Estimates on Green functions of second order differential operators with singular coefficients

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Abstract
We investigate the Green’s functions $G(x; x')$ of some second order differential operators on $\mathbb{R}^{d+1}$ with singular coefficients depending only on one coordinate $x_0$. We express the Green’s functions by means of the Brownian motion. Applying probabilistic methods we prove that when $x = (0, x)$ and $x' = (0, x')$ (here $x_0 = 0$) lie on the singular hyperplanes then $G(0, x; 0, x')$ is more regular than the Green’s function of operators with regular coefficients.

1 Introduction
We discuss Green’s functions of some second order differential operators with singular coefficients appearing in quantum physics.

As a first example consider the Lagrangian for a scalar field in $(d + 1)$-dimensions interacting with gravity
\[
\mathcal{L} = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + (m^2 + \xi R) \phi^2
\]  
(1)
where $g^{\mu\nu}$ is the metric tensor and $R$ is the scalar curvature. Such a Lagrangian with $m = 0$ and the minimal coupling $\xi = 0$ appears also in the theory of structure formation (cosmological perturbations)[1]. We discuss the Euclidean version of a spatially homogeneous metric (we write $x = (t, \mathbf{x})$ or $x = (x_0, \mathbf{x})$ depending on whether the first coordinate has an interpretation of time or space)
\[
d s^2 = d t^2 + g_{jk}(t) d x^j d x^k
\]
The Laplace-Beltrami operator resulting from the bilinear form in eq.(1) reads
\[
\Delta_g = \frac{1}{2} g^{-\frac{1}{2}} \partial_\mu (g^{\mu\nu} g^{-\frac{1}{2}} \partial_\nu)
\]  
(2)
(here \( g = \det(g_{\mu\nu}) \)). In cosmological models \( g_{jk} \simeq t^{2\alpha} \) and \( g^{jk} \simeq t^{-2\alpha} \) when \( t \to 0 \) with \( \alpha > 0 \). Such a singular behavior can appear also in models describing collapse phenomena in general relativity [2].

As a second example we consider quantum mechanics on a (topologically trivial) manifold with the Hamiltonian

\[
H = -\triangle_g + U(x_0)
\] (3)

(in some global coordinates \( x = (x_0, x) \)).

The Green functions of (Euclidean) quantum scalar fields (1) with \( m = 0 \) and the minimal coupling \( \xi = 0 \) are solutions of the equation

\[
-\triangle_g G = g^{-\frac{1}{2}} \delta
\] (4)

These Green functions are also relevant for classical field theory because they describe a propagation of disturbances. In quantum mechanics (3) we are interested in the propagator kernels

\[
\exp(-\tau H)(x_0, x; x'_0, x')
\] (5)

where \( \tau \) is purely imaginary.

In this paper we prove that if the coefficients of the Laplace-Beltrami operator have a power-law singularity at a certain point \( t = t_0 \) then the Green functions \( G(t_0, x; t_0, x') \) are more regular than the ones of operators with regular coefficients (for regular coefficients the Green function can be expressed by the geodesic distance [3][4]). In quantum field theory these Green functions have the meaning of expectation values of quantum fields at equal times. In quantum mechanics the propagator (5) will have an anomalous behavior in \( \tau \). The Green function (4) can be obtained from the propagator (5) by means of an integration over \( \tau \).

2 The Green’s functions

Let us change coordinates

\[
\frac{dt}{d\eta} = \sqrt{g}
\] (6)

The Laplace-Beltrami operator (2) takes the form

\[
2\triangle_g = g^{-1} \partial_\eta^2 + g^{jk} \partial_j \partial_k
\] (7)

The bilinear form in eq.(1) determines an operator \( A \) which is of the same form as \( H \) in quantum mechanics (eq.(3))

\[
A = -\triangle_g + w
\] (8)
Here, \( w = \frac{1}{2} m^2 + \frac{1}{2} \xi R \) for the scalar field and \( w = U \) for quantum mechanics. The Green’s function of \( \mathcal{A} \) is a solution of the equation

\[
-(\partial_\eta^2 + gg^{jk} \partial_j \partial_k - W)G = 2\delta(\eta - \eta')\delta(x - x')
\]  

(9)

where we write

\[
W = gw
\]

(10)

Together with eq.(9) we consider the differential equation

\[
-\partial_\tau P_\tau = \mathcal{A}P_\tau
\]

(11)

with the initial condition \( P_0(\eta, x; \eta', x') = \delta(\eta - \eta')\delta(x - x') \). Eq.(11) defines the transition function of a stochastic process [5].

We can formulate the problem of solving the equation

\[
\mathcal{A} G = \delta
\]

(12)

as a problem in the Hilbert space of square integrable functions \( L^2(d\eta dx) \) [6]. We assume that \( W \) is a non-negative function. The operator \( \mathcal{A} \) can be considered as a self-adjoint non-negative operator in \( L^2 \) if \( gg^{jk} \) and \( W \) are locally integrable functions (then we can define the Friedrichs extension [6] of the symmetric differential operator (8)). The transition function \( P_\tau \) of eq.(11) can be defined as the integral kernel of \( \exp(-\tau \mathcal{A}) \). Then, the kernel of the inverse

\[
\mathcal{A}^{-1} = \int_0^\infty d\tau \exp(-\tau \mathcal{A})
\]

is the solution of eq.(9). It follows that the Fourier transform \( \tilde{G} \) of \( G \) has the representation

\[
\tilde{G}(\eta, \eta'; \mathbf{p}) = \int_0^\infty d\tau \tilde{P}_\tau(\eta, \eta', \mathbf{p})
\]

(13)

where \( \tilde{P} \) is a solution of the equation

\[
-\partial_\tau \tilde{P}_\tau = \tilde{\mathcal{A}} \tilde{P}_\tau
\]

(14)

with the initial condition \( \tilde{P}_0(\eta, \eta', \mathbf{p}) = \delta(\eta - \eta') \) (the fundamental solution).

