$B_{\text{Sen}}$ VIA DISTRIBUTIONS ON WEIGHT SPACE

by

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Contents

1. Introduction ........................................... 1
2. Notation ............................................. 2
3. The quotient $W/W_{\text{tors}}$ ....................... 3
4. Distributions ........................................ 7
5. Galois action and theta operator .................... 9
6. Relation to $B_{\text{Sen}}$ ............................ 11
References ............................................. 12

1. Introduction

In an unpublished preprint [3], Kisin introduces some $p$-adic period rings with an eye towards capturing periods for a certain class of overconvergent $p$-adic modular forms. The Fontaine-style functors associated to these rings are intended to be a sort of Betti realization for these forms. With this in mind, Kisin essentially “deforms” $B_{\text{HT}}$ to force non-integral $p$-adic powers of the cyclotomic character to turn up. The rings he defines are closely related to the Iwasawa algebra and non-canonically contain a copy of $B_{\text{HT}}$.

The $p$-adic modular forms he considers are of a limited sort. In particular, they are $p$-adic limits of classical forms. As such, their weights are $p$-adic limits of classical weights, which is why $p$-adic powers of the cyclotomic character suffice for his purposes. One might naturally be led to ask what can be said outside of this realm of the eigencurve. The first thing one might try to do is to further deform $B_{\text{HT}}$ to “see” all points of $p$-adic weight space $W$. Given the interpretation of the Iwasawa algebra in terms of distributions on $\mathbb{Z}_p$, one natural approach is to consider a suitable ring of distributions on the rigid-analytic space $W$. In so doing one quickly encounters a number of unpleasant features, including zero-divisors in the ring of distributions.
and too many Galois invariants. Both of these seem to be related to the presence of torsion in \( W \) (which conveniently lies just outside the region containing the \( \mathbb{Z}_p \)-powers of the cyclotomic character considered by Kisin).

This led the author to consider the quotient of \( W \) by its torsion. We show that this quotient has the structure of a rigid space and that a certain collection of distributions on this space does indeed furnish a nice ring of periods. In fact, this ring turns out to be canonically isomorphic via a “Fourier transform” to the the ring \( B_{\text{Sen}} \) introduced by Colmez in [1].

2. Notation

Fix an odd prime \( p \). In this paper, \( W \) will denote \( p \)-adic weight space. This is a rigid space over \( \mathbb{Q}_p \) whose points with values in a complete extension \( L/\mathbb{Q}_p \) are given by

\[
W(L) = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, L^\times)
\]

Let \( \tau \) denote the Teichmüller character \( \mathbb{Z}_p^\times \rightarrow \mu_{p-1} \) and define \( \langle x \rangle = x/\tau(x) \in 1 + p\mathbb{Z}_p \). Any \( \psi \in W(K) \) can be factored as

\[
\psi(x) = \psi(\tau(x))\psi(\langle x \rangle) = \tau(x)^i\psi(\langle x \rangle)
\]

for a unique \( i \in \mathbb{Z}/(p-1)\mathbb{Z} \). The space \( W \) is the disjoint union of \( p-1 \) disks \( W^i \) associated to these values of \( i \).

Let \( \gamma \) be a generator of the pro-cyclic subgroup \( 1 + p\mathbb{Z}_p \) (for example, one could take \( \gamma = 1 + p \)). Then the restriction of \( \psi \) to \( 1 + p\mathbb{Z}_p \) is determined by \( \psi(\gamma) \). Moreover, it is not difficult to show that \( \psi(\gamma) = 1 + t \) where \( t \in L \) satisfies \( |t| < 1 \) and that moreover any such \( t \) defines a character of \( 1 + p\mathbb{Z}_p \) that we will denote by \( \psi_t \). Thus the \( L \)-valued points of \( W \) are in bijective correspondence with \( p-1 \) copies of the open unit disk in \( L \) via

\[
(i, t) \mapsto \tau^i\psi_t
\]

For each \( n \geq 0 \), let \( W^0_n \) denote the admissible affinoid in \( W^0 \) defined by the inequality

\[
|\psi(\gamma) - 1| \leq p^{-1/\mu_{p-1}(p-1)}
\]

Note that this these affinoids do not depend on the choice of \( \gamma \). The numbering is set up so that if \( \zeta \) is a primitive \( p^n \)-th root of unity and \( \psi \) satisfies \( \psi(\gamma) = \zeta \), then \( \psi \in W^0_n \setminus W^0_{n-1} \). That is, \( W^0_n \cap W^0_{n-1} = W^0[p^n] \).

In what follows \( K \) will denote a complete extension of \( \mathbb{Q}_p \). If \( K/\mathbb{Q}_p \) is finite, then for each integer \( n \geq 0 \) we define \( K_n = K(\mu_{p^n}) \subseteq \overline{\mathbb{Q}_p} \) and \( K_{\infty} = \cup_n K_n \subseteq \overline{\mathbb{Q}_p} \). The letter \( \chi \) will always denote the cyclotomic character.

