Particle Swarm Optimization: Stability Analysis using $N$-Informers under Arbitrary Coefficient Distributions

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Abstract—This paper derives, under minimal modelling assumptions, a simple to use theorem for obtaining both order-1 and order-2 stability criteria for a common class of particle swarm optimization (PSO) variants. Specifically, PSO variants that can be rewritten as a finite sum of stochastically weighted difference vectors between a particle’s position and swarm informers are covered by the theorem. Additionally, the use of the derived theorem allows a PSO practitioner to obtain stability criteria that contains no artificial restriction on the relationship between control coefficients. Almost all previous PSO stability results have provided stability criteria under the restriction that the social and cognitive control coefficients are equal; such restrictions are not present when using the derived theorem. Using the derived theorem, as demonstration of its ease of use, stability criteria are derived without the imposed restriction on the relation between the control coefficients for three popular PSO variants.

Index Terms—Particle Swarm Optimization, Stability Analysis, Stability Criteria

I. INTRODUCTION

THE particle swarm optimization (PSO) algorithm, originally developed by Kennedy and Eberhart [1], has become a widely used optimization technique [2]. Given PSO’s popularity, it has undergone a considerable amount of theoretical investigation, to list just a few, [3], [4], [5], [6], [7], [8], [9].

There are a number of aspects of PSO behaviour that can be investigated from a theoretical perspective. However, the focus of this paper is on the criteria needed for order-1 and order-2 stability of PSO particles. Specifically, order-1 and order-2 stability occurs when particle positions converge to a constant in first and second order moment respectively [10]. The vast majority of theoretical studies have focused on reducing the modelling assumption used to obtain the stability criteria for PSO with inertia (referred to as canonical PSO (CPSO) in this paper), as proposed by Shi and Eberhart [11]. A detailed discussion of the systematic weakening of these modelling assumption can be found in [9].

The focus of this paper is instead, on providing an easy to use theorem for obtaining stability criteria for PSO variants, while using the minimal modelling assumptions proposed by Cleghorn and Engelbrecht [9]. As such, the general aim of this paper is to provide a theorem that allows a researcher to still obtain stability criteria even if they have made alterations, within reason, to the fundamental PSO algorithm. Recent empirical studies have shown that selecting PSO control coefficients that are both order-1 and order-2 stable are vital to the performance of PSO [12], and as such being able to easily obtain stability criteria for a PSO variant is an important issue for the field.

The PSO variants this paper considers are those that can be rewritten as a finite sum of stochastically weighted difference vectors between a particle’s position and swarm informers. Many PSO variants can be written in this stated form. The canonical PSO is in this form naturally, with two particle informers, namely, the personal best position and the neighbourhood best position (or global best in the case of a fully connected swarm). The classic PSO variants, unified PSO (UPSO) [13] and fully informed PSO (FIPS)[14], both use multiple informers, and can be written as a finite sum of stochastically weighted difference vectors. There is also a more recent trend of adding a third informer to PSO’s update equation to guide a particle’s movement based on information external to the swarm itself. Specifically, in the work of Scheepers [15], a variant of PSO for multi-objective optimization utilizes a third informer from the pareto front archive. A similar idea was also present in the work of Meier and Kramer [16], where gradient based information was used to construct a third informer to assist PSO in the training of recurrent neural networks.

The theorem presented in this paper, for obtaining stability criteria, also removes a common restriction present in existing stability work on PSO. Specifically, previous PSO stability results have provided stability criteria under the restriction that social and cognitive control coefficients are equal [5], [8], [17], [18]; such restrictions are not present when using the provided theorem. An additional theorem is also provided for obtaining the fixed points for the expectations and variance of particle positions.

A brief description of PSO, and its general form, is given in Section [II] followed by a summary of existing relevant PSO theory in Section [III]. The theoretical derivations of criteria for stability along with the limit points for particle positions are provided in Section [IV]. Section [V] demonstrates the use of

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1Some authors have considered the stricter condition where the second order moment converges to zero, a detailed justification for using convergence to a constant second order moment is provided in [10].
the stability theorem by deriving the stability criteria for three PSO variants. Additionally, Section VI provides the first order-1 and order-2 stability criteria of CPSO and UPSO without restriction on the relationship between control coefficients. A summary of the paper’s findings is presented in Section VII.

II. PARTICLE SWARM OPTIMIZATION

Particle swarm optimization (PSO) was originally inspired by the complex movement of birds in a flock. The variant of PSO this section focuses on is the CPSO algorithm [14].

