The central charge $C_T$ is computed for scalar and Dirac fields propagating according to GJMS-type kinetic operators acting on odd $d$-dimensional spheres in the presence of a spherical monodromy. The relation of $C_T$ to the derivatives of the free energy on the conically deformed sphere via the Perlmutter factor leads to a numerical quadrature. The variation of $C_T$ with the monodromy flux, $\delta$, displays sign changes, exactly as in even dimensions. Closed forms for $C_T$ are derived when $\delta$ equals 0 or 1/2 with the derivative order either even or odd and shown to agree with existing, even $d$ expressions. The infinite $d$ limits are also derived in these special cases.

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1. Introduction.

This is a further installment of work on the effect of a codimension-2 spherical monodromy defect on certain aspects of free fields propagating on spherical space–times. The idea is to push the analysis as explicitly as possible in this simplified system. Earlier calculations on odd spheres, [1,2], are now extended to Dirac fields and to GJMS-type kinetic operators. The even–dimensional case is discussed in [3,4]. Several of the previous results are subsumed in the ones to follow. For brevity, only the $C_T$ charge is considered as the relevant ‘physical’ quantity of interest.

The treatment is partly numerical, and partly algebraic concentrating on the mechanics of evaluation especially of the closed forms for special values of the flux, $\delta$, and derivative order, $2k$.

The general setup is described in [5] where relevant references can be found e.g. [6].

2. The effective action

The basic quantity is the effective action and, as in [2], the global, spectral approach is taken. The necessary continuation of the $\zeta$–function, $\zeta(s)$, is carried out using the Bessel transform method of [7] as described for the present geometry in [8] where eqn. (11) provides an expression for $\zeta'(0)$ of sufficient generality for present application. For convenience I copy it here,

$$
\zeta'(0) = 2 \int_C \frac{dz \cosh 2(\omega - a)z \cosh 2\alpha z}{z \prod_{i=1}^{d} 2 \sinh(\omega_i z)} = \frac{1}{2^{d-2}} \int_0^\infty dx \Re \frac{\cosh 2(\omega - a)z \cosh 2\alpha z}{z \prod_{i=1}^{d} \sinh(\omega_i z)}, \quad z = x + iy.
$$

(1)

The contour $C$ runs just above the real $z$ axis i.e. $-\infty + i y \to \infty + iy$ with $0 < y < \pi/\max\{\omega_i\}$. The $\omega_i$ are parameters occurring in the eigenvalues, in particular for the orbifolded sphere, $S^d/\mathbb{Z}_q$, $\{\omega_i\} = (q, \mathbf{1}_{d-1})$ and $2\omega \equiv \sum_i \omega_i = d - 1 + q$. The eigenvalue structure is summarised in [4] and shows that, for scalars, $\omega - a = \pm q(1/2 - \delta)$. For GJMS operators, $\alpha = j + 1/2, j = 0, 1, \ldots k - 1$ and a sum over $j$ is to be performed, which is trivial to carry out using (1), being geometric.

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2 It might be argued that the fields are not technically free as they can be considered to be coupled to a non–dynamical ‘external’ gauge field which provides the monodromy.
Making these substitutions, the higher derivative scalar effective action can be written in the calculable form,

\[
A(d, q, \delta, k) = -\frac{1}{2^{d-1}} \int_0^\infty dx \, \text{Re} \, \frac{\cosh q (1 - 2\delta) z \, \sinh 2kz}{z \sinh^d z \, \sinh qz}.
\]  

(2)

For conventional scalar fields, \( k \) is integral. Furthermore, there is no need to develop the corresponding Dirac expression from first principles. It is sufficient, in order to get the effective action (per field component), to set \( k = 1/2 \), \( \delta \to 1/2 + \delta \) and to change the overall sign.

