Eliciting judgements about uncertain population means and variances

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Abstract

We propose an elicitation method for quantifying an expert’s opinion about an uncertain population mean and variance. The method involves eliciting judgements directly about the population mean or median, and eliciting judgements about the population proportion with a particular characteristic, as a means of inferring the expert’s beliefs about the variance. The method can be used for a range of two-parameter parametric families of distributions, assuming a finite mean and variance. An illustration is given involving an expert’s beliefs about the distribution of times taken to translate pages of text. The method can be implemented in R using the package SHELF.

Keywords: Prior elicitation, expert judgement, population distributions.
1 Introduction

We consider the problem of eliciting an expert’s opinion (or a group of experts’ opinions) about an uncertain mean and variance for a population of independent and identically distributed random variables $X_1, X_2, \ldots$. We suppose that either the population distribution is normal, or that a transformation can be applied so that the expert is willing to accept a normal distribution for the population, and we write

$$X_1, X_2, \ldots \mid \mu, \sigma^2 \text{iid} \sim N(\mu, \sigma^2),$$

so that the aim is to obtain the expert’s probability distribution for $\mu$ and $\sigma^2$.

An obvious application would be in Bayesian inference for the parameters of a normal distribution, though in many cases the available data would dominate any reasonable prior, and the effort in obtaining expert prior knowledge may not be worthwhile. There are, however, various scenarios when little or even no data would be available, and the prior distribution plays an important/essential role. We defer discussion of these to the next section.

Typically, elicitation methods involving asking experts to make judgements about observable quantities, rather than parameters in statistical models. However, we believe experts would be willing to make judgements directly about an uncertain measure of location such as a mean or median, and generic techniques for eliciting beliefs about a scalar quantity would normally suffice, for example, the bisection method described in [Raiffa 1968], pp161-168. See also [O’Hagan et al. 2006], Chapter 6. We would not expect experts to be willing to make direct judgements about an uncertain population variance, as this is a more abstract quantity. The challenge is then to design elicitation questions about observable quantities, that can be used to infer an expert’s beliefs about an uncertain variance.

To the best of our knowledge, there has been little work on eliciting beliefs about variances. One existing approach that can be used is based on eliciting beliefs about parameters in linear regression models. [Kadane et al. 1980] and [Al-Awadhi and Garthwaite 1998] consider elicitation for the parameters in models of the form

$$X_i = \mu + \sum_{j=1}^{p} \beta_j z_{ij} + \varepsilon_i, \quad \text{for } i = 1, 2, \ldots, n,$$

where $\varepsilon_1, \ldots, \varepsilon_n \text{iid} \sim N(0, \sigma^2)$. By setting $\beta_j = 0$ for all $j$, this reduces to our case.
Awadhi and Garthwaite (1998) proposed an elicitation method for quantifying opinions about the parameters of a multivariate normal distribution; the same elicitation method could be used for quantifying beliefs about a univariate normally distributed population. A key feature of these methods is asking the expert to update his or her judgements in light of given hypothetical data, under the assumption that the expert updates his or her beliefs using Bayes’ theorem. We think this is a difficult task: the expert may not view hypothetical data as credible and behave the same way had he/she observed real data, and it is unlikely that the expert would weight prior knowledge and hypothetical data precisely according to Bayes’ theorem in any case. The expert may be insensitive to the sample size, for example in accounting for the variability in a sample mean (Tversky and Kahneman 1971). We think it desirable to have alternative elicitation methods available to the expert.

Kadane et al. (1980) and Al-Awadhi and Garthwaite (1998) infer judgements about the parameters $\mu$ and $\sigma^2$ from judgements about the observable quantities $X_i$, by eliciting summaries from the expert’s predictive distribution. For example, suppose we wish to elicit an expert’s opinion about the variance parameter $\sigma^2$ of a random variable $X$ that follows a normal distribution with a known mean $\mu$. Since $\sigma^2$ is not directly observable then the expert is asked to make judgements about the observable quantity $X$, and we infer $p(\sigma^2)$ from these judgements. Any choice of $p(\sigma^2)$ implies a distribution

$$p_X(x) = \int_{\mathbb{R}^+} p_X(x \mid \sigma^2)p(\sigma^2)d\sigma^2,$$

and we suppose that a particular choice of $p(\sigma^2)$ will result in the above integral (approximately) matching the expert’s beliefs about $X$, so that this choice of $p(\sigma^2)$ describes the expert’s underlying beliefs about $\sigma^2$. A concern here is whether an expert really is able to account for his or her uncertainty about $\sigma^2$ when making judgements about $X$. A possibility is that the expert instead only makes judgements about $X$ conditional on some estimate of $\sigma^2$.

Kadane et al. (1980) use conjugate priors for $\mu$ and $\sigma^2$ which force the expert’s opinion about the two parameters to be dependent. However, it is possible in reality that knowledge of one parameter would not change the expert’s opinion about the other. Al-Awadhi and Garthwaite (2001) argued that, unless mathematical tractability is required, then it can be better to assume independence between the two parameters, and that this helps the expert focus on the assessments of each parameter separately. They proposed an elicitation method for the multivariate normal distribution where
the mean vector and covariance matrix are assumed to be independent, though their
method also asks the expert to update his or her judgements in the light of hypothetical
data.

