**Boids in a Loop: Self-Propelled particles within a Flexible Boundary**

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With numerical simulations, we explore the behavior of repelling and aligning self-propelled unipolar particles (boids) in 2D enclosed by a damped flexible, elastic and reflective loop-shaped boundary. Long lasting states include a disordered gas-like state, a unidirectional solid-like jammed state, and circulating states. The circulating states have circular, oval, irregular, ruffled or sprocket-shaped boundaries, depending upon the bending moment of the boundary and the total boundary to particle mass ratio. With the exception of the sprocket shapes, states resemble those exhibited by attracting self-propelled particles, but here the confining boundary acts in place of a cohesive force. We attribute the formation of ruffles on the boundary to instability mediated by pressure on the boundary when the speed of waves on the boundary approximately matches the self-propelled particle’s swim speed.

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**I. INTRODUCTION**

Active systems are non-equilibrium systems comprised of units that consume energy to generate motion. Inspired by biological systems exhibiting collective phenomena such as flocking [1], artificial systems have been designed that inject energy at the microscopic level and emulate the unique properties of their biological counterparts, including flocking, spontaneous aggregation and formation of vortex or ring-like collective motion (e.g., [2, 8]).

Collective behaviors can also emerge in confined geometries due to boundary/active particle interactions or due to complex interplay with the surrounding fluid (e.g., [9–11]). Macroscopic collective motions may be induced via the action of confining walls on the micro-scale motions. For example, confining parallel walls can induce wavelike cell migration modes [12]. Active particles can interact collectively with movable rigid or flexible objects. For example, folding configurations of a flexible polymer chain can be affected by fluctuations in an active medium [13]. Active particle motion can be harnessed to power microscopic rotating gears [14, 15]. Boundaries can be incorporated into the design of active matter based devices, for example, to generate fluid flow from confined bacteria [16]. For a review of active particles in crowded environments see Bechinger et al. [17]. We focus here on self-propelled particles that are confined by a flexible loop-shaped boundary (e.g., [18, 22]).

Soft boundaries, including loops, membranes, thin elastic rods or plates, are interesting potential components for design. Soft boundaries can influence collective motion in active matter due to particle/boundary interactions. For example, ‘swim pressure’ describes how particles push on a boundary [19, 23, 25]. Flexible materials can dynamically respond with more degrees of freedom than rigid bodies such as walls, wedges or ratchets. In this study we numerically explore the behavior of self-propelled particles that are enclosed within a flexible circular boundary. We search for forms of collective behavior involving motions in the boundary, such as growth of an oval or dumbbell shape [20, 22] or instability to growth of waves on the boundary [19]. We are motivated to understand interactions between active matter and boundaries that can respond to them so as to expand on the potential types of artificial mechanisms that can harness collective motion.

We work with the class of Dry Aligning Dilute Active Matter which is called DADAM, (see [26]). Discrete unipolar self-propelled particle models [2, 8], come in deterministic or stochastic varieties (e.g., [5, 26, 30]) and the self-propelled particles within them are sometimes called ‘boids’, following Reynolds [2]. We focus here on the deterministic variety. Our study is most similar to the numerical work by Nikola et al. [19], Paoluzzi et al. [20], Wang et al. [22] and experimental study of vibrating robotic rods by Deblais et al. [21] who also studied of repulsive active particles in 2 dimensions that interact with a flexible boundary. However our simulations lack stochastic perturbations and particles within our simulations align their direction of motion with the direction of nearby particles. Our simulations include a steering align force, as do simulations of flocking behavior (e.g., [2, 3, 5, 28]). Prior simulations of self-propelled particles within a flexible loop have focused on non-aligning self-propelled particles with stochastically perturbed directions of motion (e.g., [20, 22]).

In section III, we describe our model for simulating self-propelled particles that are enclosed inside a flexible boundary. In section III we illustrate phenomena seen with our simulations, discuss the collective behavior and the nature of instability on the boundary. A summary
and discussion follows in section IV.

II. BOID AND BOUNDARY MODEL

A system of self-propelled particles can be described at a fine-grained level taking into account the self-propulsion mechanism, the internal degrees of freedom of microswimmers, and the hydrodynamics. Alternatively the dynamics can be approximated via a coarse-grained approach where the motion of the self-propelled particles is described with effective forces [31]. We adopt the second approach and neglect background hydrodynamic-like interactions.

Our model has two particle components, a boundary that is comprised of discrete mass nodes, and a flock of self-propelled particles or boids. We describe in detail our numerical implementation as it contains more degrees of freedom than simulations of unconfined self-propelled particles (e.g., [28]) or self-propelled particles with periodic boundary conditions (e.g., [32]).

Both boundary nodes and boids can move and are massive, however boundary nodes remain in a linear chain. Particle and node positions are denoted with $x_i$ and the index identifies the particle or node. The coordinates are in two-dimensions only. The flexible boundary is initially a circular loop and encloses the boids. We first discuss the numerical description of the flock of boids, then the boundary, then we discuss interactions between boids and boundary.

A. The Flock of Boids

A boid with index $i$ has position $x_i^n$ at time denoted with index $n$. The boid velocity at the same time is $v_i^n$ and its mass is $m_i$. The total number of boids is $N_{boids}$ and the total mass in boids is $M_{boids} = N_{boids}m_i$. We update boid positions and velocities using the first order (in time) Eulerian method (as did [27]) and with a fixed time step $dt$

$$x_i^{n+1} = x_i^n + v_i^n dt$$

$$v_i^{n+1} = v_i^n + \frac{dt}{m_i} \sum F_i^n$$

$$\sum F_i^n = F_{align,i}^n + F_{repel,i}^n + F_{interact,i}^n$$

where $\sum F_i^n$ is a sum of forces that depend on boid position and velocity ($x_i^n, v_i^n$), neighboring boid positions and velocities ($x_j^n, v_j^n$ with $j \neq i$), and nearby boundary node positions. It is useful to define a vector between two boids $r_{ij} = x_i - x_j$, distance $r_{ij} = |r_{ij}|$, and direction that is the unit vector $\hat{r}_{ij} = r_{ij}/r_{ij}$.

For our self-propelled particles, we employ a Vicsek type of model [3] causing nearby particles to align but we lack stochastic perturbations that would change the direction of motion, and we include an additional inter-boid repelling force (e.g., as used by [3] [19] [20] [22] [28]). We do not apply an inter-boid attractive or cohesive force.

The repel force on boid with index $i$ caused by another boid with index $j$

$$F_{repel,i} = \frac{m_{boid}U_{repel}}{d_{repel}} e^{-r_{ij}/d_{repel}} \hat{r}_{ij}.$$  \hspace{1cm} (4)

Here $U_{repel}$ has units of the square of velocity and $d_{repel}$ characterizes the scale of the repulsive interaction. We only apply the repel force for boid pairs separated by $r_{ij} < 2d_{repel}$. The repel force is exponential (as was that adopted by [28]). We also explored a repel force proportional to the inverse interboid distance and saw similar collective phenomena.

