On a Conjecture of Lovász on Circle-Representations of Simple 4-Regular Planar Graphs∗

Michael A. Bekos1, Chrysanthi N. Raftopoulou2

1Institute for Informatics, University of Tübingen, Tübingen, Germany
bekos@informatik.uni-tuebingen.de

2School of Applied Mathematical & Physical Sciences, National Technical University of Athens, Greece
crisraft@mail.ntua.gr

Abstract

Lovász conjectured that every connected 4-regular planar graph $G$ admits a realization as a system of circles, i.e., it can be drawn on the plane utilizing a set of circles, such that the vertices of $G$ correspond to the intersection and touching points of the circles and the edges of $G$ are the arc segments among pairs of intersection and touching points of the circles. In this paper, we settle this conjecture. In particular, (a) we first provide tight upper and lower bounds on the number of circles needed in a realization of any simple 4-regular planar graph, (b) we affirmatively answer Lovász’s conjecture, if $G$ is 3-connected, and (c) we demonstrate an infinite class of simple connected 4-regular planar graphs which are not 3-connected (i.e., either simply connected or biconnected) and do not admit realizations as a system of circles.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. Given a graph $G$, we denote by $V[G]$ and $E[G]$ the set of vertices and edges of $G$, respectively. If $G$ is regular, we denote by $d(G)$ its degree.

Definition 1. Let $G$ be a connected 4-regular planar graph. We say that $G$ admits a realization as a system of circles, if it can be drawn on the plane using a set of circles such that (see Figures 1b-1d):

1. The vertex set $V[G]$ is given by the intersection and touching points of the circles.
2. The edge set $E[G]$ is defined by all circular arcs between the intersection and touching points of the circles.

In the special case where intersection points are not allowed (i.e., there are only touching circles), we say that $G$ admits a realization as a system of touching circles (see Figures 1b and 1c).

∗This draft is an update of [1], where we fix a typo in the statement of Lemma 12.
Lovász [8, pp.1175],[12, pp.426] conjectured that every simple connected 4-regular planar graph admits a realization as a system of circles. To the best of our knowledge this conjecture remained unanswered. Touching points are necessary, since if we use only crossings, we have an even number of vertices, but there are 4-regular planar graphs with an odd number of vertices [15].

In this paper, we prove that every 3-connected 4-regular planar graph admits a realization as a system of touching circles. If the input graph is not 3-connected, we demonstrate by an example that a realization as a system of circles is not always possible.

This paper is structured as follows: Section 2 overviews previous work related to this paper. In Section 3, we present bounds on the number of circles needed in a realization of a simple connected 4-regular planar graph as a system of circles. In Section 4, we prove that every 3-connected 4-regular planar graph admits a realization as a system of touching circles. In Sections 5 and 6, we prove that there exist infinitely many (either simply connected or biconnected) graphs that do not admit realizations as system of circles. We conclude in Section 7 with open problems and future work.

2 Related Work

Closely related to the problem we study is the contact graph representation problem. A contact graph is a graph whose vertices are represented by geometric objects and whose edges correspond to two objects touching in some specific predefined way. There is a rich literature related to various types of contact graphs, dating back to 1936 in Koebe’s theorem [11] which states that any planar graph can be represented as a contact graph of disks in the plane. Typical classes of such objects are curves, line segments, disks, triangles and rectangles (cf. [9, 10]). Note that Koebe’s theorem, which is also known as circle-packing theorem, is the starting point for the proof of Section 4. The main difference between the problem of representing a graph as a contact graph of disks and the problem we study is that in the former problem the vertices correspond to disks and the edges are implied by the contacts; in our problem, however, the vertices are the crossing and/or touching points of the disks and the edges are the arc-segments defined between them.

Lombardi drawings, which attempt to capture some of the visual aesthetics used by the American artist Mark Lombardi, are also closely related to our problem. Two features that stand out in Lombardi’s work are the use of circular-arc edges and their even distribution around vertices. Such even spacing of edges around each vertex (also known as perfect angular resolution) together with the usage of circular arc edges, formally de-
fine Lombardi drawings of graphs [6, 7]. Chernobelskiy et al. [5] relax the perfect angular resolution constraint in Lombardi drawings and describe functional force-directed algorithms, which produce aesthetically appealing near-Lombardi drawings.

Connected 4-regular planar graphs form a well-studied class of graphs. Manca [15] proposed four operations to generate all connected 4-regular planar graphs from the octahedron graph. As noted by Lehel [12], Manca’s construction could not generate all connected 4-regular planar graphs, however, an additional operation could fix this problem. Broersma et al [4] showed that all 3-connected 4-regular planar graphs can also be generated from the octahedron, using only three operations.

In the context of graph drawing, 4-regular planar graphs (which can be viewed as a special case of max-degree 4 planar graphs) have a long tradition, dating back to VLSI layouts and floor-planning applications. The main goal in this context is to produce drawings (referred to as orthogonal drawings) in which each vertex corresponds to a point on the integer grid and each edge is represented by a sequence of horizontal and vertical line segments. Pioneering work on orthogonal drawings was done in relation to VLSI-design by Valiant [18], Leiserson [14] and Leighton [13]. Later on the problem of constructing orthogonal drawings of maximum degree 4 planar graphs was considered by Tamassia [16], Tamassia and Tollis [17], and Biedl and Kant [2]. The objectives here have been the minimization of the used area, the total edge length, the total number of bends, the maximum number of bends per edge, and others.

3 Bounds on the Number of Circles Needed in a Circle Representation of a Simple 4-regular Planar Graph

In this section, we present upper and lower bounds on the number of circles needed in a realization of a simple connected 4-regular planar graph as a system of circles. We also prove that these bounds are tight, i.e., there exist infinitely many connected 4-regular planar graphs that admit realizations as system of circles and use the number of circles given by the two bounds.

Lemma 1. Let $G$ be a simple connected 4-regular planar graph on $n$ vertices. Then, the number of circles, say $c[G]$, that participate in a realization of $G$ as system of circles, if one exists, satisfies the following inequality:

$$(1 + \sqrt{1 + 4n})/2 \leq c[G] \leq 2n/3$$

Proof. In general, there are certain restrictions concerning the number of circles that participate in a realization of a graph as a system of circles:

- **Two circles may have at most two vertices in common**: Two crossing points if they intersect, one touching point if they are tangent, or none if they are separated.

- **There exist at least three vertices on every circle**, since we consider only simple graphs.

- **Every vertex belongs to exactly two circles**, since every vertex has degree 4.

