and a family of projections \( \{Q(t) : t \in \mathbb{R}\} \).

If \( (\gamma - \rho)/\ell \) satisfies (6), then there is a continuous function

\[
\Theta^* : \mathbb{R} \times X \to X
\]

such that \( \Theta^*(t, u) = \Theta^*(t, (I - Q(t))u) = Q(t)\Theta^*(t, u) \), and \( \Theta^*(t, 0) = 0 \) for all \( t \in \mathbb{R} \), which is uniformly Lipschitz with constant \( \kappa = \kappa(\gamma, \rho, \ell, M) > 0 \), i.e., \( \|\Theta^*(t, u) - \Theta^*(t, \tilde{u})\| \leq \kappa\|u - \tilde{u}\| \) for all \( (t, u), (t, \tilde{u}) \in \mathbb{R} \times X \).

Moreover, if \( P_{\Theta^*}(t)u := \Theta^*(t, (I - Q(t))u) + (I - Q(t))u \), for all \( (t, u) \in \mathbb{R} \times X \), the family given by

\[
\{Im(P_{\Theta^*}(t)) : t \in \mathbb{R}\} := \{P_{\Theta^*}(t, u) : u \in X\},
\]

is positively invariant such that

\[
\|T(t, \tau)P_{\Theta^*}(\tau)u\| \leq M(1 + \kappa)e^{-(\gamma-M\ell(1+\kappa))(t-\tau)}\|P_{\Theta^*}(\tau)u\|, \quad t \geq \tau, \quad u \in X,
\]

and

\[
\|u - P_{\Theta^*}(u)\| \leq M\hat{\delta}(t-\tau)\|T(t, \tau)u\|, \quad t \geq \tau, \quad u \in X,
\]

where \( \hat{\delta} = \rho + M\ell + \frac{M^2e^{(1+\kappa)(1+M)}}{\gamma - \rho - M\ell(1+\kappa)} \).

Furthermore, if \( \gamma - M\ell(1+\kappa) > 0 \), \( \{Im(P_{\Theta^*}(t)) : t \in \mathbb{R}\} \) is the stable manifold of the inertial manifold \( \{Im(P_{\Sigma^*}(t)) : t \in \mathbb{R}\} \).

**Proof.** Given \( \kappa > 0 \) consider the complete metric space

\[
\mathcal{LB}_\Theta(\kappa) = \left\{ \Theta \in C(\mathbb{R} \times X, X) : \|\Theta(t, u) - \Theta(t, \tilde{u})\| \leq \kappa\|u - \tilde{u}\|, \quad \Theta(t, 0) = 0, \right\}
\]

\[
\Theta(t, u) = \Theta(t, (I - Q(t))u) \in Im(Q(t)), \quad \forall (t, u), (t, \tilde{u}) \in \mathbb{R} \times X.
\]

with the metric \( \|\Theta - \tilde{\Theta}\| = \sup_{t \in \mathbb{R}} \sup_{u \neq 0} \frac{\|\Theta(t, u) - \tilde{\Theta}(t, u)\|}{\|u\|} \).

Next we outline the heuristic procedure that will establish the way of proving that the invariant manifold is given as a graph of a map in \( \mathcal{LB}_\Theta(\kappa) \). We are looking for \( \Theta \in \mathcal{LB}_\Theta(\kappa) \) with the property that, if \( (\tau, \eta) \in \mathbb{R} \times X \), then a solution \( u \) of (1), with initial data \( u(\tau) = \Theta(\tau, (I - Q(\tau))\eta) \), \( (I - Q(\tau))\eta \in X \), can be decomposed as \( u(t) = q(t) + p(t) \), where \( q(t) = \Theta(t, p(t)) \) for all \( t \geq \tau \). Thus, \( q \) and \( p \) must satisfy, for \( t \geq \tau \),

\[
q(t) = L(t, \tau)Q(\tau)\eta + \int_\tau^t L(t, s)Q(s)f(s, p(s) + \Theta(s, p(s)))ds,
\]

\[
p(t) = L(t, \tau)(I - Q(\tau))\eta + \int_\tau^t L(t, s)(I - Q(s))f(s, p(s) + \Theta(s, p(s)))ds.
\]

It follows that

\[
\|p(t)\| \leq \|L(t, \tau)(I - Q(\tau))\eta\| + \int_\tau^t \|L(t, s)(I - Q(s))f(s, p(s) + \Theta(s, p(s)))\|ds
\]

\[
\leq Me^{-\gamma(t-\tau)}\|\eta\| + \int_\tau^t \ell Me^{-\gamma(1+\kappa)}\|p(s)\|ds.
\]
Using Gronwall’s inequality,
\[ \|p(t)\| \leq Me^{-\gamma M(t+1)}\|p(t)\|. \]

From this and from the fact that \( q(t) = \Theta(p(t)) \), we conclude that
\[ \|L(\tau, t)Q(t)q(t)\| = \|L(\tau, t)\Theta(p(t))\| \leq \kappa M^2 e^{-\gamma M(t+1)}\|p(t)\|. \]

Applying \( L(\tau, t)Q(t) \) to (37), using that \( \Theta(\tau, (I-Q(\tau))\eta) = Q(\tau)\eta \) and making \( t \to \infty \) we have
\[ 0 = \Theta(\tau, (I-Q(\tau))\eta) + \int_\tau^{\infty} L(\tau, s)Q(s)f(s, p(s) + \Theta(s, p(s)))ds. \]

Inspired by this we define the operator \( \tilde{G} : \mathcal{L}B_\Theta(\kappa) \to \mathcal{L}B_\Theta(\kappa) \) by
\[ \tilde{G}(\Theta)(\tau, \eta) = -\int_\tau^{\infty} L(\tau, s)Q(s)f(s, p(s) + \Theta(s, y(s)))ds, \ (\tau, \eta) \in \mathbb{R} \times X. \]

The fact that \( \tilde{G} \) is a well-defined contraction is similar to Theorem 2.1, and we refrain from giving a proof. Hence \( \tilde{G} \) admits a unique fixed point \( \Theta^* \in \mathcal{L}B_\Theta(\kappa) \) satisfying the desired properties.

We now embark in the proof of (35). For any \((\tau, \eta) \in X \times t \geq \tau\),
\[ p(t) = L(t, \tau)(I - Q(\tau))\eta + \int_\tau^{\infty} L(t, s)(I - Q(s))f(s, q(s) + p(s))ds \]
and thus we wish to bound the variable \( \eta(t) := T(t, \tau)u - P_{\Theta^*}(t)T(t, \tau)u \) for any \( u \in X \) and \( t \geq \tau \).

Note that \( \eta(t) = q(t) - \Theta^*(t, p(t)) \) due to the definitions in (37).

Define \( p^*(s, t) \), for \( s \geq t \), as
\[ p^*(s, t) := L(s, t)p(t) + \int_t^{s} L(s, r)(I - Q(r))f(r, \Theta^*(r, p^*(r, t)) + p^*(r, t))dr. \]

Since \( f, \Theta^* \) are Lipschitz with respective constants \( \ell, \kappa > 0 \), we obtain
\[ \|p^*(s, t) - p(s)\| \leq M \int_t^{s} e^{-\gamma(s-r)}\|f(r, \Theta^*(r, p^*(r, t)) + p^*(r, t)) - f(r, q(r) + p(r))\|dr \]
\[ \leq M\ell \int_t^{s} e^{-\gamma(s-r)}\|(q(r) - \Theta^*(r, p(r)))\| + (1 + \kappa)\|p^*(r, t) - p(r)\|dr, \]
and, by Gronwall’s Lemma,
\[ \|p^*(s, t) - p(s)\| \leq M\ell \int_t^{s} e^{-(\gamma - M\ell(1 + \kappa))(s-r)}\|\eta(r)\|dr. \]

Also, for \( s \geq t \geq \tau \), we obtain
\[ \|p^*(s, \tau) - p^*(s, t)\| \leq \|L(s, t)(I - Q(t))[p^*(t, \tau) - p(t)]\| \]
\[ + \|\int_t^{s} L(s, r)(I - Q(r))[f(r, \Theta^*(r, p^*(r, \tau)) + p^*(r, \tau)) - f(r, \Theta^*(r, p^*(r, t)) + p^*(r, t))]|dr \]
\[ \leq M^2\ell e^{-(\gamma - M\ell(1 + \kappa))(s-\tau)}\|\eta(r)\|dr + M\ell(1 + \kappa)\int_t^{s} e^{-\gamma(s-r)}\|p^*(r, \tau) - p^*(r, t)\|dr, \]
and again by Gronwall’s Lemma,
\[ \|p^*(s, \tau) - p^*(s, t)\| \leq M^2\ell \int_t^{s} e^{-(\gamma - M\ell(1 + \kappa))(s-r)}\|\eta(r)\|dr. \]
Now, we use these inequalities to estimate $\|\eta(\tau)\|$. Note that
\[
\eta(\tau) - L(\tau, t)Q(t)\eta(t) = q(\tau) - L(\tau, t)q(t) - \Theta^*(\tau, p(\tau)) + L(\tau, t)\Theta^*(t, p(t))
\]
\[
= \int_t^\tau L(\tau, s)Q(s)[f(s, q(s) + p(s)) - f(s, \Theta^*(s, p^*(s, \tau))) + p^*(s, \tau))]ds
\]
\[
+ \int_\tau^\infty L(\tau, s)Q(s)[f(s, \Theta^*(s, p^*(s, \tau))) + p^*(s, \tau)) - f(s, \Theta^*(s, p^*(s, t))) + p^*(s, t))]ds.
\]
Thus, using (41) and (42), we obtain
\[
\|\eta(\tau) - L(\tau, t)Q(t)\eta(t)\| \leq M\ell \int_\tau^t e^{-\rho(\tau-s)}(\|q(s) - \Theta^*(s, p^*(s, \tau))\| + \|p(s) - p^*(s, \tau)\|)ds
\]
\[
+ M\ell(1 + \kappa) \int_t^\infty e^{-\rho(\tau-s)}\|p^*(s, t) - p^*(s, \tau)\|ds
\]
\[
\leq M\ell \int_\tau^t e^{-\rho(\tau-s)}\|q(s)\|ds + M^2\ell^2 (1 + \kappa) \int_\tau^t e^{-\gamma - M\ell(1 + \kappa)(\tau - r)} \int_{r}^{s} e^{-\gamma - M\ell(1 + \kappa)(\tau - r)}\|\eta(\tau)\|drds
\]
\[
+ M^3\ell^2 (1 + \kappa) \int_t^\infty e^{-\gamma - M\ell(1 + \kappa)(s - r)} \int_{r}^{s} e^{-\gamma - M\ell(1 + \kappa)(s - r)}\|\eta(\tau)\|drds.
\]
Hence
\[
\|\eta(\tau) - L(\tau, t)Q(t)\eta(t)\| \leq M\ell \int_\tau^t e^{-\rho(\tau-s)}\|q(s)\|ds
\]
\[
+ \frac{M^2\ell^2 (1 + \kappa)(1 + M)}{\gamma - \rho - M\ell(1 + \kappa)} \int_\tau^t e^{-\gamma - M\ell(1 + \kappa)(\tau - r)} e^{-\rho(\tau-r)}\|\eta(\tau)\|dr
\]
and we have that
\[
\|\eta(\tau) - L(\tau, t)Q(t)\eta(t)\| \leq \left[ M\ell + \frac{M^2\ell^2 (1 + \kappa)(1 + M)}{\gamma - \rho - M\ell(1 + \kappa)} \right] \int_\tau^t e^{-\rho(\tau-r)}\|\eta(\tau)\|dr.
\]
Thus,
\[
\|\eta(\tau)\| \leq Me^{-\rho(\tau-t)}\|\eta(t)\| + \left[ M\ell + \frac{M^2\ell^2 (1 + \kappa)(1 + M)}{\gamma - \rho - M\ell(1 + \kappa)} \right] \int_\tau^t e^{-\rho(\tau-r)}\|\eta(\tau)\|dr.
\]
By Grönwall’s Lemma, we obtain the bound in (35).

