Discrete Stein characterizations and discrete information distances

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Abstract: We construct two different Stein characterizations of discrete distributions and use these to provide a natural connection between Stein characterizations for discrete distributions and discrete information functionals.

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1. Foreword and notations

The purpose of this work is to construct an explicit connection between discrete Stein characterizations and discrete information functionals (see [10] where similar considerations are discussed for continuous distributions). In doing so we also provide two general Stein characterizations of discrete distributions, as well as a family of identities relating differences between expectations with what we call generalized score functions. In the context of Poisson approximation, our results allow in particular to construct bounds between the total variation distance and (i) the so-called scaled Fisher information used, e.g., in [8], as well as (ii) the discrete Fisher information used, e.g., in [7]. We refer the reader to [6, 11] and [1] for relevant references and similar inequalities.

Throughout the paper, we shall abuse of language and call discrete probability mass functions densities. Also, to avoid ambiguities related to division by 0, we adopt the convention that, whenever an expression involves the division by an indicator function $I_A$ for some measurable set $A$, we are multiplying the expression by the said indicator function. In particular note how ratios of the form $p(x)/p(x)$ with $p(x)$ some function do not necessarily simplify to 1. Finally, we adopt the convention that sums running over empty sets equal 0.

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2. First connection

We start with a discrete version of the so-called density approach (see [2, 10, 13] for a description in the continuous case).

**Theorem 2.1** (Discrete density approach). Let \( p \) be a density with support \( S_p \subset \mathbb{Z} \). For the sake of convenience, we choose \( S_p = [a, b] := \{a, a+1, \ldots, b\} \) with \( a < b \in \mathbb{Z}\cup\{\pm\infty\} \). Let \( \mathcal{F}_1(p) \) be the collection of all test functions \( f : \mathbb{Z} \rightarrow \mathbb{R} \) such that \( x \mapsto f(x)p(x) \) is bounded on \( S_p \) and \( f(a) = 0 \). Let \( \Delta^+_z h(x) := h(x+1) - h(x) \) be the forward difference operator and define \( \mathcal{T}_1(\cdot, p) : \mathbb{Z}^* \rightarrow \mathbb{R}^* \) through

\[
\mathcal{T}_1(f, p) : \mathbb{Z} \rightarrow \mathbb{R} : x \mapsto \mathcal{T}_1(f, p)(x) := \frac{\Delta^+_z(f(x)p(x))}{p(x)} I_{S_p}(x).
\]

(2.1)

Let \( Z \sim p \) and let \( X \) be a real-valued discrete random variable.

1. If \( X \overset{d}{=} Z \) then \( E[\mathcal{T}_1(f, p)(X)] = 0 \) for all \( f \in \mathcal{F}_1(p) \).
2. If \( E[\mathcal{T}_1(f, p)(X)] = 0 \) for all \( f \in \mathcal{F}_1(p) \), then \( X | X \in S_p \overset{d}{=} Z \).

We draw the reader’s attention to the similarity between the operator \( \mathcal{T}_1 \) and the operators introduced in [9, 10]; in the terminology of [9], our operator (2.1) allows for a discrete “location”-based parametric interpretation.

**Proof.** The first statement is trivial. To see (2), consider for \( z \in \mathbb{Z} \) the functions \( f^z_p \) defined through

\[
f^z_p : \mathbb{Z} \rightarrow \mathbb{R} : x \mapsto \frac{1}{p(x)} \sum_{k=a}^{x-1} l_z(k)p(k)
\]

with \( l_z(k) := (I_{(-\infty, z]}(k) - P_p(X \leq z)) I_{S_p}(k) \) and \( P_p(X \leq z) := \sum_{k=-\infty}^{z} p(k) \).

