Isotropic Grassmannians, Plücker and Cartan maps

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Abstract

This work is motivated by the relation between the KP and BKP integrable hierarchies, whose \( \tau \)-functions may be viewed as sections of dual determinantal and Pfaffian line bundles over infinite dimensional Grassmannians. In finite dimensions, we show how to relate the Cartan map which, for a vector space \( V \) of dimension \( N \), embeds the Grassmannian \( \text{Gr}_V^0(V + V^*) \) of maximal isotropic subspaces of \( V + V^* \), with respect to the natural scalar product, into the projectivization of the exterior space \( \Lambda(V) \), and the Plücker map, which embeds the Grassmannian \( \text{Gr}_V(V + V^*) \) of all \( N \)-planes in \( V + V^* \) into the projectivization of \( \Lambda^N(V + V^*) \). The Plücker coordinates on \( \text{Gr}_V^0(V + V^*) \) are expressed bilinearly in terms of the Cartan coordinates, which are holomorphic sections of the dual Pfaffian line bundle \( \text{Pf}^* \rightarrow \text{Gr}_V^0(V + V^*, Q) \). In terms of affine coordinates on the big cell, this is equivalent to an identity of Cauchy-Binet type, expressing the determinants of square submatrices of a skew symmetric \( N \times N \) matrix as bilinear sums over the Pfaffians of their principal minors.

1 Introduction: \( \tau \)-functions for integrable hierarchies and Grassmannians

We recall the relation between integrable hierarchies and infinite dimensional Grassmannians developed by Sato and the Kyoto school [13, 2, 3]. (For expository accounts, see [10, 4, 6].) Solutions to the KP (Kadomtsev-Petviashvili) hierarchy can be expressed in terms of KP \( \tau \)-functions \( \tau_w(t) \), parametrized by elements \( w \in \text{Gr}_{H+}(\mathcal{H}) \) of an infinite Grassmannian consisting of subspaces \( w \subset \mathcal{H} \) of a polarized Hilbert space \( \mathcal{H} = \mathcal{H}_+ + \mathcal{H}_- \),

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commensurate with the subspace $\mathcal{H}_+ \subset \mathcal{H}$, and depending on an infinite sequence of commuting flow variables
\[ t = (t_1, t_2, \ldots). \] (1.1)
These satisfy the Hirota bilinear residue relations,
\[ \text{res}_{z=0} \left( e^{\sum_{i=1}^{\infty} (t_i - s_i)z^i} \tau_w(t - [z^{-1}]) \tau(s + [z^{-1}]) \right) dz = 0, \] (1.2)
identically in $s$, where
\[ s := (s_1, s_2, \ldots), \quad [z^{-1}] := \left( \frac{1}{z}, \frac{2}{z^2}, \ldots, \frac{j}{z^j}, \ldots \right). \] (1.3)
Expanding $\tau_w(t)$ in a basis of Schur functions [11 [13]
\[ \tau_w(t) = \sum_{\lambda} \pi_{\lambda}(w) s_{\lambda}(t), \] (1.4)
with the flow parameters $(t_1, t_2, \ldots)$ interpreted as normalized power sums
\[ t_i = \frac{p_i}{i} \quad i = 1, 2, \ldots, \] (1.5)
and the labels $\lambda$ denoting integer partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0, \cdots)$ of length $\ell(\lambda)$ and weight $|\lambda|$, the coefficients $\pi_{\lambda}(w)$ may be interpreted as Plücker coordinates of the element $w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$.

The Hirota equation (1.2) is then formally equivalent to the infinite set of Plücker relations
\[ \sum_i (-1)^{i+i+1} \pi_{[\lambda^-,\lambda_i]}(w) \pi_{[\mu^+,\lambda_i-1]}(w) = 0, \] (1.6)
where $(\lambda, \mu)$ is any pair of partitions, $[\lambda^-, \lambda_i]$ is the partition of length $\ell(\lambda) - 1$ with parts
\[ [\lambda^-, \lambda_i] := (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{i-1} + 1, \lambda_{i+1}, \ldots, \lambda_{\ell(\lambda)}), \] (1.7)
and $[\mu^+, u]$ is the partition of length $\ell(\mu) + 1$ with parts
\[ [\mu^+, u] := \begin{cases} \langle u + 1, 1, \mu_2, \mu_3, \ldots, \mu_{\ell(\mu)} \rangle & \text{if } u > \mu_1 - 2 \\ \langle \mu_1 - 1, 1, \mu_2, \mu_3, \ldots, \mu_{\ell(\mu)} \rangle & \text{if } \mu_2 - 3 < u < \mu_1 - 2 \\ \langle \mu_1 - 1, \mu_2 - 1, 1, \mu_3, \ldots, \mu_{\ell(\mu)} \rangle & \text{if } \mu_3 - 4 < u < \mu_2 - 3 \\ \vdots & \text{if } \mu_4 - 5 < u < \mu_3 - 3 \\ \langle \mu_1 - 1, \mu_2 - 1, \mu_3 - 1, \ldots, \mu_{\ell(\mu)-1} - 1, 1, u + \ell(\mu) \rangle & \text{if } u < \mu_{\ell(\mu)-1} - \ell(\mu), \end{cases} \] (1.8)
which is not defined if $u = \mu_k - k - 1$ for some $1 \leq k \leq \ell(\mu)$. The summands corresponding to indices $i$ in (1.6) for which $[\mu^+, \lambda_i - i + 1]$ is not defined are omitted from the sum. The exponent $\mu(u)$ in (1.6) is defined as the position $k$ of the inserted part $u + k$ in (1.8).
The Plücker relations (1.6) determine the image of the infinite Grassmannian $\text{Gr}_{\mathcal{H}^+}(\mathcal{H})$ under the Plücker map:

$$\mathcal{P}|_{\mathcal{H}^+} : \text{Gr}_{\mathcal{H}^+}(\mathcal{H}) \to \mathbb{P}(\mathcal{F})$$

$$\mathcal{P}|_{\mathcal{H}^+} : \text{span}\{w_1, w_2, \ldots\} \mapsto [\sum_\lambda \pi_\lambda(w)|\lambda; N] \in \mathbb{P}(\mathcal{F}), \quad (1.9)$$

embedding $\text{Gr}_{\mathcal{H}^+}(\mathcal{H})$ into the projectivization of the semi-infinite wedge product space (the fermionic Fock space)

$$\mathcal{F} = \Lambda^\infty/2(\mathcal{H}) = \sum_{N \in \mathbb{Z}} \mathcal{F}_N.$$  \hspace{1cm} (1.10)

Here $\{|\lambda; N\rangle\}$ is the standard basis \cite{13} \cite{16} \cite{6} for the fermionic charge-$N$ sector $\mathcal{F}_N$ of the Fock space and $\{w_1, w_2, \ldots\}$ is any admissible basis for the subspace $w \subset \mathcal{H}$, viewed as an element of $\text{Gr}_{\mathcal{H}^+}(\mathcal{H})$. As in the finite dimensional case (detailed below in Sections \cite{2} and \cite{5}), the Plücker coordinates $\{\pi_\lambda(w)\}$ are expressible as determinants of suitably defined matrices, $W_\lambda(w)$, which are maximal minors of the homogeneous coordinate matrix $W(w)$ of the element $w$. They may be interpreted as holomorphic sections of the (dual) determinantal line bundle $\text{Det}^* \to \text{Gr}_{\mathcal{H}^+}(\mathcal{H})$ (see \cite{16} \cite{6}).

The BKP hierarchy \cite{2} \cite{3} may similarly be expressed in terms of a BKP $\tau$-function $\tau_0^B(t_B)$, parametrized by elements $w^0 \in \text{Gr}^0_{\mathcal{H}^+}(\mathcal{H}^B, Q)$ of the Grassmannian of maximal isotropic subspaces of a Hilbert space $\mathcal{H}^B$ endowed with a suitably defined complex scalar product $Q$. It depends only on the odd flow variables

$$t_B = (t_1, t_3, \ldots)$$ \hspace{1cm} (1.11)

and satisfies the Hirota bilinear residue equation

$$\text{res}_{z=0} \left( e^{\tilde{\xi}(z, s_B) - s_B^B(t_B - 2[z^{-1}])} \tau_0^B(t_B - 2[z^{-1}]) \frac{dz}{z} \right) = \tau_0^B(t_B) \tau_0^B(s_B),$$  \hspace{1cm} (1.12)

identically in

$$s_B = (s_1, s_3, \ldots), \quad (1.13)$$

where

$$\tilde{\xi}(z, t_B) := \sum_{j=1}^\infty t_{2j-1} z^{2j-1}, \quad [z^{-1}]_B := \left( z^{-1}, \frac{1}{3} z^{-3}, \frac{1}{5} z^{-5}, \ldots \right).$$ \hspace{1cm} (1.14)

We may similarly expand $\tau_0^B(t_B)$ in a series \cite{2} \cite{17} \cite{14} \cite{15}

$$\tau_0^B(t_B) = \sum_{\alpha \in \text{even strict partitions}} \kappa_\alpha(w^0) \mathcal{Q}_\alpha(t_B),$$ \hspace{1cm} (1.15)

where, up to normalization

$$\mathcal{Q}_\alpha(t_B) := \frac{1}{\sqrt{2^r}} Q_\alpha \left( \frac{t_B}{2} \right),$$  \hspace{1cm} (1.16)
are Schur’s $Q$-functions \([11]\) (also known as projective Schur functions), labelled by strict partitions $\alpha = (\alpha_1 > \alpha_2 > \cdots \geq 0)$. The coefficients $\{\kappa_\alpha(w^0)\}$ may be interpreted as Pfaffians of skew symmetric matrices $A^0(w^0)(\alpha)$, also labelled by strict partitions, which are, up to projectivization, principal minors of the skew symmetric affine coordinate matrix $A^0(w^0)$ representing the image of $w^0 \in \text{Gr}^0_H(HB)$ within the “big cell”, under the Cartan map \([11] [8] [9]\) and \([6]\), Chapt. 7, Appendix E)

$$\text{Ca}_H : \text{Gr}^0_H(HB) \to \mathbf{P}(FB)$$

(1.17)

embedding the isotropic Grassmannian $\text{Gr}^0_H(HB)$ into the projectivization of the “neutral fermion” Fock space $FB$ \([2] [3]\). (See Section \([2]\) eq. (2.18) for the definition in finite dimensions.)

