Feasibility and method of multi-step Hermitization of crypto-Hermitian quantum Hamiltonians

Miloslav Znojil\textsuperscript{1,2,3,a} \footnote{e-mail: znojil@ujf.cas.cz (corresponding author)}

\textsuperscript{1} The Czech Academy of Sciences, Nuclear Physics Institute, Hlavní 130, 250 68 Řež, Czech Republic
\textsuperscript{2} Department of Physics, Faculty of Science, University of Hradec Králové, Rokitsanského 62, 50003 Hradec Králové, Czech Republic
\textsuperscript{3} Institute of System Science, Durban University of Technology, Durban, South Africa

Received: 3 February 2022 / Accepted: 6 March 2022 © The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract Even the most economical conventional formulation of quantum mechanics called Schrödinger picture (SP) need not be always computationally or conceptually optimal. A Dyson-inspired remedy can sometimes be sought in a transfer of the calculations from the conventional and correct SP Hilbert space $H$ into a “false but friendlier” Hilbert space $R$. The difference (i.e., non-equivalence) between $R$ and $H$ is given by a judicious simplification of the inner product. Although the Hamiltonian $H$ itself appears non-Hermitian in $R$, its necessary Hermitization (mediated by a return to $H$) is usually feasible. If not, another eligible remedy is described in the present paper. Our basic idea lies in the introduction of a “chained” auxiliary manipulation multiplet of inner-product spaces $R_1, R_2, \ldots, R_N$. At arbitrary $N$, the structure and properties of the resulting generalized SP (GSP) formulation of quantum mechanics of unitary systems (characterized by “hiddenly Hermitian” observables) are described and discussed in detail.

1 Introduction

In conventional textbooks, the basics of quantum theory of unitary systems are usually explained in Schrödinger representation \textit{alias} Schrödinger picture (SP, [1]). The pure state is represented there by a time-dependent ket-vector element of a Hilbert space $L$. The evolution is assumed generated by a self-adjoint Hamiltonian $\mathcal{H}$. The one-to-one correspondence between the self-adjointness of $\mathcal{H}$ and the unitarity of the evolution was given the rigorous mathematical form by Stone [2]. In this sense, the Bender’s and Boettcher’s claim [3] that the unitary evolution could be also generated by a non-Hermitian Hamiltonian $H \neq H^\dagger$ sounded, initially, contradictory [4].

Fortunately, the apparent paradox found quickly its origin in an elementary terminological misunderstanding (see, e.g., reviews [5–7] or Sect. 2 below). It has been revealed that the Bender’s and Boettcher’s Hamiltonians $H$ are only non-Hermitian in a “theoretically redundant,” i.e., mathematically preferable but manifestly unphysical auxiliary Hilbert space $\mathcal{H}_{\text{phys}}$. In this sense, these operators have to be Hermitized in a way explained, by Scholtz et al., in review [8]. In their words, given a suitable non-Hermitian Hamiltonian $H$, one can still work in an innovated version of quantum theory which “allows for the normal quantum-mechanical interpretation”.

In such a setting, the unitary evolution is found generated by a non-Hermitian Hamiltonian $H$ because such an operator (called “quasi-Hermitian” [8] \textit{alias} “crypto-Hermitian” [9]) appears \textit{Hermitizable}. In applications, the Hermitization has to be visualized as a constructive transfer of representation of the quantum bound states of interest from the auxiliary Hilbert space $\mathcal{H}_{\text{phys}}$ to the traditional physical Hilbert space $L$ of textbooks or, better [7], to one of its other suitable representations (say, $\mathcal{H}_{\text{phys}}$).

The phenomenological relevance as well as the feasibility of the procedure of Hermitization was illustrated, using an exactly solvable model, by Buslaev and Grecchi [10]. Rigorously, these authors managed to show that the unstable anharmonic-oscillator toy-model Hamiltonian

$$H^{\text{BG}} = H(i g, j) = -\frac{d^2}{dx^2} + \frac{j^2 - 1}{4 r^2(x)} + \frac{r^2(x)}{4} - g^2 r^4(x) \frac{2}{4}, \quad r(x) = x - i \eta, \quad \eta > 0$$ \hspace{1cm} (1)

defined as acting and manifestly non-Hermitian in an unphysical Hilbert space $\mathcal{H}_{\text{phys}} = L^2(\mathbb{R})$ can be assigned, at any real $j$ and positive $g > 0$, the entirely conventional self-adjoint isospectral partner

$$\mathcal{H}^{\text{BG}} = Q(g, j) = -\frac{d^2}{dx^2} - (g x - 1/2) j + (g x - 1)^2 x^2, \quad x \in \mathbb{R}$$ \hspace{1cm} (2)

\textsuperscript{a} Received: 3 February 2022 / Accepted: 6 March 2022 © The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2022

Published online: 12 March 2022
defined as acting in another, physical Hilbert space $\mathcal{L} = L^2(\mathbb{R})$.

Remarkably enough, even such a most elementary sample of Hermitization $H^{(BG)} \rightarrow \mathcal{H}^{(BG)}$ appeared to require the introduction of several intermediate auxiliary isospectral operators (their list contains cca six items, the explicit form of which may be found in *loc. cit.*). On this ground, one may suspect that the feasible process of Hermitization should also have a multi-step, $N-$step structure in general. This is the idea which also served as the main motivation of our present study.

In the first stage of development (cf. Sect. 2), we will keep $N = 1$ and simplify and reformulate the very concept of the Hermitization. For this purpose, we will just review the two complementary existing forms of the theory. In its older version (cf. review [8] or Sect. 2.1 below), one manages to circumvent the (in general, difficultly) description of the direct change $H \rightarrow \mathfrak{h}$ of the Hamiltonian. This is mediated by introduction of another, third Hilbert space $\mathcal{H}_{phys}$ [6]. In Sect. 2.2, we will briefly review the newer and better known specific approach to the technicalities known as $\mathcal{PT}$-symmetric quantum mechanics [5]. The practical feasibility of the Hermitization process is enhanced there simply by a judicious factorization of the inner-product metric (cf. formula (9) below). Naturally, the ultimate amended physical space $\mathcal{H}_{phys}$ must be constructed as equivalent to its “user-unfriendly” textbook predecessor $\mathcal{L}$. By construction [5,7], Hilbert space $\mathcal{H}_{phys}$ only differs from manifestly unphysical $\mathcal{H}_{math}$ by an amendment and factorization of the inner-product metric (i.e., in our present notation, of operator $\Theta$) in a way illustrated by the diagram of Eq. (7) below.

In Sect. 3, we will emphasize that both of the latter amendments are essential. They will be shown to lead to a decisive ultimate simplification of correspondence between spaces $\mathcal{L}$ and $\mathcal{H}_{phys}$. In the same section, we will also recall our recent letter [11] where we described the first nontrivial implementation of the idea. Basically, we just considered $N = 2$ and proposed there a modified and slightly amended version of the Bender-inspired $\mathcal{PT}$-symmetric quantum mechanics. Although we encountered there certain conceptual obstacles (so that we simply did not think about $N > 2$), still, letter [11] can be perceived as a basic inspiration of our present message.

Briefly, our present proposal can be characterized as an implementation of the same non-Hermitian model-building strategy using arbitrary $N \geq 2$. The explicit form of the innovation will be outlined in Sect. 4, while its physical background and consequences will be discussed in Sect. 5. In a complementary Sect. 6, we will finally add a few comments on the correspondence between the present $N-$step Hermitizations of the Hilbert-space-changing form $\mathcal{H}_{math} \rightarrow \mathcal{H}_{phys}$ and the older, Dyson-inspired [8,12,13] and Hamiltonian-changing model-building flowcharts.

The summary and outline of some potential practical benefits of our present generalized SP (GSP) reformulation of quantum mechanics will finally be given in Sect. 7.

2 Quantum mechanics using two inner products ($N = 1$)

Even in the purely methodical context the Buslaev’s and Grecchi’s construction connecting the asymptotically repulsive “wrong sign” complex potential in (1) with the asymptotically well-behaved double-well potential in (2) is exceptional. Indeed, both of the related partner Hamiltonians possessing the discrete and real bound-state spectra are still just ordinary differential operators of the second order. In contrast, the analogous would-be isospectral partners $\mathcal{H}^{(BG)}$ of all of the Bender’s and Boettcher’s [3] models of the form $H^{(BB)} = -d^2/dx^2 + V^{(BB)}(x)$ have only been constructed approximatively [5,7,14]. Moreover, even this indicated that their form (which, as required, was self-adjoint in $\mathcal{L}$) was non-local and extremely complicated, represented by differential operators of infinite order. After all, such an explicit construction experience only reconfirmed the expectable contrast between a guaranteed feasibility of the Hermitization process of Hermitization should also have a $\mathcal{PT}$-symmetric quantum mechanics [5]. The practical

The same contrast can be also detected in an opposite extreme of practical realistic calculations. In the above-cited review [8], for example, the authors’ study of the Hermitization flowchart

$$H \rightarrow \mathfrak{h} = \Omega H \Omega^{-1}$$

was in fact inspired by the Dyson’s realistic many-body calculations [12,13]. In them, the arrow in Eq. (4) had only been drawn in opposite direction,

$$\mathfrak{h}^{(Dyson)} \rightarrow H^{(Dyson)} = \Omega^{-1} \mathfrak{h}^{(Dyson)} \Omega.$$ (4)

Still, the motivation of the use of the isospectral mapping $\Omega$ remained precisely the same: There was no doubt about the choice between the realistic but user-unfriendly many-fermion Hamiltonian $\mathfrak{h}^{(Dyson)}$ and its judiciously pre-conditioned isospectral bosonic partner $H^{(Dyson)}$ of Eq. (4). Indeed, as long as the Dyson’s choice of the preconditioning operator (or, in fact, matrix) $\Omega^{(Dyson)}$ was supported by intuition, the mapping (4) could really lead to a simplification of Schrödinger equation, especially when Dyson decided to work with the most general non-unitary mappings such that

$$\mathfrak{h}^{(Dyson)} \rightarrow H^{(Dyson)} = \Omega^{-1} \mathfrak{h}^{(Dyson)} \Omega.$$ (4)

