A study on some generalized multiplicative and generalized additive arithmetic functions

D. Bhattacharjee
Department of Mathematics, North-Eastern Hill University
Permanent Campus, Shillong-793022, India
e-mails: debashis_bhattacharjee@yahoo.com, dbhattacharjee@nehu.ac.in

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Abstract: In this paper by an arithmetic function we shall mean a real-valued function on the set of positive integers. We recall the definitions of some common arithmetic convolutions. We also recall the definitions of a multiplicative function, a generalized multiplicative function (or briefly a GM-function), an additive function and a generalized additive function (or briefly a GA-function). We shall study in details some properties of GM-functions as well as GA-functions using some particular arithmetic convolutions namely the Narkiewicz’s A-product and the author’s B-product. We conclude our discussion with some examples.

Keywords: Arithmetic function, Multiplicative function, Arithmetic convolution, Dirichlet convolution, Narkiewicz’s A-product, B-product, MultiplicativeB-product, GA-function, GM-function.

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1 Introduction

In this paper, by an arithmetic function we shall mean a real valued function on the set of positive integers. The early history of the theory of arithmetic functions is contained in the first volume of Dickson’s [13] monumental “History of the Theory of Numbers”. The convolution operation played a prominent role from the very beginning. Many results from early times involved the convolution of two or more particular arithmetical functions.

Early in the 19th century the Dirichlet convolution, as well as addition and multiplication, began to be viewed as binary operations on the set of arithmetic functions. In the works of Cipolla [10] and Bell [2] it was recognized that the arithmetic functions form a commutative ring with unity with respect to addition and convolution.
The study of the structure of the ring of arithmetic functions has continued and we point out papers of, Vaidyanathaswamy [25], Cohen [11], Narkiewicz [19], Subbarao [22], Scheid [20], Apostol [1], Davison [12], Gioia [14], McCarthy [18], Sivaramakrishnan [21], Haukkanen [16], Tóth [23, 24], Lehmer [17], Bhattacharjee [3–7], Bhattacharjee and Saikia [8], and others.

We now introduce some of the most common arithmetic convolutions to our readers.

1. **Natural product (usual product).** If \( f \) and \( g \) are arithmetic functions, we define the natural product \( fg \) (also written as \( f \times g \)) as the arithmetic function whose value at any positive integer \( n \) is given by:

\[
(fg)(n) = f(n)g(n)
\]

2. **Dirichlet product (convolution).** If \( f \) and \( g \) are arithmetic functions, we define the Dirichlet product (or Dirichlet convolution) \( f \ast g \) as the arithmetic function whose value at any positive integer \( n \) is given by:

\[
(f \ast g)(n) = \sum_{d|n}^{} f(d)g\left(\frac{n}{d}\right) = \sum_{d_1d_2=n}^{} f(d_1)g(d_2)
\]

3. **Unitary product (convolution).** If \( f \) and \( g \) are arithmetic functions, we define the unitary product (or unitary convolution) \( f \circ g \) as the arithmetic function whose value at any positive integer \( n \) is given by:

\[
(f \circ g)(n) = \sum_{d_1d_2=n}^{} f(d_1)g(d_2)
\]

where \((d_1, d_2)\) stands for the greatest common divisor of \(d_1\) and \(d_2\).

4. **L.C.M. product (convolution).** If \( f \) and \( g \) are arithmetic functions, we define the L.C.M product (or L.C.M convolution) \( f \oplus g \) as the arithmetic function whose value at any positive integer \( n \) is given by

\[
(f \oplus g)(n) = \sum_{[d_1,d_2]=n}^{} f(d_1)f(d_2)
\]

where \([d_1,d_2]\) stands for the least common multiple of \(d_1\) and \(d_2\).

5. **The \( k \)-product (convolution).** If \( f \) and \( g \) are arithmetic functions, we define the \( k \)-product \( f \ast_k g \) as the arithmetic function whose value at any positive integer \( n \) is given by:

\[
(f \ast_k g)(n) = \sum_{ab=n}^{} f(a)g(b)k((a,b))
\]

where \(k((a,b))\) is a function of \((a,b)\), the greatest common divisor of \(a\) and \(b\).

