Abstract

We explore the energy dependence of π mesons off the background Abelian magnetic field on the base of quenched SU(3) lattice gauge theory and calculate the magnetic dipole polarizability of charged and neutral pions for various lattice volumes and lattice spacings. The contribution of the magnetic hyperpolarizability to the neutral pion energy has been also found.

Keywords: lattice QCD, SU(3) gluodynamics, magnetic field, magnetic polarizability, pseudoscalar meson

1. Introduction

Quantum Chromodynamics in strong magnetic fields is a promising topic for research. Magnetic fields of hadronic scale could exist in the Early Uni-
verse [1] and could be formed in cosmic objects like magnetars and neutron stars. They can also be achieved in terrestrial laboratories (RHIC, LHC, FAIR, NICA) [2]. The external electromagnetic fields can be utilized as a probe of QCD properties; the recent progress obtained on this way in lattice gauge theories is discussed in [3]. The energy levels of hadrons in external magnetic field can be useful for the calculation of the cross sections [4]. The energies of mesons at the nonzero magnetic field were calculated in various phenomenological approaches [3, 6, 7, 8, 9], within the QCD sum rules [10, 11] and in the lattice gauge theories [12, 13, 14].

Background magnetic Abelian fields also enable to calculate the magnetic polarizabilities of hadrons. In order to get the dipole magnetic polarizability and the hyperpolarizability we measure the energy of a meson as a function of the magnetic field. The magnetic polarizabilities are important physical characteristics describing the distribution of quark currents inside a hadron in an external field. For the first time the concept of polarizability for the nuclear matter was used by A.B. Migdal in the analysis of the scattering of low energy gamma quanta by atomic nuclei [15]. For hadrons the notion of the polarizability was discussed in papers [16, 17].

There were some discrepancies between the experimental obtained value of the magnetic and the electric polarizabilities of the charged $\pi$ mesons and some theoretical predictions based on the chiral perturbation theory [18, 19].

Measurement of the electrical and the magnetic polarizabilities of $\pi$ mesons was performed on the spectrometer SIGMA-AJAX in Serpukhov, on the electron synchrotron Pakhra (LPI) in Moscow, on the MarkII detector at SLAC, at COMPASS (CERN) and other experiments.

According to the obtained data from these experiments, the value of the polarizability of the charged $\pi$ mesons is positive. The most precise value of the charged pion electric polarisability has been obtained experimentally by the COMPASS experiment $\alpha_\pi = (2.0 \pm 0.6\text{stat} \pm 0.7\text{syst}) \times 10^{-4} \text{fm}^3$ [20] under the assumption $\alpha_\pi = -\beta_\pi$. Comparison with the successful predictions of the experiments and the chiral perturbation theory [18, 19] is interesting for fundamental science.

In this work we consider a behaviour of the ground state energy of pions in the strong magnetic fields on the base of the SU(3) lattice gauge theory. The details of the calculations are briefly sketched in section 2. We discuss the dipole magnetic polarizabilities of $\pi^{\pm}$ and $\pi^{0}$ mesons in sections 3 and 4 accordingly. The magnetic hyperpolarizability of the neutral pion is calculated in section 5.
| Ensemble | $N_t \times N_s^3$ | $\beta_{\text{imp}}$ | $a$, fm | $N_{\text{conf}}$ |
|----------|-------------------|------------------|--------|----------------|
| $A_{16}$ | $16^4$            | 8.20             | 0.115  | 245            |
| $A_{18}$ | $18^4$            | 8.10             | 0.125  | 285            |
| $B_{18}$ | $18^4$            | 8.20             | 0.115  | 200            |
| $C_{18}$ | $18^4$            | 8.30             | 0.105  | 235            |
| $D_{18}$ | $18^4$            | 8.45             | 0.095  | 195            |
| $E_{18}$ | $18^4$            | 8.60             | 0.084  | 180            |
| $A_{20}$ | $20^4$            | 8.20             | 0.115  | 275            |

Table 1: The lattice simulations details. The lattice volume is shown in the second column, the lattice spacing and number of configurations used are presented in the fourth and the fifth columns respectively.

