On super Catalan polynomials

Emily Allen
Irina Gheorghiciuc

Abstract

We present two $q$-analogs of the super Catalan numbers, which also generalize Carlitz’s $q$-Catalan numbers $c_n(\lambda)$ for $\lambda = 0$ and $\lambda = 1$. We give a combinatorial interpretation for one of these analogs when $m = 2$.

1 Introduction

In [6] Gessel reintroduces the integers

$$S(m, n) = \frac{\binom{2m}{m} \binom{2n}{n}}{\binom{m+n}{n}} = \frac{(2m)!(2n)!}{m!n!(m+n)!},$$

which were studied by Eugene Catalan in 1874 [4]. When $m, n > 0$, the numbers $S(m, n)$ are even. Gessel refers to $T(m, n) = S(m, n)/2$ as the super Catalan numbers. An interpretation of $T(2, n)$ in terms of pairs of Dyck paths with restricted heights has been found by Gessel and Xin [7].

We will use the standard $q$-notation

$$[r]_q = 1 + q + \cdots + q^{r-1} \quad \text{and} \quad [n]_q! = \prod_{r=1}^{n} [r]_q.$$
The polynomials
\[ S_q(m, n) = \frac{[2m]!_q [2n]!_q}{[m]!_q [n]!_q [m+n]!_q} \]
have been studied by Warnaar and Zudilin [8], and by Allen [1]. Warnaar and Zudilin proved that \( S_q(m, n) \) are polynomials with nonnegative integer coefficients [8]. Allen conjectured that \( S_q(m, n) \) are unimodal [1].

Let \( S(m, n) \) denote the set of lattice paths that begin at the origin, have \( m \) up steps (drawn by ascending edges) and \( n \) down steps (drawn by descending edges). Let \( S_+(m, n) \) denote the subset of those paths which never go below the \( x \)-axis. We will refer to \( S_+(n, n) = C_n \) as the set of Catalan paths of length \( 2n \). The height of \( \pi \in S(m, n) \), which is the maximum level \( \pi \) reaches, will be denoted by \( h(\pi) \). A path of length \( \ell \) can be represented as a sequence of zeros and ones, \( \pi = \pi_1 \cdots \pi_\ell \), where zeros represent up steps and ones represent down steps.

For a lattice path \( \pi \) we define the descent set \( D(\pi) \), the major index \( maj(\pi) \), and the descent index \( des(\pi) \) to be
\[
D(\pi) = \{ i : \pi_i > \pi_{i+1}, 1 \leq i \leq \ell - 1 \}, \\
maj(\pi) = \sum_{i \in D(\pi)} i, \\
des(\pi) = |D(\pi)|.
\]

A combinatorial interpretation of \( S_q(0, n) \) is given by Andrews [3]
\[
S_q(0, n) = \left[ \frac{2^m}{n} \right]_q = \sum_{\pi \in S(n, n)} q^{maj(\pi)}.
\]

Allen showed that \( T_q(m, n) = S_q(m, n)/(1 + q^n) \) and \( U_q(m, n) = S_q(m, n)/(1 + q^m) \) are polynomials [1]. In fact
\[
T_q(1, n) = \sum_{\pi \in S_+(n, n)} q^{maj(\pi) - des(\pi)} \quad \text{and} \quad U_q(1, n) = \sum_{\pi \in S_+(n, n)} q^{maj(\pi)}
\]
are respectively Carlitz’s \( q \)-Catalan numbers \( c_n(0) \) and \( c_n(1) \) [5].

The super Catalan numbers satisfy the following identity, attributed to Dan Rubenstein [6]
\[
4T(m, n) = T(m + 1, n) + T(m, n + 1).
\]

(1)
The following $q$-analog of this identity holds

$$
(1 + q^n)(1 + q^{n-m})T_q(m,n) = q^{n-m}T_q(n,m+1) + T_q(m,n+1).
$$

(2)

In Section 2 we provide several results on $q$-analog Ballot Numbers. In Section 3 we expand on our methods in [2] to give a combinatorial interpretation of $T_q(2,n)$.