Note that Theorem 2.2 does not follow from the exponentially dominated property (10) in Theorem 2.1. Indeed, the proof follows closely the first part of the proof of the existence of the invariant manifold, and estimates (34) and (10) are closely related to the estimates (9) and (35).

Remark 2.3. Theorem 2.2 (when $\delta > 0$) provides a complementary direction of the inertial manifold that is contained in the stable manifold of $u \equiv 0$, which consisting of special solutions that decay to the trivial equilibrium $u \equiv 0$ with exponential decay rate $-(\gamma - M\ell(1 + \kappa)) < 0$ according to (34). In fact, for any global bounded solution within the inertial manifold (e.g. hyperbolic equilibrium or a normally hyperbolic periodic orbit), one can shift such solution to zero according to Remark 1.1, and hence obtain a complementary direction to the inertial manifold that its contained in the stable manifold of the aforementioned object. Moreover, in the case $\delta > 0$, solutions outside this manifold grow apart exponentially from it, according to (35).
**Remark 2.4.** The above results in Theorems 2.1 and 2.2 are proved for general operators \( A(t) \) that generate a linear evolution process and for nonlinearities \( f(t, \cdot) : X \to X \), and consequently the are well suited to be applied for hyperbolic PDEs. However, these results can also be replicated for parabolic PDEs, in the case that \(-A\) is a sectorial operator, and more importantly, if \( f : \mathbb{R} \times X^\alpha \to X \), where \( X^\alpha \) denotes the fractional power space with \( \alpha \in (0, 1) \). Since these results do not bring new insights in the proof, we refrain from giving a detailed proof. However, we mention some differences. As a preparation, note that the analytic semigroup \( e^{At} \) satisfies \( \|e^{At}u\|_{X^\alpha} \leq Me^{-\rho t}\|u\|_{X^\alpha} \) and \( \|e^{At}u\|_{X^\alpha} \leq Ne^{-\rho t}\|u\|_{X^\alpha} \) respectively for \( t \leq 0 \) and \( t < 0 \). Therefore, following an analogous proof of Theorem 2.1, the equations (20) actually become

\begin{align}
(43a) & \frac{\ell M^2(1 + \kappa) \Gamma(1 - \alpha)}{[\gamma - \rho - 2\ell N(1 + \kappa)]^{1-\alpha}} \leq \kappa, \\
(43b) & \frac{2\ell M^2 \Gamma(1 - \alpha)}{[\gamma - \rho - 2\ell N(1 + \kappa)]^{1-\alpha}} < 1,
\end{align}

where \( \Gamma(\cdot) \) is the gamma function. There are different ways to achieve (43) by appropriately choosing \( \kappa > 0 \). We present a similar choice as in Theorem 2.1 which fits our purposes in the upcoming sections. Equation (43b) yields an upper bound on the Lipschitz constant according to

\begin{equation}
\kappa < \kappa_* := \frac{\gamma - \rho - [2\ell M^2 \Gamma(1 - \alpha)]^{1-\alpha}}{2N\ell} - 1.
\end{equation}

Note that \( \kappa_* < (\gamma - \rho)/(2N\ell) - 1 \) and thus the denominators in (43) are positive. Moreover, we rewrite (43a) as \( \ell M^2(1 + \kappa) \Gamma(1 - \alpha) \leq \kappa[\gamma - \rho - 2\ell N(1 + \kappa)]^{1-\alpha} \). Note that an upper bound on \( \kappa \leq \kappa_* \), for some \( \kappa_* \in \mathbb{R}^+ \), is able to give a lower bound on the right-hand side of this inequality, \( \kappa[\gamma - \rho - 2\ell N(1 + \kappa)]^{1-\alpha} \leq \kappa[\gamma - \rho - 2\ell N(1 + \kappa_*)]^{1-\alpha} \), and if such lower bound is also an upper bound for the left-hand side of the desired inequality, \( \ell M^2(1 + \kappa) \Gamma(1 - \alpha) \leq \kappa[\gamma - \rho - 2\ell N(1 + \kappa_*)]^{1-\alpha} \), then we obtain the desired result. However, since the upper bound \( \kappa < \kappa_* \) does not yield a meaningful result, we consider the following upper bound,

\begin{equation}
\kappa \leq \kappa_* := \frac{1}{2} \frac{\gamma - \rho - [2\ell M^2 \Gamma(1 - \alpha)]^{1-\alpha}}{2N\ell} - 1,
\end{equation}

and thereby the desired inequality yields a lower bound on \( \kappa > 0 \),

\begin{equation}
\kappa \geq \kappa_* := \frac{2^{1-\alpha} M^2 \ell \Gamma(1 - \alpha)}{[\gamma - \rho + (2\ell M^2 \Gamma(1 - \alpha))^{1-\alpha}]^{1-\alpha}}.
\end{equation}

Thus for any \( \kappa \in [\kappa_* , \kappa_+ ) \), we obtain a well-defined contraction and prove Theorem 2.1 in the parabolic case. Similarly, modifying the proof accordingly, the estimate given in (10) will be satisfied with exponent

\begin{equation}
\delta_{\text{par}} := \gamma - \rho - 2N\ell(1 + \kappa) - \left[2\Gamma(1 - \alpha)\ell M \left( 1 + \frac{\ell(1 + 2M)N(1 + \kappa)}{\gamma - \rho - 2\ell N(1 + \kappa)} \right) \right]^{1-\alpha}.
\end{equation}

In applications the interplay between the size of the gap \( \gamma - \rho \) and the constant \( N \) will play a major role. See [31] for sharp conditions that guarantee the existence of inertial manifolds.
2.3. Local Inertial Manifolds. The aim of this subsection is to obtain a local version of the inertial manifold in Theorem 2.1 in the particular case that the diffusion $A(t) \equiv A$ and the non-linearity $f(t, \cdot)$ is Lipschitz in a neighborhood of an attracting set, which guarantees the existence of a local inertial manifold only in this neighborhood. For this purpose, we will need a stability property of the attracting set, which is natural when dealing with autonomous dynamical systems. Thus we will use the notion of skew product semiflow associated with our problem.

First, we recall some basic notions of skew product semiflows. Let $C(\mathbb{R} \times X, X)$ be the space of all continuous functions $g : \mathbb{R} \times X \to X$. Let $\Sigma$ be a subset of $C(\mathbb{R} \times X, X)$ which is translation invariant, that is, for any $g \in \Sigma$ we have that $\theta_t g \in \Sigma$ for all $t \in \mathbb{R}$, where $\theta_t(s, x) = g(t + s, x)$, for all $(s, x) \in \mathbb{R} \times X$. Assume that $\Sigma$ is endowed with a metric which makes it a compact metric space and that $f \in \Sigma$.

Define the set of all $t$-translations of $f$ by $B_f := \{\theta_t f : t \in \mathbb{R}\}$, and denote by $B$ the closure of $B_f$ with the metric of $\Sigma$, which is known as the Hull of $f$. The hull $B$ is compact.

We assume that for each $b \in B$ the problem
\[ \dot{u} = Au + b(t, u), \quad t \geq 0, \]
has a unique solution given by $\varphi(t, b)u_0 \in X$ for each $t \geq 0$, with initial condition $u(0) = u_0 \in X$ and $\varphi : \mathbb{R} \times X \to X$ defines a co-cycle, i.e., $\varphi(0, b) = I_X$ and $\varphi(t + \tau, b) = \varphi(t, \theta_{\tau} b) \varphi(\tau, b)$, for $t, \tau \geq 0$. In particular, since $\{T(t, \tau) : t \geq \tau\}$ is the evolution process associated to (1), we have that $T(t, \tau) = \varphi(t - \tau, \theta_{\tau} f)$ for any $t \geq \tau$.

