A 2-Approximation Algorithm for Flexible Graph Connectivity

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Abstract

We present a 2-approximation algorithm for the Flexible Graph Connectivity problem [AHM20] via a reduction to the minimum cost r-out 2-arborescence problem.

1 Introduction

In this paper, we consider the Flexible Graph Connectivity (FGC) problem which was introduced by Adjiashvili, Hommelseim and Mühlenhalter [AHM20]. In an instance of FGC, we have an undirected connected graph $G = (V, E)$, a partition of $E$ into unsafe edges $U$ and safe edges $S$, and nonnegative costs $\{c_e\}_{e \in E}$ on the edges. The graph $G$ may have multiedges, but no self-loops. A subset $F \subseteq E$ of edges is feasible for FGC if for any unsafe edge $e \in F \cap U$, the subgraph $(V, F \setminus \{e\})$ is connected. We seek a (feasible) solution $F$ minimizing $c(F) = \sum_{e \in F} c_e$. The motivation for studying FGC is two-fold. First, FGC generalizes many well-studied survivable network design problems. Most notably, the minimum-cost 2-edge connected spanning subgraph (2ECSS) problem corresponds to an instance of FGC where all edges are unsafe. Second, FGC captures a non-uniform model of survivable network design problems where a subset of edges never fail, i.e., they are always safe. Adjiashvili et al. [AHM20] gave a 2.523-approximation algorithm for FGC. Our main contribution is a simple 2-approximation algorithm for FGC. At a high level, our result is based on a straightforward extension of the 2-approximation algorithm of Khuller and Vishkin [KV94] for 2ECSS.

Theorem 1. There is a 2-approximation algorithm for FGC.

Adjiashvili et al. [AHM20] also consider the following generalization of FGC. Let $k \geq 1$ be an integer. A subset $F \subseteq E$ of edges is feasible for the $k$-FGC problem if for any edge-set $X \subseteq F \cap U$ with $|X| \leq k$, the subgraph $(V, F \setminus X)$ is connected. The goal in $k$-FGC is to find a solution of minimum cost. The usual FGC corresponds to $1$-FGC. The following result generalizes Theorem 1.

Theorem 2. There is a $(k + 1)$-approximation algorithm for $k$-FGC.

Our proof of Theorem 2 is based on a reduction from $k$-FGC to the minimum-cost $(k + 1)$-arborescence problem (see [Sch03, Chapters 52 and 53]). We lose a factor of $k + 1$ in this reduction. Fix some $k$-FGC solution $F$ and designate a vertex $r \in V$ as the root vertex. For an edge $e = uv$, we call the arc-set $\{(u, v), (v, u)\}$ as a bidirected pair arising from $e$. The key idea in our proof is that there exists an arc-set $T$ that contains $k + 1$ arc-disjoint $r \to v$ dipaths for each $v \in V \setminus \{r\}$ while satisfying the following two conditions: (i) for an unsafe edge $e = uv \in F$, $T$ uses at most 2 arcs from a bidirected pair arising from $e$; and (ii) for a safe edge $e = uv \in F$, $T$ uses at most $k + 1$ arcs from the disjoint union of $k + 1$ bidirected pairs arising from $e$. This argument is formalized in Lemma 7. Complementing this step, we show that any arc-set $T$ (consisting of appropriate orientations of edges in $E$) that contains $k + 1$ arc-disjoint $r \to v$ dipaths for every $v \in V \setminus \{r\}$ can be mapped to a $k$-FGC solution.

2 A $(k + 1)$-Approximation Algorithm for $k$-FGC

For a subset of vertices $S$ and a subgraph $H$ of $G$, we use $\delta_H(S)$ to denote the set of edges in $H$ that have one endpoint in $S$ and the other in $V \setminus S$. The following characterization of $k$-FGC solutions is straightforward.

Proposition 3. $F$ is feasible for $k$-FGC $\iff \forall \emptyset \subseteq S \subseteq V, \delta_F(S)$ contains a safe edge or $k + 1$ unsafe edges.

For the rest of the paper, we assume that the given instance of $k$-FGC is feasible: this can be easily checked by computing a (global) minimum-cut in $G$ where we assign a capacity of $k + 1$ to safe edges and a capacity of 1 to unsafe edges. Let $D = (W, A)$ be a digraph and $\{c_a\}_{a \in A}$ be nonnegative costs on the arcs. We remark that $D$ may have parallel arcs but it has no self-loops. Let $r \in W$ be a designated root vertex. For a subgraph $H$ of $D$ and a set of vertices $S \subseteq W$, we use $\delta_H^{in}(S)$ to denote the set of arcs such that the head of the arc is in $S$ and the tail of the arc is in $W \setminus S$.
Definition 1 (r-out arborescence). An r-out arborescence \((W, T)\) is a subgraph of \(D\) satisfying: (i) the undirected version of \(T\) is acyclic; and (ii) for every \(v \in W \setminus \{r\}\), there is an \(r \rightarrow v\) dipath in \((W, T)\).

Definition 2 (r-out k-arborescence). For a positive integer \(k\), a subgraph \((W, T)\) is an r-out k-arborescence if \(T\) can be partitioned into \(k\) arc-disjoint r-out arborescences.

Theorem 4 [Sch03, Chapter 53.8]. Let \(D = (W, A)\) be a digraph and let \(k\) be a positive integer. For \(r \in W\), the digraph \(D\) contains an r-out k-arborescence if and only if \(|\delta^+_D(S)| \geq k\) for every nonempty \(S \subseteq V \setminus \{r\}\).