## 2 A $q$-analog Ballot Number

Let $\mathcal{B}(n, r)$ denote the set of paths of length $2n$ which begin at the origin with an up step, end at $(2n, -2r + 2)$, and never go below the line $y = -2r + 2$. In particular $\mathcal{B}(n, 1) = \mathcal{C}_n$, the set of Catalan paths of length $2n$.

Define the $q$-analog Ballot Number

$$
B_q(n, r) = \frac{[2n-1]_q [2r]_q}{[n+r]_q ![n-r]_q} = \frac{1}{q^{n-r}} \left( \left[ \frac{2n-1}{n+r-1} \right]_q - \left[ \frac{2n-1}{n+r} \right]_q \right).
$$

**Lemma 1.** Let $\pi \in \mathcal{S}(m, n)$ ending with an up step. Reflecting $\pi$ over the x-axis gives a path $\rho \in \mathcal{S}(n, m)$ ending with a down step which satisfies $\text{maj}(\pi) = \text{maj}(\rho) + n$.

**Proof.** Given a path $\pi \in \mathcal{S}(m, n)$ ending with an up step, let $D(\pi) = \{X_1, \ldots, X_\ell\}$. We let $X_0 = 0$. Define $u_i$ and $d_i$ to be the number of up and down steps, respectively, between indices $X_i$ and $X_{i+1}$. Let $\rho$ be the reflection of $\pi$ across the x-axis. Then the descents of $\rho$ occur exactly at indices $X_i + u_i$ for $i < \ell$. Hence,

$$
\text{maj}(\rho) = \sum_{i=0}^{\ell-1} (X_i + u_i) = \text{maj}(\pi) + m - (X_\ell + u_\ell) = \text{maj}(\pi) + m - (n + m) = \text{maj}(\pi) - n.
$$

**Theorem 1.**

$$
B_q(n, r) = \sum_{\pi \in \mathcal{S}(n, r)} q^{\text{maj}(\pi) - \text{des}(\pi)}
$$

(3)

**Proof.** Let $\mathcal{S}_\geq$ denote the set of paths in $\mathcal{S}(n+r-1, n-r)$ which have height strictly greater than $2r - 1$, and let $\mathcal{S}_\leq$ denote the set of paths in $\mathcal{S}(n+r-1, n-r)$ which never go above $y = 2r - 1$. 

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We will define a bijection \( \psi : S_\rightarrow \to S(n+r, n-r-1) \) which preserves the major index. Given a path \( \pi \in S_\rightarrow \), let \( R \) be the right-most highest point on \( \pi \). Since \( \pi \) has height strictly greater than \( 2r-1 \) and ends at level \( 2r-1 \), the point \( R \) is not the last point on \( \pi \). Let \( RL \) be the \( \text{down} \) step following \( R \). Define \( \psi(\pi) \) to be the path obtained from \( \pi \) by changing the \( \text{down} \) step \( RL \) into an \( \text{up} \) step. Note that \( \psi(\pi) \in S(n+r, n-r-1) \) and \( L \) is the left-most highest point on \( \psi(\pi) \). To see that \( \psi \) is a bijection from \( S_\rightarrow \) to \( S(n+r, n-r-1) \), given a path \( \rho \) in \( S(n+r, n-r-1) \), locate the left-most highest point \( L \) on \( \rho \) and change the \( \text{up} \) step preceding it into a \( \text{down} \) step to obtain \( \pi \). Since \( \rho \) has height at least \( 2r+1 \), the path \( \pi \) will have height at least \( 2r \). Therefore \( \pi \in S_\rightarrow \) and \( \psi(\pi) = \rho \). The bijection \( \psi \) preserves the major index because the descent sets of \( \pi \) and \( \psi(\pi) \) are the same. It follows that

\[
\sum_{\pi \in S_\rightarrow} q^{\text{maj}(\pi)} - \sum_{\pi \in S(n+r, n-r-1)} q^{\text{maj}(\pi)} = \sum_{\pi \in S_\rightarrow} q^{\text{maj}(\pi)}.
\]

