Central limit theorem and estimates for the spectral radius in random matrix products theory

Aoun Richard

Abstract

Let $\mu$ be a probability measure on the general linear group $\text{GL}_d(\mathbb{R})$ whose support generates a strongly irreducible sub-semigroup. In this note, we give some new estimates for the random walk $L_n = X_n \cdots X_1$ when $\mu$ has only a finite first moment. In particular, we prove that for any sequence $(\epsilon_n)_{n \in \mathbb{N}}$ that tends to zero, the ratio between the spectral radius $\rho(L_n)$ of $L_n$ and its norm $||L_n||$ is bigger than $\epsilon_n$ with a probability tending to one. Such an information requires the study of the probability of return to neighborhoods of hyperplanes of exponential size, the regularity of the unique stationary probability measure on the projective space in the strongly irreducible and proximal case and the asymptotic independence between the right and left random walk under this finite moment assumption. More general results on the position between the Cartan and Jordan projection of a random walk on a Zariski dense sub-semigroup of the real points of an algebraic reductive group defined over $\mathbb{R}$ are stated.

As an application, and using Benoist-Quint’s central limit theorem for $\ln ||L_n||$ [BQ16a, Theorem 1.1], we deduce a central limit theorem for $\ln \rho(L_n)$ when $\mu$ has a moment of order two. This result extends [BQ16c, Theorem 13.22] where this result is proved under an exponential moment of $\mu$.

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1 Statement of the result

Let $V$ be a real vector space of dimension $d \geq 1$ and $|| \cdot ||$ a norm on $V$. For simplicity of notation, the operator norm on $\text{End}(V)$ is still denoted by $|| \cdot ||$. For every $g \in \text{GL}(V)$, we denote by $\rho(g)$ the spectral radius of $g$. If $\mu$ is a probability measure on $\text{GL}(V)$, we say that $\mu$ has moment of order $i \in \mathbb{N}$ if

$$\int (\ln ||g^{\pm 1}||)^i \, d\mu(g) < +\infty.$$  

The right random walk at time $n \in \mathbb{N}$ will be denote by $R_n = X_1 \cdots X_n$ (resp. $L_n = X_n \cdots X_1$) where $(X_i)_{i \in \mathbb{N}}$ is a family of independent and identically distributed random variables of law $\mu$. All our random variables will be defined on a probabilistic space $(\Omega, \mathcal{F}, P)$ and the expectation operator is denoted by $E$.

Let $\lambda_1(\mu)$ the top Lyapunov exponent of $\mu$, i.e.

$$\lambda_1(\mu) = \lim_{n \to +\infty} \frac{1}{n} \ln ||L_n||.$$  

1 Statement of the result
The previous limit is understood in the almost sure mode of convergence. Its existence can be proved by an application of Kingman’s ergodic subadditive theorem. We denote by $\Gamma_\mu$ the semi-group generated by the support of $\mu$. We say that $\Gamma_\mu$ is strongly irreducible if it does not stabilize a finite union of proper subspaces of $V$. Benoist-Quint proved in [BQ16a, Theorem 1.1] a central limit theorem (CLT) for $\ln ||L_n||$ under the natural moment assumption of moment of order two. One of our results in this note gives the equivalent statement for $\ln \rho(L_n)$, namely:

**Theorem 1.1.** Let $\mu$ be a probability measure on $GL(V)$ such that:

- $\mu$ has a moment of order two
- $\Gamma_\mu$ is strongly irreducible and has unbounded image in $PGL(V)$.

Then there exists $\sigma_\mu > 0$ such that the following convergence in law holds:

$$\frac{\ln \rho(L_n) - n \lambda_1(\mu)}{\sqrt{n}} \xrightarrow{n \to +\infty} N(0, \sigma_\mu).$$

This theorem extends [BQ16c, Theorem 13.22] where a CLT for $\ln$ proper subspaces of $V$ by the support of $\mu$ gives the equivalent statement for $\ln \rho(L_n)$, presumably under an exponential moment of $\mu$.

**Remark 1.2.** It will follow from the proof that the limit distribution $N(0, \sigma_\mu)$ is the same as for $\ln ||L_n||$, given by [BQ16a, Theorem 1.1].

In view of the known CLT for the $\ln ||L_n||$, Theorem 1.1 reduces to proving that $\frac{1}{\sqrt{n}} \ln \frac{\rho(L_n)}{||L_n||}$ converges in probability to zero when $\mu$ has a moment of order two. We will actually give estimates of the ratio $\frac{\rho(L_n)}{||L_n||}$ only with the assumption of a moment of order one for $\mu$. Our main result reads as follows:

**Theorem 1.3.** Let $\mu$ be a probability measure on $GL(V)$ such that:

- $\mu$ has a moment of order one
- $\Gamma_\mu$ is strongly irreducible

Then for every numerical sequence $(\epsilon_n)_{n \in \mathbb{N}^*}$ that tends to zero,

$$\mathbb{P} \left( \frac{\rho(L_n)}{||L_n||} \leq \epsilon_n \right) \xrightarrow{n \to +\infty} 0. \quad (2)$$

**Remark 1.4.**

1. The statement above is equivalent to saying that for every numerical sequence $(\epsilon_n)_{n \in \mathbb{N}^*}$ that tends to zero, the sequence of random variables $(\epsilon_n \frac{\ln ||L_n||}{\rho(L_n)})_{n \in \mathbb{N}^*}$ tends in probability to 0.

