THE GRAPHS OF NON-DEGENERATE LINEAR CODES

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ABSTRACT. We consider the Grassmann graph of $k$-dimensional subspaces of an $n$-dimensional vector space over the $q$-element field and its subgraph $\Gamma(n,k)_q$ which consists of all non-degenerate linear $[n,k]_q$ codes. We assume that $1 < k < n - 1$. It is well-known that every automorphism of the Grassmann graph is induced by a semilinear automorphism of the corresponding vector space or a semilinear isomorphism to the dual vector space and the second possibility is realized only for $n = 2k$. Our result is the following: if $q \geq 3$ or $k \neq 2$, then every isomorphism of $\Gamma(n,k)_q$ to a subgraph of the Grassmann graph can be uniquely extended to an automorphism of the Grassmann graph; in the case when $q = k = 2$, there are subgraphs of the Grassmann graph isomorphic to $\Gamma(n,k)_q$ and such that isomorphisms between these subgraphs and $\Gamma(n,k)_q$ cannot be extended to automorphisms of the Grassmann graph.

1. Introduction

Grassmann graphs are interesting for many reasons. First of all, these are classical examples of distance regular graphs [1]. Also, they are closely connected to one class of Tits buildings [2, 11, 14, 15]. The first significant result concerning Grassmann graphs was Chow’s theorem [4] which describes their automorphisms. A survey of results in that spirit can be found in [12]. Recently, Chow’s theorem was used to prove some Wigner-type theorems for Hilbert Grassmannians [5, 6, 13].

The Grassmann graph formed by $k$-dimensional subspaces of an $n$-dimensional vector space over the $q$-element field can be considered as the graph of linear $[n,k]_q$ codes, where two codes are connected by an edge if they have the maximal number of common codewords (we assume that $1 < k < n - 1$, since in the cases $k = 1, n - 1$ the Grassmann graph is complete). In practice, only non-degenerate linear codes are useful (a code is non-degenerate if every coordinate functional is not vanished on this code). For a degenerate linear $[n,k]_q$ code every generator matrix contains zero column which means that we have, in fact, a linear $[n - 1,k]_q$ code.

The graph (the subgraph of the Grassmann graph) formed by non-degenerate linear $[n,k]_q$ codes is investigated in [7, 8]. By [7], the distances in this graph coincide with the distances in the Grassmann graph if and only if

$$n < (q + 1)^2 + k - 2;$$

maximal cliques and automorphisms are determined in [8]. Subgraphs of the Grassmann graph formed by different families of linear codes are considered in [3, 9, 10].

By Chow’s theorem, every automorphism of the Grassmann graph is induced by a semilinear automorphism of the corresponding vector space or a semilinear isomorphism to the dual vector space and the second possibility is realized only for
We show that for the case when \( q \geq 3 \) or \( k \geq 3 \) every subgraph \( \Gamma \) of the Grassmann graph isomorphic to the graph of non-degenerate linear \([n,k]_q\) codes can be obtained form this graph by an automorphism of the Grassmann graph; in other words, there is a coordinate system such that \( \Gamma \) is the graph of non-degenerate codes or \( n = 2k \) and \( \Gamma \) is formed by all codes dual to non-degenerate codes.

In the case when \( q = k = 2 \), we construct a counterexample showing that the above statement fails.

## 2. Result

Let \( V \) be an \( n \)-dimensional vector space over a field \( \mathbb{F} \) (not necessarily finite) and let \( \mathcal{G}_k(V) \) be the Grassmannian consisting of all \( k \)-dimensional subspaces of \( V \). Two \( k \)-dimensional subspaces of \( V \) are called adjacent if their intersection is \((k-1)\)-dimensional, or equivalently, their sum is \((k+1)\)-dimensional. The Grassmann graph \( \Gamma_k(V) \) is the simple graph whose vertex set is \( \mathcal{G}_k(V) \) and two vertices are connected by an edge in this graph if the corresponding \( k \)-dimensional subspaces are adjacent. In the cases when \( k = 1, n-1 \), any two distinct \( k \)-dimensional subspaces are adjacent. The Grassmann graph is connected. Chow’s theorem \([4]\) provides a description of automorphisms of \( \Gamma_k(V) \) for \( 1 < k < n-1 \) (if \( k = 1, n-1 \), then every bijective transformation of \( \mathcal{G}_k(V) \) is a graph automorphism).

Recall that a transformation \( l : V \rightarrow V \) is called semilinear if

\[
l(x + y) = l(x) + l(y)
\]

for all \( x, y \in V \) and there is a homomorphism \( \sigma : \mathbb{F} \rightarrow \mathbb{F} \) such that

\[
l(ax) = \sigma(a)l(x)
\]

for all \( a \in \mathbb{F} \) and \( x \in V \). In the case when \( l \) is bijective and \( \sigma \) is a field automorphism, we say that \( l \) is a semilinear automorphism of \( V \). Every semilinear automorphism of \( V \) induces an automorphism of the graph \( \Gamma_k(V) \). Let \( \omega \) be a non-degenerate symmetric form on \( V \). For every subspace \( X \subset V \) we denote by \( X^\perp \) the orthogonal complement of \( X \) associated to the form \( \omega \). Two subspaces \( X, Y \subset V \) of the same dimension are adjacent if and only if their orthogonal complements \( X^\perp, Y^\perp \) are adjacent, i.e. the orthocomplementary map \( X \rightarrow X^\perp \) provides an isomorphism between \( \Gamma_k(V) \) and \( \Gamma_{n-k}(V) \); this is an automorphism of \( \Gamma_k(V) \) if \( n = 2k \). Chow’s theorem states that every automorphism of \( \Gamma_k(V) \), \( 1 < k < n-1 \) is induced by a semilinear automorphism of \( V \) or it is the composition of the orthocomplementary map and a graph automorphism induced by a semilinear automorphism of \( V \); the second possibility is realized only for \( n = 2k \).

Suppose that \( \mathbb{F} = \mathbb{F}_q \) is the finite field consisting of \( q \) elements and \( V = \mathbb{F}^n \). The standard basis of \( V \) is formed by the vectors

\[
\mathbf{e}_1 = (1,0,\ldots,0), \mathbf{e}_2 = (0,1,0,\ldots,0), \ldots, \mathbf{e}_n = (0,\ldots,0,1).
\]

Let \( C_i \) be the kernel of the \( i \)-th coordinate functional \( (x_1, \ldots, x_n) \rightarrow x_i \), i.e. the hyperplane of \( V \) spanned by all \( \mathbf{e}_j \) with \( j \neq i \). Every \( k \)-dimensional subspace of \( V \) is a linear \([n,k]_q\) code. Such a code \( C \) is non-degenerate if the restriction of every coordinate functional to \( C \) is non-zero, i.e. \( C \) is not contained in any coordinate hyperplane \( C_i \). \([16]\) All non-degenerate linear \([n,k]_q\) codes form the set

\[
C(n,k)_q = \mathcal{G}_k(V) \setminus \left( \bigcup_{i=1}^n \mathcal{G}_k(C_i) \right).
\]
Denote by $\Gamma(n, k)_q$ the restriction of the Grassmann graph $\Gamma_k(V)$ to the set $\mathcal{C}(n, k)_q$, i.e. $\Gamma(n, k)_q$ is the simple graph whose vertex set is $\mathcal{C}(n, k)_q$ and two vertices are connected by an edge if the corresponding $k$-dimensional subspaces are adjacent. This graph is connected [7]. By [8], every automorphism of $\Gamma(n, k)_q$, $1 < k < n - 1$ is induced by a monomial semilinear automorphism of $V$ (a semilinear automorphism sending every $e_i$ to a scalar multiple of $e_j$).

Our main result is the following.

**Theorem 1.** Suppose that $1 < k < n - 1$. If

\[
q \geq 3 \text{ or } k \geq 3,
\]

then every isomorphism of $\Gamma(n, k)_q$ to a subgraph of $\Gamma_k(V)$ can be uniquely extended to an automorphism of $\Gamma_k(V)$. In the case when $q = k = 2$, there are subgraphs of $\Gamma_k(V)$ isomorphic to $\Gamma(n, k)_q$ and such that isomorphisms between these subgraphs and $\Gamma(n, k)_q$ cannot be extended to automorphisms of $\Gamma_k(V)$.

Note that the subgraphs considered in Theorem 1 are not supposed to be induced subgraphs.

**Remark 2.** If $k = 1, n - 1$, then every injective map of $\mathcal{C}(n, k)_q$ to $\mathcal{G}_k(V)$ is an isomorphism of $\Gamma(n, k)_q$ to a subgraph of $\Gamma_k(V)$ and every bijective transformation of $\mathcal{G}_k(V)$ is an automorphism of $\Gamma_k(V)$. In this case, each isomorphism of $\Gamma(n, k)_q$ to a subgraph of $\Gamma_k(V)$ can be extended to an automorphism of $\Gamma_k(V)$, but such an extension is not unique.