We argue that the better informed the expert, the less likely a judgement of de-
pendence between the two parameters would be required. For example, consider the
distribution of running times for an individual over a distance of 5km. With no infor-
mination about the ability of the runner, one might have considerable uncertainty about
the mean, e.g. an interval of 15 minutes to 1 hour may be judged plausible, with smaller
variances of running times associated with smaller means within this interval. But if
one already has ‘expert’ knowledge about the particular runner’s ability, a much smaller
interval may be judged plausible for the mean, and one’s beliefs about the variance may
not change appreciably given different plausible means.

In this paper, we propose a new elicitation method for quantifying opinions about
an uncertain population mean and variance. Our method does not elicit judgements
using hypothetical data and Bayes’ theorem, it does not use predictive elicitation and
it assumes independence between the mean and variance. The article is organised as
follows. In the next section we briefly discuss some motivating applications where the
prior distributions will be important. In Section 3 we discuss the choice of prior families
of distributions for the uncertain mean and variance. In Section 4 we present a detailed
procedure of eliciting judgements using our proposed elicitation method. In Section 5,
we present a real elicitation example to illustrate the use of our proposed method in
practice.

2 Motivating applications

2.1 Bayesian clinical trial design

Various authors have proposed a Bayesian alternative to a power calculation in the
design of clinical trials, in which a prior distribution is elicited for a treatment effect,
and then the unconditional probability of a ‘successful trial’ (e.g. rejection of a null
hypothesis as required by a regulator) is calculated via an integral of the power function
with respect to the elicited prior [Spiegelhalter and Freedman, 1986; O’Hagan et al.]

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Power functions typically depend on population variances of patient responses, and hence a prior distribution for this variance required.

Note that the main role of this prior is in the design stage: the calculation of the unconditional probability of a successful trial before the data have been observed (and this calculation may assume that the trial data will be analysed using a frequentist approach). O’Hagan et al. (2005) considered uncertain variances, but did not propose or use formal elicitation methods.

2.2 Bayesian random-effects meta-analysis

A common scenario in a meta-analysis is that we have a number of studies, each of which has attempted to estimate the effect of some ‘treatment’ in a randomised controlled experiment, and the aim is to synthesise the data from all the studies to infer an overall treatment effect. Often, unobserved differences in the study populations cause the treatment effects to vary between studies, and this is typically handled by modelling the treatment effects as random effects, drawn from some distribution. Although sample sizes within studies may be large, the number of studies can be small, so that there is very little information in the data about the population variance of treatment effects, and the prior distribution for this variance plays an important role. There is some discussion of priors for variances of random effects in Spiegelhalter et al. (2004), Section 5.7.3, including some informal elicitation approaches when the treatment effect is measured as a log odds ratio.

2.3 Risk analysis and 2D Monte Carlo methods

In a risk analysis, a decision-maker may have to make decisions based on expert opinion only (see, for example, the discussion of the role of expert judgement in food safety risk assessment in European Food Safety Authority (2010). In particular, a risk analysis may need to consider both aleatory uncertainty caused by variability within a population, and epistemic uncertainty about the extent of this variability. For example, Clough et al. (2006, 2009) analyse the risk of contamination of farm-pasteurised milk contamination with Vero-cytotoxigenic E.coli O157. Their analysis involves the use of a mechanistic model with various uncertain inputs. One input describes the amount
of faecal contamination in bulk tank per milking, and was informed by expert judgement only. This is a quantity that would vary from one milking to the next, but the distribution of amounts of contamination would be uncertain.

The risk analysis may involve the use of a mechanistic model of the form \( Y = f(X) \), where \( X \) has a population distribution \( N(\mu, \sigma^2) \), with \( \mu, \sigma^2 \) uncertain. Given an elicited distribution for \( \mu, \sigma^2 \), analysis can proceed with a ‘2D’ or ‘second-order’ Monte Carlo simulation (see, for example, Wu and Tsang 2004): a \( \mu, \sigma^2 \) pair is sampled from its distribution, and then a sample \( X_1, \ldots, X_n \) from \( N(\mu, \sigma^2) \) is propagated through \( f \) to obtain a sample \( Y_1, \ldots, Y_N \). One can then examine how the distribution of model outputs changes as function of \( \mu, \sigma^2 \), and hence explore both the effects of aleatory uncertainty within the population, and epistemic uncertainty about the population distribution parameters. The decision-maker may consider reducing uncertainty about \( \mu, \sigma^2 \) if this source of uncertainty appears relatively important.

3 Prior families of distributions for the mean and variance

In this paper, we suppose that the expert’s uncertainty about \( \mu \) can be represented by a normal distribution
\[
\mu \sim N(m, v),
\]
and that her uncertainty about \( \sigma^2 \) can be represented by an inverse gamma distribution which we write as
\[
\sigma^2 \sim IG(a, b),
\]
with density function,
\[
p(\sigma^2) = \frac{b^a}{\Gamma(a)} \sigma^{-(a+1)} \exp \left\{ -\frac{b}{\sigma^2} \right\}, \quad a, b > 0.
\]
These are similar to the choices in Kadane et al. (1980), except that their prior for \( \mu \) was of the form \( \mu | \sigma^2 \sim N(m, \sigma^2 v) \). In some cases, alternative families of distributions may be needed for \( \mu \), and we discuss this further in Section 4.5. The SHELFL R package will fit either a gamma distribution or a lognormal distribution to the population precision \( \sigma^{-2} \), though it would be difficult to claim any single choice of family as ‘optimal’ at this stage. After the expert has provided judgements and distributions have been fitted,
we can use feedback to test whether these assumptions are acceptable to the expert; feedback is provided at several stages in our proposed method.