2. One can replace $L_n$ by $R_n$ in both theorems 1.3 and 1.1 above as they have the same law for every $n$.

Using the representation theory of reductive algebraic groups, we easily deduce a central limit theorem for the Jordan projection of a random walk in a Zariski dense sub-semigroup of a reductive real algebraic group: Theorem 1.5 below. Before stating the result, we recall standard notion of reductive groups (we refer for instance to [Kna02]). Let $G$ be a linear reductive algebraic group assumed to be Zariski connected and denote by $G = G(\mathbb{R})$ its group of real points. We denote by $K$ a maximal compact subgroup of $G$, $a$ the Lie algebra of a maximal $\mathbb{R}$-split torus $A$ with $\mathfrak{a}^+$ the positive Weil chamber, i.e. the cone in $a$ defined by the requirement that all positive roots be non-negative. Let $A^+ = \exp(\mathfrak{a}^+)$. One has that $G = KA^+K$ called Cartan or KAK decomposition. The $A^+$-component of the decomposition is an element of $G$ in this product is unique. This yields the so-called Cartan projection $\kappa : G \to a^+$.

Recall also the Jordan decomposition: any $g \in G$ can be written as a commuting product of a unipotent element, an elliptic element and a hyperbolic element (i.e. an element with a conjugate in $A$). One can then define the Jordan projection $\ell : G \to a^+$ where $\ell(g)$ is the unique element of $a^+$ such that $\exp(\ell(g))$ is conjugate to the hyperbolic part of $g$ in the Jordan decomposition of $g$.

Let now $\mu$ be a probability measure on $G$. We say that $\mu$ has a moment of order $p \geq 1$ if for some, or equivalently any, faithful representation $\phi : G \to GL_n(\mathbb{R})$ of $G$, $\phi(\mu)$ has a moment of order $p$. Let now $(L_n)_{n \geq 1}$ be the left random walk on $G$ associated to $\mu$. The equivalent formulation of (1) reads as follows: when $\mu$ has a moment of order one, the vector $\kappa(L_n)$ converges almost surely to a non random element $\lambda_\mu \in a^+$, called the Lyapunov vector of $\mu$. 

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**Theorem 1.5.** Let \( G \) be a reductive real algebraic group. \( G \) its group of real points, \( \Gamma \) a Zariski dense sub-semigroup of \( G \). Consider a probability measure \( \mu \) on \( \Gamma \) whose support generates \( \Gamma \). Assume that \( \mu \) has a moment of order one. Then, for every sequence of real numbers \( (\epsilon_n)_{n \in \mathbb{N}} \) that converges to zero, the following convergence in probability holds in \( \mathfrak{a}^+ \):

\[
\epsilon_n (\kappa(L_n) - \ell(L_n)) \xrightarrow{\mathbb{P}_{\mu}} 0. \tag{3}
\]

In particular, if \( \mu \) has a moment of order two, then the following convergence in law holds:

\[
\frac{\ell(L_n) - n\lambda^\mu}{\sqrt{n}} \xrightarrow{\mathbb{P}} N(0, \sigma^\mu), \tag{4}
\]

where \( N(0, \sigma^\mu) \) is a Gaussian distribution on the vector space \( \mathfrak{a} \) whose support is a subspace \( \mathfrak{a}_\mu \) containing the intersection of \( \mathfrak{a} \) with the Lie algebra of the derived group of \( G \). In particular, \( \mathfrak{a}_\mu = \mathfrak{a} \) if \( G \) is semisimple.

The convergence in law of \( \frac{\kappa(L_n) - n\lambda^\mu}{\sqrt{n}} \) to a Gaussian law has been proved by Benoist-Quint in [BQ16a, Theorem 4.16]. As in Remark 1.2, we will obtain that \( \frac{\kappa(L_n) - n\lambda^\mu}{\sqrt{n}} \) and \( \frac{\ell(L_n) - n\lambda^\mu}{\sqrt{n}} \) convergence in law to the same Gaussian law.

**Example 1.6.** Consider \( G = SL_d \). This is a semisimple real algebraic connected group. In this case, \( G = SL_d(\mathbb{R}) \), \( A \) (resp. \( A^+ \)) is the subgroup of \( G \) of diagonal matrices with real positive entries (resp. ordered by decreasing order). The Cartan (resp. Jordan) decomposition is the map that sends each \( g \in G \) to the \( d \)-vector of the logarithms of the singular values (resp. moduli of the eigenvalues of \( g \)) ordered by decreasing order.

Consider now a Zariski dense sub-semigroup \( \Gamma \) of \( G \), as for instance \( SL_d(\mathbb{Z}) \), a probability measure \( \mu \) on \( G \) whose support generates all of \( \Gamma \) and that has a moment of order two. In this case \( \lambda^\mu = (\lambda_1, \ldots, \lambda_d) \) is the vector of the \( d \)-Lyapunov exponents. Denote by \((\lambda_{a,1}, \ldots, \lambda_{a,d})\) the logarithms of the moduli of the eigenvalues of the left random walk \( L_n \) ordered by decreasing order. Theorem 1.5 implies then the convergence in law of

\[
\frac{1}{\sqrt{n}} (\lambda_{n,1} - n\lambda_1, \ldots, \lambda_{n,d} - n\lambda_d)
\]

to a Gaussian law in \( \mathbb{R}^d \) with full support. Note that in this case, since \( G \) acts strongly irreducibly and proximally on \( \bigwedge^k(\mathbb{R}^d) \) for any \( k = 1, \ldots, d - 1 \), we have that \( \lambda_1 > \cdots > \lambda_d \) (see next remark). In particular, \( \mathbb{P}(\forall i, \lambda_{a,i} \text{ is real}) \xrightarrow{n \to +\infty} 1 \).

**Remark 1.7.** Some remarks on the Lyapunov vector \( \lambda^\mu \in \mathfrak{a}^+ \). Adopt the same assumptions as the previous theorem.

1. The Lyapunov vector lies actually in the open Weyl chamber \( \mathfrak{a}^{++} \). This can be proved by combining Guivarc'h-Raugi’s theorem [GR85] on the simplicity of the Lyapunov spectrum together with Goldsheid-Margulis’s result [GM89] concerning the existence of proximal elements in Zariski dense subgroups of real algebraic groups with proximal elements.

