Riesz Transform on Locally Symmetric Spaces and Riemannian Manifolds with a Spectral Gap

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Abstract

In this paper we study the Riesz transform on complete and connected Riemannian manifolds \(M\) with a certain spectral gap in the \(L^2\) spectrum of the Laplacian. We show that on such manifolds the Riesz transform is \(L^p\) bounded for all \(p \in (1, \infty)\). This generalizes a result by Mandouvalos and Marias and extends a result by Auscher, Coulhon, Duong, and Hofmann to the case where zero is an isolated point of the \(L^2\) spectrum of the Laplacian.

Keywords: Riesz transform on Riemannian manifolds, Spectral gap.

1 Introduction and Main Result

Investigations concerning the question whether the Riesz transform \(\nabla \Delta^{-1/2}\) on a general complete Riemannian manifold \(M\) is a bounded operator from \(L^p(M)\) to the space of \(L^p\) vector fields, \(1 < p < \infty\), have been started with the paper [15] by Strichartz (note that we denote by \(\nabla\) the gradient and by \(\Delta\) the Laplace-Beltrami operator on \(M\)). He proved in this paper that the \(L^2\) Riesz transform is an isometry for any complete Riemannian manifold \(M\) and furthermore, he showed boundedness on \(L^p, 1 < p < \infty,\) if \(M\) is a rank

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one symmetric space of non-compact type. In general however, the Riesz transform needs not be bounded for all p > 1, see e.g. [7] where an example for a Riemannian manifold \( M \) is given such that the Riesz transform is unbounded on \( L^p \) for all \( p > 2 \). Note that the \( L^p \) boundedness of the Riesz transform on a Riemannian manifold is equivalent to the existence of a constant \( C_p > 0 \) such that

\[
\|\nabla f\|_{L^p} \leq C_p \Delta^{1/2} f \|_{L^p}
\]

for all \( f \in C_0^\infty(M) \) holds.

In [13] Mandouvalos and Marias studied the Riesz transform on certain locally symmetric spaces. Inspired by their paper and [1] we show here how the \( L^p \) boundedness of the Riesz transform for a larger class of connected Riemannian manifolds \( M \) can be obtained. Our only assumptions are a certain spectral gap in the \( L^2 \) spectrum of the Laplacian and the following properties of \( M \):

(1.) The Riemannian manifold \( M \) satisfies the exponential growth property: For all \( r_0 > 0, x \in M, \theta > 1, \) and \( 0 < r < r_0 \) the volume \( V(x, r) \) of a ball with center \( x \) and radius \( r \) satisfies

\[
V(x, \theta r) \leq C e^{c \theta} V(x, r)
\]

for some constants \( C \geq 0, c > 0 \) that only depend on \( r_0 \). Note that this condition implies the local volume doubling property, cf. [1].

(2.) If \( p(t, x, y) \) denotes the heat kernel of the heat semigroup \( e^{-t\Delta} : L^2(M) \to L^2(M) \) we assume that there exists \( C > 0 \) such that for all \( x, y \in M \) and all \( t \in (0,1) \) we have

\[
p(t, x, x) \leq \frac{C}{V(x, \sqrt{t})}
\]

and

\[
|\nabla_x p(t, x, y)| \leq \frac{C}{\sqrt{t} V(y, \sqrt{t})}.
\]

More precisely, we prove the following theorem.

