Motivated by recent experimental progress in the field of dipolar-Fermi gases, we investigate the quantum phases of dipolar fermions, on a triangular ladder at half filling. Using density matrix renormalization group method, in presence of onsite repulsion and intersite attractive interaction, we find exotic spin-triplet superfluid phase in addition to the usual spin-density and charge-density waves. We examine the stability of spin-triplet superfluid phase by varying hopping along the rungs of the triangle. Possibility of fermionic supersolidity has also been discussed, by considering three-body interaction in the Hamiltonian. We also study the effect of spin-dependent hopping on the stability of spin-triplet superfluid phase.

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I. INTRODUCTION

Recent experimental advancements in the field of dipolar Fermi gases have given opportunity to explore the quantum phases of strongly correlated fermionic systems with long-range interactions. The dipolar Fermi gas of $^{161}$ Dy and fermionic polar molecules, $^{40}$ K $^{23}$ Na $^{40}$ K, with large dipole moments have experimentally been realized in optical lattices. It has been found that the external electric and microwave fields on optical lattices can control quantum many body interactions parameters of dipolar systems and polar molecules. It has been argued that the long range and anisotropic characters of the dipolar interactions, in fact, can provide various types of exotic phases like, charge-density wave (CDW; even though the density modulation is produced by charge neutral atoms or molecules, it is called CDW in the literature) spin density wave (SDW; spin order for pseudo-spin-1/2 of dipolar fermions, shown in schematic of Fig. 2(a) liquid-crystal lower conventional and unconventional fermionic superfluids, to name a few.

Finding phases, like, triplet superfluidity and triplet superconductivity are always very challenging and interesting too as these exotic phases have connection to a number of topological phases and quantum computation. Interestingly, at low temperature, liquid $^{3}$ He forms fermionic superfluids, where $^{3}$ He atoms (or quasi particles) form pairs with p-wave symmetry in spin triplet state. Chromium based quasi-one dimensional superconductors and strontium based oxide, Sr$_2$RuO$_4$, are considered to be good candidates for triplet pairing.

Interestingly, ultra cold dipolar systems, offer intriguing possibilities to explore unconventional pairing mechanisms of the condensed-matter system. For single component fermions, a dominant $p_z$-wave superfluidity has been proposed. For two components fermions, it has been shown that there is possibility of formation of both singlet and triplet superfluidity, as both singlet and triplet pairing are allowed in such systems. In two-dimensional dipolar fermionic system, where dipoles are aligned with external electric field, it has been shown that p-wave superfluidity can be realized by varying anisotropy and geometry of the system. Unconventional spin-density waves and bond-order solids have also been shown for the two-dimensional dipolar systems.

On the other hand, more exotic phases, like, supersolid phase, has been proposed for dipolar Fermi gas in a cubic optical lattice system. Interestingly, in this, it has been shown that a p-wave superfluid is formed due to attractive interaction along the z-direction, and charge-density wave in the XY-plane due to electronic repulsions and together with the intermediate values of dipolar interactions. For a two dimensional dipolar Fermi gas, coexistence of density-wave and p-wave superfluidity has been shown. In a recent experimental study on ultra-cold three dimensional optical lattice systems, effect of multi body interaction has been demonstrated. Furthermore, in a few numerical studies, it was shown that dominant three body Coulombic interactions can give rise to a host interesting phases, like supersolid and bond-order phases. Interestingly, for polar molecules in optical lattice, the realization of three-body interactions using microwave field have been proposed, and since then there have been various theoretical studies of microscopic models with three-body interaction. These studies have shown that, with three body Coulombic interactions, the ground state can be quite exotic displaying quantum phases like, topological phases, spin liquids, fractional quantum Hall states etc.

Quasi one dimensional systems are quite unique. Due to strong quantum fluctuations, the true long range order is not possible for continuous symmetry breaking phase. In a one dimensional optical lattice, bosonization study has shown triplet superfluid (TSF) phase for dipolar fermion. TSF phase is also found in two coupled one dimensional systems for quadrupolar Fermi gas. Interestingly, mixture of triplet and singlet superfluidity has also been shown in quasi-one dimensional system with two component fermions. A recent DMRG study has also found the TSF phase in a one-dimensional dipolar Fermi gas. In presence of attractive head to tail arrangement of dipolar interactions, the one
and two dimensional dipolar fermions become unstable and they undergo either collapse or phase separation. To overcome these difficulties, bilayer system has been proposed, where dipoles are aligned perpendicular to the layers, giving more stable paired phase.\cite{27,28}

