Asymptotic expansions for the coefficients of extremal quasimodular forms and a conjecture of Kaneko and Koike

Peter J. Grabner

Received: 24 July 2020 / Accepted: 13 November 2020 / Published online: 11 February 2021
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Abstract
Extremal quasimodular forms have been introduced by Kaneko and Koike as quasimodular forms which have maximal possible order of vanishing at \( i\infty \). We show an asymptotic formula for the Fourier coefficients of such forms. This formula is then used to show that all but finitely many Fourier coefficients of such forms of depth \( \leq 4 \) are positive, which partially solves a conjecture stated by Kaneko and Koike. Numerical experiments based on constructive estimates confirm the conjecture for weights \( \leq 200 \) and depths between 1 and 4.

Keywords  Extremal quasimodular forms · Fourier coefficients

Mathematics Subject Classification  Primary: 11F30 · Secondary: 11F11, 11F25

1 Introduction
“Quasimodular forms” as a notion were introduced by Kaneko and Zagier in [12]. They have found applications in various areas of mathematics and are of interest on their own. For excellent introductions to the subject we refer to [3,16,19].

In [11] Kaneko and Koike introduced the notion of extremal quasimodular forms. These are quasimodular forms of weight \( w \) and depth \( r \), which show extremal order of vanishing at \( z = i\infty \) amongst all forms of that weight and depth. The authors conjectured certain arithmetic properties of the Fourier coefficients of these forms for depth \( r \leq 4 \). These were established in [6]; in [14,15] these were proved independently for \( r = 1 \). A second part of the conjecture stated in [11] concerned the positivity of

The author is supported by the Austrian Science Fund FWF project F5503 (part of the Special Research Program (SFB) “Quasi-Monte Carlo Methods: Theory and Applications”).

Peter J. Grabner
peter.grabner@tugraz.at

1 Institut für Analysis und Zahlentheorie, Technische Universität Graz, Kopernikusgasse 24, 8010 Graz, Austria
the Fourier coefficients of extremal quasimodular forms. In this paper we prove that for any \( w \) and \( r \leq 4 \) all but possibly finitely many Fourier coefficients are positive. Using a bound given by Jenkins and Rouse [10] we could verify the conjecture for \( 1 \leq r \leq 4 \) and \( w \leq 200 \).

2 Notation and preliminary results

In this section we collect some basic facts about modular and quasimodular forms.

2.1 Modular forms

The modular group \( \Gamma \) is the group of \( 2 \times 2 \)-matrices with integer entries and determinant 1

\[
\Gamma = \text{PSL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ac - bd = 1 \right\} /\{\pm I\}.
\]

It acts on the upper half-plane \( \mathbb{H} = \{ z \in \mathbb{C} \mid \Im z > 0 \} \) by Möbius transformation

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.
\]

The group \( \Gamma \) is generated by

\[
S_z = -\frac{1}{z}, \quad T_z = z + 1,
\]

which satisfy the relations \( S^2 = \text{id} \) and \( (ST)^3 = \text{id} \). A holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) is called a \textit{holomorphic modular form of weight} \( w \), if it satisfies

\[
(cz + d)^{-w} f \left( \frac{az + b}{cz + d} \right) = f(z)
\]

for all \( z \in \mathbb{H} \) and all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), and the limit

\[
f(i \infty) := \lim_{\Im z \to +\infty} f(z)
\]

exists. The vector space \( \mathcal{M}_w(\Gamma) \) of holomorphic modular forms is non-trivial only for even \( w \geq 4 \) and \( w = 0 \). Its dimension equals

\[
\dim \mathcal{M}_w(\Gamma) = \begin{cases} \left\lfloor \frac{w}{12} \right\rfloor & \text{for } w \equiv 2 \pmod{12}, \\ \left\lfloor \frac{w}{12} \right\rfloor + 1 & \text{otherwise}. \end{cases}
\]

Prominent examples of modular forms are the Eisenstein series
for \( k \geq 2 \), which are modular forms of weight \( 2k \). They admit a Fourier expansion (setting \( q = e^{2\pi i z} \) as usual in this context)

\[
E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,
\]

where \( \sigma_{\alpha}(n) = \sum_{d|n} d^\alpha \) denotes the divisor sum of order \( \alpha \) and \( B_{2k} \) denote the Bernoulli numbers. The defining series (3) does not converge for \( k = 1 \) in the given form. Nevertheless, the series (4) converges for \( k \geq 1 \). This entails a slightly more complicated transformation behaviour under the action of \( S \)

\[
z^{-2}E_2(Sz) = E_2(z) + \frac{6}{\pi i z}.
\]

Every holomorphic modular form can be expressed as a complex polynomial in \( E_4 \) and \( E_6 \); furthermore

\[
\bigoplus_{k=0}^{\infty} \mathcal{M}_{2k}(\Gamma) = \mathbb{C}[E_4, E_6].
\]

By the invariance under \( T \), every holomorphic modular form \( f \) has a Fourier expansion

\[
f(z) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi i n z} = \sum_{n=0}^{\infty} a_f(n) q^n.
\]

In the sequel we will follow the convention to freely switch between dependence on \( z \) and \( q \).

A holomorphic form \( f \) is called a cusp form, if \( f(i \infty) = 0 \). The prototypical example of a cusp form is

\[
\Delta = \frac{1}{1728} \left( E_4^3 - E_6^2 \right).
\]

The space of cusp forms is denoted by \( S_w(\Gamma) \). Since we only deal with modular forms for the full modular group \( \Gamma \), we will omit reference to the group in the sequel.

For a detailed introduction to the theory of modular forms we refer to \[1, 2, 5, 8, 13, 17, 18\].

### 2.2 Quasimodular forms

The vector space of quasimodular forms of weight \( w \) and depth \( \leq r \) is given by

\[
\mathcal{Q} \mathcal{M}^r_w = \bigoplus_{\ell=0}^{r} E_2^\ell \mathcal{M}_{w-2\ell}.
\]
Quasimodular forms occur naturally as derivatives of modular forms (see [3,16,19]). This aspect will be used and elaborated later.

