VANISHING GENERALIZED MORREY SPACES AND COMMUTATORS OF MARCINKIEWICZ INTEGRALS WITH ROUGH KERNEL ASSOCIATED WITH SCHRÖDINGER OPERATOR

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Abstract. Let $L = -\Delta + V(x)$ be a Schrödinger operator, where $\Delta$ is the Laplacian on $\mathbb{R}^n$, while nonnegative potential $V(x)$ belonging to the reverse Hölder class. We establish the boundedness of the commutators of Marcinkiewicz integrals with rough kernel associated with Schrödinger operator on vanishing generalized Morrey spaces.

1. Introduction and main results

Because of the need for study of the local behavior of solutions of second order elliptic partial differential equations (PDEs) and together with the now well-studied Sobolev Spaces, constitute a formidable three parameter family of spaces useful for proving regularity results for solutions to various PDEs, especially for non-linear elliptic systems, in 1938, Morrey [15] introduced the classical Morrey spaces $L_{p,\lambda}$ which naturally are generalizations of the classical Lebesgue spaces.

We will say that a function $f \in L_{p,\lambda} = L_{p,\lambda} (\mathbb{R}^n)$ if

\begin{equation}
\sup_{x \in \mathbb{R}^n, r > 0} \left( \frac{1}{r^{-\lambda}} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p} < \infty.
\end{equation}

Here, $1 < p < \infty$ and $0 < \lambda < n$ and the quantity of (1.1) is the $(p, \lambda)$-Morrey norm, denoted by $\|f\|_{L_{p,\lambda}}$. In recent years, more and more researches focus on function spaces based on Morrey spaces to fill in some gaps in the theory of Morrey type spaces (see, for example, [7, 8, 9, 10, 11, 16]). Moreover, these spaces are proved useful in harmonic analysis and PDEs. But, this topic exceeds the scope of this paper. Thus, we omit the details here. On the other hand, the study of the operators of harmonic analysis in vanishing Morrey space, in fact has been almost not touched. A version of the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ where it is possible to approximate by "nice" functions is the so called vanishing Morrey space $V L_{p,\lambda}(\mathbb{R}^n)$ has been introduced by Vitanza in [24] and has been applied there to obtain a regularity result for elliptic PDEs. This is a subspace of functions in

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L_{p,\lambda}(\mathbb{R}^n), which satisfies the condition

\[
\lim_{r \to 0} \sup_{t \in \mathbb{R}^n, 0 < t < r} \left( t^{-\lambda} \int_{B(x,t)} |f(y)|^p \, dy \right)^{1/p} = 0,
\]

where 1 < p < \infty and 0 < \lambda < n for brevity, so that

\[
VL_{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L_{p,\lambda}(\mathbb{R}^n) : \lim_{r \to 0} \sup_{x \in \mathbb{R}^n, 0 < t < r} t^{-\lambda} \|f\|_{L_p(B(x,r))} = 0 \right\}.
\]

Later in [25] Vitanza has proved an existence theorem for a Dirichlet problem, under weaker assumptions than in [13] and a W^{3,2} regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. For the properties and applications of vanishing Morrey spaces, see also [3].

After studying Morrey spaces in detail, researchers have passed to the concept of generalized Morrey spaces. Firstly, motivated by the work of [15], Mizuhara [14] introduced generalized Morrey spaces \( M_{p,\varphi} \) as follows:

**Definition 1.** [14] *(generalized Morrey space)* Let \( \varphi(x,r) \) be a positive measurable function on \( \mathbb{R}^n \times (0, \infty) \). If 0 < p < \infty, then the generalized Morrey space \( M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n) \) is defined by

\[
\left\{ f \in L^{loc}_p(\mathbb{R}^n) : \|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))} < \infty \right\}.
\]

Obviously, the above definition recover the definition of \( L_{p,\lambda}(\mathbb{R}^n) \) if we choose \( \varphi(x,r) = r^{\frac{\lambda}{p}} \), that is

\[
L_{p,\lambda}(\mathbb{R}^n) = M_{p,\varphi}(\mathbb{R}^n) \Big|_{\varphi(x,r) = r^{\frac{\lambda}{p}}}.
\]

Everywhere in the sequel we assume that \( \inf_{x \in \mathbb{R}^n, r > 0} \varphi(x,r) > 0 \) which makes the above spaces non-trivial, since the spaces of bounded functions are contained in these spaces. We point out that \( \varphi(x,r) \) is a measurable non-negative function and no monotonicity type condition is imposed on these spaces.

Throughout the paper we assume that \( x \in \mathbb{R}^n \) and \( r > 0 \) and also let \( B(x,r) \) denotes the open ball centered at \( x \) of radius \( r \), \( B^C(x,r) \) denotes its complement and \( |B(x,r)| \) is the Lebesgue measure of the ball \( B(x,r) \) and \( |B(x,r)| = v_n r^n \), where \( v_n = |B(0,1)| \).

Now, recall that the concept of the vanishing generalized Morrey spaces \( VM_{p,\varphi}(\mathbb{R}^n) \) has been introduced in [17].

**Definition 2.** [17] *(vanishing generalized Morrey space)* Let \( \varphi(x,r) \) be a positive measurable function on \( \mathbb{R}^n \times (0, \infty) \) and 1 \( \leq p < \infty \). The vanishing generalized Morrey space \( VM_{p,\varphi}(\mathbb{R}^n) \) is defined as the spaces of functions \( f \in L^p_{loc}(\mathbb{R}^n) \) such that

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \varphi(x,r)^{-1} \int_{B(x,r)} |f(y)|^p \, dy = 0.
\]
Everywhere in the sequel we assume that

\[
\lim_{t \to 0} \frac{t^n}{\varphi(x,t)} = 0,
\]

and

\[
\sup_{0 < t < \infty} \frac{t^n}{\varphi(x,t)} < \infty,
\]

which make the spaces \( VM_{p,\varphi}(\mathbb{R}^n) \) non-trivial, because bounded functions with compact support belong to this space. The spaces \( VM_{p,\varphi}(\mathbb{R}^n) \) and \( WVM_{p,\varphi}(\mathbb{R}^n) \) are Banach spaces with respect to the norm (see, for example [17])

\[
\|f\|_{VM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))},
\]

\[
\|f\|_{WVM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x,r)^{-1} \|f\|_{W_{L_p}(B(x,r))},
\]

respectively. For the properties and applications of vanishing generalized Morrey spaces, see also [1]. In [1], the boundedness of the Marcinkiewicz integrals with rough kernel associated with Schrödinger operator on vanishing generalized Morrey spaces \( VM_{p,\varphi}(\mathbb{R}^n) \) has been investigated.

