Moore-Penrose inverses of Gram matrices Leaving a Cone Invariant in an Indefinite Inner Product Space

K. Appi Reddy, T. Kurmayya
Department of Mathematics
National Institute of Technology Warangal
Warangal - 506 004, India.

Abstract

In this paper we characterize Moore-Penrose inverses of Gram matrices leaving a cone invariant in an indefinite inner product space using indefinite matrix multiplication. This characterization includes the acuteness (or obtuseness) of certain closed convex cones.

Keywords: Gram matrix; Moore-Penrose inverse; acute cones; Indefinite inner product space.

AMS Subject Classification: 46C20, 15A09.
1 Introduction

An indefinite inner product in $\mathbb{C}^n$ is a conjugate symmetric sesquilinear form $[x, y]$ together with the regularity condition that $[x, y] = 0$ $\forall y \in \mathbb{C}^n$ holds only when $x = 0$. Associated with any indefinite inner product, there exists a unique invertible hermitian matrix $N \in \mathbb{C}^{n \times n}$ (called a weight) such that $[x, y] = \langle x, Ny \rangle$, where $\langle ., . \rangle$ denotes the Euclidean inner product on $\mathbb{C}^n$ and vice versa. Motivated by the notion of Minkowski space (as studied by physicists), we also make an additional assumption on $N$, namely, $N^2 = I$. It should be remarked that this assumption also allows us to compare our results with the Euclidean case, apart from allowing us to present the results with much algebraic ease.

Investigations of linear maps on indefinite inner product spaces employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors (See for instance [3]). This causes a problem as there are two different values for the dot product of vectors. To overcome this difficulty; Kamaraj, Ramanathan and Sivakumar introduced a new matrix product called indefinite matrix multiplication and investigated some of its properties in [7]. More precisely, the indefinite matrix product of two matrices $A$ and $B$ of sizes $m \times n$ and $n \times l$ complex matrices, respectively, is defined to be the matrix $A \circ B := ANB$. The adjoint of $A$, denoted by $A^\ast$, is defined to be the matrix $NA^\ast M$, where $N$ and $M$ are weights in the appropriate spaces. Many properties of this product are similar to that of the usual matrix product (refer [7]). Moreover, it not only rectifies the difficulty indicated earlier, but also enables us to recover some interesting results in indefinite inner product spaces in a manner analogous to that of the Euclidean case. Kamaraj, Ramanathan and Sivakumar [7] also shown that in the setting of indefinite inner product spaces, Moore-Penrose inverses of certain matrices do not exist with respect to the usual matrix product where as Moore-Penrose inverses of such matrices exist with respect to the indefinite matrix product. Hence they concluded that indefinite matrix product is more appropriate than the usual matrix product.

The problem of nonnegative invertibility of matrices (or inverses of matrices leaving a cone invariant) was first studied by Collatz [5] when he applied a finite difference method for solving a class of two point boundary value problems. This idea of nonnegative invertibility has undergone a plethora of generalizations over the years. We refer the reader [2] (and the references cited there in) for a detailed survey of these extensions.

In recent years, nonnegative invertibility of Gram matrices has received, a lot of attention. This has been primarily motivated by applications in convex optimization problems. In this connection, there is a well known result that
characterizes non negative invertibility of Gram matrices in terms of obtuse-ness or acuteness of certain polyhedral cones. (See for instance Lemma 1.6 in [4]). Recently, Sivakumar [9] characterized Moore-Penrose inverses of Gram operators leaving a cone invariant over Hilbert spaces. In this paper, we follow the approach of Sivakumar [9] and discuss the Moore-Penrose inverses of Gram matrices leaving a cone invariant in an indefinite inner product space using indefinite matrix product. As the indefinite matrix product encompasses the Euclidean case as a particular example, it follows that earlier results in the finite dimensional Euclidean spaces, are easy corollaries of our main result.

The paper is organized as follows. In section 2, we introduce basic notations, definitions and results. In section 3, we prove series of lemmas and derive the main theorem.

2 Notations, Definitions and Preliminaries

In this section, we introduce notations, definitions and basic results that will be used in the rest of the paper.

Let \( \langle \cdot, \cdot \rangle \) denote the usual Euclidean inner product in \( \mathbb{R}^n \). An indefinite inner product is denoted by \( \langle x, y \rangle = \langle x, Ny \rangle \), where \( N \in \mathbb{R}^{n \times n} \) and \( N = N^{-1} \). Such a matrix \( N \) is called weight. A space with an indefinite inner product is called an indefinite inner product space. In the rest of the paper \( \mathbb{R}^m, \mathbb{R}^n \) denote indefinite inner product spaces with weights \( M, N \) respectively. Let \( A, B \) be two real matrices of order \( m \times n \) and \( n \times l \) respectively, then the indefinite matrix product of those matrices be denoted by \( A \circ B \) and defined as \( A \circ B = ANB \), where \( N \) is a weight matrix as defined earlier. For \( A \in \mathbb{R}^{m \times n} \), the adjoint \( A^\star \), of \( A \) is defined by \( A^\star = N A^* M \), where \( * \) denotes the transpose of \( A \), \( M \) and \( N \) are weights of order \( m \) and \( n \) respectively.

