A series of maximum entropy upper bounds of the differential entropy

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Abstract

We present a series of closed-form maximum entropy upper bounds for the differential entropy of a continuous univariate random variable and study the properties of that series. We then show how to use those generic bounds for upper bounding the differential entropy of Gaussian mixture models. This requires to calculate the raw moments and raw absolute moments of Gaussian mixtures in closed-form that may also be handy in statistical machine learning and information theory. We report on our experiments and discuss on the tightness of those bounds.

Keywords: Differential entropy, Gaussian mixture models, maximum entropy, absolute monomial exponential families, absolute moments.

1 Introduction

Shannon’s differential entropy [4] \( H(X) \) of a continuous random variable \( X \) following a probability density function \( p(x) \) (denoted by \( X \sim p(x) \)) on the support \( X = \{ x \in \mathbb{R} : p(x) > 0 \} \) quantifies the amount of uncertainty [4] of \( X \) by the following celebrated formula:

\[
H(X) = \int_X p(x) \log \frac{1}{p(x)} \, dx = - \int_X p(x) \log p(x) \, dx.
\]  

When the logarithm is expressed in basis 2, the entropy is measured in bits. When using the natural logarithm (basis \( e \)), the entropy is measured in nats. The entropy functional \( H(\cdot) \) is concave [4], may be negative and may be infinite when the integral of Eq. 1 diverges.

Although closed-form formula for the differential entropy are available for many common statistical distributions (see the devoted book [11] and [14]), the differential entropy of mixtures usually does not admit closed-form expressions [10,20] because the log term in Eq. 1 transforms into an untractable log-sum term when dealing with mixture densities. Let us denote by \( m(x) = \sum_{c=1}^k w_c p_c(x) \)

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1For example, when \( X \sim N(\mu, \sigma) \) is a Gaussian distribution of mean \( \mu \) and standard deviation \( \sigma > 0 \), then \( H(X) = \frac{1}{2} \log(2\pi e \sigma^2) \), and is therefore negative when \( \sigma < \frac{1}{\sqrt{2\pi e}} \).

2For example, consider \( X \sim p(x) \) with \( p(x) = \frac{\log(2)}{x \log 2} \) for \( x > 2 \) (with support \( X = (2, \infty) \)). Then \( H(X) = +\infty \). This result is to contrast with the fact that the discrete entropy on a finite alphabet \( \mathcal{X} \) is bounded by \( \log |\mathcal{X}| \).
the density of a mixture \( M \sim m(x) \) with \( k \) components \( X_i \sim p_c(x) \), where \( w \in \Delta_k \) denotes the \( k \)-dimensional open probability simplex. That is, a mixture is a convex combination of component distributions \( p_1(x), \ldots, p_k(x) \). We shall consider mixtures of Gaussians with component probability density functions \( X_i \sim N(\mu_i, \sigma_i) \) such that:

\[
p_i(x) = p(x; \mu_i, \sigma_i) = \frac{1}{\sqrt{2\pi\sigma_i}} \exp\left(-\frac{(x - \mu_i)^2}{2\sigma_i^2}\right),
\]

where \( \mu_i = E[X_i] \in \mathbb{R} \) and \( \sigma_i = \sqrt{E[(X_i - \mu_i)^2]} > 0 \) denote the mean parameter and the standard deviation of \( X_i \), respectively.

Statistical mixtures allow flexible fine modeling of arbitrary smooth densities. They are provably universal smooth density estimators. The most common mixtures are the Gaussian Mixture Models (GMMs) that are frequently used in applications. To tackle the differential entropy of continuous mixtures, various approximation techniques have been designed (see \cite{15} and references therein for a state-of-the-art). In practice, to estimate \( H(X) \) with \( X \sim p(x) \), one uses the following Monte-Carlo (MC) stochastic integration:

\[
\hat{H}_s(X) = -\frac{1}{s} \sum_{i=1}^{s} \log p(x_i), \tag{2}
\]

where \( \{x_1, \ldots, x_s\} \) is an independent and identically distributed (iid) set of variates sampled from \( X \sim p(x) \). This MC estimator \( \hat{H}_s(X) \) is consistent (ie., \( \lim_{s \to \infty} \hat{H}_s(X) = H(X) \), convergence in probability). Moshksar and Khandani \cite{13} recently considered the special case of isotropic spherical Gaussian Mixture Models (ie., GMMs with identical standard deviation), and used Taylor expansions to arbitrarily finely approximate the differential entropy of those isotropic GMMs. Interestingly, they mentioned in their paper \cite{13} the so-called Maximum Entropy Upper Bound (MEUB) that relies on the fact that the continuous distribution with prescribed variance maximizing the entropy is the Gaussian distribution of same variance. Since the entropy of a univariate Gaussian \( N(\mu, \sigma) \) is \( \frac{1}{2} \log(2\pi e\sigma^2) \), we end up with the following maximum entropy upper bound for an arbitrary random variable \( X \):

\[
H(X) \leq \frac{1}{2} \log (2\pi eV[X]), \tag{3}
\]

where \( V[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2 \) denotes the variance of \( X \). Since the variance \( V[X] \) of an arbitrary Gaussian mixture can be easily calculated in closed-form \cite{15}:

\[
V[X] = \sum_{c=1}^{k} w_c \left( (\mu_c - \bar{\mu})^2 + \sigma_c^2 \right),
\]

with \( \bar{\mu} = \sum_{c=1}^{k} w_c \mu_c = E[X] \), Eq. \ref{eq:3} yields the Gaussian MaxEnt Upper Bound:

\[
H(M) \leq \frac{1}{2} \log \left( 2\pi e \sum_{i=1}^{k} w_i ((\mu_i - \bar{\mu})^2 + \sigma_i^2) \right). \tag{4}
\]

In this work, we propose to further use the maximum entropy upper bound principle to derive an infinite series of MaxEnt upper bounds. Although our bounds will be instantiated for GMMs, they

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\[3\] Beware that the mixture random variable \( M \neq \sum_i w_i X_i \). The probability density of a weighted sum of random variables is obtained by convolution of the densities.
apply more broadly to univariate continuous mixtures. For example, our MaxEnt upper bounds also hold for mixtures of exponential families [18] that generalize the GMMs (and have always guaranteed finite entropy). For GMMs, we shall show that the Gaussian MaxEnt Upper Bound is not necessarily the best MaxEnt upper bound in closed form, and report instead a series of upper bounds.