**Theorem 1.1.** Let \( M \) denote a complete and connected Riemannian manifold satisfying properties (1), (2), and (3) from above, and assume that there is a constant \( a > 0 \) such that the following condition for the \( L^2 \) spectrum \( \sigma(\Delta) \) of \( \Delta \) holds:

\[
\sigma(\Delta) \subset \{0\} \cup [a, \infty).
\]

Then there exist for any \( p \in (1, \infty) \) constants \( c_p, C_p > 0 \) such that

\[
c_p \|\Delta^{1/2} f\|_{L^p} \leq \|\nabla f\|_{L^p} \leq C_p \|\Delta^{1/2} f\|_{L^p}
\]

for any \( f \in C_0^\infty(M) \).
This result extends [1, Theorem 1.9]. The new feature is that we allow also \(0 \in \sigma(\Delta)\). Note that similar ideas as in our proof have been used in [7, Theorem 1.3]. Important examples for manifolds that satisfy the conditions in Theorem 1.1 will be given in Section 2. Note, that properties (1), (2), and (3) are always satisfied if the Ricci curvature of \(M\) is bounded from below (cf. also the discussion in [1, Section 1.3]) and hence, these conditions are fulfilled if \(M\) is a locally symmetric space.

**Corollary 1.2.** Let \(M\) denote a complete and connected Riemannian manifold whose Ricci curvature is bounded from below, and assume that there is a constant \(a > 0\) with \(\sigma(\Delta) \subset \{0\} \cup [a, \infty)\).

Then there exist for any \(p \in (1, \infty)\) constants \(c_p, C_p > 0\) such that

\[
c_p \|\Delta^{1/2} f\|_{L^p} \leq \|\nabla f\|_{L^p} \leq C_p \|\Delta^{1/2} f\|_{L^p}
\]

holds for any \(f \in C_0^\infty(M)\).

While Mandouvalos and Marias in [13] proved boundedness of the \(L^p\) Riesz transform only for \(p \in (p_1, p_2)\) with certain \(1 < p_1 < 2 < p_2 < \infty\), we allow here \(p \in (1, \infty)\). A standing assumption in the paper [13] is that the \(L^2\) spectrum of the Laplacian on a non-compact locally symmetric space \(M = \Gamma \backslash X\) is of the form \(\{\lambda_0, \ldots, \lambda_r\} \cup [||\rho||^2, \infty)\), where \(\lambda_i < ||\rho||^2, i = 0, \ldots, r\), are eigenvalues of finite multiplicity and \(\rho\) denotes half the sum of the positive roots. Note that this assumption needs not be satisfied for all non-compact locally symmetric spaces. If e.g. the universal covering \(X\) is a higher rank symmetric space, the absolutely continuous part of the \(L^2\) spectrum of a non-compact finite volume quotient \(M = \Gamma \backslash X\) is in many cases of the form \([a, \infty)\) where \(0 < a < ||\rho||^2\), [10]. However, it is known that the above cited condition from [13] holds true for all non-compact, geometrically finite hyperbolic manifolds \(\Gamma \backslash \mathbb{H}^n\), [11].

In a first step towards a proof of the boundedness of the Riesz transform the authors show in [13] that any \(L^2\) eigenfunction \(\varphi_{\lambda_i}\) with respect to some eigenvalue \(\lambda_i < ||\rho||^2\) is contained in \(L^p(M)\) for \(p\) in an interval \((p_1, p_2)\) where

\[
p_1 = 2 \left(1 + \left(1 - \frac{\lambda_i}{||\rho||^2}\right)^{1/2}\right)^{-1},
\]

\[
p_2 = 2 \left(1 + \left(1 - \frac{\lambda_i}{||\rho||^2}\right)^{1/2}\right),
\]