Still, the motivation of the use of the isospectral mapping $\Omega$ remained precisely the same: There was no doubt about the choice between the realistic but user-unfriendly many-fermion Hamiltonian $\mathfrak{h}^{(Dyson)}$ and its judiciously pre-conditioned isospectral bosonic partner $H^{(Dyson)}$ of Eq. (4). Indeed, as long as the Dyson’s choice of the preconditioning operator (or, in fact, matrix) $\Omega^{(Dyson)}$ was supported by intuition, the mapping (4) could really lead to a simplification of Schrödinger equation, especially when Dyson decided to work with the most general non-unitary mappings such that

$$\mathfrak{h}^{(Dyson)} \rightarrow H^{(Dyson)} = \Omega^{-1} \mathfrak{h}^{(Dyson)} \Omega.$$ (4)

Still, the motivation of the use of the isospectral mapping $\Omega$ remained precisely the same: There was no doubt about the choice between the realistic but user-unfriendly many-fermion Hamiltonian $\mathfrak{h}^{(Dyson)}$ and its judiciously pre-conditioned isospectral bosonic partner $H^{(Dyson)}$ of Eq. (4). Indeed, as long as the Dyson’s choice of the preconditioning operator (or, in fact, matrix) $\Omega^{(Dyson)}$ was supported by intuition, the mapping (4) could really lead to a simplification of Schrödinger equation, especially when Dyson decided to work with the most general non-unitary mappings such that

$$\mathfrak{h}^{(Dyson)} \rightarrow H^{(Dyson)} = \Omega^{-1} \mathfrak{h}^{(Dyson)} \Omega.$$ (4)

Still, the motivation of the use of the isospectral mapping $\Omega$ remained precisely the same: There was no doubt about the choice between the realistic but user-unfriendly many-fermion Hamiltonian $\mathfrak{h}^{(Dyson)}$ and its judiciously pre-conditioned isospectral bosonic partner $H^{(Dyson)}$ of Eq. (4). Indeed, as long as the Dyson’s choice of the preconditioning operator (or, in fact, matrix) $\Omega^{(Dyson)}$ was supported by intuition, the mapping (4) could really lead to a simplification of Schrödinger equation, especially when Dyson decided to work with the most general non-unitary mappings such that

$$\mathfrak{h}^{(Dyson)} \rightarrow H^{(Dyson)} = \Omega^{-1} \mathfrak{h}^{(Dyson)} \Omega.$$ (4)

Still, the motivation of the use of the isospectral mapping $\Omega$ remained precisely the same: There was no doubt about the choice between the realistic but user-unfriendly many-fermion Hamiltonian $\mathfrak{h}^{(Dyson)}$ and its judiciously pre-conditioned isospectral bosonic partner $H^{(Dyson)}$ of Eq. (4). Indeed, as long as the Dyson’s choice of the preconditioning operator (or, in fact, matrix) $\Omega^{(Dyson)}$ was supported by intuition, the mapping (4) could really lead to a simplification of Schrödinger equation, especially when Dyson decided to work with the most general non-unitary mappings such that

$$\mathfrak{h}^{(Dyson)} \rightarrow H^{(Dyson)} = \Omega^{-1} \mathfrak{h}^{(Dyson)} \Omega.$$ (4)

Still, the motivation of the use of the isospectral mapping $\Omega$ remained precisely the same: There was no doubt about the choice between the realistic but user-unfriendly many-fermion Hamiltonian $\mathfrak{h}^{(Dyson)}$ and its judiciously pre-conditioned isospectral bosonic partner $H^{(Dyson)}$ of Eq. (4). Indeed, as long as the Dyson’s choice of the preconditioning operator (or, in fact, matrix) $\Omega^{(Dyson)}$ was supported by intuition, the mapping (4) could really lead to a simplification of Schrödinger equation, especially when Dyson decided to work with the most general non-unitary mappings such that

$$\mathfrak{h}^{(Dyson)} \rightarrow H^{(Dyson)} = \Omega^{-1} \mathfrak{h}^{(Dyson)} \Omega.$$ (4)
\[ \Omega^\dagger \Omega = \Theta \neq I. \]  

(5)

2.1 Dyson’s transformation and crypto-Hermitian Hamiltonians

During the application of the conventional textbook SP approach to certain many-particle quantum systems, the convergence of variational methods may become prohibitively slow. In these cases, the Hamiltonian is, typically, a complicated partial differential operator \( \mathfrak{h} \) acting in a complicated many-body Hilbert space \( \mathcal{L} = L^2(\mathbb{R}^d) \) with a large dimension-representing exponent \( d \). The convergence becomes particularly slow when the particles are fermions. In this case, indeed, the system has to obey the Pauli exclusion principle [15] so that “it has become customary to map Hermitian fermion operators onto non-Hermitian boson operators” [8]. Purely empirically, it has been revealed, originally by Dyson [12,13], that the calculations (typically, of the low-lying spectra) may perceivably be accelerated via a judicious \textit{a priori} simulation of the correlations. In the language of mathematics, this means that it makes sense to replace the initial “fermionic” Hilbert space \( \mathcal{L} \) by an auxiliary (and, formally, non-equivalent) “bosonic” Hilbert space, say, \( \mathcal{H}_{\text{math}} \). In the latter (and, by assumption, user-friendlier) space, the system becomes described by a new Hamiltonian (say, \( H \)). Naturally, a successful (often, just intuition-supported) estimate and simulation of the structure of the correlations of fermions has often been found to imply a decisive acceleration of the convergence of the variational treatment of the bosonic operator \( H \). In the language of numerical mathematics, one may speak about an isospectral preconditioning (4) of the Hamiltonian.

The difference between the “good” and “bad” choice of the operator \( \Theta \) mapping \( \mathcal{H}_{\text{math}} \) onto \( \mathcal{L} \) is strongly model-dependent. One of the conditions of success is that this mapping (often called Dyson mapping) has to be as general and adaptable as possible, i.e., in particular, non-unitary (cf. (5)). In a way explained in [6], the new, preconditioned, user-friendlier Hamiltonian \( H \) can be perceived as defined in another Hilbert space \( \mathcal{H}_{\text{math}} \) which is, due to property (5), not equivalent to its physical predecessor \( \mathcal{L} \). In the Dyson’s realistic calculations [12,13], for example, the fermionic states in \( \mathcal{L} \) were represented by the bosonic states in \( \mathcal{H}_{\text{math}} \).

In general, the Dyson’s rules (4) and (5) render \( H \) non-Hermitian in \( \mathcal{H}_{\text{math}} \).

\[ H \neq H^\dagger = \Theta H \Theta^{-1}. \]  

(6)

For any preselected candidate \( H \) for Hamiltonian, the constraint (6) can be read as specifying an amended, alternative, correct and physical inner product in Hilbert space \( \mathcal{H}_{\text{phys}} \) [7]. For this reason, the well-known Stone theorem [2] is not violated because the unitary evolution of the system is in fact generated, by non-Hermitian \( H \), in the auxiliary Hilbert space \( \mathcal{H}_{\text{math}} \). This space is not equivalent to its physical partner \( \mathcal{H}_{\text{phys}} \). Thus, one should rather call \( H \) crypto-Hermitian [9].

In this setting, it is probably useful to point out that the preservation or failure of the “duality” or “complementarity” between \( \mathcal{H}_{\text{math}} \) and \( \mathcal{H}_{\text{phys}} \) may be fairly sensitive to the detailed properties of the Hamiltonian in question. For a word of warning, one does not even need to go to the non-Hermitian quantum field theory where the Stone theorem need not apply. Indeed, it is fully sufficient to see that multiple subtle formal problems like, e.g., the possibility of the non-existence of the metric may emerge in a finite-dimensional model (see, e.g., section Nr. 3 in review [8]).

The main aim of transformation (4) is that the description of dynamics (originally provided by \( \mathfrak{h} \)) is now shared by the two operators (viz., by \( H \) and \( \Theta \) or \( \Omega \)). In practical calculations, this makes the Dyson-inspired SP more flexible, in principle at least. Still, all of the mathematical operations have to be performed, exclusively, in the auxiliary Hilbert space \( \mathcal{H}_{\text{math}} \). Although the latter space seems to play just a technical role, its key merit is that it can be re-read, after the mere \textit{ad hoc} redefinition of the inner product (see [7,8] or formula (24) below, with \( K = 2 \)) as a new, unitarily equivalent representation \( \mathcal{H}_{\text{phys}} \) of the “old and missing” Hilbert space \( \mathcal{L} \). The conventional probabilistic interpretation of the unitary quantum dynamics gets restored.

The non-Hermiticity of \( H \) in the working space \( \mathcal{H}_{\text{math}} \) leads to the necessity of using some less straightforward methods of solution of the underlying Schrödinger equation. In many models, the merits of the adaptability of the non-unitary mapping (often called, in the light of papers [12,13], Dyson map) proved to prevail (cf., e.g., the success and productivity of this approach in nuclear physics [15]). Still, one of the unavoidable consequences of the non-unitarity of \( \Omega \) is that the conventional interpretation of the eigenstates of \( H \) has to be modified. In the manner described in reviews [6,8], the situation may be clarified and visualized using the following diagram

\[ \text{inaccessible physical Hilbert space } \mathcal{L} \text{ of textbooks} \quad \text{(Dyson’s map } \Omega \text{)} \quad \text{friendlier representation space } \mathcal{H}_{\text{math}} \quad \text{(metric } \Theta \text{)} \quad \text{alternative physical space } \mathcal{H}_{\text{phys}} \]  

(7)

This diagram indicates that in certain specific non-Hermitian-Hamiltonian models one has to separate the probabilistic physical predictions formulated in \( \mathcal{H}_{\text{phys}} \) from the explicit calculations performed, much more efficiently, in \( \mathcal{H}_{\text{math}} \). The space \( \mathcal{L} \) and the SP Hamiltonian \( \mathfrak{h} \) have to be abandoned and replaced by the doublet \( \mathcal{H}_{\text{math/phys}} \) and by the preconditioned Hamiltonian \( H \), respectively. A very general crypto-Hermitian SP (CHSP) of review [8] is born.

The essence of the CHSP formalism has most concisely been explained by Mostafazadeh [7]. He put emphasis upon the interplay of the inner-product structures in \( \mathcal{H}_{\text{math/phys}} \). He emphasized that the standard physical role is played by \( \mathcal{H}_{\text{phys}} \). Although the merits of the introduction of another, apparently redundant Hilbert space \( \mathcal{H}_{\text{math}} \), seem less obvious, its introduction separated the
interpretations (in $\mathcal{H}_{\text{phys}}$) from calculations (in $\mathcal{H}_{\text{math}}$) and facilitated the qualitative as well as quantitative predictions (cf., once more, the above-mentioned Dyson’s study of ferromagnetism [12,13]). At the same time, during the implementation of the general CHSP theory people encountered also serious technical obstacles (some of them were listed on p. 1216 of review [7]). For this reason, only the most recent simplifications made the theory really widely known and successful [16].

