Orthogonality derivation of polarization state based on Poincaré sphere

Zhanling Wang¹,², Jiankai Huang¹ and Chen Pang¹

¹State Key Laboratory of Complex Electromagnetic Environment Effects on Electronics and Information System, College of Electronic Science and Technology, National University of Defense Technology, Changsha, China
²Corresponding author’s e-mail: zlwangnudt@outlook.com

Abstract. To make full use of the polarization information of electromagnetic waves for the polarimetric sensor array, we derive the orthogonality expression of the polarization state based on the Poincaré sphere (PS) representation. The ellipticity angle and the orientation angle of the ellipse are used to describe the polarization ellipse and the point on the surface of the Poincaré sphere. The derivation shows that two points that characterize a pair of orthogonal polarization states are distributed in the opposite positions. Compared with the conventional linear and circular polarization state, the flexible elliptical polarization state would result in the maximum separation of co-polarization and cross-polarization components, thereby enhancing the accuracy of polarization information measurement. The simulation result displays the correctness of the theoretical derivation, which could provide a basis for the further polarization state configuration.

1. Introduction

The linear and circular polarization states are commonly used in the current polarimetric sensor system [2]. For the single sensor system, the fixed polarization states above work well. However, it is not ideal for the polarimetric sensor array owing to its electronic beam steering capability [3][4]. The polarization profile of the electronically scanned beam is direction-dependent which is different from the single sensor. Specifically, one of the impacts is the rise of the cross-polarization component [5]. This will be detrimental to the acquisition accuracy of the polarization information.

To address this problem, a candidate scheme is to select the optimum polarization state according to the specific beam direction [6]. However, a single polarization state is not enough for the spatial scanning application. The requirement of obtaining the polarization information is to utilize a pair of polarization states which are orthogonal to each other [7]. The orthogonal polarization states could separate the co-polarization and the cross-polarization components of the target echo as much as possible. Therefore, the orthogonality formula of the polarization states should be deduced, which is applicable for the arbitrary polarization state.

In this paper, we derive the orthogonality expression of arbitrary polarization state based on the Poincaré sphere (PS) representation [8][9]. With the help of polarization ellipse, we use the parameters including the ellipticity angle and the orientation angle to formulate the polarization ratio. A pair of orthogonal polarization states corresponding to two different polarization ratios are represented by the opposite points on the Poincaré sphere. Then, the problem is converted to determine two points which have orthogonal polarization states. The simulation results reveal the correctness of the theoretical
derivation, which could provide the basis for polarization state configuration for the polarimetric sensor array application.

2. Orthogonality derivation of polarization state

Owing to the performance, low cost, and ease of integration, the microstrip patch antenna has been a promising candidate for the polarimetric sensor array. As shown in Figure 1, two microstrip patch antennas are taken as examples to study the orthogonality of two polarization states.

![Figure 1. Two microstrip patch antennas. (a) antenna $A_x$, (b) antenna $A_y$.](image1)

![Figure 2. Polarization ellipses corresponding to the microstrip patch antennas in Figure 1. (a) right-hand elliptical polarization (RHEP) radiated from antenna $A_x$, (b) left-hand elliptical polarization (LHEP) radiated from antenna $A_y$.](image2)

As depicted in Figure 1, the left microstrip patch antenna $A_x$ denotes the co-polarization component receiving antenna, and the right microstrip patch antenna $A_y$ denotes the cross-polarization component receiving antenna. The polarization states from $A_x$ and $A_y$ should be orthogonal to each other. The corresponding polarization ellipses are shown in Figure 2.

Let antenna $A_x$ radiate the RHEP wave as depicted in Figure 2. We assume that the excitation amplitude of the horizontally feed point 1 of antenna $A_x$ is $A_{x1}$, and the excitation amplitude of the vertically feed point 2 of antenna $A_x$ is $A_{x2}$. The amplitude ratio is expressed using $p_x$, and $p_x = A_{x2}/A_{x1}$. In the polarization characterizing, the amplitude ratio is also written as $p_x = \tan \gamma_x$ where $\gamma_x$ denotes the polarization angle. Moreover, the excitation phase of the horizontally feed point 1 of antenna $A_x$ is $\phi_x$, and the excitation phase of the vertically feed point 2 of antenna $A_x$ is $\phi_{x2}$. The phase difference $\phi_x$ is $\phi_x = \phi_{x2} - \phi_{x1}$ where $-\pi < \phi_x < 0$. The point representing the RHEP wave is on the surface of the southern hemisphere. Thus, the superimposed RHEP wave can be given by

$$\mathbf{E}_{R_H} = a_x + p_x e^{i\phi_x} a_y$$

(1)

where $a_x$ and $a_y$ are the unit vectors along the x-axis and y-axis in Figure 2, respectively.

