Stability of homogeneous steady states in a population model with both predator- and prey-taxis

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Abstract

We study the system

\[
\begin{align*}
    u_t &= D_1 \Delta u - \chi_1 \nabla \cdot (u \nabla v) + u(\lambda_1 - \mu_1 u + a_1 v) \\
    v_t &= D_2 \Delta v + \chi_2 \nabla \cdot (v \nabla u) + v(\lambda_2 - \mu_2 v - a_2 u)
\end{align*}
\]

(\#)

We prove that there exists a stable steady state \((u^*, v^*) \in [0, \infty)^2\), meaning that there is \(\varepsilon > 0\) such that, if \(u_0, v_0 \in W^{2,2}(\Omega)\) are nonnegative with \(\partial_\nu u_0 = \partial_\nu v_0 = 0\) in the sense of traces and

\[
\|u_0 - u^*\|_{W^{2,2}(\Omega)} + \|v_0 - v^*\|_{W^{2,2}(\Omega)} < \varepsilon,
\]

then there exists a global classical solution \((u, v)\) of (\#) with initial data \(u_0, v_0\) converging to \((u^*, v^*)\) in \(W^{2,2}(\Omega)\). Moreover, the convergence rate is exponential, except for the case \(\lambda_2 \mu_1 = \lambda_1 a_2\), where it is only algebraical.

Key words: double cross diffusion; large-time behavior; predator–prey; stability

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1. Introduction

Migration-influenced predator–prey interaction can mathematically be described by the system

\[
\begin{align*}
    u_t &= D_1 \Delta u + \nabla \cdot (\rho_1(u, v) \nabla v) + f(u, v), \\
    v_t &= D_2 \Delta v + \nabla \cdot (\rho_2(u, v) \nabla u) + g(u, v)
\end{align*}
\]

(1.1)

Therein, \(u\) and \(v\) model the density of predators and prey, respectively. Apart from growth, death or intra-species competition, the functions \(f\) and \(g\) model predation: Encounters are beneficial for the predators and harmful to the prey. Moreover, the species are not only assumed to move around randomly (terms \(D_1 \Delta u\)
and $D_2 \Delta v)$, but also to be able to direct their movement toward (attractive taxis, negative $\rho_1$) or away from (repulsive taxis, positive $\rho_1$) higher concentration of the other species.

The relevance of attractive prey-taxis (‘predators move towards their prey’, negative $\rho_1$) has first been biologically verified in [10]. It has been observed that such an effect may actually reduce effective biocontrol, contradicting intuitive assumptions [12]. Moreover, the presence of (sufficiently strong) prey-taxis may actually lead to a lack of pattern formation [13].

Among systems of the form (1.1), those with only attractive prey- but no predator-taxis ($\rho_1 < 0$ and $\rho_2 \equiv 0$), have been studied most extensively—perhaps because they resemble attractive chemotaxis systems from a mathematical point of view, which in turn have been studied in comparatively great detail; see for instance the survey [2].

For $\rho_1(u, v) = -\chi u$ and several $f, g$, namely, the existence of globally bounded classical solutions to (1.1) has been proved in [22], provided $\chi > 0$ is sufficiently small. In two space dimensions, the smallness condition on $\chi$ is, again for various choices of $f$ and $g$, not necessary [9, 24], while in the three-dimensional setting, one may overcome this restriction by either assuming the prey-taxis to be saturated at larger predator quantities [6, 16] or by considering weak solutions instead [21].

Moreover, a repulsive predator-taxis mechanism (‘prey moves away from their predators, positive $\rho_2$) has, for instance, been detected for crayfish seeking shelter [4, 7, 12].

While less extensively studied than those with prey-taxis, such systems have been mathematically examined as well: Now without any smallness assumptions on $\chi$, globally bounded classical solutions to (1.1) have been constructed for $\rho_1 \equiv 0, \rho_2(u, v) = \chi v$ and certain $f, g$ in [23]. The same article also considered pattern formation and shows that a strong taxis mechanism (large $\chi$) leads to the absence of stable nonconstant steady states.

Combining both these effects ($\rho_1 < 0, \rho_2 > 0$) leads to the study of so-called pursuit–evasion models which have been proposed in [19] (see also [5, 20] for the modelling of related systems featuring different taxis mechanisms). There, propagating waves differing from those in taxis-free predator–prey systems have been detected numerically.

Apparently, the mathematical analysis on systems with both predator- and prey-taxis has been, up to now, limited to the one-dimensional case. Both with zeroth order terms [17] and without [18], Tao and Winkler constructed weak solutions which (under certain conditions) converge to spatially constant equilibria.

**Main results.** Making a first step towards extending the knowledge about such systems also in the higher dimensional setting, we analyze the stability of homogeneous steady states for (1.1) with the prototypical choices $\rho_1(u, v) = -\chi_1 u, \rho_2(u, v) = \chi_2 v$, $f(u, v) = u(\lambda_1 - \mu_1 u + a_1 v)$ and $g(u, v) = v(\lambda_2 - \mu_2 v - a_2 u)$ for $u, v \geq 0$. That is, we consider

$$
\begin{aligned}
&u_t = D_1 \Delta u - \chi_1 \nabla \cdot (u \nabla v) + u(\lambda_1 - \mu_1 u + a_1 v), \quad \text{in } \Omega \times (0, \infty), \\
v_t = D_2 \Delta v + \chi_2 \nabla \cdot (v \nabla u) + v(\lambda_2 - \mu_2 v - a_2 u), \quad \text{in } \Omega \times (0, \infty), \\
\partial_n u = \partial_n v = 0, \quad &\text{on } \partial \Omega \times (0, \infty), \\
u(., 0) = u_0, v(., 0) = v_0, \quad &\text{in } \Omega
\end{aligned}
$$

in smooth, bounded domains $\Omega$ for $D_1, D_2, \chi_1, \chi_2 > 0$ and $\lambda_1, \lambda_2, \mu_1, \mu_2, a_1, a_2 \geq 0$. Our main result is

**Theorem 1.1.** Suppose $\Omega \subset \mathbb{R}^n, n \in \{1, 2, 3\}$, is a smooth, bounded domain, and let

$$
D_1, D_2, \chi_1, \chi_2 > 0.
$$

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Suppose either
\[ \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = a_1 = a_2 = 0 \quad \text{and} \quad m_1, m_2 \geq 0 \] (H1)
or\[ \lambda_1, \lambda_2 \geq 0 \quad \text{and} \quad a_1, a_2, \mu_1, \mu_2 > 0. \] (H2)

Then there exist \( \varepsilon > 0 \) and \( K_1, K_2 > 0 \) with the following properties: For any
\[ u_0, v_0 \in W^{2,2}_N(\Omega) \] being nonnegative and, if (H1) holds, with \( \int_{\Omega} u_0 = m_1 \) and \( \int_{\Omega} v_0 = m_2, \) (1.4)

where
\[ W^{2,2}_N(\Omega) := \{ \varphi \in W^{2,2}(\Omega): \partial_{\nu}\varphi = 0 \text{ in the sense of traces} \}, \] (1.5)

and fulfilling
\[ \|u_0 - u_*\|_{W^2,2(\Omega)} + \|v_0 - v_*\|_{W^2,2(\Omega)} < \varepsilon, \] (1.6)

where
\[ (u_*, v_*) := \begin{cases} \left( \frac{m_1}{\mu_1}, \frac{m_2}{\mu_2} \right), & \text{if (H1) holds,} \\ \left( \frac{\lambda_1\mu_2 + \lambda_2a_1}{\mu_1\mu_2 + a_1a_2}, \frac{\lambda_2\mu_1 - \lambda_1a_2}{\mu_1\mu_2 + a_1a_2} \right), & \text{if (H2) holds and } \lambda_2\mu_1 > \lambda_1a_2, \\ \left( \frac{\lambda_2}{\mu_2}, 0 \right), & \text{if (H2) holds and } \lambda_2\mu_1 \leq \lambda_1a_2, \end{cases} \] (1.7)

there exist nonnegative functions
\[ u, v \in C^0([0, \infty); W^{2,2}(\Omega)) \cap C^\infty(\overline{\Omega} \times (0, \infty)) \]
solving (1.2) classically and converging to \((u_*, v_*)\) in the sense that
\[ \|u(\cdot, t) - u_*\|_{W^2,2(\Omega)} + \|v(\cdot, t) - v_*\|_{W^2,2(\Omega)} \leq \begin{cases} \left( \frac{1}{\lambda_1} + K_2t \right)^{-1}, & \text{if (H2) holds and } \lambda_2\mu_1 = \lambda_1a_2, \\ K_1\varepsilon e^{-K_1t}, & \text{else} \end{cases} \] (1.8)

for all \( t > 0. \)

**Remark 1.2.** Let us give some heuristic arguments why we believe that the rates in (1.8) are, up to the values of \( K_1 \) and \( K_2 \) therein, optimal.

For the heat equation, convergence is exponentially fast (take for instance an eigenfunction as initial datum) and adding taxis terms (but no terms of zeroth order) should not dramatically speed up the convergence. Moreover, in the around \((u_*, v_*)\) linearized ODE system, \((u_*, v_*)\) is a stable fixed point, provided (H2) with \( \lambda_2\mu_1 \neq \lambda_1a_2 \) holds. Hence, also here, ‘only’ an exponential convergence rate can be expected.

The case (H2) with \( \lambda_2\mu_1 = \lambda_1a_2 \) is different. As \( u \) converges to \( \frac{\lambda_1}{\mu_1}, \) one might expect that \( v \) behaves similarly as the solution \( \tilde{v} \) to
\[ \tilde{v}' = \tilde{v} \left( \lambda_2 - \mu_2\tilde{v} - a_2 \frac{\lambda_1}{\mu_1} \right) = -\mu_2(\tilde{v})^2, \]
which is given by
\[ \tilde{v}(t) = \frac{1}{\tilde{v}(0) + \mu_1 t}, \quad t \geq 0. \]
Main ideas. After obtaining local-in-time solutions by Amann’s theory in Lemma 2.1, we will focus our analysis on estimates holding in \( \Omega \times (0, T_\eta) \) for \( \eta > 0 \) to be fixed later, where \( T_\eta \in [0, \infty] \) is the maximal time up to which \( \| u - u_* \|_{L^\infty(\Omega)} + \| v - v_* \|_{L^\infty(\Omega)} < \eta \).

In the case of (H1), that is, without any cell proliferation, one computes
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (u - \overline{u})^2 + D_1 \int_\Omega |\nabla u|^2 = \chi_1 \int_\Omega u \nabla u \cdot \nabla v \quad \text{in } (0, T_{\text{max}}).
\]
The key idea is that one can rewrite the problematic term on the right hand side as
\[
\chi_1 \int_\Omega u \nabla u \cdot \nabla v = \chi_1 \int_\Omega (u - u_*) \nabla u \cdot \nabla v + \chi_1 u_* \int_\Omega \nabla u \cdot \nabla v \quad \text{in } (0, T_{\text{max}}).
\]
and note that, as the signs for the taxis terms in (1.2) are opposite, two problematic terms cancel out in calculating
\[
\frac{d}{dt} \left( \frac{\chi_2 v_*}{2} \int_\Omega (u - \overline{u})^2 + \frac{\chi_1 u_*}{2} \int_\Omega (v - \overline{v})^2 \right) + \chi_2 D_1 v_* \int_\Omega |\nabla u|^2 + \chi_1 D_2 u_* \int_\Omega |\nabla v|^2
\]
\[
= \chi_1 \chi_2 v_* \int_\Omega (u - u_*) \nabla u \cdot \nabla v - \chi_1 \chi_2 u_* \int_\Omega (v - v_*) \nabla u \cdot \nabla v \quad \text{in } (0, T_{\text{max}}).
\]
If \( \eta > 0 \) is chosen small enough, the remaining terms on the right hand side can be absorbed by the dissipative terms—at least in \( (0, T_\eta) \).

Fortunately, for higher order terms, one can proceed similarly and thus see that the sum (norms equivalent to the \( W^{2,2}(\Omega) \) norms of both solution components is decreasing, which implies \( T_\eta = T_{\text{max}} \), provided \( \eta > 0 \) is small enough and assuming \( T_\eta > 0 \), which can be achieved by choosing \( \varepsilon > 0 \) in Theorem 1.1 sufficiently small. Due to the blow-up criterion in Lemma 2.1, one also sees that \( T_{\text{max}} = \infty \). Convergence to \( (\overline{\pi}_\eta, \overline{\nu}_\eta) \) as well as the convergence rate are then merely corollaries of the estimates already gained.

For (H2), however, this idea alone is insufficient. For instance, if \( u_* > 0 \) and \( v_* > 0 \), arguing similarly as above, for any \( A_1, A_2 > 0 \) there is \( \eta > 0 \) such that
\[
\frac{d}{dt} \left( \frac{A_1}{2} \int_\Omega (u - u_*)^2 + \frac{A_2}{2} \int_\Omega (v - v_*)^2 \right) + \frac{A_1 m_1}{2} \int_\Omega (u - u_*)^2 + \frac{A_2 m_2}{2} \int_\Omega (v - v_*)^2 + \frac{A_1 D_1}{2} \int_\Omega |\nabla u|^2 + \frac{A_2 D_2}{2} \int_\Omega |\nabla v|^2
\]
\[
\leq (A_1 a_1 u_* - A_2 a_2 v_*) \int_\Omega (u - u_*)(v - v_*) + (A_1 \chi_1 u_* - A_2 \chi_2 v_*) \int_\Omega \nabla u \cdot \nabla v \quad \text{in } (0, T_\eta), \quad (1.9)
\]
see Lemma 3.2 and (the proof of) Lemma 4.3.

