Robust Coordination in Networks
Laura Arditti, Giacomo Como, Member, IEEE, Fabio Fagnani, and Martina Vanelli, Member, IEEE

Abstract—We study the robustness of binary-action heterogeneous network coordination games equipped with an external field modeling the different players’ biases towards one action with respect to the other one. We prove necessary and sufficient conditions for global stability of consensus equilibria under best response type dynamics, robustly with respect to (constant or time-varying) values of the external field. We then apply these results to the analysis of mixed network coordination and anti-coordination games and find sufficient conditions for existence and global stability of pure strategy Nash equilibria. Our results apply to general weighted directed interaction networks and build on super-modularity properties of the coordination games in order to characterize conditions for the existence of a novel notion of robust improvement and best response paths.

Index terms: Coordination games, network games, network robustness, best response dynamics, robust stability.

I. INTRODUCTION

Robustness, meant as the ability of the system to perform effectively under a range of operating conditions absorbing possible disruptions, is undoubtedly a fundamental issue that has long been studied in control [2]. While playing a key role in several domains, robustness and the related notion of resilience have lately become central in multi-agent and network systems, such as infrastructure systems [3]–[6], financial networks [7]–[10], as well as social and economic networks [11]–[14]. In fact, the interconnectivity level of such critical societal systems has dramatically increased over the last years, due to the unprecedented technological advancements in terms of real-time sensing, communication, and decision capabilities. As a consequence, such large scale networks tend to be prone to disruptions as localized perturbations have the potential to build up through cascading mechanisms, possibly achieving detrimental systemic effects. A characteristic feature that has been recognized in the robustness analysis of network systems is the role played by their interconnection pattern topology in propagating or attenuating shocks. [15]–[19].

The increasing interest in game-theoretic models for multi-agent and network systems makes it relevant and timely to address the robustness issues above in the framework of network games [20]–[24]. These are strategic games whose players are identified with the nodes of a graph and have utilities that depend on their own and their neighbors’ actions. Typically, network games are formulated as closed systems, without explicit input or output signals. External influence is usually modeled as the modification of certain parameters appearing in the utility functions and a common problem addressed is the way such interventions can modify the Nash equilibria or their nature. This viewpoint is natural for instance in analyzing the effect of taxes or subsidies in economic models or prices and tolls in transportation systems [25], [26]. More recently, a similar approach has been proposed in the context of network quadratic games [27] to model incentive interventions. The effect of node removal in such quadratic games has also been investigated [28].

In this paper, we study network coordination games with binary action sets, intrinsically equipped with interconnection ports modeling possible interactions with the external environment. We first introduce a number of robustness notions for such games and characterize them in terms of topological properties of the network structure. We then apply these results to the analysis of network coordination dynamics embedded in a time-varying environment as well as to the study of network coordination games interconnected with different games.

Network coordination games represent one of the most popular models to describe network systems with interactions of strategic complements type [29], [30]. They find numerous applications in modeling social and economic behaviors like the emergence of social norms and conventions or the adoption of new technologies [31]–[35]. A growing literature studies various types of learning dynamics for such games [36]–[39], also as a model for contagion and cascades in networks [40]–[44]. Optimal seeding and other intervention problems for network coordination games have been studied in [45] and, in the more general setting of super-modular games, in [46]. Recently, vulnerability of network coordination games against adversarial attacks has been investigated in [47], [48], while [49] uses network coordination games as a micro-foundation for community structure in networks.

Network coordination games are a special case of super-modular games, as they satisfy the so-called increasing difference property [50], [51]. Specifically, the convenience for a player to switch from an action to the alternative action is monotone in the fraction of players in her neighborhood already playing the alternative action. Such monotonicity reflects in a threshold structure of the best response: an action becomes convenient over the alternative action for a player when a sufficient fraction of her neighbors have chosen that action. Consensus configurations (whereby all players playing the same action) are the prototypical outcome of such games. Consequently, much of the interest is in understanding which are the conditions under which a distributed learning rule leads to such configurations or, conversely, which are the possible...

Some of the results in the paper appeared in preliminary form in [1].

The authors are with the Department of Mathematical Sciences “G.L. Lagrange,” Politecnico di Torino, 10129 Torino, Italy (e-mail: {laura.arditti; giacomo.como; fabio.fagnani; martina.vanelli}@polito.it). G. Como is also with the Department of Automatic Control, Lund University, 22100 Lund, Sweden.

This research was carried on within the framework of the MIUR-funded Progetto di Eccellenza of the Dipartimento di Scienze Matematiche G.L. Lagrange, Politecnico di Torino, CUP: E11G18000350001. It received partial support from the MIUR Research Project PRIN 2017 “Advanced Network Control of Future Smart Grids” (http://vectors.dieti.unina.it), and by the Compagnia di San Paolo through the Project “SMeILE”.

CoRR 2021.1285v2 [cs.GT] 11 Oct 2021
obstacles that could prevent it. In [31], for the special case when the threshold of all nodes are equal, the existence of non-consensus Nash equilibria (to be referred to as coexistent equilibria) is studied in terms of an equivalent network topology property known as cohesiveness. In the absence of such coexistent equilibria, classical results of super-modular games show that under best response dynamics the system always converges in finite time to a consensus equilibrium.

The distinctive feature of this paper is to analyze static and dynamic properties of network coordination games on general directed graphs, parameterized by an external field that affects the utility functions. Our focus is on the structure and stability of Nash equilibria, particularly the consensus configurations, and on the asymptotic behavior of best response type learning dynamics. The external field can model either endogenous heterogeneity in the network, or an exogenous signal, to be interpreted as a perturbation or a control variable. A variation of the external field modifies the threshold of a player, in extreme cases transforming her into a stubborn agent, i.e., one whose best response is always the same action, regardless of her fellow players’ actions.

In particular, we study conditions under which the system converges to a consensus equilibrium, independently from the values taken by the external field. As it turns out, two conditions need to be satisfied for such robust stability property to hold true. The first condition, to be referred to as robust indecomposability, is a generalization of the lack of cohesive partitions [31] to parametrized families of heterogeneous network coordination games. It is equivalent (see Theorem [1]) to the lack of coexistent equilibria for any value of the external field within a certain range. On the other hand, the second condition guarantees that the external field is incapable of creating stubborn agents for both actions. While the necessity of these two conditions for convergence to a consensus is quite intuitive, the proof of sufficiency is more involved and resides on the possibility to find best response paths for the game that are robust to modifications of the external field. This is achieved in Theorem [2] that is one of our main results and uses in a crucial way the super-modularity of the game. The final convergence result is then presented in Theorem [3].

Finally, we apply the theory illustrated above to the analysis of network games where a subset of the nodes play a coordination game, while the rest of the nodes play an anti-coordination game [52], i.e., a game where the convenience for a player to switch from an action to the alternative action is anti-monotone in the fraction of players in her neighborhood already playing such alternative action. Such mixed network games have received attention in the recent literature: see, e.g., [53], where it is proven that the asynchronous best-response dynamics of pure coordination games and pure anti-coordination games will almost surely converge to Nash equilibrium for every network topology and every set of thresholds; [54], studying the more general setting of network games with the simultaneous presence of strategic complements and substitutes interactions; and [55] where a detailed analysis of the behavior of mixed network coordination/anti-coordination games is presented for the special case when the underlying graph is complete. For this class of mixed network games, in Theorem [4] and Corollary [5] we provide sufficient conditions guaranteeing convergence to a Nash equilibrium whose restriction to the set of coordinating players is a consensus.

The rest of the paper is organized as follows. Section [I] is concluded by introducing some notational conventions. Section [II] is dedicated to recall basic facts on games and, particularly, to gather a number of fundamental results on binary super-modular games. These results are not necessarily original if not for the way they are presented here, which is functional to their use in the rest of the paper. Nevertheless, we present proofs of many of these results as we were not able to find them in the literature. Section [III] is devoted to the analysis of binary network coordination games with the external field considered fixed. We propose a new classification of such games on the basis of the structure of their Nash equilibria, dependently on how many consensus Nash equilibria they possess and the possible existence of coexistent equilibria. We generalize the concept of cohesiveness and we propose a general topological property equivalent to the absence of coexistent Nash equilibria in Propositions [5] and [6]. We also recall some stability results of consensus equilibria. In Section [IV] we propose a robust version of the properties studied in Section [III] assuming the external field to take values in some bounded domain. Main result of this section is Theorem 2 that contains a robust stability analysis of consensus equilibria and is instrumental to the applications studied in the next sections. Sections [V] and Section [VI] contain the two main applications considered in this paper, both dealing with the asymptotic behavior of the asynchronous best response dynamics for network coordination games. In particular, in Section [V] we consider time-varying perturbations of the external field and find necessary and sufficient conditions for convergence to a consensus. In Section [VI] we analyze a system that is a mixture of coordinating and anti-coordinating agents and we find an original sufficient condition that guarantees that the coordinating agents will asymptotically reach a consensus equilibrium. Section [VII] closes the paper discussing some directions for future research.

We conclude this section by introducing some notational conventions to be followed throughout the paper. We will often consider real vector spaces of type \( \mathbb{R}^2 \) for some finite set \( \mathcal{I} \), equipped with the component-wise partial order \( \leq \), formally defined by

\[
x \leq y \iff x_i \leq y_i, \quad \forall i \in \mathcal{I}.
\]

For two vectors \( x \) and \( y \) in \( \mathbb{R}^2 \), the component-wise supremum \( x \vee y \) in \( \mathbb{R}^2 \) and infimum \( x \wedge y \) in \( \mathbb{R}^2 \) have entries, respectively,

\[
(x \vee y)_i = \max\{x_i, y_i\}, \quad (x \wedge y)_i = \min\{x_i, y_i\}, \quad \forall i \in \mathcal{I}.
\]

We use the notation \( \vee \) and \( \wedge \) to indicate the component-wise supremum and infimum, respectively, of a subset \( \mathcal{L} \subseteq \mathbb{R}^2 \). A function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is referred to as monotone non-decreasing (nonincreasing) if it preserves (reverses) the partial order \( \leq \) defined in (1), i.e., if \( f(x) \leq f(y) \) (\( f(x) \geq f(y) \)) for every \( x \leq y \). For a vector \( x \) in \( \mathbb{R}^2 \), \( |x|_1 \) in \( \mathbb{R}^2 \) stands for the vector with entries \( |x|_1 = |x_i| \) for every \( i \in \mathcal{I} \). The symbol \( 1 \) indicates a vector with all components equal to 1.
II. PRELIMINARIES

In this section, we first provide some background on game theory. Then, we introduce the classes of super-modular games along with some of their most relevant properties.

A. Game-theoretic notions

We consider strategic form games with a finite nonempty player set $\mathcal{V}$ and binary action set $\mathcal{A} = \{\pm 1\}$ for every player $i$ in $\mathcal{V}$. We denote by $\mathcal{X} = \mathcal{A}^\mathcal{V}$ the space of strategy profiles or configurations. The $i$-th entry $x_i$ of a configuration $x$ in $\mathcal{X}$ represents the action played by player $i$ in $\mathcal{V}$. We refer to $x = \pm 1$ as a consensus configuration, and to every $x$ in $\mathcal{X} \setminus \{\pm 1\}$ as a co-existent configuration. We use the notation

$$\mathcal{V}^+_x = \{i \in \mathcal{V} : x_i = +1\}, \quad \mathcal{V}^-_x = \{i \in \mathcal{V} : x_i = -1\}$$

to indicate the sets of players that, in the configuration $x$, are playing, respectively, +1 and −1. With a slight abuse of notation, we adopt the convention of identifying the symbols + and − with the actions +1 and −1, respectively, so that, e.g., we may write $\mathcal{V}^+_s = \{i \in \mathcal{V} : x_i = s\}$ for $s = \pm 1$.

Every player $i$ in $\mathcal{V}$ represents a rational agent whose aim is to maximize a utility function $u_i : \mathcal{X} \to \mathbb{R}$. As customary, for a strategy profile $x$ in $\mathcal{X}$ and a player $i$ in $\mathcal{V}$, we let $x_{-i}$ in $\mathcal{X}_{-i} = \mathcal{A}^\mathcal{V}\setminus\{i\}$ stand for the strategy profile of all players except for player $i$. We shall then use the slight and common abuse of notation

$$u_i(x) = u_i(x_i, x_{-i})$$

for the utility perceived by player $i$ in configuration $x$. The best response correspondence for a player $i$ in $\mathcal{V}$ is then

$$\mathcal{B}_i(x_{-i}) = \operatorname{argmax}_{x_i \in \mathcal{A}} u_i(x_i, x_{-i}).$$

For later use, we also define

$$\mathcal{B}_i^+(x_{-i}) = \max \mathcal{B}_i(x_{-i}), \quad \mathcal{B}_i^-(x_{-i}) = \min \mathcal{B}_i(x_{-i}).$$

An action $a$ in $\mathcal{A}$ is dominant for a player $i$ if

$$a \in \mathcal{B}_i(x_{-i}), \quad \forall x_{-i} \in \mathcal{X}_{-i},$$

and strictly dominant if

$$\mathcal{B}_i(x_{-i}) = \{a\}, \quad \forall x_{-i} \in \mathcal{X}_{-i}.$$

A player $i$ having a strictly dominant action $a$ is referred to as an $a$-stubborn agent.

A (pure strategy Nash) equilibrium is a configuration $x^*$ in $\mathcal{X}$ such that

$$x_i^* \in \mathcal{B}_i(x_{-i}^*), \quad \forall i \in \mathcal{V}.$$  

The set of equilibria of a game will be denoted as $\mathcal{N}$. An equilibrium $x^*$ is referred to as strict if

$$\mathcal{B}_i(x_{-i}^*) = \{x_i^*\}, \quad \forall i \in \mathcal{V}.$$  

We shall consider learning dynamics in games based on revision protocols whereby only one player modifies her action at a time and such modification never decreases the player's utility. A number of related concepts are reported below. We start with describing the type of paths that the configuration vector can follow under such revision protocols.

