PAPER

Energy current and Casimir effect with phase transition in the nonequilibrium steady-state of a quantum anisotropic XY spin chain

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Abstract

The dynamics of quantum anisotropic XY spin chain driven out of equilibrium by a transverse magnetic field, that carry an energy current \( \langle J^x \rangle \) is investigated. Considering the spin exchange and the flow of energy trapped in magnetic field between two defects in the \( xy \)-plane separated by a distance \( d \), we derive an analytical expression of the steady state Green’s function \( G_0 \) of the model using the Grassmann algebra in order to determine the Casimir force between two magnetic defects. As soon as done we find that, cooling the system as well as increasing the anisotropic parameter decrease the energy current and accordingly increase the range of the Casimir force. This symmetry is broken with a significant increase of distance between the defects. A critical temperature is then recorded exhibiting a phase transition according to the anisotropic parameter. An interesting feature of this model is that turning the anisotropic parameter as well as the temperature range will better manipulate the energy flux and control the range of the Casimir force.

1. Introduction

The Casimir force between grounded metal plates has been measured quite accurately and agrees with Casimir’s original prediction [1]. Recently, nonequilibrium electromagnetic fluctuations have been considered taking into account a fluid with a temperature gradient [2] or the two plates at different temperatures [3]. The oscillating Casimir force between impurities in one-dimensional Fermi liquids have also been investigated showing that the strength and sign of the Casimir interaction depend sensitively on the impurity separation; these oscillations in the Casimir interaction have the same period as Friedel oscillations with their maxima corresponding to the tunneling resonances tuned [4].

This paper considers the current-carrying states generated by inhomogeneous initial conditions and the Casimir effect to investigate the nonequilibrium properties in the anisotropic XY spin chain driven by a transverse magnetic field. The problem of energy transport in one-dimensional systems has been much investigated, with as main goal the Fourier’s heat law derivation and the magnetization current [5, 6]. Theoretical studies indicate for a number of classical lattice-dynamical models that the Fourier’s law condition should hold in the presence of strong nonlinearities or nonintegrability (chaoticity) of the dynamics [7]. Integrable spin systems whose Hamiltonians have an infinite number of the conserved quantities exhibit anomalous heat conduction [8, 9], which is suppressed in real materials by phonons and impurities. Some experiments showed a realization of such a ballistic transport on a small scale [10]. The nonequilibrium properties arising from this ballistic transport helps to investigate according to the accepted terminology the Casimir force between two point-like defects located at a distance \( d \) apart respectively. It emerges in isotropic spin chain that the presence of the flux decreases the magnitude of the force [11]. However, the isotropic exchange interaction, which was discussed so far, does not depend on the direction of spins with respect to the
crystal axes. Exchange interaction becomes anisotropic, if the exchange Hamiltonian includes the spin–orbit coupling which create admixed excited orbital states to the ground state of the electron making it sensitive to the local crystal environment. This admixture is accounted for in terms of the given perturbation expansion or anisotropy [12]. The anisotropic exchange contributions, however, are only accessible by means of electron spin resonance, because the spin–spin relaxation measured by the electron spin resonance linewidth is driven primarily by the corresponding effective local fields. Electron spin resonance measurements on LiCuVO₄ showed it exhibiting a strong line broadening [13], although the antisymmetric interaction should vanish for symmetry reasons. Then, another source of the line broadening was needed. The solution of this problem was found by taking into account exotic effects like quantum interference and new types of anisotropic superexchange [14]. They strongly intensify the symmetric anisotropic exchange which would be established for a number of low-dimensional systems to be the dominant source of spin relaxation in current-carrying states.

Following the considerations in [11, 15] we propose ourselves to investigate the anisotropic counterpart of the XY spin chain. This general view is as a special case of Heisenberg spin chain model where the presence of a small perturbation (an anisotropy exchange parameter) apart from the challenge led by the temperature and the distance between defects, decrease (respectively increase) the energy current at low temperature (respectively at high temperature) and accordingly, increase (respectively decrease) the magnitude of the Casimir force.

The results of this paper are presented in the following order: We start in section 2 by briefly reviewing the XY spin chain model driven by a homogeneous transverse magnetic field and we follow in section 3 by describing the properties of nonequilibrium steady states arising from the preparation of the system with a step-like temperature profile. In section 4, considering the change in the steady-state properties due to the energy flux and taking into account the magnetic field excitation at two sites of the two half transverse XY spin chain as impurities, the Casimir force is evaluated. The paper ends in section 5 with the concluding remarks.

