Wick rotation of linearized gravity in Gaussian time
and Calderón projectors

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Abstract. Motivated by the quantization of linearized gravity, we consider gauge-fixed linearized Einstein equations and their Wick rotation near a Cauchy surface. We show that Calderón projectors for the Wick-rotated equations induce Hadamard bi-solutions on the Lorentzian level. On the other hand, we find smoothing obstructions to gauge-invariance and positivity conditions needed in quantization. These obstructions are primarily due to boundary terms arising in the Wick-rotated theory and depend on the boundary conditions.

1. Introduction and summary

1.1. Introduction. The quantization of linearized gravity has remained incomplete for a long time. The structure of the classical theory needed for the quantization is nowadays relatively well understood [13, 29, 9, 6] and various possible candidates for physical states are discussed in the physics literature in some cases of highly symmetric Einstein metrics, it is however unclear if they are indeed positive. The rigorous construction of physical states has remained an open problem: in fact, there is presently no mathematically satisfactory answer even for perturbations of Minkowski space.

Beside the prerequisite of being positive, the main criterion for a state to be physical is the so-called Hadamard condition, which is needed for the renormalization of non-linear quantities (see e.g. [19, 1, 15]) and for the mathematical formulation of perturbative quantum gravity [9, 8, 43]. Unfortunately, the standard deformation argument for the existence of Hadamard states [19] does not apply to linearized gravity. In fact, the spacetime \((M, g)\) which serves as the background for the linearization must be a solution of the non-linear Einstein equations, so it cannot be arbitrarily deformed. Furthermore, more advanced techniques based on pseudodifferential calculus [35, 23] fail to preserve the gauge invariance of the equations if applied directly, whereas conformal scattering methods [10] are affected by divergent behaviour at conformal infinity [6]. A pseudodifferential construction of Hadamard states is known in the case of Yang–Mills fields linearized around a possibly non-zero solution [26], it uses however a spacetime deformation argument and various methods which are specific to differential forms.

Let us explain the problem in more detail. Let \((M, g)\) be a globally hyperbolic spacetime with \(\dim M = 4\), such that \(g\) solves the non-linear Einstein equations. We consider...
the two differential operators
\[ P = -\Box - I \circ d \circ \delta + 2 \text{Riem}_g, \quad K = I \circ d, \]
acting on symmetric \((0,2)\)-tensors, resp. \((0,1)\)-tensors, where:

- \(\Box\) is the d’Alembertian associated to \(g\), i.e. 
  \[ (\Box u)_{ab} = \nabla^c \nabla_c u_{ab}, \]
- \(I\) is the trace reversal, i.e. 
  \[ (Iu)_{ab} = u_{ab} - \frac{1}{2} \text{tr}_g (u) g_{ab}, \]
- \(d\) the symmetric differential, i.e. 
  \[ (dw)_{ab} = \nabla (u^a_b), \]
- \(\delta\) is the formal adjoint of \(d\), i.e. 
  \[ (\delta u)_a = -2 \nabla^c u_{ca}, \]
- \(\text{Riem}_g\) is the Ricci operator, i.e. 
  \[ \text{Riem}_g (u)_{ab} = R_{cd}^a b u_{cd} = R_{c d}^a_{\quad b} u_{cd}, \]

(see Subsect. 4.1.1 for more details on our conventions). Then the equation \(Pu = 0\) is equivalent to the Einstein equations linearized around \(g\) (these are usually formulated using the linearized Einstein operator \(\frac{1}{2} P \circ I\)). The identity \(PK = 0\) means that \(P\) is invariant under linear gauge transformations \(u \mapsto u + Kw\), and it is responsible for the fact that \(P\) is not hyperbolic. The de Donder gauge or harmonic gauge consists in considering the “gauge-fixed” hyperbolic operator
\[ D_2 = P + KK^*, \]
where \(K^*\) is the formal adjoint for a non-positive Hermitian form involving \(I\) (see Subsect. 4.2). Then, solutions of the linearized Einstein equations \(Pu = 0\) are obtained by solving \(D_2 u = 0\) with the gauge condition \(K^* u = 0\). An auxiliary role is played by the hyperbolic operator \(D_1 = K^* K\) acting on \((0,1)\)-tensors, which satisfies \(D_2 K = KD_1\).

In the quantization problem, of particular relevance are bi-solutions which correspond to two-point functions of states. In fact, finding a Hadamard state amounts to constructing a pair of operators \(\lambda_2^\pm\) acting on symmetric \((0,2)\)-tensors on \(M\) such that:

\[ \begin{align*}
1) & \quad D_2 \lambda_2^\pm = \lambda_2^\pm D_2 = 0 \quad \text{and} \quad i(\lambda_2^+ - \lambda_2^-) \quad \text{is the causal propagator of} \quad D_2, \\
2) & \quad (\lambda_2^\pm)^* = \lambda_2^\mp \quad \text{and} \quad \lambda_2^\pm : \text{Ran} \, K|_{C_\infty^0} \to \text{Ran} \, K|_{D'}, \\
3) & \quad \lambda_2^\pm \geq 0 \quad \text{on} \quad \text{Ker} \, K^*|_{C_\infty^0}, \\
4) & \quad \text{WF} (\lambda_2^\pm) \subset N^\pm \times N^\pm.
\end{align*} \]

The second part of (1) corresponds to the canonical commutations relations. Condition (2) is the gauge invariance and (3) the positivity. Put together, they ensure that one gets a well-defined quantum state on the physical space \(K^* h = 0\). Then, the Gelfand–Naimark–Segal construction yields quantum field operators for linearized gravity. In addition, one requires the wavefront set estimate (4): this is the celebrated Hadamard condition (see Subsect. 2.3.2 for more details) which ensures the correct short-distance behavior of fields.

The first significant difficulty as compared to the scalar Klein–Gordon equation is that the Hermitian form in (2)–(3) is not positive and in consequence the positivity condition becomes extremely delicate. Furthermore, the use of pseudodifferential calculus is helpful to get condition (4), but it interacts badly with conditions (2)–(3). The impossibility of using a spacetime deformation argument makes various previously developed methods inapplicable.
1.2. Setting. In this paper we focus on globally hyperbolic backgrounds \((M, g)\) with the following properties I. and II. First of all, we assume:

I. \((M, g)\) is a Lorentzian manifold of bounded geometry near a Cauchy surface \(\Sigma\). Roughly speaking, the bounded geometry assumption means that all relevant geometric quantities associated with \(g\) are bounded with all their derivatives relatively to some fixed reference Riemannian metric \(\hat{g}\), see Subsect. 3.1. Using Gaussian normal coordinates to \(\Sigma\) we obtain a diffeomorphism \(\chi\) defined in a neighborhood of \(\Sigma\) such that

\[ \chi^* g = -dt^2 + h_t, \]

where the function \(t \mapsto h_t\) takes values in Riemannian metrics of bounded geometry and is bounded with all its derivatives. Our second assumption is:

II. The map \(t \mapsto h_t\) is real analytic.

The precise formulation is given in Subsect. 3.3. In practice it is often easier to check the stronger condition of bounded analytic geometry, which is an analyticity condition in all variables, see Subsect. 3.4. We show that it is satisfied in examples such as Minkowski, de Sitter, Kerr–Kruskal and Schwarzschild–de Sitter spacetimes.

1.3. Plan of paper. The main idea discussed in the present paper, inspired by Euclidean gravity approaches to quantization, is to consider an elliptic operator \(\tilde{D}_2\) which is a Wick-rotated version of the hyperbolic operator \(D_2\) near a fixed Cauchy surface \(\Sigma\). We then construct the Cauchy data of \(\lambda^\pm_{2}\) in terms of the Calderón projectors \(\tilde{c}^\pm_{2}\) for \(\tilde{D}_2\). Denoting by \(s = i^{-1}t\) the imaginary time variable and fixing suitable boundary conditions, this means that \(\tilde{c}^\pm_{2}\) is the projection to Cauchy data of \(L^2\) solutions of the elliptic problem \(\tilde{D}_2 u = 0\) in \(\{\pm s > 0\}\). To put it briefly, Cauchy data for an elliptic problem are used as Cauchy data in Lorentzian signature.

The motivation is that this type of construction is known to carry good positivity and wavefront set properties in the real analytic, scalar case [27, 46]. Furthermore \(\tilde{c}^\pm_{2}\) are given by a direct formula in terms of the inverse of \(\tilde{D}_2\), so one could hope to prove the gauge invariance condition (2) using algebraic identities satisfied by \(\tilde{D}_2\).

However, it is not a priori clear how to do the Wick rotation in a way that guarantees the invertibility of a boundary value problem associated with the elliptic operator \(\tilde{D}_2\), nor how to prove the Hadamard condition without assuming full analyticity. Worst of all, the Calderón projectors \(\tilde{c}^\pm_{2}\) come with some positivity properties indeed (as expected from the scalar case), but with respect to an unphysical Euclidean version of the physical Hermitian form, so there is no particular reason for the positivity condition (3) to be satisfied.

In the present paper we tackle part of these issues in the following order.

- In Sect. 2 we recall the general structure of linear gauge theories on Lorentzian spacetimes in the formalism due to Hack–Schenkel [29], which in particular serves us to justify conditions (1)–(4). In Sect. 3 we introduce the notions of bounded geometry which enter the assumptions I. and II. and we derive various examples. As an intermediate step, in Thm. 3.11 we show that it suffices to check analytic bounded geometry assumptions for the initial data of Einstein metrics.
- In Sect. 4 we explain how exactly linearized gravity fits into the framework of Sect. 2. We then use Gaussian normal coordinates near a Cauchy surface $\Sigma$ and a parallel transport argument to reduce $D^2$ to an operator of the form $\partial_t^2 + a_2(t)$, where $a_2(t)$ is a family of elliptic operators acting in the spatial variables.

- In Sect. 5 we consider the hyperbolic operator $\partial_t^2 + a_2(t)$ and in parallel we study boundary value problems for its Wick-rotated elliptic analogue $-\partial_s^2 + a_2(is)$. Using Shubin’s pseudodifferential calculus on manifolds of bounded geometry, we construct a parametrix for Hadamard projectors $c_2^\pm$ (operators that project to Cauchy data of solutions with wavefront set in $N^\pm$) and for Calderón projectors $\tilde{c}_2^\pm$ at $s = 0$ for the operator $\tilde{D}_2$ with Dirichlet boundary conditions at some finite imaginary times $s = \pm T$. The two parametrices are deduced from similar pseudodifferential factorizations of the respective operator $\partial_t^2 + a_2(t)$ or $-\partial_s^2 + a_2(is)$, and are therefore related: we show that in fact, Hadamard and Calderón projectors coincide modulo a smoothing term. In conclusion, we can define $\lambda_2^\pm$ using $\tilde{c}_2^\pm$ as Cauchy data, and then $\lambda_2^\pm$ satisfies the Hadamard condition.

- Finally, Sect. 6 discusses gauge-invariance and positivity on the physical space. The operators $\lambda_2^\pm$ defined from the Calderón projectors $\tilde{c}_2^\pm$ are positive for an auxiliary scalar product, but this scalar product unfortunately differs from the physical Hermitian form in condition (3). However, on the Cauchy surface level, the two inner products coincide on tensors which have no mixed components and are invariant under trace reversal. The idea is then to find a gauge transformation that maps to tensors of this type, at least on the level of Cauchy data at $t = 0$. This is achieved in Lem. 6.4 in the elliptic setting in the case of Dirichlet boundary conditions. The problem with that strategy is that $\lambda_2^\pm$ turns out not to satisfy gauge invariance (2), and this results in positivity modulo a smoothing term. The obstruction to gauge invariance is due to the choice of boundary conditions for $\tilde{D}_2$—this motivates further work, possibly on different boundary conditions in Wick-rotated gravity.

These results will be used in subsequent works on quantization of linearized gravity in different settings.

1.4. Bibliographical remarks. The importance of Hadamard states for renormalization was realized already in the 1970s, cf. Allen–Folacci–Ottewill [1] for the case of linearized gravity. More recently it was re-emphasized in works by Brunetti, Fredenhagen, Rejzner and other authors [9, 17, 8, 43] on the perturbative approach to effective theories of quantum gravity.

The problem of constructing Hadamard states in linear gauge theories was considered in the simplest case of Maxwell equations by Furlani [20], Fewster–Pfenning [14], Dappiaggi–Siemssen [11] and Finster–Strohmaier [16]. The construction of Hadamard states for Yang–Mills equations linearized around the zero solution in the BRST framework is due to Hollands [33]. Later, Gérard–Wrochna [26] considered Yang–Mills equations linearized around non-zero solutions. As pointed out in the introduction, none of these constructions can be adapted to linearized Einstein equations. The case of
linearized Einstein equations on asymptotically flat spacetimes was studied by Benini–Dappiaggi–Murro [6] with methods drawing from earlier works of Ashtekar–Magnon–Ashtekar [3] and Dappiaggi–Moretti–Pinamonti [10], the quantization turns out however to be limited to a subspace of classical degrees of freedom due to divergences at null infinity.

The general structure of quantized linearized gravity and various candidates for two-point functions of states on specific spacetimes were studied in many works, too numerous to be listed here exhaustively (however, the physics literature does not appear to address the problem of positivity).

Building on works among others by Moncrief [41], the symplectic structure of linearized gravity was studied by Fewster–Hunt [13]; the latter was then built into a more general framework for linear gauge theories (used in the present paper) by Hack–Schenkel [29]. The work of Fewster–Hunt also considers the $TT$ gauge\footnote{This stands for “transverse and traceless”, though one should be aware that outside of the special case of Minkowski space one usually means by this just “traceless”} (often used in the physics literature when discussing quantization of linearized gravity) and examines if and under what assumptions it can be implemented on spaces of space-compact solutions.

In particular, many authors analyzed the question of the existence of a maximally symmetric state on de Sitter space (see e.g. [31, 40] and references therein), which appears to be extremely subtle due to infrared problems. In our construction, Dirichlet boundary conditions at finite imaginary times act as a universal infrared regularization (cf. [27, Subsect. 4.5] for an illustration in the scalar case), but they manifestly break the symmetries on top of the issues with gauge-invariance. In that particular case, however, it is possible to Wick rotate to the sphere and then boundary conditions are no longer needed, at the cost of having to deal with infrared problems directly: this will be addressed in a forthcoming paper.

Calderón projectors have been studied in various settings, including recently general fibred cusp operators by Fritzsch–Grieser–Schrohe [18]. One of the main differences between our situation and the ones typically considered in the literature is that $\{ s = 0 \}$ plays the role of an interface between two symmetric regions, rather than being the boundary of one region which can be extended in an arbitrary way. We also mention the recent construction of the Hartle–Hawking–Israel state for scalar fields with closely related techniques [21].

Finally, we remark that there has been much recent progress in the analysis of linearized Einstein equations with microlocal methods in the context of black hole stability, see e.g. Hintz–Vasy [32] and Häfner–Hintz–Vasy [30]. The techniques we use here are largely different due to the local-in-time character of the problem. However, both kinds of developments are expected to be relevant to the construction of Hadamard states with distinguished asymptotic properties, a problem which remains open for now in the case of linearized gravity.

1.5. **Notation.** Before moving on to the main part of the paper, let us introduce the relevant notation.
1.5.1. Sections of vector bundles. Let \( V \xrightarrow{\pi} M \) be a finite rank complex vector bundle over a smooth manifold \( M \).

- If \( \Sigma \subset M \) is a smooth manifold we denote by \( V|_{\Sigma} \xrightarrow{\pi} \Sigma \) the restriction of \( V \) to \( \Sigma \).
- We denote by \( C^\infty(M; V) \), resp. \( C^\infty_c(M; V) \) the space of smooth, resp. compactly supported smooth sections of \( V \).
- We denote by \( D'(M; V) \), resp. \( E'(M; V) \) the space of distributional, resp. compactly supported distributional sections of \( V \).
- If \( \Omega \subset M \) is an open set with smooth boundary and \( F(M; V) \) is one of the above spaces, we denote by \( F(\Omega; V) \subset F(M; V) \) the space of restrictions of sections of \( V \) to \( \Omega \).
- If \( \Sigma \subset M \) is a submanifold, we denote by \( F(\Sigma; V) \) the corresponding space of sections of the restriction \( V|_{\Sigma} \xrightarrow{\pi} \Sigma \) of \( V \) to \( \Sigma \).

Equipped with their natural seminorms, all the above vector spaces of sections are Fréchet spaces.

We use the same notations if \( V \) is a finite dimensional vector space, i.e. we write simply \( V \) instead of the trivial vector bundle \( M \times V \).

1.5.2. Globally hyperbolic spacetimes. We use the convention \( (-,+,...,+) \) for the Lorentzian signature. Let us recall that a globally hyperbolic spacetime is a smooth Lorentzian manifold equipped with a time orientation and having a Cauchy surface \( \Sigma \), i.e., a closed subset of \( M \) which is intersected exactly once by each maximally extended time-like curve. By the Bernal–Sánchez theorem, this definition implies that \( M \) admits smooth space-like Cauchy surfaces.

- We denote by \( J_\pm(K) \) the future/past causal shadow of \( K \subset M \).
- If \( M \) is a globally hyperbolic spacetime we denote by \( C^\infty_{sc}(M; V) \) the space of space-compact sections, i.e. sections in \( C^\infty(M; V) \) with compactly supported restriction to a Cauchy surface.

1.5.3. Distributional kernels and wavefront sets. If \( u \in D'(M; V) \) we denote by \( WF(u) \subset T^*M \setminus o \) its wavefront set, which is invariantly defined using local trivializations of \( V \).

- If \( V_i \xrightarrow{\pi_i} M_i \) are two vector bundles as above and \( A : C^\infty_c(M_1; V_1) \to D'(M_2; V_2) \) is linear continuous, then \( A \) admits a distributional kernel, still denoted by \( A \in D'(M_2 \times M_1; V_2 \boxtimes V_1) \).
- We denote by \( WF(A)' \subset (T^*M_2 \times T^*M_1) \setminus o \) its primed wavefront set, defined by
  \[ \Gamma' = \{((x_2, \xi_2), (x_1, -\xi_1)) \mid ((x_2, \xi_2), (x_1, \xi_1)) \in \Gamma \} \text{ for } \Gamma \subset T^*M_2 \times T^*M_1. \]

1.5.4. Hermitian bundles. If \( V \) is equipped with a fiberwise non-degenerate Hermitian form \( (\cdot|\cdot)_V \), we say that \( V \) is a Hermitian bundle. If the Hermitian form is positive definite, we say that \( V \) is a Hilbertian bundle.

We will always assume that \( M \) is equipped with a pseudo-Riemannian or Riemannian metric \( g \) and denote by \( d\text{vol}_g \) the associated volume form.
- If $V$ is a Hermitian bundle over $M$ and $U \subset M$ is an open set, we set
\[
(u|v)_V(U) := \int_U (u(x)|v(x))_V \, d\text{vol}_g, \quad u \in C^\infty_c(U; V).
\]
We use $\langle \cdot | \cdot \rangle_V(M)$ to inject $C^\infty_c(M; V)$, resp. $C^\infty(M; V)$, into $\mathcal{D}'(M; V)$, resp. $\mathcal{E}'(M; V)$.

- We denote by $A^*$ the formal adjoint of an operator $A$.

- If we need to consider simultaneously two different Hermitian structures, the two adjoints of $A$ will be denoted by $A^*$ and $A^\dagger$.

1.5.5. **Differential operators.** If $V_1$, $V_2$ are two vector bundles over $M$, we denote by Diff$(M; V_1, V_2)$ (resp. Diff$^m(M; V_1, V_2)$) the corresponding space of differential operators (resp. differential operators of order $m$), equipped with their Fréchet space topologies.

- We abbreviate Diff$^m(M; V, V)$ by Diff$^m(M; V)$.

- If $A \in$ Diff$(M; V_1, V_2)$ we denote by $A|_{C^\infty}$, resp. $A|_{C^\infty_c}$, its restriction to the space $C^\infty_c(M; V_1)$, resp. $C^\infty(M; V_1)$, $C^\infty_c(M; V_1)$, and we use analogous notation for distributional sections (or vector-valued distributions) $\mathcal{D}'(M; V_1)$ and compactly supported distributional sections $\mathcal{E}'(M; V_1)$.

1.5.6. **Time dependent objects.** If $I \subset \mathbb{R}$ is an interval and $\mathcal{F}$ is a Fréchet space whose topology is defined by a family of seminorms $\|\cdot\|_n$, $n \in \mathbb{N}$, we denote by $C^\infty_b(I, \mathcal{F})$ the space of maps $f : I \to \mathcal{F}$ such that $\sup_{t \in I} \|\partial_t^p f(t)\|_n < \infty$ for all $n, p \in \mathbb{N}$. Equipped with the obvious seminorms, $C^\infty_b(I, \mathcal{F})$ is a Fréchet space.

We use this notation to define for example $C^\infty_b(I; C^\infty_b(\Sigma; V))$, etc.

1.5.7. **Evolution groups.** Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D} \subset \mathcal{H}$ a dense subspace, $I \subset \mathbb{R}$ an interval and $I \ni t \mapsto H(t)$ a map with values in closed linear operators in $L(\mathcal{D}, \mathcal{H})$. Assume that for some $z_0 \in \mathbb{C}$ one has:

1. $z_0 + i\mathbb{R} \in \rho(H(t))$ and $\sup_{t \in I} \|(z_0 + i\lambda - H(t))^{-1}\|_{B(\mathcal{H})} < \infty \quad \forall t \in I$,

2. $I \ni t \mapsto H(t) \in L(\mathcal{D}, \mathcal{H})$ is strongly $C^1$.

Then by a result of Kato [36], see e.g. [47] for a recent exposition, there exists a unique propagator $I \times I \ni (t, s) \mapsto U(t, s) \in B(\mathcal{H})$ such that $U(t, s) : \mathcal{D} \to \mathcal{D}$ and:

\[
\begin{cases}
\partial_t U(t, s)u = iH(t)U(t, s)u, \\
U(s, s)u = u
\end{cases}
\]

for all $u \in \mathcal{D}$ and $t, s \in I$. Following the physics literature we will denote $U(t, s)$ by $\text{Exp}(i \int_s^t H(\sigma) d\sigma)$.

2. **Preliminaries on gauge theories**

In this section we recall an abstract formalism for the quantization of (linear) gauge theories on curved spacetimes [29]. We discuss various equivalent phase spaces that can be used for the algebraic quantization, in particular the phase space obtained by fixing a Cauchy surface and considering initial data.
We also formulate the definition of Hadamard states for gauge theories and give conditions on the Cauchy surface covariances of a state that imply the Hadamard property.

2.1. Solution spaces for hyperbolic equations. Before discussing gauge theories, we recall the setup relevant for hyperbolic equations.

Let \((M, g)\) be a globally hyperbolic spacetime, and let \(V\) be a Hermitian bundle over \(M\). We denote by \(A^\star\) the formal adjoint of \(A \in \text{Diff}(M; V)\).

One says that \(D \in \text{Diff}(M; V)\) is Green hyperbolic if \(D\) has retarded/advanced inverses \(G_{\text{ret/adv}}\) and the same holds true for \(D^\star\). This is the case in particular if the principal symbol \(\sigma_{\text{pr}}(D)\) of \(D\) equals \(\xi \cdot g^{-1}(x)\xi 1_V\), i.e. is given by the inverse metric, see \([5, \text{Thm. 3.3.1}]\).

The difference \(G := G_{\text{ret}} - G_{\text{adv}}\) is called interchangeably with its Schwartz kernel the causal propagator (or Pauli-Jordan function) of \(D\). We recall its fundamental properties below, see \([4, \text{Lem. 3.3 & Thm. 3.5}]\) for a proof at this level of generality.

**Proposition 2.1.** Suppose that \(V\) is a Hermitian bundle over \(M\) and \(D \in \text{Diff}(M; V)\) is a Green hyperbolic operator such that \(D^\star = D\). Then:

1. the induced map
   \[
   [G] : \frac{C_c^\infty(M; V)}{\text{Ran } D|_{C_c^\infty}} \rightarrow \text{Ker } D|_{C_c^\infty}
   \]
   is well defined and bijective;
2. \((G_{\text{ret/adv}})^\star = G_{\text{adv/ret}}\) and consequently, \(G^\star = -G\).

2.1.1. Green’s formula. Let us fix a Cauchy surface \(\Sigma\) of \((M, g)\). It can be shown that under the assumptions of Proposition 2.1, \(D\) has a well-posed Cauchy problem at \(\Sigma\), see e.g. \([38, \text{Subsect. 4.3}]\).

More precisely, there exists a Hilbertian bundle \(V_\Sigma\) over \(\Sigma\) and a continuous operator \(\varrho : C_c^\infty(M; V) \rightarrow C_c^\infty(\Sigma; V_\Sigma)\), mapping a smooth section to its Cauchy data on \(\Sigma\), such that

\[
\varrho : \text{Ker } D|_{C_c^\infty} \rightarrow C_c^\infty(\Sigma; V_\Sigma)
\]

is bijective.

By Prop. 2.1, there exists a unique \(q : C_c^\infty(\Sigma; V_\Sigma) \rightarrow C_c^\infty(\Sigma; V_\Sigma)\) such that \(q = q^\star\) and

\[
(\phi_1|G\phi_2)_{V(M)} = i^{-1}(gu_1|qu_2)_{V_\Sigma},
\]

for all \(\phi_i \in C_c^\infty(M; V)\) and \(u_i = G\phi_i \in \text{Ker } D|_{C_c^\infty}, \ i = 1, 2\). The l.h.s. is manifestly independent on the choice of Cauchy surface \(\Sigma\), therefore \(q\) is a conserved quantity along the Cauchy evolution, called the conserved charge or simply charge of \(D\).

Note that the literature often uses the conserved symplectic form, which is given by the r.h.s. of \((2.1)\). It defines the complex symplectic space of Cauchy data (or equivalently, the symplectic space of solutions of \(Du = 0\)) which is fundamental for field quantization.

A practical way of computing \(q\) is provided by the following elementary lemma.

**Lemma 2.2.** For \(D\) as in Proposition 2.1, the charge is the unique \(q \in \text{Diff}(\Sigma; V_\Sigma)\) such that for all \(u, v \in C_c^\infty(M; V)\):

\[
(u|Dv)_{V(J_\pm(\Sigma))} - (Du|v)_{V(J_\pm(\Sigma))} = \pm i^{-1}(gu|qv)_{V_\Sigma},
\]

(2.2)
Hypothesis 2.3. Let $J_{\pm}(K)$ be the future/past causal shadows of $K \subset M$.

**Proof.** Observe that the l.h.s. of (2.2) can be rewritten as $(u[1_{J_{\pm}(\Sigma)}], D|v)_{V(M)}$, which shows that it depends only on the traces of $u, v$ on $\Sigma$. Let $\hat{q} \in \text{Diff}(\Sigma; V_2)$ be defined by the r.h.s. of (2.2). Let also $u_i = G\phi_i \in \text{Ker} D|_{C^\infty_{sc}}$, $i = 1, 2$. Without loss of generality we can assume that $\text{supp} \phi_i$ are in the past of $\Sigma$, so that $G\phi_i = G_{\text{ret}}\phi_i$ near $\Sigma$ and $\text{supp} G_{\text{ret}}\phi_i \cap J_-(\Sigma)$ is compact. Let us fix $\chi \in C^\infty_c(M; \mathbb{R})$ such that $\chi \equiv 1$ near $\text{supp} G_{\text{ret}}\phi_i \cap J_-(\Sigma)$. Then, we have:

$$-i^{-1}(\rho G\phi_i | \hat{q}\varrho G\phi_2)_{V_\Sigma} = -i^{-1}(\chi G_{\text{ret}}\phi_1 | \hat{q}\varrho G_{\text{ret}}\phi_2)_{V_\Sigma} = (\chi G_{\text{ret}}\phi_1 | D\chi G_{\text{ret}}\phi_2)_{V(J_-(\Sigma))} - (D\chi G_{\text{ret}}\phi_1 | \chi G_{\text{ret}}\phi_2)_{V(J_-(\Sigma))} = (G_{\text{ret}}\phi_1 | \phi_2)_{V(J_-(\Sigma))} - (\phi_1 | G_{\text{ret}}\phi_2)_{V(J_-(\Sigma))} = (G_{\text{ret}}\phi_1 | \phi_2)_{V(M)} - (\phi_1 | G_{\text{ret}}\phi_2)_{V(M)} = -\langle \phi_1 | G\phi_2 \rangle_{V(M)}.$$

In the second line we use the definition of $\hat{q}$, in the third we use that $\chi \equiv 1$ near $\text{supp} G_{\text{ret}}\phi_i \cap J_-(\Sigma)$ and that $DG_{\text{ret}} = 1$ and in the fourth line that $\text{supp} \phi_i \subset J_-(\Sigma)$ and $G^*_{\text{ret}} = G_{\text{adv}}$. This shows that $\hat{q} = q$ and completes the proof of (2.2). \qed

### 2.2. Linear gauge theories

When discussing linear gauge theories it is useful to introduce an abstract framework that captures their general structure. Here we use a special case of the framework proposed by Hack–Schenkel [29], which includes examples such as electromagnetism and linearized Yang–Mills equations, and which also turns out to be well adapted to linearized gravity. We follow the presentation in [26] and refer to [51] for the relationship with the BRST formalism. The more general BV formalism on Lorentzian manifolds is discussed in [17], cf. [9] for the special case of linearized gravity.

**Hypothesis 2.3.** Suppose that we are given:

1. two Hermitian bundles $V_1, V_2$ over $M$;
2. a formally self-adjoint operator $P \in \text{Diff}(M; V_2)$;
3. an operator $K \in \text{Diff}(M; V_1, V_2)$, such that $K \neq 0$ and
   a) $PK = 0$,
   b) $D_1 := K^* K \in \text{Diff}(M; V_1)$ is Green hyperbolic,
   c) $D_2 := P + KK^* \in \text{Diff}(M; V_2)$ is Green hyperbolic,

where $K^*$ is the formal adjoint of $K$ defined as in 1.5.4 using the Hermitian structures on $V_1, V_2$.

The operator $P$ is the operator of direct physical interest (in our case, obtained by linearization of Einstein equations, see Sect. 4). The operator $K$ defines gauge transformations $u \mapsto u + Ku$, and the identity $PK = 0$ states that $P$ is invariant under gauge transformations.