However, one might want to extend \( k \) to the reals. In this case, for notational smoothness, and to avoid confusion, I will adopt the convention that all fields are scalar. If Dirac quantities, with a flux \( \delta \), are needed then one would just compute \(^3\) \(-A(d, q, 1/2 + \delta, k)\) for the chosen \( k \) with \( k = 1/2 \) giving standard Dirac, \( k = 3/2 \), cubic Dirac and so on. The other usual choice, \( k = 1, 2, \ldots \), would correspond to non–standard Dirac. The opposite is true for scalars. I note that \( A(d, q, \delta, k) \) is an odd function of \( k \).

For convergence, \( k \) is restricted by \(|k| < d/2 + q\delta\) (with \( 0 \leq \delta \leq 1/2 \)). The GJMS existence condition is \(|k| < d/2\). In terms of the conformal weight, usually denoted by \( \Delta, = d/2 - k \), this reads, \( d > \Delta > 0 \). The CFT unitarity bound is \( \Delta \geq d/2 - 1 \) which is therefore violated for \( k \) in the (allowed), positive range \( 1 < k < d/2 \), as is well known, e.g. [9,10]. Because of the monodromy, the integral converges for values of \( k \) that exceed the GJMS bound.

The GJMS operators can be continued beyond \( k = d/2 \) in odd dimensions but the effective action then acquires an imaginary part determined by the number of negative modes that appear as \( k \) is increased.

This was explicitly analysed on the round sphere in [11] for integer \( k \) and \( \delta = 0 \), using an alternative representation of the effective action, \( A \), as an integral over a hyperbolic Plancherel measure in \((d+1)\) dimensions which is more convenient than (2), but applies only for \( q = 1 \). This round sphere representation was developed in [5], with a monodromy, and used in a companion work, [1], to the present report.

\(^3\) This means that if a Dirac quantity is quoted, a minus sign has been introduced by hand.
3. Central charge

The two–point stress-energy tensor central charge, $C_T$, can be determined conveniently using the Perlmutter multiplying factor, $[12,13]$, from the derivatives of the effective action with respect to the deformation parameter, $q$, evaluated at the round sphere, $q = 1$. These are straightforwardly found from the representation, (2).

Obtained purely numerically, the figure below plots the variation of $C_T(d,\delta,k)$ with $\delta$ in 5 dimensions for three values of the $k$ parameter and shows the same behaviour as in even dimensions in that two real roots are converted to complex conjugate ones (real part 1/2) as $k$ passes through a certain value, here $k \approx 1.2981146$.

![Variation of scalar central charge with monodromy flux for three values of the higher derivative parameter, $k$, in dimension 5.](image)

The apparent triple point is only approximately so.

To elucidate this a little further, Fig.2 below shows how the the central charge, for $d = 7$, evaluated at the midpoint $\delta = 1/2$, (i.e. the $\mathbb{Z}_2$ monodromy), varies with the higher derivative parameter, $k$. It exhibits two zeros at which $C_T$ develops complex roots as a function of $\delta$. The values are $k \approx 1.194918$ and $k \approx 2.446733$.

The higher the dimension $d$, the more such zeros there are, exactly as in even dimensions. In the latter case, however, all expressions are polynomial and the number of roots can be determined from the polynomial degrees. For odd dimensions,

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4 Comments on the validity of this factor can be found in section 9.
5 Compare [14] for a hyperbolic approach.
6 For example, $C_T(7, 1/2 \pm i\gamma, 3)$ vanishes for $\gamma \approx 0.968423$ and $\gamma \approx 4.66108$, while $C_T(7, 1/2 \pm i\gamma, 2)$ only for $\gamma \approx 2.61337$. 

3
the expressions are transcendental and not so obviously analysed.

![Graph](image)

**Fig. 2.** Variation of the $\mathbb{Z}_2$ monodromy scalar central charge, $C_T(7, 1/2, k)$, with higher derivative order, $k$, for dimension 7 showing two zeros.

