Well-posedness and Critical Index Set of the Cauchy Problem for the Coupled KdV-KdV Systems on $\mathbb{T}$

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Abstract

Studied in this paper is the well-posedness of the Cauchy problem for the coupled KdV-KdV systems
\begin{align*}
\begin{aligned}
    u_t + a_1 u_{xxx} &= c_{11} u_x + c_{12} v_x + d_{11} u_x v + d_{12} u_x v, \\
    v_t + a_2 v_{xxx} &= c_{21} u_x + c_{22} v_x + d_{21} u_x v + d_{22} u_x v,
\end{aligned}
\end{align*}

(0.1)

posed on the periodic domain $T$ in the following four spaces
\begin{align*}
\mathcal{H}_k^1 := H_0^2(T) \times H_0^2(T), \quad \mathcal{H}_k^2 := H_0^1(T) \times H^1(T), \quad \mathcal{H}_k^3 := H^1(T) \times H_0^2(T), \quad \mathcal{H}_k^4 := H^1(T) \times H^1(T).
\end{align*}

The coefficients are assumed to satisfy $a_1 a_2 \neq 0$ and $\sum_{i,j} (c_{ij} + d_{ij}) > 0$.

Fix $k \in \{1, 2, 3, 4\}$. Then for any coefficients $a_1, a_2, (c_{ij})$ and $(d_{ij})$, it is shown that there exists a critical index $s^*_k \in (-\infty, +\infty]$ such that the system (0.1) is analytically locally well-posed in $\mathcal{H}_k^0$ if $s > s^*_k$ but weakly analytically ill-posed if $s < s^*_k$. Viewing $s^*_k$ as a function of the coefficients, its range $C_k$ is defined to be the critical index set for the analytical well-posedness of (0.1) in $\mathcal{H}^0_k$.

By investigating some properties of the irrationality exponents of the real numbers and by establishing some sharp bilinear estimates in non-divergence form, we manage to identify $C_1 = \left\{ -\frac{1}{2}, \infty \right\} \cup \left[ \frac{3}{2}, 1 \right]$ and $C_q = \left\{ -\frac{1}{2}, -\frac{1}{4}, \infty \right\} \cup \left[ \frac{3}{2}, 1 \right]$ for $q = 2, 3, 4$. In particular, these sets contain an open interval $(\frac{3}{2}, 1)$. This is in sharp contrast to the $\mathbb{R}$ case in which the critical index set $C$ for the analytical well-posedness of (0.1) in the space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ consists of exactly four numbers: $C = \left\{ -\frac{13}{22}, -\frac{3}{4}, 0, \frac{3}{4} \right\}$.

1 Introduction

1.1 cKdV systems and well-posedness

This paper studies the Cauchy problem for the coupled KdV-KdV (cKdV) systems posed on the periodic domain $T = \mathbb{R}/(2\pi \mathbb{Z})$ in the following general form:
\begin{align*}
\begin{aligned}
    \left( \begin{array}{c}
        u_t \\
        v_t
    \end{array} \right) + A_1 \left( \begin{array}{c}
        u_{xxx} \\
        v_{xxx}
    \end{array} \right) &= A_2 \left( \begin{array}{c}
        u_x \\
        v_x
    \end{array} \right) + A_3 \left( \begin{array}{c}
        u_x v \\
        v_x v
    \end{array} \right), \\
    \left( \begin{array}{c}
        u \\
        v
    \end{array} \right) \bigg|_{t=0} &= \left( \begin{array}{c}
        u_0 \\
        v_0
    \end{array} \right),
\end{aligned}
\end{align*}

(1.1)

where $\{A_i\}_{1 \leq i \leq 3}$ are $2 \times 2$ real constant matrices, $u = u(x, t)$ and $v = v(x, t)$ are real-valued functions of the variables $x \in \mathbb{T}$ and $t \in \mathbb{R}$. It is assumed that there exists an invertible real matrix $M$ such that $A_1 = M \left( \begin{array}{cc}
    a_1 & 0 \\
    0 & a_2
\end{array} \right) M^{-1}$ with $a_1 a_2 \neq 0$. By regarding $M^{-1} \left( \begin{array}{c}
    u \\
    v
\end{array} \right)$ as the new unknown functions

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which is called in divergence form or equivalently,

\[
\begin{pmatrix}
  u_t \\
  v_t \\
\end{pmatrix} + \begin{pmatrix}
  a_1 & 0 \\
  0 & a_2 \\
\end{pmatrix} \begin{pmatrix}
  u_{xx} \\
  v_{xx} \\
\end{pmatrix} = C \begin{pmatrix}
  uu_x \\
  vv_x \\
\end{pmatrix} + D \begin{pmatrix}
  u_xv \\
  uv_x \\
\end{pmatrix},
\]

or equivalently,

\[
\begin{align*}
  u_t + a_1 u_{xx} &= c_{11} uu_x + c_{12} vv_x + d_{11} u_x v + d_{12} u v_x, \\
  v_t + a_2 v_{xx} &= c_{21} uu_x + c_{22} vv_x + d_{21} u_x v + d_{22} u v_x, \\
  (u, v)|_{t=0} &= (u_0, v_0),
\end{align*}
\]

which is called in divergence form if \(d_{11} = d_{12}\) and \(d_{21} = d_{22}\), and in non-divergence form otherwise. Listed below are a few specializations of (1.1) appeared in the literature.

- **Majda-Biello system:**
  \[
  \begin{align*}
  u_t + u_{xx} &= -vv_x, \\
  v_t + a_2 v_{xx} &= -(uv)_x, \\
  (u, v)|_{t=0} &= (u_0, v_0),
  \end{align*}
  \]
  where \(a_2 \neq 0\). This system was proposed by Majda and Biello in [36].

- **Hirota-Satsuma system:**
  \[
  \begin{align*}
  u_t + a_1 u_{xx} &= -6a_1 uu_x + c_{12} vv_x, \\
  v_t + v_{xx} &= -3uv_x, \\
  (u, v)|_{t=0} &= (u_0, v_0),
  \end{align*}
  \]
  where \(a_1 \neq 0\). This system is in non-divergence form and was proposed by Hirota-Satsuma in [18].

- **Gear-Grimshaw system:**
  \[
  \begin{align*}
  u_t + u_{xx} + \sigma_3 v_{xxx} &= -uu_x + \sigma_1 vv_x + \sigma_2 (uv)_x, \\
  \rho_1 v_t + \rho_2 \sigma_3 u_{xxx} + v_{xxx} + \sigma_4 v_x &= \rho_2 \sigma_2 uu_x - vv_x + \rho_2 \sigma_1 (uv)_x, \\
  (u, v)|_{t=0} &= (u_0, v_0),
  \end{align*}
  \]
  where \(\sigma_i \in \mathbb{R}(1 \leq i \leq 4)\) and \(\rho_1, \rho_2 > 0\). When \(\sigma_4 = 0\), this system is a special case of (1.1) with
  \[
  A_1 = \begin{pmatrix}
  1 & \sigma_3 \\
  \rho_2 \sigma_2 & \frac{1}{\rho_1}
\end{pmatrix}.
  \]

Note that \(A_1\) in (1.7) is diagonalizable over \(\mathbb{R}\) for any \(\sigma_3 \in \mathbb{R}\) and \(\rho_1, \rho_2 > 0\). Moreover, the eigenvalues of \(A_1\) are nonzero unless \(\rho_2 \sigma_2^2 = 1\). So (1.6) can be reduced to the form (1.3) as long as \(\rho_2 \sigma_2^2 \neq 1\). This system was derived by Gear-Grimshaw in [16] (also see [6] for the physical explanation).

We are mainly concerned with the well-posedness of the Cauchy problem for the system (1.3) in some scaled Banach spaces \(X^s(\mathbb{T})\).

**Definition 1.1.** For any integer \(l \geq 0\), the Cauchy problem of the system (1.3) is said to be \(C^l\)-locally well-posed (LWP) in the space \(X^s(\mathbb{T})\) if for any \(\delta > 0\), there exists \(T > 0\) such that

(a) for any \((u_0, v_0) \in X^s(\mathbb{T})\) with \(\| (u_0, v_0) \|_{X^s(\mathbb{T})} \leq \delta\), (1.3) admits a unique solution \((u, v)\) in the space \(C([0, T]; X^s(\mathbb{T}))\) satisfying the auxiliary condition \(w \in Y^T_\delta\), where \(Y^T_\delta\) is an auxiliary metric space;
(b) the corresponding solution map \( K \) from the initial data \((u_0, v_0)\) to the solution \((u, v)\): \( K(u_0, v_0) = (u, v) \), is \( C^l \) smooth from \( \{ (u_0, v_0) : \| (u_0, v_0) \|_{X^s(T)} \leq \delta \} \) to \( C([0, T]; X^s(T)) \cap Y^T_s \).

Similarly, if the solution map \( K \) is real analytic (or uniformly continuous), then the system (1.3) is said to be analytically (or uniformly continuously) LWP in \( X^s(T) \).

In the literature, the well-posedness in Definition 1.1 is called conditional (cf. [8, 20] and the references therein). If the uniqueness holds in the space \( C \), the solution map \( K \) is said to be analytically (or uniformly continuously) IP in \( X^s(T) \). Similarly, if there does not exist a real analytic (or uniformly continuous) solution map, then (1.3) is said to be analytically (or uniformly continuously) IP in \( X^s(T) \).

Definition 1.2. For any integer \( l \geq 0 \), the Cauchy problem of the systems (1.3) is said to be \( C^l \) ill-posed (IP) in the space \( X^s(T) \) if there exists \( \delta > 0 \) such that for any \( T > 0 \), there does not exist a \( C^l \) smooth solution map \( K \) as defined in Definition 1.1. Similarly, if there does not exist a real analytic (or uniformly continuous) solution map, then (1.3) is said to be analytically (or uniformly continuously) IP in \( X^s(T) \).

In the rest of the paper, for the convenience of notations, the analytically LWP, GWP and IP will be written as A-LWP, A-GWP and A-IP respectively. Similarly, the uniformly continuously LWP, GWP and IP will be written as U-LWP, U-GWP and U-IP respectively.

1.2 Literature review

For the single KdV equation

\[
 u_t + u_{xxx} + uu_x = 0, \quad u(x, 0) = u_0(x)
\]

posed on either \( \mathbb{R} \) or \( \mathbb{T} \), the study of the its well-posedness began in the late 1960s with the work of Sjöberg [14, 45] and has culminated in the work of Killip and Visan [33]. Many significant breakthroughs were made in this study of more than fifty years (see e.g. Bona-Smith [7], Kato [22, 24], Constantin-Saut [15], Kenig-Ponce-Vega [27, 31], Bourgain [9], Christ-Colliander-Tao [12], Colliander-Keeel-Staffilani-Takaoka-Tao [13, 14], Kappeler-Topalov [21], Molinet [37, 38], Tao [46], Killip-Visan [33], to name a few).

Define \( H^s(\mathbb{R}) \) and \( H^s(\mathbb{T}) \) as the standard Sobolev spaces on \( \mathbb{R} \) and \( \mathbb{T} \) respectively. In addition, denote \( H^s_0(\mathbb{T}) \) to be the collection of the functions in \( H^s(\mathbb{T}) \) whose mean value is 0:

\[
 H^s_0(\mathbb{T}) = \left\{ f \in H^s(\mathbb{T}) : \int_0^{2\pi} f(x) \, dx = 0 \right\}.
\]

The following theorem summarizes the known results on the well-posedness of (1.8).

Theorem 1.3 ([7, 12, 13, 17, 21, 31, 33, 34, 37, 38]). The Cauchy problem (1.8) is

1. \( C^0 \)-GWP in both spaces \( H^s(\mathbb{R}) \) and \( H^s(\mathbb{T}) \) for any \( s \geq -1 \), and \( C^0 \)-IP for any \( s < -1 \);
2. A-GWP in the space \( H^s(\mathbb{R}) \) for any \( s \geq -\frac{3}{4} \), and U-IP for any \( s < -\frac{3}{4} \);
3. A-GWP in the space \( H^s_0(\mathbb{T}) \) for any \( s \geq -\frac{1}{2} \), and U-IP for any \( s < -\frac{1}{2} \);
4. U-IP in the space \( H^s(\mathbb{T}) \) for any \( s \in \mathbb{R} \).

For the cKdV systems (1.3) posed on \( \mathbb{R} \), it has also been very well studied (see e.g. [1, 6, 40, 49] and the references therein). Of particular interest is a special class of (1.3) in the following form

\[
\begin{align*}
  u_t + a_1 u_{xxx} &= d_1(uv)_x, \\
  v_t + a_2 v_{xxx} &= d_2(uv)_x, \\
  (u,v)|_{t=0} &= (u_0,v_0).
\end{align*}
\]
It was shown in [49] that it is A-LWP in the space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s \geq -\frac{13}{12}$ if $a_1 a_2 < 0$. This is rather surprising since even the single KdV equation (1.8) is $C^0$-IP in $H^s(\mathbb{R})$ for any $s \leq -1$ (see [37]). Similar phenomenon was also found in the coupled mKdV systems [11].

For the cKdV (1.3) posed on $\mathbb{T}$ in divergence form, its A-LWP has been thoroughly investigated by Oh [40]. As a specific application, [40] obtained sharp A-LWP results for the Majda-Biello system (see [37]). Similar phenomenon was also found in the coupled mKdV systems [11].

On the other hand, (1.4) is more sense to adjust the spaces $H^s(\mathbb{R})$, where $s \geq \frac{3}{2}$ is some constant only depending on $a_2$. On the other hand, (1.4) is $C^0$-IP if $s < \min\{1, \hat{s}(a_2)\}$. So the regularity threshold for the A-LWP of (1.4) in the space $H^s(\mathbb{R}) \times H^s(\mathbb{T})$ is $\min\{1, \hat{s}(a_2)\}$.

By contrast, the study for the cKdV (1.3) posed on $\mathbb{T}$ in non-divergence form is far from complete. Angulo [2, 3] studied the Hirota-Satsuma system (1.5) and obtained the following results. When $a_1 \neq 0, 1$, (1.5) is A-LWP in $H^1_0(\mathbb{T}) \times H^s_0(\mathbb{T})$ for any $s \geq 1$ and when $a_1 = 1$, (1.5) is A-LWP in $L^2_0(\mathbb{T}) \times H^1_0(\mathbb{T})$. Comparing to the known results on the cKdV systems in divergence form, there still exist a lot of room to explore and to optimize the regularity thresholds for the non-divergence cases.

### 1.3 Bilinear estimates

In Theorem 1.3, the optimal regularity index -1 for either the $\mathbb{R}$ case or the $\mathbb{T}$ case was obtained by the inverse scattering method [21, 33]. This method relies on the complete integrability of the single KdV equation and usually leads to $C^0$ well-posedness. Another method, called the bilinear estimate approach, does not require the complete integrability property and usually gives the sharp analytical well-posedness. For example, by using this method, [31] discovered the best regularity index for the A-LWP of (1.8) in $\mathbb{R}$ case (and the $\mathbb{T}$ case resp.) to be $-\frac{3}{4}$ (and $-\frac{1}{2}$ resp.).

Since most cKdV systems do not possess the complete integrability property, the bilinear estimate approach seems to be the most powerful tool to use. This method is based on the Fourier restriction spaces $X^s_{\alpha,b}$ which was first introduced by Bourgain in [9].

**Definition 1.4 ([9, 31]).** Let $G = \mathbb{R}$ (or $\mathbb{T}$) and denote $G^* = \mathbb{R}$ (or $\mathbb{Z}$) to be the dual group of $G$. For any $\alpha, s, b \in \mathbb{R}$ with $\alpha \neq 0$, the Fourier restriction space $X^s_{\alpha,b}(G \times \mathbb{R})$ is defined to be the completion of the Schwartz space $\mathcal{S}(G \times \mathbb{R})$ with respect to the norm

$$
\|w\|_{X^s_{\alpha,b}(G \times \mathbb{R})} := \|\langle k \rangle^s \langle \tau - \alpha k \rangle^b \hat{w}(k, \tau)\|_{L^2(G^* \times \mathbb{R})},
$$

where $\langle \cdot \rangle := 1 + |\cdot|$ and $\hat{w}$ refers to the space-time Fourier transform of $w$.

Then the following is the so-called bilinear estimate for the KdV equation (1.8).

$$
\|\partial_x(uv)\|_{X^s_{\alpha,b-1}} \lesssim \|u\|_{X^s_{\alpha,b}} \|v\|_{X^s_{\alpha,b}}, \quad \forall u, v \in X^1_{\alpha,b},
$$

where in the $\mathbb{T}$ case, $u$ and $v$ are also required to have zero means, i.e. $u(\cdot, t), v(\cdot, t) \in H^s_0(\mathbb{T})$ for any $t \in \mathbb{R}$.

Bourgain [9] first proved (1.11) for $s = 0$ and $b = \frac{1}{2}$ in both $\mathbb{R}$ and $\mathbb{T}$ cases, which lead to the A-LWP of (1.8) in $H^s(\mathbb{R})$ or in $H^s_0(\mathbb{T})$ for any $s \geq 0$. Later, Kenig, Ponce and Vega [31] optimized this estimate.

**Lemma 1.5 ([31]).**

- In the $\mathbb{R}$ case, (1.11) holds for any $s > -\frac{3}{4}$ with some $b = b(s) > \frac{1}{2}$, but fails for any $s < -\frac{3}{4}$ and $b \in \mathbb{R}$.