4 The proposed elicitation method

For simplicity and ease of exposition, it is supposed that there is one female expert, and that the elicitation is conducted by a male facilitator. There are, of course, various general considerations when performing elicitation such as training of the experts, and how to manage (or combine opinions from) multiple experts. The focus of this paper is solely on how to elicit judgements about a mean and variance, and we do not consider these other issues here. Guidance on these and other aspects of elicitation can be found in European Food Safety Authority (2010), O’Hagan et al. (2006), Cooke (1991) and Morgan and Henrion (1990).

4.1 Eliciting plausible bounds for \( X_i \)

We first ask the expert to provide a lower plausible bound \( L \) and an upper plausible bound \( U \) for a member of the population \( X_i \). She is told that \( P(L < X_i < U) \approx 1 \) is assumed. Though we are assuming a normal distribution for the population, limits will still be needed when plotting density functions etc. to provide graphical feedback. Providing two bounds may also reduce the effect of the anchoring and adjusting heuristic: making a judgement by starting from an additional value, the anchor, and then making a, potentially insufficient, adjustment (Tversky and Kahneman 1974) as the expert cannot use multiple anchors simultaneously.

4.2 Eliciting judgements about the population mean

The expert is asked to provide at least two quantiles (denoted as \( \mu_{(\alpha_1)} \) and \( \mu_{(\alpha_2)} \)) of her distribution of \( \mu \) such that

\[
P(\mu \leq \mu_{(\alpha_i)}) = \alpha_i,
\]

for \( i = 1, 2 \), and \( 0 < \alpha_1 < \alpha_2 < 1 \). Garthwaite and O’Hagan (2000) suggest eliciting lower and upper tertiles, and calibration studies tend to find performance worsens as
more extreme quantiles are elicited (Alpert and Raiffa, 1982; Lichtenstein et al., 1982). However, in our experience, experts may find moderate quantiles harder to judge, and so there can be a trade-off between aiming for the best calibration performance, and asking the experts questions that they are comfortable answering. We tend towards the latter, given that a population mean is a more abstract quantity than an individual observable population member. In our example, we elicited the 5th and 95th percentiles of the expert’s distribution of $\mu$.

If we denote the cumulative distribution function of the standard normal distribution by $\Phi$, we have

$$
\Phi^{-1}(\alpha_i) \sqrt{v} + m = \mu_{(\alpha_i)},
$$

and so we choose values for $m$ and $v$ as follows:

$$
\hat{m} = \frac{\mu_{(\alpha_1)} \Phi^{-1}(\alpha_2) - \mu_{(\alpha_2)} \Phi^{-1}(\alpha_1)}{\Phi^{-1}(\alpha_2) - \Phi^{-1}(\alpha_1)},
$$

$$
\hat{v} = \left( \frac{\mu_{(\alpha_2)} - \mu_{(\alpha_1)}}{\Phi^{-1}(\alpha_2) - \Phi^{-1}(\alpha_1)} \right)^2.
$$

The facilitator then feeds back different summaries of the fitted distribution, including showing the density function of $N(\hat{m}, \hat{v})$. The expert may make adjustments until she is satisfied that the fitted distribution is a reasonable representation of her opinion about $\mu$.

## 4.3 Eliciting judgements about the population variance

To elicit the expert’s judgements about the population variance, she is asked to suppose that the population mean is known to be $\hat{m}$, her median value from the previous step. This is simply to enable the expert to consider uncertainty about the population variance in isolation; she is not required to update her beliefs (in any case, we are assuming independence between $\mu$ and $\sigma^2$). There remains a concern when providing hypothetical data: the expert may think differently if she had actually observed $\mu = \hat{m}$, but as $\hat{m}$ could be thought of as her ‘best guess’, she at least is not having to condition on a value of $\mu$ she considers implausible.

The expert is now asked to consider the proportion $\theta$ (or percentage if she prefers) of the population $X_1, X_2, \ldots$ that would lie in the interval $[\hat{m}, \hat{m} + c]$, for some $c > 0$ (we consider the choice of $c$ in the next section). The expert will be uncertain about this
proportion, because she is uncertain about $\sigma^2$. To illustrate this idea, suppose we wish to elicit an expert’s beliefs about end-of-year students’ marks for an undergraduate statistics module, for a large population of students. Suppose the marks are normally distributed with a mean of 60. Then there is a true proportion of students who will get marks between 60 and 70. If this expert is certain this proportion will be less that 0.45 and more than 0.25, this would imply she is certain $\sigma$ is between 6 and 15. This is illustrated in Figure 1.