Let \( K \) be a subset of \( \mathbb{R}^n \). Then \( K \) is called cone if (i) \( x, y \in K \Rightarrow x+y \in K \) and (ii) \( x \in K, \alpha \in \mathbb{R}, \alpha \geq 0 \Rightarrow \alpha x \in K \). The dual of cone \( K \) is denoted by \( K^\star \) and is defined as \( K^\star = \{ x \in \mathbb{R}^n : \langle x, t \rangle \geq 0, \text{ for all } t \in K \} \). Let \( K^{\star \star} \) denote \( (K^\star)^\star \). If \( K = \mathbb{R}^n_+ \) then \( K^\star = I \circ \mathbb{R}^n_+ \) and \( K^{\star \star} = K \).

A cone \( C \) is said to be acute if \( \langle x, y \rangle \geq 0 \) for all \( x, y \in C \). \( C \) is said to be obtuse if \( C^\star \cap \{ \text{cl span } C \} \) is acute. In particular, let \( C = A \circ I \circ K \) then we say that \( C = \{ A \circ I \circ x : x \in K \} \) is obtuse if \( (A \circ I \circ K)^\star \cap \mathcal{R}(A \circ I) \) is acute. According to Novikoff, the acuteness of a cone \( C \) in \( \mathbb{R}^n \) is defined by the inclusion \( C \subseteq C^\star \). We can easily verify this condition in indefinite inner product spaces as \( C \subseteq C^\star \).

For \( A \in \mathbb{R}^{m \times n} \), \( A^\star \circ A \) will be called the Gram matrix of \( A \). For \( A \in \mathbb{R}^{m \times n} \), the following equations are known to have unique solution [2]:

2
A \circ X \circ A = A, \ X \circ A \circ X = X, \ (A \circ X)^{\dagger} = A \circ X, \ (X \circ A)^{\dagger} = X \circ A. 

Such an X will be denoted by A^{[l]}. If the weight matrices in indefinite inner product spaces are equal to identity then A^{[l]} = A^{\dagger}. We refer the reader \cite{10} (and the references cited there in) for a detailed study of A^{\dagger}.

Next, we collect some properties of A^{[l]}. Some of these have been proved in \cite{7} and rest can be demonstrated easily. The range space of A, \mathcal{R}(A) is defined by \mathcal{R}(A) = \{y \in \mathbb{R}^m : y = A \circ x, \ x \in \mathbb{R}^n\} and the null space of A, \mathcal{N}(A) is defined by \mathcal{N}(A) = \{x \in \mathbb{R}^n : A \circ x = 0\}. For A \in \mathbb{R}^{m \times n}, \ A^{[l]} also satisfies the following properties: \mathcal{R}(A^{[s]}) = \mathcal{R}(A^{[l]}), \mathcal{N}(A^{[s]}) = \mathcal{N}(A^{[l]}), \ A \circ A^{[l]} = P_{\mathcal{R}(A)}, \ A^{[l]} \circ A = P_{\mathcal{R}(A^{[c]})}. \ We \ also \ have \ (A^{[s]} \circ A)^{[l]} \circ (A^{[s]} \circ A) = P_{\mathcal{R}(A^{[i]} \circ A)^{[l]}} = P_{\mathcal{R}(A^{[i]} \circ A)^{[c]}} = P_{\mathcal{R}(A^{[c]})} = A^{[l]} \circ A.

Lemma 2.1. Let A \in \mathbb{C}^{m \times n}. Then
(i) A^{[s]} = A^{[s]} \circ A \circ A^{[l]} = A^{[l]} \circ A \circ A^{[s]}
(ii) A^{[l]} = A^{[s]} \circ (A \circ A^{[s]}^{[l]})^{[l]} = (A^{[s]} \circ A)^{[l]} \circ A^{[s]}
(iii) A^{[l]} \circ (A^{[l]}^{[s]} \circ A)^{[l]} = (A^{[s]} \circ A)^{[l]}
(iv) (A \circ I)^{[l]} = I \circ A^{[l]}
(v) \mathcal{R}(A \circ A^{[l]}) = \mathcal{R}(A), \mathcal{R}(A^{[l]} \circ A) = \mathcal{R}(A^{[s]}), \mathcal{N}(A \circ A^{[l]}) = \mathcal{N}(A^{[s]}), \mathcal{N}(A^{[l]} \circ A) = \mathcal{N}(A) \ where \ \mathcal{R}(X) \ and \ \mathcal{N}(X) \ denote \ the \ range \ and \ null \ spaces \ of \ X \ respectively.