The CPSO algorithm is defined as follows: Let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) be the objective function that the CPSO algorithm aims to find an optimum for, where \( d \) is the dimensionality of the objective function. For the sake of simplicity, a minimization problem is assumed from this point onwards. Specifically, an optimum \( x_0 \) is assumed from this point onwards. Let \( \Omega(t) \) be a set of \( N \) particles in \( \mathbb{R}^d \) at a discrete time step \( t \). Then \( \Omega(t) \) is said to be the particle swarm at time \( t \). The position \( x_i \) of particle \( i \) is updated using

\[
x_i(t+1) = x_i(t) + v_i(t+1)
\]

where the velocity update, \( v_i(t+1) \), is defined as

\[
v_i(t+1) = w v_i(t) + c_1 r_1(t) \otimes (y_i(t) - x_i(t)) + c_2 r_2(t) \otimes (y_c(t) - x_i(t)),
\]

where \( r_1, r_2, r_3, \ldots, r_k \sim U(0,1) \) for all \( t \) and \( 1 \leq k \leq d \). The operator \( \otimes \) is used to indicate component-wise multiplication of two vectors. The position \( y_i(t) \) represents the “best” position that particle \( i \) has visited, where “best” means the location where the particle had obtained the lowest objective function evaluation. The position \( y_c(t) \) represents the “best” position that the particles in the neighbourhood of the \( i \)-th particle have visited. The coefficients \( c_1, c_2, \) and \( w \) are the cognitive, social, and inertia weights, respectively. A full algorithm description is presented in Algorithm 1.

There are numerous PSO variants that alter equation 2 of the CPSO algorithm. The focus of this paper is on PSO variants whose velocity update equation can be rewritten into the following form:

\[
v_i(t+1) = \theta_0 \otimes v_i(t) + \sum_{i=1}^{I} \theta_i \otimes (y_i(t) - x_i(t)) + \sum_{i=1}^{I} \hat{\theta}_i \otimes (y_c(t) - x_i(t))
\]

where \( \theta_i \) and \( \hat{\theta}_i \) are arbitrary independent distributions with well defined mean and variance for each \( 0 \leq i \leq I \), and \( \xi_i \) represents each of the I particle informers. In order to make referring to this general PSO formulation easier it is referred to as N-Informer PSO (NIPSO).

III. CURRENT PSO STABILITY ANALYSIS

Almost all existing work has derived stability criteria directly for specific PSO variants. The CPSO algorithm has undergone the most theoretical stability analysis, from the earlier deterministic model works of 4, 19, 20 to the more recent stochastic works of 21, 22, 23. A number of PSO variants have been directly studied 6, 16, 24, 25. Recently, Cleghorn and Engelbrecht 9 proved Theorem 1 which allows for the derivation stability criteria for all PSO variants with the componentwise form:

\[
x_h(t+1) = x_h(t)\alpha + x_h(t-1)\beta + \gamma_t
\]

Algorithm 1 PSO algorithm

Create and initialize a swarm, \( \Omega(0) \), of \( N \) particles uniformly within a predefined hypercube of dimension \( d \). Let \( f \) be the objective function. Let \( y_i \) represent the personal best position of particle \( i \), initialized to \( x_i(0) \). Let \( y_c \) represent the neighbourhood best position of particle \( i \), initialized to \( x_i(0) \).

Initializ \( v_i(0) \) to \( 0 \).

repeat

for all particles \( i = 1, \ldots, N \) do

if \( f(x_i) < f(y_i) \) then

\( y_i = x_i \)

end if

for all particles \( i \) with particle \( i \) in their neighbourhood do

if \( f(y_i) < f(\hat{y}_i) \) then

\( y_i = \hat{y}_i \)

end if

end for

end for

for all particles \( i = 1, \ldots, N \) do

update the velocity of particle \( i \) using equation (2)

update the position of particle \( i \) using equation (1)

end for

until stopping condition is met

where \( \alpha \) and \( \beta \) are well defined random variables, and \( (\gamma_t) \) is a sequence of well defined random variables. The index \( k \) indicates the vector component. The full theorem is now stated to assist in the subsequent derivations in Section IV.

Theorem 1. The following properties hold for all PSO variants of the form described in equation 5, where \( E[\cdot] \) and \( V[\cdot] \) are the expectation and variance operator respectively, and \( \rho(\cdot) \) is the spectral radius of a matrix.

1) Assuming \( i_t \) converges, particle positions are order-1 stable for every initial condition if and only if \( \rho(A) < 1 \), where

\[
A = \begin{bmatrix} E[\alpha] & E[\beta] \\ 1 & 0 \end{bmatrix} \text{ and } i_t = \begin{bmatrix} E[\gamma_t] \\ 0 \end{bmatrix}
\]

2) The particle positions are order-2 stable if \( \rho(B) < 1 \) and \( (j_t) \) converges, where

\[
B = \begin{bmatrix} E[\alpha] & E[\beta] & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & E[\alpha^2] & E[\beta^2] & 2E[\alpha\beta] \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & E[\alpha] & 0 & E[\beta] \end{bmatrix}
\]

2In the context of this work a well defined random variable is one that has a mean and variance.
and

\[ j_t = \begin{bmatrix} E[\gamma_t] \\ 0 \\ E[\gamma_t^2] \\ 0 \end{bmatrix} \] (7)

under the assumption that the limits of \((E[\gamma_t, \alpha])\) and \((E[\gamma_t, \beta])\) exist.

