FUSION HIERARCHY AND FINITE-SIZE CORRECTIONS OF $U_q[sl(2)]$ INVARIANT VERTEX MODELS WITH OPEN BOUNDARIES

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Abstract

The fused six-vertex models with open boundary conditions are studied. The Bethe ansatz solution given by Sklyanin has been generalized to the transfer matrices of the fused models. We have shown that the eigenvalues of transfer matrices satisfy a group of functional relations, which are the $su(2)$ fusion rule held by the transfer matrices of the fused models. The fused transfer matrices form a commuting family and also commute with the quantum group $U_q[sl(2)]$. In the case of the parameter $q^h = -1$ ($h = 4, 5, \cdots$) the functional relations in the limit of spectral parameter $u \to i\infty$ are truncated. This shows that the $su(2)$ fusion rule with finite level appears for the six vertex model with the open boundary conditions. We have solved the functional relations to obtain the finite-size corrections of the fused transfer matrices for low-lying excitations. From the corrections the central charges and conformal weights of underlying conformal field theory are extracted. To see different boundary conditions we also have studied the six-vertex model with a twisted boundary condition.

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1 Introduction

In statistical mechanics the commuting transfer matrix of two-dimensional lattice models is often used. This is the case because it shows obviously that the corresponding systems are integrable and can be solved exactly. The commutativity of the transfer matrices for periodic systems is easily derived from the Yang-Baxter equation \[1, 2\]. Recently, the so-called open boundary condition has been introduced in the study of two-dimensional lattice models. To have the commuting family of transfer matrices for such systems we have to use the Yang-Baxter equation for the bulk and the reflection equation \[1\] for the boundaries.

The exactly solvable models with non-periodic boundaries have been early studied in \[3, 4, 5, 6, 7\]. After Sklyanin’s work on the algebraic Bethe ansatz of six-vertex model with the open boundary condition \[3]\, there has been increasing interest in exploring two-dimensional lattice models or integrable quantum chains with the open boundary conditions \[12 - 22\]. Recently, the boundary cross unitary has been derived in \[23\] and a bootstrap approach has been developed to two dimensional integrable field theory with boundary in \[23, 24\] (see also \[11, 25, 26\] for related works). In \[21\] a vertex operator approach has been used to solve the semi-infinite XXZ spin with a boundary magnetic field.

The spin-1/2 XXZ chain is the Hamiltonian limit of the transfer matrix of six-vertex model. Similarly, the higher spin XXZ chains are the Hamiltonian limits of the fused transfer matrices of six-vertex model \[27, 28, 29, 30\]. This corresponding relation also works for the model with the open boundary conditions \[3, 31\]. As the fusion of the Boltzmann weights of six-vertex model \[27\] so the fusion procedure of the boundary matrices has been expressed in \[37\]. The fused transfer matrices with the open boundaries are well defined and form the commuting families of the model. They satisfy a group of functional relations, which can be shown by fusion procedure. For the Andrews, Baxter and Forrester critical solid-on-solid models \[10\] it has shown that the functional relations are useful to find the Bethe ansatz solutions and the finite-size corrections of the fused transfer matrices \[38, 42\]. But, for two dimensional vertex models it seems not the case. The functional relations of the fused transfer matrices of six-vertex model with periodic boundary are given in \[38\] and have been shown that they are not closed \[59\]. However, the situation is very different for the case of the six-vertex model with the open boundary conditions. In this paper we study the six-vertex model with the open boundary conditions. The fused transfer matrices of the model commute with the quantum group $U_q[sl(2)]$. The eigenstates of the transfer matrices can then be classified according to the spin $S^z = j$. We focus on the interesting case of $q^h = -1$ ($h = 4, 5, \cdots$).
We show that the functional relations of the fused transfer matrices are truncated in braid limit. This important fact allows the functional relations to be solved in thermodynamic limit. We have found the finite-size corrections of the fused transfer matrices for the low-lying excitations, which are the spin $S$ sectors above the ground state. Therefore the central charges and the conformal weights of underlying conformal field theory have been extracted from the finite-size corrections. The central charges are given by

$$c = \frac{3p}{p + 2} - \frac{6p}{h(h - p)}, \quad h = 4, 5, \cdots \quad (1.1)$$

and the conformal weights are given by

$$\Delta_{s,\nu} = \frac{(h - (h - p)s)^2 - p^2}{4hp(h - p)} + \frac{\nu(p - \nu)}{2p(p + 2)} \quad s = 1, 3, \cdots \leq h - 1, \quad (1.2)$$

where $p = 1, 2, \cdots$ is the fusion level and $\nu$ is the integer determined by

$$\nu = s - 1 - \left\lfloor \frac{s - 1}{p} \right\rfloor p. \quad (1.3)$$

Here the brackets $[x]$ denotes the greatest integer less than or equal to $x$. For the case of $p = 1$ these conformal spectra coincide with the results of the spin-1/2 XXZ chain with the open boundary condition [11]. It is interesting to notice that for generic $p$ these are the conformal spectra of the spin-$p/2$ XXZ chain with the open boundary condition. In a similar way we also have obtained the finite-size corrections of the fused transfer matrices of the six-vertex model with a twisted boundary condition.

The outline of the paper is as follows. In subsection 1.1 and subsection 1.2 we explain the six-vertex model and the open boundary condition and recall the Bethe ansatz solution of the $U_q[sl(2)]$ invariant transfer matrix of the six-vertex model. In section 2 the functional equations of the fused transfer matrices of the model are presented and the corresponding Bethe ansatz of the transfer matrices are constructed. In section 3 we carry out the procedure of calculating the finite-size corrections of the fused transfer matrices and extract the conformal spectra of underlying conformal field theories. In section 4 a brief discussion is given. Particularly, we discuss the non-$U_q[sl(2)]$ invariant six-vertex model with boundaries. In Appendices we describe the fused Boltzmann weights of the six-vertex model and show the functional equations directly by fusion.

## 1.1 Six-vertex model

The six-vertex model is one of solvable lattice models in two dimensional statistical mechanics to prove tractable [32, 33, 34, 4]. It is ice-type model and has six nonzero
Boltzmann weights. These Boltzmann weights form a four by four $R$-matrix

$$R(z) = \begin{pmatrix} a(z) & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & c(z) & b(z) & 0 \\ 0 & 0 & 0 & a(z) \end{pmatrix}, \quad (1.4)$$

where

$$a(z) = zq - z^{-1}q^{-1}$$
$$b(z) = z - z^{-1}$$
$$c(z) = q - q^{-1}$$

(1.5)

depending on a parameter $q$ and a spectral parameter $z \in \mathbb{C}$. This $R$ matrix acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$ solves the Yang-Baxter relation

$$R^{12}(x/y)R^{13}(x/z)R^{23}(y/z) = R^{23}(y/z)R^{13}(x/z)R^{12}(x/y). \quad (1.6)$$

For the twisted boundary condition the row-to-row transfer matrix acting on a "quantum" space $\mathbb{C}^{2N} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ is defined by

$$T_m(z) = \text{tr} \left( q^{-m\sigma^z} U(z) \right), \quad (1.7)$$

where $m$ is an integer. It becomes the transfer matrix with the periodic boundary condition for $m = 0$ and for $m > 0$ it is twisted. The trace is taken in the "classical" space $\mathbb{C}^2$ and the monodromy matrix

$$U(z) = R^{c,N}(zq^{-\frac{1}{2}}) \cdots R^{c,2}(zq^{-\frac{1}{2}}) R^{c,1}(zq^{-\frac{1}{2}}) \quad (1.8)$$

is a two by two matrix in the classical space $\mathbb{C}^2$ denoted by $c$ and with their elements acting on the quantum space $\mathbb{C}^{2N}$. The Yang-Baxter equation (1.6) follows that the monodromy matrix satisfies the quadratic relation

$$R^{12}(x/y) U(x/z) U(y/z) = U(y/z) U(x/z) R^{12}(x/y) \quad (1.9)$$

and the transfer matrix forms a commuting family:

$$[ T_m(z), \ T_m(y) ] = 0. \quad (1.10)$$

This means that $T_m(z)$ is a generating function of commuting operators in quantum mechanics of one dimensional chains. Specially, the Hamiltonian

$$H_m = \sum_{n=1}^{N-1} \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \frac{q + q^{-1}}{2} \sigma_n^z \sigma_{n+1}^z \right)$$
$$+ \frac{1}{2} \left( q^{2m} \sigma_N^x \sigma_1^x + q^{-2m} \sigma_N^y \sigma_1^y + (q + q^{-1}) \sigma_N^z \sigma_1^z \right) \quad (1.11)$$
of the quantum XXZ spin-$\frac{1}{2}$ chain is contained within this family. These $\sigma_n^x, \sigma_n^y$ and $\sigma_n^z$ are Pauli matrices and $\sigma_n^\pm = \sigma_n^x \pm i\sigma_n^y$.

