On the Most Informative Boolean Functions of the Very Noisy Channel

Hengjie Yang and Richard D. Wesel
Department of Electrical and Computer Engineering
University of California, Los Angeles, Los Angeles, CA 90095, USA
Email: {hengjie.yang, wesel}@ucla.edu

Abstract

Let $X^n$ be an independently identically distributed Bernoulli $(1/2)$ random variables and let $Y^n$ be the result of passing $X^n$ through a binary symmetric channel (BSC) with crossover probability $\alpha$. For any Boolean function $f : \{0, 1\}^n \to \{0, 1\}$, Courtade and Kumar postulated that $I(f(X^n); Y^n) \leq 1 - H(\alpha)$. In this paper, we prove, in a purely mathematical point of view, that the conjecture is true in high noise regime by showing that $H(\alpha) - H(f(X^n)|Y^n) \leq 1 - H(f(X^n))$ holds for $\alpha \in \left[\frac{1}{2} - \delta, \frac{1}{2} + \delta\right]$, where $\delta > 0$ is some universally small constant. We first point out that in high noise regime, a function $f$ is more informative if the second derivative of $H(\alpha) - H(f(X^n)|Y^n)$ has a larger value at $\alpha = \frac{1}{2}$. Then we show that, the ratio spectrum of $f^{-1}(0)$, an integer sequence which characterizes the structure of $f^{-1}(0)$, is the fundamental metric that determines the second derivative of $H(\alpha) - H(f(X^n)|Y^n)$ evaluated at $\alpha = \frac{1}{2}$. With $|f^{-1}(0)|$ fixed, lex function is a locally most informative function for the very noisy BSC. The dictator function $f(X^n) = X_i, (1 \leq i \leq n)$, with $i = 1$ being a special case of lex function $|f^{-1}(0)| = 2^n - 1$, is the globally most informative function for the very noisy BSC as it is the only type of functions that maximize the second derivative of $H(\alpha) - H(f(X^n)|Y^n)$ evaluated at $\alpha = \frac{1}{2}$ to 0 over all possible $|f^{-1}(0)|$.

I. INTRODUCTION

A. Previous work

In 2013, Courtade and Kumar [1] postulated the following maximum mutual information conjecture.

Conjecture 1 (1): Let $X^n = (X_1, \ldots, X_n)$ be a sequence of $n$ i.i.d. Bernoulli(1/2) random variables, and let $Y^n$ be the result of passing $X^n$ through a memoryless binary symmetric channel (BSC) with crossover probability $\alpha$. For any Boolean function $f : \{0, 1\}^n \to \{0, 1\}$, we have

$$I(f(X^n); Y^n) \leq 1 - H(\alpha). \tag{1}$$

Although Conjecture[1] still remains open so far, several new results were obtained in a series of literature [2]–[4]. Samorodnitsky [2] proved that the conjecture holds for the high noise regime, where $\alpha \in \left[\frac{1}{2} - \delta', \frac{1}{2} + \delta'\right]$, for some universally small constant $\delta'$, by considering the entropy of the image of $f$ under the noise operator $T_\alpha$ with noise parameter $\alpha$. 
**Theorem 1** ([2]): There exists an absolute constant $\delta > 0$ such that for any noise $\alpha > 0$ with $(1 - 2\alpha)^2 \leq \delta$ and for any Boolean function $f : \{0,1\}^n \to \{0,1\}$, it holds that

$$I(f(X^n); Y^n) \leq 1 - H(\alpha). \quad (2)$$

Nevertheless, so far the best known bound that holds universally for all Boolean function $f$ is

$$I(f(X^n); Y^n) \leq (1 - 2\alpha)^2. \quad (3)$$

This bound, derived by Erkip [5], can be established through various techniques, including an application of Mrs. Gerber’s Lemma [6], the strong data-processing inequality [7], and standard Fourier analysis. Unfortunately, this bound is still strictly weaker than that in Conjecture [1]

In 2016, Ordentlich, Shayevitz, and Weinstein [3] proved a new upper bound for balanced Boolean function $f$, i.e., any Boolean function $f$ that satisfies $p(f = 0) = p(f = 1)$, which beats $(1 - 2\alpha)^2$ in $\frac{1}{2} < \alpha < \frac{1}{2}$.

**Theorem 2** ([3]): For any balanced Boolean function $f : \{0,1\}^n \to \{0,1\}$, and any $\frac{1}{2}(1 - \sqrt{2}) \leq \alpha \leq \frac{1}{2}$, we have that

$$I(f(X^n); Y^n) \leq \frac{\log_2 e}{2} (1 - 2\alpha)^2 + \log_2 e (2 - 2\alpha)^4.$$ 

In 2017, Huleihel and Ordentlich [4] studied a complementary problem of Conjecture [1] and proved the following theorem.

**Theorem 3** ([4]): For any function $f : \{0,1\}^n \to \{0,1\}^{n-1}$, we have

$$I(f(X^n); Y^n) \leq (n - 1)2^{-H(\alpha)}, \quad (5)$$

and this bound is attained with equality by, e.g., $f(x^n) = (x_1, \cdots, x_{n-1})$.

In 2018, Li and Medard [8] studied the problem of finding the optimal Boolean function that maximizes the $\delta$-th moment of $\mathbb{E}(T_\alpha f)^\delta$, where $T_\alpha$ denotes the noise operator on $f$ with noise parameter $\alpha$, in which they discussed the relationship between this problem and Conjecture [1].

**B. Our main contributions**

In this paper, we prove the same result as in Theorem [1] by directly dealing with Conjecture [1] in a purely mathematical point of view. First, we reformulate Conjecture [1] into a form where given $n$ and $f$, the left-hand side is a function $F \triangleq H(\alpha) - H(f(X^n)|Y^n)$ in terms of $\alpha \in [0,1]$ whereas the right-hand side is a constant $T = 1 - H(f(X^n))$ specified by $n$ and $|f^{-1}(0)|$. Then, we show that $F \leq T$ always holds when $\alpha \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ with $\delta > 0$ being some universally small constant by showing that $\frac{\partial F}{\partial \alpha} \big|_{\alpha = \frac{1}{2}} = 0$ and $\frac{\partial^2 F}{\partial \alpha^2} \big|_{\alpha = \frac{1}{2}} \leq 0$.

Given $|f^{-1}(0)|$, we say “$f$ is a most informative function” if $I(f(X^n); Y^n)$ is undominated in high noise regime, i.e., if $F$ for this choice of $f$ is greater than or equal to $F$ for any other choice of $f$ with the same $|f^{-1}(0)|$ near $\alpha = \frac{1}{2}$. Since for any $f$, $F = H(\alpha) - H(f(X^n))$ is equal to the constant $1 - H\left(\frac{|f^{-1}(0)|}{2^n}\right)$ at $\alpha = \frac{1}{2}$, and the first derivative of $F$ is zero at $\alpha = \frac{1}{2}$ for all $f$, it follows that $F$ is undominated if $\frac{\partial^2 F}{\partial \alpha^2} \big|_{\alpha = \frac{1}{2}}$ is maximized, which implies that the most informative functions $f$ are the ones that maximize $\frac{\partial^2 F}{\partial \alpha^2} \big|_{\alpha = \frac{1}{2}}$. 

July 31, 2018
We introduce the ratio spectrum of $f^{-1}(0)$, an integer sequence which characterizes the structure of $f^{-1}(0)$, which is sufficient to determine $\frac{\partial^2 F}{\partial \alpha^2} \Big|_{\alpha = \frac{1}{2}}$. With $|f^{-1}(0)|$ fixed, the lex function [1], a Boolean function with the first $|f^{-1}(0)|$ lexicographically ordered codewords mapped to 0, is a “locally” most informative function for the very noisy BSC as it is a function that maximizes $\frac{\partial^2 F}{\partial \alpha^2} \Big|_{\alpha = \frac{1}{2}}$ over all Boolean functions with the same $|f^{-1}(0)|$. Note that, with $|f^{-1}(0)|$ fixed, lex function is not the only locally most informative function. In general, any other function that has the same ratio spectrum as the lex function is also a locally most informative function, which will be shown in Theorem [7]. However, the dictator function $f(X^i) = X_i$, $(1 \leq i \leq n)$ is the “globally” most informative function for the very noisy BSC since it is the only type of functions that maximize $\frac{\partial^2 F}{\partial \alpha^2} \Big|_{\alpha = \frac{1}{2}}$ to 0 over all possible $|f^{-1}(0)|$, i.e., over all possible choices of Boolean functions. Note that with $i = 1$ the dictator function is a special case of the lex function with $|f^{-1}(0)| = 2^n - 1$.

This paper is organized as follows. Section II reformulates Conjecture I into the form $F \leq T$. Section III presents the lemmas that are sufficient and necessary to prove the reformulated conjecture in high noise regime. Sec. IV, V and VI present the proof details for the respective lemmas and Section VII concludes the paper.

II. Reformulation of Conjecture I

First, noting that the inequality (1) is equivalent to

$$H(\alpha) - H(f(X^n)|Y^n) \leq 1 - H(f(X^n)),$$

we reformulate Conjecture I as follows.

Let $S = \{c_0, c_1, \cdots, c_{S-1}\}$ denote the universal set of $n$-bit binary codewords sorted in lexicographical order, where $S = |S| = 2^n$, $c_i$ is the $n$-bit binary representation of index $i$. A Boolean function $f$ can be specified as follows:

$$f(c_i) = \begin{cases} 
0, & c_i \in \mathcal{U}; \\
1, & c_i \in \mathcal{V} = S \setminus \mathcal{U},
\end{cases}$$

where $\mathcal{U} = f^{-1}(0) = \{c_{t_1}, c_{t_2}, \cdots, c_{t_{M-1}}\}$ with $M \triangleq |f^{-1}(0)|$, $0 \leq M \leq S$. For brevity, we denote the above mapping by $f : \mathcal{U}, \mathcal{V} \rightarrow \{0, 1\}$. Note that when $M = 0$ or $M = 2^n$, (6) degenerates to $H(\alpha) \leq 1$ which always holds, thus it is enough to focus on the case when $1 \leq M \leq 2^n - 1$.