3) Assuming that \(x(t)\) is order-1 stable, then the following is a necessary condition for order-2 stability:

\[
1 - E[\alpha] - E[\beta] \neq 0 \quad \text{and} \quad 1 - E[\alpha^2] - E[\beta^2] - \left( \frac{2E[\alpha\beta]E[\alpha]}{1 - E[\beta]} \right) > 0
\] (8)

4) The convergence of \(E[\gamma_t]\) is a necessary condition for order-1 stability, and the convergence of both \(E[\gamma_t]\) and \(E[\gamma_t^2]\) is a necessary condition for order-2 stability.

While the generality of Theorem 1 is useful, it can make it potentially challenging for practitioners to quickly obtain stability criteria for their custom PSO variant from the theorem without a considerable amount of calculation. An example of the rigorous use of Theorem 1 can be found in [26].

In order to reduce the burden on practitioners to derive stability criteria, a specialization of Theorem 1 to the class of PSOs described by equations (3) and (4) is proposed. The intention of the specialization is to make obtaining stability criteria as easy as possible, while still maintaining a sufficient degree of generality to cater for a range of variations in the PSO update equation formulation. The overarching goal of the specialization is to reduce the need to perform full stability analysis for most simple variants of PSO. In particular if a practitioner wished to augment the PSO update equations, ideally they should be able to quickly determine what the stability criteria of their bespoke variant is. Knowledge of the stability criteria is of vital importance for parameter tuning as it has been demonstrated that the stability of PSO particles is vital to the performance of PSO [12].

At present all existing second order stability criteria published for CPSO have restricted the relationship between control coefficients. Specifically, the coefficients of CPSO have been restricted such that \(c_1\) and \(c_2\) were assumed either equal [21] or to have equal means and variances [13]. The theorem proved in the next section removes any such restriction, and can therefore produce stability criteria for arbitrary coefficient relationships.

IV. SPECIALIZATION TO N-INFORMERS

This section provides the derivation of order-1 and order-2 stability criteria for the class of PSO variants as defined in equations (3) and (4), which are collectively referred to as NIPSO. Furthermore, the order-1 and order-2 fixed points are derived.

Theorem 2. The following properties hold for all NIPSO combinations, under the non-stagnate distribution assumption for each informer:

1) Particle positions are order-1 stable for every initial condition if and only if

\[ -1 < E[\theta_0] < 1 \] (10)

and

\[ 0 < \sum_{i=1}^{I} E[\theta_i] < 2E[\theta_0] + 1 \] (11)

2) Particle positions are order-2 stable for every initial condition if and only if

\[ -1 < \frac{E[\theta_0]}{\sqrt{1 - V[\theta_0]}} < 1 \] (12)

and

\[ 0 < \psi < \frac{-2(\psi_0^2 + V[\theta_0] - 1)}{1 - E[\theta_0] + \frac{\phi(1 + E[\theta_0])}{\psi^2}} \] (13)

where \(\phi = \sum_{i=1}^{I} V[\theta_i]\) and \(\psi = \sum_{i=1}^{I} E[\theta_i]\).

Proof (1): Let the non-stagnate distribution assumption hold for each of the \(I\) informers [3]. Rewriting equations (3) and (4) into the general form of equation (5) leads to:

\[
\alpha = (1 + \theta_0) - \sum_{i=1}^{I} \theta_i
\]

\[
\beta = -\theta_0
\]

\[
\gamma_t = \sum_{i=1}^{I} \theta_i \zeta_i(t)
\]

In order to utilize part (1) of Theorem 1 to obtain the order-1 stability criteria the matrix \(A\) and the vector \(i_t\) must be constructed, as defined in equation (6). The required expectations are calculated as follows:

\[
E[\alpha] = 1 + E[\theta_0] - \sum_{i=1}^{I} E[\theta_i]
\] (14)

\[
E[\beta] = -E[\theta_0]
\] (15)

\[
E[\gamma_t] = \sum_{i=1}^{I} E[\theta_i]E[\zeta_i(t)]
\] (16)

which leads to

\[
A = \begin{bmatrix} 1 + E[\theta_0] - \sum_{i=1}^{I} E[\theta_i] & -E[\theta_0] \\ 1 & 0 \end{bmatrix}
\] (17)

3) Non-stagnate distribution assumption:

Let \(\xi_i(t)\) be an informer. It is assumed that \(\xi_i(t)\) is a random variable sampled from a time dependent distribution, such that \(\xi_i(t)\) has a well defined expectation and variance for each \(t\) and that \(\lim_{t \to \infty} E[\xi_i(t)]\) and \(\lim_{t \to \infty} V[\xi_i(t)]\) exist. A detailed justification of this modelling choice is given by Cleghorn and Engelbrecht [9].

