Hyperheavy spherical and toroidal nuclei: the role of shell structure.

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The properties of toroidal hyperheavy even-even nuclei and the role of toroidal shell structure are extensively studied within covariant density functional theory. The general trends in the evolution of toroidal shapes in the $Z \approx 130 - 180$ region of nuclear chart are established for the first time. These nuclei are stable with respect of breathing deformations. The most compact fat toroidal nuclei are located in the $Z \approx 136, N \approx 206$ region of nuclear chart, but thin toroidal nuclei become dominant with increasing proton number and on moving towards proton and neutron drip lines. The role of toroidal shell structure, its regularity, supershell structure, shell gaps as well as the role of different groups of the pairs of the orbitals in its formation are investigated in detail. The lowest in energy solutions at axial symmetry are characterized either by large shell gaps or low density of the single-particle states in the vicinity of the Fermi level in at least one of the subsystems (proton or neutron). Related quantum shell effects are expected to act against the instabilities in breathing and sausage deformations for these subsystems. The investigation with large set of covariant energy density functionals reveals that substantial proton $Z = 154$ and 186 and neutron $N = 228, 308$ and 406 spherical shell gaps exist in all functionals. The nuclei in the vicinity of the combination of these particle numbers form the islands of stability of spherical hyperheavy nuclei. The study suggests that the $N = 210$ toroidal shell gap plays a substantial role in the stabilization of fat toroidal nuclei.

I. INTRODUCTION

The studies of the nuclei at the limits are guided by human curiosity, by the need to understand new physical mechanisms governing nuclear systems in these extreme conditions and by the demand for nuclear input in nuclear astrophysics. A number of questions related to the physics at the limits emerge. These are: What are the limits of the existence of nuclei? What are the highest proton number $Z$ at which the nuclear landscape and periodic table of chemical elements cease to exist? What are the positions of proton and neutron drip lines? What types of nuclear shapes dominate these extremes of nuclear landscape? They look deceivable simple but unique answers on most of them are extremely difficult.

Recent systematic investigations of hyperheavy ($Z \geq 126$) nuclei performed in Refs. [1, 2, 4] have allowed to shed some light on these questions. Emerging new physics is summarized in Figs. 1 and 2. The increase of Coulomb interaction with increasing proton number $Z$ leads to the fact that compact nuclear shapes such as spherical, prolate and oblate (further ellipsoidal-like shapes) become either unstable against fission or energetically unfavored in hyperheavy nuclei with high $Z$ values (see Fig. 1). As a consequence, the lowest in energy solutions in such nuclei are characterized by non-compact toroidal shapes$^1$. As illustrated in Fig. 2 the boundary between ellipsoidal-like and toroidal shapes depend on the combination of proton and neutron numbers. However, spherical shapes can be stable against fission in some hyperheavy nuclei (see Refs. [1, 2] and Fig. 1). Although these states are highly excited with respect to the lowest in energy states with toroidal shapes (as obtained in axial calculations), they will become the ground states if toroidal states are not stable with respect to multifragmentation.

The state-of-the-art view on the nuclear landscape born out in Refs. [1, 2] is shown in Fig. 2. Well known nuclear structure with pronounced spherical shell gaps at particle numbers 8, 20, 28, 50, 82 (and $N = 126$) leading to the bands (shown by gray color) of spherical nuclei in the nuclear chart along the vertical and horizontal lines with these particle numbers is seen for proton numbers below $Z \approx 120$. With increasing proton number these classical features disappear and only toroidal shapes are calculated as the lowest in energy in axial relativistic Hartree-Bogoliubov (RHB) approach. This region (shown in white color in Fig. 2) is penetrated only by three islands (shown in gray color) of potentially stable spherical hyperheavy nuclei; note that spherical minima are highly excited with respect of the minima corresponding to toroidal shapes. Thus, the richness of nuclear structure seen in experimentally known part of nuclear landscape is replaced by more uniform structure of the nuclear landscape in the region of hyperheavy nuclei dominated by toroidal and spherical nuclei. Fig. 2 also reveals a substantial increase (equal to the area between extrapolated two-proton drip line for ellipsoidal shapes and the two-proton drip line for toroidal shapes) of nuclear landscape caused by the shift of two-proton drip line towards more proton-rich nuclei on transition to toroidal shapes. This transition drastically modifies the underlying single-particle structure and as a consequence lowers the energy of the Fermi level for protons (see Ref. [4]).

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$^1$ Toroidal nucleus is represented by a thin cylinder which has the ends joining together [5].
It is necessary to recognize that the physics of toroidal shapes plays an important role in classical and quantum physics, chemistry and biology. There are numerous examples but let us mention only some of them. Stable toroidal structures (micelles) play an important role in the amphiphilic polymers in large parts of the parameter space spanned by the degree of amphiphilicity, the temperature, the density and the molecular stiffness with respect to bending [6]. The wave propagation on the surface of the torus represents a vivid example of light behavior on curved surface of manifolds with interesting topologies and has potential applications in photonic structures [7]. The stability of toroidal drop freely suspended in another fluid and subjected to an electric field has been studied in Ref. [8]; this feature can play a role in a number of phenomena and applications such as thunderstorm formation, microfluids, bioimaging and effective drug delivery. Biology finds the toroidal shape at the cellular level when the reproduction of cells up to the 16th cell division creates a hollow torus called the morula [9]. On a more microscopic level, the DNA toroids are formed from individual DNA molecules of individual lengths [10].

The question of potential stability of toroidal nuclei has first been raised by J. A. Wheeler (see references in Ref. [5]). Later the toroidal shapes in atomic nuclei have been investigated in a number of the papers (see, for example, Refs. [5, 11–16] and references quoted therein). However, in absolute majority of the cases such shapes correspond to highly excited states either at extreme values of angular momentum in the nuclei across the nuclear landscape [13, 14, 17] or at spin zero in superheavy elements [12, 15]. In the former case, calculated angular momenta at which toroidal shapes appear substantially exceed the values of angular momentum presently achievable at the state-of-art experimental facilities [18]. So far, only the experimental excitation function for the 7α de-excitation of 28Si nuclei, revealing the resonance structures, may indicate the population of toroidal high-spin isomers [19]. In the latter case, such states are unstable in superheavy nuclei against returning to the shape of sphere-like geometry (Ref. [15]). This is similar to shrinking instability of uncharged toroidal droplets which are unstable due to surface tension and transform into spherical droplets [20]. The situation is different in atomic nuclei since this shrinking instability is counteracted by Coulomb repulsion of the protons which increases with proton number Z. Thus, toroidal shapes become the lowest in energy solutions in hyperheavy nuclei with Z > 130 [1, 2, 11].

The present paper extends our previous investigations of hyperheavy nuclei reported in Refs. [1, 2] and focuses on a number of issues which have not been studied so far. The presence of local minima A, B, C and D in deformation energy curve of the 466156 nucleus (see Fig. 1) is clearly due to the shell effects. So far, the underlying single-particle structure has been investigated only for spherical shapes and only for four covariant energy density functionals (CEDFs) (see Sec. V in Ref. [2]). To estimate theoretical uncertainties in the predictions of shell closures in hyperheavy nuclei at spherical shape we perform such studies with ten most widely used CEDFs. This also allows us to compare respective spherical shell
FIG. 2. The distribution of ellipsoidal and toroidal shapes in the nuclear landscape obtained in the RHB calculations with CEDF DD-PC1. The nuclei with ellipsoidal shapes are shown by the squares the color of which indicates the equilibrium quadrupole deformation $\beta_2$ (see colormap). Note that ellipsoidal shapes with the heights of fission barriers smaller than 2.0 MeV are considered as unstable (see the discussion in Sect. III of Ref. [3] and in Sect. XI in Ref. [2]). Two-proton and two-neutron drip lines for toroidal nuclei are shown by solid black lines. White region between them (as well as the islands inside this region shown in gray) corresponds to the nuclei which have toroidal shapes in the lowest in energy minimum for axial symmetry (LEMAS). The islands of relatively stable spherical hyperheavy nuclei in the $Z > 130$ nuclei, shown in light grey color, correspond to the solutions which are excited in energy with respect of the LEMAS corresponding to toroidal shapes. Note that in the same nucleus two-neutron drip lines for spherical and toroidal shapes are somewhat different. This is the reason why some islands of stability of spherical hyperheavy nuclei extend beyond the two-neutron drip line for toroidal shapes.

The extrapolation of the two-proton drip for ellipsoidal shapes, defined from its general trends seen in the $Z < 120$ nuclei, is displayed by thick orange dashed lines. Similar extrapolation for two-neutron drip line of ellipsoidal shapes is close to the two-neutron drip line of toroidal shapes (see Fig. 1 in Ref. [4]); thus it is not shown. Partially based on Fig. 24 of Ref. [2].

gaps, leading to the islands of potentially stable spherical hyperheavy nuclei, with the ones seen in experimentally known nuclei as well as with those predicted for spherical superheavy nuclei. In addition, for the first time we perform the detailed investigation of the single-particle structure of hyperheavy toroidal nuclei.

The analysis of the single-particle structure presented in Figs. 5 and 8 of Ref. [2] indicates the presence of large spherical shell gaps at $Z = 186$ and $N = 406$. However, the investigations of Ref. [2] have been restricted to the $Z \leq 180$ nuclei. Thus, to better map this region of potentially stable spherical hyperheavy nuclei, to investigate the potential role of these shell gaps as well as to search for other regions of potentially stable spherical hyperheavy nuclei we extended the calculations mapping the nuclear landscape from $Z = 180$ to $Z = 210$.