It is evident that \( x \mapsto f^z_p(x)p(x) \) is bounded and that \( f^z_p(a) = 0 \) by our convention on sums, hence \( f^z_p \in \mathcal{F}_1(p) \) for all \( z \). Moreover we have \( \Delta^+_z(f^z_p(x)p(x)) = l_z(x)p(x) \). This result is direct for \( x < b \); for \( x = b \), \( \Delta^+_z(f^z_p(x)p(x))|_{x=b} = f^z_p(b+1) - f^z_p(b)p(b) = - \sum_{k=a}^{b-1} l_z(k)p(k) = l_z(b)p(b) \) since \( \sum_{k=a}^{b} l_z(k)p(k) = 0 \) by definition of \( l_z \). It follows that this forward difference satisfies, for all \( z \), the so-called Stein equation

\[
\mathcal{T}_1(f^z_p, p)(x) = l_z(x).
\]

Consequently, we can use \( E[\mathcal{T}_1(f^z_p, p)(X)] = 0 \) to obtain

\[
P(X \leq z \cap X \in S_p) = P(X \in S_p)P_p(X \leq z)
\]

for all \( z \in \mathbb{Z} \). In other words, provided that \( P(X \in S_p) > 0 \), \( P(X \leq z | X \in S_p) = P_p(X \leq z) \) for all \( z \in \mathbb{Z} \), whence the claim. \( \Box \)

Note that the choice of a “connected” support is for convenience only, and straightforward arguments allow to adapt the result to supports of the form \([a, b] \cup \{c, d\} \) with \( c > b \). Likewise the use of a forward difference in the expression of the operator is purely arbitrary and minor adaptations (e.g., setting \( f(b) = 0 \) instead of \( f(a) = 0 \) allow to reformulate (2.1) in terms of backward differences as well.
Example 2.1. It is perhaps informative to see how the operator $\mathcal{T}_1(f,p)$ spells out in certain specific examples.

1. Take $p(x) = e^{-\lambda x^2}/x!\mathbb{I}_0(x)$ the density of a mean-$\lambda$ Poisson random variable. Then (abusing notations) $\mathcal{F}_1(Po(\lambda))$ contains the set of bounded functions $f$ with $f(0) = 0$, and simple computations show that the operator becomes

$$\mathcal{T}_1(f, Po(\lambda))(x) = \left(\frac{\lambda}{x+1}f(x+1) - f(x)\right)\mathbb{I}_N(x).$$

2. Take $p$ to be a member of Ord’s family, i.e. suppose that there exist $s(x)$ and $\tau(x)$ such that

$$\frac{p(x+1)}{p(x)} = \frac{s(x) + \tau(x)}{s(x+1)}.$$

For an explanation on these notations see [12]. The collection $\mathcal{F}_1((s, \tau))$ contains the set of all functions of the form $f(x) = f_0(x)s(x)$ with $f_0$ bounded and, for these $f$, the operator writes out

$$\mathcal{T}_1(f, (s, \tau))(x) = (s(x) + \tau(x))f_0(x+1)\mathbb{I}_{[a,b]}(x+1) - f_0(x)s(x)\mathbb{I}_{[a,b]}(x).$$

We retrieve, up to some minor modifications, the operator presented in [12], using the backward difference operator and functions $f$ of the form $f_0(x)(s(x) + \tau(x))$ yields exactly the operator proposed in that paper.

3. Write $p$ as a Gibbs measure, i.e. $p(x) = e^{V(x)}/x!\mathbb{I}_{[0,N]}(x)$ with $N$ some positive integer, $\omega > 0$ fixed, $V$ a function mapping $\mathbb{N}$ to $\mathbb{R}$ and $\mathbb{Z}$ the normalizing constant. For an explanation on the notations see [4]. The collection $\mathcal{F}_1((V, \omega))$ contains the set of all functions of the form $f(x) = x f_0(x)$ with $f_0$ bounded and, for these $f$, the operator is of the form $\mathcal{T}_1(f, (V, \omega))(x) = f_0(x+1)e^{V(x+1)-V(x)}\omega\mathbb{I}_{[0,N]}(x+1) - xf_0(x)\mathbb{I}_{[0,N]}(x)$. Supposing, as in [4], that $f_0(N+1) = 0$, the latter operator simplifies to

$$\mathcal{T}_1(f, (V, \omega))(x) = \left(f_0(x+1)e^{V(x+1)-V(x)}\omega - xf_0(x)\right)\mathbb{I}_{[0,N]}(x),$$

which corresponds to the Stein operator presented in [4].

