A solution to two party typicality using representation theory of the symmetric group

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1 Abstract

We show that, given a state on a bipartite system AB, the product of the tensor product of the typical projections for the marginal states on A and B and the typical projection for AB can be used to describe the correct asymptotics of the bipartite state itself, its square, marginals and squares of the marginals. Typicality is defined using the representation theory of the symmetric group.

This result has already been proven, but with a different notion of typicality.

2 Introduction

This work is motivated by a recent conjecture by Nicolas Dutil [3, Conjecture 3.2.7], who also gave a solution for the two-party case in the very same work. The conjecture in its original formulation reads as follows.

Conjecture 1 (Multiparty typicality conjecture - Conjecture 3.2.7 in [3]). Consider $n$ copies of an arbitrary multiparty state $ψ_{C_1...C_m}$. For any fixed $ε > 0$, $δ_T > 0$ and $n$ large enough, there exists a state $Ψ_{C_1...C_m}$ which satisfies

$$
\|Ψ_{C_1...C_m} - ψ_{C_1...C_m}\|_1 \leq ν(ε) (1)
$$

$$
\text{tr}\{(Ψ^T)^2\} \leq (1 - μ(ε))2^{−n(S(ψ_T)−δ_T)} (2)
$$

for all non-empty subsets $T \subset \{1, \ldots, m\}$. Here, $ν(ε)$ and $μ(ε)$ are functions of $ε$ which vanish by choosing arbitrarily small values for $ε$.

The main obstacle one is confronted with here is the notorious noncommutativity of different tensor products of marginals of the joint state $Ψ_{C_1...C_m}$.

3 Notation

The symbols $λ, λ', ν, ν', μ, μ'$ will be used to denote Young frames. The set of Young frames with at most $d \in \mathbb{N}$ rows and $n \in \mathbb{N}$ boxes is denoted $Y_{d,n}$.

For a Young Tableau $T$, we write $T_{ij}$ for the entry of $T$ in the $i$-th row and $j$-th column.

In the remainder, $H_A, H_B, H$ denote Hilbert spaces with dimensions $d_A, d_B, d$. The numbers $d_A, d_B$ will be arbitrary but constant, while $d$ serves as a “dummy”-dimension for intermediate
Define, for arbitrary \( \mu \) and \( \nu \) (note that these projections correspond to a specific set of these, and this set is chosen such that every two different projections are orthogonal (this may be seen as a specific choice of bases for the invariant subspaces of the standard representations of the unitary group on \((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}\)). Another constraint will be given by equation (3). Accordingly, projections onto irreducible subspaces of \( \mathbb{B}^{AB} \) get labelled \( P_{\mu,i} \otimes P_{\nu,j} \) \((\mu, \nu \in YF_{dA,dB,n}, i, j \in [m^A_\mu], [m^B_\nu])\).

Whenever it feels right, the superscripts \( A, B, AB \) will be omitted. To make up for that, in this case, the symbols \( \lambda, \lambda' \) will only be used for projections on \( AB \), while \( \mu, \mu' \) indicate that a projection on \( A \) is being used and \( \nu, \nu' \) are only subscripts for projections on the \( B \)-part.

Define, for arbitrary \( \mu \in YF_{dA,n}, \nu \in YF_{dB,n}, \lambda \in YF_{dA,dB,n} \) the projections

\[
P^A_{\mu} := \sum_{i=1}^{m^A_\mu} P_{\mu,i}^A, \quad P^B_{\nu} := \sum_{j=1}^{m^B_\nu} P_{\nu,j}^B, \quad P^{AB}_{\lambda} := \sum_{k=1}^{m^{AB}_\lambda} P_{\lambda,k}^{AB}.
\]

The choice we just made for the set \( \{P^{AB}: \lambda \in YF_{d,n}, i \in [m_\lambda]\} \) gets a little more specific now: We will choose these projections such that each \( P^A_{\mu} \otimes P^B_{\nu} \) (note that these projections correspond to subspaces which are only invariant under the action of \( \mathbb{B}^{AB} \)) can, by choosing an appropriate set \( \mathcal{M} \), be written as

\[
P^A_{\mu} \otimes P^B_{\nu} = \sum_{(\lambda,i) \in \mathcal{M}} P_{\lambda,i}.
\]

This is possible due to equation (3). Conversely, it implies that each \( P_{\lambda,i} \) obeys the inequality

\[
P_{\lambda,i} \leq P^A_{\mu} \otimes P^B_{\nu}
\]

for exactly one specific choice of \( \mu, \nu \in YF_{dA,n}, YF_{dB,n} \).

The set of states on a Hilbert space \( \mathcal{H} \) is written \( \mathcal{S}(\mathcal{H}) \). The set of probability distributions on a finite set \( X \) is denoted \( \mathcal{P}(X) \), the cardinality of \( X \) by \( |X| \).