The six-vertex model with the open boundary is given by recalling the reflection equation [4],

$$ R^{12}(x/y)K_1^1(x)R^{12}(xy)K_2^2(y) = K_2^2(y)R^{12}(xy)K_1^1(x)R^{12}(x/y) \ . \ (1.12) $$

Sklyanin has shown in [3] that the transfer matrix

$$ T(z) = \text{tr} \ K_+(zq^{1/2})U(z)K_-(zq^{-1/2})U^{-1}(z^{-1}) \ (1.13) $$

forms a commuting family

$$ [ \ T(z) , \ T(y) ] = 0 \ . \ (1.14) $$

Therefore it presents an integrable system with the boundary described by the reflection matrices $K_{\pm}(z)$. Particularly, the matrices $K_{\pm}(z)$ take the form [12]

$$ K_{\pm}(z) = \begin{pmatrix} z^\mp & 0 \\ 0 & z^\pm \end{pmatrix} , \quad (1.15) $$

then the integrable system possesses the quantum algebra $U_q[sl(2)]$ symmetry, which means

$$ T(y) S^\pm - S^\pm T(y) = 0 \quad \text{and} \quad T(y) S^z - S^z T(y) = 0 \ (1.16) $$

for any $y$. Here the operators $S^\pm$ and $S^z$ of the quantum algebra $U_q[sl(2)]$

$$ S^z = \frac{1}{2}(\sigma_1^z + \sigma_2^z + \cdots + \sigma_N^z) \quad (1.17) $$

$$ S^\pm = \sum_{n=1}^{N} q^{(\sigma_1^z + \cdots + \sigma_{n-1}^z)/2(\sigma_n^\pm / 2)} q^{-(\sigma_{n+1}^z + \cdots + \sigma_N^z)/2} \quad (1.18) $$

satisfy

$$ S^z S^\pm - S^\pm S^z = \pm S^\pm \quad \text{and} \quad S^+ S^- - S^- S^+ = [2S^z]_q \ (1.19) $$

where

$$ [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} . $$

The Hamiltonian corresponding to the integrable system is the open XXZ quantum spin-$\frac{1}{2}$ chain [3],

$$ H = \sum_{n=1}^{N-1} \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \frac{q + q^{-1}}{2} \sigma_n^z \sigma_{n+1}^z \right) + \frac{q - q^{-1}}{2} \left( \sigma_N^z + \sigma_1^z \right) \ (1.20) $$
The $U_q[sl(2)]$ invariance (1.16) implies the quantum algebra $U_q[sl(2)]$ is the "symmetric group" of the $XXZ$ chain [35]

$$H S^\pm - S^\pm H = 0 \quad \text{and} \quad H S^z - S^z H = 0 \quad (1.21)$$

For generic values of $q$ the representations of $U_q[sl(2)]$ are known to be equivalent to the ordinary $su(2)$ representations [51, 52]. The representation theory becomes more complicated for the special case of $q^h = \pm 1$ (see [52, 53, 35, 54, 55, 56] for details).

### 1.2 Bethe ansatz solution

The Bethe ansatz equations and the eigenvalues of the six-vertex model with the twisted or open boundary condition have been given using the algebraic Bethe ansatz [2, 3, 9, 34]. Set $z = e^{iu}$ and $q = e^{i\lambda}$. We recall that the eigenvalues $T_m(z)$ or $T(z)$ of the transfer matrices $T_m(z)$ or $T(z)$. For the twisted boundary case the eigenvalues are given by

$$T_m(u) = e^{-im\lambda} \sin^N(u + \frac{1}{2}\lambda) \prod_{k=1}^{M} \frac{\sin(u - v_k - \lambda)}{\sin(u - v_m)}$$

$+$

$$e^{im\lambda} \sin^N(u - \frac{1}{2}\lambda) \prod_{k=1}^{M} \frac{\sin(u - v_k + \lambda)}{\sin(u - v_m)} \quad (1.22)$$

and these $v_1, v_2, \cdots, v_M$ are the solutions of the Bethe ansatz equations

$$\frac{\sin^N(v_k + \frac{1}{2}\lambda)}{\sin^N(v_k - \frac{1}{2}\lambda)} = e^{2im\lambda} \prod_{l \neq k}^{M} \frac{\sin(v_k - v_l + \lambda)}{\sin(v_k - v_l - \lambda)}. \quad (1.23)$$

For the open boundary case the eigenvalues are given by

$$T(u) = \frac{\sin(2u + \lambda)}{\sin(2u)} \sin^{2N}(u + \frac{1}{2}\lambda) \prod_{m=1}^{M} \frac{\sin(u - v_m - \lambda) \sin(u + v_m - \lambda)}{\sin(u - v_m) \sin(u + v_m)}$$

$+$

$$\frac{\sin(2u - \lambda)}{\sin(2u)} \sin^{2N}(u - \frac{1}{2}\lambda) \prod_{m=1}^{M} \frac{\sin(u - v_m + \lambda) \sin(u + v_m + \lambda)}{\sin(u - v_m) \sin(u + v_m)} \quad (1.24)$$

and these $v_1, v_2, \cdots, v_M$ satisfy the Bethe ansatz equations

$$\left(\frac{\sin(v_m + \frac{1}{2}\lambda)}{\sin(v_m - \frac{1}{2}\lambda)}\right)^{2N} = \prod_{k \neq m}^{M} \frac{\sin(v_m - v_k + \lambda) \sin(v_m + v_k + \lambda)}{\sin(v_m - v_k - \lambda) \sin(v_m + v_k - \lambda)}. \quad (1.25)$$

### 2 Fused models and their functional equations

The fusion models can be built up by fusion [27] from the six vertex model. Suppose that $R_{(p,q)}(u)$ represents the $R$-matrix of fused vertex (see Appendix. A) and then the
relevant monodromy matrix is defined by

\[ U_{(p,q)}(u) = R^{c,1}_{(q,p)}(u)R^{c,2}_{(q,p)}(u)\cdots R^{c,N}_{(q,p)}(u), \tag{2.1} \]

where \( p \) and \( q \) are respectively the fusion levels for vertical direction and horizontal direction of the square lattice and \( p, q = 1, 2, \cdots \). With the periodic boundary condition the fused transfer matrices

\[ T^{(p,q)}(u) = \text{tr} \ U_{(p,q)}(u) \tag{2.2} \]

commute

\[ [T^{(p,q)}(u), T^{(p,q')}](v) = 0 \tag{2.3} \]

for each fusion level \( p \) fixed and any \( z \) and \( y \). These \( q \) and \( q' \) can stay in different levels. The transfer matrices satisfy the following functional relations \[38 \]

\[ T^{(p,q)}(u) T^{(p,1)}(u + q\lambda) = T^{(p,q+1)}(u) + f^p_{q-1} T^{(p,q-1)}(u), \tag{2.4} \]

where \( T^{(p,0)}(u) = I \), the identity matrix, and the \( u \)-dependent function \( f^p_q \) is generated from the antisymmetric fusion of the Boltzmann weights. These relations are the \( \text{su}(2) \) fusion rule. They mean the relationship among the eigenvalues of the fused transfer matrices. In other words, all eigenvalues \( T^{(p,q)} \) of the fused transfer matrices are determined by the relations with the initial solution \( T^{(p,1)} \). In the following subsection the similar idea is used to the transfer matrices with open or twisted boundary condition. Let \( T^{(p,q)}(u) \) be the eigenvalues of the fused transfer matrices with the open boundary condition or twisted boundary condition. We prove the following theorems.

**Theorem 2.1 (\( su(2) \) Fusion Hierarchy) Let us define**

\[ T^{(p,0)} = 1 \quad T^{(q)}_k = T^{(p,q)}(u + k\lambda) \quad f^p_q = f^p(u + q\lambda) \]

\[ f^p(u) = \omega_1(u + \lambda)\phi(u + \frac{1}{2}p\lambda + \lambda)\omega_2(u)\phi(u - \frac{1}{2}p\lambda) \tag{2.5} \]

where \( \omega_1(u), \omega_2(u) \) and \( \phi(u) \) are given by \[2.14\]-\[2.15\] and \[2.12\]. Then the \( su(2) \) fusion hierarchy follows

\[ T^{(q)}_0 T^{(1)}_q = T^{(q+1)}_0 + f^p_{q-1} T^{(q-1)}_0 \tag{2.6} \]

for \( q = 1, 2, \cdots \).