Throughout this paper we use the notation
\[ Df = \frac{1}{2\pi i} \frac{df}{dz} = q \frac{df}{dq}. \]

Higher derivatives are always expressed as powers of \( D \). Upper indices will never denote derivatives.

With this notation Ramanujan’s identities read
\[
\begin{align*}
DE_2 &= \frac{1}{12} \left( E_2^2 - E_4 \right), \\
DE_4 &= \frac{1}{3} (E_2 E_4 - E_6), \\
DE_6 &= \frac{1}{2} \left( E_2 E_6 - E_4^2 \right), \\
D\Delta &= E_2 \Delta.
\end{align*}
\]

These give rise to the definition of the Ramanujan–Serre derivative
\[ \partial_w f = Df - \frac{w}{12} E_2 f, \]
where \( w \) is (related to) the weight of \( f \). We will use the product rule
\[ \partial_{w_1+w_2}(fg) = (\partial_{w_1} f) g + f (\partial_{w_2} g) \]
and also make frequent use of the following immediate consequences of (8)
\[
\begin{align*}
\partial_1 E_2 &= -\frac{1}{12} E_4, \\
\partial_4 E_4 &= -\frac{1}{3} E_6, \\
\partial_6 E_6 &= -\frac{1}{2} E_4^2, \\
\partial_{12} \Delta &= 0.
\end{align*}
\]

From the second and third equation together with the fact that every holomorphic modular form is a polynomial in \( E_4 \) and \( E_6 \), it follows immediately that for a form \( f \in M_w \) we have \( \partial_w f \in M_{w+2} \), and for \( f \in S_w \) we have \( \partial_w f \in S_{w+2} \).

We set
\[ QS_w^r = \bigoplus_{\ell=0}^{r} E_2^\ell S_{w-2\ell} = \Delta QM_{w-12}^r. \]
for the space of quasimodular forms with cusp form coefficients for all powers of $E_2$. For the spaces of quasimodular forms we have the alternative descriptions as

$$Q \mathcal{M}_w^r = \bigoplus_{\ell=0}^r D^\ell M_{w-2\ell},$$

(10)

see for instance [3, Proposition 14.3], and

$$Q \mathcal{S}_w^r = \bigoplus_{\ell=0}^r D^\ell S_{w-2\ell}.$$  

(11)

The second decomposition follows from the last equation in (8). The direct sum in (10) can be further refined as

$$Q \mathcal{M}_w^r = \bigoplus_{\ell=0}^r D^\ell (S_{w-2\ell} \oplus C E_{w-2\ell}) = Q \mathcal{S}_w^r \oplus \bigoplus_{\ell=0}^r C D^\ell E_{w-2\ell}.$$  

(12)

We set

$$Q \mathcal{E}_w^r = Q \mathcal{M}_w^r / Q \mathcal{S}_w^r$$

(13)

the “Eisenstein space”. We write $\overline{f}$ for $f + Q \mathcal{S}_w^r$. Notice that this notation implicitly depends on $w$ and $r$. Then $D$ maps $Q \mathcal{E}_w^r$ to $Q \mathcal{E}_w^{r+1}$. Similarly, $\partial_{w-r}$ maps $Q \mathcal{E}_w^r$ to $Q \mathcal{E}_{w+2}^r$ by [6, Lemma 2.2]. Notice that it makes sense to define $\overline{f}(i \infty)$ for $\overline{f} \in Q \mathcal{E}_w^r$. Thus we can define

$$Q \mathcal{E}_w^0 = \{ \overline{f} \in Q \mathcal{E}_w^r \mid \overline{f}(i \infty) = 0 \}.$$  

Then for $w \geq 2r + 4$

$$\{ D^\ell E_{w-2\ell} \mid \ell = 0, \ldots, r \}$$

is a basis of $Q \mathcal{E}_w^r$, and

$$\{ D^\ell E_{w-2\ell} \mid \ell = 1, \ldots, r \}$$

is a basis of $Q \mathcal{E}_w^0$.

Notice that for $v, w \geq 4$

$$E_{v+w} - E_v E_w$$

is a cusp form of weight $v + w$, which allows to write

$$\overline{E_v E_w} = \overline{E_{v+w}}.$$  

(14)
Using the definition of the Serre derivative, we obtain

$$\partial_w E_w = DE_w - \frac{w}{12} E_2 E_w,$$

which is a modular form of weight $w + 2$ with constant coefficient $-\frac{w}{12}$. Thus we have

$$\partial_w E_w = -\frac{w}{12} E_{w+2} + \text{cusp form},$$

which we write as

$$\partial_w E_w = \partial_w \overline{E_w} = -\frac{w}{12} \overline{E_{w+2}}.$$  \hspace{1cm} (15)

Using this we obtain

$$\partial_{w-r} E_{w-2\ell} E_2^\ell = (\partial_{w-2\ell} E_{w-2\ell}) E_2^\ell + E_{w-2\ell} \left( \partial_{2\ell-r} E_2^\ell \right),$$

from which we derive

$$\partial_{w-r} E_{w-2\ell} E_2^\ell = -\frac{\ell}{12} E_{w-2\ell} E_2^\ell - \frac{w-2\ell}{12} E_{w-2\ell+2} E_2^\ell - \frac{\ell-r}{12} E_{w-2\ell} E_2^{\ell+1}$$

$$= -\frac{\ell}{12} E_{w-2\ell+4} E_2^{\ell-1} - \frac{w-2\ell}{12} E_{w-2\ell+2} E_2^\ell - \frac{\ell-r}{12} E_{w-2\ell} E_2^{\ell+1}.$$  \hspace{1cm} (16)