**Definition 1.** For $l \geq 0$, a length-$l$ admissible path from $x$ in $\mathcal{X}$ to $y$ in $\mathcal{X}$ is an $(l+1)$-tuple of strategy profiles

$$(x^{(0)}, x^{(1)}, \ldots, x^{(l)}) \in \mathcal{X}^{l+1}$$

such that

$$x^{(0)} = x, \quad x^{(l)} = y,$$

and for $k = 1, 2, \ldots, l$, there exists a player $i_k$ in $\mathcal{V}$ such that

$$x^{(k)}_{-i_k} = x^{(k-1)}_{-i_k}, \quad x^{(k)}_{i_k} \neq x^{(k-1)}_{i_k}.$$

A length-$l$ admissible path is referred to as

- monotone if $x^{(0)} \leq x^{(1)} \leq \ldots \leq x^{(l)}$;
- anti-monotone if $x^{(0)} \geq x^{(1)} \geq \ldots \geq x^{(l)}$.

In other words, a length-$l$ admissible path is an $(l+1)$-tuple of strategy profiles such that two consecutive strategy profiles differ just for the action of a single player $i_k$, to be referred to as the $k$-th active player. Notice that an admissible path from a configuration $x$ to another configuration $y$ is completely determined by the sequence of active players $(i_1, i_2, \ldots, i_l)$. In monotone paths, players can modify their action from −1 to +1 only, conversely only modification from +1 to −1 are allowed in anti-monotone paths.

**Definition 2.** A length-$l$ admissible path $(x^{(0)}, x^{(1)}, \ldots, x^{(l)})$ with sequence of active players $(i_1, i_2, \ldots, i_l)$ is

- an improvement path (I-path) if
  $$u_{i_k}(x^{(k)}) > u_{i_k}(x^{(k-1)}), \quad k = 1, 2, \ldots, l;$$
- a best response path (BR-path) if
  $$u_{i_k}(x^{(k)}) \geq u_{i_k}(x^{(k-1)}), \quad k = 1, 2, \ldots, l.$$

Notice that, by convention, every I-path is also a BR-path (but not vice versa) and that the singleton $(x)$ is to be considered as a length-0 I-path from $x$ to $x$ with an empty sequence of active players. Observe that, since we are assuming a binary action set, relation (5) is equivalent to

$$\mathcal{B}_{i_k}^{(k-1)}(x_{-i_k}) = \{x_{i_k}^{(k)}\}$$

while, whenever relation (6) is satisfied with equality, it means that

$$\mathcal{B}_{i_k}^{(k-1)}(x_{-i_k}) = \{x_{i_k}^{(k)}, x_{i_k}^{(k-1)}\}.$$  

**Definition 3.** For $\alpha \in \{I, BR\}$, a subset of strategy profiles $\mathcal{Y} \subseteq \mathcal{X}$ is:

- $\alpha$-reachable from strategy $x$ in $\mathcal{X}$ if there exists an $\alpha$-path from $x$ to some strategy profile $y$ in $\mathcal{Y}$;
- globally $\alpha$-reachable if it is $\alpha$-reachable from every configuration $x$ in $\mathcal{X}$;
- $\alpha$-invariant if there are no $\alpha$-paths from any $y$ in $\mathcal{Y}$ to any $z$ in $\mathcal{X} \setminus \mathcal{Y}$;
- globally $\alpha$-stable if it is globally $\alpha$-reachable and $\alpha$-invariant.

To indicate that a configuration $y$ is I-reachable (BR-reachable) from a configuration $x$, we will use the notation
x \to y (x \to y). If y is reachable from x by an monotone or anti-monotone I-path (BR-path), we will use the notation
\[ x \uparrow y, \quad x \downarrow y, \quad x \upharpoonright y, \quad x \downharpoonleft y, \]
respectively. Observe that all these relations are reflexive and transitive.

**B. Super-modular games**

Super-modular games are an important class of games and have received a considerable amount of attention in the literature \[29, 30, 50, 51\]. Below we recall the definition of super-modular games and report a number of their fundamental properties. We take advantage of the binary action setting considered here, while at the same time we recall that the theory of super-modular games can be developed in more general lattice action sets.

**Definition 4.** A binary action game is super-modular if for every player \(i \in \mathcal{V}\),
\begin{equation}
 u_i(1, x_{-i}) - u_i(-1, x_{-i}) \geq u_i(1, y_{-i}) - u_i(-1, y_{-i}), \tag{7}
\end{equation}
whenever \(x_{-i} \geq y_{-i}\).

In other words, a game is super-modular if the marginal utility \(u_i(1, x_{-i}) - u_i(-1, x_{-i})\) of every player \(i\) is a monotone nondecreasing function of the strategy profile \(x_{-i}\) of the other players. This is sometimes referred to as the increasing difference property.

It will prove useful to introduce the four maps
\begin{equation}
 f^+, f^-, g^+, g^- : \mathcal{X} \to \mathcal{X} \tag{8}
\end{equation}
on the configuration space, respectively defined by
\begin{equation*}
 f^+(x) = \bigvee \{ y \in \mathcal{X} \mid x \uparrow y \}, \quad f^-(x) = \bigwedge \{ y \in \mathcal{X} \mid x \downarrow y \},
\end{equation*}
\begin{equation*}
 g^+(x) = \bigvee \{ y \in \mathcal{X} \mid x \downarrow y \}, \quad g^-(x) = \bigwedge \{ y \in \mathcal{X} \mid x \uparrow y \},
\end{equation*}
for every \(x \in \mathcal{X}\). Thanks to Lemma \[3\] in Appendix \[A\] we have that \(f^+(x)\) and \(g^+(x)\) (\(f^-(x)\) and \(g^-(x)\)) represent the maximal (minimal) configurations that are BR-reachable and, respectively, I-reachable from \(x\) by a monotone (anti-monotone) path. Notice that all sets in the righthand side of the above are nonempty as they contain the strategy profile \(x\).

In the following statement, we gather a number of properties relating equilibria of super-modular games with the behavior of the maps \[\text{(8)}\] that will be used in the rest of the paper.

**Proposition 1.** Consider a finite super-modular game with binary action sets. Then, for every configuration \(x \in \mathcal{X}\),
\begin{enumerate}[(i)]
  \item \(f^-(f^+(x)) \in \mathcal{N}\) and \(f^+(f^-(x)) \in \mathcal{N}\) are, respectively, the greatest and least equilibria I-reachable from \(x\);
  \item \(f^-(g^+(x)) \in \mathcal{N}\) and \(f^+(g^-(x)) \in \mathcal{N}\) are, respectively, the greatest and least equilibria BR-reachable from \(x\);
  \item if \(x \in \mathcal{N}\), then \(g^+(x), g^-(x) \in \mathcal{N}\) are, respectively, the greatest and least equilibria BR-reachable from \(x\).
\end{enumerate}

Moreover,
\begin{enumerate}[(i)]
  \item the set of equilibria \(\mathcal{N}\) is a nonempty complete lattice and is globally I-stable;
  \item the two configurations
  \begin{equation}
  x^* = f^+(g^-(1)) = f^-(1), \tag{9}
  \end{equation}
  are, respectively, the least and greatest equilibria;
  \item for two equilibria \(x \in \mathcal{N}\), \(f^+(x \wedge y)\) and \(f^+(x \vee y)\) are, respectively, the least and the greatest equilibria that are, respectively, above and below both \(x\) and \(y\);
  \item if \(x^* = x^+ = x^*\), then \(\mathcal{N} = \{x^*\}\) is globally BR-stable.
\end{enumerate}

**Remark 1.** In general the set of equilibria \(\mathcal{N}\) is not BR-invariant for a super-modular game: e.g., a 2-player game with utilities \(u_1(x_1, x_2) = x_1 x_2\) and \(u_2(x_1, x_2) = 0\) for \(x_1\) and \(x_2\) in \(\{\pm 1\}\) is super-modular but its equilibrium set \(\mathcal{N} = \{\pm 1\}\) is not BR-invariant since \(B_2(1) = B_2(-1) = \{\pm 1\}\).

In order to ensure both readability and completeness, a proof of Proposition \[1\] is provided in Appendix \[A\] along with a number of other properties of super-modular games that are often scattered around in the aforementioned literature and may prove relatively difficult to be found exactly in the form used in this paper.

In the rest of this paper, we shall often consider games depending on some vector parameter \(h \in \mathbb{R}^V\) whose entries \(h_i\) are to be interpreted as endogenous or exogenous signals modifying the payoff of player \(i\). Whenever needed, we shall make the dependence on the parameter \(h\) explicit by writing, e.g., \(u_i(x_i, x_{-i}, h), B_i(x_{-i}, h), N_h, \text{ or } f^\pm(x, h)\). Particularly interesting for us will be the case when super-modularity is guaranteed also with respect to the parameter \(h\). This is captured by the following.

**Definition 5.** A binary action game with utilities \(u_i(x, h)\) that depend on a parameter \(h \in \mathcal{I}\) (where \(\mathcal{I}\) is a lattice) is referred to as a super-modular family if the game is super-modular for every \(h \in \mathcal{I}\), and
\begin{equation}
 u_i(1, x_{-i}, h) - u_i(-1, x_{-i}, h) \geq u_i(1, x_{-i}, h') - u_i(-1, x_{-i}, h'), \tag{10}
\end{equation}
for every two parameters \(h \geq h' \in \mathcal{I}\), every player \(i \in \mathcal{V}\), and every configuration \(x \in \mathcal{X}\).

In other words a parameter-dependant super-modular game is a super-modular family if it satisfies the additional increasing difference property \[10\] requiring that the marginal utility of every player in every configuration be a monotone nondecreasing function of the parameter. The following proposition gathers a number of key properties that are consequence of the additional increasing difference property \[10\] and can be proven along the same arguments used before.

**Proposition 2.** Consider a binary super-modular family of games with set of players \(\mathcal{V}\) and utility functions \(u_i(x, h)\) depending on the parameter \(h\) and assume that \[10\] holds true. Then, for every configuration \(x \in \mathcal{X}\),
\begin{enumerate}[(i)]
  \item \(B^+_i(x_{-i}, h)\) and \(B^-_i(x_{-i}, h)\) are monotone nondecreasing functions of \(h\);
  \item the maps \(f^+(x, h), f^-(x, h), g^+(x, h), g^-(x, h)\) are monotone nondecreasing functions of \(h\).
III. Network Coordination Games

In this section, we first introduce the class of network coordination games and present some of their most relevant properties. Then, we move on by introducing a classification of network coordination games based on the structure of their equilibrium set. Specifically, the proposed classification depends on two qualitatively relevant properties: the cardinality of the set of consensus equilibria and the possible presence of co-existent equilibria. We then prove necessary and sufficient conditions for such classification in terms of graph-theoretic properties of the network.

We consider directed weighted graphs $G = (\mathcal{V}, \mathcal{E}, W)$ with finite set of nodes $\mathcal{V}$, set of directed links $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and non-negative weight matrix $W$ in $\mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ such that $W_{ij} > 0$ if and only if $(i, j)$ is a link in $\mathcal{E}$. We do not allow for the presence of self-loops, i.e., we assume that the weight matrix $W$ has zero diagonal. Throughout, finite directed weighted graphs without self-loops will be simply referred to as graphs. We recover undirected graphs as a special case when the weight matrix $W = W^T$ is symmetric, so that in particular $(j, i)$ is a link in $\mathcal{E}$ whenever $(i, j)$ is. For a graph $G = (\mathcal{V}, \mathcal{E}, W)$ and a subset of nodes $\mathcal{S} \subseteq \mathcal{V}$, we denote by

$$w^\mathcal{S}_i = \sum_{j \in \mathcal{S}} W_{ij},$$

the $\mathcal{S}$-restricted out-degree of a node $i$ in $\mathcal{V}$. In the special case when $\mathcal{S} = \mathcal{V}$ coincides with the whole node set, we simply refer to

$$w_i = w^\mathcal{V}_i$$

as the out-degree of a node $i$ in $\mathcal{V}$ and let $w = W^1$ be the vector of out-degrees.

**Definition 6.** The network coordination game on a graph $G = (\mathcal{V}, \mathcal{E}, W)$ with external field $h$ in $\mathbb{R}^\mathcal{V}$ is the game with player set $\mathcal{V}$, whereby every player $i$ in $\mathcal{V}$ has binary action set $\mathcal{A} = \{\pm 1\}$ and utility function

$$u_i(x) = x_i \sum_{j \in \mathcal{V}} W_{ij} x_j + h_i x_i.$$  

(11)

Notice that it immediately follows from the computation

$$u_i(1, x_{-i}) - u_i(-1, x_{-i}) = 2 \sum_{j \in \mathcal{V}} W_{ij} x_j + 2h_i,$$

and the assumption $W_{ij} \geq 0$ for all $i, j$ in $\mathcal{V}$, that a network coordination game is a super-modular family, so that the results of Section II-B do apply to it. In fact, there are some specificities regarding network coordination games that it is worth discussing in some detail.