**Remark 3.** Let us take a basis of $V$ and consider the hyperplanes $H_1, \ldots, H_n \subset V$ spanned by vectors from this basis. The restriction of $\Gamma_k(V)$ to the set

\[
\mathcal{G}_k(V) \setminus \left( \bigcup_{i=1}^{n} \mathcal{G}_k(H_i) \right)
\]

is isomorphic to $\Gamma(n, k)_q$. If $1 < k < n - 1$ and (1) holds, then Theorem 1 states that every subgraph $\Gamma$ of $\Gamma_k(V)$ isomorphic to $\Gamma(n, k)_q$ can be obtained from $\Gamma(n, k)_q$ by an automorphism of $\Gamma_k(V)$, i.e. $\Gamma$ is the restriction of $\Gamma_k(V)$ to a subset of type (2) or $n = 2k$ and $\Gamma$ is the restriction of $\Gamma_k(V)$ to the set formed by the orthogonal complements of elements from a subset of type (2).

**Remark 4.** Suppose that $V = \mathbb{F}^n$ and $\mathbb{F}$ is an infinite field. The direct analogue of the first part of Theorem 1 does not hold for this case. This is related to the fact that $\Gamma_k(V)$ can contain proper subgraphs isomorphic to $\Gamma_k(V)$. Consider two examples. If $\sigma$ is a non-surjective endomorphism of $\mathbb{F}$ (such endomorphisms exist, for example, for the field of complex numbers), then every $\sigma$-linear embedding of $V$ to itself sending bases of $V$ to bases (strong semilinear embedding) induces an isomorphism of $\Gamma_k(V)$ to a proper subgraph of $\Gamma_k(V)$; see [12] Chapter 3 for examples of more complicated embeddings of $\Gamma_k(V)$ to itself. Suppose that $\mathbb{F} = \mathbb{R}$ and take any bijection $g$ of $\mathcal{G}_k(V)$ to a maximal clique $\mathcal{C}$ of $\Gamma_k(V)$ (such bijections exist, since $\mathcal{G}_k(V)$ and $\mathcal{C}$ are of the same cardinality). Consider the subgraph of $\Gamma_k(V)$ whose vertex set is $\mathcal{C}$ and $X, Y \in \mathcal{C}$ are connected by an edge if and only if $g^{-1}(X), g^{-1}(Y)$ are connected by an edge in $\Gamma_k(V)$. All isometric embeddings of Grassmann graphs (embedding preserving the path distance) are known, they are induced by semilinear embeddings of special type [12] Section 3.4. A classification of non-isometric embeddings is an open problem [12] Chapter 3.
3. Preliminaries

3.1. Maximal cliques in the Grassmann graph. Recall that a subset in the vertex set of a graph is called a clique if any two vertices from this subset are connected by an edge. A clique \( C \) is said to be maximal if every clique containing \( C \) coincides with \( C \).

If \( 1 < k < n - 1 \), then every maximal clique of \( \Gamma_k(V) \) is of one of the following types:

- the star \( S(X), X \in G_{k-1}(V) \) which consists of all \( k \)-dimensional subspaces containing \( X \);
- the top \( T(Y) = G_k(Y), Y \in G_{k+1}(V) \).

An \( m \)-dimensional vector space \( W \) over \( F \) contains precisely \( \frac{q^m}{q-1} = q^m - q^{m-1} + \cdots + q + 1 \) 1-dimensional subspaces; this is also the number of hyperplanes in \( W \). Since \( S(X), X \in G_{k-1}(V) \) can be identified with \( G_1(V/X) \), every star of \( \Gamma_k(V) \) consists of \( \frac{n-k+1}{q} \) elements. Every top of \( \Gamma_k(V) \) contains precisely \( \frac{k+1}{q} \) elements. Stars and tops have the same number of elements if and only if \( n = 2k \).

The intersection of two distinct maximal cliques of the same type contains at most one element. This intersection is non-empty if and only if the associated \((k-1)\)-dimensional or \((k+1)\)-dimensional subspaces are adjacent.

For a \((k-1)\)-dimensional subspace \( X \subset V \) and a \((k+1)\)-dimensional subspace \( Y \subset V \) the intersection of the associated star and top of \( \Gamma_k(V) \) is non-empty if and only if \( X \subset Y \). In this case, \( S(X) \cap T(Y) \) contains precisely \( q+1 \) elements. Every such intersection is called a line of \( G_k(V) \).

3.2. Maximal cliques in the graph of non-degenerate linear codes. As in the previous subsection, we assume that \( 1 < k < n - 1 \). For a \((k-1)\)-dimensional subspace \( X \subset V \) and a \((k+1)\)-dimensional subspace \( Y \subset V \) we denote by \( S^c(X) \) and \( T^c(Y) \) the intersection of \( C(n,k)_q \) with the star \( S(X) \) and the top \( T(Y) \), respectively. Each of these intersections is a clique of \( \Gamma(n,k)_q \) (if it is non-empty), but such a clique need not to be maximal.

Denote by \( c(X) \) the number of the coordinate hyperplanes containing a subspace \( X \subset V \). If \( X \in C(n,k-1)_q \), then \( S^c(X) = S(X) \) is a maximal clique of \( \Gamma_k(V) \) and, consequently, it is a maximal clique of \( \Gamma(n,k)_q \). If \( X \) is a \((k-1)\)-dimensional subspace which does not belong to \( C(n,k-1)_q \), then

\[ 0 < c(X) \leq n - k + 1 \]

and, by [8, Lemma 1],

\[ |S^c(X)| = (q - 1)^{c(X)-1}q^{n-k-c(X)+1} \]  \hspace{1cm} \text{(the equality does not hold if } c(X) = 0 \).  \hspace{1cm} \text{(3)}

Proposition 5. For a \((k-1)\)-dimensional subspace \( X \subset V \) the following assertions are fulfilled:

(i) If \( q \geq 3 \), then \( S^c(X) \) is a maximal clique of \( \Gamma(n,k)_q \).
(ii) For $q = 2$ the set $S^c(X)$ is a maximal clique of $\Gamma(n, k)_q$ if and only if
$$c(X) \leq n - k - 1.$$ 

(iii) If $S^c(X)$ is a maximal clique of $\Gamma(n, k)_q$, then there is no $(k+1)$-dimensional subspace $Y \subset V$ such that $S^c(X) = T^c(Y)$.

Proof. The statements (i) and (ii) are Propositions 1 and 2 from [8].

(iii). If $X$ belongs to $C(n, k-1)_q$, then $S^c(X) = S(X)$ is a star of $\Gamma_k(V)$, i.e. it is not contained in a top of $\Gamma_k(V)$. Consider the case when $X \notin C(n, k-1)_q$.

Suppose that $q \geq 3$. By [3],
$$|S^c(X)| \geq (q-1)^{n-k} \geq (q-1)^2 = q^2 - 2q + 1 \geq 3q - 2q + 1 = q + 1.$$ 

If $S^c(X) = T^c(Y)$ for a certain $(k+1)$-dimensional subspace $Y \subset V$, then $S^c(X)$ is contained in the line $S(X) \cap T(Y)$. Since $X \notin C(n, k-1)_q$, there is a coordinate hyperplane $C_i$ containing $X$. The subspace $Y \cap C_i$ is $k$-dimensional ($T^c(Y)$ is empty if $Y \subset C_i$); this is an element of the line $S(X) \cap T(Y)$ which do not belong to $C(n, k)_q$. Hence $S^c(X) = T^c(Y)$ contains not greater than $q$ elements, a contradiction.

If $q = 2$, then $c(X) \leq n - k - 1$ and [3] shows that
$$|S^c(X)| \geq q^2 = 4 > 3 = q + 1$$

which means that $S^c(X)$ is not contained in a line of $G_k(V)$. 

We say that $S^c(X)$ is a star of $\Gamma(n, k)_q$ only in the case when it is a maximal clique of $\Gamma(n, k)_q$. If $X$ belongs to $C(n, k-1)_q$, then $S^c(X) = S(X)$ is said to be a maximal star of $\Gamma(n, k)_q$.

Remark 6. If $q = 2$ and $S^c(X)$ is not a maximal clique of $\Gamma(n, k)_q$, then one of the following possibilities is realized:

- $c(X) = n - k$ and $S^c(X)$ contains precisely two elements;
- $c(X) = n - k + 1$ and $S^c(X)$ consists of a unique element.

See the proof of Proposition 2 from [8] for the details.

Let $Y$ be a $(k+1)$-dimensional subspace of $V$. Then $T^c(Y)$ is empty if $Y$ does not belong to $C(n, k+1)_q$. Consider the case when $Y \in C(n, k+1)_q$. For every $i \in \{1, \ldots, n\}$ the $k$-dimensional subspace $Y \cap C_i$ does not belong to $T^c(Y)$. Some of the subspaces
$$Y \cap C_1, \ldots, Y \cap C_n$$

may be coincident and we denote by $n(Y)$ the number of distinct elements in this collection. Then
$$k + 1 \leq n(Y) \leq n$$

(the first inequality follows from the fact that $Y^*$ is spanned by the restrictions of the coordinate functionals to $Y$) and $T^c(Y)$ contains precisely
$$[k + 1]_q - n(Y)$$
elements. We say that $T^c(Y)$ is a top of $\Gamma(n, k)_q$ if it is a maximal clique of $\Gamma(n, k)_q$. In this case, there is no $(k-1)$-dimensional subspace $X \subset V$ such that $T^c(Y) = S^c(X)$ (the statement (iii) of Proposition [8]).