2. Anisotropic spin chain model driven by a homogeneous transverse magnetic field

The transverse XY spin chain model [16–19] is one of the simplest anisotropic quantum spin systems with large fluctuations compared to the XX spin model [11, 20]. It is a one-dimensional spin chain described by the generic Hamiltonian

\[
H = -J \sum_{i=1}^{N} \left[ (1 + \gamma) s_{i}^{x} s_{i+1}^{x} + (1 - \gamma) s_{i}^{y} s_{i+1}^{y} + h s_{i}^{z} \right]
\]

where \(N \gg 1\) is the number of spins on either side of 0 arranged on a line, \(s_{i}^{\alpha} = \frac{i}{2} \sigma_{i}^{\alpha}\) and \(\sigma_{i}^{\alpha} (\alpha = x, y, z)\) denote the three Pauli matrices at sites \(i \in \{1, N\}\) where \(h\) in unit of coupling factor \(J\) (with \(J = 1\)) is a transverse magnetic field and \(\gamma (-1 \leq \gamma \leq 1)\) the anisotropic parameter having two extreme limits i.e \(\gamma = \pm 1\); the intermediate limit may be associated with the quantum XX-model [17]. Frequently system is rotationally symmetric at zero order, but anisotropy terms are present as small perturbations. For any value of \(\gamma\), a second-order quantum point transition occurs at the critical point \(h_{c} = 1\) separating a ferromagnetic ordered phase (\(h < 1\)) from a quantum paramagnetic phase (\(h > 1\)). The transition is characterized by the order parameter \((s^{z})\), the magnetization in the \(x\) direction, which has a nonzero expectation value only in the ordered ferromagnetic phase [21]. In order to find out the excitation energy spectrum of the system and subsequently obtain the energy-flux, we use the familiar diagonalization procedure often applied to the hamiltonian equation (1). The so called procedure suggests that the Hamiltonian should be rescaled and rewritten in terms of spin ladder operators \((s_{i}^{+} = s_{i}^{x} + i h s_{i}^{y}\) and \((s_{i}^{-}, s_{i}^{z}) = 2s_{i}^{z}\)) considering the SU(2) algebra which covers spin operators. In the same vein on the procedure elaboration, the study is finally carried out through the Jordan–Wigner transformations \(s_{i}^{+} = \exp \left[-i \pi \sum_{j=1}^{i-1} c_{j}^\dagger c_{j}\right] c_{i}^\dagger\) and \(s_{i}^{-} = \exp \left[-i \pi \sum_{j=i+1}^{N} c_{j}^\dagger c_{j}\right] c_{i}\) [22] which map a spin chain into a spinless fermionic chain

\[
H = - \sum_{i=1}^{N} \left[ \frac{1}{2} (c_{i}^\dagger c_{i+1} - c_{i} c_{i+1}^\dagger) + \frac{\gamma}{2} (c_{i}^\dagger c_{i+1} - c_{i+1} c_{i}^\dagger) \\
+ h (c_{i}^\dagger c_{i} - \frac{1}{2}) \right],
\]

where \(c_{i}\) and \(c_{i}^\dagger\) are annihilation and creation fermion operators satisfying the canonical anticommutative algebra [23]. Proceeding with the inverse Fourier-Bogoliubov transformation

\[
c_{i}^\dagger = \frac{1}{\sqrt{N}} \sum_{q} e^{i q} (u_{q} b_{q}^\dagger + i v_{q} b_{-q}),
\]
one gets the following Hamiltonian operator

\[ H = - \sum_q E_q b_q^\dagger b_q , \]

where \( b_q = u_q c_q + i v_q c_q^\dagger \) is the fermionic Bogoliubov operator with the functions \( u_q \) and \( v_q \) defined as follows:

\[ u_q = \frac{-\gamma \sin q}{\sqrt{2E_q(E_q + \cos q + h)}} \]

and

\[ v_q = \frac{E_q + \cos q + h}{\sqrt{2E_q(E_q + \cos q + h)}}. \]

\[ E_q = \sqrt{(\cos q + h)^2 + \gamma^2 \sin^2 q} \]

is the energy spectrum. Here \(-\pi \leq q \leq \pi\) is the variation of the fermionic wave number.