Here we have used

$$\partial_{2\ell-r} E_2^\ell = -\frac{\ell}{12} E_4 E_2^{\ell-1} - \frac{\ell-r}{12} E_2^{\ell+1}.$$

Consider the forms

$$f_w^{(k)} = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} E_{w-2\ell} E_2^\ell,$$

where we set $E_0 = 1$ and omit the term for $w - 2\ell = 2$, which only occurs if $w \leq 2k + 2$. We compute using (16)
Expanding the binomial coefficients and shifting the summation index gives

\[ \partial_{w-r} f_w^{(k)} = -\frac{1}{12} \sum_{\ell=0}^{k+1} (-1)^\ell E_{w-2\ell+2} E_2^{\ell} \]

\[ \times \left( -k \binom{k-1}{\ell} + w \binom{k}{\ell} - 2k \binom{k-1}{\ell-1} - k \binom{k-1}{\ell-2} + r \binom{k}{\ell-1} \right), \]

where we have set \( \binom{k}{m} = 0 \) for \( m < 0 \) and \( m > k \). The term in parenthesis is then equal to

\[ (w - r) \binom{k}{\ell} + (r - k) \binom{k+1}{\ell}, \]

which gives

\[ \partial_{w-r} f_w^{(k)} = -w - r f_w^{(k)} + \frac{r - k}{12} f_{w+2}^{(k+1)} \] (18)

for \( k = 0, \ldots, r \) and \( w \geq 2r + 2 \) (the case \( k = r \) and \( w = 2r + 2 \) has to be checked separately).

We also have for \( w \geq 4 \)

\[ D^k E_w = (-1)^k \frac{(w)_k}{12^k} f_w^{(k)} \] (19)

where \( (w)_k = w(w+1) \cdots (w+k-1) \) denotes the Pochhammer symbol. We prove (19) by induction. For \( k = 0 \) it obviously holds. The induction step reads as (using (18) for \( r = k \))

\[ D^{k+1} E_w = (-1)^k \frac{(w)_k}{12^k} D f_w^{(k)} \]

\[ = (-1)^k \frac{(w)_k}{12^k} \left( \partial_{w+k} f_w^{(k)} + \frac{w+k}{12} E_2 f_w^{(k)} \right) \]

\[ = (-1)^{k+1} \frac{(w)_{k+1}}{12^{k+1}} \left( (w+k) f_w^{(k)} - (w) E_2 f_w^{(k)} \right) \]

\[ = (-1)^{k+1} \frac{(w)_{k+1}}{12^{k+1}} f_{w+2k+2}^{(k+1)}. \]

Furthermore, we have

\[ E_v f_w^{(k)} = f_w^{(k)} \] (20)
for $w \geq 2k + 4$ and $v \geq 4$, which follows from (14). For later reference we notice that

$$D^k E_w(z) = \frac{2w}{B_w} \sum_{n=1}^{\infty} n^k \sigma_{w-1}(n) q^n$$

(21)

for $k \geq 1$ and even $w \geq 2$.

We will use the following convention for iterated Serre derivatives

$$\partial_w^0 f = f, \quad \partial_w^{k+1} f = \partial_w^{2k+2} \left( \partial_w^k f \right).$$

With this we get the following expressions for higher derivatives in terms of Serre derivatives, which we will need later

$$Df = \partial_{w-2} f + \frac{w - 2}{12} f$$

$$D^2 f = \left( \partial_{w-4}^2 f - \frac{w - 4}{144} E_4 f \right) + \frac{w - 3}{6} \partial_{w-4} f + \frac{(w - 3)(w - 4)}{144} f$$

$$D^3 f = \left( \partial_{w-6}^3 f - \frac{3w - 16}{144} E_4 \partial_{w-6} f + \frac{w - 6}{432} E_6 f \right)$$

$$+ \frac{w - 4}{48} \partial_{w-6}^2 f - \frac{(w - 4)(w - 6)}{576} E_4 f$$

$$+ \frac{w - 4}{48} \partial_{w-6} f + \frac{(w - 4)(w - 5)(w - 6)}{1728} f$$

$$D^4 f = \left( \partial_{w-8}^4 f - \frac{3w - 20}{72} E_4 \partial_{w-8}^2 f + \frac{2w - 15}{216} E_6 \partial_{w-8} f \right)$$

$$+ \frac{(w - 8)(w - 14)}{6192} E_4^2 f$$

$$+ \frac{(w - 5)(w - 8)}{1296} E_6 f$$

$$+ \frac{(w - 5)(w - 6)}{432} \partial_{w-8}^3 f - \frac{(3w - 22)(w - 5)}{432} E_4 \partial_{w-8} f$$

$$+ \frac{(w - 5)(w - 8)}{1296} E_6 f$$

$$+ \frac{(w - 5)(w - 6)}{432} \partial_{w-8} f$$

$$+ \frac{(w - 5)(w - 6)(w - 7)}{20736} \partial_{w-8} f.$$

(22)
Proposition 1 Let \( g \in QS^r_w \) be given by its Fourier expansion
\[
g(z) = \sum_{n=1}^{\infty} a(n)q^n.
\]
Then \( a(n) = O(n^{\frac{w-1}{2}}\sigma_0(n)) \).

Proof Let \( g \) first be in \( D^\ell S_{w-2\ell} \) for some \( \ell \geq 0 \). Then \( g \) is the \( \ell \)th derivative of a cusp form \( G \in S_{w-2\ell} \). By Deligne’s estimate [4, Théorème 8.2] (see also [9, Section 14.9]) the Fourier coefficients of \( G \) are bounded by \( O(n^{\frac{w-1}{2} - \ell} \sigma_0(n)) \). The effect of \( \ell \)-fold derivation is multiplication with \( n^\ell \), which gives the desired estimate for this special case. Since the same estimate holds for all spaces in the direct sum (11), it holds for every \( g \) in \( QS^r_w \). \( \square \)

Proposition 2 Let \( f^{(k)}_w \) be the form given by (17) and let
\[
f^{(k)}_w = \delta_{k,0} + \sum_{n=1}^{\infty} a^{(k)}_w(n)q^n
\]
be its Fourier expansion. Then the asymptotic expansion
\[
a^{(k)}_w(n) = \begin{cases} 
-\frac{2w}{B_w} \sigma_{w-1}(n) & \text{for } k = 0 \\
(-1)^{k+1} \frac{12^k}{(w-2k+1)_{k-1} B_{w-2k}} n^k \sigma_{w-2k-1}(n) & \text{for } k > 0 \\
O\left(n^{\frac{w-1}{2} - \sigma_0(n)}\right) & \end{cases}
\]
holds.