The continuous time-shift operator $\theta_t$ and the co-cycle $\varphi$ define the skew-product semigroup $\{\Pi(t) : t \geq 0\}$ on the phase-space $X := X \times B$, given by
\[ \Pi(t)(u, b) := (\varphi(t, b)u, \theta_t b), \]
where $(u, b) \in X \times B$, $t \geq 0$ and the space $X$ endowed with metric $d_X(\cdot, \cdot) = \max\{\| \cdot \|_X, d_B(\cdot)\}$.

A compact set $A \subseteq X$ is called the uniform attractor for the co-cycle $\varphi$, if for any bounded set $U \subseteq X$, we have that $\lim_{t \to +\infty} \sup_{b \in B} \text{dist}_B(\varphi(t, b)U, A) = 0$. Similarly, a compact set $\mathbb{A} \subseteq X$ is called the global attractor for the skew-product semi-flow $\{\Pi(t) : t \geq 0\}$, if it is invariant and attracts all bounded sets of $X$. The global attractor for the skew-product semi-flow is related to the uniform attractor of the co-cycle, when both exist, according to $\mathbb{A} := P_X A$, where $P_X : X \to X$ is the projection onto $X$.

We now prove that the uniform attractor satisfies certain stability property.

**Lemma 2.1.** Assume that $B$ is as above. Let $\{\Pi(t) : t \geq 0\}$ be the associated skew-product semi-flow in $X$ and assume that it has a global attractor $\mathbb{A}$. Then for a given $\epsilon > 0$, there exists $\epsilon_* \in (0, \epsilon)$ such that
\[ \bigcup_{t \geq 0} \bigcup_{b \in B} \varphi(t, b) O_{\epsilon_*}(A) \subset O_{\epsilon}(A), \]
where $O_{\epsilon}(A) \subseteq X$ is an $\epsilon$-neighborhood of the uniform attractor $A$.

**Proof.** Since $\mathbb{A}$ is the global attractor for $\{\Pi(t) : t \geq 0\}$, then given $\epsilon > 0$, there exists $\epsilon_* \in (0, \epsilon)$ such that
\[ \bigcup_{t \geq 0} \Pi(t) O_{\epsilon_*}(\mathbb{A}) \subset O_{\epsilon}(\mathbb{A}), \]
where $O_{\epsilon}(\mathbb{A}) \subseteq X$ is an $\epsilon$-neighborhood of the global attractor $\mathbb{A}$. For the proof of this fact, see for instance [19]. Next, towards a contradiction, suppose that (50) does not hold for these $\epsilon, \epsilon_*$. 
Hence there exists \( t_0 \geq 0, b \in \mathcal{B} \), and \( u \in O_{\epsilon_1}(A) \) such that \( \varphi(t, b)u \notin O_{\epsilon}(A) \). Since \( \mathbb{A} = A \times \mathcal{B} \), we have \( (u, b) \in O_{\epsilon_1}(A) \), which from (51) we have that \( \Pi(t)(u, b) \in O_{\epsilon}(A) \), for all \( t \geq 0 \). In particular, \( d(\varphi(t, b)u, A) < \epsilon \), for all \( t \geq 0 \), which contradicts the assumption that \( \varphi(t, p)u \notin O_{\epsilon}(A) \), and the proof is complete.

We are ready to define and prove the existence of local inertial manifolds for equation (1). Note that instead of requiring that \( f \) is globally Lipschitz on \( X \), as in Theorem 2.1, we now suppose that it is Lipschitz on bounded sets.

**Definition 2.1.** Let \( \epsilon > \epsilon_* > 0 \) be such that \( T(t, \tau)O_{\epsilon_*}(A) \subset O_{\epsilon}(A) \), for every \( t \geq \tau \). A family of subsets \( \{M(t) : t \in \mathbb{R}\} \subset X \) is said to be a local inertial manifold if

1. \( \mathcal{M}_{\text{loc}}(t) \) is a Lipschitz manifold for each \( t \in \mathbb{R} \).
2. \( T(t, \tau)[O_{\epsilon_*}(A) \cap \mathcal{M}_{\text{loc}}(\tau)] \subset O_{\epsilon}(A) \cap \mathcal{M}_{\text{loc}}(t), \tau \geq \tau \).
3. \( \{\mathcal{M}_{\text{loc}}(t) : t \in \mathbb{R}\} \) is exponentially attracting in the sense of Definition 1.1, part 3.

**Theorem 2.3.** Consider the equation (1) in which \( f : \mathbb{R} \times X \rightarrow X \) is Lipschitz in bounded sets uniformly in \( t \in \mathbb{R} \), i.e., \( \|f(t, u) - f(t, \tilde{u})\|_X \leq \ell\|u - \tilde{u}\|_X \), for any \( (t, u), (t, \tilde{u}) \) within bounded subsets of \( X \). Suppose that the linear evolution process \( \{L(t, \tau) : t \geq \tau\} \) has exponential splitting, with constant \( M \geq 1 \), exponents \( \gamma > \rho \) with \( \gamma > 0 \), and a family of projections \( \{Q(t) : t \in \mathbb{R}\} \) that the associated skew-product semigroup \( \{\Pi(t) : t \geq 0\} \) has a global attractor \( \mathcal{A} \) and uniform attractor \( \mathcal{A} \).

If \( \gamma \) and \( \gamma - \rho \) are sufficiently large, then there exists a function \( \Sigma_{\text{loc}} : \mathbb{R} \times O_{\epsilon}(A) \rightarrow X \) such that \( \Sigma_{\text{loc}}(t, u) \in \text{Im}(I - Q(t)) \), which is uniformly Lipschitz with constant \( \kappa > 0 \), i.e.,

\[ \|\Sigma_{\text{loc}}(t, u) - \Sigma_{\text{loc}}(t, \tilde{u})\|_X \leq \kappa\|u - \tilde{u}\|_X \quad \text{for all } (t, u), (t, \tilde{u}) \in O_{\epsilon}(A). \]

Furthermore, the local graph of \( \Sigma_{\text{loc}}(t, \cdot) \), for each \( t \in \mathbb{R} \), given by

\[ \mathcal{M}_{\text{loc}}(t) := \{u \in X : u = q + \Sigma_{\text{loc}}(t, q), q \in Q(t)O_{\epsilon}(A)\}, \]

yields a local inertial manifold \( \mathcal{M}_{\text{loc}}(t) : t \in \mathbb{R} \) for \( \{T(t, \tau) : t \geq \tau\} \).

**Proof.** Denote by \( \ell > 0 \) the Lipschitz constant of \( f \) in the bounded neighborhood \( O_{\epsilon}(A) \). Let \( \tilde{f} : \mathbb{R} \times X \rightarrow X \) be a Lipschitz extension of \( f \) out of \( O_{\epsilon}(A) \), which coincides with \( f \) on \( O_{\epsilon}(A) \) and has global Lipschitz constant \( \ell > 0 \). Conditions of Theorem 2.1 are satisfied for \( \tilde{f} \) and \( \gamma - \rho \) sufficiently large, and thus there exists an inertial manifold \( \hat{\Sigma} \in \mathcal{L}B_{\Sigma}(\kappa) \) for the equation (1), with \( f \) replaced by \( \tilde{f} \).

We then consider the restriction of the global inertial manifold \( \hat{\Sigma} \) to the neighborhood \( O_{\epsilon}(A) \), i.e., define, \( \Sigma_{\text{loc}} := \hat{\Sigma}|_{\mathbb{R} \times O_{\epsilon}(A)} \) and we obtain the sets defined in (52). In order to prove \( \mathcal{M}_{\text{loc}}(\cdot) \) yields a local inertial manifold, note that it is a Lipschitz graph, since \( \hat{\Sigma} \) is Lipschitz. Moreover, due to Lemma 2.1, there is a \( \epsilon_* \in (0, \epsilon) \) such that \( T(t, \tau)O_{\epsilon_*}(A) \subset O_{\epsilon}(A) \) for \( t \geq \tau \). Together with the fact that \( \hat{\Sigma} \) yields a global invariant set, this implies the second condition in definition 2.1.

Lastly, we prove that the graph of the local inertial manifold in (52) is exponentially attracting. Given \( u \in O_{\epsilon_*}(A) \), we have \( T(t, \tau)u \in O_{\epsilon}(A) \) for \( t \geq \tau \). Since \( \Sigma_{\text{loc}} \) coincides with \( \hat{\Sigma} \) on the neighborhood \( O_{\epsilon}(A) \), which is exponentially attracting according to (10), we obtain,

\[ \begin{align*}
\|T(t, \tau)u - P_{\Sigma_{\text{loc}}}(t)T(t, \tau)u\|_X &\leq M\|(I - P_{\Sigma_{\text{loc}}}(\tau))u\|_X e^{-\delta(t-\tau)} \\
&\leq M(1 + M + \kappa)e^{-\delta(t-\tau)}\|u\|_X, \quad \forall t \geq \tau,
\end{align*} \]

for any \( u \in O_{\epsilon_*}(A) \), where \( P_{\Sigma_{\text{loc}}}(\tau)u := Q(\tau)u + \Sigma_{\text{loc}}(\tau, u) \). Note that \( \delta > 0 \) in (10) for sufficiently large \( \gamma \).
We now prove that the family \( \{ \mathcal{M}_{\text{loc}}(t) : t \in \mathbb{R} \} \) exponentially pullback attracts any bounded subset of \( O_*(\mathcal{A}) \). Since \( \mathcal{A} \) is bounded and \( \mathcal{A} = \bigcup_{t \in \mathbb{R}} \mathcal{A}(t) \), we have that \( \mathcal{A}(t) \subset \mathcal{M}_{\text{loc}}(t) \) for each \( t \in \mathbb{R} \), and consequently \( \{ \mathcal{M}_{\text{loc}}(t) : t \in \mathbb{R} \} \) pullback attracts bounded sets of \( X \). Let us show that the rate of convergence is indeed exponential.