We will define a bijection \( \varphi : S_\leq \to \mathcal{B}(n, r) \). Let \( \pi \in S_\leq \). Define \( \varphi(\pi) \) to be the path obtained from \( \pi \) by reflecting \( \pi \) across the \( x \)-axis and then adding an \( \text{up} \) step to the beginning of the path. This is clearly a bijection from \( S_\leq \) to \( \mathcal{B}(n, r) \). By Lemma 1, reflecting \( \pi \) across the \( x \)-axis causes the major index to decrease by \( n-r \). Adding an \( \text{up} \) step to the beginning of the path increases all descents by 1, hence the major index of the reflection of \( \pi \) equals the major minus descent index of \( \varphi(\pi) \). It follows that,

\[
\frac{1}{q^{n-r}} \left[ \begin{array}{c} 2n-1 \\ n+r-1 \end{array} \right]_q - \left[ \begin{array}{c} 2n-1 \\ n+r \end{array} \right]_q = \sum_{\pi \in S_\leq} q^{\text{maj}(\pi)-(n-r)} = \sum_{\pi \in \mathcal{B}(n, r)} q^{\text{maj}(\pi)-\text{des}(\pi)}.
\]

\( \Box \)

3 Combinatorial Interpretation

The following identity is the \( q \)-analog of Eq. 2 in [2] when \( m=2 \).

\[
q^{n-1}T_q(2, n) = (1 + q^2)B_q(n, 1) - B_q(n, 2)
\]

(4)

For a path \( \pi \in \mathcal{C}_n \), let \( X \) be the last, from left to right, level one point up to and including
the right-most maximum \( R \) on \( \pi \). Let \( h_-(\pi) \) denote the maximum level that the path \( \pi \) reaches from its beginning until and including point \( X \), and \( h_+(\pi) \) denote the maximum level that the path \( \pi \) reaches after and including point \( X \). Obviously \( h_-(\pi) \leq h_+(\pi) = h(\pi) \).

Let \( \Omega_n \) denote the set of \( \pi \in C_n \) such that \( h_+^{\pi} \leq h_-^{\pi} + 2 \).

\textbf{Theorem 2.}

\[
T_q(2, n) = q^{n-1} + q^{3-n} \sum_{\pi \in \Omega_n} q^{maj(\pi) - des(\pi)}. \tag{5}
\]

\textbf{Proof.} By \( B^*(n, 2) \) we denote the set of paths in \( B(n, 2) \) which do not attain level \( y = -1 \) before their right-most maximum. Let \( B^{**}(n, 2) = B(n, 2) - B^*(n, 2) \) and

\[
B^*_q(n, 2) = \sum_{\pi \in B^*(n, 2)} q^{maj(\pi) - des(\pi)}; \quad B^{**}_q(n, 2) = \sum_{\pi \in B^{**}(n, 2)} q^{maj(\pi) - des(\pi)}.
\]

By Theorem 1 and Eq. 4

\[
q^{n-1}T_q(2, n) = (B_q(n, 1) - B^*_q(n, 2)) + (q^2B_q(n, 1) - B^{**}_q(n, 2)).
\]

First we compute \( B_q(n, 1) - B^*_q(n, 2) \). For \( \pi \in B^*(n, 2) \), let \( RQ \) be the down step that follows the right-most maximum point \( R \) of \( \pi \). We define \( f(\pi) \) to be the path obtained by substituting the down step \( RQ \) by an up step. See Figure 1. Note that \( f(\pi) \in C_n \) and, since at least two up steps precede \( Q \) on \( f(\pi) \), the height of \( f(\pi) \) is at least two. Also \( \pi \) and \( f(\pi) \) have the same set of descents, thus \( des(\pi) = des(f(\pi)) \) and \( maj(\pi) = maj(f(\pi)) \). It is important to mention that \( Q \) is the left-most maximum on \( f(\pi) \).

