Geometric analysis of minimum time trajectories for a two-level quantum system

Raffaele Romano

Department of Mathematics, Iowa State University, Ames, IA, USA

Abstract

We consider the problem of controlling in minimum time a two-level quantum system which can be subject to a drift. The control is assumed to be bounded in magnitude, and to affect two or three independent generators of the dynamics. We describe the time optimal trajectories in $SU(2)$, the Lie group of possible evolutions for the system, by means of a particularly simple parametrization of the group. A key ingredient of our analysis is the introduction of the optimal front line. This tool allows us to fully characterize the time-evolution of the reachable sets, and to derive the worst-case operators and the corresponding times. The analysis is performed in any regime: controlled dynamics stronger, of the same magnitude or weaker than the drift term, and gives a method to synthesize quantum logic operations on a two-level system in minimum time.

PACS numbers: 02.30.Yy, 03.65.Aa, 03.67.-a

Keywords: Optimal control, SU(2), Quantum dynamics

*I am grateful to D. D’Alessandro for many helpful discussions. Work supported by ARO MURI under Grant W911NF-11-1-0268
†Electronic address: rromano@iastate.edu
I. INTRODUCTION

Control theory studies how the dynamics of a system can be modified through suitable external actions called *controls* [1]. When applied to quantum systems, it provides tools for the study of the feasibility and optimization of particular operations, for instance, in quantum information processing [2], in atomic and molecular physics, and in Nuclear Magnetic Resonance (NMR) [3]. In this work, we explore the time-optimal control [15, 16] of the dynamics of a two-level system (or qubit), the basic unit in quantum information and quantum computation. For its fundamental role, the control of this system has been studied in several works, under many different assumptions (see for example [4–10] and references therein). Here, we provide a complete characterization of the time-optimal trajectories, assuming that the dynamics can contain a non-controllable part (the drift), and that the controllable part depends on two or three independent control functions.

From a mathematical point of view, we introduce some new key tools which enable a simple and comprehensive treatment of the system, and the extension of the results in [10]. In particular, our analysis holds for any relative strength between controllable and non-controllable dynamics. The drift might be a dominant contribution, a perturbation, or a comparable term with respect to the controlled part. Our analysis is relevant whenever it is not accurate to assume that quantum operations can be performed in null time, that is, through infinitely strong controls.

The system dynamics is expressed through the Schrödinger operator equation

\[
\dot{X}(t) = -i(\omega_0 S_z + u_x S_x + u_y S_y + u_z S_z)X(t)
\]

with initial condition \(X(0) = I\). The operator \(X(t)\), an element of the special unitary group \(SU(2)\), realizes the time evolution as \(\rho(t) = X(t)\rho(0)X^\dagger(t)\), where \(\rho(t)\) is the statistical operator associated to the system. The three functions of time \(u_k = u_k(t), k = x, y, z\) are the control parameters, which we assume bounded by

\[
u_x^2 + u_y^2 + u_z^2 \leq \gamma^2.
\]

Later, we will assume that only \(u_x\) and \(u_y\) can be used to affect the dynamics, i. e., we will set \(u_z = 0\). The generators \(S_k, k = x, y, z\), are given by \(S_k = \frac{1}{2} \sigma_k\), where

\[
\begin{align*}
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]
are the Pauli matrices, with commuting relations $[\sigma_k, \sigma_l] = 2i\sigma_m$, where $(k, l, m)$ is a cyclic permutation of $(x, y, z)$. The first contribution might be an arbitrary, static drift term which can always be written as in $[\Pi]$ by a suitable redefinition of the Pauli matrices and of the control functions.

In this work, we characterize the time optimal trajectories in $SU(2)$ for any final operator $X_f = X(t_f)$, where $t_f$ is the minimum time for the transition $I \to X_f$. We derive the corresponding optimal controls $u_x, u_y$ and $u_z$ for arbitrary values of $\omega_0$ and $\gamma$, and provide a complete description of the reachable sets in $SU(2)$ at any time $t$, that is, the family of operators the system evolution can be mapped to in the given time $t$ (see Definition $[\Pi]$ below). In particular, we derive worst-case operators and times. It follows from standard results in geometric control theory that the system is controllable, and every final $X_f$ can be reached at some finite time. An important ingredient of our analysis is a representation of elements of the special unitary group which solely relies on two parameters, providing a clear description of optimal trajectories and reachable sets in terms of the evolution of the boundary of the reachable sets themselves.

The plan of this work is as follows. We start our analysis by considering a system with three arbitrary controls (in control theoretical jargon, a fully actuated system). In Section $[\Pi]$ we review the Pontryagin maximum principle of optimal control $[11]$, which is the starting point of our analysis, and we derive the necessary conditions for optimal controls. By using them, in Section $[\Pi\Pi]$ we explicitly compute the candidate optimal trajectories in $SU(2)$, and represent them in the chosen parametrization of the special unitary group. We introduce the notion of optimal front line, which describes the evolution of the boundary of the reachable set. By using it, in Section $[\Pi\Pi\Pi]$ we characterize the evolution of reachable sets in the three cases $\gamma > |\omega_0|$, $\gamma = |\omega_0|$ and $\gamma < |\omega_0|$, and provide the optimal times, whenever an analytical expression is possible. We also derive the worst-case operators and the relative times. In Section $[\Pi\Pi\IV]$ we use the same ideas and formalism to fully characterize the reachable sets (and related quantities) in the case where only two controls affect the dynamics. In such a case, we find that there are different evolution for the reachable sets in the cases $\gamma \geq |\omega_0|$, $\frac{1}{\sqrt{3}}|\omega_0| < \gamma < |\omega_0|$ and $\gamma \leq \frac{1}{\sqrt{3}}|\omega_0|$. In Section $[\Pi\Pi\V]$ we provide examples of applications by particularizing our results to some special target operators: diagonal operators and the SWAP operator. This is done in both scenarios of two of three controls. In Section $[\Pi\Pi\VI]$ we compare our work to existing results on the optimal control on $SU(2)$, describe possible
II. THE PONTRYAGIN MAXIMUM PRINCIPLE

Given an arbitrary final operator $X_f$, we consider a trajectory $X(t)$ in $SU(2)$, determined by control functions $v_k = v_k(t)$, $k = x, y, z$, such that $X(0) = I$ and $X(t_f) = X_f$. A basic tool for the study of optimal control problems is given by the Pontryagin maximum principle, which, in the context of control of system (1) on the Lie group $SU(2)$, takes the following form.

**Definition 1** The Pontryagin Hamiltonian is defined as

\[ H(M, X, v_x, v_y, v_z) = i \left( \omega_0 \langle M, X^\dagger S_x X \rangle + \sum_{k=x,y,z} v_k \langle M, X^\dagger S_k X \rangle \right) \]

where $M \in su(2)$, and $\langle A, B \rangle \equiv \text{Tr} (A^\dagger B)$.

**Proposition II.1** (Pontryagin maximum principle) Assume that a control strategy $u_k = u_k(t)$, $k = x, y, z$, with $u_x^2 + u_y^2 + u_z^2 \leq \gamma^2$, and the corresponding trajectory $\tilde{X}(t)$ are optimal (that is, the final time $t_f$ is minimal). Then there exists $\tilde{M} \in su(2)$, $\tilde{M} \neq 0$, such that $H(\tilde{M}, \tilde{X}, u_x, u_y, u_z) \geq H(\tilde{M}, \tilde{X}, v_x, v_y, v_z)$ for every $v_x, v_y, v_z$ such that $v_x^2 + v_y^2 + v_z^2 \leq \gamma^2$.

Define the coefficients

\[ b_k = i \langle M, X^\dagger S_k X \rangle, \quad k = x, y, z. \]

By using the Lagrange multipliers method to maximize (4) with the bound (2), we find that the optimal controls satisfy

\[ u_k = \gamma \frac{b_k}{\sqrt{b_x^2 + b_y^2 + b_z^2}}, \quad k = x, y, z. \]

Arcs where the Pontryagin Hamiltonian is independent of the control functions are called *singular* and, on them, the controls are not constrained by equations like (6). In general, an optimal trajectory will be a concatenation of singular and non-singular arcs, and, usually, the presence of singular arcs makes the solution of the optimal control problem more difficult, and more sophisticated mathematical tools are required to face the problem (see for instance [12] for a general analysis on 2-dimensional manifolds, or [13, 14] for some recent applications.
of the Pontryagin maximum principle when singular arcs are present). In our scenario, a singular trajectory would require \( b_x = b_y = b_z = 0 \) in some interval, but this is impossible, because in this case \( \tilde{M} \) would vanish, and this is excluded by the Pontryagin principle. Therefore, we can conclude that trajectories containing singular arcs are never optimal.