- In the $\mathbb{T}$ case, (1.11) holds for any $s \geq -\frac{1}{2}$ with $b = \frac{1}{2}$, but fails if $s < -\frac{1}{2}$ or $b \neq \frac{1}{2}$.

*Actually, the mean-zero condition on $v$ is not needed, although it is needed on $u$. In fact, the mean-zero condition on $v$ is not applicable since the Hirota-Satsuma system (1.3) does not preserve the mean value of $v$. As a result, it makes more sense to adjust the spaces $H^s_0(\mathbb{T}) \times H^s_0(\mathbb{T})$ and $L^2_0(\mathbb{T}) \times H^1_0(\mathbb{T})$ in [2, 3] to be $H^s_0(\mathbb{T}) \times H^s(\mathbb{T})$ and $L^2_0(\mathbb{T}) \times H^1(\mathbb{T})$ respectively.*
For the cKdV systems \[(1.3)\], the situations are more complicated since four types of bilinear estimates, \[(1.12)-(1.15)\], need to be studied.

Divergence forms:

\[
\begin{align*}
\text{(D1):} & \quad \|\partial_x(w_1w_2)\|_{X^\alpha_{s,b,-1}} \lesssim \|w_1\|_{X^\alpha_{s,b}} \|w_2\|_{X^\alpha_{s,b}}, \\
\text{(D2):} & \quad \|\partial_x(w_1w_2)\|_{X^\alpha_{s,b,-1}} \lesssim \|w_1\|_{X^\alpha_{s,b}} \|w_2\|_{X^\alpha_{s,b}}.
\end{align*}
\]

Non-divergence forms:

\[
\begin{align*}
\text{(ND1):} & \quad \|\partial_x(w_1w_2)\|_{X^\alpha_{s,b,-1}} \lesssim \|w_1\|_{X^\alpha_{s,b}} \|w_2\|_{X^\alpha_{s,b}}, \\
\text{(ND2):} & \quad \|w_1\|_{X^\alpha_{s,b,-1}} \lesssim \|w_1\|_{X^\alpha_{s,b}} \|w_2\|_{X^\alpha_{s,b}}.
\end{align*}
\]

where \(w_1\) (or \(w_2\)) refers to \(u\) or \(v\), and \(a_1\) (or \(a_2\)) represents \(a_1\) or \(a_2\) in \[(1.3)\].

There are various ways to write out these estimates, what we adopted here is to fix the first term on the right hand side to be \(\|w_1\|_{X^\alpha_{s,b}}\).

In the \(R\) case, all the sharp regularity indices have been found for the bilinear estimates \[(1.12)-(1.15)\].

**Lemma 1.6 \[(1.16)\]**: Let \(\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}\) and denote \(r = \frac{\alpha_2}{\alpha_1}\). Then for any bilinear estimate among \[(1.12)-(1.15)\], there exists a critical index \(s^*\), as shown in Table 1, such that this bilinear estimate holds for any \(s > s^*\) with some \(b = b(s) > \frac{1}{2}\), but fails for any \(s < s^*\) and \(b \in \mathbb{R}\).

| \(s^*\) | \(r < 0\) | \(0 < r < \frac{1}{4}\) | \(r = \frac{1}{4}\) | \(r > \frac{1}{4}, r \neq 1\) | \(r = 1\) |
|---|---|---|---|---|---|
| (D1) | \(-\frac{3}{4}\) | \(-\frac{3}{4}\) | \(\frac{3}{4}\) | \(0\) | \(-\frac{3}{4}\) |
| (D2) | \(-\frac{3}{4} \frac{1}{2}\) | \(-\frac{3}{4}\) | \(\frac{3}{4}\) | \(0\) | \(-\frac{3}{4}\) |
| (ND1) | \(-\frac{3}{4}\) | \(-\frac{3}{4}\) | \(\frac{3}{4}\) | \(0\) | \(0\) |
| (ND2) | \(-\frac{3}{4}\) | \(-\frac{3}{4}\) | \(\frac{3}{4}\) | \(0\) | \(0\) |

Table 1: Sharp bilinear estimates on \(R\) \((r = \frac{\alpha_2}{\alpha_1})\)

From the above table, we can see there are four critical indices \(\left\{-\frac{13}{12}, -\frac{3}{4}, 0, \frac{3}{4}\right\}\) for the bilinear estimates associated to the cKdV systems posed on \(R\).

In the \(T\) case, Oh \[40\] studied the Majda-Biello system \[36\] whose associated bilinear estimates are \[(1.16)\] and \[(1.17)\].

\[
\begin{align*}
\|\partial_x(v^2)\|_{X^\alpha_{s,b,-1}} & \lesssim \|v\|_{X^\alpha_{s,b}} \|v\|_{X^\alpha_{s,b}}, \\
\|\partial_x(vu)\|_{X^\alpha_{s,b,-1}} & \lesssim \|v\|_{X^\alpha_{s,b}} \|u\|_{X^\alpha_{s,b}}.
\end{align*}
\]

Note that the above two estimates are special (essentially equivalent) cases of \[(1.12)\] and \[(1.13)\]. In fact,

- \[(1.16)\] corresponds to \[(1.12)\] by setting \(w_1 = w_2 = v, \alpha_1 = a_2\) and \(\alpha_2 = 1\).
- \[(1.17)\] corresponds to \[(1.13)\] by setting \(w_1 = v, w_2 = u, \alpha_1 = a_2\), and \(\alpha_2 = 1\).

\[1\] Strictly speaking, in the \(T\) case, the above bilinear estimates need to be considered in more complicated forms and some zero-mean conditions should be imposed on \(w_1\) and \(w_2\). But in order to illustrate the idea more clearly, we drop these technical details and just take the forms which are consistent with the \(R\) case. For precise bilinear estimates in the \(T\) case, please see Corollary \[1.16\] Theorem \[1.18\] and Section \[5\].
In both cases, the ratio between \(\alpha_2\) and \(\alpha_1\) is \(r = \frac{1}{a_2}\). The most interesting case is when \(a_2 \in (0, 4) \setminus \{1\}\), i.e. when \(r \in \left(\frac{1}{4}, \infty\right) \setminus \{1\}\). In this case, the key technical issue is to know how close the rational numbers can approximate a given real number. Oh [40] adopted the so-called minimal type index \(\nu(\rho)\) for any real number \(\rho\) (see Definition 1.14) to capture this approximation. Let \(c_1, c_2\) be the roots of the quadratic equation \(3a_2x^2 - 3a_2x + a_2 - 1 = 0\) and let \(d_1, d_2\) be the roots of the quadratic equation \((1-a_2)x^2 + 3a_2x - 3a_2 = 0\). More specifically, let

\[
\begin{align*}
c_1 &= \frac{1}{2} - \frac{1}{6} \sqrt{\frac{12}{a_2} - 3}, & c_2 &= \frac{1}{2} + \frac{1}{6} \sqrt{\frac{12}{a_2} - 3}, \\
d_1 &= \frac{-3a_2 - \sqrt{3a_2(4-a_2)}}{2(1-a_2)}, & d_2 &= \frac{-3a_2 + \sqrt{3a_2(4-a_2)}}{2(1-a_2)}.
\end{align*}
\]

(1.18)

Denote

\[
\nu_c = \max\{\nu(c_1), \nu(c_2)\}, \quad \nu_d = \max\{\nu(d_1), \nu(d_2)\}.
\]

(1.19)

**Lemma 1.7** [40]. Let \(a_2 \in (0, 4) \setminus \{1\}\). Define \(\nu_c\) and \(\nu_d\) as in (1.19). Then for the bilinear estimate (1.16) or (1.17), there exists a critical index \(s^*\), as shown in Table 2, such that this bilinear estimate holds for any \(s > s^*\) with \(b = \frac{1}{2}\), but fails for any \(s < s^*\) and \(b \in \mathbb{R}\).

| \(s^*\) | \(a_2 \in (0, 4) \setminus \{1\}\) |
|---|---|
| [1.16] | \(\min\{1, \frac{1}{2} + \frac{1}{2}\nu_c\}\) |
| [1.17] | \(\min\{1, \frac{1}{2} + \frac{1}{2}\nu_d\}\) |

Table 2: Sharp bilinear estimates on \(\mathbb{T}\) by Oh [40]

Based on this discovery, Oh further concludes the sharp A-LWP for the Majda-Biello system (1.4).

**Theorem 1.8** [40]. Let \(a_2 \in (0, 4) \setminus \{1\}\). Define

\[
s_1^* = \min\left\{1, \frac{1}{2} + \frac{1}{2}\max\{\nu_c, \nu_d\}\right\}.
\]

Then (1.4) is A-LWP for any \(s > s_1^*\), but \(C^3\text{-IP}\) for any \(s < s_1^*\), in the space \(H_0^s(\mathbb{T}) \times H^s(\mathbb{T})\).

### 1.4 Weakly analytical ill-posedness

For the single KdV equation (1.8), it follows from Theorem 1.3 and Lemma 1.5 that the critical indices for the bilinear estimate (1.11), match those for the A-LWP of (1.8) in both \(\mathbb{R}\) and \(\mathbb{T}\) cases. For Majda-Biello system (1.4), we can also see from Lemma 1.7 and Theorem 1.8 that (1.4) is A-LWP if and only if both the corresponding bilinear estimates (1.16) and (1.17) hold.

Inspired by these observations, it is conjectured that for any cKdV system (1.3), its regularity threshold for the A-LWP is equivalent to that for the associating bilinear estimates. In fact, by the standard argument in [9,13,31], once the associated bilinear estimates are justified for some index \(s\), then the A-LWP can also be established for the same \(s\). In other words, if a cKdV system is A-IP, then at least one of the corresponding bilinear estimates must fail. But whether the failure of the bilinear estimates implies the A-IP is not known in general. This motivates the following definition.

**Definition 1.9.** The Cauchy problem of the single KdV equation (1.8) is said to be weakly A-IP if the bilinear estimate (1.11) fails. Similarly, the cKdV systems (1.3) are said to be weakly A-IP if at least one of the corresponding bilinear estimates fails.
Based on the above definition, the aforementioned conjecture is translated to the equivalence between A-IP and weakly A-IP.

**Conjecture 1.10.** For any cKdV system (1.3), it is A-IP if and only if it is weakly A-IP.

Since A-IP always implies weakly A-IP, the conjecture further reduces to “weakly A-IP implies A-IP”. For the single KdV and the Majda-Biello system, this conjecture has been confirmed. But for some special cKdV, say (1.9) posed on $\mathbb{R}$, although the critical index for the corresponding bilinear estimate has been found to be $-\frac{13}{12}$ it is still unknown if it is A-IP in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s < -\frac{13}{12}$.

1.5 Critical index set

Consider the cKdV systems (1.3) posed on $\mathbb{R}$. Denote $H^s(\mathbb{R}) = H^s(\mathbb{R}) \times H^s(\mathbb{R})$.

**Definition 1.11.** For any cKdV system (1.3) posed on $\mathbb{R}$ with $a_1a_2 \neq 0$ and either $C = (c_{ij})$ or $D = (d_{ij})$ does not vanish.

- If there exists $s^* \in \mathbb{R}$, depending on $a_1$, $a_2$, $C$ and $D$, such that (1.3) is A-LWP in the space $H^s(\mathbb{R})$ for any $s > s^*$, but (weakly) A-IP for any $s < s^*$ in $H^s(\mathbb{R})$, then this $s^*$ is called the (weakly) analytically critical index. If there does not exist such a critical index, then we define $s^* = \infty$.

- Regarding the above critical index $s^*$ as a map: $s^* = G(a_1, a_2, C, D)$, then the range of this map, denoted as $C$, is called the (weakly) analytically critical index set of (1.3) in $H^s(\mathbb{R})$.

In the following, we will write "analytically critical index" to be "A-critical index". Based on Definition 1.11, it follows from Lemma 1.6 that the weakly A-critical index set of (1.3) in $H^s(\mathbb{R})$ is

$$C = \{-\frac{13}{12}, -\frac{3}{4}, 0, \frac{3}{4}\}.$$  

Now we consider the following (1.20) which is a variant of the single KdV equation.

$$u_t + au_{xxx} = cuu_x, \quad u(x, 0) = u_0(x), \quad ac \neq 0. \quad (1.20)$$

Similar to Definition 1.11 we can define the A-critical index set of (1.20) posed on $\mathbb{R}$ or $\mathbb{T}$. For any $l \geq 0$, we can also define the $C^l$-critical index set analogously. By some simple scaling computation, one can see that the values of $a$ and $c$ in (1.20) do not affect its well-posedness. So it follows from Theorem 1.3 that

- The $C^0$-critical index set of (1.20) in $H^s(\mathbb{R})$ or $H^s(\mathbb{T})$ is $\{-1\}$.

- The A-critical index set of (1.20) is $\{-\frac{3}{4}\}$ in the space $H^s(\mathbb{R})$, $\{-\frac{1}{2}\}$ in the space $H_0^s(\mathbb{T})$, and $\{\infty\}$ in the space $H^s(\mathbb{T})$.

Two observations can be drawn from the above results. Firstly, unlike (1.20), the coefficients of the cKdV systems (1.3) do have an effect on the A-critical index. Secondly, if the problem is posed on $\mathbb{T}$, then the critical index can be different if the underlying spaces are different (e.g. $H_0^s(\mathbb{T})$ cf. $H^s(\mathbb{T})$).

The main purpose of this paper is to investigate the weakly A-critical index set of the cKdV systems (1.3) posed on $\mathbb{T}$. We will consider the following four spaces:

$$\mathcal{H}_1^s := H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}), \quad \mathcal{H}_2^s := H_0^s(\mathbb{T}) \times H^s(\mathbb{T}), \quad \mathcal{H}_3^s := H^s(\mathbb{T}) \times H_0^s(\mathbb{T}), \quad \mathcal{H}_4^s := H^s(\mathbb{T}) \times H^s(\mathbb{T}). \quad (1.21)$$

**Definition 1.12.** Let $k \in \{1, 2, 3, 4\}$. We define $C_k$ to be the weakly A-critical index set of the cKdV systems (1.3) in the space $\mathcal{H}_k^s$.  

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From the results in Oh [40] on the Majda-Biello system (1.4), the weakly A-critical index \( s^*(a_2) \) in the space \( H_2^s \) is
\[
s^*(a_2) = \begin{cases} 
-\frac{1}{2} & \text{if } a_2 \in (-\infty, 0) \cup (4, \infty), \\
\infty & \text{if } a_2 = 1, \\
\min\{1, \tilde{s}(a_2)\} & \text{if } a_2 \in (0, 4) \setminus \{1\}.
\end{cases}
\]
where \( \tilde{s}(a_2) \geq \frac{1}{2} \) is some constant only depending on \( a_2 \). So the weakly A-critical index set for (1.4) in the space \( H_2^s \) is
\[
\{-\frac{1}{2}, \infty\} \cup \left\{ \min\{1, \tilde{s}(a_2)\} : a_2 \in (0, 4) \setminus \{1\} \right\}.
\]
But what is the precise range of \( \tilde{s}(a_2) \) over the region \((0, 4) \setminus \{1\}\)? This question was not answered in [40], we will find an answer in this paper.

### 1.6 Irrationality exponent

In order to find out the critical index set for (1.3) posed on \( T \), as we mentioned before, it is crucial to estimate how well the real number can be approximated by rational functions. In the number theory, such estimate is called the Diophantine approximation. One standard characterization is via the irrationality exponent.

**Definition 1.13** (see e.g. [5]). A real number \( \rho \) is said to be approximable with power \( \mu \) if the inequality
\[
0 < \left| \rho - \frac{m}{n} \right| < \frac{1}{|n|^\mu}
\]
holds for infinitely many \((m, n) \in \mathbb{Z} \times \mathbb{Z}^*\), and
\[
\mu(\rho) := \sup\{\mu \in \mathbb{R} : \rho \text{ is approximable with power } \mu\}
\]
is called the irrationality exponent of \( \rho \).

For any rational number \( \rho \), \( \mu(\rho) = 1 \). For any irrational number \( \rho \), \( \mu(\rho) \geq 2 \), but exact value of \( \mu(\rho) \) is difficult to find in general. The basic properties of \( \mu(\rho) \) are collected in Proposition 3.1.

Recalling the minimal type index \( \nu(\rho) \) in Oh’s paper [40], it is closely related to \( \mu(\rho) \) but is defined slightly different.

**Definition 1.14** (see e.g. [4, 40]). A real number \( \rho \) is said to be of type \( \nu \) if there exists a positive constant \( K = K(\rho, \nu) \) such that the inequality
\[
\left| \rho - \frac{m}{n} \right| \geq \frac{K}{|n|^{2+\nu}}
\]
holds for any \((m, n) \in \mathbb{Z} \times \mathbb{Z}^*\), and
\[
\nu(\rho) := \inf\{\nu \in \mathbb{R} : \rho \text{ is of type } \nu\}
\]
is called the minimal type index of \( \rho \), where the infimum is understood as \( \infty \) if the set \( \{\nu \in \mathbb{R} : \rho \text{ is of type } \nu\} \) is empty.

