BOREL WHITEHEAD GROUPS

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Abstract. We investigate the Whiteheadness of Borel abelian groups (ℵ₁-free, without loss of generality as otherwise this is trivial). We show that CH (and even WCH) implies any such abelian group is free, and always ℵ₂-free.
0.1 Definition. 1) We say that $\vec{\psi} = (\psi_0, \psi_1)$ is a code for a Borel abelian group if:

(a) $\psi_0(\ldots, \ldots)$ codes a Borel equivalence relation $E = E_{\vec{\psi}}$ on a subset $B_* = B^*_{\vec{\psi}}$ of $\omega^2$ so $[\psi_0(\eta, \eta) \leftrightarrow \eta \in B_*]$ and $[\psi_0(\eta, \nu) \rightarrow \eta \in B_* \& \nu \in B_*]$, the group will have a set of elements $B = B^*_{\vec{\psi}}/E_{\vec{\psi}}$.

(b) $\psi_1 = \psi_1(x, y, z)$ code a Borel set of triples from $\omega^2$ such that $
abla \{ (x/E_{\vec{\psi}}, y/E_{\vec{\psi}}, z/E_{\vec{\psi})} : \psi_1(x, y, z) \}$ is the graph of a function from $B \times B$ to $B$ such that $(B, +)$ is an abelian group.

2) We say Borel$^+$ if (b) is replaced by:

(b)$'$ $\psi_1$ codes a Borel function from $B_* \times B_*$ to $B_*$ which respects $E_{\vec{\psi}}$, the function is called $+$ and $(B, +)$ is an abelian group (well, we should denote the function which $+$ induces from $(B_*/E_{\vec{\psi}}) \times (B_*/E_{\vec{\psi}})$ into $B_*/E_{\vec{\psi}}$ by e.g. $+_{E_{\vec{\psi}}}$, but are not strict).

We let $B^*_{\vec{\psi}} = B_{\vec{\psi}} = (B, +)$ be the group coded by $\vec{\psi}$; abusing notation we may write $B$ for $B_{\vec{\psi}}$.

Clearly

0.2 Observation: The set of codes for Borel abelian groups is $\Pi_2^1$.

An abelian group $B$ is Borel if it has a Borel code.

An interesting problem suggested by Dave Marker is the Borel version of Whitehead’s problem: namely

0.3 Question: Is every Borel Whitehead group free?

In this paper we will give a partial answer to this question. We will show that every Borel Whitehead group is $\aleph_2$-free. In particular, the continuum hypothesis implies that every Borel Whitehead group is free. This latter result provides a contrast to the author’s proof ([Sh:98]) that it is consistent with CH that there is a Whitehead group of cardinality $\aleph_1$ which is not free.

We refer the reader to [EM] for the necessary background material on abelian groups.

Suppose $B$ is an $\aleph_1$-free abelian group. Let $S_0 = \{ G \subset B : |G| = \aleph_0 \text{ and } B/G \text{ is not } \aleph_1\text{-free} \}$. It is well known that if $B$ is not $\aleph_2$-free, then $S_0$ is stationary. We will argue that the converse is true for Borel abelian groups and the answer is quite absolute. Lastly, we deal with weakening Borel to Souslin.

0.4 Question: If $B$ is an $\aleph_2$-free Borel abelian group, what can be the $n$ in the analysis of a nonfree $\aleph_2$-free abelian subgroup of $B$ from [Sh 161] (or see [EM] or [Sh 523])?

We thank Todd Eisworth for corrections.
1.1 Hypothesis. Let $B$ be an $\aleph_1$-free Borel abelian group. Let $\bar{\psi}$ be a Borel code for $B$.

Let $S_B = S_{\bar{\psi}} = \{ K \subseteq B : K$ is a countable subgroup and $B/K$ is not $\aleph_1$-free $\}$.

1.2 Lemma. 1) If $S_B$ is stationary, then $B$ is not $\aleph_2$-free.

2) Moreover, there is an increasing continuous sequence $\langle G_i : i < \omega_1 \rangle$ of countable subgroups of $B$ such that $G_{i+1}/G_i$ is not free for each $i < \omega_1$.

Remark. On such proof in model theory see [Sh 43, §2], [BKM78] and [Sch85].

Proof. We work in a universe $V \models ZFC$. Force with $P = \{ p : p$ is a function from some $\alpha < \omega_1$ to $\omega^2 \}$.

Let $G \subseteq P$ be $V$-generic and let $V[G]$ denote the generic extension.

Since $P$ is $\aleph_1$-closed, forcing with $P$ adds no new reals. Thus $\bar{\psi}$ still codes $B$ in the generic extension, i.e. $B^V_{\bar{\psi}[G]} = B^V_{\bar{\psi}}$. Forcing with $P$ also adds no new countable subsets of $B$ hence “$B$ is $\aleph_1$-free” holds in $V$ iff it holds in $V[G]$. Similarly if $K \subseteq B$ is countable, then “$B/K$ is $\aleph_1$-free” holds in $V$ iff it holds in $V[G]$. Thus, $S^V_{\bar{\psi}} = S^V_{\bar{\psi}[G]}$. Moreover, since $P$ is proper, $S_{\bar{\psi}}$ remains stationary (see [Shf, Ch.III]).

Since $V[G] \models CH$, we can write

$$B = \bigcup_{\alpha < \omega_1} B_\alpha,$$

where $\bar{B} = \langle B_\alpha : \alpha < \omega_1 \rangle$ is an increasing continuous chain of countable subgroups.

