A Signal-Space Distance Measure for
Nondispersive Optical Fiber

Reza Rafie Borujeny, Student Member, IEEE
and Frank R. Kschischang, Fellow, IEEE

Abstract

The nondispersive per-sample channel model for the optical fiber channel is considered. Under certain smoothness assumptions, the problem of finding the minimum amount of noise energy that can render two different input points indistinguishable is formulated. Using the machinery of optimal control theory, necessary conditions that describe the minimum-energy noise trajectories are stated as a system of nonlinear differential equations. This minimum energy is taken as a distance measure. The problem of designing signal constellations with the largest minimum distance subject to a peak power constraint is formulated as a clique finding problem. As an example, a 16-point constellation is designed and compared with the conventional quadrature amplitude modulation. Based on the control-theoretic viewpoint of this paper, a new decoding scheme for such nonlinear channels is proposed.

Index Terms

Fiber-optic communications, nonlinear control, optimal control, minimum distance, constellation design.

I. INTRODUCTION

Most research in communication theory has been devoted to the study of linear communication channels, either because the communication channel of interest is a linear medium, or the medium itself is nonlinear, but can be well approximated by a linear model over the usual range of its operational parameters. The optical fiber channel belongs to this latter nonlinear class, for which various approximate linear channel models have been studied. It was not until the turn of the

The authors are with the Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4 Canada (e-mail:{rrafie,frank}@ece.utoronto.ca).

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millennium [1] that the problem of nonlinearity in the long-haul fiber-optic communications became more prominent, due chiefly to the need to operate in parameter ranges where the linear approximation is not adequate.

The optical fiber channel has been the subject of many studies in the information theory community and various mathematical channel models have been developed from an information-theoretic point of view [2]–[7]. The capacity of each model has been studied and a number of lower bounds [3]–[5] and upper bounds [6], [7] have been found.

Apart from the demand for understanding the capacity of the optical fiber in the modern “nonlinear regime” of operation, devising communication schemes that work “well” in this regime is the main engineering problem in fiber-optic communications. Here, the goodness of a scheme may be related to the complexity of its implementation [8], [9], the achievable data rates it provides [10], [11] or some mixture of the two [12]. Many transmission schemes are designed by tuning the methods suitable for linear channel models and trying to turn the fiber channel into a linear one by use of some sort of nonlinear compensation [13]. In contrast, nonlinear frequency-division multiplexing (NFDM) of [14] is based on a different school of thought: to embrace the nonlinearity rather than to compensate for it. The methodology of [14] is to consider a well-accepted nonlinear model of the fiber in a “spectral domain” that renders the input-output relation of the channel, in the noise-free scenario, seemingly straightforward. Understanding the effect of noise and its interplay with the information bearing signal in the spectral domain [15], as well as reducing the implementation complexity of the NFDM [16], are still under study by the fiber-optic community.

The problem of geometric constellation optimization is another avenue of research that has been pursued to design schemes suitable for nonlinear fiber. The development of communication schemes for the additive white Gaussian channel (AWGN) have been studied from a geometric point of view for a long while [17] (see also [18] and references therein). A communication engineer wishes to pick a set (a constellation, or a code) of points (waveforms, symbols or codewords) suitable for transmission over the channel of interest in a way that they are as far apart as possible, i.e., with the largest minimum distance possible. The appropriate measure of distance for an AWGN channel is the Euclidean distance. This type of geometric constellation optimization has been studied for some AWGN-like models of optical fiber [19], [20]. However, if one wants to take into account the effect of nonlinearity, the notion of distance between constellation points is not a clear one. The objective of this paper is to take a first step in
establishing a notion of distance between constellation points for such nonlinear channels.

We mainly focus on the per-sample nondispersive channel model of optical fiber and think of the noise as a perturbation that is caused by an adversary. We study the minimum amount of energy required by the adversary to produce the same output symbol from two distinct input symbols. This adversarial energy is considered as a measure of distance between these input symbols and can be used as a criterion for signal constellation design.

Adversarial noise affects the evolution of an input symbol as it traverses the fiber. Even if the adversarial energy is limited, the set of possible output symbols, the noise ball, for a given input symbol is difficult to describe—due to the channel nonlinearity. It is not at all straightforward to find out whether or not the noise balls corresponding to distinct input symbols intersect. Using variational methods, we find the adversarial noise trajectories with the least energy that cause a nonempty intersection of the noise balls corresponding to two input symbols. Various aspects of this adversarial distance are studied, including an upper bound and a lower bound. Using clique-finding algorithms from graph theory, we show how to design constellations of a prescribed size with largest minimum distance.

It is well-known that the per-sample channel is not necessarily of high practical relevance to the optical fiber channel (see e.g., [4] or [21]). Nevertheless, the per-sample channel seems to be the simplest nonlinear model that captures the nonlinear signal-noise interactions similar to the optical fiber—which is known to be the limiting factor in the simplest case of single user point-to-point communication over optical fiber [5], [22]. We choose this overly-simplified model to illustrate the main idea as it allows us to carry out our analysis in a rather straightforward way. We later discuss how we can readily generalize our analysis to the nondispersive waveform channel.

The rest of the paper is organized as follows. In Section II we develop the adversarial channel model that we wish to study. The problem of finding the adversarial distance between input symbols is formulated in Section III. Important properties of this distance, including a set of necessary conditions for the energy-minimizing noise trajectories, are studied in Section IV. Some aspects of the numerical calculations associated with the distance measure are discussed in Section V. A recipe for designing constellations, along with an example, are presented in Section VI. In Section VII we further outline some potential extensions and discuss the applicability of the approach of this paper for a class of linear channels. A new decoding scheme, based on the control-theoretic viewpoint of this paper, is also outlined. Section VIII concludes the paper.
II. CHANNEL MODEL

Propagation of a narrow-band optical signal over a standard single mode fiber of length \( L \) with ideal distributed Raman amplification is described by the nonlinear Schrödinger equation \[23\]

\[
\frac{\partial q(z,t)}{\partial z} = -i \beta_2 \frac{\partial^2 q(z,t)}{\partial t^2} + i \gamma |q(z,t)|^2 q(z,t) + n(z,t),
\]

\[0 \leq z \leq L, -\infty \leq t \leq \infty.\] (1)

Here, \( i = \sqrt{-1} \), \( q(z,t) \) is the complex envelope of the optical signal, \( z \) is the distance along the fiber, \( t \) is the time with respect to a reference frame moving with the group velocity, \( \beta_2 \) is the dispersion coefficient, \( \gamma \) is the nonlinearity coefficient, and \( n(z,t) \) represents the perturbation effect of the amplifier noise.

We study (1) assuming \( \beta_2 = 0 \). This assumption corresponds to setting the carrier frequency to the zero-dispersion wavelength of the fiber. The main reason for this assumption is to single out the nonlinear interaction of the optical signal \( q \) and the perturbation \( n \). The signal \( n \) is referred to as noise in most of the fiber-optic literature. We purposely avoid this terminology on purpose as it may suggest that \( n \) has a stochastic nature. To simplify our analysis further, we study the so-called per-sample channel model \[4\]. The motivation for considering the per-sample channel model comes from the fact that when \( \beta_2 = 0 \) and \( n = 0 \), the nonlinearity is localized in time in the sense that each time sample of the signal undergoes nonlinearity independently. The governing equation for the per-sample channel considered in this paper is obtained by setting \( \beta_2 = 0 \) and removing the time dependence of the signal from (1). That is,

\[
\frac{d}{dz} q(z) = i \gamma |q(z)|^2 q(z) + n(z), \quad 0 \leq z \leq L.\] (2)

The per-sample channel model, however, has its own limitations: most importantly this model does not capture the spectral broadening of the signal due to the nonlinearity, and thus may not be an accurate representation of the physics of the fiber channel (see \[21\] for a thorough discussion). Nevertheless, this model allows us to demonstrate our new approach in a relatively straightforward way as opposed to the model of (1) which requires a more elaborate treatment. As will be discussed in Section \[VII\] it is possible to extend our analysis to the more general nondispersive waveform channel case described by (1) with \( \beta_2 = 0 \).

\[1\]We occasionally drop the arguments of functions for compactness. In all such instances, the correct interpretation should be clear from the context.
The differentiability of $q$ in (2) is considered to be component-wise. That is, $q$ is not necessarily an analytic function but has differentiable real and imaginary components. To study this model, one needs to first describe the properties of the perturbation signal $n(z)$. In a probabilistic model, $n(z)$ is usually described as some random process with mathematically tractable properties that capture the physics of the amplifier noise. In this paper, however, we consider a deterministic approach, as is usually the case for adversarial channel models, and assume that $n \in F$ where the function space $F$ is a subset of functions from $[0, L]$ to $\mathbb{C}$. To make our adversarial model tractable, we impose further smoothness properties on $F$, namely, we assume that $F$ is the set of continuous functions on $[0, L]$. This may be seen as an engineering approximation of a band-limited Gaussian process, where bandwidth is defined with respect to the spatial variable. This continuity assumption is equivalent to assuming $q$ has continuously differentiable real and imaginary components (see (2)). As will be discussed later, it is possible to weaken these requirements, but we choose not to do so, so that the resulting extra complication does not overshadow the main ideas.