Here

\[
\tilde{\mathcal{A}} = -\frac{1}{2} \partial_\eta^2 + \frac{1}{2} p_j g^{jk}(\eta)g(\eta) p_k + W \equiv -\frac{1}{2} \partial_\eta^2 + V(\eta) + W(\eta)
\]

(15)

Eq.(14) is a Schrödinger-type equation with the Hamiltonian \( \tilde{\mathcal{A}} \) and the potential \( V + W \) where

\[
V(\eta) = \frac{1}{2} p_j g^{jk}(\eta)g(\eta) p_k \equiv p \tilde{\mathbf{V}}(\eta)
\]

(16)

If the potentials \( V \) and \( W \) belong to \( L^1_{loc}(d\eta) \) then \( \tilde{\mathcal{A}} \) is a well-defined essentially self-adjoint operator in \( L^2(d\eta) \)[7].
We can express the kernel ˜\(P\) by means of the Brownian motion \(b\) (the Feynman-Kac formula [8]; a discussion of the probabilistic representation for singular potentials can be found in [9])

\[
\tilde{P}_\tau(\eta, \eta', \mathbf{p}) = \frac{1}{E[\delta(\eta' - \eta - b(\tau))] \exp\left(-\int_0^\tau V(\eta + b(s)) \, ds - \int_0^\tau W(\eta + b(s)) \, ds\right)]
\]

where \(E[.]\) denotes an average over the Brownian paths. Now, the kernel of \(\exp(-\tau A)\) has the representation

\[
P_\tau(\eta, x, \eta', x') = (2\pi)^{-\frac{d}{2}} \int dp \exp(i\mathbf{p}(\mathbf{x}' - \mathbf{x})) \frac{1}{\sqrt{\tau}} \gamma(\frac{s}{\tau})
\]

In order to eliminate the \(\delta\) function in eq.(17) it is useful to express the expectation value over the Brownian motion by means of an expectation value over the Brownian bridge \(\gamma\). Let \(q\) be a path connecting \(\eta\) with \(\eta'\)

\[
q(s) = \eta + (\eta' - \eta) \frac{s}{\tau} + \sqrt{\tau} \gamma(\frac{s}{\tau})
\]

where \(\gamma\) is the Gaussian process on the interval \([0,1]\) (the Brownian bridge) starting from 0 and ending in 0 with the covariance

\[
E[\gamma(s)\gamma(s')] = s'(1-s)
\]

for \(s' \leq s\). Then, eq.(17) can be rewritten in the form [8]

\[
\tilde{P}_\tau(\eta, \eta', \mathbf{p}) = (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{1}{2\tau}(\eta' - \eta)^2\right) \exp\left(-\tau \int_0^1 ds \left(\frac{1}{2s(1-s)}V(q(s)) + W(q(s))\right)\right)
\]

Applying the Jensen inequality (see [10]-[11]) to the \(E[.]\) integral we obtain the inequality

\[
\tilde{P}_\tau(\eta, \eta', \mathbf{p}) \geq (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{1}{2\tau}(\eta' - \eta)^2\right) \exp\left(-\tau \int_0^1 ds \left(\frac{1}{2s(1-s)}(V + W)(\eta + s(\eta' - \eta) + \sqrt{\tau} y)\right)\right) \equiv \tilde{P}_L
\]

This integral is

\[
\tilde{P}_L(\eta, \eta', \mathbf{p}) = (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{1}{2\tau}(\eta' - \eta)^2\right) \exp\left(-\tau \int_0^1 ds \int dy (2\pi s(1-s))^{-\frac{d}{2}} \exp\left(-\frac{y^2}{2s(1-s)}(V + W)(\eta + s(\eta' - \eta) + \sqrt{\tau} y)\right)\right)
\]

As a simple application of the inequality (21) we note that if

\[
V + W \leq A' \mathbf{p}^2 + B'
\]
then
\[
\hat{P}_{\tau}^L(\eta, \eta', \mathbf{p}) \geq (2\pi\tau)^{-\frac{d}{2}} \exp\left(-\frac{1}{2\tau}(\eta' - \eta)^2\right) \exp\left(-\tau A'\mathbf{p}^2 - \tau B'\right)
\]
(24)

Hence, we obtain a bound from below by the transition function for the \(d\)-dimensional Brownian motion.

On the other hand we may apply the Jensen inequality in the opposite direction to the \(s\)-integral
\[
\hat{P}_{\tau}(\eta, \eta', \mathbf{p}) \leq (2\pi\tau)^{-\frac{d}{2}} \exp\left(-\frac{1}{2\tau}(\eta' - \eta)^2\right) f_0^1 ds E[\exp\left(-\tau V(q(s)) - \tau W(q(s))\right)] \equiv \tilde{P}_U
\]
(25)

This integral takes the form
\[
\hat{P}_{\tau}^U(\eta, \eta', \mathbf{p}) = (2\pi\tau)^{-\frac{d}{2}} \exp\left(-\frac{1}{2\tau}(\eta' - \eta)^2\right) f_0^1 ds \int dy (2\pi s(1-s))^{-\frac{1}{2}} \exp\left(-\frac{y^2}{2\pi(1-s)}\right) \exp\left(-\tau V + \tau W\right) (\eta + s(\eta' - \eta) + \sqrt{\tau} y)
\]
(26)

If
\[
V + W \geq A\mathbf{p}^2 + B
\]
then
\[
\hat{P}_{\tau}^U(\eta, \eta', \mathbf{p}) \leq (2\pi\tau)^{-\frac{d}{2}} \exp\left(-\frac{1}{2\tau}(\eta' - \eta)^2\right) \exp\left(-\tau A\mathbf{p}^2 - \tau B\right)
\]
(27)

Hence, we estimate the transition function from above by the Wiener transition function.