### 3. Energy flux density in the non-equilibrium steady state of an anisotropic spin chain

Since the energy is a conserved quantity as well, the local energy, the contribution of the \( i \)th spin to the total energy satisfies the following equation of continuity with the local energy current \( J^E_i \)

\[ i[H, H] = J^E_i - J^E_{i+1} \]

where its sum over \( i \) (the total energy current) has the form

\[ J^E = \sum_{q>0} [ (1 - \gamma^2) s_q^x(s_{q-1}^x s_{q+1}^x + s_{q-1}^x s_{q+1}^y) + h(s_{q-1}^y s_q^x - s_q^y s_{q+1}^x - \gamma(s_q^x s_{q+1}^x + s_q^x s_{q+1}^y))]. \]

Based on the Jordan-Wigner and Fourier transformation, the energy density spectrum is obtained as

\[ J^E = -\sin q [((1 - \gamma^2) \cos q + h)(c_q^\dagger c_q - c_q c_q^\dagger) + h\gamma (c_q^\dagger c_{-q} + c_{-q} c_q)]. \]

Using the Bogoliubov transformations, the diagonal form of the total energy current is obtained as:

\[ J^E = -\sum_q \xi_q b_q^\dagger b_q \]

where

\[ \xi_q = \sin q \sqrt{(1 - \gamma^2) \cos q + h^2 + h^2 \gamma^2} \]

To allow physical properties of the steady state of the XY spin chain, we focus our analysis on the asymptotic homogeneous steady state of the expanding central region of the system, where the fermions occupation number in the steady state is given by

\[ \langle b_q^\dagger b_q \rangle = \frac{\varphi(q)}{1 + e^\beta q} + \frac{\varphi(-q)}{1 + e^{-\beta q}} \]

where \( \varphi \) is a step function defined as \( \varphi(q) = \int_{-\infty}^{+\infty} du \frac{e^{\alpha u}}{2\pi u - \alpha \text{sgn}(q)} \) [11]. With the help of the two-point correlation function [20] we consider the anisotropic spin chain and discuss the oscillations in the energy flux. Considering the left and the right part of the chain interacting through the modulated inverse temperatures \( \beta_1 = 1/k_B T_1 \) and \( \beta_2 = 1/k_B T_2 \) respectively (with \( k_B \) the Boltzmann constant), we find that the quasi-particle with the excitation energy \( \sqrt{(\cos q + h)^2 + \gamma^2 \sin^2 q} \) runs through the spin chain at the velocity \( c = \sin q \sqrt{(\gamma^2 - 1) \cos q + h^2 + h^2 \gamma^2} \) and that the number of particles is equivalent to \( 1/(1 + e^{\beta \sqrt{(\cos q + h)^2 + \gamma^2 \sin^2 q}}) \) for \( q < 0 \) and to \( 1/(1 + e^{\beta \sqrt{(\cos q + h)^2 + \gamma^2 \sin^2 q}}) \) for \( q > 0 \). In the present consideration where the non-equilibrium steady state is affected by the anisotropy factor \( \gamma \) that affects the quantum phase transition and the energy flow, the energy current is as follows:
Considering the quasiparticle with energy $E_q$ propagating between the two halves of the chain and turning the temperature of the right part to zero ($\beta_2 \rightarrow \infty$), the renormalized interaction energy flux in equation (12) becomes

$$J^E = -\int_{-\pi}^{0} \frac{dq}{2\pi} \sin q \sqrt{(1 - \gamma^2) \cos q + h} \frac{h^2 \gamma^2}{1 + e^{\beta_2 \sqrt{(1 - \gamma^2) \cos q + h^2 + h^2 \gamma^2}}},$$

and plotted in figure 1 and in figure 2, with $k_B = 1$ and $h = 0.5$.

It is foreseeable that although the symmetry $q \rightarrow -q$ is a broken for $\gamma \neq 0$, the grounded state remains that of the XY spin model for $\gamma \leq 1$. When $\gamma \rightarrow 1$, the exchange coupling in the $y$ direction ($J_y = 1 - \gamma$) tend to zero thus the exchange interaction vanishes. As we remain in the ferromagnetic phase, the exchange interaction in $x$ dominate and favor the order in one of the $\pm x$ direction. The system remains strongly correlated in $x$ direction due to the presence of the anisotropy factor, which increases the rigidity of the ground state, and then, suppress the current (figure 1). This property of the system is lost as soon as the temperature increases (figure 2); the temperature gradient becomes large, allowing great fluctuations and accordingly strengthen the current for high values of the anisotropy factor. The system undergoes a nonequilibrium statement which will probably affect its quantum behavior. Thus in the XX spin model, the spin coupling causes the emergence of the

\begin{figure}[h]
\centering
\includegraphics[width=\columnwidth]{figure1.png}
\caption{The anisotropy dependence of the energy current in the low-temperature regime for different values of the Boltzmann temperature $\beta_2$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\columnwidth]{figure2.png}
\caption{The anisotropy dependence of the energy current in the high-temperature regime for different values of the Boltzmann temperature $\beta_2$.}
\end{figure}
energy current and probably increases entanglement while the XY spin coupling induces phase transition and contributes to reduce the effects of the energy current.