III. An Adversarial Distance Measure for the Input Alphabet

Consider the channel model that is described by the evolution equation (2). The input alphabet $\mathcal{X}$ and the output alphabet $\mathcal{Y}$ for this channel are both the complex plane $\mathbb{C}$. The channel input $x$ is described by the boundary condition $q(0) = x$. The channel output $y$ is the value of the signal at $z = L$, i.e., $y = q(L)$.

We describe the nonlinear relation between the input $x$, the output $y$, and the adversarial noise $n(z)$ by writing

$$y = N(x, n(z)),$$

for some operator $N$. That this is well-defined is proved in Theorem [1].

We consider the energy of the adversarial noise as a measure of effort that the adversary makes to transform $x$ to $y$. If $y = N(x, n(z))$ and

$$E = \int_0^L |n(z)|^2 \, dz,$$

we write

$$x \xrightarrow{E} y.$$

Define

$$S_E(x) = \left\{ y \mid x \xrightarrow{E} y \right\}.$$
The set $S_E(x)$ is the fan of possible outputs for a given input $x$ and a given effort $E$. Define

$$B_E(x) = \bigcup_{\epsilon \leq E} S_\epsilon(x).$$

For a given input $x$, the set $B_E(x)$ describes the reachable set of outputs, or the noise ball, into which the adversary can transform $x$ while making an effort of at most $E$.

From the adversary’s point of view, the channel model of (2) can be seen as a nonlinear control system. From this viewpoint, the optical signal $q$ plays the role of the state of a control system and the adversarial noise is the control signal. The distance parameter $z$ plays the role of the temporal evolution parameter of conventional control systems. The state equation for this system is

$$q' = f(q) + n$$

with $f(q) = i\gamma |q|^2q$. The output of the control system is just the final state of the system at $z = L$. For a given control signal $n$ and an initial state $q(0) = x$, the state function $q(z)$ identifies a curve in the complex plane parametrized by $z$. This locus of points is called the trajectory of the system from $x$ for the control $n$. The adversarial effort in transforming the system from an initial state to its final state along a certain trajectory, which measures the energy of the control signal for that trajectory, can be thought of as a cost function that the adversary wishes to minimize. The set of admissible control signals is $F$, i.e., the set of complex-valued component-wise continuous functions defined on $[0, L]$.

Some properties regarding the well-posedness of the control system defined in (3) are stated in Theorems 1 and 2.

**Theorem 1:** For any given control $n \in F$ and any initial state $q(0)$, the control system of (3) has a unique trajectory.

**Proof:** See Appendix A.

**Theorem 2:** For any given control $n \in F$ and any initial state $q(0)$, the unique trajectory of the system satisfies

$$q(z) = e^{i\gamma \int_0^z |q(s)|^2 ds} \left( q(0) + \int_0^z n(r)e^{-i\gamma \int_0^r |q(s)|^2 ds dr} dr \right)$$

for all $z \in [0, L]$.

**Proof:** See Appendix B.

**Remark 1:** One can use Theorem 2 to show that if $n(z) = 0$, then

$$q(L) = q(0)e^{i\gamma L |q(0)|^2}.$$  

(5)
That is, the channel with no noise only rotates the input point about the origin in the complex plane, where the amount of rotation is proportional to the squared magnitude of the input.

The next theorem establishes the local controllability of the control system (3). Intuitively, local controllability implies that small changes in the initial and final states of the control system can be achieved by small changes in the control signal. Before stating the theorem, we first define the concept of local controllability.

**Definition 1:** Let \( \hat{n} \in F \) be a control and \( \hat{q} \) be the corresponding trajectory of the system (3). The control system (3) is locally controllable along the trajectory \( \hat{q} \) if, for every \( \epsilon > 0 \), there exist a \( \delta > 0 \) such that for every \( (a, b) \in \mathbb{C}^2 \) with

\[
|\hat{q}(0) - a| < \delta, \\
|\hat{q}(L) - b| < \delta,
\]

there exists a control \( n \in F \) for the system (3) such that

\[
b = N(a, n(z))
\]

while

\[
\left| \int_0^L |\hat{n}(z)|^2 \, dz - \int_0^L |n(z)|^2 \, dz \right| \leq \epsilon.
\]

**Theorem 3:** For any given control \( n \in F \) and any initial state \( q(0) \), the control system (3) is locally controllable along the unique trajectory of the system.

**Proof:** See Appendix C.

**Remark 2:** Using Theorem 3, one can show that as the effort available to an adversary increases, the reachable set at the output of the channel inflates in all directions in the complex plane so that every reachable point with a smaller effort is an interior point of the region of reachable points with a larger effort. Intuitively, one can think of the reachable set for a given effort as a balloon. As the adversarial effort increases, the balloon inflates in every direction. We state this result in the next corollary.

**Corollary 1:** If \( E' > E > 0 \), then \( B_E(x) \) is a proper subset of \( B_{E'}(x) \). Moreover, for any boundary point \( y \) of \( B_E(x) \), there is a neighborhood of \( y \) that is contained in \( B_{E'}(x) \).

**Corollary 1** motivates the following notion of distance for any two input points. For any \( x_1 \) and \( x_2 \) in \( X \), define

\[
d(x_1, x_2) \triangleq \inf \{ E \mid B_E(x_1) \cap B_E(x_2) \neq \emptyset \}.
\]
The bivariate function $d(\cdot, \cdot)$ describes the minimum effort $E$ needed by an adversary so that

$$N(x_1, n_1(z)) = N(x_2, n_2(z))$$

with

$$E = \int_0^L |n_k(z)|^2 dz \quad k = 1, 2.$$ 

It is easy to show that $d(x_1, x_2) = d(x_2, x_1)$. Also, one can use Theorem 1 to show that $d(x_1, x_2) \geq 0$ and that equality happens if and only if $x_1 = x_2$. However, this function does not necessarily satisfy the triangle inequality and therefore it is not a metric. Nevertheless, we call $d(x_1, x_2)$ the distance between $x_1$ and $x_2$. The distance between two points in $\mathcal{X}$ measures the required adversarial effort to make them indistinguishable at the output of the channel. One of the goals of this paper is to find the value of this distance for any pair of possible input points.

IV. PROPERTIES OF THE ADVERSARIAL DISTANCE

In this section, we first formulate the problem of finding the distance between two points $x_1$ and $x_2$ in $\mathcal{X}$ as a variational problem. Some bounds for the adversarial distance are also provided. We then find the distance for the special case that one point is 0.

A. NECESSARY CONDITIONS FOR THE MINIMUM-ENERGY ADVERSARIAL NOISE

We assume that the adversarial noise that affects $x_k$ is $n_k(z)$, and the function that describes the evolution of $x_k$ over the fiber (the state of the control system) is $q_k$, for $k = 1, 2$. Then, from (9) we have

$$d(x_1, x_2) = \inf \int_0^L |n_1(z)|^2 dz ,$$

subject to 

$$q_1'(z) = i\gamma |q_1(z)|^2 q_1(z) + n_1(z),$$

$$q_2'(z) = i\gamma |q_2(z)|^2 q_2(z) + n_2(z),$$

$$q_1(0) = x_1, q_2(0) = x_2,$$

$$N(x_1, n_1) = N(x_2, n_2),$$

$$\int_0^L |n_1(z)|^2 dz = \int_0^L |n_2(z)|^2 dz.$$  \(2\)

The function $d(\cdot, \cdot)$ is called a semimetric. A metric is a semimetric that satisfies the triangle inequality.
The constraint on the equality of the two adversarial efforts is justified by Corollary [1]. If we write \( q_k(z) \) in terms of its real and imaginary components

\[
q_k(z) = a_k(z) + ib_k(z)
\]

and substitute for \( n_k(z) \) from the evolution equations, the optimization problem of (10) becomes

\[
d(x_1,x_2) = \inf \int_0^L g_1(a_1, b_1, a_1', b_1') \, dz, \tag{11}
\]

subject to

\[
a_1(0) + ib_1(0) = x_1,
\]

\[
a_2(0) + ib_2(0) = x_2,
\]

\[
a_1(L) + ib_1(L) = a_2(L) + ib_2(L),
\]

\[
\int_0^L \sum_{k=1}^2 (-1)^k g_k(a_k, b_k, a_k', b_k') \, dz = 0,
\]

with

\[
g_k(a_k, b_k, a_k', b_k') = |a_k' + ib_k' - i\gamma(a_k^2 + b_k^2)(a_k + ib_k)|^2. \tag{12}
\]

This is a variational problem with six (real) boundary conditions and one isoperimetric constraint: the trajectory of \( a_k(z) + b_k(z) \) must start from \( x_k \), and the two trajectories must end at the same point in \( \mathcal{Y} \) with the same effort. We sometimes refer to these two trajectories as optimal trajectories. Typically, to find the optimal trajectories of the problems of this sort, a system of Euler–Lagrange differential equations together with appropriate boundary conditions must be solved [25]. The main result of this section is the derivation of the associated Euler-Lagrange equations.