For the special case that \( (a_1, a_2) = \gamma(\chi_1, \chi_2) \) for some \( \gamma \geq 0 \), taking \( A_1 := \chi_2 u_* \) and \( A_2 := \chi_1 u_* \) already implies that the right hand side in (1.9) is zero. Alternatively, if \( D_1 \) and \( D_2 \) are sufficiently large compared to \( a_1, a_2, \chi_1, \chi_2, u_* \) and \( v_* \), the dissipative terms in (1.9) can be used to absorb the terms on the right hand side. In both these special cases, higher order terms can be handled similarly again so that we can conclude as above.

For arbitrary parameter values, such shortcuts are apparently unavailable. Actually, this is the reason for considering (1.2) with so many parameters: We want to emphasize that our approach does not rely on certain relationships between them.
Quite miraculously, appropriately choosing positive linear combinations of the six functionals
\[ \frac{d}{dt} \int_{\Omega} (u - u_*)^2, \quad \frac{d}{dt} \int_{\Omega} (v - v_*)^2, \quad \frac{d}{dt} \int_{\Omega} \vert \nabla u \vert^2, \quad \frac{d}{dt} \int_{\Omega} \vert \nabla v \vert^2, \quad \frac{d}{dt} \int_{\Omega} \vert u \vert^2 \quad \text{and} \quad \frac{d}{dt} \int_{\Omega} \vert v \vert^2 \] (1.10)
still allows for a cancellation of all problematic terms, see Lemma 4.3.

The remaining case, (H2) with \( \lambda_2 \mu_1 \leq \lambda_1 \alpha_2 \), is handled in Subsection 4.2. In a desire to keep the introduction at reasonable length, we just note here that the proofs also rely on the functionals in (1.10), albeit in a somewhat different fashion as in the first case, and refer for a more detailed discussion to (the beginning of) Subsection 4.2. Moreover, the in some sense degenerate case (H2) with \( \lambda_2 \mu_1 = \lambda_1 \alpha_2 \) deserves additional special treatment. We introduce a new functional in Lemma 4.6 and discuss directly beforehand why that seems to be necessary.

As a last step, in Lemma 5.1 we bring all these estimates together and prove global existence as well as convergence to \((u_*, v_*)\). Moreover, in Section 6, we discuss possible generalizations of Theorem 1.1.

Finally, in the appendix, we collect certain Gagliardo–Nirenberg-type inequalities used throughout the article. They might potentially be of independent interest and differentiate themselves from more often seen inequalities in two ways: Firstly, although we assume \( \Omega \) to be bounded, we get rid of the additional additive term on the right hand side. Secondly, instead of \( \|D^2 \varphi\|_{L^p(\Omega)} \) and \( \|D^1 \varphi\|_{L^p(\Omega)} \), our version contains only \( \|\Delta \varphi\|_{L^p(\Omega)} \) and \( \|\nabla \Delta \varphi\|_{L^p(\Omega)} \) (for certain values of \( p \in (1, \infty) \)).

2. Preliminaries

Local existence. Apparently, trying to prove local existence of classical solutions to (1.2) by following proofs for systems with a taxis term in just one equation (corresponding to either \( \chi_1 = 0 \) or \( \chi_2 = 0 \)) and thus building on the concept of mild solutions and Banach’s fixed point theorem or on Schauder’s fixed point theorem (see for instance [8] or [11], respectively) is not fruitful—at least if we want to consider both arbitrary nonnegative parameters and large initial data. Therefore, we resort to the abstract existence theory by Amann instead.

Lemma 2.1. Suppose that \( \Omega \subset \mathbb{R}^n \), \( n \in \mathbb{N} \), is a smooth, bounded domain, and let \( D_1, D_2, \chi_1, \chi_2 > 0 \) as well as \( \lambda_1, \lambda_2, \mu_1, \mu_2, a_1, a_2 \geq 0 \). Moreover, let \( p > n \) and \( u_0, v_0 \in W^{1,p}(\Omega) \) be nonnegative.

Then there exist \( T_{\text{max}} \in (0, \infty) \) and uniquely determined nonnegative \( u, v \in C^0([0, T_{\text{max}}); W^{1,p}(\Omega)) \cap C^\infty(\overline{\Omega} \times (0, T_{\text{max}})) \)
(2.1)
such that \((u,v)\) is a classical solution of (1.2) and, if \( T_{\text{max}} < \infty \), then
\[ \limsup_{t \uparrow T_{\text{max}}} \left( \|u(\cdot, t)\|_{C^\alpha(\Omega)} + \|v(\cdot, t)\|_{C^\alpha(\Omega)} \right) = \infty \quad \text{for all } \alpha \in (0, 1). \] (2.2)

Moreover, this solution further satisfies \( u, v \in C^0([0, T_{\text{max}}); W^{2,2}(\Omega)) \),
(2.3)
provided \( u_0, v_0 \) satisfy (1.4).

Proof. We will construct a solution \( U \) to
\[
\begin{align*}
U_t &= \nabla \cdot (A(U) \nabla U) + F(U), \quad \text{in } \Omega \times (0, T_{\text{max}}),
\nu \cdot A(U) \nabla U = 0, \quad \text{on } \partial \Omega \times (0, T_{\text{max}}),
U(\cdot, 0) = U_0, \quad \text{in } \Omega,
\end{align*}
\] (2.4)

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where

\[
\begin{pmatrix}
A(u) \\
F(u)
\end{pmatrix} := \begin{pmatrix}
D_1 & -\chi_1 u \\
\chi_2 v & D_2
\end{pmatrix}, \quad \begin{pmatrix}
F(u) \\
U
\end{pmatrix} := \begin{pmatrix}
(u(\lambda_1 - \mu_1 u + a_1 v)) \\
v(\lambda_2 - \mu_2 v - a_2 u)
\end{pmatrix} \quad \text{and} \quad U_0 := \begin{pmatrix}
u_0 \\
0
\end{pmatrix}
\]

for \( u, v \in \mathbb{R} \). Here and below, \( \nabla (u,v)^T := (\nabla u, \nabla v)^T, \nu \cdot (a,b)^T := (\nu \cdot a, \nu \cdot b)^T \) etc. for, say, \( u, v \in C^1(\Omega) \) and \( a, b \in \mathbb{R}^n \).

If \( u, v \geq 0 \), then \( \operatorname{tr} A((u,v)^T) = D_1 + D_2 > 0 \) and \( \det A((u,v)^T) = D_1D_2 + \chi_1 \chi_2 uv > 0 \), hence by continuity of the trace and the determinant, we may fix an (open) neighborhood \( D_0 \) of \([0, \infty)^2 \) in \( \mathbb{R}^2 \) such that the real parts of all eigenvalues of \( A((u,v)^T) \) are still positive for all \( u, v \in D_0 \).

Thus, \( (u,v) := (\nabla u, \nabla v)^T, \nu \cdot (a,b)^T := (\nu \cdot a, \nu \cdot b)^T \) etc. for, say, \( u, v \in C^1(\Omega) \) and \( a, b \in \mathbb{R}^n \).

Therefore, we may apply [1, Theorem 14.4, Theorem 14.6 and Corollary 14.7] to obtain \( T_{\max} > 0 \) and a unique \( U \in C^0([0, T_{\max}); (W^{1, p}(\Omega))^2 \cap (C^\infty(\Omega \times (0, T_{\max})))^2 \) solving (2.4) classically. Moreover, since both components of \( U \) are nonnegative by the maximum principle (for scalar equations), [1, Theorem 15.3] asserts that in the case of \( T_{\max} < \infty \) we have

\[
\limsup_{t \nearrow T_{\max}} \|U(\cdot, t)\|_{(C^\infty(\Omega))^2} = \infty \quad \text{for all} \; \alpha \in (0, 1).
\]

Thus, \( (u,v) := U^T \) satisfies the first, second and fourth equations in (1.2), if \( T_{\max} < \infty \), then (2.2) holds and, moreover, \( D_1 \partial_t u = \chi_1 u \partial_n v \) and \( D_2 \partial_t v = -\chi_2 v \partial_n u \) on \( \partial \Omega \times (0, T_{\max}) \). As \( u \) and \( v \) are nonnegative, \( \partial_t u = \frac{\partial \lambda_1}{\partial u} u \partial_n v = -\frac{\partial \chi_1}{\partial v} v \partial_n u \) on \( \partial \Omega \times (0, T_{\max}) \) implies \( \partial_t u \equiv 0 \) on \( \partial \Omega \times (0, T_{\max}) \). Analogously, we also obtain \( \partial_t v \equiv 0 \) on \( \partial \Omega \times (0, T_{\max}) \), hence \( (u,v) \) is the unique solution of regularity (2.1) to (1.2) in \( \Omega \times (0, T_{\max}) \).

Since [1, Theorem 4.1] further asserts that, for all \( t \in (0, T_{\max}) \), the operator \( \mathcal{A}(U(t)) \) in \( (L^2(\Omega))^2 \) with \( \mathcal{D}(\mathcal{A}(U(t))) = (W^{1, 2}(\Omega))^2 \) generates an analytical semigroup on \( (L^2(\Omega))^2 \), we may employ [1, Theorem 10.1] to obtain (2.3) for \( u_0, v_0 \in W^{2, 2}(\Omega) \).

Fixing parameters. In the sequel, we fix \( \Omega \subset \mathbb{R}^n, n \in \{1, 2, 3\} \), parameters as in (1.3) and (H1) or (H2), and define \((u_*, v_*)\) as in (1.7).

As we will see later in the proofs of Lemma 4.1 and Lemma 4.4, \( W^{2, 2}(\Omega) \) continuity of both solution components up to \( t = 0 \) will turn out to be crucial. By Lemma 2.1, this can be achieved if one supposes that \( u_0, v_0 \) satisfy (1.4). Given such initial data, we will denote the solution to (1.2) constructed in Lemma 2.1 by \( (u(u_0, v_0), v(u_0, v_0)) \) and its maximal existence time by \( T_{\max}(u_0, v_0) \). After fixing \((u_0, v_0)\), we will often for the sake of brevity write \((u, v)\) and \( T_{\max}\), respectively, instead. Also note that all constants below (for instance the \( c_i, i \in \mathbb{N} \), in several proofs) depend only on the parameters fixed above, not on \( u_0 \) and \( v_0 \).

The functions \( f \) and \( g \). Furthermore, we abbreviate

\[
f(u, v) := u(\lambda_1 - \mu_1 u + a_1 v) \quad \text{and} \quad g(u, v) := v(\lambda_2 - \mu_2 v - a_2 u) \quad \text{for} \; u, v > 0.
\]

Note that \( f(u_*, v_*) = 0 = g(u_*, v_*) \) and

\[
\begin{pmatrix}
f_u(u, v) \\
g_u(u, v)
\end{pmatrix} \begin{pmatrix}
f_u(u, v) \\
g_u(u, v)
\end{pmatrix} = \begin{pmatrix}
\lambda_1 - 2\mu_1 u + a_1 v & \frac{a_1 u}{-a_2 v} \\
-\frac{a_1 u}{\lambda_2 - 2\mu_2 v - a_2 u} & \lambda_2 - 2\mu_2 v - a_2 u
\end{pmatrix} \quad \text{for} \; u, v \geq 0,
\]
that is,
\[
\begin{pmatrix}
    f(u_*, v_*) \\
    g(u_*, v_*)
\end{pmatrix} =
\begin{pmatrix}
    0 & 0 \\
    0 & 0
\end{pmatrix},
\]
if (H1) holds,
\[
\begin{pmatrix}
    -\mu_1 u_* & a_1 u_* \\
    -a_2 v_* & -\mu_2 v_*
\end{pmatrix},
\]
if (H2) holds and \( \lambda_2 \mu_1 > \lambda_1 a_2 \),
\[
\begin{pmatrix}
    -\lambda_1 & a_1 u_* \\
    0 & \lambda_2 - \frac{\lambda_1 a_2}{\mu_1}
\end{pmatrix},
\]
if (H2) holds and \( \lambda_2 \mu_1 \leq \lambda_1 a_2 \).

Thus,
\[f(u_*, v_*) \leq 0\] as well as \( g(u_*, v_*) \leq 0\) \hspace{1cm} (2.5)
and
\[f(u_*, v_*) < 0\] as well as \( g(u_*, v_*) < 0\) \hspace{1cm} (2.6)

3. Estimates within \([0, T_\eta]\)

For \( u_0, v_0 \) satisfying (1.4) and \( \eta > 0 \), set
\[T_\eta(u_0, v_0) := \sup \{ t \in (0, T_{\max}(u_0, v_0)) : \|u(u_0, v_0) - u_\ast\|_{L^\infty(\Omega)} + \|v(u_0, v_0) - v_\ast\|_{L^\infty(\Omega)} < \eta \text{ in } (0, t) \} \] \hspace{1cm} (3.1)
(with the convention \( \sup \emptyset := -\infty \)). When confusion seems unlikely, we abbreviate \( T_\eta := T_\eta(u_0, v_0) \).

