Remarks on excited states of affine Toda solitons

G.M. Gandenberger*
N. J. MacKay†

Dept of Applied Maths and Theoretical Physics,
Cambridge University,
Cambridge, CB3 9EW,
England

ABSTRACT

The identification in affine Toda field theory of the quantum particle with the lowest breather allows us to re-interpret discrete modes of excitation of solitons as breathers bound to solitons, and thus to investigate them through the proposed soliton-breather S-matrices. There are implications for the physical spectrum and for the semiclassical soliton mass corrections.

1 Introduction

In a recent series of papers (to which the reader is referred for background and notation) we have constructed S-matrices invariant under a variety of quantized affine algebras, and have conjectured that they describe quantum affine Toda solitons and their bound states. This is a strong conjecture to make, because the Hamiltonian of the theory is complex, and although the solitons have real energies, momenta and higher-spin conserved charges, the quantum theory would appear non-unitary. Further, classically the solitons may be unstable, and have topological charges which do not fill the relevant multiplets. It is quite possible that only in some suitably restricted sector can the classical solitons be

*G.M.Gandenberger@damtp.cam.ac.uk.
†N.J.Mackay@damtp.cam.ac.uk. Stokes Fellow, Pembroke College
quantized, or our exact $S$-matrices describe them. Nevertheless there is much commonality of structure, and we proceed on the assumption that the spectrum of affine Toda theory (with its solitons, excited solitons, breathers and particles) is what the $S$-matrices describe, which leads in each case to the identification of the lowest scalar soliton bound state or ‘breather’ with the quantum particle. In this letter we examine the implications for the semiclassical discussion of the solitons.

In the semiclassical calculation of the soliton masses, the zero-point energies of all the modes of vibration of the solitons were summed. These included a set of discrete modes, which could be regarded as a particle bound to a soliton. Some of these discrete modes are linearizations of the excited solitons present in the spectrum. Should the rest also be identified with physical states, to be invited to the bootstrap? If the particle is identical to the lowest breather, we can attempt to provide an answer through the pole structure of the soliton-breather $S$-matrices. We describe this in the next section.

The semiclassical mass calculations had unresolved problems for certain solitons for nonsimply-laced untwisted algebras. Specifically, there was a multiplicity in the zero of the transmission coefficient of a wave incident on a soliton and thus in the frequency of the associated discrete mode. Had these been counted with multiplicity, the masses would then have agreed with the soliton $S$-matrices; yet despite several attempts only one such mode for each eigenvalue could be explicitly constructed. In the soliton-breather $S$-matrices, however, the frequencies of the discrete modes can be read off exactly, and we find that the degeneracies are split by higher-order contributions in the coupling constant, so that the full multiplicity of modes must therefore exist. We describe this in the last section.

2 Discrete modes and $S$-matrix poles

Suppose there is a discrete mode solution to the equation of motion for a small perturbation to a soliton of type $a$, and that this solution takes the asymptotic form of a particle of type $b$ at $x \to -\infty$, and has frequency $\nu = m_b \sin \phi$. Now let us identify the particle with the lowest breather state $B^{(b)}_1$ and denote the soliton of type $a$ as $A^{(a)}$. If $S_{B^{(b)}_1 A^{(a)}}$ has a pole at rapidity difference $\theta = \frac{i\pi}{2} - i\phi + \mathcal{O}(\beta^2)$ (where $\beta$ is the coupling constant) then the mass of the intermediate state is $m_b \sin \phi + \mathcal{O}(\beta^2)$ more than the mass of the original soliton, and we might expect the discrete modes to coincide with such poles.
For $d_{n+1}^{(2)}$ this is indeed what we find. The possibility of constructing a classical bound state is signalled by a zero in the transmission coefficient for a wave scattering off a soliton, and comparing the expressions for the transmission coefficients (which are the phase factors $X_{a b}(\eta)$ for solitons of rapidity difference $\eta$) with those for the S-matrices, we find that for each zero/pole in $X_{a b}(i \phi)$, there is precisely one pole/zero (plus its cross-channel partner) in $S_{B_2^{(1)} A_2^{(1)}}^{(c)} \left( \frac{i \pi}{2} - i \phi + \mathcal{O}(\beta^2) \right)$.\footnote{A more detailed discussion of the $d_3^{(2)}$ pole structure can be found in \cite{11}.}