Proof The case \( k = 0 \) is just the Fourier expansion of the Eisenstein series given in (3). For \( k > 0 \) we use (19) to obtain
\[
f^{(k)}_w = (-1)^k \frac{12^k}{(w-2k)_k} D^k E_{w-2k} + h^{(k)}_w,
\]
where \( h^{(k)}_w \in QS^{(k)}_w \). The Fourier coefficient of the first term equals
\[
(-1)^{k+1} \frac{12^k}{(w-2k)_k} \frac{2(w-2k)}{B_{w-2k}} n^k \sigma_{w-2k-1}(n)
\]
using (21). The Fourier coefficient of \( h^{(k)}_w \) is estimated using Proposition 1 to obtain (23). \( \square \)
We recall the dimension formulas for the spaces $\mathcal{QM}_w^r$ for $1 \leq r \leq 4$:

\[
\begin{align*}
\dim \mathcal{QM}_w^1 & = \left\lfloor \frac{w}{6} \right\rfloor + 1, \\
\dim \mathcal{QM}_w^2 & = \left\lfloor \frac{w}{4} \right\rfloor + 1, \\
\dim \mathcal{QM}_w^3 & = \left\lfloor \frac{w}{3} \right\rfloor + 1, \\
\dim \mathcal{QM}_w^4 & = \begin{cases} \\
\left\lfloor \frac{5w}{12} \right\rfloor & \text{if } w \equiv 10 \pmod{12}, \\
\left\lfloor \frac{5w}{12} \right\rfloor + 1 & \text{otherwise};
\end{cases}
\end{align*}
\]

(24)

see, for instance [6, Proposition 2.1].

3 Extremal quasimodular forms

The notion of an extremal quasimodular form was introduced in [11]. They are defined as quasimodular forms achieving the maximal possible order of vanishing at $z = i\infty$ for given weight $w$ and depth $r$. It follows from a simple dimension argument that the order of vanishing $\dim \mathcal{QM}_w^r - 1$ can be achieved. It was shown in [15, Theorem 1.3] and independently in [6, Remark 4.7] that for $r \leq 4$ this is actually the precise order of vanishing for such forms.

In [11] differential equations satisfied by extremal quasimodular forms are found for $r = 1$ and $r = 2$. Furthermore, two conjectures about these forms for $r \leq 4$ are stated:

- if the first non-zero Fourier coefficient of the extremal quasimodular form equals 1 (the form is called normalised then), the denominators of all Fourier coefficients are then divisible only by primes less than the weight.
- if the first non-zero Fourier coefficient of the extremal quasimodular form is positive, then all Fourier coefficients are positive.

The first conjecture has been proved for $r = 1$ in [15] and [14]. It has been proved in full generality for $1 \leq r \leq 4$ in [6].

In the course of the following section we will prove the following theorem, which partially settles the second conjecture.

**Theorem 1** Let $g_w^{(r)}$ be a normalised extremal quasimodular form of weight $w$ and depth $r \leq 4$. Then all but possibly finitely many Fourier coefficients are positive.

4 Proof of Theorem 1

The proof of Theorem 1 will be done separately for the values of the depth parameter $r$. We will make frequent use of recursive relations for extremal quasimodular forms derived in [6]. Notice that by definition an extremal quasimodular form is in $\mathcal{QM}_w^{(r)} \oplus \mathcal{QS}_w^{(r)}$. 

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The proofs follow the general scheme

1. express the form $g^{(r)}_w$ in terms of a linear recurrence obtained in [6],
2. use this recurrence to obtain a linear recurrence for the coefficients of $f^{(r)}_w (\ell = 1, \ldots, r)$ in the decomposition of $g^{(r)}_w$,
3. rewrite this decomposition in terms of $DE_{w, -2\ell}$ and observe the positivity of the asymptotic main term originating from $DE_{w, -2}$.

The recursions obtained in [6, Section 6] contain a positive factor, which ensures that the forms are normalised, which is important in the context there. In this section we use these recursions without this factor and at some occasions change this factor, which does not affect the sign of the coefficients.

### 4.1 Depth 1

Using [6, Proposition 6.1] we define a sequence of quasimodular forms by

\[
\begin{align*}
    g^{(1)}_6 &= E_2 E_4 - E_6 = -f^{(1)}_6 = 3DE_4, \\
    g^{(1)}_{w+6} &= E_4 \partial_{w-1}g^{(1)}_w - \frac{w + 1}{12}E_6g^{(1)}_w, \\
    g^{(1)}_{w+2} &= \frac{12}{w-1}\partial_{w-1}g^{(1)}_w, \\
    g^{(1)}_{w+4} &= E_4g^{(1)}_w
\end{align*}
\]

for $w \equiv 0 \pmod{6}$. These forms are then extremal quasimodular forms of weight $w$ and depth 1 with positive coefficient of the first non-vanishing term of its Fourier expansion.

