NON-LINEAR EXTENSION OF INTERVAL ARITHMETIC AND EXACT RESOLUTION OF INTERVAL EQUATIONS OVER SQUARE REGIONS

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ABSTRACT

The interval numbers is the set of compact intervals of \( \mathbb{R} \) with addition and multiplication operation, which are very useful for solving calculations where there are intervals of error or uncertainty, however, it lacks an algebraic structure with an inverse element, both additive and multiplicative. This fundamental disadvantage results in overestimation of solutions in an interval equation or also overestimation of the image of a function over square regions. In this article we will present a solution to this problem, through a morphism that preserves both the addition and the multiplication between the space of the interval numbers to the space of square diagonal matrices.

Keywords Interval Arithmetic · Interval Equation · Interval Computation

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1 Introduction

In this article we propose a general method to solve equations with interval variables, that is to say where the unknown of the problem is a compact interval, through a \( \varphi \) application that takes intervals and takes them to 2 times 2 square matrices. In the first part of this article we will give a brief introduction to the Artificial Interval. In the second part we will state and demonstrate the fundamental theorem of the interval functions, which gives us the necessary conditions so as not to have lost points, when computing the image of a square region (finite Cartesian producer of compact intervals) through the \( \varphi \) application. And finally in the third and last part of this article, through the fundamental theorem of the interval functions, we will state and demonstrate the fundamental theorem of the interval equations, that will give us necessary and sufficient conditions to solve the interval equations.
1.1 Basic Terms and Concepts the interval arithmetic

Recall that the closed interval denoted by \([a, b]\) is the real numbers given by
\[
[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}
\]  
(1)

We say that an interval is Degenerate if \(a = b\). Such an interval contains a single real number \(a\). By convention, we agree to identify a degenerate interval \([a, a]\).

We will denote by \(K_c(\mathbb{R})\) the set of compact intervals real.

We are about to define the basic arithmetic operations between intervals. The key point in these definition is that computing with set. For example when we add two interval, the resulting interval is set containing the sums of all pair of number, one form each of the initial sets. By definition then, the sum of two intervals \(X\) and \(Y\) is the set
\[
\{x + y; x \in X, y \in Y\}
\]
(3)

The difference of two intervals \(X\) and \(Y\) is the set
\[
\{x - y; x \in X, y \in Y\}
\]
(4)

The product of \(X\) and \(Y\) is given by
\[
\{xy; x \in X, y \in Y\}.
\]
(5)

Finally, the quotient \(X/Y\) with \(0 \notin Y\) is defined as
\[
\{x/y; x \in X, y \in Y\}
\]
(6)

1.2 Endpoint formulas for the arithmetic Operations

**Addition** Let us that an operational way to add intervals. Since \(x \in X = [x_1, x_2]\) means that \(x_1 \leq x \leq x_2\) and \(y \in Y = [y_1, y_2]\) means that \(y_1 \leq y \leq y_2\), we see by addition of inequalities that the numerical sums \(x + y \in X + Y\) must satisfy \(x_1 + y_1 \leq x + y \leq x_2 + y_2\). Hence, the formula \(X + Y = [x_1 + y_1, x_2 + y_2]\)

**Example 1** Let \(X = [0, 2]\) and \(Y = [-1, 1]\). Then \(X + Y = [0 - 1, 1 + 2] = [-1, 3]\)

**Subtraction** Let \(X = [x_1, x_2]\) and \(Y = [y_1, y_2]\). We add the inequalities
\[
x_1 \leq x \leq x_2 \quad \text{and} \quad -y_2 \leq -y \leq -y_1
\]
to get \(x_1 - y_2 \leq -y \leq x_2 - y_1\). It follows that \(X - Y = [x_1 - y_2, x_2 - y_1]\). Note that \(X - Y = X + (-Y)\) where \(-Y = [-y_2, -y_1]\)

**Example 2** Let \(X = [-1, 0]\) and \(Y = [1, 2]\). Then \(X - Y = [-1 - 2, 0 - 1] = [-3, -1]\)

**Multiplication** In terms of endpoint, the product \(XY\) of two intervals \(X\) and \(Y\) is given by
\[
XY = \{\min S, \max S\}, \quad \text{where} \quad S = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}
\]
(8)