2. Assume now that \( \mu \) has a moment of order two. Breuillard and Sert refined recently the previous result. Indeed, they proved in [BS18, Theorem 1.9] that \( \lambda^\mu_a \) lies in the interior of the Benoist cone of \( \Gamma \) introduced by Benoist [Ben97] (it is the closure in \( \mathfrak{a}^+ \) of the positive linear combinations of \( f(g) \), \( g \in \Gamma \)). Actually the authors proved that \( \lambda^\mu_a \) lies in the relative interior of the joint spectrum of \( \Gamma \) (a convex body of \( \mathfrak{a}^+ \) that the authors define).

**Remark 1.8.** All the methods used in this paper remain true over arbitrary local fields without any substantial difficulty. However, the non-degeneracy of the gaussian law in the limit is more delicate as it is explained in [BQ16c, Example 12.21]. More precisely:

1. Theorems 1.3 remains true over any local field.

2. Theorems 1.1 and 1.5 remain true over any local field (once the Cartan and Jordan projection are well defined), except that the Gaussian law can be degenerate (i.e. in Theorem 1.1 \( \sigma_\mu \) can be zero and the last sentence of Theorem 1.5 concerning the support of \( \sigma_\mu \) may no longer be true).

3. Also, when the local field is not \( \mathbb{R} \), the Lyapunov vector \( \lambda^\mu_a \) may no longer be in \( \mathfrak{a}^{++} \).
2 Preliminary reduction

In this section, we reduce the proof of Theorem 1.3 and then of Theorem 1.1 to the following Theorem 2.1 below which says essentially that the distance between the attracting point of \( L_n \) is fairly far from its repelling hyperplane.

First, we introduce our notation. Without loss of generality, we take \( V = \mathbb{R}^d \). Let \( P(V) \) be the projective space of \( V \). For every non zero vector \( v \) (resp. non zero subspace \( E \)) of \( V \), we denote by \([v] = \mathbb{R}v \) (resp. \([E] \)) its projection onto \( P(V) \). The action of \( g \in \mathrm{GL}(V) \) on a vector \( v \) will be simply denoted by \( gv \), while the action of \( g \) on a point \( x \in P(V) \) will be denoted by \( g \cdot x \).

We endow \( V \) with the canonical basis \((e_1, \ldots, e_d)\) and the usual Euclidean dot product and norm. Let \( K = O_d(\mathbb{R}) \) be the orthogonal group. Denote by \( \Lambda \subset \mathrm{GL}_d(\mathbb{R}) \) the subgroup of diagonal matrices. The KAK decomposition (or polar decomposition) states that \( \mathrm{GL}_d(\mathbb{R}) = K A^+ K \). For every \( g \in \mathrm{GL}(V) \), we denote by \( g = k_g a_g u_g \) a KAK decomposition of \( g \) in a fixed basis \((e_1, \ldots, e_d)\) of \( g \). We call attracting point and repelling hyperplane the following respective point in \( P(V) \) and projective hyperplane of \( P(V) \):

\[
v_g^+ = k_g [e_1] , \quad H_g^- = \ker[ u_g^{-1} e_1 ] = (\mathbb{R} [ u_g^{-1} e_1 ])^{\perp}.
\]

In the definitions above, \((e_1, \ldots, e_d)\) denotes the dual basis of \((e_1, \ldots, e_d)\) in the dual vector space \( V^* \) of \( V \). Also \( \mathrm{GL}(V) \) acts on \( V^* \) by \((g f)(x) = f(g^{-1} x), g \in \mathrm{GL}(V), f \in V^* \) and \( x \in V \).

Endow the vector space \( \Lambda^2 V \) with the canonical norm associated to the basis \((e_i \wedge e_j)_{1 \leq i < j \leq d} \). We endow \( P(V) \) with the Fubini-Study metric \( \delta \) defined by:

\[
\forall x = [v], y = [w] \in P(V), \delta(x, y) := \|v \wedge w\|/\|v\||\|w\|.
\]

Finally, an endomorphism \( g \in \mathrm{End}(V) \) is said to be proximal if it has a unique eigenvalue with maximal modulus and a sub-semigroup \( \Gamma \) of \( \mathrm{GL}(V) \) is said to be proximal if it contains a proximal element.

We are now able to state our main technical result:

**Theorem 2.1.** Assume that \( \mu \) has a moment of order one and that \( \Gamma_n \) is strongly irreducible and proximal. Then for any sequence of real numbers \((\epsilon_n)\) such that \( \epsilon_n \rightarrow 0 \),

\[
\mathbb{P} \left( \delta(v_g^+, H_{\epsilon_n}^-) \leq \epsilon_n \right) \rightarrow 0.
\]

In order to prove Theorem 1.3 modulo Theorem 2.1, we need the following geometric lemma borrowed from Breuillard [Bre08, Lemma 4.7] and Benoist-Quint [BQ16c, Lemma 13.14].

**Lemma 2.2.** ([Bre08, Lemma 4.7] and Benoist-Quint [BQ16c, Lemma 13.14])

Let \( g \in \mathrm{GL}(V) \). If \( \delta(v_g^+, H_g^-) > 2 \sqrt{\frac{\pi}{12}} \), then

\[
\rho(g) \geq \frac{\delta(v_g^+, H_g^-)}{2}.
\]

Moreover, in this case, \( g \) is necessary a proximal element.

For the convenience of the reader, we include a proof.

**Proof.** Fix \( g \in \mathrm{GL}(V) \). To simplify the notation, let \( \delta_g := \delta(v_g^+, H_g^-) \). For every \( \epsilon > 0 \), let \( U_\epsilon \subset P(V) \) be the complement of the \( \epsilon \)-neighborhood around \( H_g^- \), i.e.

\[
U_\epsilon := \{x \in P(V) ; \delta(x, H_g^-) \geq \epsilon\}.
\]

The following statements are easy to verify using the definition of the Cartan decomposition and the Fubini-Study metric (except the statement i. which is trivial and follows just from the reversed triangular inequality).
By (5) and the admitted Lemma (2.1), we deduce that the quantity above converges to zero.