The coefficients $\{\kappa_\alpha(w^0)\}$, which we refer to as Cartan coordinates (cf. \([6]\), Chapt. 7), are similarly interpreted as sections of a holomorphic line bundle: the dual Pfaffian line bundle $\text{Pf}^* \to \text{Gr}^0_H(HB)$ over the isotropic Grassmannian $\text{Gr}^0_H(HB)$, which will be defined below in the finite dimensional setting originally studied by Cartan. They also satisfy quadratic relations that determine the image of $\text{Gr}^0_H(HB)$ under the Cartan map, the Cartan relations \([11] [2] [8] [9]\) and \([6]\), Chap. 7 and Appendix C) (or the Pfaffian Plücker relations, as they are called in \([12, 15]\)):

\[
\sum_{i=1}^{\ell(\alpha)} (-1)^{i+\beta(\alpha_i)}K_{(\alpha^-,-\alpha_i)}K_{(\alpha^+,\alpha_i)} + \sum_{i=1}^{\ell(\beta)} (-1)^{i+\alpha(\beta_i)}K_{(\beta^-,-\beta_i)}K_{(\beta^+,\beta_i)} = \frac{1}{2} \left((-1)^{\ell(\alpha)+\ell(\beta)} - 1\right) \kappa_\alpha \kappa_\beta.
\]

(1.18)

Here $(\alpha, \beta)$ is any pair of strict partitions of lengths $(\ell(\alpha),\ell(\beta))$. For a strict partition $\alpha = (\alpha_1 > \alpha_2 > \cdots > \alpha_{\ell(\alpha)} \geq 0)$ and any nonnegative integer $m$ lying between $\alpha_i$ and $\alpha_{i+1}$:

$$\alpha_i > m > \alpha_{i+1},$$

(1.19)

$(\alpha^+, m)$ is defined to be the strict partition of length $\ell(\alpha) + 1$ obtained from $\alpha$ by adding the part $m$:

$$(\alpha^+, m) := (\alpha_1, \ldots, \alpha_i, m, \alpha_{i+1}, \ldots \alpha_{\ell(\alpha)})$$

(1.20)

while, for any $\alpha_i$, $(\alpha^-, \alpha_i)$ is defined as the strict partition of length $\ell(\alpha) - 1$ obtained from $\alpha$ by omitting the part $\alpha_i$:

$$(\alpha^-, \alpha_i) = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots \alpha_{\ell(\alpha)}).$$

(1.21)

For $m \in \mathbf{N}$, the integers $\alpha(m), \beta(m)$ are defined to be the number of parts of $\alpha$ and $\beta$, respectively, greater than $m$. 

4
It is a standard result \cite{2,3} that the square \((\tau_{w_0}^B(t_B))^2\) of a BKP \(\tau\)-function is equal to the restriction of a KP \(\tau\)-function \(\tau_w(t)\) to the values
\[
\mathbf{t'} := (t_1, 0, t_3, 0, t_5, 0, \ldots).
\] (1.22)
This implies an identity expressing the Plücker coordinates \(\pi_\lambda(w^0)\) quadratically in terms of the Cartan coordinates \(\kappa_\alpha(w^0)\). In the setting of finite dimensional Grassmannians, this quadratic relation is given by Theorem 2.1 of Section 2. It is closely related to Cartan’s identification of maximal isotropic subspaces of a complex Euclidean vector space with projectivized pure spinors \cite{1}.

For a vector space \(V\) of dimension \(N\), the dual determinantal and Pfaffian line bundles
\[
\text{Det}^* \to \text{Gr}_V(V + V^*) \quad \text{and} \quad \text{Pf}^* \to \text{Gr}_V^0(V + V^*)
\] (1.23)
over the Grassmannian \(\text{Gr}_V(V + V^*)\) of \(N\)-dimensional subspaces of the \(2N\)-dimensional space \(V + V^*\) and the maximal isotropic Grassmannian \(\text{Gr}_V^0(V + V^*)\) with respect to the tautologically defined scalar product, respectively, are defined in Section 2, as well as the Plücker and Cartan maps embedding these into \(\mathbf{P}(\Lambda^N(V + V^*))\) and \(\mathbf{P}(\Lambda(V))\), respectively,
\[
\mathcal{P}_V : \text{Gr}_V(V + V^*) \hookrightarrow \mathbf{P}(\Lambda^N(V + V^*)),
\] (1.24)
\[
\mathcal{C}_V : \text{Gr}_V^0(V + V^*) \hookrightarrow \mathbf{P}(\Lambda(V)),
\] (1.25)
with images determined by the Plücker relations and the Cartan relations, respectively.

Section 2 reviews the construction of (dual) determinantal and Pfaffian lines bundles \(\text{Det}^* \to \text{Gr}_V(V + V^*)\) and \(\text{Pf}^* \to \text{Gr}_V^0(V + V^*)\) over the Grassmannian \(\text{Gr}_V(V + V^*)\) of \(N\)-planes in the direct sum of a (complex) \(N\)-dimensional vector space \(V\) and its dual \(V^*\), and the Grassmannian \(\text{Gr}_V^0(V + V^*)\) of maximal isotropic subspaces of \(V + V^*\) with respect to the canonically defined scalar product \(Q\) associated to the dual pairing, respectively. The latter is related to the hyperplane section bundle over the (projectivized) irreducible Clifford module associated to the scalar product \(Q\), and its connected components over the irreducible 1/2-spinor modules. Theorem 2.1 gives the main result, expressing the Plücker map (1.24) bilinearly in terms of the Cartan map (1.25), and thereby effectively inverting the Cartan embedding. Section 3 provides a direct proof of this theorem, without the use of Plücker or Cartan coordinates. Proposition 4.1, Section 4, gives a factorization of the Plücker map
\[
\mathcal{P} : \text{Gr}_k(V) \to \mathbf{P}(\Lambda^k(V))
\] (1.26)
as the composition of the tautological embedding map:
\[
\iota_V : \text{Gr}_k(V) \to \text{Gr}_V^0(V + V^*)
\] (1.27)
with the Cartan map. The interpretation of Plücker and Cartan coordinates as holomorphic sections of \( \text{Det}^* \to \text{Gr}_V(V + V^*) \) and \( \text{Pf}^* \to \text{Gr}_V^0(V + V^*) \) is reviewed in Section 5 and the Cartan coordinates \( \kappa_\alpha(w^0) \) are defined and expressed as Pfaffians of principal minors of the affine coordinate matrix \( A^0(w^0) \) on the “big cell” of the isotropic Grassmannian \( \text{Gr}_V^0(V + V^*) \). Theorem 6.3, Section 6, interprets Theorem 2.1 in coordinate for \( m \) as a Pfaffian analog of the Cauchy-Binet identity \([11]\) and gives an alternative proof, using inner and outer products on the exterior algebra \( \Lambda(V + V^*) \).

2 Plücker and Cartan maps: the determinantal and Pfaffian line bundles

Let \( V \) be a complex vector space of dimension \( N \), \( V^* \) its dual space and \( \text{Gr}_V(V + V^*) \) the Grassmannian of \( N \)-planes in \( V + V^* \). The \textit{Plücker map} \([7]\)

\[
\mathfrak{P}l_V : \text{Gr}_V(V + V^*) \to \mathbb{P}(\Lambda^N(V + V^*))
\]  
(2.1)

is the \( GL(V + V^*) \) equivariant embedding of \( \text{Gr}_V(V + V^*) \) in the projectivization \( \mathbb{P}(\Lambda^N(V + V^*)) \) of the exterior space \( \Lambda^N(V + V^*) \) defined by:

\[
\mathfrak{P}l_V : w \mapsto [w_1 \wedge \cdots \wedge w_N] \in \mathbb{P}(\Lambda^N(V + V^*)),
\]  
(2.2)

where \( \{w_1, \ldots, w_N\} \) is a basis for the subspace \( w \in \text{Gr}(V + V^*) \). Its image is cut out by the intersection of a number of quadrics, the \textit{Plücker quadrics}, defined by the \textit{Plücker relations} \([7] \text{ Chapt. I.5 and eq. (1.6) above})

Let \( \{e_i\}_{i=1, \ldots, N} \) be a chosen basis for \( V \) and \( \{f_i\}_{i=1, \ldots, N} \) the dual basis for \( V^* \),

\[
f_i(e_j) = \delta_{ij},
\]  
(2.3)

Denote by \( (e_1, \ldots, e_{2N}) \) the basis for \( V + V^* \) in which

\[
e_{N+i} := f_i, \quad 1 \leq i \leq N.
\]  
(2.4)

Define the basis \( \{|\lambda\} \) for \( \Lambda^N(V + V^*) \) by

\[
|\lambda\rangle := e_{\ell_1} \wedge \cdots \wedge e_{\ell_N},
\]  
(2.5)

where \( \lambda \) is any partition whose Young diagram fits in the \( N \times N \) square diagram, and

\[
l_j := \lambda_j - j + N + 1, \quad 1 \leq j \leq N
\]  
(2.6)

are the \textit{particle positions} (see, e.g., \([9]\) Chapt. 11, Sec. 11.3) associated to the partition

\[
\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0, \ldots).
\]  
(2.7)
Thus \( l_1 > \cdots > l_N \) is a strictly decreasing sequence of positive integers between 1 and 2\( N \). The (complex) scalar product on \( \Lambda^N(V + V^*) \) is defined, in bra/ket notation, by requiring the \( \{ | \lambda \rangle \} \) basis to be orthonormal

\[
\langle \lambda | \mu \rangle = \delta_{\lambda \mu}.
\]  

(2.8)

Following Cartan \[1\], we have a natural complex scalar product \( Q \) on \( V + V^* \), defined by

\[
Q((X, \mu), (Y, \nu)) = \nu(X) + \mu(Y),
\]

(2.9)

where \( X, Y \in V, \mu, \nu \in V^* \). The standard irreducible representation

\[
\Gamma : C_{V+V^*,Q} \to \text{End}(\Lambda(V)),
\]

\[
\Gamma : \sigma \mapsto \Gamma_\sigma
\]

(2.10)

of the Clifford algebra \( C_{V+V^*,Q} \) on \( V + V^* \) determined by the scalar product \( Q \) is generated by the linear elements, which are defined by exterior and interior multiplication:

\[
\Gamma_{v+\mu} := v \wedge + i_\mu \in \text{End}(\Lambda(V))
\]

\[
v \in V, \mu \in V^*.
\]

(2.11)

The Clifford representations of the basis elements are denoted

\[
\psi_i := \Gamma_{e_i} = e_i \wedge, \quad \psi^\dagger_i := \Gamma_{f_i} = i_{f_i}, \quad i = 1, \ldots, N,
\]

(2.12)

and viewed as finite dimensional fermionic creation and annihilation operators, satisfying the anticommutation relations

\[
[\psi_i, \psi_j]_+ = 0, \quad [\psi_i^\dagger, \psi_j^\dagger]_+ = 0, \quad [\psi_i, \psi_j^\dagger]_+ = \delta_{ij}.
\]

(2.13)

Let

\[
\text{Gr}_V^0(V + V^*) \subset \text{Gr}_V(V + V^*)
\]

(2.14)

be the sub-Grassmannian of \( N \)-planes in \( V + V^* \) that are totally isotropic with respect to \( Q \). That is, if \( \{ w_1, \ldots, w_N \} \) is a basis for an element \( w^0 \in \text{Gr}_V^0(V + V^*) \), then

\[
Q(w_i, w_j) = 0, \quad 1 \leq i, j \leq N.
\]

(2.15)

It then follows from (2.15) and the Clifford algebra relations that

\[
\Gamma_{w_i} \Gamma_{w_j} + \Gamma_{w_j} \Gamma_{w_i} = 0, \quad 1 \leq i, j \leq N.
\]

(2.16)

Together with the linear independence of the \( w_j \)'s, this implies \[1\] that

\[
\text{rank} \left( \prod_{j=1}^N \Gamma_{w_j} \right) = 1.
\]

(2.17)
Definition 2.1. The Cartan map \( \text{Ca}_V : \text{Gr}_V^0(V + V^*) \to \mathbf{P}(\Lambda(V)) \), defined \([1, 8, 9]\) by:

\[
\text{Ca}_V : w^0 \mapsto \text{Image} \left( \prod_{j=1}^{N} \Gamma_{w_j} \right) \in \mathbf{P}(\Lambda(V)).
\] (2.18)

This gives an equivariant embedding of the isotropic Grassmannian \( \text{Gr}_V^0(V + V^*) \) into the projectivization \( \mathbf{P}(\Lambda(V)) \) of the exterior space \( \Lambda(V) \) (the irreducible Clifford module).

