Chiral plasma instability and primordial Gravitational wave

Sampurn Anand,* Jitesh R. Bhatt,† and Arun Kumar Pandey‡
Physical Research Laboratory, Theoretical Physics Division, Ahmedabad 380 009, India

It is known that cosmic magnetic field, if present, can generate anisotropic stress in the plasma and hence, can act as a source of gravitational waves. On the other hand, cosmic magnetic can be generated even at very temperature, much above electroweak scale, due to gravitational anomaly. Such magnetic field, in its due course of evolution, generates instability in the chiral plasma. In this article we discuss the generation of gravitational waves due to turbulence in the chiral plasma sourced by the magnetic field created due to gravitational anomaly. Such a gravitational wave will have unique spectrum. We estimate the amplitude and frequency of such gravitational waves which may be detected in the Pulsar Timing Array (PTA) or Square Kilometer Array (SKA) experiments.

I. INTRODUCTION

Gravitational wave (GW) once generated, propagates almost unhindered through the space time. This property makes GW a very powerful probe of the source which produces it as well as the medium through which it propagates (see references [1–3] and references therein). From the cosmological point of view, the most interesting gravitational radiation are the stochastic gravitational wave (SGW) backgrounds. Such gravitational radiations are produced by events in the early stages of the Universe and hence, may decipher the physics of those epochs. Several attempts have already been made in this regard and various sources of SGW have been considered. List of SGW source includes quantum fluctuations during inflation [4–7], bubble wall collision during phase transition [8–13], cosmological magnetic fields [14–16] and turbulence in the plasma [16–18].

In the early universe, before electroweak phase transition, many interesting phenomenon could have taken place. For instance, it has been shown by several authors [19–24] that in presence of asymmetry in the left handed and right handed particles in the early Universe, there will be an instability which in turn leads to the generation of turbulence in the plasma as well as (hyper-) magnetic fields. The magnetic field generated via this mechanism are helical in nature. It is well known that the magnetic field as well as turbulence can generate anisotropic stress which can source the tensor perturbations in the metric. Recently, it has been shown that the seed magnetic field can be generated even in absence of net chiral charge but due to gravitational anomaly [25]. However, in presence of chiral imbalance, the magnetic field can produces instability in the plasma during its subsequent evolution in spacetime. This can lead to the generation of anisotropic stress and hence the GW. The underlying physics of GW generation is completely different from previously considered. Therefore, it is important to investigate the generation and evolution of GW in this context.

In this article, we compute the metric tensor perturbation due to chiral magnetic field. Since chiral magnetic field which source the tensor perturbation has a unique spectrum, the GWs generated is expected to have a unique signature in its spectrum as well. Moreover, we compute the amplitude and frequency of the GW and show its dependences on the model parameters. Consequently, any detection of SGW in future measurements like Pulsar Timing Array (PTA) and Square Kilometer Array (SKA) will constrain or rule out such theoretical constructs.

This paper is organized as follows: in section II we outline the generation and evolution of magnetic field due to gravitational anomaly and chiral imbalance. We discuss the generation of SGW in section III. We present our results in section IV and finally conclude in V. Throughout this work, we have used $\hbar = c = k_B = 1$ unit. We have also considered that, in an expanding universe, the line element for the background space time is described by Friedman-Robertson-Walker metric

$$ds^2 = a^2(\eta) \left(-d\eta^2 + \delta_{ij} dx^i dx^j\right),$$

where scale factor $a(\eta)$ have dimension of length, whereas conformal time $\eta$ and conformal coordinate $x^i$ are dimensionless quantities. In the radiation dominated epoch $a = 1/T$, we can define conformal time $\eta = M_* T$, where

$$M_* = \left(\frac{90}{8\pi^2 g_{\text{eff}}}\right)^{1/2} M_{\text{pl}} \cdot g_{\text{eff}}$$

and $M_{\text{pl}} = 1/\sqrt{G}$ are effective relativistic degree of freedom and Planck mass respectively.

II. GRAVITATIONAL ANOMALY AND MAGNETIC FIELDS IN THE EARLY UNIVERSE

Although origin of large scale magnetic field is still an unsolved issue in cosmology, several attempts have been made in this regard. It has been discussed in the literature that there are processes in the early universe, much above electroweak scale, which can lead to more number
of right-handed particle than the left-handed ones and remains in thermal equilibrium via its coupling with the hypercharge gauge bosons [26, 27]. Furthermore, if the plasma has rotational flow or external gauge field present, there could be a current in the direction parallel to the vorticity due rotational flow or parallel to the external field. The current parallel to the vorticity is known as chiral vortical current the phenomenon is called chiral vortical effects (CVE) [28–32]. Similarly, the current parallel to the external magnetic field is known chiral magnetic current and the phenomenon is called chiral magnetic effect (CME) [33–37]. CVE and CME are characterized by the coefficients $\xi$ and $\xi^{(B)}$ respectively. Form of these coefficients can be obtained by demanding the consistency with second law of thermodynamics ($\partial_{\mu} s^\mu \geq 0$, with $s^\mu$ being the entropy density). Thus, in presence of chiral imbalance and gravitational anomaly, which arises due to coupling of spin with gravity [38], the coefficients for each right and left particle have the following form [31, 32, 35]