2.2 $\mathcal{PT}$—symmetric quantum mechanics

The CHSP approach can be perceived as equivalent to the standard quantum mechanics, being distinguished just by the conversion of the conventional SP Hamiltonian $\mathcal{H}$ into its less standard isospectral representation $\mathcal{H}$ defined as acting in $\mathcal{H}_{\text{math}}$ and amenable to necessary Hermitization. Nevertheless, the CHSP formalism has only recently been converted, via decisive simplifications, into one of the most influential and popular versions of SP, widely known as $\mathcal{PT}$—symmetric quantum mechanics (PTQM, [5]).

In a slightly less general PTQM setting, the key role is still played by the details of the metric-mediated transition from the auxiliary Hilbert space $\mathcal{H}_{\text{math}}$ to its correct physical partner $\mathcal{H}_{\text{phys}}$. In a way indicated by the horizontal arrow in diagram (7), the fundamental message that the Hamiltonian $\mathcal{H}$ must be Hermitian in $\mathcal{H}_{\text{phys}}$ remains unchanged. In the literature, unfortunately, this message is often obscured by the diversity of notation conventions (see their sample in Table Nr. 1 of [17]). We will use here, therefore, the properly modified Dirac’s bra-ket formalism [1] and the maximally compact abbreviations for $\mathcal{H}_{\text{phys}} = \mathcal{R}_0$ and $\mathcal{H}_{\text{math}} = \mathcal{R}_N$. This will enable us to distinguish easily between the quantum mechanics of textbooks (in which we may put $N = 0$ to imply that $\mathcal{H}_{\text{math}} \equiv \mathcal{H}_{\text{phys}} \equiv \mathcal{L}$) and the menu of the two-space CHSP scenarios in which one may choose $N \geq 1$ emphasizing the non-equivalence between $\mathcal{H}_{\text{math}} = \mathcal{R}_N$ and $\mathcal{H}_{\text{phys}} = \mathcal{R}_0$.

One of the sources of the user-friendliness of the general CHSP formalism as described by Scholtz et al. [8] is that once we put, for simplicity, $N = 1$, we just have to consider the doublet

$$\{\mathcal{H}_{\text{math}}, \mathcal{H}_{\text{phys}}\} = \{\mathcal{R}_1, \mathcal{R}_0\}$$

of relevant Hilbert spaces. This clearly differs from the conventional quantum theory in which one sets $N = 0$. After innovation (8), the Hilbert space $\mathcal{H}_{\text{math}}$ becomes unphysical and the Hamiltonian $\mathcal{H}$ becomes manifestly non-Hermitian in this space.

During the necessary Hermitization of $\mathcal{H}$, the main technical task is the construction of the correct physical metric operator $\Theta$ which would be compatible with the Hamiltonian crypto-Hermiticity alias pseudo-Hermiticity [7] alias quasi-Hermiticity [8] condition (6). It is worth a comment that the key role of operator $\Theta$ (with the upper-case Greek-letter symbol proposed in [6]) is in a sharp contrast with a lack of its sufficiently widely accepted denotation. In Refs. [12,13], [8] or [7], for example, these physical Hilbert-space metrics may be found denoted by the very different symbols like $F$, $T$ or $\eta_+$, respectively.

In the most straightforward Bender-inspired PTQM approach [5], the metric operator is constructed in the form of product

$$\Theta_{(\text{Bender})} = \mathcal{P}\mathcal{C}$$

where $\mathcal{P}$ is parity and where the symbol $\mathcal{C}$ denotes the so called charge [18]. The introduction of such an ansatz proved well motivated. In many models, it appeared to offer a perceivable simplification of the calculations as well as of the subsequent necessary extraction of the model-dependent testable and also, in principle, falsifiable predictions [7,16].

3 Quantum mechanics using three inner products ($N = 2$)

Our present paper will be devoted to a certain conceptual and methodical extension and completion of the CHSP and PTQM approaches. We felt motivated by the observation that in a broader area of physics both of these recipes appeared comparatively difficult to implement. Various authors offered remedies involving the restriction of attention to the bounded-operator Hamiltonians [8] or the use of specific models in which the SP Hamiltonian $\mathcal{H}$ is $\mathcal{PT}$—symmetric (see [5]). It is worth adding that the latter approach was based on a parity-related (i.e., $\mathcal{P}$—related) and time-reversal-related ($\mathcal{T}$—related) assumption $\mathcal{H}\mathcal{PT} = \mathcal{PT}\mathcal{H}$ known as $\mathcal{PT}$—symmetry of the Hamiltonian. This was an intuitively appealing feature which made the PTQM approach particularly popular, often even beyond its original scope and restriction to the closed and unitary quantum systems [19].

3.1 Intermediate space

Strictly speaking, several differences between the closely related CHSP and PTQM unitary models are non-trivial [7]. This fact happened to be obscured by the slightly misleading current terminology. Pars pro toto, let us mention that the Hamiltonian operators $\mathcal{H}$ which are all required to possess the real spectrum and which are all required self-adjoint in the correct physical Hilbert space $\mathcal{H}_{\text{phys}}$ are still called non-Hermitian in the literature. Presumably, the reasons are psychological: All of the necessary two-space calculations are, naturally, performed in just one of the frames, viz., in the auxiliary and manifestly unphysical Hilbert space $\mathcal{H}_{\text{math}}$ in which the Hamiltonians really are non-Hermitian.
In our recent letter [11], we paid more attention to the terminology. We decided to choose \( N = 2 \) and to replace the CHSP Hilbert-space doublet (8) by triplet
\[
\{ \mathcal{H}_{\text{math}}, \mathcal{H}_{\text{intermediate}}, \mathcal{H}_{\text{phys}} \} = \{ \mathcal{R}_2, \mathcal{R}_1, \mathcal{R}_0 \}. \tag{10}
\]
The introduction of the intermediate inner-product space helped us to throw new light on the terminology as well as on the fundamental operator-product (9). In the resulting intermediate-space SP (ISP) version of the formalism, the correct probabilistic physical interpretation of a given set of some preselected candidates \( \Lambda_m \) for the operators of observables with \( m = 0, 1, \ldots, M \) has been clarified.

On these grounds, the Hermitization did not proceed directly from \( \mathcal{H}_{\text{math}} \) to \( \mathcal{H}_{\text{phys}} \) (as, for example, in Ref. [8]) but rather indirectly, via the third inner-product space (i.e., Hilbert or Krein space) \( \mathcal{H}_{\text{intermediate}} \). A key to the comparison of ISP and PTQM has been found in the charge. Indeed, in the PTQM framework the theory only becomes consistent after one guarantees the \( \mathcal{PCT} \) symmetry of the Hamiltonian [18]. In parallel, in the ISP context the charge appeared to play a double role, i.e., not only the original role of component of the correct physical inner-product metric (9) which determines the geometry in \( \mathcal{H}_{\text{phys}} = \mathcal{R}_0 \), but also a new role of an auxiliary metric operator in \( \mathcal{H}_{\text{intermediate}} = \mathcal{R}_1 \).

For the sake of an enhancement of clarity, we will now change the notation and abbreviate \( C = Z_1 \) (emphasizing the geometry-determining role of the charge in \( \mathcal{R}_1 \)) and \( \mathcal{P} = Z_2 \) (underlining the analogous auxiliary-metric-operator role of the – possibly, generalized – parity in \( \mathcal{H}_{\text{math}} = \mathcal{R}_2 \)). Another abbreviation will be used to emphasize the privileged status of the specific physical metric \( \Theta_{(\text{Bender})} = \mathcal{P}C = Z_2Z_1 = Y_2 \) of Eq. (9). Marginally, we might add that in the literature the reference to the “parity” survived even when the operator \( \mathcal{P} = Z_2 \) itself (re-denoted by Greek lower-case \( \eta \) in [7] and required to be self-adjoint in \( \mathcal{R}_2 \), \( Z_2 = Z_2^\dagger \)) ceased to be immediately connected with spatial reflection (check also [20]).

### 3.2 Hermitian conjugation as an ambiguous, inner-product-dependent concept

Needless to emphasize, the unmodified Dirac’s notation conventions can only be used in the mathematical-space extreme of a maximal Hilbert-space subscript \( j = N \). Thus, at \( N = 2 \), the conventional Hermitian conjugation of an operator can be written in the conventional Dirac’s form \( \Lambda \to \Lambda^\dagger \) in \( \mathcal{H}_{\text{math}} = \mathcal{R}_2 \). Otherwise, in every other space \( \mathcal{R}_j \) with \( j < N \) we will mark the generalized Hermitian conjugation of operators as follows,
\[
\Lambda \to \Lambda^\dagger(j)
\]
i.e., by a dedicated space-dependent superscript.

Concerning the Hermitian conjugations of the ket vectors, the basic inspiration of its present, modified-Dirac denotation lies in a rather elementary fact that any Hilbert space (with the inner product \( \langle \psi_a | \psi_b \rangle \) linear in the second, ket-vector argument) can be perceived as an ordered pair \( \{ \mathcal{V}, \mathcal{V}' \} \) of a linear topological vector space \( \mathcal{V} \) (of conventional ket-vector elements \( |\psi\rangle \)) and of the dual vector space of linear functionals marked by the prime, \( \mathcal{V}' \) (see, e.g., p. 246 in textbook [1] where, incidentally, also one of the best detailed explanations of the standard bra-ket notation conventions can be found).

As long as the three Hilbert spaces (10) have to share the same linear ket-vector-space component \( \mathcal{V} \) (cf. Ref. [11]), it is necessary to individualize, by the notation, the respective duals \( \mathcal{V}' \), i.e., the respective invertible antilinear correspondences (we will write \( T : \mathcal{V} \to \mathcal{V}' \)). It would be insufficient to use the same Dirac-recommended bra-vector symbol \( \langle \chi \rangle \) for the denotation of the elements of all of the different dual spaces of our interest.

