On the Formation Rejoin Problem

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Flight lead frame

- **Flight lead frame** is parametrized at time $t \in \mathbb{R}$ by its position and orientation, $x_d(t) \in \mathbb{R}^3$ and $R_d(t) \in SO(3)$, in the spatial frame.

- **Admissible lead trajectories** restricted to those such that the velocity, acceleration and angular velocity, $\dot{x}_d(t)$, $\ddot{x}_d(t)$ and $\omega_d(t)$ are bounded, with $\omega_d(t)$ satisfying

  $$\dot{R}_d(t) = R_d(t)\hat{\omega}_d(t).$$

  (1)

- **Hat operator**: matrix representation of the cross product

  $$\hat{\omega} v = \omega \times v, \quad \hat{\omega} = \begin{bmatrix}
  0 & -\omega_3 & \omega_2 \\
  \omega_3 & 0 & -\omega_1 \\
  -\omega_2 & \omega_1 & 0
\end{bmatrix}$$

- **Rotation matrix** $R_d(t)$ maps local vectors to the spatial frame

  $$y_{spatial} = R_d(t)y_{local}.$$
Relative Dynamics

- **Flight lead frame** wingman’s position, velocity and acceleration
  
  \[ p = R_d^T(t) [x - x_d(t)] \]
  
  \[ v = R_d^T(t) [\dot{x} - \dot{x}_d(t)] \]
  
  \[ a = R_d^T(t) [\ddot{x} - \ddot{x}_d(t)] \]

- **Local dynamics**: state \((p, v)\), input \(a\)
  
  \[ \dot{p} = v - \hat{\omega}_d(t) p \]
  
  \[ \dot{v} = a - \hat{\omega}_d(t) v \]
Energy Considerations

We expect the energy of the system to be unaffected by $\hat{\omega}_d(t)$ since it does no work.

- Let $T = \frac{1}{2} v^T v$ be some sort of kinetic energy, then
  $$\dot{T} = -v^T \hat{\omega}_d(t) v + v^T a = v^T a$$

- Let the potential be a pure quadratic, $U = \frac{1}{2} k_p p^T p$, $k_p > 0$, then the total energy is
  $$E = \frac{1}{2} k_p p^T p + \frac{1}{2} v^T v$$
  and the derivative is
  $$\dot{E} = k_p p^T (-\hat{\omega}_d(t) p + v) + v^T a = k_p p^T v + v^T a$$

- What property does the kinetic and potential energy posses that allows the energy to evolve independent of $\hat{\omega}_d(t)$?
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  Rotational Invariance
Rotational Invariance

**Definition**

\[ U : \mathbb{R}^3 \rightarrow \mathbb{R} \text{ is rotationally invariant if} \]

\[ U(Rp) = U(p) \text{ for all } p \in \mathbb{R}^3 \text{ and } R \in SO(3). \]
Rotational Invariance

**Lemma**

$U : \mathbb{R}^3 \rightarrow \mathbb{R}$ is rotationally invariant if and only if

$$DU(p) \cdot \hat{\omega} p = 0, \quad \forall p, \omega \in \mathbb{R}^3.$$  

**Proof.**

$(\Rightarrow)$ Let $p, \omega \in \mathbb{R}^3$ be arbitrary and set $R(t) = e^{t\hat{\omega}} \in SO(3), \ t \in \mathbb{R}$. Then, since $U(R(t)p) = U(p), \ t \in \mathbb{R}$, by rotational invariance, we see that

$$0 = \frac{d}{dt}\{U(R(t)p)\} = DU(R(t)p) \cdot \hat{\omega} R(t)p$$

$(\Leftarrow)$ Let $\bar{R} \in SO(3)$ and $p \in \mathbb{R}^3$ be arbitrary, choose $\omega \in \mathbb{R}^3$ such that $\bar{R} = e^{\hat{\omega}}$ and define $R(t) = e^{t\hat{\omega}}, \ t \in \mathbb{R}$, so that $R(1) = \bar{R}$. Then

$$U(R(t)p) = U(p) + \int_0^1 DU(R(\tau)p) \cdot \hat{\omega} R(\tau)p \ d\tau = F(p)$$
Level sets of a rotationally invariant function are spherical.