The dynamics of the \( b_k \) coefficients can be derived by differentiating (5) with respect to \( t \), and by using the commutation relations among Pauli matrices. We find

\[
\begin{align*}
\dot{b}_x &= -(\omega_0 + u_z) b_y + u_y b_z, \\
\dot{b}_y &= (\omega_0 + u_z) b_x - u_x b_z, \\
\dot{b}_z &= u_x b_y - u_y b_x.
\end{align*}
\]

By considering the form of optimal controls (6), we obtain that \( \dot{b}_z = 0 \), that is, \( b_z \) is constant. Moreover, using (6) in (7),

\[
\begin{align*}
b_x &= \mu \cos (\omega_0 t + \varphi), \\
b_y &= \mu \sin (\omega_0 t + \varphi),
\end{align*}
\]

where \( \mu \geq 0 \) and \( \varphi \) are constants. Therefore, the candidate optimal controls are given by

\[
\begin{align*}
u_x &= \gamma \sqrt{1 - \alpha^2} \cos (\omega_0 t + \varphi) \\
u_y &= \gamma \sqrt{1 - \alpha^2} \sin (\omega_0 t + \varphi) \\
u_z &= \gamma \alpha,
\end{align*}
\]

and \( \alpha \in [-1, 1] \) is given by

\[
\alpha = \frac{b_z}{\sqrt{b_x^2 + b_y^2 + b_z^2}}.
\]

Because of the special form of the candidate optimal controls, the dynamics (11) can be integrated, as we prove in the next section.

III. EXTREMAL TRAJECTORIES IN SU(2)

We substitute the extremals (6) in (11), and find the corresponding extremal trajectories in \( SU(2) \). To proceed, it is convenient to counter-evolve the drift of the system, by passing to the interaction picture of the dynamics,

\[
Z = e^{i\omega_0 S_z t} X, \quad Z(0) = I.
\]
With this substitution the differential evolution for $Z$ is given by

$$\dot{Z} = -i\gamma \left( \sqrt{1 - \alpha^2}(\cos \varphi S_z + \sin \varphi S_y) + \alpha S_z \right) Z,$$

(12)

which is simpler to integrate because the generator is time-independent. In the adopted representation we find that

$$Z = \begin{pmatrix} \cos \gamma \tau - i\alpha \sin \gamma \tau & -i\sqrt{1 - \alpha^2} e^{-i\varphi} \sin \gamma \tau \\ -i\sqrt{1 - \alpha^2} e^{i\varphi} \sin \gamma \tau & \cos \gamma \tau + i\alpha \sin \gamma \tau \end{pmatrix},$$

(13)

where we have defined $\tau = t^2$ to simplify the notation. Throughout this paper, we will switch between $t$ and the re-scaled time $\tau$, possibly with subscripts, without further comments. By using (11), we compute

$$X = \begin{pmatrix} e^{-i\omega_0 \tau} \left( \cos \gamma \tau - i\alpha \sin \gamma \tau \right) & -i\sqrt{1 - \alpha^2} e^{-i(\omega_0 \tau + \varphi)} \sin \gamma \tau \\ -i\sqrt{1 - \alpha^2} e^{i(\omega_0 \tau + \varphi)} \sin \gamma \tau & e^{i\omega_0 \tau} \left( \cos \gamma \tau + i\alpha \sin \gamma \tau \right) \end{pmatrix}.$$  

(14)

This is the form of the extremal trajectories in $SU(2)$. They depend on the two parameters $\alpha$ and $\varphi$ (which can be tuned via $u_x$, $u_y$, $u_z$) as well as on the fixed parameters $\omega_0$ and $\gamma$. To find the optimal trajectory for a given final state $X_f$, one has to determine the values of $\alpha$ and $\varphi$ such that the transition $I \rightarrow X_f$ takes the minimal time. This is conveniently done by choosing a suitable representation of $SU(2)$, described in the following.

**Remark III.1** An arbitrary operator $X \in SU(2)$ can be given the following representation:

$$X = \begin{pmatrix} r e^{i\psi} & \sqrt{1 - r^2} e^{i\phi} \\ -\sqrt{1 - r^2} e^{-i\phi} & r e^{-i\psi} \end{pmatrix}.$$  

(15)

Therefore, $X$ is described in terms of three parameters: $r$, $\psi$ and $\phi$. It turns out that, in the control scenario at hand, the optimal time does not depend on the parameter $\phi$. In other words, all the operators in $SU(2)$ which differ only for the value on $\phi$ are reached in the same optimal time. In fact, in (14) it is possible to arbitrarily change the phase of the off-diagonal terms by suitably choosing $\varphi$. This parameter enters the analysis only in the phase of the off-diagonal terms, and it is independent of the choice of $\alpha$. Therefore, to fully characterize the optimal trajectories in $SU(2)$ and the reachable sets, we can limit our attention to the upper diagonal element of $X(t)$, which is sufficient to determine $r$ and $\psi$.

This result can also be proven by adopting the argument in Proposition 2.1 in [10], where it is shown that the minimum time to reach $X$ and $e^{i\xi \sigma_z} X e^{-i\xi \sigma_z}$ is the same for all real $\xi$. 

6
when there are two controls $u_x$ and $u_y$. The two operators differ only for the phase of the off-diagonal entries. The proof is valid also for a fully actuated system.

According to the previous remark, we shall parameterize $SU(2)$ solely by $r$ and $\psi$, or $x$ and $y$ in the equivalent representation $re^{i\psi} = x + iy$. A point in the unit disk in the $(x, y)$ plane represents a family of matrices in $SU(2)$ which only differ by the phase of the anti-diagonal elements. These matrices are reached in the same minimum time. Moreover, every candidate optimal trajectory can be represented by its projection onto the unit disk with the understanding that any trajectory corresponds to a family of trajectories only differing by the phase $\varphi$. Points on the border of the unit disk ($r = 1$) correspond to diagonal matrices, and the initial point, the identity matrix, corresponds to the point $x = 1$, $y = 0$.

By direct inspection of (14) we have

$$
x_\alpha(\tau) \equiv x(\tau) = \cos \omega_0 \tau \cos \gamma \tau - \alpha \sin \omega_0 \tau \sin \gamma \tau,
$$

$$
y_\alpha(\tau) \equiv y(\tau) = -\sin \omega_0 \tau \cos \gamma \tau - \alpha \cos \omega_0 \tau \sin \gamma \tau.
$$

with $\alpha \in [-1, 1]$. For $\alpha = -1$ we find

$$
x_{-1}(\tau) = \cos (\gamma - \omega_0) \tau, \quad y_{-1}(\tau) = \sin (\gamma - \omega_0) \tau,
$$

and for $\alpha = 1$

$$
x_{1}(\tau) = \cos (\gamma + \omega_0) \tau, \quad y_{1}(\tau) = -\sin (\gamma + \omega_0) \tau.
$$

These trajectories lie on the border of the unit disk. Moreover, by multiplying the first equation in (16) by $\cos \omega_0 \tau$, and the second equation by $\sin \omega_0 \tau$, and subtracting the results, we eliminate the parameter $\alpha$, and obtain

$$
y(\tau) \sin \omega_0 \tau - x(\tau) \cos \omega_0 \tau + \cos \gamma \tau = 0.
$$

This relation is a constraint on the terminal points of the candidate optimal trajectories at time $t = 2\tau$, for arbitrary $\alpha$: they lie on a line with time-dependent slope and intercept. This can also be seen by noticing that we can recast (16) in the form

$$
x_\alpha(\tau) + iy_\alpha(\tau) = \frac{1 - \alpha}{2} e^{i(\gamma - \omega_0)\tau} + \frac{1 + \alpha}{2} e^{-i(\gamma + \omega_0)\tau},
$$

which explicitly shows the special role played by the trajectories with $|\alpha| = 1$. 