4. Casimir effect and phase transition: the effective force between the impurities

Recent realization of a one-dimensional system made in the domain of ultra-cold Fermi gases in a strong 2D optical lattice has provided a novel opportunity to study Luttinger–liquid effects in a setup with cold gases—e.g., spin-charge separation [24]. The objective was to study whether the Casimir interactions might be observed in these systems. In this way, the atomic gas of fermions in two hyperfine states was considered where the two internal states play the role of (iso)spin-1/2 states. In this consideration, it has been shown in principle that both the sign and strength of the interaction can be controlled using scattering resonances—e.g., a confinement-induced resonance [25].

We propose in this part of the work to extend the study to the XY spin chain model. We also take the walls of the Casimir setup as two magnetic impurities at lattice sites \( \pm d/2 \), where their strength \( \Lambda h_{L/R} \) enters the spinless fermionic transformation of the Hamiltonian through the following term

\[
L = - \Lambda h_{L/R} c_{d/2}^\dagger c_{d/2} - \Lambda h_{R/L} c_{d/2}^\dagger c_{d/2}.
\]  

(14)

In order to compute the Casimir force between these impurities, we will use the definition [11] which express the force on a lattice as the discontinuity of the energy across the defect.

To avoid non-coherent states, we use the fermionic Grassmann numbers \( \psi_i \) and \( \bar{\psi}_i \) which obey an ordinary but anticommutative algebra. The steady-state Green’s function \( G(t, t') = \langle c_i(t) c_j^\dagger(t') \rangle \) of the model is calculated using the Grassmann algebra where its Fourier transform is given as:

\[
G_{ij}(\omega) = \left( \psi_i(\omega) \bar{\psi}_j(\omega) \right) = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \bar{\psi}_j(\omega) \psi_i(\omega) e^{-S[\bar{\psi}, \psi]}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S[\bar{\psi}, \psi]}}
\]  

(15)

and

\[
S = S_0 + AS
\]  

(16)

denotes the action, with

\[
S_0[\bar{\psi}, \psi] = \sum_{i, \omega} \left[ (-i\omega - \hbar) \bar{\psi}_i(\omega) \psi_i(\omega) - \frac{1}{2} \left( \langle \bar{\psi}_i(\omega) \psi_{i+1}(\omega) \rangle - \psi_i(\omega) \bar{\psi}_{i+1}(\omega) \right) - \frac{1}{2} \gamma \left( \langle \bar{\psi}_i(\omega) \psi_{i+1}(\omega) \rangle - \psi_i(\omega) \psi_{i+1}(\omega) \right) \right]
\]  

(17)

and

\[
AS[\bar{\psi}, \psi] = - \sum_{i, \omega} \Lambda h_{L/R} \bar{\psi}_i(\omega) \psi_i(\omega).
\]  

(18)

From the Green’s function in absence of defects, we introduce the function

\[
\chi(\omega, r) = \int \frac{dq}{2\pi} e^{iqr} \chi_q(\omega)
\]  

(19)

with the Fourier transform

\[
\chi_q(\omega) = \frac{1}{-i\omega + E'_q}
\]  

(20)

\( E'_q \) is the renormalized form of the energy spectrum equation (5) given by

\[
E'_q = \varphi(q) \frac{\beta_1}{\beta} E_q + \varphi(-q) \frac{\beta_2}{\beta} E_q
\]  

(21)

Where \( \beta = \frac{\beta_1 + \beta_2}{2} \) is the temperature of the central region. As far as transport properties are concerned, all relevant processes take place near the Fermi points. The consequence of this is that only relevant Fourier components are those with small wave vectors and wave vectors of the order of Fermi wave vector. We shall take the approximation of relativistic fermions moving with velocity

\[ v = \frac{\beta E}{\beta E} \]
respectively in the left and the right part of the chain.