In a network coordination game, players are to be identified with the nodes of a graph $G$, and the utility of a player depends only on her own action and on the actions of those players identified with her out-neighbors in $G$. Specifically, player $i$’s utility function may be rewritten as

$$u_i(x) = x_i \left( h_i + w^+_i(x) - w^-_i(x) \right),$$  

(12)

where

$$w^+_i(x) = w^+_i(x), \quad w^-_i(x) = w^-_i(x).$$

Equation (12) highlights the decomposition of the utility of a player $i$ in $\mathcal{V}$ as the sum of a term $h_i x_i$ that depends only on her own action and rewards its alignment with the corresponding entry of the external field, plus a term $x_i (w^+_i(x) - w^-_i(x))$ that is the difference between the aggregate weight of links pointing to out-neighbors with the same action and the aggregate weight of links pointing to out-neighbors with the opposite action.

The following statement gathers a few specific simple results for coordination games.

**Proposition 3.** Consider the network coordination game on a graph $G = (\mathcal{V}, \mathcal{E}, W)$ with external field $h$ in $\mathbb{R}^\mathcal{V}$. Then, for every strategy profile $x$ in $\mathcal{X}$ and every player $i$ in $\mathcal{V}$,

$$x_i \in \mathcal{B}_i(x_{-i}) \iff u_i(x) \geq 0;$$  

(13)

(ii) the best response correspondence has the threshold form

$$\mathcal{B}_i(x_{-i}) = \begin{cases} 
\{+1\} & \text{if } w^+_i(x) > r_i w_i \\
\{-1\} & \text{if } w^-_i(x) = r_i w_i \\
\{-1\} & \text{if } w^-_i(x) < r_i w_i,
\end{cases}$$

(14)

for every $i$ in $\mathcal{V}$, where

$$r_i = \frac{1}{2} - \frac{h_i}{2w_i},$$

is the threshold of player $i$;

(iii) action $a = \pm 1$ is a strictly dominant strategy for a player $i$ in $\mathcal{V}$ if and only if $ah_i > w_i$.

Proof: (i) This follows directly from the equivalent form of the utility function.

(ii) By substituting the identity $w_i = w^+_i(x) + w^-_i(x)$ into (12), we have that

$$u_i(x) = x_i \left( h_i + 2w^+_i(x) - w^-_i(x) \right),$$

from which (14) follows directly.

(iii) This follows directly from point (ii).

To emphasize the dependance on the external field $h$, we shall use the notation $\mathcal{N}_h$ for the set of equilibria of the network coordination game with external field $h$, and let

$$\mathcal{N}^*_h = \mathcal{N}_h \cap \{\pm 1\}, \quad \mathcal{N}_h^c = \mathcal{N}_h \setminus \{\pm 1\},$$

be the subsets of consensus and, respectively, co-existent equilibria. We then introduce the following notion.

**Definition 7.** A network coordination game on a graph $G$ with external field $h$ is

- regular if $|\mathcal{N}^*_h| = 2$;
- biased if $|\mathcal{N}^*_h| = 1$; more precisely, for $a = \pm 1$, the network coordination game is $a$-biased if $\mathcal{N}^*_{ha} = \{a\}$;
- frustrated if $|\mathcal{N}_h^c| = 0$.

Notice that, in a frustrated network coordination game, neither of the consensus configurations is an equilibrium. Since network coordination games are super-modular, by Proposition I the set of equilibria $\mathcal{N}_h$ is never empty, hence a frustrated
network coordination game always admits at least one co-existent equilibrium. In contrast, when the game is not frustrated (either regular or biased) at least a consensus configuration is an equilibrium. Besides consensus, there might or might not exist co-existent equilibria. To distinguish these cases, the following further classification proves useful.

**Definition 8.** A network coordination game on a graph $G$ with external field $h$ is

- unpolarizable if $N^*_h = \emptyset$;
- polarizable if $N^*_h \neq \emptyset$.

The set of equilibria of an unpolarizable regular network coordination game contains both consensus configurations $\pm 1$ and no other configurations, whereas unpolarizable biased network coordination games admit a single (consensus) equilibrium: $x^* = +1$ in the positively biased case and $x^* = -1$ in the negatively biased one. On the other hand, polarizable network coordination games have always co-existent equilibria plus possibly consensus ones (if they are regular or biased).

In the sequel, we shall identify necessary and sufficient conditions for a network coordination game to be regular, biased, or frustrated, and for it to be polarizable or unpolarizable.

We start introducing two sets that will play a key role in our analysis:

$$S_a(h) = \{i \in V | a_i - h_i > 0\}, \quad a = \pm 1.$$  \hspace{1cm} (16)

By Proposition 3(iii), $S_a(h)$ coincides with the set of players for which $a$ is a strictly dominant action, i.e., the set of $a$-stubborn agents. The following simple result relates the presence of $a$-stubborn players with that of the consensus equilibrium $-a1$.

**Lemma 1.** Consider a network coordination game on a graph $G = (V, E, W)$ with external field $h$ and let $x^*(h)$ and $\pi^*(h)$ be its least and greatest equilibria, respectively. Then,

(i) $x^*(h) = -1 \iff S_{-1}(h) = \emptyset \iff h \leq w$

(ii) $\pi^*(h) = +1 \iff S_{+1}(h) = \emptyset \iff h \geq -w$

**Proof:** If $x^*(h) = -1$, then there cannot be $-1$-stubborn players, i.e., $S_{-1}(h) = \emptyset$, so that $h \leq w$. On the other hand, if $h \leq w$, then, using the threshold form of the best response in Proposition 3(ii), we deduce that $-1$ is an equilibrium, namely $x^*(h) = -1$. This proves (i), while (ii) can be proven analogously.

In fact, Lemma 1 directly implies the following result.

**Proposition 4.** Let $G$ be a graph with out-degree vector $w$. Then, the network coordination game on $G$ with external field $h$ is

(i) regular if and only if

$$|h| \leq w;$$  \hspace{1cm} (17)

(ii) $a$-biased for $a = \pm 1$ if and only if

$$w \geq -ah, \quad w \not\geq ah;$$  \hspace{1cm} (18)

(iii) frustrated if and only if

$$w \not\geq -h, \quad w \not\geq h.$$  \hspace{1cm} (19)

**Remark 2.** The necessary and sufficient conditions in Proposition 4 can be readily interpreted in terms of the presence of stubborn agents, as introduced in Section 2. In fact, Lemma 1 implies that (17) is equivalent to the fact that no player is stubborn, (18) is equivalent to the existence of at least one $a$-stubborn agent but no $-a$-stubborn agents, and (19) is equivalent to the existence of both $+1$- and $-1$-stubborn agents. Hence, Proposition 4 states that a network coordination game is regular if and only if there are no stubborn agents, biased if and only if it contains stubborn agents of one type only, and frustrated if it contains stubborn agents of both types.

In contrast to the relative simplicity of the characterization above, necessary and sufficient conditions for polarizability of network coordination games as per Definition 8 are in general more involved. First, we introduce the following notion.

**Definition 9.** A graph $G = (V, E, W)$ is $h$-indecomposable for a vector $h$ in $\mathbb{R}^V$ if for every nontrivial binary partition of the node set

$$V = V^+ \cup V^-, \quad V^- \cap V^+ = \emptyset, \quad V^+ \neq \emptyset, \quad V^- \neq \emptyset,$$  \hspace{1cm} (20)

there exist $s$ in $\{\pm 1\}$ and a node $i$ in $V^s$ such that

$$w^*_i + sh_i < w^*_{i^-},$$  \hspace{1cm} (21)

where $w^*_i = w^*_{i^1}$.

We have the following result.

**Proposition 5.** The network coordination game on a graph $G$ with external field $h$ is unpolarizable if and only if $G$ is $h$-indecomposable.

**Proof:** We start with a general consideration. Given any configuration $x^*$ in $\mathcal{X}$, for any player $i$ such that $x_i^* = s$, from (12) we can write that

$$u_i(x^*) = \begin{cases} s(h_i + w^*_i(x^*) - w^-_{i^1}(x^*)) = s(h_i + sw^*_i(x^*) - sw^-_{i^1}(x^*)) & \text{if } h \leq w \\ sh_i + w^*_i(x^*) - w^-_{i^1}(x^*) & \text{if } h \geq -w \end{cases}$$  \hspace{1cm} (22)

We now argue as follows. If the network coordination game on $G$ with external field $h$ is polarizable, there exists an equilibrium $x^* \neq \pm 1$. From (22) and Proposition 3(i) we derive that, for every $s$ and $i$ such that $x_i^* = s$,

$$sh_i + w^*_i(x^*) - w^-_{i^1}(x^*) \geq 0.$$  \hspace{1cm} (23)

This implies that relatively to the nontrivial binary partition $V = V^+_i \cup V^-_i$, (21) does not hold true for any $s$ in $\{\pm 1\}$ and $i$ in $V^s$. Hence, $G$ is not $h$-indecomposable.

On the other hand, if $G$ is not $h$-indecomposable, then there exists a nontrivial binary partition $V = V^+_i \cup V^-_i$ such that for every $s$ in $\{\pm 1\}$ and $i$ in $V^s$,

$$w^*_i + sh_i \geq w^-_{i^1},$$  \hspace{1cm} (23)

where $w^*_i = w^*_{i^1}$. Let $x^*$ in $\mathcal{X}$ be the configuration with entries $x_i^* = s$ for every $i$ in $V^s$. Then, (22) and (23) imply that

$$u_i(x^*) = sh_i + w^*_i(x^*) - w^-_{i^1}(x^*) \geq 0,$$  \hspace{1cm} (24)

for every $i$ in $V^s$. Proposition 3(i) implies that $x^*$ is an equilibrium for the network coordination game with external
field \( h \). This equilibrium is necessarily co-existent since \( V^- \) and \( V^+ \) are both nonempty.

Remark 3. Given a graph \( G = (V, E, W) \) and \( v \in [0, 1] \), a
subset of nodes \( S \subseteq V \) is called \( r \)-cohesive \([57]\) if
\[
w_i^S \geq rv_i, \quad \forall i \in S,
\]
and \( r \)-closed if its complement \( V \setminus S \) is \((1-r)\)-cohesive. Notice
that, using the identity \( w_i = w_i^r + w_i^{1-r} \), condition \((21)\) can be
rewritten as \( 2w_i^r < w_i - sh_i \). In the special case when players
have homogeneous thresholds \( r_i = r \in [0, 1] \), equivalently
when the external field is proportional to the node degree
vector, i.e., \( h = (1 - 2r)w \), \((21)\) is equivalent to \( w_i^r < rw_i \).
Hence, in this special case, \( h \)-indecomposability of a graph \( G \)
is equivalent to the non-existence of nonempty proper subsets
of nodes \( S \) that are both \( r \)-cohesive and \( r \)-closed. In this sense,
Proposition 5 generalizes \([56, \text{Proposition 9.7}]\) to network
coordination games with heterogeneous thresholds.

In the following examples, we shall illustrate how Propositions 4 and 5 can be applied to classify the behavior of network coordination games on some simple graph structures.

Example 1. Let \( G = (V, E, W) \) be a graph with two
nodes \( V = \{1, 2\} \) connected by two directed links of weight
\( W_{12} = w_1 \) and \( W_{21} = w_2 \), respectively. Consider a network
coordination game on \( G \) with external field \( h = (h_1, h_2) \).

As illustrated in Figure 7 by Proposition 2 the network
coordination game is: regular if \( |h| \leq w \) (white region); \(+1-
biased if \( h \geq -w \) and \( h \leq w \) (\( +1 \)-shaded region); \(-1\)-biased if
\( h \leq w \) and \( h \geq -w \) (\( -1 \)-shaded region); frustrated if \( h_1 > w_1 \)
and \( h_2 < -w_2 \) or \( h_1 < -w_1 \) and \( h_2 > w_2 \) (dotted region).

On the other hand, Proposition 5 ensures that the network
coordination game on \( G \) is indecomposable if and only if
one of the following holds true: (i) \( h < w \), (ii) \( h > -w \), (iii)
\( |h_1| < w_1 \), (iv) \( |h_2| < w_2 \), as illustrated in Figure 7 Notice
that, for the special case of only two-players, the network
coordination game is polarizable if and only if it is frustrated.

Example 2. Consider the network coordination game on a
complete graph \( G = (V, E, W) \) with \( n = |V| \) nodes and weight
matrix \( W = 11' - I \), as in Figure 2

First, consider the network coordination game on \( G \) with
external field \( h = 1 \). Consider any binary partition \( V = V^- \cup V^+ \)
such that \( |V^-| = m \) for \( 1 \leq m < n \). If \( m < n/2 \), then
\((21)\) reduces to \( m = w_i^- + h_i < w_i = n - m \) for every \( i \in V^+ \),
hence it is satisfied. On the other hand, if \( m \geq n/2 \),
then \((21)\) reduces to \( m = n - m - 2 = w_i^- - h_i < w_i^- \) for every \( i \in V^- \),
which is also satisfied. This shows that \( G \)
is \( 1 \)-indecomposable, so that, by Proposition 5 the network
coordination game is indecomposable.

Now, consider an external field \( h \) such that \( h_i = 1 \) for
\( 1 \leq i \leq \lfloor n/2 \rfloor \) and \( h_i = -1 \) for \( \lceil n/2 \rceil < i \leq n \). Then, \( G \)
is \( 1 \)-indecomposable if and only if \( n \) is odd. In fact, for even \( n \),
the binary partition \( V = V^- \cup V^+ \) with \( V^+ = \{1, \ldots, n/2\} \)
and \( V^- = \{n/2 + 1, n\} \), is such that \((21)\) is violated by every
node \( i \) in \( V \). Then, by Proposition 5 the network coordination
game is polarizable for odd \( n \) and indecomposable for even \( n \).

Biased unipolarizable coordination games can be character-
ized in an equivalent simpler form.