Finally, because of numerical limitations the studies of toroidal shapes in hyperheavy nuclei have been with a single exception restricted to the $Z \leq 138$ nuclei in Refs. [1, 2]. Thus, we performed detailed investigation of toroidal shapes corresponding to the lowest in energy solution at axial symmetry in extremely large basis for isotopic chains with $Z = 136, 146, 156, 166$ and 176. This allows us to better understand their evolution with particle numbers and to get some understanding about their potential stability with respect of different types of distortions.

The manuscript is organized as follows. The details of theoretical calculations are discussed in Sec. II. Section III is devoted to the analysis of the role of shell structure and large shell gaps at spherical shape. The distribution of the shapes of toroidal hyperheavy nuclei across the nuclear landscape and major features of their shell structure are discussed in Sec. IV. Finally, Sec. V summarizes the results of our work.

II. THE DETAILS OF THE THEORETICAL CALCULATIONS.

The investigations of the properties of hyperheavy even-even nuclei are performed within the axial reflection symmetric Hartree-Bogoliubov (RHB) framework (see Ref. [21]). Until specified otherwise, the calculations are performed with the DD-PC1 covariant energy density functional (CEDF) [22]. This functional is considered to
be one of the best CEDFs today based on systematic and
global studies of different physical observables related to
the ground state properties and fission barriers [21, 23–
28].

The constrained calculations in the RHB code perform
the variation of the function

$$E_{RHB} + \frac{C_{20}}{2}(\langle \hat{Q}_{20} \rangle - q_{20})^2,$$

where $E_{RHB}$ is the total energy and $\langle \hat{Q}_{20} \rangle$ denotes the
expectation value of the mass quadrupole operator,

$$\hat{Q}_{20} = 2z^2 - x^2 - y^2.$$  \hspace{1cm} (2)

Here $q_{20}$ is the constrained value of the multipole
moment, and $C_{20}$ the corresponding stiffness constant [29].
In order to provide the convergence to the exact value
of the desired multipole moment we use the method sug-
gested in Ref. [30]. Here the quantity $q_{20}$ is replaced by
the parameter $q_{20}^{eff}$, which is automatically modified dur-
ing the iteration in such a way that we obtain $\langle \hat{Q}_{20} \rangle = q_{20}$
for the converged solution. This method works well in our
constrained calculations.

The $\beta_2$ quantity is extracted from the quadrupole mo-
ments:

$$Q_{20} = \int d^3r \rho(r) (2z^2 - x^2 - y^2),$$  \hspace{1cm} (3)

via

$$\beta_2 = \sqrt{\frac{5}{16\pi}} \frac{4\pi}{3AR_0^2} Q_{20},$$  \hspace{1cm} (4)

where $R_0 = 1.2A^{1/3}$. The $\beta_2$ values have a standard
meaning of the deformations of ellipsoid-like density dis-
tributions only for $|\beta_2| \lesssim 1.0$ values. At higher $\beta_2$ values
they should be treated as dimensionless and particle nor-
malized measures of the $Q_{20}$ moments. This is because of
the presence of toroidal shapes at large negative $\beta_2$
values and necking degree of freedom at large positive $\beta_2$
values.

For each nucleus under study, the deformation energy
curves are calculated in the $-5.0 < \beta_2 < 3.0$ range; such
large range is needed for a reliable definition of the type
of shape (toroidal or ellipsoidal) representing the lowest
in energy minimum for axial symmetry (LEMAS). Two
truncation schemes are used in the calculations based on
the analysis presented in Sec. III of Ref. [2] and addi-
tional analysis performed in this manuscript. All states
belonging to major shells up to $N_F=30$ fermionic shells
for the Dirac spinors are taken into account when detailed
analysis of toroidal shapes in the $Z = 136 – 176$ region
and their underlying shell structure is performed. Note
that these calculations are extremely time-consuming.
As discussed in detail in Sec. III of Ref. [2] on the
example of the $^{466}156$ nucleus and verified by a similar
analysis of a pair of the $Z = 176$ nuclei, this basis
provides sufficient numerical accuracy of the calculations
of toroidal shapes. However, the analysis of numerical
convergence in the $^{210}616$ nucleus reveals that the de-
scription of higher $Z$ nuclei requires even large $N_F$
for a proper description of LEMAS corresponding to toroidal
shapes. These facts were the reasons why we perform de-
tailed study of toroidal shape only up to $Z = 176$ and for
$Z > 176$ nuclei we focus mainly on ellipsoidal-like shapes
which require smaller basis as compared with toroidal
shapes (see Sec. III of Ref. [2] for a detailed comparison
of numerical convergence for toroidal and ellipsoidal-like
shapes). To save computational time the extension (as
compared with the results presented in Ref. [2]) of nu-
clear landscape to the $Z = 182 – 210$ region is performed
with $N_F = 26$; this truncation scheme allows accurate
description of spherical and ellipsoidal shapes, reliable
definition of toroidal shapes as corresponding to LEMAS
but does not provide accurate enough description of their
energies and shapes in LEMAS.

To avoid the uncertainties connected with the defini-
tion of the size of the pairing window [40], we use the
separable form of the finite-range Gogny pairing inter-
action introduced in Ref. [41]. Its matrix elements in r-space
have the form

$$V(r_1, r_2, r_1', r_2') = -G\delta(R - R')P(r)P(r') \frac{1}{2}(1 - P^x)$$  \hspace{1cm} (5)

with $R = (r_1 + r_2)/2$ and $r = r_1 - r_2$ being the center
of mass and relative coordinates. The form factor $P(r)$
is of Gaussian shape,

$$P(r) = \frac{1}{(4\pi a^2)^{3/2}} e^{-r^2/4a^2}.$$  \hspace{1cm} (6)

The parameters of this interaction have been derived by a
mapping of the $^1S_0$ pairing gap of infinite nuclear matter
to that of the Gogny force D1S. The resulting parameters
are: $G = 728$ fm$^3$ and $a = 0.644$ fm [41]. This pairing
provides a reasonable description of pairing properties
in heaviest nuclei in which pairing properties can be ex-
tracted from experimental data [21, 42, 43].

III. SPHERICAL HYPERHEAVY NUCLEI: THE
ROLE OF SHELL STRUCTURE

Hyperheavy nuclei are stabilized by shell effects, i.e.,
by the large shell gap(s) or at least a considerably re-
duced density of the single-particle states in the vicinity
of the Fermi level. To better understand the impact of
shell gaps on the underlying structure of spherical nuclei
in the context of global description of nuclear structure,
Fig. 3 shows their evolution across nuclear chart. It starts
from well known gaps in doubly magic $^{56}$Ni, $^{100,132}$Sn and
$^{208}$Pb nuclei and extends to the gaps in the hyperheavy
nuclei. In addition, it provides the evaluation of theore-
tical uncertainties in their predictions by comparing the
results obtained with ten most widely used CEDFs.
TABLE I. Thin and thick vertical lines are used to show the spread of the sizes of calculated shell gaps; the tops and bottoms of these lines correspond to the upper and lower shell gaps among the considered set of CEDFs. Thin lines show this spread for all employed CEDFs, while thick lines are used for the subset of four globally-tested CEDFs (NL3*, DD-ME2, DD-PC1, and PC-PK1). (c) and (d) The same as in panels (a) and (b) respectively, but with the sizes of the shell gaps and the spreads in their predictions scaled with mass factor $A^{1/3}$.

Figs. 3a and b show that the average sizes of proton $Z = 154, 186$ and neutron $N = 228, 308$ and 406 gaps obtained in the calculations are larger than those ($Z = 120$ and $N = 184$) in classical region of superheavy nuclei. This suggests that spherical hyperheavy nuclei may be more stable as compared with spherical superheavy nuclei (see the discussion of fission barriers in Refs. [1, 2]). It is also interesting that theoretical uncertainties in the sizes of shell gaps in hyperheavy nuclei are smaller than those in experimentally known nuclei and in classical region of superheavy nuclei.

The absolute values of shell gaps do not tell full story about their potential stabilizing effect since the single-particle level density increases with mass number $A$. This is a reason why scaled shell gap $\Delta E_{\text{gap}} A^{1/3}$ provides a better measure (see discussion in Sect. III of Ref. [23]). Scaled proton and neutron shell gaps are shown in Figs. 3(c) and (d). One can see that scaled proton $Z = 154$ and 186 shell gaps are significantly larger than scaled $Z = 120$ shell gap in superheavy nuclei and that they are close to the scaled $Z = 82$ shell gap in $^{208}$Pb (see Fig. 3(c)). On the contrary, scaled $N = 228, 308$ and 406 shell gaps are on average only slightly larger than scaled $N = 184$ gap in superheavy nuclei but they are smaller by a factor of approximately two than scaled $N = 126$ shell gap in $^{208}$Pb (see Fig. 3(d)).

Large uncertainties in the predictions of the $Z = 120$ and $N = 184$ shell gaps and softness of potential en-
energy surfaces leads to substantial differences in the predic-
tions of ground state properties of superheavy nuclei (see Ref. [23]). For many nuclei it is even impossible to reli-
ably predict whether the ground state will be spheri-
cal or oblate [23]. The situation is different in hyperheavy
nuclei where for ellipsoidal type shapes only potentially
stable spherical minima appear in the calculations be-
cause of larger scaled spherical shell gaps seen in Figs.
5(c) and (d).

