Euclidean simplices generating
discrete reflection groups

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Introduction

Let $P$ be a convex polytope in the spherical space $\mathbb{S}^n$, in the Euclidean space $\mathbb{E}^n$, or in the hyperbolic space $\mathbb{H}^n$. Consider the group $G_P$ generated by reflections in the facets of $P$. We call $G_P$ a reflection group generated by $P$. The problem we consider in this paper is to list polytopes generating discrete reflection groups.

The answer is known only for some combinatorial types of polytopes. Already in 1873, Schwarz [10] listed spherical triangles generating discrete groups. In 1998, E. Klimenko and M. Sakuma [9] solved the problem for hyperbolic triangles. In [2], [4], [3], [5] the problem was solved for hyperbolic quadrilaterals, compact hyperbolic pyramids and triangular prisms, hyperbolic simplices, and Lambert cubes in $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$. In [6] the problem was solved for spherical simplices.

In this paper, we use the method of [6] to classify Euclidean simplices generating discrete reflection groups.

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1 Preliminaries

A convex polytope in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ is called a Coxeter polytope if all its dihedral angles are submultiples of $\pi$. The group $G_F$ generated by reflections in the facets of any Coxeter polytope $F$ is discrete, and $F$ is a fundamental domain of $G_F$.

On the other hand, any discrete group $G$ generated by reflections coincides with $G_F$ for some Coxeter polytope $F$. If $G = G_P$ for some non-Coxeter polytope $P$, then $P$ consists of several copies of $F$, and any two copies containing a facet in common are symmetric to each other with respect to this facet.

Spherical reflection groups. Let $G$ be a reflection group acting on $\mathbb{S}^n$. Suppose that $G$ acts on $\mathbb{S}^n$ discretely, i.e. $G$ is a finite group. Then $G$ is generated by reflections in the facets of some spherical Coxeter polytope. It is shown by H. S. M. Coxeter [11] that any spherical Coxeter polytope containing no pair of antipodal points of $\mathbb{S}^n$ is a simplex.

To describe Coxeter polytopes we use Coxeter diagrams. A Coxeter diagram of a Coxeter polytope $F$ is a graph whose nodes $v_i$ correspond to the facets $\Pi_i$.
Nodes $v_i$ and $v_j$ are joined by a $(k - 2)$-fold edge if the dihedral angle formed up by $\Pi_i$ and $\Pi_j$ equals $\frac{\pi}{k}$ (if $\Pi_i$ is orthogonal to $\Pi_j$, $v_i$ and $v_j$ are disjoint). Indecomposable spherical and Euclidean Coxeter simplices were classified by Coxeter [1]. The list of their Coxeter diagrams is represented in Table 1.

Euclidean reflection groups. Let $G$ be a discrete reflection group acting on $E^n$. Let $F$ be a fundamental chamber of $G$. As it is shown in [1], $F$ is a direct product of several simplices and simplicial cones.

Suppose that $G$ is generated by a simplex (the simplex may not be a Coxeter simplex). Then $G$ is an indecomposable group and the fundamental chamber of $G$ is compact. In this case the fundamental chamber of $G$ is one of the simplices $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4$ and $\tilde{G}_2$ (see Table 1).

See [11] for more information about discrete reflection groups.

We use the notation $A_n, \tilde{A}_n, B_n$ and so on for Coxeter simplices as well as for the groups generated by these simplices. Finite root systems are denoted by $A_n, B_n$ and so on. See [8] for background on root systems.

Let $P$ be a simplex generating discrete reflection group ($P$ may not be a Coxeter simplex). Clearly, in this case all the dihedral angles of $P$ are of the type $\frac{\pi m}{k}$. Hence, for any simplex generating discrete reflection group we can construct the following generalized Coxeter diagram: the nodes $v_i$ of the diagram correspond to the facets $\Pi_i$ of $P$; the nodes $v_i$ and $v_j$ are joined by a $(k - 2)$-fold edge that is decomposed into $m$ parts if the dihedral angle formed up by $\Pi_i$ and $\Pi_j$ equals to $\frac{\pi m}{k}$.

2 Families of simplices

Spherical simplices generating discrete reflection groups. Let $E^n$ be the $n$-dimensional Euclidean space, and let $S^{n-1}$ be the unit sphere centered at the origin. Any hyperplane of $S^{n-1}$ is a section of $S^{n-1}$ by some hyperplane of $E^n$ containing the origin. Hence, any $(n - 1)$-dimensional simplex in $S^{n-1}$ is an intersection of $S^{n-1}$ with an interior of some cone centered at the origin.

So, suppose that $P$ is a spherical $(n - 1)$-dimensional simplex. Then we can define $P$ by unit outward normal vectors $f_1, ..., f_n$ to the facets of the corresponding cone.