Since \( \mathcal{A} \) is an uniform attractor, for each bounded subset \( U \subseteq X \), there exists \( t_* = t_*(U, \epsilon_*) \geq 0 \) such that \( T(t_* + \tau, \tau) U \subset O_*(\mathcal{A}) \), for every \( \tau \in \mathbb{R} \). Let \( t \in \mathbb{R} \) be fixed and choose any \( \tau \leq \tau_* := t - t_* \), then from Lemma 2.1, we have that \( T(t, \tau) U \subset O_*(\mathcal{A}) \), for every \( \tau \leq \tau_\ast \). Thus, for any \( u_0 \in U \), the evolution \( T(t_* + \tau, \tau) u_0 \in O_*(\mathcal{A}) \), for every \( \tau \leq \tau_\ast \). Thus applying the inequality (53) at the point \( T(t_* + \tau, \tau) u_0 \), we obtain

\[
\|T(t, \tau) u_0 - P_{\Sigma_{\text{loc}}}(t) T(t, \tau) u_0\|_X \leq M(1 + M + \kappa)\|T(t_0 + \tau, \tau) u_0\|_X e^{\delta t_0} e^{-\delta (t-\tau)} \\
\leq Ce^{-\delta (t-\tau)}, \quad \forall \tau \leq \tau_\ast,
\]

for any \( u_0 \in U \), where \( C > 0 \) is some constant independent of \( \tau \). Thus,

\[
dist_H(T(t, \tau) U, \mathcal{M}_{\text{loc}}(t)) \leq Ce^{-\delta (t-\tau)}, \quad \forall \tau \leq \tau_\ast,
\]

which proves that \( \{ \mathcal{M}_{\text{loc}}(t) : t \in \mathbb{R} \} \) is pullback exponentially attracting. Analogous arguments yield forward exponentially attraction, and consequently exponential attraction.

\[\square\]

3. Applications

3.1. The Saddle Point Property. We now obtain the saddle point property as an immediate consequence of Theorems 2.1 and 2.2.

We respectively define the unstable and stable sets of a hyperbolic global solution \( u_* \) of (1) as

\[
W^u(u_*) := \left\{ (\tau, u_0) \in \mathbb{R} \times X : \text{there is a solution } u : (-\infty, \tau] \to X \text{ such that } u(\tau) = u_0 \text{ and } \lim_{t \to -\infty} \|u(t) - u_*(t)\|_X = 0 \right\},
\]

\[
W^s(u_*) := \left\{ (\tau, u_0) \in \mathbb{R} \times X : \text{there is a solution } u : [\tau, \infty) \to X \text{ such that } u(\tau) = u_0 \text{ and } \lim_{t \to +\infty} \|u(t) - u_*(t)\|_X = 0 \right\}.
\]

The existence, continuity and local characterization as graphs of the unstable and stable manifolds for non-autonomous hyperbolic asymptotic profiles was proved in [8]. We obtain their results as a corollary of Theorems 2.1 and 2.2. We only construct the unstable and stable manifolds for the trivial equilibria \( u_* = 0 \), as any other global hyperbolic solution can be shifted to zero. See Remark 1.1.

**Corollary 3.1.** Suppose that the linear evolution process \( \{ L(t, \tau) : t \geq \tau \} \) has exponential dichotomy, with constant \( M \geq 1 \), exponent \( \gamma > 0 \) and a family of projections \( \{ Q(t) : t \in \mathbb{R} \} \).

Suppose that \( \ell > 0 \) is sufficiently small, then there are continuous functions \( \Sigma^u \in \mathcal{LB}_\Sigma(\kappa) \) and \( \Theta^s \in \mathcal{LB}_\Theta(\kappa) \) such that the unstable and stable manifolds of \( u_* = 0 \) are given by

\[
W^u(0) = \left\{ (\tau, u) \in \mathbb{R} \times X : u = Q(\tau) u + \Sigma^u(\tau, Q(\tau) u) \right\},
\]

\[
W^s(0) = \left\{ (\tau, u) \in \mathbb{R} \times X : u = \Theta^s(\tau, (I - Q(\tau)) u + (I - Q(\tau)) u) \right\}.
\]

Moreover, solutions within the unstable (resp. stable) manifold exponentially decay to zero backwards (resp. forwards) in time, according to (9) and (34).

**Proof.** For \( \ell > 0 \) sufficiently small, the condition (6) is satisfied and \( \delta > 0 \), and thus we obtain the graph of \( \Sigma^* \) from Theorem 2.1. We now prove that the unstable set \( W^u(0) \) defined in (56a) coincides with the graph of \( \Sigma^* := \Sigma^* \). On one hand, the graph of \( \Sigma^* \) is contained in the unstable
set by (9). On the other hand, any solution \( z : ( - \infty, t] \rightarrow X \) which backwards converges to zero satisfies, from (10),
\[
\| z(t) - P_{\Sigma^s}(t) z(t) \| = \| (I - Q(t)) z(t) - \Sigma^u(t, Q(t) z(t)) \| \leq M \| (I - P_{\Sigma^s}(\tau)) z(\tau) \| e^{-\delta (t - \tau)}, \quad t \geq \tau.
\]
Since \( \delta > 0 \), we obtain that \( (I - Q(t)) z(t) = \Sigma^u(t, Q(t) z(t)) \) for all \( t \in \mathbb{R} \) as \( \tau \rightarrow -\infty \), and thus any element in the unstable set lies in the graph of \( \Sigma^u \). The case of stable manifold is analogous applying Theorem 2.2.

\[ \Box \]

**Remark 3.1.** Corollary 3.1 assumes that the nonlinearity \( f(t, \cdot) \) is globally uniformly (in \( t \)) Lipschitz, and thereby yields global unstable and stable manifolds. However, in general, we have that the nonlinearity is only locally Lipschitz on a neighborhood of some equilibrium (here assumed to be zero after suitable translation), \( O_\epsilon(0) \), and thus by means of a cut-off outside this region, we obtain maps, \( \Sigma^u : \mathbb{R} \times O_\epsilon(0) \rightarrow X \) and \( \Theta^s : \mathbb{R} \times O_\epsilon(0) \rightarrow X \), whose graphs contain the local unstable and stable manifolds, \( W^u_{loc}(0) \) and \( W^s_{loc}(0) \), respectively. Indeed, due to the respective growth bounds (9) and (34), there is a neighborhood \( O_{\epsilon^*}(0) \) contained in the cut-off region \( O_\epsilon(0) \) such that orbits in the smaller neighborhood \( O_{\epsilon^*}(0) \) do not leave the bigger neighborhood \( O_\epsilon(0) \) in the appropriate time direction. See the proof of Theorem 2.3 for more details on such cut-off of the nonlinearity to obtain local graphs, bearing in mind that after the procedure of shifting a solution to zero and obtaining a nonlinearity \( g(t, u) \) in Remark 1.1, one can construct a cut-off function with sufficiently small \( \ell > 0 \) on some neighborhood \( O_\epsilon(0) \), since \( g(t, 0) = 0 \) and \( g_u(t, 0) = 0 \). See [8, Theorem 6.1] for more details.

### 3.2. Fine Description Within Invariant Manifolds.

We describe a finer growth and decay structure within invariant manifolds obtained in Section 3.1, in case of an additional exponential splitting. This allows the comparison between two different growth (resp. decay) rates within the unstable (resp. stable) manifold, which dictates the directions and the preferred directions along which solutions may approach the equilibria backwards in time (resp. forwards in time). We then apply this result to the example of asymptotically autonomous PDEs, which generalizes the well-known case of autonomous equations, see [7, Lemma 2.2] and [1, Lemma 6].

Often, exponential splitting for a linear evolution process \{ \( L(t, \tau) : t \geq \tau \) \} occurs with several different exponents and families of projections. The splitting given by the different projections are well behaved (nested) as it is proved in the next lemma.

**Lemma 3.1.** Suppose that the linear evolution process \{ \( L(t, \tau) : t \geq \tau \) \} has exponential splitting, with constant \( M \geq 1 \), exponents \( \gamma > \rho \), and family of projections \{ \( Q(t) : t \in \mathbb{R} \) \}. Assume further that \{ \( L(t, \tau) : t \geq \tau \) \} has another exponential splitting with constant \( M_* \geq 1 \), with exponents \( \gamma_* > \rho_* \) such that \( \gamma > \gamma_* \) and family of projections \{ \( Q_*(t) : t \in \mathbb{R} \) \}. Then \( \text{Im}(Q_*(t)) \subset \text{Im}(Q(t)) \) and \( \text{Ker}(Q(t)) \subset \text{Ker}(Q_*(t)) \).

**Proof.** For \( \mu = \min\{ \gamma, \frac{\gamma_* - \rho_*}{2} \} \), we have that
\[
\begin{align*}
(58a) \quad & \text{Im}(Q_*(\tau)) = \{ u \in X : e^{(\gamma_* - \mu)(t - \tau)} L(t, \tau) u \in X \text{ is defined for all } t \leq \tau \text{ and is bounded} \}, \\
(58b) \quad & \text{Im}(Q(\tau)) = \{ u \in X : e^{(\gamma - \mu)(t - \tau)} L(t, \tau) u \in X \text{ is defined for all } t \leq \tau \text{ and is bounded} \}, \\
(58c) \quad & \text{Ker}(Q_*(t)) = \{ u \in X : e^{(\gamma_* - \mu)(t - \tau)} L(t, \tau) u \in X \text{ is bounded for all } t \geq \tau \}, \\
(58d) \quad & \text{Ker}(Q(t)) = \{ u \in X : e^{(\gamma - \mu)(t - \tau)} L(t, \tau) u \in X \text{ is bounded for all } t \geq \tau \},
\end{align*}
\]
see [9, Theorem 7.12]. Clearly \( \text{Im}(Q(\tau)) \supset \text{Im}(Q_*(\tau)) \) and \( \text{Ker}(Q(t)) \subset \text{Ker}(Q_*(t)) \). \[ \Box \]
Corollary 3.2. Suppose that the linear evolution process \( \{L(t, \tau) : t \geq \tau\} \) has exponential splitting, with constant \( M \geq 1 \), exponents \( \gamma > \rho \), and family of projections \( \{Q(t) : t \in \mathbb{R}\} \). Assume further that \( \{L(t, \tau) : t \geq \tau\} \) has another exponential splitting with constant \( M_\ast \geq 1 \), with exponents \( \gamma_\ast > \rho_\ast \) such that \( \rho \geq \gamma_\ast \) and family of projections \( \{Q_\ast(t) : t \in \mathbb{R}\} \).