Thanks to the assumptions on $D_1, D_2$, the non-hyperbolic equation $Pu = 0$ can be reduced by gauge transformations to the subspace $K^* u = 0$ of solutions of the hyperbolic problem $D_2 u = 0$. The equation $K^* u = 0$ is sometimes called *subsidiary condition*. The term $KK^*$ in the definition of $D_2 = P + KK^*$ is the so-called *gauge-fixing term*. 


Hypothesis 2.3 immediately implies the identities
\[ K^*P = 0, \quad K^*D_2 = D_1K^*, \quad D_2K = KD_1. \]
We apply the notations from 2.1 to the hyperbolic operators \( D_1 \) and \( D_2 \), with the addition of a subscript \( i = 1, 2 \) to distinguish between the two. In particular \( G_2 \) is the causal propagator of \( D_2 \).

**Proposition 2.4** ([26, Prop. 2.7]). Assume Hypothesis 2.3. Then the induced maps
\[ a) \quad [G_2]: \frac{\text{Ker} K^*|_{C_c^\infty}}{\text{Ran} P|_{C_c^\infty}} \to \frac{\text{Ker} P|_{C_c^\infty}}{\text{Ran} K|_{C_c^\infty}}, \]
\[ b) \quad [G_2]: \frac{\text{Ker} K^*|_{C_c^\infty}}{\text{Ran} P|_{C_c^\infty}} \to \frac{\text{Ker} D_2|_{C_c^\infty} \cap \text{Ker} K^*|_{C_c^\infty}}{\text{Ran} G_2K|_{C_c^\infty}}, \]
are well defined and bijective.

The isomorphism a) justifies the use of the first quotient space by saying it is equivalent to the space of solutions of \( Pu = 0 \), modulo gauge transformations. The isomorphism b) is of practical importance as an intermediary step to get an isomorphism with a space of Cauchy data for the Green hyperbolic operator \( D_2 \).

Note that in both cases \([G_2]\) maps to a quotient space. The possibility of choosing different representatives of an element of that quotient is called the *residual gauge freedom*.

**Definition 2.5.** The *physical phase space* is the Hermitian space \((V_P, q_P)\), where:
\[ V_P = \frac{\text{Ker} K^*|_{C_c^\infty}}{\text{Ran} P|_{C_c^\infty}}, \quad [u] \cdot q_P[v] = i(u|G_2v)_{V_2}, \quad [u], [v] \in \frac{\text{Ker} K^*|_{C_c^\infty}}{\text{Ran} P|_{C_c^\infty}}. \]

By [26, Prop. 2.6], \( q_P \) is a well-defined Hermitian form on \( V_P \).

### 2.2.1 Phase space of Cauchy data
Identifying the correct space of Cauchy data requires several more definitions.

For \( i = 1, 2 \) we denote by \( U_i \) the operator which assigns to a Cauchy datum the corresponding solution in \( \text{Ker} D_i|_{C_c^\infty} \), i.e. \( U_i \) is the inverse of \( \varrho_i: \text{Ker} D_i|_{C_c^\infty} \to C_c^\infty(\Sigma; V_i, \Sigma) \).

In other terms,
\[
\begin{cases}
D_i U_i = 0, \\
\varrho_i U_i = 1.
\end{cases}
\]
(2.3)

To the operator \( K \) we associate an operator \( K_\Sigma \in \text{Diff}(\Sigma; V_1, \Sigma, V_2, \Sigma) \) acting on Cauchy data by setting
\[ K_\Sigma = \varrho_2 K U_1. \]

We introduce the notation \( K_\Sigma^\dagger \in \text{Diff}(\Sigma; V_2, \Sigma, V_1, \Sigma) \) for the adjoint w.r.t. the Hermitian forms \( \langle \cdot | q_2 \cdot \rangle_\Sigma \) and \( \langle \cdot | q_1 \cdot \rangle_\Sigma \), i.e.
\[ q_1 K_\Sigma^\dagger := K_\Sigma^* q_2. \]
(2.4)

The notation \( \dagger \) is used to avoid confusion with the formal adjoint \( * \) w.r.t. the Hermitian structures on the bundles \( V_1, \Sigma, V_2, \Sigma \).

**Lemma 2.6** ([26, Lem. 2.9]). Assume Hypothesis 2.3. Then:
(1) $K U_1 = U_2 K_\Sigma$ and $K^* U_2 = U_1 K^\dag_\Sigma$.
(2) $\varrho_2 K = K_{\Sigma} \varrho_1$ on $D_2|_{C^\infty_\omega}$, and $\varrho_1 K^* = K^\dag_{\Sigma} \varrho_2$ on $\text{Ker} D_2|_{C^\infty_\omega}$.
(3) $\text{Ker} K^\dag_{\Sigma}|_{C^\infty_\omega} = \varrho_2 G_2\text{Ker} K^*|_{C^\infty_\omega}$.
(4) $\text{Ran} K^\Sigma|_{C^\infty_\omega} = \varrho_2 G_2 \text{Ran} K|_{C^\infty_\omega}$.
(5) $K^\dag_{\Sigma} K_{\Sigma} = 0$.

**Proposition 2.7** ([26, Prop. 2.10]). The induced map

$$[\varrho_2]: \frac{\text{Ker} D_2|_{C^\infty_\omega} \cap \text{Ker} K^*|_{C^\infty_\omega}}{\text{Ran} G_2 K|_{C^\infty_\omega}} \rightarrow \frac{\text{Ker} K^\dag_{\Sigma}|_{C^\infty_\omega}}{\text{Ran} K_{\Sigma}|_{C^\infty_\omega}}$$

is well defined and bijective.

By 2.1 applied to $D_2$, we obtain the Hermitian form $(\cdot | q_2 \cdot)_{V_2 \Sigma}$ which is well defined on the quotient space $\text{Ker} K^\dag_{\Sigma}|_{C^\infty_\omega}/\text{Ran} K_{\Sigma}|_{C^\infty_\omega}$. This is an easy consequence of (2.4); indeed, if $u, v \in \text{Ker} K^\dag_{\Sigma}|_{C^\infty_\omega}$ then for all $w \in C^\infty_\omega(\Sigma; V_{1, \Sigma})$,

$$(u + K_{\Sigma} w | q_2 v)_\Sigma = (u | q_2 v)_\Sigma + (w | q_2 K^\dag_{\Sigma} v)_\Sigma = (u | q_2 v)_\Sigma,$$

and similarly for gauge transformations in the second argument. Thus, $(u | q_2 v)_\Sigma$ depends only on the equivalence classes of $u$ and $v$.

Summarizing we have:

**Proposition 2.8.** The map

$$[\varrho_2 G_2]: \frac{\text{Ker} K^*|_{C^\infty_\omega}}{\text{Ran} P|_{C^\infty_\omega}}, i(\cdot | G_2 \cdot)_{V_2} \rightarrow \frac{\text{Ker} K^\dag_{\Sigma}|_{C^\infty_\omega}}{\text{Ran} K_{\Sigma}|_{C^\infty_\omega}}, (\cdot | q^2 \cdot)_{V_2 \Sigma}$$

is pseudo-unitary.

### 2.3. Quantization

The algebraic quantization of linear gauge theories is discussed in detail in [26, Sect. 3]. The algebraic framework reduces the quantization problem to showing the existence of physically relevant quantum states on the CCR $\ast$-algebra $\mathcal{V}P(q_P)$ associated to the Hermitian space $(\mathcal{V}P, q_P)$ defined in Sect. 2.2. The notions of quasi-free states and covariances (or two-point functions) are explained in [26, Sect. 3] and references therein.

#### 2.3.1. Two-point functions

A quasi-free state on $\mathcal{V}P(q_P)$ is determined by a pair $\Lambda^\pm$ of *covariances*, i.e. of Hermitian forms on $\mathcal{V}P$ such that

$$
\begin{cases}
\Lambda^\pm \geq 0, \\
\Lambda^+ - \Lambda^- = q_P.
\end{cases}
$$

We will consider quasi-free states $\omega$ on $\mathcal{V}P(q_P)$ with covariances obtained from a pair of maps $\lambda^\Sigma_2: C^\infty_\omega(M; V_2) \rightarrow C^\infty_\omega(M; V_2)$ (called the pseudo-covariances of $\omega$) by:

$$[\bar{u}] \cdot \Lambda^\pm [v] = (u | \lambda^\Sigma_2 v)_{V_2}, \quad [u], [v] \in \frac{\text{Ker} K^*|_{C^\infty_\omega}}{\text{Ran} P|_{C^\infty_\omega}}. \tag{2.6}$$

The following lemma [26, Lem. 3.16] is straightforward.
Lemma 2.9. Suppose \( \lambda_i^\pm : C_c^\infty(M; V_2) \to C_c^\infty(M; V_2) \) are such that:

i) \( D_2 \lambda_2^\pm = \lambda_2^\pm D_2 = 0 \) and \( \lambda_2^+ - \lambda_2^- = i^{-1}G_2 \),

ii) \( (\lambda_2^\pm)^* = \lambda_2^\mp \) for \( (\cdot|\cdot)_{V_2} \) and \( \lambda_2^\pm : \text{Ran} K|_{C_c^\infty} \to \text{Ran} K|_{\mathcal{V}_P} \), \( \lambda_2^\pm \geq 0 \) for \( (\cdot|\cdot)_{V_2} \) on \( \text{Ker} K^*|_{C_c^\infty} \).

Then \( \lambda_2^\pm \) are the pseudo-covariances of a quasi-free state on \( \mathcal{V}_P, q_P \).

In fact it is easy to check that conditions i) and ii) above imply that \( \Lambda^\pm \) are well defined on the quotient in (2.6). The name ‘pseudo-covariance’ comes from the fact that \( \lambda_2^\pm \) are not required to be positive for \( (\cdot|\cdot)_{V_2} \) on \( C_c^\infty(M; V_2) \), but only on the subspace \( \text{Ker} K^*|_{C_c^\infty} \).

2.3.2. Hadamard condition. We use the following definition of Hadamard states [45], cf. [26, Subsect. 3.4] and references therein. The general consensus is that only states satisfying the Hadamard condition (the Hadamard states) are physical. We recall that

\[
\mathcal{N} = \{(x, \xi) \in T^* M \setminus o \mid \xi \cdot g^{-1}(x)\xi = 0\}
\]

is the characteristic set of the wave operator on \( (M, g) \), and

\[
\mathcal{N}^\pm = \mathcal{N} \cap \{(x, \xi) \in T^* M \setminus o \mid \pm v \cdot \xi > 0 \ \forall v \in T_x M \text{ future-directed time-like}\}
\]

are its two connected components.

Definition 2.10. A quasi-free state \( \omega \) on \( \mathcal{V}_P, q_P \) given by pseudo-covariances \( \lambda_2^\pm \) as in Lem. 2.9 is Hadamard if in addition to (2.7) it satisfies:

\[
WF(\lambda_2^\pm)' \subset \mathcal{N}^\pm \times \mathcal{N}^\pm.
\]

(2.8)

2.3.3. Hadamard condition on a Cauchy surface. One can equivalently consider pseudo-covariances \( \lambda_{\Sigma}^\pm \) for \( D_2 \) acting on Cauchy data on a Cauchy surface \( \Sigma \), see e.g. [26, Subsect. 3.3]. Namely we can set

\[
\lambda_{\Sigma}^\pm := \varphi_2^\pm q_2 \lambda_2^\pm \varphi_2^\pm q_2,
\]

and then one also has

\[
\lambda_{\Sigma}^\pm = (q_2 G_2)^* \lambda_{\Sigma}^\pm (q_2 G_2).
\]

(2.9)

The maps \( \lambda_{\Sigma}^\pm \) are correspondingly called Cauchy surface pseudo-covariances. Since \( q_2 \) is non-degenerate, we can set

\[
\lambda_{\Sigma}^\pm := \pm q_2 c_2^\pm,
\]

(2.10)

and we now formulate conditions on \( c_2^\pm \) which imply that \( \lambda_2^\pm \) satisfy the conditions in Lem. 2.9 and Def. 2.10.

Proposition 2.11. Suppose \( c_2^\pm : C_c^\infty(\Sigma; V_2, \Sigma) \to C_c^\infty(\Sigma; V_2, \Sigma) \) is a pair of operators such that:

1) \( c_2^+ + c_2^- = 1 \);
2) \( q_2 c_2^- = (q_2 c_2^+)^* \) for the scalar product \( (\cdot|\cdot)_{V_2, \Sigma} \);
3) \( \pm (f | q_2 c_2^\pm f)_{\Sigma} \geq 0 \) for all \( f \in \text{Ker} K_{\Sigma}^\dagger|_{C_c^\infty} \);
4) \( c_2^\pm K_{\Sigma} = K_{\Sigma} c_1^\pm \) for some \( c_1^\pm : C_c^\infty(\Sigma; V_1, \Sigma) \to C_c^\infty(\Sigma; V_1, \Sigma) \).
Then $\lambda^\pm_2$ given by \textcolor{red}{(2.9)} and \textcolor{red}{(2.10)} are the pseudo-covariances of a quasi-free state on $\text{CCR}(\mathcal{F}, q_{\mathcal{F}})$. Furthermore, suppose that the principal symbol of $D_2$ is $\xi \cdot g^{-1}(x)\xi 1_{V_2}$. Then, if for some neighborhood $\mathcal{U}$ of $\Sigma$ in $M$ we have:

\textcolor{red}{(5)} $\text{WF}(U_2 \circ c^\pm) \subset (\mathcal{N}^\pm \cup \mathcal{F}) \times T^*M$ over $\mathcal{U} \times \Sigma$, where $\mathcal{F} \subset T^*M$ is a conic set with $\mathcal{F} \cap \mathcal{N} = \emptyset$, then the associated state is Hadamard.

Let us briefly outline the physical significance of conditions (1)–(4). Condition (4) is interpreted as gauge-invariance of the state. It ensures that $\pm (\cdot | q_{\mathcal{F}}c^\pm)_{\Sigma}$ is well-defined on the quotient space $\text{Ker}\ K_\Sigma^{{\dagger}|c^\infty} / \text{Ran}\ K_\Sigma|c^\infty$ by an argument analogous to \textcolor{red}{(2.5)}. Condition (1) expresses the canonical commutation relations, while (2) and (3) are responsible for the positivity of the state. We point out that positivity is only required to hold on the physical space $\text{Ker}\ K_\Sigma^{{\dagger}|c^\infty}$.

Our objective in the next chapters will be to show that conditions (1)–(5) can be satisfied simultaneously in the case of linearized gravity. In view of Prop. 2.11 this will imply the existence of a Hadamard state.

**Proof of Prop. 2.11.** The proof that (1)–(4) imply \textcolor{red}{(2.7)} is straightforward. Let us prove that (5) implies \textcolor{red}{(2.8)}, removing the subscripts for ease of notation. We know that the operator $U$ defined in \textcolor{red}{(2.3)} can be expressed as: $U = i^{-1}(gG)^*q$, hence $\lambda^\pm = \pm i^{-1}Uc^\pm \circ (gG)$. Note that we are allowed to compose the kernels $Uc^\pm$ and $\rho G$ since $\rho G : C^\infty_c(M; \mathcal{V}) \to C^\infty_c(\Sigma; V\Sigma)$. It is well known that $\text{WF}(G)' \subset \mathcal{N} \times \mathcal{N}$. Using also (5) and [34, Thm. 8.2.14] we obtain

$$\text{WF}(\lambda^\pm)' \subset ((\mathcal{N}^\pm \cup \mathcal{F}) \times \mathcal{N}) \cup ((\mathcal{N}^\pm \cup \mathcal{F}) \times o) \text{ over } U \times M,$$

where we recall that $o \subset T^*M$ is the zero section. Condition (2) implies that $\lambda^\pm = \lambda^\mp*$ for $(\cdot | \cdot)_\mathcal{V}$ and hence we obtain that $(X, X') \in \text{WF}(\lambda^\pm)'$ if and only if $(X', X) \in \text{WF}(\lambda^\mp)'$. Using that $\mathcal{F} \cap \mathcal{N} = \emptyset$, we then deduce from \textcolor{red}{(2.11)} that

$$\text{WF}(\lambda^\pm)' \subset (\mathcal{N}^\pm \times \mathcal{N}) \cup (\mathcal{N}^\pm \times o) \cup (o \times \mathcal{N}^\pm) \text{ over } U \times M.$$

Since $\lambda^+ - \lambda^- = iG$, using once more that $\text{WF}(G)' \subset \mathcal{N} \times \mathcal{N} \times \mathcal{N}$ and the hypothesis $\mathcal{F} \cap \mathcal{N} = \emptyset$, this implies that

$$\text{WF}(\lambda^\pm)' \cap ((\mathcal{N}^\pm \times o) \cup (o \times \mathcal{N}^\pm)) = \emptyset,$$

which proves \textcolor{red}{(2.8)} over $U \times M$. To extend \textcolor{red}{(2.8)} to $M \times M$ we use that $\text{DA}^\pm = 0$ and argue by propagation of singularities as in [12, Lem. 6.5.5].
In practice the analyticity in time is easier to deduce from an analyticity condition in all variables, which leads to the notion of spacetimes of bounded analytic geometry, defined in Subsects. 3.4, 3.5.

Existence of many Einstein manifolds of bounded analytic geometry is proved in Subsect. 3.6. Finally in Subsect. 3.7 we give concrete examples, such as the Kerr–Kruskal and maximal Schwarzschild–de Sitter spacetimes.

3.1. Spacetimes of bounded geometry.

3.1.1. Riemannian manifolds of bounded geometry. We recall that an \( n \)-dimensional Riemannian manifold \((M, \hat{g})\) is of bounded geometry if its injectivity radius \( r_{\hat{g}} \) is strictly positive and \( \nabla^k R \) are bounded tensors, where \( R \) is the Riemann curvature tensor and \( \nabla \) the covariant derivative associated to \( \hat{g} \), see e.g. [48, App. 1].

Although we are mainly interested in Lorentzian manifolds \((M, g)\), we will need an auxiliary Riemannian metric \( \hat{g} \) of bounded geometry to define various function spaces, like spaces of bounded tensors, Sobolev spaces, etc.

An equivalent characterization (see e.g. [23, Thm. 2.2]) is given in Prop. 3.1 below: let us denote by \( B_n(0, 1) \) the unit ball in \( \mathbb{R}^n \), by \( \delta \) the flat metric on \( \mathbb{R}^n \) and by \( BT_q^p(B_n(0, 1), \delta) \) the space of \((q, p)\)-tensors on \( B_n(0, 1) \) which are bounded on \( B_n(0, 1) \) together with all their derivatives.

For \( p = q = 0 \), ie for functions, we use often the notation \( C_0^\infty(B_n(0, 1)) \) instead of \( BT_0^0(B_n(0, 1), \delta) \).

Then \( BT_q^p(B_n(0, 1), \delta) \) is a Fréchet space and one can hence define a bounded family of \((q, p)\)-tensors in \( BT_q^p(B_n(0, 1), \delta) \). For example, a family \( (f_i)_{i \in I} \) of functions on \( B_n(0, 1) \) is bounded if \( \sup_{i \in I, x \in B_n(0, 1)} |\partial^\alpha_x f_i(x)| < \infty \) for all \( \alpha \in \mathbb{N}^n \).

Proposition 3.1. \((M, \hat{g})\) is of bounded geometry if and only if for each \( x \in M \) there exists \( U_x \subset M \) open neighborhood of \( x \) and

\[
\psi_x : U_x \rightarrow B_n(0, 1)
\]

a diffeomorphism with \( \psi_x(x) = 0 \) such that if \( \hat{g}_x := (\psi_x^{-1})^*\hat{g} \) then:

1. the family \( \{\hat{g}_x\}_{x \in M} \) is bounded in \( BT_0^0(B_n(0, 1), \delta) \),
2. there exists \( c > 0 \) such that:

\[
c^{-1} \delta \leq \hat{g}_x \leq c\delta, \quad x \in M.
\]

It is known that one can find a sequence \( (x_i)_{i \in \mathbb{N}} \) of points in \( M \) such that setting \( (U_i, \psi_i) := (U_{x_i}, \psi_{x_i}) \), \( (U_i, \psi_i)_{i \in \mathbb{N}} \) is an atlas of \( M \) with the additional property that there exists \( N \in \mathbb{N} \) such that \( \bigcap_{i \in J} U_i = \emptyset \) if \( \sharp J > N \). Such atlases are called bounded atlases of \( M \).

The standard choice for \((U_i, \psi_i)\) is \( (B^{\hat{g}}(x_i, r), \exp^{\hat{g}}_{x_i}) \) for \( 0 < r < r_{\hat{g}} \), i.e. the geodesic ball and exponential map at \( x_i \). One can associate to a bounded atlas \((U_i, \psi_i)_{i \in \mathbb{N}}\) a partition of unity

\[
1 = \sum_{i \in \mathbb{N}} \chi_i^2, \quad \chi_i \in C_0^\infty(U_i),
\]
such that \( \{(\psi_i^{-1})^*\chi_i\}_{i \in \mathbb{N}} \) is a bounded sequence in \( C_b^\infty(B_n(0,1)) \). Such a partition of unity is called a bounded partition of unity.

3.1.2. Bounded tensors, bounded differential operators, Sobolev spaces. We recall now several notions due to Shubin [48], see also [23, Subsect. 2.3].

- If \((M, \hat{g})\) is a Riemannian manifold of bounded geometry, we denote by \( BT^p_q(M, \hat{g}) \) the space of bounded \((q,p)\)-tensors on \( M \). Concretely \( T \in BT^p_q(M, \hat{g}) \) if, given a bounded atlas \( \{(U_i, \psi_i)\}_{i \in \mathbb{N}} \), the seminorms of the push-forwards of \( T \) to \( U_i \) by \( \psi_i \) in \( BT^p_q(B_n(0,1), \delta) \) are bounded uniformly in \( i \in \mathbb{N} \). The definition is independent on the choice of the bounded atlas.
- As above we will sometimes use the notation \( C_b^\infty(M) \) for \( BT^0_0(M, \hat{g}) \).
- If \( U \subset M \) is an open set, we define similarly the space \( BT^p_q(U, \hat{g}) \) by requiring the above uniform bound only over the \( i \) such that \( U \cap U_i \neq \emptyset \).
- If \( I \subset \mathbb{R} \) is an interval we use the notation introduced in 1.5.6 to define the spaces \( C_b^\infty(I, BT^p_q(M, \hat{g})) \).
- The space \( \text{Diff}(M, \hat{g}) \) is the space of bounded differential operators on \( M \), i.e. differential operators which form a bounded family of differential operators on \( B_n(0,1) \) when expressed in a bounded atlas of \( M \).
- Finally, we denote by \( H^s(M, \hat{g}) \) the Sobolev space of order \( s \in \mathbb{R} \).

3.1.3. Vector bundles of bounded geometry. Let \((M, g)\) be a Riemannian manifold of bounded geometry. We recall the definition of vector bundles of bounded geometry, see [48].

A vector bundle \( E \xrightarrow{\pi} M \) of rank \( N \) is of bounded geometry if there exists a bounded covering \( \{(U_i)_{i \in \mathbb{N}} \) of \( M \) which forms a bundle atlas of \( E \) such that the transition maps \( t_{ij} : U_{ij} \rightarrow M(\mathbb{C}) \) are a bounded family of matrices.

Clearly if \((M, \hat{g})\) is of bounded geometry, then all tensor bundles over \( M \) (like the bundles \( \otimes_g^k T^*M \) that will be used in Sect. 4) are also of bounded geometry.

The space of bounded sections of \( E \) (analogous to the spaces of bounded tensors) is denoted by \( C_b^\infty(M; E) \). More precisely, if \( u \) is a section of \( E \), we denote by \( u_i : U_i \rightarrow \mathbb{C}^N \) its local trivializations over \( U_i \), and then \( u \in C_b^\infty(M; E) \) if and only if the family \( (u_i)_{i \in \mathbb{N}} \) is bounded in \( C_b^\infty(U_i; \mathbb{C}^N) \).

The notion of bounded differential operators acting on smooth sections of \( E \) can now defined similarly as before, and so does the notion of bounded Hermitian forms on (the fibers of) \( E \).

3.1.4. Bounded Hilbert space structures. Using a partition of unity one can equip a vector bundle \( E \xrightarrow{\pi} M \) of bounded geometry with a positive definite bounded Hermitian form \( \beta \), i.e. a Hilbert space structure on the fibers of \( E \). This means that if \( \beta_i : U_i \rightarrow L_h(\mathbb{C}^N, \mathbb{C}^{N*}) \) are its local trivializations, then \( \beta_i > 0 \) and the families \( (\beta_i)_{i \in \mathbb{N}} \) and \( (\beta_i^{-1})_{i \in \mathbb{N}} \) are bounded in \( C_b^\infty(U_i; L_h(\mathbb{C}^N, \mathbb{C}^{N*})) \) and \( C_b^\infty(U_i; L_h(\mathbb{C}^{N*}, \mathbb{C}^N)) \).

Any two of these bounded Hilbert space structures are equivalent, in the sense of the bounded geometry.
3.1.5. Sobolev spaces. For $s \in \mathbb{R}$, one defines the Sobolev space $H^s(M; E)$ in the natural way, for example using the norm

$$
\|u\|_s^2 = \sum_{i \in \mathbb{N}} \|T_i \circ \psi_i \chi_i u\|_s^2,
$$

where $(U_i, \psi_i)_{i \in \mathbb{N}}$ is a bounded atlas, $T_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^N$ are local trivializations, $1 = \sum \chi_i^2$ is a bounded partition of unity subordinate to $(U_i)_{i \in \mathbb{N}}$ and $\|\cdot\|_s$ is the usual Sobolev norm on $H^s(\mathbb{R}^n; \mathbb{C}^N)$.

The topology of $H^s(M; E)$ is independent of the above choices.

3.1.6. Spacetimes of bounded geometry.

**Definition 3.2.** Let $(M, g)$ be a globally hyperbolic spacetime and $\Sigma \subset M$ a smooth spacelike Cauchy surface. We say that $(M, g)$ is of bounded geometry near $\Sigma$ if the following conditions hold:

1. there exists a reference Riemannian metric $\hat{g}$ such that $(M, \hat{g})$ is of bounded geometry;
2. there exists an open neighborhood $U$ of $\Sigma$ in $M$ such that $g \in BT^0_2(U, \hat{g})$ and $g^{-1} \in BT^0_2(U, \hat{g})$;
3. the embedding $\Sigma \hookrightarrow U$ is of bounded geometry for $\hat{g}$, i.e. there exists a bounded atlas $\{(U_i, \psi_i)\}_{i \in \mathbb{N}}$ for $M$ such that if $\Sigma_i = \psi_i(\Sigma \cap U_i)$, we have

$$
\Sigma_i = \{(v', v_n) \in B_n(0, 1) \mid v_n = F_i(v')\},
$$

where the seminorms of $F_i$ in $BT^0_2(B_n(0, 1), \delta)$ are uniformly bounded for $i \in \mathbb{N}$ such that $U_i \cap U \neq \emptyset$;
4. if $n(y)$ for $y \in \Sigma$ is the future directed unit normal for $g$ to $\Sigma$, one has:

$$
\sup_{y \in \Sigma} n(y) \cdot \hat{g}(y)n(y) < \infty.
$$

If $(M, g)$ is of bounded geometry near $\Sigma$, then the Gaussian normal coordinates to $\Sigma$ are well adapted to the bounded geometry framework. We recall a result in this direction, see [23, Thm. 3.5].

**Theorem 3.3.** Let $(M, g)$ be a Lorentzian manifold of bounded geometry near a Cauchy surface $\Sigma$. Then the following holds:

1. there exists $\delta > 0$ such that the normal geodesic flow to $\Sigma$:

$$
\chi : [-\delta, \delta] \times \Sigma \to M \quad \chi : (t, x) \mapsto \exp^g_\Sigma(tn(x))
$$

is well defined and is a smooth diffeomorphism on its range;
2. $\chi^* g = -dt^2 + h_t$, where $\{h_t\}_{t \in [-\delta, \delta]}$ is a smooth family of Riemannian metrics on $\Sigma$ such that:

i) $(\Sigma, h_0)$ is of bounded geometry,

ii) $t \mapsto h_t \in C^\infty([-\delta, \delta], BT^0_2(\Sigma, h_0))$,

iii) $t \mapsto h_t^{-1} \in C^\infty([-\delta, \delta], BT^2_0(\Sigma, h_0))$. 

We recall that the spaces $C^\infty_b(I, BT^p_q(M, \hat{g}))$ are defined in 3.1.2.

### 3.2. Analyticity in Gaussian time

Let $(M, g)$ be a Lorentzian spacetime of bounded geometry near a Cauchy surface $\Sigma$ with respect to a reference Riemannian metric $\hat{g}$ on $M$. Then by Thm. 3.3, we can assume that $M = I \times \Sigma$ for $I$ an open interval and $g = -dt^2 + h_t$, the function $t : M \to \mathbb{R}$ being called the Gaussian time associated to $\Sigma$.

In later sections, we will perform the Wick rotation in $t$, corresponding to replacing $t$ by $is$, which requires that the metric $g$ is real analytic in $t$. We will need the property that $g$ extends holomorphically in $t$ in a strip in the complex plane, with estimates in the space variables $x$ adapted to the bounded geometry of $(\Sigma, h_0)$.

To formulate precisely our hypotheses we first introduce the relevant Banach spaces of bounded analytic functions.

#### 3.2.1. Bounded analytic functions

Let $D_1(0, r) \subset \mathbb{C}$ be the open disk of radius $r$ and $C_1(0, r) = [-r, r]$ its intersection with the real line. We denote by $\mathcal{A}(C_1(0, r))$ the Banach space of functions $u \in C^\infty(C_1(0, r))$ such that

$$
\|u\|_r := \sup_{\alpha \in \mathbb{N}^n} |r|^{\alpha!}(\alpha!)^{-1}\partial_\alpha^r u(0) < \infty.
$$

The space $\mathcal{A}(C_1(0, r))$ coincides with the Banach space of analytic functions on $C_1(0, r)$ which extend holomorphically to $D_1(0, r)$ with

$$
\|u\|_{r,1} := \sup_{z \in D_1(0, r)} |(1 - r^{-1}|z|)u(z)| < \infty, \quad m_r(z) = (1 - r^{-1}|z|).
$$

The norms $\|\cdot\|_r$ and $\|\cdot\|_{r,1}$ are equivalent, as follows easily from the Cauchy integral formula.

