I. INTRODUCTION

Global Navigation Satellite Systems (GNSSs) have become an indispensable tool in modern life. From civil aviation to ride-sharing, the applications of GNSSs continue to increase in scope and usage. The increasing dependence of our modern economy on GNSSs has led to the development of an expansive infrastructure aimed at achieving more reliable, accurate, and precise location systems.

Traditional GNSSs are based on a Newtonian framework, particularly on the simple principle of trilateration in Euclidean space, i.e. the use of three sources to determine the position of a given user. However, a purely Newtonian framework is not enough; when one proceeds naïvely with the calculation of the position of the user employing standard Newtonian mechanics, even neglecting sources of errors associated with the signal transmission, one is faced with large accumulative errors [1, 2]. Such errors are mainly sourced by two effects. The first is the difference in clock rates due to the relative motion of the user and the satellites, while the second is due to the gravitational time dilation effects; the latter contribution is more than six times larger than the former. Combined with other relativistic effects, they amount to about a 40 microsecond delay per day. Translated into location error, this offset would amount to an error of about a 40 microsecond delay per day. Translated into a 40 microsecond delay per day.

In recent years, efforts in the definition and development of a consistent RPS have led to the development of a number of different approaches [2, 8–32]. Much effort has been devoted to establishing a transformation between positioning system [1, 2]. Relativistic corrections therefore increase the size of the ground GNSS infrastructure (see e.g. [3, 4] for some details in this matter) which in turn increases the general cost and maintenance burden of the system itself.

In this context, it makes sense to design a positioning system based directly on relativistic principles. The concept of a Relativistic Positioning System (RPS) employs emission coordinates as the primary coordinates for spacetime [5, 6]. Emission coordinates are formed from the timestamps of proper time broadcasts for a system of satellites, so that the location of the user in emission coordinates is immediately established upon signal reception. Moreover, the satellite coordinate positions become trivial in emission coordinates.

The simplicity of an RPS based on emission coordinates offers several advantages over traditional implementations of GNSSs. Since the user and satellite positions in emission coordinates are expressed directly in terms of proper time broadcasts received by the users and satellites, an implementation of an RPS in terms of emission coordinates has the potential to reduce the post processing, number of emitters, and number of ground stations, which would permit a significant reduction in the size and scope of the infrastructure required without compromising (and possibly improving) the performance and accuracy of the service. Additionally, an RPS might be employed equally well for positioning in space. Finally, RPSs can be used as key scientific tools; there are, for instance, proposals to use an RPS network for relativistic geodesy as well as the detection of gravitational waves (see e.g. [7]).

In this article, we describe and numerically implement a method for relativistic positioning in slightly curved, but otherwise generic spacetimes. For terrestrial positioning, such an algorithm can, in addition to gravitational corrections, incorporate atmospheric and ionospheric effects by way of the analogue Gordon metric. The methods are implemented in the squirrel.j1 code, written in the Julia language, which employs a quasi-Newton Broyden algorithm in conjunction with automatic differentiation of null geodesic solutions obtained by numerical integration. Though optimization is not our primary focus, our implementation is already fast enough for practical use, establishing a position from five emission points in < 1 s on a desktop computer for reasonably simple spacetime geometries. In a vacuum, our implementation can achieve submillimeter accuracy in the Kerr metric (with terrestrial parameters) for $n \geq 5$ emission points. With atmospheric and ionospheric effects included, our implementation can achieve submeter accuracy for horizontal terrestrial positioning, even allowing for up to a 10% uncertainty in the ionospheric electron density profile.
emission coordinates and a standard coordinate system. The majority of the approaches in this direction are limited to a small class of geometrical backgrounds and require the inversion of transcendental equations. One exception is that of [33], which is applicable for general backgrounds, but this approach still requires numerically solving the (curved spacetime) Eikonal equation, a partial differential equation (PDE). Thus a key point in the development of RPSs is the development of calculational methods applicable to more general spacetime geometries that are efficient enough to be performed on standard hardware such as that available in handheld devices or satellites.

Another issue that is often neglected in the development of RPSs is the modeling of nongravitational effects, such as the interaction of the signal with the atmosphere and ionosphere. These phenomena are typically thought to require methods independent of the general relativistic formalism. For this reason, despite the relevance of the phenomena to the performance of the positioning system, and the fact that they are among the largest contributors to typical GNSS error budgets [34] (see for instance tables 24 & 25 for typical error budgets), they are often excluded in the framework of RPSs.

In this paper, we propose a new approach to the relativistic location problem, applicable in generic, slightly curved, spacetimes, which can be thought of as a way of analogue gravity models. These spacetimes incorporate the interaction of light signals with the atmosphere and ionosphere in a fully relativistic framework. We will then compare the performance of our method with respect to the standard performance of the GALILEO system, showing that our method can in principle achieve similar results. Our approach requires solving at minimum four ordinary differential equations (ODEs), greatly reducing the computational complexity of calculations compared to PDE-based approaches. Our work can be seen as complementary to the recent work [20] and earlier works [35–38] that address instead satellite ephemeris errors (another large contributor to GNSS error budgets), which we neglect here.