Following the methodology introduced in [10], the next step consists in uncovering a factorization property of the operator (2.1) for two densities $p$ and $q$. It will be fruitful to consider distributions $p$ and $q$ having non-equal supports. We choose to fix, for the sake of convenience (and for this sake only), $S_p = \mathbb{N}$ and $S_q = [0, \ldots, N]$, for $N \in \mathbb{N} \cup \{\infty\}$. Then, since we can always write $1 = q(x)/q(x) + \mathbb{I}_{[N+1,\ldots]}(x)$, we get

$$\mathcal{T}_1(f,p)(x) = \frac{\Delta^+_z(f(x)p(x)\frac{q(x)}{q(x)})}{p(x)} + \frac{\Delta^+_z(f(x)p(x)\mathbb{I}_{[N+1,\ldots]}(x))}{p(x)}.$$

Now recall the product rule for discrete derivatives

$$\Delta^+_z(h(x)q(x)) = h(x)\Delta^+_zq(x) + q(x)\Delta^+_zh(x).$$
Applying this and keeping in mind that we have set $S_p \subset S_q$, the first term on the rhs of (2.2) becomes

$$\frac{f(x + 1)q(x + 1)}{p(x)} \Delta^+ \left( \frac{p(x)}{q(x)} \right) + \frac{\Delta^+ (f(x)q(x))}{p(x)} \frac{p(x)}{q(x)}$$

$$= f(x + 1) \left( \frac{p(x + 1)}{p(x)} - \frac{q(x + 1)}{q(x)} \right) I_{[0,...,N]}(x) + \frac{\Delta^+ (f(x)q(x))}{q(x)}.$$ 

Therefore, letting

$$r_1(p,q)(x) := \left( \frac{p(x + 1)}{p(x)} - \frac{q(x + 1)}{q(x)} \right) I_{[0,...,N]}(x),$$

we have just shown that, for all $f \in \mathcal{F}_1(p) \cap \mathcal{F}_1(q)$, we have the factorization property

$$\mathcal{T}_1(f,p)(x) = \mathcal{T}_1(f,q)(x) + f(x + 1)r_1(p,q)(x) + e^N_{f,p}(x),$$

where

$$e^N_{f,p}(x) := f(x + 1)p(x + 1)/p(x)I_{[N,...]}(x) - f(x)I_{[N+1,...]}(x).$$

**Remark 2.1.** The statements above (and their consequences) are easily adapted to situations where $S_p \subset S_q$; having in mind the context of a Poisson target $p$ explains our willingness to restrict our choice.

Now let $l : \mathbb{Z} \to \mathbb{R}$ be a function such that $E_p[l(X)]$ and $E_q[l(X)]$ exist, with $E_r[l(X)] := \sum_{k \in S_r} l(k)r(k)$ for a density $r$ with support $S_r$. Still following [10], it is immediate that the function

$$f^p_{1,l} : \mathbb{Z} \to \mathbb{R} : x \mapsto \frac{1}{p(x)} \sum_{k=0}^{x-1} (l(k) - E_p[l(X)]) p(k)$$

is solution of the so-called Stein equation $\mathcal{T}_1(f,p)(x) = l(x) - E_p[l(X)]$, so that, taking expectations and using (2.4), we get

$$E_q[l(X)] - E_p[l(X)] = E_q[f^p_{1,l}(X + 1)r_1(p,q)(X)] + e^N_{p,q}(l),$$

with $e^N_{p,q}(l) := q(N)f^p_{1,l}(N + 1)p(N + 1)/p(N)$.

**Remark 2.2.** The error term $e^N_{p,q}(l)$ in (2.6) will be negligible as $N$ tends to infinity since, in general, the Stein solution $f^p_{1,l}$ will be bounded over $\mathbb{N}$. This latter fact also ensures that $f^p_{1,l}$ belongs to $\mathcal{F}_1(q)$.