Corollary

\[ \nabla U(p) \parallel p \text{ and } DU(p) \cdot p > 0 \text{ (away from } p = 0) \text{ for an increasing rotationally invariant function } U(p). \]

Summary: If the kinetic and potential energy are rotationally invariant the total energy

\[ E(p, v) = T(v) + U(p) \]

will evolve independently of \( \omega_d(t) \) according to

\[ \dot{E} = DU(p) \cdot v + DT(v) \cdot a \]
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general rotational invariance: \( E(p, v) = E(Rp, Rv) \)
Feedback Stabilization

Open loop structure looks like a double integrator, suggesting the use of a PD controller to stabilize the system.

- For the moment let’s entertain the controller:

\[ a = -K_p p - K_v v \]

with \( K_p \) and \( K_v \) symmetric and positive definite.

- Provides exponential stability for \( \omega_d(t) = 0 \). What about a general constant \( \omega_d \)?
Consider a constant turning maneuver in the XY plane with
- $\omega_d = [0 \ 0 \ 1]^T$
- $K_v = I$
- $K_p = \text{diag}(\begin{bmatrix} a & b & 1 \end{bmatrix})$ with $a, b > 0$

For initial conditions in horizontal plane, the system evolves within that plane as $\dot{x} = Ax$ with

$$A = \begin{bmatrix}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
-a & 0 & -1 & 1 \\
0 & -b & -1 & -1
\end{bmatrix}$$

giving a characteristic polynomial of

$$\chi(s) = s^4 + 2s^3 + (a + b + 3)s^2 + (a + b + 2)s + ab - a - b + 2.$$ 

The system is unstable if $a + b > ab + 2$, for instance, $a = 4$ and $b = 1/2$. 
General Stabilizing Feedback

Why doesn’t the general $Kp > 0$ work?

- the associated potential $U(p) = \frac{1}{2} p^T K_p p$ (for proportional feedback $-K_p p = -\nabla U(p)$) is not rotationally invariant.

One solution:

- Use a rotationally invariant potential $U(p)$ that is positive definite and a control law of the form
  \[ a = -\nabla U(p) - \bar{k}_v(v) \]  
  \hspace{1cm} (2)

Then the energy

\[ E(p, v) = U(p) + \frac{1}{2} v^T v \]  
  \hspace{1cm} (3)

evolves according to

\[ \dot{E} = -v^T \bar{k}_v(v). \]  
  \hspace{1cm} (4)

If $\bar{k}(v) \equiv 0$ then the energy is conserved, and if $v^T \bar{k}_v(v) > 0$ for nonzero $v$ then the energy will dissipate over time.
Exponential Stability in LTI case

Suppose $\omega_d(t) = \omega_d$ is constant.

If

- Potential: $U(p) = \frac{1}{2} k_p p^T p$, \quad $k_p > 0$
- Feedback:

$$a = -k_p p - K_v v,$$

where $K_v$ is positive definite (and usually symmetric).

Then the energy evolves according to $\dot{E} = -v^T K_v v \leq 0$ and the closed loop system is exponentially stable.

Pf. LaSalle’s invariance principle.
Exponential Stability of LTV

We wish to augment the energy in order to convert it from a weak Lyapunov function to a strict one.

- Extend the notion of rotational invariance to functions with two arguments.