Fig. 4 presents the extension of the map of the heights
of fission barriers around spherical shape from earlier
published range of $Z = 120 − 180$ (see Fig. 6a in Ref.
[1]) to the range of proton numbers from $Z = 120$ up to
$Z = 210$. The value of the fission barrier height $E_B$ is
defined as the lowest value of the barriers located on the
oblate and prolate sides with respect to spherical state
in the deformation energy curves obtained in axial RHB
calculations. One can see that the island of spherical hy-
perheavy nuclei previously labeled as “$Z ≈ 174, N ≈ 410$
island” in Ref. [1] has been considerably extended up to
$Z ≈ 206$. In a given isotope chain of this island, the max-
imum of fission barriers heights is located at $N = 406$.
The highest fission barriers with the heights between
$≈ 7.5$ and $≈ 8.5$ MeV are found in the $Z = 186, 184,
182$ and $180$ isotopic chains. They are higher than those
obtained in the classical region of superheavy nuclei (see
Ref. [24]). Based on these results for fission barriers and
for the sizes of the $Z = 186, N = 406$ spherical shell
gaps, we relabel this island as “$Z ≈ 186, N ≈ 406$ island
of spherical hyperheavy nuclei”. The extension of upper
boundary of nuclear landscape from $Z = 180$ to $Z = 210$
does not reveal other islands of spherical hyperheavy nu-
clei.

Similar to the results presented in Fig. 6 of Ref. [1]
the size of the $Z ≈ 186, N ≈ 406$ island of spherical hy-
perheavy nuclei and the stability of the elements in it are
expected to depend strongly on employed functional. We
have not attempted to map this region with other than
DD-PC1 functionals but some insight on this issue can be
obtained from the analysis of the heights of fission
barrier $E_B$ of the central nucleus ($^{592}$186) of this region
calculated with different functionals. These results are
summarized in Table I. The FSUGold and next three
functionals (DD-ME2, DD-ME5 and DD-PC1) produce the
highest calculated fission barriers: at 10.66 MeV for
FSUGold and clustered around $E_B ≈ 7.7$ MeV for other
three functionals. These barriers are higher than those
produced in the CDFT framework in the classical region
of superheavy nuclei (see Fig. 10 in Ref. [24]). These
functionals are also expected to produce the island of
spherical hyperheavy nuclei which is comparable in size
to that shown in Fig. 4. The next five functionals (PC-
PK1, NL3, PC-F1, TM1 and NL3*) produce the cluster
with $E_B ≈ 4$ MeV (see Table I); this value is not far away
from what is obtained in the $Z ≈ 116, Z ≈ 180$ region
of superheavy nuclei (see Fig. 10 in Ref. [24]). For these
functionals the $Z ≈ 186, N ≈ 406$ island of stability of
spherical hyperheavy nuclei is expected to be substan-
tially smaller than the one shown in Fig. 4. Finally, the
lowest fission barrier is produced by the NL1 functional;
its value indicates the instability of spherical hyperheavy
nuclei. However, the predictions of this functional have
to be considered as least reliable because of well known
problems in its isovector properties (see Ref. [32]).

The difference in the predictions of $E_B$ is in part
related to the fact that the first group of functionals pre-
dicts the $Z = 186$ and $N = 406$ shell gaps which are
on average larger by $≈ 0.1$ MeV and $≈ 0.5$ MeV, re-
spectively, than those produced by the second group of
CEDFs (see Table I). Note also that the nuclear matter
properties and the density dependence are substantially
better defined for density-dependent (DD*) functionals
as compared with non-linear (NL* and TM1) and point-
coupling (PC-PK1 and PC-F1) ones [25]. As a conse-
quence, in general, they are expected to perform better
for large extrapolations from known regions.

Note that the axial RHB calculations for deformation
energy curves in the vicinity of spherical minimum indi-
cate nearly symmetric barriers with saddles at $β_2 ≈ ±0.2$
(similar to Fig. 17(b) below). The experience in actinides
and superheavy nuclei tells us that octupole deformation
in fission barrier area typically does not develop for such
low deformations [3, 26, 44] (corresponding to inner fis-
sion barrier in actinides and superheavy nuclei) and this
result has been confirmed in octupole deformed RHB
calculations with CEDF DD-PC1 for spherical minimum
of several hyperheavy nuclei in Ref. [2]. The results pre-
sent in Fig. 6 for the $^{592}$186 nucleus are in line with
these expectations; the saddle of fission barrier is located
at $β_3 = 0.0$ and octupole deformation does not affect
the spherical minimum in the calculations with DD-PC1 and
NL3* functionals.

The analysis of Ref. [2] indicates that the impact of tri-
axial deformation on the fission barriers around spherical
minima is relatively modest. This is the consequence of
the topology of potential energy surfaces which is similar
to those of volcanos (see Figs. 7). The central area around
spherical minimum is similar to caldera, the rim of which is
represented by the fission barrier. The area beyond the
rim (fission barrier) is fast down-sloping as a function of
quadrupole deformation $β_2$. The saddles of axial fission
barriers (on oblate and prolate sides of spherical mini-
num) are located at modest quadrupole deformation of
$β_2 ≈ 0.2$. As a result, the distance between these two
saddles in the $(β_2, γ)$ plane plane is relatively small, so
that large changes in binding energy due to triaxialty
for nearly constant $β_2$ values could not develop. As a
consequence, the lowest fission barrier around spherical
minimum obtained in axial RHB calculations is a good
approximation to the barrier obtained in the TRHB cal-
culations. For example, this is a case in the calculations
with CEDF DD-PC1 (see Fig. 7a). Even if the saddle of
fission barrier is located at $γ \neq 0^\circ$ and $γ \neq 60^\circ$, the
energy lowering in fission barrier height as compared with
the lowest fission barrier at these $γ$ values is rather mod-
est. For example, in the calculations with the NL3* func-
FIG. 4. The fission barrier heights $E_B$ [in MeV] as a function of proton and neutron numbers. Only the nuclei with fission barriers higher than 2 MeV are shown. Partially based on the results presented in Fig. 6a of Ref. [1].

![Fission barrier heights](image)

**TABLE I.** The heights of the fission barriers $E_B$ and the sizes $\Delta E$ of spherical $N = 406 (\Delta E_{N=406})$ and $Z = 186 (\Delta E_{Z=186})$ shell gaps in the $^{592}_{186}$ nucleus obtained with indicated CEDFs. The functionals are ordered in such a way that $E_B$ is decreasing.

| CEDF  | $E_B$ [MeV] | $\Delta E_{N=406}$ [MeV] | $\Delta E_{Z=186}$ [MeV] |
|-------|-------------|--------------------------|---------------------------|
| FSUGold | 10.66       | 1.84                     | 2.17                      |
| DD-ME2  | 7.73        | 2.11                     | 2.43                      |
| DD-MEδ  | 7.72        | 1.98                     | 2.68                      |
| DD-PC1  | 7.59        | 1.93                     | 2.45                      |
| PC-PK1  | 4.35        | 1.61                     | 2.37                      |
| NL3     | 4.28        | 1.43                     | 2.15                      |
| PC-F1   | 3.87        | 1.41                     | 2.45                      |
| TM1     | 3.86        | 1.38                     | 2.29                      |
| NL3*    | 3.59        | 1.45                     | 2.37                      |
| NL1     | 1.27        | 1.27                     | 2.34                      |

not been completely covered in that study because of the restriction to the $Z \leq 180$ nuclei. To fill this gap in our knowledge, Fig. 5 compares proton and neutron density distributions of the $^{584}_{174}$ nucleus (studied in Ref. [2]) with those of doubly magic $^{592}_{186}$ one. Neutron densities of these two nuclei are very similar; they are slightly larger for the $^{584}_{174}$ nucleus because of the occupation of the $2j_{13/2}$ orbitals by four additional neutrons. The differences are more visible for proton densities because 12 additional protons in the doubly magic $^{592}_{186}$ nucleus (8 in the $2g_7/2$ and 4 in $1j_{13/2}$ orbitals) occupy the orbitals which fill the density either in surface region (the $1j_{13/2}$ orbitals) or in-between central and surface regions (the $2g_7/2$ orbitals) (see Ref. [45]). The increase of the Coulomb repulsion in the $Z = 186$ nucleus as compared with the $Z = 174$ one also plays a role in an enhancement of proton density near the surface. As a consequence, the semi-bubble structure becomes more pronounced in the proton densities of the $^{592}_{186}$ nucleus as compared with the $^{584}_{174}$ one.
FIG. 6. Potential energy surfaces of the $^{592}_{186}$ nucleus obtained in the reflection asymmetric (octupole deformed) RHB calculations with indicated CEDFs. The energy difference between two neighboring equipotential lines is equal to 1.0 MeV. Spherical minimum is indicated by a circle and the saddle point of the barrier around spherical minimum by solid black square. The colormaps show the excitation energies (in MeV) with respect to the energy of the deformation point with largest (in absolute value) binding energy. The calculations are performed with $N_F = 20$. Note that the topology of potential energy surfaces is almost the same in the calculations with $N_F = 20$ and $N_F = 26$. Thus, to save computational time these figures are plotted with $N_F = 20$.

FIG. 7. The same as in Fig. 6 but for potential energy surfaces obtained in triaxial RHB calculations.