\[
\xi_i = C \mu_i^2 \left[ 1 - \frac{2n_i \mu_i}{3(\rho + p)} \right] + \frac{D T^2}{2} \left[ 1 - \frac{2n_i \mu_i}{(\rho + p)} \right],
\]

(2)

\[
\xi_i^{(B)} = C \mu_i \left[ 1 - \frac{n_i \mu_i}{2(\rho + p)} \right] - \frac{D}{2} \frac{n_i T^2}{(\rho + p)}.
\]

(3)

The constants $C$ and $D$ are related to those of the chiral anomaly and mixed gauge-gravitational anomaly as $C = \pm 1/4\pi^2$ and $D = \pm 1/12$ for right and left handed chiral particles respectively.

Using the effective Lagrangian for the standard model, one can derive the generalized Maxwell’s equation [39],

\[
\vec{\nabla} \times \vec{B} = \vec{j},
\]

(4)

where $\vec{j} = j_e + j_5$ is the total current with $j_e$ being the vector current while $j_5$ is the axial current. Vector and axial currents respectively takes the following form:

\[
j_e^\mu = n_e u^\mu + \sigma E^\mu + \xi_e \omega^\mu + \xi_e^{(B)} B^\mu,
\]

(5)

\[
j_5^\mu = n_5 u^\mu + \xi_5 \omega^\mu + \xi_5^{(B)} B^\mu.
\]

(6)

In the above equations, the quantity $x_{e,5} = x_R \pm x_L$ denotes the sum (difference) of the quantities pertaining to right and left handed particles. We have ignored the displacement current in these equations. Taking $u^\mu = (1, \vec{v})$ and using eq.(5)- eq.(6), one can show that

\[
\vec{j}^0 = \eta n_e + n_5
\]

\[
\vec{j} = n \vec{v} + \sigma (\vec{E} + \vec{v} \times \vec{B}) + \xi \vec{w} + \xi^{(B)} \vec{B},
\]

(7)

with $\xi = \xi_e + \xi_5$ and $\xi^{(B)} = \xi_e^{(B)} + \xi_5^{(B)}$. Assuming the velocity field to be divergence free field, i.e. $\vec{\nabla} \cdot \vec{v} = 0$ and taking curl of eq.(4) along with the expression for current from eq.(7) we get,

\[
\frac{\partial \vec{B}}{\partial \eta} = \frac{n}{\sigma} \vec{v} \times \vec{v} + \frac{1}{\sigma} \nabla^2 \vec{B} + \vec{v} \times (\vec{v} \times \vec{B}) + \frac{\xi}{\sigma} \vec{v} \times \vec{w} + \frac{\xi^{(B)}}{\sigma} \vec{v} \times \vec{B}.
\]

(8)

At this stage, we define comoving variables such as magnetic fields $B_c = a^\prime(\eta)B(\eta)$, chemical potential $\mu_c = a(\eta) \mu$ and temperature $T_c = a(\eta) T$. In terms of these comoving variables, the evolution equations of fluid and electromagnetic fields are form invariant [40–42]. Therefore, we will work with the above defined comoving quantities and omit the subscript “c” in our further discussion.

In our previous work [25], we discussed that the seed magnetic field (for which $\vec{B}$ in the right hand side of eq. (8) is zero) can be generated even if $n = 0$. The $T^2$ term in $\xi$ (see eq. (2)), which arises due to gravitational anomaly, acts as a source for the generation of seed field. On the other hand, presence of finite chiral imbalance such that $\mu/T \ll 1$, $T^2$ term in $\xi$ still acts as source of seed magnetic field but non-zero $\xi^{(B)}$ triggers instability in the system. This result is in agreement with the previous studies where it was shown that in presence of chiral imbalance in the plasma, much above the Electroweak scale ($T > 100$ GeV), there can be instability known as chiral instability [43].