Proposition 7. The following two conditions are equivalent:

- $T^c(Y)$ is a top of $\Gamma(n, k)_q$ for every $Y \in C(n, k+1)_q$;
\[ |k + 1|_q - (q + 1) > n. \]

**Proof.** See [8, Corollary 1]. \[\Box\]

Since
\[ |k + 1|_q - (q + 1) = q^k + \cdots + q^2 \geq q^k \geq 2k, \]
the second condition from Proposition 7 holds, for example, if \( 2k > n \).

**Remark 8.** In the case when the second condition from Proposition 7 fails, there are \( Y \in C(n, k + 1)_q \) such that \( T^c(Y) \) is a proper subset in a certain star of \( \Gamma(n, k)_q \) or empty [8, Propositions 4 and 5].

4. **General Lemmas**

Let \( f \) be an isomorphism of \( \Gamma(n, k)_q \) to a certain subgraph \( \Gamma \) of \( \Gamma_k(V) \) and let \( 1 < k < n - 1 \). Then \( f \) is an adjacency preserving injection of \( C(n, k)_q \) to \( G_k(V) \), i.e. if \( X, Y \in C(n, k)_q \) are adjacent, then \( f(X) \) and \( f(Y) \) are adjacent. It is clear that \( f \) transfers maximal cliques of \( \Gamma(n, k)_q \) to maximal cliques of \( \Gamma \). Every maximal clique of \( \Gamma \) is a (not necessarily maximal) clique of \( \Gamma_k(V) \), i.e. it is a subset of a star or a top of \( \Gamma_k(V) \). It must be pointed out that \( f \) need not to be adjacency preserving in both directions: \( f(X) \) and \( f(Y) \) may be adjacent for non-adjacent \( X, Y \in C(n, k)_q \) (two adjacent \( k \)-dimensional subspaces of \( V \) are not necessarily connected by an edge of \( \Gamma \)). For this reason the intersection of a maximal clique of \( \Gamma_k(V) \) with the vertex set of \( \Gamma \) need not to be a clique of \( \Gamma \) and we cannot state that \( f \) sends distinct maximal cliques of \( \Gamma(n, k)_q \) to subsets of distinct maximal cliques of \( \Gamma_k(V) \).

**Lemma 9.** One of the following possibilities is realized:

- \( f \) sends every maximal star of \( \Gamma(n, k)_q \) to a star of \( \Gamma_k(V) \);
- \( n = 2k \) and \( f \) transfers all maximal stars of \( \Gamma(n, k)_q \) to tops of \( \Gamma_k(V) \).

**Proof.** (1). If \( 2k < n \), then the number of elements in a star of \( \Gamma_k(V) \) and, consequently, in a maximal star of \( \Gamma(n, k)_q \) is greater than the number of elements in a top of \( \Gamma_k(V) \). Therefore, \( f \) transfers all maximal stars of \( \Gamma(n, k)_q \) to stars of \( \Gamma_k(V) \).

(2). In the case when \( n = 2k \), stars and tops of \( \Gamma_k(V) \) have the same number of elements. Thus, \( f \) transfers every maximal star of \( \Gamma(n, k)_q \) to a star or a top of \( \Gamma_k(V) \). Suppose that \( X \in C(n, k - 1)_q \) and \( f \) sends \( S(X) \) to a certain top of \( \Gamma_k(V) \). Note that \( C(n, k - 1)_q \) consists of one element if \( q = 2 \) and \( n = 2k = 4 \) (the 1-dimensional subspace containing the vector \( (1, \ldots, 1) \)); for this case the statement is trivial.

In the general case, for any \( Y \in C(n, k - 1)_q \) adjacent to \( X \) we take a \( (k + 1) \)-dimensional subspace \( Z \) containing \( X + Y \). Then \( Z \in C(n, k + 1)_q \) and \( T^c(Z) \) contains the lines
\[ S(X) \cap T(Z) \quad \text{and} \quad S(Y) \cap T(Z) \]
which means that \( T^c(Z) \) is a top of \( \Gamma(n, k)_q \). Let \( \mathcal{X} \) be a maximal clique of \( \Gamma_k(V) \) containing \( f(T^c(Z)) \). It is clear that \( f(S(X)) \) and \( \mathcal{X} \) are distinct. The intersection of \( S(X) \) and \( T^c(Z) \) contains more that one element and, consequently,
\[ |f(S(X)) \cap \mathcal{X}| > 1. \]
Since \( f(S(X)) \) is a top of \( \Gamma_k(V) \), the latter implies that \( \mathcal{X} \) is a star of \( \Gamma_k(V) \). The intersection of \( S(Y) \) and \( T^c(Z) \) contains more than one element, i.e. \( f(S(Y)) \) and
$X$ are distinct maximal cliques of $\Gamma_k(V)$ whose intersection contains more than one element. Since $X$ is a star of $\Gamma_k(V)$, $f(S(Y))$ is a top of $\Gamma_k(V)$.

So, $f(S(Y))$ is a top of $\Gamma_k(V)$ for every $Y \in \mathcal{C}(n, k-1)_q$ adjacent to $X$. Using the connectedness of $\Gamma(n, k-1)_q$, we establish that $f$ sends every maximal star of $\Gamma(n, k)_q$ to a top of $\Gamma_k(V)$.

(3) Suppose that $2k > n$. In this case, $T^c(Y)$ is a top of $\Gamma(n, k)_q$ for every $Y \in \mathcal{C}(n, k+1)_q$ (Proposition 7). The number of elements in every top of $\Gamma(n, k)_q$ is greater than the number of elements in a star of $\Gamma_k(V)$. Indeed, a top of $\Gamma(n, k)_q$ contains not less than

$$[k+1]_q - n$$

elements, a star of $\Gamma_k(V)$ contains precisely

$$[n-k+1]_q$$

 elements and

$$[k+1]_q - n - [n-k+1]_q = \frac{q^{k+1} - q^{n-k+1}}{q-1} - n = \frac{q^n-1(q^{2k-n}-1)}{q-1} - n = q^{n-k+1}(q^{2k-n-1} + \cdots + q + 1) - n \geq q^k - n \geq 2k - n > 0.$$ 

Therefore, $f$ transfers tops of $\Gamma(n, k)_q$ to subsets in tops of $\Gamma_k(V)$.

Let $X \in \mathcal{C}(n, k-1)_q$. We take any $(k+1)$-dimensional subspace $Y \subset V$ containing $X$. Then $Y$ belongs to $\mathcal{C}(n, k+1)_q$ and

$$|S(X) \cap T^c(Y)| > 1.$$ 

Since $f(T^c(Y))$ is contained in a top of $\Gamma_k(V)$, for $f(S(X))$ one of the following possibilities is realized:

(a) $f(S(X))$ is a star of $\Gamma_k(V)$;

(b) $f(S(X))$ and $f(T^c(Y))$ both are contained in a certain top of $\Gamma_k(V)$.

We need to show that (b) is impossible. Suppose that $f(S(X))$ and $f(T^c(Y))$ are contained in $T(Z)$ for a certain $(k+1)$-dimensional subspace $Z \subset V$. For any $Y' \in \mathcal{C}(n, k+1)_q$ containing $X$ and distinct from $Y$ we have

$$|S(X) \cap T^c(Y')| > 1$$

which implies that a top of $\Gamma_k(V)$ containing $f(T^c(Y'))$ intersects $T(Z)$ in a subset containing more than one element and, consequently,

$$f(T^c(Y')) \subset T(Z).$$

Since each of $T^c(Y), T^c(Y')$ contains not less than $[k+1]_q - n$ elements and their intersection is one-element,

$$|T^c(Y) \cup T^c(Y')| \geq 2|k+1]_q - 2n - 1 = [k+1]_q + (q^k + \cdots + q + 1) - 2n - 1.$$ 

We have $2k > n$ and $1 \leq k < n - 1$ which implies that $k \geq 3$. Therefore,

$$|T^c(Y) \cup T^c(Y')| \geq [k+1]_q + q^k + q^{k-1} + q^{k-2} + q^{k-3} - 2n - 1 \geq [k+1]_q + 2k + 2(k-1) + 2(k-2) + 2(k-3) - 2n - 1 = [k+1]_q + 8k - 2n - 13 \geq [k+1]_q + 8k - 2(k-1) - 13 = [k+1]_q + 4k - 11 > [k+1]_q = |T(Z)|.$$ 

which means that the case (b) is not realized. $\square$
In the case when the second possibility from Lemma 9 is realized, we consider the composition of the orthocomplementary map and \( f \). This is an isomorphism of \( \Gamma(n, k) \) to a subgraph of \( \Gamma_k(V) \) sending every maximal star of \( \Gamma(n, k) \) to a star of \( \Gamma_k(V) \). If this map is extendable to an automorphism of \( \Gamma_k(V) \), then the same holds for \( f \).

From this moment, we assume that \( f \) transfers every maximal star of \( \Gamma(n, k) \) to a star of \( \Gamma_k(V) \). Then \( f \) induces an injective map

\[
f_{k-1} : \mathcal{C}(n, k-1)_q \to \mathcal{G}_k(V)
\]

such that

\[
f(S(X)) = S(f_{k-1}(X))
\]

for every \( X \in \mathcal{C}(n, k-1)_q \). Denote by \( \Gamma_{k-1} \) the restriction of the Grassmann graph \( \Gamma_k(V) \) to the image of \( f_{k-1} \). Since the intersection of two distinct stars

\[
S(X), S(Y), \ X, Y \in \mathcal{G}_k(V)
\]

is non-empty if and only if \( X, Y \) are adjacent, the map \( f_{k-1} \) is an isomorphism of \( \Gamma(n, k-1)_q \) to \( \Gamma_{k-1} \).