From the latter two properties, it follows that in any realization of $G$ as a system of circles, the number of vertices of $G$ defined by all circles is at least $3c[G]/2$, which
immediately implies the desired upper bound, i.e., $c[G] \leq 2n/3$. On the other hand, if every pair of circles defines exactly two vertices (i.e., every pair of circles intersects), the corresponding realization of $G$ has the minimum number of circles. However, in this case a total of $c[G](c[G] - 1)/2$ pairs of circles define at most $c[G](c[G] - 1)$ vertices. Hence, it follows that $n \leq c[G](c[G] - 1)$ or equivalently that $(1 + \sqrt{1 + 4n})/2 \leq c[G]$. □

In the following, we prove that the bounds of Lemma 1 are tight.

Lemma 2. There exist infinitely many connected 4-regular $n$-vertex planar graphs that admit realizations as system of $(1 + \sqrt{1 + 4n})/2$ and $2n/3$ circles.

Proof. In order to prove this lemma, it is enough to show that there exist two classes of graphs that admit realizations as system of circles, in which (a) every pair of circles intersect and (b) every circle has exactly three vertices.

a. We aim to create a set of non-coincident circles all with the same radius and all containing the same interior point. More formally, let $c \geq 3$ be an arbitrary integer. Let also $C$ be an auxiliary geometric circle of radius $R > 0$ (refer to the dashed circle of Fig.2a). We proceed to draw $c$ circles of the same radius $r > 0$ centered at the vertices of a regular $c$-gon inscribed at circle $C$, such that $r > R$ (see Fig.2a). It is not difficult to see that every pair of circles intersects, since their centers are at distance less than $2r$. One can also appropriately choose $r$, so that no three circles pass through the same point. Since $c$ is arbitrarily chosen, the class of graphs derived by the circle representations corresponding to different values of integer $c$, $c \geq 3$, has obviously the property that its members admit realizations in which every pair of circles intersect.

b. In order to prove that there exist infinitely many graphs that admit realizations as system of circles, in which every circle has exactly three vertices, we follow a similar approach as in the previous case (refer to Fig.2b). Let $m > 2$ be an odd integer number. We proceed to draw a “chain of circles” consisting of $m$ circles of equal radius respectively, which touch with each other and also touch the interior of an “enclosing circle”, as illustrated in Fig.2b. The construction ensures that every circle has exactly three vertices. Hence, the class of graphs derived by the circle representations corresponding to different values of $m > 2$, has obviously the property that its members admit realizations in which every circle has exactly three vertices.

□

4 The case of 3-connected 4-regular planar graphs

We prove that a 3-connected 4-regular planar graph admits a realization as a system of touching circles. Our starting point is the circle packing theorem [11]. A circle packing is a “connected collection” of touching circles with disjoint interiors. The intersection graph (also known as tangency or contact graph) of a circle packing is the graph having a vertex for each circle and an edge for every pair of circles that are tangent. A graph that admits a realization as a system of touching circles is called coin graph (see Fig.3). Coin graphs are always simple, connected and planar. The circle packing theorem states the following.
Figure 2: Circle representations in which (a) every pair of circles intersects and (b) every circle has exactly three vertices.

**Theorem 1** (Circle packing theorem [11]). For every simple connected planar graph $G$, there is a circle packing in the plane with $G$ as its intersection graph.

![Figure 3](image)

**Figure 3**: (a) A planar graph. (b) Its representation as a system of touching circles.

We will use Theorem 1 on an auxiliary graph that can be constructed based on any 3-connected 4-regular planar graph.

It is well known that a connected planar graph is Eulerian if and only if its dual is bipartite [3, pp.172]. Let $G$ be an embedded 4-regular planar graph. Since $G$ is obviously Eulerian, its dual $G^*$ is bipartite. Hence, we can color the faces of $G$ using two colors, say gray and white, so that any two adjacent faces are of different colors. For convenience, we assume that the outer face of $G$ is always colored white. We proceed to construct a new graph $IL(G)$ as follows. We associate a vertex of $IL(G)$ with every gray face of $G$. We join two vertices of $IL(G)$ with an edge if and only if the corresponding faces of $G$ have at least one vertex in common (refer to the black colored graph of Fig.4a).

**Lemma 3.** If $G$ is a 3-connected 4-regular planar graph, then $IL(G)$ is simple.

**Proof.** Suppose that $G$ is 3-connected and assume for the sake of contradiction that $IL(G)$ is not simple. W.l.o.g., we further assume that $IL(G)$ contains a multiple edge, say a double edge between $f$ and $g$, where $f, g \in V[IL(G)]$ (see Fig.5a). The case where $IL(G)$ contains selfloops is treated similarly. By definition, $f$ and $g$ correspond to gray faces of $G$ that have exactly two common vertices, say $u, v \in V[G]$. Then, $u, f, v$ and $g$ define a separating simple closed curve which intersects $G$ at exactly two vertices (i.e.,
Figure 4: (a) Constructing graph $IL(G)$. (b) A realization of $G$ as system of circles.

Figure 5: (a) Vertices $u$, $f$, $v$ and $g$ define a separating simple closed curve. (b) If $G$ is biconnected, then $IL(G)$ is not necessarily simple (refer to bold edges).

vertices $u$ and $v$; see Fig. 5a). Note that since $G$ is simple there is at least one vertex of $G$ that lies in the interior of this curve and one in its exterior. Hence, $G$ is not 3-connected, which implies the claimed contradiction.

We are now ready to state the main theorem of this section.

**Theorem 2.** Every 3-connected 4-regular planar graph admits a realization as a system of touching circles.

**Proof.** By Lemma 3, $IL(G)$ is simple. So, we can apply Theorem 1 on it. This leads to a drawing in which each gray-colored face of $G$ corresponds to a circle and two circles meet at a point if and only if the corresponding faces are vertex-adjacent (see Fig. 4b).

By construction of $IL(G)$ we have that the vertices of $G$ are the points where circles touch. Also, every circle contains as arcs all the edges of the gray face it corresponds to. Since the gray-colored faces contain all edges of the graph, it follows that the constructed representation is indeed a system of touching circles for $G$.

5 The case of connected 4-regular planar graphs

In this section, we will demonstrate an infinite class of connected 4-regular planar graphs that do not admit a realization as a system of circles. The base of our constructive proof is the octahedron graph $G_{oct}$ (see Fig. 1a), which is a 3-connected 4-regular planar graph.
and hence, by Theorem 2, it admits a realization as a system of touching circles. Note that in the case where $G$ is not 3-connected, $ IL(G) $ is not necessarily simple (for an example refer to Fig.5b). Hence, Theorem 1 cannot be applied directly.