An appropriately adapted notation will be used here, therefore. In essence, we will first define the inner product in \( \mathcal{H}_{\text{phys}} = \mathcal{R}_0 \) via an explicit specification of the correct physical mappings \( T_{\text{phys}} \). Only \textit{after} such a preparatory step one can formulate the phenomenological predictions deduced from the evaluation of the corresponding matrix elements of the relevant observables. For this purpose, we will use the matrix-multiplication-resembling left-action convention to define
\[
T_{\text{phys}} : |\psi\rangle \to \langle \psi | \Theta, \quad \Theta = \Theta^\dagger \tag{11}
\]
Here, the Hilbert-space-metric operator \( \Theta \) is precisely the one entering the crypto-Hermiticity constraint (6). In this context, it is important to re-emphasize that the choice of the metric \( \Theta \) only becomes admissible when one guarantees that this operator is self-adjoint in \( \mathcal{H}_{\text{math}} \) [8].

In connection with Eq. (11), we should note that multiple nontrivial technical problems may emerge when the dimension of our topological vector space \( \mathcal{V} \) of kets is infinite. This aspect of the theory must carefully be discussed in the framework of functional analysis [21]. As a complementary further reading in this direction, let us mention the old Dieudonné’s paper [22], or the recent Refs. [23,24] and [25]. This being said, it is useful to add, explicitly, that in any case, the Hilbert-space metric operator \( \Theta \) must necessarily be chosen invertible and positive definite and bounded, with bounded inverse [8,21].

One of the other and most important related technical obstacles is that in the general CHSP framework our choice of the “physical” inner product (i.e., of the invertible antilinear Hermitian-conjugation correspondence (11)) is, given the Hamiltonian, \textit{non unique}. An exhaustive discussion of this topic can be found in the literature [1,7,26]. The ambiguity of the inner product is only irrelevant...
in \( \mathcal{H}_{\text{math}} = \mathcal{R}_N \) where we may use the conventional Dirac notation
\[ \mathcal{T}_{\text{math}} : |\psi\rangle \rightarrow \langle \psi| \] (12)
and where we may define the Hermitian conjugation operation as a mapping in which \( |\psi\rangle \) is a column vector while \( \langle \psi| \) is perceived as its row-vector transposition containing complex-conjugate elements.

Let us now fix \( N = 2 \) and let us consider the triplet \( (\mathcal{R}_j) \) of the inner-product spaces (10) as in [11]. Then, the mathematical extreme of Eq. (12) might optionally be marked by a subscripted bracketed \( j = N \),
\[ \mathcal{T}_N : |\psi\rangle \rightarrow \langle \psi| \equiv \langle |N\rangle \psi| . \] (13)
At the other two \( j < N \), the subscripted index becomes obligatory. Thus, at the correct and physical \( j = 0 \) extreme of Eq. (11) with \( \Theta = Y_2 \) we will write
\[ \mathcal{T}_0 : |\psi\rangle \rightarrow \langle |0\rangle \psi| \equiv \langle Y_2 | \psi \rangle . \] (14)
As long as the auxiliary metric \( Z_2 = Z_2^\dagger \) in \( \mathcal{H}_{\text{math}} = \mathcal{R}_2 \) is tractable as the (possibly, generalized) parity [20], we will finally define the intermediate-space Hermitian conjugation as follows,
\[ \mathcal{T}_1 : |\psi\rangle \rightarrow \langle |1\rangle \psi| \equiv \langle Z_2 | \psi \rangle . \] (15)
The specific PTQM (or rather, with \( N = 2 \), ISP) realization of the general CHSP framework can be now characterized by formula (9). It suppresses the ambiguity of the inner-product metric \( \Theta_j, (\text{Bender}) = Y_2 = Z_2 Z_1 \). It also implies that besides the Hamiltonian \( H \) (alternatively denoted here as \( Z_0 \)), also the charge \( Z_1 \) is kept observable.

In the original PTQM proposal [18], the formula for \( \Theta_j, (\text{Bender}) \) was required to contain a self-adjoint operator of charge. In the alternative ISP framework of paper [11], the same factorization ansatz admitted a more general charge. In contrast to its status in PTQM, it was admitted non-Hermitian in \( \mathcal{H}_{\text{math}} \). In our present terminology, the charge \( Z_1 \) acquired the physical meaning of another non-Hermitian and crypto-Hermitian but Hermitizable observable-representing operator.

### 4 Quantum mechanics using \( K-plets \) of inner products

The choice of \( N = 2 \) which characterizes the ISP Hermitization of Ref. [11] can be generalized. The number \( N \) of the auxiliary, manifestly unphysical inner-product spaces \( \mathcal{R}_j \) with \( j > 0 \) can be, in the resulting GSP extension of the ISP formulation of quantum mechanics, arbitrarily large. The ISP triplet (10) of the inner-product spaces sharing the same ket-vector elements of \( \mathcal{V} \) will be replaced by the \( K \)-plet
\[ \{ \mathcal{H}_{\text{math}}, \mathcal{R}_{K-2}, \mathcal{R}_{K-3}, \ldots, \mathcal{R}_1, \mathcal{H}_{\text{phys}} \} \] (16)
with \( K = N + 1 \) and with alternative symbols for the extremely mathematical inner-product space \( \mathcal{H}_{\text{math}} = \mathcal{R}_{K-1} \) and for the extremely physical Hilbert space \( \mathcal{H}_{\text{phys}} = \mathcal{R}_0 \).

#### 4.1 The inner-product-dependent Hermitian conjugations

For any preselected operator \( \Lambda \) defined as acting upon kets \( |\psi\rangle \in \mathcal{V} \), its Hermitian conjugate partners will be inner-product-dependent. At any \( K \), these partners will form the \( K \)-plet
\[ \Lambda^{i(0)}, \Lambda^{i(1)}, \ldots, \Lambda^{i(K-2)}, \Lambda^{i(K-1)} \] (17)
with elements marked by the inner-product-dependent superscripts. In every space \( \mathcal{R}_j \), we will consider an operator \( Z_j \) which will be required self-adjoint in the respective space,
\[ Z_j = Z_j^{i(j)}, \quad j = 0, 1, \ldots, K - 1 . \] (18)
This assumption will enable us to treat, in the GSP framework, every \( Z_j \) as an analogue of the ISP metric \( \Theta \) of Eq. (11). This means that every such an operator will have to satisfy the consistency conditions as listed, e.g., in Eq. Nr. (2.1) of the physics-oriented review [8]. These mathematical conditions involve the domain-completeness (cf. Nr. (2.1a)), Hermiticity (2.1b) (equivalent to our present Eq. (18)), positive definiteness (2.1c) and the metric-boundedness constraint Nr. (2.1d) plus, let us add, the metric-inverse boundedness constraint missing in [8]. Still, the authors of review [8] formulated a consistent theory because they managed to circumvent certain formal difficulties as mentioned by Dieudonné [22] by restricting their attention, drastically, just to the not too realistic models possessing bounded-operator Hamiltonians. In contrast, Dieudonné himself revealed that once the class of Hamiltonians remains sufficiently general, the ISP metric \( \Theta \) need not exist at all.

In 2015, the situation has been reconsidered and summarized in the mathematically oriented monograph [21] where the readers may find that for unbounded operators, a sufficiently general and still mathematically satisfactory formulation of sufficient conditions
of the applicability of the theory is still an open problem. At the same time, there are no formal obstacles in all of the phenomenological applications working with finite matrices. In our present paper, we decided to pay attention, predominantly, just to the algebraic aspects and amendments of the theory. This means that our readers will either keep in mind the sufficiently elementary (and, typically, finite-dimensional) implementations of the theory or, alternatively, they will recall the highly sophisticated specialized literature (pars pro toto, let us recommend Ref. [23,24] for introductory reading).

Somewhere in between the two extremes, one may consider the separable (i.e., still sufficiently general) Hilbert spaces $\mathcal{R}_j = [\mathcal{V}, \mathcal{V}']$ marked by a subscript $j = 0, 1, \ldots, K − 1$. We will assume that the set $\mathcal{V}$ of the ket vectors remains the same (i.e., $j$–independent) while the dual sets $\mathcal{V}'$ of the linear functionals [1] become $j$–dependent, $\mathcal{V}' = \mathcal{V}'_j$. Every space $\mathcal{R}_j$ is self-dual. The correspondence between a ket $|\psi\rangle \in \mathcal{V}$ and its dual can be perceived as a result of action of an invertible antilinear operator $T_{[j]}$. The $K$–plet of Hermitian conjugations yields the subscript-dependent bra-vectors,

$$T_{[j]} : |\psi\rangle \rightarrow \langle j | \psi \rangle, \quad j = 0, 1, \ldots, K − 1. \quad (19)$$

We will drop again the highest subscript $j = K − 1$ as redundant, $\langle [K − 1] | \psi \rangle \equiv \langle \psi \rangle$. This underlines that the Hilbert space $\mathcal{R}_{K−1} = \mathcal{H}_{math}$ is a privileged one, preferred in the calculations.

Next, we may generalize the two ISP rules (15) and (14) and set $\langle [K − 2] | \psi \rangle = \langle \psi | Z_{K−1}$ and $\langle [K − 3] | \psi \rangle = \langle \psi | Y_{K−1}$, respectively. These bra-vectors are just the first two transforms of the same preselected element. The list is easily completed at any $K > 3$. Using an antilexicographically ordered set of arbitrary left-acting operators, we may define

$$\langle [K−2] | \psi \rangle = \langle \psi | Z_{K−1}, \quad \langle [K−3] | \psi \rangle = \langle \psi | Y_{K−1}, \quad \langle [K−4] | \psi \rangle = \langle \psi | X_{K−1}, \ldots, \quad (20)$$

with the last-item $\langle [0] | \psi \rangle = \langle \psi | \emptyset$. In GSP, it makes sense to replace the $(K−1)$–subscripted operators entering Eq. (20) by the sequence of the $Z_j$ operators with $j = K−2, K−3, \ldots$. Such a replacement can be realized via the sequence of linear equations

$$Y_j = Z_j Z_{j−1}, \quad X_j = Z_j Y_{j−1}, \quad W_j = Z_j X_{j−1}, \quad \ldots, \quad j = K−1, K−2, \ldots, \quad (21)$$
i.e., via recurrences

$$Z_{j−1} = Z_j^{−1} Y_j, \quad Y_{j−1} = Z_j^{−1} X_j, \quad X_{j−1} = Z_j^{−1} W_j, \quad \ldots, \quad j = K−1, K−2, \ldots. \quad (22)$$

The bra-vectors of Eq. (19) can be then given their ultimate, exclusively $Z_j$–dependent form,

$$T_{[j]} : |\psi\rangle \rightarrow \langle j | \psi \rangle = \langle \psi | Z_{K−1} Z_{K−2} \ldots Z_{j+1}, \quad j = K−2, K−3, \ldots, 1, 0. \quad (23)$$

All of these, by assumption, mutually non-parallel bra-vectors have the same structure. The formula reproduces the $K = 3$ and $K = 2$ special cases appearing in the quasi-Hermitian quantum mechanics of reviews [7,8,27]. The “non-metric” operator $Z_0$ playing the role of the physical SP Hamiltonian does not enter our final definition (23) of course.