We recall some standard facts about the \( p \)-adic logarithm and exponential for future use. See [4] for a nice account of this material. Let

\[
\log_p(1 + x) = \sum_{k \geq 1} (-1)^{k+1} \frac{x^k}{k}
\]
and

\[ \exp_p(x) = \sum_{k \geq 0} \frac{x^k}{k!} \]

These series have radii of convergence 1 and \( p^{-1/(p-1)} \), respectively. If \( |x| < p^{-1/(p-1)} \), then \( |\log_p(1 + x)| = |x| \) and we have

\[ \exp_p(\log_p(1 + x)) = x \]

and

\[ \log_p(\exp_p(x)) = x \]

Also, the identity

\[ \log_p(uv) = \log_p(u) + \log_p(v) \]

holds whenever it makes sense (\(|u - 1|, |v - 1| < 1\)). One consequence of the last property is that \( \log_p(\zeta) = 0 \) for any \( p \)-power root of unity \( \zeta \). In fact, the converse of this statement holds as well in the sense that the \( p \)-power roots of unity are the only roots of \( \log_p \) in its domain of convergence.

### 3. The quotient \( W/W_{\text{tors}} \)

The space \( W \) has \((p-1)\)-torsion given by the powers \( \tau^i \) of the Teichmuller character. This torsion is all rational over \( \mathbb{Q}_p \) and the quotient of \( W \) by this torsion subgroup is canonically identified (via Teichmuller) with the identity component \( W^0 \). The space \( W^0 \) has only \( p \)-power torsion.

We wish to describe the quotient \( W^0/W_{\text{tors}}^0 \). Looking only at the \( \mathbb{C}_p \)-points of these spaces, the properties of \( \log_p \) described above ensure that it defines an injective map

\[ W^0(\mathbb{C}_p)/W_{\text{tors}}^0(\mathbb{C}_p) \rightarrow \mathbb{C}_p \]

\[ \psi \mapsto \log_p(\psi(\gamma)) \]

It is not difficult to see that this map is also surjective (see the proof of Theorem \( 3.2 \) below). This description has two deficiencies. The first is that it is only a description on points and does not identify \( W^0/W_{\text{tors}}^0 \) as a rigid space. The second is that it depends on the choice of \( \gamma \). The latter is easily remedied as follows. Identify \( W^0 \) with the disk defined by \( |t - 1| < 1 \) using \( \gamma \) as explained in the previous section, and define an analytic function \( \vartheta \) on \( W^0 \) by

\[ \vartheta = \frac{1}{\log_p(\gamma)} \log_p(1 + t) \]

In terms of characters, this functions is given by

\[ \vartheta(\psi) = \frac{\log_p(\psi(\gamma))}{\log_p(\gamma)} \]

It follows immediately from the latter description and properties of \( \log_p \) that the function \( \vartheta \) does not depend on the choice of \( \gamma \) and thus furnishes a canonical analytic function on \( W^0 \) that is defined over \( \mathbb{Q}_p \) (say, by taking \( \gamma = 1 + p \)) and satisfies

\[ \vartheta(\psi\psi') = \vartheta(\psi) + \vartheta(\psi') \quad \text{for all} \quad \psi, \psi' \in W^0 \]
The first difficulty is more subtle, as the notion of quotient in the setting of rigid spaces is not well-known. We shall adopt the following definition of quotient in our context (see [2] for a much more general discussion of such quotients). Let $G$ be a rigid-analytic group (such as $W^0$ or $W_n^0$) and $H$ be a subgroup (such as $W_{tor}^0$ or $W_n^0[p^n]$, respectively) and let $H \times G \rightarrow G$ denote the equivalence relation defined by multiplication and projection. We will say that a rigid space $X$ equipped with a surjective map $G \rightarrow X$ is the quotient of $G$ by $H$ if the compositions

$$H \times G \rightarrow G \rightarrow X$$

coincide and the resulting map

$$H \times G \rightarrow G \times_X G$$

is an isomorphism.

**Lemma 3.1.** — If $\psi \in W_n^0$, we have $|\vartheta(\psi)| \leq p^{n-1/(p-1)}$.

**Proof.** — If $\psi \in W_n^0$ then $|\psi(\gamma) - 1| \leq p^{-p/(p-1)} < p^{-1/(p-1)}$, so

$$|\vartheta(\psi)| = \frac{|\log_p(\psi(\gamma))|}{|\log_p(\gamma)|} = p|\log_p(\psi(\gamma))| \leq p^{1-p/(p-1)} = p^{-1/(p-1)}$$

which is the claim for $n = 0$.

