On the Difference between Consecutive Primes and Estimates of the Number of Primes in the Interval \((n, 2n)\)

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Abstract

Using evaluations of the difference between consecutive primes we develop another way of estimating the number of primes in the interval \((n, 2n)\). We also discuss the ultra Cramer conjecture, \(p_{n+1} - p_n = O(\log^{1+\epsilon} p_n)\) where \(\epsilon > 0\), in the context of the results we have obtained in our paper.

1 Introduction

The difference between consecutive primes is an important characteristic of the distribution of the prime numbers [3]. However, as it will be illustrated, it is also closely linked with the estimates of the number of primes in the interval \((n, 2n)\). Using the best available evaluation \(p_{n+1} - p_n = O(p_n^{0.525})\) [1] and also the hypothetical evaluations of the difference between consecutive primes \(p_{n+1} - p_n = O(\sqrt{p_n}), p_{n+1} - p_n = O(\ln^2(p_n))\) [2] we develop another way evaluating the number of primes in the interval \((n, 2n)\).

2 Estimates of the number of primes in the interval \((n, 2n)\)

There is the well-known estimate for the number of the prime numbers in an interval \((n, 2n)\) [6]:

\[
\frac{1}{3} \frac{n}{\log(2n)} < \pi(2n) - \pi(n) < \frac{7}{5} \frac{n}{\log(n)} \text{ where } n > 1
\]

(1)

The left side is a lower bound for the number of primes within \((n, 2n)\), and the right side is the upper bound.

**Proposition 2.1.** Let \(k, N_k\) be such that for every prime \(p_m \geq N_k\), \(\sqrt{p_m} - \sqrt{p_{m-1}} < \frac{1}{k}\) is true. Then the interval \((n, 2n)\) contains no less than \(\lfloor (\frac{k}{4}) \sqrt{2n} \rfloor\) primes for every \(n > N_k\).

**Lemma 2.2.** Let an integer \(k\) be such that the condition of proposition [2] is satisfied. Then the interval \((\frac{p_n}{2}, p_n)\), for every \(p_n\) where \(\frac{p_n}{2} > N_k\), contains no less than \(\lfloor (\frac{k}{4}) \sqrt{p_n} \rfloor\) prime numbers.
Corollary 2.4. Let \( p_n \) be such a prime, \( \frac{2}{\sqrt{2}} > N_k \), that \( \left( \frac{2}{\sqrt{2}}, p_n \right) \) contains less than \( \left\lfloor \left( \frac{k}{2} \right) \sqrt{p_n} \right\rfloor \) primes. Let \( \{p_i\} \) be the set of all primes inside \( \left( \frac{2}{\sqrt{2}}, p_n \right) \). Let \( \left( \frac{2}{\sqrt{2}}, p_n \right) = \bigcup_{j=1}^{\|p_i\|} I_j = \left( \frac{2}{\sqrt{2}}, p_1 \right) \cup \left( \cup_{j=1}^{\|p_i\|} (p_j, p_{j+1}) \right) \).

Some \( I_i \) has length no less than \( 2 \sqrt{p_n} < \frac{(p_0 - m_n)}{\left\lfloor \left( \frac{k}{2} \right) \sqrt{p_n} \right\rfloor} < p_i - p_{i-1} \). This contradicts the condition \( 2.1 \) since \( p_i - p_{i-1} < 2 \sqrt{p_n} < \frac{2 \sqrt{p_n}}{k} \). Thus lemma \( 2.2 \) is true. \( \square \)