4. Special fluxes. Closed forms

As in even dimensions, a check of the numerical work can be made by explicit analysis at special values of the flux, $\delta$. In particular, of course, at $\delta = 0$ but also at the $\mathbb{Z}_2$ value $\delta = 1/2$ (zero Dirac flux). The arising integrals produce closed forms. This is possible generally in terms of polygamma functions, but the expressions are not especially enlightening and so I restrict these particular evaluations, in section 5, to the cases when $k$ is either an integer or a half–integer.

Numerical integration confirms at $\delta = 0$ and $k = 1$ that $C_T = 2d/(d – 1)$, the known value for ordinary complex scalars. The corresponding value for a higher, $2k$, derivative scalar was given by Osborn and Stergiou [9] in even dimensions using a local CFT approach based on the flat space four point function (see also Gliozzi et al, [10]) and is,

$$
C_T(d, 0, k) = (-1)^{k+1} \frac{8k(d/2 + k)! (d/2 - k)!}{(d + 2)(d - 1)(d/2 - 1)!^2}, \quad k \in \mathbb{Z}.
$$

(3)

The same formula was derived in [15] on the basis of a *global* spectral argument (as here) on even–dimensional spheres using properties of generalised Bernoulli polynomials and the Perlmutter factor.

I likewise also found the corresponding spinor general expression to be, per
component, in even dimensions,\(^7\)

\[-C_T(d, 1/2, k) = (-1)^l \frac{(8l(l + 1) - (d + 2)(d - 1))(d/2 + l)!(d/2 - l - 1)!}{(d + 2)(d - 1)((d/2 - 1)!)^2}, \; l \in \mathbb{Z}.
\]

(4)

For notational uniformity the half–integer (for spinors) \(k, = \frac{l}{2} + \frac{1}{2}\), is used as the argument. The value \(l = 0\) gives conventional Dirac, \(l = 1\) cubic Dirac, etc.

5. Equivalence of odd and even dimensions

The expression for \(C_T\) for scalars on an odd–dimensional sphere (with zero \(\delta\) and \(k\) an integer) as it emerges from the calculation of the derivatives of the free energy, (2), using the basic integral,

\[\int_0^\infty dx \frac{\cosh(2ux)}{\cosh^{2n} x} = 2^{2n-2} \frac{\Gamma(n + u)\Gamma(n - u)}{\Gamma(2n)} = 2^{2n-2} B(n + u, n - u), \quad (5)\]

is,

\[C_T(d, 0, k) = (-1)^k k \frac{2^{2d+1}(d + 1)! ((d - 1)/2)!^2 \Gamma(d/2 + k + 1)\Gamma(d/2 - k + 1)}{\pi (d - 1)!^2 \Gamma(d + 3)}. \quad (6)\]

In order to confirm, for a given value of \(k\), that this is the same function of \(d\) that arises in even dimensions, (3), the expression has to be rewritten so as to be valid for even \(d\). This means that the factor of \(\pi\) has to be eliminated. (An evaluation for any specific \(k\) could be made but a more general demonstration is preferable.)

It is convenient to set \(d = 2e + 1\) and (6) is

\[C_T(d, 0, k) = (-1)^k k \frac{2^{4e+3}(2e + 2)! e^2 \Gamma(2e + 3/2)\Gamma(2e - k + 3/2)}{\pi (2e)!^2 \Gamma(2e + 4)}, \quad k \in \mathbb{Z}. \quad (7)\]

Using the double factorial, this can be written,

\[C_T(d, 0, k) = (-1)^k k \frac{(2e + 2)! (2e + 2k + 1)!! (2e - 2k + 1)!!}{e (2e + 3)! (2e - 1)!!^2}
\]

\[= (-1)^k k \frac{1}{e (2e + 3) (2e - 1)(2e - 3)\ldots (2e - 2k + 3)} \frac{(2e + 2k + 1)\ldots (2e + 3)(2e + 1)}{(2e + 3)\ldots (2e + 1)}
\]

\[= (-1)^k k \frac{2k}{(d - 1)(d + 2)} \frac{(d + 2k)\ldots (d + 2)d}{(d - 2)(d - 4)\ldots (d - 2k + 2)}, \quad k > 0, \quad (8)\]

\(^7\) I have not seen this elsewhere.
which can now be extended to all $d$, in particular agreeing with the even $d$ formula, (3). $C_T$ has poles at those even dimensions, $d$, that violate the GJMS existence condition.