![Figure 1: f substitutes the down step RQ by an up step](image1)

We will show that \( f \) is a bijection between \( B^*(n, 2) \) and the set of paths \( \rho \) in \( C_n \) of height \( h(\rho) > 1 \). Let \( Q \) be the left-most maximum on \( \rho \) and \( RQ \) be the up step that precedes \( Q \).
Substitute the up step $RQ$ by a down step, which makes $R$ the right-most maximum of the resulting path, call it $\pi$. Note that $\pi \in \mathcal{B}^*(n, 2)$ and $f(\pi) = \rho$.

It follows that
\[
B_q(n, 1) - B_q^*(n, 2) = \sum_{\pi \in C_n \atop h(\pi) = 1} q^{\text{maj}(\pi) - \text{des}(\pi)} = q^{n-1}.
\]

We define a descent point to be a point on a path which is preceded by a down step, and followed by an up step. We define a down wedge sequence to be a portion of a path that starts with a down step, alternates between down steps and up steps, and ends with an up step. See Figure 2.

![Figure 2: A down wedge sequence](image)

We find a combinatorial interpretation for $q^2 B_q(n, 1) - B_q^{**}(n, 2)$ by establishing an injection $g$ from $\mathcal{B}^{**}(n, 2)$ to $C_n$. A path $\pi$ in $\mathcal{B}^{**}(n, 2)$ attains $y = -1$ before its right-most maximum point $R$. Let $N$ be the first point before $R$ at which $\pi$ attains $y = -1$. We consider two cases: when $N$ is immediately followed by a down step, and when $N$ is immediately followed by an up step.

**Case one: $N$ is immediately followed by a down step.** Since $N$ is the left-most point on $\pi$ on level $y = -1$, $N$ is preceded by two down steps and is followed by one down step, then one up step. See Figure 3. Let $MN$ be the down step that precedes $N$ and $NY$ be the down step that follows $N$. Substitute $MY$ by two up steps. The resulting path is a ballot path of length $2n$ that ends at level two. Rename $N$ to be $X$. From left to right, $X$ is the last level one point on this ballot path. The maximum level that this ballot path reaches up to and including point $X$ is less than the maximum level it reaches after and including point $X$ by at least 4.

Let $L$ be the left-most maximum point of this ballot path and $QL$ be the up step that precedes $L$. Substitute the up step $QL$ by a down step. See Figure 3. The resulting path $g(\pi)$ is in $C_n$ and $Q$ is its right-most maximum. Point $X$ is the last level one point on $g(\pi)$ before its right-most maximum $Q$ and the point on $g(\pi)$ before $X$ is a decent. Note that $h_+(g(\pi)) \geq h_-(g(\pi)) + 3$. Also $\text{des}(\pi) = \text{des}(g(\pi))$ and $\text{maj}(\pi) = \text{maj}(g(\pi)) + 2$. Thus $\text{maj}(\pi) - \text{des}(\pi) = \text{maj}(g(\pi)) - \text{des}(g(\pi)) + 2$. 

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Case two: \( N \) is immediately followed by an up step. Since \( N \) is the left-most point on \( \pi \) on level \( y = -1 \), \( N \) is preceded by two down steps which form a segment we denote by \( XN \). See Figure 4. Let \( \sigma \) be the longest, possibly empty, down wedge sequence that precedes \( X \). Let \( Y \) be the first point of the sequence \( \sigma \). Note that either \( Y \) is the second point on \( \pi \) or \( Y \) is preceded by a down step. Remove \( \sigma \) from its original position and insert it immediately after \( N \). Then substitute \( XN \) by two up steps. The resulting path is a ballot path of length \( 2n \) that ends at level two. From left to right, \( X \) is the last level one point on this ballot path. The maximum level that this ballot path reaches up to and including point \( X \) is less than the maximum level it reaches after and including point \( X \) by at least 4.