cf. [13, Theorem 1]. Note that the authors use \(\lambda_r\) instead of \(\lambda_i \leq \lambda_r\) in the statement of their result. The proof however shows, that \(p_1\) and \(p_2\) from above can be used. Note further, that \(p_1' \geq p_2\), where \(p_1'\) denotes the conjugate of \(p_1\), i.e. \(\frac{1}{p_1} + \frac{1}{p_1'} = 1\). In the proof of this result the authors state that the volume of balls grows exponentially (formula 2.11) with respect to the radius. Unfortunately, this assumption is often not satisfied if \(M\) is a locally symmetric space with non-trivial fundamental group. In the case of
finite volume it is certainly false. And even if the volume is infinite the volume growth may be linear (the following example was communicated to us by Richard D Canary): if $M$ is a compact hyperbolic 3-manifold that fibers over the circle we may ‘unwrap’ this manifold along the circle such that only the fundamental group coming from the fibers survives. The resulting hyperbolic manifold $\tilde{M}$ is a covering of $M$ that has an infinite cyclic subgroup of isometries with compact quotient and hence linear volume growth. The existence of hyperbolic 3-manifolds that fiber over the circle follows from Thurston’s work in [19] and, of course, from Perelman’s proof of Thurston’s geometrization conjecture using Ricci flow. However, a detailed look at the proof of [13, Theorem 1] shows that only an exponential upper bound on the volume growth is needed and this condition is always satisfied for complete locally symmetric spaces $M$. In our proof of the boundedness of the Riesz transform we do not need to make any of the above assumptions and we will not need specific information about the $L^2$ eigenfunctions. Nevertheless, we want to mention briefly that in some cases it is possible to enlarge the interval $(p_1, p_2)$: If $M$ is a locally symmetric space with $\mathbb{Q}$-rank one it had been shown in [10, Theorem 4.1] that $p_1 = 1$ and $p_2 = 2(1 + (1 - \lambda_i ||\rho||^2)^{1/2})^{-1}$ can be chosen.

If the injectivity radius of $M$ is strictly positive, it follows immediately from Taylor’s work on the $L^p$ spectrum of the Laplacian [17, Proposition 3.3] that the interval can be enlarged to the right hand side, more precisely, we can choose $p_2 = p_1' = 2(1 - (1 - \lambda_i ||\rho||^2)^{1/2})^{-1}$. Note that the condition on the injectivity radius implies in particular that $M$ has infinite volume.

2 Proof and Examples

The following theorem due to Auscher, Coulhon, Duong, and Hofmann is a main ingredient in our proof of Theorem 1.1.

**Theorem 2.1.** ([1, Theorem 1.7]). Let $M$ denote a complete Riemannian manifold satisfying the properties (1), (2), and (3), and let $p \in (1, \infty)$. Then there exists a constant $C_p > 0$ such that the inequality

$$\|\nabla f\|_{L^p} + \|f\|_{L^p} \leq C_p \left(\|\Delta^{1/2} f\|_{L^p} + \|f\|_{L^p}\right),$$

holds for all $f \in C_0^\infty(M)$.

The following lemma can basically be found in the proof of [7, Theorem 1.3]. For the sake of completeness and because we need a variant of the argument below, we recall its short proof.

**Lemma 2.2.** Assume that $0 \notin \sigma(\Delta)$. Then for any $\alpha \in (0, 1)$ the operator $\Delta^{-\alpha}$ defines a bounded operator on $L^p(M), p \in (1, \infty)$.
Proof. Since $\Delta$ is a selfadjoint operator on $L^2(M)$ and $\sigma(\Delta) \subset [a, \infty)$ for some $a > 0$ it follows from the spectral theorem $\|e^{-t\Delta}\|_{L^2 \to L^2} \leq e^{-at}$. Furthermore, the semigroup $e^{-t\Delta} : L^2(M) \to L^2(M)$ is positive and $L^\infty(M)$ contractive. Hence, this semigroup extends to a strongly continuous contraction semigroup $e^{-\alpha \Delta} : L^p(M) \to L^p(M)$ for any $p \in [1, \infty)$. By interpolation and duality it follows that for any $p \in [1, \infty)$ we have $\|e^{-t\Delta}\|_{L^p \to L^p} \leq e^{-2\min(1/p, 1/p')at}$ and hence, the integral

$$\Delta^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{1-\alpha} e^{-t\Delta} dt.$$ 

defines for all $\alpha \in (0, 1)$ a bounded operator on $L^p(M), p \in (1, \infty)$.

Before we proceed with the proof of Theorem 1.1, we recall the following general result that has been proved in [15]. Note that we denote by $-\Delta_p$ the generator of the heat semigroup $T_p(t) = e^{-t\Delta}$ on $L^p(M)$.