Let $S = \{ \alpha < \omega_1 : B/B_\alpha$ is not $\aleph_1$-free $\}$. Since $S_{\bar{\psi}}$ is stationary (as a subset of $[B]^{\aleph_0}$) necessarily, $S$ is a stationary subset of $\omega_1$. So $V[G] \models "B$ is not free$"$.

By Pontryagin’s criteria for each $\alpha \in S$ there are $n_\alpha \in \omega$ and $a_\alpha^0, \ldots, a_\alpha^{n_\alpha}$ such that

$$PC(B_\alpha \cup \{ a_\alpha^0, \ldots, a_\alpha^{n_\alpha} \})/B_\alpha$$

is not free, where $PC(X) = PC(X, B)$ is the pure closure of the subgroup of $B$ which $X$ generates. We choose $n_\alpha$ minimal with this property.

Work in $V[G]$. Let $\kappa$ be a regular cardinal such that $\mathcal{H}(\kappa)$ satisfies enough axioms of set theory to handle all of our arguments, and let $<^*$ be a well ordering of $\mathcal{H}(\kappa)$. Let $N \preceq (\mathcal{H}(\kappa), e, <^*)$ be countable such that $\bar{\psi}, S, \langle B_\alpha : \alpha < \omega_1 \rangle$ and $\langle \langle a_\alpha^0, \ldots, a_\alpha^{n_\alpha} \rangle : \alpha < \omega_1 \rangle$ belong to $N$.

The model $N$ has been built in $V[G]$, but since forcing with $P$ adds no new reals, there is a transitive model $N_0 \in V$ isomorphic to $N$ and let $h$ be an isomorphism from $N$ onto $N_0$. Clearly $h$ maps $\bar{\psi}$ to $\bar{\psi}$. From now on we work in $V$.

We build an increasing continuous elementary chain $\langle N_\alpha : \alpha < \omega_1 \rangle$, choosing $N_\alpha$ by induction on $\alpha$, as follows. Note the $N_\alpha$’s are not necessarily transitive or even well founded.

Let $\Gamma = \Gamma_\alpha = \{ \varphi(v) : N_\alpha \models "\delta \in h(S) : \varphi(\delta)" \}$ is stationary” and $\varphi \in \Phi_\alpha$ where $\Phi_\alpha$ is the set of first order formulas with parameters from $N_\alpha$ in the vocabulary.
\[ \{ \in, <^* \} \] and the only free variable \( v \). Let \( \leq_{\Gamma} \) be the following partial order of \( \Gamma : \theta \leq_{\Gamma} \varphi \) iff \( N_{\alpha} \models "(\forall x)(\varphi(x) \rightarrow \theta(x))" \). Let \( t_{\alpha} \) be a subset of \( \Gamma_{\alpha} \) such that:

(a) \( t_{\alpha} \) is downward closed, i.e. if \( \theta \leq_{\Gamma} \varphi \) and \( \varphi \in t_{\alpha} \) then \( \theta \in t_{\alpha} \)

(b) \( t_{\alpha} \) is directed

(c) for some countable \( M_{\alpha} \prec \langle \mathcal{H}(\kappa), \in, <^* \rangle \) to which \( N_{\alpha} \) belongs, if \( \Gamma \in M_{\alpha}, \Gamma \subseteq \Gamma_{\alpha} \) is a dense subset of \( \Gamma_{\alpha} \) then \( t_{\alpha} \cap \Gamma \neq \emptyset \).

Clearly by the density if \( \varphi \in \Gamma_{\alpha} \) and \( \theta \in \Phi_{\alpha} \), then \( \varphi \wedge \theta \in \Gamma_{\alpha} \) or \( \varphi \wedge \neg \theta \in \Gamma_{\alpha} \). Thus, \( t_{\alpha} \) is a complete type over \( N_{\alpha} \). Since \( N_{\alpha} \) has definable Skolem functions, we can let \( N_{\alpha+1} \) be the Skolem hull of \( N_{\alpha} \cup \{ b_{\alpha} \} \) where \( N_{\alpha} \prec N_{\alpha+1}, b_{\alpha} \in N_{\alpha+1} \) realizes \( t_{\alpha} \).

We claim that \( N_{\alpha+1} \) has no “new natural numbers”, i.e. if \( N_{\alpha+1} \models "c \text{ is a natural numbers}" \) then \( c \in N_{\alpha} \). Why? As \( c \in N_{\alpha+1} \) clearly for some \( f \in N_{\alpha} \) we have \( N_{\alpha} \models "f \text{ is a function with domain } \omega_1, \text{ the countable ordinals}" \) and \( N_{\alpha+1} \models "f(b_{\alpha}) = c" \). Let

\[ D_f = \{ \varphi(v) \in \Gamma : N_{\alpha} \models "(\forall x)(\varphi(x) \rightarrow f(x) \text{ is not a natural number})" \text{ or for some } d \in N_{\alpha} \text{ we have } N_{\alpha} \models "(\forall x)(\varphi(x) \rightarrow f(x) = d)" \} . \]

It is easy to check that \( D_f \) is a subset of \( \Gamma_{\alpha} \), it belongs to \( M_{\alpha} \) and it is a dense subset of \( \Gamma_{\alpha} \); hence \( t_{\alpha} \cap D_f \neq \emptyset \). Let \( \varphi(x) \in D_f \cap t_{\alpha} \), so \( N_{\alpha+1} \models \varphi[b_{\alpha}] \), and by the definition of \( D_f \) we get the desired conclusion.

