The Ultraproducts of Quasirandom Groups

Yilong Yang
email: yy26@math.ucla.edu

University of California, Los Angeles

February 10, 2015

Abstract

In this paper, we shall prove that an ultraproduct of non-abelian finite simple groups is either finite simple, or has no finite dimensional unitary representation other than the trivial one. Then we shall generalize this result for other kinds of quasirandom groups. A group is called $D$-quasirandom if all of its nontrivial representations over the complex numbers have dimensions at least $D$. We shall study the question of whether a non-principal ultraproduct of a given sequence of quasirandom groups remains quasirandom, and whether an ultraproduct of increasingly quasirandom groups becomes minimally almost periodic (i.e. no non-trivial finite-dimensional unitary representation at all). We answer this question in the affirmative when the groups in question are simple, quasisimple, semisimple, or when the groups in question have bounded number of conjugacy classes in their cosocles (the intersection of all maximal normal subgroups), or when the groups are arbitrary products (not necessarily finite) of the groups just listed.

We shall also present with an ultraproduct of increasingly quasirandom groups with a non-trivial one-dimensional representation. We also obtain some results in the case of semi-direct products and short exact sequences of quasirandom groups. Finally, two applications of our results are given, one in triangle patterns of quasirandom groups and one in self-Bohrifying groups.

Our main tools are some variations of the covering number for groups, different kinds of length functions on groups, and the classification of finite simple groups.

1 Introduction

In this paper, the following theorem about non-abelian finite simple groups is proven.

Theorem 1.1. An ultraproduct of non-abelian finite simple groups is either finite simple, or has no finite dimensional unitary representation other than the trivial one.

Non-abelian finite simple groups are not the only kind of groups exhibiting such a behavior. In this paper, the author will prove the existence of many classes of groups with similar results. Such a class is called a *quasiuniformly periodic* class (see Definition 1.14 and Theorem 1.20). It turns out that such a behavior has a very close link to the notion of quasirandom groups, defined by Gowers [9].

2010 Mathematics subject classification. Primary 20D06, Secondary 20D05, 03C20, 43A65

Key words and phrases. Quasirandom Group, Finite Simple Group, Minimally Almost Periodic Group, Self-Bohrifying Group, Ultraproduct.
Definition 1.2. For a positive integer \( D \), a group \( G \) is \( D \)-quasirandom if it has no non-trivial unitary representation of dimension less than \( D \).

Remark 1.3. The term “quasirandom” is due to the fact that a group is \( D \)-quasirandom for some large \( D \) iff all its Cayley graphs by any subset of generators are \( c \)-quasirandom graphs for some large \( c \). (See Gowers [9])

In this paper, we allow the group \( G \) to be infinite, and all representations considered in this paper are over \( \mathbb{C} \). We shall informally say that a group is quasirandom when the group is \( D \)-quasirandom for some large \( D \).

Example 1.4 (Gowers [9]).

1. A group (not necessarily finite) is \( 2 \)-quasirandom iff it is perfect. The reason is that a non-perfect group has a non-trivial abelian quotient, which in turn has a non-trivial homomorphism into the circle group, which is isomorphic to \( U_1(\mathbb{C}) \). A perfect group, on the other hand, can only have a trivial homomorphism into the abelian group \( U_1(\mathbb{C}) \).

2. A finite perfect group with no normal subgroup of index less than \( n \) is \( \sqrt{\log n}/2 \)-quasirandom. In fact, using a form of Jordan’s theorem [8], a finite perfect group with no normal subgroup of index less than \( n \) is \( \Theta(\log n) \)-quasirandom.

3. In particular, a nonabelian finite simple group \( G \) is \( \Theta(\log n) \)-quasirandom if it has \( n \) elements.

4. The alternating group \( A_n \) is \((n−1)\)-quasirandom for \( n > 5 \), and the special linear group \( SL_2(F_p) \) is \( \frac{p−1}{2} \)-quasirandom for any prime \( p \).

5. In the converse direction of the statement 2 of Example [17], any \( D \)-quasirandom group must have more than \( (D−1)^2 \) elements.

In the case of infinite groups, it might happen that a group has no nontrivial finite dimensional representation at all.

Definition 1.5. An infinite group is minimally almost periodic if it has no nontrivial finite dimensional unitary representation.

Remark 1.6. The term “minimally almost periodic” is initially used by von Neumann and Wigner [16], due to the fact that their only almost periodic functions are the constants. Minimally almost periodic groups have nice consequences in ergodic theory and topological group theories, see e.g. [3] and [17].

A group is minimally almost periodic iff it is \( D \)-quasirandom for all \( D \). Then it is natural to wonder whether some sort of limits of increasingly quasirandom groups would give us a minimally almost periodic group. One of such limit to consider is the ultraproduct, which can be defined as the following. These definitions can be found in any set theory text books, e.g. [15].

Definition 1.7. A filter on \( \mathbb{N} \) is a collection \( \omega \) of subsets of \( \mathbb{N} \) such that:

1. \( \emptyset \notin \omega \);

2. If \( X \in \omega \) and \( X \subset Y \), then \( Y \in \omega \);
3. If \( X, Y \in \omega \), then \( X \cap Y \in \omega \).

An **ultrafilter** is a filter that is maximal with respect to the containment order. A **non-principal ultrafilter** is an ultrafilter which contains no finite subset of \( \mathbb{N} \).

**Definition 1.8.** Given a sequence of groups \( (G_i)_{i \in \mathbb{N}} \), let \( G \) be their direct product. Given an ultrafilter \( \omega \) on \( \mathbb{N} \), let \( N := \{ g = (g_i)_{i \in \mathbb{N}} \in G : \{ i \in \mathbb{N} : g_i = e \} \in \omega \} \), which is clearly a normal subgroup of \( G \). Then we call \( G/N \) the **ultraproduct** of the groups \( (G_i)_{i \in \mathbb{N}} \) by \( \omega \), denoted by \( \prod_{i \to \omega} G_i \).

**Remark 1.9.** An ultrafilter \( \omega \) is principal (i.e. not non-principal) iff we can find an element \( n \in \mathbb{N} \) such that for all subsets \( A \subseteq \mathbb{N} \), we have \( A \in \omega \) iff \( n \in A \). In this case, the corresponding ultraproduct of groups \( (G_i)_{i \in \mathbb{N}} \) is isomorphic to \( G_n \). Therefore, in practice, the useful ultrafilters are usually non-principal.

The particular choice of ultraproduct is not that important. As long as we fix a non-principal ultrafilter, then all the discussion for the rest of the paper will be true for the ultraproduct of this ultrafilter.

One should morally interpret an ultrafilter as a way to define “most”, and one would expect that certain statements are true for the ultraproduct group iff they are true for “most” of the groups forming the ultraproduct. This turns out to be true for first order statements by Loś’s Theorem (see e.g. [14]). Here we shall state the special case of the theorem relevant to our discussion.

**Theorem 1.10** (A special case of Loś’s Theorem). Let \( \omega \) be an ultrafilter. Let \( G_n \) be groups, and let \( G = \prod_{i \to \omega} G_i \) be their ultraproduct. Let \( \phi(a_1, a_2, \ldots, a_n) \) be a property that, under specific arrangement of quantifiers for \( a_1, \ldots, a_n \), certain collection of identities are true. Then \( \phi(a_1, a_2, \ldots, a_n) \) is true for \( G \) iff the set of numbers \( \{ i \in \mathbb{N} : \phi(a_1, \ldots, a_n) \text{ is true for } G_i \} \) is in \( \omega \). (i.e. iff \( \phi(a_1, a_2, \ldots, a_n) \) is true for “most” \( G_i \).)

**Example 1.11.**

1. A group \( G \) is abelian if the following property is true: for any \( x, y \in G \), we have the identity \( xy = yx \). So by Loś’s Theorem, the ultraproduct of groups is abelian iff “most” groups in this ultraproduct are abelian.

2. A group \( G \) has commutator length \( \ell \) if the following property is true: for any \( x \in G \), there exist \( a_1, b_1, a_2, b_2, \ldots, a_{\ell}, b_{\ell} \in G \), such that \( x = \prod (a_i b_i a_i^{-1} b_i^{-1}) \). Therefore, by Loś’s Theorem, the ultraproduct of groups have commutator length \( \ell \) iff “most” groups in this ultraproduct have commutator length \( \ell \).

3. For any group \( G \), the following property is true: there exists \( e \in G \) such that for all \( x \in G \), we have \( ex = xe = x \). Therefore by Loś’s Theorem, the ultraproduct of groups have an identity element. Similar argument will show that, in particular, the ultraproduct of groups is a group.

4. Let \( G_n \) be a group of \( n \) elements for each \( n \). Then all \( G_n \) are finite groups. But the ultraproduct \( G = \prod_{n \to \omega} G_n \) is not finite. In fact, \( G \) is uncountable.

5. Let \( G_n = \mathbb{Z}/n\mathbb{Z} \), the cyclic group of \( n \) elements. Then all \( G_n \) are cyclic groups. But the ultraproduct \( G = \prod_{n \to \omega} G_n \) is uncountable, and thus cannot be generated by one element.
Ultraproducts preserve all local properties at the scale of elements. In particular, all element-wise identities are preserved. But global properties of a group, like being finite or finitely generated, might be lost after taking ultraproducts.

In view of the ultraproduct construction, one may wonder if a non-principal ultraproduct of increasingly quasirandom group is minimally almost periodic. This turns out to be false. In particular, we have the following counterexample, pointed out by László Pyber.