A similar analysis holds when $\delta = 1/2$ and $k$ is a half-integer, corresponding to conventional and higher derivative Dirac fields. Some of the details are sketched in order to bring out a curious parallel between the odd and even $d$ intermediate algebraic manipulations.

Taking the real part of the integrand that results from the application of the derivatives with respect to $q$ to the effective action, (2), at $q = 1$ and with $k$ half–integral, yields two terms in the numerator which, up to overall factors, has the schematic general form $\cosh \cosh - 3\cosh$. Expansion of the product gives,

$$\frac{1}{2} \cosh + \frac{1}{2} \cosh - 3 \cosh$$

Integration, using the integral, (5), yields,

$$\frac{1}{2} B(\delta + 3 + k, \delta + 1 - k) + \frac{1}{2} B(\delta + 3 - k, \delta + 1 + k) - 3B(\delta + 2 + k, \delta + 2 - k)$$

which is to be compared with the derivatives of the Bernoulli polynomials that arise in the even $d$ case. This gives equn.(33) in [15], which is, schematically,

$$\frac{1}{2} B(\delta) + \frac{1}{2} B(\delta) + 3B(\delta)$$.  

Continuing with the present calculation, a translation of the arguments in (9) produces, on multiplying by the Perlmutter, and the other dropped factors,

$$C_T(d, 1/2, k) = (-1)^{l+1} \frac{1}{2^3} \frac{(1 + 2e)(1 + e) - 4k^2}{e(2e + 3)((1 + e)^2 - k^2)} \frac{(2e + 2 + 2k)!!}{(2e - 1)!!(2e - 1)!!}$$

$$= (-1)^{l+1} \frac{1}{2^3} \frac{e(2e + 3)(1 + 2e)(1 + e) - 4k^2}{e(2e + 3)((1 + e)^2 - k^2)} \frac{(2e + 2 + 2k)\ldots(2e + 1)}{(2e - 1)\ldots(2e + 4 - 2k)}$$

In this equation, $k = l + 1/2$, with $l = 0, 1, 2, \ldots$. It is antisymmetric in $k$ from the $(-1)^l$ factor.

The final transformation in (10) is valid only for $k \geq 5/2$. The values $k = 1/2$ and $k = 3/2$ should be treated separately. For ordinary Dirac, $-C_T(d, 1/2, 1/2) = d/2$ while for cubic Dirac,

$$-C_T(d, 1/2, 3/2) = -\frac{d(d^2 + d - 18)}{2(d - 2)(d - 1)}$$

holding for odd and even $d$. Note the pole at $d = 2$.

For any $l$, the infinite $d$ limit is $C_T \rightarrow (-1)^l \frac{d}{2}$. Formally summed over all $l$ then produces $d/4$.  

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8 I note that Mathematica does not completely simplify the sum over $l$ of the full spinor expressions (4) and (10) as it does for the scalar case, [10].

6
I now turn to the $\mathbb{Z}_2$ case for $k$ integral, for example standard scalar fields at the midpoint. The algebra is a little more extensive. Some intermediate algebraic details are given which might prove of value.

The $k = 1$ value was found in [2] and, similar to there, two contributions arise which here come from taking the real part of the integrand in (2). (At $q = 1$, the choice $y = \pi/2$ is most convenient.)