It intertwines the action of the orthogonal group \( SO(V + V^*, Q) \) on \( \text{Gr}_V^0(V + V^*) \) with the (projectivized) representation of the spin group \( \text{Spin}(V + V^*, Q) \) on \( \mathbf{P}(\Lambda(V)) \) determined by the Clifford representation. Its image \( \text{Ca}_V(\text{Gr}_V^0(V + V^*)) \) consists of (the projectivization of) pure spinors \([1]\), which are the elements of the lowest dimensional stratum of the \( \text{Spin}(V + V^*, Q) \) representation on \( \Lambda(V) \). Similarly to the Plücker map, its image is cut out by a set of homogeneous quadratic relations, which we refer to as the Cartan relations. (Cf. \([1]\), Secs. 106-107, eq. (1.18) above, and \([6]\), Appendix E.)

The irreducible Clifford module \( \Lambda(V) \) is the direct sum of the two irreducible \( \frac{1}{2} \)-spinor modules (Weyl spinors)

\[
\Lambda(V) = \Lambda_+(V) \oplus \Lambda_-(V),
\] (2.19)

consisting of even (\( + \)) and odd (\( - \)) multivectors \( v \in \Lambda(V) \), denoted \( v_+ \) and \( v_- \), respectively. The isotropic Grassmannian \( \text{Gr}_V^0(V + V^*) \) is the disjoint union of two connected components

\[
\text{Gr}_V^0(V + V^*) = \text{Gr}_V^0(V + V^*) \sqcup \text{Gr}_V^0(V + V^*),
\] (2.20)

such that

\[
\text{Ca}_V(w^0)(\text{Gr}_V^0(V + V^*)) \subset \mathbf{P}(\Lambda_\pm(V)).
\] (2.21)

In particular the Cartan image of the element \( V \in \text{Gr}_V^0(V + V^*) \) is

\[
\text{Ca}_V(V) = [\Omega_V],
\] (2.22)

where \( \Omega_V \in \Lambda^N V \) is a volume form on \( V \).

We then have six standard holomorphic line bundles:

1. The hyperplane section bundle \( \mathcal{O}(1) \to \mathbf{P}(\Lambda^N(V + V^*)) \) dual to the tautological bundle.
2. The pair of hyperplane section bundles \( \mathcal{O}(1) \to \mathbf{P}(\Lambda_\pm(V)) \).
3. The dual determinantal line bundle \( \text{Det}^* \to \text{Gr}_V(V + V^*) \).
4. The pair of dual Pfaffian line bundles \( \text{Pf}^*_\pm \to \text{Gr}^0_{V\pm}(V + V^*) \).
Bundles 1 and 3, and 2 and 4 are related by pullback under the Plücker and Cartan maps, respectively:

\[ \text{Det}^* = \Phi_V^* (\mathcal{O}(1)(\mathcal{P}(\Lambda^N(V + V^*))))|_{\text{Gr}_V(V + V^*)}, \]
\[ \text{Pf}^*_\pm = \text{Ca}_V^* \left( \mathcal{O}(1)(\mathcal{P}(\Lambda(V)))|_{\text{Ca}_V(\text{Gr}_V^0(V + V^*))} \right). \]  

(2.23)

The dimensions of their spaces of holomorphic sections are

\[ h^0(\text{Gr}_V(V + V^*), \text{Det}^*) = h^0(\mathcal{P}(\Lambda^N(V + V^*)), \mathcal{O}(1)(\mathcal{P}(\Lambda^N(V + V^*)))) = \binom{2N}{N}, \]
\[ h^0(\text{Gr}_V^0(V + V^*), \text{Pf}^*_\pm) = h^0(\mathcal{P}(\Lambda_\pm(V)), \mathcal{O}(1)(\mathcal{P}(\Lambda_\pm(V)))) = 2^{N-1}. \]  

(2.24)

Bundles 3 and 4 are related by

\[ (\text{Pf}^*_\pm)^{\otimes 2} \simeq \text{Det}^*|_{\text{Gr}_V^0(V + V^*)}. \]  

(2.25)

From the viewpoint of geometric representation theory ([5], Chapt. 9), the space \( \Lambda^N(V + V^*) \) is the representation module of \( \text{SL}(V + V^*, \mathbb{C}) \) (and, by restriction, \( \text{Spin}(V + V^*, Q) \)) obtained from the Borel-Weil theorem by identifying it with the space of holomorphic sections of the line bundle \( \text{Det}^* \to \text{Gr}_V(V + V^*) \). Denote by \( V_\pm \) the subspaces:

\[ V_+ := \begin{cases} V = \text{span}(e_1, \ldots, e_N) \in \text{Gr}_V^0 & \text{if } N \text{ is even}, \\ \text{span}(e_1, \ldots, e_{N-1}, f_N) \in \text{Gr}_V^0 & \text{if } N \text{ is odd}, \end{cases} \]
\[ V_- := \begin{cases} V = \text{span}(e_1, \ldots, e_N) \in \text{Gr}_V^0 & \text{if } N \text{ is odd}, \\ \text{span}(e_1, \ldots, e_{N-1}, f_N) \in \text{Gr}_V^0 & \text{if } N \text{ is even}. \end{cases} \]  

(2.26)

Let \( P^\pm \subset \text{SO}(V + V^*, Q) \) be the stabilizers of \( V_\pm \), and \( \tilde{P}^\pm \subset \text{Spin}(V + V^*, Q) \) the corresponding stabilizers of their images \( \text{Ca}(V_\pm) \subset \Lambda_\pm(V) \) under the Cartan map. The components \( \text{Gr}_V^0(V + V^*) \) are the orbits of \( V_\pm \) under \( \text{SO}(V + V^*, Q) \), and hence are identified with the homogeneous spaces

\[ \text{Gr}_V^0(V + V^*) = \text{SO}(V + V^*, Q)/P^+ = \text{Spin}(V + V^*, Q)/\tilde{P}^+ \]  

(2.27)

\[ \text{Gr}_V^0(V + V^*) = \text{SO}(V + V^*, Q)/P^- = \text{Spin}(V + V^*, Q)/\tilde{P}^- \]  

(2.28)

As a representation of \( \text{SO}(V + V^*, Q) \) (and \( \text{Spin}(V + V^*, Q) \)), the module \( \Lambda^N(V + V^*) \) decomposes into the direct sum of two irreducible modules

\[ \Lambda^N(V + V^*) = \Lambda^N_+(V + V^*) \oplus \Lambda^N_-(V + V^*), \]  

(2.29)

such that

\[ \mathfrak{H}(V_\pm) \subset \mathcal{P}(\Lambda^N_\pm(V + V^*)). \]  

(2.30)
Let \( \text{Det}_\pm^* \) be the restriction of \( \text{Det}^* \) to \( \text{Gr}_V^0 (V + V^*) \subset \text{Gr}_V (V + V^*) \). Then

\[
\Psi(\text{Gr}_V^0 (V + V^*)) \subset P(\Lambda^N_{\pm} (V + V^*))
\]  

(2.31)

and

\[
\text{Det}_\pm^* = \Psi_{V^*} (\mathcal{O}(1)(P(\Lambda^N_{\pm} (V + V^*))|_{\Psi_{V^*}(\text{Gr}_V (V + V^*))}).
\]  

(2.32)

The standard diagonal Cartan subalgebra \( \mathfrak{h} \) of \( \text{SO}(V + V^*, Q) \) is isomorphic to \( V \), by the map \( v \mapsto \text{diag}(v, -v) \). We can take the element

\[
e_1 \wedge e_2 \wedge \cdots \wedge e_N \in \Lambda^N_{(-1)^N} (V) \subset \Lambda^N_{(-1)^N} (V + V^*)
\]

(2.33)
as a generator of the highest weight space for \( \Lambda^N_{(-1)^N} (V) \), and

\[
e_1 \wedge e_2 \wedge \cdots \wedge e_{N-1} \wedge f_N \in \Lambda^N_{(-1)^{N+1}} (V + V^*)
\]

(2.34)
as a generator of the highest weight space for \( \Lambda^N_{(-1)^{N+1}} (V) \). The highest weight on \( \mathfrak{h} \) corresponding to \( e_1 \wedge e_2 \wedge \cdots \wedge e_N \) is then \( (f_1 + \cdots + f_N) \), and the highest weight on \( \mathfrak{h} \) corresponding to \( e_1 \wedge e_2 \wedge \cdots \wedge e_{N-1} \wedge f_N \) is \( (f_1 + \cdots + f_{N-1} - f_N) \). If \( \chi_{\pm} \) are the characters on the Cartan subgroup associated to these weights, the line bundles \( \text{Det}_\pm^* \) over the homogeneous spaces \( \text{Gr}_V^0 (V + V^*) = \text{Spin}(V + V^*, Q)/\tilde{P}^\pm \) can be constructed in the standard way, as quotients by \( \tilde{P}^\pm \) of the trivial bundle \( \text{Spin}(V + V^*, Q) \times \tilde{P}^\pm \mathbb{C} \rightarrow \text{Spin}(V + V^*, Q) \), where the parabolic subgroups \( \tilde{P}^\pm \) act on \( \text{Spin}(V + V^*, Q) \) by right multiplication and on \( \mathbb{C} \) by multiplication by the inverses of the characters \( \chi_{\pm}^{-1} \).

Likewise, the spin module \( \Lambda(V) \) decomposes into the sum \( (2.19) \) of the even and odd irreducible \( \frac{1}{2} \)-spinor modules (Weyl spinors) corresponding to sections of the line bundles \( \text{Pf}_\pm^* \) over the two connected components \( \text{Gr}_V^0 (V + V^*) \) of the isotropic Grassmannian. The highest weight of \( \Lambda_{(-1)^N} (V) \) is \( (f_1 + \cdots + f_N)/2 \), and that of \( \Lambda_{(-1)^{N+1}} (V) \) is \( (f_1 + \cdots + f_{N-1} - f_N)/2 \). If \( \chi_{0, \pm} \) are the corresponding characters, the line bundles \( \text{Pf}_\pm^* \) are built in the same way as \( \text{Det}_\pm^* \), but using \( \chi_{0, \pm} \) instead of \( \chi_{\pm} \), and the identification of \( \text{Gr}_V^0 (V + V^*) \) as \( \text{Spin}(V + V^*, Q)/\tilde{P}^\pm \).