**Example 1.12.** We recall that a group $G$ (not necessarily finite) is $2$-quasirandom iff $G$ is perfect. We claim that there is a sequence of $D_i$-quasirandom groups $(G_i)_{i \in \mathbb{Z}^+}$ with $\lim_{i \to \infty} D_i = \infty$, whose ultraproduct by a non-principal ultrafilter is not even perfect.

Using the construction of Holt and Plesken, one may construct a finite perfect group $G_{p,n}$ for each prime $p \geq 5$ and positive integer $n$, such that an element of $G_{p,n}$ cannot be written as a product of less than $n$ commutators, and that the only simple quotient of $G_{p,n}$ is $\text{PSL}_2(\mathbb{F}_p)$, the projective special linear group of rank $2$ over a field of $p$ elements. Then by the second statement of Example 1.4, any $D_i$, $G_{p,n}$ is $D_i$-quasirandom for large enough $p$.

Let $G_i$ be $G_{p_i,i}$, where $(p_i)_{i \in \mathbb{Z}^+}$ is a strictly increasing sequence of primes. Then $G_i$ is $D_i$-quasirandom for some $D_i$ with $\lim_{i \to \infty} D_i = \infty$. Let $g_i \in G_i$ be an element which cannot be written as a product of less than $i$ commutators. Then $g_i = (g_i)_{i \in \mathbb{N}}$ corresponds to an element of the ultraproduct $G = \prod_{i \to \omega} G_i$ by any non-principal ultrafilter $\omega$. Then clearly $g$ cannot be written as a product of finite number of commutators in $G$. So $g$ is not in the commutator subgroup of $G$, and thus $G$ is not perfect.

However, a recent paper by Bergelson and Tao showed the following theorem, which shed some new light on this inquiry:

**Theorem 1.13 (Bergelson and Tao [5, Theorem 49 (i)]).** Let $\text{SL}_2(\mathbb{F}_{p_i})$ be a sequence of special linear groups of rank $2$ over fields of increasing prime order $p_i$. Then the ultraproduct $\prod_{i \to \omega} \text{SL}_2(\mathbb{F}_{p_i})$ by a non-principal ultrafilter $\omega$ is minimally almost periodic.

**Definition 1.14.** A class $\mathcal{F}$ of groups is a **q.u.p. (quasirandom ultraproduct property) class** if for any sequence of groups in $\mathcal{F}$ with quasirandom degree going to infinity, its non-principal ultraproducts will be minimally almost periodic.

**Definition 1.15.** A class $\mathcal{F}$ of groups is a **Q.U.P. class** if there is an unbounded non-decreasing function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that any ultraproduct of any sequence of $D$-quasirandom groups in $\mathcal{F}$ is $f(D)$-quasirandom.

**Remark 1.16.** A Q.U.P class is automatically a q.u.p. class. It is like an effective version of q.u.p. class, where it is able to keep track of the amount of quasirandomness passed down to the ultraproduct.

It turns out that in the case of finite simple groups, quasirandomness is equivalent to a local property, in the sense that Łos’s Theorem can be applied. Thus the class of simple groups is Q.U.P. And in the general cases, we can measure the “globalness” of the quasirandomness by looking at the size of the cosocle (defined below) of a group.

**Definition 1.17.** The **cosocle** of a group is the intersection of all its maximal normal subgroups.

**Remark 1.18.** The **socle** of a group is the subgroup generated by all minimal normal subgroups, and the cosocle is in a sense the opposite of the socle (and thus the name). It has important applications in finite group theory, for example see [2]. A cosocle can be seen as the smallest normal subgroup whose corresponding quotient group is a semisimple group. (i.e. a direct product of simple groups.)
Example 1.19.

1. The cosocle of a semisimple group is the trivial subgroup.
2. The cosocle of a quasisimple group is its center.
3. The cosocle of the symmetric group $S_n$ is the alternating group $A_n$.

In this paper, the following results will be proven.

Theorem 1.20. Let $n$ be any positive integer. Let $C_n$ be the class of groups that are arbitrary direct product (not necessarily finite) of finite quasisimple groups and finite groups whose cosocle has at most $n$ conjugacy classes. Then this class is a Q.U.P. class.

Corollary 1.21. The following classes are Q.U.P.

1. The class $C_A$ of finite alternating groups.
2. The class $C_S$ of finite simple groups.
3. The class $C_{QS}$ of finite quasisimple groups.
4. The class $C_{SS}$ of finite semisimple groups.
5. The class $C_{CS(n)}$ of finite groups with at most $n$ conjugacy classes in their cosocles.

There are some other results by the author, compiled here for the sake of completeness.

Theorem 1.22. Let $K$ be any positive integer, let $F, F'$ be q.u.p. (or Q.U.P.) classes. The following classes of groups are q.u.p. (or Q.U.P.):

1. $F \cup F'$.
2. Any subclass of $F$.
3. The class of groups that are direct products of at most $K$ groups in $F$.
4. The class of semi-direct product groups $G = N \rtimes H$ where $H \in F$, $N$ is finite, and an element $h \in H$ acts on $N$ by conjugation with no nontrivial fixed point.
5. The class of semi-direct product groups $G = N \rtimes H$ where $H \in F$, and every element of $N$ can be written as the product of at most $K$ conjugates of elements of $H$.
6. The class of semi-direct product groups $G = N \rtimes H$ where $H \in F$, and $N$ is abelian, and $G$ has commutator width $\leq K$.
7. A sequence of groups $G_i$ each with a normal subgroup $N_i$ such that $N_i, G_i/N_i \in F$, and $N_i$ is $D_i$-quasirandom with $\lim_{i \to \infty} D_i = \infty$.

All classes listed above have bounded commutator width. In view of this, the following conjecture was suggested by László Pyber.

Conjecture 1.23. For any integer $n$, the class of perfect groups with commutator width $\leq n$ (i.e. every element of these groups can be written as a product of $n$ commutators) is Q.U.P.
So far, we do not know if there is a non-Q.U.P but q.u.p. class of groups. Some applications of our results are already found. In a paper in preparation by Bergelson, Robertson and Zorin-Kranich [4, Theorem 1.12], it is shown that a sufficiently quasirandom group in a q.u.p. class will have many “triangles”. One may also use our method to find many self-Bohrifying groups. Both applications will be rigorously stated and explained in Section 8 of this paper.

Here we shall briefly outline the sections of this article:

1. A model case of the alternating groups to illustrate the general idea. (Section 2)
2. A group with some nice covering property is very quasirandom. (Section 3)
3. Covering property can ignore small cosocles. (Section 4)
4. Quasirandom finite quasisimple groups have nice covering properties. (Section 5)
5. Proof of Theorem 1.20. (Section 6)
6. Proof of Theorem 1.22. (Section 7)
7. Applications of our results. (Section 8)

Acknowledgement

I would like to thank Professor Terence Tao for introducing me to this area and for his patient guidance. I would also like to thank Professor Richard Schwartz, Professor Vitaly Bergelson and Professor Nikolay Nikolov for their helpful inputs, and thank Professor László Pyber for his helpful inputs and for pointing me to a number of very useful references.

2 The Class of Alternating Groups

Let $A_n$ denote the alternating group of rank $n$, and $S_n$ denote the symmetry group of rank $n$. We shall show that $C_A$ is a Q.U.P. class, as a simple illustration of the general idea to attack Theorem 1.20.

2.1 Quasirandom Alternating Groups have a nice Covering Property

Definition 2.1.

1. For any subset $A$ of a group $G$, the product set $A^n := \{a_1a_2 \ldots a_n : a_1, ..., a_n \in A\}$.
2. An element $g$ of a group $G$ is said to have covering number $K$ if its conjugacy class $C(g)$ has $(C(g))^K = G$.
3. Let $m$ be any positive integer or $\infty$. Then an element $g \in G$ has the covering property $(K, m)$ if $g^i$ has covering number $K$ for all $1 \leq i \leq m$.
4. A group $G$ has the covering property $(K, m)$ if it has an element with the covering property $(K, m)$.

Definition 2.2. An even permutation $\sigma \in A_n$ is exceptional if its cycles in the cycle decomposition have distinct odd lengths, or equivalently, if its conjugacy class in $A_n$ is distinct from its conjugacy class in $S_n$. 
Theorem 2.3 (Brenner [6, Lemma 3.05]). If an even permutation $\sigma \in A_n$ is fixed-point free and non-exceptional, then $A_n = C(\sigma)^4$.

Theorem 2.4. For any $m \in \mathbb{Z}^+$, $A_n$ has the covering property $(4, m)$ for $n$ large enough.

Proof. Pick any odd prime $p > m$, and pick another prime $q > p$.

Since $p, q$ are necessarily coprime, for any large enough integer $n$, we can find positive integers $a, b$ such that $n = ap + bq$. Then let $\sigma \in S_n$ be a permutation composed of $a$ $p$-cycles and $b$ $q$-cycles, where all cycles are disjoint.

Since $p, q$ are odd, $\sigma$ is an even permutation in $A_n$. Furthermore, for $n$ large enough, $a$ or $b$ can be chosen to be larger than 1, so $\sigma$ will be non-exceptional. $\sigma$ is also fixed-point free by construction. So $A_n = C(\sigma)^4$.

Now clearly $\sigma^i$ will also have a cycle decomposition of $a$ $p$-cycles and $b$ $q$-cycles for all $1 \leq i \leq p - 1$, and this implies that $A_n = C(\sigma^i)^4$ for all $1 \leq i \leq p - 1$. So $A_n$ has the covering property $(4, p - 1)$. Since $p - 1 \geq m$, $A_n$ has the covering property $(4, m)$. \hfill \Box

Corollary 2.5. For any $m \in \mathbb{Z}^+$, any $D'$-quasirandom alternating group has the covering property $(4, m)$ for large enough $D'$.

