Calculus of Variations

Two short closed geodesics on a sphere of odd dimension

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Abstract

We show that for an open and dense set of non-reversible Finsler metrics on a sphere $S^n$ of odd dimension $n = 2m - 1 \geq 3$ there is a second closed geodesic with Morse index $\leq 4(m + 2)(m - 1) + 2$.

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1 Introduction

In this paper we consider the sphere $S^n$ of dimension $n \geq 2$ carrying a non-reversible Finsler metric $f$. Hence the length of a curve in general depends on the orientation. The reversibility $\lambda = \max\{f(-X); f(X) = 1\}$ was introduced in [17]. Then $\lambda \geq 1$ and $\lambda = 1$ if and only if the Finsler metric is reversible, i.e. $f(-X) = f(X)$ for all tangent vectors $X$. For a tangent vector $X \in TS^n$ we denote by $f_0(X) = \sqrt{g_0(X,X)}$ the length of a vector with respect to the standard Riemannian metric $g_0$ of constant sectional curvature 1 on $S^n$. Let $D = D(f)$ be the smallest positive number such that

$$D^{-1}f_0(X) \leq f(X) \leq Df_0(X)$$

holds for all tangent vectors $X$. We call this invariant the distortion of the Finsler metric $f$. Obviously $D^2 \geq \lambda$. Let $L = L(f)$ be the critical value of a generator of the non-trivial homology class $H_{n-1}(\Lambda S^n/S^1; \mathbb{Q}) \cong \mathbb{Q}$ in dimension $(n - 1)$ in the free loop space $\Lambda S^n$. Lyusternik and Fet [12] used an idea by Birkhoff to show the existence of a closed geodesic $c_1$ whose length $l(c_1)$ equals $L$ and whose Morse index satisfies $\text{ind}(c_1) \leq n - 1$. Inequality (1) implies that $2\pi/D \leq L = l(c_1) \leq 2\pi D$. It follows from a result by Fet [6] that there exists a second closed geodesic for a reversible Finsler metric which is bumpy, i.e. all its closed geodesics are non-degenerate.

In this paper we consider the existence of a second closed geodesic for a non-reversible Finsler metric. On a 2-sphere with a bumpy metric there always exists a second closed
geodesic $c_2$ geometrically distinct from $c_1$ as shown in [16, (4.1)]. Bangert and Long were able to show in [2] that this statement holds for any non-reversible Finsler metric. There is a family $f_\mu$, $\mu \in [0, 1)$, $\mu \notin \mathbb{Q}$ of Katok metrics on $S^2$ which are bumpy, have constant flag curvature 1 and carry exactly two geometrically distinct closed geodesics $c_1, c_2$ with $\text{ind}(c_1) = 1$, $\lim_{\mu \to 1} \text{ind}(c_2) = \infty$, $L(c_1) < 2\pi$, and $\lim_{\mu \to 1} L(c_2) = \infty$. Hence there exists in general only one short closed geodesic on $S^2$.

In higher dimensions there are many results on the existence of a second closed geodesic, cf. for example [4, 19, 20], [5, Cor. 1.2], and [1, Cor. 1.14]. Compare also the recent survey [11]. For existence results for closed geodesics in Riemannian and Finsler geometry we also refer to the surveys [13, 22]. Under curvature assumptions one can give bounds for the index of the second closed geodesic, cf. for example [18]. But we are not aware of estimates for the index of the second closed geodesic holding on an open and dense subset of metrics on an $n$-dimensional sphere with $n \geq 3$.

We state our main result which shows in particular that for an odd-dimensional sphere of dimension $n = 2m - 1 \geq 3$ endowed with a bumpy metric there are two geometrically distinct short closed geodesics, with index $\leq 4(m + 2)(m - 1) + 2$. More precisely we show:

**Theorem 1.1** Let $f$ be a non-reversible Finsler metric on the odd-dimensional sphere $S^n$ of dimension $n = 2m - 1 \geq 3$ with distortion $D = D(f)$. Let $p_m$ be the smallest prime number which is neither a divisor of $(m - 1)$ nor of $m$. cf. Lemma 3.3, in particular $3 \leq p_m \leq m + 2$ for all $m \geq 2$. Assume that all closed geodesics with length $\leq L_3 := 2\pi p_m D^3$ are non-degenerate. Then there are two geometrically distinct closed geodesics with index $\leq 4p_m(m - 1) + 2$ and of length $\leq L_3$.

For $n = 3, m = 2, p_3 = 3$ we obtain for the second closed geodesic $c_2$ on $S^3$ : $\text{ind} c_2 \leq 14$. For $n = 6k + 3 = 2m - 1$ resp. $m \equiv 2 \pmod{3}$ we have $p_m = 3$, hence for the second closed geodesic $c_2$ on $S^{2m-1}$ : $\text{ind}(c_2) \leq 12m - 10$. The proof of Theorem 1.1 is given in Sect. 4. We use the computation of the homomorphism in homology induced by the projection of the free loop space $\Lambda S^n$ onto the quotient space $\Lambda S^n/S^1$ as given in Lemma 2.1 for $n = 2m - 1$. An analogous result is not available for even dimension $n$. Recall that $f_0$ is the Finsler metric defined by the standard Riemannian metric of constant sectional curvature 1. There is a one-parameter family $f_\mu$, $\mu \in [0, 1)$ of Finsler metrics on $S^n$ starting at the standard metric $f_0$ with the following properties: For every irrational $\mu$ the metric is non-reversible and bumpy and carries exactly $2m$ geometrically distinct closed geodesics. For $n = 2m - 1$ of these closed geodesics the index is at most $6(m - 1)$ but the index of one of these closed geodesics can be arbitrarily large. This example is explained in detail in Sect. 5, these metrics were first studied by Katok, cf. [23].

The set of metrics satisfying the assumptions of Theorem 1.1 contains an open and dense subset. This follows from the following

**Theorem 1.2** Let $M$ be a compact manifold endowed with a Finsler metric $f_0$. For an arbitrary non-reversible Finsler metric $f$ the distortion $D = D(f)$ is the smallest positive number satisfying Eq. (1) for all tangent vectors. For a positive number $L$ let $\mathcal{F}_1(L)$ be the set of Finsler metrics $f$ on $M$ all of whose closed geodesics of length $\leq D^3(f) L$ are non-degenerate. Then $\mathcal{F}_1(L)$ is an open and dense subset of the space $\mathcal{F}(M)$ of all Finsler metrics on $M$ with respect to the (strong) $C^r$-topology for $r \geq 4$.

We give the proof in Sect. 6. The essential ingredient is the bumpy metrics theorem for Finsler metrics, cf. [21, Thm. 4].

Using Theorem 1.2 we obtain from Theorem 1.1 the following
Corollary 1.3 Let \( p_m \) be the smallest prime number which is neither a divisor of \((m - 1)\) nor of \(m\). Then there is an open and dense subset of non-reversible Finsler metrics on the sphere \( S^n \) of odd dimension \( n = 2m - 1 \geq 3 \) carrying two geometrically distinct closed geodesics with index \( \leq 4p_m(m - 1) + 2 \).