IV. TOROIDAL NUCLEI

A. Distribution of shapes of toroidal nuclei across the nuclear landscape

In our calculations the truncation of basis is performed in such a way that all states belonging to the major shells up to $N_F$ fermionic shells for the Dirac spinors are taken into account. Accurate calculations of LEMAS require extremely large fermionic basis and its size, defined by $N_F$, increases with the raise of proton and neutron numbers (see discussion in Sect. III of Ref. [2]). As a result, the $\beta_2$ values (and, thus, respective density distributions) of the lowest in energy toroidal states have only been partially mapped in the $Z = 122 – 138$ region (see Fig. 3 in Ref. [1]) in the axial RHB calculations with $N_F = 26$. For higher $Z$ nuclei, existing calculations only confirm that the lowest in energy solutions have always toroidal shapes (see the discussion of Fig. 3 in Ref. [2]) but do not provide accurate $\beta_2$ values.

To fill this gap in our knowledge, additional calculations are performed in the $N_F = 30$ basis which provides quite accurate description of toroidal shapes in the
$Z \geq 140$ hyperheavy nuclei (see Sect. III in Ref. [2]). Such calculations are extremely time-consuming even in axial RHB framework and thus they are carried out only for restricted set of nuclei displayed in Fig. 8. These are $Z = 136, 146, 156, 166$ and 176 nuclei. Apart of few regions, the calculations are performed in step of $\Delta N = 10$ to save computational time. Despite these limitations they allow to understand the general features of the distribution of toroidal shapes as well as the evolution of underlying single-particle structure across the nuclear chart.

The results of these calculations are presented in Fig. 8. To facilitate the discussion we are using here the definitions of tori as thin and fat employed in the physics of toroidal liquid droplets [46]. Large/small ratio of the radius $R$ of toroid (called as "major radius" in some publications see, for example, Ref. [5]) to the radius $d$ of its tube (called as "minor radius" in Ref. [5]) corresponds to thin/fat tori. The lowest $\beta_2$ values ($\beta_2 \approx -2.2$) are obtained in the $Z \approx 136, N \approx 206$ region (see Fig. 8) and these nuclei can be defined as fat toroidal nuclei because of small aspect ratio $R/d$. The absolute $\beta_2$ values increase on moving away from this region. Especially large values of $|\beta_2|$ are obtained in proton-rich nuclei with $Z > 140$ in the vicinity of two-proton drip line. These toroidal shapes are characterized by very large radius of the torus and small radius of the torus tube and thus these nuclei are described as thin toroidal nuclei. Slightly smaller values of $|\beta_2|$ are seen in neutron-rich $N \geq 310$ nuclei. The aspect ratios $R/d$ for these nuclei are slightly smaller as compared with the ones in proton-rich nuclei but these nuclei are still the representatives of thin toroidal nuclei. Remaining nuclei shown by cyan, dark and light green as well as grey colors in Fig. 8 are characterized by $\beta_2$ ranging from $-2.5$ to $-3.7$. A general trend of the increase of torus radius $R$ and the aspect ratio $R/d$ with increasing proton number is seen in Fig. 8. It is a consequence of Coulomb repulsion: toroidal shapes provide less compact distribution of charge as compared with spherical ones and thus the Coulomb energy is substantially reduced for toroidal shapes as compared with spherical ones (see discussion in Sect. XII in Ref. [2]). The increase of proton number requires the increase of torus radius in order to minimize the Coulomb energy by creating less compact distribution of charge. Observed features in the distribution of toroidal shapes, which are the result of the competition of different energy minima similar to the minima A and B shown in Fig. 1 (see also Fig. 16 in Ref. [2]), have a root in underlying shell structure of toroidal hyperheavy nuclei (see Sec. IV B).
To get a better understanding of the relative properties of proton and neutron density distributions, we compare them in Fig. 9 for the $^{348}\text{138}$ and $^{466}\text{156}$ nuclei. Similar to the situation at spherical shape (see, for example, Fig. 5), the maximum of proton density distribution $\rho_{p}^{\text{max}}$ is significantly smaller ($\rho_{p}^{\text{max}} \approx \frac{2}{3}\rho_{n}^{\text{max}}$) than the neutron one $\rho_{n}^{\text{max}}$ and those maxima do not necessary appear at the same distance from the center of toroid. The outer edges of the proton and neutron density distributions appear at approximately the same distances from the center of toroid. However, the diameter of the hole in the center of proton density distribution is visibly larger than the one in the case of neutrons. This is most likely the consequence of the Coulomb repulsion acting on protons. Thus, the diameter of toroid tube is smaller in the case of neutrons as compared with the one for neutrons. Note also that the density distribution in toroid tube is not necessary symmetric with respect of its geometrical axis of symmetry: this is especially visible in the case of proton density distributions presented in Figs. 9(b) and (d). Detailed analysis reveals that this is a consequence of the occupation of the single-particle orbitals characterized by different spatial distributions of the single-particle densities.

Because of the presence of well pronounced minima (similar to the minimum D in Fig. 1), the present axial RHB calculations in extremely large basis confirm for the first time the stability of toroidal $Z \geq 140$ nuclei shown in Fig. 8 with respect of so-called breathing deformations. The breathing deformation [5] preserves the azimuthal symmetry of the torus and it is defined by the radius of torus and the radius of its tube. In our calculations, this type of deformation is related to the $\beta_{2}$ values (see discussion in Ref. [1]). This result is clearly different as compared with the ones obtained for classical uncharged toroidal liquid droplets which are unstable with respect of shrinking instabilities [20, 46, 47]. Because of surface tension such droplet starts from toroidal shape but then gradually shrinks by closing its interior hole and transforms into spherical droplet [20, 46, 47]. In atomic nuclei, this shrinking instability is counteracted by the Coulomb force: the transition to a more compact spherical configuration leads to a substantial increase of the Coulomb energy and thus it is not energetically favored in hyper-heavy nuclei [2].

Another class of potential instabilities of toroidal nuclei is related to so-called sausage deformations [5]: they make a torus thicker in one section(s) and thinner in another section(s). This class of the instabilities is much more difficult to describe in the density functional theories since their consideration requires, in general, symmetry unrestricted computer codes. This fact combined with the requirement for extremely large basis in high-$Z$ systems makes this problem numerically intractable with existing computer codes for absolute majority of toroidal nuclei. The only exception are fat toroidal nuclei located in the $Z \approx 136$, $N \approx 210$ region for which (as illustrated by the examples of the $^{354}\text{134}$ and $^{348}\text{138}$ nuclei discussed in Refs. [1, 2]) the calculations for even-multipole sausage deformations within the triaxial RMF+BCS codes are possible [1]. However, even such calculations are extremely time-consuming and can be performed only for a few nuclei.

In such a situation it is useful to get some insight from the studies of classical liquid droplets. Thin toroidal droplets exhibit Plateau-Rayleigh instabilities: when the outer circumferences of toroid is equal to an integer ($n$) times of the wavelength $\lambda_{c}$ of unstable mode, the toroidal droplet will eventually fission into $n$ spherical droplets [46] (see also Ref. [48] for the results obtained for liquid toroidal droplets suspended in another liquid). Note that in classical toroidal liquid droplets the Plateau-Rayleigh instability disappears for sufficiently fat tori ($R/d < 2$) while the shrinking mode is present for all aspect ratios [46]. These features have been confirmed in experimental studies of stability of both toroidal droplets in a viscous liquid [49] as well as melted polymer rings [50]. The instability with respect of so-called sausage deformations [5] in nuclear physics leading to multifragmentation\footnote{In this context it would be interesting to see whether the ob-} is an analog of the Plateau-Rayleigh instabilities. Thus,
FIG. 10. Proton and neutron single-particle energies, i.e., the diagonal elements of the single-particle Hamiltonian $h$ in the canonical basis [29], for the lowest in total energy solution in the $^{348}_{138}$ nucleus calculated as a function of the $\beta_2$ quantity. Black solid and red dashed lines are used for positive- and negative-parity states, respectively. The dominant components $\Omega[N, n_z, \Lambda]$ of the wave functions (as calculated at LEMAS) are shown by blue and green colors for the positive- and negative-parity orbitals, respectively. The energies $E_F$ of the respective Fermi levels are shown by blue dotted lines. The vertical orange lines and orange arrows are drawn at the $\beta_2$ value corresponding to LEMAS. Shell gaps are indicated by encircled numbers.
these results suggest that such instabilities are less important for fat toroidal nuclei [characterized by low (in absolute sense) values of $\beta_2 > -2.5$ and located in the $Z \approx 134, N \approx 210$ region (see Fig. 8)] but become more critical (and probably fatal) for thin toroidal nuclei characterized by large (in absolute sense) values of $\beta_2$. The latter type of nuclei become dominant both with increasing proton number $Z$ and in proton- and neutron-rich nuclei (see Fig. 8). The former suggestion is in line with the observed multifragmentation of high-spin configurations of $^{28}\text{Si}$ into $7\alpha$-particles [19] represents the analog of Plateau-Rayleigh instabilities of toroidal droplets in nuclear physics.

FIG. 11. The same as Fig. 10 but for the $^{466}_{156}$ nucleus.
results of triaxial RMF+BCS calculations for the $^{354\,134}$ and $^{348\,138}$ nuclei, which have 4.4 and 8.54 MeV fission barriers for non-axial distortions, respectively (see Ref. [1]).