Denote by $\Pi_1, ..., \Pi_n$ the facets of $P$. The hyperplanes containing $\Pi_1, ..., \Pi_n$ decompose $S^{n-1}$ into $2^n$ simplices $P_1, ..., P_{2^n}$ encoded by $n$-tuples of vectors $\{\pm f_1, ..., \pm f_n\}$. In this paper the set of simplices $P_1, ..., P_{2^n}$ is called a family. Each of the simplices $P_1, ..., P_{2^n}$ generates the same reflection group as $P$ does. Thus, we can study the families instead of studying of simplices themselves. (In fact, any family contains at most $2^{n-1}$ simplices up to isometry: the simplex $\{f_1, ..., f_n\}$ is always congruent to $\{-f_1, ..., -f_n\}$).

Let $P$ be a simplex generating a discrete reflection group $G_P$. Clearly, the dihedral angles of $P$ are rational numbers multiplied by $\pi$. Moreover, if $G$ is an
Table 1: Coxeter diagrams. Connected elliptic and parabolic Coxeter diagrams are listed in left and right columns respectively. Special nodes are marked.

| Coxeter Type (n ≥ k) | Diagram |
|----------------------|---------|
| \( A_n \) \( (n ≥ 1) \) | ![Diagram] |
| \( \tilde{A}_1 \) | ![Diagram] |
| \( \tilde{A}_n \) \( (n ≥ 2) \) | ![Diagram] |
| \( B_n = C_n \) \( (n ≥ 2) \) | ![Diagram] |
| \( \tilde{B}_n \) \( (n ≥ 3) \) | ![Diagram] |
| \( \tilde{C}_n \) \( (n ≥ 2) \) | ![Diagram] |
| \( D_n \) \( (n ≥ 4) \) | ![Diagram] |
| \( \tilde{D}_n \) \( (n ≥ 4) \) | ![Diagram] |
| \( G_2^{(m)} \) \( m \) | ![Diagram] |
| \( \tilde{G}_2 \) | ![Diagram] |
| \( F_4 \) | ![Diagram] |
| \( \tilde{F}_4 \) | ![Diagram] |
| \( E_6 \) | ![Diagram] |
| \( \tilde{E}_6 \) | ![Diagram] |
| \( E_7 \) | ![Diagram] |
| \( \tilde{E}_7 \) | ![Diagram] |
| \( E_8 \) | ![Diagram] |
| \( \tilde{E}_8 \) | ![Diagram] |
| \( H_3 \) | ![Diagram] |
| \( H_4 \) | ![Diagram] |

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indecomposable spherical reflection group different from $G_2^{(m)}$, $H_3$ and $H_4$, then any dihedral angle of $P$ is either $\frac{\pi}{k}$ or $\frac{\pi(k-1)}{k}$, where $k = 2, 3$ or 4 (cf. Table \[1\]).

Analogously to Coxeter simplices, any simplex whose dihedral angles equal either $\frac{\pi}{k}$ or $\frac{\pi(k-1)}{k}$ can be represented by the following family diagram: the nodes $v_i$ correspond to the facets of $P$; the nodes $v_i$ and $v_j$ are joined by a $(k-2)$-fold edge if the angle formed up by $\Pi_i$ and $\Pi_j$ is either $\frac{\pi}{k}$ or $\frac{\pi(k-1)}{k}$ (if $\Pi_i$ is orthogonal to $\Pi_j$, $v_i$ and $v_j$ are disjoint).

Note, that if $P$ is a Coxeter simplex then the diagram corresponding to $P$ is a Coxeter diagram of $P$.

Any two simplices in one family have the same family diagram. So, we can assign to a family the diagram of any simplex contained in the family. We call this diagram a family diagram. It is proved in \[4\] (Lemma 2) that any graph having no edge of multiplicity greater than two is a family diagram for at most one family of simplices. The same statement for the dihedral group $G_2 = G_2^{(6)}$ is evident.

**Euclidean simplices generating discrete reflection groups.** Let $P$ be a simplex in $\mathbb{E}^n$ generating discrete reflection group $G_P$. Let $\Pi_0, ..., \Pi_n$ be the facets of $P$, and $f_0, ..., f_n$ be the outward unit normal vectors to $\Pi_0, ..., \Pi_n$. The hyperplanes containing $\Pi_0, ..., \Pi_n$ decompose $\mathbb{E}^n$ into $2^{n+1} - 1$ domains $P_1, ..., P_{2^{n+1} - 2}$ encoded by the vectors $\{\pm f_0, ..., \pm f_n\}$. All these domains except the initial simplex $P$ are non-compact. Simplex $P$ is encoded by the vectors $\{f_0, ..., f_n\}$. Each of the domains $P_i$ generates the same reflection group as $P$ does.