4) The sufficient condition is not theoretically derived, as the inequality problem becomes intractable. Rather it is supported by extensive experimental evidence in line with the approach used by Bonyadi and Michalewicz [12] and Cleghorn and Engelbrecht [9] up to \(I = 25\).

5) Strictly speaking, only a well defined expectation and limit point of the informer is needed to prove part 1.
and
\[ i_t = \left[ \sum_{i=1}^{I} E[\theta_i] E[\zeta_i(t)] \right]. \tag{18} \]

Since \( E[\theta_i] \) is well defined for each \( i \) and \( E[\zeta_i(t)] \) is well defined and convergent for each \( i \), by the non-stagnate distribution assumption, it follows that \( i_{t,0} \) is convergent and therefore \( i_t \) is convergent. In order to find the criteria needed to satisfy the condition \( \rho(A) < 1 \), the eigenvalues of \( A \) are required and are calculated to be:
\[ \lambda_1, \lambda_2 = \frac{\eta \pm \sqrt{\eta^2 - 4E[\theta_0]}}{2} \tag{19} \]

where \( \eta = 1 + E[\theta_0] - \sum_{i=1}^{I} E[\theta_i] \). After some simplification it is found that \( \rho(A) < 1 \) holds if and only if
\[ -1 < E[\theta_0] < 1 \text{ and } 0 < \sum_{i=1}^{I} E[\theta_i] < 2(E[\theta_0] + 1). \tag{20} \]

It follows from part (1) of Theorem 1 that NIPS0 is order-1 stable if and only if the criteria of equations (20) and (11) hold.

Proof (2): Let the non-stagnate distribution assumption hold for each of the \( I \) informers. In order to obtain the necessary conditions for order-2 stability, part 3 of Theorem 1 is utilized. A number of expectations are required to construct the matrix \( B \) and the vector \( j \). Specifically, \( E[\alpha^2] \), \( E[\beta^2] \), and \( E[\alpha \beta] \) are required, and calculated as
\[ E[\alpha^2] = E \left[ \left( 1 + \theta_0 - \sum_{i=1}^{I} \theta_i \right)^2 \right] = 1 + 2E[\theta_0] - 2 \sum_{i=1}^{I} E[\theta_i] \]
\[ - 2E[\theta_0] \sum_{i=1}^{I} E[\theta_i] + E \left[ \left( \sum_{i=1}^{I} \theta_i \right)^2 \right], \tag{21} \]

where
\[ E \left[ \left( \sum_{i=1}^{I} \theta_i \right)^2 \right] = V \left[ \sum_{i=1}^{I} \theta_i \right] + \sum_{i \neq j} \text{cov} (\theta_i, \theta_j) + \left( \sum_{i=1}^{I} E[\theta_i] \right)^2 \]
\[ = \sum_{i=1}^{I} V[\theta_i] + \sum_{i \neq j} \text{cov} (\theta_i, \theta_j) + \left( \sum_{i=1}^{I} E[\theta_i] \right)^2, \tag{22} \]

since each \( \theta_i \) is independent. Substituting equation (22) back into equation (21) leads to,
\[ E[\alpha^2] = 1 + 2E[\theta_0] - 2(1 + [\theta_0]) \sum_{i=1}^{I} E[\theta_i] \\
\[ + \sum_{i=1}^{I} V[\theta_i] + \left( \sum_{i=1}^{I} E[\theta_i] \right)^2. \tag{23} \]

The expectation of \( \beta^2 \) and \( \alpha \beta \) are easily calculated as:
\[ E[\beta^2] = E[\theta_0^2] = V[\theta_0] + E[\theta_0]^2 \tag{24} \]
\[ E[\alpha \beta] = E \left[ -\theta_0 \left( 1 + \theta_0 \right) - \sum_{i=1}^{I} \theta_i \right] = -E[\theta_0] - V[\theta_0] - E[\theta_0]^2 - E[\theta_0] \sum_{i=1}^{I} E[\theta_i]. \tag{25} \]

For equation (8), in part 3 of Theorem 1 to be satisfied the following condition must hold:
\[ \psi = \sum_{i=1}^{I} E[\theta_i] \neq 0 \tag{26} \]

For equation (9), in part 3 of Theorem 1 to be satisfied the following condition must hold:
\[ 1 + 2E[\theta_0] + 2 \left( 1 + E[\theta_0] \right) \psi - \phi - \psi^2 - V[\theta_0] + E[\theta_0]^2 - \left( \frac{(1 + E[\theta_0] - V[\theta_0] - E[\theta_0]^2 - E[\theta_0] \psi) (1 + E[\theta_0] - \psi)}{1 + E[\theta_0]} \right) > 0, \]

which is simplified using a method similar to that of Bonyadi and Michalewicz [18], to equal the criteria of equations (12) and (13). The necessary condition of part 2 of Theorem 2 is therefore proved.

