RESEARCH ARTICLE

Linnik’s large sieve and the $L^1$ norm of exponential sums

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Abstract
The elementary method of Balog and Ruzsa and the large sieve of Linnik are utilized to investigate the behaviour of the $L^1$ norm of an exponential sum over the primes. A new proof of a lower bound due to Vaughan for the $L^1$ norm of an exponential sum formed with the von Mangoldt function is furnished.

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1 | INTRODUCTION

The $L^1$ norm of exponential sums whose coefficients are taken to be arithmetical functions, such as the Möbius $\mu$ and the von Mangoldt $\Lambda$ functions, as well as the characteristic function of smooth numbers, arises in many interesting problems in analytic number theory. Balog and Perelli [1], for
example, have published a work in which they proved that, for some positive constant $A$,
\[
\exp\left(\frac{A \log N}{\log \log 2N}\right) \ll \int_0^1 \left| \sum_{n=1}^N \mu(n)e(n\alpha) \right| d\alpha \ll N^{1/2},
\] (1)
where $e(x) = e^{2\pi i x}$.

McGehee et al. [9] have solved entirely a problem of Littlewood [7] in classical Fourier analysis concerning a lower bound for the $L^1$ norm of a certain exponential sum, namely,
\[
\int_0^1 \left| \sum_{n=1}^N a_n e(n\alpha) \right| d\alpha \gg \log N
\]
whenever the coefficients $a_n$ are arbitrary complex numbers subject to
\[
\sum_{n=1}^N |a_n|^2 \gg N.
\]

It was shown by Balog and Ruzsa [2] that a modest generalization to the core assumptions underlying McGehee et al.’s result improves the lower bound for the $L^1$ norm to a power of $N$. Balog and Ruzsa have invented an ingenious but beautifully simple method that enabled them to prove that
\[
\int_0^1 \left| \sum_{n=1}^N a_n e(n\alpha) \right| d\alpha \gg \frac{1}{N^{3/8} \log N} \left( \sum_{n=1}^N |a_n|^2 \right)^{1/2}
\] (2)
whenever the $a_n$ are arbitrary complex numbers subject to $a_n = 0$ for non-square-free integers $n$, a condition we shall henceforth understand to mean that the $a_n$ are supported on the square-free integers $n$.

With regard to the case in which the $a_n$ are the values of $\mu$, the method yields a substantial improvement over (1), particularly,
\[
\int_0^1 \left| \sum_{n=1}^N \mu(n)e(n\alpha) \right| d\alpha \gg \frac{N^{1/8}}{\log N}.
\] (3)

Balog and Ruzsa [3] improved on (3) to $\gg N^{1/6}$, using additional ideas, counting (17), but these ideas do not improve on (2). A variation of the method in its principal element of novelty may be found in the construction of exponential sums that are pointwise close to (7) and supported off of the square-free integers [2, p. 417]. It produces a square root saving for the logarithmic factor in (2) and (3).

The behaviour of the $L^1$ norm of a different type of exponential sum over the primes was investigated by Vaughan [15], who showed that there exists a positive constant $B$ such that
\[
\int_0^1 \left| \sum_{n=1}^N \Lambda(n)e(n\alpha) \right| d\alpha \geq BN^{1/2}.
\] (4)
As Vaughan remarked and as follows from a very simple calculation, (4) is close to the best possible result. “It seems quite likely,” Vaughan wrote, that there exists a positive constant $C$ such that the $L^1$ norm of $\sum_{n=1}^{N} \Lambda(n)e(n\alpha)$ is asymptotically $C(N \log N)^{1/2}$ when $N \to \infty$, “but if true this must lie very deep.” This problem remained—and is still—unsolved.