If \( N_{\alpha} \models "b \text{ is a countable ordinal}" \) then \( N_{\alpha+1} \models "b < b_{\alpha} \& b_{\alpha} \text{ is a countable ordinal}" \). Also \( N_{\alpha+1} \models "b_{\alpha} \in h(S)" \).

We claim that \( b_{\alpha} \) is the least ordinal of \( N_{\alpha+1} \setminus N_{\alpha} \) in the sense of \( N_{\alpha+1} \). Assume \( N_{\alpha+1} \models "c \text{ is a countable ordinal}, c < b_{\alpha}" \) so for some \( f \in N_{\alpha} \) we have \( N_{\alpha} \models "f : \omega_1 \rightarrow \omega_1 \text{ is a function}" \) and \( N_{\alpha+1} \models "c = f(b_{\alpha})", N_{\alpha+1} \models "f(b_{\alpha}) < b_{\alpha}" \). Then \( N_{\alpha} \models "(\exists \beta \in h(S) : f(\beta) < \beta) \text{ is a stationary subset of } \omega_1" \). Let \( D = \{ \varphi(v) \in \Gamma : (\exists \gamma < \omega_1)(\forall \upsilon)((\varphi(v) \rightarrow f(\upsilon) = \gamma) \lor (\forall \upsilon)((\varphi(v) \rightarrow f(\upsilon) \geq \upsilon)) \} \). By Fodor’s lemma (which \( N_{\alpha} \) satisfies) \( D \) is a dense subset of \( \Gamma_{\alpha} \) and clearly \( D \subseteq M_{\alpha} \). Since \( t_{\alpha} \) is sufficiently generic, there is a \( \gamma \in N_{\alpha} \) such that \( N_{\alpha+1} \models "f(b_{\alpha}) = \gamma" \).

Now \( N_{\alpha} \) is not necessarily wellfounded but it has standard \( \omega \) and without loss of generality \( N_{\alpha} \models "a \subseteq \omega" \) implies \( a = \{ n < \omega : N_{\alpha} \models "n \in a" \} \) so as \( h(\psi) = \tilde{\psi} \) clearly \( N_{\alpha} \models "x/E^\psi \in B^\psi \Rightarrow x/E^\psi \in B, \text{ and } N_{\alpha} \models "x, y, z \in B, x/E^\psi + y/E^\psi = z/E^\psi \Rightarrow x/E^\psi + y/E^\psi = z/E^\psi" \) .

For each \( \alpha < \omega_1 \), if \( N_{\alpha} \models "b < \omega_1" \), let \( B_{\alpha}^b \) be the group \( (h(B))_b \) as interpreted in \( N_{\alpha} \), i.e. \( N_{\alpha} \) thinks that \( B_{\alpha}^b \) is the \( b \)-th group in the increasing chain \( h(B) \).

Clearly \( B_{\alpha}^b \subseteq B \) is the equality, otherwise let \( j_{\alpha}^\psi \) map \( (x/E^\psi)^{N_{\alpha}} \to x/E^\psi \), so \( j_{\alpha}^\psi \) embeds \( B_{\alpha}^b \) into \( B^\psi \); let this image be called \( G_{\alpha}^\psi \). Also in \( N_{\alpha} \) there is a bijection between \( B_{\alpha}^b \) and \( \omega \). If \( \gamma > \alpha \), since \( N_{\alpha} \leq N_{\gamma} \) have the same natural numbers, clearly \( B_{\alpha}^b = B_{\gamma}^\psi \) when \( E^\psi \) is equality or \( j_{\alpha}^\psi = j_{\gamma}^\psi \) and \( G_{\alpha}^\psi = G_{\gamma}^\psi \) in the general case. In particular, \( G_{\alpha+1}^{b_{\alpha}} \) is the union of \( \{ G_{\alpha}^\psi : N_{\alpha} \models "b < \omega_1" \} \).

For \( \alpha < \omega_1 \), let \( G_{\alpha} = G_{\alpha+1}^{b_{\alpha}} \) and let \( h((j_{\alpha}^\psi : \ell \leq m_{\alpha} : \alpha \in S)) \in N_{\alpha+1} \) be \( \langle (\alpha_{\alpha}^\psi/E^\psi)^{N_{\alpha}} : \ell \leq m_{\alpha} \rangle \), so \( N_{\alpha+1} \) thinks that \( \langle \alpha_{\alpha}^\psi/E^\psi : \ell \leq m_{\alpha} \rangle \) witness that \( h(B)/B_{\alpha+1}^{b_{\alpha}} \) is not free. Clearly \( \alpha_{\alpha}^\psi/E^\psi, \ldots, \alpha_{m_{\alpha}}^\psi/E^\psi \in G_{\alpha+1} \) and
is not free. So $G_{\alpha+1}/G_{\alpha}$ is not free. Let $G = \bigcup_{\alpha < \omega_1} G_{\alpha}$. Then $G$ is not free. But $G$ is a subgroup of $B$, thus $B$ is not $\aleph_2$-free.

Remark. Instead of the forcing we could directly build the $N_\alpha$’s but we have to deal with stationary subsets of $\omega_2$ instead of $\omega_1$.