In the sequel, we will derive several estimates within \((0, T_\eta)\). Obviously, if \( (0, T_\eta) = \emptyset \), the statements below are trivially true. Thus upon reading the proofs, the reader might as well assume that \((0, T_\eta)\) is not empty. The only exception is Lemma 5.1, where we finally choose \( \varepsilon > 0 \) in (1.6) sufficiently small and guarantee that \( T_\eta > 0 \) for certain \( \eta > 0 \).

Note that \( T_{\eta_1} \leq T_{\eta_2} \) for \( \eta_1 \leq \eta_2 \). Moreover,
\[
\|u - u_\ast\|_{L^\infty(\Omega)} \leq \|u - u_\ast\|_{L^\infty(\Omega)} + \|v - v_\ast\|_{L^\infty(\Omega)} = \|u - u_\ast\|_{L^\infty(\Omega)} + \frac{1}{|\Omega|} \left| \int_\Omega (u - u_\ast) \right| \leq 2\eta \quad \text{in } (0, T_\eta)
\]
and likewise
\[
\|v - v_\ast\|_{L^\infty(\Omega)} \leq 2\eta \quad \text{in } (0, T_\eta)
\]
for all \( \eta > 0 \), where \((u, v, T_{\max}) = (u(u_0, v_0), v(u_0, v_0), T_{\max}(u_0, v_0))\) for any \( u_0, v_0 \) complying with (1.4).

In the remainder of this section, we derive estimates in \((0, T_\eta)\) for positive linear combinations of
\[
\begin{align*}
\frac{d}{dt} \int_\Omega (u - u_\ast)^2 & \quad \text{and} \quad \frac{d}{dt} \int_\Omega (v - v_\ast)^2, \\
\frac{d}{dt} \int_\Omega |\nabla u|^2 & \quad \text{and} \quad \frac{d}{dt} \int_\Omega |\nabla v|^2 \quad \text{as well as} \\
\frac{d}{dt} \int_\Omega |\Delta u|^2 & \quad \text{and} \quad \frac{d}{dt} \int_\Omega |\Delta v|^2.
\end{align*}
\]

We begin by treating the first pair in
Lemma 3.1. There is $\eta_0 > 0$ such that if $u_0, v_0$ comply with (1.4) and $(u, v) = (u(u_0, v_0), v(u_0, v_0))$ denotes the corresponding solution, then
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (u - u_*)^2 + \frac{3D_1}{4} \int_\Omega |\nabla u|^2 + (-f_u(u_*, v_* - \eta(a_1 + \mu_1)) \int_\Omega (u - u_*)^2 \\
\leq a_1 u_* \int_\Omega (u - u_*)(v - v_*) + \chi_1 u_* \int_\Omega \nabla u \cdot \nabla v + \frac{\eta \chi_1}{2} \int_\Omega |\nabla v|^2
\]
and
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (v - v_*)^2 + \frac{3D_2}{4} \int_\Omega |\nabla v|^2 + (-g_v(u_*, v_* - \eta(a_2 + \mu_2)) \int_\Omega (v - v_*)^2 \\
\leq -a_2 v_* \int_\Omega (u - u_*)(v - v_*) - \chi_2 v_* \int_\Omega \nabla u \cdot \nabla v + \frac{\eta \chi_2}{2} \int_\Omega |\nabla u|^2
\]
hold in $(0, T_\eta)$ for all $\eta \in (0, \eta_0)$, where $T_\eta$ is given by (3.1).

Proof. Let
\[
\eta_0 := \frac{1}{2} \min \left\{ \frac{D_1}{\chi_1}, \frac{D_2}{\chi_2} \right\}. 
\]  

Fixing $u_0, v_0$ satisfying with (1.4), by a direct calculation, we see that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (u - u_*)^2 + D_1 \int_\Omega |\nabla u|^2 = \chi_1 \int_\Omega u \nabla u \cdot \nabla v + \int_\Omega f(u, v)(u - u_*)
\]
holds in $(0, T_{\text{max}})$.

For any $\eta > 0$, we have therein by Young’s inequality
\[
\chi_1 \int_\Omega u \nabla u \cdot \nabla v = \chi_1 u_* \int_\Omega \nabla u \cdot \nabla v + \chi_1 \int_\Omega (u - u_*) \nabla u \cdot \nabla v \\
\leq \chi_1 u_* \int_\Omega \nabla u \cdot \nabla v + \frac{\eta \chi_1}{2} \int_\Omega |\nabla v|^2 + \frac{\eta \chi_1}{2} \int_\Omega |\nabla u|^2 \quad \text{in (0, } T_\eta).
\]

Moreover, as $f(u_*, v_*) = 0$,
\[
\int_\Omega f(u, v)(u - u_*) = \int_\Omega f(u_*, v_*)(u - u_*) + a_1 \int_\Omega u(v - v_*)(u - u_*) \\
= f_u(u_*, v_*) \int_\Omega (u - u_*)^2 + \frac{f_{uu}(u_*, v_*)}{2} \int_\Omega (u - u_*)^3 \\
+ a_1 \int_\Omega (u - u_*)^2(v - v_*) + a_1 u_* \int_\Omega (u - u_*)(v - v_*) \quad \text{in (0, } T_{\text{max}}).
\]

Since $f_{uu}(u_*, v_*) = -2\mu_1$, we may further estimate
\[
\frac{f_{uu}(u_*, v_*)}{2} \int_\Omega (u - u_*)^3 \leq \eta \mu_1 \int_\Omega (u - u_*)^2 \quad \text{in (0, } T_\eta) \quad \text{for all } \eta > 0
\]
and
\[
a_1 \int_\Omega (u - u_*)^2(v - v_*) \leq \eta a_1 \int_\Omega (u - u_*)^2 \quad \text{in (0, } T_\eta) \quad \text{for all } \eta > 0.
\]

Noting that (3.7) implies $D_1 \frac{\eta \chi_1}{2} \geq \frac{3}{4} D_1$, we may combine these estimates to obtain (3.5), while (3.6) follows from an analogous computation. \qed
For sufficiently small $\eta$ and suitable linear combinations of (3.5) and (3.6), the terms $\frac{\eta_0}{2} \int_{\Omega} |\nabla v|^2$ and $\frac{\eta_0}{2} \int_{\Omega} |\nabla u|^2$ can be absorbed by the dissipative terms therein.

**Lemma 3.2.** For any $A_1, A_2 > 0$, there is $\eta_0 > 0$ such that whenever $u_0, v_0$ satisfy (1.4), then the corresponding solution $(u, v) = (u(u_0, v_0), v(u_0, v_0))$ satisfies

\[
\frac{d}{dt} \left( \frac{A_1}{2} \int_{\Omega} (u - u_*)^2 + \frac{A_2}{2} \int_{\Omega} (v - v_*)^2 \right) + \frac{A_1 D_1}{2} \int_{\Omega} |\nabla u|^2 + \frac{A_2 D_2}{2} \int_{\Omega} |\nabla v|^2 \\
+ A_1 (f(u, v) - \eta(a_1 + \mu_1)) \int_{\Omega} (u - u_*)^2 + A_2 (-g_2(u, v, \eta(a_2 + \mu_2)) \int_{\Omega} (v - v_*)^2 \\
\leq (A_1 \chi_1 u_0 - A_2 \chi_2 v_0) \int_{\Omega} (u - u_*) (v - v_*) + (A_1 \chi_1 u_0 - A_2 \chi_2 v_0) \int_{\Omega} \nabla u \cdot \nabla v \quad \text{in} \ (0, T_\eta) \quad (3.8)
\]

for all $\eta < \eta_0$, where $T_\eta$ is as in (3.1).

**Proof.** Lemma 3.1 allows us to choose $\eta_1$ such that (3.5) and (3.6) hold in $(0, T_\eta)$. Let moreover $A_1, A_2 > 0$, fix $\eta_0 > 0$ sufficiently small such that

\[
\frac{A_2 \eta_2 \chi_2}{2} \leq \frac{A_1 D_1}{4} \quad \text{and} \quad \frac{A_1 \eta_2 \chi_1}{2} \leq \frac{A_2 D_2}{4}
\]

and set $\eta_0 := \min\{\eta_0, \eta_1\}$.

The statement then immediately follows upon multiplying (3.5) and (3.6) with $A_1$ and $A_2$, respectively, and adding these inequalities together. \qed

Next, we handle the second pair in (3.4), this time only in a coupled version.

**Lemma 3.3.** Let $B_1, B_2 > 0$. There is $\eta > 0$ such that for any $u_0, v_0$ complying with (1.4) we have

\[
\frac{d}{dt} \left( \frac{B_1}{2} \int_{\Omega} |\nabla u|^2 + \frac{B_2}{2} \int_{\Omega} |\nabla v|^2 \right) + \frac{B_1 D_1}{2} \int_{\Omega} |\Delta u|^2 + \frac{B_2 D_2}{2} \int_{\Omega} |\Delta v|^2 \\
\leq (B_1 \chi_1 u_0 - B_2 \chi_2 v_0) \int_{\Omega} \nabla u \cdot \nabla v + (B_1 \chi_1 u_0 - B_2 \chi_2 v_0) \int_{\Omega} \Delta u \Delta v \quad \text{in} \ (0, T_\eta),
\]

where again $(u, v) := (u(u_0, v_0), v(u_0, v_0))$ and $T_\eta := T_\eta(u_0, v_0)$ is given by (3.1).

**Proof.** Let $B_1, B_2 > 0$. We begin by fixing some parameters: By the Gagliardo–Nirenberg inequality A.3, there is $c_1 > 0$ such that

\[
\int_{\Omega} |\nabla \varphi|^4 \leq c_1 |\varphi|^2 \quad \text{and} \quad |\nabla \varphi|^2 \quad \text{for all} \ \varphi \in C^2(\overline{\Omega}) \text{ with} \ \partial_\Omega \varphi = 0 \text{ on} \ \partial \Omega. \quad (3.9)
\]

Choose $\eta > 0$ so small that

\[
M_1(\eta) := \frac{B_1 \eta \chi_1}{2} + \frac{B_2 \eta \chi_2}{2} + \frac{2 B_1 \eta^2 \chi_1^2 c_1}{D_1} + \frac{2 B_2 \eta^2 \chi_2^2 c_1}{D_2} + B_1 C_P \eta (2 \mu_1 + a_1) + \frac{B_1 C_P a_1 \eta}{2} + \frac{B_2 C_P a_2 \eta}{2}
\]

and

\[
M_2(\eta) := \frac{B_1 \eta \chi_1}{2} + \frac{B_2 \eta \chi_2}{2} + \frac{2 B_1 \eta^2 \chi_1^2 c_1}{D_1} + \frac{2 B_2 \eta^2 \chi_2^2 c_1}{D_2} + B_2 C_P \eta (2 \mu_2 + a_2) + \frac{B_1 C_P a_1 \eta}{2} + \frac{B_2 C_P a_2 \eta}{2},
\]

where $C_P$ is as in Lemma A.1, fulfill

\[
M_1(\eta) < \frac{B_1 D_1}{4} \quad \text{and} \quad M_2(\eta) < \frac{B_2 D_2}{4}. \quad (3.10)
\]
Fixing \( u_0, v_0 \) as in (1.4), we calculate

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 + D_1 \int_\Omega |\Delta u|^2 = \chi_1 \int_\Omega u \Delta u \Delta v + \chi_1 \int_\Omega \nabla u \cdot \nabla v \Delta u + \int_\Omega f(u, v)|\nabla u|^2 + a_1 \int_\Omega u \nabla u \cdot \nabla v
\]

\[
=: I_1 + I_2 + I_3 + I_4 \quad \text{in } (0, T_{\text{max}}).
\]

Therein is

\[
I_1 = \chi_1 u_* \int_\Omega \Delta u \Delta v + \chi_1 \int_\Omega (u - u_*) \Delta u \Delta v
\]

\[
\leq \chi_1 u_* \int_\Omega \Delta u \Delta v + \frac{\eta_1}{2} \int_\Omega |\Delta u|^2 + \frac{\eta_1}{2} \int_\Omega |\Delta v|^2 \quad \text{in } (0, T_\eta).
\]

Furthermore, by (3.9), (3.2) and Young’s inequality,

\[
I_2 \leq \frac{D_1}{4} \int_\Omega |\Delta u|^2 + \frac{\chi_1^2}{D_1} \int_\Omega |\nabla u|^2 |\nabla v|^2
\]

\[
\leq \frac{D_1}{4} \int_\Omega |\Delta u|^2 + \frac{\chi_1^2}{2D_1} \int_\Omega |\nabla u|^4 + \frac{\chi_1^2}{2D_1} \int_\Omega |\nabla v|^4
\]

\[
\leq \frac{D_1}{4} \int_\Omega |\Delta u|^2 + \frac{2\eta^2 \chi_1^2 c_1}{D_1} \int_\Omega |\Delta u|^2 + \frac{2\eta^2 \chi_1^2 c_1}{D_1} \int_\Omega |\Delta v|^2 \quad \text{in } (0, T_\eta).
\]

Moreover, due to (2.5), by the mean value theorem, as \( f_{uu} \equiv 2\mu_1 \) and \( f_{uv} \equiv a_1 \) and because of the Poincaré inequality A.1 (with \( C_P > 0 \) as in that lemma),

\[
I_3 \leq \int_\Omega (f_u(u, v) - f_u(u_*, v_*)) |\nabla u|^2
\]

\[
\leq \int_\Omega (\|f_{uu}\|_{L^\infty((0, \infty)^2)} |u - u_*| + \|f_{uv}\|_{L^\infty((0, \infty)^2)} |v - v_*|) |\nabla u|^2
\]

\[
\leq \eta (2\mu_1 + a_1) C_P \int_\Omega |\Delta u|^2 \quad \text{in } (0, T_\eta).
\]

Finally, by Young’s inequality and the Poincaré inequality A.1 (again with \( C_P > 0 \) as in that lemma),

\[
I_4 = a_1 u_* \int_\Omega \nabla u \cdot \nabla v + a_1 \int_\Omega (u - u_*) \nabla u \cdot \nabla v
\]

\[
\leq a_1 u_* \int_\Omega \nabla u \cdot \nabla v + \frac{\eta a_1 C_P}{2} \left( \int_\Omega |\Delta u|^2 + \int_\Omega |\Delta v|^2 \right) \quad \text{in } (0, T_\eta).
\]

Along with an analogue computation for \( v \), these estimates imply

\[
\frac{d}{dt} \left( \frac{B_1}{2} \int_\Omega |\nabla u|^2 + \frac{B_2}{2} \int_\Omega |\nabla v|^2 \right)
\]

\[
+ \left( \frac{3B_1 D_1}{4} - M_1(\eta) \right) \int_\Omega |\Delta u|^2 + \left( \frac{3B_2 D_2}{4} - M_2(\eta) \right) \int_\Omega |\Delta v|^2
\]

\[
\leq (B_1 a_1 u_* - B_2 a_2 v_*) \int_\Omega \nabla u \cdot \nabla v + (B_1 \chi_1 u_* - B_2 \chi_2 v_*) \int_\Omega \Delta u \Delta v \quad \text{in } (0, T_\eta).
\]

The statement follows due to (3.10).\[\square\]
At last, we deal with the third pair in (3.4).

