A POISSON MANIFOLD OF STRONG COMPACT TYPE

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1. Introduction

Poisson structures make precise the notion of (possibly singular) foliations by symplectic leaves. A Poisson structure on a manifold \( M \) is given by a bi-vector \( \pi \) closed under the Schouten Bracket, or more classically, by a bracket \( \{ \cdot, \cdot \} \) on smooth functions such that \( (C^\infty(M), \{ \cdot, \cdot \}) \) becomes a Lie algebra over the reals and \( \{ f, \cdot \} \) is a derivation for every function \( f \).

Poisson manifolds may have very complicated characteristic foliation and transverse geometry, and this results in the lack of global or semilocal structural theorems in Poisson geometry.

A natural way to exert control on a Poisson structure is to require the existence of a well-behaved ‘Lie group-like’ object integrating in a suitable sense the infinite dimensional Lie algebra \( (C^\infty(M), \{ \cdot, \cdot \}) \). To pursue this point of view, it is more convenient to regard at the Lie algebroid structure induced on \( T^*M \) by the Poisson structure: specifically, the Lie algebroid bracket on 1-forms is determined by the Lie algebra bracket on functions \( \left[ df, dg \right] := d\{ f, g \} \); the anchor map \( \rho \) sends \( df \) to its Hamiltonian vector field \( X_f := \{ f, \cdot \} \).

Recall that a Lie algebroid \( (A \to M, [\cdot, \cdot], \rho) \) is said to be integrable if there exists a Lie groupoid whose Lie algebroid is isomorphic to \( (A \to M, [\cdot, \cdot], \rho) \); such a Lie groupoid is called an integration of \( (A \to M, [\cdot, \cdot], \rho) \). Also, if a Lie algebroid is integrable, up to isomorphism it has a unique integration with 1-connected source fibers, its so-called ‘canonical integration’.

In analogy with Lie theory and group actions, it is natural to focus one’s attention on Poisson manifolds whose associated Lie algebroid is integrable, and the canonical integration is compact.

Definition 1. \cite{7} We say that \( (M, \pi) \) is a Poisson manifold of strong compact type (henceforth PMSCT) if its associated Lie algebroid structure \( (T^*M, [\cdot, \cdot]_\pi, \rho_\pi) \) is integrable, and its canonical integration is Hausdorff and compact.

If the Lie algebroid associated to a Poisson structure is integrable, then the canonical integration \( \Sigma(M) \to M \) carries a canonical symplectic structure \( \Omega_\Sigma \) compatible with the groupoid operations (a multiplicative symplectic structure) \cite{6, 5}. Thus, one can think of a PMSCT as being ‘desingularized’ by a compact symplectic manifold.

By their very definition, PMSCT should have common features with both compact semisimple Lie groups and compact symplectic manifolds, and, in fact, it is possible to lay down a rich general theory for them confirming the above expectation \cite{7}. However, also because the definition of PMSCT is rather demanding, the construction of PMSCT beyond the trivial case of compact symplectic manifolds with finite fundamental group, is a rather challenging problem. In this respect, in \cite{7} we describe a connection between PMSCT and quasi-Hamiltonian \( \mathbb{T}^n \)-spaces.

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Quasi-Hamiltonian \(\mathbb{T}^n\)-spaces appeared for the first time in [12], related to a possible characterization of Hamiltonian circle actions as those symplectic circle actions with fixed points (however, the name quasi-Hamiltonian space was coined in [3], where these spaces were analyzed under the perspective of symplectic reduction). In [11], D. Kotschick gave an example of a quasi-Hamiltonian \(S^1\)-space with contractible orbits and free \(S^1\)-action, answering a question raised in [13]. According to [7], the Poisson reduced space of such quasi-Hamiltonian \(S^1\)-space –which is a bundle over the circle with fiber diffeomorphic to a K3 surface– must be a PMSCT.

Kotschick’s construction relies on properties of the moduli space of marked hyperKahler K3 surfaces. In this note we revisit the moduli theory of marked K3 surfaces from a Poisson theoretic perspective, namely, making emphasis on the (Poisson) universal family. Then we show how appropriate equivariant mappings into the moduli of marked pairs give rise to PMSCT. Finally, we construct such an equivariant mapping building on Kotschick’s work (the most delicate part being the ‘holes’ in the moduli of hyperKahler metrics associated to roots, an issue which was overlooked in [11]).

