Einstein-Maxwell-dilaton theory in Newman-Penrose formalism

Wen-Di Tan

Center for Joint Quantum Studies and Department of Physics, School of Science, Tianjin University, 135 Yaguan Road, Tianjin 300350, China

ABSTRACT. In this paper, we study four dimensional Einstein-Maxwell-dilaton (EMD) theories in the Newman-Penrose (NP) formalism. The equations of motion of the EMD theories are adapted into the NP formalism. The solution space that approaching asymptotic flatness is derived. As an application of the solution space, we investigate the gravitational and electromagnetic memory effects in EMD theories.

1 Introduction

In 1960s, to understand the gravitational radiation in full Einstein theory, Bondi and his collaborators established an elegant framework for axisymmetric isolated systems and demonstrated that gravitational waves exist in the full Einstein theory rather than an artifact of linearization [1]. In this framework, they chose a suitable coordinates system and expanded the metric fields in inverse powers of the radial coordinate r. Imposing the proper boundary conditions that the space-times should asymptotically approach flatness, the equations of motion can be solved order by order in $1/r$ expansions. In this framework, the gravitational radiation is characterized by the news functions and the mass of the system decreases whenever the news function exists. Shortly, this framework was extended to asymptotically flat space-times by Sachs [2]. Meanwhile, Newman and Penrose [3] developed a new approach to understand gravitational radiation by means of a tetrad or spinor formalism. They derived a compact set of first order differential equations which are linear combinations of the equations for the Riemann tensor expressed in Ricci rotation coefficients or the spinor affine connection and equivalent to the empty space Einstein equations. From these equations, one can investigate the asymptotic behavior of the field specifically and systematically, under the condition that the space-time should approach to flatness at infinity. This condition of asymptotic flatness is imposed on empty-space Riemann tensor rather than the metric. This formalism is motivated by the strong belief that the essential element of a space-time is its light-cone structure and it is
the most effective way for grasping the inherent symmetries of the space-times such as the black-hole solutions of general relativities. In this formalism, the geometrical property of the space-times is more transparent and it is the most satisfactory way to study fermion coupled theories. The asymptotically flat solutions of the empty Newman-Penrose equations were first derived by Newman and Unti [4]. News functions and the mass-loss formula were successfully recovered in NP formalism. When matter fields are coupled, the equations of motion of the matter fields should be also adapted into NP formalism. This can be done easily when electromagnetic source is coupled to gravity. The Einstein-Maxwell theory was studied in [5, 6]. However other theories are less addressed in NP formalism.

In this paper, we study the four dimensional Einstein-Maxwell-dilaton theories in Newman-Penrose formalism. Including Kaluza-Klein theory which arises from five dimensional Einstein gravity reduced on a circle, the four-dimensional EMD theories are a class of theories that can be embedded into various kinds of supergravities which originate from string theories or M-theory. In this class of theories, the matter sector includes Maxwell field $A$ and dilatonic scalar $\varphi$, both of which are massless and minimal coupled to gravity. Nevertheless, the dilaton is non-minimally coupled to the Maxwell kinetic term in the form of an exponential function $e^{a\varphi}$ where $a$ is the dilaton coupling constant. In [7], the authors used Newman-Penrose formalism to analyze the perturbations of the Kerr-Newman dilatonic black hole background, which is a kind of application of Einstein-Maxwell-dilaton theories in Newman-Penrose formalism. However, EMD theories have not been fully studied in NP formalism elsewhere. In the present work, we adapt the equations of motion of EMD theories into NP formalism and the solution space that approaching flatness asymptotically is obtained in NP formalism. After that, we examine the mass-loss formula and charge conservation in EMD theories. As a direct application of the solution space, we also study the memory effect.

First reported by Zel’dovich and Polnarev [8] in linearized gravity and further studied by Christodoulou in full Einstein gravity [9], gravitational memory effects are a large group of observational effects for gravitational radiation which characterized by the change of the asymptotic shear $\Delta \sigma^0$ [10] (see also [11–17] for relevant developments). The gravitational memory effects have analogue in Maxwell theory which are the electromagnetic memory effects [18, 19]. This kind of memory effects is called “kick memory effects” since they are described by the change of the velocity of the particle. We should notice that these kinds of memory effects are classified by the observational effects. In the recent years, there have been renewed interests on memory effects from a theoretical viewpoint. Strominger and Zhiboedov [20] discovered that there is a kind of relation among BMS supertranslation symmetry, leading soft graviton theorem and a kind of displacement gravitational memory effect. This memory effect is a displacement of two parallel inertial detectors caused by the radiative energy flux which is mathemati-
cally equivalent to Weinberg’s soft graviton theorem \[21\] by a Fourier transformation or inverse Fourier transformation. Pasterski, Strominger and Zhiboedov \[22\] discovered a kind of spin memory effect which characterized by the relative time delay between different orbiting light rays caused by the radiative angular momentum. This kind of spin memory effect is mathematically equivalent to subleading soft graviton theorem \[23\]. In \[24\], the authors found another kind of spin memory effect represented by the proper time delay of a free-falling massive particle constrained on a time-like \( r = r_0 \) hypersurface near the null infinity. In \[25\], the authors considered the motion of a charged observer and investigated the gravitational memory effects and electromagnetic memory effects in a unified manner in Einstein-Maxwell theory. It is definitely of interest to study the memory effect in EMD theories to see what is the observational effect from the non-minimal coupling of the scalar field and electromagnetic field. We believe that a fully understood memory effect in EMD theories can also help us to understand the memory effect in string theories \[26\] and M-theory.

The main structure of this paper is as follow. In section 2, we will give a brief introduction of NP formalism and derive the NP equations of four dimensional Einstein-Maxwell-dilaton theories. The asymptotically flat solution space of these theories will be derived in section 3. We also examine the charge conservation and the mass-loss formula in EMD theories. In section 4, we will consider the gravitational and electromagnetic memory effects based on the investigation of \[25\] in EMD theories as an application of the solution space. We will give briefly conclude in section 5.

2 Einstein-Maxwell-dilaton theory in the NP formalism

The Newman-Penrose formalism is a special tetrad formalism with two real null basis vectors \( e_1 = l, e_2 = n \), and two complex null basis vectors \( e_3 = m, e_4 = \bar{m} \). These basis vectors have the orthogonality relations

\[
 l \cdot m = l \cdot \bar{m} = n \cdot m = n \cdot \bar{m} = 0. \tag{2.1}
\]

and are normalized as

\[
 l \cdot n = 1, \quad m \cdot \bar{m} = -1. \tag{2.2}
\]

The metric is obtained from the basis vectors as

\[
 g_{\mu \nu} = \eta_{ab}(e^a_{\mu})(e^b_{\nu}) = n_\mu l_\nu + l_\mu n_\nu - m_\mu \bar{m}_\nu - m_\nu \bar{m}_\mu. \tag{2.3}
\]

where \( e^a_{\mu} \) represents the basis vector \( l, n, m, \bar{m} \), \( \mu \) is the coordinate index, while \( a \) is the tetrad index, \( \eta_{ab} \) is the metric component under the tetrad form. The connection coefficients, called spin coefficients in the NP formalism with special Greek symbols (we
will follow the convention of [27], are presented as follows

\[ \kappa = \Gamma_{311} = l^\nu m^\mu \nabla_\nu l_\mu, \quad \pi = -\Gamma_{421} - l^\nu \bar{m}^\mu \nabla_\nu n_\mu, \]  
\[ \epsilon = \frac{1}{2} (\Gamma_{211} - \Gamma_{431}) = \frac{1}{2} (l^\nu n^\mu \nabla_\nu l_\mu - l^\nu \bar{m}^\mu \nabla_\nu m_\mu). \]  
(2.4)

\[ \tau = \Gamma_{312} = n^\nu m^\mu \nabla_\nu l_\mu, \quad \nu = -\Gamma_{422} = -n^\nu \bar{m}^\mu \nabla_\nu n_\mu, \]  
\[ \gamma = \frac{1}{2} (\Gamma_{212} - \Gamma_{432}) = \frac{1}{2} (n^\nu n^\mu \nabla_\nu l_\mu - n^\nu \bar{m}^\mu \nabla_\nu m_\mu). \]  
(2.5)

\[ \sigma = \Gamma_{313} = m^\nu m^\mu \nabla_\nu l_\mu, \quad \mu = -\Gamma_{423} = -m^\nu \bar{m}^\mu \nabla_\nu n_\mu, \]  
\[ \beta = \frac{1}{2} (\Gamma_{213} - \Gamma_{433}) = \frac{1}{2} (m^\nu n^\mu \nabla_\nu l_\mu - m^\nu \bar{m}^\mu \nabla_\nu m_\mu). \]  
(2.6)

\[ \rho = \Gamma_{314} = \bar{m}^\nu m^\mu \nabla_\nu l_\mu, \quad \lambda = -\Gamma_{424} = -\bar{m}^\nu \bar{m}^\mu \nabla_\nu n_\mu, \]  
\[ \alpha = \frac{1}{2} (\Gamma_{214} - \Gamma_{434}) = \frac{1}{2} (\bar{m}^\nu n^\mu \nabla_\nu l_\mu - \bar{m}^\nu \bar{m}^\mu \nabla_\nu m_\mu). \]  
(2.7)

We use five complex scalars to represent ten independent components of the Weyl tensors

\[ \Psi_0 = -C_{1313}, \quad \Psi_1 = -C_{1213}, \quad \Psi_2 = -C_{1342}, \quad \Psi_3 = -C_{1242}, \quad \Psi_4 = -C_{2424}. \]  
(2.8)