The fact that the restriction of \( (\chi_{0, \pm})^2 \) to \( \text{Gr}_V^0 (V + V^*) \) is equal to \( \chi_{\pm} \),

\[
(\chi_{0, \pm})^2|_{\text{Gr}_V^0 (V + V^*)} = \chi_{\pm} \tag{2.35}
\]

then gives the isomorphisms \( (2.25) \). On the spaces of holomorphic sections of these bundles, i.e., the representation modules, there are corresponding maps

\[
\Lambda_{\pm} (V) \otimes \Lambda_{\pm} (V) \rightarrow \Lambda^N_{\pm} (V + V^*),
\]

(2.36)

which are defined by restricting the bilinear form \( \beta_N \) defined below in eq. \( (2.46) \) to the subspaces \( \Lambda_{\pm} (V) \).
The main result of this paper is to show how the Plücker and Cartan maps are related, and to express this relation in terms of Plücker and Cartan coordinates. For an \( r \)-element subset \( I = (I_1, \ldots, I_r) \subset \{1, \ldots, N\}, 0 \leq r \leq N \), arranged in decreasing order \( I_1 > \cdots > I_r \), we denote the complementary subset by \( \tilde{I} = (\tilde{I}_1, \ldots, \tilde{I}_{N-r}) \subset \{1, \ldots, N\} \), also arranged in decreasing order \( \tilde{I}_1 > \cdots > \tilde{I}_{N-r} \).

**Definition 2.2.** Let \( \text{sgn}(I) \) be the sign of the permutation \((I, \tilde{I})\) of \((1, \ldots, N)\)
\[
\text{sgn}(I) := \text{sgn}(I_1, \cdots, I_r, \tilde{I}_1, \ldots, \tilde{I}_{N-r}).
\] (2.37)

Recall the Hodge star automorphism \( \ast : \Lambda(V) \to \Lambda(V) \), defined relative to the basis \((e_1, \ldots, e_N)\), as follows:

**Definition 2.3.** \( \ast : \Lambda(V) \to \Lambda(V) \) is defined on basis elements by:
\[
\ast e_{I_1} \wedge \cdots \wedge e_{I_r} := \text{sgn}(I) e_{\tilde{I}_1} \wedge \cdots \wedge e_{\tilde{I}_{N-r}},
\] (2.38)
and extended to \( \Lambda(V) \) by linearity.

This coincides with the usual definition of \( \ast \) with respect to the metric in which \((e_1, \ldots, e_N)\) is a positively oriented, orthonormal basis, and the volume form is
\[
\Omega_V := e_1 \wedge \cdots \wedge e_N \in \Lambda^N(V).
\] (2.39)

Following Cartan, [1], we also introduce the closely related automorphism of \( \Lambda(V) \) defined by forming the product of \( N \) orthogonal linear elements
\[
C : \Lambda(V) \to \Lambda(V)
C : v \mapsto (\psi_1 - \psi_1^\dagger) \cdots (\psi_N - \psi_N^\dagger)v.
\] (2.40)
When acting on homogeneous elements \( v \in \Lambda^r(V) \), this is related to \( \ast \) by
\[
Cv = (-1)^{\frac{1}{2}N(N+1)}v.
\] (2.41)
Up to a sign, its square is the identity element
\[ C^2 = (-1)^{\frac{1}{2}N(N+1)}I, \] (2.42)
and it leaves invariant the scalar product \((v, w)\) on \( \Lambda(V) \) in which the basis elements
\[
e_{I} = e_{I_1} \wedge \cdots \wedge e_{I_r}, \quad N \geq I_1 > \cdots > I_r \geq 1
\] (2.43)
are orthonormal:
\[
(e_I, e_J) := \delta_{IJ}
\]
\[
(C(v), C(w)) = (v, w).
\] (2.44)
**Definition 2.4.** Following Cartan [1], we define the scalar-valued bilinear form $\beta_0$ on $\Lambda(V)$ as

$$\beta_0 : \Lambda(V) \times \Lambda(V) \rightarrow \mathbb{C}$$

$$\beta_0(v, w) = (v, Cw),$$

(2.45)

and the bilinear form

$$\beta_N : \Lambda(V) \times \Lambda(V) \rightarrow \Lambda^N(V + V^*)$$

(2.46)

with values in $\Lambda^N(V + V^*)$ such that, for $\sigma \in \Lambda^N(V + V^*)$,

$$\langle \sigma | \beta_N(v, w) \rangle := \beta_0(v, \Gamma_\sigma w) = (v, C\Gamma_\sigma w), \quad v, w \in \Lambda(V).$$

(2.47)

**Definition 2.5.** The diagonal value of the product Cartan map is denoted

$$Ca^D_V : Gr^0_V(V + V^*) \rightarrow P(\Lambda^N(V)) \times P(\Lambda^N(V))$$

$$Ca^D_V(w^0) = (Ca_V(w^0), Ca_V(w^0)).$$

(2.48)

We then have:

**Theorem 2.1.** Up to projectivization,

$$\beta_N \circ Ca^D_V = \Psi|_{Gr^0_V(V + V^*)}.$$  

(2.49)

This may be deduced as a consequence of results given by Cartan ([1], Sections 105-109) in somewhat implicit form. A self-contained proof is given in the next section. Theorem 6.3, Section 6, gives an equivalent relation, in coordinate form, consisting of the bilinear identity (6.23), which expresses the determinants of all minors of any skew matrix in terms of the Pfaffians of its principal minors. The proof of this equivalence is given there, as well as a direct proof of the identity (6.23).

**3 Proof of Theorem 2.1**

**Proof.** Since $C$ preserves the scalar product $(\ , \ )$ on $\Lambda(V)$, eq. (2.49) is equivalent to

$$(C(Ca_V(w^0), C\Gamma_\sigma (Ca_V(w^0)))) = \langle \sigma | \Psi(w^0) \rangle, \quad \forall \sigma \in \Lambda^N(V + V^*).$$

(3.1)

But, as noted in [1],

$$C\Gamma_\sigma C = \Gamma_\sigma^T,$$

(3.2)

so this is equivalent to

$$(\Gamma_\sigma C(Ca_V(w^0)), Ca_V(w^0)) = \langle \sigma | \Psi(w^0) \rangle.$$  

(3.3)
It is sufficient to prove this for all basis elements of the form
\[ \sigma = f_I \wedge * e_J, \]  
for pairs \((I, J)\) of decreasingly ordered subsets \(I, J \subset \{1, \ldots, N\}\) of equal cardinality \(r = |I| = |J|\), \(r = 0 \ldots N\),

for which (3.3) becomes
\[ (\mathcal{A}(\Gamma_{f_I} \Gamma_{e_J}) C(C_{w^0}(w^0)), C_{w^0}(w^0))) = (f_I \wedge * e_J | \mathfrak{P}_{w^0}(w^0)), \]

where, for \(\{w_i \in V + V^*\}_{i=1,...,m}\)
\[ \mathcal{A}(\Gamma_{w_1} \cdots \Gamma_{w_m}) := \frac{1}{m!} \sum_{\tau \in S_m} sgn(\tau) \Gamma_{w_{\tau(1)}} \cdots \Gamma_{w_{\tau(m)}} \]
is the antisymmetrization map. Since \(\Gamma_{f_I}\) and \(\Gamma_{e_J}\) are already antisymmetric, we have
\[ A(\Gamma_{f_I} \Gamma_{e_J}) = \Gamma_{f_I \wedge * e_J} = \Gamma_{\sigma}. \]

Therefore (3.6) is equivalent to
\[ (\Gamma_{f_I \wedge * e_J} C(C_{w^0}(w^0)), C_{w^0}(w^0))) = (f_I \wedge * e_J | \mathfrak{P}_{w^0}(w^0)). \]

By the equivariance of the Cartan and Plücker maps under \(\text{Spin}(V + V^*, Q)\) and \(SO(V + V^*, Q)\), it is sufficient to prove this for just one element \(w^0\) in each of the two connected components of \(\text{Gr}_V^0(V + V^*)\). These may be chosen as
\[ w^0 = V = \text{span}\{e_1, \ldots, e_N\}, \]
which lies in the component \(\text{Gr}_V^0(-1)^N(V + V^*)\), and
\[ \tilde{w}^0 = \text{span}\{e_1, \ldots, e_{N-1}, f_N\}, \]
which lies in \(\text{Gr}_V^0(-1)^{N+1}(V + V^*)\).

For the first, we have
\[ C_{w^0}(w^0) = C_{w^0}(V) = [\Omega_V], \quad C(C_{w^0}(w^0)) = [1], \quad \mathfrak{P}_{w^0}(w^0) = [\Omega_V], \]
and
\[ (\Gamma_{f_I \wedge * e_J} 1, \Omega_V) = \delta_{I,\emptyset} \delta_{J,\emptyset} = \langle f_I \wedge * e_J | \Omega_V \rangle. \]

For the second,
\[ C_{\tilde{w}^0}(\tilde{w}^0) = [e_1 \wedge \cdots \wedge e_{N-1}], \quad C(C_{\tilde{w}^0}(\tilde{w}^0)) = [e_N], \quad \mathfrak{P}_{\tilde{w}^0}(\tilde{w}^0) = [e_1 \wedge \cdots \wedge e_{N-1} \wedge f_N], \]
and
\[ (\Gamma_{f_I \wedge * e_J} e_N, e_1 \wedge \cdots \wedge e_{N-1}) = \delta_{I,(N)} \delta_{J,(N)} = \langle f_I \wedge * e_J | e_1 \wedge \cdots \wedge e_{N-1} \wedge f_N \rangle. \]
4 Factorization of the Plücker map

\[ \mathcal{P}l_{k,N} : \text{Gr}_k(V) \rightarrow \mathbb{P}(\Lambda^k V) \]

The dual Pfaffian line bundles \( \text{Pf}^*_{\pm} \rightarrow \text{Gr}_V^0(V + V^*) \) may also be related to the dual determinantal line bundles \( \text{Det}^* \rightarrow \text{Gr}_k(V) \) over the Grassmannian of \( k \)-planes in \( V \), for \( k = 1, \ldots N - 1 \), by composition of the tautological embedding of \( \text{Gr}_k(V) \) in the isotropic Grassmannian \( \text{Gr}_V^0(V + V^*) \)

\[
\begin{align*}
\iota_{V,k} : \text{Gr}_k(V) & \rightarrow \text{Gr}_V^0(V + V^*) \\
\iota_{V,k} : v & \mapsto v + v^\perp \in \text{Gr}_V^0(V + V^*),
\end{align*}
\]

where \( v^\perp \subset V^* \) is the \( N - k \) dimensional annihilator of the \( k \)-dimensional subspace \( v = \text{span}\{v_1, \ldots, v_k\} \in \text{Gr}_k(V) \), with the Plücker map

\[
\begin{align*}
\mathcal{P}l_{k,N} : \text{Gr}_k(V) & \rightarrow \mathbb{P}(\Lambda^k V) \\
\mathcal{P}l_{k,N} : v & \mapsto [v_1 \wedge \cdots \wedge v_k]
\end{align*}
\]

embedding the Grassmannians \( \text{Gr}_k(V) \) of \( k \)-planes in \( V \) into the projectivization of the \( k \)th exterior power \( \Lambda^k(V) \).