6. **Davison’s product (convolution).** If \( f \) and \( g \) are arithmetic functions, we define the Davison’s product \( f \ast_D g \) as the arithmetic function whose value at any positive integer \( n \) is given by:

\[
(f \ast_D g)(n) = \sum_{ab=n}^{} f(a)g(b)A(a,b)
\]

where \(A(a,b)\) is a function of two variables \(a\) and \(b\), instead of being a function of their greatest common divisor as in 5.
7. **The Lehmer $\psi$-product (convolution).** Let $\psi(x,y)$ be a positive integral-valued function defined for a prescribed set $T$ of ordered pairs $(x,y)$ such that $x,y \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. Then the $\psi$-product $f \circ g$ of $f$ and $g$ is defined by:

$$ (f \circ g)(n) = \sum_{\psi(a,b)=n} f(a)g(b), $$

where $n \in \mathbb{N}$.

8. **Narkiewicz’s $A$-product (convolution).** For every positive integer $n$ fix the set $A_n$ of some divisors of $n$. For arithmetic functions $f$ and $g$ their $A$-product $f \ast_A g$ is given by

$$ (f \ast_A g)(n) = \sum_{d \in A_n} f(d) g\left(\frac{n}{d}\right) $$

for $n = 1, 2, 3, \ldots$.

We now recall the definitions of a multiplicative function, a generalized multiplicative function (or briefly a $GM$-function), an additive function and a generalized additive function (or briefly a $GA$-function). We shall study in details some properties of $GM$-functions as well as $GA$-functions using some particular arithmetic convolutions namely the Narkiewicz’s $A$-product and the author’s $B$-product as multiplicative binary operations.

An arithmetic function $F$ is said to be multiplicative if $F(1) = 1$ and $F(mn) = F(m)F(n)$, whenever $(m,n) = 1$, where $(m,n)$ stands for the greatest common divisor of $m$ and $n$.

There are several generalizations of multiplicative functions which can be found in McCarthy [18], Sivaramakrishnan [21]. Zafrullah in his paper [26] introduced a new generalization where he defined an arithmetic function $F$ to be a Generalized Multiplicative function (or briefly a $GM$-function) if $F(1) = 1$ and there exists a multiplicative function $f$ such that:

$$ F(mn) = F(m)f(n)F(n)f(m) $$

(1)

whenever $(m,n) = 1$.

An arithmetic function $F$ is said to be an additive function if $F(mn) = F(m) + F(n)$ whenever $(m,n) = 1$. We call an arithmetic function $F$ to be a Generalized Additive function (or briefly a $GA$-function) if there exists a multiplicative function $f$ such that

$$ F(mn) = F(m)f(n) + F(n)f(m) $$

(2)

whenever $(m,n) = 1$.

Haukkanen in his paper [15] studied some properties of $GM$-functions and $GA$-functions considering Dirichlet convolution as the multiplicative binary operation. The present author in this paper tries to study some properties of $GM$-functions and $GA$-functions by considering the Narkiewicz’s $A$-product [19] and the author’s $B$-product [3] as multiplicative binary operation.

We conclude our discussion with some examples.

As mentioned earlier, Narkiewicz in his paper [19] defined $A$-product as follows: For every positive integer $n$ fix the set $A_n$ of some divisors of $n$. For arithmetic functions $f$ and $g$ their $A$-product $f \ast_A g$ is given by:

$$ (f \ast_A g)(n) = \sum_{d \in A_n} f(d) g\left(\frac{n}{d}\right) $$

(I)

for $n = 1, 2, 3, \ldots$.

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The author in his paper [3] defined $B$-product as follows: For every positive integer $n$, fix the set $B_n$ of some pairs of divisors of $n$. For arithmetic functions $f$ and $g$ their $B$-product $f \ast_B g$ is given by:

$$(f \ast_B g)(n) = \sum_{(u,v) \in B_n} f(u)g(v) \quad (\text{II})$$

for $n = 1, 2, 3, \ldots$.

