A METRIC ON THE MODULI SPACE OF BODIES

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Abstract. We construct a metric on the moduli space of bodies in Euclidean space. The moduli space is defined as the quotient space with respect to the action of integral affine transformations. This moduli space contains a subspace, the moduli space of Delzant polytopes, which can be identified with the moduli space of symplectic toric manifolds. We also discuss related problems.

1. Introduction

An n-dimensional Delzant polytope is a convex polytope in \( \mathbb{R}^n \) which is simple, rational and smooth\(^1\) at each vertex. There exists a natural bijective correspondence between the set of n-dimensional Delzant polytopes and the equivariant isomorphism classes of 2n-dimensional symplectic toric manifolds, which is called the Delzant construction [3]. Motivated by this fact Pelayo-Pires-Ratiu-Sabatini studied in [11] the set of n-dimensional Delzant polytopes\(^2\) \( \mathcal{D}_n \) from the view point of metric geometry. They constructed a metric on \( \mathcal{D}_n \) by using the n-dimensional volume of the symmetric difference. They also studied the moduli space of Delzant polytopes \( \tilde{\mathcal{D}}_n \), which is constructed as a quotient space with respect to natural action of integral affine transformations. It is known that \( \tilde{\mathcal{D}}_n \) corresponds to the set of equivalence classes of symplectic toric manifolds with respect to weak isomorphisms [7], and they call the moduli space the moduli space of toric manifolds. In [11] they showed that, the metric space \( \mathcal{D}_2 \) is path connected, the moduli space \( \tilde{\mathcal{D}}_n \) is neither complete nor locally compact. Here we use the metric topology on \( \mathcal{D}_n \) and the quotient topology on \( \tilde{\mathcal{D}}_n \). They also determined the completion of \( \mathcal{D}_2 \).

Though in [11] they did not consider a metric on \( \tilde{\mathcal{D}}_n \) it is natural to ask whether \( \mathcal{D}_n \) has a natural metric or not. In this paper we give an answer for this question. Our main result is a construction of a metric on the moduli space \( \tilde{\mathcal{D}}_n \). We also show that its metric topology is homeomorphic to the quotient topology. Strictly speaking our construction does not rely on any conditions of the Delzant polytope so

\(^1\)A vertex of a convex polytope in \( \mathbb{R}^n \) is called smooth if directional vectors of edges at the vertex can be chosen as a basis of \( \mathbb{Z}^n \).

\(^2\)In [11] they use the notation \( \mathcal{D}_T \) for the set of Delzant polytopes instead of \( \mathcal{D}_n \).
that it is possible to apply our proof for all bodies, that is, subsets in $\mathbb{R}^n$ obtained as the closure of bounded open subsets. Such a generalization for non-Delzant case has been increasing in importance in recent years in several areas. For example there exists an integrable system on a flag manifold, called the Gelfand-Cetlin system [4], which is studied from several context such as representation theory, algebraic geometry including mirror symmetry and so on (See [1], [5] or [10] for example). The Gelfand-Cetlin system is equipped with a torus action but it is not a symplectic toric manifold, however, it associates a non-simple convex polytope (as a base space of the integrable system), called the Gelfand-
Cetlin polytope. More generally the theory of Newton-Okounkov bodies associated to divisors on varieties is developed recently, and it has great success in representation theory and algebraic geometry, especially for toric degeneration of varieties (See [6], [8] or [9] for example).

This paper is organized as follows. In Section 2 we introduce the moduli space of bodies with respect to the group of integral affine transformations. In Section 3 we construct a metric $\tilde{d}$ on the moduli space following [11]. We also show that the metric topology and the quotient topology are homeomorphic to each other. In Section 4 we give a proof of our main theorem (Theorem 3.2), that is, $\tilde{d}$ is a metric on the moduli space. In Section 5 we propose several related problems.

2. The moduli space of bodies

Let $\mathcal{B}_n$ be the set of all bodies (i.e., compact subsets obtained as the closure of open subsets) in $\mathbb{R}^n$. Now we introduce the moduli space of bodies following [11]. Let $G_n := \text{AGL}(n, \mathbb{Z})$ be the integral affine transformation group. Namely $G_n = \text{GL}(n, \mathbb{Z}) \times \mathbb{R}^n$ as a set and the multiplication is defined by

$$(A_1, t_1) \cdot (A_2, t_2) = (A_1A_2, A_1t_2 + t_1)$$

for each $(A_1, t_1), (A_2, t_2) \in G_n$. This group $G_n$ acts on $\mathcal{B}_n$ in a natural way, and $A \in \mathcal{B}_n$ and $B \in \mathcal{B}_n$ are called $G_n$-congruent if $A$ and $B$ are contained in the same $G_n$-orbit.