Let \( L \) be the left-most maximum point of this ballot path and \( QL \) be the up step that precedes \( L \). Substitute the up step \( QL \) by a down step. See Figure 4. The resulting path \( g(\pi) \) is in \( C_n \) and \( Q \) is its right-most maximum. Note that \( X \) is the last level one point on \( g(\pi) \) before its right-most maximum \( Q \) and the point on \( g(\pi) \) before \( X \) is NOT a decent. Also \( h_+(g(\pi)) \geq h_-(g(\pi)) + 3 \). If \( Y \) is the second point on the original path \( \pi \), then \( g \) removes the original decent of \( \pi \) that occurs immediately after \( Y \) and moves the decent that originally corresponds to \( N \) one unit to the left. Thus \( \text{des}(\pi) = \text{des}(g(\pi)) + 1 \) and \( \text{maj}(\pi) = \text{maj}(g(\pi)) + 3 \). If \( Y \) is NOT the second point on the original path \( \pi \) and \( \sigma \) is not empty, then \( g \) moves the decent that originally occurs immediately after \( Y \) and the one that corresponds to \( N \) one unit to the left. If \( Y \) is NOT the second point on the original path \( \pi \) and \( \sigma \) is empty, then \( g \) moves the decent that corresponds to \( N \) two
units to the left. Thus $$\text{des}(\pi) = \text{des}(g(\pi))$$ and $$\text{maj}(\pi) = \text{maj}(g(\pi)) + 2$$. In all the cases $$\text{maj}(\pi) - \text{des}(\pi) = \text{maj}(g(\pi)) - \text{des}(g(\pi)) + 2$$.

If a path $$\rho$$ is in the image of $$g$$, then $$h_+(\rho) \geq h_-(\rho) + 3$$, thus $$\rho \in C_n - \Omega_n$$. We will show that the image of $$g$$ is $$C_n - \Omega_n$$. Let $$\rho$$ be in $$C_n$$ and $$h_+(\rho) \geq h_-(\rho) + 3$$. Let $$Q$$ be the right-most maximum on $$\rho$$ and $$QL$$ be the down step that follows $$Q$$. Substitute the down step $$QL$$ by an up step. The result is a ballot path of length $$2n$$ that ends at level two. Note that $$L$$ is the left-most maximum on this ballot path. Let $$R$$ denote the right-most maximum on this ballot path. From left to right, let $$X$$ be the last level one point on this ballot path. The maximum level that this ballot path reaches up to and including point $$X$$ is less than the maximum level it reaches after and including point $$X$$ by at least 4. We consider two cases: when the point before $$X$$ is a decent, and when the point before $$X$$ is not a decent.

If the point before $$X$$ is a decent, let $$M$$ be that decent. Let $$Y$$ be the point that follows $$X$$. Since $$X$$ is the last point on level one before $$R$$, $$XY$$ is an up step. Substitute $$MY$$ with two down steps. We will call the resulting path $$\pi$$. Note that $$\pi$$ attains level $$y = -1$$ before its right-most maximum $$R$$ and, immediately after attaining level $$y = -1$$ for the first time, it attains level $$y = -2$$. Thus $$\pi$$ is in $$B^{**}(n, 2)$$, falls into Case 1, and $$g(\pi) = \rho$$.

Next we consider the case when the point before $$X$$ is NOT a decent. Note that, since $$X$$ is the last level one point before $$R$$, $$X$$ is followed by two up steps. Let $$XY$$ be the segment that consists of these two up steps. Let $$\sigma$$ be the longest, possibly empty, down wedge sequence that starts at $$Y$$. Note that $$\sigma$$ is followed by an up step. Remove $$\sigma$$ from its original position and insert it immediately before $$X$$, then substitute $$XY$$ with two down steps. We will call
the resulting path $\pi$. Note that $\pi$ attains level $y = -1$ for the first time at $Y$, before its right-most maximum $R$, and $Y$ is followed by an up step. Thus $\pi$ is in $B^{**}(n, 2)$, falls into Case 2, and $g(\pi) = \rho$.

It follows that

$$q^2 B_q(n, 1) - B_q^{**}(n, 2) = q^2 \sum_{\pi \in \Omega_n} q^{\text{maj}(\pi) - \text{des}(\pi)}.$$

\[\square\]

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