2. Details of calculations

2.1. The improved gauge action

For the generation of quenched $SU(3)$ lattice configurations we used the tadpole improved Lüscher-Weisz action \([21]\), which reduces the ultraviolet lattice artifacts. The action has the form

$$S = \beta_{\text{imp}} \sum_{\text{pl}} S_{\text{pl}} - \frac{\beta_{\text{imp}}}{20u_0^2} \sum_{\text{rt}} S_{\text{rt}},$$

where $S_{\text{pl,rt}} = (1/2)\text{Tr} (1 - U_{\text{pl,rt}})$ is the plaquette (denoted by pl) or $1 \times 2$ rectangular loop term (rt), $u_0 = (W_{1 \times 1})^{1/4} = \langle (1/2)\text{Tr} U_{\text{pl}} \rangle^{1/4}$ is the input tadpole factor computed at zero temperature \([28]\). Our simulations have been carried out on the symmetrical lattices. The parameters of the lattice ensembles and number of the configurations are listed in the Table 1.

2.2. Fermionic spectrum

We solve the Dirac equation numerically

$$D\psi_k = i\lambda_k \psi_k, \quad D = \gamma^\mu (\partial_\mu - iA_\mu)$$

and found the eigenfunctions $\psi_k$ and the eigenvalues $\lambda_k$ for a test quark in an external gauge field $A_\mu$.  

3
For this goal we use the massive overlap operator \[22\]. It has the following form

\[
M_{ov} = \left(1 - \frac{am_q}{2\rho}\right) D_{ov} + m_q,
\]

where \(m_q\) is the quark mass, \(\rho = 1.4\) is the parameter in our calculations, \(D_{ov}\) is the massless overlap Dirac operator, which preserves chiral invariance even at the finite lattice spacing \(a\). It may be written as

\[
D_{ov} = \frac{\rho}{a} \left(1 + \frac{D_W}{\sqrt{D_W^\dagger D_W}}\right) = \frac{\rho}{a}(1 + \gamma_5 \text{sign}(H)),
\]

where \(D_W = M - \rho/a\) is the Wilson-Dirac operator with the negative mass term \(\rho/a\), \(M\) is the Wilson hopping term, \(H = \gamma_5 D_W\) is the Hermitian Wilson-Dirac operator.

We construct the polynomial approximation for the function

\[
\text{sign}(H) = \frac{H}{\sqrt{H^\dagger H}}.
\]

This approximation should be valid on the entire spectrum of \(H\) matrix \([\lambda_{\text{min}}, \lambda_{\text{max}}] \subseteq \mathbb{R}\). Since

\[
\text{sign}(H) \equiv \text{sign}(H/\|H\|) = \text{sign}(W)
\]

and \(\|H\| = \lambda_{\text{max}}\), then \(\text{spec}(W) \in [\lambda_{\text{min}}/\lambda_{\text{max}}; 1] \subseteq \mathbb{R}\). The MinMax polynomial approximation is used for the function \(1/\sqrt{H^2}\) at \(\sqrt{\epsilon} \leq H < 1\), where \(\epsilon = \lambda_{\text{min}}^2/\lambda_{\text{max}}^2\).

The polynomial \(P_n(H^2)\), \(H^2 \in [\epsilon; 1]\) of a degree \(n\) can be the best approximation, if it minimizes the maximal relative error

\[
\delta = \max|h(H^2)|
\]

where \(h(H^2) = 1 - \sqrt{H^2}P_n(H^2)\). The polynomial is represented by the series

\[
P_n(H^2) = \sum_{k=0}^{n} c_k T_k(z), \quad z = \frac{2H^2 - 1 - \epsilon}{1 - \epsilon},
\]

where \(T_n(z), n = 0, 1, 2, \ldots\) are the Chebyshev polynomials defined in the range \([-1; 1]\). The detailed description of the algorithm used for the calculation of \(P_n\) can be found in \[29\]. The resulting polynomial of the matrix
has the same eigenfunctions $\psi_k$ as the original matrix. The eigenvalues can be found from the eigenfunctions using the formula $\lambda_k = \frac{\psi_k^\dagger Q \psi_k}{\psi_k^\dagger \psi_k}$ for a some operator $Q$. From the polynomial approximation for the sign function we get the approximation for the overlap Dirac operator. We find its eigenfunctions and eigenvalues, which are used for the calculation of the propagators and the correlators.

