Continued fractions in non-Euclidean imaginary quadratic fields

Daniel E. Martin

Abstract. In the Euclidean imaginary quadratic fields, continued fractions have been used to give rational approximations to complex numbers since the late 19th century. A variety of algorithms have been proposed in the 130 years following their introduction, but none are applicable outside of the same five fields. Here we overcome the non-Euclidean obstacle. We show how continued fractions can be produced in any imaginary quadratic field, and we prove that they share many of the properties enjoyed by their classical forebear. The inspiration for the algorithm is a fractal arrangement of circles arising from subsets of GL_2(C) acting on the Riemann sphere. The geometry of these arrangements reveals an analog of the Euclidean algorithm that points us toward a more general continued fraction.

1. Introduction

In 1887 A. Hurwitz introduced complex continued fractions [3] when he investigated the nearest integer algorithm over \( \mathbb{Z}[i] \). Briefly stated, we begin with a point to be approximated, say \( z = z_0 \in \mathbb{C} \), and let \( a_n \) (the \( n \)th coefficient) be the nearest integer to \( z_n = 1/(z_{n-1} - a_{n-1}) \). Among his results is that the approximations,

\[
\frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_{n-1}}}},
\]

converge exponentially to \( z \), and their denominators grow exponentially in norm. The proofs of these facts rely heavily on the size of \( z_n \). Open discs of radius 1 centered at the lattice points of \( \mathbb{Z}[i] \) cover \( \mathbb{C} \), so \( z_n \) is necessarily inside the one centered at some \( a_{n-1} \). Thus when \( z_{n-1} - a_{n-1} \) is reflected over the unit disc at the origin, it moves from interior to exterior, meaning \( \|z_n\| \) is larger than some fixed constant greater than 1 (in the Gaussian case it is 2). The same is true whenever unit discs on the ring of integers cover the plane, motivating use of the algorithm in each of the five Euclidean imaginary quadratic fields. See [5], for example.

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Our purpose is to apply an algorithm with the same subtract and reciprocate structure in any imaginary quadratic field, selecting $p_n$ and $q_n$ from its ring of integers. This is achieved with Algorithm 5.1. The majority of our effort is spent remediying the lack of a covering by unit discs, which has been the obstacle to generalization since the complex continued fraction endeavor began. Once this is achieved in Section 4, we prove in Section 5 that our continued fractions satisfy the following properties, which form a standard litmus test [2] for quality of approximation. The constants do not depend on $n$ or $z$.

- The sequence $(\|q_n\|)_n$ is bounded below by an exponential function, and though it may not increase monotonically, $\|q_{n-1}\| < c\|q_n\|$ (Proposition 5.9).
- Each $p_n/q_n$ satisfies $c/\|a_nq_n^2\| < \|z - p_n/q_n\| < c'/\|a_nq_n^2\|$ (Proposition 5.9),
- and it is a best approximation of the second kind up to constants. That is, for $p, q \in \mathcal{O}_K$, if $\|q\| < c\|q_{n+1}\|$ then $\|q_nz - p_n\| < c'\|qz - p\|$ (Proposition 5.8).

Our coefficients are also shown to retain several classical properties.

- The sequence $(a_n)_n$ is finite if and only if $z \in K$ (Corollary 5.4),
- $(a_n)_n$ is eventually periodic if and only if $[K(z) : \overline{K}] = 2$ (Proposition 5.11),
- and $(a_n)_n$ is bounded if and only if $z$ is badly approximable (Corollary 5.10).

To our knowledge, such an algorithm only exists in the literature for six number fields—the rational numbers and the Euclidean imaginary quadratic fields. Let us further examine the problem in the remaining imaginary quadratic fields.

To maintain the structure of the algorithm, the point $z_n$ is reflected over the boundary of a disc centered at an integer. Covering $\mathbb{C}$ with discs of radius larger than 1 seems to be a natural way to achieve this while maintaining $\|z_n\| > 1$, which is a key to exponential convergence. The issue is gleaned from the identity

$$z - \frac{p_n}{q_n} = \frac{p_nz_n + p_{n-1}}{q_nz_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{p_{n-1}q_n - p_nq_{n-1}}{q_n(q_nz_n + q_{n-1})}.$$ 

The denominator resembles the desired expression, $a_nq_n^2$, in the second bullet above, so the numerator must stay small to guarantee good approximations (classically, it remains $\pm 1$). Unfortunate for this attempt is that a reflection over the disc of radius $b$ inflates $p_{n-1}q_n - p_nq_{n-1}$ by $b^2$ at every stage, seemingly disallowing the use of larger discs. We will do this nevertheless, just with a catch. Larger discs are only given to those lattice points that earn them in a way that will be made precise at the heart of these notes in Section 4.
Figure 1. The Schmidt arrangements of $\mathbb{Q}(\sqrt{-2})$ (left) and $\mathbb{Q}(\sqrt{-19})$ (right) centered on a fundamental region for $\mathcal{O}_K$. Circles with curvature at most 18 are shown.

The source of this idea is surprising. It so happens that the arithmetic behind the Hurwitz algorithm lives geometrically in an ornate arrangement of circles contained in the complex plane. The cases $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-19})$ are displayed in Figure 1.

For an imaginary quadratic field $K$ with ring of integers $\mathcal{O}_K$, the Schmidt arrangement of $K$ is produced by stereographically projecting the orbit of $\mathbb{RP}^1$ under the action of $\text{SL}_2(\mathcal{O}_K)$ via the correspondence to Möbius transformations,

$$
\begin{bmatrix}
\alpha & \gamma \\
\beta & \delta
\end{bmatrix}
\mapsto z \mapsto \frac{\alpha z + \gamma}{\beta z + \delta}.
$$

The study of these arrangements over a general imaginary quadratic field was pioneered by Stange [13]. In the Euclidean cases they were originally employed by A. Schmidt in [9, 12, 11, 10] to provide a continued fraction algorithm, but not of the nearest integer variety. Instead an arrangement is used to partition the complex plane into successively smaller pieces. Schmidt showed that the intersection points around the boundary of each piece approximate the points

Figure 2. The initial partition of the Schmidt arrangement of $\mathbb{Q}(\sqrt{-11})$. 

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in its interior. In [14] Stange provides details for $\mathbb{Q}(\sqrt{-11})$, among others. For this field, the initial partition can be seen in Figure 2 to consist of hexangles and circles with either two or three highlighted intersections along their boundaries.

A similar idea applies to the Schmidt arrangement of $\mathbb{Q}(\sqrt{-2})$. The initial partition can be visualized in Figure 1 to consists of circles and quadrangles. In $\mathbb{Q}(\sqrt{-19})$, the arrangement (also shown in Figure 1) partitions the plane, but not usefully. One piece in particular is unbounded to its left and right, making it unhelpful in narrowing down the location of its interior points. This issue occurs in all of the non-Euclidean fields [13].

To see the relevance to the nearest integer algorithm, note that

$$\left[ \begin{array}{cc} p_n & p_{n-1} \\ q_n & q_{n-1} \end{array} \right] \quad \text{and} \quad \left[ \begin{array}{cc} p_{n+1} & p_n \\ q_{n+1} & q_n \end{array} \right]$$

both map a point of $\mathbb{RP}^1$, namely 0 and $\infty$, respectively, to $p_n/q_n$. This means the circles corresponding to consecutive matrices of this form intersect at the approximations output by the nearest integer algorithm. So when $\mathcal{O}_K$ is Euclidean, the Schmidt arrangement becomes a road map to our destination of choice in the complex plane. At each fork in the road we find an approximation that meets the standard previously outlined.

![Figure 3](image-url)

**Figure 3.** The convergents for the Hurwitz continued fraction expansion of $z = \frac{4}{7-2i}$ are intersection points in the Gaussian Schmidt arrangement. Details are given in Section 3.

In the non-Euclidean fields the failure of the nearest integer algorithm manifests in a disconnected Schmidt arrangement [13]. There are points in the plane that
cannot be accessed by traveling along the connected component of the real line.

When the class group is nontrivial the Schmidt arrangement exhibits another insufficiency—it does not cover every point in $K$. Thinking of $p/q \in \mathbb{Q}$ as a column vector, the ideal $(p, q) \subseteq \mathcal{O}_K$ is preserved when multiplying by a matrix in $\text{SL}_2(\mathcal{O}_K)$. But ideals in $\mathbb{Z}$ are principal, so for every $\alpha/\beta \in K$ on the Schmidt arrangement, $(\alpha, \beta)$ represents the identity class. A continued fraction algorithm should terminate in finitely many steps given any rational input, not just those corresponding to a principal ideal.

Our goal is to extend the Schmidt arrangement until it is connected and covers every point of $K$. This is done in the next section, which is likely the one containing the most foreign content. The reader is encouraged to pursue, however, as understanding an extended Schmidt arrangement makes the work that follows far more intuitive. Our study of its geometry ultimately shows us how to loosen the requirement for being Euclidean, and this furnishes the algorithm.

As an auxiliary result, the search for these arrangements also provides a new approach to a question answered in [7]. To our knowledge, Corollary 5.15 gives the smallest known set $S \subset \mathcal{O}_K$ for which $S^{-1}\mathcal{O}_K$ is norm-Euclidean. This result, as well as the last three propositions, is intended to highlight the first half of Section 4 as a natural generalization of Euclideaneity.

A note on the figures. They were produced with software created by the author, hosted at math.colorado.edu/~dmartin/arrangements.

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2. Extended Schmidt arrangements

Let us fix notation and definitions. We will take $\Delta$ as the discriminant of $K$ and $\tau$ as either $\frac{1}{2}\sqrt{\Delta}$ or $\frac{1}{2}(1 + \sqrt{\Delta})$, depending on $\Delta \mod 2$. In Section 5, $p_n$ and $q_n$ will denote the entries of our matrices, but until then the following is more convenient.

Notation 2.1. A generic matrix $M \in \text{GL}_2(K)$ will be written

$$\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}.$$ 

Subscripts and further decoration of the symbol “$M$” will be inherited by its entries. Let $(M)$ denote the ideal $(\alpha, \beta, \gamma, \delta)$. 


The arrangements we will study arise by post-composing such matrices with the projection \([z : 1] \mapsto z\) and collecting the images of \(\mathbb{RP}^1\). With respect to this projection we give \(\mathbb{RP}^1\) positive orientation (the upper-half plane is to the left of the direction of travel) thereby giving everything in its orbit an orientation. We will maintain the convention of calling counterclockwise orientation positive.

**Notation 2.2.** For a matrix \(M \in \text{GL}_2(K)\), let \(M^o \subset \mathbb{C}\) denote the projected image of \(\mathbb{RP}^1\) under \(M\). If \(S\) is a set of matrices then \(S^o\) will denote \(\{M^o \mid M \in S\}\).

So \(M^o\) consists of points of the form \((\alpha x + \gamma)/(\beta x + \delta)\) for \(x \in \mathbb{R}\) (when it is well-defined) as well as \(\alpha/\beta\). In particular \(\text{GL}_2(\mathbb{Q})^o = \{\mathbb{R}\}\), and in fact for any matrix \(M \in \text{GL}_2(K)\), \(M^o = \mathbb{R}\) if and only if \(M\) is a scalar multiple of a matrix in \(\text{GL}_2(\mathbb{Q})\). We also see a piece of \(\text{SL}_2(\mathcal{O}_K)^o\) for four different fields in Figures 1, 2, and 3.

**Definition 2.3.** The *cocurvature* of \(M^o\) is its (Gaussian) curvature after post-composition with the projection \([1 : z] \mapsto z\) instead of \([z : 1] \mapsto z\). If the curvature of \(M^o\) is nonzero, the *curvature-center* is the product of its curvature and center.

Each of these values has a formula in terms of the entries of \(M\). The following was proved in [13] for the case \(\text{SL}_2(\mathcal{O}_K)\), and it applies more generally to \(\text{GL}_2(\mathbb{C})\).

**Proposition 2.4.** For \(M \in \text{GL}_2(K)\) the curvature, cocurvature, and curvature-center (when applicable) of \(M^o\) are given by

\[
\frac{i(\beta \bar{\delta} - \bar{\beta} \delta)}{\sqrt{||\det M||}}, \quad \frac{i(\alpha \bar{\gamma} - \bar{\alpha} \gamma)}{\sqrt{||\det M||}}, \quad \text{and} \quad \frac{i(\alpha \bar{\delta} - \beta \gamma)}{\sqrt{||\det M||}}.
\]

**Proof.** We have \(z \in M^o\) if and only if the Möbius transformation corresponding to \(M^{-1}\) maps \([z : 1]\) to a point in \(\mathbb{RP}^1\). So an equation defining \(M^o\) is

\[
\Im \left( \frac{\delta z - \gamma}{-\beta z + \alpha} \right) = 0.
\]

This defines a line in the complex plane (with curvature 0) if and only if \(\Im(\beta \delta) = 0\). Otherwise, multiplying by \(-\beta z + \alpha\), completing the squares for \(\Re(z)\) and \(\Im(z)\), and using

\[
||\det M|| = ||\alpha \bar{\delta} - \beta \gamma|| + (\alpha \bar{\gamma} - \bar{\alpha} \gamma)(\beta \delta - \bar{\beta} \delta)
\]

allow us to rewrite (3) in the standard form for the equation of a circle as follows:

\[
\left| \frac{z - \alpha \bar{\delta} - \beta \gamma}{\beta \delta - \bar{\beta} \delta} \right| = \left| \frac{\det M}{i(\beta \delta - \bar{\beta} \delta)} \right|.
\]
Recalling that the magnitude of its curvature is the reciprocal of a circle’s radius, we
now need only verify the sign. The point of $C$ (expressed as a column vector) that
maps to the center of $M^o$ is

$$M^{-1} \begin{bmatrix} \alpha \delta - \beta \gamma \\ \beta \delta - \bar{\beta} \bar{\delta} \end{bmatrix} = \begin{bmatrix} -\beta \delta \\ \| \beta \| \end{bmatrix}.$$  

This is in the interior of the real axis, which is the upper half-plane, if and only if
$-2\Im(\beta \delta) = i(\beta \delta - \beta \delta)$ is positive. The sign of the curvature-center must then be
correct as well since the quotient is the center of the circle.

Finally, the cocurvature is the curvature after swapping the rows of $M$, so we
replace $\beta$ and $\delta$ with $\alpha$ and $\gamma$ in the formula.

For those $M^o$ with curvature 0, we take the expression in Proposition 2.4 as the
definition of its curvature-center.

We will consider the projected orbits of $\mathbb{RP}^1$ under sets of matrices with entries
from $O_K$ that, in a sense, all share the same determinant. Corresponding to the
original Schmidt arrangement is the set with fixed determinant 1, $\text{SL}_2(O_K)$. Since
scaling $M$ has no effect on $M^o$, we could take the set more broadly to be those
matrices whose entries are divisible by some $a \in O_K$ that squares to the determinant.
In general $(M)$ may not be principal, motivating the replacement of the element $a$
with an ideal $a \subseteq O_K$ containing the entries of $M$.

**Definition 2.5.** For an ideal $\mathcal{D}$, let $\mathcal{M}_\mathcal{D} \subset \text{GL}_2(K)$ consist of those matrices, $M$,
for which $(\det M)/\mathcal{D}$ is a perfect square that contains $(M)^2$.

The perfect square (since here the determinant is a quadratic polynomial in the
matrix entries) is “$a$” from the discussion above, and $\mathcal{D}$ is the “shared determinant.”
Morally, we want matrices

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix},$$

whose integral ideal entries belong to the same class and satisfy $ad - bc = \mathcal{D}$. Definition 2.5 is just a working version of this.

**Proposition 2.6.** The set $\mathcal{M}_\mathcal{D}$ is nonempty if and only if $\mathcal{D}$ is an integral representa-
tive of an ideal class in the principal genus.