Suppose that the claim holds for some $n$ and let $\psi \in W_{n+1}^0$. Observe that

$$|\psi(\gamma)^p - 1| = |((\psi(\gamma) - 1)^p - 1| \leq |(\psi(\gamma) - 1)p + p(\psi(\gamma) - 1)(1 + \cdots)| \leq p^{-1/p} p^{n-1}(p-1)$$

so that $\psi^p \in W_n^0$. Thus

$$p^{-1}|\vartheta(\psi)| = |p\vartheta(\psi)| = |\vartheta(\psi^p)| \leq p^{-1/(p-1)}$$

which establishes the claim for $n + 1$ and thus for all $n$ by induction.

The upshot of this lemma is that $\vartheta$ defines a canonical analytic function

(1) \[ \vartheta : W_n^0 \rightarrow \mathbb{A}_{p^{n-1/(p-1)}}^1 \]

for each $n \geq 0$, where $\mathbb{A}_{p^{n-1/(p-1)}}^1$ denotes the affinoid ball of radius of $p^{n-1/(p-1)}$ centered at 0 in the rigid-analytic affine line $\mathbb{A}^1$.

**Theorem 3.2.** — The map $\vartheta$ in (1) identifies $\mathbb{A}_{p^{n-1/(p-1)}}^1$ with the quotient of $W_n^0$ by $W_n^0[p^n]$

**Proof.** — Let us first show that the map $\vartheta$ in (1) is surjective. Suppose that $x \in \mathbb{C}_p$ has $|x| \leq p^{n-1/(p-1)}$. Then

$$|p^n \log_p(\gamma)x| \leq p^{-n-1}p^{n-1/(p-1)} = p^{-1-1/(p-1)} < p^{-1/(p-1)}$$

so $y = \exp_p(p^n \log_p(\gamma)x)$ is defined. Let $z \in \mathbb{C}_p$ satisfy $z^p = y$ and let $\psi$ be the unique point of $W^0_n$ with $\psi(\gamma) = z$. Then $|z - 1| \leq p^{-1/p^{n-1/(p-1)}}$, as is easy to see
using an inductive binomial theorem argument akin to the one in the proof of Lemma 3.1. Thus \( \psi \in W_n^0 \) and

\[
p^n \vartheta(\psi) = \frac{\log_p(\psi(\gamma)^n)}{\log_p(\gamma)} = \frac{\log_p(\exp_p(p^n \log_p(\gamma)x))}{\log_p(\gamma)} = p^n x
\]

so \( \vartheta(\psi) = x \) and \( \vartheta \) is surjective.

Let \( A \) denote the affinoid algebra of \( W_n^0 \) which we will identify with the Tate algebra of power series in \( t \) with coefficients in \( \mathbb{Q}_p \) that are strictly convergent for \( |t| \leq p^{-1/p^n - 1} \). That is,

\[
A = \left\{ \sum a_k t^k \mid |a_k|p^{-k/p^n - 1} \to 0 \text{ as } k \to \infty \right\}
\]

Similarly, we let \( B \) denote the affinoid algebra of \( A_{p^n-1/(p-1)}^1 \), which we identify with the power series over \( \mathbb{Q}_p \) in the variable \( s \) that are strictly convergent for \( |s| \leq p^{n-1/(p-1)} \). The map \( \vartheta \) corresponds to a function \( B \to A \), and by suggestive abuse of notation, we will denote the image of \( s \) under this pull-back by \( \vartheta(1 + t) \). Explicitly,

\[
\vartheta(1 + t) = \frac{\log_p(1 + t)}{\log_p(\gamma)} = \frac{1}{\log_p(\gamma)} \sum_{k \geq 0} (-1)^{k+1} \frac{t^k}{k}
\]

which lies in \( A \), as is evident by taking \( \gamma = 1 + p \), for example.

Note that the functional properties of \( \log_p \) correspond to (in fact, result from) formal properties of this series, which we use with only minor comments below.

The two maps \( W_n[p^n] \times W_n \to W_n^0 \) comprising the equivalence relation correspond to the maps

\[
A \quad \longrightarrow \quad A / ((1 + t)^{p^n} - 1) \circledast A
\]

\[
t \quad \longmapsto \quad 1 \otimes t
\]

\[
t \quad \longmapsto \quad 1 \otimes t + t \otimes 1 + t \otimes t
\]

We claim that the compositions of these maps with the map \( B \to A \) coincide. Indeed

\[
\vartheta((1 + (1 \otimes t + t \otimes 1 + t \otimes t)) = \vartheta((1 + 1 \otimes t)(1 + t \otimes 1))
\]

\[
= \vartheta(1 + 1 \otimes t) + \vartheta(1 + t \otimes 1)
\]

\[
= \vartheta(1 + 1 \otimes t) + p^{-n} \vartheta((1 + t \otimes 1)^{p^n})
\]

\[
= \vartheta(1 + 1 \otimes t)
\]

where the last equality follows because

\[
\vartheta((1 + t \otimes 1)^{p^n}) = \vartheta(1 + ((1 + t \otimes 1)^{p^n} - 1))
\]