As we consider pure lattice gauge theory, the Abelian magnetic field is introduced only into the overlap Dirac operator. In the symmetric gauge the magnetic field parallel to 'z' axis has the form

$$A^B_\mu = \frac{B}{2} (x\delta_{\mu,2} - y\delta_{\mu,1}).$$

(9)

The total gauge field is the sum of the non-Abelian $SU(3)$ gluonic field and the Abelian $U(1)$ field of magnetic photons

$$A_{\mu ij} = A_{\mu ij}^g + A_{\mu i}^B \delta_{ij},$$

(10)

where $i,j = 1,..N^2_c - 1$ are the colour indices, $\mu = 1,2,3,4$ are the Lorentz indices. Quark fields obey periodic boundary conditions in space and antiperiodic boundary conditions in time. In order to match with the periodic boundary conditions we apply the additional x-dependent boundary twist for fermions.

In the finite lattice volume the magnetic flux trough any two-dimensional face of the hypecube is quantized. So the magnetic field value is

$$eB = \frac{6\pi n_B}{(aN_s)^2}, \quad n_B \in \mathbb{Z},$$

(11)

where $e$ is the elementary charge and $N_s$ is the numbers of lattice sites in spatial directions.

### 2.3. Calculation of correlation functions

To observe the ground state energy for a meson we construct the interpolating operator creating the state with the corresponding quantum numbers. In case the pseudoscalar charged $\pi$ meson the interpolating operator is described by the equations

$$O(\pi^+) = \bar{\psi}_d(n)\gamma_5\psi_u(n), \quad O(\pi^-) = \bar{\psi}_u(n)\gamma_5\psi_d(n)$$

(12)
The interpolating operator for the neutral pion is
\[ O(\pi^0) = (\bar{\psi}_u(n)\gamma_5\psi_u(n) - \bar{\psi}_d(n)\gamma_5\psi_d(n))/\sqrt{2}. \] (13)

It should be mentioned that in Euclidean space \( \bar{\psi} = \psi^\dagger \). We are interested in the 2-point lattice correlation function of the interpolating operators
\[ \langle O_1 O_2 \rangle_A = \langle \psi^\dagger(n)\Gamma_1\psi(n)\psi^\dagger(n')\Gamma_2\psi(n') \rangle_A, \] (14)
where \( \Gamma_1, \Gamma_2 = \gamma_5, \gamma_\mu \) are Dirac gamma matrices, \( n = (n, n_t) \) and \( n' = (n', n'_t) \) are lattice coordinates. The spatial lattice coordinate \( n, n' \in \Lambda_3 = \{(n_1, n_2, n_3)|n_i = 0, 1, ..., N - 1 \} \) and \( n_t, n'_t \) are the numbers of lattice sites in time direction.

The correlation function can be represented as the sum of connected and disconnected parts
\[ \langle O_1 O_2 \rangle_A = -\text{Tr} [\Gamma_1D^{-1}(n, n')\Gamma_2D^{-1}(n', n)] + \text{Tr} [\Gamma_1D^{-1}(n, n')\text{Tr} [\Gamma_2D^{-1}(n', n')]], \] (15)
where \( D^{-1}(n, n') \) is the Dirac propagator. For the isovector currents the disconnected part of the correlation function has to be zero due to cancelation of \( u \) and \( d \) quarks contributions, and we verify this on the lattice (see Sec. 4). The massive Dirac propagator is calculated using the lowest \( M = 50 \) Wilson-Dirac eigenmodes
\[ D^{-1}(n, n') = \sum_{k<M} \frac{\psi_k(n)\psi_k^\dagger(n')}{i\lambda_k + m}. \] (16)

We perform a discrete Fourier transformation of (15) numerically and set \( \langle \mathbf{p} \rangle = 0 \) because we are interested in the ground state energy. To obtain the masses we expand the correlation function to the exponential series
\[ \langle O_1 O_2 \rangle_A = \sum_k \langle 0|\hat{O}_1|k\rangle\langle k|\hat{O}_2^\dagger|0\rangle e^{-n_t\alpha E_k}, \] (17)
where \( \hat{O} \) and \( \hat{O}^\dagger \) are the corresponding Hilbert space operators.

For the large \( n_t \) the main contribution to the correlator (17) comes from the term corresponding to the ground state. Due to the periodic boundary conditions the correlator in the leading order has the following form
\[ \tilde{C}(n_t) = 2A_0e^{-N_T\alpha E_0/2} \text{cosh}((N_T/2 - n_t)\alpha E_0), \] (18)
where $A_0$ is a constant, $E_0$ is the ground state energy, $a$ is the lattice spacing. We found the ground state energies fitting the lattice correlators by the equation (18) at $5 \leq n_t \leq N_t - 5$.