Define

$$F(\alpha, n, M, f) \triangleq H(\alpha) - H(f(X^n)|Y^n),$$

and

$$T(n, M) \triangleq 1 - H(f(X^n)),$$

where $H(f(X^n)) = H(\frac{M}{2^n}) = H(\frac{M}{n})$ and the logarithm base is 2. Note that in Sec. IV, V and VI the logarithm base is $e$. Therefore, (6) is equivalent to

$$F(\alpha, n, M, f) \leq T(n, M).$$
Fig. 1. An example of $F(\alpha,n,M,f)$ and $T(n,M)$, with $n = 4, M = 4$. We show two typical shapes of $F(\alpha,n,M,f)$: the quasi-concave shape for $F(\alpha,n,M,f_1)$ with $f_1^{-1}(0) = \{0, 1, 2, 3\}$ and the “single-peak wave” shape for $F(\alpha,n,M,f_2)$ with $f_2^{-1}(0) = \{0, 1, 2, 4\}$, where the index in the curly bracket represents the corresponding $n$-bit codeword.

Note that given pair $(n,M)$, $n \in \mathbb{Z}^+$, $0 \leq M \leq 2^n$, $T$ is a constant specified by $(n,M)$ whereas $F(\alpha,n,M,f)$ still depends on the choice of $f$. Later we will show that the second derivative of $F(\alpha,n,M,f)$ is uniquely determined by the ratio spectrum of $U$, a fundamental quantity that characterizes the structure of codewords in $U$. Thus, Conjecture 1 translates to the following conjecture.

**Conjecture 2:** Let $\alpha^* \triangleq \frac{1}{2}$. Given pair $(n,M)$, $n \in \mathbb{Z}^+$, $0 \leq M \leq 2^n$, for any function $f : \{U,V\} \to \{0,1\}$ and $\alpha \in [0,1]$, we have

$$\max_{\alpha \in [0,1]} F(\alpha,n,M,f) = F(\alpha^*,n,M,f) = T(n,M), \quad (11)$$

where

$$\alpha^* = \operatorname{argmax}_{\alpha \in [0,1]} \{F(\alpha,n,M,f)\}. \quad (12)$$

Note that it is trivial to show $F(\alpha^*,n,M,f) = T(n,M)$ due to the fact that when $\alpha = \alpha^*$, $X$ and $Y$ are independent so that $H(f(X^n)|Y^n) = H(f(X^n))$. Thus, $F(\alpha^*,n,M,f) = H(\alpha^*) - H(f(X^n)|Y^n) = 1 - H(f(X^n)) = T(n,M)$. Hence, the crucial part is to show $F(\alpha^*,n,M,f) = \max_{\alpha \in [0,1]} F(\alpha,n,M,f)$.

An interesting fact is that the dictator function $f(X^n) = X_i, (1 \leq i \leq n)$ satisfies $F(\alpha,n,M,f) = T(n,M) = 0$ for any $\alpha \in [0,1]$. Let $f(X^n) = X_i$, $H(f(X^n)|Y^n) = H(X_i|Y^n) = H(X_i|Y_i) = H(\alpha)$. Therefore, $F(\alpha,n,M,f) = H(\alpha) - H(f(X^n)|Y^n) = 0$ and $T(n,M) = 1 - H(f(X^n)) = 1 - H\left(\frac{1}{2}\right) = 0$, i.e., $F(\alpha,n,M,f) = T(n,M) = 0$.

In this paper, we will also show that the dictator function is the globally most informative function for the very
noisy BSC as it is the only type of functions that maximize the second derivative of \( F(\alpha, n, M, f) \) evaluated at \( \alpha = \frac{1}{2} \) to 0 over all possible \(|f^{-1}(0)|\).

As an example of the reformulation, Fig. 1 shows that \( F(\alpha, n, M, f) \leq T(n, M) \) for \( n = 4 \) and \( M = 4 \). Meanwhile, Fig. 1 also depicts two typical shapes of \( F(\alpha, n, M, f) \): a quasi-concave shape as shown in \( F(\alpha, n, M, f_1) \), and a “single-peak wave” shape as shown in \( F(\alpha, n, M, f_2) \). In fact, we conjecture that these are the only two possible shapes of \( F(\alpha, n, M, f) \) given \((n, M, f)\). Note that even for dictator function \( f(X^n) = X_i \), \( 1 \leq i \leq n \), \( F(\alpha, n, M, f) = 0 \) is still quasi-concave.

### III. Main Results

Note that for any \((n, M, f)\), \( F(\alpha, n, M, f) \) is a continuous, twice differentiable function of \( \alpha \). We prove that Conjecture 2 holds in the high noise regime, as stated in Theorem 4, which is equivalent to Theorem 1.

**Theorem 4:** Let \( \alpha^* \triangleq \frac{1}{2} \). Given pair \((n, M)\), \( n \in \mathbb{Z}^+, 0 \leq M \leq 2^n \), for any function \( f : \{U, V\} \rightarrow \{0, 1\} \), there exists a universally small constant \( \delta > 0 \) such that

\[
\max_{\alpha \in [\alpha^* - \delta, \alpha^* + \delta]} F(\alpha, n, M, f) = F(\alpha^*, n, M, f) = T(n, M),
\]

where

\[
\alpha^* = \arg\max_{\alpha \in [\alpha^* - \delta, \alpha^* + \delta]} \{F(\alpha, n, M, f)\}.
\]

The entire paper is to establish Theorem 4 by proving the following 3 lemmas.

**Lemma 1:** Given pair \((n, M)\), \( n \in \mathbb{Z}^+, 0 \leq M \leq 2^n \), for any function \( f : \{U, V\} \rightarrow \{0, 1\} \), \( F(\alpha, n, M, f) \) is symmetric with respect to \( \alpha^* = \frac{1}{2} \).

**Lemma 2:** Given pair \((n, M)\), \( n \in \mathbb{Z}^+, 0 \leq M \leq 2^n \), for any function \( f : \{U, V\} \rightarrow \{0, 1\} \), we have

\[
\frac{\partial F(\alpha, n, M, f)}{\partial \alpha} \bigg|_{\alpha = \alpha^*} = 0.
\]

**Lemma 3:** Given pair \((n, M)\), \( n \in \mathbb{Z}^+, 0 \leq M \leq 2^n \), for any function \( f : \{U, V\} \rightarrow \{0, 1\} \), we have

\[
\frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} \leq 0,
\]

where equality holds if and only if \( f \) is a dictator function.

In the proof of Lemma 3, we introduce several new concepts such as lex function, ratio spectrum, etc., which play an important role in the proof. We also show that, with \(|f^{-1}(0)|\) fixed, the lex function is a locally most informative function for the very noisy BSC as it maximizes the \( \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} \) over the set of Boolean functions with the same \(|f^{-1}(0)|\). The dictator function is the globally most informative function as it is the only type of functions that maximize \( \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} \) to 0 over all possible \( M = |f^{-1}(0)| \), i.e., over all possible choices of Boolean functions.
IV. PROOF OF LEMMA 1

The expansion of $F(\alpha, n, M, f)$ is as follows.

$$F(\alpha, n, M, f) = H(\alpha) - H(f(X^n)|Y^n)$$

$$= H(\alpha) - \sum_{j=0}^{S-1} \Pr(Y^n = c_j)H(f(X^n)|Y^n = c_j)$$

$$= H(\alpha) + \sum_{j=0}^{S-1} \Pr(Y^n = c_j) \frac{1}{f} p(f|c_j) \log p(f|c_j)$$

$$= H(\alpha) - \frac{1}{S} \sum_{j=0}^{S-1} H(\lambda_j), \quad (20)$$

where

$$\lambda_j \triangleq p(f = 0|c_j)$$

$$= \sum_{i=0}^{M-1} \Pr(X^n = c_i|Y^n = c_j)$$

$$= \sum_{i=0}^{M-1} \alpha^{k_{i,j}}(1 - \alpha)^{n-k_{i,j}} \quad (23)$$

$$H(p) \triangleq p \log \frac{1}{p} + (1 - p) \log \frac{1}{1 - p},$$

and $k_{i,j}$ denotes the Hamming distance between $c_i$ and $c_j$. Note that when $\alpha = 1/2$, $\lambda_0 = \lambda_1 = \cdots = \lambda_{S-1} = M/S$.

Consider $\alpha \in [0, \frac{1}{2}]$ and its symmetric part $\overline{\alpha} = 1 - \alpha \in [\frac{1}{2}, 1]$. Note that $c_j$ and its mirror codeword $c_{S-1-j}$ satisfy $k_{i,j} + k_{i,S-1-j} = n$, we have

$$\lambda'_j = \sum_{i=0}^{M-1} \overline{\alpha}^{k_{i,j}}(1 - \overline{\alpha})^{n-k_{i,j}}$$

$$= \sum_{i=0}^{M-1} (1 - \alpha)^{n-k_{i,S-1-j}} \alpha^{k_{i,S-1-j}}$$

$$= \lambda_{S-1-j}. \quad (27)$$

Thus, it follows from (27) that

$$F(\overline{\alpha}, n, M, f) = H(\overline{\alpha}) - \frac{1}{S} \sum_{j=0}^{S-1} H(\lambda'_j)$$

$$= H(\alpha) - \frac{1}{S} \sum_{j=0}^{S-1} H(\lambda_{S-1-j})$$

$$= F(\alpha, n, M, f),$$

which implies that $F(\alpha, n, M, f)$ is symmetric with respect to $\alpha^* = \frac{1}{2}$.

Similarly, an additional symmetry property for complementary function $f^c$ is presented as follows.
**Theorem 5:** Given pair \((n, M)\), \(n \in \mathbb{Z}^+, 0 \leq M \leq 2^n\), for each function \(f : \{\mathcal{U}, \mathcal{V}\} \to \{0, 1\}\), define its complementary function \(f^c : \{\mathcal{U}^c, \mathcal{V}^c\} \to \{0, 1\}\) with \(\mathcal{U}^c = \mathcal{V}\) and \(\mathcal{V}^c = \mathcal{U}\). Then, \(F(\alpha, n, M, f) = F(\alpha, n, S - M, f^c)\).

**Proof:** It is equivalent to expressing \(f^c : \{\mathcal{U}, \mathcal{V}\} \to \{1, 0\}\). Define

\[
\lambda_j^c \triangleq p(f = 1|e_j) = \sum_{i=0}^{M-1} \Pr(X^n = e_i|Y^n = e_j)
\]

(31)

\[
= \sum_{i=0}^{M-1} \alpha^{k_{i,j}} (1 - \alpha)^{n-k_{i,j}}.
\]

(32)

From (31), we notice that \(\lambda_j^c\) is exactly the same as the definition of \(\lambda_j\). Thus, \(f^c\) and \(f\) have the same \(F(\alpha, n, M, f)\).

The implication of Theorem 5 is that it suffices to focus on \(1 \leq M \leq 2^{n-1}\).