1.3 Corollary. If $B$ is an $\aleph_1$-free Borel abelian group, then $B$ is $\aleph_2$-free if and only if $\{K \subseteq B : |K| = \aleph_0 \text{ and } B/K \text{ is } \aleph_1\text{-free}\}$ is not stationary.

1.4 Fact: If $2^{\aleph_0} < 2^{\aleph_1}$ then every Borel Whitehead group $B$ is $\aleph_2$-free.

Proof. By [DvSh 65] (or see [EM]) as $2^{\aleph_0} < 2^{\aleph_1}$ we have: if $G$ be a Whitehead group of cardinality $\aleph_1$ and $G = \bigcup_{\alpha < \omega_1} G_{\alpha}$ is such that $\langle G_{\alpha} : \alpha < \omega_1 \rangle$ is an increasing continuous chain of countable subgroups, then $\{\alpha : G_{\alpha+1}/G_{\alpha} \text{ is not free}\}$ does not contain a closed unbounded set (see [EM, Ch.XII,1.8]). Thus, if $B$ is not $\aleph_2$-free, then the subgroup $G$ constructed in the proof of lemma 1.2 is not Whitehead. Since being Whitehead is a hereditary property (see [EM]), $B$ is not Whitehead.

The lemma shows that

1.5 Conclusion. For Borel abelian groups $B^\psi$, “$B^\psi$ is $\aleph_2$-free” is absolute (in fact it is a $\Sigma_1^1$ property of $\psi$).

Proof. The formula will just say that there is a model of a suitable fragment of ZFC (e.g. ZC) with standard $\omega$ to which $\psi$ belongs and it satisfies “$B^\psi$ is $\aleph_2$-free”.
§2 On $\aleph_2$-free Whitehead

2.1 Theorem. If $B$ is a Borel Whitehead group, then $B$ is $\aleph_2$-free.

2.2 Conclusion: (CH) Every Whitehead Borel abelian group is free.

Before we prove we quote [Sh 44, Definition 3.1].

2.3 Definition. 1) If $L$ is a subset of the $\aleph_1$-free abelian group, $G$, $PC(L, G)$ is the smallest pure subgroup of $G$ which contains $L$. Note that if $H$ is a pure subgroup of $G, L \subseteq H$ then $PC(L, G) = PC(L, H)$. We omit $G$ if it is clear.

2) If $H$ is a subgroup of $G, L$ a finite subset of $G, a \in G$, we say that $\pi(a, L, H, G)$ means that: $PC(H \cup L) = PC(\pi(a, L, H, G))$ but for no $b \in PC(H \cup L \cup \{a\})$ is $PC(\pi(a, L, H, G)) = PC(H \cup L \cup \{a\})$.

Proof. Assume $B$ is not $\aleph_2$-free. We repeat the proof of Lemma 1.2. So in $V^P, B$ is a non-free $\aleph_1$-free abelian group of cardinality $\aleph_1$. Hence by [Sh 44, p.250.3.1(3)], $B$ satisfies possibility I or possibility II where we have chosen $B = (B_\alpha : \alpha < \omega_1)$ increasing continuous with $B_\alpha$ countable, $B = \bigcup_{\alpha < \omega_1} B_\alpha$; the possibilities are explained below. The proof splits into the two cases.

Possibility I: By [Sh 44, p.250].

So we can find (still in $V^P$) an ordinal $\delta < \omega_1$ and $a_\ell^i \in B$ for $i < \omega_1, \ell < n_i$ such that

(A) $\{a_\ell^i + \delta B : \ell < \omega_1, \ell \leq n_i\}$ is independent in $B/B_\delta$

(B) $\pi(a_\ell^i, L_i, B_\delta, B)$ where $L_i$ is the subgroup of $B$ generated by $\{a_\ell^i : \ell < n_i\}$.

This situation does not survive well under the process and the proof of Lemma 1.2 but after some analysis a revised version will.

Without loss of generality $n_i = n_*(v) = n^*$ (by the pigeon hole principle). Let $N \prec (H(\chi), \in, <^*)$ be countable such that $\mu, B_\delta, B, (B_\alpha : \alpha < \omega_1), \langle a_0^i, \ldots, a_{n_i}^i : i < \omega_1 \rangle$ belong to $N$. We can find $M \subseteq V, M \cong N$; without loss of generality $M$ is transitive (so $M \models \text{“}n \text{ is a natural number” iff } n$ is a natural number).

Let $\mathfrak{B} \prec (H(\chi), \in, <^*)$ be countable, $M \in \mathfrak{B}$. Let $\Phi_M$ be the set of f.o. formulas $\varphi(v)$ in the vocabulary $\{\in, <^*\}$ and parameters from $M$ and the only free variable $v$. Now we imitate the proof of [Sh 202]. Let $\Gamma = \{\varphi(v) \in \Phi_M : M \models \text{“}\alpha < \omega_1 : \varphi(\alpha)\text{ is uncountable”}\}$ (equivalently $\Gamma$ is $\{a \subseteq \omega_1 : |a| = \aleph_1\}|M\}$. We can find $\langle t_\eta(v) : \eta \in \omega_2 \rangle$ such that:

(a) each $t_\eta(v)$ a suitable generic subset of $\Gamma$, i.e. $\Gamma$, is ordered by $\varphi_1(v) \leq \varphi_2(v)$ if $M \models (\forall v)(\varphi_2(v) \rightarrow \varphi_1(v))$ so $t_\eta(v)$ is directed, downward closed and is not disjoint to any dense subset of $\Gamma$ from $\mathfrak{B}$

(b) for $k < \omega, \eta_0, \ldots, \eta_{k-1} \in \omega_2$ which are pairwise distinct $\langle t_{\eta_0}(v), \ldots, t_{\eta_{k-1}}(v) \rangle$ is generic too (for $\Gamma^k$), i.e. if $D \in \mathfrak{B}$ is a dense subset of $\Gamma^k$ then $\prod_{\ell < k} t_{\eta_\ell}(v)$ is not disjoint to $D$.  