The first contribution is,

$$(-1)^{e+1} (-1)^k 2^{3-d} \int_0^\infty dx \frac{\sinh x \sinh 2kx}{\cosh^{d+2} x} \Gamma(e + 2 + k) \Gamma(e + 1 - k) - (k \to -k)$$

$$= (-1)^{e+1} (-1)^k 2^2 \frac{\Gamma(e + 2 + k) \Gamma(e + 1 - k) - (k \to -k)}{\Gamma(2e + 3)}$$

$$= (-1)^{e+1} (-1)^k 2^3 \frac{-k}{(e + 1)^2 - k^2} \frac{\Gamma(e + 2 + k) \Gamma((e + 2 - k)}{\Gamma(2e + 3)},$$

and the second,

$$(-1)^{e+1} (-1)^k 2^{2-d+1} \int_0^\infty dx \frac{\cosh 2x - 3 \sinh 2kx}{\cosh^{d+3} x} \cosh 2x - 3 \cosh 2kx$$

$$= (-1)^{e+1} (-1)^k 2^{-d} \frac{\partial}{\partial k} \int_0^\infty dx \frac{\cosh 2x - 3 \cosh 2kx}{\cosh^{d+3} x}$$

$$= (-1)^{e+1} (-1)^k \frac{\partial}{\partial k} \left( \frac{\Gamma(e + 3 + k) \Gamma(e + 1 - k) + (k \to -k)}{\Gamma(2e + 4)} - \frac{6 \Gamma(e + 2 + k) \Gamma(e + 2 - k)}{\Gamma(2e + 4)} \right)$$

$$= (-1)^{e+1} (-1)^k 2 \left( \frac{2(1 + e)(3 + 2e)k}{(1 + e)^2 - k^2} - \frac{(1 + e)(1 + 2e) - 4k^2}{(1 + e)^2 - k^2} \Delta \psi(e, k) \right) \times$$

$$\frac{\Gamma(e + 2 + k) \Gamma(e + 2 - k)}{\Gamma(2e + 4)}.$$  \hfill (12)

The difference of the $\psi$–functions, $\Delta \psi(e, k)$, can be written in terms of harmonic numbers as,

$$\Delta \psi(e, k) \equiv \psi(e + 2 + k) - \psi(e + 2 - k)$$

$$= H_{e+1+k} - H_{e+1-k} = \sum_{j=1}^{2k} \frac{1}{e + 1 + k - j}.$$  \hfill (13)

Adding (11) and (12) gives an expression of the form,

$$(-1)^{e+1} (-1)^k \frac{P_1(e, k)}{Q_1(e, k)} \frac{\Gamma(e + 2 + k) \Gamma(e + 2 - k)}{\Gamma(2e + 4)},$$  \hfill (13)
where,
\[
P_1(e, k) = 2k(3 + 2e)((e + 1)(2e + 1) - 2k^2) + 
+ ((1 + e)^2 - k^2)((e + 1)(2e + 1) - 4k^2)\Delta\psi(e, k),
\]
and
\[
Q_1(e, k) = ((1 + e)^2 - k^2)^2.
\]
For later use it is convenient to translate the arguments of the $\Gamma$–functions by,
\[
\frac{\Gamma(e + 2 + k)\Gamma(e + 2 - k)}{\Gamma(2e + 4)} = \frac{(e + 1 + k)\ldots(e + 1)}{e\ldots(e + 2 - k)} \frac{1}{(2e + 3)(2e + 2)(2e + 1)} \frac{\Gamma(e + 1)^2}{\Gamma(2e + 1)},
\]
so that (13) reads,
\[
(-1)^{e+1}(-1)^k \frac{P_2(e, k)\Gamma(e + 1)^2}{Q_2(e, k)\Gamma(2e + 1)} \quad (14)
\]
the reason being that there is still the multiplying Perlmutter factor which is of similar form,
\[
Pf(e) = 2^{4e+1} \frac{(-1)^{e+1}(e + 1)(2e + 1)\Gamma(e + 1)^2}{\pi^2 e \Gamma(2e + 1)}.
\]
Finally, therefore, the $\mathbb{Z}_2 C_T$ can be written,
\[
C_T(d, 1/2, k) = \frac{(-1)^k P_3(e, k)\Gamma(e + 1)^4}{\pi^2 Q_3(e, k)\Gamma(2e + 1)^2}, \quad k \in \mathbb{Z},
\]
where $P_3$ and $Q_3$ are explicit functions of $e$ (i.e. $d$) and $k$ obtained by combining the factors in the previous equations, a process best left to the machine. For given values of $k$ and $d$, the result is a rational multiple of $1/\pi^2$, a sample value being,
\[
C_T(19, 1/2, 3) = -\frac{5773509787648}{43616234025\pi^2} \approx -13.411952.
\]
The infinite $d$ limit can be found very simply by inspection using,
\[
\frac{\Gamma(1 + e)^4}{\Gamma(1 + 2e)^2} \to 2^{-4e}\pi e, \quad \text{and} \quad H_{e+1+k} - H_{e+1-k} \to \frac{2k}{e}.
\]
The Perlmutter multiplier gives an extra $2^{4e+2}e/\pi^2$. Also,
\[
\frac{P_1(e, k)}{Q_1(e, k)} \to \frac{12k}{e}.
\]
and
\[
\frac{(e+1+k)\ldots(e+1)}{e\ldots(e+2-k)} \cdot \frac{1}{(2e+3)(2e+2)(2e+1)} \to \frac{1}{8e}.
\]

