Instability of the $\left(\frac{\varphi}{\varphi}^2\right)^3$ model with spontaneously broken scale symmetry

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Abstract

The viability of the phase with spontaneous breaking of scale symmetry observed in the infinite N limit of the O(N) symmetric phi-six theory in three dimensions is scrutinized against quantum corrections at the next-to-leading order in 1/N expansion. It is shown that inclusion of such corrections disrupt the flatness and the global real-valuedness of the effective potential which develops an imaginary part for some values of the fields in a strong coupling regime that covers the BMB and the nontrivial ultraviolet fixed points. This signals instability of the phase with spontaneously broken approximate scale invariance which is doomed to decay by higher 1/N radiative corrections.

1. Introduction

The tri-critical O(N) symmetric $\eta(\varphi^2)^3$-theory in three space–time dimensions has the attractive feature of being scale-invariant. This model offers a unique platform to study the spontaneous breaking of scale symmetry which occurs in the large N limit. The phase diagram of this model is of importance in studying several condensed matter systems, such as liquid helium, metamagnets, and graphene [1]. The model contains no dimensional parameters at the classical level, the $\eta(\varphi^2)^3$ interaction is scale invariant in three space-time dimensions and remains exactly marginal at the leading order in the large N expansion. At the next-to-leading order, the interaction becomes a marginally irrelevant slowly running coupling making the theory approximately scale invariant. The beta function is known to exhibit a trivial infrared fixed point at $\eta = 0$ along with a nontrivial ultraviolet fixed point at $\eta \approx 192$ [2–5]. Yet, more surprisingly, as explored in [6], the infinite N limit of $\eta(\varphi^2)^3$ model shows a quantum phase transition to a phase in which the O(N) singlet composite operator $\varphi^2$ develops a nonzero expectation value giving rise to a broken scale symmetry phase with a massless dilaton. This so-called BMB phase occurs when the coupling constant is tuned at the fixed point (FP) to a critical value $\eta_c = 16\pi^2$, which is situated below the ultraviolet fixed point. The analysis of [6] advocated that in the large but not infinite N limit, where the scale symmetry becomes approximate, the dilaton becomes a quasi-Goldstone boson, acquiring a mass of order 1/N. Several aspects of that phenomenon including 1/N corrections were analyzed in [7]. Other relevant investigations included models with abelian and non-Abelian Chern–Simons gauge fields [8, 9]. Also recent work [10, 11] in the context of the functional renormalization group interpreted the model’s BMB fixed point as the intersection between a line of regular FPs and another line made of singular FPs. This latter study delineated the dependence on N and on the space–time dimension d to preserve the critical behavior, and showed the existence of non-perturbative FPs with which the BMB FP can collide in the (N, d) phase space as N varies from infinity to finite values. On the other hand, the likelihood of BMB phenomenon was examined in [12, 13] which inferred that for all finite N the BMB phenomenon does not survive and that it is regularization scheme dependent which hints to an instability of the BMB phase. Also [14] pointed out the lack of strict continuum limit of the theory due to the unboundedness of the potential. Additionally, the fate of the dilaton was investigated in [15] revealing that it is in fact a tachyon and indicating instability of the phase with spontaneously broken approximate scale invariance.

Motivated by these works which deemed the $\eta(\varphi^2)^3$ BMB phase as just some unstable state, we re-examine the robustness of the phase by carefully investigating the fields’ space domain and the range of coupling.
constants where the effective potential is driven to instability by developing an imaginary part from which the
decay rate of its ‘false vacuum’ state can be obtained. Our analysis is carried out in the framework of the
constraint effective potential method [16–18] and 1/N expansion technique [19]. The constraint effective
potential method fixes dynamical variables by imposing a constraint on a measure of the partition function so
that the dynamical variables take some given value. This is alternative to the conventional way that defines the
effective potential through external sources coupled to dynamical variables followed by the Legendre
transformation [20]. The connection between the two methods is clarified in [16].

The 1/N expansion technique offers a basis to explore non-perturbative features of the model. This is
achieved by means of auxiliary fields followed by an integration over the original degrees of freedom which yields
an effective theory in which all-N dependence is fully explicit, enabling the analysis of higher order corrections in
a systematic 1/N expansion [19]. This technique has a long history in the study of critical phenomena, starting
with the seminal work of Stanley [21] on the N vector model and the analysis of Ma [22] of the 1/N corrections to
the critical exponents which elucidated the Wilson renormalization group ideas. Large-N technique was also
applied successfully to the Gross-Neveu and the CPN-1 models [23, 24], as well as to the topological Ginzburg–
Landau theory of self-dual Josephson junction arrays [25].