In this article, we consider dipolar fermions in a triangular ladder system at half-filling. We study the stability of various exotic phases, like, spin-density wave, charge density wave and triplet-superfluid phases. In the ladder, the dipolar fermions are considered to be polarized along the rungs of the triangles (as shown in schematic of Fig.1). The strength and direction of polarization can be controlled by external electric field or by varying distance between lattice sites. Due to alignment of dipolar fermions along the rungs, attractive interaction is generated on alternative rungs (odd rungs). It is also possible to generate repulsive interaction in each of the chains and diagonal rungs of triangle, by alignment of dipoles. In the presence of attractive dipolar interaction and on-site Hubbard repulsion, a stable TSF phase gets generated. We have checked the stability of the TSF phase thoroughly, by tuning in the inter-chain hopping strength and the repulsive interaction parameters. Additionally, We have also examined the effect of spin-dependent interchain hopping on the stability of the TSF phase. Interestingly, due to triangular geometry, three-body interactions can also play important role in identifying new quantum phase, like, fermionic super-solid phase of dipolar fermions.\cite{40}

The remaining part of the article is organized as follows. In sec.II we have discussed the model Hamiltonian and the method used to solve it. Subsequently, we have discussed the results obtained from DMRG calculations. This is divided into four subsections, where in each subsection the details of phase and phase transition is discussed. In last section, we have summarized all our results.

II. THE MODEL

We consider two-component (pseudo-spin-1/2) dipolar fermions on a two-leg triangular ladder at half-filling.

The effective Hamiltonian of the system can be written as,

\[ H = -\sum_{\sigma,i} \left( t c_{\sigma,i}^\dagger c_{\sigma,i+2} + t' c_{\sigma,i}^\dagger c_{\sigma,i+1} + H.c. \right) + U \sum_i \tilde{n}_{i,\uparrow} \tilde{n}_{i,\downarrow} + \sum_{\langle i \neq j \rangle} V(i,j) \tilde{n}_i \tilde{n}_j - W \sum_i \tilde{n}_{i+1,\uparrow} \tilde{n}_{i,\downarrow} \]

where \( c_{\sigma,i} \) is annihilation operator with spin \( \sigma = \uparrow, \downarrow \) at site \( i \). Here \( \uparrow \) and \( \downarrow \) states refer to two hyperfine states of dipolar atoms or molecules. \( \tilde{n} = (n - \langle n \rangle) \) where \( n \) is the number operator and \( \langle n \rangle = 1 \). \( t \) and \( t' \) are the hopping terms and \( U \) is the onsite interaction term between the fermion with opposite spins; \( V(i,j) \) is the two-body nearest-neighbour intersite interaction term. The last term in the Hamiltonian, \( W \), represents attractive three body interactions between the fermions, which act on the fermions belonging to the same triangle (as shown in the Fig.1). The two-body interaction term depend on direction and distance between the dipoles. When the two dipoles are parallel to each other, the interaction becomes repulsive, while when they align to each other along the rungs, interaction become attractive. The most dominating interactions arise from the nearest-neighbour terms\cite{42,43} and also in optical lattice by adjusting the distance between sites, one can make other subdominating interactions quite small.\cite{43} Thus, we restrict ourself to only nearest-neighbour terms of \( V(i,j) \) in the Hamiltonian.\cite{41} The two body nearest-neighbour term, \( V(i,j) \), can be described as

\[ V(i,j) = \begin{cases} V_r & \text{Intersite repulsive term on each chain.} \\ V_d & \text{Intersite repulsive term for even rungs.} \\ -V_a & \text{Intersite attractive term for odd rungs.} \end{cases} \]

Since the dipolar interaction depends on angle and distance between the dipoles, it allows tuning of magnitude and sign of these interaction parameters to a wide range to explore rich quantum many-body phases. The dipolar interactions can be tuned by external electric field or changing the distance between sites. The above Hamiltonian preserves \( U(1) \) and \( SU(2) \) symmetry, which is related to conservation of total charge and spin degrees of freedom. Note that, for nonzero next nearest neighbor terms, \( t \) and \( W \), the Hamiltonian does not have particle-hole symmetry.

