Differential geometry

On the rank of a product of manifolds

Sur le rang d’un produit de variétés

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Abstract
This note gives an example of closed smooth manifolds M and N for which the rank of $M \times N$ is strictly greater than rank $M + \text{rank } N$.

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Résumé
Cette note donne un exemple de deux variétés compactes M et N pour lesquelles le rang de $M \times N$ est strictement plus grand que rang $M + \text{rang } N$.

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1. Introduction

Milnor defined the rank of a smooth manifold M as the maximal number of commuting vector fields on M that are linearly independent at each point.

One of the questions raised by Milnor at the Seattle Topology Conference of 1963, and echoed by Novikov [2], was

$$\text{is rank}(M \times N) = \text{rank}(M) + \text{rank}(N)$$

whenever $M$ and $N$ are smooth closed manifolds?

In this note we give a negative answer to this question.

2. The main result

We need a simple result about mapping tori.
Let $f : X \to X$ be a diffeomorphism of a manifold X and let
\[
M(f) = \frac{I \times X}{(0, x)(1, f(x))}
\]
be the mapping torus of \( f \) where \( I = [0, 1] \).

Equivalently, \( M(f) = \frac{\mathbb{R} \times X}{\mathbb{Z}} \) where the action of \( \mathbb{Z} \) on \( \mathbb{R} \times X \) is given by \( \alpha(k)(t, x) = (t + k, f^k(x)) \). \( M(f) \) is a fiber bundle over \( S^1 \) with fiber \( X \). We note that \( \pi_1(M(f)) = \pi_1(X) * f \mathbb{Z} \) where * denotes the semi-direct product and \( f_* : \pi_1(X) \to \pi_1(X) \).

**Proposition 2.1.** Consider two periodic diffeomorphisms \( f : X \to X \) and \( g : Y \to Y \) with periods \( m \) and \( n \) respectively. Assume \( m \) and \( n \) are relatively prime, i.e., there are integers \( c, d \) such that \( mc + nd = 1 \).

Then \( M(f) \times M(g) \) is diffeomorphic to \( M(h) \) where \( h : S^1 \times X \to Y \) is defined by \( h(\theta, x, y) = (\theta, f^{-d}(x), g^c(y)) \). Moreover \( h^{m-n} = (id, f, g) \).

**Proof.** \( M(f) \times M(g) \) can be identified with the quotient of \( \mathbb{R}^2 \times X \times Y \) under the action of \( \mathbb{Z}^2 \) given by \( \beta(z)(u, x, y) = (u + z, f^z(x), g^z(y)) \), where \( z = (z_1, z_2) \in \mathbb{Z}^2, u = (u_1, u_2) \in \mathbb{R}^2 \) and \( (x, y) \in X \times Y \).

Set \( \lambda = (m, n) \) and \( \mu = (-d, c) \). Since \( mc + nd = 1 \), \( B = \{\lambda, \mu\} \) is at the same time a basis of \( \mathbb{Z}^2 \) as a \( \mathbb{Z} \)-module and a basis of \( \mathbb{R}^2 \) as a vector space. On the other hand

\[
\beta(\lambda)(u, x, y) = (u + \lambda, x, y) \quad \text{and} \quad \beta(\mu)(u, x, y) = (u + \mu, f^{-d}(x), g^c(y)).
\]

Therefore the action \( \beta \) referred to the new basis \( B \) of \( \mathbb{Z}^2 \) and \( \mathbb{R}^2 \) is written now:

\[
\beta(k, r)(a, b, x, y) = (a + k, b + r, \varphi^k(x), \gamma^r(y))
\]

where \( \varphi = f^{-d} \) and \( \gamma = g^c \).

As the action of the first factor of \( \mathbb{Z}^2 \) on \( X \times Y \) is trivial, identifying \( S^1 \) with \( \mathbb{R} / \mathbb{Z} \) shows that \( M(f) \times M(g) \) is diffeomorphic to \( M(h) \).

Finally from \((-n)(-d) = 1 - cm \) and \( cm = 1 - dn \) follows that \( h^{m-n} = (id, f, g) \). \( \square \)

On the other hand:

**Lemma 2.1.** Let \( f : N \to N \) be a diffeomorphism and let \( X_1, \ldots, X_k \) be a family of commuting vector fields on \( N \) that are linearly independent everywhere. Assume \( f, X_i = \sum_{j=1}^k a_{ij} X_j, i = 1, \ldots, k \), where the matrix \( (a_{ij}) \in GL(k, \mathbb{R}) \). Then rank(M(f)) \( \geq k \).

**Proof.** It suffices to construct \( k \) commuting vector fields \( \tilde{X}_1, \ldots, \tilde{X}_k \) on \( I \times N \) that are linearly independent at each point and such that every \( \tilde{X}_i(t, x) \) equals \( X_i(x) \) if \( t \) is close to zero and \( f, X_i(x) \) when \( t \) is close to 1 (\( X_1, \ldots, X_k \) are considered vector fields on \( I \times N \) in the obvious way).