The align or steer force exerted on boid $i$ due to other boids is

$$F_{align,i} = \alpha_{align} m_i(v_0\hat{w} - v_i)$$  \hspace{1cm} (5)

$$w = \sum_{i \neq j, r_{ij} < d_{align}} v_j$$  \hspace{1cm} (6)

$$\hat{w} = \frac{w}{|w|}.$$  \hspace{1cm} (7)

Here $\alpha_{align}$ has units of inverse time and $v_0$ is the boid speed, equal to the ‘terminal velocity’ in the model by Touma et al. [28]. The unit vector $\hat{w}$ is computed from the average of the velocities of nearby boids. It is multiplied by $v_0$ so that the boid accelerates if its speed is slower than $v_0$ and it decelerates if it is going faster than $v_0$. The average velocity $\hat{w}$ is computed for boids within a distance $d_{align}$ which characterizes the scale of the alignment interactions. For boids with no neighbors within $d_{align}$, we propel them using the boid’s own current velocity direction $\hat{v}_i$ in place of the average $\hat{w}$ in equation 5. The align or steer force also serves to propel the boids at a velocity that is approximately $v_0$.

B. The Flexible Elastic Boundary

The numerical description of our flexible boundary is similar to that used by Nikola et al. [19] (see VI of their supplements). The boundary is described with a chain of mass nodes each of mass $m_{node}$ and each initially separated from its two nearest neighbors by a distance $\Delta s$. The chain is closed by connecting its two endpoints so that it forms a loop. A node at position $x_i$ has neighbors $x_{i+1}$ and $x_{i-1}$ with indices given modulo the total number of nodes in the chain, $N_{nodes}$. The total mass in the boundary is $M_{nodes} = N_{nodes}m_{node}$. To maintain boundary length, each consecutive pair is separated by a spring with rest length $\Delta s = 2\pi R/N_{nodes}$ where $R$ is the initial loop radius. Using a thin elastic beam approximation, we apply forces to the nodes that allow the boundary to resist bending. We first discuss the bending forces and then the spring forces.
We update node positions and velocities using equations 8 and 9 but with \( m_{\text{boid}} \) replaced with \( m_{\text{node}} \). Instead of equation 3, the sum of forces on node \( i \) at time step \( n \) is

\[
\sum F_i^n = F_{\text{bend},i}^n + F_{\text{spring},i}^n + F_{\text{interact},i}^n + F_{\text{damp},i}^n
\]

(8)

and the forces depend on node positions and velocity \((x_i^n, v_i^n)\), neighboring node positions and velocities \((x_j^n, v_j^n)\) with \( j \neq i \), and nearby boid positions.

The Euler-Bernoulli theory of thin elastic beams describes the centerline of a beam with a curve \( X(s) \) where \( ds \) gives length along the boundary. The elastic potential energy depends on

\[
U_{\text{bend}} = \int ds \frac{\alpha_{\text{bend}}}{2} (X''(s))^2
\]

(9)

where \( X'' = \frac{\partial^2 X(s)}{\partial s^2} \) is the curvature. The coefficient \( \alpha_{\text{bend}} = EI \) is known as the bending moment or flexural rigidity, with \( E \) the elastic modulus and \( I \) is the second moment of area integrated on the beam’s cross-section. For a linear beam oriented on the \( x \) axis with linear mass density \( \mu \), and displacement from the \( x \) axis \( w(x, t) \), the above potential energy gives equation of motion

\[
\mu \frac{\partial^2 w}{\partial t^2} = -\alpha_{\text{bend}} \frac{\partial^4 w}{\partial x^4}.
\]

(10)

We discretize our boundary by putting its mass into a consecutive set of mass nodes \( x_i \), each separated by distance \( \Delta s \). The curvature at a node

\[
x_i'' \approx (\Delta s)^{-2} (x_{i+1} + x_{i-1} - 2x_i).
\]

(11)

The potential energy for the discrete chain

\[
U_{\text{bend}} = \sum_i \frac{\alpha_{\text{bend}}}{(\Delta s)^3} (3|x_i|^2 + x_i \cdot x_{i+2} - 4x_i \cdot x_{i+1}).
\]

(12)

Taking the derivative of potential energy \( U \) with respect to node position \( x_i \) gives the force on a node

\[
F_{\text{bend},i} = -\frac{\partial U}{\partial x_i} = -\frac{\alpha_{\text{bend}}}{(\Delta s)^3} (x_{i-2} - 4x_{i-1} + 6x_i - 4x_{i+1} + x_{i+2}).
\]

(13)

The equation of motion

\[
m_{\text{node}} \frac{d^2 x_i}{dt^2} = -\frac{\alpha_{\text{bend}}}{(\Delta s)^3} (x_{i-2} - 4x_{i-1} + 6x_i - 4x_{i+1} + x_{i+2})
\]

(14)

is a discrete approximation to the equation of motion from Euler-Bernoulli elastic beam theory (e.g., [33, 34]).

We insert a spring between each consecutive node on the boundary. The springs are intended to maintain a nearly constant length boundary. The total potential energy due to springs is

\[
U_{\text{spring}} = \sum_i \frac{k_s}{2} (v_{i,i-1} - \Delta s)^2
\]

(15)
where \( r_{i,i-1} = |x_i - x_{i-1}| \) is the distance between two consecutive nodes, \( \Delta s \) is the rest spring length and \( k_s \) the spring constant. The force exerted on each node due to the springs is

\[
F_{\text{spring},i} = -k_s \frac{(x_i - x_{i-1})}{r_{i,i-1}} (r_{i,i-1} - \Delta s) - k_s \frac{(x_i - x_{i+1})}{r_{i,i+1}} (r_{i,i+1} - \Delta s).
\]

This follows common implementations of N-body mass/spring models (e.g., \[35\]).

To mimic an external viscous or friction like boundary interaction, we add a velocity dependent damping force on each boundary node

\[
F_{\text{damp},i} = -m_{\text{node}} \gamma_{\text{damp}} v_i,
\]

where damping parameter \( \gamma_{\text{damp}} \) is in units of inverse time and \( v_i \) is velocity of the node.

### C. Boundary Node/Boid interactions

We apply an equal and opposite repulsive force to each pair of boundary and boid particles. The force on particle \( i \) (either a boundary node or boid) from a particle \( j \) (of the opposite type)

\[
F_{\text{interact},i} = -F_{\text{interact}} e^{-r_{ij}/d_{\text{interact}}} \hat{r}_{ij}.
\]

The distance \( d_{\text{interact}} \) describes the range of the interaction. We only apply the force at distances \( r_{ij} < 3d_{\text{interact}} \). The parameter \( F_{\text{interact}} \) determines the strength of the interaction. This size and distance are not important as long as the interaction force causes accelerations that exceed those from other forces and so causes reflection off the boundary faster than interboid distance travel times.