7
As $\alpha$ varies in $[-1, 1]$, Eq. (20) describes a segment connecting two points on the unit circle. The end points rotate with uniform speed on the disk border, unless $|\gamma| = \omega_0$, since in this case one of them is fixed in $(1, 0)$. Extremal trajectories are parameterized by $\alpha \in [-1, 1]$. In general, since there is a one-to-one correspondence between points of the segment and $\alpha \in [-1, 1]$, two extremal trajectories cannot reach the same point in exactly the same time. Consequently, there are not overlaps points where two (or more) different extremal trajectories intersect. The only exception to this behavior is when the aforementioned segment collapses to a point. This scenario will arise only when $\gamma > \omega_0$, at the worst case time.

The segment we have just described, and its generalization to the case of two controls only in Sec. V will be a fundamental ingredient in the analysis of reachable sets and optimal times. Therefore, we find it convenient to assign a specific name to it: the optimal front-line $F_t$. More precisely, we can write $F_t \equiv F_t(-1 \leq \alpha \leq 1)$ and we will use this notation to represent subsets of the front line, as for instance $F_t(\alpha_1 \leq \alpha \leq \alpha_2)$.

Given an arbitrary final state $X_f$, represented in the unit disk by $r_f e^{i\psi_f} = x_f + iy_f$, in order to find the optimal trajectory leading to it, we have to require that $x_\alpha(\tau_f) = x_f$ and $y_\alpha(\tau_f) = y_f$. The minimal time $t_f = 2\tau_f$ is the smallest $t_f$ such that $(x_f, y_f)$ is in the optimal front line. The corresponding $\alpha$ determines the optimal control strategy. The optimal minimum time can also be calculated analytically or numerically as follows. From (16) we find that

$$
\begin{align*}
  r_f \cos \psi_f &= \cos \omega_0 \tau_f \cos \gamma \tau_f - \alpha \sin \gamma_0 \tau_f \sin \gamma \tau_f, \\
  r_f \sin \psi_f &= -\sin \omega_0 \tau_f \cos \gamma \tau_f - \alpha \cos \gamma_0 \tau_f \sin \gamma \tau_f.
\end{align*}
$$

In (21), if we multiply the first equation by $\cos \omega_0 \tau_f$, the second by $\sin \omega_0 \tau_f$, and then we subtract them, we obtain

$$
  r_f \cos \left(\omega_0 \tau_f + \psi_f\right) = \cos \gamma \tau_f.
$$

The minimum time is the smallest $t_f = 2\tau_f$ for which this equation is valid. Furthermore, by squaring the two equations in (21) and summing them, we find

$$
  r_f^2 = \cos^2 \gamma \tau_f + \alpha^2 \sin^2 \gamma \tau_f,
$$

from which $\alpha$ can be found, given the prior knowledge of $t_f$. In principle, this approach can be used to find the optimal strategy for any final target operation. However, a geometrical
analysis of the optimal front line provides much more information on how the states are reached, further insights on the optimal times, and the geometry of the reachable sets.

IV. PROPERTIES OF THE REACHABLE SETS AND OPTIMAL TIMES

**Definition 2** The reachable set at time \( t \) is the set \( \mathcal{R}_t \) of operators \( Y \in SU(2) \) such that there are control strategies \( v_k \) \( (k = x, y, z) \), with \( v_x^2 + v_y^2 + v_z^2 \leq \gamma \), driving \( X(0) = I \) to \( X(t) = Y \) at time \( t \), under the evolution (1). The reachable set until time \( t \) is the set \( \mathcal{R}_{\leq t} \) of operators \( Y \in SU(2) \) such that there are control strategies \( v_k \) \( (k = x, y, z) \), with \( v_x^2 + v_y^2 + v_z^2 \leq \gamma \), driving \( X(0) = I \) into \( X(s) = Y \) with \( s \leq t \), under the evolution (1).

The two sets are related by

\[
\mathcal{R}_{\leq t} = \bigcup_{s \leq t} \mathcal{R}_s. \tag{24}
\]

The structure of these sets is a direct consequence of the evolution of the aforementioned optimal front-line \( \mathcal{F}_t \). It is a known fact in optimal control theory that, if a trajectory is optimal for \( X_f \) at time \( t \), then \( X_f \) belongs to the boundary of the reachable set until time \( t \), i.e. \( X_f \in \partial \mathcal{R}_{\leq t} \). Therefore, if a point of the unit disk is reached by \( \mathcal{F}_t \) for the first time at time \( t \), it belongs to \( \partial \mathcal{R}_{\leq t} \). However, in general not all points of the optimal front line belong to \( \partial \mathcal{R}_{\leq t} \), because they might be included in front lines corresponding to earlier times. Therefore, our strategy is to study the evolution of the front lines, and an important role will be played by the curve where \( \mathcal{F}_t \) intersects \( \mathcal{F}_{t+dt} \). This curve contains the points where optimal trajectories loose their optimality. We illustrate the procedure in the three different scenarios, depending on the relative values of \( \omega_0 \) and \( \gamma \). A generalization of this idea will be used in Sec. V as well, with the difference that we will find several intersection curves between \( \mathcal{F}_t \) and \( \mathcal{F}_{t+dt} \).

**A. The case \( \gamma > |\omega_0| \)**

This is the case where the control action is assumed to be more powerful than the natural evolution of the system. In this case, \( \gamma - \omega_0 \) and \( \gamma + \omega_0 \) are both positive. Therefore, following (17) and (18), the extremal points of the optimal front-line rotate in opposite directions along the unit-circle, with constant angular speed. This shows that \( \mathcal{F}_t \) and \( \mathcal{F}_{t+dt} \) do not intersect.
FIG. 1: (Color online) Time evolution of the reachable sets in the unit disk, when $\gamma = 3$ and $\omega_0 = 1$. We have represented the optimal-front line and the optimal trajectories for several values on $\alpha$, at growing time. The worst-case operator is marked by a small circle.

on the unit disk, although this fact could be proved by a direct computation. Therefore, all the trajectories ending on the front line are optimal. During its evolution, $F_t$ spans all the unit disk, and eventually collapses to a point on the border of the disk, defined by the condition

$$e^{i(\gamma - \omega_0)\tau} = e^{-i(\gamma + \omega_0)\tau}. \tag{25}$$

The corresponding worst-case time is $t_{max} = \frac{2\pi}{\gamma}$, and $(-\cos \frac{\pi \omega_0}{\gamma}, \sin \frac{\pi \omega_0}{\gamma})$ is the collapsing point. The worst-case time is independent of $\omega_0$ because the relative angular velocity between the extremal points of the optimal front-line depends only on $\gamma$. Notice that $I$ and the worst-case operator are conjugate points, since there is a one-parameter family of geodesics connecting them (the parameter is $\alpha$).

All the points in the unit disk are reached in an optimal time $t_f \leq t_{max}$. See Fig. 1 for a graphical representation of the evolution of the reachable sets in a specific case. Notice that, as a special case, we can consider $\omega_0 = 0$, that is, there is not drift in the dynamics of the system. The corresponding worst-case operator is represented by the point $(-1, 0)$.

B. The case $\gamma = |\omega_0|$

In this case, the strength of the control action is the same as the free evolution of the system. One of the extremal points of the optimal front-line is fixed at $(1, 0)$, and the optimal front-line rotates about it. This point corresponds to $\alpha = \pm 1$ when $\omega_0 = \pm \gamma$, respectively. The analysis is analogous to the previous case, with the optimal trajectories ending on
FIG. 2: (Color online) Time evolution of the reachable sets in the unit disk, when $\gamma = 1$ and $\omega_0 = 1$. As before, we have represented the optimal-front line, some optimal trajectories and the worst-case operator.