**Theorem 4:** If the trajectories \( a_k \) and \( b_k \), \( k = 1, 2 \), minimize the distance between \( x_1 \) and \( x_2 \), they satisfy the following system of equations

\[
(1 - \lambda) \left(-4\gamma b_1'(a_1^2 + b_1^2) + 3\gamma^2 a_1(a_1^2 + b_1^2)^2 - a_1''\right) = 0,
\]

\[
(1 - \lambda) \left(4\gamma a_1'(a_1^2 + b_1^2) + 3\gamma^2 b_1(a_1^2 + b_1^2)^2 - b_1''\right) = 0,
\]

\[
\lambda \left(-4\gamma b_2'(a_2^2 + b_2^2) + 3\gamma^2 a_2(a_2^2 + b_2^2)^2 - a_2''\right) = 0,
\]

\[
\lambda \left(4\gamma a_2'(a_2^2 + b_2^2) + 3\gamma^2 b_2(a_2^2 + b_2^2)^2 - b_2''\right) = 0,
\]

\[
c'(z) + g_1(a_1, b_1, a_1', b_1') - g_2(a_2, b_2, a_2', b_2') = 0,
\]
together with the boundary conditions at $z = 0$ given by

\[ a_k(0) + ib_k(0) = x_k, \]
\[ c(0) = 0, \]
\[ c(L) = 0, \]

and at $z = L$ given by

\[ a_1(L) + ib_1(L) = a_2(L) + ib_2(L), \]
\[ (1 - \lambda)a'_1(L) + \lambda a'_2(L) + \gamma b_1(L) \left( a_1^2(L) + b_1^2(L) \right) = 0, \]
\[ (1 - \lambda)b'_1(L) + \lambda b'_2(L) - \gamma a_1(L) \left( a_1^2(L) + b_1^2(L) \right) = 0. \]

**Proof:** See Appendix D.

Theorem 4 describes a system of differential equations, together with one unknown Lagrange multiplier $\lambda$, with a consistent number of boundary conditions and may be solved by numerical methods. The additional helper function $c(z)$ in Theorem 4 changes the constraint on the equality of the adversarial efforts into the Mayer form [26], which allows this constraint to be incorporated into the optimization procedure. In Section V, we use Theorem 4 to find the distance between pairs of points in the input alphabet.

**B. Bounds on the Adversarial Distance**

It is straightforward to show that $d(\cdot, \cdot)$ is rotationally invariant, meaning that

\[ d(x_1, x_2) = d(x_1e^{i\Theta}, x_2e^{i\Theta}), \quad x_1, x_2 \in \mathbb{C}, \Theta \in [-\pi, \pi). \]

We refer to this property as rotational symmetry. Rotational symmetry can reduce the computational complexity of finding $d(\cdot, \cdot)$ on certain sets of points, subject to certain symmetries.

We find it convenient to introduce the following notion of distance. The radial distance between two points $x_1, x_2$ is defined by

\[ d_R(x_1, x_2) = \inf \{d(x, y) \mid |x| = |x_1|, |y| = |x_2| \}. \]

This corresponds to the minimum adversarial distance between the circle centered at the origin of radius $|x_1|$ and the circle centered at the origin of radius $|x_2|$. Rotational symmetry guarantees that the radial distance is equal to

\[ d_R(x_1, x_2) = \inf \{d(|x_1|, |x_2|e^{i\Theta}) \mid \Theta \in [-\pi, \pi) \}. \]
It is helpful to rewrite the state equation in the polar coordinates

\[ q(z) = R(z)e^{i\theta(z)}. \]

Let the real part and the imaginary part of \( n(z) \) be \( n_1(z) \) and \( n_2(z) \), respectively. The state equation in polar coordinates becomes

\[ R' \cos(\theta) - \theta' R \sin(\theta) = -\gamma R^3 \sin(\theta) + n_1, \tag{16} \]
\[ R' \sin(\theta) + \theta' R \cos(\theta) = \gamma R^3 \cos(\theta) + n_2. \tag{17} \]

If we multiply (16) by \( \cos(\theta) \) and (17) by \( \sin(\theta) \), and add up the results, we get

\[ R' = n_1 \cos(\theta) + n_2 \sin(\theta). \tag{18} \]

That is, the rate of change in the radial direction is equal to the projection of the adversarial noise \( n(z) \) on the unit vector pointing out from the state of the system at \( z \) in the radial direction. With similar algebraic manipulations, we can show that

\[ \theta' = \gamma R^2 + \frac{n_2 \cos(\theta) - n_1 \sin(\theta)}{R}, \tag{19} \]

which shows that the rate of change of \( \theta \) comes from two sources: the first term on the right hand side of (19) captures the nonlinearity of the system and the second term is the projection of the adversarial noise on the azimuthal direction.

From (18), one can show that

\[ |n(z)| \geq |R'(z)|. \tag{20} \]

The Cauchy–Schwarz inequality, then, gives

\[ E = \int_0^L |n(z)|^2 \, dz \geq \frac{1}{L} \left( \int_0^L |n(z)| \, dz \right)^2 \geq \frac{(|y| - |x|)^2}{L}, \tag{21} \]

where the trajectory starts at \( q(0) = x \) and ends at \( q(L) = y \). If we consider a control signal of the form\(^3\)

\[ n(z) = Ce^{i\theta(z)} \tag{22} \]

\(^3\)A noise of this form is always orthogonal to the azimuthal direction.
with $C$ being a real constant, one can see that the unique trajectory that starts from $x$ and ends at a point on the circle centered at the origin of radius $|y|$ requires an effort of

$$E = \frac{(|y| - |x|)^2}{L}.$$  

To find the radial distance $d_R(x_1, x_2)$, assume that the optimal trajectories corresponding to $x_1$ and $x_2e^{i\theta}$ reach $y(\Theta)$ at $z = L$. The effort $E(\Theta)$ required to move $x_1$ to $y(\Theta)$ satisfies

$$E(\Theta) \geq \frac{(|y| - |x_1|)^2}{L}. \quad (23)$$

Similarly, the effort required to move $x_2e^{i\theta}$ to $y$ satisfies

$$E(\Theta) \geq \frac{(|y| - |x_2|)^2}{L}. \quad (24)$$

Using controls of the form (22), for any $y$, one can see that there exist a pair of trajectories, one connecting the two concentric circles centered at 0 of radii $|x_1|$ and $|y(\Theta)|$ and the other connecting the two concentric circles centered at 0 of radii $|x_2|$ and $|y|$, with efforts exactly equal to the right hand sides of (23) and (24). Using rotational symmetry, one can prove the following theorem.

**Theorem 5:** For any pair of points $x_1, x_2$,

$$d_R(x_1, x_2) = \frac{(|x_1| - |x_2|)^2}{4L}. \quad (25)$$

By definition, $d_R(x_1, x_2)$ gives a lower bound for $d(x_1, x_2)$. That is,

$$d(x_1, x_2) \geq \frac{(|x_1| - |x_2|)^2}{4L}. \quad (26)$$

To find an upper bound for the adversarial distance, we find two control signals $n_1, n_2$, corresponding to the initial states $x_1, x_2$, so that the final state of the two system is the same. In particular, we consider the control system when the control signal has a constant magnitude. We then use two trajectories of this type to confuse the two initial states $x_1, x_2$. The result is summarized in Theorem 6.