Proposition 6. The network coordination game on a graph \( G \)
with external field \( h \) is \( a \)-biased and unipolarizable if and only
if the following conditions are both satisfied:

(a) \( w \nless than ah \);
(b) every non-empty subset \( R \subset V \setminus S_a(h) \) contains
some node \( i \) such that
\[
w_i^R \leq w_i^{V \setminus R} + ah_i .
\]

Proof: (Only if) Assume that the network coordination
game is \( a \)-biased and unipolarizable. Then, condition (a)
follows from Proposition 3(ii). To prove (b), assume by con-
tradiction that there exists a non-empty subset \( R \subset V \setminus S_a(h) \)
such that
\[
w_i^R \geq w_i^{V \setminus R} + ah_i ,
\]
for every \( i \) in \( R \) and let \( x \) be a configuration such that
\( x_i = a \) for every \( i \in V \setminus R \) and \( x_i = -a \) for every \( i \) in \( R \).
Notice that \((26)\) and \((27)\) imply that \( u_i(x) \geq 0 \), so that,
by Proposition 3(i), \(-a = x_i \in B_1(x_i) \), for every \( i \) in \( R = V^- \).
If \( a = +1 \) (or \( -1 \)), this implies that there are no monotone
(anti-monotone) \( I \)-paths of positive length starting at \( x \), so that
in particular \( f^a(x) \) is. Then, by Proposition 1(i), we get that
\[
x^* = f^{-a}(x) = f^{-a}(f^a(x))
\]
is an equilibrium. Notice that on the one hand \( x^*_i = -a \) for
every \( i \) in \( R \) (since \( x_i = -a \) and \( x^* = f^{-a}(x) \)), on the other
hand \( x^*_i = a \) for every \( i \) in \( S_a(h) \) (since those are stubborn
players). Hence, \( x^* \) is a co-existent equilibrium, thus
contradicting the assumption that the game is unipolarizable.
Therefore, if the network coordination game is \( a \)-biased and
unipolarizable, both conditions (a) and (b) must be satisfied.
Since \( a_i + w_i \geq ah_i + w_i^{\forall R} > w_i^R \geq 0 \).

Since \( a_i + w_i > ah_i - w_i \geq 0 \) for every \( i \) in \( S_a(h) \), we deduce that \( ah + w \geq 0 \). Together with assumption (a), by Proposition 3 (ii), this yields that the game is \( a \)-biased. We finally prove that the game is unpolarizable. By contradiction, suppose there exists a co-existent equilibrium \( x^* \). Necessarily \( x^*_i = a \) for every \( i \) in \( S_a(h) \). Put \( R = V \setminus a \) and notice that (22) yields
\[
0 \leq u_i(x^*) = -ah_i + w_i^{-a}(x^*) - w_i^a(x^*) = -ah_i + w_i^R - w_i^{\forall R}
\]
for every \( i \) in \( R \) thus contradicting condition (b). The proof is then complete.

**Remark 4.** In the special case \( h = (1 - 2r)w \) considered in Remark 3, i.e., when players have homogeneous thresholds \( r_i = r \in [0, 1] \), (25) is equivalent to \( w_i^{\forall R} < (1 - r)w_i \), so condition (b) of Proposition 6 reduces to the non-existence of \((1 - r)\)-cohesive subsets of \( V \setminus S_a(h) \). In the literature \cite{31}, such property is referred to as the set \( V \setminus S_a(h) \) being uniformly not \((1 - r)\)-cohesive. In this sense, Proposition 6 generalizes \cite{56} Proposition 9.8 to network coordination games with heterogeneous thresholds.

As argued previously in this paper, in many applications it is of interest to study the conditions under which consensus configurations are the natural outcome of a (distributed) learning process. We conclude this section with the statement below, gathering some results on global I- and BR-stability of consensus equilibria for network coordination games that directly follow from the analysis just developed.

**Corollary 1.** For a graph \( G \) with out-degree vector \( w \), consider the network coordination game on it with external field \( h \). Assume that \( G \) is \( h \)-indecomposable. Then:
(i) if \( |h| \leq w \), then \( N_h = \{ \pm 1 \} \) is globally I-stable;
(ii) if \( |h| < w \), then \( N_h = \{ \pm 1 \} \) is globally BR-stable;
(iii) if \( w \nmid ah \), then \( N_h = \{ a1 \} \) is globally BR-stable.

**Proof:** (i) Proposition 5 (i) implies that when \( |h| \leq w \), the game is regular so that the two consensus configurations \(-1 \) and \(+1 \) are both equilibria. Since \( G \) is \( h \)-indecomposable, Proposition 5 guarantees that the game is unpolarizable. Then, the set of equilibria is \( N_h = \{ \pm 1 \} \) and Proposition 6 (iv) guarantees that it is globally I-stable.

(ii) We already know from point (i) that \( N_h = \{ \pm 1 \} \) is globally I-stable. The BR-stability follows from the fact that when \( |h| < w \) these two equilibria are strict.

(iii) By Lemma 1, \( S_a(h) \neq \emptyset \), so that the configuration \(-1 \) is not an equilibrium. Since \( G \) is \( h \)-indecomposable, Proposition 5 implies that \( N_h^c = \emptyset \). Hence, \( N_h = \{ a1 \} \). BR-stability then follows from Proposition 1 (vii).

**IV. ROBUST NETWORK COORDINATION**

In this section, we introduce and characterize robust versions of the notions introduced in Definitions 7 and 8. For a graph \( G = (V, E, W) \) and a subset of vectors \( H \subseteq \mathbb{R}^V \), we say that a property of the network coordination game on \( G \) is \( H \)-robustly if it is satisfied for every external field \( h \) in \( H \). In what follows we concentrate on the special case when, for two vectors \( h^- \) and \( h^+ \) in \( \mathbb{R}^V \) such that \( h^- \leq h^+ \), the set \( H \) is the hyper-rectangle
\[
H = \{ h : h^- \leq h \leq h^+ \} = \bigcap_{i \in V} \left[ h_i^-, h_i^+ \right].
\]

In this case, the verification that certain important properties are \( H \)-robustly satisfied can be significantly simplified with respect to checking the property for every single value of \( h \) in \( H \). Below we report the results in this sense. We start with the following robust version of Proposition 8.

**Corollary 2.** Let \( G \) be a graph with out-degree vector \( w \) and let \( H \) be as in (27) for two vectors \( h^- \leq h^+ \). Then, the network coordination game on \( G \) is:
(i) \( H \)-robustly regular if and only if
\[
w \geq -h^- \land h^+;
\]
(ii) \( H \)-robustly \( a \)-biased for an action \( a = \pm 1 \) if and only if
\[
ah^{-a} + w \geq 0, \quad w \nmid ah^{-a};
\]
(iii) \( H \)-robustly frustrated if and only if
\[
w \nmid ah^{-a}, \quad \forall a = \pm 1.
\]

Another interesting property is the robust unpolarizability, which can be interpreted as the resilience of a network coordination game against getting co-existent equilibria. By virtue of Proposition 8, this can equivalently be expressed as a robust indecomposability of the graph \( G \). However, it is useful to reformulate this in a form analogous to Definition 3 as in the following result.

**Theorem 1.** Let \( G = (V, E, W) \) be a graph and let \( H \) be as in (27) for two vectors \( h^- \leq h^+ \). Then, the following conditions are equivalent:
(a) the network coordination game on \( G \) is \( H \)-robustly unpolarizable;
(b) \( G \) is \( H \)-robustly indecomposable;
(c) for every nontrivial binary partition of the node set as in (20), there exists \( s \) in \( \{ \pm \} \) and a node \( i \) in \( V^s \) such that
\[
w_i^s + sh_i^s < w_i^{-s}
\]
where \( w_i^s = w_i^{V^s} \).

**Proof:** Clearly, equivalence between conditions (a) and (b) directly follows from Proposition 8. We shall now prove equivalence between conditions (a) and (c).

First, assume that the network coordination game on \( G \) is not \( H \)-robustly unpolarizable, i.e., there exists an external field \( h \) in \( H \) such that the set of equilibria \( N_h \) contains a co-existent configuration \( x^* \neq \pm 1 \). Notice that \( h^-i \leq h_i \leq h_i^+ \) implies
that $x_i^t h_i \leq s_i h_s^t$, where $s_i = \text{sgn}(x_i^t)$, for every player $i$ in $\mathcal{V}$. Then, Proposition 3 (i) and (12) imply that
\[
0 \leq w_i(x^*) = x_i^t h_i + x_i^s \omega_i^t(x^*) - x_i^t \omega_i^-(x^*) \leq s_i h_s^t + w_i^t - w_i^-.
\]
Hence, there exists no node $i$ in $\mathcal{V}$ satisfying (30) for the nontrivial binary partition $\mathcal{V} = \mathcal{V}_x^t \cup \mathcal{V}_x^-$. This proves that condition (c) is not satisfied.

On the other hand, if condition (c) is not satisfied, then there exists a nontrivial binary partition as in (20) such that
\[
w_i^t + s_i h_s^t \geq w_i^-,
\]
where $s_i$ in $\{\pm\}$ is such that $i$ belongs to $\mathcal{V}_s^t$. Let $h$ in $[h^-, h^+]$ have entries
\[
h_i = \begin{cases} h_i^+ & \text{if } x_i^t = +1 \\ h_i^- & \text{if } x_i^t = -1 \end{cases} \quad \forall i \in \mathcal{V}.
\]
Then, $\mathcal{G}$ is not $h$-indecomposable. Thus, Proposition 5 implies that the network coordination game on $\mathcal{G}$ with external field $h$ is polarizable so that condition (a) is not satisfied.

Theorem 1 shows that testing $\mathcal{H}$-robust unipolarizability of a network coordination game on a graph $\mathcal{G}$ can be done by verifying that (30) holds true for every nontrivial binary partition of the node set, which has the same complexity as testing of $h$-indecomposability of $\mathcal{G}$, i.e., verifying that (21) holds true for every nontrivial binary partition of the node set. Of course, in the worst case such complexity remains exponential in the graph order, since so is the number of nontrivial binary partition of the node set. However, for specific graphs, the task can be much simplified as in the following example.

Example 2 (continued). Let $\mathcal{H} = \{h \in \mathbb{R}^\mathcal{V} : -1 \leq h \leq 1\}$. Then, the complete graph with $n$ nodes $\mathcal{G}$ is $\mathcal{H}$-indecomposable if and only if $n$ is odd. Indeed, for even $n$, we have already shown that $\mathcal{G}$ is $h$-decomposable with respect to the external field $h$ in $\mathcal{H}$ with entries $h_i = 1$ for $1 \leq i \leq n/2$ and $h_i = 1$ for $n/2 < i \leq n$. On the other hand, for odd $n$, consider any nontrivial binary partition $\mathcal{V} = \mathcal{V}_x^t \cup \mathcal{V}_x^-$ and let $m = |\mathcal{V}_x^+|$. If $m < n/2$, then $w_i^+ + h_i m < n - m = w_i^-$ for every player $i$ in $\mathcal{V}_x^+$, so that (30) is satisfied. If $m > n/2$, then $w_i^- - h_i^+ = n - m = m i < w_i^+$ for every player $i$ in $\mathcal{V}_x^-$, so that (30) is satisfied.

In fact, in an analogous way as Theorem 1 has just been derived as a robust generalization of Proposition 5 one can prove a robust generalization of Proposition 6. While refraining from doing that explicitly for the sake of conciseness, we focus on the stability results in Corollary 1 for which, besides their straightforward robust generalization, some deeper consequences can be derived as reported below. These will in turn prove instrumental for the analysis carried on in next section.

Theorem 2. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph of order $n = |\mathcal{V}|$ and let $\mathcal{H}$ be as in (27) for two vectors $h^- \leq h^+$. Assume that $\mathcal{G}$ is $\mathcal{H}$-robustly indecomposable. Then,

(i) for every configuration $x$ in $\mathcal{X}$ there exists an $\mathcal{H}$-robust $I$-path from $x$ to $\{\pm 1\}$ of length at most $n$. In particular, $\{\pm 1\}$ is $\mathcal{H}$-robustly $I$-reachable;

(ii) if the network coordination game on $\mathcal{G}$ is $\mathcal{H}$-robustly regular then the equilibrium set $\mathcal{N} = \{\pm 1\}$ is $\mathcal{H}$-robustly globally I-stable;

(iii) if there exists an action $a = \pm 1$ such that (29) holds true, then there exists an $\mathcal{H}$-robust $I$-path from every configuration $x$ in $\mathcal{X}$ to $a$. In particular, in this case, the equilibrium set $\mathcal{N} = \{a1\}$ is $\mathcal{H}$-robustly globally BR-stable.

Proof: (i) Given an arbitrary configuration $x$ in $\mathcal{X}$, consider the external field $h^x$ in $\mathcal{H}$ with entries
\[
h_i^x = h_i^x^t, \quad i \in \mathcal{V},
\]
and let
\[
x = f^+(f^-(x, h^x), h^x), \quad x = f^-(f^+(x, h^x), h^x).
\]
By Proposition 1 (i), the above are the least and greatest equilibria of the network coordination game on $\mathcal{G}$ with external field $h^x$ that are I-reachable from configuration $x$. As the graph $\mathcal{G}$ is $\mathcal{H}$-robustly indecomposable, it follows from Theorem 1 that the network coordination game on $\mathcal{G}$ with external field $h^x$ is unpolarizable, so that its equilibria $x$ and $x$ are both consensus configurations. Since clearly $x \leq x$, there are three possible alternative cases:

(a) $x = x = +1$;
(b) $-1 = x < x = +1$;
(c) $-1 = x = x = -1$.