**Example 3** Let \(X = [-1, 0]\) and \(Y = [1, 2]\). Then \(S = \{-1, -2, 0\}\) and \(XY = [-2, 0]\).

The multiplication of intervals is given in terms of the minimum and maximum of four products of endpoint, this can be broken into nine spacial cases. Let \(X = [x_1, x_2]\) and \(Y = [y_1, y_2]\) then \(XY = Z = [z_1, z_2]\), then

| case                        | \(z_1\)  | \(z_2\)  |
|-----------------------------|----------|----------|
| \(0 \leq x_1, y_1\)        | \(x_1y_1\) | \(x_2y_2\) |
| \(x_1 < 0 < x_2\) and \(0 \leq y_1\) | \(x_1y_2\) | \(x_2y_2\) |
| \(x_2 \leq 0\) and \(0 \leq y_1\) | \(x_1y_2\) | \(x_2y_1\) |
| \(0 \leq x_1\) and \(y_1 < 0 < y_2\) | \(x_2y_1\) | \(x_2y_2\) |
| \(x_2 \leq 0\) and \(y_1 < 0 < y_2\) | \(x_1y_2\) | \(x_1y_1\) |
| \(0 \leq x_1\) and \(y_2 \leq 0\) | \(x_2y_1\) | \(x_1y_2\) |
| \(x_1 < 0 < x_2\) and \(y_2 \leq 0\) | \(x_2y_1\) | \(x_1y_1\) |
| \(x_2 \leq 0\) and \(y_2 \leq 0\) | \(x_2y_2\) | \(x_1y_2\) |
| \(x_1 < 0 < x_2\) and \(y_1 < 0 < y_2\) | \(\min\{x_1y_2, x_2y_1\}\) | \(\max\{x_1y_1, x_2y_2\}\) |
Division As with real number, division can accomplished via multiplication by the reciprocal of the second operand. That is, we can implement equation using
\[ X/Y = X \left( \frac{1}{Y} \right), \quad (9) \]
where
\[ \frac{1}{Y} = \left\{ y; \frac{1}{y} \in Y \right\}. \quad (10) \]
Again, this assume \( 0 \notin Y \). For more information about interval numbers, see [1]

2 Interval Function Several Variable and his Fundamental Theorem

The main result of this section is the fundamental theorem of the interval functions, which allows us to calculate the value of a function on a square region, that is, on a finite Cartesian product of compact intervals, this can be interpreted as determining the image of a Interval function where each variable detects a unique value under the same symbol. Consider \( M_{2x2}(\mathbb{R}^n) \) as the space of the square matrices two by two over \( \mathbb{R}^n \). Let \( ||.||_n \) a norm in \( \mathbb{R}^n \) and \( ||.||_s \) the norm of sum in \( M_{2x2}(\mathbb{R}) \) then we define the following norm in \( M_{2x2}(\mathbb{R}^n) \) as
\[ \left\| \left( \begin{array}{cc}
(a_1, \ldots, a_n) & (b_1, \ldots, b_n) \\
(c_1, \ldots, c_n) & (d_1, \ldots, d_n)
\end{array} \right) \right\| = ||(a_1, \ldots, a_n)||_n + ||(b_1, \ldots, b_n)||_n + ||(c_1, \ldots, c_n)||_n + ||(d_1, \ldots, d_n)||_n \quad (11) \]
We have this application, it satisfies the norms of norm, then the space \( \mathbb{R}^n \) is a normed space. Let \( ||.|| \) be a matrix norm in \( M_{2x2}(\mathbb{R}^n) \). In this section we will consider \( D_2(\mathbb{R}^n) \) the space of the square diagonal matrices two by two over \( \mathbb{R}^n \) with the following norm \( ||.||_s = \frac{1}{2}||.|| \). Obviously \( ||.||_s \) is a norm in \( D_2(\mathbb{R}^n) \).

Proposition 4 \( M_{2x2}(\mathbb{R}^n) \) with a norm defined is complete.

Proof Let \( \{A_j\}_j \) an Cauchy sequence in \( M_{2x2}(\mathbb{R}^n) \) and let \( \varepsilon > 0 \), then exist a \( N > 0 \) such that for \( p, m > N \) we have \( ||A_p - A_m|| < \varepsilon \), then
\[ \left\| \left( \begin{array}{cc}
(a_{p1}, \ldots, a_{pn}) - (a_{m1}, \ldots, a_{mn}) & (b_{p1}, \ldots, b_{pn}) - (b_{m1}, \ldots, b_{mn}) \\
(c_{p1}, \ldots, c_{pn}) - (c_{m1}, \ldots, c_{mn}) & (d_{p1}, \ldots, d_{pn}) - (d_{m1}, \ldots, d_{mn})
\end{array} \right) \right\| \]
\[ = ||(a_{p1}, \ldots, a_{pn}) - (a_{m1}, \ldots, a_{mn})||_n + ||(b_{p1}, \ldots, b_{pn}) - (b_{m1}, \ldots, b_{mn})||_n + ||(c_{p1}, \ldots, c_{pn}) - (c_{m1}, \ldots, c_{mn})||_n + ||(d_{p1}, \ldots, d_{pn}) - (d_{m1}, \ldots, d_{mn})||_n < \varepsilon \]
\[ (12) \]
\[ (13) \]
\[ (14) \]
this implique that \( ||(x_{p1}, \ldots, x_{pn}) - (x_{m1}, \ldots, x_{mn})||_n < \varepsilon \) for \( x = a, b, c, d \) as \( \mathbb{R}^n \) is complete, then \( (x_{p1}, \ldots, x_{pn}) \) is convergent. Therefore \( \{A_j\}_j \) is convergent. Which proves that \( M_{2x2}(\mathbb{R}^n) \) is complete.