Now, we show that \( f_{k-1} \) sends every maximal star of \( \Gamma(n, k-1)_q \) to a star of \( \Gamma_{k-1}(V) \) if \( k \geq 3 \). By Lemma 9 we only need to consider the case when \( n = 2(k-1) \). Let us take a \( k \)-dimensional subspace \( Y \subset V \) containing the 1-dimensional subspaces

\[
Q_1 = \mathbb{F}(1, \ldots, 1), \ Q_2 = \mathbb{F}(1, 0, 1, \ldots, 1), \ Q_3 = \mathbb{F}(0, 1, 1, \ldots, 1).
\]

For any \((k - 3)\)-dimensional subspace \( Z \subset Y \) satisfying

\[
Z \cap (Q_1 + Q_2 + Q_3) = 0
\]

the subspaces

\[
Z + Q_1 + Q_2, \ Z + Q_1 + Q_3, \ Z + Q_2 + Q_3
\]

form a clique in \( \Gamma(n, k-1)_q \) which is not contained in a star. Therefore, \( T^c(Y) \) is a top of \( \Gamma(n, k-1)_q \). If \( M \in T^c(Y) \), then

\[
Y \in S(M) \quad \text{and} \quad f(Y) \in S(f_{k-1}(M))
\]

which implies that \( f_{k-1}(M) \subset f(Y) \), i.e.

\[
f_{k-1}(T^c(Y)) \subset T(f(Y)).
\]

If a \((k - 2)\)-dimensional subspace \( X \subset Y \) contains \( Q_1 \), then \( X \in \mathcal{C}(n, k-2)_q \) and

\[
|S(X) \cap T^c(Y)| > 1;
\]

consequently,

\[
|f_{k-1}(S(X)) \cap T(f(Y))| > 1.
\]

Since \( f_{k-1}(S(X)) \) and \( T(f(Y)) \) are distinct maximal cliques of \( \Gamma_{k-1}(V) \) and the intersection of two distinct tops of \( \Gamma_{k-1}(V) \) contains at most one element, \( f_{k-1}(S(X)) \) is a star of \( \Gamma_{k-1}(V) \).

So, if \( k \geq 3 \), then \( f_{k-1} \) sends every maximal star of \( \Gamma(n, k-1)_q \) to a star of \( \Gamma_{k-1}(V) \), i.e. it induces an isomorphism \( f_{k-2} \) of \( \Gamma(n, k-2)_q \) to a certain subgraph \( \Gamma_{k-2}(V) \). We construct recursively a sequence

\[
f_k = f, f_{k-1}, \ldots, f_1,
\]

where

\[
f_i : \mathcal{C}(n, i)_q \to \mathcal{G}_i(V)
\]
is an isomorphism of $\Gamma(n,i)q$ to a certain subgraph $\Gamma_i$ of $\Gamma_i(V)$ and

$$f_i(S(X)) = S(f_{i-1}(X))$$

for every $X \in C(n,i-1)_q$ and $i \geq 2$.

**Lemma 10.** The isomorphism $f_i$, $i \in \{2,\ldots,k\}$ sends every star of $\Gamma(n,i)_q$ to a subset in a star of $\Gamma_i(V)$.

**Proof.** We prove the statement for $i = k$ (the general case is similar). Let $X$ be a $(k - 1)$-dimensional subspace of $V$ such that $S^c(X)$ is contained in $T^c(Z)$ for a certain $(k + 1)$-dimensional subspace $Z \subset V$ which is impossible. Observe that every $T^c(Z_j)$ is a top of $\Gamma(n,k)_q$. Indeed, $T^c(Z_j)$ contains $X_j, Q + X$ and $Q + X'$, where $X'$ is a $(k - 1)$-dimensional subspace of $X_j$ distinct from $X$; these subspaces form a clique of $\Gamma(n,k)_q$ which is not contained in a star.

Each $T^c(Z_j)$ intersects the star $S^c(X)$ and the maximal star $S(Y)$ in subsets containing more than one element. Since $f(S(Y))$ is a star of $\Gamma_k(V)$, every $f(T^c(Z_j))$ is contained in a top of $\Gamma_k(V)$. This means that for $f(S^c(X))$ one of the following possibilities is realized:

(a) $f(S^c(X))$ is contained in a star of $\Gamma_k(V)$;

(b) there is a top of $\Gamma_k(V)$ containing $f(S^c(X))$ and both $f(T^c(Z_j)), j \in \{1,2\}$.

Show that the case (b) is impossible.

Suppose that $q \geq 3$. Then

$$|S^c(X)| \geq (q - 1)^{n-k}$$

by (2) and

$$|T^c(Z_j)| \geq [k+1]_q - n = q^k + \cdots + q + 1 - n.$$ 

Observe that

$$T^c(Z_1) \cap T^c(Z_2) = S^c(X) \cap T^c(Z_1) \cap T^c(Z_2) = \{X + Y\}$$

and

$$|S^c(X) \cap T^c(Z_j)| \leq q + 1.$$ 

By the inclusion-exclusion principle, we have

$$|S^c(X) \cup T^c(Z_1) \cup T^c(Z_2)| \geq (q - 1)^{n-k} + 2(q^k + \cdots + q + 1 - n) - 2(q + 1) - 1 + 1 \geq 2(n - k) + 2(q^k + \cdots + q^2) - 2n = (q^k + \cdots + q + 1) + (q^k + \cdots + q^2) - q - 1 - 2k > q^k + \cdots + q + 1 = [k+1]_q$$

which contradicts (b).

Suppose that $q = 2$. In this case, $S^c(X)$ contains not less than 4 elements (by the statement (ii) from Proposition 5 and 3). If $j \in \{1,2\}$, then for every $(k - 1)$-dimensional subspace $N \subset X_j$ the subspace $N + Q$ belongs to $T^c(Z_j)$; this subspace belongs to $S^c(X)$ only in the case when $N = X$. Therefore, each $T^c(Z_j)$ contains

$$[k]_2 - 1 = 2^{k-1} + \cdots + 2$$

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elements which do not belong to $S^c(X)$. This means that the union of our three maximal cliques contains not less than
\[ 4 + 2(2^{k-1} + \cdots + 2) > [k + 1]_2 \]
elements and (b) is impossible. □

5. Proof of Theorem 1. The case $q \geq 3$

Suppose that $q \geq 3$. In this case, $S^c(X)$ is a star of $\Gamma(n, i)_q$ for every $(i - 1)$-dimensional subspace $X \subset V$ (the statement (i) of Proposition 1). By Lemma 10 the isomorphism $f_i$, $i \in \{2, \ldots, k\}$ transfers every star of $\Gamma(n, i)_q$ to a subset in a star of $\Gamma_i(V)$. Since the intersection of two distinct stars of $\Gamma_i(V)$ contains at most one element, the image of every star of $\Gamma(n, i)_q$ is contained in a unique star of $\Gamma_i(V)$. Therefore, for every $j \in \{1, \ldots, k - 1\}$ the isomorphism $f_j$ can be extended to a map $g_j : \mathcal{G}_j(V) \to \mathcal{G}_j(V)$ such that
\[ f_{j+1}(S^c(X)) \subset S(g_j(X)) \]
for every $X \in \mathcal{G}_j(X)$. Note that
\[ g_j(\mathcal{G}_j(Y)) \subset \mathcal{G}_j(f_{j+1}(Y)) \]
for every $Y \in \mathcal{C}(n, j + 1)_q$.

**Lemma 11.** The map $g_1$ is bijective.

**Proof.** It is sufficient to show that $g_1$ is injective. Since $q \geq 3$, we can take $P_1, P_2, P_3 \in \mathcal{C}(n, 1)_q$ whose sum is 3-dimensional. For example, the 1-dimensional subspaces
\[ \mathbb{F}(1, \ldots, 1), \mathbb{F}(\alpha, 1, \ldots, 1), \mathbb{F}(\alpha, \alpha, 1, \ldots, 1), \alpha \neq 0, 1 \]
are as required. If
\[ g_1(P) = g_1(Q) = P' \]
for distinct 1-dimensional subspaces $P, Q \subset V$, then $f_2$ sends $S^c(P)$ and $S^c(Q)$ to subsets of $S(P')$. There is $i \in \{1, 2, 3\}$ such that $P_i \not \subset P + Q$. The maximal star $S(P_i)$ intersects $S^c(P)$ and $S^c(Q)$ in distinct elements ($P_i + P$ and $P_i + Q$, respectively). Then $f_2(S(P_i))$ intersects $S(P')$ in two distinct elements. Since $f_2(S(P_i))$ is a star of $\Gamma_2(V)$, we have
\[ f_2(S(P_i)) = S(P') \]
which contradicts the injectivity of $f_2$ (since $f_2(S^c(P))$ and $f_2(S^c(Q))$ both are contained in $S(P')$). □

**Lemma 12.** For every 2-dimensional subspace $X \not \in \mathcal{C}(n, 2)_q$ there are $X_1, X_2 \in \mathcal{C}(n, 1)_q$ such that $X + X_1$ and $X + X_2$ are distinct elements of $\mathcal{C}(n, 3)_q$ and each of them defines a top of $\Gamma(n, 2)_q$.