We then have the following sequence of embeddings and pull-backs

\[
\begin{array}{ccccc}
\text{Det}_k^* & \downarrow & \text{Pf}^*_{\{(-1)^k \} \text{Gr}_k(V)} & \downarrow & \mathcal{O}(1)(\mathbb{P}(\Lambda(-1)^k(V))) \\
\downarrow & & \downarrow & & \\
\text{Gr}_k(V) & \iota_{V,k} & \text{Gr}_V^0(V(-1)^k(V + V^*)) & \overset{\text{Ca}_V}{\rightarrow} & \mathbb{P}(\Lambda(V(-1)^k)),
\end{array}
\]

giving the dual determinantal line bundle \( \text{Det}_k^* \rightarrow \text{Gr}_k(V) \) as the pull-back under \( \iota_{V,k} \) of the dual Pfaffian line bundle \( \text{Pf}^*_{\{(-1)^k \} \text{Gr}_k(V)} \) over \( \text{Gr}_V^0(V + V^*) \) restricted to the image \( \iota_{V,k}(\text{Gr}_k(V)) \) of \( \text{Gr}_k(V) \) under \( \iota_{V,k} \).

**Proposition 4.1.**

\[
\text{Det}_k^* = \iota_{V,k}^*([\text{Pf}^*_{\{(-1)^k \} \text{Gr}_k(V)}],)
\]

and the Plücker map

\[
\mathcal{P}l_{k,N} : \text{Gr}_k(V) \rightarrow \mathbb{P}(\Lambda^k(V))
\]

factorizes through the Cartan map

\[
\mathcal{P}l_{k,N} = \text{Ca}_V \cdot \iota_{V,k}.
\]

**Proof.** This follows immediately from the definitions

\[
\mathcal{P}l_{k,N} : \text{Gr}_k(V) \rightarrow \mathbb{P}(\Lambda^k(V)) \\
\mathcal{P}l_{k,N} : \text{span}\{v_1, \ldots, v_k\} \rightarrow [v_1 \wedge \cdots \wedge v_k]
\]

of the Plücker map, \( (2.11) \), \( (2.18) \) of the Cartan map \( \text{Ca}_V \) and \( (4.1) \) of the tautological embedding \( \iota_{V,k} \).
5 Plücker and Cartan coordinates

The Plücker coordinates \( \{ \pi_\lambda(w) \} \) of an element \( w \in \text{Gr}_V(V + V^*) \) are, up to projectivization, the coefficients in the expansion of the image of the Plücker map in the basis \( \{ |\lambda \rangle \} \):

\[
\mathfrak{P}_V(w) = \left[ \sum_{\lambda \subset (N)^N} \pi_\lambda(w) |\lambda \rangle \right].
\]

Equivalently, if we define \( W \) to be the \( 2N \times N \) dimensional rectangular matrix whose \( j \)th column is the \( j \)th basis vector \( w_j \) for \( w \in \text{Gr}_V(V + V^*) \), expressed as a column vector relative to the basis \( (e_1, \ldots, e_N, \ldots, e_{2N}) \), and \( W_\lambda \) to be the \( N \times N \) submatrix whose \( i \)th row is the \( l_i \)th row of \( W \), we have, up to projectivization,

\[
\pi_\lambda(w) = \det(W_\lambda).
\]

Thus, each Plücker coordinate \( \pi_\lambda(w) \) is in fact a holomorphic section \( \pi_\lambda \in \Gamma(\text{Det}^*[V]) \) of the dual determinantal line bundle \( \text{Det}^* \to \text{Gr}_V(V + V^*) \), and the full set of Plücker coordinates provides a basis for the space \( H^0(\text{Gr}_V(V + V^*), \text{Det}^*) \) of holomorphic sections (cf. [7]).

Recall also that, by the (generalized) Giambelli identity (cf. [6], Appendix C.8), if we express \( \lambda \) in Frobenius notation [11] as

\[
\lambda(\alpha, \beta) = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r),
\]

with

\[
N > \alpha_1 > \cdots > \alpha_r \geq 0, \quad N > \beta_1 > \cdots > \beta_r \geq 0,
\]

we have

\[
(\pi_\emptyset(w))^{r-1} \pi_\lambda(w) = \det \left( \pi_{(\alpha_i | \beta_j)}(w) \right)_{1 \leq i, j \leq r}.
\]

**Definition 5.1.** For \( \alpha = (\alpha_1, \ldots, \alpha_r) \), let \( \alpha' = (\alpha'_1, \ldots, \alpha'_r) \) denote the complement in the \( r \times N \) rectangle (with columns labelled from 0 to \( N - 1 \)), in reversed order:

\[
\alpha'_i := N - 1 - \alpha_{r-i+1}, \quad 1 \leq i \leq r.
\]

We denote by

\[
\lambda(\alpha) := (\alpha_1, \ldots, \alpha_r | \alpha'_1, \ldots, \alpha'_r)
\]

the *pseudosymmetric* partitions, whose Frobenius indices satisfy \( \beta = \alpha' \).

**Remark 5.1.** If we reverse the ordering in the second half of our basis, choosing instead:

\( \{ e_1, \ldots, e_N, e_{2N}, \ldots, e_{N+1} \} \), these partitions would correspond to \( \beta = \alpha \), and hence would, in fact, be symmetric in the usual sense.
In the following, we adopt the notation $I(\alpha)$ to denote the ordered subset
\[ \{I_1(\alpha), \ldots, I_r(\alpha)\} \subset \{1, \ldots, N\} \] (5.8)
obtained from $\alpha = (\alpha_1, \ldots, \alpha_r)$, viewed as a strict partition, by adding 1 to each of its parts
\[ I_i(\alpha) := \alpha_i + 1, \quad 1 \leq I_i(\alpha) \leq N, \quad i = 1, \ldots, r. \] (5.9)
Define the bases $\{e_I\}$ and $\{f_I\}$ for $\Lambda V$ and $\Lambda V^*$, respectively, as:
\[ e_I := e_{I_1} \wedge \cdots \wedge e_{I_r} \in \Lambda V, \] (5.10)
\[ f_I := f_{I_1} \wedge \cdots \wedge f_{I_r} \in \Lambda V^*, \] (5.11)
where
\[ I = (I_1 > \cdots I_r > 0) \subset \{1, \ldots, N\}, \quad 1 \leq r \leq N. \] (5.12)
In particular, the volume form $\Omega_V$ on $V$ is
\[ \Omega_V := (-1)^{\frac{N}{2}(N-1)} e_{(N,N-1,\ldots,1)}. \] (5.13)

**Definition 5.2.** Let $\tilde{\alpha}$ denote the complement of $\alpha$ in $(0, \ldots, N-1)$, with parts $\{\tilde{\alpha}_i\}_{i=1,\ldots,N-r}$ in decreasing order forming a strict partition of length $N-r$, and let
\[ \tilde{I}(\alpha) = I(\tilde{\alpha}) \] (5.14)
denote the corresponding complement of $I(\alpha)$ in $\{1, \ldots, N\}$.

On $\Lambda^N(V + V^*)$, we then have
\[ |\lambda(\alpha, \beta)\rangle = f_{I(\alpha)} \wedge \ast e_{I(\beta')}. \] (5.15)
In particular, for pseudosymmetric partitions $\lambda(\alpha)$, we have
\[ |\lambda(\alpha)\rangle = f_{I(\alpha)} \wedge \ast e_{I(\alpha)}, \] (5.16)
which is the image, under the Plücker map, of the isotropic subspace spanned by the basis vectors $\{e_{I_1(\alpha)}, \ldots, e_{I_{N-r}(\alpha)}; f_{I_1(\alpha)}, \ldots, f_{I_r(\alpha)}\}$.

**Definition 5.3.** The sign of the strict partition $\alpha$, denoted $\text{sgn}(\alpha)$ is defined to be the same as that of $I(\alpha)$:
\[ \text{sgn}(\alpha) := \text{sgn}(I(\alpha)) = (-1)^{r(N-r)} \text{sgn}(\tilde{\alpha}). \] (5.17)
**Definition 5.4.** The Cartan coordinates $\kappa_\alpha(w^0)$ of $w^0 \in \text{Gr}^0_V(V + V^*)$ are defined as:

$$\kappa_\alpha(w^0) := (\text{Ca}_V(w^0), *e_{I(\alpha)}) = (-1)^{r(r-1)+rN} \beta_0 (\text{Ca}_V(w^0), e_{I(\alpha)}).$$ (5.18)

Therefore we may express its Cartan image as

$$\text{Ca}_V(w^0) = \sum_{\alpha \in \{\text{strict partitions of parity } \pm (-1)^N, \text{ where } w^0 \in \text{Gr}^0_{\leq (V + V^*)}\}} \kappa_\alpha(w^0) * e_{I(\alpha)}. \tag{5.19}$$

On the intersection

$$U^0_{\lambda(\alpha)} := U_{\lambda(\alpha)} \cap \text{Gr}^0_V(V + V^*) \tag{5.20}$$

of the coordinate neighbourhood $U^0_{\lambda(\alpha)} \subset \text{Gr}_V(V + V^*)$ on which

$$\det(W_{\lambda(\alpha)}) \neq 0, \tag{5.21}$$

with $\text{Gr}^0_V(V + V^*)$, let $A^{\lambda(\alpha)}(w^0)$ denote the $N \times N$ submatrix of $WW_{\lambda(\alpha)}^{-1}$ whose rows are in the complementary positions to those of $W_{\lambda(\alpha)}$. The condition that $w^0$ is isotropic is equivalent to the fact that this is a skew symmetric matrix

$$(A^{\lambda(\alpha)}(w^0))^T = -A^{\lambda(\alpha)}(w^0). \tag{5.22}$$

In particular, for the null partition $\alpha = \emptyset$, $A^\emptyset(w^0)$ is the affine coordinate matrix of the element $w^0 \in \text{Gr}^0_V(V + V^*)$ on the “big cell” $U^0_\emptyset$.

By the generalized Giambelli identity \([5.5]\), the Plücker coordinates of $w^0 \in U^0_\emptyset$ may be expressed as determinants of minors of the affine coordinate matrix $A^\emptyset(w^0)$.

**Lemma 5.1.** For strict partitions $\alpha, \beta$ of length $r$, let $A_{(I(\alpha)|I(\beta))}$ denote the submatrix of the matrix $A^\emptyset(w^0)$ with rows $I(\alpha)$ and columns $I(\beta)$. Then, up to projective equivalence, we have the following expression for the Plücker coordinates of $w^0$:

$$\pi_{\lambda(\alpha|\beta)}(w^0) = \det(A_{(I(\alpha)|I(\beta))}) = (f_{I(\alpha)} \wedge *e_{I(\beta)}, \mathcal{P}_V(w^0)). \tag{5.23}$$

The Cartan coordinates $\{\kappa_\alpha(w^0)\}$ may similarly be expressed as Pfaffians of the principal minors of $A^\emptyset(w^0)$.

**Definition 5.5.** For any skew symmetric $N \times N$ matrix $A$, and any strict partition $\alpha$ of length $r = \ell(\alpha)$, between 1 and $N$, let $A(\alpha)$ denote the $r \times r$ principal minor of $A$ with rows and columns in the positions $(I_1(\alpha), \ldots, I_r(\alpha))$.