2 Homology of the free loop space

Closed geodesics on \( S^n \) with a Finsler metric \( f \) are the critical points of the functional

\[
F : \Lambda S^n \to \mathbb{R}; \quad F(\sigma) := \left( \int_0^1 f^2(\sigma'(t)) \, dt \right)^{1/2},
\]
cf. [10, Sect. 1] and [16, ch. 1]. We denote by \( \Lambda = \Lambda S^n \) the free loop space, i.e. the space of \( H^1 \)-maps \( \sigma : S^1 = \mathbb{R}/\mathbb{Z} \to S^n \). The function \( F \) is up to a factor \( 1/2 \) the square root of the energy functional \( E(\sigma) = 1/2 \int_0^1 f^2(\sigma'(t)) \, dt \). The functional \( F \) agrees with the length functional \( l(\gamma) = \int_0^1 f(\gamma'(t)) \, dt \) on loops parametrized proportional to arc length. The Morse index \( \text{ind}(c) \) is the maximal dimension of a subspace of the tangent space \( T_c \Lambda S^n \) on which the hessian \( d^2 F \) is negative definite, cf. for example [16, ch. 1]. For a closed geodesic \( c \) the iterations \( c^k, k \geq 1 \) with \( c^k(t) = c(kt) \) are closed geodesics, too. These closed geodesics are geometrically equivalent. Note that in general the curve \( c^{-1} \) with opposite orientation, i.e. \( c^{-1}(t) = c(-t) \), is not a closed geodesic since the metric is assumed to be non-reversible.

For \( f = f_0 \) we use the following notation:

\[
F_0(\sigma) := \left( \int_0^1 f^2_0(\sigma'(t)) \, dt \right)^{1/2}; \quad l_0(\sigma) = \int_0^1 f_0(\sigma'(t)) \, dt.
\]

For the sublevel sets of the functional \( F \) we use the following notation: \( \Lambda^R = \{ \sigma \in \Lambda; F(\sigma) \leq R \} \). The free loop space \( \Lambda \) carries a canonical \( S^1 \)-action by linear reparametrization of the curves, i.e. shift of the initial point. We use the following notation for quotient spaces with respect to the \( S^1 \)-action and its sublevel spaces: \( \overline{\Lambda} = \Lambda/S^1 \) and \( \overline{\Lambda}^R = \{ \sigma \in \overline{\Lambda}; F(\sigma) \leq R \} \). For the sublevel sets with respect to the functional \( F_0 \) we use the following notation: \( \Lambda^R_0 = \{ \sigma \in \overline{\Lambda}; F_0(\sigma) \leq R \} \) and \( \overline{\Lambda}^R_0 = \{ \sigma \in \overline{\Lambda}; F_0(\sigma) \leq R \} \). The set of prime closed geodesics of positive length of the standard metric \( f_0 \) equals the subset \( BS^n \subset \Lambda S^n \) of great circles which can be identified with the unit tangent bundle \( T^1 S^n \).

Then the set of closed geodesics equals the union \( \bigcup_{j \geq 1} B^j \). Here \( B^j := \{ c_j^i, c_0 \in BS^n \} \) is the set of \( j \)-fold covered great circles, i.e. great circles \( c_0 \) parametrized proportional to arc length with \( l_0(c_j^i) = j l_0(c_0) = 2\pi j \). The functional \( F_0 : \Lambda S^n \to \mathbb{R} \) is a Morse-Bott function, i.e. the subsets \( B^j \) are non-degenerate critical submanifolds. This follows since the dimension of the kernel of the hessian of a great circle equals the dimension \( 2n - 1 \) of the manifold \( BS^n = T^1 S^n \). For \( n = 2m - 1 \geq 3 \) we have

\[
H_j \left( T^1 S^{2m-1}; \mathbb{Z} \right) \cong \begin{cases} \mathbb{Z} & j = 0, 2m - 2, 2m - 1, 4m - 3 \\ 0 & \text{otherwise} \end{cases}.
\]

If \( v_k : N_k \to B^k \) is the negative normal bundle of the critical submanifold \( B^k \) of dimension \( \text{ind}(c^k) = (4k - 2)(m - 1) \) with the associated disc bundle \( v_k : DN_k \to B^k \), resp. sphere bundle \( SN_k \to B^k \), then the generalized Morse lemma implies

\[
H_j \left( \Lambda_0^{2\pi k}, \Lambda_0^{2\pi(k-1)}; \mathbb{Z} \right) \cong H_j(DN_k, SN_k; \mathbb{Z}),
\]
This follows from the Gysin sequence of the Beweis Satz 4.9. Hence we obtain:

\[ H_j \left( \Lambda_0^{2\pi k}, \Lambda_0^{2\pi (k-1)}; \mathbb{Z} \right) \cong H_{j-(4k-2)(m-1)} \left( T^1 S^{2m-1}; \mathbb{Z} \right). \] (2)

The functional \( F_0 \) is perfect, i.e.

\[ H_j(\Lambda, \Lambda^0; \mathbb{Z}) \cong \bigoplus_{k \geq 1} H_j \left( \Lambda_0^{2\pi k}, \Lambda_0^{2\pi (k-1)}; \mathbb{Z} \right) \] (3)

which follows for \( m \geq 2 \) from the long exact homology sequence. Hence

\[ H_j (\Lambda, \Lambda^0; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} ; & j = 2r(m-1); r \geq 1 \\ \mathbb{Z} ; & j = 2r(m-1) + 1, r \geq 2 \\ 0 ; & \text{otherwise} \end{cases}, \] (4)

and the homomorphism

\[ H_j \left( \Lambda_0^{2\pi k}, \Lambda_0^0; \mathbb{Q} \right) \rightarrow H_j (\Lambda, \Lambda^0; \mathbb{Q}) \] (5)

induced by the inclusion is an isomorphism for all \( k \geq 1 \) and \( j < i(k+1) = (4k+2)(m-1) \).

This follows since \( i(k+1) = \text{ind}(c_{k+1}^0) = i(k) + 4(m-1) \).

The quotient space \( T^1 S^n / \mathbb{S}^1 \) of unparametrized oriented great circles can be identified with the Grassmannian \( \widetilde{G}(2, 2m-2) \) of oriented two-dimensional linear subspaces of \( \mathbb{R}^{2m} \).

The equivariant Morse Lemma implies

\[ H_j \left( \Lambda_0^{2\pi k}, \Lambda_0^{2\pi (k-1)}; \mathbb{Z} \right) \cong H_j \left( \overline{D \Lambda_0^{k}}, \overline{D \Lambda_0^{k-1}}; \mathbb{Z} \right), \]

cf. [15, Sect. 4]. Here the quotient bundle \( v_k : \overline{D \Lambda_0^k} \rightarrow \overline{B_k} \) resp. \( v_k : \overline{S \Lambda_0^k} \rightarrow \overline{B_k} \) is a bundle with fibre \( D^{(k)}/\mathbb{Z}_k \) resp. \( S^{(k)-1}/\mathbb{Z}_k \). Here \( i(k) = \text{ind}(c_{k}^0) = (4k-2)(m-1) \) is the Morse index of a \( k \)-fold covered great circle \( c_0^k \) as a closed geodesic of the standard metric \( f_0 \).