One further result in this direction is known. A delicate study by Goldston [6] shows that, for every positive $\epsilon$ and $N \geq N_0(\epsilon)$,

$$\int_0^1 \left| \sum_{n=1}^{N} \Lambda(n)e(n\alpha) \right| \, d\alpha \leq \left( \frac{1}{2} + \epsilon \right) N \log N \right)^{1/2}.$$  

The main objective of the study presented here is twofold. First of all, we undertake a line of inquiry begun by Balog and Ruzsa and consider the $L^1$ norm of the exponential sum

$$S_N(\alpha) = \sum_{n=1}^{N} 1_p(n)a_n e(n\alpha), \quad (5)$$

where the $a_n$ are arbitrary complex numbers, $p$ denotes a prime and $1_p$ is the indicator function of the primes. We extend their approach to the construction of exponential sums that are pointwise close to (7) but supported off of the primes greater than $N^{1/2}$. This has the following consequences: With (14) thus obtained in the manner of Balog and Ruzsa, we are led to a result that is only a power of the logarithm weaker than (4). Our main new result is attributed to this construction.

**Theorem 1.** For $N \geq 2$

$$\int_0^1 \left| S_N(\alpha) \right| \, d\alpha \gg \frac{1}{N^{1/4}(\log N)^{1/2}} \left( \sum_{n=1}^{N} 1_p(n)|a_n|^2 \right)^{1/2}.$$  

We then give a new proof of (4) inspired by Vaughan’s investigation. It does not involve any difficulty of principle, other than Linnik’s large sieve [4], [8], [12], and [14], which has an explicit role in the form of (12), and this idea: Vaughan calculates the contributions of all largish peaks at fractions $\alpha$ in (4), whereas we incorporate how these peaks are transformed from the single peak at zero.

## 2 | SOME NOTATIONS AND AUXILIARY THEOREMS

We begin by considering an exponential sum

$$F_N(\alpha) = \sum_{n=1}^{N} e(n\alpha) \quad (6)$$

and its Fejér kernel

$$T_N(\alpha) = \frac{1}{N} \left| F_N(\alpha) \right|^2 = \sum_{|k| \leq N} \left( 1 - \frac{|k|}{N} \right) e(k\alpha), \quad (7)$$
so that

\[
\sum_{a=1}^{q} T_N\left(\alpha - \frac{a}{q}\right) = q \sum_{|k| \leq N, \frac{|k|}{q} \perp k} \left(1 - \frac{|k|}{N}\right) e(k\alpha)
= \sum_{|k| \leq N} \left(1 - \frac{|k|}{N}\right) \varepsilon_q(k)e(k\alpha),
\]

where

\[
\varepsilon_q(k) = \sum_{a=1}^{q} e\left(-\frac{ka}{q}\right) = \begin{cases}
q & \text{if } q \mid k,
0 & \text{if } q \nmid k.
\end{cases}
\]

We define particularly

\[
H_N(\alpha) = \frac{1}{\pi(P)} \sum_{p \leq P} \sum_{|k| \leq N} \left(1 - \frac{|k|}{N}\right) \varepsilon_p(k)e(k\alpha)
= T_N(\alpha) + \frac{1}{\pi(P)} \sum_{p \leq P} \sum_{a=1}^{p-1} T_N\left(\alpha - \frac{a}{p}\right),
\]

where \(\pi(P)\) gives the number of primes not exceeding \(P\). The method accordingly suggests to us to take the alternative representation

\[
H_N(\alpha) = \sum_{|k| \leq N} \left(1 - \frac{|k|}{N}\right) d_k e(k\alpha),
\]

where

\[
d_k = \frac{1}{\pi(P)} \sum_{p \leq P} \varepsilon_p(k).
\]

It follows that we must have \(d_k \neq 0\) if and only if there exists a prime \(p\) with \(p \leq P\) such that \(p \mid k\). It is important to further realize that \(d_k = 0\) if and only if the smallest prime factor of \(k\) is greater than \(P\) or \(k = 1\). The value of \(P\) with which we shall be dealing is \(N^{1/2}\), and thus \(H_N\) is supported off of the primes \(p\) in the range \(P < p \leq N\). We wish to prove that \(H_N\) gives a very good approximation of \(T_N\) in the following sense.