By a family of Euclidean simplices we call a set of simplices having the same set of vectors $\{\pm f_0, ..., \pm f_n\}$. Note, that in the Euclidean case any two simplices contained in one family are mutually similar (moreover, any two of these simplices are homothetic).

We classify Euclidean simplices generating discrete reflection groups up to similarity.

Furthermore, note that any dihedral angle of $P$ is either $\frac{\pi}{k}$ or $\frac{\pi(k-1)}{k}$, where $k = 2, 3, 4$ or 6 (see Table \[1\]). Hence, we can define family diagrams for Euclidean simplices in the same way as for the spherical ones.

**Lemma 1.** Let $\Phi$ and $\Psi$ be two different families of Euclidean simplices generating discrete reflection groups. Then their family diagrams are distinct.

The proof of the lemma follows the proof of Lemma 2 from paper \[6\].

### 3 Special vertices

In this section we prove several auxiliary facts.

A hyperplane $\alpha$ is called a mirror of the group $G$ if $G$ contains a reflection with respect to $\alpha$.

**Lemma 2.** Let $P$ be a simplex in $\mathbb{E}^n$ generating discrete reflection group $G_P$. Then no mirror of $G_P$ decomposing $P$ is parallel to a facet of $P$. 


Proof. Let \( \Pi_0, ..., \Pi_n \) be the facets of \( P \) and let \( V_0 \) be the vertex opposite to \( \Pi_0 \). Suppose that there exists a mirror of \( G_P \) that is parallel to \( \Pi_0 \) and decomposes \( P \). Since \( G_P \) is discrete, there exist only finitely many of such mirrors. Let \( m \) be one of them closest to \( V_0 \).

Denote by \( h \) the homothety with center \( V_0 \) taking \( \Pi_0 \) to \( m \). Denote by \( r_i \) the reflection with respect to \( \Pi_i \) \((i = 0, ..., n)\), and by \( r \) the reflection with respect to \( m \). Since \( r \in G_P \), \( r = r_i \) for some \( l \). From the other hand, \( r = hr_0h^{-1} \).

Consider the reflection \( hrh^{-1} = (hr_ih^{-1})...(hr_ih^{-1}) \). Since \( hr_ih^{-1} \in G_P \) for any \( i = 0, ..., n \), we have \( hrh^{-1} \in G_P \). Furthermore, \( hrh^{-1} \) is a reflection with respect to some hyperplane \( m' \) parallel to \( m \). Moreover, \( m' \) decomposes \( P \) and goes closer to \( V_0 \) than \( m \). This contradicts to the choice of \( m \).

\( \square \)

Lemma 3. Let \( P \) be a simplex generating discrete group \( G_P \). Let \( V \) be a vertex of \( P \) and \( \Pi_1, ..., \Pi_n \) be the facets of \( P \) containing \( V \). Then the stabilizer \( Fix(V, G_P) \) of \( V \) in \( G_P \) coincides with the group \( G \) generated by the reflections with respect to \( \Pi_1, ..., \Pi_n \).

Proof. Suppose that the lemma is false. Then there exists a non-empty set \( M \) of simplices for which the statement of the lemma is broken. We may assume that \( P \in M \) is a simplex minimal by the inclusion. Let \( V \) be a vertex of \( P \) for which the statement of the lemma is false. Then among the dihedral angles formed up by \( \Pi_1, ..., \Pi_n \) there exists a dihedral angle \( \pi \frac{k}{m} \), where \( 1 < k < m \) are mutually co-prime integers. Suppose that this angle is formed up by \( \Pi_1 \) and \( \Pi_n \).

Consider a mirror \( \Pi \) of \( G \) such that \( \Pi \) contains \( \Pi_1 \cap \Pi_n \), and the angle formed up by \( \Pi \) and \( \Pi_1 \) is equal to \( \frac{k}{m} \). This mirror decomposes \( P \) into two simplices \( P_1 \) and \( P_2 \). Let \( P_1 \) be the simplex having a facet \( \Pi_1 \). Denote by \( G_{P_1} \) the group generated by \( P_1 \). Clearly, \( G_{P_1} = G_P \). Furthermore, the group generated by the reflections with respect to the facets \( \Pi, \Pi_1, ..., \Pi_{n-1} \) coincides with the group generated by the reflections with respect to the facets \( \Pi_1, ..., \Pi_n \). From the other hand, the stabilizer \( Fix(V, G_{P_1}) \) of \( V \) in \( G_{P_1} \) coincides with \( Fix(V, G_P) \).

By the assumption, \( Fix(V, G_P) \) does not coincide with the group generated by the reflections with respect to \( \Pi_1, ..., \Pi_n \). Hence, \( Fix(V, G_{P_1}) \) differs from the group generated by the reflections with respect to \( \Pi, \Pi_1, ..., \Pi_{n-1} \). Thus, the statement of the lemma is broken for \( P_1 \). This contradicts to the assumption that \( P \in M \) is the minimal simplex, and the lemma is proved.