If \( \ell > 0 \) is sufficiently small, then there are graphs corresponding to the fast and slow submanifolds represented by \( W^u_{\text{fast}}(0), W^u_{\text{slow}}(0) \) (\( W^u_{\text{fast}}(0)(t) \) is tangent to \( \text{Im}(Q_\ast(t)) \) and \( W^u_{\text{slow}}(0)(t) \) is tangent to \( \text{Ker}(Q_\ast(t)Q(t)) \)) of the invariant manifold represented by \( W^u(0) \) (\( W^u(0)(t) \) is tangent to \( \text{Im}(Q(t)) \)) such that

\[
\lim_{t \to -\infty} \frac{\| (I - P_{\text{fast}}(\tau)) T(\tau, t) u \|}{\| (I - P_{\text{fast}}(\tau)) T(\tau, t) u \|} = 0,
\]

for any \( (\tau, u) \in W^u(0) \setminus W^u_{\text{fast}}(0) \), where \( P_{\text{fast}}(\cdot) \) and \( P_{\text{slow}}(\cdot) \) denote the respective nonlinear projections onto \( W^u_{\text{fast}}(0) \) and \( W^u_{\text{slow}}(0) \).

**Figure 1.** The local dynamics inside the unstable manifold of \( u_\ast \equiv 0 \). Note that solutions in \( W^u(0) \setminus W^u_{\text{fast}}(0) \) are tangent space of \( W^u_{\text{slow}}(0) \) as \( t \to -\infty \).

**Proof.** For \( (\tau, u_0) \in W^u(0) \setminus W^u_{\text{fast}}(0) \), we obtain from (10) and (35) that

\[
\| (I - P_{\text{fast}}(t)) T(t, \tau) u_0 \| \leq M e^{-\delta_\ast (t-\tau)} \| (I - P_{\text{fast}}(\tau)) u_0 \|, \quad t \geq \tau.
\]

where \( \delta_\ast = \gamma_\ast - M \ell - \frac{M^2 \ell^2 (1+\kappa)(1+M)}{\gamma_\ast - \rho_\ast - \ell M(1+\kappa)} \). Similarly, from and (35) we have that

\[
\| (I - P_{\text{slow}}(\tau)) u_0 \| \leq M e^{\tilde{\delta}_\ast (t-\tau)} \| (I - P_{\text{slow}}(\tau)) T(t, \tau) u_0 \|, \quad t \geq \tau.
\]

where \( \tilde{\delta}_\ast = \rho_\ast + M \ell + \frac{M^2 \ell^2 (1+\kappa)(1+M)}{\gamma_\ast - \rho_\ast - \ell M(1+\kappa)} \).

Therefore, the bounds (60) and (61) applied to \( u_0 = T(\tau, t) u \) yield

\[
\frac{\| (I - P_{\text{slow}}(\tau)) T(\tau, t) u \|}{\| (I - P_{\text{fast}}(\tau)) T(\tau, t) u \|} \leq M^2 e^{-\tilde{\delta}_\ast (t-\tau)} \frac{\| (I - P_{\text{slow}}(t)) u_0 \|}{\| (I - P_{\text{fast}}(t)) u_0 \|}, \quad t \geq \tau.
\]

where \( \tilde{\delta}_\ast = \gamma_\ast - \rho_\ast - 2 M \ell - \frac{2M^2 \ell^2 (1+\kappa)(1+M)}{\gamma_\ast - \rho_\ast - \ell M(1+\kappa)} \). The limit \( \tau \to -\infty \) yields the desired claim, since \( \tilde{\delta}_\ast > 0 \), for suitably small \( \ell > 0 \). \( \square \)
Remark 3.2. Consider a small non-autonomous perturbation of a scalar semilinear parabolic equation in one spatial dimension,

\begin{equation}
\begin{aligned}
    u_t &= u_{xx} + f_\nu(t, x, u), \quad x \in (0, \pi), \\
    u(t, 0) &= u(t, \pi) = 0,
\end{aligned}
\end{equation}

for \( x \in (0, \pi) \) with Dirichlet boundary conditions, \( \nu \in [0, 1) \) and \( f_\nu(t, x, 0) = 0 \) for all \( t \in \mathbb{R} \), \( x \in (0, \pi) \) and \( \nu \in [0, 1) \). Assume that

\begin{equation}
\limsup_{\nu \to 0} \sup_{t \in \mathbb{R}} \sup_{x \in [0, \pi]} \sup_{|u| \leq R} \{|f_\nu(t, x, u) - f_0(x, u)| + |\partial_u f_\nu(t, x, u) - \partial_u f_0(x, u)|\} = 0
\end{equation}

for all \( R > 0 \). Assume further that \( f_0(x, u) \) satisfy dissipative growth conditions and thus the limiting autonomous equation with nonlinearity \( f_0(x, u) \) possess a gradient global attractor. Suppose that all equilibria of the limiting autonomous equations are hyperbolic. If \( \nu \) is sufficiently small, associated to each hyperbolic equilibria there is a hyperbolic global solution and the non-autonomous evolution process associated to (63) is dynamically gradient, see [5].

Due to Remark 1.1, without loss of generality, we analyse the local dynamics nearby \( u_* \equiv 0 \). Consider the linear operator \( A := \partial_{xx} + f'_0(x, 0)I \), which is Sturm-Liouville. Hence its spectrum consists of simple eigenvalues \( \{\lambda_k\}_{k \in \mathbb{N}} \) with corresponding eigenfunctions denoted by \( \{\phi_k\}_{k \in \mathbb{N}} \). Let \( n \in \mathbb{N} \) be the Morse index of the hyperbolic equilibrium \( u_* \equiv 0 \) and thus \( \lambda_1, \ldots, \lambda_n > 0 \), whereas \( \lambda_{n+1}, \lambda_{n+2}, \ldots < 0 \).

For \( k \leq n \), consider the splitting \( \sigma(A) := \sigma_k^+ \cup \sigma_k^- \), where \( \sigma_k^+ = \{\lambda_1, \ldots, \lambda_k\} \) and \( \sigma_k^- = \{\lambda_{k+1}, \lambda_{k+2}, \ldots\} \). Let \( Q_k^0 \) be the spectral projection associated to \( \sigma_k^+ \). For sufficiently small \( \nu \in [0, 1) \), there are projections \( Q_k^\nu(t) \) (which are suitably close to the orthogonal projections \( Q_k^0 \)) associated to the exponential splitting of the linear process described by

\begin{equation}
\begin{aligned}
    u_t &= u_{xx} + f'_0(x, 0)u + [f'_\nu(t, x, 0) - f'_0(x, 0)]u, \quad x \in (0, \pi) \\
    u(0) &= u(\pi) = 0.
\end{aligned}
\end{equation}

After an appropriate change of coordinates, we can apply Corollary 3.1 for any \( k = 1, \ldots, n \), yielding the \( k \)-fast unstable manifold, \( W_k^u(0) \subseteq W^u(0) \), which is tangent to \( \text{Im}(Q_k^\nu(t)) \). Moreover, we can also apply Corollary 3.2 to determine that solutions in \( W^u(0) \setminus W_k^u(0) \) are tangent to the lower dimensional subspaces \( \text{Im}(Q^\nu(t)) \setminus \text{Im}(Q_k^\nu(t)) \), where \( Q^\nu(t) \) is the projection associated with the exponential dichotomy that splits the positive and negative eigenstructure. Due to the continuity of the projections we conclude that all solutions in the unstable manifold approach the equilibria in directions with are suitably close to the directions of the eigenfunctions of the corresponding linearization.

As a consequence, solutions of equations with asymptotically autonomous nonlinearities approach the equilibria in the direction of an eigenfunction associated to the linearization of the limiting problem around the associated equilibria.

3.3. Roughness of Exponential Dichotomy. In this section, we prove that the roughness of exponential dichotomy, i.e., that exponential dichotomies are preserved under small perturbations.

We first assume that the linear evolution process \( \{L(t, \tau) : t \geq \tau\} \) associated to the problem

\begin{equation}
\dot{u} = Au(t), \quad t \geq \tau, \quad u(\tau) = u_0,
\end{equation}

has exponential dichotomy with constant \( M \) and exponent \( \gamma > 0 \) and then consider the linear evolution process \( \{T(t, \tau) : t \geq \tau\} \), associated to a perturbation of it, given by the following linear
equation,
\begin{equation}
\dot{u} = A(t)u + B(t)u, \ t \geq \tau, \ u(\tau) = u_0.
\end{equation}
where the map \( t \mapsto B(t) \in \mathcal{L}(X) \) is strongly continuous for \( t \in \mathbb{R} \) and \( \sup_{t \in \mathbb{R}} \| B(t) \|_{\mathcal{L}(X)} \leq \ell \), for some suitably small \( \ell > 0 \). Recall that, as in (3), the evolution process \( \{ T(t, \tau) : t \geq \tau \} \subset \mathcal{L}(X) \) associated to (66) is given by
\begin{equation}
T(t, \tau) = L(t, \tau) + \int_{\tau}^{t} L(t, s)B(s)T(s, \tau)\, ds, \ t \geq \tau.
\end{equation}
We wish to prove that (66) has exponential dichotomy for suitably small \( \ell \).

This result can be obtained by firstly applying Theorems 2.1 and 2.2 in a linear setting, which are suitable in order to establish the existence of the linear invariant manifold and its stable manifold (see Corollary 3.3) and then apply it to (66) with \( \gamma > 0 \) and \( \rho = -\gamma \).