If \( \rho \in \mathbb{Q} \), then it is easy to see \( \nu(\rho) = \infty \) and \( \mu(\rho) = 1 \). If \( \rho \in \mathbb{R} \setminus \mathbb{Q} \), then it will be shown in Proposition 3.5 that \( \nu(\rho) = \mu(\rho) - 2 \).

\[\text{‡}\] The interested readers are referred to S. Lang [35] and Y. Bugeaud [10] for a nice introduction to this subject.
1.7 Main results

Recall the irrationality exponent $\mu(\cdot)$ in Definition 1.13. For $r \geq \frac{1}{4}$, define

$$\sigma_r = \mu(\sqrt{12r} - 3) \quad \text{and} \quad s_r = \begin{cases} 1 & \text{if } \sigma_r = 1 \text{ or } \sigma_r \geq 3, \\ \frac{\sigma_r - 1}{2} & \text{if } 2 \leq \sigma_r < 3. \end{cases} \quad (1.26)$$

The basic properties of $s_r$ are collected in Proposition 3.6.

For the Majda-Biello system (1.4) with $a_2 \in (0, 4) \setminus \{1\}$, Oh [40] showed the critical regularity indexes for the associated bilinear estimates to be $\min \left\{ 1, \frac{1}{2} + \frac{1}{2} \nu_c \right\}$ and $\min \left\{ 1, \frac{1}{2} + \frac{1}{2} \nu_d \right\}$, see Table 2. The first main result of this paper is to demonstrate these two indexes are actually the same.

**Theorem 1.15.** Let $a_2 \in (0, 4) \setminus \{1\}$ and define $\nu_c$ and $\nu_d$ as in (1.19). Then $\nu_c = \nu_d$ and

$$\min \left\{ 1, \frac{1}{2} + \frac{1}{2} \nu_r \right\} = s_{\frac{1}{4}},$$

where $s_{\frac{1}{4}}$ is defined as in (1.26).

The key observation in the proof of this theorem is the invariance of the irrationality exponent under the reciprocal operation, that is $\mu(\rho) = \mu(1/\rho)$, see Proposition 3.4.

Combining Theorem 1.15 with Lemma 1.7 (also see Oh [40]), we are able to write out the sharp regularity indexes for the bilinear estimates (1.12) and (1.13) in a more unified way.

**Corollary 1.16.** Let $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ and denote $r = \frac{\alpha_2}{\alpha_1}$. Define $s_r$ as in (1.26) for $r \geq \frac{1}{4}$. Then for the bilinear estimate (1.12) or (1.13), there exists a critical index $s^*$, as shown in Table 3, such that this bilinear estimate holds for any $s > s^*$ with $b = \frac{1}{4}$, but fails for any $s < s^*$ and $b \in \mathbb{R}$.

| $s^*$ | $r < \frac{1}{4}, r \neq 0$ | $r \geq \frac{1}{4}, r \neq 1$ | $r = 1 (\hat{w}_1(0, \cdot) = \hat{w}_2(0, \cdot) = 0)$ |
|---|---|---|---|
| (D1) with $\hat{w}_2(0, \cdot) = 0$ | $-\frac{1}{2}$ | $\min\{1, s_r\}$ | $-\frac{1}{2}$ |
| (D2) | $-\frac{1}{2}$ | $\min\{1, s_r\}$ | $-\frac{1}{2}$ |

Table 3: Sharp bilinear estimates on $\mathbb{T}$ in divergence form ($r = \frac{\alpha_2}{\alpha_1}$)

Let $\mathcal{U}$ be the range of $s_r$: $\mathcal{U} = \left\{ s_r : \frac{1}{4} \leq r < \infty \right\}$. Then it will be shown in Proposition 3.6 that $\mathcal{U} = \left[ \frac{1}{2}, 1 \right]$. As a result, we obtain the following conclusion (Recall that $H_{a_1}^s := H_{a_1}^0(\mathbb{T}) \times H^s(\mathbb{T})$ in (1.21)).

**Corollary 1.17.** For the Majda-Biello systems (1.4) posed on $\mathbb{T}$ with $a_2 \neq 0$, its weakly $A$-critical index set in the space $H_{a_1}^s$ is $\left\{ -\frac{1}{2}, \infty \right\} \cup \left[ \frac{1}{2}, 1 \right]$.

Next, we study the cKdV systems (1.3) in non-divergence form. The main result is the following sharp bilinear estimates.

**Theorem 1.18.** Let $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ and denote $r = \frac{\alpha_2}{\alpha_1}$. Define $s_r$ as in (1.26) for $r \geq \frac{1}{4}$. Then for the bilinear estimate (1.14) or (1.16), there exists a critical index $s^*$, as shown in Table 4, such that this bilinear estimate holds for any $s > s^*$ with $b = \frac{1}{4}$, but fails for any $s < s^*$ and $b \in \mathbb{R}$.

| $s^*$ | $r < \frac{1}{4}, r \neq 0$ | $r \geq \frac{1}{4}, r \neq 1$ | $r = 1$ |
|---|---|---|---|
| (ND1) ($\hat{w}_2(0, \cdot) = 0$) | $-\frac{1}{4}$ | $\min\{1, s_r\}$ | $\frac{1}{2}$ |
| (ND2) | $-\frac{1}{4}$ | $\min\{1, s_r\}$ | $\frac{1}{2}$ ($\hat{w}_1(0, \cdot) = 0$) |

Table 4: Sharp bilinear estimates on $\mathbb{T}$ in non-divergence form ($r = \frac{\alpha_2}{\alpha_1}$)
As a corollary, consider the Hirota-Satsuma systems (1.5).

**Corollary 1.19.** For the Hirota-Satsuma systems (1.5) posed on \( T \) with \( a_1 \neq 0 \) and \( c_{12} \in \mathbb{R} \), its weakly A-critical index set in the space \( H^s_\lambda \) is \( \{ -\frac{1}{4}, \infty \} \cup \left[ \frac{1}{2}, 1 \right] \).

The more detailed well-posedness results on the Hirota-Satsuma system are shown in the appendix. Based on the results in Table 3 and 4, we can actually study the well-posedness of cKdV systems (1.3) in the space \( H^s_\lambda \) for any \( k \in \{ 1, 2, 3, 4 \} \). The following is a summary of their weakly A-critical index sets.

**Theorem 1.20.** The weakly A-critical index sets \( C_k (k = 1, 2, 3, 4) \) in Definition 1.12 are

\[
C_1 = \{ -\frac{1}{2}, \infty \} \cup \left[ \frac{1}{2}, 1 \right] \quad \text{and} \quad C_q = \{ -\frac{1}{2}, -\frac{1}{3}, \infty \} \cup \left[ \frac{1}{2}, 1 \right], \quad q = 2, 3, 4.
\]

Fix any \( k \in \{ 1, 2, 3, 4 \} \). The above result enables us to provide a complete classification of the cKdV systems (1.3) in \( H^s_\lambda \).

**Theorem 1.21** (Classification of the systems (1.3)). Assume \( a_1a_2 \neq 0 \) and either \( C \) or \( D \) does not vanish. Fix \( k \in \{ 1, 2, 3, 4 \} \) and define \( H^s_\lambda \) as in (1.21). Then the systems (1.3) are completely classified into a family of \( k \) classes, each of which corresponds to a unique index \( s^* \in C_k \) such that any system in this class is A-LWP in the space \( H^s_\lambda \) if \( s > s^* \) while it is weakly A-IP if \( s < s^* \).

### 1.8 Organization of the paper

The organization of the rest of the paper is as follows. In Section 2, we will present the definitions of the Fourier restriction spaces and the resonance functions. Section 3 is devoted to explore properties of the irrationality exponents and prove Theorem 1.15. Some linear estimates will be introduced in Section 4. Theorem 1.18 will be broken into Lemma 5.1 and 5.2 and Proposition 5.3 and 5.4 in Section 5. We will justify Lemma 5.1 and 5.2 in Section 6 and prove Proposition 5.3 and 5.4 in Section 7. Finally, Appendix A includes the analytical well-posedness results about the Hirota-Satsuma systems (1.5) which can be obtained as a corollary of Theorem 1.18.

### 2 Fourier restriction spaces on \( T \)

To study the LWP of (1.3), we adopt the similar treatment as in (1.3) to deal with more general periodic problem posed on \( T_\lambda = \mathbb{R}/(2\pi \lambda \mathbb{Z}) \) for \( \lambda \geq 1 \) and thus consider the system

\[
\begin{align*}
\begin{cases}
(u_t + a_1 &\frac{u_{xx} + a_2}{v_{xx}}) = C(u_{xx} + D(u_{xx} + u_{xx} + u_{xx}), \quad x \in T_\lambda, \ t \in \mathbb{R}, \\
(u_t)|_{t=0} &\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in H^s(T_\lambda) \times H^s(T_\lambda).
\end{cases}
\end{align*}
\]

We use \( F_x \), \( F_t \) and \( F \) to denote the spatial, temporal and space-time Fourier transform respectively. However, when there is no confusion, we simply use *\* to denote any of these three types of Fourier transforms. On the other hand, \( F_x^{-1} \), \( F_t^{-1} \) and \( F^{-1} \) represent the corresponding inverse Fourier transforms.

The temporal Fourier transform and its inverse are defined standardly. The definitions of the spatial Fourier transform and its inverse are more complicated. Denote the frequency space corresponding to \( T_\lambda \) by \( \mathbb{Z}_\lambda = \{ k | k = n/\lambda \text{ for some } n \in \mathbb{Z} \} \). The normalized counting measure \( dk^\lambda \) on \( \mathbb{Z}_\lambda \) is defined by

\[
\int_{\mathbb{Z}_\lambda} a(k) dk^\lambda = \frac{1}{\lambda} \sum_{k \in \mathbb{Z}_\lambda} a(k). \quad (2.2)
\]

For any function \( f(x) \) on \( T_\lambda \),

\[
(F_x f)(k) := \int_0^{2\pi \lambda} e^{-ikx} f(x) \ dx, \quad \forall k \in \mathbb{Z}_\lambda. \quad (2.3)
\]
On the other hand, for any function $g(k)$ on $\mathbb{Z}_\lambda$, \[
(F^{-1}_x g)(x) := \frac{1}{2\pi} \int_{\mathbb{Z}_\lambda} e^{ikx} g(k) \, dk, \quad \forall x \in \mathbb{R}. \tag{2.4}\]
Consequently, the norm for the Sobolev space $H^s(\mathbb{T}_\lambda)$ is defined as \[
\|f\|_{H^s(\mathbb{T}_\lambda)} := \|\langle k \rangle^s \hat{f}(k)\|_{L^2(\mathbb{Z}_\lambda)} = \left( \frac{1}{\lambda} \sum_{k \in \mathbb{Z}_\lambda} \langle k \rangle^{2s} \langle \hat{f}(k) \rangle^2 \right)^{\frac{1}{2}}, \tag{2.5}\]
where $\langle \cdot \rangle := 1 + |\cdot|$. The homogeneous subspace $H^s_0(\mathbb{T}_\lambda)$ is defined as $H^s_0(\mathbb{T}_\lambda) = \{ f \in H^s(\mathbb{T}_\lambda) : \hat{f}(0) = 0 \}$. Finally, the space-time Fourier transform and its inverse are defined as $\mathcal{F} = F_t F_x$ and $\mathcal{F}^{-1} = F^{-1}_x F^{-1}_t$.

For any $\alpha \neq 0$ and $\lambda \geq 1$, consider \[
\begin{cases}
\partial_t w + \alpha \partial^3_x w = 0, \\ w(0) = w_0 \in H^s(\mathbb{T}_\lambda).
\end{cases} \tag{2.6}
\]
The solution to (2.6) is given explicitly by \[
w(x,t) = \int_{\mathbb{Z}_\lambda} e^{ikx} e^{i\phi^s(k)t} \hat{w}_0(k) \, dk, \tag{2.7}\]
with \[
\phi^s(k) := \alpha k^3. \tag{2.8}\]
The following is a generalized version of (1.4) for the definition of the Fourier restriction spaces on $\mathbb{T}$.

**Definition 2.1** ([9, 13, 31]). For any $\alpha, s, b, \lambda \in \mathbb{R}$ with $\alpha \neq 0$ and $\lambda \geq 1$, the Fourier restriction space $X^\alpha_{s,b,\lambda}$ is defined to be the completion of the Schwartz space $\mathcal{S}(\mathbb{T}_\lambda \times \mathbb{R})$ with respect to the norm \[
\|w\|_{X^\alpha_{s,b,\lambda}} := \|\langle k \rangle^s (\tau - \phi^s(k)) \hat{w}(k,\tau)\|_{L^2(\mathbb{Z}_\lambda \times \mathbb{R})}, \tag{2.9}\]
where $\hat{w}$ refers to the space-time Fourier transform of $w$.

It has been pointed out in [31] that one needs to take $b = \frac{1}{2}$ for the periodic case. However, this space barely fails to be in $C(\mathbb{R}; H^s_x)$. To ensure the continuity of the time flow of the solution, a smaller space $Y^\alpha_{s,\lambda}$ will be used via the norm \[
\|w\|_{Y^\alpha_{s,\lambda}} := \|w\|_{X^\alpha_{s,b,\lambda}} + \|\langle k \rangle^s \hat{\tilde{w}}(k,\tau)\|_{L^2(\mathbb{Z}_\lambda ; L^1(\mathbb{R}))}. \tag{2.10}\]
Since the second term $\|\langle k \rangle^s \hat{\tilde{w}}(k,\tau)\|_{L^2(\mathbb{Z}_\lambda ; L^1(\mathbb{R}))}$ has already dominated the $L^\infty_t H^s_x$ norm of $w$, it follows that $Y^\alpha_{s,\lambda} \subseteq C(\mathbb{R}; H^s_x)$. The companion spaces $Z^\alpha_{s,\lambda}$ via the norm (2.11) is then introduced to control the $Y^\alpha_{s,\lambda}$ norm of the integral term from the Duhame principle (see Lemma 4.1).

\[
\|w\|_{Z^\alpha_{s,\lambda}} := \|w\|_{X^\alpha_{s,b,\lambda}} + \|\langle k \rangle^s \hat{\tilde{w}}(k,\tau)\|_{L^2(\mathbb{Z}_\lambda ; L^1(\mathbb{R}))}. \tag{2.11}\]

For convenience, we will drop $\lambda$ when it equals 1. That is, $X^\alpha_{s,b} := X^\alpha_{s,b,1}$, $Y^\alpha_{s} := Y^\alpha_{s,1}$ and $Z^\alpha_{s} := Z^\alpha_{s,1}$. Throughout this paper, \[
\mathbb{R}^* := \mathbb{R} \setminus \{0\}, \quad \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}, \quad \mathbb{Z}^*_\lambda := \mathbb{Z}_\lambda \setminus \{0\}, \quad \mathbb{Q}^* := \mathbb{Q} \setminus \{0\}.
\]

**Definition 2.2** ([40]). Let $(\alpha_1, \alpha_2, \alpha_3)$ be a triple in $(\mathbb{R}^*)^3$. Define the resonance function $H$ associated to this triple by \[
H(k_1, k_2, k_3) = \sum_{i=1}^{3} \phi^{\alpha_i}(k_i), \quad \forall \sum_{i=1}^{3} k_i = 0. \tag{2.12}\]
where $\phi^*(k_i)$ is as defined in (2.8). The resonance set of $H$ is defined to be the zero set of $H$, that is
\[ \{(k_1, k_2, k_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} k_i = 0, H(k_1, k_2, k_3) = 0 \}. \]

### 3 Proof of Theorem 1.15

We first collect some classical results about the irrationality exponent $\mu(\rho)$.

**Proposition 3.1.**

(a) If $\rho \in \mathbb{Q}$, then $\mu(\rho) = 1$;
(b) If $\rho \in \mathbb{R} \setminus \mathbb{Q}$, then $\mu(\rho) \geq 2$;
(c) If $\rho$ is an irrational algebraic number, then $\mu(\rho) = 2$;
(d) For almost every $\rho \in \mathbb{R}$, $\mu(\rho) = 2$;
(e) The function $\mu$ maps $\mathbb{R}$ onto $\{1\} \cup [2, \infty)$.

**Proof.** Part (a) and (b) are standard. Part (c) is the famous Thue-Siegel-Roth theorem [42, 43, 48]. Part (d) is the Khintchine theorem [32]. Part (e) was proved by Jarnik [19, 20] using the theory of continued fractions. \[ \square \]

Next, we present two technical lemmas concerning the irrationality exponent function $\mu$.

**Lemma 3.2.** If a real number $\rho$ is approximable with power $\mu$, then there exist $\{(m_j, n_j)\}_{j=1}^{\infty} \subset \mathbb{Z} \times \mathbb{Z}^*$ with the following two properties.

1. $\{n_j\}_{j=1}^{\infty}$ is an increasing positive sequence and $\lim_{j \to \infty} n_j = \infty$;
2. For each $j \geq 1$, $(m_j, n_j)$ satisfies
   \[ 0 < \left| \rho - \frac{m_j}{n_j} \right| < \frac{1}{n_j^{\mu}}. \]

**Proof.** The conclusion follows from Definition 1.13 and the observation that for any fixed $n \in \mathbb{Z}^*$, there are at most finitely many $m \in \mathbb{Z}$ such that
   \[ 0 < \left| \rho - \frac{m}{n} \right| < \frac{1}{|n|^\mu}. \] \[ \square \]

**Lemma 3.3.** Let $\rho \in \mathbb{R} \setminus \mathbb{Q}$ with $\mu(\rho) < \infty$. Then for any $\epsilon > 0$, there exists a constant $K = K(\rho, \epsilon)$ such that the inequality
   \[ \left| \rho - \frac{m}{n} \right| \geq \frac{K}{|n|^{\mu_\epsilon}} \] (3.1)
holds for any $(m, n) \in \mathbb{Z} \times \mathbb{Z}^*$, where $\mu_\epsilon = \mu(\rho) + \epsilon$.