This $B$-product generalizes simultaneously the $A$-product of Narkiewicz [19] and the L.C.M. product and it has a non-void intersection with the $\Psi$-product of Lehmer [17]. The $\tau$-product of Scheid [20] is also a particular case of $B$-product.

Narkiewicz in his paper [19] called an $A$-product to be multiplicative if and only if $A_{mn} = A_m \times A_n$ for every pair $(m, n)$ of relatively prime positive integers (here $B \times C$ denotes the set of all integers, which can be represented in the form $bc, b \in B, c \in C$).

The present author in his paper [4] defined a $B$-product to be multiplicative if and only if $B_{mn} = \{(r_1 r_2, s_1 s_2) \mid (r_1, s_1) \in B_m, (r_2, s_2) \in B_n\}$ for every pair $(m, n)$ of relatively prime positive integers; in other words a $B$-product is multiplicative if and only if the following conditions hold:

For every pair $(m, n)$ of relatively prime positive integers we have $(r, s) \in B_{mn}$ if and only if $(r^{(m)}, s^{(m)}) \in B_m$ and $(r^{(n)}, s^{(n)}) \in B_n$, where $r^{(k)}$ stands for the greatest common divisor of $r$ and $k$.

In the next section, we shall study in details some properties of $GM$-functions and $GA$-functions using Narkiewicz’s $A$-product and the author’s $B$-product as multiplicative binary operations.

## 2 Main results

### 2.1 $GM$-Function and its properties

As mentioned in our discussion earlier, Zafrullah in his paper [26] introduced a new generalization of multiplicative functions where he defined an arithmetic function $F$ to be a generalized multiplicative function (or $GM$-function) if $F(1) = 1$ and there exists a multiplicative function $f$ such that $F(mn) = F(m)^{f(n)} F(n)^{f(m)}$, whenever $(m, n) = 1$. We now prove some results related to $GM$-functions using Narkiewicz’s $A$-product and the author’s $B$-product as the multiplicative binary operations.

**Theorem 2.1.1.** Let $F$ be a $GM$-function and $h$ be a multiplicative function and let the $A$-product be multiplicative, then the arithmetic function $H$ defined by

$$H(n) = \prod_{d \in A_n} F(d)^{h(n)}$$

for all $n \in \mathbb{N}$ is a $GM$-function.

**Proof:** We have

$$H(1) = \prod_{d \in A_1} F(d)^{h(1)} = F(1)^{h(1)} = F(1) = 1$$

Let $(m, n) = 1$, then
\[
H(mn) = \prod_{d \in A_{mn}} F(d)^{h(mn)}
\]

\[
= \prod_{d_1 \in A_m, d_2 \in A_n} F(d_1 d_2)^{h(mn)}
\]

\[
= \prod_{d_1 \in A_m} \prod_{d_2 \in A_n} [F(d_1)F(d_2)f(d_1)]^{h(mn)}
\]

(\text{where } f \text{ is a multiplicative function})

\[
= \prod_{d_1 \in A_m} F(d_1)^{h(m)} \prod_{d_2 \in A_n} F(d_2)^{h(n)}
\]

\[
= K(m)^{v(m) \times h(n)^{(n)}}
\]

Since \( f \) and \( h_1 \) are multiplicative functions and the \( A \)-product is multiplicative, therefore \( v \) is a multiplicative function.

Thus, \( H \) is a \textit{GM}-function.

\textbf{Theorem 2.1.2.} Let \( F \) be a \textit{GM}-function and \( h_1 \) and \( h_2 \) be multiplicative functions and let the \( A \)-product be multiplicative, then the arithmetic function \( H \) defined by

\[
H(n) = \prod_{d \in A_n} F(d)^{h_1(d)h_2(d/n)}
\]

for all \( n \in \mathbb{N} \) is a \textit{GM}-function.