We use the following lemma frequently in this paper.

Lemma 2.2. Let A \in \mathbb{R}^{m \times n} and b \in \mathbb{R}^m. Then, the linear equation A \circ X = b has a solution iff b \in \mathcal{R}(A). In this case, the general solution is given by x = A^{[l]} \circ b + z \ where \ z \in \mathcal{N}(A).

3 Main Results

For given A \in \mathbb{R}^{m \times n}, Ramanathan and Sivakumar \cite{9} derived a set of necessary and sufficient conditions for a cone to be invariant under (A^{[s]} \circ A)^{[l]}. These conditions include pairwise acuteness (or pairwise obtuseness) of certain cones. In this article, we avoid pairwise acuteness of cones and characterize Moore-Penrose inverses of Gram matrices leaving a cone invariant in the approach of Sivakumar \cite{9}. These results generalize the existing results of Sivakumar \cite{9} in the finite dimensional setting from Euclidean spaces to indefinite inner product spaces. First we prove series of lemmas that lead up to the main theorem (Theorem 3.10).

As mentioned earlier; \mathbb{R}^m, \mathbb{R}^n denote indefinite inner product spaces with weights M, N respectively. Let A \in \mathbb{R}^{m \times n} be such that I \circ A = A \circ I \ that \ is \ MA = AN \ and \ let \ K \ be \ a \ closed \ cone \ in \ \mathbb{R}^n.
Lemma 3.1. \([A \circ x, y] = [x, A[x] \circ y]\).

Proof. \([A \circ x, y] = \langle A \circ x, My \rangle = \langle ANx, My \rangle = \langle x, NA^*My \rangle = [x, A^*My] = [x, I \circ (NA^*)M \circ I \circ y] = [x, I \circ A[x] \circ I \circ y] = [x, (I \circ A[I])^x \circ y] = [x, A[I] \circ y]. \) ✷

Lemma 3.2. \(u \in (A \circ I \circ K)^x \Rightarrow (A \circ I)^x \circ u \in K^x.\)

Proof. Let \(u \in (A \circ I \circ K)^x\) and \(r \in K\). Then \(0 \leq [u, AoIor] = [(A \circ I)^x \circ u, r]\), by Lemma 3.1. Thus \((A \circ I)^x \circ u \in K^x.\) ✷

Next, we show that \(K\) is invariant under \(A[I] \circ A\) if and only if \(K^x\) is invariant under \(A[I] \circ A.\)

Lemma 3.3. \(A[I] \circ A \circ K \subseteq K \iff A[I] \circ A \circ K^x \subseteq K^x.\)

Proof. Let \(A[I] \circ A \circ K \subseteq K, y = A[I] \circ A \circ x\) with \(x \in K^x, u \in K\) and \(u^1 = A[I] \circ A \circ u \in K.\) Then \([y, u] = [A[I] \circ A \circ x, u] = [x, (A[I] \circ A)^x \circ u] = [x, A[I] \circ A \circ u] = [x, u^1] \geq 0.\) This shows that \(y \in K^x.\) Hence \(A[I] \circ A \circ K^x \subseteq K^x.\)

Similarly one can easily prove the converse part. ✷

In the next result, we determine the set \((A \circ I \circ K)^x\) in the presence of an additional condition.

Theorem 3.4. \((A \circ I \circ K)^x \subseteq (A[I] \circ I \circ K^x + NA[(A \circ I)^x]).\) If \(K\) is invariant under \(A[I] \circ A,\) then equality holds.

Proof. Let \(y \in (A \circ I \circ K)^x.\) Then by Lemma 3.2, \(z = (A \circ I)^x \circ y \in K^x.\)

By Lemma 3.2, \(y = ((A \circ I)^x[I] \circ z + w\) for some \(w \in NA[(A \circ I)^x]).\) Then \(y \in ((A \circ I)^x[I] \circ K^x + NA[(A \circ I)^x]) = (A[I] \circ I \circ K^x + NA[(A \circ I)^x]).\) This proves the first part.