**Proof.** Since $\mu_\epsilon = \mu(\rho) + \epsilon > \mu(\rho)$, it follows from Definition 1.13 that there exist at most finitely many $(m, n) \in \mathbb{Z} \times \mathbb{Z}^*$ such that
   \[ 0 < \left| \rho - \frac{m}{n} \right| < \frac{1}{|n|^{\mu_\epsilon}}. \]
In addition, since $\rho$ is an irrational number, $|\rho - \frac{m}{n}|$ is never zero. Therefore, by choosing a sufficiently small constant $K = K(\rho, \epsilon)$, (3.1) holds for any $(m, n) \in \mathbb{Z} \times \mathbb{Z}^*$. \[ \square \]

Now some invariant properties of the irrationality exponent $\mu$ will be justified.
Proposition 3.4.

(a) For any $\sigma \in \mathbb{Q}^*$ and $\rho \in \mathbb{R}$, $\mu(\sigma \rho) = \mu(\rho)$.
(b) For any $\sigma \in \mathbb{Q}$ and $\rho \in \mathbb{R}$, $\mu(\rho + \sigma) = \mu(\rho)$.
(c) For any $\rho \in \mathbb{R}^*$, $\mu\left(\frac{1}{\rho}\right) = \mu(\rho)$.

Proof. As part (a) and (b) are obvious, we will only prove part (c). First, by taking advantage of (a), we may assume $\rho > 0$. Then due to symmetry, it reduces to prove

$$\mu(\rho) \leq \mu\left(\frac{1}{\rho}\right).$$

If $\rho \in \mathbb{Q}$, then it is trivial. So we further assume $\rho$ is an irrational number, which implies $\mu\left(\frac{1}{\rho}\right) \geq 2$.

Let $\rho$ be approximable with some power $\mu \geq 2$ and fix any $\epsilon > 0$. It follows from Lemma 3.2 that there exists a sequence $\{(m_j, n_j)\}_{j=1}^{\infty} \subset \mathbb{Z} \times \mathbb{Z}^*$ such that $n_j > 0$, $\lim_{j \to \infty} n_j = \infty$ and

$$0 < \left| \frac{1}{\rho} - \frac{m_j}{n_j} \right| < \frac{1}{n_j^\mu}.$$ 

Since $\rho > 0$ and $\mu \geq 2$, it immediately yields that $\lim_{j \to \infty} m_j = \infty$. In addition, for sufficiently large $j$, we have $0 < \frac{m_j}{n_j} < 2\rho$. Hence, for any such $j$, noticing

$$0 < \left| \frac{1}{\rho} - \frac{m_j}{n_j} \right| = \frac{n_j}{\rho m_j} \left| \frac{m_j}{n_j} - \rho \right| < \frac{1}{\rho m_j n_j^{\mu-1}} = \frac{1}{\rho} \left( \frac{m_j}{n_j} \right)^{\mu-1} \frac{1}{m_j^\mu},$$

we obtain

$$0 < \left| \frac{1}{\rho} - \frac{m_j}{n_j} \right| < \frac{(2\rho)^{\mu-1}}{\rho} \frac{1}{m_j^\mu}.$$ 

Finally, since $\lim_{j \to \infty} m_j = \infty$, when $j$ is large enough, $m_j^\mu > (2\rho)^{\mu-1}/\rho$. Therefore,

$$0 < \left| \frac{1}{\rho} - \frac{m_j}{n_j} \right| < \frac{1}{m_j^{\mu-\epsilon}}.$$ 

Since the above inequality is valid for any large $j$, $\frac{1}{\rho}$ is approximable with the power $\mu - \epsilon$. \qed

Finally, we discuss the relation between the irrationality exponent $\mu(\rho)$ in Definition 1.13 and the minimal type index $\nu(\rho)$ in Definition 1.14.

Proposition 3.5.

(a) If $\rho \in \mathbb{Q}$, then $\nu(\rho) = \infty$ and $\mu(\rho) = 1$;
(b) If $\rho \in \mathbb{R} \setminus \mathbb{Q}$, then $\nu(\rho) = \mu(\rho) - 2$.

Proof. Part (a) is obvious, so we will only prove part (b). Let $\rho \in \mathbb{R} \setminus \mathbb{Q}$. We will first show $\nu(\rho) \geq \mu(\rho) - 2$. If $\rho$ is approximable with power $\mu$, then it follows from Lemma 3.2 that there exists a sequence $\{(m_j, n_j)\}_{j=1}^{\infty} \subset \mathbb{Z} \times \mathbb{Z}^*$ such that $n_j > 0$, $\lim_{j \to \infty} n_j = \infty$ and

$$\left| \rho - \frac{m_j}{n_j} \right| < \frac{1}{n_j^\mu}.$$
As a result, for any \( \epsilon > 0 \), there does not exist \( K = K(\rho, \epsilon) \) such that

\[
|\rho - \frac{m_j}{n_j}| \geq \frac{K}{n_j^{\mu - \epsilon}}
\]

holds for all \( j \geq 1 \). In other words, \( \rho \) is not of type \( \mu - 2 - \epsilon \) according to Definition \( 1.14 \). Therefore, \( \nu(\rho) \geq \mu - 2 - \epsilon \). Sending \( \epsilon \to 0^+ \) and then taking supremum with respect to \( \mu \) leads to \( \nu(\rho) \geq \mu(\rho) - 2 \).

Now if \( \mu(\rho) = \infty \), then it follows from \( \nu(\rho) \geq \mu(\rho) - 2 \) that \( \nu(\rho) = \infty \). So in the following, we just assume \( \mu(\rho) < \infty \) and intend to show \( \nu(\rho) \leq \mu(\rho) - 2 \). According to Lemma \( 3.3 \) for any \( \epsilon > 0 \), there exists a constant \( K = K(\rho, \epsilon) > 0 \) such that

\[
|\rho - \frac{m}{n}| \geq \frac{K}{|n|^{|\mu(\rho)|+\epsilon}}
\]

holds for any \( (m, n) \in \mathbb{Z} \times \mathbb{Z}^* \). Hence, \( \rho \) is of type \( \mu(\rho) + \epsilon - 2 \), which implies \( \nu(\rho) \leq \mu(\rho) + \epsilon - 2 \). Sending \( \epsilon \to 0^+ \) yields \( \nu(\rho) \leq \mu(\rho) - 2 \). \( \square \)

For \( \rho \geq \frac{1}{4} \), denote \( \sigma_r \) and \( s_r \) as in \( 1.26 \). That is,

\[
\sigma_r = \mu(\sqrt{12r-3}) \quad \text{and} \quad s_r = \begin{cases} 
\frac{1}{2} & \text{if } \sigma_r = 1 \text{ or } \sigma_r \geq 3, \\
\frac{\sigma_r - 1}{2} & \text{if } 2 \leq \sigma_r < 3.
\end{cases}
\]

(3.2)

The properties of \( s_r \) can be derived from Proposition \( 3.1 \).

**Proposition 3.6.** Let \( \rho \geq \frac{1}{4} \) be given. Then

(a) \( s_r = 1 \) if \( \sqrt{12r-3} \in \mathbb{Q} \);

(b) \( \frac{1}{2} \leq s_r \leq 1 \) for any \( r \geq \frac{1}{4} \);

(c) \( s_r = \frac{1}{2} \) if \( r \) is an algebraic number and \( \sqrt{12r-3} \notin \mathbb{Q} \).

(d) \( s_r = \frac{1}{2} \) for almost every \( r \geq \frac{1}{4} \);

(e) The range of \( s_r \) over \( r \in [\frac{1}{4}, \infty) \) is \( [\frac{1}{2}, 1] \).

**Proof.** (a) When \( \sqrt{12r-3} \in \mathbb{Q} \), it follows from Proposition \( 3.1(\alpha) \) that \( \sigma_r = 1 \). Therefore, \( s_r = 1 \).

(b) This part is obviously due to the definition \( 1.26 \) for \( s_r \).

(c) When \( r \) is an algebraic number, \( \sqrt{12r-3} \) is also an algebraic number. Now if \( \sqrt{12r-3} \notin \mathbb{Q} \), then it follows from Proposition \( 3.1(\alpha) \) that \( \sigma_r = 2 \). Hence, \( s_r = \frac{1}{2} \).

(d) By Proposition \( 3.1(\alpha) \), \( \sigma_r = 2 \) for almost every \( r \geq \frac{1}{4} \). Thus, \( s_r = \frac{1}{2} \) for almost every \( r \geq \frac{1}{4} \).

(e) Combining Proposition \( 3.1(\alpha) \) with Proposition \( 3.4(\alpha) \), we conclude the range of \( \sigma_r \) is \( \{1\} \cup [2, \infty) \).

As a result, the range of \( s_r \) is \( [\frac{1}{2}, 1] \). \( \square \)

Now we are ready to prove Theorem \( 1.15 \).

**Proof of Theorem 1.15** Let \( a_2 \in (0, 4] \setminus \{1\} \). Recall \( \nu_c = \max\{\nu(c_1), \nu(c_2)\} \), where

\[
c_1 = \frac{1}{2} - \frac{1}{6}\sqrt{\frac{12}{a_2}} - 3, \quad c_2 = \frac{1}{2} + \frac{1}{6}\sqrt{\frac{12}{a_2}} - 3,
\]

and recall \( \nu_d = \max\{\nu(d_1), \nu(d_2)\} \), where

\[
d_1 = \frac{-3a_2 - \sqrt{3a_2(4 - a_2)}}{2(1 - a_2)}, \quad d_2 = \frac{-3a_2 + \sqrt{3a_2(4 - a_2)}}{2(1 - a_2)}.
\]
Denote $r = \frac{1}{a_2}$ and $\rho_r = \sqrt{12r - 3}$. Then $r \geq \frac{1}{4}$ and it follows from (3.2) that $\sigma_r = \mu(\rho_r)$. In addition, based on Proposition 3.1

$$
\mu(c_1) = \mu\left(\sqrt{\frac{12}{a_2} - 3}\right) = \sigma_r.
$$

Similarly, $\mu(c_2) = \sigma_r$. On the other hand, noticing that $d_1 = \frac{1}{c_1}$ and $d_2 = \frac{1}{c_2}$, so it follows from Proposition 3.4 that $\mu(d_1) = \mu(c_1)$ and $\mu(d_2) = \mu(c_2)$. Hence,

$$
\mu(d_1) = \mu(d_2) = \mu(c_1) = \mu(c_2) = \sigma_r.
$$

Then according to Proposition 3.5, $\nu(d_1) = \nu(d_2) = \nu(c_1) = \nu(c_2)$. In particular, $\nu_d = \nu_c = \nu(c_1)$.

Next, we will justify (1.27). Denote

$$
\bar{s} = \min\left\{1, \frac{1}{2} + \frac{1}{2} \nu(c_1)\right\}.
$$

Then it suffices to prove $\bar{s} = s_r$.

- **Case 1:** $\rho_r \in \mathbb{Q}$. In this case, $c_1 \in \mathbb{Q}$, so it follows from Proposition 3.5 that $\nu(c_1) = \infty$. Therefore, $\bar{s} = 1$. On the other hand, by Proposition 3.6(a), we also have $s_r = 1$.

- **Case 2:** $\rho_r \in \mathbb{R} \setminus \mathbb{Q}$. In this case, $c_1 \in \mathbb{R} \setminus \mathbb{Q}$, so it follows from Proposition 3.5 that $\nu(c_1) = \mu(c_1) - 2$, which implies $\nu(c_1) = \sigma_r - 2$. Putting this into (3.4) yields

$$
\bar{s} = \min\left\{1, \frac{\sigma_r - 1}{2}\right\}.
$$

Since $\rho_r \in \mathbb{R} \setminus \mathbb{Q}$, then Proposition 3.1(b) implies $\sigma_r = \mu(\rho_r) \geq 2$. Thus, we conclude from (3.2) that

$$
s_r = \min\left\{1, \frac{\sigma_r - 1}{2}\right\} = \bar{s}.
$$

\(\square\)

### 4 Linear estimates

Let $\psi \in C^\infty_0(\mathbb{R})$ be a bump function supported on $[-2, 2]$ and $\psi = 1$ on $[-1, 1]$. We first present two linear estimates, one is for the solution to the homogeneous linear KdV equation (2.6), and another one is for the solution to the forced linear KdV equation (2.6) with the right hand side being $F$ instead of 0. Recall the notation $S^s_\lambda$ as defined in (2.7).

**Lemma 4.1.** There exists a constant $C$ which only depends on the bump function $\psi$ such that for any $\alpha, s, \lambda \in \mathbb{R}$ with $\alpha \neq 0$ and $\lambda \geq 1$,

$$
\|\psi(t)S^s_\lambda(t)w_0\|_{Y^s_{\psi, \lambda}} \leq C\|w_0\|_{H^s(T_\lambda)}
$$

and

$$
\left\|\psi(t) \int_0^t S^s_\lambda(t - t')F(t')dt'\right\|_{Y^s_{\psi, \lambda}} \leq C\|F\|_{C^s_{\psi, \lambda}}
$$

The proof of the above lemma is almost the same as those for Lemma 7.1 and Lemma 7.2 in [13], so we omit it. Next, we provide two well-known embedding results.

**Lemma 4.2.** There exists a universal constant $C$ such that for any $\alpha \in \mathbb{R}^*$, $\lambda \geq 1$ and for any function $g$ on $T_\lambda \times \mathbb{R}$,

$$
\|g\|_{L^4_t L^2_x} \leq C\|g\|_{X^s_{\psi, \lambda}}.
$$

The proof of this estimate can be found in (39, Lemma 2.3.2).
Lemma 4.3. Let $\alpha \in \mathbb{R}^*$. Then there exists a constant $C = C(\alpha)$ such that for any $\lambda \geq 1$ and for any function $g$ on $\mathbb{T}_\lambda \times \mathbb{R}$,
\[
\|g\|_{L^4(\mathbb{T}_\lambda \times \mathbb{R})} \leq C\|g\|_{X_{0,\frac{s}{2},\lambda}^\alpha}.
\] (4.4)

When $\alpha = 1$ and $\lambda = 1$, Lemma 4.3 was first proved by Bourgain in [9] for a version when the left hand side of (4.4) is localized in time. Then Tao removed such a restriction in ([46], Proposition 6.4). Later, a more elementary proof was provided by Oh in his online note [41]. Actually, similar method had been applied earlier to the Schrödinger equation (see e.g. [47], Proposition 2.13).

5 Bilinear estimates in non-divergence form

This section will present the rigorous version of the bilinear estimates (1.14) and (1.15) in non-divergence form. Meanwhile, Theorem 1.18 will be broken into Lemma 5.1, 5.2 and Proposition 5.3 in more general settings. We denote $s_\tau$ as in (1.26) for any $r \geq \frac{1}{4}$.

Lemma 5.1. Let $\lambda \geq 1$ and $\alpha_1, \alpha_2 \in \mathbb{R}^*$ with $r := \frac{\alpha_2}{\alpha_1}$. Assume one of the conditions below is satisfied.

(a) $r < \frac{1}{4}$ and $s \geq -\frac{1}{2}$ and $p > 0$;

(b) $r = 1$, $s \geq \frac{1}{2}$ and $p > 0$;

(c) $r \in [\frac{1}{4}, \infty) \setminus \{1\}$, $s \geq 1$ and $p > 0$;

(d) $r \in [\frac{1}{4}, \infty) \setminus \{1\}$ with $s_r < 1$, $s > s_r$ and $p > s_r$.

Then there exist constants $\epsilon = \epsilon(\alpha_1, \alpha_2)$ and $C = C(\alpha_1, \alpha_2, s, p)$ such that
\[
\| (\partial_x w_1) w_2 \|_{X_{r, \lambda}^{\alpha_1}} \leq C \lambda^p \| w_1 \|_{X_{r, \lambda}^{\alpha_1}} \| w_2 \|_{X_{r, \lambda}^{\alpha_2}},
\] (5.1)
for any $w_1 \in X_{s_\lambda, \lambda}^{\alpha_1}$ and $w_2 \in X_{s_\lambda, \lambda}^{\alpha_2}$ with $\hat{w}_2(0, \cdot) = 0$.

Lemma 5.2. Let $\lambda \geq 1$ and $\alpha_1, \alpha_2 \in \mathbb{R}^*$ with $r := \frac{\alpha_2}{\alpha_1}$. Assume one of the conditions below is satisfied.

(a) $r < \frac{1}{4}$ and $s \geq -\frac{1}{2}$ and $p > 0$;

(b) $r = 1$, $s \geq \frac{1}{2}$, $p > 0$ and $\hat{w}_1(0, \cdot) = 0$;

(c) $r \in [\frac{1}{4}, \infty) \setminus \{1\}$, $s \geq 1$ and $p > 0$;

(d) $r \in [\frac{1}{4}, \infty) \setminus \{1\}$ with $s_r < 1$, $s > s_r$ and $p > s_r$.

Then there exist constants $\epsilon = \epsilon(\alpha_1, \alpha_2)$ and $C = C(\alpha_1, \alpha_2, s, p)$ such that
\[
\| w_1 (\partial_x w_2) \|_{X_{r, \lambda}^{\alpha_1}} \leq C \lambda^p \| w_1 \|_{X_{r, \lambda}^{\alpha_1}} \| w_2 \|_{X_{r, \lambda}^{\alpha_2}},
\] (5.2)
for any $w_1 \in X_{s_\lambda, \lambda}^{\alpha_1}$ and $w_2 \in X_{s_\lambda, \lambda}^{\alpha_2}$.