3. Dipole polarizability of $\pi^\pm$ meson

Figure 1: The ground state energy squared of $\pi^\pm$ meson versus the field value squared for the lattice volumes $16^4, 18^4, 20^4$, various lattice spacings and the quark masses 17.13 MeV and 34.26 MeV. Fits of the lattice data correspond to theoretical formula (20) at $eB \in [0, 2]$ GeV$^2$.

The energy levels of a free charged pointlike particle in a constant Abelian magnetic field parallel to z axis are described by the following equation:

$$E^2 = p_z^2 + (2n + 1)|qH| - gs_z qH + E^2 (H = 0),$$

(19)

where $p_z = 0$ is the particle momentum along 'z' axis, $n$ is the principal quantum number, $q$ is the particle charge, $s_z$ is its spin projection, $g$ is the dimensionless quantity, that characterizes the magnetic moment of the particle and $E(H = 0) \equiv m$ is the energy at zero magnetic field (the mass).

For the charged pion in the ground state in the rest frame it is necessary to utilize $p_z = 0$, $n = 0$, $q = \pm 1$ and $s_z = 0$. In the strong external magnetic field the charged pion is not a pointlike particle anymore and its internal
structure can be described by the magnetic polarizabilities. If we take into account the dipole magnetic polarizability $\beta_m$ and the first order magnetic hyperpolarizability $\beta_{1h}^m$, then in the relativistic case the pion energy squared has the following form

$$E^2 = |qH| + m^2 - 4\pi m\beta_m H^2 - 4\pi m\beta_{1h}^m H^4, \quad H = eB.$$  \hspace{1cm} (20)

The energy of a charged pion was calculated from the correlation function

$$C^{PSPS} = \langle \bar{\psi}_d(\vec{0}, n_t)\gamma_5\psi_u(\vec{0}, n_t)\bar{\psi}_u(\vec{0}, 0)\gamma_5\psi_d(\vec{0}, 0) \rangle.$$  \hspace{1cm} (21)

The energies squared are shown in Fig.1 for the lattice volumes $16^4$, $18^4$, $20^4$, lattice spacings $0.086$ fm, $0.105$ fm, $0.115$ fm, quark masses $17.13$ MeV and $34.26$ MeV. We fit our lattice data by formula (20) at $eB \in [0, 2]$ GeV$^2$, where $m$, $\beta_m$ and $\beta_{1h}^m$ are the fit parameters. The fitting curves are also depicted in Fig.1. At the magnetic fields $eB \lesssim 0.5$ GeV$^2$ we observe a linear dependence of the energy squared versus the magnetic field $eB$ and it increases with the field for the all sets of lattice data. At higher fields the nonlinear terms in the magnetic field contribute to the pion energy.

Fitting the lattice data we do not consider the term with the second order magnetic hyperpolarizability $4\pi m\beta_{2h}^m H^6$ in (20), because from the data fits it follows that its relative contribution is small in comparison with the terms proportional to $H^2$ and $H^4$.

In Table 2 we show the values of the magnetic dipole polarizability and hyperpolarizability for several lattice data sets. The best values obtained are $\beta_m = (-1.15 \pm 0.31) \times 10^{-4}$ fm$^3$ for the lattice volume $20^4$, lattice spacing $0.115$ fm and $\beta_m = (-2.06 \pm 0.76) \times 10^{-4}$ fm$^3$ for the lattice volume $18^4$ and lattice spacing $0.086$ fm.

In two loops of the chiral perturbation theory it was predicted that the magnetic polarizability of $\pi^\pm$ equals to $-2.77 \times 10^{-4}$ fm$^3$ [19, 23], that is close to our result. The polarizabilities for the $V = 18^4$ and $a = 0.115$ fm lattice at quark mass $m_q = 34.26$ MeV and $m_q = 17.13$ MeV are not represented in Table 2 because of the poor relative accuracy. It is a very subtle effect but the work in this direction is carried out, including the smaller bare quark masses.