To solve the above Hamiltonian and to find quantum phases in the parameter space, we have used density-matrix renormalization group (DMRG)\cite{42,43,44} method. We have used open boundary conditions and vary the DMRG cut-off (max = \( n \)) from 300 to 600, for consistency in results. Most of the results presented in the article are obtained using max=520, unless otherwise stated. To calculate the error, we have checked the truncation error, \( e = 1 - \sum_i \rho_i \), where \( \rho_i \) is the eigenvalues corresponding to the reduced density matrix. We found that depending upon the interaction parameters and system size, truncation error \( e \) varies from \( 10^{-5} \) to \( 10^{-4} \). We have verified energy and excitations for some parameters with those
from exact diagonalization for smaller system sizes. To characterize different phases, namely SDW, TSF, and CDW phases, we have calculated corresponding correlation functions and also spin and charge density profiles. For showing plots of correlation functions, unless stated explicitly, we have considered system size \( L = 128 \). To determine phase boundary between different phases and to minimize the finite size effect, we have done finite-size scaling of order-parameters, of the system with size \( L \) up to 160.

### III. RESULTS

#### A. SDW to TSF to CDW transition

![Schematic of the SDW, TSF and CDW phases on a triangular lattice](image)

![Plot of spin-density \( \langle s^z_i \rangle \) with site index \( i \), for \( V_a = 1.6 \) (triangle) and \( V_a = 2.5 \) (star).](image)

![Plot of charge density \( \langle n_i \rangle \) for \( V_a = 2.4 \) (star) and \( V_a = 3.2 \) (square).](image)

FIG. 2. (a) Schematic of the SDW, TSF and CDW phases on a triangular lattice (here arrows indicate electronic spins of fermions). (b) Plot of spin-density \( \langle s^z_i \rangle \) with site index \( i \), for \( V_a = 1.6 \) (triangle) and \( V_a = 2.5 \) (star). (c) Plot of charge density \( \langle n_i \rangle \) for \( V_a = 2.4 \) (star) and \( V_a = 3.2 \) (square).

We first consider a simple case, where \( t' = 0 \), the intersite repulsive dipolar term, \( V_r = 0 \), \( V_d = 0 \) and three body term, \( W = 0 \). Due to long range of dipolar interactions, two chains of triangular ladder can couple through attractive dipolar interaction, \( V_a \), even though the tunneling between the chains remain zero. For finding TSF phase, we take onsite Hubbard interaction \( U = 2 \), and vary the attractive interaction, \( V_a \) (0 to 4), along the rungs (odd rungs). For \( U = 2 \) and lower values of \( V_a \), we find that to minimize repulsive onsite interaction, fermions stay put in each site and form spin density wave, \( \uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \downarrow, \uparrow \ldots \) (as shown in schematic of Fig.2(a)). In order to show spin density profile of the system, in Fig.2(b), we have plotted spin-density \( \langle s^z_i \rangle \) of system, with site index, \( i \). With increase in attractive interaction, \( V_a \), the fermions form intersite pairs along the rungs of the ladder, where the electronic spins form triplet symmetry \( \langle s^z = 0 \rangle = | \uparrow \downarrow \rangle + | \downarrow \uparrow \rangle \)62. This phase remains so for moderate values of \( V_a \). For large value of attractive interaction, fermions with up and down spin prefer to sit together and form CDW-phase, where the state appears like, \( | \uparrow \uparrow, \uparrow \downarrow, \downarrow \uparrow, \downarrow \downarrow, 0, 0, 0, 0 \ldots \) (as shown in schematic of Fig2(a)). To show this, in Fig.2(c), we have plotted charge density profile of fermions, \( \langle n_i \rangle \), with site index \( i \). Interestingly, this CDW-phase appears even without any intersite-repulsive terms. Thus is precisely due to the triangular geometry and the attractive interaction along leg-direction62. However, in strictly one dimensional case, for large values of attractive interaction, the system goes to either phase-separated phase or it collapses60.

In order to characterize SDW, TSF and CDW phases and their boundaries, we vary \( V_a \) with fixed value of \( U = 2 \), and we look into the behavior of corresponding correlation functions. For SDW phase, we have calculated correlation function, \( S(r) = \langle s^z_i s^z_{i+r} \rangle \), where \( r \) (even distances) is the distance from the middle site of the ladder to the one end of the ladder. We found that with increase in \( r \), fluctuations appears in the correlation function (Appendix Fig.15). To reduce these fluctuations, we have calculated average correlation function, \( S(r) = 1/N(r) \sum_i \langle s^z_i s^z_{i+r} \rangle \). Here, we have summed over the all correlations, which are separated by the same distance \( r \) from sites \( i \) and divided by the numbers \( N(r) \) of such same distances correlations62. As shown in Fig.3(a), for lower values of \( V_a \), the correlation function, \( S(r) \), decays algebraically, while for larger values of \( V_a \gtrsim 2.0 \), it decays exponentially.