**Proof.** Without loss of generality we can assume that $X \subset C_1$, i.e. the first coordinate of every vector belonging to $X$ is zero. Then $X$ contains a non-zero vector $x$ such that the first and the second coordinates of $x$ are zero.

Let $\alpha$ be a non-zero element of the field $\mathbb{F}$ distinct from 1. First, we consider the case when $X$ contains a 1-dimensional subspace
\[ Q = \mathbb{F}(0, 0, x_3, \ldots, x_n), \]
where every \( x_i \) is zero or \( \alpha \). The 1-dimensional subspaces

\[ P_1 = \mathbb{F}(1, \ldots, 1), \quad P_2 = \mathbb{F}(\alpha, 1, \ldots, 1), \quad P_3 = \mathbb{F}(\alpha, \alpha, 1, \ldots, 1) \]

and

\[ P'_1 = \mathbb{F}(1, 1 - x_3, \ldots, 1 - x_n), \]
\[ P'_2 = \mathbb{F}(\alpha, 1 - x_3, \ldots, 1 - x_n), \]
\[ P'_3 = \mathbb{F}(\alpha, \alpha, 1 - x_3, \ldots, 1 - x_n) \]

belong to \( C(n, 1)_q \) and

\[ P'_i \subset Q + P_i \]

for every \( i \in \{1, 2, 3\} \). Each \( X + P_i \) belongs to \( C(n, 3)_q \). If \( T \) is a 1-dimensional subspace of \( X \) distinct from \( Q \), then

\[ Q + P_i, \quad T + P_i, \quad T + P'_i \]

form a clique of \( \Gamma(n, 2)_q \) which is not contained in a star. Therefore, \( T^c(X + P_i) \) is a top of \( \Gamma(n, 2)_q \).

The subspace \( P_1 + P_2 + P_3 \) is 3-dimensional. If \( X \) is not contained in \( P_1 + P_2 + P_3 \), then we have

\[ X + P_i \neq X + P_j \]

for some distinct \( i, j \). If \( X \) is contained in \( P_1 + P_2 + P_3 \), then \( X \) intersects \( P_2 + P_3 \) in a 1-dimensional subspace \( Q' \) distinct from \( P_2, P_3 \). This subspace contains the vector

\[ v = (\alpha, 1, \ldots, 1) + a(\alpha, \alpha, 1, \ldots, 1) = (\alpha + a\alpha, 1 + a\alpha, 1 + a, \ldots, 1 + a) \]

for a certain non-zero \( \alpha \in \mathbb{F} \). Since \( Q' \subset X \subset C_1 \), we have \( \alpha + a\alpha = 0 \). Then \( a + 1 = 0 \) and all coordinates of \( v \), except the second, are zero. So, \( X \) contains the 1-dimensional subspace \( Q' = \mathbb{F}e_2 \). The 1-dimensional subspaces

\[ P_4 = \mathbb{F}(\alpha, \alpha, \alpha, 1, \ldots, 1) \quad \text{and} \quad P'_4 = \mathbb{F}(\alpha, \alpha - 1, \alpha, 1, \ldots, 1) \]

belong to \( C(n, 1)_q \). Also,

\[ P'_i \subset Q' + P_4 \]

and \( X + P_4 \in C(n, 3)_q \). If \( T \) is a 1-dimensional subspace of \( X \) distinct from \( Q' \), then

\[ Q' + P_4, \quad T + P_4, \quad T + P'_4 \]

form a clique of \( \Gamma(n, 2)_q \) which is not contained in a star. Thus \( X + P_4 \) defines a top of \( \Gamma(n, 2)_q \). Since \( P_4 \) is not contained in \( P_1 + P_2 + P_3 \),

\[ X + P_i \neq X + P_4 \]

for every \( i \in \{1, 2, 3\} \).

In the general case, there is a monomial linear automorphism \( l \) of \( V \) such that \( l^{-1}(X) \) contains a non-zero vector \((0, 0, x_3, \ldots, x_n)\), where every \( x_i \) is zero or \( \alpha \). Then all \( l(P_i) \) belong to \( C(n, 1)_q \), each \( X + l(P_i) \) defines a top of \( \Gamma(n, 2)_q \) and

\[ X + l(P_i) \neq X + l(P_j) \]

for some distinct \( i, j \).

Recall that the projective space associated to \( V \) is the point-line geometry whose points are 1-dimensional subspaces of \( V \) and the lines are subsets of type \( G_1(X) \), where \( X \) is a 2-dimensional subspace of \( V \). By the Fundamental Theorem of Projective Geometry, every bijective transformation of \( G_1(V) \) sending lines to lines is induced by a semilinear automorphism of \( V \).\[ \square \]
Lemma 13. The bijective transformation $g_1$ sends lines to lines.

Proof. If $X \in \mathcal{C}(n, 2)_q$, then

$$g_1(\mathcal{G}_1(X)) \subset \mathcal{G}_1(f_2(X)).$$

Since $g_1$ is bijective, we obtain that

$$g_1(\mathcal{G}_1(X)) = \mathcal{G}_1(f_2(X)).$$

Let $X$ be a 2-dimensional subspace of $V$ which does not belong to $\mathcal{C}(n, 2)_q$. By Lemma 12 there are $X_1, X_2 \in \mathcal{C}(n, 1)_q$ such that $Z_1 = X + X_1$ and $Z_2 = X + X_2$ are distinct elements of $\mathcal{C}(n, 3)_q$ and each of them defines a top of $\Gamma(n, 2)_q$. We have

$$|S(X_i) \cap T^c(Z_i)| > 1$$

for $i \in \{1, 2\}$ which implies that $f_2(T^c(Z_i))$ is contained in a top of $\Gamma_2(V)$, i.e.

$$f_2(T^c(Z_i)) \subset T(Z_i')$$

for some 3-dimensional subspace $Z_i' \subset V$, $i \in \{1, 2\}$. Every 1-dimensional subspace $P \subset Z_i$ is contained in a certain $Y \in T^c(Z_i)$ and

$$g_1(P) \subset f_2(Y) \subset Z_i'$$

which means that $g_1(\mathcal{G}_1(Z_i))$ is contained in $\mathcal{G}_1(Z_i')$. Since $g_1$ is bijective,

$$g_1(\mathcal{G}_1(Z_i)) = \mathcal{G}_1(Z_i')$$

and $Z_1' \neq Z_2'$. We have $X = Z_1 \cap Z_2$ which implies that $g_1$ sends $\mathcal{G}_1(X)$ to $\mathcal{G}_1(X')$, where $X' = Z_1' \cap Z_2'$.

So, $g_1$ is induced by a semilinear automorphism $l$ of $V$. Then $f_2$ also is induced by $l$ and the statement is proved for $k = 2$. Consider the case when $k \geq 3$.

Lemma 14. If $X \in \mathcal{C}(n, k)_q$ contains $P \subset \mathcal{C}(n, 1)_q$, then

$$f(X) = l(X).$$

Proof. Let $Q$ be a 1-dimensional subspace of $X$. Consider a sequence

$$P \subset X_2 \subset \cdots \subset X_{k-1} \subset X_k = X,$$

where every $X_i$ is an $i$-dimensional subspace and $X_2$ contains $Q$. Then $X_i \in \mathcal{C}(n, i)$ for every $i \in \{2, \ldots, k\}$ and

$$f_{i-1}(X_{i-1}) \subset f_i(X_i)$$

if $i \geq 3$. This implies that

$$g_1(Q) \subset f_2(X_2) \subset f(X).$$

Since $g_1$ is bijective, we have

$$g_1(\mathcal{G}_1(X)) = \mathcal{G}_1(f(X)).$$

On the other hand, $g_1$ is induced by $l$ and

$$g_1(\mathcal{G}_1(X)) = \mathcal{G}_1(l(X))$$

which gives the claim. \qed

Lemma 15. For every $(k - 1)$-dimensional subspace $X \subset V$ there are $Z_1, Z_2 \in \mathcal{C}(n, k)_q$ such that $X = Z_1 \cap Z_2$ and each $Z_i$ contains an element of $\mathcal{C}(n, 1)_q$. 

These subspaces are distinct and, consequently, we have
\[ P_1 = \mathbb{F}(1, \ldots, 1), \quad P_2 = \mathbb{F}(\alpha, 1, \ldots, 1), \ldots, \quad P_{k+1} = \mathbb{F}(\alpha, \ldots, \alpha, 1, \ldots, 1) \]

is not contained in \( X \), i.e. every \( X + P_i \) is an element of \( \mathcal{C}(n, k) \). Since the subspace \( P_1 + \cdots + P_{k+1} \) is \( (k+1) \)-dimensional, there are at least two distinct \( X + P_i \).

Let \( X \) be a \((k-1)\)-dimensional subspace of \( V \) and let \( Z_1, Z_2 \) be as in Lemma 15. Then \( f(Z_i) = l(Z_i) \) (by Lemma 14) and
\[ l(X) = l(Z_1) \cap l(Z_2) = f(Z_1) \cap f(Z_2) = g_{k-1}(X). \]
Therefore, \( g_{k-1} \) is induced by \( l \). On the other hand,
\[ g_{k-1}(g_{k-1}(Y)) \subset g_{k-1}(f(Y)) \]
for every \( Y \in \mathcal{C}(n, k) \) and we obtain that \( f(Y) = l(Y) \). So, \( f \) is induced by \( l \).