On the other hand, suppose that \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \) \((n \geq 2)\) equipped with the normalized Lebesgue measure \( d\sigma = d\sigma(x') \).

In [19], Stein has defined the Marcinkiewicz integral for higher dimensions. Suppose that \( \Omega \) satisfies the following conditions.

(a) \( \Omega \) is the homogeneous function of degree zero on \( \mathbb{R}^n \setminus \{0\} \), that is,

\[
\Omega(\mu x) = \Omega(x), \text{ for any } \mu > 0, x \in \mathbb{R}^n \setminus \{0\}.
\]

(b) \( \Omega \) has mean zero on \( S^{n-1} \), that is,

\[
\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,
\]

where \( x' = \frac{x}{|x|} \) for any \( x \neq 0 \).

(c) \( \Omega \in Lip_\gamma(S^{n-1}), 0 < \gamma \leq 1 \), that is there exists a constant \( M > 0 \) such that

\[
|\Omega(x') - \Omega(y')| \leq M|x' - y'|^{\gamma} \quad \text{for any } x', y' \in S^{n-1}.
\]

(d) \( \Omega \in L_1(S^{n-1}) \).

The Marcinkiewicz integral operator of higher dimension \( \mu\Omega \) is defined by

\[
\mu\Omega(f)(x) = \left( \int_0^\infty \left| F_{\Omega,t}(f)(x) \right|^2 \frac{dt}{t^3} \right)^{1/2},
\]

where

\[
F_{\Omega,t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.
\]

Since Stein’s work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [20, 21, 23].
Remark 1. We easily see that the Marcinkiewicz integral operator of higher dimension $\mu_{\Omega}$ can be regarded as a generalized version of the classical Marcinkiewicz integral in the one dimension case. Also, it is easy to see that $\mu_{\Omega}$ is a special case of the Littlewood-Paley $g$-function if we take

$$g(x) = \Omega(x') |x|^{-n+1} \chi_{|x| \leq 1} (|x|).$$

When $\Omega$ satisfies some size conditions, the kernel of the operator $\mu_{\Omega}$ has no regularity, and so the operator $\mu_{\Omega}$ is called rough Marcinkiewicz integral operator. The theory of Operators with rough kernel is a well studied area (see [7, 9, 10, 11] for example).

For simplicity of notation, $\Omega$ is always homogeneous function of degree zero and satisfies

$$\Omega \in L_q(S^{n-1}), \quad 1 < q \leq \infty$$

and (1.8) throughout this paper if there are no special instructions.

Now we give the definition of the commutator generalized by $\mu_{\Omega}$ and $b$ by

$$\mu_{\Omega,b}(f)(x) = \left( \int_0^\infty |F_{\Omega,t,b}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t,b}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y) dy.$$ 

Let $f \in L^{1}_{loc}(\mathbb{R}^n)$. The rough Hardy-Littlewood maximal operator $M_{\Omega}$ and commutator of the Hardy-Littlewood maximal operator with rough kernel are defined by

$$M_{\Omega} f(x) = \sup_{t > 0} \frac{1}{|B(x,t)|} \int_{B(x,t)} |\Omega(x-y)| |f(y)| dy,$$

$$M_{\Omega,b} (f)(x) = \sup_{t > 0} |B(x,t)|^{-1} \int_{B(x,t)} |b(x) - b(y)| |\Omega(x-y)| |f(y)| dy,$$

respectively.

The following results concerning the boundedness of commutator operators $\mu_{\Omega,b}$ and $M_{\Omega,b}$ on $L_p$ space are known.

Theorem 1. (see [4]) Let $1 \leq p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$ satisfies (1.7), (1.8) and $b \in BMO(\mathbb{R}^n)$. Then, for $p > 1$ $\mu_{\Omega,b}$ is bounded on $L_p(\mathbb{R}^n)$ and for $p = 1$ from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.

Theorem 2. (see [2]) Let $1 < p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$ satisfies (1.7) and $b \in BMO(\mathbb{R}^n)$. Then, for every $q' < p < \infty$ or $1 < p < q$, there is a constant $C$ independent of $f$ such that

$$\|M_{\Omega,b} (f)\|_{L_p} \leq C \|f\|_{L_p}.$$ 

Moreover, for $p > 1$ $M_{\Omega,b}$ is bounded on $L_p(\mathbb{R}^n)$ and for $p = 1$ from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. 

On the other hand, in this paper we consider the Schrödinger operator

\[ L = -\Delta + V (x) \] on \( \mathbb{R}^n, \quad n \geq 3 \]

where \( V (x) \) is a nonnegative potential belonging to the reverse Hölder class \( RH_q \), for some exponent \( q \geq \frac{\alpha}{2} \); that is, there exists a constant \( C \) such that the reverse Hölder inequality

\[ \left( \frac{1}{|B|} \int_B V (x)^q \, dx \right)^{\frac{1}{q}} \leq C \left( \frac{1}{|B|} \int_B V (x) \, dx \right), \quad (1.9) \]

holds for every ball \( B \subset \mathbb{R}^n \); see [18].

We introduce the definition of the reverse Hölder index of \( V \) as \( q_0 = \sup \{ q : V \in RH_q \} \). It is worth pointing out that the \( RH_q \) class is that, if \( V \in RH_q \) for some \( q > 1 \), then there exists \( \varepsilon > 0 \), which depends only on \( n \) and the constant \( C \) in (1.9), such that \( V \in RH_{q+\varepsilon} \). Therefore, under the assumption \( V \in RH_{q_0} \), we may conclude \( q_0 > \frac{n}{2} \).

Throughout this paper, we always assume that \( 0 \neq V \in RH_n \). In particular, Shen [18] has considered \( L_p \) estimates for Schrödinger operators \( L \) with certain potentials which include Schrödinger Riesz transforms \( R_j^i = \frac{\partial}{\partial \sigma_j} L^{-\frac{1}{2}} \), \( j = 1, \ldots, n \). Then, Dziubański and Zienkiewicz [5] has introduced the Hardy type space \( H^1 L (\mathbb{R}^n) \) associated with the Schrödinger operator \( L \), which is larger than the classical Hardy space \( H^1 (\mathbb{R}^n) \).