In the following, lists of symbols contained in the curly brackets \( \{x_1, x_2, \ldots \} \) denote sets, and lists of more than two symbols contained in the round brackets \((v_1, v_2, \ldots)\) denote vectors, with \(v_i\) either representing components or lower-dimensional vectors. In the latter case, \((v_1, v_2, \ldots)\) represents a vector formed from the concatenation of vectors \(v_1, v_2, \ldots\). Greek indices represent spacetime coordinate indices and take values from the set \{0, 1, 2, 3\}. Lowercase latin indices from the middle of the alphabet \(i, j, k, l\) represent spatial coordinate indices and take values from the set \{1, 2, 3\}. Unless otherwise indicated, Einstein summation convention is employed on coordinate indices. Uppercase latin indices \(I\) (for instance) and the lowercase latin indices \(a, b\) are not treated as tensor indices, and are used to label emitters and emission points; the uppercase indices take values from the set \{1, 2, ..., \(N\)\}, and the lowercase indices \(a, b\) take values from the set \{1, 2, 3\}. Lowercase bold latin letters (such as \(b, v\) and \(x\)) are reserved for three-component quantities; when components of such letters are displayed explicitly (for instance \(x^1, x^2\)), raised indices always represent the value of the coordinate index and the lowered indices represent the value of the emission point label. Uppercase bold latin letters (such as \(A\) and \(J\)) are reserved for matrices.

In Sec. II, we discuss the problem of relativistic location in flat spacetime. Our algorithm for relativistic location in curved spacetime is described in Sec. III. In Secs. IV and V, we describe the spacetime metrics and index of refraction models used in tests of our implementation of the algorithm. Tests and benchmarks of our implementation are described in Sec. VI. We conclude with a summary and brief discussion in VII.

II. RELATIVISTIC LOCATION IN FLAT SPACETIME

A. Relativistic positioning and relativistic location

Relativistic positioning systems are based on the concept of emission coordinates (a detailed discussion of which may be found in [5, 6]; see also [39] for the two-dimensional case), which correspond to the broadcasted proper times of a system of at least four satellites. Each value of proper time \(\tau_I\) broadcast by a satellite \(I\) defines a (null) hypersurface corresponding to events at which an observer receives the broadcasted value \(\tau_I\); this surface forms the future pointing light cone for the spacetime position \(X^I_I\) of satellite \(I\) at the moment the broadcast is emitted. Given four satellites, each with a single broadcast of proper time (which we collectively write as \(\tau = \{\tau_1, \tau_2, \tau_3, \tau_4\}\), one may define four such hypersurfaces, the intersection of which is (generically) a single point in an appropriate region of a well-behaved spacetime geometry. Locally, points in such regions are distinguished by different values of proper time broadcasts; the collection of proper times \(\tau\) broadcasted by the four satellites may then be used as coordinates in certain regions of spacetime.

A central problem in relativistic positioning system is that of transforming between emission coordinates \(\tau\) and a more standard coordinate system in a given spacetime geometry. If the ephemerides of the satellites are known in a standard coordinate system (Cartesian coordinates for flat spacetime, for instance), the emission coordinates \(\tau\) may be converted into the coordinates for the emission points \(X_I\) (which we collectively write as \(X = \{X_1, X_2, X_3, X_4\}\)), or the spacetime positions of the satellites at the moments when the broadcasted values \(\tau\) were emitted. To perform the coordinate transformation, one must find in the standard coordinate system the coordinates for the intersection point \(X_c\) of the future light cones of four emission points \(X_i\), assuming a unique point \(X_c\) exists in some appropriate region of spacetime. We refer to the problem of finding the coordinates \(X_c\), given
the coordinates of the emission points $X$ as the relativistic location problem.

In Cartesian coordinates $t, x, y, z$ on flat spacetime, the coordinates for the intersection point $X_c$ must satisfy the following constraint, which can in principle be solved using root-finding methods in a brute-force approach:

$$ (X_I^\mu - X_J^\mu) (X_I^\nu - X_J^\nu) \eta_{\mu\nu} = 0, \quad (1) $$

where here, $I \in \{1, 2, 3, 4\}$, and $\eta_{\mu\nu}$ are the components of the Minkowski metric:

$$ \eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2) $$

In flat spacetime, several methods for computing such points, which avoid brute-force root-finding methods, may be found in the literature, for instance in [40, 41] (implemented in [42]) and [36, 38].

### B. Transformation algorithm

Here, we describe an algorithm for computing the intersection point $X_c$ from four emission points based on Lorentz transformations. To our knowledge, this algorithm has not been explicitly described in the literature before, though some of the methods may in principle be inferred from the diagrams presented in [41] (which we reproduce here in Figs. 1 and 2). While these algorithms are not the most optimal, we present them here because they are physically intuitive and of conceptual utility.