$F_t(-1 < \alpha \leq 1)$ when $\omega_0 = \gamma$, and on $F_t(-1 \leq \alpha < 1)$ when $\omega_0 = -\gamma$. The worst-case time is again $t_{max} = \frac{2\pi}{\gamma}$, and the worst-case operator is represented by $(1, 0)$. However, in this case it is possible to derive analytically the values for $t_f$ and $\alpha$ for a given final state $X_f$, since the optimal trajectories are circles. In fact, for $\omega_0 = \pm \gamma$ a direct computation shows that (16) is consistent with

$$
(x(\tau) - \frac{1 \mp \alpha}{2})^2 + y(\tau)^2 = \left(\frac{1 \pm \alpha}{2}\right)^2,
$$

and then, for any value of $\alpha$, the trajectory is a circle of radius $\frac{1 \pm \alpha}{2}$, centered in $(\frac{1 \mp \alpha}{2}, 0)$. In the two cases, the optimal controls for a target $X_f$, represented by $(x_f, y_f)$, are given by

$$
\alpha = \mp \left(x_f + \frac{y_f^2}{x_f - 1}\right),
$$

and the optimal times are

$$
 t_f = \frac{2\pi}{\gamma} - \frac{1}{\gamma} \arctan \frac{2y_f(1 - x_f)}{y_f^2 - (1 - x_f)^2}
$$

for $\omega_0 = \gamma$, and

$$
 t_f = \frac{1}{\gamma} \arctan \frac{2y_f(1 - x_f)}{y_f^2 - (1 - x_f)^2}
$$

for $\omega_0 = -\gamma$. See Fig 2 for a pictorial representation of the evolution of the reachable sets in a special case.
FIG. 3: (Color online) Time evolution of the optimal front line for $\gamma = 1$ and $\omega_0 = 3$. The front lines are shown in even increments of time in the interval $t \in [1.2, 1.55]$. The critical optimal trajectory is the locus of self-intersections of these lines, where they lose their optimality. The dashed segments are the non-optimal parts of the front lines, since these points have already been reached at a former time.

C. The case $\gamma < |\omega_0|$

In this case, the strength of the control action is smaller than that of the free evolution. In the limit of $\gamma$ small, the control can be seen as a perturbation to the dynamics. The analysis is more complicated, because the optimal front-lines at time $t$ have a self-intersection during their evolution. Therefore, some trajectories ending on the optimal front-line $F_t$ will not be optimal. The geometric explanation of this behavior is that, in this case, the end points of $F_t$ rotate in the same direction, generating at each time a rotation of this segment about one of its points. To determine this point at time $t = 2\tau$ we have to require that it is in both $F_t$ and $F_{t+dt}$. According to (19), we have to impose that $(x(\tau), y(\tau))$ satisfies

$$
\begin{align}
\begin{cases}
y(\tau) \sin \omega_0 \tau - x(\tau) \cos \omega_0 \tau + \cos \gamma \tau = 0, \\
y(\tau) \sin \omega_0 (\tau + d\tau) - x(\tau) \cos \omega_0 (\tau + d\tau) + \cos \gamma (\tau + d\tau) = 0,
\end{cases}
\end{align}
$$

(30)

and the second condition can be replaced by

$$
y(\tau) \omega_0 \cos \omega_0 \tau + x(\tau) \omega_0 \sin \omega_0 \tau - \gamma \sin \gamma \tau = 0.
$$

(31)
We find that the unique solution at time $t = 2\tau$ is given by

$$x_c(\tau) \equiv \frac{\gamma}{\omega_0} \sin \omega_0 \tau \sin \gamma \tau + \cos \omega_0 \tau \cos \gamma \tau,$$

$$y_c(\tau) \equiv \frac{\gamma}{\omega_0} \cos \omega_0 \tau \sin \gamma \tau - \sin \omega_0 \tau \cos \gamma \tau,$$

(32)

and, by comparing (32) and (16), we notice that the locus of self-intersections of the optimal front-line, described by (32), is itself an extremal trajectory for the system, corresponding to $\alpha_c = -\frac{\gamma}{\omega_0}$, which we call the critical trajectory. The value $\alpha_c$ is critical, in the sense that trajectories can be optimal only for $\alpha \in [-1, \alpha_c]$ when $\omega_0 < 0$, and $\alpha \in [\alpha_c, 1]$ when $\omega_0 > 0$. This can be understood by considering that for $\omega_0 < 0$ the end point of the optimal front-line corresponding to $\alpha = -1$ foreruns the other end point, and similarly for $\omega_0 > 0$. Fig. 3 shows how the optimal front lines generate the critical trajectory during their evolution.

The critical trajectory is a well-known concept in optimal geometric control theory, where it is called the cut locus. In fact, for a given initial point, the cut locus is defined as the set of points where the extremal trajectories lose their optimality. We will shortly see that, in the regime under investigation, all the optimal trajectories lose their optimality on the critical trajectory. Therefore, our analysis of the optimal front-line represents a simple approach for determining the cut locus. Notice that, when $\gamma \geq \omega_0$, the cut locus reduces to a point, corresponding to the worst-case operator. This is the conjugate point to the initial point $(1, 0)$.

The critical trajectory has a singular point when $\dot{x}_c(\tau) = \dot{y}_c(\tau) = 0$. This point is a cusp singularity, whose appearance can be geometrically understood by considering the evolution of the optimal front-line [18]. From

$$\dot{x}_c(\tau) = \left(\frac{\gamma^2 - \omega_0^2}{2\omega_0^2}\right) \sin \omega_0 \tau \cos \gamma \tau,$$

$$\dot{y}_c(\tau) = \left(\frac{\gamma^2 - \omega_0^2}{2\omega_0^2}\right) \cos \omega_0 \tau \cos \gamma \tau,$$

(33)

we find that $t_c = 2\tau_c = \frac{\pi}{\gamma}$, and the singular point of the critical trajectory is

$$x_c(\tau_c) = \frac{\gamma}{\omega_0} \sin \frac{\pi \omega_0}{2\gamma}, \quad y_c(\tau_c) = \frac{\gamma}{\omega_0} \cos \frac{\pi \omega_0}{2\gamma}.$$

(34)

It turns out that this is the point where the critical trajectory loses optimality. In fact, when $t > t_c$, the points of the critical trajectory are in the reachable set until time $t$, and then they have already been reached at a former time.
FIG. 4: (Color online) Time evolution of the reachable sets in the unit disk, when $\gamma = 1$ and $\omega_0 = 3$. The evolution of the optimal-front line generates the critical optimal trajectory. The worst-case operator is marked by a small circle. Every trajectory loses optimality when intersecting the critical trajectory, which then represents a cut locus for the system.

Any other optimal trajectory loses optimality at some time, when it intersects the reachable set until that time. The boundary of the reachable set until time $t$, $\partial R_{\leq t}$, is given by the optimal front-line and the critical trajectory. Since the self-intersections of the optimal front line form themselves an extremal trajectory for the system, an optimal trajectory can lose optimality only by intersecting the critical trajectory. For this reason, as mentioned before, the critical trajectory is a cut locus for this system.

If we denote by $t_i = 4\pi/(\gamma + |\omega_0|)$ the time when the optimal front-line will come back to the point $(1, 0)$, we can conclude that, for $t < \min(t_i, t_c)$, $\mathcal{F}_t(-1 \leq \alpha \leq 0)$ describes the terminal points of the optimal trajectories when $\omega_0 < 0$. Analogously, these terminal points are given by $\mathcal{F}_t(\alpha_c \leq \alpha \leq 1)$ when $\omega_0 > 0$. For $t > \min(t_i, t_c)$, the extremal trajectories which are still optimal end on $\mathcal{F}_t(\alpha_1 \leq \alpha \leq \alpha_2)$, where $\alpha_1$ and $\alpha_2$ are determined by the intersection of $\mathcal{F}_t$ and the critical trajectory. In general, their analytical derivation is not possible. However, we can determine the worst-case time $t_{\text{max}}$ and the corresponding $\alpha$: these are obtained by requiring that $\mathcal{F}_t$ becomes tangent to the critical trajectory at some point. If we assume that this point is reached at time $\bar{t} = 2\bar{\tau}$, we can write it as $(x_c(\bar{\tau}), y_c(\bar{\tau}))$. The tangent to the critical trajectory in this point is given by

$$\dot{x}_c(\bar{\tau})(y - y_c(\bar{\tau})) = \dot{y}_c(\bar{\tau})(x - x_c(\bar{\tau})), \quad (35)$$

which, considering the explicit expressions of $x_c$, $y_c$, $\dot{x}_c$ and $\dot{y}_c$ from (32) and (33), can be
recast in the form

\[ y \sin \omega_0 \bar{\tau} - x \cos \omega_0 \bar{\tau} + \cos \gamma \bar{\tau} = 0, \]  

(36)

with \( \bar{t} \in [0, t_c] \). We require that this line coincides with the optimal front-line \( \text{(19)} \) at some later time \( t > t_c \). Therefore

\[ \sin \omega_0 \tau = \pm \sin \omega_0 \bar{\tau}, \quad \cos \omega_0 \tau = \pm \cos \omega_0 \bar{\tau}, \quad \cos \gamma \tau = \pm \cos \gamma \bar{\tau}, \]  