lies in the ideal generated by \( (1 + t \otimes 1)^{p^n} - 1 \), which is 0. The upshot of the equality of these two functions is that we have a well-defined map

\[
A \circledast_B A \quad \longrightarrow \quad A / ((1 + t)^{p^n} - 1) \circledast A
\]

\[
t \otimes 1 \quad \longrightarrow \quad 1 \otimes t + t \otimes 1 + t \otimes t
\]
To complete the proof, we must show that this map is an isomorphism. Let us try to define an inverse map via

\[
A/(1 + t)^{p^n} \otimes A \longrightarrow A \otimes_B A
\]

\[
1 \otimes t \longmapsto 1 \otimes t
\]

\[
t \otimes 1 \longmapsto \frac{1 + t \otimes 1}{1 + 1 \otimes t} - 1 = (1 + t \otimes 1) \left( \sum_{k \geq 0} (-1)^k (1 \otimes t)^k \right) - 1
\]

If this map is well-defined, then it is a simple formal matter to check that it is the inverse of the map defined above. To see that it is well-defined, first note that the series in the definition has radius of convergence 1 and thus lies in \( A \). Finally, we must show that \((1 + t \otimes 1)^{p^n} - 1\) gets sent to 0 under the above proposed map. Thus, we need to check

\[
\left( \frac{1 + t \otimes 1}{1 + 1 \otimes t} \right)^{p^n} - 1 = 0
\]

in \( A \otimes_B A \), which is in turn equivalent to \((1 + t)^{p^n} \otimes 1 = 1 \otimes (1 + t)^{p^n}\) in \( A \otimes_B A \). By definition of the tensor product, this will follow if we can find an element \( b \in B \) whose image in \( A \) under \( B \longrightarrow A \) is \((1 + t)^{p^n}\). We claim that

\[
b = \exp_p(p^n \log_p(\gamma)s) = \sum_{k \geq 0} \frac{(p^n \log_p(\gamma))^k}{k!} s^k
\]

is such an element. First, since \( \exp_p(x) \) is strictly convergent on \( |x| \leq p^{-1-(p-1)} < p^{-1/(p-1)} \), the given series is strictly convergent for \( |s| \leq p^{n-1/(p-1)} \) and thus lies in \( B \). The image of \( b \) under the map \( B \longrightarrow A \) is

\[
\exp_p(p^n \log_p(\gamma) \vartheta(1 + t)) = \exp_p \left( p^n \log_p(\gamma) \frac{\log_p(1 + t)}{\log_p(\gamma)} \right) = (1 + t)^{p^n}
\]

as desired.

\( \square \)

**Definition 3.3.** — For each \( n \geq 0 \), let \( X_n \) denote the quotient \( \mathcal{W}_n^0/\mathcal{W}_n^0[p^n] \). By the previous theorem, \( X_n \) is an affinoid defined over \( \mathbb{Q}_p \) equipped with a canonical isomorphism \( \vartheta : X_n \overset{\sim}{\longrightarrow} \mathbb{A}_{p^{n-1/(p-1)}}^1 \).

Theorem 3.2 has the following immediate consequence.

**Corollary 3.4.** — The ring \( \mathcal{O}(X_n \otimes K) \) consists of functions of the form

\[
f = \sum a_k \vartheta^k
\]

with \( a_k \in K \) and \( |a_k| p^{k(n-1/(p-1))} \longrightarrow 0 \). In terms of this expansion we have

\[
\|f\|_{\sup} = |a_k| p^{k(n-1/(p-1))}
\]

for all such \( f \).
For each $n \geq 0$ the natural map

\begin{equation}
X_n \longrightarrow X_{n+1}
\end{equation}

is injective and identifies $X_n$ with an admissible affinoid open in $X_{n+1}$. Gluing over increasing $n$, we conclude that the function $\vartheta$ furnishes a canonical isomorphism

$$W^0/W^0_{\text{tars}} \sim K^1$$

\section{Distributions}

To the affinoids $X_n$, we associate spaces of bounded distributions as follows.

\textbf{Definition 4.1.} — Let $K/\mathbb{Q}_p$ be a complete extension. A bounded $K$-valued distribution on $X_n$ is a $\mathbb{Q}_p$-linear map $\mu : \mathcal{O}(X_n) \longrightarrow K$ such that there exists $C$ such that $|\mu(f)| \leq C\|f\|_{\sup}$ for all $f \in \mathcal{O}(X_n)$. We denote the space of all such distributions by $\mathcal{D}(X_n, K)$. The map (2) induces a map

\begin{equation}
\mathcal{D}(X_n, K) \longrightarrow \mathcal{D}(X_{n+1}, K)
\end{equation}

for each $n$, and we define

$$\mathcal{D}(X_\infty, K) = \lim_{\longrightarrow} \mathcal{D}(X_n, K)$$

The maps in (3) are injective for all $n \geq 0$, so the injective limit is simply a union. This injectivity is not obvious from the definition, but is a simple consequence of Lemma 4.3 below.