The algorithm we describe requires that the emission points $X$ are spacelike separated, or that:

$$ (X_I^\mu - X_J^\mu) (X_I^\nu - X_J^\nu) \eta_{\mu\nu} \geq 0, \quad (3) $$

for all $I, J$. The frame in which the emission points $X$ are defined will be called $A$. From these points, one may construct three spacelike vectors $E_1, E_2, E_3$ in the following manner:

$$ E_1 = X_2 - X_1 $$

$$ E_2 = X_3 - X_1 $$

$$ E_3 = X_4 - X_1. \quad (4) $$

These three vectors span a hyperplane $\Sigma$, called the configuration hyperplane; from these three vectors, one may construct a vector normal to the configuration hyperplane $\Sigma$ in the following manner ($\epsilon_{\alpha\beta\gamma\delta}$ being the Levi-Civita tensor):

$$ N^\mu = \eta^{\mu\alpha} \epsilon_{\alpha\beta\gamma\delta} E_1^\alpha E_2^\beta E_3^\gamma, \quad (5) $$

and a unit normal vector:

$$ n^\mu = \frac{q}{\sqrt{N_N^\sigma N_N^\sigma}} N^\mu, \quad (6) $$

where $q = \pm 1$, with the sign specified by the requirement that $n$ be future pointing if timelike.

\[ \text{FIG. 1.} \quad 2 + 1 \text{ illustration for the relativistic location algorithm in the case of a spacelike configuration hyperplane } \Sigma \text{ in an adapted frame. If the } \Sigma \text{ is spacelike, one can perform a Lorentz transformation so that the emission points lie on a surface of constant time coordinate } t = X^0 \text{ (the corresponding } t \text{-axis is vertical) with a value } t = t_0. \text{ In this frame, the emission points lie on a sphere } S \text{ (represented here as a circle) in the configuration hyperplane } \Sigma. \text{ The problem consists of first finding the circumcenter } x_c \text{ and the circumradius } r_c. \text{ In this frame, the time it takes for light to travel the distance } r_c \text{ is } \Delta t = r_c \text{ in units where the speed of light is unity. The intersection point is then given by } X_c = (t_0 + \Delta t, x_c). \]

1. **Spacelike configuration hyperplane**

We first consider the case where the configuration hyperplane $\Sigma$ is spacelike, so that the normal vector is timelike. In this case, one may write:

$$ n = (\gamma, \beta \hat{r}), \quad (7) $$

where $\hat{r} = (\hat{r}_x, \hat{r}_y, \hat{r}_z)$ is a unit vector. $\gamma := n^0$ and $\beta = \sqrt{1 - 1/\gamma^2}$. One may then perform a Lorentz transformation to a frame in which the spatial components of the unit normal $n$ vanish. The Lorentz transformation matrix takes the form:

$$ \Lambda = \begin{bmatrix} \gamma & -\beta \gamma \hat{r} \\ -\beta \gamma \hat{r} & (I + (\gamma - 1) \hat{r} \otimes \hat{r}) \end{bmatrix}, \quad (8) $$

where $I$ is the identity matrix and $\otimes$ denotes a tensor product.

Since the Lorentz transformation $\Lambda$ transforms to a frame in which $n' = \Lambda \cdot n$ has no spatial components, it follows that since the vectors $E_1, E_2$ and $E_3$ are orthogonal to $n$, the time component of their transformed counterparts $E_1', E_2'$ and $E_3'$ must vanish. From Eq. (4), it follows that the time components of the transformed emission points $X' = \{X'_1, X'_2, X'_3, X'_4\}$ are all equal; in this frame, the emission points all lie on the same constant time slice $t = t_0$. The primed frame will be called $B$.

The problem of finding the intersection point of the light cones from four points is simply a matter of finding the point spatially equidistant from the four emission points. To see this, consider four signals emitted from four points at the same instant. We seek the spatial point at which the signals simultaneously arrive. The preceding analysis establishes that one can find a reference frame (frame $B$) where the four emission points all...
lie on the same time slice. Since the speed of light is constant in all frames, the point where the signals simultaneously arrive must be spatially equidistant from the four emission points. If the distance between the simultaneous arrival point and each of the emission points is \( r_c \), then the time coordinate in frame \( B \) is given by the time it takes for light to travel a distance \( r_c \) (which has a value \( r_c \) in units where the speed of light is \( c = 1 \)).

In a three-dimensional Euclidean space, this is a straightforward task. Generically, four points that do not all lie in the same plane form the corners of a tetrahedron. It is well-known that for any tetrahedron, one can construct a circumsphere \( S \) which passes through all the corners of a tetrahedron. The coordinates of the circumcenter \( x_c \) specify the spatial coordinates of the intersection point in frame \( B \), and the circumradius determines the time coordinate. Given four (spatial) points \( \{x_1, x_2, x_3, x_4\} \), one can compute the coordinates of the circumcenter \( x_c \) using the following formulas [43]:

\[
x_c = A^{-1}u,
\]

where the \( 3 \times 3 \) matrix \( A \) and the vector \( u \) are defined:

\[
A := \begin{bmatrix}
x_2 - x_1 \\
x_3 - x_1 \\
x_4 - x_1
\end{bmatrix}^T
\quad
u := \frac{1}{2} \begin{bmatrix}
x_2^2 - x_1^2 \\
x_3^2 - x_1^2 \\
x_4^2 - x_1^2
\end{bmatrix}.
\]