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2. Construction of the PMSCT

The PMSCT is going to be a fibration over the circle with fiber diffeomorphic to the smooth manifold underlying a K3 surface.

2.1. Families of marked K3 surfaces. In what follows we will recall the aspects of the moduli theory of marked K3 surfaces, making emphasis on the universal families. Our reference is [3], Chapter VIII.

2.1.1. The objects. A K3 surface is a closed complex surface with trivial canonical bundle and trivial fundamental group. K3 surfaces are all diffeomorphic to a fixed 4-dimensional simply connected manifold \(F\). The intersection form in \(H^2(F;\mathbb{Z})\) is isomorphic to the rank 22 lattice

\[
L := 3H \oplus 2(-E_8),
\]

where \(H\) is the intersection form of \(S^2 \times S^2\) and \(E_8\) is the intersection form with matrix the absolute values of the entries of the Cartan matrix of \(E_8\) (see [10] for background on intersection forms).

A marked K3 surface is defined to be a K3 surface \(S\) together with an isomorphism of lattices \(H^2(S;\mathbb{Z}) \cong L\).

2.1.2. The spaces. Let \(L_C\) denote the complexification of the unimodular lattice \(L\).

The period domain is defined to be

\[
\Omega = \{ [v] \in P(L_C) \mid \langle v, v \rangle = 0, \langle v, \bar{v} \rangle > 0 \}.
\]

For marked K3 surfaces there exists a (non-Hausdorff) moduli space \(M_1\) together with a so called period map

\[
\tau : M_1 \to \Omega.
\]

This is an étale holomorphic surjection, sending each marked K3 in \(M_1\) to the line spanned in \(L_C\) by any of its holomorphic symplectic forms (via the marking).

The moduli space relevant for our purposes is the moduli space of marked pairs or marked K3 surfaces together with a Kahler class/Kahler form, denoted by \(K\Omega\). Let \(\Delta = \{ \delta \in L \mid \delta^2 = -2 \}\) be the so called set of roots and let \(L_{\mathbb{R}}\) be the tensorization of the unimodular lattice \(L\) with the reals. Then

\[
K\Omega = \{ (z, v) \in L_{\mathbb{R}} \times \Omega \mid \langle z, z \rangle > 0, z \in v^+, \langle z, \delta \rangle \neq 0, \forall \delta \in v^+ \cap \Delta \}.
\]
There is a forgetful map $K\Omega^0 \to M_1$ which for a fixed complex structure collapses
the Kahler cone to a point; there is also a first projection $\text{pr}_1: K\Omega^0 \to L_\mathbb{R}$.

2.1.3. The families. The two aforementioned moduli spaces have a corresponding
universal family.

Firstly, there exist $U \to M_1$ a universal marked family.

The second universal family is perhaps less standard. We say that $(E, \pi) \to W$
is a Poisson structure of hyperKahler type or a hyperKahler family, if $E$
is a real analytic family of K3 surfaces and $\pi$ is a (smooth) Poisson structure
with characteristic foliation the fibration $E \to W$, and such that $\pi(w)$ is a Kahler form
for a hyperKahler metric for $E_w$ (here we understand $\pi$ as a fiberwise symplectic
form).

The pullback of $U$ by the real analytic submersion $K\Omega^0 \to M_1$
gives rise to a real analytic family $KU \to K\Omega^0$. This family is endowed with a Poisson structure
$\pi_{KU}$ such that $\pi_{KU}(\kappa, v)$ is the Kahler form representing the class $\kappa$
associated to the unique hyperKahler metric whose space of self-dual classes is the positive 3-plane
in $L_\mathbb{R}$ spanned by $\kappa, v$. As a result $(KU, \pi_{KU})$ is a universal marked Poisson space
of hyperKahler type.

Remark that the integrability criteria in [6] and [1] imply that $(KU, \pi_{KU})$
is integrable and that the canonical integration is Hausdorff.

2.1.4. The action. The lattice $L$ has an index 2 subgroup of automorphisms $O^+$:
these are the elements which act on $L_\mathbb{R}$ preserving the orientation of positive 3-planes [4].