2.2 The Covering Property passes to Ultraproducts and implies Quasirandomness

Theorem 2.6. Let $G_n$ be a sequence of groups such that all but finitely many of them have covering property $(K, m)$. Then any ultraproduct of them by a non-principal ultrafilter will have covering property $(K, m)$.

Proof. A group $G$ has covering property $(K, m)$ if it satisfy the following property: there exists $g \in G$ such that for all $x \in G$, there exist $a_{i,j} \in G$ for all $1 \leq i \leq m, 1 \leq j \leq K$ with $x = \prod_j (a_{i,j}g^{a_{i,j}^{-1}})$ for all $i$. So by Los Theorem, such a property is passed to the ultraproduct group. \hfill \Box

We now state a special case of Theorem 3.4, proven in Section 3.

Theorem 2.7. Any group $G$ (not necessarily finite) with the covering property $(K, m)$ is $D$-quasirandom when $m > c_DK^{D^2}$ for some constant $c_D$ depending only on $D$.

Theorem 2.8. $C_A$ is a Q.U.P. class.

Proof. For any $D \in \mathbb{Z}^+$, find $m > c_D4^{D^2}$ and find $D' \in \mathbb{Z}$ such that any $D'$-quasirandom alternating group has the covering property $(4, m)$. Let $G$ be an ultraproduct of $D'$-quasirandom alternating groups. Then $G$ will also have the covering property $(4, m)$. Then by Theorem 2.7, $G$ is $D$-quasirandom. \hfill \Box

3 Covering Properties Implies Quasirandomness

This section is devoted to obtaining some local properties (in the sense that we can apply Los’s Theorem) that guarantee the quasirandomness of a group.

Definition 3.1.
1. An element $g$ of a group $G$ is said to have **symmetric covering number** $K$ if $(C(g) \cup C(g^{-1}))^K = G$.

2. Let $m$ be a positive integer or $\infty$. Then an element $g \in G$ has the **symmetric covering property** $(K, m)$ if $g^i$ has symmetric covering number $K$ for all $1 \leq i \leq m$.

3. A group $G$ has the **symmetric covering property** $(K, m)$ if it has an element $g \in G$ with the symmetric covering property $(K, m)$.

4. A group $G$ has the (symmetric) covering property $(K, m)$ mod $N$ for some normal subgroup $N$ if $G/N$ has the (symmetric) covering property $(K, m)$.

**Definition 3.2.**

1. A pair of elements $(g, g')$ of a group $G$ is said to have **symmetric double covering number** $(K_1, K_2)$ if we have $(C(g) \cup C(g^{-1}))^{K_1} (C(g') \cup C((g')^{-1}))^{K_2} = G$.

2. Let $m_1, m_2$ be positive integers or $\infty$. A pair of elements $(g, g')$ in $G$ has the symmetric double covering property $[[K_1, m_1], [K_2, m_2]]$ if $(g^i, (g')^j)$ has symmetric double covering number $(K_1, K_2)$ for all $1 \leq i \leq m_1, 1 \leq j \leq m_2$.

3. A group $G$ has the symmetric double covering property $[[K_1, m_1], [K_2, m_2]]$ if it has a pair of elements $(g, g')$ in $G$ with the symmetric double covering property $[[K_1, m_1], [K_2, m_2]]$.

4. A group $G$ has the symmetric double covering property $[[K_1, m_1], [K_2, m_2]]$ mod $N$ for some normal subgroup $N$ if $G/N$ has the symmetric double covering property $[[K_1, m_1], [K_2, m_2]]$.

**Remark 3.3.**

1. Suppose $K < K'$. Then an element with covering number $K$ has covering number $K'$. In general, the (symmetric) covering property $(K, m)$ implies the (symmetric) covering property $(K', m')$ when $K' \geq K, m' \leq m$. A similar statement is also true for the symmetric double covering property.

2. Any symmetric covering property is always weaker than the corresponding non-symmetric covering property.

3. Any group with the symmetric covering property $(K, m)$ has the symmetric double covering property $[[1, \infty], (K, m)]$. This is easily seen by taking $g$ to be the identity, and taking $g'$ to be the element with the symmetric covering property $(K, m)$.

4. By imitating the definition of the symmetric double covering property, one can in fact define the symmetric $n$-tuple covering property for groups. As $n$ grows larger and larger, the corresponding covering property will become weaker and weaker. Note that most results throughout this paper would still hold by replacing the symmetric double covering property by the symmetric $n$-tuple covering property, though for our purpose here, the symmetric double covering property is enough.

**Theorem 3.4** (Local criterion for quasirandomness). Any group $G$ (not necessarily finite) with the symmetric double covering property $[[K_1, m_1], [K_2, m_2]]$ is $D$-quasirandom when $m_i > c_D K_i^{D^2}$ for $i = 1, 2$, for some constant $c_D$ depending only on $D$. 

8
**Corollary 3.5.** Any group $G$ (not necessarily finite) with the symmetric covering property $(K, m)$ is $D$-quasirandom when $m > c_DK^2$, for some constant $c_D$ depending only on $D$.

**Corollary 3.6.** Any group $G$ (not necessarily finite) with the covering property $(K, m)$ is $D$-quasirandom when $m > c_DK^2$, for some constant $c_D$ depending only on $D$.

**Remark 3.7.** We note here that a partial converse, Corollary [6,4], of the above result is true, i.e. quasirandomness implies a nice covering property mod cosocle. The proof of this converse will be presented in Section [6].

The proof Theorem [3,4] will be the main part of this section. We shall first explore some geometric structure of $U_D(\mathbb{C})$.

**Definition 3.8.** The Hilbert-Schmidt norm of an $n$ by $n$ complex matrix $A$ is $||A|| = \sqrt{\text{Tr}(A^*A)}$.

**Lemma 3.9.**

1. The Lie group $U_D(\mathbb{C})$ has a Riemannian metric $d : U_D(\mathbb{C}) \times U_D(\mathbb{C}) \to \mathbb{R}$ such that $d(A, B) = ||B - A||$ for all $A, B \in U_D(\mathbb{C})$. The norm here is the Hilbert-Schmidt norm.

2. This metric is bi-invariant in the sense that $d(AB, AC) = d(BA, CA) = d(B, C)$ for all $A, B, C \in U_D(\mathbb{C})$.

3. This metric induces a Haar measure, and the volume of $U_D(\mathbb{C})$ under this Haar measure is finite, and $\text{vol}(U_D(\mathbb{C})) = \frac{(2\pi)^{N(N+1)/2}}{N!^{2N}}$.

4. Under this metric, $U_D(\mathbb{C})$ has non-negative Ricci curvature everywhere.

5. A geodesic ball of radius $r$ in $U_D(\mathbb{C})$ will have volume at most $c_Dr^{N^2}$ for some constant $c_D$ depending only on $D$.

**Proof.** These are very standard facts. See e.g. [18] and [7].

**Definition 3.10.** Let $G$ be any group. A non-negative function $\ell : G \to \mathbb{R}$ is called a **length function** if it has the following properties.

1. $\ell(g) = 0$ iff $g$ is the identity element.

2. $\ell$ is symmetric, i.e. $\ell(g) = \ell(g^{-1})$ for all $g \in G$.

3. $\ell$ is conjugate invariant, i.e. $\ell(ghg^{-1}) = \ell(h)$ for all $g, h \in G$.

4. $\ell$ satisfies the triangle inequality, i.e. $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in G$.

A **pseudo length function** is a non-negative function $\ell : G \to \mathbb{R}$ satisfying 2, 3 and 4 above.

**Lemma 3.11.** Let $G$ be a group, and suppose $g, g' \in G$ have symmetric double covering number $(K_1, K_2)$. Let $\phi : G \to H$ be any homomorphism and let $\ell$ be a length function of $H$. Then for all $h \in G$, we have $\ell(\phi(h)) \leq K_1\ell(\phi(g)) + K_2\ell(\phi(g'))$.

**Proof.** For any $h \in G$, $h$ can be written as the product of $K_1$ conjugates of $g$ or $g^{-1}$ multiplied with the product of $K_2$ conjugates of $g'$ or $(g')^{-1}$. So by triangle inequality, we have $\ell(\phi(h)) \leq K_1\ell(\phi(g)) + K_2\ell(\phi(g'))$. 


Proposition 3.12. The function $\ell : U_D(\mathbb{C}) \to \mathbb{R}$ defined by $\ell(A) = d(A, I)$ is a length function.

Proof. Let $A, B$ be any unitary matrices. Clearly $\ell(A) = 0$ iff $A = I$. We also have $\ell(A) = d(A, I) = d(AA^{-1}, I) = \ell(A^{-1})$, and $\ell(BAB^{-1}) = d(BAB^{-1}, I) = d(BA, B) = d(A, I) = \ell(A)$.

Finally, $\ell(AB) = d(AB, I) \leq d(AB, B) + d(B, I) = d(A, I) + d(B, I) = \ell(A) + \ell(B)$. \hfill \square

Lemma 3.13. For any $\epsilon > 0$, any $m > \frac{v_D}{\epsilon^2}$ points in $U_D(\mathbb{C})$ will have two points with distance smaller than $\epsilon$, where $v_D$ is a constant depending only on $D$.

Proof. This follows from a volume packing argument. Since our metric is bi-invariant, each ball of radius $\epsilon/2$ in $U_D(\mathbb{C})$ has the same volume $\text{vol}(B_{\epsilon/2})$. Since $U_D(\mathbb{C})$ has finite volume, let $m$ be an integer larger than $\text{vol}(U_D(\mathbb{C}))/\text{vol}(B_{\epsilon/2}) \leq \frac{c_D}{\epsilon^2}$ for some constant $c_D$.