Similar to the Marcinkiewicz integral operator with rough kernel \( \mu_\Omega \), we define the Marcinkiewicz integral operator with rough kernel \( \mu^{L}_j, \Omega \) associated with the Schrödinger operator \( L \) by

\[ \mu^{L}_j,\Omega f (x) = \left( \int_0^\infty \left| \int_{|x-y| \leq \varepsilon} |\Omega (x-y)| K^{L}_j (x, y) f (y) \, dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}, \]

where \( K^{L}_j (x, y) = \widetilde{K}^{L}_j (x, y) |x-y| \) and \( \widetilde{K}^{L}_j (x, y) \) is the kernel of \( R_j = \frac{\partial}{\partial \sigma_j} L^{-\frac{1}{2}} \), \( j = 1, \ldots, n \). In particular, when \( V = 0 \), \( K^{\Delta}_j (x, y) = \widetilde{K}^{\Delta}_j (x, y) |x-y| = ((x_j - y_j) / |x-y|) / |x-y|^{n-1} \) and \( \widetilde{K}^{\Delta}_j (x, y) \) is the kernel of \( R_j = \frac{\partial}{\partial \sigma_j} \Delta^{-\frac{1}{2}}, \ j = 1, \ldots, n \). In this paper, we write \( K_j (x, y) = K^{\Delta}_j (x, y) \) and \( \mu_j, \Omega = \mu^{\Delta}_j, \Omega \) and so, \( \mu^{\Delta}_j, \Omega \) is defined by

\[ \mu^{\Delta}_j,\Omega f (x) = \left( \int_0^\infty \left| \int_{|x-y| \leq \varepsilon} |\Omega (x-y)| K_j (x, y) f (y) \, dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}. \]

Obviously, \( \mu_j \) are classical Marcinkiewicz functions. Therefore, it will be an interesting thing to study the properties of \( \mu^{L}_j, \Omega \).

Given an operator \( \mu^{L}_j, \Omega \), and a function \( b \), we define the commutator of \( \mu^{L}_j, \Omega \) and \( b \) by

\[ \mu^{L}_j,\Omega, b f (x) = [b, \mu^{L}_j, \Omega] f (x) = b(x) \mu^{L}_j,\Omega f (x) - \mu^{L}_j,\Omega (bf) (x). \]

If \( \mu^{L}_j, \Omega \) is defined by integration against a kernel for certain \( x \), such as when \( \mu^{L}_j, \Omega \) is Marcinkiewicz integral operator with rough kernel associated with the Schrödinger
operator $L$, we have that this becomes

$$
\mu_{j,\Omega,b}^L f (x) = [b, \mu_{j,\Omega}^L] f (x) = \left( \int_0^\infty \int_{|x-y| \leq t} |\Omega (x-y)| K_j^L (x,y) \left| [b(x) - b(y)] f(y) \right| dy \right)^{2 \over t^3},
$$

for all $x$ for which the integral representation of $\mu_{j,\Omega}^L$ holds. It is worth noting that for a constant $C$, if $\mu_{j,\Omega}^L$ is linear, we have,

$$
[b + C, \mu_{j,\Omega}^L] f = (b + C) \mu_{j,\Omega}^L f - \mu_{j,\Omega}^L ((b + C) f)
= b\mu_{j,\Omega}^L f + C\mu_{j,\Omega}^L f - \mu_{j,\Omega}^L (bf) - C\mu_{j,\Omega}^L f
= [b, \mu_{j,\Omega}^L] f.
$$

This leads one to intuitively look to spaces for which we identify functions which differ by constants, and so it is no surprise that $b \in BMO$ (bounded mean oscillation space) has had the most historical significance.

Now, for a given potential $V \in RH_q$, with $q > \frac{3}{2}$, we introduce the auxiliary function

$$
\rho (x) = \frac{1}{m_v (x)} = \sup_{r > 0} \left\{ r : \frac{1}{m_v r^{n-2}} \int_{B(x,r)} V (y) \, dy \leq 1 \right\}, \quad x \in \mathbb{R}^n.
$$

The above assumptions $\rho (x)$ are finite, for all $x \in \mathbb{R}^n$. Obviously, $0 < m_v (x) < \infty$ if $V \neq 0$. In particular, $m_v (x) = 1$ with $V = 1$ and $m_v (x) \sim (1 + |x|)$ with $V = |x|^2$.

**Proposition 1.** (see [18]) There exist $C$ and $k_0 \geq 1$ such that

$$
C^{-1} \rho (x) \left( 1 + \frac{|x-y|}{\rho (x)} \right)^{-k_0} \leq \rho (y) \leq C \rho (x) \left( 1 + \frac{|x-y|}{\rho (x)} \right)^{k_0},
$$

for all $x, y \in \mathbb{R}^n$.

In particular, $\rho (x) \sim \rho (y)$, if $|x-y| < C \rho (x)$. A ball $B (x, \rho (x))$ is called critical.

**Proposition 2.** (see [5]) There exist a sequence of points $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^n$, so that the family $B_k = B (x_k, \rho (x_k))$, $k \geq 1$, satisfies the following:

1. $\bigcup_k B_k = \mathbb{R}^n$;
2. $\exists N$ such that, for every $k \in N$, $\text{card} \{j : 4B_j \cap 4B_k \neq \emptyset\} \leq N$.

**Lemma 1.** (see [22]) For any $l > 0$, there exists $C_l > 0$ such that

$$
K_j^L (x,y) \leq \frac{C_l}{\left( 1 + \frac{|x-y|}{\rho (y)} \right)^l \frac{1}{|x-y|^{n-1}}},
$$

and

$$
|K_j^L (x,y) - K_j (x,y)| \leq C \frac{\rho (x)}{|x-y|^{n-2}},
$$

where $\rho$ is the auxiliary function.
Tang and Dong [22] have shown that Marcinkiewicz integral $\mu^{L}_{j}$ is bounded on $L_{p}(\mathbb{R}^{n})$, for $1 < p < \infty$, and are bounded from $L_{1}(\mathbb{R}^{n})$ to $WL_{1}(\mathbb{R}^{n})$.

Shen [18] has given the following kernel estimate that we need.

**Theorem 4.** Let $\mu^{L}_{j}$ be a Marcinkiewicz integral and we use the convention $1' < p < \infty$ everywhere.

**Lemma 2.** If $V \in RH_{n}$, then, one has

(i) for every $N$ there exists a constant $C$ such that

$$|K^{L}_{j}(x,z)| \leq C \left(1 + \frac{|x-z|}{\rho(x)}\right)^{-N},$$

(ii) for every $N$ and $0 < \delta < \min\left\{1, 1 - \frac{m}{q_{0}}\right\}$, there exists a constant $C$ such that

$$|K^{L}_{j}(x,z) - K^{L}_{j}(y,z)| \leq C \left|\frac{x-y}{|x-z|}\right|^\delta \left(1 + \frac{|x-z|}{\rho(x)}\right)^{-N},$$

where $|x-y| \leq \frac{2}{3} |x-z|$, $\rho(x)$ is the distance from $x$ to the boundary of $\Omega$, and $\rho(x)$ denotes the dual or conjugate exponent of $p$.