**V. PROOF OF LEMMA 2**

According to (23), The first derivative of \(\lambda_j\) with respect to \(\alpha\) is

\[
\frac{\partial \lambda_j}{\partial \alpha} = \sum_{i=0}^{M-1} \frac{\partial \alpha^{k_{i,j}} (1 - \alpha)^{n-k_{i,j}}}{\partial \alpha}
\]

(33)

\[
= \sum_{i=0}^{M-1} (k_{i,j} - n\alpha)\alpha^{k_{i,j}-1}(1 - \alpha)^{n-k_{i,j}-1}.
\]

(34)

Therefore, from (20), the first derivative of \(F(\alpha, n, M, f)\) evaluated at \(\alpha^*\) is

\[
\frac{\partial F(\alpha, n, M, f)}{\partial \alpha} \bigg|_{\alpha=\alpha^*} = \frac{\partial (H(\alpha) - \frac{1}{S} \sum_{j=0}^{S-1} H(\lambda_j))}{\partial \alpha} \bigg|_{\alpha=\alpha^*}
\]

(35)

\[
= \left( \log \frac{1 - \alpha}{\alpha} + \frac{1}{S} \sum_{j=0}^{S-1} \frac{\partial \lambda_j}{\partial \alpha} \log \frac{\lambda_j}{1 - \lambda_j} \right) \bigg|_{\alpha=\alpha^*}
\]

(36)

\[
= 0 + \frac{1}{S} \left( \sum_{j=0}^{S-1} \frac{\partial \lambda_j}{\partial \alpha} \bigg|_{\alpha=\alpha^*} \right) \log \frac{\lambda_0}{1 - \lambda_0}
\]

(37)

\[
= C \frac{S}{S} \left( \sum_{j=0}^{S-1} \sum_{i=0}^{M-1} (k_{i,j} - \frac{1}{2} n) \left( \frac{1}{2} \right)^{n-2} \right)
\]

(38)

\[
= \frac{4C}{S^2} \left( \sum_{i=0}^{M-1} \sum_{j=0}^{S-1} k_{i,j} - \frac{1}{2} n M S \right)
\]

(39)

\[
= \frac{4C}{S^2} \left( \sum_{i=0}^{M-1} \sum_{j=0}^{\frac{S}{2} - 1} (k_{i,j} + k_{i,j+S-1}) - \frac{1}{2} n M S \right)
\]

(40)

\[
= \frac{4C}{S^2} \left( \sum_{i=0}^{M-1} \sum_{j=0}^{\frac{S}{2} - 1} n - \frac{1}{2} n M S \right)
\]

(41)

\[
= 0,
\]

(42)

where \(C = \log \frac{\lambda_0}{1 - \lambda_0}\) and (36) to (37) follows from that \(\lambda_0 = \lambda_1 = \cdots = \lambda_{S-1} = M/S\) when \(\alpha = \alpha^* = \frac{1}{2}\).
VI. PROOF OF LEMMA 3

The proof of Lemma 3 proceeds as follows: First we compute \( \frac{\partial^2 F(\alpha,n,M,f)}{\partial \alpha^2} \bigg|_{\alpha=\frac{1}{2}} \) for any \((n, M, f)\), then we find that, with \((n, M)\) fixed, the lex function maximizes \( \frac{\partial^2 F(\alpha,n,M,f)}{\partial \alpha^2} \bigg|_{\alpha=\frac{1}{2}} \). The proof concludes by showing that \( \max \left\{ \frac{\partial^2 F(\alpha,n,M,f)}{\partial \alpha^2} \bigg|_{\alpha=\frac{1}{2}} \right\} \leq 0 \), i.e., the second derivative is nonpositive, when \( f \) is lex.

We first introduce several new definitions which play an important role in proving Lemma 3.

Definition 1: (lex function) We define \( f \) to be lex when \( U = f^{-1}(0) = \{ c_0, c_1, \cdots, c_{M-1} \} \), i.e., the first \( M \) lexicographically ordered codewords.

Definition 2: (codeword matrix \( C_U \)) Given pair \((n, M)\) and any function \( f : \{ U, V \} \rightarrow \{ 0, 1 \} \), define the codeword matrix \( C_U \) as an \( M \times n \) matrix in which the \( i \)-th row is \( c_i \), \( 0 \leq i \leq M - 1 \).

Definition 3: (0–1 ratio) Assume the \( l \)-th column of \( C_U \) has \( i \) bit 1’s and \( M - i \) bit 0’s, where \( 0 \leq i \leq M \). Define the 0–1 ratio of the \( l \)-th column of \( C_U \) as \( \max \{ \frac{M-i}{1}, \frac{i}{M-1} \}, (\frac{M}{0} \equiv +\infty) \).

Definition 4: (ratio spectrum) The ratio spectrum of \( C_U \) is defined by a sequence \( R_U = \{ r_0, r_1, \cdots, r_{\frac{M}{2}} \} \), where \( r_i \) denotes the number of columns in \( C_U \) that have the 0–1 ratio, max \( \{ \frac{M-i}{1}, \frac{i}{M-1} \} \).

Definition 5: (lexicographic ordering of ratio spectra) The ratio spectrum \( R_U = \{ r_0, r_1, \cdots, r_{\frac{M}{2}} \} \) is said to be (strictly) lexicographically greater than \( R_{U'} = \{ r'_0, r'_1, \cdots, r'_{\frac{M}{2}} \} \), denoted by \( R_U \triangleright R_{U'} \), if and only if \( r_j > r'_j \) for some \( j \) and \( r_i = r'_i \) for all \( i < j \).

With these definitions established, we begin the proof of Lemma 3. According to (34) and (36), the second derivative of \( \lambda_j \) and \( F(\alpha,n,M,f) \) evaluated at \( \alpha^* = \frac{1}{2} \) are given as follows.

\[
\frac{\partial^2 \lambda_j}{\partial \alpha^2} = \sum_{i=0}^{M-1} \frac{\partial (k_{i,j} - n\alpha) \alpha^{k_{i,j}-1} (1 - \alpha)^{n-k_{i,j}-1}}{\partial \alpha} \tag{43}
\]

\[
= \sum_{i=0}^{M-1} \left[ k_{i,j} (k_{i,j} - 1) + 2(1 - n)k_{i,j} \alpha + (n^2 - n)\alpha^2 \right] \alpha^{k_{i,j}-2} (1 - \alpha)^{n-k_{i,j}-2} \tag{44}
\]

Thus,

\[
\frac{\partial^2 \lambda_j}{\partial \alpha^2} \bigg|_{\alpha=\alpha^*} = \frac{16}{S} \sum_{i=0}^{M-1} \left[ k_{i,j}(k_{i,j} - 1) + (1 - n)k_{i,j} + \frac{1}{4} (n^2 - n) \right]. \tag{45}
\]

Therefore,

\[
\frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \bigg|_{\alpha=\alpha^*} = \frac{\partial}{\partial \alpha} \left( \log \frac{1 - \alpha}{\alpha} + \frac{1}{S} \sum_{j=0}^{S-1} \frac{\partial \lambda_j}{\partial \alpha} \log \frac{\lambda_j}{1 - \lambda_j} \bigg|_{\alpha=\alpha^*} \right) \tag{46}
\]

\[
= \left\{ \frac{-1}{(1 - \alpha)} + \frac{1}{S} \sum_{j=0}^{S-1} \left[ \frac{\partial^2 \lambda_j}{\partial \alpha^2} \log \frac{\lambda_j}{1 - \lambda_j} + \left( \frac{\partial \lambda_j}{\partial \alpha} \right)^2 \right] \right\} \bigg|_{\alpha=\alpha^*} \tag{47}
\]

\[
= -4 + C \sum_{j=0}^{S-1} \frac{\partial^2 \lambda_j}{\partial \alpha^2} \bigg|_{\alpha=\alpha^*} + \frac{S}{(S-M)M} \sum_{j=0}^{S-1} \left( \frac{\partial \lambda_j}{\partial \alpha} \right)^2 \bigg|_{\alpha=\alpha^*} \tag{48}
\]

\[
= -4 + L_1 + L_2, \tag{49}
\]
where

\[
C \triangleq \log \frac{\lambda_0}{1 - \lambda_0} \quad (50)
\]

\[
L_1 \triangleq \frac{C}{S} \sum_{j=0}^{S-1} \frac{\partial^2 \lambda_j}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} \quad (51)
\]

\[
L_2 \triangleq \frac{S}{(S - M)M} \sum_{j=0}^{S-1} \left( \frac{\partial \lambda_j}{\partial \alpha} \right)^2 \bigg|_{\alpha = \alpha^*} \quad (52)
\]

and (47) to (48) follows from that \( \lambda_0 = \lambda_2 = \cdots = \lambda_{S-1} = M/S \) when \( \alpha = \alpha^* \).

Next, we show that \( L_1 = 0 \):

\[
L_1 = \frac{C}{S} \sum_{j=0}^{S-1} \frac{\partial^2 \lambda_j}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} \quad (53)
\]

\[
= \frac{16C}{S^2} \sum_{j=0}^{S-1} \sum_{i=0}^{M-1} \left[ k_{i,j}(k_{i,j} - 1) + (1 - n)k_{i,j} + \frac{1}{4}(n^2 - n) \right] \quad (54)
\]

\[
= \frac{16C}{S^2} \left[ \sum_{i=0}^{M-1} \sum_{j=0}^{S-1} k_{i,j}(k_{i,j} - n) + \frac{1}{4}(n^2 - n)MS \right] \quad (55)
\]

\[
= \frac{16C}{S^2} \left[ - \sum_{i=0}^{M-1} \sum_{j=0}^{S-1} k_{i,j}(n - k_{i,j}) + \frac{1}{4}(n^2 - n)MS \right] \quad (56)
\]

\[
= \frac{16C}{S^2} \left[ - \sum_{i=0}^{M-1} \left( \sum_{k=0}^{n} \binom{n}{k} k(n - k) \right) + \frac{1}{4}(n^2 - n)MS \right] \quad (57)
\]

\[
= \frac{16C}{S^2} \left[ M \left( \sum_{k=0}^{n} \binom{n}{k} k^2 - n \sum_{k=0}^{n} \binom{n}{k} k \right) + \frac{1}{4}(n^2 - n)MS \right] \quad (58)
\]

\[
= \frac{16C}{S^2} \left[ M \left( n(n + 1)2^{n-1} - n^22^{n-1} \right) + \frac{1}{4}(n^2 - n)MS \right] \quad (59)
\]

\[
= \frac{16C}{S^2} \left[ \frac{1}{4}MS (-n^2 + n) + \frac{1}{4}(n^2 - n)MS \right] \quad (60)
\]

\[
= 0, \quad (61)
\]

where (58) to (59) is in Appendix A.

\[
L_2 = \frac{S}{(S - M)M} \sum_{j=0}^{S-1} \left( \frac{\partial \lambda_j}{\partial \alpha} \right)^2 \bigg|_{\alpha = \alpha^*} \quad (62)
\]

\[
= \frac{S}{(S - M)M} \sum_{j=0}^{S-1} \left[ \sum_{i=0}^{M-1} \left( k_{i,j} - \frac{1}{2}n \right) \left( \frac{1}{2} \right)^{n-2} \right]^2 \quad (63)
\]

\[
= \frac{4}{(S - M)MS} \left[ \sum_{j=0}^{S-1} \left( \sum_{i=0}^{M-1} k_{i,j} \right)^2 - 4nM \sum_{i=0}^{M-1} \sum_{j=0}^{S-1} k_{i,j} + n^2M^2S \right] \quad (64)
\]

\[
= \frac{4}{(S - M)MS} \left[ \sum_{j=0}^{S-1} \left( \sum_{i=0}^{M-1} k_{i,j} \right)^2 - 4nM \sum_{i=0}^{M-1} \sum_{k=0}^{n} \binom{n}{k} k + n^2M^2S \right] \quad (65)
\]
According to (73), it is obvious that, with bit ratio spectrum defined in Definition 4 as to show that not only the ratio spectrum determines \(L\), but also the largest ratio spectrum maximizes \(\sum_{i=1}^{n} d(c_{i,t} \oplus c_{j,l})\), as stated in Theorem 6 and 7 respectively.