Next, suppose that \(A[I] \circ A \circ K \subseteq K.\) Let \(u = u^1 + u^2,\) where \(u^1 = (A[I] \circ I \circ K^x)\) and \(u^2 \in NA[(A \circ I)^x].\)

Let \(v = AoIot, t \in K\) and set \(t' = A[I] \circ AoIot \in K.\) Then \([u, v] = [u^1 + u^2, v] = [u^1, v] + [u^2, v] = [u^1, AoIot] = [(A[I] \circ I \circ K^x[I] \circ I \circ t, A \circ I \circ t] = [l, t'] \geq 0,\) since \([u^2, v] = [u^2, A \circ I \circ t] = 0.\) Thus \(u \in (A \circ I \circ K)^x.\) ✷

Remarks 3.5. The following example shows that in the absence of the condition \(A[I] \circ A \circ K \subseteq K,\) the reverse inclusion may not hold in Theorem 3.4. Let \(A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\) and \(N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.\) Then \(A[I] = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}\) and \(A[I] = NA[I] = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}.\) Let \(K = \mathbb{R}^3_+\) then
Let $A$ be given as in Remark 3.5. Then $A^{[1]} \circ A \circ x = (1, \frac{-1}{2}, \frac{1}{2}) \notin K$. So, $A^{[1]} \circ A \circ K \nsubseteq K$. Also $A \circ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. So $N((A \circ I)^{[n]})$ contains only the zero vector. Let $y = (1, 2, 3)^t \in K$ then $y^1 = Ny = (1, -2, 0) \in K^{[n]}$. Then $u = (A^{[1]})^{[n]} \circ I \circ y^1 = (1, 1)^t \in (A^{[1]})^{[n]} \circ I \circ K^{[n]}$. But $u \notin (A \circ I \circ K)^{[n]}$, since $[u, A \circ I \circ v] = \langle u, MAv \rangle < 0$ for $v = (1, 4, 8)^t$.

The next result is analogous to Theorem 3.3. This will be used later.

**Lemma 3.6.** $(A^{[1]})^{[n]} \circ I \circ K^{[n]} \subseteq A \circ I \circ K + N((A \circ I)^{[n]}).$ If $A^{[1]} \circ A \circ K \subseteq K$, then equality holds.

**Proof.** The proof follows from Lemma 3.3 and Theorem 3.4 by replacing $A$ by $(A^{[1]})^{[n]}$ and $K$ by $K^{[n]}$. $\square$

**Remarks 3.7.** Let $A$ be given as in Remark 3.3. Then $A^{[1]} \circ A \circ K \nsubseteq K$. Let $y = (2, 5, 8)^t \in K$ and set $y^1 = A \circ I \circ y = Ay = (2, 3)^t \in A \circ I \circ K$. Let $v = N(1, 2, 0)^t = (1, -2, 0)^t \in K^{[n]}$ and $z = (A^{[1]})^{[n]} \circ I \circ v = (1, 1)^t \in (A^{[1]})^{[n]} \circ I \circ K^{[n]}$. Then $y^1, z = y^1, Mz = \langle (2, 3)^t, (1, -1)^t \rangle < 0$, so that $y^1 \notin ((A^{[1]})^{[n]} \circ I \circ K^{[n]})^{[n]}$. This shows that the condition $A^{[1]} \circ A \circ K \subseteq K$ is essential for the reverse inclusion to hold in Lemma 3.6.

**Lemma 3.8.** $(A \circ I \circ K)^{[n]} \cap \mathcal{R}(A \circ I) \subseteq (A^{[1]})^{[n]} \circ I \circ K^{[n]}$. If $A^{[1]} \circ A \circ K \subseteq K$, then equality holds.

**Proof.** Let $y = A \circ I \circ x \in (A \circ I \circ K)^{[n]}$. Then by Lemma 3.2, $(A \circ I)^{[n]} \circ y \in K^{[n]}$. Also, $y = (A \circ I) \circ (A \circ I)^{[n]} \circ y = ((A \circ I) \circ (A \circ I)^{[n]})^{[n]} \circ y = ((A \circ I)^{[n]} \circ I \circ (A \circ I)^{[n]} \circ y \in (A^{[1]})^{[n]} \circ I \circ K^{[n]}$, proving that $(A \circ I \circ K)^{[n]} \cap \mathcal{R}(A \circ I) \subseteq (A^{[1]})^{[n]} \circ I \circ K^{[n]}$.