The formula (1) can be rewritten in vector form as

$$\mathbf{E}_{R_H} = \begin{bmatrix} 1 & p_x e^{i\phi_x} \end{bmatrix}^T$$

(2)
In order to make the polarization state of antenna $A_x$ be orthogonal to that of antenna $A_y$, the antenna $A_x$ should radiate the LHEP wave. The derivation method to determine the excitation of antenna $A_x$ is given below.

According to Poincaré’s representation, a polarization state can be described as a point on the surface of a unit sphere [8]. The longitude of the point is taken as $2\beta$, and the latitude of the point is taken as $2\alpha$, as shown in Figure 3.

![Figure 3. Polarization representation on Poincaré sphere.](image)

Polarization ratio can be expressed as $P = Y/X$ where $X$ and $Y$ are the complex excitation voltage of the horizontally feed point 1 and vertically feed point 2, respectively. The polarization ratio is also written in the form $P = \tan \gamma e^{\varphi}$ where $\varphi = \varphi_x - \varphi_c$.

The spherical trigonometry formula is expressed as [8]

$$\begin{align*}
\cos 2\gamma &= \cos 2\alpha \cos 2\beta \\
\tan \varphi &= \tan 2\alpha \csc 2\beta
\end{align*}$$

where $\alpha$ denotes the ellipticity angle, and $\beta$ denotes the orientation angle.

Accordingly, the formula for the antenna $A_y$ radiating RHEP wave can be written where $-\pi < \varphi_x < 0$ and $-\pi/4 < \alpha_x < 0$. The point is on the surface of the southern hemisphere. Similarly, the formula for the antenna $A_x$ radiating LHEP wave is given by

$$\begin{align*}
\cos 2\gamma_x &= \cos 2\alpha_x \cos 2\beta_x \\
\tan \varphi_x &= \tan 2\alpha_x \csc 2\beta_x
\end{align*}$$

where $0 < \varphi_x < \pi$ and $0 < \alpha_x < \pi/4$. The point is on the surface of the northern hemisphere.

Two polarization states which are represented by the opposite point on the sphere are known to be orthogonal [8]. According to Figure 3, it can be deduced that the parameters relation can be written as

$$\begin{align*}
\alpha_x &= -\alpha_c \\
\beta_x &= \beta_c + \pi/2
\end{align*}$$

Due to that the magnitude $\tan \gamma$ of polarization ratio $P$ cannot be negative, the polarization angle $\gamma$ should meet that $0 < \gamma_x < \pi/2$ and $0 < \gamma_x < \pi/2$. The formula can be rewritten as

$$\begin{align*}
\cos 2\gamma_x &= \cos(-2\alpha_x) \cos(2\beta_x + \pi) = -\cos 2\alpha_x \cos 2\beta_c = -\cos 2\gamma_x = \cos(\pi - 2\gamma_x) \\
\tan \varphi_x &= \tan(-2\alpha_x) \csc(2\beta_c + \pi) = (-\tan 2\alpha_x)(-\csc 2\beta_c) = \tan 2\alpha_x \csc 2\beta_c = \tan \varphi_c
\end{align*}$$

Thus,

$$\begin{align*}
2\gamma_x &= \pi - 2\gamma_c \\
\varphi_x &= \varphi_c + \pi
\end{align*}$$

The polarization ratio of RHEP is $P_x = \tan \gamma_x e^{i\varphi_x}$, and the polarization ratio of LHEP is expressed as...
\[ P_x = \tan \gamma_x e^{i\varphi_x} = \tan(\pi / 2 - \gamma_x) e^{i(\phi_x + \psi)} = -e^{i\varphi} \]  

Let \( p = \tan \gamma \), it has been apparent that \( p_x = 1 / p_c \). The LHEP wave can be given by

\[ E_{tx} = a_n + p_x e^{i\varphi} a_e = a_n - \frac{1}{p_c} e^{i\varphi} a_e \]  

The formula (9) can be rewritten in vector form as

\[ E_{tx} = \left[1 \quad p_x e^{i\varphi}\right] = \left[1 \quad -\frac{1}{p_c} e^{i\varphi}\right]^T \]  

The inner product of two complex vectors \( E_{tx}^R \) and \( E_{tx}^L \) is utilized to verify the orthogonality of formulas (2) and (10).

\[ \{E_{tx}^R, E_{tx}^L\} = E_{tx}^{R*} E_{tx}^L = \left[1 \quad p_x e^{i\varphi}\right] \left[\frac{1}{p_c} e^{i\varphi}\right] = 0 \]  

Therefore, the RHEP wave radiated by antenna \( A_c \) and the LHEP wave radiated by antenna \( A_c \) are always orthogonal to each other.