**Lemma 3.4.** For any $C_1, C_2 > 0$, there exists $\eta > 0$ such that $(u, v, T_\eta) := (u(u_0, v_0), v(u_0, v_0), T_\eta(u_0, v_0))$, where $T_\eta$ is defined in (3.1), satisfies

$$\frac{d}{dt} \left( \frac{C_1}{2} \int_\Omega |\Delta u|^2 + \frac{C_2}{2} \int_\Omega |\Delta v|^2 \right) + \frac{C_1 D_1}{2} \int_\Omega |\nabla \Delta u|^2 + \frac{C_1 D_2}{2} \int_\Omega |\nabla \Delta v|^2$$

$$\leq (C_1 a_1 u_\nu - C_2 a_2 v_\nu) \int_\Omega \Delta u \Delta v + (C_1 \chi_1 u_\nu - C_2 \chi_2 v_\nu) \int_\Omega \nabla \Delta u \cdot \nabla \Delta v$$

provided $u_0, v_0$ fulfill (1.4).

**Proof.** Fix $C_1, C_2 > 0$. Let us again begin by fixing some constants: By Lemma A.4 and Lemma A.2, there is $c_1 > 0$ such that

$$6 \max \left\{ \frac{\lambda^2}{D_1}, \frac{\lambda^2}{D_2} \right\} \int_\Omega |\nabla \varphi|^6 \leq c_1 \|\varphi - \psi\|^4_{L^\infty(\Omega)} \int_\Omega |\nabla \Delta \varphi|^2$$

for all $\varphi \in C^3(\Omega)$ with $\partial_\nu \varphi = 0$ on $\partial \Omega$ (3.11)

as well as

$$12 \max \left\{ \frac{\lambda^2}{D_1}, \frac{\lambda^2}{D_2} \right\} \int_\Omega |D^2 \varphi|^3 \leq c_1 \|\varphi - \psi\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla \Delta \varphi|^2$$

for all $\varphi \in C^3(\Omega)$ with $\partial_\nu \varphi = 0$ on $\partial \Omega$ (3.12)

and Lemma A.3 provides us with $c_2 \geq 1$ such that

$$\int_\Omega |\nabla \varphi|^4 \leq c_2 \|\varphi - \psi\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla \Delta \varphi|^2$$

for all $\varphi \in C^2(\Omega)$ with $\partial_\nu \varphi = 0$ on $\partial \Omega$. (3.13)

Fix furthermore $C_P$ as in Lemma A.1 and choose $\eta > 0$ so small that

$$M_1(\eta) := \frac{C_1 \eta \chi_1}{2} + \frac{C_2 \eta \chi_2}{2} + (C_1 + C_2) c_1 (2\eta + 16\eta^2) + \frac{C_1 C_P c_2 \eta (9a_1 + 14\mu_1)}{2} + \frac{5 C_2 C_P a_2 c_2 \eta}{2}$$

and

$$M_2(\eta) := \frac{C_1 \eta \chi_1}{2} + \frac{C_2 \eta \chi_2}{2} + (C_1 + C_2) c_1 (2\eta + 16\eta^2) + \frac{C_2 C_P c_2 \eta (9a_2 + 14\mu_2)}{2} + \frac{5 C_1 C_P a_1 c_2 \eta}{2}$$

satisfy

$$M_1(\eta) < \frac{C_1 D_1}{4} \quad \text{and} \quad M_2(\eta) < \frac{C_2 D_1}{4}. \quad (3.14)$$

Fix also $u_0, v_0$ complying with (1.4). Since $\partial_\nu u = 0$ on $\partial \Omega \times (0, T_{\max})$ implies $\partial_\nu u = 0$ on $\partial \Omega \times (0, T_{\max})$ and as $|\Delta \varphi| \leq \sqrt{m} D^2 \varphi$ for all $\varphi \in C^2(\Omega)$, we may calculate

$$\frac{d}{dt} \int_\Omega |\Delta u|^2$$

$$= - \int_\Omega \nabla u \cdot \nabla \Delta u + \int_{\partial \Omega} (\partial_\nu u) \Delta u$$

$$= -D_1 \int_\Omega |\nabla \Delta u|^2 + \chi_1 \int_\Omega \nabla (u \Delta u + \nabla u \cdot \nabla v) \cdot \nabla \Delta u - \int_\Omega \nabla (f(u, v)) \cdot \nabla \Delta u$$

$$\leq -D_1 \int_\Omega |\nabla \Delta u|^2 - \int_\Omega \nabla (f(u, v)) \cdot \nabla \Delta u$$

$$+ \chi_1 \int_\Omega u \nabla \Delta u \cdot \nabla v + \chi_1 \int_\Omega (D^2 u |\nabla v| + (1 + \sqrt{n}) |D^2 v | |\nabla u|) |\nabla \Delta u|$$

in $(0, T_{\max})$. \quad (3.15)
Therein is by Young’s inequality
\[
\chi_1 \int_\Omega u \nabla \Delta v \cdot \nabla \Delta u = \chi_1 u_* \int_\Omega \nabla \Delta v \cdot \nabla \Delta u + \chi_1 \int_\Omega (u - u_*) \nabla \Delta v \cdot \nabla \Delta u
\]
\[
= \chi_1 u_* \int_\Omega \nabla \Delta v \cdot \nabla \Delta u + \frac{\eta \chi_1}{2} \int_\Omega |\nabla \Delta u|^2 + \frac{\eta \chi_1}{2} \int_\Omega |\nabla \Delta v|^2 \quad \text{in } (0, T_\eta).
\]
Again by Young’s inequality combined with \(\sqrt{n} \leq 2\), (3.11), (3.12), (3.2) and (3.3), we further estimate
\[
\chi_1 \int_\Omega \left( |D^2 u| |\nabla v| + (1 + \sqrt{n}) |D^2 v| |\nabla u| \right) |\nabla \Delta u|
\]
\[
\leq \frac{D_1}{4} \int_\Omega |\nabla \Delta u|^2 + \frac{2 \chi_1^2}{D_1} \int_\Omega |D^2 u|^2 |\nabla v|^2 + \frac{18 \chi_1^2}{D_1} \int_\Omega |D^2 v|^2 |\nabla u|^2
\]
\[
\leq \frac{D_1}{4} \int_\Omega |\nabla \Delta u|^2 + \frac{4 \chi_1^2}{3 D_1} \int_\Omega |D^2 u|^2 + \frac{2 \chi_1^2}{D_1} \int_\Omega |\nabla v|^6 + \frac{12 \chi_1^2}{D_1} \int_\Omega |D^2 v|^3 + \frac{6 \chi_1^2}{D_1} \int_\Omega |\nabla u|^6
\]
\[
\leq \left( \frac{D_1}{4} + 2 c_1 \eta + 16 c_1 \eta^4 \right) \int_\Omega |\nabla \Delta u|^2 + (2 c_1 \eta + 16 c_1 \eta^4) \int_\Omega |\nabla \Delta v|^2 \quad \text{in } (0, T_\eta).
\]
Regarding the remaining term in (3.15), we first note that
\[
D^2 f(u, v) = \begin{pmatrix}
-2 \mu_1 & a_1 \\
a_1 & 0
\end{pmatrix}
\]
and that (2.5) implies
\[
f_u(u, v) = f_u(u, v_*) + a_1 (v - v_*)
\]
\[
= f_u(u, v_*) + f_{uu}(u_*, v_*) (u - u_*) + a_1 (v - v_*)
\]
\[
\leq -2 \mu_1 (u - u_*) + a_1 (v - v_*) \quad \text{in } (0, T_{\max}).
\]
Therefore, an integration by parts and applications of Young’s inequality as well as Poincaré’s inequality A.1 yield
\[
\int_\Omega \nabla (f(u, v)) \cdot \nabla \Delta u
\]
\[
= - \int_\Omega f_u(u, v) \nabla u \cdot \nabla \Delta u - \int_\Omega f_v(u, v) \nabla v \cdot \nabla \Delta u
\]
\[
= \int_\Omega f_u(u, v) |\Delta u|^2 + \int_\Omega f_{uu}(u, v) |\nabla u|^2 |\Delta u| + 2 \int_\Omega f_{uv}(u, v) \nabla u \cdot \nabla v |\Delta u|
\]
\[
+ \int_\Omega f_v(u, v) |\Delta u| |\Delta v| + \int_\Omega f_{vv}(u, v) |\nabla v|^2 |\Delta u|
\]
\[
\leq \eta (a_1 + 2 \mu_1) \int_\Omega |\Delta u|^2 + 2 \mu_1 \int_\Omega |\nabla u|^2 |\Delta u| + 2 a_1 \int_\Omega \nabla u \cdot \nabla v |\Delta u|
\]
\[
+ a_1 \int_\Omega (u - u_*) |\Delta u| |\Delta v| + a_1 u_* \int_\Omega |\Delta u| |\Delta v|
\]
\[\leq C_P \eta (a_1 + 2 \mu_1) \int_\Omega |\nabla \Delta u|^2 + a_1 u_* \int_\Omega \Delta u \Delta v + \eta \mu_1 \int_\Omega |\Delta u|^2 + \frac{\mu_1}{\eta} \int_\Omega |\nabla u|^4 + a_1 \eta \int_\Omega |\nabla u|^2 + \frac{a_1 \eta}{2} \int_\Omega |\nabla v|^4 + \frac{a_1 \eta}{2} \int_\Omega |\nabla v|^2 \]

Thus, due to \(c_2 \geq 1\),

\[-\int \nabla (f(u,v)) \cdot \nabla u \leq \frac{C_P c_2 \eta (9a_1 + 14 \mu_1)}{2} \int_\Omega |\nabla \Delta u|^2 + \frac{5C_P c_2 \eta a_1 \mu_1}{2} \int_\Omega |\nabla \Delta v|^2 + a_1 u_* \int_\Omega \Delta u \Delta v \]

holds in \((0, T_\eta)\).

As usual, we now combine the estimates above with analogous computations for \(v\) to obtain

\[
\frac{d}{dt} \left( \frac{C_1}{2} \int_\Omega |\Delta u|^2 + \frac{C_2}{2} \int_\Omega |\Delta v|^2 \right) + \left( \frac{3C_1 D_1}{4} - M_1(\eta) \right) \int_\Omega |\nabla \Delta u|^2 + \left( \frac{3C_2 D_2}{4} - M_2(\eta) \right) \int_\Omega |\nabla \Delta v|^2 \leq (C_1 a_1 u_* - C_2 a_2 v_* ) \int_\Omega \Delta u \Delta v + (C_1 \chi_1 u_* - C_2 \chi_2 v_* ) \int_\Omega \nabla \Delta u \cdot \nabla \Delta v \quad \text{in } (0, T_\eta),
\]

which in virtue of (3.14) implies the statement. ∎

4. Deriving \(W^{2,2}(\Omega)\) bounds for \(u\) and \(v\)

In this section, we will make use of the estimates gained in the previous section to finally obtain \(W^{2,2}(\Omega)\) bounds for both solution components. That is, we will aim to bound \(\|u - u_*\|_{W^{2,2}(\Omega)} + \|v - v_*\|_{W^{2,2}(\Omega)}\) by, say, \(\frac{\eta}{2}\) in \((0, T_\eta)\) (for a certain \(\eta > 0\)), as then \(T_\eta = T_{\text{max}} = \infty\) can be concluded—provided \(T_\eta > 0\) which in turn can be achieved by requiring \(\|u_0 - u_*\|_{W^{2,2}(\Omega)} + \|v_0 - v_*\|_{W^{2,2}(\Omega)}\) to be sufficiently small.

