Gauge theory on fuzzy spaces

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Abstract. The formulation of gauge theory on fuzzy spaces is reviewed, with emphasis on the matrix model point of view. We first discuss classical and quantum aspects of gauge theory on the fuzzy sphere, and give the appropriate generalizations for the 4-dimensional space fuzzy $\mathbb{CP}^2$. A different approach based on localization and equivariant cohomology is also briefly presented.

1. Introduction

Gauge theories provide the best known description of the fundamental forces in nature. At very short distances however, physics is not known, and it seems unlikely that spacetime is a perfect continuum down to arbitrarily small scales. Indeed, physicists have started to learn in recent years how to formulate field theory on quantized, or noncommutative (NC) spaces, see [1, 2] for reviews. This is expected to incorporate to some extent effects of (quantum) gravity in field theory.

The quantization of non-commutative field theories however turns out to be rather nontrivial. The reason is a new phenomenon called UV/IR mixing, which appears to be very generic in NC field theories, both for scalar and for gauge field theories. In essence it means that the UV divergences not only lead to the usual infinite renormalization of the mass and couplings, but also to new divergences in the infrared behaviour of the propagator (hence the name), which are likely to signal new physics. It is therefore important to develop appropriate techniques for the quantization of NC field theories, and to find models which are set up in a very clean way in order to avoid possible mathematically artifacts.

Fuzzy spaces are a nice class of noncommutative spaces based on finite-dimensional algebras of “functions”, with the same symmetries as their classical counterparts. This means that field theory on fuzzy spaces is regularized, but compatible with a geometrical symmetry group unlike lattice field theory. A large family of such spaces is given by the quantization of (co)adjoint orbits $O$ of a Lie group in terms of certain finite matrix algebras $O_N$. They are labeled by a noncommutativity parameter $\frac{1}{N}$, and the classical space is recovered in the large $N$ limit. The simplest example is the fuzzy sphere $S_N^2$, which has been studied in great detail; see e.g. [5, 10, 11, 6, 9, 3] and references therein. The simplest 4-dimensional fuzzy spaces are $S_N^2 \times S_N^2$ [13] and $\mathbb{CP}_N^2$ [14, 16, 17, 18, 19].

2. Fuzzy spaces: the fuzzy sphere and beyond

The fuzzy sphere $S_N^2$ [5] is a matrix approximation of the usual sphere $S^2$. The algebra of functions on $S^2$, spanned by the spherical harmonics, is truncated at a given frequency. The
algebra then becomes the finite dimensional algebra of $N \times N$ matrices. More precisely, recall that the algebra of functions on the ordinary sphere can be generated by the coordinates of $\mathbb{R}^3$ modulo the relation $\sum_{a=1}^{3} x_a x_a = r^2$. Similarly, the fuzzy sphere $S_N^2$ is the non-commutative manifold whose coordinate functions

$$x_a = \frac{1}{\sqrt{C_2(N)}} X_a$$

are $N \times N$ hermitian matrices proportional to the generators of the $N$-dimensional representation $X_a$ of SU(2). They satisfy the condition $\sum_{a=1}^{3} x_a x_a = 1$ and the commutation relations

$$[X_a, X_b] = i\varepsilon_{abc} X_c.$$  

(2)

It can be proven that one obtains the usual commutative sphere for $N \to \infty$. The best way to see this is to decompose the space of functions on $S_N^2$ in irreps under the SU(2) rotations, which is obtained from

$$S_N^2 \cong (N) \otimes (N) = (1) \oplus (3) \oplus \ldots \oplus (2N - 1) = \{Y^{0,0}\} \oplus \ldots \oplus \{Y^{(N-1),m}\}.$$  

(3)

This provides at the same time the definition of the fuzzy spherical harmonics $Y^{lm}$.

The following useful observation was made in [3]: If one combines the generators into a bigger hermitian $N \times N$ matrix

$$C = \frac{1}{2} \mathbb{1}_N \otimes \sigma_0 + X_i \otimes \sigma_i$$  

(4)

where $\sigma_0 = \mathbb{1}_2$ and $N = 2n$, one easily checks

$$C^2 = \frac{N^2}{4}, \quad TrC = N.$$  

(5)

This means that $C$ has eigenvalues $\pm \frac{N}{2}$ with multiplicities $n_+ = N + 1, n_- = N - 1$.