We are now prepared to define, in all of our Hilbert spaces $\mathcal{R}_j$, the respective inner products $(\psi_1, \psi_2)_{\mathcal{R}_j} \equiv \langle j | \psi_1 | \psi_2 \rangle$ in recursive manner,

$$\langle j | \psi_1 | \psi_2 \rangle = \langle [j+1] | \psi_1 | Z_{j+1} | \psi_2 \rangle, \quad j = K−2, K−3, \ldots, 1, 0. \quad (24)$$

Only the “friendliest,” $j = K−1$ item is allowed to be written in the conventional Dirac’s bra-ket form without subscripts again, $(\psi_1, \psi_2)_{\mathcal{R}_{K−1}} = \langle \psi_1 | \psi_2 \rangle$.

4.2 Systematic replacements of Hermiticities by pseudo-Hermiticities

In every inner-product space $\mathcal{R}_j$, we postulated the $j$–dependent Hermiticity property (18) of $Z_j$ (cf. also the third column in Table 1). At $j = 0$, this relation represents the most important dynamical property of the Hamiltonian while the rest of the list

| $j$ | Space | Metric | Operator product |
|-----|-------|--------|-----------------|
| 1   | $\mathcal{R}_1$ | $Z_1 = Z_1^{(1)}$ | $Y_1 = Y_1^{(1)}$ |
| 2   | $\mathcal{R}_2$ | $Z_2 = Z_2^{(2)}$ | $Y_2 = Y_2^{(2)}$ |
| 3   | $\mathcal{R}_3$ | $Z_3 = Z_3^{(3)}$ | $Y_3 = Y_3^{(3)}$ |

}\tablecaption{List of auxiliary Hermiticity relations}
Table 2 Pseudo-Hermiticity properties of operators $Z_k$ at $k < j$

| $j$ | Space | $k = j - 1$ | $k = j - 2$ | $k = j - 3$ | $k = j - 4$ | ... |
|-----|-------|-------------|-------------|-------------|-------------|-----|
| 1   | $\mathcal{R}_1$ | $Z_0^{(1)} Z_1 = Z_1 Z_0$ | $Z_0^{(2)} (Z_2 Z_1) = (Z_2 Z_1) Z_0$ | $Z_0^{(3)} (Z_3 Z_2 Z_1) = (Z_3 Z_2 Z_1) Z_0$ | ... |
| 2   | $\mathcal{R}_2$ | $Z_2^{(1)} Z_2 = Z_2 Z_1$ | $Z_1^{(2)} (Z_3 Z_2) = (Z_3 Z_2) Z_1$ | $Z_2^{(3)} (Z_4 Z_3 Z_2) = (Z_4 Z_3 Z_2) Z_1$ | ... |
| 3   | $\mathcal{R}_3$ | $Z_3^{(1)} Z_3 = Z_3 Z_2$ | $Z_2^{(2)} (Z_4 Z_3) = (Z_4 Z_3) Z_2$ | $Z_3^{(3)} (Z_5 Z_4 Z_3) = (Z_5 Z_4 Z_3) Z_2$ | ... |
| ... | ... | ... | ... | ... | ... | ... |

concerns the $j$-dependent metrics. From the point of view of the users working in $\mathcal{H}_{math}$, only the $j = N$ item

$$Z_N = Z_N^{(N)} = Z_N^2$$

can directly be tested and verified. The rest of the list with $j < N$ will only be accessible to the verification after its pull-down to $\mathcal{H}_{math}$.

In the first step of such a process, in the manner proposed in [11], every Hermiticity relation for a metric (18) becomes re-interpreted as a crypto-Hermiticity requirement

$$Z_j^{(j+1)} Z_{j+1} = Z_{j+1} Z_j, \quad j = 0, 1, \ldots, K - 2$$

imposed upon the same operator $Z_j$ in the different inner-product context of the neighboring, more friendly space $\mathcal{R}_{j+1}$. The series of transformations of the picture to the more friendly space can be iterated. For every initial choice of the subscript $j$, the ultimate goal of the iterations is to reach the formally equivalent representation of property (18) in the mathematically optimal space $\mathcal{H}_{math} = \mathcal{R}_{K-1}$.

For illustration purposes, let us now return to the $K = 3$ scenario as discussed in [11]. The first-step quasi-Hermiticity (25) has been shown there to imply the charge-pseudo-Hermiticity of the Hamiltonian as well as the parity-pseudo-Hermiticity of the charge (see Table Nr. 1 in loc. cit.). The proof was based on the antilexicographically ordered relations (21) and on the observation that $Z_1 Z_0 = Y_1 = Y_1^{(1)}$. The latter relation is interesting also per se because the operator-product quantity $Y_1$ is of an immediate phenomenological interest in the random matrix theory [28–30] or in the open-system physical context (cf. Ref. [31,32] where such a quantity has been found conserved).

In the closed-system setting of Ref. [11], the construction has been completed by the quasi-Hermiticity $Y_j^{(2)} Z_2 = Z_2 Y_1$ holding in $\mathcal{R}_2 = \mathcal{H}_{math}$. This relation proves equivalent to the parity-charge pseudo-Hermiticity of the admissible Hamiltonians. In this sense, the explicit reference to the auxiliary space $\mathcal{R}_1$ and even to the physical Hilbert space $\mathcal{R}_0$ itself becomes, from the practical user’s point of view, redundant.

4.2.1 $K > 3$

Once we move to $K > 3$, every round of the formal pseudo-Hermitization process can be perceived as a relegation of Hermiticity to a more user-friendly space. After a completion of the whole process, one obtains an $N$-plet of the relevant relations as sampled, in Table 2, at $j = N$. The step-by-step derivation of these relations is straightforward. Indeed, the second round of the relegations involves the $(K - 1)$-plet of operator products

$$Z_j Z_{j-1} = Y_j = Y_j^{(j)}, \quad j = 1, 2, \ldots, K - 1.$$  \hspace{1cm} (26)

The manifest Hermiticity of these products in $\mathcal{R}_j$ is equivalent to their quasi-Hermiticity property valid in the next space $\mathcal{R}_{j+1}$,

$$Y_j^{(j+1)} Z_{j+1} = Z_{j+1} Y_j, \quad j = 1, 2, \ldots, K - 2.$$  \hspace{1cm} (27)

In full analogy with the special case of ISP, each operator $Z_{j+1}$ plays here the role of an intermediate, subscript-dependent Hilbert- or Krein-space inner-product metric.

In the subsequent round of algebraic manipulations, we start from the the $Y_{j-1}$-containing operator products

$$Z_j Y_{j-1} = X_j = X_j^{(j)}, \quad j = 2, 3, \ldots, K - 1.$$  \hspace{1cm} (28)

We replace their Hermiticity valid in $\mathcal{R}_j$ by the quasi-Hermiticity

$$X_j^{(j+1)} Z_{j+1} = Z_{j+1} X_j, \quad j = 2, 3, \ldots, K - 2.$$  \hspace{1cm} (29)
postulated in the next Hilbert space $\mathcal{R}_{j+1}$. This is in fact the key consequence of our assumption of existence of the chain of Hermitian conjugations, i.e., of several inner-product spaces.

The core of our message is the recommendation of a systematic and exhaustive iterative transfer of the description of all of the operators of interest to $\mathcal{R}_{K-1}$. Such a process is, therefore, naturally continued by the fourth round, with Hermiticities

$$Z_j X_{j-1} = W_j = W_j^{(j)}, \quad j = 3, 4, \ldots, K - 1$$  \hspace{1cm} (30)

replaced by quasi-Hermiticities

$$W_j^{(j+1)} Z_{j+1} = Z_{j+1} W_j, \quad j = 3, 4, \ldots, K - 2,$$  \hspace{1cm} (31)

e etc. Tables 1 (implicitly) and 2 (explicit when setting $j = N$) offer a sample of the ultimate results of these systematic replacements.

5 Physics behind the multi-space mathematics

The practical implementation of the GSP formalism beyond its ISP special case where we had $N = K - 1 = 2$ is a truly challenging task. In an indirect support of its potential relevance, let us mention our older paper [33]. At the time of its publication, there was, naturally, no consistent SP theory with $K = 4$ at our disposal. As long as the model in question exhibited a nonlinear supersymmetry, we needed the charge $C$ expressed as a product of two operators. In the light of ansatz (9), the physical Hilbert-space metric would have to be triply factorized, therefore, $\Theta_3 = Z_j Z_j Z_1$. In principle, the $j = N = 3$ line of our present Table 2 would apply. In retrospective, the non-availability of a consistent GSP quantum theory with $K = 4$ was precisely the reason why the results of our study [33] remained incomplete.

5.1 Observability status of the charge at arbitrary $K$

In the general—$K$ multiplet (16), one has to treat the first item $\mathcal{H}_{\text{math}} = \mathcal{R}_{K-1}$ of the sequence as the mathematically optimal, preferred and manipulation-friendly Hilbert space. The last item $\mathcal{H}_{\text{phys}} = \mathcal{R}_0$ of the same sequence is, in contrast, the only Hilbert space in which the mean values of all of the observables carry the standard probabilistic interpretation, i.e., in which the inner product is correct and physical. All of the other inner-product spaces play just an interpolative and auxiliary role. In Table 2, in particular, one has to select and, in practice, work just with the single row where $j = N = K - 1$.

At any $K$, in the manner fully consistent with Stone theorem [2], Hamiltonians $H = Z_0$ have to be self-adjoint in $\mathcal{H}_{\text{phys}} = \mathcal{R}_0$. Simultaneously, all of the relevant mathematical constructions have to be performed in working space $\mathcal{R}_N = \mathcal{H}_{\text{math}}$. Contextually, the same Hamiltonian can be called Hermitian or non-Hermitian. In the Hermitian case, one simply returns to the conventional probabilistic interpretation of the unitary system in question. In this sense, the physical interpretation of the predictions of the GSP theory using $K > 1$ remains unchanged. Only our mathematical working space ceases to be equivalent to the correct physical Hilbert space so that the operators $\Lambda$ representing the observables are defined, in $\mathcal{R}_N = \mathcal{H}_{\text{math}}$, as non-Hermitian.