As $\alpha$ varies over the $2^N$ strict partitions $\alpha$ for which $\lambda(\alpha) \subset (N)^N$, applying \((5.18)\), the Cartan coordinates $\{\kappa_\alpha(w^0)\}$ for $w^0 \in U^\emptyset$ are given by the following.
Proposition 5.2. On the big cell $U^0_{\emptyset}$, the Cartan coordinates are

$$\kappa_{\alpha}(w^0) = (-1)^{\frac{r}{2}} \text{Pf}(A^\theta(w^0)(\alpha)),$$

up to projective equivalence.

Proof. For $w^0 \in U^0_{\emptyset}$, we may choose the basis

$$w_i = e_i + \sum_{j=1}^{N} A^\theta_{ji} f_j, \quad i = 1, \ldots, N.$$  \hspace{1cm} (5.25)

and hence

$$\Gamma_{w_i} = \psi_i + \sum_{j=1}^{N} A^\theta_{ji} \psi_j^\dagger, \quad i = 1, \ldots, N.$$ \hspace{1cm} (5.26)

The homogeneous $2N \times N$ coordinate matrix representing $w^0$ in the basis $(e_1, \ldots, e_N, f_1, \ldots, f_N)$ is thus

$$\begin{pmatrix} I_N \\ A^\theta(w^0) \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 \\ A^\theta(w^0) & 0 \end{pmatrix} \begin{pmatrix} I_N \\ 0 \end{pmatrix} = \begin{pmatrix} I_N \\ A^\theta(w^0) \\ I_N \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \hspace{1cm} (5.27)$$

Eq. (2.22), together with the equivariance of the Cartan map, imply that

$$\text{Ca}(w^0) = \left[e^{\sum_{1 \leq i < j \leq N} (A^\theta)_{ij}(w^0) \psi_i^\dagger \psi_j^\dagger} \cdot \Omega_V\right]$$

$$= \left[\sum_{n=0}^{[N/2]} \frac{1}{n!} \left(\sum_{1 \leq i < j \leq N} (A^\theta)_{ij} \psi_i^\dagger \psi_j^\dagger\right)^n \Omega_V\right]$$

$$= \left[\sum_{\alpha \in \{\text{strict partitions of even cardinality } r\}} \kappa_{\alpha}(w^0) * e_{I(\alpha)} \right], \hspace{1cm} (5.28)$$

where

$$\kappa_{\alpha}(w^0) := (-1)^{\frac{r}{2}} \text{Pf} \left(A^\theta(w^0)(\alpha)\right).$$ \hspace{1cm} (5.29)

6 Coordinate interpretation: a Pfaffian analog of the Cauchy-Binet identity

Choose two pairs

$$I, J \subseteq \{1, \ldots, N\}, \quad K, L \subseteq \{1, \ldots, N\}$$ \hspace{1cm} (6.1)

of decreasingly ordered subsets

$$I = (I_1, \ldots, I_i), \quad J = (J_1, \ldots, J_j)$$
\[ K = (K_1, \ldots, K_k), \quad L = (L_1, \ldots, L_l) \quad (6.2) \]

with cardinalities
\[ |I| = i, \quad |J| = j, \quad |K| = k, \quad |L| = l. \quad (6.3) \]
satisfying
\[ i + j = k + l = 2r. \quad (6.4) \]
for some \( r \), with
\[ 1 \leq r \leq N. \quad (6.5) \]

**Remark 6.1.** In Theorem 6.3 below, only the case where \( I \) and \( J \) have equal cardinalities
\[ i = j = r \quad (6.6) \]
will be needed. But for the moment, we do not require this restriction.

Let \( \mathcal{I}_{(IJ|KL)} \) be the \( 2r \times 2r \) submatrix of the \( 2N \times 2N \) matrix with block form
\[ \mathcal{I}_{2N} := \begin{pmatrix} I_N & I_N \\ I_N & -I_N \end{pmatrix}, \quad (6.7) \]
whose first \( i \) rows are the rows of \( \mathcal{I}_{2N} \) in positions \( \{I_i\}_{i=1,\ldots,r} \) and next \( j \) rows are those in positions \( \{J_{i+N}\}_{i=1,\ldots,r} \), and whose first \( k \) columns are those of \( \mathcal{I}_{2N} \) in positions \( \{K_i\}_{i=1,\ldots,k} \) and next \( l \) columns are those in positions \( \{L_{i+N}\}_{i=1,\ldots,l} \). Now define
\[ \Delta_{(IJ|KL)} := \det(\mathcal{I}_{(IJ|KL)}). \quad (6.8) \]

**Lemma 6.1.** \( \Delta_{(IJ|KL)} \) vanishes unless the set theoretic conditions
\[ K \cup L = I \cup J \quad \text{and} \quad K \cap L = I \cap J \quad (6.9) \]
are satisfied.

**Proof.** Elements of \( K \cup L \) that are not in \( I \cup J \) give vanishing columns of \( \mathcal{I}_{(IJ|KL)} \), and elements of \( I \cup J \) that are not in \( K \cup L \) give vanishing rows. In either case, the determinant \( \Delta_{(IJ|KL)} \) vanishes. Therefore, if it does not vanish, the first equality of \( (6.9) \) must hold. The rows and columns of \( \mathcal{I}_{(IJ|KL)} \) each contain either no nonzero entries, or one, equal to \( \pm 1 \), or two, equal to \( \pm 1 \). The elements of \( I \cap J \) give pairs of rows, in the top and bottom half, which either vanish, or have a single element \( \pm 1 \) that is nonzero in the same place, or a pair of elements with nonzero entries \( (1, 1) \) and \( (1, -1) \). In the first two cases, the determinant vanishes. If the element is not in \( K \cap L \), it cannot be the third case, so we must have \( I \cap J \subset K \cap L \) for the determinant not to vanish. The same consideration, interchanging rows and columns, shows we must have \( K \cap L \subset I \cap J \) for it not to vanish. Therefore, if it does not vanish, the second equality of \( (6.9) \) is true. \( \square \)
Remark 6.2. Let
\[ T := I \cup J = K \cup L \] (6.10)
The equalities (6.9) are equivalent to the fact that the following are disjoint subsets of \( T \):
\[ S := I \cap J = K \cap L, \quad A := (I \cap K) \setminus S, \quad B := (I \cap L) \setminus S, \]
\[ C := (J \cap K) \setminus S, \quad D := (J \cap L) \setminus S, \] (6.11)
whose union is \( T \),
\[ T = A \cup B \cup C \cup D \cup S. \] (6.12)
It follows that
\[ I = A \cup B \cup S, \quad J = C \cup D \cup S, \]
\[ K = A \cup C \cup S, \quad L = B \cup D \cup S. \] (6.13)
Denote the cardinalities of \((A, B, C, D, S, T)\)
\[ |A| =: a, \quad |B| =: b, \quad |C| =: c, \quad |D| =: d, \quad |S| =: s, \quad |T| =: t. \] (6.14)
Then
\[ t + s = 2r, \]
\[ t - s = 2t - 2r = a + b + c + d, \]
\[ i = a + b + s, \quad j = c + d + s, \]
\[ k = a + c + s, \quad l = b + d + s, \] (6.15)
and therefore \( a + b + c + d \) must be even.

Remark 6.3. Note also that, for given \((I, J)\), conditions (6.9) imply that \( L \) is uniquely determined once \( K \) is given, and vice versa:
\[ L = ((I \cup J) \setminus K) \cup (I \cap J), \quad K = ((I \cup J) \setminus L) \cup (I \cap J). \] (6.16)
The determinant \( \Delta_{(I,J|KL)} \) is independent of the overall ordering \( \{1, \ldots, N\} \), since both rows and columns are transformed in the same way under a permutation. We may therefore, without loss of generality, choose the ordering such that
\[ A = (1, \ldots, a), \quad B = (a + 1, \ldots, a + b), \]
\[ C = (a + b + 1, \ldots, a + b + c), \quad D = (a + b + c + 1, \ldots, a + b + c + d), \]
\[ T = (1, \ldots, t), \quad S = (a + b + c + d + 1, \ldots, a + b + c + d + s = t). \] (6.17)
We then have the following expression for \( \Delta_{(I,J|KL)} \).

Lemma 6.2.
\[ \Delta_{(I,J|KL)} = (-1)^{(b+s)c+bs+d+s}2^s \delta_{I \cup J,K \cup L} \delta_{I \cap J,K \cap L}. \] (6.18)
Moreover, if \( l = b + d + s \) is even, and hence so is \( k = a + c + s \), this simplifies to
\[ \Delta_{(I,J|KL)} = (-1)^{jd}2^s \delta_{I \cup J,K \cup L} \delta_{I \cap J,K \cap L}. \] (6.19)
Remark 6.4.

Proof. It follows from (6.15) that, up to conjugation by a permutation, $\mathcal{I}_{(I,J|KL)}$ has the block form

$$
\mathcal{I}_{(I,J|KL)} = \begin{pmatrix}
I_a & 0_{ac} & 0_{as} & 0_{ab} & 0_{ad} & 0_{as} \\
0_{ba} & 0_{bc} & 0_{bs} & I_b & 0_{bd} & 0_{bs} \\
0_{sa} & 0_{sc} & I_s & 0_{sb} & 0_{sd} & I_s \\
0_{ca} & I_c & 0_{cs} & 0_{cb} & 0_{cd} & 0_{cs} \\
0_{da} & 0_{dc} & 0_{ds} & 0_{db} & -I_d & 0_{ds} \\
0_{sa} & 0_{sc} & I_s & 0_{sb} & 0_{sd} & -I_s
\end{pmatrix},
$$

where $I_n$ denotes the $n \times n$ identity matrix and $0_{mn}$ denotes the $m \times n$ matrix with vanishing entries. By successively eliminating unit matrix blocks, the determinant reduces to

$$
\Delta_{(I,J|KL)} = (-1)^{(b+s)c+bs+d}\det \begin{pmatrix}
I_s & I_s \\
I_s & -I_s
\end{pmatrix} = (-1)^{(b+s)c+bs+d+sp^s}. 
$$

Adding the Kronecker $\delta$’s that assure the relations (6.9) are satisfied gives (6.18). To obtain (6.19) when $l$ is even, substitute $c = j - d - s$, $b + s \equiv d (\text{mod } 2)$ in the exponent of $-1$ in (6.18) and reduce (mod 2).

Now let $A$ be a skew symmetric $N \times N$ matrix and, for any pair $K, L \subseteq \{1, \ldots, N\}$ of ordered subsets of cardinalities $k, l$, respectively, with $k + l = 2r$, let $A_{(K|L)}$ denote the $k \times l$ submatrix of $A$ whose rows and columns are the restriction of those of $A$ in positions $K$ and $L$, respectively. For $w^0 \in \text{Gr}_r(V + V^*)$ in the big cell, choosing $A$ to equal the affine coordinate matrix

$$
A := A^0(w^0),
$$

Theorem 2.1 is then equivalent to the following identity.