Ricci tensors are defined by four real and three complex scalars as follows

\[ \Phi_{00} = -\frac{1}{2} R_{11}, \quad \Phi_{22} = -\frac{1}{2} R_{22}, \quad \Phi_{02} = -\frac{1}{2} R_{33}, \quad \Phi_{20} = -\frac{1}{2} R_{44}, \]  
\[ \Phi_{11} = -\frac{1}{4} (R_{12} + R_{34}), \quad \Phi_{01} = -\frac{1}{2} R_{13}, \quad \Phi_{12} = -\frac{1}{2} R_{23}, \]  
\[ \Lambda = \frac{1}{24} R = \frac{1}{12} (R_{12} - R_{34}), \quad \Phi_{10} = -\frac{1}{2} R_{14}, \quad \Phi_{21} = -\frac{1}{2} R_{24}. \]  
(2.9)

where \( \Lambda \) is the cosmological constant. Considered as directional derivatives, the basis vectors are represented by special symbols:

\[ D = l^\mu \partial_\mu, \quad \Delta = n^\mu \partial_\mu, \quad \delta = m^\mu \partial_\mu. \]  
(2.10)

The equations that describing NP formalism include three classes:

1) The commutation relations of the basis vectors and the structure constants

\[ [e_a, e_b] = (\Gamma_{cba} - \Gamma_{cab}) e^c = C^c_{ab} e_c. \]  
(2.11)

Where \( e_a \) is the basis vector, and \( C^c_{ab} \) is the structure constant. The general tensorial formalism does not consider these relations since the coordinate basis is commutative. An example is as follow

\[ [\Delta, D] = [\eta, l] = [e_2, e_1] = (\Gamma_{c12} - \Gamma_{c21}) e^c \]  
\[ = -\Gamma_{121} \Delta + \Gamma_{212} D - (\Gamma_{312} - \Gamma_{321}) \delta - (\Gamma_{412} - \Gamma_{421}) \delta. \]  
(2.12)
Giving the spin coefficients their symbols, we get
\[
\Delta D - D\Delta = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\bar{\gamma} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta}.
\]

(2) The Ricci identities, similar to using the coordinate basis to calculate the Riemann curvature tensor in the general tensorial formalism, i.e.
\[
- \Psi_0 = C_{1313} = R_{1313} = \Gamma_{133,1} - \Gamma_{131,3} + \Gamma_{121} + \Gamma_{431} - \Gamma_{413} + \Gamma_{431} + \Gamma_{134}
\]
\[
- \Gamma_{131}(\Gamma_{433} + \Gamma_{123} - \Gamma_{213} + \Gamma_{231} + \Gamma_{132}).
\]

Substituting for the directional derivatives and the spin coefficients their designated symbols, we obtain
\[
D\sigma - \delta\kappa = \sigma(3\epsilon - \tau + \rho + \bar{\rho}) + \kappa(\bar{\pi} - \tau - 3\beta - \bar{\alpha}) + \Psi_0.
\]

(3) The Bianchi identities. It is similar to the Bianchi identities in the tensorial form, i.e.
\[
R_{1313|4} + R_{1334|1} + R_{1341|3} = 0.
\]
Where "|" represents the covariant derivative in tetrad form. It can be rewritten in the following form
\[
- \bar{\delta}\Psi_0 + D\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 + [Ricci] = 0.
\]

As for Maxwell theory, in NP formalism the antisymmetric Maxwell-tensor \( F_{\mu\nu} \) is replaced by the three complex scalars
\[
\phi_0 = F_{13} = F_{\mu\nu}l^\mu m^\nu,
\]
\[
\phi_1 = \frac{1}{2}(F_{12} + F_{43}) = \frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{n}^\mu m^\nu),
\]
\[
\phi_2 = F_{42} = F_{\mu\nu}\bar{n}^\mu n^\nu.
\]
Correspondingly, the Maxwell equations in tetrad form
\[
F_{[ab|c]} = 0, \quad \eta^{nm}F_{an|m} = 0.
\]
can be replaced by those equations
\[
\phi_{1|1} - \phi_{0|4} = 0, \quad \phi_{2|1} - \phi_{1|4} = 0,
\]
\[
\phi_{1|3} - \phi_{0|2} = 0, \quad \phi_{2|3} - \phi_{1|2} = 0.
\]
Expanding these equations into the terms of the ordinary derivatives and spinor coefficients, then expressing them into the symbols above, we can get the Maxwell equations in NP formalism. Similar disposition can be used to deal with the Klein-Gorden equation for scalar field, where we define \( \Omega_1 = D\varphi, \Omega_2 = \Delta \varphi, \Omega_3 = \delta \varphi, \Omega_4 = \bar{\delta} \varphi. \)

The freedom of the rotation of the basis vectors, see e.g. in [3], will allow us to set \( \pi = \kappa = \epsilon = 0, \rho = \bar{\rho}, \tau = \bar{\alpha} + \beta. \) (2.22)

From those conditions, one can find that \( l \) is tangent to a null geodesic with an affine parameter. Also, the congruence of the null geodesic is hypersurface orthogonal, that is, \( l \) is proportional to the gradient of a scalar field. So it is convenient to choose the scalar field as coordinate \( u = x^1, \) and the affine parameter as \( r = x^2. \) Thus, the basis vectors and the co-tetrad must have the form

\[
\begin{align*}
n^\mu \partial_\mu &= \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X^A \frac{\partial}{\partial x^A}, & \quad l^\mu \partial_\mu &= \frac{\partial}{\partial r}, & m^\mu \partial_\mu &= \omega \frac{\partial}{\partial r} + L^A \frac{\partial}{\partial x^A},
\end{align*}
\]

(2.23)

where \( L^A L^A = 0, L^A \bar{L}^A = -1. \) We will use the standard stereographic coordinates \( z = e^{i\phi} \cot \frac{\theta}{2} \) and \( \bar{z} = e^{-i\phi} \cot \frac{\theta}{2} \) in this work.

The Lagrangian of four-dimensional Einstein-Maxwell-dilaton theories are

\[
\mathcal{L} = \sqrt{-g} \left[ R - \frac{1}{4} e^{a\varphi} F^2 + \frac{1}{2} (\partial \varphi)^2 \right], \quad F = dA.
\]

(2.24)

This class of theories is generalized from the Einstein-Maxwell theory to include a real dilatonic scalar. When the dilaton coupling constant \( a \) takes the following specific values \( a = 0, \frac{1}{\sqrt{3}}, 1, \sqrt{3}, \) the EMD theories can be embedded into the \( N = 2 \) STU supergravity. Einstein-Maxwell theory, which is the bosonic sector of \( N = 2 \) supergravity, can be reduced from the \( a = 0 \) case, while the \( a = \sqrt{3} \) case can be Kaluza-Klein theory. Now we suppose that \( a \) is an arbitrary real constant.

The dilaton, Maxwell and Einstein equations can be derived from Lagrangian (2.24)

\[
\begin{align*}
\partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi) + a \frac{\sqrt{-g} e^{a\varphi}}{4} F^2 &= 0. & \quad (2.25)
\end{align*}
\]

\[
\begin{align*}
\partial_\nu (\sqrt{-g} e^{a\varphi} F_{\mu\nu}) &= 0. & \quad (2.26)
\end{align*}
\]

\[
\begin{align*}
R_{\mu\nu} = \frac{1}{2} e^{a\varphi} F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{8} g_{\mu\nu} e^{a\varphi} F^2 - \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi. & \quad (2.27)
\end{align*}
\]

According to these equations of motion in the tensorial form, we can easily recast them into the NP formalism. We divide these equations into three groups [4]:

\footnote{Note that we use the signature \((+, -, -, -)\). Hence the third term in the Lagrangian is \( \frac{1}{2} (\partial \varphi)^2 \) rather than most used convention \( -\frac{1}{2} (\partial \varphi)^2 \), see e.g. in [28].}
I. Radial equations

This group of equations can be integrated to find the radial dependence of all the variables, up to an proper order of magnitude. Each integration gives an arbitrary function of three nonradial coordinates (integration constant).

II. Nonradial equations

This group of equations give the relations among these integration constants so that most of the functions can be expressed in terms of two basic functions $\sigma^0(u, z, \bar{z})$ and $P(u, z, \bar{z})$.

III. The u-derivative equations

This group of equations characterizes the propagation of the components of Weyl tensor, the dilatonic scalar field and the Maxwell fields off the hypersurface in the u-direction (time direction), from null surface to null surface.