*Proof. Suppose $\mathcal{D} \subseteq O_K$ and that $a^2 \mathcal{D} = (\eta)$ for some $\eta \in O_K$. Choosing $\alpha$ and $\beta$
to generate $a$ means we can find $\gamma, \delta \in a \mathcal{D}$ so that det $M = \alpha \delta - \beta \gamma = \eta$. Then since
$a = (\alpha, \beta) \subseteq (M) \subseteq a \mathcal{D}$ we get $a = (M)$ giving $(\det M)/(M)^2 \mathcal{D} = (\eta)/a^2 \mathcal{D} = O_K$.
Thus $M \in \mathcal{M}_\mathcal{D}$. Conversely, if $M \in \mathcal{M}_\mathcal{D}$ then $\mathcal{D} \subseteq (\det M)/(M)^2 \subseteq O_K$ and
$[\mathcal{D}/(\det M)] = [\mathcal{D}]$ is a perfect square. 

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Definition 2.7. Provided it is nonempty, the set $M^o_D$ is called an extended Schmidt arrangement.

While these sets are not groups under multiplication when $\mathcal{D}$ is not $\mathcal{O}_K$, they have left and right $\mathcal{M}_{\mathcal{O}_K}$ group actions which we frequently employ. This is a consequence of the next proposition.

Proposition 2.8. If $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{O}_K$ are coprime then $(MM') = (M)(M')$ for any $M \in M_{\mathcal{D}}$ and $M' \in M_{\mathcal{D}'}$. In particular, $M_{\mathcal{D}}M_{\mathcal{D}'} = M_{\mathcal{D}\mathcal{D}'}$.

Proof. Scale $M$ and $M'$ so that $(M), (M')$, and $\mathcal{D}\mathcal{D}'$ are pairwise coprime. Then

$$(MM') \supseteq ((M) \det M', (M') \det M) \supseteq (M)(M')(\mathcal{D}', (M')\mathcal{D}) = (M)(M') \supseteq (MM'),$$

which gives $(MM') = (M)(M')$. Now scale back to get the original two matrices, preserving this equality of ideals.

For the second claim, the inclusion $M_{\mathcal{D}}M_{\mathcal{D}'} \subseteq M_{\mathcal{D}\mathcal{D}'}$ now follows since the determinant and the ideal generated by the entries are multiplicative. The reverse inclusion is true even without coprimality. For $M' \in M_{\mathcal{D}\mathcal{D}'}$, we can take $M \in M_{\mathcal{D}}$.
congruent to the adjugate of $M'$ modulo $\mathfrak{D}$. Assuming without loss of generality (scale if necessary) that $(\det M/\mathfrak{D}, \mathfrak{D}) = (\det M'/\mathfrak{D}', \mathfrak{D}) = \mathcal{O}_K$, we see that

$$(MM')^2 \subseteq \mathfrak{D}^2(M)^2(M')^2 \subseteq \mathfrak{D}^2$$

$$= (\det M') \frac{\mathfrak{D}}{\mathfrak{D}'},$$

implying $MM' \in \mathcal{M}_{\mathfrak{D}'}$. Thus $M' = M^{-1}MM' \in \mathcal{M}_{\mathfrak{D}}\mathcal{M}_{\mathfrak{D}'}$.

Taking this proposition alongside the observation that $(M')^2 = (\det M')$ for any $M' \in \mathcal{M}_{\mathfrak{D}K}$, and we see that the left and right actions of $\mathcal{M}_{\mathfrak{D}K}$ on $M \in \mathcal{M}_\mathfrak{D}$ preserve the perfect square $\mathfrak{D}(M)^2/(\det M)$. Among those matrices with $(\det M)/\mathfrak{D} = (M)^2$, it can be checked that an orbit under this action is exactly the annihilator of an element of the projective line $\mathbb{P}^1(\mathcal{O}_K/\mathfrak{D})$ (viewed as either a row or column vector, depending on the action side). When $(\det M)/\mathfrak{D}$ contains $(M)^2$ properly, the matrix is degenerate in a sense, having already occurred as an element of $\mathcal{M}_{\mathfrak{D}'}$ for some $\mathfrak{D}'$ properly containing $\mathfrak{D}$, namely $\mathfrak{D}' = (\det M)/(M)^2$. The orbits of such elements are then parameterized by $\mathbb{P}^1(\mathcal{O}_K/\mathfrak{D}')$ instead. The reason for allowing these degenerate matrices in $\mathcal{M}_\mathfrak{D}$ is a convenience regarding the corresponding arrangements that is mentioned after Proposition 2.10.

We remark that $\mathcal{M}_{\mathfrak{D}K}$ has been studied before [15]. It is called the extended Bianchi group. For those fields with nontrivial 2-torsion in the class group it creates a proper extension of the Schmidt arrangement, though $\mathcal{M}_{\mathfrak{D}K}$ still does not in general cover every point in $K$. Indeed, if $M \in \mathcal{M}_{\mathfrak{D}K}$ then

$$(\alpha, \beta)(\gamma, \delta) \subseteq (\alpha, \beta)(M) \subseteq (M)^2 = (M)^2\mathcal{O}_K \subseteq (\det M) = (\alpha\delta - \beta\gamma) \subseteq (\alpha, \beta)(\gamma, \delta).$$

This implies $(\alpha, \beta) = (M)$ as each containment, in particular the second one, must be equality. Turning to the third containment we then see that $[(\alpha, \beta)]$ is 2-torsion. Even if this does account for the whole class group, all rational points on $M^o$ correspond to the same class, namely $[(M)]$. Indeed, all such points occur as numerator and denominator composing the column of something in $\text{MSL}_2(\mathbb{Z})$, and these matrices’ entries all generate the same ideal. Thus the different elements of the class group manifest as distinct components of the arrangement that do not intersect rationally (or at all). Figure 4 displays our observations for $\mathbb{Q}(\sqrt{-5})$. The yellow component is no aid in attempting a walk across the plane like in Figure 3.

Note here that $\mathcal{M}_{\mathfrak{D}K}$ gives the maximal discrete subgroup of $\text{GL}_2(K)$ containing $\text{SL}_2(\mathcal{O}_K)$ [1], meaning our choice of extension is not a group by force.

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1These arguments are generalized to arbitrary $\mathfrak{D}$ in Propositions 2.19 and 2.20.
2Deduced formally in Section 3.
3By finiteness of the class group, every element of $\mathcal{M}_\mathfrak{D}$ can be scaled to have integer entries and a bounded determinant, so $\mathcal{M}_\mathfrak{D}$ is discrete.
Definition 2.9. Let $M \in \mathcal{M}_D$. The reduced curvature, reduced cocurvature, and reduced curvature-center of $M^o$ are, respectively, the integers

$$r = \frac{\sqrt{\|D\|(\beta \delta - \beta \delta)}}{\sqrt{\Delta \| \det M \|}}, \quad r' = \frac{\sqrt{\|D\|(\alpha \gamma - \alpha \gamma)}}{\sqrt{\Delta \| \det M \|}}, \quad \text{and} \quad \omega = \frac{\sqrt{\|D\|(\alpha \delta - \beta \gamma)}}{\sqrt{\| \det M \|}}.$$

The next proposition provides an alternative definition for extended Schmidt arrangements in which the underlying set of matrices need not be considered. The matrix-free perspective is behind the code that produced these figures.

Proposition 2.10. Associating $M^o$ to its reduced curvature and reduced curvature-center gives a one-to-one correspondence between $M^o_D$ and pairs $(r, \omega) \in \mathbb{Z} \times O_K$ for which $\Delta(r \| \omega \| - \| D \|)$.

Proof. For one direction of the correspondence, scale (4) by $\| D \|/\| \det M \|$ to get $\| D \| = \| \omega \| + \Delta r r'$. This shows that the reduced curvature and reduced curvature-center of $M^o$ satisfy the divisibility condition, and, moreover, the quotient is the reduced cocurvature.

Now suppose $\Delta \| r \| \| \omega \| - \| D \|$ with $D$ integral. This implies $4 \| D \| \equiv 4 \Re(\omega)^2 \mod \Delta$, meaning $[D]$ belongs to the principal genus. So suppose $p$ has prime norm $p$ not dividing $\Delta \| r \| \| \omega \|$ and that $(\eta) = p^2$. For any prime $q \supset (\sqrt{\Delta})$ we have

$$(\eta - r \omega)(\eta + r \omega) \equiv p^2(\| D \| - \| \omega \|) \equiv 0 \mod q.$$ 

Recalling that $q = \bar{q}$, this gives $\varepsilon_q \eta - r \omega \in q$ with $\varepsilon_q$ as 1 or $-1$. We want $\varepsilon_q = 1$ for all $q$, but we may have to adjust $\eta$ to get it.

By weak approximation for quadratic forms there exists a primitive integer solution, $(X_0, Y_0, Z_0)$, to $X^2 - \Delta Y^2 = Z^2$ with $\gcd(Z_0, \| D \| r) = 1$ and $X_0 - \varepsilon_q Z_0 \in q$. If necessary use weak approximation to relate $X_0$ and $Z_0$ modulo 8 as well, so that after scaling $\eta$ by $(X_0 + Y_0 \sqrt{\Delta})$ we have $\eta - Z_0 p \omega = \sqrt{\Delta}(x + y \tau)$ for some $x + y \tau \in O_K$. Our choice of solution also gives $(X_0 - \Delta Y_0)p^2 = \alpha^2$ for some integral ideal $\alpha$ coprime to $\Delta$ with no rational divisors. Indeed, its norm is a square, it cannot have inert or ramified divisors since $\gcd(X_0, Y_0, Z_0) = 1$, and the possibility of containment in $(p)$ is eliminated by $\gcd(Z_0, p) = 1$. Also useful will be $(\eta, \bar{\alpha}) = O_K$ (here we use $\gcd(Z_0, \| D \| p) = 1$) which gives $\gcd(\| \alpha \|, x, y) = 1$.

Now let $s + t \tau = \eta$, set $d' = \gcd(r, x, y)$, and let $d$ be a solution to the following system of congruences.

$$dy \equiv -d'(x - t + 2y \Re(\tau)) \mod \| \alpha \| r \quad \text{and} \quad dx \equiv d'(y \| \tau \| + s) \mod \| \alpha \| r. \quad (5)$$
Note that such a \( d \) exists since cross-multiplying coefficients of \( d \) and \( d' \) and subtracting gives

\[
y(y\|\tau\| + s) + x(x - t + 2y\Re(\tau)) = \|x + y\tau\| + sy - tx
\]

\[
= \frac{1}{|\Delta|}(\sqrt[|\Delta|]{x + y\tau} - \eta - \|\eta\|) = \frac{\|a\|^2}{|\Delta|}(\|\omega\| - \|\mathcal{D}\|),
\]

which is a multiple of \( \|a\| \) by assumption. Now recall that \( \gcd(\|a\|, x, y) = 1 \) and \( d' = (r, x, y) \). So the system is seen to be solvable modulo any prime power dividing \( \|a\| \) by considering the left congruence when it is coprime to \( y/d' \) and the right congruence when it is coprime to \( x/d' \).

Set \( a = x/d' \), \( a' = y/d' \), \( b = \|a\|r/d' \),

\[
c = \frac{dx - d'(y\|\tau\| + s)}{\|a\|r}, \quad \text{and} \quad c' = \frac{dy + d'(x - t + 2y\Re(\tau))}{\|a\|r}.
\]

These are integers thanks to the congruence restrictions imposed on \( d \). Let

\[
M = \begin{bmatrix} a + a'\tau & c + c'\tau \\ b & d + d'\tau \end{bmatrix},
\]

and compute that \( \det M = \eta \) and that \( M^0 \) has reduced curvature \( r \) and reduced curvature-center \( \omega \). We can check that \( \|a\|/\|d + d'\tau\| \) by using the left congruence in (5) for the primes in \( \|a\| \) not dividing \( y \) and the right congruence for those not dividing \( x \). Since \( b \) is in \( \mathfrak{a} \) and \( (\eta, \mathfrak{a}) = \mathcal{O}_K \), it must be that \( d + d'\tau \in \mathfrak{a} \). But we also have \( a + a'\tau \in \mathfrak{a} \) (recall that \( d' \) divides \( r \) and \( \gcd(r, \|a\|) = \gcd(r, \mathbb{Z}[p]) = 1 \)) and \( (b, \|a\|^2) = \mathfrak{a} \), which implies \( c + c'\tau \in \mathfrak{a} \). Thus \( (M) \subseteq \mathfrak{a} \) giving \( \det M)/(M)^2\mathcal{D} \supseteq (\eta)/\|a\|^2\mathcal{D} = \mathcal{O}_K \), so that \( M \in \mathcal{M}_\mathcal{D} \).

The last line of the proof is the reason the requirement of \( M \) in Definition 2.5 is not tightened to read \( \det M)/(M)^2\mathcal{D} = \mathcal{O}_K \). This proposition and the forthcoming remark would be false. Similar statements could be made, however, by restricting attention to those elements of \( \mathcal{M}_\mathcal{D}^0 \) not belonging to any \( \mathcal{M}_\mathcal{D}^0 \) with \( \mathcal{D}' \) properly containing \( \mathcal{D} \). These are precisely the elements for which \( (r, r', \omega, \mathcal{W}) = \mathcal{O}_K \).

A partial illustration of the proposition with \( K = \mathbb{Q}(i) \) and \( D = 1 \) is given in Figure 5. There are two forms of reduced curvature-centers for which there is a nonempty set of reduced curvatures satisfying the requirement \( 4r \|\omega\| - 1 \). They are \( \omega = 2x + (2y + 1)i \) and \( \omega = 2x + 1 + 2yi \) for \( x, y \in \mathbb{Z} \). For clarity only the original Schmidt arrangement, \( \text{SL}_2(\mathbb{Z}[i])^\circ \), is on display. It consists of those elements with reduced curvature-center fitting the first form. To give a full illustration of Proposition 2.10 for \( \mathcal{M}_{\omega(i)}^0 \), we could draw the same pictures rotated 90 degrees and
superimpose them onto these. This accounts for the other admissible form of reduced curvature-centers and corresponds to those matrices in $\text{GL}_2(\mathbb{Z}[i])$ (everything in $\mathcal{M}_{\mathbb{Z}[i]}$ is a scalar multiple of such a matrix, as is generally true of $\mathcal{M}_{\mathcal{O}_K}$ when the class group is trivial) with determinant $\pm i$.

![Figure 5](image_url)

**Figure 5.** Left: circles of radius 1 on the points $\omega = 2x + (2y + 1)i$ for $x, y \in \mathbb{Z}$ labeled with the value $x^2 + y^2 + y$, highlighting multiples of 12. Right: circles of reduced curvature 12 in $\text{SL}_2(\mathbb{Z}[i])^\circ$.

Notice that the ideal $\mathfrak{D}$ is lost in the defining divisibility condition of Proposition 2.10, where only its norm is needed. So distinct integral ideals of the same norm define the same extended Schmidt arrangements despite having different underlying sets of matrices. This means that given some $M \in \mathcal{M}_{\mathfrak{D}}$ and any ideal $\mathfrak{D}' \subseteq \mathcal{O}_K$ with $\|\mathfrak{D}'\| = \|\mathfrak{D}\|$, there should be a way to produce $M' \in \mathcal{M}_{\mathfrak{D}'}$ that represents the same element of the arrangement.

**Definition 2.11.** Two matrices $M, M' \in \text{GL}_2(K)$ are equivalent if $M^o = M'^o$. The same will be said of sets of matrices. The matrices $M$ and $M'$ are called representatives of $M^o$.

Let us find the matrix $M' \in \mathcal{M}_{\mathfrak{D}'}$, to which we alluded before the definition. The procedure we use here will be ubiquitous in the last two sections.