(37)

which, with the further constraint \( 0 \leq \bar{t} \leq t_c \leq t \), is solved by

\[ \bar{t} = t_c \left(1 - \frac{\gamma}{|\omega_0|}\right), \quad t = t_c \left(1 + \frac{\gamma}{|\omega_0|}\right). \]  

(38)

Therefore, the worst-case time for \( \gamma < |\omega_0| \) is

\[ t_{\text{max}} = 2\tau_{\text{max}} = \frac{\pi}{\gamma} \left(1 + \frac{\gamma}{|\omega_0|}\right), \]  

(39)

which is consistent with the result found when \( \gamma = |\omega_0| \). The worst-case point in the unit disk is arbitrarily close to \( (x_c(\bar{\tau}), y_c(\bar{\tau})) \), and it is approached through the optimal trajectory characterized by \( \alpha = -\alpha_c \). This can be seen by requiring that

\[ r^2(\tau_{\text{max}}) = r_c^2(\bar{\tau}) \]  

(40)

and using that

\[ \sin \gamma \bar{\tau} = \sin \gamma \tau_{\text{max}}, \]  

(41)

a direct consequence of (38) and (39). It turns out that (40) is equivalent to \( \alpha = \alpha_c^2 \), and \( \alpha = \alpha_c \) is not admitted since it corresponds to the critical optimal trajectory. In Fig 4 we provide a graphical representation of the evolution of the reachable sets in a special case.

As \( \gamma \) decreases, the critical trajectory stretches and spirals around the center of the unit disk. Eventually, when \( \gamma \to 0 \), the singular point of the critical trajectory approaches the center of the unit disk. In this limit, this point represents the worst case operator, which is reached only asymptotically \( (t_{\text{max}} \to \infty) \).

V. THE CASE WITH TWO CONTROLS

In this section we consider the case where \( u_z = 0 \) in (11), that is, the control action enters only through \( S_y \) and \( S_z \). This is not the most general case of dynamics with two controls
and a drift term, which could contain also contributions along $S_y$ and $S_z$. However, the general scenario cannot be described with the $SU(2)$ representation adopted in this work, since, in this case, to operators differing by the phase $\phi$ in the off-diagonal elements there usually correspond different optimal times.

A. Optimal controls and trajectories

This problem has been recently considered in [10], and, by using a different approach, the optimal trajectories have been derived under the condition $\frac{1}{\sqrt{3}}\omega_0 \leq \gamma \leq \omega_0$. Following the procedure outlined in the previous sections, we are able to fully characterize the reachable sets (and related properties) for arbitrary values of $\omega_0$ and $\gamma$. In particular, $\omega_0$ can be both positive, negative, or null. Under the constraint $u_x^2 + u_y^2 \leq \gamma^2$, we find that the optimal controls must satisfy

$$u_k = \gamma \frac{b_k}{\sqrt{b_x^2 + b_y^2}}, \quad k = x, y$$

and $b_k$, $k = x, y, z$ are defined as in (5). Their dynamics is given by

$$\dot{b}_x = \omega_0 b_y - u_y b_z,$$
$$\dot{b}_y = -\omega_0 b_x + u_x b_z,$$
$$\dot{b}_z = -u_x b_y + u_y b_x,$$

and, by using (42) in (43), we obtain that $b_z$ is constant. Moreover, we find

$$b_x = \mu \cos (\omega t + \phi), \quad b_y = \mu \sin (\omega t + \phi),$$

where $\mu$ and $\phi$ are two constants, and $\omega$ is given by

$$\omega = \omega_0 - \frac{\gamma b_z}{\mu}.$$ 

The candidate optimal controls have the form

$$u_x = \gamma \cos (\omega t + \phi), \quad u_y = \gamma \sin (\omega t + \phi).$$

Since $b_z$ is unconstrained, $\omega$ can assume any real value. Singular arcs are given by $b_x = b_y = 0$ on some interval, which implies $\dot{b}_z = 0$ and $u_x = u_y = 0$ in that interval. Following the argument of [10], it is possible to prove that, also in this case, singular arcs can never contribute to an optimal trajectory.
Integration of the dynamics (1) follows the same lines outlined before (with the intermediate operator $Z = e^{i\omega S_z t}X$), and the final result is

$$X = \begin{pmatrix} e^{-i\omega \tau} \left( \cos a\tau - i\frac{b}{a} \sin a\tau \right) & -i\frac{\gamma}{a} e^{-i(\omega\tau + \phi)} \sin a\tau \\ -i\frac{\gamma}{a} e^{i(\omega\tau + \phi)} \sin a\tau & e^{i\omega \tau} \left( \cos a\tau + i\frac{b}{a} \sin a\tau \right) \end{pmatrix}, \quad (47)$$

where we have defined $b = b(\omega) \equiv \omega_0 - \omega$, $a = a(\omega) \equiv \sqrt{b^2 + \gamma^2}$, and $\tau = \frac{t}{2}$. The candidate optimal trajectories, in the adopted representation of $SU(2)$ (see Remark III.1), are obtained by taking the real and imaginary parts of the upper diagonal element in (47):

$$x_\omega(\tau) \equiv x(\tau) = \cos \omega\tau \cos a\tau - \frac{b}{a} \sin \omega\tau \sin a\tau,$$

$$y_\omega(\tau) \equiv y(\tau) = -\sin \omega\tau \cos a\tau - \frac{b}{a} \cos \omega\tau \sin a\tau.$$

In analogy with the case of a fully actuated system, one could numerically solve these equations for an arbitrary final operator $X_f$ reached in minimal time $t_f = 2\tau_f$. However, in this work we are mainly interested in studying the evolution of the reachable sets by introducing the optimal front line and studying its evolution.

**B. The optimal front-line**

As before, we define the optimal front-line as the set of terminal points for a candidate optimal trajectory at time $t = 2\tau$:

$$\mathcal{F}_t(-\infty < \omega < \infty) \equiv \{(x_\omega(\tau), y_\omega(\tau)), -\infty < \omega < \infty\}. \quad (49)$$

It is possible to verify that there is a one-to-one correspondence between $\omega$ and points on $\mathcal{F}_t$. This can be seen, for instance, by rewriting (48) in polar coordinates

$$r^2_\omega(\tau) = x^2_\omega(\tau) + y^2_\omega(\tau) = 1 - \frac{\gamma^2}{a^2} \sin^2 a\tau \quad (50)$$

and

$$\psi_\omega(t) = \begin{cases} \omega \tau + \arctan \left( \frac{b}{a} \tan a\tau \right), & \text{if } 0 \leq \tau < \frac{\pi}{2a}, \\ \pi + \omega \tau + \arctan \left( \frac{b}{a} \tan a\tau \right), & \text{if } \frac{\pi}{2a} < \tau < \frac{\pi}{a}. \end{cases} \quad (51)$$

Although it is possible to obtain $r^2_{\omega_1}(\tau) = r^2_{\omega_2}(\tau)$ with $\omega_1 \neq \omega_2$, this necessarily implies $\psi_f(\omega_1) \neq \psi_f(\omega_2)$. Therefore, the correspondence $\omega \leftrightarrow (x_\omega(\tau), y_\omega(\tau))$ is one-to-one at any $\tau$. 17
Not all the extremal trajectories are optimal. Following the discussion of the previous sections, we have to consider the self-intersections of $\mathcal{F}_t$, as well as the intersections of $\mathcal{F}_t$ and $\partial R_{\leq t}$, in order to determine critical values of $\omega$ for which the trajectories lose optimality. In this case we must use the parametric expressions for the points of $\mathcal{F}_t$ since it is not possible to solve for $\omega$ one of the two equations in (48), and obtain a closed expression of the optimal front line in terms of $x$ and $y$ alone. Therefore, we cannot directly rely on the procedure developed in the previous sections. However, the optimal front line can be considered as the envelope of its tangent lines. Therefore, if there is a self-intersection of $\mathcal{F}_t$ in some point, there must also be a self-intersection of the tangent line to $\mathcal{F}_t$ in that point. Consequently, we can find the intersections of $\mathcal{F}_t$ and $\mathcal{F}_{t+dt}$ by considering, for each $\omega$, the intersections of the tangent lines to the optimal front-line at time $t$ and $t + dt$. If they are on $\mathcal{F}_t$, they correspond to the desired intersection of $\mathcal{F}_t$ and $\mathcal{F}_{t+dt}$, and the corresponding $\omega$ is a critical value, relevant for determining where the trajectories are optimal.

