Supplementary material for the paper:
Comparative study of $L_1$ regularized logistic regression methods for variable selection

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CASE I: $n > p$

Table 3: Simulation results of mean of BIAS.

| $\beta$ | $n$ | $p$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ | $r = 0.9$ | $r = 0.99$ | $r = 0.99$ CC* |
|--------|-----|-----|--------|----------|----------|----------|----------|----------|----------|------------|----------------|
| 4      | 1000| 400 | IRLS   | 3.59     | 0.83     | 0.96     | 1.43     | 2.63     | 4.37     | 9.59      | 10.12      |
| 4      | 5000| 400 | IRLS   | 1.22     | 0.82     | 0.96     | 1.42     | 2.64     | 4.36     | 9.57      | 10.10      |
| 4      | 10000| 400 | IRLS   | 0.80     | 0.78     | 0.93     | 1.39     | 2.59     | 4.30     | 9.55      | 10.09      |
| 4      | 50000| 400 | IRLS   | 0.80     | 0.78     | 0.93     | 1.39     | 2.59     | 4.30     | 9.55      | 10.09      |

Simulation results of mean of BIAS.
| $\beta$ | $n$ | $p$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ | $r = 0.9$ | $r = 0.95$ | $r = 0.99$ | $r = \cdot \cdot CC^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1000 | 4 | 100 | IRLS | 1.44 | 1.51 | 1.84 | 2.13 | 3.47 | 3.37 | 9.75 | 9.75 | 1.88 |
| | | | Glmnet | 0.19 | 0.19 | 0.19 | 0.29 | 0.42 | 0.66 | 0.84 | 1.00 | 0.43 |
| | 6 | | IRLS | 2.17 | 2.90 | 2.89 | 2.13 | 3.47 | 3.47 | 9.75 | 9.75 | 1.88 |
| | | | Glmnet | 0.26 | 0.27 | 0.32 | 0.38 | 0.58 | 0.91 | 1.08 | 1.22 | 0.45 |
| | 8 | | IRLS | 2.68 | 3.20 | 2.89 | 2.13 | 3.47 | 3.37 | 9.75 | 9.75 | 1.88 |
| | | | Glmnet | 0.29 | 0.33 | 0.39 | 0.49 | 0.65 | 1.04 | 1.33 | 1.46 | 0.56 |
| 5000 | 4 | 100 | IRLS | 0.44 | 0.65 | 0.50 | 0.67 | 0.78 | 1.71 | 1.71 | 2.11 | 0.82 |
| | | | Glmnet | 0.11 | 0.11 | 0.12 | 0.15 | 0.20 | 0.36 | 0.53 | 0.99 | 0.12 |
| | 6 | | IRLS | 0.54 | 0.60 | 0.58 | 0.98 | 0.93 | 1.61 | 1.61 | 2.70 | 0.84 |
| | | | Glmnet | 0.17 | 0.17 | 0.18 | 0.22 | 0.32 | 0.47 | 0.76 | 1.25 | 0.23 |
| | 8 | | IRLS | 0.50 | 0.56 | 0.77 | 1.13 | 1.34 | 2.15 | 2.15 | 3.37 | 0.99 |
| | | | Glmnet | 0.20 | 0.21 | 0.25 | 0.31 | 0.45 | 0.63 | 0.87 | 1.37 | 0.26 |
| 5000 | 4 | 400 | IRLS | 0.56 | 1.21 | 1.40 | 1.47 | 1.88 | 3.89 | 3.89 | 5.32 | 1.52 |
| | | | Glmnet | 0.11 | 0.10 | 0.12 | 0.16 | 0.21 | 0.41 | 0.60 | 0.96 | 0.36 |
| | 6 | | IRLS | 1.12 | 1.52 | 1.71 | 2.10 | 3.49 | 7.23 | 7.23 | 7.28 | 1.86 |
| | | | Glmnet | 0.16 | 0.16 | 0.17 | 0.24 | 0.32 | 0.60 | 0.75 | 1.13 | 0.43 |
| | 8 | | IRLS | 1.37 | 1.75 | 2.12 | 3.61 | 3.49 | 7.23 | 7.23 | 7.28 | 2.00 |
| | | | Glmnet | 0.20 | 0.18 | 0.25 | 0.32 | 0.39 | 0.72 | 0.89 | 1.31 | 0.31 |
| 10000 | 4 | 400 | IRLS | 1.38 | 0.75 | 0.77 | 1.07 | 1.35 | 2.08 | 2.08 | 2.52 | 2.42 |
| | | | Glmnet | 0.10 | 0.09 | 0.09 | 0.12 | 0.15 | 0.30 | 0.38 | 0.82 | 0.17 |
| | 6 | | IRLS | 0.70 | 0.95 | 0.95 | 1.32 | 1.43 | 1.99 | 1.99 | 3.10 | 2.49 |
| | | | Glmnet | 0.14 | 0.14 | 0.13 | 0.18 | 0.23 | 0.42 | 0.57 | 1.11 | 0.38 |
| | 8 | | IRLS | 0.91 | 1.22 | 1.10 | 1.94 | 2.51 | 4.22 | 0.42 | 5.44 | 2.48 |
| | | | Glmnet | 0.18 | 0.16 | 0.19 | 0.24 | 0.32 | 0.56 | 0.75 | 1.36 | 0.40 |

Table 4: Simulation results of standard deviation of BIAS.
Table 5: Simulation results of mean of Adjusted BIAS ($BIAS_{Adj}$).