In both cases (a) and (b) we have
\[
+1 = x = f^-(f^+(x, h^x), h^x) = f^+(x, h^x),
\]
so that $x \uparrow +1$ for the network coordination game on $\mathcal{G}$ with external field $h^x$. We now show that in fact a monotone I-path from $x$ to $+1$ exists $\mathcal{H}$-robustly by proving that $f^+(x, h^-) = f^+(x, h^x)$. Indeed, since in any monotone path only players originally playing action $-1$ can get activated, and since $h_i^- = h_i^x$ for every such player, we have that every monotone path is an I-path with respect to $h^-$ if and only if it is an I-path with respect to $h^x$. By the way the function $f^+$ is defined, this yields that
\[
f^+(x, h^-) = f^+(f^+(x, h^x), h^x) = f^+(x, h^x),
\]
which shows that $+1$ is reachable from $x$ by a monotone $I$-path for the network coordination game on $\mathcal{G}$ with external field $h^x$. We can then consider any monotone path $(x^0, x^1, \ldots, x^l)$ from $x^0 = x$ to $x^l = f^+(x, h^x) = 1$ that is an I-path for the network coordination game on $\mathcal{G}$ with external field $h^x$, where clearly $l \leq |\mathcal{V}|$. Since network coordination games are super-modular families, by Proposition 2 (i), this is also a monotone I-path for the network coordination game on $\mathcal{G}$ with any external field $h$ in $\mathcal{H}$.

Similarly, in both cases (b) and (c) we have
\[
-1 = x = f^-(f^+(x, h^x), h^x) = f^-(x, h^x),
\]
so that by an argument completely analogous to the one developed above we can find an $\mathcal{H}$-robust anti-monotone $I$-path from $x$ to $-1$. The proof of point (i) is then completed by...
the observation that the length of monotone and anti-monotone is never larger than \( n \).

(ii) From point (i), the equilibrium set \( \mathcal{N} = \{ \pm 1 \} \) is \( \mathcal{H} \)-robustly globally \( I \)-reachable. Due to the \( \mathcal{H} \)-robust regularity assumption, \( \mathcal{N} = \{ \pm 1 \} \) is also \( \mathcal{H} \)-robustly \( I \)-invariant. Hence, \( \mathcal{N} = \{ \pm 1 \} \) is \( \mathcal{H} \)-robustly globally \( I \)-stable.

(iii) Existence of an \( \mathcal{H} \)-robust \( I \)-path from every configuration \( x \) in \( \mathcal{X} \) to the equilibrium \( a_1 \) follows from point (i) and Corollary \([\text{2}](\text{ii})\). On the other hand, Corollary \([\text{1}](\text{iii})\) ensures BR-invariance of \( a_1 \) for the network coordination game on \( \mathcal{G} \) with any external field \( h \) in \( \mathcal{H} \). Therefore, the equilibrium set \( \mathcal{N} = \{ a_1 \} \) is \( \mathcal{H} \)-robustly globally \( BR \)-stable.  

Example 2 (continued). Let \( \mathcal{H} = \{ h \in \mathbb{R}^V : -1 \leq h \leq 1 \} \). As we have seen, the complete graph \( \mathcal{G} \) of odd order \( n \) is \( \mathcal{H} \)-robustly indecomposable. In fact, every anti-monotone paths from a configuration \( x \) in \( \mathcal{X} \) with at most \((n-1)/2 + 1\)-players to the consensus equilibrium \(-1\) is an \( \mathcal{H} \)-robust \( I \)-path, so that \( x \downarrow -1 \): notice that \(-1\) is the only Nash equilibrium that is \( \mathcal{H} \)-robustly reachable from \( x \). Symmetrically, every monotone path from a configuration \( x \) in \( \mathcal{X} \) with at least \((n+1)/2 + 1\)-players to the consensus equilibrium \(+1\) is an \( \mathcal{H} \)-robust \( I \)-path, so that \( x \uparrow +1 \): in this case, \(-1\) is the only Nash equilibrium that is \( \mathcal{H} \)-robustly reachable from \( x \). In particular, this shows that the robust global reachability result of Theorem \([\text{2}]\) cannot be sharpened: even though both consensus configurations are robust equilibria, for each configuration \( x \), only one of them is \( \mathcal{H} \)-robustly reachable from it.

V. COORDINATION DYNAMICS WITH TIME-VARYING EXTERNAL FIELDS

In this section and in the next one, we apply the results of Section \([\text{IV}]\) to the analysis of interconnections of network coordination games with other systems. In particular, in this section we study asynchronous best response dynamics with a possibly time-varying external field modeling an exogenous signal and derive sufficient conditions for finite time absorption in consensus configurations.

Consider a network coordination game embedded in a time-varying environment with the coupling described by an external field \( h(t) \) that is a measurable function of time \( t \geq 0 \). Assume that a learning process takes place among the players, modeled as an asynchronous best response dynamics. Precisely, let every player \( i \in \mathcal{V} \) be equipped with an independent Poisson clock of rate \( \lambda_i > 0 \). If the clock of player \( i \) ticks at time \( t \), player \( i \) modifies her current action by choosing any value in her best response correspondence \( B_i(x_{-i}(t), h(t)) \) uniformly at random, i.e., playing her best response action if this is unique and choosing one of the two actions uniformly at random if these give the same utility.

This mechanism generates a continuous-time Markov chain \( X(t) \) on the configuration space \( \mathcal{X} \) to be referred to as the coordination dynamics with input field \( h(t) \). We shall denote the transition rate matrix of \( X(t) \) by \( \Lambda(t) \) in \( \mathbb{R}^{n \times n} \), whose entries \( \Lambda_{xy}(t) \) stand for the transition rates from configuration \( x \) in \( \mathcal{X} \) to configuration \( y \) in \( \mathcal{X} \) at time \( t \geq 0 \). Notice that \( \Lambda(t) \) depends on the external field \( h(t) \) and for this reason it is time-varying. We have that \( \Lambda_{xy}(t) = 0 \) whenever \( x \) and \( y \) differ in more than one entry, reflecting the fact that with probability \( 1 \) no two players will modify their action simultaneously. On the other hand, if there exists \( i \) in \( \mathcal{V} \) such that \( x_{-i} = y_{-i} \) and \( x_i \neq y_i \), then \( \Lambda_{xy}(t) > 0 \) if and only if \( y_i \in B_i(x_{-i}(t), h(t)) \), and in this case \( \Lambda_{xy}(t) = \lambda_i / |B_i(x_{-i}(t), h(t))| \). Finally, the diagonal entries of \( \Lambda(t) \) are nonpositive and such that the all row sums of \( \Lambda(t) \) are null, i.e., \( \Lambda_{xx}(t) = -\sum_{y \neq x} \Lambda_{xy}(t) \).

The following result establishes sufficient conditions for the process \( X(t) \) to converge to a consensus configuration, and shows that such conditions are essentially necessary when the time-varying external field is an arbitrary exogenous signal \( h : \mathbb{R}^+ \to \mathcal{H} \) and \( \mathcal{G} \) is \( \mathcal{H} \)-robustly indecomposable.

Theorem 3. Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W}) \) be a graph and let \( \mathcal{H} \) be as in \([\text{2}]\) for two vectors \( h^- \leq h^+ \). For a measurable exogenous signal \( h : \mathbb{R}^+ \to \mathcal{H} \), consider the coordination dynamics \( X(t) \) on \( \mathcal{G} \) with input field \( h(t) \) for \( t \geq 0 \). Assume that \( \mathcal{G} \) is \( \mathcal{H} \)-robustly indecomposable. Then:

(i) if \( ah^- + w > 0 \) and \( w \not\approx ah^- \) for some action \( a = \pm 1 \), then
\[
P(\exists t_0 | X(t) = a_1, \forall t \geq t_0) = 1 ;
\]
(ii) if \( w > h^+ \) and \( w \not\approx -h^- \), then
\[
P(\exists t_0 | X(t) \in \{ \pm 1 \}, \forall t \geq t_0) = 1 .
\]
(iii) if \( w \not\approx h^+ \) and \( w \not\approx -h^- \), then there exists a measurable signal \( h : \mathbb{R}^+ \to \mathcal{H} \) such that with probability 1 the coordination dynamics \( X(t) \) on \( \mathcal{G} \) with input field \( h(t) \) visits both consensus configuration \(+1\) and \(-1\) infinitely often, hence in particular, it fluctuates forever.

Proof: Notice that the definition of best response dynamics and the fact that the actions set is binary imply that the following uniform bounds hold true for the transition rates of \( X(t) \) at any time \( t \geq 0 \):

\[
\Lambda_{xy}(t) > 0 \Rightarrow \Lambda_{xy}(t) \geq \frac{\lambda_\ast}{2} , \quad \forall x, y \in \mathcal{X} , \quad (34)
\]

\[
\sum_{y \neq x} \Lambda_{xy}(t) \leq n \bar{\lambda} , \quad \forall x \in \mathcal{X} , \quad (35)
\]

where
\[
n = |\mathcal{V}| , \quad \bar{\lambda} = \min_{i \in \mathcal{V}} \lambda_i > 0 , \quad \bar{\lambda} = \frac{1}{n} \sum_{i \in \mathcal{V}} \lambda_i .
\]

(i)-(ii) We prove the two results at once. First, observe that the assumptions on \( w, h^+ \), and \( h^- \) imply that in case (i) \( a_1 \) is a strict equilibrium and in case (ii) \(+1\) and \(-1\) are both strict equilibria, for every \( h^- \leq h \leq h^+ \). Put \( \mathcal{N} = \{ a_1 \} \) in the former case and \( \mathcal{N} = \{ \pm 1 \} \) in the latter. Then, in both cases, \( \Lambda_{xy}(t) = 0 \) for every \( x \in \mathcal{N}, y \not\approx x, t \geq 0 \), so that \( \mathcal{N} \) is a trapping set for the Markov chain \( X(t) \).

On the other hand, for every initial profile \( X(0) = x^{(0)} \), Theorem \([\text{2}](\text{i})\) guarantees the existence of an \( \mathcal{H} \)-robust \( I \)-path \( (x^{(0)}, x^{(1)}, \ldots, x^{(l)}) \) of length \( l \leq n \) from \( x^{(0)} \) to \( \mathcal{N} \). Consider now the discrete-time jump process associated to \( X(t) \), defined by
\[
Y(k) = X(T_k) , \quad k = 0, 1, \ldots
\]
where \(0 = T_0 < T_1 < T_2 < \ldots\) are the random times when the value of \(X(t)\) changes. Using (34) and (35), we can estimate the probability that the best response dynamics follows this path at some time as follows
\[
P(Y_{s+l} = x^{(l)}, \ldots, Y_{s+1} = x^{(1)} | T_s = t, Y_s = x^{(0)}) \geq \alpha^l,
\]
for every \(s \geq 0\), where
\[
\alpha = \frac{\lambda_s}{2n^X} \in (0, 1).
\]

Since \(\mathcal{N}\) is trapping, this implies that
\[
P(Y_{s+n} \in \mathcal{N} | T_s = t, X_s = x_0) \geq \alpha^l \geq \alpha^n,
\]
for every \(x_0 \in \mathcal{X}\), \(t \geq 0\) and \(s \geq 0\). A standard induction argument now yields that, for every initial condition \(x^{(0)}\) in \(\mathcal{X}\) and for every \(h = 1, 2, \ldots\),
\[
P(Y_{hn} \notin \mathcal{N} | X(0) = x^{(0)}) \leq (1 - \alpha)^{hn}.
\]

Since \(P(Y_k \notin \mathcal{N} | X(0) = x^{(0)})\) is non-increasing in \(k\), we have that
\[
P(X_k \notin \mathcal{N} | X_0 = x^{(0)}) \leq (1 - \alpha)^{k/n}, \quad k \geq 0.
\]

Let now \(T_N = \inf\{t \geq 0 : X(t) \in \mathcal{N}\}\) be the first entrance time of the Markov chain \(X(t)\) in \(\mathcal{N}\). Then,
\[
P(T_N < +\infty) = 1 - \lim_{k \to +\infty} P(X(t) \notin \mathcal{N}) = 1 - \lim_{k \to +\infty} P(Y_k \notin \mathcal{N}) \geq 1 - \lim_{k \to +\infty} (1 - \alpha)^{k/n} = 1,
\]
thus proving that, with probability 1, the Nash equilibrium set \(\mathcal{N}\) is reached in finite time. Since we have already shown that \(\mathcal{N}\) is a trapping set, this proves that (32) and (33) hold true in case (i) and case (ii), respectively.

(iii) By assumption, there exist two players \(i\) and \(j\) in \(\mathcal{Y}\) such that \(u_i < h_i^+\) and \(u_j < h_j^-\). Since \(i \in S_+(h^+)^-\) and \(G\) is \(\mathcal{H}\)-robustly indecomposable, Corollary 1 implies that \(\mathcal{N}_{h^+} = \{+1\}\) is BR-stable for the network coordination game on \(G^+\) with external field \(h^+\). This implies that, for every \(\tau > 0\), the coordination dynamics with an input field that is constant \(h(t) = h^+\) in the interval \([0, \tau]\) is such that
\[
\alpha_+ = \min_{x \in \mathcal{X}} P(X(\tau) = +1 | (X(0) = x)) > 0.
\]

Analogously, \(j \in S_-(h^-)^-\) and \(G\) is \(\mathcal{H}\)-robustly indecomposable, we get that the coordination dynamics with an input field that is constant \(h(t) = h^-\) in the interval \([0, \tau]\) is such that
\[
\alpha_- = \min_{x \in \mathcal{X}} P(X(\tau) = -1 | (X(0) = x)) > 0.
\]

For the coordination dynamics with periodic piece-wise constant input field defined as follows
\[
h(t) = \begin{cases} 
  h^+ & \text{if } 2k\tau \leq t < (2k + 1)\tau, \quad k \in \mathbb{Z}^+ \\
  h^- & \text{if } (2k + 1)\tau \leq t < (2k + 2)\tau, \quad k \in \mathbb{Z}^+,
\end{cases}
\]
we then have that
\[
P(X((2k+1)\tau) = +1, X((2k+2)\tau) = -1 | X(2k\tau) = x) \geq \alpha,
\]
for every \(k \in \mathbb{Z}^+\) and \(x \in \mathcal{X}\), where
\[
\alpha = \alpha_+\alpha_- > 0.
\]

It then follows that with probability 1 there exist infinitely many values of \(k \in \mathbb{Z}^+\) such that \(X((2k + 1)\tau) = +1\) and \(X((2k + 2)\tau) = -1\), thus proving that \(X(t)\) keeps fluctuating forever.