Now for any $m$ points in $U_D(\mathbb{C})$, suppose any two of them have distance larger than $\epsilon$. Then the balls of radius $\epsilon/2$ centered at these $m$ points will be disjoint and contained in $U_D(\mathbb{C})$, which is impossible. So two of the points have distance smaller than $\epsilon$. \hfill \square

Lemma 3.14. For any non-trivial cyclic subgroup of $U_D(\mathbb{C})$, there is always an element of it with length larger than $\sqrt{2}$.

Proof. Let $A$ be any nontrivial element of $U_D(\mathbb{C})$ of finite order. Let $\lambda_1, ..., \lambda_D$ be its eigenvalues, and WLOG let $\lambda_1 \neq 1$ be a primitive $n$-th root of unity. Then replacing $A$ by a proper power of itself, we may assume that $\lambda_1$ is an $n$-th root of unity closest to $-1$. Then in particular, $|\lambda_1 - 1| > \sqrt{2}$.

Then we know $\ell(A)^2 = \text{Tr}(A - I)^*(A - I) = \sum_{i=1}^{D} |\lambda_i - 1|^2 \geq |\lambda_1 - 1|^2 > 2$.

Now suppose $A$ has infinite order. Let $\lambda_1, ..., \lambda_D$ be its eigenvalues, and WLOG let $\lambda_1 \neq 1$ be an element of infinite order on the unit circle. Then replacing $A$ by a proper power of itself, we may assume that $\lambda_1$ is arbitrarily close to $-1$. Then in particular, $|\lambda_1 - 1| > \sqrt{2}$. Then we are done by the same computation. \hfill \square

Proof of Theorem 3.3 For any $\epsilon_1, \epsilon_2 > 0$, pick $m_1 > \frac{v_D}{\epsilon_1^2}$ and $m_2 > \frac{v_D}{\epsilon_2^2}$. For any unitary representation $\phi : G \to U_D(\mathbb{C})$ of a group $G$ with the symmetric double covering property $[(K_1, m_1), (K_2, m_2)]$, we may find elements $h, h' \in G$ for this symmetric double covering property.

Now consider the points $I, \phi(h), \phi(h^2), ..., \phi(h^{m_1})$. By Lemma 3.13 since $m_1 > \frac{v_D}{\epsilon_1^2}$, we can find two points with distance less than $\epsilon_1$. Say $d(\phi(h^s), \phi(h^t)) < \epsilon_1$ for some $1 \leq s < t \leq m_1$. Then $\ell(\phi(h^{t-s})) = \ell(\phi(h^{t-s}), I) = d(\phi(h^t), \phi(h^s)) < \epsilon_1$. So we have $\ell(\phi(h^i)) < \epsilon_1$ for some $1 \leq i \leq m_1$. Similarly we have $\ell(\phi(h^{t-j})) < \epsilon_2$ for some $1 \leq j \leq m_2$.

To sum up, there are elements $g, g' \in G$ with symmetric double covering number $(K_1, K_2)$, and $\ell(\phi(g)) < \epsilon_1$, $\ell(\phi(g')) < \epsilon_2$. So by Lemma 3.11 all elements of $\phi(G)$ would have length smaller than $K_1\epsilon_1 + K_2\epsilon_2$.

Now pick $\epsilon_1, \epsilon_2$ small enough so that $K_1\epsilon_1 + K_2\epsilon_2 < \sqrt{2}$. (Say $\epsilon_1 < \frac{\sqrt{2}}{2K_1}$ and $\epsilon_2 < \frac{\sqrt{2}}{2K_2}$) Then all elements of $\phi(G)$ would have length smaller than $\sqrt{2}$. But by Lemma 3.14 this means $\phi(G)$ is trivial.

Therefore, a group with symmetric double covering property $[(K_1, m_1), (K_2, m_2)]$ will be $D$-quasirandom if $m_1 \geq c_D K_1^D$ and $m_2 \geq c_D K_2^D$. \hfill \square
4 Covering Property and the Cosocle

In this section, we will show that a nice covering property mod cosocle is equivalent to a weaker covering property of the whole group.

**Proposition 4.1.** Let \( G \) be a group with the symmetric double covering property \([((K_1, m_1), (K_2, m_2))]\) mod \( N \) for a normal subgroup \( N \) contained in the cosocle, and suppose that \( N \) contains exactly \( n \) conjugacy classes of \( G \). Then \( G \) has the symmetric double covering property \([((3n - 2)K_1, m_1), ((3n - 2)K_2, m_2)]\).

**Proof.** Find \( g, g' \in G \) such that \((gN, g'N)\) has symmetric double covering number \((K_1, K_2)\) in \( G/N \). Let \( C := (C(g) \cup C(g^{-1}))K_1(C(g') \cup C((g')^{-1}))K_2 \). Then by our construction, \( C \) is mapped onto \( G/N \) through the quotient map. So \( CN = G \).

Now \( N \) contains exactly \( n \) conjugacy classes of \( G \), so let them be \( C_1 = \{e\}, C_2, \ldots, C_n \). I claim that, with some permutation of the indices of these conjugacy classes, \( C^{3t - 2} \supset C(\bigcup_{i=1}^t C_i) \), which would imply that \( C^{3n - 2} \supset CN = G \), finishing our proof.

We proceed by induction. When \( t = 1 \), clearly \( C^{3-2} = C = C\{e\} = CC_1 \). Now assume the statement is true for some \( t < n \). If \( C^{3t-1} \) is disjoint from \( C(N - \bigcup_{i=1}^t C_i) \), then \( C^{3t-1} \subset C(\bigcup_{i=1}^t C_i) \subset C^{3t-2} \). By Lemma 4.2, it follows that \( C^{3t-1} = C^{3t-2} = G \), and \( G \) would be disjoint from \( N - \bigcup_{i=1}^t C_i \), which is a contradiction. So \( C^{3t-1} \) contains an element \( h \) of \( CN \) outside of \( C(\bigcup_{i=1}^t C_i) \). We may permute the index of the conjugacy classes so that \( h \in C f \) for some \( f \in C_{t+1} \).

Since \( C \) is a symmetric set, \( h \in C f \) implies that \( f \in C h \subset C^{3t} \). Since \( C^{3t} \) is conjugacy invariant, and it contains an element of \( C_{t+1} \), it follows that \( C_{t+1} \subset C^{3t} \). So \( CC_{t+1} \subset C^{3t+1} \). This concludes our induction.

**Lemma 4.2.** Let \( G \) be a group with the symmetric double covering property \([((K_1, m_1), (K_2, m_2))]\) mod \( N \) for a normal subgroup \( N \) contained in the cosocle. Let \( g, g' \in G \) be a pair of elements such that \((gN, g'N)\) has symmetric double covering number \((K_1, K_2)\) in \( G/N \). Let \( C := (C(g) \cup C((g')^{-1}))K_1(C(g') \cup C((g')^{-1}))K_2 \). Then for any conjugate-invariant subset \( S \subset G \), \( SC = S \) implies that \( S = G \).

**Proof.** Suppose the normal closure of \( \{g, g'\} \) is not all of \( G \). Then the normal closure of \( \{g, g'\} \) will be contained in a maximal normal subgroup \( M \) in \( G \), which must contain the cosocle and thus must contain \( N \). Then \( gN, g'N \) would be contained in \( M/N \), a maximal normal subgroup of \( G/N \), contradicting the fact that the normal closure of \( \{gN, g'N\} \) is all of \( G/N \). So he normal closure of \( \{g, g'\} \) is all of \( G \).

Now \( g, g' \in C \). If \( SC = S \), then \( S \) would contain the normal closure of \( C \), and thus containing the normal closure of \( \{g, g'\} \). So \( S = G \).

**Proposition 4.3.** Let \( G \) be a group with the symmetric covering property \((K, m)\) mod \( N \) for a normal subgroup \( N \) contained in the cosocle, and suppose that \( N \) contains exactly \( n \) conjugacy classes of \( G \). Then \( G \) has the symmetric covering property \(((3n - 2)K, m)\).

**Proof.** Same strategy as Proposition 4.1.
5 Quasirandom Finite Simple Groups have Nice Covering Property

This section will show that, for finite quasisimple groups, large quasirandomness will imply a nice covering property. We shall first deal with finite simple groups of bounded rank in Subsection 5.1. Then we shall deal with the case of alternating groups in Subsection 5.2. Finally, we shall deal with the finite simple groups with large rank by embedding alternating groups into them in Subsection 5.3. The classification of finite simple groups is used throughout this section.

Definition 5.1. For a finite quasisimple group $G$, we define its **rank** $r(G)$ as the following:

1. When $G$ is abelian or when $G$ is a sporadic group, then $r(G) = 1$.
2. When $G$ is an extension of the alternating group $A_n$, then $r(G) = n$.
3. When $G$ is an extension of a finite simple group of Lie type, then $r(G)$ is the rank of that simple group in the usual sense.

5.1 Finite simple groups of bounded rank

**Theorem 5.2** (Stolz and Thom [19, Proposition 3.8]). For any finite simple group of Lie type of rank $\leq r$, any non-identity element will have covering number $K_r$ for some constant depending only on $r$.

**Theorem 5.3** (Babai, Goodman and Pyber [1, Proposition 5.4]). Let $k$ be any positive integer. Then for all finite simple groups $G$ whose order has no prime divisor greater than $k$, $|G| < k^k$.

**Proposition 5.4.** Let $G$ be a finite simple group of rank $\leq r$. Let $K_r$ be the constant depending on $r$ in Theorem 5.2. For any $m < \infty$, $G$ has covering property $(K_r, m)$ if $G$ is $D$-quasirandom for $D$ large enough.