In the sequel, we distinguish between multiple cases. More concretely, we will handle
• \((H1)\) in Lemma 4.2,
• \((H2)\) with \(\lambda_2 \mu_1 > \lambda_1 \alpha_2\) in Lemma 4.3,
• \((H2)\) with \(\lambda_2 \mu_1 < \lambda_1 \alpha_2\) in Lemma 4.4 and Lemma 4.5
• \((H2)\) with \(\lambda_2 \mu_1 = \lambda_1 \alpha_2\) and \(\lambda_1 > 0\) in Lemma 4.7 (ii) and Lemma 4.8 as well as
• \((H2)\) with \(\lambda_1 = \lambda_2 = 0\) in Lemma 4.9.

These five cases can be divided into two groups, the first of which we deal with in the following subsection.

4.1. The cases \((H1)\) and \((H2)\) with \(\lambda_2 \mu_1 > \lambda_1 \alpha_2\)

If either \((H1)\) holds with \(m_1, m_2 > 0\) or \((H2)\) holds with \(\lambda_2 \mu_1 > \lambda_1 \alpha_2\), \(u_\star\) and \(v_\star\) are positive—which is the reason these cases can be handled in a similar fashion. In both cases, we will aim to apply the following elementary lemma.

Lemma 4.1. For \(A, B, C > 0\) and \(\varphi \in W^{2,2}(\Omega)\) set

\[
\phi_{A,B,C}(\varphi) := \frac{A}{2} \int_{\Omega} \varphi^2 + \frac{B}{2} \int_{\Omega} |\nabla \varphi|^2 + \frac{C}{2} \int_{\Omega} |\Delta \varphi|^2
\]  

and let \(A_1, A_2, B_1, B_2, C_1, C_2 > 0, \eta > 0\) and \(K_1 > 0\).

There is \(K_2 > 0\) such that, if \(u_0, v_0\) comply with (1.4), \(T_\eta\) is as in (3.1) and

\[
y: [0, T_\eta) \to \mathbb{R}, \quad t \mapsto \phi_{A_1,B_1,C_1}(u(\cdot,t) - u_\star) + \phi_{A_2,B_2,C_2}(v(\cdot,t) - v_\star)
\]

fulfills

\[
y'(t) \leq -2K_1 y(t) \quad \text{in } (0, T_\eta),
\]

then

\[
\|u(\cdot,t) - u_\star\|_{W^{2,2}(\Omega)} + \|v(\cdot,t) - v_\star\|_{W^{2,2}(\Omega)} \leq K_2 e^{-K_1 t} \left( \|u_0 - u_\star\|_{W^{2,2}(\Omega)} + \|v_0 - v_\star\|_{W^{2,2}(\Omega)} \right)
\]  

for all \(t \in (0, T_\eta)\).

Proof. As \(W^{2,2}(\Omega)\) continuity of \(u\) and \(v\) up to \(t = 0\) is ensured by (2.3), we may make use of an ODE comparison argument to obtain

\[
y(t) \leq e^{-2K_1 t} y(0) \quad \text{for all } t \in (0, T_\eta).
\]

The statement follows by taking square roots on both sides and noting that \(\|\varphi\| := \sqrt{\phi_{A,B,C}(\varphi)}\) defines for \(A, B, C > 0\) a norm on \(W^{2,2}_N(\Omega)\), which is equivalent to the usual one by Lemma A.2. \(\square\)

For both cases covered in this subsection, we will now choose \(A_1, A_2, B_1, B_2, C_1, C_2 > 0\) appropriately so that Lemma 4.1 is applicable.

Lemma 4.2. Suppose \((H1)\). Then there are \(\eta > 0\) and \(K_1, K_2 > 0\) such that (4.4) holds for all \(u_0, v_0\) satisfying (1.4).
Proof. In the case of (H1) with \( m_1 = 0 \) or \( m_2 = 0 \), that is, if at least one of the initial data is trivial, the uniqueness statement in Lemma 2.1 asserts that one solution component is constantly zero while the other solves the heat equation. As in that case the statement becomes trivial, we may assume \( m_1 > 0 \) and \( m_2 > 0 \).

Then \( u_*, v_* > 0 \) and hence \( A_1 = B_1 = C_1 := \chi_2 v_* \) as well as \( A_2 = B_2 = C_2 := \chi_1 u_* \) are positive as well. Because of

\[ A_1 \chi_1 u_* - A_2 \chi_2 v_* = 0, \quad B_1 \chi_1 u_* - B_2 \chi_2 v_* = 0, \quad C_1 \chi_1 u_* - C_2 \chi_2 v_* = 0 \]

and (H1), Lemma 3.2, Lemma 3.3 and Lemma 3.4 assert that there is \( \eta > 0 \) such that

\[
\left( A_1 \chi_1 u_* - A_2 \chi_2 v_* \right) \int_{\Omega} \nabla u \cdot \nabla v + \left( B_1 \chi_1 u_* - B_2 \chi_2 v_* \right) \int_{\Omega} \nabla u \Delta v + \left( C_1 \chi_1 u_* - C_2 \chi_2 v_* \right) \int_{\Omega} \nabla u \cdot \nabla \Delta v = 0 \quad \text{in} \ (0, T_\eta),
\]

whenever \( u_0, v_0 \) comply with (1.4), where \( \phi \) and \( T_\eta \) are as in (4.1) and (3.1), respectively.

As integrating the first two equations in (1.2) implies \( u_*= \overline{u}_0 = \overline{u} \) and \( v_* = \overline{v}_0 = \overline{v} \) in \((0, T_{\text{max}})\), we further obtain by Poincaré’s inequality A.1 that (4.3) is fulfilled for some \( K_1 > 0 \), hence the statement follows by Lemma 4.1.

Somewhat surprisingly, also in the case (H2) with \( \lambda_2 \mu_1 > \lambda_1 \alpha_2 \), suitably choosing \( A_1, A_2, B_1, B_2, C_1, C_2 \) in Lemma 3.2, Lemma 3.3 and Lemma 3.4 allows for a cancellation of all problematic terms.

Lemma 4.3. Suppose (H2) holds and \( \lambda_2 \mu_1 > \lambda_1 \alpha_2 \). Then we can find \( \eta > 0 \) and \( K_1, K_2 > 0 \) with the property that (4.4) holds whenever \( u_0, v_0 \) satisfy (1.4).

Proof. Positivity of \( u_* \) and \( v_* \) implies that the constants

\[ A_1 := a_2 v_*, \quad A_2 := a_1 u_*, \quad B_1 := (a_2 + \chi_2) v_*, \quad B_2 := (a_1 + \chi_1) u_*, \quad C_1 := \chi_2 v_* \quad \text{and} \quad C_2 := \chi_1 u_* \]

are all positive, hence we may apply Lemma 3.2, Lemma 3.3 and Lemma 3.4 to obtain \( \eta_1 > 0 \) such that

\[
\begin{aligned}
&\left( A_1 \chi_1 u_* - A_2 \chi_2 v_* \right) \int_{\Omega} \nabla u \cdot \nabla v + \left( B_1 \chi_1 u_* - B_2 \chi_2 v_* \right) \int_{\Omega} \nabla u \Delta v + \left( C_1 \chi_1 u_* - C_2 \chi_2 v_* \right) \int_{\Omega} \nabla u \cdot \nabla \Delta v \\
&\quad + \left( (A_1 \chi_1 B_1 a_1 u_*) - (A_2 \chi_2 B_2 a_2 v_*) \right) \int_{\Omega} \nabla u \cdot \nabla v \\
&\quad + \left( (B_1 \chi_1 C_1 a_1 u_*) - (B_2 \chi_2 C_2 a_2 v_*) \right) \int_{\Omega} \nabla u \Delta v \\
&\quad + \left( (C_1 \chi_1 u_*) - (C_2 \chi_2 v_*) \right) \int_{\Omega} \nabla u \cdot \nabla \Delta v \\
&\quad \leq (A_1 \chi_1 u_* - A_2 \chi_2 v_*) \int_{\Omega} (u - u_*) (v - v_*) \\
&\quad + [(A_1 \chi_1 B_1 a_1 u_*) - (A_2 \chi_2 B_2 a_2 v_*)] \int_{\Omega} \nabla u \cdot \nabla v \\
&\quad + [(B_1 \chi_1 C_1 a_1 u_*) - (B_2 \chi_2 C_2 a_2 v_*)] \int_{\Omega} \nabla u \Delta v \\
&\quad + (C_1 \chi_1 u_* - C_2 \chi_2 v_*) \int_{\Omega} \nabla u \cdot \nabla \Delta v \\
&\quad \text{holds in} \ (0, T_\eta) \quad \text{for all} \ \eta \leq \eta_1,
\end{aligned}
\]

provided \( u_0, v_0 \) satisfy (1.4), where again \( \phi \) and \( T_\eta \) are defined in (4.1) and (3.1), respectively.
Setting further \( \eta_2 := \min \left\{ \frac{f_u(u,v)}{2(a_1 + \mu_1)} \cdot \frac{g_v(u,v)}{2(a_2 + \mu_2)} \right\} \), which is positive by (2.6), and noting that

\[
\begin{align*}
A_1a_1u_* - A_2a_2v_* = 0, \\
(A_1\chi_1 + B_1a_1)u_* - (A_2\chi_2 + B_2a_2)v_* = 0, \\
(B_1\chi_1 + C_1a_1)u_* - (B_2\chi_2 + C_2a_2)v_* = 0
\end{align*}
\]

we obtain

\[
\begin{align*}
\frac{d}{dt} \left( \phi_{A_1,B_1,C_1}(u(t) - u_*) + \phi_{A_2,B_2,C_2}(v(t) - v_*) \right) \\
+ \frac{C_1D_1}{2} \int_\Omega |\nabla u|^2 + \frac{C_2D_2}{2} \int_\Omega |\nabla v|^2 \\
- \frac{A_1f_u(u_*,v_*)}{2} \int_\Omega (u - u_*)^2 - \frac{A_2g_v(u_*,v_*)}{2} \int_\Omega (v - v_*)^2 \\
\leq 0 \quad \text{in} \ (0,T_\eta)
\end{align*}
\]

for \( \eta := \min\{\eta_1,\eta_2\} \), provided \( u_0, v_0 \) comply with (1.4).

In virtue of Poincaré's inequality A.1, this first asserts (4.3) for some \( K_1 > 0 \) and then also (4.4) for some \( K_2 > 0 \) by Lemma 4.1. \( \square \)

### 4.2. The case (H2) with \( \lambda_2\mu_1 \leq \lambda_1\mu_2 \)

The condition (H2) with \( \lambda_2\mu_1 \leq \lambda_1\mu_2 \) implies \( v_* = 0 \), hence for any choice of \( A_1, A_2, B_1, B_2, C_1, C_2 > 0 \) in Lemma 3.2, Lemma 3.3 and Lemma 3.4, unlike as in the previous subsection, no cancellation of problematic terms occurs (except if also \( u_* = 0 \), but then we will rely on a different functional, see Lemma 4.9 below).

However, the disappearance of \( v_* \) can also be used to our advantage. As the coefficients of the problematic terms no longer depend on \( A_2, B_2 \) and \( C_2 \), we can choose (one of) these parameters comparatively large and thus obtain stronger dissipative terms. This idea first manifests itself in the following

**Lemma 4.4.** Suppose (H2) holds and \( \lambda_2\mu_1 \leq \lambda_1\mu_2 \). There are \( \eta > 0 \) as well as \( K > 0 \) and \( C_2 > 0 \) such that whenever \( u_0, v_0 \) comply with (1.4) and \( T_\eta \) is as in (3.1),

\[
\int_\Omega |\Delta u(\cdot,t)|^2 + C_2 \int_\Omega |\Delta v(\cdot,t)|^2 \leq e^{-Kt} \left( \int_\Omega |\Delta u_0|^2 + C_2 \int_\Omega |\Delta v_0|^2 \right) \quad \text{for all} \ t \in (0,T_\eta).
\]

**Proof.** Set \( K := \min\left\{ \frac{D_1D_2}{2} \right\} > 0 \), \( C_1 := 1 \) and

\[
C_2 := \frac{16 \max \{C_2 a_1^2, \chi_1^2\} (u_*)^2 + 1}{D_1D_2} > 0,
\]

where \( C_P > 0 \) denotes the constant given in Lemma A.1.

By Lemma 3.4, there is \( \eta > 0 \) with the property that

\[
\begin{align*}
\frac{d}{dt} \left( \int_\Omega |\Delta u|^2 + C_2 \int_\Omega |\Delta v|^2 \right) + D_1 \int_\Omega |\nabla \Delta u|^2 + C_2D_2 |\nabla \Delta v|^2 \\
\leq 2a_1u_* \int_\Omega \Delta u \Delta v + 2\chi_1u_* \int_\Omega \nabla \Delta u \cdot \nabla \Delta v \quad \text{in} \ (0,T_\eta),
\end{align*}
\]

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provided the (henceforth fixed) initial data \( u_0, v_0 \) satisfy (1.4).