\textit{Proof:} Let \( F(d)^{h_1(d)} = K(d) \). Then

\[
H(n) = \prod_{d \in A_n} K(d)^{h_2(n)}
\]

and \( K(1) = F(1)^{h_1(1)} = 1 \). Let \( (m, n) = 1 \), then

\[
K(mn) = F(mn)^{h_1(mn)}
\]

\[
= [F(m)^{f(n)}F(n)^{f(m)}]^{h_1(m)h_1(n)}
\]

(\text{where } f \text{ is a multiplicative function})

\[
= K(m)^{f^{h_1(n)}}K(n)^{f^{h_1(m)}}
\]

\[
= K(m)^{v(m)}K(n)^{v(m)}
\]

Since \( f \) and \( h_1 \) are multiplicative functions, then \( v = fh_1 \) is also a multiplicative function. Therefore, \( K \) is a \textit{GM}-function.

Therefore, by Theorem 2.1.1 we conclude that \( H \) is a \textit{GM}-function.
Theorem 2.1.3. Let $F$ be a $GM$-function and $h$ be a multiplicative function and let the $B$-product be multiplicative. Then the arithmetic function $J(n)$ given by

$$J(n) = \prod_{(r,s) \in B_n} F(r)^{h(s)}$$

for all $n \in \mathbb{N}$ is a $GM$-function.

Proof: We have

$$J(1) = \prod_{(r,s) \in B_1} F(r)^{h(s)} = F(1)^{h(1)} = 1$$

Let $(m,n) = 1$. Then

$$J(mn) = \prod_{(r,s) \in B_{mn}} F(r)^{h(s)}$$

$$= \prod_{(r,m), (s,n) \in B_m} \prod_{(r,n), (s,m) \in B_n} F(r)^{h(s)} f(r)^{f(r)} f(r)^{f(r)} h(s)^{h(s)} h(s)^{h(s)}$$

(where $f$ is a multiplicative function)

$$= \left( \prod_{(r,m), (s,n) \in B_m} \prod_{(r,n), (s,m) \in B_n} \left[ F(r)^{h(s)} \right] \right) \times \left( \prod_{(r,m), (s,n) \in B_m} \prod_{(r,n), (s,m) \in B_n} \left[ F(r)^{h(s)} \right] \right)$$

$$= \prod_{(r,m), (s,n) \in B_m} \prod_{(r,n), (s,m) \in B_n} \left[ F(r)^{h(s)} \right] \times \left[ \prod_{(r,m), (s,n) \in B_m} \prod_{(r,n), (s,m) \in B_n} \left[ F(r)^{h(s)} \right] \right]$$

$$= f^{(m)^{h^*(h)(m)}} \times (n)^{(m)^{h^*(h)(m)}}$$

$$= J(m)^{f^*B_f(h)(m)} \times J(n)^{(m)^{h^*(h)(m)}}$$

where $v = f \ast_B h$.

Since $f$ and $h$ are multiplicative functions and the $B$-product is multiplicative, therefore $v$ is a multiplicative function.

Hence, $J$ is a $GM$-function.

Theorem 2.1.4. Let $F$ be a $GM$-function and let $h_1$ and $h_2$ be multiplicative functions and let the $B$-product be multiplicative. Then the arithmetic function $J$ defined by

$$J(n) = \prod_{(r,s) \in B_n} F(r)^{h_1(r)h_2(s)}$$

for all $n \in \mathbb{N}$ is a $GM$-function.

Proof: Let $U(r) = F(r)^{h_1(r)}$. Therefore,
\[ J(n) = \prod_{(r,s) \in B_n} U(r)^{h_2(s)} \]

Clearly,

\[ U(1) = F(1)^{h_1(1)} = 1. \]

Let \((m, n) = 1\). Then

\[ U(mn) = F(mn)^{h_1(mn)} = (F(m)^{f(n)}F(n)^{f(m)})^{h_1(m)n_1(n)} \]

(where \(f\) is a multiplicative function)

\[ = [F(m)^{f(n)}]^{h_1(m)n_1(n)} [F(n)^{f(m)}]^{h_1(m)n_1(n)} \]

\[ = [F(m)^{h_1(m)}]^{f(n)}^{h_1(n)} [F(n)^{h_1(n)}]^{f(m)}^{h_1(m)} \]

\[ = U(m)^{v(n)}U(n)^{v(m)} \]

where \(v = fh_1\), is a multiplicative function.