Theorem 3 ensures that, for any coordination dynamics with time-varying external field \(h(t)\) on a graph \(G\) that is \(\mathcal{H}\)-robustly indecomposable, finite-time absorption in the set of consensus equilibria is guaranteed provided that either (i) there exist some \(a\)-stubborn players and no \(-a\)-stubborn players even with the most \(-a\)-favorable external field \(h^-\) (and in this case there is a unique, consensus equilibrium \(a\)), or (ii) no player has a dominant action with any external field in the range \(\mathcal{H}\) (in which case both consensus configurations \(\pm 1\) are equilibria).

VI. NETWORK COORDINATION WITH NON-COORDINATING AGENTS

In this section, we consider a scenario of a strategic form game where players are of mixed nature: a subset of them have the utility of a coordination game while the rest have different utilities. We determine conditions for the existence and stability of equilibria of the overall game.

Specifically, we fix a graph \(G = (\mathcal{V}, \mathcal{E}, W)\) and two nonempty subsets of nodes \(R, S \subseteq \mathcal{V}\) such that
\[
R \cap S = \emptyset, \quad \mathcal{V} = R \cup S.
\]

We consider a strategic form game with player set \(\mathcal{V}\), binary action set \(A = \{\pm 1\}\) and utilities \(u_i(x)\) as in (11) for every player \(i \in R\), and of an arbitrary form (different from (11)) for players \(i \in S\). We shall refer to such a game as a mixed network game with set of coordinating players \(R\) and set of alien agents \(S\). One can think of the alien agents as outsiders that are perturbing the coordination game. The main point is to understand the extent to which the coordinating players can be affected by the alien agents: in particular, under which conditions do equilibria for this game exist and correspond to globally I-stable consensus configurations for the coordinating players?

In fact, in this context, equilibria may fail to exist. E.g., the two-player game with binary action set \(A = \{\pm 1\}\) and utilities \(u_1(x_1, x_2) = x_1 x_2\) and \(u_2(x_1, x_2) = -x_1 x_2\), which is commonly referred to as the dis-coordination game, is known not to admit any equilibrium. On the other hand, even for games admitting equilibria whose restriction to the set of coordinating players are consensus configurations, such equilibria may fail to be I-stable, as illustrated in the following example.

Example 3. Let \(G = (\mathcal{V}, \mathcal{E}, W)\) be a graph as in Figure 3 with all links with weight 1. Consider a mixed network game with set of coordinating players \(R = \{1, \ldots, 7\}\) having utility \(u_1(x) = x_i \sum_j W_{ij} x_j\) and a single alien agent \(S = \{8\}\) with utility \(u_8(x) = -x_8 \sum_j W_{8j} x_j\). Then, it can be readily verified that the equilibrium set is \(\mathcal{N} = \{x^*\}\), where \(x^*\)
is the configuration with entries $x_i^* = -1$ and $x_i^* = +1$ for $i = 1, \ldots, 7$. However, $\mathcal{N}$ is not globally BR-stable (hence, not I-reachable), as the subset of configurations

$$\mathcal{Y} = \{ y \in \mathcal{X} : y_1 = y_2 = y_3 = -y_5 = -y_6 = -y_7 \}$$

is BR-invariant.

To analyze mixed network games with set of coordinating players $\mathcal{R}$ and set of alien players $\mathcal{S} = \mathcal{V} \setminus \mathcal{R}$, it proves convenient to canonically identify the configuration space as the external product $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$ of the strategy profiles spaces $\mathcal{Y} = \mathcal{A}^{\mathcal{R}}$ and $\mathcal{Z} = \mathcal{A}^{\mathcal{S}}$ of players in $\mathcal{R}$ and in $\mathcal{S}$, respectively. Correspondingly, we may decompose every strategy profile $x$ in $\mathcal{X}$, as $x = (y, z)$ whereby $y = x_{\mathcal{R}}$ in $\mathcal{Y}$ is the strategy profile of coordinating players and $z = x_{\mathcal{S}}$ in $\mathcal{Z}$ is the strategy profile of alien agents. We also introduce the induced subgraph $G_{\mathcal{R}} = (\mathcal{R}, \mathcal{E}_{\mathcal{R}}, W_{\mathcal{R}, \mathcal{R}},)$ where $\mathcal{E}_{\mathcal{R}} = \mathcal{E} \cap (\mathcal{R} \times \mathcal{R})$ and $W_{\mathcal{R}, \mathcal{R}}$ is the submatrix of $W$ restricted to rows and columns in the coordinating player set $\mathcal{R}$.

For every strategy profile $z$ in $\mathcal{Z}$ of the alien agents, we may consider the induced restricted game with player set $\mathcal{R}$ and utility functions

$$u^{(z)}_i(y) = y_i \sum_{j \in \mathcal{R}} W_{ij} y_j + y_i \sum_{j \in \mathcal{S}} W_{ij} z_j + y_i h_i,$$  \hspace{1cm} (37)

for every coordinating player $i$ in $\mathcal{R}$ and coordinating players’ strategy profile $y$ in $\mathcal{Y}$. This can be interpreted as a network coordination game on the induced subgraph $G_{\mathcal{R}}$ with external field $h^{(z)} = h_{\mathcal{R}} + (Wz)_{\mathcal{R}} \in \mathbb{R}^{\mathcal{R}}$.

If we define

$$\mathcal{H} = \{ h \in \mathbb{R}^{\mathcal{R}} : h_{\mathcal{R}} - w^{\mathcal{S}}_{\mathcal{R}} \leq h \leq h_{\mathcal{R}} + w^{\mathcal{S}}_{\mathcal{R}} \},$$  \hspace{1cm} (38)

we have that $h^{(z)}$ belongs to $\mathcal{H}$ for every alien agents’ strategy profile $z$ in $\mathcal{Z}$. Analogously, consider, for a given coordinating players’ strategy profile $y$ in $\mathcal{Y}$, the induced restricted game with player set $\mathcal{S}$ and utility functions

$$u^{(y)}_i(z) = u_i(y, z),$$  \hspace{1cm} (39)

for every alien agent $i$ in $\mathcal{S}$ and strategy profile $z$ in $\mathcal{Z}$. We denote by $\mathcal{N}^{(z)}_{\mathcal{R}}$ and $\mathcal{N}^{(y)}_{\mathcal{S}}$, the set of equilibria, of, respectively, the restricted coordination game with player set $\mathcal{R}$ when the alien agents are playing $z$ and the restricted game with player set $\mathcal{S}$ when the coordinating agents are playing $y$. We are interested in investigating the existence and stability properties of equilibria of the mixed network game whose projection on the set of coordinating players $\mathcal{R}$ is a consensus configuration.

We denote the set of such configurations as

$$\mathcal{T} = \mathcal{T}_{+1} \cup \mathcal{T}_{-1},$$

where

$$\mathcal{T}_a = \{(a1, z) : z \in \mathcal{Z} \}, \hspace{1cm} a = \pm1.$$

We have the following result.

**Theorem 4.** For a graph $G = (\mathcal{V}, \mathcal{E}, W)$, consider a mixed network game on $G$ with sets of coordinating players $\mathcal{R}$ and alien agents $\mathcal{S}$ satisfying (36). Assume that

(a) $G_{\mathcal{R}}$ is $\mathcal{H}$-robustly indecomposable for $\mathcal{H}$ as in (35),

(b) the sets $\mathcal{N}^{(z)}_{\mathcal{R}}$ and $\mathcal{N}^{(y)}_{\mathcal{S}}$ are globally BR-stable for the game restricted to the set of alien agents,

(c) $w^{\mathcal{R}}_{\mathcal{R}} > w^{\mathcal{S}}_{\mathcal{R}} + |h_{\mathcal{R}}|.$

Then, $\mathcal{N} \cap \mathcal{T}$ is globally BR-stable.

**Proof:** Using Theorem 2 (i), one has that $\{ \pm1 \}$ is $\mathcal{H}$-robustly globally I-reachable for the restricted coordination game with player set $\mathcal{R}$. In particular, this implies that, for every $z \in \mathcal{Z}$, there exists a in $\{ \pm1 \}$ such that $a1 \in \mathcal{N}^{(z)}_{\mathcal{R}}$ and the restricted coordination game with player set $\mathcal{R}$ and utilities as in (37) admits an I-path $(y = y^{(0)}, y^{(1)}, \ldots, y^{(l)} = a1)$ from $y$ to $a1$. On the other hand, the assumption on the $\mathcal{S}$-restricted game implies that there exists a BR-path $(z = z^{(0)}, z^{(1)}, \ldots, z^{(m)} = z^*)$ from $z$ to an equilibrium $z^*$ in $\mathcal{N}^{(a1)}_{\mathcal{S}}$ for the $\mathcal{S}$-restricted game with utilities as in (39) with $y = a1$. It then follows that $x^* = (a1, z^*) \in \mathcal{N}$ is an equilibrium and that the concatenation

$$((y^{(0)}z), (y^{(1)}z), \ldots, (a1z), (a1z^{(1)}), \ldots, (a1, z^*))$$

is a BR-path for the mixed network game on $G$ with set of coordinating players $\mathcal{R}$ and set of alien agents $\mathcal{S}$. This proves that $\mathcal{N} \cap \mathcal{T}$ is globally BR-reachable.

Now, observe that (40) guarantees that the consensus configurations $a1$, for $a = \pm1$, are strict equilibria for the $\mathcal{R}$-restricted coordination game with utilities as in (37). Combined with the assumption that $\mathcal{N}^{(a1)}_{\mathcal{S}}$ is BR-stable for the $\mathcal{S}$-restricted game, this implies that $\mathcal{N} \cap \mathcal{T}_a$ is BR-invariant for the mixed network game for $a = \pm1$. Thus, we have proved that $\mathcal{N} \cap \mathcal{T}$ is globally BR-stable.

**Remark 5.** It is worth pointing out that variations of the conclusions of Theorem 2 hold true under somewhat different combinations of assumptions.

In particular, notice that if

$$w^{\mathcal{R}}_{\mathcal{R}} > w^{\mathcal{S}}_{\mathcal{R}} - ah_{\mathcal{R}},$$  \hspace{1cm} (41)

then coordinating players are in strict equilibrium, when all playing action $a$, regardless of what alien agents play. Therefore, if, for some $a$ in $\{ \pm1 \}$, condition (41) holds true and $\mathcal{N}^{(a)}_{\mathcal{S}} \neq \emptyset$, i.e., the restricted game with player set $\mathcal{S}$ when the coordinating players are playing $a1$ admits an equilibrium, then there exist equilibria of the whole game where coordinating agents are at consensus, i.e., $\mathcal{N} \cap \mathcal{T}_a \neq \emptyset$. If $\mathcal{N}^{(a)}_{\mathcal{S}}$ is also BR-invariant for the game restricted to the set of alien agents, then BR-invariance of $\mathcal{N} \cap \mathcal{T}_a$ for the mixed network game is ensured.

On the other hand, if (41) holds true and

$$w^{\mathcal{R}}_{\mathcal{R}} \not\geq w^{\mathcal{S}}_{\mathcal{R}} + ah_{\mathcal{R}},$$

we have the following result.
for some $a \in \{\pm 1\}$, then Theorem 2 (iii) implies that $a1$ is $\mathcal{H}$-robustly globally BR-stable for the restricted coordination game with player set $\mathcal{R}$. If $\mathcal{N}^{(3)}_{S}$ is globally BR-stable for the game restricted to the set of alien agents, then arguing as in the proof of Theorem 2 one gets that $\mathcal{N}_{\cap T_{a}}$ is globally BR-stable for the mixed network game.

We notice that, in Theorem 4 the assumption of global BR-stability of the sets $\mathcal{N}^{(3)}_{S}$ and $\mathcal{N}^{(-3)}_{S}$ for the game restricted to the set of alien agents is automatically verified when this is a potential game [57], or its a super-modular game itself. A remarkable example for this former case is that of an anti-coordination game on an undirected graph. This is the case when alien agents have utilities

$$u_{i}(x) = -x_{i} \sum_{j} W_{ij} x_{j} + h_{i} x_{i}, \quad i \in S.$$ (42)

The ensuing mixed network coordination/anti-coordination game has received considerable attention in the recent literature. For the special case of an undirected network connecting two players, this reduces to the already mentioned literature. For arbitrary undirected graphs we have the following result.

**Corollary 3.** Let $G = (\mathcal{V}, \mathcal{E}, W)$ be an undirected graph and $h$ in $\mathbb{R}^{\mathcal{V}}$ a vector. Consider the mixed network coordination/anti-coordination game on $G$ with external field $h$, and sets of coordinating and anti-coordinating players $\mathcal{R}$ and $S$, respectively, satisfying (36). Let (40) hold true. Then,

(i) for $a = \pm 1$, the set $N \cap T_{a}$ is non-empty, i.e., there exists an equilibrium $x^{*}$ such that $x_{R}^{*} = a1$;

(ii) if $G_{R}$ is $\mathcal{H}$-robustly indecomposable for $\mathcal{H}$ as in (38), then the set of equilibria $\mathcal{N}_{\cap T}$ is globally BR-stable.