\textbf{Remark 4.2.} — If $K/L$ is an extension of complete extensions of $\mathbb{Q}_p$, then any $\mu \in \mathcal{D}(X_n, K)$ induces an $L$-linear map

$$\mathcal{O}(X_n \hat{\otimes} L) \longrightarrow K$$

by extension of scalars in the obvious manner.

The following lemma allows us to characterize our distributions on $X_n$ via their “moments” under the isomorphism of Theorem 3.2.

\textbf{Lemma 4.3.} —

1. If $\mu \in \mathcal{D}(X_n, K)$, then $|\mu(\vartheta^k)|p^{-k(n-1)/(p-1)}$ is bounded in $k$.

2. Conversely, if $x_k \in K$ is a sequence such that $|x_k|p^{-k(n-1)/(p-1)}$ is bounded, then there exists a unique $\mu \in \mathcal{D}(X_n, K)$ such that $\mu(\vartheta^k) = x_k$.

\textbf{Proof.} —

1. Since $\mu \in \mathcal{D}(X_n, K)$, there exists $C$ such that $|\mu(f)| \leq C\|f\|_{\sup}$ for all $f \in \mathcal{O}(X_n)$. Thus

$$|\mu(\vartheta^k)| \leq C\|\vartheta^k\|_{\sup} = Ck^{n-1/(p-1)}$$


2. Let \( x_k \) be as in the statement. For \( f \in \mathcal{O}(X_n) \) we may write
\[
f = \sum_k a_k \vartheta^k
\]
with \( |a_k| p^{k(n-1/(p-1))} \to 0 \) by Corollary 3.4. The hypotheses on \( a_k \) and \( x_k \) ensure that \( a_k x_k \to 0 \), so we may define \( \mu(f) = \sum_k a_k x_k \). Note that
\[
|\mu(f)| \leq \sup_k |a_k| |x_k| = \sup_k (|a_k| p^{k(n-1/(p-1))} (|x_k| p^{-k(n-1/(p-1))})
\]
\[
\leq \|f\| \sup_k |x_k| p^{-k(n-1/(p-1))}
\]
again by Corollary 3.4. Thus \( \mu \) is bounded and defines an element of \( \mathcal{D}(X_n, K) \).

The uniqueness of \( \mu \) follows because \( \mu(\sum b_k \vartheta^k) = \sum b_k \mu(\vartheta^k) \) holds for any \( \mu \in \mathcal{D}(X_n, K) \) and any \( \sum b_k \vartheta^k \in \mathcal{O}(X_n) \) by the boundedness of \( \mu \), so any such \( \mu \) is determined by its moments.

The group structure on \( X_n \) endows \( \mathcal{D}(X_n, K) \) with a convolution product. To define this product, we will need a lemma. For \( f \in \mathcal{O}(X_n) \) and \( \varphi \in X_n \), define a function \( T_{\varphi}f \) on \( X_n \) by
\[
T_{\varphi}f(\psi) = f(\varphi \psi)
\]
Note that if \( \varphi \) is a \( K \)-valued point of \( X_n \), then \( T_{\varphi}f \) is naturally an element of \( \mathcal{O}(X_n \hat{\otimes} K) \).

**Lemma 4.4.** — Let \( f \in \mathcal{O}(X_n) \) and let \( \mu \in \mathcal{D}(X_n, K) \). The function
\[
\varphi \mapsto \mu(T_{\varphi}f)
\]
is an element of \( \mathcal{O}(X_n \hat{\otimes} K) \).

**Proof.** — First note that this function makes sense by Remark 3.2 and the comment preceding the lemma. By Corollary 3.4 \( f \) may be written as \( f = \sum a_k \vartheta^k \) with \( |a_k| p^{k(n-1/(p-1))} \to 0 \). We have
\[
(T_{\varphi}f)(\psi) = f(\varphi \psi) = \sum_k a_k \vartheta(\varphi \psi)^k
\]
\[
= \sum_k a_k (\vartheta(\varphi) + \vartheta(\psi))^k
\]
\[
= \sum_k \sum_{m \leq k} a_k \binom{k}{m} \vartheta(\varphi)^m \vartheta(\psi)^{k-m}
\]
Thus, since \( \mu \) is bounded we have
\[
\mu(T_{\varphi}f) = \sum_k \sum_{m \leq k} a_k \binom{k}{m} \mu(\vartheta^{k-m}) \vartheta(\varphi)^m = \sum_m \left( \sum_{k \geq m} a_k \binom{k}{m} \mu(\vartheta^{k-m}) \right) \vartheta(\varphi)^m
\]
According to Corollary 3.4 in order to complete the proof we must show that
\[ \left| \sum_{k \geq m} a_k \left( \frac{k}{m} \right) \mu(g^{k-m}) \right| p^{m(n-1/(p-1))} \to 0 \]
as \( m \to \infty \). This expression is bounded by the supremum over \( k \geq m \) of
\[ |a_k| \mu(g^{k-m})|p^{m(n-1/(p-1))} = |a_k|p^{k(n-1/(p-1))}|\mu(g^{k-m})|p^{(m-k)(n-1/(p-1))} \]
By Corollary 3.4 there exists \( C \) such that \( |\mu(g^{k-m})| \leq Cp^{k(n-1/(p-1))} \) for all \( k \geq m \), so the previous bound is at most \( C|a_k|p^{k(n-1/(p-1))} \). This quantity tends to zero in \( k \), and therefore in \( m \) since \( m \leq k \).