The circumradius \( r_c \) may then be computed using the formula:

\[
r_c = |x_c - x_I|^{1/2}.
\]

The intersection of light cones in the frame \( B \) is then given by:

\[
X_c' = (t_0 + r_c, x_c).
\]

To obtain the intersection of the light cones \( X_c \) in the original frame \( A \), simply invert the Lorentz transformation:

\[
X_c = A^{-1}X_c'.
\]

There are instances in which this algorithm fails. For instance, the algorithm may fail when the matrix \( A \) becomes degenerate, which can occur if the emission points are coplanar (in which case \( r_c \) diverges) [41]; these cases are discussed in detail in [44, 45].

2. **Timelike configuration hyperplane**

We now turn to the case in which the configuration hyperplane \( \Sigma \) is timelike, which corresponds to a spacelike unit normal vector \( n^\mu \). In this case, the adapted frame is constructed differently; one first performs a Lorentz transformation such that the spacelike \( n^\mu \) is tangent to a surface of constant \( t = X^0 \) (here, \( t = X^0 \), \( x = X^1 \), \( y = X^2 \), \( z = X^3 \)). Then, one performs a spatial rotation so that \( n^\mu \) is aligned with the \( z \)-axis. The configuration hyperplane \( \Sigma \) in the resulting frame is characterized by a constant \( z = z_0 \) coordinate. The points will lie on an elliptic hyperboloid\(^1\) \( H \) formed by the intersection of the past light cone for the solution point and the configuration hyperplane \( \Sigma \) (see Fig. 2). The main task is to find the coordinates of the vertex \( v_c \) in this frame and the distance \( R \), the latter being the distance between the vertex \( v_c \) and a point on \( H \). The distance between the \( z = z_0 \) plane and the intersection point \( X_c \) is given by \( R = \Delta z \). With these quantities in hand, one may obtain the intersection point \( X_c = (v_c, z_0 \pm \Delta z) \).

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\( \Delta z \) From here on, all hyperboloids will be assumed to be elliptic.
In this case, one does not have a unique point for the intersection of light cones—this is the bifurcation problem, which is discussed in detail in [41].

C. Five emission points

The bifurcation problem may be solved by including an additional point. If five emission points are available, one can obtain the intersection point in a straightforward way. Following [21], one begins with the constraint function:

$$\Psi_I := (X_I^\mu - X_c^\mu) (X_J^\nu - X_c^\nu) \eta_{\mu\nu},$$  \hspace{1cm} (18)

where now $I \in \{1, 2, 3, 4, 5\}$. One can take the differences to form five unique equations of the following form:

$$\Psi_I - \Psi_J = \eta_{\mu\nu} \left[ 2X_c^\mu (X_J^\nu - X_c^\nu) + X_I^\mu X_J^\nu - X_J^\mu X_I^\nu \right],$$  \hspace{1cm} (19)

which are linear in $X_c^\nu$; one can reduce this to a straightforward linear algebra problem. Though this method requires an additional emission point, it is preferred due to its computational simplicity and accuracy.

D. Implementation, evaluation, and discussion

The new algorithms we have presented here, as well as the formulas described in [40, 41] and the five-point algorithm of [21] (which we have described in Eqs. (18) and (19)), have been implemented in the cereal.jl code, available at [47]. We have written cereal.jl to accommodate abstract datatypes; this allows user-specified floating point precision. In the tests we perform, we consider two types of floating point variables, the default Float64 double precision, and the Double64 “double double” precision variables implemented in the DoubleFloats library [48].

Included in cereal.jl are test routines which perform tests of the code by stochastically generating a set of $N$ emission points on the past light cone of some intersection point $X_c$, and comparing the results $X_c$ generated by the algorithms in cereal.jl with the true value for $X_c$. The points $X_c$ and $X_c$ are compared according to Euclidean $L_2$ norms (with $|V| = \sqrt{V \cdot V}$ for some vector $V$):

$$\varepsilon = |X_c - X_c|/|X_c|$$  \hspace{1cm} (20)

In our tests, the most accurate algorithm is the five-point formula of [21], which for $10^6$ test cases satisfies $\varepsilon < 10^{-9}$ with double precision (Float64), and $\varepsilon < 10^{-23}$ with extended precision (Double64). The accuracy of the formula in [40, 41] and the new algorithm presented in Sec. II B are comparable to each other, but both are less accurate than the formula of [21]. For $10^6$ test cases, the errors for the methods presented in [40, 41] and in Sec. II B typically satisfy $\varepsilon < 10^{-7}$ with double precision (Float64), and $\varepsilon < 10^{-14}$ with extended precision (Double64).

Execution times differ greatly between the algorithms. On a standard desktop computer (with an Intel i5-7500 processor), the five-point formula of [21] typically performs the computation in $< 1.5 \mu s$. The four-point algorithms that we have implemented in cereal.jl are significantly slower, despite only requiring four emission points. For four emission points, our implementation of the formula in [40, 41] typically requires $\sim 150 \mu s$ to perform the computation. The algorithm we have presented here has improved performance, requiring a computation time of $\sim 36 \mu s$. Since the five-point formula is faster and yields results with significantly higher accuracy, we employ it when computing the initial guess for the curved spacetime algorithm that we will describe in the next section.