Proof of proposition 2.1 Let the condition of proposition 2.1 be satisfied. Let \( n_0 > N_1 \) be such that the interval \( (n_0, 2n_0) \) contains less than \( \left\lfloor \left( \frac{k}{2} \right) \sqrt{2n_0} \right\rfloor \) primes. Let \( p_{n-1}, p_n \) be such that \( p_{n-1} < 2n_0 < p_n \). Then \( \left( \frac{2}{\sqrt{2}}, p_n \right) \) contains primes less than \( \left\lfloor \left( \frac{k}{2} \right) \sqrt{p_n} \right\rfloor \). Indeed, the interval \( (n_0, p_n) = (n_0, 2n_0) \cup [2n_0] \cup (2n_0, p_n) \) contains primes less than \( \left\lfloor \left( \frac{k}{2} \right) \sqrt{2n_0} \right\rfloor \). Furthermore \( \left( \frac{2}{\sqrt{2}}, p_n \right) \subset (n_0, p_n) \) as \( \frac{2}{\sqrt{2}} > n_0 \) so \( \left( \frac{2}{\sqrt{2}}, p_n \right) \) contains primes less than \( \left\lfloor \left( \frac{k}{2} \right) \sqrt{p_n} \right\rfloor \), contradicting lemma 2.2. \( \square \)

Corollary 2.3 (Bertrand’s Postulate). Let \( \sqrt{p_m} - \sqrt{p_{m-1}} < 1 \) be satisfied for every integer \( m \geq 2 \). Then \( (n, 2n - 2) \) contains no less than two primes for every integer \( n \geq 8 \).

Proof. According to proposition 2.1 where \( k = 1, N_1 = 2 \) an interval \( (n, 2n) \) contains no less than three prime numbers for every integer \( n \geq 72 \). Corollary 2.3 is true for all values of \( n \) no less than 72; by direct verification we find that it is true for smaller values. Thus corollary 2.3 is true for \( n \geq 8 \). \( \square \)

Corollary 2.4. Let \( \sqrt{p_m} - \sqrt{p_{m-1}} < \frac{1}{2} \) be satisfied for every integer \( m \geq 32 \). Then an interval \( (n, 2n) \) contains no less than \( \left\lfloor \frac{1}{2} \sqrt{2n} \right\rfloor \) prime numbers for every integer \( n \geq 2 \).

Proof. This is a particular case of proposition 2.1 where \( k = 2, N_2 = 131 \). Since corollary 2.4 is true for all values of \( n \) not less than 131; by direct verification we find that it is true for smaller values. Thus corollary 2.4 is true for \( n \geq 2 \). \( \square \)

The theorem: “An interval \( (n, 2n) \) contains not less than \( \left\lfloor \frac{1}{2} \sqrt{2n} \right\rfloor \) primes for every integer \( n \geq 2 \)” has been proved by H. Karcher using Tschebyschef - Erdos approach [4].

The following statement is based on using Cramer’s conjecture in the form \( p_{n+1} - p_n = O(\ln^2 p_n) \):

Proposition 2.5. Let Cramer’s conjecture be true then there exist such constants \( k, N \) such that for every integer \( n > N \), an interval \( (n, 2n) \) contains no less than \( \left\lfloor \frac{n}{k \log^2 2n} \right\rfloor \) primes.

Lemma 2.6. Let there exist such integers \( k, N \) such that \( p_m - p_{m-1} < k \log^2 p_m \) is true for every \( p_m \geq N \). Then \( \left( \frac{2}{\sqrt{2}}, p_n \right) \) contains no less than \( \left\lfloor \frac{1}{k \log^2 2n} \right\rfloor \) primes for every \( p_n \) where \( \frac{2}{\sqrt{2}} > N \).

Proof. The proof is the same as in lemma 2.2 \( \square \)