**Definition 2.1.** The moduli space of the bodies in $\mathbb{R}^n$ with respect to the $G_n$-congruence $\tilde{\mathcal{B}}_n$ is defined by the quotient

$$\tilde{\mathcal{B}}_n := \mathcal{B}_n / G_n.$$  

**Remark 2.2.** $\mathcal{B}_n$ contains an important subset, the set of $n$-dimensional Delzant polytopes $\mathcal{D}_n$. It is well known that there exists a bijective correspondence, the Delzant construction, between the set of all equivariant isomorphism classes of $2n$-dimensional symplectic toric manifolds.

On the other hand, the quotient space $\tilde{\mathcal{D}}_n := \mathcal{D}_n / G_n$ corresponds to the weak equivalence classes of $2n$-dimensional symplectic toric manifolds. It is called the moduli space of toric manifolds in [11].
3. Construction of the metric

Let \( d : \mathcal{B}_n \times \mathcal{B}_n \to \mathbb{R} \) be the function defined by

\[
d(A, B) := \text{vol}_n(A \triangle B) = \int_{\mathbb{R}^n} \chi_{A \triangle B} d\lambda = \int_{\mathbb{R}^n} |\chi_A - \chi_B| d\lambda
\]

for each \( A, B \in \mathcal{B}_n \), where \( A \triangle B := (A \setminus B) \cup (B \setminus A) \) is the symmetric difference of \( A \) and \( B \), \( \text{vol}_n(X) \) denotes the \( n \)-dimensional volume of \( X \) with respect to the \( n \)-dimensional Lebesgue measure \( d\lambda \) and \( \chi_X : \mathbb{R}^n \to \mathbb{R} \) denotes the characteristic function of a subset \( X \) of \( \mathbb{R}^n \). Then one can see that \( d \) is a metric on \( \mathcal{B}_n \).

Note that the function \( \tilde{d} : \tilde{\mathcal{B}}_n \times \tilde{\mathcal{B}}_n \ni ([P_1], [P_2]) \mapsto d(P_1, P_2) \) is not well-defined. So we introduce the function by taking the infimum among the values of \( d \).

**Definition 3.1.** Define a function \( \tilde{d} : \tilde{\mathcal{B}}_n \times \tilde{\mathcal{B}}_n \to \mathbb{R} \) by

\[
\tilde{d}(\alpha, \beta) := \inf \{d(P_1, P_2) \mid [P_1] = \alpha, [P_2] = \beta\}
\]

for \( (\alpha, \beta) \in \tilde{\mathcal{B}}_n \times \tilde{\mathcal{B}}_n \).

For \( g = (A, t) \in G_n \), since \( \det A = \pm 1 \) we have \( d(gP_1, gP_2) = d(P_1, P_2) \) for any \( P_1, P_2 \in \mathcal{B}_n \). When we fix representatives \( P_1 \in \mathcal{B}_n \) of \( \alpha \in \tilde{\mathcal{B}}_n \) and \( P_2 \in \mathcal{B}_n \) of \( \beta \in \tilde{\mathcal{B}}_n \) it follows that

\[
\tilde{d}(\alpha, \beta) = \inf \{d(g_1 P_1, g_2 P_2) \mid g_1, g_2 \in G_n\} = \inf \{d(P_1, gP_2) \mid g \in G_n\}.
\]

Clearly \( \tilde{d} \) is a symmetric function. The following is the main theorem of the present paper.

**Theorem 3.2.** \( \tilde{d} \) is a metric on \( \tilde{\mathcal{B}}_n \).

We show this theorem by proving the triangle inequality in Proposition 4.1 and the positiveness\(^3\) in Proposition 4.6.

Due to Theorem 3.2, we can consider the metric topology \( \mathcal{O}_\tilde{d} \) of \( \tilde{\mathcal{B}}_n \). On the other hand by using the metric topology of \( \mathcal{B}_n \) and the natural projection \( \pi : \mathcal{B}_n \to \tilde{\mathcal{B}}_n \), we also have the quotient topology \( \mathcal{O}_\pi \) of \( \tilde{\mathcal{B}}_n \). We have the following.

**Theorem 3.3.** Two topological spaces \( (\tilde{\mathcal{B}}_n, \mathcal{O}_\tilde{d}) \) and \( (\tilde{\mathcal{B}}_n, \mathcal{O}_\pi) \) are homeomorphic to each other. In particular \( (\tilde{\mathcal{B}}_n, \mathcal{O}_\pi) \) is a Hausdorff space.

This theorem follows from the following general proposition.