For \( \mu, \nu \in \mathcal{D}(X_n, K) \) we define the convolution of \( \mu \) and \( \nu \) by
\[ (\mu * \nu)(f) = \mu(\varphi \mapsto \nu(T_\varphi f)) \]
This is well-defined by the previous lemma, and clearly defines an element of \( \mathcal{D}(X_n, K) \). It is trivial to check that the distribution \( \mu_1 \in \mathcal{D}(X_0, Q_p) \) given by \( \mu_1(f) = f(1) \), that is, the Dirac distribution associated to the identity character, is a two-sided identity for the convolution product. As we will see below, the rings \( \mathcal{D}(X_n, K) \) are in fact commutative integral domains.

**Example 4.5.** — Let \( \psi, \psi' \in \mathcal{W} \). The convolution of the Dirac distributions associated to these two characters is
\[ (\mu_\psi * \mu_\psi')(f) = \mu_\psi(\varphi \mapsto \mu_\psi(T_\varphi f)) = \mu_\psi(\varphi \mapsto f(\varphi \psi')) = f(\psi \psi') \]
which is to say \( \mu_\psi * \mu_\psi = \mu_{\psi \psi'} \).

5. Galois action and theta operator

The group \( G = \text{Gal}(\mathbb{Q}_p/Q_p) \) acts on the \( \mathbb{C}_p \)-valued points of \( X_n \) with \( g \in G \) acting on the class of a character \( \psi \) to give the class of \( g \circ \psi \). We define an action of \( G \) on \( \mathcal{O}(X_n \otimes \mathbb{C}_p) \) via
\[ f^g(\psi) = g(f(g^{-1} \circ \psi)) \]
In terms of the expansion of Corollary 3.4 this action is simply given by the action of \( G \) on the coefficients \( a_k \) since \( g \) is defined over \( Q_p \), and hence the isomorphism of Theorem 3.2 intertwines this action with the usual one (given by action on coefficients) on analytic functions on the ball \( \mathbb{A}_p^{1(n-1/(p-1))} \).

Let \( G_n = G_{Q_p,n} \subset G \) be the subgroup \( G_n = \text{Gal}(\mathbb{Q}_p/Q_p(\mu_{p^n})) \). Then for \( g \in G_n \) we have \( v(\log_p(\chi(g))) = v(\chi(g) - 1) \geq n \) and hence
\[ p^{k(n-1/(p-1))} \left| \frac{(\log_p(\chi(g)))^k}{k!} \right| \leq 1 \]
It follows from Corollary 5.4 that
\[ \exp_p(\vartheta \log_p(\chi(g))) = \sum_k \frac{(\log_p(\chi(g)))^k}{k!} \vartheta^k \]
is an analytic function on $X_n$. This function allows us to define an action of $G_n$ on $\mathcal{D}(X_n, \mathbb{C}_p)$ by the formula

$$ (g \cdot \mu)(f) = g(\mu(f^{g^{-1}} \cdot \exp_p(\vartheta \log_p(\chi(g)))) ) $$

It is an easy matter to check that this preserves the boundedness condition and defines an action of $G_n$. The collection $\{ \mathcal{D}(X_n, \mathbb{C}_p) \}$ together with the respective actions by $G_n$ is an instance of a “$G_{\mathbb{Q}_p, \infty}$-module” in the sense of Colmez in \cite{Colmez}.

**Remark 5.1.** — In the special case $n = 0$, the above recipe actually defines an action the entire group $G$ on $\mathcal{D}(X_0, \mathbb{C}_p)$ if we agree that $\log_p(\chi(g))$ is to be interpreted as $\log_p(\chi(g)\tau(\chi(g))^{-1})$ (equivalently, if we agree that $\log_p$ be extended to $\mathbb{Z}_p^\times$ so that the $(p-1)^{st}$ roots of unity be sent to zero).

Let us now turn to the $\Theta$ operator. For $\mu \in \mathcal{D}(X_n, \mathbb{C}_p)$, define a distribution $\Theta\mu$ by the formula $(\Theta\mu)(f) = \mu(f \cdot \vartheta)$. The operator $\Theta$ so defined preserves the space of bounded distributions and evidently commutes with the action of $G_n$.