| $\beta$ | $n$ | $p$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ | $r = 0.9$ | $r = 0.95$ | $r = 0.99$ | $r = \infty$ |
|---------|-----|-----|--------|----------|----------|----------|----------|----------|----------|-----------|-------------|-------------|
| 10000   | 5000| 200 | IRLS   | 0.65     | 0.68     | 0.74     | 0.82     | 1.11     | 1.23     | 2.60     | 2.60      | 1.05        |
|         |     |     | Glmnet | 0.38     | 0.38     | 0.40     | 0.48     | 0.59     | 0.78     | 1.08     | 1.43      | 0.58        |
| 5000    | 5000| 100 | IRLS   | 0.66     | 0.67     | 0.80     | 0.82     | 1.11     | 1.21     | 2.60     | 2.60      | 1.05        |
|         |     |     | Glmnet | 0.42     | 0.43     | 0.45     | 0.52     | 0.64     | 0.90     | 1.08     | 1.38      | 0.59        |
| 10000   | 400 | 100 | IRLS   | 0.66     | 0.65     | 0.80     | 0.82     | 1.11     | 1.21     | 2.60     | 2.60      | 1.05        |
|         |     |     | Glmnet | 0.43     | 0.44     | 0.48     | 0.56     | 0.66     | 0.91     | 1.14     | 1.37      | 0.62        |
| 10000   | 5000| 200 | IRLS   | 0.24     | 0.27     | 0.25     | 0.35     | 0.44     | 0.77     | 0.77     | 1.13      | 0.56        |
|         |     |     | Glmnet | 0.24     | 0.24     | 0.24     | 0.28     | 0.33     | 0.46     | 0.59     | 1.09      | 0.30        |
| 5000    | 400 | 100 | IRLS   | 0.25     | 0.31     | 0.30     | 0.35     | 0.44     | 0.74     | 0.74     | 1.15      | 0.48        |
|         |     |     | Glmnet | 0.25     | 0.26     | 0.27     | 0.31     | 0.38     | 0.49     | 0.64     | 1.15      | 0.33        |
| 10000   | 400 | 100 | IRLS   | 0.25     | 0.42     | 0.30     | 0.40     | 0.48     | 0.78     | 0.78     | 1.14      | 0.40        |
|         |     |     | Glmnet | 0.26     | 0.27     | 0.29     | 0.35     | 0.40     | 0.54     | 0.67     | 1.11      | 0.37        |
| 10000   | 100 | 5000| IRLS   | 0.27     | 0.50     | 0.48     | 0.61     | 0.74     | 1.31     | 1.31     | 1.75      | 0.95        |
|         |     |     | Glmnet | 0.26     | 0.26     | 0.26     | 0.33     | 0.38     | 0.59     | 0.75     | 1.13      | 0.42        |
| 5000    | 100 | 5000| IRLS   | 0.47     | 0.54     | 0.57     | 0.59     | 0.88     | 1.53     | 1.53     | 1.98      | 0.96        |
|         |     |     | Glmnet | 0.27     | 0.28     | 0.30     | 0.37     | 0.44     | 0.66     | 0.82     | 1.35      | 0.52        |
| 10000   | 100 | 400 | IRLS   | 0.51     | 0.62     | 0.60     | 0.71     | 0.88     | 1.53     | 1.53     | 1.98      | 0.99        |
|         |     |     | Glmnet | 0.29     | 0.32     | 0.34     | 0.40     | 0.49     | 0.71     | 0.87     | 1.37      | 0.56        |
| 10000   | 400 | 100 | IRLS   | 0.53     | 0.32     | 0.33     | 0.42     | 0.55     | 0.85     | 0.85     | 1.05      | 1.00        |
|         |     |     | Glmnet | 0.21     | 0.21     | 0.21     | 0.25     | 0.30     | 0.45     | 0.54     | 1.04      | 0.28        |
| 5000    | 100 | 400 | IRLS   | 0.29     | 0.39     | 0.39     | 0.43     | 0.50     | 0.77     | 0.77     | 1.03      | 1.00        |
|         |     |     | Glmnet | 0.22     | 0.23     | 0.24     | 0.29     | 0.35     | 0.50     | 0.63     | 1.14      | 0.37        |
| 10000   | 400 | 100 | IRLS   | 0.30     | 0.38     | 0.38     | 0.46     | 0.60     | 0.94     | 0.94     | 1.22      | 1.00        |
|         |     |     | Glmnet | 0.23     | 0.25     | 0.27     | 0.32     | 0.39     | 0.54     | 0.66     | 1.19      | 0.44        |
| $\beta$ | $n$ | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ | $r = 0.9$ | $r = 0.95$ | $r = 0.99$ | $r = \cdot CC^*$ |
|-----|-----|-----|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 1000 | 4   | 100 | 8   | IRLS   | 0.36   | 0.38   | 0.49   | 0.53   | 0.87   | 0.73   | 2.44   | 2.44   | 0.47   |
|     |     | 6   |     | Glmnet | 0.05   | 0.05   | 0.05   | 0.07   | 0.10   | 0.16   | 0.21   | 0.25   | 0.11   |
|     |     | 8   |     | IRLS   | 0.36   | 0.48   | 0.48   | 0.53   | 0.87   | 0.77   | 2.44   | 2.44   | 0.47   |
|     |     |     |     | Glmnet | 0.04   | 0.04   | 0.05   | 0.06   | 0.10   | 0.15   | 0.18   | 0.20   | 0.08   |
|     | 5000| 100 | 6   | IRLS   | 0.11   | 0.14   | 0.12   | 0.17   | 0.20   | 0.43   | 0.43   | 0.53   | 0.20   |
|     |     |     |     | Glmnet | 0.03   | 0.03   | 0.03   | 0.04   | 0.05   | 0.09   | 0.13   | 0.25   | 0.03   |
|     |     | 8   |     | IRLS   | 0.09   | 0.08   | 0.10   | 0.15   | 0.16   | 0.27   | 0.27   | 0.45   | 0.14   |
|     |     |     |     | Glmnet | 0.03   | 0.03   | 0.03   | 0.04   | 0.05   | 0.08   | 0.13   | 0.21   | 0.04   |
|     |     | 8   |     | IRLS   | 0.06   | 0.07   | 0.10   | 0.14   | 0.17   | 0.27   | 0.27   | 0.42   | 0.12   |
|     |     |     |     | Glmnet | 0.03   | 0.03   | 0.03   | 0.04   | 0.06   | 0.08   | 0.11   | 0.17   | 0.03   |
|     | 10000| 400 | 6   | IRLS   | 0.28   | 0.25   | 0.29   | 0.35   | 0.85   | 1.20   | 1.20   | 1.21   | 0.39   |
|     |     |     |     | Glmnet | 0.03   | 0.03   | 0.03   | 0.04   | 0.05   | 0.10   | 0.15   | 0.24   | 0.09   |
|     |     | 8   |     | IRLS   | 0.23   | 0.21   | 0.26   | 0.45   | 0.58   | 1.20   | 1.20   | 1.21   | 0.40   |
|     |     |     |     | Glmnet | 0.02   | 0.02   | 0.02   | 0.03   | 0.04   | 0.09   | 0.11   | 0.16   | 0.04   |
|     | 10000| 100 | 4   | IRLS   | 0.17   | 0.19   | 0.19   | 0.27   | 0.34   | 0.52   | 0.52   | 0.63   | 0.38   |
|     |     |     |     | Glmnet | 0.03   | 0.02   | 0.02   | 0.03   | 0.04   | 0.08   | 0.09   | 0.21   | 0.04   |
|     |     | 6   |     | IRLS   | 0.17   | 0.16   | 0.16   | 0.22   | 0.24   | 0.13   | 0.33   | 0.52   | 0.39   |
|     |     |     |     | Glmnet | 0.02   | 0.02   | 0.20   | 0.03   | 0.04   | 0.07   | 0.09   | 0.19   | 0.06   |
|     |     | 8   |     | IRLS   | 0.15   | 0.19   | 0.14   | 0.24   | 0.31   | 0.53   | 0.53   | 0.68   | 0.40   |
|     |     |     |     | Glmnet | 0.02   | 0.02   | 0.02   | 0.03   | 0.04   | 0.07   | 0.09   | 0.17   | 0.05   |

Table 6: Simulation results of standard deviation of $BIAS_{Adj.}$.
| $\beta$ | $n$  | $p$  | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ | $r = 0.9$ | $r = 0.95$ | $r = 0.99$ | $r = \cdot CC'$ |
|--------|------|------|-----|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|------------|
| 1.00  | 100  | 4    |     | IRLS   | 1.00      | 1.00      | 1.00      | 1.00      | 1.00      | 0.82      | 0.82      | 0.79      |            |
|       |      | 6    |     | Glmnet | 1.00      | 1.00      | 1.00      | 1.00      | 0.98      | 0.79      | 0.29      | 0.97      |            |
|       |      | 8    |     | IRLS   | 1.00      | 1.00      | 1.00      | 1.00      | 0.91      | 0.72      | 0.29      | 0.96      |            |
|       |      |      |     | Glmnet | 1.00      | 1.00      | 1.00      | 1.00      | 0.87      | 0.61      | 0.27      | 0.95      |            |
| 5.00  | 100  | 4    |     | IRLS   | 1.00      | 1.00      | 1.00      | 1.00      | 1.00      | 0.82      | 0.82      | 0.79      |            |
|       |      | 6    |     | Glmnet | 1.00      | 1.00      | 1.00      | 1.00      | 1.00      | 0.82      | 0.82      | 0.79      |            |
|       |      | 8    |     | IRLS   | 1.00      | 1.00      | 1.00      | 1.00      | 0.87      | 0.61      | 0.27      | 0.95      |            |
|       |      |      |     | Glmnet | 1.00      | 1.00      | 1.00      | 1.00      | 0.87      | 0.61      | 0.27      | 0.95      |            |
| 10.00 | 100  | 4    |     | IRLS   | 1.00      | 1.00      | 1.00      | 1.00      | 1.00      | 0.82      | 0.82      | 0.79      |            |
|       |      | 6    |     | Glmnet | 1.00      | 1.00      | 1.00      | 1.00      | 1.00      | 0.82      | 0.82      | 0.79      |            |
|       |      | 8    |     | IRLS   | 1.00      | 1.00      | 1.00      | 1.00      | 0.87      | 0.61      | 0.27      | 0.95      |            |
|       |      |      |     | Glmnet | 1.00      | 1.00      | 1.00      | 1.00      | 0.87      | 0.61      | 0.27      | 0.95      |            |

Table 7: Simulation results of mean of TPR.
Table 8: Simulation results of standard deviation of TPR.

| β   | n   | p   | q   | Method | r = 0.0 | r = 0.1 | r = 0.2 | r = 0.5 | r = 0.7 | r = 0.9 | r = 0.99 | r = 0.99 CC |
|-----|-----|-----|-----|--------|---------|---------|--------|---------|---------|---------|---------|-----------|
| 1000| 4   | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.19 | 0.19 | 0.09 |
|     | 6   | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 5000| 4   | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|     | 6   | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 10000| 4  | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|     | 6   | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

Note: The table continues with similar entries.
Table 9: Simulation results of mean of FPR.

