Experimental study of Hong-Ou-Mandel interference using independent phase randomized weak coherent states

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Hong-Ou-Mandel interferometers are valuable tools in many Quantum Information and Quantum Optics applications that require photon indistinguishability. The theoretical limit for the Hong-Ou-Mandel visibility is 0.5 for indistinguishable weak coherent photon states, but several device imperfections may hinder achieving this value experimentally. In this work, we examine the dependence of the interference visibility on various factors, including (i) detector side imperfections due to after-pulses, (ii) mismatches in the intensities and states of polarization of the input signals, and (iii) the overall intensity of the input signals. We model all imperfections and show that theoretical modeling is in good agreement with experimental results.

I. INTRODUCTION

The interference of two photons at a beam splitter was first examined by Hong, Ou, and Mandel [1] (HOM interference). As the input photons (Figure 1) become increasingly indistinguishable in all degrees of freedom, the coincidence rate of the beam splitter output photons exhibits a characteristic dip, the depth of which depends on the degree of indistinguishability of the input photons [2]. Various applications have been proposed and demonstrated, utilizing the interference of single photons created through Spontaneous Parametric Down Conversion (SPDC), including clock synchronization [3], quantum teleportation [4], and quantum logic gates [5].

A convenient alternative to SPDC heralded photons is an input state consisting of weak coherent states [6], implemented as attenuated laser light. Studies have been conducted to examine the HOM visibility using coherent states, including the effect of the frequency chirp and time jitter [7, 8], the optical delay between the inputs and detection time differences [9], and frequency mismatch [10].

Because HOM interference can be used for experimental Bell state analysis [11] [12], it lies at the heart of Measurement-Device-Independent (MDI) Quantum Key Distribution (QKD) [13], which is a novel quantum communication protocol resistant in all possible detector-side attacks. The applicability of the protocol has been demonstrated in multiple experiments [14]–[19]. In MDI QKD, the interference visibility significantly affects the final key generation rate [17]–[19], [20]. The use of coherent states instead of single-photon states could open up a potential vulnerability due to the non-zero probability of multiple photon pulses, but the implementation of the decoy-state method [21]–[23] can overcome such a threat.

Wang, et al., [24] examined how realistic imperfections of the devices used in an HOM interference experiment affect the HOM visibility. In particular, they considered possible imperfections of the beam splitter, mismatches in the input intensities, and studied the effect of the after-pulses in single-photon avalanche detectors. In this work, we provide experimental measurements and extend the work to include possible mismatches in the state of polarization of the inputs and examine the effect of the overall intensity of the inputs on the HOM visibility. We also discuss modeling of imperfections and show good agreement of experimental results with the theoretical modeling.

II. PARAMETRIZING THE HONG-OU-MANDEL INTERFERENCE VISIBILITY

The set-up for our Hong-Ou-Mandel interference measurements consists of two independent input laser pulses, interfering at a beam splitter (BS) and with each output directed to a single-photon avalanche detector (SPAD) (Figure 1).

We model the input to the beam-splitter state as two weak coherent states:

\[ |\Psi_{in}\rangle = |\alpha\rangle \otimes |\beta\rangle = e^{-\frac{i\alpha \beta}{2}} e^{i\theta_\alpha \hat{a}^\dagger + i\theta_\beta \hat{b}^\dagger} |0\rangle \]  

created by creation operators \( \hat{a}^\dagger \) and \( \hat{b}^\dagger \), and of parameters \( \alpha \) and \( \beta \), respectively. The coherent-state parameters are complex and include a phase, \( \alpha = \sqrt{\mu_a} e^{i\theta_a} \), \( \beta = \sqrt{\mu_b} e^{i\theta_b} \), where \( \mu_{a,b} \) are the corresponding average photon numbers of the two beams. In our experimental setup, the phases are randomized. Therefore, the initial
Given this output state, the probability \( P \) transforms into the output state 

\[
P_{mn} = \sum_{m,n=0}^{\infty} P_{mn} m^n n^m \tag{10}
\]

Nevertheless, we will continue to work with the state \([1]\) and average over the phases at the end.