We note that the algorithms described here may be used in conformally flat spacetimes, since flat spacetimes and conformally flat spacetimes share the same null cone and null geodesic structure on regions where the conformal factor remains nonsingular. In particular, the intersection point for four null cones in a conformally flat spacetime will be the same as that for the underlying flat spacetime (underlying in the sense that the metric for the conformally flat spacetime differs from the flat spacetime by a conformal factor). This class of spacetimes include cosmological spacetimes, such as de Sitter, anti de Sitter and the more general Friedmann-Lemaître-Robertson-Walker spacetimes.

III. RELATIVISTIC LOCATION IN CURVED SPACETIME

A. Geodesics

For general spacetime geometries, described by a metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, the problem of finding the intersection point $X_c$ of four future pointing light cones (provided that such a point exists) amounts to finding the intersection of four null geodesics from the emission points $X$: this follows from the fact that for some emission point $X_I$, a point $X_p$ in the future pointing null cone lies on a geodesic connecting $X_p$ and $X_L$. Note also that the emission points $X$ lie on the past light cone of $X_c$. Given some inverse metric $g^{\mu\nu} = g^{\mu\nu}(x)$ describing the spacetime geometry, an affinely parameterized null geodesic may be described by the Hamiltonian:

$$H := \frac{1}{2} g^{\mu\nu} p_\mu p_\nu, \hspace{1cm} (21)$$

where the four-momenta are given by:

$$p_\mu = g_{\mu\nu} \frac{dx^{\nu}}{d\lambda}, \hspace{1cm} (22)$$

and the associated Hamilton equations:

$$\frac{dx^{\mu}}{d\lambda} = \frac{\partial H}{\partial p_\mu}, \hspace{1cm} \frac{dp_\mu}{d\lambda} = -\frac{\partial H}{\partial x^{\mu}}. \hspace{1cm} (23)$$
For null geodesics, the initial data at \( \lambda = 0 \) is given by an initial point \( x_0^\mu \) and an initial three-velocity \( v_i \), with the initial four-momentum \( p_{\mu}(\lambda=0) \) satisfying the following (with \( i \in \{1, 2, 3\} \)):

\[
\delta^i_\mu \frac{dx^\mu}{d\lambda} \bigg|_{\lambda=0} = v_i, \quad g_{\mu\nu}(x_0) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \bigg|_{\lambda=0} = 0. \tag{24}
\]

The solution to Hamilton's equations is formally given by \( x^\mu = x^\mu(\lambda, x_0, v) \). Since \( \lambda \) is an affine parameter, one can redefine \( \lambda \) up to linear transformations—it is therefore always possible to rescale \( \lambda \) so that it takes values in the domain \( \lambda \in [0, 1] \), with \( \lambda = 1 \) being the final point.

### B. Geodesic intersection

The problem of finding the intersection of light cones in a slightly curved spacetime may be reformulated in terms of null geodesics [29]. Consider four formal solutions to Hamilton’s equations (23), distinguished by the indices \( I \in \{1, 2, 3, 4\} \), that have endpoints \( X_I^\mu \) which are functions of the initial data \( X_I \) and \( v_I \):

\[
X_I^\mu = X_I^\mu(X_I, v_I) = x_1^\mu(1, X_I, v_I). \tag{25}
\]

Then define the following vector valued function:

\[
F := (X_1 - X_2, X_1 - X_3, X_1 - X_4). \tag{26}
\]

where \( F = F(X, v) \) (with \( v = (v_1, v_2, v_3, v_4) \)). Observe that upon evaluation, the function \( F \) yields a 12 component vector. The intersection of four null geodesics is given by the condition

\[
F(X, v) = 0. \tag{27}
\]

The problem of solving the system of 12 equations in Eq. (27) is a standard root-finding problem. In particular, given a set of four emission points \( X = \{X_1, X_2, X_3, X_4\} \), one solves Eq. (27) for the 12 quantities \( v = (v_1, v_2, v_3, v_4) \) that constitute the initial data.

### C. Initial data

From \( v \) and \( X \), one may construct the initial data for the geodesics by first constructing the vector \( V_I \) for each geodesic:

\[
V_I := (V_I^0, v_I^1, v_I^2, v_I^3), \tag{29}
\]

where \( V_I^0 \) is determined by the condition

\[
V_I^0 V_I^\nu g_{\mu\nu}(X_I) = 0. \tag{30}
\]

The conjugate momenta are given by:

\[
p_I^\mu = V_I^\nu g_{\mu\nu}(X_I). \tag{31}
\]

The initial positions \( X_I \) and initial conjugate momenta \( p_I^\mu \) provide initial data for Eq. (23), which may then be solved to compute the value of \( f(v) \) according to Eq. (26).