| β   | n   | p   | Method | r = 0.0 | r = 0.1 | r = 0.2 | r = 0.5 | r = 0.7 | r = 0.9 | r = 0.95 | r = 0.99 | r = CC' |
|-----|-----|-----|--------|---------|---------|---------|---------|---------|---------|----------|----------|---------|
| 0.33| 100 | 4   | IRLS   | 0.30    | 0.28    | 0.09    | 0.25    | 0.25    | 0.15    | 0.25     | 0.25     | 0.30    |
|     |     |     |        | 0.01    | 0.02    | 0.03    | 0.06    | 0.08    | 0.08    | 0.09     | 0.07     | 0.09    |
|     |     |     |        | 0.03    | 0.04    | 0.07    | 0.09    | 0.11    | 0.12    | 0.12     | 0.08     | 0.12    |
| 0.07| 100 | 6   | IRLS   | 0.42    | 0.38    | 0.39    | 0.25    | 0.25    | 0.15    | 0.25     | 0.25     | 0.30    |
|     |     |     |        | 0.01    | 0.02    | 0.03    | 0.06    | 0.08    | 0.08    | 0.09     | 0.07     | 0.09    |
|     |     |     |        | 0.03    | 0.04    | 0.07    | 0.09    | 0.11    | 0.12    | 0.12     | 0.08     | 0.12    |
| 0.17| 100 | 8   | IRLS   | 0.50    | 0.47    | 0.39    | 0.25    | 0.25    | 0.25    | 0.25     | 0.25     | 0.30    |
|     |     |     |        | 0.05    | 0.08    | 0.09    | 0.12    | 0.14    | 0.14    | 0.15     | 0.09     | 0.15    |
|     | 5000| 100 | IRLS   | 0.21    | 0.23    | 0.19    | 0.21    | 0.21    | 0.22    | 0.22     | 0.24     | 0.27    |
|     |     |     |        | 0.00    | 0.00    | 0.01    | 0.04    | 0.06    | 0.08    | 0.09     | 0.09     | 0.03    |
|     |     |     |        | 0.01    | 0.02    | 0.03    | 0.07    | 0.10    | 0.11    | 0.12     | 0.12     | 0.07    |
|     |     |     |        | 0.02    | 0.04    | 0.07    | 0.11    | 0.13    | 0.17    | 0.15     | 0.14     | 0.12    |
| 0.10| 5000| 100 | IRLS   | 0.12    | 0.17    | 0.17    | 0.13    | 0.16    | 0.15    | 0.15     | 0.16     | 0.15    |
|     |     |     |        | 0.00    | 0.01    | 0.02    | 0.03    | 0.04    | 0.05    | 0.05     | 0.04     | 0.07    |
|     |     |     |        | 0.01    | 0.02    | 0.03    | 0.04    | 0.05    | 0.06    | 0.06     | 0.06     | 0.09    |
|     | 10000| 100 | IRLS   | 0.25    | 0.09    | 0.09    | 0.09    | 0.10    | 0.10    | 0.10     | 0.09     | 0.25    |
|     |     |     |        | 0.00    | 0.00    | 0.00    | 0.01    | 0.02    | 0.02    | 0.03     | 0.03     | 0.03    |
|     |     |     |        | 0.00    | 0.00    | 0.01    | 0.02    | 0.03    | 0.04    | 0.05     | 0.05     | 0.04    |
|     |     |     |        | 0.01    | 0.02    | 0.03    | 0.04    | 0.05    | 0.06    | 0.06     | 0.06     | 0.08    |
|     | 10000| 100 | IRLS   | 0.28    | 0.25    | 0.23    | 0.23    | 0.23    | 0.23    | 0.21     | 0.17     | 0.28    |
|     |     |     |        | 0.01    | 0.01    | 0.02    | 0.05    | 0.07    | 0.08    | 0.09     | 0.09     | 0.06    |
|     |     |     |        | 0.02    | 0.03    | 0.05    | 0.09    | 0.11    | 0.11    | 0.11     | 0.07     | 0.10    |
|     |     |     |        | 0.04    | 0.06    | 0.08    | 0.12    | 0.13    | 0.14    | 0.13     | 0.08     | 0.13    |
|     | 5000| 100 | IRLS   | 0.18    | 0.18    | 0.16    | 0.17    | 0.19    | 0.19    | 0.21     | 0.20     | 0.23    |
|     |     |     |        | 0.00    | 0.00    | 0.00    | 0.03    | 0.05    | 0.07    | 0.09     | 0.08     | 0.02    |
|     |     |     |        | 0.00    | 0.01    | 0.03    | 0.06    | 0.09    | 0.10    | 0.12     | 0.11     | 0.04    |
|     |     |     |        | 0.01    | 0.03    | 0.05    | 0.09    | 0.12    | 0.13    | 0.14     | 0.13     | 0.08    |
|     | 5000| 100 | IRLS   | 0.10    | 0.10    | 0.09    | 0.09    | 0.09    | 0.09    | 0.09     | 0.08     | 0.13    |
|     |     |     |        | 0.00    | 0.00    | 0.00    | 0.01    | 0.02    | 0.03    | 0.03     | 0.03     | 0.02    |
|     |     |     |        | 0.16    | 0.15    | 0.14    | 0.13    | 0.14    | 0.13    | 0.14     | 0.10     | 0.17    |
|     |     |     |        | 0.00    | 0.01    | 0.03    | 0.04    | 0.05    | 0.05    | 0.05     | 0.04     | 0.04    |
|     |     |     |        | 0.01    | 0.02    | 0.03    | 0.04    | 0.05    | 0.06    | 0.06     | 0.05     | 0.07    |
|     | 10000| 100 | IRLS   | 0.07    | 0.08    | 0.07    | 0.07    | 0.08    | 0.08    | 0.08     | 0.08     | 0.13    |
|     |     |     |        | 0.11    | 0.10    | 0.11    | 0.11    | 0.12    | 0.11    | 0.10     | 0.08     | 0.16    |
|     |     |     |        | 0.00    | 0.00    | 0.01    | 0.02    | 0.03    | 0.04    | 0.04     | 0.04     | 0.02    |
|     |     |     |        | 0.15    | 0.14    | 0.15    | 0.14    | 0.14    | 0.13    | 0.13     | 0.08     | 0.18    |
Table 10: Simulation results of standard deviation of FPR.

| \( \beta \) | \( n \) | \( p \) | \( q \) | Method | \( r = 0.0 \) | \( r = 0.1 \) | \( r = 0.2 \) | \( r = 0.5 \) | \( r = 0.7 \) | \( r = 0.9 \) | \( r = 0.95 \) | \( r = 0.99 \) | \( r = ^{'}C \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1000 | 100 | 6 | IRLS | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| | | | glmnet | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| 5000 | 100 | 6 | IRLS | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| | | | glmnet | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| 10000 | 400 | 6 | IRLS | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| | | | glmnet | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| 10000 | 100 | 6 | IRLS | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| | | | glmnet | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| 5000 | 100 | 6 | IRLS | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| | | | glmnet | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| 10000 | 100 | 6 | IRLS | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| | | | glmnet | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| 10000 | 400 | 6 | IRLS | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| | | | glmnet | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |

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### Table 11: Simulation results of mean of BCR.