3 Scale invariant metrics

We consider in this section a power-law cosmological expansion. Such an expansion is an exact solution of coupled Einstein equations for a metric and for the scalar field with an exponential self-interaction. Some consequences for a structure formation with such an expansion are discussed in [12][13]. If \(g_{jk}(t)\) has an isotropic power-law behavior then \(V\) is scale invariant. Let us assume here that \(V\) and \(W\) are nonnegative and scale invariant around \(\eta = 0\) (there is nothing special in the choice of \(\eta = 0\) as a singular point, see a discussion at eq.(44))
\[
V^{jk}(\lambda\eta) = \lambda^{2\nu}V^{jk}(\eta)
\]
(28)
and
\[
W(\lambda\eta) = \lambda^{2\sigma}W(\eta)
\]
(29)
Let us denote $\theta = \tau^{-\frac{1}{2}} \eta$. We apply the scaling properties of the Brownian bridge (19). Then, for $V$ of the form (28) and $W$ (29) we obtain

$$\tilde{P}_\tau(\eta, \eta', \mathbf{p}) = (2\pi\tau)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\tau}(\eta' - \eta)^2\right)$$

$$E[\exp\left(-\tau^{1+\nu} \int_0^1 \mathbf{p} \tilde{V} \mathbf{p} \left(\theta + s(\theta' - \theta) + \gamma(s)\right) ds\right) \exp\left(-\tau^{1+\sigma} \int_0^1 W \left(\theta + s(\theta' - \theta) + \gamma(s)\right) ds\right)] \quad (30)$$

The bounds (22) and (26) become simple if $\eta = \eta' = 0$. Then, the bound (22) reads

$$\tilde{P}_\tau(0, 0, \mathbf{p}) = (2\pi\tau)^{-\frac{1}{2}} \exp\left(-\tau^{1+\nu} \int_0^1 ds \int dy (2\pi s(1-s))^{-\frac{1}{2}} V(y) \exp\left(-\frac{y^2}{2\pi(1-s)}\right) \right)$$

$$-\tau^{1+\sigma} \int_0^1 ds \int dy (2\pi s(1-s))^{-\frac{1}{2}} W(y) \exp\left(-\frac{y^2}{2\pi(1-s)}\right)$$

$$= (2\pi\tau)^{-\frac{1}{2}} \exp\left(-\tau^{1+\nu} \mathbf{p} \mathbf{h} \mathbf{p} \int_0^1 ds (s(1-s))^{\nu} - B \tau^{1+\sigma} \int_0^1 ds (s(1-s))^{\sigma}\right) \quad (31)$$

where the bilinear form $h$ in $\mathbf{p}$ is defined by

$$\mathbf{p} \mathbf{h} \mathbf{p} = (2\pi)^{-\frac{1}{2}} \int dy \exp\left(-\frac{y^2}{2}\right) \tilde{V}(y) \mathbf{p} \quad (32)$$

and the constant $B$ in eq.(31) is

$$B = (2\pi)^{-\frac{1}{2}} \int dy \exp\left(-\frac{y^2}{2}\right) W(y) \quad (33)$$

The integral (32) is finite if $\nu > -\frac{1}{2}$ and (33) is finite if $\sigma > -\frac{1}{2}$. In such a case the lower bound (31) is non-trivial. The upper bound (26) takes the form

$$\tilde{P}_\tau'(0, 0, \mathbf{p}) = (2\pi\tau)^{-\frac{1}{2}} \int dy (2\pi s(1-s))^{-\frac{1}{2}} V(y) \exp\left(-\frac{y^2}{2\pi(1-s)}\right)$$

$$\exp\left(-\tau^{1+\nu} V(y) - \tau^{1+\sigma} W(y)\right) \quad (34)$$

We are interested in the Green functions (9) of the operator $\mathcal{A}$ which according to eqs.(13) and (18) are expressed by an $\tau$ integration upon $P_\tau$. As the simplest example of the integral (13) let $V + W = A\mathbf{p}^2 + B$ then performing the $\tau$ integration upon the rhs of eq.(13) we obtain

$$\tilde{G}_\tau(0, 0; \mathbf{p}) = (2A\mathbf{p}^2 + 2B)^{-\frac{1}{2}} \quad (35)$$

This is the standard behavior of equal-time Green’s functions for the quantum free field.

In eq.(31) let us first discuss the case $W = B = 0$. Then, the integral over $\tau$ of eq.(31) gives the lower bound on the Green’s function

$$\tilde{G}(0, 0; \mathbf{p}) \geq K_1(\mathbf{p} \mathbf{h} \mathbf{p})^{-\omega} \quad (36)$$
where

$$\omega = \frac{1}{2(1+\nu)}$$  \hspace{1cm} (37)

In order to estimate the upper bound (34) (for $W = 0$) let us assume a lower bound $|\tilde{V}|_0$ on $\tilde{V}$, i.e., for $p \neq 0$

$$p\tilde{V}p \geq p^2|\tilde{V}|_0 > 0$$  \hspace{1cm} (38)

Now, we change variables in eqs. (13) and (34) $(\tau, y) \to (\rho, u)$ where

$$\rho = \tau |p|^{\frac{1}{1+\nu}}|\tilde{V}(y)|_0^{\frac{1}{1+\nu}}$$

$$u = y(s(1-s))^{-\frac{1}{2}}$$

Then, the upper bound takes the form

$$\tilde{G}(0, 0; p) \leq |p|^{-2\omega} \int_0^\infty d\rho (2\pi \rho)^{-\frac{1}{2}} \int_0^1 ds \int du (2\pi)^{-\frac{1}{2}}(s(1-s))^{\frac{-1}{2}}\exp(-\nu^2) \exp (-\rho^{1+\nu})$$  \hspace{1cm} (39)

We can see that the integral on the rhs of eq.(39) is finite if $-1 < \nu < \infty$.