By the fact that $\mathcal{Q}^{0}_{E^{1}}_w$ is one dimensional we set

\[
\overline{g^{(1)}_w} = C_w \overline{f^{(1)}_w}.
\]

Inserting this into (25) and using (18) and (20) gives

\[
\begin{align*}
    \overline{g^{(1)}_{w+6}} &= C_w \left( E_4 \partial_{w-1}f^{(1)}_w - \frac{w + 1}{12}E_6f^{(1)}_w \right) \\
    &= -C_w \left( \frac{w - 1}{12}E_4f^{(1)}_{w+2} + \frac{w + 1}{12}E_6f^{(1)}_{w} \right) = -\frac{w}{6}C_w \overline{f^{(1)}_{w+6}},
\end{align*}
\]

from which we derive

\[
C_w = (-1)^{w/6} \left( \frac{w}{6} - 1 \right) !.
\]
Together with (19) this gives
\[ g_{6k}^{(1)} = (-1)^{k-1} \frac{6(k - 1)!}{3k - 1} DE_{6k-2}. \]

Applying the third equation in (25) we obtain
\[ g_{6k+2}^{(1)} = \frac{12}{6k - 1} \partial_{6k-1} g_{6k}^{(1)} = (-1)^{k-1} \frac{2(k - 1)!}{k} DE_{6k} \]
and
\[ g_{6k+4}^{(1)} = E_4 g_{6k}^{(1)} = (-1)^{k-1} \frac{6(k - 1)!}{3k + 1} DE_{6k+2}, \]
which gives
\[ g_{w}^{(1)} = (-1)^{w-1} \frac{12 \left( \left\lfloor \frac{w}{6} \right\rfloor - 1 \right)!}{w - 2} DE_{w-2}. \]

The Fourier coefficients of \( g_{w}^{(1)} \) are then given by
\[ \frac{24 \left( \left\lfloor \frac{w}{6} \right\rfloor - 1 \right)!}{|B_{w-2}|} n^{w-3}(n) + \mathcal{O} \left( n^{w-1} \sigma_0(n) \right). \]

Notice that the first term is of order \( n^{w-2} \). Thus we have proved Theorem 1 for \( r = 1 \).

### 4.2 Depth 2

Using [6, Proposition 6.2] we define a sequence of quasimodular forms by
\[ g_4^{(2)} = E_4 - E_2^2 = -12DE_2 = 2f_4^{(1)} - f_4^{(2)}, \]
\[ g_w^{(2)} = w(w + 1)E_4 g_w^{(2)} - 36\partial_{w-2}^2 g_w^{(2)}, \]
\[ g_{w+2}^{(2)} = \frac{12}{w - 2} \partial_{w-2} g_w^{(2)} \]
for \( w \equiv 0 \pmod{4} \). The form \( g_w^{(2)} \) is then an extremal quasimodular form of weight \( w \) and depth 2 with positive coefficient of the first non-vanishing term in its Fourier expansion.

We make the ansatz
\[ g_{4k}^{(2)} = a_{4k} f_{4k}^{(1)} + b_{4k} f_{4k}^{(2)} \]
with \( a_4 = 2 \) and \( b_4 = -1 \). Applying (18) twice gives

\[
\frac{\partial^2}{\partial y_{4k-2}} f_{4k}^{(1)} = \frac{2k(2k-1)}{36} f_{4k+4}^{(1)} + \frac{4k-1}{72} f_{4k+4}^{(2)},
\]

\[
\frac{\partial^2}{\partial y_{4k-2}} f_{4k}^{(2)} = \frac{2k(2k-1)}{36} f_{4k+4}^{(2)}.
\]

Inserting this into the recurrence (27) then gives

\[
\begin{pmatrix} a_{4k+4} \\ b_{4k+4} \end{pmatrix} = \begin{pmatrix} 6k(2k+1) & 0 \\ -\frac{4k-1}{2} & 6k(2k+1) \end{pmatrix} \begin{pmatrix} a_{4k} \\ b_{4k} \end{pmatrix}.
\] (28)

This recurrence has the solutions

\[
a_{4k} = 2 \cdot 3^{k-1}(2k-1)!,
\]

\[
b_{4k} = -3^k (2k-1)! \left( \frac{1}{3} + \sum_{\ell=1}^{k-1} \frac{4\ell - 1}{18\ell(2\ell + 1)} \right),
\]

which can be seen from

\[
a_{4k+4} = 6k(2k+1)a_{4k} = 3(2k+1)2k \cdot 2 \cdot 3^{k-1}(2k-1)! = 2 \cdot 3^k (2k+1)!
\]

and

\[
b_{4k+4} = 6k(2k+1)b_{4k} = \frac{4k-1}{2} a_{4k},
\]

\[
= -3^{k+1} (2k+1)! \left( \frac{1}{3} + \sum_{\ell=1}^{k-1} \frac{4\ell - 1}{18\ell(2\ell + 1)} \right) - (4k-1)3^{k-1}(2k-1)!,
\]

\[
= -3^{k+1} (2k+1)! \left( \frac{1}{3} + \sum_{\ell=1}^{k-1} \frac{4\ell - 1}{18\ell(2\ell + 1)} + \frac{4k-1}{18k(2k+1)} \right)
\]

Similarly we obtain

\[
a_{4k+2} = -2 \cdot 3^{k-1}(2k-1)!,
\]

\[
b_{4k+2} = 3^k (2k-1)! \left( \frac{1}{3} + \sum_{\ell=1}^{k-1} \frac{4\ell - 1}{18\ell(2\ell + 1)} - \frac{1}{3(2k-1)} \right).
\]

Thus we have

\[
\frac{g_w^{(2)}}{w w_{w-2}} = - \frac{12}{w-2} a_w D E_{w-2} + \frac{144}{(w-4)(w-3)} b_w D^2 E_{w-4},
\]
where we have used (19) to rewrite $f^{(k)}_w$ $(k = 1, 2)$ in terms of $DE_{w-2}$ and $D^2E_{w-4}$.