| $\beta$ | $n$   | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ | $r = 0.9$ | $r = 0.95$ | $r = 0.99$ | $r = \infty$ CC$^*$ |
|---------|-------|-----|-----|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------------|
| $\beta_1$ | 1000  | 100 | 4   | IRLS   | 0.85      | 0.89      | 0.95      | 0.88      | 0.88      | 0.78      | 0.78      | 0.78      | 0.74      |
|          |       |     | 6   | Glmnet | 0.99      | 0.90      | 0.97      | 0.95      | 0.95      | 0.85      | 0.85      | 0.84      | 0.81      |
|          |       |     | 8   | IRLS   | 0.99      | 0.89      | 0.97      | 0.88      | 0.84      | 0.78      | 0.78      | 0.78      | 0.74      |
|          |       |     |     | Glmnet | 0.79      | 0.81      | 0.88      | 0.88      | 0.82      | 0.78      | 0.78      | 0.78      | 0.74      |
| $\beta_2$ | 5000  | 100 | 4   | IRLS   | 0.79      | 0.89      | 0.94      | 0.88      | 0.84      | 0.78      | 0.78      | 0.78      | 0.74      |
|          |       |     | 6   | Glmnet | 0.98      | 0.90      | 0.96      | 0.88      | 0.84      | 0.78      | 0.78      | 0.78      | 0.74      |
|          |       |     | 8   | IRLS   | 0.75      | 0.96      | 0.95      | 0.93      | 0.87      | 0.73      | 0.59      | 0.90      | 0.90      |
|          |       |     |     | Glmnet | 0.98      | 0.96      | 0.95      | 0.94      | 0.87      | 0.73      | 0.59      | 0.90      | 0.90      |
| $\beta_3$ | 5000  | 400 | 4   | IRLS   | 0.94      | 0.95      | 0.96      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      |
|          |       |     | 6   | Glmnet | 0.99      | 0.99      | 0.99      | 0.98      | 0.98      | 0.98      | 0.98      | 0.98      | 0.98      |
|          |       |     | 8   | IRLS   | 0.86      | 0.96      | 0.96      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      |
|          |       |     |     | Glmnet | 0.83      | 0.95      | 0.96      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      |
| $\beta_4$ | 5000  | 400 | 4   | IRLS   | 0.94      | 0.95      | 0.96      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      |
|          |       |     | 6   | Glmnet | 0.99      | 0.99      | 0.99      | 0.98      | 0.98      | 0.98      | 0.98      | 0.98      | 0.98      |
|          |       |     | 8   | IRLS   | 0.93      | 0.96      | 0.96      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      |
|          |       |     |     | Glmnet | 0.93      | 0.96      | 0.96      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      | 0.95      |

Note: The table presents the simulated mean of BCR for different values of $\beta$, $n$, $p$, and $q$, using IRLS and Glmnet methods. The results are given for various values of $r$.
Table 12: Simulation results of standard deviation of BCR

| β   | n   | p   | Method | r = 0.0 | r = 0.1 | r = 0.2 | r = 0.5 | r = 0.7 | r = 0.9 | r = 0.95 | r = 0.99 | r = ' CC* |
|-----|-----|-----|--------|---------|---------|---------|---------|---------|---------|----------|----------|-----------|
| 1000| 100 | 4   | IRLS   | 0.08    | 0.08    | 0.08    | 0.07    | 0.07    | 0.05    | 0.11     | 0.11     | 0.07      |
|     |     | 6   | Glmnet | 0.01    | 0.01    | 0.01    | 0.01    | 0.01    | 0.04    | 0.10     | 0.11     | 0.05      |
|     |     | 8   |        | 0.02    | 0.02    | 0.02    | 0.02    | 0.02    | 0.06    | 0.09     | 0.08     | 0.05      |
| 5000| 100 | 4   | IRLS   | 0.06    | 0.07    | 0.06    | 0.05    | 0.06    | 0.06    | 0.06     | 0.05     | 0.08      |
|     |     | 6   | Glmnet | 0.00    | 0.00    | 0.01    | 0.01    | 0.01    | 0.01    | 0.02     | 0.10     | 0.01      |
|     |     | 8   |        | 0.01    | 0.01    | 0.01    | 0.02    | 0.02    | 0.02    | 0.09     | 0.09     | 0.02      |
| 10000| 400 | 4   | IRLS   | 0.07    | 0.04    | 0.04    | 0.04    | 0.04    | 0.04    | 0.04     | 0.04     | 0.04      |
|     |     | 6   | Glmnet | 0.00    | 0.00    | 0.00    | 0.00    | 0.01    | 0.01    | 0.12     | 0.03     |           |
|     |     | 8   |        | 0.00    | 0.00    | 0.00    | 0.00    | 0.01    | 0.02    | 0.09     | 0.05     | 0.03      |
| 10000| 400 | 4   | IRLS   | 0.07    | 0.03    | 0.03    | 0.03    | 0.03    | 0.03    | 0.03     | 0.03     | 0.03      |
|     |     | 6   | Glmnet | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00    | 0.00     | 0.07     | 0.01      |
|     |     | 8   |        | 0.00    | 0.00    | 0.00    | 0.00    | 0.01    | 0.02    | 0.09     | 0.08     | 0.02      |
| 10000| 400 | 4   | IRLS   | 0.07    | 0.02    | 0.02    | 0.02    | 0.02    | 0.02    | 0.02     | 0.09     | 0.09      |
|     |     | 6   | Glmnet | 0.00    | 0.00    | 0.00    | 0.00    | 0.01    | 0.01    | 0.02     | 0.09     | 0.09      |
|     |     | 8   |        | 0.00    | 0.00    | 0.00    | 0.00    | 0.01    | 0.01    | 0.02     | 0.09     | 0.09      |

Note: The table continues with similar entries for different values of β and different sample sizes.
## Table 13: Simulation results of Elapsed Time (in minutes).

| β  | n  | p  | Method | \( r = 0.0 \) | \( r = 0.1 \) | \( r = 0.2 \) | \( r = 0.5 \) | \( r = 0.7 \) | \( r = 0.9 \) | \( r = 0.95 \) | \( r = 0.99 \) | \( r = 0.99 \) |
|----|----|----|--------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1000 | 100 | 4  | IRLS   | 1.14        | 1.50        | 0.38        | 0.40        | 0.33        | 2.34        | 0.85        | 4.90        | 3.77        |
|      |    | 4  | glmnet | 2.71        | 3.10        | 3.69        | 4.50        | 5.00        | 7.28        | 10.25       | 15.15       | 6.79        |
|      |    | 6  | IRLS   | 1.21        | 1.60        | 1.58        | 1.10        | 0.60        | 0.65        | 0.67        | 0.67        | 1.62        |
|      |    | 6  | glmnet | 2.96        | 3.04        | 3.27        | 4.45        | 5.80        | 8.57        | 10.86       | 14.86       | 6.19        |
|      |    | 8  | IRLS   | 1.42        | 1.80        | 1.87        | 1.85        | 2.32        | 0.30        | 0.40        | 0.40        | 0.37        |
|      |    | 8  | glmnet | 3.12        | 3.35        | 3.67        | 5.41        | 7.63        | 13.25       | 16.24       | 17.28       | 5.78        |
| 5000 | 100 | 4  | IRLS   | 1.30        | 1.20        | 14.25       | 15.88       | 16.70       | 20.94       | 20.93       | 22.87       | 23.38       |
|      |    | 4  | glmnet | 9.41        | 9.82        | 11.22       | 11.70       | 17.49       | 21.17       | 27.87       | 20.95       | 24.47       |
|      |    | 6  | IRLS   | 14.76       | 14.28       | 17.47       | 17.18       | 18.13       | 22.29       | 22.28       | 25.20       | 25.84       |
|      |    | 6  | glmnet | 10.09       | 10.37       | 10.71       | 15.62       | 17.99       | 26.77       | 30.67       | 39.39       | 21.47       |
|      |    | 8  | IRLS   | 15.04       | 20.20       | 17.55       | 19.27       | 19.80       | 21.02       | 23.02       | 26.00       | 24.40       |
|      |    | 8  | glmnet | 10.70       | 11.13       | 11.94       | 17.48       | 19.96       | 28.13       | 32.74       | 39.93       | 20.05       |
| 5000 | 400 | 4  | IRLS   | 77.09       | 82.15       | 82.90       | 90.52       | 102.95      | 134.23      | 134.23      | 155.40      | 176.90      |
|      |    | 4  | glmnet | 45.42       | 45.40       | 47.49       | 56.93       | 72.94       | 93.07       | 111.88      | 150.18      | 129.73      |
|      |    | 6  | IRLS   | 90.60       | 93.09       | 103.73      | 94.57       | 118.22      | 133.14      | 133.13      | 164.23      | 206.61      |
|      |    | 6  | glmnet | 48.85       | 50.83       | 51.83       | 75.48       | 80.32       | 101.43      | 126.90      | 144.98      | 103.35      |
|      |    | 8  | IRLS   | 93.34       | 104.82      | 114.02      | 108.77      | 127.38      | 44.28       | 44.42       | 14.75       | 245.54      |
|      |    | 8  | glmnet | 51.84       | 51.36       | 55.04       | 84.58       | 95.09       | 121.67      | 119.41      | 150.50      | 96.12       |
| 10000| 400 | 4  | IRLS   | 151.42      | 139.67      | 144.17      | 164.93      | 190.90      | 234.47      | 234.47      | 254.92      | 262.96      |
|      |    | 4  | glmnet | 73.75       | 70.84       | 74.91       | 80.51       | 118.30      | 157.79      | 160.21      | 80.98       | 214.14      |
|      |    | 6  | IRLS   | 154.24      | 153.90      | 178.83      | 185.22      | 193.50      | 236.81      | 236.80      | 270.03      | 310.97      |
|      |    | 6  | glmnet | 76.00       | 74.92       | 75.30       | 116.05      | 139.66      | 162.99      | 185.53      | 195.12      | 169.96      |
|      |    | 8  | IRLS   | 160.14      | 170.40      | 189.57      | 210.45      | 234.55      | 275.12      | 275.10      | 309.80      | 320.17      |
|      |    | 8  | glmnet | 78.56       | 82.03       | 80.44       | 128.25      | 145.04      | 178.39      | 183.28      | 225.87      | 164.69      |

\[ r = 0.0, r = 0.1, r = 0.2, r = 0.5, r = 0.7, r = 0.9, r = 0.95, r = 0.99 \]

\[ \beta = 0, 0.25, 0.5, 0.75, 1.0 \]
CASE II: $n = p$

Table 14: Simulation results of mean of BIAS.