The are obvious actions of $O^+$ on $L_\mathbb{R}$ and $\Omega$. The strong form of Torelli’s theorem
says that $O^+$ also acts on $M_1$ and that the obvious action on $L_\mathbb{R} \times \Omega$ preserves $K\Omega^0$.

The induced action on universal families $U$ and $KU$ is a consequence on the one
hand of the strong form of Torelli’s theorem, which for each $\gamma \in O^+$
grants the existence of a unique biholomorphisms between any two fibers in the orbit of $\gamma$
in the base of the corresponding universal family, and the absence of holomorphic
vector fields on any K3, which implies that the fiberwise biholomorphism fits into
an automorphism (and the latter property also holds for real analytic families [13],
so there is an induced action on $KU$). It also follows easily that the Poisson tensor
$\pi_{KU}$ is preserved.

All the maps described between spaces and families are $O^+$-equivariant.

2.1.5. Universal family of hyperKahler type and PMSCT. The universal family
$(KU, \pi_{KU})$ provides a way of constructing PMSCT. Before stating our result, recall
that an affine subspace $V$ of $H^2(F; \mathbb{R})$ is called integral affine, if its vector space of
directions $\overrightarrow{V}$ intersects the integral cohomology lattice $H^2(F; \mathbb{Z})$ in a lattice whose
rank is the dimension of $V$ (a lattice of full rank).

**Theorem 1.** Let $V^0$ be a subset of $K\Omega^0$ and $\Gamma$ a subgroup of $O^+$ with the following
properties:

1. $V^0$ is $\Gamma$-invariant;
2. $V := \text{pr}_1(V^0)$ is an integral affine subspace with a section $f: V \to V^0$;
3. The action of $\Gamma$ on $V_0$ is free, proper and co-compact;

Then

$$(M, \pi) := f^*(KU, \pi_{KU})/\Gamma$$

is a PMSCT.

**Proof.** Because $V^0$ is $\Gamma$-invariant the hyperKahler family $f^*(KU, \pi_{KU})$ is also acted
upon by $\Gamma$. Because the action of $\Gamma$ on $V_0$ is free, proper and co-compact, the action
of $\Gamma$ on $f^*(KU, \pi_{KU})$ has the same properties. Therefore

$$(M, \pi) := f^*(KU, \pi_{KU})/\Gamma$$

is a well-defined compact Poisson manifold.

The integrability of a Poisson structure is controlled by the behavior of the so-called monodromy lattices $N_x, x \in M$. In case the Poisson manifold is a fibration with simply connected fiber $F$ and leaf space $B$, near a given fiber $F_b$, the Poisson structure induces a map

$$\text{ch} := U_{b0} \subset B \rightarrow H^2(F; \mathbb{R}), \quad b \mapsto [\pi(b)],$$

and the monodromy lattice at a point $x$ in a fiber $F_b$ is the lattice dual to $\text{ch}^*(b)H^2(F; \mathbb{Z}) \subset \nu_z(F_b)$.

Back to our Poisson manifold $(M, \pi)$, its is identified with $V/\Gamma$. Therefore, for $x \in M$ in the fiber corresponding to $b \in V/\Gamma$, the dual of the monodromy lattice $N_x^*$ can be identified with $H^2(F; \mathbb{Z}) \cap \bar{V}$. Because by assumption $V$ is integral affine, we conclude that $N_x \subset \nu_z(F_b)^*$ is a discrete lattice of full rank.

It can be proven that discrete monodromy lattices of full rank must vary smoothly. Therefore $\bigsqcup_{x \in M} N_x$ is closed subset of the conormal bundle to the fibration, and in particular it is uniformly discrete. Hence, according to [6] $(M, \pi)$ is integrable.

The symplectic fibers of $(M, \pi)$ are simply connected, and therefore the isotropy group of the canonical integration $\Sigma(M)$ at $x$ can be identified with $\nu_x(F_b)^*/N_x$. Because $N_x$ has full rank this isotropy group is a torus, and in particular it is compact. As a consequence the source fibers of $\Sigma(M) \Rightarrow M$ are principal bundles with compact fiber over a compact base, and thus compact. Since $M$ is compact, we conclude that $\Sigma(M)$ is compact.