Thus all the discrete modes (including those putative bound states whose existence was forbidden classically by their bad asymptotic behaviour at $x \to \infty$) are contained among, though generally they by no means exhaust, the relevant (i.e. $0 < \phi < \frac{\pi}{2}$) simple poles in the S-matrix. Among the poles associated with the allowed classical bound states, in turn, are found those already included in the bootstrap as excited solitons. (Recall that a pole’s being simple is not alone enough for its inclusion as a physical state: a generalized Coleman-Thun mechanism can lead to on-shell higher-order diagrams which nevertheless correspond to simple poles in the S-matrix.)

Consider, as an example, the $d_3^{(2)}$ theory. There are two species. The transmission coefficients are

$$
X_{11}(\eta) = \{1\}\{-1/2\}
$$

$$
X_{12}(\eta) = X_{21}(\eta) = \{\sqrt{3}/2\}
$$

$$
X_{22}(\eta) = \{1\}\{1/2\}
$$

where

$$\{a\} = \frac{\cosh \eta - a}{\cosh \eta + a},$$

and the condition for a zero in $X$ to give a normalizable bound state is that $m_b \cos \phi < m_a$. Thus there are two allowed discrete modes, for $a = 2$ and $b = 1, 2$, whilst that for $a = 1$ and $b = 2$ is disallowed.

The necessary S-matrices are

$$
S_{B_1^{(1)} A_1^{(1)}}(\theta) = \left( \frac{\omega}{2} \right) \left( \frac{-5\omega}{2} - 1 \right) \left( \frac{3\omega}{2} \right) \left( \frac{3\omega}{2} + 1 \right)
$$

\footnote{This is in contrast to soliton-soliton scattering, where, at least for $d_3^{(2)}$, the correspondence is between the zeros/poles in $X_{a b}(i \phi)$ and poles/zeros in $S_{A_2^{(a)} A_2^{(b)}}(i \phi)$.}
\[ S_{B_1^{(1)}A^{(2)}}(\theta) = S_{B_1^{(2)}A^{(1)}}(\theta) = (2\omega)(\omega + 1) \]

\[ S_{B_1^{(2)}A^{(2)}}(\theta) = \left( \frac{5\omega}{2} \right) \left( \frac{\omega}{2} + 1 \right) \left( \frac{3\omega}{2} \right) \left( \frac{3\omega}{2} + 1 \right), \]

in which we have used the notation
\[ (y) \equiv \frac{\sin \left( \frac{\pi}{h\lambda}(\mu + y) \right)}{\sin \left( \frac{\pi}{h\lambda}(\mu - y) \right)}, \quad \lambda \equiv \frac{4\pi}{\beta^2} - \frac{4}{3}, \quad \omega \equiv \frac{2\pi}{\beta^2} - 1, \quad \mu \equiv -i \frac{h\lambda}{2\pi} \theta \quad \text{and} \quad h = 3, \]

and we see that for each \( \{\sin(a\pi/6)\} \) in \( X \) we find one factor \( (a\omega/2 + \mathcal{O}(1)) \), and its cross channel partner, in \( S \).

Of the relevant poles, first note that those at \( \mu = \frac{3\omega}{2} + 1 \), in \( S_{B_1^{(1)}A^{(1)}} \) and \( S_{B_1^{(2)}A^{(2)}} \), correspond to the fusion into elementary solitons \( A^{(1)} \) and \( A^{(2)} \) respectively, whilst the poles at \( \mu = \frac{3\omega}{2} \) are their cross channel partners. Classically they correspond to zero modes, at \( \phi = 0 \). Of the poles at \( \mu = \omega + 1 \), that in \( S_{B_1^{(1)}A^{(2)}} \) corresponds to the excited soliton \( A_1^{(2)} \), whilst that in \( S_{B_1^{(2)}A^{(1)}} \) is classically the discrete mode disallowed by its bad asymptotic behaviour. In the quantum theory this pole was not included in the bootstrap but could be explained by the following higher-loop-order diagram, in which the S-matrix denoted by the black dot has a simple zero:

The interesting pole is that at \( \mu = \frac{\omega}{2} + 1 \) in \( S_{B_1^{(2)}A^{(2)}} \), since this corresponds to an allowed classical bound state and yet was not included in the bootstrap as a physical state, but was explained by the higher-order diagram shown below. (The pole at \( \mu = \frac{3\omega}{2} \) corresponds to the same process in the cross channel, as indicated by the arrows in the diagram.)
Of course, this does not forbid the pole’s inclusion in the bootstrap; it merely shows it not to be necessary. In the absence of a way of summing diagrams and relating them to S-matrix residues, as is possible for the particle S-matrices\cite{12}, we cannot prove that these poles should not be invited to the bootstrap. Nevertheless we believe that they are not, and that the spectrum given previously\cite{2} is complete. Poles in our S-matrices seem to correspond to precisely one on-shell diagram: for example, in those poles which are invited to the bootstrap we do not find further higher-order diagrams to confuse the issue.

In the above diagram we note that all the internal states are expected to be quantized forms of classical solutions. The classical bound state might therefore be expected to be the linearized form of the classical solution at any time during the process \(i.e.\) corresponding to any vertical slice through the diagram. Presumably the same then applies to the pole considered on the previous page, at \(\mu = \omega + 1\) in \(S_{B_1^{(2)} A_1^{(1)}}\), for which the classical mode is disallowed. It is not clear what distinguishes the two at S-matrix level.

### 3 Lifting of degeneracy in discrete modes

The theories based on untwisted nonsimply-laced algebras are more complicated in several ways. The construction of S-matrices was only possible relatively recently, with the advent of solutions of the Yang-Baxter equation for twisted algebras\cite{13,14}, and there are outstanding problems with the semiclassical soliton masses.

In such theories the solitons are obtained by ‘folding’ the Dynkin diagram of a (‘parent’) simply-laced theory. Most of the solitons have single parents, but some are obtained by
folding parent multisolitons, and (solely) in the case of particles traversing such multisoli-
tons, the zero in $X$ occurs with multiplicity. However, the corresponding multiplicity of
classical solutions could not be found, so the contributions were counted only once. As
pointed out in the notes to the original papers[33], the full multiplicity is needed for agree-
ment with the soliton mass ratios predicted by the exact S-matrices, and somehow the
degeneracy has to be broken.

Using the breather-particle correspondence, we can check whether this happens, since
the S-matrix poles are exact to all orders in $\beta^2$. Let us take as our example the $b_n^{(1)}$ theory,
where the $n$th soliton is folded from the $n$th and $n+1$th solitons in the $d_{n+1}^{(1)}$ theory. The
transmission coefficient for particle $a$ through soliton $n$ is
$$\{ \sin(\frac{a\pi}{2n}) \}^2,$$
and the mode
with frequency $m_a \cos(\frac{a\pi}{2n})$ thus appears with a problematic double zero.

The relevant S-matrix is
$$S_{B_n^{(a)}} = \prod_{k=1}^a \left( \frac{\omega}{2}(2k-a) + \frac{3}{4} \right) \left( \frac{\omega}{2}(2n-2k+a) - \frac{1}{4} \right) \times \left( \frac{\omega}{2}(2-2k+a) + \frac{1}{4} \right) \left( \frac{\omega}{2}(2n-2+2k-a) + \frac{1}{4} \right),$$
where now
$$\lambda \equiv \frac{4\pi}{\beta^2} - \frac{2n-1}{2n}, \quad \omega \equiv \frac{4\pi}{\beta^2} - 1, \quad \mu = -i\frac{\lambda}{2\pi} \theta \quad \text{and} \quad h = 2n,$$
and we wish to examine the poles at $\theta = \frac{a\pi}{2n} + \mathcal{O}(\beta^2)$. What we find is not a double pole
but two closely separated simple poles, at $\mu = \frac{a}{2} + \frac{1}{4}$ and $\mu = \frac{a}{2} + \frac{3}{4}$ (which for general $a$
we expect to correspond to higher-order diagrams). Thus we expect that the frequencies,
degenerate at $\mathcal{O}(1)$, are separated at next order and should therefore be counted with
multiplicity.