Assume now that a graph $ G $ admits a realization, say $ R $, as a system of circles. In general, $ R $ is not uniquely defined, as one can construct infinitely many realizations of $ G $ as a system of circles based on $ R $, e.g., by scaling $ R $ or by translating $ R $ over the plane. With the same spirit, if we slightly change the radii of the circles or even the centers of the circles of the realization of the octahedron graph depicted in Fig.1d, then we can obtain a new realization of the octahedron graph (again as a system of three mutually crossing circles), which will be more or less “equivalent” to the one depicted in Fig.1d. The same actually holds, if we simply change the triple of the vertices delimiting its outerface (as the octahedron graph is symmetric). Intuitively, two realizations $ R $ and $ R' $ of $ G $ are equivalent if there is a bijective function from the faces of $ R $ to the faces of $ R' $, which maps each face of $ R $ to a face of $ R' $ of the same shape, where the shape of a face is determined by the convexity of the arcs it consists of, i.e., towards or away from the interior of the face.

More formally, given a realization $ R $ of $ G $ as a system of circles, first we smooth out $ G $ by eliminating vertices of degree two, and then construct the dual, say $ G^*_R $, and orient its edges as follows: For an edge $ e \in E[G] $ incident to two faces $ f_e $ and $ f'_e $ of $ R $, edge $ (f_e, f'_e) \in E[G^*_R] $ is oriented from $ f_e $ to $ f'_e $ if and only if every straight-line segment with endpoints on the arc of $ R $ corresponding to $ e $ does not lie entirely in $ f'_e $. Otherwise, $ (f_e, f'_e) $ is directed from $ f'_e $ towards $ f_e $. Note that for any circle of $ R $, say $ c $, a face of $ G $ lies either in the interior of $ c $ or in its exterior. Then, orienting edge $ (f_e, f'_e) \in E[G^*_R] $ from $ f_e $ to $ f'_e $ implies that $ f_e $ lies in the interior of $ c $ and $ f'_e $ in its exterior, where $ c $ is a circle of $ R $ and $ e $ is an arc-segment of $ c $. We say that two realizations $ R $ and $ R' $ of $ G $ are equivalent if and only if $ G^*_R $ and $ G^*_{R'} $ are isomorphic, that is, there is a bijective function $ g : V[G^*_R] \rightarrow V[G^*_R'] $ such that $ (v, v') \in E[G^*_R] $ if and only if $ (g(v), g(v')) \in E[G^*_R'] $. Observe that in the aforementioned definition degree two vertices affect neither the set of circles that participate in the realization nor their relative positions. This is the reason why they are omitted. The following lemma describes all non-equivalent realizations of $ G_{oct} $ as a system of circles.

**Lemma 4.** The octahedron graph has exactly three non-equivalent realizations as a system of circles, which are shown in Figures 1b-1d.

**Proof.** Lemma 1 implies that any realization of the octahedron graph $ G_{oct} $ as a system of circles consists of either three or four circles. Easy considerations show that in the former case the three circles are mutually crossing, while in the latter one the four circles are mutually touching. Consider first the case of three mutually crossing circles (see Fig.1d), in which there are six vertices of degree four and every face is a triangle. Since $ G_{oct} $ is the only fully-triangulated 4-regular planar graph on six vertices, this is indeed a realization of $ G_{oct} $ as a system of circles. There is exactly one face of $ G_{oct} $ that belongs to the interior of all three circles. It follows that in $ G^*_R $ the corresponding vertex has out-degree three (refer to the innermost vertex of Fig.6a). Also, for every pair of circles, there is exactly one face that belongs to the interior of both circles and to the exterior of the third circle. The corresponding vertices of $ G^*_R $ have out-degree two and in-degree one (refer to vertices at distance one from the innermost vertex of Fig.6a). For every circle, there is exactly one face that belongs to its interior and to the exterior of the other two circles. The corresponding vertices of $ G^*_R $ have out-degree
one and in-degree two (refer to vertices at distance two from the innermost vertex of Fig.6a). Finally, the vertex corresponding to the outerface has in-degree three. From the above, it follows that the oriented dual $G^*_{oct}$ that corresponds to a realization of $G_{oct}$ as a system of three mutually crossing circles is isomorphic to the one given in Fig.6a, where the vertex corresponding to the outerface is omitted.

In the case of four touching circles, by Lemma 1 it follows that every circle has exactly three vertices. Since vertices are defined by touching points and two circles can have at most one common point, it follows that a circle touches all three other circles of the representation. Let $c_1, \ldots, c_4$ be the four circles. If $c_2$ lies in the interior of $c_1$, then $c_3$ and $c_4$ are also in the interior of $c_1$ (since they touch with $c_2$). This implies that either all circles have empty interiors or one circle contains all three other circles in its interior (see Figures 1c and 1b, respectively), in which there are six vertices of degree four defined, and every face is a triangle. Since $G_{oct}$ is the only fully-triangulated 4-regular planar graph on six vertices, these are indeed realizations of $G_{oct}$ as system of circles. For the case where all circles have empty interior, circles $c_1$ to $c_4$ are faces of $G_{oct}$ and the corresponding vertices of $G^*_{oct}$ have out-degree three (refer to the innermost vertex of Fig.6b and to vertices at distance two from it). All other faces of $G_{oct}$ lie in the exterior of all circles and have therefore in-degree three and out-degree zero (refer to vertices at distance one from the innermost vertex of Fig.6b; the outerface vertex also fits to this case). Therefore, the oriented dual $G^*_{oct}$ that corresponds to a realization of $G_{oct}$ as four touching circles with disjoint interiors is isomorphic to the one given in Fig.6b, where the vertex corresponding to the outerface is omitted.

Suppose now that one circle, say circle $c_1$, contains in its interior all three other circles. Then, circles $c_2, c_3$ and $c_4$ are faces of $G_{oct}$ and the corresponding vertices of $G^*_{oct}$ have out-degree three (refer to vertices at distance one from the innermost vertex of Fig.6c). There is exactly one face delimited by $c_2, c_3$ and $c_4$ (in the interior of $c_1$) that corresponds to a vertex of $G_{oct}$ with in-degree three (refer to the innermost vertex of Fig.6c). Three distinct faces share an edge with $c_1$ and lie in its interior corresponding to three vertices of $G^*_{oct}$ with in-degree two and out-degree one (refer to vertices at distance two from the innermost vertex of Fig.6c). The outerface corresponds to a vertex of in-degree three. Similarly to the previous cases, the oriented dual $G^*_{oct}$ that corresponds to a realization of $G_{oct}$ as four touching circles where one circle contains in its interior all three other circles, is isomorphic to the one given in Fig.6c, where the vertex corresponding to the outerface is omitted.
It is not hard to see that the three realizations presented are not equivalent: In the realization of Fig.6a there is only one vertex with out-degree three, while in the realization of Fig.6b there are four vertices with out-degree three, and in the realization of Fig.6c there are three. Since the degree sequences of the three dual graphs are different they can’t be isomorphic.