The physics behind the “non-Hermitian” theory acquires the universal, $K$—independent CHSP form, with its various phenomenological aspects extensively discussed in Ref. [8]. In our present notation, one merely represents $\mathcal{H}_{\text{phys}} = \mathcal{R}_0$ in $\mathcal{R}_N$ using a formal replacement of the trivial metric in $\mathcal{R}_N = \mathcal{H}_{\text{math}}$ by its amended physical alternative. This enables us to work, simultaneously, with the two inner products such that

$$\langle \psi_a | \psi_b \rangle = \langle \langle N \rangle \psi_a | \psi_b \rangle \neq \langle \langle 0 \rangle \psi_a | \psi_b \rangle = \langle \langle N \rangle \psi_a | \Theta | \psi_b \rangle = \langle \psi_a | \Theta | \psi_b \rangle,$$

keeping in mind that only the latter inner product has the standard probabilistic interpretation.

In the most elementary nontrivial scenario, we set $N = 2$ and get the ISP formalism with $\Theta = Y_2$. Equally well, we may choose any larger $N > 2$ and get the genuine GSP formalism. Under both of these arrangements, our Hamiltonian only carries its conventional physical meaning of an observable in $\mathcal{R}_0 = \mathcal{H}_{\text{phys}}$. Hence, even the predictions of the generalized theory remain probabilistic. The model-building process does not proceed in the conventional physical Hilbert space $\mathcal{H}_{\text{phys}}$ but rather, step-by-step, along the sequence of manifestly unphysical but perceptively user-friendlier mathematical representation spaces $\mathcal{R}_j$ with the decreasing subscript $j$.

There exist two main distinguishing features of the GSP models with $K > 2$. The first one is formal: the correct physical metric $\Theta$ which makes our Hamiltonian $H$ self-adjoint is equal to an $N$—term operator product which generalizes Eq. (9). The first few illustrative examples are displayed as the $k = 0$ items in Table 2.

The second distinguishing feature of the GSP models is rather serendipitous: One can notice that after a tentative premultiplication by the charge $C = Z_1$ from the right, also the $k = 1$ pseudo-Hermiticities as sampled in Table 2 acquire, unexpectedly, the form equivalent to the precise crypto-Hermiticity criterion

$$C \neq C^\dagger = \Theta C \Theta^{-1}.$$  \hspace{1cm} (32)
Once we compare this relation with Eq. (6) above, we see that the most unexpected but strictly physical second characteristics of the GSP theory is that it guarantees the observability of the crypto-Hermitian Hamiltonian (i.e., energy) together with the observability of the crypto-Hermitian charge.

5.2 Recurrences for conjugations

The \( j \)-th-Hilbert-space Hermiticity [e.g., (18)] and the related \((j + 1)\)-th-Hilbert-space quasi-Hermiticity of an arbitrary linear operator \( \Lambda \) can be made explicit in recurrent manner,

\[
\Lambda^{(j)} = (Z_{j+1})^{-1} \Lambda^{(j+1)} Z_{j+1}, \quad j = 0, 1, \ldots, K - 2.
\]  

This pull-down of the conjugation connecting the neighboring Hilbert spaces is an elementary consequence of definition (24) which played just a marginal role in Ref. [11] at \( K = 3 \) (cf. equation Nr. 10 in loc. cit.). The importance of relation (33) grows with the growth of \( K \). At any \( K \geq 3 \), it leads to the closed-form definition

\[
\Lambda^{(j)} = \Theta^{-1}_{(K-1,j)} \Lambda^{\mathbit{\mathcal{I}}(K-1,j)}, \quad \Theta_{(K-1,j)} = Z_{K-1} Z_{K-2} \ldots Z_{j+2} Z_{j+1}
\]  

of the \( j \)-th conjugation in terms of the conventional one as defined in \( \mathcal{H}_{\text{math}} \).

For illustration, one could return to Tables 1 and 2. A detailed inspection of Tables indicates that at any preselected \( K \) the relegation of the Hermiticity will terminate only after we manage to exhaust the whole set of products (21). The systematically iterated step-by-step replacements

\[
Z_0 (= H) \rightarrow Y_1 \rightarrow X_2 \rightarrow W_3 \rightarrow \ldots \rightarrow \Theta
\]

will then guarantee the self-adjointness of the Hamiltonian in the correct space \( \mathcal{H}_{\text{phys}} = \mathcal{R}_0 \) re-expressed as its quasi-Hermiticity in \( \mathcal{H}_{\text{math}} = \mathcal{R}_{K-1} \).

An analogous chain of relegations of Hermiticity remains applicable to the charge and/or to any other operator of phenomenological relevance. In the spaces which are infinite-dimensional, it would be necessary to discuss multiple technical questions concerning the domains of operators, etc. Here, we skipped these questions and restricted our attention just to the detailed discussion of what could be called the underlying algebraic relations and symmetries.

5.2.1 \( K = 3 \)

At \( K = 3 \), the closed-form solution of the GSP recurrences was given in [11]. For methodical purposes, it still makes sense to start the presentation of this solution by the tutorial re-derivation of the first nontrivial \( K = 3 \) ISP pattern. Indeed, in our present notation the correspondence between \( \mathcal{H}_{\text{phys}} \) and \( \mathcal{H}_{\text{math}} \) can be seen as mediated either by a single-step simplification of the inner product \( \langle \psi_a | \psi_b \rangle \rightarrow \langle \psi_a | \psi_b \rangle \) (at \( K = 2 \)), or by a two-step realization in which the preparatory step \( \langle \psi_a | \psi_b \rangle \rightarrow \langle \psi_a | \psi_b \rangle \) is followed by the final step \( \langle \psi_a | \psi_b \rangle \rightarrow \langle \psi_a | \psi_b \rangle \) (at \( K = 3 \)).

Naturally, after an elementary algebraic exercise one can show that the phenomenological consequences of the two recipes are equivalent. Indeed, although the latter, ISP pattern with \( K = 3 \) is prescribed by the three inner-product-dependent Hermiticity relations (18) which read

\[
Z_0 = Z_0^{(0)}, \quad Z_1 = Z_1^{(1)}, \quad Z_2 = Z_2^{(2)},
\]

only the last, underlined item is in its final form living in \( \mathcal{H}_{\text{math}} = \mathcal{R}_2 \). For the other two relations, we still have to find their representation using the \( \mathbit{\mathcal{I}}(2) \)–marked Hermitian conjugation defined in the user-friendliest and preferred Hilbert space \( \mathcal{R}_2 = \mathcal{H}_{\text{math}} \). For this purpose, let us recall Eq. (25) and re-express the two former relations in the respective quasi-Hermitian forms,

\[
Z_0^{(1)} Z_1 = Z_1 Z_0, \quad Z_1^{(2)} Z_2 = Z_2 Z_1.
\]

The second, underlined formula is final as it already lives in \( \mathcal{H}_{\text{math}} \). The former item requires a further translation using the first formula in (26),

\[
Z_1 Z_0 = Y_1 = Y_1^{(1)}.
\]

This is easily transferred to \( \mathcal{H}_{\text{math}} \) via Eq. (27),

\[
Y_1^{(2)} Z_2 = Z_2 Y_1.
\]

What remains to be done is the insertion of definition (37),

\[
Z_0^{(2)} Z_1^{(2)} Z_2 = Z_2 Z_1 Z_0
\]
and the incorporation of Eq. (36) yielding our final underlined equation

\[ Z_{0}^{(2)} \Theta_{2} = \Theta_{2} Z_{0}, \quad \Theta_{2} = Z_{2} Z_{1}. \]  

(40)

This is the ultimate form of the quasi-Hermiticity of the Hamiltonian written in terms of the Hilbert space metric \( \Theta_{K-1} \) at \( K = 3 \).

We see that all of the auxiliary, intermediate Hilbert spaces \( R_{j} \) with \( 0 < j < K - 1 \) have been successfully eliminated. In terms of the operator-product metric \( \Theta_{2} \), the observability of \( H \) is fully guaranteed, strictly in the spirit of review [8], by its quasi-Hermiticity property (40) in \( H_{\text{math}} \). A consistent quantum model can be constructed in which all observables \( \Lambda \) would have to satisfy the same quasi-Hermiticity relation as \( H \) itself does. Besides \( Z_{0} \), as we already mentioned, also operator \( Z_{1} \) represents an observable quantity [for proof, it is sufficient to pre-multiply the underlined equation in Eq. (36) by \( Z_{1} \) from the right]. Analogously, the product \( Z_{2} Z_{1} \) can represent another observable (the proof is similar) while \( Z_{2} \), when standing alone, cannot.

5.2.2 \( K = 4 \)

At \( N = 3 \), the quadruplet of the Hermiticity relations (18) can be split in the underlined final rule for \( Z_{3} = Z_{3}^{(3)} \) in \( H_{\text{math}} \) and the remaining auxiliary triplet

\[ Z_{0} = Z_{0}^{(0)}, \quad Z_{1} = Z_{1}^{(1)}, \quad Z_{2} = Z_{2}^{(2)}, \]  

(41)

to be replaced by its quasi-Hermitian analogue (25). This yields, first of all, the last, underlined relation \( Z_{2}^{(3)} \) \( Z_{3} = Z_{3} Z_{2} \) for \( Z_{2} \) which is already in its desired final form. In \( H_{\text{math}} = R_{3} \), this makes the quasi- or pseudo-Hermiticity of \( Z_{2} \) perceived as mediated, under appropriate mathematical conditions, by metric \( Z_{3} \). The remaining doublet of relations

\[ Z_{0}^{(1)} Z_{1} = Z_{1} Z_{0} (= Y_{1}), \quad Z_{1}^{(2)} Z_{2} = Z_{2} Z_{1} (= Y_{2}) \]  

(42)

is restricted by the respective Hermiticity requirements \( Y_{1} = Y_{1}^{(1)} \) and \( Y_{2} = Y_{2}^{(2)} \) [cf. Eq. (26)]. Using relation (27), they may be both quasi-Hermitized,

\[ Y_{1}^{(2)} Z_{2} = Z_{2} Y_{1} (= X_{2}), \quad Y_{2}^{(3)} Z_{3} = Z_{3} Y_{2} (= X_{3}). \]  

(43)

The second, underlined item is already written in the correct space \( H_{\text{math}} = R_{3} \). The insertion of \( Y_{2} \) from the second line of Eq. (42) gives \( Z_{1}^{(3)} (Z_{2} Z_{3}) = (Z_{2} Z_{3}) Z_{1} \), i.e., the correct final rule for \( Z_{1} \) formulated in \( H_{\text{math}} \).