### D. Numerical Implementation

All boundary node masses are equivalent and all boid masses are equivalent, however node mass is usually not equal to boid mass. The total number of boids and nodes remains fixed during the simulation. For visualization, we translate the viewing window so that it is centered on the center of mass of the boundary.

The simulations are initialized with boids initially confined within a circle with radius of 0.9 the initial boundary radius, \( R \). Boids are initially uniformly and randomly distributed within this circle. We explored two types of initial conditions for the boids, an initially rotating flock and a nearly stationary flock. In both cases we also added a small initial random velocity, uniformly distributed in angle, of size 0.1 \( v_0 \), where \( v_0 \) is the boid swim speed. The rotating swarm has boids initially rotating about the boundary center at a velocity of 0.8 \( v_0 \). Circulating initial conditions are chosen when we study the circulating states, whereas random initial conditions without mean rotation are chosen when we study the transitions between gaseous-like, circulating and jammed states.

The boundary nodes are initially placed in a circle of radius \( R \), equally spaced and at zero velocity. Springs between neighboring nodes are initially set to their rest length and all springs have the same spring constant. The bending moment does not vary as a function of position on the boundary.

We work in units of boid speed \( v_0 \), initial boundary radius, \( R \) and total boid mass \( M_{\text{boids}} \). A unit of time is

\[
t_R = R/v_0,
\]

which is the time for a lone boid moving at \( v_0 \) to cross the radius \( R \) of the boundary. After choosing these units, the free parameters are the total boundary mass \( M_{\text{nodes}} \) which is also the boid to boundary mass ratio, the number of nodes and boids \( N_{\text{nodes}} \) and \( N_{\text{boids}} \), the alignment force strength and length scale, \( \alpha_{\text{align}} \) and \( d_{\text{align}} \), the repel force strength and length scale \( U_{\text{repel}} \) and \( d_{\text{repel}} \), the bending moment, \( \alpha_{\text{bend}} \), the node damping parameter \( \gamma_{\text{damp}} \), the node-boid interaction strength and length scale, \( F_{\text{interact}} \) and \( d_{\text{interact}} \), and the spring constant \( k_s \). To run a simulation we also require a time step \( dt \), which is fixed during the simulation, and a maximum length of time \( t_{\text{max}} \) to integrate. This is a large parameter space, but not all these parameters should affect the collective dynamics. As long as number of nodes is high enough that the boids are confined and they smoothly interact with the boundary, the dynamics should not depend on the number of nodes in the boundary or the parameters describing the boid/node interactions. The springs are used to set the boundary length so the spring constant should not affect the dynamics. The dynamics could depend upon the number of and mass of boids as the swim pressure, or pressure exerted by boids on the boundary, depends on their number density.

### E. The time step

The speed of compression waves traveling in a linear mass/spring chain is

\[
v_c = \sqrt{\frac{k_s}{m_{\text{node}}} \Delta s} = \sqrt{\frac{k_s}{m_{\text{node}}} \frac{2\pi R}{N_{\text{nodes}}}}.
\]

For numerical stability, a CFL-like condition for the time step is that it must be less than the time it takes a compression wave to travel between nodes or

\[
dt < \sqrt{\frac{m_{\text{node}}}{k_s}}.
\]

In the continuum limit, equation \[14\] gives a dispersion relation for bending waves equivalent to that from Euler-Bernoulli beam theory

\[
\omega^2 = \alpha_{\text{bend}} \frac{2\pi}{\mu} k_s^4.
\]
where $\alpha_{\text{bend}}$ is the bending moment or flexural rigidity, $\mu = m_{\text{node}}/\Delta s$ is the linear mass density, $\omega$ is angular wave frequency and $k$ the wavevector. The simulation time step should be chosen so that small corrugations in the boundary are not numerically unstable. Taking the wave speed for wavevector $k = 1/\Delta s$, from the node separation, a condition on the time step for numerical stability is

$$dt < \sqrt{m_{\text{node}}/\alpha_{\text{bend}}}(\Delta s)^2. \quad (23)$$

The time step should be shorter than the time it takes a boid to travel between boundary nodes, the mean distance between boids, and the repel, align and boundary interaction distances,

$$dt < \min \left( \frac{\Delta s}{v_0}, \frac{1}{v_0} \sqrt{\frac{\pi R^2}{N_{\text{boids}}} \frac{d_{\text{repel}}}{v_0} \frac{d_{\text{align}}}{v_0} \frac{d_{\text{interact}}}{v_0}} \right). \quad (24)$$

We chose time step satisfying the conditions of equations 21, 23 and 24 with equation 23 usually the most restrictive.

The springs are present to keep the boundary length nearly constant. We would like the springs to be strong enough that the choice of spring constant does not affect the simulation collective behavior. Because they must turn, boids circulating near a circular boundary exert a pressure on the boundary. The force per unit length on the boundary is $p \sim M_{\text{boids}} v_0^2 \frac{\pi}{2\pi R}$. This pressure is balanced by a tension in the boundary (sometimes called wall tension and related to hoop stress) that depends on its curvature, $p \sim T/R$. Balancing these two estimates, we estimate the tension on the boundary

$$T \sim M_{\text{boids}} v_0^2 \frac{1}{R} \frac{\pi}{2\pi R}. \quad (25)$$

This tension can stretch each spring by $\delta x$ from its rest length with tension $T = k_s \delta x$. The spring strain is $\epsilon = \delta x/\Delta s$ with spring rest length $\Delta s = 2\pi R/N_{\text{nodes}}$. Setting tension from wall strain equal to that from spring tension, we solve for the spring strain to give a dimensionless parameter

$$\epsilon_{k_s} = M_{\text{boids}} v_0^2 \frac{N_{\text{nodes}}}{(2\pi R)^2} \frac{1}{k_s}. \quad (26)$$

As long as this parameter is small, the springs should remain near their rest length and the choice of spring constant should not affect the behavior of the simulations. We ensure that our spring constant $k_s$ is large enough that $\epsilon_{k_s} < 1$ is satisfied.

### F. Other constraints on parameters

The boundary/boid interaction should primarily cause boids to reflect off the boundary. The acceleration on the boids from the boundary nodes should exceed the interboid repel force

$$\frac{F_{\text{interact}}}{m_{\text{boid}}} \frac{d_{\text{interact}}}{\Delta s} \gtrsim \frac{U_{\text{repel}}}{d_{\text{repel}}}. \quad (27)$$

where the factor $d_{\text{interact}}/\Delta s$ describes the number of nodes that push away a single boid as it approaches the boundary. We also require internode distance to be similar or less than the boundary interaction distance, $\Delta s \lesssim d_{\text{interact}} \ll R$. The interaction force should not be so large that boids on the boundary move a large distance during a single time step, giving an upper bound

$$\frac{F_{\text{interact}}}{m_{\text{boid}}} \frac{dt}{v_0} \lesssim 1. \quad (28)$$

We maintain these conditions so that the parameters describing the boid/node interaction force should not significantly affect the boid collective behavior. We have halved the time step and doubled the spring constant to check that these did not affect our simulations. We repeated simulations to check that boid distribution and boundary morphologies look similar at the end. There is sensitivity to initial conditions with some simulations freezing or jamming in a bullet-like state and others with the same parameters remaining in a circulating state. This is discussed in more detail in section III.