**Theorem 6:** The adversarial distance $d(x_1, x_2)$ is upper bounded by

$$\min_y \max_{k \in \{1, 2\}} \frac{(|y| - |x_k|)^2}{L} \left[ 1 + \left( \frac{\Delta(x_k, y)}{\ln(\frac{|y|}{|x_k|})} \right)^2 \right] \quad (27)$$

where $\Delta(\cdot, \cdot)$ is defined in (100).

**Proof:** See Appendix E.  

\[\blacksquare\]
In case of singularities, the upper bound of Theorem 6 is understood as a limit (see the proof). This upper bound provides a tight estimate for \( d(x, -x) \) when \(|x|\) is not too large, but becomes loose when \(|x| \to \infty\). Similar to the proof of Theorem 6, one can consider a special functional form for the control signal and obtain various other upper bounds. It seems that the numerical evaluation of such upper bounds is usually more difficult than solving the system of equations given in Theorem 4.

C. Distance From the Origin

Although the general solution of the optimization problem of (10) may not have a closed form, it may be possible to find a closed form in some special cases. Finding the distance \( d(x, 0) \) of an arbitrary point \( x \) from the origin is one such case. This special case corresponds to the design of the on-off keying transmission scheme in which one looks for a point \( x^* \) of minimum energy whose adversarial distance from the origin is larger than a given value. The minimum energy requirement means that the Euclidean distance of \( x^* \) from the origin is required to be minimum, while the adversarial distance is kept larger than the available effort.

Using (21) with \( x = 0 \), one can see by substitution that

\[
n(z) = \frac{y}{L} e^{\frac{y^{2}|y|^2}{2L^2}(z^2 - L^2)}
\]

(28)
gives the trajectory from 0 to \( y \) with the minimum energy. Hence, the inequality in (21) is attainable with equality and

\[
E = \frac{|y|^2}{L}.
\]

(29)

Note that the effort remains the same for all final points \( q(L) \) on the circle

\[
ye^{i\Theta}, \quad \Theta \in [-\pi, \pi).
\]

As the circle centered at the origin of radius 0 contains only one point, namely the origin itself, rotational symmetry guarantees that

\[
d(x, 0) = d_R(x, 0).
\]

(30)

One can then use Theorem 5 to show that

\[
d(x, 0) = \frac{|x|^2}{4L}.
\]

(31)
V. NUMERICAL CALCULATION OF THE ADVERSARIAL DISTANCE

The problem of finding the adversarial distance is one instance of an optimal control problem [25]–[30]. There are two main types of numerical methods for finding the distance between two points, namely direct methods and indirect methods. In a direct method, first the state equation is discretized and the distance problem is expressed as a nonlinear programming problem. The problem can then be treated by means of well developed nonlinear programming numerical methods. For this reason, direct methods are sometimes referred as “discretize, then optimize.” There are many numerical computing packages that implement various types of direct methods (see [30]). We use the direct methods of dynamic optimization of [31].

In an indirect method, on the other hand, the main ingredient is the necessary conditions of Theorem 4, i.e., we “optimize, then discretize.” These necessary conditions form a 2-point boundary value problem and we use bvp4c (see [32], [33]) to solve this system of ordinary differential equations (ODEs). When one wants to solve the equations of Theorem 4, usually a good initial guess is needed. We use various initial data obtained by perturbing the initial states and using the direct methods with low spatial resolution to solve the system of ODEs in...
Theorem 4. The ODE solver is then provided with these initial guesses.

We use both direct and indirect methods to find \(d(x, 0)\) and compare the results with the exact solution given in (31). The parameters of the model are \(L = 2000\) km and \(\gamma = 1.27\) W\(^{-1}\)km\(^{-1}\). The results are depicted in Fig. 1. Our conclusion is that the use of an indirect method usually results in a more accurate estimate of the distance. Hereinafter, we only use the indirect method of Theorem 4 for our numerical computations.

It is interesting to look at the optimum trajectories described by Theorem 4. We have solved the equations of Theorem 4 numerically for three pairs of \((x_1, x_2)\). In all three cases, we have chosen two antipodal input points, i.e., \(x_2 = -x_1 = x\). Three different magnitudes for \(x\) are considered, corresponding to different input powers. The trajectories of evolution of \(x_1\) and \(x_2\) that are obtained by the optimum adversarial noise are depicted in Fig. 2. It is evident that the strategy of the adversary varies as the input power is increased. In particular, for lower input powers, the optimum trajectory is obtained by confusing the two points at the origin. The adversary, in this case, needs to make enough effort to bring each point to the origin. At very high input powers, on the contrary, confusing the two points \(x, -x\) through phase changes requires less effort (see Theorem 7, also see the argument on Fermat’s spiral in [34]).

VI. Constellation Design

One important application of the distance defined in Section III is to design signal constellations. Just as Euclidean distance can be used to position input signals for an AWGN channel, the adversarial distance of Section III can be used to provide guidelines when designing a signal constellation for the per-sample channel. Unlike the classical AWGN channel, where one can design a constellation for unit input peak/average power and then scale up/down the constellation points by a constant scalar based on the required power constraint, the design of a constellation for a nonlinear channel, such as the one we consider in this paper, is drastically different. In particular, the design of a constellation for a given average input power seems to be more difficult for the channel model of this paper as it requires the knowledge of the distance \(d(\cdot, \cdot)\) for practically all points of \(\mathbb{C}^2\). Finding a way to alleviate this problem is out of the scope of

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\(^4\)The spatial resolution required to obtain the initial guess using the direct method is much lower than the resolution used in finding the distance using the direct method itself. Hence, the time required to find the initial guess is negligible compared to the overall time complexity of the indirect method. The parameters used for both direct and indirect methods are chosen such that both solvers can converge in a comparable amount of time.
this introductory paper and is left for the future research. Henceforth, we consider a peak power constraint for our constellation design problems.

In our first example, we calculate the distance between the two points of a binary antipodal constellation and an on-off keying constellation, and compare these two constellations for various peak powers. Then, we explain how we can design larger constellations with the largest minimum distance possible for a given peak power using clique-finding algorithms. A 16-point constellation with maximum minimum-distance for a fixed peak power is found to illustrate the ideas. The performance, in terms of symbol error rate (SER), of our proposed constellation is compared with that of the standard quadrature amplitude modulation (QAM) of the same size and peak power when the amplifier noise is assumed white (in space) and Gaussian.

### A. Antipodal Versus On-Off Keying

We use numerical tools to find the distance $d(x, -x)$ for binary antipodal constellations with different input powers, as well as the closed-form equation for $d(x, 0)$ corresponding to on-off keying constellations. The optical fiber parameters are the same as in Section V. The results are plotted in Fig. 3. For small input powers, $d(x, -x)$ matches the upper bound (27), that is, in the “linear regime” the adversarial distance agrees with the Euclidean distance. One can see that the distance measure for an antipodal constellation shows a phase transition at around $x = 0.03$. Eventually, at high input powers, it is seen that the points of the on-off keying constellation
require a higher amount of adversarial effort to become indistinguishable. Hence, at high input powers, on-off keying is preferred over the antipodal scheme of the same peak power. From Fig. 3 it seems
\[ \lim_{|x| \to \infty} d(x, -x) = 0. \] (32)
The following theorem generalizes this observation.

**Theorem 7:** Let \( \phi \in [-\pi, \pi) \). Then
\[ \lim_{|x| \to \infty} d(x, xe^{i\phi}) = 0. \] (33)

**Proof:** See Appendix F. \( \square \)

### B. Constellation Design

The minimum distance of a constellation \( C \) is defined by
\[ d(C) \triangleq \min \{ d(x_1, x_2) \mid x_1, x_2 \in C, x_1 \neq x_2 \} \]. (34)

Having the distances of all pairs of points on a grid, subject to a certain peak power, one can find a multi-point constellation with the largest minimum distance possible. The procedure we outline here is not specific to the adversarial distance of this paper and can be used to find a constellation with a prescribed size from a finite set of points equipped with a semimetric [35]. Let \( G \) be a grid of points and assume that \[ d : G \times G \rightarrow D \subset \mathbb{R} \] (35)
where \( D \) is the range of \( d(\cdot, \cdot) \) when restricted to \( G \times G \). We form a sorted list of all elements of \( D \). We then consider a threshold distance \( d_{th} \) in \( D \) and form a simple graph with vertex set \( G \). Two vertices are connected by an edge if and only if their distance is at least \( d_{th} \). In this graph, we then find a maximal clique [36]. If the size of the maximal clique is larger (smaller) than the prescribed constellation size, we choose a larger (smaller) \( d_{th} \) from \( D \). If the size of the maximal clique obtained this way is exactly equal to the prescribed value, and choosing a larger \( d_{th} \) results in a strictly smaller maximal clique, then the obtained constellation has the largest minimum distance possible.