Putting these factors together, the limit is a constant,
\[
C_T(d, 1/2, k) \to (-1)^k \frac{6k}{\pi} \quad \text{as} \quad d \to \infty, \quad k \in \mathbb{Z}.
\]  

For numerical comparison, \(C_T(10^6+1, 1/2, 10) \approx 19.102391\) and \(60/\pi \approx 19.098593.\)

Formally summing (16) over all integer \(k \geq 0\) gives a negative value, \(-3/2\pi\).

The final special values are zero scalar flux, \(\delta = 0,\) and \(k\) a half–integer, \(k = l + 1/2.\) The calculation could apply either to a scalar field, with zero flux, propagating according to non–standard (pseudo) operators or, with a sign change, to a standard Dirac field with flux 1/2.

The algebra is slightly more tractable and leads to the integral,
\[
\left(1 - \frac{1}{2} \frac{\partial}{\partial l}\right) \int_0^\infty dx \frac{\cosh 2lx}{\cosh^{2e+4} x} + (l \to -1 - l),
\]
which, after a little work, reduces to,
\[
2^{2e+2}(e+1+l)!(e-l)!
(2e+3)!
\left(2e + 2 - (l + \frac{1}{2}) (H_{e+1+l} - H_{e-l})\right), \tag{17}
\]
again involving harmonic numbers.

Translating the prefactor gives,
\[
\frac{(e+1+l)!(e-l)!}{(2e+3)!} = \frac{1}{(2e+3)(2e+2)(2e+1)} \frac{(e+1+l)\ldots(e+1)}{e\ldots(e-1+l)} \frac{e!^2}{(2e)!}. \tag{18}
\]

Combining (17), (18) and the Perlmutter factor, (15), again produces the general form,
\[
C_T(d, 0, k) = \frac{(-1)^l}{\pi^2} \frac{P_4(e, l)}{Q_4(e, l) (2e)!^2}, \quad k = l + \frac{1}{2}, \quad l \in \mathbb{Z},
\]
with explicit polynomials, \(P_4\) and \(Q_4,\) which it is not necessary to write out. A typical value is,
\[
C_T(11, 0, 3/2) = \frac{15895712}{165375\pi^2} \approx -9.7168544.
\]

As before, it is a simple matter to take the infinite \(d\) limit. This time one finds a dimension dependence,
\[
C_T(d, 0, k) \to (-1)^l \frac{4d}{\pi}, \quad d \to \infty, \quad k = l + \frac{1}{2}.
\]
Summing over all \(l,\) this gives \(2d/\pi.\)

I remark that, unlike a previous case, the term involving harmonic numbers, in (17), contributes only at a sub–leading level.

\[9\) The rational factor has over 600,000 digits in both numerator and denominator.\]
6. Going up a dimension. A resolvent

The next three sections contain brief remarks on the general formalism of this report outside the main computation.