(iii) if $K$ denotes the $\mathbb{R}^{n}$ vector valued kernel of the classical Riesz operator, for every $0 < \delta < 2 - \frac{m}{q_{0}}$, we have

$$|K^{L}_{j}(x,z) - K_{j}(x,z)| \leq C \left|\frac{x-z}{\rho(z)}\right|^\delta,$$

where $K_{j}(x,z) = K(x,z) |x-z|$.

Inspired by [1], we give BMO estimates for commutators of Marcinkiewicz integrals with rough kernel associated with Schrödinger operator on vanishing generalized Morrey spaces $V_{M}^{\infty}A_{p,q}^{\infty}(\mathbb{R}^{n})$.

We now make some conventions. Throughout this paper, we use the symbol $A \lesssim B$ to denote that there exists a positive constant $C$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \approx B$ and say that $A$ and $B$ are equivalent. For a fixed $p \in [1, \infty)$, $p'$ denotes the dual or conjugate exponent of $p$, namely, $p' = \frac{p}{p-1}$ and we use the convention $1' = \infty$ and $\infty' = 1$.

Our main results can be formulated as follows.

**Theorem 3.** Let $1 < p < \infty$, $\Omega \in L_{q}(S^{n-1})$, $1 < q \leq \infty$ satisfies (1.7). Also, let $V \in RH_{n}$ and $b \in BMO(\mathbb{R}^{n})$. Then, for every $q' < p < \infty$ or $1 < p < q$, there is a constant $C$ independent of $f$ such that

$$\|\mu^{L}_{j,\Omega,b,f}\|_{L_{p}} \leq C \|f\|_{L_{p}}.$$  

**Theorem 4.** Let $x_{0} \in \mathbb{R}^{n}, 1 < p < \infty$ and $b \in BMO(\mathbb{R}^{n})$. Let $\Omega \in L_{q}(S^{n-1})$, $1 < q \leq \infty$ satisfies (1.7) and $V \in RH_{n}$. Then, for $q' \leq p$ the inequality

$$\|\mu^{L}_{j,\Omega,b,f}\|_{L_{p}(B(x_{0},r))} \lesssim \|b\|_{s} r^{-\frac{n}{p'}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right)^{-\frac{n}{p'}-1} \|f\|_{L_{p}(B(x_{0},t))} dt$$

holds for any ball $B(x_{0},r)$ and for all $f \in L_{p}^{\infty}(\mathbb{R}^{n})$.

Also, for $p < q$ the inequality

$$\|\mu^{L}_{j,\Omega,b,f}\|_{L_{p}(B(x_{0},r))} \lesssim \|b\|_{s} r^{-\frac{n}{p'}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right)^{-\frac{n}{p'}-1} \|f\|_{L_{p}(B(x_{0},t))} dt$$
holds for any ball $B(x_0, r)$ and for all $f \in L^p_{\text{loc}}(\mathbb{R}^n)$.

**Theorem 5.** Let $\Omega \in L_q(S^{n-1}), 1 < q \leq \infty$, satisfies (1.7) and $V \in RH_n$. Let $1 < p < \infty$ and $b \in \text{BMO}(\mathbb{R}^n)$. For $q' \leq p$ if the pair $(\varphi_1, \varphi_2)$ satisfies conditions (1.3)-(1.4) and

\[
(1.12) \quad c_3 := \int_0^\infty \left(1 + \ln \frac{t}{r} \right) \sup_{x \in \mathbb{R}^n} \varphi_1(x, t) t^{-\frac{p}{p-1}} dt < \infty
\]

for every $\delta > 0$, and

\[
(1.13) \quad \int_r^\infty \left(1 + \ln \frac{t}{r} \right) \frac{\varphi_1(x, t)}{t^{\frac{p}{p-1}+1}} dt \leq C_0 \frac{\varphi_2(x, r)}{r^{\frac{p}{p-1}}},
\]

and for $p < q$ if the pair $(\varphi_1, \varphi_2)$ satisfies conditions (1.3)-(1.4) and also

\[
(1.14) \quad c_{3'} := \int_0^\infty \left(1 + \ln \frac{t}{r} \right) \sup_{x \in \mathbb{R}^n} \varphi_1(x, t) t^{-\frac{p}{p}+\frac{q}{q}-1} dt < \infty
\]

for every $\delta' > 0$, and

\[
(1.15) \quad \int_r^\infty \left(1 + \ln \frac{t}{r} \right) \frac{\varphi_1(x, t)}{t^{\frac{p}{p}+\frac{q}{q}}+1} dt \leq C_0 \frac{\varphi_2(x, r)}{r^{\frac{p}{p}+\frac{q}{q}}},
\]

where $C_0$ does not depend on $x \in \mathbb{R}^n$ and $r > 0$, then the operators $\mu_{j, \varphi_1, \varphi_2}^L$ for $j = 1, \ldots, n$ are bounded from $\text{VM}_{p, \varphi_1}$ to $\text{VM}_{p, \varphi_2}$. Moreover,

\[
(1.16) \quad \|\mu_{j, \varphi_1, \varphi_2}^L\|_{\text{VM}_{p, \varphi_2}} \lesssim \|b\|_* \|f\|_{\text{VM}_{p, \varphi_1}}.
\]

2. Some preliminaries

We begin with some properties of $\text{BMO}(\mathbb{R}^n)$ spaces which play a great role in the proofs of our main results.

Let us recall the definition of the space of $\text{BMO}(\mathbb{R}^n)$.