**Example 5.2.** — Let us determine explicitly the action of $G_n$ on the Dirac distribution $\mu_\psi$ for $\psi$ a point of $\mathcal{W}_0^\circ$. 

$$ (g \cdot \mu_\psi)(f) = g(\mu_\psi(f^{g^{-1}} \cdot \exp_p(\vartheta \log_p(\chi(g)))) ) $$

$$ = g(f^{g^{-1}}(\psi) \exp_p(\vartheta(\psi) \log_p(\chi(g)))) $$

$$ = f(g \circ \psi) \exp_p(\vartheta(g \circ \psi) \log_p(\chi(g))) $$

so

$$ g \cdot \mu_\psi = \mu_{g \circ \psi} \cdot \exp_p(\vartheta(g \circ \psi) \log_p(\chi(g))) $$

In particular, consider the $\mathbb{Q}_p$-valued character $\psi(x) = x^k \tau(x)^{-k} \in \mathcal{W}_0^\circ$. By Remark 5.1 it makes sense to consider $g \cdot \mu_\psi$ for any $g \in G$, and the above computation shows that

$$ g \cdot \mu_\psi = \mu_{g \circ \psi} \cdot \exp_p(k \log_p(\chi(g)\tau(\chi(g))^{-1})) $$

Thus the $\mathbb{C}_p$-subalgebra (= $\mathbb{C}_p$-submodule) of $\mathcal{D}(X_0, \mathbb{C}_p)$ generated by these characters for all $k \in \mathbb{Z}$ is rather akin to the ring $B_{HT}$, but with all characters twisted by a Teichmüller power to lie in $\mathcal{W}_0^\circ$. Alternatively, one may choose a nonzero $\alpha \in \mathbb{C}_p$ with the property that $g(\alpha) = \tau(\chi(g))\alpha$. Then we have (see Example 4.5)

$$ g \cdot (\alpha \mu_{\tau(\chi)^{-1}} f) = g \cdot (\alpha^k \mu_{\tau(\chi)^{-k}} f) $$

$$ = g(\alpha^k)(\mu_{\tau(\chi)^{-k}} f) $$

$$ = (\tau(\chi(g))^k \alpha^k)(\mu_{\tau(\chi)^{-k}} f) $$

$$ = \chi(g)^k(\alpha^k \mu_{\tau(\chi)^{-k}} f) $$

It follows that the $\mathbb{C}_p$-subalgebra (= $\mathbb{C}_p$-submodule) of $\mathcal{D}(X_0, \mathbb{C}_p)$ generated by the $\alpha^k \mu_{\tau(\chi)^{-k}} f$ for $k \in \mathbb{Z}$ is isomorphic to $B_{HT}$ as a $G$-module, so we have a non-canonical injection $B_{HT} \hookrightarrow \mathcal{D}(X_0, \mathbb{C}_p)$ corresponding to the choice of $\alpha$. This is reminiscent of the non-canonical injections of $B_{HT}$ into the rings considered in \cite{HT} and \cite{Colmez}. 


6. Relation to $B_{\text{Sen}}$

In [1], Colmez introduces the ring $B_{\text{Sen}}$ defined as the collection of power series in $\mathbb{C}_p[[T]]$ with positive radius of convergence. This ring has an ascending filtration $\{B^n_{\text{Sen}}\}$ where $B^n_{\text{Sen}}$ consists of power series of radius of convergence at least $p^{-n}$. Colmez defines a “$G_{\mathbb{Q}_p,\infty}$” action on this filtered ring, meaning a compatible collection of actions of $G_n$ on $B^n_{\text{Sen}}$, by acting in the natural way on coefficients and setting $g \cdot T = T + \log_p(\chi(g))$.

**Definition 6.1.** — Let $\mu \in \mathcal{D}(X_n, \mathbb{C}_p)$. The Fourier transform of $\mu$ is the formal series

$$F_n(\mu) = \mu(\exp(T\vartheta)) := \sum k \frac{\mu(\vartheta^k)}{k!} T^k$$

Using the usual estimate for the $p$-divisibility of $k!$ we have

$$\left| \frac{\mu(\vartheta^k)}{k!} T^k \right| \leq |\mu(\vartheta^k)| p^{-k(n-1/(p-1))} (p^n|T|)^k$$