**Theorem 6:** Given pair \((n, M)\) and any function \(f: \{\mathcal{U}, \mathcal{V}\} \rightarrow \{0, 1\}\), 

\[
\frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*}
\]

can be uniquely determined by the ratio spectrum \(\{r_0, r_1, \ldots, r_{4^M}\}\) of \(C_{\mathcal{U}}\).

**Proof:** Let us define three operations on columns of \(C_{\mathcal{U}}\):

- Permutation: to re-arrange the positions of bits in a column;
- Flipping: to flip all bits in a column;
- Switching: to switch the \(i\)-th column with the \(j\)-th column;

According to (73), it is obvious that, with bit \(c_{j,l}\) fixed, the permutation operation on the \(l\)-th column will not change \(h_{j,l}\) and the flipping operation on the \(l\)-th column will simply change \(h_{j,l}\) to \(M - h_{j,l}\).

Define the mapping \(G_{\mathcal{U}}: \{0, 1\}^n \rightarrow \{0, 1, 2, \ldots, M\}^n\), which maps a codeword \(c_j = (b_1 b_2 \ldots b_n)\) to a Hamming distance sequence \(\{h_{j,1}, h_{j,2}, \ldots, h_{j,n}\}\) through \(C_{\mathcal{U}}\), that is,

\[
G_{\mathcal{U}}(c_j) = \{h_{j,1}, h_{j,2}, \ldots, h_{j,n}\}.
\]
Given \( C_\mathcal{U} \) and a codeword \( c = (b_1 b_2 \ldots b_n) \), since any other \( \mathcal{U}' \) that has the same ratio spectrum with \( C_\mathcal{U} \) can only be obtained from \( \mathcal{U} \) with permutation, flipping and switching. After fixing the operation mapping from \( C_\mathcal{U} \) to \( C_\mathcal{U}' \) (which means that the operation mapping from columns of \( C_\mathcal{U} \) to columns of \( C_\mathcal{U}' \) is bijective and fixed), for each \( c_j = (b_0 b_1 \cdots b_n) \in \mathcal{S} \) associated with \( C_\mathcal{U} \), we implement the following codeword-matching scheme to find one, unique \( c_j' = (b_0' b_1' \cdots b_n') \in \mathcal{S} \) associated with \( C_\mathcal{U}' \).

\[
b_j' = \begin{cases} 
b_i, & C_{\mathcal{U}'\mathcal{J}} \text{ is obtained from } C_{\mathcal{U}\mathcal{J}} \text{ by permutation (and switching)}, \\
1 - b_i, & C_{\mathcal{U}'\mathcal{J}} \text{ is obtained from } C_{\mathcal{U}\mathcal{J}} \text{ by flipping (and switching)}, 
\end{cases}
\]  

(76)

where \( C_{\mathcal{U}\mathcal{J}} \) denotes the \( j \)-th column of \( C_\mathcal{U} \). In this way, it can be verified that

\[
\{h_{j',1}, h_{j',2}, \ldots, h_{j',n}\} = G_{\mathcal{U}'}(c_j') = G_{\mathcal{U}}(c_j) = \{h_{j,1}, h_{j,2}, \ldots, h_{j,n}\},
\]

(77)

where \( h_{j,l} \) is defined in (73).

Let \( \mathcal{T}' = \{c_j'\}_{j=0}^{2^s-1} \) be the set of codewords chosen with the codeword-matching scheme for set \( \mathcal{T} = \{c_j\}_{j=0}^{2^s-1} \). Due to the one-to-one mapping property of the codeword-matching scheme, all codewords in \( \mathcal{T}' \) are different. Therefore, \( \frac{\partial^2 F(\alpha, n, M, f')}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} \) for \( f' : \{\mathcal{U}', \mathcal{V}'\} \to \{0, 1\} \), where \( \mathcal{U}' \) has the same ratio spectrum with \( \mathcal{U} \), can be computed as follows.

\[
L_2' = \frac{4}{(S - M)MS} \left[ n^2 M^2 S - 8 \cdot \frac{1}{2} \sum_{j=0}^{S-1} \left( \sum_{l=1}^{n} h_{j,l}' \right) \left( \sum_{l=1}^{n} h_{S-1-j,l}' \right) \right] 
\]

(78)

\[
= \frac{4}{(S - M)MS} \left[ n^2 M^2 S - 8 \sum_{c_j \in \mathcal{T}'} \left( \sum_{l=1}^{n} h_{j,l} \right) \left( \sum_{l=1}^{n} h_{S-1-j,l} \right) \right] 
\]

(79)

\[
= \frac{4}{(S - M)MS} \left[ n^2 M^2 S - 8 \sum_{c_j \in \mathcal{T}'} \left( \sum_{l=1}^{n} h_{j,l} \right) \left( \sum_{l=1}^{n} h_{S-1-j,l} \right) \right] 
\]

(80)

\[
= \frac{4}{(S - M)MS} \left[ n^2 M^2 S - 8 \sum_{j=0}^{2^s-1} \left( \sum_{l=1}^{n} h_{j,l} \right) \left( \sum_{l=1}^{n} h_{S-1-j,l} \right) \right] 
\]

(81)

\[
= L_2,
\]

(82)

where (78) to (79) is due to the fact that any \( S/2 \) different codewords in \( \mathcal{S} \) can result in the same sum. Hence,

\[
\frac{\partial^2 F(\alpha, n, M, f')}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} = -4 + L_2' 
\]

(83)

\[
= -4 + L_2
\]

(84)

\[
= \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*}. 
\]

(85)

Theorem 7: Given pair \((n, M)\), any function \( f : \{\mathcal{U}, \mathcal{V}\} \to \{0, 1\} \) whose \( C_\mathcal{U} \) has the largest lexicographical ratio spectrum \( R^*_\mathcal{U} = \{r^*_0, r^*_1, \cdots, r^*_n\} \) maximizes \( \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} \).

Proof: It is enough to prove that, given pair \((n, M)\), for \( f : \{\mathcal{U}, \mathcal{V}\} \to \{0, 1\} \) and \( f' : \{\mathcal{U}', \mathcal{V}'\} \to \{0, 1\} \), if \( R_\mathcal{U} > R_\mathcal{U}' \), \( \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} > \frac{\partial^2 F(\alpha, n, M, f')}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} \).

July 31, 2018 DRAFT
Let $R_d = \{r_0, \cdots, r_i, \cdots, r_j, \cdots, r_{\lceil M/2 \rceil} \}$ and $R_d' = \{r_0, \cdots, r_i - 1, \cdots, r_j + 1, \cdots, r_{\lceil M/2 \rceil} \}$, where $0 \leq i < j \leq \lfloor M/2 \rfloor$. Since many functions can have the same ratio spectrum, we consider a “canonical form” of $C_d$ that satisfies the following conditions.

- for each column, the number of bit 0’s is greater or equal to the number of bit 1’s;
- the columns (which are numbered increasingly from the leftmost to the rightmost in $C_d$) are sorted so that the first $r_0$ columns have $0 - 1$ ratio $\frac{M}{2}$, the next $r_1$ columns have $0 - 1$ ratio $\frac{M-1}{2}$, etc.

Note that the above ”canonical form” of $C_d$ can be obtained from any other codeword matrix with the same ratio spectrum by permutation, flipping and switching. Also, define $C_{d'}$ as changing the $0 - 1$ ratio $\frac{M-1}{2}$ at the $\delta$-th $(\sum_{l=0}^{i-1} r_l + 1 \leq \delta \leq \sum_{l=0}^{i} r_l)$ column of $C_d$ to $\frac{M-j}{2}$ by flipping $j - i$ bit 0’s to bit 1’s.

Obviously, $R_d \succ R_{d'}$, and any other ratio spectrum inequalities can be established by successively implementing the above operation. We examine the difference of the second derivative values, that is,

$$\frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*} - \frac{\partial^2 F(\alpha, n, M, f')}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*}$$

$$= L_2 - L_2'$$

$$= C \left[ \sum_{j=0}^{\frac{M}{2} - 1} \left( \sum_{l=1}^{n} h_{j,l} \right) \left( \sum_{i=1}^{\frac{M}{2} - 1} h_{S-1-j,l} \right) - \sum_{j=0}^{\frac{M}{2} - 1} \left( \sum_{l=1}^{n} h_{j,l} \right) \left( \sum_{i=1}^{\frac{M}{2} - 1} h_{S-1-j,l} \right) \right],$$

where $C = \frac{32}{(S-M)MS}$.