To account for the action of the beam splitter, we introduce a pair of orthogonal directions, named horizontal and vertical, respectively, and express the polarization vectors of the incoming beams \( \hat{\varepsilon}_a, \hat{\varepsilon}_b \) in terms of unit vectors in the chosen directions, \( \hat{\varepsilon}_H, \hat{\varepsilon}_V \). The creation operators are similarly expressed as linear combinations:

\[
\begin{align*}
\hat{a}^\dagger &= \hat{\varepsilon}_a \cdot \hat{\varepsilon}_H a_H^\dagger + \hat{\varepsilon}_a \cdot \hat{\varepsilon}_V a_V^\dagger \\
\hat{b}^\dagger &= \hat{\varepsilon}_b \cdot \hat{\varepsilon}_H b_H^\dagger + \hat{\varepsilon}_b \cdot \hat{\varepsilon}_V b_V^\dagger
\end{align*}
\]

The action of a beam splitter with reflectivity \( R = r^2 \) and transmissivity \( T = t^2 \), with \( R + T = 1 \), is described by the unitary transformation:

\[
\begin{align*}
a_H^\dagger &= t c_H^\dagger + r d_H^\dagger \\
b_H^\dagger &= t d_H^\dagger - r c_H^\dagger
\end{align*}
\]

where \( c_H^\dagger \) and \( d_H^\dagger \) are the creation operators of the respective output beams, with \( i = H, V \). The input state \([1]\) transforms into the output state

\[
|\Psi_{\text{out}}\rangle = e^{-\mu + \mu} \prod_{i=H, V} e^{\alpha(c_H^\dagger + r d_H^\dagger)\hat{\varepsilon}_i} e^{\beta(r c_H^\dagger - t d_H^\dagger)\hat{\varepsilon}_i} |0\rangle \tag{5}
\]

Given this output state, the probability \( P_{mn} \) that \( m \) (\( n \)) photons emerge at output port \( c \) (\( d \)) is found to be (see Appendix A for details)

\[
P_{mn} = e^{- \mu_c - \mu_d} \frac{\mu_c^m \mu_d^n}{m! n!} \tag{6}
\]

where \( \mu_{c,d} \) are the corresponding mean photon numbers at the two output ports of the beam splitter,

\[
\begin{align*}
\mu_c &= \mu_a t^2 + \mu_b r^2 + 2r \Re(\alpha \beta^* \hat{\varepsilon}_a \cdot \hat{\varepsilon}_b) \\
\mu_d &= \mu_a r^2 + \mu_b t^2 - 2r \Re(\alpha \beta^* \hat{\varepsilon}_a \cdot \hat{\varepsilon}_b)
\end{align*}
\]

Notice that the mean photon numbers of the beams obey the conservation law

\[
\mu_a + \mu_b = \mu_c + \mu_d \tag{8}
\]

which is a consequence of the unitarity of the beam-splitter transformation \([1]\). \( R + T = r^2 + t^2 = 1 \).

Our real detectors at the two beam-splitter ports have efficiencies \( \eta_c \) and \( \eta_d \), and dark-count probabilities \( d_c \) and \( d_d \), respectively. Therefore the probability that the detectors click is given by

\[
P_{mn} = P_{mn}^{(\text{out})} \times (1 - (1 - \eta_c)^m (1 - d_c)) \\
(1 - (1 - \eta_d)^n (1 - d_d)) \tag{9}
\]

The total coincidence probability is given by

\[
P_{\text{coin}} = \sum_{m,n=0}^{\infty} P_{mn} \tag{10}
\]