**Lemma 1.** \(|H_N(\alpha) - T_N(\alpha)| \ll N^{1/2} \log N\) uniformly for every \(\alpha\).

**Proof.** We obtain from (7) and (8) that

\[
0 \leq H_N(\alpha) - T_N(\alpha) = \frac{1}{\pi(P)} \sum_{p \leq P} \sum_{a=1}^{p-1} \frac{1}{N} \left|F_N\left(\alpha - \frac{a}{p}\right)\right|^2.
\]
We consider this optimal formulation of the large sieve: Let \( \alpha_r \), where \( 1 \leq r \leq R \) and \( R \geq 2 \), be distinct points modulo one. Let \( \delta \) be positive such that \( \| \alpha_r - \alpha_s \| \geq \delta \) for \( r \neq s \). Then, for arbitrary complex numbers \( a_n \),

\[
\sum_{r=1}^{R} \left| \sum_{n=M+1}^{M+N} a_n e(n \alpha_r) \right|^2 \leq \left( N + \frac{1}{\delta} - 1 \right) \sum_{n=M+1}^{M+N} |a_n|^2,
\]

(12)

where \( M \) and \( N \) are integers and \( N \) is positive. The stated constant \( N + 1/\delta - 1 \) is sharp and was contributed by Selberg [5, Chapter 27].

We can apply (12) with \( \delta = 1/P^2 \), so that we obtain

\[
\sum_{p \leq P} \sum_{a=1}^{P-1} \left| F_N \left( \alpha - \frac{a}{p} \right) \right|^2 \leq (N + P^2 - 1)N.
\]

(13)

Then from (11) and (13) we get, for sufficiently large \( N \),

\[
0 \leq H_N(\alpha) - T_N(\alpha) \leq \frac{1}{\pi(P)}(N + P^2 - 1) < N^{1/2} \log N
\]

if \( P = N^{1/2} \) with \( \pi(P) > P / \log P \) holding true for \( P \geq 17 \) [13, Corollary 1, inequality (3.5)]. We see that \( H_N \) has the stated property, which proves Lemma 1.

Having thus defined \( H_N \), we shall now define

\[
H_{N,P}(\alpha) = \frac{1}{\pi(P)} \sum_{p \leq P} \sum_{p \leq |k| \leq N} \left( 1 - \frac{|k|}{N} \right) \epsilon_p(k) e(k\alpha)
\]

(14)

A similar result holds, of course, for \( H_{N,P} \). According to the definition of \( H_{N,P} \), it is supported off of all primes. From (9), (10), and (14) we have

\[
|H_N(\alpha) - H_{N,P}(\alpha)| \leq \sum_{|k| \leq P} |d_k| = \frac{1}{\pi(P)} \sum_{|k| \leq P} \sum_{p \leq P} \epsilon_p(k)
\]

\[
= \frac{1}{\pi(P)} \left( \sum_{p \leq P} p + \sum_{1 \leq |k| \leq P \atop p \nmid k} P \right)
\]

\[
= \frac{1}{\pi(P)} \left( \sum_{p \leq P} p + \sum_{1 \leq |p| \leq P} 1 \right)
\]

\[
= \frac{1}{\pi(P)}(P \pi(P) + 2P \pi(P)) = 3N^{1/2}
\]
if we reassert our earlier choice that $P = N^{1/2}$. Using this estimate and Lemma 1, we will have, then, produced the following result.

**Lemma 2.** If $P = N^{1/2}$, then

$$|H_{N,P}(\alpha) - T_N(\alpha)| \ll N^{1/2} \log N$$

uniformly for every $\alpha$.