Again, by means of this simple analysis we are able to fully characterize the cut loci for this system. We will find not trivial cut loci for any value of $\omega_0$ and $\gamma$.

The slope of the tangent line to the optimal front line at time $t$, in the point labeled by $\omega$, is given by

$$\frac{dy}{dx} = \frac{dy}{d\omega} \left(\frac{dx}{d\omega}\right)^{-1}. \quad (52)$$

Since

$$\frac{dx}{d\omega} = -\frac{\gamma^2}{a^2} \sin \omega \tau \cos a \tau - \frac{1}{a} \sin a \tau, \quad \frac{dy}{d\omega} = -\frac{\gamma^2}{a^2} \cos \omega \tau \cos a \tau - \frac{1}{a} \sin a \tau, \quad (53)$$

we find that

$$\frac{dy}{dx} = \cot \omega \tau. \quad (54)$$

Therefore, the tangent line to $\mathcal{F}_t$ in the point $(x_\omega(\tau), y_\omega(\tau))$, at time $t = 2\tau$, is given by

$$y(\tau) \sin \omega \tau - x(\tau) \cos \omega \tau + \cos a \tau = 0. \quad (55)$$

The intersections of tangent lines to $\mathcal{F}_t$ and $\mathcal{F}_{t+dt}$ are obtained by solving the system

$$\begin{cases}
y(\tau) \sin \omega \tau - x(\tau) \cos \omega \tau + \cos a \tau = 0, \\
y(\tau) \sin \omega(\tau + d\tau) - x(\tau) \cos \omega(\tau + d\tau) + \cos a(\tau + d\tau) = 0,
\end{cases} \quad (56)$$
whose solution follows the same steps which have been detailed in the previous section.

When \( \omega \neq 0 \), we find the unique solution 19

\[
\begin{align*}
    x(\tau) &= \frac{a}{\omega} \sin \omega \tau \sin a\tau + \cos \omega \tau \cos a\tau, \\
    y(\tau) &= \frac{a}{\omega} \cos \omega \tau \sin a\tau - \sin \omega \tau \cos a\tau.
\end{align*}
\]

(57)

However, since the intersection point must be on \( F_t \), also (48) must be satisfied. Therefore

\[
\begin{align*}
    \frac{a}{\omega} \sin \omega \tau \sin a\tau &= -\frac{b}{a} \sin \omega \tau \sin a\tau, \\
    \frac{a}{\omega} \cos \omega \tau \sin a\tau &= -\frac{b}{a} \cos \omega \tau \sin a\tau,
\end{align*}
\]

(58)

which has several solutions. If \( \sin a\tau \neq 0 \), we find that \( a^2 + b\omega = 0 \), solved by

\[
\omega_c = \frac{\omega_0^2 + \gamma^2}{\omega_0}.
\]

(59)

Since this critical value is time-independent, this locus of self-intersections of the optimal front-line is by itself a critical optimal trajectory \((x_c(\tau), y_c(\tau))\). It loses its optimality at a critical time \( t_c = 2\tau_c \) such that \( \dot{x}_c(\tau) = \dot{y}_c(\tau) = 0 \). Since

\[
\begin{align*}
    \dot{x}_c(\tau) &= -\left(\omega_0^2 + \gamma^2\right) \sin \omega_c \tau \cos a\tau, \\
    \dot{y}_c(\tau) &= -\left(\omega_0^2 + \gamma^2\right) \cos \omega_c \tau \cos a\tau,
\end{align*}
\]

(60)

we find that the critical time is

\[
t_c = \frac{\pi |\omega_0|}{\gamma \sqrt{\omega_0^2 + \gamma^2}}.
\]

(61)

This trajectory is a cut locus for the system, analogous to that described in the case of three controls, when \( \gamma < \omega_0 \).

Additional solutions to (58) are found when \( \sin a\tau = 0 \). In this case the critical frequencies are implicitly defined by \( a(\omega_c')\tau = k\pi \), where \( k \) is an integer. The corresponding points are on the boundary of the unit disk: \( x_c'(\tau) = \cos \omega_c' \tau, y_c'(\tau) = -\sin \omega_c' \tau \). These cut loci are not optimal trajectories for the system since the critical frequencies are time-dependent.

The explicit expressions of these critical frequencies are

\[
\omega^+_c(k, \tau) = \omega_0 + \sqrt{\left(\frac{k\pi}{\tau}\right)^2 - \gamma^2}, \quad \omega^-_c(k, \tau) = \omega_0 - \sqrt{\left(\frac{k\pi}{\tau}\right)^2 - \gamma^2},
\]

(62)

and they are defined for \( \tau \leq \frac{k\pi}{\gamma} \), that is, \( k \geq \frac{\pi}{\gamma} \). It turns out that \( \omega^+_c(k, \tau) \geq \omega^-_c(k, \tau) \), and equality holds only when \( \tau = \frac{k\pi}{\gamma} \). If we write \( x_c'(\tau) + iy_c'(\tau) = e^{i\psi_c(\tau)} \), and require that
ψ_ε(0) = 0, we have

\[\begin{align*}
\psi_+^+(k, \tau) &= -\omega_+^+(k, \tau) + k\pi = -\omega_0 \tau - \sqrt{(k\pi)^2 - (\gamma \tau)^2} + k\pi, \\
\psi_-^-(k, \tau) &= -\omega_-^-(k, \tau) - k\pi = -\omega_0 \tau + \sqrt{(k\pi)^2 - (\gamma \tau)^2} - k\pi.
\end{align*}\]

(63)

It is possible to prove that \(\psi_+^+(k, \tau) \geq \psi_-^-(k, \tau)\) for all \(\tau \leq \frac{k\pi}{\gamma}\). Moreover, if \(k_2 > k_1\), it follows that \(\psi_2^+(k_2, \tau) < \psi_1^+(k_1, \tau)\) and \(\psi_2^- (k_2, \tau) > \psi_1^- (k_1, \tau)\). Since \(63\) are continuous functions, with \(\psi_0^+(k, 0) = \psi_-^-(k, 0) = 0\), for the study of optimal trajectories we have to take \(k = 1\) in \(62\) and \(63\). In the following, we suppress the parameter \(k\) to simplify the notation.

It turns out that, when \(\omega_0 > 0\), \(\psi^+_c(\tau)\) is monotonically decreasing for all \(\tau\), and \(\psi^+_c(\tau)\) decreasing for \(\tau < 2\tau_c\) and increasing for \(\tau > 2\tau_c\). Viceversa, when \(\omega_0 < 0\), \(\psi^+_c(\tau)\) is monotonically increasing for all \(\tau\), and \(\psi^+_c(\tau)\) increasing for \(\tau < 2\tau_c\) and decreasing for \(\tau > 2\tau_c\). If \(\omega_0 = 0\), we have \(t_c = 0\), and \(\psi^+_c(\tau)\), \(\psi^+_c(\tau) = -\psi^-_c(\tau)\) are monotonically decreasing and increasing for all \(\tau\), respectively. Furthermore, for \(\tau \in [0, 2\tau_c)\) we have \(\psi^-_c(\tau) < \omega_c < \psi^+_c(\tau)\). Notice that \(\omega^+_c(2\tau_c) = \omega_c\) if \(\omega_0 > 0\), and \(\omega^-_c(2\tau_c) = \omega_c\) if \(\omega_0 < 0\).

C. Reachable sets and optimal times

The evolution of the reachable set \(R_{< t}\) is determined by the dynamics of the optimal front line, and the treatment goes after that presented in the case of three controls. To start with, we consider the case \(\omega_0 < 0\). Following the discussion of the previous subsection, for \(t \in [0, t_c]\) the optimal trajectories at time \(t\) end on \(F_t(\omega_c \leq \omega \leq \omega^+_c(\tau))\). The trajectory characterized by \(\omega^+_c(\tau)\) ceases to be optimal since it reaches the cut locus on the border of the unit disk. The other trajectories cannot intersect, since the optimal front line has not self intersections in this range of values for \(\omega\). For \(t > t_c\) the situation is more complicated. In a neighborhood of \(t_c\), the optimal trajectories are determined by \(F_t(\omega_1 \leq \omega \leq \omega^+_c(\tau))\), where \(\omega_1 > \omega_c\) can be found by intersecting the optimal front-line and the critical optimal trajectory. The analytical expression of \(\omega_1\) cannot be generically found.