**Theorem 2.2 (\( su(2) \) TBA) If we define**

\[ t^0_0 = 0 \tag{2.7} \]

\[ t^q_0 = T^{(q+1)}_0 T^{(q-1)}_1 / \prod_{k=0}^{q-1} f^p_k, \tag{2.8} \]

then it follows that \( su(2) \) TBA equations

\[ t^q_0 t^q_1 = (1 + t^q_0)(1 + t^{-1}_1). \tag{2.9} \]
2.1 Bethe ansatz for fused models

The Bethe ansatz solutions (1.22) and (1.24) can be written in the form of
\[ T(u)Q(u) = \omega_1(u)\phi(u + \frac{1}{2}\lambda)Q(u - \lambda) + \omega_2(u)\phi(u - \frac{1}{2}\lambda)Q(u + \lambda) \] (2.10)
using Baxter’s auxiliary matrix \( Q \) which commutes with the transfer matrix \( T(u) \). The eigenvalue \( Q \) of the auxiliary matrix \( Q \) is given by
\[ Q(u) = \begin{cases} \prod_{m=1}^{M} \{\sin(u - v_m)\} & \text{for twisted boundary} \\ \prod_{m=1}^{M} \{\sin(u - v_m)\sin(u + v_m)\} & \text{for open boundary} \end{cases} \] (2.11)
and \( \phi(u) \) is given by
\[ \phi(u) = \sin^N(u) \] (2.12)
with
\[ N' = \begin{cases} N & \text{for twisted boundary} \\ 2N & \text{for open boundary} \end{cases} \] (2.13)
The functions \( Q(u) \) and \( \phi(u) \) are not directly related to the boundary. The boundary terms come in the expression of eigenvalues through the factors \( \omega_1(u) \) and \( \omega_2(u) \)
\[ \omega_1(u) = \omega_2(-u) = \frac{\sin(2u + \lambda)}{\sin(2u)} \] (2.15)
Here the simple case \( m = 1 \) has been taken for the twisted boundary.

The fusion procedure [27, 37] (see [36, 51, 11, 18, 49] for related works) shows that the fused transfer matrices \( T^{(p,1)}(u) \) can be constructed directly from the unfused ones by applying the fusion projectors to the "quantum" space (see Appendix. B). The procedure implies obviously that the eigenvalue \( T^{(p,1)}(u) \) for the transfer matrix \( T^{(p,1)}(u) \) has the following form
\[ T^{(p,1)}(u)Q(u) = \omega_1(u)\phi(u + \frac{1}{2}p\lambda)Q(u - \lambda) + \omega_2(u)\phi(u - \frac{1}{2}p\lambda)Q(u + \lambda) \] (2.16)
The boundary conditions take the position in the "classical" space and thus \( \omega_1(u) \) and \( \omega_2(u) \) are not affected by the fusion procedure in "quantum" space. The function \( Q(u) \) is however dependent on the fusion level \( p \). For example, the ground state corresponds to
take $M = pN/2$. The Bethe ansatz equations determining all these $v_1, v_2, \cdots$ are given by setting

$$T^{(p,1)}(v_k) = 0 \ . \quad (2.17)$$

To show the $su(2)$ fusion hierarchy let us use semi-standard Young tableaux \[38, 39, 40, 41\]. Define

$$\begin{align*}
1 \kappa &= \omega_2(u + k\lambda) \phi(u + k\lambda - \frac{1}{2}p\lambda) \frac{Q(u + k\lambda + \lambda)}{Q(u + k\lambda)} \\
2 \kappa &= \omega_1(u + k\lambda) \phi(u + k\lambda + \frac{1}{2}p\lambda) \frac{Q(u + k\lambda - \lambda)}{Q(u + k\lambda)} \quad (2.18)
\end{align*}$$

for a single Young tableau so that

$$T^{(1)}_0 = \begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
2 \\
1
\end{array}
\end{array} = \sum \begin{array}{c}
\begin{array}{c}
2 \\
2
\end{array}
\end{array} \quad (2.19)$$

For a general one-row Young tableau, the numbers must not decrease moving to the right along the row, e.g.

$$\begin{array}{cccccc}
1 & 1 & 2 & 2 & 2 & 0
\end{array} \quad (2.20)$$

Such a Young tableau denotes the product of the five labeled boxes defined by (2.18) where it is understood that the relative shifts in the spectral parameters are given by

$$\begin{array}{ccccccc}
u + 4\lambda & v + 3\lambda & u + 2\lambda & u + \lambda & u & 0
\end{array} \quad (2.21)$$

and the zero superscript gives the shift in the most right box. Filling the numbers 1 and 2 in this five-box Young tableau according to the rule that the numbers must not decrease moving to the right along the row, we get six numbered Young tableaux. Then taking sum of these six Young tableaux with the correct spectral parameter shifts (2.21), we obtain the eigenvalues $T^{(5)}(u)$.

By a similar way the eigenvalues of the fused row transfer matrix at level $q$ can be written as

$$T^{(q)}_0 = T^{(p,q)}_0(u) = \sum \begin{array}{c}
\begin{array}{c}
\cdots \end{array}
\end{array} q \quad (2.22)$$

where the number of terms in the sum is given by the dimension of the irreducible representations of $su(2)$

$$\dim(q) = (q + 1) \ . \quad (2.23)$$
For example, the fusion level $q = p = 2$ case gives the eigenvalues of transfer matrix of the 19-vertex model

$$T_0^{(2)} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$  \hspace{1cm} (2.24)

where it is understood that the relative shifts in the arguments are given by

$$\begin{bmatrix} u + \lambda \\ u \end{bmatrix}^0$$  \hspace{1cm} (2.25)

It is straightforward to show that set

$$f^p(u) := \begin{bmatrix} 1 \\ 2 \end{bmatrix} \equiv \omega_1(u + \lambda)\phi(u + \frac{1}{2} p \lambda + \lambda)\omega_2(u)\phi(u - \frac{1}{2} p \lambda)$$,

then $T_0^{(q)}$ given by (2.22) satisfy (2.6). This leads the theorem 2.1. The proof also shows that the fusion hierarchy is compatible with the $su(2)$ fusion rule

$$\begin{bmatrix} q \\ q \end{bmatrix} \otimes \begin{bmatrix} q \\ q \end{bmatrix} = \begin{bmatrix} q - 1 \\ q - 1 \end{bmatrix} \oplus \begin{bmatrix} q + 1 \\ q + 1 \end{bmatrix}$$ \hspace{1cm} (2.26)

To show theorem 2.2 let us consider the triple

$$T_0^{(p,q)}(T_1^{(p,q-1)}T_1^{(p,1)}) = (T_0^{(p,q)}T_1^{(p,1)})T_1^{(p,q-1)}$$.

Inserting the fusion hierarchy into the terms in parentheses this equation gives new functional equations

$$T_0^{(q)}T_1^{(q)} = \prod_{k=0}^{q-1} f_k^p + T_0^{(q+1)}T_1^{(q-1)}$$ \hspace{1cm} (2.27)

which corresponds to the following $su(2)$ fusion rule

$$\begin{bmatrix} q \\ q \end{bmatrix} \otimes \begin{bmatrix} q \\ q \end{bmatrix} = \begin{bmatrix} q - 1 \\ q - 1 \end{bmatrix} \oplus \begin{bmatrix} q + 1 \\ q + 1 \end{bmatrix} \oplus \phi$$ \hspace{1cm} (2.28)

Then it is easy to see the theorem 2.2 by rewriting the fusion rule according to the definition of $t^q(u)$.

The functional equations (2.6) and (2.8) in form are the same as the $su(2)$ functional equations of $A$–$D$–$E$ models [39] and the dilute $A$–$D$–$E$ models [40].
2.2 Zeros and poles of eigenvalues

The functional relations have been shown to be very useful to calculate the finite size corrections of the fused transfer matrices \[42, 44\]. To solve the fusion hierarchy (2.7) and (2.9) we need to know the distribution of zeros and poles of these transfer matrices $T^{(q)}$ and $t^{(q)}$. For $q = p$ these $T^{(q)}$ possess the physical strip of the model. Inside the strip the ground state eigenvalues $T^{(q)}$ do not possess any zero apart from those which are imposed by the fusion of the Boltzmann weights and the boundary. The zeros contributed by the Boltzmann weights are of order $N$ and those by the boundary are only of order 1. They list them as follows.