We start off by fixing a polar grid of points as candidates for our constellation points. We consider twenty different radii equally spaced between 0 and 0.05 together with forty different

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\(^5\)The two headed arrow \( \rightarrow \) indicates an onto map.
phases at each radius. The peak power of 0.5 is selected so that the effect of nonlinearity becomes prominent (see Fig. 3). We use rotational symmetry to reduce the number of times the differential equations of Theorem 4 needs to be solved. Fig. 4 shows a 16-point constellation with maximum minimum-distance.

C. Gaussian Noise Model

In this subsection, we study the performance of some of the constellations given in Fig. 4 in terms of SER. To set up the simulations, we consider the channel model

$$\frac{d}{dz} q(z) = i\gamma |q(z)|^2 q(z) + N(z), \quad 0 \leq z \leq L.$$  \hspace{2cm} (36)

where $N(z)$ is a complex white Gaussian process with autocorrelation function

$$E[N(z)N^*(z')] = \sigma^2 \delta(z - z').$$  \hspace{2cm} (37)

The signal constellations that we can design based on the geometric approach of this paper are not necessarily optimum in terms of SER for the stochastic channel model of (36). Such constellation optimization has been considered before [37], [38]. In particular, in [38], for a
target constellation size, a bank of amplitude phase-shift keying constellations are considered. The best constellation in terms of SER is then selected based on the results of simulations for each average input power $\sigma^2$. The size of the collection of constellations that is considered in [38] grows exponentially with the constellation size which renders their method impractical for larger constellations. Nevertheless, our objective in this section is not to compare the performance of the schemes designed in this paper with the exhaustive method of [38]. One should also note that we consider a peak power constraint as opposed to the average power constraint of [38]. We prefer to compare our design with the standard QAM constellations which seem to be a more natural baseline for us.

The SER of the optimal 16-point constellation of Fig. 4 and a conventional 16-QAM of the same peak power are illustrated in Fig. 5. To obtain SER of each constellation, the channel model of (36) is simulated by considering a fiber of length 2000 km as a concatenation of noise-free fibers of length 1 km each and injecting a Gaussian noise with variance $\sigma^2$ at the output of each fiber segment of length 1 km. Each constellation point is transmitted a total of 250,000 times. A fine 2 dimensional histogram is used to capture the empirical conditional distribution of the channel. All points of a constellation are chosen with the same probability. The mutual information for the two constellation under study is also estimated and is depicted in Fig. 6.

VII. DISCUSSION

In this section, we outline potential extensions of the variational approach considered in this paper. We first discuss the problem of minimum distance decoding based on the distance measure introduced in this paper. We then explain how we can readily extend the analysis of this paper to the nondispersive waveform channel. It is also shown how one can use the approach of this paper for a class of linear channels. We also briefly review the possibility of extending our model to the general case of (1). Other discussions include the possibility of extending the set $F$ of possible adversarial noise trajectories. The problem of designing input signal spaces based on the

\[ \text{Moreover, perfect knowledge of noise power spectral density is required to decide on the optimal constellation. If the noise is not Gaussian, the method of [38] becomes irrelevant. Our design, however, does not require the knowledge of the noise power spectral density, nor the exact statistics of the noise.} \]

\[ \text{The conditional distribution of the output given the input for the channel model of (36) is known [4], [39]. We do not use this conditional distribution as it is computationally expensive to obtain the results in the range of noise powers that we wish to consider.} \]
proposed adversarial distance is also briefly discussed. Finally, we comment on the analogy of the adversarial concepts of this paper and their relation to concepts in non-stochastic information theory.

A. Minimum Distance Decoder

Having a notion of distance, we can consider a minimum distance decoder which produces the constellation point that requires the least amount of adversarial effort to reach to the received point at the output of the channel. Let $y$ denote the point received at the output of the channel. Define

$$E(x, y) = \inf \{ \varepsilon \mid y \in B_\varepsilon(x) \}.$$  \hspace{1cm} (38)

That is, $E(x, y)$ is the minimum adversarial effort needed to transform $x$ to $y$ through the nonlinear channel of (2). The minimum distance decoder, for a constellation $\mathcal{C}$, then decides on

$$\hat{x} = \text{DEC}_{\text{MD}}(y) \triangleq \arg \min_{x \in \mathcal{C}} E(x, y),$$  \hspace{1cm} (39)
where ties are broken arbitrarily. Minimum distance decoding, therefore, requires calculation of $E(x, y)$ for all points $x$ in $C$. Using techniques similar to the proof of Theorem 4, one can prove the following Theorem.

**Theorem 8:** If the trajectory

$$q(z) = a(z) + ib(z)$$

(40)

minimizes the adversarial effort needed to transform $x$ to $y$, then

$$-4\gamma b'(a^2 + b^2) + 3\gamma^2 a(a^2 + b^2)^2 - a'' = 0,$$

$$4\gamma a'(a^2 + b^2) + 3\gamma^2 b(a^2 + b^2)^2 - b'' = 0,$$

together with the boundary conditions at $z = 0$ given by

$$a(0) + ib(0) = x,$$

and at $z = L$ given by

$$a(L) + ib(L) = y.$$

Theorem 8 implies that a system of ODEs needs to be solved to find out the minimum adversarial effort that has caused the received symbol from any of the constellation points. This is a numerically expensive process. However, once the constellation is fixed these calculations need to be done only once. One way is to first quantize the complex plane using a fine grid and compute the distance of each point of the grid from the constellation points. Each grid point is then labeled with the constellation point closest to it in terms of the adversarial distance. These labels can be stored in a look-up table and can be read at the time of decoding.

**B. From the Per-Sample Channel to the Waveform Channel**

Consider the channel model that is described by the evolution equation

$$\frac{\partial}{\partial z} q(z,t) = i\gamma |q(z,t)|^2 q(z,t) + n(z,t),$$

$$0 \leq z \leq L, \; -T \leq t \leq T.$$  (41)

The input alphabet $\mathcal{X}$ and the output alphabet $\mathcal{Y}$ for this channel are the set of component-wise continuously differentiable complex functions defined on $[-T, T]$. The channel input $x(t)$ is described by the boundary condition $q(0, t) = x(t)$. Similarly, the channel output $y(t)$ is the
Fig. 5. The SER for the 16-point constellation proposed in this paper (solid) and a 16-QAM of the same peak power (dashed) is plotted. Fiber length is assumed \( L = 2000 \) km and \( \gamma = 1.27 \).

signal at \( z = L \), i.e., \( y(t) = q(L, t) \). Similar to Section \[II\] we describe the nonlinear relation between the input \( x \), the output \( y \), and the adversarial noise \( n(z, t) \) by writing

\[
y = N(x, n).
\]

If \( y = N(x, n) \) and

\[
E = \int_0^L \int_{-T}^T |n(z, t)|^2 \, dt \, dz,
\]

we write

\[
x \xrightarrow{E} y.
\]

Define

\[
S_E(x) = \{ y \mid x \xrightarrow{E} y \}.
\]

The noise balls are defined in a same way as in Section \[\text{III}\] by

\[
B_E(x) = \bigcup_{\varepsilon \leq E} S_\varepsilon(x).
\]

Finally, for any \( x_1 \) and \( x_2 \) in \( \mathcal{X} \), define

\[
D(x_1, x_2) \triangleq \inf \{ E \mid B_E(x_1) \cap B_E(x_2) \neq \emptyset \}.
\]  (42)
Because the channel acts on different time-samples of the signal independently, the waveform distance \( D(\cdot, \cdot) \) is related to the per-sample distance \( d(\cdot, \cdot) \) by

\[
D(x_1, x_2) = \int_{-T}^{T} d(x_1(t), x_2(t)) \, dt. \tag{43}
\]

Although moving from the per-sample channel to the waveform channel is completely described by (43), the problem of designing constellations with maximum minimum-distance in this case is slightly more complicated. We do not intend to address this problem here, but one may consider further restriction on the input set so that the problem becomes feasible. For instance, the input set may be limited to a set of waveforms of particular shapes (e.g., square root raised cosines or rectangular pulses).

C. Application to Linear Channels

Consider the class of channels defined by the linear evolution equation

\[
\frac{\partial}{\partial z} q(z, t) = \sum_{j=0}^{J} a_j \frac{\partial^j}{\partial D} q(z, t) + n(z, t),
\]

\[
0 \leq z \leq L, \quad -T \leq t \leq T, \tag{44}
\]
where \( a_j \) are complex constants. The input alphabet \( \mathcal{X} \) and the output alphabet \( \mathcal{Y} \) are the set of \( N \) times component-wise continuously differentiable complex functions defined on \([-T, T]\).