The case with \(\omega_0 > 0\) is analogous, and the evolution of the optimal trajectories are described by \(F_t(\omega^-_c(\tau) \leq \omega \leq \omega_c)\) for \(t \in [0, t_c]\), and by \(F_t(\omega^-_c(\tau) \leq \omega \leq \omega_2)\) for \(t > t_c\), with a suitable value \(\omega_2 < \omega_c\) determined by the intersection of the optimal front-line and the critical optimal trajectory.

For the subsequent evolution, we can recognize several regimes, depending on the relative
FIG. 5: (Color online) Time evolution of the reachable sets in the unit disk, when $\gamma = 3$ and $\omega_0 = 1$. We have represented the optimal-front line, the critical optimal trajectory and the optimal trajectories for several values of $\omega$ at successive times. The worst-case operator is marked by a small circle.

strength of controlled and free dynamics. It is convenient to characterize where the intersection loci described by $\omega_0^+$ and $\omega_0^-$ converge. Therefore, we impose $\psi_0^+(\tau) - \psi_0^-(\tau) = 2\pi$, which is solved by

$$t_{\text{max}} = 2\tau = \frac{2\pi}{\gamma}. \hspace{1cm} (64)$$

This is the worst-time case if the point of convergence is outside the reachable set. This requirement reads $\psi_0^+(\tau_{\text{max}}) < 2\pi$ for $\omega_0 < 0$ or rather $\psi_0^+(\tau_{\text{max}}) > 0$ for $\omega_0 > 0$, leading to

$$\gamma \geq |\omega_0|. \hspace{1cm} (65)$$

The corresponding worst-case operator is a diagonal operator represented by $(\cos \psi_{\text{max}}, \sin \psi_{\text{max}})$, where

$$\psi_{\text{max}} = \psi_0^+(\tau_{\text{max}}) = \pi \left(1 - \frac{\omega_0}{\gamma}\right). \hspace{1cm} (66)$$

It is reached along the trajectory described by $\omega = \omega_0$, which corresponds to the critical frequencies at the final time: $\omega_0^+(\tau_{\text{max}}) = \omega_0^-(\tau_{\text{max}}) = \omega_0$.

Notice that this analysis applies as well to the case $\omega_0 = 0$, that is, controlled dynamics without drift. In this case the critical trajectory collapses to the initial point $(1,0)$, which is self-conjugate.

We consider now the case $\gamma < |\omega_0|$. The worst case state, and the corresponding time, can be found by considering the evolution of the optimal front line. In particular, they can
be determined by requiring that $\mathcal{F}_t$ and the critical optimal trajectory are tangent to each other, that is, their tangent lines overlap. In principle, this is a necessary but not sufficient condition, since there could be several points satisfying this requirement. Nonetheless, we find that, for all $\gamma$, there is only one possible solution. Therefore, it must correspond to the worst-case operator.

The tangent line to the critical optimal trajectory in the point $(x_c(\bar{\tau}), y_c(\bar{\tau}))$, with $\bar{\tau} < \tau_c$, is found by using the same argument discussed in the previous section:

$$y \sin \omega_c \bar{\tau} - x \cos \omega_c \bar{\tau} + \cos a_c \bar{\tau} = 0,$$

where $a_c = a(\omega_c)$. The tangent line to $\mathcal{F}_t$ at a later time $t = 2\tau > \tau_c$ is given by (55), and it coincides with (67) if and only if

$$\sin \omega \tau = \pm \sin \omega_c \bar{\tau}, \quad \cos \omega \tau = \pm \cos \omega_c \bar{\tau}, \quad \cos a \tau = \pm \cos a_c \bar{\tau}.$$  

This system of equations must be solved for $\omega$, $\tau$ and $\bar{\tau}$, with the time hierarchy $0 \leq \bar{\tau} \leq \tau_c \leq \tau$, and $\tau$ is the minimal time. $\tau_c$ is given by $\tau_c = \frac{t_c}{2}$ with $t_c$ as in (61). It turns out that the only possible solution to (68) which satisfies the required constraints is

$$\omega \tau = \pi + \omega_c \bar{\tau}, \quad a \tau = \pi - a_c \bar{\tau},$$  

which gives the optimal frequency

$$\omega = \frac{\omega_0^2 - \gamma^2}{\omega_0},$$

and the times

$$\tau_{max} = \frac{\pi}{2|\omega_0|} \left( 1 + \frac{\sqrt{\omega_0^2 + \gamma^2}}{\gamma} \right), \quad \bar{\tau} = \frac{\pi}{2a_c} \frac{|\omega_c| - 2a_c}{|\omega_c| - a_c}.$$
where \(a_c = a(\omega_c)\) and \(\omega_c\) is given in (59). The worst case state in the adopted representation is arbitrary close to by the point \((x_c(\bar{\tau}), y_c(\bar{\tau}))\), approached through the optimal trajectory with \(\omega\) as in (70). From \(\bar{\tau} \geq 0\), we find that these results holds for

\[
\gamma \leq \frac{1}{\sqrt{3}}|\omega_0|.
\]  

(72)

We can complete our analysis in the case

\[
\frac{1}{\sqrt{3}}|\omega_0| < \gamma < |\omega_0|
\]  

(73)

by considering that the optimal front line is described by continuous functions, and then it has a smooth evolution. It turns out that, in this regime, the optimal front line is always tangent to the critical optimal trajectory in the point \((1,0)\), and this is the only point of intersection. Therefore, it also represents the worst case operator. The worst case time is defined by \(\psi_c^+(\tau) = 2\pi\) if \(\omega_0 < 0\), or \(\psi_c^-(\tau) = -2\pi\) if \(\omega_0 > 0\), which are solved by

\[
\tau_{\max} = \frac{2\pi|\omega_0|}{\omega_0^2 + \gamma^2}.
\]  

(74)

When \(\omega_0 < 0\) one finds \(\omega_c^+(\tau_{\max}) = \frac{\omega}{2}\), and similarly, when \(\omega_0 > 0\), \(\omega_c^-(\tau_{\max}) = \frac{\omega}{2}\). Therefore, \(\omega = \frac{\omega}{2}\) characterizes the optimal trajectory for the worst case state when (73) holds.

VI. EXAMPLES

In this section, we derive the optimal strategy for three different target operations. This is done for generic control strength \(\gamma\) and drift \(\omega_0\), in the two cases of two or three controls affecting the dynamics.

A. Diagonal operators

Assume that the target operator is given by \(X_f = e^{i\lambda z}\), with \(\lambda \in [0,2\pi]\) without loss of generality. In the case of three controls, following the former discussion, we find that the optimal control strategy is given by \(\alpha = 1\) or \(\alpha = -1\), depending on the relative values of \(\lambda, \gamma\) and \(\omega_0\). The optimal time is given by

\[
t_f = \begin{cases} 
\frac{4\pi - 2\lambda}{\gamma + \omega_0}, & \text{if } \omega_0 \geq \frac{\pi - \lambda}{\pi} \gamma \\
\frac{2\lambda}{\gamma - \omega_0}, & \text{if } \omega_0 < \frac{\pi - \lambda}{\pi} \gamma
\end{cases}
\]  

(75)
and, in particular, \( t_f = \frac{1}{2} \) if \( \omega_0 = -\gamma \) or \( t_f = \frac{1}{2}(2\pi - \lambda) \) if \( \omega_0 = \gamma \).

In the case of two controls, the diagonal operators are always the terminal points of optimal trajectories determined by \( \omega_0^c \) or \( \omega_0^d \). In general, the optimality conditions are \( \psi_0^c(\tau) = \lambda \) or \( \psi_0^d(\tau) = \lambda - 2\pi \), depending on the specific values of \( \lambda, \gamma \) and \( \omega_0 \). The optimal time is generically given by

\[
    t_f = \frac{2}{\omega_0^2 + \gamma^2} \left( (\pi - \lambda)\omega_0 + \Omega \right),
\]

where \( \Omega \equiv \sqrt{\pi^2\omega_0^2 + (2\pi\lambda - \lambda^2)\gamma^2} \). In the specific case \( \omega_0 = 0 \) we obtain

\[
    t_f = \frac{2}{\gamma} \sqrt{2\pi\lambda - \lambda^2},
\]

in accordance with the result of [8]. The expression (76) can be made more precise if a specific diagonal operator is specified. For instance, if \( X_f = i\sigma_z \), i.e., \( \lambda = \frac{\pi}{2} \), we find the expression

\[
    t_f = \frac{\pi}{\omega_0^2 + \gamma^2} \left( \omega_0 + \sqrt{4\omega_0^2 + 3\gamma^2} \right),
\]

which is valid for any \( \omega_0 \).