Define

$$s_j \triangleq \sum_{l=1}^{n} h_{j,l}$$

$$\Delta r \triangleq j - i > 0.$$  

Also, we assume $0 - 1$ ratio of the $\delta$-th column is changed from $\frac{M-1}{2}$ to $\frac{M-j}{2}$. Thus, when $\delta > 1$,

$$L_2 - L_2' = C \left( \sum_{j=0}^{\frac{M}{2} - 1} s_j s_{S-1-j} - \sum_{j=0}^{\frac{M}{2} - 1} s_j s_{S-1-j} \right)$$

$$= C \left( \sum_{j \in D_1} (s_j + \Delta r)(s_{S-1-j} - \Delta r) + \sum_{j \in D_2} (s_j - \Delta r)(s_{S-1-j} + \Delta r) - \sum_{j=0}^{\frac{M}{2} - 1} s_j s_{S-1-j} \right)$$

$$= C \left( \Delta r \left( \sum_{j \in D_2} s_j - \sum_{j \in D_1} s_j \right) + \Delta r \left( \sum_{j \in D_1} s_{S-1-j} - \sum_{j \in D_2} s_{S-1-j} \right) - \frac{S}{2} \Delta r^2 \right)$$

$$= C \left( \frac{S}{4} (M-2i) \Delta r + \frac{S}{4} (M-2i) \Delta r - \frac{S}{2} \Delta r^2 \right)$$

$$= \frac{CS}{2} \Delta r (M-2i-\Delta r)$$

$$= \frac{CS}{2} \Delta r (M-i-j)$$

$$> 0,$
where \( D_1, D_2 \) denote the set of indices of codewords in which bit 0 or bit 1 is at position \( \delta \), respectively. Obviously, \(|D_1| = |D_2| = \frac{S}{2}, D_1 \cap D_2 = \emptyset, D_1 \cup D_2 = \{0, 1, \ldots, \frac{S}{2} - 1\}\). From the canonical form, we know \( s'_j - s_j = \Delta r \) if \( j \in D_1 \), and \( s'_j - s_j = -\Delta r \) if \( j \in D_2 \). (93) to (94) follows from that, since \( C_U \) only differs from \( C_{U'} \) at position \( \delta \),

\[
\sum_{j \in D_2} s_j - \sum_{j \in D_1} s_j = \sum_{j \in D_2} h_{\delta,j} - \sum_{j \in D_1} h_{\delta,j} = \frac{S}{4}(M - i) - \frac{S}{4}i = \frac{S}{4}(M - 2i).
\]

It is straightforward to show that if \( \delta = 1 \), the difference is still the same as in (96).

Therefore,

\[
\left. \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \right|_{\alpha = \alpha^*} - \left. \frac{\partial^2 F(\alpha, n, M, f')}{\partial \alpha^2} \right|_{\alpha = \alpha^*} = L_2 - L'_2 > 0.
\]

Next, we show that lex function is one type of function that has the largest (lexicographical) ratio spectrum given pair \((n, M)\), as stated in Theorem 8.

**Theorem 8:** Given pair \((n, M)\), when \( f : \{U, V\} \to \{0, 1\} \) is lex, \( R_{U'} \) has the largest lexicographical ratio spectrum.

**Proof:** It is enough to prove that, when \( f \) is lex, \( R_{U'} \geq R_{U''} \) always holds, where \( U'' = f'^{-1}(0) \), and \( f' \) is any other possible function.

Given pair \((n, M)\), when \( f \) is lex, assume that there exists another function \( f' \) that has \( R_{U'} > R_{U''} \). This is only possible by first deleting bit 1’s in \( C_{U'} \), and then performing permutation, flipping and switching. Since only deletion can change the ratio spectrum and note that deleting bits in a lexicographical ordering will result in repetitive codewords, which violates the definition of \( f \) which requires \( U' \) to contain different codewords. Therefore, the assumption is not valid, which follows that \( C_{U'} \) has the largest lexicographical ratio spectrum among any other function \( f' \).

Combining Theorem 8, 7 and 8, we have the following most important corollary which shows that, with \( M = |f^{-1}(0)| \) fixed, lex function is a locally most informative function in high noise regime since it maximizes \( \left. \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \right|_{\alpha = \alpha^*} \) over the set of Boolean functions with the same \( |f^{-1}(0)| \).

**Corollary 1:** Given pair \((n, M)\), lex function maximizes \( \left. \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \right|_{\alpha = \alpha^*} \).

Since the ratio spectrum \( R_{U'} \) solely determines the second derivative stated in Theorem 8 we give the explicit expression for \( \left. \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \right|_{\alpha = \alpha^*} \) in terms of ratio spectrum \( R_{U'} = \{r_0, r_1, \ldots, r_{\frac{M}{2}}\} \).

**Theorem 9:** Given pair \((n, M)\) and the ratio spectrum \( R_{U'} = \{r_0, r_1, \ldots, r_{\frac{M}{2}}\} \) of \( C_{U'} \),

\[
\left. \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \right|_{\alpha = \alpha^*} = -4 + \frac{4}{(S - M)M} \left[ nM^2 - 4 \sum_{t=0}^{\frac{M}{2}} (M - t)r_t \right].
\]
Proof: We consider the canonical form of $C_{n,t}$ as in the proof of Theorem 7, in which the columns are labeled independently in each block with $0-1$ ratio $\frac{M-t}{t}$. According to (72),

\[ L_3 \equiv \sum_{j=0}^{\frac{n}{2}-1} \left( \sum_{t=1}^{n} h_{j,t} \right) \left( \sum_{t=1}^{n} h_{S-1-j,t} \right) \]

\( = \sum_{j=0}^{\frac{n}{2}-1} \left( \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{t(l)=1}^{r_t} h_{j,t(l)} \right) \left( \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{t(l)=1}^{r_t} h_{S-1-j,t(l)} \right) \)

\( = \sum_{j=0}^{\frac{n}{2}-1} \left( M a_j + \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t(r_t - 2a_{t,j}) \right) \left( M(n-a_j) - \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t(r_t - 2a_{t,j}) \right) \)

\( = \sum_{j=0}^{\frac{n}{2}-1} \left[ M^2 a_j(n - a_j) + M(n - 2a_j) \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t(r_t - 2a_{t,j}) - \left( \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t(r_t - 2a_{t,j}) \right)^2 \right] \)

\( = \frac{1}{8}(n^2 - n)M^2S + \sum_{j=0}^{\frac{n}{2}-1} \left[ M(n - 2a_j) \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t(r_t - 2a_{t,j}) - \left( \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t(r_t - 2a_{t,j}) \right)^2 \right], \)

where $a_j$ denotes the number of bit 1’s in the $j$-th codeword, and $a_{t,j}$ denotes the number of bit 1’s in the $j$-th codeword in the block with $0-1$ ratio $\frac{M-t}{t}$. Clearly, for this block,

\[ \sum_{t(l)=1}^{r_t} h_{j,t(l)} = (M - t)a_{t,j} + t(r_t - a_{t,j}) \]

\[ = Ma_{t,j} + t(r_t - 2a_{t,j}), \]

and

\[ a_j = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} a_{t,j}. \]

Now we break (107) by computing each individual part. Note that

\[ \sum_{j=0}^{\frac{n}{2}-1} M(n - 2a_j) \left( \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t(r_t - 2a_{t,j}) \right) \]

\[ = \sum_{j=0}^{\frac{n}{2}-1} M(n - 2a_j) \left( \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} tr_t \right) - \sum_{j=0}^{\frac{n}{2}-1} M(n - 2a_j) \left( \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} 2ta_{t,j} \right) \]

\[ = M \left( \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} tr_t \right) \left( \frac{nS}{2} - 2 \sum_{j=0}^{\frac{n}{2}-1} a_j \right) - 2M \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t \left( \sum_{j=0}^{\frac{n}{2}-1} (n_{a,j} - 2a_{j}a_{t,j}) \right) \]

\[ = M \left( \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} tr_t \right) \left( \frac{nS}{2} - 2(n - 1)2^{n-2} \right) - 2nM \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t \left( \sum_{j=0}^{\frac{n}{2}-1} a_{t,j} \right) + 4M \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t \sum_{j=0}^{\frac{n}{2}-1} a_{j}a_{t,j} \]

\[ = \frac{1}{2}MS \left( \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} tr_t \right) - 2nM \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t \left( \frac{Sr_t}{4} \right) + 4M \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} t \sum_{k=0}^{n-1} k \left( \frac{n - 2}{k - 1} \right) r_t \]

\[ = \frac{1}{2}MS \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} tr_t, \]
where \( \sum_{k=0}^{n-1} \binom{n-2}{k} k = \frac{1}{8} nS. \)

Also,

\[
\sum_{j=0}^{2} \left( \left( \sum_{t=0}^{j} t r_t - 2 a_{t,j} \right)^2 \right) = \sum_{j=0}^{2} \left[ \left( \sum_{t=0}^{j} tr_t \right)^2 + 4 \left( \sum_{t=0}^{j} ta_{t,j} \right)^2 - 4 \left( \sum_{t=0}^{j} tr_t \right) \left( \sum_{t=0}^{j} ta_{t,j} \right) \right]
\]

\[
= S \left( \sum_{t=0}^{j} tr_t \right)^2 - 4 \left( \sum_{t=0}^{j} tr_t \right) \left( \sum_{t=0}^{j} t \sum_{j=0}^{s/2-1} a_{t,j} \right) + 4 \left( \sum_{t=0}^{j} t^2 \sum_{j=0}^{s/2-1} a^2_{t,j} \right) + 4 \left( \sum_{t_1=0}^{j} \sum_{t_2 \neq t_1} t_{t_1} \sum_{t_2 \neq t_1} t_{t_2} a_{t_1,j} a_{t_2,j} \right)
\]

Up to this point, we use the following combinatorial fact (in Appendix C, Theorem 14) that

\[
\sum_{j=0}^{s-1} a^2_{t,j} = \sum_{k=0}^{n-1} \sum_{k_1=0}^{k} \binom{n-1-r_t}{k-k_1} \binom{r_t}{k_1} k_1^2
\]

\[
= r_t (r_t + 1) \cdot 2^{n-1-2}
\]

\[
= \frac{1}{8} r_t (r_t + 1) S
\]

Similarly (in Appendix D, Theorem 15),

\[
\sum_{j=0}^{s-1} a_{t_1,j} a_{t_2,j} = \sum_{k=0}^{n-1} \sum_{k_1=0}^{k} \sum_{k_2=0}^{k-k_1} \binom{n-1-r_{t_1}-r_{t_2}}{k-k_1-k_2} \binom{r_{t_1}}{k_1} \binom{r_{t_2}}{k_2} k_1 k_2
\]

\[
= r_{t_1} r_{t_2} \cdot 2^{n-1-2}
\]

\[
= \frac{1}{8} r_{t_1} r_{t_2} S
\]