![Plot of correlation function \( S(r) \), (b) correlation function \( P(r) \), for \( U = 2 \) and varying \( V_a < 2.3 \). (c) Plot of correlation function \( P(r) \), (d) correlation function \( C(r) \), for \( U = 2 \) and varying \( V_a \) (2.3 to 2.9).](image)

FIG. 3. (a) Plot of correlation function \( S(r) \), (b) correlation function \( P(r) \), for \( U = 2 \) and varying \( V_a < 2.3 \). (c) Plot of correlation function \( P(r) \), (d) correlation function \( C(r) \), for \( U = 2 \) and varying \( V_a \) (2.3 to 2.9).

With increase in attractive attraction along the rungs of the triangle, interchain fermions form bound pairs.
along the rung, giving rise to interchain spin-triplet superfluid phase, which is quite interesting. In general, the TSF phase can be characterized by pair correlation function\textsuperscript{55} \( P(r) = \langle \Delta_{i}^{\dagger} \Delta_{i+r} \rangle \), where \( \Delta(l) = \left( c_{i_{\downarrow}}^{\dagger} c_{i+1_{\uparrow}}^{\dagger} + C_{i_{\uparrow}}^{\dagger} c_{i+1_{\downarrow}}^{\dagger} \right) \), creates a fermionic pair in spin triplet state on a rung (labeled \( l \)) and \( r \) (even distance) is the distance from the rung \( l \) (near to the center of the triangular ladder). This correlation function \( P(r) \), is also called \( p_{z} \) wave like superfluid correlation function, because of spin triplet pairing along \( z \) direction. For \( P(r) \) also, fluctuations appear with increase in \( r \). To smooth out these fluctuations, we have calculated average correlation function, \( P(r) = 1/N(r) \sum_{n} \langle \Delta_{i}^{\dagger} \Delta_{i+r} \rangle \), where we have summed over the correlations which are separated by same distances \( r \) from rung \( l \), divided by the numbers, \( N(r) \), with such same distances correlations.

To characterize the phase boundary accurately between SDW and TSF phases, we have calculated the exponent of the correlation function, \( S(r) \). The exponent, \( K \), can be obtained by fitting the correlation function with algebraic decay function of the form, \( S(r) \sim \cos(2kp_{F}r)/(1/r)^{1+K} \) (as shown in Fig. 4(c))\textsuperscript{55,57}. To get rid of short range correlation functions and the finite size effects, we have fitted the correlation function, \( S(r) \), from distance \( r=10 \) to 70, for system size length \( L=160 \). We find that the correlation function, \( S(r) \), fits very well in the SDW phase, however, near the phase boundary close to the TSF phases, the fitting error increases. From Luttinger liquid theory, for \( K < 1 \), the SDW phase dominates, while for \( K > 1 \), the TSF phase dominates\textsuperscript{54,57}. The transtion point for SDW to TSF phase is expected to be at \( K = 1 \). As shown in the Fig. 4(b), at \( V_a = 1.9 \pm 0.06 \), the exponent \( K \) of the correlation function \( S(r) \) takes the value \( K \sim 1 \), which signifies the transition from SDW phase to TSF phase.

To characterize CDW-phase, we have calculated correlation function, \( C(r) = \langle (n(i) - \bar{n})(n(j) - \bar{n}) \rangle \), where \( r \) is the distance from middle site of the ladder to other on one side of the ladder. As shown in Fig.3(d), the correlation function, \( C(r) \), for \( V_a > 2.5 \) has nearly long range order, while \( P(r) \) decays exponentially (as shown in Fig.3(c)). Thus, for \( V_a > 2.5 \), the system is in the CDW phase. To calculate the phase boundary between TSF and CDW phase, we have done finite size scaling of order-parameter, \( O_p = 1/L \sum_{r=1}^{L} |C(r)| \). In the density wave phase order-parameter \( O_p \), takes non-zero values in the thermodynamic limit\textsuperscript{55}. To obtain the thermodynamic value of \( O_p \), we have done finite-size scaling for systems with length \( L \) up to 160, by fitting the finite-size \( O_p \) values with a function, \( O_p + O_{1}/L + O_{2}/L^{2} \). As shown in the Fig.4(a), TSF to CDW transition occurs at \( V_a = 2.55 \pm 0.05 \) as \( O_p \) takes finite non-zero values for \( V_a = 2.55 \pm 0.05 \).