**Proof.** By choosing $D$ to be larger than some absolute constant, a $D$-quasirandom group $G$ is neither abelian nor sporadic. And when $D > r$, $G$ cannot be an alternating group of rank $\leq r$. So we only need to consider finite simple groups of Lie type.

Recall that any $D$-quasirandom group must have more than $(D - 1)^2$ elements. For any $m \in \mathbb{Z}^+$, let $D$ be an integer $> 1 + \sqrt{mm^2}$. Then all $D$-quasirandom finite simple groups of rank $\leq r$ will have order $> mm^2$, and thus have an element $g$ of prime order $p > m$. Then $g^i$ are non-identity for all $1 \leq i \leq p - 1$, and thus all these elements have covering number $K_r$, by Theorem 5.2. So $G$ has the covering property $(K_r, m)$.

**Corollary 5.5.** Let $G$ be a finite quasisimple group of rank $\leq r$. Let $K_r$ be the constant depending on $r$ in Theorem 5.2. For any $m < \infty$, $G$ has symmetric covering property $(K_r \max(r, 12), m)$ if $G$ is $D$-quasirandom for $D$ large enough.

**Proof.** A quasisimple group is $D$-quasirandom iff the simple group it covers is $D$-quasirandom. Therefore, it is enough to show that, if a finite simple group $G$ has covering property $(K, m)$, then any of its perfect central extension $G'$ will have covering property $(K \max(r, 12), m)$.

Let $Z$ be the center of $G'$. Then $Z$ will be the cosocle of $G'$, and its size is bounded by the Schur multiplier of the simple group $G$. Since $G$ has bounded rank at most $r$, by going through the list of finite simple groups, its Schur multiplier has a size at most $\max(r, 12)$. So if $G$ has covering property $(K, m)$, by Proposition 4.3, $G'$ will have symmetric covering property $(K \max(r, 12), m)$.
5.2 Alternating groups

**Proposition 5.6.** Let $G$ be a quasisimple group over an alternating group. Then for any $m < \infty$, $G$ has symmetric covering property $(8, m)$ if $G$ is $D$-quasirandom for $D$ large enough.

**Proof.** If $G$ is $D$-quasirandom for some large $D$, then the alternating group it covers must be $A_n$ for some large $n$. Then by Theorem 2.4, $A_n$ has covering property $(4, m)$. Now when $n > 7$, $A_n$ will have a Schur multiplier of 2. So $G$ has covering property $(8, m)$. □

5.3 Finite simple groups of large rank

The goal of this section is to prove the following proposition.

**Proposition 5.7.** There is an absolute constant $K_0$, such that for any $m < \infty$, all finite quasisimple groups of rank $\geq r$ will have symmetric covering property $(K_0, m)$ for $r$ large enough.

By the classification of finite simple groups, a finite simple group of rank larger than some absolute constant will have to be a classical finite simple group of Lie type or an alternating group. Any classical finite simple group of Lie type is in one of the following four classes:

1. The projective special linear groups $PSL_n$. For $n$ large enough, $SL_n$ are their universal perfect central extensions.
2. The projective symplectic groups $PSp_n$. For $n$ large enough, $Sp_n$ are their universal perfect central extensions.
3. The projective special unitary groups $PSU_n$. For $n$ large enough, $SU_n$ are their universal perfect central extensions.
4. The projective Omega groups $P\Omega^+_2$, $P\Omega^-_2$, or $P\Omega^{2n+1}$. Here $\Omega_n$ are the commutator subgroups for the special orthogonal groups $SO_n$, and $P\Omega_n = \Omega_n/Z(\Omega_n)$. The plus or minus signs indicate different quadratic forms used to obtain the groups in even dimensions, by the convention. For $n$ large enough, $\Omega_n$ are the universal perfect central extensions of $P\Omega_n$.

The above statements can be found in any standard textbook in classical groups. (e.g. See [10]) It is enough to show Proposition 5.7 for $SL_n$, $Sp_n$, $SU_n$, and $\Omega_n$, since they are the universal perfect central extensions of the simple groups they cover.

We start by analyzing a length function for groups of Lie type over finite fields.

**Definition 5.8.** Let $g$ be a matrix of rank $n$ over a finite field $F$. Let $m_g := \sup_{a \in F^\times} \dim(\ker(a-g))$. Then the **Jordan length** of $g$ is $\ell_J(g) := \frac{n - m_g}{n}$

**Proposition 5.9.** Let $G$ be any subgroup of $GL_n(F)$ for some finite field $F$. The function $\ell_J$ on $G$ is a pseudo length function.

**Proof.** **Non-negativity:** For any $g \in G$, $m_g = \sup_{a \in F^\times} \dim(\ker(a-g)) \leq n$. So $\ell_J(g) = \frac{n - m_g}{n} \geq 0$.

**Symmetry:** For any $g \in G$, any $a \in F^\times$, for a vector $v \in F^n$, we have

$$v \in \ker(a-g) \iff av = gv \iff g^{-1}v = a^{-1}v \iff v \in \ker(a^{-1} - g^{-1}).$$

So $m_g = \sup_{a \in F^\times} \dim(\ker(a-g)) = \sup_{a \in F^\times} \dim(\ker(a - g^{-1})) = m_{g^{-1}}$. So $\ell_J(g) = \ell_J(g^{-1})$. □
Conjugate-invariance: For any $g, h \in G$, any $a \in F^\times$, a vector $v \in F^n$, we have

$$v \in \ker(a - g) \iff av = gv \iff abv = hgh^{-1}v \iff hv \in \ker(a - hgh^{-1}).$$

So $m_g = \sup_{a \in F^\times} \dim(\ker(a - g)) = \sup_{a \in F^\times} \dim(\ker(a - hgh^{-1})) = m_{hgh^{-1}}$. So $\ell_J(g) = \ell_J(hgh^{-1})$.

Triangle inequality: For any $g, h \in G$, any $a, b \in F^\times$, we have

$$\ker(a - g) \cap \ker(a - ab^{-1}h^{-1}) \subset \ker(ab^{-1}h^{-1} - g).$$

So $\dim(\ker(a - g) + \dim(\ker(b - h)) - n \leq \dim(\ker(ab^{-1} - gh)) \leq m_{gh}$. Since this is true for all $a, b \in F^\times$, therefore $m_g + m_h - n \leq m_{gh}$. So $\ell_J(g) + \ell_J(h) \leq \ell_J(gh) \leq \ell_J(g) + \ell_J(h)$.

**Proposition 5.10.** Given an $n_1 \times n_1$ matrix $A$ over a finite field $F$, and an $n_2 \times n_2$ matrix $B$ over the same finite field, then $\ell_J(A \oplus B) \geq \frac{n_1}{n_1 + n_2} \ell_J(A) + \frac{n_2}{n_1 + n_2} \ell_J(B)$.

**Proof.** For any $a \in F^\times$, $\ker(a - A \oplus B) = \ker((a - A) \oplus (a - B)) = \ker(a - A) \oplus \ker(a - B)$. So $\dim(\ker(a - A \oplus B)) \leq m_A + m_B$. Since this is true for all $a \in F^\times$, therefore $m_{A\oplus B} \leq m_A + m_B$. So $\ell_J(A \oplus B) \geq \frac{n_1}{n_1 + n_2} \ell_J(A) + \frac{n_2}{n_1 + n_2} \ell_J(B)$.

**Lemma 5.11** (Stolz and Thom [19, Lemma 3.11]). There is an absolute constant $c$, such that for any finite classical quasisimple group of Lie type $G$, and for any $g \in G \setminus Z(G)$, where $Z(G)$ is the center of $G$, then $C(g)^m = G$ for all $m \geq \frac{\ell_J(g)}{c}.$

In short, element of large Jordan length will automatically have small covering number.

The next step is to identify subgroups of these quasisimple groups of Lie type isomorphic to the alternating groups. A key step is to treat elements in alternating groups as matrices, namely the permutation matrices. These are the matrices with exactly one entry of value 1 in each column and in each row, and 0 in all other entries. Such an $n \times n$ matrix will act on the standard orthonormal basis of a $n$-dimensional vector space by permutation, and thus will give an embedding of $S_n$ into $GL_n(F)$ for any field $F$. Any such matrix is in $A_n$ iff it has determinant 1.

**Proposition 5.12.** If $P$ is an $n \times n$ permutation matrix where its cycle decomposition has $k$ cycles, then we have $\ell_J(P) \geq \frac{k - n}{n}$.

**Proof.** By cycle decomposition, after a change of basis in the vector space, $P$ will be a direct sum of many cyclic permutation matrices. By Proposition 5.10, it's enough to prove the case when $P$ is a single cycle of length $n$, and show that $\ell_J(P) \geq \frac{n}{n}$.

Since $P$ is a single cycle of length $n$, its eigenvalues in the algebraic closure of $F$ are precisely all the $n$-th roots of unity, with multiplicity 1 for each root of unity. So $\dim(\ker(\lambda - P)) \leq 1$ for all $\lambda \in F^\times$. So $\ell_J(P) \geq \frac{n}{n}$. 

**Proposition 5.13.** There is an absolute constant $K_0$ such that for any $m < \infty$, any finite quasisimple group of Lie type of rank $n$ containing $A_n$ as permutation matrices will have the covering property $(K_0, m)$, for $n$ large enough.

**Proof.** Let $K_0 > 2c$ for the absolute constant $c$ in Lemma 5.11. Then any element $a$ of Jordan length $\geq \frac{1}{4}$ will have covering number $K_0$ in any finite quasisimple group of Lie type.