Thus, from (120),

\[
\sum_{j=0}^{s-1} \left( \sum_{t=0}^{j} t (r_t - 2 a_{t,j}) \right)^2 = S \left( \sum_{t=0}^{j} tr_t \right)^2 - S \left( \sum_{t=0}^{j} tr_t \right)^2 + 4 \left( \sum_{t=0}^{j} t^2 \cdot \frac{1}{8} r_t (r_t + 1) S \right) + 4 \left( \sum_{t_1=0}^{j} \sum_{t_2 \neq t_1} t_{t_1} \sum_{t_2 \neq t_1} t_{t_2} \cdot \frac{1}{8} r_{t_1} r_{t_2} S \right)
\]

\[
= S \left( \sum_{t=0}^{j} tr_t \right)^2 + \frac{S}{2} \sum_{t=0}^{j} (tr_t)^2 + \frac{S}{2} \sum_{t_1=0}^{j} \sum_{t_2 \neq t_1} t_{t_1} r_{t_1} t_{t_2} r_{t_2}
\]

\[
= S \sum_{t=0}^{j} t^2 r_t.
\]
Hence, it follows from (107) that,

$$L_3 = \frac{1}{8}(n^2 - n)M^2S + \sum_{j=0}^{\frac{n}{2}-1} M(n - 2a_j) \sum_{t=0}^{\frac{M}{2}} t(r_t - 2a_{t,j}) - \left( \sum_{t=0}^{\frac{M}{2}} t(r_t - 2a_{t,j}) \right)^2 \tag{131}$$

$$= \frac{1}{8}(n^2 - n)M^2S + \frac{1}{2} MS \sum_{t=0}^{\frac{M}{2}} t - S \frac{\frac{M}{2}}{2} \sum_{t=0}^{\frac{M}{2}} t^2 r_t \tag{132}$$

$$= \frac{1}{8}(n^2 - n)M^2S + \frac{S}{2} \sum_{t=0}^{\frac{M}{2}} (M - t)tr_t. \tag{133}$$

As a result, the second derivative can be simplified as

$$\frac{\partial^2 F(\alpha,n,M,f)}{\partial \alpha^2} \bigg|_{\alpha=\alpha^*} = -4 + \frac{4}{(S - M)MS} \left( n^2M^2S - 8L_3 \right) \tag{135}$$

$$= -4 + \frac{4}{(S - M)MS} \left[ n^2M^2S - 8 \left( \frac{1}{8}(n^2 - n)M^2S + \frac{S}{2} \sum_{t=0}^{\frac{M}{2}} (M - t)tr_t \right) \right] \tag{136}$$

$$= -4 + \frac{4}{(S - M)M} \left[ nM^2 - 4 \sum_{t=0}^{\frac{M}{2}} (M - t)tr_t \right]. \tag{137}$$

Note that in the above proof, we directly omit the fact that \( \{c_0, \cdots, c_{\frac{n}{2} - 1} \} \) all have bit 0 at the first column as it can be easily verified that the final result does not depend on it.

Last, we finish the proof of Lemma 3 by showing that when \( f \) is lex, \( \frac{\partial^2 F(\alpha,n,M,f)}{\partial \alpha^2} \bigg|_{\alpha=\alpha^*} \) is nonpositive. Meanwhile, we also show that the dictator function \( f(X^n) = X_i, (1 \leq i \leq n) \) is the globally most informative function as it maximizes \( \frac{\partial^2 F(\alpha,n,M,f)}{\partial \alpha^2} \bigg|_{\alpha=\alpha^*} \) to 0 over all possible \( M = |f^{-1}(0)| \). In the proof, we fix \( M \) and let \( n \geq \log_2(M) + 1 \) be a variable.

**Theorem 10:** Given \( M \geq 1 \) and \( f \) being lex, \( \frac{\partial^2 F(\alpha,n,M,f)}{\partial \alpha^2} \bigg|_{\alpha=\alpha^*} \) is a strictly decreasing function of \( n \) with \( n \geq \log_2(M) + 1 \) (since it suffices to focus on \( 1 \leq M \leq 2^{n-1} \)), i.e.,

$$\frac{\partial^2 F(\alpha,n,M,f)}{\partial \alpha^2} \bigg|_{\alpha=\alpha^*} \leq \frac{\partial^2 F(\alpha,\log_2(M) + 1, M, f)}{\partial \alpha^2} \bigg|_{\alpha=\alpha^*} \leq 0, \tag{138}$$

where the first equality holds if and only if \( n = \log_2(M) + 1 \) and the second equality holds if and only if \( M = 2^{k-1}, k \in \mathbb{Z}^+ \).

**Proof:** Let the columns in \( C_M \) be numbered increasingly from right to left. Define the weight \( W(M,i) \) of the \( i \)-th (\( 1 \leq i \leq n \)) column of \( C_M \) as the number of bit 1’s at the \( i \)-th column. Since \( f \) is lex, we have

$$W(M,i) = \sum_{k=0}^{2^{i-1} - 1} \left\lfloor \frac{M+k}{2^i} \right\rfloor, \tag{139}$$
where $W(M,i) = 0$ for $\lceil \log_2(M) \rceil < i \leq n$. It follows that

$$C \equiv 4 \sum_{i=0}^{\lceil \frac{M}{2^i} \rceil} (M - t) r_t$$

$$= 4 \sum_{i=1}^{n} \left[ M - W(M,i) \right] W(M,i)$$

$$= 4 \sum_{i=1}^{n} \left( M - \sum_{k=0}^{2^{i-1}-1} \left\lfloor \frac{M + k}{2^i} \right\rfloor \right) \left( \sum_{k=0}^{2^{i-1}-1} \left\lfloor \frac{M + k}{2^i} \right\rfloor \right),$$

which is a constant when $M$ is fixed and $n \geq \lceil \log_2(M) \rceil$.

Next, according to (102), we examine the first derivative of $n$, i.e., making $n$ a continuous variable. Define

$$g(n) \equiv \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \bigg|_{\alpha = \alpha^*},$$

$$= -4 + \frac{4(nM^2 - C)}{(2^n - M)^2}.$$

Thus,

$$g'(n) = 4 \cdot \frac{(2^n - M)M^2 - (nM^2 - C)2^n \ln 2}{(2^n - M)^2 M}.$$  

In order to show that $g'(n) < 0$ for $n \geq \log_2(M) + 1$, we need to show

$$(2^n - M)M^2 - (nM^2 - C)2^n \ln 2 < 0,$$

that is,

$$C < \frac{M^3}{2^n \ln 2} + \left( n - \frac{1}{\ln 2} \right) M^2, \ \forall n \geq \log_2(M) + 1.$$  

(147)

Since $f(x) = \frac{M^3}{2^n \ln 2} + \left( x - \frac{1}{\ln 2} \right) M^2$ is an increasing function for $x \geq \log_2(M)$, which implies that it is enough to show that

$$C < \min_{n \geq \log_2(M)+1} \left\{ \frac{M^3}{2^n \ln 2} + \left( n - \frac{1}{\ln 2} \right) M^2 \right\} = M^2 \left( \log_2(M) + \frac{2 \ln 2 - 1}{2 \ln 2} \right).$$  

(148)

In Appendix E, Theorem 16, we prove that for any $M \in \mathbb{Z}^+$ and $n \geq \lceil \log_2(M) \rceil$,

$$M^2 \log_2(M) \leq 4 \sum_{i=1}^{n} \left( M - \sum_{k=0}^{2^{i-1}-1} \left\lfloor \frac{M + k}{2^i} \right\rfloor \right) \left( \sum_{k=0}^{2^{i-1}-1} \left\lfloor \frac{M + k}{2^i} \right\rfloor \right) < M^2 \left( \log_2(M) + \frac{2 \ln 2 - 1}{2 \ln 2} \right),$$  

(149)

where the equality holds if and only if $M = 2^{k-1}, k \in \mathbb{Z}^+$.

With the above results, $g'(n) < 0$, which concludes that $\left. \frac{\partial^2 F(\alpha,n,M,f)}{\partial \alpha^2} \right|_{\alpha = \alpha^*}$ is a strictly decreasing function of $n$ with $n \geq \log_2(M) + 1$. Thus, according to (102),

$$\left. \frac{\partial^2 F(\alpha,n,M,f)}{\partial \alpha^2} \right|_{\alpha = \alpha^*} \leq \left. \frac{\partial^2 F(\alpha, \log_2(M)+1, M, f)}{\partial \alpha^2} \right|_{\alpha = \alpha^*}$$

$$= -4 + \frac{4}{(\log_2(M)+1 - M)M} \left( \log_2(M) + 1 \right)M^2 - 4 \sum_{t=0}^{\lceil \frac{M}{2^i} \rceil} (M - t) r_t$$

$$\leq -4 + \frac{4}{M^2} \left( \log_2(M) + 1 \right)M^2 - M^2 \log_2(M)$$

$$= 0.$$  

(150)
where the first equality holds if and only if $n = \log_2(M) + 1$ and the second equality holds if and only if $M = 2^{k-1}, k \in \mathbb{Z}^+$, which implies that \( \frac{\partial^2 F(\alpha, n, M, f)}{\partial \alpha^2} \big|_{\alpha=\alpha^*} = 0 \) if and only if \( f \) is a dictator function. Here we implicitly use the fact that only the dictator functions \( f(X^n) = X_i \) \( (0 \leq i \leq n-1) \) have the same ratio spectrum as \( f(X^n) = X_0 \).

VII. Conclusion

In this paper, we proved that for the very noisy BSC, i.e., when \( \alpha \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta] \) where \( \delta > 0 \) is some universally small constant, Conjecture 1 holds by directly dealing with the inequality in a purely mathematical point of view. We first pointed out that in high noise regime, a function \( f \) is more informative if the second derivative of \( H(\alpha) - H(f(X^n)|Y^n) \) has a larger value at \( \alpha = \frac{1}{2} \). Then we showed that, the ratio spectrum is the fundamental metric that determines the second derivative of \( H(\alpha) - H(f(X^n)|Y^n) \) evaluated at \( \alpha = \frac{1}{2} \). With \( |f^{-1}(0)| \) fixed, lex function is a locally most informative function for the very noisy BSC as it maximizes the second derivative of \( H(\alpha) - H(f(X^n)|Y^n) \) evaluated at \( \alpha = \frac{1}{2} \) over the set of Boolean functions with the same \( |f^{-1}(0)| \). The dictator function \( f(X^n) = X_i \), \( (1 \leq i \leq n) \), with \( i = 1 \) being a special case of lex function \( |f^{-1}(0)| = 2^{n-1} \), is the globally most informative function for the very noisy BSC as it is the only type of functions that maximize the second derivative of \( H(\alpha) - H(f(X^n)|Y^n) \) evaluated at \( \alpha = \frac{1}{2} \) to 0 over all possible \( |f^{-1}(0)| \), i.e., over all possible choices of Boolean functions. Future work will be focused on proving that \( \alpha^* = \frac{1}{2} \) is also the global maximum point of \( H(\alpha) - H(f(X^n)|Y^n) \) for any \( n \) and \( f \) so as to solve Conjecture 1 completely.
**APPENDIX A**

**THE EXPANSION OF** \((1 + x)^n\)

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n, \tag{154}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} k = n \cdot 2^{n-1}, \tag{155}
\]

\[
\sum_{k=0}^{n} \binom{n}{k} k^2 = n(n+1) \cdot 2^{n-2}. \tag{156}
\]

**Proof:** All above identities can be derived from

\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \tag{157}
\]

by taking derivatives and setting \(x = 1\). ■

**APPENDIX B**

**VANDERMONDE’S IDENTITY**

**Theorem 11:** For any \(m, n \in \mathbb{Z}^+\),

\[
\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} \tag{158}
\]

**Proof:** The identity can be established by the expansion of \((1 + x)^{m+n}\),

\[
(1 + x)^{m+n} = \sum_{r=0}^{m+n} \binom{m+n}{r} x^r \tag{159}
\]

Also, note that

\[
(1 + x)^{m+n} = (1 + x)^m (1 + x)^n \tag{160}
\]

\[
= \left( \sum_{i=0}^{m} \binom{m}{i} x^i \right) \left( \sum_{j=0}^{n} \binom{n}{j} x^j \right) \tag{161}
\]

\[
= \sum_{r=0}^{m+n} \left( \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} \right) x^r \tag{162}
\]

By comparing the coefficient, we conclude that

\[
\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} \tag{163}
\]

■

The generalized Vandermonde’s identity is stated as follows.