As in 3.1.2 it is straightforward to extend this definition to functions with values in a Fréchet space. Namely, if $\mathcal{F}$ is a Fréchet space whose topology is defined by a family of seminorms $\|\cdot\|_n, n \in \mathbb{N}$, we denote by $\mathcal{A}(C_1(0, r), \mathcal{F})$ the space of maps $f : C_1(0, r) \to \mathcal{F}$ such that

$$
\sup_{\alpha \in \mathbb{N}^n} \|r|^{\alpha!}(\alpha!)^{-1}\partial_\alpha^r f(0)\|_n < \infty,
$$

for all $n \in \mathbb{N}$. Equipped with the obvious seminorms it is again a Fréchet space.

In particular, we use this convention to define the spaces $\mathcal{A}(C_1(0, r); BT^p_q(M, \hat{g}))$.

### 3.3. Main hypotheses

We can now state precisely our hypotheses on the Einstein metric $g$.

**Hypothesis 3.4.** We assume that:

I. $(M, g)$ is a Lorentzian manifold of bounded geometry near a Cauchy surface $\Sigma$.

II. The map $[-\delta, \delta] \ni t \mapsto h_t$ in Thm. 3.3 belongs to $\mathcal{A}(C_1(0, r); BT^2_2(\Sigma, h_0))$ for some $0 < \delta \ll 1$.

**Remark 3.5.** While it is rather easy to check I in examples, it is more difficult to check II. In the next subsection, we will show that it follows from a stronger condition of ‘bounded analyticity’ in all variables.
3.4. Manifolds of bounded analytic geometry. We now define the analog of the notions in Subsect. 3.1 in the analytic category.

3.4.1. Banach spaces of analytic tensors. We start by defining the analog of the spaces $BT_q^p(B_n(0,1),\delta)$ in the analytic case, by extending the definitions in 3.2.1 to several variables.

For well-known reasons, it is convenient to work with polydisks instead of balls. Accordingly, we denote by $D_n(0,0) \subset \mathbb{C}^n$ the polydisk $D_n(0,r) = \{ z \in \mathbb{C}^n \mid |z| < r \}$ and by $C_n(0,r)$ the cube $D_n(0,r) \cap \mathbb{R}^n$. We set $m_r(z) = \prod_{i=1}^n (1 - r^{-1}|z_i|)$ for $z \in D_n(0,r)$.

We denote by $A(C_n(0,r))$ the Banach space of functions $u \in C^\infty(C_n(0,r))$ such that

$$\|u\|_r := \sup_{\alpha \in \mathbb{N}^n} |r|^{\alpha}|(\alpha!)^{-1}\partial^\alpha_x u(0)| < \infty.$$ 

$A(C_n(0,r))$ coincides with the Banach space of analytic functions on $C_n(0,r)$ which extend holomorphically to $D_n(0,r)$ with

$$\|u\|_{r,1} := \sup_{x \in D_n(0,r)} |m_r(z)u(z)| < \infty,$$

and the norms $\|\cdot\|_r$ and $\|\cdot\|_{r,1}$ are again equivalent.

We denote by $AT_q^p(C_n(0,r),\delta)$ the Banach space of analytic $(q,p)$-tensors on $C_n(0,r)$, equipped with the canonical norm obtained from the metric $\delta$ and the norm $\|\cdot\|_r$ on $A(C_n(0,r))$.

By Cauchy estimates $AT_q^p(C_n(0,r),\delta)$ injects continuously into $BT_q^p(\mathcal{U},\delta)$ if $\mathcal{U} \subset C_n(0,r)$.

3.4.2. Riemannian manifolds of bounded analytic geometry. Let $(M, \hat{g})$ be a Riemannian manifold. Assume that $M$ has been given the structure of a real analytic manifold, compatible with its $C^\infty$ structure, and that $\hat{g}$ is an analytic metric.

Definition 3.6. $(M, \hat{g})$ is of bounded analytic geometry if for each $x \in M$ there exists an open neighborhood $\mathcal{U}_x$ of $x$ and

$$\psi_x : \mathcal{U}_x \xrightarrow{\sim} C_n(0,1)$$

an analytic diffeomorphism with $\psi_x(x) = 0$ such that if $\hat{g}_x := (\psi_x^{-1})^*\hat{g}$ then:

1. the family $\{\hat{g}_x\}_{x \in M}$ is bounded in $AT_2^0(C_n(0,1),\delta)$,
2. there exists $c > 0$ such that

$$c^{-1}\delta \leq \hat{g}_x \leq c\delta, \quad x \in M.$$ 

A family $\{(\mathcal{U}_x, \psi_x)\}_{x \in M}$ as above will be called a bounded analytic atlas of $M$.

3.4.3. Bounded analytic tensors. If $(M, \hat{g})$ is a Riemannian manifold of bounded analytic geometry, one can naturally define spaces of bounded analytic tensors.

Definition 3.7. Let $(M, \hat{g})$ be of bounded analytic geometry. We denote by $AT_q^p(M, \hat{g})$ the spaces of smooth $(q,p)$-tensors $T$ on $M$ such that for a bounded analytic atlas $\{(\mathcal{U}_x, \psi_x)\}_{x \in M}$, the family $\{\psi_x^{-1}*T\}_{x \in M}$ is bounded in $AT_q^0(C_n(0,\epsilon),\delta)$ for some $0 < \epsilon \leq 1$. We equip $AT_q^p(M, \hat{g})$ with the Banach space topology given by $\|T\| = \sup_{x \in M} \|T_x\|$, where $T_x = (\psi_x^{-1})^*T$ and $\|T_x\|$ is the norm of $T_x$ in $AT_q^p(C_n(0,\epsilon),\delta)$.
One can show that the spaces \( \mathcal{A}^{T_0^p}(M, \hat{g}) \) are independent on the choice of the bounded analytic atlas \( \{(U_x, \psi_x)\}_{x \in M} \), see Subsect. A.2.

If \( \mathcal{U} \subset M \) is an open set, the spaces \( \mathcal{A}^{T_0^p}(\mathcal{U}, \hat{g}) \) are defined as in 3.1.2.

Noting that \( C_1(0, \alpha) = ]-\alpha, \alpha[ \), the spaces \( \mathcal{A}(]-\alpha, \alpha[; \mathcal{A}^{T_0^p}(M, \hat{g})) \) are defined as in 3.2.1.

3.5. Spacetimes of bounded analytic geometry.

**Definition 3.8.** Let \((M, g)\) be a globally hyperbolic analytic spacetime and \(\Sigma \subset M\) an analytic spacelike Cauchy surface. We say that \((M, g)\) is of bounded analytic geometry near \(\Sigma\) if the following conditions hold:

1. there exists a reference analytic Riemannian metric \(\hat{g}\) such that \((M, \hat{g})\) is of bounded geometry;
2. there exists an open neighborhood \(\mathcal{U}\) of \(\Sigma\) in \(M\) such that \(g \in \mathcal{A}^{T_0^2}(\mathcal{U}, \hat{g})\) and \(g^{-1} \in \mathcal{A}^{T_0^2}(\mathcal{U}, \hat{g})\);
3. the embedding \(\Sigma \hookrightarrow \mathcal{U}\) is analytic of bounded geometry for \(\hat{g}\), i.e. there exists a bounded analytic atlas \(\{(U_i, \psi_i)\}_{i \in \mathbb{N}}\) for \(M\) such that if \(\Sigma_i = \psi_i(\Sigma \cap U_i)\), we have
   \[
   \Sigma_i = \{(v', v_n) \in B_n(0, 1) \mid v_n = F_i(v')\},
   \]
   where the seminorms of \(F_i\) in \(\mathcal{A}^{T_0^0}(B_n(0, 1), \delta)\) are uniformly bounded for \(i \in \mathbb{N}\) with \(U_i \cap \mathcal{U} \neq \emptyset\);
4. if \(n(y)\) for \(y \in \Sigma\) is the future directed unit normal for \(g\) to \(\Sigma\) one has:
   \[
   \sup_{y \in \Sigma} n(y) \cdot \hat{g}(y)n(y) < \infty.
   \]

**Theorem 3.9.** Let \((M, g)\) be a Lorentzian manifold of bounded analytic geometry near a smooth spacelike Cauchy surface \(\Sigma\). Then the following holds:

1. there exists \(\delta > 0\) such that the normal geodesic flow to \(\Sigma\):
   \[
   \chi : [-\delta, \delta] \times \Sigma \to M, \quad (t, x) \mapsto \exp_\hat{g}(tn(x))
   \]
   is well defined and is a smooth diffeomorphism on its image;
2. there exists \(\epsilon_1 > 0\) such that \(\chi^*g = -dt^2 + h_t\), where \(\{h_t\}_{t \in ]-\epsilon_1, \epsilon_1[}\) is a smooth family of Riemannian metrics on \(\Sigma\) such that:
   \[
   \begin{align*}
   i) & \quad (\Sigma, h_0) \text{ is of bounded analytic geometry,} \\
   ii) & \quad t \mapsto h_t \in \mathcal{A}(]-\epsilon_1, \epsilon_1[; \mathcal{A}^{T_0^2}(\Sigma, h_0)), \\
   iii) & \quad t \mapsto h_t^{-1} \in \mathcal{A}(]-\epsilon_1, \epsilon_1[; \mathcal{A}^{T_0^2}(\Sigma, h_0)).
   \end{align*}
   \]

The proof will be given in Appendix A.2.

From Thm. 3.9 we immediately obtain the following corollary.

**Corollary 3.10.** Assume that \((M, g)\) is of bounded analytic geometry near a Cauchy surface \(\Sigma\). Then \((M, g)\) satisfies the hypotheses in Subsect. 3.3.
3.6. Einstein manifolds of bounded analytic geometry. Let Σ be a smooth 3-dimensional manifold and let $\Lambda \in \mathbb{R}$. The initial data on Σ for the non-linear Einstein equations

$$\text{Ric}_{ab} - \Lambda g_{ab} = 0$$

are the Riemannian metric $h$ induced by $g$ on Σ, and the second fundamental form $k$, i.e. the symmetric $(0, 2)$-tensor on Σ defined by

$$u \cdot kv = \nabla_u n \cdot g v, \quad u, v \in T\Sigma,$$

with $n$ the forward unit normal to Σ. Above, Ric is the Ricci tensor of $g$ and $\Lambda$ the cosmological constant.

They have to satisfy the constraint equations, see e.g. [44, Prop. 13.3]:

$$\begin{cases}
\text{Scal}(h) - \text{tr}((kh^{-1})^2) + \text{tr}(kh^{-1})^2 = 2\Lambda, \\
\text{div}(k - \text{tr}(kh^{-1})h) = 0.
\end{cases}$$

(3.1)

We give below conditions on $(\Sigma, h, k)$ that imply that $g$ is of bounded analytic geometry near the initial surface Σ.

Theorem 3.11. Suppose that $(\Sigma, \hat{h}_0)$ is a Riemannian manifold of analytic bounded geometry, and that

$$h, k \in AT^0_2(\Sigma, \hat{h}_0), \quad h^{-1} \in AT^2_0(\Sigma, \hat{h}_0),$$

satisfy the constraint equations (3.1). Then there exists $\delta > 0$ and a solution $g$ of the Einstein equations

$$\text{Ric} - \Lambda g = 0$$

(3.2)

on $M = [-\delta, \delta] \times \Sigma$ with Cauchy data $(h, k)$, such that $(M, g)$ is of bounded analytic geometry near Σ.

Proof. The result follows easily from the local existence for the Einstein equations given in [44, Chap. 14] and the Cauchy–Kowalevski theorem. Let us first recall the arguments in [44, Chap. 14]. Let us recall that $G_{ab} = \text{Ric}_{ab} - \frac{1}{2} R g_{ab}$ is the Einstein tensor and that (3.2) is equivalent to $G + \Lambda g = 0$.

One equips $\mathbb{R} \times \Sigma$ with the Lorentzian metric $\tilde{g} = -dt^2 + h$ and considers the equation

$$\text{Ric}_{ab} + \nabla_c D_c g_{ab} = 0,$$

(3.3)

where the auxiliary $(0, 1)$-tensor $D$ is defined by

$$D_{\nu} = -g_{\mu\alpha} g^\alpha {}^\beta (\Gamma^{\mu}_{\alpha\beta} - \tilde{\Gamma}^{\mu}_{\alpha\beta})$$

and $\Gamma^{\mu}_{\alpha\beta}, \tilde{\Gamma}^{\mu}_{\alpha\beta}$ are the Christoffel symbols for $g$ and $\tilde{g}$. The equation (3.3) with unknown $g$ is a quasilinear hyperbolic system. If $g$ solves (3.3) in some neighborhood $\Omega$ of $\Sigma$ in $\mathbb{R} \times \Sigma$ then

$$G_{ab} = -\nabla_c D_c g_{ab} + \frac{1}{2} \nabla^c D_c g_{ab} \text{ in } \Omega,$$

(3.4)

hence since $\nabla^a G_{ab} = 0$

$$-\Box D_a + \text{Ric}^b_a D_b = 0 \text{ in } \Omega,$$

see [44, (14.8)].
3.7. Examples. We now give several examples of spacetimes satisfying the hypotheses in Subsect. 3.3.

3.7.1. Warped products. Let \((\Sigma, h_0)\) be a Riemannian manifold of bounded analytic geometry, \(0 \in I \subset \mathbb{R}\) an open interval and \(f : I \to \mathbb{R}\) an analytic function with \(f(t) > 0\) for \(t \in I\). Take \(M = I \times \Sigma\) and \(g = -dt^2 + f(t)h\). Then \((M, g)\) is of bounded analytic geometry near \(\{0\} \times \Sigma\).

This applies for example to the de Sitter spacetime

\[ M = \mathbb{R}_t \times S^3, \quad g = -dt^2 + 3\Lambda^{-1}\cosh((\Lambda/3)^{\frac{1}{2}})t)d^2\omega. \]

3.7.2. The Kerr–Kruskal spacetime. Let us prove that the maximal globally hyperbolic extension of the slowly rotating exterior Kerr spacetime (i.e. with parameters \(a, M\) such that \(0 < |a| < M\), which we call for the sake of brevity the Kerr–Kruskal spacetime, satisfies the hypotheses in Subsect. 3.3. We will do this by applying Thm. 3.11.

The Kerr metric is given on \(\mathbb{R}_t \times \mathbb{R}_r \times S^2_{\theta, \varphi}\) in Boyer–Lindquist coordinates \((t, r, \theta, \varphi)\) by

\[ g = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4aMr \sin^2 \theta}{\rho^2}dtd\varphi + \frac{\rho^2}{\Delta}dr^2 + \rho^2d\theta^2 + \frac{\sigma^2}{\rho^2}\sin^2 \theta d\varphi^2, \]

for

\[ \Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \]
\[ \sigma^2 = (r^2 + a^2)\rho^2 + 2aMr \sin^2 \theta. \]

If \(r_- < r_+\) are the two roots of \(\Delta\), the regions \(I := \{r_+ < r\}, \quad \Pi := \{r_- < r < r_+\}\) are the first two Boyer–Lindquist blocks. The apparent singularity of \(g\) at \(r = r_+\) can be removed using Kruskal–Boyer–Lindquist coordinates \((U, V, \theta, \varphi^2)\), see [42, Sect. 3.5].
One obtains in this way the *Kerr–Kruskal spacetime* $\text{KK}$, equal to $\mathbb{R}_U \times \mathbb{R}_V \times S^2_{\theta, \phi}$. The expression of the metric $g$ on $\text{KK}$ is given in [42, Prop. 3.5.3] and is manifestly analytic. $\text{KK}$ contains isometrically the four Boyer-Lindquist blocks: $\text{I}$, the blackhole exterior, $\text{II}$, the black hole interior, and $\text{I}'$, $\text{II}'$, where $M'$ equals $M$ with the reversed time orientation.

It is shown in [22, Prop. C.12] that $(\text{KK}, g)$ is globally hyperbolic, with $\Sigma = \{U = V\}$ as an analytic spacelike Cauchy surface.

**Figure 1.** The Kerr–Kruskal spacetime. The shaded regions represent the neighborhoods $\mathcal{U}$ and $\mathcal{V}$.

To check the hypotheses of Thm. 3.11 we can remove an arbitrary compact neighborhood $\mathcal{V}$ of the bifurcation sphere $S(\mathcal{r}_+^2) = \{U = V = 0\}$.

If $\mathcal{U}$ is a neighborhood of $\Sigma$, then $\mathcal{U} \setminus \mathcal{V}$ is included in $\text{I} \cup \text{I}'$ (see Fig. 1), so it suffices to consider the Kerr metric in block $\text{I}$, where the Cauchy surface $\Sigma$ equals $\{t = 0\}$, and we can use the original Boyer–Lindquist coordinates.

As a reference Riemannian metric on $\Sigma$ we choose $\hat{h}_0 = dr^2 + r^2d\omega$, which is clearly analytic of bounded geometry if we identify $]r_0, +\infty[ \times S^2_{\theta, \phi}$ with $\mathbb{R}^3 \setminus B(0, r_0)$ by spherical coordinates.

It follows that to check that a $(0, 2)$-tensor $f$ on $\Sigma$ belongs to $\mathcal{A}_{T^2}(\Sigma, \hat{h}_0)$, it suffices to check that the functions

$$f_{rr}, r^{-1}f_{r\theta}, r^{-1}f_{r\phi}, r^{-2}f_{\phi\phi}, r^{-2}f_{\theta\phi}, r^{-2}f_{\theta\theta}$$

extend as bounded holomorphic functions in $(r, \theta, \phi)$ in some strip $\{|\text{Im}r| + |\text{Im}\theta| + |\text{Im}\phi| < \delta\}$.

From this observation we immediately obtain that the first fundamental form $h$ belongs to $\mathcal{A}_{T^2}(\Sigma, \hat{h}_0)$. Let us now consider the second fundamental form $k$.

We note that if $X, Y \in T\Sigma$ then $X \cdot kY = (-dt \cdot g^{-1}dt)^{-\frac{1}{2}}\nabla_X dt \cdot Y$, since the future directed unit normal $n$ equals $(-dt \cdot g^{-1}dt)^{-\frac{1}{2}}g^{-1}dt$. Using the coordinates $r, \theta, \phi$ on $\Sigma$,.
we obtain that
\[ k_{r\varphi} = \frac{1}{2}(g^{tt})^{-\frac{1}{2}}(g^{tt}\partial_r g_{r\varphi} + g^{\varphi\varphi}\partial_r g_{r\varphi}), \]
\[ k_{\theta\varphi} = \frac{1}{2}(g^{tt})^{-\frac{1}{2}}(g^{tt}\partial_\theta g_{r\varphi} + g^{\varphi\varphi}\partial_\theta g_{r\varphi}), \]
all other components being 0. We use that \( g^{tt} = -\frac{\sigma^2}{\Delta r^2} \) and obtain similarly that \( k \in AT_2(S, \hat{h}_0). \)

3.7.3. The Schwarzschild–de Sitter spacetime. The Schwarzschild–de Sitter metric is given on \( \mathbb{R}_t \times [0, +\infty) \times S^2_\alpha, \varphi \) by
\[ g = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2d\omega^2, \]
where \( d\omega^2 \) is the round metric on \( S^2 \) and \( F(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 \). For \( 9\Lambda^2m^2 < 1 \), \( F \) has two positive \( 0 < r_+ < r_{++} \), with \( F'(r_+), F'(r_{++}) < 0 \) and \( F(r) > 0 \) on \( ]r_+, r_{++}[ \).

We now have three Boyer–Lindquist blocks \( I := \{ r_+ < r < r_{++} \} \) and \( \Pi_H := \{ 0 < r < r_+ \} \) and \( \Pi_C := \{ r_+ < r < +\infty \} \). Block I is called the static region of S-dS, \( \mathcal{H}_{H/C} = \{ r = r_{++/+} \} \) being the black hole, resp. cosmological horizon.

The apparent singularities at \( t = r_+, r_{++} \) are removed as before by introducing Kruskal-type coordinates near \( \mathcal{H}_H \), resp. \( \mathcal{H}_C \) see e.g. [28, Appendix]. One first defines the Regge–Wheeler coordinate \( r^* \) by \( \frac{dr^*}{dr} = F^{-1}(r) \) (the integration constant being irrelevant). Near \( \mathcal{H}_H \) one sets \( U_H = -e^{-\alpha(t-r^*)} \), \( V_H = e^{\alpha(t+r^*)} \), so that \( r^* = (2\alpha)^{-1}\ln(-U_HV_H) \), \( t = -(2\alpha)^{-1}\ln(-U_HV_H^{-1}) \). If \( \alpha = \frac{1}{2}F'(r_+) \), one obtains that \( U_HV_H = (r - r_+)f_1(r) \), where \( f_1 \) is analytic near \( r = r_+ \) and \( f_1(r_+) \neq 0 \).

Expressed in the coordinates \( (U_H, V_H, \omega) \), the metric becomes
\[ g = -f_2(r)du_Hdv_H + r^2d\omega^2, \]
where again \( f_2 \) is analytic near \( r = r_+ \) and \( f_2(r_+) > 0 \). The coordinates \( (U_H, V_H) \) allow to glue the blocks \( I, \Pi_H, \Pi' \) and \( \Pi_H' \) along \( r = r_+ \).

One can perform a similar change of coordinates near \( r = r_C \) setting now \( U_C = e^{\alpha(t-r^*)} \), \( V_C = -e^{-\alpha(t+r^*)} \), \( \alpha = \frac{1}{2}F'(r_{++}) \), with similar conclusions with \( r_+ \) replaced by \( r_{++}. \) The coordinates \( (U_C, V_C) \) allow to glue the blocks \( I, \Pi_C, \Pi' \) and \( \Pi_C' \) along \( r = r_{++}. \)

![Figure 2. The extended Schwarzschild–de Sitter spacetime.](image-url)

The extended S–dS spacetime is obtained by a bi-infinite sequence of these two gluing procedures, see Fig. 2.
Let us now consider as initial surface \( \Sigma = \{t = 0\} \) in block I. In the Kruskal coordinates (near \( r = r_+ \) or \( r = r_{++} \)), \( \Sigma = \{U_{H/C} + V_{H/C} = 0\} \) and we can extend \( \Sigma \) to the extended S–dS spacetime.

The metric \( g \) is clearly analytic and bounded in each of the three shaded open sets in Fig. 2. From this we obtain that if \( h, k \) are the Cauchy data of \( g \) on \( \Sigma \), \((\Sigma, h)\) is of bounded analytic geometry and \( k \in \mathcal{A} T^0_2(\Sigma, h) \).

**Remark 3.12.** We expect that the Kerr–de Sitter metric can be handled similarly. We refer the reader to [7] for a recent investigation on maximal extensions of the Kerr–de Sitter spacetime.

4. **Linearized gravity**

In this section we formulate linearized gravity as a classical gauge theory, following [29]. We also rewrite the various operators occurring in linearized gravity after using Gaussian normal coordinates to a Cauchy surface and parallel transport. This reduced setting will be used in later sections.

4.1. **Notation.** We start by fixing notation. Let \((M, g)\) be a 4-dimensional Lorentzian manifold.

4.1.1. **Convention for the Riemann tensor.** We use the same convention as in e.g. [44, 13, 6] for the sign of the Riemann tensor \( R_{abcd} = R^e_{abc} g_{ed} \), i.e.

\[
(\nabla_a \nabla_b - \nabla_b \nabla_a)u_c = R_{abc}^d u_d
\]

on (0,1)-tensors in terms of the Levi-Civita connection \( \nabla_a \) on \((M, g)\). Let us recall that the Ricci tensor is the symmetric tensor

\[
\text{Ric}^a_b = R^e_{aebc} = R^e_{acb},
\]

and the scalar curvature is \( R = g^{ab}\text{Ric}_{ab} \). If \( \dim M = 4 \) then the non-linear Einstein equations with cosmological constant \( \Lambda \geq 0 \), i.e. \( \text{Ric} - \frac{1}{2} g \text{Ric} + \Lambda g = 0 \), are equivalent to

\[
\text{Ric} = \Lambda g,
\]

and as a consequence of (4.1) one gets \( R = 4\Lambda \).

4.1.2. **Hermitian forms on tensors.** We denote by

\[
V_k := \mathbb{C} \otimes^k T^* M
\]

the complex bundle of symmetric \((0,k)\)-tensors. We will only need the cases \( k = 1, 2 \). \( V_k \) is equipped with the non-degenerate Hermitian form

\[
(u|u)_{V_k} := \overline{u} \cdot k!(g^\otimes k)^{-1} u.
\]

In abstract index notation,

\[
(u|u)_{V_k} = k! g^{a_1 b_1} \cdots g^{a_k b_k} \overline{u}_{a_1 \cdots a_k} u_{b_1 \cdots b_k}.
\]

For example for \( k = 2 \) we have

\[
(u|u)_{V_2} = 2\text{tr}(u^* g^{-1} u g^{-1}).
\]
The $k!$ normalization differs from the most common convention, it has however the advantage that various expressions involving adjoints look more symmetric.

Sometimes it will be convenient to indicate explicitly the metric used to define the Hermitian form on $V_k$, for $k = 1, 2$. Accordingly, we will set

$$
(u|u)_{g} = \bar{u} \cdot g^{-1} u = (u|u)_{V_1},
(u|u)_{g^2} = \bar{u} \cdot (g^2)^{-1} u = \frac{1}{2}(u|u)_{V_2}.
$$

As in Sect. 2, for $U \subset M$ open, the Hermitian form (4.2) on fibers induces a Hermitian form

$$
(u|v)_{V_k}(U) = \hat{U}(u(x)|v(x))_{V_k} d\text{vol}_g, \quad u, v \in C^\infty_c(U; V_k).
$$

The adjoint of $A : C^\infty(M; V_k) \to C^\infty(M; V_l)$ for those Hermitian forms will be denoted by $A^*$.

4.1.3. The differential and its adjoint. Let

$$
d : C^\infty(M; V_k) \to C^\infty(M; V_{k+1})
$$

$$(du)_{a_1, \ldots, a_{k+1}} = \nabla_{a_1} u_{a_2 \ldots a_{k+1}},$$

where $u_{(a_1 \ldots a_k)}$ is the symmetrization of $u_{a_1 \ldots a_k}$, and

$$
\delta : C^\infty(M; V_k) \to C^\infty(M; V_{k-1})
$$

$$(\delta u)_{a_1, \ldots, a_{k-1}} = -k \nabla^a u_{a_1 \ldots a_{k-1}}.$$

With these conventions, we have $d^* = \delta$ w.r.t. the Hermitian form (4.4).

4.1.4. Operators on tensors. The operator of trace reversal $I$ is given by

$$
I := 1 - \frac{1}{4} |g)(g|,
$$

i.e. $I$ is the orthogonal symmetry w.r.t. the line $\mathbb{C}g$. Equivalently

$$(I u)_{ab} = u_{ab} - \frac{1}{2} \text{tr}_g(u) g_{ab}, \quad \text{tr}_g(u) := g^{ab} u_{ab} = \frac{1}{2} (g|u)_{V_2}.
$$

It satisfies

$$
I^2 = 1, \quad I = I^* \text{ on } C^\infty(M; V_2).
$$

The d’Alembertian is

$$
\Box = \nabla^c \nabla_c, \text{ acting on } (p, q)-\text{tensors},
$$

and the Ricci operator is

$$
\text{Riem}_g(u)_{ab} := R_{a}^{cd} g_{bcd} = R^{c}_{cd} u_{cd}, \quad u \in C^\infty(M; V_2).
$$

The fact that $\text{Riem}_g$ preserves symmetric $(0, 2)$-tensors follows from the symmetries of the Riemann tensor.

The proofs of the next two lemmas are easy and left to the reader.

**Lemma 4.1.** The d’Alembertian satisfies:

$$
\text{i) } \Box \circ I = I \circ \Box,
$$

$$
\text{ii) } \Box = \Box^*.
$$

(4.5)
Lemma 4.2. The Ricci operator satisfies:

\begin{enumerate}
  \item $\text{Riem}_g g = -\text{Ric}$,
  \item $\text{Riem}_g \circ I = I \circ \text{Riem}_g$, if $g$ is Einstein, \hspace{1cm} (4.6)
  \item $\text{Riem}_g = \text{Riem}_g^*$.
\end{enumerate}

Lemma 4.3. If $(M, g)$ is Einstein then:

\begin{enumerate}
  \item $(\Box - 2\text{Riem}_g) \circ d = d \circ (\Box + \Lambda)$ on $C^\infty(M; V_1)$,
  \item $\delta \circ (\Box - 2\text{Riem}_g) = (\Box + \Lambda) \circ \delta$ on $C^\infty(M; V_2)$,
  \item $\delta \circ I \circ d = \Box + \Lambda$ on $C^\infty(M; V_1)$,
  \item $(\Box - 2\text{Riem}_g) \circ I = I \circ (\Box - 2\text{Riem}_g)$ on $C^\infty(M; V_2)$,
  \item $\text{tr}_g \circ (\Box - 2\text{Riem}_g) = (\Box - 2\Lambda) \circ \text{tr}_g$ on $C^\infty(M; V_2)$.
\end{enumerate}

Proof. ii) is [13, Lem. 2.6] and it implies i) by taking adjoints w.r.t. $(|\cdot|)_{V_2(M)}$. Next,

\[
(I \circ dw)_{ab} = \frac{1}{2} \left( \nabla_a w_b + \nabla_b w_a - \frac{1}{2} g^{cd} (\nabla_c w_d + \nabla_d w_c) g_{ab} \right)
= \frac{1}{2} \left( \nabla_a w_b + \nabla_b w_a - (\nabla_d w_d) g_{ab} \right),
\]

\[
(\delta \circ I \circ dw)_b = (\nabla^a \nabla_a w_b + \nabla^a \nabla_b w_a - \nabla_b \nabla^a w_a)
= - (\Box w)_b - \Lambda w_b.
\]

Furthermore, iv) follows from (4.5) ii) and (4.6) iii). Finally,

\[
\text{tr}_g(\text{Riem}_g(u)) = \frac{1}{2} (g|\text{Riem}_g(u)) = \frac{1}{2} (\text{Riem}_g(g)|u)_{V_2} = \frac{\Lambda}{2} (g|u)_{V_2} = \Lambda \text{tr}_g u,
\]

since $\text{Riem}_g = \text{Riem}_g^*$ and $\text{Riem}_g(g) = \Lambda g$, which implies v). \hfill \Box

4.2. Linearized gravity as a gauge theory. Let us now explain how linearized gravity fits in the framework introduced in Sect. 2.2.