Pick any odd prime $p > m$, and pick another prime $q > p$. For any $n$ large enough, we have $n = ap + bq$ for some integers $a > 1$, $0 < b < p + 1$. Then find $\sigma \in A_n$ made up of exactly a $p$-cycles
and $b$ $q$-cycles, where all cycles are disjoint. This element will be fixed-point free and non-exceptional, and it will have roughly $\frac{b}{2}$ cycles.

For any finite quasisimple group of Lie type of rank $n$ containing $A_n$ as permutation matrices, let $P$ be the matrix corresponding to $\sigma$. Then $\ell_J(P)$ will be roughly $\geq \frac{n-1}{n} = \frac{n-1}{p} > \frac{1}{2}$. So this element will have covering number $K_0$ in $G$. It clearly has order $pq$, and all of its power coprime to $pq$ will also have the same covering number. So $G$ has the covering property $(K_0, p - 1)$ and thus the covering property $(K_0, m)$.

\textbf{Corollary 5.14.} For any $m < \infty$, all finite special linear groups of rank $n$ for large enough $n$ will have the covering property $(K_0, m)$. Here $K_0$ is the absolute constant in Proposition 5.13.

\textbf{Proposition 5.15.} There is an absolute constant $K_0$, such that for any $m < \infty$, we have the following:

1. Any finite quasisimple group of Lie type of rank $2n$ containing $A_n$ as $\{P \oplus P : P \in A_n\}$ is a permutation $n \times n$ matrix will have the covering property $(K_0, m)$ for $n$ large enough.

2. Let $I_1$ be the $1 \times 1$ identity matrix. Then any finite quasisimple group of Lie type of rank $2n + 1$ containing $A_n$ as $\{P \oplus P \oplus I_1 : P \in A_n\}$ is a permutation $n \times n$ matrix will have the covering property $(K_0, m)$ for $n$ large enough.

3. Let $I_2$ be the $2 \times 2$ identity matrix. Then any finite quasisimple group of Lie type of rank $2n + 2$ containing $A_n$ as $\{P \oplus P \oplus I_2 : P \in A_n\}$ is a permutation $n \times n$ matrix will have the covering property $(K_0, m)$ for $n$ large enough.

\textbf{Proof.} The strategy is identical to Proposition 5.13. Just take $\sigma \oplus \sigma$, $\sigma \oplus \sigma \oplus I_1$ or $\sigma \oplus \sigma \oplus I_2$ instead of $\sigma$, and use Proposition 5.10.

\textbf{Proposition 5.16.} Let $V$ be any finite dimensional vector space over a finite field $F$, and let $B$ be any non-degenerate symmetric bilinear form, or any non-degenerate alternating bilinear form, or any non-degenerate Hermitian form. Then we have an orthogonal decomposition $V = W \oplus (\bigoplus_{i=1}^n H_i)$ where $W$ is anisotropic of dimension at most 2, and $H_i$ are hyperbolic planes.

\textbf{Proof.} These are standard facts in the geometry of classical groups (e.g. See [10]).

\textbf{Proposition 5.17.} Any finite symplectic or unitary or orthogonal group contains $A_n$ as the subset in one of the ways described by Proposition 5.14.

\textbf{Proof.} Let $V$ be any finite dimensional vector space over any finite field $F$, with a non-degenerate Hermitian, symmetric bilinear or alternating bilinear form $B : V \times V \to F$. Then we have an orthogonal decomposition $V = W \oplus H$ with an anisotropic space $W$ of dimension at most 2, and an orthogonal sum of hyperbolic planes $H = \bigoplus_{i=1}^n H_i$.

Then let $(v_i, w_i)$ be the hyperbolic pair generating $H_i$ for each $i$. For any $\sigma \in A_n$, we can let $\sigma$ act by permutation on the set $\{v_1, ..., v_n, w_1, ..., w_n\}$ such that $\sigma(v_i) = v_{\sigma(i)}$ and $\sigma(w_i) = w_{\sigma(i)}$.

Now clearly $\{v_1, ..., v_n, w_1, ..., w_n\}$ is a basis of $H$. So the above action of $\sigma$ induces a linear transformation $P \oplus P$ on $H$, where $P$ is the $n \times n$ permutation matrix for $\sigma$. And this $P \oplus P$ clearly preserves the form $B$ by construction. Now take the direct sum of $P \oplus P$ on $H$ and the identity matrix on $W$, we obtain our desired embedding of $A_n$ in the symplectic or unitary or orthogonal group.

\textbf{Corollary 5.18.} For any $m < \infty$, any finite symplectic or special unitary group of rank $n$ has the covering property $(K_0, m)$ for $n$ large enough. $K_0$ is the absolute constant in Proposition 5.15.
Corollary 5.19. For any odd prime $p$, any $\Omega_{2n}^+, \Omega_{2n+1}^+$ or $\Omega_{2n}^-$, for large enough $n$ has the covering property $(K_0, p - 1)$. $K_0$ is the absolute constant in Proposition 5.15.

Proof. Embed $A_n$ in $SO_2^+, SO_2^-$, and $SO_{2n+1}$ in the ways described by Proposition 5.15. After taking commutator subgroup, the groups $\Omega_{2n}^+$, $\Omega_{2n}^-$ and $\Omega_{2n+1}$ will still contain $A_n$ through this embedding, because $A_n$ is its own commutator subgroup. So we may apply Proposition 5.15 to $\Omega_{2n}^+$, $\Omega_{2n}^-$ and $\Omega_{2n+1}$ and obtain the desired result. □

Proposition 5.15 is proven by putting Corollary 5.14, Corollary 5.18 and Corollary 5.19 together.

6 Proof of Theorem 1.20

The results of Section 5 can be summarized into the following useful lemma.

Lemma 6.1. For any integer $D$ and any constant $c$, we can find integers $D', K_1, K_2, m_1, m_2$ such that all $D'$-quasirandom finite quasisimple groups have symmetric double covering property $[(K_1, m_1), (K_2, m_2)]$ such that $m_1 > cK_1^{D'}$, $m_2 > cK_2^{D'}$.

Proof. Let $K_1$ be the absolute constant $K_0$ as in Proposition 5.7. Pick some $m_1 > cK_1^{D'}$. Find $r$ large enough such that, according to Proposition 5.7, all finite quasisimple groups of rank $\geq r$ will have symmetric covering property $(K_1, m_1)$.

Let $K_2$ be the constant $K_0 \max(r, 12)$ as in Corollary 5.3 and pick some $m_2 > cK_2^{D'}$. Then for $D'$ large enough, all $D'$-quasirandom finite quasisimple groups will have symmetric covering property $(K_2, m_2)$.

In any cases, a $D'$-quasirandom finite quasisimple group will have symmetric double covering property $[(K_1, m_1), (K_2, m_2)]$. □

Corollary 6.2. Let $\mathcal{C}_{QS}$ be the class of finite quasisimple groups. Then $\mathcal{C}_{QS}$ is a Q.U.P. class.

Proof. For any integer $D$, and for the constant $c_D$ as in Theorem 3.4, we can find $D', K_1, K_2, m_1, m_2$ as in Lemma 6.1.

Let $G_n$ be a sequence of $D'$-quasirandom groups in $\mathcal{C}_{QS}$. Then $G_n$ all have symmetric double covering property $[(K_1, m_1), (K_2, m_2)]$. Then any ultraproduct $G = \prod_{n \rightarrow \omega} G_n$ will have symmetric double covering property $[(K_1, m_1), (K_2, m_2)]$ by Loš’s Theorem.

Since $m_1 > c_D K_1^{D'}$, $m_2 > c_D K_2^{D'}$, $G$ is $D$-quasirandom by Theorem 3.4. □

Proposition 6.3. Let $G$ be a group with the symmetric double covering property for some parameters, and let $(G_i)_{i \in I}$ be an arbitrary family of groups with the symmetric double covering property for some uniform parameters. Then the following are true:

1. For any normal subgroup $N$, $G$ has the symmetric double covering property for the same parameters mod $N$.
2. Any quotient group of $G$ has the symmetric double covering property for the same parameters.
3. The group $\prod_{i \in I} G_i$ has the symmetric double covering property for the same parameters.
4. As a result of the statement 2 and statement 3, any ultraproduct $\prod_{i \rightarrow \omega} G_i$ has the symmetric double covering property for the same parameters. (This also follows directly from Loš’s Theorem.)

16
We shall omit the proof of Proposition 6.3 as it is straightforward.

**Corollary 6.4.** (Quasirandomness implies a Nice Covering Property). For any integer $D$, and any constant $c$, we can find integers $D', K_1, K_2, m_1, m_2$ such that all finite $D'$-quasirandom groups have symmetric double covering property $\langle [K_1,m_1], (K_2,m_2) \rangle$ mod cosocle, with $m_1 > cK_1^{D'}$, $m_2 > cK_2^{D'}$.

**Proof.** Let $D', K_1, K_2, m_1, m_2$ be exactly as in Lemma 6.1. Let $G$ be any finite $D'$-quasirandom group.

Let $N$ be the cosole of $G$. Then $G/N$ is a direct product of $D'$-quasirandom finite simple groups. These simple groups all have symmetric double covering property $\langle [K_1,m_1], (K_2,m_2) \rangle$. So by Proposition 6.3 their product $G/N$ will have this same symmetric double covering property.

**Corollary 6.5.** Let $\mathcal{C}_{CS(n)}$ be the class of finite groups with at most $n$ conjugacy classes in their cosocles. Then $\mathcal{C}_{CS(n)}$ is a Q.U.P. class.

**Proof.** Let $c = c_D(3n - 2)^{D^2}$ where $c_D$ is the constant in Theorem 3.4.

For any integer $D$, and for the constant $c$, we can find $D', K_1, K_2, m_1, m_2$ as in Corollary 6.3.