\[
\text{zero}[T^{(p,q)}(u)] = \emptyset \quad \text{for } q \leq p \quad (2.29)
\]
\[
\text{zero}[T^{(p,q)}(u)] = \bigcup_{k=0}^{q-p-1} \{-k\lambda - \frac{1}{2}p\lambda\}^N \quad \text{for } q > p. \quad (2.30)
\]

for the twisted boundary condition and

\[
\text{zero}[T^{(p,q)}(u)] = \bigcup_{k=0}^{q-2} \{-k\lambda - \frac{1}{2}\lambda\} \quad \text{for } q \leq p \quad (2.31)
\]
\[
\text{zero}[T^{(p,q)}(u)] = \bigcup_{k=0}^{q-2} \{-k\lambda - \frac{1}{2}\lambda\} \bigcup_{k=0}^{q-p-1} \{-k\lambda - \frac{1}{2}p\lambda\}^2 N \quad \text{for } q > p. \quad (2.32)
\]

for the open boundary condition. The zeros and poles of $t^{(q)}$ are determined by (2.8). So we have

(I) $q \leq p - 1$ :
\[
\text{zero}[t^{(p,q)}(u)] = \emptyset
\]
\[
\text{pole}[t^{(p,q)}(u)] = \bigcup_{k=0}^{q-1} \{-k\lambda - \frac{1}{2}p\lambda - \lambda\}^N \bigcup_{k=0}^{q-1} \{\frac{1}{2}p\lambda - k\lambda\}^N \quad (2.33)
\]

(II) $q = p$ :
\[
\text{zero}[t^{(p,p)}(u)] = \{-\frac{1}{2}p\lambda\}^N
\]
\[
\text{pole}[t^{(p,p)}(u)] = \bigcup_{k=0}^{p-1} \{-k\lambda - \frac{1}{2}p\lambda - \lambda\}^N \bigcup_{k=0}^{p-1} \{\frac{1}{2}p\lambda - k\lambda\}^N \quad (2.34)
\]

(III) $q \geq p + 1$ :
\[
\text{zero}[t^{(p,q)}(u)] = \emptyset
\]
\[
\text{pole}[t^{(p,q)}(u)] = \bigcup_{k=0}^{p-1} \{\frac{1}{2}p\lambda - k\lambda\}^N \bigcup_{k=1}^{p} \{-k\lambda + \frac{1}{2}p\lambda - q\lambda\}^N \quad (2.35)
\]

for the twisted boundary condition and

(I) $q \leq p - 1$ :
\[
\text{zero}[t^{(p,q)}(u)] = \text{set}_q
\]
\[
\text{pole}[t^{(p,q)}(u)] = \{-q\lambda - \frac{1}{2}\lambda\}\{\frac{1}{2}\lambda\}
\]

for the twisted boundary condition.
\[
\bigcup_{k=0}^{q-1} \{ -k\lambda - \frac{1}{2}p\lambda - \lambda \}^{2N} \bigcup_{k=0}^{q-1} \{ \frac{1}{2}p\lambda - k\lambda \}^{2N} \quad (2.36)
\]

(II) \( q = p \) :

\[
\begin{align*}
\text{zero}[t^{(p,p)}(u)] &= \{ -\frac{1}{2}p\lambda \}^{2N} \\
\text{pole}[t^{(p,p)}(u)] &= \{ -p\lambda - \frac{1}{2}\lambda \} \{ \frac{1}{2}\lambda \} \\
&\quad \bigcup_{k=0}^{p-1} \{ -k\lambda - \frac{1}{2}p\lambda - \lambda \}^{2N} \bigcup_{k=0}^{p-1} \{ \frac{1}{2}p\lambda - k\lambda \}^{2N} \\
(2.37)
\end{align*}
\]

(III) \( q \geq p + 1 \) :

\[
\begin{align*}
\text{zero}[t^{(p,q)}(u)] &= \{ \frac{1}{2}p\lambda - q\lambda \}^{2N} \\
\text{pole}[t^{(p,q)}(u)] &= \{ -q\lambda - \frac{1}{2}\lambda \} \{ \frac{1}{2}\lambda \} \\
&\quad \bigcup_{k=0}^{p-1} \{ \frac{1}{2}p\lambda - k\lambda \}^{2N} \bigcup_{k=1}^{p} \{ -k\lambda + \frac{1}{2}p\lambda - q\lambda \}^{2N} \\
(2.38)
\end{align*}
\]

for the open boundary condition, where \( \text{set} \ q = \{ -\frac{1}{2}\lambda \} \) for \( q = 1 \) and \( \text{set} \ q = \emptyset \) for \( q > 1 \).

The zeros or poles with order 1 have less contribution than those of order \( N \) when the system size \( N \) becomes large. Especially, in the thermodynamic limit \( N \to \infty \) only these zeros or poles with order \( N \) are important.

### 3 Functional relations in \( N \to \infty \)

The finite-size corrections for the eigenvalues \( T^{(p)} \) can be obtained by solving functional relations \((2.6)\) and \((2.9)\) in the physical strip,

\[-\lambda < \text{Re} \ u + \frac{1}{2}p\lambda < \lambda \, . \quad (3.1)\]

Denote the finite-size corrections of \( T^{(p)} \) by \( T^{(p)}_{\text{finite}}(u) \) and write

\[
T^{(p)}(u) = T^{(p)}_{\text{finite}}(u)T^{(p)}_{\text{bulk}}(u) \, . \quad (3.2)
\]

The bulk and the surface energies determined by the unitary conditions of \( R \) and \( K \) matrices and satisfy

\[
T^{(p)}_{\text{bulk}}(u)T^{(p)}_{\text{bulk}}(u + \lambda) = \prod_{k=0}^{p-1} f_k^p \quad (3.3)
\]

Inserting \((3.2)\) into \((2.27)\) we find that

\[
T^{(p)}_{\text{finite}}(u)T^{(p)}_{\text{finite}}(u + \lambda) = 1 + t^{(p)}(u) \, . \quad (3.4)
\]
So the finite-size corrections for $T^{(p)}(u)$ are represented by the hierarchy $t^{(p)}(u)$ ($t$-system or $y$-system are also called). In the following subsections the analytical treatment of (3.4) and (2.9) is given. We will see that the finite-size corrections in scaling limit are only dependent on the braid asymptotics and bulk behavior of the functional relations.

### 3.1 Nonlinear integral equations of real variable

The Bethe ansatz equations (2.17) render $T^{(p)}(u)$ to be analytic. Since all functions are $\pi$-periodic, the analyticity domains for $T^{(p)}(u)$ are not unique. It is useful to introduce functions of a real variable by restricting the eigenvalue functions to certain lines in the complex plane,

$$T^q(x) := T^{(q)} \left( \frac{i}{\pi} x \lambda + \frac{p - q + 1}{2} \lambda - \frac{i}{2} p \lambda \right),$$

$$\alpha^q(x) := t^{(q)} \left( \frac{i}{\pi} x \lambda + \frac{p - q}{2} \lambda - \frac{i}{2} p \lambda \right) \quad \text{and} \quad A^q(x) := 1 + \alpha^q(x).$$

The functional relation (3.4) can then be rewritten in terms of the new functions as

$$T^p(x - \frac{1}{2} \pi i) T^p(x + \frac{1}{2} \pi i) = A^p(x).$$

For the ground state the functions $A^{(p)}(x)$ and $T^{(p)}(x)$ are analytic, non-zero\(^4\) in $-3\pi/2 < \text{Im} \ x < \pi/2$ and possess constant asymptotics for $\text{Re} \ x \to \pm \infty$ (ANZC), which can be seen from the eigenvalues directly. Taking the logarithmic derivative of the above equation and introducing Fourier transforms

$$B^p(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left[ \ln T^p(x) \right]' e^{-i k x},$$

$$\left[ \ln T^p(x) \right]' = \int_{-\infty}^{\infty} dk \ B^p(k) e^{i k x}$$

with analogous equations for $A^q$ and its Fourier transform $A^q$, then we have

$$B^p(k) = \frac{A^p(k)}{e^{(\pi/2) k} + e^{-(\pi/2) k}}.$$  

Transforming back and defining the kernel $k(x)$

$$k(x) := \frac{1}{2\pi \cosh(2x)},$$

we are able to express $T^q$ in terms of $A^q$,

$$\ln T^p = k \ast \ln A^p + C^p,$$
where $C_p$ are integration constants. The convolution $f * g$ of two functions $f$ and $g$ is defined by

$$(f * g) := \int_{-\infty}^{\infty} f(x-y)g(y) \, dy = \int_{-\infty}^{\infty} g(x-y)f(y) \, dy.$$ (3.12)

In case of the low-lying excitations states we have to take care of zeros in the analyticity strips so that the simple ANZC properties hold. The result (3.11) is still correct if we change integration path $\mathcal{L}$ so that $T^p(x)$ has an ANZC area and Cauchy theorem can be applied like the discussion in [42]. The integration constants in (3.11) can be evaluated from the asymptotics of $A^q$ and $T^q$. In this limit (3.11) becomes

$$\ln T^q_\infty = \frac{1}{2} \ln A^q_\infty + C^q.$$ (3.13)

It can be seen that the constants are just the multiple of $i\pi$ and do not contribute to the $\frac{1}{N}$ corrections.