If the interboid alignment force is too weak, then many boid crossing travel times would be required for collective phenomena to develop. We maintain alignment strength $\alpha_{\text{align}} t_R > 1$ so that self-propelled particles align on a timescale shorter than the travel time across the enclosed region. This condition also ensures that transient behavior decays within a few dozen domain travel times, $t_R$. Likewise we expect that the repel strength divided by the square of the swim speed $U_{\text{repel}}/v_0^2$ should be of order 1 for the boids to effectively repel one another during a simulation extending a few dozen crossing times $t_R$. There is some degeneracy between $\alpha_{\text{align}}$ and $d_{\text{align}}$ in how they affect collective behavior as both affect boid alignment. There is also a degeneracy between $U_{\text{repel}}$ and $d_{\text{repel}}$ in how these affect the collective behavior as they determine interboid repulsion. Consequently we usually set the alignment and repel strengths $\alpha_{\text{align}}$ and $U_{\text{repel}}$, and vary their length scales $d_{\text{align}}$ and $d_{\text{repel}}$ in our numerical exploration of collective phenomena.

The damping parameter $\gamma_{\text{damp}}$ mimics friction or viscous interaction with a background substrate or fluid. If the damping parameter $\gamma_{\text{damp}} t_R \gg 1$ then the boundary is over-damped and will not be sensitive to boid pressure. If $\gamma_{\text{damp}} t_R$ is extremely small, then circulating boids within the boundary will cause the boundary to rotate, eventually matching the boid rotation speed. We set $\gamma_{\text{damp}} t_R = 0.1$, an intermediate value, so that transient behavior will decay within a few dozen crossing times.

To allow transient behavior to decay, we run each simulation for $t_{\text{max}} = 50 t_R$. We show in section III C that the
growth of structure on the boundary usually saturates by this time.

We checked our classification of collective behavior and phenomena with two independently written codes. One version is written in C, uses an openGL display and nearest neighbor searches are accelerated with a 2D quad-tree search algorithm based on the Barnes-Hut algorithm [36]. This code can be found here: https://github.com/jsmucker/boids-in-a-boundary. Another version of our code is written in Javascript using the p5.js library (see https://p5js.org/). This code displays in a web-browser and nearest neighbor searches are not accelerated. This code is available on github at https://github.com/aquillen/boids_in_a_loop. The figures in this manuscript were made with the Javascript code.

III. COLLECTIVE PHENOMENA

In Figure 1 each row shows a series of 11 simulations. Each panel is a simulation snap shot that shows the boid distribution and boundary morphology at the end of a simulation. Boundary particles are shown in red. Each boid is marked with a navy blue isosceles triangle. The vertex with narrowest angle marks the direction of motion. In each simulation series, parameters are identical except for one parameter which is consecutively increased in each simulation. Common parameters for these simulations are listed in Table I. Additional parameters for the series of simulations are listed in Table II. These series have been done with \( N_{\text{boids}} = 400 \), however we saw similar phenomena with \( N_{\text{boids}} = 100, 200 \) and 800. A live animation showing a circulating state can be seen here https://aquillen.github.io/boids_in_a_loop/. This animation is part of the first series of simulations and has bending moment \( \alpha_{\text{bend}}/(M_{\text{boids}}v_0^2R) = 10^{-3} \). The 5-th panel (from the left) in Fig 1a, 7-th panels in Fig 1c and d and 4-th panel in Fig 1e all have parameters approximately the same as this animation.

Below we describe the different types of boid and boundary behavior seen in our simulations. In section IIIA we discuss boundaries between gaseous, circulating and jammed states. In section IIIB we discuss the sensitivity of boundary morphology to simulation parameters. In section IIIC we discuss the nature of the instability that cause the boundary to be ruffled or corrugated.

We see three types of collective phenomena:

The disordered gaseous state. Boids are not aligned with each other, there is little circulation or rotation and the boid velocity dispersion is high. This state is characterized by a weak or very short range alignment force. An example of this state is in the leftmost panel of Figure 1 (fourth row from top). This particular simulation has a very short alignment distance, \( d_{\text{align}} = 0.01R \). Numerically we find that \( d_{\text{align}}/\alpha_{\text{align}}/v_0 \lesssim 0.01 \) tends to give a gaseous state. We see disordered gas-like behav-
ior with little to no align forces, as encountered in simulations of 2-dimensional swarms of unconfined unipolar self-propelled particles. Our model lacks stochastic perturbations, however, billiards within a non-round but convex boundary can be chaotic. Even if our boundary was smooth instead of discrete and boids did not repel, ergodic behavior is introduced via boids reflecting off the boundary and repelling each other. These interactions occur frequently because the boids are confined and would account for the persistence of a disordered phase.

The solid-like jammed bullet state. All boids are moving in the same direction. Boid positions and velocities appear frozen in a frame moving with along with them. The boid velocity dispersion is low and boids do not move relative to each other. This state is characterized by a strong or long range alignment force and a lower mass boundary that is easily pushed by the boids. A low damping rate on the boundary aids in forming this state. An example of this state is in the rightmost panel of Figure 1 (fourth panel from top) with $d_{\text{align}}/R = 1.1$. Numerically we find that this state is likely when $d_{\text{align}}/v_0 \gtrsim 1$. Even though our simulations lack an interboid attractive force, the confinement caused by the boundary gives us a jammed state. This state is similar to the jammed state seen previously in simulations of confined soft repelling self-propelled particles at high density. The simulations by lack an alignment force and their soft boundary was fixed. This state is perhaps also similar to moving cohesive groups or droplet states seen in simulations of unconfined unipolar self-propelled particles that attract each other (e.g., Touma et al. [28], Gregoire and Hugues [32]).

The rotating or circulating states. The boids are circulating within the boundary. The boundary can be rotating but is usually moving more slowly than the boids which all circulate in the same direction. The boundary shape can be circular, oval, irregular or sprocket shaped. Oval loop-shaped flexible boundaries were previously seen in simulations of non-aligning self-propelled particles. We use the word ‘sprocket’ rather than ‘gear’ or ‘ratchet’ to describe states with more than a few radial projections. A sprocket is usually used to engage a chain and is distinguished from a gear in that sprocket teeth are never meshed together. A ‘ratchet’ is part of a mechanical device used for turning objects that allows continuous linear or rotary motion in only one direction.

For the irregular and sprocket shapes, the boundary is deformed by groups of boids. As the boids circulate, bulges in the boundary travel along the boundary. Irregular or sprocket boundaries are more likely if the boundary mass exceeds the total boid mass but the boundary is not so massive that the boids cannot push it. Irregular or sprocket boundaries are more likely with a more flexible rather than stiff boundary. As is true for the bullet states, the circulating states arise in the absence of interboid attraction. The confining boundary serves in place of attractive forces that cause circulating states in unconfined self-propelled particles. Because there is no attraction force between boids, we do not see multiple separate flocks, though we do see clumps of boids in pockets moving along the boundary.