Claim 5. Let \((W, T)\) be an r-out k-arborescence for an integer \(k \geq 1\). Let \(u, v \in W\) be any two vertices. Then, the number of arcs in \(T\) that have one endpoint at \(u\) and the other endpoint at \(v\) (counting multiplicities) is \(\leq k\).

Proof. Since an r-out k-arborescence is a union of \(k\) arc-disjoint r-out 1-arborescences, it suffices to prove the result for \(k = 1\). The claim holds for \(k = 1\) because the undirected version of \(T\) is acyclic, by definition.

Theorem 6 [Sch03, Theorem 53.10]. In polynomial time, we can obtain an optimal solution to the minimum \(c^*\)-cost r-out k-arborescence problem on \(D\), or conclude that there is no r-out k-arborescence in \(D\).

The following lemma shows how a \(k\)-FGC solution \(F\) can be used to obtain an r-out \((k+1)\)-arborescence (in an appropriate digraph) of cost at most \((k+1)c(F)\).

Lemma 7. Let \(F\) be a \(k\)-FGC solution. Consider the digraph \(D = (V, A)\) where the arc-set \(A\) is defined as follows: for each unsafe edge \(e \in F \cap U\), we include a bidirected pair of arcs arising from \(e\), and for each safe edge \(e \in F \cap S\), we include \(k+1\) bidirected pairs arising from \(e\). Consider the natural extension of the cost vector \(c\) to \(D\) where the cost of an arc \((u, v) \in A\) is equal to the cost of the edge that gives rise to it. Then, there is an r-out \((k+1)\)-arborescence in \(D\) with cost at most \((k+1)c(F)\).

Proof. Let \((V, T)\) be a minimum-cost r-out \((k+1)\)-arborescence in \(D\). First, we argue that \(T\) is well-defined. By Theorem 4 it suffices to show that for any nonempty \(S \subseteq V \setminus \{r\}\), we have \(|\delta^+_D(S)| \geq k+1\). Fix some nonempty \(S \subseteq V \setminus \{r\}\). By feasibility of \(F\), \(\delta_F(S)\) contains a safe edge or \(k+1\) unsafe edges (see Proposition 3). If \(\delta_F(S)\) contains a safe edge \(e = uv\) with \(u \in S\), then by our choice of \(A\), \(\delta^+_D(S)\) contains \(k+1\) \((u, v)\)-arcs. Otherwise, \(\delta_F(S)\) contains \(k+1\) unsafe edges, and for each such unsafe edge \(uv\) with \(v \in S\), \(\delta^+_D(S)\) contains the arc \((u, v)\). Since \(|\delta^+_D(S)| \geq k+1\) in both cases, \(T\) is well-defined.

Finally, we use Claim 5 to show that \(T\) satisfies the required cost-bound. For each unsafe edge \(e \in F\), \(T\) contains at most 2 arcs from the bidirected pair arising from \(e\), and for each safe edge \(e \in F\), \(T\) contains at most \(k+1\) arcs from the (disjoint) union of \(k+1\) bidirected pairs arising from \(e\). Thus, \(c(T) \leq 2c(F \cap U) + (k+1)c(F \cap S) \leq (k+1)c(F)\).

Lemma 7 naturally suggests a strategy for Theorem 6 via minimum-cost \((k+1)\)-arborescences.

Proof of Theorem 6. Fix some vertex \(r \in V\) as the root vertex. Consider the digraph \(D = (V, A)\) obtained from our FGC instance as follows: for each unsafe edge \(e \in U\), we include a bidirected pair arising from \(e\), and for each safe edge \(e \in S\), we include \(k+1\) bidirected pairs arising from \(e\). For each edge \(e \in E\), let \(R(e)\) denote the multi-set of all arcs in \(D\) that arise from \(e\). For any edge \(e = uv \in E\) and arc \((u, v) \in R(e)\), we define \(c_{uv} := c_e\). Let \((V, T)\) denote a minimum \(c\)-cost r-out \((k+1)\)-arborescence in \(D\). By Lemma 7, \(c(T) \leq (k+1)OPT\), where \(OPT\) denotes the optimal value for the given instance of k-FGC.

We finish the proof by arguing that \(T\) induces a \(k\)-FGC solution \(F\) with cost at most \(c(T)\). Let \(F := \{e \in E : R(e) \cap T \neq \emptyset\}\). By definition of \(F\) and our choice of arc-costs in \(D\), we have \(c(F) \leq c(T)\). It remains to show that \(F\) is feasible for \(k\)-FGC. Consider a nonempty set \(S \subseteq V \setminus \{r\}\). Since \(T\) is an r-out \((k+1)\)-arborescence, Theorem 4 gives \(|\delta^+_F(S)| \geq k+1\). If \(\delta^+_F(S)\) contains a safe arc (i.e., an arc that arises from a safe edge), then that safe edge belongs to \(\delta_F(S)\). Otherwise, \(\delta^+_F(S)\) contains some \(k+1\) unsafe arcs (that arise from unsafe edges). Since both orientations of an edge cannot appear in \(\delta^+_F(S)\), we get that \(|\delta_F(S) \cap U| \geq k+1\). Thus, \(F\) is a feasible solution for the given instance of \(k\)-FGC, and \(c(F) \leq (k+1)OPT\).

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