Proposition 5 Let \( \{A_j\}_j \) a sequence in \( M_{2x2}(\mathbb{R}^n) \). If the series \( \sum_{j=0}^{\infty} ||A_j|| \) converges for any norm in, so the series \( \sum_{j=0}^{\infty} A_j \) is converges.

Proof Let \( \varepsilon > 0 \), We will see that there is an integer \( N > 0 \) such that \( p, q \geq N \), then \( ||S_p - S_q|| < \varepsilon \), where \( S_p = \sum_{j=0}^{p} A_j \). This shows that the succession \( S_n \) of partial sums is a Cauchy sequence, As \( M_{2x2}(\mathbb{R}^n) \) it’s complete, then \( \{S_n\} \) is converges. Indeed:
\[ ||S_p - S_q|| = || \sum_{j=p+1}^{q} A_j || \leq \sum_{j=p+1}^{q} ||A_j|| = \sum_{j=0}^{p} ||A_j|| - \sum_{j=0}^{q} ||A_j|| \]
\[ (15) \]
Now, if \( \sum_{j=0}^{\infty} \|A_j\| \) converges, exist \( N > 0 \) such that \( p, q \geq N \), then

\[
\left| \sum_{j=0}^{p} \|A_j\| - \sum_{j=0}^{p} \|A_j\| \right| < \varepsilon.
\] (16)

what we wanted to demonstrate.

\begin{definition}
Let \( f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R} \) analytic function and \( a = (a_1, \ldots, a_n) \in \text{int}(\mathcal{X}) \) and \( \varepsilon > 0 \) such that \( B(a, \varepsilon) \subset \text{int}(\mathcal{X}) \), \( (a_1, \ldots, a_n), (\beta_1, \ldots, \beta_n) \in B(a, \varepsilon) \), we defined

\[
f \left( \prod_{j=1}^{n} \begin{pmatrix} x_j \\ y_j \end{pmatrix} \right) = f \left( \prod_{j=1}^{n} \varphi(a_j, b_j) \right) = f \left( \prod_{j=1}^{n} \begin{pmatrix} a_j \\ b_j \end{pmatrix} \right)
\] (17)

where \( (j_{j_1 \ldots j_n}) \) is a multinomial coefficient.

\begin{definition}
Let \( f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R} \) analytic function and \( a = (a_1, \ldots, a_n) \in \text{int}(\mathcal{X}) \) and \( \varepsilon > 0 \) such that \( B(a, \varepsilon) \subset \text{int}(\mathcal{X}) \), \( (a_1, \ldots, a_n), (\beta_1, \ldots, \beta_n) \in B(a, \varepsilon) \), we defined \( \varphi : Kc(\mathbb{R}) \rightarrow D_2(\mathbb{R}) \) by \( \varphi([a, b]) = \left( \begin{array}{c} a \\ 0 \\ b \end{array} \right) \) and

\[
\overline{\varphi} : Kc(\mathbb{R}) \rightarrow D_2(\mathbb{R}) \text{ by }
\overline{\varphi} f \left( \prod_{j=1}^{n} [a_j, b_j] \right) = f \left( \prod_{j=1}^{n} [a_j, b_j] \right)
\] (18)

\begin{proposition}
Let \( f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R} \) analytic function and \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) such that \( B(x_0, \varepsilon) \subset \mathcal{X} \) for any \( \varepsilon > 0 \) and \( \prod_{j=1}^{n} [a_j, \beta_j] \subseteq B(x_0, \varepsilon) \), then

\[
f \left( \prod_{j=1}^{n} \begin{pmatrix} a_j \\ 0 \\ 0 \end{pmatrix} \right) = f \left( \prod_{j=1}^{n} \begin{pmatrix} a_j \\ 0 \\ 0 \end{pmatrix} \right)
\] (19)

\begin{proof}
Consider any norm \( \| \cdot \|_n \) of \( \mathbb{R}^n \). Let \( a \in \text{int} \mathcal{X} \), as \( f \) is analytic in \( \mathcal{X} \), then exist \( \varepsilon > 0 \) such that \( B(x_0, \varepsilon) \subset \mathcal{X} \) such that for all \( x \in B(x_0, \varepsilon) \) there is a series of Taylor that converges for \( f(x) \). Let \( \prod_{j=1}^{n} [a_j, \beta_j] \subseteq B(x_0, \varepsilon) \), consider a matrix \( \begin{pmatrix} (a_1, \ldots, a_n) \\ 0 \\ (b_1, \ldots, b_n) \end{pmatrix} \) that is matrix representation associated with the interval \( \prod_{j=1}^{n} [a_j, \beta_j] \) and \( \begin{pmatrix} (a_1, \ldots, a_n) \\ 0 \\ (a_1, \ldots, a_n) \end{pmatrix} \) the representation of the matrix associated with the interval \( \prod_{j=1}^{n} [a_j, a_j] \). Consider a norm \( d \) in space \( M_{2 \times 2}(\mathbb{R}^n) \). Note that:

\[
\left\| \left( \begin{array}{c} (a_1, \ldots, a_n) \\ 0 \\ (a_1, \ldots, a_n) \end{array} \right) - \left( \begin{array}{c} (a_1, \ldots, a_n) \\ 0 \\ (\beta_1, \ldots, \beta_n) \end{array} \right) \right\|_d
\] (20)

\[
= \frac{1}{2} \left( \| (a_1, \ldots, a_n) - (a_1, \ldots, a_n) \|_n + \| (a_1, \ldots, a_n) - (\beta_1, \ldots, \beta_n) \|_n \right) < \varepsilon.
\] (21)
then by proposition 5, the series of matrix power
\[
\sum_{j=0}^{\infty} \frac{1}{j!} \sum_{j_1 + \ldots + j_n = j} \left( \begin{array}{c} j \\ j_1 \ldots j_n \end{array} \right) \partial^n f(a_1, \ldots, a_n) \left( \begin{array}{ccc} a_1 - \alpha_1 & 0 & 0 \\ 0 & a_1 - \beta_1 & 0 \\ 0 & 0 & a_n - \alpha_n \end{array} \right)^{j_1} \left( \begin{array}{ccc} a_n - \alpha_n & 0 & 0 \\ 0 & a_n - \beta_n & 0 \\ 0 & 0 & a_n - \beta_n \end{array} \right)^{j_n}
\]
(23)
is convergent.

Now consider the partial sum
\[
\sum_{j=0}^{m} \frac{1}{j!} \sum_{j_1 + \ldots + j_n = j} \left( \begin{array}{c} j \\ j_1 \ldots j_n \end{array} \right) \partial^n f(a_1, \ldots, a_n) \left( \begin{array}{ccc} a_1 - \alpha_1 & 0 & 0 \\ 0 & a_1 - \beta_1 & 0 \\ 0 & 0 & a_n - \alpha_n \end{array} \right)^{j_1} \left( \begin{array}{ccc} a_n - \alpha_n & 0 & 0 \\ 0 & a_n - \beta_n & 0 \\ 0 & 0 & a_n - \beta_n \end{array} \right)^{j_n}
\]
(24)
\[
= \left( \Gamma_m(a_1 - \alpha_1)^{j_1} \ldots (a_n - \alpha_n)^{j_n} \right)
\]
(25)
where \( \Gamma_m = \sum_{j=0}^{m} \frac{1}{j!} \sum_{j_1 + \ldots + j_n = j} \left( \begin{array}{c} j \\ j_1 \ldots j_n \end{array} \right) \partial^n f(a_1, \ldots, a_n) \). As \((a_1, \ldots, a_n), (\beta_1, \ldots, \beta_n) \in B(a, \varepsilon),\) then
\[
\Gamma_m(a_1 - \alpha_1)^{j_1} \ldots (a_n - \alpha_n)^{j_n} \rightarrow f \left( \prod_{j=1}^{n} \alpha_j \right) \quad \text{and} \quad \Gamma_m(a_1 - \beta_1)^{j_1} \ldots (a_n - \beta_n)^{j_n} \rightarrow f \left( \prod_{j=1}^{n} \beta_j \right) \quad \text{as} \ n \rightarrow \infty,
\]
then
\[
\left( \begin{array}{ccc} \Gamma_m(a_1 - \alpha_1)^{j_1} \ldots (a_n - \alpha_n)^{j_n} \\ 0 \\ \Gamma_m(a_1 - \beta_1)^{j_1} \ldots (a_n - \beta_n)^{j_n} \end{array} \right) \rightarrow \left( \begin{array}{ccc} f \left( \prod_{j=1}^{n} \alpha_j \right) & 0 \\ 0 & f \left( \prod_{j=1}^{n} \beta_j \right) \end{array} \right)
\]
(26)
as \( m \rightarrow \infty, \) then
\[
f \left( \prod_{j=1}^{n} \left( \begin{array}{c} \alpha_j \\ \beta_j \end{array} \right) \right) = f \left( \prod_{j=1}^{n} \alpha_j \right)\left( \begin{array}{c} 0 \\ \beta_j \end{array} \right)
\]
(27)

**Definition 9** Let \( f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R} \) differential function and \( \prod_{j=1}^{n} [a_j, b_j] \subset \mathcal{X} \), we say that \( \prod_{j=1}^{n} [a_j, b_j] \) is free of singularity if the components of the gradient vector are different from zero in all \( \prod_{j=1}^{n} [a_j, b_j] \), that is to say
\[
\frac{\partial f(x)}{\partial x_j} \neq 0 \quad \text{for all} \ x \in \prod_{j=1}^{n} [a_j, b_j] \text{and} \ j = 1, \ldots, n.
\]
(28)