Next, we will address the sharpness of Lemma 5.1 and Lemma 5.2. Without loss of generality, we take $\lambda = 1$.

Proposition 5.3. The bilinear estimate
\[
\| (\partial_x w_1) w_2 \|_{X_{r, \lambda}^{\alpha_1}} \leq C \lambda^p \| w_1 \|_{X_{r, \lambda}^{\alpha_1}} \| w_2 \|_{X_{r, \lambda}^{\alpha_2}},
\] (5.3)
fails for any $b \in \mathbb{R}$ (and hence (5.1) fails by taking $b = \frac{1}{2}$) under any of the following conditions.

(a) $r < \frac{1}{4}$ and $s < -\frac{1}{4}$;
(b) $r = 1$ and $s < \frac{1}{2}$;
(c) $r \in \left[ \frac{1}{2}, \infty \right) \setminus \{1\}$ and $s < s_r$;
(d) $r \in \mathbb{R}^*$, $s \in \mathbb{R}$, but without the restriction $\tilde{w}_2(0, \cdot) = 0$ in \(5.3\).

**Proposition 5.4.** The bilinear estimate

$$
\| w_1(\partial_x w_2) \|_{X^{s_1}_{s,b}} \leq C \| w_1 \|_{X^{s_1}_{s,b}} \| w_2 \|_{X^{s_2}_{s,b}}
$$

(5.4)

fails for any $b \in \mathbb{R}$ (and hence \(5.2\) fails by taking $b = \frac{1}{2}$) under any of the following conditions.

(a) $r < \frac{1}{4}$ and $s < -\frac{1}{4}$;
(b) $r = 1$, $s < \frac{1}{2}$ and $\tilde{w}_1(0, \cdot) = 0$;
(c) $r = 1$, $s \in \mathbb{R}$, but without the restriction $\tilde{w}_1(0, \cdot) = 0$;
(d) $r \in \left[ \frac{1}{4}, \infty \right) \setminus \{1\}$ and $s < s_r$.

6 Proofs of the bilinear estimates

The goal of this section is to prove Lemma \(5.1\) and Lemma \(5.2\). Since their proofs are similar, we will only justify Lemma \(5.2\).

6.1 Idea of the proof

Without loss of generality, we consider the following simpler version of \(5.2\) with $\lambda = 1$,

$$
\| w_1(\partial_x w_2) \|_{X^{s_1}_{s,-\frac{1}{2}}} \lesssim \| w_1 \|_{X^{s_1}_{s,-\frac{1}{2}}} \| w_2 \|_{X^{s_2}_{s,-\frac{1}{2}}}.
$$

(6.1)

By duality and Plancherel identity, in order to verify \(6.1\), it suffices to prove (see \(16\) or Lemma 6.1)

$$
\int_{A} \frac{|k_2| \langle k_3 \rangle^{s} \prod_{i=1}^{3} |f_i(k_i, \tau_i)|}{\langle k_1 \rangle^{s} \langle k_2 \rangle^{s} \langle L_1 \rangle^{\frac{1}{2}} \langle L_2 \rangle^{\frac{1}{2}} \langle L_3 \rangle^{\frac{1}{2}}} \leq C \prod_{i=1}^{3} \| f_i \|_{L^2(Z \times \mathbb{R})}, \quad \forall \{f_i\}_{1 \leq i \leq 3},
$$

(6.2)

where $\langle \cdot \rangle = 1 + |\cdot|$, $A = \left\{ (k_1, k_2, k_3, \tau_1, \tau_2, \tau_3) \in \mathbb{Z}^3 \times \mathbb{R}^3 : \sum_{i=1}^{3} k_i = \sum_{i=1}^{3} \tau_i = 0 \right\}$ and

$$
L_1 = \tau_1 - \phi^{\alpha_1}(k_1), \quad L_2 = \tau_2 - \phi^{\alpha_2}(k_2), \quad L_3 = \tau_3 - \phi^{\alpha_3}(k_3),
$$

where $\phi^{\alpha}(k) = \alpha k^3$ is as defined in \(2.8\).

In \(6.2\), the loss of the spatial derivative in the bilinear estimate \(6.1\) is reflected via the term $|k_2| \langle k_3 \rangle^{s} \langle k_2 \rangle^{s}$, and the gain of the time derivative is reflected via the term $\langle L_1 \rangle^{\frac{1}{2}} \langle L_2 \rangle^{\frac{1}{2}} \langle L_3 \rangle^{\frac{1}{2}}$. How to compensate the loss of the spatial derivative from the gain of the time derivative is the key point. Denote

$$
R_1 = \frac{|k_2| \langle k_3 \rangle^{s} \langle k_2 \rangle^{s}}{\langle k_1 \rangle^{s} \langle k_2 \rangle^{s}} \quad \text{and} \quad R_2 = \langle L_1 \rangle^{\frac{1}{2}} \langle L_2 \rangle^{\frac{1}{2}} \langle L_3 \rangle^{\frac{1}{2}}.
$$

Then we need to control $R_1$ by $R_2$. Since $\sum_{i=1}^{3} k_i = 0$, then $\langle k_3 \rangle \leq \langle k_1 \rangle \langle k_2 \rangle$. As a result, $R_1$ is decreasing in $s$, which means the larger $s$ is, the more likely the bilinear estimate will hold. So the interest lies in the search for the smallest $s$ such that the bilinear estimate holds. Noticing that $L_i$ contains the time...
Since $R_{2.2}$ is a function of $\{k_i\}_{i=1,2,3}$ only. Because of this, we define the resonance function $H_2$ as in Definition 2.2.

$$H_2(k_1, k_2, k_3) := \phi_{a_1}^3(k_1) + \phi_{a_2}^3(k_2) + \phi_{a_3}^3(k_3). \quad (6.3)$$

Since $R_2 \geq (H_2)^{1/2}$, then it is easier to control $R_1$ by $R_2$ in the region where $H_2$ is large. The situation becomes more complicated near the region where $H_2$ vanishes. The zero set of $H_2$ is called the resonance set as in Definition 2.2.

By writing $k_3 = -k_1 - k_2$ in (6.3) and simplifying,

$$H_2(k_1, k_2, k_3) = (\alpha_2 - \alpha_1)k_3^3 - 3\alpha_1 k_1 k_2^2 - 3\alpha_1 k_1^2 k_2, \quad \forall \sum_{i=1}^3 k_i = 0. \quad (6.4)$$

If $k_2 = 0$, then $H_2(k_1, k_2, k_3) = 0$. If $k_2 \neq 0$, then $H_2$ can be rewritten as

$$H_2(k_1, k_2, k_3) = -3\alpha_1 k_2^3 h_r\left(\frac{k_1}{k_2}\right), \quad \forall \sum_{i=1}^3 k_i = 0 \text{ with } k_2 \neq 0, \quad (6.5)$$

where

$$h_r(x) = x^2 + x + \frac{1-r}{3}. \quad (6.6)$$

The following is the classification of the roots of $h_r$ depending on the values of $r$.

1. $r < \frac{1}{4}$: $h_r(x)$ does not have real roots;
2. $r = \frac{1}{4}$: $h_r(x)$ has a unique root $-\frac{1}{2}$;
3. $r > \frac{1}{4}$ but $r \neq 1$: $h_r(x)$ has two roots, neither of which equals -1 or 0;
4. $r = 1$: $h_r(x)$ has two roots -1 and 0.

Among the above four situations, Case (3) is the most interesting one, so we will first focus on this case. Assume $r \in \left(\frac{1}{4}, \infty\right) \setminus \{1\}$ and denote the two roots of $h_r(x)$ as $x_{1r}$ and $x_{2r}$, i.e.

$$x_{1r} := -\frac{1}{2} - \frac{\sqrt{12r-3}}{6}, \quad x_{2r} := -\frac{1}{2} + \frac{\sqrt{12r-3}}{6}. \quad (6.7)$$

Define $\sigma_r$ and $s_r$ as in (4.26). Then it follows from Proposition 3.4 that

$$\mu(x_{1r}) = \mu(x_{2r}) = \mu(\sqrt{12r-3}) = \sigma_r. \quad (6.8)$$

In addition, the resonance function $H_2$ can be written as

$$H_2(k_1, k_2, k_3) = -3\alpha_1 k_2^3 \left(\frac{k_1}{k_2} - x_{1r}\right) \left(\frac{k_1}{k_2} - x_{2r}\right).$$

As a result, the resonance set consists of three lines: $k_2 = 0$, $k_1 = x_{1r} k_2$ and $k_1 = x_{2r} k_2$. The most difficult estimate is near the line $k_1 = x_{1r} k_2$ or $k_1 = x_{2r} k_2$. Without loss of generality, let us consider the region near the line $k_1 = x_{1r} k_2$. In this situation, $|\frac{k_1}{k_2} - x_{2r}| \approx |x_{1r} - x_{2r}|$ which is a positive constant. Hence,

$$|H_2(k_1, k_2, k_3)| \sim |k_2|^3 |x_{1r} - \frac{k_1}{k_2}|. \quad (6.9)$$
In addition, when $\frac{k_1}{k_2}$ is very close to $x_{1r}$, we have $|k_1| \sim |k_2| \sim |k_3|$. Consequently,

$$R_1 \sim |k_2|^{1-s}.$$  \hfill (6.10)

On the other hand, noticing that both $k_1$ and $k_2$ are integers, the estimate of $|x_{1r} - \frac{k_1}{k_2}|$ reduces to the problem of the Diophantine approximation of $x_{1r}$.

- If $x_{1r} \in \mathbb{Q}$ or if its irrationality exponent $\sigma_r = \mu(x_{1r}) \geq 3$, then $s_r = 1$ and there exist infinitely many $(k_1, k_2)$ such that

$$|x_{1r} - \frac{k_1}{k_2}| = 0 \quad \text{or} \quad |x_{1r} - \frac{k_1}{k_2}| \lesssim \frac{1}{k_2^r}.$$  

So $\langle H_2(k_1, k_2, k_3) \rangle \sim 1$ due to (6.9). Then in order to have $|R_1| \lesssim \langle H_2 \rangle^{\frac{1}{2}}$, it follows from (6.10) that $s \geq 1 = s_r$.

- If $2 \leq \sigma_r < 3$, then $s_r = \frac{\sigma_r - 1}{2}$ and it follows from Lemma 3.3 that for any $\epsilon > 0$ and for any integers $k_1$ and $k_2$,

$$|x_{1r} - \frac{k_1}{k_2}| \gtrsim \frac{1}{|k_2|^\sigma_r + \epsilon}.$$  

So it follows from (6.9) that $|H_2(k_1, k_2, k_3)| \gtrsim |k_2|^{3 - \sigma_r - \epsilon}$. Then in order to have $|R_1| \lesssim \langle H_2 \rangle^{\frac{1}{2}}$, we need $1 - s \leq (3 - \sigma_r - \epsilon)/2$, that is

$$s \geq \frac{\sigma_r - 1}{2} + \frac{\epsilon}{2} = s_r + \frac{\epsilon}{2}.$$  

The above argument explains why the critical index is $s_r$ when $r \in (\frac{1}{4}, \infty) \setminus \{1\}$. Next, we will briefly discuss the rest cases (1), (2) and (4).

- When $r = \frac{1}{4}$, the argument is similar to the above. The two roots of $h_r(x)$ are the same: $x_{1r} = x_{2r} = -\frac{1}{2}$. So $s_r = \sigma_r = 1$. Meanwhile, $H_2(k_1, k_2, k_3) = -3\alpha k_3^2 \left(\frac{k_1}{k_2} + \frac{1}{2}\right)^2$. So there exist infinitely many integer pairs $(k_1, k_2)$ such that $\frac{k_1}{k_2} + \frac{1}{2} = 0$, which implies $|H_2(k_1, k_2, k_3)| = 0$. In order to have $|R_1| \lesssim \langle H_2 \rangle^{\frac{1}{2}}$, it follows from (6.10) that $s \geq 1 = s_r$.

- When $r < \frac{1}{4}$, $|h_r(x)| \gtrsim 1 + x^2$, so it follows from (6.5) that

$$|H_2(k_1, k_2, k_3)| \gtrsim |k_2^2 k_3^3|.$$  \hfill (6.11)

For $s < 0$, the worst situation is when $|k_1| \sim |k_2| \gg 1$ and $|k_3| \lesssim 1$. In this situation, $R_1 \sim |k_2|^{1-2s}$ and $|H_2| \sim |k_2|^s$. So in order to ensure $R_1 \lesssim \langle H_2 \rangle^{\frac{1}{2}}$, $s$ needs to be at least $-\frac{1}{4}$.

- When $r = 1$, $|H_2(k_1, k_2, k_3)| \sim |k_1 k_2 k_3|$. The worst situation occurs when $k_1 = -k_2$ and $k_3 = 0$, which implies $H_2 = 0$ and $R_1 \sim |k_2|^{1-2s}$. So in order to have $R_1 \lesssim \langle H_2 \rangle^{\frac{1}{2}} = 1$, $s$ should be at least $\frac{1}{2}$.

### 6.2 Auxiliary results

For any vector $(\vec{k}, \vec{\tau}) \in \mathbb{Z}_\lambda^3 \times \mathbb{R}^3$, we denote it as $(\vec{k}, \vec{\tau}) = (k_1, k_2, k_3, \tau_1, \tau_2, \tau_3)$. For any $\lambda \geq 1$,

$$A_\lambda := \left\{ (\vec{k}, \vec{\tau}) \in \mathbb{Z}_\lambda^3 \times \mathbb{R}^3 : \sum_{i=1}^{3} k_i = \sum_{i=1}^{3} \tau_i = 0 \right\},$$  \hfill (6.12)

and for any given $(k_3, \tau_3) \in \mathbb{Z}_\lambda \times \mathbb{R}$,

$$A_\lambda(k_3, \tau_3) := \left\{ (k_1, k_2, \tau_1, \tau_2) \in \mathbb{Z}_\lambda^2 \times \mathbb{R}^2 : (\vec{k}, \vec{\tau}) \in A_\lambda \right\}. \hfill (6.13)$$
Lemma 6.1. Let $\lambda \geq 1$, $s \in \mathbb{R}$ and $\alpha_i \in \mathbb{R}^*$ for $1 \leq i \leq 3$. The bilinear estimate
\[
\|w_1(\partial_x w_2)\|_{Z^s_{x,\lambda}} \leq C\lambda^p \|w_1\|_{X^{\alpha_1}_{s,\frac{1}{\alpha} \lambda}} \|w_2\|_{X^{\alpha_2}_{s,\frac{1}{\alpha} \lambda}}, \quad \forall \{w_i\}_{i=1,2} \tag{6.14}
\]
holds if and only if the following two estimates hold,
\[
\int_{A_\lambda} \frac{k_2(k_3)^s}{(k_1)^s (k_2)^s} \frac{3}{\langle L_i \rangle^{\frac{1}{2}}} \prod_{i=1}^3 f_i(k_i, \tau_i) \leq C\lambda^p \prod_{i=1}^3 \|f_i\|_{L^2(\mathbb{Z}_\lambda \times \mathbb{R})}, \quad \forall \{f_i\}_{i=1,2,3} \tag{6.15}
\]
and
\[
\left\| \frac{1}{\langle L_3 \rangle} \int_{A_\lambda(k_3, \tau_3)} \frac{k_2(k_3)^s}{(k_1)^s (k_2)^s} \frac{2}{\langle L_i \rangle^{\frac{1}{2}}} \prod_{i=1}^2 f_i(k_i, \tau_i) \right\|_{L^2_{\lambda} L^1_L} \leq C\lambda^p \prod_{i=1}^2 \|f_i\|_{L^2(\mathbb{Z}_\lambda \times \mathbb{R})}, \quad \forall \{f_i\}_{i=1,2} \tag{6.16}
\]
where $L_i = \tau_i - \phi^{\alpha_i}(k_i)$.

This lemma is essentially proved in [13, 40] by using duality, Plancherel identity and definition (2.11), so we omit the details here.

Lemma 6.2. Let $\lambda \geq 1$, $q \geq \frac{1}{4}$ and $\alpha_i \in \mathbb{R}^*$ for $1 \leq i \leq 3$. Then there exists a constant $C = C(\alpha_1, \alpha_2, \alpha_3, p)$ such that for any functions $\{f_i\}_{i=1}^3$ on $\mathbb{Z}_\lambda \times \mathbb{R}$,
\[
\int_{A_\lambda} M^q \prod_{i=1}^3 \left| \frac{f_i(k_i, \tau_i)}{\langle L_i \rangle^q} \right| \leq C \prod_{i=1}^3 \|f_i\|_{L^2(\mathbb{Z}_\lambda \times \mathbb{R})}, \tag{6.17}
\]
where $L_i = \tau_i - \phi^{\alpha_i}(k_i)$ and $M := \max \{\|L_1\|, \|L_2\|, \|L_3\|\}$. In particular, if $H(k_1, k_2, k_3)$ denotes the resonance function as defined in Definition 2.12 then
\[
\int_{A_\lambda} \langle H(k_1, k_2, k_3) \rangle^q \prod_{i=1}^3 \left| \frac{f_i(k_i, \tau_i)}{\langle L_i \rangle^q} \right| \leq C \prod_{i=1}^3 \|f_i\|_{L^2(\mathbb{Z}_\lambda \times \mathbb{R})}.
\]

This lemma is also standard (e.g. see [9, 13, 40]), but the statement is a little bit different from that in the literature, so we will briefly include a proof here for the convenience of the readers.