However, it is necessary to recognize that fully quantum mechanical calculations based on the density functional theory are needed for establishing the stability of toroidal nuclei with respect of sausage deformations. Toroidal liquid droplets have a uniform density and the tube of torus has a cylindrical form [46]. On the contrary, the DFT calculations paint much more complicated picture. First, the density rapidly changes across the tube of the torus with considerable mismatch between proton and neutron densities (see Fig. 9 in the present paper, Fig. 2(c) and (d) in Ref. [1] and Fig. 9 in Ref. [51]) which are defined by the occupation of underlying proton and neutron single-particle orbitals. The description of such a situation on the level of liquid-drop model would require the model based on two (proton and neutron) fluids with the specification of functional dependencies of their densities on the position in the tube of the torus. Second, not in all cases the tube of the torus is represented by a perfect cylinder (see Fig. 2 in Ref. [2]). This may lead to an enhanced stability against sausage deformations since experimental studies of toroidal liquid droplets show that oblong cross section of the torus tube suppresses Plateau-Rayleigh instabilities as compared with circular one [47]. Because of above mentioned reasons the analysis of Ref. [5] indicating the instability of toroidal nuclei with respect of sausage deformations in the liquid drop model should not be taken at face value. Note also that this analysis considers only the nuclei with $Z < 120$ in which toroidal shapes are formed at high excitation energies with respect of the ground states while the toroidal shapes in the majority of hyperheavy nuclei are expected to be the ground states. Moreover, the quantum shell effects can counterbalance the potential instabilities towards sausage deformations at some combinations of proton and neutron numbers and deformations [1, 5].

B. Shell structure of toroidal hyperheavy nuclei

It is well known that the presence of large gaps in proton and neutron single-particle energies leads to an extra stability of nuclear systems. So far, the analysis of toroidal shell structure at spin $I = 0$ has been performed in light nuclei [5, 51], in the intermediate mass region nuclei [52] and in superheavy $Z \approx 120$ nuclei [15, 17]. Such an analysis was based either on phenomenological toroidal single-particle potential (see Refs. [5, 51, 52]) or on Skyrme DFT calculations (see Refs. [15, 17, 51]). Large gaps in the single-particle energies have been found at toroidal shapes in all these regions. For example, in light nuclei these energy gaps give rise to “toroidal shells” at “magic” nucleon numbers $N = 2(2m+1)$ with $m$ being integer satisfying the condition $m \geq 1$ [5]. The extra stability associated with toroidal shells leads to local energy minima at toroidal shapes in many nuclei either at spin zero [5, 53] or in some high spin isomer states [51]. However, in all these nuclei such minima are located at high excitation energies with respect of ellipsoidal-like ground state.

However, the situation changes in hyperheavy nuclei in which the ground states are expected to have toroidal shapes. Thus, it is very important to investigate shell structure of toroidal hyperheavy nuclei. In particular, it would be interesting to see whether there are large shell gaps or reduced density of the single-particle states at specific particle numbers which could provide an extra stability with respect of potential instabilities originating from sausage deformations. One should also remember that even if hyperheavy nuclei are unstable with respect of sausage deformations in the liquid drop model, they can be stabilized by quantum shell corrections. The best known example of such a situation are superheavy nuclei: they are unstable in the liquid drop model but are relatively stable in fully quantum mechanical picture which includes shell corrections [54, 55].

The analysis presented in Sec. IV A suggests that it is more likely to get potentially stable toroidal nuclei when their shapes in corresponding minima are characterized by small absolute $\beta_2$ values (or small aspect ratio $R/d$). The toroidal $^{354\,134}$ and $^{348\,138}$ nuclei are representative cases of such shapes (see Fig. 1 in supplemental material to Ref. [1] and Fig. 19 in Ref. [2]). Triaxial RMF+BCS calculations of Refs. [1, 2] suggest that these two nuclei are expected to be relative stable with respect of non-axial distortions (even-multipole sausage deformations) with calculated fission barriers being equal to 4.4 and 8.54 MeV, respectively. Enhanced stability of the $^{348\,138}$ nucleus is a reason why we start the analysis of toroidal shell structure from this nucleus which is characterized by moderately compact toroidal shapes [see Figs. 9(a) and (b)]. We also consider toroidal shell structure in the $^{466\,156}$ nucleus. The LEMAS of this nucleus is characterized by non-compact toroidal shapes with large $R/d$ aspect ratio [see Figs. 9(c) and (f)], but there is also an excited minimum B (see Fig. 1) which is characterized by very compact toroidal shapes with very small hole in the center [see Figs. 9(c) and (d)].

The Nilsson diagrams for these nuclei are shown in Figs. 10 and 11. In order to illustrate the differences between shell structure of toroidal and ellipsoidal-like nuclei, bottom panels display proton and neutron single-particle states in the very large energy and $\beta_2$ ranges. They are shown from the bottom of respective potentials up to 4 MeV energy above the continuum threshold and from $\beta_2 = -5.1$, corresponding to toroidal nuclei with large $R/d$ aspect ratio, up to $\beta_2 = +3.5$ in the $^{348\,138}$ nucleus and up to $\beta_2 = 2.25$ in the $^{466\,156}$ nuclei. These large positive $\beta_2$ values correspond to pre-fissioning configurations with well pronounced neck (see, for example, density distribution at the position F of Fig. 1). Middle and top panels of Fig. 10 show the regions of interest in
FIG. 12. The same as Fig. 10 but for the single-particle states located in the bottom part of nucleonic potential and in the $\beta_2$ range from $-5.1$ up to 0.0. Toroidal shell gaps are shown by bold blue numbers. Encircled letters and green arrows are used to indicate the pairs of the single-particle states which are almost degenerate in energy. The structure of these states are shown in Table II.

TABLE II. The dominant components of the wave functions of nearly-degenerate pairs of the single-particle states of same parity indicated by the letters in Fig. 12. They are defined at the $\beta_2$ value corresponding to LEMAS. The states forming the pair are shown in the columns labelled as "1st state" and "2nd state".

|       | 1st state          | 2nd state          |
|-------|--------------------|--------------------|
| (a)   | $\frac{1}{2}[9,0,1]$ | $\frac{3}{2}[9,0,1]$ |
| (b)   | $\frac{3}{2}[10,0,2]$ | $\frac{5}{2}[10,0,2]$ |
| (c)   | $\frac{5}{2}[9,0,3]$ | $\frac{7}{2}[9,0,3]$ |
| (d)   | $\frac{7}{2}[10,0,4]$ | $\frac{9}{2}[10,0,4]$ |
| (e)   | $\frac{9}{2}[9,0,5]$ | $\frac{11}{2}[9,0,5]$ |
| (f)   | $\frac{11}{2}[10,0,6]$ | $\frac{13}{2}[10,0,6]$ |
| (g)   | $\frac{13}{2}[11,0,7]$ | $\frac{15}{2}[11,0,7]$ |
| (h)   | $\frac{15}{2}[10,0,8]$ | $\frac{17}{2}[10,0,8]$ |
| (i)   | $\frac{17}{2}[11,0,9]$ | $\frac{19}{2}[11,0,9]$ |
| (j)   | $\frac{19}{2}[12,0,10]$ | $\frac{21}{2}[12,0,10]$ |
| (k)   | $\frac{21}{2}[13,0,11]$ | $\frac{23}{2}[13,0,11]$ |
| (l)   | $\frac{23}{2}[12,0,12]$ | $\frac{25}{2}[14,1,12]$ |

blown-up scale. The analysis of these figures reveals the general features which are discussed below.

Toroidal shell structure (especially the one for the shapes with large $R/d$ aspect ratio) has much more pronounced regular features as compared with the shell structure of ellipsoidal-like shapes in the range of the $\beta_2$ values from $\approx -1.15$ up to $\approx 1.5$ which looks quite chaotic for deformed shapes [see Fig. 10(e) and (f) and Fig. 11(e) and (f)]. At higher $\beta_2$ values typical features of shell structure of two-center shell model (see, for example, Ref. [56]) are seen.

The bunching of the pairs of the orbitals of the same parity with dominant structure of $\Omega[N,n_z,\Lambda]$ and $(\Omega+1)[N,n_z,\Lambda]$ with $N \geq 9$ and $n_z = 0$ (see Table II)\(^4\) leads to the appearance of toroidal shell gaps at particle numbers 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46 at the bottom of proton and neutron potentials (see Fig. 12). These gaps exist in a large range of the $\beta_2$ values; this is contrary to the case of shell gaps for ellipsoid-like shapes which are localized in deformation. They are also consistent with the ones obtained in the study of toroidal shapes in light nuclei within toroidal harmonic oscillator shell model\(^5\) and Skyrme DFT (see Figs. 1, 5 and 4 The only exception is the last pair of the states shown in Table II for which the 2nd state has $n_z = 1$.

5 This type of the model has been described before either as shell
FIG. 13. The same as Fig. 10 but for the single-particle states located in the intermediate energy range of nucleonic potential of the $^{348}_{138}$ nucleus and in the $\beta_2$ range from $-3.0$ up to $-1.5$. Encircled letters and green arrows are used to indicate the pairs of single-particle states of opposite parity which are almost degenerate in energy. The structure of these states is shown in Table III.