All that remains is to prove that satisfying the criteria of equations (12) and (13) is in fact sufficient, and not only necessary, for order-2 stability. This is achieved by verifying that if the criteria of equations (12) and (13) are satisfied then \( \rho(B) < 1 \), from Theorem 1 part 2. All the expectations needed to construct matrix \( B \) have already been obtained while deriving the necessary condition. In order to verify that \( \rho(B) < 1 \) the empirical approach of Bonyadi and Michalewicz [18] and Cleghorn and Engelbrecht [9] is used. Specifically, for \( I = 1, 2, \cdots, 50 \) informers the experimental procedure followed is: \( I \times 10^8 \) random configurations representing \( \{ E[\theta_0], V[\theta_0], \cdots, E[\theta_i], V[\theta_i] \} \) are generated such that equations (12) and (13) are satisfied. In all of the cases it was found that if equations (12) and (13) were satisfied, then the condition \( \rho(B) < 1 \) held. This finding is strong evidence that the criteria is sufficient for order-2 stability.

Theorem 3. The following properties hold for all NIPS0 combinations:

1) Under order-1 stability the fixed points of the particle position expectations are:
\[ E_{x_{1,k}} = \frac{\sum_{i=1}^{I} E[\theta_i] E[\zeta_{1,k}]}{\sum_{i=1}^{I} E[\theta_i]} \tag{27} \]

where \( E[\zeta_{1,k}] \) is the limit of \( E[\zeta_{1,k}(t)] \).

2) Under order-1 and order-2 stability, the fixed points of the particle position variances are:
\[ V_{x_{1,k}} = \frac{2 \psi (1 + E[\theta_0]) \left( \kappa_1 - 2 \kappa_2 E_{x_{1,k}} + \kappa_3 E_{x_{1,k}}^2 \right)}{2 \psi (1 + E[\theta_0]) - V[\theta_0] - \phi (1 + E[\theta_0]) + \psi^2 (E[\theta_0] - 1)} \tag{28} \]

6While the experimental verification was only done up to 50 informers, there is no clear reason why it would fail to hold for higher informer counts. Practically speaking, a variant with more than 50 informers seems unlikely.
where

\[ \kappa_1 = \sum_{i=1}^{f} (E^2[\theta_i]V[\zeta_{i,k}] + E^2[\zeta_{i,k}]V[\theta_i] + V[\theta_i]V[\zeta_{i,k}]) \],

\[ \kappa_2 = \sum_{i=1}^{f} V[\theta_i]E[\zeta_{i,k}], \quad \phi = \sum_{i=1}^{f} V[\theta_i], \quad \psi = \sum_{i=1}^{f} E[\theta_i], \]

with \( E[\zeta_{i,k}] \) and \( V[\zeta_{i,k}] \) as the the limit of \( E[\zeta_{i,k}(t)] \) and \( V[\zeta_{i,k}(t)] \) respectively.

**Proof (1):** Under the assumption of order-1 stability each particle \( i \) converges to a fixed point in expectation. Let such a fixed point be called \( E_{\zeta_i} \). The fixed point is calculated by rewriting equations (3) and (4) into the following component-wise second order recurrence relation form:

\[ x_{i,k}(t + 1) = x_{i,k}(t) (1 + \theta_0) - \theta_0 x_{i,k}(t - 1) \]

\[ + \sum_{i=1}^{f} \theta_i (\zeta_{i,k}(t) - x_{i,k}(t)). \]  

(29)

Applying the expectation operator leads to

\[ E[x_{i,k}(t + 1)] = E[x_{i,k}(t)] (1 + E[\theta_0]) - E[\theta_0]E[x_{i,k}(t - 1)] \]

\[ + \sum_{i=1}^{f} E[\theta_i] (E[\zeta_{i,k}(t)] - E[x_{i,k}(t))]. \]  

(30)

Then by setting \( E[x_{i,k}(t - 1)] = E[x_{i,k}(t)] = E[x_{i,k}(t + 1)] = E_{\zeta_i} \) and \( E[\zeta_{i,k}(t)] \) to its limits \( E[\zeta_{i,k}] \), equation (30) can be rearranged to find an explicit expression for \( E_{\zeta_i} \), thus obtaining equation (27).