Therein are by Young’s inequality and Poincaré’s inequality A.1, with \( C_P > 0 \) as in that lemma,

\[
2a_1 u_* \int_\Omega \Delta u \Delta v \leq \frac{D_1}{4C_P} \int_\Omega |\Delta u|^2 + \frac{4C_P a_1^2 u_*^2}{D_1} \int_\Omega |\Delta v|^2
\]

\[
\leq \frac{D_1}{4} \int_\Omega |\nabla \Delta u|^2 + \frac{C_2 D_2}{4} \int_\Omega |\nabla \Delta v|^2 \quad \text{in } (0, T_{\text{max}})
\]

and, again by Young’s inequality,

\[
2\chi_1 u_* \int_\Omega \nabla u \cdot \nabla v \leq \frac{D_1}{4} \int_\Omega |\nabla \Delta u|^2 + \frac{4\chi_1^2 u_*^2}{D_1} \int_\Omega |\Delta v|^2
\]

\[
\leq \frac{D_1}{4} \int_\Omega |\nabla \Delta u|^2 + \frac{C_2 D_2}{4} \int_\Omega |\nabla \Delta v|^2 \quad \text{in } (0, T_{\text{max}}).
\]

Thus, the statement follows upon an integration over \((0, T_\eta)\) due to (2.3), the \(W^{2,2}(\Omega)\) continuity of \(u\) and \(v\) up to \(t = 0\).

In the case (H2) with \( \lambda_2 \mu_1 < \lambda_1 a_2 \), by a similar argument we also obtain bounds for \( \int_\Omega (u - u_\ast)^2 \) and \( \int_\Omega v^2 \).

**Lemma 4.5.** If (H2) holds with \( \lambda_2 \mu_1 < \lambda_1 a_2 \), then there are \( \eta > 0, K > 0 \) and \( A_2 > 0 \) such that

\[
\int_\Omega (u - u_\ast)^2 + A_2 \int_\Omega v^2 \leq e^{-Kt} \left( \int_\Omega (u_0 - u_\ast)^2 + A_2 \int_\Omega (v_0 - v_\ast)^2 \right) \quad \text{for all } t \in (0, T_\eta),
\]

provided \( u_0, v_0 \) satisfy (1.4) and \( T_\eta \) is as in (3.1).

**Proof.** Since \( \lambda_2 \mu_1 < \lambda_1 a_2 \), both \( f_u(u_\ast, v_\ast) \) and \( g_v(u_\ast, v_\ast) \) are negative, hence there is \( \eta_1 > 0 \) such that

\[
K := \min \{-f_u(u_\ast, v_\ast) - \eta_1 (a_1 + \mu_1), -g_v(u_\ast, v_\ast) - \eta_1 (a_2 + \mu_2)\} > 0.
\]

Set moreover \( A_1 := 1 \) and

\[
A_2 := \max \left\{ \frac{a_1^2}{K^2}, \frac{\chi_1^2}{D_1 D_2} \right\} u_\ast > 0.
\]

Then Lemma 3.2 provides us with \( \eta \in (0, \eta_1) \) such that

\[
\frac{d}{dt} \left( \int_\Omega (u - u_\ast)^2 + A_2 \int_\Omega (v - v_\ast)^2 \right)
\]

\[
+ D_1 \int_\Omega |\nabla u|^2 + A_2 D_2 \int_\Omega |\nabla v|^2
\]

\[
+ 2K \int_\Omega (u - u_\ast)^2 + 2A_2 K \int_\Omega v^2
\]

\[
\leq 2a_1 u_* \int_\Omega (u - u_\ast)v + 2\chi_1 u_* \int_\Omega \nabla u \cdot \nabla v \quad \text{in } (0, T_\eta),
\]

whenever \( u_0, v_0 \) comply with (1.4).

Henceforth fixing such initial data, two applications of Young’s inequality give

\[
2a_1 u_* \int_\Omega (u - u_\ast)v \leq K \int_\Omega (u - u_\ast)^2 + \frac{a_1^2 u_*^2}{K} \int_\Omega v^2 \leq K \int_\Omega (u - u_\ast)^2 + A_2 K \int_\Omega v^2
\]
and
\[ 2 \chi_1 u_\ast \int \Omega \nabla u \cdot \nabla v \leq D_1 \int \Omega |\nabla u|^2 + \frac{\chi_1^2 u_\ast^2}{D_1} \int \Omega |\nabla v|^2 \leq D_1 \int \Omega |\nabla u|^2 + A_2 D_2 \int \Omega |\nabla v|^2 \]
in \((0, T_{\text{max}})\), so that the statement follows by the comparison principle for ordinary differential equations. 

The case \((H2)\) with \(\lambda_2 \mu_1 = \lambda_1 a_2\) cannot be handled in a similar fashion as then \(g_v(u_\ast, v_\ast) = 0\) resulting in the term \(A_2(-g_v(u_\ast, v_\ast) - \eta(\sigma_2 - \mu_1))\) \(\int_\Omega v^2\) in \((3.8)\) having an unfavorable sign. Similarly, if \(\lambda_1 = 0\), then \(f_u(u_\ast, v_\ast) = 0\) and \(A_1(-f_u(u_\ast, v_\ast) - \eta(a_1 + \mu_1)) < 0\). Thus, we introduce an additional functional to counter these terms.

**Lemma 4.6.** Suppose that \(u_0, v_0\) comply with \((1.4)\). If \(\lambda_1 = 0\), then
\[ \frac{d}{dt} \int \Omega u = -\mu_1 \int \Omega u^2 + a_1 \int \Omega uv \quad \text{in } (0, T_{\text{max}}) \]
and if \((H2)\) holds with \(\lambda_2 \mu_1 = \lambda_1 a_2\), then
\[ \frac{d}{dt} \int \Omega v = -\mu_2 \int \Omega v^2 - a_2 \int \Omega (u - u_\ast)v \quad \text{in } (0, T_{\text{max}}). \]

**Proof.** The first statement immediately follows by integrating the first equation in \((1.2)\).

Furthermore, the assumptions \((H2)\) and \(\lambda_2 \mu_1 = \lambda_1 a_2\) imply \((u_\ast, v_\ast) = (\frac{\lambda_1}{\mu_1}, 0) = (\frac{\lambda_1 a_2}{\mu_1}, 0)\) and hence
\[ g(u, v) = v(\lambda_2 - \mu_2 v - a_2 u) = v(\lambda_2 - \mu_2 v - a_2 u_\ast) + a_2(u - u_\ast)v = -\mu_2 v^2 - a_2(u - u_\ast)v \quad \text{in } (0, T_{\text{max}}). \]

Thus, the second statement follows also due to integrating.

With the help of this lemma, we can now handle the remaining case, namely \((H2)\) with \(\lambda_2 \mu_1 = \lambda_1 a_2\). The proof is split into three lemmata; before dealing with the (in some sense) fully degenerate case, in the following two lemmata, we first handle the half-degenerate case, where at least \(u_\ast > 0\) and \(f_u(u_\ast, v_\ast) > 0\).

**Lemma 4.7.** Suppose \((H2)\), \(\lambda_2 \mu_1 = \lambda_1 a_2\) as well as \(\lambda_1 > 0\) and, for \(\eta > 0\), let \(T_\eta\) be as in \((3.1)\).

(i) There are \(\eta > 0\) and \(K_1, K_2 > 0\) such that
\[ \|v(\cdot, t)\|_{L^1(\Omega)} \leq \left( K_1 \left( \|u_0 - u_\ast\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)} \right)^{-1} + K_2 t \right)^{-1} \]
whenever \(u_0, v_0\) are such that \((1.4)\) holds.

(ii) We can find \(\eta' > 0\) and \(K'_1, K'_2 > 0\) such that
\[ \|v(\cdot, t)\|_{W^{2,1}(\Omega)} \leq \left( K'_1 \left( \|u_0 - u_\ast\|_{W^{2,1}(\Omega)} + \|v_0\|_{W^{2,1}(\Omega)} \right)^{-1} + K'_2 t \right)^{-1} \]
for all \(t \in (0, T_{\eta'})\), if \(u_0, v_0\) comply with \((1.4)\).

**Proof.** Setting \(A_1 := 1, X_2 := \frac{\lambda_1 a_1}{\mu_1 a_2} > 0, A_2 := \frac{\lambda_1 a_2^2}{\mu_1 a_2^2} > 0\), by Lemma 3.2 and Lemma 4.6 we find \(\eta_0 > 0\) such that
\begin{align*}
\frac{d}{dt} \left( \frac{A_1}{2} \int_\Omega (u - u_\ast)^2 + \frac{A_2}{2} \int_\Omega v^2 + X_2 \int_\Omega v \right) \\
+ \frac{A_1 D_1}{2} \int_\Omega |\nabla u|^2 + \frac{A_2 D_2}{2} \int_\Omega |\nabla v|^2 \\
+ (-A_1 f_u(u_\ast, v_\ast) - A_1 \eta(a_1 + \mu_1)) \int_\Omega (u - u_\ast)^2 + (X_2 \mu_2 - A_2 \eta(a_2 + \mu_2)) \int_\Omega v^2 \\
\leq (A_1 a_1 u_\ast - X_2 a_2) \int_\Omega (u - u_\ast)v + A_1 \chi_1 u_\ast \int_\Omega \nabla u \cdot \nabla v \quad \text{in } (0, T_{\eta}) \text{ for all } \eta \leq \eta_0,
\end{align*}

(4.7)
whenever $u_0, v_0$ comply with (1.4).

Set $c_1 := \frac{A_1 \eta u_0}{2} > 0$, $c_2 := \frac{X_0 u_0}{2} > 0$, $c_3 := \min \left\{ \frac{4c_1}{A_1}, \frac{2c_2}{A_2}, -\frac{c_3}{X_0^2} \right\} > 0$ as well as

$$\eta := \min \left\{ 1, \eta_0, |\Omega|^{-\frac{1}{2}}, \frac{c_1}{A_1(a_1 + \mu_1)}, \frac{c_2}{A_2(a_2 + \mu_2)} \right\} > 0$$

and fix $u_0, v_0$ satisfying (1.4).

As the term $A_1 a_1 u_* - X_0 v_2$ vanishes due to the definition of $A_1$ and $X_0$, and Young’s inequality as well as the definition of $A_2$ imply

$$A_1 \chi_{1\Omega} \int_{\Omega} \nabla u \cdot \nabla v \leq \frac{A_1 D_1}{2} \int_{\Omega} |\nabla u|^2 \leq \frac{A_2 D_2}{2} \int_{\Omega} t u^2$$

we may conclude from (4.7) that

$$\frac{d}{dt} \left( \frac{A_1}{2} \int_{\Omega} (u - u_*)^2 + \frac{A_2}{2} \int_{\Omega} v^2 + X_0 \int_{\Omega} v \right) \leq -c_1 \int_{\Omega} (u - u_*)^2 - c_2 \int_{\Omega} v^2$$

holds in $(0, T)$. Since $\eta \leq |\Omega|^{-\frac{1}{2}}$ implies $\int_{\Omega} (u - u_*)^2 \leq 1$ as well as $\int_{\Omega} v^2 \leq 1$ in $(0, T)$ and due to Hölder’s inequality as well as the elementary inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ for $a, b, c \in \mathbb{R}$, we further obtain

$$\frac{d}{dt} \left( \frac{A_1}{2} \int_{\Omega} (u - u_*)^2 + \frac{A_2}{2} \int_{\Omega} v^2 + X_0 \int_{\Omega} v \right) \leq -c_1 \int_{\Omega} (u - u_*)^2 - c_2 \left( \int_{\Omega} v^2 \right)^2 - c_2 \frac{2}{|\Omega|} \left( \int_{\Omega} v \right)^2$$

$$\leq -c_3 \left( \frac{A_1}{2} \int_{\Omega} (u - u_*)^2 + \frac{A_2}{2} \int_{\Omega} v^2 + X_0 \int_{\Omega} v \right)$$

in $(0, T)$. Because of $\eta \leq 1$ and since without loss of generality $\|u_0 - u_*\|_{L^\infty(\Omega)} \leq \eta$ and $\|v_0\|_{L^\infty(\Omega)} \leq \eta$, this implies

$$X_0 \|v(\cdot, t)\|_{L^1(\Omega)} \leq \left( \left( \frac{A_1}{2} \int_{\Omega} (u_0 - u_*)^2 + \frac{A_2}{2} \int_{\Omega} v_0^2 + X_0 \int_{\Omega} v_0 \right)^{-1} + c_3 t \right)^{-1}$$

and hence proves part (i) for certain $K_1, K_2 > 0$.

Part (ii) follows then from Lemma 4.4, part (i) and the observation that

$$\|v\|_{W^{2,2}(\Omega)} \leq \|v - \nabla v\|_{W^{2,2}(\Omega)} + \|\nabla v\|_{L^2(\Omega)} \leq C \|\Delta v\|_{L^2(\Omega)} + |\Omega|^{-\frac{1}{2}} \|v\|_{L^2(\Omega)}$$

holds in $(0, T)$ due to Lemma A.2 (with $C > 0$ as in that lemma).

Next, we proceed to gain similar estimates also for the first equation.
Lemma 4.8. Assume (H2) holds and $\lambda_2 \mu_1 = \lambda_1 a_2$ as well as $\lambda_1 > 0$. Then there are $\eta > 0$ and $K_1, K_2 > 0$ such that

$$\|u(\cdot, t) - u_\ast\|_{W^{2,2}(\Omega)} \leq \left( K_1 \left( \|u_0 - u_\ast\|_{W^{2,2}(\Omega)} + \|v_0\|_{W^{2,2}(\Omega)} \right)^{-1} + K_2 t \right)^{-1} \quad \text{for all } t \in (0, T_\eta),$$

if $u_0, v_0$ satisfy (1.4) and $T_\eta$ is as in (3.1).