The previous convergence is in particular true in the probability mode of convergence. Hence for every $$\epsilon > 0$$, we have

$$\sup_{x,y \in U_r} \frac{\delta(g \cdot x, g \cdot y)}{\delta(x,y)} \leq \frac{a_{2,g}}{a_{1,g}} \frac{1}{\epsilon^2}.$$  

Since the family $$(U_r)_{r>0}$$ is decreasing, we deduce from observations i. and ii. above that $$U_r$$ is stabilized by $$g$$ as soon as

$$\frac{a_{2,g}}{a_{1,g}} \frac{2}{\delta_g} \leq \epsilon \leq \frac{\delta_g}{2}.$$  

From now, we assume that $$\delta_g > 2\sqrt{\frac{a_{2,g}}{a_{1,g}}}$$ and we set $$\epsilon = \delta_g/2$$. With these assumptions, inequality iv. implies that the action of $$g$$ on $$U_r$$ is contracting. By completeness of the closed subset $$U_r$$ of the complete metric space $$P(V)$$, we deduce that this action has a unique fixed point $$x^*_r \in U_r$$. This fixed point corresponds to an eigenvalue $$\lambda$$ of $$g$$. By iii. we have

$$\left| \frac{\lambda}{\|g\|} \right| \geq \epsilon = \delta_g/2.$$  

A fortiori, the spectral radius $$\rho(g)$$ of $$g$$ satisfies the desired inequality. Actually, one can prove using the Jordan decomposition and following [Bre08], that $$g$$ is proximal and that $$\lambda$$ is the top eigenvalue of $$g$$. 

Proof of Theorem 1.3 modulo Theorem 2.1: First, we show that we can assume without loss of generality that $$\Gamma_\mu$$ is strongly irreducible and proximal (i-p to abbreviate). Indeed, let $$p \in \{1, \ldots, d\}$$ be the index of $$\Gamma_\mu$$, i.e. the least integer $$k \in \{1, \ldots, d\}$$ such that there exists a sequence of scalar $$\lambda_n \in \mathbb{R}$$ and of elements $$g_n \in \Gamma_\mu$$ such that $$\lambda_n g_n$$ converges in $$\text{End}(V)$$ to an endomorphism of rank $$k$$. By [BQ16a, Lemma 4.13 ] there exist $$\Gamma_\mu$$-invariant subspaces $$U$$ and $$W$$ of $$\bigwedge^p V$$ such that $$U \subset W \subset \bigwedge^p V$$ such that the action of $$\Gamma_\mu$$ on $$W' := W/U$$ is i-p and that such that $$\{ ||g||^{p} \}_{g \in \Gamma_\mu}$$ is bounded, where $$\pi$$ is the morphism action of $$\text{GL}(V)$$ on $$W'$$. Since $$\rho(\pi(g)) \leq \rho(\bigwedge^p g) \leq \rho(g)^p$$, then proving (2) for $$\Gamma_{\pi(\mu)}$$ is enough to prove the same estimate for $$\Gamma_\mu$$.

For now on $$\Gamma_\mu$$ is assumed to be i-p. For every $$n \in \mathbb{N}$$, let $$L_n = k_n a_n u_n$$ be a KAK decomposition of $$L_n$$, $$v_n^*$$ the attracting point of $$L_n$$, $$H_n^\ast$$ its repelling hyperplane and $$\delta_n := \delta(v_n^*, H_n^\ast)$$. Let $$\Omega_n \subseteq \Omega$$ be the following event

$$\Omega_n := \left\{ \omega \in \Omega; \delta_n^2(\omega) > 4 \frac{a_{2,n}(\omega)}{a_{1,n}(\omega)} \right\}.$$  

First, we check that

$$\mathbb{P}(\Omega_n) \longrightarrow 1.$$  

(5)

Indeed, by definition of the Lyapunov exponents, the following almost sure convergence holds:

$$\frac{1}{n} \ln \frac{a_{2,n}}{a_{1,n}} \xrightarrow{n \to +\infty} \lambda_2 - \lambda_1.$$  

The previous convergence is in particular true in the probability mode of convergence. Hence for $$\gamma := (\lambda_1(\mu) - \lambda_2(\mu))/2$$, the following inequality holds:

$$\mathbb{P} \left( \frac{a_{2,n}}{a_{1,n}} \leq \exp(-n\gamma) \right) \longrightarrow 1.$$  

Since $$\Gamma_\mu$$ is i-p, Guivarc'h-Raugi’s theorem [GR85] insures that $$\gamma > 0$$. Applying now the admitted Lemma 2.1 for $$\epsilon_n = 3 \exp(-n\gamma/2)$$, gives that with probability tending to one, $$\delta_n^2 > 9 \exp(-n\gamma) > 4 \frac{a_{2,n}}{a_{1,n}}$$, i.e. $$\mathbb{P}(\Omega_n) \longrightarrow 1.$$  

Let now $$(\epsilon_n)_{n \in \mathbb{N}}$$ be an arbitrary sequence that converges to zero. By Lemma (2.2), we have for every $$n \in \mathbb{N}^\ast$$,

$$\mathbb{P} \left( \frac{\rho(L_n)}{||L_n||} < \epsilon_n \right) = \mathbb{P}(\Omega \setminus \Omega_n) + \mathbb{P} \left( \delta(v_n^*, H_n^\ast) < 2\epsilon_n \right).$$  

By (5) and the admitted Lemma (2.1), we deduce that the quantity above converges to zero.
We easily deduce the proof of Theorem 1.1. We recall the classical Slutsky lemma in probability theory which asserts that if \((X_n)_{n \in \mathbb{N}}\) and \((Y_n)_{n \in \mathbb{N}}\) are two sequence of random variables such that \((X_n)_n\) converges in probability to a random variable \(X\) and \((Y_n)_{n \in \mathbb{N}}\) converges in probability to a constant \(c \in \mathbb{R}\), then the joint vector \((X_n, Y_n)\) converges in law to \((X, c)\).