**Definition**

$W : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is *rotationally invariant* if and only if

$$W(Rp, Rv) = W(p, v) \text{ for all } p, v \in \mathbb{R}^3 \text{ and all } R \in SO(3).$$

**Lemma**

$W(p, v)$ is rotationally invariant if and only if

$$D_1 W(p, v) \cdot \hat{\omega} p + D_2 W(p, v) \cdot \hat{\omega} v = 0 \text{ for all } p, v, \omega \in \mathbb{R}.$$
Strict Lyapunov Function

Introduce the rotationally invariant cross term $W(p, v) = p^T v$ with $\omega_d(t)$ free derivative

$$\dot{W}(p, v, a) = v^T v + p^T a.$$ 

**Lyapunov function**

$$V(p, v) = E(p, v) + \epsilon W(p, v) = \frac{1}{2}k_p p^T p + \frac{1}{2}v^T v + \epsilon p^T v$$

- time independent
- quadratic form $V(p, v) = \frac{1}{2}x^T P x$, $x = [p; v]$.
- positive definite provided that

$$P = \begin{bmatrix} k_p I & \epsilon I \\ \epsilon I & I \end{bmatrix} > 0$$

which occurs when $\epsilon^2 < k_p$. 
Strict Lyapunov Function

Derivative of the Lyapunov function

$$\dot{V}(p, v) = \dot{E}(p, v) + \epsilon \dot{W}(p, v) = v^T K_v v - \epsilon p^T K_v v^T - \epsilon k_p p^T p + \epsilon v^T v$$

- time independent
- quadratic form $$\dot{V}(p, v) = -\frac{1}{2} x^T Q x$$
- for general (possibly non-symmetric) $$K_v$$, $$K_v^s = \frac{K_v + K_v^T}{2} > 0$$

$$Q = \begin{bmatrix} 2\epsilon k_p I & \epsilon K_v \\ \epsilon K_v^T & 2(K_v^s - \epsilon I) \end{bmatrix} > 0$$

when

$$\epsilon < \lambda_{\text{min}}(4k_p K_v^s, (K_v^T K_v + 4k_p I)).$$

- for symmetric $$K_v$$

$$\epsilon < 1/\lambda_{\text{max}}(K_v^{-1} + K_v/4k_p).$$
Proposition

For each $k_p > 0$ and $K_v \in \mathbb{R}^{3 \times 3}$ with $K_v^s = (K_v + K_v^T)/2 > 0$, the linear feedback (5) exponentially stabilizes the maneuvering system (3) where lead maneuvers with a locally bounded angular velocity $\omega_d(\cdot)$.

- local boundedness condition ensures uniqueness.
- a uniform bound on $\omega_d(\cdot)$ is not required
- $\omega_d(\cdot)$ need not be differentiable.
Decay Rate Estimate

- If the LTI system $\dot{x} = Ax$ with quadratic Lyapunov function $V(x) = x^T P x$, $P^T = P > 0$ is stable, then there exist an $\alpha > 0$ such that

$$A^T P + PA + 2\alpha P \leq 0$$

- and $\|x(t)\| \leq \kappa(P)^{\frac{1}{2}} \|x(0)\| e^{-\alpha t}$

The actual decay rate $\alpha = -\max_k \text{real } \lambda_k(A)$ can be determined by solving the generalized eigenvalue problem in $P$ and $\alpha$

$$\begin{align*}
\text{maximize} & \quad \alpha \\
\text{subj to} & \quad P > 0 \\
& \quad A^T P + PA + 2\alpha P \leq 0
\end{align*}$$

which provides a Lyapunov function proving that decay rate.
Decay Rate Estimate

- LTV case, let the closed loop system be described by
  \[ \dot{x} = (A + A_{\omega_d}(t)) x, \]
  \[ A = \begin{bmatrix} 0 & I \\ -k_p I & -K_v \end{bmatrix}, \quad A_{\omega_d}(t) = \begin{bmatrix} -\hat{\omega}_d(t) & 0 \\ 0 & -\hat{\omega}_d(t) \end{bmatrix} \]

- Using the Lyapunov function \( V(p, v) = \frac{1}{2} x^T P(\epsilon) x, \quad x = [p; v] \) with
  \[ P(\epsilon) = \begin{bmatrix} k_p I & \epsilon I \\ \epsilon I & I \end{bmatrix} \]

then by rotational invariance, \( A_{\omega_d}(t)^T P = PA_{\omega_d}(t) = 0_{6 \times 6} \)