Observing that the sign of $-\frac{a_w}{B_{w-2}}$ is always positive, whereas the sign of $\frac{b_w}{B_{w-4}}$ is always negative, we derive the n-th Fourier coefficient of $g^{(2)}_w$ using (19) and (21)

\[
\frac{24|a_w|}{|B_{w-2}|} n\sigma_{w-3}(n) - \frac{288|b_w|}{(w-3)|B_{w-4}|} n^2\sigma_{w-5}(n) + \mathcal{O}(n^{\frac{w-1}{2}\sigma_0(n)}).
\]

Notice that the first term is asymptotically dominating and positive, whereas the second term is negative. This proves Theorem 1 for $r = 2$.

### 4.3 Depth 3

Using [6, Proposition 6.3] we define a sequence of quasimodular forms by

\[
\begin{align*}
    g^{(3)}_6 &= 5E^3_2 - 3E_2E_4 - 2E_6 = -12f^{(1)}_6 + 15f^{(2)}_6 - 5f^{(3)}_6, \\
    g^{(3)}_{w+6} &= 48(7w^2 + 42w + 60)\partial_{w-3}g^{(3)}_w \\
    &\quad - (15w^4 + 96w^3 + 151w^2 - 30w - 116)E_4\partial_{w-3}g^{(3)}_w \\
    &\quad - \frac{1}{6}(w+1)(9w^4 + 45w^3 + 40w^2 + 24w + 144)E_6g^{(3)}_w, \\
    g^{(3)}_{w+2} &= \partial_{w-3}g^{(3)}_w, \\
    g^{(3)}_{w+4} &= (w+1)(3w+1)E_4g^{(3)}_w - 48\partial_{w-3}g^{(3)}_w
\end{align*}
\]

for $w \equiv 0 \pmod{6}$. These forms are then extremal quasimodular forms of weight $w$ and depth 3 with positive coefficient of the first non-vanishing term of its Fourier expansion.

We make the ansatz

\[
\overline{g^{(3)}_w} = a_w\overline{f^{(1)}_w} + b_w\overline{f^{(2)}_w} + c_w\overline{f^{(3)}_w}
\]

with $a_6 = -12$, $b_6 = 15$, and $c_6 = -5$ and first consider the case $w = 6k$. Applying (18) thrice gives

\[
\begin{align*}
    \partial_{6k-3}^3 f^{(1)}_{6k} &= -\frac{(6k+1)(6k-1)(2k-1)}{576} f^{(1)}_{6k+6} - \frac{108k^2 - 36k - 1}{864} f^{(2)}_{6k+6} \\
    &\quad - \frac{6k-1}{288} f^{(3)}_{6k+6}, \\
    \partial_{6k-3}^3 f^{(2)}_{6k} &= -\frac{(6k+1)(6k-1)(2k-1)}{576} f^{(2)}_{6k+6} - \frac{108k^2 - 36k - 1}{1728} f^{(3)}_{6k+6}, \\
    \partial_{6k-3}^3 f^{(3)}_{6k} &= -\frac{(6k+1)(6k-1)(2k-1)}{576} f^{(3)}_{6k+6}.
\end{align*}
\]
Then a computation similar to the one which gave (28) gives the recurrence

\[
\begin{pmatrix}
    a_{6k+6} \\
    b_{6k+6} \\
    c_{6k+6}
\end{pmatrix} =
\begin{pmatrix}
    -96(2k+1)^2(3k+1)(3k+2) & 0 & 0 \\
    8(2k+1) (108k^3+99k^2+17k-2) & -96(2k+1)^2(3k+1)(3k+2) & 0 \\
    -6k(2+4k^2+4k+10) & 4(2k+1)(108k^3+99k^2+17k-2) & -96(2k+1)^2(3k+1)(3k+2)
\end{pmatrix}
\begin{pmatrix}
    a_{6k} \\
    b_{6k} \\
    c_{6k}
\end{pmatrix}.
\]

Notice that this recurrence implies that \((-1)^ka_{6k}, (-1)^{k-1}b_{6k}, \text{and } (-1)^kc_{6k}\) are positive for all \(k \geq 1\). Applying the third equation in (29) and using (18) gives

\[
\frac{g_{6k+2}^{(3)}}{g_{6k+2}^{(3)}} = -\frac{2k-1}{4} a_{6k} f_{6k+2}^{(1)} - \left(\frac{2k-1}{4} b_{6k} + \frac{1}{6}a_{6k}\right) f_{6k+2}^{(2)}
\]

\[
- \left(\frac{2k-1}{4} c_{6k} + \frac{1}{12}b_{6k}\right) f_{6k+2}^{(3)};
\]

similarly, the fourth equation in (29) gives

\[
\frac{g_{6k+4}^{(3)}}{g_{6k+4}^{(3)}} = \frac{1}{48} \left(5172k^2 + 1160k + 47\right) a_{6k} f_{6k+4}^{(1)}
\]

\[
+ \left(\frac{1}{48} \left(5172k^2 + 1160k + 47\right) b_{6k} - \frac{1}{36}a_{6k}\right) f_{6k+4}^{(2)}
\]

\[
+ \left(\frac{1}{48} \left(5172k^2 + 1160k + 47\right) c_{6k} - \frac{1}{144}b_{6k}\right) f_{6k+4}^{(3)}.
\]

Together with (19) and (21) this gives

\[
a_w \frac{24}{B_{w-2}}n\sigma_{w-3}(n) + b_w \frac{288}{(w-3)B_{w-4}}n^2\sigma_{w-5}(n)
\]

\[
+c_w \frac{3456}{(w-4)(w-5)B_{w-6}}n^3\sigma_{w-7}(n) + \mathcal{O}(n^{w-1} \sigma_0(n)).
\]

The first term is positive by our discussion of the sign of \(a_{6k}\) and the signs of the Bernoulli numbers. It is of order \(n^{w-2}\) and thus dominates the other terms. This implies the assertion of Theorem 1 for \(r = 3\).