The $A^q$ can be solved from the hierarchy from (2.9), which can be rewritten in terms of $\alpha^q$ as

$$\alpha^q(x - \frac{1}{2} \pi i)\alpha^q(x + \frac{1}{2} \pi i) = A^{q-1}(x)A^{q+1}(x).$$ (3.14)

According to section 2.2 the analyticity strip (3.1) for $t^{(p)}(u)$ contains a zero of order $N$ at $u = -\frac{1}{2}p\lambda$ and a pole of order $N$ at $u = -\frac{1}{2}p\lambda + \lambda$ or $u = -\frac{1}{2}p\lambda - \lambda$. All other functions $t^{(b,a)}$ are analytic and non-zero in their analyticity strips $-\frac{1}{2}p\lambda - \lambda < u < -\frac{1}{2}p\lambda + \lambda$. We introduce finite-size correction terms $l^q(x)$ by writing $\alpha^q(x)$ as

$$\alpha^q(x) = \begin{cases} l^q(x), & q \neq p \\ \tanh^{N}(\frac{1}{2}x)l^q(x), & q = p. \end{cases}$$ (3.15)

The factor $\tanh^{N}(\frac{1}{2}x)$ gives the right zero and poles and all the functions $l^q(x)$ therefore are ANZC in $-\pi < \text{Im} \ x < \pi$. They satisfy the functional equations

$$l^q(x - \frac{1}{2} \pi i)l^q(x + \frac{1}{2} \pi i) = A^{q-1}(x)A^{q+1}(x).$$ (3.16)

Again applying Fourier transforms to the logarithmic derivative of the equations and then integrating the equations back we obtain the nonlinear integral equations

$$\ln \alpha^q = \ln \epsilon^q + k * \ln A^{q-1} + k * \ln A^{q+1} + D^q,$$ (3.17)

where

$$\epsilon^q(x) := \begin{cases} 1, & q \neq p \\ \tanh^{N}(\frac{1}{2}x), & q = p. \end{cases}$$ (3.18)

$D^q$ are the integral constants. For the same reason we have to take care of the extra zeros in the analyticity strips so that the ANZC is held in (3.17).
3.2 Finite-size correction and scaling limit

The information of finite-size corrections can be extracted from the nonlinear integral equations (3.17) and (3.11). The system size \( N \) enters the nonlinear equations (3.17) through (3.18). The function \( e^p \) has three asymptotic regimes with transitions in scaling regimes when \( x \) is of the order of \(-\ln N\) or \(\ln N\). We suppose that \(\alpha^q\) and \(A^q\) scale similarly. So in the following scaling limits,

\[
\begin{align*}
\ell a^q(x) &:= \lim_{N \to \infty} e^q \left( \pm (x + \ln N) \right),
\ell e^q(x) &:= \begin{cases} 0, & q \neq p, \\ -2e^{-x}, & q \neq p \end{cases},
\ell A^q(x) &:= \lim_{N \to \infty} A^q \left( \pm (x + \ln N) \right) = 1 + \ell a^q(x).
\end{align*}
\]

In this scaling limits, (3.17) takes the form

\[
\ell a^q = \ell e^q + k \ast \ell A^{q-1} + k \ast \ell A^{q+1} + D^q,
\]

where we use the abbreviations

\[
\begin{align*}
\ell a^q(x) &:= \ln a^q(x), \quad \ell A^q(x) := \ln A^q(x),
\ell e^q(x) &:= \begin{cases} 0, & q \neq p, \\ -2e^{-x}, & q \neq p \end{cases},
\end{align*}
\]

and suppress the subscripts \(\pm\). The transfer matrix \(T^p(x)\) in \(N \to \infty\) now becomes

\[
\ln T^p(x) = (k \ast \ln A^p)(x)
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\ln A^p(y + \ln N)}{\cosh(x + y + \ln N)} + \frac{\ln A^p(-y - \ln N)}{\cosh(x - y - \ln N)} \right) dy + o\left( \frac{1}{N} \right)
= \frac{e^x}{N\pi} \int_{-\infty}^{\infty} e^{-y}(\ell A^p(y)) dy + \frac{e^{-x}}{N\pi} \int_{-\infty}^{\infty} e^{-y}(\ell A^p(y)) dy + o\left( \frac{1}{N} \right)
= \frac{2 \cosh x}{N\pi} \int_{-\infty}^{\infty} e^{-y}(\ell A^p(y)) dy + o\left( \frac{1}{N} \right).
\]

Above equation converges and actually can be evaluated explicitly with the help of the dilogarithmic function

\[
L(x) = -\int_0^x dy \frac{\ln(1-y)}{y} + \frac{1}{2} \ln x \ln(1 - x).
\]

Multiplying the derivative of (3.20) with \(\ell A^q\) and (3.20) itself with \((\ell A^q)'\), taking the difference, summing over \(q\), and finally integrating we find

\[
\sum_q \int_{-\infty}^{\infty} \left[ (\ell a^q)'\ell A^q - \ell a^q(\ell A^q)' \right] dx
= \sum_q \int_{-\infty}^{\infty} \left[ (\ell e^q)'\ell A^q - (\ell e^q - D^q)(\ell A^q)' \right] dx,
\]

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where the sum is over all fusion levels \( q \) and the contribution of the kernel cancel due to the symmetry
\[
k(-x) = k(x) .
\] (3.25)

Then inserting (3.21) into the right-hand side and integrating the left-hand side of (3.24), we are able to obtain
\[
2 \int_{-\infty}^{\infty} e^{-y \ell A^p(y)} \, dy = - \sum_{q} L \left( \frac{1}{A^q} \right) \bigg|_{-\infty}^{\infty} + \frac{1}{2} \sum_{q} D^q \ell A^q \bigg|_{-\infty}^{\infty}
\] (3.26)

where the constants \( D^q \) are given in terms of
\[
D^q = \ell a^q - \frac{1}{2} \ell A^q - \frac{1}{2} \ell A^q + 1
\] (3.27)
in asymptotics \( x \to \infty \).

3.3 Asymptotics and bulk behavior

The nonlinear integral equations (3.20) can be easily solved for the limit \( x \to \pm \infty \) and
\[
\lambda = \frac{\pi}{h} \quad h = 3, 4, \ldots .
\] (3.28)

For different \( h \) it corresponds to different models. The equation (3.26) shows that these asymptotic solutions are enough to obtain the finite-size corrections of the transfer matrix \( T^{(p)}(u) \). Before discuss the asymptotic solutions it is useful to observe that
\[
T^{p,h-1}(\pm i\infty) = 0 \quad \text{for } \lambda = \pi/h ,
\] (3.29)

which can be easily seen from the eigenvalues (2.16) and the theorem 2.1.

It is obvious to see that the asymptotics \( x \to \infty \) corresponds to the braid limit of \( u \to \pm i\infty \). In this limit (2.4) reduces to
\[
(t_{\infty}^{(q)})^2 = (1 + t_{\infty}^{(q-1)})(1 + t_{\infty}^{(q+1)}) .
\] (3.30)

This equation in turn means
\[
2 \ell a^q = \ell A^{q-1} + \ell A^{q+1} + D^q
\] (3.31)
in terms of the functions \( a^q \). Where the constants \( D^q \) can be zero or non-zero because the different branches can be taken for the dilogarithmic functions in the nonlinear integral equations.
To solve the $t^{(q)}$ let us write $t^{(1)}_{\infty}$ as

$$t^{(1)}_{\infty} = \frac{\sin(3\theta)}{\sin(\theta)}$$

with the parameter $\theta$ to be determined. The recursion relation (3.30) implies

$$t^{(q)}_{\infty} = \frac{\sin(q\theta)\sin((q+2)\theta)}{\sin^2 \theta}$$

$$t^{(q)}_{\infty} + 1 = \frac{\sin^2((q+1)\theta)}{\sin^2 \theta}$$

(3.32)

for all $q = 1, 2 \cdots$. This solution have to be consistent with the braid limit of $T^{(q)}(u)$. To fix the constant parameter $\theta$ let us consider the "ground" state $M = \frac{1}{2}pN$, 

$$\lim_{\text{Im} u \to \pm \infty} T^{(1)}(u)/\phi(u) = 2 \cos \left( \frac{\pi}{h} \right).$$

(3.33)

By the relation

$$\frac{\sin(3\theta)}{\sin(\theta)} = t^{(1)}_{\infty} = \lim_{\text{Im} u \to \pm \infty} T^{(2)}_{0}/f_{0}^{p} =$$

$$\lim_{\text{Im} u \to \pm \infty} \frac{T^{(1)}_{0}T^{(1)}_{1}}{f_{0}^{p}} - 1 = 4 \cos^2 \left( \frac{\pi}{h} \right) - 1$$

(3.34)

we have

$$\theta = \lambda = \frac{\pi}{h}.$$  

(3.35)

Moreover the special value of $\theta$ leads the closure condition

$$t^{(h-2)}_{\infty} = 0.$$  

(3.36)

For the sector $j = \frac{1}{2}pN - M$ we have to modify $\theta$ to be

$$\theta = m\lambda = \frac{m\pi}{h}$$

(3.37)

where $m = 2j + 1 = 1, 3, \cdots \leq h - 1$. For the periodic case $m = 1, 2, 3, \cdots, h - 1$.