After averaging over the phases, we obtain the total coincidence probability corresponding to the state \([2]\) (see Appendix A for details) in terms of Bessel functions:

\[
P_{\text{coin}} = 1 - C I_0(2\eta_c \sqrt{\mu_a \mu_b} r \cos \Phi) \\
- D I_0(2\eta_d \sqrt{\mu_a \mu_b} t \cos \Phi) + C D I_0(2(\eta_c - \eta_d) \sqrt{\mu_a \mu_b} r \cos \Phi) \tag{11}
\]

where

\[
C = e^{-\eta_d(\mu_a t^2 + \mu_b r^2)} (1 - d_c) \\
D = e^{-\eta_d(\mu_a r^2 + \mu_b t^2)} (1 - d_d) \tag{12}
\]

and \( \Phi \) is a measure of the polarization mismatch between the two incoming beams defined by

\[
\cos \Phi = |\hat{\varepsilon}_a \cdot \hat{\varepsilon}_b| \tag{13}
\]

The total probability that the detector at port \( c \) clicks, after averaging over phases, is also expressed similarly in terms of a Bessel function,

\[
P^{(c)} = 1 - C I_0(2\eta_c \sqrt{\mu_a \mu_b} tr \cos \Phi) \tag{14}
\]

The total probability that the detector at port \( d \) clicks is found similarly:

\[
P^{(d)} = 1 - D I_0(2\eta_d \sqrt{\mu_a \mu_b} tr \cos \Phi) \tag{15}
\]

Details can be found in Appendix A.

We define the Hong-Ou-Mandel visibility by

\[
V_{\text{HOM}} = 1 - \frac{P_{\text{coin}}}{P^{(c)} P^{(d)}} \tag{16}
\]

Using the explicit expressions \([11], [14], \) and \([15]\), we find that \( V_{\text{HOM}} \in [0, 0.5] \). We aim at maximizing the value of \( V_{\text{HOM}} \).
III. EXPERIMENTAL SETUP

Our experimental setup is shown on Figure 2. Two independent continuous wave (CW) lasers (Wavelength References) at 1542 nm were employed to prepare weak coherent states. The frequency difference between the two lasers stayed below 10 MHz without performing any feedback control. Note, in all experiments, the phase difference between the two lasers swept through a multi-2π range within the data acquisition time. This is equivalent to the phase averaging process assumed in the theoretical analysis. To generate laser pulses, two LiNbO₃ intensity modulators were used to modulate the outputs of the two lasers. The two intensity modulators were driven by the same digital delay generator (Stanford Research Systems) and their DC bias voltages were carefully adjusted to achieve high extinction ratios. The polarization state of each pulse can be randomly changed with a home-made high-speed polarization modulator, which is driven by a Keysight arbitrary waveform generator (AWG). Details about the polarization modulator can be found in ref. [16]. It is imperative that the AWGs controlling the polarizations of Alice and Bob and the SRS share the same timebase. Once the polarization is applied to the pulse, it passes through a beam circulator and enters free space. Upon returning to fiber, the pulses enter an additional polarization controller where HOM visibility is fine-tuned prior to data collection. The pulse is then digitally attenuated in order to reach the single-photon level to interfere at the beam splitter and be read by single-photon avalanche detectors (SPAD). The detectors are both IdQuantique 210 with one being an ultra-low noise model. For most runs, the detectors are in free-gating mode with the SRS triggers starting times of each gate. Timestamps of detection events are recorded on a time-interval analyzer (TIA) with a resolution of 81 ps.

IV. RESULTS

Here we report on our experimental results. We begin with the characterization of our two SPADs to evaluate their limitations in certain applications. Subsequently, we consider these device imperfections with regards to the HOM visibility. Lastly, we present results characterizing the HOM visibility considering state preparation imperfections.

A. Detector Characterization

We begin with a characterization of our SPADs, which are commercial InGaAs-based single-photon avalanche photodiodes.

Detector characterization is critical to understanding performance in certain applications and several methods for SPADs have been developed over the years [25,27].

Here we adapt some of these methods to evaluate detector imperfections and their effect on the Hong-Ou-Mandel visibility under certain conditions. Three important criteria for SPAD characterization are dark-count rates (carriers generated from thermal fluctuations or tunneling that trigger false detection events), the detection efficiency, and the after-pulse probability (due to trapped carriers from a previous avalanche inducing new avalanches).