Long lived states can depend on the initial boid velocity distribution. When alignment is strong and the boundary is lower mass, initially rotating boids are less likely to go into the bullet state. Once a system goes into a bullet state, we find that it stays there. Circulating states can nevertheless be long lived and even after long integrations, with $t_{\text{max}} > 100\tau$, the simulation won’t fall into a bullet state even if a different initial velocity distribution would put the system in such a state.

The most interesting of the states seen in our simulations are those where the boundary becomes corrugated. Sokolov et al. [14] and Leonardo et al. [38] describe an asymmetric gear shaped boundary that is spun by bacteria. We find that a flexible loop-shaped boundary can become corrugated and the corrugations can rotate because of unipolar self-propelled particles that move within it. We are probably seeing a modulational instability due to swim pressure inhomogeneities near the boundary that was predicted for non-aligning self-propelled particles by Nikola et al. [19].

Increased boid density near the boundary is particularly noticeable in the simulation with higher repel distance $d_{\text{repel}}$ (Figure 1 or second row). The interplay of self-propulsion, confinement and stochastic processes is often sufficient to explain accumulation of self-propelled particles on a boundary. Here we lack stochastic perturbations, however boundary-boid and boid-boid interactions serve as a source of chaotic behavior that might aid in increasing the boid density near the boundary. Boids on the boundary only feel repulsion from other boids on one side so they can be closer together than boids in the interior.

A. Phase diagrams

In Figure 2, we include phase plots delineating gaseous, circulating and bullet states. Figure 2 shows phases as a function of repel and alignment distances, $d_{\text{repel}}$ and $d_{\text{align}}$, Figure 2b as a function of boundary to boid mass ratio and align distance and Figure 2c as a function of number of boids and align distance. For this last figure we set $d_{\text{repel}} \propto \sqrt{N_{\text{boids}}}$ so that the repel distance divided by mean boid number density remains constant in the different simulations. Otherwise the high number density simulations would be at high pressure as boid repulsion would be pushing them up against the boundary. The parameters for the simulations shown in Figure 2 are listed in Table I and in the rightmost columns in Table II. In this figure, red circles represent simulations giving gases states, green triangles represent those giving circulating states, and blue squares are simulations that ended in bullet states. Classification for this plot was done by eye from simulations run in the browser. We have shaded
the different regions to give a rough view of the locations of the different phases.

The transition between circulating and gas states is not sensitive to parameters other than align force strength and distance and the boid number density. The gas/circulating dividing line on Figure 2 has slope consistent with alignment distance scaling with the mean distance between boids or \( d_{\text{align}} \propto 1/\sqrt{N_{\text{boids}}} \). If the boid number density is higher, a smaller alignment distance can be enough to allow them to circulate.

However, we have noticed that the dividing line between bullet and circulating states is sensitive to some of the other parameters. More flexible, less damped and lower mass boundaries are more likely to elongate and trap boids, aiding in formation of a jammed state. Confined self-propelled soft particles at high density jam [29], and unconfined self-propelled particle with strong cohesion have moving solid-like droplets [28, 32]. The sensitivity of the bullet/circulating phase to \( \alpha_{\text{bend}}, \gamma_{\text{damp}} \) and \( M_{\text{nodes}}/M_{\text{boids}} \) would be consistent with a picture where strong alignment pushes the boids into the boundary, increasing their density, but the jammed state is maintained only when the boundary can fold and trap them.

### B. Sensitivity of boundary corrugations on simulation parameters

We discuss the 5 series of simulations shown in Figure 1 and with parameters listed in Tables I and II. In Figure 1 (top panel) we show a series of simulations, all with the same parameters except that bending moment \( \alpha_{\text{bend}} \) increases from simulation to simulation. The factors used to increase the varied parameter, here \( \alpha_{\text{bend}} \), in each series are also listed in Table II. The varied parameter is computed as follows. The lowest value of \( \alpha_{\text{bend}}/(M_{\text{boids}}v_0^2R) \) in the first series is \( 10^{-4} \). The factor used to vary this parameter is 1.7. The 11th simulation has bending moment \( \alpha_{\text{bend}}/(M_{\text{boids}}v_0^2R) = 10^{-4} \times (1.7)^{10} = 0.02 \). This set has \( d_{\text{repel}} = 0.1 \) so has a fairly short range repulsive force. With a very flexible boundary (on the left in Figure 1) and small \( \alpha_{\text{bend}} \), the boundary has many corrugations. As the bending moment increases, the wavelength of the boundary corrugations increases.

The second series of simulations shown in Figure 1 (second row) is similar to the first series except the repel distance \( d_{\text{repel}} = 0.35 \) is larger. The repel distance is large enough that boids are pushed against the boundary by their repulsion alone. This differs from the simulations at lower \( d_{\text{repel}} \) where only the centrifugal force due to their circulation pushes them up against the boundary. Despite being in a different regime, we also see boundary corrugations in the series shown in Figure 1, and again with wavelength increasing with increasing bending moment. In this regime a single angular Fourier mode often dominates, whereas at lower repel distance \( d_{\text{repel}} \) the boundary corrugations were more irregular. With higher \( d_{\text{repel}} \) and lower bending moment \( \alpha_{\text{bend}} \), the boundary looks like a sprocket or a gear.

We were most surprised by the third series of simulations, shown in Figure 1 (third row). In this series of simulations, the boundary mass is increased, with low mass boundaries on the left and high mass boundaries on the right. We had expected that a lower mass boundary would show more corrugations because it would be easier for the boids to push the boundary. However, we find that the opposite is true; the higher mass boundaries have boundaries with more corrugations.

In Figure 1 (fourth row), we vary the alignment distance \( d_{\text{align}} \). This set of simulations shows the transition from a gas-like state, at low \( d_{\text{align}} \), on the left to the jammed bullet-like state at high \( d_{\text{align}} \), on the right. In some of the intermediate simulations we saw a circulating flock of boids that moved back and forth from one side of a boundary to the other.

In Figure 1 (fifth row), we vary the repel force strength \( U_{\text{repel}} \). This parameter affects the boid density. We find that the boundary is more likely to be corrugated when the boid density is higher near the boundary and at lower repel strength, \( U_{\text{repel}} \).

### C. Instability on the boundary

Prior studies have described the types of collective motion as phases and delineated boundaries between these phases in parameter space, similar to phase transitions (e.g., [3, 28, 29]). The higher number of free parameters present in our system and sensitivity to initial conditions makes it more challenging to delineate transitions between gas-like, solid-like and circulating collective motion. The most novel phenomena illustrated by our dynamical system is corrugations in the boundary that grew during the simulations. The dynamics of the boundary is coupled to the collective motions. Instead of examining in more detail the sensitivity of the gas-like/circulation and circulation/bullet phases to system parameters, we examine the nature of the instability leading to the growth of corrugations on the boundary.