The paper is organized as follows: Section 2 introduces the general principle for building Maximum Entropy (MaxEnt) Upper Bounds. It is followed by Section 3 that construct a series of MaxEnt upper bounds derived from a special family of MaxEnt distributions that we termed Absolute Monomial Exponential Families. Those generic bounds are instantiated for Gaussian Mixture Models (GMMs) in Section 4. Section 5 report on our experiments and discusses the tightness of the bounds. Finally, Section 6 wrap ups the results and conclude the work. Besides, an appendix provides the detailed calculation of the raw absolute moment of a non-centered normal distribution that is used in Section 4 to get closed-form MaxEnt upper bounds for GMMs.

2 Maximum entropy upper bounds on the differential entropy

The MaxEnt distribution principle was investigated by Jaynes [7, 8] to infer a distribution given several “moment constraints.” MaxEnt asks to solve the following constrained optimization problem:

$$\max_p H(p) : E[t_i(X)] = \eta_i, \quad i \in [D] = \{1, \ldots, D\}. \tag{5}$$

When an iid sample set \(\{x_1, \ldots, x_s\}\) is given, we may choose, for example, the raw geometric sample moments \(\eta_i = \frac{1}{s} \sum_{j=1}^s x_j^i\) for setting up the constraint \(E[X^i] = \eta_i\) (ie., taking \(t_i(X) = X^i\) in Eq. 5). The distribution \(p(x)\) maximizing the entropy under those moment constraints is unique and termed the MaxEnt distribution. The constrained optimization of Eq. 5 is solved by means of Lagrangian multipliers [4,12]. It is well-known [6,12] that the MaxEnt distribution \(p(X)\) different from the MaxEnt distribution \(p(X) = p(x; \theta)\) (with \(\sum_{x} p(x; \theta) = 1\)) will have necessarily smaller entropy: \(H(p(x)) \leq H(p(x; \theta))\) with \(p(x) = p(x; \theta)\). However, depending on the choosing sufficient statistics \(t_i\)'s, neither \(\theta\) nor \(F(\theta)\) may be available in closed-forms, and thus need to be approximated numerically [12].

In the remainder, we upper bound the differential entropy of a continuous random variable (eg., finite mixtures) by building a collection of upper bounds derived from MaxEnt distributions which
admit closed-form expressions for their differential entropy. Those bounds prove handy in practice for GMMs since the differential entropy of a GMM is not available in closed-form \cite{13,20}. Besides, we report closed-form formula for calculating the arbitrary raw absolute moments of a univariate Gaussian Mixture Model that may prove useful in other areas of statistical machine learning and information theory.

3 MaxEnt upper bounds from raw absolute moment constraints:

The Absolute Monomial Exponential MaxEnt distributions

Consider the univariate uni-order \((D = 1)\) family of Absolute Monomial Exponential Family (AMEF) induced by the absolute value of a monomial of degree \(l \in \mathbb{N}\) defined over the full support \(X = \mathbb{R}\):

\[
p_l(x; \theta) = \exp\left(\theta |x|^l - F_l(\theta)\right), \quad x \in \mathbb{R}
\]

for \(\theta < 0\). The natural parameter space is \(\Theta = (-\infty, 0)\). The log-normalizer\footnote{This integral can be computed using any Computer Algebra Systems (CASs) like Maxima that can be downloaded at \url{http://maxima.sourceforge.net/}. CASs implement the semi-algorithm of Risch \cite{17} for symbolic integration, and can therefore determine whether the integral admits a closed-form or not in terms of elementary functions. See \cite{2,5} for recent developments. Some CAS code snippets are provided in the appendix.} is:

\[
F_l(\theta) = \log 2 + \log \Gamma\left(\frac{1}{l}\right) - \log l - \frac{1}{l} \log(-\theta),
\]

where the Gamma function \(\Gamma(u) = \int_0^{\infty} x^{u-1} \exp(-x) dx\) generalizes the factorial \((\Gamma(n) = (n-1)!\) for \(n \in \mathbb{N}\)). The Gamma function can be approximated finely in a few constant operations. In fact, even better, it is the function \(\log\Gamma(u)\) that can be calculated quickly (see the numerical recipe in \cite{16}), so that the log-normalizer of Eq. \((8)\) can be calculated fast for any \(l \in \mathbb{N}\) and \(\theta < 0\). Note that those AMEF distributions are unimodal distributions with the unique mode located at \(x = 0\).

Since \(p_l(x; \theta) = p_l(-x; \theta)\), the mean \(E_{p_l(x; \theta)}[X]\) of an AMEF is always zero.

Now, the key element is to notice that the differential entropy \(H_l(\theta) = H(p_l(x; \theta))\) of an AMEF admits the following closed-form formula:

\[
H_l(\theta) = \log 2 + \log \Gamma\left(\frac{1}{l}\right) - \log l + \frac{1}{l}(1 + \log(-\theta)),
\]

where \(a_l = \log 2 + \log \Gamma\left(\frac{1}{l}\right) - \log l + \frac{1}{l}\) is a constant independent of \(\theta\). The entropy can be expressed equivalently using the Legendre convex conjugate \cite{14} \(F^*(\eta)\) as:

\[
H_l(\eta) = -F^*_l(\eta) = F_l(\theta) - \theta F'_l(\theta),
\]

with \(F'_l(\theta) = -\frac{1}{\eta} = \eta\) and \(\theta = -\frac{1}{\eta}\). Therefore the entropy formula expressed using the \(\eta\)-parameter is:

\[
H_l(\eta) = \log 2 + \log \Gamma\left(\frac{1}{l}\right) - \log l + \frac{1}{l}(1 + \log l + \log \eta),
\]

\[
= b_l + \frac{1}{l} \log \eta,
\]
with \( b_l = \log \frac{2\Gamma((\frac{1}{2}+l))}{l^{\frac{1}{2}}} \) a constant at prescribed \( l \), independent of \( \eta \). We readily check Young-Fenchel equality: \( F_l(\theta) + F_l^*(\eta) - \theta \eta = 0 \) since \( \theta \eta = -\frac{1}{l} \), and observe that \( H_l(\theta) = -F_l^*(\eta) = F_l(\theta) + \frac{1}{2} \)). Let \( H = \{ \eta(\theta) : \theta \in \Theta \} \) denote the expectation parameter space. For AMEFs, the dual natural/expectation parameter spaces are thus \( \Theta = (-\infty, 0) \) and \( H = (0, +\infty) \). To avoid confusion, let us denote by \( H_l^\theta(\cdot) \) and by \( H_l^\eta(\cdot) \) the entropy formula of Eq. 9 and Eq. 11 with respect to the natural and expectation parameters, respectively.