The general features of the pairs of orbitals with dominant structure of $\Omega[N,n_z,\Lambda]$ and $(\Omega + 1)[N,n_z,\Lambda]$ with $n_z = 0$ changes drastically in the energy ranges between $-20$ MeV and $0$ MeV for protons and between $-25$ MeV and $0$ MeV for neutrons (see Figs. 10, 11 and 13) for $\beta_2 \leq -1.8$. First, their energies decrease almost linearly with increasing absolute value of $\beta_2$. Second, there is a periodic pattern in the change of the orbitals: with increasing energy two positive parity orbitals are followed by two negative parity orbitals and then by two positive parity orbitals and so on. Third, these orbitals form the grating-like structure with almost equidistant in energy spacing between them.

In the same energy range as discussed in previous paragraph, there are other single-particle structures dictated by the symmetries of the toroid. These are almost degenerate in energy single-particle states of opposite parity (see Fig. 13 and Figs. 10 and 11) with dominant structures of the wave functions given by $\Omega[N,n_z,\Lambda]$ and $\Omega[N',n'_z,\Lambda']$ where the following conditions $N' = N \pm 1$, $|\Lambda' - \Lambda| = 0$ or $1$ and $|n'_z - n_z| = 0$ or $1$ are typically satisfied (see Table III). These states change their energy very slowly when the $\beta_2$ value is varied. Note that such pairs of the states are also present in the Skyrme DFT
This leads to the existence of many gaps in the single-particle spectra which are quite large. These are proton $Z = 120, 130, 134, 138, 140, 144$ and $148$ shell gaps with typical size of approximately $1$ MeV and neutron $N = 206, 210$ and $214$ shell gaps which are larger than $1$ MeV in the $^{348}138$ nucleus [see Figs. 10(a) and (b)]. Similar situation is also seen in the $^{466}156$ nucleus. In this nucleus the bands of proton $(Z = 130, 132, 134, 136), (Z = 142, 144, 146, 148)$, $(Z = 156, 158)$ and $(Z = 168, 170)$ shell gaps are formed because of the presence of the bunches of four single-particle states with relatively low $\Lambda$ values located between them. The energies of these bunches slightly decrease with increasing absolute value of $\beta_2$ [see Fig. 11(a)]. Note that some of these gaps reach almost $2$ MeV in size. Smaller neutron gaps with size of around $1$ MeV and below are seen at $N = 294, 296, 302, 314, 318$ and contrary to proton subsystem they do not form the bands of shell gaps [see Fig. 11(b)].

The obtained results for shell structure of toroidal nuclei allow us to understand its contribution into the stability of toroidal shapes with respect of breathing deformations. For example, LEMAS in the $^{348}138$ nucleus corresponds to the situation in which proton and neutron Fermi levels are located in the middle of the region of low density of single-particle states in the vicinity of the $Z = 134$ and $N = 210$ gaps, respectively [see Figs. 10(a) and (b)]. Any increase or decrease of the $\beta_2$ value from the one corresponding to LEMAS will lead to the increase of the density of the single-particle states in the vicinities of respective Fermi levels. This effect is especially pronounced for the neutron subsystem. As a consequence, the LEMAS corresponds to the largest or near-largest (in absolute sense) negative proton and neutron shell correction energies, while the deviation (in terms of $\beta_2$) from total energy minimum will lead to the reduction of these energies. This contributes to the stability of toroidal shapes with respect of breathing deformations.
However, as illustrated by the case of the \( ^{466}_{156} \) nucleus, the contribution of shell correction effects to the stability of the nuclei is expected to depend on proton and neutron numbers. In this nucleus, the neutron Fermi level at LEMAS is located at high density of the neutron single-particle states [see Fig. 11(b)], which likely leads to positive neutron shell correction energies. On the contrary, shell correction energies will be large and negative in the proton subsystem since the proton Fermi level is located in the vicinity of large \( Z = 156 \) gap [see Fig. 11(a)]. Note that this gap is so large that proton pairing collapses at the \( \beta_2 \) values near LEMAS; this is seen from the fact that the energy of the proton Fermi level coincides with the energy of the single-particle state located below the \( Z = 156 \) gap.

The same features are also active in respect of the stability of toroidal nuclei in sausage deformation degree of freedom. This is because of two factors. First, the shell gaps in breathing degree of freedom are also the shell gaps in the sausage degree of freedom (see Sect. IVD in Ref. [5]). Second, as shown in toroidal harmonic oscillator shell model for particle numbers of interest, the increase of sausage deformations \( \sigma_\Lambda \) of multipolarities \( \lambda = 1, 2 \) and 3 from zero to some finite values leads to washing out of these shell gaps and an increase of the density of the single-particle states in the vicinity of the Fermi level (see Figs. 21, 22 and 23 in Ref. [5]). Let us consider the nuclei in which the proton and neutron Fermi levels of the LEMAS solution are located in the region of low density of the single-particle states. In these nuclei the shell correction energy is negative at \( \beta_2 = 0 \) but it will either be reduced in absolute value or become positive when sausage deformations become non-zero. Thus, the instability in the breathing degree of freedom which exists on the level of liquid drop is counterbalanced in these nuclei by the quantum shell effects. The balance of these two contributions defines whether the toroidal nucleus is stable with respect of sausage deformations or not. Fully quantum mechanical calculations based on DFT are needed to establish the stability of a given nucleus with respect of sausage deformations. However, the analysis of the shell structure and the level density of the single-particle states in the vicinity of the proton and neutron Fermi levels provides a useful information on whether a given nucleus could potentially be stable with respect of sausage deformations. For example, as discussed above such an analysis for the \( ^{348}_{138} \) nucleus shows low densities of the single-particle states in the vicinity of proton and neutron Fermi levels and indeed the RMF+BCS calculations of Refs. [1, 2] reveal the stability of toroidal shapes in this nucleus with respect of even-multipole sausage deformations.

Figures 10 and 11 reveal some global bunching of the pairs of almost degenerate in energy single-particle states of opposite parities. For example, such bunches of the states are seen in the proton subsystem of the \( ^{348}_{138} \) nucleus at the energies \( \approx -19 \) MeV, \( \approx -14 \) MeV, \( \approx 0 \) MeV and \( \approx 4 \) MeV for \( \beta_2 = -5.0 \) (see Fig. 10(c) and (e)). The density of the single-particle states is high in these bunches and thus it is reasonable to expect that the shell correction energy \( E_{shell} \) will be positive when the Fermi level is located near or within these bunches. For such a situation it is reasonable to expect that the quantum shell effects will not help to stabilize toroidal shapes with respect of sausage deformations. With decreasing absolute value of \( \beta_2 \) the energies of these bunches of the single-particle states go down (see Fig. 10(c) and (e)). However, these bunches and the low density single-particle structure between them persist down to \( \beta_2 \) values corresponding to the transition from toroidal to concave disk shapes. The density of the single-particle states is low between these bunches and it is reasonable to expect that the majority of the combinations of particle number and \( \beta_2 \) the \( E_{shell} \) values will be negative when the Fermi level is located in this region. These features are the manifestation of so-called supershell structure which has been discussed in the case of ellipsoidal-like shapes in Ref. [57].
There is a drastic difference in the behavior of neutron and proton Fermi levels as a function of the $\beta_2$ value (see Fig. 10 and Fig. 11). The neutron Fermi level is more or less constant as a function of $\beta_2$. As a consequence, the calculated two-neutron drip line for toroidal shapes is close to the extrapolation of this line for ellipsoidal-like shapes (see Fig. 2). On the contrary, the proton Fermi level dives deeper into nucleonic potential with increasing absolute value of $\beta_2$; it is lower by approximately 5 MeV for toroidal shapes with large aspect ratio as compared with its position for biconcave disk shapes. As a consequence, the transition to toroidal shapes in hyperheavy nuclei creates a substantial expansion [the area between black solid and orange dashed lines in Fig. 2] of the nuclear landscape.

There are drastic changes in the single-particle structure of the $^{348}\text{138}$ nucleus at $\beta_2 \approx -1.15$ and $\beta_2 \approx -1.85$ (see Figs. 10(e), (d), (e) and (f)). The first change is related to the transition from biconcave disk shape to toroidal one (which is equivalent to an opening of the hole in the center of biconcave disk shape). The second one is associated with the redistribution of the proton density in the torus caused by the change of the occupation of the single-particle orbitals. This density is asymmetric with respect of the axis of torus tube and has a maximum closer to an outer edge of the torus for the $\beta_2$ values ranging from $\approx -1.15$ down to $\approx -1.85$. However, it becomes almost symmetric with respect of the axis of the torus tube for $\beta_2 \leq -1.85$. Note that similar changes in the single-particle structure are seen at $\beta_2 \approx -1.1, \beta_2 \approx -1.8$ and $\beta_2 \approx -2.7$ in the $^{466}\text{156}$ nucleus (see Figs. 11(c), (d), (e) and (f)) and their origins are similar to the ones discussed above in the $^{348}\text{138}$ nucleus.