Following the analysis presented in Section [V] we find that, when \( \omega_0 > 0 \), the optimal trajectories leading to diagonal operators are characterized by \( \omega \leq \frac{\omega_0}{2} \), and, when \( \omega_0 < 0 \), by \( \omega \geq \frac{\omega_0}{2} \). The critical frequency \( \omega_c \) has been defined in (59). The analysis applies as well when \( \omega_0 = 0 \), but in this case \( \omega_c \) diverges to \( \infty \) and any \( \omega \) produces a trajectory ending on the border of the unit disk. Therefore we conclude that, for \( \omega_0 \neq 0 \), the value \( \omega = \frac{\omega_0}{2} \) corresponds to a limit trajectory, mapping the point \((1,0)\) to itself, and separating the trajectories leading to diagonal operators to the others trajectories, in accordance with the result found in [10]. This trajectory is given by

\[
    x_{\omega_c}(\tau) = \frac{\omega_0^2}{\omega_0^2 + \gamma^2} \cos \frac{\omega_0^2 + \gamma^2}{\omega_0} \tau + \frac{\gamma^2}{\omega_0^2 + \gamma^2} \tau,
\]

\[
    y_{\omega_c}(\tau) = -\frac{\omega_0^2}{\omega_0^2 + \gamma^2} \sin \frac{\omega_0^2 + \gamma^2}{\omega_0} \tau,
\]

therefore it is a circle of radius \( \frac{\omega_0^2}{\omega_0^2 + \gamma^2} \) centered in \( \left( \frac{\gamma^2}{\omega_0^2 + \gamma^2}, 0 \right) \).

### B. SWAP operator

The SWAP operator is given by \( X_f = i\sigma_y \), and it is represented by \((0,0)\), the center of unit disk. When there are three controls, by imposing \( x_\alpha(\tau) = y_\alpha(\tau) = 0 \) in (16), we find the
optimal time $t_f = \frac{\pi}{\gamma}$ and $\alpha = 0$. But $\alpha = 0$ is equivalent to $b_z = 0$, from the definition (10) of $\alpha$. This in turn implies $u_z = 0$ from (6). Therefore, the optimal trajectories leading to the SWAP operator are the same in the case of two or three controls, and also the optimal time is the same. This can be directly seen by imposing $x_\omega(\tau) = y_\omega(\tau) = 0$ in (48), which is solved by $\tau = \frac{\pi}{2a}$ and $b = 0$. This latter condition reads $\omega = \omega_0$, and then $a = \gamma$ and $t_f = \frac{\pi}{\gamma}$, as expected. The trajectory, obtained with $\alpha = 0$ in (16) or rather $\omega = \omega_0$ in (48), is given by

$$x(\tau) + iy(\tau) = |\cos \gamma \tau| e^{-i\omega_0 \tau},$$

and, in general, it does not lose optimality after reaching the SWAP operator. When $\omega_0 = 0$, this trajectory is a segment connecting $(1,0)$ to $(-1,0)$, the worst-case operator.

Notice that, in the case of two controls, $\omega = \omega_0$ defines the optimal worst-case trajectory when $\gamma > |\omega_0|$. The worst time is twice the time to reach the SWAP operator. Interestingly, in the same regime, the same worst time is obtained when three controls can be used. Nonetheless, this result does not hold only for the trajectory with $\alpha = 0$, but for any $\alpha$.

VII. CONCLUSIONS

In this work we have studied the minimum time control of $SU(2)$ quantum operations for a two-level quantum system. We have assumed that the system dynamics, which possibly contains a time-independent drift term, could be externally modified by means of two or three independent control actions. The total strength of the control is bounded, and of arbitrary magnitude when compared to the free dynamics of the system. By using an especially simple parametrization of the Lie group of special unitary operations, and by studying the dynamics of the boundary of the reachable set through the evolution of the optimal front line, we are able to provide a comprehensive description of the dynamics of the reachable sets for any relative magnitude of the free and controlled dynamics.

Our results complement and extend former results on the behavior of optimal trajectories [8, 10]. We provide a complete description of the critical trajectories in $SU(2)$, which in our context arise as loci of self-intersections of the optimal front line. Whenever possible, we analytically derive the optimal control strategies and the corresponding optimal times, and in each case characterize the worst-case operator and time (the so-called diameter of the system). We provide a geometrical description of the optimal control problem, which
makes clear the existence of different regimes depending on the relative strength of drift and control terms. In table I we summarize the worst-case time in all the cases.

Our results are relevant whenever quantum operations on qubits have to be engineered in the shortest possible time, preeminently in quantum information processing, quantum communication, atomic and molecular physics, and Nuclear Magnetic Resonance. While the case of a fully actuated system is not of primary relevance for real applications, with its simplicity it provides the ideal framework for illustrating our technique. This approach, based on the study of the optimal front line, strongly simplifies the analysis of the system, and, when suitably generalized, it might represent a promising tool for the investigation of other optimal control problems. As an example, the present analysis of minimum time evolutions applies, with minor modifications, to the Lie group $SO(3)$, since $SU(2)$ is a double cover of it.

![Table I: Diameter of the system in several regimes](image)

| Case | Three controls | Two controls |
|------|----------------|--------------|
|      | $\gamma < |\omega_0|$ | $\gamma \geq |\omega_0|$ | $\frac{1}{\sqrt{3}} |\omega_0| < \gamma < |\omega_0|$ | $\gamma \geq |\omega_0|$ |
| $t_{max}$ | $\frac{\pi}{\gamma} \left( 1 + \frac{\gamma}{|\omega_0|} \right)$ | $\frac{2\pi}{\gamma}$ | $\frac{\pi}{|\omega_0|} \left( 1 + \frac{\sqrt{\omega_0^2 + \gamma^2}}{\gamma} \right)$ | $\frac{4\pi |\omega_0|}{\omega_0^2 + \gamma^2}$ | $\frac{2\pi}{\gamma}$ |

[1] D. D’Alessandro, Introduction to Quantum Control and Dynamics, CRC Press, Boca Raton FL (2007)
[2] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, U.K., New York (2000)
[3] M. H. Levitt, Spin dynamics: basics of nuclear magnetic resonance, John Wiley and sons, New York-London-Sydney (2008)
[4] R. Wu, C. Li and Y. Wang, Phys. Lett. A 295, 20 (2002)
[5] U. Boscain and P. Mason, J. Math. Phys. 47, 062101 (2006)
[6] M. Wenin and W. Pötz, Phys. Rev. A 74, 022319 (2006)
[7] E. Kirillova, T. Hoch and K. Spindler, WSEAS Trans. Math. 7, 687 (2008)
[8] A. Garon, S. J. Glaser and D. Sugny, Phys. Rev. A 88 043422 (2013)
[9] G. C. Hegerfeldt, Phys. Rev. Lett. 111, 260501 (2013)
[10] F. Albertini and D. D'Alessandro, arXiv:1407.7491
[11] L. Pontryagin et al., Mathematical theory of optimal processes, Mir, Moscou (1974)
[12] U. Boscain and B. Piccoli, Optimal Syntheses for Control Systems on 2-D Manifolds, Springer SMAI 43 (2004)
[13] R. Wu et al., Phys. Rev. A 86, 013405 (2012)
[14] M. Lapert et al., Phys. Rev. Lett. 104, 083001 (2010)
[15] D. D'Alessandro and M. Dahleh, IEEE Trans. A. C. 46, 866 (2001)
[16] N. Khaneja, R. Brockett and S. J. Glaser, Phys. Rev. A 63, 032308 (2001)
[17] Candidate optimal controls and trajectories are called extremals in optimal control theory language.
[18] Generally, the optimal front line undergoes a time-dependent roto-translation in the $(x, y)$ plane. The cusp singularity appears when the translational contribution vanishes. Therefore, it represents the instantaneous rotation center of $F_t$
[19] When $\omega = 0$, the only solution to (56) is given by $|x(\tau)| = 1, y(\tau) = 0$, with the constraint $\sin a\tau = 0$. This solution is already accounted for in the case $\omega \neq 0$. 

27