As shown in schematic of Fig.4(d), for fixed values of onsite interaction, \( U = 2 \) and by varying \( V_a \), we found SDW phase for \( V_a \lesssim 1.9 \), TSF phase for \( 1.9 \lesssim V_a \lesssim 2.55 \) and CDW-phase for \( V_a \gtrsim 2.55 \).

**B. Effect of Onsite Repulsive Interaction**

![Fig. 4](image-url)  
**FIG. 4.** Finite-size scaling of (a) order parameter \( O_p \) (b) exponent \( K \) of the correlation function \( S(r) \), at \( U = 2 \) and different values of \( V_a \). (c)Power law fitting of \( S(r) \) at \( V_a = 1.6 \), on a log-log scale for system size \( L = 128 \). (d) phase diagram for fixed value of \( U = 2 \) with varying \( V_a \).

![Fig. 5](image-url)  
**FIG. 5.** (a) Plot of correlation function \( C(r) \), (b) correlation function \( P(r) \), for \( V_a = 1.8 \) and varying \( U < 1.5 \). (c) Plot of correlation function \( P(r) \), (d) correlation function \( C(r) \), for \( V_a = 1.8 \) and varying \( U \) (1.5 to 3.0).

To find the role of onsite interaction, \( U \), in the triplet pairing and formation of other phases, we varied the \( U \) values from \((U = 0.0 \) to 3.0\)), for fixed values of attractive interaction \( V_a = 1.8 \). As shown in Fig.5(a) and Fig.5(b), initially for lower values of \( U \), the correlation function, \( C(r) \), shows nearly long range order, while \( P(r) \) decays exponentially, indicating CDW phase in the system. On
the other hand, for $U \gtrsim 1.1$, the correlation function, $P(r)$, shows algebraic decay behaviour, displaying TSF phase in the system. To find out the phase boundary between the CDW and TSF phase, we have done finite size scaling of order-parameter $O_p$. As shown in Fig.6(a), $O_p$ takes finite non-zero values for $U = 1.1 \pm 0.05$, indicate transition from CDW phase to TSF phase.

As shown in Fig.5(c) and Fig.5(d), with increase in $U$, initially $P(r)$ shows power law behaviour, while $S(r)$ decays exponentially. On the other hand, for large values of $U$, $S(r)$ shows power law behaviour, while $P(r)$ decays exponentially. For moderate values of $U$, TSF and SDW phases compete with each other. To find the phase boundary between TSF and SDW phase, we have done finite size scaling of exponent of correlation function $S(r)$, as discussed in previous section. Fig.6(b), shows transition from TSF to SDW phase at $U = 1.9 \pm 0.06$, as exponent of $S(r)$, takes the value $K = 1$. As shown in schematic of Fig.6(d), we find CDW-phase for $U \lesssim 1.1$, TSF phase for $1.1 \lesssim U \lesssim 1.9$ and SDW phase for $U \gtrsim 1.9$, for a fixed value of attractive interaction, $V_a = 1.8$

![FIG. 6. Finite-size scaling of (a) order parameter $O_p$ (b) exponent $K$ of the correlation function $S(r)$, at $V_a = 1.8$ and different values of $U$. (c) Power law fitting of $S(r)$ at $U = 2.2$, on a log-log scale, for $L = 128$. (d) phase diagram for fixed value of $V_a = 1.8$ with varying $U$.](image)

C. Effect of inter chain hopping

Here, we study the effect of inter chain hopping, $t'$ on the triangular ladder. We find that, as the interchain hopping is turned on, the SDW phase becomes unstable and disappears quickly with increase in $t'$. On the other hand, TSF phase becomes prominent with nonzero $t'$ values, however, as the $t'$ becomes larger, the prominence decreases. The spin triplet pairs formed due to $V_a$ term along the rung, gets higher stability with introduction of $t'$, as it promotes the antiferromagnetic exchange between the electrons on the rungs. This results in increase in pair-correlation, $P(r)$. Interestingly, for large values of attractive interaction, $V_a$, when the system is in the CDW-phase, it gets hardly affected by inter chain hopping term, as the charge ordered state arrests the effective hopping between the chains. However, close to the phase boundary between TSF and CDW phases, when the system is near the CDW phase boundary, for finite values of $t'$, system can again make transition to the TSF phase.