**Theorem 12 (Generalized Vandermonde’s Identity):** For any \(n_1, \ldots, n_p \in \mathbb{Z}^+\),

\[
\sum_{k_1 + \cdots + k_p = m} \binom{n_1}{k_1} \binom{n_2}{k_2} \cdots \binom{n_p}{k_p} = \binom{n_1 + \cdots + n_p}{m} \tag{164}
\]
APPENDIX C

THE EXPANSION OF \((1 + y)^m(x + y)^n\)

Consider the expansion of

\[
(1 + y)^m(x + y)^n = \left( \sum_{i=0}^{m} \binom{m}{i} y^i \right) \left( \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \right)
\]

(165)

\[
= \sum_{r=0}^{m+n} \left( \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} x^{n-r+k} \right) y^r.
\]

(166)

Setting \(y = 1\), we have

\[
\sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} x^{n-r+k} = (1 + x)^m 2^m.
\]

(167)

By (167), we can derive the following identities.

**Theorem 13 (First-order identity):** For any \(m, n \in \mathbb{Z}^+\), it holds that

\[
\sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = m \cdot 2^{m+n-1}.
\]

(168)

**Proof:** Taking the first derivative of \(x\) in (167) and setting \(x = 1\), we have

\[
\sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} (n-r+k) = n \cdot 2^{m+n-1}.
\]

(169)

Note that

\[
\sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} (n-r) = \sum_{r=0}^{m+n} \binom{m+n}{r} (n-r) = n \cdot 2^{m+n} - (m+n) \cdot 2^{m+n-1}
\]

(170)

\[
= (n-m) \cdot 2^{m+n-1}.
\]

(171)

Plugging (172) in (169), it follows that

\[
\sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} k = n \cdot 2^{m+n-1} - (n-m) \cdot 2^{m+n-1}
\]

(173)

\[
= m \cdot 2^{m+n-1}.
\]

(174)

**Theorem 14 (Second-order identity):** For any \(m, n \in \mathbb{Z}^+\), it holds that

\[
\sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} k^2 = m(m+1) \cdot 2^{m+n-2}.
\]

(175)

**Proof:** Taking the second derivative of \(x\) in (167) and setting \(x = 1\), we have

\[
n(n-1) \cdot 2^{m+n-2} = \sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} (n-r+k)(n-r+k-1)
\]

(176)

\[
= \sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} \left[ k^2 + (2n-1)k - 2r + (n-r)(n-r-1) \right].
\]

(177)
Also, taking the first derivative of \( y \) in \((166)\) and setting \( y = 1 \), then taking the first derivative of \( x \) and setting \( x = 1 \), we have

\[
\sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} (nr - r^2 + rk) = [m + (m + n)(n - 1)] \cdot 2^{m+n-2}.
\]  \((178)\)

Thus, it follows from \((178)\) that

\[
\sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} r^k = [m + (m + n)(n - 1)] \cdot 2^{m+n-2} - \sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} (nr - r^2)
\]

\[
= [m + (m + n)(n - 1) - (n^2 - m^2 - m - n)] \cdot 2^{m+n-2}
\]

\[
= (m^2 + mn + m) \cdot 2^{m+n-2}.
\]  \((179)\)

Therefore, by plugging \((181)\) and \((168)\) in \((177)\), we derive that

\[
\sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} k^2 = n(n-1) \cdot 2^{m+n-2} - \sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} [(2n-1)k - 2rk + (n-r)(n-r-1)]
\]

\[
= n(n-1) \cdot 2^{m+n-2} - (2n-1)m \cdot 2^{m+n-1} + 2(m^2 + mn + m) \cdot 2^{m+n-2}
\]

\[
- (n^2 + m^2 - 2mn - n + 3m) \cdot 2^{m+n-2}
\]

\[
=m(m+1) \cdot 2^{m+n-2}.
\]  \((180)\)

\[
=m(m+1) \cdot 2^{m+n-2}.
\]  \((181)\)

\[\]

**APPENDIX D**

**The Expansion of \((1 + x)^m(y + x)^n(z + x)^t\)**

Consider the expansion of \((1 + x)^m(y + x)^n(z + x)^t\),

\[
(1 + x)^m(y + x)^n(z + x)^t = \left( \sum_{i=0}^{m} \binom{m}{i} x^i \right) \left( \sum_{j=0}^{n} \binom{n}{j} y^n j x^j \right) \left( \sum_{k=0}^{t} \binom{t}{k} z^l x^k \right)
\]

\[
= \sum_{r=0}^{m+n+t} \sum_{k=0}^{r} \sum_{l=0}^{r-k} \binom{m}{r-k} \binom{n}{r-k-l} \binom{t}{l} y^n z^l x^r.
\]  \((186)\)

By setting \( x = 1 \), we have

\[
2^m(y + 1)^n(z + 1)^t = \sum_{r=0}^{m+n+t} \sum_{k=0}^{r} \sum_{l=0}^{r-k} \binom{m}{r-k-l} \binom{n}{r-k-l} \binom{t}{l} y^n z^l x^r.
\]

\[
= \sum_{r=0}^{m+n+t} \sum_{k=0}^{r} \sum_{l=0}^{r-k} \binom{m}{r-k-l} \binom{n}{r-k-l} \binom{t}{l} y^n z^l x^r.
\]  \((187)\)

\[\]

**Theorem 15:** For any \( m, n, t \in \mathbb{Z}^+ \),

\[
\sum_{r=0}^{m+n+t} \sum_{k=0}^{r} \sum_{l=0}^{r-k} \binom{m}{r-k-l} \binom{n}{r-k-l} \binom{t}{l} k l = nt \cdot 2^{m+n+t-2}.
\]  \((188)\)
Proof: Taking the first derivative of \( y \) in (188) and setting \( y = 1 \), then repeating the same procedure for \( z \), we have

\[
nt \cdot 2^{m+n+t-2} = \sum_{r=0}^{m+n+t} \sum_{k=0}^{r-k} \left( \sum_{l=0}^{r-k} \left( \binom{m}{r-k-l} \binom{n}{k} \binom{t}{l} (n-k)(t-l) \right) \right) \kappa = n \cdot 2^{m+n+t-1}
\]

(190)

Also, note that by taking the first derivative of \( y \) and \( z \) respectively in (188) and then setting \( y = z = 1 \), we can derive that

\[
\sum_{r=0}^{m+n+t} \sum_{k=0}^{r-k} \left( \sum_{l=0}^{r-k} \left( \binom{m}{r-k-l} \binom{n}{k} \binom{t}{l} \right) \right) l = t \cdot 2^{m+n+t-1}.
\]

(193)

Therefore, by plugging (192) and (193) in (191), we have

\[
\sum_{r=0}^{m+n+t} \sum_{k=0}^{r-k} \left( \sum_{l=0}^{r-k} \left( \binom{m}{r-k-l} \binom{n}{k} \binom{t}{l} \right) \right) kl = nt \cdot 2^{m+n+t-2} - \sum_{r=0}^{m+n+t} \sum_{k=0}^{r-k} \left( \binom{m}{r-k-l} \binom{n}{k} \binom{t}{l} (nt-nl-tk) \right) \kappa = nt \cdot 2^{m+n+t-2}.
\]

(195)

(196)

(197)

\section*{APPENDIX E}

\section*{MISCELLANEOUS}

\textbf{Theorem 16:} For any \( M \in \mathbb{Z}^+ \) and \( n \geq \lceil \log_2(M) \rceil, n \in \mathbb{Z} \),

\[
M^2 \log_2(M) \leq 4 \sum_{i=1}^{n} \left( M - \sum_{k=0}^{2^{i-1}-1} \left\lfloor \frac{M+k}{2^i} \right\rfloor \right) \left( \sum_{k=0}^{2^{i-1}-1} \left\lfloor \frac{M+k}{2^i} \right\rfloor \right) < M^2 \left( \log_2(M) + \frac{2 \ln 2 - 1}{2 \ln 2} \right), \tag{198}
\]

where the equality holds if and only if \( \log_2(M) \in \mathbb{Z} \).

\textbf{Proof:} First, it can be verified that, for each \( i = 1, 2, \ldots, n \),

\[
\sum_{k=0}^{2^{i-1}-1} \left\lfloor \frac{M+k}{2^i} \right\rfloor = \begin{cases} 2^{i-1} \left\lfloor \frac{M}{2^i} \right\rfloor, & \text{if } \left\lfloor \frac{M}{2^i} \right\rfloor \text{ mod } 2 = 0; \\ M - 2^{i-1} \left\lfloor \frac{M}{2^i} \right\rfloor, & \text{if } \left\lfloor \frac{M}{2^i} \right\rfloor \text{ mod } 2 = 1, \end{cases} \tag{199}
\]

since we can count the number of bit 1’s by chunk of length \( 2^i \) which contains \( 2^{i-1} \) bit 1’s.
Thus, if $M = 2^n$,

$$
4 \sum_{i=1}^{n} \left( M - \sum_{k=0}^{2^i-1} \left\lfloor \frac{M + k}{2^i} \right\rfloor \right) \left( \sum_{k=0}^{2^i-1} \left\lfloor \frac{M + k}{2^i} \right\rfloor \right) \tag{200}
$$

$$
= 4 \sum_{i=1}^{[\log_2(M)]} \left( M - 2^{i-1} \left\lfloor \frac{1}{2} + \frac{M}{2^i} \right\rfloor \right) \left( 2^{i-1} \left\lfloor \frac{1}{2} + \frac{M}{2^i} \right\rfloor \right) \tag{201}
$$

$$
= 4 \sum_{i=1}^{[\log_2(M)]} \left( M - \frac{1}{2} M \right) \left( \frac{1}{2} M \right) \tag{202}
$$

$$
= M^2 \log_2(M), \tag{203}
$$

which shows that the lower bound is attained when $M = 2^n$.