We can summarize our results as

**Theorem 1**

Assume that $W = 0$ and the potential $V$ in eq.(15) is nonnegative and scale invariant with $\nu > -\frac{1}{2}$ (eq.(28)). Then, the operator $\tilde{A}$ is essentially self-adjoint and the integral kernel of $\exp(-\tau\tilde{A})$ has the probabilistic representation (20). Assume that the potential $V$ satisfies the lower bound (38) then the Fourier transform $\tilde{G}(\eta, \eta'; p)$ of $G(\eta, \eta'; x - x')$ at $\eta = \eta' = 0$ for any $p$ satisfies the inequalities

$$K_1(p|p|^{-\omega}) \leq \tilde{G}(0, 0; p) \leq K_2|p|^{-2\omega}$$  \hspace{1cm} (40)

where $h$ is defined in eq.(32), $K_1$ and $K_2$ are some positive constants.

For $\nu < 0$ the Fourier transform $\tilde{G}$ is decaying to zero faster than the Green function for operators with constant coefficients. As a consequence $G$ is less singular than the one for operators with constant coefficients (see eq.(43) below).

In the configuration space if $W = 0$ then we can extract the $\tau$ dependence from $V$ using its scale invariance. Then, changing the integration variable in eq.(18) $p = \tau^{-\frac{1}{2}(1+\nu)}k$ we can conclude that $P$ has the form

$$P_\tau(\eta, \xi, \xi') = \tau^{-\frac{1}{2}(1+\nu)d-\frac{d}{4}}F(\tau^{-\frac{1}{2}\eta}, \tau^{-\frac{1}{2}\eta'}, \tau^{-\frac{1}{2}(1+\nu)}(\xi - \xi'))$$  \hspace{1cm} (41)

with a certain function $F$. Integration over $\tau$ with a rescaled $\tau = \tau|x - x'|^{1+\nu}$ brings the Green’s function at equal time to the form

$$G(\eta, x, \eta', x') = |x - x'|^{-d+\frac{1}{2}+\nu}f(|x - x'|^{1+\nu}\eta, (x - x')|x - x'|^{-1})$$  \hspace{1cm} (42)
It follows
\[ G(0, x, 0, x') = |x - x'|^{-d + \frac{1}{1 + \nu}} f((x - x')|x - x'|^{-1}) \quad (43) \]
We obtain such a behavior in $|x - x'|$ if we apply the inverse Fourier transform to the functions on both sides of the inequalities (40).

Let us note that if $V$ is singular at $\eta_0 \neq 0$ (e.g., $V \approx |\eta - \eta_0|^{2\nu}$) then all our results concerning the transition functions and Green functions still hold true but instead of setting $\eta = \eta' = 0$ we set $\eta = \eta' = \eta_0$ (this conclusion follows directly from eq.(30)). So, e.g., the formula (43) reads
\[ G(\eta_0, x, \eta_0, x') = |x - x'|^{-d + \frac{1}{1 + \nu}} f((x - x')|x - x'|^{-1}) \quad (44) \]
We admit now $W \neq 0$

**Theorem 2**

Let $W \geq 0$ be scale invariant (eq.(29)) and $\sigma > -\frac{1}{2}$ then (under the assumptions of Theorem 1 concerning $V$) for any $\Lambda > 0$ if $|p| > \Lambda$ then there exist positive constants $K_1$ and $K_2$ such that the inequalities (40) hold true.

**Proof:** setting $W = 0$ in eq.(34) we obtain the upper bound (39). For the lower bound we note that the exponential in eq.(31) is dominated by the terms quadratic in the momenta. We change the integration variable in eqs.(13) and (34)
\[ \tau = r(p)h^{-\frac{1}{1 + \nu}} \]
Then, we can see that for any $\Lambda > 0$ if $|p| > \Lambda$ then there exists a constant $C$ such that in the exponential of eq.(31) $B\tau^{1+\sigma} < C r^{1+\sigma}$. Then
\[ \int d\tau \bar{P}_\tau^L(0, 0, p) \geq (ph_2 p)^{-\omega} \int dr (2\pi r)^{-\frac{1}{2}} \exp \left(-r^{1+\nu} \int_0^1 ds (1 - s)^{\nu} - C r^{1+\sigma} \int_0^1 ds (1 - s)^{\sigma} \right) \]
From this lower bound and from the upper bound (39) we obtain the results of the theorem.

If $W > 0$ then the lower bound in eq.(40) cannot be true for arbitrarily small $p$ because as follows from eq.(34) ($V = 0$ for $p = 0$)
\[ \tilde{G}(0, 0, 0) \leq \int_0^\infty d\tau (2\pi \tau)^{-\frac{1}{2}} \int_0^1 ds \int dy (2\pi s(1 - s))^{-\frac{1}{2}} \exp(-\frac{y^2}{2(1-s)}) \exp(-r^{1+\sigma} W(y)) < \infty \quad (45) \]
If we imposed the condition that $t \geq 0$ (which is quite artificial in the Euclidean framework) then we would need to impose boundary conditions at $\eta = 0$ on the Brownian motion in the path integral (17). The Dirichlet boundary conditions can easily be imposed in the functional integration framework. We just insert the characteristic function of the positive real axis in the path integral (17) rejecting all the Brownian paths which leave the positive real axis. With the Dirichlet boundary conditions our estimates on the upper bound remain
unchanged whereas the estimates on the lower bound require some minor mod-
ifications.