Now, using DMRG, we demonstrate the effect of $t'$ by considering two values of $V_a$, 1.6 and 2.8, and for a fixed value of $U = 2$. These $V_a$ values correspond to SDW and CDW phases respectively, without any inter chain hopping term, $t'$. As we turn on $t'$, we look at the variation in $SDW$ and $CDW$ phases. As shown in Fig.7(a), for $V_a = 1.6$, the spin-spin correlation function, $S(r)$, starts decaying exponentially for $t' \gtrsim 0.1$ (Fig.7(a)), whereas, the pair correlation function, $P(r)$, initially increases with $t'$, for even small values of it. It clearly shows that the system makes transition from SDW phase to TSF phase in presence of interchain hopping $t'$. On the other hand, as we increase the $t'$ value, for larger values of $t'$ ($t' \sim t$), the pair correlation function, $P(r)$, starts decreasing (Fig.7(b)). Interestingly, there the system show a density profile, $\langle n(i) \rangle$, which is oscillatory in nature (as shown in the inset of Fig.7(a), for $t' = 1.2$ and 1.6). In fact, at very large values of $t'$ ($t' \gtrsim 2.0$), the system enters into a metallic phase, where the density becomes homogeneous and takes values around one (see inset of
We find that the CDW-phase is quite robust against the interchain hopping term, \( t' \). As shown in Fig.8(a), the charge-charge correlation function, \( C(r) \), shows nearly long range order for \( t' \lesssim 0.5 \). On the other hand, as shown in Fig.8(b), the pair correlation function, \( P(r) \), decays exponentially for lower values of \( t' \lesssim 0.5 \), while shows powerlaw behaviour for \( t' > 0.5 \). Such behavior of the correlation functions indicate a phase transition from CDW-phase to TSF phase for \( t' \approx 0.5 \pm 0.05 \). For moderate values of \( t' \), the \( P(r) \) shows power law behaviour, while for larger values of \( t' \gtrsim 1.2 \), it starts decaying exponentially and the system again enters into a density wave phase. For large values of \( t' (t' \sim 2.0) \), the density wave phase enters into a metallic phase. For \( V_a \gtrsim 3 \), the CDW-phase is quite stable and it requires a really large values of \( t' \) to destroy the CDW-phase.

**D. Effect of Intersite Repulsive Interactions**

When the dipolar fermions are aligned along the rungs of the triangle, repulsive interactions can be generated along each chain direction (\( V_r \)) as well as along the diagonal (\( V_d \)) of the triangular ladder (as shown in schematic Fig.1). For demonstrating the effect of repulsive interactions, \( V_r \) and \( V_d \), we chose interaction parameters \( U = 2, t' = 0.4, V_a = 1.8 \) and vary the intersite repulsive parameters, \( V_r \) and \( V_d \). As discussed in the previous section, in absence of repulsive intersite interactions, for these parameter values, the system remains in the TSF phase. On the other hand, with increase of intersite repulsive interactions, the fermions try to avoid each other and form a CDW state with structure, like \( |2, 0, 0, 2.. \rangle \).

In the presence of attractive interaction, \( V_a \), along the rungs of the triangles, the fermions in each of the chain become correlated with each other. We also found that in presence of \( V_d \), small values of repulsive interaction \( V_r \) is enough to produce a CDW-phase. As shown in Fig.10(a), the pair correlation function, \( P(r) \), shows power law behaviour up to \( V_r \approx 0.24 \), while for larger values of \( V_r \), it decays exponentially. On the other hand, the charge charge correlation function \( C(r) \), shows nearly long range behaviour for \( V_r \gtrsim 0.24 \) (Fig.10(b)). To find the phase boundary, we have done finite size scaling of order-parameter, \( O_p \). As shown in the inset of Fig.10(a), \( O_p \) takes finite value for \( V_r = 0.24 \pm 0.02 \), which clearly shows the phase transition from the TSF phase to the CDW phase at \( V_d = 0.75 \pm 0.06 \).
E. Effect of Three-body interaction

Due to triangular geometry and dipolar interactions, an additional three-body interaction term may appear in each of the triangular plaquette, as suggested by others on similar grounds. Three body term can break the particle hole symmetry of the Hamiltonian. In optical lattices, the three body and two body interactions can be tuned independently. Here, we demonstrate the consequences of attractive three body interaction, along with two body interactions and ask whether the three body term can generate new phases or combine several phases. To show the effect of three body interactions, we choose the system parameters, $U = 2.0$, $V_a = 1.8$, $V_d = 0.3$, $V_e = 0.1$ and $t' = 0.4$ and varied the $W$. Without $W$ term, the system exists in TSF phase for these parameters. As we turn on the attractive three body interaction, $W$, both TSF and CDW phases coexist and the system remains so up to moderate values of $W$.