**Proof of Theorem 1.1:** By Benoist-Quint’s central limit theorem [BQ16a, Theorem 1.1] for \(\ln\|L_n\|\) and Slutsky lemma, all we need to show is the following convergence in probability:

\[
Y_n := \frac{1}{\sqrt{n}} \ln \frac{\|L_n\|}{\rho(L_n)} \xrightarrow{\mathbb{P}} 0.
\]

This convergence is guaranteed by Theorem 1.3. 

We easily deduce Theorem 1.5 using the well established theory of representations of reductive groups. We refer for instance to [BT72].

**Proof of Theorem 1.5:** Let \(d\) be the real rank of \(G\) and \(d_S\) its semisimple rank. There exists a basis \(\{\chi_1, \ldots, \chi_d\}\) of the dual \(\mathfrak{n}^*\) such that each \(\chi_i\) is a highest weight of some irreducible representation \(V_i\) of \(G\) (all of these representations are also strongly irreducible by Zariski connectedness of \(G\)). Indeed, it is enough to concatenate the \(d_S\) fundamental weights to \(d - d_S\) characters of the abelianization \(G/[G, G]\) of \(G\). But if \((\psi, V)\) is an irreducible representation of \(G\) and \(\chi\) is a highest weight, then by using Mostow’s theorem [Mos55], we can find a norm \(\| \cdot \|\) (depending on \(\psi\) and \(V\)) on each \(V\) such that for every \(g \in G(\mathbb{R})\), \(\chi(k(g)) = \ln(||\psi(g)||)\) and \(\chi(t(g)) = \ln(\rho(\psi(g)))\). Applying now Theorem 1.3 on each \((V_i, \rho_i)\) proves (3).

We deduce (4) from (3) in the same way we deduced Theorem 1.1 from Theorem 1.3, i.e. using (3), Slutsky lemma and Benoist-Quint’s central limit theorem for the Cartan projection [BQ16a, Theorem 4.16].

\[\square\]

## 3 Estimates with a moment of order one

Here we check the validity of equivalent of Theorem 2.1 when \(H_{\Gamma_\mu}\) is replaced by a deterministic hyperplane \(H\) and make sure that our estimates are uniform on that hyperplane. In the next section, we prove Theorem 2.1. Although its simplicity, the passage from point 3. to point 4. in Proposition 3.3 below is one of the main points.

We recall that when \(\Gamma_\mu\) is strongly irreducible and proximal, Guivarc’h and Raugi proved in [GR85] that there exists a unique \(\mu\)-stationary probability measure \(\nu\) on \(P(V)\), i.e. a unique probability measure on \(P(V)\) such that for every continuous function on \(P(V)\), \(\int_{P(V)} f d\nu = \int_{GL(V) \times P(V)} f(g \cdot x) d\mu(g) d\nu(x)\). We will prove that \(\nu\) has some regularity in the following sense:

**Proposition 3.1.** Assume that \(\mu\) has a moment of order one and that \(\Gamma_\mu\) is strongly irreducible and proximal. Let \(\nu\) be the unique \(\mu\)-stationary probability measure on \(P(V)\). Then,

\[
\sup_{H \text{ projective hyperplane of } P(V)} \int_{P(V)} \nu(x \in P(V); \delta(x, H) \leq t) \xrightarrow{t \to 0} 0.
\]

Equivalently, there exists a proper map \(\phi : [1, +\infty) \to [1, +\infty)\) such that

\[
\sup_{H \text{ projective hyperplane of } P(V)} \int_{P(V)} \phi \left( \frac{1}{\delta(x, H)} \right) d\nu(x) < +\infty.
\]

**Remark 3.2.**

1. The same result as above but without the uniformity in the hyperplanes is straightforward since we know by Furstenberg [Fur73] that \(\nu\) is non-degenerate, i.e. it does not charge any projective hyperplane of \(P(V)\).

2. The result stated in Proposition 3.1 is known when \(\mu\) has an exponential moment by Guivarc’h [Gui90]. In this case, \(\phi(x) = x^\alpha\) works for all \(\alpha > 0\) small enough. In other terms, \(\nu\) has Holder regularity. In particular, \(\nu\) has positive Hausdorff dimension.
Proposition 3.3. Assume that \( \mu \) has a moment of order \( p > 1 \) thanks to Benoist-Quint [BQ16a]. In this case, \( \phi(x) = (\ln(x))^{p-1} \) works. In particular, \( \nu \) is log-regular (i.e. \( \phi(x) = \ln x \) works). We note that proving the log-regularity of \( \nu \) when \( p = 2 \) was crucial for Benoist-Quint to prove the CLT for \( \ln ||L_n|| \).

4. The existence of such a function \( \phi \) when \( \mu \) has a moment of order one is new. However, Theorem 3.1 does not give an explicit example of such an \( \phi \). It is then interesting to provide an explicit example of \( \phi \).

5. (Continuation of the previous remark) If \( \Gamma \) is a non-elementary subgroup of \( SL_2(\mathbb{R}) \), more can be said about the regularity of \( \nu \) when \( \mu \) has a moment of order one. Indeed, using the work of Benoist and Quint [BQ16b, Section 5] on central limit theorems on hyperbolic groups, one can deduce that the unique \( \mu \)-stationary probability measure on the projective line is log-regular, even when \( \mu \) has a moment of order one.

We begin first with the following estimates.