### D. Root finding

In general, one does not possess analytical solutions to the geodesic equation (23) for a generic metric \( g_{\mu\nu} \). To evaluate the function (26), one must therefore solve the geodesic equation (23) numerically for each emission point. One might expect a root finding algorithm for Eq. (27) to be computationally expensive, particularly in the computation of a Jacobian.

However, libraries for efficiently computing the Jacobian of generic functions have become available in recent years, in particular those which employ automatic differentiation methods. Automatic differentiation refers to a set of methods which, by way of the chain rule, exploit the fact that all numerical computations can in principle be broken down into finite compositions of elementary arithmetic operations. These methods can in principle be used to numerically compute the derivatives of programs to machine precision with a minimal computational overhead. A detailed discussion of automatic differentiation may be found in [49, 50]. It should be mentioned that while automatic differentiation methods have been previously proposed for differentiating analytical geodesic solutions in the Schwarzschild spacetime [51], the approach presented here differs in that we perform automatic differentiation of numerical solutions to the geodesic equation in a generic slightly curved spacetime.\(^2\)

The specific root finding algorithm we employ is based on an iterative quasi-Newton Broyden method [53, 54], which we summarize here. The task at hand is to obtain the root of some function \( f(v) \). In the initial iteration, the Jacobian of \( f(v) \) is computed using automatic differentiation methods. We also employ automatic differentiation in computing the gradient of the Hamiltonian.

\(^2\) We note that in [52], automatic differentiation has been previously proposed as a method for obtaining Taylor expansions of the initial value problem for the geodesic equation.
a strategy also employed in [55] for solving the geodesic equation in Hamiltonian form. Given the Jacobian $J$ and its inverse $J^{-1}$, at some iteration $i$, one can update $v$ according to the Newton prescription:

$$v_{i+1} = v_i + J_i^{-1} f(v_i). \quad (32)$$

In the standard Broyden method (alternatively referred to as the “good” Broyden method), the first iteration is given by Eq. (32), with the Jacobian computed by differentiation. For the subsequent iterations, one computes the following:

$$\Delta v_i = v_i - v_{i-1}$$
$$\Delta f_i = f(v_i) - f(v_{i-1}). \quad (33)$$

The inverse Jacobian $J^{-1}$ is then updated according to the Sherman-Morrison formula:

$$J_{i+1}^{-1} = J_i^{-1} + \frac{\Delta v_i^T - J_i^{-1} \Delta f_i \Delta v_i^T}{\Delta v_i^T J_i^{-1} \Delta f_i}. \quad (34)$$

One may then use Eq. (34) in conjunction with (32) to iteratively solve for the root of $f(v)$. The termination of the algorithm is determined by the behavior of $f_i$; if a local minimum is detected within a specified range of iterations, the algorithm terminates and the results corresponding to the minimum are returned. In case the algorithm does not converge, a hard termination limit is used.

Given a root for $f(v)$, one can obtain the intersection point by solving the geodesic equations once more with the updated values for the initial data constructed from $v$ and $X$, and averaging over the endpoints (which are assumed to be close).

**E. The squirrel algorithm**

We now summarize the curved spacetime algorithm employed in the squirrel.jl code:

1. First, apply a flat spacetime algorithm (either that of Secs. II B or II C) to the emission points $X$ to obtain a guess for the intersection point and initial velocities.

2. Apply a root finding algorithm to the function $f(v) = F(X, v)$ to obtain the initial velocities $v$ for subsets of four emission points.

3. Integrate the geodesics with the resulting initial velocities $v$ and emission points $X$ to find the intersection point.

As indicated, steps 2. and 3. of the above algorithm are applied to a subset of four emission points. If additional emission points are available, an outlier algorithm, described in the next subsection, is employed to exclude large errors.

**F. Outlier detection**

There are instances in which the algorithm described in this section can generate large errors, which can result from a combination of large errors in the initial guesses provided by the flat spacetime algorithm and convergence failures in the Broyden algorithm. One might expect such errors to occur, since the function $F(X, v)$ is generally nonlinear. To increase the reliability of the algorithm, we describe here methods which can mitigate the effects of these errors when additional emission points are available.

As discussed before, given $N > 4$ emission points, one can choose up to $C(N, 4)$ combinations of four emission points $X$, and for each set $X$, the previously described algorithm can be applied to obtain a total of $C(N, 4)$ intersection points. Since there is only one receiver for the emission data, all $C(N, 4)$ intersection points should agree. If errors in the algorithm are assumed to be rare, one can employ an outlier detection algorithm which can identify the intersection points that strongly deviate from the others.

We employ a simple outlier detection algorithm, which begins by first computing the median values for the intersection points, and then computes the deviation of each intersection point from the median. The points which deviate from the median beyond a user-specified threshold are then discarded. The final intersection point is then computed from the remaining intersection points.