Figure 1: Density plots for exam marks assumed to be normally distributed with mean 60. The grey area represents the true proportion of students who get marks between 60 and 70. If the expert is certain this proportion is between 0.25 and 0.45, she is certain $\sigma$ is between 6 and 15.

Again, to ease the elicitation process, we suggest eliciting tail quantiles of $\theta$ because the expert may be more willing to judge tail quantiles as argued earlier. Hence the expert is asked for the 5th and 95th percentiles of the population proportion, which we denote by $\theta_{(0.05)}$ and $\theta_{(0.95)}$ respectively, and these are described to the expert as approximate lower and upper limits.

It may help the expert to explicitly consider the two intervals $[\hat{m}, \hat{m} + c]$ and $(\hat{m} + c, U]$ and consider how the population is distributed between these two intervals. For
example, for an appropriate choice of \( c \), she might judge a split of 2%-48% across the two intervals highly unlikely, which can help prompt judgements of more plausible allocations (though the facilitator should remain cautious of anchoring effects). If the expert prefers to report judgements on the 0-1 scale rather than the 0-0.5 scale, she could instead be asked to consider the proportion in \([\hat{m}, \hat{m} + c]\), out of those known to be greater than \( \hat{m} \).

The facilitator uses the expert’s judgements \( \theta_{(0.05)} \) and \( \theta_{(0.95)} \) to infer the corresponding quantiles of her distribution for the variance, \( \sigma^2_{(0.95)} \) and \( \sigma^2_{(0.05)} \). It is straightforward to show that

\[
\sigma^2_{(\alpha)} = \left( \frac{c}{\Phi^{-1}(\theta_{(1-\alpha)} + \frac{1}{2})} \right)^2,
\]

for \( \alpha \in (0, 1) \).

The facilitator chooses \( a \) and \( b \) to be

\[
\hat{a}, \hat{b} = \arg\min_{a,b} \left[ \left\{ \frac{F_{\sigma^2}(\hat{\sigma}^2_{(0.05)}; a, b) - 0.05}{2} + \left\{ \frac{F_{\sigma^2}(\hat{\sigma}^2_{(0.95)}; a, b) - 0.95}{2} \right\}^2 \right\} \right],
\]

where \( F_{\sigma^2}(.; a, b) \) is the CDF of the IG distribution. The minimisation cannot be done analytically so numerical optimisation is applied.

The expert would may find it hard to visualise her judgements from the fitted IG distribution since \( \sigma^2 \) is not directly observable. Therefore, we suggest showing the expert two density plots of the population distribution with a fixed mean and the two calculated quantiles of the variance, \( N(\hat{m}, \sigma^2_{(0.05)}) \) and \( N(\hat{m}, \sigma^2_{(0.95)}) \). The expert visualises the variation in the distribution of \( X_i \) based on her judgements about \( \theta \), and then confirms whether the fitted IG distribution is a reasonable representation of her opinion about \( \sigma^2 \).

To summarise, the expert is asked to make six judgements, as set out in Table 1.
Table 1: The elicitation questions with the corresponding notation.

| Notation | Elicitation Question / Definition |
|----------|----------------------------------|
| $L$      | What is your lower plausible bound for the quantity of interest where is it mostly unlikely for any $X_i$ to take values less than $L$? |
| $U$      | What is your upper plausible bound for the quantity of interest where is it mostly unlikely for any $X_i$ to take values more than $U$? |
| $\mu_{\alpha_1}$ | What is your $\alpha_1$ quantile of the population mean? |
| $\mu_{\alpha_2}$ | What is your $\alpha_2$ quantile of the population mean? |
| $\theta$ | Suppose the population mean is known to be $\hat{m}$. Then let $\theta$ denote the proportion of members of population that will lie in the interval $[\hat{m}, \hat{m} + c]$. |
| $\theta_{(0.05)}$ | What is your 5th percentile of the unknown proportion $\theta$ that will lie in the given interval? |
| $\theta_{(0.95)}$ | What is your 95th percentile of the unknown proportion $\theta$ that will lie in the given interval? |

4.3.1 Choice of $c$

Theoretically, regardless of the choice of $c$, the expert will have a 5th and 95th percentile for her corresponding proportion $\theta$, from which the hyperparameters $a$ and $b$ can be determined. However, the expert may find the elicitation task particularly difficult for certain choices of $c$, for example, for sufficiently large $c$, the expert may simply be certain that $\theta$ will be 0.5. In any case, there is likely to be some vagueness to the expert’s judgements: for example, she may state $\theta_{(0.95)} = 0.4$, but express indifference to an alternative value such as 0.35. Hence, in practice, some choices of $c$ may be better than others, both in helping the expert to consider her percentiles for $\theta$, and in making the produce more robust to vaguely expressed judgements.

Given the relationship between $\theta$ and $\sigma^2$ in equation 4.1, we consider relative
changes in $\sigma$ as $\theta$ varies and so consider $\log \sigma$ as a function of $\theta$:

$$\log \sigma = \log c - \Phi^{-1}(\theta + 0.5).$$

We plot this relationship (ignoring the $\log c$ term) in Figure 2. The gradient is fairly constant and at its minimum over the range $[0.2, 0.45]$, so to make the procedure as robust as possible it would be desirable for the expert to report her quantiles for $\theta$ outside this interval. In particular, if the expert reports $\theta_{(0.05)}$ of say 0.05 or 0.1, the implied 95th percentile for $\sigma$ could change more appreciably given minor adjustments to $\theta_{(0.05)}$. In this case, the facilitator could consider repeating the process with a larger value of $c$.