**Definition 10** Let \( f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R} \) analytic function, \( \prod_{j=1}^{n} [a_j, b_j] \subset \mathcal{X} \) free of singularity and \([a_j, b_j]\) an interval of the \( j \)-variable. Defined the switch \( \left( \prod_{j=1}^{n} [a_j, b_j] \right) \) of the interval as \([a_j, b_j] \) if \( \frac{\partial f}{\partial x_j} > 0 \) and \([b_j, a_j] \) if \( \frac{\partial f}{\partial x_j} < 0 \), and denote by
\[
\phi f \left( \prod_{j=1}^{n} [a_j, b_j] \right) = \varphi f \left( \prod_{j=1}^{n} [a_j, b_j] \right).
\]
Note that \( \varphi \) is well defined, since the swapper depends only on the sign of the directional derivative on the square region, and this does not depend on how the function is formulated and also how the square region is free of singularities, then
where this sign is constant in the square region. And on the other hand, since \( f \) is an analytical function, we have that the matrix series is convergent.

**Theorem 11 (Fundamental Theorem of Interval Functions)** Let \( f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R} \) analytic function and \( x_0 \in \mathbb{R}^n \) such that \( B(x_0, \varepsilon) \subset \mathcal{X} \) for any \( \varepsilon > 0 \) and \( \prod_{j=1}^{n} [a_j, b_j] \subset B(x_0, \varepsilon) \) free of singularity, then

\[
f \left( \prod_{j=1}^{n} [a_j, b_j] \right) = \varphi^{-1} \phi f \left( \prod_{j=1}^{n} (a_j, b_j) \right)
\]

(29).

**Proof** Let \( f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R} \) analytic function and \( x_0 \in \mathbb{R}^n \) such that \( B(x_0, \varepsilon) \subset \mathcal{X} \) for any \( \varepsilon > 0 \) and \( \prod_{j=1}^{n} [a_j, b_j] \subset B(x_0, \varepsilon) \) free of singularity, then

\[
\phi f \left( \prod_{j=1}^{n} [a_j, b_j] \right) = \overline{\varphi} f \left( \prod_{j=1}^{n} [x_j, y_j] \right)
\]

(30)

where \([x_j, y_j]\) is the result of applying the switch to \([a_j, b_j]\), then by Proposition 8 since \( f \) is analytic function, we have

\[
\overline{\varphi} f \left( \prod_{j=1}^{n} [x_j, y_j] \right) = \begin{pmatrix}
f \left( \prod_{j=1}^{n} x_j \right) & 0 \\
0 & f \left( \prod_{j=1}^{n} y_j \right)
\end{pmatrix}
\]

(31)

Applying \( \varphi^{-1} \), we have the following interval,

\[
\begin{bmatrix}
f \left( \prod_{j=1}^{n} x_j \right) \\
f \left( \prod_{j=1}^{n} y_j \right)
\end{bmatrix}
\]

(32)

Now we will prove that the interval above corresponds to the image of \( f \) on \( R = \prod_{j=1}^{n} [a_j, b_j] \). First we observe that both \( f \left( \prod_{j=1}^{n} x_j \right) \) and \( f \left( \prod_{j=1}^{n} x_j \right) \) are elements of \( f(R) \), since \( R \) is connected and closed, we have that

\[
\begin{bmatrix}
f \left( \prod_{j=1}^{n} x_j \right) \\
f \left( \prod_{j=1}^{n} y_j \right)
\end{bmatrix}
\]

is a subset of \( f(R) \).

On the other hand. As we have that the gradient is different from zero within \( R \), we have that the function reaches its extreme points at the border of \( R \). First we will prove that the maximum and minimum point are not in the edges of \( R \), but are in one of the vertex. Indeed, since the gradient has non-zero components in all \( R \) in particularly in the edge of \( R \), we have that the extreme points of \( f \) can not be in the edges, since \( f \) restricted to the edges, it is a continuous function over a compact interval where its derivative is not null, then its extreme points are the extremes of the interval. Since the argument is valid for all edges, we conclude that the extreme points of \( f \) are in the vertex.

Let \((z_1, \ldots, z_j, \ldots, z_n)\) be a vertex of \( R \). Since \( f \) is monotonous on the edges, we have the following inequality in \( x_j \) leaving the other variables fixed

\[
f(z_1, \ldots, x_j, \ldots, z_n) \leq f(z_1, \ldots, z_j, \ldots, z_n) \leq f(z_1, \ldots, y_j, \ldots, z_n).
\]

(33)

Taking this inequality inductively on each variable, we have

\[
f(z_1, \ldots, z_j, \ldots, z_n) \geq f(x_1, \ldots, z_j, \ldots, z_n) \geq f(x_1, \ldots, x_j, \ldots, z_n) \geq f(x_1, \ldots, x_j, \ldots, x_n)
\]

(34)
and
\[ f(z_1, \ldots, z_j, \ldots, z_n) \leq f(y_1, \ldots, y_j, \ldots, z_n) \leq f(y_1, \ldots, y_j, \ldots, y_n). \] (35)