The production and evolution of the magnetic field can be seen through the evolution equation given in eq. (8). In order to do so, we decompose the divergence free vector fields, e.g. magnetic field, in the orthonormal helicity basis, $\varepsilon_i^\pm$, defined as

\[
\varepsilon_i^\pm(k) = \frac{-i}{\sqrt{2}} [\varepsilon_1(k) \pm i \varepsilon_2(k)],
\]

(9)

where $(\varepsilon_1, \varepsilon_2, \varepsilon_3 = \hat{k})$ form a right-handed orthonormal basis with $\varepsilon_2 = \hat{k} \times \varepsilon_1$. We choose $\varepsilon_1$ to remain invariant under the transformation $\vec{k} \rightarrow -\vec{k}$ while $\varepsilon_2$ flip its sign. In this basis, the magnetic field can be decomposed as

\[
B_i(k) = B^+(k) \varepsilon_i^+(k) + B^-(k) \varepsilon_i^-(k).
\]

(10)

In this basis, the evolution equation for the $|B^+|^2$ can be obtained from eq. (8)

\[
\frac{\partial |\vec{B}^+|^2}{\partial \eta} = \frac{2}{\sigma} \left( k^2 + \xi^{(B)} k \right) |\vec{B}^+|^2 + \frac{2}{\sigma} \left( \pm nk + \xi k^2 \right)^2 |\varepsilon_i^\pm|^2 F_i^a(\eta, k)
\]

(11)

where $F_i^a(\eta) = \eta - \eta_0$ for $\eta - \eta_0 \leq 2\pi/(k\nu)$ and zero for $\eta - \eta_0 \geq 2\pi/(k\nu)$. In order to obtain the velocity profile, we used the scaling symmetry [44, 45] rather than solving the Navier-Stokes equation. We also showed that the scaling law allows more power in the magnetic field. For $n = 0$, $E_B \propto k^2$ at larger length scale whereas for $n \neq 0$, $E_B \propto k^5$ instead of $k^7$ [22]. However, in both the scenarios $E_B$ is more than that of the case without considering the scaling symmetry [22]. This aspect is important for this analysis as more power in the magnetic field can generated larger anisotropic stress. Moreover, the magnetic field generated by this mechanism is helical in nature.

In the case of helical magnetic field, the two point correlation function is gives as

\[
\langle B_i(k) B_j^*(k') \rangle = \frac{(2\pi)^3}{2} \delta(k - k') \left[ P_{ij} S(k) + i \epsilon_{ijk} \mathcal{A}(k) \right]
\]

(2)
where $S(k)$ is the symmetric and $A(k)$ is the helical part of the magnetic field power spectrum. \( P_{ij} = \delta_{ij} - k_i k_j \) is the transverse plane projector which satisfies: \( P_{ij} k_j = 0 \), \( P_{ij} P_{jk} = P_{ik} \) with $\hat{k}_i = k_i / k$ and $\epsilon_{ijk}$ is the totally antisymmetric tensor. Using eq. (12) and the reality condition $B^\ast(k) = -B(k)^*$, one can show that

\[
\langle B^\ast(k) B^\ast(k') + B^-(k) B^- (k') \rangle = - (2\pi)^3 S(k) \delta(k - k')
\]

and

\[
\langle B^\ast(k) B^\ast(k') - B^-(k) B^- (k') \rangle = (2\pi)^3 A(k) \delta(k - k') .
\]

Note that, $A(k)$ represents the difference in the power of left handed and right handed magnetic fields, however a maximally helical magnetic fields configuration can be achieved when $A(k) = S(k)$.

1. Anisotropic stress

Tensor component of metric perturbation, which results in the gravitational waves, are sourced by the transverse-traceless part of the stress-energy tensor. In this work, we assume that the anisotropic stress is generated by the magnetic stress energy tensor which is given by

\[
T_{ij} = \frac{1}{a^2} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) ,
\]

where $a(q)$ is the scale factor. Note that the spatial indices are raised, lowered and contracted by the Kronecker delta such that $B^2 = \delta^{ij} B_i B_j$. The magnetic field component $B_i$ then coincide with the comoving magnetic field which in our notation is $B_c = a^2 B$. In Fourier space, the stress energy tensor for the magnetic field takes the following form [46]

\[
T_{ij}(k) = \frac{a^{-2}}{2(2\pi)^2} \int d^3q \left( B_i(q) B^*_j(q-k) - \frac{1}{2}B_i(q) B^*_j(q) \delta_{ij} \right) .
\]

We are interested in the generation of GW and hence, we need to extract the transverse traceless component of the stress energy tensor given in eq. (14). This can be done by using the projection operator which leads to [47]

\[
\Pi_{ij}(k) = \left( P_{ij}(k) P_{jm}(k) - \frac{1}{2} P_{ij}(k) P_{lm}(k) \right) T_{lm}(k) .
\]