Any realization of the octahedron graph as a system of circles will use either three mutually crossing circles or four touching circles. Since we proved that the oriented dual in the former case is isomorphic to the one of Fig.6a, while in the latter case isomorphic either to the one of Fig.6b or to the one of Fig.6c, it follows that it is always isomorphic to one of the digraphs of Fig.6, giving a total of three non-equivalent realizations of $G_{\text{oct}}$ as a system of circles.

Initially, we will exhibit a specific connected 4-regular planar graph that does not admit a realization as a system of circles. This graph will be constructed based on $G_{\text{oct}}$, augmented by appropriately “attaching” a specific gadget-subgraph to its edges, leading thus to a graph, say $G_{\text{oct}}^{\text{aug}}$, that contains cut-vertices and separation pairs (note that any connected 4-regular planar graph is bridgeless [19, pp.34]). The gadget-subgraph is illustrated in Fig.7a. Observe that it contains exactly two vertices of degree two, namely $v_1$ and $v_2$, which are its endpoints. Now we replace every edge $e = (u,v)$ of $G_{\text{oct}}$ by a path consisting of 8 internal vertices. Clearly, the graph, say $G_{\text{oct}}^{\text{sub}}$, that is obtained in this manner is a subdivision of $G_{\text{oct}}$. Let $u \to z_1 \to z_2 \to \cdots \to z_8 \to v$ be the path replacing edge $(u,v)$. We associate four copies of the gadget-subgraph having vertices $z_1, \ldots, z_8$ as their endpoints: the first gadget-subgraph connects vertices $z_1$ and $z_3$, the second connects $z_2$ and $z_5$, the third connects $z_3$ and $z_8$, and the last connects $z_4$ and $z_7$ (see Fig.8a, in which the gadget-subgraphs are drawn with dashed curves joining the end-vertices; Fig.8b depicts the resulting graph $G_{\text{oct}}^{\text{aug}}$).

The skeleton of the gadget-subgraph consists of vertices $v_1$, $v_2$, $w_1$, $w_2$ and $w$ (see Fig.7) and edges $(w_i, v_i)$, $(w_i, w)$ and $(v_i, w_i)$, $i = 1, 2$. If we remove the edges of the skeleton, the remaining graph consists of three isolated vertices (namely $v_1$, $v_2$ and $w$) and two disjoint graphs that are subdivisions of the octahedron graph (refer to the gray-shaded graphs of Fig.7a). In this section we will exhibit some properties of the gadget-subgraphs. These properties are not actually due to the structure of the gadget-subgraphs. In fact, any graph in which every vertex has degree four except for exactly one degree-2 vertex on the outerface can be used instead of the gadget-subgraphs still guaranteeing the same properties. The general situation is shown in Fig.7b, where the subgraphs are drawn as self-loops at vertices $w_1$ and $w_2$. For convenience, we will refer to these subgraphs as loop-subgraphs.

Lemma 5. Let $G$ be a 4-regular planar graph that contains at least one copy of the
gadget-subgraph. Suppose that there is a realization of $G$ as a system of circles. Then, the skeleton of each gadget-subgraph in this realization consists of two circles $C_1$ and $C_2$ tangent at a point $w$, where circle $C_i$ contains vertices $\{v_i, w, w_i\}$ and the arc-segments realizing edges $(v_i, w_i)$, $(w_i, w)$, and $(v_i, w)$, for $i = 1, 2$.

**Proof.** Suppose that there is a realization of $G$ as a system of circles and consider a copy of the gadget-subgraph in this realization. Since every vertex is defined by exactly two circles and $w_i$ is a cut-vertex, it follows that one of the two circles defining $w_i$ contains vertices that belong only to its loop-subgraph, $i = 1, 2$. Hence, the edges $(v_i, w_i)$ and $(w_i, w)$ belong to the same circle, $i = 1, 2$. Let $C_i$ be the circle that contains $(v_i, w_i)$ and $C'_i$ the circle that contains $(v_i, w)$, $i = 1, 2$. We claim that $C_i = C'_i$, $i = 1, 2$. Observe that this implies the lemma. For the sake of contradiction, assume that $C_1 \neq C'_1$. Since vertex $w$ is defined by exactly two circles, we have that $\{C_1, C'_1\} = \{C_2, C'_2\}$, which also implies that $C_2 \neq C'_2$. Then, $C_1$ and $C'_1$ have at least three points in common, namely vertices $v_1, v_2$ and $w$, from which we obtain $C_1 = C'_1$; a contradiction. \qed

**Lemma 6.** Let $G$ be a 4-regular planar graph and $G^{\text{sub}}$ a subdivision of $G$. Let $v_1$ and $v_2$ be two subdivision vertices of $G^{\text{sub}}$, i.e. $v_1$ and $v_2$ are degree-2 vertices. Attach a gadget-subgraph, so that $v_1$ and $v_2$ are its endpoints, and such that the resulting graph is planar. Then, in any realization of the resulting graph as a system of circles, the realization of the gadget-subgraph and the realization of $G^{\text{sub}}$ are independent, i.e., any circle contains edges that belong exclusively either to the gadget-subgraph or to $G^{\text{sub}}$.

**Proof.** Refer to Fig. 9. Since the resulting graph is planar, $v_1$ and $v_2$ lie on the boundary of a face of $G^{\text{sub}}$. By Lemma 5, it follows that edges $(v_i, w_i)$, $(v_i, w)$ and $(w_i, w)$ belong to the same circle, say $C_i$, $i = 1, 2$. Let $u_j, u'_j$ be the two neighbors of vertex $v_j$ in $G^{\text{sub}}$, $j = 1, 2$ (Note that $u'_1 = v_2$ and $u'_2 = v_1$ are possible). Since every vertex belongs to exactly two circles, it follows that edges $(u_j, v_j)$ and $(v_j, u'_j)$ belong to a circle different from $C_1$ and $C_2$. Therefore, if we remove $C_1$ and $C_2$ and the circles representing the loop-subgraphs of the gadget-subgraph, we have a representation of the remaining graph (namely of graph $G^{\text{sub}}$), as a system of circles. \qed

From the above, it follows that in any realization of $G^{\text{aug}}_{\text{occl}}$ as a system of circles the realization of each gadget-subgraph and the realization of $G^{\text{sub}}_{\text{occl}}$ are independent. This is summarized in the following corollary.