Hence
\[ f(x_1, \ldots, x_j, \ldots, x_n) \leq f(z_1, \ldots, z_j, \ldots, z_n) \leq f(y_1, \ldots, y_j, \ldots, y_n). \] (36)

Finally, we have for all \( u = (u_1, \ldots, u_n) \) in \( R \),
\[ f \left( \prod_{j=1}^{n} x_j \right) \leq f \left( \prod_{j=1}^{n} u_j \right) \leq f \left( \prod_{j=1}^{n} y_j \right) \] (37)
or equivalent
\[ f(R) \subset \left[ f \left( \prod_{j=1}^{n} x_j \right), f \left( \prod_{j=1}^{n} y_j \right) \right]. \] (38)

Therefore
\[ f(R) = \left[ f \left( \prod_{j=1}^{n} x_j \right), f \left( \prod_{j=1}^{n} y_j \right) \right] \] (39)
which means that \( f(R) = \varphi^{-1} \phi f \left( R \right) \).

An observation to the above theorem, is that it remains true if the gradient is annulled at the most in two vertexes so that gradient does not cancel out in other points of \( R \). For example, there is no problem if it is only annulled in a single vertex, in the case of two vertex, one of the points must correspond to a local maximum and the other a local minimum, or also that one of the two points is a saddle point. Now it can not happen that there are two maximum points or two minimum points, because, this would imply that there must exist a point of \( R \) between those points such that the gradient is zero. To this condition we will call free of singularity except for the most in two vertexes.

**Corollary 11.1** Under the same hypothesis of the above theorem. Let \( R = \bigcup_{j=1}^{m} R_j \) where \( R_j \) are free of singularity except for the most in two vertexes, then
\[ f(R) = \bigcup_{j=1}^{m} \varphi^{-1} \phi f \left( R_j \right). \] (40)

**Proof** Indeed \( f(R) = f \left( \bigcup_{j=1}^{m} R_j \right) = \bigcup_{j=1}^{m} f(R_j) = \bigcup_{j=1}^{m} \varphi^{-1} \phi f \left( R_j \right) \).
A consequence of the theorem is the following corollary

**Corollary 11.2** Let the following sets:

1. $K_c(\mathbb{R})_0^+ = \{ X \in K_c(\mathbb{R}), x_1 \geq 0 \}$,
2. $K_c(\mathbb{R})_0^- = \{ X \in K_c(\mathbb{R}), x_2 \leq 0 \}$,
3. $K_c(\mathbb{R})_0^+ = \{ X \in K_c(\mathbb{R}), x_1 > 0 \}$,
4. $K_c(\mathbb{R})_0^- = \{ X \in K_c(\mathbb{R}), x_2 < 0 \}$.

Then

1. $\varphi(X + Y) = \varphi(X) + \varphi(Y)$ for all $X, Y \in K_c(\mathbb{R})$,
2. $\varphi(kX) = k\varphi(X)$ for all $X \in K_c(\mathbb{R})$ and $k \geq 0$,
3. $\varphi(kX) = k\varphi(\hat{X})$ for all $X \in K_c(\mathbb{R})$ and $k \leq 0$,
4. $\varphi(XY) = \varphi(X)\varphi(Y)$ for all $X, Y \in K_c(\mathbb{R})_0^+$,
5. $\varphi(XY) = \varphi(X)\varphi(\hat{Y})$ for all $X \in K_c(\mathbb{R})_0^-$ and $Y \in K_c(\mathbb{R})_0^+$,
6. $\varphi(XY) = \varphi(\hat{Y})\varphi(\hat{Y})$ for all $X, Y \in K_c(\mathbb{R})_0^-.$

**Proof** By simple inspection.

We can observe that the $\varphi$ application preserves the addition and multiplication for intervals, intervals with a single sign. Besides that we can observe the first point is that due to 2) and 3) we have that $\varphi$ is nonlinear. And on the other hand, in general these properties are not combinable, for example by combining the properties of additivity and multiplication, we obtain the following inconsistency $\varphi([1, 2](1 - [0, 1])) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ and on the other hand $\varphi([1, 2] - [1, 2][0, 1]) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, from where we come to a contradiction.

3 Resolution of Interval Equations

Suppose we have an interval equation, for example a linear equation $AX + B = C$, where all the components are intervals, what should be the procedure to solve this equation?, assuming that there is some solution. We could for example consider the equation $ax + b = c$, where the values of this equation are defined over their corresponding intervals, that is to say that $a \in A$, and clear $x$ of the equation and then determine the image of the square region, using the fundamental theorem, however, what we will obtain is a region that contains the solution of the equation.

**Example 12** Let the interval linear equation $[1, 2][X + [0, 1]] = [1, 3]$ we can verify that the solution is $[1, 1]$ however, the the image of $f(u, b, c) = \frac{c - b}{a}$ of the square region $[1, 2] \times [0, 1] \times [1, 3]$ is $[0, 3]$.