Proof of proposition 2.3 Let the constants \( k, N \) of proposition 2.5 such that \( p_m - p_{m-1} < k \log^2 p_m \) is true for every \( p_m \geq N \). However, there is such an integer \( n_0 > N \) that an interval \( (n_0, 2n_0) \) contains primes less than \( \left\lfloor \frac{1}{k \log^2 2n_0} \right\rfloor \). Let \( p_{n-1}, p_n \) be such primes that \( p_{n-1} < 2n_0 < p_n \). Then
\((\frac{np}{2}, p_n)\) contains primes less than \(\lfloor \frac{np}{2 \log p_n} \rfloor\). Since \((n_0, p_n) = (n_0, 2n_0) \cup [2n_0] \cup (2n_0, p_n)\) contains primes less than \(\lfloor \frac{2n_0}{2 \log^2 2n_0} \rfloor\). Furthermore \((\frac{np}{2}, p_n) \subset (n_0, p_n)\) since \(\frac{np}{2} > n_0 > N\) so \((\frac{np}{2}, p_n)\) contains primes less than \(\lfloor \frac{np}{2 \log p_n} \rfloor\), contradicting lemma 2.6. 

We would like to note that Cramer’s conjecture \(p_{n+1} - p_n = O(\ln^2 p_n)\) is consistent with the admissible estimate of the lower bound for the number of primes in the interval \((n, 2n)\), \(\sim \frac{n}{\log 2n}\), and the hypothesis \(\sqrt{p_{n+1}} - \sqrt{p_n} = o(1)\) which has the experimental support [5].

It is surprising that it is impossible to obtain the lower bound for the number of primes in \((n, 2n)\) in the classical form \(\sim \frac{n}{\log 2n}\) by evaluations of the difference of primes. It is a real fact due to E. Westzynthius, \(p_{m+1} - p_m = O(\log p_m)\) is not true. However, after works of P. Erdös and R. Rankin it is expected that for any real \(\epsilon > 0\) the relation \(p_{m+1} - p_m = O(\log^{1+\epsilon} p_m)\) is true (if this is really so?) then the evaluation of the difference between consecutive primes permits to obtain the lower bound as \((\frac{1}{k(\epsilon)})\frac{n}{\log^{1+2\epsilon} 2n}\) where \(\frac{n}{\log^{1+2\epsilon} 2n} = O(\frac{n}{\log 2n})\) under \(\epsilon \to 0\) while \(k(\epsilon) = O(1)\) is not true. The conjecture \(p_{n+1} - p_n = O(\log^{1+\epsilon} p_n)\) is consistent both with the hypothesis \(\sqrt{p_{n+1}} - \sqrt{p_n} = o(1)\) and with the admissible estimate of the lower bound for the number of primes in \((n, 2n)\).

**Proposition 2.7.** There exists a constant \(C\) such that for every integer \(n > C\) the interval \((n, 2n)\) contains not less than \(\lfloor \frac{1}{2}(2n)^{0.475} \rfloor\) prime numbers.

**Proof.** According to [1] “Theorem 1. For all \(x > x_0\), the interval \([x - x_0^{0.525} , x]\) contains a prime number. With enough effort, the value of \(x_0\) could be determined effectively”. We have that if \(p_m > C = x_0\) then \(p_m - p_{m-1} < P_m^{0.525}\). Further the proof goes like in proposition 2.5.

Nowadays the estimate \(\sim n^{0.475}\) of the lower bound for the number of primes in \((n, 2n)\) is obtained by the evaluations of the difference between consecutive primes is the best available result under such an approach.

### 3 Discussion and Conclusions

We have shown that by the evaluations of the difference between consecutive primes one can obtain the estimates of the lower bound for the number of primes in an interval \((n, 2n)\). Nowadays the best available result under such an approach is \(\lfloor \frac{1}{2}(2n)^{0.475} \rfloor\).

Our results permit us to conclude that the relations \(p_{n+1} - p_n = O(\ln^2 p_n)\) (Cramer’s conjecture) and \(p_{n+1} - p_n = O(\log^{1+\epsilon} p_n)\) (ultra Cramer’s conjecture) have real reasons to be valid as they are consistent both with the admissible estimate of the lower bound for a number of primes in \((n, 2n)\) \(\sim \frac{n}{\log 2n}\) and with the conjecture \(\sqrt{p_{n+1}} - \sqrt{p_n} = o(1)\) which has the experimental support [5] and do not conflict with the results of the works of E. Westzynthius, P. Erdös and R. Rankin.
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