Radial equations

\[
D\rho = \rho^2 + \sigma\sigma + \frac{1}{2}e^{a}\phi_0\phi_1 - \frac{1}{4}(\Omega_1)^2, \quad (2.28)
\]
\[
D\sigma = 2\rho\sigma + \Psi_0, \quad (2.29)
\]
\[
D\tau = \tau\rho + \tau\sigma + \Psi_1 + \frac{1}{2}e^{a}\phi_0\phi_1 - \frac{1}{4}\Omega_1\Omega_3, \quad (2.30)
\]
\[
D\alpha = \rho\alpha + \beta\sigma + \frac{1}{2}e^{a}\phi_1\phi_0 + \frac{1}{4}\Omega_1\Omega_4, \quad (2.31)
\]
\[
D\beta = \alpha\sigma + \rho\beta + \Psi_1, \quad (2.32)
\]
\[
D\gamma = \tau\alpha + \tau\beta + \Psi_2 + \frac{1}{2}e^{a}\phi_1\phi_0 + \frac{1}{6}\Omega_1\Omega_2 + \frac{1}{12}\Omega_3\Omega_4, \quad (2.33)
\]
\[
D\lambda = \rho\lambda + \sigma\mu + \frac{1}{2}e^{a}\phi_2\phi_0 + \frac{1}{4}(\Omega_4)^2, \quad (2.34)
\]
\[
D\mu = \rho\mu + \sigma\lambda + \Psi_2 + \frac{1}{12}(\Omega_3\Omega_4 - \Omega_1\Omega_2), \quad (2.35)
\]
\[
D\nu = \tau\mu + \tau\lambda + \Psi_3 + \frac{1}{2}e^{a}\phi_2\phi_1 + \frac{1}{4}\Omega_2\Omega_4, \quad (2.36)
\]
\[
DU = \tau\omega + \tau\overline{\omega} - (\gamma + \overline{\gamma}), \quad (2.37)
\]
\[
DX^A = \tau L^A + \tau\overline{L}^A, \quad (2.38)
\]
\[
D\omega = \rho\omega + \sigma\overline{\omega} - \tau, \quad (2.39)
\]
\[
DL^A = \rho L^A + \sigma\overline{L}^A, \quad (2.40)
\]
\[
D\Psi_1 - \delta\Psi_0 = 4\rho\Psi_1 - 4\alpha\Psi_0 + \frac{1}{2}e^{a}\phi_1\phi_0 \quad \frac{1}{2}(a\phi_1 - \frac{a}{2}e^{a}\phi_1 - \frac{a}{2}e^{a}\phi_1)\phi_0\Omega_1
\]
\[-(a\phi_0 - \frac{a}{2}e^{a}\phi_0 - \frac{a}{2}e^{a}\phi_0)\phi_0\Omega_3 + \phi_1 D\phi_0 - \phi_0\delta\phi_0 - 2\sigma\phi_1\phi_0
\]
\[+2\beta\phi_0\phi_0] - \frac{1}{2}\Omega_1\delta\Omega_1 + \frac{1}{4}\Omega_3 D\Omega_1 + \frac{1}{4}\Omega_1 D\Omega_3 - \frac{1}{2}\rho\Omega_1\Omega_3 - \frac{1}{2}\sigma\Omega_1\Omega_4
\]
\[+ \frac{1}{2}(\tau + \beta)(\Omega_1)^2, \quad (2.41)
\]
\[
D\Psi_2 - \delta\Psi_1 = 3\rho\Psi_2 - 2\alpha\Psi_1 - \lambda\Psi_0
\]
\[+ \frac{1}{2}e^{a}\phi_1\phi_0 + \frac{1}{2}e^{a}\phi_1 + \frac{1}{2}e^{a}\phi_1\phi_4 \phi_0\Omega_2 - (a\phi_0 - \frac{a}{2}e^{a}\phi_0)\phi_0\Omega_2
\]
\[
\begin{align*}
-\frac{1}{2} ae^{a\varphi} \phi_2 \phi_0 \Omega_1 &+ \phi_1 \overline{d} \phi_0 - \phi_0 D\phi_0 - 2\alpha \phi_0 \phi_1 + 2\rho \phi_1 \phi_0 + 2\gamma \phi_0 \phi_0 \\
-2\tau \phi_1 \phi_0 &\bigg]+ \frac{1}{4} \Omega_2 \overline{d} \Omega_1 + \frac{1}{4} \Omega_1 \overline{d} \Omega_3 - \frac{1}{2} \Omega_1 \Delta \Omega_1 - \frac{1}{2}(\alpha + \tau) \Omega_1 \Omega_3 \\
+ \frac{1}{4} \rho \Omega_1 \Omega_2 + \Omega_3 \Omega_4 &\bigg] + \frac{1}{4} \overline{\sigma}(\Omega_3)^2 - \frac{1}{4}(\overline{\mu} - 2\gamma - 2\overline{\tau})(\Omega_1)^2 - \frac{1}{2} \tau \Omega_1 \Omega_4 \\
+ \frac{1}{12} (\Omega_2 D\Omega_1 + \Omega_1 D\Omega_2 - \Omega_4 D\Omega_3 - \Omega_3 D\Omega_4),
\end{align*}
\]
\[(2.42)\]
\[
D\Psi_3 - \overline{\sigma} \Psi_2 = 2\rho \Psi_3 - 2\lambda \Psi_1 + \frac{1}{2} e^{a\varphi} [(a\phi_1 - \frac{1}{2} ae^{a\varphi} \phi_1 - \frac{1}{2} ae^{a\varphi} \phi_0) \phi_2 \Omega_1 \\
- (a\phi_0 - \frac{1}{2} ae^{a\varphi} \phi_0 - \frac{1}{2} ae^{a\varphi} \phi_0) \phi_2 \Omega_3 &+ \phi_1 D\phi_2 - \phi_0 D\phi_2 + 2\mu \phi_1 \phi_0 \\
-2\beta \phi_2 \phi_0 &\bigg] + \frac{1}{4}(\Omega_4 D\Omega_2 + \Omega_2 D\Omega_4) - \frac{1}{2} \Omega_4 \delta \Omega_4 - \frac{1}{2} \rho \Omega_2 \Omega_2 + \frac{1}{2} \mu \Omega_1 \Omega_4 \\
+ \frac{1}{2}(\overline{\mu} - \beta)(\Omega_4)^2 - \frac{1}{12} (\Omega_2 \overline{d} \Omega_1 + \Omega_1 \overline{d} \Omega_2 - \Omega_4 \overline{d} \Omega_3 - \Omega_3 \overline{d} \Omega_4),
\end{align*}
\]
\[(2.43)\]
\[
D\Psi_4 - \overline{\sigma} \Psi_3 = \rho \Psi_4 + 2\alpha \Psi_3 - 3\lambda \Psi_2 + \frac{1}{2} e^{a\varphi} [(\frac{1}{2} ae^{a\varphi} \phi_0 - a\phi_0) \phi_2 \Omega_2 \\
- \frac{1}{2} ae^{a\varphi} \phi_2 \phi_0 \Omega_1 &- (\frac{1}{2} ae^{a\varphi} \phi_1 - \frac{1}{2} ae^{a\varphi} \phi_1 - a\phi_0) \phi_2 \Omega_4 \\
- \phi_0 \Delta \phi_2 + \phi_1 \phi_2 &+ 2\alpha \phi_1 \phi_2 + 2\nu \phi_1 \phi_0 - 2\gamma \phi_0 \phi_2 - 2\lambda \phi_1 \phi_1 \\
+ \frac{1}{4} \Omega_1 \Omega_4 - \frac{1}{4} \Omega_2 \delta \Omega_2 - \frac{1}{4} \Omega_2 \delta \Omega_4 - \frac{1}{2}(\alpha + \tau) \Omega_2 \Omega_4 \\
- \frac{1}{2} \nu \Omega_1 \Omega_4 &\bigg] + \frac{1}{4}(\Omega_1 \Omega_2 + \Omega_3 \Omega_4) + \frac{1}{4}(\overline{\mu} + 2\gamma - 2\overline{\tau})(\Omega_4)^2,
\end{align*}
\]
\[(2.44)\]
\[
D\phi_1 - \overline{\sigma} \phi_1 = 2\rho \phi_1 - 2\alpha \phi_0 - \frac{1}{2} ae^{a\varphi} (\phi_1 + \phi_1) \Omega_1 + \frac{1}{2} ae^{a\varphi} \phi_0 \Omega_4 \\
+ \frac{1}{2} ae^{a\varphi} \phi_0 \Omega_3,
\end{align*}
\]
\[(2.45)\]
\[
D\phi_2 - \overline{\sigma} \phi_2 = \rho \phi_2 - \lambda \phi_0 + \frac{1}{2} ae^{a\varphi} \phi_0 \Omega_2 - \frac{1}{2} ae^{a\varphi} \phi_1 \Omega_1 \\
+ \frac{1}{2} ae^{a\varphi} (\phi_1 - \phi_1) \Omega_4.
\end{align*}
\]
\[(2.46)\]

**Non-radial equations**

\[
\begin{align*}
\Delta \lambda &= \overline{\omega} - (\mu + \overline{\mu}) \lambda - (3\gamma - \overline{\tau}) \lambda + 2\alpha \nu - \Psi_4, \\
\Delta \rho &= \overline{\tau} - \rho \overline{\tau} - \sigma \lambda - 2\alpha \tau + (\gamma + \overline{\tau}) \rho - \Psi_2 + \frac{1}{12} \Omega_1 \Omega_2 - \frac{1}{12} \Omega_3 \Omega_4, \\
\Delta \alpha &= \overline{\sigma} \gamma + \rho \nu - (\tau + \beta) \lambda + (\overline{\tau} - \gamma - \overline{\mu}) \alpha - \Psi_3, \\
\Delta \mu &= \delta \nu - \mu^2 - \lambda \overline{\lambda} - (\gamma + \overline{\tau}) \mu + (\sigma + \beta) \nu - \frac{1}{2} e^{a\varphi} \phi_2 \phi_2 - \frac{1}{4}(\Omega_2)^2, \\
\Delta \beta &= \delta \gamma - \mu \tau + (\sigma + \beta) \gamma - \overline{\tau} - \mu) - \alpha \overline{\lambda} - \frac{1}{2} e^{a\varphi} \phi_1 \phi_2 - \frac{1}{4} \Omega_2 \Omega_3, \\
\Delta \sigma &= \delta \tau - \tau \mu - \rho \lambda - 2\beta \tau + (3\gamma - \overline{\tau}) \sigma - \frac{1}{2} e^{a\varphi} \phi_0 \phi_0 - \frac{1}{4}(\Omega_3)^2, \\
\Delta \omega &= \delta U - \overline{\lambda} - \overline{\lambda} \omega + (\gamma - \overline{\tau} - \mu) \omega, \\
\Delta L^A &= \delta X^A - \overline{\lambda} L^A + (\gamma - \overline{\tau} - \mu) L^A, \\
\delta \rho - \overline{\sigma} \sigma &= \rho \tau - \sigma (3\alpha - \beta) - \Psi_1 + \frac{1}{2} e^{a\varphi} \phi_0 \phi_1 + \frac{1}{4} \Omega_1 \Omega_3,
\end{align*}
\]
\[(2.47)\]
\[ \begin{align*}
\delta \alpha - \delta \beta &= \mu \rho - \lambda \sigma + \alpha \overline{\alpha} + \beta \overline{\beta} - 2\alpha \beta - \Psi_2 + \frac{1}{2} e^{a \varphi} \phi_1 \phi_1 \\
+ \frac{1}{12} \Omega_1 \Omega_2 + \frac{1}{6} \Omega_3 \Omega_4, \\
\delta \lambda - \delta \mu &= \mu \overline{\tau} + \lambda (\overline{\alpha} - 3\beta) - \Psi_3 + \frac{1}{2} e^{a \varphi} \phi_2 \phi_1 + \frac{1}{4} \Omega_2 \Omega_4, \\
\delta \overline{\alpha} - \delta \omega &= \mu - \overline{\mu} - (\alpha - \overline{\beta}) \omega + (\alpha - \overline{\beta}) \overline{\omega}, \\
\delta \overline{L}^A - \overline{\delta} L^A &= (\overline{\alpha} - \overline{\beta}) \overline{L}^A - (\alpha - \overline{\beta}) L^A.
\end{align*} \] (2.56)