\( \square \)

Let \( P \) be any Euclidean polytope generating discrete reflection group. Suppose that there exists a vertex \( V \) of \( P \) such that the stabilizer of \( V \) contains a linear part of any element of \( G_P \). We call such a vertex a special vertex of \( P \). It is known (see [8], Ch. 6) that any Euclidean Coxeter simplex has at least one special vertex.

Lemma 4. Let \( P \) be a simplex generating discrete reflection group \( G_P \). Then \( P \) has at list one special vertex.
Proof. Let \( F \) be a fundamental polytope of \( G_P \) contained in \( P \). Since \( G_P \) is indecomposable, \( F \) is a simplex. Any fundamental polytope of discrete reflection group is Coxeter polytope, thus \( F \) is a Coxeter simplex. Let \( V \) be a special vertex of \( F \).

Suppose that \( F \) is either an inner point of \( P \) or an inner point of some face of \( P \). Let \( \Pi \) be any facet of \( P \) not containing \( V \). Let \( m \) be the mirror of \( G_P \) parallel to \( \Pi \) and containing \( V \). Then \( m \) cuts \( P \), that contradicts to Lemma 2.

Thus, \( V \) is a vertex of \( P \). Let \( \Pi_1, \ldots, \Pi_n \) be the facets of \( P \) containing \( V \). By Lemma 3, the reflections with respect to \( \Pi_1, \ldots, \Pi_n \) generate the stabilizer \( \text{Fix}(V, G_P) \) of \( V \) in \( G_P \). Thus, \( V \) is a special vertex of \( P \).

\[ \square \]

4 Simplices generating given reflection group

Let \( G \) be an indecomposable Euclidean reflection group. Let \( P \) be a simplex generating \( G \), and \( f_0, f_1, \ldots, f_n \) be the vectors orthogonal to the facets of \( P \). By Lemma 4 there exists a special vertex \( V \) of \( P \). Let \( \Pi_0, \ldots, \Pi_n \) be facets of \( P \), such that \( \Pi_i \) is orthogonal to \( f_i \) for \( i = 0, \ldots, n \). We may assume that \( V \) is contained in \( \Pi_1, \ldots, \Pi_n \).

The stabilizer \( \text{Fix}(V, G) \) of \( V \) in \( G \) is a Weyl group of some finite root system \( \Delta \). Suitably normalizing the vectors \( f_0, f_1, \ldots, f_n \), we may assume that these vectors belong to \( \Delta \). Since \( V \) is a special vertex of \( P \), there exists a mirror of \( G \) through \( V \) parallel to \( \Pi_0 \). By Lemma 5 suitably normalizing \( f_0 \) we may assume that \( f_0 \) is contained in \( \Delta \), Note, that \( f_0, f_1, \ldots, f_n \) are linearly dependent vectors, however any \( n \) of these vectors are linearly independent.

Now we are able to find all families of simplices generating given group \( G \).

Let \( F \) be a fundamental simplex of \( G \), and let \( U \) be a special vertex of \( F \). Denote by \( W \) the stabilizer \( \text{Fix}(U, G) \), and let \( \Delta \) be the corresponding root system. Let \( v_0, \ldots, v_n \) be any set containing \( n + 1 \) vectors of \( \Delta \), such that any \( n \) of these vectors are linearly independent. Let \( \Pi_1, \ldots, \Pi_n \) be the mirrors of \( G \) containing \( U \) and orthogonal to \( v_1, \ldots, v_n \). Let \( \Pi_0 \) be a mirror of \( G \) closest to \( U \), orthogonal to \( v_0 \) and not containing \( U \). Then the mirrors \( \Pi_0, \Pi_1, \ldots, \Pi_n \) are the facets of some simplex \( P \) generating a finite index subgroup of \( G \). If, in addition, the reflections with respect to \( \Pi_1, \ldots, \Pi_n \) generate \( W \), then some simplex similar to \( P \) generates \( G \) (the only exception is the group \( W = B_n = C_n \); in this case we can obtain either \( \tilde{B}_n \) or \( \tilde{C}_n \) depending on the vector \( v_0 \), see Section 4.2).

In other word, to classify families generating \( G \) it is sufficient to follow the algorithm:

1) Find all linearly independent systems \( f_1, \ldots, f_n \) in \( \Delta \), generating \( W \) (we say that \( f_1, \ldots, f_n \) generate \( W \), if the reflections with respect to the hyperplanes through the origin orthogonal to these vectors generate \( W \)).

2) For each system \( f_1, \ldots, f_n \) obtained on the previous step, add a vector \( f_0 \in \Delta \) in such a way that any \( n \) of vectors \( f_0, f_1, \ldots, f_n \) are linearly independent. This vector \( f_0 \) should be chosen by all possible ways.
3) If \( W = B_n = C_n \), examine which of the groups \( \tilde{B}_n \) and \( \tilde{C}_n \) is generated by \( f_0, f_1, ..., f_n \).