Therefore, \(U\) is a **GM**-function.

Hence, by Theorem 2.1.3 we conclude that \(J\) is a **GM**-function.

\[ \square \]

### 2.2 **GA-Function and its properties**

An arithmetic function \(F\) is said to be additive if \(F(mn) = F(m) + F(n)\) whenever \((m, n) = 1\).

We define an arithmetic function \(F\) to be a generalized additive function (or briefly a **GA**-function) if there exists a multiplicative function \(f\) such that \(F(mn) = F(m)f(n) + F(n)f(m)\) whenever \((m, n) = 1\). Chawla [9] calls such a function a distributive function. One can notice that the notion of **GA**-function is an additive analogue of **GM**-function. We now prove some results on **GA**-function in the context of Narkiewicz’s A-product and the author’s B-product.

**Theorem 2.2.1.** Let \(F\) be a **GA**-function and \(h\) be a multiplicative function and let the A-product be multiplicative. Then the arithmetic function \(H\) defined by

\[ H(n) = \sum_{d \in A_n} F(d)h(d) \]

for all \(n \in \mathbb{N}\) is a **GA**-function.

**Proof:** Let \(U(d) = F(d)h(d)\) and \((m, n) = 1\). Then

\[ U(mn) = F(mn)h(mn) \]

\[ = [F(m)f(n) + F(n)f(m)]h(m)h(n) \]

(where \(f\) is a multiplicative function)

\[ = F(m)h(m)f(n)h(n) + F(n)h(n)f(m)h(m) \]

\[ = U(m)v(n) + U(n)v(m) \]

(where \(v = fh\) is a multiplicative function).

Therefore, \(U\) is a **GA**-function. Now
As $d_1 \in A_m, d_2 \in A_n$ so $(d_1, d_2) = 1$, and since $U$ is a GA-function, therefore there exists a multiplicative function $f$ such that $U(d_1 d_2) = U(d_1) f(d_2) + U(d_2) f(d_1)$. Therefore,

$$H(mn) = \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} \left[ U(d_1) f(d_2) + U(d_2) f(d_1) \right]$$

$$= \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_1) f(d_2) + \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_2) f(d_1)$$

$$= H(m) \omega(n) + H(n) \omega(m),$$

where

$$\omega(m) = \sum_{d_1 \in A_m} f(d_1), \; \omega(n) = \sum_{d_2 \in A_n} f(d_2)$$

are multiplicative functions.

Hence, $H$ is a GA-function.

Theorem 2.2.2. Let $F$ be a GA-function and $h_1$ and $h_2$ be multiplicative functions and let the A-product be multiplicative. Then the arithmetic function $H$ defined by

$$H(n) = \sum_{d \in A_n} F(d) h_1(d) h_2\left(\frac{n}{d}\right)$$

for all $n \in \mathbb{N}$, is a GA-function.

Proof: Let $U(d) = F(d) h_1(d)$. Then from the proof of Theorem 2.2.1, $U$ is a GA-function. Now

$$H(n) = \sum_{d \in A_n} U(d) h_2\left(\frac{n}{d}\right)$$

Let $(m, n) = 1$. Then

$$H(mn) = \sum_{d \in A_{mn}} U(d) h_2\left(\frac{mn}{d}\right)$$

$$= \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_1 d_2) h_2\left(\frac{mn}{d_1 d_2}\right)$$

As $d_1 \in A_m, d_2 \in A_n$ so $(d_1, d_2) = 1$, and since $U$ is a GA-function, therefore there exists a multiplicative function $f$ such that $U(d_1 d_2) = U(d_1) f(d_2) + U(d_2) f(d_1)$.