To combat the false events, the SPADs offer a dead-time setting: after a successful photon detection a dead time is applied during which the detector is blind to all signals (real or false).

For each detector, when operated in gated mode, the detection probability is given by

$$P = 1 - e^{-\eta \mu} = \frac{R_{det} T_{gat}}{1 - R_{det} T_{dt} + R_{det} T_{gat}}$$

(17)
where \( \eta \) is the efficiency of the detector, and \( \mu \) is the average photon number of the input. \( R_{det} \) is the detector count rate, \( T_{dt} \) is the time until a gate becomes available again after a detection, and \( T_{gat} \) is the gating period. Since the detectors are operated in gated mode, according to the above definition, the minimum value of \( T_{dt} \) is \( T_{gat} \), so it is necessary to include the term \( R_{det}T_{gat} \) in the denominator of eq. (17).

In figure 3, we plot the detector count rates vs photon number for our two detectors. From the data shown in figure 3, the detection efficiencies of both SPADs have been determined to be 13%.

The absorption of a photon in the active area of an InGaAs detector results in an avalanche of photo-electrons which in turn is registered as a digital signal. These energetic charge carriers may remain active in the detector’s depletion zone for a significant time interval, in the range of several microseconds, and trigger subsequent avalanches which are registered as false detection events \([27]\).

The events that contribute to the after-pulse probability occur when the gate is open, i.e., at times \( t = T_{dt}, T_{dt} + T_{gat}, T_{dt} + 2T_{gat}, \ldots \), where \( T_{dt} \) is the dead time and \( T_{gat} \) is the gating period. We assume that the probability for these after-pulse events can be fitted with a single exponential decay, \( P(t) = P_0e^{-bt} \), with the parameters \( P_0 \) and \( b \) determined experimentally. The total after-pulse probability is

\[
P^{(aft)} = P(T_{dt}) + P(T_{dt} + T_{gat}) + \ldots
= P_0 \frac{e^{-bT_{dt}}}{1 - e^{-bT_{gat}}}
\] (18)

Our method is based on a procedure outlined in \([27]\) which is applicable to detectors operating in gated mode and characterized using a cw source.

The recorded data consist of a histogram of detection events binned into time intervals between successive detection events. Since our detectors operate in gated mode, it is assumed no detection events occur between successive gates (dead time). The gate period is fixed to \( T_{gat} \), and each successive gate occurs at a time \( nT_{gat} \) from the initial detection. We record for a time long enough to attain sufficient counting statistics in terms of total counts. From this histogram, a probability density function of detection time intervals can be extracted to estimate the total after-pulse probability specific to each detector.

The model for after-pulse probability as a function of the number of gates between detections is based on the assumption that trapped carriers decay mostly due to a single exponential to first order.

For our setup, the gating frequency was fixed so that the period is shorter than the assumed de-trapping lifetime of charged carriers triggering after-pulses. We use a gating frequency of 6 MHz and 20 MHz, while the dead time was set to our minimum at 0.1 \( \mu \)s.

In figure 4, a histogram of the detection events time-binned corresponding to each gate with the natural log of detection probability per gate. There are two distinct regions in the fit resulting from events originating from the cw source photons and the region where the after-pulsing effect is clearly dominant. The high slope region starting from \( m = 1 \) is mostly due to after-pulsing events dominant within the first subsequent gates after an initial detection. The lower value slope for the rest of the fit corresponds to source photons as these events occur much later after a first detection and are assumed to occur after nearly all trapped carriers decay.

We fit both regions of this plot to extract the parameters in eq. (20). These values are then inserted into our model for HOM visibility as a function of dead time.