**Definition 3.** [12] The space $\text{BMO}(\mathbb{R}^n)$ of functions of bounded mean oscillation consists of locally summable functions with finite semi-norm

\[
(2.1) \quad \|b\|_* \equiv \|b\|_{\text{BMO}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - b_{B(x, r)}| dy < \infty,
\]

where $b_{B(x, r)}$ is the mean value of the function $b$ on the ball $B(x, r)$. The fact that precisely the mean value $b_{B(x, r)}$ figures in (2.1) is inessential and one gets an equivalent seminorm if $b_{B(x, r)}$ is replaced by an arbitrary constant $c$:

\[
(2.2) \quad \|b\|_* \approx \sup_{r > 0} \inf_{c \in \mathbb{C}} \frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - c| dy.
\]

Indeed, it is obvious that (2.1) implies (2.2). If (2.2) holds, then

\[
|b_{B(x, r)} - c| = \left| \frac{1}{|B(x, r)|} \int_{B(x, r)} (b(y) - c) dy \right| \leq C,
\]
Each bounded function $b \in BMO$. Moreover, $BMO$ contains unbounded functions, in fact $\log|x|$ belongs to $BMO$ but is not bounded, so $L_\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$.

In 1961 John and Nirenberg [12] established the following deep property of functions from $BMO$.

**Theorem 6.** [12] If $b \in BMO(\mathbb{R}^n)$ and $B(x,r)$ is a ball, then

$$\left| \{ x \in B(x,r) : |b(x) - b_{B(x,r)}| > \xi \} \right| \leq |B(x,r)| \exp\left( -\frac{\xi}{C\|b\|_*} \right), \quad \xi > 0,$$

where $C$ depends only on the dimension $n$.

By Theorem 6, we can get the following results.

**Corollary 1.** [12] Let $b \in BMO(\mathbb{R}^n)$. Then, for any $q > 1$,

$$\|b\|_* \approx \sup_{x \in \mathbb{R}^n, r > 0} \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy \right)^{\frac{1}{p}}$$

is valid.

**Corollary 2.** Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$|b_{B(x,r)} - b_{B(x,t)}| \leq C\|b\|_*\left( 1 + \ln \frac{t}{r} \right) \text{ for } 0 < 2r < t,$$

and for any $q > 1$, it is easy to see that

$$\|b - (b)_{B}\|_{L_q(B)} \leq Cr^{\frac{n}{q}}\|b\|_*\left( 1 + \ln \frac{t}{r} \right).$$

where $C$ is independent of $b$, $x$, $r$ and $t$.

### 3. Proofs of the main results

3.1. **Proof of Theorem 3.** In the proof we have used the idea in [6]. It suffices to show that

$$\mu^f_{\Omega,B}(x) \leq \mu_{\Omega,B}(x) + CM_{\Omega,B}f(x), \text{ a.e. } x \in \mathbb{R}^n,$$

where $M_{\Omega,B}$ denotes commutator of the Hardy-Littlewood maximal operator with rough kernel.
Fix $x \in \mathbb{R}^n$ and let $r = \rho(x)$. 

\[
\begin{align*}
\mu_{J,\Omega,b} f (x) & \leq \left( \int_0^r \int_{|x-y| \leq t} |\Omega(x-y)| K^L_{f_{J}} (x, y) |b(x) - b(y)| f(y) \, dy \, \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
& \quad + \left( \int_r^\infty \int_{|x-y| \leq r} |\Omega(x-y)| K^L_{f_{J}} (x, y) |b(x) - b(y)| f(y) \, dy \, \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
& \quad + \left( \int_0^r \int_{r<|x-y| \leq t} |\Omega(x-y)| K^L_{f_{J}} (x, y) |b(x) - b(y)| f(y) \, dy \, \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
& \quad + \left( \int_0^\infty \int_{r<|x-y| \leq r} |\Omega(x-y)| K^L_{f_{J}} (x, y) |b(x) - b(y)| f(y) \, dy \, \frac{dt}{t^3} \right)^{\frac{1}{2}} \\
& \quad = E_1 + E_2 + E_3 + E_4.
\end{align*}
\]
For $E_1$, by Lemma 2, we have

$$E_1 \leq C \left( \int_0^r \left| \int_{|x-y| \leq t} \frac{1}{|x-y|^{n-1}} \left( \frac{|x-y|}{\rho (x)} \right)^\delta |\Omega (x-y)| \|b (x) - b (y)\| f (y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$\leq Cr^{-\delta} \left( \int_0^r \sum_{k=-\infty}^0 \frac{1}{(2^{k-1}r)^{n-1}} \int_{|x-y| \leq 2^k t} |\Omega (x-y)| \|b (x) - b (y)\| f (y) dy \right)^2 \frac{dt}{t^3}^{\frac{1}{2}}$$

$$\leq Cr^{-\delta} \left( \int_0^r \sum_{k=-\infty}^0 \frac{(2^k)^{\delta+1} t^{\delta+1}}{(2^k t)^n} \int_{|x-y| \leq 2^k t} |\Omega (x-y)| \|b (x) - b (y)\| f (y) dy \right)^2 \frac{dt}{t^3}^{\frac{1}{2}}$$

$$\leq Cr^{-\delta} \left( \int_0^r \sum_{k=-\infty}^0 (2^k)^{\delta+1} t^{\delta+1} M_{\Omega,b} f (x) \right)^2 \frac{dt}{t^3}$$

$$\leq C M_{\Omega,b} f (x)$$

Obviously,

$$E_2 \leq \mu_{j,\Omega,b} f (x)$$

For $E_3$, using Lemma 2 again, we get

$$E_3 \leq C \left( \int_r^\infty \left| \int_{|x-y| \leq r} \frac{1}{|x-y|^{n-1}} |\Omega (x-y)| \|b (x) - b (y)\| f (y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$\leq C \left( \int_r^\infty \sum_{k=-\infty}^0 \frac{1}{(2^{k-1}r)^{n-1}} \int_{|x-y| \leq 2^k r} |\Omega (x-y)| \|b (x) - b (y)\| f (y) dy \right)^2 \frac{dt}{t^3}^{\frac{1}{2}}$$

$$\leq C \left( \int_r^\infty \sum_{k=-\infty}^0 \frac{2^k r}{(2^k r)^n} \int_{|x-y| \leq 2^k r} |\Omega (x-y)| \|b (x) - b (y)\| f (y) dy \right)^2 \frac{dt}{t^3}$$

$$\leq C \left( \int_r^\infty \sum_{k=-\infty}^0 2^k r M_{\Omega,b} f (x) \right)^2 \frac{dt}{t^3}$$

$$\leq C r \left( \int \frac{dt}{t^3} \right)^{\frac{1}{2}} M_{\Omega,b} f (x)$$

$$\leq C M_{\Omega,b} f (x)$$
It remains to estimate $E_4$. By Lemma 2, we obtain

$$E_4 \leq C \left( \int_r^\infty \left( \int_{r < |x-y| \leq t} |\Omega (x-y)| |b(x) - b(y)| \frac{|f(y)|}{|x-y|^\alpha} dy \right)^{2} \left( \frac{dt}{t^3} \right)^{\frac{1}{2}} \right)$$

$$\leq Cr \left( \int_r^\infty \left( \log_2 \left( \frac{t}{r} \right) + 1 \right) M_{\Omega,b}f(x) \left( \frac{2}{t^3} \right)^{\frac{1}{2}} \right)$$

$$\leq Cr \left( \int_r^\infty \left( \frac{t}{r} \right)^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$\leq CM_{\Omega,b}f(x).$$

Thus, Theorem 3 is proved.