At this stage, we will evaluate the two point correlation function of the energy momentum tensor which will be used in the later part of the calculation. The two point correlation of the stress energy tensor takes the form

\[
\langle T_{ij}(k) T^*_{lm}(k') \rangle = \frac{1}{(2\pi)^6} \frac{1}{(4\pi^2 a^4) \int d^3p \int d^3q \left[ (B_i(p) B^*_j(p-k) B^*_m(q) B_m(q-k)) + \ldots \delta_{ij} + \ldots \delta_{lm} + \ldots \delta_{ij} \delta_{lm} \right]}
\]

It was shown in ref. [48] that only first term in the angular bracket will have a non-vanishing contribution in the two-point correlation function of the anisotropic stress ($\Pi_{ij} \Pi^{*}_{lm}$). Therefore, we will not evaluate other terms. Moreover, around the chiral instability the magnetic field profile is Gaussian and the major contribution to the anisotropic stress comes from this regime only. Therefore, we can safely assume that the magnetic fields are Gaussian field and hence four point correlator in the integrand can be expressed, using Wick’s theorem, in terms of two point correlators as

\[
\langle B_i(k_i) B_j(k_j) B_l(k_l) B_m(k_m) \rangle = \\
\langle B_i(k_i) B_j(k_j) \rangle \langle B_l(k_l) B_m(k_m) \rangle + \langle B_i(k_i) B_l(k_l) \rangle \langle B_j(k_j) B_m(k_m) \rangle + \langle B_i(k_i) B_m(k_m) \rangle \langle B_j(k_j) B_l(k_l) \rangle .
\]

(17)

After a bit of lengthy but straight forward calculation, we obtain

\[
\langle T_{ij}(k) T^*_{lm}(k') \rangle = \frac{1}{4(4\pi)^2 a^4} \delta(k - k') \int d^3p \int d^3q \left[ S(p) S(k-p) \right] \left[ P_{ij}(\hat{p}) P_{jm}(k-p) + P_{im}(\hat{p}) P_{pj}(k-p) \right] - \mathcal{H}(p) \mathcal{H}(k-p) \epsilon_{ita} \epsilon_{jm b} \hat{p}_a (k-p)_b + \epsilon_{imc} \epsilon_{jld} \hat{p}_c (k-p)_d + i \mathcal{H}(p) S(k-p) \epsilon_{ita} P_{jm}(k-p) \hat{p}_a + i \mathcal{H}(k-p) S(p) \epsilon_{jm b} P_{il}(\hat{p}) (k-p)_b + i \mathcal{H}(k-p) S(p) \epsilon_{jld} P_{im}(\hat{p}) (k-p)_d \right] .
\]

(18)

and

\[
\langle \Pi_{ab}(k) \Pi^*_{cd}(k') \rangle = \mathcal{P}_{abij}(k) \mathcal{P}_{cdlm}(k') \langle T_{ij}(k) T^*_{lm}(k') \rangle ,
\]

(19)

where $\mathcal{P}_{abij}(k) = \left[ P_{ai}(k) P_{bj}(k) - \frac{1}{2} P_{ab}(k) P_{ij}(k) \right]$. Above equation can also be written in terms of a most general isotropic transverse traceless fourth rank tensor which obeys $\mathcal{P}_{abcd} = \mathcal{P}_{bacd} = \mathcal{P}_{abcd} = \mathcal{P}_{cdab}$ as [49, 50]

\[
\langle \Pi_{ab}(k) \Pi^*_{cd}(k') \rangle = \frac{1}{4a^4} [ M_{ab} c d f (k) + i \mathcal{A}_{abcd} g (k) ] \delta(k - k')
\]

with

\[
M_{abcd} = P_{ac} P_{bd} + P_{ad} P_{bc} - P_{ab} P_{cd}
\]

\[
\mathcal{A}_{abcd} = \frac{1}{2} \hat{k}_e (P_{bd} \epsilon_{ace} + P_{ac} \epsilon_{bde} + P_{ad} \epsilon_{bec} + P_{bc} \epsilon_{ade})
\]

(20)
and follows following properties:

\[ M_{abcd} = M_{cdab} = M_{abdc} = M_{bacd} \]
\[ A_{abcd} = A_{cdab} = -A_{abdc} = -A_{badc} \]
\[ M_{abab} = 4 \]
\[ M_{abcd} = M_{abcc} = 0 \]
\[ P_{ea}M_{abcd} = M_{ebcd} \]
\[ P_{ea}A_{abcd} = A_{ebcd} \]
\[ M_{abcd}M_{abcd} = M_{abcd}M_{abcd} = 8 \]
\[ A_{abcd}A_{abcd} = 0 \]
\[ A_{abab} = A_{abac} = A_{abcc} = 0. \]  

(22)