For \( \lambda \in YF_{d,n}, \lambda \in \mathcal{P}([d]) \) is defined by \( \lambda(i) := \lambda_i/n \). If \( \rho \in \mathcal{S}(\mathcal{H}) \) with dim \( \mathcal{H} = d \) has spectrum \( s \in \mathcal{P}([d]) \), then it will always be assumed that \( s(1) \geq \ldots \geq s(d) \) holds and the distance between a spectrum \( s \) and a Young frame \( \lambda \in YF_{d,n} \) is measured by \( \|\lambda - s\| := \sum_{i=1}^{d} |\lambda(i) - s(i)| \).

We now define two important entropic quantities, both of which use the base two logarithm. Throughout this work, this function will be written \( \log \). Given a finite set \( X \) and two probability distributions \( r, s \in \mathcal{P}(X) \), we define the relative entropy \( D(r||s) \) by

\[
D(r||s) := \begin{cases} 
\sum_{x \in X} r(x) \log(r(x)/s(x)), & \text{if } s \gg r \\
\infty, & \text{else}
\end{cases}
\]
In case that $D(r||s) = \infty$, for a positive number $a > 0$, we use the convention $2^{-aD(r||s)} = 0$. The relative entropy is connected to $\| \cdot \|$ by the Pinsker’s inequality $D(r||s) \leq \frac{1}{2\ln(2)} \| r - s \|^2$.

The entropy of $r \in \mathcal{P}(X)$ is defined by the formula

$$H(r) := - \sum_{x \in X} r(x) \log(r(x)).$$ (8)

4 Result

Our result is a collection of estimates concerning the state

$$\Phi := (P_A^x \otimes P_B^x) P_D \rho_{AB}^{\otimes n} (P_A^x \otimes P_B^x).$$ (9)

A proper definition is included in the preliminaries of Theorem [11]. It is understood that this is our version of the sought-after state $\Psi_{C_1, C_2}$.

The necessary estimates for $\Phi$ to fulfill the requirements of The multiparty typicality conjecture, Conjecture [11], are given in inequalities (11), (12) and the right hand inequality of (13). The proof of these inequalities is almost trivial.

The remaining inequalities are stated only for sake of completeness, although especially the left hand inequality in (13) is comparably hard to prove.

**Theorem 1.** Let $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ have spectrum $r_A, r_B$. For every $\varepsilon, \delta > 0$ and $n \in \mathbb{N}$, define the projections

$$P_A^x := \sum_{\mu : \| \mu - r_A \| \leq \varepsilon} P_{\mu}^A, \quad P_B^x := \sum_{\nu : \| \nu - r_B \| \leq \varepsilon} P_{\nu}^B \quad \text{and} \quad P_{AB}^x := \sum_{\lambda : \| \lambda - r \| \leq \delta} P_{\lambda}^{AB}$$ (10)

The dependence of these projections on the parameter $n$ will not be written out in order to simplify notation.

Set $\Phi := (P_A^x \otimes P_B^x) P_{AB}^{\otimes n} (P_A^x \otimes P_B^x)$.

There is a function $\gamma : \mathbb{N} \mapsto \mathbb{R}_+$ satisfying $\lim_{n \to \infty} \gamma(n) = 0$, a function $\varphi : (0, 1/2) \mapsto \mathbb{R}_+$ satisfying $\lim_{\delta \to 0} \varphi(\delta) = 0$ and an $N \in \mathbb{N}$ such that for all $n \geq N$

$$\text{tr} \{ \Phi^2 \} \leq 2^{-n(H(r_A) - \gamma(n))}$$ (11)

$$\| \Phi - \rho_{AB}^{\otimes n} \|_1 \leq 3 \cdot 2^{-n(\min\{\varepsilon, \delta\})^2 + \gamma(n)/2}$$ (12)

$$2^{-nH(r) - 2^{-n(\varphi(\delta) + \gamma(n))}(1 - 2^{-2n\varepsilon^2})} \leq \text{tr} \{ \Phi^2 \} \leq 2^{-n(H(r) - \gamma(n))}$$ (13)

$$\text{tr} \{ \Phi \} \geq 1 - 2^{-n(2\min\{\varepsilon, \delta\})^2 + \gamma(n))}$$ (14)

$$\| (P_A^x \otimes P_B^x) \rho_{AB}^{\otimes n} (P_A^x \otimes P_B^x) - \rho_{AB}^{\otimes n} \|_1 \leq 3 \cdot 2^{-n(\varepsilon^2 + \gamma(n)/2)}$$ (15)

$$\text{tr} \{ [(P_A^x \otimes P_B^x) \rho_{AB}^{\otimes n}]^2 \} \geq 2^{-n(H(r) + \gamma(n))}(1 - 2^{-2n\varepsilon^2})$$ (16)

$$\text{tr} \{ (P_A^x \otimes P_B^x) \rho_{AB}^{\otimes n} \} \geq 1 - 2^{-n(2\varepsilon^2 + \gamma(n))}$$ (17)

**Remark 1.** Note that $P_{AB}^{\otimes n}$ commutes with $P_A^x \otimes P_B^x$ as well as with $\rho_{AB}^{\otimes n}$.