In order to find potentially most stable toroidal nuclei, two-proton $S_{2p}(Z,N)$ and two-neutron $S_{2n}(Z,N)$ separation energies

$$S_{2n}(Z,N) = B(Z,N) - B(Z,N - 2),$$

$$S_{2p}(Z,N) = B(Z,N) - B(Z - 2,N),$$

and the $\delta_{2n}(Z,N)$ and $\delta_{2p}(Z,N)$ quantities defined as

$$\delta_{2n}(Z,N) = S_{2n}(Z,N) - S_{2n}(Z,N - 2),$$

$$\delta_{2p}(Z,N) = S_{2p}(Z,N) - S_{2p}(Z - 2,N),$$

are plotted in Fig. 15 for the region with ($Z = 132 - 144, N = 204 - 228$). Here $B(Z,N)$ is the binding energy. The separation energies show a sudden drop at the shell gaps, if they are large. If the variations of the level density are less pronounced, the $\delta_{2n}(Z,N)$ and $\delta_{2p}(Z,N)$ quantities related to the derivatives of the separation energies are more sensitive indicators of the localizations of the shell gaps (see discussion in Appendix of Ref. [58]). They also provide the information on average density of the single-particle states.

The presence of the neutron gap at $N = 210$ for toroidal shapes is visible in Figs. 15(a) and Fig. 16(a) in the $Z = 126 - 136$ nuclei. The $\delta_{2n}(Z,N)$ values for neutron numbers away from $N = 210$ are low which are indicative of high density of neutron single-particle states below and above the $N = 210$ shell gap. These features correlate with the ones seen in the Nilsson diagram [see Fig. 10(b)].

On the contrary, the $S_{2p}(Z,N)$ and $\delta_{2p}(Z,N)$ values (see Figs. 15(b) and Fig. 16(b)) are relatively smooth functions of proton number which indicates that the average density of proton single-particle states remains more or less constant. However, on average the $\delta_{2p}(Z,N)$ values are substantially higher than the $\delta_{2n}(Z,N)$ ones; only in the region of the peak of $\delta_{2n}(Z,N)$ at $N = 210$ they are comparable (see Fig. 16). This clearly indicates that the density of proton single-particle states is low in a wide range of proton numbers and this observation is supported by the comparison of Figs. 10(a) and (b). Note that the peak of $\delta_{2p}(Z,N) \approx 1.3$ MeV is seen for neutron numbers $N = 204 - 212$ (see Fig. 16(b)) suggesting an extra stability of these nuclei.

The combination of proton and neutron shell effects should lead to an enhanced stability of specific nuclei. As a result, discussed above features are most likely reasons why fission barrier is higher in the $N = 210$ $^{348}\text{138}$ nucleus as compared with the $N = 220$ $^{354}\text{134}$ one.
C. Functional dependence of the results

When considering the predictions for toroidal hyper-heavy nuclei and their shell structure it is important to evaluate their dependence on the employed functional. So far all predictions for such nuclei presented in Refs. [1, 2] and in the present paper were obtained with the CEDF DD-PC1. To study functional dependence of the predictions we perform additional calculations for the \(^{348}138\) and \(^{466}156\) nuclei with the NL3* [32], PC-PK1 [37], DD-ME2 [35] and DD-ME\(\delta\) [36] functionals and compare their results with the ones obtained with DD-PC1 earlier. These five state-of-the-art functionals represent three major classes of CDFT models [21] and have been globally tested in Refs. [21, 23, 25, 59, 60]. Note that in this set of the functionals the CEDF DD-PC1 and PC-PK1 provide better description of binding energies on a global scale as compared with other functionals.

The deformation energy curves obtained with these functionals are presented in Fig. 17. In both nuclei and in terms of relative energies of the minima corresponding to toroidal and ellipsoidal-like shapes there is a large similarity of the results obtained with point-coupling models DD-PC1 and PC-PK1 as well as with nonlinear meson-nucleon coupling model NL3* on the one hand and those obtained with density-dependent meson-exchange models DD-ME2 and DD-ME\(\delta\) on the other hand. In the latter type of the models, the toroidal shapes are less energetically favored with respect of ellipsoidal-like shapes as compared with former models. For example, in the \(^{348}138\) nucleus the fat toroidal shapes corresponding to the minimum A are more (less) energetically favored as compared with biconcave disk shapes corresponding to minimum B in the calculations with DD-PC1 and PC-PK1 (DD-ME2 and DD-ME\(\delta\)) functionals. Note that these two minima are located at approximately the same energies in the calculations with the NL3* functional [see Fig. 17(a)]. However, this difference in the predictions of relative energies of the minima A and B is not principal because the minimum B is not stable with respect of triaxial distortions in the calculations with DD-PC1 functional (see Ref. [1]) and the same situation is expected for other functionals because of the similarity of underlying shell structure. On the other hand, the minimum A is relatively stable with respect of even-multipole sausage deformations in the calculations with DD-PC1 (see Refs. [1, 2]) and because of similarity of underlying toroidal shell structure (see discussion of Fig. 18 below) it is reasonable to expect that this is also the case for remaining functionals.

Similar situation to the \(^{348}138\) nucleus holds also in the \(^{466}156\) one. This is because toroidal shapes are more energetically favored as compared with ellipsoidal-like ones in the calculations with CEDFs DD-PC1, PC-PK1 and NL3* than in those employing DD-ME2 and DD-ME\(\delta\) functionals (see Fig. 17(b)). For example, the energy difference \(\Delta E_{\text{diff}}\) between the minimum A corresponding to thin toroidal shapes and the minimum D corresponding to spherical shapes is approximately 117 MeV in the calculations with the first group of the functionals and only approximately 67 MeV in the calculations with the second group. Note that these differences cannot be explained by the differences in nuclear matter properties of the functionals since they are similar (quite different) in the pair of the DD-PC1 and DD-ME2 (DD-PC1 and PC-PK1) functionals (see Ref. [25]) which provide the \(\Delta E_{\text{diff}}\) values which differ by 53.4 MeV (by only 6.5 MeV).

These differences between the functionals, related to the relative energies of the minima corresponding to toroidal and ellipsoidal-like shapes, are expected to affect the position of the boundary between ellipsoidal-like and toroidal shapes in the nuclear landscape (see Fig. 2 in the present manuscript and the discussion in Sec. XII of Ref. [2]). However, this boundary depends not only on relative energies of these two types of the shapes but also on the stability of ellipsoidal-like shapes with respect of fission (see Ref. [2]). There is a quite substantial dependence of the fission barrier heights for ellipsoidal-like shapes on CEDF with the PC-PK1 and NL3* (DD-ME2 and DD-PC1) functionals providing the lowest (highest) barrier heights for superheavy nuclei among the CEDFs considered in Ref. [4] and a similar situation is also expected in the hyperheavy nuclei.

Despite above mentioned differences there are large similarities between the results of the calculations obtained with five functionals. For the first time, the results presented in Fig. 17 confirm that the transition from ellipsoidal-like to toroidal shapes with increasing proton number \(Z\) does not depend on CEDF. The presence of similar local minima in deformation energy curves (such as the minima A, B, C and D in the \(^{466}156\) nucleus and the minima A and B in the \(^{348}138\) nucleus) with similar equilibrium \(\beta_2\) values presented in Fig. 17 clearly suggest the similarity of underlying shell structure in all employed functionals. Note that in a few cases such minima are shoulder-like in deformation energy curves without a sufficient barrier on one side: these are the minimum C in the calculations with NL3* and PC-PK1 and the minimum B in the calculations with NL3* [see Fig. 17(b)].

The analysis of toroidal shell structure of the \(^{348}138\) and \(^{466}156\) nuclei obtained with the NL3*, PC-PK1, DD-ME2 and DD-ME\(\delta\) functionals reveals the same general features as those discussed in Sec. IV B for the DD-PC1 functional. Thus we will focus on fine details of the shell structure of these nuclei in the vicinity of the respective Fermi levels at the LEMAS of the minimum A in these two nuclei (see Fig. 17) since they are responsible for potential stability of respective toroidal shapes. The Nilsson diagrams for these four CEDFs are shown in Figs. 18 and 19: they can be compared with those obtained for DD-PC1 and presented in Figs. 10(a) and (b) and Figs. 11(a) and (b). This comparison reveals significant similarities between the results of the calculations obtained with different functionals.

For example, in the \(^{348}138\) nucleus the proton Fermi
level $E_{F}$ at LEMAS is located in the region of reduced density of proton single-particle states between shell gaps at $Z = 134$ and $Z = 140$ (see Fig. 10(a)) in the calculations with the DD-PC1 functionals. Similar situation exists also in the calculations with NL3*, PC-PK1, DD-ME2 and DD-ME$\delta$ CEDFs [see Figs. 18(a), (c), (e) and (g)]. In this nucleus, the neutron Fermi level is located in the middle of substantial $N = 210$ toroidal shell gap in the calculations with DD-PC1 [see Fig. 10(a)], DD-ME2 and DD-ME$\delta$ (see Figs. 18(e) and (g)) but it is shifted to the region of somewhat higher density of the neutron single-particle states below the $N = 214$ toroidal shell gap in the calculations with NL3* and PC-PK1 [see Fig. 18(b) and (d)]. These results suggest that two-proton separation energies $S_{2p}(Z, N)$ and the $\delta_{2p}(Z, N)$ quantities (see the discussion in the end of Sec. IV B) should be very similar for all five employed functionals. The same is true for related neutron $S_{2n}(Z, N)$ and $\delta_{2n}(Z, N)$ values obtained in the calculations with DD-PC1, DD-ME2 and DD-ME$\delta$ which are expected to reveal the presence of the $N = 210$ toroidal shell gap [see Fig. 16(b)]. However, it is quite likely that the peak in the $\delta_{2n}(Z, N)$ values visible at $N = 210$ in the calculations with DD-PC1 [see Fig. 16(a)] will be moved to $N \approx 214$ and substantially washed out in the calculations with NL3* and PC-PK1.