We will then give a theorem that gives us the procedure to determine the solution of an interval equation, however, this solution does not always exist, since, the matrix we obtain as a solution to the matrix equation associated with the equation does not always satisfy the condition of have the first entry less than or equal to the last entry.

**Definition 13** Let $f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ analytic function, $\prod_{i=1}^{n}[a_j, b_j] \subset \mathcal{X}$ free of singularity, $X = [a_j, b_j]$ a compact interval and $M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ a diagonal matrix. Defined the switch of $M$ respect to $X$ as $\sigma^X M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ if $\hat{X} = [a_j, b_j]$ and $\sigma^X M = \begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}$ if $\hat{X} = [b_j, a_j]$ and we say that $\varphi \hat{X} = M$ defines a matrix with the first
entry less than or equal to the last entry if $\varphi X = \sigma^X M$ is a matrix with the first entry less than or equal to the last entry. If there is no ambiguity, we will denote the switch of $M$ with respect to $X$ as $\sigma^X M = \hat{M}$.

Below we present the main theorem of this article

**Theorem 14 (Fundamental Theorem of Interval Equations)** Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ analytic function, $\prod_{i=1}^{n} X_j \subset \mathcal{X}$ free of singularity with $X_j = [a_j, b_j]$, and $X_0 \subset f(\mathcal{X})$ an compact interval. Suppose it exists a function $g : \prod_{i=2}^{n} X_j \to \mathbb{R}$ be such a function that for all $x_0 \in X_0$ exists $(x_2, \ldots, x_n) \in \prod_{i=2}^{n} X_j$ such that $f \left( g \left( \prod_{i=2}^{n} x_j \right), \prod_{i=2}^{n} x_j \right) = x_0$.

Then the equation $f \left( \prod_{i=1}^{n} X_j \right) = X_0$ has solution in $X_1$ if and only if $\varphi \hat{X}_1 = g \left( \prod_{i=2}^{n} \varphi \hat{X}_j \right)$ defines an matrix with the first entry less than or equal to the last entry.

**Proof** Consider the following interval equation in $X_1$:

$$f \left( X_1, \prod_{i=2}^{n} X_j \right) = X_0. \quad (41)$$

As $\prod_{i=1}^{n} X_j \subset \mathcal{X}$ is free of singularity, then

$$f \left( X_1, \prod_{i=2}^{n} X_j \right) = \varphi^{-1} f \left( \varphi \hat{X}_1, \prod_{i=2}^{n} \varphi \hat{X}_j \right) \quad (42)$$

Then $\varphi^{-1} f \left( \varphi \hat{X}_1, \prod_{i=2}^{n} \varphi \hat{X}_j \right) = X_0$ or the equivalent

$$f \left( \varphi \hat{X}_1, \prod_{i=2}^{n} \varphi \hat{X}_j \right) = \varphi X_0 \quad (43)$$

On the other hand we have by hypothesis, that there exists a function $g : \prod_{i=2}^{n} X_j \to \mathbb{R}$ be such a function that for all $x_0 \in X_0$ exists $(x_2, \ldots, x_n) \in \prod_{i=2}^{n} X_j$ such that $f \left( g \left( \prod_{i=2}^{n} x_j \right), \prod_{i=2}^{n} x_j \right) = x_0$, that is, we can clear $X_1$ of the matrix equation, then

$$\varphi \hat{X}_1 = g \left( \prod_{i=2}^{n} \varphi \hat{X}_j \right) \quad (44)$$

Suppose there is a solution to the equation (41), then (44) define a matrix solution for the equation (43) where the first entry less than or equal to the last entry.

Let’s suppose that $\varphi \hat{X}_1 = g \left( \prod_{i=2}^{n} \varphi \hat{X}_j \right)$ defined a matrix such that the first entry less than or equal to the last entry, by (43), we have

$$f \left( g \left( \prod_{i=2}^{n} \varphi \hat{X}_j \right), \prod_{i=2}^{n} \varphi \hat{X}_j \right) = \varphi X_0 \quad (45)$$

then

$$\varphi^{-1} f \left( g \left( \prod_{i=2}^{n} \varphi \hat{X}_j \right), \prod_{i=2}^{n} \varphi \hat{X}_j \right) = X_0 \quad (46)$$

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by theorem \([\text{[11]}]\) we have
\[
X_0 = \varphi^{-1} f \left( g \left( \prod_{i=2}^{n} \varphi \hat{X}_j \right), \prod_{i=2}^{n} \varphi \hat{X}_j \right) = f \left( \varphi^{-1} g \left( \prod_{i=2}^{n} \varphi \hat{X}_j \right), \prod_{i=2}^{n} X_j \right) \tag{47}
\]
as \(\varphi \hat{X}_1 = g \left( \prod_{i=2}^{n} \varphi \hat{X}_j \right)\) defines a matrix with the first entry less than or equal to the last entry, we have to
\[
\varphi^{-1} g \left( \prod_{i=2}^{n} \varphi \hat{X}_j \right) \in K_r(\mathbb{R}), \text{ then, we have that the latter corresponds to a solution of the equation (41).}
\]

we can generalize the previous theorem for square regions with an arbitrary amount of singularities (that is, points with some null component of the gradient).