Let $(M, g)$ be a globally hyperbolic spacetime of dimension 4. Let us introduce the differential operators

\[
P := -\Box - I \circ d \circ \delta + 2\text{Riem}_g \in \text{Diff}^2(M; V_2),
K := I \circ d \in \text{Diff}^1(M; V_1, V_2).
\]

From now on we assume that $(M, g)$ is Einstein. Then, $Pu = 0$ is the linearized Einstein equation. The condition $K^* u = 0$, where $K^*$ is defined below, is the linearized de Donder or harmonic gauge.

Remark 4.4. We use the same formalism as in [29, Example 3.8]. In [13], $P$ is replaced by $L = \frac{1}{2} P \circ I$ and $K$ by $d = I \circ K$, which corresponds to replacing $u$ by $Iu$.

We consider $V_1$ resp. $V_2$ as Hermitian bundles, where the Hermitian forms on fibers is now

\[
(u|u)_{I,V_1} := (u|u)_{V_1}, \text{ for } u \in V_1, \text{ resp. } (u|u)_{I,V_2} := (Iu|u)_{V_2}, \text{ for } u \in V_2.
\]
The corresponding Hermitian form on smooth sections of $V_i$, $i = 1, 2$, is

$$(u|u)_{I,V_k(U)} = \int_U (u(x)|u(x))_{I,V_k} d\text{vol}_g, \quad u, v \in C_\infty(U; V_k).$$

We denote by $A^\star$ the corresponding formal adjoint of $A$ for $(\cdot | \cdot)_{I,V_k(M)}$ to distinguish it from the formal adjoint $A^*$ for $(\cdot | \cdot)_{V_k(M)}$. The two are related as follows:

$A^\star = IA^*I$ if $A : C_\infty(M; V_2) \to C_\infty(M; V_2),$

$A^\star = A^*I$ if $A : C_\infty(M; V_1) \to C_\infty(M; V_2),$

$A^\star = IA^*$ if $A : C_\infty(M; V_2) \to C_\infty(M; V_1),$

$A^\star = A^*$ if $A : C_\infty(M; V_1) \to C_\infty(M; V_1).$

In particular,

$K^\star = K^*I = \delta \circ I \circ I = \delta. \quad (4.9)$

**Proposition 4.5.** Let

$D_2 = P + KK^* : C_\infty(M; V_2) \to C_\infty(M; V_2),$

$D_1 = K^*K : C_\infty(M; V_1) \to C_\infty(M; V_1).$

Then

$D_2 = -\Box + 2\text{Riem}_g,$

$D_1 = -\Box - \Lambda. \quad (4.10)$

Furthermore, Hypothesis 2.3 is satisfied, i.e. $P^* = P$, $PK = 0$ and $D_1, D_2$ are Green hyperbolic.

**Proof.** The first identity in (4.10) follows from $KK^* = I \circ d \circ \delta$. Using (4.9) and Lem. 4.3 we get $K^*K = \delta \circ I \circ d = -\Box - \Lambda$, hence the second identity.

Next, by Lem. 4.1 and 4.2 we have:

$P^* = -\Box - d \circ \delta \circ I + 2\text{Riem}_g,$

$P^* = IP^*I = P.$

The identity $PK = 0$ follows from

$$(-\Box - I \circ d \circ \delta + 2\text{Riem}_g)I \circ d = I \circ (-\Box + 2\text{Riem}_g) \circ d - I \circ d \circ (\delta \circ I \circ d) = I \circ d(-\Box - \Lambda) + I \circ d(\Box + \Lambda) = 0,$$

by Lem. 4.3. Finally, Green hyperbolicity of $D_i$ for $i = 1, 2$ follows from [5, Thm. 3.3.1], since their principal symbol is $(\xi \cdot g^{-1}(x)\xi)_{1_{V_i}}$. □

**Remark 4.6.** The following alternative expression for $D_i$ can be found in [52]:

$D_1 = \delta \circ d - d \circ \delta,$

$D_2 = \delta \circ d - d \circ \delta + 4\text{Riem}_g + 2\Lambda.$
4.3. **Gaussian normal coordinates.** We assume that \( M = I_t \times \Sigma \), where \( I \subset \mathbb{R} \) is an interval with \( 0 \in I \) and \( \Sigma \) a smooth \( d \)-dimensional manifold. We denote by \( \xi = (\tau, k) \) the dual variables to \( x = (t, x) \).

We set \( \Sigma_t = \{ t \} \times \Sigma \) and identify \( \Sigma_0 \) with \( \Sigma \). We assume that

\[
g = -dt^2 + h(t, x)\, dx^2,
\]

where \( h \in C^\infty(M; \otimes^2 T^*\Sigma) \), is a \( t \)-dependent Riemannian metric on \( \Sigma \). If the normal geodesic flow to \( \Sigma \subset M \) is defined for a uniform time interval, as is the case for spacetimes of bounded geometry, we can reduce ourselves to this model situation.

Let

\[
r := \frac{1}{2} \partial_t h h^{-1} \in C^\infty(M; L(T^*\Sigma)).
\]

By computing the Christoffel symbols for \( g \), see (4.14), one can check that the second fundamental form of \( g \) is \( k = \frac{1}{2} \partial_t h \), hence \( r = kh^{-1} \).

We will use the notation introduced in (4.3) for the Hermitian forms on \( V_1, V_2 \).

4.3.1. **Decomposition of \((0,1)\)-tensors.** We identify

\[
C^\infty(M; T^*M) \xrightarrow{\sim} C^\infty(M) \oplus C^\infty(M; T^*\Sigma)
\]

by

\[
w \mapsto (w_1, w_\Sigma), \quad w =: w_t dt + w_\Sigma.
\]

The scalar product \( (\cdot|\cdot)_g \) reads then

\[
(w|w)_g = -|w_t|^2 + (w_\Sigma|w_\Sigma)_h,
\]

i.e.,

\[
g^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & h^{-1} \end{pmatrix}.
\]

**Lemma 4.7.** If \( w_1, w_2 \) are \((0,1)\)-tensors on \( \Sigma \) we have:

\[
(w_1|rw_2)_h = (rw_1|w_2)_h,
\]

i.e. \( r \) is selfadjoint for \( (\cdot|\cdot)_h \).

4.3.2. **Decomposition of \((0,2)\)-tensors.** Similarly we identify

\[
C^\infty(M; \otimes^2 T^*M) \xrightarrow{\sim} C^\infty(M) \oplus C^\infty(M; T^*\Sigma) \oplus C^\infty(M; \otimes^2 T^*\Sigma)
\]

by

\[
u \mapsto (u_t, u_\Sigma, u_{\Sigma\Sigma}), \quad u =: u_t dt \otimes dt + w_\Sigma \otimes dt + dt \otimes w_\Sigma + w_{\Sigma\Sigma}.
\]

The scalar product \( (\cdot|\cdot)_{g \otimes^2} \) reads

\[
(u|u)_{g \otimes^2} = |u_t|^2 - 2(u_\Sigma|u_\Sigma)_h + (u_{\Sigma\Sigma}|u_{\Sigma\Sigma})_h \otimes^2.
\]

We have the following analog of Lem. 4.7. In (4.13) below, \( r^\# \in L(T\Sigma) \) is the transpose of \( r \in L(T^*\Sigma) \).
Lemma 4.8. If \( u_1, u_2 \) are \((0,2)\)-tensors on \( \Sigma \) we have:
\[
(u_1 \circ u_2)_{h \otimes 2} = (r \circ u_1 | u_2)_{h \otimes 2}, \\
(u_1 | u_2 \circ r^\#)_{h \otimes 2} = (u_1 \circ r^\# | u_2)_{h \otimes 2},
\]
i.e. the operators \( r \circ \cdot \) and \( \cdot \circ r^\# \) are self-adjoint for \((\cdot | \cdot)_{h \otimes 2}\).

Proof. Since \((u_1 | u_2)_{h \otimes 2} = \text{tr}(u_1^* h^{-1} u_2 h^{-1})\), we have
\[
(u_1 | r \circ u_2)_{h \otimes 2} = \text{tr}(u_1^* h^{-1} \partial_t h h^{-1} u_2 h^{-1}) \\
= \text{tr}((\partial_t h h^{-1} u_1)^* h^{-1} u_2 h^{-1}) = (r \circ u_1 | u_2)_{h \otimes 2},
\]
\[
(u_1 | u_2 \circ r^\#)_{h \otimes 2} = \text{tr}(u_1^* h^{-1} u_2 h^{-1} \partial_t h h^{-1}) \\
= \text{tr}((u_1 \partial_t h h^{-1})^* h^{-1} u_2 h^{-1}) = (u_1 \circ r^\# | u_2)_{h \otimes 2}. \quad \Box
\]

4.3.3. Covariant derivatives. We recall that \( \nabla, d, \delta \) are the covariant derivative, differential and codifferential for \( g \). Similarly we denote by \( \nabla_\Sigma, d_\Sigma, \delta_\Sigma \) the corresponding operators for \( h \), acting on tensors on \( \Sigma \). We set
\[
\nabla_0 = \nabla_{\partial_t}.
\]

Let us denote by \( \Gamma^i_{ab}(g) \) resp. \( \Gamma^i_{ij}(h) \) the Christoffel symbols of \( g \) resp. \( h \). A routine computation gives
\[
\Gamma^0_{ij}(g) = \frac{1}{2} \partial_t h_{ij}, \\
\Gamma^0_{ij}(g) = \Gamma^i_{j0}(g) = r^i_j, \\
\Gamma^i_{jk}(g) = \Gamma^i_{jk}(h),
\]
all other entries being equal to 0. From (4.14) we obtain that
\[
(\nabla_0 w)_t = \partial_t w_t, \\
(\nabla_0 w)_\Sigma = \partial_t w_\Sigma - r w_\Sigma, \\
(\nabla_0 u)_\Sigma = \partial_t u_\Sigma - r u_\Sigma, \\
(\nabla_0 u)_\Sigma = \partial_t u_\Sigma - r \circ u_\Sigma - u_\Sigma \circ r^\#.
\]
Furthermore,
\[
(dt)_t = \partial_t w_t, \\
(dt)_\Sigma = \frac{1}{2} \partial_t h w_\Sigma - r w_\Sigma + \frac{1}{2} d_\Sigma w_t, \\
(dw)_\Sigma = d_\Sigma w_\Sigma - \frac{1}{2} \partial_t h w_\Sigma.
\]

4.4. Reduced setting. We now perform a reduction which corresponds to identify tensors on \( \Sigma_t \) with tensors on \( \Sigma \) by parallel transport along \( \partial_t \). This allows us to simplify the expressions of the operators \( D_t, \nabla_0, d, \delta \). Let
\[
\mathbf{u}(t) = T_{\exp}(\int_t^0 r(s) ds) \in C^\infty(I; C^\infty(\Sigma; \mathcal{L}(T^* \Sigma))),
\]
i.e. \( \mathbf{u}(t) \) is the unique solution of
\[
\begin{cases}
\partial_t \mathbf{u}(t) = -\mathbf{u}(t) r(t), \\
\mathbf{u}(0) = 1.
\end{cases}
\]
Proof. We compute using (4.17):

\[ U_1(w_t, w_\Sigma) := (w_t, uw_\Sigma), \]

where we use the identification in (4.11) and

\[ U_2(u_{tt}, u_{t\Sigma}, u_{\Sigma \Sigma}) := (u_{tt}, uw_{t\Sigma}, u \circ u_{\Sigma \Sigma} \circ u^*), \]

using the identification in (4.12). Let us set

\[ g_0 := -dt^2 + h_0(x)dx^2, \]

where \( h_0(x) = h(0, x) \).

**Proposition 4.9.** We have:

1. \( U^*_1g_0^{-1}U_1 = g^{-1}, \)
2. \( U^*_2(g_0^{\otimes 2})^{-1}U_2 = (g^{\otimes 2})^{-1}, \)
3. \( U_2g = g_0, \)
4. \( I_0 := U_2IU_2^{-1} \) is the trace reversal w.r.t. \( g_0, \)
5. \( U_i \circ \nabla_0 = \partial_t \circ U_i, \ i = 1, 2. \)

**Proof.** We compute using (4.17):

\[ \partial_t(u(t)h(t)u^*(t)) = u(t)(-r(t)h(t) + \partial_t h(t)) - h(t)r^*(t)) u^*(t) = 0. \]

This implies that \( u(t)h(t)u^*(t) = h_0, \) hence \( U^*_1g_0^{-1}U_1 = g^{-1} \) or equivalently, \( U_2g = g_0, \)

which proves (1) and (3). Similarly

\[ (U_2u|U_2u)_{g^{\otimes 2}} = \text{tr}(g_0^{-1}U_2\nabla g_0^{-1}U_2u) = \text{tr}(g_0^{-1}U_1\nabla U^*_1g_0^{-1}U_1uU^*_1) \]

\[ = \text{tr}(U^*_1g_0^{-1}U_1\nabla U^*_1g_0^{-1}U_1u) = \text{tr}(g_0^{-1}g_0^{-1}u) = (u|u)_{g^{\otimes 2}}, \]

which proves (2). From (2) and (3) we obtain

\[ (g|u)_g = (U_2g|U_2u)_{g_0} = (g_0|U_2u)_{g_0}, \]

which implies (4). Finally, by (4.15) we obtain (5). \( \square \)

**Lemma 4.10.** We have:

\[ (U_2dU_1^{-1}w)_{tt} = \partial_t w_t, \]

\[ (U_2dU_1^{-1}w)_{t\Sigma} = \frac{1}{2}(\partial_t w_\Sigma - u\nabla u^{-1}w_\Sigma + ud_\Sigma w_t), \]

\[ (U_2dU_1^{-1}w)_{\Sigma \Sigma} = u(d_\Sigma u^{-1}w_\Sigma)u^* - \frac{1}{2}u\partial_t huu^* w_t. \]

**Proof.** This follows from (4.16), using that \( \partial_t u^{-1} = ru^{-1}. \) \( \square \)

4.4.1. **Reduced operators.** Let us set for \( u \in C^\infty_c(M; V_i), \ i = 1, 2: \)

\[ Tu = s_t u \]

for \( s_t = |h_t|^{-\frac{1}{2}}|h_0|^{-\frac{1}{2}}. \)

Then

\[ (u|u)_{V_i(M)} = \int_M (u|u)V_i dvol_g = \int_M (Tu|Tu)V_i dvol_{g_0}, \quad (4.18) \]
and hence
\[ (u|u)_{V_1(M)} = \int_M (TU_1 u | TU_1 u)_{g_0} d\text{vol}_{g_0}, \]
\[ (u|u)_{V_2(M)} = 2 \int_M (TU_2 u | TU_2 u)_{g_0} \otimes d\text{vol}_{g_0}. \]

We define the analogues of \( D \), \( d \) and \( I \) in the reduced setting:
\[ \hat{D}_i := (TU_i) \circ D_i \circ (TU_i)^{-1}, \]
\[ \hat{d} := (TU_2) \circ d \circ (TU_1)^{-1}, \]
\[ \hat{I} := (TU_2) \circ I \circ (TU_2)^{-1}. \] (4.19)

**Proposition 4.11.** The operators \( \hat{D}_i \), \( \hat{d} \) and \( \hat{I} \) have the following properties.
1. \( \hat{D}_i = \partial_t^2 + \hat{a}_i(t) \), where \( \hat{a}_i \in C^\infty(\Sigma; \Psi^2(\Sigma; V_i)) \) has principal symbol \( k \cdot h_t^{-1} k \mathbf{1}_{V_i} \) and is self-adjoint for the Hermitian form \( (\cdot | \cdot)_{V_i(\Sigma)} \) defined by:
   \[ (u|u)_{V_1(\Sigma)} := \int_{\Sigma}(u|u)_{g_0} d\text{vol}_{g_0}, \quad (u|u)_{V_2(\Sigma)} := 2 \int_{\Sigma}(u|u)_{g_0} \otimes d\text{vol}_{g_0}. \]
2. We have
   \[ (\hat{d}w)_{tt} = \partial_t w_t - \frac{1}{2} \text{tr}(r) w_t, \]
   \[ (\hat{d}w)_{t\Sigma} = \frac{1}{2} \left( \partial_t w_{\Sigma} - \frac{1}{2} \text{tr}(r) w_{\Sigma} - w u^{-1} w_{\Sigma} + (su) \circ d_{\Sigma} \circ (su)^{-1} w_t \right), \]
   \[ (\hat{d}w)_{\Sigma \Sigma} = (su) \circ d_{\Sigma} \circ (su)^{-1} w_{\Sigma} u'^\# - \frac{1}{2} u \partial_t h u'^\# w_t. \]
3. \( \hat{I} = I_0 = 1 - \frac{1}{2}|g_0)(g_0| \).
4. We have \( \hat{D}_2 \hat{d} = \hat{d} \hat{D}_1 \) and \( \hat{I} \hat{D}_2 = \hat{D}_2 \hat{I} \).

**Proof.** (4) is straightforward.

Observe that \( \hat{D}_i \) has the same principal symbol as \( D_i \), i.e. \( \xi \cdot g^{-1} \xi \mathbf{1}_{V_i} \). Therefore,
\[ \hat{D}_i = \partial_t^2 + b_i(t,x) \partial_t + a_i(t,x, \partial_x), \]
where \( \sigma_{pr}(a_i)(t,x,k) = k_i h^{ij}(t,x)k_j \mathbf{1} \) and \( \hat{a}_i \in C^\infty(\Sigma; \Psi^2(\Sigma; V_i)), b_i \in C^\infty(\Sigma; \Psi^0(\Sigma; V_i)) \).

Since \( D_i \) is self-adjoint for \( (\cdot | \cdot)_{V_i(M)} \), we deduce from Prop. 4.9 and (4.18) that \( \hat{D}_i \) is self-adjoint for the Hermitian form:
\[ (u|u)_{\hat{V}_i(M)} = \int_M (u|u)_{g_0} d\text{vol}_{g_0}. \]

This implies that \( b_i(t,x) = 0 \) and that \( a_i(t,x, \partial_x) \) is self-adjoint for the scalar product \( (\cdot | \cdot)_{V_i(\Sigma)} \), which proves (1).

We have \( \partial_t h_t = 2 |h_t| \text{tr}(r) \) hence \( \partial_t s = \frac{1}{2} \text{tr}(r) s \). Using Lem. 4.10 we obtain (2). Finally (3) follows from Prop. 4.9 (4). \( \square \)
4.4.2. **Gauge invariance.** We can write \( \hat{d} \) in the form
\[
\hat{d} = \hat{d}_0(t) \partial_t + \hat{d}_1(t),
\]
where \( \hat{d}_i \in C^\infty_0(I; \text{Diff}^j(\Sigma; V_1, V_2)) \). An easy computation shows that the gauge identity
\[
\hat{D}_2 \hat{d} = \hat{d} \hat{D}_1
\]
is equivalent to
\[
\begin{align*}
&i) \quad \partial_t \hat{d}_0 = 0, \\
&ii) \quad 2 \partial_t \hat{d}_1 + \hat{a}_2 \hat{d}_0 - \hat{d}_0 \hat{a}_1 = 0, \\
&iii) \quad \partial_t^2 \hat{d}_1 + \hat{a}_2 \hat{d}_1 - \hat{d}_1 \hat{a}_1 - \hat{d}_0 \partial_t \hat{a}_1 = 0.
\end{align*}
\]

5. **Hadamard and Calderón projectors**

In this section we first revisit the construction of Hadamard projectors for second order hyperbolic equations acting on sections of Hermitian vector bundles.

Hadamard projectors act on distributional Cauchy data and project on Cauchy data whose solutions have wavefront sets in one of the two energy shells \( \mathcal{N}^\pm \). If the Hermitian bundle is Hilbertian, i.e. if the fiber scalar product is positive definite, they produce Hadamard states for the associated quantum fields. In general they produce only Hadamard pseudo-states.

We then consider second order elliptic equations, typically obtained by Wick rotation of the hyperbolic equations in the time variable, and we construct the associated Calderón projectors. We also study Dirichlet-to-Neumann maps, which will be important in later sections. Finally we show that Hadamard and Calderón projectors coincide modulo smoothing operators.

5.1. **ΨDO calculus on manifolds of bounded geometry.** The constructions in this section rely on a global pseudodifferential calculus on a Cauchy surface \( \Sigma \). Namely, we use Shubin’s calculus which we now quickly recall.

Let \((M, \hat{g})\) be a Riemannian manifold of bounded geometry and \( V \xrightarrow{\pi} M \) a finite rank complex vector bundle of bounded geometry.

One can then define for \( m \in \mathbb{R} \) the symbol classes \( S^m(T^*M; L(V)) \) of poly-homogeneous symbols, see [48] or [24, Sect. 5]. Using a bounded atlas \( \{ (U_i, \psi_i) \}_{i \in \mathbb{N}} \) and associated local trivializations of \( V \) and partition of unity \( 1 = \sum_i \chi_i^2 \) one can define a quantization map
\[
\text{Op} : S^m(T^*M; L(V)) \to L(C^\infty_c(M; V)),
\]
\( \text{Op}(a) \) being a (classical) pseudodifferential operator of order \( m \). Choosing a different atlas, trivializations or partition of unity produces of course in general a different quantization map \( \text{Op}' \). However,
\[
\text{Op}(a) - \text{Op}'(a) \in \mathcal{W}^{-\infty}(M; V),
\]
where
\[
\mathcal{W}^{-\infty}(M; V) := \bigcap_{m \in \mathbb{N}} B(H^{-m}(M; V), H^m(M; V)),
\]
is an ideal of smoothing operators. Similarly if \( \Omega \subset M \) is an open set we set
\[
\mathcal{W}^{-\infty}(\Omega; V) := \bigcap_{m \in \mathbb{N}} B(H^{-m}(\Omega; V), H^m(\Omega; V)),
\]
and if \((M_i, \tilde{g}_i)\) and \(V_i \xrightarrow{\pi} M_i\) are of bounded geometry:
\[
\mathcal{W}^{-\infty}(M_1, M_2; V_1, V_2) := \bigcap_{m \in \mathbb{N}} B(H^{-m}(M_1; V_1), H^m(M_2; V_2)).
\]
One sets then
\[
\Psi^m(M; V) = \text{Op}(S^m(T^*M; L(V))) + \mathcal{W}^{-\infty}(M; V),
\]
and \(\Psi^\infty(M; V) = \bigcup_{m \in \mathbb{R}} \Psi^m(M; V).\)

We refer the reader to [48, App. 1] and [24, Sect. 5] for more details.

5.2. **Lorentzian case.** We set \(M = I_t \times \Sigma_x\), where \(I \subset \mathbb{R}\) is an interval with 0 \(\in \mathring{I}\) and \((\Sigma, h_0)\) a \(d\)-dimensional Riemannian manifold of bounded geometry. We set \(\Sigma_t = \{t\} \times \Sigma\) and identify \(\Sigma_0\) with \(\Sigma\). The dual variables to \((t, x)\) are denoted by \((\tau, k)\).

We fix a \(t\)-dependent Riemannian metric on \(\Sigma\),
\[
h : I \ni t \mapsto h(t) \in C^\infty_b(I; BT^0(\Sigma, h_0)).
\]
We assume that \(h(0) = h_0\) and for ease of notation we often denote \(h(t)\) by \(h_t\).

We equip \(M\) with the Lorentzian metric
\[
g := -dt^2 + h_t dx^2.
\]

5.2.1. **Hermitian bundle.** We fix a finite rank complex vector bundle \(V \xrightarrow{\pi} \Sigma\) of bounded geometry over \((\Sigma, h_0)\). We still denote by \(V\) the vector bundle over \(M\) : \(I \times V \xrightarrow{\pi} M\) which is a vector bundle with the same fibers as \(V\). We have for example
\[
C^\infty_b(M; V) \approx C^\infty_b(I; C^\infty_b(\Sigma; V)).
\]
We assume that \(V \xrightarrow{\pi} M\) is equipped with a non-degenerate fiberwise Hermitian structure \((\cdot | \cdot)_V\), which is assumed to be independent of \(t\).

We fix a reference fiberwise Hilbertian structure \((\cdot | \cdot)_\tilde{V}\) on the fibers of \(V\) which is also independent of \(t\).

We denote by \((\cdot | \cdot)_{\tilde{V}}, (\cdot | \cdot)_V\) the same Hermitian structures acting on the fibers of \(V \xrightarrow{\pi} \Sigma\). If \(x \in M\) and \(u, v \in V_x\) we have
\[
(u | v)_V = (u | \tau_x v)_{\tilde{V}}, \quad \tau_x \in L(V_x),
\]
and we denote by \(\tau \in C^\infty(M; L(V))\) the corresponding section, which is independent on \(t\).

Note that \(\tau = \tau^*\). By polar decomposition, after possibly changing \((\cdot | \cdot)_{\tilde{V}}\), we can assume that
\[
\tau^* \tau = 1, \quad \text{i.e.} \quad \tau \text{ is unitary for } (\cdot | \cdot)_{\tilde{V}}.
\]
We assume that the Hermitian structures \((\cdot | \cdot)_V\) and \((\cdot | \cdot)_{\tilde{V}}\), and hence \(\tau\), are of bounded geometry.

If \(a \in L(V_x)\) for \(x \in M\) we denote by \(a^*\), resp. \(a^{\star}\), the adjoints of \(a\) for \((\cdot | \cdot)_{\tilde{V}}\), resp. \((\cdot | \cdot)_V\).

Then,
\[
a^* = \tau_x^{-1} a^* \tau_x
\]
for some \(\tau_x \in L(V_x)\).
For \( u, v \in C^\infty_{sc}(M; V) \) we set
\[
(u|v)_{V(\Sigma_i)} := \int_{\Sigma_i} (u|v)_{\tilde{V}}|h_0|^{\frac{3}{2}}dx,
\]
\[
(u|v)_{V(\Sigma)} := \int_{\Sigma} (u|v)_{\tilde{V}}|h_0|^{\frac{3}{2}}dx = (u|\tau v)_{\tilde{V}(\Sigma)},
\]
\[
(u|v)_{V(M)} := \int_M (u(t)|v(t))_{\tilde{V}}|h_0|^{\frac{3}{2}}dt dx,
\]
\[
(u|v)_{V(\Omega)} := \int_{\Omega} (u(t)|v(t))_{\tilde{V}}|h_0|^{\frac{3}{2}}dt dx = (u|\tau v)_{\tilde{V}(\Omega)}.
\]

If \( \Omega \subset M \) is some open set, we also denote
\[
(u|v)_{\tilde{V}(\Omega)} := \int_\Omega (u(t)|v(t))_{\tilde{V}}|h_0|^{\frac{3}{2}}dt dx,
\]
\[
(u|v)_{\tilde{V}(\Omega)} := \int_\Omega (u(t)|v(t))_{\tilde{V}}|h_0|^{\frac{3}{2}}dt dx = (u|\tau v)_{\tilde{V}(\Omega)}.
\]

We denote by \( L^2(\Sigma; \tilde{V}) \) the \( L^2 \) space obtained from the Hilbertian scalar product \((\cdot|\cdot)_{\tilde{V}(\Sigma)}\).

5.2.2. Adjoints. If \( a \in C^\infty_c(I; \text{Diff}(\Sigma; V)), \) resp. \( A \in \text{Diff}(M; V) \), we denote by \( a^* \) resp. \( A^* \) its formal adjoint for \((\cdot|\cdot)_{\tilde{V}(\Sigma)}\) resp. \((\cdot|\cdot)_{\tilde{V}(M)}\). We set \( \Re a = \frac{1}{2}(a + a^*) \).

We denote by \( a^* \) resp. \( A^* \) its formal adjoint for \((\cdot|\cdot)_{\tilde{V}(\Sigma)}\) resp. \((\cdot|\cdot)_{\tilde{V}(M)}\). As above we have:
\[
a^* = \tau^{-1}a^*\tau, \quad A^* = \tau^{-1}A^*\tau.
\]

5.2.3. Hyperbolic operator. We fix a \( t \)-dependent differential operator \( a = a(t, x, D_x) \) belonging to \( C^\infty_c(I; \text{Diff}^2(\Sigma; V)) \) and denote by \( \sigma_{pr}(a) \in C^\infty(T^*\Sigma; L(V)) \) its principal symbol.

We assume the following properties:
\[
\text{(H1)} \quad a(t) = a^*(t), \quad t \in I,
\]
\[
\text{(H2)} \quad \sigma_{pr}(a)(t)(x, k) = k \cdot h_t^{-1}(x)k 1_V, \quad t \in I.
\]

We set
\[
D := \partial_t^2 + a(t) \text{ acting on } C^\infty_{sc}(M; V),
\]
which is a hyperbolic operator with scalar principal part. Note that \( D = D^* \), but of course \( D \neq D^* \) in general.

5.2.4. Green’s formula. We set
\[
gu = \begin{pmatrix} u(0) \\ i^{-1}\partial_t u(0) \end{pmatrix}, \quad u \in C^\infty_{sc}(M; V),
\]
\[
\text{and equip } C^\infty_c(\Sigma; V \otimes \mathbb{C}^2) \text{ with the Hilbertian scalar product } (\cdot|\cdot)_{\tilde{V}(\Sigma)\otimes\mathbb{C}^2} \text{ defined by}
\]
\[
(f|f)_{\tilde{V}(\Sigma)\otimes\mathbb{C}^2} := (f_0|f_0)_{\tilde{V}(\Sigma)} + (f_1|f_1)_{\tilde{V}(\Sigma)}.
\]

The Green identity in Lem. 2.2 takes the form:
\[
(u|Dv)_{V(J^\pm(\Sigma))} - (Du|v)_{V(J^\pm(\Sigma))} = \pm i^{-1}(gu|gu)_{\tilde{V}(\Sigma)\otimes\mathbb{C}^2},
\]
where
\[
q = \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}.
\]
The Hilbertian vector bundle $V^\Sigma$ in 2.1.1 equals $\tilde{V}(\Sigma) \otimes \mathbb{C}^2$, since the operator $D$ is of second order.

5.2.5. *Symplectic adjoint.* If $A$ is an operator acting on $C^\infty_c(\Sigma; \tilde{V} \otimes \mathbb{C}^2)$ we denote by $A^\dagger$ its adjoint for $q$, i.e.:

$$A^\dagger = q^{-1} A^* q.$$

5.2.6. *Square root of $a(t)$.* We first construct an approximate square root $a(t)$ adapted to our future needs.

**Lemma 5.1.** There exist $r_{\infty} \in C^\infty_b(I; \Psi^{-\infty}(\Sigma; V))$ and $\epsilon \in C^\infty_b(I; \Psi^1(\Sigma; V))$ such that:

1. $\epsilon = \epsilon^*$, $\epsilon^2 = a + r_{\infty}$,
2. $\sigma_{pr}(\epsilon) = (k \cdot h^{-1}(x) k)^{1/2} 1_V$,
3. $\epsilon$ with domain $H^1(\Sigma; \tilde{V})$ is m-accretive with $\Re \epsilon \geq 1$.