Then

\[ H_*(D^{(k)}/\mathbb{Z}_k, S^{(k)-1}/\mathbb{Z}_k; \mathbb{Q}) \cong H_*(D^{(k)}, S^{(k)-1}; \mathbb{Q}) \]

and the Thom isomorphism implies

\[ H_*(\Lambda_0^{2\pi k}, \Lambda_0^{2\pi (k-1)}; \mathbb{Q}) \cong H_{*-i(k)} \left( \widetilde{G}(2, 2m-2), \mathbb{Q} \right). \]

Non-trivial homology only occurs in even dimensions since

\[ H_j (\widetilde{G}(2, 2m-2); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} ; & j = 0, 4m-4 \\ \mathbb{Z} \oplus \mathbb{Z} ; & j = 2m-2 \\ 0 ; & \text{otherwise} \end{cases}. \] (6)

This follows from the Gysin sequence of the \( S^1 \)-bundle \( T^1 S^{2m-1} \rightarrow \widetilde{G}(2, 2m) \), cf. [14, Beweis Satz 4.9]. Hence we obtain:

\[ H_* \left( \overline{\Lambda}, \Lambda^0; \mathbb{Q} \right) \cong \bigoplus_{k \geq 1} H_* \left( \Lambda_0^{2\pi k}, \Lambda_0^{2\pi (k-1)}; \mathbb{Q} \right), \]
which implies
\[
H_j \left( \bar{\Lambda}, \bar{\Lambda}^0; \mathbb{Q} \right) \cong \begin{cases} 
\mathbb{Q} & ; j \geq 2(m-1), j \text{ even}, j \neq 2k(m-1), k \geq 2 \\
\mathbb{Q} \oplus \mathbb{Q} & ; j = 2k(m-1), k \geq 2 \\
0 & ; \text{otherwise}
\end{cases}, \quad (7)
\]

[16, Rem. 2.5(a)]. Therefore the functional \( F_0 : \bar{\Lambda} \to \mathbb{R} \) can be seen as a perfect Morse Bott function for rational coefficients, too. In particular the homomorphism
\[
i_* : H_j \left( \bar{\Lambda}^{2\pi k}, \bar{\Lambda}^0; \mathbb{Q} \right) \to H_j \left( \bar{\Lambda}, \bar{\Lambda}^0; \mathbb{Q} \right)
\]
induced by the inclusion is an isomorphism for all \( k \geq 1 \) and \( j < i(k+1) = (4k+2)(m-1) \).

This follows since \( i(k+1) = \text{ind}(c_0^{k+1}) = i(k) + 4(m-1) \).

**Lemma 2.1** \( n = 2m - 1, m \geq 2 \). Let \( a_k \in H_{4k(m-1)}(\Lambda, \Lambda^0; \mathbb{Z}) \cong \mathbb{Z}, k \geq 1 \) be a generator. Then the canonical projection \( \rho : (\Lambda, \Lambda^0) \to (\bar{\Lambda}, \bar{\Lambda}^0) \) induces an injective homomorphism
\[
\rho_* : H_{4k(m-1)}(\Lambda, \Lambda^0; \mathbb{Z}) \cong \mathbb{Z} \to H_{4k(m-1)}(\bar{\Lambda}, \bar{\Lambda}^0; \mathbb{Z})
\]
with \( \rho_*(a_k) = k\tilde{a}_k \neq 0 \) and \( \tilde{a}_k \) is not a torsion element.

**Proof** The projection \( \rho : \Lambda \to \bar{\Lambda} \) induces the homomorphism
\[
\rho_* : H_{4k(m-1)}(\Lambda, \Lambda^0; \mathbb{Z}) \cong \mathbb{Z} \to H_{4k(m-1)}(\bar{\Lambda}, \bar{\Lambda}^0; \mathbb{Z})
\]
The homomorphism
\[
\rho_* : H_{4k(m-1)}(\Lambda^{2\pi k}, \Lambda^{2\pi(k-1)}; \mathbb{Z}) \to H_{4k(m-1)}(\bar{\Lambda}^{2\pi k}, \bar{\Lambda}^{2\pi(k-1)}; \mathbb{Z})
\]
can be expressed by the homomorphism
\[
\rho_* : H_{4k(m-1)}(DN_k, SN_k; \mathbb{Z}) \cong \mathbb{Z} \to H_{4k(m-1)}(\bar{DN}_k, \bar{SN}_k; \mathbb{Z})
\]
which is a multiplication with the number \( k \), i.e. for a generator \( a'_k \) with \( 0 \neq a'_k \in H_{4k(m-1)}(DN_k, SN_k; \mathbb{Z}) \cong \mathbb{Z} \) we have \( \rho_*(a'_k) = ks\tilde{a}_k \) for a generator \( \tilde{a}_k \in H_{4k(m-1)}(\bar{DN}_k, \bar{SN}_k; \mathbb{Z}) \) and an integer \( s > 0 \). This follows since the homomorphism
\[
H_{4k(m-1)}(D^{4k(m-1)}, S^{4k(m-1)-1}; \mathbb{Z}) \cong \mathbb{Z} \to H_{4k(m-1)}(D^{4k(m-1)}/Z_k, S^{4k(m-1)-1}/Z_k; \mathbb{Z}) \cong \mathbb{Z}
\]
induced by the canonical projection is a multiplication by \( k \). This follows since the isometric \( \mathbb{Z}_k \)-action on the disc \( D^{4k(m-1)} \) is free on an open and dense subset, which we see as follows: For any divisor \( d|k, d < k \) we have the following inequality for the indices of coverings \( c_0^k \) of a great circle \( c_0 \) : \( \text{ind}(c_0^d) < \text{ind}(c_0^k) \). Actually one can show \( s = 1 \), i.e. \( \rho_*(a_k) = k\tilde{a}_k \).

This follows from the Gysin sequence of the \( S^1 \)-bundle \( T^1S^{2m-1} \to \bar{G}(2, 2m - 2) \) and Eq. (6).

**Remark 2.2** The \( S^1 \)-action on \( \Lambda \) induces the homomorphism
\[
\Delta : H_{4k(m-1)}(\Lambda, \Lambda^0; \mathbb{Z}) \cong \mathbb{Z} \cdot a_k \to H_{4k(m-1)+1}(\Lambda, \Lambda^0; \mathbb{Z})
\]
cf. [8, (17.1)]. The homomorphism is used to define the *Batalin Vilkovisky algebra*, cf. [3, Thm. 5.4].
It can be expressed as composition $\Delta = \tau \circ \rho_\ast$ of the homomorphism

$$\rho_\ast : H_{4k(m-1)}(\Lambda, \Lambda^0; \mathbb{Z}) \cong \mathbb{Z} \cdot a_k \rightarrow H_{4k(m-1)}(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Z})$$

induced by the canonical projection and the transfer map

$$\tau : H_{4k(m-1)}(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Z}) \rightarrow H_{4k(m-1)+1}(\Lambda, \Lambda^0; \mathbb{Z}) \cong \mathbb{Z} \cdot \tilde{a}_k .$$

Hence we obtain $\Delta(a_k) = \tau(\rho_\ast(a_k)) = k\tilde{s}\tilde{a}_k$ for a positive integer $\tilde{s}$ and a generator $\tilde{a}_k \in H_{4k(m-1)+1}(\Lambda, \Lambda^0; \mathbb{Z})$. The homomorphism can be computed, cf. [14, Satz 4.13] resp. [10, Lem. 6.2], it follows that $\tilde{s} = 2$.