Conversely, suppose that $x \in (A^{[1]})^{[n]} \circ I \circ K^{[n]}$. Then $x = ((A \circ I)^{[n]})^{[n]} \circ u$ for some $u \in K^{[n]}$. This implies $x \in \mathcal{R}(A \circ I)$. Let $w \in K$, $v = A \circ I \circ w \in A \circ I \circ K$ and $w^1 = A^{[1]} \circ A \circ w \in K$. Then we have $[u, v] = [u, A^{[1]} \circ A \circ w]^t = [u, w^1] \geq 0$. Thus $x \in (A \circ I \circ K)^{[n]}$. $\square$

**Remarks 3.9.** Let $A$ be given as in Remark 3.3. Then $A^{[1]} \circ A \circ K \nsubseteq K$. Let $y = (2, 5, 8)^t \in K$, and $y^1 = A \circ I \circ y = Ay = (2, 3)^t \in A \circ I \circ K$. Let $v = N(1, 2, 0)^t = (1, -2, 0)^t \in K^{[n]}$ and $z = (A^{[1]})^{[n]} \circ I \circ v = (1, 1)^t \in (A^{[1]})^{[n]} \circ I \circ K^{[n]}$. But $y^1, z = \langle y^1, Mz \rangle = \langle (2, 3)^t, (1, -1)^t \rangle < 0$. Thus $z \notin (A \circ I \circ K)^{[n]} \cap \mathcal{R}(I \circ A)$. Hence the condition $A^{[1]} \circ A \circ K \subseteq K$ is necessary for the reverse inclusion to hold in Lemma 3.8.

Next, we obtain an equivalent condition for the acuteness of the cone $(A \circ I \circ K)^{[n]} \cap \mathcal{R}(A \circ I)$.
Lemma 3.10. Let \( A[l] \circ A \circ K \subseteq K \). Then \( (A \circ I \circ K)[s] \cap R(A \circ I) \) is acute \( \iff (A \circ I \circ k)[s] \cap R(A \circ I) \subseteq A \circ I \circ K \).

Proof. Suppose that \( L = (A \circ I \circ K)[s] \cap R(A \circ I) \) is acute. Then \( L \subseteq L[s] \). By Lemma 3.6 and Lemma 3.8 it follows that \( L[s] = ((A \circ I \circ K)[s] \cap R(A \circ I))[s] = ((A[l])[s] \circ I \circ K[s])[s] = A \circ I \circ K + N((A \circ I)[s]) \). So, \( (A \circ I \circ K)[s] \cap R(A \circ I) \subseteq A \circ I \circ K + N((A \circ I)[s]) \). But, we have to show that \( (A \circ I \circ K)[s] \cap R(A \circ I) \subseteq A \circ I \circ K \). Let \( x \in (A \circ I \circ K)[s] \cap R(A \circ I) \). Then \( x = A \circ I \circ u + z \), with \( u \in K \), \( z \in N((A \circ I)[s]) \). But since \( A \circ I \circ u \in R(A \circ I) \), it follows that \( z \in R(A \circ I) \cap N((A \circ I)[s]) = \{0\} \). Thus \( x \in A \circ I \circ K \).

Conversely, let \( x, y \in (A \circ I \circ K)[s] \cap R(A \circ I) \subseteq A \circ I \circ K \). Then \( x = A \circ I \circ u, \ u \in K \). We also have \( (A \circ I)[s] \circ y \in K[s] \). Now, \([x, y] = [A \circ I \circ u, y] = [u, (A \circ I)[s] \circ y] \geq 0 \). Thus \( (A \circ I \circ k)[s] \cap R(A \circ I) \) is acute. \( \square \)

We next obtain a necessary and sufficient condition for a cone to be invariant under \((A[s] \circ A)[l]\) (See Lemma 3.11).

Lemma 3.11. \((A[l])[s] \circ I \circ K[s] \subseteq A \circ I \circ K + N((A \circ I)[s]) \iff (A[s] \circ A)[l] \circ K[s] \subseteq K + N(A \circ I)\)

Proof. For \( x \in K[s] \), let \( y = (A[s] \circ A)[l] \circ x = ((A \circ I)[s] \circ (A \circ I))[l] \circ x = (A \circ I)[l] \circ ((A \circ I)[l])[s] \circ x \). Then

\[
A \circ I \circ y = (A \circ I) \circ ((A \circ I)[l]) \circ ((A \circ I)[l])[s] \circ x
\]

\[
= ((A \circ I)[l] \circ (A \circ I) \circ ((A \circ I)[l])[s]) \circ x
\]

\[
= ((A \circ I)[l])[s] \circ x
\]

\[
= (A[l])[s] \circ I \circ x \in (A[l])[s] \circ I \circ K[s]
\]

\[
\subseteq A \circ I \circ K + N((A \circ I)[s])
\]

Therefore \( A \circ I \circ y = A \circ I \circ v + w, \ v \in k, \ w \in N((A \circ I)[s]) \). So, \( A \circ I \circ (y - v) \in R(A \circ I) \cap N((A \circ I)[s]) = \{0\} \). Then \( A \circ I \circ (y - v) = 0 \). This implies, \( y - v = u \in N(A \circ I) \). Then \( y = u + v, \ v \in K, \ u \in N(A \circ I) \). This shows that \((A[s] \circ A)[l] \circ K[s] \subseteq K + N(A \circ I)\).