**Proof.** Note first that $a$ is uniformly elliptic in $C^\infty_b(I; \Psi^2(\Sigma; V))$ hence has a parametrix $a(-1) \in C^\infty_b(I; \Psi^{-2}(\Sigma; V))$. Therefore $a(t)$ is closed with domain $H^2(\Sigma; \tilde{V})$ and $\text{Dom} a^*(t) = H^2(\Sigma; \tilde{V})$. We set

$$2a_{\text{ref}} := \frac{1}{2}(a + a^*) + \tau^{-1} \frac{1}{2}(a + a^*) \tau.$$

It satisfies:

1. $a - a_{\text{ref}} \in C^\infty_b(I; \text{Diff}^1(\Sigma; V))$,
2. $a_{\text{ref}} = a^*_{\text{ref}} = a_{\text{ref}}^*$.

In fact, (5.3) i) follows from (H2), and (5.3) ii) from the fact that $\tau = \tau^* = \tau^{-1}$.

Clearly $a_{\text{ref}}$ with domain $H^2(\Sigma; \tilde{V})$ is self-adjoint for $(\cdot | \cdot)_{\tilde{V}(\Sigma)}$. We fix $\chi \in C^\infty_c(\mathbb{R})$ with $\chi(0) = 1$ and set $\chi_R(\lambda) = \chi(R^{-1} \lambda)$ for $R \gg 1$. Since $a_{\text{ref}}$ is elliptic we know that $\chi_R(a_{\text{ref}}) \in C^\infty_b(I; \Psi^{-\infty}(\Sigma; V))$.

We set now

$$r_{-\infty} = R \chi_R(a_{\text{ref}}),$$

where $R \gg 1$ will be chosen below. From (5.3) we deduce that

$$r_{-\infty} = r^*_{-\infty} = r^*_{-\infty}$$

and

$$\Re(a + r_{-\infty}) = a_{\text{ref}} + R \chi_R(a_{\text{ref}}) + a_1$$

for some $a_1 \in C^\infty_b(I; \text{Diff}^1(\Sigma; V))$.

By the self-adjoint functional calculus we can find $R \gg 1$ such that:

$$a_{\text{ref}} + R \chi(R^{-1} a_{\text{ref}}) \geq \frac{1}{2} a_{\text{ref}} + 1,$$

and hence by (5.4)

$$\Re(a + r_{-\infty}) \geq 1.$$
we obtain that $\epsilon$ is closed, using that $a$ is closed. Therefore $a + r_{-\infty}$ is $m$-accretive. By [37, Thm. V.3.35] $a + r_{-\infty}$ has a unique $m$-accretive square root:
\[
\epsilon := \pi^{-1} \int_0^{+\infty} \lambda^{-\frac{1}{2}} (a + r_{-\infty} + \lambda)^{-1} (a + r_{-\infty}) d\lambda
\]
\[(5.7)\]
where the integrals are strongly convergent on $\text{Dom } a = H^2(\Sigma; \tilde{V})$. By [37, Pb. V.3.39] we have
\[
\Re \epsilon \geq 1.
\]
(5.8)
Arguing as in [24, Subsect. 5.3], using the representation of $\epsilon$ in the second line of (5.7), we obtain that $\epsilon \in C^\infty_b(I; \Psi^1(\Sigma; V))$ with
\[
\sigma_{pr}(\epsilon) = (k \cdot h^{-1}_t(x)k)^\frac{1}{2} 1_V.
\]
The operator $\epsilon(t)$ with domain $H^1(\Sigma; \tilde{V})$ is closed, elliptic, $m$-accretive and invertible by (5.8), hence $\epsilon^{-1} \in C^\infty_b(I; \Psi^{-1}(\Sigma; V))$. From (5.7) we obtain that $\epsilon = \epsilon^*$. $\square$

5.2.7. Factorization of $D$.

**Proposition 5.2.** There exists $b \in C^\infty_b(I; \Psi^1(\Sigma; V))$ unique modulo $C^\infty_b(I; \Psi^{-\infty}(\Sigma; V))$ such that

\begin{itemize}
  \item[i)] $b = \epsilon + C^\infty_b(I; \Psi^0(\Sigma; V))$,
  \item[ii)] $i \partial_t b - b^2 + a = r_{-\infty} \in C^\infty_b(I; \Psi^{-\infty}(\Sigma; V))$,
  \item[iii)] $b + b^* = (2\epsilon)^\frac{1}{2} (1 + r_{-1})^2 (2\epsilon)^\frac{1}{2}$,
\end{itemize}

\[
r_{-1} \in C^\infty_b(I; \Psi^{-1}(\Sigma; V)), \quad r_{-1} = r_{-1}^*, \quad \|r_{-1}\| \leq \frac{2}{3}.
\]

**Proof.** We first solve i) and ii). Let $b_0 \in C^\infty_b(I; \Psi^0(\Sigma; V))$. A routine computation shows that

\[
i \partial_t (\epsilon + b_0) - (\epsilon + b_0)^2 + a = 0
\]
iff

\[
b_0 = (2\epsilon)^{-1} i \partial_t \epsilon + F(b_0),
\]
(5.9)
for

\[
F(b_0) = (2\epsilon)^{-1} (i \partial_t b_0 + [\epsilon, b_0] - b_0^2).
\]

By Prop. A.1 we find $b_0 \in C^\infty_b(I; \Psi^0(\Sigma; V))$, unique modulo $C^\infty_b(I; \Psi^{-\infty}(\Sigma; V))$ solving (5.9) modulo $C^\infty_b(I; \Psi^{-\infty}(\Sigma; V))$.

Next we take $\chi$ as in (5.5) and set

\[
b = \epsilon + b_0 (1 - \chi_{R}(a_{\text{ref}})),$
\]
where $R \gg 1$ will be fixed below. Since $\chi_{R}(a_{\text{ref}}) \in C^\infty_b(I; \Psi^{-\infty}(\Sigma; V))$, we see that $b$ satisfies i) and ii).

It remains to check iii). By the same argument as in Lem. 5.1 we can construct the square root $(2\epsilon)^\frac{1}{2}$, which belongs to $C^\infty_b(I; \Psi^\frac{1}{2}(\Sigma; V))$ and is invertible with $(2\epsilon)^\frac{1}{2} = (2\epsilon)^\frac{1}{2}$. We have

\[
b + b^* = (2\epsilon)^\frac{1}{2} (1 - s_{-1})(2\epsilon)^\frac{1}{2},
\]
where
\[ s_{-1} = -(2\varepsilon)^{-\frac{1}{2}}(b_0(1 - \chi_R(a_{ref}))) + (1 - \chi_R(a_{ref}))b_0^*(2\varepsilon)^{-\frac{1}{2}}. \]

We see that \( s_{-1} = s_{-1}^* \), \( s_{-1} \in C^\infty_b(I; \Psi^{-1}(\Sigma; V)) \) and since \((1 - \chi_R(a_{ref}))(2\varepsilon)^{-\frac{1}{2}} \) tends to 0 in norm when \( R \to \infty \) we can fix \( R \gg 1 \) such that \( \|s_{-1}\| \leq \frac{1}{3} \). We set now
\[ 1 + r_{-1} = (1 - s_{-1})^{\frac{1}{2}} = \sum_{n=0}^{\infty} c_n(s_{-1})^n, \]
where \( c_n = \frac{f^{(n)}(0)}{n!} \) with \( f(x) = (1 - x)^{\frac{1}{2}} \) satisfies \( |c_n| \leq 2 \) for \( n \in \mathbb{N} \). It follows that
\[ \|r_{-1}\| \leq 2 \sum_{n=1}^{\infty} \|s_{-1}\|^n = 2\|s_{-1}\|(1 - \|s_{-1}\|)^{-1} \leq \frac{2}{3}. \]

Moreover
\[ r_{-1} \in C^\infty_b(I; \Psi^{-1}(\Sigma; V)), \quad r_{-1} = r_{-1}^*, \quad (1 + r_{-1})^2 = (1 - s_{-1}). \]
This proves \( iii) \). \[ \square \]

We now set
\[ b^+ := b, \quad b^- = -b^*, \quad (5.10) \]
and obtain that
\[ b^\pm = \pm \varepsilon + C^\infty_b(I; \Psi^0(\Sigma; V)), \]
\[ i\partial_t b^\pm - (b^\pm)^2 + a = r_{\pm \infty}, \]
for \( r_{\pm \infty} = r_{-\infty}, \quad r_{-\infty} = r_{+\infty} \). This is equivalent to the two factorizations of \( D \) modulo smoothing error terms:
\[ (\partial_t + ib^\pm)(\partial_t - ib^\pm) = D - r_{\pm \infty}. \quad (5.11) \]

5.2.8. Cauchy evolution. For \( s \in I \) the Cauchy problem
\[ \begin{cases} Du = 0 \text{ in } M \\ \partial_s u = f \in C_c^\infty(\Sigma; V \otimes \mathbb{C}^2) \end{cases} \quad (5.12) \]
is well-posed, where
\[ \partial_s u = \left( \begin{array}{c} u(s) \\ i^{-1} \partial_t u(s) \end{array} \right). \]

We denote by \( u = U_s f \) the unique solution of (5.12), so that \( U_s : C_c^\infty(\Sigma; V \otimes \mathbb{C}^2) \to C_{sc}^\infty(M; V) \). For \( t \in I \) we denote by
\[ U(t, s) := \partial_t \circ U_s : C_c^\infty(\Sigma; V \otimes \mathbb{C}^2) \to C_c^\infty(\Sigma; V \otimes \mathbb{C}^2) \]
the Cauchy evolution of \( D \). The Green identity (5.2) implies that \( U(t, s) \) is pseudo-unitary for \((\cdot|\cdot)_{V(\Sigma) \otimes \mathbb{C}^2}\):
\[ q = U(t, s)^* q U(t, s), \quad \text{or equivalently } U(t, s) = U(s, t) \quad t, s \in I, \]
where as before \( A^* \) denotes the adjoint of \( A \) for \((\cdot|\cdot)_{V(\Sigma) \otimes \mathbb{C}^2}\).
5.2.9. Factorization of the Cauchy evolution. For a solution \( u \in C^\infty_{sc}(M; V) \) of \( Du = 0 \) we set \( \psi(t) = \begin{pmatrix} u(t) \\ 1^{-1} \partial_t u(t) \end{pmatrix} \) so that

\[
\partial_t \psi(t) = iA(t)\psi(t), \quad A(t) = \begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix},
\]

and \( \psi(t) = U(t, s)\psi(s) \). Next, we define \( S(t) \) by

\[
S^{-1}(t)\psi(t) = \begin{pmatrix} (\partial_t - ib^-(t))u(t) \\ (\partial_t - ib^+(t))u(t) \end{pmatrix},
\]

which yields

\[
S = i^{-1} \begin{pmatrix} 1 & -1 \\ b^+ & b^- \end{pmatrix} (b^+ - b^-)^{-1}, \quad S^{-1} = i \begin{pmatrix} -b^- & 1 \\ -b^+ & 1 \end{pmatrix}.
\]

From (5.11) we obtain that

\[
\partial_t S^{-1}(t)\psi(t) = iB(t)S^{-1}(t)\psi(t),
\]

for

\[
B = \begin{pmatrix} -b^- & 0 \\ 0 & b^+ \end{pmatrix} + B_{-\infty},
\]

where

\[
B_{-\infty} = \begin{pmatrix} r^-_{-\infty} & -r^-_{-\infty} \\ r^+_{-\infty} & -r^+_{-\infty} \end{pmatrix} (b^+ - b^-)^{-1} \in C_b^\infty(I; \Psi^{-\infty}(\Sigma; V \otimes C^2)).
\]

We find

\[
S^* q S = \tau (b^+ - b^-)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix}.
\]

Let \( c = (2\epsilon)^{1/2}(1 + r_{-1}) \), where \( r_{-1} \) is as in Prop. 5.2 iii). Then \( c^*(b^+ - b^-)^{-1}c = 1 \), hence \( c^*\tau(b^+ - b^-)^{-1}c = \tau \). Setting

\[
T := S \circ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix},
\]

we have

\[
T^* q T = \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix}.
\]

Moreover we have

\[
\partial_t T^{-1}(t)\psi(t) = iC(t)T^{-1}(t)\psi(t),
\]

for

\[
C = \begin{pmatrix} \epsilon^+ & 0 \\ 0 & \epsilon^- \end{pmatrix} + C_{-\infty}(t)
\]

where

\[
\epsilon^\pm = \pm c^{-1}b^\pm + ic^{-1}\partial_t c = \pm \epsilon + C_b^\infty(I; \Psi^0(\Sigma; V)),
\]

and

\[
C_{-\infty} = c^{-1}B_{-\infty} c \in C_b^\infty(I; \Psi^{-\infty}(\Sigma; V \otimes C^2)).
\]

In the next proposition we use the notation recalled in 1.5.7. The hypotheses of Kato’s theorem are easy to check using \( \Psi \)DO calculus.
Proposition 5.3. For all $t, s \in I$ we have
\[ U(t, s) = T(t)\text{Exp}(i \int_s^t C(\sigma) d\sigma)T^{-1}(s) \]
\[ = T(t) \left( \text{Exp}(i \int_s^t \epsilon^+(\sigma) d\sigma) \begin{pmatrix} 0 & \text{Exp}(i \int_s^t \epsilon^-(\sigma) d\sigma) \\ 0 & 0 \end{pmatrix} T^{-1}(s) + R_\infty(t, s) \right). \]

5.2.10. Hadamard projectors. We set $\pi^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \pi^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and
\[ c^\pm := T(0)\pi^\pm T^{-1}(0) = \begin{pmatrix} \mp(b^+ - b^-)^{-1}b^\mp & \pm(b^+ - b^-)^{-1} \\ \mp(b^+ - b^-)^{-1}b^- & \pm(b^+ - b^-)^{-1} \end{pmatrix}(0). \tag{5.14} \]

We recall that $A^\dagger$ is the symplectic adjoint of $A$, see 5.2.5.

Proposition 5.4. The operators $c^\pm$ defined in (5.14) satisfy:
1. $c^+ + c^- = 1$,
2. $(c^\pm)^\dagger = c^\pm$,
3. $\text{WF}(U(\cdot, 0)c^\pm)' \subset (\mathcal{N}^\pm \cup \mathcal{F}) \times T^*\Sigma$ for $\mathcal{F} = \{ k = 0 \} \subset T^*M$.

Proof. (1) is straightforward; (2) follows from (5.13). We set $Q_\pm = (\partial_t - ic^\pm(t, x, \partial_x))$, considered as an operator acting on $M \times \Sigma$ on the first group of variables and let $A(t, x, x')$ the distributional kernel of $U(\cdot, 0)c^\pm$.

Prop. 5.3 it follows that $Q_\pm A \in C^\infty(M \times \Sigma, L(V \otimes \mathbb{C}^2, V))$. If $Q_\pm$ were classical $\Psi$DOs on $M \times \Sigma$, this would imply that $\text{WF}(U(\cdot, 0)c^\pm)' \subset \mathcal{N}^\pm \times T^*\Sigma$ by elliptic regularity. We reduce ourselves to this situation by an argument from [12, Lem. 6.5.5], see for example [24, Prop. 6.8] for details.

We call the maps $c^\pm$ Hadamard projectors.

Remark 5.5. From (5.14) we obtain immediately that $c^\pm$ are projections indeed. The terminology is justified by the fact that Prop. 5.4 implies that $\lambda^\pm = \pm q \circ c^\pm$ are a pair of Cauchy surface Hadamard pseudo-covariances for $D$. Note however that the positivity condition $\lambda^\pm > 0$ for $(\cdot | \cdot)_V(\Sigma \otimes \mathbb{C}^2)$ is in general not satisfied. Moreover, different choices of $b$ in Prop. 5.2 lead to different projections $c^\pm$, differing by a term in $\Psi^{-\infty}(\Sigma; V \otimes \mathbb{C}^2)$.

5.3. Euclidean case. We now consider a Euclidean analogue of the setting considered so far. We set $\tilde{M} = I_s \times \Sigma$, where $I \subset \mathbb{R}$ is an interval with $0 \in I$ and $\Sigma$ a $d$-dimensional manifold. As before we identify $\{0\} \times \Sigma$ with $\Sigma$. We fix an $s$-dependent sesquilinear form:
\[ \tilde{h} : I \ni s \mapsto C_b^\infty(I; L(T\Sigma, T\Sigma^*)), \]
such that $\tilde{h}(0)$ is a Riemannian metric on $\Sigma$, i.e. $\tilde{h}(0) = \tilde{h}(0)^*, \tilde{h}(0) > 0$. For ease of notation $\tilde{h}(0)$ is often denoted by $\tilde{h}_s$.

We assume that $\tilde{h}$ is uniformly coercive, i.e. there exists $C > 0$ such that:
\[ C^{-1}\tilde{h}(0) \leq \text{Re} \tilde{h}(s) \leq C\tilde{h}(0) \]
\[ |\text{Im} \tilde{h}(s)| \leq C\text{Re} \tilde{h}(s), \quad s \in I. \]
5.3.1. **Hilbertian bundle.** We equip $\tilde{M}$ with the Hilbertian bundle $\tilde{V}$ as in 5.2.1. For $u, v \in C^\infty_c(\Sigma; \tilde{V})$ resp. $C^\infty_c(\tilde{M}; \tilde{V})$ we set
\[
(u|v)_{\tilde{V}((\Sigma)} := \int_{\Sigma} (u|v)_{\tilde{\Psi}}|\tilde{h}_0|^{\frac{1}{2}} dx,
\]
\[
(u|v)_{\tilde{V}(\tilde{M})} := \int_{\tilde{M}} (u|v)_{\tilde{\Psi}}|\tilde{h}_0|^{\frac{1}{2}} d\tilde{xd}t.
\]

5.3.2. **Adjoints.** As in 5.2.2 if $\tilde{a}(s) \in C^\infty_b(I; \text{Diff}(\Sigma; \tilde{V}))$, resp. $\tilde{A} \in \text{Diff}(\tilde{M}; \tilde{V})$ we denote by $\tilde{a}^*(s)$ resp. $\tilde{A}^*$ its formal adjoint for $(\cdot|\cdot)_{\tilde{V}((\Sigma)}$ resp. $(\cdot|\cdot)_{\tilde{V}(\tilde{M})}$.

5.3.3. **Elliptic operator.** We fix an $s$-dependent differential operator $\tilde{a}(s) = \tilde{a}(s, x, D_x)$ belonging to $C^\infty_b(I; \text{Diff}^2(\Sigma; \tilde{V}))$ and denote by $\sigma_{pr}(\tilde{a})(s)$ its principal symbol. We assume the following property:
\[
(\tilde{H}1) \quad \sigma_{pr}(\tilde{a})(s)(x, k) = k \cdot \tilde{h}_s^{-1}(x)k 1_{\tilde{V}}.
\]
We set
\[
\tilde{D} := -\partial^2_s + \tilde{a}(s) \text{ acting on } C^\infty_c(\tilde{M}; \tilde{V}),
\]
which is an elliptic differential operator.

5.3.4. **Factorization of $\tilde{D}$.** As in Lem. 5.1, we see that $\tilde{a}(s)$ is closed with domain $H^2(\Sigma; \tilde{V})$ and Dom $\tilde{a}^*(s) = H^2(\Sigma; \tilde{V})$.

We add to $\tilde{a}(s)$ a self-adjoint term $\tilde{r}_{-\infty} \in C^\infty_b(I; \Psi^{-\infty}(\Sigma; \tilde{V}))$ such that
\[
\text{Re} \tilde{a}(s) + r_{-\infty}(s) \geq \delta 1, \quad \delta > 0,
\]
and $\tilde{a}(s) + r_{-\infty}(s)$ is $m$-accretive. We denote by
\[
\tilde{\epsilon} := (\tilde{a} + r_{-\infty})^{\frac{1}{2}} \in C^\infty_b(I; \Psi^1(\Sigma; \tilde{V}))
\]
its unique $m$-accretive square root given by (5.7), which satisfies:
\[
\text{Re} \tilde{\epsilon}(s) \geq \delta^{\frac{1}{2}} 1.
\]
As in 5.2.6 we have:
\[
\sigma_{pr}(\tilde{\epsilon}(s)) = (\sigma_{pr}(\tilde{a}(s)))^{\frac{1}{2}}.
\]
The operator $\tilde{\epsilon}(s)$ with domain $H^1(\Sigma; \tilde{V})$ is closed, elliptic and invertible by (5.16), hence $\tilde{\epsilon}^{-1}(s) \in C^\infty_b(I; \Psi^{-1}(\Sigma; \tilde{V}))$.

**Proposition 5.6.** There exists $\tilde{b}^\pm(s) \in C^\infty_b(I; \Psi^1(\Sigma; \tilde{V}))$, unique modulo a term in $C^\infty_b(I; \Psi^{-\infty}(\Sigma; \tilde{V}))$, such that:
\[
\begin{align*}
&i) \quad \tilde{b}^\pm(s) = \pm \tilde{\epsilon}(s) + C^\infty_b(I; \Psi^0(\Sigma; \tilde{V})), \\
&ii) \quad \partial_s \tilde{b}^\pm(s) - (\tilde{b}^\pm)^2(s) + \tilde{a}(s) = \tilde{r}^\pm_{-\infty}(s) \in C^\infty_b(I; \Psi^{-\infty}(\Sigma; \tilde{V})), \\
&iii) \quad \pm \text{Re} \tilde{b}^\pm \geq 1, \\
&iv) \quad \tilde{b}^+ - \tilde{b}^- : H^1(\Sigma; \tilde{V}) \to L^2(\Sigma; \tilde{V}) \text{ is invertible, } (\tilde{b}^+ - \tilde{b}^-)^{-1} \in C^\infty_b(I; \Psi^{-1}(\Sigma; \tilde{V})).
\end{align*}
\]
the equation
\[ \tilde{\tilde{b}} = (2\tilde{\epsilon})^{-1}\partial_s\tilde{\epsilon} + \tilde{F}(\tilde{b})_0, \]
for
\[ \tilde{F}(\tilde{b})_0 = (2\tilde{\epsilon})^{-1}(\pm\partial_s\tilde{b}_0 + [\tilde{\epsilon}, \tilde{b}]_0 - \tilde{b}_0^2). \]
We use the same fixed point argument as in Prop. 5.2 and obtain \( \tilde{b} \) satisfying i) and ii). To obtain iii) we use that \( \tilde{b} = \tilde{\epsilon} + C^\infty(I; \Psi^0(\Sigma; \tilde{V})) \) and add to \( \tilde{b} \) elements \( r^\pm_{1,-\infty} \in C^\infty_b(I, \Psi^{-\infty}(\Sigma; \tilde{V})) \) so that \( \pm \Re \tilde{b}^\pm \geq 1 \). Then \( \tilde{b}^+ - \tilde{b}^- \) is \( m \)-accretive with \( 0 \not\in (\tilde{b}^+ - \tilde{b}^-) \), which implies iv).

We obtain the following factorization of \( \tilde{D} \), analogous to (5.11):
\[ (-\partial_s + \tilde{b}^+)(\partial_s + \tilde{b}^-) = \tilde{D} - \tilde{r}^\pm_{-\infty}. \]

**Remark 5.7.** Suppose that the interval \( I \) is symmetric with respect to 0 and that
\[ a^*(s) = \tau a(-s)\tau^{-1}, \quad s \in I. \]
Then from (5.7) we have \( \tilde{\epsilon}^*(s) = \tau \tilde{\epsilon}(-s)\tau^{-1} \). We set
\[ \tilde{F}(\tilde{b})_0 = (2\tilde{\epsilon})^{-1}(\partial_s\tilde{b}_0 + [\tilde{\epsilon}, \tilde{b}]_0 - \tilde{b}_0^2), \]
we solve the fixed point equation
\[ \tilde{b}_0 = (2\tilde{\epsilon})^{-1}\partial_s\tilde{\epsilon} + \tilde{F}(\tilde{b}_0), \]
and construct \( \tilde{b} = \tilde{\epsilon} + \tilde{b}_0 \) such that
\[ \partial_s\tilde{b} - \tilde{b}^2 + \tilde{a} = \tilde{r}_{-\infty} \in C^\infty_b(I; \Psi^{-\infty}(\Sigma; \tilde{V})), \]
\[ \Re \tilde{b} \geq 1. \]
Then we can take:
\[ \tilde{b}^+(s) = \tilde{b}(s), \quad \tilde{r}^+_{-\infty}(s) = \tilde{r}_{-\infty}(s), \]
\[ \tilde{b}^-(s) = -\tau^{-1}\tilde{b}^*(s), \quad \tilde{r}^-_{-\infty}(s) = \tau^{-1}\tilde{r}^+_{-\infty}(s) \tau. \]

**5.3.5. Parametrix for \( \tilde{D} \).** By Prop. 5.6 we know that \( \pm\tilde{b}^\pm(s) \) is \( m \)-accretive with \( 0 \in \text{rs}(\tilde{b}^\pm(s)) \) (the resolvent set) and \( \text{Dom} \tilde{b}^\pm(s) = H^1(\Sigma; \tilde{V}) \) for all \( s \in I \). Since \( \tilde{b}^\pm \in C^\infty_b(\Sigma; \tilde{V}) \), we can check the hypotheses of [36] (in the version presented in [47]) and conclude that
\[ V^\pm(s, s') := \text{Texp}(\int_s^\sigma \tilde{b}^\pm(\sigma)d\sigma) \text{ exists for } \mp(s - s') \geq 0, \quad s, s' \in I. \]

**Lemma 5.8.**
1. \( V^\pm(s, s') : H^m(\Sigma; \tilde{V}) \to H^m(\Sigma; \tilde{V}) \) is uniformly bounded for \( s, s' \in I \) and \( \pm(s - s') \geq 0; \)
2. \( V^\pm(s, s') \in \Psi^{-\infty}(\Sigma; \tilde{V}) \) for \( \mp(s - s') \geq \delta > 0 \).

**Proof.** To prove (1) it suffices to apply Kato’s theorem in [36] to the Hilbert space \( H^m(\Sigma; \tilde{V}) \). The hypotheses follow from \( \Psi DO \) calculus. If we denote by \( V^\pm_m(s, s') \) the resulting semi-group on \( H^m(\Sigma; \tilde{V}) \), then \( V^\pm_m \) is an extension resp. restriction of \( V^\pm \) if \( m < 0 \) resp. \( m > 0 \). Let us prove (2) in the + case. We fix \( s' \in I \), \( \chi \in C^\infty_b(I) \) with \( \text{supp} \chi \subset \{ s \leq s' - \delta/2 \} \), \( \chi = 1 \) in \( \{ s \leq s' - \delta \} \). Then \( (\partial_s - \tilde{b}^+(s))\chi(s)V^+(s, s')u = \chi'(s)V^+(s, s')u \). The operator \( \partial_s - \tilde{b}^+(s) \) has principal symbol \((i\sigma - \sigma_{\text{Re}}(\tilde{a}(s))(\kappa^2)^{1/2})I_{\tilde{V}}\).
hence is elliptic in $\Psi^1(I \times \Sigma; \tilde{\nabla})$. If $u \in H^{m_0}(\Sigma; \tilde{\nabla})$, then $V^+(\cdot, s')u \in H^{m_0}(I \times \Sigma; \tilde{\nabla})$ for some $n_0$ and by elliptic regularity $\chi(\cdot)V^+(\cdot, s')u \in H^{n_0+1}(I \times \Sigma; \tilde{\nabla})$. By iterating this argument we obtain that $\chi(\cdot)V^+(\cdot, s')u \in H^n(I \times \Sigma; \tilde{\nabla})$ for any $n$ so $V^+(s, s')u \in H^m(\Sigma; \tilde{\nabla})$ for any $m \in \mathbb{N}$.

For $v \in C_b^\infty(I; C_c^\infty(\Sigma; \tilde{\nabla}))$ we set

$$T^\pm v(s) := \pm \int_\mathbb{R} H(\mp(s - s'))V^\pm(s, s')v(s')ds',$$

where $H(t) = 1_{\mathbb{R}^+}(t)$ is the Heaviside function, so that

$$( -\partial_s + \tilde{b}^\pm )T^\pm = T^\pm( -\partial_s + \tilde{b}^\pm ) = 1. \quad (5.18)$$

**Proposition 5.9.** Let

$$\hat{D}(-1) = \left( \tilde{b}^+ - \tilde{b}^- \right)^{-1}(T^+ - T^-).$$

Then

$$\hat{D} \circ \hat{D}(-1) = 1 + R_{-\infty},$$

for some $R_{-\infty} \in W^{-\infty}(\tilde{\nabla}; \tilde{\nabla})$.