**Remark 2.3** Since $c$ is prime and since for all divisors $q$ of $r$ with $q < r$ the inequality $\text{ind}(c^q) < \text{ind}(c^r)$ holds, we can conclude that for $r \geq 1$ the following holds: There are generators

$$s_r, t_r \in H_\ast(\Lambda^{rL}, \Lambda^{(r-1)L}; \mathbb{Z}) ; \quad s_r \in H_\ast(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}; \mathbb{Z})$$

with $\deg(s_r) = \deg(S_r) = \deg(t_r) - 1 = \text{ind}(c^r) = j$ such that the induced projection

$$\rho_\ast : H_j(\Lambda^{rL}, \Lambda^{(r-1)L}; \mathbb{Z}) \cong \mathbb{Z} \cdot s_r \rightarrow H_j(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}; \mathbb{Z}) \cong \mathbb{Z} \cdot S_r$$

satisfies

$$\rho_\ast(s_r) = r \cdot S_r ,$$

cf. [15, Sect. 3]. This will be crucial in the Proof of Theorem 1.1 given in Sect. 4. For the transfer homomorphism

$$\Delta : H_j(\Lambda^{rL}, \Lambda^{(r-1)L}; \mathbb{Z}) \cong \mathbb{Z} \cdot s_r \rightarrow H_{j+1}(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}; \mathbb{Z}) \cong \mathbb{Z} \cdot t_r$$

one obtains $\Delta(s_r) = r \cdot t_r$.

### 3 Morse theory for a metric with only one closed geodesic

In this section we study non-reversible Finsler metrics on $S^{2m-1}$ for which all closed geodesics with length $\leq 2\pi p_m D^3(f)$ are geometrically equivalent to the closed geodesic $c$ of length $L = l(c)$. We will show that this assumption determines the sequence $\text{ind}(c^r)$, $rL \leq 2\pi p_m D^3$ completely.

**Lemma 3.1** Let $f$ be a non-reversible Finsler metric on the sphere $S^n$, $n \geq 2$ with distortion $D = D(f)$. We assume that all closed geodesics with length $\leq 2\pi D$ are non-degenerate. Then there exists a prime closed geodesic $c$ whose length satisfies $L := l(c) \leq 2\pi D$ and with $\text{ind}(c) \leq n - 1$.

**Proof** Equation (1) implies that $\Lambda^\ast \subset \Lambda^D$. Since

$$H_{n-1}(\Lambda^\ast, \Lambda^0; \mathbb{Q}) \rightarrow H_{n-1}(\Lambda, \Lambda^0; \mathbb{Q}) \cong \mathbb{Q}$$

is an isomorphism, cf. Eq. (5), we conclude that the homomorphism

$$H_{n-1}(\Lambda^D, \Lambda^0; \mathbb{Q}) \rightarrow H_{n-1}(\Lambda, \Lambda^0; \mathbb{Q}) \cong \mathbb{Q}$$

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is surjective, i.e. \( \dim H_{n-1}(\Lambda^{2 \pi D}, \Lambda^0; \mathbb{Q}) \geq 1 \). It follows from the Morse inequalities for the space \( \Lambda^{2 \pi D} \) that there is a closed geodesic \( c \) with length \( l(c) \leq 2 \pi D \) and index \( \text{ind}(c) \leq n - 1 \), cf. [16, Sect. 2].

We later use the following

**Assumption 3.2** For \( m \geq 2 \) let \( p_m \) be the smallest prime number which does not divide \((m - 1)\) nor \( m \), cf. Lemma 3.3. Given a non-reversible Finsler metric \( f \) on a sphere of dimension \( n = 2m - 1 \geq 3 \) with distortion \( D = D(f) \) we assume that all closed geodesics \( \gamma \) with \( L(\gamma) \leq L_3 := 2 \pi p_m D^3 \) are non-degenerate and that all closed geodesics with length \( \leq L_3 = 2 \pi p_m D^3 \) and index \( \leq 4 p_m (m - 1) + 2 \) are geometrically equivalent.

Hence we conclude from Lemma 3.1 that there is a prime closed geodesic \( c \) such that every closed geodesic \( \gamma \) with \( l(\gamma) \leq L_3 = 2 \pi p_m D^3 \) and \( \text{ind}(\gamma) \leq 4 p_m (m - 1) + 2 \) is up to the choice of the initial point a covering of the closed geodesic \( c \), i.e. there is a positive integer \( r \geq 1 \) and an element \( z \in S^1 = \mathbb{R}/\mathbb{Z} = [0, 1]/\{0, 1\} \) such that \( \gamma = z.c^r \). Here, \( z.c(t) = c(t + z) \) defines the canonical \( S^1 \)-action on the free loop space \( \Lambda = \Lambda S^n \) leaving the functional \( F \) invariant.

**Lemma 3.3** For \( m \geq 2 \) denote by \( p_m \) the smallest prime number, which is neither a divisor of \((m - 1)\) nor of \( m \). Then \( p_2 = 3 \), \( p_3 = 5 \), and for \( m \geq 4 \): \( 3 \leq p_m \leq m + 1 \).

**Proof** For \( m \leq 5 \) we have: \( p_2 = 3 \), \( p_3 = 5 \), \( p_4 = 5 \). Assume \( m \geq 5 \). If \( m \equiv 2 \pmod{3} \) then \( p_m = 3 \). If \( m - 2 \neq 2^s \) for some \( s \) choose a prime factor \( q \geq 3 \) of \( m - 2 \geq 4 \). If \( m - 2 = 2^s \), choose a prime factor \( q \geq 3 \) of \( m + 1 \). Then \( p_m \leq q \leq m + 1 \), and hence \( 3 \leq p_m \leq m + 2 \) for all \( m \geq 2 \). \( \square \)

The invariant \( \gamma_c \in \{\pm 1/2, \pm 1\} \) of a prime closed geodesic is defined as follows: \( \gamma_c = \pm 1 \) if and only if \( \text{ind}(c^2) - \text{ind}(c) \) is even and \( \gamma_c > 0 \) if and only if \( \text{ind}(c) \) is even, cf. [16, Def. 1.6].

**Lemma 3.4** Let Assumption 3.2 be satisfied, i.e. there exists a prime closed geodesic \( c \) with \( L = l(c) \leq 2 \pi D \) such that all closed geodesics \( \gamma \) with length \( l(\gamma) \leq 2 \pi p_m D^3 \) and index \( \text{ind}(\gamma) \leq 4 p_m (m - 1) + 2 \) are geometrically equivalent to \( c \), cf. Lemma 3.1.

We use the following notation for Betti numbers of the quotients \( \Lambda^{2 \pi D p_m} \) and \( \Lambda^{2 \pi D^3 p_m} \) of the sublevel sets by the canonical \( S^1 \)-action:

\[
\overline{\beta}_j := \dim H_j \left( \Lambda^{2 \pi D p_m}, \Lambda^0; \mathbb{Q} \right), \quad \overline{\beta}^*_j := \dim H_j \left( \Lambda^{2 \pi D^3 p_m}, \Lambda^0; \mathbb{Q} \right).
\]

Let \( L_1 = 2 \pi p_m D \), \( L_3 = 2 \pi p_m D^3 = L_1 D^2 \) and let

\[
v_j := \# \{1 \leq r \leq L_1/L; \text{ind}(c^r) = j, r \equiv 1 \pmod{2} \text{ or } \gamma_c = \pm 1\}
\]

\[
v_j^* := \# \{1 \leq r \leq L_3/L; \text{ind}(c^r) = j, r \equiv 1 \pmod{2} \text{ or } \gamma_c = \pm 1\}.
\]

Then for all even \( j \leq 4 p_m (m - 1) + 2 \):

\[
\overline{\beta}_j = v_j; \quad \overline{\beta}^*_j = v_j^*
\]

and \( \overline{\beta}_j = \overline{\beta}^*_j = 0 \) for all odd \( j \leq 4 p_m (m - 1) + 2 \).
Proof We conclude from \cite[Sec. 2]{16} and \cite[Def. 1.6]{16} or \cite[Sec. 2]{19}:

Let \( v_j(c') = b_j(\Lambda^L, \Lambda^{r-1}; \; \mathbb{Q}) \). If \( l(c') = rl(c) \leq L_3 \) we conclude from Assumption 3.2 for all \( j \leq 4p_m(m-1) + 2 \) : \( v_j(c') \in \{0, 1\} \) with \( v_j(c') = 1 \) if and only if \( j = \text{ind}(c') \) and \( r \) is odd or \( \text{ind}(c^2) \equiv \text{ind}(c) \pmod{2} \). It follows that for \( 1 \leq r \leq L_3/L : \)

\[
 v_j(c') = 1 \Rightarrow j = \text{ind}(c') \equiv \text{ind}(c) \pmod{2}. \tag{11}
\]

The Morse inequalities for the functional \( F \) on the space \( \Lambda^{L_1} = \Lambda^{L_3} S^n \) resp. \( \Lambda^{L_3} S^n \) give a relation between the number of (homologically visible) critical points \( v_j \), resp. \( v^*_j \) with index \( j \) and length \( l \leq L_1 \), resp. \( \leq L_3 \) with the Betti numbers \( \overline{b}_j \), resp. \( \overline{b}^*_j \). We obtain:

\[
 v_j = \overline{b}_j + q_j + q_{j-1} \quad \text{resp.} \quad v^*_j = \overline{b}^*_j + q^*_j + q^*_{j-1}
\]

for a non-negative sequence \( q_j, j \geq 0 \), resp. \( q^*_j, j \geq 0 \), cf. \cite[Sec. 2]{16}. Equation (11) implies the following for all \( j \leq 4p_m(m-1) + 2, j \equiv 1 \pmod{2} \)

\[
 v_j = v^*_j = 0 \tag{12}
\]

and \( q_j = q^*_j = 0 \) for all \( j \). Here we have used that under the assumptions of the Lemma there is up to geometric equivalence only one closed geodesic of length \( \leq 2\pi p_m D^3 \), and that an iterate \( c' \) can have non-trivial local homology in degree \( j \) only for even \( j \), cf. Eq. (11).

Hence

\[
 v_j = \overline{b}_j; \quad v^*_j = \overline{b}^*_j \tag{13}
\]

for all \( j \leq 4p_m(m-1) + 2 \).

For a topological pair \((X, A)\) with singular homology \( H_j(X, A; \mathbb{Z}) \) with integer coefficients let \( \text{Tor}_j \subset H_j(X, A) \) be the torsion submodule. We denote by \( FH_j(X, A; \mathbb{Z}) = H_j(X, A; \mathbb{Z})/\text{Tor}_j \) the associated free module. Then \( H_j(X, A; \mathbb{Q}) \cong H_j(X, A; \mathbb{Z}) \otimes \mathbb{Q} \cong FH_j(X, A; \mathbb{Z}) \otimes \mathbb{Q} \).

Lemma 3.5 If the Finsler metric \( f \) on \( S^{2m-1} \) satisfies Assumption 3.2 and \( p = p_m \) then the homomorphism

\[
 H_j \left( \Lambda^{2\pi p D}, \Lambda^0; \mathbb{Q} \right) \longrightarrow H_j \left( \Lambda, \Lambda^0; \mathbb{Q} \right)
\]

induced by the inclusion is an isomorphism for all \( j \leq 4p(m-1) + 2 \). Using the notation from Lemma 3.4 we obtain for the Betti numbers \( \overline{b}_j := \dim H_j(\Lambda, \Lambda^0, \mathbb{Q}) \) for all \( j \leq 4p(m-1) + 2 \) :

\[
 \overline{b}_j = \overline{b}_j. \tag{14}
\]

Proof From the definition of the distortion given in Eq. (1) we obtain the following inclusions:

\[
 \Lambda_0^{2\pi p} \subset \Lambda^{2\pi p D} \subset \Lambda_0^{2\pi p D^2} \subset \Lambda^{2\pi p D^3} \tag{15}
\]

and

\[
 \overline{\Lambda}_0^{2\pi p} \subset \overline{\Lambda}^{2\pi p D} \subset \overline{\Lambda}_0^{2\pi p D^2} \subset \overline{\Lambda}^{2\pi p D^3}.
\]

It follows that the composition

\[
 H_j \left( \overline{\Lambda}_0^{2\pi p}, \overline{\Lambda}^0; \mathbb{Q} \right) \longrightarrow H_j \left( \overline{\Lambda}^{2\pi p D}, \overline{\Lambda}^0; \mathbb{Q} \right) \longrightarrow H_j \left( \overline{\Lambda}_0^{2\pi p D^2}, \overline{\Lambda}^0; \mathbb{Q} \right) \cong H_j \left( \Lambda, \Lambda^0; \mathbb{Q} \right) \tag{16}
\]
is an isomorphism for \( j \leq 4p(m - 1) + 2 \), cf. Eq. (7) and the arguments below. Therefore we conclude that the homomorphism

\[
i_{1*} : H_j \left( \overline{\Lambda}^{2\pi p D}, \overline{\Lambda}^0; \mathbb{Q} \right) \longrightarrow H_j \left( \overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Q} \right)
\]

induced by the inclusion is surjective for \( j \leq 4p(m - 1) + 2 \) since \( 4p(m - 1) + 2 < i(p + 1) = (4p + 2)(m - 1) \). From Assumption 3.2 and Lemma 3.4 we conclude

\[
H_j \left( \overline{\Lambda}^{2\pi p D}, \overline{\Lambda}^0; \mathbb{Q} \right) = H_j \left( \overline{\Lambda}^{2\pi p D^3}, \overline{\Lambda}^0; \mathbb{Q} \right) = 0
\]

for all odd \( j \leq 4p(m - 1) + 2 \).

If the homomorphism given in Eq. (17) is not injective for some \( j \leq 4p(m - 1) + 2 \) then there is a non-trivial class

\[
Z \in H_j \left( \overline{\Lambda}^{2\pi p D} S^n, \overline{\Lambda}^0 S^n; \mathbb{Q} \right)
\]

with \( \deg(Z) = j \leq 4p(m - 1) + 2 \) such that \( i_{1*}(Z) = 0 \).

We consider the homomorphisms induced by the respective inclusions

\[
i_{2*} : H_j \left( \overline{\Lambda}^{2\pi p D}, \overline{\Lambda}^0; \mathbb{Q} \right) \longrightarrow H_j \left( \overline{\Lambda}^{2\pi p D^2}, \overline{\Lambda}^0; \mathbb{Q} \right)
\]

\[
i_{3*} : H_j \left( \overline{\Lambda}^{2\pi p D^2}, \overline{\Lambda}^0; \mathbb{Q} \right) \longrightarrow H_j \left( \overline{\Lambda}^{2\pi p D^3}, \overline{\Lambda}^0; \mathbb{Q} \right)
\]

\[
i_{4*} : H_j \left( \overline{\Lambda}^{2\pi p D^3}, \overline{\Lambda}^0; \mathbb{Q} \right) \longrightarrow H_j \left( \overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Q} \right)
\]