The functions \( f(k) \) and \( g(k) \) are integrals and defined as:

\[ \delta(k - k') f(k) = \frac{1}{2} M_{abcd} \langle T_{ab}(k) T_{cd}'(k') \rangle \]
\[ \delta(k - k') g(k) = -i \frac{1}{2} A_{abcd} \langle T_{ab}(k) T_{cd}'(k') \rangle. \]  

(23)

The product of symmetric part in the integrand of eq. (18), \( f_s(k) \), can be given as:

\[ f_s(k) = 2 \int d^3 p S(p) S(k - p)(1 + \gamma^2)(1 + \beta^2) \]  

(24)

where \( p = |p|, (k - p) = |k - p|, \gamma = \hat{k} \cdot \hat{p} \) and \( \beta = \hat{k} \cdot (\hat{k} - \hat{p}) = (k - py)/\sqrt{k^2 + p^2 - 2ypk} \). Similarly one can evaluate the product of antisymmetric parts which reads

\[ f_a(k) = 4 \int d^3 p \gamma \beta \mathcal{H}(p) \mathcal{H}(k - p) \]  

(25)

Hence, \( f(k) = f_s(k) + f_a(k) \) and \( g(k) \) are given by [48–51]

\[ f(k) = \frac{1}{4(4\pi)^2} \int d^3 p \left[(1 + \gamma^2)(1 + \beta^2) S(p) S(k - p) + 4 \gamma \beta \mathcal{H}(p) \mathcal{H}(k - p) \right] \]  

(26)

\[ g(k) = \frac{1}{2(4\pi)^2} \int d^3 p \left[(1 + \gamma^2) \beta S(p) \mathcal{H}(k - p) \right]. \]  

(27)

III. GRAVITATIONAL WAVES FROM CHIRAL MAGNETIC FIELDS

We have seen that the chiral magnetic field generated at very temperature can produce anisotropic stress which leads to tensor perturbation in the metric. In this paper, we consider flat FLRW metric and the tensor perturbation of the metric eq. (1) can be written as:

\[ ds^2 = a^2(\eta)[-d\eta^2 + (\delta_{ij} + 2h_{ij})dx^i dx^j]. \]  

(28)

where the tensor perturbation satisfies the following conditions \( h_{ij} = 0 \) and \( \partial^i h^i = 0 \). In this gauge, these tensor perturbations describe the GW whose evolution equation can be obtained by solving Einstein’s equation which, to the linear order in \( h_{ij} \), is given as:

\[ h''_{ij} + 2H h'_{ij} + k^2 h_{ij} = 16\pi G a^2 T_{ij}. \]  

(29)

where prime denotes the derivative with respect to the conformal time and \( H = \frac{1}{a(\eta)} \frac{\partial a(\eta)}{\partial \eta} \). The time dependence in the right hand side of the eq. (29) comes from the fact that the magnetic field is frozen in the plasma. Therefore, \( \Pi_{ij}(k, \eta) \) is a coherent source, in the sense that each mode undergoes the same time evolution. Assuming that the tensor perturbations has a Gaussian distribution function, the statistical properties can be described by the two point correlation function given as,

\[ \langle h'_{ij}(k, \eta) h'_{lm}(k', \eta) \rangle = \frac{1}{4} \delta^3(k - k')[M_{ijlm}S_{GW}(k, \eta) + iA_{ijlm}H_{GW}(k, \eta)] \]  

(30)

where \( S_{GW} \) and \( A_{GW} \) characterizes the amplitude and polarization of the GWs. With the above definition, we can write,

\[ \delta(k - k') S_{GW} = \frac{1}{(2\pi)^3} \langle \Pi_{ijlm}(k) \Pi'_{ijlm}(k') \rangle \]  

(31)

\[ \delta(k - k') H_{GW} = \frac{1}{(2\pi)^3} \langle \mathcal{A}_{ijlm}(k) h'_{ijlm}(k') \rangle \]  

(32)

We now choose a coordinate system, for which unit vectors are \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \), in which GW propagates in the \( \hat{e}_3 \) direction. Further, we introduce

\[ e^t_i = -\sqrt{\frac{3}{2}} (e^t_i \times e^t_j) \]  

(33)

which forms basis for a tensor perturbations and satisfy the following properties: \( \delta_{ij} e^t_i = 0, \delta_{ij} e^t_i = 0 \) and \( e^t_i e^t_i = 3/2 \). The right handed and left handed circularly polarized state of the GWs are represented by + and − sign respectively. In this basis, polarization tensor and modes of the GWs can be written as follows:

\[ \Pi_{ij}(k) = e^t_i \Pi^t(k) + e^t_i \Pi^t(k), \]  

(34)

\[ h_{ij}(k, \eta) = h^t(k, \eta) e^t_i + h^t(k, \eta) e^t_i. \]  

(35)