### 4.4 Depth 4

Using [6, Proposition 6.4] we define a sequence of quasimodular forms by

\[
g_{12}^{(4)} = 13025E_3^3 - 12796E_6^2 + 3852E_2E_4E_6 - 2706E_2^2E_4^2
\]

\[
+ 27500E_2^3E_6 - 28875E_2^4E_4.
\]
\[
\begin{align*}
  g_{w+12}^{(4)} &= -p_0(w)E_4 \partial_{w-4}^2 g_w^{(4)} + \frac{(w + 4)^4}{12} p_1(w) E_6 \partial_{w-4}^3 g_w^{(4)} \\
  &= + \frac{1}{720} p_2(w) E_4^2 \partial_{w-4}^2 g_w^{(4)} + \frac{1}{8640} p_3(w) E_4 E_6 \partial_{w-4}^4 g_w^{(4)} \\
  &= + \left( \frac{w + 1}{25920} p_4(w) E_4^3 + \frac{(w + 1)(w + 4)^4}{15} p_5(w) \Delta \right) g_w^{(4)},
\end{align*}
\]

\[
\begin{align*}
  g_{w+2}^{(4)} &= \partial_{w-4} g_w^{(4)}, \\
  g_{w+4}^{(4)} &= (w + 1)(2w + 1) E_4 g_w^{(4)} - 18 \partial_{w-4}^2 g_w^{(4)}, \\
  g_{w+6}^{(4)} &= \left( 17 w^2 + 78 w + 90 \right) \partial_{w-4}^3 g_w^{(4)} \\
  &= - \frac{1}{144} \left( 191 w^4 + 1008 w^3 + 1504 w^2 + 192 w - 576 \right) E_4 \partial_{w-4}^4 g_w^{(4)} \\
  &= - \frac{1}{432} (w + 1) \left( 81 w^4 + 376 w^3 + 560 w^2 + 528 w + 576 \right) E_6 g_w^{(4)}.
\end{align*}
\]

\[
\begin{align*}
  g_{w+8}^{(4)} &= - \left( 1313 w^6 + 28678 w^5 + 255122 w^4 + 1183008 w^3 \\
  &+ 3016512 w^2 + 4012416 w + 2177280 \right) \partial_{w-4}^4 g_w^{(4)} \\
  &= + \frac{1}{144} \left( 13423 w^8 + 295800 w^7 + 2645368 w^6 + 12166080 w^5 \\
  &+ 29311504 w^4 + 29020416 w^3 - 15653376 w^2 \\
  &- 56692224 w - 33094656 \right) E_4 \partial_{w-4}^5 g_w^{(4)} \\
  &= + \frac{1}{432} \left( 6561 w^9 + 136994 w^8 + 1139536 w^7 + 4759344 w^6 \\
  &+ 10294016 w^5 + 11541472 w^4 + 14671104 w^3 \\
  &+ 41398272 w^2 + 63016704 w + 31974912 \right) E_6 \partial_{w-4}^6 g_w^{(4)} \\
  &= + \frac{1}{2592} (w + 1)(2048 w^9 + 38685 w^8 + 287792 w^7 + 1130616 w^6 \\
  &+ 3110288 w^5 + 8497968 w^4, \\
  &+ 18484992 w^3 + 14141952 w^2 - 20570112 w - 30855168 \right) E_4 g_w^{(4)} \\
  g_{w+10} &= \left( 293 w^4 + 4332 w^3 + 22968 w^2 + 51192 w + 40824 \right) E_4 \partial_{w-4}^3 g_w^{(4)} \\
  &= - \frac{4}{3} \left( w^5 + 15 w^4 + 90 w^3 + 270 w^2 + 405 w + 243 \right) E_6 \partial_{w-4}^2 g_w^{(4)} \\
  &= - \frac{1}{144} \left( 3311 w^6 + 51234 w^5 + 291550 w^4 + 731040 w^3 \\
  &+ 717696 w^2 - 2592 w - 256608 \right) E_4 \partial_{w-4}^3 g_w^{(4)}
\end{align*}
\]
\[-\frac{1}{432}(w + 1) \left(1313w^6 + 19430w^5 + 104354w^4 + 251616w^3 + 310464w^2 + 300672w + 248832\right) E_4E_6g_w^{(4)},\]

for \(w \equiv 0 \pmod{12}\). These forms are then extremal quasimodular forms of weight \(w\) and depth 4 with positive coefficient of the first non-vanishing term of its Fourier expansion. The polynomials \(p_0, \ldots, p_5\) are given by

\[
p_0(w) = 53567w^{14} + 4499628w^{13} + 173318340w^{12} + 4055616864w^{11} + 64374205218w^{10} + 732790207224w^9 + 6165100658404w^8 + 38914973459904w^7 + 185044363180416w^6 + 659055640624128w^5 + 1729058937394176w^4 + 3237068849283072w^3 + 4084118362128384w^2 + 3105388005949440w + 1072718335180800\]

\[
p_1(w) = 21257w^{11} + 1465884w^{10} + 45186990w^9 + 821051740w^8 + 9759703548w^7 + 79588527156w^6 + 453687847200w^5 + 1804779218520w^4 + 4900200364800w^3 + 8628400143360w^2 + 8845395333120w + 3990767616000\]

\[
p_2(w) = 2662740w^{16} + 224120550w^{15} + 8648003840w^{14} + 202621853220w^{13} + 3217542322665w^{12} + 36586266504480w^{11} + 306658234963680w^{10} + 1919356528986240w^9 + 8970889439482816w^8 + 30866477857195008w^7 + 75319919247624192w^6 + 118664936756305920w^5 + 83296021547483136w^4 - 82769401579438080w^3 - 258790551639293952w^2 - 245119018746249216w - 86822757140004864\]