We will need a few preliminary results before proving this theorem. First, a few estimates are needed:

With $h(i, j)$ denoting Hook-lengths, the dimensions of the irreducible subspaces of any representation of $S_n$ on $(\mathbb{C}^d)^{\otimes n}$ ($d > 0$) obey the following estimates.

$$\prod_{i=1}^n (\lambda_i + d + 1)! \leq \prod_{(i, j) \in \lambda} h(i, j) = \dim F_{\lambda} \leq \frac{n!}{\prod_{i=1}^d \lambda_i!} \quad (\lambda \in YF_{d, n}).$$ (18)
Also, we are going to employ the following estimate taken from \[1\], Lemma 2.3:

\[
\frac{1}{(n+1)^d} 2^{nH(\lambda)} \leq \frac{n!}{\prod_{i=1}^d \lambda_i!} \leq 2^{nH(\lambda)} \quad (\lambda \in YF_{d,n}) \tag{19}
\]

as well as, with \(\lambda + d + 1(i) := \frac{1}{n+d(d+1)}(\lambda_i + d + 1)\),

\[
\|\lambda - \lambda + d + 1\| = \sum_{i=1}^d \frac{\lambda_i}{n} = \frac{\lambda_i + d + 1}{n + d(d + 1)} \tag{20}
\]

\[
\leq d \cdot \max_{i=1,\ldots,d} \left| \frac{\lambda_i}{n} - \frac{\lambda_i + d + 1}{n + d(d + 1)} \right| \tag{21}
\]

\[
= d \cdot \max_{i=1,\ldots,d} \left| \frac{\lambda_i d(d + 1) - n(d + 1)}{n(n + d(d + 1))} \right| \tag{22}
\]

\[
\leq \frac{d(d + 1)}{n^2} \cdot \max_{i=1,\ldots,d} |\lambda_i d - n| \tag{23}
\]

\[
\leq \frac{d(d + 1)^2}{n} \tag{24}
\]

\[
\leq \frac{(d + 1)^3}{n} \tag{25}
\]

\[
\leq \frac{8d^3}{n} \tag{26}
\]

\[
\text{(if } d \geq 2) \leq \frac{d^6}{n} \tag{27}
\]

and, at last, Lemma 2.7 from \[1\]:

**Lemma 1.** If, for \(A\) a finite alphabet and \(p, q \in \mathfrak{P}(A)\) we have \(|p - q| \leq \Theta \leq 1/2\), then

\[
|H(p) - H(q)| \leq -\Theta \log \frac{\Theta}{|A|}. \tag{28}
\]

Combining equations (18) and (19) leads to the estimate

\[
\dim F_\lambda \leq 2^{nH(\lambda)} \quad (\lambda \in YF_{d,n}). \tag{29}
\]

Deriving a lower bound on \(\dim F_\lambda\) is slightly more involved: Let \(n \geq 2d^2\). Then

\[
\dim F_\lambda = \frac{n!}{\prod_{i=1}^d (\lambda_i + d + 1)!} \tag{30}
\]

\[
= \frac{1}{(n + d(d + 1)) \cdots (n + 1) \prod_{i=1}^d (\lambda_i + d + 1)!} \tag{31}
\]

\[
\geq \frac{1}{(2n)^{2d^2} \prod_{i=1}^d (\lambda_i + d + 1)!} \tag{32}
\]

\[
\geq \frac{1}{(2n)^{2d^2}} \frac{1}{(2n)^d} 2^{(n+d(d+1))H(\lambda+d+1)} \tag{33}
\]

\[
\geq \frac{1}{(2n)^{3d^2}} 2^{nH(\lambda+d+1)} \tag{34}
\]

\[
\geq \frac{1}{(2n)^{5d^2}} 2^{nH(\lambda+d+1)+d^6 \log \frac{d^6}{n}} \tag{35}
\]

\[
= 2^{nH(\lambda)+\frac{d^6}{n} \log \frac{d^6}{n} - \frac{5d^2}{2} \log(2n)} \tag{36}
\]
Set $\gamma_1(n) := -\frac{d^2}{n} \log \frac{d^2}{n} + \frac{5d^2}{n} \log(2n)$, then there is an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have
\[
\dim F_\lambda \geq 2^{n(H(\lambda) - \gamma_1(n))}.
\] (37)

An important step in the application of the representation theory of the symmetric group to quantum information theory was the following theorem:

**Theorem 2**. For $\lambda \in YF_{d,n}$ and $\sigma \in S(H)$ ($\dim H = d$) with spectrum $s$ it holds
\[
\text{tr}\{P_\lambda \sigma \otimes n\} \leq (n+1)^{d(d-1)/2 - nD(\mathfrak{X}(s))}.
\] (38)

Now let us state the first essential ingredient to our theorem.