Since anything in $\text{GL}_2(\mathbb{Q})$ is equivalent to the identity, right multiplication by
these matrices preserves equivalence. Most relevant to us are those of the form
\[
\begin{bmatrix}
1 & 0 \\
0 & b
\end{bmatrix},
\]
(6)
because in the special case when \(b\) divides \(\| (\alpha, \beta) \|\), this matrix reveals the connection between \(M_D\) and \(M_{D'}\) as follows. For some \(M \in M_D\), let \(a^2 = (\det M)/D\) and set \(b = \| (\alpha, \beta)/a \| \in \mathbb{Z}\). Now scale the right column of \(M\) by \(b\) to obtain a new matrix, \(M'\). Then
\[
\mathcal{D}' := \frac{(\det M')}{(M')^2} = \frac{b \det M}{(\alpha, \beta)^2} = \frac{a(\alpha, \beta)}{\overline{a}(\alpha, \beta)} \mathcal{D}
\]
(7)
is integral with norm \(\| \mathcal{D} \|\). The ideal \((\alpha, \beta) = (M')\) has essentially become waste for \(M'\) since its square can be discarded as a divisor of \((\det M')\) without losing any part of \(\mathcal{D}'\). For our purposes, \(\alpha' = \alpha\) and \(\beta' = \beta\) are now as good as coprime. More formally, the system of congruences,
\[
a\alpha' + \gamma', \ a\beta' + \delta' \in \mathcal{D}'(M'),
\]
now has an integer solution \(a\). In particular, we could take \(a \in \mathbb{Z}\) that solves this system except possibly modulo the rational divisors of \(\mathcal{D}'\). Then
\[
M' \begin{bmatrix} 1 & a' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & a\alpha + b\gamma \\ \beta & a\beta + b\delta \end{bmatrix}
\]
is still equivalent to \(M\). Now just as we did with \((\alpha, \beta)/a \supseteq \mathcal{D}\) in (7), we can conjugate our favorite factor of \(\mathcal{D}'\) (which now divides both right column entries) to obtain any determinant ideal of norm \(\| \mathcal{D} \|\). This confirms the implication of Proposition 2.10 regarding an arrangement’s independence on the particular ideal of a given norm.

We introduce the following notation in light of this discussion.

**Notation 2.12.** An integer subscript, like \(M^a_D\), will denote the norm of the ideals underlying the equivalent sets of the form \(M_D\). That is, \(M^a_D = M_D^a\) where \(D = \| \mathcal{D} \|\). The notation \(M_D\) is not defined.

Among our options for representatives of an element of \(M^a_D\), we highlight one in particular.

---

4If the rows of this matrix are swapped, it represents a reflection over the disc of radius \(\sqrt{b}\). Note that with \(b = \| (\alpha, \beta)/a \|\) this paragraph shows that there is no inflation in the norm of the determinant ideal \(\mathcal{D}\), even if \(b > 1\). This hints at a solution to the non-Euclidean dilemma.
**Definition 2.13.** A matrix $M \in \mathcal{M}_D$ is *trimmed* if $(\alpha, \beta)^2 \mathcal{D} = (\det M)$ (in particular $(\alpha, \beta) = (M)$). If, furthermore, $(\gamma, \delta) = (M) \mathcal{D}$, we will call $M$ *fully trimmed*.

**Proposition 2.14.** Every element in $\mathcal{M}^0_D$ has a trimmed representative $M \in \mathcal{M}_D$ for some $\mathcal{D}$ of norm $D$. The left column, $\alpha/\beta$, can be taken as any point on $\mathcal{M}^0_D$.

**Proof.** Taking a representative $M \in \mathcal{M}_D$ for some $M^0 \in \mathcal{M}^0_D$, let $a^2 = (\det M)/\mathcal{D}$. Let $p/q = M^{-1}(\alpha/\beta)$ with $p, q \in \mathbb{Z}$ coprime, and right multiply $M$ by a matrix in $\text{SL}_2(\mathbb{Z})$ (preserving equivalence and the ideal $a$) with left column entries $p$ and $q$. This puts $\alpha/\beta$ in the left column of $M$, and we saw in (7) that scaling the right column of $M$ by $\|((\alpha, \beta)/a)\|$ renders it trimmed. \hfill $\Box$

It is not true in general that every element of $\mathcal{M}^0_D$ has a *fully* trimmed representative with a given left column. If the determinant ideal $\mathcal{D}$ of a given trimmed matrix $M$ is divisible by a rational integer, then fully trimming via right multiplication by an element of $\text{SL}_2(\mathbb{Z})$ may not be possible. This is because there are always congruence classes modulo such an ideal that have no rational representative. As an example, suppose that

$$M = \begin{bmatrix} 1 & -2 + \sqrt{-2} \\ 1 & 1 + \sqrt{-2} \end{bmatrix} \in \mathcal{M}_{(3)} \subset \text{GL}_2(\mathbb{Z}[\sqrt{-2}])$$

is equivalent to some other trimmed matrix $M' \in \mathcal{M}_{\mathcal{D}'}$ with $\|\mathcal{D}'\| = 9$ and $\alpha'/\beta' = 1$. Without loss of generality assume that $\alpha' = \beta' = 1$. Then $M^{-1}M'$ has top-left entry 1 and must be a scalar (from $\mathbb{Q}(\sqrt{-2})$) multiple of a matrix in $\text{GL}_2(\mathbb{Q})$ because it represents the real line. Thus $M^{-1}M'$, whose determinant has norm 1, is of the form

$$\begin{bmatrix} 1 & a \\ 0 & \pm 1 \end{bmatrix}$$

for some $a \in \mathbb{Q}$, meaning $\mathcal{D}' = (3)$. But then $\delta' = a \pm (1 + \sqrt{-2})$ makes $\delta' \in (3)$ impossible, so that $M'$ cannot be fully trimmed.

The fact that $\mathcal{D}'$ was forced to be (3) is an example of a general phenomenon for trimmed matrices with a given left column. In using our technique of scaling the right column to obtain a trimmed matrix, we had no choice in the resulting determinant ideal, $\mathcal{D}'$. Corollary 2.17 ii) after the next proposition shows that this limitation is unavoidable. By trimming we lose control over the ideal of norm $D$.

**Definition 2.15.** Define the “reflect,” “shift,” and “turn” matrices to be

$$R(b) = \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix}, \quad S(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad T(u) = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}.$$
Proposition 2.16. The elements of \( \mathcal{M}_D^\circ \) that contain \( \alpha/\beta \in K \) break into distinct families, one for each pairing of a unit in \( \mathcal{O}_K/\{\pm 1\} \) with an ideal in \( [(\alpha, \beta)]^{-2} \) of norm \( D \). The angle of intersection between elements from families corresponding to \( \mathcal{D} \) and \( \mathcal{D}' \) is the argument of a generator for \( \mathcal{D}'/\mathcal{D} \), with tangential intersection occurring whenever the two families coincide. The set of reduced curvatures from each family is a congruence class modulo \( \|\beta/(\alpha, \beta)\| \).

Proof. If there exists an ideal in \( [(\alpha, \beta)]^{-2} \) of norm \( D \), say \( \mathcal{D} \), then we can find \( \gamma, \delta \in (\alpha, \beta)\mathcal{D} \) to produce a matrix \( M \) with \( (\det M) = (\alpha, \beta)^2 \mathcal{D} = (M)^2 \mathcal{D} \). So \( M \) is fully trimmed with left column entries \( \alpha \) and \( \beta \). Now for \( u \in \mathcal{O}_K/\{\pm 1\} \) (call one of its two representative units \( u \) as well) and \( \mathcal{D}' \in [(\alpha, \beta)]^{-2} \) of norm \( D \), fix a generator \( \eta \) for \( \mathcal{D}'/\mathcal{D} \). We have the family of matrices \( \{MT(u\eta)S(a)\vert a \in \mathcal{O}_K\} \subset \mathcal{M}_{\mathcal{D}'}. \) Möbius transformations are conformal, so the angle of intersection between \( M^\circ \) and an element represented by one of these families’ matrices is that of the real line and \( (T(u\eta)S(a))^\circ \). This is the angle we claimed. The reduced curvature of \( MT(u\eta)S(a) \) is computed to be \( r + a\|\beta/(M)\| \), where \( a' = 23(a)/\sqrt{-\Delta} \) and \( r \) is the reduced curvature of \( MT(u\eta)^\circ \). Since \( (M) = (\alpha, \beta) \), this shows that the set of reduced curvatures contains a congruence class modulo \( \|\beta/(\alpha, \beta)\| \).

To complete the proof we must show that every element of \( \mathcal{M}_D^\circ \) passing through \( \alpha/\beta \) is in one of the families we have defined. For such an element we can take a trimmed representative with left column entries \( \alpha' \) and \( \beta' \) that satisfy \( \alpha'/\beta' = \alpha/\beta \) by Proposition 2.14. This matrix can then be shifted by an integer (perhaps not rational) to get a fully trimmed \( M' \in \mathcal{M}_{\mathcal{D}'} \) for some \( \mathcal{D}' \in [(\alpha, \beta)]^{-2} \) of norm \( D \). But here \( [(\alpha', \beta')] = [(\alpha, \beta)] \), so scaling \( M' \) by the appropriate generator for \( (\alpha, \beta)/(\alpha', \beta') \) gives \( \alpha = \alpha' \) and \( \beta = \beta' \). Then, taking \( \eta \) as the same fixed generator for \( \mathcal{D}'/\mathcal{D} \), we have \( M^{-1}M' = T(u\eta)S(a) \) for some \( u \in \mathcal{O}_K^\circ \) and \( a \in \mathcal{O}_K \). \( \square \)

Corollary 2.17. Suppose \( M \in \mathcal{M}_D \) and \( M' \in \mathcal{M}_{\mathcal{D}'} \) are equivalent with \( \|\mathcal{D}\| = \|\mathcal{D}'\| \).

i) If \( \mathcal{D} = \mathcal{D}' \), then \([\mathbf{a}] = [\mathbf{a'}] \), where \( \mathbf{a}^2 \mathcal{D} = (\det M) \) and \( \mathbf{a'}^2 \mathcal{D} = (\det M') \).

ii) If \( M \) and \( M' \) are both trimmed and \( \alpha/\beta = \alpha'/\beta' \), then \( \mathcal{D} = \mathcal{D}' \).

Proof. By shifting and scaling (which do not change \([\mathbf{a}] \) or \([\mathbf{a'}]\)) we may assume that \( \alpha = \alpha' \) and \( \beta = \beta' \). Then \( M^{-1}M = T(u)S(a) \) for some \( u \in K \) satisfying \( u \det M = \det M' \). Now, \( M \) and \( M' \) are equivalent, and in particular they represent elements of the same family. So by Proposition 2.16 (which proves formally that only scalar multiples of \( \text{GL}_2(\mathbb{Q}) \) are equivalent to the identity), \( u \in \mathbb{Q} \). But \( ua^2 = \mathbf{a'}^2 \) means \( (u) \) is the square of some ideal, which must in turn be generated by a rational number. Thus \([\mathbf{a}] = [\mathbf{a'}] \).
For the second claim recall from the last paragraph of the previous proof that we may assume $M^{-1}M' = T(u)S(a)$ for some $u \in \mathcal{O}^{-1}$ with norm 1. Since $M^e$ and $M^{e'}$ are in the same family, the argument of $u$ is 0. This means $u = \pm 1$, implying $\mathcal{O}' = u\mathcal{O} = \mathcal{O}$.

Referring to $i)$ above, we have grappled with the uniqueness of $[a]$ before, though it was disguised by switching from ideals to quadratic forms then handled with the tool of weak approximation. In the proof of Proposition 2.10 our initial choice for $p$ may not have belonged to the unique class in the particular coset of the 2-torsion subgroup we needed. Indeed, if $(\sqrt{\Delta})$ has multiple prime divisors, implying the 2-torsion subgroup is nontrivial, then the signs of our $\varepsilon_q$’s may not match. Scaling $\eta$ was not a matter of convenience for the remainder of the proof, it was out of necessity according to $i)$ of the previous corollary.

Proposition 2.16 says that the angles of intersection at $\alpha/\beta$ are determined by the units of $\mathcal{O}_K$ as well as the additional “units” introduced by any of the ideals in $[(\alpha, \beta)]^{-2}$ of norm $D$.

**Notation 2.18.** For an ideal $\mathcal{O} \subseteq \mathcal{O}_K$ let $\mathcal{O}_K^*(\mathcal{O}) = \{ u \in K \mid \|u\| = 1, u\mathcal{O} \subseteq \mathcal{O}_K \}$.

These units are on display in Figure 6. The class group of $\mathbb{Q}(\sqrt{-47})$ has order 5, and 2 and 3 split. Up to conjugation there are four ideals of norm 24—the principal ideal $(3 + \tau)$, the ideals $p_2^2p_3$ and $2p_2p_3$ in $[p_2]$, and $2p_2p_3$ in $[p_2]^2$. From Proposition 2.16 the points $\alpha/\beta$ for which $(\alpha, \beta)$ is principal possess two families due to the additional unit $(3 + \tau)/(3 + \tau)$. The intersection angle is the argument of the unit, which is $2 \arctan(\sqrt{47}/7) \approx 0.49\pi$. These are the points featured prominently along the bottom. Then there are those across the middle that look agitated. The angle between these two families is $\arctan(\sqrt{47}/23) \approx 0.09\pi$, the argument of a generator for $p_2^2p_3/2p_2p_3$.

So $[(\alpha, \beta)]$ is either $\sqrt{[p_2]} = [p_2]^{-2}$ or $[p_2]^2$ in this case. Finally, the points corresponding to $\sqrt{[p_2]}^{\pm2} = [p_2]^{\pm1}$ possess only one family, implying tangential intersection, and four of them with relatively small $\|\beta/(\alpha, \beta)\|$ can be seen across the top.
Proposition 2.19. The set of ideal classes corresponding to points covered by \( \mathcal{M}_D^0 \) consists of those whose square contains an integral ideal of norm \( D \), and any point corresponding to such a class is covered. In particular, \( \mathcal{M}_D^0 \) covers all of \( K \) if and only if every class in the principal genus contains an integral ideal of norm \( D \).

Proof. For \( \alpha/\beta \in \mathcal{M}_D^0 \), Proposition 2.14 gives a trimmed \( M \in \mathcal{M}_D \) with left column entries \( \alpha \) and \( \beta \) for some integral ideal \( \mathfrak{d} \) of norm \( D \). Then \( (\alpha, \beta)^2 \mathfrak{d} = (\det M) \mathfrak{d} \) gives \( \mathfrak{d} \in [(\alpha, \beta)]^2 \). We already observed at the start of the proof of Proposition 2.16 that the points in question are covered by \( \mathcal{M}_D^0 \).

This shows that if \( \mathcal{M}_D^0 \) covers points in \( K \) corresponding to a given class then it also covers points corresponding to every 2-torsion multiple of that class.

Proposition 2.20. The set of ideal classes corresponding to points covered by \( \mathcal{M}_D^0 \) is \( \{[(M)\mathfrak{d}] \mid (M)^2\mathfrak{d} \supseteq (\det M)\} \).

Proof. Right multiplication by elements of \( \text{SL}_2(\mathbb{Z}) \) puts any point on \( \mathcal{M}_D^0 \) in the left column, and \( (M) \) and \( \det M \) are preserved in such a product. Thus \( (\det M)/(M) \subseteq (\gamma, \delta) \subseteq (\alpha, \beta) \subseteq (M) \) shows that the ideal classes of \( \mathcal{M}_D^0 \) are contained in \( \{[(M)\mathfrak{d}] \mid (M)^2\mathfrak{d} \supseteq (\det M)\} \). For the reverse containment, the Chinese remainder theorem gives \( p, q \in \mathbb{Z} \) with \( (\alpha p + \gamma q, \beta p + \delta q) = (M)\mathfrak{d} \) for any \( \mathfrak{d} \supseteq (\det M)/(M)^2 \).

The previous two propositions are the generalization of our argument involving the extended Bianchi group.

In Section 4 we will use our understanding of extended Schmidt arrangements to produce one in which the every point in \( K \), regardless of its corresponding ideal class, lies on the same connected component. It is interesting to note that the opposite goal is impossible in general. Only 2-torsion ideal classes need not share the elements of \( \mathcal{M}_D^0 \) as stated precisely below.

Proposition 2.21. If a circle or line in \( \mathbb{C} \) contains three points of \( K \), then it covers the ideal class corresponding to each of the three points and their conjugates densely.