3. Gauge theory on the fuzzy sphere

3.1. Matrix model formulation

The formulation of gauge theory as multi-matrix model has at least 2 notable features, which are not present in the classical case: First, it leads to a very simple picture of nontrivial gauge sectors such as monopoles, which arise as nontrivial solutions in the matrix configuration space. This was noted in [15] and further explored in [3] for the fuzzy sphere. The concepts of fiber bundles are not required but arise automatically, in an intrinsically noncommutative way. Second, the matrix-model formulation allows a nonperturbative quantization in terms of a finite “path” integral, which in the case of $U(n)$ Yang-Mills on $S_N^2$ can be carried out explicitly in the large $N$ limit [3]. Some of these features can be extended to $\mathbb{C}P^2_N$. For example, one can again find monopole and (generalized) instanton solutions on $\mathbb{C}P^2_N$, generalizing the approach of [3].

Here we briefly review the construction of YM gauge theory on $S_N^2$ as multi-matrix model [3]. Consider the matrix model for 3 hermitian $N \times N$ matrices $C_i$, with action

$$S = Tr\left(a^2(C_a C_a - \frac{1}{4}(N^2 - 1))^2 + \frac{1}{g^2} F^a_{ab} F_a^{ab}\right)$$  

(6)

and define

$$F_{ab} = [C_a, C_b] - i\varepsilon_{abc} C_c.$$  

(7)
This action is invariant under the $U(N)$ “gauge” symmetry acting as

$$C_a \rightarrow U^{-1} C_a U.$$ 

A priori, we do not assume any underlying geometry, which arises dynamically. We claim that it describes $U(1)$ YM gauge theory on the fuzzy sphere $S^2_N$.

To see this, we first note that the action is positive definite, with global minimum $S = 0$ for the “vacuum” solution

$$C_a = X_a$$

where are the generators of $S^2_N$. This is a first indication that the model “dynamically generates” its vacuum geometry, which is the fuzzy sphere $S^2_N$. It is then natural to write a general field $C_a$ in the form

$$C_a = X_a + A_a,$$

The gauge transformation then takes the form

$$A_a \rightarrow U^{-1} A_a U + U^{-1} [X_a, U],$$

which is the transformation rule of a $U(1)$ gauge field. The field strength becomes

$$F_{ab} = [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - i \varepsilon_{abc} A_c$$

which looks like the field strength of a $U(1)$ gauge field, with the caveat that we seem to have $3$ degrees of freedom rather than $2$. To solve this puzzle, consider again the action, writing it in the form

$$S = Tr \left( a^2 \varphi^2 + \frac{1}{g^2} F_{ab}^\dagger F_{ab} \right),$$

where we introduce the scalar field

$$\varphi := C_a C_a - \frac{1}{4} (N^2 - 1) = X_a A_a + A_a X_a + A_a A_a.$$ 

Since only configurations where $\varphi$ is small will significantly contribute to the action, it follows that

$$x_a A_a + A_a x_a = O\left( \frac{\varphi}{N} \right)$$

is small. This means that $A_a^2$ is tangential in the (commutative) large $N$ limit, and $2$ tangential gauge degrees of freedom survive. Equivalently, one can use the scalar field $\phi = N \varphi$, which would acquire a mass of order $N$ and decouple from the theory.

We have thus established that the matrix model (6) is indeed a fuzzy version of pure $U(1)$ YM theory on the sphere. Without the term $(C_a C_a - \frac{1}{4} (N^2 - 1))^2$, there would be an additional massless scalar degree of freedom, which completely alters its meaning and destabilizes the vacuum.

Monopoles. One can easily find new, non-trivial solutions of this model using the ansatz [3]

$$C_i = \alpha_m X_i^{(M)}$$

for suitable normalization constant $\alpha_m$. Here $X_i^{(M)}$ is the generator of the $M$-dimensional irrep of $su(2)$, which can be embedded in the configuration space of $N \times N$ matrices if $m = N - M > 0$. It turns out that (15) describes monopole solutions with monopole charge $m$, and the corresponding
gauge potential can be calculated explicitly [3]. Notice that negative monopole charges \( m < 0 \) can also be obtained by admitting matrices \( C_i \) of size \( N' \times N' \) with \( N < N' \ll 2N \).

As a subtlety, it should be noted that the correct action \( S = \frac{m^2}{2g^2} \) of the above monopoles is recovered only upon a slight modification of the constraint term in the action (6) as indicated in [3], which does not affect the classical limit. The reason is that the "empty" blocks in (15) if embedded in \( N \times N \) matrices give a large contribution due to the first term in (6). This is certainly unphysical (it could be interpreted as action of a Dirac string), and can be avoided by the slightly modified action (78) in [3]. Then the energy of all monopoles is correctly reproduced in the commutative limit \( N \to \infty \). All this extends immediately to the non-abelian case:

**Non-abelian case** This model is readily extended to the nonabelian case by using matrices of size \( M = nN \), i.e.

\[
C_i = C_{i,\alpha}t^\alpha = X_i t^0 + A_{i,0} t^0 + A_{i,a} t^a
\]

where \( t^a \) denote the Gell-Mann matrices of \( \text{su}(n) \). The action then reduces to the usual \( U(n) \) Yang-Mills action

\[
S = \frac{1}{g^2} \int F_{mn,0}F_{mn,0} + F_{mn,a}F_{mn,a}
\]

in the commutative limit. Again, all "instanton" sectors are recovered if one admits matrices of arbitrary size \( M \approx nN \) for the above action.