Aside from its higher derivative and conformal weight interpretations, $k$ can also be thought of as a resolvent parameter in one higher dimension. This is most clear in the alternative representation which uses Barnes $\zeta$–functions as described in [16,11] on the full sphere but can be brought out here by differentiating the effective action, $\mathcal{A}$, with respect to $k$. On the round sphere this gives, $(\delta' \equiv \delta - 1/2)$,

$$\frac{\partial \mathcal{A}}{\partial k} = \cos \pi (k - \delta') \frac{\Gamma \left( \frac{d+1}{2} + k + \delta' \right) \Gamma \left( \frac{d+1}{2} - k - \delta' \right)}{\Gamma(d+1)} + (k \rightarrow -k).$$ (19)

Taking $k$ positive, this quantity has poles, from each term, at,

$$k = \frac{d + 1}{2} \pm \delta' + n, \quad n = 0, 1, \ldots,$$

or,

$$k \in \left( \frac{d}{2} + \delta + n \right) \cup \left( \frac{d}{2} + 1 - \delta + n \right), \quad n \in \mathbb{Z}.$$

The $k$ are the eigenvalues of the pseudo–operator $(Y_{d+1} + 1/4)^{1/2}$ ($Y$ is the Yamabe–Penrose conformally invariant Laplacian) on an $S^{d+1}$, with flux. The pole residues determine the corresponding degeneracies. This sphere is the Cartan dual of a bulk hyperbolic manifold in an AdS/CFT construction, with the $S^d$ as boundary. [16].

Integrating back gives another representation for the effective action, $\mathcal{A}$. This manoeuvre occurs in AdS/CFT theory, e.g. [17–19].

7. The current average

The back integration device can also be applied to the derivative with respect to the flux and was used in [1] to find the defect contribution to the effective action in a very efficient way.

The quantity, $\partial \mathcal{A}/\partial \delta$, corresponds to the derivative with respect to the external potential driving the monodromy and so equals the vacuum average of the field current, $\langle J_\theta \rangle$. It is given by the companion integral to (19),

$$\frac{\partial \mathcal{A}}{\partial \delta} = \cos \pi (k - \delta') \frac{\Gamma \left( \frac{d+1}{2} + k + \delta' \right) \Gamma \left( \frac{d+1}{2} - k - \delta' \right)}{\Gamma(d+1)} - (k \rightarrow -k).$$ (20)

\(^{10}\) I have not yet managed to obtain the eigenvalues on a conically deformed, fluxed sphere in this way.
For standard fields, a straightforward local computation of the current average can be, and has been, performed.

8. Some incidental relations

When \( k \) is an integer, or half-integer, trigonometric expansion of the \( \sinh 2kx \) factor in (2), allows the effective action, \( A(d, q, \delta, k) \), to be determined as a combination of the standard field quantities \( A(d, q, \delta, 1) \) or \( A(d, q, \delta, 1/2) \) for varying dimension \( d \). Aspects of these relations have been systematically explained elsewhere, so only two simple illustrations will be given here.

The expansion of \( \sinh 4z \), substituted into (2) easily yields,

\[
A(d, q, \delta, 2) = 2A(d, q, \delta, 1) + A(d - 2, q, \delta, 1).
\]

To find the relation between the central charges, the Perlmuter factor comes in and then one finds, for example,

\[
C_T(9, \delta, 2) = 2C_T(9, \delta, 1) - \frac{270}{49}C_T(7, \delta, 1).
\]

Similarly the expansion of \( \sinh 3z \) leads to

\[
C_T(11, \delta, 3/2) = 3C_T(11, \delta, 1/2) - \frac{704}{135}C_T(9, \delta, 1/2).
\]

These relations are not much use computationally, but can form handy confirmatory tests. As \( d \to \infty \), e.g., the ratio of Permutter factors, \( Pf(e)/Pf(e - 1) \), tends to -4 and one can check that these relations are consistent with the infinite limits found earlier.