\(^3\)It is known that the function defined as in Definition 3.1 is a pseudo-metric and it satisfies the positiveness if and only if each \( G_n \)-orbit is a closed subset in \( \mathcal{B}_n \). See [12, p.80] for example. In fact our proof of Proposition 4.6 is a proof to show the closedness of the orbit.
Proposition 3.4. Suppose that \((X,d)\) is a metric space and a group \(G\) acts on \(X\) in an isometric way. Let \(
abla := X/G\) be the quotient space and assume that the function \(\tilde{d} : \nabla \times \nabla \to \mathbb{R}\) defined by
\[
\tilde{d}([x_1],[x_2]) = \inf \{d(x_1,gx_2) \mid g \in G\}
\]
is a metric on \(\nabla\). Then the quotient topology \(\mathcal{O}_\pi\) and the metric topology \(\mathcal{O}_{\tilde{d}}\) on \(\nabla\) are homeomorphic to each other.

Proof. Suppose that \(U \in \mathcal{O}_\pi\) and take \(\alpha \in U\). Fix a representative \(x \in \pi^{-1}(\alpha) \subset \pi^{-1}(U)\). Since \(\pi^{-1}(U)\) is an open subset in \(X\) there exists \(\delta > 0\) such that \(B_\delta(x)(\subset X)\), the open ball of radius \(\delta\) centered at \(x\), is contained in \(\pi^{-1}(U)\). For this \(\delta\) let \(\tilde{B}_\delta(\alpha) \subset \nabla\) be the open ball of radius \(\delta\) centered at \(\alpha\). For each \(\beta \in \tilde{B}_\delta(\alpha)\) and a representative \(x' \in \pi^{-1}(\beta)\) since
\[
\tilde{d}(\alpha,\beta) = \inf_g \{d(x,gx')\} < \delta
\]
there exists \(g \in G\) such that \(d(x,gx') < \delta\). It implies that \(gx' \in B_\delta(x) \subset \pi^{-1}(U)\), and hence, \(\beta = \pi(gx') \in U\). So we have \(\tilde{B}_\delta(\alpha) \subset U\), and it means \(U \in \mathcal{O}_{\tilde{d}}\).

Conversely suppose that \(U \in \mathcal{O}_{\tilde{d}}\) and take \(x \in \pi^{-1}(U)\). Then there exists \(\delta > 0\) such that \(\tilde{B}_\delta(\pi(x)) \subset U\). For this \(\delta > 0\) consider the open ball \(B_\delta(x)\) in \(X\). Take \(x' \in B_\delta(x)\) then we have \(\tilde{d}(\pi(x),\pi(x')) \leq d(x,x') < \delta\), and hence, \(\pi(x') \in \tilde{B}_\delta(\pi(x)) \subset U\). It implies that \(x' \in \pi^{-1}(U)\). So we have \(B_\delta(x) \subset \pi^{-1}(U)\), and it means \(U \in \mathcal{O}_\pi\). \(\square\)

4. PROOF OF THE MAIN THEOREM

Proposition 4.1. The function \(\tilde{d}\) defined in Definition 3.1 satisfies the triangle inequality.

Proof. For \(\alpha, \beta, \gamma \in \tilde{B}_n\) we take and fix their representatives \(P_1, P_2, P_3 \in B_n\). By the triangle inequality of \(d\) we have,
\[
d(P_1,gP_2) \leq d(P_1,g'P_3) + d(g'P_3,gP_2) = d(P_1,g'P_3) + d(g^{-1}g'P_3, P_2)
\]
for any \(g,g' \in G_n\). By taking the infimum for \(g\) we have
\[
\tilde{d}(\alpha,\beta) \leq d(P_1,g'P_3) + \inf_g \{d(g^{-1}g'P_3, P_2)\}
\]
\[
= d(P_1,g'P_3) + \inf_g \{d(g^{-1}g'P_3, g''P_2)\}
\]
\[
\leq d(P_1,g'P_3) + d(P_3,g''P_2)
\]
for any \(g'' \in G_n\). By taking the infimum for \(g', g'' \in G_n\) we have the triangle inequality
\[
\tilde{d}(\alpha,\beta) \leq \tilde{d}(\alpha,\gamma) + \tilde{d}(\gamma,\beta).
\]
\(\square\)
Remark 4.2. The above proof also works for the same general situation in Proposition 3.4.

To show the positiveness of $\tilde{d}$ we use the following two elementary lemmas.