On using eq. (34), eq. (23) can be expressed as

\[ \delta(k - k') f(k) = \frac{3}{2} \langle \Pi^t(k) \Pi^t(k') + \Pi^t(k) \Pi^t(k') \rangle \]  

(36)

\[ \delta(k - k') g(k) = \frac{3}{2} \langle \Pi^t(k) \Pi^t(k') - \Pi^t(k) \Pi^t(k') \rangle \]  

(37)

Adding and subtracting eq. (36) and eq. (37) we obtain,

\[ f(k) + g(k) = 3(\Pi^t(k) \Pi^t(k')) \approx 3(|\Pi^t|^2) \]  

(38)

\[ f(k) - g(k) = 3(\Pi^t(k) \Pi^t(k')) \approx 3(|\Pi^t|^2). \]  

(39)
Similarly, we can write eq. (32) as:
\[ \delta(k - k')\mathcal{S}_{GW} = \frac{3}{2} \left\{ h^+(k, \eta)h^+(k', \eta) + h^-(k, \eta)h^-(k', \eta) \right\} \]
\[ \delta(k - k')\mathcal{H}_{GW} = -\frac{3}{2} \left\{ h^+(k, \eta)h^-(k', \eta) - h^-(k, \eta)h^+(k', \eta) \right\} \]

Therefore, components of the GWs evolve as
\[ h^{\pm}(k, \eta) + 2 \frac{a'}{a} h^{\pm}(k, \eta) + k^2 h^{\pm}(k, \eta) = 8\pi G \frac{\Pi_{\pm}(k)}{a^2}, \quad (40) \]

here \( \Pi_{ij} \) is the square root mean value of the transverse traceless part of the energy momentum tensor. In terms of dimensionless variable \( x = k\eta \), the above equation reduces to
\[ h^{\pm} + 2 \frac{\alpha'}{\alpha} h^{\pm} + h^{\pm} = \frac{s^\pm}{k^2}, \quad (41) \]

where \( s^\pm(k, \eta) = \left( \frac{8\pi G}{a^2} \right) \sqrt{\frac{f(k) \mp g(k)}{3}} \). And parameters \( \alpha = 1 \) and \( \alpha = 2 \) indicates the radiation dominated and the matter dominated epoch respectively. The homogeneous solution of the equation (41) in the radiation dominated are the spherical Bessel function \( j_0 \) and \( y_0 \). In our case, magnetic field is generated at \( \eta_{in} \) in the radiation dominated epoch due to chiral instability leading to anisotropic stress which in turn generates the gravitational waves. Thus, the general solution of eq. (41) can be given as,
\[ h^+(x) = c^+_1(x) j_0(x) + c^+_2(x) y_0(x) \quad (42) \]

where \( c_1(x) \) and \( c_2(x) \) are undetermined coefficients which is given as
\[ c^+_1(x) = -\int_{x_n}^x dx' \frac{s^+(x')}{w(x')k^2} y_0(x') \quad (43) \]
\[ c^+_2(x) = \int_{x_n}^x dx' \frac{s^+(x')}{w(x')k^2} j_0(x') \quad (44) \]

where \( w(x) = j_0(x)y'_0(x) - y_0(x)j'_0(x) = \frac{1}{x^2} \). We have calculated \( c^+_1(x) \) and \( c^+_2(x) \) using equations eq. (43) and eq. (44) under the limits of \( x > 1 \). In this limit, the second term with \( y_0 \) diverges, therefore, first term dominates over second one. In this case, in the radiation dominated epoch, the two polarizations of the tensor perturbations can be written as:

\[ h^+(x) = c^+_1(1) j_0(x) = -\frac{90}{\pi^2 g_{eff}} \sqrt{\frac{f - g}{3}} j_0(x) \log(x_{in}) \quad (45) \]

\[ h^-(x) = c^+_1(1) j_0(x) = -\frac{90}{\pi^2 g_{eff}} \sqrt{\frac{f + g}{3}} j_0(x) \log(x_{in}) \quad (46) \]

here \( x = 1 \) in \( c_1(1) \) signifies the value at the time of horizon crossing. After horizon crossing, these gravitational waves propagates without any hindrance. However their energy and polarization stretched by scale factor, similar to the magnetic radiation energy.