### Table I. Common parameters for simulation series

| Parameter | Value |
|-----------|-------|
| \( N_{\text{nodes}} \) | 150 |
| \( \alpha_{\text{align}}/R \) | 3 |
| \( \gamma_{\text{damp}}/R \) | 0.1 |
| \( \kappa_{s}v_0^{-1}R \) | \( 2 \times 10^4 \) |
| \( F_{\text{interact}}M_{\text{boids}}v_0^{-2}R \) | 1.5 |
| \( d_{\text{interact}}/R \) | 0.02 |
| \( dt/t_R \) | 0.005 |
| \( t_{\text{max}}/t_R \) | 50 |
| \( \epsilon_{k_s} \) | 0.03 |

The parameter \( \epsilon_{k_s} \) is defined in equation 20.
TABLE II. Simulation series

| Varying | bending moment | bending moment | boundary mass | align distance | repel strength | align+repel distances | align distance, boundary mass | align distance, boid number |
|---------|----------------|----------------|---------------|---------------|---------------|----------------------|-----------------------------|-----------------------------|
| Figure | ![image](image1.png) | ![image](image2.png) | ![image](image3.png) | ![image](image4.png) | ![image](image5.png) | ![image](image6.png) | ![image](image7.png) | ![image](image8.png) |
| Factor | 1.7 | 1.7 | 1.5 | 1.6 | 1.5 | - | - | - |
| \( \alpha_{\text{bend}}/(M_{\text{boids}}v_0^2R) \) | \([10^{-4}, 0.01]\) | \([10^{-4}, 0.01]\) | \(10^{-3}\) | \(10^{-3}\) | \(10^{-3}\) | \(10^{-3}\) | \(10^{-3}\) |
| \( M_{\text{nodes}}/M_{\text{boids}} \) | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| \( d_{\text{align}}/R \) | 0.2 | 0.2 | 0.2 | [0.01,1.1] | 0.2 | [0.01,3.3] | [0.01,3.3] | [0.01,1.3] |
| \( U_{\text{repel}}v_0^2 \) | 0.1 | 0.1 | 0.1 | 0.1 | [0.03,1.6] | 0.1 | 0.1 | 0.1 |
| \( d_{\text{repel}}/R \) | 0.1 | 0.35 | 0.1 | 0.1 | 0.1 | [0.04,0.4] | 0.1 | 0.1 \( \sqrt{N_{\text{boids}}/400} \) |
| \( N_{\text{boids}} \) | 400 | 400 | 400 | 400 | 400 | 400 | 400 | [100,800] |

Initial conditions rotating rotating rotating not rotating not rotating not rotating not rotating not rotating

The first row gives the parameter or parameters varied for the series. Each column gives parameters for simulations that are shown in the Figure listed in the second row of the table. Additional parameters for these simulations are listed in Table I. Numbers in brackets give the range for the parameter that is varied. The third row, labelled ‘Factor’ gives the multiplicative factor used to increase the varied parameter for each consecutive simulation in Figure 1.

Hydrodynamic analogies for our boundary corrugations include ripples excited on a flag by wind, or the Kelvin-Helmholtz instability which is driven by the velocity difference across an interface between two fluids. Classically, instabilities can be studied by linearizing equations of motion and deriving a dispersion relation for wave-like solutions. Frequencies that have complex parts are moving next to a curved surface their trajectory must curve. The pressure on the boundary due to the boids depends on the curvature of the boundary and the boid density \( p_{\text{swim}} \propto \rho_{\text{boid}} \frac{\partial^2 w}{\partial x^2} \). The pressure force is opposite that due to tension in the boundary, as it would push in the same direction as a bulge in the boundary, rather than counter it. In this sense, the boid swim pressure acts like pressure variations in an incompressible fluid near a boundary that is derived from linearization of Bernoulli’s equation. We set the pressure on the boundary to

\[
p_{\text{swim}} \sim -\beta_{\text{swim}} M_{\text{boids}} v_0^2 \frac{\partial^2 w}{\partial x^2}.
\]  

The tension related \( k^2 \) and bending rigidity related \( k^4 \) terms are consistent with discussion on the instability discussed by Nikola et al. [19].

If the boids are moving parallel to a straight surface, they will not interact with the boundary. However if they are moving next to a curved surface their trajectory must curve. The pressure on the boundary due to the boids is a dimensionless factor that we can adjust. This gives a simple approximate model for variations in boid pressure exerted along a corrugated boundary and is in a similar form to that predicted in equation 27 by Nikola et al. [19]. This form for the swim pressure gives a term in the wave equation similar to the tension term (see equation 29 for tension) but with the opposite sign (and this is also consistent with the discussion by Nikola et al. [19] in their supplements). The dispersion relation (in equation 30) becomes

\[
\omega^2 = \frac{\alpha_{\text{bend}}}{\mu} k^4 + \frac{T(1 - \beta_{\text{swim}})}{\mu} k^2.
\]

In Figure 3 we have plotted the phase velocity \( \omega/(kv_0) \), computed using equation 32 as a function of wavelength for different boundary to boid mass ratios, bending moments and for two different values for the dimensionless coefficient \( \beta_{\text{swim}} \). The values of boundary to total boid mass ratio and bending moments are the same as used in our simulation series. In Figure 3, velocities
Maroon solid lines of increasing thickness have mass ratio $\frac{M_{\text{nodes}}}{M_{\text{boids}}}$ = 3, 10, 25, respectively, and bending moment $\alpha_{\text{bend}} / (M_{\text{boids}} v_0^2 R) = 10^{-3}$. Thin cyan and thick blue dotted lines have $\alpha_{\text{bend}} / (M_{\text{boids}} v_0^2 R) = 10^{-2}$ and $10^{-4}$, respectively, and mass ratio $M_{\text{nodes}} / M_{\text{boids}} = 10$. Horizontal grey lines are at velocity $v_0$ and $3/v_0$. Wavelengths to the right of where the curved lines cross a horizontal line have phase velocity below the value of the horizontal line. If instability depends on matching boid speed to the velocity of waves on the boundary, then smaller wavelengths are unstable for higher mass and more flexible boundaries.

Higher boundary mass gives lower wave velocity on the boundary. Likewise weaker boundaries, (with lower $\alpha_{\text{bend}}$) have lower wave velocity. The trends we see in Figure 1 showing that corrugation wavelengths decrease with increasing boundary mass and decreasing bending moment, are matched by the trends we see in wave velocity. This suggests that the instability on the boundary grows when the wave speed on the boundary is similar to boid speed. Horizontal grey lines on Figure 3 show constant velocities. Wavelengths to the right of where the curved lines cross a horizontal grey line have phase velocity below the value of the horizontal line. If instability depends on matching boid speed to the velocity of waves on the boundary, then smaller wavelengths are unstable with higher mass and more flexible boundaries.