To upper bound the entropy of any arbitrary univariate continuous random variable \( X \) (let it be a mixture or not), we simply calculate the \( l \)-th raw absolute geometric moment \( A_l(X) = E_X[|X|^l] \), and deduce the following MaxEnt entropy Upper Bound (MEUB) \( U_l \):

\[
H(X) \leq H_l^\eta\left(E_X\left[|X|^l\right]\right)
\]

We thus obtain an infinite countable series of MaxEnt Upper Bounds (MEUBs) that we summarize in the following theorem:

**Theorem 1 (AMEF MaxEnt Upper Bounds)** Let \( X \) be a continuous random variable with support \( X = (-\infty, \infty) \). Then the differential entropy \( H(X) \) of \( X \) is upper bounded by the following series of MaxEnt upper bounds:

\[
H(X) \leq U_l(X) = b_l + \frac{1}{l} \log E_X[|X|^l], \quad \forall l \in \mathbb{N},
\]

where \( b_l = \log 2 + \log \Gamma\left(\frac{1}{l}\right) - \log l + \frac{1}{l}(1 + \log l) \).

Note that for even integer \( l \), \( A_l(X) = E_X[|X|^l] = E_X[X^l] \). That is, the absolute geometric moments coincide with the geometric moments for even integer \( l \).

Let us give two well-known MaxEnt distributions that are AMEF MaxEnt distributions in disguise, with their corresponding differential entropies:

- Consider \( l = 2 \). Since \( \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \), we get \( F_2(\theta) = \frac{1}{2} \log \frac{\pi}{\theta} \). Thus \( p_2(x; \theta) = \frac{\sqrt{\theta}}{\pi} \exp\left(-\frac{x^2}{2\theta}\right) \).

  By setting \( \theta = -\frac{1}{2\pi^2} \), we get the usual canonical *standard Gaussian density*: \( \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \).

  Since \( \theta = -\frac{1}{2\pi^2} \), we recover the usual entropy of a Gaussian: \( H(p_2(x; \theta)) = \frac{1}{2} \log 2\pi\sigma^2 \).

  It follows that:

  \[
  H(p_2(x; \theta)) = \sqrt{\frac{-\theta}{\pi}} \frac{\sqrt{\pi}}{2\sqrt{-\theta}} + \log \sqrt{\frac{\pi}{-\theta}} = \frac{1}{2} \log \frac{\pi}{-\theta} e.
  \]

  Since \( \theta = -\frac{1}{2\pi^2} \), and considering the location-scale family, we get back the usual entropy expression of a Gaussian \( X \sim N(\mu, \sigma) \): \( H(p_2(x; \theta)) = \frac{1}{2} \log 2\pi\sigma^2 \).

- Consider \( l = 1 \). The MaxEnt distribution is the *standard Laplacian distribution* \[6\] with density written canonically as \( p(x; \theta) = \exp\left(\theta|x| - \log\left(-\frac{1}{\theta}\right) \right) \) with \( F_1(\theta) = \log\left(-\frac{1}{\theta}\right) \). The differential entropy can be expressed in *either* the natural or expectation coordinate system as \( H(p(x; \theta)) = 1 + \log \left(\frac{2}{\theta}\right) \) or \( H(p(x; \eta)) = 1 + \log(2\eta) \) with \( \eta = F_1'(\theta) = -\frac{1}{\theta} \), respectively.

In general, the differential entropy of an AMEF distribution of degree \( l \) is negative when:

\[
\eta < \frac{l!}{le(2\Gamma(\frac{1}{l}))^l},
\]
and non-negative otherwise. Note that when \( l = 2 \), the AMEF is the Gaussian family, and since 
\( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \), we recover the negative entropy condition \( \eta = \sigma^2 < \frac{1}{2\pi e} \); and conclude that \( \sigma < \frac{1}{\sqrt{2\pi e}} \), as already claimed above.

Finally, we can extend the AMEF differential entropy formula to location-scale AMEF distributions with \( \mu \) a location parameter and \( \sigma \) a dispersion parameter. Let \( y = \mu + \sigma x \), and 
\[ p_l(x;\theta,\mu,\sigma) = \frac{1}{\sigma} p_l\left(\frac{x-\mu}{\sigma};\theta\right) \] (where \( p_l(x;\theta) \) denotes the standard AMEF distribution). We have 
\( dy = \sigma dx \) and by making a change of variable in the integral of Eq. 1 (see Appendix), it follows 
that \( H(Y) = H(X) + \log \sigma \) (thus always independent of the location parameter).

**Lemma 1** The differential entropy of a location-scale absolute monomial exponential family of degree \( l \) and location parameter \( \mu \) and dispersion parameter \( \sigma > 0 \) is available in closed-form as:

\[
H^\theta_{l,\mu,\sigma}(\theta) = a_l - \frac{1}{l} \log(-\theta) + \log \sigma, \quad (13)
\]

\[
H^\eta_{l,\mu,\sigma}(\eta) = b_l + \frac{1}{l} \log \eta + \log \sigma, \quad (14)
\]

where 
\( a_l = \log 2 + \log \Gamma\left(\frac{1}{l}\right) - \log l + \frac{1}{l} \)
\( \) and 
\( b_l = \log 2\Gamma\left(\frac{1}{l}\right)\left(e^{\frac{1}{l}}\right)^\frac{1}{l} \).

Note that scaling an AMEF amounts to scale its natural parameter since 
\( \theta | x | \sigma^l = \theta | x |^l \) (see Eq. 7). When the support is restricted to \( X = [0, \infty) \) instead of \( \mathbb{R} \), we subtract the 
\( \log 2 \) from \( F \) and \( H \) formula. For example, this is useful when considering mixtures of Rayleigh distributions instead of Gaussian distributions.

## 4 MaxEnt upper bounds for GMMs

In order to apply the MaxEnt upper bounds \( U_l \) for a GMM with probability density function 
\( m(x) = \sum_{c=1}^{k} w_c p(x;\mu_c,\sigma_c) \), we need to compute its absolute raw moment and plug this value into 
formula Eq. 12. By linearity of the expectation operator, we have:

\[
E_{m(x)} [||X||^r] = \sum_{i=1}^{k} w_i E_{p(x;\mu_i,\sigma_i)} [||X||^r].
\]

The raw geometric moments and absolute raw geometric moments for a centered Gaussian distribution (ie. \( \mu = 0 \)) are reported in [21]: Closed-form formula are reported for the (absolute) moments for real-valued \( r > -1 \) using the Kummer’s confluent hypergeometric functions [21].