**Corollary 3.3.** Suppose that \( \{ L(t, \tau) : t \geq \tau \} \) has exponential splitting with constant \( M \), exponents \( \gamma > \rho \) and family of projections \( \{ Q(t) : t \in \mathbb{R} \} \) and that (6) is satisfied. The following holds:

- There are maps \( \Sigma^*, \Theta^* : \mathbb{R} \times X \to X, \Sigma^*(t, \cdot), \Theta^*(t, \cdot) \in \mathcal{L}(X) \) and \( \| \Sigma^*(t, u) \| \leq \kappa \| u \|_X \), \( \| \Theta^*(t, u) \|_X \leq \kappa \| u \|_X \) for all \( (t, u) \in \mathbb{R} \times X \) and for some \( \kappa = \kappa_\ell > 0 \);
- The graph \( G(\Sigma^*) \) of \( \Sigma^* \) is an invariant family and (10) holds, the graph \( G(\Theta^*) \) of \( \Theta^* \) is a positively invariant family;
- The evolution process \( \{ T(t, \tau) : t \geq \tau \} \) given by (67) satisfies
\begin{align}
\| T(t, \tau)P_{\Sigma^*}(\tau) \|_{\mathcal{L}(X)} &\leq M(1 + \kappa)e^{-(\rho + M(1 + \kappa))(t-\tau)}, \ t \leq \tau, \\
\| T(t, \tau)P_{\Theta^*}(\tau) \|_{\mathcal{L}(X)} &\leq M(1 + \kappa)e^{-(\gamma - M(1 + \kappa))(t-\tau)}, \ t \geq \tau,
\end{align}
where \( P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u) \) and \( P_{\Theta^*}(t)u := \Theta^*(t, (I - Q(t))u) + (I - Q(t))u, t \in \mathbb{R} \).

**Proof.** The proof is a direct consequence of Theorems 2.1 and 2.2 in the case that \( f(t, \cdot) \) is linear and uniformly (with respect to \( t \)) bounded. Note that, the linearity of \( \Sigma(t, \cdot) \) follows since \( f(t, \cdot) \) is linear, and thereby \( G(\Sigma) \) given by equation (16) is also linear. Consequently, the fixed point, \( G(\Sigma^*)(t, u) = \Sigma^*(t, u) \), is linear. Similarly, \( G \) in equation (38) is also linear and so is \( \Theta^* \).

Next, we show the robustness of the exponential dichotomy.

**Corollary 3.4.** Suppose that \( \{ L(t, \tau) : t \geq \tau \} \) has exponential dichotomy with constant \( M \geq 1 \), exponent \( \gamma > 0 \) and family of projections \( \{ Q(t) : t \in \mathbb{R} \} \). Moreover, assume that \( \sup_{t \in \mathbb{R}} \| B(t) \|_{\mathcal{L}(X)} \leq \ell \), where \( \ell > 0 \) satisfies
\begin{equation}
\ell < \frac{2\gamma}{3M(M + 1)}.
\end{equation}
Then \( \{ T(t, \tau) : t \geq \tau \} \) has exponential dichotomy, i.e., there is a family of projections \( \{ Q_\ell(t) : t \in \mathbb{R} \} \) such that \( T(t, \tau) : \text{Im}(Q_\ell(\tau)) \to \text{Im}(Q_\ell(t)) \) is an isomorphism for \( t \geq \tau \), with inverse denoted by \( T(\tau, t) \), and
\begin{align}
\| T(t, \tau)Q_\ell(\tau) \|_{\mathcal{L}(X)} &\leq M_\ell e^{\gamma_\ell (t-\tau)}, \ t \leq \tau, \\
\| T(t, \tau)(I - Q_\ell(\tau)) \|_{\mathcal{L}(X)} &\leq M_\ell e^{-\gamma_\ell (t-\tau)}, \ t \geq \tau,
\end{align}
where \( M_\ell := M(1 + \kappa_\ell)/(1 - 2\kappa_\ell) > 1 \) and \( \gamma_\ell := \gamma - \ell M(1 + \kappa_\ell) > 0 \) for the Lipschitz constant \( \kappa_\ell \) obtained in Corollary 3.3. Moreover,
\begin{equation}
\sup_{t \in \mathbb{R}} \| Q(t) - Q_\ell(t) \|_{\mathcal{L}(X)} \leq \frac{2\kappa_\ell}{1 - 2\kappa_\ell}.
\end{equation}
Remark 3.3. Note that Corollary 3.4 explicitly provides the constant and exponent of the exponential dichotomy for \( \{ T(t, \tau) : t \geq \tau \} \) depending of \( \ell \). Moreover, the constant \( \kappa_\ell \) in inequality (71) is given by smallest possible Lipschitz constant \( \kappa_- \) explicitly obtained in (21), which occurs in the equality of equation (20). In particular, the hypothesis (69) holds in case that \( \ell > 0 \) is suitably small. Furthermore, note that \( \kappa_\ell \to 0, M_\ell \to M, \gamma_\ell \to \gamma, Q_\ell \to Q \) as \( \ell \to 0 \), uniformly in \( t \).

Proof. Consider the linear maps \( P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u) \) and \( P_{\Theta^*}(t)u := (I - Q(t))u + \Theta^*(t, (I - Q(t))u) \), for \( (t, u) \in \mathbb{R} \times X \), which were obtained in Corollary 3.3, where \( \Sigma^* \) and \( \Theta^* \) are bounded linear maps that do not exceed \( \kappa_\ell > 0 \).

We will prove that \( X = \text{Im}(P_{\Sigma^*}(t)) \oplus \text{Im}(P_{\Theta^*}(t)) \), for every \( t \in \mathbb{R} \). Equivalently, we show that, for each \( (t, u) \in \mathbb{R} \times X \), the operator \( \mathcal{I}_u(t) \), defined by

\[
\mathcal{I}_u(t) : X \to X
\]

\[
v \mapsto \mathcal{I}_u(t)v := u - \Sigma^*(t, v) - \Theta^*(t, v),
\]

admits a unique fixed point on \( X \). In fact, if that is the case, for each \( (t, u) \in \mathbb{R} \times X \), there exists a unique \( v_u \in X \) such that \( \mathcal{I}_u(t)v_u = v_u \), that is,

\[
u - \Sigma^*(t, v_u) - \Theta^*(t, v_u) = v_u = Q(t)v_u + (I - Q(t))v_u,
\]

or

\[
u = Q(t)v_u + \Sigma^*(t, v_u) + (I - Q(t))v_u + \Theta^*(t, v_u) = P_{\Sigma^*}(t)v_u + P_{\Theta^*}(t)v_u.
\]

Which is the unique representation of \( u \) as a sum of elements of \( \text{Im}(P_{\Sigma^*}(t)) \) and \( \text{Im}(P_{\Theta^*}(t)) \) and proves the desired decomposition.

In order to show that \( \mathcal{I}_u(t) \) has a unique fixed point, note that \( \mathcal{I}_u(t) \) is a contraction on \( X \), since

\[
\|\mathcal{I}_u(t)v - \mathcal{I}_u(t)\tilde{v}\| = \|\Sigma^*(t, v) - \Sigma^*(t, \tilde{v}) + \Theta^*(t, v) - \Theta^*(t, \tilde{v})\|,
\]

\[
\leq 2\kappa\|v - \tilde{v}\|,
\]

for any \( (t, v), (t, \tilde{v}) \in \mathbb{R} \times X \), as \( \Sigma^*(t, \cdot), \Theta^*(t, \cdot) \) are Lipschitz with constant \( \kappa = \kappa_\ell > 0 \), for all \( t \in \mathbb{R} \). Thus, \( \mathcal{I}_u(t) \) is a contraction, for each \( t \in \mathbb{R} \) and \( \kappa \in [\kappa_-, \min\{1/2, \min\{\kappa_+, \kappa_*\}\}] \), where \( \kappa_- \) is given by (21), since the hypothesis (69) implies that \( \kappa_- < 1/2 \). Without loss of generality, we may choose \( \kappa_\ell := \kappa_- \).

![Figure 2. Given a point \( u \in X \), we find a unique point \( v_u \in X \) such that \( P_{\Sigma^*}(t)u = Q(t)v_u + \Sigma^*(t, Q(t)v_u) \) and \( P_{\Theta^*}(t)u = (I - Q(t))v_u + \Theta^*(t, (I - Q(t))v_u) \).](image-url)
Note that, for each \( u \in X \), since \( v_u \) is the unique element of \( X \) satisfying \( v_u = u - \Sigma^*(t, v_u) - \Theta^*(t, v_u) \), the map \( u \mapsto v_u \) is a bounded linear operator such that

\[
\|v_u\|_X \leq \frac{\|u\|_X}{1 - 2\kappa},
\]

due to the triangle inequality.

For each \( t \in \mathbb{R} \), define \( Q_t(t) \in \mathcal{L}(X) \). The linear projection onto \( \text{Im}(P_{\Sigma^*}(t)) \) along \( \text{Im}(P_{\Theta^*}(t)) \), which can be written as \( Q_t(t)u := P_{\Sigma^*}(t)v_u \) due to the first part of the proof. Its complementary projection is given by \( (I - Q_t(t))u = P_{\Theta^*}(t)v_u \), for each \( (t, u) \in \mathbb{R} \times X \).

From Corollary 3.3, we have that \( \{\text{Im}(Q_t(t)) : t \in \mathbb{R}\} \) is invariant and \( \{\text{Im}(I - Q_t(t)) : t \in \mathbb{R}\} \) is positively invariant. Thus \( T(t, \tau)Q_t(\tau) = Q_t(t)T(t, \tau) \), for every \( t \geq \tau \). Equations (68) and (76) imply the bounds in (70). This proves that \( \{T(t, \tau) : t \geq \tau\} \) has exponential dichotomy with constant \( M_t := M(1 + \kappa)/(1 - 2\kappa) \) and exponent \( \gamma_\ell := \gamma - \ell M(1 + \kappa) > 0 \).