**Lemma 2.3.** For any $1 < p < \infty$ and any complete Riemannian manifold $M$, the space $C_0^\infty(M)$ is a core for the operator $\Delta_p$ in $L^p(M)$, i.e. $C_0^\infty(M)$ is dense in $\mathcal{D}(\Delta_p)$ with respect to the graph norm. In particular, $C_0^\infty(M)$ is a core for the operator $\Delta_p^{1/2}$.

**Proof.** Letting $L_p$ denote the $L^p(M)$ closure of $\Delta$, defined on $C_0^\infty(M)$, it is shown in [15], beginning of the proof of Theorem 3.5, that $-L_p$ is dissipative and generates a contraction semigroup in $L^p(M)$. This semigroup coincides with the heat semigroup, which implies $L_p = \Delta_p$.

**Proof of Theorem 1.1.** It is well known (cf. [2, 3]) that by duality the first inequality in (4) is implied by the second and hence we will concentrate on the second.

The case $0 \notin \sigma(\Delta)$ was already proved in [1, Theorem 1.9] (cf. also [7, Theorem 1.3]): By Lemma 2.2 the inverse square root $\Delta^{-1/2}$ defines a bounded operator on $L^p(M)$ for any $p \in (1, \infty)$ and hence, the claim follows from Theorem 2.1 together with $\|f\|_{L^p} \leq C\|\Delta^{1/2}f\|_{L^p}$ for $f \in C_0^\infty(M)$.

Assume now $0 \in \sigma(\Delta)$. Then 0 is an isolated point in the spectrum of a self-adjoint operator and hence an eigenvalue. If $f \in L^2(M)$ is a corresponding eigenfunction we may conclude that $f = \text{const.}$ as $0 = \langle \Delta f, f \rangle = \langle \nabla f, \nabla f \rangle$ and thus $\text{vol}(M) < \infty$. From Hölder’s inequality it follows in particular $L^p(M) \hookrightarrow L^q(M)$ if $1 \leq q \leq p \leq \infty$.

Let us define for $p \in [1, \infty)$

$$L_0^p(M) = \left\{ f \in L^p(M) : \langle 1, f \rangle = \int_M f \, dx = 0 \right\}.$$

Then $L_0^p(M)$ is an invariant subspace for the semigroup $T_p(t) = e^{-t\Delta} : L^p(M) \to L^p(M)$ and $L^p(M) = L_0^p(M) \oplus \text{span}\{1\}$. Furthermore, if we denote by $-\Delta_0$ the generator of $e^{-t\Delta} : L_0^2(M) \to L_0^2(M)$ we have $\Delta_0 = \Delta|_{\mathcal{D}(\Delta_0) \cap L_0^2(M)}$. As $\sigma(\Delta_0) \subset [a, \infty)$ for some $a > 0$ we can prove as in Lemma 2.2 that $\Delta_0^\alpha$ is bounded on $L_0^p(M)$ for all $\alpha \in (0, 1)$ and $p \in (1, \infty)$. Note that we need here to assure that the spaces $L_0^p(M), p \in [1, \infty)$,
interpolate correctly, i.e. \([L^p_0(M),\sigma(\Delta)]_{\theta} = L^p_\theta(M)\) (see [20]), where \(\frac{1}{p_0} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}\). This however follows from the fact that

\[
P : L^p(M) \to L^p(M), f \mapsto f - \frac{1}{\text{vol}(M)} \int_M f \, dx
\]

is a continuous projection onto \(L^p_0(M)\) together with [20, 1.2.4]. If \(M\) is compact then inequality (4) follows from Theorem 2.1 since \(P f \in C_0^\infty(M) \cap L^p_0(M)\) and (4) holds for the constant function \(g := f - P f\).