Proof. Choose $\eta_1 > 0$ so small that $c_1 := \lambda_1 - (a_1 + \mu_1)\eta_1 > 0$ and set $c_2 := \max \left\{ \frac{\alpha_1^2 a_1^2}{c_1}, \frac{2 \alpha_1^2 a_1^2}{3D_1}, \chi_1 \right\}$. By Lemma 3.1 and Lemma 4.7, there are moreover $\eta_2, \eta_3 > 0$ and $c_3, c_4 > 0$ such that

$$\frac{d}{dt} \int_\Omega (u - u_\ast)^2 + \frac{3D_1}{2} \int_\Omega |\nabla u|^2 + 2 ( - f_u(u_\ast, v_\ast) - \eta (a_1 + \mu_1)) \int_\Omega (u - u_\ast)^2 \leq 2a_1 u_\ast \int_\Omega (u - u_\ast) v + 2 \chi_1 u_\ast \int_\Omega \nabla u \cdot \nabla v + \eta_1 \int_\Omega |\nabla v|^2 \quad \text{in } (0, T_\eta) \text{ for all } \eta \in (0, \eta_2)$$

and

$$\|v(\cdot, t)\|_{W^{2,2}(\Omega)} \leq \left( \frac{\sqrt{2} c_3}{\sqrt{2}} \left( \|u_0 - u_\ast\|_{W^{2,2}(\Omega)} + \|v_0\|_{W^{2,2}(\Omega)} \right)^{-1} + \sqrt{2} c_4 t \right)^{-1} \quad \text{in } (0, T_\eta),$$

provided $u_0, v_0$ comply with (1.4).

Thus, fixing $\eta := \min \{\eta_1, \eta_2, \eta_3, 1\}$ as well as $u_0, v_0$ satisfying (1.4) and noting that $f_u(u_\ast, v_\ast) = -\lambda_1$, we obtain

$$\frac{d}{dt} \int_\Omega (u - u_\ast)^2 \leq - \frac{3D_1}{2} \int_\Omega |\nabla u|^2 - 2c_1 \int_\Omega (u - u_\ast)^2 + 2a_1 u_\ast \int_\Omega (u - u_\ast)^2 v + 2 \chi_1 u_\ast \int_\Omega \nabla u \cdot \nabla v + \eta_1 \int_\Omega |\nabla u|^2$$

which by the variation-of-constants formula implies

$$\int_\Omega (u - u_\ast)^2(\cdot, t) \leq e^{-c_1 t} \int_\Omega (u_0 - u_\ast)^2(\cdot, t) + \int_0^t e^{-c_1 (t-s)}(c_3 I_0^{-1} + c_4 s)^{-2} \, ds \quad \text{for all } t \in (0, T_\eta),$$

where we abbreviated $I_0 := \|u_0 - u_\ast\|_{W^{2,2}(\Omega)} + \|v_0\|_{W^{2,2}(\Omega)}$. Noting that $[0, \infty) \ni s \mapsto (c_3 I_0^{-1} + c_4 s)^{-2}$ is decreasing, we further calculate

$$\int_0^t e^{-c_1 (t-s)}(c_3 I_0^{-1} + c_4 s)^{-2} \, ds = \int_0^{t/2} e^{-c_1 (t-s)}(c_3 I_0^{-1} + c_4 s)^{-2} \, ds + \int_{t/2}^t e^{-c_1 (t-s)}(c_3 I_0^{-1} + c_4 s)^{-2} \, ds$$

$$\leq \frac{I_0^2}{c_3} \int_0^{t/2} e^{-c_1 s} \, ds + \left( c_3 I_0^{-1} + \frac{c_4 t}{2} \right)^{-2} \int_0^{t/2} e^{-c_1 s} \, ds$$

Combining these estimates with Lemma 4.4 and Lemma A.2 yields the statement for certain $K_1, K_2 > 0$. □

Finally, we deal with the aforementioned fully degenerate case.
Lemma 4.9. Suppose (H2) and \( \lambda_1 = \lambda_2 = 0 \). Then there are \( \eta > 0 \) and \( K_1, K_2 > 0 \) such that
\[
\|u(\cdot, t)\|_{W^{2,2}(\Omega)} + \|v(\cdot, t)\|_{W^{2,2}(\Omega)} \leq \left(K_1 \left(\|u_0\|_{W^{2,2}(\Omega)} + \|v_0\|_{W^{2,2}(\Omega)}\right)^{-1} + K_2 t\right)^{-1}
\] (4.8)
for all \( t \in (0, T_\eta) \), where \( T_\eta \) is defined in (3.1), provided \( u_0, v_0 \) satisfy (1.4).

Proof. Set \( c_1 := \frac{\min(\mu_1, \mu_2)}{2} \) and fix \( u_0, v_0 \) complying with (1.4).

By multiplying (4.5) and (4.6) with \( a_2 \) and \( a_1 \), respectively, we obtain
\[
\frac{d}{dt} \left(a_2 \int_{\Omega} u + a_1 \int_{\Omega} v\right) = -\mu_1 a_2 \int_{\Omega} u^2 - \mu_2 a_1 \int_{\Omega} v^2 \quad \text{in } (0, T_{\max}).
\]

Hence, along with Hölder’s inequality this implies
\[
\frac{d}{dt} \left(a_2 \int_{\Omega} u + a_1 \int_{\Omega} v\right) \leq -c_1 \left(a_2 \int_{\Omega} u + a_1 \int_{\Omega} v\right)^2 \quad \text{in } (0, T_{\max}),
\]
which upon integrating results in
\[
a_2 \int_{\Omega} u(\cdot, t) + a_1 \int_{\Omega} v(\cdot, t) \leq \left(\left(a_2 \int_{\Omega} u_0 + a_1 \int_{\Omega} v_0\right)^{-1} + c_1 t\right)^{-1} \quad \text{for all } t \in (0, T_{\max}). \quad (4.9)
\]

As in the proof of Lemma 4.7, we now apply Lemma A.2 (with \( C > 0 \) as in that lemma) to see that
\[
\|\varphi\|_{W^{2,2}(\Omega)} \leq \|\varphi - \nabla \varphi\|_{W^{2,2}(\Omega)} + \|\nabla \varphi\|_{L^2(\Omega)} \leq C \|
\Delta \varphi\|_{L^2(\Omega)} + |\Omega|^{-\frac{1}{2}} \|\varphi\|_{L^1(\Omega)} \quad \text{for all } \varphi \in C^2(\Omega) \text{ with } \partial_\nu \varphi = 0,
\]
which applied to \( \varphi = u \) and \( \varphi = v \) and combined with (4.9) and Lemma 4.4 implies (4.8) for certain \( K_1, K_2 > 0 \) and \( \eta > 0 \).

\[ \square \]

5. Proof of Theorem 1.1

The various lemmata from Section 4 allow us now to find \( \varepsilon > 0 \) such that if \( u_0, v_0 \) satisfy ((1.4) and) (1.6) then \( T_{\max} = \infty \) and \( (u, v) \) converges to \((u_*, v_*)\).

Lemma 5.1. For \( \varepsilon > 0 \) and \( K_1, K_2 > 0 \), define
\[
y_{\varepsilon, K_1, K_2} : [0, \infty) \to \mathbb{R}, \quad t \mapsto \begin{cases} \left(\frac{1}{K_1} + K_2 t\right)^{-1}, & \text{if (H2) holds and } \lambda_2 \mu_1 = \lambda_1 a_2, \\ K_1 \varepsilon e^{-K_2 t}, & \text{else}. \end{cases}
\]

Then there are \( \varepsilon > 0 \) and \( K_1, K_2 > 0 \) such that \( T_{\max}(u_0, v_0) = \infty \),
\[
\|u(u_0, v_0))(\cdot, t) - u_*\|_{W^{2,2}(\Omega)} + \|v(u_0, v_0))(\cdot, t) - v_*\|_{W^{2,2}(\Omega)} \leq y_{\varepsilon, K_1, K_2}(t) \quad \text{for all } t \geq 0,
\]
whenever \( u_0, v_0 \) satisfy (1.4) and (1.6).
Proof. Lemma 4.2, Lemma 4.3, Lemma 4.4, Lemma 4.5 Lemma 4.7 (ii) Lemma 4.8 and Lemma 4.9 imply that there are \( \eta > 0 \) and \( K_1, K_2 > 0 \) with the following property: Let \( \epsilon' > 0 \). If \( u_0, v_0 \) comply with (1.4) and (1.6) with \( \epsilon \) replaced by \( \epsilon' \), then

\[
\|u(\cdot, t) - u_*\|_{W^{2,2}(\Omega)} + \|v(\cdot, t) - v_*\|_{W^{2,2}(\Omega)} \leq y_{\epsilon'K_1K_2}(t) \quad \text{for all } t \in [0, T_{\eta}),
\]

where \( (u, v) := (u(u_0, v_0), v(u_0, v_0)) \) and \( T_{\eta} := T_{\eta}(u_0, v_0) \) is as in (3.1).

Thanks to the restriction \( n \leq 3 \), Sobolev’s embedding theorem asserts that there are \( \alpha \in (0, 1) \) and \( c_1 > 0 \) such that

\[
\|\varphi\|_{C^{\alpha}(\overline{\Omega})} \leq c_1\|\varphi\|_{W^{2,2}(\Omega)} \quad \text{for all } \varphi \in W^{2,2}(\Omega).
\]

Fix an arbitrary \( \epsilon \in \left(0, \frac{\eta}{c_1 \max\{K_1, K_2\}}\right) \) and \( u_0, v_0 \) complying not only with (1.4) but also with (1.6). As then

\[
\|u_0 - u_*\|_{L^{\infty}(\Omega)} + \|v_0 - v_*\|_{L^{\infty}(\Omega)} \leq c_1 \left(\|u_0 - u_*\|_{W^{2,2}(\Omega)} + \|v_0 - v_*\|_{W^{2,2}(\Omega)}\right) \leq c_1 \epsilon < \eta,
\]

we infer \( T_{\eta} > 0 \) from \( u, v \in C^0([0, T_{\max}] \times [0, T_{\max}]) \). Moreover,

\[
\|u(\cdot, t) - u_*\|_{L^{\infty}(\Omega)} + \|v(\cdot, t) - v_*\|_{L^{\infty}(\Omega)} \leq \|u(\cdot, t) - u_*\|_{C^{0}(\Omega)} + \|v(\cdot, t) - v_*\|_{C^{0}(\Omega)}
\leq c_1 \left(\|u(\cdot, t) - u_*\|_{W^{2,2}(\Omega)} + \|v(\cdot, t) - v_*\|_{W^{2,2}(\Omega)}\right)
\leq c_1 y_{\epsilon K_1 K_2}(t)
\leq c_1 y_{\epsilon K_1 K_2}(0)
= K_1 c_1 \epsilon < \eta \quad \text{for all } t \in (0, T_{\eta}),
\]

hence the definition (3.1) of \( T_{\eta} \) asserts \( T_{\eta} = T_{\max} \). In that case, (5.2) further implies \( T_{\max} = \infty \) because of the blow-up criterion (2.2). Finally, as then \( T_{\eta} = T_{\max} = \infty \), the statement is equivalent to (5.1).

Theorem 1.1 is now a direct consequence of already proved lemmata.

Proof of Theorem 1.1. Local existence and the regularity statements were already part of Lemma 2.1, while global extensibility, convergence to \( (u_*, v_*) \) as well as the claimed convergence rates were the subject of Lemma 5.1.

6. Possible generalizations of Theorem 1.1

At last, let us discuss whether the methods used above, could potentially be used to derive more general versions of Theorem 1.1.

Remark 6.1. Recall that the limitation on the space dimension, namely that \( n \in \{1, 2, 3\} \), has only been used at one place: In the proof of Lemma 5.1 we made use of the embedding \( W^{2,2}(\Omega) \rightarrow C^2(\overline{\Omega}) \) (for some \( \alpha \in (0, 1) \)), which only holds in said space dimensions. Thus, it is conceivable that replacing \( W^{2,2}(\Omega) \) by \( W^{m,2}(\Omega) \) for suitable \( m \in \mathbb{N} \) in Theorem 1.1 allows for certain generalizations of our main result.

Indeed, if \( n = 1 \), Theorem 1.1 remains correct if one replaces \( W^{2,2}(\Omega) \) by \( W^{1,2}(\Omega) \) in all occurrences (and \( W^{2,2}_N(\Omega) \) also by \( W^{1,2}_N(\Omega) \)). This can be seen by a straightforward modification of the proofs above: Combine Lemma 3.2 only with Lemma 3.3 and not also with Lemma 3.4. However, a detailed proof would lead to either a considerably longer or an unreasonably more complicated exposition (or to both) and is hence omitted.

At first glance, similar arguments as above appear to imply an analogon of Theorem 1.1 (with \( W^{2,2}(\Omega) \) replaced by \( W^{m,2}(\Omega) \) for sufficiently large \( m \in \mathbb{N} \)) even for higher dimensions. The main problem, however, is, that
during the computations several boundary terms would appear, which apparently cannot be dealt with easily. Let us emphasize that the question whether (a suitably modified version of) Theorem 1.1 holds also in the higher dimensional setting is purely of mathematical interest. The biologically relevant dimensions are covered in Theorem 1.1.

**Remark 6.2.** The prototypical choices of $\rho_1, \rho_2, f$ and $g$ in (1.1) are mainly made for simplicity. We leave it to further research to determine more general conditions on these functions allowing for a theorem of the form of Theorem 1.1.