Proof. Three rational points along with an application of Proposition 2.14 gives a trimmed matrix representative \( M \in \mathcal{M}_D \) for some \( \mathfrak{d} = (\det M)/(M)^2 \). The inability to fully trim \( M \) (for a fixed left column) through right multiplication by \( S(a) \) for some \( a \in \mathbb{Z} \) is the fault of rational integer divisors of \( \mathfrak{d} \), and these are all principal. So at least we may assume \( (\gamma, \delta) = (M)\mathfrak{d}/d \) for some \( d \in \mathbb{Z} \). Thus \( (\alpha, \beta)(\gamma, \delta) = (M)^2\mathfrak{d}/d = (\det M/d) \), meaning \( [(\gamma, \delta)] = [(\alpha, \beta)]^{-1} \). Right multiplication with elements of \( \text{SL}_2(\mathbb{Z}) \) proves the density claim.
This proposition asserts that we could never hope to part an ideal class from its inverse. We can get close to isolating each class, however, by taking $D$ to be prime or 1. If $M \in \mathcal{M}_D$ with $\|D\|$ prime or 1, then $(\det M)/(M)^2 = D$, so by Proposition 2.19 $M^\circ$ covers points corresponding to classes in the set $\{[(M)D] \mid D \supseteq \mathfrak{D}\}$. But there are at most two ideals dividing $\mathfrak{D}$. If there are two, which happens when $\mathfrak{D}$ is prime, then they are inverses of each other since their product is in the class $\[(M)^2 \mathfrak{D} \] = \[(\det M)\]$. If there is only one, which happens when $\mathfrak{D} = \mathcal{O}_K$, then it is the 2-torsion class, $\[(M)\]$, whose square is $\[(\det M)\]$.

The examples below show choices for $D$ that are as good as possible at separating ideal classes (and in a sense as bad as possible for producing continued fractions).

Figure 7. From left to right: $\mathcal{M}_1^\circ$ and $\mathcal{M}_{11}^\circ$ in $\mathbb{Q}(\sqrt{-55})$ (cyclic group of order 4); $\mathcal{M}_{1}^\circ$, $\mathcal{M}_2^\circ$, $\mathcal{M}_3^\circ$, and $\mathcal{M}_5^\circ$ in $\mathbb{Q}(\sqrt{-71})$ (class number 7); $\mathcal{M}_1^\circ$ and $\mathcal{M}_{13}^\circ$ in $\mathbb{Q}(\sqrt{-29})$ (class number 6); and $\mathcal{M}_{25}^\circ$ and $\mathcal{M}_{13}^\circ$ in $\mathbb{Q}(\sqrt{-53})$ (class number 6).
3. Motivating examples

Let us now show how Proposition 2.4 allows us to draw the nearest integer algorithm. We will reproduce the data displayed in Figure 3.

To obtain the continued fraction expansion of \( z = z_0 = 4/(7 - 2i) \), the algorithm dictates that we take \( a_0 = 1 \), which centers the disc containing \( z_0 \), the point labeled “0” in Figure 8. Thus \( z_1 = 1/(z - a_0) = (-25 - 8i)/13 \) so \( a_1 = -2 - i \). Continuing this way gives \( z_2 = (1 - 5i)/2, a_2 = 1 - 2i \), and \( z_3 = a_3 = -1 + i \).

We could now find the rational approximations \( p_n/q_n \) using (1), but the convergents admit a recursive definition that is more computationally convenient. The formula in (1) is equivalent to taking \( p_{-1} = q_0 = 0 \) and \( p_0 = q_{-1} = 1 \) and setting \( p_{n+1} = a_np_n + p_{n-1} \) and \( q_{n+1} = a_nq_n + q_{n-1} \). Or otherwise expressed, let \( p_{n+1} \) and \( q_{n+1} \) be the left column entries of the matrix \( M_{n+1} = M_nS(a_n)R(1) \) (a shift by \( a_n \) and a reflection over the unit disc), and start with \( M_0 \) as the identity matrix. This gives us

\[
M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -1 - i & 1 \\ -2 - i & 1 \end{bmatrix},
\]

\[
M_3 = \begin{bmatrix} -2 + i & -1 - i \\ -3 + 3i & -2 - i \end{bmatrix}, \quad \text{and} \quad M_4 = \begin{bmatrix} -4i & -2 + i \\ -2 - 7i & -3 + 3i \end{bmatrix}.
\]

We see that right multiplication by \( S(a_n)R(1) \) sends the left column of \( M_n \) to the right column of \( M_{n+1} \) so that \( p_n/q_n \) is a point on both \( M_n \) and \( M_{n+1} \). Thus we can draw the Gaussian Schmidt arrangement and watch the algorithm approach \( 4/(7 - 2i) \). This is Figure 3. At each stage the approximations manifest as intersections of consecutive circles.

Proposition 2.10 is nicer for drawing the arrangement holistically, but we apply Proposition 2.4 to our select four elements above. As an example, \( M_4 \) has curvature \( i(q_4q_3 - q_3q_4) = -54 \), making its radius \( 1/54 \), and curvature-center \( i(p_4q_3 - q_4p_3) = -28 - 9i \), making its center \((28 + 9i)/54 \). This is the smallest circle in Figure 3 as well as the first and third images in Figure 9.
Note that $z_2 = (1 - 5i)/2$ creates a four-way tie for the nearest integer. The choices $1 - 2i$, $-2i$, $1 - 3i$, and $-3i$ for $a_2$ lead to the four shortest paths to $z$ beginning at the point $p_2/q_2$ on $M_2^2$.

![Figure 9](image1.png)

**Figure 9.** The convergents at the end of four nearest integer expansions for $\frac{4}{7-2i}$. From left to right: $a_2 = 1 - 2i$, $-2i$, $1 - 3i$, and $-3i$.

Not only will we pay no attention in Section 5 to a tie like this, we will not even require that the choice $a_n$ be closest to $z_n$. Returning to Figure 8, our results will apply equally well in taking $a_0 = 0$ since $z = z_0$ is also in the unit disc at the origin.

An attempt to execute this algorithm in a non-Euclidean field may fail given the lack of a covering by unit discs. In Figure 10 we see this for $z = (46 + 9\tau)/25$ in $\mathbb{Q}(\sqrt{-15})$.

![Figure 10](image2.png)

**Figure 10.** Left: The points $z_n$, labeled by their index. No integer in $\mathbb{Q}(\sqrt{-15})$ is within 1 of $z_3$. Right: The approximations in $M_1^2$ to $z = (46 + 9\tau)/25$ produced by the nearest integer algorithm.
The coefficients are $a_0 = 2$, $a_1 = 1 - \tau$, and $a_2 = -\tau$, which are the nearest integers to $z_0 = (46 + 9\tau)/25$, $z_1 = (125 - 225\tau)/304$, and $z_2 = (-100 - 79\tau)/141$. There is no integer within 1 of $z_3 = (-19 - 31\tau)/68$. The convergents are $p_1/q_1 = 2$, $p_2/q_2 = (3 - 2\tau)/(1 - \tau)$, and $p_3/q_3 = (6 + \tau)/3$.

Both images in Figure 10 are illustrations of non-Euclideaneity. The first is more classical. Regarding the second image, Stange proved that a disconnected Schmidt arrangement (in blue) is equivalent to not having a Euclidean algorithm [13].

In this particular example, which shows $\mathcal{M}_1^0$ in $\mathbb{Q}(\sqrt{-15})$, Proposition 2.16 tells us that the angles of intersection in the arrangement are the arguments of elements in $\mathcal{O}_K^* = \{\pm 1\}$. So we have proved that intersections are tangential. But a tangential intersection of circles with rational centers and radii can only occur at a rational point. Since $\mathcal{O}_K$ has only one divisor, Proposition 2.20 implies that the rational points on a chain of intersecting circles all correspond to the same ideal class.

This shows that the connected component of the real line (on which every possible $p_n/q_n$ will occur) cannot enter the yellow region that is bounded by circles corresponding to the nontrivial class. So $z = (46 + 9\tau)/25$ never stood a chance. The yellow shading highlights points in $\mathbb{C}$ that will cause the nearest integer algorithm to terminate prematurely, just as $z_3$ did above.

What we aim to do in Section 5 is replace $\mathcal{M}_1^0$ in the nearest integer algorithm with a larger arrangement like $\mathcal{M}_4^0$, shown in Figure 11. It appears to have access to the entire complex plane. Both images in Figure 10 display the non-Euclideaneity of the ring of integers, the first by failing to cover and the second by failing to connect. In contrast, we will show how Figure 11 displays a newfound Euclideaneity with its connectivity, begging the existence of some corresponding covering of the complex plane. This is the missing piece in generalizing continued fractions to any imaginary quadratic field.

To this end, we start by proving that these arrangements are actually connected and that such an arrangement always exists for every $K$.

5Multiple colors are no longer used since every circle covers both ideal classes.

6Another example is found by comparing $\mathcal{M}_1^0$ in Figure 1 to $\mathcal{M}_4^0$ in Figure 4, both in $\mathbb{Q}(\sqrt{-19})$. 

Figure 11. The set $\mathcal{M}_4^0$ in $\mathbb{Q}(\sqrt{-15})$. 

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4. Connectivity

For Euclidean imaginary quadratic fields the Schmidt arrangement is shown to be connected by expressing the Euclidean algorithm as a product of matrices. By definition of norm-Euclidean, if \( M' = MS(a)R(1) \) we can always choose \( a \in \mathcal{O}_K \) to either get \( \|\alpha'\| = \|\gamma + aa\| < \|\alpha\| \) or \( \|\beta'\| = \|\delta + ba\| < \|\beta\| \) (provided \( \alpha \neq 0 \) or \( \beta \neq 0 \)). Every time this procedure is repeated we take another step in the Schmidt arrangement toward 0 if we reduce \( \|\alpha\| \) or \( \infty \) if we reduce \( \|\beta\| \).

We would like to realize such a relationship between connectivity and Euclideaneity in the sets \( \mathcal{M}_D^0 \). Then we can proceed to search for one that is path connected in the same fashion as a Euclidean Schmidt arrangement. Specifically, a path must traverse the elements of \( \mathcal{M}_D^0 \), and when it crosses from one element to another it should do so over rational intersections only, like the point \( \alpha/\beta \) on \( M^0 \) and \( M^1 \) in the previous paragraph. In traveling along the path we hope to perform some kind of reduction of norm, again as above. Proposition 2.16 indicates that the geometric analog to \( \|\beta\| \) in an extended Schmidt arrangement is \( \|\beta/(\alpha, \beta)\| \), so this will be the norm of interest. The definition below encapsulates all of this.

**Definition 4.1.** The set \( \mathcal{M}_D^0 \) is **connected** if for any rational point \( \alpha/\beta \) that it covers there is a sequence,

\[
\frac{\alpha_0}{\beta_0} = \frac{\alpha_1}{\beta_1}, ..., \frac{\alpha_N}{\beta_N} = \infty,
\]

with \( \alpha_{n-1}/\beta_{n-1} \) and \( \alpha_n/\beta_n \) composing the right and left columns\(^7\) of some \( M_n \in \mathcal{M}_D \) with \( \|\mathcal{D}_n\| = D \) for each \( n = 1, ..., N \). If the sequence can always be chosen to contain a subsequence indexed by \( n_0 = 0, n_1, ..., n_I = N \) satisfying \( \|\beta_{n_i}/(\alpha_{n_{i-1}}, \beta_{n_{i-1}})\| > \|\beta_{n_i}/(\alpha_{n_i}, \beta_{n_i})\| \) and \( n_i - n_{i-1} \leq k \) for each \( i = 1, ..., I \), then \( \mathcal{M}_D^0 \) is **k-connected**, or **monotonically connected** if \( k = 1 \).

Let us see what is required to replicate the monotonic connectivity found in the Euclidean cases. Let \( M \in \mathcal{M}_D \) be fully trimmed with left column entries \( \alpha \) and \( \beta \) (recall that in trimming we have lost control over the ideal of norm \( \|\mathcal{D}\| \)). For some \( a \in \mathcal{O}_K \) and \( u \in \mathcal{O}_K^*(\mathcal{D}) \), let \( M' = MT(u)S(a)R(1) \). Then

\[
(\alpha', \beta') = (a\alpha + u\gamma, a\beta + u\delta) = (M)(a, u\mathcal{D}).
\]

Thus the desired decrease, which is \( \|\beta/(M)\| > \|\beta'/(\alpha', \beta')\| \), can be rewritten as \( \|\delta/\beta + u^{-1}a\| < \|(a, u\mathcal{D})\| \). That is, \( -\delta/\beta \in B(u^{-1}a, \sqrt{\|(a, u\mathcal{D})\|}) \). So monotonically

\(^7\)Insisting that \( \alpha_{n-1}/\beta_{n-1} \) and \( \alpha_n/\beta_n \) actually compose the columns of \( M_n \) rather than just lie on \( M_n \), recreates the relationship shared by the matrices in (2).
decreasing $||\beta/ (\alpha, \beta)||$ is always possible if $C$ is covered by such discs. This is the generalization of the covering by unit discs that we need.

**Definition 4.2.** For $\mathcal{D} \subseteq \mathcal{O}_K$ and $\varepsilon > 0$ let

$$
\varepsilon \mathcal{D}^\bullet = \bigcup_{a \in \mathcal{O}_K, u \in \mathcal{O}_K^* (\mathcal{D})} B(u^{-1}a, \varepsilon \sqrt{||(a, u\mathcal{D})||}).
$$

This is a union of sets, one for each $u \in \mathcal{O}_K^* (\mathcal{D})$, which are periodic with fundamental region equal to that of the ideal $u\mathcal{D}$. So the small window displayed in the left image of Figure 12, for example, is enough to prove that $(2)^\bullet = C$. This confirms that $M_0^\circ$ in $\mathbb{Q}(\sqrt{-19})$ is connected (in fact monotonically connected). The procedure described in the last paragraph is exhibited below.

![Figure 12. In $\mathbb{Q}(\sqrt{-19})$, the set $(2)^\bullet$ (left) and a monotonic chain in $M_0^\circ$ (right).](image)

The first circle in the chain containing the point

$$
\frac{\alpha_0}{\beta_0} = \frac{1 + 6\tau}{3 + 4\tau}
$$

is $M_0^\circ = \begin{bmatrix} 1 + 6\tau & 13 + 3\tau \\ 3 + 4\tau & 11 + \tau \end{bmatrix}^\circ$.

The progression from $\alpha_{n-1}/\beta_{n-1}$ to $\alpha_n/\beta_n$ stipulated by Definition 4.1 is defined by a shift, which we called $a$ above. The shifts are

$$
a_0 = -1 + \tau, \quad a_1 = -2 + 2\tau, \quad a_2 = -\tau, \quad a_3 = -1 + \tau, \quad \text{and} \quad a_4 = 2 - \tau.
$$
The matrix $M_n$ is found by trimming $M_{n-1}$ and multiplying on the right by $S(a_{n-1})R(1)$. Computing this way gives

\[ M_1 = \begin{bmatrix} -18 + 4\tau & 1 + 6\tau \\ -12 + 4\tau & 3 + 4\tau \end{bmatrix}, \quad M_2 = \begin{bmatrix} 6\tau & 9 - 2\tau \\ 2 + 4\tau & 6 - 2\tau \end{bmatrix}, \]

\[ M_3 = \begin{bmatrix} -3 + \tau & 3\tau \\ -2 + \tau & 1 + 2\tau \end{bmatrix}, \quad M_4 = \begin{bmatrix} 2 & -3 + \tau \\ 2 & -2 + \tau \end{bmatrix}, \quad \text{and} \quad M_5 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}. \]

At every stage we have $\gamma_n/\delta_n = \alpha_{n-1}/\beta_{n-1}$, as Definition 4.1 requires, and $-\delta_{n-1}/\beta_{n-1}$ is contained in the disc centered at $a_{n-1}$ of radius $\sqrt{\|\alpha_n, \beta_n\|}$. Despite monotonicity, the reduced curvature magnitudes in this chain of circles do not decrease at every stage. The reduced curvature of the circle hiding behind labels 0 and 1 is larger in magnitude than that of its predecessor. Indeed, all reduced curvatures up to 80 are shown in Figure 12, so by observing the “floating” circles we see that a chain with monotonically decreasing reduced curvature magnitudes does not always exist for a given starting point. However, without the presence of nontrivial units, as is the case here, we can guarantee that a chain of circles can be taken in which every other reduced curvature (in both the subsequence starting with 0 and the subsequence starting with 1) is strictly decreasing in magnitude. This is because of Proposition 2.16, which states that $\|\beta/(\alpha, \beta)\|$ is the gap between consecutive reduced curvatures in a family. For three consecutive circles, the families at the two intersection points have reduced curvatures that form congruence classes, each containing the reduced curvature of the middle circle. The modulus is the only thing that distinguishes them, and this is the quantity referred to with the word “monotonic” in Definition 4.1.\(^8\)

**Definition 4.3.** If $\epsilon < 1$ and the closure of $\epsilon\mathcal{D}^*$ covers $\mathbb{C}$ for every $\mathcal{D} \subseteq \mathcal{O}_K$ of norm $D$ then $\mathcal{O}_K$ is $D$-Euclidean. For such a $D$, the smallest admissible $\epsilon$ is called padding.