**Proof:** The key observation is that, since the graph $G$ is undirected, for every $y \in \mathcal{Y}$ the $S$-restricted anti-coordination game with utilities

$$u_{i}^{(y)}(z) = -z_{i} \sum_{j \in \mathcal{R}} W_{ij} y_{j} - z_{i} \sum_{j \in S} W_{ij} z_{j} + h_{i} z_{i}, \quad i \in S$$

is a potential game with potential function

$$\Phi^{(y)}(z) = -\sum_{i,j \in S} W_{ij} z_{i} z_{j} - \sum_{i \in \mathcal{R}} z_{i} \sum_{j \in \mathcal{R}} W_{ij} y_{j} + \sum_{i \in S} h_{i} z_{i}.$$ 

As a consequence, $\mathcal{N}^{(y)}_{S}$ is nonempty and globally BR-stable for every $y \in \mathcal{Y}$ [57]. Claim (i) then follows from Remark 5 and claim (ii) from Theorem 4. Q.E.D.

**Remark 6.** It is clear from the proof above that the assumption that the graph $G = (\mathcal{V}, \mathcal{E}, W)$ is undirected in Corollary 3 can be relaxed to just the restriction of $\mathcal{G}$ to the set of anti-coordinating players $\mathcal{S}$ be undirected, i.e.,

$$W_{ij} = W_{ji}, \quad \forall i, j \in S.$$ 

**Remark 7.** It is worth comparing our results with [53], where convergence to Nash equilibria is proven for the asynchronous best-response dynamics in pure network coordination games and pure network anti-coordination games for every network topology and every set of thresholds. While the setup in

Fig. 4: The graph considered in Example 3 with set of coordinating players $\mathcal{R} = \{1, \ldots, 7\}$ and set of alien agents $\mathcal{S} = \{7, \ldots, 11\}$ in grey.

**Example 3.** Consider the graph $G = (\mathcal{V}, \mathcal{E}, W)$ as in Figure 4. Then, for the mixed network coordination/anti-coordination game on $G$ with set of coordinating players $\mathcal{R} = \{1, \ldots, 7\}$ and external field $h = 0$, both assumptions in Corollary 3 are satisfied. In particular, $\mathcal{H}$-robust indecomposability of $G_{R}$ for $\mathcal{H}$ as in (38) is equivalent to the robust indecomposability of the complete graph subject to an external field $-1 \leq h \leq 1$ discussed in Example 2. It then follows from Corollary 3 that the set of equilibria $\mathcal{N}_{\cap T}$ is globally BR-stable.

**VII. Conclusion**

We have studied heterogeneous network coordination games on general weighted directed interaction graphs, equipped with an external field modeling exogenous interventions or individual biases towards specific actions. We have proved necessary and sufficient conditions for global stability of consensus equilibria under best response type dynamics, robustly with respect to a (constant or time-varying) external field. We have then applied these results to the analysis of mixed network coordination and anti-coordination games and determined sufficient conditions for existence and global stability of pure strategy Nash equilibria.

A key step in our analysis has consisted in the introduction of novel robust notions of improvement and best response paths. Our analysis have strongly used the fact that coordination games are super-modular, but also their peculiar threshold structure of best response correspondences. Extension of such concepts and results to more general super-modular games is a challenging problem that deserves further investigation.

**APPENDIX A**

**PROPERTIES OF FINITE SUPER-MODULAR GAMES**

In this Appendix, we prove the technical results of Section II-B on super-modular games with binary actions. Throughout,
we shall assume to have fixed a super-modular game with finite player set \( V \) and configuration space \( X = \{ \pm 1 \}^V \).

First, we state the following direct, though crucial, consequence of the increasing difference property (7).

**Lemma 2.** For every player \( i \) in \( V \), both \( B_i^+(x_{-i}) \) and \( B_i^-(x_{-i}) \) are monotone nondecreasing in \( x_{-i} \).

We now move on with the next result characterizing properties of monotone and anti-monotone I- and BR-paths of super-modular games.

**Lemma 3.** For \( x, y, z \) in \( X \), the following relations hold true:

(i) \( x \uparrow y, x \uparrow z \Rightarrow x \uparrow (y \lor z) \)

(ii) \( x \downarrow y, x \downarrow z \Rightarrow x \downarrow (y \land z) \)

(iii) \( x \uparrow y, x' \geq x \Rightarrow x' \uparrow (y \lor x') \)

(iv) \( x \downarrow y, x' \leq x \Rightarrow x' \downarrow (y \land x') \)

(v) \( x \rightarrow y \Rightarrow x \uparrow y', x \downarrow y' \) for some \( y'' \leq y \).

Moreover, the analogous results hold true for the BR-case.

**Proof:** We prove (i). Let \((y^{(0)}, y^{(1)}, \ldots, y^{(l)})\) and \((z^{(0)}, z^{(1)}, \ldots, z^{(r)})\) be two monotone I-paths from \( x \) to, respectively, \( y \) and \( z \). Let \((i_1, \ldots, i_l)\) and \((j_1, \ldots, j_r)\) be the two corresponding sequences of active players. Let \((s_1, \ldots, s_k)\) be the subsequence of \((j_1, \ldots, j_r)\) consisting of exactly those players that are not in the sequence \((i_1, \ldots, i_l)\). We claim that the sequence \((x^{(0)}), x^{(l+r)}\) is

\[
x^{(h)} = y^{(h)} \quad \text{for} \ h = 0, 1, \ldots, l,
\]

\[
x^{(h+l)} = x^{(h+l-1)} + \delta s_h \quad \text{for} \ h = 1, \ldots, k
\]

is a monotone I-path from \( x \) to \( y \lor z \). By construction, the path is admissible and monotone. Moreover, \( x^{(h+l)} \geq z_{s_h} \)

for every \( h = 1, \ldots, k \). Since \( \{+1\} = B_{j_{s_h}}(z^{(s_h-1)}) \), by Lemma 2 \( \{+1\} = B_{j_{s_h}}(x^{(h+l-1)}) \). This implies that it is an I-path. Proof of (ii) is completely analogous.

We prove (iii). Let \((x^{(0)}, x^{(1)}, \ldots, x^{(l)})\) be a monotone I-path from \( x \) to \( y \) with set of active players \((i_1, \ldots, i_l)\). Consider the subsequence \((s_1, \ldots, s_k)\) of those players for which \( x \) and \( x' \) coincide. Then, \((x^{(0)}, x^{(l)} \lor x', x^{(i_1)} \lor x', \ldots, x^{(i_k)} \lor x')\) is a monotone I-path from \( x \) to \( y \lor x' \). Indeed, notice that, by construction, \( x^{(i_k)} \lor x' = x^{(l)} \lor x' = y \lor x' \). We only need to show that it is an I-path. Since \( x^{(i_k)} \lor x' - x_{s_h} \geq z_{s_h} \) and using the increasing difference property (7) we obtain that

\[
0 \leq u_{s_h}(x^{(i_k)} \lor x') - u_{s_h}(x^{(i_k-1)} \lor x'),
\]

The proof of (iii) is complete. The proof of (iv) is completely analogous.

(v): If \( x \uparrow y \), then \( y \lor x = x \). If \( x \downarrow y \), then \( x \geq y \) and \( y \lor x = x \). In both cases the result is evident. The general case can be proven by induction on the length of a minimal I-path from \( x \) to \( y \). Indeed, by definition of an I-path, for sure we can find an intermediate configuration \( z \) for which one of the two possible cases hold: \( x \uparrow z \rightarrow y \) or \( x \downarrow z \rightarrow y \). In the first case, using the induction hypothesis \( z \uparrow y' \geq y \), we obtain by transitivity that \( x \uparrow y' \geq y \). In the second case, using the induction hypothesis \( z \uparrow y' \geq y \) and point (iii), we obtain that \( x \uparrow (x' \lor y') \geq y \). Similarly, we prove the other relation.

Finally, the proofs for the analogous results in the BR-case can be obtained by the same identical arguments.

The following result gathers some elementary key facts connected to the maps (8).

**Lemma 4.** The following facts hold true:

(i) \( f^+, f^-, g^+, g^- \) are monotone nondecreasing maps;

(ii) a configuration \( x \) in \( X \) is an equilibrium if and only if \( f^+(x) = x = f^-(x) \);

(iii) a configuration \( x \) in \( X \) is a strict equilibrium if and only if \( g^+(x) = x = g^-(x) \).

**Proof:** (i) It follows from Lemma 3(i) that \( x \uparrow f^+(x) \) for every configuration \( x \) in \( X \). If \( x' \geq x \), Lemma 3(iii) yields \( x' \uparrow f^+(x) \lor x' \). Therefore \( f^+(x') \geq f^+(x) \lor x' \geq f^+(x) \). Proofs for \( f^- \), \( g^+, g^- \) are analogous.

(ii) and (iii) coincide with the definitions of equilibrium and, respectively, strict equilibrium.

We are now ready to prove Proposition 1, which we restate below for the reader’s convenience.

**Proposition 1.** Consider a finite super-modular game with binary action sets. Then, for every configuration \( x \) in \( X \),

(i) \( f^-(f^+(x)) \in \mathcal{N} \) and \( f^+(f^-(x)) \in \mathcal{N} \) are, respectively, the greatest and least equilibria BR-reachable from \( x \); and \( f^+(g^+(x)) \in \mathcal{N} \) and \( f^+(g^+(x)) \in \mathcal{N} \) are, respectively, the greatest and least equilibria BR-reachable from \( x \);

Moreover,

(iv) the set of equilibria \( \mathcal{N} \) is a complete lattice and is globally stable;

(v) the two configurations

\[
\pi^* = f^+(g^(-1)) = f^+(-1),
\]

\[
\pi^* = f^-(g^+(1)) = f^-(-1),
\]

are, respectively, the least and greatest equilibria;

(vii) if \( \pi^* = \pi^* \) and \( \pi^* \) are, respectively, the least and greatest equilibria that are, respectively, above and below both \( x \) and \( y \).

(vii) if \( \pi^* = \pi^* \) and \( \pi^* \) are, respectively, the least and greatest equilibria that are, respectively, above and below both \( x \) and \( y \).

**Proof:** (i) Put \( \mathcal{X}^+ = \{ x \in \mathcal{X} | f^+(x) = x \} \). We notice that for every \( x \in \mathcal{X} \), \( f^+(x) \in \mathcal{X}^+ \). Moreover, \( \mathcal{X}^+ \) is closed with respect to anti-monotone I-path. Namely, if \( x \in \mathcal{X}^+ \) and \( x \downarrow y \), then also \( y \in \mathcal{X}^+ \). To see this, by induction, it is sufficient to prove it when \( x \) and \( y \) are connected by an anti-monotone I-path of length 1, namely there exists \( i \in \mathcal{V} \) such that \( y_i < x, y_{i-1} = x_{i-1}, \) and \( B_i(y) = \{-1\} \). If \( y \notin \mathcal{X}^+ \), then it would exist \( z \in \mathcal{X} \) and a player \( j \in \mathcal{V} \) such that \( z_j > y_j, \) \( z_{j-1} = y_{j-1}, \) and \( B_j(y) = \{+1\} \). Evidently \( j \neq i \) and from Lemma 3(iii) applied to \( y \uparrow z \) and \( x \geq y \) we obtain \( x \uparrow z \). Since by construction \( x \downarrow z \), this would imply that \( f^+(x) \neq x \) contrarily to the assumption that \( x \in \mathcal{X}^+ \). Similarly, \( \mathcal{X}^- = \{ x \in \mathcal{X} | f^-(x) = x \} \) is closed with respect to monotone I-path. Notice that \( \mathcal{N} = \mathcal{X}^+ \cap \mathcal{X}^- \).

Consider now \( y = f^-(f^+(x)) \). Being in the image of \( f^- \), necessarily \( y \in \mathcal{X}^- \). On the other hand, since \( f^+(x) \in \mathcal{X}^+ \),
by the fact that $X^+$ is closed with respect to anti-monotone I-path, we have that also $y \in X^+$. Hence $y$ is a Nash. The argument for $f^+(f^-(x))$ is completely analogous.

If now $y \in N$ is any equilibrium reachable from $x$, namely $x \to y$, by Lemma 4 (v) it follows that $x \uparrow y \geq y$. By definition of $f^+(x)$ we have that $f^+(x) \geq y \geq y$. Therefore, by Lemma 4 (i), we have that

$$f^-(f^+(x)) \geq f^-(y) = y,$$

with the last equality above following from Lemma 4 (ii). Similarly, we can show that $f^+(f^-(x)) \leq y$. This concludes the proof of point (i).

(ii) and (iii) can be proven along the same lines as point (i). We omit the details.

(v) It follows from point (ii) that the configurations $x^+$ and $\pi^+$ defined in (43) are indeed equilibria. Given any equilibrium $x$ in $N$, let $x^* = g^+(x)$ and $x^+ = g^+(x)$. Point (iii) implies that both $x^- = x^+$ and $x^+$ are equilibria, hence $f^+(x^-) = x^- = x$ and $f^+(x^+) = x^+$ by Lemma 4 (ii). Since $x^- \leq x \leq x^+$, it follows from Lemma 4 (i) that

$$x^* = f^+(x^-) \leq f^+(g^-(x)) = x^- \leq x,$$

$$x \leq x^* = f^+(g^+(x)) \leq f^+(+1) = \pi^+,$$

so that $x^* \leq x \leq \pi^+$.