![Figure 8: Each dashed edge corresponds to the gadget-subgraph of Fig. 7.](image)
Corollary 1. In any realization of $G_{\text{aug}}^{\text{oct}}$ as a system of circles, the realization of each gadget-subgraph and the realization of $G_{\text{sub}}^{\text{oct}}$ are independent.

Corollary 1 is the key element of our proof. It implies that the realization of $G_{\text{sub}}^{\text{oct}}$ obtained from a realization of $G_{\text{aug}}^{\text{oct}}$ by removing all vertices of the gadget-subgraphs except for their endpoints will be equivalent to one of the realizations of $G_{\text{oct}}$ depicted in Figures 1b-1d (recall that the definition of two equivalent realizations ignores vertices of degree-2).

In any planar embedding of $G_{\text{oct}}$, there is always a triangular face that shares no vertex and no edge with the outerface. Hence, the gadget-subgraphs attached to the edges of this triangular face have to be realized as in Fig.10a (in fact a realization as in Fig.10b is only possible if the gadget-subgraph is incident to the outerface). So, there is a total of six gadget-subgraphs attached along the edges of this triangular face, each with a realization as in Fig.10a.

In the following, we state two useful geometric results regarding tangent circles. We denote by $C(O, r)$ a circle with center $O$ and radius $r$.

Lemma 7. Let $C_1(O_1, r_1)$ and $C_2(O_2, r_2)$ be two circles, so that $C_1$ is tangent to $C_2$ at point $p_1$ and $C_2$ lies entirely in the interior of $C_1$. Let $C(O, r)$ be another circle that is tangent to $C_1$ at point $p_2$ ($p_2 \neq p_1$), tangent to $C_2$ and lies in the interior of $C_1$ (see Fig.11a). If $\phi$ is the angle $O_1p_1p_2$, then the radius $r$ of $C$ is an increasing function of
\[ \phi, \phi \in (0, \pi]. \]

**Proof.** W.l.o.g., we assume that \( O_1 \) coincides with the origin of the Cartesian coordinate system and point \( p_1 \) lies on the x-axis, i.e., at point \((r_1,0)\). Then, the center of circle \( C_2 \) is at point \((r_1-r_2,0)\), while the center of circle \( C_1 \) is at point \(((r_1-r) \cos \phi, (r_1-r) \sin \phi)\), as shown in Fig.11a. Since \( C_2 \) and \( C_1 \) are tangent the distance between their centers equals to the sum of their radii, i.e.:

\[
((r_1-r) \cos \phi - (r_1-r_2))^2 + ((r_1-r) \sin \phi)^2 = (r_2 + r)^2
\]

\[
\Rightarrow (r_1-r_2)^2 + (r_1-r)^2 - 2(r_1-r)(r_1-r) \cos \phi = (r_2 + r)^2
\]

\[
\Rightarrow (r_1 + r_2)(r_1-r) - 2r_1r_2 - (r_1-r)(r_1-r) \cos \phi = 0
\]

\[
\Rightarrow r = r_1 - \frac{r_1+r_2-(r_1-r_2) \cos \phi}{r_1-r}
\]

By the above equation, when \( \phi \) is increasing in the interval \((0, \pi]\), \( \cos \phi \) is decreasing and \( r \) is increasing. Hence, circle \( C_1 \) has maximum radius for angle \( \phi = \pi \).

A similar result holds if circles \( C \) and \( C_2 \) lie outside circle \( C_1 \).

**Lemma 8.** Let \( C_1(O_1, r_1) \) and \( C_2(O_2, r_2) \) be two circles, so that \( C_1 \) is tangent to \( C_2 \) at point \( p_1 \) and \( C_1 \) lies entirely in the exterior of \( C_2 \). Let \( C(O, r) \) be another circle that is tangent to \( C_1 \) at point \( p_2 \) \((p_2 \neq p_1)\), tangent to \( C_2 \) and lies in the exterior of \( C_1 \) (see Fig.11b). If \( \phi \) is the angle \( p_1O_1p_2 \), then the radius \( r \) of \( C \) is an increasing function of \( \phi \), \( \phi \in (0, \arccos(\frac{r_1-r_2}{r_1+r_2})] \).

**Proof.** The proof of Lemma 8 is similar to the one of Lemma 7. So, we omit the details. We simply mention the corresponding equation for \( r \), which is the following:

\[
r = \frac{2r_1r_2}{r_2 - r_1 + (r_2 + r_1) \cos \phi} - r_1
\]

Note that circle \( C \) does not always exist. For an example, refer to Fig.11b, when \( \phi = \pi/2 \) and \( r_1 > r_2 \). In particular, for given radii \( r_1 \) and \( r_2 \), angle \( \phi \) is bounded from above by value \( \arccos(\frac{r_1-r_2}{r_1+r_2}) \), which corresponds to the angle in the extreme case where circle \( C \) is of infinite radius and is therefore reduced to the common tangent of circles \( C_1 \) and \( C_2 \).

**Lemma 9.** Consider a circle \( C(O, r) \) and an arc \( \hat{AB} \) of \( C \) with \( \hat{AOB} = \phi < \pi \). Let \( C_1(O_1, r_1) \) and \( C_2(O_2, r_2) \) be two tangent circles, that are both tangent to \( C \) at points \( A \) and \( B \). Then, the radius of the circle \( C \) is given by:

\[
r = \frac{2r_1r_2}{r_2 - r_1 + (r_2 + r_1) \cos \phi} - r_1
\]

Note that circle \( C \) does not always exist. For an example, refer to Fig.11b, when \( \phi = \pi/2 \) and \( r_1 > r_2 \). In particular, for given radii \( r_1 \) and \( r_2 \), angle \( \phi \) is bounded from above by value \( \arccos(\frac{r_1-r_2}{r_1+r_2}) \), which corresponds to the angle in the extreme case where circle \( C \) is of infinite radius and is therefore reduced to the common tangent of circles \( C_1 \) and \( C_2 \).