**Estimated Decay Rate of LTV**

\[
\begin{align*}
\text{maximize} & \quad \alpha \\
\text{subj to} & \quad P(\epsilon) > 0 \\
& \quad \epsilon > 0 \\
& \quad A^T P(\epsilon) + P(\epsilon) A + 2\alpha P(\epsilon) \leq 0
\end{align*}
\]
Analytic decay rate for scalar $K_v$

Letting $K_v = k_v I$ we can consider the scalar system $x = (p, v)^T \in \mathbb{R}^2$ so that the corresponding quadratic form matrices are

$$P(\epsilon) = \begin{bmatrix} k_p & \epsilon \\ \epsilon & 1 \end{bmatrix} \quad \text{and} \quad Q(\epsilon) = \begin{bmatrix} 2\epsilon k_p & \epsilon k_v \\ \epsilon k_v & 2(k_v - \epsilon) \end{bmatrix}$$

and the decay rate, as a function of $\epsilon$ feasible, is given by

$$\alpha(\epsilon) = \frac{1}{2} \min_{\|y\|=1} \frac{y^T Q(\epsilon)y}{y^T P(\epsilon)y}.$$ 

Letting $R(\epsilon)$ be the Cholesky factor of $P(\epsilon) > 0$ ($P(\epsilon) = R^T(\epsilon)R(\epsilon)$), we see that

$$\alpha(\epsilon) = \frac{1}{2} \lambda_{\min}(R^{-T}(\epsilon)Q(\epsilon)R^{-1}(\epsilon))$$

where

$$R^{-T}(\epsilon)Q(\epsilon)R^{-1}(\epsilon) = \begin{bmatrix} 2\epsilon & \epsilon \frac{k_v - 2\epsilon}{\sqrt{k_p - \epsilon^2}} \\ \epsilon \frac{k_v - 2\epsilon}{\sqrt{k_p - \epsilon^2}} & 2(k_v - \epsilon) \end{bmatrix}$$
Equations of Motion

Energy Considerations

Feedback Stabilization

Example: figure eight rejoin

\[
\begin{bmatrix}
2\epsilon \\
\epsilon \frac{k_v - 2\epsilon}{\sqrt{k_p - \epsilon^2}} \\
\epsilon \frac{k_v - 2\epsilon}{\sqrt{k_p - \epsilon^2}} \\
2(k_v - \epsilon)
\end{bmatrix}
\]

characteristic polynomial:
\[\lambda^2 - 2k_v \lambda + \frac{4k_p k_v - \epsilon(4k_p + k_v^2)}{k_p - \epsilon^2}\]

re-parametrize:
- \(k_p = \omega_n^2\), \(\omega_n > 0\)
- \(k_v = 2\zeta \omega_n\), \(\zeta \in [0, 1)\)

\(\epsilon\) constraints:
\[\epsilon < k_v \min \left\{ \frac{1}{2\zeta}, 1, \frac{1}{1+\zeta^2} \right\} = \frac{k_v}{1+\zeta^2}\]

parametrize feasible \(\epsilon\):
\[\epsilon = \delta \cdot \frac{k_v}{1+\zeta^2}\] with \(\delta \in (0, 1)\)

discriminant:
\[\frac{(\zeta^2 - 2\delta + 1)^2}{\zeta^4 + (2 - 4\delta^2)\zeta^2 + 1} k_v^2\]

Discriminant is zero when \(\delta = (1 + \zeta^2)/2\) at which point the minimum eigenvalue is \(k_v\), giving a decay rate of \(\alpha = k_v/2\) for the LTV system.
Coordinate free

- Note that, when $K_v$ is take to be scalar, then the control action acceleration is a simple linear combination of the position and velocity vectors.
- This means that the feedback is essentially coordinate free, meaning that $p$, $v$, and $a$ can be expressed in any coordinate system as long as the same is used for all three.
- In particular, the maneuvering vehicle pilot can determine his control action within any convenient frame.
Nonlinear Feedback

- **Motivation:** Address the saturation limits of the achievable acceleration (and velocity).
- **framework:** Already exists!