Let us consider an example of a threedimensional space. By a change of
coordinates we can diagonalize the matrix \((g_{jk})\)

\[
g_{jk} = \delta_{jk} a_j^2
\]

Let \(a = (a_1 a_2 a_3)\) and

\[
\delta_j = a_j^{-1} a^{-2} \partial_\eta a_j
\]

\[
\delta = a^{-3} \partial_\eta a
\]

\[
Q = \frac{1}{18} \sum_{j<k} (\delta_j - \delta_k)^2
\]

Then, in the potential \(W\) of eq.(9)[14]

\[
g R = 6 a^4 (a^{-2} \partial_\eta \delta + \delta^2 + Q)
\]

and

\[
m^2 g = m^2 a^6
\]

We obtain a scale invariant \(V\) and \(W\) if \(a_j\) are scale invariant. Let us consider
the simplest case when all \(a_j\) are equal, \(t \in \mathbb{R}\) and

\[
a(t) = |t|^\alpha
\]

We have

\[
\eta = (1 - 3\alpha)^{-1} t |t|^{-3\alpha}
\]

Note that for \(\alpha > \frac{1}{3}\) the point \(t = 0\) corresponds to \(\eta = -\infty\) and \(t = \infty\) to
\(\eta = 0\).

Then

\[
V(y) = \kappa p^2 |y|^{2\nu}
\]

where \(\kappa > 0\) is a certain constant and

\[
\nu = 2\alpha (1 - 3\alpha)^{-1}
\]

For a scale invariant metric

\[
W = m^2 g(\eta) + \xi g R = C_1 m^2 |\eta|^{\frac{12}{3\alpha}} + \xi C_2 \eta^{-2}
\]

De Sitter space can be obtained as a limit \(\alpha \to \infty\). Then, we have \(V(\eta) = \kappa p^2 |\eta|^{-\frac{2}{3}}\) and \(m^2 g = c' \eta^{-2}\), hence \(W(\eta) = \frac{c}{\eta} \eta^{-2}\). This is a singular pertur-
aption which goes beyond our analysis. It can be treated by means of the path
integral methods. However, in such a case \(W\) needs a regularization, then a
renormalization and a subsequent removal of the regularization [9]. The \(\eta^{-2}\)
singularity comes also from the term $gR$. Hence, the results of this section apply only to $\xi = 0$. Then, in eq.(29) $\sigma = 3\alpha(1 - 3\alpha)^{-1}$. $B$ in eq.(33) is finite if $|\alpha| < \frac{1}{3}$.

In quantum mechanics $x_0$ is interpreted as a space variable. The metric takes the form $(d + 1 = 3)$

$$ds^2 = dx_0^2 + |x_0|^{2\alpha}(dx_1^2 + dx_2^2)$$

Then, $\eta = (1 - 2\alpha)^{-1}x_0|x_0|^{-2\alpha}$. The Hamiltonian (3) is symmetric in $L^2(\sqrt{\nu}dx)$. The change of coordinates $x_0 \to \eta$ associates with $H$ the operator $\tilde{A} = gH$ which is symmetric in $L^2(d\eta dx)$

$$\tilde{A} = -\partial^2_\eta + V + W$$

where

$$V(\eta) = C_1|\eta|^{\frac{2\alpha}{1 - 2\alpha}}$$

(51)

with $p^2 = p_1^2 + p_2^2$ and

$$W = gU(\eta) = C_2|\eta|^{\frac{1}{1 - 2\alpha}}U(\eta)$$

(52)

The anomalous behavior of $\hat{P}_r$ has as a consequence

**Corollary 3**

Let $\hat{P}_r(\eta, \eta', p)$ be the fundamental solution of eq.(14) with $W = 0$ and $V$ defined in eq.(51). If $\nu = \frac{\alpha}{1 - 2\alpha} > -\frac{1}{2}$ then for any $\tau \geq 0$

$$\int dx P_r(0, x, 0, x')|x - x'|^2 = (-\Delta_p)\hat{P}_r(0, 0, p)|_{p=0} = B_1\tau^{\frac{1}{2}+\nu}$$

(53)

and

$$\int dxd\eta' P_r(0, x, \eta', x')|x - \eta'|^2 = \int d\eta' (-\Delta_p)\hat{P}_r(0, \eta', p)|_{p=0} = B_2\tau^{1+\nu}$$

(54)

If $W(\eta) \geq 0$ defined in eq.(52) belongs to $L^1_{\text{loc}}(d\eta)$ then instead of the equalities in eqs.(53)-(54) we have bounds from above by $B_1\tau^{\frac{1}{2}+\nu}$ in eq.(53) and $B_2\tau^{1+\nu}$ in eq.(54).