### 3.2. Quantization: matrix model approach

The quantization of \( U(n) \) Yang-Mills gauge theory on the usual 2-sphere is well known, see e.g. [4, 7]. In particular, the partition function and correlation functions of Wilson loops have been calculated. Our goal is to calculate the partition function for the YM action (6) on the fuzzy sphere, taking advantage of the formulation as matrix model. This can be achieved by collecting the 3 matrices \( C_i \) into a single \( 2M \times 2M \) matrix

\[
C = C_0 + C_i \sigma^i
\]

The main observation is that the above action (6) can be rewritten simply as

\[
S(B) = TrV(C)
\]

imposing the constraint \( C_0 = \frac{1}{2} \), for the potential

\[
V(C) = \frac{1}{g^2 N} (C^2 - (\frac{N}{2})^2)^2
\]

Then we proceed as

\[
Z = \int dC \exp(-S(C)) = \int dC \delta(C_0 - \frac{1}{2}) \exp(-TrV(C)) = \int d\Lambda_i \Delta(\Lambda_i) \exp(-TrV(\Lambda)) \int dU \delta((U^{-1}U)_0 - \frac{1}{2})
\]

where \( dU \) is the integral over \( 2M \times 2M \) unitary matrices, \( C = U^{-1}U \), and \( \Delta(\Lambda_i) \) is the Vandermonde-determinant of the eigenvalues \( \Lambda_i \). Here \( \delta(C_0 - \frac{1}{2}) \) is a product over \( M^2 \) delta
functions, which can be calculated by introducing $J = \left( \begin{array}{cc} K & 0 \\ 0 & K \end{array} \right) = K \sigma^0$ where $K$ is a $N \times N$ matrix. Then
\[
\delta((U^{-1}CU)_0 - \frac{1}{2}) = \int dK \exp(iTr(U^{-1}(C - \frac{1}{2}UJ)))
\]

By gauge invariance, the r.h.s. depends only on the eigenvalues $\Lambda_i$ of $C$. Hence
\[
Z = \int dK \int d\Lambda_i \Delta^2(\Lambda_i) \exp(-TrV(\Lambda)) \int dU \exp(iTr(U^{-1}\Lambda UJ - \frac{1}{2}J))
\]
\[
= \int dK \ Z[J] \ e^{-\frac{1}{2}TrJ}
\] (18)

where
\[
Z[J] := \int dC \ \exp(-TrV(C) + iTechtron(CJ))
\] (19)

depends only on the eigenvalues $J_i$ of $J$. Diagonalizing $K = V^{-1}kV$, we get
\[
Z = \int d\kappa \Delta^2(\kappa) \int d\Lambda_i \Delta^2(\Lambda_i) \exp(-TrV(\Lambda)) \int dU \exp(iTr(U^{-1}(\Lambda - \frac{1}{2}UJ)))
\]

where $\int dV$ was absorbed in $\int dU$. The integral over $\int dU$ can now be done using the Itzykson-Zuber-Harish-Chandra formula [8]
\[
\int dU \exp(iTr(U^{-1}CUJ)) = const \ \frac{\det(e^{i\Lambda_i J_j})}{\Delta(\Lambda_i)\Delta(J_j)},
\]
which also depends only on the eigenvalues of $J$ and $C$.

In this step the number of integrals has been reduced from $N^2$ to $2N$. This basically means that the integral over fields on $S^2_0$ is reduced to the integral over functions in one variable. This is a huge step, just like in the usual matrix models. The constraint however forces us to evaluate in addition the integral over $\kappa_i$, which is quite complicated due to the rapid oscillations in $\det(e^{i\Lambda_i J_j})$; note that $\Lambda_i \approx \pm \frac{N}{2}$. Nevertheless, the integrals can be evaluated for large $N$ [3], with the result
\[
Z_m = \sum_{m_1 + \ldots + m_n = m} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \Delta^2(\kappa) e^{i\kappa_{m_1} \ldots \kappa_{m_n}} \exp(-\frac{g^2}{2} \sum \kappa_i^2).
\]