We will apply (2.6) in the context of a Poisson target distribution in Section 4. In particular we will show how our approach provides a connection between the so-called total variation distance (as well as many other probability distances) and the scaled Fisher information in use for information theoretic approaches to Poisson approximation problems (see [1, 8, 11]).
3. A second connection

The construction from the previous section (i.e. the factorization (2.4), the score function (2.3) and the identity (2.6)) is by no means unique, nor is the initial characterization from Theorem 2.1. There are, in fact, an infinite number of variations on the different steps outlined above, each providing a connection between probability distances and different forms of information distances. Now it appears that, in the world of Poisson approximation, the scaled Fisher information is not the only “natural” measure of discrepancy and [7] (followed later by [1]) make use of another information distance which they call the discrete Fisher information. We choose to show how this specific distance can be obtained from our Stein characterizations as well.

In [9] we propose a construction of Stein characterizations tailored for parametric densities, that is densities depending on some real-valued parameter. In what follows, we shall denote by $p_θ(x)$ the parametric density with parameter $θ$ belonging to the parameter space $Θ$. For the sake of simplicity we consider families with support $S_p = [a, b] := \{a, a+1, \ldots, b\}$ with $a < b \in \mathbb{Z} \cup \{\pm \infty\}$ not depending on $θ$; we also suppose that, for all $x$, the function $θ \mapsto p_θ(x)$ is continuously differentiable. (A similar result can also be obtained for integer-valued parameters $θ$.) We then obtain the following result, whose proof is omitted because it is directly inspired from [9] and runs along the same lines as the proof of Theorem 2.1.

**Theorem 3.1 (Parametric discrete density approach).** For $θ$ an interior point of $Θ$, let $p(x) := p_θ(x)$ be a parametric density with support $S_p \subset \mathbb{Z}$ and define $\tilde{p}(x) := \partial_θ(p_θ(x)/p_θ(a))$. Let $F_2(p)$ be the collection of all test functions $f : \mathbb{Z} \to \mathbb{R}$ such that $x \mapsto f(x)\tilde{p}(x)$ is bounded on $S_p$. Define the operator $T_2(\cdot, p) : \mathbb{Z}^* \to \mathbb{R}^*$ through

$$T_2(f, p) : \mathbb{Z} \to \mathbb{R} : x \mapsto T_2(f, p)(x) := \frac{\Delta^+(f(x)\tilde{p}(x))}{p(x)} I_{S_p}(x).$$

Let $Z \sim p$ and let $X$ be a real-valued discrete random variable.

1. If $X \leq Z$ then $E[T_2(f, p)(X)] = 0$ for all $f \in F_2(p)$.
2. If $E[T_2(f, p)(X)] = 0$ for all $f \in F_2(p)$, then $X | X \in S_p \leq Z$.

We attract the reader’s attention to the fact that, contrarily to $F_1(p)$ in Theorem 2.1, the class of test functions $F_2(p)$ here does not ask that $f(a) = 0$. This comes from the fact that, by definition, $\tilde{p}(a) = 0$, hence this requirement on the $f$ can be dropped.

Theorem 3.1 allows to recover the well-known Stein operators and characterizations of the Poisson, geometric, binomial distributions, to cite but these; we refer the reader to [9] for intuition about the perhaps unusual form of the operator, as well as for explicit computations and examples.

From here onwards we restrict our attention to distributions $p$ and $q$ with full support $\mathbb{N}$. Note that this entails that $\tilde{p}(x)$ and $q(x-1)$ share the same
support \( N_0 \). While not strictly necessary, this assumption will yield considerable simplifications. It is, moreover, in line with the related literature when a Poisson target is to be considered (see [1]).