The u-derivative equations

\[ \begin{align*}
\Delta \Psi_0 - \delta \Psi_1 &= (4\gamma - \mu) \Psi_0 - (4\tau + 2\beta) \Psi_1 + 3\sigma \Psi_2 \\
&- \frac{1}{2} \left[ (a \varphi_2 - \frac{1}{2} a e^{a \varphi} \phi_2) \phi_0 \Omega_1 + \frac{1}{2} a e^{a \varphi} \phi_0 \phi_2 \Omega_2 + \right. \\
&\left. \left( \frac{1}{2} a e^{a \varphi} \phi_2 - \frac{1}{2} a e^{a \varphi} \phi_1 \right) \phi_2 \Omega_3 + \phi_2 D \phi_0 - \phi_1 \delta \phi_2 + 2\beta \phi_0 \phi_1 \right] \\
&+ 2\sigma (\Omega_1 \Omega_2 + \Omega_3 \Omega_4) + \frac{1}{4} \rho (\Omega_3)^2 \\
\Delta \Psi_1 - \delta \Psi_2 &= \nu \Psi_0 + (2\gamma - 2\mu) \Psi_1 - 3\tau \Psi_2 + 2\sigma \Psi_3 \\
&+ \frac{1}{2} \left[ (a \varphi_2 - \frac{1}{2} a e^{a \varphi} \phi_2) \phi_0 \Omega_2 + \frac{1}{2} a e^{a \varphi} \phi_0 \phi_2 \Omega_3 + \right. \\
&\left. \left( \frac{1}{2} a e^{a \varphi} \phi_2 - \frac{1}{2} a e^{a \varphi} \phi_1 \right) \phi_0 \Omega_3 + \phi_0 D \phi_2 - \phi_1 \delta \phi_2 + 2\beta \phi_0 \phi_1 \right] \\
&+ \frac{2\mu \phi_0 \phi_1}{2} (\Omega_1 \Omega_2 + \Omega_3 \Omega_4) + \frac{1}{4} \left( \tau - 2\beta + 2\alpha \right) (\Omega_3)^2 - \frac{1}{12} (\Omega_2 \delta \Omega_1 + \Omega_1 \delta \Omega_2 \\
&- \Omega_3 \delta \Omega_4 - \Omega_4 \delta \Omega_3). \\
\Delta \Psi_2 - \delta \Psi_3 &= 2\nu \Psi_1 - 3\mu \Psi_2 + (2\beta - 2\tau) \Psi_3 + \sigma \Psi_4 \\
&+ \frac{1}{2} \left[ (a \varphi_2 - \frac{1}{2} a e^{a \varphi} \phi_2) \phi_0 \Omega_3 + \frac{1}{2} a e^{a \varphi} \phi_0 \phi_2 \Omega_2 + \right. \\
&\left. \left( \frac{1}{2} a e^{a \varphi} \phi_2 - \frac{1}{2} a e^{a \varphi} \phi_1 \right) \phi_0 \Omega_2 + \phi_0 D \phi_3 - \phi_1 \delta \phi_2 + 2\beta \phi_0 \phi_2 \right] \\
&+ 2\mu (\Omega_1 \Omega_2 + \Omega_3 \Omega_4) - \frac{1}{4} \lambda (\Omega_4)^2 - \frac{1}{4} \rho (\Omega_2)^2 + \frac{1}{12} (\Omega_2 \delta \Omega_1 + \Omega_1 \delta \Omega_2 \\
&- \Omega_4 \delta \Omega_3 - \Omega_3 \delta \Omega_4). \\
\Delta \Psi_3 - \delta \Psi_4 &= 3\nu \Psi_2 - (2\gamma + 4\mu) \Psi_3 + (4\beta - \tau) \Psi_4 \\
&+ \frac{1}{2} \left[ (a \varphi_2 - \frac{1}{2} a e^{a \varphi} \phi_2) \phi_0 \Omega_4 + \phi_2 D \phi_2 - \phi_1 \delta \phi_2 + 2\beta \phi_0 \phi_1 \right] \\
&+ \frac{1}{4} \left( \Omega_1 \Omega_2 + \Omega_3 \Omega_4 - \frac{1}{2} \lambda (\Omega_2)^2 - \frac{1}{2} (\overline{\mu} + \gamma) \Omega_2 \Omega_4 \\
&- 2\alpha \phi_2 \phi_2 \right] + \frac{1}{4} \left( \Omega_2 \delta \Omega_2 + \frac{1}{2} \Omega_2 \overline{\delta} \Omega_2 + \frac{1}{2} (\overline{\mu} + \gamma) \Omega_2 \Omega_4 \\
\[-\frac{1}{4} \nu (\Omega_1 \Omega_2 + \Omega_3 \Omega_4) - \frac{1}{4} \psi (\Omega_4)^2 + \frac{1}{2} \lambda \Omega_2 \Omega_3 - \frac{1}{4} (\alpha + \beta) (\Omega_2)^2. \quad (2.63)\]
\[\Delta \phi_0 - \delta \phi_0 = (2 \gamma - \mu) \phi_0 - 2 \tau \phi_1 + \sigma \phi_2 - \frac{1}{2} a e^{a \varphi} \phi_0 \Omega_2 + \frac{1}{2} a e^{a \varphi} \phi_2 \Omega_1 + \frac{1}{2} a e^{a \varphi} \phi_1 \Omega_3 - \frac{1}{2} a e^{a \varphi} \phi_1 \Omega_3. \quad (2.64)\]
\[\Delta \phi_1 - \delta \phi_2 = \nu \phi_0 - 2 \mu \phi_1 - \frac{1}{2} a e^{a \varphi} \phi_1 \Omega_2 - \frac{1}{2} a e^{a \varphi} \phi_2 \Omega_2 + \frac{1}{2} a e^{a \varphi} \phi_2 \Omega_2. \quad (2.65)\]
\[\Delta \Omega_1 + D \Omega_2 = \delta \Omega_3 + \delta \Omega_4 + (\gamma + \gamma - \mu - \mu) \Omega_1 + 2 \rho \Omega_2 - 2 \mu \Omega_4 - 2 \alpha \Omega_3 + a e^{a \varphi} (\phi_1^2 + \phi_2^2 - \phi_0 \bar{\phi}_2 - \phi_0 \bar{\phi}_2). \quad (2.66)\]

3 The solution space

The main conditions of approaching flatness at infinity are \(\Psi^0 = \frac{\Psi^0}{r} + \mathcal{O}(r^{-6}).\) Newman and Unti [4] listed the fall-off conditions of the rest quantities by solving the empty space Newman-Penrose equations

\[\rho = -r^{-1} + \mathcal{O}(r^{-3}), \quad \sigma = \mathcal{O}(r^{-2}), \quad \alpha = \mathcal{O}(r^{-1}), \quad \beta = \mathcal{O}(r^{-1}),\]
\[\tau = \mathcal{O}(r^{-3}), \quad \lambda = \mathcal{O}(r^{-1}), \quad \mu = \mathcal{O}(r^{-1}), \quad \gamma = \mathcal{O}(1),\]
\[\nu = \mathcal{O}(1), \quad U = \mathcal{O}(r), \quad X^z = \mathcal{O}(r^{-3}), \quad \omega = \mathcal{O}(r^{-1}),\]
\[L^z = \mathcal{O}(r^{-2}), \quad L^z = \mathcal{O}(r^{-1}),\]
\[\Psi_1 = \mathcal{O}(r^{-4}), \quad \Psi_2 = \mathcal{O}(r^{-3}), \quad \Psi_3 = \mathcal{O}(r^{-2}), \quad \Psi_4 = \mathcal{O}(r^{-1}). \quad (3.1)\]

The fall-off of the matter fields can not violate the asymptotic conditions above, so we choose

\[\phi_0 = \frac{\phi_0^0}{r^3} + \mathcal{O}(r^{-4}). \quad (3.2)\]
\[\varphi = \frac{\varphi_1}{r} + \frac{\varphi_2}{r^2} + \mathcal{O}(r^{-3}). \quad (3.3)\]

and

\[\phi_1 = \frac{\phi_1^0}{r^2} + \mathcal{O}(r^{-3}), \quad \phi_0 = \frac{\phi_0^0}{r} + \mathcal{O}(r^{-2}). \quad (3.4)\]

Using the conditions above, we work out the asymptotically flat solution space of Newman-Penrose equations in Einstein-Maxwell-dilaton theory. Here are the solutions of the radial equations. One should notice that here the boundary topology is an arbitrary 2 surface but not \(S^2\)