Define

$$
a(m) \triangleq \sum_{i=1}^{m+1} \left( m + 1 - 2^{i-1} \left\lfloor \frac{1}{2} + \frac{m + 1}{2^i} \right\rfloor \right) \left( 2^{i-1} \left\lfloor \frac{1}{2} + \frac{m + 1}{2^i} \right\rfloor \right), \tag{204}
$$

thus we have in [9] that

$$
a(0) = 0 \tag{205}
$$

$$
a(2m) = 2a(m) + 2a(m-1) + m(m+1) \tag{206}
$$

$$
a(2m+1) = 4a(m) + (m+1)^2. \tag{207}
$$

So, it is equivalent to proving that, for $m \geq 0, m \in \mathbb{Z}$,

$$
\frac{1}{4}(m+1)^2 \log_2(m+1) \leq a(m) < \frac{1}{4}(m+1)^2 \left( \log_2(m+1) + \frac{2\ln 2 - 1}{2\ln 2} \right). \tag{208}
$$

When $m = 0, 1$ and $m = 2$, apparently, the inequality holds. Assume when $m \leq M, M \geq 3$, the inequality holds for any $m$. Consider $m = M + 1$. We first establish the lower bound.

1) If $M$ is odd, then $m$ is even and we apply (206).

$$
a(M+1) = 2a \left( \frac{M+1}{2} \right) + 2a \left( \frac{M-1}{2} \right) + \left( \frac{M+1}{2} \right) \left( \frac{M+3}{2} \right) \tag{209}
$$

$$
\geq \frac{1}{2} \left( \frac{M+3}{2} \right)^2 \log_2 \left( \frac{M+3}{2} \right) + \frac{1}{2} \left( \frac{M+1}{2} \right)^2 \log_2 \left( \frac{M+1}{2} \right) + \left( \frac{M+1}{2} \right) \left( \frac{M+3}{2} \right) \tag{210}
$$

$$
\geq \frac{1}{4}(M+2)^2 \log_2(M+2), \tag{211}
$$

where (210) to (211) holds by the induction hypothesis, (211) to (212) follows from the following fact.

Define

$$
f(x) \triangleq \frac{1}{2} \left( \frac{x+3}{2} \right)^2 \log_2 \left( \frac{x+3}{2} \right) + \frac{1}{2} \left( \frac{x+1}{2} \right)^2 \log_2 \left( \frac{x+1}{2} \right) + \left( \frac{x+1}{2} \right) \left( \frac{x+3}{2} \right)
$$

$$
- \frac{1}{4}(x+2)^2 \log_2(x+2) \tag{213}
$$

$$
= \frac{1}{8}(x+3)^2 \log_2(x+3) + \frac{1}{8}(x+1)^2 \log_2(x+1) - \frac{1}{4}(x+2)^2 \log_2(x+2) - \frac{1}{2}. \tag{214}
$$
Thus,
\[ f'(x) = \frac{1}{4}(x + 3) \log_2(x + 3) + \frac{1}{4}(x + 1) \log_2(x + 1) - \frac{1}{2}(x + 2) \log_2(x + 2) \geq 0, \quad (215) \]
which follows from Jensen’s inequality and \( x^2 \log_2 x \) is convex. Hence,
\[ f(x) \geq f(0) = \frac{3}{8} \log_2 \left( \frac{27}{16} \right) > 0. \quad (216) \]

2) If \( M \) is even, then \( m \) is odd and we apply (207),
\[ a(M + 1) = \frac{4}{2}a \left( M \right) + \left( \frac{M + 2}{2} \right)^2 \]
\[ \geq \left( \frac{M + 2}{2} \right)^2 \log_2 \left( \frac{M + 2}{2} \right) + \left( \frac{M + 2}{2} \right)^2 \]
\[ = \frac{1}{4} (M + 2)^2 \log_2 (M + 2). \quad (219) \]
Therefore, the lower bound also holds when \( m = M + 1 \).

Next, we establish the upper bound by proving an even tighter upper bound for \( m \geq 6 \). (For \( 1 \leq m \leq 5 \), we can verify that the original upper bound in (208) holds)
\[ a(m) \leq \frac{1}{4} (m + 1)^2 \left( \log_2(m + 1) + \frac{(m - 1)b}{m} \right). \quad (220) \]
where \( b \triangleq \frac{2 \ln 2 - 1}{2 \ln 2} \).

First, it can be verified that when \( m = 6 \) and \( m = 7 \), (220) holds. Assume (220) holds when \( m \leq M, M \geq 7 \). Consider \( m = M + 1 \).

1) If \( M \) is odd, then \( m \) is even and we apply (206).
\[ a(M + 1) \]
\[ = 2a \left( \frac{M + 1}{2} \right) + 2a \left( \frac{M - 1}{2} \right) + \left( \frac{M + 1}{2} \right) \left( \frac{M + 3}{2} \right) \]
\[ \leq \frac{1}{2} \left( \frac{M + 3}{2} \right)^2 \log_2 \left( \frac{M + 3}{2} \right) + \left( \frac{M - 1}{M + 1} \right) \]
\[ + \left( \frac{M + 1}{2} \right) \left( \frac{M + 3}{2} \right) \]
\[ \leq \frac{1}{4} (M + 2)^2 \log_2 (M + 2) + \frac{Mb}{M + 1}, \quad (224) \]
where (223) to (224) follows from the following fact.

Define
\[ f(x) \triangleq \frac{1}{2} \left( \frac{x + 3}{2} \right)^2 \log_2 \left( \frac{x + 3}{2} \right) + \left( \frac{x - 1}{x + 1} \right) \]
\[ + \left( \frac{x + 1}{2} \right) \left( \frac{x + 3}{2} \right) - \frac{1}{4} (x + 2)^2 \log_2(x + 2) + \frac{bx}{x + 1} \]
\[ = \frac{1}{8} (x + 3)^2 \log_2(x + 3) + \frac{1}{8} (x + 1)^2 \log_2(x + 1) - \frac{1}{4} (x + 2)^2 \log_2(x + 2) \]
\[ - \left[ (1 - b)x + (1 + b) \right] (x + 3)^2 \]
\[ - \left[ (1 - b)x + (3b - 1) \right] (x + 1)^2 \]
\[ - \frac{bx(x + 2)^2}{4(x + 1)} + \frac{1}{4} (x + 1)(x + 3). \quad (225) \]
Thus,
\[
    f'(x) = \frac{1}{4}(x+3)\log_2(x+3) + \frac{1}{4}(x+1)\log_2(x+1) - \frac{1}{2}(x+2)\log_2(x+2) \\
    - \frac{1}{4}(1-b)x^2 + (2-b)x + (1-2b)(x+3) - \frac{1}{2}(1-b)x^2 + (3b-2)x - 4b \quad (x+1) \\
    - \frac{1}{4}(2bx^2 + 3bx + 2b)(x+2) + x + \frac{2}{2}. 
\]

(227)

It can be verified that \( f'(x) \) is a monotonically decreasing function of \( x \) and \( f'(x) < 0 \) as \( x \geq 7 \). Thus, for any \( m \in \mathbb{Z}^+ \), we have
\[
    f(m) \leq f(7) \approx -0.00574 < 0. 
\]

(228)

2) If \( M \) is even, then \( m \) is odd and we apply (207),
\[
    a(M+1) = 4a\left(\frac{M}{2}\right) + \left(\frac{M+2}{2}\right)^2 \leq \left(\frac{M+2}{2}\right)^2 \left(\log_2\left(\frac{M+2}{2}\right) + \frac{(M-2)b}{M}\right) + \left(\frac{M+2}{2}\right)^2 \\
    = \frac{1}{4}(M+2)^2 \left(\log_2(M+2) + \frac{(M-2)b}{M}\right) \\
    
\]

(230)

(231)

Therefore, (220) holds when \( m = M+1 \). Thus for \( m \geq 6 \),
\[
    a(m) \leq \frac{1}{4}(m+1)^2 \left(\log_2(m+1) + \frac{(m-1)b}{m}\right) < \frac{1}{4}(m+1)^2 \left(\log_2(m+1) + b\right). 
\]

(233)

In summary, both the lower bound and upper bound hold, which follows that (198) holds.

\[\square\]

REFERENCES

[1] G. R. Kumar and T. A. Courtade, “Which boolean functions are most informative?” in 2013 IEEE International Symposium on Information Theory, July 2013, pp. 226–230.

[2] A. Samorodnitsky, “On the entropy of a noisy function,” IEEE Transactions on Information Theory, vol. 62, no. 10, pp. 5446–5464, Oct 2016.

[3] O. Ordentlich, O. Shayevitz, and O. Weinstein, “An improved upper bound for the most informative boolean function conjecture,” in 2016 IEEE International Symposium on Information Theory (ISIT), July 2016, pp. 500–504.

[4] W. Huleihel and O. Ordentlich, “How to quantize n outputs of a binary symmetric channel to n-1 bits?” in 2017 IEEE International Symposium on Information Theory (ISIT), June 2017, pp. 91–95.

[5] E. Erkip and T. M. Cover, “The efficiency of investment information,” IEEE Transactions on Information Theory, vol. 44, no. 3, pp. 1026–1040, May 1998.

[6] A. Wyner and J. Ziv, “A theorem on the entropy of certain binary sequences and applications–i,” IEEE Transactions on Information Theory, vol. 19, no. 6, pp. 769–772, November 1973.

[7] V. Anantharam, A. A. Gohari, S. Kamath, and C. Nair, “On hypercontractivity and the mutual information between boolean functions,” in 2013 51st Annual Allerton Conference on Communication, Control, and Computing (Allerton), Oct 2013, pp. 13–19.

[8] J. Li and M. Medard, “Boolean functions: noise stability, non-interactive correlation, and mutual information.” [Online]. Available: [http://arxiv.org/abs/1801.0462v3]

[9] A. Kundgen, “The on-line encyclopedia of integer sequences, 2003, sequence a022560.”