The foregoing approximation can be used in the following way. We write, for a fixed $\alpha$,

$$\int_0^1 H_{N,P}(\alpha - \beta)S_N(\beta)d\beta = \sum_{n=1}^{N} 1_p(n)a_n \sum_{P < |k| \leq N} \left(1 - \frac{|k|}{N}\right) d_k e(k\alpha) \int_0^1 e((n-k)\beta)d\beta$$

$$= \sum_{P < n \leq N} \left(1 - \frac{n}{N}\right) 1_p(n)a_n e(n\alpha) = 0.$$  

In like manner we find

$$\int_0^1 T_N(\alpha - \beta)S_N(\beta)d\beta = \sum_{n=1}^{N} \left(1 - \frac{n}{N}\right) 1_p(n)a_n e(n\alpha).$$

We have therefore

$$\sum_{n=1}^{N} \left(1 - \frac{n}{N}\right) 1_p(n)a_n e(n\alpha) = \int_0^1 (T_N(\alpha - \beta) - H_{N,P}(\alpha - \beta))S_N(\beta)d\beta. \quad (15)$$

We obtain the ensuing result on the basis of Lemma 2.

**Theorem 2.** For $N \geq 2$

$$\left|\sum_{n=1}^{N} \left(1 - \frac{n}{N}\right) 1_p(n)a_n e(n\alpha)\right| \ll N^{1/2} \log N \int_0^1 |S_N(\beta)|d\beta$$

uniformly for every $\alpha$.

We conclude with one simple application. With the substitutions $a_n = 1$ and $\alpha = 0$, it is by means of the prime number theorem that the left side of (15) becomes

$$\sum_{n=1}^{N} \left(1 - \frac{n}{N}\right) 1_p(n) \sim \sum_{p \leq N} \left(1 - \frac{p}{N}\right)$$

$$\sim \int_2^N \left(1 - \frac{u}{N}\right) \frac{du}{\log u}$$

$$\sim \frac{N}{2 \log N}$$

when $N \to \infty$. This estimate and Theorem 2 give the next result.
Theorem 3. For $N \geq 2$

$$\int_0^1 \left| \sum_{p \leq N} e(p\alpha) \right| d\alpha \gg \frac{N^{1/2}}{(\log N)^{2}}. \quad (16)$$

3 \quad PROOF OF THEOREM 1

What Vaughan [15] proved was that the lower bound for the $L^1$ norm in (16) is $\gg N^{1/2}/\log N$. His proof only seems to work for (5) with the $a_n$ being smooth continuous functions. The lower bound (4) depends on the sum $\sum_{n=1}^{N} 1_p(n)a_n$ not cancelling out. Exactly as in [2] we now prove the corresponding result for (5).

We define, for arbitrary complex numbers $b_n$,

$$g_N(\alpha) = \sum_{n=1}^{N} b_n e(n\alpha).$$

We have

$$\int_0^1 \left| g_N(\beta) g_N(\alpha + \beta) d\beta = \sum_{n=1}^{N} |b_n|^2 e(n\alpha). \right.$$ 

Then

$$\int_0^1 \left| \sum_{n=1}^{N} |b_n|^2 e(n\alpha) \right| d\alpha \leq \int_0^1 \int_0^1 |g_N(\beta)||g_N(\alpha + \beta)| d\beta d\alpha$$

$$= \left( \int_0^1 |g_N(\alpha)| d\alpha \right)^2. \quad (17)$$

From this we obtain

$$\int_0^1 \left| \sum_{n=1}^{N} b_n e(n\alpha) \right| d\alpha \gg \left( \int_0^1 \sum_{n=1}^{N} |b_n|^2 e(n\alpha) \right)^{1/2} \left. d\alpha \right).$$