Then \( i_{1*} = i_{4*} \circ i_{3*} \circ i_{2*} \). Since the homomorphism

\[
i_{4*} \circ i_{3*} : H_j \left( \overline{\Lambda}^{2\pi p D^2}, \overline{\Lambda}^0; \mathbb{Q} \right) \longrightarrow H_j \left( \overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Q} \right)
\]

is an isomorphism for all \( j \leq 4p(m - 1) + 2 \), cf. Eq. (16), we conclude that \( Z \) lies in the kernel of the homomorphism

\[
i_{3*} \circ i_{2*} : H_j \left( \overline{\Lambda}^{2\pi p D}, \overline{\Lambda}^0; \mathbb{Q} \right) \longrightarrow H_j \left( \overline{\Lambda}^{2\pi p D^3}, \overline{\Lambda}^0; \mathbb{Q} \right),
\]

i.e. \( (i_3 \circ i_2)_* \)(Z) = 0. The exactness of the long homology sequence of the triple \( (\overline{\Lambda}^{2\pi p D^3} S^n, \overline{\Lambda}^{2\pi p D} S^n, \overline{\Lambda}^0 S^n) \) implies that there exists a non-trivial class

\[
Y \in H_{j+1} \left( \overline{\Lambda}^{2\pi p D^3} S^n, \overline{\Lambda}^{2\pi p D} S^n; \mathbb{Q} \right)
\]

with \( \partial_s Y = Z \). Here \( \partial_s \) is the boundary operator of the long homology sequence of the triple. But since \( j \) is even this leads to a contradiction to Eq. (18).

Let \( L = F(c) = l(c) \) be the length of the prime closed geodesic \( c \). Then we obtain for the Betti numbers \( b_j(c') = \text{rk} H_j(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}; \mathbb{Z}) \) of the critical group of \( c' \), \( r \leq L_3/L \):

\[
b_k(c') = \begin{cases} 1 & : k = \text{ind}(c'), r \text{ odd, or } \gamma = 1 \\ 1 & : k = \text{ind}(c') + 1, r \text{ odd, or } \gamma = 1 \\ 0 & : \text{otherwise} \end{cases}
\]
The Betti numbers $\overline{b}_k(c^r) = \text{rk} H_k(\Lambda^{r} L, \Lambda^{(r-1)L}; \mathbb{Z}) = \dim H_k(\Lambda^{r} L, \Lambda^{(r-1)L}; \mathbb{Q})$ of the $S^1$-critical group of $c^r$ for $r \leq L_3/L$:

$$\overline{b}_k(c^r) = \begin{cases} 
1 & ; k = \text{ind}(c^r), \gamma_c = 1 \text{ or } r \text{ odd} \\
0 & ; \text{otherwise} 
\end{cases}.$$ 

The Betti numbers $b_k = \text{rk} H_k(\Lambda S^n, \Lambda^0 S^n; \mathbb{Z}) = \dim H_k(\Lambda S^n, \Lambda^0 S^n; \mathbb{Q})$ are given by

$$b_k = \begin{cases} 
1 & ; k = 2s(m - 1), s \geq 1 \\
1 & k = 2s(m - 1) + 1, s \geq 2 \\
0 & k \text{ otherwise} 
\end{cases}.$$ 

cf. Eq. (4). The Betti numbers $\overline{b}_k = \text{rk} H_k(\Lambda S^{2m-1}, \Lambda^0 S^{2m-1}; \mathbb{Z}) = \dim H_k(\Lambda S^{2m-1}, \Lambda^0 S^{2m-1}; \mathbb{Q})$ of the $S^1$-quotient space are as follows:

$$\overline{b}_k = \begin{cases} 
2 & k = 2s(m - 1), s \geq 2 \\
1 & k \geq 2m - 2, k \text{ even}, k \neq 2s(m - 1), s \geq 2 \\
0 & k \text{ otherwise} 
\end{cases}.$$ 

\text{(21)} 

cf. Eq. (7).

Bott’s formula for the sequence $(\text{ind}(c^r))_{r \geq 1}$ of indices of the iterates $c^r$ implies (cf. for example [19]):

$$\text{ind}(c^r) \geq \text{ind}(c), r \geq 1.$$ 

\text{(22)}

Lemma 3.4, Eqs. (22) and (21) imply that $\text{ind}(c) = n - 1$ and that the sequence $\text{ind}(c^r)$ is monotone increasing, i.e. for all $r \geq 1$:

$$\text{ind}(c^{r+1}) \geq \text{ind}(c^r).$$

\text{(23)}

cf. [20] or [19]. Bott’s formula implies that $v_j > 0$ resp. $v_j^* > 0$ for $j \leq 4p(m - 1) + 2$ holds only for even $j$. Since

$$v_j = v_j^* = \overline{\beta}_j = \overline{\beta}_j^* = 0$$

\text{(24)}

for all odd $j \leq 4p(m - 1) + 2$ the Morse inequalities take the simple form for all $j \leq 4p(m - 1) + 2$, cf. Eq. (13):

$$v_j = \overline{\beta}_j = \overline{\beta}_j; v_j^* = \overline{\beta}_j^*.$$ 

\text{(25)}

If $\gamma_c = 1/2$, i.e. if $\text{ind}c^2 = 2m - 1 = \text{ind}c + 1$, we obtain from Bott’s formula for $\text{ind}(c^r)$ that the sequence $\text{ind}(c^r), r \geq 1$ is strictly monotone increasing. But $\overline{b}_{4m-4} = 2$, cf. Eq. (21). This contradicts Eq. (25). Hence $\gamma_c = 1$, resp. $\text{ind}(c^2) = 2m$. The sequence $\text{ind}(c^r)_{1 \leq r \leq L_1/L}$ is uniquely determined by Eq. (23) and Eq. (25):

$$(\text{ind}(c^r))_{r \geq 1} = (2m - 2, 2m, 2m + 2, \ldots, 4m - 6, 4m - 4, 4m - 4, 4m - 2, \ldots, 6m - 8, 6m - 6, 6m - 6, 6m - 4, \ldots)$$

From Lemma 3.5 and Eq. (21) we conclude

$$\overline{b}_{4p(m - 1)} = \dim H_{4p(m - 1)}\left(\Lambda^{2\pi p D}, \Lambda^0; \mathbb{Q}\right) = \overline{b}_{4p(m - 1)} = 2.$$ 

Therefore we obtain the following, cf. [20, Eq. (13)]:
Lemma 3.6 If Assumption 3.2 holds then for \( p = p_m \) we have
\[
\text{ind} \left( c(2p-1)m \right) = \text{ind} \left( c(2p-1)m+1 \right) = 4p(m - 1). \tag{26}
\]
and \( l(c(2p-1)m+1) = ((2p - 1)m + 1)L \leq 2\pi pD. \)

Lemma 3.7 If Assumption 3.2 holds, \( p = p_m \) then for all \( j \leq 4p(m - 1) + 1 : \)
\[
H_j \left( \Lambda^{2\pi pD^3}, \Lambda^{2\pi pD}; \mathbb{Q} \right) = 0; \quad H_j \left( \prod^{2\pi pD^3}, \Lambda^{2\pi pD}; \mathbb{Q} \right) = 0, \tag{27}
\]
and the homomorphism
\[
H_j \left( \Lambda^{2\pi pD}, \Lambda^0; \mathbb{Q} \right) \rightarrow H_j(\Lambda, \Lambda^0, \mathbb{Q}) \tag{28}
\]
induced by the inclusion is an isomorphism for all \( j \leq 4p(m - 1). \)