**Lemma 4.3.** For $a > 0$ let $C$ be a cube $C := [-2a, 2a]^n = [-2a, 2a] \times [-2a, 2a] \times \cdots \times [-2a, 2a] \subset \mathbb{R}^n$. Then for any affine hyperplane $H$ in $\mathbb{R}^n$ there exists a vertex $v$ of $C$ such that

$$\text{dist}(v, H) > a.$$ 

**Proof.** Suppose that $H$ is an affine hyperplane with defining equation

$$f(x) := \nu \cdot x - 2c = 0 \quad (x \in \mathbb{R}^n),$$

where $\cdot$ denotes the Euclidean inner product, $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$ is a unit normal vector of $H$ and $c \in \mathbb{R}$. Let $v_i$ ($i = 1, 2, \ldots, 2^n$) be the vertices of $C$. Suppose that

$$\text{dist}(v_i, H) \leq a$$

for any $i = 1, 2, \ldots, 2^n$. Then for any $v_i$ we have

$$a \geq \text{dist}(v_i, H) = |f(v_i)| = 2 \left| a \sum_{k=1}^{n} (\pm \nu_k) - c \right|.$$

By taking the square we have

$$a^2 \geq 4 \left( a^2 \left( 1 + 2 \sum_{k \neq l} (\pm \nu_k \nu_l) \right) - 2ac \sum_{k=1}^{n} (\pm \nu_k) + c^2 \right),$$

and hence, we have inequalities

$$8a^2 \sum_{k \neq l} (\pm \nu_k \nu_l) - 8ac \sum_{k=1}^{n} (\pm \nu_k) \leq -3a^2 - 4c^2 < 0$$

for all $i = 1, \ldots, 2^n$. On the other hand the sum of the left hand of the inequality for all $i$ is equal to 0, we have a contradiction

$$0 \leq 2^n (-3a^2 - 4c^2) < 0.$$ 

So we have $\text{dist}(v_i, H) > a$ for some $i$. \qed

**Lemma 4.4.** Let $C$ be an $n$-dimensional cube in $\mathbb{R}^n$. For a sequence $\{g_m\}_m \subset G_n$ we put $C_m := g_mC$. If $\text{diam}(C_m)$ goes to $\infty$ as $m \to \infty$, then there exist a sequence $\{F_m\}_m$ of $n-1$-dimensional faces of $C_m$ such that

$$\text{dist}(H_m, H'_m) \to 0 \quad (m \to \infty),$$

where $H_m$ is the $n-1$-dimensional affine subspace containing $F_m$ and $H'_m$ is the other affine hyperplane containing the other $n-1$-dimensional face in $C_m$ which is parallel to $F_m$. 

To show this lemma by induction we generalize it in the following way as in Figure 1.

**Lemma 4.5.** Let \( C \) be an \( n \)-dimensional cube in \( \mathbb{R}^n \). For a sequence \( \{g_m\}_m \subset G_n \) we put \( C_m := g_mC \). For a sequence \( \{F_m\}_m \) of \( k(\geq 2) \)-dimensional faces of \( C_m \), if the \( k \)-dimensional volumes (with respect to the \( k \)-dimensional Hausdorff measure) \( \{\text{vol}_k(F_m)\}_m \) is bounded and the diameters \( \text{diam}(F_m) \) goes to \( \infty \) as \( m \to \infty \), then there exist a sequence \( \{E_m\}_m \) of \( k-1 \)-dimensional faces of \( F_m \) such that

\[
\text{dist}(H_m, H'_m) \to 0 \quad (m \to \infty),
\]

where \( H_m \) is the \( k-1 \)-dimensional affine subspace containing \( E_m \) and \( H'_m \) is the other \( k-1 \)-dimensional affine subspace containing the other \( k-1 \)-dimensional face in \( F_m \) which is parallel to \( E_m \).

**Proof.** We show this lemma by induction on \( k \). For \( k = 2 \) since \( \text{diam}(F_m) \to \infty \) one can see by the triangle inequality that the length of the diagonal of \( F_m \) goes to \( \infty \), and hence, \( \text{vol}_1(E_m) \to \infty \) for some edge \( E_m \) of \( F_m \). Since for the line \( H_m \) containing \( E_m \) and the other line \( H'_m \) containing the edge which is parallel to \( E_m \), we have

\[
\text{vol}_2(F_m) = \text{dist}(H_m, H'_m)\text{vol}_1(E_m)
\]

and it is bounded. It implies that \( \text{dist}(H_m, H'_m) \to 0 \).