The photon number of the source photons is measured simultaneously using a Thorlabs PM100D intensity meter and the known attenuation factor.
Next, we consider here the effect on HOM visibility due to detector imperfections. Ref. [24] highlights the after-pulse effect as a significant source of error in an experimental implementation of the Hong-Ou-Mandel interference. The authors of [24] employed a non-Markovian model and showed that the coincidence probability, after taking into account the after-pulse effect, can be written as:

\[ P(\text{coin}) = P(\text{coin}_a) + \left[ P(c) - P(\text{coin}) \right] P(d) P(\text{aft}) + \left[ P(d) - P(\text{coin}) \right] P(c) P(\text{aft}) \]  

(19)

where \( P(\text{coin}) \) is the coincidence probability given by (11), and \( P(c), P(d) \) are the detection probabilities for the detectors at ports \( c \) and \( d \), respectively, given by eqs. (14) and (15). In eq. (19) the after-pulse probabilities for the two detectors, \( P(\text{aft})_c \) and \( P(\text{aft})_d \) are calculated using eq. (18) with experimentally determined fitting parameters \( P_o \) and \( b \), as outlined above.

We studied the after-pulse effect on the Hong-Ou-Mandel probability at different settings of dead time. The coincidence probability is experimentally determined as the ratio of the number of coinciding detections to the number of coinciding gates with a 5-ns coincidence window. Furthermore, we calculated the experimental HOM visibility using eq. (16) and the experimentally determined single-count rates. We used a gate width of \( \sim 6 \) ns, and a pulse width of \( \sim 2 \) ns. Using the extracted values of \( P(\text{aft})_c \) and \( P(\text{aft})_d \), as discussed above, the HOM visibility as a function of varying dead time was applied to the model to produce the resulting theoretical curve (calculated using eqs. (11), (14), and (15)) depicted in figure 5. The experimental data are in good agreement with the theoretical curve.

For this measurement, the photon number was fixed at 0.15 for both input arms. Both the pulse and gate triggering frequency were set at 6 MHz.

C. Source Effects on HOM Visibility

By lowering the total input intensity, one improves the HOM visibility. However, reaching very low intensities may render the experiment vulnerable to dark counts, and increases the required time to perform a measurement. This in turn renders the experiment vulnerable to various drifts (e.g., the drift in the state of polarization, or in the DC offset of the modulators). We examined the effect of the overall input intensity on the HOM visibility. Setting the intensities of the input beams equal to each other, \( \mu_a = \mu_b = \mu \) in eqs. (11), (14), and (15), we studied the dependence of the HOM visibility on the average input photon number \( \mu \). Theoretically, the HOM visibility approaches the limit value 0 at large input intensities, whereas it approaches the maximum value 0.5 at weak intensities. In our measurements, we used pulses of a 2.2-ns width. Our detectors were running in external gating mode at 1-MHz trigger frequency with an effective gate width of approximately 3 ns (nominal gate width set to 8.25 ns) and \( \sim 10\% \) efficiency. The dark counts of the two detectors were recorded approximately as \( 2.5 \times 10^{-5} \) and \( 1.5 \times 10^{-5} \) per gate, respectively. The dead time on the detectors was set to 10.5 \( \mu s \), so given the 1-\( \mu s \) trigger period, eleven gates were blanked after a detection. To
make sure that the beam splitter inputs were equal, the free-space path of one arm was blocked, and the intensity of the unblocked armed was digitally attenuated until the detection rate reached the desired value. The average photon number input to the beam splitter is related to the observed detection rate by

$$\mu = \frac{2}{\eta} \ln \left( \frac{1 - R_{\text{det}}T_{\text{dt}} + R_{\text{det}}T_{\text{gat}}}{1 - R_{\text{det}}T_{\text{det}}} \right)$$ \hspace{1cm} (20)$$

where $\eta$ is the detector efficiency, $R_{\text{det}}$ is the detector rate of each unblocked input, $T_{\text{dt}}$ is the time until a gate becomes available again after a detection (11 $\mu$s in our case), and $T_{\text{gat}}$ is the gating period. The factor of 2 in (20) accommodates the intensity splitting at the 50:50 beam splitter. In figure 6, the measurement results of the HOM visibility as a function of the input photon number are presented and compared with the theoretical model (calculated using eqs. (11), (14), and (15)), showing good agreement between theory and experiment.