Proof of Lemma 6.2. Since $H(k_1, k_2, k_3) = -\sum_{i=1}^3 L_i$, it is obvious that $\langle H \rangle \lesssim M$, so it suffices to prove (6.17).

- Let's first focus on the region where $M = \langle L_1 \rangle$. In this region, $\langle L_1 \rangle^q \gtrsim M^q$. Define $g_1 = \mathcal{F}^{-1}(f_1(k_1, \tau_1))$ and
\[
g_i = \mathcal{F}^{-1}\left( \frac{f_i(k_i, \tau_i)}{\langle L_i \rangle^q} \right), \quad i = 2, 3.
\]
Then
\[
\text{LHS of (6.17)} \lesssim \int |\mathcal{F}(g_1)(k_1, \tau_1)||\mathcal{F}(g_2)(k_2, \tau_2)||\mathcal{F}(g_3)(k_3, \tau_3)|\lesssim \|g_1\|_{L^2(\mathbb{Z}_\lambda \times \mathbb{R})}\|g_2\|_{L^4(\mathbb{Z}_\lambda \times \mathbb{R})}\|g_3\|_{L^4(\mathbb{Z}_\lambda \times \mathbb{R})}.
\]
As a result, it follows from Lemma 4.3 and $q \geq \frac{1}{3}$ that
\[
\|g_1\|_{L^2} \|g_2\|_{L^4} \|g_3\|_{L^4} \lesssim \|g_1\|_{L^2} \|g_2\|_{X^{\alpha_2}_{0,\frac{1}{2} \lambda}} \|g_3\|_{X^{\alpha_3}_{0,\frac{1}{2} \lambda}} \lesssim \prod_{i=1}^3 \|f\|_{L^2(\mathbb{Z}_\lambda \times \mathbb{R})}.\]
• In the regions where $M = \langle L_2 \rangle$ or $\langle L_3 \rangle$, the argument is similar, so (6.17) is justified.

\[ \square \]

6.3 Proof of Lemma 5.2

As we have seen in Section 6.1, the main ideas in the proofs for part (a)–part (d) in Lemma 5.2 are analogous. So we will only carry out the detailed proof for part (d) which is the most technical case. The framework of the following proof is similar to that in [40].

Proof of Part (d). First, we recall from Proposition 3.6(b) that $s_r > 1$ and Proposition 3.6(a) that $r \neq \frac{3}{4}$. The case of $\frac{1}{4} < r < 1$ is analogous to the case of $r > 1$, so we will just assume $\frac{1}{4} < r < 1$ in the rest of the proof.

Fix $s > s_r$ and $p > s_r$, according to Lemma 6.1 it remains to prove

\[ \int_{A_{\lambda}} \left| \frac{\langle k_2 \rangle^{3}}{\langle k_1 \rangle^{3}} \prod_{i=1}^{3} \frac{|f_i(k_i, \tau_i)|}{\langle L_i \rangle^{\frac{1}{2}}} \right| \leq C \lambda^{3} \prod_{i=1}^{3} \| f_i \|_{L^2(\mathbb{Z} \times \mathbb{R})}, \quad (6.18) \]

and

\[ \left\| \frac{1}{\langle L_3 \rangle} \int_{A_{\lambda}(k_3, \tau_3)} \frac{|\langle k_2 \rangle^{3}}{\langle k_1 \rangle^{3}} \prod_{i=1}^{2} \frac{|f_i(k_i, \tau_i)|}{\langle L_i \rangle^{\frac{1}{2}}} \right\|_{L_{\lambda}^{2} L_{1}} \leq C \lambda^{2} \prod_{i=1}^{2} \| f_i \|_{L^2(\mathbb{Z} \times \mathbb{R})}, \quad (6.19) \]

where

\[ L_1 = \tau_1 - \phi_1(k_1), \quad L_2 = \tau_2 - \phi_2(k_2), \quad L_3 = \tau_3 - \phi_1(k_3). \]

Since $\frac{\langle k_3 \rangle}{\langle k_1 \rangle} \langle k_2 \rangle \leq 1$, it suffices to consider the case when $s$ is sufficiently close to $s_r$. So we can just assume

\[ s_r < s < p < 1. \quad (6.20) \]

Next, we will prove (6.18) first and then briefly mention the proof for (6.19).

Proof of (6.18). First, in the region where $k_2 = 0$, the integrand on the LHS of (6.18) vanishes, so we only need to focus on the region where $k_2 \neq 0$. Recall the resonance function $H_2$ in (6.5):

\[ H_2(k_1, k_2, k_3) = -3\alpha_1 k_2^3 h_r \left( \frac{k_1}{k_2} \right), \quad \forall \sum_{i=1}^{3} k_i = 0 \text{ with } k_2 \neq 0, \quad (6.21) \]

where $h_r$ is defined in (6.6). When $\frac{1}{4} < r < 1$, it follows from (6.7) that the two roots $x_{1r}$ and $x_{2r}$ of $h_r$ satisfy $-1 < x_{1r} < x_{2r} < 0$. Hence, there exists a constant $d_r \in (0, \frac{1}{8})$, which only depends on $r$, such that (see Figure 1)

\[ -1 < x_{1r} - 2d_r < x_{1r} + 2d_r < x_{2r} - 2d_r < x_{2r} + 2d_r < 0. \quad (6.22) \]

![Figure 1: location of $x_{1r}$ and $x_{2r}$](image)

On the other hand, there exists a constant $\epsilon_r$, which also only depends on $r$, such that

\[ |h_r(x)| \geq \epsilon_r (1 + x^2) \text{ if } |x - x_{1r}| \geq d_r \text{ and } |x - x_{2r}| \geq d_r. \quad (6.23) \]

In the following, we denote $\text{MAX} = \max \{ \langle L_1 \rangle, \langle L_2 \rangle, \langle L_3 \rangle \}$. 

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Since $|H_2(k_1, k_2, k_3)| = \left| \sum_{i=1}^{3} \langle L_i \rangle \right|$, then $\langle H_2(k_1, k_2, k_3) \rangle \lesssim \text{MAX}$.

**Case 1.** $\frac{k_1}{k_2} - x_{1r} \geq d_r$ and $\frac{k_1}{k_2} - x_{2r} \geq d_r$. In this case, it follows from (6.21) and (6.23) that

$$|H_2(k_1, k_2, k_3)| \sim |k_2|^{3} h_r \left( \frac{k_1}{k_2} \right) \gtrsim |k_2| (k_1^2 + k_2^2).$$

So $|k_2|^3 \lesssim |H_2| \lesssim \text{MAX}$. In addition, $k_2 \neq 0$ implies that $|k_2| \geq \frac{1}{\lambda}$. Hence, $|k_2|^2 \lesssim \lambda(\text{MAX})$. Recalling $\frac{(k_3)^s}{(k_1)^s (k_2)^s} \leq 1$, so

$$\text{LHS of (6.18)} \lesssim \lambda^\frac{1}{2} \int (\text{MAX}) \frac{3}{2} \prod_{i=1}^{3} \frac{|f_i(k_i, \tau_i)|}{\langle L_i \rangle} \lesssim \lambda^\frac{1}{2} \sum_{i=1}^{3} \|f_i\|_{L^2(\mathbb{Z}_+, \mathbb{R})},$$

where the last inequality is due to Lemma 6.2. This verifies (6.18) since $p > s_r \geq \frac{1}{2}$.

**Case 2.** $\frac{k_1}{k_2} - x_{1r} < d_r$ or $\frac{k_1}{k_2} - x_{2r} < d_r$. Without loss of generality, it is assumed that $|\frac{k_1}{k_2} - x_{1r}| < d_r$. Then it can be seen from (6.22) or Figure 1 that $|k_1| \sim |k_2| \sim |k_3|$ and $|\frac{k_1}{k_2} - x_{2r}| > 3d_r$. Meanwhile, if $|k_2| \lesssim 1$, then (6.18) is obviously true. So we will just assume $|k_2| \gg 1$. Thus, (6.18) reduces to

$$\int |k_2|^{1-s} \prod_{i=1}^{3} \frac{|f_i(k_i, \tau_i)|}{\langle L_i \rangle} \lesssim C \lambda^p \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{Z}_+, \mathbb{R})}. \quad (6.24)$$

Denote

$$\epsilon_s = 2s - 2s_r, \quad (6.25)$$

then $\epsilon_s > 0$. Next, we will split the range of $|\frac{k_1}{k_2} - x_{1r}|$ further.

**Case 2.1.** $\frac{d_r}{\lambda |k_2|} \leq \left| \frac{k_1}{k_2} - x_{1r} \right| < d_r$. In this case,

$$|H_2(k_1, k_2, k_3)| \sim |k_2|^3 \left| \frac{k_1}{k_2} - x_{1r} \right| \left| \frac{k_1}{k_2} - x_{2r} \right| \gtrsim \frac{k_2^2}{\lambda}.$$

Then it follows from $|H_2| \lesssim \text{MAX}$ and $s > s_r \geq \frac{1}{2}$ that $|k_2|^{1-s} \lesssim (\lambda |H_2|)^{\frac{s_2}{s}} \lesssim \lambda^\frac{1}{2} (\text{MAX})^\frac{s}{2}$. Hence,

$$\text{LHS of (6.24)} \lesssim \lambda^\frac{1}{4} \int (\text{MAX}) \frac{3}{2} \prod_{i=1}^{3} \frac{|f_i(k_i, \tau_i)|}{\langle L_i \rangle} \lesssim C \lambda^p \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{Z}_+, \mathbb{R})} \lesssim \text{RHS of (6.24)}.$$

**Case 2.2.** $\frac{k_1}{k_2} - x_{1r} < \frac{d_r}{\lambda |k_2|}$.

Since $s_r < 1$, it follows from (1.26) that $s_r = \frac{2s_r - 1}{2}$. In addition, by taking advantage of (6.8) and Lemma 3.3, there exists a constant $K = K(r, s)$ such that

$$|x_{1r} - \frac{k_1}{k_2}| = |x_{1r} - \frac{\lambda k_1}{\lambda k_2}| \geq \frac{K}{|\lambda k_2|^{s_r + \epsilon_s}}.$$

Combining $s_r = \frac{2s_r - 1}{2}$ and (6.25) yields $\sigma_r + \epsilon_s = 2s + 1$. Therefore,

$$\left| x_{1r} - \frac{k_1}{k_2} \right| \gtrsim \frac{1}{|\lambda k_2|^{2s+1}} \quad (6.26)$$

which implies

$$|H_2(k_1, k_2, k_3)| \sim |k_2|^3 \left| \frac{k_1}{k_2} - x_{1r} \right| \left| \frac{k_1}{k_2} - x_{2r} \right| \gtrsim \lambda^{1-2s} |k_2|^{2-2s}.$$
Then we obtain
\[ |k_2|^{1-s} \lesssim \lambda^{\frac{2}{3} + s}|H_2|^\frac{1}{2} \lesssim \lambda^{\frac{2}{3} + s}(\text{MAX})^{\frac{1}{2}}. \]  
(6.27)

So (6.24) reduces to
\[ \lambda^{\frac{2}{3}} \int (\text{MAX})^{\frac{1}{2}} \prod_{i=1}^{3} \frac{|f_i(k_i, \tau_i)|}{(L_i)^{\frac{1}{2}}} \leq C \lambda^{p-s} \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{Z}_\lambda \times \mathbb{R})}. \]  
(6.28)

- **Case 2.2.1.** $\langle L_1 \rangle = \text{MAX}$. Then (6.28) reduces to
\[ \lambda^{\frac{2}{3}} \int \frac{1}{(L_2)^{\frac{1}{2}}(L_3)^{\frac{1}{2}}} \prod_{i=1}^{3} |f_i(k_i, \tau_i)| \leq C \lambda^{p-s} \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbb{Z}_\lambda \times \mathbb{R})}. \]  
(6.29)

For fixed $k_1$, it follows from $|k_2| < \frac{d_{x_1}}{|x_1|}$ that $k_2 \in E_r(k_1)$ which is defined as follows.
\[ E_r(k_1) = \left\{ k_2 \in \mathbb{Z}_\lambda : \left| k_2 - \frac{k_1}{x_1} \right| < \frac{d_r}{|x_1|} \right\}. \]  
(6.30)

It is easily seen that the size $|E_r(k_1)|$ is at most $\frac{2d_r}{|x_1|}$. Define
\[ g_1(k_1, \tau_1) = |f_1(k_1, \tau_1)|, \quad g_2(k_2, \tau_2) = \frac{|f_2(k_2, \tau_2)|}{(L_2)^{\frac{1}{2}}}, \quad g_3(k_3, \tau_3) = \frac{|f_3(k_3, \tau_3)|}{(L_3)^{\frac{1}{2}}} \]  
(6.31)

Since $(k_3, \tau_3) = -(k_1 + k_2, \tau_1 + \tau_2)$, the LHS of (6.29) is bounded above by
\[ \lambda^{\frac{2}{3}} \int_{\mathbb{Z}_\lambda} \int_{\mathbb{R}} g_1(k_1, \tau_1) \left( \int_{\mathbb{Z}_\lambda} \mathbb{1}_{E_r(k_1)}(k_2) \left[ g_2(k_2, \cdot) \ast \tau \ g_3(-k_1 - k_2, \cdot) \right](\tau_2) \ d k_2^\lambda \right) d \tau_1 \ d k_1^\lambda. \]

By the change of variable $(k_1, \tau_1) \rightarrow (-k_1, -\tau_1)$ and the definition (2.2) for the measure $dk_2^\lambda$, the above quantity is seen to be dominated by
\[ \lambda^{-\frac{1}{2}} \|g_1\|_{L^2_{k_1} L^2_{\tau_1}} \left\| \sum_{k_2 \in E_r(-k_1)} [g_2(k_2, \cdot) \ast \tau \ g_3(k_1 - k_2, \cdot)](\tau_2) \right\|_{L^2_{k_2} L^2_{\tau_2}}. \]  
(6.32)

Since the size of the set $E_r(-k_1)$ is bounded by the constant $\frac{2d_r}{|x_1|}$, then by applying the Plancherel identity, we obtain
\[ \|g_1\|_{L^2_{k_1} L^2_{\tau_1}} \leq \lambda^{-\frac{1}{2}} \left\| \sup_{k_2 \in E_r(-k_1)} \mathcal{F}_t^{-1} g_2(k_2, t) \mathcal{F}_t^{-1} g_3(k_1 - k_2, t) \right\|_{L^2_{k_2} L^2_{\tau_2}}. \]  
(6.33)

Noticing $\mathcal{F}_t^{-1} = \mathcal{F}_x^{-1} \mathcal{F}^{-1}$, then it follows from the definition of $\mathcal{F}_x$ and Hölder’s inequality that
\[ \|\mathcal{F}_t^{-1} g_2(k_2, t)\|_{L^2_{x,t}} \lesssim \|\mathcal{F}^{-1} g_2(x, t)\|_{L^2_{x}} \lesssim \lambda^{\frac{4}{3}} \|\mathcal{F}^{-1} g_2(x, t)\|_{L^2_{x}}. \]  
(6.34)

In addition, it is not difficult to show that for any fixed $t$,
\[ \left\| \sup_{k_2 \in E_r(-k_1)} |\mathcal{F}_t^{-1} g_3(k_1 - k_2, t)| \right\|_{L^2_{k_2}} \lesssim \|\mathcal{F}^{-1} g_3(x, t)\|_{L^2_{x}}. \]  
(6.35)

Then plugging (6.34) and (6.35) into (6.33) yields
\[ \text{RHS of (6.33)} \lesssim \|g_1\|_{L^2_{k_1} L^2_{\tau_1}} \left\| \|\mathcal{F}^{-1} g_2(x, t)\|_{L^2_{x}} \right\|_{L^2_{\tau}} \left\| \|\mathcal{F}^{-1} g_3(x, t)\|_{L^2_{x}} \right\|_{L^2_{\tau}}. \]
Finally, by Holder’s inequality and Lemma 4.2

\[
\text{RHS of (6.33)} \lesssim \|g_1\|_{L^2_tL^2_x} \|F^{-1}g_2(x,t)\|_{L^1_tL^2_x} \|F^{-1}g_3(x,t)\|_{L^1_tL^2_x} \\
\lesssim \|g_1\|_{L^2_tL^2_x} \|F^{-1}g_2\|_{X^{1,1}_0} \|F^{-1}g_3\|_{X^{1,1}_0} \\
\lesssim \prod_{i=1}^3 \|f_i\|_{L^2(Z \times \mathbb{R})},
\]

where the last inequality is due to (6.31).

- **Case 2.2.2.** MAX = \(\langle L_2 \rangle\) or \(\langle L_3 \rangle\). The argument is similar to that in Case 2.2.1, therefore omitted.

**Proof of (6.19).** We basically reduce the proof of (6.19) to the following two cases. In different regions, we have the flexibility to choose which case to estimate.

