A Note on the Computational Complexity of Unsmoothened Vertex Attack Tolerance

Gunes Ercal

Southern Illinois University Edwardsville

Abstract

We have previously introduced vertex attack tolerance (VAT) and unsmoothened VAT (UVAT), denoted respectively as $\tau(G) = \min_{S \subseteq V} \frac{|S|}{|V - S - C_{\max}(V - S)| + 1}$ and $\hat{\tau}(G) = \min_{S \subseteq V} \frac{|S|}{|V - S - C_{\max}(V - S)|}$, where $C_{\max}(V - S)$ is the largest connected component in $V - S$, as appropriate mathematical measures of resilience in the face of targeted node attacks for arbitrary degree networks. Here we prove the hardness of approximating $\hat{\tau}$ under various plausible computational complexity hypotheses.

1. Definitions and Preliminaries

Given a connected, undirected graph $G = (V, E)$, the Vertex Attack Tolerance of $G$ is denoted by $\tau(G)$ defined as follows:\cite{3,7,2}

$$\tau(G) = \min_{S \subseteq V, S \neq \emptyset} \frac{|S|}{|V - S - C_{\max}(V - S)| + 1}$$

where $C_{\max}(V - S)$ is the largest connected component in $V - S$. As in \cite{2}, we refer to connected, undirected graphs $G = (V, E)$ with more than one node ($|V| \geq 2$) as non-trivial.

Remark 1.1. \cite{2} For nontrivial $G = (V, E)$, $0 < \tau(G) \leq 1$.

VAT was originally introduced as $\hat{\tau}$ (UVAT for “unsmoothened VAT”), of which $\tau$ is a smoothened variation, defined as follows\cite{3,7}:

$$\hat{\tau} = \min_{S \subseteq V, S \neq \emptyset} \frac{|S|}{|V - S - C_{\max}(V - S)|}$$

where $C_{\max}(V - S)$ is the largest connected component in $V - S$. Note that for any graph $G = (V, E)$ such that $G$ is not a clique, the pair of nodes $u, v$
which are not adjacent may be disconnected by attacking all of the other $n-2$ nodes. However, for cliques $K_n$, no such pair exists. Therefore:

**Remark 1.2.** $\hat{\tau}$ is undefined for cliques $K_n$ and defined for all other graphs. Moreover, when $G = (V, E) \neq K_n$, $\hat{\tau}(G) \leq n-2$. Furthermore, when $G = (V, E)$ is connected, $\hat{\tau}(G) > 0$. Therefore, when $G = (V, E)$ is connected and not complete, $S(\hat{\tau})$ is a vertex separator.

For notational convenience: For any graph $G = (V, E)$, and any real function $f$ defined on subsets of $V$, if $h = \min_{S \subset V} f$, we define $h_S(G) = f(S)$ and $S(h(G)) = \operatorname{argmin}_{S \subset V} f(S)$. In particular, when $h$ is a resilience measure on a graph, then $S(h)$ denotes the critical attack set.

We refer to the optimization problem corresponding to computing $\tau(G)$ and $\hat{\tau}(G)$ as simply VAT and UVAT, respectively. It is assumed that any approximation algorithm for UVAT returns a candidate critical attack set that is a valid vertex separator when the input is not a clique (as finding some vertex separator is easy).

The reduction in this work extends the techniques for the NP-Hardness proof for the vertex integrity of co-bipartite graphs presented in [1]. Similarly, our computational hardness results for VAT and other measures involve reductions with the Balanced Complete Bipartite Subgraph problem (BCBS). The BCBS problem is defined as:

**Definition 1.3.** Instance: A balanced bipartite graph $G = (V_1, V_2, E)$ with $n = |V_1| = |V_2|$ and an integer $0 < k \leq n$. Question: Does there exist $A \subset V_1$ and $B \subset V_2$ such that $|A| = |B| = k$ and $(A, B)$ form a $k \times k$ complete bipartite graph?

The maximization version of the problem can be referred to as MAX-BCBS. The following three theorems regard the hardness of approximating MAX-BCBS under various plausible complexity theoretic assumptions:

**Theorem 1.4.** [4] It is NP-hard to approximate the MAX-BCBS problem within a constant factor if it is NP-hard to approximate the maximum clique problem within a factor of $n/2^{\sqrt{\log n}}$ for some small enough $c > 0$.

**Theorem 1.5.** [6] Let $\epsilon > 0$ be an arbitrarily small constant. Assume that SAT does not have a probabilistic algorithm that runs in time $2^{n^c}$ on an instance of size $n$. Then there is no polynomial time (possibly randomized) algorithm for MAX-BCBS that achieves an approximation ratio of $N^{\epsilon'}$ on graphs of size $N$ where $\epsilon' = \frac{1}{2^{O(1/c \log (1/c))}}$. 
**Theorem 1.6.** [3] MAX-BCBS is R4SAT-Hard to approximate within a factor of $n^{\delta}$ where $n$ is the number of vertices in the input graph, and $0 < \delta < 1$ is some constant. More specifically, under the random 4-SAT hardness hypothesis: There exists two constants $\epsilon_1 > \epsilon_2 > 0$ such that no efficient algorithm is able to distinguish between bipartite graphs $G = (V_1, V_2, E)$ with $|V_1| = |V_2| = n$ which have a clique of size $\geq (n/16)^2(1 + \epsilon_1)$ and those in which all bipartite cliques are of size $\leq (n/16)^2(1 + \epsilon_2)$.