This does not settle the question of the validity of the semiclassical approximation; it
merely shows the S-matrix and our interpretation of it to be self-consistent. We still
do not know whether the correct multiplicity of distinct solutions can be constructed
semiclassically[4] or whether instead some demisemiclassical method is required, producing
solutions indistinguishable at semiclassical order. Nor, if some such method were to be

---

4Some effort has failed to produce them[4], but not yet enough for us to suggest that they
do not exist. In particular, our boundary conditions may have been too constraining, with the
problematic cases being especially sensitive to the regularizing of the system by its being placed
in a box.
used, could we yet characterize from S-matrix information alone which poles correspond
to discrete modes, or whether they have allowed asymptotic behaviour.

Acknowledgments

NJM would like to thank Robert Weston and Gérard Watts for useful conversations,
and Gérard Watts for a critical reading of the manuscript. GMG acknowledges financial
support from the Cambridge Kurt Hahn Trust.

References

[1] G.M. Gandenberger, Exact S-matrices for bound states of $a_2^{(1)}$ affine Toda solitons,
Nucl. Phys. B449 (1995), 375, hep-th/9501136

[2] G.M. Gandenberger and N.J. MacKay, Exact S-matrices for $d_n^{(2)}$ affine Toda solitons
and their bound states, Nucl. Phys. B457 (1995), 240, hep-th/9506163

[3] G.M. Gandenberger, N.J. MacKay and G.M.T. Watts, Twisted algebra R-matrices
and exact S-matrices for $b_n^{(1)}$ affine Toda solitons and their bound states, Nucl. Phys.
B465 (1996), 329, hep-th/9509007

[4] T.J. Hollowood, Solitons in affine Toda field theory, Nucl. Phys. B384 (1992), 523;
Quantizing $sl(N)$ solitons and the Hecke algebra, Int.Jour.Mod.Phys. A8 No.5 (1993),
947, hep-th/9203076

[5] D.I. Olive, N. Turok and J.W.R. Underwood, Solitons and the energy-momentum
tensor for affine Toda theory, Nucl.Phys. B401 (1993), 663; and Affine Toda solitons
and vertex operators, Nucl. Phys. B409 (1993) [FS], 509, hep-th/9305160.
M. Freeman, Conserved charges and soliton solutions in affine Toda theory, Nucl.
Phys. B433 (1995), 657, hep-th/9408092

[6] S. Pratik Khastgir and R. Sasaki, Instability of solitons in imaginary-coupling affine
Toda field theory, Prog. Theor. Phys. 95 (1996),485, hep-th/9507001

[7] T.J. Hollowood, Quantum soliton mass corrections in $sl(n)$ affine Toda field theory,
Phys. Lett. B300 (1993), 73. hep-th/9209024
[8] G.W. Delius and M.T. Grisaru, *Toda soliton mass corrections and the particle-soliton duality conjecture*, Nucl. Phys. B441 (1995), 259, hep-th/9411176

[9] N.J. MacKay and G.M.T. Watts, *Quantum mass corrections for affine Toda solitons*, Nucl. Phys. B441 (1995), 277, hep-th/9411169.

  G.M.T. Watts, *Quantum mass corrections for $c_2^{(1)}$ affine Toda solitons*, Phys. Lett. B338 (1994), 40, hep-th/9404065

[10] A.Fring, P.R. Johnson, M.A.C.Kneipp and D.I. Olive, *Vertex operators and soliton time delays in affine Toda field theory*, Nucl. Phys. B430 (1994), 597, hep-th/9405034

[11] G.M. Gandenberger, *Exact $S$-matrices for quantum affine Toda solitons and their bound states*, Ph.D. thesis, Cambridge University 1996, unpublished

[12] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Affine Toda field theory and exact $S$-matrices*, Nucl. Phys. B338 (1990), 689; and *Multiple poles and other features of affine Toda field theory*, Nucl. Phys. B356 (1991), 469

[13] G.W. Delius, M.D. Gould and Y.-Z. Zhang, *Twisted quantum affine algebras and solutions to the Yang–Baxter equation*, Int. J. Mod. Phys. A11 (1996), 3415, q-alg/9508012

[14] N.J. MacKay and G.M.T. Watts, *unpublished*