Proof Since \( \text{ind}(c^r) \geq 4p(m - 1) + 2 \) for all \( r \geq (2p - 1)m + 2 \) it also follows that for \( j \leq 4p(m - 1) + 1 : v_j = v^*_j \), hence Eq. (27) follows, cf. Lemma 3.6. The inclusion Eq. (15) together with the isomorphism (3) imply that the homomorphism (28) is an isomorphism for \( j \leq 4p(m - 1). \)

\[ \square \]

4 Proof of Theorem 1.1

In this proof we use as coefficient ring for homology the ring \( \mathbb{Z} \) of integers if not otherwise stated. We assume that Assumption 3.2 holds and derive a contradiction. Let \( p = p_m \). Because of the Morse inequalities (25) and Lemma 3.5 we obtain for \( j \leq 4p(m - 1) + 2 : \)
\[
FH_j(\Lambda, \Lambda^0; \mathbb{Z}) \cong FH_j(\Lambda^{2\pi pD^3}, \Lambda^0) \cong \bigoplus_{rL \leq 2\pi pD} FH_j(\Lambda^rL, \Lambda^{(r-1)L}). \]

We have shown that \( \text{ind}(c) = 2m - 2 \), \( \text{ind}(c^2) = 2m \) and \( \text{ind}(c(2p-1)m) = \text{ind}(c(2p-1)m+1) = 4p(m - 1), \) cf. Eq. (26). It also follows that \( ((2p - 1)m + 1)L \leq 2\pi pD \).

Let \( s_{(2p-1)m}, s_{(2p-1)m+1} \) denote generators of the local critical groups, cf. Eq. (8). It follows that
\[
H_{4p(m-1)}(\Lambda^{((2p-1)m+1)L}, \Lambda^{((2p-1)m-1)L})
\]
\[
\cong H_{4p(m-1)}(\Lambda^{((2p-1)m+1)L}, \Lambda^{(2p-1)mL}) \oplus H_{4p(m-1)}(\Lambda^{(2p-1)mL}, \Lambda^{((2p-1)m-1)L})
\]
\[
\cong \mathbb{Z} \cdot s_{(2p-1)m+1} \oplus \mathbb{Z} \cdot s_{(2p-1)m}.
\]

We consider the following commutative diagram, the vertical homomorphisms are induced by inclusions, the horizontal ones by the canonical projection with respect to the \( S^1 \)-action:
\[
\begin{array}{ccc}
H_{4p(m-1)}(\Lambda^{((2p-1)m+1)L}, \Lambda^{((2p-1)m-1)L}) & \xrightarrow{\rho_1} & FH_{4p(m-1)}(\Lambda^{((2p-1)m+1)L}, \Lambda^{((2p-1)m-1)L}) \\
\uparrow h_{1*} & & \uparrow j_1* \cong \\
H_{4p(m-1)}(\Lambda^{((2p-1)m+1)L}, \Lambda^0) & \xrightarrow{\rho_2} & FH_{4p(m-1)}(\Lambda^{((2p-1)m+1)L}, \Lambda^0) \\
\cong h_{2*} & & \cong j_2* \\
H_{4p(m-1)}(\Lambda, \Lambda^0) & \xrightarrow{\rho_{0*}} & FH_{4p(m-1)}(\Lambda, \Lambda^0) \\
\end{array}
\]

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Here we also allow $\alpha = 0$, which implies $\beta = \pm 1$ resp. $\beta = 0$, which implies $\alpha = \pm 1$. By Eq. (10) we have

$$\rho_1(s_{(2p-1)m+1}) = (\alpha = 0), \quad \rho_1(s_{(2p-1)m}) = (\alpha = 1).$$

Since $\bar{s}_{(2p-1)m+1}, \bar{s}'_{(2p-1)m}$ form a basis for $FH_{p(m-1)}(\Lambda, \Lambda^0)$ there are integers $w, z \in \mathbb{Z}$ with

$$\bar{a}_p = w\bar{s}_{(2p-1)m+1} + z\bar{s}'_{(2p-1)m}.$$

We obtain the following explicit description of the last commutative diagram with respect to the given basis elements

$$\begin{array}{c}
\mathbb{Z}s_{(2p-1)m+1} \oplus \mathbb{Z}s_{(2p-1)m} \\
\mathbb{Z}a'_p \\
\mathbb{Z}d_p
\end{array} \xrightarrow{(\alpha, \beta)} \begin{array}{c}
\mathbb{Z}s_{(2p-1)m+1} \oplus \mathbb{Z}s_{(2p-1)m} \\
\mathbb{Z}s'_{(2p-1)m+1} \oplus \mathbb{Z}s'_{(2p-1)m}
\end{array} \xrightarrow{\rho_1} \begin{array}{c}
\mathbb{Z}s_{(2p-1)m+1} \oplus \mathbb{Z}s_{(2p-1)m} \\
\mathbb{Z}s'_{(2p-1)m+1} \oplus \mathbb{Z}s'_{(2p-1)m}
\end{array} \xrightarrow{\rho_2} \begin{array}{c}
\mathbb{Z}s'_{(2p-1)m+1} \oplus \mathbb{Z}s'_{(2p-1)m}
\end{array} \xrightarrow{\rho} \begin{array}{c}
\mathbb{Z}s''_{(2p-1)m+1} \oplus \mathbb{Z}s''_{(2p-1)m}
\end{array} \xrightarrow{\rho} \begin{array}{c}
\mathbb{Z}s''_{(2p-1)m+1} \oplus \mathbb{Z}s''_{(2p-1)m}
\end{array}

We conclude from this diagram

$$\rho_*(a_p) = p\bar{a}_p = p\left(w\bar{s}'_{(2p-1)m+1} + z\bar{s}'_{(2p-1)m}\right).$$
\[ j_{2\star} \rho_{2\star} h_{2\star}^{-1}(a_p) = j_{2\star} \rho_{2\star}(a_p) = j_{2\star} j_{1\star}^{-1} \rho_{1\star} h_{1\star}(a_p) \]
\[ = j_{2\star} j_{1\star}^{-1} \rho_{1\star} (\alpha \cdot s_{(2p-1)m+1} + \beta \cdot s_{(2p-1)m}) \]
\[ = \alpha((2p-1)m+1) \cdot s_{(2p-1)m+1} + \beta(2p-1)m \cdot s_{(2p-1)m}. \]

Since \( s_{(2p-1)m+1} \) and \( s_{(2p-1)m} \) form a basis we obtain:
\[ pw = ((2p-1)m+1)\alpha ; \quad pz = (2p-1)m\beta \]

which is equivalent to
\[ p(2m\alpha - w) = (m-1)\alpha ; \quad p(2m\beta - z) = m\beta. \quad (29) \]

Equation (29) implies that \( p \) is a common divisor of the numbers \( \alpha \) and \( \beta \) since \( p = p_m \) neither divides \( m \) nor \( m-1 \), cf. Lemma 3.3.

But the numbers \( \alpha, \beta \) are by assumption coprime, hence we arrive at a contradiction. Note that this argument is also valid for the cases \( \alpha = 0, \beta = \pm 1 \), resp. \( \alpha = \pm 1, \beta = 0 \).

5 Katok metrics

Choose numbers \( p_1 < \ldots < p_m \) which are relatively prime and let \( p = p_1 \cdots p_m \).