**Proof.** We obtain using (5.18) and Prop. 5.6 ii):

$$\partial_s \left( (\tilde{b}^+ - \tilde{b}^-)^{-1}(T^+ - T^-) \right)$$

$$= (\tilde{b}^+ - \tilde{b}^-)^{-1} \left( \tilde{b}^+ T^+ - \tilde{b}^- T^- - (\tilde{b}^{+2} - \tilde{b}^{-2})(\tilde{b}^+ - \tilde{b}^-)^{-1}(T^+ - T^-) \right) + r_{1,-\infty},$$

for

$$r_{1,-\infty} = -(\tilde{b}^+ - \tilde{b}^-)^{-1}(\tilde{r}^+_{-\infty} - \tilde{r}^-_{-\infty})(\tilde{b}^+ - \tilde{b}^-)^{-1}(T^+ - T^-).$$

Next,

$$\tilde{b}^+ - (\tilde{b}^{+2} - \tilde{b}^{-2})(\tilde{b}^+ - \tilde{b}^-)^{-1} = - (\tilde{b}^+ - \tilde{b}^-)\tilde{b}^- (\tilde{b}^+ - \tilde{b}^-)^{-1}$$

$$= -\tilde{b}^- + (\tilde{b}^{+2} - \tilde{b}^{-2})(\tilde{b}^+ - \tilde{b}^-)^{-1} = (\tilde{b}^+ - \tilde{b}^-)^{-1}(\tilde{b}^+ - \tilde{b}^-)^{-1},$$

hence

$$\partial_s \left( (\tilde{b}^+ - \tilde{b}^-)^{-1}(T^+ - T^-) \right)$$

$$= -\tilde{b}^- (\tilde{b}^+ - \tilde{b}^-)^{-1}T^+ + \tilde{b}^+ (\tilde{b}^+ - \tilde{b}^-)^{-1}T^- + r_{1,-\infty} \quad (5.19)$$

$$= T^\pm - \tilde{b}^- \left( (\tilde{b}^+ - \tilde{b}^-)^{-1}(T^+ - T^-) \right) + r_{1,-\infty}. $$

Using again (5.18) we obtain

$$( -\partial_s + \tilde{b}^+)(\partial_s + \tilde{b}^+ ) \left( (\tilde{b}^+ - \tilde{b}^-)^{-1}(T^+ - T^-) \right) = 1 - (\partial_s + \tilde{b}^+ )r_{1,-\infty}. $$

Hence, setting

$$\hat{D}(-1)v(s) = (\tilde{b}^+ (s) - \tilde{b}^- (s))^{-1}(T^+ - T^-)v(s), \quad (5.20)$$

by (5.17) we get

$$\hat{D} \hat{D}(-1) = 1 + R_{-\infty}, \quad R_{-\infty} \in C_b^\infty(I; \Psi^{-\infty}(\Sigma; \tilde{\nabla})), $$

for

$$R_{-\infty} = ( -\partial_s + \tilde{b}^+ r_{1,-\infty}) + \tilde{r}^+_{-\infty} \hat{D}(-1). $$
Using (5.18) and Lem. 5.8 (1) we obtain that if \( r_{-\infty} \in C_0^\infty(I; W^{-\infty}(\Sigma; \tilde{V})) \), then \( r_{-\infty} T^\pm \in W^{-\infty}(\tilde{M}; \tilde{V}) \). This implies that \( R_{-\infty} \in W^{-\infty}(\tilde{M}; \tilde{V}) \) as claimed. □

5.3.6. Dirichlet realization of \( \tilde{D} \). Let us fix \( T \) such that \(-T, T] \subset I \) and set \( \Omega = ]-T, T[ \times \Sigma \). We denote by \( H_0^1(\Omega; \tilde{V}) \) the closure of \( C_c^\infty(\Omega, \tilde{V}) \) for the norm 
\[
\| u \|^2_{H^1_0(\Omega; \tilde{V})} = \int_\Omega \left( (\partial_s u | \partial_s \bar{u})_{\tilde{V}} + (u - \Delta_{R_0} u | \bar{u})_{\tilde{V}} + (u | u)_{\tilde{V}} \right) |\tilde{h}|_0^{1/2} \, dt \, dx. \tag{5.21}
\]
We denote by \( L^2(\Omega, \tilde{V}) \) the \( L^2 \) space defined using the scalar product \((\cdot | \cdot)_{\tilde{V}(\Omega)}\).

We consider the sesquilinear form 
\[
Q_\Omega(v, u) := (v | \tilde{D} u)_{\tilde{V}(\Omega)}, \quad \text{with domain } \text{Dom} \, Q_\Omega = C_c^\infty(\Omega; \tilde{V}).
\]

**Proposition 5.10.** There exist \( T_0 > 0 \) such that for \( 0 < T \leq T_0 \) one has:
1. \( Q_\Omega \) and \( Q_\Omega^{\ref} \) are closeable on \( L^2(\Omega; \tilde{V}) \);
2. their closures \( Q_\Omega, Q_\Omega^{\ref} \) are sectorial with domain \( H_0^1(\Omega; \tilde{V}) \);
3. the closed operators \( \tilde{D}_\Omega, \tilde{D}_\Omega^{\ref} \) associated to \( Q_\Omega, Q_\Omega^{\ref} \) satisfy \( 0 \in \text{rs}(\tilde{D}_\Omega), 0 \in \text{rs}(\tilde{D}_\Omega^{\ref}) \);
4. \( \tilde{D}_\Omega^{\ref} \) is the adjoint of \( \tilde{D}_\Omega \).

**Proof.** Let \( Q_{\ref} \) be the sesquilinear form associated to (5.21) with domain \( C_c^\infty(\Omega; \tilde{V}) \). By (5.15) and the Poincaré inequality, we can find \( T_0 > 0 \) such that for \( 0 < T \leq T_0 \) one has
\[
C^{-1} Q_{\ref} \leq \text{Re} \, Q_\Omega \leq C Q_{\ref}, \quad |\text{Im} \, Q_\Omega| \leq C Q_{\ref}, \quad C > 0. \tag{5.22}
\]
This implies (2). Then, (3) follows from [37, Sect. VI.2.1] and (4) from [37, Thm. VI.2.5]. □

The operator \( \tilde{D}_\Omega \) is the Dirichlet realization of \( \tilde{D} \). We denote by \( \tilde{D}_\Omega^{-1} : L^2(\Omega, \tilde{V}) \to \text{Dom} \, \tilde{D}_\Omega \) its inverse.

5.3.7. Parametrix for the Dirichlet problem. Let us fix \( T > 0 \) such that \([ -T, T ] \subset I \). We want to find a parametrix for the Dirichlet problem
\[
\begin{aligned}
\tilde{D} u &= f \quad \text{in } \Omega, \\
u |_{\partial \Omega} &= 0.
\end{aligned} \tag{5.23}
\]

Let
\[
W^{\pm} (s, s') = \text{Exp}(-\int_{s'}^s \tilde{b}^\pm(\sigma) \, d\sigma), \quad \text{for } \pm (s - s') \geq 0,
\]
and
\[
R_{1, -\infty} = \begin{pmatrix} 0 & W^{-}(-T, T) \\ W^{+}(T, -T) & 0 \end{pmatrix}.
\]

Since \( \pm \text{Re} \, \tilde{b}^{\pm} \geq 1 \) by Prop. 5.6, we have \( \| R_{1, -\infty} \|_{B(L^2(\Sigma; \tilde{V} \otimes \mathbb{C}^2))} \leq e^{-2T} \) and by Lem. 5.8 (2), \( R_{1, -\infty} \in W^{-\infty}(\Sigma; \tilde{V} \otimes \mathbb{C}^2) \). Therefore \( 1 + R_{1, -\infty} \) is invertible in \( B(L^2(\Sigma; \tilde{V} \otimes \mathbb{C}^2)) \) and by spectral invariance, \( (1 + R_{1, -\infty})^{-1} \in 1 + W^{-\infty}(\Sigma, \tilde{V} \otimes \mathbb{C}^2) \).

Let us set for \( g \in C_0^\infty(I; C_0^\infty(\Sigma; \tilde{V})) \):
\[
\mathbb{g} \mathbb{g} := \begin{pmatrix} g(-T) \\ g(T) \end{pmatrix},
\]
and for \( v^\pm \in L^2(\Sigma; \tilde{V}) \):

\[
S \begin{pmatrix} v^+ \\ v^- \end{pmatrix} (s) := W^+(s, -T)v^+ + W^-(s, T)v^-,
\]

so that

\[
\varrho_{\partial \Omega} \circ S = 1 + R_{1,-\infty}.
\]

The following proposition gives a construction of a parametrix for the Dirichlet problem (5.23).

**Proposition 5.11.** For \( f \in C^\infty_b(I; C^\infty_b(\Sigma; \tilde{V})) \) let

\[
\tilde{D}^{(-1)}_\Omega = \tilde{D}^{(-1)}_\Omega - S \circ (1 + R_{1,-\infty})^{-1} \circ \varrho \circ \tilde{D}^{(-1)}_\Omega.
\]

Then

\[
\begin{cases}
    \tilde{D} \circ \tilde{D}^{(-1)}_\Omega = 1 + R_{-\infty}, & R_{-\infty} \in W^{-\infty}(\tilde{M}; \tilde{V}), \\
    \varrho_{\partial \Omega} \circ \tilde{D}^{(-1)}_\Omega = 0.
\end{cases}
\]

**Proof.** By (5.17) and the analog of Lem. 5.8 for \( W^\pm \) we obtain that:

\[
\tilde{D}S \begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \tilde{r}^+_{-\infty}W^+(\cdot, -T)v^+ + \tilde{r}^-_{-\infty}W^-(\cdot, T)v^- = R_{2,-\infty} \begin{pmatrix} v^+ \\ v^- \end{pmatrix}
\]

where \( R_{2,-\infty} \in C^\infty_b(I; W^{-\infty}(\Sigma; \tilde{V})) \). On the other hand by (5.20) we have

\[
\tilde{D}^{(-1)}_\Omega v(\pm T) = (\tilde{b}^+ - \tilde{b}^-)^{-1}(\pm T) \int_{-T}^T V^\mp(\pm T, s')v(s')ds'.
\]

Therefore if we set

\[
u = \tilde{D}^{(-1)}_\Omega v - S(1 + R_{1,-\infty})^{-1} \varrho \tilde{D}^{(-1)}_\Omega v,
\]

we get

\[
\begin{cases}
    \tilde{D}u = v + R_{3,-\infty}v, \\
    \varrho_{\partial \Omega}u = 0,
\end{cases}
\]

for some \( R_{3,-\infty} \in W^{-\infty}(\tilde{M}; \tilde{V}) \).

**Proposition 5.12.** Let \( \tilde{D}^{(-1)}_\Omega \) be as Prop. 5.11. Then,

\[
\tilde{D}^{-1}_\Omega - \tilde{D}^{(-1)}_\Omega \in W^{-\infty}(\Omega; \tilde{V}).
\]

**Proof.** From the sesquilinear form associated to \( \tilde{D}_\Omega \) we know that \( \tilde{D}_\Omega^{-1} : H^{-1}(\Omega; \tilde{V}) \rightarrow H^1(\Omega; \tilde{V}) \). By the usual argument of commuting tangential derivatives with \( \tilde{D} \) and using the equation \( \tilde{D}u = v \) to control \( \partial_s \) derivatives, we obtain that \( \tilde{D}^{-1}_\Omega = H^s(\Omega; \tilde{V}) \rightarrow H^{s+2}(\Omega; \tilde{V}) \) for all \( s \in \mathbb{R} \). By Prop. 5.11 we obtain \( \tilde{D}^{(-1)}_\Omega = \tilde{D}^{-1}_\Omega + R_{-\infty}\tilde{D}^{-1}_\Omega \), which proves the proposition.

Let us fix an extension map \( e : C^\infty(\partial \Omega; \tilde{V}) \rightarrow \overline{C^\infty}(\Omega; \tilde{V}) \) such that \( (ef)|_{\Omega} = f \). We can assume moreover that \( \varrho^+ef = 0 \) by choosing \( e \) such that \( ef = 0 \) near \( s = 0 \).

**Lemma 5.13.** We have

\[
\varrho^+ \tilde{D}^{-1}_\Omega \tilde{D}u = \varrho^+u + \varrho^+ \tilde{D}^{-1}_\Omega \tilde{D}e(u|_{\Omega}), \quad u \in H^1(\Omega; \tilde{V}).
\]
Proof. Let \( v = \tilde{D}^{-1} \bar{D}u \). We set \( v = u - eu \partial_{\Omega} + w \) where
\[
\begin{aligned}
\tilde{D}w &= \tilde{D} eu |_{\partial \Omega} \text{ in } \Omega, \\
w |_{\partial \Omega} &= 0,
\end{aligned}
\]
i.e. \( w = \tilde{D}^{-1} \tilde{D} eu |_{\partial \Omega} \). Applying \( \tilde{\sigma} \pm \) to this identity we obtain the lemma.

\[\square\]

5.3.8. Green’s formula. Let \( \Omega^\pm = \Omega \cap \{ \pm s > 0 \} \). For \( u \in C^\infty(\Omega; \bar{V}) \) we set
\[
\tilde{\rho} u := \left( \begin{array}{l} u|_{\Sigma} \\ -\partial_s u|_{\Sigma} \end{array} \right) = \left( \begin{array}{l} u(0) \\ -\partial_s u(0) \end{array} \right).
\]
We denote by \( \tilde{\sigma} \pm \) the analogous trace operators defined on \( \overline{C^\infty(\Omega^\pm; \bar{V})} \):
\[
\tilde{\sigma} \pm u := \left( \begin{array}{l} u(0^\pm) \\ -\partial_s u(0^\pm) \end{array} \right),
\]
and by \( \tilde{\sigma} \pm T \) the trace operator at \( \pm T \), i.e.
\[
\tilde{\sigma} \pm T u := \left( \begin{array}{l} u|_{\partial \Omega \cap \Sigma} \\ -\partial_s u|_{\partial \Omega \cap \Sigma} \end{array} \right) = \left( \begin{array}{l} u(\pm T) \\ -\partial_s u(\pm T) \end{array} \right).
\]
The definitions extend of course to spaces with sufficient Sobolev regularity.

Let us denote
\[
\tilde{\sigma} = \left( \begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array} \right), \quad \tilde{\rho} = \left( \begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array} \right).
\]

Proposition 5.14. Let \( u, v \in \overline{H^1(\Omega^\pm; \bar{V})} \) with \( \tilde{D}^* u, \tilde{D} v \in L^2(\Omega^\pm; \bar{V}) \). Then
\[
(u|\tilde{D} v)_{\tilde{V}(\Omega^\pm)} - (\tilde{D}^* u|v)_{\tilde{V}(\Omega^\pm)} = \pm (\tilde{\sigma} \pm u|\tilde{\sigma} \pm v)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2}
\]
\[= (\tilde{\sigma} \pm T u|\tilde{\sigma} \pm T v)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2}. \tag{5.26} \]

Let \( u, v \in \overline{H^1(\Omega^\pm; \bar{V})} \) with \( \tilde{D} u, \tilde{D} v \in L^2(\Omega^\pm; \bar{V}) \). Then
\[
(u|\tilde{D} v)_{\tilde{V}(\Omega^\pm)} + (\tilde{D} u|v)_{\tilde{V}(\Omega^\pm)} = 2 \eta_{\Omega^\pm}(u, v) \mp (\tilde{\rho} \pm u|\tilde{\rho} \pm v)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2}
\]
\[\mp (\tilde{\sigma} \pm T u|\tilde{\rho} \pm T v)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2}, \tag{5.27} \]
where
\[
\eta_{\Omega^\pm}(u, v) = (\partial_s u|\partial_s v)_{\tilde{V}(\Omega^\pm)} + (u|\text{Re } \tilde{a}) v)_{\tilde{V}(\Omega^\pm)}.
\]

Proof. By elliptic regularity we know that \( u, v \in H^2(\Omega^\pm; \mathbb{C}^2) \), hence \( \tilde{\sigma} \pm u, \tilde{\rho} \pm v \) belong to \( H^1(\Sigma; \bar{V}) \oplus L^2(\Sigma; \bar{V}) \) and in consequence the r.h.s. in (5.26), (5.27) are well-defined. The identities follow then by integration by parts in \( s \).

In agreement with the notation introduced in 1.5.1 we denote by \( \overline{H^1_0(\Omega^\pm; \bar{V})} \) the space of restrictions to \( \Omega^\pm \) of elements of \( H^1_0(\Omega; \bar{V}) \), i.e.
\[
u \in \overline{H^1_0(\Omega^\pm; \bar{V})} \text{ iff } u \in H^1(\Omega^\pm; \bar{V}) \text{ and } u|_{\partial \Omega \cap \Sigma} = 0.
\]

Note that in the special case when \( u, v \in \overline{H^1_0(\Omega^\pm; \bar{V})} \), we have
\[
(\tilde{\sigma} \pm T u|\tilde{\sigma} \pm T v)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2} = 0 = (\tilde{\rho} \pm T u|\tilde{\rho} \pm T v)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2} \tag{5.28}
\]
and the Green’s formulas in Prop. 5.14 become simpler.
5.3.9. Calderón projectors. We denote by \( \tilde{\varrho}^* : \mathcal{E}'(\Sigma; \tilde{V}) \to \mathcal{D}'(\Omega; \tilde{V}) \) the formal adjoint of \( \tilde{\varrho} \) in \((5.25)\), where \( C^\infty(\Omega; \tilde{V}) \), resp. \( C^\infty_c(\Sigma; \tilde{V}) \otimes \mathbb{C}^2 \), are equipped with the scalar products \( (\cdot, \cdot)_{\tilde{V}(\tilde{M})} \), resp. \( (\cdot, \cdot)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2} \). Explicitly we have

\[
\tilde{\varrho}^* f = \delta(s) \otimes f_0 + \delta'(s) \otimes f_1, \quad f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in C^\infty_c(\Sigma; \tilde{V}) \otimes \mathbb{C}^2.
\]

**Definition 5.15.** The Calderón projectors for the Dirichlet realization \( \hat{D}_\Omega \) of \( \hat{D} \) are the maps

\[
\hat{c}^\pm := \mp \hat{\varrho}^{\pm} \hat{D}_\Omega^{-1} \hat{\varrho}^* \hat{\sigma}.
\]

We will see in Prop. 5.17 that \( \hat{c}^\pm \) are indeed projections if we consider them as operators acting on the spaces

\[
\mathcal{H}^s(\Sigma; \tilde{V} \otimes \mathbb{C}^2) := H^s(\Sigma; \tilde{V}) \oplus H^{s-1}(\Sigma; \tilde{V}), \quad s \in \mathbb{R}.
\]

**Proposition 5.16.** The Calderón projectors satisfy

\[
\hat{c}^\pm = \begin{pmatrix} \mp(\tilde{b}^+ - \tilde{b}^-)^{-1}\tilde{b}^\mp & \pm(\tilde{b}^+ - \tilde{b}^-)^{-1} \\ \mp\tilde{b}^-(\tilde{b}^+ - \tilde{b}^-)^{-1}\tilde{b}^- & \pm\tilde{b}^+(\tilde{b}^+ - \tilde{b}^-)^{-1} \end{pmatrix} (0) + R^{\pm}_{-\infty}.
\]

for some \( R^{\pm}_{-\infty} \in \mathcal{W}^{-\infty}(\Sigma; \tilde{V} \otimes \mathbb{C}^2) \).

**Proof.** We first claim that we can replace \( \hat{D}_\Omega^{-1} \) by \( \hat{D}^{-1} \) in \((5.29)\) modulo an error term in \( \mathcal{W}^{-\infty}(\Sigma; \tilde{V} \otimes \mathbb{C}^2) \). By Prop. 5.12 we can replace \( \hat{D}_\Omega^{-1} \) by the parametrix \( \hat{D}_\Omega^{-1} \) in \((5.29)\), modulo an error term in \( \mathcal{W}^{-\infty}(\Sigma; \tilde{V} \otimes \mathbb{C}^2) \). Next, by Lem. 5.8 (2) with \( V^\pm \) replaced by \( W^\pm \), we obtain that \( \varrho^\pm S \), where \( S \) is defined in \((5.24)\), belongs to \( \mathcal{W}^{-\infty}(\Sigma; \tilde{V} \otimes \mathbb{C}^2) \). Using the expression of \( \hat{D}_\Omega^{-1} - \hat{D}^{-1} \) given in Prop. 5.11 this proves our claim.

Furthermore, if \( v = \delta(s) \otimes f_1 \) then:

\[
T^\pm v(s) = \pm \int_{\mathbb{R}} H(\mp(s - s'))V^\pm(s, s')\delta(s') \otimes f_1 ds'
\]

\[
= \pm H(\mp s)V^\pm(s, 0) f_1.
\]

Therefore

\[
T^+ v(0^+) = 0, \quad T^- v(0^+) = - f_1
\]

and

\[
(\hat{D}^{-1}(0^+) = (\tilde{b}^+ - \tilde{b}^-)^{-1}(0) f_1.
\]

By \((5.19)\) we have

\[
\partial_s(\hat{D}^{-1}(0^+) = T^+ v(0^+) - \tilde{b}^+(\hat{D}^{-1}(0^+) + r_{1,-\infty} f_1, \quad r_{1,-\infty} \in \mathcal{W}^{-\infty}(\Sigma; \tilde{V}). \quad (5.31)
\]

If \( v = -\delta'(s) \otimes f_0 \), we obtain similarly

\[
T^\pm v(s) = \mp \int_{\mathbb{R}} H(\mp(s - s'))V^\pm(s, s')\delta'(s') \otimes f_0 ds'
\]

\[
= \mp \int_{\mathbb{R}} \left( \delta(\mp(s - s'))V^\pm(s, s') \pm H(\mp(s - s'))V^\pm(s, s')\tilde{b}^\pm(s') \right) \delta(s') \otimes f_0 ds'
\]

\[
= \delta(s) f_0 \mp H(\mp s)V^\pm(s, 0)\tilde{b}^\pm(0) f_0,
\]
using that $\partial_s V^\pm(s, s') = -V^\pm(s, s') b^\pm(s')$. Therefore,
\[
T^+ v(0^+) = 0, \quad T^- v(0^+) = b^-(0) f_0
\]
and
\[
(\tilde{D}(-1)v)(0^+) = -(\tilde{b}^+ - \tilde{b}^-)^{-1}(0^+) \tilde{b}^-(0) f_0.
\]
Using again (5.31) we obtain:
\[
(\partial_s \tilde{D}(-1)v)(0^+) = \tilde{b}^+(0)(\tilde{b}^+ - \tilde{b}^-)^{-1}(0) \tilde{b}^-(0) f_0 + r_{1,-\infty} f_0.
\]
Therefore we obtain
\[
c^+ = \begin{pmatrix}
-\tilde{b}^+ - \tilde{b}^-)^{-1}\tilde{b}^- & (\tilde{b}^+ - \tilde{b}^-)^{-1} \\
-\tilde{b}^+ (\tilde{b}^+ - \tilde{b}^-)^{-1}\tilde{b}^- & \tilde{b}^+ (\tilde{b}^+ - \tilde{b}^-)^{-1}
\end{pmatrix} (0) + R_{-\infty}.
\]
The proof for $\tilde{c}^-$ is analogous. \qed

**Proposition 5.17.** The Calderón projectors $c^\pm$ are bounded on $\mathcal{H}^s(\Sigma; \tilde{V} \otimes \mathbb{C}^2)$ for $s \in \mathbb{R}$ and satisfy
\[
c^+ + c^- = 1, \quad c^\pm = (\tilde{c}^\pm)^2 \text{ on } \mathcal{H}^s(\Sigma; \tilde{V} \otimes \mathbb{C}^2).
\]

**Proof.** The boundedness property follows immediately from Prop. 5.16. We prove that $c^+ + c^- = 1$ as in [27, Thm. 4.5]. To prove the last statement we apply [27, Prop. 4.8] which generalizes easily to the vector bundle setting. Let us explain the argument in [27, Prop. 4.8]. If $\Sigma$ is compact the proof is elementary.

If $\Sigma$ is not compact, we first choose a convenient sequence $(\psi_n)_{n \in \mathbb{N}}$ of cutoff functions on $\Sigma$. We fix some reference point $x_0 \in \Sigma$ and denote by $d(x_0, x)$ the geodesic distance for $\tilde{h}_0$. Since $(\Sigma, \tilde{h}_0)$ is of bounded geometry, there exists a function $r \in C^\infty(\Sigma)$ such that
\[
C^{-1}d(x_0, x) \leq r(x) \leq C d(x_0, x), \quad \nabla r \in C^\infty_0(\Sigma; T\Sigma).
\]
Next, we set $\psi_n(x) = F(n^{-1}r(x))$ for $F \in C^\infty_0(\mathbb{R})$ equal to 1 near 0. Then:
\[
\begin{array}{ll}
i) & \psi_n = 1 \text{ on } K \subset \Sigma \text{ for } n \text{ large enough}, \\
ii) & \nabla \psi_n \in O(n^{-1}) \text{ in } C^\infty_0(\Sigma; T\Sigma).
\end{array}
\]
Clearly, $s - \lim_{n \to \infty} \psi_n = 1$ on $C^\infty_0(\Sigma; \tilde{V}) \otimes \mathbb{C}^2$. Arguing as in [27, Prop. 4.8] we obtain that $\psi_n \tilde{c}^\pm f - \tilde{c}^\pm \psi_n \tilde{c}^\pm f$ tends to 0 in $\mathcal{D}'(\Sigma; \tilde{V}) \otimes \mathbb{C}^2$ as $n \to \infty$ for all $f \in C^\infty_0(\Sigma; \tilde{V}) \otimes \mathbb{C}^2$.

By (5.32) the sequence $\psi_n$ tends also strongly to 1 on $\mathcal{H}^s(\Sigma; \tilde{V} \otimes \mathbb{C}^2)$. Therefore $\tilde{c}^\pm = (\tilde{c}^\pm)^2$ on $C^\infty_0(\Sigma; \tilde{V}) \otimes \mathbb{C}^2$ and thus on $\mathcal{H}^s(\Sigma; \tilde{V} \otimes \mathbb{C}^2)$ by density. \qed

5.3.10. **Dirichlet-to-Neumann maps.** Let us now consider the following inhomogeneous boundary value problem:
\[
\begin{cases}
\tilde{D}u = 0 \text{ in } \Omega^\pm, \\
u|_{\partial \Omega^+ \setminus \Sigma} = 0, \\
u|_{\Sigma} = v.
\end{cases}
\]
We claim that (5.33) has a unique solution $u =: P_{\Omega^\pm} v$. In fact, let $\chi \in C^\infty_0(]0, T[)$ with $\chi = 1$ on $]0, T/2[$ and $\chi^\pm(s) = \chi(\pm s)$. Then for
\[
u_1 = \chi^\pm(s) W^\pm(s, 0)v,
\]
we have $u_1(\pm T) = 0$, $u_1(0) = v$ and by (5.17)
\[
\hat{D} u_1 = (-\partial_s + \hat{b}_1^\pm ) (\partial_s \chi) \hat{W}^\pm (\cdot, 0)v + \hat{r}_{-\infty}^\pm \hat{W}^\pm (\cdot, 0)v := m_{-\infty}^\pm v,
\]
where $m_{-\infty}^\pm \in W^{-\infty}(\Sigma, \tilde{\Omega}^\pm, \tilde{V})$.

By the same arguments as in 5.3.6, we can consider the Dirichlet realization $\hat{D}_{\Omega^\pm}$ of $\hat{D}$ in $\Omega^\pm$, which for $0 < T \ll 1$ is invertible. Therefore we can solve (5.33) by
\[
P_{\Omega^\pm} v = \chi_{W^\pm} (\cdot, 0)v - \hat{D}_{\Omega^\pm}^{-1} m_{-\infty}^\pm v.
\]
(5.34)

**Definition 5.18.** The Dirichlet-to-Neumann maps are
\[
N_{\Omega^\pm} : = -\partial_s P_{\Omega^\pm} v|_\Sigma.
\]

**Proposition 5.19.** The Dirichlet-to-Neumann maps satisfy
\[
N_{\Omega^\pm} = \hat{b}_1^\pm (0) + r_{-\infty}^\pm, \quad r_{-\infty}^\pm \in W^{-\infty}(\Sigma; \tilde{V}).
\]

**Proof.** By (5.34) we have
\[
N_{\Omega^\pm} v = -\partial_s \chi \hat{W}^\pm (\cdot, 0)v|_\Sigma + \partial_s \hat{D}_{\Omega^\pm}^{-1} m_{-\infty}^\pm v|_\Sigma
\]
\[
= : \hat{b}_1^\pm (0)v + r_{-\infty}^\pm v.
\]
To prove that $r_{-\infty}^\pm \in W^{-\infty}(\Sigma; \tilde{V})$, we use Prop. 5.12 and the explicit form of the parametrix $\hat{D}_{\Omega}^{-1} \Omega^\pm$ with $\Omega$ replaced by $\Omega^\pm$. □

5.3.11. Positivity of the Dirichlet to Neumann maps.

**Proposition 5.20.** The Dirichlet-to-Neumann maps satisfy
\[
\pm \Re N_{\Omega^\pm} \sim (-\Delta_{\tilde{h}_0} + 1)^{\frac{1}{2}} \text{ on } C_c^\infty(\Sigma; \tilde{V}),
\]
for the scalar product $(\cdot|\cdot)_{\tilde{V}(\Sigma)}$.

**Proof.** By Prop. 5.19 we have $\pm \Re N_{\Omega^\pm} \leq c_0 (-\Delta_{\tilde{h}_0} + 1)^{\frac{1}{2}}$, so it suffices to prove the other inequality. Let $u = P_{\Omega^\pm} v$ for $v \in C_c^\infty(\Sigma; \tilde{V})$. We have $u \in H^1_0(\Omega^\pm; \tilde{V})$ and $\hat{D} u = 0$ in $\Omega^\pm$ so we can apply (5.27). We obtain:
\[
\pm (v| \Re N_{\Omega^\pm} v)_{\tilde{V}(\Sigma)} = \eta_{\Omega^\pm} (u, u) \sim \|u\|^2_{H^1(\Omega^\pm)}.
\]
By the continuity properties of $u \mapsto u|_\Sigma$ between Sobolev spaces the rhs is equivalent to $\|v\|^2_{H^\frac{1}{2}(\Sigma)}$. □

5.4. Relation between Lorentzian and Euclidean projectors. We are ready to prove that if $\tilde{a}(s)$ formally coincides with the Wick rotation of $a(t)$ in the sense of Taylor coefficients at 0, then the Hadamard projectors $c^\pm$ for $D = \partial_s^2 + a(t)$ differs by a smoothing term from the Calderón projectors $\tilde{c}^\pm$ for $\tilde{D} = -\partial_s^2 + \tilde{a}(s)$.

**Proposition 5.21.** Suppose that
\[
(i\partial_t)^n a(0) = \partial_s^n \tilde{a}(0) \forall n \in \mathbb{N}.
\]
Then
\[
c^\pm - \tilde{c}^\pm \in W^{-\infty}(\Sigma; \tilde{V} \otimes \mathbb{C}^2).
\]
Proof. By comparing formulas (5.14) and (5.30), we see that it suffices to prove that

$$b^\pm(0) - \tilde{b}^\pm(0) \in W^{-\infty}(\Sigma; \hat{V}),$$  \hspace{1cm} (5.36)

where $b^\pm$ are defined in (5.10) and $\tilde{b}^\pm$ in Prop. 5.6. The proof of (5.36) is divided into two steps.