4) Among the obtained systems one should find the systems corresponding to different families, i.e. families having different family diagrams.

The order of vectors in the systems is not important for us. The Weyl group acts on \( E^n \), hence, it acts on \((n+1)\)-tuples of vectors. We do not differ \((n+1)\)-tuples equivalent with respect to this action.

4.1 Simplices generating \( \tilde{A}_n \)

Let \( P \) be a simplex generating the group \( \tilde{A}_n \). We may assume that the vectors \( f_0, f_1, ..., f_n \) belong to the root system \( A_n = \{ \pm (h_i - h_j) \}, 0 \leq i < j \leq n \), where \( h_0, ..., h_n \) is a standard basis of \( E^{n+1} \).

For any simplex \( P = \{ f_0, f_1, ..., f_n \} \) generating \( \tilde{A}_n \) we construct the following graph \( \Gamma(P) \): the nodes \( v_0, ..., v_n \) of \( \Sigma \) correspond to the vectors \( h_0, ..., h_n \); the nodes \( v_i \) and \( v_j \) are joined by an edge if one of vectors \( (h_i - h_j) \) and \( -(h_i - h_j) \) belongs to the set \( \{ f_0, ..., f_n \} \). Clearly, two simplices from one family have the same graph.

**Theorem 1.** There exists a unique family of simplices generating \( \tilde{A}_n \). This family consists of Coxeter simplices \( \tilde{A}_n \).

**Proof.** Let \( P = \{ f_0, f_1, ..., f_n \} \) be a simplex generating \( \tilde{A}_n \) and \( \Gamma(P) \) be the corresponding graph. Then \( \Gamma(P) \) contains exactly \( n + 1 \) nodes and the same number of edges. Since any \( n \) vectors contained in the set \( \{ f_0, ..., f_n \} \) are linearly independent, \( \Gamma(P) \) has no cycles containing less than \( n + 1 \) nodes. Hence, \( \Gamma(P) \) is a cycle with \( n + 1 \) nodes.

Note, that the graph \( \Gamma(P) \) determines the family diagram of \( P \). In more details, the nodes of the family diagram correspond to the edges of \( \Gamma(P) \), two nodes are joined if the corresponding edges of \( \Gamma(P) \) have a common node. Hence, in case of \( \tilde{A}_n \) the family diagram is a cycle, too. By Lemma \( \Pi \) the family of simplices is completely determined by a family diagram. Thus, \( P \) is a Coxeter simplex \( A_n \).

4.2 Simplices generating \( \tilde{B}_n \) and \( \tilde{C}_n \)

Let \( P \) be a simplex generating the group \( \tilde{B}_n \) or \( \tilde{C}_n \). We may assume that the vectors \( f_0, f_1, ..., f_n \) belong to the root system \( B_n = \{ \pm h_i, \pm h_i \pm h_j \}, 1 \leq i < j \leq n \), where \( h_1, ..., h_n \) is a standard basis of \( E^n \).

For any simplex \( P = \{ f_0, f_1, ..., f_n \} \) generating \( \tilde{B}_n \) or \( \tilde{C}_n \) we construct the following graph \( \Gamma(P) \): the nodes \( v_0, ..., v_n \) of \( \Sigma \) correspond to the vectors \( h_1, ..., h_n \); the nodes \( v_i \) and \( v_j \) are joined by an edge if one of vectors \( \pm (h_i - h_j) \) and \( \pm (h_i + h_j) \) belongs to the set \( \{ f_0, ..., f_n \} \); the node \( v_i \) is marked if \( \{ f_0, f_1, ..., f_n \} \)
contains \( \pm h_i \). If \( \{f_0, f_1, \ldots, f_n\} \) contains both \( \pm (h_i + h_j) \) and \( \pm (h_i - h_j) \), the nodes \( v_i \) and \( v_j \) are joined by two edges.

Since the system of vectors \( f_0, f_1, \ldots, f_n \) is indecomposable, \( \Gamma(P) \) is connected. Clearly, \( \Gamma(P) \) contains at least one marked node (otherwise \( f_i \) belongs to \( D_n \) for any \( i \), where \( D_n \) is embedded in \( B_n \) as a set of long roots).

**Lemma 5.** Let \( P \) be a simplex generating \( \tilde{B}_n \) or \( \tilde{C}_n \).

1) If \( \Gamma(P) \) contains more than one marked node then \( P \) is a Coxeter simplex \( \tilde{C}_n \).

2) If \( \Gamma(P) \) contains a unique marked node then \( \Gamma(P) \) is one of the graphs shown in the left column of Table 2.