| $\beta_1$ | $n$ | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.75$ |
|-----------|-----|-----|-----|--------|-----------|-----------|-----------|-----------|-----------|
| 100       | 100 |     | 4   | IRLS   | 15.25     | 23.39     | 25.09     | 27.36     | 4.45      |
|           |     |     | 6   | Glmnet | 3.05      | 3.06      | 3.38      | 3.71      | 4.45      |
|           |     |     | 8   | IRLS   | 27.47     | 23.39     | 25.09     | 27.36     | 6.38      |
|           |     |     |     | Glmnet | 6.52      | 6.81      | 7.26      | 8.16      | 8.05      |
| 400       | 400 |     | 4   | IRLS   | 27.47     | 23.39     | 44.71     | 27.36     | 3.86      |
|           |     |     | 6   | Glmnet | 3.63      | 3.75      | 4.05      | 5.12      | 5.82      |
|           |     |     | 8   | IRLS   | 27.47     | 23.39     | 44.71     | 27.36     | 7.48      |
|           |     |     |     | Glmnet | 5.09      | 5.59      | 5.97      | 7.25      | 7.48      |
| 1000      | 1000|     | 4   | IRLS   | 27.47    | 23.39     | 44.71     | 27.36     | 5.00      |
|           |     |     | 6   | Glmnet | 1.81      | 1.87      | 2.04      | 2.41      | 3.60      |
|           |     |     | 8   | IRLS   | 27.47    | 23.39     | 44.71     | 27.36     | 5.10      |
|           |     |     |     | Glmnet | 2.97      | 3.24      | 3.55      | 4.20      | 5.10      |

Table 15: Simulation results of standard deviation of BIAS.

| $\beta_2$ | $n$ | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.75$ |
|-----------|-----|-----|-----|--------|-----------|-----------|-----------|-----------|-----------|
| 100       | 100 |     | 4   | IRLS   | 11.20     | 11.14     | 15.40     | 0.46      | 0.60      |
|           |     |     | 6   | Glmnet | 0.46      | 0.60      | 0.97      | 0.59      | 0.59      |
|           |     |     | 8   | IRLS   | 11.00     | 9.07      | 15.40     | 0.57      | 0.60      |
|           |     |     |     | Glmnet | 0.59      | 0.59      | 0.63      | 0.63      | 0.63      |
| 400       | 400 |     | 4   | IRLS   | 10.17    | 7.84      | 15.40     | 0.46      | 0.62      |
|           |     |     | 6   | Glmnet | 0.59      | 0.59      | 0.75      | 0.75      | 0.84      |
|           |     |     | 8   | IRLS   | 10.17    | 7.84      | 17.99     | 0.40      | 0.33      |
|           |     |     |     | Glmnet | 0.40      | 0.40      | 0.54      | 0.54      | 0.54      |
| 1000      | 1000|     | 4   | IRLS   | 10.17    | 9.27      | 17.99     | 0.46      | 0.46      |
|           |     |     | 6   | Glmnet | 0.46      | 0.46      | 0.69      | 0.69      | 0.69      |
|           |     |     | 8   | IRLS   | 10.17    | 9.27      | 17.99     | 0.46      | 0.46      |
|           |     |     |     | Glmnet | 0.46      | 0.46      | 0.69      | 0.69      | 0.69      |

* Blank cells denote scenarios that IRLS method could not converge.
Table 16: Simulation results of mean of $BIAS_{\text{Adj}}$.

| $\beta$ | $n$ | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = \infty$ |
|---------|-----|-----|-----|--------|---------|---------|---------|---------|-----------|
| 4       | 100 | 100 |     | IRLS   | 3.13    | 6.00    | 7.69    | 7.69    | 7.69      |
|         |     |     |     | Glmnet | 0.65    | 0.67    | 0.75    | 0.75    | 0.75      |
| 6       |     |     |     | IRLS   | 3.97    | 4.24    | 7.69    | 7.69    | 7.69      |
|         |     |     |     | Glmnet | 0.60    | 0.68    | 0.75    | 0.75    | 0.75      |
| 8       |     |     |     | IRLS   | 3.06    | 3.27    | 4.24    | 7.69    | 7.69      |
|         |     |     |     | Glmnet | 0.65    | 0.64    | 0.69    | 0.77    | 0.77      |
| 400     | 100 | 100 |     | IRLS   | 3.06    | 3.27    | 4.24    | 7.69    | 7.69      |
|         |     |     |     | Glmnet | 0.47    | 0.49    | 0.63    | 0.86    | 0.86      |
| 6       |     |     |     | IRLS   | 3.06    | 3.27    | 4.24    | 9.08    | 9.08      |
|         |     |     |     | Glmnet | 0.48    | 0.52    | 0.67    | 0.82    | 0.82      |
| 8       |     |     |     | IRLS   | 3.06    | 6.24    | 4.24    | 9.08    | 9.08      |
|         |     |     |     | Glmnet | 0.36    | 0.39    | 0.50    | 0.80    | 0.80      |
| 1000    | 100 | 100 |     | IRLS   | 3.06    | 6.24    | 4.24    | 9.08    | 9.08      |
|         |     |     |     | Glmnet | 0.38    | 0.45    | 0.56    | 0.74    | 0.74      |

$*$ Blank cells denote scenarios that IRLS method could not converge.

Table 17: Simulation results of standard deviation of $BIAS_{\text{Adj}}$.

| $\beta$ | $n$ | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = \infty$ |
|---------|-----|-----|-----|--------|---------|---------|---------|---------|-----------|
| 4       | 100 | 100 |     | IRLS   | 1.72    | 2.79    | 3.85    | 3.85    | 3.85      |
|         |     |     |     | Glmnet | 0.12    | 0.10    | 0.15    | 0.15    | 0.15      |
| 6       |     |     |     | IRLS   | 1.83    | 1.51    | 2.59    | 2.59    | 2.59      |
|         |     |     |     | Glmnet | 0.10    | 0.08    | 0.10    | 0.10    | 0.10      |
| 8       |     |     |     | IRLS   | 1.27    | 0.98    | 1.51    | 1.51    | 1.51      |
|         |     |     |     | Glmnet | 0.06    | 0.06    | 0.08    | 0.09    | 0.09      |
| 400     | 100 | 100 |     | IRLS   | 1.27    | 0.98    | 1.51    | 1.51    | 1.51      |
|         |     |     |     | Glmnet | 0.07    | 0.07    | 0.09    | 0.09    | 0.09      |
| 6       |     |     |     | IRLS   | 1.27    | 1.16    | 1.51    | 1.51    | 1.51      |
|         |     |     |     | Glmnet | 0.05    | 0.05    | 0.06    | 0.09    | 0.09      |
| 8       |     |     |     | IRLS   | 1.27    | 1.16    | 1.51    | 1.51    | 1.51      |
|         |     |     |     | Glmnet | 0.03    | 0.04    | 0.05    | 0.07    | 0.07      |

$*$ Blank cells denote scenarios that IRLS method could not converge.
Table 18: Simulation results of mean of TPR.

| $\beta$ | $n$  | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = CC^2$ |
|---------|------|-----|-----|--------|---------|---------|---------|---------|-----------|
|         | 100  | 100 | 4   | IRLS   | 1.00   | 0.99    | 0.99    | 0.97    | 0.58      |
|         |      |     |     | Glmnet | 0.92   | 0.93    | 0.90    | 0.82    | 0.54      |
| 100      | 6    |     | 6   | IRLS   | 1.00   | 0.99    | 0.99    | 0.97    | 0.54      |
|         |      |     |     | Glmnet | 0.95   | 0.90    | 0.84    | 0.68    | 0.54      |
| 6        | 8    |     | 8   | IRLS   | 0.99   | 0.99    | 0.93    | 0.97    | 0.53      |
|         |      |     |     | Glmnet | 0.86   | 0.88    | 0.80    | 0.64    | 0.53      |