Therefore,

$$H(mn) = \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} \left[ U(d_1) f(d_2) + U(d_2) f(d_1) \right] \times \left[ h_2\left(\frac{m}{d_1}\right) h_2\left(\frac{n}{d_2}\right) \right]$$

$$= \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} \left[ U(d_1) f(d_2) h_2\left(\frac{m}{d_1}\right) h_2\left(\frac{n}{d_2}\right) + U(d_2) f(d_1) h_2\left(\frac{m}{d_1}\right) h_2\left(\frac{n}{d_2}\right) \right]$$

$$= \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_1) h_2\left(\frac{m}{d_1}\right) f(d_2) h_2\left(\frac{n}{d_2}\right) + \sum_{d_1 \in A_m} \sum_{d_2 \in A_n} U(d_2) h_2\left(\frac{n}{d_2}\right) f(d_1) h_2\left(\frac{m}{d_1}\right)$$
\[
= \sum_{d_1 \in A_m} U(d_1)h_2\left(\frac{m}{d_1}\right)\sum_{d_2 \in A_n} f(d_2)h_2\left(\frac{n}{d_2}\right) + \sum_{d_2 \in A_n} U(d_2)h_2\left(\frac{n}{d_2}\right)\sum_{d_1 \in A_m} f(d_1)h_2\left(\frac{m}{d_1}\right)
\]

\[
= [(U \ast_A h_2)(m)] \cdot [(f \ast_A h_2)(n)] + [(U \ast_A h_2)(n)] \cdot [(f \ast_A h_2)(m)]
\]

\[
= H(m)f(n) + H(n)f(m)
\]

where \( v(k) = (f \ast_A h_2)(k) \).

Now since \( f \) and \( h_2 \) are multiplicative functions and the \( A \)-product is multiplicative, therefore \( v = f \ast_A h_2 \) is a multiplicative function. Consequently \( H \) is a \( GA \)-function.

**Theorem 2.2.3.** Let \( F \) be a \( GA \)-function and let \( h_1 \) and \( h_2 \) be multiplicative functions and let the \( B \)-product be multiplicative. Then the arithmetic function \( H \) defined by

\[
H(n) = \sum_{(r,s) \in B_n} F(r) h_1(r) h_2(s)
\]

for all \( n \in \mathbb{N} \) is a \( GA \)-function.

**Proof:** Let \( U(d) = F(d) h_1(d) \). Then, from the proof of Theorem 2.2.1, \( U \) is a \( GA \)-function. Therefore,

\[
H(n) = \sum_{(r,s) \in B_n} U(r) h_2(s)
\]

Let \( (m,n) = 1 \). Then

\[
H(mn) = \sum_{(r,s) \in B_m} \sum_{(m,n) \in B_n} U(r) h_2(s)
\]

\[
= \sum_{(r^{(m)}, s^{(m)}) \in B_m} \sum_{(r^{(n)}, s^{(n)}) \in B_n} U(r^{(m)} r^{(n)}) h_2(s^{(m)} s^{(n)})
\]

As \( (r^{(m)}, s^{(m)}) \in B_m \) and \( (r^{(n)}, s^{(n)}) \in B_n \), so \( (r^{(m)}, r^{(n)}) = 1 \) and \( (s^{(m)}, s^{(n)}) = 1 \) and since \( U \) is a \( GA \)-function, therefore, there exists a multiplicative function \( f \) such that

\[
U(r^{(m)} r^{(n)}) = U(r^{(m)}) f(r^{(n)}) + U(r^{(n)}) f(r^{(m)})
\]

Therefore

\[
H(mn) = \sum_{(r^{(m)}, s^{(m)}) \in B_m} \sum_{(r^{(n)}, s^{(n)}) \in B_n} \left[ U(r^{(m)}) f(r^{(n)}) + U(r^{(n)}) f(r^{(m)}) \right] [h_2(s^{(m)} s^{(n)})]
\]

\[
= \sum_{(r^{(m)}, s^{(m)}) \in B_m} \sum_{(r^{(n)}, s^{(n)}) \in B_n} \left[ U(r^{(m)}) f(r^{(n)}) h_2(s^{(m)}) h_2(s^{(n)}) \right]
\]

\[
+ \sum_{(r^{(n)}, s^{(n)}) \in B_n} \left[ U(r^{(n)}) f(r^{(m)}) h_2(s^{(m)}) h_2(s^{(n)}) \right]
\]