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(See explanation in the end of the proof of case II).
So for each \( \eta, t_\eta(v) \) is a complete type over \( M \) hence we can find \( M_\eta, M \prec M_\eta, M_\eta \)
the Skolem hull of \( M \cup \{ y_\eta \} \) such that \( y_\eta \) realizes \( t_\eta(v) \) in \( M_\eta \). So \( M_\eta \models \{ y_\eta \) a
countable ordinal\}. Without loss of generality if \( M_\eta \models \{ \rho \in \omega^2 \) then \( \rho \in \omega^2 \) and \( \rho(n) = i \iff M_\eta \models \rho(n) = i \) when \( n < \omega, i < 2 \).
Let \( h : N \to M \) be the isomorphism from \( N \) onto \( M \). We still use \( B_3 \)!
As \( \bar{a} = \langle \langle a^\ell_\eta : \ell \leq n^* : i < \omega_1 \rangle \in N \rangle \) we can look at \( \bar{a} \) and \( h(\bar{a}) \) as a two-place function (with variables written as superscript and subscript). So we can let \( a^\ell_\eta(\ell \leq n^*, \eta \in \omega^2) \) be
reals such that: \( M_\eta \models \{ h(\bar{a})^\ell_\eta = a^\ell_\eta \} \). By absoluteness \( a^\ell_\eta \in B \) (more exactly \( a^\ell_\eta \in B \) \( \bar{a} = B^\equiv_a, a^\ell_\eta/E^\equiv \in B \) and \( \pi(a^\ell_\eta, a^\ell_\eta : \ell < n^*), B_3, B \) )
If we can prove that \( \{ a^\ell_\eta : \eta \in \omega^2, \ell \leq n^* \} \) is independent over \( B_3 (= h(B_3)) \), then
the proof of [Sh:98, 3.3] finish our case: proving \( B \) is not Whitehead group. But
independence is just a demand on every finite subset. So it is enough to prove
\[ \oplus \text{ if } k < \omega, \eta_0, \ldots, \eta_{k-1} \in \omega^2 \text{ are distinct, then } \{ a^\ell_{\eta_m} : \ell \leq n^*, m \leq k \} \text{ is independent over } B_3. \]
We prove this by induction on \( k \). For \( k = 0 \) this is vacuous, for \( k = 1 \) it is part of
the properties of each \( \{ a^\ell_\eta : \ell \leq n^* \} \). So let us prove it for \( k + 1 \). Remember that
\( \langle t_{\eta_0}(v), \ldots, t_{\eta_k}(v) \rangle \) (more exactly \( \prod_{\ell < k} t_{\eta_\ell}(v) \) ) is a generic subset of \( \Gamma^k \).
Assume the desired conclusion fails. So by absoluteness we can find \( \varphi_\ell(v) \in t_{\eta_\ell}(v) \) and \( s^\ell_m \in \mathbb{Z} \) for \( m \leq k, \ell \leq n^* \) such that:
\[ \ominus \text{ if } t^m_{\eta_m}(v) \subseteq \Gamma \text{ is generic over } B \text{ for } m \leq k, \text{ moreover } \langle t^m_{\eta_m}(v) : m \leq k \rangle \text{ is a generic subset of } \Gamma \text{ over } B \text{ and } \varphi_\ell(v) \in t^m_{\eta_m}(v), \text{ then (defining } M^\ell_{\eta_m} \text{ by } t^m_{\eta_m}(v) \text{ and } a^\ell_{\eta_m} \text{ as before) } \sum_{\ell \leq n^*, m \leq k} s^m_\ell a^\ell_{\eta_m} = t \in B_3. \]
Clearly for \( m \leq k \) we have \( M \models \{ v : M \models \{ \varphi_m(v) \lor v \text{ a countable ordinal} \} \) has order type \( \omega_1^\ast \) and without loss of generality \( M \models \{ v : M \models \{ \neg \varphi_m(v) \lor v \text{ a countable ordinal} \} \) has order type \( \omega_1^\ast \).
So in \( M \) there are \( g_0, \ldots, g_k \in M \) such that: \( M \models \{ g_i \text{ is a permutation of } \omega_1, \text{ for } i \leq k \} \) we have \( (v)(\forall \varphi(v) \leftrightarrow \varphi_0(g_0(v)) \) and \( g_0(v), g_1(v), \ldots, g_k(v) \) are pairwise
distinct\}. Let for \( m \leq k, t^m_{\eta_0}(v) = \{ \varphi(v) \in \Gamma : \varphi(g_i(v)) \in t^m_{\eta_i}(v) \} \). Let in \( M^\eta_{\eta_0}, y^\eta_{\eta_0} = [g_i(y^\eta_{\eta_0})]^{M^m_{\eta_0}}, a^\eta_{\eta_0} \} \text{ is generic over } B \text{ and } \varphi_0(v) \in t^m_{\eta_0}(v), \text{ pairwise } t^m_{\eta_0}(v) \subseteq \Gamma^k + 1 \text{ is generic over } B \)
and \( \varphi(v) \in t^m_{\eta_0}(v), \varphi_0(v) \in t^m_{\eta_0}(v), \ldots, \varphi_k(v) \in t^m_{\eta_0}(v) \). Hence for each \( i \leq k \) in \( B \) we have
\[ \sum_{\ell \leq n^*, m \leq k} s^m_\ell a^\ell_{\eta_0} + \sum_{0 < m \leq k \leq n^*} \sum_{\ell \leq n^*} s^m_\ell a^\ell_{\eta_0} = t \in B_3. \]
By linear algebra \( \{ a^\ell_{\eta_0} : i \leq k, \ell \leq n^* \} \) is not independent (actually, \( i = 0, 1 \) suffices - just subtract the equations). By absoluteness this holds in \( M^\eta_{\eta_0} \). But the
formula saying this is false holds in \( (\mathcal{M}(\chi), \in, <^*) \) hence in \( N \), hence in \( M \), hence in \( M^\eta_{\eta_0} \) (it speaks on \( \bar{a}, B, B_3 \) ), contradiction. So \( \ominus \) fails hence \( \oplus \) holds so we have
finished Possibility I.
Possibility II of [Sh 44, p.250]: In this case we have “not possibility I” but $S = \{\delta < \omega_1 : \delta \text{ a limit ordinal and there are } a^\delta_\ell \text{ for } \ell \leq n_\delta \text{ such that } \pi(a^\delta_n, a^\delta_\ell : \ell < n_\delta, B, B_\delta, B) \}$ is stationary; all in $VP$. Now without loss of generality we can find $\langle a^\delta_n : n < \omega \rangle$ such that: $a^\delta_n < a^\delta_{n+1}, \delta = \bigcup_{n<\omega} a^\delta_n$, and there are $y^\delta_m \in B_{\delta+1}, t^\delta_m \in B_{\alpha^\delta_{n+1}}$ and $s^\delta_{m,\ell} \in \mathbb{Z}$, (for $\ell < n_\delta$) such that:

$\exists(s)_0 y_0^\delta = a^\delta_{n_\delta}$ and

$(s)_2 s^\delta_{m,n_\delta} y^\delta_{m+1} = \sum_{\ell < n^*} s^\delta_{m,\ell} a^\delta_\ell + y^\delta_m + t^\delta_m$

$(s)_3 s^\delta_{m,n_\delta} > 1$, moreover if $s$ is a proper divisor of $s^\delta_{m,n_\delta}$ (e.g. 1) then $s y^\delta_{m+1,n_\delta}$ is not in $B_\delta + \{(a^\delta_\ell : \ell < n_\delta) \cup \{y^\delta_{n_\delta}\}B$

$(s)_4$ if $\alpha \in \delta \setminus \{a^\delta_n : n < \omega \}$ then $PC_B(B_{\alpha+1} \cup \{a^\delta_0, \ldots, a^\delta_{n_\delta}\}) = PC_B(B_{\alpha} \cup \{a^\delta_0, \ldots, a^\delta_{n_\delta}\}) + B_{\alpha+1}$

[why? known, or see later.]

Without loss of generality $\delta \in S \Rightarrow n_\delta = n^*$. So as in the proof of Lemma 1.2 we can choose countable $N < (\mathcal{H}(\chi), \epsilon, <^*)$ such that $\alpha = \langle a^\delta_\ell : \ell \leq n^* : \delta \in S \rangle$, $\alpha = \langle \langle \delta^\alpha_n : n < \omega : \delta \in S \rangle, \langle \langle s^\delta_{m,\ell} : \ell \leq n^* : y^\delta_m, t^\delta_m \rangle : \delta \in S \rangle \rangle$ belongs to $N$, then define $M$ and choose $\mathcal{B}$ as before. We let this time $\Gamma = \Gamma_M$ be as in the proof of Lemma 1.2, that is $\{\varphi(v) : M \models \{\{\delta \in S : \varphi(\delta)\} \} \text{ stationary}\}$. We can find $\langle t_{\eta}(v) : \eta \in \omega_2 \rangle$ such that:

(a) each $t_{\eta}(v) \subseteq \Gamma$ is generic over $\mathcal{B}$ as before hence

(b) for $k < \omega$ and pairwise distinct $\eta_0, \ldots, \eta_{k-1} \in \omega_2, \langle t_{\eta_0}, \ldots, t_{\eta_{k-1}} \rangle$ is generic over $\mathcal{B}$

(c) letting $M_{n^*}, y_{\eta}$ be such that: $M < M_{n^*}, y_{n^*}$ the Skolem hull of $M_{n^*} \cup \{y_{n^*}\}, y_{\eta}$ realizes $t_{\eta}(v)$ in $M_{n^*}$ we have

(i) $M_{n^*} \models \"y_{n^*} \text{ is a countable ordinal } \in S^n\"$

(ii) $M \models \"a \text{ is a countable ordinal}\" \Rightarrow M_{n^*} \models \"a < y_{n^*}\"$

(iii) if $y \in M_{n^*}$ satisfies (i) + (ii) then $M_{n^*} \models \"y_{\eta} < y\"$

So looking at $h : N \rightarrow M$ the isomorphism, then $a^\eta_n = [h(\alpha)]^\eta_n$ for $n < \omega$ satisfies:

$M_{n^*} \models \"a^\eta_n \text{ a countable ordinal}\"$

$M_{n^*} \models \"a^\eta_n < a^\eta_{n+1} < y_{\eta}\"$

$M_{n^*} \models \"[h(\alpha)]^\eta_n \text{ is unbounded below } y_{\eta}\"$

hence $\{a^\eta_n : n < \omega \} \subseteq M$ is unbounded among the countable ordinals of $M$.