Suppose that $N$ is an even integer, and let us consider

$$b_n = \begin{cases} 1_p(n)a_n & \text{if } n \leq N/2, \\ 0 & \text{if } N/2 < n \leq N. \end{cases}$$
If we take $M = N/2$, we can have from Theorem 2
\[
\int_0^1 |S_M(\alpha)| d\alpha \geq \left( \int_0^1 \left| \sum_{n=1}^M 1_p(n)|a_n|^2 e(n\alpha) \right| d\alpha \right)^{1/2} \\
\gg \frac{1}{M^{1/4}(\log M)^{1/2}} \left( \sum_{n=1}^M \left( 1 - \frac{n}{2M} \right) 1_p(n)|a_n|^2 e(n\alpha) \right)^{1/2}.
\]

We choose $\alpha = 0$ particularly on the extreme right side. The factor $1 - n/(2M)$ is greater than or equal to $1/2$ in the sum. It may be removed from the lower bound. Hence, Theorem 1 is entirely proved.

## 4 PROOF OF (4)

The present proof of (4) rests upon the following lemma.

**Lemma 3.** Let
\[
V = \int_0^1 \sum_{n=1}^N \Lambda(n)e(n\alpha)K_{Q,N}(\alpha) d\alpha,
\]
where
\[
K_{Q,N}(\alpha) = \sum_{q \leq Q} \mu(q) \sum_{a=1 \atop (a,q)=1}^q \left| F_N \left( \alpha - \frac{a}{q} \right) \right|^2
\]
with the notation of (6). We assume
\[
Q = f(N) \to \infty
\]
when $N \to \infty$ and
\[
f(N) = O \left( \frac{N}{(\log N)^2} \right).
\]
Then
\[
V \sim \frac{3}{\pi^2} QN^2.
\]  

**Proof.** Since, obviously, by (7)
\[
|F_N(\alpha)|^2 = \sum_{|k| \leq N} (N - |k|)e(k\alpha),
\]
then

\[ V = \sum_{q \leq Q} \mu(q) \sum_{n=1}^{N} (N - n)\Lambda(n)c_q(-n), \]  

(19)

where \( c_q \) is Ramanujan’s sum [11] defined by

\[ c_q(n) = \sum_{(a,q)=1}^{q} e\left(\frac{an}{q}\right). \]

Now, \( c_q(-n) = c_q(n) \) and \( c_q(n) = \mu(q) \) if \((n, q) = 1\). It is trivially true that \(|c_q(n)| \leq \phi(q) \leq q\), where \( \phi \) denotes Euler’s totient function. Thus, we have

\[ \sum_{q \leq Q} \mu(q)c_q(n) = \sum_{q \leq Q} \mu(q)^2 + \sum_{q \leq Q \ (n,q)>1} \mu(q)(c_q(n) - \mu(q)). \]

Since

\[ \sum_{q \leq Q} \mu(q)^2 = \frac{6}{\pi^2} Q + O(Q^{1/2}) \]

for \( Q \geq 1 \) [10, Theorem 2.2], it follows that

\[ \sum_{q \leq Q} \mu(q)c_q(n) = \frac{6}{\pi^2} Q + O(Q^{1/2}) + O \left( \sum_{q \leq Q \ (n,q)>1} \mu(q)^2q \right). \]

Substitution of this in (19) yields

\[ V = \left( \frac{6}{\pi^2} Q + O(Q^{1/2}) \right) \sum_{n=1}^{N} (N - n)\Lambda(n) + O \left( \sum_{q \leq Q \ (n,q)>1} \mu(q)^2q \sum_{n=1}^{N} (N - n)\Lambda(n) \right). \]

In the second error term, we see that \( n = p^m \) and \( q \) is square-free. It is thus that the condition \((n, q) > 1\) implies that \( p \mid q \). We have by the prime number theorem

\[ \sum_{n=1}^{N} (N - n)\Lambda(n) = \frac{1}{2} N^2(1 + o(1)). \]