Taking our dispersion relation of equation 32 the wavevector that gives $\omega = k v_0$ (and matching wave phase velocity to boid speed) is

$$k_{\text{crit}} = \sqrt{\frac{\mu v_0^2 - T (1 - \beta_{\text{swim}})}{\alpha_{\text{bend}}}}, \quad (33)$$

For $M_{\text{nodes}} > M_{\text{boids}}$ and the regime giving us interesting boundary morphology, the critical wave vector

$$k_{\text{crit}} R \approx \sqrt{\frac{R v_0^2 M_{\text{boids}}}{2\pi \alpha_{\text{bend}}}} \sqrt{\frac{M_{\text{nodes}}}{M_{\text{boids}}}} \quad (34)$$

In terms of a critical wavelength $\lambda_{\text{crit}} = 2\pi / k_{\text{crit}}$,

$$\frac{\lambda_{\text{crit}}}{R} \approx 0.16 \left( \frac{\alpha_{\text{bend}}}{10^{-3} M_{\text{boids}} v_0^2} \right)^{\frac{1}{2}} \left( \frac{10}{M_{\text{nodes}}/M_{\text{boids}}} \right)^{\frac{1}{2}}. \quad (35)$$

The scaling and approximate values for the critical wavelength are consistent with the wavelengths giving phase velocity of $v_0$ in Figure 3.

As long as the coefficient giving swim pressure strength $\beta_{\text{swim}} < 1$, the dispersion relation in equation 32 always gives real frequencies $\omega$ when the wavevectors are real. Only wavelike solutions would be present and perturbations on the boundary would not grow. If the dispersion relation has regions where frequency $\omega$ is complex for real $k$, then perturbations at these wavelengths would grow exponentially giving instability on the boundary. If the $k^2$ term in the dispersion is negative then there is an instability at small wavelengths. This is the setting discussed by Nikola et al. [19] for instability of a filament embedded in a medium containing self-propelled particles. A modified form for the swim pressure might give a larger negative term in the dispersion relation and show instability.

Using a linearized version of Bernoulli’s equation, a two-dimensional incompressible fluid approximation for boids moving at $v_0$ would give boid pressure perturbations with amplitude $p_0 \propto M_{\text{boids}} (\omega - k v_0)^2 / k$ for a perturbation $\propto e^{i(\omega t - kx)}$ on the boundary. However unstable regions in the dispersion relation then occur at larger wavelengths for heavier boundaries which is opposite to what is seen in our simulations (see Figure 1). A model where swim pressure is proportional to boid density and boid density is proportional to the local boundary curvature (e.g., [19]) also would predict this trend that is...
not consistent with our simulations. If the local swim pressure is large and $\beta_{\text{swim}} > 1$ in equation 32 unstable regions would also give this incorrect trend. These types of instability models also predict rapid growth rates for the instability, also in contradiction to what we see in the simulations, where corrugations in the boundary take 5 to 10 crossing times $t_R$ to grow.

The models discussed in the previous paragraph and equation 31 (and by [19]) have boid swim pressure perturbations, exerted on the boundary, that are in phase with the boundary perturbation. However, we see a difference in the boid motions between leeward and windward sides of corrugations in our simulations. This is most extreme for the massive boundaries on the right hand side of Figure 1 (third row) where boids are pushed outward toward the center of the enclosed region after they pass a convex region of the boundary. The difference between leeward and windward sides in the boid motions implies there is an asymmetry in the response of the boids to perturbations in the boundary. The response of the boids slightly lags behind the perturbation, giving a phase shift in the pressure response.

We consider a model where the boid swim pressure is slightly out phase with a small perturbation on the boundary. For a perturbation $\propto e^{i(\omega t - kx)}$ on the boundary, we assume that the sign of the phase shift depends on $\bar{v} - \omega/k$ where $\bar{v}$ is the mean speed of boids that are next to the boundary. We approximate $\bar{v} \sim v_0$ even though the mean speed $\bar{v}$ is usually lower than $v_0$ because the boids are slowed by bouncing against the boundary. The phase shift gives an additional complex component to the amplitude of the boid pressure perturbation $p_{\text{swim,k}}$. We assume that the phase shift in boid pressure is in the same form as equation 31 contributing a complex component

$$\text{Im}(p_{\text{swim,k}}) = i\delta_{\text{lag}}TK^2\text{sign}(kv_0 - \omega)$$

(36)

to the swim pressure perturbation amplitude. Here $\delta_{\text{lag}}$ is a small dimensionless parameter describing the size of the lag. Modifying equation 32 the resulting dispersion relation is

$$\omega^2 = \frac{\alpha_{\text{bend}}}{\mu}k^4 + \frac{T}{\mu}k^2 (1 - \beta_{\text{swim}} + i\delta_{\text{lag}}\text{sign} (kv_0 - \omega)).$$

(37)

Assuming that the parameter $\delta_{\text{lag}}$ is small, we find that the perturbation only grows if the imaginary term on the right hand is positive. An instability is present if $v_0 > \omega/k$, so only boundaries with slow wave speeds would be unstable to the growth of corrugations. As heavier boundaries have slower bending wave speeds, the delay could account for the relation between corrugation and boundary mass we see in Figure 1.

With small $\delta_{\text{lag}}$, we estimate an instability growth rate from the imaginary component of the frequency

$$\gamma(k) = \text{Im}(\omega) = \frac{\delta_{\text{lag}}TK^2}{2\mu \text{Re}(\omega(k))}.$$  

(38)

Unstable perturbations would have amplitudes that increase exponentially with time, $\propto e^{\gamma(k)t}$. While all wave-lengths larger than the critical one (where wave speed matches boid speed) would be unstable (due to the sign of the phase lag), the growth rate is maximum near the smallest unstable wavelength which is the critical one. Using equation 34 for the critical wavevector, we estimate the the growth rate for this wavelength,

$$\gamma(k_{\text{crit}})t_R \approx \frac{T\delta_{\text{lag}}}{2\mu v_0^2}k_{\text{crit}}R \approx \frac{\delta_{\text{lag}}}{2} \sqrt{\frac{Rv_0^2M_{\text{boids}}}{2\pi\alpha_{\text{bend}}}} \sqrt{\frac{M_{\text{boids}}}{M_{\text{nodes}}}}$$

$$\approx 2\delta_{\text{lag}} \left( \frac{10^{-3}M_{\text{boids}}v_0^2}{\alpha_{\text{bend}}} \right)^{\alpha_{\text{bend}}} \left( \frac{10}{M_{\text{nodes}}/M_{\text{boids}}} \right)^{\beta_{\text{bend}}}.  

(39)

We can test this phase-lag model by examining the rate that boundary perturbations grow in our simulations. In 5 simulations we measure Fourier amplitudes $A_m(t) > 0$ as a function of time, where integer $m$ gives the angular frequency of radius $R(\theta, t) = \sum_m A_m(t) \cos(m\theta + \phi_m(t))$ as a function of angle $\theta$ along the boundary. For example, a triangular perturbation gives an amplitude $A_3$. The angles $\phi_m$ depend on the orientation of the perturbation. The 5 simulations have parameters taken from Table I and Table II but with the boundary to boid mass ratio and bending moments chosen to be the same as the phase velocities plotted in Figure 3. These simulations are the part of the first and third series listed in Table II and shown in the first and third rows of Figure 4. In Figure 4k, we plot $\ln(\sum_{m=3}^{7} A_m/R)$ as a function of time and in Figure 4b we plot $\ln(\sum_{m=10}^{20} A_m/R)$. Lines have the same colors and styles as in Figure 5.