**G. Remarks on implementation**

The algorithm described here is implemented in the squirrel.jl code (available at [56]). The squirrel.jl code is written in the Julia language, which is ideal for implementing the squirrel algorithm due to the state of the art automatic differentiation and ODE solver libraries available. Automatic differentiation is handled using the ForwardDiff.jl forward-mode automatic differentiation library [57], and geodesics are integrated using the recommended Verner 7th order Runge-Kutta integrator AutoVern7 [58] in OrdinaryDiffEq.jl [59], which features stiffness detection and automated switching to a specified stiff integrator (we use the fifth order Rosenbrock method integrator Rodas5 [60]). Though our system is Hamiltonian, we have avoided symplectic integrators in favor of integrators with adaptive timestepping in order to minimize execution time.

The Broyden algorithm is implemented directly, depending only on standard Julia libraries. The default ter-
nomination limit is set to 24. The initial guess is provided by one of the flat spacetime algorithms implemented in the cereal.jl code, depending on the number of emission points available; if \( N = 4 \) emission points are available, the flat spacetime algorithm presented in Sec. II B is employed (in which case, our implementation returns two points), but if \( N \geq 5 \) emission points are available, the formula of [21] reviewed in Sec. II C is employed. The outlier detection algorithm becomes active for \( N \geq 5 \) emission points, and is applied to the location algorithm of squirrel.jl to remove results with large errors.

Since there is now widespread availability of devices with multithreading capabilities, the squirrel.jl code employs multithreading on loops containing the integration of geodesics and the automatic differentiation of geodesic solutions. With multithreading enabled on a desktop computer with four cores and four threads (Intel i5-7500), the squirrel.jl code can establish a position from five emission points in under 1 s for reasonably simple spacetime geometries—benchmarks will be discussed in Sec. VIC.

IV. CURVED SPACETIME GEOMETRIES

A. The Kerr-Schild metric

In this section, we describe some specific choices for the spacetime metric \( g_{\mu \nu} \) used in our tests. In general relativity, the spacetime geometry surrounding a stationary rotating object in a vacuum is given by the Kerr geometry [61], described by the following metric in Kerr-Schild coordinates [62]:

\[
g_{\mu \nu} = \eta_{\mu \nu} + f k_\mu k_\nu
\]

\[
k_\mu = \left( 1, \frac{r x + a y}{r^2 + a^2}, \frac{r y - a x}{r^2 + a^2}, \frac{z}{r} \right)
\]

\[
f = \frac{2 G M r^3}{r^4 + a^2 z^2}
\]

where \( G \) is the gravitational constant, \( M \) is the mass, and \( a \) is the spin parameter. The radius \( r \) may be compactly expressed as the solution to:

\[
\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.
\]

At the surface of the Earth, the spacetime curvature is small, so one might ask whether the flat spacetime algorithms suffice for positioning, neglecting atmospheric and ionospheric corrections. To determine whether this is indeed the case, we test the five-point algorithm in Eqs. (18) and (19) against the Kerr geometry. Recalling that emission points lie on the past light cone of the intersection point \( X_c \), we stochastically generate intersection points \( X_c \), then construct initial data for past directed null geodesics. To obtain the emission points \( X = \{X_1, X_2, X_3, X_4, X_5\} \), we integrate the geodesic Hamilton equations for five sets of initial data for the null geodesics. The resulting emission points \( X \) are used with the formula of [21] to obtain \( X_c \), the intersection point in flat spacetime. The positioning error is given by (here the absolute symbols denote the \( L_2 \) Euclidean distance norm):

\[
\epsilon_{\text{KS}} = |\hat{P}(x_{\text{tar}} - x_c)|.
\]

Here, \( \hat{P}(\cdot) \) is a projection operator (projecting to horizontal or vertical directions relative to some surface), \( x_c \)
form the spatial components of $X$, and $X_{\text{tar}}$ denotes the spatial components of $X_{\text{tar}}$ which is the true intersection point for the future light cones of $X$ with respect to the Kerr-Schild metric.

We consider a Kerr-Schild metric with parameter choices $M = 1$ and $a = 738$ (the latter corresponding to the angular momentum for the Earth). We perform a test with $10^5$ randomly generated target points $X_{\text{tar}}$ on the WGS-84 reference ellipsoid [63] (setting $M = 1$ to be the Earth mass) and randomly generated initial data sets for null geodesics. The result, illustrated in Figs. 3 and 4, indicates that the error satisfies $\epsilon_{\text{KSc}} \leq 2 \text{ cm}$ for 95% of the points in the vertical direction (the direction orthogonal to the reference ellipsoid), and $\epsilon_{\text{KSc}} \leq 3 \text{ mm}$ in the horizontal direction. In a vacuum, the five emission point algorithm in flat spacetime suffices for positioning to an accuracy on the order of a centimeter. This result is consistent with those of [64], where it is also argued that the dominant errors from spacetime curvature come from the determination of satellite orbits, rather than the bending of photon trajectories.

**B. The Gordon metric**

If one seeks centimeter-scale accuracy in a vacuum on terrestrial scales, flat spacetime algorithms suffice. However, for terrestrial positioning, atmospheric and ionospheric effects significantly affect the propagation of electromagnetic signals and introduce errors in the computed position. From a general relativistic perspective, one might be tempted to dismiss atmospheric and ionospheric effects as ancillary (practical considerations aside), as the underlying spacetime geometry does not depend to a significant degree on atmospheric and ionospheric profiles. However, in a positioning system based on the exchange of electromagnetic signals, such effects will deform the emission coordinates, and in this sense, atmospheric and ionospheric effects should still be taken into consideration even if one insists on a fundamentally relativistic approach.