Hence, this error term is at most

\[ O \left( N \log N \sum_{p \leq Q} \sum_{q \leq Q \ p \mid q} q \right) = O \left( Q^2 N \log N \sum_{p \leq Q} \frac{1}{p} \right) \]

\[ = O(Q^2 N \log N \log \log Q), \]
since
\[ \sum_{p \leq Q} \frac{1}{p} = \log \log Q + D + O\left(\frac{1}{\log Q}\right) \]

for \( Q \geq 2 \), where \( D \) is a constant [10, Theorem 2.7, part (d)]. Altogether

\[ V = \frac{3}{\pi^2} Q N^2 (1 + o(1)) + O(Q^2 N \log N \log \log Q), \]

and Lemma 3 follows. \( \square \)

Let us now come quickly to the crucial point. We have
\[
V \leq \max_{0 \leq \alpha \leq 1} \left| K_{Q,N}(\alpha) \right| \int_0^1 \left| \sum_{n=1}^N \Lambda(n)e(n\alpha) \right| d\alpha. \tag{20}
\]

Applying (12) with \( \delta = 1/Q^2 \), we get

\[
\left| K_{Q,N}(\alpha) \right| \leq \sum_{q \leq Q} \sum_{a=1 \atop (a,q)=1}^q \left| F_N \left( \alpha - \frac{a}{q} \right) \right|^2 \]

\[
\leq (N + Q^2 - 1) \sum_{n=1}^N 1 \]
\[
< N(N + Q^2). \]

In virtue of this estimate, (18) and (20),

\[
\int_0^1 \left| \sum_{n=1}^N \Lambda(n)e(n\alpha) \right| d\alpha \geq \frac{V}{N(N + Q^2)} \geq \left( \frac{3}{\pi^2} - \epsilon \right) \frac{QN}{N + Q^2}.
\]

We have only to choose \( Q = N^{1/2} \) and obtain finally the desired result (4).

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REFERENCES

1. A. Balog and A. Perelli, *On the $L^1$ mean of the exponential sum formed with the Möbius function*, J. Lond. Math. Soc. 57 (1998), no. 2, 275–288.
2. A. Balog and I. Z. Ruzsa, *A new lower bound for the $L^1$ mean of the exponential sum with the Möbius function*, Bull. Lond. Math. Soc. 31 (1999), 415–418.
3. A. Balog and I. Z. Ruzsa, *On the exponential sum over $r$-free integers*, Acta Math. Hungar. 90 (2001), no. 3, 219–230.
4. E. Bombieri, *On the large sieve*, Mathematika 12 (1965), 201–225.
5. H. Davenport, *Multiplicative number theory*, 3rd ed., Graduate Studies in Mathematics, No. 74, Springer, New York, 2000.
6. D. A. Goldston, *The major arcs approximation for an exponential sum over primes*, Acta Arith. 92 (2000), no. 2, 169–179.
7. G. H. Hardy and J. E. Littlewood, *A new proof of a theorem on rearrangements*, J. Lond. Math. Soc. 23 (1948), 163–168.
8. Y. V. Linnik, *The large sieve*, Dokl. Akad. Nauk SSSR 30 (1941), 292–294.
9. O. C. McGehee, L. Pigno, and B. Smith, *Hardy’s inequality and the $L^1$ norm of exponential sums*, Ann. of Math. 113 (1981), 613–618.
10. H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory, Vol. I, Classical theory*, Cambridge Studies in Advanced Mathematics, No. 97, Cambridge University Press, Cambridge, 2007.
11. S. Ramanujan, *On certain trigonometrical sums and their applications in the theory of numbers*, Trans. Cambridge Philos. Soc. 22 (1918), 259–276.
12. A. Rényi, *On the large sieve of Ju. V. Linnik*, Compos. Math. 8 (1950), 68–75.
13. J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), 64–94.
14. K. F. Roth, *On the large sieves of Linnik and Rényi*, Mathematika 12 (1969), 1–9.
15. R. C. Vaughan, *The $L^1$ mean of exponential sums over primes*, Bull. Lond. Math. Soc. 20 (1988), 121–123.