For \( l = 2 \), we can thus recover the well-known MaxEnt Variance GMM upper bound [15]:

\[
H(X) \leq U_2 = \frac{1}{2} \log \left( 2\pi e \sum_{i=1}^{k} w_i ((\mu_i - \bar{\mu})^2 + \sigma_i^2) \right), \quad (15)
\]

with \( \bar{\mu} = \sum_{i=1}^{k} w_i \mu_i \).

Surprisingly, we did not find the general formula for the raw absolute moments of a non-centered Gaussian. We carried out the calculations reported in the Appendix. Fortunately, the raw absolute moments of a Gaussian admit closed-form formula expressed equivalently either using the
Cumulative Distribution Function (CDF) \( \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^{2}}{2}} \, dt \), the error function \( \text{erf}(x) \) or the complementary error function \( \text{erfc}(x) = 1 - \text{erf}(x) \). Those basic CDF, erf and erfc functions are related to each other by the following identities:

\[
\Phi(x) = \frac{1}{2} \left(1 + \text{erf} \left( \frac{x}{\sqrt{2}} \right) \right) = \frac{1}{2} \text{erfc} \left( -\frac{x}{\sqrt{2}} \right).
\]

Based on our calculations, we state the series of MaxEnt upper bounds for the differential entropy of a GMM \( X \sim m(x) = \sum_{c=1}^{k} w_c p(x; \mu_c, \sigma_c) \) in the following corollary of Theorem 1:

**Corollary 1** The differential entropy \( H(X) \) of a Gaussian mixture model \( X \sim m(x) = \sum_{c=1}^{k} w_c p(x; \mu_c, \sigma_c) \) is upper bounded by:

\[
H(X) \leq U_l(X) = b_l + \frac{1}{l} \log A_l(X), \quad \forall l \in \mathbb{N},
\]

where \( b_l = \log 2 + \log \Gamma \left( \frac{1}{l} \right) - \log l + \frac{1}{4} \log (1 + \log l) \) and

\[
A_l(X) = \begin{cases} 
\sum_{c=1}^{k} w_c \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor} \mu_c^{l-2i} \sigma_c^{2i} \frac{(1+2i)}{\sqrt{\pi}} & \text{for even } l, \\
\sum_{c=1}^{k} w_c \sum_{i=0}^{l} \left( \mu_c - (1) \sigma_c ^{i} \Gamma_i \left( \frac{1}{2} \right) \right) & \text{for odd } l.
\end{cases}
\]

where

\[
I_i(a) = \frac{1}{\sqrt{2\pi}} \int_{a}^{+\infty} x^i \exp \left( -\frac{1}{2} x^2 \right) \, dx,
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( a^{i-1} \exp \left( -\frac{1}{2} a^2 \right) \right) + (i-1)I_{i-2}(a),
\]

with the terminal recursion cases:

\[
I_0(a) = 1 - \Phi(a) = \frac{1}{2} \left(1 - \text{erf} \left( \frac{a}{\sqrt{2}} \right) \right) = \frac{1}{2} \text{erfc} \left( \frac{a}{\sqrt{2}} \right),
\]

\[
I_1(a) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} a^2 \right).
\]

In particular, the first two MaxEnt upper bounds (corresponding to the Laplacian and Gaussian MaxEnt distributions, respectively) are given as:

**Corollary 2** (Laplacian maximum entropy upper bound) The differential entropy of a Gaussian mixture model \( X \sim \sum_{c=1}^{k} w_c p(x; \mu_c, \sigma_c) \) is upper bounded by:

\[
H(X) \leq U_1(X) = \log \left( 2e \left( \sum_{c=1}^{k} w_c \left( 1 - 2 \Phi \left( -\frac{\mu_c}{\sigma_c} \right) \right) + \sigma_c \sqrt{\frac{2}{\pi}} \exp \left( -\frac{1}{2} \left( \frac{\mu_c}{\sigma_c} \right)^2 \right) \right) \right).
\]

**Corollary 3** (Gaussian maximum entropy upper bound) The differential entropy of a
GMM $X \sim \sum_{c=1}^{k} w_c p(x; \mu_c, \sigma_c)$ is upper bounded by:

$$H(X) \leq U_2(X) = \frac{1}{2} \log \left( 2\pi e \sum_{c=1}^{k} w_c ((\mu_c - \bar{\mu})^2 + \sigma_c^2) \right),$$

with $\bar{\mu} = \sum_{c=1}^{k} w_c \mu_c$.

Thus we can bound the differential entropy by $H(X) \leq \min(U_1(X), U_2(X))$, and for our series of upper bounds by:

$$H(X) \leq \min_{i \in \mathbb{N}} U_i(X).$$

Since the differential entropy does not change by changing the location parameter, we may consider without loss of generality that the GMM is centered to zero (that is, its expectation $E[X]$ is zero). If not, we simply translate the GMM by setting the component means to $\mu'_i = \mu_i - \bar{\mu}$ so that the expectation of the GMM matches the expectation of the AMEF. This alignment of the GMM to the AMEF preserves the MaxEnt upper bounds.

In general, we may shift the GMM $X$ by $\delta \in \mathbb{R}$ by setting $\mu'_i = \mu_i - \bar{\mu}$. Let $X_\delta$ denote this shifted GMM, $A_i(\delta) = E[|X_\delta|^l]$ and $U_i(\delta) = U_i(X_\delta)$. We can further refine the MEUBs by minimizing the MaxEnt upper bounds:

$$U_i(\delta) = b_l + \frac{1}{l} \log A_i(\delta),$$

where $b_l = \log 2 + \log \Gamma \left( \frac{1}{l} \right) - \log l + \frac{1}{l} (1 + \log l)$.

When $l = 2$, the optimal shift is obtained for $\delta = \bar{\mu}$. However, when $l = 1$, the optimization problem is non-trivial and requires numerical optimization procedures. (In the remainder, we consider $\delta = \bar{\mu}$ when carrying experiments.)