6. Proof of Theorem 4. The case \( q = 2, \ k \geq 3 \)

Suppose that \( q = 2 \). In this case, \( \mathcal{C}(n, 1) \) consists of one element, the 1-dimensional subspace containing the vector \((1, \ldots, 1)\). By the statement (ii) of Proposition 5 for a 1-dimensional subspace \( P \subset V \) the set \( S^c(P) \) is a star of \( \Gamma(n, 2) \) if and only if the weight of \( P \) (the number of non-zero coordinates of the non-zero vector belonging to \( P \)) is greater than 2. Let \( \mathcal{G}_1'(V) \) be the set of all such 1-dimensional subspaces of \( V \). By Lemma 10, there is a map
\[ g_1 : \mathcal{G}_1'(V) \rightarrow \mathcal{G}_1(V) \]
such that
\[ f_2(S^c(P)) \subset S(g_1(P)) \]
for every \( P \in \mathcal{G}_1'(V) \).

**Lemma 16.** If \( k \geq 3 \), then \( g_1 \) is injective.

**Proof.** Let \( T \) be the 1-dimensional subspace containing the vector \((1, \ldots, 1)\), i.e. the unique element of \( \mathcal{C}(n, 1) \). Then \( f_2(S(T)) \) is a star of \( \Gamma_2(V) \). If \( g_1(T) = g_1(P) \) for a certain \( P \in \mathcal{G}_1'(V) \) distinct from \( T \), then \( f_2(S^c(P)) \) is contained in \( f_2(S(T)) \), which contradicts the injectivity of \( f_2 \).

Let \( P \) and \( Q \) be distinct elements of \( \mathcal{G}_1'(V) \) different from \( T \). If \( T \) is not contained in \( P + Q \), then \( T + P \) and \( T + Q \) are distinct elements of \( \mathcal{C}(n, 2) \). Since
\[ g_1(P) \neq g_1(T) \neq g_1(Q), \]
we have
\[ f_2(T + P) = g_1(T) + g_1(P) \quad \text{and} \quad f_2(T + Q) = g_1(T) + g_1(Q). \]
These subspaces are distinct and, consequently, \( g_1(P) \neq g_1(Q) \).

Suppose that \( T \subset P + Q \) and \( i \) is the weight of \( P \). First, we consider the case when \( P \) contains the vector
\[ (\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0). \]
Then $Q$ contains the vector

$$Q(0, \ldots, 0, 1, \ldots, 1).$$

Since $P$ and $Q$ belong to $G_1\mathcal{C}(V)$, the both $i$ and $n - i$ are not less than 3. Let $P'$ and $Q'$ be the 1-dimensional subspaces containing the vectors

$$(0, 1, \ldots, 1) \text{ and } (1, \ldots, 1, 0),$$

respectively. The subspaces

$$X = P + P', \ Y = P' + Q', \ Z = Q + Q'$$

are mutually distinct elements of $C(n, 2)_2$, each of them does not contain $T$. Therefore,

$$f_2(X) = g_1(P) + g_1(P'), \ f_2(Y) = g_1(P') + g_1(Q'), \ f_2(Z) = g_1(Q) + g_1(Q')$$

are mutually distinct. The subspaces $X + Y$ and $Y + Z$ are distinct elements of $C(n, 3)_2$. Note that $X + Y$ and $Y + Z$ are distinct elements of $C(n, 3)_2$ even if the both $i$ and $n - i$ are not less than 2. This fact will be used to prove Lemma 17. We have

$$f_3(X + Y) = f_2(X) + f_2(Y) = g_1(P) + g_1(P') + g_1(Q'),$$

$$f_3(Y + Z) = f_2(Y) + f_2(Z) = g_1(Q) + g_1(P') + g_1(Q')$$

are distinct. The latter implies that $g_1(P) \neq g_1(Q)$. In the general case, there is a monomial linear automorphism $l$ of $V$ transferring (5) and (6) to the non-zero vectors belonging to $P$ and $Q$, respectively. Applying the above arguments to the 2-dimensional subspaces

$$P + l(P'), \ l(P') + l(Q'), \ Q + l(Q')$$

we establish that $g_1(P) \neq g_1(Q)$. \hfill \Box

From this moment, we suppose that $k \geq 3$.

We extend $g_1$ to a transformation of $G_1\mathcal{C}(V)$. Let $P$ be a 1-dimensional subspace of $V$ which does not belong to $G_1\mathcal{C}(V)$. This is equivalent to the fact that the weight of $P$ is not greater than 2. In the case when $P$ is of weight 2, there are precisely two elements of $C(n, 2)_2$ containing $P$ (Remark 6). If $X$ and $Y$ are such elements of $C(n, 2)_2$, then $f_2(X)$, $f_2(Y)$ are adjacent and we define $g_1(P)$ as their intersection.

In the case when the weight of $P$ is 1, the 2-dimensional subspace $X$ spanned by $P$ and the vector $(1, \ldots, 1)$ is the unique element of $C(n, 2)_2$ containing $P$. Let $P'$ and $P''$ be the 1-dimensional subspaces of $X$ distinct from $P$. They belong to $G_1\mathcal{C}(V)$ and $g_1(P') \neq g_1(P'')$. We define $g_1(P)$ as the 1-dimensional subspace of $f_2(X)$ distinct from $g_1(P')$ and $g_1(P'')$.

So, we obtain a transformation $g_1$ of $G_1\mathcal{C}(V)$ such that (4) holds for every 1-dimensional subspace $P \subset V$ and

$$g_1(G_1(X)) \subset G_1(f_2(X))$$

for every $X \in C(n, 2)_2$.

**Lemma 17.** The extended transformation $g_1$ is bijective.
Consider a 2-dimensional subspace $X$. The bijective transformation

Lemma 18. If $g_1(P) \neq g_1(T)$ by the definition. If $P$ is a 1-dimensional subspace of weight 2 and $g_1(P) = g_1(T)$, then $f_2(S(P))$ is contained in $f_2(S(T))$ which contradicts the injectivity of $f_2$.

Let $P$ and $Q$ be distinct 1-dimensional subspaces of $V$ different from $T$ and such that the weight of at least one of them is not greater than 2. In the case when $T \not\subset P + Q$, we use reasonings from the proof of Lemma 16. Suppose that $T \subset P + Q$. If one of $P, Q$ is of weight 1, then $g_1(P) \neq g_1(Q)$ by the definition. For the case when the weights of $P$ and $Q$ are not less than 2 we apply arguments used in the proof of Lemma 16.

Now, we prove the direct analogue of Lemma 13.

Lemma 18. The bijective transformation $g_1$ sends lines to lines.

Proof. If $X \in C(n, 2)_2$, then $g_1(G_1(X))$ is contained in $G_1(f_2(X))$. Since $g_1$ is bijective, we have

$$g_1(G_1(X)) = G_1(f_2(X)).$$

Consider a 2-dimensional subspace $X \subset V$ which does not belong to $C(n, 2)_2$. Let $Q_1, Q_2, Q_3$ be the 1-dimensional subspaces of $X$.

For every 1-dimensional subspace $P \subset V$ we denote by $N(P)$ the set of all indices $i \in \{1, \ldots, n\}$ such that the $i$-coordinate of the non-zero vector of $P$ is zero. The weight of $P$ is equal to $n - |N(P)|$. Since $X$ does not belong to $C(n, 2)_2$,

$$N(Q_1) \cap N(Q_2) \cap N(Q_3) \neq \emptyset.$$ If $t \in \{1, 2, 3\}$ and \{1, 2, 3\} ∖ \{t\} = \{i, j\}, then

$$N(Q_i) = (N(Q_i) \cap N(Q_j)) \cup (N(Q_i) \cup N(Q_j))^c$$

(for $I \subset \{1, \ldots, n\}$ we denote by $I^c$ the complement $\{1, \ldots, n\} \setminus I$). Therefore, if there are distinct $i, j \in \{1, 2, 3\}$ such that

(7) \quad $N(Q_i) \not\subset N(Q_j)$, $N(Q_j) \not\subset N(Q_i)$ and $N(Q_i) \cup N(Q_j) \neq \{1, \ldots, n\}$,

then the same holds for any distinct $i, j \in \{1, 2, 3\}$. On the other hand, if for some distinct $i, j \in \{1, 2, 3\}$ we have

(8) \quad $N(Q_i) \cup N(Q_j) = \{1, \ldots, n\}$ and \{1, 2, 3\} ∖ \{i, j\} = \{t\}, then

$$N(Q_t) = N(Q_i) \cap N(Q_j).$$

Conversely, if $N(Q_i) \subset N(Q_t)$ for some distinct $i, t \in \{1, 2, 3\}$, then [s] holds for $j \in \{1, 2, 3\} \setminus \{i, t\}$.

So, one of the following two possibilities is realized:

(i) \quad \{7\} holds for any distinct $i, j \in \{1, 2, 3\}$;

(ii) \quad there are distinct $i, j \in \{1, 2, 3\}$ satisfying [s].