- **Firstly,** by Cauchy-Schwartz inequality and duality, the proof of (6.19) reduces to establish

\[
\int_{A_{\lambda}(k_3, \tau_3)} |k_2| \langle k_3 \rangle^s \langle k_1 \rangle^s \langle L_2 \rangle^\frac{3}{2} \langle L_3 \rangle^\frac{1}{2} \lesssim C \lambda \prod_{i=1}^3 \|f_i\|_{L^2(Z \times \mathbb{R})}.
\]

(6.36)

- **Secondly,** for fixed \(k_3\), if \(L_3\) is restricted to some set \(\Omega(k_3)\), then it follows from Holder’s inequality in \(\tau_3\) that

\[
\text{LHS of (6.19)} \lesssim \left( \int_{\mathbb{R}} \frac{1}{\langle L_3 \rangle} \frac{1}{\langle L_3 \rangle} \right)^\frac{1}{2} \int_{A_{\lambda}(k_3, \tau_3)} |k_2| \langle k_3 \rangle^s \langle k_1 \rangle^s \langle L_1 \rangle^\frac{1}{2} \langle L_2 \rangle^\frac{1}{2} \langle L_3 \rangle^\frac{1}{2} \lesssim \lambda^p.
\]

So if we can prove

\[
\int_{\mathbb{R}} \frac{1}{\langle L_3 \rangle} d\tau_3 \lesssim \lambda^p,
\]

then (6.19) will follow from (6.37), duality and (6.18).

If \(\langle L_1 \rangle = \text{MAX}\) or \(\langle L_2 \rangle = \text{MAX}\) or \(\langle L_3 \rangle = \text{MAX}\) with \(\langle L_1 \rangle \langle L_2 \rangle \gtrsim \langle L_3 \rangle \lambda^p\), then we can apply the similar argument as that in the proof of (6.18) to justify (6.36).

If \(\langle L_3 \rangle = \text{MAX}\) with \(\langle L_1 \rangle \langle L_2 \rangle \ll \langle L_3 \rangle \lambda^p\), then we will justify (6.37). Similar to the proof of (6.18), the most delicate situation is when \(\frac{k_3}{k_2} = \text{constant}\) very near the root \(x_{1r}\) or \(x_{2r}\). Without loss of generality, we will only investigate the region where \(\frac{k_3}{k_2} - x_{1r} < \frac{d_r}{\lambda |k_2|}\) in the rest argument (see the above Case 2).

Firstly, it follows from \(\left| \frac{k_3}{k_2} - x_{1r} \right| < \frac{d_r}{\lambda |k_2|}\) that

\[
\left| k_1 + \frac{x_{1r} k_3}{1 + x_{1r}} \right| < \frac{d_r}{(1 + x_{1r}) \lambda}.
\]

So when \(k_3\) is fixed, the choice of \(k_1\) is at most \(\frac{2d_r}{1 + x_{1r}}\). Secondly,

\[
\langle L_3 + H_2 \rangle = \langle L_3 - \sum_{i=1}^3 L_i \rangle = \langle L_1 + L_2 \rangle \ll \langle L_4 \rangle \lambda^p.
\]

As a result, \(|H_2| \sim |L_3| \gg 1\) and \(\langle L_3 + H_2 \rangle \ll |H_2| \lambda^p\). So there exists a small constant \(\delta \in (0, \frac{1}{10})\).
such that $\langle L_3 + H_2 \rangle \leq \delta |H_2|^\frac{2}{\max}$. Fix this $\delta$. For any $\kappa \in \mathbb{Z}_\lambda$, define

$$\Omega^\delta(\kappa) = \left\{ \eta \in \mathbb{R} : \exists k_1, k_2 \in \mathbb{Z}_\lambda \text{ such that } \sum_{i=1}^3 k_i = 0, \left| k_1 + \frac{x_{1r}k_3}{1 + x_{1r}} \right| < \frac{d\tau}{(1 + x_{1r})\lambda} \right\}$$

and $\langle \eta + H_2(k_1, k_2, k_3) \rangle \leq \delta |H_2(k_1, k_2, k_3)|^\frac{2}{\max}$.

Then $\tau_3 - \phi^\alpha(k_3) = L_3 \in \Omega^\delta(k_3)$. For any $M \geq 1$, define $\Omega^\delta_M(k_3) = \{ \eta \in \Omega^\delta(k_3) : M/2 \leq |\eta| \leq 2M \}$. For any $\eta \in \Omega^\delta_M(k_3)$, the choices of $k_1$, as mentioned above, is at most $\frac{2d\tau}{1 + x_{1r}}$. On the other hand, it follows from

$$\langle \eta + H_2(k_1, k_2, k_3) \rangle \leq \delta |H_2(k_1, k_2, k_3)|^\frac{2}{\max}$$

that $|H_2| \sim \eta \sim M$. So $\langle \eta + H_2(k_1, k_2, k_3) \rangle \leq \delta M^\frac{2}{\max}$. As a result, $|\Omega^\delta_M(k_3)| \lesssim M^\frac{1}{\max}$. By the change of variable $\eta = \tau_3 - \phi^\alpha(k_3) = L_3$,

$$\int_{\mathbb{R}} \frac{1}{(L_3)} \frac{d\eta}{\langle \eta \rangle} = \int_{\Omega^\delta(k_3)} \frac{d\eta}{\langle \eta \rangle} = \int_{|\eta| \leq \lambda^6} \frac{d\eta}{\langle \eta \rangle} + \sum_{M \text{ dyadic, } M > \lambda^6} \int_{\Omega^\delta_M(k_3)} \frac{d\eta}{\langle \eta \rangle} \lesssim \ln(1 + \lambda^6) + \sum_{M \text{ dyadic, } M > \lambda^6} \frac{|\Omega^\delta_M(k_3)|}{M} \lesssim \ln(1 + \lambda^6) + \sum_{M \text{ dyadic, } M > \lambda^6} M^{-\frac{1}{2}} \lesssim \lambda^p,$$

which verifies (6.37). \qed

7 Sharpness of the bilinear estimates

In this section we will prove Propositions 5.3 and 5.4 which justify the sharpness of the bilinear estimates in Lemmas 5.1 and 5.2. Since their proofs are similar, we will only prove Proposition 6.4. Throughout this section,

$$A := \left\{ (\bar{k}, \bar{\tau}) \in \mathbb{Z}^3 \times \mathbb{R}^3 : \sum_{i=1}^3 k_i = \sum_{i=1}^3 \tau_i = 0 \right\}.$$

The following lemma will be frequently used in this section.

Lemma 7.1. Let $E_i \subseteq \mathbb{Z} \times \mathbb{R} (1 \leq i \leq 3)$ be bounded regions such that $E_1 + E_2 \subseteq -E_3$, i.e.,

$$-(k_1 + k_2, \tau_1 + \tau_2) \in E_3, \quad \forall (k_i, \tau_i) \in E_i, 1 \leq i \leq 2.$$

Then

$$\int_A \prod_{i=1}^3 1_{E_i}(k_i, \tau_i) = |E_1| |E_2|.$$

Proof. Rewriting the left hand side as

$$\int_{E_1} \int_{E_2} \prod_{i=1}^3 1_{E_i}(-(k_1 + k_2), -(\tau_1 + \tau_2)) \, dk_2 \, d\tau_2 \, dk_1 \, d\tau_1,$$

then the conclusion follows from (7.1). \qed
7.1 Proof of Proposition 5.4

Similar to Lemma 6.1, (5.4) is equivalent to

\[
\int_{A} \frac{k_{2}(k_{3})^{s}}{(k_{1})^{s}(k_{2})^{s}(L_{1})^{b}(L_{2})^{b}(L_{3})^{1-b}} \prod_{i=1}^{3} f_{i}(k_{i}, \tau_{i}) \leq C \prod_{i=1}^{3} \| f_{i} \|_{L^{2}(\mathbb{R})}, \quad \forall \{ f_{i} \}_{i=1,2,3}
\]  

(7.2)

where

\[
L_{1} = \tau_{1} - \alpha_{1}k_{1}^{3}, \quad L_{2} = \tau_{2} - \alpha_{2}k_{2}^{3}, \quad L_{3} = \tau_{3} - \alpha_{1}k_{3}^{3}.
\]

The corresponding resonance function \( H_{2} \) is given by (6.4):

\[
H_{2}(k_{1}, k_{2}, k_{3}) = -3\alpha_{1}k_{2}\left[k_{1}^{2} + k_{1}k_{2} + \frac{1-r}{3}k_{2}^{2}\right],
\]

(7.3)

where \( r = \frac{\alpha_{2}}{\alpha_{1}} \). If \( k_{2} \neq 0 \), then \( H_{2} \) can be rewritten as in (6.5):

\[
H_{2}(k_{1}, k_{2}, k_{3}) = -3\alpha_{1}k_{2}^{2}h_{r}\left(\frac{k_{1}}{k_{2}}\right),
\]

(7.4)

where \( h_{r}(x) = x^{2} + x + \frac{1-r}{3} \).

**Proof of Part (a).** Suppose there exist \( r < \frac{1}{4}, s < -\frac{1}{4} \) and \( b \in \mathbb{R} \) such that (7.2) holds.

- For large \( N \), define \( f_{i} = -\mathds{1}_{B_{i}} \) for \( 1 \leq i \leq 3 \), where

\[
B_{1} = \{(k_{1}, \tau_{1}) : k_{1} = N, \quad |\tau_{1} - \alpha_{1}N^{3}| \leq 1\},
\]

\[
B_{2} = \{(k_{2}, \tau_{2}) : k_{2} = -N, \quad |\tau_{2} + \alpha_{2}N^{3}| \leq 1\},
\]

\[
B_{3} = \{(k_{3}, \tau_{3}) : k_{3} = 0, \quad |\tau_{3} + (\alpha_{1} - \alpha_{2})N^{3}| \leq 2\}.
\]

Then \( B_{1} + B_{2} \subseteq -B_{3} \). In addition, for any \( (k_{i}, \tau_{i}) \in B_{i}, 1 \leq i \leq 3, \) \( \langle L_{1} \rangle \sim 1, \langle L_{2} \rangle \sim 1 \) and it follows from (7.4) that \( |H_{2}(k_{1}, k_{2}, k_{3})| \sim N^{3} \). Therefore, \( \langle L_{A} \rangle \sim N^{3} \) and we conclude from (7.2) that

\[
\frac{N}{N^{2s-1/2}} \int_{A} \prod_{i=1}^{3} \mathds{1}_{B_{i}}(k_{i}, \tau_{i}) \lesssim C \prod_{i=1}^{3} |B_{i}|^{\frac{1}{2}}.
\]

Noticing \( |B_{i}| \sim 1 \), applying Lemma 7.1 leads to \( N^{3b-2s-2} \lesssim C \). In other words,

\[
3b - 2s - 2 \leq 0.
\]

(7.5)

- Similarly, define \( f_{i} = -\mathds{1}_{B_{i}} \) for \( 1 \leq i \leq 3 \), where

\[
B_{1} = \{(k_{1}, \tau_{1}) : k_{1} = N, \quad |\tau_{1} - \alpha_{1}N^{3}| \leq 1\},
\]

\[
B_{3} = \{(k_{3}, \tau_{3}) : k_{3} = 0, \quad |\tau_{3}| \leq 1\},
\]

\[
B_{2} = \{(k_{2}, \tau_{2}) : k_{2} = -N, \quad |\tau_{2} + \alpha_{1}N^{3}| \leq 2\}.
\]

Then \( B_{1} + B_{3} \subseteq -B_{2} \) and by similar argument, we conclude

\[
1 - 2s - 3b \leq 0.
\]

(7.6)

(7.5) and (7.6) together yields \( s \geq -\frac{1}{3} \), which contradicts to the assumption \( s < -\frac{1}{4} \). In addition, when \( s = -\frac{1}{4}, \) \( b \) has to be exactly \( \frac{1}{2} \).

**Proof of Part (b).** Under the additional assumption \( \tilde{w}_{1}(0, \cdot) = 0 \), (5.4) is equivalent to (7.2) with the additional restriction \( f_{1}(0, \cdot) = 0 \). When \( r = 1 \), writing \( \alpha_{1} = \alpha_{2} = \alpha \), then \( H_{2}(k_{1}, k_{2}, k_{3}) = 3\alpha k_{1}k_{2}k_{3} \).
Define \( f_i = -\mathds{1}_{B_i} \) for \( 1 \leq i \leq 3 \), where

\[
B_1 = \{(k_1, \tau_1) : k_1 = N, \ |\tau_1 - \alpha N^3| \leq 1\},
\]
\[
B_2 = \{(k_2, \tau_2) : k_2 = -N, \ |\tau_2 + \alpha N^3| \leq 1\},
\]
\[
B_3 = \{(k_3, \tau_3) : k_3 = 0, \ |\tau_3| \leq 2\}.
\]

Then \( B_1 + B_2 \subseteq -B_3 \). In addition, for any \((k_i, \tau_i) \in B_i, 1 \leq i \leq 3\), \( \langle L_1 \rangle \sim 1, \langle L_2 \rangle \sim 1 \) and \( H_2(k_1, k_2, k_3) = 0 \), which implies \( \langle L_3 \rangle \sim 1 \). Then it follows from (7.2) that

\[
\frac{N}{N^{2s}} \int_A \prod_{i=1}^{3} \mathds{1}_{B_i}(k_i, \tau_i) \lesssim C \prod_{i=1}^{3} |B_i|^\frac{1}{2}.
\]

Noticing \( |B_i| \sim 1 \), applying Lemma 7.1 leads to \( s \geq \frac{1}{2} \).

**Proof of Part (c).** Analogous to part (b) but without the restriction \( \Omega_{r, \tau} \), so \( f_1(0, \cdot) \) is not required to be zero anymore. So it is valid to define \( f_i = -\mathds{1}_{B_i} \), where

\[
B_1 = \{(k_1, \tau_1) : k_1 = 0, \ |\tau_1| \leq 1\},
\]
\[
B_2 = \{(k_2, \tau_2) : k_2 = N, \ |\tau_2 - \alpha N^3| \leq 1\},
\]
\[
B_3 = \{(k_3, \tau_3) : k_3 = -N, \ |\tau_3 + \alpha N^3| \leq 2\}.
\]

Then \( B_1 + B_2 \subseteq -B_3 \). In addition, for any \((k_i, \tau_i) \in B_i, 1 \leq i \leq 3\), \( \langle L_1 \rangle \sim 1, \langle L_2 \rangle \sim 1 \) and \( H_2(k_1, k_2, k_3) = 0 \), which implies \( \langle L_3 \rangle \sim 1 \). Then it follows from (7.2) that

\[
N \int_A \prod_{i=1}^{3} \mathds{1}_{B_i}(k_i, \tau_i) \lesssim C \prod_{i=1}^{3} |B_i|^\frac{1}{2},
\]

which implies \( N \lesssim C \) due to Lemma 7.1. But this is impossible since \( N \) can be arbitrarily large. \( \square \)

**Proof of Part (d).** The following proof is in the similar spirit as that in (40, Proposition 3.9). The proofs for the case \( r \in [\frac{1}{4}, 1) \) and the case \( r \in (1, \infty) \) are analogous, so we will just assume \( r \in [\frac{1}{4}, 1) \). Suppose there exist \( s < s_r \) and \( b \in \mathbb{R} \) such that (5.4) holds. Then (7.2) holds for some constant \( C = C(\alpha_1, \alpha_2, s, b) \). Since \( r \in [\frac{1}{4}, 1) \), the function \( h_r \) has two roots

\[
x_{1r} = \frac{1}{2} - \frac{\sqrt{12r - 3}}{6}, \quad x_{2r} = \frac{1}{2} + \frac{\sqrt{12r - 3}}{6}.
\]

In addition, \(-1 < x_{1r} \leq x_{2r} < 0 \) and \( \mu(x_{1r}) = \mu(x_{2r}) = \mu(\sqrt{12r - 3}) = \sigma_r \). When \( k_2 \neq 0 \), the resonance function \( H_2 \) can be written as

\[
H_2(k_1, k_2, k_3) = -3\alpha_1 k_2^3 \left( \frac{k_1}{k_2} - x_{1r} \right) \left( \frac{k_1}{k_2} - x_{2r} \right).
\]

In the following, we will first show that \( \frac{1}{4} \leq b \leq \frac{2}{3} \) and then derive a contradiction to \( s < s_r \).

For large positive \( N \), define \( f_i = \mathds{1}_{B_i} \) for \( 1 \leq i \leq 3 \), where

\[
B_1 = \{(k_1, \tau_1) : k_1 = 0, \ |\tau_1| \leq 1\},
\]
\[
B_2 = \{(k_2, \tau_2) : k_2 = N, \ |\tau_2 - \alpha_2 N^3| \leq 1\},
\]
\[
B_3 = \{(k_3, \tau_3) : k_3 = -N, \ |\tau_3 + \alpha_2 N^3| \leq 2\}.
\]

Then \( B_1 + B_2 \subseteq -B_3 \). In addition, for any \((k_i, \tau_i) \in B_i, 1 \leq i \leq 3\), \( \langle L_1 \rangle \sim 1, \langle L_2 \rangle \sim 1 \) and \( |H_2(k_1, k_2, k_3)| = |\alpha_2 N^3 - \alpha_1 N^3| \sim N^3 \).
which implies \((L_3) \sim N^3\). Then it follows from (7.2) that
\[
\frac{N^{1+s}}{N^{2s}N^{3(1-b)}} \int_A \prod_{i=1}^3 \mathbb{I}_{B_i}(k_i, \tau_i) \lesssim C \prod_{i=1}^3 |B_i|^{1/2}.
\]
Noticing \(|B_1| \sim 1\), applying Lemma 7.1 leads to \(b \leq \frac{2}{3}\). On the other hand, define \(f_i = \mathbb{I}_{B_i}\) for \(1 \leq i \leq 3\), where
\[
B_1 = \{(k_1, \tau_1) : k_1 = 0, \ |\tau_1| \leq 1\},
B_3 = \{(k_3, \tau_3) : k_3 = -N, \ |\tau_3 + \alpha_1 N^3| \leq 1\},
B_2 = \{(k_2, \tau_2) : k_2 = N, \ |\tau_2 - \alpha_1 N^3| \leq 2\}.
\]
Then \(B_1 + B_3 \subseteq -B_2\) and by applying similar arguments, we conclude that \(b \geq \frac{1}{3}\). Thus, \(\frac{1}{3} \leq b \leq \frac{2}{3}\).