In a realistic experimental setup, two independent laser beams are independently attenuated. In practice, perfect intensity balance may be not possible. Using eqs. (11), (14), and (15), we can model the HOM visibility theoretically as a function of the ratio of the input photon numbers $\frac{n_a}{n_b}$.

The deadtime for each detector was set to 9 $\mu$s with efficiency 10%. As in the previous measurements, the pulse width and gate width were under the same conditions. Each free space arm was blocked for either Alice/Bob between data points to record detector count-rates. The count-rates were controlled via digital attenuation and set to desired count-rates to within 2%. From the detector count-rates and using formula (20), the photon number can be extracted for each count-rate. In Figure 7, the measurement results of the HOM visibility as a function of the ratio of input photon numbers are plotted with the theoretical model (using eqs. (11), (14), and (15)), showing good agreement. We fixed the photon number for the input arm at port $a$ at $\mu_a = 0.47$, while varying the attenuation on the input at port $b$ digitally. We sent weak coherent pulses at 1 MHz with pulse widths of $\sim$2 ns through the beam splitter and to our detectors. Each detector’s gate width was approximately 6 ns to mitigate the detection of background source photons outside the intended pulse width.

The detector count rates were recorded and photon numbers were extracted using eq. (20). The dead time for this particular data set was 10 $\mu$s. In figure 7 the measurement results of the HOM visibility as a function of the imbalance of the input photon numbers are presented and compared with the theoretical model (using eqs. (11), (14), and (15)), showing good agreement between theory and experiment. Photon number for input $a$ was fixed to 0.47. Weak coherent pulses were sent at 1 MHz with pulse widths of $\sim$2 ns. Each detector’s gate width was $\sim$6 ns.

Next, we consider the effect of the polarization misalignment of the incoming beams on the HOM visibility. Eqs. (11), (14), and (15) show the dependence of the HOM visibility on the polarization misalignment $\Phi$ (eq. (13)). Assuming that the bases of the two inputs are
perfectly aligned, we can write the polarization vectors $\hat{\varepsilon}_a$ and $\hat{\varepsilon}_b$ in terms of the transverse-electric (TE) and transverse-magnetic (TM) modes of the phase modulator’s waveguide as:

$$
\hat{\varepsilon}_a = \cos \phi_a |TE\rangle + \sin \phi_a e^{i\phi_0} |TM\rangle
$$

$$
\hat{\varepsilon}_b = \cos \phi_b |TE\rangle + \sin \phi_b e^{i\phi_0} e^{i\phi_M} |TM\rangle
$$

(21)

where $\phi_M = \frac{V_g}{V_\pi} \pi$ is the modulation phase caused by the driving generator, $V_g$ is the voltage applied by the generator, and $V_\pi$ is the constant voltage that causes a $\pi$ phase shift. Using a manual polarization controller we carefully arrange the input to the waveguide to be at 45° with respect to the waveguide’s axis, so $\cos \phi_a = \cos \phi_b = \frac{1}{\sqrt{2}}$. The polarization misalignment angle $\Phi$ (eq. (13)) can then be related to the applied voltage as:

$$
\cos \Phi = |\hat{\varepsilon}_a \cdot \hat{\varepsilon}_b^\ast| = \frac{\pi V_g}{2V_\pi}
$$

(22)

For our measurements, we controlled the state of polarization using the Polarization Modulation set-up described in ref. [28]. We used a Keysight waveform generator to drive an EOSpace Phase Modulator. The $V_\pi$ voltage of the phase modulator was determined to be 5.25 V. Pulses of width 2.1 ns and average photon number $\mu = 0.45 \pm 0.05$ interfered at a 50:50 beam splitter. The outputs were directed to two SPADs operated at a free-gated mode at $\sim$10% efficiency with gate period 1 $\mu$s, dead time set at 7.5 $\mu$s, and gate width of 6.5 ns. The dark counts were recorded approximately $5.5 \times 10^{-5}$ and $2.0 \times 10^{-5}$ per gate, respectively. The coincidence window was set at 5 ns.