Conversely, let \( y = (A[l])[s] \circ I \circ x, \ x \in K[s] \). Then \( y = ((A \circ I)[l])[s] \circ x \) and \((A \circ I)[l] \circ y = (A \circ I)[l] \circ ((A \circ I)[l])[s] \circ x = ((A \circ I)[s] \circ (A \circ I))[l] \circ x = (A[s] \circ A)[l] \circ x = u + v, \ u \in K, \ v \in N(A \circ I) \). Then \( y = ((A \circ I)[l])[s] \circ (u + v) + w, \ w \in N((A \circ I)[l]) \) Then \( y = A \circ I \circ u + w \in A \circ I \circ K + N((A \circ I)[s]) \). \( \square \)

We also have stronger one-way implication, given below. The proof follows from necessity part of Lemma 3.11.

Lemma 3.12. \((A[l])[s] \circ I \circ K[s] \subseteq A \circ I \circ K \Rightarrow (A[s] \circ A)[l] \circ K[s] \subseteq K + N(A \circ I)\).
Lemma 3.13. \((A^{[*]} \circ A)^{[I]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I) \Rightarrow K^{[*]} \cap \mathcal{R}(A \circ I)^{[*]} \subseteq A^{[*]} \circ A \circ K + \mathcal{N}(A \circ I)\)

Proof. Let \(y = (A \circ I)^{[*]} \circ x \in K^{[*]}\). Then \((A^{[*]} \circ A)^{[I]} \circ y = u + z, u \in K, z \in \mathcal{N}(A \circ I)\). From this \(y = (A^{[*]} \circ A) \circ (u + z) + w, w \in \mathcal{N}(A^{[*]} \circ A)^{[I]}\). Since \(A^{[*]} \circ A = (A \circ I)^{[*]} \circ (A \circ I)\) and \(z \in \mathcal{N}(A \circ I)\), we get \(y = A^{[*]} \circ A \circ u + w \in A^{[*]} \circ A \circ K + \mathcal{N}(A \circ I)\) \(\Box\)

Lemma 3.14. Suppose that \(A^{[I]} \circ A \circ K \subseteq K\). Then \((A^{[*]} \circ A)^{[I]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I) \Leftrightarrow (A^{[*]} \circ A)^{[I]} \circ K^{[*]} \subseteq K\)

Proof. It is enough to show the necessity part. Let \(x \in K^{[*]}\) and \(y = (A^{[*]} \circ A)^{[I]} \circ x\). Then \((A^{[*]} \circ A)^{[I]} \circ x = u + v\) where \(u \in K, v \in \mathcal{N}(A \circ I)\). This implies \(x = (A^{[*]} \circ A) \circ (u + v) + w, w \in \mathcal{N}(A \circ I)\), so that \(y = (A^{[*]} \circ A)^{[I]} \circ (A^{[*]} \circ A) \circ (u = A^{[I]} \circ A \circ u \in K\) \(\Box\)

Remarks 3.15. Let \(A\) be as given in Remark 3.5 and let \(K = \mathbb{R}^{3,3}_+\). Then \(K^{[*]} = NK^* = \mathbb{N}\mathbb{R}^{3,3}_+, A \circ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \mathcal{N}(A \circ I) = \text{span}\{(0, 1, 1)^t\}\) and \(\mathcal{R}(A \circ I)^{[*]} = \{(x, -y, y)^t : x, y \in \mathbb{R}\}\). Also, \(A^{[*]} \circ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\). So,

\[(A^{[*]} \circ A)^{[I]} = \frac{1}{4} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\] \(\Rightarrow\) \(K^*\).

Let \(x^1 = (x, y, z)^t \in K^*\), then \((A^{[*]} \circ A)^{[I]} \circ N x^1 = \frac{1}{4} 4(4x, -y + z, y - z) \in (A^{[*]} \circ A)^{[I]} \circ K^{[*]}\).

Since \((4x, -y + z, y - z)^t = (4x, b, c)^t - \left(\frac{b + c}{2}\right)(0, 1, 1)^t\) where \(b, c \geq 0\) such that \(\frac{b - c}{2} = -y + z, nn (4x, -y + z, y - z)^t \in K + \mathcal{N}(A \circ I)\). Thus \((A^{[*]} \circ A)^{[I]} \circ K^{[*]} \subseteq K + \mathcal{N}(A \circ I)\). But for \(x^1 = (1, 2, 3) \in K^*\), \(N x^1 \in K^{[*]}\) and \((A^{[*]} \circ A)^{[I]} \circ N x^1 = \frac{1}{4} 4(4, 1, -1)^t \notin K\) Thus \((A^{[*]} \circ A)^{[I]} \circ K^{[*]} \notin K\). Hence we can conclude that in the absence of the condition \(A^{[I]} \circ A \circ K \subseteq K\), Lemma 3.14 may not be true.