**Proof:** we prove eq.(54) (eq.(53) is simpler and proved in a similar way). Let us calculate

$$(-\Delta_p)\int d\eta' \hat{P}_r(0, \eta', p)|_{p=0} = \int d\eta'(2\pi\tau)^{-\frac{1}{2}} \exp(-\frac{1}{2\tau}(\eta')^2) E[\tau^{1+\nu}\int_0^1 Tr\hat{V}\left(s\tau^{-\frac{1}{2}}\eta' + \gamma(s)\right)ds] = B_2\tau^{1+\nu}$$

(55)

If $W \geq 0$ then instead of the expectation value (55) we have

$$\int d\eta'(2\pi\tau)^{-\frac{1}{2}} \exp(-\frac{1}{2\tau}(\eta')^2) E[\tau^{1+\nu}\int_0^1 Tr\hat{V}\left(s\tau^{-\frac{1}{2}}\eta' + \gamma(s)\right)ds \exp(-\int_0^\tau W(s\tau^{-1}\eta' + \sqrt{\tau}\gamma(\frac{s}{\tau}))ds)] \leq B_2\tau^{1+\nu}$$
where the inequality follows from \( W \geq 0 \).

Corollary 3 means that if \( \nu < 0 \) then the sample paths of diffusions generated by operators with singular coefficients have worse continuity properties than the Brownian paths (for Brownian paths see [8]).

4 More general metrics

We study the lower bound on \( G \) following from eq.(22)

\[
\tilde{G}^L(0, 0, p) = \int_0^\infty d\tau (2\pi\tau)^{-\frac{1}{2}} \\
\exp \left( -\tau \int_0^1 ds \int dy (2\pi s(1-s))^{-\frac{1}{2}} \exp \left( -\frac{y^2}{2(1-s)} \right) V \left( \sqrt{\tau y} \right) \right)
\]

and the upper bound following from eq.(26)

\[
\tilde{G}^U(0, 0, p) = \int_0^\infty d\tau (2\pi\tau)^{-\frac{1}{2}} \\
\int_0^1 ds \int dy (2\pi s(1-s))^{-\frac{1}{2}} \exp \left( -\frac{y^2}{2(1-s)} \right) \exp \left( -\tau V \left( \sqrt{\tau y} \right) \right)
\]

for more general \( V \)

A generalization of Theorem 1 reads

**Theorem 4**

Let us consider \( V = p\tilde{V} p \) which is not scale invariant but of the form

\[
\tilde{V}(\eta) = \tilde{V}(\eta)f(\eta) + l(\eta)
\]

where \( \tilde{V} \) is a matrix scale invariant function (28) satisfying the conditions of Theorem 1 with \(-\frac{1}{2} < \nu < 0\), \( f \) is a bounded function with a strictly positive lower bound, \( l \) is a nonnegative bounded matrix function. Assume in addition that

\[
\int dy \exp \left( -\frac{y^2}{2} \right)f(y)\tilde{V}(y) \geq cI > 0
\]

where \( c \) is a positive number. Under our assumptions (56)-(57) for any \( \Lambda > 0 \) if \( |p| > \Lambda \) then there exist a positively definite bilinear form \( h_2 \) and constants \( K_1 \) and \( K_2 \) such that

\[
K_1(p h_2 p)^{-\omega} \leq \tilde{G}(0, 0; p) \leq K_2 |p|^{-2\omega}
\]

If \( \nu \geq 0 \) for \( \tilde{V} \) in eq.(56) then for \( |p| > \Lambda \) the inequalities (58) hold true with \( \omega = \frac{1}{2} \).

**Proof:** our assumptions (56) on \( \tilde{V} \) mean that it satisfies the inequalities

\[
\tau^\nu p \tilde{V}_1(y)p + pl_1 p \leq V(\sqrt{\tau y}) \leq \tau^\nu p \tilde{V}_2(y)p + pl_2 p
\]
with certain matrix functions $\tilde{V}_1$ and $\tilde{V}_2$ independent of $\tau$ and bilinear forms $l_1$ and $l_2$ (independent of $y$). It follows that the integral of $\tilde{P}_\tau$ satisfies the bounds

$$
\int d\tau \tilde{P}_{\tau}^{L_2} \exp(-\tau p l_2 p) \leq \int d\tau \tilde{P}_{\tau} \leq \int d\tau \tilde{P}_{\tau}^{U_1} \exp(-\tau p l_1 p)
$$

(60)

where in the lower bound $\tilde{P}_{\tau}^{L_2}$ the potential $V_2$ from the rhs of eq.(59) is applied and in $\tilde{P}_{\tau}^{U_1}$ the one from the lhs of eq.(59). The integral (57) defines $h$ of eq.(32) (and the $h_2$ from the upper bound (59)). Let us change the integration variable $\tau = r (p h_2 p)^{-\frac{1}{2}}$ on the lhs of eq. (60) and $\tau = \rho |p|^{-2 \omega} |\tilde{V}_1(y)|_0^{2 \omega}$ on the rhs. Then, the lower and upper bounds read (from eqs.(31),(34) and (38))

$$
(h p h_2 p)^{-\omega} \int_0^\infty dr (2\pi r)^{-\frac{1}{2}} \exp(-\rho^{1+\nu} - r (p h_2 p)^{-2\omega} p l_2 p) \\
\leq \tilde{G}(0,0, p) \leq |p|^{-2\omega} \int_0^\infty d\rho (2\pi \rho)^{-\frac{1}{2}} \\
\int_0^1 ds \int du (2\pi)^{-\frac{1}{2}} (s(1-s))^{-\frac{1}{4}} |\tilde{V}_1(u)|_0^{-\omega} \exp(-\frac{u^2}{2}) \exp\left(-\rho^{1+\nu} - \rho |p|^{-2\omega} |\tilde{V}_1(u \sqrt{s(1-s)})|_0^{2\omega} p l_1 p\right)
$$

(61)