Here we consider matrices of size $M = nN - m$, which corresponds to the monopole sectors with total $U(1)$ charge $m = m_1 + \ldots + m_n$. This can be rewritten in the “localized” form as a weighted sum of saddle-point contributions, as advocated by Witten [7]:
\[
Z_m = \sum_{m_1 + \ldots + m_n = m} P(m_i, g) \ \exp\left(-\frac{1}{2g^2} \sum m_i^2\right)
\]

where $P(m_i, g)$ is a totally symmetric polynomial in the $m_i$ which can be given explicitly. In order to include all monopole configurations, we should simply sum over matrices of different sizes $M = nN - m$, for the same action given by $V(C)$. One can indeed find corresponding saddle-points of the action (6) which have the form [3]
\[
A_i = \left( \begin{array}{cccc} m_1 A_i & 0 & \ldots & 0 \\ 0 & m_2 A_i & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & m_n A_i \end{array} \right)
\]
where
\[ \vec{A}_i^{(c)} = \vec{r} \times \vec{A} \approx m_2 \left( \begin{array}{c} x_2 \\ -x_1 \\ 0 \end{array} \right) \]
becomes the usual monopole field for large \( N \), and action becomes
\[ S(C^{(m_1, \ldots, m_n)}) = \frac{1}{2g^2} \sum_i m_i^2 \]
for large \( N \). This is a standard result on the classical sphere.

Hence the full partition function is obtained by summing over all \( Z_m \),
\[ Z = \sum_m Z_m = \sum_{m_1, \ldots, m_n = -\infty}^{\infty} \int d\kappa \Delta^2(\kappa) e^{i\kappa; m_i} e^{-\frac{g^2}{2} \sum \kappa_i^2}. \]
Using a Poisson resummation, this can be rewritten in the form
\[ Z = \sum_{p_1, \ldots, p_n \in \mathbb{Z}} \Delta^2(p) \exp(-2\pi^2 g^2 \sum_i p_i^2) \]
or equivalently
\[ Z = \sum_C (d_R)^2 \exp(-4\pi^2 g^2 C_2R). \]
Here the sum is over all representations of \( U(n) \), \( d_R \) is the dimension of the representation and \( C_2R \) the quadratic casimir. This form was found in [4] for the partition function of a \( U(n) \) Yang-Mills theory on the ordinary 2-sphere.

We see that the limit \( N \to \infty \) of the partition function for \( U(n) \) YM on the fuzzy sphere is well-defined, and reproduces the result for YM on the classical sphere. This strongly suggests that the same holds for the full YM theory on the fuzzy sphere, and that there is no UV/IR mixing for pure gauge theory on \( S^2_N \). This is unlike the case of a scalar field, which exhibits a “non-commutative anomaly” [9] related to UV/IR mixing.

### 3.3. Localization and equivariant cohomology for gauge theory on \( S^2_N \)

We now discuss an alternative approach to the quantization of gauge theory on \( S^2_N \), using the ideas of localization and equivariant cohomology following Witten [7]. This is a short preview of an ongoing collaboration with R. Szabo [12].

**Degrees of freedom** To introduce gauge fields, consider again the covariant coordinates
\[ C_i = X_i + A_i, \quad C_0 = \frac{1}{2} + A_0 \] (20)
which transform under the gauge group \( U(N) \) as \( C_\mu \to U^{-1} C_\mu U \) for \( U \in U(N) \). We can again assemble them into a bigger \( \mathcal{N} \times \mathcal{N} \) matrix,
\[ C = C_\mu \otimes \sigma_\mu. \] (21)
These would be 4 independent fields, and we have to reduce them to 2 tangential fields. There are several ways to do this; one possibility has been discussed in section 3.1. Here, we will use a different approach and impose the constraints
\[ C^2 = N^2/4, \quad Tr(C) = N \] (22)
which is equivalent to saying that $C$ has eigenvalues $\pm N/2$ with multiplicities $n_\pm$ fixed as $n_+ = N + 1, n_- = N - 1$. In terms of components, this amounts to

$$C^2 = (C_i C_i + C_0 C_0) \otimes \sigma_0 + (i \varepsilon_{ijk} C_j C_k + \{C_0, C_i\}) \otimes \sigma_i = \frac{N^2}{4}. \quad (23)$$

This is satisfied for $A_\mu = 0$. We can then consider the action

$$C \to U^{-1} C U \quad (24)$$

for $U \in U(2N)$, which generates a (co)adjoint orbit of $U(2N)$, and clearly preserves the constraint (22). The gauge group $U(N)$ and the rotation group $SU(2)$ are subgroups of this larger group $U(2N)$. In particular, the generators of the gauge group are given by $\phi = \phi_0 \otimes \sigma_0$.