We are now in a position to prove the main result of this article.

Theorem 3.16. (Main Result) Let \(A \in \mathbb{R}^{m \times n}\) with \(\mathcal{R}(A \circ I)\) closed, \(K\) be a closed in \(\mathbb{R}^n\) with \(A^{[I]} \circ A \circ K \subseteq K\). Let \(C = A \circ I \circ K\) and \(D = (A^{[I]})^{[*]} \circ I \circ K^{[*]}\). Then the following conditions are equivalent:

8
(i) $D$ is acute.
(ii) $(A^{[s]} \circ A)^{[l]} \circ K^{[s]} \subseteq K + \mathcal{N}(A \circ I)$.
(iii) $C$ is obtuse.

Proof. (i) $\Rightarrow$ (ii):
Suppose $D$ is acute then by definition, $D \subseteq D^{[s]}$. By Lemma 3.6, $D^{[s]} = A \circ I \circ K + \mathcal{N}(A \circ I)^{[s]}$. Thus $D \subseteq A \circ I \circ K + \mathcal{N}(A \circ I)^{[s]}$. Now, by Lemma 3.11 we obtain $(A^{[s]} \circ A)^{[l]} \circ K^{[s]} \subseteq K + \mathcal{N}(A \circ I)$.

(ii) $\Rightarrow$ (i):
Suppose $(A^{[s]} \circ A)^{[l]} \circ K^{[s]} \subseteq K + \mathcal{N}(A \circ I)$. By Lemma 3.11, $D \subseteq A \circ I \circ K + \mathcal{N}(A \circ I)^{[s]}$. But by Lemma 3.6, $A \circ I \circ K + \mathcal{N}(A \circ I)^{[s]} = D^{[s]}$. So, $D \subseteq D^{[s]}$. Hence $D$ is acute.

(ii) $\Rightarrow$ (iii) Suppose $(A^{[s]} \circ A)^{[l]} \circ K^{[s]} \subseteq K + \mathcal{N}(A \circ I)$. Note that $C = A \circ I \circ K$ is obtuse if $C^{[s]} \cap \mathcal{R}(A \circ I)$ is acute. By Lemma 3.10, it is enough to show that $C^{[s]} \cap \mathcal{R}(A \circ I) \subseteq C$.

Let $y \in C^{[s]} \cap \mathcal{R}(A \circ I)$. Then $y = A \circ I \circ x$ and by Lemma 3.2, $(A \circ I)^{[s]} \circ y \in K^{[s]}$. So, $(A \circ I)^{[s]} \circ y \in K^{[s]} \cap \mathcal{R}(A \circ I)^{[s]}$. By Lemma 3.13, $(A \circ I)^{[s]} \circ y = A^{[s]} \circ A \circ u + z$ with $u \in K, z \in \mathcal{N}(A \circ I)$. Since $A^{[s]} \circ A = (A \circ I)^{[s]} \circ (A \circ I)$, it follows that $(A \circ I)^{[s]} \circ y, A^{[s]} \circ A \circ u \in \mathcal{R}(A \circ I)^{[s]}$. Thus $z \in \mathcal{R}(A \circ I)^{[s]} \cap \mathcal{N}(A \circ I) = \{0\}$. This implies $z = 0$. Then $(A \circ I)^{[s]} \circ y = A^{[s]} \circ A \circ u$. From this,

\[
y = ((A \circ I)^{[l]}{[s]} \circ (A \circ I)^{[s]} \circ A \circ I \circ u) + w
= ((A \circ I) \circ (A \circ I)^{[l]}{[s]} \circ (A \circ I) \circ u) + w
= (A \circ I) \circ (A \circ I)^{[l]} \circ (A \circ I) \circ u + w
= (A \circ I) \circ u + w,
\]

where $w \in \mathcal{N}((A \circ I)^{[s]}).$

Since $y \in \mathcal{R}(A \circ I)$, it follows that $w \in \mathcal{R}(A \circ I) \cap \mathcal{N}(A \circ I)^{[s]} = \{0\}$. Thus $y \in A \circ I \circ K = C$.

(iii) $\Rightarrow$ (ii):
Let $C = A \circ I \circ K$ be obtuse. Then by definition, $C^{[s]} \cap \mathcal{R}(A \circ I) \subseteq C$. By Lemma 3.8, $(A^{[l]}{[s]} \circ I \circ K^{[s]} \subseteq C$. Now by Lemma 3.12, $(A^{[s]} \circ A)^{[l]} \circ K^{[s]} \subseteq K + \mathcal{N}(A \circ I)$.