The case (i). The sets

$$(N(Q_1) \cup N(Q_2))^c, (N(Q_1) \cup N(Q_3))^c, (N(Q_2) \cup N(Q_3))^c$$

are non-empty. Furthermore, these sets are mutually disjoint (otherwise, there is $s \in \{1, \ldots, n\}$ such that the $s$-coordinate of the non-zero vector of $Q_i$ is non-zero for
every $i \in \{1, 2, 3\}$ which is impossible). Without loss of generality we can assume that

$$3 \notin \mathcal{N}(Q_1) \cup \mathcal{N}(Q_2), \ 2 \notin \mathcal{N}(Q_1) \cup \mathcal{N}(Q_3), \ 1 \notin \mathcal{N}(Q_2) \cup \mathcal{N}(Q_3).$$

Then

$$1, 2 \notin \mathcal{N}(Q_3), \ 1, 3 \notin \mathcal{N}(Q_2), \ 2, 3 \notin \mathcal{N}(Q_1)$$

which implies that $i \in \mathcal{N}(Q_i)$ for every $i \in \{1, 2, 3\}$ (otherwise, there is $i \in \{1, 2, 3\}$ such that the $i$-coordinates of the non-zero vectors of $Q_1, Q_2, Q_3$ are non-zero which is impossible). Let $P, P_1, P_2, P_3$ be the 1-dimensional subspaces of $V$ containing the vectors

$$(1, \ldots, 1), (0, 1, \ldots, 1), (1, 0, 1, \ldots, 1), (1, 1, 0, 1, \ldots, 1),$$

respectively. Then

$$Z = P + X \text{ and } Z_i = P_i + X, \ i \in \{1, 2, 3\}$$

belong to $\mathcal{C}(n, 3)_2$; furthermore, the following assertions are fulfilled:

- $P + Q_i$ belongs to $\mathcal{C}(n, 2)_2$ for every $i \in \{1, 2, 3\}$;
- $P_j + Q_i$ belongs to $\mathcal{C}(n, 2)_2$ if and only if $i \neq j$.

This implies that

$$g_1(Q_i) \subset f_2(P + Q_i) \subset f_3(Z), \ i \in \{1, 2, 3\}$$

and

$$g_1(Q_i) \subset f_2(P_j + Q_i) \subset f_3(Z_j), \ i \neq j.$$

It is clear that $Z \neq Z_i$ for every $i \in \{1, 2, 3\}$, but we cannot state that $Z_i \neq Z_j$ for distinct $i, j$. The subset of $\mathcal{C}(n, 3)_2$ formed by $Z$ and all $Z_i$, $i \in \{1, 2, 3\}$ contains at least 2 elements and it is contained in a star of $\Gamma(n, 3)_2$. Since $f_3$ sends stars of $\Gamma(n, 3)_2$ to subsets in stars of $\Gamma_3(V)$, the intersection of $f_3(Z)$ and all $f_3(Z_i)$ is a 2-dimensional subspace $X'$. By (9) and (10), every $g_1(Q_i)$ is contained in $f_3(Z)$ and $f_3(Z_j)$ if $j \neq i$ which means that $g_1(Q_i) \subset X'$. So, $g_1$ sends $\mathcal{G}_1(X)$ to $\mathcal{G}_1(X')$.

**The case (ii).** Without loss of generality we assume that

$$\mathcal{N}(Q_1) \cup \mathcal{N}(Q_2) = \{1, \ldots, n\}.$$

As above, $P$ is the 1-dimensional subspace containing the vector $(1, \ldots, 1)$ and $Z = P + X$. Let $Q'_i, i \in \{1, 2, 3\}$ be the 1-dimensional subspace of $P + Q_i$ distinct from $P$ and $Q_i$. Then

$$\mathcal{N}(Q'_i) = \mathcal{N}(Q_i)^c$$

and

$$\mathcal{N}(Q'_1) \cap \mathcal{N}(Q'_2) = (\mathcal{N}(Q_1) \cup \mathcal{N}(Q_2))^c = \emptyset.$$

The latter implies that $Q'_1 + Q'_2$ belongs to $\mathcal{C}(n, 2)_2$. Note that $Q_3$ is the third 1-dimensional subspace of $Q'_1 + Q'_2$.

By (11), we have

$$\mathcal{N}(Q_3) = \mathcal{N}(Q_1) \cap \mathcal{N}(Q_2)$$

and, consequently,

$$\mathcal{N}(Q'_3) = \mathcal{N}(Q'_1) \cup \mathcal{N}(Q'_2).$$

Then

$$\mathcal{N}(Q_1) \cap \mathcal{N}(Q'_2) \cap \mathcal{N}(Q'_3) = \mathcal{N}(Q_1) \cap \mathcal{N}(Q'_2) = \mathcal{N}(Q_1) \cap \mathcal{N}(Q_2)^c \neq \emptyset$$

and

$$\mathcal{N}(Q_2) \cap \mathcal{N}(Q'_1) \cap \mathcal{N}(Q'_3) = \mathcal{N}(Q_2) \cap \mathcal{N}(Q'_1) = \mathcal{N}(Q_2) \cap \mathcal{N}(Q_1)^c \neq \emptyset.$$
which means that $Q_1 + Q_2'$ and $Q_1' + Q_2$ do not belong to $C(n, 2)_2$. Note that the intersection of these 2-dimensional subspaces is $Q_3'$.

So, $Z$ contains precisely four elements of $C(n, 2)_2$; these are

$$P + Q_i, \ i \in \{1, 2, 3\} \text{ and } Q_1' + Q_2.'$$

The isomorphism $f_2$ transfers them to some 2-dimensional subspaces of $f_3(Z)$. Observe that $g_1(P)$ is contained in all $f_2(P + Q_i)$ and $f_2(Q_1' + Q_2')$ intersects

$$f_2(P + Q_1), \ f_2(P + Q_2), \ f_2(P + Q_3)$$

in $g(Q_1'), g(Q_2'), g(Q_3)$ (respectively), see Fig. 1. This implies that all $g_1(Q_i)$ form a line of the projective space associated to $V$. □

![Figure 1](image_url)

So, $g_1$ is induced by a semilinear automorphism $l$ of $V$. Since

$$g_1(G_1(X)) = G_1(f_2(X))$$

for every $X \in C(n, 2)_2$, $f_2$ also is induced by $l$.

**Lemma 19.** If $X \in C(n, k)_2$ contains

$$(1, \ldots, 1) \text{ or } v_i = (1, \ldots, 1) + e_i,$$

then

$$f(X) = l(X).$$

**Proof.** If $X$ contains $(1, \ldots, 1)$, then we use arguments from the proof of Lemma 14.

Suppose that $X$ contains $v_i$. Since $X$ belongs to $C(n, k)_2$, the subspace $X \cap C_i$ is a hyperplane of $X$. For every vector $x \in X \setminus C_i$ the 2-dimensional subspace $Y$ containing $x$ and $v_i$ belongs to $C(n, 2)_2$. We consider a sequence of subspaces

$$Y = Y_2 \subset \cdots \subset Y_{k-1} \subset Y_k = X,$$

where $Y_i \in C(n, i)_2$ for every $i \in \{2, \ldots, k\}$, and establish that $l(Y) \subset f(X)$. Therefore,

$$l(X \setminus C_i) \subset f(X)$$

which gives the claim. □

Denote by $G'_k(V)$ the set of all $(k-1)$-dimensional subspaces $X \subset V$ satisfying

$$c(X) \leq n - k - 1.$$
The statement (ii) of Proposition \[\text{(5)}\] says that \(S^c(X)\) is a star of \(\Gamma(n, k)\) if and only if \(X\) belongs to \(G'_{k-1}(V)\). By Lemma \[\text{(10)}\] there is a map
\[
g_{k-1} : G'_{k-1}(V) \rightarrow G_{k-1}(V)
\]
such that
\[
f(S^c(X)) \subset S(g_{k-1}(X))
\]
for every \(X \in G'_{k-1}(V)\). Note that \(g_{k-1}\) is an extension of \(f_{k-1}\).

Let \(G''_{k-1}(V)\) be the set of all \((k-1)\)-dimensional subspaces \(X \subset V\) such that
\[
c(X) \leq n - k.
\]

If a \((k-1)\)-dimensional subspace \(X \subset V\) does not belong to \(G''_{k-1}(V)\), then \(c(X) = n - k + 1\) which implies that \(X\) is spanned by some \(e_{i_1}, \ldots, e_{i_{k-1}}\).

We extend \(g_{k-1}\) to a map of \(G''_{k-1}(V)\). By Remark \[\text{(6)}\] for a \((k-1)\)-dimensional subspace \(X \subset V\) satisfying \(c(X) = n - k\) there are precisely two elements of \(C(n, k)\) containing \(X\). If \(Y\) and \(Y'\) are such elements of \(C(n, k)\), then \(f(Y), f(Y')\) are adjacent and we define \(g_{k-1}(X)\) as their intersection.

**Lemma 20.** For every \(X \in G''_{k-1}(V)\) there are \(Y, Y' \in C(n, k)\) such that \(X = Y \cap Y'\) and each of \(Y, Y'\) contains \((1, \ldots, 1)\) or \(v_i\) for some \(i\) (\(v_i\) is as in Lemma \[\text{(7)}\]).