(vi) For $x, y \in N$ we have $f^-(x) = x$ and $f^-(y) = y$ by Lemma 4 (ii). It then follows from Lemma 4 (i) that

$$x = f^-(x) \leq f^-(x \vee y), \quad y = f^-(y) \leq f^-(x \vee y),$$

so that

$$x \vee y \leq f^-(x \vee y) \leq x \vee y.$$ Hence, $f^-(x \vee y) = x \vee y$, so that

$$f^+(x \vee y) = f^+(f^-(x \vee y)) \in N.$$ On the other hand, every $z \in N$ such that $z \geq x$ and $z \geq y$ is such that $z \geq x \vee y$ and thus $z = f^+(z) \geq f^+(x \vee y)$ proving that $f^+(x \vee y)$ is the smallest of the Nash above both $x$ and $y$. The claim on $f^+(x \vee y)$ is completely analogous.

(iv) Point (vi) implies that the set of equilibria $N$ is a complete lattice. By definition, $N$ is invariant, while global reachability follows from point (i).

(vii) Point (iii) that implies $N = \{x^+\}$ is globally I-reachable and thus also globally BR-reachable. Moreover, notice that point (iii) implies that both $g^-(x^*)$ and $g^+(x^*)$ are equilibria, so that the assumption $N = \{x^+\}$ implies that $g^-(x^*) = x^* = g^+(x^*)$. Hence $\{x^+\}$ is BR-invariant.

REFERENCES

[1] L. Arditti, G. Como, F. Fagnani, and M. Vanelli, “Equilibria and learning dynamics in mixed network coordination/anti-coordination game,” in Proceedings of the 60th IEEE Conference on Decision and Control, 2021.

[2] K. Zhou, J. C. Doyle, and K. Glover, Robust and Optimal Control. Prentice Hall, 1996.

[3] S. Rinaldi, J. P. Perenboom, and T. K. Kelly, “Identifying, understanding, and analyzing critical infrastructure interdependencies,” IEEE Control Systems Magazine, vol. 21, no. 6, pp. 11–25, 2001.

[4] Disaster Resilience: a National Imperative. The National Academies Press, 2012.

[5] K. Savla, G. Como, and M. A. Dahleh, “Robust network routing under cascading failures,” IEEE Transactions on Network Science and Engineering, vol. 1, no. 1, pp. 53–66, 2014.

[6] G. Como, “On resilient control of dynamical flow networks,” Annual Reviews in Control, vol. 43, pp. 80–90, 2017.

[7] L. Eisenberg and T. H. Noe, “Systemic risk in financial networks,” Management Science, vol. 47, no. 2, pp. 237–249, 2001.

[8] A. Haldane and R. May, “Systemic risk in banking ecosystems,” Nature, vol. 469, pp. 351–355, 2011.

[9] D. Acemoglu, A. Ozdaglar, and A. Tahbaz-Salehi, “Systemic risk and stability in financial networks,” American Economic Review, vol. 105, no. 2, pp. 656–608, 2015.

[10] L. Massai, G. Como, and F. Fagnani, “Equilibria and systemic risk in saturated networks,” Mathematics of Operations Research, 2021.

[11] D. Acemoglu, M. Dahleh, I. Lobel, and A. Ozdaglar, “Bayesian learning in social networks,” Review of Economic Studies, vol. 78, no. 4, pp. 1201–1236, 2011.

[12] D. Acemoglu, G. Como, F. Fagnani, and A. Ozdaglar, “Opinion fluctuations and disagreement in social networks,” Mathematics of Operations Research, vol. 38, no. 1, pp. 1–27, 2013.

[13] D. Acemoglu, V. Carvalho, A. A. Ozdaglar, and Tahbaz-Salehi, “The network origins of aggregate fluctuations,” Econometrica, vol. 80, pp. 1977–2016, 2012.

[14] D. R. Baqee, “Cascading failures in production networks,” Econometrica, vol. 86, no. 5, pp. 1819–1838, 2018.

[15] D. Acemoglu, A. Ozdaglar, and A. Tahbaz-Salehi, “Networks, shocks and systemic risk,” in The Oxford Handbook on the Economics of Networks, ch. 21, pp. 569–607, Oxford University Press, 2016.

[16] G. Como, K. Savla, D. Acemoglu, M. A. Dahleh, and E. Frazzoli, “Robust distributed routing in dynamical networks - part II: Strong resilience, equilibrium selection and cascaded failures,” IEEE Trans. Automatic Control, vol. 58, no. 2, pp. 333–348, 2013.

[17] G. Como, E. Lovisari, and K. Savla, “Throughput optimality and overloaded behavior of dynamical flow networks under monotone distributed routing,” IEEE Transactions on Control of Network Systems, vol. 2, no. 1, pp. 57–67, 2015.

[18] T. Sarkar, M. Roozbehani, and M. A. Dahleh, “Asymptotic network robustness,” IEEE Transactions on Systems, vol. 6, no. 2, pp. 812–821, 2019.

[19] K. Savla, J. S. Shammas, and M. A. Dahleh, “Network effects on the robustness of dynamic systems,” Annual Review of Control, Robotics, and Autonomous Systems, vol. 3, no. 1, pp. 115–149, 2020.

[20] A. Galeotti, S. Goyal, M. Jackson, F. Vega-Redondo, and L. Yariv, “Network games,” The Review of Economic Studies, vol. 77, no. 1, pp. 218–244, 2010.

[21] Y. Bramouillé, R. Kranton, and M. D’Amours, “Strategic interaction and networks,” American Economic Review, vol. 104, no. 3, pp. 898–930, 2014.

[22] M. O. Jackson and Y. Zenou, Handbook of game theory with economic applications, vol. 4, ch. Games on networks, pp. 95–163. Elsevier, 2015.

[23] Y. Bramouillé and R. Kranton, Games played on networks, vol. The Oxford Handbook of the Economics of Networks, ch. 5. Oxford University Press, 2016.

[24] F. Parise and A. Ozdaglar, “Analysis and interventions in large network games,” Annual Review of Control, Robotics, and Autonomous Systems, vol. 4, pp. 455–486, 2021.

[25] A. C. Pigou, The economics of welfare. Palgrave Macmillan, 1920.

[26] M. Beckman, C. B. McGuire, and C. B. Winsten, Studies in the economics of transportation. New Haven: Yale University Press, 1956.

[27] A. Galeotti, B. Golub, and S. Goyal, “Targeting interventions in networks,” Econometrica, vol. 88, no. 6, pp. 2445–2471, 2020.

[28] C. Ballester, A. Calvó-Arregón, and Y. Zenou, “Who’s who in networks: wanted: The key player,” Econometrica, vol. 74, no. 5, pp. 1403–1417, 2006.

[29] P. Milgrom and J. Roberts, “Rationalizability, learning, and equilibrium in games with strategic complementarities,” Econometrica, vol. 58, no. 6, pp. 1255–1277, 1990.

[30] X. Vives, “Nash equilibrium with strategic complementarities,” Journal of Mathematical Economics, vol. 19, pp. 305–321, 1990.

[31] S. Morris, “Contagion,” The Review of Economic Studies, vol. 67, no. 1, pp. 57–78, 2000.

[32] H. P. Young, Individual Strategy and Social Structure: An Evolutionary Theory of Institutions. Princeton University Press, 2001.

[33] H. P. Young, “The evolution of conventions,” Econometrica: Journal of the Econometric Society, pp. 57–84, 1993.
K. R. Apt, S. Simon, and D. Wojtczak, “Coordination games on weighted networks,” Econometrica, vol. 61, no. 1, 1993.

L. Blume, “The statistical mechanics of best response strategy revision,” Games and Economic Behavior, vol. 11, no. 2, pp. 111–145, 1995.

M. Granovetter, “Threshold models of collective behavior,” American Journal of Sociology, vol. 83, no. 6, pp. 1420–1443, 1978.

M. Lelarge, “Diffusion and cascading behavior in random graphs,” Games and Economic Behavior, vol. 75, pp. 752–775, 2012.

E. Adam, M. A. Dahleh, and A. Ozdaglar, “On the behavior of threshold models over finite networks,” in Proceedings of 51st IEEE Conference on Decision and Control, CDC’12, pp. 2672–2677, 2012.

W. S. Rossi, G. Como, and F. Fagnani, “Threshold models of cascades in large-scale networks,” IEEE Transactions on Network Science and Engineering, vol. 6, no. 2, pp. 158–172, 2017.

K. R. Apt, S. Simon, and D. Wojtczak, “Coordination games on weighted directed graphs,” Mathematics of Operations Research, 2021.

D. Kempe, J. Kleinberg, and E. Tardos, “Maximizing the spread of influence through a social network,” in Proceedings of SIGKDD’03, pp. 137–146, 2003.

G. Como, S. Durand, and F. Fagnani, “Optimal targeting in supermodular games,” IEEE Transactions on Automatic Control, vol. https://arxiv.org/abs/2009.09946, 2021.

K. Paarporn, B. Canty, P. N. Brown, M. Alizadeh, and J. R. Marden, “The impact of complex and informed adversarial behavior in graphical coordination games,” IEEE Transactions on Control of Network Systems, vol. 8, no. 1, pp. 200–211, 2021.

K. Paarporn, M. Alizadeh, and J. R. Marden, “A risk-security tradeoff in graphical coordination games,” IEEE Transactions on Automatic Control, vol. 66, no. 5, pp. 1973–1985, 2020.

M. Jackson and E. Storms, “Behavioral communities and the atomic structure of networks,” Available at SSRN 3049748, 2019.

D. M. Topkis, “Equilibrium points in nonzero-sum n-person submodular games,” SIAM Journal on Control and Optimization, vol. 17, no. 6, pp. 773–787, 1979.

D. M. Topkis, Supermodularity and Complementarity. Princeton University Press, 1998.

Y. Bramoullé, “Anti-coordination and social interactions,” Games and Economic Behavior, vol. 58, no. 1, pp. 30–49, 2007.

P. Ramazzi, J. Richl, and M. Cao, “Networks of conforming and nonconforming individuals tend to reach satisfactory decisions,” Proceedings of the National Academy of Sciences of the United States of America, vol. 113, no. 46, pp. 12985–12990, 2016.

A. Monaco and T. Sabarwal, “Games with strategic complements and substitutes,” Economic Theory, vol. 62, no. 1, pp. 65–91, 2016.

M. Vanelli, G. Como, and F. Fagnani, “On games with coordinating and anti-coordinating agents,” in Proceedings of the 21st IFAC World Congress, 2020.

M. O. Jackson, Social and Economic Networks. Princeton University Press, 2008.

D. Monderer and L. Shapley, “Potential games,” Games and Economic Behavior, vol. 14, pp. 124–143, 1996.

Laura Arditti is a PhD student in Applied Mathematics at the Department of Mathematical Sciences, Politecnico di Torino, Italy. She received the B.Sc. in Physics Engineering in 2016 and the M.S. in Mathematical Engineering in 2018, both magna cum laude from Politecnico di Torino. Her research is concentrated on game theory, its relationship with graphical models, and its applications to infrastructure, social, economic, and financial networks.

Fabio Fagnani received the Laurea degree in Mathematics from the University of Pisa and the Scuola Normale Superiore, Pisa, Italy, in 1986. He received the PhD degree in Mathematics from the University of Groningen, Groningen, The Netherlands, in 1991. From 1991 to 1998, he was an Assistant Professor of Mathematical Analysis at the Scuola Normale Superiore. In 1997, he was a Visiting Professor at the Massachusetts Institute of Technology (MIT), Cambridge, MA. Since 1998, he has been with the Politecnico of Torino, where since 2002 he has been a Full Professor of Mathematics. He received the B.Sc., M.S., and Ph.D. degrees in Applied Mathematics from Politecnico di Torino, in 2002, 2004, and 2008, respectively. He was a Visiting Assistant in Research at Yale University in 2006–2007 and a Postdoctoral Associate at the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, from 2008 to 2011. He currently serves as Associate Editor of the IEEE Transactions on Network Science and Engineering and of the IEEE Transactions on Control of Network Systems and as chair of the IEEE-CS Technical Committee on Networks and Communications. He was the IPC chair of the IFAC Workshop NecSys’15 and a plenary speaker at the International Symposium MTNS’16. He is recipient of the 2015 George S. Axelby Outstanding Paper Award. His research interests are in dynamics, information, and control in network systems with applications to cyber-physical systems, infrastructure networks, and social and economic networks.

Giacomo Como is a Professor at the Department of Mathematical Sciences, Politecnico di Torino, Italy, and a Senior Lecturer at the Automatic Control Department of Lund University, Sweden. He received the B.Sc., M.S., and Ph.D. degrees in Applied Mathematics from Politecnico di Torino, in 2002, 2004, and 2008, respectively. He was a Visiting Assistant in Research at Yale University in 2006–2007 and a Postdoctoral Associate at the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, from 2008 to 2011. He currently serves as Associate Editor of the IEEE Transactions on Network Science and Engineering and of the IEEE Transactions on Control of Network Systems and as chair of the IEEE-CS Technical Committee on Networks and Communications. He was the IPC chair of the IFAC Workshop NecSys’15 and a plenary speaker at the International Symposium MTNS’16. He is recipient of the 2015 George S. Axelby Outstanding Paper Award. His research interests are in dynamics, information, and control in network systems with applications to cyber-physical systems, infrastructure networks, and social and economic networks.

Martina Vanelli is a PhD student in Applied Mathematics at the Department of Mathematical Sciences (DISMA), Politecnico di Torino, Italy. She received the B.Sc. and M.S. degrees in Applied Mathematics from Politecnico di Torino, in 2017 and 2019, respectively. From October 2018 to March 2019, she was a visiting student at Technion, Israel. Her research interests include game theory, auction theory, and network systems with applications to social and economic networks and power markets.