The condition (57) implies that the bilinear form $h_2$ is strictly positive. Hence, there exists a constant $K$ such that

$$K p h_2 p \geq p l_2 p
$$

Then, for $-\frac{1}{2} < \nu < 0$ and $|p| > \Lambda$ there exists $c_1$ such that

$$r (p h_2 p)^{-\frac{1}{2}} p l_2 p < rc_1
$$

in the exponential on the lhs of eq.(61). The $l_1$ term can be set zero for the upper bound. In such a case for each $\Lambda > 0$ there exist constants $c_1$ and $c_2$ such that if $|p| > \Lambda$ then the inequalities (61) take the form

$$
(h p h_2 p)^{-\omega} \int_0^\infty dr (2\pi r)^{-\frac{1}{2}} \exp(-\rho^{1+\nu} - rc_1) \\
\leq \tilde{G}(0,0, p) \leq |p|^{-2\omega} \int_0^\infty d\rho (2\pi \rho)^{-\frac{1}{2}} \\
\int_0^1 ds \int du (2\pi)^{-\frac{1}{2}} (s(1-s))^{-\frac{1}{4}} |\tilde{V}_1(u)|_0^{-\omega} \exp(-\frac{u^2}{2}) \exp\left(-\rho^{1+\nu}\right)
$$

(62)

The inequalities (62) coincide with (58) because under our assumptions the behavior of $\tilde{P}_\tau$ for a small $\tau$. If $\nu > 0$ then in eq.(60) $\tau^{1+\nu} < A\tau$ for any $A$ and a sufficiently small $\tau$. Hence, we obtain the same behavior of $\tilde{G}$ for large momenta as in the case $\tilde{V} = 1$.

We would like to note that the restrictive form (56) of $V$ is not necessary. As an example we could consider $V$ which has singularities at several points, e.g.

$$
V(\eta) = p^2 (a_0 |\eta_0|^{2\omega} + \kappa |\eta|^{2\omega})
$$

(63)
with $|\nu_0| < |\nu|$ (only negative indices are non-trivial). An application of the lower and upper bounds (31) and (34) to the potential (63) leads to the conclusion that after an integration upon $\tau$ the inequalities (40) hold true for $|p| > \Lambda$. Hence, the leading singularity $\nu$ determines the behavior at large momenta.

5 Discussion and summary

As we pointed out in the Introduction our results concerning the Green functions can find an application to quantum field theory in an expanding universe. The stronger damping in momenta (eq.(40)) in the inflationary models ($\alpha > 1$) at $\eta = \eta' = 0$ indicates that it would be promising to start quantization at this time ($\eta = 0$ corresponds to $t = \infty$ in cosmological models with $\alpha > \frac{1}{3}$). The exponential inflation can be obtained as a limit $\alpha \to \infty$ which corresponds to $\nu = -\frac{1}{4}$. This limit is beyond our rigorous approach but it could be treated by means of more sophisticated methods of ref.[9]. By a formal scaling argument we obtain again the behavior (40) which in inflationary cosmological models is known as the Harrison-Zeldovich spectrum of scalar fluctuations [1][15]. The Green functions can be applied in order to derive a solution of Einstein equations linearized around the homogeneous background [12][13]. In such a case in addition to the scalar Green function the tensor Green function must be studied as well. Further consequences of our estimates concerning the spectrum of $\tilde{G}$ for the complete theory still need to be explored. For this purpose a detailed dependence of the Green function on $\eta$ and $\eta'$ would be useful. It is much harder to derive such estimates than the ones for the time zero case. In the Appendix we investigate the upper bound $G_U$ for general $\eta$. In particular, calculations performed there suggest that it is only the behavior of $\mathcal{V}(y)$ for small $y$ which is relevant for Theorem 4 and that the upper bound is valid for all $\nu > -1 + \frac{1}{d}$. For the lower bound $G_L$ we can also obtain an integral representation. However, it is quite complicated.

Another motivation for a study of the $(d+1)$-dimensional Green functions comes from the problem of a dimensional reduction of quantum fields defined on a brane [16]. In such a case we restrict ourselves to a $d$-dimensional submanifold imposing the condition $\eta = \eta' = 0$. We have proved here that if the metric has a power-law behavior then the Green functions of the restricted quantum field theory are decaying faster in the momentum space than the standard $|p|^{-1}$. In particular, for $\nu = -\frac{1}{2}$ we obtain the propagator $|p|^{-2}$ in $d$-dimensions which is the same as the one of the Euclidean massless free field.

6 Appendix

We calculate the upper bound for the Green’s function $\tilde{G}$ following form eqs.(26) and (13) in more detail. Set $u = \sqrt{\tau}y$ and let us perform the integration upon
\[ \tau \text{ in eq.(26) with } W = 0. \text{ Then} \]
\[ \tilde{G}^U(\eta, \eta', \mathbf{p}) = 2(2\pi)^{-\frac{1}{2}} \int_0^1 ds \int du (2\pi s(1-s))^{-\frac{1}{2}} K_0 \left( \sqrt{2Mp\tilde{V}(u)} \right) \] (64)

where
\[ M = (\eta' - \eta)^2 + (s(1-s))^{-1} (u - \eta - s(\eta' - \eta))^2 \] (65)

and \( K_0 \) is the modified Bessel function of order \( \rho \) [17].

The integral is simpler if \( \eta = \eta' \)
\[ \tilde{G}^U(\eta, \eta, \mathbf{p}) = \pi^{-1} \int_0^1 ds \int du K_0 \left( N \sqrt{2p\tilde{V}(u)} \right) \] (66)

where
\[ N = (s(1-s))^{-\frac{1}{2}} \eta - u \] (67)

If \( \eta' = \eta = 0 \) then the integral (64) further simplifies. If additionally \( V \) is scale invariant then we can calculate it exactly as in sec.3.