**Lemma 2.** Let $\mu \in YF_{dA,n}$, $\nu \in YF_{dB,n}$, $\lambda \in YF_{dAdb,A,n}$.

1. If $\|\lambda - r\| < \varepsilon$, $\|\mu - r\| > \delta$, then for $P_{\lambda,i}, P_{\lambda,j}$ ($i, j \in [m_\lambda]$) with $P_{\lambda,i} \leq P_\mu \otimes P_\nu$ and $P_{\lambda,j} \leq P_\mu \otimes P_\nu$ it holds
\[
\text{tr}\{P_{\lambda,i} \rho_{AB} \otimes n P_{\lambda,j} \rho_{AB} \otimes n\} \leq 2^{-n(2\delta^2 + H(\lambda) + \varepsilon \log(c_d c_d) - \gamma_2(n))}.
\] (39)

2. If $\|\lambda - r\| > \varepsilon$, then for $P_{\lambda,i}$ and $P_{\lambda,j}$ with $i, j \in [m^{AB}_\lambda]$ it holds
\[
\text{tr}\{P_{\lambda,i} \rho_{AB} \otimes n P_{\lambda,j} \rho_{AB} \otimes n\} \leq 2^{-n(4\varepsilon^2 + H(\lambda) - \gamma_2(n))}.
\] (40)

The function $\gamma_2$ is given by $\gamma_2(n) := \frac{(d + d^2)^2}{n} \log(n+1) + \gamma_1(n)$ and the function $\gamma_2$ through the formula $\gamma_2(n) := \frac{2(d + d^2)^2}{n} \log(n+1) + \gamma_1(n)$.

**Remark 2.** The lemma can w.l.o.g. be read with the roles of $A$ and $B$ interchanged.

**Proof.** We consider the first statement first. Let us take a look at $P_{\lambda} \rho_{AB} \otimes n$ first. Observe that the two operators in this product commute. Since $P_\lambda \rho_{AB} \otimes n$ is invariant under permutations, we can write it as
\[
P_{\lambda} \rho_{AB} \otimes n = \sum_{i,j=1}^{m_\lambda} c_{ij} Y_{ij},
\] (41)

where the operators $Y_{ij} \in \mathcal{B}(H_{AB}^\otimes)$ satisfy
\[
P_{\lambda,i} Y_{ij} P_{\lambda,j} = Y_{ij}, \quad Y_{ij} Y_{kl} = \delta(j, k) Y_{il} \quad \text{and} \quad Y_{jj} = P_{\lambda,j}.
\] (42)

Since $P_\lambda \rho_{AB} \otimes n$ is self-adjoint, we get
\[
\sum_{i,j=1}^{m_\lambda} c_{ij} Y_{ij} = \sum_{i,j=1}^{m_\lambda} \tilde{c}_{ij} Y_{ij}^\dagger
\] (43)
\[
= \sum_{i,j=1}^{m_\lambda} \tilde{c}_{ij} Y_{ji},
\] (44)

from which it follows that $\tilde{c}_{ij} = c_{ji}$. Also, for every $i, j \in [m^{AB}_\lambda]$ we know that
\[
c_{ii} Y_{ii} + c_{ij} Y_{ij} + c_{ji} Y_{ji} + c_{jj} Y_{jj} = (P_{\lambda,i} + P_{\lambda,j}) \rho_{AB} \otimes n (P_{\lambda,i} + P_{\lambda,j}) \geq 0.
\] (45)

By choosing appropriate bases for supp($P_{\lambda,i}$) and supp($P_{\lambda,j}$), this translates to the statement
\[
\begin{pmatrix}
c_{ii} & c_{ij} \\
c_{ji} & c_{jj}
\end{pmatrix} \geq 0.
\] (46)
This now shows us that \(|c_{ij}|^2 \leq |c_{ii}| \cdot |c_{jj}|\) has to hold and that all the \(c_{ii}, i = 1, \ldots, [m^{AB}],\) are nonnegative real numbers. We now prove the promised inequality:

\[
\tr\{P_{\lambda,i} \rho_{AB}^n \rho_{AB}^n P_{\lambda,j} \rho_{AB}^n\} = \sum_{k,l=1}^{[m^{AB}]} c_{kl} \tr\{P_{\lambda,i} Y_{kl} P_{\lambda,j} Y_{xy}\} = \sum_{k,l=1}^{[m^{AB}]} c_{kl} \tr\{Y_{ii} Y_{kl} Y_{jj} Y_{xy}\} = |c_{ij}|^2 \tr\{Y_{ii}\} = |c_{ij}|^2 \dim(F_{\lambda}) \leq |c_{ii}| \cdot |c_{jj}| \dim(F_{\lambda}).
\]