2. Results

Our main theorem is as follows:

**Theorem 2.1.** All of the following statements hold even when UVAT is restricted to co-bipartite graphs.

(I) It is NP-Hard to approximate UVAT within a constant factor if it is NP-hard to approximate the maximum clique problem within a factor of $n/2^{c\sqrt{\log n}}$ for some small enough $c > 0$.

(II) Let $\epsilon, \epsilon'$ be as in Theorem 1.5. If SAT has no probabilistic algorithm that runs in time $2^{n^\epsilon}$ on instances of size $n$, then there is no polynomial time (possibly randomized) algorithm for UVAT that achieves an approximation ratio of $N^{\epsilon'}$ on graphs of size $N$.

(III) UVAT is R4SAT-Hard to approximate within a factor of $n^{\delta}$ where $n$ is the number of vertices in the input graph, and $0 < \delta < 1$ is some constant.

The theorem follows directly from part (III) of the following Lemma and Theorems 1.3, 1.5, 1.6.

**Lemma 2.2.** Let $G = (V_1, V_2, E)$ with $|V_1| = |V_2| = n$ be a bipartite graph with $E \neq \emptyset$, and let $\overline{G} = (V_1, V_2, \overline{E})$ be the co-bipartite complement of $G$. Let $BK(G) = \{(A, B)|A \times B \text{ is a bipartite clique in } G \text{ with } |A| \leq |B|\}$. Moreover, let $BBK(G) = \{(A, B) \in V_1 \times V_2|A \times B \text{ is a bipartite clique of } G \text{ with } |A| = |B|\}$, and let $(\hat{A}, \hat{B}) = \arg\max_{(A,B) \in BBK(G)}|A|$ be the maximum balanced bipartite clique of $G$ with corresponding size $k = |\hat{A}|$. Then, the following hold:

(I) $\hat{\tau}(\overline{G}) = \min_{(A,B) \in BK(G)}\frac{2n - |A| - |B|}{|A|} = \min_{(A,B) \in BK(G)}\frac{2n - |B|}{|A|} - 1$

(II) $\frac{n}{k} - 1 \leq \hat{\tau}(\overline{G}) \leq 2(\frac{n}{k} - 1)$
(III) If UVAT can be approximated to factor $\alpha$ in polynomial time, then MAX-BCBS can be approximated to factor $2\alpha$ in polynomial time, even when restricted to co-bipartite graphs.

Proof of Lemma 2.2. Let $S = S(\tau(G))$, $U = S(\hat{\tau}(G))$, $R = S(I(G))$, and $C = S(T(G))$ be the critical attack sets corresponding to $\tau$, $\hat{\tau}$, $I$, and $T$ for $G$, respectively. Furthermore, let $S_i = V_i \cap S$, $U_i = V_i \cap U$, $R_i = V_i \cap R$, and $C_i = V_i \cap C$. For $X \in \{S, U, R, C\}$, let $A_X = \min\{V_1 - X_1, V_2 - X_2\}$ and $B_X = \max\{V_1 - X_1, V_2 - X_2\}$.

Note that $G$ is not a clique as $E \neq \emptyset$. Moreover, because $V_1$ and $V_2$ must both be cliques in $G$, $A_X$ and $B_X$ must each be cliques in $G$, for any $X \in \{S, U, R, C\}$. Namely, the removal of $X$ results in exactly two cliques $A_X$ and $B_X$ in $G$. Clearly, there can be no edge between $A_X$ and $B_X$ in $G$ as such an edge would have remained upon the removal of $X$. Therefore, $(A_X, B_X)$ forms a bipartite clique in $G$. Part (I) of the lemma now follows from the definitions of $\hat{\tau}$ and the fact that $|A_X| \leq |B_X|$.

Now note that for any $(A, B) \in BK(G)$, any subset $B_A \subset B$ such that $|B_A| = |A|$ forms a balanced bipartite clique with $A$. Also clearly, $BBK(G) \subset BK(G)$. Therefore, by (I) and fact that $|A_X| \leq |B_X| \leq n$, (II) follows as well.

For part (III): Let $M$ be an algorithm that gives a constant factor approximation for UVAT with approximation factor $\alpha > 1$. Let $q$ such that $\hat{\tau}(G) \leq q \leq \alpha \hat{\tau}(G)$ be the approximation to $\hat{\tau}$ computed via $M$ on the input. Simplifying and rearranging Lemma 2.2 part (II.b):

$$\frac{n}{q+1} \leq k \leq \frac{n}{1 + q/(2\alpha)}$$

Similarly, let $r = (\frac{n}{q+1})/(\frac{n}{1 + q/(2\alpha)})$ denote the ratio between the right hand side and left hand side of the inequality, so:

$$r = \frac{q + 1}{1 + q/(2\alpha)}$$

If $r > 2\alpha$ then $1 > 2\alpha$ resulting in a contradiction. Therefore,

$$\frac{n}{q+1} \leq k \leq 2\alpha \frac{n}{q+1}$$

And, $\frac{n}{q+1}$ is thus a $\frac{1}{2\alpha}$ approximation for the MAX-BCBS problem with corresponding approximation ratio $2\alpha$. \qed
References

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