3.2. Proof of Theorem 4. For $x \in B(x_0, t)$, notice that $\Omega$ is homogenous of degree zero and $\Omega \in L_q(S^{n-1}), 1 < q \leq \infty$. Then, we obtain

$$\left( \int_{B(x_0, t)} |\Omega (x-y)|^q dy \right)^{\frac{1}{q}} = \left( \int_{B(x-x_0, t)} |\Omega (z)|^q dz \right)^{\frac{1}{q}}$$

$$\leq \left( \int_{B(0, t+|x-x_0|)} |\Omega (z)|^q dz \right)^{\frac{1}{q}}$$

$$\leq \left( \int_{B(0, 2t)} |\Omega (z)|^q dz \right)^{\frac{1}{q}}$$

$$= \left( \frac{2t}{S^{n-1}} \int_{S^{n-1}} |\Omega (z')|^q d\sigma (z') r^{n-1} dr \right)^{\frac{1}{q}}$$

(3.1) $$= C \|\Omega\|_{L_q(S^{n-1})} \|B(x_0, 2t)\|^{\frac{1}{q}}.$$

Let $1 < p < \infty$ and $q' \leq p$. For any $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at $x_0$ and of radius $r$ and $2B = B(x_0, 2r)$. We represent $f$ as

$$f = f_1 + f_2, \quad f_1 (y) = f (y) \chi_{2B} (y), \quad f_2 (y) = f (y) \chi_{(2B)^c} (y), \quad r > 0$$

and have

$$\|\mu_{j,\Omega,b}f\|_{L_p(B)} \leq \|\mu_{j,\Omega,b}f_1\|_{L_p(B)} + \|\mu_{j,\Omega,b}f_2\|_{L_p(B)}.$$
Since \( f_1 \in L_p(\mathbb{R}^n) \), \( \mu_{j, \Omega, b}^L f_1 \in L_p(\mathbb{R}^n) \), from the boundedness of \( \mu_{j, \Omega, b}^L \) on \( L_p(\mathbb{R}^n) \) (see Theorem 3) it follows that:

\[
\| \mu_{j, \Omega, b}^L f_1 \|_{L_p(B)} \leq \| \mu_{j, \Omega, b}^L f_1 \|_{L_p(\mathbb{R}^n)} \leq \| b \|_\ast \| f_1 \|_{L_p(\mathbb{R}^n)} = \| b \|_\ast \| f \|_{L_p(2B)}.
\]

It is known that \( x \in B \), \( y \in (2B)^C \), which implies \( \frac{1}{2} |x_0 - y| \leq |x - y| \leq \frac{3}{2} |x_0 - y| \). Then for \( x \in B \), we have

\[
| \mu_{j, \Omega, b}^L f_2 (x) | \lesssim \int_{\mathbb{R}^n} \frac{\Omega (x - y)}{|x - y|^n} |b(y) - b(x)| |f(y)| dy \approx \int_{(2B)^C} \frac{\Omega (x - y)}{|x - y|^n} |b(y) - b(x)| |f(y)| dy.
\]

Hence we get

\[
\| \mu_{j, \Omega, b}^L f_2 \|_{L_p(2B)} \lesssim \left( \int_{\mathbb{B}} \left( \int_{(2B)^C} \frac{\Omega (x - y)}{|x - y|^n} |b(y) - b(x)| |f(y)| dy \right)^p dx \right)^{\frac{1}{p}}
\]

\[
\lesssim \left( \int_{\mathbb{B}} \left( \int_{(2B)^C} \frac{\Omega (x - y)}{|x - y|^n} |b(y) - b_B| |f(y)| dy \right)^p dx \right)^{\frac{1}{p}}
\]

\[
+ \left( \int_{\mathbb{B}} \left( \int_{(2B)^C} \frac{\Omega (x - y)}{|x - y|^n} |b(x) - b_B| |f(y)| dy \right)^p dx \right)^{\frac{1}{p}}
\]

\[
= J_1 + J_2.
\]

We have the following estimation of \( J_1 \). When \( q' \leq p \) and \( \frac{1}{p} + \frac{1}{q} + \frac{1}{q} = 1 \), by the Fubini’s theorem

\[
J_1 \approx r^\frac{n}{p} \int_{(2B)^C} \frac{\Omega (x - y)}{|x - y|^n} |b(y) - b_B| |f(y)| dy
\]

\[
\approx r^\frac{n}{p} \int_{(2B)^C} \frac{\Omega (x - y)}{|x - y|^n} |b(y) - b_B| |f(y)| \int_{|x_0 - y|}^{\infty} \frac{dt}{t^{n+1}} dy
\]

\[
\approx r^\frac{n}{p} \int_{2r}^{\infty} \int_{2r \leq |x_0 - y| \leq t} \frac{\Omega (x - y)}{|x - y|^n} |b(y) - b_B| |f(y)| dy \frac{dt}{t^{n+1}}
\]

\[
\lesssim r^\frac{n}{p} \int_{2r}^{\infty} \int_{B(x_0, t)} \frac{\Omega (x - y)}{|x - y|^n} |b(y) - b_B| |f(y)| dy \frac{dt}{t^{n+1}} \text{ holds.}
\]
Applying the Hölder’s inequality and by (3.1), (2.3), (2.4) and (2.5), we get

\[ J_1 \lesssim r^p \int_0^\infty \int_{B(x_0,t)} |\Omega(x-y)| |b(y) - b_{B(x_0,t)}| |f(y)| \, dy \, dt \frac{dt}{t^{n+1}} \]

\[ + r^p \int_0^\infty |b_{B(x_0,t)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |\Omega(x-y)||f(y)| \, dy \, dt \frac{dt}{t^{n+1}} \]

\[ \lesssim r^p \int_0^\infty \|\Omega(-y)\|_{L_p(B(x_0,t))} \|b(-y) - b_{B(x_0,t)}\|_{L_p(B(x_0,t))} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}} \]

\[ + r^p \int_0^\infty |b_{B(x_0,t)} - b_{B(x_0,t)}| \|\Omega(-y)\|_{L_p(B(x_0,t))} \|f\|_{L_p(B(x_0,t))} |B(x_0,t)|^{1 - \frac{1}{p} - \frac{1}{q}} \frac{dt}{t^{n+1}} \]

\[ \lesssim |b|_* r^p \int_0^\infty \left(1 + \ln \frac{t}{r} \right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}}. \]