If $|T| < p^{-n}$, then this tends to 0 as $k \to \infty$ by Lemma 4.3, so the series defining $F_n(\mu)$ has radius of convergence at least $p^{-n}$. Thus $F_n$ is a $\mathbb{C}_p$-linear map

$$F_n : \mathcal{D}(X_n, \mathbb{C}_p) \to B^n_{\text{Sen}}$$

**Proposition 6.2.** — The Fourier transform $F_n$ is an injective ring homomorphism.

**Proof.** — That $F_n$ is injective follows from Corollary 3.4. Observe that

$$F_n(\mu \ast \nu) = (\mu \ast \nu)(\exp(T\vartheta))$$

$$= \mu(\varphi \mapsto \nu(T\varphi \exp(T\vartheta)))$$

$$= \mu(\varphi \mapsto \nu(\exp(T\vartheta(\varphi))))$$

$$= \mu(\varphi \mapsto \nu(\exp(T\vartheta(\varphi)) \exp(T\vartheta(\psi))))$$

$$= \mu(\exp(T\vartheta)) \nu(\exp(T\vartheta)) = F_n(\mu) F_n(\nu)$$

Lastly, note that

$$F_n(\mu_1) = \mu_1(\exp(T\vartheta)) = \exp(T\vartheta(1)) = \exp(0) = 1$$

Thus $F_n$ is indeed a ring homomorphism. \qed

It follows immediately from this that the rings $\mathcal{D}(X_n, K)$ are commutative integral domains. In fact, we can say rather more. Let $P(T) = \sum a_k T^k$ have positive radius of convergence. Then there exists a positive integer $m$ such that $|a_k| p^{-km} \to 0$. The standard estimates on the $p$-divisibility of $k!$ imply that

$$|k! a_k| p^{-k(m+1-1/(p-1))} = |a_k| p^{-km} |k!| p^{-kp/(p-1)} \to 0$$

so by Lemma 4.3 there exists $\mu \in \mathcal{D}(X_{m+1}, \mathbb{C}_p)$ with $\mu(\vartheta^k) = k! a_k$. Thus we have $F_{m+1}(\mu) = P(T)$, and, although the individual $F_n$ are not isomorphisms (it is not difficult to check that they are not individually surjective), the limit

$$F : \mathcal{D}(X_{\infty}, \mathbb{C}_p) = \lim_n \mathcal{D}(X_n, \mathbb{C}_p) \sim B_{\text{Sen}}$$
is an isomorphism of rings.

We claim that the Fourier transform $\mathcal{F}_n$ intertwines the action of $G_n$ that we have defined on bounded distributions with the action defined by Colmez. The following calculations are somewhat formal, but can be made rigorous with power series expansions with no difficulty. We have

$$\mathcal{F}_n(g \cdot \mu)(T) = (g \cdot \mu)(\exp(T \vartheta))$$
$$= g(\mu(\exp(g^{-1}(T) \vartheta) \exp(\vartheta \log_p(\chi(g))))))$$
$$= g(\mu(\exp((g^{-1}(T) + \log_p(\chi(g))) \vartheta)))$$
$$= g(\mathcal{F}_n(\mu)(g^{-1}(T) + \log_p(\chi(g))))$$

which is equal to $g$ applied to $\mathcal{F}_n(\mu)$ in the sense of Colmez, since $\log_p(\chi(g)) \in \mathbb{Q}_p$ is Galois-invariant.

In a similar manner, we can determine the way in which $\Theta$ interacts with the Fourier transform.

$$\mathcal{F}_n(\Theta \mu) = (\Theta \mu)(\exp(T \vartheta)) = \mu(\vartheta \exp(T \vartheta)) = \frac{d}{dT} \mathcal{F}_n(\mu)$$

Note that this is the negative of the operator called $\Theta$ in [1].

Recall that for a $G_{K_{\infty}}$-module $M = \bigcup_n M_n$ in the sense of [1] we define $M^{G_{K_{\infty}}} = \bigcup_n M_n^{G_{K_n}}$.

**Proposition 6.3.** —

1. $\mathcal{D}(X_{\infty}, \mathbb{C}_p)^{G_{\mathbb{Q}_{p,\infty}}} = \mathbb{Q}_{p,\infty}$
2. If $V$ is $d$-dimensional $\mathbb{C}_p$-representation of $G$, then

$$(V \otimes_{\mathbb{C}_p} \mathcal{D}(X_{\infty}, \mathbb{C}_p))^{G_{\mathbb{Q}_{p,\infty}}} := \lim_{\to} (V \otimes_{\mathbb{C}_p} \mathcal{D}(X_n, \mathbb{C}_p))^{G_n}$$

is isomorphic to $D_{\text{Sen}}(V)$ as a $\mathbb{Q}_{p,\infty}$ vector space equipped with an endomorphism $\Theta$. In particular, it is $d$-dimensional over $\mathbb{Q}_{p,\infty}$.

**Proof.** — We have already shown that we have a compatible sequence of $G_n$-equivariant injections

$$\mathcal{F}_n : \mathcal{D}(X_n, \mathbb{C}_p) \to B_{\text{Sen}}^n$$

that are an isomorphism in the injective limit. It follows easily that they induce an isomorphism on invariants, so the result follows from Théorème 2 of [1].

**Remark 6.4.** — This result holds more generally with $\mathbb{Q}_p$ replaced by any finite extension $K/\mathbb{Q}_p$, as Théorème 2 of [1] is proven in this generality.

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