Now we assume that the statement holds for any sequence of \( k-1 \)-dimensional faces of \( C_m \). Suppose that a sequence \( \{F_m\}_m \) of \( k \)-dimensional faces of \( C_m \) satisfies that \( \text{diam}(F_m) \to \infty \) and \( \{\text{vol}_k(F_m)\}_m \) is bounded. For any \( k-1 \)-dimensional face \( E_m \) of \( F_m \) we have

\[
\text{vol}_k(F_m) = \text{dist}(H_m, H'_m)\text{vol}_{k-1}(E_m).
\]

If there exists a sequence \( \{E_m\}_m \) with \( \text{vol}_{k-1}(E_m) \to \infty \), then we have \( \text{dist}(H_m, H'_m) \to 0 \), and hence the sequence \( \{E_m\}_m \) is the required one. We assume that the \( k-1 \)-dimensional volumes of any \( k-1 \)-dimensional face of \( F_m \) is bounded. The assumption \( \text{diam}(F_m) \to \infty \)
and the triangle inequality imply that there exists a sequence of edges \( \{e_m\}_m \) of \( F_m \) with \( \text{vol}_1(e_m) \to \infty \). For any \( k - 1 \)-dimensional face \( E_m \) containing \( e_m \), the assumption of the induction implies that there exists sequences of \( k - 2 \)-dimensional affine subspaces \( \{f_{1,m}\}_m \) and \( \{f'_{1,m}\}_m \) containing \( k - 2 \)-dimensional face of \( E_m \) such that
\[
\text{dist}(f_{1,m}, f'_{1,m}) \to 0 \quad (m \to \infty).
\]
For the other \( k - 1 \)-dimensional face \( E'_m \) of \( F_m \) which is parallel to \( E_m \), let \( f_{2,m} \) and \( f'_{2,m} \) be the affine subspaces which corresponds to \( f_{1,m} \) and \( f'_{1,m} \) by the translation which maps to \( E_m \) to \( E'_m \). Let \( G_m \) be the \( k - 1 \)-dimensional face of \( F_m \) containing \( f_{1,m} \) and \( f_{2,m} \). Let \( \tilde{H}_m \) be the affine subspace containing \( G_m \) and \( \tilde{H}'_m \) the other \( k - 1 \)-dimensional affine subspace containing the \( k - 1 \)-dimensional face of \( F_m \) which is parallel to \( G_m \). Then we have
\[
\text{dist}(\tilde{H}_m, \tilde{H}'_m) = \text{dist}(f'_{1,m}, f'_{2,m}) = \text{dist}(f_{1,m}, f_{2,m}) \to 0 \quad (m \to \infty).
\]
It implies that this \( G_m \) is the required one, and we complete the proof.

\[\square\]

**Proposition 4.6.** The function \( \tilde{d} \) defined in Definition 3.1 satisfies the positiveness.

**Proof.** Suppose that \( \tilde{d}(\alpha, \beta) = 0 \) for \( \alpha, \beta \in \tilde{B}_n \). Take representatives \( P_1, P_2 \in B_n \) of \( \alpha, \beta \) and a minimizing sequence \( \{g_m\}_m \subset G_n \), i.e.,
\[
d(P_1, g_m P_2) \to \tilde{d}(\alpha, \beta) = 0 \quad (m \to \infty).
\]
We show that \( \{g_m\}_m \) is a bounded sequence in \( G_n \) with respect to the direct product metric of the Euclidean metrics on \( G_n = \text{GL}(n, \mathbb{Z}) \times \mathbb{R}^n \subset \mathbb{R}^{n^2} \times \mathbb{R}^n \). If \( \{g_m\}_m \) is bounded, then by taking a limit \( g \in G_n \) of a convergent subsequence of \( \{g_m\}_m \) we have \( d(P_1, g P_2) = 0 \), and hence,
\[\alpha = [P_1] = [g P_2] = \beta.
\]
We may take representatives \( P_1 \) and \( P_2 \) of \( \alpha \) and \( \beta \) so that \( P_1 \cap P_2 \) contains the origin of \( \mathbb{R}^n \) as an interior point. For \( a > 0 \) small enough we take an \( n \)-dimensional cube \( C := [-2a, 2a]^n \subset P_1 \subset \mathbb{R}^n \). We also take an \( n \)-dimensional large cube \( C' \) so that \( P_2 \subset C' \). See Figure 2. Consider a sequence of polytopes \( \{C_m := g_m C'\}_m \).

Hereafter for any unbounded sequence we may consider it goes to \( \infty \) by taking a subsequence. Suppose that \( \{g_m = (A_m, t_m)\}_m \) is unbounded. Then the norm \( |g_m| = |A_m| + |t_m| \) goes to \( \infty \).