Lastly, we prove the bound in equation (71), i.e., the continuous dependence of the projections \( \{Q(t) : t \in \mathbb{R}\} \) and \( \{Q_{\ell}(t) : t \in \mathbb{R}\} \), corresponding to the exponential dichotomies of the respective evolution processes \( \{L(t, \tau) : t \geq \tau\} \) and \( \{T(t, \tau) : t \geq \tau\} \).

Consider \( u \in X \), which can be uniquely decomposed as \( u = v_u + \Sigma^*(t, v_u) + \Theta^*(t, v_u) \). Hence, \( Q(t)u = Q(t)v_u + \Theta^*(t, v_u) \), since \( Q(t)\Sigma^*(t, v_u) = 0 \), and \( Q_{\ell}(t)u = Q(t)v_u + \Sigma^*(t, v_u) \), by definition of \( Q_{\ell}(t) \). Therefore,

\[
Q(t)u - Q_{\ell}(t)u = \Theta^*(t, v_u) - \Sigma^*(t, v_u).
\]

Since \( \Sigma^*, \Theta^* \) are Lipschitz with constant \( \kappa_\ell > 0 \), and due to equation (76), we obtain (71). \hfill \square

### 3.4. Hyperbolic Solutions for a Nonautonomous PDE

We establish hyperbolicity for global solutions a non-autonomous parabolic PDE by reducing the infinite dimensional problem to a two-dimensional non-autonomous ODE with hyperbolic solutions.

Consider the scalar parabolic non-autonomous equation

\[
\begin{align*}
  u_t &= (a_\nu(x)u_x)_x + f(t, u), & x \in (0, 1), & t > \tau \\
  u_x(t, 0) &= u_x(t, 1) = 0, & t > \tau \\
  u(\tau, x) &= u_0(x), & x \in (0, 1),
\end{align*}
\]

where the reaction is given by \( f(t, u) := u - \beta(t)u^2 \), where \( \beta : \mathbb{R} \to \mathbb{R} \) is an uniformly positive and bounded globally Lipschitz function. The diffusivity \( a_\nu \in C^2(\mathbb{R}) \) satisfies \( a_\nu(x) > 0 \), for all \( x \in [0, 1] \) and \( \nu > 0 \), it is small in a neighborhood of a point \( x_\ast \in (0, 1) \) and it is large outside this neighborhood for sufficiently small \( \nu > 0 \), according to [11]. More precisely,

\[
\begin{align*}
  a_\nu(x) &\geq \frac{1}{\nu}, & x \in (0, x_\ast - \nu\beta_\nu) \cup (x_\ast + \nu\beta_\nu, 1), \\
  a_\nu(x) &\geq \nu\alpha_0, & x \in (x_\ast - \nu\beta_\nu, x_\ast + \nu\beta_\nu), \\
  a_\nu(x) &\leq \nu\beta_\nu, & x \in (x_\ast - \nu\beta_0, x_\ast + \nu\beta_0),
\end{align*}
\]

for some \( x_\ast \in (0, 1) \), \( \alpha_\nu, \beta_\nu \in (0, \infty) \) satisfying \( \alpha_\nu \searrow \alpha_0 > 0, \beta_\nu \searrow \beta_0 > 0 \), as \( \nu \to 0 \).

The spectral properties of the operator \( A_\nu u := -(a_\nu(x)u_x)_x \), defined by \( A_\nu : D(A_\nu) \subset L^2(0, 1) \to L^2(0, 1), \) where \( D(A_\nu) := \{u \in H^2(0, 1) : u_x(0) = u_x(1) = 0\} \) can be found in [11, Lemma 1.1]. In particular, it was shown that the spectrum consists of a sequence of eigenvalues, \( \lambda_1 < \lambda_2 < \lambda_3 < \ldots \), with corresponding normalized eigenfunctions, \( \phi_1, \phi_2, \phi_3, \ldots \), such that

\[
\begin{align*}
  \lambda_1 &= 0, & \lambda_2 &\xrightarrow{\nu \to 0} \frac{\alpha_0}{2\beta_0(1 - x_\ast)}, & \lambda_3 &\xrightarrow{\nu \to 0} + \infty,
\end{align*}
\]
and

\[(81) \quad \phi_1(x) \equiv 1, \quad \phi_2(x) = \begin{cases} -k_1 + O(\nu^\epsilon), & x \in [0, x_* - \nu \beta_0] \\ O(1), & x \in [x_* - \nu \beta_0, x_* + \nu \beta_0] \\ k_1^{-1} + O(\nu^\epsilon), & x \in [x_* + \nu \beta_0, 1] \\ \end{cases} \quad \nu \to 0 \xrightarrow{\nu \to 0} \begin{cases} -k_1, & x \in [0, x_*) \\ k_1^{-1}, & x \in (x_*, 1], \end{cases}
\]

for some \( \epsilon \in (0, 1) \) and \( k_1 := \sqrt{(1-x_*)/x_*} \).

Therefore, (78) is associated with a skew-product semiflow in the phase-space \( X^\alpha \times \mathcal{B} \), where \( X^\alpha \) is the fractional power spaces associated with \( A \), for some \( \alpha \in (0, 1) \), and \( \mathcal{B} \) is the base space consisting of the closure of time translations of \( \beta(\cdot) \), with respect to the metric of the uniform convergence in bounded intervals. Moreover, this skew-product semiflow is dissipative and hence has a global attractor \( A_\nu \), with associated uniform attractor \( A_\nu \) for all \( \nu > 0 \), see [6, Chapter 6]. Next, we present a result regarding the existence of non-trivial hyperbolic global solutions of (78).

**Corollary 3.5.** For sufficiently small \( \nu > 0 \), there are four non-trivial hyperbolic global solutions of (78).

**Proof.** We rewrite solutions \( u(t, x) \) of equation (78) according to the eigenprojections in the first two eigendirections and a remainder term, as

\[(82) \quad u(t, x) = u_1(t)\phi_1 + u_2(t)\phi_2 + w(t, x),\]

which evolve according to the following system of differential equations

\[(83) \quad \begin{align*}
\dot{u}_1 &= h_1(t, u_1, u_2, w), & t > \tau, \\
\dot{u}_2 &= -\lambda_2 u_2 + h_2(t, u_1, u_2, w), & t > \tau, \\
\dot{u}_1 &= (\alpha_{\nu}(x)w_x)_x + h_3(t, u_1, u_2, w), & t > \tau,
\end{align*}\]

where \( w \) satisfies Neumann boundary conditions, \( w_x(t, 0) = w_x(t, 1) = 0 \) for all \( t > \tau \), with initial condition \( w(\tau, x) = u_0(x) - u_1(\tau, x)\phi_1(x) - u_2(\tau, x)\phi_2(x) \), and the vector field is given by

\[(84) \quad \begin{align*}
h_1(t, u_1, u_2, w) &= \int_0^1 f(t, u_1(t)\phi_1(y) + u_2(t)\phi_2(y) + w(t, y))dy, \\
h_2(t, u_1, u_2, w) &= \int_0^1 f(t, u_1(t)\phi_1(y) + u_2(t)\phi_2(y) + w(t, y))\phi_2(y)dy, \\
h_3(t, u_1, u_2, w) &= f(t, u_1\phi_1 + u_2\phi_2 + w) - h_1(t, u_1, u_2, w) - h_2(t, u_1, u_2, w)\phi_2.
\end{align*}\]

As a consequence of Theorem 2.3, we obtain the non-autonomous version of the inertial manifold reduction in [11, Theorem 1.2], due to the arbitrarily large gap \( \gamma - \rho := \lambda_3 - \lambda_2 \) in (80) for sufficiently small \( \nu > 0 \). Note that we obtain a two-dimensional local inertial manifold \( \{\mathcal{M}_{\nu}(t) : t \in \mathbb{R}\} \) according to (52) for each sufficiently small \( \nu > 0 \), which is a graph over \( \text{Im}(Q(t)) = \text{span}\{\phi_1, \phi_2\} \). Moreover, the graph describing the local inertial manifold satisfies \( \Sigma_\nu \to 0 \), as \( \nu \to 0 \), in \( C^1 \)-norm, in a similar manner as in [11], due to the properties of the inertial manifold in Theorem 2.1. Thus it is sufficient to analyze the corresponding two-dimensional vector field projected to \( \text{span}\{\phi_1, \phi_2\} \) for sufficiently small \( \nu > 0 \), as the remaining of the solutions can be obtained in terms of the graph. Furthermore, the solution in the inertial manifold is given by \( u(t, x) = u_1(t) + u_2(t)\phi_2(x) + \Sigma_\nu(t, u_1(t) + u_2(t)\phi_2) \), where \( (u_1(t), u_2(t)) \) is the solution of

\[(85) \quad \begin{align*}
\dot{u}_1(t) &= h_1(t, u_1, u_2, \Sigma_\nu(t, u_1, u_2)), \\
\dot{u}_2(t) &= -\lambda_2 u_2 + h_2(t, u_1, u_2, \Sigma_\nu(t, u_1, u_2)).
\end{align*}\]
Since $\Sigma_{\nu} \to 0$, as $\nu \to 0$, the limiting ordinary differential equation, as $\nu \to 0$, is given by

$$
\begin{align*}
\dot{u}_1(t) &= f_1(t, u_1, u_2), \\
\dot{u}_2(t) &= -\frac{\alpha_0}{2\beta_0} \left(1 - x_+\right) u_2 + f_2(t, u_1, u_2),
\end{align*}
$$

where

$$
\begin{align*}
f_1(t, u_1, u_2) := x_* f(t, u_1 - k_1 u_2) + (1 - x_+) f(t, u_1 + k_1^{-1} u_2), \\
f_2(t, u_1, u_2) := -x_* f(t, u_1 - k_1 u_2) + (1 - x_+) k_1^{-1} f(t, u_1 + k_1^{-1} u_2).
\end{align*}
$$

Changing the variables, $z_1(t) := u_1(t) - k_1 u_2(t)$ and $z_2(t) := u_1(t) + k_1^{-1} u_2(t)$, the new variables $z_1, z_2$ evolve according to the following ODE

$$
\begin{align*}
\dot{z}_1 &= \frac{\alpha_0}{2\beta_0 x_*} (z_2 - z_1) + f(t, z_1) \\
\dot{z}_2 &= -\frac{\alpha_0}{2\beta_0 (1 - x_+)} (z_2 - z_1) + f(t, z_2)
\end{align*}
$$

For $x_* = 1/2$ and $\alpha_0/\beta_0 \in (\frac{1}{3}, \frac{1}{2})$, the 2-dimensional ODE (88) is considered in [10]. Taking advantage of the invariant subsets $E_1 = \{(z_1, z_2) : z_1 = z_2\}$ and $E_2 = \{(z_1, z_2) : z_1 = -z_2\}$, the authors in [10, Theorem 3.4] prove that there are exactly four non-autonomous global solutions which remain away from zero hyperbolic solutions $\xi_{1,\pm}$ and $\xi_{2,\pm}$ of (88), corresponding to the asymptotic profiles for the longtime behaviour.