Proceeding as in Section 2 (and keeping all supports implicit) we readily obtain
\[
\mathcal{T}_2(f, p)(x) = \frac{\Delta^+_x \left( f(x)q(x-1) \frac{\tilde{p}(x)}{q(x-1)} \right)}{p(x)}
\]
\[
= \frac{\Delta^+_x (f(x)q(x-1) \frac{\tilde{p}(x)}{q(x)})}{p(x)} + f(x) \frac{q(x-1)}{p(x)} \Delta^+_x \left( \frac{\tilde{p}(x)}{q(x-1)} \right).
\]

Straightforward simplifications then yield for \( f \in \mathcal{F}_2(p) \cap \mathcal{F}_2(q) \) the factorization
\[
\mathcal{T}_2(f, p)(x) = f(x)r_2(p, q)(x) + \frac{\Delta^+_x (f(x)q(x-1) \frac{\tilde{p}(x)}{q(x-1)})}{p(x)} \Delta^+_x \left( \frac{\tilde{p}(x)}{q(x-1)} \right).
\]

with
\[
r_2(p, q)(x) := \frac{\tilde{p}(x+1)}{p(x)} q(x-1) \frac{\tilde{p}(x)}{q(x)} - \frac{\tilde{p}(x)}{p(x)}.
\]

Now let \( l : \mathbb{Z} \to \mathbb{R} \) be a function such that \( E_p[l(X)] \) and \( E_q[l(X)] \) exist, and define
\[
f^p_{2,l}(x) : \mathbb{Z} \to \mathbb{R} : x \mapsto \frac{1}{\tilde{p}(x)} \sum_{k=0}^{x-1} (l(k) - E_p[l(X)])p(k).
\]

Then clearly \( \mathcal{T}_2(f^p_{2,l}, p)(x) = l(x) - E_p[l(X)] \) so that, taking expectations on both sides of (3.1) for this choice of test function, we obtain
\[
E_q[l(X)] - E_p[l(X)] = E_q[f^p_{2,l}(X)r_2(p, q)(X)]
\]
\[
+ E_q \left[ \Delta^+_x \left( f^p_{2,l}(x)q(x-1) \right) \bigg|_{x=X} \frac{\tilde{p}(X+1)}{p(X)} \right].
\]

Finally suppose that \( \tilde{p}(X+1)/p(X) \) simplifies to a constant (as is the case for a Poisson target). Then straightforward calculations lead to the analog of (2.6) for the score function \( r_2 \), namely
\[
E_q[l(X)] - E_p[l(X)] = E_q[f^p_{2,l}(X)r_2(p, q)(X)].
\]

As will be shown in Section 4, specifying a Poisson distribution for the target \( p \) in (3.4) yields the scaled score function whose variance is the so-called discrete Fisher information introduced in [7].

4. Applications to a Poisson target

Working as in [10] it is easy to obtain, from (2.6) and (3.4), inequalities of the form
\[
\forall h \in \mathcal{H}_p(q) := \sup_{h \in \mathcal{H}} |E_q[h(X)] - E_p[h(X)]| \leq \kappa(p, q)\mathcal{J}(p, q),
\]
where $\mathcal{H}$ is, as usual, a suitably chosen class of functions, $\kappa(p, q)$ are constants depending on both $p$ and $q$ and $\mathcal{J}(p, q)$ is a so-called information distance between $p$ and $q$, which is given by the variance of one of the score functions (2.3) or (3.2) introduced in the two previous sections. The main difficulty then resides in computing the constants appearing in these inequalities and in putting the information distance $\mathcal{J}$ to good use. Such computations are not the primary purpose of the present paper. Hence we choose to focus on a Poisson target, for which much is already known. From here onwards we therefore only consider $p = Po(\lambda)$, the mean-$\lambda$ Poisson density.