(i) Suppose that \( |A_m| \) goes to \( \infty \). We may assume that the \((1,1)\)-entry of \( A_m \) goes to \( \infty \) and put \( a_m \) the \((1,1)\)-entry of \( A_m \). Then for some \( r > 0 \) small enough we have
\[
\text{diam}(C_m) \geq r |a_m| \to \infty \quad (m \to \infty).
\]
Figure 2

Since \( \text{diam}(C_m) \to \infty \) and \( \text{vol}(C_m) = \text{vol}(C') \) is bounded, by Lemma 4.4, there exists an \( n - 1 \)-dimensional face \( F_m \) of \( C_m \) such that

\[
\text{dist}(H_m, H'_m) \to 0 \quad (m \to \infty),
\]

where \( H_m \) is the \( n - 1 \)-dimensional affine subspace containing \( F_m \) and \( H'_m \) is the \( n - 1 \)-dimensional affine subspace containing the \( n - 1 \)-dimensional face of \( C_m \) which is parallel to \( F_m \). In particular we have

\[
(4.2) \quad \text{dist}(H_m, H'_m) < \frac{a}{2}
\]

for \( m \gg 1 \). On the other hand by Lemma 4.3, there exists a vertex \( v_m \) of \( C \) for each \( H_m \) such that

\[
(4.3) \quad \text{dist}(v_m, H_m) > a.
\]

By (4.2) and (4.3) we have

\[
\text{dist}(v_m, C_m) \geq \text{dist}(v_m, H_m) - \frac{a}{2} > \frac{a}{2},
\]

and hence,

\[
(4.4) \quad B_{a/4}(v_m) \cap C \cap C_m = \emptyset,
\]

where \( B_{a/4}(v_m) \) is the open ball in \( \mathbb{R}^n \) of radius \( a/4 \) centered at \( v_m \). See Figure 3. Since \( P_1 \triangle g_m P_2 \supseteq P_1 \setminus g_m P_2 \supseteq C \setminus C_m \supseteq B_{a/4}(v_m) \cap C \), we have

\[
d(P_1, g_m P_2) \geq \text{vol}_n(C \setminus C_m) \geq \text{vol}_n(B_{a/4}(v_m) \cap C) = \frac{1}{2^n} \text{vol}(B_{a/4}(0)) > 0
\]

by (4.4). This contradicts to \( d(P_1, g_m P_2) \to 0 \). So we have that \( \{A_m\}_m \) is a bounded sequence\(^4\).

\(^4\)Since \( \text{GL}(n, \mathbb{Z}) \) is a discrete space without accumulation points, a convergent subsequence of \( \{A_m\}_m \) in \( \text{GL}(n, \mathbb{Z}) \) is a constant sequence for \( m \gg 1 \).
Figure 3

(ii) Suppose that \( \{t_m\}_m \) is unbounded and \( \{A_m\}_m \) is bounded. In this case since \( r_m := \text{diam}(C_m) \) is a bounded sequence, by fixing a point \( p_m \in C_m \) we have

\[
C_m \subset B_{r_m}(p_m) \subset B_R(p_m)
\]

for some \( R \gg 0 \) and any \( m \). Since \( \{t_m\}_m \) is unbounded and \( \{A_m\}_m \) is bounded, we have \( \text{dist}(p_m, C) \to \infty \). In particular we have \( \text{dist}(p_m, C) > 2R \), \( C \cap B_R(p_m) = \emptyset \), and hence,

\[
C \cap C_m = \emptyset
\]

for \( m \gg 1 \). It implies that \( C \subset P_1 \triangle g_m P_2 \) and

\[
d(P_1, g_m P_2) > \text{vol}_n(C) > 0.
\]

This contradicts to \( d(P_1, g_m P_2) \to 0 \). So we also have \( \{t_m\}_m \) is a bounded sequence.

By (i) and (ii) we have that \( \{g_m\}_m \) is a bounded sequence and complete the proof. \( \square \)

5. Further problems

In this section we focus on the moduli space of the Delzant polytopes \( \tilde{D}_n = D_n/G_n \). As we noted in Remark 2.2, this moduli space can be identified with the set of all equivalence classes of symplectic toric manifolds with respect to the weak isomorphisms [7].

5.1. Completion. As it is shown in [11], the metric space \( (D_n, d) \) is not complete. Namely there exists a Cauchy sequence\(^5\) \( \{T_m\}_m \) in \( D_n \)

\(^5\)In [11] they constructed such a sequence in \( D_2 \). By taking the product with cubes \([0, 1]^{n-2}\) the sequence gives a sequence with the same property in \( D_n \).
which does not converge in $\mathcal{D}_n$. In [11] they determined the completion of $(\mathcal{D}_2, d)$.