Since we now know there are four non-trivial hyperbolic global non-autonomous solutions of equation (88) in the variables $(z_1, z_2)$, we can reverse gear and obtain corresponding hyperbolic solutions for (86) in the variables $(u_1, u_2)$. Moreover, since $\Sigma_{\nu} \to 0$ as $\nu \to 0$, in $C^1$-norm uniformly in $t \in \mathbb{R}$, we apply the persistence of hyperbolic solutions in [9, Lemma 8.3] for each non-trivial hyperbolic solution of the limiting case $\nu = 0$ in (86), which ensures the existence of a hyperbolic solution of (85) for any sufficiently small $\nu > 0$. Therefore, we obtain the four hyperbolic global solutions of (78) for any sufficiently small $\nu > 0$.

\[
\square
\]

**Remark 3.4.** The family of pullback attractors $\{A_{\nu}(t) : \nu \geq 0\}$ of (78) and (86) are continuous at $\nu = 0$, i.e., $d_H(A_{\nu}(t), A_0(t)) \to 0$ as $\nu \to 0$, see [8, Theorem 7.1]. Moreover, for $x_* = 1/2$, $\alpha_0/\beta_0 \in (1/3, 1/2)$ and $\beta : \mathbb{R} \to [1, 2]$, the skew product associated to (78) is dynamically gradient for suitably small $\nu > 0$, since this is a small perturbation of a dynamically gradient system, see [8, 10].

**References**

[1] S. Angenent. The Morse-Smale property for a semi-linear parabolic equation. *J. Diff. Eq* 62, 427-442, (1986).

[2] L. Barreira, C. Silva and C. Valls. *Stability of Nonautonomous Differential Equations*. Lecture Notes in Math. 1926, Springer-Verlag (2008).

[3] L. Barreira and C. Valls. Robustness of non-invertible dichotomies. *J. Math. Soc. Japan* 67 (1), 293-317, (2015).

[4] L. Barreira and C. Valls. On the Robustness of exponential dichotomies. *Fixed Point Theory* 16 (1), 31-48, (2015).

[5] M. C. Bortolan, C. A. E. N. Cardoso, A. N. Carvalho and L. Pires. Lipschitz perturbations of Morse-Smale semigroups. *J. Diff. Eq. 269* (3), 1904-1943, (2020).

[6] M. Bortolan, A. N. Carvalho, J. A. Langa. *Attractors Under Autonomous and Non-autonomous Perturbations*. Mathematical Surveys and Monographs, AMS, (2020).
INERTIAL MANIFOLDS, SADDLE POINT PROPERTY AND DICHOTOMY

7. P. Brunovský and B. Fiedler. Numbers of Zeros on Invariant Manifolds in Reaction-diffusion Equations. Nonlinear Analysis: TMA 10, 179-193, (1986).

8. A.N. Carvalho and J.A. Langa. Non-autonomous perturbation of autonomous semilinear differential equations: Continuity of local stable and unstable manifolds. J. Diff. Eq 233, 622-653, (2007).

9. A.N. Carvalho, J.A. Langa and J.C. Robinson. Attractors for infinite-dimensional non-autonomous dynamical systems. Applied Mathematical Sciences 182, Springer-Verlag 2013.

10. A.N. Carvalho, J. A. Langa, R. Obaya and L.R.N. Rocha. Structure of non-autonomous attractors for a class of diffusively coupled ODE. Preprint, (2021).

11. A.N. Carvalho and A.L. Pereira. A Scalar Parabolic Equation Whose Asymptotic Behavior Is Dictated by a System of Ordinary Differential Equations. J. Diff. Eq. 112, 81-130, (1994).

12. V. Chepyzhov and M. Vishik. Attractors for Equations of Mathematical Physics, volume 49 of American Mathematical Society Colloquium Publications. Amer. Math. Soc., 2002.

13. S.-N. Chow, K. Lu and G. Sell. Smoothness of Inertial Manifolds. J. Math. An. App. 169, 283-312, (1992).

14. W. A. Coppel. Dichotomies and reducibility. J. Diff. Eq. 3, 500-521, (1967).

15. W. A. Coppel. Dichotomies in stability theory. Lecture Notes in Mathematics 629, Springer-Verlag, Berlin-New York, (1978).

16. J. L. Dalecki˘ı and M.G. Kre˘ın. Stability of solutions of differential equations in Banach space. American Math. Soc. Transl., Providence RI, 1974.

17. C. Foias, G. Sell and R. Temam. Inertial manifolds for nonlinear evolutionary equations. J. Diff. Eq. 73 (2), 309-353, (1988).

18. J. K. Hale. Ordinary differential equations. Second edition. Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., (1980).

19. J. K. Hale. Asymptotic Behavior of Dissipative Systems. American Mathematical Society, (1988).

20. D. Henry. Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics 840, Springer-Verlag, Berlin, (1981).

21. D. Henry. Semigroups. Handwritten Notes. IME-USP, São Paulo SP, Brazil, (1981).

22. N. Koksch and S. Siegmund. Pullback Attracting Inertial Manifolds for Nonautonomous Dynamical Systems. J. Dyn. Diff. Eq. 14, 889-941, (2002).

23. A. Kostianko and S. Zelik. Smooth extensions for inertial manifolds of semilinear parabolic equations. arXiv:2102.03473, (2021).

24. Y. Latushkin, T. Randolf and R. Schnaubelt. Exponential Dichotomy and Mild Solutions of Nonautonomous Equations in Banach Spaces. J. Dyn. Diff. Eq. 10, 489-510, (1998).

25. A. Lunardi. Analytic Semigroups and Optimal Regularity in Parabolic Problems. Birkhäuser Verlag, Basel, (1995).

26. A. M. Lyapunov. Obščaya zada˘ı ob ustoo˘čivosti dviženiya. (Russian) General Problem of the Stability of Motion, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 471 pp., (1950).

27. J. L. Massera and J. J. Schäffer. Linear differential equations and functional analysis. Academic Press, New York (1966).

28. J. L. Massera and J. J. Schäffer. Linear differential equations and functional analysis I. Ann. of Math. 67, 517-573, (1958).

29. J. L. Massera and J. J. Schäffer. Linear differential equations and functional analysis III. Ann. of Math. 69, 535-574, (1959).

30. A. Mielke. Locally Invariant Manifolds for Quasilinear Parabolic Equations. Rocky Mountain J. Math. 21, 707-714, (1991).

31. M. Mikuševič. A sharp condition for existence of an inertial manifold. J. Dyn. Diff. Eq. 3, 437-456, (1991).

32. O. Perron. Die Stabilitätsfrage bei Differentialgleichungen. Math. Z. 32, (1930).

33. V. A. Pliss. A reduction principle in the theory of stability of motion. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 28, 1297-1324, (1964).

34. R. J. Sacker and G. R. Sell. Existence of dichotomies and invariant splittings for linear differential equations I. J. Diff. Eq. 15, 429-458, (1974).

35. R. J. Sacker and G. R. Sell. Existence of dichotomies and invariant splittings for linear differential equations II. J. Diff. Eq. 22, 478-496, (1976).
[36] R. J. Sacker and G. R. Sell. Existence of dichotomies and invariant splittings for linear differential equations III. *J. Diff. Eq.* **22**, 497-522, (1976).

[37] J. J. Schäffer. Norms and determinants of linear mappings. *Math. Z.* **118**, 331-339, (1970).

[38] G. R. Sell. Nonautonomous Differential Equations and Topological Dynamics I. The Basic Theory. *Trans. Amer. Math. Soc.* **127**, 241 – 262, (1967).

[39] G. R. Sell and Y. You. *Dynamics of Evolutionary Equations*. Springer, New York, (2000).

[40] E. Vesentini. *Introduction to continuous semigroups*. Pisa, Scuola Normale Superiore, (2002).

[41] A. Yagi. *Abstract Parabolic Evolution Equations and their Applications*. Berlin: Springer Verlag, (2010).

[42] S. Zelik. Inertial manifolds and finite-dimensional reduction for dissipative PDEs. *Proc. Royal Soc. Edinburgh: Sec. A Math.* **144** (6), 1245-1327, (2014).

(A. N. Carvalho, P. Lappicy, E. M. Moreira, A. N. Oliveira-Sousa) INSTITUTO DE CIÊNCIAS MATEMÁTICAS E DE COMPUTAÇÃO, UNIVERSIDADE DE SÃO PAULO-CAMPUS DE SÃO CARLOS, CAIXA POSTAL 668, SÃO CARLOS SP, BRAZIL

Email address, A. Carvalho: andcarva@icmc.usp.br

Email address, P. Lappicy: lappicy@usp.br

Email address, E. Moreira: estefani@usp.br

Email address, A. Oliveira-Sousa: alexandrenosousa@gmail.com