Next, we discuss two situations depending on whether \(\sqrt{12r} - 3\) is a rational number or not.

**Case 1.** \(\sqrt{12r} - 3 \in \mathbb{Q}\).
In this case, \(x_{1r} \in \mathbb{Q}, \sigma_r = 1\) and \(s_r = 1\). In addition, since \(-1 < x_{1r} < 0\), we can assume \(x_{1r} = \frac{p}{q}\) for some \(p_1, q_1 \in \mathbb{Z}\) with \(p_1 < 0, q_1 > 0\) and \(p_1 + q_1 > 0\). For any large positive \(N\), define \(f_i = -\mathbb{I}_{B_i}\) for \(1 \leq i \leq 3\), where
\[
B_1 = \{(k_1, \tau_1) : k_1 = p_1 N, \ |\tau_1 - \alpha_1 (p_1 N)^3| \leq 1\},
B_2 = \{(k_2, \tau_2) : k_2 = q_1 N, \ |\tau_2 - \alpha_2 (q_1 N)^3| \leq 1\},
B_3 = \{(k_3, \tau_3) : k_3 = -(p_1 + q_1) N, \ |\tau_3 + \alpha_1 (p_1 N)^3 + \alpha_2 (q_1 N)^3| \leq 2\}.
\]
Then \(B_1 + B_2 \subseteq -B_3\). In addition, for any \((k_i, \tau_i) \in B_i, 1 \leq i \leq 3, \langle L_1 \rangle \sim 1, \langle L_2 \rangle \sim 1\) and \(k_2^i = \frac{p_1}{q_1} = x_{1r}\). As a result,
\[
H_2(k_1, k_2, k_3) = -3 \alpha_1 k_3^3 \left(\frac{k_1}{k_2} - x_{1r}\right) \left(\frac{k_1}{k_2} - x_{2r}\right) = 0,
\]
which implies \((L_3) \sim 1\). Then it follows from (7.2) that
\[
\frac{N^{1+s}}{N^{2s}N^{3(1-b)}} \int_A \prod_{i=1}^3 \mathbb{I}_{B_i}(k_i, \tau_i) \lesssim C \prod_{i=1}^3 |B_i|^{1/2}.
\]
Noticing \(|B_i| \sim 1\), applying Lemma 7.1 leads to \(s \geq 1\), which contradicts to \(s < s_r = 1\).

**Case 2.** \(\sqrt{12r} - 3 \notin \mathbb{Q}\).
Since \(s < s_r\) by the assumption, there exists \(\varepsilon > 0\) such that \(s + \varepsilon < s_r\). Recalling \(\mu(x_{1r}) = \sigma_r\), so \(x_{1r}\) is approximable with power \(\sigma_r - \varepsilon_0\). Hence, it follows from Lemma 3.2 that there exists a sequence \(\{(m_j, n_j)\}_{j=1}^\infty \subset \mathbb{Z} \times \mathbb{Z}^*\) such that \(n_j > 0, \lim_{j \to \infty} n_j = \infty\) and
\[
0 < \left|\frac{m_j}{n_j} - x_{1r}\right| < \frac{1}{n_j^{\sigma_r - \varepsilon_0}}.
\]
When \(j\) is large enough, \(\frac{m_j}{n_j}\) will be very close to \(x_{1r}\). Since \(-1 < x_{1r} < 0\), it implies \(m_j < 0, m_j + n_j > 0\) and \(m_j \sim m_i + n_j \sim n_j\). In addition, \(\left|\frac{m_j}{n_j} - x_{2r}\right| \approx |x_{1r} - x_{2r}|\) which is a positive constant only depending on \(r\). As a result,
\[
\left|H_2(m_j, n_j, -(m_j + n_j))\right| = \left|3 \alpha_1 n_j^3 \left(\frac{m_j}{n_j} - x_{1r}\right) \left(\frac{m_j}{n_j} - x_{2r}\right)\right| \lesssim n_j^{3-\sigma_r + \varepsilon_0}.
\]
Denote

\[ \zeta_r = \max\{\epsilon_0, 3 - \sigma_r + \epsilon_0\}. \tag{7.7} \]

Then no matter \( \sigma_r < 3 \) or \( \sigma_r \geq 3 \), it always holds that

\[ \langle H_2(m_j, n_j, -(m_j + n_j)) \rangle = 1 + |H_2(m_j, n_j, -(m_j + n_j))| \lesssim n_j^{\zeta_r}. \tag{7.8} \]

- For any large \( j \) as in the above discussion, define \( f_i = \mathbb{1}_{B_i} \) for \( 1 \leq i \leq 3 \), where

\[
\begin{align*}
B_1 &= \{(k_1, \tau_1) : k_1 = m_j, \quad |\tau_1 - \alpha_1 m_j^3| \leq 1\}, \\
B_2 &= \{(k_2, \tau_2) : k_2 = n_j, \quad |\tau_2 - \alpha_2 n_j^3| \leq 1\}, \\
B_3 &= \{(k_3, \tau_3) : k_3 = -(m_j + n_j), \quad |\tau_3 + \alpha_1 m_j^3 + \alpha_2 n_j^3| \leq 2\}.
\end{align*}
\]

Then \( B_1 + B_2 \subseteq -B_3 \). In addition, for any \((k_i, \tau_i) \in B_i, 1 \leq i \leq 3, (L_1) \sim 1, (L_2) \sim 1\) and

\[ \langle H_2(k_1, k_2, k_3) \rangle = \langle H_2(m_j, n_j, -(m_j + n_j)) \rangle \lesssim n_j^{\zeta_r}, \]

which implies \( \langle L_3 \rangle \lesssim n_j^{\zeta_r} \). Then it follows from (7.2) and \( |m_j| \sim m_j + n_j \sim n_j \) that

\[
\frac{n_j^{1+s}}{n_j^{2s(1-b)}} \prod_{i=1}^{3} \mathbb{1}_{B_i}(k_i, \tau_i) \lesssim C \prod_{i=1}^{3} |B_i|^\frac{1}{2}.
\]

Noticing \( |B_i| \sim 1 \), applying Lemma 7.1 leads to

\[ 1 - s - \zeta_r(1 - b) \leq 0. \tag{7.9} \]

- On the other hand, define \( f_i = \mathbb{1}_{B_i} \) for \( 1 \leq i \leq 3 \), where

\[
\begin{align*}
B_1 &= \{(k_1, \tau_1) : k_1 = m_j, \quad |\tau_1 - \alpha_1 m_j^3| \leq 1\}, \\
B_2 &= \{(k_2, \tau_2) : k_2 = n_j, \quad |\tau_2 + \alpha_1 m_j^3| \leq 1\}, \\
B_3 &= \{(k_3, \tau_3) : k_3 = -(m_j + n_j), \quad |\tau_3 + \alpha_1 m_j^3 + \alpha_2 n_j^3| \leq 2\}.
\end{align*}
\]

Then \( B_1 + B_3 \subseteq -B_2 \). In addition, by applying similar arguments, we conclude that

\[ 1 - s - \zeta_r b \leq 0. \tag{7.10} \]

Adding (7.9) and (7.10) together yields \( s \geq 1 - \frac{\zeta_r}{2} \). Recalling \( s + \epsilon_0 < s_r \), so

\[ 1 - \frac{\zeta_r}{2} \leq s < s_r - \epsilon_0. \tag{7.11} \]

If \( \sigma_r \geq 3 \), then \( s_r = 1 \) and \( \zeta_r = \epsilon_0 \), which contradicts to (7.11). If \( 2 \leq \sigma_r < 3 \), then \( s_r = \frac{\sigma_r - 1}{2} \) and \( \zeta_r = 3 - \sigma_r + \epsilon_0 \), which also contradicts to (7.11).

\[ \square \]

### A Well-posedness for the Hirota-Satsuma systems (1.5)

In this appendix, we summarize the analytical well-posedness results on the Hirota-Satsuma systems (1.5).

**Theorem A.1.** The Hirota-Satsuma system (1.5) is A-LWP in \( \mathcal{H}^2 \) if one of the following conditions is satisfied.

1. \( a_1 \in (-\infty, \frac{1}{4}) \setminus \{0\} \) and \( s \geq -\frac{1}{4} \);
(2) $a_1 = 1$, $c_{12} = 0$ and $s \geq \frac{1}{2}$;

(3) $a_1 \in \left[\frac{1}{4}, \infty \right) \setminus \{1\}$, and $s \geq 1$ or $s > s_{a_1}$ (equivalently $s \geq \min\{1, s_{a_1} + 1\}$).

Due to the following conserved energies for (1.5), the A-GWP follows directly from Theorem A.1.

$$E_1(u, v) = \int u^2 + \frac{c_{12}}{3} v^2 \, dx,$$

$$E_2(u, v) = \int \left(1 - a_1 \right) u_x^2 + c_{12} v_x^2 - 2(1 - a_1) u^3 - c_{12} uv^2 \, dx.$$ (A.1)

**Theorem A.2.** Let $c_{12} > 0$. Then the Hirota-Satsuma system (1.5) is A-GWP in $H^s_2$ if one of the following conditions is satisfied.

(1) $a_1 \in (-\infty, \frac{1}{4}) \setminus \{0\}$ and $s \geq 0$;

(2) $a_1 \in [\frac{1}{4}, 1)$ and $s \geq 1$.

**References**

[1] B. Alvarez and X. Carvajal. On the local well-posedness for some systems of coupled KdV equations. *Nonlinear Anal.*, 69(2):692–715, 2008.

[2] J. Angulo. Stability of cnoidal waves to Hirota-Satsuma systems. *Mat. Contemp.*, 27:189–223, 2004.

[3] J. Angulo. Stability of dnoidal waves to Hirota-Satsuma system. *Differential Integral Equations*, 18(6):611–645, 2005.

[4] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*. Fundamental Principles of Mathematical Sciences, vol.250. Springer-Verlag, New York, second edition, 1988.

[5] V. Becher, Y. Bugeaud, and T. A. Slaman. The irrationality exponents of computable numbers. *Proc. Amer. Math. Soc.*, 144(4):1509–1521, 2016.

[6] J. L. Bona, G. Ponce, J.-C. Saut, and M. M. Tom. A model system for strong interaction between internal solitary waves. *Comm. Math. Phys.*, 143(2):287–313, 1992.

[7] J. L. Bona and R. Smith. The initial-value problem for the Korteweg-de Vries equation. *Philos. Trans. Roy. Soc. London Ser. A*, 278(1287):555–601, 1975.

[8] J. L. Bona, S.-M. Sun, and B.-Y. Zhang. Conditional and unconditional well-posedness for nonlinear evolution equations. *Adv. Differential Equations*, 9(3-4):241–265, 2004.

[9] J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation. *Geom. Funct. Anal.*, 3(3):209–262, 1993.

[10] Y. Bugeaud. *Approximation by algebraic numbers*, volume 160 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2004.

[11] X. Carvajal and M. Panthee. Sharp well-posedness for a coupled system of mKdV-type equations. *J. Evol. Equ.*, 19(4):1167–1197, 2019.

[12] Michael Christ, James Colliander, and Terrence Tao. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. *Amer. J. Math.*, 125(6):1235–1293, 2003.

[13] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and T. *J. Amer. Math. Soc.*, 16(3):705–749, 2003.

[14] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Multilinear estimates for periodic KdV equations, and applications. *J. Funct. Anal.*, 211(1):173–218, 2004.

[15] P. Constantin and J.-C. Saut. Local smoothing properties of dispersive equations. *J. Amer. Math. Soc.*, 1(2):413–439, 1988.

[16] J. A. Gear and R. Grimshaw. Weak and strong interactions between internal solitary waves. *Stud. Appl. Math.*, 70(3):235–258, 1984.

[17] Z. Guo. Global well-posedness of Korteweg-de Vries equation in $H^{-3/4}(\mathbb{R})$. *J. Math. Pures Appl.* (9), 91(6):583–597, 2009.
[18] R. Hirota and J. Satsuma. Soliton solutions of a coupled Korteweg-de Vries equation. *Phys. Lett. A*, 85(8-9):407–408, 1981.

[19] V. Jarnik. Zur metrischen theorie der diophantischen approximationen. *Prace Mat.- Fiz.*, 36:91–106, 1928/1929.

[20] V. Jarnik. Über die simultanen diophantischen Approximationen. *Math. Z.*, 33(1):505–543, 1931.

[21] T. Kappeler and P. Topalov. Global wellposedness of KdV in $H^{-1}(T, \mathbb{R})$. *Duke Math. J.*, 135(2):327–360, 2006.

[22] T. Kato. Quasi-linear equations of evolution, with applications to partial differential equations. pages 25–70. Lecture Notes in Math., Vol. 448, 1975.

[23] T. Kato. On the Korteweg-de Vries equation. *Manuscripta Math.*, 28(1-3):89–99, 1979.

[24] T. Kato. The Cauchy problem for the Korteweg-de Vries equation. In *Nonlinear partial differential equations and their applications.*, volume 53 of *Res. Notes in Math.*, pages 293–307. Pitman, Boston, Mass.-London, 1981.

[25] T. Kato. On the Cauchy problem for the (generalized) Korteweg-de Vries equation. In *Studies in applied mathematics*, volume 8 of *Adv. Math. Suppl. Stud.*, pages 93–128. Academic Press, New York, 1983.

[26] T. Kato. On nonlinear Schrödinger equations. II. $H^s$-solutions and unconditional well-posedness. *J. Anal. Math.*, 67:281–306, 1995.

[27] C. E. Kenig, G. Ponce, and L. Vega. On the (generalized) Korteweg-de Vries equation. *Duke Math. J.*, 59(3):585–610, 1989.

[28] C. E. Kenig, G. Ponce, and L. Vega. Oscillatory integrals and regularity of dispersive equations. *Indiana Univ. Math. J.*, 40(1):33–69, 1991.

[29] C. E. Kenig, G. Ponce, and L. Vega. Well-posedness of the initial value problem for the Korteweg-de Vries equation. *J. Amer. Math. Soc.*, 4(2):323–347, 1991.

[30] C. E. Kenig, G. Ponce, and L. Vega. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Comm. Pure Appl. Math.*, 46(4):527–620, 1993.

[31] C. E. Kenig, G. Ponce, and L. Vega. A bilinear estimate with applications to the KdV equation. *J. Amer. Math. Soc.*, 9(2):573–603, 1996.

[32] A. Khintchine. Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. *Math. Ann.*, 92(1-2):115–125, 1924.

[33] Rowan Killip and Monica Vişan. KdV is well-posed in $H^{-1}$. *Ann. of Math. (2)*, 190(1):249–305, 2019.

[34] N. Kishimoto. Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity. *Differential Integral Equations*, 22(5-6):447–464, 2009.

[35] S. Lang. *Introduction to Diophantine approximations*. Springer-Verlag, New York, second edition, 1995.

[36] A. J. Majda and J. A. Biello. The nonlinear interaction of barotropic and equatorial baroclinic Rossby waves. *J. Atmospheric Sci.*, 60(15):1809–1821, 2003.

[37] L. Molinet. A note on ill posedness for the KdV equation. *Differential Integral Equations*, 24(7-8):759–765, 2011.

[38] L. Molinet. Sharp ill-posedness results for the KdV and mKdV equations on the torus. *Adv. Math.*, 230(4-6):1895–1930, 2012.

[39] T. Oh. Well-posedness theory of a one parameter family of coupled KdV-type systems and their invariant Gibbs measures. ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–University of Massachusetts Amherst.

[40] T. Oh. Diophantine conditions in well-posedness theory of coupled KdV-type systems: local theory. *Int. Math. Res. Not.*, (18):3516–3566, 2009.

[41] T. Oh. Periodic L4-Strichartz estimate for KdV. unpublished online note.

[42] K. F. Roth. Rational approximations to algebraic numbers. *Mathematika*, 2:1–20, 1955.

[43] C. Siegel. Approximation algebraischer Zahlen. *Math. Z.*, 10(3-4):173–213, 1921.
[44] A. Sjöberg. On the Korteweg-de Vries equation: existence and uniqueness. Department of Computer Sciences, Uppsala University, Uppsala, Sweden, 1967. 3

[45] A. Sjöberg. On the Korteweg-de Vries equation: existence and uniqueness. J. Math. Anal. Appl., 29:569–579, 1970. 4

[46] T. Tao. Multilinear weighted convolution of $L^2$-functions, and applications to nonlinear dispersive equations. Amer. J. Math., 123(5):839–908, 2001. 3 11 16 17 20

[47] T. Tao. Nonlinear Dispersive Equations: Local and Global Analysis. CBMS Regional Conference Series in Mathematics, vol.106. Published by the American Mathematical Society, Providence, RI, 2006. 16

[48] A. Thue. Über Annäherungswerte algebraischer Zahlen. J. Reine Angew. Math., 135:284–305, 1909. 12

[49] X. Yang and B.-Y. Zhang. Local well-posedness of the KdV-KdV systems on $\mathbb{R}$. Submitted, arXiv:1812.08261. 3 4 5

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