**Proof (2):** Under the assumption of order-1 and order-2 stability each particle \( i \) converges to a fixed point for each of the following sequences: \( E[x_{i,k}(t)], E[x_{i,k}(t)x_{i,k}(t - 1)], \) and \( E[x_{i,k}^2(t)] \). Let such fixed points be called \( E_{\zeta_i}, E_{\zeta_i}, \) and \( E_{\zeta_i} \) respectively as we will be working in the limit. First define

\[ \partial x_{i,k}(t) = x_{i,k}(t) - E[x_{i,k}(t)] = x_{i,k}(t) - E_{\zeta_i}. \]  

(31)

It follows that \( V[x_{i,k}(t)] = E[\partial^2 x_{i,k}(t)] \), where \( \partial^2 x_{i,k}(t) \) denotes the square of equation (31). In order to obtain \( V[x_{i,k}(t)] \), consider the class of recurrence relations as defined in equation (3), and that \( \partial x_{i,k}(t) \) can be rewritten as

\[ \partial x_{i,k}(t) = \alpha \partial x_{i,k}(t - 1) + \beta \partial x_{i,k}(t - 2) + d_{i,k}(t - 1) \]  

(32)

\[ d_{i,k}(t - 1) = \gamma_{t-1} - E_{\zeta_i}(1 - \alpha - \beta). \]  

(33)

Squaring and applying the expectation operator to equation (32) leads to

\[ E[\partial^2 x_{i,k}(t)] \]

\[ = E[\alpha^2]E[\partial^2 x_{i,k}(t - 1)] + 2E[\alpha \beta]E[\partial x_{i,k}(t - 1) \partial x_{i,k}(t - 2)] \]

\[ - E[d_{i,k}(t - 1)] (2E[\alpha]E[\partial x_{i,k}(t - 1)] + 2E[\beta]E[\partial x_{i,k}(t - 2)] \]

\[ + E[\beta^2]E[\partial^2 x_{i,k}(t - 2)] + E[d_{i,k}(t - 1)]^2. \]  

(34)

In order to simplify equation (34) consider that

\[ E[\partial x_{i,k}(t) \partial x_{i,k}(t - 1)] = E[\partial x_{i,k}(t - 1) \partial x_{i,k}(t - 2)] \]  

(35)

and

\[ E[\partial x_{i,k}(t - 2)] = E[\partial x_{i,k}(t - 1)] = E[\partial E_{\zeta_i,k}] = 0. \]  

(36)

Now

\[ E[\partial x_{i,k}(t) \partial x_{i,k}(t - 1)] \]

\[ = E[\alpha]E[\partial^2 x_{i,k}(t - 1)] + E[\beta]E[\partial x_{i,k}(t - 2) \partial x_{i,k}(t - 1)] \]

\[ + E[d_{i,k}(t - 1)]E[\partial x_{i,k}(t - 1)]. \]  

(37)

Using equations (35) and (36), equation (37) can be rearranged to yield,

\[ E[\partial x_{i,k}(t) \partial x_{i,k}(t - 1)] = \frac{E[\alpha]E[\partial^2 x_{i,k}(t - 1)]}{1 - E[\beta]} \]  

(38)

Now all that remains is to substitute equation (38) into equation (34), and utilize the fact that \( E[\partial^2 x_{i,k}(t - 1)] = E[\partial^2 x_{i,k}(t)] \) (once again this is permissible in the limit), to obtain

\[ E[\partial^2 x_{i,k}(t)] = \frac{E[d_{i,k}(t - 1)^2]}{1 - E[\alpha^2] - E[\beta^2] - 2E[\alpha \beta]E[\alpha^2]}. \]  

(39)

Equation (39) represents the variance fixed point for the large class of PSOs. However, our focus is on the case where

\[ \alpha = (1 + \theta_0) - \sum_{i=1}^{f} \theta_i \]

\[ \beta = -\theta_0 \]

\[ \gamma_t = \sum_{i=1}^{f} \theta_i \zeta_{i,k}(t). \]

Substituting these specific \( \alpha, \beta, \) and \( \gamma_t \) into equation (39) and performing a substantial amount of simplification (which is omitted for the sake of brevity), leads to equation (28) as was required to be proved.

V. APPLICATION OF STABILITY RESULTS

In this section a number of existing stability criteria are re-derived to demonstrate how Theorem 2 can be easily applied to rapidly obtain stability criteria. Furthermore, previous stability results have considerably restricted the allowable relationship between control coefficients, for example \( c_1 = c_2 \), this limitation is present in this section. All derived criteria contained in the section have no restriction on the coefficient relations, and as such are a novel contribution in addition to illustrating the ease of using Theorem 2.

Stability criteria for CPSO, fully informed PSO, and unified PSO are derived in sections V.A, V.B, and V.C respectively.