**Proposition 4.4.** $\mathcal{M}_D^*$ is monotonically connected if and only if $\mathcal{O}_K$ is $D$-Euclidean.

*Proof.* The argument preceding Definition 4.2 proves monotonic connectivity assuming $D$-Euclideaneity. For the other direction, assume that $\mathcal{M}_D^*$ is monotonically connected and take some $M \in \mathcal{M}_D$ that is fully trimmed. In the proof of Proposition 2.16, we saw that up to scaling of columns any matrix with left column $\alpha/\beta$ can be written as $MT(u)S(a)$, where $a \in \mathcal{O}_K$ and $u \in \mathcal{O}_K^*(\mathcal{D})$. Thus monotonic connectivity fails if $-\delta/\beta$ is not covered by $\mathcal{D}^*$. But the points $-\delta/\beta$ with $\delta \in \mathcal{D}(\beta, \delta)$ are

\(^8\)We note this example’s proximity to a continued fraction algorithm. If the Möbius transformation $M_0^{-1}$ were applied to Figure 12, the images of the labeled points would be the convergents in a continued fraction expansion of $-\delta_0/\beta_0$. The point at infinity cannot be seen, but it is labeled “5.”
dense in $\mathbb{C}$, and these all extend with some $\alpha$ and $\gamma$ to make a fully trimmed element of $\mathcal{M}_D$.

The “smallest” three extended Schmidt arrangements in $\mathbb{Q}(\sqrt{-23})$ are shown above. The small gaps in $p_2^\ast$, $p_3^\ast$, and their conjugates result in a spectacular failure in connectivity. Not only are the arrangements not monotonically connected, they are topologically disconnected. Indeed, if the Schmidt arrangement (left) were superimposed on either $\mathcal{M}_2^\circ$ or $\mathcal{M}_3^\circ$ there would be no intersection. Regarding $\mathcal{M}_2^\circ$, the angle of intersection would have to be $\arccos \frac{n}{2\sqrt{1\cdot 2}}$ for some $n \in \mathbb{Z}$ with $n^2 \equiv 4(1\cdot 2) \mod 23$. 

Figure 13. Top, from left to right: $O_k^\ast$, $p_2^\ast$, and $p_3^\ast$. Bottom, from left to right: $M_1^\circ$, $M_2^\circ$, and $M_3^\circ$. All in $\mathbb{Q}(\sqrt{-23})$. 

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For $\mathcal{M}_3^2$ it would have to be $\arccos \frac{n}{2\sqrt{3}}$ for some $n \in \mathbb{Z}$ with $n^2 \equiv 4(1 \cdot 3) \mod 23$. In each case there is no $n$ that makes arccosine’s argument at most 1 in magnitude. We will not go further into these statements, but the reader is referred to [13] where Stange found “ghost circles,” which she used to prove the disconnectedness of the non-Euclidean Schmidt arrangements [13].

The first value of $D$ that makes a monotonically connected arrangement in $\mathbb{Q}(\sqrt{-23})$ is 4. There are three ideals of norm 4. They are $(2)$, $p_2^2$, and $\overline{p}_2^2$. According to Proposition 4.4, all three must provide a covering. We display only two, since the third is the conjugate of the second.

![Figure 14. Left to right: $(2)^*$, $(p_2^2)^*$, and $\mathcal{M}_4^2$ in $\mathbb{Q}(\sqrt{-23})$.](image)

The set $\mathcal{M}_4^2$ contains the original Schmidt arrangement, which has been highlighted in blue. When $\alpha_n/\beta_n$ is touching a blue circle it corresponds to the principal class, so in the procedure outlined after Definition 4.1, the relevant covering at such a point is the blue one. When $\alpha_n/\beta_n$ does not touch a blue circle, the green covering or its conjugate is relevant.

Monotonic connectivity is not required to write a nearest integer-type continued fraction algorithm. There are extended Schmidt arrangements which are $k$-connected only for $k > 1$. An example of this is given in Figure 15. The arrangement $\mathcal{M}_4^2$ in $\mathbb{Q}(\sqrt{-31})$ cannot be monotonically connected because of the gaps in the almost-covering by $(p_2^2)^*$, shown in orange. But we can prove that the arrangement is 2-connected as follows. The gaps are bounded by two circles of radius $\sqrt{2}$ and two circles of radius 2. Should $-\delta/\beta$ lie in such a gap, reflecting over a circle of radius $\sqrt{2}$ conjugates only one factor of $p_2$ in $p_2^2$, changing the determinant ideal to $\mathfrak{D} = (2)$.
This makes the green covering relevant. The loss in $\|\beta/(\alpha, \beta)\|$ suffered by reflecting over a disc that did not contain $-\delta/\beta$ (specifically the nearest one of radius $\sqrt{2}$) is at most $(35 - \sqrt{713})/8 \approx 1.037$. The boundaries of these two discs are inflated by approximately $\sqrt{1.037}$ in the bottom, middle image to demonstrate this. But now we are in the green covering, so at the very next step in our algorithm we can take advantage of the padding shown in the top, middle image. Each disc here can be scaled by the square root of $12 - 2\sqrt{31} \approx 0.864$, and their closures will still cover $\mathbb{C}$. Since $(12 - 2\sqrt{31})(35 - \sqrt{713})/8$ is much smaller than 1, we decrease overall.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure15.png}
\caption{The sets $(2)^\bullet$ (top, left), $(p_2^2)^\bullet$ (bottom, left), and $M_4^\circ$ (right) in $\mathbb{Q}(\sqrt{-31})$.}
\end{figure}

All of the aforementioned arithmetic plays out in $M_4^\circ$. Consider the matrix
\[
M = \begin{bmatrix}
17 - 28\tau & -78 - 58\tau \\
25 & 57 - 13\tau
\end{bmatrix} \in M_{p_2^2}.
\]
It is fully trimmed, and $-\delta/\beta$, the small dot in the bottom, left image, is not covered. The centered point in the extended Schmidt arrangement is $\alpha/\beta$. The circle to its left passing through it is $M^\circ$, which has reduced curvature $-65$. If $M_4^\circ$ were monotonically connected there would be a circle neighboring the family at $\alpha/\beta$ with reduced curvature strictly smaller than 65.\footnote{Recall that we can always make a chain of circles starting with $M^\circ$ at $\alpha/\beta$ in which every other reduced curvature decreases in magnitude.} The two suspiciously large circles highlighted...
in green, however, have reduced curvature $66$ and $-66$, and the one highlighted in orange has reduced curvature $-68$. So the connectivity cannot be monotonic. But if we shift and reflect $-\delta/\beta$ with one of the nearby discs of radius $\sqrt{2}$, thereby switching to the green covering, we move to one of the green circles in $\mathcal{M}_c'$. On the very next step the reduced curvature can drop drastically to $-6$ or $1$. This is an admissible chain for 2-connectivity. Using the nearby disc of radius $2$ does not create an admissible chain for 2-connectivity. This moves to the circle highlighted in orange that has nowhere to go.

The top, middle image also shows the idea behind $\varepsilon$ introduced in Definition 4.2. It measures the minimal padding between a point $z_n$ and the boundary of the disc over which it is reflected.

On one hand an algorithm that works over a $k$-connected arrangement benefits from allowing smaller values of $D$. This makes for a faster continued fraction algorithm and improves the approximation constants in the proofs of Section 5. On the other hand it presents the task of searching for some sequence of up to $k$ moves that, only when combined, improve approximation quality. This seems difficult when $k > 1$ so we will not do it unless we have to, and Proposition 4.7 says we do not have to. For the interested reader there is a study of “$k$-stage Euclidean algorithms” in [6]. Note, however, that the generalization to $k$-connectivity alone is not enough to create nearest integer-type continued fractions in a non-Euclidean field. Stange proved [13] that the non-Euclidean Schmidt arrangements are topologically disconnected—they do not just lack the monotonicity.

Given two ideals of the same norm we see that $D^* = D'^*$ whenever $[D] = [D']$, so the number of coverings needed to satisfy the hypothesis of Proposition 4.4 is at most the size of the principal genus. This observation provides a useful strategy for finding monotonically connected arrangements. Namely, if we can produce just one ideal, say $d$, with $d^* = \mathbb{C}$, then it can be multiplied by some $f \subseteq \mathcal{O}_K$ whose factors can be conjugated (recall our technique) to maneuver among ideal classes. That is, we can set $D = \|df\|$ where the sole purpose of an ideal of norm $\|f\|$ is to adjust the determinant, making the cover by $d^*$ ever-relevant.

**Lemma 4.5.** If an ideal $f \subseteq \mathcal{O}_K$ is contained in at least one ideal from each class, then the same is true for every ideal of norm $\|f\|$.

**Proof.** Fix a prime $p$ containing $f$, and let $e \leq f = \text{ord}_p f$. We must show that any ideal class, say $[a]$, contains a divisor of $p^e f/p^e$. Take $g \supseteq f$ that represents $[ap^e]$, and let $g = \text{ord}_p g$. If $e \leq g$, then $p^e f/p^e \subseteq g/p^e \subseteq \mathcal{O}_K$, and $[g/p^e] = [a]$. In the case $e > g$, we have $p^e f/p^e \subseteq p^e g/p^g \subseteq p^e g/\|p\|^g \subseteq \mathcal{O}_K$, and $[p^e g/\|p\|^g] = [p^e g] = [a]$. \qed
Proposition 4.6. If $\mathfrak{d}^* = \mathbb{C}$ and $\mathfrak{f}$ is contained in at least one ideal from each class, then $\mathcal{O}_K$ is $D$-Euclidean with $D = \|\mathfrak{d}\|$.

Proof. Suppose $\|\mathfrak{d}\| = D$ and $\mathfrak{f} \supseteq \mathfrak{D}$. If $\mathfrak{g} \supseteq \mathfrak{f}$ is an ideal in $\sqrt{[\mathfrak{df}/\mathfrak{D}]}$, then with $u = \mathfrak{df}/\mathfrak{Dg}$ we have $u\mathfrak{D} \subseteq \mathfrak{d}$. This implies $\mathfrak{d}^* \supseteq \mathfrak{d}^* = \mathbb{C}$.

The proof that a monotonically connected Schmidt arrangement exists has now been reduced to the search for one ideal $\mathfrak{d}$ that produces a covering $\mathfrak{d}^*$.

Proposition 4.7. Fix $\varepsilon > 0$ and let $\mathfrak{d}$ be such that there are at least $\sqrt{\frac{|\Delta|}{\varepsilon^2\sqrt{3}}}$ elements of $\mathcal{O}_K$ in $B(0, \mathfrak{d})$. If $\mathfrak{d}$ is divisible by every integer in $B(0, 2\mathfrak{d})$, then $\varepsilon\mathfrak{d}^* = \mathbb{C}$.

Proof. The covolume of the lattice in $\mathbb{C}$ defined by $\mathfrak{d}$ is $\|\mathfrak{d}\|\sqrt{|\Delta|}/2$, and it is well-known that the optimal packing (arrangement without intersections) of discs of radius $\varepsilon\sqrt{\|\mathfrak{d}\|}/2$ has covolume $\varepsilon^2\sqrt{3}\|\mathfrak{d}\|/2$ (it is the hexagonal lattice packing). So for any $z \in \mathbb{C}$ if we consider as centers for such discs points of the form $bz + a$ for $b \in B(0, \mathfrak{d})$ and $a \in \mathfrak{d}$, we see that $\mathfrak{d}$ has been chosen to give this set a covolume less than the minimum. Thus there are distinct $b, b' \in \mathcal{O}_K \cap B(0, \mathfrak{d})$ along with $a, a' \in \mathfrak{d}$ satisfying

$$\|(b - b')z - (a - a')\| < \varepsilon^2\|\mathfrak{d}\|,$$

meaning $z \in B\left(\frac{\varepsilon\sqrt{\|\mathfrak{d}\|}}{\sqrt{\|b - b'\|}}, \frac{a - a'}{b - b'}\right)$. Since $\mathfrak{d}/(b - b') \supseteq ((a - a')/(b - b'), \mathfrak{d})$ we have $z \in \varepsilon\mathfrak{d}^*$.

The strategy of taking $\|\mathfrak{d}\|$ is convenient for an existence proof, but it is not computationally effective. Smaller values of $D$ are preferred for continued fractions to increase the guaranteed quality of approximation. In the table below, the largest ideal $\mathfrak{d}$ satisfying the hypothesis of Proposition 4.7 with $\varepsilon = 1$ is given (we do not compute the padding). This is compared to the smallest $D$ for which $\mathcal{O}_K$ is $D$-Euclidean. These integers were found experimentally.

| $|\Delta|$ | 3     | 4     | 7     | 8     | 11    | 15    | 19    | 20    | 23    | 24    | 31    | 35    |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\mathfrak{d}$ | (6)   | (2)   | (4)   | (6)   | (6)   | (4)   | (2)   | (2)   | (2)   | (840) | (1980) |       |
| $D$      | 1     | 1     | 1     | 1     | 2     | 4     | 2     | 4     | 4     | 8     | 8     | 5     |
|          | 39    | 40    | 43    | 47    | 51    | 52    | 55    | 56    | 59    | 67    | 68    |       |
|          | (240) | (4620)| (1716)| (84)  | (780) | (1092)| (336) | (420) | (60)  | (12)  | (12)  |       |
|          | 8     | 8     | 36    | 12    | 33    | 44    | 8     | 12    | 36    | 144   | 24    |       |
|          | 71    | 79    | 83    | 84    | 87    | 88    | 91    | 95    |       |       |       |       |
|          | (12)  | (68640)| (956340)| (23100)| (157080)| (6117540)| (1400700)| (4680) |       |       |       |       |
|          | 24    | 40    | 81    | 24    | 48    | 144   | 140   | 48    |       |       |       |       |
The coverings associated to this table permit a continued fraction algorithm with many of the desired attributes—the approximations converge exponentially and are the best of the second kind up to constants, a finite expansion is equivalent to rationality, and an eventually periodic expansion is equivalent to quadracticity. But there are two proofs in Section 5 that require an additional assumption not satisfied by many of the table’s values.

**Notation 4.8.** Let \( \rho \) denote the maximal ratio of any two radii in a fixed subset of \( \bigcup_{\|\mathcal{D}\|=D} \varepsilon \mathcal{D}^* \).

Propositions 5.9 and its corollary require that \( \rho \) not be too large in relation to \( \varepsilon \). That is, if our discs just barely cover the plane then we will require them to have similar radii. Here the use of \( \mathfrak{d} \) and \( \mathfrak{f} \) makes for a straightforward existence proof again. The fact that \( d \) from Proposition 4.7 is related linearly to \( 1/\varepsilon \) is more than enough. We will make this explicit for the purpose of an example in the last section.

**Lemma 4.9.** Let \( S \subseteq \mathbb{R}^n \) be origin-symmetric, bounded, and convex. For a lattice \( L \subset \mathbb{R}^n \), the size of \( S \cap L \) is at least \( \frac{\text{vol} S}{2^n \det L} \).

**Proof.** Define \( f : \mathbb{R}^n \to \mathbb{Z} \) by letting \( f(x) \) be the size of \( \{ l \in L \mid x \in l + \frac{1}{2} S \} \). Then

\[
\int_{\frac{1}{2} S} \frac{dV}{f(x)} \leq \det L
\]

(with equality if and only if \( \bigcup_L (l + \frac{1}{2} S) = \mathbb{R}^n \)). But \( f(x) \leq \#(S \cap L) \).

**Proposition 4.10.** For any \( c > 1 \) there exist \( \varepsilon > 0 \) and \( D \in \mathbb{N} \) so that for any integral ideal \( \mathcal{D} \) of norm \( D \), the set \( \varepsilon \mathcal{D}^* \) admits a subcovering that makes \( \varepsilon^c \rho \) arbitrarily small.