Step 1. In the first step we prove that

$$\hat{b}^-(0) = -\tau^{-1} \tilde{b}^+(0) \tau \mod C_b^\infty(I; W^{-\infty}(\Sigma; \hat{V})).$$  \hspace{1cm} (5.37)

Let us prove (5.37). Let us abbreviate $\tilde{b}^+$ by $\hat{b}$. We know that $\hat{b} = \hat{\epsilon} + \hat{b}_0$, where $\hat{b}_0$ solves the fixed point equation

$$\hat{b}_0 = \hat{c}_0 + \hat{F}(\hat{b}_0), \quad \hat{c}_0 = (2\hat{\epsilon})^{-1}(\partial_s \hat{\epsilon})$$  \hspace{1cm} (5.38)

for

$$\hat{F}(\hat{d}) = (2\hat{\epsilon})^{-1}(\partial_s \hat{d} + [\hat{\epsilon}, \hat{d}] - \hat{d}^2).$$

Let us recall that (5.38) is solved symbolically by

$$\hat{b}_0 = \hat{c}_0 + \sum_{k \geq 1} \hat{d}_k - \hat{d}_{k-1}, \quad \hat{d}_0 = \hat{c}_0, \quad \hat{d}_k = \hat{c}_0 + \hat{F}(\hat{d}_{k-1}).$$  \hspace{1cm} (5.39)

Let us also set

$$\hat{b}(s) = -\tau^{-1} \tilde{b}^-(s) \tau.$$

Since

$$\partial_s \hat{b}(s) - \hat{b}^2(s) + a(s) = 0 \mod C_b^\infty(I; W^{-\infty}(\Sigma; \hat{V})), $$

we obtain that

$$\partial_s \hat{b}(s) - \hat{b}^2(s) + \hat{a}(s) = 0 \mod C_b^\infty(I; W^{-\infty}(\Sigma; \hat{V}))$$

for $\hat{a}(s) = \tau^{-1} \hat{a}^*(s) \tau$. Note that $\hat{a}(s)$ has the same properties as $\tilde{a}(s)$. Moreover,

$$\hat{b}(s) = -\hat{\epsilon}(s) \mod C_b^\infty(I; \Psi^0(\Sigma; \hat{V})),$$

where $\hat{\epsilon}(s) = \hat{\epsilon}^\frac{1}{2}(s)$ is the $m$-accretive square root of $\hat{a}(s)$. By the uniqueness statement in Prop. 5.6, it follows that

$$\hat{b}(s) = \hat{b}^-(s) \mod C_b^\infty(I; W^{-\infty}(\Sigma; \hat{V})), $$

where $\hat{b}^-(s)$ is the solution in Prop. 5.6 with $a(s)$ replaced by $\hat{a}(s)$. Therefore we have

$$\hat{b} = -\hat{\epsilon} + \hat{b}_0,$$

where $\hat{b}_0$ solves the fixed point equation:

$$\hat{b}_0 = \hat{c}_0 + \hat{F}^-(\hat{b}_0), \quad \hat{c}_0 = (2\hat{\epsilon})^{-1}\partial_s \hat{\epsilon},$$  \hspace{1cm} (5.40)

where

$$\hat{F}^-(\hat{d}) = (2\hat{\epsilon})^{-1}(-\partial_s \hat{d} + [\hat{\epsilon}, \hat{d}] - \hat{d}^2).$$

Next, (5.40) is solved symbolically by (5.39), with $\hat{c}_0, \hat{d}_k, \hat{F}$ replaced by $\tilde{c}_0, \tilde{d}_k, \tilde{F}$.

A direct inspection of (5.39) shows that modulo $C_b^\infty(I; W^{-\infty}(\Sigma; \hat{V}))$, $\tilde{b}_0(0)$ depends only on the Taylor expansion of $s \mapsto \tilde{\epsilon}(s)$ at $s = 0$, i.e. on the Taylor expansion of $s \mapsto \tilde{a}(s)$ at $s = 0$. The same result holds also for $\hat{b}_0(0)$, replacing $\tilde{a}(s)$ by $\hat{a}(s)$.

We claim that

$$(\partial_s)^n \hat{a}(0) = \partial_s^n \hat{a}(0), \quad \forall n \in \mathbb{N},$$  \hspace{1cm} (5.41)
which by the discussion above implies (5.37). We first note that since \(a(t) = a^*(t)\) by (H1), we have \(a^*(t) = \tau a(t)\tau^{-1}\), hence

\[
((i\partial_t)^n a)^*(0) = (-1)^n \tau (i\partial_t)^n a(0)\tau^{-1}, \quad n \in \mathbb{N}.
\]

By (5.35) this implies that

\[
\partial_s^n \tilde{a}^*(0) = (-1)^n \tau \partial_s^n \tilde{a}(0)\tau^{-1}, \quad n \in \mathbb{N},
\]

and hence (5.41). This completes Step 1.

**Step 2.** In Step 2 we complete the proof of the proposition. Let \(b\) be the operator from Prop. 5.2. We know that \(b = \epsilon + b_0\), where \(b_0\) solves the fixed point equation

\[
b_0 = c_0 + F(c_0), \quad c_0 = (2\epsilon)^{-1}(i\partial_t\epsilon).
\]

(5.42)

for

\[
F(d) = (2\epsilon)^{-1}(i\partial_t d + [\epsilon, d] - d^2).
\]

The equation (5.42) is solved symbolically as before by

\[
b_0 = c_0 + \sum_{k \geq 1} d_k - d_{k-1}, \quad d_0 = c_0, \quad d_k = c_0 + F(d_{k-1}).
\]

From the definitions of \(F, \tilde{F}\) we get that if \(d \in C^\infty_b(I; \Psi^0(\Sigma; V))\), \(\tilde{d} \in C^\infty_b(I; \Psi^0(\Sigma; \tilde{V}))\) satisfy

\[
(i\partial_t)^n d(0) = (\partial_s)^n \tilde{d}(0), \quad n \in \mathbb{N},
\]

then

\[
(i\partial_t)^n F(d)(0) = (\partial_s)^n \tilde{F}(\tilde{d})(0), \quad n \in \mathbb{N}.
\]

Since \((i\partial_t)^n a(0) = \partial_s^n \tilde{a}(0)\) \(\forall n \in \mathbb{N}\), we obtain by (5.7) that

\[
(i\partial_t)^n \epsilon(0) = \partial_s^n \tilde{\epsilon}(0), \quad n \in \mathbb{N}, \quad \text{i.e.} \quad (i\partial_t)^n c_0(0) = \partial_s^n \tilde{c}_0(0), \quad n \in \mathbb{N}.
\]

Therefore \((i\partial_t)^n d_k(0) = (\partial_s)^n \tilde{d}_k(0)\) \(\forall n, k \in \mathbb{N}\), which implies that

\[
b(0) - \tilde{b}(0) = b^+(0) - \tilde{b}^+(0) \in \mathcal{W}^{-\infty}(\Sigma; \tilde{V}).
\]

We have \(b^-(0) = -b^*(0) = -\tau^{-1}b^*(0)\tau\), see (5.10), and \(\tilde{b}^-(0) = -\tau^{-1}\tilde{b}^*(0)\tau\) mod \(C^\infty_b(I; \mathcal{W}^{-\infty}(\Sigma; \tilde{V}))\) by (5.37). Therefore

\[
b^-(-0) - \tilde{b}^-(-0) \in \mathcal{W}^{-\infty}(\Sigma; \tilde{V}),
\]

which implies the proposition.

\[\square\]

6. **Wick rotation in Gaussian time**

In this section we prove the main result of this paper, namely the existence of Hadamard states for linearized gravity on Einstein spacetimes \((M, g)\) satisfying the analyticity hypotheses in Subsect. 3.3.

The idea of constructing states by Wick rotation in Gaussian time \(t\) is inspired by the earlier work in the real analytic case [27], see [46, 50] for further analyticity properties in related frameworks; we also remark that these results are largely consistent with the program recently outlined by Kontsevich–Segal [39].
6.1. **Framework.** We assume that the metric \( g \) satisfies the hypotheses in Subsect. 3.3 and we apply the framework of Sect. 5 to the reduced operators \( \hat{D}_i, i = 1, 2 \), constructed in Subsect. 4.4.

6.1.1. **Hyperbolic operators.** From now on the operators \( \hat{D}_i, \hat{d}, \hat{I} \) introduced in 4.4.1 will simply be denoted by \( D_i = \partial_t^2 + a_i(t), d \) and \( I_2 \) respectively. We can write \( d \) as

\[
d = d_0(t)\partial_t + d_1(t), \quad d_i \in C^\infty_b(I; \text{Diff}(\Sigma; V_1, V_2)).
\]

In order to have more uniform formulas with respect to the index \( i = 1, 2 \) we set \( I_1 = 1 \), which corresponds to the notation introduced in (4.8).

The trace operators \( \varrho_i \) are defined as in (5.1).

6.1.2. **Hermitian bundles.** The Hermitian bundles are

\[
V_i = I \times \left( \sum_{k=0}^i \mathbb{C} \otimes_{\mathbb{R}}^k T^*\Sigma \right),
\]

see the identifications (4.11) and (4.12), equipped with the Hermitian structures

\[
(\cdot|\cdot)_V^i = (\cdot|\cdot)_{g_0}, \quad (\cdot|\cdot)V_2 = 2(\cdot|\cdot)_{g_0^2}.
\]

The physical Hermitian forms are:

\[
(u|u)_{I,V_i} = (u|I_iu)V_i.
\]

6.1.3. **Hilbertian bundles.** The corresponding Hilbertian bundles are \( \tilde{V}_i \), equal to \( V_i \) as complex vector bundles but equipped with the Hilbertian structures defined by

\[
(u|\tau_i v)_{\tilde{V}_i} = (u|v)V_i,
\]

for

\[
\tau_0 = 1, \quad \tau_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

As in Subsect. 4.3.1 and 4.3.2, we write an element of \( \tilde{V}_1 \), resp. \( \tilde{V}_2 \), as

\[
w = (w_s, w_\Sigma), \quad \text{resp.} \quad u = (u_{ss}, u_{s\Sigma}, u_{\Sigma\Sigma}),
\]

which gives:

\[
(w|w)_{\tilde{V}_1} = \overline{w_s}w_s + \overline{w_\Sigma}h_0^{-1}w_\Sigma,
\]

\[
(u|u)_{\tilde{V}_2} = 2(\overline{u_{ss}}u_{ss} + 2\overline{u_{s\Sigma}}h_0^{-1}u_{s\Sigma} + \overline{u_{\Sigma\Sigma}}(h_0^{-2})^{-1}u_{\Sigma\Sigma}).
\]

6.1.4. **Gauge invariance.** We recall that the gauge operator is \( K = I \circ d \), see (4.7). It will be convenient to treat the factors \( I \) and \( d \) in \( K \) separately. We recall from Lem. 4.3 \( ii) \) that \( d = d_0 \partial_t + d_1 \) satisfies \( D_2d = dD_1 \). We can hence define the Cauchy surface version of \( d \) by

\[
\varrho_2 \circ d =: T_\Sigma \circ \varrho_1 \text{ on } \ker D_1|_{C^\infty_b(M; V_1)}.
\]

Using (4.20) we obtain that

\[
T_\Sigma = \begin{pmatrix} d_1(0) \\ \frac{1}{2}(d_0a_1 + a_2d_0)(0) \end{pmatrix}.
\]
6.1.5. Trace reversal. Recall that the trace reversal operator \( \hat{I} \) in (4.19) is denoted by \( I_2 \). Explicitly,
\[
I_2 u = u - \frac{1}{4}(g_0|u)v_2 g_0, \quad u \in C^\infty(M;V_2),
\]
and we have also set
\[
I_1 u = u, \quad u \in C^\infty(M;V_1).
\]
Since \( D_i I_i = I_i D_i \), we can define \( I_\Sigma : C_c^\infty(\Sigma;V_i(\Sigma) \otimes \mathbb{C}^2) \) by
\[
q_i \circ I_i = I_\Sigma \circ q_i \text{ on } \ker D_i|C^\infty(M;V_i).
\]
A routine computation shows that
\[
I_\Sigma = I_i \otimes 1_{\mathbb{C}^2},
\]
and since \( K = I \circ d \) we have
\[
K_\Sigma = I_\Sigma \circ T_\Sigma.
\]

6.1.6. Physical charge on a Cauchy surface. The (unphysical) charges for \( D_i \) on the Cauchy surface \( \Sigma \) corresponding to the (unphysical) Hermitian forms \( (\cdot|\cdot)_{V_i} \) in (2.2) are given by
\[
q_i = \begin{pmatrix} 0 & \tau_i \\ \tau_i & 0 \end{pmatrix}.
\]
(6.3)
For linearized gravity one has to use the physical charges corresponding to \( (\cdot|\cdot)_{I,V_i} \), see (6.1). On the Cauchy surface \( \Sigma \) they are given by
\[
q_{i,\text{phys}} = q_i \circ I_\Sigma = \begin{pmatrix} 0 & \tau_i I_i \\ \tau_i I_i & 0 \end{pmatrix}.
\]
(6.4)
The extra subscript in \( q_{i,\text{phys}} \) (absent in previous parts of the paper) is used throughout this section to disambiguate from (6.3).

6.2. Wick rotation. By the hypotheses in Subsect. 3.3, we know that the maps \( t \mapsto a_i(t), d_0(t), d_1(t) \) extend holomorphically to some disk \( D_1(0,\epsilon) \) with values in differential operators on \( \Sigma \), i.e.
\[
a_i \in \mathcal{A} T(C_1(0,\epsilon); \text{Diff}^2(\Sigma;V_i)), \quad d_j \in \mathcal{A} T(C_1(0,\epsilon); \text{Diff}^j(\Sigma;V_1,V_2)),
\]
i = 1, 2, j = 0, 1. Therefore we can define the Wick rotated operators
\[
\tilde{a}_i(s) = a_i(is), \quad i = 1, 2, \quad \tilde{d}_0(s) = -id_0(is), \quad \tilde{d}_1(s) = d_1(is),
\]
(6.5)
which for \( \tilde{I} = ]-\epsilon,\epsilon[, \quad 0 < \epsilon \ll 1 \) satisfy
\[
\tilde{a}_i \in C^\infty(\tilde{I};\text{Diff}^2(\Sigma;\tilde{V}_i)), \quad \tilde{d}_j \in C^\infty(\tilde{I};\text{Diff}^j(\Sigma;\tilde{V}_1,\tilde{V}_2)),
\]
i = 1, 2, j = 0, 1.
6.2.1. **Wick rotated operators.** The elliptic operators
\[ \tilde{D}_i = -\partial_s^2 + \tilde{a}_i(s), \]
satisfy the conditions in Subsect. 5.3. The trace operators \( \tilde{\varrho}_i \) are defined as in (5.25).

Since \( \tilde{a}_i(s) = a_i(is) \) we have
\[ (i\partial_t)^n a_i(0) = \partial_s^n \tilde{a}_i(0), \quad n \in \mathbb{N}, \ i = 1, 2. \quad (6.6) \]
i.e. condition (5.35) in Prop. 5.21 is satisfied. This fact will be crucial to establish the Hadamard property.

Note that since \( a_i(t) = a_i^*(t) \) and \( a_i^*(t) = \tau_i^{-1} a^* \tau_i \) we have \( a_i^*(t) = \tau_i a_i(t) \tau_i^{-1} \) hence by analytic continuation in \( t \) we have
\[ \tilde{a}_i^*(s) = \tau_i \tilde{a}_i(-s) \tau_i^{-1}, \quad s \in I, \ i = 1, 2. \quad (6.7) \]

Setting
\[ \kappa_i u(s) := \tau_i u(-s), \]
(6.7) is equivalent to
\[ \tilde{D}_i^* = \kappa_i \tilde{D}_i \kappa_i^{-1} = -\partial_s^2 + \tau_i \tilde{a}_i(-s) \tau_i^{-1}. \]
Since the open set \( \Omega = ]-T, T[ \times \Sigma \) is invariant under \( s \mapsto -s \) we obtain that
\[ \tilde{D}_i^*_{\Omega} = \kappa_i \tilde{D}_i_{\Omega} \kappa_i^{-1}, \]
where we recall that \( \tilde{D}_i_{\Omega} \) is the Dirichlet realization of \( \tilde{D}_i \) in \( \Omega \).

6.2.2. **Gauge invariance for Wick rotated operators.** Setting
\[ \tilde{d} = \tilde{d}_0(s) \partial_s + \tilde{d}_1(s), \]
we deduce from \( D_2 d = d D_1 \) and analytic continuation in \( t \) that
\[ \tilde{D}_2 \tilde{d} = \tilde{d} \tilde{D}_1. \quad (6.8) \]

Therefore, we can we define \( \tilde{T}_\Sigma \) by
\[ \tilde{\varrho}_2^\pm \circ \tilde{d} =: \tilde{T}_\Sigma \tilde{\varrho}_1^\pm \text{ on } \text{Ker } \tilde{D}_1|_{C^\infty(\Omega^\pm; \tilde{\nu}_i)}. \]

**Lemma 6.1.** Let \( T_\Sigma \) be the operator defined in (6.2). Then \( T_\Sigma = \tilde{T}_\Sigma \).

**Proof.** Property (6.8) is equivalent to:
\[
\begin{align*}
\phantom{i) \quad i) & \quad \partial_s \tilde{d}_0 = 0, \\
& \quad ii) \quad 2\partial_s \tilde{d}_1 - \tilde{a}_2 \tilde{d}_0 + \tilde{d}_0 \tilde{a}_1 = 0, \\
& \quad iii) \quad \partial_s^2 \tilde{d}_1 - \tilde{a}_2 \tilde{d}_1 + \tilde{d}_1 \tilde{a}_1 + \tilde{d}_0 \partial_s \tilde{a}_1 = 0.
\end{align*}
\]
(6.9)

From (6.9) we obtain that:
\[ T_\Sigma = \begin{pmatrix}
\tilde{d}_1(0) & -\tilde{d}_0(0) \\
-\frac{1}{2} (\tilde{d}_0 \tilde{a}_1 + \tilde{a}_2 \tilde{d}_0)(0) & \tilde{d}_1(0)
\end{pmatrix}. \]
Using (6.6) for \( n = 0 \) and the fact that \( \tilde{d}_1(0) = d_1(0), \ \tilde{d}_0(0) = -id_0(0) \), we obtain \( T_\Sigma = \tilde{T}_\Sigma \). □
6.2.3. Trace reversal for Wick rotated operators. Let us also define a ‘trace reversal’ for $\tilde{D}_2$. We set

$$\tilde{I}_2 u = u - \frac{1}{4} (g_0|u)_{V_2} g_0, \ u \in C^\infty(M; V_2).$$

Note that $\tilde{I}_2 = I_2$ and the expression above immediately shows that $\tilde{I}_2$ is selfadjoint for $(\cdot|\cdot)_{V_2}$. For coherence of notation we set again $\tilde{I}_1 = 1$.

We deduce from $D_2 \circ I = I \circ D_2$ and analyticity in $t$ that

$$\tilde{I}_2 \tilde{D}_2 = \tilde{D}_2 \tilde{I}_2,$$

We then define $\tilde{I}_{2\Sigma}$ by

$$\tilde{\rho}^\pm_{2\Sigma} \circ \tilde{I}_2 := \tilde{I}_{2\Sigma} \circ \tilde{\rho}^\pm_{2},$$

and a routine computation shows that $\tilde{I}_{2\Sigma} = \tilde{I}_2 \otimes 1_{C^2}$ hence

$$\tilde{I}_{2\Sigma} = I_{2\Sigma}. \tag{6.10}$$

In a similar vein we set

$$\tilde{K} = \sqrt{2} \tilde{I} \circ \tilde{d},$$

so that

$$\tilde{D}_2 \circ \tilde{K} = \tilde{K} \circ \tilde{D}_1.$$

The Cauchy surface version of $\tilde{K}$ is

$$\tilde{K}_\Sigma = \tilde{I}_\Sigma \circ \tilde{T}_\Sigma = K_\Sigma \tag{6.11}$$

by Lem. 6.1 and (6.10).

6.3. Calderón projectors. We now show a list of properties of the Calderón projectors $\tilde{c}_i^\pm$ constructed in 5.3.9 for the Dirichlet realizations of the operators $\tilde{D}_i$.

In Prop. 6.2 below, the adjoints $(\tilde{c}_i^\pm)^\dagger$ are computed with respect to the physical charges $g_i,_{\text{phys}}$, see (6.4).

**Proposition 6.2.** The Calderón projectors $\tilde{c}_i^\pm$ for $i = 1, 2$ satisfy:

1. $\tilde{c}_i^\pm \in \Psi^1(\Sigma; \tilde{V}_i \otimes C^2)$;
2. $\tilde{c}_i^\pm = (\tilde{c}_i^\pm)^2$ and $\tilde{c}_i^+ + \tilde{c}_i^- = 1$ on $H^s(\Sigma; \tilde{V}_i \otimes C^2)$;
3. $(I_i \otimes 1)\tilde{c}_i^\pm = \tilde{c}_i^\pm (I_i \otimes 1)$ on $C^\infty_c(\Sigma; \tilde{V}_i \otimes C^2)$;
4. $\tilde{c}_i^\pm = (\tilde{c}_i^\pm)^\dagger$;
5. $\tilde{c}_2^\pm K_\Sigma = K_\Sigma \tilde{c}_1^\pm \pm K_{-\infty}$ on $C^\infty_c(\Sigma; \tilde{V}_1 \otimes C^2)$, where

$$K_{-\infty} \in W^{-\infty}(\Sigma; \tilde{V}_1 \otimes C^2, \tilde{V}_2 \otimes C^2).$$

**Proof.** (1) is proved in Prop. 5.16, (2) in Prop. 5.17. (3) follows from the definition of $\tilde{c}_i^\pm$ and the fact that $\tilde{I}_i \tilde{D}_\Omega = \tilde{D}_\Omega \tilde{I}_i$, since $\tilde{I}_i$ preserves the Dirichlet boundary condition on $\partial \Omega$.

Let us now prove (4). We drop the $i$ index. Let us set for $f, g \in C^\infty_c(\Sigma; \tilde{V} \otimes C^2)$:

$$u = -r^+ \tilde{D}^{-1}_{\Omega} \tilde{g}^* \tilde{\sigma} f, \ v = -r^+ \kappa \tilde{D}^{-1}_{\Omega} \tilde{g}^* \tilde{\sigma} g,$$
where \( r^+ \) is the operator of restriction to \( \Omega^+ \). We use now [27, Lem. A.1] which extends to the case of vector bundles, and we obtain that \( u, v \in \overline{H}_0^1(\Omega^+; \tilde{V}) \). Since \( \tilde{D}u = \tilde{D}^+ v = 0 \) in \( \Omega^+ \) we deduce from the Green’s formula (5.26) and (5.28) that

\[
(\tilde{\varphi}^+ v | \tilde{\varphi}\tilde{\varphi}^+ u)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2} = 0.
\]

Using that \( \tilde{\varphi}^+ \kappa = \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix} \tilde{\varphi}^- \) we have

\[
\tilde{\varphi}^+ u = \tilde{c}^+ f, \quad \tilde{\varphi}^+ v = \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix} \tilde{c}^- g.
\]

Since \( \tilde{\sigma}^* \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix} = - \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \tilde{\varphi}^- \), we obtain that \( \tilde{c}^{-*} \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \tilde{c}^+ = 0 \) hence \( \tilde{c}^{-*} \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \tilde{c}^+ \). Since \( I_{\Sigma}^* = I_{\Sigma} \), we obtain that \( \tilde{c}^{-*} I_{\Sigma} \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} = I_{\Sigma} \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \tilde{c}^+ \), i.e. \( \tilde{c}^{-*} q_{\text{phys}} = q_{\text{phys}} \tilde{c}^+ \), which proves (4).

We now prove (5). Let us set for \( f_i \in C^\infty(\Sigma; \tilde{V}_i \otimes \mathbb{C}^2) \):

\[
u_2 = r^+ \tilde{D}_{21}^{-1} \tilde{K} \tilde{\varphi}^* \tilde{\varphi} \tilde{\varphi}^* f_1, \quad \nu_2 = r^+ \kappa_2 \tilde{D}_{21}^{-1} \tilde{\varphi}^* \tilde{\varphi} f_2.
\]

We have \( \tilde{D}_{2}^* \nu_2 = 0 \) in \( \Omega^+ \), \( \tilde{D}_{2} \nu_2 = 0 \) in \( \Omega^+ \).

As before we can apply (5.26) and obtain

\[
(\tilde{\varphi}^+ \nu_2 | \tilde{\varphi}^* \tilde{\varphi}^+ \nu_2)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2} = 0.
\]

(6.12)

Since \( \tilde{D}_{2} \tilde{K} = \tilde{K} \tilde{D}_{1} \) as differential operators, we have

\[
\begin{aligned}
\tilde{\varphi}^+ u_2 &= \tilde{\varphi}^* \tilde{D}_{21}^{-1} \tilde{K} \tilde{\varphi}^* \tilde{\varphi} \tilde{\varphi}^* f_1 = \tilde{\varphi}^* \tilde{D}_{21}^{-1} \tilde{K} \tilde{D}_{1} \tilde{D}_{1}^{-1} \tilde{\varphi}^* \tilde{\varphi} \tilde{\varphi}^* f_1 \\
&= \tilde{\varphi}^* \tilde{D}_{21}^{-1} \tilde{D}_{21} \tilde{K} \tilde{D}_{1} \tilde{D}_{1}^{-1} \tilde{\varphi}^* \tilde{\varphi} \tilde{\varphi}^* f_1 \\
&= \tilde{\varphi}^* \tilde{D}_{21}^{-1} \tilde{D}_{21} \tilde{K} \tilde{D}_{1} \tilde{D}_{1}^{-1} \tilde{\varphi}^* \tilde{\varphi} \tilde{\varphi}^* f_1 + \tilde{\varphi}^* \tilde{D}_{21}^{-1} \tilde{D}_{21} \tilde{K} \tilde{D}_{1} \tilde{D}_{1}^{-1} \tilde{\varphi}^* \tilde{\varphi} \tilde{\varphi}^* f_1 |_{\partial \Omega}
\end{aligned}
\]

where we have used Lem. 5.13 and \( \tilde{K}_\Sigma = K_\Sigma \) in the last line. Using the parametrix in 5.3.7, see Prop. 5.12, we obtain that \( (\tilde{K} \tilde{D}_{1}^{-1} \tilde{\varphi}^* \tilde{\varphi} f_1) |_{\partial \Omega} := T_{-\infty} f_1 \) is smoothing, i.e. \( T_{-\infty} \in \mathcal{W}^{-\infty}(\Sigma; \tilde{V}) \). Therefore \( r_{-\infty}^+ := \tilde{\varphi}^* \tilde{D}_{21}^{-1} \tilde{D}_{21} \tilde{K} \tilde{D}_{1} \tilde{D}_{1}^{-1} \tilde{\varphi}^* \tilde{\varphi} f_1 \) belongs to \( \mathcal{W}^{-\infty}(\Sigma; \tilde{V} \otimes \mathbb{C}^2) \).

Furthermore,

\[
\tilde{\varphi}^+ v_2 = \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix} \tilde{c}^- f_2.
\]

Thus, (6.12) gives the identity

\[
\tilde{c}^{-*} \begin{pmatrix} 0 & \tau_2 \\ \tau_2 & 0 \end{pmatrix} (K_\Sigma \tilde{c}^+ - r_{-\infty}^+) = 0,
\]

which implies that \( \tilde{c}^- (K_\Sigma \tilde{c}^+ - r_{-\infty}^+) = 0 \) since \( \tilde{c}^{-*} \begin{pmatrix} 0 & \tau_2 \\ \tau_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tau_2 \\ \tau_2 & 0 \end{pmatrix} \tilde{c}^- \) by the proof of (4). In a similar vein we get

\[
\tilde{c}^+ (K_\Sigma \tilde{c}^- - r_{-\infty}^-) = 0.
\]
where \( r_{\infty}^- \in \mathcal{W}^{-\infty}(\Sigma; \tilde{V} \otimes \mathbb{C}^2) \). Using (2) we obtain
\[
\tilde{c}_2^+ K_{\Sigma} = K_{\Sigma} \tilde{c}_1^+ - \tilde{c}_2^- K_{\Sigma} \tilde{c}_1^- + \tilde{c}_2^+ K_{\Sigma} \tilde{c}_1^-
= K_{\Sigma} \tilde{c}_1^+ - \tilde{c}_2^- r_{\infty}^- + \tilde{c}_2^+ r_{\infty}^-
= K_{\Sigma} \tilde{c}_1^+ + K_{-\infty},
\]
where \( K_{-\infty} \in \mathcal{W}^{-\infty}(\Sigma; \tilde{V}_1 \otimes \mathbb{C}_2, \tilde{V}_2 \otimes \mathbb{C}^2) \).

**Remark 6.3.** We remark that one can attempt to modify \( \tilde{c}_i^+ \) (and \( \tilde{\pi}_i^+ \) at the same time) by changing the boundary conditions for \( \tilde{D}_1 \) and \( \tilde{D}_2 \) at \( s = T \) and \( s = -T \) (in particular a well-motivated candidate is provided by boundary conditions studied in [2, 49]). If this gives a pair of invertible operators \( \tilde{D}_{1,\text{mod}} \) and \( \tilde{D}_{2,\text{mod}} \) such that
\[
\tilde{K} \text{ Dom } \tilde{D}_{1,\text{mod}} \subset \text{ Dom } \tilde{D}_{2,\text{mod}}, \tag{6.13}
\]
then one can show (under some further assumptions on the boundary conditions) that the modified operators \( \tilde{c}_{i,\text{mod}}^+ \) satisfy gauge invariance \( \tilde{c}_{2,\text{mod}}^+ K_{\Sigma} = K_{\Sigma} \tilde{c}_{1,\text{mod}}^+ \). This motivates a broader study of boundary conditions in Wick-rotated linearized gravity. One of the main difficulties is that it is not clear how to obtain operators satisfying (6.13) and which are at the same time invertible.

The next lemma states that there exists a gauge transformation in the Wick-rotated setting mapping to tensors which have no mixed components and are symmetric with respect to the trace reversal at \( \Sigma \). Furthermore, the gauge transformation preserves boundary conditions and is given in terms of a solution \( w \) rather than some arbitrary \((0,1)\)-tensor.