**Proof.** The proof of Lemma 8 is similar to the one of Lemma 7. So, we omit the details. We simply mention the corresponding equation for \( r \), which is the following:

\[
r = \frac{2r_1r_2}{r_2 - r_1 + (r_2 + r_1) \cos \phi} - r_1
\]

Note that circle \( C \) does not always exist. For an example, refer to Fig.11b, when \( \phi = \pi/2 \) and \( r_1 > r_2 \). In particular, for given radii \( r_1 \) and \( r_2 \), angle \( \phi \) is bounded from above by value \( \arccos(\frac{r_1-r_2}{r_1+r_2}) \), which corresponds to the angle in the extreme case where circle \( C \) is of infinite radius and is therefore reduced to the common tangent of circles \( C_1 \) and \( C_2 \).
and B respectively (see Fig.12). Let \( C'_1(O'_1, r'_1) \) and \( C'_2(O'_2, r'_2) \) be another such pair of tangent circles that are both tangent to \( C \) at points \( A' \) and \( B' \) respectively (with \( A' \) and \( B' \) on the arc between \( A \) and \( B \)), so that the two pairs of circles have no crossing and no touching points. Assuming that \( C_i \) and \( C'_i \), \( i = 1, 2 \) are all in the interior of \( C \) or in the exterior of \( C \) and \( A', A', B', B' \), and \( B \) occur in this order on the arc between \( A \) and \( B \), then:

\[
|\hat{A'B'}| < |\hat{AA'}| \quad \text{and} \quad |\hat{A'B'}| < |\hat{B'B}|.
\]

**Proof.** Consider the circle \( C_1 \). There exist two circles \( C''_i \) for \( i = 1, 2 \) with radius \( r''_i \), that are both tangent to \( C_1 \) and also tangent to \( C \) at points \( A' \) and \( B' \) respectively. Note that for \( i = 1, 2 \) circle \( C''_i \) contains circle \( C'_i \), and so \( r''_i \geq r'_i \). We have that

\[
\hat{AOA'} < \hat{AOB'} < \hat{AOB}.
\]

and therefore, by Lemmas 7 and 8, \( r'_1 \leq r''_1 < r_2 \) for \( i = 1, 2 \). Similarly, starting from circle \( C_2 \) we have that \( r'_1 \leq r''_1 < r_1 \) for \( i = 1, 2 \). Now, for circle \( C'_1 \) we have that \( r'_2 < r_1 \) and \( \hat{AOA'}, \hat{AOB'} < \pi \). Then by Lemmas 7 and 8 it follows that \( \hat{AOB'} < \hat{AOA'} \). Similarly, we conclude that \( \hat{AOB'} < \hat{B'OB} \). So, we have that:

\[
|\hat{A'B'}| < |\hat{AA'}| \quad \text{and} \quad |\hat{A'B'}| < |\hat{B'B}|.
\]

\[\square\]

Note that Lemma 9 is still true when the four circles lie either in the interior or on the exterior of circle \( C \). Let \( e = (u, v) \) be an edge of the innermost interior face of the octahedron graph \( G_{oct} \), as shown in Fig.8b. Assume that in a realization of the octahedron as a system of circles, \( e \) is drawn as an arc of a circle \( C(O, r) \), with \( \hat{uOv} = \phi < \pi \). The next lemma proves that this assumption leads to a contradiction to the existence of a realization of graph \( G'_{oct} \) as a system of circles.

**Lemma 10.** Consider a circle \( C(O, r) \) and assume that edge \( e = (u, v) \in E[G_{oct}] \) is drawn as an arc segment \( \hat{uv} \) of \( C \) such that \( \hat{uOv} = \phi < \pi \). If we attach two pairs of gadget-subgraphs along \( e \), as shown in Fig.8a, then the resulting subgraph of \( G'_{oct} \) does not admit a realization as a system of circles.

**Proof.** By Lemma 5 we have that each gadget-subgraph is drawn as a pair of tangent circles that are also tangent to the arc \( uv \) at points \( z_i, z_j \). Furthermore, the two tangent

\[\square\]
circles have disjoint interiors since they are not on the outerface of $G_{oct}^{aug}$, as in Fig.10a.

By applying Lemma 9 to the pair of gadget-subgraphs with endpoints $z_1$, $z_6$ and $z_2$, $z_5$, we have:

$$\hat{z}_2 \hat{z}_5 < \hat{z}_1 \hat{z}_2 \text{ and } \hat{z}_2 \hat{z}_5 < \hat{z}_5 \hat{z}_6$$

Similarly, for the pair of gadget-subgraphs with endpoints $z_3$, $z_8$ and $z_4$, $z_7$, we have:

$$\hat{z}_4 \hat{z}_7 < \hat{z}_3 \hat{z}_4 \text{ and } \hat{z}_4 \hat{z}_7 < \hat{z}_7 \hat{z}_8$$

Combining those inequalities and the fact that $\hat{z}_i \hat{z}_j \leq \hat{z}_i' \hat{z}_j'$ for $i' \leq i \leq j \leq j'$, we have:

$$\hat{z}_4 \hat{z}_7 < \hat{z}_3 \hat{z}_4 \leq \hat{z}_2 \hat{z}_5 < \hat{z}_5 \hat{z}_6 \leq \hat{z}_4 \hat{z}_7$$

that is $\hat{z}_4 \hat{z}_7 < \hat{z}_4 \hat{z}_7$, which is a contradiction. \qed

In order to complete the proof that graph $G_{oct}^{aug}$ does not admit a realization as a system of circles, it suffices to show that in any realization of $G_{oct}$ (and therefore of $G_{oct}^{sub}$), at least one edge of the innermost interior face meets the requirements of Lemma 10.

**Lemma 11.** In any realization of the octahedron as a system of circles, at least one edge, say $e = (u,v)$, of the innermost interior face is drawn as an arc of a circle $C(O,r)$ so that $uOv = \phi < \pi$.

**Proof.** By Lemma 4, it suffices to show that the lemma holds for the three non-equivalent representations shown in Fig.1. For the first two representations of Fig.1, the result is almost straightforward. More precisely, let $C_1(O_1,r_1)$, $C_2(O_2,r_2)$ and $C_3(O_3,r_3)$ be the circles (white-colored in Fig.1b) that define the innermost interior face (refer to the innermost gray-shaded face of Fig.1b) of the first representation. The three points of this face lie on the edges of the triangle defined by points $O_1$, $O_2$ and $O_3$, since circles $C_1$, $C_2$ and $C_3$ are mutually tangent. Then, at least one of the angles of the triangle is less than $\pi$, as desired. In the second representation, the innermost interior face is a circle (refer to the innermost gray-shaded circle of Fig.1c) with three distinct points on its boundary. Trivially, at least one of the arcs defined by those points corresponds to an angle that is smaller than $\pi$.