For arbitrary \( \eta \) and \( \eta' \) the behavior of \( \tilde{G}^U \) is much more complicated because the decay of \( \tilde{G}^U \) substantially depends on \( \eta \). In the simplest case when \( \tilde{V} = \frac{1}{2} \)
\[ \tilde{G}(\eta, \eta', \mathbf{p}) = |\mathbf{p}|^{-1} \exp(-|\mathbf{p}||\eta' - \eta|) \] (68)

Let us consider an arbitrary \( V \) and assume that it behaves as
\[ V(\lambda y) = \lambda^{2\nu} B_\lambda(y) \] (69)

when \( \lambda \to 0 \) with a certain \( B_\lambda \) which as a function of \( \lambda \) is bounded from above and from below, i.e., \( C_2(y) \geq B_\lambda(y) \geq C_1(y) > 0 \) for a small \( \lambda \). Let us change variables in eq.(64)
\[ u = |\mathbf{p}|^{-\frac{1}{1+\nu}} y \] (70)

Assume that \( |\mathbf{p}| \to \infty \), \( \eta \to 0 \) and \( \eta' \to 0 \) in such a way that \( \theta = |\mathbf{p}|^{-\frac{1}{1+\nu}} \eta \) and \( \theta' = |\mathbf{p}|^{-\frac{1}{1+\nu}} \eta' \) remain finite. In such a case from eq.(64) we can conclude that
\[ \tilde{G}(|\mathbf{p}|^{-\frac{1}{1+\nu}} \theta, |\mathbf{p}|^{-\frac{1}{1+\nu}} \theta', \mathbf{p}) \simeq |\mathbf{p}|^{-2\omega} \] (71)

for large \( \mathbf{p} \) in agreement with eq.(40) and eq.(68) (\( \nu = 0 \) for a constant \( \tilde{V} \)).

Eqs.(64)-(67) give an integral representation of the upper bound which is expected to approximate the exact Green function \( \tilde{G} \) for large \( \mathbf{p} \). We suppose that the Fourier transform \( G^U \) of \( \tilde{G}^U \) is a reliable approximation to \( G \) at short distances. After the Fourier transform of eq.(26) with \( W = 0 \) we can calculate the \( \tau \) integral in eq.(13) exactly. We obtain
\[ G^U(\eta, \mathbf{x}, \eta', \mathbf{x}') = (2\pi)^{-\frac{d+2}{2}} \int_0^1 ds (s(1-s))^{-\frac{1}{2}} \int dy \left( \det \tilde{V}(y) \right)^{-\frac{1}{2}} \]
\[ \left( (\mathbf{x} - \mathbf{x}') \tilde{V}^{-1}(y) (\mathbf{x} - \mathbf{x}') + (\eta - \eta')^2 + (s(1-s))^{-1} (y - \eta - s(\eta' - \eta))^2 \right)^{-\frac{d}{2}} \] (72)
If $\tilde{V}^{ij}(y) = \delta^{ij}v(y)$ then the formula (72) can be expressed in a simpler form

$$G^U(\eta, x, \eta', x') = (2\pi)^{-\frac{d+1}{2}} \int_0^1 ds \left( |x - x'|^2 + v(y) (\eta - \eta')^2 + (s (1-s))^{-1} v(y) (y - \eta - s (\eta' - \eta))^2 \right)^{-\frac{d+1}{2}}$$

(73)

If $v = 1$ then eq.(73) gives

$$(2\pi)^{-\frac{d+1}{2}} \left( |x - x'|^2 + (\eta - \eta')^2 \right)^{-\frac{d+1}{2}}$$

(74)

as it should.

The integrals (72)-(73) suggest some generalizations of the theorems proved in the main part. First, assume that $V(y) \simeq |y|^2 \rho$ for a large $|y|$ then the integrals (72)-(73) are finite (for $|x - x'| \neq 0$) if $\rho > -1 + \frac{1}{d}$. Next, it can be shown from eq.(73) that if $v(y) \simeq |y|^{2\nu}$ for $y \to 0$ and $\eta = \eta' = 0$ then

$$G^U(0, x, 0, x') \simeq |x - x'|^{-d+\frac{1}{1+\nu}}$$

(75)

for $x - x' \to 0$. The derivation of the result (44) based on eq.(73) suggests that for Theorem 4 only the behavior of $V(y)$ for a small $y$ is relevant (assuming the integral (73) is finite).

For general $v$ and arbitrary $x, x', \eta$ and $\eta'$ it is harder to obtain usable estimates. Let us mention some special cases. It follows directly from eq.(41) that

$$G(\eta, x, 0, x) \simeq |\eta|^{-d(1+\nu)+1}$$

and

$$G(0, x, \eta', x) \simeq |\eta'|^{-d(1+\nu)+1}$$

whereas from eq.(73) we obtain that if $v(\eta) \neq 0$ then

$$G^U(\eta, x, \eta, x') \simeq v(\eta)^{-\frac{d}{2}} |x - x'|^{-d+1}$$

(75)

when $x \to x'$.

If $V(y) \geq c|y|^{2\nu}$ with $c > 0$ and $\rho > -1 + \frac{1}{d}$ for large $y$ then changing the integration variable $y = |x - x'|^{\frac{1}{1+\nu}} z$ we can show that for any $\eta$ and $\eta'$ there exists $A$ such that if $|x - x'| \geq A$ then

$$G^U(\eta, x, \eta', x') \leq K |x - x'|^{-d+\frac{1}{1+\nu}}$$

(76)

When $\rho > 0$ then eq.(76) gives a non-trivial estimate saying that the Green’s function has a stronger decay for large distances than the one for operators with constant coefficients. However, such a decay at large distances will be changed by most perturbations $W$ whereas the behavior for short distances is remarkably stable with respect to perturbations.
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