**Proof.** Fix some \( \mathfrak{f} \subseteq \mathcal{O}_K \) contained in at least one ideal from each class. With \( \mathfrak{d} \) as in Proposition 4.7, we saw that only discs of radius at least \( \sqrt{\|\mathfrak{d}/b\|} \) with \( b \in B(0,2d) \) were needed to form a covering. Thus we may take \( \rho = 2d \sqrt{\|\mathfrak{f}\|} \). But by the lemma,

\[
d = \frac{1}{\varepsilon} \sqrt{\frac{2|\Delta|}{\pi \sqrt{3}}} \quad (8)
\]

is enough to satisfy the hypothesis of Proposition 4.4. So let \( \varepsilon \) be sufficiently small to prove the claim. \( \square \)
5. Continued fractions

We are set to perform a geometric procedure similar to that used in producing continued fractions in the five Euclidean rings. However, instead of writing a nearest integer algorithm we will write a close enough integer algorithm. “Nearest” in the Euclidean case means partitioning the complex plane into either rectangles or hexagons centered at lattice points. But when $D > 1$ the pieces of the partition are not polygons. They are shown here for the coverings from Figure 14.

![Figure 16. The nearest integer partition using $(2)^\bullet$ (left) and $(p_2)^\bullet$ (right) in $\mathbb{Q}(\sqrt{-23})$.](image)

Avoiding such complexity, the results in this section are valid for any method of selecting a particular disc containing a point. This reader’s choice occurs in step 3 of Algorithm 5.1. The effect can be observed throughout this section, particularly in Figure 19.

In any case, the following algorithm produces continued fractions in an imaginary quadratic field, regardless of Euclideaneity. Its functioning depends on the parameters $D$ and $\varepsilon$, to which we devoted Section 4. A pair satisfying Definition 4.3 can be found by drawing $\varepsilon \mathfrak{D}^\bullet$ for each ideal $\mathfrak{D} \subseteq \mathcal{O}_K$ of norm $D$ and checking for a covering.\textsuperscript{10} Alternatively, Propositions 4.6 and 4.7 implicitly provide a working $D$ and $\varepsilon$ for any discriminant. Once a pair is selected, an initial matrix, $M_0$, is the final parameter required. Classically (with $D = 1$ in a Euclidean field) we take $M_0$ to be the identity, and the fact that other choices exist is often ignored. But its im-

\textsuperscript{10}This procedure was used to create the final row in the table at the end of Section 4 for $|\Delta| < 100$. 

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impact on runtime and approximation quality are notable from Proposition 5.3 through Proposition 5.9. We remark further on this matrix following its appearance in (11).

**Algorithm 5.1.** Set parameters $D \in \mathbb{N}$ with $(D, \Delta) = 1$ and $\varepsilon < 1$ for which $\mathcal{O}_K$ is $D$-Euclidean (see Definition 4.3) as well as a trimmed matrix $M_0 \in \mathcal{M}_{D_0}$ (see Definitions 2.5 and 2.13) for some integral ideal $\mathfrak{D}_0$ of norm $D$. Input $z \in \mathbb{C}$ to be approximated and a termination level $N \in \mathbb{N}$. Set $n = 0$.

1. Set $z_n = M_n^{-1}(z)$. Let $p_n$ (top) and $q_n$ (bottom) be the left column of $M_n$.

2. Solve modulo $\mathfrak{D}_n = \det M_n/(M_n)^2$ for some $a' \in \mathcal{O}_K$ so that $M_n S(a')$ is fully trimmed (see Definitions 2.13 and 2.15).

3. (Euclidean step) Search among the finitely many $u_n \in \mathcal{O}_K^*(\mathfrak{D}_n)$ (see Notation 2.18) and $a \in \mathcal{O}_K \cap B(u_n z_n - u_n a', \sqrt{D})$ until $u_n z_n \in B(a_n, \varepsilon \sqrt{\|b_n\|})$, where $a_n = u_n a' + a$ and $b_n = (a, u_n \mathfrak{D}_n)$.

4. Set $M_{n+1} = M_n T(u_n) S(a_n) R(\|b_n\|)$ (see Definition 2.15).\(^{11}\)

5. Terminate if $u_n z_n = a_n$ or if $n = N$, and output the sequence of approximations $(p_n/q_n)_n$. Otherwise, increment $n$ and return to 1.

Let us simplify the algorithm to something more familiar when $\mathcal{O}_K$ is Euclidean and we choose $D$ to be 1 and $M_0$ to be the identity. Step 1 is true to tradition, and step 2 is immaterial as we will remain in $\text{SL}_2(\mathcal{O}_K)$ where matrices are already fully trimmed (let $a' = 0$). So we focus on step 3. Here the choice of $u_n \in \mathcal{O}_K^*(\mathfrak{D}_n) = \mathcal{O}_K^*(\mathcal{O}_K) = \mathcal{O}_K^*$ amounts to taking a rotation of discs centered on the possible choices of $a_n = a \in \mathcal{O}_K$. For a general $D$ it may be that only one rotation covers $z_n$, but when $D = 1$ these discs have uniform radius $\varepsilon \sqrt{\|b_n\|} = \varepsilon \sqrt{\|(a, u_n \mathcal{O}_K)\|} = \varepsilon$. Such a covering is unaffected by unit rotation, so we can follow Hurwitz and always use $u_n = 1$. Now our search is for an integer within $\varepsilon$ of $z_n$. Taking $a_n$ to be the closest and noting that step 4 becomes $M_{n+1} = M_n T(u_n) S(a_n) R(\|b_n\|) = M_n S(a_n) R(1)$, we have recovered the Hurwitz algorithm.

The biggest adjustments when $D > 1$ occur in steps 2 and 3. In our discussion of a generalized Euclidean algorithm in Section 4, we saw that if

$$M_n = \begin{bmatrix} p_n & \|b_n-1\| p_{n-1} \\ q_n & \|b_{n-1}\| q_{n-1} \end{bmatrix} \in \mathcal{M}_{D_n}$$

\(^{11}\)If $N$ is large, at this step we could divide the entries of $M_{n+1}$ by their greatest common principal divisor to avoid overwhelming growth.
is fully trimmed, then a covering by $\varepsilon D^\bullet_n$ can be utilized. The point is that

$$a_p + \|b_{p-1}\|_p, \quad a_p + \|b_{p-1}\|_p \in (M_n)D_n$$

needs to be a solvable system of congruences to have an existence guarantee in step 3. This is what the definition of fully trimmed, and hence step 2, is designed to give.

Then, in the most time consuming step, we look for a some disc in $\varepsilon D^\bullet_n$ that contains $z_n$. Step 3 gives the first incentive to choose $D$ as small as possible. At worst, an exhaustive search in a disc of radius $\sqrt{D}$ will check $O(D/\sqrt{\Delta})$ integers. We have $D$ as a crude bound for $|\mathcal{O}_K^\ast(D_n)|$ (the set can be precomputed), and we can compute $\|b_n\|$ with $O(\log D)$ basic arithmetic operations as it must divide $D$. Thus this step, and therefore a loop through steps 1 to 5, requires $O(D^2 \log D/\sqrt{\Delta})$ operations for $D > 1$, adding appeal to the bottom row of Section 4’s table.

Finally in step 4, we rotate $z_n$ by $u_n$, shift by $a_n$, and reflect over the disc of radius $\sqrt{\|b_n\|}$ centered at the origin. Note that arithmetically, the reflection serves to trim $M_{n+1}$ in case $b_n \neq \mathcal{O}_K$. This was observed in Section 2.

As an example we take $D = 12$, $\varepsilon = 0.96$, $z = \pi + \sqrt{-2}$, and $M_0 = R(\tau)$, all in $\mathbb{Q}(\sqrt{-47})$. In Figure 17 we see that the three ideals (up to conjugation) of norm 12 all provide a covering, and this holds even after scaling radii by 0.96. So we may execute Algorithm 5.1. To avoid including 17 different images, $z_n$ and $a_n$ are displayed after being shifted by $a'$ from step 2. So the location of $z_n$ in the complex plane is not displayed, but rather its position relative to $a_n$ and the covering by $\mathcal{D}_n^\bullet$ in general. The values of $\sqrt{\|b_n\|}$ can also be seen as the radii of discs.

![Figure 17](image-url)

**Figure 17.** The points $z_n$ after shifting by $a'$ in the same expansion of $\pi + \sqrt{-2}$. From the left to right: $(\tau)^\ast$, $2p_3^\ast$, and $p_3^2p_3^\ast$. Discs of unused radii are omitted for clarity.

The “farthest integer” version is used here, meaning that in step 3, $a_n$ and $u_n$ are chosen to maximize $\|(u_n z_n - a_n)/b_n\|$ under the stipulation that it is less than
0.96^2. The resulting approximations are shown in Figure 18. The algorithm bumbles through the extended Schmidt arrangement, clearly missing better routes, but eventually making its way to \( z \). The nearest integer version approaches \( z \) more directly, but it makes for a less interesting picture since within four or five steps the curvatures become too large to draw accurately on a modest machine.

![Figure 18. Convergents in a continued fraction expansion of \( \pi + \sqrt{-2} \) using \( \mathcal{M}_{12} \) in \( \mathbb{Q}(\sqrt{-47}) \). Colors indicate the ideal class of \((p_n, q_n)\) up to conjugation, and correspond to Figure 17. The increase in zoom with each successive image is roughly tenfold.](image)

We note that the only ideal class with a nontrivial unit is the principal one, \((\tau)\), and the argument of the unit is \( \arctan(\sqrt{47}/23) \approx 0.09\pi \). This is the angle of intersection at the green (the color used when \((p_n, q_n)\) is principal) points labeled 3, 4, and 16. All other intersections are tangential.
The data behind these figures is displayed in the table below. It can be checked that \( \|u_n z_n - a_n\| \) is never more than 0.96^2\(\|b_n\|\).

| n \(\approx z_n\) | 0  | 1  | 2  | 3  | 4  |
|-----------------|----|----|----|----|----|
| \(a_n\)        | −3.3 + 11.5\(i\) | 0.8 + 2.4\(i\) | −1.9 + 2.7\(i\) | −1 + 2\(i\) | −3 − 5.2\(i\) |
| \(\|b_n\|\)    | 6  | 4  | 2  | 12 | 6  |
| \(u_n\)        | 1  | 1  | 1  | −1 + \(\frac{1}{12}\)\(\tau\) | −1 + \(\frac{1}{12}\)\(\tau\) |
| \(p_n\)        | ∞  | \(-\frac{6+4\tau}{\tau}\) | \(-\frac{21+\tau}{-6+\tau}\) | \(-\frac{15-2\tau}{5}\) | \(\frac{7+5\tau}{4+\tau}\) |
| \(q_n\)        |     |     |     |     |     |

| 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|----|
| −2.5 + 0.8\(i\) | −1.2 + 1.8\(i\) | −1.8 + 1.8\(i\) | 1.5 − 2.3\(i\) | −1.8 − 2\(i\) | 3.5 − 1.9\(i\) |
| −2 | \(\tau\) | −3 | \(3 - \tau\) | −4 − \(\tau\) | 5 − \(\tau\) |
| 2 | 6 | 6 | 4 | 12 | 4 |
| 1 | 1 | 1 | 1 | 1 | 1 |
| \(6+25\tau\) | \(15-10\tau\) | \(42+10\tau\) | \(58+8\tau\) | \(3+5\tau\) | \(98-59\tau\) |
| \(12+6\tau\) | \(-3\tau\) | \(16+\tau\) | \(-13+4\tau\) | \(23+13\tau\) | \(3-18\tau\) |

| 11 | 12 | 13 | 14 | 15 | 16 |
|----|----|----|----|----|----|
| −1.2 − 1.8\(i\) | −0.9 − 2.1\(i\) | −4.5 − 4.4\(i\) | −2.4 + 1.2\(i\) | −0.5 − 1.6\(i\) | 3.2 − 3\(i\) |
| −\(\tau\) | 1 − \(\tau\) | −2 − \(\tau\) | −2 | −2 − \(\tau\) | \(\tau\) |
| 4 | 12 | 6 | 2 | 12 | 12 |
| 1 | 1 | 1 | 1 | 1 | −1 + \(\frac{1}{12}\)\(\tau\) |
| \(175−32\tau\) | \(-148+138\tau\) | \(-252+55\tau\) | \(-40-51\tau\) | \(-292+56\tau\) | \(181-108\tau\) |
| \(36-14\tau\) | \(-96-27\tau\) | \(-69+10\tau\) | \(-10-14\tau\) | \(-59+24\tau\) | \(6-33\tau\) |

Now we begin our analysis of the output of Algorithm 5.1. From the Möbius transformations used to move from \(M_n\) to \(M_{n+1}\) in step 4, we have

\[
z_{n+1} = \frac{\|b_n\|}{u_n z_n - a_n},
\]

which is the familiar recursion relation between consecutive \(z_n\)'s if \(b_n = O_K\) and \(u_n = 1\). Taking these values for \(b_n\) and \(u_n\) also highlights the unchanged relation among convergents,

\[
p_{n+1} = a_n p_n + \|b_{n-1}\|u_{n} p_{n-1} \quad \text{and} \quad q_{n+1} = a_n q_n + \|b_{n-1}\|u_{n} q_{n-1}.
\]
And letting \( b_n = u_n\|b_{n-1}\| \) gives

\[
u_0 M_0^{-1} \left( \frac{p_n}{q_n} \right) = a_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_{n-1}}{a_{n-1}},
\]

justifying our persistent use of the name “continued fraction.”

As mentioned, traditional continued fractions take the parameter \( M_0 \) to be the identity. In general we need to start with a matrix in \( M_{\mathfrak{D}_0} \) for an integral ideal \( \mathfrak{D}_0 \) of norm \( D \). At the end of this section we will see that \( M_0 = T(\delta_0) \) is always possible for some \( \delta_0 \in \mathcal{O}_K \), so that \( p_n/q_n \) is just a multiple of the continued fraction on the right side of (11). That is, when \( \mathcal{O}_K \) is \( D \)-Euclidean and \((D, \Delta) = 1\) there is necessarily a principal ideal \( (\delta_0) \) of norm \( D \). In fact (11) can be further simplified by reworking this paper with matrix entry denominators that are not units in place of determinants that are not units, allowing for the classic \( M_0 = \text{Id} \). Such a perspective makes it more natural to work over unions of extended Schmidt arrangements,\(^{12}\) which was avoided here for simplicity.

As \( n \) increases, \( \| \det M_n \| \) may grow exponentially when \( D \) is not 1—a consequence of using larger discs which was forewarned in Section 1. However, the growth all occurs in the waste factor \( (M_n) \), which can be scaled to have norm no more than a Minkowski bound [8]. When assessing quality of approximation, it is convenient to eliminate this ideal. For example, the monotonic connectivity of the underlying extended Schmidt arrangement translates directly into the monotonicity of the sequence \((\|q_n z - p_n\|/\|(M_n)\|)\)\(_n\), whereas the sequence \((\|q_n z - p_n\|)\)\(_n\) need not decrease monotonically. We will show now that \( M_n \) is always trimmed (otherwise step 2 may not be executable), meaning \((M_n) = (p_n, q_n)\). This reveals the more natural looking sequence \((\|q_n z - p_n\|/\|(p_n, q_n)\|)\)\(_n\).

**Lemma 5.2.** For any \( n \geq 0 \) the matrix \( M_n \) is trimmed in \( M_{\mathfrak{D}_n} \), which is to say \((M_n) = (p_n, q_n)\), and \( \mathfrak{D}_n \) has norm \( D \). Moreover, \((M_{n+1}) = b_n(M_n)\).

**Proof.** Assume both statements for some \( n \geq 0 \). By definition of \( b_n \), the greatest common divisor of the left column entries in \( M_n T(u_n) S(a_n) \), which are \( p_{n+1} \) and \( q_{n+1} \), is \( b_n(M_n) = b_n(p_n, q_n) \). Then right multiplication by \( R(\|b_n\|) \) leaves \( M_{n+1} \) trimmed, as discussed in Section 2. Finally, \( \|\mathfrak{D}_{n+1}\| = \| \det M_{n+1}/(M_{n+1})^2\| = \|\|b_n\| \det M_n/b_n^2(M_n)^2\| = \|\mathfrak{D}_n\| = D \).