In order to estimate \( J_2 \) note that

\[ J_2 = \|b(-y) - b_{B(x_0,t)}\|_{L_p(B(x_0,t))} \int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |f(y)| \, dy. \]

By (2.3), we get

\[ J_2 \lesssim |b|_* r^p \int_{(2B)^C} \frac{\|\Omega(x-y)\|}{|x_0-y|^n} |f(y)| \, dy. \]

Applying the Hölder’s inequality, we get

\[ J_2 \lesssim |b|_* r^p \int_{2r}^\infty \|f\|_{L_p(B(x_0,t))} \|\Omega(x-\cdot)\|_{L_q(B(x_0,t))} |B(x_0,t)|^{1 - \frac{1}{p} - \frac{1}{q}} \frac{dt}{t^{n+1}}. \]

Thus, by (3.1) we get

\[ J_2 \lesssim |b|_* r^p \int_{2r}^\infty \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}}. \]

Summing up \( J_1 \) and \( J_2 \), for all \( p \in (1, \infty) \) we get

\[ \|\mu_j^{L,\Omega_b} f\|_{L_p(B)} \lesssim |b|_* r^p \int_{2r}^\infty \left(1 + \ln \frac{t}{r} \right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}}. \]

Finally, we have the following

\[ \|\mu_j^{L,\Omega_b} f\|_{L_p(B)} \lesssim |b|_* \|f\|_{L_p(2B)} + |b|_* r^p \int_{2r}^\infty \left(1 + \ln \frac{t}{r} \right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}}. \]

On the other hand, we have
\[ \|f\|_{L^p(2B)} \approx r^\frac{p}{q} \|f\|_{L^p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{p+1}} \]

(3.2) \[ \leq r^\frac{p}{p'} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{p+1}}. \]

By combining the above inequalities, we obtain

\[ \|\mu^L_{f,\Omega,B}f\|_{L^p(B(x_0,r))} \lesssim \|b\|_{\ast} r^\frac{\mu}{p} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-\frac{\mu}{p'}-1} \|f\|_{L^p(B(x_0,t))} dt, \]

which completes the proof of first statement.

Similarly to (3.1), when \( y \in B(x_0,t) \), it is true that

(3.3) \[ \left( \int_{B(x_0,r)} \Omega(x-y)^{\frac{\mu}{p}} \, dy \right)^{\frac{1}{\frac{\mu}{p}}} \leq C \|\Omega\|_{L^q(S^{n-1})} \left| B \left( x_0, \frac{3}{2}t \right) \right|^{\frac{1}{q}}. \]

On the other hand when \( p < q \), by the Fubini’s theorem and the Minkowski inequality, we get

\[ J_1 \lesssim \left( \int_B \int_{2r}^{\infty} \left( b(y) - b_{B(x_0,t)} \right) \|f(y)\| \Omega(x-y) \, dy \frac{dt}{t^{n+1}} \right)^{\frac{p}{p'}} \]

\[ + \left( \int_B \int_{2r}^{\infty} \left( b_{B(x_0,r)} - b_{B(x_0,t)} \right) \|f(y)\| \Omega(x-y) \, dy \frac{dt}{t^{n+1}} \right)^{\frac{p}{p'}} \]

\[ \lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} \left( b(y) - b_{B(x_0,t)} \right) \|f(y)\| \Omega(x-y) \, dy \frac{dt}{t^{n+1}} \]

\[ + \int_{2r}^{\infty} \int_{B(x_0,t)} \left( b_{B(x_0,r)} - b_{B(x_0,t)} \right) \|f(y)\| \Omega(x-y) \, dy \frac{dt}{t^{n+1}} \]

\[ \lesssim |B|^{\frac{1}{p'}-\frac{1}{p'}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left( b(y) - b_{B(x_0,t)} \right) \|f(y)\| \Omega(x-y) \, dy \frac{dt}{t^{n+1}} \]

\[ + |B|^{\frac{1}{p'}-\frac{1}{p'}} \int_{2r}^{\infty} \int_{B(x_0,r)} \left( b_{B(x_0,r)} - b_{B(x_0,t)} \right) \|f(y)\| \Omega(x-y) \, dy \frac{dt}{t^{n+1}}. \]
Applying the Hölder’s inequality and by (3.3), (2.3), (2.4) and Lemma ??, we get

\[ J_1 \lesssim r^{\frac{n}{q} - \frac{n}{q'}} \int_{2r}^{\infty} \left\| \left( b(\cdot) - b_{B(x_0,t)} \right) f \right\|_{L_1(B(x_0,t))} B \left( x_0, \frac{3t}{2} \right) \frac{dt}{t^{n+1}} \]

\[ + r^{\frac{n}{2} - \frac{n}{q'}} \int_{2r}^{\infty} \left\| b_{B(x_0,t)} \right\|_{L_q(B(x_0,t))} \left\| f \right\|_{L_p(B(x_0,t))} B \left( x_0, \frac{3t}{2} \right) \frac{dt}{t^{n+1}} \]

\[ \lesssim r^{\frac{n}{q} - \frac{n}{q'}} \int_{2r}^{\infty} \left\| \left( b(\cdot) - b_{B(x_0,t)} \right) \right\|_{L_p(B(x_0,t))} \left\| f \right\|_{L_q(B(x_0,t))} t^{\frac{n}{2q'}} \frac{dt}{t^{n+1}} \]

\[ + r^{\frac{n}{2} - \frac{n}{q'}} \int_{2r}^{\infty} \left\| b_{B(x_0,t)} \right\|_{L_q(B(x_0,t))} \left\| f \right\|_{L_p(B(x_0,t))} t^{\frac{n}{2q'}} \frac{dt}{t^{n+1}} \]

\[ \lesssim \|b\|_r r^{\frac{n}{p} - \frac{n}{q'}} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) t^{\frac{n}{q'} - \frac{n}{q'}} \left\| f \right\|_{L_p(B(x_0,t))} dt. \]