Table 19: Simulation results of standard deviation of TPR.

| $\beta$ | $n$  | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = CC^2$ |
|---------|------|-----|-----|--------|---------|---------|---------|---------|-----------|
|         | 100  | 100 | 4   | IRLS   | 0.02   | 0.05    | 0.06    | 0.09    | 0.09      |
|         |      |     |     | Glmnet | 0.15   | 0.10    | 0.16    | 0.17    | 0.17      |
| 100      | 6    |     | 6   | IRLS   | 0.02   | 0.05    | 0.06    | 0.09    | 0.09      |
|         |      |     |     | Glmnet | 0.19   | 0.13    | 0.15    | 0.14    | 0.18      |
| 6        | 8    |     | 8   | IRLS   | 0.05   | 0.05    | 0.00    | 0.02    | 0.00      |
|         |      |     |     | Glmnet | 0.20   | 0.13    | 0.13    | 0.14    | 0.17      |
| 8        | 400  | 400 | 400 | IRLS   | 0.05   | 0.05    | 0.00    | 0.09    | 0.09      |
|         |      |     |     | Glmnet | 0.00   | 0.00    | 0.00    | 0.02    | 0.00      |
| 400      | 1000 | 1000| 1000| IRLS   | 0.05   | 0.05    | 0.00    | 0.09    | 0.09      |
|         |      |     |     | Glmnet | 0.00   | 0.00    | 0.00    | 0.02    | 0.00      |
| 1000     | 1000 | 1000| 1000| IRLS   | 0.05   | 0.05    | 0.00    | 0.09    | 0.09      |
|         |      |     |     | Glmnet | 0.00   | 0.00    | 0.00    | 0.02    | 0.00      |

* Blank cells denote scenarios that IRLS method could not converge.
### Table 20: Simulation results of mean of FPR.

| $\beta$ | $n$ | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ | $r = 0.9$ |
|--------|-----|-----|-----|--------|---------|---------|---------|---------|---------|---------|
| $\beta_0$ | 100 | 100 | 4  | IRLS   | 0.71    | 0.72    | 0.74    | 0.72    | 0.03    | 0.03    |
|         |     |     |    | Glmnet | 0.03    | 0.03    | 0.06    | 0.06    | 0.06    | 0.08    |
|         | 400 | 400 | 4  | IRLS   | 0.78    | 0.72    | 0.74    | 0.72    | 0.05    | 0.05    |
|         |     |     |    | Glmnet | 0.05    | 0.05    | 0.07    | 0.10    | 0.10    | 0.09    |
|         | 1000| 1000| 4  | IRLS   | 0.78    | 0.72    | 0.76    | 0.72    | 0.06    | 0.07    |
|         |     |     |    | Glmnet | 0.06    | 0.07    | 0.09    | 0.11    | 0.11    | 0.09    |
| $\beta_1$ | 400 | 400 | 4  | IRLS   | 0.78    | 0.72    | 0.60    | 0.72    | 0.01    | 0.02    |
|         |     |     |    | Glmnet | 0.01    | 0.02    | 0.02    | 0.03    | 0.03    | 0.04    |
|         | 1000| 1000| 6  | IRLS   | 0.78    | 0.72    | 0.60    | 0.72    | 0.02    | 0.03    |
|         |     |     |    | Glmnet | 0.02    | 0.04    | 0.05    | 0.05    | 0.05    | 0.06    |
| $\beta_2$ | 400 | 400 | 4  | IRLS   | 0.78    | 0.72    | 0.60    | 0.72    | 0.00    | 0.01    |
|         |     |     |    | Glmnet | 0.01    | 0.01    | 0.01    | 0.01    | 0.01    | 0.03    |
|         | 1000| 1000| 6  | IRLS   | 0.78    | 0.72    | 0.60    | 0.72    | 0.01    | 0.02    |
|         |     |     |    | Glmnet | 0.01    | 0.02    | 0.02    | 0.03    | 0.03    | 0.04    |

* Blank cells denote scenarios that IRLS method could not converge.

### Table 21: Simulation results of standard deviation of FPR.

| $\beta$ | $n$ | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ | $r = 0.9$ |
|--------|-----|-----|-----|--------|---------|---------|---------|---------|---------|---------|
| $\beta_0$ | 100 | 100 | 4  | IRLS   | 0.10    | 0.24    | 0.19    | 0.17    | 0.05    | 0.04    |
|         |     |     |    | Glmnet | 0.07    | 0.05    | 0.04    | 0.03    | 0.04    | 0.03    |
|         | 400 | 400 | 4  | IRLS   | 0.14    | 0.24    | 0.26    | 0.17    | 0.06    | 0.05    |
|         |     |     |    | Glmnet | 0.02    | 0.02    | 0.01    | 0.01    | 0.01    | 0.01    |
|         | 1000| 1000| 6  | IRLS   | 0.14    | 0.24    | 0.26    | 0.17    | 0.02    | 0.01    |
|         |     |     |    | Glmnet | 0.03    | 0.02    | 0.01    | 0.01    | 0.01    | 0.01    |
| $\beta_1$ | 400 | 400 | 4  | IRLS   | 0.14    | 0.24    | 0.26    | 0.17    | 0.01    | 0.01    |
|         |     |     |    | Glmnet | 0.01    | 0.01    | 0.01    | 0.01    | 0.00    | 0.00    |
|         | 1000| 1000| 6  | IRLS   | 0.14    | 0.24    | 0.26    | 0.17    | 0.01    | 0.01    |
|         |     |     |    | Glmnet | 0.01    | 0.01    | 0.01    | 0.01    | 0.01    | 0.01    |
| $\beta_2$ | 400 | 400 | 4  | IRLS   | 0.21    | 0.26    | 0.26    | 0.26    | 0.05    | 0.04    |
|         |     |     |    | Glmnet | 0.05    | 0.04    | 0.04    | 0.04    | 0.04    | 0.04    |
|         | 1000| 1000| 6  | IRLS   | 0.28    | 0.17    | 0.26    | 0.26    | 0.05    | 0.04    |
|         |     |     |    | Glmnet | 0.05    | 0.04    | 0.04    | 0.04    | 0.04    | 0.04    |

* Blank cells denote scenarios that IRLS method could not converge.
### Table 22: Simulation results of mean of BCR.

| $\beta$ | $n$ | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.5$ CC\% |
|---------|-----|-----|-----|--------|-----------|-----------|-----------|-----------|----------------| |
|         | 100 | 100 |     |        | 0.06     | 0.64      | 0.62      | 0.63      | 0.94           | |
|         | 4   |     |     | IRLS   | 0.94     | 0.95      | 0.92      | 0.88      | 0.75           | |
|         | 6   |     |     | Glmnet | 0.91     | 0.92      | 0.89      | 0.79      | 0.72           | |
|         | 8   |     |     |        | 0.90     | 0.90      | 0.85      | 0.77      | 0.72           | |
| $\beta_1$ | 400 | 400 |     |        | 0.60     | 0.64      | 0.70      | 0.63      | 0.99           | |
|         | 4   |     |     | IRLS   | 0.99     | 0.99      | 0.99      | 0.98      | 0.85           | |
|         | 6   |     |     | Glmnet | 0.99     | 0.99      | 0.98      | 0.97      | 0.88           | |
|         | 8   |     |     |        | 0.99     | 0.98      | 0.98      | 0.95      | 0.89           | |
|         | 1000| 1000|     |        | 0.60    | 0.64      | 0.70      | 0.63      | 0.99           | |
|         | 4   |     |     | IRLS   | 1.00     | 1.00      | 1.00      | 0.99      | 0.86           | |
|         | 6   |     |     | Glmnet | 0.60     | 0.64      | 0.70      | 0.63      | 0.99           | |
|         | 8   |     |     |        | 0.99     | 0.99      | 0.99      | 0.99      | 0.90           | |
| $\beta_2$ | 400 | 400 |     |        | 0.60    | 0.64      | 0.70      | 0.63      | 0.99           | |
|         | 4   |     |     | IRLS   | 0.62    | 0.58      | 0.59      | 0.59      | 0.99           | |
|         | 6   |     |     | Glmnet | 0.78    | 0.83      | 0.79      | 0.74      | 0.67           | |
|         | 8   |     |     |        | 0.77    | 0.80      | 0.78      | 0.72      | 0.68           | |
|         | 1000| 1000|     |        | 0.60    | 0.64      | 0.70      | 0.63      | 0.99           | |
|         | 4   |     |     | IRLS   | 0.96    | 0.96      | 0.97      | 0.92      | 0.81           | |
|         | 6   |     |     | Glmnet | 0.97    | 0.97      | 0.96      | 0.91      | 0.82           | |
|         | 8   |     |     |        | 0.97    | 0.96      | 0.94      | 0.89      | 0.82           | |

* Blank cells denote scenarios that IRLS method could not converge.