Figure 8 depicts the measured HOM visibility as a function of the polarization angle mismatch. Pulses of width 2.1 ns and frequency 1 MHz interfered at a 50:50 beam splitter. The relative polarization angle was modulated by a phase modulator with $V_\pi = 5.25$ V. SPADs of $\sim$10% efficiency and 7.5-$\mu$s dead time, 6.5-ns gate width operated at 1-MHz frequency. The theoretical curve was calculated using eqs. (11), (14), and (15); it is in good agreement with experimental data.

In conclusion, Hong-Ou-Mandel-type measurements are easily implemented in a laboratory environment using standard commercially available optical components and single-photon detectors.

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Appendix A: Details of the calculation of HOM visibility

Here we provide details of the theoretical model for the HOM visibility.

We assume that the two detectors have efficiencies $\eta_{c,d}$ and dark count probabilities $d_{c,d}$, respectively, and are blind to the photon number, i.e., a single-photon event cannot be distinguished from a multi-photon event.

Let $P_{m}^{(\text{out})}$ be the probability that $m$ ($n$) photons arrive at the detector at port $c$ ($d$). Since the ports $c$ and $d$ are separate, the output state $|\Psi_{\text{out}}\rangle$ (eq. (5)) can be factorized into coherent states:

$$|\Psi_{\text{out}}\rangle = |\gamma_{H}\rangle \otimes |\gamma_{V}\rangle \otimes |\delta_{c}\rangle \otimes |\delta_{V}\rangle \quad (A1)$$

where the coherent states with parameter $\gamma_i$ ($\delta_i$) are in output port $c$ ($d$), $i = H, V$, and

$$\gamma_i = (\alpha i \hat{\epsilon}_a + \beta r \hat{\epsilon}_b) \cdot \hat{\epsilon}_i \quad , \quad \delta_i = (\alpha r \hat{\epsilon}_a - \beta t \hat{\epsilon}_b) \cdot \hat{\epsilon}_i \quad (A2)$$

Therefore, we can write the probability $P_{m}^{(\text{out})}$ as a product:

$$P_{m}^{(\text{out})} = P_{m}^{(\text{out},c)} P_{m}^{(\text{out},d)} \quad (A3)$$

where

$$P_{m}^{(\text{out},c)} = \sum_{m_{H} + m_{V} = m} P_{m_{H}} P_{m_{V}}$$

$$P_{m}^{(\text{out},d)} = \sum_{n_{H} + n_{V} = n} P_{n_{H}} P_{n_{V}} \quad (A4)$$

The probabilities on the right-hand side are easily deduced from the corresponding coherent states. We obtain

$$P_{m_{i}} = e^{-|\gamma_{i}|^2} \frac{|\gamma_{i}|^{2m_{i}}}{m_{i}!} , \quad P_{n_{i}} = e^{-|\delta_{i}|^2} \frac{|\delta_{i}|^{2n_{i}}}{n_{i}!} \quad (A5)$$

Using the binomial theorem, we deduce

$$P_{m}^{(\text{out},c)} = e^{-\mu_{c}} \frac{\mu_{c}^{m}}{m!} , \quad P_{n}^{(\text{out},d)} = e^{-\mu_{d}} \frac{\mu_{d}^{n}}{n!} \quad (A6)$$

and therefore

$$P_{m_{n}}^{(\text{out})} = e^{-\mu_{c} - \mu_{d}} \frac{\mu_{c}^{m} \mu_{d}^{n}}{m! n!} \quad (A7)$$

where

$$\mu_{c} = \sum_{i=H,V} |\gamma_{i}|^2 = |\alpha i \hat{\epsilon}_a + \beta r \hat{\epsilon}_b|^2$$

$$\mu_{d} = \sum_{i=H,V} |\delta_{i}|^2 = |\alpha r \hat{\epsilon}_a - \beta t \hat{\epsilon}_b|^2$$

Notice that $\phi_0$ is an irrelevant phase, because we need to average over the phases.