Figure 4 shows that corrugation growth rates are faster with lower values of bending moment (comparing blue dotted, red dot-dashed and thin teal dotted lines), as expected from equation 39. The inverse dependence of growth rate on boundary to total boid mass ratio is less evident, but the mass ratio varies by a factor of about 3 rather than 10 as for the bending moment. The low mass boundary only grows larger wavelength perturbations (with lower Fourier index $m$) and its growth rate is slower than for the higher mass boundaries with the same bending moment (comparing thin orange to thick red and maroon lines). The trends we see in Figure 4 are consistent with those predicted by equation 39.

We use our numerically measured growth rate to estimate the size of the pressure lag. In equation 39 we have estimated the growth rate of the critical wavelength for the mass ratio 10 and $\alpha_{\text{bend}}/(M_{\text{boids}}v_0^2R) = 10^{-3}$ simulation which is shown with a dot-dashed red line in Figure 4k. The slope of the red line gives a growth rate of $\gamma t_R \sim 0.2$. Equating this to the growth rate in equation 39 we estimate $\delta_{\text{lag}} \sim 0.1$. The required lag for the pressure is small enough to be consistent with the appearance of the simulations. This implies that a small delay in boid response moving over boundary perturbations can account for the instability.

Throughout the discussion in this section we have assumed that tension on the boundary was that estimated by equation 25. However if the boid separation
is shorter than the repel distance, $\sqrt{\frac{\pi R^2}{N_{\text{boids}}}} < d_{\text{repel}}$, then there is more tension on the boundary because the boids are pushed against the boundary by their repulsion. An increase in tension increases the wave speed and would reduce the wavelength of corrugations on the boundary. The second series of simulations shown in Figure 1b (second row) is in this regime and shows weaker boundary perturbations. Comparison of this simulation to that with identical parameters but lower $d_{\text{repel}}$ (Figure 1a, top row) shows that the corrugations in the higher tension simulations tend to be shorter wavelength, confirming our expectation. A single Fourier perturbation tends to dominate in these simulations, but we lack an explanation for this.

What accounts for the size of the phase lag parameter $\delta_{\text{lag}}$? The phase lag may be due to the time it takes other boids to push near-boundary boids back onto the boundary. This time might be governed by the strength of the interboid repel force. We have noticed that a weaker repel force $U_{\text{repel}}$ gives larger density contrasts in the boids. We would expect this to give a larger asymmetry between windward and leeward sides of corrugations in the boid distribution, leading to faster corrugation growth rates and larger amplitude corrugations but not necessarily a change in the wavelengths that are unstable. In Figure 1f (fifth row), the simulations with lower $U_{\text{repel}}$ do seem to have smaller wavelength corrugations and with larger $U_{\text{repel}}$, the boundary instability is suppressed. The variation in the wavelengths of instability must be due to another cause, perhaps because changing $U_{\text{repel}}$ also affects boid density near the boundary and also the tension on the boundary which would change the speed of boundary waves. Boids are slowed down near the boundary and if the mean speed depends on $U_{\text{repel}}$, this too could affect the wavelengths that are unstable. We lack a straightforward way to predict the delay parameter, $\delta_{\text{lag}}$. Better understanding of the boid’s continuum dynamics near the boundary may make it possible to predict the phase lag from the repel force law and mean boid number density.

In summary, we have explored simple models for boid swim pressure, exerted onto the boundary, that would give instability on the boundary. A model with boid swim pressure dependent on the boundary curvature and slightly lagging its corrugations is most successful at matching sensitivity of boundary corrugation wavelength to boundary mass and bending moment and the corrugation growth rates. Perturbations on the boundary that move with wave speed slower than but near the boid speed are most likely to grow and this determines the wavelengths that grow on the boundary.

### IV. SUMMARY AND DISCUSSION

We have carried out a numerical exploration in 2-dimensions of self-propelled particles with alignment and repelling forces that are enclosed in a flexible elastic loop. We primarily find three types of long lived states: a stochastic gas-like state, a solid-like or jammed bullet state where the boids align and push the boundary in a single direction and rotating or circulating states. The gaseous and circulating states resemble those exhibited by unconfined unipolar self-propelled particles with cohesive or attractive interactions [5, 28]. The solid-like state resembles the jammed state seen in simulations of confined soft repulsive self-propelled particles at high density [29] and the moving droplets seen in simulations of unconfined self-propelled particle with strong cohesion [28, 32]. We recover these three types of states without cohesion due to the confining nature of the boundary.

The most of interesting and novel of the states exhibited by our simulations are the circulating states as they include rotating ovals and sprocket shaped and irregular or ruffled boundaries. Oval shaped boundaries are similar to those seen in simulations of non-aligning stochastically perturbed self-propelled particles [20, 22]. The ruffled or sprocket shaped rotated boundaries mimic the rotating ratchet that was achieved by placing a rigid ratchet in an solution of active particles [14, 16, 39], but here the collective motion of the self-propelled particles and instability on the boundary drive the rotation. The
instability is likely mediated by boid pressure inhomogeneities, as predicted by Nikola et al. [19]. However, the instability is most noticeable in the simulations with more massive and flexible boundaries. The wavelength of corrugations on the boundary is near the wavelength of elastic waves on the boundary that have phase velocity equal to the particle swim speed. We suspect that the instability depends on a lag between boid swim pressure exerted on the boundary and boundary shape perturbations. In this sense our instability model differs from that by Nikola et al. [19] who did not adopt a phase lag.

It may be possible to devise an experiment giving an instability on a flexible boundary that is mediated by active particles. Here we considered a uniform loop boundary, but a boundary could be designed to be more flexible on one side than the other. For examples, if the instability is fast, waves might be excited on one side only, making it possible to fix the other side to another surface. States with rotating or fluttering boundaries might be used to generate flow or vorticity or to create a swimmer. They could more efficiently use power from self-propelled particles as the particles are in proximity to the moving boundary rather than distributed in a solution, though providing the particles with an energy source for propulsion could be more difficult as their fuel must be stored within or cross the boundary.

In this study we ignored stochastic perturbations and cohesion in the self-propelled particles and the hydrodynamics of the medium in which the self-propelled particles move. Our simulations were restricted to a few hundred boids. Future studies could extend and vary the physical model and explore dynamics in three dimensions. Future work could also explore other types of active materials that are enclosed by flexible boundaries, such as active self-propelled rods or active neumatics. With unipolar self-propelled particles, we did not see long lived bending oscillations. Perhaps other types of active materials enclosed in a flexible boundary could exhibit this type of phenomena.

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