**Proof.** If \(X\) contains \((1, \ldots, 1)\) or \(v_i\), then any two distinct \(k\)-dimensional subspaces containing \(X\) are as required.

Suppose that \((1, \ldots, 1)\) and all \(v_i\) do not belong to \(X\). Let \(I\) be the set of all indices \(i \in \{1, \ldots, n\}\) such that \(X\) contains a vector whose \(i\)-coordinate is non-zero. Since \(c(X) \leq n - k\), this set contains at least \(k\) indices. For every \(i \in I\) we denote by \(Y_i\) the subspace containing \(X\) and \(v_i\). Let also \(Y\) be the subspace containing \(X\) and \((1, \ldots, 1)\). By our assumption, \(Y\) and all \(Y_i\) are \(k\)-dimensional; furthermore, these subspaces belong to \(C(n, k)\).

We assert that there is \(i \in I\) such that \(Y_i \neq Y\). Indeed, if \(Y = Y_i\) for all \(i \in I\), then \(Y\) contains the vector \((1, \ldots, 1) + v_i = e_i\) for every \(i \in I\). Since \(|I| \geq k\), the subspace \(Y\) is spanned by some \(e_{i_1}, \ldots, e_{i_k}\) which contradicts the fact that \((1, \ldots, 1)\) belongs to \(Y\).

Let \(X \in G''_{k-1}(V)\) and let \(Y, Y' \in C(n, k)\) be as in Lemma \[\text{(20)}\]. Then
\[
f(Y) = l(Y), \quad f(Y') = l(Y')
\]
(by Lemma \[\text{(19)}\]) and
\[
g_{k-1}(X) = f(Y) \cap f(Y') = l(Y) \cap l(Y') = l(Y \cap Y') = l(X).
\]

Therefore, \(g_{k-1}\) is induced by \(l\).

Every \(X \in C(n, k)\) contains two distinct \(Y, Y' \in G''_{k-1}(V)\) (otherwise, there are distinct \((k-1)\)-dimensional subspaces \(Z, Z' \subset X\) satisfying \(c(Z) = c(Z') = n - k + 1\); then \(X = Z + Z'\) and \(X\) is spanned by some \(e_{i_1}, \ldots, e_{i_k}\) which is impossible). Since
\[
g_{k-1}(Y) = l(Y) \quad \text{and} \quad g_{k-1}(Y') = l(Y')
\]
are distinct \((k-1)\)-dimensional subspaces of \(f(X)\), we have
\[
f(X) = l(Y) + l(Y') = l(Y + Y') = l(X).
\]

So, \(f\) is induced by \(l\).
7. Proof of Theorem 1. The case $q = 2$, $k = 2$

As in the previous section, we suppose that $q \geq 3$. For a non-empty subset $I \subset \{1, \ldots, n\}$ we denote by $P_I$ the 1-dimensional subspace of $V$ containing the vector whose $i$-coordinate is 1 for every $i \in I$ and 0 if $i \notin I$. Let $H$ be the hyperplane of $V$ spanned by all $P_I$, where $I$ is an $(n-1)$-element subset of $\{1, \ldots, n\}$ containing $n$. It is easy to see that $P_{\{1, \ldots, n\}}$ is not contained in $H$. We decompose $C(n, 2)_2$ in three subsets $A, B, C$ as follows:

- $A$ consists of all elements of $C(n, 2)_2$ containing $P_{\{1, \ldots, n\}}$;
- $B$ is formed by all elements of $C(n, 2)_2$ contained in $H$;
- $C = C(n, 2)_2 \setminus (A \cup B)$.

Consider $X \in C$. This subspace intersects $H$ in a 1-dimensional subspace and we suppose that $P_I, P_J$ are the remaining two 1-dimensional subspaces of $X$. Then

\[ I \cup J = \{1, \ldots, n\} \]

and $P_{I \cup J^c}$ is the 1-dimensional subspace of $X$ contained in $H$ (recall that for every $S \subset \{1, \ldots, n\}$ we denote by $S^c$ the complement of $S$ in $\{1, \ldots, n\}$). Then $P_{I^c}, P_{J^c}$ are contained in $H$. Indeed, the 2-dimensional subspaces

\[ P_I + P_{I^c}, \ P_J + P_{J^c} \]

both contain $P_{\{1, \ldots, n\}}$ and each of the 1-dimensional subspaces $P_{\{1, \ldots, n\}}, P_I, P_J$ is not in $H$. Denote by $X^c$ the 2-dimensional subspace of $V$ containing $P_{I^c}$ and $P_{J^c}$.

It is clear that $X^c \subset H$. By \((\ref{12})\), we have $I^c \cap J^c = \emptyset$ and the third 1-dimensional subspace of $X^c$ is $P_{I \cup J^c}$. Since $P_{\{1, \ldots, n\}}$ is not contained in $H$ and, consequently, in $X^c$, we have

\[ I^c \cup J^c \neq \{1, \ldots, n\} \]

which shows that $X^c$ does not belong to $C(n, 2)_2$. Observe that $P_{I \cup J^c}$ is the intersection of $X$ and $X^c$.

If $Y \in C$ is distinct from $X$, then $X^c$ and $Y^c$ are distinct. Indeed, if $X^c = Y^c$ for a certain $Y \in C$ distinct from $X$, then $Y$ contains $P_{I \cap J}$ and one of the subspaces $P_I, P_J$ which contradicts the fact that $Y \in C(n, 2)_2$.

Let $h$ be the map of $C(n, 2)_2$ to $\mathcal{G}_2(V)$ which leaves fixed every element of $A \cup B$ and sends every $X \in C$ to $X^c$. Since $X^c$ does not belong to $C(n, 2)_2$ for each $X \in C$ and $X^c \neq Y^c$ for distinct $X, Y \in C$, this map is injective.

Lemma 21. The map $h$ is adjacency preserving.

Proof. Let $X$ and $Y$ be adjacent elements of $C(n, 2)_2$. We need to show that $h(X), h(Y)$ are adjacent. Since the restriction of $h$ to $A \cup B$ is identity, the statement holds if $X, Y$ both belong to $A \cup B$.

We will use the following properties of $Z \in C$:

- the 1-dimensional subspace $Z \cap H$ is contained in $Z^c$;
- if $P_I \subset Z$ is not contained in $H$, then $P_{I^c} \subset Z^c$.

Suppose that $X, Y \in C$. If the 1-dimensional subspace $X \cap Y$ is contained in $H$, then

\[ X \cap Y = X^c \cap Y^c \]

which means that $h(X) = X^c, h(Y) = Y^c$ are adjacent. If $X \cap Y = P_I$ is not contained in $H$, then

\[ X^c \cap Y^c = P_{I^c} \]
and \( h(X) = X^c, h(Y) = Y^c \) are adjacent.

Consider the case when \( X \in \mathcal{C} \) and \( Y \in \mathcal{A} \cup \mathcal{B} \). If the 1-dimensional subspace \( X \cap Y \) is contained in \( H \), then \( X^c \) contains \( X \cap Y \) and \( h(X) = X^c, h(Y) = Y \) are adjacent. Assume that \( X \cap Y = P_J \) is not contained in \( H \). Then \( Y \in \mathcal{A} \). Since \( P_{\{1,\ldots,n\}} \subset Y \), the third 1-dimensional subspace of \( Y \) is \( P_I \), and we have
\[
X^c \cap Y = P_I^c
\]
which implies that \( h(X) = X^c, h(Y) = Y \) are adjacent. \( \square \)

Consider the restriction of \( \Gamma_2(V) \) to the image of \( h \). Removing from this restriction all edges connecting \( h(X) \) and \( h(Y) \) if \( X, Y \in \mathcal{C}(n,2) \) are not adjacent, we obtain a subgraph of \( \Gamma_2(V) \) isomorphic to \( \Gamma(n,2)_2 \).

Now, we show that \( h \) is adjacency preserving only in one direction. This means that \( h \) cannot be extended to an automorphism of \( \Gamma_2(V) \).

The 2-dimensional subspaces
\[
X = P_{\{1,3,\ldots,n\}} + P_{\{2,\ldots,n\}} \quad \text{and} \quad Y = P_{\{1,2,4,\ldots,n\}} + P_{\{3,\ldots,n\}}
\]
belong to \( \mathcal{C}(n,2)_2 \). The third 1-dimensional subspaces of \( X \) and \( Y \) are \( P_{\{1,2\}} \) and \( P_{\{1,2,3\}} \), respectively. Therefore, \( X \) and \( Y \) are not adjacent. We have \( X \in \mathcal{B} \) (since \( H \) contains \( P_{\{1,3,\ldots,n\}} \) and \( P_{\{2,\ldots,n\}} \)). Observe that
\[
P_{\{1,\ldots,n\}} \subset P_{\{1,2\}} + P_{\{3,\ldots,n\}}, \quad P_{\{1,\ldots,n\}} \not\subset H, \quad P_{\{1,2\}} \subset H,
\]
i.e. \( H \) does not contain \( P_{\{3,\ldots,n\}} \) and, consequently, \( Y \in \mathcal{C} \). The subspaces
\[
h(X) = X = P_{\{1,3,\ldots,n\}} + P_{\{1,2\}} \quad \text{and} \quad h(Y) = Y^c = P_{\{1,2,4,\ldots,n\}} + P_{\{1,2\}}
\]
are adjacent.

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