\[
= \sum_{(r^{(m)}, s^{(m)}) \in B_m} \sum_{(r^{(n)}, s^{(n)}) \in B_n} U(r^{(m)}) f(r^{(n)}) h_2(s^{(m)}) h_2(s^{(n)})
\]

\[
+ \sum_{(r^{(m)}, s^{(m)}) \in B_m} \sum_{(r^{(n)}, s^{(n)}) \in B_n} U(r^{(n)}) f(r^{(m)}) h_2(s^{(m)}) h_2(s^{(n)})
\]
\[
\sum_{(r^{(m)},s^{(m)}) \in B_m} U(r^{(m)}) h_2(s^{(m)}) \sum_{(r^{(n)},s^{(n)}) \in B_n} f(r^{(n)}) h_2(s^{(n)})
\]
\[
+ \sum_{(r^{(n)},s^{(n)}) \in B_n} U(r^{(n)}) h_2(s^{(n)}) \sum_{(r^{(m)},s^{(m)}) \in B_m} f(r^{(m)}) h_2(s^{(m)})
\]
\[
= H(m)(f \ast_B h_2)(n) + H(n)(f \ast_B h_2)(m).
\]

Now since \(f\) and \(h_2\) are multiplicative functions and the \(B\)-product is multiplicative, therefore, \(f \ast_B h_2\) is a multiplicative function. Consequently, \(H\) is a \(\text{GM}\)-function. \(\square\)

We conclude our discussion with some examples.

3 Examples

Example 1. If
\[
H(n) = \prod_{d \in A_n} d^d
\]
for all \(n \in \mathbb{N}\), and the \(A\)-product is multiplicative, then \(H\) is a \(\text{GM}\)-function.

Proof. We have
\[
H(n) = \prod_{d \in A_n} d^d.
\]
Let \(F(d) = d^d\), then for every relatively prime positive integers \(m, n\) we have
\[
F(mn) = (mn)^{mn} = m^m n^m = (m^m)^n (n^m)^m = F(m)^n F(n)^m = F(m)^f(n) F(n)^{f(m)}.
\]
Where \(f(u) = u\) for every \(u \in \mathbb{N}\), is a multiplicative function. Therefore, \(F\) is a \(\text{GM}\)-function. Now
\[
H(mn) = \prod_{d \in A_{mn}} F(d) = \prod_{d_1 d_2 \in A_{mn}} F(d_1 d_2),
\]
(where \(d = d_1 d_2\), \(d_1 | m\) and \(d_2 | n\))
\[
= \prod_{d_1 d_2 \in A_{mn}} F(d_1)^{g(d_2)} F(d_2)^{g(d_1)}
\]
(where \(g\) is a multiplicative function)
\[
= \{ \prod_{d_1 \in A_m} F(d_1)^{\sum_{d_2 \in A_n} g(d_2)} \} \times \{ \prod_{d_2 \in A_n} F(d_2)^{\sum_{d_1 \in A_m} g(d_1)} \}
\]
\[
= H(m)^{u(n)} \times H(n)^{u(m)}
\]
where \(u(m) = \sum_{d_1 \in A_m} g(d_1)\) and \(u(n) = \sum_{d_2 \in A_n} g(d_2)\)

Since \(g\) is a multiplicative function and the \(A\)-product is multiplicative, therefore, \(u\) is a multiplicative function. Hence, \(H\) is a \(\text{GM}\)-function.

Example 2. If
\[
H(n) = \prod_{(r,s) \in B_n} r^s
\]
for all \( n \in \mathbb{N} \), and the \( B \)-product is multiplicative, then \( H \) is a \textit{GM}-function.

\textbf{Proof.} Let \( F(r) = r \) and \( h(s) = s \), then \( F(mn) = mn = m^{\eta(n)}n^{\eta(m)} = F(m)^{\eta(n)}F(n)^{\eta(m)} \), where \( \eta(k) = 1 \) for every \( k \in \mathbb{N} \) is a multiplicative function and therefore \( F \) is a \textit{GM}-function. Now since \( h(s) = s \) is a multiplicative function, and the \( B \)-product is multiplicative, therefore by Theorem 2.1.3

\[ H(n) = \prod_{(r,s) \in B_n} r^s = \prod_{(r,s) \in B_n} F(r)^{h(s)} \]

is a \textit{GM}-function.