□

Remarks 3.17.

(i) The following example illustrates Theorem 3.10. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$,
\[
M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \mathbb{R}^3_+.
\]
Then $A^+ = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. 

9
\[A^\dagger = NA^\dagger M = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\] and \[K[^*] = N\mathbb{R}^3_+\]. Note that for \(x^1 = (x, y, z)^t \in K\), \(A^\dagger \circ A \circ x^1 = A^\dagger Ax^1 = \frac{1}{2}(x + z, 0, x + z)^t \in K\).

Thus \(A^\dagger 0 A \circ K \subseteq K\). And \((A[^*] \circ A)^\dagger = \frac{1}{16} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}\). Therefore \((A[^*] \circ A)^\dagger \circ K[^*] = \mathbb{N}(A[^*] \circ A)^\dagger N K[^*] \subseteq K\).

Note that for \(x^1 = (x, y, z)^t \in K\), \(A^\dagger \circ 0 A \circ x^1 = A^\dagger Ax^1 = \frac{1}{2}(x + z, 0, x + z)^t \in K\).

Thus \(A^\dagger 0 A \circ K \subseteq K\). And \((A[^*] \circ A)^\dagger = \frac{1}{16} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}\). Therefore \((A[^*] \circ A)^\dagger \circ K[^*] = \mathbb{N}(A[^*] \circ A)^\dagger N K[^*] \subseteq K\). Also one can easily verify that \(C = A \circ I \circ K\) is obtuse and \(D = (A^\dagger[^*] \circ A)^\dagger \circ I \circ K[^*]\) is acute.

(ii) Here, we show by an example that in the absence of the condition \(A \circ I = I \circ A\), Theorem 3.16 may not hold. Let \(A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\), \(M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) = \(N\). Then clearly \(A \circ I \neq I \circ A\). Let \(K = \{(x, 0) : x \geq 0\}\) then \(K[^*] = \{(x, y) : x \geq 0, y \in \mathbb{R}\}\) and \(K[^*] = \{(y, x) : x \geq 0, y \in \mathbb{R}\}\).

Also, \(A^\dagger = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\) and \(A[^*] = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\). Clearly \(A^\dagger \circ A \circ K \subseteq K\) and \(D = \{(x, x/2) : x \geq 0\}\) is acute but \((A[^*] \circ A[^*])^\dagger \circ K[^*] \notin K\) where \((A[^*] \circ A[^*])^\dagger = \frac{1}{4} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\).

(iii) For given \(A \in \mathbb{R}^{m \times n}\), Ramanathan and Sivakumar [8] derived a set of necessary and sufficient conditions for a cone to be invariant under \((A[^*] \circ A[^*])^\dagger\) in terms of pairwise acuteness of cones \(D\) and \(I \circ D\) in indefinite inner product space. We would like to remark here that pairwise acuteness of \(D\) and \(I \circ D\) is same as acuteness of the cone \(D\) in usual inner product space.

Acknowledgements: We thank Prof.K.C. Sivakumar for his valuable comments and suggestions to improve this article.

References

[1] Ben-Israel, A., and Greville, T.N.E., Generalized Inverses:Theory and Applications, 2nd edition, Springer Verlag, New York, 2003.

[2] Berman, A., and Plemmons, R.J., Nonnegative Matrices in the Mathematical Sciences, Classics in Applied Mathematics, SIAM, 1994.
[3] Bognar, J., *Indefinite inner product spaces*, Springer Verlag, 1974.

[4] Cegielski, A., *Obtuse cones and Gram matrices with non-negative inverse*, Lin. Alg. Appl., 335, 167-181, 2001.

[5] Collatz, L., *Functional Analysis and Numerical Mathematics*, Academic Press, New York, 1966.

[6] Kamaraj, K., Sivakumar, K.C., *Moore-Penrose inverse in an Indefinite Inner Product space*, Journal of Applied Mathematics and computing, 19(1-2), 297-310, 2005.

[7] Ramanathan, K., Kamaraj, K and Sivakumar, K.C., *Indefinite product of matrices and applications to indefinite inner product spaces*, Journal of Analysis, 12, 135-142, 2004.

[8] Ramanathan, K., Sivakumar, K.C., *Nonnegative Moore-Penrose Inverse of Gram Matrices in an Indefinite Inner Product Space*, J Optim Theory Appl., 140, 189-196, 2009.

[9] Sivakumar, K.C., *A new characterization of nonnegativity of Moore-Penrose inverses of Gram operators*, Positivity, 13, 277-286, 2009.