The remainder of this paper is organized as follows. In section 2, I briefly review the O(N)-symmetric (\(\tilde{\varphi}^2\))^N
model on a lattice in order to define the constraint effective potential and a formulation of its 1/N expansion. In
section 3, I focus on the continuum limit of the model and the renormalization effects of the effective potential.
In section 4, I investigate the effective potential at the next-to-leading order and extract its imaginary part from
which I determine the decay rate of its ‘false vacuum’ state. Section 5 contains some concluding remarks.

2. The model and 1/N expansion

The O(N)-symmetric (\(\tilde{\varphi}^2\))^N on a 3-dimensional cubic lattice (the lattice spacing is set to be unity for now) has
N-scalar fields \(\varphi^a_x, a = 1, \ldots, N\), defined on each lattice site \(x\). We impose periodic boundary conditions:

\[\varphi^a_x = \varphi^a_{x + \text{L} e^\mu}, \quad \mu = 1, 2, 3\]

where \(e^\mu\) is the unit vector pointing to the \(\mu\)-direction and \(L\) is an integer representing
the length of the side of the cubic lattice. The model is defined by the action

\[S_0 = \sum_x \left\{ \frac{1}{2} \varphi^a_x (\Delta) \varphi^a_x + \frac{\eta}{6N^2} \langle \varphi^a_x \varphi^a_{x+1} \rangle^2 \right\},\]

where the summation over \(a = 1, \ldots, N\) is implicitly taken. The action of the lattice Laplace operator \(\Delta\) on fields is as follows

\[\Delta \varphi^a_x = \sum_{\mu=1}^3 \left( \varphi^a_{x+\ell_\mu} + \varphi^a_{x-\ell_\mu} - 2 \varphi^a_x \right).\]

Introducing auxiliary fields \(\sigma_x\), transforms the action into

\[S = \sum_x \left\{ \frac{1}{2} \varphi^a_x (\Delta + \varphi^a_x) \right\} - \frac{N}{2} \sigma^2_x + NV (\sigma^2_x)\]

where \(V (\sigma_x) = \eta \sigma^4_x / 6\). This is possible since the functional integration over \(\sigma_x\) in the partition function yields
the constraint \(\varphi^a_x \varphi^a_{x+1} = N \sigma^2_x\), and the resulting action equals the original \(S_0\) up to a constant. The effective
potential \(U\) of the model is defined by

\[\exp \{ -\Omega U (\tilde{\varphi}^2) \} = \int \prod_{x,a} d\varphi^a_x d\sigma_x \exp \{ -S \},\]

where \(\tilde{\varphi}^2 = \phi^a \phi^a\) and \(\Omega\) is the total number of the lattice sites and

\[\varphi^a_x = \frac{1}{\Omega} \sum_x \varphi^a_x.\]

Due to the O(N) symmetry, \(U\) is a function of \(\tilde{\varphi}^2\) and the partition function \(Z\) of the system is given by

\[Z = \int \prod_a d\phi^a \exp \{ -\Omega U (\tilde{\varphi}^2) \}.\]

To proceed with the construction of the effective potential \(U\), we first implement a Fourier representation of
the \(\delta\)-function

\[\delta (\varphi^a_x - \sqrt{N} \tilde{\varphi}^2) = \int d\xi \exp \left\{ i \xi a \left( \sum_x \varphi^a_x - \sqrt{N} \tilde{\varphi}^2 \right) \right\}.\]
Given that the action in (3) is quadratic in the $\varphi^a_x$ scalar fields, we perform a functional integration over $\varphi^a_x$ and $\xi^a$ resulting into

$$
\exp \{ -\Omega U(\phi^2) \} = \int \prod_x d\chi_x d\sigma_x \exp \{ -NS_1 \}
$$

$$
S_1 = \frac{1}{2} \text{Tr} \ln(-\Delta + \sigma_x) + \frac{1}{2} \ln \Gamma + \frac{i\Omega}{2} \Gamma^{-1}\phi^2
$$

$$
+ \sum_x \left\{ -\frac{1}{2} \sigma_x \chi_x + V(\chi_x) \right\},
$$

where the functional $\Gamma$ is defined as

$$
\Gamma = \sum_{x,y} \left\{ \chi \left| \frac{1}{-\Delta + \sigma_x} \right\} \gamma \right\}.
$$