**Proof.** Suppose that \( \Gamma(P) \) contains more than one marked node. Then \( \Gamma(P) \) contains a path from one marked node to another. Consider such a path that does not intersect itself. The vectors corresponding to the edges and marked nodes of this path are linearly dependent. Hence, the path contains all edges of \( \Gamma(P) \), and \( \Gamma(P) \) is the graph shown in Fig. 1. In this case \( P = \tilde{C}_n \).

Now suppose that \( \Gamma(P) \) contains a unique marked node. Since the number of edges of \( \Gamma(P) \) equals \( n \) and the number of nodes equals \( (n + 1) - 1 = n \), \( \Gamma(P) \) contains a unique cycle (it may consist of two nodes). Since any \( n \) vectors are linearly independent, if the cycle contains less than \( n \) edges then it does not contain the marked node. Hence, \( \Gamma(P) \) have a subgraph shown in the left column of Table 2. From the other hand, this subgraph corresponds to linearly dependent system of vectors. Hence, this subgraph coincides with \( \Gamma(P) \).

![Figure 1: Graph \( \Gamma(\tilde{C}_n) \).](image)

**Theorem 2.** There exists exactly \( n - 1 \) families of simplices generating the group \( \tilde{B}_n \). Family diagrams and generalized Coxeter diagrams of these simplices are shown in Table 2.

**Proof.** By the second statement of Lemma 5 any simplex \( P \) generating \( \tilde{B}_n \) corresponds to a graph \( \Gamma(P) \) shown in the left column of Table 2.

Let us show that any graph \( \Gamma \) shown in the left column of Table 2 corresponds to a simplex generating \( \tilde{B}_n \). It is easy to see that any \( n \) vectors \( f_0, f_1, \ldots, f_n \) are linearly independent and all these vectors are linearly dependent. Hence, \( \Gamma \) corresponds to some simplex \( P \) in \( \mathbb{E}^n \). The group generated by \( P \) is a maximal rank indecomposable subgroup of \( \tilde{B}_n \), and the fundamental simplex of this subgroup has a dihedral angle equal to \( \frac{\pi}{4} \). By \( \varphi \), \( P \) generates either \( \tilde{B}_n \) or \( \tilde{C}_n \).

Let \( \varphi \) and \( \psi \) be two dihedral angles equal to \( \frac{\pi}{4} \) formed up by mirrors of \( G_P \). Then there exists an element \( \gamma \) of \( G_P \) such that \( \gamma(\varphi) = \psi \). The group \( \tilde{C}_n \) contains two equivalency classes of such dihedral angles. Hence, \( G_P \neq \tilde{C}_n \) and \( P \) generates \( \tilde{B}_n \).

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Furthermore, let us show that each graph shown in the left column of the Table 2 corresponds to a unique family of simplices generating $\tilde{B}_n$. Indeed, the family diagram of the family containing $P$ can be easily recovered from $\Gamma(P)$: all but one nodes of the diagram correspond to the edges of $\Gamma(P)$, two nodes are adjacent if the corresponding edges have a common point; the rest node corresponds to the marked node of $\Gamma(P)$, this node is joined by a 2-fold edge with all the nodes that correspond to edges of $\Gamma(P)$ incident to the marked node. Thus, any graph shown in the left column of Table 2 corresponds to a unique family diagram, and, by Lemma 1, it corresponds to a unique family.

An explicit calculation shows that the generalized Coxeter diagram of simplex generating $\tilde{B}_n$ is one of the diagrams shown in the right column of Table 2.

Thus, families of simplices generating $\tilde{B}_n$ are in one-to-one correspondence with graphs shown in the left column of Table 2. To find the number of these families note, that $\Gamma(P)$ is uniquely determined by the number of edges in the cycle. The latter is any integer number $N$ satisfying $2 \leq N \leq n$.

\[ \square \]

Table 2: Simplices generating $\tilde{B}_n$.

| $\Gamma(P)$ | Family diagram | Generalized Coxeter diagram |
|-------------|----------------|-----------------------------|
|             | ![Diagram](image1) | ![Diagram](image2)          |
|             | ![Diagram](image3) | ![Diagram](image4)          |
|             | ![Diagram](image5) | ![Diagram](image6)          |
|             | ![Diagram](image7) | ![Diagram](image8)          |

As a corollary of Theorem 2 and the first statement of Lemma 5 we obtain the following theorem:
**Theorem 3.** There exists a unique family of simplices generating $\tilde{C}_n$. This family consists of Coxeter simplices $\tilde{C}_n$.

### 4.3 Simplices generating $\tilde{D}_n$

Let $P$ be a simplex generating the group $\tilde{D}_n$. We may assume that the vectors $f_0, f_1, ..., f_n$ belong to the root system $D_n = \{ \pm h_i \pm h_j \}, 1 \leq i < j \leq n$, where $h_1, ..., h_n$ is a standard basis of $\mathbb{R}^n$.