Figure 2: The solid line shows the relationship between $\log \sigma$ and $\theta$, with the dashed line indicating that the gradient of this function is fairly constant and minimised over the interval $[0.2, 0.45]$. If the expert provides lower and upper quantiles of the proportion $\theta$ outside this interval, the implied quantiles for $\sigma$ may be less robust to rounding or imprecision in her judgements.

If we were to choose $c$ to be the midpoint of $\hat{m}$ and $U$ then, assuming a unimodal population distribution, the expert should be certain $\theta$ will be at least 0.25, but could judge $\theta$ to be almost certainly 0.5 and find it difficult to consider her uncertainty.
Consequently, we suggest a smaller value of $c$: a third of the distance from $\hat{m}$ to $U$. In any case, at the training stage, the choice of $c$ should be stated in advance to the expert, so that she does not interpret the given interval as a source of information (Schwarz, 1999).

4.4 Feedback

To help the expert visualise the fitted prior distribution, we suggest plotting point-wise density estimates of the cumulative distribution function, and obtaining intervals for population quantiles. We use the following algorithm.

1. Sample $\mu_1, ..., \mu_K$ from $N(\hat{m}, \hat{v})$ and $\sigma_1^2, ..., \sigma_K^2$ from IG($\hat{a}, \hat{b}$).

2. Choose evenly spaced values $x_1, ..., x_J$ between $L$ and $U$.

3. For $k = 1, ..., K$ and $j = 1, ..., J$ calculate
   \[ F_k(x_j) = \Phi\left(\frac{x_j - \mu_k}{\sigma_k}\right), \]

4. For $k = 1, ..., K$ obtain the $k$th sampled value of the population $\alpha$ quantile $X(\alpha)$ as
   \[ X(\alpha),k = \mu_k + \sigma_k \Phi^{-1}(\alpha). \]

5. At each $x_j$, show an empirical 95% interval for $P(X \leq x_j | \mu, \sigma^2)$ given the sample $F_1(x_j), ..., F_K(x_j)$.

6. Report an empirical 95% interval for $X(\alpha)$ given the sample $X(\alpha),1, ..., X(\alpha),K$.

4.5 Other population distributions

In this section, we consider transformations of the normal distribution and discussion modifications to the elicitation method. We suppose that

\[ g(X_1), g(X_2), ..., | \mu, \sigma^2 \overset{iid}{\sim} N(\mu, \sigma^2), \]

where $g$ is a monotone (increasing) function. Two likely choices for $g$ would be $g(X) = \log X$ for skewed distributions, and $g(X) = \log\{X/(1 - X)\}$ for variables bounded between 0 and 1.
Assuming a non-symmetric population distribution, making judgements about the population mean is likely to be difficult, so we suggest to instead elicit judgements about the population median, which we write as $\phi = g^{-1}(\mu)$. It is possible that a non-symmetric distribution would also be needed to represent the expert’s beliefs about $\phi$, and so we should elicit at least three quantiles $\phi(\alpha_1), \phi(\alpha_2), \phi(\alpha_3)$ (with $P(\phi \leq \phi(\alpha_i)) = \alpha_i$) from the expert’s distribution for $\phi$. If we then choose a (two-parameter) family of distributions with parameters $m$ and $v$ and cumulative distribution function $F_{m,v}$, we can obtain $\hat{m}$ and $\hat{v}$ numerically by minimising

$$\sum_{i=1}^{3} \left( F_{m,v}(\phi(\alpha_i)) - \alpha_i \right)^2$$

with respect to $m$ and $v$. In practice, one out of a normal, log-normal, or (possibly scaled) beta distribution is likely to be adequate as a distribution for $\phi$.

To elicit the prior for $\sigma^2$, the expert is asked to suppose that $\phi = \hat{m}$, and is asked to consider her uncertainty about the population proportion $\theta$ lying in the interval $[k_1, k_2]$, with $k_1 = g^{-1}(\hat{m})$, and $k_2 = g^{-1}(\hat{m} + c)$ for some appropriate value of $c$. The method then proceeds as before, following Equations (4.1) and (4.2).

5 Example: eliciting beliefs about translation times

We describe an elicitation exercise regarding the time needed to translate one page from a book from English to Arabic, for a given translator. The translation time will vary from one page to the next, resulting in an (unknown) population distribution of translation times. The expert was asked to consider a book called *The art of creative writing* by Lajos Egri, and similar types of books. The expert believes that the length of time will be symmetrically distributed around the mean. Hence the facilitator assumes the expert’s uncertainty about the population $X_1, X_2, \ldots$ can be represented by a normal distribution with uncertain mean $\mu$ and variance $\sigma^2$ as follows:

$$X_1, X_2, \ldots \mid \mu, \sigma^2 \sim \text{N}(\mu, \sigma^2),$$

$$\mu \sim \text{N}(m,v),$$

$$\sigma^2 \sim \text{IG}(a,b).$$
A description of the elicitation is given below. Commands for reproducing the analysis in R (R Core Team 2016) are given in the Appendix.