**Corollary 14.1** Let \(f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R} \) analytic function, \(\prod_{i=1}^{n} X_j \subset \mathcal{X} \) with \(X_j = [a_j, b_j]\), and \(X_0 \subset f(\mathcal{X})\) an compact interval. Suppose it exists a function \(g : \prod_{i=2}^{n} X_j \rightarrow \mathbb{R} \) be such a function that for all \(x_0 \in X_0\) exists
\[
(x_2, \ldots, x_n) \in \prod_{i=2}^{n} X_j \text{ such that } f \left( \prod_{i=2}^{n} x_j, \prod_{i=2}^{n} X_j \right) = x_0. \text{ Let } \Gamma \text{ the interval in the larger variable } x_1 \text{ such that } \Gamma \times \prod_{i=2}^{n} X_j \subset \mathcal{X} \text{ and let } \{R_\alpha\} \subset \Gamma \text{ and } \{R_\beta\} \subset \prod_{i=2}^{n} X_j \text{ with such that:}
\]
1. \(\bigcup_{\alpha, \beta} R_\alpha \times R_\beta = \Gamma \times \prod_{i=2}^{n} X_j,\)
2. each \(\left\{ \frac{\partial f}{\partial x_j} \right\}\) has a constant sign not null in \(\text{int}(R_\alpha \times R_\beta),\)
3. \(R_\alpha \times R_\beta\) is free of singularity except for the most in two vertexes.

let’s denote by \(X_{0, \alpha} = f(R_\alpha \times R_\beta) \cap X_0\) and \(X_\alpha = X_1 \cap R_\alpha,\) let’s take the following equation in \(X_\alpha\)
\[
f(X_\alpha \times R_\beta) = X_{0, \alpha}^{\alpha, \beta}. \tag{48}
\]

If \(X_0 \subset f \left( \Gamma \times \prod_{i=2}^{n} X_j \right),\) the following equation has a solution not empty
\[
f \left( X_1 \times \prod_{i=2}^{n} X_j \right) = X_0 \tag{49}
\]
if there is any solution of not empty for \([48]\) some \(\alpha\) and \(\beta\). Additionally, we have the solution of \([49]\) if it exists, it is equal to \(X_1 = \bigcup_{\beta} \varphi^{-1} \hat{\phi}_\alpha (R_\beta)\) where \(f(g_\alpha(R_\beta), R_\beta) = X_{0, \alpha}^{\alpha, \beta}.\)

**Proof** Let’s prove that \(\bigcup_{\alpha} X_\alpha\) is a solution to the equation \([49]\), for simplicity, we will say that in the case that the equation \([49]\) some \(\alpha\) and \(\beta\) has no solution then we will say that \(X_\alpha = \emptyset\), under the hypothesis of the corollary, we have
\[
f \left( \bigcup_{\alpha} X_\alpha \times \prod_{i=2}^{n} X_j \right) = f \left( \bigcup_{\alpha} X_\alpha \times \bigcup_{\beta} R_\beta \right) = f \left( \bigcup_{\alpha, \beta} X_\alpha \times R_\beta \right) \tag{50}
\]
\[
= \bigcup_{\alpha, \beta} f(X_\alpha \times R_\beta) = \bigcup_{\alpha, \beta} X_{0, \alpha}^{\alpha, \beta} = \bigcup_{\alpha} f(R_\alpha \times R_\beta) \cap X_0 = f \left( \bigcup_{\alpha, \beta} R_\alpha \times R_\beta \right) \cap X_0 = X_0 \tag{51}
\]
then \( \bigcup \alpha X_\alpha \) is solution of \( f \left( X_1, \prod_{i=2}^{n} X_j \right) = X_0 \). On the other hand as \( R_\alpha \times R_\beta \) is singularity except for the most in two vertexes, then by theorem [14] we have \( X_\alpha = \varphi^{-1} \hat{g}_\alpha (R_\beta) \). Therefore \( X_1 = \bigcup_\beta \varphi^{-1} \hat{g}_\alpha (R_\beta) \).

\[ \square \]

4 Another Approach

There is another approach the resolution of interval equations apart from using square matrices, and it is to use the conscious set of polynomials of a variable over \((x^2 - x)\), it is easy to observe that this set is a ring with the operations of addition and multiplication of polynomials. The elements of this set are of the form \( a + bh \), where \( h^2 = h \), which we will call pseudo complexes because of their similarity to complex numbers. In this case the application between the sets of the interval numbers and the complex pseudo numbers is given by \([a, b] \mapsto a + (b - a)h\) and analogously as we did for the case of the square matrices, we can redo them for the set of the complex pseudo number.

References

[1] Ramon E. Moore. Method and applications of interval analysis. *Sism*, (1979)