In the limit of $x \to -\infty$ $t^{(q)}$ can be considered as the bulk behavior in large $N$. According to section 2.2 the analyticity strip for $t^{(p)}(u)$ contains a zero of order $N$ at $u = -\frac{1}{2}p\lambda$ and poles of order $N$ at $u = -\frac{1}{2}p\lambda \pm \lambda$. All other functions $t^{(q)}$ are analytic and non-zero in their analyticity strips in $-\frac{1}{2}p\lambda - \lambda < u < \frac{1}{2}p\lambda + \lambda$. For large $N$ the leading bulk behavior to the $t^{(q)}$ we find that

$$t^{(q)}_{\text{bulk}}(u) = \begin{cases} 
\text{constant} & q \neq p, \\
\text{constant} \left( \tan \left( \frac{1}{2}hu \right) \right)^N & q = p.
\end{cases}$$

(3.38)
The constants are fixed by the functional equations (2.9) and can be calculated similarly to the asymptotics of these $t^{(q)}$. Like $A_L$ model [12], it is easy to see that the limit

$$
\lim_{x \rightarrow -\infty} \lim_{N \rightarrow \infty} t^{(p)} \sim \lim_{x \rightarrow -\infty} \exp(-2e^{-x}) \rightarrow 0.
$$

(3.39)

Therefore the functional equations (2.9) are divided into two parts and we find the constants for $1 \leq q \leq p - 1$

$$
t^{(q)}_{\text{bulk}} = \frac{\sin(q\sigma)\sin((q+2)\sigma)}{\sin^2 \sigma},
$$

$$
t^{(q)}_{\text{bulk}} + 1 = \frac{\sin^2((q+1)\sigma)}{\sin^2 \sigma},
$$

(3.40)

where

$$
\sigma = \frac{m\pi}{p+2}, \quad m = 1, 2, \ldots, p+1.
$$

(3.41)

Similarly, for $p + 1 \leq q \leq h - 3$ we suppose

$$
t^{(q)}_{\text{bulk}} = \frac{\sin((q-p)\tau)\sin((q-p+2)\tau)}{\sin^2 \tau},
$$

$$
t^{(q)}_{\text{bulk}} + 1 = \frac{\sin((q-p+1)\tau)}{\sin^2 \tau},
$$

(3.42)

with

$$
\tau = \frac{m^*\pi}{h-p},
$$

(3.43)

which is consistent with the closure condition (3.36). The eigen-spectra of the transfer matrices have only one "quantum" number $M$, which is related to the braid limit. These $m^*$ and $m^*$ can not be free parameters. In [12] an interpolate method is applied to compute the finite size corrections of transfer matrices for ABF models. This follows that the exponents $m'$ from the bulk behavior are no longer independent,

$$
m' = m - m^* + 2n + 1
$$

(3.44)

with the integer $n$ given by

$$
n = \left\lfloor \frac{m - m^*}{p} \right\rfloor,
$$

(3.45)

where the brackets $[x]$ denotes the greatest integer less than or equal to $x$. Here for the periodic boundary system we need two parameters $m, m^*$ and we suppose that $m^* = 1, 2, \ldots, h - p - 1$ and $m'$ is given by (3.44). For the largest eigenvalue (or the ground
state), the appropriate choices are \( m = m' = m'' = 1 \) and \( m, m', m'' > 1 \) give the low-lying excited states. The open boundary systems possess the \( U_q[sl(2)] \) invariance. According to the study of \( XXZ \)-chain \([11]\) we modify (3.44) to be

\[
m' = m + 2n \quad n = \left\lfloor \frac{m - 1}{p} \right\rfloor \quad (3.46)
\]
or suppose that \( m'' = 1 \) and then \( m' \) is determined by \( m \). The low-lying excited states are given by \( m, m > 1 \).

The solution \( t_{\text{bulk}}^{(p)}(u) \) is given by

\[
t_{\text{bulk}}^{(p)}(u + \lambda) = (1 + t_{\text{bulk}}^{(p+1)})(1 + t_{\text{bulk}}^{(p-1)}) \nonumber = 16 \cos^2 \sigma \cos^2 \tau \quad (3.47)
\]

Thus we find lastly that

\[
t_{\text{bulk}}^{(p)}(u) = \pm 4 \cos \sigma \cos \tau \left( \tan \left( \frac{1}{2} hu \right) \right)^N. \nonumber (3.48)
\]

### 3.4 Central charge and conformal weights

The finite-size corrections are only dependent on the braid and bulk limits of the models. In these limits the functional relations are truncated and therefore the sum in (3.26) can be replaced with the finite sum

\[
2 \int_{-\infty}^{\infty} e^{-y} L^p(y) \, dy = - \sum_{q=1}^{h-3} L \left( \frac{1}{A^q} \right) \left[ \int_{-\infty}^{\infty} + \frac{1}{2} \sum_{q=1}^{h-3} D^q L^p \right] \quad (3.49)
\]

where the constants \( D^{(b,q)} \) can be zero or non-zero because the different branches can be taken for the dilogarithmic functions in the nonlinear integral equations. The choice of branches have to give the right finite size corrections. Simply taking \( D^{(b,q)} = 0 \) is consistent with the asymptotics solutions given in subsection \( 3.3 \). To take nonzero \( D^{(b,q)} \) we have to single out the right branches of the dilogarithm for the asymptotic solutions of the equations, which have been shown for ABF models in \([12]\).

The following useful dilogarithm identity has been established by Kirillov \([15]\). Consider the functions

\[
y^{(q)}(j, r) := \frac{\sin(q + 2) \varphi \sin(q \varphi)}{\sin^2(\varphi)} \quad 1 \leq b \leq n - 1, \quad 1 \leq q \leq r \quad (3.50)
\]

with

\[
\varphi = \frac{(1 + j) \pi}{2 + r} \quad 0 \leq j \leq r. \quad (3.51)
\]
It is obvious that they are the asymptotic solutions of the functions equations (3.20) with \( r = h - 2 \) or the bulk behavior of the functions equations with \( r = p \) and \( r = h - 2 - p \).

Then the following dilogarithmic function identity holds,

\[
s(j, r) := \sum_{q=1}^{r} \frac{1}{1 + y^{q}(j, r)} = L(1) \left( 3r \frac{2r}{2 + r} - \frac{6j(j + 2)}{2 + r} + 6j \right). \tag{3.52}
\]

In terms of the dilogarithm function the finite-size corrections (3.22) are expressed as

\[
\ln T^{(p)}(x) = \cos \frac{x}{\sqrt{N}} \left( \sum_{q=1}^{h-3} L \left( \frac{1}{A^q} \right) \right) + \frac{1}{2} \sum_{q=1}^{h-3} D^q \ell A^q + o \left( \frac{1}{N} \right). \tag{3.53}
\]

Here we take the case of \( D^q \neq 0 \). Note that the nonlinear integral equations (3.20) including the closure condition (3.36) and their solutions presented in subsection 3.3 are the same as ones of the ABF models studied in [42]. Therefore we can calculate the finite size corrections in the same way. Similarly to [42, 45], it can be shown that in terms of the functions \( s(j, r) \) the finite-size corrections (3.53) for the open boundary systems can be expressed as

\[
\ln T^{(p)}(x) = \frac{\pi \cos x}{6N} \left( s(0, h - 2 - p) + s(\nu, p) - s(m - 1, h - 2) - \frac{6(1-m)(p+1-m) + 6\nu(p-\nu)}{p} \right) + o \left( \frac{1}{N} \right), \tag{3.54}
\]

where \( \nu \) is an unique integer determined by

\[
\nu = m - 1 - \left\lfloor \frac{m - 1}{p} \right\rfloor p. \tag{3.55}
\]

Inserting (3.52) into (3.54) we have the finite-size correction

\[
\ln T^{(p)}(x) = \frac{\pi \cos x}{6N} \left( c - 24 \Delta_{m,\nu} \right) \cosh x + o \left( \frac{1}{N} \right), \tag{3.56}
\]

where the center charges \( c \) are given by

\[
c = \frac{3p}{p + 2} - \frac{6p}{h(h - p)} \tag{3.57}
\]

and the conformal weights are given by

\[
\Delta_{m,\nu} = \frac{(h - (h - p)m)^2 - p^2}{4hp(h - p)} + \frac{\nu(p - \nu)}{2p(p + 2)} \tag{3.58}
\]

where \( p = 1, 2, \cdots, h - 2, \ m = 1, 3, \cdots \leq h - 1 \). Taking into account the geometrical factor \( \cosh x = \sin(uh) \) and \( \sinh x = i \cos(hu) \) we obtain the expression of finite-size...
correction to the energy given in [16, 17]. For the special case of \( p = 2 \) and \( m - 1 = \text{even} \) the same result has been claimed in [35].