A. Canonical PSO

Consider the case of the CPSO algorithm, as defined by equations (1) and (2). After dropping the particle and component indices, without loss of generality, the stability criteria for CPSO can be obtained by using two informers with \( \theta_0 = w, \theta_1 = c_1 r_1, \) and \( \theta_2 = c_2 r_2 \). In order to utilize Theorem 2, \( \psi \) and \( \phi \) are required and calculated as:

\[ \psi = \sum_{i=1}^{2} E[\theta_i] = \frac{c_1}{2} + \frac{c_2}{2} \]

\[ \phi = \sum_{i=1}^{2} V[\theta_i] = \frac{c_1^2}{12} + \frac{c_2^2}{12}. \]  

(40)

Substituting \( \psi \) and \( \phi \) into the criteria of equations (10), (11), (12), and (13) the following order-1 and order-2 stability are obtained:

\[ -1 < w < 1 \quad \text{and} \quad 0 < c_1 + c_2 < \frac{4 (1 - w^2)}{1 - w + \frac{c_1^2 + c_2^2 (1 + w^2)}{\beta (c_1 + c_2)^2}}. \]  

(41)

The criteria in equation (41) is the first time CPSO’s full order-1 and order-2 stability criteria has not been simplified to the
case where $c_1 = c_2$. If the simplification is reimposed, the following commonly reported form reappears:

$$-1 < w < 1 \text{ and } 0 < c_1 + c_2 < \frac{24}{7 - 5w}.$$  (42)

It is interesting to observe that the weighting between $c_1$ and $c_2$ has a direct influence of the size and shape of the stability region as illustrated in Figure 1, where the cross-sections of the stability region, with fixed inertia values, are shown. Additionally, Figure 1 demonstrates that using equation (42) without the knowledge of the restrictions of $c_1 = c_2$ can lead to the misclassification of stable parameter configurations.

velocity equation of CPSO is altered such that each particle is influenced by all its neighbours. Specifically, the velocity update equation for FIPS is:

$$v_i(t + 1) = w v_i(t) + \sum_{m=1}^{N_p} \gamma_m \odot \frac{(y_m(t) - x_i(t))}{|N_p|},$$  (43)

where $N_p$ is the set of particles in particle $i$’s neighbourhood, $y_m(t) \in N_p$, and $\gamma_m, k \sim U(0, \hat{c})$, were $c_1 + c_2 = \hat{c}$.

After dropping the particle and component indices, without loss of generality, the stability criteria for FIPS can be obtained by considering $I = |N|$ informers and setting $\theta_0 = w$ and $\theta_1 = \frac{c_1}{|N|}$ for $1 \leq i \leq |N|$. The following calculations are required to use Theorem 2.

$$\psi = \sum_{i=1}^{I} E[\theta_i] = \frac{|N|}{|N|} E[\gamma] = \sum_{i=1}^{N} \frac{\hat{c}^2}{2|N|} = \frac{\hat{c}^2}{2}$$  (44)

and

$$\phi = \sum_{i=1}^{I} V[\gamma_i] = \sum_{i=1}^{|N|} \gamma_i^2 = \frac{c^2}{12|N|}$$  (45)

Substituting $\psi$ and $\phi$ into the criteria of equations (10), (11), (12), and (13) the following criteria for order-1 and order-2 stability are obtained:

$$-1 < w < 1 \text{ and } 0 < \hat{c} < \frac{12 (1 - w^2)}{3|N| + 1 + w(1 - 3|N|)).}$$  (46)

The derived criteria is in agreement with existing criteria of both Cleghorn and Engelbrecht [24] and García-Gonzalo and Fernández-Martínez [27], but are obtained with minimal calculations, and under a weaker modelling assumption.

C. Unified PSO

The USPSO algorithm was designed by Parsopoulos and Vrahatis [13] as a weighted merger between the local best PSO and the global best PSO. The PSO variants utilizes the additional control parameter, $\eta \in [0, 1]$, called the unification factor, to control the importance placed on either the global best PSO update or the local best PSO. Specifically, the update equation for USPSO are:

$$g_i(t + 1) = w v_i(t) + c_1 r_1 \odot (y_i(t) - x_i(t))$$
$$+ c_2 r_2 \odot (g(t) - x_i(t))$$  (47)

$$l_i(t + 1) = w v_i(t) + c_1 r'_1 \odot (y_i(t) - x_i(t))$$
$$+ c_2 r'_2 \odot (\hat{y}_i(t) - x_i(t))$$  (48)

$$v_i(t + 1) = u g_i(t + 1) + (1 - u) l_i(t + 1)$$  (49)

$$x_i(t + 1) = x_i(t) + v_i(t + 1)$$  (50)

where $r_{1,k}, r_{2,k}, r'_{1,k}, r'_{2,k} \sim U(0, 1)$, and both $y_i(t)$ and $\hat{y}_i(t)$ are defined as before with the addition of $g(t)$ as the global best position with the swarm at time step $t$.