Now by easy manipulation (see proof below):

(c) if $\eta_1 \neq \eta_2$ then $\{a^\eta_1 : n < \omega \} \cap \{a^\eta_2 : n < \omega \}$ is finite.
(We can be lazy here demanding just that no \( \{ \alpha_n^\eta : n < \omega \} \) is included in the union of a finite set with the union of finitely many sets of the form \( \{ \alpha_n^\eta : n < \omega \} \) which follows from pairwise generic, and one has to do slightly more abelian group theory work below).

Now we can let \( a_\ell^n = [(h(\bar{a}))_\ell]^{M_n} \). By linear algebra we get the independence hence a contradiction to our being in possibility II (or directly get \( \otimes \) in the proof in the case possibility I holds).

An alternative is the following:

We are assuming that in \( V^P \), possibility I fails. So also in \( V \), letting \( A = M \cap \bar{B^\mathsf{\dagger}} \)

the following set is countable:

\[
K[A] =: \{ \langle a_\ell : \ell \leq n \rangle : n < \omega, a_\ell \in B, (a_\ell : \ell \leq n) \text{ independent over } A \text{ in } B \text{ and } \pi(a_n, \langle a_\ell : \ell < n \rangle_B, A, B) \} \text{ (see proof later).}
\]

For each such \( \bar{a} = \langle a_\ell : \ell \leq n \rangle \) we can look at a relevant type it realizes over \( A \)

\[
t(\bar{a}, A) = \{ (\exists y)(s_y = \sum_{\ell \leq n} s_\ell x_\ell) : B \models (\exists y)(s_y = \sum s_\ell a_\ell), \]

\[
\text{ s, } s_\ell \text{ integers} \}
\]

so \( \{ t(\bar{a}, A) : \bar{a} \in K[A] \} \) is countable. But for the \( \eta \in {}^\omega 2 \) the types \( t(\langle a_\ell^n : \ell < n_\eta \rangle, A) \) are pairwise distinct, contradiction, so actually case II never occurs.

We still have some debts in the treatment of possibility II.

**Why do clauses (b) and (c) hold?** For each \( n \) we let

\[
\Gamma_{M,n} = \left\{ \varphi(v) : \right. \\
(i) \quad \varphi(v) \text{ is a first order formula with parameters from } M \\
(ii) \quad \text{for some } \beta_\ell^\eta \in M \cap \omega_1 \text{ for } \ell < n \text{ we have} \\
\quad \quad \quad \quad M \models "(\forall v)(\varphi(v) \rightarrow v \in h(S)) \& \bigwedge_{\ell < n} (h(\bar{a}))_\ell^\nu = \beta_\ell^\nu" \\
(iii) \quad \quad \quad M \models "(\forall \beta < \omega_1)(\exists v \in \mathbb{N}_1)[(\varphi(v) \& \beta < (h(\bar{a}))_\nu)]" \}
\]

Now note:

\[ \otimes_0 \Gamma_{M,n} \subseteq \Gamma_M \]
\[ \otimes_1 \text{ if } \varphi(v) \in \Gamma_M \text{ and } n < \omega \text{ then for some } m \in [n, \omega) \text{ and } \beta_\ell \in M \cap \omega_1 \text{ for } \ell < m \text{ we have } "\varphi(v) \& \bigwedge_{\ell < m} "(h(\bar{a}))_\ell^\nu = \beta_\ell^\nu" \text{ belongs to } \Gamma_{M,m} \]
\[ \otimes_2 \text{ if } \varphi(v) \in \Gamma_{M,n} \text{ and } \beta \in M \cap \omega_1 \text{ then } \varphi'(v) = \varphi(v) \& \beta < (h(\bar{a}))_\nu \text{ belongs to } \Gamma_{M,n}. \]

Now let \( \langle \mathcal{D}_n : n < \omega \rangle \) be the family of dense open subsets of \( \Gamma_M \) which belong to \( \mathcal{B} \). We choose by induction on \( n, \langle \varphi_\eta(v) : \eta \in {}^n 2 \rangle, k_\eta < \omega \) such that:

\[ (a) \ \varphi_n(v) \in \Gamma_{M,k_\eta} \]
(β) $\varphi_\eta(v) \in D_\ell$ if $\ell < \ell g(\eta)$

(γ) $\varphi_\eta(v) \leq_\Gamma \varphi_{\eta \cdot i}(v)$ for $i = 0, 1$

(δ) if $\eta_0 \neq \eta_1 \in {}^n2, \eta_i < \nu_i \in {}^{n+1}2$ for $i = 0, 1$ and $k_{n_0} \leq k < k_{\nu_0}$ and $M \models (\forall v)(\varphi_{\nu_0}(v) \rightarrow (h(\bar{\alpha}))^v_k = \beta)$ then $M \models (\forall v)[\varphi_{\nu_1}(v) \rightarrow \bigwedge_{\ell < k_{\nu_1}} (h(\bar{\alpha}))^v_\ell \neq \beta]$.