For any simplex $P = \{ f_0, f_1, ..., f_n \}$ generating $\tilde{D}_n$ we construct the following graph $\Gamma(P)$: the nodes $v_1, ..., v_n$ of $\Sigma$ correspond to the vectors $h_1, ..., h_n$; the nodes $v_i$ and $v_j$ are joined by an edge if one of vectors $\pm (h_i - h_j)$ and $\pm (h_i + h_j)$ belongs to the set $\{ f_0, ..., f_n \}$; if $\{ f_0, f_1, ..., f_n \}$ contains both $\pm (h_i + h_j)$ and $\pm (h_i - h_j)$ then $v_i$ and $v_j$ are joined by two edges.

The system of vectors $f_0, f_1, ..., f_n$ is indecomposable, hence, $\Gamma(P)$ is connected. Since $\Gamma(P)$ has $n$ nodes and $n + 1$ edges, $\Gamma(P)$ contains at least two cycles $C_1$ and $C_2$.

Suppose that $C_1$ and $C_2$ have no common nodes. Since the graph containing two disjoint cycles corresponds to a linearly dependent system of vectors, $\Gamma(P)$ is one of the graphs shown in the left column of Table 3. If $C_1$ and $C_2$ have a unique common node then $\Gamma(P)$ is the graph shown at the bottom of left column of Table 3.

**Lemma 6.** Suppose that $\Gamma(P)$ contains two cycles having at least two common nodes. Then the system of vectors $f_0, f_1, ..., f_n$ contains $n$ linearly dependent vectors.

**Proof.** Consider the graph $\Gamma(P)$ colored by the following way: edges corresponding to vectors $\pm (h_i + h_j)$ are red, and the rest edges, i.e. the edges corresponding to $\pm (h_i - h_j)$, are black. Note that substituting the vector $h_j$ by $-h_j$ we change the color of all edges incident to $v_i$. Thus, preserving the vectors $f_0, f_1, ..., f_n$, we can make all but one edge of a given cycle black (the rest edge is either red or black). Vectors $h_1 - h_2, h_2 - h_3, ..., h_k - h_1$ are linearly dependent. Hence, each cycle containing an even number of red edges corresponds to a system of linearly dependent vectors.

Consider common nodes of two cycles contained in $\Gamma(P)$. There are at least three paths $L_1, L_2$ and $L_3$ joining these nodes in $\Gamma(P)$. Denote by $c(L_i)$ the number of red edges in $L_i$. Then $c(L_i) + c(L_j)$ is even for some $i \neq j, i, j = 1, 2, 3$. We assume that $c(L_1) + c(L_2)$ is even. Then the cycle $C = L_1 \cup L_2$ contains even number of red edges, so, it corresponds to some linearly dependent vectors. Since some edges of $\Gamma(P)$ do not belong to $C$, the number of these linearly dependent vectors is less than $n + 1$. The contradiction proves the lemma.

**Corollary 1.** $\Gamma(P)$ contains exactly two cycles and coincides with one of graphs shown in the left column of Table 3.

Thus, we obtain the following theorem:
Theorem 4. The group $\widetilde{D}_n$ is generated by exactly $\frac{1}{2}n(n-2)$ families of simplices if $n$ is even, and by exactly $\frac{1}{4}(n-1)^2$ families if $n$ is odd. Family diagrams and generalized Coxeter diagrams of these simplices are presented in Table 3.

Table 3: Simplices generating $\widetilde{D}_n$.

| $\Gamma(P)$ | Family diagram | Generalized Coxeter diagram |
|-------------|----------------|-----------------------------|
| ![Diagram](image1.png) | ![Diagram](image2.png) | ![Diagram](image3.png) |
| ![Diagram](image4.png) | ![Diagram](image5.png) | ![Diagram](image6.png) |
| ![Diagram](image7.png) | ![Diagram](image8.png) | ![Diagram](image9.png) |

Proof. By Cor. 1, a simplex $P$ generating $\widetilde{D}_n$ corresponds to a graph $\Gamma(P)$ shown in the left column of Table 3.

Let us show that any graph $\Gamma$ shown in the left column of Table 3 corresponds to a simplex generating $\widetilde{D}_n$. For each cycle of $\Gamma$ choose an edge $v_i v_j$ and put the vector $h_i - h_j$ in correspondence with this edge (in other words, suppose these edges to be red). For all other edges $v_k v_l$ take vectors $h_k + h_l$. It is easy to see that any $n$ of these vectors are linearly independent and all these vectors are linearly dependent. Hence, $\Gamma$ corresponds to some simplex $P$ in $\mathbb{R}^n$. By [7], the group $\widetilde{D}_n$ has no finite index indecomposable subgroups different from $\tilde{D}_n$. Therefore, $P$ generates $\tilde{D}_n$. 