The parameter $N$ seen in (8) plays the role of $1/h$. Therefore, for large $N$, one can use the saddle-point method to expand the path integral around translationally invariant solutions $\sigma$ and $\chi$ which extremize the effective action $S_1$. This approach was implemented in [26] and yields the gap equation which takes the following form in momentum space

$$
\phi^2 = \chi - \frac{1}{2} \sum_{p=0}^{13} \frac{1}{-\Delta(p) + m^2},
$$

where $m^2 = \sigma$. The leading-order effective potential $U(\phi^2)$ is obtained by inserting the gap equation solution into $S_1$,

$$
U(\phi^2) = \frac{N}{2\Omega} \sum_{p=0}^{13} \ln(-\Delta(p) + m^2) + \frac{N}{2} m^2 (\phi^2 - \chi) + NV(\chi).
$$

Higher-order $1/N$ corrections are obtained by integrating out fluctuations around the saddle point using, $\chi_x \rightarrow \chi + \delta\chi/\sqrt{N}$, $\sigma_x \rightarrow m^2 + i\delta\sigma/\sqrt{N}$ and expanding the $\text{Tr} \ln(\cdots)$ term in (8) as

$$
N\text{Tr} \ln \left( -\Delta + m^2 + \frac{i\delta\sigma}{\sqrt{N}} \right) = \frac{1}{2} \sum_{x_1,x_2} S^{(2)}(x_1, x_2) \delta\sigma(x_1) \delta\sigma(x_2)
$$

$$
- \frac{i}{3\sqrt{N}} \sum_{x_1,x_2,x_3} S^{(3)}(x_1, x_2, x_3) \delta\sigma(x_1) \delta\sigma(x_2) \delta\sigma(x_3)
$$

$$
- \frac{1}{4N} \sum_{x_1,x_2,x_3,x_4} S^{(4)}(x_1, x_2, x_3, x_4) \delta\sigma(x_1) \delta\sigma(x_2) \delta\sigma(x_3) \delta\sigma(x_4)
$$

$$
+ \cdots,
$$

where $S^{(n)}(x_1,\ldots,x_n) = \delta^n\Sigma/\delta\sigma(x_1) \cdots \delta\sigma(x_n)$. The quadratic, cubic, and quartic interaction terms in (12) have a simple graphical interpretation as shown in the diagrams of figure 1.

### 3. Effective potential in the continuum and renormalization effects

The effective potential in the continuum quantum field theory is obtained, in a standard way, by introducing a lattice spacing parameter which measures the dimensions of physical quantities (Volume $V = a^4\Omega$) and the canonical dimension of the scalar fields $\chi \rightarrow a\chi$, $\sigma \rightarrow a^2\sigma$, $\varphi^2 \rightarrow a^4\varphi^2$. The imposed periodic boundary conditions on the system lead to a quantized momentum by $2\pi/V^{1/3}$ which runs from $-\pi/a$ to $\pi/a$. In the infinite volume limit ($V \rightarrow \infty$) and the continuum limit ($a \rightarrow 0$), the lattice Laplace operator is replaced by the momentum squared $-a^2\Delta \rightarrow p^2$ and all summations over the lattice momentum are replaced by continuum momentum integrations. Integrating over quadratic fluctuations while ignoring the cubic and higher-order
vertices in (12) gives the effective potential at the next-to-leading order. In the continuum limit this is given by

\[
\frac{1}{N} U(\phi^2) = \frac{1}{2} m^2 (\phi^2 - \chi) + V(\chi) - \frac{1}{12\pi} m^3 + \frac{1}{2N} \int dp \ln\left(1 + 2V''(p)\right)
\]

\[
+ \frac{1}{2N} \int dp \ln\left(1 + \frac{4\phi^2 V''}{(p^2 + m^2)(1 + 2V''(p))}\right)
\]