5.1 Training

At the first elicitation session, the expert was given general knowledge about the concept of probability and the purpose of elicitation. Additionally, he was told how people may make biased judgements, for example by unintentionally applying the anchoring and adjusting heuristic (Tversky and Kahneman 1974). The facilitator discussed with the expert different examples of heuristics and the expert was encouraged to consider them when making his judgements.

Additionally, the expert received intensive training about probability judgements, including three practice examples: (1) the distance from the Hicks building at Sheffield University to Manchester city centre, in kilometres, using the MATCH software tool (Morris et al. 2014); (2) the monthly rent of one-bedroom properties in Sheffield city centre; and (3) the prices in pounds per hour charged by Saudi translators for translating English texts into Arabic. Practice examples 2 and 3 were conducted using our proposed method to familiarise the expert with it. The facilitator used templates from the SHELF package (O’Hagan and Oakley 2010) to complete a ‘SHELF 1 (Context)’ form that records basic details about the elicitation session, such as the date, purpose of elicitation, training given and the definition of the quantity of interest.

5.2 Eliciting plausible bounds for $X_i$

The expert judged the shortest plausible length of time required to translate a randomly selected page into Arabic to be $L = 5$ minutes, confirming that even for a short page with known and repeated expressions it was mostly unlikely to take less than 5 minutes. He judged the longest plausible length of time he would require to translate a randomly selected page into Arabic to be $U = 70$ minutes. He confirmed that he believed that it was mostly unlikely for any page of the book to take more than 70 minutes.
5.3 Eliciting judgements about the population mean

The facilitator defined $\mu$ as the mean length of time (in minutes) required by the expert to translate a randomly selected page into Arabic. The expert made his judgements about the 5th and 95th percentiles of the mean as $\mu_{(0.05)} = 30$ and $\mu_{(0.95)} = 40$ and confirmed that he was 90% certain that the true value of $\mu$ would fall between these values. In addition, he confirmed that he thought the true value of the mean was equally likely to fall below or above 35 minutes.

The facilitator fitted the expert’s judgements about $\mu$ to a normal distribution with mean $\hat{\mu} = 35$ and variance $\hat{\sigma}^2 = 9.24$. The PDF of the fitted normal distribution was shown to the expert and feedback was given in the form of the 1st and 99th percentiles of the N(35, 9.24) distribution, respectively 28 and 42 minutes. The expert confirmed these as appropriate. Additionally, the expert was also asked if he believed it is possible that someone with the same expertise as him would say the mean is 20 or 50 minutes; he confirmed that he did not think it possible, and that he would be surprised if this happened. The expert then agreed that the fitted distribution was a reasonable representation of his knowledge about $\mu$.

5.4 Eliciting judgements about the population variance

The facilitator defined $\sigma^2$ as the variance of the length of time required by the expert to translate a randomly selected page from the book into Arabic. The expert was told to suppose that the population mean was known to be 35 minutes, based on his earlier judgements. The facilitator then elicited the expert’s judgements about the percentage of pages whose translating time lies within a given interval $[k_1, k_2]$. The expert was told to set the first endpoint to be his judgement about the median value from the previous step, that is, $k_1 = 35$ and to add 10 minutes to his median value for the second endpoint so that $k_2 = 45$. The facilitator explained to the expert that based on the definition of the median, 50% of the pages in the book would take less than 35 minutes to be translated whereas the other half of the pages will take longer than 35 minutes. Now the elicitation method focuses on the percentage ($\theta$) of pages that will take between 35 and 45 minutes. As we have illustrated earlier that the expert may feel more comfortable when making judgements about the 5th and 95th percentiles and told to interpret them as the approximate lower and upper limits, respectively.
The expert judged the 5th percentile $\theta_{(0.05)}$ of the percentage of pages that would take between 35 and 45 minutes of translation to be 33%; he confirmed that he was 95% certain that the percentage of pages taking times in [35, 45] will be more than 33%. Then he judged the 95th percentile $\theta_{(0.95)}$ of the percentage of pages that would take time between 35 and 45 minutes of translation to be 40%; he confirmed that he was 95% certain that the percentage of pages taking times in [35, 45] will be less than 40%.

The facilitator obtained the corresponding quantiles of the distribution of $\sigma^2$:

$$
\sigma^2_{(0.95)} = \left( \frac{10}{\Phi^{-1}(0.4 + \frac{1}{2})} \right)^2,
\sigma^2_{(0.05)} = \left( \frac{10}{\Phi^{-1}(0.4 + \frac{1}{2})} \right)^2,
$$

The facilitator numerically fitted an IG distribution to the expert’s judgements using a least squares approach to minimise

$$
\hat{a}, \hat{b} = \arg\min_{a,b} \left[ \{ F_{\sigma^2}(\sigma^2_{(0.05)}; a, b) - 0.05 \}^2 + \{ F_{\sigma^2}(\sigma^2_{(0.95)}; a, b) - 0.95 \}^2 \right],
$$

where $F_{\sigma^2}(\cdot; a, b)$ is the CDF of an IG distribution, obtaining the IG(31.5, 2514) distribution.