In 2015 the COMPASS collaboration at CERN has investigated the pion Compton scattering $\pi^-\gamma \rightarrow \pi^-\gamma$ [20]. They have found the pion electric polarizability equal to $\alpha_{\pi^\pm} = (2.0 \pm 0.6_{\text{stat}} \pm 0.7_{\text{syst}}) \times 10^{-4}$ fm$^3$ under the assumption $\alpha_{\pi^\pm} = -\beta_{\pi^\pm}$ of the ChPT, which is true in the exact chiral limit.
We have observed agreement of this value with our lattice results. This is also in accordance with the earlier analysis of MARK II group data of the cross section of the process $e^+e^- \rightarrow e^+e^-\pi^+\pi^-$ \cite{25} which was made in \cite{26} and gave the value $\alpha_{\pi\pm} = -\beta_{\pi\pm} = (2.2 \pm 1.6_{\text{stat+syst}}) \times 10^{-4}$ fm$^3$.

The Serpukhov group have found the value $\alpha_{\pi\pm} = (6.8 \pm 1.4_{\text{stat}} \pm 1.2_{\text{syst}}) \times 10^{-4}$ fm$^3$ exploring the radiative pion nucleon scattering $\pi^-Z \rightarrow \pi^-Z\gamma$ in 1983 \cite{27}. Without using the relation $\alpha_{\pi\pm} + \beta_{\pi\pm} = 0$ they obtained $\alpha_{\pi\pm} + \beta_{\pi\pm} = (1.4 \pm 3.1_{\text{stat}} \pm 2.5_{\text{syst}}) \times 10^{-4}$ fm$^3$.

The analysis of the experimental data on helicity amplitudes of the reaction $\gamma\gamma \rightarrow \pi^+\pi^-$ gave the value $\alpha_{\pi\pm} = -\beta_{\pi\pm} = 6.41 \times 10^{-4}$ fm$^3$ \cite{32}.

4. Dipole polarizability of $\pi^0$ meson

We calculate the energy of neutral pion from the correlation function

$$C^{PSPS} = \left( \langle \bar{\psi}_d(\vec{0}, n_t)\gamma_5\psi_d(\vec{0}, n_t) \bar{\psi}_d(\vec{0}, 0)\gamma_5\psi_d(\vec{0}, 0) \rangle \right)$$

$$+ \langle \bar{\psi}_u(\vec{0}, n_t)\gamma_5\psi_u(\vec{0}, n_t) \bar{\psi}_u(\vec{0}, 0)\gamma_5\psi_u(\vec{0}, 0) \rangle / 2.$$ \hspace{1cm} (22)
It is taken into account that the \( d \) and \( u \) quarks interact with the external magnetic field differently because of their non-equal charges.

We have checked on the lattice that the disconnected part of the correlator (15) is zero within the error range. In Fig.3 it is shown for three values of the magnetic field. In Fig.4 the connected correlators are represented with the fits (18) for comparison. In the case of exact isospin symmetry the disconnected part has to be zero so in what follows we neglect it at all.

Fig.2 shows the ground state energy of the \( \pi^0 \) meson depending on the field value squared for ensembles \( A_{20} \) at the quark mass 34.26 MeV and \( B_{18} \) at the quark masses 11.99 MeV, 17.13 MeV, 25.70 MeV, 34.26 MeV (see Table 1). If the neutral pion would consist of one type of quark, \( d\bar{d} \) (\( u\bar{u} \)), then its energy decreases slower (faster) than for the real pion, see Fig.2.

For the pion each lattice data set is fitted at \((eB)^2 \in [0, 0.15 \text{ GeV}^2]\) using the function

\[
E = E(B = 0) - 2\pi \beta_m(eB)^2,
\]

where we determine \( E(B = 0) \) and \( \beta_m \) as the fit parameters. \( E(B = 0) \) is the energy at zero magnetic field and \( \beta_m \) is the magnetic polarizability of neutral pion which is presented in Table 3. We observe the linear energy dependence.
on the magnetic field squared at small values of the field.

In Fig.5 we represent the $\beta_m$ values of the $\pi^0$ meson as a function of the quark mass for ensemble $B_{18}$, the $\beta_m$ values for ensembles $A_{16}$, $A_{20}$ at quark mass 34.26 MeV are also depicted.

The value of $\beta_m(\pi^0)$ diminishes with the quark mass and extrapolation to the chiral limit gives the number $(3.3 \pm 0.4) \cdot 10^{-4}$ fm$^3$ for the lattice volume $18^4$ and lattice spacing 0.115 fm. It is a bit higher than the value of magnetic polarizability obtained in our previous work [13], because we considered $\pi^0$ meson consisting of one type of quarks $d$. In the framework of the chiral perturbation theory the value of the $\beta_m(\pi^0)$ is equal to $\beta_m = 0.5(1.5 \pm 0.3) \cdot 10^{-4}$ fm$^3$ in one (two) loops [18].