**Theorem 5.1** ([11], Theorem 8). The completion $(\tilde{\mathcal{D}}_2, \tilde{d})$ of $(\mathcal{D}_2, d)$ is isometric to

$$(C'_2/\sim) \cup \{\emptyset\},$$

where $C'_2$ is the set of all compact convex subsets in $\mathbb{R}^2$ with positive Lebesgue measure and $A \sim B$ if $\text{vol}_2(A \triangle B) = 0$ for $A, B \in C'_2$.

The Cauchy sequence $\{T_m\}_m$ in $\mathcal{D}_n$ gives a Cauchy sequence in $\tilde{\mathcal{D}}_n$ which is not a convergent sequence\(^6\) in $\tilde{\mathcal{D}}_n$. In particular $(\tilde{\mathcal{D}}_n, \tilde{d})$ is not complete. One may consider the following problem.

**Problem 5.2.** Determine the completion of $(\tilde{\mathcal{D}}_n, \tilde{d})$. For instance whether the completion of $\tilde{\mathcal{D}}_2$ is equal to $\mathcal{D}_2/G_2$ or not?

5.2. Dimension of the moduli space. Each metric space has an important invariant, the *Hausdorff dimension*. Then one may ask the following.

**Problem 5.3.** Estimate the Hausdorff dimension of $(\mathcal{D}_n, d)$ or $(\tilde{\mathcal{D}}_n, \tilde{d})$.

This may be a naive problem because even if we fix $n$, the dimension of polytopes, the number of vertices increase any number, and hence, the dimension may become infinity. One candidate to have a reasonable answer is to introduce a stratification of $\mathcal{D}_n$ or $\tilde{\mathcal{D}}_n$ by the numbers of vertices.

**Definition 5.4.** For each natural number $l$ define $\mathcal{D}_n^{(l)}$ and $\tilde{\mathcal{D}}_n^{(l)}$ by

$$\mathcal{D}_n^{(l)} := \{P \in \mathcal{D}_n \mid P \text{ has } l \text{ vertices}\}$$

and

$$\tilde{\mathcal{D}}_n^{(l)} := \mathcal{D}_n^{(l)}/G_n.$$ 

Note that $d$ and $\tilde{d}$ induce metrics on $\mathcal{D}_n^{(l)}$ and $\tilde{\mathcal{D}}_n^{(l)}$ respectively.

**Problem 5.5.** Estimate the Hausdorff dimension of $(\mathcal{D}_n^{(l)}, d)$ or $(\tilde{\mathcal{D}}_n^{(l)}, \tilde{d})$.

5.3. Relation with Gromov-Hausdorff distance. For each Delzant polytope $P \in \mathcal{D}_n$, one can canonically associate a compact symplectic toric manifold $M_P$ by the Delzant construction procedure. In fact the symplectic manifold $M_P$ carries canonical Kähler structure, in particular it is equipped with a Riemannian metric. One can see that if two Delzant polytopes $P$ and $P'$ are $G_n$-congruent each other, then two compact Riemannian manifolds $M_P$ and $M_{P'}$ are isometric. In this way we have a natural map between two metric spaces

$$\text{Del} : \tilde{\mathcal{D}}_n \to \mathcal{M}, \quad [P] \mapsto \text{Del}([P]) := M_P,$$

\(^6\)In fact it converges in $\tilde{\mathcal{B}}_n$. 


where $\mathcal{M}$ is the metric space consisting of all (isometry classes of) compact Riemannian manifolds equipped with the Gromov-Hausdorff distance $d_{GH}$. See [2, Chapter 7] for its definition.

It would be interesting to investigate the map Del from both viewpoints of symplectic geometry and metric geometry, however, the moduli space $\tilde{\mathcal{D}}_n$ would not be suitable for this problem as seen in the following example.

**Example 5.6.** Let $\{P_m\}_m$ be a sequence in $\mathcal{D}_2$ defined by rectangles $P_m := [0, 1] \times [0, 1/m]$. It is known that the corresponding Riemannian manifolds by Del is a sequence $\{(S^2 \times S^2, g_{FS} \oplus \frac{1}{m} g_{FS})\}_m$ in $\mathcal{M}$, where $g_{FS}$ is the Fubini-Study metric on $\mathbb{C}P^1 = S^2$. Its limit with respect to the Gromov-Hausdorff distance is $(S^2, g_{FS})$, which is a 2-dimensional symplectic toric manifold constructed from the closed interval $[0, 1] \in \mathcal{D}_1$ by the Delzant construction. On the other hand the limit of $\{P_m\}_m$ with respect to the metric $d$ (in the completion $\bar{\mathcal{D}}_2$) is the empty set $\emptyset$. See Figure 4.

To overcome this problem we first extend the moduli space in the following way.