The expert was given feedback in the form of visualising the effect of his judgements about $\theta$ on the PDF plots of $X_i$ with a fixed mean and the two calculated quantiles of the variance: $N(\hat{m}, \hat{\sigma}^2_{(0.05)})$ and $N(\hat{m}, \hat{\sigma}^2_{(0.95)})$ The expert confirmed how his judgements about $\theta$ affect the fitted distribution of $X_i$ and agreed that the fitted distribution was a reasonable representation of his uncertainty about $\sigma^2$.

5.5 Feedback

The expert’s elicited judgements are given in Table 2.
Figure 3: The PDF plots of the population distribution with fixed mean $E[\mu] = \hat{m}$ and variance at the elicited 5th percentile (top) and the 95th percentile (bottom).
Table 2: Elicitation questions with the corresponding notations for the translation example.

| Notation | Elicited | Elicitation Question / Definition |
|----------|----------|----------------------------------|
| $L$      | 5        | What is your lower plausible bound for the length of time you require to translate a randomly selected page into Arabic? |
| $U$      | 70       | What is your upper plausible bound for the length of time you require to translate a randomly selected page into Arabic? |
| $\mu_{(0.05)}$ | 30       | What is your 5th percentile of the mean length of time you require to translate a randomly selected page from the book? |
| $\mu_{(0.95)}$ | 40       | What is your 95th percentile of the mean length of time you require to translate a randomly selected page from the book? |
| $\theta$ | $\hat{m} = 35$ | Suppose the population mean is known to be $\hat{m} = 35$. Then let $\theta$ denote the proportion of the pages that will take time in $[35, 45]$ to be translated. |
| $\theta_{(0.05)}$ | 33%      | What is your 5th percentile of the percentage of pages that will take times in $[35, 45]$ minutes to be translated? |
| $\theta_{(0.95)}$ | 40%      | What is your 95th percentile of the proportion of pages that will take times in $[35, 45]$ minutes to be translated? |

The feedback was given in the form of a CDF plot of $X_i \mid \mu, \sigma^2$ as explained earlier. For $k = 1, ..., K$ and $j = 1, ..., J$, we sample $\mu_1, ..., \mu_K$ from $N(35, 9.24)$ and $\sigma^2_1, ..., \sigma^2_K$ from $IG(31.5, 2514)$. We calculate

$$F_k(X_j) = \Phi \left( \frac{X_j - \mu_k}{\sigma_k} \right),$$

for $X_1, ..., X_J$ evenly spaced between 5 and 70, and we set $K = J = 300$.

The expert was asked to consider the 5th quantile of the population distribution, interpreted as the 5th quickest page to translate assuming a book with 100 pages. The fitted 90% interval for the 5th quickest page was (12 minutes, 23 minutes). He was also asked to consider the 95th quantile of the population distribution: the 5th slowest page to translate assuming a book with 100 pages. The fitted 90% interval for the 5th quickest page was (47 minutes, 58 minutes). Given the fitted intervals, the expert chose to alter his judgements: he thought the times in the interval for the 5th quickest page were too long. He modified his elicited values until he was satisfied with the feedback, obtaining $\theta_{(0.05)} = 30\%$ and $\theta_{(0.95)} = 35\%$, with the facilitator obtaining $\sigma^2 \sim IG(62.8, 7114)$. 

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The expert then agreed that the fitted population distribution was a reasonable representation of his judgements about the population distribution, and the elicitation session was concluded.

Figure 4: The CDF plot of the fitted population distribution of the length of times required by the expert to translate a randomly selected page from the book into Arabic. The centre line shows the pointwise median CDF, and the shaded region indicates pointwise 95% intervals for the CDF.

6 Summary

We have proposed a novel method for eliciting beliefs about a population mean and variance, with the main novelty in how beliefs about a variance are elicited. The method avoids asking the expert to update his or her beliefs given hypothetical data, or to provide summaries from the predictive distribution which would require ‘mentally integrating out’ the uncertain parameters. The elicitation tasks are still likely to be difficult for the expert, but we think some difficulty is unavoidable; considering one’s uncertainty about population variability is always likely to be hard.
In our case study, we were able to obtain a distribution that was satisfactory to the expert, after a process of feedback in which the expert was able to challenge some of the fitted probability judgements, suggesting the expert was able to ‘take ownership’ of the elicited distribution. Nevertheless, the task was not easy for the expert, and several practice elicitations were needed first until the expert was comfortable with the procedure.

Code for the implementing the method is available as part of the R package SHELF, which enables the facilitator to provide feedback to the expert in real time. We will continue to develop this, and welcome suggestions for improvements from users.