For the twisted boundary case we have

\[
\ln \mathcal{T}^{(p)}(x) = \frac{\pi \cosh x}{6N} \left( s(m - 1, h - 2 - p) + s(\nu, p) - s(m - 1, h - 2) \right.
\]

\[
- \frac{6(m - m')(p + m' - m) + 6\nu(p - \nu)}{p} + o\left(\frac{1}{N}\right)
\]

\[
= \frac{\pi}{6N} \left( c - 24\Delta_{m-m',\nu,m} \right) \cosh x + o\left(\frac{1}{N}\right), \tag{3.59}
\]

where \( \nu \) is an unique integer determined by

\[
\nu = m - m' - \left\lfloor \frac{m - m'}{p} \right\rfloor p \tag{3.60}
\]

The conformal weights are given by the following standard expression

\[
\Delta_{t,\nu,s} = \frac{\left(h \tilde{t} - (h - p)s\right)^2 - p^2}{4hp(h - p)} + \frac{\nu(p - \nu)}{2p(p + 2)}, \tag{3.61}
\]

with

\[
\nu = s - t - \left\lfloor \frac{s - t}{p} \right\rfloor p \tag{3.62}
\]

given in [31, 12]. Comparing with the standard expression we find that \( t = 1, s = 1, 3, \cdots \leq h - 1 \) for the fused six-vertex models with the open boundary or \( t = 1, 2, \cdots, h - p - 1, s = 1, 2, \cdots, h - 1 \) for the fused six-vertex models with the twisted boundary.

4 Discussion

We have constructed the functional relations among the fused transfer matrices of the six-vertex model with the open boundary conditions. The fusion procedure shows that the fusion level of the model can be any positive integers and therefore the fusion hierarchy of the six-vertex model with periodic boundary conditions is infinite. The functional relations correspond to the su(2) fusion rule without truncation. It means that the underlying algebra su(2) has infinite level. This theory has the central charge \( 3p/(2 + p) \).

By open boundaries, however, the situation can be dramatically changed. The functional relations for the six vertex model with the open boundary in braid limit are truncated for \( q^h e^{h\lambda} = -1 \). So the su(2) fusion rule with a finite level appears again for the model. This shows that the open boundary lower the central charges to be less than \( 3p/(2 + p) \).
The central charges of underlying conformal field theories for the fused six-vertex models with the open boundary condition have been found to be the same as those for the fused ABF models. The open boundaries ensure \( U_q[sl(2)] \) invariance and also play the role of the charge at infinite in the Feigin Fuchs construction \[57\]. The conformal weights of the models now take only subset of the Kac formula (3.61) or \( \Delta_{s,\nu} = \Delta_{1,\nu,s} \). The quantum number \( s \) from the braid solutions of the model is specified by the symmetric algebra \( U_q[sl(2)] \). That means that the spin \( S \) enters the calculation through the asymptotics of inversion identity hierarchies \( t^q \). Suppose that \( S_z = j \). Then we have \( s = 2j + 1 \) \[35\]. Specially, we have known that the partition function is a single form in Virasoro characters \[11, 58, 35\].

For the models with the twisted boundary we have found the same central charges and the conformal weights. But \( U_q[sl(2)] \) is no longer the "symmetric group". With the similar analysis to \[35\] we can find that the states of \( T_m(u) \) with different \( m \) are mapped on each other by the \( U_q[sl(2)] \). We need two integers \( s \) and \( t \) for the representations of Virasoro algebras. As we know the partition function is a sesquilinear form in Virasoro characters.

The six-vertex model with the boundary specified by the reflection matrices \( K_+(u) = K(-u - \lambda, \xi_+) \) and \( K_-(u) = K(u, \xi_-) \)

\[
K(u) = \begin{pmatrix}
\sin(u + \xi) & 0 \\
0 & \sin(-u + \xi)
\end{pmatrix}
\]

is the original model studied by Sklyanin \[3\]. This model does not possess \( U_q[sl(2)] \) invariance. In the case of \( \xi_+ + \xi_- = \frac{\pi}{h} \) we can solve the finite size corrections similarly and thus the similar conformal spectra (3.57) and (3.58). One fact to see this is to quickly check the braid limit of the transfer matrices, which are the same as those of the model with the open boundary condition (1.15).

The conformal spectra given in this paper is for the fusion hierarchy and therefore is more general. For the case of \( p = 1 \) the underlying model is the unfused six-vertex model. The conformal spectra coincides with that of spin-\( \frac{1}{2} \) \( XXZ \) chain with open boundary \[1, 34\]. The conformal spectra for the general fusion level \( p \) gives the spectra of spin-\( \frac{1}{2}p \) \( XXZ \) chains with the open boundaries.

The analytic method to calculate the finite size corrections of transfer matrix by solving the functional equations has been described for study of the ABF models \[12\]. Here we have generalized the method to find the finite size corrections of transfer matrix with open boundaries. We like to mention that there is other method available to calculate the finite size corrections of transfer matrix with open boundaries, which generalizes the method described in \[60, 71\].
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Appendix A: Fused weights of the six-vertex model

We give the explicit expression for the fused weights of the six-vertex model in this section. Let $Y_p$ be the projector on the space of symmetric tensor in $\mathbb{C}^{2p}$,

$$Y_p = \frac{1}{p!}(P^{1,p} + \cdots + P^{p-1,p} + I) \cdots (P^{1,2} + I)$$

$$P^{i,j} = R^{i,j}(0)/\sin(\lambda).$$

(A.1)

The fused weight $R_{p,q}(u) \in \mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}$ is defined by

$$R_{(p,q)}(u) = Y_q R_{(p,q)}(u - q\lambda + \lambda) \cdots R_{(p,2)}(u - \lambda) R_{(p,1)}(u) Y_q$$

(A.2)

$$R_{(p,j)}(u) = Y_p R^{1,j}((u + p\lambda - \lambda) \cdots R^{p,j}(u) Y_p$$

(A.3)

The derivation of the fused $R$ matrix is straightforward and it is $(p + 1)^2$ by $(p + 1)^2$ matrix with the following elements

$$R_{(p,q)}(u)_{i,j}^{k,l} = C(u) \sum_n \left(F(n)_{i,j}^{k,l} F(n, u)_{i,j}^{k,l}\right) \quad i, j, k, l = -s, -s + 1, \ldots, s$$

(A.4)

$$C(u) = \prod_{m=0}^{2s-1} \left(\sin^{-1}(2s\lambda - m\lambda) \prod_{n=1}^{2s-1} \sin(u + n\lambda - m\lambda)\right)$$

(A.5)

$$F(n)_{i,j}^{k,l} = \prod_{m=1}^{l+n} \frac{\sin(s + i + n - m + 1)\lambda}{\sin(m\lambda)} \prod_{m=1}^{n} \frac{\sin(s - i - m + 1)\lambda}{\sin(m\lambda)} \prod_{m=1}^{s-j-1} \frac{\sin(s - j - m)\lambda}{\sin(2s - m)\lambda}$$

(A.6)

$$F(n, u)_{i,j}^{k,l} = \prod_{m=1}^{s-l-n} \frac{\sin(u + s\lambda - i\lambda - n\lambda - m\lambda + \lambda)}{\sin(m\lambda)} \prod_{m=1}^{s-j-n} \frac{\sin(u + i\lambda + j\lambda - m\lambda + \lambda)}{\sin(m\lambda)}$$

(A.7)

where $s = \frac{1}{2}p$ and the sum over $n$ is from $\max(0, j - l)$ to $\min(s + j, s - l, s - i)$. The matrix $R_{p,1}(u)$ can be written as 2 by 2 matrix

$$R_{p,1}(u) = \prod_{n=1}^{p-1} \sin(u + n\lambda) \sum_{m=0}^{p-1} \left(\frac{\sin(u + p\lambda - m\lambda)e_{m,m}}{\sin(m\lambda)e_{m-1,m}} \frac{\sin(p\lambda - m\lambda)e_{m+1,m}}{\sin(u + m\lambda)e_{m,m}}\right)$$