(13)

where \(V'' = \eta \chi\) and \(\Pi(p)\) arises from the quadratic term in (12) which is visualized as the first Feynman diagram in figure 1. In \(d = 3\) momentum space, it is given by [26]

\[
\Pi(p) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2 (k + p)^2 + m^2} = \frac{1}{4\pi p} \arctan\left(\frac{p}{2m}\right).
\]

(14)

We should note that the passage to the continuum introduces ultraviolet divergences in the momentum integral in the leading-order effective potential (11). This requires a cut-off \(\Lambda\) and a non-multiplicative renormalization of the field as \(\chi_R = \chi - \Lambda/2\pi^2\). Additional divergences arise from the momentum integral of the logarithm term in the first line of (13) as follows

\[
\frac{1}{2} \int \ln\left(1 + 2V''(\Pi(p))\right) = V'' \int \Pi(p) - (V'')^2 \int \Pi^2(p) + \frac{4(V'')^3}{3} \int \Pi^3(p) + \cdots
\]

\[
= \frac{V'' \Lambda^2}{32\pi^2} - \frac{V'' \Lambda}{128\pi^2}\left(\frac{32m}{\pi} + V''\right)
\]

\[
+ \frac{(V'')^2}{16\pi^3} \left(\frac{m}{\Lambda} + \frac{\pi V''}{48}\right) \ln\left(\frac{\Lambda}{m}\right) + \text{Finiteterms.}
\]

(15)

These are handled by expressing the effective potential (13) in terms of a renormalized mass \(M\) defined through the self-energy \(\Sigma\) represented by the diagram of figure 2 where the dashed line stands for the propagator \(D(q)\) of the \(\delta\sigma\) fluctuations [26]

\[
M^2 = m^2 + \frac{1}{N} \sum_i (0, m^2) - \frac{m^2 \partial \Sigma(p^2, m^2)}{\partial p^2}\bigg|_{p^2 = 0},
\]

(16a)

with

\[
\sum_i (p, m^2) = \int \frac{d^d q}{(2\pi)^d} \frac{D(q)}{(p + q)^2 + m^2} = \int \frac{d^d q}{(2\pi)^d} \frac{4V''}{((p + q)^2 + m^2)(1 + 2V''\Pi(q))}
\]

\[
= \int \frac{d^d q}{(2\pi)^d} \frac{4V''}{q^2 + m^2 (1 - 2V''\Pi(q)) + (2V'')^2 \Pi^2(q) + \cdots}
\]

\[
= \frac{2V''}{\pi^2} \Lambda - \frac{(V'')^2}{2\pi^2} \ln\left(\frac{\Lambda}{m}\right) + \text{finite terms.}
\]

(16b)

It’s worth noting that \([\partial \Sigma / \partial p^2]_{p^2 = 0}\) in (16a) has no divergences and hence there is no wave-function renormalization at the next-to-leading order.

Expressing the effective potential in (13) in terms of the renormalized mass \(M\), gets rid of all divergent terms involving powers of \(M\). The remaining divergent terms involving the \(\chi\) field alone are cancelled by adding counter terms to \(V(\chi)\). This step yields a relation between the bare coupling constant \(\eta\) and the renormalized one \(\eta_R\) at a scale \(\mu\), and the freedom in defining the coupling constant at scale \(\mu\) leads to the known renormalization group beta function \(\beta(\eta_R) = d\eta_R / d \ln \mu [3-5,26]\)

\[
\beta(\eta_R) = \frac{3\eta_R^2}{2\pi^2 N} - \frac{\eta_R^3}{2\pi^2 N}.
\]

(17)

The effective potential to the next-to-leading order expressed now in terms of renormalized quantities become
For ease of notation I dropped the subscript R from the renormalized coupling $\eta$ and field $\chi$. Note that the combined last two terms in the second line of (18a) give a divergence free contribution. This is easily seen by transforming them in the following form

$$
\frac{1}{N} U(\phi^2) = \frac{1}{2} M^2(\phi^2 - \chi) - \frac{1}{12\pi} M^3 + V(\chi) - \frac{1}{6} \chi^3 \beta(\eta) \ln \left( \frac{\mu}{M} \right) \\
- \frac{\phi^2}{2N} \Sigma(M) + \frac{1}{2N} \int \frac{d^4p}{(2\pi)^4} \ln \left( 1 + \frac{4\phi^2 V''(\rho)}{(\rho^2 + m^2)(1 + 2V''(\rho))} \right)
$$