We now turn our attention to the more interesting case where the realization of the octahedron graph as a system of circles is implied by three mutually crossing circles (refer to Fig.1d). First, consider two circles $C_1(O_1,r_1)$ and $C_2(O_2,r_2)$ intersecting at points $A$ and $B$ and assume w.l.o.g. that their centers lie along the x-axis, such that $O_1$ is to the left of $O_2$ (see Fig.13). We are interested in the angles that correspond to the two arcs of $C_1$ and $C_2$ that “confine” the common points of the two circles (refer to the
dashed drawn arcs incident to $A$ and $B$ in Fig.13). It is not difficult to see that $\widehat{AO_1B}$ and $\widehat{AO_2B}$ cannot be both greater than $\pi$. Consider now two of the crossing circles of the realization of the octahedron of Fig.1d. From the above, it follows that at least one of the two arcs that “confine” their common points, corresponds to an angle that is less than $\pi$. Since the innermost interior face is also confined by the two arcs, it follows that at least one edge of the innermost interior face has the desired property.

**Theorem 3.** There exists a connected 4-regular planar graph that does not admit a realization as a system of circles.

*Proof.* Lemma 11 states that in any realization of the octahedron graph as a system of circles, at least one edge of the innermost interior face is drawn as an arc segment of angle less than $\pi$. Hence, by Lemma 10 it follows that $G^{aug}$ does not admit a realization as a system of circles.

**Theorem 4.** There exists an infinite class of connected 4-regular planar graphs that do not admit a realization as a system of circles.

*Proof.* Recall that in order to obtain $G^{aug}$, each edge of the octahedron graph was augmented by two pairs of gadget-subgraphs. However, Theorem 3 trivially holds if more than two pairs of gadget-subgraphs are attached to each edge of $G_{oct}$, defining thus an infinite class of connected 4-regular planar graphs that do not admit a realization as a system of circles. An alternative (and more interesting) class of such graphs can be derived by replacing the octahedron graph of the loop-subgraph of each gadget-subgraph by any 4-regular planar graph, in which one of the edges on its outerface is replaced by a path of length two and the additional vertex implied by this procedure is identified by vertices $w_1$ and/or $w_2$ of the gadget-subgraph (refer to Fig.7).

6 The case of biconnected 4-regular planar graphs

In this section, we consider the case of biconnected 4-regular planar graphs. More precisely, we will prove that there exist infinitely many biconnected 4-regular planar graphs that do not admit realizations as system of circles. To do so, we follow a similar approach as the one presented in Section 5. Recall that graph $G^{aug}_{oct}$ that we constructed in Section 5 was not biconnected, since each gadget-subgraph defines two cut-vertices. In fact, all cut-vertices of $G^{aug}_{oct}$ belong to the gadget-subgraphs. So, for the case of biconnected 4-regular planar graphs, we will construct a new gadget-subgraph, referred to as bigadget-subgraph, that does not contain cut-vertices and simultaneously has the same properties as the corresponding ones of Section 5 (in particular the properties implied by Lemma 5).

The bigadget-subgraph is illustrated in Fig.14a. Again, it contains exactly two vertices of degree two, namely $v_1$ and $v_2$, which are its endpoints. However, its skeleton now consists of seven vertices (i.e., $v_i$, $w_i$, $w'_i$ and $w$, $i = 1, 2$). If we remove the edges of the skeleton except for the edges $(w_i, w'_i)$, $i = 1, 2$, the remaining graph again consists of three isolated vertices and two disjoint biconnected graphs, which we call biloop-subgraphs (refer to the grey-shaded graphs of Fig.14a). The properties of the bigadget-subgraph are again independent of the biloop-subgraphs, i.e., any simple biconnected planar graph satisfying the following degree condition can be used instead: Every vertex is of degree 4 except for exactly two vertices on the outerface that are of
The bigadget-subgraph.

Abstraction of the bigadget-subgraph.

Figure 14: Illustrations of the new gadget-subgraph.

degree 3. The general situation is shown in Fig.14b, where the subgraphs are drawn as “bi-loops” at vertices \(w_i\) and \(w'_i\), \(i = 1, 2\).

Having specified the bigadget-subgraph, we can augment the octahedron graph similarly to Section 5. This will result in a biconnected graph, which we denote by \(BG_{\text{aug}}^{\text{oct}}\).

Now, we are in position to prove the analogue of Lemma 5.

**Lemma 12.** Let \(G\) be a 4-regular planar graph that contains at least one copy of the bigadget-subgraph. Suppose that there is a realization of \(G\) as a system of circles. Then, the skeleton of each bigadget-subgraph in this realization consists of two circles \(C_1\) and \(C_2\) tangent at a point \(w\), where circle \(C_i\) contains vertices \(\{v_i, w_i, w'_i, w\}\) and the arc-segments realizing edges \((v_i, w), (v_i, w_i)\) and \((w'_i, w)\), for \(i = 1, 2\).

**Proof.** Suppose that there is a realization of \(G\) as a system of circles and consider a copy of the bigadget-subgraph in this realization. Note that the edges \((v_i, w_i)\) and \((w'_i, w)\) belong to the same circle, \(i = 1, 2\) (otherwise, one of the circles would contain vertex \(w_i\) twice and the other would contain vertex \(w'_i\) twice, \(i = 1, 2\), which is not possible since every vertex belongs to exactly two different circles). Let \(C_i\) be the circle that contains \((v_i, w_i)\) and \((w'_i, w)\) and \(C'_i\) the circle that contains \((v_i, w)\), \(i = 1, 2\). We claim that \(C_i = C'_i\), \(i = 1, 2\). For the sake of contradiction, assume that \(C_1 \neq C'_1\). Since vertex \(w\) is defined by exactly two circles, we have that \(\{C_1, C'_1\} = \{C_2, C'_2\}\), which also implies that \(C_2 \neq C'_2\). Then, \(C_1\) and \(C'_1\) have at least three points in common, namely vertices \(v_1, v_2\) and \(w\), from which we obtain \(C_1 = C'_1\); a contradiction. □

**Remark 1.** In the statement of Lemma 12 in [1], we erroneously wrote that circle \(C_i\) the arc-segments \((v_i, w_i), (w, w'_i)\) and \((w'_i, w)\), for \(i = 1, 2\). This was clearly a typo, which we fix in this version.