We further assume that the functions in \( \mathcal{X} \) and \( \mathcal{Y} \) satisfy Dirichlet conditions (so that they have a Fourier series representation) and that the functions themselves and all of their derivatives vanish at the boundaries \( t = \pm T \) (so that the derivative of their Fourier series is the Fourier series of their derivative). We wish to find a set of necessary conditions similar to Theorem 4 that characterizes the adversarial noise trajectories of least energy that confuse two input waveforms \( x_1(t) \) and \( x_2(t) \). The linearity of the evolution equation (44) greatly simplifies the analysis as opposed to the nonlinear evolution of the per-sample channel. Let the Fourier series representation of \( x_k(t) \) be

\[
x_k(t) = \sum_m X_m^{(k)} e^{i\omega_m t}, \quad k = 1, 2,
\]

where

\[
\omega_m = \frac{m\pi}{T}.
\]

Also, let the state variable that describes the evolution of \( x_1 \) be \( q(z, t) \) and the state variable that describes the evolution of \( x_2 \) be \( p(z, t) \). Let the Fourier series coefficients of \( q(z, t) \) and \( p(z, t) \) be \( Q_m(z) \) and \( P_m(z) \), respectively. The channel law in (44) can be identified by a channel polynomial

\[
R(x) = \sum_{j=0}^{J} a_j x^j.
\]

With these notations, we summarize the results in Theorem 9.

**Theorem 9:** The trajectories \( q(z, t) \) and \( p(z, t) \) that minimize the effort needed to confuse \( x_1 \) and \( x_2 \) satisfy the following system of equations:

\[
Q_m(z) = \begin{cases} 
(A_m + B_m z) e^{R(i\omega_m) z} & \text{if } \text{Re} R(i\omega_m) = 0, \\
A_m e^{R(i\omega_m) z} + B_m e^{-R^*(i\omega_m) z} & \text{otherwise},
\end{cases}
\]

\[
P_m(z) = \begin{cases} 
(C_m + D_m z) e^{R(i\omega_m) z} & \text{if } \text{Re} R(i\omega_m) = 0, \\
C_m e^{R(i\omega_m) z} + D_m e^{-R^*(i\omega_m) z} & \text{otherwise},
\end{cases}
\]

with the boundary conditions at \( z = 0 \)

\[
Q_m(0) = X_m^{(1)},
\]

\[
P_m(0) = X_m^{(2)},
\]
and at $z = L$

$$Q_m(L) = P_m(L),$$

(50)

together with

$$(1 - \mu) B_m + \mu D_m = 0,$$

(51)

$$\sum_m f(R(i\omega_m)) (|B_m|^2 - |D_m|^2) = 0,$$

(52)

with

$$f(x) = \begin{cases} 1 & \text{if } \Re x = 0, \\ (x + x^*) (e^{-L(x+x^*)} - 1) & \text{otherwise.} \end{cases}$$

(53)

**Proof:** The proof is very similar to the proof of Theorem 4. \[\blacksquare\]

Note that Theorem 9 describes the optimal trajectories as a system of algebraic equations. If we assume that both $q$ and $p$ are bandlimited and we only have $2M + 1$ nonzero frequency taps in their Fourier series representations, i.e.,

$$q(z, t) = \sum_{m=-M}^{M} Q_m(z) e^{i\omega_m t},$$

(54)

$$p(z, t) = \sum_{m=-M}^{M} P_m(z) e^{i\omega_m t},$$

(55)

then, Theorem 9 gives $4 \times (2M + 1) + 1$ equations in the unknowns

$$A_m, B_m, C_m, D_m,$$

and the Lagrange multiplier $\mu$. This is much easier to solve than the system of differential equations that appears in the nonlinear case.

**Example 1:** In this example, we consider the channel described by the nonlinear Schrödinger equation (1) when $\gamma = 0$. For this dispersive channel, the channel polynomial is

$$R(x) = -i \frac{\beta_2}{2} x^2.$$ 

(56)

One can easily show that $\mu = 1/2$ and

$$Q_m(z) = \left( X_m^{(1)} + \frac{X_m^{(2)} - X_m^{(1)}}{2L} z \right) e^{i \frac{\beta_2}{2} \omega_m^2 z},$$

(57)

$$P_m(z) = \left( X_m^{(2)} - \frac{X_m^{(2)} - X_m^{(1)}}{2L} z \right) e^{i \frac{\beta_2}{2} \omega_m^2 z},$$

(58)
and that
\[
\begin{align*}
    d(x_1, x_2) &= \sum_m \frac{|X_m^{(2)} - X_m^{(1)}|^2}{4L} \\
    &= \int_0^L \left| \frac{x_2(z) - x_1(z)}{2L} \right|^2 dz,
\end{align*}
\]
(59)
which is proportional to the squared Euclidean distance between \(x_1\) and \(x_2\).

\[D. \text{ Extension to the Nonlinear Schrödinger Equation}\]

It is possible to extend the adversarial model of this paper to the general case of the optical fiber described by (1). Instead of a complex number, the input of the channel is a complex function described by the boundary condition at \(z = 0\), i.e.,
\[x(t) = q(0, t), \quad t \in [-T, T].\]
The input alphabet may be restricted to the functions that decay sufficiently rapidly within the time frame \([-T, T]\). The number \(T\) should be chosen large enough to capture the dispersive effect of the fiber. One can also think of letting \(T \to \infty\). The output of the channel, then, is
\[y(t) = q(L, t), \quad t \in [-T, T].\]
The adversarial effort can be generalized to
\[e = \int_0^L \int_{-T}^T |n(z, t)|^2 dt dz.\]
One can then formulate the distance between two input signals as a more general variational problem. To extend Theorem 4 and find the distance between any two input signals is a subject for future research.

We should also mention that it is possible to consider other types of adversarial effort. We chose energy as at seems to be the most natural quantity. One may also relate the common probabilistic model to the adversarial model by considering the maximum effort of the “typical” noise trajectories in the probabilistic model and consider the adversaries with limited effort accordingly.
E. Generalizing Adversarial Noise Trajectories

In defining the distance $d(\cdot, \cdot)$, we assumed that the adversarial noise trajectories are continuous functions of $z$. It is possible to extend the class of possible adversarial noise trajectories $F$ so that they have a finite number of discontinuities. That is, $F$ is the set of piecewise continuous functions from $[0, L]$ to $\mathbb{C}$. We will not pursue this assumption here. We only mention that it is possible to solve the variational problem (11) by considering extra Weierstrass–Erdmann conditions at the points of discontinuity [25].

F. Code Design

The average power of a constellation $C$ is defined by

$$P(C) \triangleq \frac{1}{|C|} \sum_{x \in C} |x|^2.$$  \hfill (61)

The following design question can be asked:

- Given two positive numbers $d_{\text{min}}$ and $P_{\text{ave}}$, design a constellation $C$ having $d(C) \geq d_{\text{min}}$ and $P(C) \leq P_{\text{ave}}$, with $|C|$ as large as possible.

This question can be thought of as a packing problem. Naturally, a Gilbert–Varshamov-type argument may be used to find a lower bound on the size of a constellation.

G. Relation to Non-stochastic Information Theory

The adversarial noise model of this paper is closely related to the non-stochastic framework of [40]–[42]. The input and output of the channel model of this paper can be thought of as two uncertain variables (UVs) [40]. The peak power constraint together with the adversarial distance considered in this paper define a bounded semimetric space for the range $[X]$ of the input UV $X$. The noise ball $B_E(x)$ is equivalent to the conditional range $[Y \mid x]$, where $Y$ is the output UV. Similarly, finding the largest signal constellation with $d_{\text{min}} > E$ is equivalent to the $(E, 0)$-capacity of [41], [42] or the Kolmogorov $2E$-capacity [43]. This analogy shows the intimate connection between reachability analysis of bounded perturbation in control theory and non-stochastic information theory. It would be interesting to see whether or not one can use the framework of non-stochastic information theory to estimate the capacity of nonlinear channels such as the one considered in this paper (see also [34]).
VIII. Conclusions

We have proposed an adversarial model for the nondispersive optical fiber channel, and given necessary conditions for the energy-minimizing adversarial noise. By means of numerical methods, we have shown that the optimum noise trajectories show different trends in different input-power regimes.

This paper outlines only the very first steps toward a new way of studying the nonlinear interaction of the signal and noise in optical fiber. It remains to see whether this model can be used to design new fiber-optic communication schemes.