To facilitate the expression of the orthogonal polarization state, we unitize the two complex vectors \( E_{tx}^R \) and \( E_{tx}^L \). The modulus of \( E_{tx}^R \) can be written as \( |E_{tx}^R| = \|E_{tx}^R\| = \sqrt{1 + p_c^2} \) where \( \|E\| \) denotes the 2-norm of a complex vector. Then, the unit vector \( e_{tx}^R \) along with \( E_{tx}^R \) is

\[ e_{tx}^R = \frac{a_n + p_x e^{i\varphi} a_e}{\sqrt{1 + p_c^2}} \]  

Similarly, \( |E_{tx}^L| = \|E_{tx}^L\| = \sqrt{1 + \frac{1}{p_c^2}} \), and the unit vector \( e_{tx}^L \) of \( E_{tx}^L \) is

\[ e_{tx}^L = \frac{p_x a_n - e^{i\varphi} a_e}{\sqrt{1 + \frac{1}{p_c^2}}} \]  

A set of orthogonal bases is composed of \( e_{tx}^R \) and \( e_{tx}^L \). It could be written in matrix form as

\[ \begin{bmatrix} e_{tx}^R \\ e_{tx}^L \end{bmatrix} = \frac{1}{\sqrt{1 + p_c^2}} \begin{bmatrix} 1 & p_x e^{i\varphi} \\ p_x e^{i\varphi} & -e^{i\varphi} \end{bmatrix} \begin{bmatrix} a_n \\ a_e \end{bmatrix} \]  

3. Simulation results

The simulation is made to verify the effectiveness of the theoretical derivation method. The Poincaré representation method is to obtain a pair of points on the surface of a unit sphere, while the two points are origin-symmetric. For the antenna implementation, the problem is to modulate the amplitude ratio and phase difference of the feed points 1 and 2. According to the formula (14), the simulation is conducted, as shown in Figure 4. Figure 4(a) displays the case when the excitation parameter of antenna \( A \) is \( p_c = 0.5, \varphi_c = \pi / 6 \). And Figure 4(b) displays the case when that of antenna \( A \) is \( p_c = 2, \varphi_c = -5\pi / 6 \). The simulation results reveal that the two points are origin-symmetric. Therefore, the derivation based on the Poincaré sphere is feasible. The Poincaré representation method is effective to determine the orthogonal polarization state.
Figure 4. The orthogonality representation of the polarization state on the Poincaré sphere when (a) $p_c = 0.5$, $\varphi_c = \pi/6$, (b) $p_c = 2$, $\varphi_c = -5\pi/6$.

In order to further assess the derivation of the orthogonal polarization state, the orthogonality distribution along with the amplitude ratio $p_c$ and phase difference $\varphi_c$ is shown in Figure 5. The orthogonality corresponding to the region indicated by the black arrow is close to zero, thereby having better orthogonality for the two polarization states. The excitation parameter in Figure 5(a) is that $p_c$ rounds towards 2, and $\varphi_c$ towards $-5\pi/6$. Figure 5(b) calculates the issue when $p_c = 2$, $\varphi_c = -5\pi/6$. The corresponding excitation parameter is that $p_c$ rounds towards 0.5, and $\varphi_c$ towards $\pi/6$. The polarization parameters are consistent with that deduced in Figure 4. The physical meaning of z axis in Figure 5 is the inner product between $E_{s_{z}}$ or $E_{s_{z}}$ and $e_{s_{z}}$ or $e_{s_{z}}$.

Figure 5. The orthogonality distribution along with the amplitude ratio $p_c$ and phase difference $\varphi_c$. (a) $p_c = 0.5$, $\varphi_c = \pi/6$, (b) $p_c = 2$, $\varphi_c = -5\pi/6$.

4. Conclusion

In this paper, the derivation method of the orthogonal polarization state is presented. The theoretical deduction and simulation results certify that the Poincaré representation method is feasible. The orthogonal polarization states can be characterized by the opposite points on the surface of the Poincaré sphere, which is described by the longitude $2\beta$ and latitude $2\alpha$. The simulation results reveal the correctness of the orthogonality derivation, which would be helpful to reduce the cross-polarization level, thereby enhancing the accuracy of polarization information measurement for the polarimetric sensor array.

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