\textbf{Example 3.} If

\[ H(n) = \sum_{d \in A_n} d \log d \]

for all \( n \in \mathbb{N} \), and the \( A \)-product is multiplicative, then \( H \) is a \textit{GA}-function.

\textbf{Proof.} Let \((m,n) = 1\), then

\[ H(mn) = \sum_{d \in A_{mn}} d \log d = \sum_{d_1 \in A_m, d_2 \in A_n} (d_1 d_2) \log(d_1 d_2) = \sum_{d_1 \in A_m, d_2 \in A_n} (d_1 d_2) [\log d_1 + \log d_2] \]

\[ = \sum_{d_1 \in A_m, d_2 \in A_n} (d_1 \log d_1) d_2 + \sum_{d_1 \in A_m, d_2 \in A_n} (d_2 \log d_2) d_1 \]

\[ = \sum_{d_1 \in A_m} d_1 \log d_1 \sum_{d_2 \in A_n} d_2 + \sum_{d_1 \in A_m} d_2 \log d_2 \sum_{d_1 \in A_n} d_1 \]

\[ = H(m) \sum_{d_2 \in A_n} d_2 + H(n) \sum_{d_1 \in A_m} d_1 \]

\[ = H(m)\alpha(n) + H(n)\alpha(m) \]

where

\[ \alpha(k) = \sum_{d \in A_k} d = \sum_{d \in A_k} h(d) \]

and \( h(s) = s \) for every \( s \in \mathbb{N} \). Since \( h \) is a multiplicative function and the \( A \)-product is multiplicative, therefore \( \alpha \) is a multiplicative function. Hence \( H \) is a \textit{GA}-function.

\textbf{Example 4.} If

\[ H(n) = \sum_{(r,s) \in B_n} r \log s \]

for all \( n \in \mathbb{N} \), and the \( B \)-product is multiplicative, then \( H \) is a \textit{GA}-function.

\textbf{Proof.} Let \((m,n) = 1\). Then

\[ H(mn) = \sum_{(r,s) \in B_{mn}} r \log s \]
\[
\sum_{(r^{(m)}, s^{(m)}) \in B_m, (r^{(n)}, s^{(n)}) \in B_n} r^{(m)} r^{(n)} \log (s^{(m)} s^{(n)})
\]

\[
= \sum_{(r^{(m)}, s^{(m)}) \in B_m, (r^{(n)}, s^{(n)}) \in B_n} r^{(m)} r^{(n)} \log (s^{(m)}) + \log (s^{(n)})
\]

\[
= \sum_{(r^{(m)}, s^{(m)}) \in B_m, (r^{(n)}, s^{(n)}) \in B_n} r^{(m)} + \sum_{(r^{(m)}, s^{(m)}) \in B_m, (r^{(n)}, s^{(n)}) \in B_n} r^{(n)} \log (s^{(n)}) r^{(m)}
\]

\[
= \sum_{(r^{(m)}, s^{(m)}) \in B_m} r^{(m)} \log (s^{(m)}) \sum_{(r^{(n)}, s^{(n)}) \in B_n} r^{(n)} + \sum_{(r^{(n)}, s^{(n)}) \in B_n} r^{(n)} \log (s^{(n)}) \sum_{(r^{(m)}, s^{(m)}) \in B_m} r^{(m)}
\]

\[
= H(m) \alpha(n) + H(n) \alpha(m)
\]

where

\[
\alpha(k) = \sum_{(r^{(k)}, s^{(k)}) \in B_k} r^{(k)} = \sum_{(r^{(k)}, s^{(k)}) \in B_k} h(r^{(k)})
\]

and \( h(s) = s \) for every \( s \in \mathbb{N} \).

Since \( h \) is a multiplicative function and the \( B \)-product is multiplicative, therefore \( \alpha \) is a multiplicative function. Hence \( H \) is a \( GA \)-function.

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