There is no problem to do it and $t_\eta(v) = \{\varphi(v) \in \Gamma_M : \varphi(v) \leq_{r_M} \varphi_{\eta \cap n}(v)$ for some $n < \omega\}$ for $\eta \in {}^{\omega^2}$ are as required.

**Why does $\boxtimes$ hold?**

For $\delta \in S$ let $w_\delta = \{\alpha < \delta : PC_B(B_\alpha + 1 \cup \{a_0^\delta, \ldots, a_n^\delta, B_\alpha + 1 \subseteq B\}).$

Let $S' = \{\delta \in S : (\forall \alpha < \delta)(|w_\delta \cap \alpha| < \aleph_0)\}$, if $S'$ is stationary we get $\boxtimes$, otherwise $S \setminus S'$ is stationary, and for $\delta \in S \setminus S'$ let $\alpha_\delta = \text{Min}\{\alpha : w_\delta \cap \alpha \text{ is infinite}\}$. By Fodor’s lemma for some $\alpha(*) < \omega_1, S'' = \{\delta \in S \setminus S' : \alpha_\delta = \alpha(*)\}$ is stationary hence uncountable and we can get possibility I, contradiction. $\Box$
§3 Refinements

We may wonder if we can weaken the demand “Borel”.

3.1 Definition. 1) We say \( \overline{\psi} \) is a code for a Souslin abelian group if in Definition 0.1 we weaken the demand on \( \psi_0, \psi_1 \) to being a \( \sum_1^1 \) relation.
2) A model \( M \) of a fragment of ZFC is essentially transitive if:

(a) if \( M \models \text{“} x \text{ is an ordinal”} \) and \( \{ \{ y : y <^M x \}, \in^M \} \) is well ordered then \( x \) is an ordinal and \( M \models \text{“} y \in x \iff y \in x \) 
(b) if \( \alpha \) is an ordinal, \( \{ \{ y : y <^M x \}, \in^M \} \) is well ordered and \( M \models \text{“} \alpha \text{ an ordinal, }rk(x) = \alpha^\prime \)”, then \( M \models \text{“} y \in x \iff y \in x \.

3) For \( M \) essentially transitive with standard \( \omega \) such that \( \overline{\psi} \in M \) let \( B^M \) is \( B^{\overline{\psi}} \) as interpreted in \( M \) and \( \text{trans}(M) = \{ x \in M : x \text{ as in (b) of part (2)} \} \).

3.2 Fact. 1) “\( \overline{\psi} \) codes a Souslin abelian group” in a \( \Pi_2 \) property.
2) If \( M \) is a model of a suitable fragment of set theory (comprehension is enough), then \( M \) is isomorphic to an essentially transitive model.
3) If \( M \) is an essentially transitive model with standard \( \omega \) of a suitable fragment of ZFC and \( \overline{\psi} \in M \), (note \( \overline{\psi} \) is really a pair of subsets of \( \mathcal{P}(\mathbb{N}_0) \)), then letting \( B^{\overline{\psi}} = (B^{\overline{\psi}})^M \cap \text{trans}(M) \) there is a homomorphism \( j_M \) from \( B^M \) into \( B = B^{\overline{\psi}} \) such that \( M \models \text{“} t = x/E^{\overline{\psi}} \) implies \( j_M(t) = x/E^\psi \).
4) If \( M \prec N \) are as in (3), then \( j_M \subseteq j_N \).

Proof. Straightforward.

3.3 Claim. 1) In 1.2, 2.1 we can assume that \( B = B^{\overline{\psi}} \) is only Souslin.
2) If \( B = B^{\overline{\psi}} \) is not \( \aleph_2 \)-free, then case I of [Sh 44](3.1) holds, more of the conclusion of case I in the proof of 2.1 holds.

Remark. If only \( \psi_1 \) is Souslin, i.e. is \( \sum_1^1 \), just repeat the proofs.

Proof. For both we imitate the proof of 2.1.

In both possibilities, for each \( \eta \in \text{“} 2 \), let \( G_\eta \) be the group which \( \overline{\psi} \) defines in \( M_\eta \), (the \( M_\eta \)'s chosen as there). So \( j_{M_\eta} \) is a homomorphism from \( G_\eta \) into \( B \). However, \( j_M \subseteq j_{M_\eta} \) and \( j_M \) is one to one. Now in defining \( \pi(x, L, B_3, B) \) we can add that we cannot find \( L' \cup \{ x' \} \subseteq PC(B_3 \cup L \cup \{ x \}) \) such that \( \pi(x', L', B_3, B) \) and \( |L'| < |L| \), i.e. the \( n \) is minimal. As \( B \) is \( \aleph_1 \)-free, this implies that \( j_M \upharpoonright B(\text{PC}(B_3 \cup \{ a_\ell : \ell \leq n^* \})^{M_\eta} \) is one to one and by easy algebraic argument, we can get, for 2.1, non-Whiteheadness and for 1.2, non-\( \aleph_2 \)-freeness. \( \square_3.3 \)

3.4 Fact. 1) “\( B^{\overline{\psi}} \) is non-\( \aleph_2 \)-free” is a \( \sum_1^1 \)-property of \( \overline{\psi} \), assuming \( B^{\overline{\psi}} \) is a \( \aleph_1 \)-free Souslin abelian group.
2) “\( \overline{\psi} \) codes a \( \aleph_1 \)-free Souslin abelian group” is a \( \Pi_1^1 \)-property of \( \overline{\psi} \).

Proof. Just check.
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