We thus claim that a possible configuration space of gauge fields is given by the single (co)adjoint orbit

$$O = \{U^{-1}(\frac{1}{2} I_N \otimes \sigma_0 + X_i \otimes \sigma_i)U, \ U \in U(2N)\} \quad (25)$$

A similar proposal for $CP^2$ was given in [18]. Thus

$$O \cong u(2N)/U(N + 1) \times U(N - 1) \quad (26)$$

with $D := \dim O = 2(N^2 - 1)$. This is essentially the number of degrees of freedom of 2 scalar fields on the fuzzy sphere. It is possible to check that the $A_i$ defined in this way are indeed tangential vector fields, at least in the commutative limit.

The YM action. We claim that the following action

$$S = \frac{N}{g} Tr((C_0 - \frac{1}{2})(C_0 - \frac{1}{2})) \quad (27)$$

for $C \in O$ reduces in the commutative limit $N \to \infty$ to the usual Yang-Mills action on $S^2$, and can therefore be taken as a definition of the Yang-Mills action on the fuzzy sphere $S^2_N$. To see this, consider again the 3-component field strength

$$F_i = i \varepsilon_{ijk} C_j C_k + C_i = i \varepsilon_{ijk} [\xi_j, A_k] + i \varepsilon_{ijk} A_j A_k + A_i. \quad (28)$$

At the “north pole” where $X_3 \approx \frac{N}{2} x_3 = \frac{N}{2}$ (setting the radius $R = 1$), one can replace

$$i[X_i, \cdot] \to -\varepsilon_{ij} \frac{\partial}{\partial x_j} \quad (29)$$

in the commutative limit for $i, j = 1, 2$. Hence upon identifying the commutative gauge fields $A_i^{(cl)}$ via

$$A_i^{(cl)} = -\varepsilon_{ij} A_i \quad (30)$$

the “radial” component $F_3$ of the field strength (28) reduces in the commutative limit to the standard expression

$$F_3 \approx \partial_i A_2^{(cl)} - \partial_2 A_1^{(cl)} + i[A_1^{(cl)}, A_2^{(cl)}], \quad (31)$$

while the others turn out to be irrelevant here. We now note that using (23), the constraint $C^2 = N^2/4$ implies

$$F_i + \{C_0 - \frac{1}{2} C_i\} = F_i + \{A_0, C_i\} = 0 \quad (32)$$
Therefore
\[ F_3 = \{-A_0, C_3\} \approx -NA_0 \] (33)

near the north pole (using \( X_3 = O(N) \) while \( A_3 = O(1) \)). All this can be made global by considering the “radial” field strength \( F_r = x_i F_i \), which reduces to the usual field strength scalar on \( S^2 \). Therefore the action (27) indeed reduces indeed to the usual YM action in the commutative limit,
\[ S \approx \frac{1}{Ng} Tr(F_r F_r) \approx \frac{1}{4\pi g} \int F_r F_r. \] (34)

**Critical points.** The critical points of (27) are easy to find: Since the most general variation of \( C \in \mathcal{O} \) is given by \( C = [C, \phi] \), the critical points satisfy
\[ 0 = Tr(C_0(C_0 - \frac{1}{2})) = Tr([C, \phi]C_0) = -Tr(\phi[C, C_0]) \] (35)

for arbitrary \( \phi \). Therefore they are given by solutions of
\[ [C_0, C] = 0. \] (36)

This agrees with the known saddle-points in the formulation of section 3.1: \( [C_0, C_i] = 0 \) together with \( C^2 = \frac{N^2}{4} \) (23) implies that
\[ [C_i, C_j] = i\epsilon_{ijk}(2C_0) C_k, \]
\[ C_0^2 = \frac{N^2}{4} - \sum_i C_i^2. \] (37)

This means that \( C_i \) is given by a sum of irreps of (37) characterized by partitions \( \vec{n} = (n_1, \ldots, n_l) \) of the integer \( N = n_1 + \ldots + n_l \), where \( n_i \in \{1, 2, 3, \ldots\} \) is the dimension of the \( i \)-th irreducible subrepresentation in the representation \( \pi_N \). Therefore each critical point is labeled (up to gauge transformations) by the set of dimensions \( n_i \) of the irreps, supplemented by a “sign”, which is defined by \( s_i = \text{sign}(C_0) = \pm 1 \) (in that irrep) provided \( C_0 \neq 0 \), and \( s_i = 0 \) if \( C_0 = 0 \).