\[
a = -\nabla U(p) - \tilde{k}_v(v)
\]  \hspace{1cm} (\diamondsuit 2)

\[
E(p, v) = U(p) + \frac{1}{2}v^T v
\]  \hspace{1cm} (\diamondsuit 3)

\[
\dot{E} = -v^T \tilde{k}_v(v)
\]  \hspace{1cm} (\diamondsuit 4)

- **Question:** How do we choose \( U(p) \) and \( \tilde{k}_v(v) \) to ensure stability and attractiveness of the nonlinear system?
Conditions on $U(p)$

- rotationally invariant
- positive definite ($U(0) = 0$ and $U(p) > 0$, $p \neq 0$) locally
- radially unbounded (for global results)

Nice to haves:

- $\nabla U(p) \neq 0$ for all $p \neq 0$ (potential energy grows)
- quadratic in a neighborhood of the origin

Conditions on $\bar{k}_v(v)$

- $v^T \bar{k}_v(v) > 0$ whenever $v \neq 0$.
- $\bar{k}_v(0) = 0$.

Nice to have:

- $v^T \bar{k}_v(v)$ is rotationally invariant
Theorem

Suppose that

- $U : \mathbb{R}^3 \to \mathbb{R}$ is a $C^2$ rotationally invariant, positive definite and radially unbounded function such that $\nabla U(p) \neq 0$ for all $p \neq 0$,
- $\tilde{k}_v : \mathbb{R}^3 \to \mathbb{R}^3$ is a $C^1$ mapping such that $\tilde{k}_v(0) = 0$ and $v^T \tilde{k}_v(v) > 0$ for all $v \neq 0$,
- $\omega_d : \mathbb{R} \to \mathbb{R}^3$ is bounded.

Then, the closed loop system is uniformly globally asymptotically stable. If the Hessian $D^2 U(0)$ and the Jacobian $D \tilde{k}_v(0)$ are both full rank, then the closed loop system is locally exponentially stable.
Proof.

Local exponential stability:
- apply Proposition 7 to the linearization at zero
- local quadratic approximation of $U(p)$.

Nonlinear part:
- Lyapunov function $V(x) = E(p, v)$ (uniformly globally stable)
- let $E = \{x = (p, v) : v = 0\}$ be the $\dot{V} = 0$ set
- set $W(x) = p^T v$ and $\dot{W}(x) = -p^T \nabla U(p) + v^T v - p^T \bar{k}_v(v)$
- on the set $E \setminus \{0\}$, $\dot{W}(p, 0) = -p^T \nabla U(p) < 0$ so that we can’t stay near $E$ and away from the origin for very long.

These properties of $V$ and $W$ and the fact that $\omega_d(\cdot)$ is bounded allow us to conclude, by Matrosov’s Theorem, that the origin is a uniformly globally asymptotically stable equilibrium for the closed loop system.
Example: Figure Eight Rejoin

Maneuver Parameters:
- lead velocity: $10 \text{ m/s}$
- max lateral acc.: $7.2 \text{ m/s}^2$ (0.73 $g'$s)
- lead in coordinated flight (accelerations constrained to normal and axial directions along)

Figure: Rejoin to a *figure eight* in the N-E spatial frame.
Potential Function:

\[ U(p) = \frac{a_m^2}{k_p} \log \cosh \left( \frac{k_p}{a_m} \|p\| \right) \]

Position Feedback:

\[ \nabla U(p) = \left( \frac{a_m}{\|p\|} \tanh \left( \frac{k_p}{a_m} \|p\| \right) \right) p \]

Velocity Feedback: analogous to position feedback

**Figure**: Rejoin to a figure eight maneuver: flight lead angular velocity \( \omega_d(t) \) (top) together with the resulting relative position (mid) and velocity (bottom) trajectories.

**Figure**: Acceleration feedback component magnitudes.
Figure: Lyapunov and auxiliary function values for figure eight rejoin maneuver.

- auxiliary function $W$ ensures that the system trajectory does not stay close to the $\dot{V} = 0$ for long when there is energy to burn