### Table 23: Simulation results of standard deviation of BCR.

| $\beta$ | $n$ | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.5$ CC\% |
|---------|-----|-----|-----|--------|-----------|-----------|-----------|-----------|----------------| |
|         | 100 | 100 |     |        | 0.05    | 0.11      | 0.10      | 0.08      | 0.09           | |
|         | 4   |     |     | IRLS   | 0.07    | 0.07      | 0.08      | 0.08      | 0.09           | |
|         | 6   |     |     | Glmnet | 0.05    | 0.11      | 0.10      | 0.08      | 0.08           | |
|         | 8   |     |     |        | 0.06    | 0.06      | 0.07      | 0.07      | 0.08           | |
| $\beta_1$ | 400 | 400 |     |        | 0.06    | 0.11      | 0.13      | 0.08      | 0.00           | |
|         | 4   |     |     | IRLS   | 0.01    | 0.01      | 0.01      | 0.01      | 0.00           | |
|         | 6   |     |     | Glmnet | 0.06    | 0.11      | 0.13      | 0.08      | 0.03           | |
|         | 8   |     |     |        | 0.06    | 0.11      | 0.13      | 0.08      | 0.03           | |
|         | 1000| 1000|     |        | 0.06   | 0.11      | 0.13      | 0.08      | 0.00           | |
|         | 4   |     |     | IRLS   | 0.01   | 0.00      | 0.00      | 0.00      | 0.00           | |
|         | 6   |     |     | Glmnet | 0.06   | 0.11      | 0.13      | 0.08      | 0.00           | |
|         | 8   |     |     |        | 0.06   | 0.11      | 0.13      | 0.08      | 0.00           | |

* Blank cells denote scenarios that IRLS method could not converge.
Table 24: Simulation results of Elapsed Time (in minutes).

| β | n    | p    | q    | Method | \( r = 0.0 \) | \( r = 0.1 \) | \( r = 0.2 \) | \( r = 0.5 \) | \( r = \beta \) CC* |
|---|------|------|------|--------|-------------|-------------|-------------|-------------|----------------|
| 100 | 4    | 3.38 | 6    | IRLS   | 1.14        | 3.07        | 3.07        | 2.48        |                |
|    |      |      | 8    |        | 1.13        | 0.90        | 1.45        | 0.05        |                |
|    |      |      |      |        | 0.59        | 0.60        | 0.56        | 0.57        |                |
|    |      |      |      |        | 4.45        | 1.75        | 3.62        | 0.57        |                |
|    |      |      |      |        | 0.61        | 0.58        | 0.57        | 0.57        |                |
| 200 | 4    | 3.01 | 6    | IRLS   | 0.78        | 3.07        | 8.45        | 0.45        |                |
|    |      |      |      |        | 3.79        | 3.30        | 3.47        | 3.29        |                |
|    |      |      |      |        | 0.25        | 11.62       | 3.85        | 0.40        |                |
|    |      |      |      |        | 3.73        | 3.07        | 2.81        | 2.99        |                |
|    |      |      |      |        | 3.33        | 10.92       | 0.35        | 0.37        |                |
|    |      |      |      |        | 3.49        | 2.96        | 2.72        | 2.87        |                |
| 1000| 4    | 25.55| 6    | IRLS   | 4.18        | 25.55       | 16.72       | 9.20        |                |
|    |      |      |      |        | 22.59       | 19.13       | 16.55       | 15.32       |                |
|    |      |      |      |        | 4.83        | 35.63       | 22.83       | 6.43        |                |
|    |      |      |      |        | 21.62       | 16.63       | 13.75       | 14.05       |                |
|    |      |      |      |        | 4.88        | 52.13       | 17.33       | 6.08        |                |
|    |      |      |      |        | 20.33       | 14.84       | 12.57       | 15.47       |                |

* Blank cells denote scenarios that IRLS method could not converge.
**CASE III: \( n < p \)**

Table 25: Simulation results of mean of BIAS.

| \( \beta \) | \( n \) | \( p \) | \( q \) | Method | \( r = 0.0 \) | \( r = 0.1 \) | \( r = 0.2 \) | \( r = 0.5 \) | \( r = 0.7 \) |
|------------|------|------|------|--------|---------|---------|---------|---------|---------|
| 4          | 400  | 5000 | 4    | IRLS   | 37.80   | 2.76    | 3.04    | 3.15    | 4.12    | 4.89    |
| 6          | IRLS | 37.80 | 4.64 | 5.00  | 5.77    | 6.86    | 8.00    |
| 8          | IRLS | 37.80 | 6.65 | 7.28  | 8.26    | 9.76    | 10.68   |
| 4          | 400  | 10000| 6    | IRLS   | 37.80   | 4.61    | 4.88    | 5.37    | 6.91    | 8.30    |
| 8          | IRLS | 37.80 | 6.80 | 7.43  | 7.53    | 9.93    | 11.05   |
| 4          | 400  | 5000 | 6    | IRLS   | 3.59    | 3.80    | 4.36    | 5.20    | 5.90    | 6.52    |
| 8          | IRLS | 5.07  | 5.59 | 6.19  | 7.44    | 8.40    |
| 4          | 400  | 10000| 6    | IRLS   | 2.24    | 2.32    | 2.63    | 3.49    | 4.12    | 4.83    |
| 8          | IRLS | 3.53  | 3.83 | 4.17  | 5.47    | 6.35    |
| 4          | 400  | 5000 | 6    | IRLS   | 39.22   | 36.33   | 37.97   | 46.57   |
| 8          | IRLS | 5.14  | 5.39 | 5.90  | 7.47    | 8.50    |

* Blank cells denote scenarios that IRLS method could not converge.

Table 26: Simulation results of standard deviation of BIAS.

| \( \beta \) | \( n \) | \( p \) | \( q \) | Method | \( r = 0.0 \) | \( r = 0.1 \) | \( r = 0.2 \) | \( r = 0.5 \) | \( r = 0.7 \) |
|------------|------|------|------|--------|---------|---------|---------|---------|---------|
| 4          | 400  | 5000 | 4    | IRLS   | 10.34   | 0.40    | 0.42    | 0.44    | 0.46    | 0.51    |
| 6          | IRLS | 10.34 | 0.51 | 0.56  | 0.47    | 0.57    | 0.67    |
| 8          | IRLS | 10.34 | 0.65 | 0.48  | 0.51    | 0.59    | 0.61    |
| 4          | 400  | 10000| 6    | IRLS   | 10.34   | 0.41    | 0.41    | 0.48    | 0.45    | 0.50    |
| 8          | IRLS | 10.34 | 0.65 | 0.47  | 0.49    | 0.55    | 0.64    |

* Blank cells denote scenarios that IRLS method could not converge.

Table 27: Simulation results of mean of BIAS_{Adj}.

| \( \beta \) | \( n \) | \( p \) | \( q \) | Method | \( r = 0.0 \) | \( r = 0.1 \) | \( r = 0.2 \) | \( r = 0.5 \) | \( r = 0.7 \) |
|------------|------|------|------|--------|---------|---------|---------|---------|---------|
| 4          | 400  | 5000 | 4    | IRLS   | 9.45    | 0.69    | 0.76    | 0.79    | 1.03    | 1.22    |
| 6          | IRLS | 9.45  | 0.77 | 0.96  | 1.14    | 1.33    |
| 8          | IRLS | 9.45  | 0.83 | 0.91  | 1.04    | 1.22    | 1.34    |
| 4          | 400  | 10000| 6    | IRLS   | 9.45    | 0.70    | 0.74    | 0.81    | 1.08    | 1.31    |
| 8          | IRLS | 9.45  | 0.77 | 0.81  | 0.99    | 1.15    | 1.38    |
| 4          | 400  | 5000 | 6    | IRLS   | 9.45    | 0.56    | 0.60    | 0.62    | 0.82    | 0.95    |
| 8          | IRLS | 9.45  | 0.63 | 0.73  | 0.87    | 0.98    |
| 4          | 400  | 10000| 6    | IRLS   | 4.90    | 4.54    | 7.76    |
| 8          | IRLS | 4.90  | 4.90 | 4.90  | 4.90    | 4.90    | 4.90    |

* Blank cells denote scenarios that IRLS method could not converge.