We are left with the first item in (43). It can readily be quasi-Hermitized in \( H_{\text{math}} \),

\[ X_{2}^{(3)} Z_{3} = Z_{3} X_{2}. \]  

(44)

After a series of elementary insertions, we finally get

\[ Z_{0}^{(3)} \Theta_{3} = \Theta_{3} Z_{0}, \quad \Theta_{3} = Z_{3} Z_{2} Z_{1}. \]  

(45)

These conclusions are again summarized in Table 2. The role of an observable which would be manifestly self-adjoint in \( H_{\text{phys}} \) can now be played by \( Z_{1} \), by the product \( Z_{2} Z_{1} \) and by the product \( Z_{3} Z_{2} Z_{1} \), but not by the product \( Z_{3} Z_{2} \) or by the operators \( Z_{2} \) or \( Z_{3} \) standing alone (the proofs are analogous to the ones given above). Thus, in the more conventional notation one can conclude that the “hidden Hermiticity” (i.e., the observability) status holds for \( H \) and for the charge \( C = Z_{1} \). The observability status of the “other charge” \( D = Z_{2} \) (i.e., the validity of condition \( D^\dagger \Theta_{3} = \Theta_{3} D \)) could only be achieved, in a sufficiently elementary manner, under an additional commutativity requirement \( D C = C D \). Otherwise, the observability status of \( D \) would require a complicated analysis as sampled in Ref. [34].

5.2.3 General \( K \)

Among the \( K \) Hermiticity relations (18) valid (or postulated) for the operators \( Z_{j} \), the last one (with \( j = K - 1 \)) is a \( H_{\text{math}} \)-space property of \( Z_{K-1} \). Among the \( K - 1 \) Hermiticity relations (25) for the two-term products of \( Z_{j} \)s, the last one (with \( j = K - 2 \)) is always a valid Hermiticity property of \( Y_{K-1} \) (i.e., a valid quasi- or pseudo-Hermiticity property of \( Z_{K-2} \)) in the same \( H_{\text{math}} \)-space, etc. Along this line, we finally arrive at \( j = 0 \) and Hamiltonian \( H = Z_{0} \). An elementary proof by mathematical induction really yields, in the space \( R_{K-1} = H_{\text{math}} \), the quasi-Hermiticity relation

\[ Z_{0}^{(K-1)} \Theta_{K-1} \equiv Z_{0}^{(1)} \Theta_{K-1} = \Theta_{K-1} Z_{0}, \quad \Theta_{K-1} = Z_{K-1} Z_{K-2} \ldots Z_{2} Z_{1}. \]  

(46)

This is our ultimate, experimentally relevant property of the Hamiltonian of the system in question.

In the latter relation, the ultimate metric operator \( \Theta_{K-1} \) is factorized in a way which is compatible with the last \( j = 0 \) item in formula (23) defining the ultimate and correct physical bra-vector. An internal consistency of the GSP formalism is confirmed. The
observation also opens the possibility of a removal, in the nearest future, of our present methodical limitation of attention to the specific SP formulation of quantum mechanics.

In this direction, two next steps of possible methodical development seem most promising. In the first one, the present stationary GSP theory could be replaced by its non-stationary generalization. This would lead to the implementation of the present metric-factorization idea in the non-SP context of the so-called non-Hermitian interaction picture (interested readers might find its \( K = 2 \) description in [17]).

In the second, alternative direction of research our present factorization of the metric could also inspire a further progress in the context of quantum statistical mechanics. In it, one would merely replace a pure state characterized, in our present notation, by the elementary projector

\[
\pi(t) = \frac{1}{\langle \psi(0)(t) | \psi(t) \rangle} \langle \psi(0)(t) | \psi(t) \rangle
\]  

(47)

by the non-Hermitian density matrix

\[
\rho(t) = \sum_{k} \frac{p_k}{\langle \psi(k)(t) | \psi(k)(t) \rangle} \langle \psi(k)(t) | \psi(t) \rangle, \quad \sum_k p_k = 1
\]  

(48)

which would describe the probability distribution of a statistical mixture of states. This density matrix would have the well-known physical meaning combining the probability-based experimental preparation of a statistical quantum system with its generalized \( K > 1 \) theoretical description.

6 Generalized Dyson maps

In the majority of applications of conventional quantum theory, the information about dynamics (i.e., about the realistic self-adjoint Hamiltonian \( \mathcal{H} \)) is extracted, directly or indirectly, from the principle of correspondence. All interest in some unconventional, crypto-Hermitian Hamiltonians emerges only after one encounters some unsurmountable technical difficulties. One then imagines that the conventional formulations of quantum dynamics need not be optimal. In such a situation, Dyson [12,13] found, purely empirically, that his calculations prove perceptibly simplified after a reformulation of the SP theory in which one would work with two inner-product spaces (i.e., in our present notation, with \( K = 2 \)).

The motivation of the Dyson’s construction was pragmatic, based on a good intuitive guess of the operator \( \Omega \) representing correlations. In fact, the strong dependence of the success on this guess was one of the main weak points of the strategy. Thus, one has to ask how should one amend this aspect, and how could one make the choice of the correlators \( \Omega \) more robust and flexible.

6.1 Multi-step preconditionings

In the most elementary \( K = 2 \) CHSP scenario, it is not easy to keep the physics given by the respective candidates \( \mathcal{H} \) and \( \Omega \) for the Hamiltonian and mapping fully under control. One must be even more careful when using the inverted flowchart \( H \rightarrow \Theta \rightarrow \Omega \) as discussed by Scholtz et al. [8]. In an ideal situation, the determination of an optimal \( \Omega \) should not be based on the mere educated guess. One may expect an improvement of the results, in particular, after the above-described multi-step relegation of the Hermiticity of operators from \( R_j \) to \( R_{j+1} \).

What seems lost is the reference to the idea of preconditioning, i.e., an intuition-based insight in the dynamics. This is a shortcoming of the theory which damages the appeal and popularity of the CHSP formalism in applications. A step towards recovery came with PTQM, i.e., with the proposal of factorization (9). Still, a certain theoretical weakness of the recipe survived, lying in a somewhat mysterious status of the charge [7] as well as in a manifestly non-Dysonian nature of the Bender’s factorization \( \Theta_2 = PC \) of the metric.

The gap has partially been filled in [11]. We found there that the charge \( C \) becomes in fact a very natural component of the formalism. Still, our satisfaction remained limited because the operator-product metric \( \Omega = PC \) did not seem to be easily re-factorized (i.e., in effect, interpreted) in terms of a single Dyson map \( \Omega \). Also, the most common re-factorization \( \Omega = \sqrt{\Theta} \) of the metric (as recommended, e.g., in [7]) may often prove too artificial.

In [11], precisely the latter artificiality impression caused our loss of interest in the Hermiticity-to-quasi-Hermiticity relegations at \( K \geq 4 \). We did not imagine that a consistent concept of the Dyson maps and of their compositions might very naturally be connected with the notion of the space-dependent Hermiticity.

6.2 Composition laws

In a way inspired by Eq. (5), let us postulate a factorization

\[
Z_j = \Omega_j^{1(j)} \Omega_j, \quad j = 1, \ldots, K - 1.
\]  

(49)
Such an ansatz is, first of all, compatible with our present $j$—dependent self-adjointness (18) of $Z_j$ in $\mathcal{R}_j$. It allows us to reclassify the new family of the invertible operators $\Omega_j$ as an upgraded model-building-information input.

**Lemma 1** At $K = 3$, let us assume that both $Z_2$ and $Z_1$ are positive definite and factorized via Eq. (49). In $\mathcal{H}_{\text{math}}$, the physical Hilbert-space metric $\Theta_2 = Z_2 Z_1$ is then factorized as follows,

$$\Theta_2 = \Omega_2^{(1)} \Omega_2^{(2)} = \Omega_2 \Omega_2^{\dagger} \Omega_1^{\dagger}, \quad \Omega_2 = \Omega_2 \Omega_1.$$  

(50)

**Proof** Relation (33) implies that in the factorization postulate $Z_1 = \Omega_1 \Omega_1$, we can relegate the Hermiticity of $\Omega_1^{\dagger} = -Z_2^{-1} \Omega_1^{\dagger}$ to $Z_2$. The insertion in $\Theta_2 = Z_2 Z_1$ yields the result. $\square$

**Lemma 2** At $K = 4$, let us assume that operators $Z_j$ with $j = 1, 2, 3$ are positive definite and factorized via Eq. (49). In $\mathcal{H}_{\text{math}}$, the physical Hilbert-space metric is then factorized as follows,

$$\Theta_3 = Z_3 Z_2 Z_1 = \Omega_3^{(3)} \Omega_3^{(2)} \Omega_3^{(1)} \Omega_1.$$  

(51)

**Proof** As long as the superscript $1(j)$ denotes the usual Hermitian conjugation in our unique representation space $\mathcal{H}_{\text{math}}$ at $j = K - 1 = 3$, we have to eliminate, from the definition of $\Theta_3 = Z_3 Z_2 Z_1$, just the two unusual Hermitian conjugation superscripts $1(j)$ with $j = 1$ and $j = 2$. We need, for this purpose, just the two items in relations (33), viz.,

$$Z_2 \Lambda_1^{(1)} = \Lambda_2^{(2)} Z_2$$  

(52)

and

$$Z_3 \Lambda_3^{(2)} = \Lambda_3^{(3)} Z_3 = \Lambda_3 \Lambda_3^{\dagger},$$  

(53)

respectively. With $\Lambda = \Omega_1$, we get

$$\Theta_3 = Z_3 \Omega_1^{(2)} Z_2 \Omega_1$$

from the former identity, while we further get

$$\Theta_3 = \Omega_1 \Omega_2 \Omega_1$$

from the latter identity. In the third step, the factorization (49) at $j = 2$ and the use of Eq. (53) with $\Lambda = \Omega_2$ lead immediately to the final result. $\square$

The structure of the latter proof reveals its recursive nature. Along the same lines, it is now easy to prove the general result (the task is left to the readers).