\[\Psi_0 = \frac{\Psi_0^0(u, z, \bar{z})}{r^3} + \frac{\Psi_0^1(u, z, \bar{z})}{r^6} + \mathcal{O}(r^{-7}),\]
\[\phi_0 = \frac{\phi_0^0(u, z, \bar{z})}{r^3} + \frac{\phi_0^1(u, z, \bar{z})}{r^4} + \mathcal{O}(r^{-5}),\]
φ = \frac{\dot{\varphi}_1}{r} + \frac{\dot{\varphi}_2}{r^2} + O(r^{-3}),
\rho = -\frac{1}{r} + \frac{-\varphi_1^2 - 4\sigma^0\sigma^0}{4r^3} \frac{\varphi_1\varphi_2}{4r^4} + \frac{1}{48r^5}(-8\phi_0^0\phi_0^0 - \varphi_1^4 - 16\varphi_2^2 - 24\varphi_1\varphi_3
+ 8\sigma^0\Psi_0^0 + 8\sigma^0\Psi_0^0 - 16\sigma^0\sigma^0\varphi_1^2 - 48(\sigma\sigma^0) + O(r^{-6}),
\sigma = \frac{\sigma^0(u, z, \bar{z})}{r^2} + \frac{-2\Psi_0^0 + \sigma^0\varphi_1^2 + 4\sigma_0\sigma^0\varphi_1^2}{4r^4} + \frac{1}{3r^5}(-\Psi_1^0 + \sigma^0\varphi_1\varphi_2) + O(r^{-6}),
L^z = -\frac{\sigma^0}{r^2}(4\tilde{P}\Psi_0^0 - 5\tilde{P}\sigma^0\varphi_1^2 - 24\tilde{P}\sigma^0(\sigma^0)^2) + \frac{1}{12r^5}(\tilde{P}\Psi_1^0
- 3\tilde{P}\varphi_1\varphi_2\varphi_0^0) + O(r^{-6}),
L^z = \frac{P(u, z, \bar{z})}{r} + \frac{1}{8r^3}(P\varphi_1^2 + 8P\sigma^0\sigma^0) + \frac{1}{6r^4}P\varphi_1\varphi_2 + \frac{1}{384r^5}(16P\phi_0^0\phi_0^0
+ 5P\varphi_1^4 + 32P\varphi_2^2 + 48P\varphi_1\varphi_3 - 64P\sigma^0\Psi_0^0 - 32P\sigma^0\Psi_0^0 + 112P\sigma^0 \sigma^0 \varphi_1^2
+ 384P(\sigma^0)^2(\sigma^0)^2) + O(r^{-6}),
L_z = -\frac{\varphi_1^2}{8Pr} + \frac{\varphi_1\varphi_2}{6Pr^2} + \frac{1}{384Pr^3}(16\phi_0^0\phi_0^0 - \varphi_1^4 + 32\varphi_2^2 + 48\varphi_1\varphi_3 + 32\sigma^0\Psi_0^0)
+ O(r^{-4}),
L_z = -\frac{\sigma^0}{P} + \frac{1}{2Pr^2}(4\Psi_0^0 + \sigma^0\varphi_1^2) + \frac{1}{12Pr^3}(\Psi_1^0 + \sigma^0\varphi_1\varphi_2) + O(r^{-4}),
\alpha = \frac{\alpha^0}{r} + \frac{\sigma^0\alpha^0}{r^2} + \frac{1}{8r^3}(\alpha^0\varphi_1^2 + 8\alpha^0\sigma^0\sigma^0 + \varphi_1\bar{\alpha}\varphi_1)
+ \frac{1}{24r^4}(2\sigma^0\sigma^0(\sigma^0)^2 - 4\sigma^0\Psi_0^0 + 4\sigma^0\Psi_0^0 - 4\phi_0^0\phi_0^0 - 2\sigma^0\varphi_1^2 + 4\alpha^0\varphi_1\varphi_2
+ 5\alpha^0\sigma^0(\varphi_1)^2 - 2\sigma^0\varphi_1\bar{\varphi}_1 + 4\varphi_2\bar{\varphi}_1 + 2\varphi_1\bar{\varphi}_2) + O(r^{-5}),
\beta = -\frac{\sigma^0}{r} + \frac{\sigma^0\sigma^0}{r^2} + \frac{1}{8r^3}(-\sigma^0\varphi_1^2 - 4\Psi_1^0 - 8\sigma^0\sigma^0\alpha^0)
+ \frac{1}{24r^4}(8\phi_0^0\phi_0^0 + 4\alpha^0\Psi_0^0 - 24\sigma^0(\sigma^0)^2\sigma^0 - 12\phi_0^0\phi_0^0 + 2\omega^0\varphi_1^2 - 4\varphi_1\varphi_2
- 5\alpha^0\sigma^0\varphi_1^2 + 4\varphi_2\bar{\varphi}_1 - 2\varphi_1\bar{\varphi}_2 + \sigma^0\varphi_1\bar{\varphi}_1) + O(r^{-5}),
\tau = \frac{1}{8r^3}(-4\Psi_1^0 + \varphi_1\bar{\varphi}_1) + \frac{1}{24r^4}(8\Psi_0^0 + 4\sigma^0\Psi_1^0 - 16\phi_0^0\phi_0^0 - \sigma^0\varphi_1\bar{\varphi}_1 + 8\varphi_2\bar{\varphi}_1) + O(r^{-5}),
\omega = \frac{\omega^0}{r} + \frac{1}{8r^2}(4\Psi_1^0 - 8\sigma^0\sigma^0 + \varphi_1\bar{\varphi}_1) + \frac{1}{24r^4}(4\Psi_0^0 + 24\sigma^0(\sigma^0)^2\omega^0 + 8\sigma^0\Psi_0^0 - 8\phi_0^0\phi_1^0
+ 3\omega^0\varphi_1^2 - 2\sigma^0\varphi_1\bar{\varphi}_1 + 4\varphi_2\bar{\varphi}_1) + O(r^{-4}),
\Psi_1^0 = \frac{\Psi_1^0(u, z, \bar{z})}{r^4} + \frac{1}{4r^5}(6\phi_0^0\phi_0^0 - 4\Phi_0^0\Phi_0^0 - \omega^0\varphi_1^2 - 2\varphi_2\bar{\varphi}_1 + \varphi_1\bar{\varphi}_2 - \sigma^0\varphi_1\bar{\varphi}_1) + O(r^{-6}),
X^z = \frac{1}{24r^3}(4\tilde{P}\Phi_0^0 - \tilde{P}\varphi_1\bar{\varphi}_1) + \frac{1}{24r^4}(-2\tilde{P}\Phi_0^0 - 4P\sigma^0\Psi_0^0 + 4P\phi_0^0\phi_1^0
+ \tilde{P}\sigma^0\varphi_1\bar{\varphi}_1 - 2\tilde{P}\varphi_2\bar{\varphi}_1) + O(r^{-5}),
\[ \gamma = \gamma^0 + \frac{1}{12r^2}(\gamma^0 \varphi_1^2 + \gamma^0 \varphi_2^2 - 6\Psi_0^1 + \varphi_1 \partial_u \varphi_1) + \frac{1}{24r^3}(8\Psi_0^2 + 4\alpha^0 \Psi_1^0) - 4\sigma(\Psi_0^1 - 12\varphi_1^0) - 2\mu^0 \varphi_1^2 + 2U^0 \varphi_1^2 + 4\varphi_2 \partial_u \varphi_1 - 2\varphi_1 \partial_u \varphi_2 - 2\overline{\partial} \varphi_1 \partial \varphi_1 - 2\varphi_2 \overline{\partial} \varphi_1 - \alpha^0 \varphi_1 \partial \varphi_1 + \alpha^0 \varphi_1 \overline{\partial} \varphi_1) + O(r^{-4}), \]

\[ \lambda = \frac{\lambda^0}{r} - \frac{\sigma^0 \mu^0}{r^2} + \frac{1}{24r^3}(24\sigma^0 \sigma^0 \lambda^0 + 12\sigma^0 \Psi_0^2 - 6\phi_2 \phi_0^0 - 3\gamma^0 \varphi_1^2 + (\gamma^0 + \overline{\gamma}^0) \sigma^0 \varphi_1^2 - 3(\overline{\gamma} \varphi_1^2 + \sigma^0 \varphi_1 \partial \varphi_1) + O(r^{-5}), \]

\[ \mu = \frac{\mu^0}{r} + \frac{1}{12r^2}(-12\Psi_1^0 - 12\sigma^0 \lambda^0 + U^0 \varphi_1^2 - \varphi_1 \partial_u \varphi_1) + \frac{1}{8r^3}(8\sigma^0 \sigma^0 \mu^0 + 4\overline{\gamma} \text{Psi}_1^0 - 4\phi_1 \phi_0^0) - 4(\gamma^0 + \overline{\gamma}^0) \varphi_1 \varphi_2 + 2U^0 \varphi_1^2 - 2\overline{\partial} \varphi_1 - \varphi_1 \overline{\partial} \varphi_1 - 2\varphi_1 \partial_u \varphi_2) + O(r^{-4}), \]

\[ U = -r(\gamma^0 + \overline{\gamma}^0) + U^0 + \frac{1}{6r}(\gamma^0 \varphi_1^2 + \gamma^0 \varphi_1^2 - 3\Phi_2^1 - 3\Phi_2^0 + \varphi_1 \partial_u \varphi_1) + \frac{1}{24r^2}(-12\phi_2 \phi_1 + 3\mu^0 \varphi_1^2 + 8(\gamma^0 + \overline{\gamma}^0) \varphi_1 \varphi_2 - 12\overline{\partial} \Psi_0^1 - 5\varphi_1^2 U^0) - 2\varphi_1 \partial_u \varphi_1 + 5\varphi_1 \partial_u \varphi_2 + 2\overline{\partial} \varphi_1 \partial \varphi_1 + 3\varphi_1 \overline{\partial} \varphi_1) + O(r^{-5}), \]

\[ \nu = \nu^0 - \frac{\Psi_0^0}{r} + \frac{1}{24r^2}(-12\phi_2 \phi_1 + 12\overline{\partial} \Psi_0^2 + \overline{\partial} \gamma^0 \varphi_1^2 + \overline{\partial} \gamma^0 \varphi_1^2 - 4\overline{\gamma} \varphi_1 \overline{\partial} \varphi_1 - 4\gamma^0 \varphi_1 \overline{\partial} \varphi_1 - 5\overline{\partial} \varphi_1 \partial_u \varphi_1 + \varphi_1 \overline{\partial} \varphi_1) + O(r^{-3}), \]

\[ \Psi_2 = \frac{\Psi_0^0}{r^3} + \frac{1}{12r^4}(12\phi_2 \phi_1 + 3\mu^0 \varphi_1^2 + 8(\gamma^0 + \overline{\gamma}^0) \varphi_1 \varphi_2 - 12\overline{\partial} \Psi_1^1 - 5\varphi_1^2 U^0) - 2\varphi_1 \partial_u \varphi_1 + 5\varphi_1 \partial_u \varphi_2 + 2\overline{\partial} \varphi_1 \partial \varphi_1 + 3\varphi_1 \overline{\partial} \varphi_1) + O(r^{-5}), \]

\[ \nu = \nu^0 - \frac{\Psi_0^0}{r} + \frac{1}{24r^2}(-12\phi_2 \phi_1 + 12\overline{\partial} \Psi_0^2 + \overline{\partial} \gamma^0 \varphi_1^2 + \overline{\partial} \gamma^0 \varphi_1^2 - 4\overline{\gamma} \varphi_1 \overline{\partial} \varphi_1 - 4\gamma^0 \varphi_1 \overline{\partial} \varphi_1 - 5\overline{\partial} \varphi_1 \partial_u \varphi_1 + \varphi_1 \overline{\partial} \varphi_1) + O(r^{-5}), \]