We first adapt the results from Section 2. The score function (2.3) becomes

$$r_1(Po(\lambda), q)(x) = \frac{\lambda}{x + 1} \left(1 - \frac{(x + 1)q(x + 1)}{\lambda q(x)}\right) \mathbb{1}_{[0, \ldots, N]}(x)$$

so that (2.6) yields

$$E_q[l(X)] - E_{Po}[l(X)] = E_q\left[\left(\frac{\sqrt{\lambda} f^p_{1, h}(X + 1)}{X + 1}\right) \sqrt{X} \left(1 - \frac{(X + 1)q(X + 1)}{\lambda q(X)}\right)\right] + e^N_{p, q}(l).$$

One recognizes, in the rhs of (4.1), the scaled score function whose variance yields the scaled Fisher information

$$K_1(Po(\lambda), q) := \lambda E_q \left[\left(\frac{(X + 1)q(X + 1)}{\lambda q(X)} - 1\right)^2\right].$$

This information distance is subadditive over convolutions; this is useful when computing rates of convergence for sums towards the Poisson distribution (see, e.g., [1, 8]). Using a Poincaré inequality, [8] show that, for $q$ a discrete distribution with mean $\lambda$,

$$\|q - Po(\lambda)\|_{TV} \leq \sqrt{2K_1(Po(\lambda), q)},$$

with $\| \cdot \|_{TV}$ indicating the total variation distance. From (4.1) and Hölder’s inequality we obviously recover a much more general result, namely

$$d_\mathcal{H}(Po(\lambda), q) = \sup_{h \in \mathcal{H}} \left| E_q[h(X)] - E_{Po(\lambda)}[h(X)] \right|$$

$$\leq H_{1, \mathcal{H}}(Po(\lambda), q) \sqrt{K_1(Po(\lambda), q)},$$

where the constant

$$H_{1, \mathcal{H}}(Po(\lambda), q) := \sup_{h \in \mathcal{H}} \left( E_q \left[\left(\frac{\sqrt{\lambda} f^p_{1, h}(X + 1)}{X + 1}\right) \sqrt{X} \left(1 - \frac{(X + 1)q(X + 1)}{\lambda q(X)}\right)\right]^2 + e^N_{p, q}(h) \sqrt{K_1(Po(\lambda), q)} \right)$$

is some kind of general Stein (magic) factor. The notation $H$ for these constants is borrowed from [1] where similar relationships are obtained, within the context of compound Poisson approximation.
Likewise, in the notations of Section 3, we have \( \tilde{p}(x) = \lambda x^{-1}/(x - 1) \Pi_{\mathbb{N}}(x) \) so that \( \tilde{p}(x + 1)/p(x) = e^\lambda \Pi_{\mathbb{N}}(x) \) and \( \tilde{p}(x)/p(x) = e^\lambda x/\lambda \Pi_{\mathbb{N}}(x) \). Hence for all \( q \) with full support \( \mathbb{N} \) we get

\[
r_2(Po(\lambda), q)(x) = e^\lambda \left( \frac{q(x - 1)}{q(x)} - \frac{x}{\lambda} \right) \Pi_{\mathbb{N}}(x)
\]

so that (3.4) yields

\[
E_q[l(X)] - E_p[l(X)] = E_q \left[ \left( \lambda^{-1} e^\lambda f_{2,l}(X) \right) \left( \frac{\lambda q(X - 1)}{q(X)} - X \right) \right]. \tag{4.3}
\]

One recognizes, in the rhs of (4.3), a special instance of the Katti-Panjer score function introduced in [1, equation (3.1)] and whose variance yields our second information distance, namely the discrete Fisher information

\[
K_2(Po(\lambda), q) := E_q \left[ \left( \frac{\lambda q(X - 1)}{q(X)} - X \right)^2 \right]. \tag{4.4}
\]

This is easily shown to be related to the discrete Fisher information distance \( I(q) := E_q[(qX - 1)/(qX - 1)^2] \) introduced in [7]. The information distance (4.4) has been shown to be subadditive over convolutions (see [1]). From (4.3) and Hölder’s identity we obviously recover the following general relationship

\[
d_H(Po(\lambda), q) = \sup_{h \in \mathcal{H}} \left| E_q[h(X)] - E_{Po(\lambda)}[h(X)] \right|
\leq H_{2,H}(Po(\lambda), q) \sqrt{K_2(Po(\lambda), q)}, \tag{4.5}
\]

where the constant

\[
H_{2,H}(Po(\lambda), q) := \sqrt{\sup_{h \in \mathcal{H}} \left[ \left( \lambda^{-1} e^\lambda f_{2,l}(X) \right)^2 \right]}
\]

is, again, some kind of general Stein (magic) factor.