**Theorem 6.3.**

$$
\det(A_{(I|J)}) = \frac{(-1)^{\frac{1}{2}r(r-1)}}{2^r} \sum_{(K, L) \subseteq \{1, \ldots, N\}} (-1)^{l/2}\Delta_{(I,J|KL)}\text{Pf}(A_{(K|K)})\text{Pf}(A_{(L|L)}).
$$

**Remark 6.4.** Here $(I, J)$ are related to the strict partitions $(\alpha, \beta)$ that determine the partition $\lambda(\alpha, \beta)$ by

$$
I = I(\alpha), \quad J = I(\beta),
$$

and hence have equal cardinality $|I| = |J| = r$, and $(K, L)$ are related to the strict partitions $(\gamma, \delta)$ that determine the pseudosymmetric partitions $\lambda(\gamma)$ and $\lambda(\delta)$ by

$$
K = I(\gamma), \quad L = I(\delta),
$$

so

$$
\det(A_{(I|J)}) = \pi_{\lambda(\alpha, \beta')}^r(w), \quad \text{Pf}(A_{(K|K)}) = \kappa_{\gamma}(w), \quad \text{Pf}(A_{(L|L)}) = \kappa_{\delta}(w).
$$
For $I = J$ the only admissible pair in the above sum is $(K, L) = (I, I)$, and therefore (6.23) reduces to the standard identity

$$\det(A_{(I|I)}) = \begin{cases} \text{Pf}(A_{(I|I)})^2 & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd.} \end{cases} \quad (6.27)$$

In view of the fact that, by (6.13), $I$, $J$ and $K$ uniquely determine $L$, and $I$, $J$ and $L$ uniquely determine $K$, eq. (6.23) really consists only of a single sum, over either the variable $K$ or $L$, and hence is equivalent to the following. Using

$$\Delta_{(I|J|K|L)} = (-1)^{rd} 2^{|I\cup J|} \delta_{I\cup J, K\cup L} \delta_{I\cap J, K\cap L}, \quad (6.28)$$

for even $k$ and $l$, this gives

**Corollary 6.4.**

$$\det(A_{(I|J)}) = (-1)^{\frac{1}{2}r(r+1)} \sum_{K, |K| \text{ even} \atop L = ((I\cup J)\setminus K)\cup (I\cap J)} (-1)^{k/2+rd} \text{Pf}(A_{(K|K)}) \text{Pf}(A_{(L|L)}))$$

$$= (-1)^{\frac{1}{2}r(r-1)} \sum_{L, |L| \text{ even} \atop K = ((I\cup J)\setminus L)\cup (I\cap J)} (-1)^{l/2+rd} \text{Pf}(A_{(K|K)}) \text{Pf}(A_{(L|L)}). \quad (6.29)$$

Before proceeding to a direct proof of eq. (6.23) we show that, under the identifications (6.24), (6.25), it is just the coordinate expression of eq. (3.3), and hence equivalent to Theorem 2.1 expressed in affine coordinates.

**Proposition 6.5.** Under the identifications (6.24), (6.25), eqs. (3.3) and (6.23) are equivalent (up to projectivization).

**Proof.** As shown in Section 3, eq. (3.3) is equivalent to verifying

$$(\Gamma f_I \wedge *e_J C(Ca_V(w^0), Ca_V(w^0))) = \langle f_I \wedge *e_J | \Psi_V(w^0) \rangle, \quad (6.30)$$

for all $\sigma$ of the form

$$\sigma = f_I \wedge *e_J, \quad (6.31)$$

Lemma (5.1) gives

$$\langle f_I \wedge *e_J | \Psi_V(w^0) \rangle = (-1)^{r(N-r)} N_N \det(A_{(I|J)}), \quad (6.32)$$

where $N_N \neq 0$ is any projective normalization factor. Making the identifications (6.24), (6.25), substituting (6.32) into the RHS of eq. (6.30), and (5.19) twice into the LHS, using
eq. (5.24) to express the Cartan coordinates as Pfaffians, and (2.41), (2.42), to relate the map $C$ to Hodge $\ast$, and using the fact that $k$ and $l$ are even gives:

$$\det(A_{(I|J)}) = \frac{(-1)^{\frac{1}{2}N(N+1)}}{N_N} \sum_{(K,L)} (-1)^{\frac{1}{2}K \Delta_{(I|J|KL)}} \text{Pf}(A_{(K|K)}) \text{Pf}(A_{(L|L)}),$$

(6.33)

where

$$\Delta_{(I|J|KL)} := (\Gamma f_i \wedge e_j e_K, * e_L).$$

(6.34)

Lemma 6.6 below shows that this is equivalent to:

$$\det(A_{(I|J)}) = \frac{(-1)^{\frac{1}{2}N(N+1)}}{N_N} \sum_{(K,L)} (-1)^{\frac{1}{2}K \Delta_{(I|J|KL)}} \text{Pf}(A_{(K|K)}) \text{Pf}(A_{(L|L)}).$$

(6.35)

To obtain the correct normalization, it is sufficient to evaluate this for one specific choice of $(I|J)$. But for any $I = J$, we have

$$\det(A_{(I|I)}) = \text{Pf}(A_{(I|I)})^2,$$

(6.36)

so

$$N_N = (-1)^{\frac{1}{2}N(N+1)},$$

(6.37)

and we obtain eq. (6.23).

Lemma 6.6. Let $I$ and $J$ have the same cardinality $i = j = r$, and let $K$, and hence also $L$, have even cardinalities $(k, l)$. Then $\Delta_{(I|J|KL)}$ is nonzero if and only if eq. (6.9) holds and, in that case

$$\Delta_{(I|J|KL)} = \frac{(-1)^{r(r-1)/2}}{2^r} \Delta_{(I|J|KL)}.$$  

(6.38)

Proof. First note that the product $\Gamma f_i \Gamma e_j$ acting on $e_K$ gives a multiple of a homogeneous basis element

$$\Gamma f_i \Gamma e_j e_K = -\Gamma e_j \Gamma f_i e_K = \pm e_M$$

(6.39)

for some $M$ of the same cardinality as $K$ if $i \neq j$. If $i = j$, either $\Gamma f_i \Gamma e_j e_K$ or $\Gamma e_j \Gamma f_i e_K$ vanishes, and the other equals $e_K$. It follows that $\Gamma f_i \wedge e_j e_K$ is a monomial, and hence a multiple

$$\Gamma f_i \wedge e_j e_K = \gamma e_P$$

(6.40)

for some scalar $\gamma$ and some ordered subset $P \subset \{1, \ldots, N\}$. We next show that the assumption that $\Delta_{(I|J|KL)}$ does not vanish is equivalent to the set theoretic equalities (6.9). If $\gamma \neq 0$ we must have $K \subset I \cup J$. If $\gamma \neq 0$ and $l \in P$, then $l \notin \tilde{P}$ and either $l \in I$
(since $e_l$ is removed by $\Gamma_{f_l}$) or $l \notin \tilde{J}$ (since otherwise $e_l$ would have been added by $\Gamma_{e_j}$), which implies $l \in I \cup J$. If $l \in K \cup P$ and $l \notin K \cap P$, then either $l \notin K$ and $l \in P$ or $l \in K$ and $l \notin P$. In either case $l \in I \cup J$. Therefore $I \cup J = K \cup P$. It follows similarly that $I \cap J = K \cap P$, and hence

$$P = (I \cup J) \setminus K) \cup (I \cap J) = L,$$

and eq. (6.9) is satisfied.

This means that we can decompose the subsets $I, J, K, L$ in the same way as for the case $\Delta_{(I,J|KL)}$ above. We can again choose the ordering of $\{1, \ldots, N\}$, since the expression $\Delta_{(I,J|KL)}$ is invariantly defined, and we choose the same order as above. If

$$M = \{m_1 < m_2 < \ldots < m_s\},$$

we have

$$\Gamma_{f_M} = \Gamma_{f_1} \Gamma_{f_2} \ldots \Gamma_{f_s}, \quad \Gamma_{e_M} = \Gamma_{e_1} \Gamma_{e_2} \ldots \Gamma_{e_s},$$

so that there is just one product.

Changing $*e_J, *e_L$ to $e_J, e_L$, we first consider

$$\tilde{\Delta}_{(I,J|KL)} := (\Gamma_{f_I \wedge e_J} e_K, e_L).$$

Applying $\Gamma_{f_I \wedge e_J}$ to $e_K$ gives the same result as the antisymmetrisation of

$$\Gamma_{f_A} \Gamma_{f_B} \Gamma_{f_S} \Gamma_{e_A} \Gamma_{e_B} \Gamma_{e_T}$$

applied to $e_K$. Now note that antisymmetrization gives

$$\mathcal{A}(\Gamma_{f_A} \Gamma_{f_B} \Gamma_{f_S} \Gamma_{e_A} \Gamma_{e_B} \Gamma_{e_T}) = (-1)^{(s+a)b} \mathcal{A}(\Gamma_{f_A} \Gamma_{f_S} \Gamma_{e_A} \Gamma_{f_B} \Gamma_{e_B} \Gamma_{e_T})$$

and apply $\Gamma_{f_S} \Gamma_{e_A} \Gamma_{f_A} \Gamma_{f_B} \Gamma_{e_B} \Gamma_{e_T}$ to $e_K = e_A \wedge e_C \wedge e_S$:

$$(-1)^{(s+a)(b+a)} \Gamma_{f_S} \Gamma_{e_A} \Gamma_{f_A} \Gamma_{f_B} \Gamma_{e_B} \Gamma_{e_T} (e_A \wedge e_C \wedge e_S)$$

$$= (-1)^{(s+a)(b+a)+\ell(a+c+s)+a(a-1)/2+b(b-1)/2} \Gamma_{f_S} \Gamma_{e_A} \Gamma_{f_A} \Gamma_{f_B} \Gamma_{e_B} (e_A \wedge e_C \wedge e_S \wedge e_T)$$

$$= (-1)^{(s+a)(b+a)+\ell(a+c+s)+a(a-1)/2+b(b-1)/2+(a+c)s+s(s-1)/2} (e_A \wedge e_C \wedge e_T)$$

Antisymmetrizing this would give the same result, except that we can only retain permutations in which the factors $\Gamma_{e_A}$ precede the factors $\Gamma_{f_A}$, and the factors $\Gamma_{f_B}$ precede the factors $\Gamma_{e_B}$. This reduces the sum by a total factor of $2^{-(a+b)}$, giving

$$\tilde{\Delta}_{(I,J|KL)} = \frac{(-1)^{(s+a)(b+a)+\ell(a+c+s)+a(a-1)/2+b(b-1)/2+(a+c)s+s(s-1)/2}}{2^{a+b}}. 
(6.48)$$
The quotient is therefore
\[ \frac{\tilde{\Delta}_{(IJ|KL)}}{\Delta_{(IJ|KL)}} = \frac{(-1)^{(s+a)(b+a)+\ell(a+c+s)+a(a-1)/2+b(b-1)/2+(a+c)s+s(s-1)/2+(b+s)c+bs+d+s}}{2^{(a+b)+s}}. \] (6.49)

It remains only to simplify the sign. Using \( m^2 \equiv m \) (mod 2) and the fact that \( a+c+s = k \) and \( b + d + s = \ell \) are even, we have
\[ (-1)^{(s+a)(b+a)+a(a-1)/2+b(b-1)/2+(a+c)s+s(s-1)/2+(b+s)c+bs+b} = (-1)^{a+ab+b+bc+a(a-1)/2+b(b-1)/2+s(s-1)/2}, \] (6.50)
and also
\[ (-1)^{a(a-1)/2+b(b-1)/2+s(s-1)/2} = (-1)^{r(r-1)/2+ab+as+bs}. \] (6.51)