Still, the methods employed should be robust enough to also allow for (certain) nonlinear taxis sensitivities, for instance. At least for the case (H2) with $\lambda_2 \mu_1 > \lambda_1 a_2$, however, the signs of $\rho_1$ and $\rho_2$ are important: Our approach demands, that, roughly speaking, predators move towards their prey and the prey flees from them.

Likewise, the methods presented here should, in general, also work for different functional responses. Again, there is one caveat: The species moving towards (away from) the other one needs to benefit from (be harmed by) inter-species encounters.

### A. Gagliardo–Nirenberg inequalities

Throughout the appendix, we fix a smooth, bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ and, for $m \in \mathbb{N}$ and $p \in [1, \infty)$, set $W^{m,p}_N(\Omega) := \{ \varphi \in C^\infty(\Omega) : \partial_\nu \varphi = 0 \text{ on } \partial \Omega \}$. (As can be seen easily, for $m = p = 2$, this definition is consistent with the definition of $W^{2,2}_N(\Omega)$ given in (1.5).)

We begin by stating Poincaré’s inequality and straightforward consequences thereof.

**Lemma A.1.** There exists $C_P > 0$ such that

$$\int_{\Omega} (\varphi - \overline{\varphi})^2 \leq C_P \int_{\Omega} |\nabla \varphi|^2 \quad \text{for all } \varphi \in W^{1,2}(\Omega)$$

$$\int_{\Omega} |\nabla \varphi|^2 \leq C_P \int_{\Omega} |\Delta \varphi|^2 \quad \text{for all } \varphi \in W^{2,2}_N(\Omega) \quad \text{and}$$

$$\int_{\Omega} |\Delta \varphi|^2 \leq C_P \int_{\Omega} |\nabla \Delta \varphi|^2 \quad \text{for all } \varphi \in W^{3,2}_N(\Omega).$$

**Proof.** By Poincaré’s inequality (cf. [14, Corollary 12.28]), there is $C_P > 0$ such that

$$\int_{\Omega} (\varphi - \overline{\varphi})^2 \leq C_P \int_{\Omega} |\nabla \varphi|^2 \quad \text{for all } \varphi \in W^{1,2}(\Omega). \quad (A.1)$$

By straightforward approximation/normalization arguments, it is sufficient to prove the remaining two inequalities for all $\varphi \in C^\infty(\Omega)$ with $\int_{\Omega} \varphi = 0$ and $\partial_\nu \varphi = 0$ on $\partial \Omega$. Thus, we fix such a $\varphi$.

An integration by parts, Hölder’s inequality and (A.1) give

$$\int_{\Omega} |\nabla \varphi|^2 = -\int_{\Omega} \varphi \Delta \varphi + \int_{\partial \Omega} \varphi \partial_\nu \varphi \leq \left( \int_{\Omega} \varphi^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\Delta \varphi|^2 \right)^{\frac{1}{2}} + 0 \leq \left( C_P \int_{\Omega} |\nabla \varphi|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\Delta \varphi|^2 \right)^{\frac{1}{2}},$$

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hence, in both cases \( \int_\Omega |\nabla \varphi|^2 = 0 \) and \( \int_\Omega |\nabla \varphi|^2 > 0 \),

\[
\int_\Omega |\nabla \varphi|^2 \leq C_p \int_\Omega |\Delta \varphi|^2.
\]

Similarly, we have

\[
\int_\Omega |\Delta \varphi|^2 = -\int_\Omega \nabla \varphi \cdot \nabla \Delta \varphi + \int_\Omega \Delta \varphi \partial_\nu \varphi \leq \left( C_p \int_\Omega |\Delta \varphi|^2 \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla \Delta \varphi|^2 \right)^{\frac{1}{2}} + 0 \leq C_p \int_\Omega |\nabla \varphi|^2.
\]

The following lemma should also be well-known. However, failing to find a suitable reference, we choose to give a short proof.

**Lemma A.2.** Let \( p \in (1, \infty) \). There exists \( C > 0 \) such that

\[
\| \varphi - \varphi_k \|_{W^{2,p}(\Omega)} \leq C \| \Delta \varphi \|_{L^p(\Omega)} \quad \text{for all } \varphi \in W^{2,p}_N(\Omega).
\]

**Proof.** Suppose this is not the case. By an approximation/normalization argument, there exists \( (\varphi_k)_{k \in \mathbb{N}} \subset C^\infty(\Omega) \) with \( \int_\Omega \varphi_k = 0 \) as well as \( \partial_\nu \varphi_k = 0 \) on \( \partial \Omega \) and

\[
\| \varphi_k \|_{W^{2,p}(\Omega)} > k \| \Delta \varphi_k \|_{L^p(\Omega)} \quad \text{for all } k \in \mathbb{N}.
\]

Without loss of generality, we may assume \( \| \varphi_k \|_{W^{2,p}(\Omega)} = 1 \) for all \( k \in \mathbb{N} \). Thus, there are \( \varphi_\infty \in W^{2,p}(\Omega) \) and \( (k_j)_{j \in \mathbb{N}} \subset \mathbb{N} \) with \( k_j \to \infty \) for \( j \to \infty \) such that

\[
\varphi_{k_j} \rightharpoonup \varphi_\infty \quad \text{in } W^{2,p}(\Omega) \quad \text{as } j \to \infty.
\]

Since \( W^{2,p}(\Omega) \hookrightarrow \hookrightarrow L^p(\Omega) \), this implies

\[
\varphi_{k_j} \to \varphi_\infty \quad \text{in } L^p(\Omega) \quad \text{as } j \to \infty
\]

and thus also \( \int_\Omega \varphi_\infty = 0 \).

As

\[
\left| \int_\Omega \nabla \varphi_\infty \cdot \nabla \psi \right| = \lim_{j \to \infty} \left| \int_\Omega \nabla \varphi_{k_j} \cdot \nabla \psi \right| = \lim_{j \to \infty} \left| \int_\Omega \Delta \varphi_{k_j} \psi \right| \leq \limsup_{j \to \infty} \frac{1}{k_j} \| \psi \|_{L^{\frac{2p}{p-2}}(\Omega)} = 0 \quad \text{for all } \psi \in C^\infty(\overline{\Omega})
\]

by Hölder’s inequality, we further conclude that \( \varphi_\infty \) is constant and because of \( \int_\Omega \varphi_\infty = 0 \) we have \( \varphi_\infty \equiv 0 \).

However, as \([3, \text{Theorem 19.1}]\) asserts

\[
\| \psi \|_{W^{2,p}(\Omega)} \leq C \| \Delta \psi \|_{L^p(\Omega)} + C \| \psi \|_{L^p(\Omega)} \quad \text{for all } \psi \in C^2(\overline{\Omega}) \quad \text{with } \partial_\nu \psi = 0 \text{ on } \partial \Omega
\]

for some \( C > 0 \), we derive

\[
1 = \lim_{j \to \infty} \| \varphi_{k_j} \|_{W^{2,p}(\Omega)} \leq C \limsup_{j \to \infty} \left( \| \Delta \varphi_{k_j} \|_{L^p(\Omega)} + \| \varphi_{k_j} \|_{L^p(\Omega)} \right) = 0,
\]

a contradiction. \( \square \)

These lemmata immediately imply the following version of the Gagliardo–Nirenberg inequality.
Lemma A.3. Let $j \in \{0, 1\}$ and suppose $p, q \in [1, \infty], r \in (1, \infty)$ are such that
\[
\theta := \frac{\frac{1}{p} - \frac{p}{n} - \frac{1}{q}}{\frac{1}{p} - \frac{p}{n} - \frac{1}{q}} \in \left[\frac{j}{2}, 1\right).
\]
Then there exists $C > 0$ such that
\[
\|\varphi - \overline{\varphi}\|_{W^{j, p}(\Omega)} \leq C\|\varphi\|_{L^p(\Omega)}^\theta \|\varphi - \overline{\varphi}\|_{L^q(\Omega)}^{1-\theta} \quad \text{for all } \varphi \in W_N^{2, r}(\Omega).
\] (A.2)

In particular, for any $r \in (1, \infty)$, we may find $C' > 0$ such that
\[
\|\nabla \varphi\|_{L^r(\Omega)}^{2\theta} \leq C'\|\Delta \varphi\|_{L^r(\Omega)}^{r\theta} \|\varphi - \overline{\varphi}\|_{L^q(\Omega)}^{1-r\theta} \quad \text{for all } \varphi \in W_N^{2, r}(\Omega).
\] (A.3)

Proof. The usual Gagliardo–Nirenberg inequality [15] gives $c_1 > 0$ such that
\[
\|\varphi - \overline{\varphi}\|_{W^{2, p}(\Omega)} \leq c_1\|D^2\varphi\|_{L^p(\Omega)}^\theta \|\varphi - \overline{\varphi}\|_{L^q(\Omega)}^{1-\theta} + c_1\|\varphi - \overline{\varphi}\|_{L^q(\Omega)} \quad \text{for all } \varphi \in W_N^{2, r}(\Omega).
\]

As Hölder’s inequality asserts
\[
\|\psi\|_{L^1(\Omega)} \leq c_2\|\psi\|_{L^p(\Omega)}^\theta \|\psi\|_{L^q(\Omega)}^{1-\theta} \quad \text{for all } \psi \in L^r(\Omega) \cap L^q(\Omega)
\]
for some $c_2 > 0$, we find $c_3 > 0$ such that
\[
\|\varphi - \overline{\varphi}\|_{W^{2, p}(\Omega)} \leq c_3\|\varphi - \overline{\varphi}\|_{W^{2, r}(\Omega)}^\theta \|\varphi - \overline{\varphi}\|_{L^q(\Omega)}^{1-\theta} \quad \text{for all } \varphi \in W_N^{2, r}(\Omega).
\]
In conjunction with Lemma A.2, this proves (A.2).

Moreover, for any $r \in (1, \infty)$, letting $j := 1$, $p := 2r$ and $q := \infty$, we see that
\[
\theta := \frac{\frac{1}{p} - \frac{p}{n} - \frac{1}{q}}{\frac{1}{p} - \frac{p}{n} - \frac{1}{q}} = \frac{\frac{1}{r} - \frac{r}{n} - \frac{1}{\infty}}{\frac{1}{r} - \frac{r}{n} - \frac{1}{\infty}} = \frac{1}{2} \in \left[\frac{j}{2}, 1\right).
\]
Hence, (A.3) follows from (A.2). \qed

In order to avoid any discussions how $\int_\Omega |D^3\varphi|^2$ and $\int_\Omega |\nabla \Delta \varphi|^2$ relate for $\varphi \in W_N^{3, 2}(\Omega)$, we choose to prove the following Gagliardo–Nirenberg-type inequalities, which have been used in the proof of Lemma 3.4, by hand.

Lemma A.4. There exists $C > 0$ such that for all $\varphi \in W_N^{3, 2}(\Omega)$ the estimates
\[
\int_\Omega |\nabla \varphi|^6 \leq C\|\varphi - \overline{\varphi}\|^4_{L^\infty(\Omega)} \int_\Omega |\nabla \Delta \varphi|^2
\]
and
\[
\int_\Omega |\Delta \varphi|^3 \leq C\|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)} \int_\Omega |\nabla \Delta \varphi|^2
\]
hold.

Proof. By Lemma A.3, there is $c_1 > 0$ such that
\[
\int_\Omega |\nabla \varphi|^6 \leq c_1\|\varphi - \overline{\varphi}\|^3_{L^\infty(\Omega)} \int_\Omega |\Delta \varphi|^3 \quad \text{for all } \varphi \in W_N^{2, 3}(\Omega).
\] (A.4)
Let $\varphi \in C^3(\Omega)$ with $\partial_\nu \varphi = 0$ on $\partial \Omega$. Noting that $(|\xi|\xi)' = 2|\xi|$ for $\xi \in \mathbb{R}$, by an integration by parts, Hölder’s inequality and (A.4) we obtain

$$\int_\Omega |\Delta \varphi|^3 = \int_\Omega |\Delta \varphi| \Delta \varphi \Delta \varphi$$

$$= -\int_\Omega \nabla(|\Delta \varphi| \Delta \varphi) \cdot \nabla \varphi$$

$$= -2\int_\Omega |\Delta \varphi| \nabla \varphi \cdot \nabla \Delta \varphi$$

$$\leq 2 \left( \int_\Omega |\Delta \varphi|^3 \right)^{\frac{1}{3}} \left( \int_\Omega |\nabla \varphi|^6 \right)^{\frac{1}{6}} \left( \int_\Omega |\nabla \Delta \varphi|^2 \right)^{\frac{1}{2}}$$

$$\leq 2c_1 \|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)} \left( \int_\Omega |\Delta \varphi|^3 \right)^{\frac{1}{3}} \left( \int_\Omega |\nabla \Delta \varphi|^2 \right)^{\frac{1}{2}},$$

hence

$$\int_\Omega |\Delta \varphi|^3 \leq c_2 \|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)} \int_\Omega |\nabla \Delta \varphi|^2,$$

where $c_2 := 4c_1^{\frac{1}{2}}$.

Plugging this into (A.4) yields

$$\int_\Omega |\nabla \varphi|^6 \leq c_1 c_2 \|\varphi - \overline{\varphi}\|_{L^\infty(\Omega)} \int_\Omega |\nabla \Delta \varphi|^2.$$  

The statement follows by an approximation procedure and by setting $C := \max\{c_1, c_1 c_2\}$. 

\[\square\]

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