As shown in the Fig.11, triplet pair correlation function, $P(r)$ with increase in $W$, shows power law behavior, with slight changes in exponent. Additionally, with increase in $W$, a periodic modulation appeared in the charge charge correlation function, $C(r)$. To see the appearance of CDW order in the thermodynamic limit, we have done finite size scaling of order-parameter, $O_p$. As shown in inset of Fig.12, $O_p$ takes finite nonzero values for $W = 0.6 \pm 0.1$. Periodic modulation in density correlation, $C(r)$, and algebraic decay of $P(r)$, give signature of fermionic supersolid phase in the system for $0.6 \lesssim W \lesssim 1.7$, where both CDW and TSF phases coexist. This supersolid phase is different from the supersolid phase formed due to coexistence of onsite pairing of fermions (s-wave superfluid), and charge density wave of the system. Here, fermions form pairs in spin-triplet state ($p_\uparrow$-wave superfluid), which coexist with CDW phase of the system. For large values of $W \gtrsim 1.7$, the system becomes unstable and thereby become phase separated. In the phase separated state, density distribution is inhomogeneous, while correlation function, $P(r)$ decay exponentially. Note that, in the phase separated state, there is generally convergence problem, which we found for $W > 2.0$. In the inset of Fig.11, plot of charge density profile $\langle n_i \rangle$ has been shown for $W = 1.8$, with site index $i$ (also see in Appendix Fig.16, the plot of $\langle n_i \rangle$ for different values of $W$).

F. Effect of spin-dependent hopping

In this section, we analyze the effect of spin dependent hopping on the TSF phase. We apply spin dependent hopping along the rungs of the triangle. We considered up-spin hopping term to be more stronger than the down-
spin hopping term The corresponding change in hopping term in the Hamiltonian can be written as

\[ H_{ts} = \sum_i \left( t_i^+ c_{i+1,\uparrow} + h.c. \right) + \left( \alpha t_i^+ c_{i+1,\downarrow} + h.c. \right) \]

where, \( \alpha \) is an anisotropic term (\( \alpha = 1 \) make the Hamiltonian same as Eq.1). Spin dependent hopping term breaks the spin rotational symmetry, \( SU(2) \) and also the time reversal symmetry of the Hamiltonian\[24\]. As the \( SU(2) \) symmetry is broken, ground state is no more in \( s^z_{\text{tot}} = 0 \) sector, while the number sector is still fixed. In such a situation, we have checked our DMRG results with those from exact diagonalization results with the same setup for smaller system sizes. As the results compare fairly well, we have set up DMRG calculations with fixed number of particles without considering \( s^z_{\text{tot}} \) quantum number. Since the matrix dimension in each of the DMRG iteration increases quite considerably (\( \sim 10^6 \)), we have carried out DMRG calculations with \( \text{max}=150 \) and for system length, \( L = 96 \). We have verified the results for \( \alpha = 1 \) by running DMRG calculations with \( s^z_{\text{tot}} = 0 \) and without considering \( s^z_{\text{tot}} \) quantum number upto \( L = 96 \) with \( \text{max}=450 \) and found the results compare quite well. We thus have carried out DMRG calculations with the parameters, \( U = 2, V_a = 2, V_\tau = 0.1, V_d = 0.2, t' = 0.4 \) with varying \( \alpha \) values. Note that, for these parameter values with \( \alpha = 1 \), the system is known to be in TSF phase (see Fig.13(a)).