**Definition 5.7.** For each non-negative integer $k$ with $0 \leq k \leq n$ let $\iota_k : \mathbb{R}^k \to \mathbb{R}^n$ be the inclusion, $\mathbb{R}^k \ni (x_1, x_2, \ldots, x_k) \mapsto (x_1, x_2, \ldots, x_k, 0, \ldots, 0) \in \mathbb{R}^n$. We define the set $\mathcal{D}_{\leq n}$ by

$$\mathcal{D}_{\leq n} := \{g(\iota_k(P)) \mid P \in \mathcal{D}_k, \ g \in G_n\},$$

and $\tilde{\mathcal{D}}_{\leq n}$ by

$$\tilde{\mathcal{D}}_{\leq n} := \mathcal{D}_{\leq n}/G_n.$$

Since the metric $d$ on $\mathcal{D}_n$ cannot be extended to $\mathcal{D}_{\leq n}$ we would like to use the Hausdorff metric $d_H$ on $\mathcal{D}_{\leq n}$ as a substitute of $d$. See [2, Chapter 7] for its definition. The limit of the sequence $\{P_m\}_m$ in Example 5.6 converges to $\iota_1([0, 1])$ in $\mathcal{D}_{\leq 2}$. 

**Figure 4**

\[\begin{array}{c}
\mathcal{D}_2 \ni \\
\downarrow \\
\mathcal{M} \ni \left(S^2 \times S^2, g_{FS} \oplus \frac{1}{m} g_{FS}\right) \quad \xrightarrow{m \to \infty} \quad (S^2, g_{FS}) \in \mathcal{M}
\end{array}\]

\[\xrightarrow{1/m} \quad \emptyset \in \bar{\mathcal{D}}_2\]
Note that $D_n$ is a dense subset in $D_{≤n}$ with respect to the metric topology of $d_H$. In fact for each $g(ι_k(P)) \in D_{≤n}$ the sequence $P_m := g(ι_k(P) \times [0,1/m]^{n-k}) \in D_n \ (m \in \mathbb{N})$ converges to $g(ι_k(P))$.

**Problem 5.8.** Define a function $\tilde{d}_H : \tilde{D}_{≤n} \times \tilde{D}_{≤n} \to \mathbb{R}$ by

$$\tilde{d}_H(α, β) := \inf \{ d_H(P_1, P_2) \mid [P_1] = α, [P_2] = β \}$$

for $α, β \in \tilde{D}_{≤n}$. Then does this $\tilde{d}_H$ become a metric on $\tilde{D}_{≤n}$ or not?

Now we extend the map Del to a map from $\tilde{D}_{≤n}$. For given equivalence class $[g(ι_k(P))] \in \tilde{D}_{≤n}$ let $M_P$ be a 2k-dimensional Riemannian manifold constructed from $P \in D_k$ by the Delzant construction.

**Lemma 5.9.** For each $[g(ι_k(P))] \in \tilde{D}_{≤n}$ the equivalence class of $M_P$ in $M$ does not depend on a choice of the representative $P \in D_k$.

**Proof.** We show that if $ι_k(P_1) = g(ι_k(P_2)) \in D_{≤n}$ for $g = (A, t) \in G_n$ and $P_1, P_2 \in D_k$, then $P_1 = g'P_2$ for some $g' \in G_k$. Since $ι_k(P_2)$ contains interior points in $\mathbb{R}^k(\subset \mathbb{R}^n)$ one has that $g(\mathbb{R}^k) = \mathbb{R}^k$, and hence, we have a block decomposition of $A$ and $t$ such as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \in \text{GL}(n, \mathbb{Z}), \quad t = \begin{pmatrix} t' \\ 0 \end{pmatrix} \in \mathbb{R}^k \oplus \mathbb{R}^{n-k} = \mathbb{R}^n$$

for some $A_{11} \in \text{GL}(k, \mathbb{Z}), A_{22} \in \text{GL}(n-k, \mathbb{Z})$ and $t' \in \mathbb{R}^k$. It implies that $P_1 = g'P_2$ for $g' := (A_{11}, t') \in \text{AGL}(k, \mathbb{Z})$. \hfill \Box

Due to Lemma 5.9 the following holds.

**Proposition 5.10.** The map Del : $\tilde{D}_n \to M$ can be extended to a map Del : $\tilde{D}_{≤n} \to M$, $[g(ι_k(P))] \mapsto M_P$.

**Problem 5.11.** Study the map Del : $(\tilde{D}_{≤n}, \tilde{d}_H) \to (M, d_{GH})$ as a map between metric spaces (if Problem 5.8 is solved affirmatively).

**Remark 5.12.** It would be interesting to study the map Del by considering $M$ as a metric space by the intrinsic flat metric $d_F$, which is introduced in [13].

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