The probability of detection if $m$ ($n$) photons reach detector $c$ ($d$) is $1 - (1 - \eta_{c})^{m}(1 - \eta_{d})^{n}(1 - d_{c})(1 - d_{d}))$. Therefore, the probability of detection given $m$ ($n$) photons coming out of beam splitter port $c$ ($d$) is

$$P_{m_{n}} = [1 - (1 - \eta_{c})^{m}(1 - d_{c})] [1 - (1 - \eta_{d})^{n}(1 - d_{d})] P_{m}^{(\text{out})} \quad (A9)$$

The total coincidence probability is

$$P^{(\text{coin})} = \sum_{m,n=0}^{\infty} P_{m_{n}} = (1 - e^{-\eta_{c} \mu_{c}} (1 - d_{c})) (1 - e^{-\eta_{d} \mu_{d}} (1 - d_{d})) \quad (A10)$$

showing that the effective average photon number is the average photon number of the output beam that reaches the detector multiplied by the detector efficiency.

After averaging over the phases $\theta_{a,b}$, we obtain an expression in terms of Bessel functions,

$$P^{(\text{coin})} \rightarrow \int_{0}^{2\pi} d\theta_{a} \int_{0}^{2\pi} d\theta_{b} P^{(\text{coin})} = 1 - C I_{0}(2\eta_{c}\sqrt{\mu_{a}\mu_{b}}r \cos \Phi)$$

$$D I_{0}(2\eta_{d}\sqrt{\mu_{a}\mu_{b}}r \cos \Phi) + C D I_{0}(2(\eta_{c} - \eta_{d})\sqrt{\mu_{a}\mu_{b}}r \cos \Phi) \quad (A11)$$

where

$$C = e^{-\eta_{c} (\mu_{a} r^{2} + \mu_{b} r^{2})} (1 - d_{c}) , \quad D = e^{-\eta_{d} (\mu_{a} r^{2} + \mu_{b} r^{2})} (1 - d_{d}) \quad (A12)$$

For the HOM visibility, we also need to calculate the probabilities for one of the two detectors to click. The probability for detector at port $c$ to click, after averaging over phases, is

$$P^{(c)} = \sum_{m=0}^{\infty} (1 - e^{-\eta_{c} \mu_{c}} (1 - d_{c})) P_{m}^{(\text{out},c)}$$

$$= 1 - C I_{0}(2\eta_{c}\sqrt{\mu_{a}\mu_{b}}r \cos \Phi) \quad (A13)$$

Similarly, for the other detector, we obtain

$$P^{(d)} = \sum_{n=0}^{\infty} (1 - e^{-\eta_{d} \mu_{d}} (1 - d_{d})) P_{n}^{(\text{out},d)}$$

$$= 1 - D I_{0}(2\eta_{d}\sqrt{\mu_{a}\mu_{b}}r \cos \Phi) \quad (A14)$$

We define the HOM visibility by

$$V_{\text{HOM}} = 1 - \frac{P^{(\text{coin})}}{P^{(c)} P^{(d)}} \quad (A15)$$

Notice that $V_{\text{HOM}} = 0$, for $\Phi = \frac{\pi}{2}$ (orthogonal polarizations).
In the limit $\alpha, \beta \to 0$ (small average photon number), and in the ideal case of no dark counts ($d_c = d_d = 0$), the HOM visibility is approximately

$$V_{HOM} \approx \frac{2RT\mu_a\mu_b\cos^2 \Phi}{(T\mu_a + R\mu_b)(R\mu_a + T\mu_b)}$$  \hspace{1cm} (A16)$$

Its maximum value of $\frac{1}{2}$ is attained for $\Phi = 0$, and $\frac{\mu_b}{\mu_a} = \frac{T}{R}$. For a 50:50 beam splitter, it reduces to

$$V_{HOM} \approx \frac{2\mu_a\mu_b\cos^2 \Phi}{(\mu_a + \mu_b)^2}$$  \hspace{1cm} (A17)$$

which vanishes for $\Phi = \frac{\pi}{2}$ (orthogonal polarizations), and for $\Phi = 0$ (parallel polarizations), it has maximum $\frac{1}{2}$ at $\mu_a = \mu_b$. 