Let $G_i$ be a sequence of $D'$-quasirandom groups in $\mathcal{C}_{QS}$. Then $G_i$ all have symmetric double covering property $\langle [K_1,m_1], (K_2,m_2) \rangle$ mod cosocle. Since the cosocle contains at most $n$ conjugacy classes, by Proposition 4.4 $G_i$ all have symmetric double covering property $\langle [(3n - 2)K_1,m_1], ((3n - 2)K_2,m_2) \rangle$. Then any ultraproduct $G = \prod_{n \to \omega} G_n$ will have symmetric double covering property $\langle [(3n - 2)K_1,m_1], ((3n - 2)K_2,m_2) \rangle$ by Łos’s Theorem.

Since $m_1 > c_D[(3n - 2)K_1]^{D^2}$, $m_2 > c_D[(3n - 2)K_1]^{D^2}$, $G$ is $D$-quasirandom by Theorem 3.4.

**Proof of Theorem 7.20.** For any integer $D$, let $c = c_D(3n - 2)^{D^2}$ where $c_D$ is the constant in Theorem 3.4. We can find $D', K_1, K_2, m_1, m_2$ as in Corollary 6.3 and Lemma 6.4.

Let $G_i$ be a sequence of $D'$-quasirandom groups in $\mathcal{C}_n$. Then each $G_i$ is a direct product of $D'$-quasirandom groups in $\mathcal{C}_{QS} \cup \mathcal{C}_{CS(n)}$. These factor groups must then have symmetric double covering property $\langle [(3n - 2)K_1,m_1], ((3n - 2)K_2,m_2) \rangle$. By Proposition 6.3 $G_i$ must also have this symmetric double covering property $\langle [(3n - 2)K_1,m_1], ((3n - 2)K_2,m_2) \rangle$. Then any ultraproduct $G = \prod_{n \to \omega} G_n$ will have symmetric double covering property $\langle [(3n - 2)K_1,m_1], ((3n - 2)K_2,m_2) \rangle$ by Łos’s Theorem.

Since $m_1 > c_D[(3n - 2)K_1]^{D^2}$, $m_2 > c_D[(3n - 2)K_1]^{D^2}$, $G$ is $D$-quasirandom by Theorem 3.4.

**7 Proof of Theorem 1.22**

The goal of this section is to prove Theorem 1.22.

**Proposition 7.1.** $(G_i)_{i \in I}$ be an arbitrary family of groups. Let $N_i \triangleleft G_i$. Then the ultraproduct $\prod_{i \to \omega} N_i$ is a normal subgroup of $\prod_{i \to \omega} G_i$ with quotient group $\prod_{i \to \omega} G_i/N_i$.

**Proposition 7.2.** Let $\mathcal{F}$, $\mathcal{F}'$ be Q.U.P. (or q.u.p.), then $\mathcal{F} \cup \mathcal{F}'$, $\mathcal{F} \times \mathcal{F}'$ and subclasses of $\mathcal{F}$ are Q.U.P. (or q.u.p.), where $\mathcal{F} \times \mathcal{F}'$ is the class of groups which are direct products of a group in $\mathcal{F}$ and a group in $\mathcal{F}'$.

**Proof.** We shall only prove the Q.U.P. case, as the case for q.u.p. is essentially the same.

The statement for subclasses is trivial.

For the case of $\mathcal{F} \cup \mathcal{F}'$, for any $D \in \mathbb{Z}^+$, we can find $D''$, $D''' \in \mathbb{Z}^+$ such that ultraproducts of $D''$-quasirandom groups in $\mathcal{F}$ and ultraproducts of $D'''$-quasirandom groups in $\mathcal{F}'$ are $D$-quasirandom. Let $D' = \max(D'', D''')$. For any ultraproduct $G$ of $D'$-quasirandom groups $G_i$ in $\mathcal{F} \cup \mathcal{F}'$ by an
Let $I := \{ i \in \mathbb{N} : G_i \in \mathcal{F} \}$. By maximality of $\omega$, either $I \in \omega$ or $\mathbb{N} \setminus I \in \omega$. If $I \in \omega$, then $G$ equals to an ultraproduct of $D'$-quasirandom groups in $\mathcal{F}$, and is thus $D$-quasirandom. If $\mathbb{N} \setminus I \in \omega$, then $G$ equals to an ultraproduct of $D'$-quasirandom groups in $\mathcal{F}'$, and is thus $D$-quasirandom.

For the case of $\mathcal{F} \times \mathcal{F}'$, for any $D \in \mathbb{Z}^+$, we can find $D'', D''' \in \mathbb{Z}^+$ such that ultraproducts of $D''$-quasirandom groups in $\mathcal{F}$ and ultraproducts of $D'''$-quasirandom groups in $\mathcal{F}'$ are $D$-quasirandom. Let $D' = \max\{D'', D'''\}$. For any ultraproduct $G$ of $D'$-quasirandom groups $G_i = H_i \times H'_i$ in $\mathcal{F} \times \mathcal{F}'$ by an ultrafilter $\omega$, then $H_i$ and $H'_i$ are $D'$-quasirandom. Then $H = \prod_{i \in \omega} H_i$ and $H' = \prod_{i \in \omega} H'_i$ are $D$-quasirandom. Then $G = H \times H'$ is $D$-quasirandom.

**Proposition 7.3.** Let $\mathcal{F}$ be a Q.U.P. class (or q.u.p. class), and let $K$ be a positive integer. Then the class of semi-direct product groups $G = N \rtimes H$ where $H \in \mathcal{F}$ and $C(H)^K \supset N$ is Q.U.P. (or q.u.p.)

**Proof.** We shall only prove the Q.U.P. case, as the case for q.u.p. is essentially the same.

For any $D \in \mathbb{Z}^+$, we can find $D' \in \mathbb{Z}^+$ such that ultraproducts of $D'$-quasirandom groups in $\mathcal{F}$ are $D$-quasirandom.

Let $G_i$ be $D'$-quasirandom and $G_i = N_i \rtimes H_i$ where $H_i \in \mathcal{F}$ and $C(H_i)^K \supset N_i$. Let $G, H, N$ be the ultraproduct for the sequences $G_i, H_i, N_i$ respectively. Then $G = N \rtimes H$, $C(H)^K \supset N$, and $H$ is an ultraproduct of $D'$-quasirandom groups and is therefore $D$-quasirandom.

For any representation of $G$ with dimension $< D$, since $H$ is $D$-quasirandom, this subgroup $H$ must be contained in the kernel. But $C(H)^K \supset N$, so $N$ is also contained in the kernel. So this representation would induce a representation on $G/N = H$ with dimension $< D$, which has to be trivial. So $G$ is $D$-quasirandom.

**Proposition 7.4.** Let $\mathcal{F}$ be a Q.U.P. class (or q.u.p. class), and let $K$ be a positive integer. Consider the class of semi-direct product groups $G = N \rtimes H$ where $H \in \mathcal{F}$, and $N$ is abelian, and $G$ has commutator width $\leq K$. Then this class is Q.U.P. (or q.u.p.)

**Proof.** A commutator in $G$ must be of the form $(nh)(mg)(nh)^{-1}(mg)^{-1}$, where $m, n \in N$ and $g, h \in H$. Now we can do the following computation:

$$(nh)(mg)(nh)^{-1}(mg)^{-1} = (nm^{-1})(nm^{-1}gh^{-1}m^{-1}n^{-1})(nm^{-1}g^{-1}nm^{-1}n^{-1})(nm^{-1}m^{-1})$$

$$\in C(H)^3$$ since $N$ is abelian.

Therefore since every element of $G$ can be written as a product of $K$ commutators, we conclude that $G \subset C(H)^{3K}$. Then we are done by Proposition 7.3.

If we know more about the action of $H$ on $N$, we can drop the abelian condition on $N$.

**Lemma 7.5.** Let $G := N \rtimes H$, where $N$ is finite and an element $h \in H$ acts on $N$ by conjugation with no nontrivial fixed point. Then $G = C(H)^3$.

**Proof.** For any $n \in N$, let $f(n) := [n, h] = nhn^{-1}h^{-1}$.

Then $f(n)$ is equal to $f(n')$ if and only if $(n')^{-1}n = h(n')^{-1}nh^{-1}$, which is a fixed point of the action of $h$ on $N$, if and only if $n = n'$.

As a result, since $N$ is finite, $f : N \rightarrow N$ is a bijection. Now since $n hn^{-1} = f(n)h$ for all $n \in N$, we see that all elements in $\{nh : n \in N\}$ are conjugate to $h$, and thus in $C(H)$. Finally, for any $g \in G$, $g = nh'$ for some $n \in N, h' \in H$. Then $g = (nh)(h^{-1}h') \in C(H)^2$. So $G = C(H)^2$. \qed
Proposition 7.6. Let $F$ be a Q.U.P. class (or q.u.p.), and let $K$ be a positive integer. Consider the class of semi-direct products $G = N \rtimes H$ where $H \in F$, $N$ is finite, and an element $h \in H$ acts on $N$ by conjugation with no nontrivial fixed point. This class is Q.U.P. (or q.u.p.)

Proof. Combine Proposition 7.3 with Lemma 7.5.

Corollary 7.7. Let $F$ be a Q.U.P. class (or q.u.p. class). Consider the class of groups $G = N \rtimes H$ where $N$ is a vector space over a finite field treated as an additive abelian group, $H$ is a subgroup of $GL(N)$ acting in the obvious way and $H \in F$, and an element of $H$ has no eigenvalue $1$. Then this class is Q.U.P. (or q.u.p.)