**Lemma 6.4.** Let \( u \in H^s(\Omega^\pm; \tilde{V}) \cap \overline{H^T(\Sigma^\pm; \tilde{V}_2)} \) for some \( s > \frac{1}{2} \). Then there exists \( w \in H^{s+1}(\Omega^\pm; \tilde{V}_1) \cap \overline{H^T_0(\Omega^\pm; \tilde{V}_1)} \) such that \( \tilde{D}_1 w = 0 \) in \( \Omega^\pm \) and \( v = u - \tilde{K} w \) satisfies
\[
\begin{align*}
\{ (v\Sigma) |_{\Sigma} &= 0, \\
(\tilde{I}_2 v)|_{\Sigma} &= v|_{\Sigma}.
\end{align*}
\tag{6.14}
\]

**Proof.** It suffices to prove the ‘+’ case. We rewrite (6.14) as
\[
\begin{align*}
\{ (\tilde{K} w) s|_{\Sigma} &= u_{\Sigma}, \\
(\tilde{g}_0 | \tilde{K} w)|_{\tilde{v}_2} &= u_s,
\end{align*}
\]
for \( u_{\Sigma} = u_{s\Sigma} |_{\Sigma} \in H^{s-\frac{1}{2}}(\Sigma; T^s \Sigma) \) and \( u_s = (\tilde{g}_0 | u)|_{\tilde{v}_2} |_{\Sigma} \in H^{s-\frac{1}{2}}(\Sigma; \mathbb{C}) \).

Using that \( \tilde{K} = \tilde{I}_2 \tilde{d}, \tilde{I}_2 \tilde{d} = \tilde{I}_2 \tilde{v}_2 = \tilde{g}_0 = -\tilde{g}_0 \) and the fact that \( \tilde{I} \) does not act on the \( s\Sigma \) components, this is equivalent to
\[
\begin{align*}
\{ (\tilde{d} w) s|_{\Sigma} &= u_{\Sigma}, \\
(\tilde{g}_0 | \tilde{d} w)|_{\tilde{v}_2} |_{\Sigma} &= -u_s.
\end{align*}
\tag{6.15}
\]

Let \( w \in H^{s+1}(\Omega^\pm; \tilde{V}_1) \cap \overline{H^T_0(\Omega^\pm; \tilde{V}_1)} \) such that \( \tilde{D}_1 w = 0 \) in \( \Omega^+ \). Let us compute \( \tilde{d} w |_{\Sigma} \). We replace \( \partial_t \) by \(-i\partial_s\) and set \( t = 0 \) in the expression of \( d \) in Prop. 4.11, which
corresponds to the relationship between $d$ and $\tilde{d}$ at $s = 0$ given by (6.5). Keeping in mind that $u(0) = 1$ and $s(0) = 1$, we obtain

$$(\tilde{d}w)_{ss}|_{\Sigma} = -i\partial_s w_s|_{\Sigma} - \frac{1}{2} \text{tr}(r_0) w_s|_{\Sigma},$$

$$(\tilde{d}w)_{s\Sigma}|_{\Sigma} = \frac{1}{2}(-i\partial_s w_{\Sigma}|_{\Sigma} - \frac{1}{2} \text{tr}(r_0) w_{\Sigma}|_{\Sigma} - r_0 w_{\Sigma}|_{\Sigma} + d_{\Sigma} w_s|_{\Sigma}),$$

$$(\tilde{d}w)_{\Sigma\Sigma}|_{\Sigma} = d_{\Sigma} w_{\Sigma}|_{\Sigma} - \frac{1}{2} \partial_t h_0 w_s|_{\Sigma},$$

and hence

$$\frac{1}{2}\langle g_0 | \tilde{d}w | V_2 \rangle_{\Sigma} = -(\tilde{d}w)_{ss}|_{\Sigma} + (h_0 | (\tilde{d}w)_{s\Sigma}|_{\Sigma} \rangle_{h_0}$$

$$= i\partial_s w_s|_{\Sigma} + \frac{1}{2} \text{tr}(r_0) w_s|_{\Sigma} + (h_0 | d_{\Sigma} w_{\Sigma}|_{\Sigma} \rangle_{h_0} - \frac{1}{2} (h_0 | \partial_t h_0)_{h_0} w_s|_{\Sigma}$$

$$= i\partial_s w_s|_{\Sigma} - \frac{1}{2} \text{tr}(r_0) w_s|_{\Sigma} - \delta_{\Sigma} w_{\Sigma}|_{\Sigma}.$$

Since $\tilde{D}_1 w = 0$ in $\Omega^+$ and $w|_{\partial \Omega^+ \cap \Sigma} = 0$, we have $(-\partial_s w)|_{\Sigma} = N_{\Omega^+} w|_{\Sigma}$ where $N_{\Omega^+}$ is the Dirichlet-to-Neumann map for $\tilde{D}_1$, see 5.3.10. Setting

$$y := w|_{\Sigma}, \quad H = N_{\Omega^+} + i \begin{pmatrix} -\frac{1}{2} \text{tr}(r_0) & -\delta_{\Sigma} \\ -d_{\Sigma} & \frac{1}{2} \text{tr}(r_0) + r_0 \end{pmatrix},$$

we can rewrite (6.15) as

$$H y = \begin{pmatrix} -\frac{1}{2} u_s \\ 2iu_{\Sigma} \end{pmatrix}. \quad (6.16)$$

Next, $H$ belongs to $\Psi^1(\Sigma; \tilde{V}_1)$. By Prop. 5.19 its principal symbol is

$$\sigma_{pr}(H) = (k \cdot h_0^{-1} k)^\frac{3}{2} 1 + i \begin{pmatrix} 0 & -(k|\cdot)|h_0 \\ -k & 0 \end{pmatrix},$$

where we recall that $(x, k)$ are the variables in $T^*\Sigma$. It follows that $H$ is elliptic in $\Psi^1$.

We claim that $H : H^{s+\frac{3}{2}}(\Sigma; \tilde{V}_1) \to H^{s-\frac{3}{2}}(\Sigma; \tilde{V}_1)$ is invertible. Let $Q_H$ be the sesquilinear form $\langle \cdot | H \cdot \rangle_{\tilde{V}_1(\Sigma)}$ with domain $H^{\frac{3}{2}}(\Sigma; \tilde{V}_1)$. Since $H$ is elliptic, $Q_H$ is closed, and in view of Prop. 5.20 $Q_H$ is coercive. By the Lax–Milgram theorem this implies that $H : H^{\frac{3}{2}}(\Sigma, \tilde{V}_1) \to H^{-\frac{3}{2}}(\Sigma; \tilde{V}_1)$, which proves our claim for $s = 0$. For arbitrary $s$ we use the standard argument of commutation with tangential derivatives.

Therefore there exists a unique solution $y \in H^{s+\frac{3}{2}}(\Sigma; \tilde{V}_1)$ of (6.16). We choose $w = P_{\Omega^+} y$, see 5.3.10, i.e. we take $w$ to be the unique solution of the boundary value problem

$$\begin{cases}
\tilde{D}_1 w = 0 \text{ in } \Omega^+, \\
w|_{\partial \Omega^+ \cap \Sigma} = 0, \\
w|_{\Sigma} = v.
\end{cases}$$

By the arguments in 5.3.10, $w$ belongs to $H^{s+1}(\Omega^+; \tilde{V}_1) \cap \overline{H^0(\Omega^+; \tilde{V}_1)}$. This completes the proof of the lemma. \qed
We now give a version of Lem. 6.4 expressed in terms of Cauchy data. Let us introduce the operator
\[ J_2 = \begin{pmatrix} \tilde{I}_2 & 0 \\ 0 & 1 \end{pmatrix}, \]
which satisfies
\[ J_2^* = J_2, \quad J_2^2 = 1, \quad J_2 q_{2, \text{phys}} J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tilde{q}_2. \] (6.17)
In particular, \( J_2 \) transforms the physical charge \( q_{2, \text{phys}} \) into the Euclidean charge \( \tilde{q}_2 \).

Below, we set \( H^\infty(\Omega^\pm; \tilde{V}_i) = \bigcap_{s \in \mathbb{R}} H^s(\Omega^\pm; \tilde{V}_i) \) and we define \( H^\infty(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2) \) similarly.

**Lemma 6.5.** For all \( f \in \ker K_{2,1}|_{C^\infty} \) there exists \( h \in H^\infty(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2) \) and \( k \in H^\infty(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \) such that
\[ \tilde{c}_2^+ f = k + K\Sigma \tilde{c}_1^+ h, \quad J_2 k = k. \]

**Proof.** We prove only the + case. Let \( f \in \ker K_{2,1}|_{C^\infty} \) and
\[ u = -r^+ \tilde{D}_{20}^{-1} \tilde{q}_2^+ \tilde{q}_2 f \in H_0^1(\Omega^+; \tilde{V}_2). \]
We see that \( u \) is the solution of
\[ \begin{cases} \tilde{D}_2 u = 0 \text{ in } \Omega^+, \\ u|_{\partial \Omega^+ \setminus \Sigma} = 0, \\ u|_{\Sigma} = \pi_0 \tilde{c}_2^+ f, \end{cases} \]
where \( \pi_0 : \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \mapsto f_0 \) is the projection on the first component. Commuting tangential derivatives and using that \( \tilde{c}_2^+ f \in H^\infty(\Sigma; \tilde{V} \otimes \mathbb{C}^2) \), we obtain that \( u \in H^\infty(\Omega^+; \tilde{V}) \). Let \( w \) be as in Lem. 6.4 and \( v = u - \tilde{K} w, h = \tilde{q}_1^+ w, k = \tilde{q}_2^+ v \). We have
\[ \tilde{c}_2^+ f = \tilde{q}_2^+ u = k + K\Sigma h \]
since \( K\Sigma = \tilde{K}\Sigma \), see (6.11). The conditions on \( v|_{\Sigma} \) in Lem. 6.4 mean that \( J_2 k = k \).
Moreover \( h \in H^\infty(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2) \) hence \( k \in H^\infty(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \). Finally we claim that \( h = \tilde{c}_1^+ h \).
In fact let \( ew \) the extension of \( w \) by 0 in \( \Omega^- \). Then \( \tilde{D}_1\tilde{c}_1 w = \tilde{q}_1^+ \tilde{q}_1 h \) and \( ew|_{\partial \Omega} = 0 \) so
\[ \tilde{D}_1 \tilde{c}_1 w = \tilde{q}_1^+ \tilde{q}_1 h \text{, hence } h = \tilde{q}_1^+ w = \tilde{q}_1^+ \tilde{D}_1 \tilde{c}_1 w = \tilde{c}_1^+ h. \] This completes the proof of the lemma. \( \square \)

We now prove the positivity of Calderón projectors for the physical charge \( q_{2, \text{phys}} \) on \( \ker K_{2,1}|_{C^\infty} \), modulo a smoothing term arising.

**Proposition 6.6.** Let \( \tilde{c}_2^\pm \) be the Calderón projectors for \( \tilde{D}_2 \). Then
\[ \pm (f|_{q_{2, \text{phys}}} (\tilde{c}_2^\pm + \tilde{r}_2^\pm, -\infty) f) \tilde{v}(\Sigma) \otimes \mathbb{C}^2 \geq 0 \quad \forall f \in \ker K_{2,1}|_{C^\infty} \]
where \( \tilde{r}_2^\pm \in \mathcal{W}^{-\infty}(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \).
Proof. We only prove the ‘+’ case. Let \( f \in \text{Ker} K_\Sigma^1|_{C^\infty} \) and let \( k, h \) be as in Lem. 6.5. We recall that
\[
\tilde{c}_2^+ f = k + K_\Sigma \tilde{c}_1^+ h,
\]
\[
K_\Sigma^1 \tilde{c}_2^+ f = \tilde{c}_1^+ K_\Sigma^1 f + K_{-\infty}^1 f = K_{-\infty}^1 f,
\]
\[
K_\Sigma^1 k = K_\Sigma^1 (\tilde{c}_2^- f - K_\Sigma \tilde{c}_1^+ h) = K_{-\infty}^1 f.
\]
Since \((\tilde{c}_2^+)^2 = \tilde{c}_2^+ \) on \( \mathcal{H}^\infty(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \) and \( \tilde{c}_2^- = (\tilde{c}_2^+)^\dagger \) we obtain
\[
(f \mid q_2, \text{phys} \tilde{c}_2^+ f)_{\tilde{V}_2(\Sigma) \otimes \mathbb{C}^2} = (\tilde{c}_2^+ f \mid q_2, \text{phys} \tilde{c}_2^+ f)_{\tilde{V}_2(\Sigma) \otimes \mathbb{C}^2}.
\]
Next, \( \tilde{k} := \tilde{c}_2^+ (f - K_\Sigma \tilde{c}_1^+ h) = \tilde{c}_2^+ (f - K_\Sigma h) \), hence \( \tilde{k} = \tilde{c}_2^+ v \) for \( v = \tilde{D}_2 \tilde{\sigma}_2 (f - K_\Sigma h) \).
Since \( \tilde{D}_2 v = 0 \) in \( \Omega^+ \) with \( v|_{\partial \Omega^+ \cap \Sigma} = 0 \) we obtain by Green’s formula (5.27) that
\[
(\tilde{k} \mid \tilde{q}_2 \tilde{k})_{\tilde{V}_2(\Sigma)} = 2 \text{Re} Q_{\Omega^+}(v, v) \geq 0,
\]
where the positivity follows from coercivity of \( Q_{\Omega^+} \), see (5.22). Finally,
\[
k = \tilde{k} - \tilde{c}_2^+ K_\Sigma \tilde{c}_1^+ h = \tilde{k} - \tilde{c}_2^- K_{-\infty} h.
\]
In conclusion, \((f \mid q_2, \text{phys} \tilde{c}_2^+ f)_{\tilde{V}_2(\Sigma) \otimes \mathbb{C}^2} \) equals (6.18) modulo
\[
(f \mid q_2, \text{phys} \tilde{r}_{2, -\infty}^+ f)_{\tilde{V}_2(\Sigma) \otimes \mathbb{C}^2},
\]
where \( \tilde{r}_{2, -\infty}^+ \in \mathcal{W}^{-\infty}(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \) is the sum of terms obtained by composition of \( K_{-\infty} \) with \( \tilde{c}_i^+, \) \( i = 1, 2, K_\Sigma, J_2 \), and the Poisson operator from Lem. 6.4. \( \square \)

APPENDIX A.

A.1. Symbolic fixed point. We recall a useful way to solve recursive equations that are often encountered in symbolic calculus. We refer the reader to [25, Lem. A.1] for the proof.

Proposition A.1. Suppose \( F : C^\infty_b(I; \Psi^\infty(M; V)) \to C^\infty_b(I; \Psi^\infty(M; V)) \) is a map such that:

i) \( F : C^\infty_b(I; \Psi^0(M; V)) \to C^\infty_b(I; \Psi^{-1}(M; V)) \),

ii) \( b_1 - b_2 \in C^\infty_b(I; \Psi^{-j}(M; V)) \implies F(b_1) - F(b_2) \in C^\infty_b(I; \Psi^{-j-1}(M; V)) \) \( \forall j \in \mathbb{N} \).

Moreover, let \( a \in C^\infty_b(I; \Psi^0(M; V)) \). Then there exists a solution \( b \in C^\infty_b(I; \Psi^0(M; V)) \), unique modulo \( C^\infty_b(I; \Psi^{-\infty}(M; V)) \), of the equation
\[
b = a + F(b) \mod C^\infty_b(I; \Psi^{-\infty}(M; V)).
\]
A.2. **Bounded analytic geometry.** Let us first prove the claim in 3.4.3 about the independence of the spaces \( AT^p_q(M, \hat{g}) \) in Def. 3.7 on the choice of the bounded analytic atlas \( \{(U_x, \psi_x)\}_{x \in M} \).

Clearly to define an manifold \((M, \hat{g})\) of bounded analytic geometry, there is some freedom in the ranges of \( \psi_x \) and the holomorphy domain of \( \hat{g}_x \). In fact we can require equivalently in Def. 3.6 that \( \psi_x : U_x \cong B_n(0, 1) \) and

1. \( \{\hat{g}_x\}_{x \in M} \) is bounded in \( BT(B_n(0, 1), \delta) \),
2. \( c^{-1} \delta \leq \hat{g}_x \leq c \delta \) uniformly in \( x \in M \),
3. \( \{\hat{g}_x\}_{x \in M} \) is bounded in \( AT(C_n(0, \epsilon), \delta) \) for some \( \epsilon > 0 \).

The next proposition is the analog of [23, Thm. 2.4]. It implies that as \( \{(U_x, \psi_x)\}_{x \in M} \) one can take \( U_x = B^g_M(x, \epsilon) \), for some \( \epsilon > 0 \) small enough, and \( \psi^{-1}_x = \exp_{\hat{x}}^g \circ e_x \), where \( e_x : (\mathbb{R}^n, \delta) \to (T_M, \hat{g}(x)) \) is a linear isometry. It follows from Prop. A.2 that the definition of the spaces \( AT^p_q(M, \hat{g}) \) in Def. 3.7 is indeed independent on the choice of the bounded analytic atlas \( \{(U_x, \psi_x)\}_{x \in M} \), since we can always take the geodesic maps \( \exp_{\hat{x}}^g \circ e_x \) instead of \( \psi^{-1}_x \).

**Proposition A.2.** There exists \( \epsilon, \epsilon_1, \epsilon_2 > 0 \) such that if

\[
\chi_x := \psi_x \circ \exp_{\hat{x}}^g \circ e_x : B_n(0, \epsilon) \to \psi_x(B^g_M(x, \epsilon))
\]

then the family \( \{\chi_x\}_{x \in M} \) is bounded in \( AT(C_n(0, \epsilon_1)) \), \( C_n(0, \epsilon_2) \subset \chi_x(C_n(0, \epsilon_1)) \) and the family \( \{\chi^{-1}_x\}_{x \in M} \) is bounded in \( AT(C_n(0, \epsilon_2)) \).

**Proof.** We choose \( \epsilon \) as in the proof of [23, Thm. 2.4], and \( \epsilon_1 \) such that \( C_n(0, 2\epsilon_1) \subset B_n(0, \epsilon) \) and (3) above holds for \( 2\epsilon_1 \).

We have \( \psi_x \circ \exp_{\hat{x}}^g \circ e_x = \exp_{\hat{x}}^g \circ T_x \), for \( T_x = D_x \psi_x \circ e_x \in L(\mathbb{R}^n) \). Using that \( T_x \hat{g}_x(0)T_x = \delta \) and (2) we obtain that \( \{T_x\}_{x \in M} \) is bounded in \( L(\mathbb{R}^n) \).

Denoting by \( \Gamma^k_{ij} \) the Christoffel symbols of the metric \( \hat{g}_x \), \( x(t) = \exp_{\hat{x}}^g (tv) \) solves the geodesic equations:

\[
\begin{align*}
\dot{x}^k(t) &= \Gamma^k_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t), \\
x(0) &= 0, \quad \dot{x}(0) = v.
\end{align*}
\]

Since \( \{\hat{g}_x\}_{x \in M} \) is bounded in \( AT(C_n(0, \epsilon_1), \delta) \), the family \( \{\Gamma^k_{ij}\}_{x \in M} \) is bounded in \( AT(C_n(0, 2\epsilon_1)) \) and we deduce from the Cauchy–Kowalevski theorem that the family of diffeomorphisms \( \{\exp_{\hat{x}}^g\}_{x \in M} \) is bounded in \( AT(C_n(0, \epsilon_1)) \). The same is true for \( \chi_x = \exp_{\hat{x}}^g \circ T_x \), since \( \{T_x\}_{x \in M} \) is bounded in \( L(\mathbb{R}^n) \). Since by (2) \( D\chi_x(0) \) is uniformly bounded in \( L(\mathbb{R}^n) \), we can find \( \epsilon_2 > 0 \) such that \( C_n(0, \epsilon_2) \subset \chi_x(C_n(0, \epsilon_1)) \) and the family \( \{\chi^{-1}_x\}_{x \in M} \) is bounded in \( AT(C_n(0, \epsilon_2)) \).

**A.2.1. Proof of Thm. 3.9.** Without loss of generality we can assume that \( U = M \). All statements are already proved in Thm. 3.3, except for the analyticity.

For \( x \in \Sigma \) we choose \( U_x, \psi_x \) such that if \( \Sigma_x = \psi_x(\Sigma \cap U_x) \), then \( \Sigma_x \cap C_n(0, \epsilon) = C_n(0, \epsilon) \cap \{v_n = 0\} \). If \( g_x = (\psi^{-1}_x)_{*}^* g \) and \( n_x \) is the future unit normal vector field to \( \Sigma_x \) for \( g_x \), then we can decompose \( n_x \) as \( n_x = n_x' + \lambda_x e_n \). The families \( \{g_x\}_{x \in \Sigma}, \{g^{-1}_x\}_{x \in \Sigma}, \{n_x'\}_{x \in \Sigma} \) and \( \{\lambda_x\}_{x \in \Sigma} \) are bounded in \( AT^2_2(C_{n-1}(0, \epsilon)), AT^2_0(C_{n-1}(0, \epsilon), \delta), AT^2_0(C_{n-1}(0, \epsilon), \delta) \) and
\[ \mathcal{A}T_0^\delta(C_{n-1}(0, \epsilon), \delta) \] respectively. We introduce the normal geodesic flow to \( \Sigma_x \) for \( g_x \):

\[
\chi_x : -\epsilon_1, \epsilon_1 [\times C_{n-1}(0, \epsilon_1) \to C_n(0, \epsilon) \]

\[
(t, v') \mapsto \exp_{g_x}^{g_x}(t n_x(v', 0))
\]

By the Cauchy–Kowalevski theorem we obtain that for \( \epsilon_1 \) small enough \( \{\chi_x\}_{x \in \Sigma} \) is bounded in \( \mathcal{A}T(C_n(0, \epsilon_1), \delta) \). We have \( \chi_x^* g_x = -dt^2 + h_x(t, v')dv'^2 \), and \( \{h_x\}_{x \in \Sigma}, \{h_x^{-1}\}_{x \in \Sigma} \) are bounded in \( \mathcal{A}T_0^\delta(C_n(0, \epsilon_1), \delta) \) and \( \mathcal{A}T_0^\delta(C_n(0, \epsilon_1), \delta) \) respectively, using the same properties of \( \chi_x, g_x, g_x^{-1} \) recalled above.

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REFERENCES

[1] B. Allen, A. Folacci, and A. C. Ottewill. Renormalized graviton stress-energy tensor in curved vacuum space-times. Phys. Rev. D, 38(4):1069–1082, 1988.
[2] M. T. Anderson. On boundary value problems for Einstein metrics. Geom. Topol., 12(4):2009–2045, 2008.
[3] A. Ashtekar and A. Magnon-Ashtekar. On the symplectic structure of general relativity. Commun. Math. Phys., 86(1):55–68, 1982.
[4] C. Bär and N. Ginoux. Classical and Quantum Fields on Lorentzian Manifolds. In C. Bär, J. Lohkamp, and M. Schwarz, editors, Glob. Differ. Geom. Springer Proc. Math. 17, pages 359–400. Springer-Verlag, 2012.
[5] C. Bär, N. Ginoux, and F. Pfäffle. Wave Equations on Lorentzian Manifolds and Quantization. European Mathematical Society Publishing House, Zürich, 2007.
[6] M. Benini, C. Dappiaggi, and S. Murro. Radiative observables for linearized gravity on asymptotically flat spacetimes and their boundary induced states. J. Math. Phys., 55(8):082301, 2014.
[7] J. Borthwick. Maximal Kerr–de Sitter spacetimes. Class. Quantum Gravity, 35(21):215006, 2018.
[8] R. Brunetti, K. Fredenhagen, T.-P. Hack, N. Pinamonti, and K. Rejzner. Cosmological perturbation theory and quantum gravity. J. High Energy Phys., 2016(8):32, 2016.
[9] R. Brunetti, K. Fredenhagen, and K. Rejzner. Quantum gravity from the point of view locally covariant quantum field theory. Commun. Math. Phys., 345(3):741–779, 2016.
[10] C. Dappiaggi, V. Moretti, and N. Pinamonti. Hadamard States from Light-like Hypersurfaces, volume 25 of SpringerBriefs in Mathematical Physics. Springer International Publishing, Cham, 2017.
[11] C. Dappiaggi and D. Siemssen. Hadamard states for the vector potential on asymptotically flat spacetimes. Rev. Math. Phys., 25(01):1350002, 2013.
[12] J. J. Duistermaat and L. Hörmander. Fourier integral operators. II. Acta Math., 128:183–269, 1972.
[13] C. J. Fewster and D. S. Hunt. Quantization of linearized gravity in cosmological vacuum spacetimes. Rev. Math. Phys., 25(02):1330003, 2013.
[14] C. J. Fewster and M. J. Pfennig. A quantum weak energy inequality for spin-one fields in curved space-time. J. Math. Phys., 44(10):4480, 2003.
[15] C. J. Fewster and R. Verch. The necessity of the Hadamard condition. Class. Quantum Gravity, 30(23):235027, 2013.
[16] F. Finster and A. Strohmaier. Gupte–Bleuler quantization of the Maxwell field in globally hyperbolic space-times. Ann. Henri Poincaré, 16(8):1837–1868, 2015.
[17] K. Fredenhagen and K. Rejzner. Batalin-Vilkovisky formalism in perturbative Algebraic Quantum Field Theory. Commun. Math. Phys., 317(3):697–725, 2013.
[18] K. Fritzsch, D. Grieser, and E. Schrohe. The Calderón projector for fibred cusp operators. *J. Funct. Anal.*, 285(10):110127, 2023.

[19] S. Fulling, F. Narcowich, and R. M. Wald. Singularity structure of the two-point function in quantum field theory in curved spacetime, II. *Ann. Phys. (N. Y.)*, 136(2):243–272, 1981.

[20] E. P. Furlani. Quantization of the electromagnetic field on static space–times. *J. Math. Phys.*, 36(3):1063–1079, 1995.

[21] C. Gérard. The Hartle–Hawking–Israel state on spacetimes with stationary bifurcate Killing horizons. *Rev. Math. Phys.*, 33(08):2150028, 2021.

[22] C. Gérard, D. Häfner, and M. Wrochna. The Unruh state for massless fermions on Kerr spacetime and its Hadamard property. *arXiv:2008.10993*, 2020.

[23] C. Gérard, O. Oulghazi, and M. Wrochna. Hadamard states for the Klein–Gordon equation on Lorentzian manifolds of bounded geometry. *Commun. Math. Phys.*, 2017.

[24] C. Gérard and T. Stoskopf. Hadamard states for quantized Dirac fields on Lorentzian manifolds of bounded geometry. *Rev. Math. Phys.*, 34(04), 2022.

[25] C. Gérard and M. Wrochna. Construction of Hadamard states by pseudo-differential calculus. *Commun. Math. Phys.*, 325(2):713–755, 2014.

[26] C. Gérard and M. Wrochna. Hadamard states for the linearized Yang–Mills equation on curved spacetime. *Commun. Math. Phys.*, 337(1):253–320, 2015.

[27] C. Gérard and M. Wrochna. Analytic Hadamard states, Calderón projectors and Wick rotation near analytic Cauchy surfaces. *Commun. Math. Phys.*, 366(1):29–65, 2019.

[28] J. Guven and D. Núñez. Schwarzschild-de Sitter space and its perturbations. *Phys. Rev. D*, 42(8):2577–2584, 1990.

[29] T.-P. Hack and A. Schenkel. Linear bosonic and fermionic quantum gauge theories on curved spacetimes. *Gen. Relativ. Gravit.*, 45(5):877–910, 2013.

[30] D. Häfner, P. Hintz, and A. Vasy. Linear stability of slowly rotating Kerr black holes. *Invent. Math.*, 223(3):1227–1406, 2021.

[31] A. Higuchi, D. Marolf, and I. A. Morrison. de Sitter invariance of the dS graviton vacuum. *Class. Quantum Gravity*, 28(24):245012, 2011.

[32] P. Hintz and A. Vasy. The global non-linear stability of the Kerr–de Sitter family of black holes. *Acta Math.*, 220(1):1–206, 2018.

[33] S. Hollands. Renormalized quantum Yang-Mills fields in curved spacetime. *Rev. Math. Phys.*, 20(09):1033–1172, 2008.

[34] L. Hörmander. *The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis*. Springer Verlag, Berlin, second edition, 1990.

[35] W. Junker. Hadamard states, adiabatic vacua and the construction of physical states for scalar quantum fields on curved spacetime. *Rev. Math. Phys.*, 08(08):1091–1159, 1996.

[36] T. Kato. Integration of the equation of evolution in a Banach space. *J. Math. Soc. Japan*, 5(2):208–234, 1953.

[37] T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin Heidelberg, 2nd edition, 1995.

[38] I. Khavkine. Characteristics, conal geometry and causality in locally covariant field theory. *arXiv:1211.1914*, 2012.

[39] M. Kontsevich and G. Segal. Wick rotation and the positivity of energy in quantum field theory. *Q. J. Math.*, 72(1-2):673–699, 2021.

[40] S. P. Miao, P. J. Mora, N. C. Tsamis, and R. P. Woodard. Perils of analytic continuation. *Phys. Rev. D*, 89(10):104004, 2014.

[41] V. Moncrief. Decompositions of gravitational perturbations. *J. Math. Phys.*, 16(8):1556–1560, 1975.

[42] B. O’Neill. *The Geometry of Kerr Black Holes*. Dover Publications, 2014.

[43] K. Rejzner. *Perturbative Algebraic Quantum Field Theory*. Mathematical Physics Studies. Springer International Publishing, Cham, 2016.

[44] H. Ringström. *The Cauchy Problem in General Relativity*. European Mathematical Society Publishing House, Zürich, 2009.

[45] H. Sahlmann and R. Verch. Microlocal spectrum condition and Hadamard form for vector-valued quantum fields in curved spacetime. *Rev. Math. Phys.*, 13(10):1203–1246, 2001.
[46] P. Schapira. Wick rotation for D-modules. *Math. Physics, Anal. Geom.*, 20(3):21, 2017.

[47] J. Schmid and M. Griesemer. Kato’s theorem on the integration of non-autonomous linear evolution equations. *Math. Physics, Anal. Geom.*, 17(3-4):265–271, 2014.

[48] M. A. Shubin. Spectral theory of elliptic operators on non-compact manifolds. In *Méthodes semi-classiques Vol. 1 - École d’Été (Nantes, juin 1991)*, number 207 in Astérisque. Société mathématique de France, 1992.

[49] E. Witten. A note on boundary conditions in Euclidean gravity. *Rev. Math. Phys.*, 33(10), 2021.

[50] M. Wrochna. Wick rotation of the time variables for two-point functions on analytic backgrounds. *Lett. Math. Phys.*, 110(3):585–609, 2020.

[51] M. Wrochna and J. Zahn. Classical phase space and Hadamard states in the BRST formalism for gauge field theories on curved spacetime. *Rev. Math. Phys.*, 29(04):1750014, 2017.

[52] V. Wünsch. Cauchy’s problem and Huygens’ principle for relativistic higher spin wave equations in an arbitrary curved space-time. *Gen. Relativ. Gravit.*, 17(1):15–38, 1985.