In Fig.15 two non-equivalent realizations are shown. Note that these realizations actually depend on the relative position of the two touching circles \(C_1\) and \(C_2\) in the planar embedding of \(G\). In particular, in Fig.15a the two circles \(C_1\) and \(C_2\) contain only the biloops-subgraphs in their interior, while in Fig.15b \(C_1\) is the outerface and therefore contains the entire graph. This implies that in any realization of \(BG_{\text{aug}}^{\text{oct}}\) as a system of circles, all bigadget-subgraphs are drawn as in Fig.15a except for at most one bigadget-subgraph if one of its two circles is the outerface of \(BG_{\text{aug}}^{\text{oct}}\). Hence, we can similarly prove the analogue of Corollary 1.

**Corollary 2.** In any realization of \(BG_{\text{aug}}^{\text{oct}}\) as a system of circles, the realizations of each bigadget-subgraph and the realization of \(G_{\text{sub}}^{\text{oct}}\) are independent.

Lemma 12 and Corollary 2 allow us to prove an analogous of Lemma 10 (where bigadget-subgraphs are used instead of gadget-subgraphs). That, together with Lemma 11, allows us to give an analogue of Theorem 3.
Figure 15: Non-equivalent realizations of the bigadget-subgraph as system of circles: (a) the two circles representing the skeleton have disjoint interior, and (b) one circle lies in the interior of the other.

**Theorem 5.** There exists a biconnected 4-regular planar graph that does not admit a realization as a system of circles.

In order to prove that there exist infinitely many biconnected 4-regular planar graphs that do not admit realizations as system of circles, one can attach more than two pairs of bigadget-subgraphs to every edge of $G_{oct}$, or replace the biloops-subgraphs by any simple biconnected planar graph (in which every vertex is of degree 4 except for exactly two vertices on the outerface that are of degree 3), which leads to the following conclusion.

**Theorem 6.** There exists an infinite class of biconnected 4-regular planar graphs that do not admit a realization as a system of circles.

### 7 Conclusion - Open Problems

In this paper, we proved that every 3-connected 4-regular planar graph admits a realization as a system of touching circles. We also demonstrated that there exist 4-regular planar graphs which are not 3-connected (i.e., either connected or biconnected) and do not admit realizations as system of circles. However, our work raises several open problems. In the following, we name only few of them:

- What is the computational complexity of the corresponding decision problem, i.e., does a given connected 4-regular planar graph admit a realization as a system of circles?

- Which is the smallest connected 4-regular planar graph not admitting a realization as a system of circles? The ones we manage to construct consist of more than 100 vertices.

- The octahedron graph admits non-equivalent realizations as system of circles, in which the number of circles participating in the corresponding realizations also differs. In general, an $n$-vertex 4-regular planar graph needs at least $(1 + \sqrt{1 + 4n})/2$ and at most $2n/3$ circles in order to be realized as a system of circles, as shown in Section 3. So, what is the range of the number of circles needed in order to realize a given (3-connected) 4-regular planar graph as a system of circles?
• In the context of graph realizations as system of circles, it would be interesting to study the class of Eulerian planar graphs. Obviously, certain vertices would be defined as the intersection of more than two circles.

Acknowledgments

The authors would like to thank the anonymous reviewers for useful suggestions and comments that helped in improving the readability of the paper.

References

[1] Michael A. Bekos and Chrysanthi N. Raftopoulou. On a conjecture of lovász on circle-representations of simple 4-regular planar graphs. JoCG, 6(1):1–20, 2015.

[2] T. Biedl and G. Kant. A better heuristic for orthogonal graph drawings. In Proc. 2nd Annual European Symposium on Algorithms (ESA ’94), volume 855 of LNCS, pages 24–35. Springer-Verlag, 1994.

[3] Bela Bollobás. Modern Graph Theory. Springer, 1998.

[4] H.J. Broersma, A.J.W. Duijvestijn, and F. Göbel. Generating all 3-connected 4-regular planar graphs from the octahedron graph. Journal of Graph Theory, 17(5):613–620, 1993.

[5] Roman Chernobelskiy, Kathryn Cunningham, Michael Goodrich, Stephen G. Kobourov, and Lowell Trott. Force-Directed Lombardi-Style Graph Drawing. In Proc. of 19th International Symposium on Graph Drawing, volume 7034 of LNCS, pages 310–321, 2011.

[6] Christian A. Duncan, David Eppstein, Michael T. Goodrich, Stephen G. Kobourov, and Martin Nöllenburg. Lombardi Drawings of Graphs. Journal of Graph Algorithms and Applications, 16(1):85–108, 2011.

[7] Christian A. Duncan, David Eppstein, Michael T. Goodrich, Stephen G. Kobourov, and Martin Nöllenburg. Drawing Trees with Perfect Angular Resolution and Polynomial Area. Discrete & Computational Geometry, 49(2):157–182, 2013.

[8] Paul Erdős, Alfred Rényi, and Vera T. Sós. Combinatorial theory and its applications. North-Holland Amsterdam, 1970.

[9] Petr Hlinený and Jan Krátcochvíl. Representing graphs by disks and balls (a survey of recognition-complexity results). Discrete Mathematics, 229(1–3):101–124, 2001.

[10] Petr Hlinený. Classes and recognition of curve contact graphs. Journal of Combinatorial Theory, Series B, 74(1):87–103, 1998.

[11] Paul Koebe. Kontaktprobleme der konformen Abbildung. Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Physikalische Klasse, 88:141–164, 1936.

[12] J. Lehel. Generating all 4-regular planar graphs from the graph of the octahedron. Journal of Graph Theory, 5(4):423–426, 1981.
[13] F. T. Leighton. New lower bound techniques for VLSI. In Proc. 22nd Annual IEEE Symposium on Foundations of Computer Science, pages 1–12. IEEE Computer Society, 1981.

[14] Charles E. Leiserson. Area-efficient graph layouts (for VLSI). In Proc. 21st Annual IEEE Symposium on Foundations of Computer Science, volume 1547, pages 270–281. IEEE Computer Society, 1980.

[15] Paolo Manca. Generating all planar graphs regular of degree four. Journal of Graph Theory, 3(4):357–364, 1979.

[16] R. Tamassia. On embedding a graph in the grid with the minimum number of bends. SIAM Journal of Computing, 16:421–444, 1987.

[17] R. Tamassia and I.G. Tollis. Planar grid embedding in linear time. Circuits and Systems, IEEE Transactions on, 36(9):1230–1234, 1989.

[18] Leslie G. Valiant. Universality considerations in VLSI circuits. IEEE Transaction on Computers, 30(2):135–140, 1981.

[19] Douglas B. West. Introduction to Graph Theory. Prentice Hall, 2000.