\(^{12}\) This significantly improves approximation constants. There would be no need for taking a least common multiple in Proposition 4.7.
Proposition 5.3. For any \( n \) and \( z \), the output of Algorithm 5.1 satisfies
\[
\left\| q_{n+1}z - p_{n+1} \right\| < \varepsilon^2 \frac{\left\| q_nz - p_n \right\|}{\left\| (p_n, q_n) \right\|}.
\]

Proof. By the lemma and (9) we have
\[
\frac{\left\| (p_n, q_n) \right\| \cdot \left\| p_{n+1} - q_{n+1}z \right\|}{\left\| (p_n, q_n) \right\|} = \frac{\left\| b_n \right\| \cdot \left\| p_{n+1} - q_{n+1}z \right\|}{\left\| (q_nz - p_n) \right\|} = \frac{\left\| b_n \right\|}{\left\| M_{n+1}^{-1}(z) \right\|} = \frac{\left\| b_n \right\|}{\left\| z_{n+1} \right\|} = \frac{\left\| u_nz_n - a_n \right\|}{\left\| b_n \right\|}.
\]
Our choices of \( a_n \) and \( u_n \) make this expression at most \( \varepsilon^2 \). 

Corollary 5.4. If \( z = p/q \) for \( p, q \in \mathcal{O}_K \), then Algorithm 5.1 terminates in at most \(-\frac{1}{2} \log_{\varepsilon} \left\| (q_0p - p_0q)/(p_0, q_0) \right\| \) steps.

Proof. We have a sequence of nonnegative integers, \( \left\| (q_0p - p_0q)/(p_n, q_n) \right\|_n \), whose consecutive terms decrease by a factor of \( \varepsilon^2 \). 

The overlap among discs in \( \varepsilon D^* \) can grant a variety of choices for \( a_n \) and \( u_n \). The last corollary is asserting more than the existence of a sequence of choices terminating the algorithm at \( p_n/q_n = z \). It asserts that the algorithm is forced to terminate at such a point. So necessarily there is some \( n \) for which \( z_n \) equals \( u_n^{-1}a_n \) and is not contained in any other disc from the shifted copy of \( \varepsilon D_n^* \).

Corollary 5.5. If \( n > -\frac{1}{2} \log_{\varepsilon} \left\| (q_0z - p_0)/(p_0, q_0) \right\| \) then \( q_n \neq 0 \).

Proof. For those values of \( n > 0 \) with \( q_n = 0 \) we have
\[
1 = \left\| (q_nz - p_n)/(p_n, q_n) \right\| < \varepsilon^2 \left\| (q_0z - p_0)/(p_0, q_0) \right\|.
\]

Corollary 5.6. The convergents converge to \( z \). In particular, if \( q_n \neq 0 \) then \( \left\| z - p_n/q_n \right\| < c\varepsilon^2n \) for some constant independent of \( z \) and \( n \).

We now wish to show that the convergents produced by Algorithm 5.1 are essentially the best possible. Even in the last example with \( M_{12}^0 \) in \( \mathbb{Q}(\sqrt{-47}) \), where we tried to sabotage the results by taking the farthest admissible integer \( a_n \) at each stage, our approximations were the best of the second kind up to constants according to the next proposition.
Lemma 5.7. For any $n \geq \| q_0 / (p_0, q_0) \|$, if $q_{n+1} \neq 0$ we have
\[
\left\| 1 + \frac{\| b_n \| q_n}{q_{n+1} z_{n+1}} \right\| > c_0^2,
\]
where $c_0 = (1 - \varepsilon) / \max \{ 1, \rho \varepsilon \}$ with $\rho$ as in Notation 4.8.

Proof. Since $\| b_n \| / \| z_{n+1} \| < \varepsilon^2$, if $\| b_n q_n / q_{n+1} \| \leq 1$ we are done by the triangle inequality, so suppose this is not true. By rewriting $q_{n+1}$ and $z_{n+1}$ using (9) and (10) we get
\[
1 + \frac{\| b_n \| q_n}{q_{n+1} z_{n+1}} = \frac{q_{n+1} + (u_n z_n - a_n)q_{n+1} u_n z_n}{q_{n+1}} \left( 1 + \frac{\| b_{n-1} \| q_{n-1}}{q_{n+1} z_n} \right).
\]
(12)
We keep applying this as long as $\| b_i \| / q_{i+1} \| > 1$, which must stop at some point since $n \geq \| q_0 / (p_0, q_0) \|$, and the nonnegative integers $\| q_i / (p_i, q_i) \|$ cannot keep decreasing. Supposing this goes on for $m_0 < i \leq n$, we combine the resulting equation with $\| u_i z_i / b_i \| = \| b_{i-1} / b_i \| \cdot \| b_{i-1} / (u_{i-1} z_{i-1} - a_{i-1}) \| > \| b_{i-1} / b_i \| / \varepsilon^2$ giving
\[
\left\| 1 + \frac{\| b_n \| q_n}{q_{n+1} z_{n+1}} \right\| > \frac{1}{\varepsilon (n-m_0)} \frac{\| b_{m_0} \|}{\| b_n \|} \cdot \left\| 1 + \frac{\| b_{m_0} \| q_{m_0}}{q_{m_0+1} z_{m_0+1}} \right\| > \frac{1}{\rho \varepsilon (n-m_0)} \left( 1 + \frac{\| b_{m_0} \| q_{m_0}}{q_{m_0+1} z_{m_0+1}} \right).
\]
As noted before, the norm on the right is bounded by the triangle inequality.

Proposition 5.8. Let $c_1 = c_0 \sqrt{3/|D|}$. For any constant $c$ and $p, q \in \mathcal{O}_K$,
\[
(c_1 - \varepsilon)^2 \frac{\| q_0 z - p \|}{\| (p_0, q_0) \|} \leq \| qz - p \| \quad \text{whenever} \quad \| q \| < c^2 \frac{\| q_{n+1} \|}{\| (p_{n+1}, q_{n+1}) \|}.
\]
provided $n \geq \| q_0 / (p_0, q_0) \|$.

Proof. If $q_{n+1} = 0$ then the statement is vacuously true, so suppose $q_{n+1} \neq 0$. Take some $p, q \in \mathcal{O}_K$ and choose $m \in (M_{n+1})$ with minimal norm. Let
\[
\begin{bmatrix} x \\ y \end{bmatrix} = m M_{n+1}^{-1} \begin{bmatrix} p \\ q \end{bmatrix},
\]
so $x, y \in \mathcal{O}_{n+1}$. We can replace $p$ and $q$ with their expressions in terms of $x$ and $y$ and use $M_{n+1}^{-1}(z) = z_{n+1}$ to see that
\[
qz - p = (q_0 z - p_0) \left( \frac{\| b_n \| y}{m} - \frac{\| b_n \| x}{m z_{n+1}} \right).
\]
Now substitute $x = (mq - \|b_n\|y_{n+1})/q_{n+1}$ in the expression on the right to get

$$qz - p = (q_n z - p_n) \left( \frac{\|b_n\|y}{m} \left( 1 + \frac{\|b_n\|q_n}{q_{n+1}z_{n+1}} \right) - \frac{\|b_n\|q}{q_{n+1}z_{n+1}} \right).$$

The lemma takes care of the expression in the inner parentheses, so to bound the ratio $(qz - p)(q_n, p_n)/(q_n z - p_n)$ we focus on the other two pieces. For the first, the norm of $(p_n, q_n)\|b_n\|y/m$, which equals $(M_{n+1})\|b_n y/m\}$ by Lemma 5.2, is only smaller than $1/\|D_{n+1}\| = 1/D$ by at most a Minkowski bound [8], like $\sqrt{\Delta}/3$. For the second,

$$\| (p_n, q_n) \| b_n \| q \| \| q_{n+1} z_{n+1} \| < \| (p_{n+1}, q_{n+1}) \| b_n q \| \| q_{n+1} z_{n+1} \| < \frac{1}{\varepsilon^2} \| (p_{n+1}, q_{n+1}) q \| \| q_{n+1} \| < \frac{1}{\varepsilon^2 \varepsilon^2},$$

where the last inequality is our hypothesis.

To reemphasize, the quality of approximation does not suffer by an exponential function of $n$ when we permit choice on overlapping discs. Our bound depends only on $\varepsilon$, $\rho$, and $D$. An example of this phenomenon is shown in Figure 19, where Algorithm 5.1 has produced two different sequences of convergents for $z = -0.798 + 0.473i$ using $M^4$ in $\mathbb{Q}(\sqrt{-23})$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure19.png}
\caption{Convergents for expansions of $-0.798 + 0.473i$ using $M^4$ in $\mathbb{Q}(\sqrt{-23})$.}
\end{figure}
In both cases we began with $M_0 = T(2)$. The blue path shows the output from the farthest integer algorithm with $\varepsilon = 0.96$. The red path is from the nearest integer algorithm using the partitions in Figure 16, essentially taking $\varepsilon^2 = (13 - \sqrt{115})/3$ (so $\varepsilon \approx 0.87$). The result of loosening $\varepsilon$ is that more approximations are found of lower (but boundedly so) quality. Both paths ultimately find the approximation $-\frac{(4 + \tau)}{(2 + 2\tau)}$, but the latter finds it at $n = 3$, and the former not until $n = 6$.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
\approx -\frac{1}{2} \log_\varepsilon \left\| \frac{q_n - p_n}{(p_n, q_n)} \right\| & 0 & 2 & 3 & 21 & 25 & 27 & 53 & 57 & 137 & 139 \\
\approx \frac{1}{2} \log_\varepsilon \left\| \frac{q_n}{(p_n, q_n)} \right\| & -\infty & 0 & 17 & 17 & 45 & 47 & 34 & 64 & 66 & 137 \\
\approx -\frac{1}{2} \log_\varepsilon \left\| \frac{q_n - p_n}{(p_n, q_n)} \right\| & 0 & 5 & 8 & 16 & 20 & 41 & 45 & 52 & 55 & 72 \\
\approx \frac{1}{2} \log_\varepsilon \left\| \frac{q_n}{(p_n, q_n)} \right\| & -\infty & 0 & 6 & 10 & 17 & 20 & 43 & 48 & 55 & 56 \\
\hline
\end{array}$$

Mind that different values of $\varepsilon$ are used for the blue and red rows. With no logarithms the red rows are shrinking and growing exponentially faster. By Proposition 5.3, the first and third row increase by at least 1 at every stage.

The data set is too small (or our bound is too imprecise) for a meaningful application of Proposition 5.8, since $\varepsilon = 0.96$ gives $c_1 \approx 0.0038$. For any $c < c_1/\varepsilon$, even the blue approximations listed are much better than what the proposition guarantees.

Note from the second row that $\left\| \frac{q_n}{(p_n, q_n)} \right\|$ need not grow monotonically as it does for the nearest integer version in the Gaussian case [3]. Still, the table suggests that we can do much better than the bound $\left\| \frac{q_n}{(p_n, q_n)} \right\| \geq 1$ that we used to get Corollary 5.6. It appears as if this norm grows roughly like $1/\varepsilon^{2n}$. It turns out such growth is always exhibited by a subsequence of $(q_n)_n$ (at least every other one, as can be ascertained by (16) in the proof below). But to guarantee it for every $n$, the technique that follows requires that $\rho$ not be too large. The explicit bound will be seen in the proof and highlighted afterward.

**Proposition 5.9.** If $\rho$ is sufficiently small in relation to $\varepsilon$ there are positive constants so that for any $z \in \mathbb{C}$ and any $n > \left\| \frac{q_0}{(p_0, q_0)} \right\|$ we have

1. $c_2 \left\| (p_n, q_n)^2/a_nq_n^2 \right\| < \left\| z - \frac{p_n}{q_n} \right\| < c_3 \left\| (p_n, q_n)^2/a_nq_n^2 \right\|,$
2. $c_4/\varepsilon^{2n} < \left\| \frac{q_n}{(p_n, q_n)} \right\| \left\| \frac{q_n + 1}{(p_n + 1, q_n + 1)} \right\|,$
3. and $\left\| z - \frac{p_n}{q_n} \right\| < c_6/\varepsilon^{4n}.$

**Proof.** To prove 1), we begin with the identity

$$z - \frac{p_n}{q_n} = M_n(z_n) - \frac{p_n}{q_n} = \frac{\det M_n}{z_nq_n^2} \left( 1 + \frac{\|b_n-1\|q_n-1}{q_nz_n} \right)^{-1}. \quad (13)$$
An application of Lemma 5.7 then shows that
\[
\left\| z - \frac{p_n}{q_n} \right\| < \frac{D}{c_0^2} \left\| (p_n, q_n)^2 \right\|,
\]
which is almost the \( c_3 \) inequality, just with \( z_n \) in place of \( a_n \). But
\[
\left\| 1 - \frac{a_n}{u_n z_n} \right\| < \varepsilon^2 \frac{\| b_n \|}{\| z_n \|} < \varepsilon^4 \left\| \frac{b_n}{b_{n-1}} \right\| < \varepsilon^4 \rho^2.
\]
This means \( 1 - \varepsilon^2 \rho < \sqrt{\| a_n/z_n \|} < 1 + \varepsilon^2 \rho \), and the upper bound completes the proof of the \( c_3 \) inequality. To use (13) for the \( c_2 \) inequality we need the the upper bound version of Lemma 5.7, which can be found by first proving the \( c_5 \) inequality.

To this end, we have
\[
\frac{\det M_{n+1}}{\| b_n \|} = p_{n+1}q_n - p_nq_{n+1} = q_n(q_nz - p_n) \left( \frac{q_{n+1}}{q_n} + \frac{p_n - q_{n+1}z}{q_nz - p_n} \right).
\]
Now we take norms and apply (14) to the factor \( q_n(q_nz - p_n) \). The triangle inequality splits the last sum above, so that after applying Proposition 5.3 to the second summand and scaling everything by \( \sqrt{\| z_n/(p_n, q_n)(p_{n+1}, q_{n+1}) \|} \) we get
\[
\sqrt{\frac{\| z_n \| \| \det M_{n+1} \|}{\| b_n \| \| (p_n, q_n)(p_{n+1}, q_{n+1}) \|}} < \frac{\sqrt{D}}{c_0} \left( \sqrt{\frac{q_{n+1}(p_n, q_n)}{q_n(p_{n+1}, q_{n+1})}} + \varepsilon \right).
\]
Finally, we use Lemma 5.2 to rewrite the expression on the left as \( \sqrt{\| z_n \| \| \mathfrak{D}_{n+1}/b_n \|} > \sqrt{D}/\varepsilon \rho \) so that \( \sqrt{\| q_{n+1}(p_n, q_n)/q_n(p_{n+1}, q_{n+1}) \|} > c_0/\rho \varepsilon - \varepsilon \), which is the \( c_5 \) inequality. This shows the missing half of Lemma 5.7:
\[
\left\| 1 + \frac{\| b_n \| q_n}{q_{n+1}z_{n+1}} \right\| < \frac{c_0^2}{(c_0 - \rho \varepsilon)^2}, \quad \text{(15)}
\]
thus proving \( i \) in its entirety.

For the \( c_4 \) inequality we scale identity (12) by \( q_{n+1}/(p_{n+1}, q_{n+1}) \) and apply it along with Lemma 5.2 repeatedly to get
\[
\frac{q_{n+1}z_{n+1} + \| b_n \| q_n}{z_{n+1}(p_{n+1}, q_{n+1})} = \frac{q_{n+1}z_{n+1} + \| b_n \| q_n}{z_{n+1}(p_{n+1}, q_{n+1})} \prod_{i=n_0-1}^{n} \frac{u_i z_i}{b_i}.
\]
Then (15) gives an upper bound for the norm of the first expression as a multiple of \( \| q_{n+1}/(p_{n+1}, q_{n+1}) \| \), and the product on the right has norm at least \( 1/\rho^2 \varepsilon^{2(n-n_0)} \). So take \( n_0 = \| q_0/(p_0, q_0) \| \), allowing for Lemma 5.7 to bound the factor in front of the product.