We conclude the paper with explicit computations.

**Proposition 4.1.** Take \( p = Po(\lambda) \) and \( q \) a pdf with support \([0, \ldots, N]\). Then

\[
\|p - q\|_{TV} \leq \sqrt{N} H(\lambda) \sqrt{K_1(Po(\lambda), q)} + e^N_q \quad \text{and} \quad \|p - q\|_{TV} \leq H(\lambda) \sqrt{K_2(Po(\lambda), q)}, \tag{4.6}
\]

where the error term \( e^N_q \) is of order \( q(N)/(N + 1) \) and \( H(\lambda) = 1 \wedge \sqrt{\frac{2}{e^\lambda}} \). The second bound in (4.6) only holds if \( N = \infty \).

**Proof.** Choose

\[
h(x) := I[p(x) \leq q(x)] - I[p(x) \geq q(x)] = 2I[p(x) \leq q(x)] - 1.
\]

Then obviously \( E_p[h(X)] \) and \( E_q[h(X)] \) exist, and

\[
\sum_x |p(x) - q(x)| = E_q[h(X)] - E_p[h(X)]
\]
so that, by definition of the total variation distance, we get
\[ \|p - q\|_{TV} = \frac{1}{2} (E_q [h(X)] - E_p [h(X)]).\]

It now suffices to apply (4.2) and (4.5), respectively, to obtain the announced relationships. All that remains is to compute bounds on the constants.

In the first case, known results on the properties of \( f_{1,h} \) show that the claim on the error term is evident. The expression for the constant \( \sqrt{\lambda H(\lambda)} \) is derived from the quantity
\[ E_q \left( \frac{\sqrt{\lambda f_{1,h}(X + 1)}}{X + 1} \right)^2, \]
with \( h \) specified (and bounded by 2). Indeed, from (2.5) and [5, Theorem 2.3], we get
\[ \left\| \frac{f_{1,h}(x + 1)}{x + 1} \right\| = \left\| \frac{x!}{\lambda^{x+1}} \sum_{k=0}^{x} (h(k) - E_p[h(X)]) \frac{\lambda^k}{k!} \right\| \]
\[ \leq \left( 1 \wedge \sqrt{\frac{2}{\lambda}} \right) \left( \sup_{i \in \mathbb{N}} h(i) - \inf_{i \in \mathbb{N}} h(i) \right). \]

The claim follows.

For \( \lambda < 2/e \), \( H(\lambda) = 1 \) and hence the bounding constant for the scaled Fisher information \( K_1(P\theta(\lambda), q) \) becomes \( \sqrt{\lambda} < \sqrt{2/e} \); in case \( \lambda > 2/e \), this constant equals \( \sqrt{2/e} \). Since the error term \( e_q^N \) is either null for \( N = \infty \) or negligible in comparison to the term involving the scaled Fisher information, our bounds on the total variation distance corresponding to the first inequality in (4.6) improve on those proposed in [8], where the bounding constant is given by \( \sqrt{2} \), while ours are inferior to \( \sqrt{2/e} \). For the sake of illustration we conclude this section by applying Proposition 4.1 to the three examples studied in [8].
Example 4.1. Take $X_i$ i.i.d. Bernoulli ($\lambda/n$) random variables and let $S_n = \sum_{i=1}^n X_i$. Put $q = P_{S_n}$, the density associated with the sum $S_n$. Then straightforward calculations reveal that $K_1(Po(\lambda), q) = \lambda^2/(n(n - \lambda))$ and $e_q^i$ is of order $\lambda^n/n^{n+1}$. Consequently, we have

$$\|P_{S_n} - Po(\lambda)\|_{TV} \leq \sqrt{\frac{2}{e}} \frac{\lambda}{\sqrt{n(n - \lambda)}} + c \frac{\lambda^n}{n^{n+1}}$$

for some positive constant $c$ and sufficiently large $n$. This is an improvement over the $(2 + \epsilon)\lambda/n$, $\epsilon > 0$, bound obtained in [8].