## 5 Experiments and tightness of the bounds

### 5.1 Laplacian versus Gaussian MaxEnt upper bounds

First, we consider the following experiment repeated $t = 1000$ times: We draw of a GMM $X \sim m$ with two components with $\mu_i, \sigma_i \sim_{\text{iid}} U(0, 1)$ and $w_i \sim_{\text{iid}} U(0, 1)$ chosen as uniform weights renormalized to 1, we recenter the GMM so that $\bar{\mu}' = 0$ (setting $\delta = \bar{\mu}$), and compute the stochastic approximation $\hat{H}$ of $H(X)$ (for $s = 10^6$ samples), and the first order and second order maximum entropy upper bounds $H_1^0(E_m[|X|])$ and $H_2^0(E[X^2])$. We report average approximations $\left| \frac{H_i^0 - \hat{H}}{\hat{H}} \right|$, and the percentage of times MEUB $U_1(X) < U_2(X)$: Gaussian MEUB is on average 40% above $\hat{H}$ and Laplacian MEUB is on average 10% above $\hat{H}$. Laplacian MEUB bound beats the Gaussian MEUB 32.9% on average. Thus we recommend practitioners to upper bound the differential entropy of a GMM $X$ by $H(X) \leq \min(U_1(X), U_2(X))$.

### 5.2 Series of MaxEnt upper bounds

A question one may ponder is whether all MEUBs $U_i(X)$ are useful of not? We performed an experiment by drawing at random GMMs $X$ with $k = 2$ components and checking among the first $n$ bounds $U_1(X), \ldots, U_n(X)$. We found experimentally that most of the time the bounds
$U_1(X)$ and $U_2(X)$ suffices, but sometimes the tightest bound could be $U_n(X)$. This is the case when one component is almost a Dirac ($\sigma = o(1)$) while the other component has significant standard deviation, and the two Gaussian components far apart. For example, let us choose $X \sim m(x) = \frac{1}{2}p(x, -\frac{1}{2}, 10^{-5}) + \frac{1}{2}p(x, \frac{1}{2}, 10^{-1})$. Then $U_{i+1}(X) \leq U_i(X)$ for $i \leq 37$. We get NaN numerical errors when computing $U_{38}$ using the closed-form formula.

We shall make more precise those arguments in the following section.

### 5.3 Tightness analysis of MaxEnt upper bounds

First, let us show that bound $U_1$ (the Laplacian MEUB) may be better than $U_2$ (the Gaussian MEUB). To derive analytic conditions, we consider the restricted case of zero-centered GMMs [19]. We have $A_1(X) = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{k} w_i \sigma_i = \sqrt{\frac{2}{\pi}} \bar{\sigma}$, and therefore get the upper bound $U_1$ on the differential entropy of the mixture as:

$$H(X) \leq U_1(X) = 1 + \log 2 \sqrt{\frac{2}{\pi}} \left( \sum_{i=1}^{k} w_i \sigma_i \right). \quad (16)$$

This bound is strictly better than the traditional Gaussian bound [13]:

$$U_2(X) = \log \sqrt{2\pi e} \left( \sum_{i=1}^{k} w_i \sigma_i^2 \right),$$

provided that $U_1(X) < U_2(X)$. Note that when $l = 2$ and $k = 1$, the $U_2$ bound matches precisely the entropy of the single-component Gaussian mixture. Let $\bar{\sigma}_1$ be the arithmetic weighted mean and $\bar{\sigma}_2 = \sqrt{\sum_{i=1}^{k} w_i \sigma_i^2}$ be the quadratic mean of the weighted standard deviations, respectively. Then $U_1(X) < U_2(X)$ if and only if:

$$\log 2e \sqrt{\frac{2}{\pi}} \sigma_1 \leq \log \sqrt{2\pi e \bar{\sigma}_2}. \quad (17)$$

That is, we need to have $\frac{\sigma_1}{\bar{\sigma}_2} \leq \frac{\pi}{2\sqrt{\pi}} \approx 0.9527$. Observe that the weighted quadratic mean dominates\(^5\) the weighted arithmetic mean, and therefore $\frac{\sigma_1}{\bar{\sigma}_2} \leq 1$. Equality of arithmetic/quadratic means only happens when all the $\sigma_i$’s coincide (since we have zero-centered GMMs, that means that the GMM collapses to a Gaussian). To summarize our illustrating example, bound $U_1$ may be better or worse than $U_2$ depending on the set of $\sigma_i$’s. For the degenerate case $k = 1$ (single component $X = N(0, \sigma)$), the condition of $U_1 < U_2$ ($U_1$ tighter than $U_2$) writes as $\sigma > 2\sqrt{\pi}$ (that is, $\sigma > 1.0496$).

Now, for $l \in \mathbb{N}$, we built a MaxEnt upper bound on the differential entropy of a GMM $X \sim m(x)$. How does this infinite sequence of bounds $U_1, U_2, \ldots, U_q, \ldots$ relate to each others? For a prescribed value $A$, $H^q_l(A)$ decreases as $l$ increases, but the absolute raw moment $A_l(X)$ also varies. Does there always exist a GMM $X$ so that there exists $l' > l$ such that $H^q_{l'}(A_l(X)) < H^q_{l'}(A_{l'}(X))$, or not?

\(^5\)In general, we denote for a strictly increasing function $f(x)$ the quasi-arithmetic weighted mean by $\bar{\sigma}_{f(x)} = f^{-1}\left(\sum_{i=1}^{k} w_i f(\sigma_i)\right)$, see [9].
yield the tightest one. Here, to answer negatively this question when considering the family of zero-centered Gaussian mixtures, we shall consider even integers \( l \) and \( l' = l + 2 \). Let \( N \sim N(\mu = 0, \sigma) \). Then the geometric raw moments coincide with the central geometric moments, and by the linearity of the expectation operator, we have \([21]\):

\[
E_X[X^l] = 2\pi \frac{\Gamma\left(\frac{1+l}{2}\right)}{\sqrt{\pi}} \left( \sum_{i=1}^k w_i \sigma_i^l \right) = A_l(X). \tag{18}
\]

Then we have the following MaxEnt upper bound \( U_l \):

**Lemma 2 (Zero-centered GMMs)** The differential entropy of a zero-centered GMM \( X \sim \sum_{c=1}^k w_c p(x; \sigma_c) \) is upper bounded by:

\[
H(X) \leq H_l^\eta(A_l(X)) = b_l + \frac{1}{l} \log z_l + \log \bar{\sigma}_l, \tag{19}
\]

where \( \bar{\sigma}_l \) is the \( l \)-th power mean:

\[
\bar{\sigma}_l = \left( \sum_{i=1}^k w_i \sigma_i^l \right)^{\frac{1}{l}}. \tag{20}
\]