Observe that

\[
|c_{ii}| = c_{ii} = \tr\{Y_{ii} \sum_{k,l} c_{kl} Y_{kl}\} / \dim F_{\lambda} = \tr\{P_{\lambda,i} P_{\lambda}(\rho_{AB}^n)^n\} / \dim F_{\lambda} = \tr\{P_{\lambda,i} \rho_{AB}^n \rho_{AB}^n\} / \dim F_{\lambda},
\]

so by the preceding

\[
\tr\{P_{\lambda,i} \rho_{AB}^n \rho_{AB}^n\} \leq \frac{1}{\dim(F_{\lambda})} \tr\{P_{\lambda,i} \rho_{AB}^n\} \frac{1}{\dim(F_{\lambda})} \tr\{P_{\lambda,j} \rho_{AB}^n\} \dim(F_{\lambda})
\]

(by assumption) \leq \tr\{(P_{\mu} \otimes P_{\nu})_{AB}^n\} \tr\{P_{\lambda} \rho_{AB}^n\} \dim(F_{\lambda})^{-1}

(since \(P_{\nu} \leq 1_{\mathcal{H}_{AB}^{(1)}}\))

(\text{Theorem 2 inequality 57}) \leq (n + 1)^2 d_2^2 2^{-nD(\|\mu\|_{AB})} (n + 1)^2 d_2^2 2^{-nD(\|\nu\|_{AB})} 2^{-n(D(\|\lambda\|_{AB}) - \gamma_1(n))}

(Pinsker’s inequality) \leq (n + 1)^2 d_2^2 2^{-nD(\|\lambda\|_{AB})} 2^{-n(D(\|\lambda\|_{AB}) - \gamma_1(n))}

(Lemma 1) \leq 2^{-nD(\|\lambda\|_{AB})} 2^{-n(D(\|\lambda\|_{AB}) - \gamma_1(n))} + \varepsilon \log(d_A d_B / \varepsilon)

\leq 2^{-n(2D(\|\lambda\|_{AB}) + H(\|\lambda\|_{AB}) - \gamma_2(n) + \varepsilon \log(d_A d_B))}

\leq 2^{-n(2D(\|\lambda\|_{AB}) + H(\|\lambda\|_{AB}) - \gamma_2(n))}.

For the second statement, just read the proof until equation (67), then use the estimate given in Theorem 2 apply Pinsker’s inequality and inequality 57. It follows

\[
\tr\{P_{\lambda,i} \rho_{AB}^n P_{\lambda,j} \rho_{AB}^n\} \leq \frac{1}{\dim(F_{\lambda})} \frac{1}{\dim(F_{\lambda})} \tr\{P_{\lambda,i} \rho_{AB}^n\} \tr\{P_{\lambda,j} \rho_{AB}^n\} \dim(F_{\lambda})
\]

(\text{Theorem 2 inequality 57}) \leq (n + 1)^2 d_2^2 2^{-nD(\|\lambda\|_{AB})} 2^{-n(D(\|\lambda\|_{AB}) - \gamma_2(n))}

\leq 2^{-n(4\varepsilon^2 + H(\|\lambda\|_{AB}) - \gamma_2(n))}.

\]

\textbf{Lemma 3.} For a type \(N(\cdot)\) on \([d_A d_B]^n\) and its corresponding typeclass \(t \subset [d_A d_B]^n\) and \(e_1, \ldots, e_{d_A d_B}\) a basis in which \(\rho_{AB}^n\) is diagonal, let \(\mathcal{H}_t := \text{span}\{e_{x_t} \otimes \ldots \otimes e_{x_t} : x^t \in t\}\). Denote the projection onto \(\mathcal{H}_t\) by \(p_{\mathcal{H}_t}\).

Then \(\mathcal{H}_t\) is invariant under the action \(\mathbb{B}^{AB}\) of \(S_n\) and for every \(\lambda = N^t\), \(P_{\lambda,i}^t \leq p_{\mathcal{H}_t}\), for at least one \(i \in [m^{AB}]\) (here, \(P_{\lambda,i}^t\) denotes the projection onto a copy of the irreducible subspace \(F_{\lambda}\) within \(\mathcal{H}_{\otimes^n}\).
Remark 3. $P'_{\lambda,i}$ is not necessarily equal to any of the $P_{\lambda,j}$ that were defined in the introduction.

Proof. To a given $N(\cdot)$, take $T$ to be the standard tableaux for $\lambda = N^i$ which has entries $T_{1i} = i$, $T_{2i} = \lambda_1 + i$ and so on, until finally $T_{d_{A}d_{B}+i} = \lambda_1 + \ldots \lambda_{n-1} + i$. Let $v = \otimes_{i=1}^{d_{A}d_{B}} e_i \otimes N(i)$. Denote the set of all permutations belonging to $T$ by $R_T$, the column permutations by $C_T$ and set $E_T := \{\pi \circ \tau : \pi \in C_T, \tau \in R_T\}$. Note that $V_{\lambda} := \text{span}\{(E(T)v : v \in \mathcal{H}^{\otimes n}, T - \text{standard tableaux for } \lambda)\}$ is the isotypical vectorspace belonging to $\lambda$, it holds $\text{supp}(P_{\lambda}) = V_{\lambda}$.