**Localization of the path integral** The partition function can now be rewritten as
\[ Z = \int_{\mathcal{O} \times su(N)} [d\phi] \exp(\omega - iTr(C\phi) - \frac{g'}{2} Tr(\phi\phi)) \] (38)

where \( \phi \equiv \phi\lambda_0 \) is an auxiliary scalar field, which after integrating over \( \phi \) reproduces the action (27). Here \( \omega \) is the (Kirillov-Kostant) symplectic form on the coadjoint orbit \( \mathcal{O} \), which provides the volume form. Let \( V_\phi \) denote the vector field on \( \mathcal{O} \) defined by \( V_\phi \equiv [\phi, \cdot] \). One can now show that
\[ dTr(C_0\phi) = i \iota_{V_\phi}\omega \] (39)

and one defines the “BRST operator”
\[ D = d + i\iota_{V_\phi} \] (40)

where \( d \) is the exterior derivative on \( \mathcal{O} \), and \( \iota_{V_\phi} \) acts trivially on \( \phi \). It satisfies
\[ D^2 = i\{d, \iota_{V_\phi}\} = i\mathcal{L}_{V_\phi} \] (41)
so that \( D^2 = 0 \) exactly on \( SU(N) \)-invariant forms taking values in symmetric functions \( \mathbb{C}[[su(N)]] \).

By construction, we have

\[
D(\omega - i Tr(C_0 \phi)) = 0
\]

using (39), and

\[
D Tr(\phi \phi) = 0
\]

trivially. Therefore the integrand

\[
\exp(\omega - i Tr(C \phi) - g' \frac{1}{2} Tr(\phi \phi))
\]

defines an equivariant cohomology class. Using standard arguments, \( Z \) is unchanged if we add any \( D \)-exact term to the action. Hence we can replace it by

\[
Z = \int_{O \times su(N)} [d\phi] \exp(\omega - i Tr(C \phi) - g' \frac{1}{2} Tr(\phi \phi) + t D\alpha)
\]

which is independent of \( t \) for any \( SU(N) \)-invariant 1-form on \( O \), where

\[
D\alpha = d\alpha + i(\alpha, V_{\phi}).
\]

It follows that for \( t \to \infty \) the integral localizes at the stationary points of \( \langle \alpha, V_{\phi} \rangle \). Using the ansatz (cp. [7, 2])

\[
\alpha = Tr(C_0 [C, dC]_0),
\]

\[
S = \frac{1}{2} Tr(C_0 C_0)
\]

it turns out that the critical points are the same as for the original action.

Thus we have realized the ideas of [7] for applying equivariant cohomology in the context of 2-dimensional gauge theory. Moreover, in this approach based on the fuzzy sphere everything is finite-dimensional and completely well-defined, as opposed to the commutative case. This therefore allows to give a rigorous realization of the ideas in [7]. The details of evaluating the contributions from the various saddle-points are complicated however, which is work in progress [12].

4. Multi-Matrix Models for Yang-Mills on fuzzy \( \mathbb{C}P^2 \)

4.1. Fuzzy \( \mathbb{C}P^2 \)

We briefly describe how the formulation of gauge theory can be generalized to 4 dimensions, in the example of \( \mathbb{C}P^2 \) following [18]. From a physical point of view, \( \mathbb{C}P^2 \) should be considered here as compactified \( \mathbb{R}^4 \). Fuzzy \( \mathbb{C}P^2 \) is a quantization of \( \mathbb{C}P^2 \) as (co)adjoint orbit; for the basic properties of this space we refer to [14, 16, 17, 20]. We only recall here that this space is defined in terms of non-commutative coordinates \( x_a, a = 1, \ldots, 8 \) which describe the embedding of \( \mathbb{C}P^2 \subset \mathbb{R}^8 \), and satisfy [16]

\[
if_{\alpha}^{ab} x_a x_b = -3 \frac{R}{\sqrt{\frac{1}{3}N^2 + N}} x_c,
\]

\[
g^{ab} x_a x_b = R^2,
\]

\[
d_{\alpha}^{ab} x_a x_b = R \frac{2N/3 + 1}{\sqrt{\frac{1}{3}N^2 + N}} x_c.
\]

Here \( R \) is an arbitrary radius, which will usually be 1 here. This generates the algebra of functions on fuzzy \( \mathbb{C}P^2_8 \), which is simply \( Mat(D_N, \mathbb{C}) \) for \( D_N \) being the dimension of a certain irrep of \( su(3) \).
4.2. Degrees of freedom and field strength