When \( l \to \infty \), we have \( \sigma_l \to \max_i \sigma_i \). Thus for a tighter bound \( U_{l+2} < U_l \), we need to find a zero-centered GMM so that:

\[
\log \left( \frac{\bar{\sigma}_{l+2}}{\bar{\sigma}_l} \right) \leq b_l + \frac{1}{l} \log z_l - b_{l+2} - \frac{1}{l+2} \log z_{l+2} - \Delta_l.
\]

Since the \((l + 2)\)-power mean dominates the \( l \)-power mean (ie., \( \bar{\sigma}_{l+2} > \bar{\sigma}_l \)), we have \( \frac{\bar{\sigma}_{l+2}}{\bar{\sigma}_l} \geq 1 \) and therefore \( \log \frac{\bar{\sigma}_{l+2}}{\bar{\sigma}_l} \geq 0 \). It turns out that \( \Delta_l < 0 \) when \( l > 2 \) with \( \lim_{l \to \infty} \Delta_l = 0 \). So we conclude that only \( U_1 \) and \( U_2 \) are necessary for zero-centered GMMs (and we define the MEUB as \( \min(U_1, U_2) \)).

When considering arbitrary GMMs, the situation is analytically more complex to decide.

Last but not least, whether bound \( U_l \) proves useful or not depends on the mixture family (eg., mixtures of Pareto distributions \([1]\)). A Pareto distribution has density \( \alpha x^{\alpha-1} \) for \( x > 0 \) and \( \alpha \) the shape parameter. The raw moments of a Pareto distribution is \( A_l = \frac{\alpha l^{-\alpha}}{\alpha-1} \) for \( \alpha > l \) and \( \infty \) otherwise.

### 6 Conclusion

We considered the novel parametric family of Absolute Monomial Exponential Families (AMEFs), and reported a closed-form differential entropy formula for these AMEFs. We then considered a collection of Maximum Entropy Upper Bounds (MEUBs) for an arbitrary continuous random variable based on its raw geometric absolute moments (Theorem \([1]\)), and show how to apply those generic bounds to the specific case of Gaussian Mixture Models (GMMs). Interestingly, we showed that the Laplacian MaxEnt upper bound may potentially be tighter than the traditionally used

\[\text{A mean } \overline{\sigma}_{g(x)} \text{ dominates another mean } \overline{\sigma}_{f(x)} \text{ (that is, } \overline{\sigma}_{g(x)} \geq \overline{\sigma}_{f(x)} \text{) when } g(x) \geq f(x).\]
Gaussian MaxEnt upper bound. Therefore, we recommend in practice to take the minimum of these Laplacian and Gaussian MEUBs. This new series of MaxEnt upper bounds proves useful in practice since the differential entropy of mixtures does not admit a closed-form formula \cite{20}. Besides, we report in the Appendix closed-form formula for calculating the raw absolute moments of a univariate Gaussian Mixture Model. The method can be extended to any location-scale univariate continuous distribution.

A Java\textsuperscript{TM} source code for reproducible research with test experiments is available at:

https://www.lix.polytechnique.fr/~nielsen/MEUB/

A Differential entropy of a location-scale distribution

Let \( p(x; \mu, \sigma) = \frac{1}{\sigma} p_0\left(\frac{x-\mu}{\sigma}\right) \) denote the density of a location-scale distribution on the full support \( \mathbb{R} \), where \( \mu \in \mathbb{R} \) denotes the location parameter and \( \sigma > 0 \) the dispersion parameter. For example, a normal distribution has location parameter its mean and dispersion parameter its standard deviation. Let us prove that the entropy \( H(X) \) is \( H(X_0) + \log \sigma \) with \( X \sim p(x; \mu, \sigma) \) and \( X_0 \sim p_0(x) \), a quantity always independent of the location parameter \( \mu \). We shall make use of a change of variable \( y = \frac{x-\mu}{\sigma} \) (with \( dy = \frac{dy}{\sigma} \)) in the integral to get:

\[
H(X) = \int_{x=-\infty}^{+\infty} -\frac{1}{\sigma} p_0\left(\frac{x-\mu}{\sigma}\right) \left( \log \frac{1}{\sigma} p_0\left(\frac{x-\mu}{\sigma}\right) \right) dx, \tag{21}
\]

\[
= \int_{y=-\infty}^{+\infty} -p_0(y)(\log p_0(y) - \log \sigma), \tag{22}
\]

\[
= H(X_0) + \log \sigma. \tag{23}
\]

B Raw absolute moments of a non-centered normal distribution

Let \( X \sim N(\mu, \sigma) \) be a normal random variable of mean \( \mu \in \mathbb{R} \) and standard deviation \( \sigma > 0 \). Let us express the density of the normal distribution as a location-scale density: \( p(x; \mu, \sigma) = \frac{1}{\sigma} p_0\left(\frac{x-\mu}{\sigma}\right) \), where \( p_0(x) \) denotes the density of the standard normal distribution \( X_0 \):

\[
p_0(x) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{1}{2} x^2 \right).
\]

Define the raw (uncentered) \( l \)-th absolute moment \( E[|X|^{l}] \) for a continuous univariate location-scale family with standard density \( p_0 \):

\[
A_l = E[|X|^{l}] = \int_{-\infty}^{+\infty} |x|^l \frac{1}{\sigma} p_0\left(\frac{x-\mu}{\sigma}\right) dx.
\]

We first consider the calculation of \( A_l = E[|X|^{l}] = E[X^l] \) for even integer \( l \), and then proceed with the computation of \( E[|X|^{l}] \) for odd \( l \).
### B.1 Raw even (absolute) moments

The raw absolute geometric moment amounts to the raw geometric moment for even integer $l$: $a_l = E[|X|^l] = E[X^l]$. It follows after a change of variable $y = \frac{x-\mu}{\sigma}$ (so that $x = \sigma y + \mu$) with $dy = \frac{dx}{\sigma}$ (and $dx = \sigma dy$) that:

$$A_l = \int_{-\infty}^{+\infty} (\sigma y + \mu)^l p_0(y) dy.$$  

Performing the binomial expansion $(\sigma y + \mu)^l = \sum_{i=0}^{l} \binom{l}{i} (\sigma y)^i \mu^{l-i}$, we get:

$$A_l = \sum_{i=0}^{l} \binom{l}{i} \mu^{l-i} \sigma^i \int_{-\infty}^{+\infty} y^i p_0(y) dy,$$

where $\int_{-\infty}^{+\infty} y^i p_0(y) dy$ is the $i$-th raw moment of the standard normal distribution $X_0$:

$$E[|X_0|^l] = \begin{cases} (l-1)!! \sigma^l & \text{even } l, \\ \sqrt{\frac{2}{\pi}} \frac{l-1}{2} (\frac{l-1}{2})! \sigma^l & \text{odd } l. \\ \end{cases}$$

with $n!!$ the double factorial: $n!! = \sqrt{2^{n+1}/\pi} \Gamma(\frac{n+1}{2}) + 1$.