We calculate the overlap of $v$ with a suitably chosen element of $V_{\lambda}$:

$$
\langle v, B(E_T)v \rangle = \sum_{\pi \in C_T} \text{sgn}(\pi) \sum_{\tau \in R_T} \langle B(\pi)B(\tau)v, v \rangle = |R_T| \sum_{\pi \in C_T} \text{sgn}(\pi) |\langle B(\pi)v, v \rangle| = |R_T|,
$$

since $\langle B(\pi)v, v \rangle = 0$ for every $C_T \ni \pi \neq e$. Now assume that $\mathcal{H}_t$ contains no irreducible subspace corresponding to $\lambda$. Then, of course, for every vector $w \in \text{supp}(P_{\lambda})$ we have $w \perp \mathcal{H}_t$. But by the preceding, the vector $w := \langle B(E_T)v \rangle v \in \text{supp}(P_{\lambda})$ is not perpendicular to $\mathcal{H}_t$.

Thus, there must be at least one copy of $F_\lambda$ in $\mathcal{H}_t$, which is what we set out to prove. \hfill $\square$

Lemma 4. For any $\lambda \in YF_{d_{A}d_{B},n}$, it holds that

$$
\dim(F_\lambda)2^{n2}\sum_{i=1}^{k} \lambda_i \log \lambda_i \leq \text{tr}\{P_{\lambda}(\rho_{AB}^{2})^{\otimes n}\}.
$$

In case that $r_i = 0$ holds for an $i \in [d_{A}d_{B}]$, we again use the convention $2^{-\infty} = 0$, so that the formula is valid also in this case.

Remark 4. This result can, with a subexponentially different lower bound, also be found in \[ equation (6.20).\]

Proof. By Lemma 3 for the subspace $\mathcal{H}_t$ defined by the typeclass corresponding to $\lambda$, we have $P_{\lambda,i} \leq p_{H_t}$ for at least one $i \in [m^\lambda]$. Also, $\langle v, (\rho_{AB}^{2})^{\otimes n}v \rangle = \prod_{j=1}^{d_{A}d_{B}} r_j^{2\lambda_j}$ for every $v \in \mathcal{H}_t$. Thus,

$$
\text{tr}\{(P_{\lambda}(\rho_{AB}^{2})^{\otimes n})^2\} = \text{tr}\{P_{\lambda}(\rho_{AB}^{2})^{\otimes n}\} \geq \text{tr}\{P_{\lambda,i}(\rho_{AB}^{2})^{\otimes n}\} = \text{tr}\{P_{\lambda,i}^p(\rho_{AB}^{2})^{\otimes n}p_{H_t}\} = \prod_{j=1}^{d_{A}d_{B}} r_j^{2\lambda_j}\text{tr}\{P_{\lambda,i}^p(\rho_{AB}^{2})^{\otimes n}\} = 2^{n2}\sum_{i=1}^{k} \lambda_i \log \lambda_i \dim(F_\lambda).
$$

Corollary 1. It is an immediate consequence of Lemma 4 and Lemma 7 that for any $\varepsilon > 0$ we have

$$
\text{tr}\{P_{\varepsilon}^A \otimes P_{\varepsilon}^B \rho_{AB}^{2} P_{\varepsilon}^A \otimes P_{\varepsilon}^B \rho_{AB}^{2}\} \geq 2^{-(n(H(r)+2\varepsilon)+c_1+c_0 \log(d_{A}d_{B}/\varepsilon))}
$$

with the constant $c_1 = c_1(r)$ given by $c_1 := -\log \min\{r_i : r_i \neq 0, i \in [d_{A}d_{B}]\}$.

Proof. (Of Theorem 1): We first prove the promised lower bounds. Doing so, we use the definition $B_{\varepsilon}(r_X) := \{p \in P([\text{dim } X]) : ||p - r_X|| \leq \varepsilon\}$, $X \in \{A, B, AB\}$. It obviously holds

$$
\text{tr}\{[(P_{\varepsilon}^A \otimes P_{\varepsilon}^B)\rho_{AB}^{2}]^2\} \geq \text{tr}\{[(P_{\varepsilon}^A \otimes P_{\varepsilon}^B)P_{\varepsilon}^A \otimes P_{\varepsilon}^B)]^2\}.
$$
Now, set
\[ X := \text{tr}\{P_\delta^{AB}(\rho_{AB}^2)^{\otimes n}\} - \text{tr}\{[(P_\varepsilon^A \otimes P_\varepsilon^B)P_\delta^{AB}(\rho_{AB}^2)]^2\}. \] 
(79)