\[ \phi_2 = \frac{\phi_0^0}{r^3} + \frac{1}{12r^4}(-a\phi_0 \varphi_1 - 2\overline{\partial} \phi_1^1 + \frac{1}{8r^3}(4\lambda^0 \phi_0^0 + 8\varphi_0^0 \phi_0^1 + 4\alpha^0 \sigma^0 \phi_0^2) + 4\sigma^0 \phi_0^0 + 4\phi_0^0 \phi_0^1 - 2a(\gamma^0 + \overline{\gamma}^0) \phi_0 \varphi_1 + \phi_0 \varphi_1^2 - a^2 \psi \varphi_1^2 - 4a \phi_0 \varphi_2 + 2a \phi_0 \phi_1 \varphi_1 + 4a \phi_1 \phi_0 \varphi_1 - 2a \phi_0 \partial_u \varphi_1) + O(r^{-4}), \]

\[ \phi_1 = \frac{\phi_0^0}{r^2} + \frac{1}{2r^3}(-2\overline{\partial} \phi_0^0 - a \phi_0 \varphi_1 - a \phi_1 \varphi_1) + O(r^{-4}), \]

\[ \Psi_4 = \frac{\Psi_0^0}{r} + \frac{1}{2r^2} + O(r^{-3}). \]

The solutions of non-radial equations are as follow

\[ \Psi_4 = \overline{\partial} \nu^0 - \partial_u \lambda^0 - 4\gamma^0 \lambda^0, \]

\[ U^0 = \mu^0, \quad \gamma^0 = -\frac{1}{2} \partial_u \ln P, \quad \phi^0 = \overline{\partial} \sigma^0, \quad \nu^0 = \overline{\partial} \gamma^0, \quad \alpha^0 = \frac{1}{2} \overline{\partial} \partial_u \ln P. \]
\[ \lambda^0 = \partial_u \sigma^0 + \bar{\sigma}^0 (3\gamma^0 - \bar{\gamma}^0), \]
\[ \Psi_3^0 = \bar{\sigma} \mu^0 - \bar{\partial} \lambda^0, \]
\[ \mu^0 = -\frac{1}{2} P \bar{P} \partial_2 \partial_3 \ln P \bar{P}, \]
\[ \Psi_2^0 - \bar{\Psi}_2^0 = \bar{\sigma}^0 \sigma^0 - \bar{\sigma}^2 \sigma^0 + \sigma^0 \lambda^0 - \sigma^0 \lambda^0. \] (3.6)

And we obtain the solutions of the u-derivative equations which determine the propagation of the fields off the null hypersurface
\[ \partial_u \Psi_0^0 = \frac{3}{2} \phi_0 \phi_2 - \frac{1}{4} \gamma^0 \phi_2 - (\gamma^0 + 3\gamma^0) \Psi_0^0 + \bar{\sigma} \Psi_1^0 - \frac{1}{4} (\gamma^0 + \bar{\gamma}^0) \sigma^0 \phi_1^2 + 3 \sigma^0 \phi_2^0 \]
\[ - \frac{1}{4} \sigma^0 \phi_1 \partial_u \phi_1 + \frac{1}{2} (\partial \phi_1)^2. \] (3.7)
\[ \partial_u \Psi_1^0 = \phi_1 \phi_2 - \frac{1}{4} \gamma^0 \phi_2 - 2 (\gamma^0 + 2\gamma^0) \Psi_1^0 + 2 \sigma^0 \Psi_3^0 + \frac{1}{12} (\bar{\sigma}^2 \gamma^0 + \sigma^0 \phi_1^2) \]
\[ - \frac{1}{3} \gamma^0 \phi_1 \partial_u \phi_1 + \frac{1}{6} \gamma^0 \phi_1 \partial_u \phi_1 + \bar{\sigma} \Psi_2^0 - \frac{1}{4} \phi_1 \partial_u \phi_1 + \frac{1}{3} \partial_u \phi_1 \partial_u \phi_1 + \frac{1}{2} \partial \phi_1 \partial_u \phi_1. \] (3.8)
\[ \partial_u \Psi_2^0 = \frac{1}{2} \phi_2 \phi_2 - 3 (\gamma^0 + \bar{\gamma}^0) \Psi_2^0 + \sigma^0 \Psi_4^0 + \bar{\sigma} \phi_3^0 - \frac{1}{12} (\partial_u \gamma^0 + \partial_u \bar{\gamma}^0) \phi_1^2 \]
\[ + \frac{1}{12} (\gamma^0 + \bar{\gamma}^0) \phi_1 \partial_u \phi_1 + \frac{1}{6} (\partial_u \phi_1)^2 - \frac{1}{12} \phi_1 \partial_u \phi_1. \] (3.9)
\[ \partial_u \Psi_3^0 = \bar{\sigma} \Psi_3^0 - 2 (\gamma^0 + \bar{\gamma}^0) \Psi_3^0. \] (3.10)
\[ \partial_u \phi_0^0 = -(\gamma^0 + 3\bar{\gamma}^0) \phi_0^0 + \sigma^0 \phi_2^0 + \bar{\sigma} \phi_1^0 - \frac{1}{2} \phi_2 \partial_u \phi_1. \] (3.11)
\[ \partial_u \phi_1^0 = \bar{\sigma} \phi_2^0 - 2 (\gamma^0 + \bar{\gamma}^0) \phi_1^0. \] (3.12)
\[ \partial_u \phi_2^0 = -2 (\gamma^0 + \bar{\gamma}^0) \phi_2 - \bar{\sigma} \phi_1^0. \] (3.13)

The “\(\partial\)” operator is defined as
\[ \partial \eta^s = PP^{-s} \partial_z (P^s \eta^s) = P \partial_z \eta^s + 2s \bar{\sigma} \eta^s, \]
\[ \bar{\partial} \eta^s = \bar{P} P^{-s} \partial_z (P^s \eta^s) = \bar{P} \partial_z \eta^s - 2s \sigma^0 \eta^s. \] (3.14)

where \(s\) is the spin weight of the field \(\eta\). The spin weights of relevant fields are listed in Table 1.

| \(s\) | \(\bar{\sigma}\) | \(\partial_u\) | \(\gamma^0\) | \(\nu^0\) | \(\mu^0\) | \(\sigma^0\) | \(\lambda^0\) | \(\Psi_0^0\) | \(\Psi_1^0\) | \(\Psi_2^0\) | \(\Psi_3^0\) | \(\phi_0^0\) | \(\phi_1^0\) | \(\phi_2^0\) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 0   | 0   | -1  | 0   | 2   | -2  | -2  | -1  | 0   | 1   | 2   | -1  | 0   | 1   |

From the solutions above, we find that there is no constraint at the order \(O(\frac{1}{r^3})\) of \(\phi_2\) and \(\varphi\), and at the order \(O(\frac{1}{r^2})\) of \(\sigma\). So \(\sigma^0\), \(\phi_2^0\) and \(\varphi_1\) are related to the news functions in
the system which indicate gravitational, electromagnetic and scalar radiations. $\sigma^0$ has a special geometric meaning that it represents the asymptotic shear of $l$ (see [3] and [24]), the change of which at early time $u_i$ and late time $u_f$ is equivalent to the time integration of the asymptotic shear of $n$, i.e. $\lambda^0$ when we set the boundary topology to be $S^2 \times R$, i.e. $P = \tilde{P} = P_s = \frac{1 + \bar{z}^2}{\sqrt{2}}$ (see [10] and [24]). The memory effect [10, 24], which will be discussed in the following section, is controlled by the time integration of the asymptotic shear of $n$, i.e.$\lambda^0$, thus the change of the asymptotic shear at early time and late time $\Delta \sigma^0$ is a very important quantity to characterize gravitational memory effect.

According to eq(3.11), we can find that this time evolution equation involves the coupling constant $a$ which represents the non-minimal coupling of the electromagnetic field and the scalar field. $\phi_0^0$ is related to the electric dipole [29]. Our result is consistent with the result in [28]. The phenomenon that the coupling constant $a$ do not appear in the time evolution functions of four tetrad components of Weyl tensor reflects that the scalar field is minimally coupled to gravity.

Here from eq(3.9) and eq(3.12) we can consider the conservation laws and the loss of mass in EMD theory. Here we work in the unit 2-sphere case. From eq(3.9), i.e. $P = \tilde{P} = P_s = \frac{1 + \bar{z}^2}{\sqrt{2}}$, we can find

$$\partial_u \Psi_2^0 = \frac{1}{2} \varphi_2^0 \varphi_0^0 - \sigma^0 \partial_u \varphi_0^0 - \sigma^0 \partial_u \varphi_0^0 + \delta \psi^0_3 + \frac{1}{6} (\partial_u \varphi_1)^2 - \frac{1}{12} \varphi_1 \partial_u^2 \varphi_1. \quad (3.15)$$

Define the mass density

$$M = \frac{1}{2} (\Psi_2^0 + \tilde{\Psi}_2^0) + \frac{1}{2} (\sigma^0 \partial_u \sigma^0 + \sigma^0 \partial_u \sigma^0) + \frac{1}{2} (\bar{\sigma} \bar{\sigma} + \bar{\sigma} \bar{\sigma}) + \frac{1}{12} \varphi_1 \partial_u \varphi_1. \quad (3.16)$$

We can obtain

$$\partial_u M = \frac{1}{2} \varphi_2^0 \varphi_0^0 + \partial_u \sigma^0 \partial_u \sigma^0 + \frac{1}{4} (\partial_u \varphi_1)^2. \quad (3.17)$$

Considering the signature convention in NP formalism, we get the mass loss theorem in EMD theories: The mass density at any angle of the system can never increase. It is a constant if and only if there is no news. Our mass-loss formula generalizes the one in [28] by removing the constraint of axisymmetry.