If \(M\) is non-compact but of finite volume then \(P f \in C_0^\infty(M)\) is no longer true in general. But by Lemma 2.3, Theorem 2.1 holds by approximation for all \(g \in D(\Delta_1^{1/2})\) with the same constant. In particular, it holds for all \(g \in C_0^\infty(M) + \text{span}\{1\}\). Now we can finish the proof as in the compact case since \(f \in C_0^\infty(M)\) implies \(P f \in (C_0^\infty(M) + \text{span}\{1\}) \cap L^p_0(M)\).

Examples

The conditions in Theorem 1.1 are satisfied for the following manifolds.

**Compact manifolds.** If \(M\) is compact, its \(L^2\) spectrum is discrete and \(0 \in \sigma(\Delta)\).

**Locally symmetric spaces.** Let \(M = \Gamma \backslash G/K\) denote a locally symmetric space whose universal covering is a symmetric space of non-compact type.

If the critical exponent \(\delta(\Gamma)\) satisfies \(\delta(\Gamma) < 2|\rho|\) (\(\rho\) denotes half the sum of the positive roots) then the bottom of the \(L^2\) spectrum is bounded from below by a strictly positive constant [12, 21]. Note that in this case \(\text{vol}(M) = \infty\).

If \(M\) is non-compact with finite volume, the \(L^2\) spectrum equals \(\sigma(\Delta) = \{0, \lambda_1, \ldots, \lambda_r\} \cup [b, \infty)\) for some \(b > 0\), see e.g. M"uller’s article in [18].

Important examples in this context are all non-compact hyperbolic manifolds \(M = \Gamma \backslash \mathbb{H}^n\) where \(\Gamma\) is geometrically finite. Then, in the case of finite and infinite volume of \(M\), the \(L^2\) spectrum is of the form \(\{\lambda_1, \ldots, \lambda_r\} \cup \left[\frac{(n-1)^2}{4}, \infty\right)\), cf. [11].

**Manifolds with cusps of rank one.** These non-compact manifolds with finite volume satisfy also the condition \(\sigma(\Delta) = \{0, \lambda_1, \ldots, \lambda_r\} \cup [b, \infty)\) for some \(b > 0\), cf. [14].

**Homogeneous Spaces.** Consider \(M = \Gamma \backslash G\) where \(G\) is a semisimple Lie group endowed with an invariant metric and \(\Gamma \subset G\) is a non-uniform arithmetic lattice in \(G\). Then, the spectrum equals \(\sigma(\Delta) = \{0, \lambda_1, \ldots, \lambda_r\} \cup [b, \infty)\) for some \(b > 0\), cf. [4].

Note that the continuous \(L^2\) spectrum of the Laplacian on a complete Riemannian manifold is invariant under compact perturbations of the Riemannian metric. This follows from the so-called decomposition principle in [8]. Hence, the spectral gap condition in the above mentioned cases is still satisfied after compact perturbations.
Is zero in the spectrum? This question was raised in various contexts. Let us recall the following results which immediately give further examples.

**Theorem 2.4** ([6]). Let $M$ have infinite volume. Suppose that there is a constant $\kappa \geq 0$ such that $\text{Ricci}_M \geq -\kappa^2$. Then $0 \notin \sigma(\Delta)$ if and only if $M$ is open at infinity.

Recall that a manifold with infinite volume is called open at infinity if and only if the Cheeger constant is strictly positive.

**Theorem 2.5** ([5]). Let $X$ be a normal covering of a compact manifold $M$ with covering group $\Gamma$. Then $0 \in \sigma(\Delta)$ if and only if $\Gamma$ is amenable.

**Theorem 2.6** ([16] or [9]). Let $M$ denote a Riemannian manifold with Ricci curvature bounded from below and with empty cut locus. If there is a point $x_0 \in M$ such that the volume form $\sqrt{g(x_0, \zeta)}$ of the Riemannian metric grows exponentially in every direction (with respect to geodesic normal coordinates $(x, \zeta)$), the bottom of the $L^2$ spectrum is strictly positive.

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