It is clear that \( X \geq 0 \) holds. Define the sets
\[ \mathcal{A} := \{ P_{\lambda,i} : \lambda \in B_\delta(r) \}, \quad P_{\lambda,i} \leq P_\mu \otimes P_\nu \quad \text{for} \quad \mu, \nu \text{ with } \|\varpi - r_\lambda\| > \varepsilon, \|\varpi - r_B\| \leq \varepsilon \] 
(80)
\[ \mathcal{B} := \{ P_{\lambda,i} : \lambda \in B_\delta(r) \}, \quad P_{\lambda,i} \leq P_\mu \otimes P_\nu \quad \text{for} \quad \mu, \nu \text{ with } \|\varpi - r_A\| \leq \varepsilon, \|\varpi - r_B\| > \varepsilon \] 
(81)
\[ \mathcal{AB} := \{ P_{\lambda,i} : \lambda \in B_\delta(r) \}, \quad P_{\lambda,i} \leq P_\mu \otimes P_\nu \quad \text{for} \quad \mu, \nu \text{ with } \|\varpi - r_A\| > \varepsilon, \|\varpi - r_B\| > \varepsilon \] 
(82)
\[ \mathcal{E} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{AB}. \] 
(83)

Then according to Lemma 2 for \( P_{\lambda,i} \in \mathcal{E} \) and \( P_{\lambda,j} \) with \( \lambda \in B_\delta(r) \) the estimate
\[ \text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}P_{\lambda,j}\rho_{AB}^{\otimes n}\} \leq 2^{-n(2\varepsilon^2 + H(r) + \delta \log(\delta d_A d_B) - \gamma_2(n))} \] 
(84)
holds and, therefore,
\[ X = \sum_{\lambda \in B_\delta(r)} \sum_{j=1}^{m^{AB}_{\lambda}} \left( \sum_{P_{\lambda,i} \in \mathcal{E}} \text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}P_{\lambda,j}\rho_{AB}^{\otimes n}\} + \sum_{P_{\lambda,i} \in \mathcal{E}} \text{tr}\{P_{\lambda,j}\rho_{AB}^{\otimes n}P_{\lambda,i}\rho_{AB}^{\otimes n}\} \right) \] 
(85)
\[ = 2 \sum_{\lambda \in B_\delta(r)} \sum_{j=1}^{m^{AB}_{\lambda}} \sum_{P_{\lambda,i} \in \mathcal{E}} \text{tr}\{P_{\lambda,i}\rho_{AB}^{\otimes n}P_{\lambda,j}\rho_{AB}^{\otimes n}\} \] 
(86)
\[ \leq 2 \cdot |B_\delta(r)| \cdot (m^{AB}_{\lambda})^2 \cdot 2^{-n(2\varepsilon^2 + H(r) + \delta \log(\delta d_A d_B) - \gamma_2(n))} \] 
(87)
\[ \leq 2 \cdot (n + 1)^{d_A d_B} \cdot (n + 1)^{(d_A d_B)^4} \cdot 2^{-n(2\varepsilon^2 + H(r) + \delta \log(\delta d_A d_B) - \gamma_2(n))} \] 
(88)
\[ = 2^{-n(2\varepsilon^2 + H(r) + \delta \log(\delta d_A d_B) - \gamma_3(n))}, \] 
(89)
where the last inequality follows from equation (1.22) in [2] and the type counting bound Lemma 2.2 in [1] and the function \( \gamma_3 \) is defined by \( \gamma_3(n) := \frac{(d_A d_B)^5}{n} \log(2(n + 1)) + \gamma_2(n) \) and satisfies \( \lim_{n \to \infty} \gamma_3(n) = 0 \).

We arrive at
\[ \text{tr}\{[(P_\varepsilon^A \otimes P_\varepsilon^B)\rho_{AB}^{\otimes n}]^2\} \geq \text{tr}\{P_\delta^{AB}(\rho_{AB}^2)^{\otimes n}\} - X \] 
(91)
\[ \geq 2^{-n(H(r) + 3\delta c_1 + \delta \log(d_A d_B / \delta))} - 2^{-n(2\varepsilon^2 + H(r) + \delta \log(\delta d_A d_B) - \gamma_3(n))} \] 
(92)
\[ = 2^{-nH(r)(2^{-n(2\delta c_1 + \delta \log(d_A d_B / \delta))} - 2^{-n(2\varepsilon^2 + \delta \log(\delta d_A d_B) - \gamma_3(n)))}, \] 
(93)

By reading only the r.h.s. of the above inequalities and remembering the definition of \( X \), this proves the l.h.s. estimate of Inequality 13 of Theorem 4 \( \varphi(\delta) := 2\delta c_1 + \delta \log(d_A d_B / \delta) \) and for every function \( \gamma : \mathbb{N} \to \mathbb{R}_+ \) with \( \gamma \geq \gamma_3 \).

To show inequality 14, choose \( \delta = d_A d_B / n \), define \( \gamma_4(n) := 2c_1 d_A d_B / n + d_A d_B \log(n) / n, \gamma_5(n) := -d_A d_B \log(n) / n + \gamma_3(n) \) and \( \gamma_6(n) := \max\{\gamma_4(n), \gamma_5(n)\} \).