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Now, show that each graph \( \Gamma \) shown in the left column of Table 3 corresponds to a unique family of simplices generating \( \tilde{D}_n \). Indeed, a family diagram of the family containing \( P \) can be recovered from \( \Gamma(P) \): nodes of diagram correspond to edges of \( \Gamma \), two nodes are adjacent if the corresponding edges have a common point. Therefore, the family does not depend on the choice of initial edges. Moreover, in the beginning of the procedure we could make red not one but several edges: the family diagram would not be changed. Hence, family diagram of the family containing \( P \) can be recovered from \( \Gamma(P) \). Nodes of diagram correspond to edges of \( \Gamma \), two nodes are adjacent if the corresponding edges have a common point. Therefore, the family does not depend on the choice of initial edges.

Moreover, in the beginning of the procedure we could make red not one but several edges: the family diagram would not be changed. Hence, family diagram of simplices generating \( \tilde{D}_n \) are in one-to-one correspondence with graphs shown in the left column of Table 3. An explicit calculation shows that generalized Coxeter diagram of simplex generating \( \tilde{D}_n \) is one of diagrams shown in the right column of Table 3.

To find the number of families note, that \( \Gamma(P) \) is uniquely determined by the numbers \( N_1 \) and \( N_2 \) of edges in two cycles. Numbers \( N_1 \) and \( N_2 \) are any two integers satisfying \( N_1 + N_2 \leq n + 1 \) and \( 2 \leq N_1 \leq N_2 \leq n \). The number of such pairs \( (N_1, N_2) \) equals either \( \frac{1}{4}n(n-2) \) or \( \frac{1}{4}(n-1)^2 \) if \( n \) is even or odd respectively.

4.4 Simplices generating other groups

In Sections 4.1–4.3 we have described all families generating the groups \( \tilde{A}_n, \tilde{B}_n, \tilde{C}_n \) and \( \tilde{D}_n \). Now, we are left to classify families generating finitely many of other indecomposable Euclidean reflection groups. Namely, we are left with the groups \( \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4 \), and \( \tilde{G}_2 \). We can find the complete answer following the algorithm contained in the beginning of Section 4. As the result we obtain the lists which are rather large: there exist

- 17 families of simplices generating \( \tilde{E}_6 \),
- 142 families of simplices generating \( \tilde{E}_7 \),
- 1736 families of simplices generating \( \tilde{E}_8 \),
- 11 families of simplices generating \( \tilde{F}_4 \),
- 2 families of simplices generating \( \tilde{G}_2 \).

Appendix contains the complete list of these families.

Appendix

Appendix contains the list of families generating \( \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4 \), and \( \tilde{G}_2 \). Families generating \( \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \) are encoded in the following way. For a family \( \pm f_0, \pm f_1, ..., \pm f_n \) construct a symmetrical matrix \( G^+ = \{g_{i,j}\} \), where \( g_{i,j} = 2|\langle f_i, f_j \rangle| \). This is a doubled unsigned Gram matrix of the system \( f_0, f_1, ..., f_n \). The upper triangle of \( G^+ \) is filled up by 0 and 1. Let \( p \) be a decimal number which is equal to the binary number \( g_1, g_2, g_3, ..., g_n, g_{n+1}, ..., g_{n+1}, g_{n+2}, ..., g_{n+1} \) of the number \( p \) depends on the ordering of vectors \( f_0, f_1, ..., f_n \). We choose \( p \) the smallest possible.
Families generating $\tilde{E}_6$
515583 128885 104443 104126 104021 104011 64447 40671 39646 39583 39573
35838 35695 35629 35622 35131 35128

Families generating $\tilde{E}_7$
33554431 33554375 33148285 33148159 16576223 16574039 16540255
15062365 15062363 8287019 7273232 7273249 6813566 6813547 6813536

Families generating $\tilde{E}_8$
16642998271 16575889407 16575348735 8036015852 8036015839 8036015609
8027492222 8027491834 8023230399 8023229951 8023229630 7834757047

Families generating $\tilde{F}_4$

Families generating $F_4$ are encoded in the following way. For a family determined by $\pm f_0, \pm f_1, \ldots, \pm f_4$ construct a symmetric matrix $G^+ = \{g_{i,j}\}$, where $g_{i,j} = 0$ if $f_i$ is orthogonal to $f_j$, $g_{i,j} = 1$ if $\angle f_i f_j = \frac{\pi}{3}$ or $\frac{2\pi}{3}$, $g_{i,j} = 2$ if $\angle f_i f_j = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. Let $p$ be a decimal number which is equal to the base three number $g_1, g_2, g_3, \ldots, g_{n-1}, g_n$. The number $p$ depends on the numbering of vectors $f_1, \ldots, f_n$. We choose $p$ the smallest possible. Then $p$ depends only on the family of simplices.

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