Our result coincides in sign with the prediction of the ChPT and differs in value, and there can be various reasons for that, including the contributions of higher chiral loops. In turn, from the lattice side, the dynamical quark loops may be relevant.

5. Magnetic hyperpolarizability of neutral pion

The terms of the higher degrees of the magnetic field give contribution to the energy of the pseudoscalar meson at very high magnetic field. Fig.6
Figure 5: The dipole magnetic polarizability of $\pi^0$ meson for the lattice volumes $16^4$, $18^4$, $20^4$ and lattice spacing $0.115$ fm depending on the quark mass.

shows the $\pi^0$ ground state energy depending on the magnetic field squared at $eB \in [0, 1.7]$ GeV$^4$. The lattice data fits are described by the formula

$$E = E(B = 0) - 2\pi \beta_m(eB)^2 - 2\pi \beta^h_m(eB)^4 - k(eB)^6,$$

(24)

where $\beta^h_m$ is the magnetic hyperpolarizability and $k$ is the hyperpolarizability of higher order, $\beta_m$, $\beta^h_m$ and $k$ are the fit parameters. The term proportional to $B^3$ is parity forbidden: the physical interpretation is that $\pi^0$ can’t decay to three (magnetic) photons.

The decay $\pi^0 \rightarrow 4\gamma_M$ is allowed, so the term ~ $(eB)^4$ presents in formula (24). The values $\beta^h_m$ are negative for the all sets of the lattice data and showed in Table 4.

We found the value $\beta^h_m = (-6.7 \pm 1.7) \cdot 10^{-7}$ fm$^7$ at lattice spacing $a = 0.095$ fm and $\beta^h_m = (-6.8 \pm 1.5) \cdot 10^{-7}$ fm$^7$ at $a = 0.115$ fm for the lattice volume $18^4$, and quark mass $34.26$ MeV.

At the lowest quark mass $17.13$ MeV we obtained $\beta^h_m(\pi^0) = (-7.8 \pm 1.7) \cdot 10^{-7}$ fm$^7$ using for fitting formula (24).

Note also that the energy shows the qualitative tendency of flattening at large magnetic field so that it should not turn to zero and cause the phase transition, in complete similarity to $\rho$–meson case studied earlier [14]. The
Figure 6: The ground state energy of $\pi^0$ meson versus the field value squared for the lattice volumes $18^4$, $20^4$, the lattice spacings 0.095 fm, 0.105 fm, 0.115 fm and quark masses 17.13 MeV, 34.26 MeV. Fits of the lattice data were carried out using the equation (24) at $eB \in [0, 1.7]$ GeV$^2$.

decrease at very large fields is likely to be compensated by higher hyperpolarizabilities. Still, the quantitative analysis of this behaviour requires more investigations.

6. Conclusions

We have calculate the magnetic dipole polarizability of the charged pion for several lattice spacings, lattice volumes $16^4$, $18^4$, $20^4$ and bare lattice quark mass 34.26 MeV. For the lattice volume $20^4$ and the lattice spacing 0.115 fm we obtain the value $\beta_m = (-1.15 \pm 0.31) \cdot 10^{-4}$ fm$^3$. At finest lattice with the lattice spacing 0.086 fm and the volume $18^4$ we get $\beta_m = (-2.06 \pm 0.76) \cdot 10^{-4}$ fm$^3$. We found the agreement of this result with the observation of COMPASS collaboration.

We have explored the energy dependence of neutral pion off the external Abelian magnetic field and have found its dipole magnetic polarizability and hyperpolarizability. The dipole magnetic polarizability depends on the lattice quark mass and the chiral extrapolation was performed. In the chiral limit we have obtained the magnetic dipole polarizability of the $\pi^0$ is equal to
(3.3±0.4)·10^{-4} \text{ fm}^3 \text{ for the lattice volume } V_{latt}^{18} \text{ and the lattice spacing } a = 0.115 \text{ fm, that is close to the prediction of the ChPT [19].}

The contribution of the hyperpolarizability $\beta_{hm}^h$ have been also revealed to the $\pi^0$ energy at considered strong magnetic field. This is very tiny effect and for the all sets $\beta_{hm}^h < 0$. For the finest lattice with spacing $a = 0.095 \text{ fm}$ we found $\beta_{hm}^h = (-6.7 \pm 1.7) \cdot 10^{-7} \text{ fm}^7$. We have observed qualitative behaviour of energy which is not favouring the appearance of tachyonic mode and phase transition.