We end-up with the following raw moment direct formula for an even integer $l$:

$$A_l = \sum_{i=0}^{l} \binom{l}{i} \mu^{l-2i} \sigma^{2i} \frac{\Gamma(\frac{l+2i}{2})}{\sqrt{\pi}}.$$

In particular, we recover the second (absolute) moment:

$$E[|X|^2] = \mu^2 + \sigma^2.$$

### B.2 Raw absolute moments of odd order

We get rid of the cumbersome absolute value by splitting the integral onto the positive and negative support as follows:

$$A_l = \int_{-\infty}^{0} -x^l 1/\sigma p_0 \left( \frac{x-\mu}{\sigma} \right) dx + \int_{0}^{+\infty} x^l 1/\sigma p_0 \left( \frac{x-\mu}{\sigma} \right) dx.$$

Consider the change of variable $y = \frac{x-\mu}{\sigma}$. We get:

$$A_l = \int_{-\infty}^{\frac{\mu}{\sigma}} -(\sigma y + \mu)^l p_0(y) dy + \int_{\frac{\mu}{\sigma}}^{+\infty} (\sigma y + \mu)^l p_0(y) dy.$$

By performing binomial expansion and sliding the integral inside the binomial sum, we get:

$$A_l = \sum_{i=0}^{l} \binom{\frac{n}{i}}{l} \mu^{l-i} \sigma^i \left( I_i \left( \frac{\mu}{\sigma} \right) - J_i \left( \frac{-\mu}{\sigma} \right) \right),$$

where $I_i$ and $J_i$ are the modified Bessel functions of the first and second kind, respectively.
with
\[
I_i(a) = \frac{1}{\sqrt{2\pi}} \int_a^{+\infty} x^i \exp \left( -\frac{1}{2} x^2 \right) \, dx, \quad (24)
\]
\[
J_i(b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^b x^i \exp \left( -\frac{1}{2} x^2 \right) \, dx. \quad (25)
\]

By a change of variable \( y = -x \) (with \( dy = -dx \)), we find that:
\[
J_i(b) = (-1)^i I_i(-b).
\]

Thus it is enough to consider the computation of \( I_i \) for all non-negative integers \( i \geq 0 \), and get the raw absolute moment as:
\[
A_l = \sum_{i=0}^l \binom{n}{l} \mu^{l-i} \sigma^i \left( I_i \left( -\frac{\mu}{\sigma} \right) - (-1)^i I_i \left( \frac{\mu}{\sigma} \right) \right),
\]

Consider the integration by parts \(^7\) for calculating integral:
\[
I_i(a) = \frac{1}{\sqrt{2\pi}} \int_a^{+\infty} x^i \exp \left( -\frac{1}{2} x^2 \right) \, dx, \quad i \in \mathbb{N}
\]

with \( v'(x) = x \exp \left( -\frac{1}{2} x^2 \right) \) (and antiderivative \( v(x) = -\exp \left( -\frac{1}{2} x^2 \right) + c \), where \( c \) is a constant) and \( u(x) = \frac{1}{\sqrt{2\pi}} x^{i-1} \) (and derivative \( u'(x) = \frac{1}{\sqrt{2\pi}} (i-1)x^{i-2} \)):
\[
I_i(a) = \frac{1}{\sqrt{2\pi}} a^{i-1} \exp \left( -\frac{1}{2} a^2 \right) - (i-1) \frac{1}{\sqrt{2\pi}} \int_a^{+\infty} x^{i-2} \exp \left( -\frac{1}{2} x^2 \right) \, dx.
\]

Thus we end up with the following recursive formula:
\[
I_i(a) = \frac{1}{\sqrt{2\pi}} a^{i-1} \exp \left( -\frac{1}{2} a^2 \right) + (i-1)I_{i-2}(a),
\]

with the terminal recursion cases:
\[
I_0(a) = 1 - \Phi(a), \quad (26)
\]
\[
I_1(a) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} a^2 \right). \quad (27)
\]

Equivalent expressions may be obtained using the error function or the complementary error function by using the following identities:
\[
\Phi(x) = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{x}{\sqrt{2}} \right) \right) = \frac{1}{2} \text{erfc} \left( -\frac{x}{\sqrt{2}} \right).
\]

\(^7\)Recall that \( \int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx \).
B.3 Numerical robustness

We checked experimentally the numerical robustness of those formula by comparing the exact moment formula with the Monte-Carlo estimated ones. In general, the moment $E_X[g(X)] = \int g(x)p(x;\mu,\sigma)dx$ can be approximated stochastically using Monte-Carlo sampling as:

$$\frac{1}{s} \sum_{i} g(x_i),$$

with $x_1,\ldots,x_s \sim_{\text{iid}} X$. MC estimators are consistent (ie., tend to the true value when $s \rightarrow \infty$). We can also discretize the moment integral using various quadratic rules to approximate the moment values.

The table below reports the first ten (10) raw absolute moments $E[|X|^l]$ of a Gaussian random variable $X \sim N(\mu = 1, \sigma = 2)$:

| Order $l$ | MC estimation ($s = 10^8$) | formula       | error in percent |
|-----------|----------------------------|---------------|-----------------|
| 0         | 1.0                        | 0.9999999999811017 | 1.889327347471877E-9 |
| 1         | 1.791120828723894          | 1.7911862296052241 | 0.003651385868581992 |
| 2         | 5.001255187253235          | 4.99999999971595 | 0.02509745023866034 |
| 3         | 17.65125885616714          | 17.65237556639124 | 0.006174453685761998 |
| 4         | 73.02031796881072          | 72.9999999342585 | 0.027823852597777112 |
| 5         | 339.9322718149212          | 339.9650189004305 | 0.009633414720237885 |
| 6         | 1740.1120429182856         | 1740.9999997924972 | 0.051028718399213205 |
| 7         | 9659.30994586874           | 9649.665608394966 | 0.099845498253121 |
| 8         | 57314.11839220495           | 57232.999991840465 | 0.14153301603174934 |
| 9         | 359089.900922202495         | 360173.109411100196 | 0.30165384379096866 |
| 10        | 2400947.7149176745         | 2389140.9996155496 | 0.4917522871846192 |

The error percentage is defined as $\frac{|MC-\text{formula}|}{MC} \times 100$. Note that although the formula of the raw absolute moment is exact, we use a fast approximation of the gamma function (see the numerical receipe in [16]).