Appendix A

Proof of Theorem 1

To prove the uniqueness of the solution of the state equation (3), we use the following theorem (see [44, p. 94]):

Theorem 10: Let \( g(q, z) \) be continuous in \( z \) and locally Lipschitz in \( q \) for all \( z \in [0, L] \) and all \( q \) in a domain \( D \subset \mathbb{C} \). Let \( W \) be a compact subset of \( D \), \( x \in W \), and suppose that it is known that every solution of

\[
q' = g(q, z), \quad q(0) = x
\]

lies entirely in \( W \). Then, there is a unique solution that is defined for all \( z \in [0, L] \).

For us, the function \( g(\cdot, \cdot) \) is given by

\[
g(q, z) = f(q) + n(z). \tag{63}
\]

The continuity of \( n(z) \) guarantees the continuity of \( g(q, z) \) in \( z \). It is also straightforward to show that the function

\[
f(q) = i \gamma |q|^2 q \tag{64}
\]

is locally Lipschitz for all \( q \in \mathbb{C} \). We only need to show that, for any given control signal \( n(z) \), any solution of (62) lies in a compact subset of \( C \). Equivalently, we show that any solution has a bounded magnitude.

To prove the boundedness of \( q \), we rewrite the state equation in polar coordinates. Let

\[
q(z) = R(z)e^{i \theta(z)}. \tag{65}
\]

If

\[
N_0 = \|n\| := \max_{z} |n(z)|, \tag{66}
\]
then by using the Cauchy–Schwarz inequality and (18), one can show
\[ R' \leq N_0. \]  
(67)

From this, the Grönwall–Bellman lemma \[45\] implies that
\[ R(z) \leq R(0)e^{zN_0} \leq R(0)e^{LN_0}. \]  
(68)

APPENDIX B

PROOF OF THEOREM 2

After the uniqueness of the solution is established (see Theorem 1), one can multiply both sides of (3) by the integrating factor
\[ \int_0^z e^{-\gamma|q(s)|^2} ds \]  
(69)
and integrate over \( z \).

APPENDIX C

PROOF OF THEOREM 3

The linearized control system corresponding to the state equation (3) along the trajectory of \( q(0) = x \) and \( n(z) \) is defined by
\[ Q' = f'(q)Q + N \]  
(70)
where \( Q(z) \) is the state of the linearized system and \( q(z) \) is the unique solution of (3) with the initial condition \( q(0) = x \) and the control signal \( n(z) \). The nonlinear system is locally controllable along \( q \), if the linear time-variant system (70) is controllable (see [24, Theorem 3.6]). However, the control signal in the linearized system is acting additively. Hence, from [24, Theorem 1.16], the linearized system is controllable.

APPENDIX D

PROOF OF THEOREM 4

This theorem is an example of problems in optimal control theory with some extra boundary conditions [46]. We sketch a proof for the sake of completeness. To follow all of the steps, some familiarity with calculus of variations may be needed (see [25]).

\(^8\)To be more accurate, we should say space-variant; recall that here \( z \) plays the role of the evolution parameter.
First, we rewrite the energy constraint
\[ \int_0^L \sum_{k=1}^{2} (-1)^k g_k(a_k, b_k, a'_k, b'_k) dz = 0 \] (71)
as a differential equation and then incorporate this condition into the optimization using a Lagrange multiplier. Define
\[ c(z) = \int_0^z \sum_{k=1}^{2} (-1)^k g_k(a_k, b_k, a'_k, b'_k) \, dz. \] (72)
Then
\[ c'(z) = \sum_{k=1}^{2} (-1)^k g_k(a_k, b_k, a'_k, b'_k), \] (73)
with the boundary conditions
\[ c(0) = c(L) = 0. \] (74)
Now we form the augmented Lagrangian
\[ \mathcal{L} = g_1(a_1, b_1, a'_1, b'_1) \]
\[ -\mu(z) \left( c' - \sum_{k=1}^{2} (-1)^k g_k(a_k, b_k, a'_k, b'_k) \right), \]
where \( \mu(z) \) is the Lagrange multiplier. Consider the action \( s \) defined by
\[ s = \int_0^L \mathcal{L}(a_1, a_2, b_1, b_2, a'_1, a'_2, b'_1, b'_2, \mu, c') \, dz, \] (76)
subject to the boundary conditions
\[ a_k(0) + ib_k(0) = x_k, \] (77)
\[ a_1(L) = a_2(L), \] (78)
\[ b_1(L) = b_2(L), \] (79)
\[ c(0) = c(L) = 0. \] (80)
We consider the variations of \( s \), denoted by \( \delta s \), caused by varying \( a_k(z), b_k(z), \mu(z) \) and \( c(z) \) while all boundary conditions are kept satisfied. Due to the energy constraint (73), the variations of \( a_k(z), b_k(z) \) and \( c(z) \) are not independent and finding the explicit relation between them, for all \( z \), is not easy. The Lagrange multiplier \( \mu \) allows us to avoid this issue—similarly to the case of optimization problems in multi-variable calculus with nontrivial constraints.
We expand $\delta s$ in terms of $\delta a_k, \delta b_k, \delta \mu$ and $\delta c$ to get
\[
\delta s = \int_0^L \left[ \mathcal{L}(a_1 + \delta a_1, a_2 + \delta a_2, b_1 + \delta b_1, b_2 + \delta b_2, \\
a'_1 + \delta a'_1, a'_2 + \delta a'_2, b'_1 + \delta b'_1, b'_2 + \delta b'_2, \\
\mu + \delta \mu, c' + \delta c') \\
- \mathcal{L}(a_1, a_2, b_1, b_2, a'_1, a'_2, b'_1, b'_2, \mu, c') \right] dz.
\] (81)

A Taylor series expansion to first order gives
\[
\delta s = \int_0^L (1 - \mu) \left( \frac{\partial g_1}{\partial a_1} \delta a_1 + \frac{\partial g_1}{\partial a'_1} \delta a'_1 \right) dz
+ \int_0^L (1 - \mu) \left( \frac{\partial g_1}{\partial b_1} \delta b_1 + \frac{\partial g_1}{\partial b'_1} \delta b'_1 \right) dz
+ \int_0^L \mu \left( \frac{\partial g_2}{\partial a_2} \delta a_2 + \frac{\partial g_2}{\partial a'_2} \delta a'_2 \right) dz
+ \int_0^L \mu \left( \frac{\partial g_2}{\partial b_2} \delta b_2 + \frac{\partial g_2}{\partial b'_2} \delta b'_2 \right) dz
- \int_0^L \mu \delta c' dz
- \int_0^L \left( c' - \sum_{k=1}^{2} (-1)^k g_k(a_k, b_k, a'_k, b'_k) \right) \delta \mu dz.
\] (82)

Note that because of the energy constraint (73), the coefficient of $\delta \mu$ is zero. We integrate the terms having $\delta a'_k, \delta b'_k$ and $\delta c'$ by parts and use the boundary conditions (77) and (80) to get
\[
\delta s = \int_0^L \left( (1 - \mu) \frac{\partial g_1}{\partial a_1} - \frac{d}{dz} \left( (1 - \mu) \frac{\partial g_1}{\partial a'_1} \right) \right) \delta a_1 \ dz
+ \int_0^L \left( (1 - \mu) \frac{\partial g_1}{\partial b_1} - \frac{d}{dz} \left( (1 - \mu) \frac{\partial g_1}{\partial b'_1} \right) \right) \delta b_1 \ dz
+ \int_0^L \mu \left( \frac{\partial g_2}{\partial a_2} - \frac{d}{dz} \left( \mu \frac{\partial g_2}{\partial a'_2} \right) \right) \delta a_2 \ dz
+ \int_0^L \mu \left( \frac{\partial g_2}{\partial b_2} - \frac{d}{dz} \left( \mu \frac{\partial g_2}{\partial b'_2} \right) \right) \delta b_2 \ dz
+ \int_0^L \mu' \delta c \ dz
+ (1 - \mu) \frac{\partial g_1}{\partial a'_1} \delta a_1 \bigg|_{z=L} + (1 - \mu) \frac{\partial g_1}{\partial b'_1} \delta b_1 \bigg|_{z=L}
+ \frac{\partial g_2}{\partial a'_2} \delta a_2 \bigg|_{z=L} + \mu \frac{\partial g_2}{\partial b'_2} \delta b_2 \bigg|_{z=L}.
\] (83)
If \(a^*_k(z), b^*_k(z), \mu^*(z)\) and \(c^*(z)\) are minimizers of the action \(s\), then

\[
\delta s \bigg|_{a_k=a^*_k, \ b_k=b^*_k, \ \mu=\mu^*, \ c=c^*} = 0. \quad (84)
\]