\[
p_3(w) = 4272785w^{17} + 351970350w^{16} + 13234823080w^{15} + 300533087760w^{14} + 4592608729932w^{13} + 49787752253076w^{12}\]
As before we make the ansatz
\[ g_w^{(4)} = a_w f_w^{(1)} + b_w f_w^{(2)} + c_w f_w^{(3)} + d_w f_w^{(4)}, \]
which gives a recurrence
\[
\begin{pmatrix}
 a_{12(k+1)} \\
 b_{12(k+1)} \\
 c_{12(k+1)} \\
 d_{12(k+1)}
\end{pmatrix} = \begin{pmatrix}
 \lambda_k & 0 & 0 & 0 \\
 -\star & \lambda_k & 0 & 0 \\
 +\star & -\star & \lambda_k & 0 \\
 -\star & +\star & -\star & \lambda_k
\end{pmatrix} \begin{pmatrix}
 a_{12k} \\
 b_{12k} \\
 c_{12k} \\
 d_{12k}
\end{pmatrix},
\]
where \( \lambda_k \) is a polynomial of degree 18, which factors into rational linear factors and \( \pm \ast \) denotes positive/negative entries. This together with the signs of the initial values \( a_{12} = 34560, b_{12} = -93456, c_{12} = 88000 \), and \( d_{12} = -28875 \) shows that \( a_{12k}, -b_{12k}, c_{12k}, \) and \( -d_{12k} \) are all positive. From this it follows that \((-1)^{\frac{w}{2}}a_w\) is positive. Finally, this gives the asymptotic formula

\[
\frac{24a_w}{B_{w-2}}n\sigma_{w-3}(n) + \frac{288b_w}{(w-3)B_{w-4}}n^2\sigma_{w-5}(n)
\]

\[
+ \frac{3456c_w}{(w-4)(w-5)B_{w-6}}n^3\sigma_{w-7}(n)
\]

\[
- \frac{41472d_w}{(w-5)(w-6)(w-7)B_{w-8}}n^4\sigma_{w-9}(n) + O\left(n^{w-1}2\sigma_0(n)\right)
\]

for the Fourier coefficients of \( g_w^{(4)} \), where we have used (19) and (21) for the explicit expression of the terms coming from \( f_w^{(k)} (k = 1, \ldots, 4). \) The first term asymptotically dominates and is positive by our discussion of the sign of \( a_w \) and the sign of the Bernoulli number. This implies the theorem for \( r = 4. \)

5 Numerical experiments

In [10] an explicit bound for the Fourier coefficients of cusp forms has been derived.

**Theorem 2** (Theorem 1 in [10]) Let

\[ G(z) = \sum_{n=1}^{\infty} g(n)q^n \]

be a cusp form of weight \( w \). Then

\[
|g(n)| \leq \sqrt{\log w} \left( 11 \sum_{m=1}^{\ell} \frac{|g(m)|^2}{m^{w-1}} + e^{18.72(41.41)^{w/2}} \left| \sum_{m=1}^{\ell} g(m)e^{-7.288m} \right| n^{w-1}2\sigma_0(n) \right) \]

(30)

where \( \ell \) is the dimension of the space of cusp forms of weight \( w \).

For an application of this theorem we write an extremal quasimodular form of depth \( r \) as

\[ g_w^{(r)} = \sum_{\ell=1}^{r} c_\ell D^\ell E_{w-2\ell} + \sum_{\ell=0}^{r} D^\ell \alpha_{w-2\ell}, \]

(31)
where $c_1, \ldots, c_r$ are the coefficients computed in Sections 4.1 to 4.4 for the according values of $r$, and $\alpha_{w-2r}, \ldots, \alpha_w$ are cusp forms of weights $w-2r, \ldots, w$. Rewriting the forms $g_w^{(r)}$ is done using the expressions for derivatives given in (22). In order to make this more clear, we give the according conversion formula for the case $r = 1, 2$

$$A_w + E_2B_{w-2} = \left( A_w - \frac{12}{w-2} \partial_{w-2}B_{w-2} \right) + D \left( \frac{12}{w-2}B_{w-2} \right)$$

$$A_w + B_{w-2}E_2 + C_{w-4}E_2^2$$

$$= \left( A_w - \frac{12}{w-2} \partial_{w-2}B_{w-2} + \frac{144}{(w-2)(w-3)} \partial_{w-4}^2C_{w-4} + \frac{1}{w-3}E_4C_{w-4} \right)$$

$$+ D \left( \frac{12}{w-2}B_{w-2} - \frac{288}{(w-2)(w-4)} \partial_{w-4}C_{w-4} \right)$$

$$+ D^2 \left( \frac{144}{(w-3)(w-4)}C_{w-4} \right).$$

The cases $r = 3, 4$ are much more complex; the computations were done using Mathematica. The Mathematica source code is available at [7].

Theorem 2 can then be applied to the forms $\alpha_{w-2\ell}$ ($\ell = 0, \ldots, r$) to derive bounds of the form $C_\ell n^{w-2\ell-1} \sigma_0(n)$ for the Fourier coefficients of the forms $D^\ell \alpha_{w-2\ell}$. This gives the bound $(C_0 + \cdots + C_r)n^{w-2\ell-1} \sigma_0(n)$ for the Fourier coefficient of the second sum in (31).

The Fourier coefficients of the terms in the first sum are

$$c_{\ell} \frac{2(w-2\ell)}{B_{w-2\ell}} n^{w-2\ell-1} \sigma_{w-2\ell-1}(n).$$

For these we use the bounds

$$n^{w-2\ell-1} \leq \sigma_{w-2\ell-1}(n) \leq n^{w-2\ell-1} \sum_{d|n} d^{2\ell+1-w} \leq \xi(w-2\ell-1)n^{w-2\ell-1}$$

and $\sigma_0(n) \leq 2\sqrt{n}$ to derive an explicit lower bound for the Fourier coefficients of $g_w^{(r)}$. This bound is positive for $n \geq N_0$ for an explicitly computable value $N_0$.

For the remaining finitely many Fourier coefficients positivity can be checked with the help of a computer. We have performed these computations for $1 \leq r \leq 4$ and $w \leq 200$.

Acknowledgements The author is grateful to an anonymous referee for the many valuable comments that improved the readability of the paper.

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