Without loss of generality the particle and component wise index are dropped again. In order to rewrite USPSO into the NIPSO form, substitute equations (47) and (48) into the velocity update equation (49) to arrive at

$$v(t + 1) = uv(t) + c_1 (ur_1 + (1 - u)r'_1)(y(t) - x(t))$$
$$+ c_2 ur_2(g(t) - x(t)) + c_2 (1 - u)r'_2(\hat{y}(t) - x(t)).$$  (51)

Now equation (51) is in the NIPSO form with $I = 3$ and $\theta_0 = w$, $\theta_1 = c_1 (ur_1 + (1 - u)r'_1)$, $\theta_2 = c_2 ur_2$, and $\theta_3 = \hat{c}$. Fig. 1: Order-2 stable regions of CPSO under fixed inertia values. The interior region of the elliptic shapes correspond to where equation (51) is satisfied for a given $w$.
In order to calculate $\psi$ the following additional terms are required:

$$E[\theta_i] = c_1 u E[r_1] + c_1 (1-u) E[r'_1]$$

$$= \frac{c_1 u}{2} + \frac{c_1 (1-u)}{2} = \frac{c_1}{2}$$

$$E[\theta_2] = c_2 u E[r_2] = \frac{c_2 u}{2}$$

$$E[\theta_3] = c_2 (1-u) E[r'_2] = \frac{c_2 (1-u)}{2}$$

The summation of equations (52), (53), and (54) leads to

$$\psi = \sum_{i=1}^{3} E[\theta_i] = \frac{c_1}{2} + \frac{c_2 u}{2} + \frac{c_2 (1-u)}{2} = \frac{c_1 + c_2}{2}.$$  (55)

In order to calculate $\psi$ the following additional terms are required:

$$V[\theta_1] = c_1^2 V[ur_1 + (1-u)r'_1]$$

$$= c_1^2 u^2 V[r_1] + (1-u)^2 V[r_2] + 2u(1-u)COV[r_1, r'_2]$$

$$= \frac{c_1^2 u^2}{12} + \frac{(1-u)^2}{12} = \frac{c_1^2}{12} \left( u^2 + (1-u)^2 \right)$$  (56)

$$V[\theta_2] = V[c_2 ur_2] = \frac{c_2^2 u^2}{12}$$  (57)

$$V[\theta_2] = V[c_2 (1-u)r'_2] = \frac{c_2^2 (1-u)^2}{12}.$$  (58)

The summation of equations (56), (57), and (58) leads to

$$\phi = \sum_{i=1}^{3} V[\theta_i] = \frac{(c_1^2 + c_2^2) (u^2 + (1-u)^2)}{12}.$$  (59)

Substituting $\psi$ and $\phi$ into the criteria of equations (10), (12), and (13) the following criteria for order-1 and order-2 stability are obtained:

$$0 < c_1 + c_2 < \frac{4(1-w^2)}{1-w + \frac{(c_1^2 + c_2^2)(u^2 + (1-u)^2)(1+w)}{3c_1 + c_2 r^2}}.$$  (60)

The criteria of equations (60) and (61) is the first derivation of full USPO stability criteria without artificial restrictions on the control coefficients. As with the CPSO case, in Section V-A the weighting between $c_1$ and $c_2$ has a clear influence on the size and shape of stability region, as illustrated in Figure 2 where the cross-sections of the stability region, with fixed inertia values is shown.

In the restricted case where $c_1 = c_2$ is considered, the following criteria are obtained:

$$0 < c_1 + c_2 < \frac{24(1-w^2)}{7-5w + 2(u^2 - w)(1+w)}.$$  (62)

which is in agreement with the derived criteria of Cleghorn and Engelbrecht [23] with minimal calculations needed, and under a weaker modelling assumption.

VI. CONCLUSION

This paper derives general theorems for rapidly obtaining order-1 and order-2 stability criteria and fixed points for a class of PSO variants. Specifically, PSO variants that can be rearranged into a sum of difference vectors between informers and the current particle positions, are catered for. From this general derivation, stability criteria can be obtained for a set of custom PSO variants in a direct manner without substantial mathematical calculation. Given the direct linkage between PSO performance and the satisfaction of order-1 and order-2 stability criteria, the theorems provided in this paper will be directly applicable to the PSO community as a whole.

Furthermore, the proved theorems allows for stability criteria to be derived without unnecessary restrictions on the relationship between control coefficients. In this vein, stability criteria for both the canonical PSO and the unified PSO are, for the first time, derived without restrictions on the relationship between control coefficients in this paper.

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