7. Acknowledgements

O.T. is indebted to A. Guskov and M. Ivanov for useful discussions.

The authors are grateful to FAIR-ITEP supercomputer center where these numerical calculations were performed. This work is completely supported by a grant from the Russian Science Foundation (project number 16-12-10059).

| $V_{latt}$ | $m_q$ (MeV) | $a$ (fm) | $\beta_m$ (GeV$^{-3}$) | $\beta_{hm}^h$ (GeV$^{-7}$) | $\chi^2$/d.o.f. |
|---------|------------|---------|----------------|----------------|---------------|
| 18$^4$  | 34.26      | 0.086   | $-0.027 \pm 0.010$ | 0.007 ± 0.004 | 4.828         |
| 20$^4$  | 34.26      | 0.115   | $-0.015 \pm 0.004$ | 0.011 ± 0.002 | 5.121         |
| 18$^4$  | 34.26      | 0.105   | $-0.011 \pm 0.007$ | 0.005 ± 0.003 | 3.414         |
| 16$^4$  | 34.26      | 0.115   | $-0.021 \pm 0.015$ | 0.015 ± 0.009 | 3.935         |

Table 2: The magnetic dipole polarizability $\beta_m$ and hyperpolarizability $\beta_{hm}^h$ of $\pi^\pm$, its errors and $\chi^2$/d.o.f. of fit (20) to the data at $eB \in [0,2]$ GeV$^2$ for the bare quark masses $m_q = 34.26$ MeV, various lattice volumes and spacings.

References

[1] D. Grasso and H.R. Rubinstein, Phys. Rept. 348, 163 (2001), arXiv: astro-ph/0009061.

[2] V. Skokov, A. Illarionov and V. Toneev, Int. J. Mod. Phys. A 24, 5925 (2009), arXiv: 0907.1396[nucl-th].

[3] Massimo D’Elia, PoS LATTICE2014 020 (2015), IFUP-TH-2015-03, arXiv:1502.06047 [hep-lat].
Table 3: The magnetic dipole polarizability of $\pi^0$, its error and $\chi^2$/d.o.f. obtained from fit (23) to the lattice data at $(eB)^2 \in [0,0.15]$ GeV$^4$ for various bare quark masses, lattice volume $18^4$ and lattice spacing $0.115$ fm. For the $16^4$ lattice the fit was done at $(eB)^2 \in [0,0.2]$ GeV$^4$.

| $V_{latt}$ | $m_d$ (MeV) | $a$ (fm) | $\beta_m$ (GeV$^{-3}$) | $\chi^2$/d.o.f. |
|-----------|-------------|--------|----------------|-----------------|
| $16^4$    | 34.26       | 0.115  | 0.064 $\pm$ 0.006 | 0.828           |
| $18^4$    | 11.99       | 0.115  | 0.046 $\pm$ 0.009 | 0.947           |
| $18^4$    | 17.13       | 0.115  | 0.061 $\pm$ 0.003 | 0.137           |
| $18^4$    | 25.70       | 0.115  | 0.067 $\pm$ 0.002 | 0.200           |
| $18^4$    | 34.26       | 0.115  | 0.071 $\pm$ 0.009 | 3.670           |
| $18^4$    | $m_d = 0$ extr. | 0.115  | 0.043 $\pm$ 0.005 | 0.576           |

Table 4: The magnetic dipole polarizability and hyperpolarizability of $\pi^0$, its errors and $\chi^2$/d.o.f. obtained from the fit (24) to the lattice data at $(eB)^2 \in [0,1.7]$ GeV$^4$ for various quark masses, lattice volumes $18^4$, $20^4$ and lattice spacings $0.095$ fm, $0.115$ fm.

| $V_{latt}$ | $m_d$ (MeV) | $a$ (fm) | $\beta_m$ (GeV$^{-3}$) | $\beta_h$ (GeV$^{-7}$) | $\chi^2$/d.o.f. |
|-----------|-------------|--------|----------------|----------------|-----------------|
| $18^4$    | 17.13       | 0.115  | 0.056 $\pm$ 0.007 | $-0.068 \pm 0.015$ | 1.433           |
| $18^4$    | 34.26       | 0.115  | 0.061 $\pm$ 0.007 | $-0.059 \pm 0.013$ | 7.622           |
| $18^4$    | 34.26       | 0.095  | 0.064 $\pm$ 0.009 | $-0.058 \pm 0.015$ | 0.673           |
| $20^4$    | 34.26       | 0.115  | 0.069 $\pm$ 0.007 | $-0.066 \pm 0.011$ | 10.846          |

[4] S.R. Beane et al., NPLQCD Collaboration, Phys.Rev.Lett. 115 no.13, 132001 (2015), [arXiv:1505.02422] [hep-lat].