Our basic degrees of freedom are 8 hermitian matrices $C_a \in Mat(D_N, \mathbb{C})$ transforming in the adjoint of $su(3)$, which are naturally arranged as a single $3D_N \times 3D_N$ matrix

$$C = C_a \tau^a + C_0 \mathbb{1}$$

(51)

where $C_0 = 0$ in much of the following. Here $\tau^a$ are the conjugated Gell-mann matrices. The size $D_N$ of these matrices will be relaxed later. We want to find a multi-matrix model in terms these $C_a$, which for large $N$ reduces to Yang-Mills gauge theory on $\mathbb{C}P^2$. The idea is again to interpret the $C_a$ as suitably rescaled “covariant coordinates” on fuzzy $\mathbb{C}P^2_N$, with the gauge transformation

$$C_a \rightarrow U^{-1}C_a U$$

(52)

for unitary matrices $U$ of the same size. The $C_a$ can also be interpreted as components of a one-form if desired [18]. Following the above model for the fuzzy sphere, we look for an action which has the “vacuum” solution

$$C_a = \xi_a$$

(53)

corresponding to $\mathbb{C}P^2_N$, and forces $C_a$ to be at least approximately the corresponding representation $V_{N\lambda_2}$ of $su(3)$. Then the fluctuations

$$C_a = \xi_a + A_a$$

(54)

are small, and describe the gauge fields. By inspection, these gauge fields $A_a$ transform as

$$\delta A_a = i[\xi_a + A_a, \Lambda] = iL_a\Lambda + i[A_a, \Lambda]$$

(55)

for $U = e^{i\Lambda}$, which is the appropriate formula for a gauge transformation. Since the $C_a$ resp. $\xi_a$ correspond to “global” coordinates in the embedding space $\mathbb{R}^8$, we can hope that nontrivial solutions such as instantons can also be described in this way.

A suitable definition for the field strength is then given by

$$F_{ab} = i[C_a, C_b] + \frac{1}{2} f_{abc}C_c = i(L_aA_b - L_bA_a + [A_a, A_b]) + \frac{1}{2} f_{abc}A_c.$$

(56)

We will also need

$$F_a = if_{abc}C_bC_c + 3C_a = \frac{1}{2} f_{abc}F_{ab},$$

$$D_a = d_{abc}C_aC_b - (\frac{2N}{3} + 1) C_c.$$

(57)

Under gauge transformations, the field strength transforms as

$$F_{ab} \rightarrow U^{-1}F_{ab}U.$$  

(58)

$F$ can also be interpreted as 2-form

$$F = dA + AA$$

(59)

if one considers the fields $C_a$ as one-forms $C = C_a\theta_a = \Theta + A$, using the differential calculus introduced in [18]. Furthermore, one can show that $F_{ab}$ is (approximately) tangential if $C_a$ satisfies (approximately) the constraints of $\mathbb{C}P^2$. Assuming that $A_a$ tend to well-defined functions on $\mathbb{C}P^2$ in the large $N$ limit, this implies that $F_{ab}$ are the components of the usual field strength 2-form in the commutative (large $N$) limit. This justifies the above definition of $F_{ab}$, and it is a matter of taste whether one works with the components or with forms.
4.3. Constraints

In order to describe fuzzy $\mathbb{C}P^2$, the fields $C_a$ should satisfy at least approximately the constraints (49), (50) of $\mathbb{C}P^2_N$, 

\[ D_a = 0, \]
\[ g_{ab} C_a C_b = \frac{1}{3} N^2 + N \]

which are gauge invariant. These constraints ensure that $C_a$ can be interpreted as describing a ("dynamical" or fluctuating) $\mathbb{C}P^2_N$. These constraints are analyzed in considerable detail in [18] in the non-commutative case.

4.4. The Yang-Mills action

Assume that the $C_a$ satisfy the constraints (60), (61) of $\mathbb{C}P^2_N$ exactly or approximately. This implies that $F_{ab}$ is tangential in the commutative limit, as shown in [18]. Then one can define the "Yang-Mills" action as

\[ S_{YM} = \frac{1}{g} \int F_{ab} F_{ab} = \frac{1}{g D_N} Tr (\left(-[C_a, C_b]\right)^2 + 2i f_{abc} C_a C_b C_c + 3 C_a C_a), \]

(62)

It reduces to the classical Yang-Mills action on $\mathbb{C}P^2$, because only the tangential indices contribute in the commutative limit. The corresponding equation of motion is

\[ 2[C_b, F_{ab}] - i F_a = 0 \]

(63)

We now have to impose the constraints (60), (61) either exactly or approximately, and there are several possibilities how to proceed. Imposing both of them exactly seems too restrictive, since they are not independent even classically. One can hence either

(i) consider all 8 fields $C_a$ as dynamical and add something like

\[ S_D = \frac{1}{g D_N} Tr \left( \mu_1 (dCC - \left(\frac{2N}{3} + 1\right)C)^2 + \mu_2 (C \cdot C - \left(\frac{N^2}{3} + N\right))^2 \right) \]

(64)

to the action. This will impose the constraint dynamically for suitable $\mu_1 > 0$ and $\mu_2 \geq 0$, by giving the 4 transversal fields a large mass $m \to \infty$. Or,

(ii) impose the constraint $D = dCC - \left(\frac{2N}{3} + 1\right)C = 0$ exactly, or a slightly modified version.