To have admissible variations, we must ensure that (77)–(80) are satisfied by all of the variations considered. We consider all \(\delta \mu\) for which \(\mu'\) is orthogonal to \(\delta c\). This allows us to pick arbitrary variations for \(a_k\) and \(b_k\) without violating the energy constraint (73). The boundary conditions (78)–(79) at \(z = L\) imply

\[
\delta a_1(L) = \delta a_2(L),
\]

and

\[
\delta b_1(L) = \delta b_2(L).
\]

The trick is now to pick variations in such a way that all but one of the terms in (83) vanish. For instance, consider all admissible variations for which

\[
\delta a_1(L) = 0
\]

and

\[
\delta a_2(z) = \delta b_1(z) = \delta b_2(z) = 0, \ z \in [0, L].
\]

We then have

\[
\int_0^L \left( (1 - \mu) \frac{\partial g_1}{\partial a_1} - \frac{d}{dz} \left( (1 - \mu) \frac{\partial g_1}{\partial a'_1} \right) \right) \delta a_1 \, dz = 0. \quad (85)
\]

From [25] Lemma 1 of Sec 1.3, we conclude that the integrand is zero, i.e.,

\[
(1 - \mu) \frac{\partial g_1}{\partial a_1} - \frac{d}{dz} \left( (1 - \mu) \frac{\partial g_1}{\partial a'_1} \right) = 0. \quad (86)
\]

With appropriate selection of variations, one can show that the other terms with integrals in (83) are zero. Thus, (83) is simplified and we get

\[
+(1 - \mu) \frac{\partial g_1}{\partial a_1} \delta a_1 \bigg|_{z=L} + \mu \frac{\partial g_2}{\partial a'_2} \delta a_1 \bigg|_{z=L} = 0 \quad (87)
\]

\[
+(1 - \mu) \frac{\partial g_1}{\partial b_1} \delta b_1 \bigg|_{z=L} + \mu \frac{\partial g_2}{\partial b'_2} \delta b_1 \bigg|_{z=L} = 0.
\]

If we consider those variations for which

\[
\delta a_1(L) = 0,
\]
from (87) we get
\[
\left. \left( (1 - \mu) \frac{\partial g_1}{\partial b_1} + \mu \frac{\partial g_2}{\partial b_2'} \right) \right|_{z=L} = 0. \quad (88)
\]
Similarly, one can get
\[
\left. \left( (1 - \mu) \frac{\partial g_1}{\partial a_1} + \mu \frac{\partial g_2}{\partial a_2'} \right) \right|_{z=L} = 0. \quad (89)
\]
If we consider variations \( \delta \mu \) for which \( \mu' \) is not necessarily orthogonal to \( \delta c \), we can now consider arbitrary variations \( \delta c \) and, with similar argument as in the previous paragraph, we must have
\[
\int_0^L \mu' \delta c = 0. \quad (90)
\]
Again, from [25, Lemma 1 of Sec 1.3], we conclude that
\[
\mu' = 0, \quad (91)
\]
i.e., the optimal Lagrange multiplier is a constant (as expected). Let \( \mu^*(z) = \lambda \). The Euler-Lagrange conditions can now be simplified. These equations, together with the required energy constraint, become
\[
(1 - \lambda) \left( \frac{\partial g_1}{\partial a_1} - \frac{d}{dz} \left( \frac{\partial g_1}{\partial a_1'} \right) \right) = 0,
\]
\[
(1 - \lambda) \left( \frac{\partial g_1}{\partial b_1} - \frac{d}{dz} \left( \frac{\partial g_1}{\partial b_1'} \right) \right) = 0,
\]
\[
\lambda \left( \frac{\partial g_2}{\partial a_2} - \frac{d}{dz} \left( \frac{\partial g_2}{\partial a_2'} \right) \right) = 0,
\]
\[
\lambda \left( \frac{\partial g_2}{\partial b_2} - \frac{d}{dz} \left( \frac{\partial g_2}{\partial b_2'} \right) \right) = 0,
\]
\[
c'(z) = \sum_{k=1}^2 (-1)^k g_k(a_k, b_k, a'_k, b'_k). \quad (92)
\]
The required boundary conditions are

\[ a_k(0) + ib_k(0) = x_k, \]
\[ a_1(L) = a_2(L), \]
\[ b_1(L) = b_2(L), \]
\[ c(0) = c(L) = 0, \]
\[ \left. \left( (1 - \lambda) \frac{\partial g_1}{\partial a_1'} + \lambda \frac{\partial g_2}{\partial a_2'} \right) \right|_{z=L} = 0, \]
\[ \left. \left( (1 - \lambda) \frac{\partial g_1}{\partial b_1'} + \lambda \frac{\partial g_2}{\partial b_2'} \right) \right|_{z=L} = 0. \]

(93)

There are four differential equations of second order and one of first order in (92). There are also ten boundary conditions in (93) together with one unknown \( \lambda \). Hence, at least in principle, it is possible to solve these equations.

From these, after some algebraic manipulation, one obtains the equations given in Theorem 4.

Appendix E

Proof of Theorem 6

Consider the state equation in polar coordinates \( q(z) = R(z)e^{i\theta(z)} \), with \( q(0) = x \) and \( q(L) = y \), and consider a control \( n(z) \) with the functional form

\[ n(z) = Ce^{i\theta(z)}, \]

(94)

where \( C = a + ib \) is a complex constant. With this control, the state equation in polar coordinates becomes

\[ R' = a, \]
\[ \theta' = \gamma R^2 + \frac{b}{R}. \]

(95) \hspace{1cm} (96)

Solving (95) with the boundary conditions \( R(0) = |x| \) and \( R(y) = |y| \), we get

\[ R(z) = \frac{|y| - |x|}{L} z + |x|. \]

(97)

In particular,

\[ a = \frac{|y| - |x|}{L}. \]

(98)
Having $R(z)$, one can use (96) to solve for $\theta(z)$. It is straightforward to show that we get
\[
b^2 = \left( \frac{|y| - |x|}{L \ln \left( \frac{|y|}{|x|} \right)} \right)^2 \Delta^2(x, y)
\] (99)

where
\[
\Delta(x, y) = \left( \arg \frac{y}{x} - \frac{\gamma L |y|^3 - |x|^3}{3 |y| - |x|} \right) \pmod{2\pi}.
\] (100)

Here, the operator $\pmod{2\pi}$ returns an angle in $[-\pi, \pi)$.

It follows that the minimum energy needed to move $x$ to $y$ with a control of the form (94) is
\[
\int_0^L |n(z)|^2 dz = \int_0^L a^2 + b^2 dz
\] (101)
\[
= \frac{(|y| - |x|)^2}{L} \left[ 1 + \left( \frac{\Delta(x, y)}{\ln(|y|/|x|)} \right)^2 \right].
\] (102)

In case of singularities, (102) is understood as a limit—these are $|x| \to |y|, |x| \to 0$ or $|y| \to 0$.

Note that if we pick any final state $y$ such that the adversary requires at most the effort $E$ for going to $y$ from both $x_1, x_2$, then $E$ is an upper bound for $d(x_1, x_2)$. Hence, $d(x_1, x_2)$ is upper bounded by
\[
\min_y \max_k \left\{ \frac{(|y| - |x_k|)^2}{L} \left[ 1 + \left( \frac{\Delta(x_k, y)}{\ln(|y|/|x_k|)} \right)^2 \right] \right\}.
\]

APPENDIX F
PROOF OF THEOREM 7

Let $x_1 = x$ and $x_2 = xe^{i\phi}$. Consider the control acting on $x_1$ of the form
\[
n_1(z) = -a \left( z - \frac{L}{2} \right) e^{i\theta_1(z)},
\] (103)

where $a$ is a positive real number. Let the control acting on $x_2$ be just $n_2 = 0$. After some straightforward algebra, one can find a solution for $a$ that satisfies
\[
a = O \left( \frac{1}{|x|} \right).
\] (104)

and results in
\[
q_1(L) = q_2(L).
\] (105)

\[\text{Here, } O \text{ represents Landau’s big O notation.}\]
Therefore, the adversarial effort for $n_1$ is

$$E_1 = \mathcal{O}\left(\frac{1}{|x|^2}\right). \quad (106)$$

Note that with this choice of adversarial noise, we have

$$B_{E_1}(x_1) \cap B_0(x_2) \neq \emptyset. \quad (107)$$

Therefore, the noise balls of the points $x_1$ and $x_2$ with an effort

$$E = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad (108)$$

intersect. The result follows by allowing

$$|x| \to \infty. \quad (109)$$

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