Table 28: Simulation results of standard deviation of BIAS_{Adj}.

| \( \beta \) | \( n \) | \( p \) | \( q \) | Method | \( r = 0.0 \) | \( r = 0.1 \) | \( r = 0.2 \) | \( r = 0.5 \) | \( r = 0.7 \) |
|------------|------|------|------|--------|---------|---------|---------|---------|---------|
| 4          | 400  | 5000 | 4    | IRLS   | 2.58    | 0.10    | 0.11    | 0.11    | 0.11    | 0.13    |
| 6          | IRLS | 2.58  | 0.09 | 0.09  | 0.08    | 0.09    | 0.11    |
| 8          | IRLS | 2.58  | 0.08 | 0.06  | 0.06    | 0.07    | 0.08    |
| 4          | 400  | 10000| 6    | IRLS   | 2.58    | 0.10    | 0.10    | 0.12    | 0.12    | 0.13    |
| 8          | IRLS | 2.58  | 0.08 | 0.07  | 0.08    | 0.10    | 0.10    |
| 4          | 400  | 5000 | 6    | IRLS   | 2.58    | 0.10    | 0.11    | 0.12    | 0.12    | 0.13    |
| 8          | IRLS | 2.58  | 0.08 | 0.06  | 0.06    | 0.07    | 0.08    |

* Blank cells denote scenarios that IRLS method could not converge.
**Table 29: Simulation results of mean of TPR.**

| $\beta$ | $n$   | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ |
|---------|-------|-----|-----|--------|------------|------------|------------|------------|------------|
|         | 400   | 5000| 6   | IRLS   | 1.00       | 1.00       | 1.00       | 0.94       | 0.79       |
|         |       |     |     | Glmnet | 1.00       | 1.00       | 1.00       | 0.92       | 0.51       |
| $\beta_1$ | 8   |     |     | IRLS   | 1.00       | 0.99       | 0.96       | 0.68       | 0.36       |
|         |       |     |     | Glmnet | 1.00       | 0.99       | 0.96       | 0.57       | 0.36       |
|         | 400   | 10000| 6 | IRLS   | 1.00       | 1.00       | 1.00       | 0.92       | 0.63       |
|         |       |     |     | Glmnet | 1.00       | 1.00       | 1.00       | 0.85       | 0.43       |
| $\beta_2$ | 8   |     |     | IRLS   | 1.00       | 0.99       | 0.98       | 0.68       | 0.27       |
|         |       |     |     | Glmnet | 1.00       | 0.98       | 0.98       | 0.57       | 0.27       |

* Blank cells denote scenarios that IRLS method could not converge.

**Table 30: Simulation results of standard deviation of TPR.**

| $\beta$ | $n$   | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ |
|---------|-------|-----|-----|--------|------------|------------|------------|------------|------------|
|         | 400   | 5000| 6   | IRLS   | 0.00       | 0.00       | 0.00       | 0.00       | 0.11       |
|         |       |     |     | Glmnet | 0.00       | 0.00       | 0.00       | 0.11       | 0.20       |
| $\beta_1$ | 8   |     |     | IRLS   | 0.02       | 0.00       | 0.05       | 0.15       | 0.20       |
|         |       |     |     | Glmnet | 0.02       | 0.03       | 0.07       | 0.15       | 0.15       |
|         | 400   | 10000| 6 | IRLS   | 0.00       | 0.00       | 0.00       | 0.13       | 0.23       |
|         |       |     |     | Glmnet | 0.00       | 0.06       | 0.00       | 0.13       | 0.18       |
| $\beta_2$ | 8   |     |     | IRLS   | 0.00       | 0.04       | 0.05       | 0.14       | 0.15       |
|         |       |     |     | Glmnet | 0.01       | 0.04       | 0.05       | 0.16       | 0.13       |

* Blank cells denote scenarios that IRLS method could not converge.

**Table 31: Simulation results of mean of FPR.**

| $\beta$ | $n$   | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ |
|---------|-------|-----|-----|--------|------------|------------|------------|------------|------------|
|         | 400   | 5000| 6   | IRLS   | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
|         |       |     |     | Glmnet | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
| $\beta_1$ | 8   |     |     | IRLS   | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
|         |       |     |     | Glmnet | 0.01       | 0.01       | 0.01       | 0.01       | 0.01       |
|         | 400   | 10000| 6 | IRLS   | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
|         |       |     |     | Glmnet | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
| $\beta_2$ | 8   |     |     | IRLS   | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
|         |       |     |     | Glmnet | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |

* Blank cells denote scenarios that IRLS method could not converge.

**Table 32: Simulation results of standard deviation of FPR.**

| $\beta$ | $n$   | $p$ | $q$ | Method | $r = 0.0$ | $r = 0.1$ | $r = 0.2$ | $r = 0.5$ | $r = 0.7$ |
|---------|-------|-----|-----|--------|------------|------------|------------|------------|------------|
|         | 400   | 5000| 6   | IRLS   | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
|         |       |     |     | Glmnet | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
| $\beta_1$ | 8   |     |     | IRLS   | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
|         |       |     |     | Glmnet | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
|         | 400   | 10000| 6 | IRLS   | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
|         |       |     |     | Glmnet | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
| $\beta_2$ | 8   |     |     | IRLS   | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |
|         |       |     |     | Glmnet | 0.00       | 0.00       | 0.00       | 0.00       | 0.00       |

* Blank cells denote scenarios that IRLS method could not converge.
### Table 33: Simulation results of mean of BCR.

| β   | p  | q  | Method | r = 0.0 | r = 0.1 | r = 0.2 | r = 0.5 | r = 0.7 |
|-----|----|----|--------|---------|---------|---------|---------|---------|
| 400 | 5000 | 6 | IRLS    | 0.97    | 1.00    | 1.00    | 1.00    | 0.89    |
|     |      | 8 | Glmnet  | 1.00    | 0.99    | 0.98    | 0.78    | 0.68    |
| 400 | 10000 | 6 | IRLS    | 0.97    | 1.00    | 1.00    | 0.96    | 0.81    |
|     |      | 8 | Glmnet  | 1.00    | 1.00    | 1.00    | 0.92    | 0.71    |

*Blank cells denote scenarios that IRLS method could not converge.

### Table 34: Simulation results of standard deviation of BCR.

| β   | p  | q  | Method | r = 0.0 | r = 0.1 | r = 0.2 | r = 0.5 | r = 0.7 |
|-----|----|----|--------|---------|---------|---------|---------|---------|
| 400 | 5000 | 6 | IRLS    | 0.01    | 0.00    | 0.00    | 0.06    | 0.10    |
|     |      | 8 | Glmnet  | 0.01    | 0.01    | 0.03    | 0.08    | 0.10    |
| 400 | 10000 | 6 | IRLS    | 0.01    | 0.00    | 0.00    | 0.07    | 0.11    |
|     |      | 8 | Glmnet  | 0.01    | 0.01    | 0.03    | 0.07    | 0.11    |

*Blank cells denote scenarios that IRLS method could not converge.

### Table 35: Simulation results of Elapsed Time (in minutes).

| β   | p  | q  | Method | r = 0.0 | r = 0.1 | r = 0.2 | r = 0.5 | r = 0.7 |
|-----|----|----|--------|---------|---------|---------|---------|---------|
| 400 | 5000 | 6 | IRLS    | 106.26  | 12.31   | 12.25   | 12.47   | 15.59   | 24.68   |
|     |      | 8 | Glmnet  | 13.78   | 11.75   | 12.31   | 15.40   | 32.11   |
| 400 | 10000 | 6 | IRLS    | 4.52    | 160.13  | 131.95  |
|     |      | 8 | Glmnet  | 13.38   | 12.44   | 13.20   | 16.05   | 33.66   |

*Blank cells denote scenarios that IRLS method could not converge.