[5] M.A. Andreichikov, B.O. Kerbikov, V.D. Orlovsky, Yu.A. Simonov, Phys.Rev. D 87 (2013) 094029 (2013), arXiv: 1304.2533 [hep-ph].

[6] H. Liu, L. Yu and M. Huang, Phys. Rev. D 91, 014017 (2015), [arXiv:1408.1318] [hep-ph].

[7] H. Taya, Phys.Rev. D 92, no.1, 014038 (2015), [arXiv:1412.6877] [hep-ph].

[8] M. Kawaguchi, S. Matsuzaki, Phys.Rev. D 93, no.12, 125027 (2016), [arXiv:1511.06990].
[9] K. Hattori, T. Kojo and N. Su, Nucl.Phys. A 951 1 (2016), arXiv:1512.07361 [hep-ph].

[10] S. Cho, K. Hattori, S.H. Lee, K. Morita and Sho Ozaki, Phys.Rev. D 91, no.4, 045025 (2015), arXiv:1411.7675 [hep-ph].

[11] Ph. Gubler et. al., Phys.Rev. D 93, no.5, 054026 (2016), arXiv:1512.08864.

[12] G. Bali, B.B. Brandt, G. Endrodi and B. Glaessle, arXiv:1510.03899 [hep-lat].

[13] E.V. Luschevskaya, O.E. Solovjeva, O.E. Kochetkov and O.V. Teryaev, Nucl. Phys. B 898, 627 (2015).

[14] E.V. Luschevskaya, O.E. Kochetkov, O.V. Teryaev and O.E. Solovjeva, JETP Letters 101, Issue 10, 674 (2015).

[15] A.B. Migdal, J.Phys. USSR 8, 331 (1944)

[16] A. Klein, Phys. Rev. 99, 998 (1955)

[17] A.M. Baldin, Nucl. Phys. 18, 310 (1960)

[18] J. Gasser, M.A. Ivanov and M.E. Sainio, Nucl. Phys. B 728, 31 (2005).

[19] A. Aleksejevs and S. Barkanova, Nucl. Phys. Proc. Suppl. 245, 17 (2013), arXiv:1309.3313 [hep-ph].

[20] C. Adolph et al. (COMPASS Collaboration), Phys. Rev. Lett. 114, 062002 (2015).

[21] M. Lüscher and P. Weisz, Commun. Math. Phys. 97 59 (1985).

[22] H. Neuberger, Phys. Lett. B 417, 141 (1998), arXiv: hep-lat/9707022

[23] M.A. Ivanov, Int.J.Mod.Phys.Conf.Ser. 39 1560104 (2015), e-Print: arXiv:1509.05225

[24] J.F. Donoghue, B.R. Holstein, Phys.Rev.D 40, 2378 (1989).

[25] J. Boyer et al., Phys.Rev.D 42, 1350 (1990)
[26] D. Babusci, S. Bellucci, G. Giordano, G. Matone, A. M. Sandorfi, M.A. Moinester, Phys.Lett.B 277, 158 (1992)

[27] Yu. M. Antipov et al., Phys.Lett.B 121, 445 (1983).

[28] V.G. Bornyakov, E.-M. Ilgenfritz, and M. Müller-Preussker, Phys. Rev. D 72, 054511 (2005), hep-lat/0507021.

[29] L. Giusti, C. Hoelbling, M. Lüscher, and H. Wittig, Comp. Phys. Commun. 153, 31 (2003).

[30] H. Neff, N. Eicker, Th. Lippert, J.W. Negele, and K. Schilling, Phys. Rev. D 64, 114509 (2001), arXiv: hep-lat/0106016.

[31] M.H. Al-Hashimi and U.J. Wiese, Annals Phys. 324 343 (2009), arXiv: 0807.0630 [quant-ph].

[32] L.V. Fil'kov, V.L. Kashevarov, Phys.Rev.C 73 035210 (2006), arXiv:nucl-th/0512047.