**Proposition 3.3.** Assume that \( \mu \) has a moment of order one and that \( \Gamma_\mu \) is strongly irreducible and proximal. Denote by \( \mu^* \) the pushforward probability measure on \( GL(V^*) \) of \( \mu \) by the map \( g \mapsto g^\bullet \). Then

1. For every \( \varepsilon > 0 \),
   \[
   \sup_{||v||=1} \mathbb{P} \left( \frac{||L_n v||}{||L_n||} \leq \exp(-\varepsilon n) \right) \to 0. \tag{\text{n} \to +\infty}
   \]

2. There exists \( C > 0 \) such that
   \[
   \sup_{x,y \in P(V)} \mathbb{P} (\delta(L_n \cdot x, L_n \cdot y) \geq \exp(-Cn)) \to 0. \tag{\text{n} \to +\infty}
   \]

3. For every \( \varepsilon > 0 \),
   \[
   \sup_{x \in P(V)} \mathbb{P} (\delta(L_n \cdot x, H) \leq \exp(-\varepsilon n)) \to 0. \tag{\text{n} \to +\infty}
   \]

4. (Stronger property than 3.) For every numerical sequence \( (\varepsilon_n)_n \) that tends to zero,
   \[
   \sup_{x \in P(V)} \mathbb{P} (\delta(L_n \cdot x, H) \leq \varepsilon_n) \to 0. \tag{\text{n} \to +\infty}
   \]

5. For every numerical sequence \( (\varepsilon_n)_n \) that tends to zero,
   \[
   \sup_{H \text{ projective hyperplane of } P(V)} \mathbb{P} (\delta(v^\perp_{L_n}, H) \leq \varepsilon_n) \to 0. \tag{\text{n} \to +\infty}
   \]

6. There exists \( C > 0 \), a random variable \( Z \in P(V) \) of law the unique \( \mu \)-invariant probability measure on \( P(V) \) such that
   \[
   \sup_{x \in P(V)} \mathbb{P} (\delta(R_n \cdot x, Z) \geq \exp(-Cn)) \to 0 \quad \text{and} \quad \mathbb{P} (\delta(v^+_{R_n}, Z) \geq \exp(-Cn)) \to 0.
   \]

7. Similarly, there exists \( C > 0 \), a random variable \( Z^* \in P(V^*) \) of law the unique \( \mu^* \)-invariant probability measure on \( P(V^*) \) such that if \( H^\perp_{L_n} := [\ker(f_{L_n})] \), then
   \[
   \mathbb{P} (\delta(f_{L_n}, Z^*) \geq \exp(-Cn)) \to 0, \tag{\text{n} \to +\infty}
   \]
   where \( \delta \) denotes by abuse of notation the Fubini-Study metric on \( P(V^*) \).

**Remark 3.4.** In parts 1, 2, 3, the speed of convergence is

- exponential when \( \mu \) has an exponential moment [BL85], [Gui90], [Aou11];
- of order \( C_n \) for some sequence \( (C_n)_n \in \mathbb{N} \) that satisfies \( \sum_n n^{p-2} C_n < +\infty \), when \( \mu \) has a moment of order \( p > 1 \) [BQ16a].

**Remark 3.5.** The role of \( R_n \) and \( L_n \) is interchangeable in the statements of Proposition 3.3 except for estimates 6 and 7 where the result fails if we interchange \( R_n \) and \( L_n \).

**Proof.** We will use in all the proof that if \( (A_n)_n \) and \( (B_n)_n \) are two sequences of subsets of \( \Omega \) such that
   \[
   \mathbb{P}(A_n) = 1 - o(1) \quad \text{and} \quad \mathbb{P}(B_n) = 1 - o(1),
   \]
   then \( \mathbb{P}(A_n \cap B_n) = 1 - o(1) \).
1. By [BL85, Corollary 3.4 item (iii)], we know that for any sequence \((v_n)_n\) in \(V\) of norm one, 
\[
\frac{1}{n} \mathbb{E} (\ln \|L_n v_n\|) \xrightarrow{n \to \infty} \lambda_1.
\]
Hence \(\frac{1}{n} \mathbb{E} \left( \ln \frac{\|L_n v_n\|}{\|v_n\|} \right) \xrightarrow{n \to \infty} 0\) for every such sequence \((v_n)_n\). Thus,
\[
\sup_{v \in P(V)} \frac{1}{n} \mathbb{E} \left( \ln \frac{\|L_n v\|}{\|v\|} \right) \xrightarrow{n \to \infty} 0.
\]
It is enough now to apply Markov’s inequality in order to have the estimate of 1.

2. Let \(x = [v], y = [w] \in P(V)\). Without loss of generality \(\|v\| = \|w\| = 1\). We have by the definition of the metric \(\delta\):
\[
\forall g \in \text{GL}(V), \quad \delta(g \cdot x, g \cdot y) \leq \frac{\|A^2 g\|}{\|g\|^2} \times \frac{\|g\|^2}{\|g v\| \|gw\|}.
\]
(9)
On the one hand, we know by Guivarc’h-Raugi theorem [GR85] that with our assumptions on the semi-group generated by the support of \(\mu\), the first Lyapunov exponent is simple. Hence the following almost sure convergence holds:
\[
\frac{1}{n} \ln \frac{\|A^2 L_n\|}{\|L_n\|^2} \xrightarrow{n \to \infty} \lambda_2 - \lambda_1 < 0.
\]
Since the previous convergence is true in the probability mode of convergence, we deduce that for \(C := (\lambda_2 - \lambda_1)/2 > 0\), we have that
\[
\mathbb{P} \left( \frac{\|A^2 L_n\|}{\|L_n\|^2} \leq \exp(-Cn) \right) = 1 - o(1).
\]
(10)
On the other hand, applying estimate 1. for \(\epsilon = C/4\) to \(\tilde{\delta}\) get that
\[
\mathbb{P} \left( \frac{\|L_n\|^2}{\|L_n v\| \|L_n w\|} \leq \exp(Cn/2) \right) = 1 - o(1).
\]
(11)
Moreover the previous estimate is uniform in \(v\) and \(w\). Combining (9), (10) and (11) and the remark at the beginning of the proof, we get the desired estimate.

3. Fix a basis \((e_1, \ldots, e_d)\) of \(V\) and use the max norm on \(V\). Let \(H\) be a projective hyperplane of \(P(V)\). Then it is easy to see (see for instance [Aou11, Lemma 4.19]) that there exists \(i = 1, \ldots, d\) such that
\[
\delta(g \cdot [e_i], H) \geq \frac{\|ge_i\|}{\|g\|}.
\]
In particular, for every \(x \in P(V)\),
\[
\delta(g \cdot x, H) \geq \min_{i=1,\ldots,d} \left\{ \frac{\|ge_i\|}{\|g\|} - \delta(g \cdot x, g \cdot [e_i]) \right\} \vee 0.
\]
We easily deduce the desired estimate by combining estimates 1. and 2.