As for Maxwell part, we work in retarded radial gauge $A_r = 0$. The Maxwell-tensor is constructed as

$$F_{\mu \nu} = (\phi_1 + \bar{\phi}_1)(n_\mu l_\nu - l_\mu n_\nu) + (\phi_1 - \bar{\phi}_1)(m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu) + \varphi_2 (l_\mu m_\nu - m_\mu l_\nu) + \bar{\varphi}_2 (l_\mu \bar{m}_\nu - \bar{m}_\mu l_\nu) + \phi_0 (m_\mu n_\nu - n_\mu m_\nu) + \bar{\phi}_0 (m_\mu n_\nu - n_\mu m_\nu). \quad (3.18)$$

We represent the Newman-Penrose variables of Maxwell parts in terms of the gauge fields $A_\mu$,

$$A_0^0 = -(\phi_1 + \bar{\phi}_1), \quad \partial_u A_0^0 = -\frac{\phi_0^0}{P}, \quad A_1^1 = -\frac{\bar{\phi}_0}{P}, \quad (\partial_z A_0^0 - \partial_z A_0^0) = \frac{\phi_1 - \bar{\phi}_1}{PP}. \quad (3.19)$$
\[ \partial_u \left( \frac{A_0^u}{P^u} \right) = \partial_u (\partial_z A_0^u + \partial_{\bar{z}} A_0^u). \tag{3.20} \]

where
\[ A_u = \frac{A_0^u(u, \bar{z})}{r} + \mathcal{O}(r^{-2}), \quad A_z = \frac{A_0^z(u, \bar{z})}{r} + \mathcal{O}(r^{-2}). \tag{3.21} \]

From eq (3.12), we find
\[ \partial_u \phi_0^1 = \partial \phi_0^2 = P_s \partial_{\bar{z}} \phi_0^2 - \partial_{\bar{z}} P_s \phi_0^2 = P_s^2 \partial_{\bar{z}} (\phi_0^2 / P_s). \tag{3.22} \]

Taking the real part of eq (3.22), we get
\[ \partial_u \left( \frac{\phi_0^1 + \bar{\phi}_0^1}{2} \right) = \frac{P_s^2}{2} (\partial_{\bar{z}} (\phi_0^2 / P_s) + \partial_{\bar{z}} (\bar{\phi}_0^2 / P_s)). \tag{3.23} \]

Substituting eq (3.19) into eq (3.23), we find
\[ \partial_u A_0^u = \partial_u (P_s^2 \partial_{\bar{z}} A_0^u + P_s^2 \partial_{\bar{z}} A_0^u). \tag{3.24} \]

Defining the flux,
\[ \Phi = A_0^u - P_s^2 (\partial_{\bar{z}} A_0^u + \partial_{\bar{z}} A_0^u). \tag{3.25} \]

we find
\[ \partial_u \Phi = 0. \tag{3.26} \]

This means that the flux does not change with time. According to Gauss’s Law, we can conclude that the charge is conserved, which again generalizes the result in [28] to asymptotically flat case.

### 4 The memory effects

According to the solution space in the previous section, we can derive the memory effect as an application. We will examine the motion of a charged time-like particle to specify the observational effects in a unified expression [25], that is to say, we consider the contributions of the gravitational radiation, electromagnetic radiation and scalar radiation at the same time by considering the affects of the motion of a charged particle caused by the radiations in EMD theories. Gravitational memory effects, characterized by the non-linear contribution to the overall change in the shear of outgoing null surfaces at the future null infinity [10], are a large group of observational effects of the gravitational radiation. Displacement memory effects are a kind of observational effects about a location displacement of the observers, i.e. [20] describes a distance shift of two parallel inertial detectors near the null infinity caused by the radiative energy flux. Spin memory effects [22,24] are
a kind of memory effects that the radiation causes the observers to rotate. i.e. [22] is about a relative time delay of two beams of light on clockwise and counterclockwise orbits induced by the radiative angular momentum flux. [24] discovers a kind of spin memory effect characterized by the time delay of a free-falling massive particle constrained on a time-like, \( r = r_0 \) hypersurface. Gravitational memory effects have analogue in Maxwell theory, which are called electromagnetic memory effects, i.e. [18, 19, 30, 31]. [19] is a change of the velocity (a “kick”) of a charged particle. Here we will consider all these three kinds of memory effects. The charged particle will be constrained on a time-like, \( r = r_0 \) hypersurface. \( r = r_0 \) is a fixed radial distance which is very large, means that the particle is very far from the gravitational and electromagnetic source. The induced metric of this hypersurface can be derived by inserting the solution space in the previous section into eq(2.3) and eq(2.23), which in series expansions is given by

\[
\begin{align*}
\frac{ds^2}{dt} &= \left[ 1 + \frac{1}{r_0} (\Psi_0^2 + \bar{\Psi}_0^2 - \frac{1}{3} \varphi_1 \partial_0 \varphi_1) + \frac{1}{12 r_0^2} (12 \phi_0^2 \phi_1^0 - 4 \bar{\partial} \Psi_1^0 - 4 \bar{\partial} \bar{\Psi}_1^0 + 2 \bar{\partial} \varphi_1 \partial_0 \varphi_1 \\
&\quad + 2 \varphi_1 \partial_0 \varphi_1 - 4 \varphi_2 \partial_0 \varphi_2 + 2 \varphi_1 \partial_0 \varphi_2) + O(r_0^{-3}) \right] du^2 + 2\left[ - \frac{\partial \sigma_0^0}{P_s} + \frac{1}{6 P_s r_0} (4 \bar{\Psi}_1^0 - \varphi_1 \partial_0 \varphi_1) \\
&\quad + \frac{1}{3 P_s^2 r_0} (\Psi_0^0 + \bar{\sigma}_0^0 \varphi_1^2) + O(r_0^{-2}) \right] dz^2 + 2\left[ \frac{1}{3 P_s^2} + \frac{1}{3 P_s^2 r_0} (\Psi_0^0 + \sigma_0^0 \varphi_1^2) \\
&\quad + \frac{1}{3 P_s^2 r_0} (\Psi_0^0 + \bar{\sigma}_0^0 \varphi_1^2) + O(r_0^{-2}) \right] dz \bar{z}.
\end{align*}
\]

(4.1)

One should notice that here we fix the topology of the 2-surface which is the time slice of the null infinity as the unit 2-sphere case, meaning that \( P = \bar{P} = P_s = \frac{1 + \bar{z}}{\sqrt{2}} \). The induced Maxwell field on the \( r = r_0 \) hypersurface is

\[
\begin{align*}
F_{uz} &= -\frac{\partial_0 \bar{\phi}_2}{P_s} + \frac{1}{2 P_s r_0} (a \phi_2^0 \varphi_1 - 2 \bar{\sigma}_0^0 \phi_2^0 + 2 \bar{\partial} \phi_1^0) + O(r_0^{-2}), \\
F_{u\bar{z}} &= -\frac{\partial_0 \bar{\phi}_2}{P_s} + \frac{1}{2 P_s r_0} (a \bar{\phi}_2^0 \varphi_1 - 2 \sigma_0^0 \phi_2^0 + 2 \partial \bar{\phi}_1^0) + O(r_0^{-2}), \\
F_{\bar{z}\bar{z}} &= \frac{\partial_0 \bar{\phi}_1}{P_s^2} + \frac{\partial \sigma_0^0 - \bar{\partial} \sigma_0^0}{P_s^2 r_0} + O(r_0^{-2}).
\end{align*}
\]

(4.2)

The dilaton field on the hypersurface is

\[
\varphi(u, r_0, z, \bar{z}) = \frac{\varphi_1(u, z, \bar{z})}{r_0} + \frac{\varphi_2(u, z, \bar{z})}{r_0^2} + O(r_0^{-3}).
\]

(4.3)

Where

\[
\partial_0 \varphi_2 + \bar{\partial} \varphi_1 = 0.
\]

(4.4)
The equation of motion of free falling charged particle on this hypersurface is

\[ V^\nu (\nabla_\nu V^\mu + qF^\nu_\mu) = 0. \tag{4.5} \]

where \( V \) is the tangent vector of the particle worldline, \( \nabla \) is the covariant derivative on this 3 dimensional hypersurface and \( q \) is the charge of the particle.

According to [24], we impose that \( V \) has the following asymptotic expansion

\[ V^u = 1 + \sum_{a=1}^{\infty} \frac{V^u_a}{r^a}, \quad V^z = \sum_{a=2}^{\infty} \frac{V^z_a}{r^a}. \tag{4.6} \]

Then we solve (4.5) order by order. The solution up to relevant order is

\[ V^u_1 = -\frac{1}{2}(\Psi_2^0 + \Psi_1^0) + \frac{1}{6} \phi_1 \partial_u \phi_1, \]
\[ V^z_2 = -P_s \partial^0 \sigma^0 + qP_s^2 A_0^0, \]
\[ V^u_2 = q^2 P_s^2 A_0^0 A_0^0 + \frac{1}{6} (\partial \Psi_1^0 + \partial \Psi_2^0) + \frac{3}{8} (\Psi_0^0 + \Psi_0^0)^2 - \partial \sigma^0 \partial \sigma^0 - \frac{1}{2} \phi_1^0 \phi_1^0 \\
- \frac{1}{4} (\Psi_2^0 + \Psi_2^0) \partial_u \phi_1 - \frac{1}{12} \phi_1 \partial_u \phi_2 + \frac{1}{6} \phi_2 \partial_u \phi_1 + \frac{1}{24} (\phi_1 \partial_u \phi_1)^2 \\
- \frac{1}{12} (\partial \phi_1 \partial \partial \phi_1 + g_1 \partial \partial \phi_1), \]
\[ V^z_3 = P_s [2 \partial^0 \sigma^0 \sigma^0 + \frac{2}{3} \Psi_1^0 + \frac{1}{2} \partial \sigma^0 (\Psi_2^0 + \Psi_2^0)] - P_s \int dv \frac{1}{2} (\partial \Psi_2^0 + \partial \Psi_2^0 + 2q \partial A_0^0), \]
\[ - 2qP_s^2 \sigma^0 A_0^0 + qP_s^2 A_1^0 - \frac{1}{6} \partial \sigma^0 \partial \phi_1 \partial_u \phi_1, \tag{4.7} \]

where we have set all integration constants of \( u \) to zero since we require that the charged particle is static initially. At \( r_0^{-2} \) order, we can see that \( V \) has angular components. In other words, gravitational and electromagnetic radiations characterized by \( \sigma^0 \) and \( A_0^0 \) cause free falling charged particle to rotate over some tiny angle about the “center” of the spacetime \( r = 0 \). The leading memory effect is the velocity kick of the charged particle

\[ \Delta V^z = -\frac{1}{r_0^2} (P_s \overline{\Phi} \Delta \sigma^0 - qP_s^2 \Delta A_0^0) + O(r_0^{-3}). \tag{4.8} \]