9. Discussion and conclusion

The fact that \( C_T \) is negative (‘ghost–like’ field) for a range of fluxes for the standard (canonical) cases (i.e. \( k = 1 \) for scalars and \( k = 1/2 \) for Dirac) raises questions regarding the underlying field theory because these systems are normally assumed to be unitary. For higher derivatives, a negative \( C_T \) is not usually considered to be a problem although one might wonder about the significance of the alternating sign.

Furthermore, a vanishing \( C_T \) (‘null–like’ field) would seem to signify that an energy–momentum tensor does not exist. Precisely how the monodromy influences
the field theory to accomplish this at specific fluxes is not clear to me. For canonical fields, the one–point vacuum average, \(\langle T_0^0 \rangle\), computed in [20] in flat space-time, displays no peculiarities \(^{12}\) and vanishes only for zero flux, \(\delta = 0\).

The energy–momentum tensors for higher derivative fields are somewhat complicated, \([9,23,24]\), but their vacuum averages on a conically deformed manifold, with an Aharonov-Bohm flux, could be determined after insertion of the field correlators into a monodromy modified Sommerfeld–Carslaw contour integral, in the manner of \([25,26,20]\).\(^{13}\) I do not wish to predict the outcome but I expect it would show an innocuous dependence on \(\delta\).

A possible loophole is that the Perlmutter factor, which plays a key role in the calculation leading to the two point function, \(\langle TT \rangle\), might not be universal in the presence of a flux or, as stated in \([6]\), of a chemical potential, and has been applied beyond its domain of validity so that the \(C_T\) computed here is just a definition without operational significance in terms of just \(\langle TT \rangle\).

I do not address this vital question in any fundamental way\(^{14}\), but a rough circumstantial argument in favour of the present results is the following.

The approach here yields a single, continuous function, \(C_T(d, \delta, k)\), which, for \(k = 1\) and \(\delta = 0\), gives the known value, \(2d/(d − 1)\), for canonical scalars and, for \(k = 1/2\) and \(\delta = 1/2\), equals (with a statistics sign change) the known value, \(d/2\), for canonical Dirac (per component).

This last fact makes it difficult (but not impossible) to believe that the Perlmutter factor, as it stands, works only for the non–zero flux of \(\delta = 1/2\) (as well as for \(\delta = 0\)). Also, even if it were so limited, there would still be a problem because confirmed calculation shows that \(C_T(d, 1/2, 1)\) is negative implying that \(C_T(d, \delta, 1)\), whatever this might be, must vanish somewhere in the range \(0 \leq \delta \leq 1/2\).

A relevant technical point, of some importance, is that \(\delta\), in say (2), is the flux through the \(q\)–deformed sphere and is, therefore, held constant when taking \(q\)–derivatives. This conforms to requirement (ii) in \([27]\) for the applicability of a replica–based formula, eqn.(2.48) there, for the entanglement entropy and accounts for the agreement found, modulo a choice of defect, between the entropy values in \([28]\) and \([4]\).

\(^{11}\) More recent derivations can be found in \([21,22]\)

\(^{12}\) Apart from some derivative discontinuities at the unit cell boundary.

\(^{13}\) The calculation is much simplified in the absence of a conical singularity.

\(^{14}\) One could try to repeat Perlmutter’s argument using a grand canonical hyperbolic average, See [6] for a perturbative treatment involving mixed higher point functions such as \(\langle T J \rangle\).
As mentioned in [3], this independence of $\delta$ on the conical deformation has the consequence of making derived field theory quantities invariant under $\delta \to 1 - \delta$ which is a property of the basic eigenvalue spectrum.

By contrast, in [6], it is the chemical potential, $\mu_E \equiv 2\pi q\delta$, that is held constant. The dependence of, say, the entanglement entropy on the flux is then quite different. For example, in the polynomial case of even $d$ dimensions, if $\mu$ is held constant the highest power of $\delta$ is $d - 1$, whereas, if $\delta$ is held constant, it is $d$.

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