4. Since \(R_n\) and \(L_n\) have the same law for every \(n \in \mathbb{N}^*\), it is enough to prove the desired estimate for the right random walk \((R_n)_n\). Let \((\epsilon_n)_n\) be positive numerical sequence such that \(\epsilon_n \xrightarrow{n \to \infty} 0\).
Without loss of generality assume that \(\epsilon_n \leq \epsilon^{-1}\) for all \(n\). For every \(n \in \mathbb{N}^*\), consider the following quantities
\[
I_n := \sup_{x \in P(V)} \mathbb{P} (\delta(R_n \cdot x, H) \leq -\epsilon(n)) \quad \text{and} \quad J_n := \sup_{x \in P(V)} \mathbb{P} (\delta(R_n \cdot x, H) \leq \epsilon_n).
\]
Fix \(n \in \mathbb{N}\), \(x \in P(V)\) and \(H\) a projective hyperplane of \(P(V)\). If \(\epsilon_n \leq \exp(-n)\), then \(J_n \leq I_n\). Otherwise, \(-\ln \epsilon_n < n\), so that we can write
\[
R_n x = R_{[-\ln \epsilon_n ]} \cdot (X_{[-\ln \epsilon_n ]+1} \cdots X_n \cdot x).
\]
Using the independence of the \((X_i)_i\)'s, we deduce that
\[
\mathbb{P} (\delta(R_n \cdot x, H) \leq \epsilon_n) \leq \sup_{y \in P(V)} \mathbb{P} (\delta(R_{[-\ln \epsilon_n ]} \cdot y, H) \leq \epsilon_n) \leq I_{[-\ln \epsilon_n ]}.
\]
Hence for every \( n \in \mathbb{N}^* \),
\[
J_n \leq \max\{I_n, I_{-\ln \epsilon_n}\}.
\]
As \( \epsilon_n \xrightarrow{n \to +\infty} 0 \), \( -\ln \epsilon_n \xrightarrow{n \to +\infty} +\infty \). Since by the previous part \( I_k \xrightarrow{k \to +\infty} 0 \), we deduce that \( J_n \xrightarrow{n \to +\infty} 0 \).

5. Let \( x = [v] \in P(V) \) and \( g \in \text{GL}(V) \). We have that
\[
\delta(v^*_g, g \cdot x) = \delta(e_1, a_g u_g \cdot x) = O \left( \frac{a_2(g)}{a_1(g)} \right) \times \frac{\|v\| \|v\|}{\|v^g\|}.
\]
Apply now estimates 1. and 2. gives some \( C > 0 \) such that:
\[
\sup_{x \in P(V)} \mathbb{P}(\delta(v^*_R, R_n \cdot x) \geq \exp(-Cn)) \xrightarrow{n \to +\infty} 0. \tag{12}
\]
Let \( (\epsilon_n)_{n \in \mathbb{N}^*} \) be a numerical sequence that converges to 0, \( x \in P(V) \) and \( H \) a projective hyperplane. Applying estimate 4, for \( \epsilon' := \epsilon_n + \exp(-Cn) \) that still tends to zero gives that, with probability tending to one uniformly on \( H \), \( \delta(v^*_R, H) \geq \delta(R_n \cdot x, H) - \delta(v^*_R, R_n \cdot x) \geq \epsilon_n \).

6. Let \( x \in P(V) \). We know from [BL85, Theorem 4.3] that there exists a random variable \( Z \) on \( P(V) \) independent of \( x \) of law the unique \( \mu \)-invariant probability measure on \( P(V) \) such that the sequence of random variables \( (R_n \cdot x)_{n \in \mathbb{N}} \) converges in probability to \( Z \). Hence, there exists a non random subsequence \( (n_k)_{k \in \mathbb{N}} \) such that \( (R_{n_k} \cdot x)_{k \in \mathbb{N}} \) converges almost surely to \( Z \). Fix now \( n \in \mathbb{N} \) and denote by \( C \) the positive constant given in estimate 2. On the one hand, we have by Fatou’s lemma that:
\[
\mathbb{P}(\delta(R_n \cdot x, Z) < \exp(-Cn)) \leq \liminf_{k \to +\infty} \mathbb{P}(\delta(R_n \cdot x, R_{n_k} \cdot x) < \exp(-Cn)). \tag{13}
\]
On the other hand, writing \( R_{n_k} \cdot x = R_n \cdot (X_{n+1} \cdots X_{n_k}) \cdot x \) for all \( k > n \) and using the independence of the \( X_i \)'s, we get that for all \( k > n \),
\[
\mathbb{P}(\delta(R_n \cdot x, R_{n_k} \cdot x) < \exp(-Cn)) \leq \sup_{a, b \in P(V)} \mathbb{P}(\delta(R_n \cdot a, R_n \cdot b) < \exp(-Cn)). \tag{14}
\]
Combining (13) and (14), we deduce that for every \( n \in \mathbb{N} \),
\[
\mathbb{P}(\delta(R_n \cdot x, Z) < \exp(-Cn)) \leq \sup_{a, b \in P(V)} \mathbb{P}(\delta(R_n \cdot a, R_n \cdot b) < \exp(-Cn)).
\]
By estimate 2. and the fact that \( R_n \) and \( L_n \) have the same law for every \( n \in \mathbb{N} \), we deduce that the quantity above goes to one as \( n \) tends to infinity. This proves the first inequality. The second one follows then from (12).

7. Apply the previous estimate for the probability measure \( \mu^* \) which satisfies the same assumptions as \( \mu \).