This \( c_4 \) inequality pairs with the \( c_3 \) inequality to prove \( iii \).
The harshest restriction on \( \rho \) in this proof is found in (15). Recalling the value of \( c_0 \), it requires \( \rho^2 \varepsilon^3 < 1 - \varepsilon \) to be effective. Such coverings exist by Proposition 4.10, but \( \mathfrak{d} \) becomes significantly larger in norm than the ideals in the table from Section 4. For example, in \( \mathbb{Q}(\sqrt{-15}) \) the smallest ideal with a divisor from each class is \( \mathfrak{p}_2 \). Using \( \|f\| = \|p_2\| = 2 \) in (8) and its preceding formula for \( \rho \), the inequality \( \rho^2 \varepsilon^3 < 1 - \varepsilon \) becomes

\[
\varepsilon < \frac{\pi \sqrt{3}}{\pi \sqrt{3} + 240} \approx 0.022.
\]

We will not compute the smallest (in norm) \( \mathfrak{d} \) that works in Proposition 4.7 for this \( \varepsilon \), as the digits in a generator would likely fill a page.

The coefficients of these continued fraction expansions carry information outside of (11). For the nearest integer algorithm in the real numbers, boundedness of the coefficients is equivalent to \( z \) being badly approximable (the set \( \{\|qz - p\| \mid p, q \in \mathbb{Z}\} \) has a positive infimum). Also, periodicity of the coefficients indicates whether \( z \) is quadratic over \( \mathbb{Q} \). Both of these hold more generally with our algorithm.

**Corollary 5.10.** If \( z \in \mathbb{C} \) is badly approximable then all of its continued fraction expansions have coefficients bounded by an absolute constant. If \( \rho \) and \( \varepsilon \) satisfy the hypothesis of Proposition 5.9, then \( z \in \mathbb{C} \) is badly approximable only if it admits a continued fraction expansion with bounded coefficients.

**Proof.** If \( z \) is badly approximable then the coefficients in any of its expansions are bounded by the \( c_3 \) inequality in Proposition 5.9. This requires no assumption on \( \rho \).

Now suppose \( z \) admits a continued fraction expansion in which \( a_n \) is bounded. This assumption bounds the first summand of \( (a_n q_n + \|b_{n-1}\| u_{n-1} q_{n-1})/q_n = q_{n+1}/q_n \), and the second is bounded from above by the \( c_5 \) inequality of Proposition 5.9. So \( \|q_{n+1} (p_n, q_n)/q_n (p_{n+1}, q_{n+1})\| \) is bounded from above. This means that for any \( q \in \mathcal{O}_K \) we are guaranteed the existence of an \( n \) for which \( \|q_{n+1}/(p_{n+1}, q_{n+1})\| \) is barely big enough to apply Proposition 5.8. That is,

\[
c^2 \|q\| < \left\| \frac{q_{n+1}}{(p_{n+1}, q_{n+1})} \right\| < c' \|q\|
\]

for some \( c' \), where neither \( c' \) nor \( c \) (using the notation from Proposition 5.8) depend on \( q \). Therefore

\[
c_5 c_2 \left\| \frac{(q_{n+1}, p_{n+1})}{a_n q_{n+1}} \right\| < c_2 \left\| \frac{(p_n, q_n)}{a_n q_n} \right\| < \left\| \frac{q_n z - p_n}{(p_n, q_n)} \right\| < \left\| \frac{qz - p}{(c_1 - \varepsilon c)^2} \right\|
\]

(17)
for any $p \in \mathcal{O}_K$. But
\[
\frac{\|q(qz - p)\|}{\|q\|} < c' \frac{\|q(qz - p)\| \|(p_{n+1}, q_{n+1})\|}{\|q_{n+1}\|}.
\]
Comparing the left of (17) with the right of (18) shows that $\|q(qz - p)\|$ is not too small.

For roots of quadratic polynomials over $K$, the term “periodic” can only be used to describe an expansion once a method for choosing among a collection of overlapping discs has been settled upon. Without any such convention, if for distinct $m < n$ we get $z_m = z_n$ lying in the overlap of at least two discs in $\mathfrak{D}_m^\bullet$ (assume the same shifted covering happens to be relevant at both stages, so that this is also $\mathfrak{D}_n^\bullet$), then at the $n^{th}$ we may select the same disc used at the $m^{th}$ stage or not. What we aim to prove is that this choice is always available. Since the number of ideals of the given norm $D$ is finite, as is the number of shifts of their corresponding coverings (by $a'$ in Step 2 of Algorithm 5.1), it suffices to show that $\{z_n\}_n$ is finite. Our proof resembles Lagrange’s original 1770 proof [4] of periodicity for $\mathbb{Q}$.

**Proposition 5.11.** The set $\{z_n\}_n$ is finite if and only if $[K(z) : K] \leq 2$.

**Proof.** If $\{z_n\}_n$ is finite and $(z_n)_n$ is not then there are distinct $m, n \in \mathbb{N}$ with $M_m^{-1}(z) = z_m = z_n = M_n^{-1}(z)$. Note that $M_m$ cannot equal $M_n$ as Möbius transformations since this would violate the monotonicity of $\frac{\|q_n z - p_n\|}{\|(p_n, q_n)\|}$ shown in Proposition 5.3. Thus $M_m M_n^{-1}(z) = z$ shows that $z$ satisfies a quadratic (irreducible by Corollary 5.4) polynomial in $K$.

For the converse suppose $[K(z) : K] = 2$. Let $(w + x)/y = z$ with $w, x^2, y \in \mathcal{O}_K$, and set $v = (w^2 - x^2)/y \in \mathcal{O}_K$. Taking some expansion of $z$ with Algorithm 5.1, we apply $\text{adj} M_0$ to $z$ and rationalize the denominator to get $(w_0 + x_0)/y_0 = z_0$ where
\[
w_0 = \alpha_0 \beta_0 v + (\alpha_0 \delta_0 + \beta_0 \gamma_0)w + \gamma_0 \delta_0 y, \tag{19}\]
\[x_0 = x \det M_0 \quad \text{and} \quad y_0 = \beta_0^2 v + 2 \beta_0 \delta_0 w + \delta_0^2 y.\]
Now by applying $R(\|b_n\|)S(-a_n)T(\overline{u_n})$ to $z_n$ and rationalizing the denominator again, we can recursively define
\[
w_{n+1} = \|b_n\|(u_n w_n - a_n y_n), \quad x_{n+1} = -u_n \|b_n\| x_n, \tag{20}\]
and
\[y_{n+1} = \frac{(u_n w_n - a_n y_n)^2 - u_n^2 x_n^2}{y_n}\]
to get $z_{n+1} = (w_{n+1} + x_{n+1})/y_{n+1}$ and $x_{n+1} = x \det M_{n+1}$. Then

$$\frac{1}{\varepsilon} \left\| \frac{y_{n+1} \det M_n}{y_n \det M_{n+1}} \right\| < \left\| \frac{y_{n+1} z_{n+1}}{y_n b_n} \right\| = \left\| \frac{1}{b_n} \left( u_n z_n - a_n - \frac{2u_n x_n}{y_n} \right) \right\| \leq \left( \varepsilon + 2 \sqrt{\left\| \frac{x_n}{y_n} \right\|} \right)^2.$$  

Thus $\|y_{n+1}/\det M_{n+1}\|$ is bounded provided $\|x_n/\det M_n\|$ is bounded, which it is. This means the following expression is also bounded by a multiple of $\|\det M_{n+1}\|$: 

$$\left\| b_n \left( u_n z_n - a_n \right) + x_{n+1} \right\| = \left\| b_n \left( u_n w_n - a_n y_n \right) \right\| = \|w_{n+1}\|.$$  

But if instead of (19) and (20) we had used $\text{adj} M_n(z)$ to define $w_n$, $x_n$, and $y_n$ directly, we would have $x_n = x \det M_n$, which matches our recursive definition. Since $\{1, x\}$ is a basis for $K(z)/K$, the two definitions of $w_n$ and $y_n$ must match as well. This shows $w_n, x_n/x, y_n \in (M_n)^2$. Since $\det M_n/(M_n)^2 = \mathcal{O}_n$ always has norm $D$ (in particular its norm is bounded) and $\mathcal{O}^*_K$ is finite, our triples $(w_n, x_n, y_n)_n$ all come from a finite set.  

We remark that aside from being overly complicated, this proof applies equally well to continued fractions over $\mathbb{Q}$. The author is not aware of such a perspective (absent of choice on overlaps) in the literature.  

The set $\{z_n\}_n$ is shown in Figure 20 for $z = (3 + 5i)/4$ using $D = 1$ in $\mathbb{Q}(\sqrt{-11})$.

![Figure 20](image-url)
The covering is centered at the origin, and \( z \) is labeled 0. As it lies in both a red disc (centered at \( \tau \)) and a blue disc (centered at \( 1 + \tau \)), there are two admissible shifts and reflections with which we may begin. Their results are indicated by the red and blue arrows to 1 and 2 in the diagram. Dashed lines indicate negation. For example, when the point labeled \(-5\) is shifted and reflected over the blue disc centered at \(-1 + \tau\), it lands on the point labeled 4, not \(-4\).

To conclude we address a remark made in this section about the option to take \( M_0 = T(\delta_0) \), thereby simplifying the arithmetic relating \( p_n/q_n \) to the classic continued fraction in (11).

**Proposition 5.12.** Choose \( D \) with \( (\Delta, D) = 1 \). If \( \mathcal{O}_K \) is \( D \)-Euclidean, then every element of the principal genus contains an ideal of norm \( D \).

**Proof.** By Corollary 5.4 our algorithm terminates in finitely many steps given any rational input. This means \( \mathcal{M}_D \) covers all of \( K \), so we can apply Proposition 2.19.

The class of principal ideals belongs to the principal genus, verifying our claim on the existence of \( \delta_0 \in \mathcal{O}_K \) with norm \( D \).

The proof of Corollary 5.4 and the way it was just employed bear resemblance to the classical proof of the statement "Euclidean implies principle ideal domain." In a similar spirit, we have the following two propositions.

**Proposition 5.13.** Choose \( D \) with \( (\Delta, D) = 1 \). If \( \mathcal{O}_K \) is \( D \)-Euclidean, then every element of the class group contains an integral ideal whose norm divides \( D \).

**Proof.** Fix trimmed matrices \( M, M' \in \mathcal{M}_D \) with \( \alpha/\beta = \alpha'/\beta' \). Then since \( [(M)] = [(\alpha, \beta)] = [(\alpha', \beta')] = [(M')] \) we can scale \( M' \) to get \( (M) = (M') \). Note that this ideal is preserved by the action of \( \text{SL}_2(\mathcal{O}_K) \).

Now take any \( M'' \in \mathcal{M}_D \) and use Proposition 4.4 to get a rationally connected chain of elements from \( M'' \) back to \( M \). Applying Proposition 2.14 and the previous paragraph repeatedly along this chain allows us to assume \( (M'') = (M) \) and \( \| \det M''/M''^2 \| = D \). As \( \mathcal{M}_D \) covers all of \( K \), we use Proposition 2.20 to see that every ideal class has a representative \( \mathfrak{a} \) satisfying \( (M) \mathfrak{O}' \subseteq \mathfrak{a} \subseteq (M) \), where \( \mathfrak{O}' \) depends on the point representing \( [\mathfrak{a}] \). But if the collection of all such \( \mathfrak{a} \) represents every ideal class, then so do the ideals of the form \( \mathfrak{a}/(M) \) for our fixed \( (M) \). These have norm dividing \( D \).

In the definition of \( D \)-Euclidean, multiple coverings are required when the principal genus of the class group is nontrivial. Just one covering by \( \mathfrak{O}'^* \), however, still has an immediate connection to the Euclidean algorithm as demonstrated by the next proposition. In connection with our work in Section 4, it gives a geometric perspective to Markanda’s object of study in [7].
Proposition 5.14. Fix $\mathfrak{D} \subseteq \mathcal{O}_K$, and let $S \subset \mathcal{O}_K$ consist of those elements that generate ideals dividing some power of $\mathfrak{D}$. If $\mathfrak{D}^* = \mathbb{C}$ then $S^{-1}\mathcal{O}_K$ is norm-Euclidean. In particular, the ideal classes represented by the divisors of $\mathfrak{D}$ generate the class group of $\mathcal{O}_K$.

Proof. Even in the absence of $\mathcal{M}_\mathfrak{D}$ (since it is not assumed that $[\mathfrak{D}]$ is a square), we can still employ the Euclidean-like algorithm of Section 4.

Define a norm on $S^{-1}\mathcal{O}_K$ by $\|\alpha/s\|_\mathfrak{D} = \|\alpha/a\|$ where $a \supseteq (\alpha)$ is the maximal ideal for which $(\alpha/a, \mathfrak{D}) = \mathcal{O}_K$. Now take some $\alpha/s, \beta/t \in S^{-1}\mathcal{O}_K$ and let $b \subseteq (\beta)$ satisfy $\|\beta/b\| = \|\beta\|_\mathfrak{D}$. Let $b$ generate $b^n$ for some $n$ and pick some $\gamma \in b^{-1}$ with $(\gamma b, \mathfrak{D}) = \mathcal{O}_K$. A shifted cover of $C$ by $\mathfrak{D}^*$ is still a cover, allowing us to choose $a \in \mathcal{O}_K$ with $bt\alpha/\beta \in B(a, \sqrt{\|(bt\alpha\gamma - a\beta\gamma, \mathfrak{D})\|})$. Since $\|bt\alpha\gamma - a\beta\gamma, \mathfrak{D}\| = \|(bt\alpha - a\beta, b\mathfrak{D})/b\|$, this gives

$\left\| \frac{\alpha - a}{s} \cdot \frac{\beta}{t} \right\|_\mathfrak{D} = \|bt\alpha - a\beta\|_\mathfrak{D} \leq \frac{\|bt\alpha - a\beta\|}{\|(bt\alpha - a\beta, b\mathfrak{D})\|} < \frac{\|\beta\|}{\|t\|_\mathfrak{D}}$. \hfill $\Box$

Corollary 5.15. If $S \subset \mathcal{O}_K$ is the multiplicatively closed set generated by elements of norm at most $(8|\Delta|/\pi\sqrt{3})^h$, where $h$ is the class number of $\mathcal{O}_K$, then $S^{-1}\mathcal{O}_K$ is norm-Euclidean.

Proof. In (8) we saw that $d = \sqrt{2|\Delta|/\pi\sqrt{3}}$ was large enough to satisfy the hypothesis of Proposition 4.7 with $\varepsilon = 1$. This means that in taking $\mathfrak{D}$ to be the largest ideal divisible by the integers in $B(0, 2d)$, we get $\mathfrak{D}^* = \mathbb{C}$. To make sure that our set $S$ satisfies the hypothesis of the previous proposition for this $\mathfrak{D}$, suppose $(\eta) = p_1 \cdots p_k \supseteq \mathfrak{D}^n$ for some $\eta \in \mathcal{O}_K, n \in \mathbb{N}$, and prime ideals $p_1, \cdots, p_k$. If $\eta$ is not irreducible we are done by induction. Otherwise, we must have $k \leq h$, which completes the proof as $\|p_i\| \leq 8|\Delta|/\pi\sqrt{3}$ by construction of $\mathfrak{D}$. \hfill $\Box$

Propositions 5.13 and 5.14 are also generalizations of “Euclidean implies principal ideal domain.” For the former, the assertion that $\mathcal{O}_K$ is 1-Euclidean is exactly the statement that it is an honest Euclidean ring, and the only integral ideal with norm dividing 1 is principal. For the latter, $\mathcal{O}_K^* = \mathbb{C}$ is again the Euclidean condition, and the only integral divisor of $\mathcal{O}_K$ is principal. Also similar is that the converses of these two propositions frequently fail (as does the converse of Proposition 5.12). Just as many principal ideal domains are non-Euclidean, we have seen cases like the second and third images in Figure 13 from $\mathbb{Q}(\sqrt{-23})$, where we lack coverings yet still satisfy the conclusions of the last two propositions. The suggestion is that for an arbitrary imaginary quadratic field, our Euclidean-like algorithm and the resulting continued fractions are natural analogs of those belonging to the Euclidean rings.
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