(18a)
\[
\frac{1}{N} U(\phi^2) = \frac{1}{2} M^2 (\phi^2 - \chi) - \frac{1}{12\pi} M^3 + V(\chi) - \frac{1}{6} \chi^3 \beta(\eta) \ln \left( \frac{\mu}{M} \right) - \frac{1}{2N} \int_{0}^{4\phi^2/V} dz \int d^3p \frac{1}{(2\pi)^3 (p^2 + M^2)(1 + 2V''\Pi(p))(z + (p^2 + M^2))} [z + (p^2 + M^2)(1 + 2V''\Pi(p))].
\]  

(18b)

4. Imaginary part of the effective potential

Let us examine the shape of the effective potential in (18a). At the leading order, the gap equation \( \delta U/\delta M = 0 \), gives \( \chi = \phi^2 - M/4\pi \). We substitute the solution into the effective action and then obtain

Figure 5. The next to leading radiative correction of \( U \) as a function of \( \eta = 4\pi\phi^2/M \) and \( k = \eta/16\pi^2 \).

Figure 6. The leading order effective potential (dot curve), the next-to-leading order correction (dashed curve) and their collective effect (solid curve). Here \( k = 1 \) and \( N = 10 \).
The potential at this order is depicted in figure 3. The next to leading radiative correction as expressed by the second line in (18b) modifies the shape of the effective potential in figure 3. To verify that we note that only the small momentum part of the integrand in the second line of (18b) contributes significantly, and thus we can approximate it as follows:

\[
\frac{U}{N} = \frac{M^3}{24\pi} + \frac{\eta}{6}\left(\phi^2 - \frac{M}{4\pi}\right)^3.
\]  

(19)
where $\tilde{f}^2 = 4\pi f^2/M$ and $k = \eta/16\pi^2$. In particular for $k = 1$, this expression reduces to

$$\frac{2M^3}{3\pi^2N} \left\{ 4\left(1 - \tilde{f}^2\right) + \ln \left(4\tilde{f}^2 - 3\right) \right\}.$$ (20b)

Because of the logarithm term, this radiative correction takes on large negative values as $\tilde{f}^2$ approaches 0.75 from the right. This is validated by a numerical computation of the integral in the second line of (18b) as depicted in figure 4.

Furthermore, the 3d plot in figure 5 shows how the flatness around $\phi = 0$ of the leading order potential is lifted for various values of $k = \eta/16\pi^2$.

This clearly shows that the next-to-leading order corrections change drastically the shape of the leading order effective potential. Figure 6 illustrates the marked change on the total effective potential.

Figure 9. (a) $x_1$ versus $q$, (b) $x_2$ versus $q$, and (c) the imaginary part of the effective potential for $1/k = 0.4$.

Figure 10. (a) $x_1$ versus $q$ and (b) the imaginary part of the effective potential for $k = 1$.  

\begin{equation}
\begin{aligned}
-\frac{1}{2N} \int_0^{\Delta^V} z dz \int_{p<2M} \frac{d^3p}{(2\pi)^3} \frac{1}{M^2(1 + 2V''(0))(z + M^2(1 + 2V''(0)))} \\
\end{aligned}
\end{equation}
As seen in figure 6, the $1/N$ radiative correction disrupts the vacuum stability at the leading order. Bearing in mind that the effective potential has the physical meaning of an energy density, the leading order vacuum at the origin in figure 6 can be viewed as a ‘false vacuum’ since it has a higher energy than the much lower energy state produced by next-to-leading order radiative corrections. From this viewpoint, a field configuration in the initial higher energy state, would inevitably decay to the much lower energy states generated by the $1/N$ radiative correction. The probability of decay of this state per unit volume $G V$ is determined by the imaginary part of the effective potential $\text{Im}(U) = -\Gamma/2V$ [27, 28]. Note that adding a negative imaginary term to the effective potential is a standard device for representing unstable particles or field configurations [29].