The leading memory effect consists of two parts, namely the gravitational part \(-P_s \overline{\Phi} \Delta \sigma^0\) and electromagnetic part \(qP_s^2 \Delta A_0^0\). They are mathematically equivalent to leading soft graviton theorem [20] and leading soft photon theorem [31] respectively by a Fourier transformation. That is why we call this a unified expression of leading gravitational memory effect and leading electromagnetic memory effect. It is the same as the result in Einstein-Maxwell theory [25], which means that the scalar field has no contribution to the leading memory effect. Besides this, we can not see the coupling effect in the leading memory effect. We should notice that the change of the velocity of the charged particle (the velocity kick) is considered as distinct effect from the displacement memory effect.
The gravitational memory is a property of gravitational wave characterized by the change of the asymptotic shear $\Delta \sigma^0$ [10]. The velocity kick we discuss in this paper and the relative displacement of nearby observers (e.g. in [20]) are different observational effects of the gravitational wave with memory.

According to the treatment in electromagnetism [30], the sub-leading memory effect is a position displacement of the charged particle:

$$\Delta z = \int V^i du = -\frac{1}{r_0^2} \int du (P_s \overline{\sigma^0} - qP_s^2 A^0_z) + \mathcal{O}(r_0^{-3}). \quad (4.9)$$

The gravitational contribution $-\int (P_s \overline{\sigma^0}) du$ has a relevance to sub-leading soft graviton theorem (see [22] for specific discussion), and the electromagnetic contribution $\int (qP_s^2 A^0_z) du$ has a relevance to subleading soft photon theorem (see [30] for further discussion). So we call this a unified expression of subleading gravitational memory effect and subleading electromagnetic memory effect. The result is the same as [25], we do not see the coupling effect in this sub-leading memory effect either.

Another sub-leading observational memory effect is a time delay of the observer [24, 32]. It is a kind of spin memory effect. The time delay of a charge particle will also have contributions from the electromagnetic radiation and the scalar radiation. Since $V$ is time-like, the infinitesimal change of the proper time can be derived from the co-vector:

$$d\chi = \left\{ 1 + \frac{1}{6r_0} (3\Psi_2^0 + 3\overline{\Psi}_2^0 - \varphi_1 \partial_u \varphi_1) + \frac{1}{r_0^2} \left[ -\frac{1}{8} (\Psi_2^0 + \overline{\Psi}_2^0)^2 - \frac{1}{6} (\overline{\sigma})_1^0 + \overline{\sigma}^0_1 \right] \right\} \overline{\sigma}^0 - \frac{1}{12} \overline{\varphi}_1 \varphi_1 - \frac{1}{12} \overline{\varphi}_1 \varphi_1$$. \quad (4.10)

The electromagnetic contribution $\frac{1}{12} (\Psi_2^0 + \overline{\Psi}_2^0)\varphi_1 \partial_u \varphi_1$ in the $\frac{1}{r_0}$ expansion, but the scalar contribution $-\frac{1}{6} \varphi_1 \partial_u \varphi_1$ appears the same order as the gravitational contribution, which means that the scalar effect is stronger than the electromagnetic effect and it is of the same order as the gravitational effect. The coupling constant $a$ does not show in eq(4.10) which means that we can not find the effect of the non-minimal coupling of the scalar field and the electromagnetic field at this order. We can find a scalar-gravitational coupled term $\frac{1}{12} (\Psi_2^0 + \overline{\Psi}_2^0)\varphi_1 \partial_u \varphi_1$ which is the same order as the electromagnetic contribution. Except this, we can not find any other term about the coupling of the gravitation and matter fields at this order in this spin memory effect.

---

2 We have used the fact that $du = d\chi + \mathcal{O}(r_0^{-1})$, where $\chi$ is the proper time.

3 We have used the fact that $dz = \Delta z + \mathcal{O}(r_0^{-1})$. 
5 Conclusion

In this work, we study the Einstein-Maxwell-dilaton theory in Newman-Penrose formalism. We derive the NP equations of EMD theories and give the asymptotically flat solution space. The solution space can be considered as an extension of [28] to the asymptotically flat case in NP formalism. Based on the solution space, we investigate some observational effects, i.e. the memory effects, near the null infinity.

Acknowledgements

The author thanks Pujian Mao for useful discussion. This work is supported in part by the NSFC (National Natural Science Foundation of China) under the Grant Nos. 11905156 and 11935009.

References

[1] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, “Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems,” Proc. Roy. Soc. Lond. A269 (1962) 21–25.

[2] R. K. Sachs, “Gravitational waves in general relativity. 8. Waves in asymptotically flat space-time,” Proc. Roy. Soc. Lond. A270 (1962) 103–126.

[3] E. Newman and R. Penrose, “An approach to gravitational radiation by a method of spin coefficients,” J. Math. Phys. 3 (1962) 566–278.

[4] E. Newman and T. W. J. Unti, “Behavior of asymptotically flat empty spaces,” J. Math. Phys. 3 (1962) 891–901.

[5] B. Kozarzewski, “Asymptotic properties of the electromagnetic and gravitational fields,” Acta Phys. Polon 27 (1965) 775.

[6] A. R. Exton, E. Newman, and R. Penrose, “Conserved quantities in the Einstein-Maxwell Theory,” J. Math. Phys 10 (1969) 1566–1570.

[7] R. Casadio, B. Harms, Y. Leblanc, and P. H. Cox, “Perturbations in the Kerr-Newman dilatonic black hole background: 1. Maxwell waves,” Phys. Rev. D 56 (1997) 4948–4961.

[8] Y. B. Zel’dovich and A. G. Polnarev, “Radiation of gravitational waves by a cluster of superdense stars,” Soviet. Astronomy 18 (Aug. 1974) 17.
[9] D. Christodoulou, “Nonlinear nature of gravitation and gravitational wave experiments,” *Phys. Rev. Lett* **67** (1991) 1486–1489.

[10] J. Frauendiener, “Note on the memory effect,” *Class. Quant. Grav* **9** (1992) 1639–1641.

[11] V. B. Braginsky and L. P. Grishchuk, “Kinematic resonance and memory effect in free mass gravitational antennas,” *Sov. Phys. JETP* **62** (1985) 427–430.

[12] V. B. Braginsky and K. S. Thorne, “Gravitational-wave bursts with memory and experimental prospects,” *Nature* **327** (May, 1987) 123–125.

[13] A. G. Wiseman and C. M. Will, “Christodoulou’s nonlinear gravitational wave memory: Evaluation in the quadrupole approximation,” *Phys. Rev. D* **44** (1991) R2945–R2949.

[14] K. S. Thorne, “Gravitational-wave bursts with memory: The Christodoulou effect,” *Phys. Rev D* **45** (1992) 520–524.

[15] P. D. Lasky, E. Thrane, Y. Levin, J. Blackman, and Y. Chen, “Detecting gravitational-wave memory with LIGO: implications of GW150914,” *Phys. Rev. Lett* **117** (2016) 061102.

[16] D. A. Nichols, “Spin memory effect for compact binaries in the post Newtonian approximation,” *Phys. Rev. D* **95** (2017) 084048.

[17] H. Yang and D. Martynov, “Testing gravitational memory generation with compact binary mergers,” *Phys. Rev. Lett* **121** (2018) 071102.

[18] L. Susskind, “Electromagnetic memory,” [arXiv:1507.02584 [hep-th]].

[19] L. Bieri and D. Garfinkle, “An electromagnetic analogue of gravitational wave memory,” *Class. Quant. Grav* **30** (2013) 195009.

[20] A. Strominger and A. Zhiboedov, “Gravitational memory, BMS supertranslation and soft theorems,” *JHEP* **01** (2016) 086.

[21] S. Weinberg, “Infrared photons and gravitons,” *Phys. Rev* **140** (1965) B516–B524.

[22] S. Pasterski, A. Strominger, and A. Zhiboedov, “New gravitational memories,” *JHEP* **12** (2016) 053.

[23] F. Cachazo and A. Strominger, “Evidence for a new soft graviton theorem,” [arXiv:1404.4091 [hep-th]].

[24] P. Mao and X. Wu, “More on gravitational memory,” *JHEP* **05** (2019) 058.
[25] P. Mao and W. Tan, “On gravitational and electromagnetic memory,” arXiv:1912.01840 [gr-qc].

[26] H. Afshar, E. Esmaeili, and M. M. Sheikh-Jabbari, “String memory effect,” JHEP 02 (2019) 053.

[27] Chandrasekhar, “The Newman-Penrose formalism,” The mathematical theory of black holes pp.40–55.

[28] H. Lu, P. Mao, and J. Wu, “Asymptotic structure of Einstein-Maxwell-dilaton theory and its five dimensional origin,” JHEP 11 (2019) 005.

[29] A. I. Janis and E. T. Newman, “Structure of gravitational sources,” J. Math. Phys 6 (1965) 902–914.

[30] P. Mao, H. Ouyang, J. B. Wu, and X. Wu, “New electromagnetic memories and soft photon theorems,” Phys. Rev. D 95 (2017) 125011.

[31] S. Pasterski, “Asymptotic symmetries and electromagnetic memory,” JHEP 09 (2017) 154.

[32] E. E. Flanagan, A. M. Grant, A. I. Harte, and D. A. Nichols, “Persistent gravitational wave observables: general framework,” Phys. Rev. D. 99 (2019) 084044.