In the second approach, it is not clear whether there are sufficiently many solutions of $D = 0$ in the noncommutative case to admit 4 tangential gauge fields. This concern could be circumvented by modifying the constraint, which is discussed in [18]. Therefore we concentrate on the first approach here. It offers the additional possibility to give physical meaning to the non-tangential degrees of freedom.

Therefore our action is

\[ S = S_{YM} + S_D. \]

(65)

It is shown in [18] that this reproduces the classical Yang-Mills action on $\mathbb{C}P^2$ in the large $N$ limit provided

\[ \mu_1 = o\left(\frac{1}{N}\right), \quad \mu_2 \leq o\left(\frac{1}{N^3}\right). \]

(66)

Here $o\left(\frac{1}{N}\right)$ stands for a function which scales exactly like $\frac{1}{N}$. These constraints on the scaling of $\mu_{1,2}$ ensure that the vacuum which defines the geometry of $\mathbb{C}P^2_N$ is stable (note that the
geometry is determined dynamically in noncommutative gauge theories!), and the monopole- and instanton solutions survive [18]. Imposing e.g. \( \mu_1 = 0 \) strictly would suppress the instanton solutions, hence in some sense fix the topology of the gauge fields. These issues certainly need further investigations; similarly, one may or may not fix the size of the matrices to be exactly \( D_N \), which also has some influence on the existence of certain non-trivial solutions. Our choices are such that the conventional Yang-Mills theories emerge in the large \( N \) limit. These issues are discussed in more detail in [18].

We proceed to find the “vacuum”, i.e. the minimum of the action. Assume first that the size of the matrices is \( D_N \). Then the absolute minima of the action are given by solutions of \( F_{ab} = 0 \) and \( D_a = 0 \), which means that \( C_a \) is a representation of \( su(3) \) with \( D_a = 0 \). The latter implies that the only allowed irreps are \( V_{N\Lambda_2} \) or the trivial representation. Ignoring the latter (it has a smaller “phase space” of fluctuations), the vacuum solution is therefore

\[
(C_{vac})_a = \xi_a
\]

in a suitable basis. These arguments go through if we allow the size of the matrices \( C_a \) to be somewhat bigger than \( D_N \), say

\[
C_a \in Mat(D_N + N, \Psi)
\]

(anything much smaller that \( 2D_N \) will do), which is needed to accomodate all the nontrivial solutions found in [18]. Any configuration with finite action is therefore close to (67), and can hence be written as

\[
C_a = \xi_a + A_a
\]

with “small” \( A_a \).

It is quite straightforward to include scalar fields in this construction. Assume that we have an additional complex scalar field \( \phi \). Without gauge coupling, a natural action would be \( \int (|\xi_a, \phi\rangle\langle\xi_a, \phi| - \int \phi^\dagger \Delta \phi \). If we assume that \( \phi \) is charged and transforms under gauge transformations as

\[
\phi \rightarrow U\phi,
\]

then a natural gauge-invariant action would be

\[
S[\phi] = \int (C_a \phi - \phi \xi_a)^\dagger (C_a \phi - \phi \xi_a).
\]

This reduces to \( \int (D_a \phi)\dagger D_a \phi \), where \( D_a = [\xi_a, \_] + A_a \). Fermions are much more difficult to handle since \( \Psi^P \) is a spin² manifold but not spin; and a fully satisfactory treatment in the fuzzy case is still lacking; for some possible approach see [16].

5. Fuzzy extra dimensions

Finally, I would like to point out a realization of the ideas and results of section 3 within the context of ordinary, renormalizable 4-dimensional gauge theory [21]. We start with a \( SU(N) \) gauge theory on 4-dimensional Minkowski space \( M^4 \), and add 3 antihermitian scalar fields with the most general renormalizable action invariant under an additional global \( SU(2) \) symmetry. It then turns out that this potential is closely related to the actions considerd in section 3, and indeed the model dynamically generated an extra dimension in the form or a fuzzy sphere. The appropriate interpretation is then as an \( U(n) \) gauge theory on \( M^4 \times S^2 \). The details of this will be published elsewhere [21].

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