Similar situation exists also in the $^{466}156$ nucleus. The bands of proton ($Z = 130; 132; 134; 136$), ($Z = 142; 144; 146; 148$), ($Z = 156; 158; 160$) and ($Z = 168; 170$) shell gaps, formed because of the presence of the bunches of single-particle states with relatively low A values located between them, exist in all five functionals [see Figs. 11(a) and 19(a), (c), (e) and (g)]. Note that some of these gaps reach almost 2 MeV in size. The proton Fermi level at LEMAS is located either in the middle of large $Z = 156$ shell gap in the NL3*, PC-PK1, DD-ME2 and DD-ME$\delta$ functionals or at the bottom of this shell gap in the DD-PC1 CEDF and thus shell correction energies will be large and negative in proton subsystem in all functionals.

Smaller neutron shell gaps with the size of around 1 MeV and below are seen at $N = 294, 296, 302, 314, 318$ in DD-PC1 [Fig. 11(a)], at $N = 278, 290, 294, 300, 302, 314, 318$ in NL3* [Fig. 19(b)], at $N = 296, 300, 302, 314, 326$ in PC-PK1 [Fig. 19(b)], at $N = 292, 296, 298, 302, 308$ in DD-ME2 [Fig. 19(f)], and at $N = 282, 298, 304, 306, 312$ in DD-ME$\delta$ [Fig. 19(h)] and contrary to proton subsystem they do not form the bands of shell gaps. Considering relatively small size of neutron shell gaps, larger (as compared with proton subsystem) dependence of the predictions for neutron shell gaps on the functional is expected. These differences are not critical since in all functionals the neutron Fermi level at LEMAS is located at high density of the neutron single-particle states, which likely leads to positive neutron shell correction energies.

The results presented in Fig. 17 clearly indicate the stability of the nuclei under discussion with respect of breathing deformation in all employed functionals. The similarity of the shell structure in all five functionals strongly suggests that the considerations provided in Sec. IV B on potential stability with respect of sausage deformations of the nuclei under study in the case of CEDF DD-PC1 are also applicable for the NL3*, PC-PK1, DD-ME2 and DD-ME$\delta$ functionals.

**V. CONCLUSIONS**

In conclusion, the detailed investigation of the properties of spherical and toroidal hyperheavy even-even nuclei and their underlying shell structure have been performed in the framework of covariant density functional theory. The following conclusions have been obtained:

- Proton $Z = 154, 186$ and neutron $N = 228, 308$ and 406 spherical shell gaps exist in all employed CEDFs. Their combinations define the islands of stability of spherical hyperheavy nuclei. The sizes of these gaps (both actual $\Delta E_{gap}$ and scaled $\Delta E_{gap}\alpha^{4/3}$) are larger than those of $Z = 120$ and $N = 184$ in superheavy nuclei. This sug-

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**FIG. 17.** The deformation energy curves obtained in axial RHB calculations with indicated CEDFs. The local and global minima are indicated by the arrows with letters. The same labelling of minima as shown in Fig. 1 is used for the $^{466}156$ nucleus.
FIG. 18. The same as Figs. 10(a) and (b) but for the results obtained with indicated CEDFs. The vertical dashed orange lines are drawn at the $\beta_2$ values corresponding to LEMAS.
suggests that some spherical hyperheavy nuclei may be more stable than superheavy ones. Systematic theoretical uncertainties in the predictions of the sizes of spherical shell gaps in hyperheavy nuclei are smaller than those in superheavy nuclei and experimentally known nuclei.

- Detailed calculations in extremely large basis have allowed to establish for the first time the general trends of the evolution of toroidal shapes in the $Z \approx 130 – 180$ region of nuclear chart. Although they have been performed only for selected $Z = 136, 146, 156, 166$ and 176 nuclei with the step in neutron number of $\Delta N = 10$, their distribution in the nuclear chart between two-proton and two-neutron drip lines and deformation energy curves of these nuclei are such that they allow to safely extrapolate major conclusions to all nuclei in above mentioned region. The most compact fat toroidal nuclei are located in the $Z \approx 136, N \approx 206$ region (see Fig. 8). Thin toroidal nuclei with large $R/d$ aspect ratio become dominant with increasing proton number and on moving towards proton and neutron drip lines.

- All the nuclei in the $Z \approx 130 – 180$ region located between neutron and proton drip lines are expected to be stable with respect of breathing deformations. Because of numerical difficulties it is much more problematic to answer the question on their stability with respect of sausage deformations. However, the analysis of theoretical and experimental studies of toroidal liquid droplets as well as the results on the stability of the $^{354}134$ and $^{348}138$ nuclei with respect of even-multipole sausage deformations obtained in Refs. [1, 2] suggest that fat toroidal nuclei located in the $Z \approx 136, N \approx 210$ region are potentially more stable with respect of sausage deformations than thin toroidal nuclei located outside of this region. Nevertheless, future fully quantum mechanical calculations based on DFT are needed to establish the stability of specific toroidal nuclei since the quantum shell effects can counterbalance the instabilities with respect of sausage deformations [5].

- Toroidal shell structure (especially the one for the shapes with large $R/d$ aspect ratio) has much more pronounced regular features as compared with the shell structure of deformed ellipsoidal-like nuclei. Global bunching of the pairs of almost degenerate single-particle states of opposite parities leads to an appearance of supershell structure. These features are mostly driven by the existence of the two classes of the pairs of the orbitals at toroidal shapes. The pairs of the orbitals with dominant structure of $\Omega[N,n_z,\Lambda]$ and $(\Omega + 1)[N,n_z,\Lambda]$ with $n_z = 0$ belong to the first class. The second class is formed by almost degenerate in energy single-particle states of opposite parities with dominant structures of the wave functions given by $\Omega[N,n_z,\Lambda]$ and $\Omega[N',n_z',\Lambda']$ for which the conditions $N' = N \pm 1$, $|N' - \Lambda| = 0$ or 1 and $|n_z' - n_z| = 0$ or 1 are typically satisfied.

- As illustrated by discussed cases, at LEMAS large shell gaps and/or low density of the single-particle states appear at least in one of the subsystems (proton and/or neutron) in the vicinity of its Fermi level. These shell gaps are also the gaps in breathing and sausage degrees of freedom [5]. If the Fermi level in a given subsystem is located in the vicinity of the large shell gap or low density of the single-particle states, quantum shell effects will act against the instabilities in breathing and sausage deformations. These stabilizing effects will be definitely enhanced if both proton and neutron subsystems are characterized by such features.

- However, the analysis of the Nilsson diagrams for all nuclei calculated in Fig. 8 shows that in many of these nuclei the level densities are high near the proton and neutron Fermi levels at LEMAS. In reality, such a situation becomes much more frequent with increasing proton and neutron numbers and respective rise of the single-particle level densities. The reason is quite simple: the $\beta_2$ value of LEMAS is defined mostly by the competition of nuclear surface tension and Coulomb interaction and the shell correction effects play only a secondary role here. As a result, such nuclei are expected to be unstable with respect of sausage deformations. Thus, it is reasonable to expect the existence of the "continent" of stability of toroidal nuclei in low-$Z$ systems which is replaced by the "isolated islands" of their stability in higher-$Z$ nuclei located in the "sea of the instability".

The problem of the stability of toroidal nuclei with respect of sausage deformations emerges as a major obstacle in their study. There are several possible ways to investigate such instabilities. One is based on the analysis of time evolution of the toroidal nucleus after some external disturbance of equilibrium shape in time-dependent Hartree-(Fock)-Bogoliubov framework formulated in coordinate representation. However, the sizes of thin toroidal nuclei are significantly larger that those of ellipsoidal ones and the tube of the torus of such nuclei is characterized by a small radius and rapid change of the densities. These factors would require very large three-dimensional box with small step in each direction. At present, it is not clear whether such calculations are numerically feasible.

An alternative possibility is to rewrite existing RHB computer codes in the basis of toroidal harmonic oscillator potential and to study "fission" barriers in respective sausage deformations. Since this is a native basis for toroidal shapes, it is reasonable to expect that suf-
FIG. 19. The same as Figs. 11(a) and (b) but for the results obtained with indicated CEDFs. The vertical dashed orange lines are drawn at the $\beta_2$ values corresponding to LEMAS.
ficient numerical accuracy could be achieved at significantly lower size of the toroidal harmonic oscillator basis as compared with existing computers codes formulated in the traditional harmonic oscillator basis which is more suitable for ellipsoidal-like shapes. For example, in the latter codes the $N_F = 20$ fermionic shells are sufficient for the description of spherical and ellipsoidal shapes in the $^{466}_{156}$ nucleus but $N_F = 30$ is required for the description of toroidal shapes [2]. The use of toroidal harmonic oscillator basis would reverse the situation and hopefully the basis with $N_F = 20$ will be sufficient for the description of toroidal shapes near LEMAS and their instabilities with respect of sausage deformations. Our experience tells us that numerical calculations in such a basis are feasible with existing high performance computers. The instabilities of toroidal nuclei with respect of sausage deformations can potentially be studied by means of three-dimensional lattice (3D lattice) method suggested in Ref. [61]. For example, this method has been used for the investigation of the stability of linear chain structure of three $\alpha$ clusters in $^{12}_{C}$ against bending and fission in the framework of cranking CDFT in Ref. [62].

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