Lastly, the delicate issue of the separability of $\Sigma(M)$ follows from the closedness of $\bigsqcup_{x \in M} N_x$ in the conormal bundle [1].

2.2. Construction of the PMSCT. We are going to construct a PMSCT with leaf space $\mathbb{R}/\mathbb{Z}$ by applying theorem [4] Our construction builds on an original idea by Kotschick [11].

The subgroup $\Gamma \cong \mathbb{Z}$ of $O^+$ is going to be defined by giving a generator $\phi$: let $\{u, v\}, \{x, y\}$ and $\{z, t\}$ be standard basis of each of the three copies of the hyperbolic intersection form $H$. Consider the automorphism

$$\phi: 2H \rightarrow 2H, \quad u \mapsto u, \quad v \mapsto v + y, \quad x \mapsto x - u, \quad y \mapsto y,$$

and extend it to $L$ by the identity on $H \oplus 2(-E_8)$. It is not hard to check that $\phi \in O^+$.

To define the subset $V^0$ we consider the map

$$f: \mathbb{R} \rightarrow L_R \oplus \mathbb{P}(L_C)$$

$$s \mapsto (2u + v + sy, [x - su + 2y + e, z + 2t + f]),$$

where $e = (e', 0), f = (0, e') \in 2(-E_8)$. We assume that the components of $e'$ and $1$ are linearly independent over the rationals, and that $e$ is small enough. We define

$$V^0 := f(\mathbb{R}).$$

Note that

1. $V^0$ is $\phi$-invariant and therefore $\Gamma$-invariant;
2. the projection $\pi_1: V^0 \rightarrow V$ is a diffeomorphism onto an integral affine line and therefore it has a section;
3. the action of $\Gamma$ on $V$ is obviously free, proper and co-compact.
Therefore, to conclude that we are in the hypotheses of theorem 1 we just need to make sure that
\[ V^0 \subset K\Omega^0, \]
which amounts to solving an arithmetic problem.

The three vectors \( 2u + v + sy, x - su + 2y + e, z + 2t + f \in L_\mathbb{R} \) are orthogonal and have positive square provided \( e' \) is small. The difficulty is checking that the positive 3-plane they span is not orthogonal to a root for all \( s \in \mathbb{R} \).

We argue by contradiction: we assume that the exist such a root \( \delta = \delta_1 + \delta_21 + \delta_22 \), where \( \delta \in 3H \) and \( \delta_21 \) belongs to the \( i \)-th copy of \(-E_8\).

Because \( \delta \) is orthogonal to \( 2u + v + sy \) we have
\[
\delta_1 = Ax + Bz - (2D + sA)u + Dv + Ey + Ft, \quad A, B, 2D + sA, D, E, F \in \mathbb{Z}.
\]
Because \( \delta \) is orthogonal to \( x - su + 2y + e, z + 2t + f \) we have
\[
F + 2B + (\delta_{22}, e') - Ds = 0, \quad E + 2A + (\delta_{21}, e') = 0. \tag{1}
\]
The second equation in (1) implies \( \delta_{21} = 0 \), because otherwise the irrationality assumption on the components of \( e' \) and 1 would not hold.

Since \(-E_8\) is an even, negative definite lattice, we may write \( (\delta_{22}, \delta_{22}) = -2n, \quad n \in \mathbb{N} \). The root condition for \( \delta \) becomes
\[
2D^2 + 2A^2 + 2B^2 - (B\delta_{22}, e') = 1 - n. \tag{2}
\]
Once more the irrationality assumption on the components of \( e' \) and 1 implies that (2) can only have solutions if \( \delta_{22} = 0 \), and any such solution must also solve
\[
2D^2 + 2A^2 + 2B^2 = 1,
\]
which is a contradiction since \( A, B, D \) must be integers.

Therefore we conclude that such root \( \delta \) cannot exist and thus \( V^0 \subset K\Omega^0 \), finishing the construction of the non-trivial PMSCT.

**Remark 1.** The previous construction can be arranged to produce PMSCT with 2-dimensional base, the arithmetic problem associated to the roots being more involved.

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