**Proof of Proposition 3.1:** Consider any numerical sequence \( (\epsilon_n)_{n \in \mathbb{N}} \) that converges to zero. Consider a projective hyperplane \( H \) of \( P(V) \) and let \( n \in \mathbb{N}^* \). Since \( \nu \) is \( \mu^* \)-invariant, we have
\[
\nu(x \in P(V); \delta(x, H) \leq \epsilon) = \int_{P(V)} \mathbb{P}(\delta(L_n \cdot x, H) \leq \epsilon_n) \, d\nu(x) \leq \sup_{x \in P(V)} \mathbb{P}(\delta(L_n \cdot x, H) \leq \epsilon_n).
\]
Using now estimate 4. of Proposition 3.3, we deduce that \( \nu \{x \in P(V); \delta(x, H) \leq \epsilon_n\} \) converges to zero uniformly on \( H \). This proves (7).

Now we check that (7) is equivalent to (8). Assume first that (7) holds. We can then find a decreasing sequence \( (a_n)_{n \in \mathbb{N}} \) in \( (0, 1) \) that converges to zero and verifying:
\[
\forall k \in \mathbb{N}, \nu(x \in P(V); \delta(x, H) \leq a_k) < e^{-k}.
\]
For every \( k \in \mathbb{N}^* \), denote by \( U_k \) the interval \([\frac{1}{a_k}, \frac{1}{a_{k+1}}]\) with the convention \( U_0 = [1, \frac{1}{a_1}] \). Let \( \phi: [1, +\infty) \to [1, +\infty) \) be any proper function such that \( \phi_{|U_k} \leq e^{k/2} \) for every \( k \in \mathbb{N} \) (for instance affine.
on each $U_k$ with $\phi(\frac{1}{n}) = e^{(k-1)/2}$. Let $H$ be a projective hyperplane and $A_k = \{ x \in P(V) \setminus H; \frac{1}{n+k} \in U_k \}$, $k \in \mathbb{N}$. Since $(A_k)_{k \geq 0}$ covers $P(V) \setminus H$ and since $\nu$ is not degenerate on $P(V)$, we deduce that
\[
\int_{P(V)} \phi \left( \frac{1}{\delta(x,H)} \right) \, d\nu(x) = \sum_{k=0}^{\infty} \int_{A_k} \phi \left( \frac{1}{\delta(x,H)} \right) \, d\nu(x)
\leq \sum_{k=0}^{\infty} e^{-k} e^{k/2} < +\infty.
\]

The finite sum above being independent of $H$, the forward implication is proved.

Conversely, assume that (8) holds and let $C := \sup_H \int_{P(V)} \phi \left( \delta^{-1}(x,H) \right) \, d\nu(x) < +\infty$.

Let $\epsilon > 0$. By properness of $\phi$ we can find $\delta > 0$ such that $\phi(\frac{1}{\delta}) > \frac{\epsilon}{2}$ for every $0 < t < \delta$.

Hence, for every $t \in (0, \delta)$ and for every projective hyperplane $H$, $\nu(\{x \in P(V); \delta(x,H) < t\}) \leq \nu(\{x \in P(V); \phi(\delta(x,H)^{-1}) > \frac{\epsilon}{2}\})$. By Markov’s inequality, we deduce that for every $H$ one has that $\nu(\{x \in P(V); \delta(x,H) < t\}) \leq \epsilon$ whenever $t \in (0, \delta)$. This proves the backward implication. \hfill \square

4 End of the proof

The end of the proof is based on a usual trick about the asymptotic independence of the right of the left random walk. We refer for instance to [Tut65], [Gui90] and [Aou13, Lemme 4.3] for a general statement.

Proof of Theorem 2.1: By estimates 6. and 7. of proposition 3.3, there exists random variables $Z \in P(V)$ and $Z^* \in P(V^*)$, $C > 0$, $n_0$ such that for every $n \geq n_0$:

i. $\mathbb{P}(\delta(v_{X_1}^+ \cdots X_n, Z) \geq \exp(-Cn)) = o(1)$.

ii. $\mathbb{P}(\delta(f_{X_1} \cdots X_n, f_{X_{[n/2]}+1}^+ \cdots X_n) \geq \exp(-Cn)) = \mathbb{P}(\delta(f_{X_n} \cdots x_1, f_{X_{n-[n/2]}+1}^+ \cdots X_1) \geq \exp(-Cn)) = o(1)$.

The independence of the $X_i$’s is used in the middle equality above. We deduce that, for $n \geq n_0$,
\[
\mathbb{P}(\delta(v_{X_1}^+ \cdots X_n, H_{X_{[n/2]}+1} \cdots X_n) \leq \epsilon_n) \leq o(1) + \mathbb{P}(\delta(v_{X_1}^+ \cdots X_{[n/2]}^+, H_{X_{[n/2]}+1} \cdots X_n) \leq \epsilon_n + 2 \exp(-Cn)) \leq o(1) + \sup_{H \text{ hyperplane of } V} \mathbb{P}(\delta(v_{X_1}^+ \cdots X_{[n/2]}^+, H) \leq \epsilon_n + 4 \exp(-Cn)) \leq o(1) + \sup_{H \text{ hyperplane of } V} \mathbb{P}(\delta(Z, H) \leq \epsilon_n + 5 \exp(-Cn)).
\]

Estimates (15) and (18) follow immediately from estimate i. at the beginning of the proof. In line (16), we used estimate ii. above and the following identity true for every $x \in V$ of norm one and every non zero $f, f' \in V^*$,
\[
\delta([x], \ker f) - \delta([x], \ker f') = \left| |f(x)| - |f'(x)| \right| \leq \min\{||f - f'||, ||f + f'||\} \leq \sqrt{2} \delta([f], [f']).
\]

Identity (17) is due to the independence of the $(X_i)$’s. By Proposition 3.1, we deduce that the previous quantity goes to zero. Since $R_n$ and $L_n$ have the same for every $n \in \mathbb{N}^*$, we deduce that $\mathbb{P}(\delta(v_{X_n}^+, H_{X_n}^\perp) \leq \epsilon_n) = o(1)$ as desired. \hfill \square

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