Proposition 7.8. Let $F$ be a Q.U.P. class (or q.u.p. class). Consider a sequence of groups $G_i$ each with a normal subgroup $N_i$ such that $N_i, G/N_i \in F$, and $N_i$ is $D_i$-quasirandom with $\lim_{i \to \infty} D_i = \infty$. Then this sequence is Q.U.P. (or q.u.p.)

Proof. This is trivial since ultraproducts of a short exact sequence is a short exact sequence of ultraproducts, and quasirandomness is preserved through short exact sequences.

Corollary 7.9. Let $n$ be any positive integer. Consider a sequence of groups $G_i$ with composition series of length $\leq n$, such that each simple factor is $D_i$-quasirandom with $\lim_{i \to \infty} D_i = \infty$. This class is Q.U.P.

8 Applications

8.1 Triangles in a quasirandom group

A quasirandom group usually contains many patterns. For example, Gowers has shown the following result:

Theorem 8.1 (Gowers [9, Theorem 5.1]). Pick any $\epsilon_1, \epsilon_2 > 0, 0 < \alpha < 1$. If $G$ is a $D$-quasirandom group for some large enough $D$, then for any subset $A$ of $G$ such that $|A| \geq \alpha |G|$, there are more than $(1 - \epsilon_1)\alpha^2 |G|$ elements $x \in G$ such that $|A \cap xA| \geq (1 - \epsilon_2)\alpha^2 |G|$.

Morally, if we define an $x$-pair to be a set $\{y, xy\}$ for some $y \in G$, then this theorem means that any large enough subset of a quasirandom group $G$ will contain many $x$-pairs for many $x$.

If $G$ is not only quasirandom, but also contained in a q.u.p. class. Then passing $G$ through an ultraproduct, we will obtain a minimally almost periodic group. By applying ergodic theory on such a group, a pattern similar to that of Theorem 8.1 emerges. It is proven by Bergelson, Robertson and Zorin-Kranich [4] that, for a quasirandom group $G$ in a q.u.p class, any large enough subset of $G \times G$ will contain many $x$-triangle for many $x$.

Definition 8.2. Let $g$ be an element of a group $G$. Then a $g$-triangle is the set $\{(x, y), (gx, y), (gx, gy)\} \subset G \times G$ for some $x, y \in G$.

Theorem 8.3 (Bergelson, Robertson and Zorin-Kranich [4, Theorem 1.12]). Let $G$ be contained in a q.u.p. class. For any $\epsilon > 0, 0 < \alpha < 1$, there are integers $D, K$ such that, if $G$ is $D$-quasirandom, then for any subset $A$ of $G \times G$ with $|A| \geq \alpha |G|^2$, the set $T_A = \{g \in G : A \text{ contains more than } (\alpha^4 - \epsilon)|G|^2 \text{ triangles}\}$ can cover $G$ with at most $K$ of its left shifts.
8.2 Self-Bohrifying groups

Definition 8.4. A Bohr compactification of a discrete group $G$ is a homomorphism $b : G \to bG$ such that any homomorphism from $G$ to a compact group factors uniquely through $b$.

Remark 8.5.

1. The Bohr compactification exists for any group by the work of Holm [12]. It is obviously unique up to a unique isomorphism.

2. Clearly, a discrete group is minimally almost periodic iff it has trivial Bohr compactification.

Definition 8.6. A discrete group $G$ is said to be self-Bohrifying if its Bohr compactification $bG$ is the same group as $G$, but with a compact topology.

By the result and technique of this paper, one can find many examples of self-Bohrifying groups. In particular, we have the following theorem.

Theorem 8.7. Let $n$ be a positive integer. Let $G_i$ be a sequence of increasingly quasirandom groups in $C_n$, the class defined as in Theorem 6.7. Then $\prod_{i \in \mathbb{N}} G_i$ is self-Bohrifying as a discrete group.

Corollary 8.8. Let $G_n$ be a sequence of non-abelian finite simple groups of increasing order. Then $\prod_{i \in \mathbb{N}} G_n$ is self-Bohrifying as a discrete group.

Theorem 8.7 is proven by first showing that $\prod_{i \in \mathbb{N}} G_i/\prod_{i \in \mathbb{N}} G_i$ is minimally almost periodic, and then use a theorem by Hart and Kunen [11].

Definition 8.9. Let $G_i$ be a sequence of groups.

1. Their sum is the group $\prod_{i \in \mathbb{N}} G_i = \{g \in \prod_{i \in \mathbb{N}} G_i : \text{only finitely many coordinates of } g \text{ is nontrivial}\}$.

2. Their reduced product is the group $\prod_{i \in \mathbb{N}} G_i/\prod_{i \in \mathbb{N}} G_i$.

Theorem 8.10 (Hart and Kunen [11, Lemma 3.8]). Let $\{G_i\}_{i \in \mathbb{N}}$ be a sequence of finite groups. Then $\prod_{i \in \mathbb{N}} G_i$ is self-Bohrifying if and only if finitely many $G_i$ are perfect groups, and $\prod_{i \in \mathbb{N}} G_i/\prod_{i \in \mathbb{N}} G_i$ has trivial Bohr compactification. (i.e. $\prod_{i \in \mathbb{N}} G_i/\prod_{i \in \mathbb{N}} G_i$ is minimally almost periodic).

Proof of Theorem 8.7. All 2-quasirandom groups are perfect. So it is enough to show that the reduced product of $G_i$ is minimally almost periodic, i.e. it is $D$-quasirandom for all $D$.

For any integer $D$, let $c = c_D(3n - 2)^2$ where $c_D$ is the constant in Theorem 3.4. We can find $D', K_1, K_2, m_1, m_2$ as in Corollary 6.4 and Lemma 6.1.

Let $G_i$ be a sequence of increasingly quasirandom groups in $C_n$. Then all but finitely many $G_i$ will be $D'$-quasirandom. Since we are interested in the reduced product, which is invariant under the change of finitely many coordinates, we may WLOG assume that all $G_i$ are $D'$-quasirandom.

Since $G_i \in C_n$, each $G_i$ is a direct product of $D'$-quasirandom groups in $C_{QS} \cup C_{CS(n)}$. These factor groups must then have symmetric double covering property $[((3n - 2)K_1, m_1), ((3n - 2)K_2, m_2)]$. By Proposition 6.3, $G_i$ must also have this symmetric double covering property $[((3n - 2)K_1, m_1), ((3n - 2)K_2, m_2)]$.

Now by Proposition 6.3, covering properties are preserved by arbitrary products and quotients. So $\prod_{i \in \mathbb{N}} G_i$ will have this covering property, and the reduced product $\prod_{i \in \mathbb{N}} G_i/\prod_{i \in \mathbb{N}} G_i$ will also have this covering property.

Since $m_1 > c[(3n - 2)K_1]^{D^2}$, $m_2 > c[(3n - 2)K]^{D^2}$, the reduced product is $D$-quasirandom by Theorem 3.4. So we are done by Theorem 8.10. □
References

[1] L. Babai, A. J. Goodman, and L. Pyber. “Groups without Faithful Transitive Permutation: Representations of Small Degree”. *Journal of Algebra* 195 (1997), pp. 1–29.

[2] A. Ballester-Bolinches and L. M. Ezquerro. *Classes of Finite Groups*. Vol. 584. Springer, 2006.

[3] V. Bergelson and H. Furstenberg. “WM groups and Ramsey theory”. *Topology and its Applications* 156(16) (2009), pp. 2572–2580.

[4] V. Bergelson, D. Robertson, and P. Zorin-Kranich. “Triangles in Cartesian Squares of Quasirandom Groups”. arXiv:1410.5385 (2014).

[5] V. Bergelson and T. Tao. “Multiple recurrence in quasirandom groups”. arXiv:1211.6372 (2012).

[6] J. L. Brenner. “Covering theorems for FINASIGs. VIII. Almost all conjugacy classes in $A_n$ have exponent $\leq 4$”. *J. Austral. Math. Soc.* A 25(02) (1978), pp. 210–214.

[7] J. Cheeger and D. G. Ebin. *Comparison Theorems in Riemannian Geometry*. American Mathematical Society, 1975.

[8] M. Collins. “On Jordan’s theorem for complex linear groups”. *J. Group Theory* 10(4) (2007), pp. 411–423.

[9] W. T. Gowers. “Quasirandom groups”. *Combinatorics, Probability and Computing* 17(3) (2008), pp. 363–387.

[10] L. C. Grove. *Classical groups and geometric algebra*. American Mathematical Soc., 2002.

[11] J. E. Hart and K. Kunen. “Bohr Compactifications of Non-Abelian Groups”. *Topology Proceedings* 26(2) (2002), pp. 593–626.

[12] P. Holm. “On the Bohr Compactification”. *Math. Annalen* 156 (1964), pp. 34–46.

[13] D. Holt and W. Plesken. *Perfect Groups*. Oxford: Clarendon Press, 1989.

[14] A. E. Hurd and P. A. Loeb. *An introduction to nonstandard real analysis*. Vol. 118. Academic Press, 1985.

[15] T. Jech. *Set Theory*. Springer, 2002.

[16] J. von Neumann and E. P. Wigner. “Minimally almost periodic groups”. *Annals of Math.* 41(2) (1940), pp. 746–750.

[17] J. Nienhuys. “A solenoidal and monothetic minimally almost periodic group”. *Fundamenta Mathematicae* 73(2) (1971), pp. 167–169.

[18] M. R. Sepanski. *Compact Lie Group*. Vol. 235. Springer Science and Business Media, 2007.

[19] A. Stolz and A. Thom. “On the lattice of normal subgroups in ultraproducts of compact simple groups”. *Proceedings of the London Mathematical Society* 108(1) (2013), pp 73–102.