Let \( \frac{1}{p} = \frac{1}{q'} + \frac{1}{q} \), then for \( J_2 \), by the Fubini’s theorem, the Minkowski inequality, the Hölder’s inequality and from (3.3), we get

\[ J_2 \lesssim \left( \int \int_{B(x_0,t)} \int_{2r}^{\infty} \left\| f(y) \right\| \left\| b(x) - b_{B} \right\| \Omega(x - y) \frac{dy}{t^{n+1}} \frac{dx}{t^{n+1}} \right)^{\frac{1}{p}} \]

\[ \lesssim \int \int_{2r B(x_0,t)} \left\| f(y) \right\| \left\| b(\cdot) - b_{B} \right\| \Omega(\cdot - y) \frac{dy}{t^{n+1}} \]

\[ \lesssim \int \int_{2r B(x_0,t)} \left\| f(y) \right\| \left\| b(\cdot) - b_{B} \right\|_{L_q(B)} \left\| \Omega(\cdot - y) \right\|_{L_q(B)} \frac{dy}{t^{n+1}} \]

\[ \lesssim \|b\|_r \left\| B \right\|^{\frac{1}{p} - \frac{1}{q'}} \int_{2r B(x_0,t)} \int \left\| f(y) \right\| \left\| \Omega(\cdot - y) \right\|_{L_q(B)} \frac{dt}{t^{n+1}} \]

\[ \lesssim \|b\|_r r^{\frac{n}{p} - \frac{n}{q'}} \int_{2r}^{\infty} \left\| f \right\|_{L_1(B(x_0,t))} B \left( x_0, \frac{3t}{2} \right) \frac{dt}{t^{n+1}} \]

\[ \lesssim \|b\|_r r^{\frac{n}{p} - \frac{n}{q'}} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right) t^{\frac{n}{q'} - \frac{n}{q'}} \left\| f \right\|_{L_p(B(x_0,t))} dt. \]

By combining the above estimates, we complete the proof of Theorem 4.

3.3. **Proof of Theorem 5.** The statement is derived from inequalities (1.10) and (1.11). Let \( q' \leq p \). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space follows from Theorem 4 and condition
\[ (1.13) \]
\[
\| \mu_{j, \Omega, b}^f \|_{VM_{p \cdot \varphi}^2} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} \| \mu_{j, \Omega, b}^f \|_{L_p(B(x, r))} \\
\lesssim \| b \|_* \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} r^\frac{p}{p+1} \int_r^\infty \left( 1 + \ln \frac{t}{r} \right) f(\| L_p(B(x, t)) \|) \frac{dt}{t^{\frac{n}{p+1}}} \\
\lesssim \| b \|_* \| f \|_{VM_{p \cdot \varphi}^2} \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} r^\frac{p}{p+1} \int_r^\infty \left( 1 + \ln \frac{t}{r} \right) \varphi_1(x, t) \left( \varphi_1(x, t)^{-1} \| f \|_{L_p(B(x, t))} \right) \frac{dt}{t^{\frac{n}{p+1}}} \\
\lesssim \| b \|_* \| f \|_{VM_{p \cdot \varphi}^2}.
\]

So we only have to prove that
\[ (3.4) \]
\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \varphi_1(x, r)^{-1} \| f \|_{L_p(B(x, r))} = 0 \implies \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \varphi_2(x, r)^{-1} \| \mu_{j, \Omega, b}^f \|_{L_p(B(x, r))} = 0.
\]

To show that \( \sup_{x \in \mathbb{R}^n} \varphi_2(x, r)^{-1} \| \mu_{j, \Omega, b}^f \|_{L_p(B(x, r))} < \epsilon \) for small \( r \), we split the right-hand side of (1.10):
\[ (3.5) \]
\[
\varphi_2(x, r)^{-1} \| \mu_{j, \Omega, b}^f \|_{L_p(B(x, r))} \leq C \left[ I_{\delta_0}(x, r) + J_{\delta_0}(x, r) \right],
\]
where \( \delta_0 > 0 \) (we may take \( \delta_0 < 1 \)), and
\[
I_{\delta_0}(x, r) := \| b \|_* \frac{r^\frac{p}{p+1}}{\varphi_2(x, r)} \left( \int_{\delta_0}^r \left( 1 + \ln \frac{t}{r} \right) \varphi_1(x, t) t^{-\frac{n}{p+1}} \left( \varphi_1(x, t)^{-1} \| f \|_{L_p(B(x, t))} \right) dt \right),
\]
and
\[
J_{\delta_0}(x, r) := \| b \|_* \frac{r^\frac{p}{p+1}}{\varphi_2(x, r)} \left( \int_{\delta_0}^\infty \left( 1 + \ln \frac{t}{r} \right) \varphi_1(x, t) t^{-\frac{n}{p+1}} \left( \varphi_1(x, t)^{-1} \| f \|_{L_p(B(x, t))} \right) dt \right)
\]
and \( r < \delta_0 \). Now we choose any fixed \( \delta_0 > 0 \) such that
\[
\sup_{x \in \mathbb{R}^n} \varphi_1(x, r)^{-1} \| f \|_{L_p(B(x, t))} < \frac{\epsilon}{2CC_0},
\]
where \( C \) and \( C_0 \) are constants from (1.13) and (3.5), which is possible since \( f \in VM_{p \cdot \varphi} \). This allows to estimate the first term uniformly in \( r \in (0, \delta_0) \):
\[
\| b \|_* \sup_{x \in \mathbb{R}^n} CI_{\delta_0}(x, r) < \frac{\epsilon}{2}, \quad 0 < r < \delta_0.
\]

The estimation of the second term may be obtained by choosing \( r \) sufficiently small. Indeed, by (1.3) we have
\[
J_{\delta_0}(x, r) \leq \| b \|_* c_{\delta_0} \| f \|_{VM_{p \cdot \varphi}} \frac{r^\frac{p}{p+1}}{\varphi(x, r)},
\]
where $c_0$ is the constant from (1.12). Then, by (1.3) it suffices to choose $r$ small enough such that
\[
\sup_{x \in \mathbb{R}^n} r^\frac{\delta}{\beta} \varphi(x, r) \leq \epsilon \frac{2 \|b\|_{c_0} \|f\|_{V_{M_p, \varphi}}}{\|c\|_{\delta}}.
\]
which completes the proof of (3.4).

For the case of $p < q$, we can also use the same method, so we omit the details. Thus, we obtain (1.16), which completes the proof of Theorem 5.

**Remark 2.** Conditions (1.12) and (1.14) are not needed in the case when $\varphi(x, r)$ does not depend on $x$, since (1.12) follows from (1.13) and similarly, (1.14) follows from (1.15) in this case.

**References**

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