The time derivative of the equations eq. (45) and eq. (46) is
\[ h^+(x) = -\frac{90}{\pi^2 g_{eff}} \sqrt{\frac{f - g}{3}} \left[ j'_0(x) \log(x_{in}) \right] \quad (47) \]
\[ h^-(x) = -\frac{90}{\pi^2 g_{eff}} \sqrt{\frac{f + g}{3}} \left[ j'_0(x) \log(x_{in}) \right] \quad (48) \]

In real space, the energy density of the gravitational waves is defined as
\[ \rho_{GW} = \frac{1}{16 \pi G a^2} \langle h'_i h'_i \rangle. \quad (49) \]

Note that a factor of \( a^2 \) in the denominator comes from the fact that \( h' \) is the derivative respect to conformal time. In Fourier space, the energy density of the gravitational wave is given as
\[ S_{GW}(k) = \int \frac{dk}{k} \frac{dS_{GW}}{d\log k} \quad (50) \]

with
\[ \frac{d S_{GW}(k)}{d \log k} = \frac{k^3}{2 M^4 a^2 (2\pi)^6 G} (|h^+|^2 + |h^-|^2). \quad (51) \]

With this definition, we can define power spectrum evaluated at the time of generation as
\[ \frac{d \Omega_{GW,s}}{d \log k} = \frac{1}{\rho_{c,s}/M^4} \frac{dS_{GW,s}}{d \log k} \quad (52) \]
\[ = \frac{16\pi k^3}{3(2\pi)^6 a^2} \left( \frac{90}{\pi^2 g_{eff}} \right)^2 f(k) \left[ \frac{j'_0(x) \log(x_{in})}{H_0} \right]^2. \]

where \( \rho_{c,s} \) is the critical density of the universe at the time of generation of GW. Once gravitational waves are produced, they are decoupled from rest of the universe. This implies that the energy density of the gravitational waves will fall as \( a^{-4} \) and frequency red shifts as \( a^{-1} \). Hence, the power spectrum at today’s epoch can be given as
\[ \frac{d \Omega_{GW,0}}{d \log k} = \frac{d \Omega_{GW,s}}{d \log k} \left( \frac{a_s}{a_0} \right)^4 \left( \frac{\rho_{c,s}}{\rho_{c,0}} \right). \quad (53) \]

Assuming that the universe has expanded adiabatically which implies that the entropy per comoving volume is conserved leads to
\[ \frac{a_s}{a_0} = \left( \frac{g_{eff,0}}{g_{eff,s}} \right)^{1/3} \left( \frac{T_0}{T_s} \right), \quad (54) \]

where we have used \( g_{eff} \) for the effective degrees of freedom that contributes to the entropy density also. This is due to the fact that effective degrees of freedom that contribute to the energy and entropy densities are same at
very high temperature. Therefore, Eq.(53), using equation (54) reads as

\[
\frac{d\Omega_{\text{GW}_s}}{d\log k} = \left(\frac{g_{\text{eff},0}}{g_{\text{eff},s}}\right)^4 \left(\frac{T_0}{T_s}\right)^4 \left(\frac{H_s}{H_0}\right)^2 \frac{d\Omega_{\text{GW}_s}}{d\log k}
\]

\[
= \frac{16\pi k^3}{3(2\pi)^6 a_s^2} \times \left(\frac{90}{\pi^2 g_{\text{eff}}}\right)^2 \frac{g_{\text{eff},0}}{g_{\text{eff},s}}^{4/3} \left(\frac{T_0}{T_s}\right)^4 \frac{f(k)}{H_0^2} \left[j_0'(\chi)\log(x_{\text{ins}})\right]^2
\]

(55)

We have shown in figure (1-2), variation of GW spectrum with respect to k for different temperature at fix number density and for different number density at fix temperature. In these plots we have shown power spectrum at today’s epoch.

IV. RESULTS AND DISCUSSION

It has been shown in [24, 25], that the instability in the chiral plasma occurs at \(k_{\text{ins}} \sim \frac{\xi B(T)}{\sigma}\). Therefore, one would expect that the anisotropic stress generated due to chiral instability will peak around \(k_{\text{ins}}\) which in turn leads to a peak in GW. Since gravitational waves are generated due to instability in the plasma sourced by chiral imbalance above electroweak scale, therefore the dominant frequency of GW would be that of the maximally grown mode of the magnetic field which is \(\omega_m \approx \frac{\xi_B^2}{4\sigma}\). Thus, we can write the frequency of the gravitational waves as

\[\nu = \frac{2\pi}{\omega} = \frac{\xi_B^2}{8\pi\sigma}\]

Since \(\xi_B = \frac{1}{4\pi^2} \delta T\) where \(\delta = \mu_3/T\), this leads to \(\nu = \frac{1}{128\pi^3} \frac{\delta^2 T^2}{\sigma}\). Thus, at \(T = 10^8\) GeV where \(\delta \sim 10^{-6}\) and \(\sigma/T \sim 100\), we obtain the frequency of GW turns out to be \(\nu \approx \frac{10^{-4}}{128\pi}\) GeV. Redshifted value of the frequency at today’s epoch would be \(\nu_0 = \nu \left(\frac{T_0}{T_s}\right) \approx 10^{-8}\) Hz, which is in the range of the SKA and PTA sensitivity [54, 55].