**Theorem 3** At any integer $K \geq 2$, the existence of decompositions (49) of the positive definite operators $Z_j$ at $j = 1, 2, \ldots, K - 1$ is formally equivalent to the existence of a refactorization

$$\Theta_{K-1} = \Omega_{K-1}^{\dagger} \Omega_{K-1} \Omega_{K-1}^{(1)} \Omega_{K-1}^{(2)} \ldots \Omega_{K-1}^{(2)} \Omega_1 \Omega_1^{\dagger}, \quad \Omega_{K-1} \Omega_{K-1}^{(1)} \Omega_{K-1}^{(2)} \ldots \Omega_{K-1}^{(2)} \Omega_1 \Omega_1^{\dagger},$$  

(54)

of the Hilbert-space metric

$$\Theta_{K-1} = Z_{K-1} Z_{K-2} \ldots Z_2 Z_1$$  

(55)

which is responsible for the quasi-Hermiticity (46) of the Hamiltonian $H = Z_0$ in $\mathcal{H}_{\text{math}}$.

It makes good sense to emphasize that the latter result is purely algebraic, i.e., guaranteed as valid in the finite-dimensional Hilbert spaces. For the infinite-dimensional Hilbert spaces, the statement had to be complemented by several necessary technical assumptions like, typically, those concerning the domains and ranges of operators, etc. In this direction of the further development of the theory, the recommended preparatory reading may be found, e.g., in the recent dedicated monograph [21].

### 7 Conclusions

In the manner proposed and verified by Dyson [12,13], a decisive simplification of difficult calculations of certain properties of complicated quantum systems can sometimes be achieved via introduction of an additional, auxiliary inner product [11]. Our present paper offers, in essence, just a nontrivial $K$—space technical extension and amendment of the Dyson-inspired CHSP quantum theory. We felt motivated by the observation that the fairly complicated CHSP-related mathematics weakened the appeal and visibility of the innovative methodical aspects of the approach [35]. For this reason, the current return of attention to the physics with non-Hermiticities [36] was obviously motivated by the simplification of some technicalities provided by the most popular PTQM philosophy [5,7].
Applications of the updated PTQM theory profited, first of all, from the factorization of metric. Simultaneously, these applications suffered from the loss of a direct contact with the principle of correspondence as offered, initially, by Eq. (4). In our recent brief note [11], we addressed the problem. We pointed out that the somewhat mysterious PTQM-mediated charge-based relegation of the Hermiticity from the physical Hilbert space $H_{phys}$ to auxiliary $H_{math}$ may be given a very natural conceptual background when one decides to treat, on equal footing, the observability of the Hamiltonian $H$ and the observability of the charge $C$.

More recently, we revealed that the strict restriction of the resulting ISP formalism to the $K = 3$ kinematical scenario given by Eq. (10) was in fact based on an elementary misunderstanding. We felt discouraged by the apparently counterintuitive observation that at $K = 3$ the parity $P = Z_2$ seems to have lost its physical observability status. Only later we imagined that such a loss is in fact mathematically very natural, especially when one builds the models without some suitable ad hoc additional constraints (see, e.g., an extensive discussion of this point in [34]).

With the latter mental barrier removed, we still felt discouraged by another, more technical observation that whenever one tries to move to any nontrivial $K > 2$ (i.e., to the kinematics using the larger inner-product-space multiplets (16)), the properly generalized Dyson maps seem to “cease to behave nicely.” After a more detailed analysis of the latter problem (leading, finally, to the results sampled here by Theorem 3), fortunately, also the latter doubts have been eliminated.

We can conclude that the present systematic $K > 3$ GSP extension of the scope of the most economical and popular SP formulation of quantum mechanics will find efficient implementations in all of the contexts in which one finds the reasons for the replacement of the Dyson-mapping-based metric (5) by its generalizations, be it the metrics (50) or (51) or the entirely general metric-factorization formulae (54) or (55) in which the number of the auxiliary generalized-charge factors can be an arbitrary positive integer $N = K - 1$.

Acknowledgements The author is grateful to Excellence project PIIF UHK 2211/2022-2023 for the financial support.

Data Availability Statement Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

Conflicts of Interest The author declares no conflict of interest.

References

1. A. Messiah, Quantum Mechanics I (North Holland, Amsterdam, 1961)
2. M.H. Stone, On one-parameter unitary groups in Hilbert Space. Ann. Math. 33, 643 (1932)
3. C.M. Bender, S. Boettcher, Real spectra in non-Hermitian Hamiltonians having PT symmetry. Phys. Rev. Lett. 80, 5243 (1998)
4. R.F. Streeter, Lost Causes in and Beyond Physics (Springer, Berlin, 2007)
5. C.M. Bender, Making sense of non-Hermitian Hamiltonians. Rep. Prog. Phys. 70, 947 (2007)
6. M. Znojil, Symm. Integ. Geom. Methods and Appl. 5, 001 (2009) (e-print overlay: arXiv:0901.0700)
7. A. Mostafazadeh, Pseudo-Hermitian quantum mechanics. Int. J. Geom. Meth. Mod. Phys. 7, 1191 (2010)
8. F.G. Scholtz, H.B. Geyer, F.J.W. Hahne, Quasi-Hermitian operators in quantum mechanics and the variational principle. Ann. Phys. (NY) 213, 74 (1992)
9. A.V. Smilga, Pseudo-Hermitian quantum mechanics. Int. J. Geom. Meth. Mod. Phys. 7, 1191 (2010)
10. R. El-Ganainy, K.G. Makris, M. Khajavikhan, Z.H. Musslimani, S. Rotter, D.N. Christodoulides, Non-Hermitian physics and PT symmetry. Nat. Phys. 18, 012007 (2019)
11. M. Znojil, Quantum mechanics using two auxiliary inner products. Phys. Lett. A 421, 12792 (2022)
12. F.J. Dyson, General theory of spin-wave interactions. Phys. Rev. 102, 1217 (1956)
13. F.J. Dyson, Thermodnamic behavior of an ideal ferromagnet. Phys. Rev. 102, 1230 (1956)
14. A. Mostafazadeh, Metric operator in pseudo-Hermitian quantum mechanics and the imaginary cubic potential. J. Phys. A Math. Gen. 39, 10171 (2006)
15. D. Janssen, P. Schuck, Quantum fluctuations within a symmetry conserving generalization of mean field theory for finite Fermi systems: translational invariance. Nucl. Phys. A 501, 270 (1989)
16. C.M. Bender, PT Symmetry in Quantum and Classical Physics (World Scientific, Singapore, 2018)
17. M. Znojil, Non-Hermitian interaction representation and its use in relativistic quantum mechanics. Ann. Phys. (NY) 385, 162 (2017)
18. C. M. Bender, D. C. Brody, H. F. Jones, Phys. Rev. Lett. 89 (2002) 270401 and ibid. 92 (2004) 119902 (erratum)
19. R. El-Ganainy, K.G. Makris, M. Khajavikhan, Z.H. Musslimani, S. Rotter, D.N. Christodoulides, Non-Hermitian physics and PT symmetry. Nat. Phys. 14, 11 (2018)
20. M. Znojil, H.B. Geyer, Smere quantum lattices exhibiting PT-symmetry with positive P. Fortschr. Phys. Prog. Phys. 61, 111 (2013)
21. F. Bagarello, J.-P. Gazeau, F. Szafraniec, M. Znojil (eds.), Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects (Wiley, Hoboken, 2015)
22. J. Dieudonne, Proc. Int. Symp. Lin. Spaces (Pergamon, Oxford, 1961), pp. 115–122
23. J.-P. Antoine, C. Trapani, Metric operators, generalized hermiticity and lattices of Hilbert spaces. In Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects (Wiley, Hoboken, 2015), pp. 345–402
24. J.-P. Antoine, Beyond Hilbert space: RHS, PIP and all that. J. Phys. Conf. Ser. 1194, 012007 (2019)
25. F. Bagarello, N. Hatano, A chain of solvable non-Hermitian Hamiltonians constructed by a series of metric operators. Ann. Phys. (NY) 430, 168511 (2021)
26. D. Krejciřík, V. Lottoreichik, M. Znojil, The minimally anisotropic metric operator in quasi-hermitian quantum mechanics. Proc. Roy. Soc. A Math. Phys. Eng. Sci. 474, 20180264 (2018)
27. M. Znojil, On the role of the normalization factors $K_n$ and of the pseudo-metric $P$ in crypto-Hermitian quantum models. Symm. Integ. Geom. Methods Appl. 4, 001 (2008) (e-print overlay: arXiv:0710.4432v3)
28. J. Feinberg, A. Zee, Non-hermitian random matrix theory: method of hermitian reduction. Nucl. Phys. B 504, 579 (1997)
29. J. Feinberg, R. Riser, Dynamics of disordered mechanical systems with large connectivity, free probability theory, and quasi-Hermitian random matrices. Ann. Phys. (NY) 168456 (2021) https://doi.org/10.1016/j.aop.2021.168456
30. J. Feinberg, R. Riser, Pseudo-hermitian random matrix theory: a review. J. Phys. Conf. Ser. 2038, 012009 (2021)
31. Z. Bian, L. Xiao, K. Wang, X. Zhan, F.A. Onanga, F. Růžička, W. Yi, Y.N. Joglekar, P. Xue, Conserved quantities in parity-time symmetric systems. Phys. Rev. Res. 2, 022039 (2020)
32. F. Růžička, K.S. Agarwal, Y.N. Joglekar, Conserved quantities, exceptional points, and antilinear symmetries in non-Hermitian systems. J. Phys. Conf. Ser. 2038, 012021 (2021). arXiv: 2104.11265
33. B. Bagchi, A. Banerjee, E. Caliceti, F. Cannata, H.B. Geyer, C. Quesne, M. Znojil, CPT-conserving Hamiltonians and their nonlinear supersymmetrization using differential charge-operators C. Int. J. Mod. Phys. A 20, 7107 (2005)
34. M. Znojil, I. Semorádová, F. Růžička, H. Moulla, I. Leghrib, Problem of the coexistence of several non-Hermitian observables in PT-symmetric quantum mechanics. Phys. Rev. A 95, 042122 (2017)
35. M. Znojil, Non-self-adjoint operators in quantum physics: ideas, people, and trends. In Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects (Wiley, Hoboken, 2015), pp. 7–58
36. D. Christodoulides, J.-K. Yang (eds.), Parity-time Symmetry and Its Applications (Springer, New York, 2018)