Example 4.2. Consider the same situation as above, but with $\lambda$ replaced by $\mu \sqrt{n}$ for some $\mu > 0$. From the previous example, we directly deduce that

$$\|P_{S_n} - Po(\mu \sqrt{n})\|_{TV} \leq \sqrt{\frac{2}{e}} \frac{\mu}{\sqrt{n(\sqrt{n} - \mu)}} + \frac{\mu^n}{n^{n/2+1}}$$

for some positive constant $c$ and sufficiently large $n$. Although the rate is good and the constant above is again an improvement over the one obtained in [8], it is still not as good as the optimal constant $\sqrt{1/(2\pi e)}$ derived in [3].

Example 4.3. Finally take $X_i$ independent geometric random variables with respective distributions $P_i(x) = (1 - q_i)^x q_i \mathbb{1}_{x}(x)$, where $0 \leq q_i \leq 1$ for all $i = 1, \ldots, n$. Let $S_n = \sum_{i=1}^n X_i$ and $q = P_{S_n}$, the density associated with the sum $S_n$. Put $\lambda = \mathbb{E}[S_n]$. The subadditivity property of $K_1(Po(\lambda), q)$ states that (see [8, Proposition 3])

$$K_1(Po(\lambda), P_{S_n}) \leq \sum_{i=1}^n \frac{(1 - q_i)}{\lambda q_i} K_1(Po(e_i), P_{X_i}),$$

where $P_{X_i}$ is the density associated with $X_i$ and $e_i = \mathbb{E}[X_i]$. Straightforward computations show that $K_1(Po(e_i), P_{X_i}) = (1 - q_i)^2/q_i$. Since here $e_q^\infty = 0$, it follows that

$$\|P_{S_n} - Po(\lambda)\|_{TV} \leq \sqrt{\frac{2}{\lambda e}} \left( \sum_{i=1}^n \frac{(1 - q_i)^2}{q_i} \right)^{1/2}$$

for sufficiently large $n$. Again we improve on the constant obtained in [8]. Note that restricting, as in [8], to the case where $q_i = n/(n + \lambda)$ yields a rate of $\sqrt{2/e}(\lambda/\sqrt{n(n + \lambda)})$.

Next consider the second information functional $K_2$. Direct computations yield an expression for $K_2(Po(\lambda), P_{S_n})$ which we will dispense of here, and hence an explicit bound on $\|P_{S_n} - Po(\lambda)\|_{TV}$ can also easily be obtained in terms of this functional as well. The general expression appears inscrutable, and hence we restricted our attention to the case where $q_i = n/(n + \lambda)$. There, numerical evaluations in Mathematica 7 encourage us to suggest that the second information distance provides a better rate than the $\sqrt{2/e}(\lambda/\sqrt{n(n + \lambda)})$ mentioned above, at least for moderate values of $\lambda$ and large values of $n$ (that is, $n \geq 100$).
5. Final comments

The results reported in the present work are to be read in conjunction with those reported in [10]. The main message of these two papers is that all the so-called Fisher information functionals used in the literature on Gaussian and Poisson approximation bear an interpretation in terms of a specific Stein characterization. As concluding remark to the present paper we wish to stress the fact that our method applies to many more distributions than just the Gaussian or the Poisson (e.g., the compound Poisson, allowing comparisons with the results of [1]), and in particular provides generalized scaled Fisher information distances between any two (nice) distributions. Of course much remains to be explored, in particular on the properties of these generalized information functionals. However the freedom of choice for the densities as well as for the test functions in (2.6), (3.4) and [10, Theorem 2.3] makes us confident that there remains much to be gained from a crafty usage of such identities.

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