(A.8)

where the matrix $e_{i,j} \in \text{End}(\mathbb{C}^{2s+1})$ has only non-zero entry $(i, j)$ which is 1.
Appendix B: Eigenvalue problem of $T^{(p,1)}$

Here we would like to show the Bethe ansatz solutions of $T^{(p,1)}$. The transfer matrix $T^{(p,1)}$

$$T^{(p,1)}(z) = \text{tr } K_+(zq^{1/2}) U^{(p,1)}(z) K_-(zq^{-1/2}) U^{-1}_{(p,1)}(z^{-1})$$  \hspace{1cm} (B.1)

with the monodromy matrix

$$U^{(p,1)}(z) = R_{(1,p)}^c(z) \cdots R_{(1,p)}^{c,2}(z) R_{(1,p)}^{c,1}(z)$$  \hspace{1cm} (B.2)

is given by the fusion in the "quantum" space. Removing the zeros generated from fusion and replacing $u$ by $u - \frac{1}{2} p \lambda$ the fused $R$ matrix reads

$$R_{(1,p)}(u) = \sum_{m=0}^{p-1} \begin{pmatrix} \sin(u + \frac{1}{2} p \lambda - m \lambda) e_{m,m} & \sin(p \lambda - m \lambda) e_{m+1,m} \\ \sin(m \lambda) e_{m-1,m} & \sin(u - \frac{1}{2} p \lambda + m \lambda) e_{m,m} \end{pmatrix}$$  \hspace{1cm} (B.3)

which can be rewritten as

$$R_{(1,p)}(z) = \begin{pmatrix} z q^{S^z} - z^{-1} q^{-S^z} & (q - q^{-1}) S^- \\ (q - q^{-1}) S^+ & z q^{-S^z} - z^{-1} q^{S^z} \end{pmatrix}$$  \hspace{1cm} (B.4)

where the operators $S^z, S^\pm$ are generators of $U_q[sl(2)]$

$$q^{S^z} S^+ q^{-S^z} = q^+ S^\pm \quad S^+ S^- - S^- S^+ = [2S^z]_q$$  \hspace{1cm} (B.5)

The matrix (B.4) is just the $L$-matrix used in [12]. Therefore the Bethe ansatz solutions of $T^{(p,1)}(z)$ defined in (B.1) have been given exactly in [12] (also see [13]), which are the equations (2.16) and (2.17).

Appendix C: Functional equations from fusion procedure

In this section we explain how the functional equations of the fused transfer matrices come out from fusion procedure. As a simple example, we only show that

$$T_0^{(1)} T_1^{(1)} = T_0^{(2)} + f_0^p I$$  \hspace{1cm} (C.1)

graphically. For this purpose let us represent the $R$- and $K$-matrices by

$$R^{12}(u) = \quad \quad K_-(u) = \quad \quad K_+(u) = \quad \quad$$  \hspace{1cm} (C.2)

Therefore we can represent the Yang-Baxter equation (1.6) by

$$1 \quad 2 \quad 3$$

and

$$1 \quad 2 \quad 3$$

\hspace{1cm} (C.3)
the reflection equation (1.12) by

\[
\begin{array}{c}
\begin{array}{ccc}
1 & u & 2 \\
& u & 2 \\
1 & u & 2 \\
\end{array}
\end{array}
\quad \quad \quad
\begin{array}{c}
\begin{array}{ccc}
1 & u & 2 \\
& u & 2 \\
1 & u & 2 \\
\end{array}
\end{array}
\quad \quad \quad
\begin{array}{c}
\begin{array}{ccc}
1 & u & 2 \\
& u & 2 \\
1 & u & 2 \\
\end{array}
\end{array}
\quad \quad \quad
\begin{array}{c}
\begin{array}{ccc}
1 & u & 2 \\
& u & 2 \\
1 & u & 2 \\
\end{array}
\end{array}
\end{array}
\]

and the unitary relation \( R^{1,2}(u)R^{1,2}(-u) \sim I \) by

\[
\begin{array}{c}
\begin{array}{ccc}
1 & u & 2 \\
& u & 2 \\
1 & u & 2 \\
\end{array}
\end{array}
\sim \begin{array}{c}
\begin{array}{ccc}
1 & u & 2 \\
& u & 2 \\
1 & u & 2 \\
\end{array}
\end{array}
\sim \begin{array}{c}
\begin{array}{ccc}
1 & u & 2 \\
& u & 2 \\
1 & u & 2 \\
\end{array}
\end{array}
\]

The transfer matrix \( T(u) \) with open boundary (1.13) is then represented by

\[
\begin{array}{c}
\begin{array}{ccc}
K_+(u+\frac{1}{2}\lambda) & \cdots & u \\
& u & \cdots \\
& u & \cdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
K_-(u-\frac{1}{2}\lambda) & \cdots & u \\
& u & \cdots \\
& u & \cdots \\
\end{array}
\end{array}
\end{array}
\]

(C.6)

Let us consider \( T_0^{(1)} \), \( T_1^{(1)} \)

\[
\begin{array}{c}
\begin{array}{ccc}
K_+(u+\frac{3}{2}\lambda) & \cdots & u+\lambda \\
& u+\lambda & \cdots \\
& u+\lambda & \cdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
K_-(u+\frac{1}{2}\lambda) & \cdots & u \\
& u & \cdots \\
& u & \cdots \\
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
K_+(u+\frac{1}{2}\lambda) & \cdots & u \\
& u & \cdots \\
& u & \cdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
K_-(u-\frac{1}{2}\lambda) & \cdots & u \\
& u & \cdots \\
& u & \cdots \\
\end{array}
\end{array}
\end{array}
\]

(C.7)

Inserting the identical operator into the position \( a, b \) and using the unitary condition (C.3) and the Yang-Baxter equation (C.3) (the spectra parameter \( u \) is shifted to be \( u - \frac{1}{2}\lambda \) in the monodromy matrix (1.8)), we are able to obtain

\[
\begin{array}{c}
\begin{array}{ccc}
K_+(u+\frac{3}{2}\lambda) & \cdots & u+\lambda \\
& u+\lambda & \cdots \\
& u+\lambda & \cdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
K_-(u+\frac{1}{2}\lambda) & \cdots & u \\
& u & \cdots \\
& u & \cdots \\
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
K_+(u+\frac{1}{2}\lambda) & \cdots & u \\
& u & \cdots \\
& u & \cdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
K_-(u-\frac{1}{2}\lambda) & \cdots & u \\
& u & \cdots \\
& u & \cdots \\
\end{array}
\end{array}
\end{array}
\]

(C.8)

The fully symmetric and antisymmetric operators are given by the \( R \) matrix,

\[
Y_2 = \frac{\lambda}{2}(I + P^{1,2}) = D^+R^{1,2}(\lambda) \quad (C.9)
\]

\[
Y^- = \frac{\lambda}{2}(I - P^{1,2}) = D^-R^{1,2}(-\lambda) \quad (C.10)
\]
where
\[
D^\pm = \begin{pmatrix}
\sin^{-1}(2\lambda) & 0 & 0 & 0 \\
0 & \pm \frac{1}{2} \sin^{-1}(\lambda) & 0 & 0 \\
0 & 0 & \pm \frac{1}{2} \sin^{-1}(\lambda) & 0 \\
0 & 0 & 0 & \sin^{-1}(2\lambda)
\end{pmatrix}
\] (C.11)

Then inserting the identical operator into the position \(a, c\) of (C.8) and using \(Y_2 + Y^- = I\) and the fusion of the \(R\) matrices

\[
\begin{pmatrix} u+\lambda \\ u \end{pmatrix} \sim \begin{pmatrix} u \end{pmatrix}
\] (C.12)

and the \(K\) matrices

\[
K_+(u+\frac{3}{2}\lambda) \sim K_+(1,2)(u+\frac{1}{2}\lambda)
\] (C.13)

\[
K_+(u+\frac{1}{2}\lambda) \sim K_+(1,2)(u-\frac{1}{2}\lambda)
\] (C.14)

The term involved by \(Y_2\) in (C.8) gives the transfer matrix with the open boundary of fusion level 2

\[
K_+(1,2)(u+\frac{1}{2}\lambda) \sim K_-(1,2)(u-\frac{1}{2}\lambda)
\] (C.15)

which is \(T_0^{(2)}\). The another term with the antisymmetric projector \(Y^-\), which collapses the matrix, is proportional to the identical matrix

\[
f_0^1(u) \sim \cdots
\] (C.16)

Therefore we have the equation (C.1). It can be seen that the proof is correct also for the non-diagonal reflection \(K\) matrices. So it is very clear that the \(\text{su}(2)\) fusion rule also works for the six-vertex or the eight-vertex models with non-diagonal reflection matrices. These will be published elsewhere.
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