In the vicinity of the Fermi level the Fermi fields exist as wave packets with average wave vectors $\pm q_F$, then, $\chi(\mathbf{r}, \omega)$ can be expressed as follows

$$
\chi_{\mathbf{r}}(\omega) = \chi_{L}(\omega, \mathbf{r}) \exp \left\{ i q_F \mathbf{r} \right\} + \chi_{R}(\omega, \mathbf{r}) \exp \left\{-i q_F \mathbf{r} \right\}
$$

where

$$
\chi_{L}(\omega, \mathbf{r}) = i \frac{\exp \left\{ -\frac{i\omega}{\alpha} \right\}}{c_L} \left[ \varphi(r) \varphi(\omega) - \varphi(-r) \varphi(-\omega) \right],
$$

$$
\chi_{R}(\omega, \mathbf{r}) = i \frac{\exp \left\{ -\frac{i\omega}{\alpha} \right\}}{c_R} \left[ \varphi(-r) \varphi(\omega) - \varphi(r) \varphi(-\omega) \right].
$$

The expression of the force experienced at one defect site is obtained over the Matsubara frequency as:

$$
F = \frac{2 \Delta h_L \Delta h_R}{\beta} \left[ 2 q_F \sin(2 q_F d) d \sum_{\omega} \chi_{L}(\omega, d) \chi_{R}(\omega, -d) + \cos(2 q_F d) d \sum_{\omega} \varrho \chi_{L}(\omega, d) \chi_{R}(\omega, -d) \right]
$$

with $\varrho = \omega \left( \frac{1}{2} + \frac{1}{\beta} \right)$.

This expression of the force is $\gamma$-dependant through the velocities $c_L$ and $c_R$. The specific case of previous founds [11] emerges by using the limit $\gamma \to 0$ for which the velocity of moving fermions tend to $\sin q$. Since our interest is for the current-carrying chain, the graphs are plotted for different values of the temperature gradient.

The curves showing the plots of the oscillating Casimir force, considering $\beta_1 \to \infty$ are presented in figures 3 and 4. These curves show an enharmonic decreasing of the Casimir force as the elongation increases. In figure 4 where $\gamma \to 1$, the oscillating force compared to the case $\gamma \to 0$ corresponding to the XX chain (figure 3) drastically decreases with the elongation $\phi = q_F d$ taken in the unit of the lattice parameter $a$ (a have been set equal to 1). As expected, the decay is exponential with a characteristic length $\frac{\beta \omega}{\pi \beta}$ and agree with previously found [11]. However, in both cases, for low temperatures, the decay is algebraic.

More recently, the transverse field in a quantum XY spin system has been used to investigate transition effect [27] where the quantum phase transition is shown to be a sudden change occurring in the ground states of many-body systems when one or more of the physical parameters of the system (such as anisotropic...
parameter, transverse magnetic field) are continuously varied at absolute zero temperature. Following then the non-equilibrium anisotropic XY spin chain driven by a weak homogeneous transverse magnetic field, we made a drawing closer between the problem of Casimir and that of the quantum phase transitions in chains.

Figures 5, 6 shows the Casimir force $|F|$ recorded with the anisotropic parameter $\gamma$ for various values of $\beta_2$. The results in figure 5 shows a significant increase of the force for low temperatures. When the temperature increases, the energy current increases for high values of anisotropy (figure 2) and accordingly decrease the Casimir force for the corresponding values as shown in red and blue curves of figure 5. The difference between low and high temperatures behaviors of the energy current according to the anisotropy is then recorded on the anisotropy dependence of the Casimir force. By pointing up the role played by the anisotropy, this also confirms the general picture [11, 28, 29] that nonequilibrium fluctuations lead to stiffer systems. This symmetry is broken when the distance between impurities increases significantly (figure 6).

On the other hand, the temperature dependence of the Casimir force (figure 7) show a critical point above which the order of curves according to different values of anisotropy is inverted. This characterizes a change in state of the system and following the accepted terminology, is allowed to a phase transition.

5. Concluding remarks

The effect of anisotropy factor on the energy-flux driven non-equilibrium quantum XY spin chain has been investigated. It pointed out that cooling the system as well as increasing the anisotropy factor, reduce
considerably the range of the current. On the other hand, the investigation of the resulting Casimir force between two pointlike defects and the study of its dependence on the anisotropy shows different behaviors of the interplay between anisotropy and temperature (high and low) in the chain. In fact, it clearly emerges according to the previous found [11] how the quantum fluctuations are challenged by the energy flux. However, this picture is no longer valid when the distance between defects increases. It has also been recorded a critical temperature according to anisotropy factor which exhibits a phase transition. An interesting feature of this model is that turning the anisotropic parameter as well as the temperature range will better manipulate the energy flux and control the range of the Casimir force.

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![Figure 6. Anisotropy dependence of the Casimir force for a distance $d = 10$ and for different values of the Boltzmann temperature $\beta_2$.](image1)

![Figure 7. Temperature dependence of the Casimir force for different values of the anisotropy factor $\gamma$.](image2)
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