A R code

The following commands show the fitting and feedback for the example in Section 4. The SHELF R package is available on CRAN.

```
install.packages("SHELF")
library(SHELF)
# Fit distribution to the expert’s judgements about the population mean
meanfit <- fitdist(vals = c(30, 40), probs = c(0.05, 0.95), lower = 5, upper = 70)
# Plot fitted normal distribution and 1st and 99th quantiles
plotfit(meanfit, ql = 0.01, qu = 0.99, d = "normal")
# Obtain 1st and 99th quantiles from fitted normal distribution
feedback(meanfit, quantiles=c(0.01, 0.99))

# Fit distribution to expert’s judgments about the precision = 1/variance
# (Final judgements used in this example)
prfit <- fitprecision(interval = c(35, 45), propvals = c(0.3, 0.35))
# Report fitted 90% intervals for the population 5th and 95th percentiles
cdffeedback(meanfit, prfit, quantiles = c(0.05, 0.95), alpha = 0.1)
# Plot pointwise 95% intervals for the population CDF
cdfplot(meanfit, prfit)
```
References

Al-Awadhi, S. and Garthwaite, P. (1998). An elicitation method for multivariate normal distributions, *Communications In Statistics-Theory And Methods*, 27 (5): 1123–1142.

Al-Awadhi, S. and Garthwaite, P. (2001). Prior distribution assessment for a multivariate normal distribution: an experimental study, *Journal of Applied Statistics*, 28 (1): 5–23.

Alpert, M. and Raiffa, H. (1982). A progress report on the training of probability assessors, in *Judgment under Uncertainty: Heuristics and Biases*, edited by Kahneman, D., Slovic, P. and Tversky, A., pp. 294–305, Cambridge: Cambridge University Press.

Clough, H. E., Clancy, D. and French, N. P. (2006). Vero-cytotoxigenic escherichia coli O157 in pasteurized milk containers at the point of retail: a qualitative approach to exposure assessment, *Risk analysis*, 26 (5): 1291–1309.

Clough, H. E., Clancy, D. and French, N. P. (2009). Quantifying exposure to vero-cytotoxigenic *Escherichia coli* O157 in milk sold as pasteurized: a model-based approach, *International journal of food microbiology*, 131 (2): 95–105.

Cooke, R. (1991). *Experts in Uncertainty: Opinion and Subjective Probability in Science*, New York: Oxford University Press.

European Food Safety Authority (2010). Guidance on expert knowledge elicitation in food and feed safety risk assessment, *EFSA Journal*, 12 (6).

Garthwaite, P. and O’Hagan, A. (2000). Quantifying expert opinion in the UK water industry: an experimental study, *Journal of the Royal Statistical Society Series D (The Statistician)*, 49 (4): 455–477.

Kadane, J., Dickey, J., Winkler, R., Smith, W. and Peters, S. (1980). Interactive elicitation of opinion for a normal linear model, *Journal of the American statistical association*, 75 (372): 845–854.

Lichtenstein, S., Fischhoff, B. and Phillips, L. D. (1982). Calibration of probabilities: the state of the art to 1980., in *Judgment under Uncertainty: Heuristics and Biases*, edited by Kahneman, D., Slovic, P. and Tversky, A., pp. 306–334, Cambridge: Cambridge University Press.
Morgan, M. G. and Henrion, M. (1990). *Uncertainty: A Guide to Dealing with Uncertainty in Quantitative Risk and Policy Analysis*, Cambridge: Cambridge University Press.

Morris, D., Oakley, J. and Crowe, J. (2014). A web-based tool for eliciting probability distributions from experts, *Environmental Modelling & Software*, **52**: 1–4.

O’Hagan, A., Buck, C. E., Daneshkhah, A., Eiser, J. E., Garthwaite, P. H., Jenkinson, D. J., Oakley, J. E. and Rakow, T. (2006). *Uncertain Judgements: Eliciting Expert Probabilities*, England: John Wiley and Sons Ltd.

O’Hagan, A. and Oakley, J. E. (2010). SHELF: the Sheffield Elicitation Framework (version 2.0), *School of Mathematics and Statistics, University of Sheffield*, [http://tonyohagan.co.uk/shelf](http://tonyohagan.co.uk/shelf).

O’Hagan, A., Stevens, J. W. and Campbell, M. J. (2005). Assurance in clinical trial design, *Pharmaceutical Statistics*, **4**: 187–201.

R Core Team (2016). *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria.

Raiffa, H. (1968). *Decision Analysis: Introductory Lectures on Choice Under Uncertainty*, Reading, Mass: Addison-Wesley.

Ren, S. and Oakley, J. E. (2014). Assurance calculations for planning clinical trials with time-to-event outcomes, *Statistics in Medicine*, **33** (1): 31–45.

Schwarz, N. (1999). Self-reports: How the questions shape the answers., *American Psychologist.*, **54**: 93–105.

Spiegelhalter, D. J., Abrams, K. R. and Myles, J. P. (2004). *Bayesian Approaches to Clinical Trials and Health-Care Evaluation*, Chichester: Wiley.

Spiegelhalter, D. J. and Freedman, L. S. (1986). A predictive approach to selecting the size of a clinical trial based on subjective clinical opinion, *Statistics in Medicine*, **5**: 1–13.

Tversky, A. and Kahneman, D. (1971). Belief in the law of small numbers., *Psychological bulletin*, **76** (2): 105.
Tversky, A. and Kahneman, D. (1974). Judgment under uncertainty - heuristics and biases, *Science, 185* (4157): 1124–1131.

Wu, F.-C. and Tsang, Y.-P. (2004). Second-order monte carlo uncertainty/variability analysis using correlated model parameters: application to salmonid embryo survival risk assessment, *Ecological Modelling, 177* (34): 393 – 414.