We report below the first fifteen (15) raw absolute moments $E[|X|^l]$ of a Gaussian mixture random variable $X \sim m(x) = \frac{1}{2}p(x;-1,1) + \frac{1}{2}p(x;1,1)$:

| Order $l$ | MC estimation ($s = 10^8$) | formula       | error in percent |
|-----------|----------------------------|---------------|-----------------|
| 0         | 1.0                        | 1.0           | 0.0             |
| 1         | 1.023962043303955          | 1.02298206195441 | 0.01116005850390587 |
| 2         | 1.586851647073658          | 1.5870832593136879 | 0.0145954970643931 |
| 3         | 3.0642511302367           | 3.065376036966404 | 0.024423596384612 |
| 4         | 6.86813166292202          | 6.869191248884848 | 0.01542757178583545 |
| 5         | 17.23233174871346         | 17.231432948993978 | 0.04764564201920164 |
| 6         | 47.33241385186923         | 47.348515902828 | 0.0340238457465065 |
| 7         | 140.3877346937668         | 140.442927244606705 | 0.0393143677586867 |
| 8         | 444.72048147704123        | 444.9348623596309 | 0.0482057587890772 |
| 9         | 1493.545072510505         | 1493.4618390491788 | 0.005582928671066997 |
| 10        | 5273.89170145998          | 527.5759633479758 | 0.0698640270280147 |
| 11        | 19493.5806050307          | 1953.828198789193 | 0.2064656528070466 |
| 12        | 7564.96071848647          | 7540.1994120814818 | 0.316947187993167 |
| 13        | 302766.1898278726         | 302530.0169101444 | 0.0780090499999941 |
| 14        | 1260210.5985202221        | 1257605.9222757513 | 0.2066857487502786 |
| 15        | 5354365.384185081         | 5402772.331681476 | 0.904065076308983 |

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B.4 Raw sufficient statistic moments of exponential families

For univariate mixtures of natural EFs with a polynomial sufficient statistic $t(X)$, we may easily calculate moments using the Moment Generating Function (MGF) [3]:

$$M(t(u)) = E[\exp(\langle u, X \rangle)] = \exp(F(\theta + u) - F(\theta)).$$  \hspace{1cm} (28)

Thus for uni-order EFs, the geometric moments are given by the higher-order derivatives $E_X[t(X)^l] = M^{(l)}(0)$. For uni-order exponential families, it follows that $E[t(X)] = F'(\theta) = \eta$, and $V[t(X)] = F''(\theta) > 0$ (since $F$ is strictly convex). It follows that EFs have always all finite order moments expressed using the higher-order derivatives of the MGF. Thus we can always explicitly calculate the geometric moments of the sufficient statistic $t(X)$ from the MGF provided that the log-normalizer $F(\theta)$ is available in closed-form. For example, we may consider mixtures of Rayleigh distributions (with $t(x) = x^2$, see [3]) instead of GMMs, and get closed-form MaxEnt upper bounds. The geometric raw moments of a Rayleigh mixture $X$ is $A_l(X) = 2^l \Gamma(1 + \frac{l}{2}) \sum_{i=1}^{k} \sigma_i^l$.

Symbolic integration using a computer algebra system

The examples below show how definite integration is performed using the Computer Algebra System (CAS) Maxima:

```maxima
/* Example of a density that has infinite Shannon entropy */
p(x) := log(2)/(x*log(x)**2);
/* check it integrates to 1 */
integrate(p(x),x,2,inf);
/* check integral diverges */
integrate(-p(x)*log(p(x)),x,2,inf);
```

To handle AMEFs, we need to enforce that $\theta < 0$. For example, to compute the log-normalizer of an AMEF of order $l = 5$ in Maxima, we may use the following script:

```maxima
assume (theta<0);
F(theta) := integrate(exp(theta*abs(x)^5),x,-inf,inf);
integrate(exp(theta*abs(x)^5-F(theta)),x,-inf,inf);
```

To program a binomial expansion in Maxima, we write the following recursive function:

```maxima
binomialExpansion(i,p,q) := if i = 1 then p+q
else expand((p+q)*binomialExpansion(i-1,p,q)) ;
expand(binomialExpansion(10,x,y));
factor(%);
```

We get at the console the following output:

[^5]: http://maxima.sourceforge.net/
Thus to obtain direct formulae for the raw absolute moments, we may use the following symbolic
program in Maxima:

\[
\begin{align*}
\text{binomialExpansion}(i,p,q) &= \text{if } i = 1 \text{ then } p+q \\
&\quad \text{else expand}((p+q)\text{binomialExpansion}(i-1,p,q)) ; \\
\text{p0(y)} &= \exp(-y^2/2)/\sqrt{2\pi} ; \\
\text{absMoment}(\mu,\sigma,l) &= \text{ratsimp}\left(\int \text{factor}((\text{expand}((\text{binomialExpansion}(l,\mu,y*\sigma))))*\text{p0(y)},y,-\mu/\sigma,\infty) \right. \\
&\quad \left. -\int \text{factor}((\text{expand}((\text{binomialExpansion}(l,\mu,y*\sigma))))*\text{p0(y)},y,-\infty,-\mu/\sigma)\right) ; \\
\text{assume}(\sigma>0) ; \\
\text{absMoment}(\mu,\sigma,1) ; \\
\text{absMoment}(\mu,\sigma,3) ;
\end{align*}
\]

We get at the console the following output:

\[
\begin{align*}
\text{binomialExpansion}(i,p,q) &= \text{if } i = 1 \text{ then } p+q \\
&\quad \text{else expand}((p+q)\text{binomialExpansion}(i-1,p,q)) ; \\
p0(y) &= \exp(-y^2/2)/\sqrt{2\pi} ; \\
\text{absMoment}(\mu,\sigma,1) &= \text{ratsimp}\left(\int \text{factor}((\text{expand}((\text{binomialExpansion}(1,\mu,y*\sigma))))*\text{p0(y)},y,-\mu/\sigma,\infty) \right. \\
&\quad \left. -\int \text{factor}((\text{expand}((\text{binomialExpansion}(1,\mu,y*\sigma))))*\text{p0(y)},y,-\infty,-\mu/\sigma)\right) ; \\
\text{absMoment}(\mu,\sigma,3) ;
\end{align*}
\]

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