With spin-dependent hopping, we find that the TSF phase is suppressed, while the SDW phase starts dominating. As shown in Fig.13(a), the pair correlation function, \( P(r) \), decays algebraically for \( \alpha \gtrsim 0.6 \pm 0.1 \), showing clearly that the TSF phase is sustained by spin dependent hopping, while for \( \alpha \lesssim 0.6 \pm 0.1 \), the pair correlation decays exponentially. With lower values of \( \alpha \), spin-spin correlation function, \( S(r) \), has nearly quasi long range order for \( \alpha \lesssim 0.6 \pm 0.1 \) (Fig.13(b)). In the inset of Fig.13(a), we show spin density profile, \( \langle s_i^z \rangle \) with site index \( i \). As can be seen, the \( \langle s_i^z \rangle \) takes finite values for \( \alpha = 0.4 \), however it vanishes for \( \alpha = 1.0 \). For lower values of \( \alpha \), the down-spin becomes reluctant to hop between legs of triangle, thus promoting SDW phase while suppressing TSF phase in the ladder system.

IV. CONCLUSION

In summary, we have investigated the SDW, TSF and CDW phases of dipolar fermions, at half filling, on a triangular ladder. In presence of moderate values of repulsive onsite interaction and attractive intersite interactions, the fermions form exotic spin triplet superfluid phase. In presence of intersite attractive interactions, and onsite repulsive interaction, a charge density wave phase is found even without any intersite repulsive interactions. We have demonstrated the stability of spin triplet phase, by introducing inter leg hopping, which effectively enhances the spin triplet superfluid phase region by replacing the spin density wave phase. In presence of repulsive interactions, we show transition between TSF phase and a CDW phase. We also have looked at the effect of three body interactions on the TSF and CDW phases. We find that the three body term can introduce a fermionic supersolid phase, where both TSF and CDW coexist. We strongly believe that our study, which unravel the rich physics of exotic phases of dipolar-fermionic systems in ultra-cold systems would show inroads for further experiments.

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VI. APPENDIX

To check the accuracy of our DMRG calculations, we have calculated truncation error of the system. In DMRG, the effective basis is truncated by keeping the \( m \) largest eigenvectors of the reduced density matrix corresponding to the \( m \) largest eigenvalues. The error caused by the truncation can be measured by calculating \( \epsilon = 1 - \sum_i \rho_i \), where \( \rho_i \) is the eigenvalues corresponding to the reduced density matrix. Fig.14 shows plot of truncation error with max values \( m \), for system size \( L = 128 \) and for interaction parameters values \( U = 2, V_a = 1.8, \) keeping all the other parameters, \( t', V_\tau, V_d \) and \( W \), as zero. With increase in max value \( m > 420 \), truncation error changes very slowly.
To check the behaviour of correlation function, $P(r)$, with max values, we have calculated the correlation function with different max values (Fig.15). As shown in inset of Fig.15, $P(r)$ almost overlaps for $m = 450$ and $m = 520$. This proves that $m$ value of 450, is large enough to obtain accurate correlation function, $P(r)$.

We have used open boundary condition for our calculations in DMRG. To remove the edge effects, we have computed correlation functions from central site to one side of the triangular ladder. In case of correlation functions $S(r)$ and $P(r)$, we found that with increase in distance $r$, rapid fluctuations appeared in correlation functions. As shown in Fig.16, to smoothen these fluctuations, we have calculated average correlation function, $S(r) = 1/N(r) \sum_r | \langle s_i^z s_{i+r}^z \rangle |$, where we took the sum over the correlations, which are separated by same distance $r$ from the sites $i$ from the other side of the ladder. This is then divided by the number, $N(r)$, of such same distance correlations. While averaging, we excluded lattice sites within distance $L/4$ from both the end of the ladder (of system size, $L$). We have calculated the average correlation function, $P(r) = 1/N(r) \sum_r | \langle \Delta_{t}^{z} \Delta_{t+r}^{z} \rangle |$, by summing over the correlations, which are separated by same distance $r$ and divide the sum by $N(r)$.

As discussed in section III.E, for large values of $W \gtrsim 1.7$, the system enters into a phase separated state. In Fig.17, the plot of charge density profile $\langle n_i \rangle$, for interaction parameters, $U = 2.0$, $V_a = 1.8$, $V_r = 0.1$, $V_d = 0.3$, $t' = 0.4$ and different values of $W$. For $W = 1.7$, the system shows a periodic density modulation, while for $W = 1.8$, inhomogenous feature appears in the density profile. Interestingly, for $W = 1.8$, the $\langle n_i \rangle$ takes the maximum possible values ($\sim 2$) near the center of the ladder, and for $W = 1.9$, it shifts to one side of the ladder. For $W \gtrsim 2.0$, we found convergence problem.
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