If \( |a_{ij}| > 0 \) consider an interval \( [a, b] \subset (0, 1) \) and a (differentiable) map \( (\varphi_{ij}) : I \to GL(k, \mathbb{R}) \) such that \( \varphi_{ij}([0, a]) = \delta_{ij} \) and \( \varphi_{ij}([b, 1]) = a_{ij} \), and set \( \tilde{X}_i(t, x) = \sum_{j=1}^k \varphi_{ij}(t) X_j(x) \).

When \( |a_{ij}| < 0 \) first take an interval \( [c, d] \subset (0, 1/2) \) and a function \( \rho : [0, 1/2] \to \mathbb{R} \) such that \( \rho([0, c]) = 1, \rho([d, 1/2]) = -1 \), and on \([0, 1/2] \times N \) set \( \tilde{X}_i(t, x) = \rho(t) X_i(x) + (1 - \rho^2(t)) \frac{\partial}{\partial t} \) and \( \tilde{X}_i(t, x) = X_i(x), i = 2, \ldots, k \).

The matrix of coordinates of \( f, X_1, \ldots, f, X_k \) with respect to the basis \( \{-X_1, X_2, \ldots, X_k\} \) has positive determinant, so by doing as before we can extend \( \tilde{X}_1, \ldots, \tilde{X}_k \) to \([1/2, 1) \times N \) by means of an interval \([a, b] \subset (1/2, 1) \) and a suitable map \( (\tilde{\varphi}_{ij}) : [1/2, 1) \to GL(k, \mathbb{R}) \). \( \square \)

**Proposition 2.1** and **Lemma 2.1** quickly yield a counterexample.

Assume \( X \) is a torus \( \mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k \) and \( f \) is the map induced by a nontrivial element of \( GL(k, \mathbb{Z}) \). Then by the above lemma applied to \( \frac{\partial}{\partial t} \), \( j = 1, \ldots, k \), rank\( (M(f)) \geq k \). But \( M(f) \) has non-Abelian fundamental group, so it is not a torus and rank\( (M(f)) = k \). (If \( M \) is a closed connected \( n \)-manifold of rank \( n \), then \( M \) is diffeomorphic to the \( n \)-torus.)

For the same reason, if \( Y = \mathbb{T}^r \) and \( g \) is induced by a nontrivial element of \( GL(r, \mathbb{Z}) \), then rank\( (M(g)) = r \).

If \( f \) and \( g \) are periodic with relatively prime periods \( m \) and \( n \), respectively, then by **Proposition 2.1**, \( M(f) \times M(g) = M(h) \) where \( h : \mathbb{T}^{k+r+1} \to \mathbb{T}^{k+r+1} \) is induced by a nontrivial element of \( GL(k + r + 1, \mathbb{Z}) \). Moreover rank\( (M(h)) = k + r + 1 \). Therefore:

\[
\text{rank}(M(f) \times M(g)) > \text{rank}(M(f)) + \text{rank}(M(g)).
\]

For instance, set \( k = r = 2 \) and consider \( f, g \) induced by the elements in \( SL(2, \mathbb{Z}) \subset GL(2, \mathbb{Z}) \)

\[
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}
\]

respectively, so \( M(f) \) and \( M(g) \) are orientable. Then the period of \( f \) is 2 and that of \( g \) equals 3.
An even simpler but non-orientable counterexample can be constructed as follows. Take \( r \) and \( g \) as before, \( k = 1 \) and \( f \) induced by \((-1)\). Then \( M(f) \) is the Klein bottle which has rank 1 and \( M(g) \) has rank 2; however, \( M(f) \times M(g) \) is diffeomorphic to \( M(h) \) and hence has rank 4.

**Remark 1.** The *file* of a manifold \( M \) was defined by Rosenberg [3] to be the largest integer \( k \) such that \( \mathbb{R}^k \) acts locally free on \( M \). When \( M \) is closed file(\( M \)) equals rank(\( M \)) but file(\( \mathbb{R} \times S^2 \)) = 1, [3], while rank(\( \mathbb{R} \times S^2 \)) = 3.

The analog of Milnor's question for the file of a product of noncompact manifolds also fails. Indeed, let \( \mathbb{R}^4_\text{e} \) be any exotic \( \mathbb{R}^4 \). Then file(\( \mathbb{R}^4_\text{e} \)) \( \leq 3 \) otherwise \( \mathbb{R}^4_\text{e} \times \mathbb{R}^4 = \mathbb{R}^4 \). But \( \mathbb{R}^4_\text{e} \times \mathbb{R} = \mathbb{R}^5 \), because there in no exotic \( \mathbb{R}^5 \), so file(\( \mathbb{R}^4_\text{e} \times \mathbb{R} \)) \( = 5 > \) file(\( \mathbb{R}^4_\text{e} \)) + file(\( \mathbb{R} \)).

Orientable closed connected \( n \)-manifolds of rank \( n - 1 \) are completely described in [4,1,5].

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