Then our estimate reads
\[ \text{tr}\{[(P_\varepsilon^A \otimes P_\varepsilon^B)\rho_{AB}^{\otimes n}]^2\} \geq 2^{-nH(r)(2^{-n(2\gamma_4(n))} - 2^{-n(2\varepsilon^2 - \gamma_5(n)))}} \] 
(94)
\[ \geq 2^{-nH(r)2^{-n\gamma_6}(1 - 2^{-n2\varepsilon^2})}. \] 
(95)

This shows \( \gamma \) for every function \( \gamma : \mathbb{N} \to \mathbb{R}_+ \) with \( \gamma \geq \gamma_6 \). The r.h.s. inequality of 13 can be obtained by the estimate \( P_\varepsilon^A \otimes P_\varepsilon^B \leq 1_{H_{AB}^{\otimes n}} \) followed by application of equation (57), the estimate
\( \text{tr}\{P_{A} \rho_{AB}^{\otimes n}\} \leq 1 \), equation (29) and Lemma 1.

Inequality (11) follows by application of Theorem 2 (see, for example [2], Corollaries 2.14 and 2.15). More precisely, this inequality will, again, be valid for all functions \( \gamma : \mathbb{N} \to \mathbb{R}_+ \) with \( \gamma \geq \gamma_7 \), where \( \gamma_7 \) is a nonnegative function on the natural numbers which vanishes for \( n \) going to infinity, as do all the other functions \( \gamma_i \), \( i \in \mathbb{N} \).

Inequality (17) follows from (14) by choosing \( \delta > \varepsilon \).

The inequalities (12) and (15) both follow from Winters gentle measurement lemma [10].

Inequality (11) can be obtained using the following chain of inequalities:

\[
\begin{align*}
\text{tr}\{\text{tr}_{B}\{\Phi\}^2\} & \leq \text{tr}\{\text{tr}_{B}\{(P_{\varepsilon}^A \otimes P_{\varepsilon}^B)\rho_{AB}^{\otimes n}(P_{\varepsilon}^A \otimes P_{\varepsilon}^B)\}^2\} \\
& \leq \text{tr}\{P_{\varepsilon}^A\text{tr}_{B}\{(1_A^{\otimes n} \otimes P_{\varepsilon}^B)\rho_{AB}^{\otimes n}(1_A^{\otimes n} \otimes P_{\varepsilon}^B)\}P_{\varepsilon}^A\rho_{AB}^{\otimes n}\} \\
& \leq \text{tr}\{P_{\varepsilon}^A P_{\varepsilon}^A P_{\varepsilon}^A\rho_A^{\otimes n}\} \\
& \leq 2^{-n(H(r_{A})-\gamma(n))},
\end{align*}
\]

where, at last, \( \gamma \) is defined via \( \gamma(n) := \max\{\gamma_i(n)\}_{i=1}^7 \) and the r.h.s. estimate of inequality (13) is used. Note that inequality (90) is valid because \( P_{\delta}^{AB} \) commutes with \( \rho_{AB}^{\otimes n} \), while the step from (96) to (97) is possible because of the estimate \( \text{tr}_{B}\{(1_A \otimes P_{B})X_{AB}(1_A \otimes P_{B})\} \leq \text{tr}_{B}\{X_{AB}\} \), which is valid for any nonnegative operator \( X_{AB} \) on a composite system \( AB \) and projection \( P_{B} \) on the \( B \)-part of the system. \( \square \)

5 Conclusion and Outlook

We proved 2-party typicality by application of representation theory of the symmetric group. Most of the work we did involved the l.h.s. estimate of (13), an estimate which was not needed to prove 2-party typicality. Indeed, the necessary estimates for two-party typicality follow almost trivially from the fact that \((P_{A}^A \otimes P_{B}^B)P_{\delta}^{AB}\) is a projection, which in turn is due to the structure of the representation of involved groups (see Remark 1).

Despite several attempts we were not able to generalize our result to 3-party typicality. We suspect that this is due to noncommutativity: Extending our notation to tripartite systems \( ABC \) in the obvious way, we believe that the operators \( P_{AB}^{\lambda} \otimes P_{C}^{\nu} \) and \( P_{A}^{\lambda} \otimes P_{BC}^{\nu} \) do not commute in general. This, in turn, forbids a standard application of the gentle measurement Lemma.

We hope that future research will enable us to give a more precise formulation of the multiparty typicality conjecture in terms of representation theoretic objects.

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[5] Theorem 2 seems to have a long history, including independent rediscoveries. This history is explained in more detail in [2]. According to [2], it first appeared in [7], was then independently proven in [9], later appeared in [8] with a shortened proof, and was, at last, restated in [6], again with the (obtained independently from [8]) shortened version of the proof.

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