Let
\[ R(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \]

be the rotation in \( \mathbb{R}^2 \) with angle \( \phi \). Let \( \mathbb{R}^{2m} = V_1 \oplus \ldots \oplus V_m \) be an orthogonal decomposition into 2-dimensional subspaces and let \( A(t) \in SO(2m) \) be the 1-parameter family of isometries of \( S^{2m-1} \) with \( A(t)|V_j = R(pt/p_j), \ j = 1, \ldots, m \). This one-parameter group of isometries generates a Killing field \( V \) on \( S^{2m-1} \) with norm
\[ \|V\| = a(p_1, \ldots, p_m) = p \left( \sum_{j=1}^{m} p_j^{-2} \right)^{1/2}. \]

For \( \mu < a(p_1, \ldots, p_m)^{-1} \) we define the Killing field \( V_\mu = \mu V \) with \( \|V_\mu\| < 1 \). Then the sphere bundle determined by \( \{ \xi \in T_1 S^n : \|\xi - V_\mu(x)\| = 1 \} \) determines the unit sphere bundle of a non-reversible Finsler metric \( f_\mu \). These metrics are called Katok metrics, cf. [23, p.139] or Zermelo deformation of the standard metric [7]. For the flow \( \psi_t \) of the Killing field \( V_\mu \) the geodesics of the Finsler metric \( f_\mu \) are of the form \( t \mapsto \psi_t(c(t)) \) for a great circle \( c(t) \) on \( S^n \). For irrational \( \mu \) the Katok metric \( f_\mu \) determined by the Killing field \( V_\mu \) has exactly \( 2m \) closed geodesics \( c_j^\pm \), \( j = 1, \ldots, m \) with \( c_j^\pm(t) = c_j^\pm(-t) \) for all \( t \), cf. [23, p.139] and [7]. These are the great circles invariant under the flow \( \psi_t \). Since \( \mu \) is irrational these metrics are bumpy. As remarked in [7, 17] these metrics have constant flag curvature 1. The lengths \( l(c_j^\pm) \), \( j = 1, \ldots, m \) of the closed geodesics \( c_j^\pm \) are given by
\[ l(c_j^\pm) = \frac{2\pi}{1 \pm \frac{1}{p_j \mu}}, \ j = 1, \ldots, m . \]

The distortion is given by:
\[ D = \lambda = \frac{1}{1 - \max\{\|V_\mu(x)\|, x \in S^n\}} = \frac{1}{1 - \mu a(p_1, \ldots, p_m)}. \]
If we choose \( p_1 = 1 \) then for an arbitrary \( N \in \mathbb{N} \) one can choose \( N < p_2 < \ldots < p_m \) and an irrational \( \mu \) satisfying
\[
\frac{1}{p} \frac{N-1}{N} < \mu < \frac{1}{a(1, p_2, \ldots, p_m)}.
\]
This implies that \( l(c_{j_1}^-) \geq 2\pi N \), \( \text{ind}(c_{j_1}^-) \geq 2N(m-1) \) and the distortion satisfies \( D \geq N \). One can also show that \( l(c_{j_2}^+) < 2\pi \), \( \text{ind}(c_{j_2}^+) \leq 4(m-1) \), \( 1 \leq j \leq m \) and for \( 2 \leq j \leq m \) we obtain \( l(c_{j_1}^-) < 2\pi N/(N-1) \), \( \text{ind}(c_{j_1}^-) \leq 6(m-1) \).

For \( n = 3 \), \( m = 2 \) we obtain for any \( N \) a bumpy Katok metric \( f_\mu \) with exactly four closed geodesics \( c_1^+, c_2^+ \) with the following (in)equalities for the indices resp. lengths of these closed geodesics: \( l(c_1), l(c_2) < 2\pi \), \( \text{ind}(c_1) = 2 \), \( \text{ind}(c_2) = 4 \), \( l(c_2^{-1}) \leq 2\pi N/(N-1) \); \( \text{ind}(c_2^{-1}) \in \{4, 6\} \), and \( l(c_2^{-1}) \geq 2\pi N \), \( \text{ind}(c_2^{-1}) \geq 2N(m-1) \).

In a certain sense one can say that these examples show that the minimal number of short closed geodesics on a sphere of dimension \( n = 2m-1 \) resp. \( n = 2m \) is \( 2m-1 \). Here short closed geodesics possess an a priori bound for the index.

6 Genericity statement

The set \( \mathcal{F}(T) \) of Finsler metrics on a compact manifold \( M \) for which all closed geodesics of length \( \leq T \) are non-degenerate is an open and dense subset of the space \( \mathcal{F} = \mathcal{F}(M) \) of Finsler metrics on \( M \) with the strong \( C^r \)-topology for \( r \geq 4 \), cf.[21, Thm. 4].

Proof of Theorem 1.2 Let \( f_1 \in \mathcal{F}_1(L) \), hence by definition all closed geodesics of the Finsler metric \( f_1 \) of length \( \leq D^3(f_1) L \) are non-degenerate. Let \( \phi_1 : TM \rightarrow TM \) be the geodesic flow of the Finsler metric \( f_1 \). If \( \tau : TM \rightarrow M \) is the tangent bundle projection then \( \tau(\phi_1(v)) \) is the geodesic \( c_v \) determined by the initial condition \( c_v(0) = v \). Let \( HM = (TM - M)/\mathbb{R}^+ \) be the bundle of oriented directions in the tangent bundle \( TM \). We consider instead of the geodesic flow \( \phi_1 : TM \rightarrow TM \) the map \( \Phi_1 : HM \rightarrow HM \) with \( \Phi_1(v) = \phi_1(v/\|v\|) \), hence the geodesic \( t \in [0, 1] \mapsto \tau(\Phi_1(v)) \in M \) is parametrized by arc length. If \( \Phi_1(v) \) is a periodic flow line of period \( a \), then \( t \in [0, a] \mapsto \tau(\Phi_1(v)) \) is a closed geodesic of length \( a \). The minimal period is then the length of the underlying prime closed geodesic.

Let \( \Phi_i(v_1), \ldots, \Phi_i(v_N) \) be the periodic flow lines of the geodesic flow \( \Phi_1 : HM \rightarrow HM \) corresponding to the closed geodesics of period (resp. length) \( a_1, \ldots, a_N \) which satisfy \( a_i \leq D(f_1)^3 L \).

Then there is an open neighborhood \( U \subset \mathcal{F} \) of \( f_1 \) such that the following holds: There are continuous maps \( v_i : f \in U \mapsto v_i(f) \in HS^n \), \( a_i : f \in U \mapsto a_i(f) \in (0, \infty), i = 1, 2, \ldots, N \) with \( v_i = v_i(f_1), a_i = a_i(f_1), i = 1, \ldots, N \) such that for all \( f \in U \) the sets \( \Phi_i(v_i(f)): t \in [0, a_i(f)], i = 1, 2, \ldots, N \) are periodic and non-degenerate flow lines of the geodesic flow of \( f \) of period \( a_1(f), \ldots, a_N(f) \) and there are no further periodic flow lines of \( f \) of length \( \leq D^3(f) L \). This holds since the distortion
\[
f \in \mathcal{F} \mapsto D(f) \in (0, \infty)
\]
is a continuous function. Hence the set \( \mathcal{F}_1(L) \) is an open subset of \( \mathcal{F} \).

Choose \( T = 2D^3(f) L \). Since \( \mathcal{F}(T) \) is a dense subset of \( \mathcal{F} \) we find a sequence \( (f_k)_{k \geq 2} \subset \mathcal{F}(T) \) converging to \( f_1 \). Since the function given in Eq. (30) is continuous it follows that also \( \mathcal{F}_1(L) \) is dense in \( \mathcal{F} \).

\( \square \)
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