A close examination of the effective potential in (18a) reveals that it develops an imaginary part for those field values that make the argument of the logarithm in the second line of (18a) negative. To determine the imaginary part of $\text{Im}(U)$, it is merely necessary to add an infinitesimal negative imaginary constant to the $+ \pi M^2$ term in the denominator of the second line of (18b) and use the familiar device

$$\int_{-\infty}^{\infty} dp = \int_{-\infty}^{0} dp + \int_{0}^{\infty} dp$$

for $\varepsilon \rightarrow 0$. We obtain from (18b)

$$\text{Im}(U) = \frac{-\pi}{2N} \int_{0}^{\pi} dv \int_{0}^{2\pi} d\varphi \delta(v - (p^2 + M^2)(1 + 2V\Pi(p)))$$

$$= -\frac{2M^2}{3\pi N} (x_2^3 - x_1^3 H(q^* - q)),$$

where $\tilde{\varphi} = 4M^2k\tilde{\varphi}^2 (1 - \tilde{\varphi}^2), q = \tilde{\varphi}^2 = 4\pi \tilde{\varphi}^2/M$ and $H(q)$ is the Heaviside step function. The conditions for the occurrence of a nonzero imaginary part are shown in figure 7 - namely, $1/k < (1 - q)(4q + 1)$. As seen in figure 7, for $1/k > 1$, the region in the field space over which an imaginary part of the effective potential exists is bounded by $q_d$ and $q_0$, which are given by $q_{d,2} = [3 \pm (25 - 16/k)^{1/2}] / 8$ and gets narrower as $1/k$ approaches 25/16; for $1/k \leq 1$, this region is much wider and is between zero and $q_d$.

The other parameters entering the second line of equation (21) are defined in figure 8.

For small values of $x = p/2M$, $x_1$ and $x_2$ can be approximated by using $\arctan(x) \approx x - x^3/3$ as

$x_0 = [3(1 - k(1 - q)^{-1})]^{1/2}, x_2 = [k(1 - q)(1 + 4q) - 1]^{1/2} / [4 - 11k(1 - q)/3]^{1/2}$. A more accurate depiction of their variation with $q = \tilde{\varphi}^2 = 4\pi \tilde{\varphi}^2/M$ is obtained by a numerical computation whose results are shown in figures 9, 10 and 11 for particular values of $1/k$. 

![Figure 11. (a) $x_2$ versus $q$, and (b) the imaginary part of the effective potential for $1/k = 1.4$.](image-url)
5. Conclusion

In this paper I have re-examined the phi-six $O(N)$ symmetric vector model in the phase that exhibits spontaneous braking of scale invariance where a fundamental mass scale naturally emerges accompanied by a massless dilaton which is a Goldstone boson. Working in the framework of the constraint effective potential method and using the $1/N$ expansion technique, I investigated the effects of the quantum corrections in order to inspect whether the viability of the phase at the leading order phase is preserved. I found that although at the leading order in $1/N$, the effective potential has a flat bottom, the next-to-leading radiative corrections disrupt that flatness. Furthermore, the effective potential ceases to be globally real-valued since it develops an imaginary part. The conditions for such occurrence are obtained in terms of the domain of dangerous $\phi$-field configurations and the range of coupling constants. This turns out to correspond to the strong coupling regime ($\eta > (16\pi/5)^2 \approx 101$) which encompasses the BMB fixed point at $\eta = 16\pi^2 \approx 158$ and the nontrivial ultraviolet fixed point at $\eta = 192$ where supposedly the $1/N$ corrections are required to stabilize the theory. The occurrence of the imaginary part in the effective potential signals instability of the $\eta(\phi^2)$ BMB phase which should be considered as a state doomed to decay at finite $N$ by dominant radiative corrections. Moreover, given that the effective potential has the physical meaning of an energy density, when it acquires an imaginary part, the system’s phase develops a certain probability of disappearing per unit volume $\Gamma/2\sqrt{N}$ which is determined by the imaginary part of the effective potential $\text{Im}(U) = -\Gamma/2\sqrt{N}$ [27–29].

In the spirit of a $1/N$ expansion, the appearance of an imaginary part in the effective potential at the next-to-leading order is sufficient to signal an instability of the theory. This instability observed at the next-to-leading order should remain valid since any corrections at the next-to-next-to-leading order are suppressed by $1/N^2$ and cannot nullify it. However, to go actually beyond this approximation, a more rigorous numerical lattice calculation is required.

These findings shed more light on the instability, at finite $N$, of the phase of the theory with spontaneously broken approximate scale invariance and support the result of [15] which reached a similar conclusion by a different approach that found the accompanying dilaton acquired a tachyonic mass.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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