Recalling that \( a + b + s = r \), the sign becomes
\[ (-1)^{a+bc+as+bs+r(r-1)/2} = (-1)^{(r-s)+(r-s)s+bc+r(r-1)/2}, \] (6.52)
and therefore
\[ \frac{\tilde{\Delta}_{(IJ|KL)}}{\Delta_{(IJ|KL)}} = \frac{(-1)^{(r-s)+(r-s)s+bc+r(r-1)/2}}{2^r}. \] (6.53)

Replacement of \( e_J \) by \(*e_J \) involves multiplication by
\[ \text{sgn}(J) = (-1)^{r(r-s)}. \] (6.54)
and replacement of \( e_L \) by \(*e_L \) introduces the sign
\[ \text{sgn}(L) = (-1)^{bc}. \] (6.55)

Combining with (6.53), we obtain
\[ (-1)^{(r+s+1)(r-s)+r(r-1)/2} = (-1)^{(r+s+1)(r+s)+r(r-1)/2} = (-1)^{r(r-1)/2}, \] (6.56)
and so
\[ \frac{\tilde{\Delta}_{(IJ|KL)}}{\Delta_{(IJ|KL)}} = \frac{(-1)^{r(r-1)/2}}{2^r}. \] (6.57)

As preparation for a direct proof of Theorem \[6.3\], we introduce a second set of basis vectors for \( V + V^* \)
\[ g_i := e_i + f_i, \quad h_i := e_i - f_i, \quad i = 1, \ldots, N, \] (6.58)
and define the \( 2N \) component row vectors
\[ (e, f) = (e_1, e_2, \ldots, e_N, f_1, f_2, \ldots, f_N), \quad (g, h) = (g_1, g_2, \ldots, g_N, h_1, h_2, \ldots, h_N), \] (6.59)
whose entries are the basis elements \(\{e_i, f_i\}_{1=1,\ldots,N}\) and \(\{g_i, h_i\}_{1=1,\ldots,N}\), respectively. These are related by

\[
(g, h) = (e, f) \mathcal{I}_{2N}, \\
(e, f) = \frac{1}{2}(g, h) \mathcal{I}_{2N}.
\] (6.60)

Let \(g_I, h_J, e_K\) and \(f_L\) denote the exterior forms in \(\Lambda(V + V^*)\) associated to ordered subsets \(I, J, K\) and \(L\) of \(\{1, 2, \ldots, N\}\), defined by:

\[
g_I := g_{i_1} \wedge \cdots \wedge g_{i_r}, \quad h_J := h_{j_1} \wedge \cdots \wedge h_{j_r}, \\
e_K := e_{k_1} \wedge \cdots \wedge e_{k_k}, \quad f_L := f_{l_1} \wedge \cdots \wedge f_{l_l},
\] (6.61)

By the usual formulae for changes of bases, we then have

**Lemma 6.7.** For all ordered subsets \(I, J \subset \{1, 2, \ldots, N\}\) whose lengths sum up to \(2r\)

\[
g_I \wedge h_J = \sum_{K,L} \Delta_{(I|J|KL)} e_K \wedge f_L,
\] (6.62)

and for the dual basis,

\[
(g_I \wedge h_J)^* = \frac{1}{2^r} \sum_{K,L} \Delta_{(I|J|KL)} (e_K \wedge f_L)^*.
\] (6.63)

We now proceed to the proof of Theorem 6.3.

**Proof of Theorem 6.3.** Define the 2-form

\[
\omega =: \sum_{k,l=1}^{N} A_{kl} g_k \wedge h_l.
\] (6.64)

The skew symmetry of \(A\) implies

\[
\omega = 2 \sum_{1 \leq k < l \leq N} A_{kl} [e_k \wedge e_l - f_k \wedge f_l] = 2(\omega_1 - \omega_2),
\] (6.65)

where

\[
\omega_1 := \sum_{1 \leq i < j \leq N} A_{ij} e_i \wedge e_j, \quad \omega_2 := \sum_{1 \leq i < j \leq N} A_{ij} f_i \wedge f_j.
\] (6.66)

For any \(1 \leq r \leq N\), the \(r\)th wedge power of \(\omega\) can be written as

\[
\omega^\wedge r = \left( \sum_{i,j=1}^{N} A_{ij} g_i \wedge h_j \right)^r
\]
\[
\sum_{i_1, i_1', \ldots, i_r, j_r} \prod_{k=1}^r A_{i_k j_k} g_{i_1} \wedge h_{j_1} \wedge \cdots \wedge g_{i_r} \wedge h_{j_r} = (-1)^{\binom{r}{2}} \sum_{i_1, i_1', \ldots, i_r, j_r} \prod_{k=1}^r A_{i_k j_k} g_{i_1} \wedge \cdots \wedge g_{i_r} \wedge h_{j_1} \wedge \cdots \wedge h_{j_r} \]

\[
= (-1)^{\binom{r}{2}} \sum_{I, J} \left( \sum_{\pi, \rho \in S_r} \text{sgn}(\pi) \text{sgn}(\rho) \prod_{k=1}^r A_{i_{\pi(k)} j_{\rho(k)}} \right) g_I \wedge h_J \]

\[
= (-1)^{\binom{r}{2}} r! \sum_{I, J} \text{det}(A_{(I|J)}) g_I \wedge h_J, \quad (6.67)
\]

where the sign factor on the third line is obtained by moving the \(h_j\) factors to the right, and the sums in the fourth and fifth lines are over all pairs of subsets \((I, J)\) of \{1, 2, \ldots, N\} of cardinality \(r\).

On the other hand, since 2-forms commute, the binomial formula implies

\[
\omega^{\wedge r} = 2^r (\omega_1 - \omega_2)^{\wedge r} = 2^r \sum_{m=0}^r (-1)^{r-m} \binom{r}{k} \omega_1^{\wedge m} \wedge \omega_2^{\wedge (r-m)} = 2^r r! \sum_{m=0}^r (-1)^{r-m} \frac{\omega_1^{\wedge m}}{m!} \wedge \frac{\omega_2^{\wedge (r-m)}}{(r-m)!} \]

\[
= 2^r r! \sum_{m=0}^r \left( \sum_{|K|=2m} \text{Pf}(A_{(K|K)}) e_K \right) \wedge \left( \sum_{|L|=2(r-m)} (-1)^{|L|/2} \text{Pf}(A_{(L|L)}) f_L \right) \]

\[
= 2^r r! \sum_{(K, L) \text{ even}} (-1)^{|L|/2} \text{Pf}(A_{(K|K)}) \text{Pf}(A_{(L|L)}) e_K \wedge f_L, \quad (6.68)
\]

where the last sum is over all pairs of subsets \((K, L)\) of \{1, 2, \ldots, N\} of even cardinalities \(k\) and \(l\) with

\[
k + l = 2r. \quad (6.69)
\]

We therefore have the identity

\[
\text{det}(A_{(I|J)}) = (-1)^{\binom{r}{2}} 2^r \sum_{(K, L) \text{ even}} (-1)^{|L|/2} \text{Pf}(A_{(K|K)}) \text{Pf}(A_{(L|L)}) (g_I \wedge h_J)^* \omega(e_K \wedge f_L) \quad (6.70)
\]

and Lemma 6.7 implies

\[
\text{det}(A_{(I|J)}) = \left( -\frac{1}{2} \right)^{\binom{r}{2}} \sum_{(K, L) \text{ even}} (-1)^{\binom{r}{2}} \Delta_{(I|J)(K,L)} \text{Pf}(A_{(K|K)}) \text{Pf}(A_{(L|L)}). \quad (6.71)
\]

Eq. (6.23) then follows from the fact that \(\Delta_{(I|J)(K,L)}\) vanishes unless condition (6.9) holds.

\[\square\]
Remark 6.5. If the cardinalities of $I$ and $J$ are different, the same calculation that leads to (6.23) yields the vanishing quadratic relations

$$
\sum_{L, |L| \text{ even} \atop K = ((I \cup J) \setminus L) \cup (I \cap J)} (-1)^{|L|/2} \Delta_{(IJ|KL)} \text{Pf}(A_{(K|K)}) \text{Pf}(A_{(L|L)}) = 0,
$$

(6.72)

which are satisfied by the Pfaffians of principal minors of any skew $N \times N$ matrix $A$, valid for any pair $(I, J)$ of different cardinality. In fact, this is nothing but another way to express the Cartan relations (1.18).

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References

[1] E. Cartan, The Theory of Spinors, Dover Publications Inc, Mineola N.Y., 1981.

[2] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, “Transformation groups for soliton equations IV. A new hierarchy of soliton equations of KP type”, Physica 4D 343-365 (1982).

[3] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, “Transformation groups for soliton equations”, In: Nonlinear integrable systems - classical theory and quantum theory, World Scientific (Singapore), eds. M. Jimbo and T. Miwa (1983).

[4] L. A. Dickey, “Soliton Equations and Hamiltonians Systems”, 2nd ed. Advanced Series in Mathematical Physics 26, World Scientific (new Jersey, Lodon, Singapore) (2003).

[5] W. Fulton, Young Tableaux. London Math. Soc. Student Texts 35, Cambridge Univ. Press, Cambridge U.K., 1997.

[6] J. Harnad and F. Balogh, “Tau functions and their applications”, Monographs on Mathematical Physics series, Cambridge University Press (in press, 2020).

[7] P. Griffiths and J. Harris, Principles of algebraic geometry, Chapt. I.5 Wiley-Interscience, John Wiley and Sons, New York, 1978.

[8] J. Harnad and S. Shnider, “Isotropic geometry and twistors in higher dimensions I. The generalized Klein correspondence and spinor flags in even dimensions,” J. Math. Phys. 33, 3191-3208 (1992);

[9] J. Harnad and S. Shnider, “Isotropic geometry and twistors in higher dimensions II. Odd dimensions, reality conditions and twistor superspaces”, J. Math. Phys., 36 1945-1970 (1995).

[10] M. Jimbo and T. Miwa. “Solitons and infinite-dimensional lie algebras”, Publ. Res. Inst. Math. Sci., 19 943-1001 (1983)
[11] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, (1995).

[12] Y. Ohta, “Bilinear theory of solitons with Pfaffian labels”, *Kokyuroku, RIMS*, no. 822, 197-205 (1993).

[13] M. Sato. “Soliton equations as dynamical systems on infinite dimensional Grassmann manifold” *Kokyuroku, RIMS* 30-46, (1981).

[14] Y. Shigyo, “On Addition Formulae of KP, mKP and BKP Hierarchies”, SIGMA 9 035 (2013).

[15] Y. Shigyo, “On the expansion coefficients of Tau-function of the BKP hierarchy”, *J. Phys. A* 49 295201 (2016).

[16] G. Segal and G. Wilson, “Loop groups and equations of KdV type ”, *Publ. Math. IHÉS* 6, 5-65 (1985).

[17] Y. You, “Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups”, in: *Infinite-Dimensional Lie Algebras and Groups*, *Adv. Ser. Math. Phys.* 7 (1989). World Sci. Publ., Teaneck, NJ.