In Fig. (1), we have shown that the GW spectrum indeed has a peak around \(k_{\text{ins}}\). It is also evident from this plot that at high temperature, when chiral imbalance are generated, the amplitude and frequency of the resulting GWs lies in the sensitivity range of SKA and PTA experiments. However, as temperature decreases, the amplitude of GWs generated also decrease and hence, they may not be observed in the proposed SKA and PTA observations. Further, it is to be noted that as temperature falls, the source of the GWs transfer its energy from lower length scale to large scale which is known as inverse cascade. Shift in the peak towards lower k, when temperature decreases, is a consequence of inverse cascade. This transfer is maximum, when magnetic fields are in maximally helical configuration.

Further, the strength of magnetic field changes when chiral charge density \(n\) change. Fig. (2) shows the effect of \(n\) on GW spectrum. It is apparent that the \(k_{\text{ins}}\) is not effected by the the number density and hence, peak does not shift. However, the power in a particular k mode enhances with increase in \(n\). This happens due to the fact that for larger value of \(n\), magnetic field strength is higher at larger k [25].

V. CONCLUSION

In the present work, we have extended our earlier works on the generation of primordial magnetic fields in a chi-
eral plasma [24, 25] to the generation of GWs. This kind of source may exist much above electroweak scale. We have shown that the gravitational anomaly generates the seed magnetic field which evolves and create instability in the system. This instability acts as a source of anisotropic stress which leads to the production of gravitational waves. The production and evolution of the magnetic field has been studied using eq. (8). In order to obtain the velocity profile, we have used scaling properties [44, 45] rather than solving the Navier-Stokes equation. This scaling properties results in more power in the magnetic field as smaller k as compared to that of the case without scaling symmetries (see [25]). We have calculated power spectrum of the produced GWs and shown that the spectrum has a distinct peak at $k_{\text{ins}}$ and hence correspond to the dominant frequency of GW. The GW generated via afore described method is potentially detectable through SKA and PTA observations.

In this work, we have considered massless electrons much above electroweak scale and discussed the production of gravitational waves due to chiral instabilities in presence of Abelian fields belonging to $U(1)_{\gamma}$ group. However, similar situation can arise in the case of Quark-Gluon Plasma (QGP) at $T \gtrsim 100$ MeV where quarks are not confined and interact with gluons which may results in instabilities. Thus, GW can be produced in QGP as well.

To conclude, the study of relic GWs can open the door to explore energy scales beyond our current accessibility and give insight into exotic physics.

ACKNOWLEDGEMENTS

We would like to acknowledge late Prof. P. K. Kaw for his insight and motivation towards the problem. We also thank Abhishek Atreya for discussions.
[40] K. A. Holcomb and T. Tajima, Phys. Rev. D40, 3809 (1989).
[41] R. M. Gailis, N. E. Frankel, and C. P. Dettmann, Phys. Rev. D52, 6901 (1995).
[42] C. P. Dettmann, N. E. Frankel, and V. Kowalenko, Phys. Rev. D48, 5655 (1993).
[43] Y. Akamatsu and N. Yamamoto, Phys. Rev. Lett. 111, 052002 (2013), arXiv:1302.2125 [nucl-th].
[44] P. Olesen, Phys. Lett. B398, 321 (1997).
[45] N. Yamamoto, Phys. Rev. D93, 125016 (2016).
[46] J. D. Jackson, Classical Electrodynamics (Wiley, 1998).
[47] R. Durrer and M. Kunz, Phys. Rev. D 57, R3199 (1998).
[48] A. Mack, T. Kahniashvili, and A. Kosowsky, Phys. Rev. D65, 123004 (2002), arXiv:astro-ph/0105504 [astro-ph].
[49] C. Caprini and R. Durrer, Phys. Rev. D 65, 023517 (2001).
[50] C. Caprini, R. Durrer, and T. Kahniashvili, Phys. Rev. D 69, 063006 (2004).
[51] R. Durrer, P. G. Ferreira, and T. Kahniashvili, Phys. Rev. D 61, 043001 (2000).
[52] C. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (W. H. Freeman, San Francisco, 1973), Sec. VIII, 1973).
[53] W. Hu and M. White, Phys. Rev. D 56, 596 (1997).
[54] C. J. Moore, R. H. Cole, and C. P. L. Berry, Class. Quant. Grav. 32, 015014 (2015), arXiv:1408.0740 [gr-qc].
[55] G. Janssen et al., Proceedings, Advancing Astrophysics with the Square Kilometre Array (AASKA14): Giardini Naxos, Italy, June 9-13, 2014, PoS AASKA14, 037 (2015), arXiv:1501.00127 [astro-ph.IM].