Fortunately, as indicated in [65], the framework of general relativity can by way of analogue spacetime geometries incorporate the effects of dielectric media on electromagnetic signal propagation. In dielectric media, light propagation may under certain conditions be described with the geodesics of the Gordon metric [66–68], which has the form:

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \left(1 - \frac{1}{n^2}\right) u^\mu u^\nu, \quad (38)$$

where $u^\mu$ corresponds to the four-velocity of the medium, and $n$ is an effective index of refraction. The atmospheric index of refraction has a sea level value of $n_{\text{atm}} - 1 \sim 2.7 \times 10^{-4}$, and the effective index of refraction for the ionosphere has a maximum value on the order of $n_{\text{ion}} - 1 \sim 4.0 \times 10^{-5}$. It follows that the atmospheric and ionospheric corrections to the metric are of the respective orders $10^{-4}$ and $10^{-5}$. In contrast, the difference $f_k^\mu k^\nu$ between the components of the Kerr-Schild metric and the Minkowski metric is roughly on the order of $10^{-9}$ at the surface of the Earth, so atmospheric and ionospheric effects dominate.

**C. Weak field metric**

At this point we emphasize that when performing tests with the analogue Gordon metric, we incorporate gravitational effects with the weak-field metric, rather than the Kerr metric. The weak field metric has the form:

$$g_{\mu\nu} = \eta_{\mu\nu} - 2 V \delta_{\mu\nu}, \quad (39)$$

where $V$ is the gravitational potential of the Earth, which takes the form:

$$V = -\frac{GM}{r} \left[1 - J_2 \left(\frac{a_{\text{ell}}}{r} P_2(\cos \theta)\right)\right], \quad (40)$$

where $r^2 = x^2 + y^2 + z^2$, $P_2$ is a Legendre polynomial of degree 2, $J_2$ is a quadrupole moment of the Earth, which takes a value [69]:

$$J_2 = 1.0826300 \times 10^{-3}. \quad (41)$$

The quantity $a_{\text{ell}}$ is the equatorial radius of the Earth, and is one of the parameters of the reference ellipsoid, which approximates the Earth’s geoid up to roughly 100 m. Following [69], the reference ellipsoid we use is the WGS-84 standard [63], which corresponds to the following values for the semimajor axis $a_{\text{ell}}$ and the semiminor axis $b_{\text{ell}}$:

$$a_{\text{ell}} = 6378.137 \text{ km}, \quad b_{\text{ell}} = 6356.752314245 \text{ km}. \quad (42)$$

**V. INDEX OF REFRACTION MODELS**

We now turn to the construction of models for the effective index of refraction, which we will use in the analogue Gordon metric (38) for our tests of the algorithm. The effective index of refraction is then given by the following expression

$$n_{\text{eff}} = 1 + \Delta n_{\text{atm}} + \Delta n_{\text{ion}}. \quad (43)$$

In this section, we describe the construction of simplified profiles for $\Delta n_{\text{atm}}$ and $\Delta n_{\text{ion}}$ which we will use in evaluating the squirrel.jl code.
A. Atmospheric model

Given the pressure and temperature profiles for the atmosphere, the profile for the atmospheric index of refraction can be computed from the revised Edlén equation [70] for the refractive index of air:

\[
\Delta n_{\text{air}} = \frac{\Delta n_s [P/\text{Pa}]}{96095.43} \times \frac{1 + 10^{-8} (0.601 - 0.00072 [T/^\circ \text{C}]) [P/\text{Pa}]}{1 + 0.0036610 [T/^\circ \text{C}]},
\]

where \( \Delta n_{\text{air}} = n_{\text{air}} - 1 \), and \( \Delta n_s \) is given by

\[
\Delta n_s \times 10^8 = 8342.54 + \frac{2406147}{130 - 1/\lambda^2} + 15998 - 38.9 - 1/\lambda^2. \tag{45}
\]

Following [71], one may obtain standard atmospheric temperature and pressure profiles from one of several atmospheric models, for instance the U.S. Standard Atmosphere model [72], or the more detailed NRLMSISE-00 model [73]. Using the former, atmospheric index of refraction profiles up to 80 km are computed, and we fit the computed values to a function of the form:

\[
\Delta n_{\text{atm}} = \frac{A_1}{B_1 + C_1 (h - H_1)} + \frac{A_2}{B_2 + C_2 (h - H_2)}, \tag{46}
\]

with \( h \) denoting altitude (in km) from an appropriate reference ellipsoid. The fitted parameter values are:

\[
\begin{align*}
A_1 &= -222.666, \\
B_1 &= 99.0621, \\
C_1 &= 0.157823 \text{ km}^{-1}, \\
H_1 &= -7.1541 \text{ km}, \\
A_2 &= -253.499, \\
B_2 &= 112.757, \\
C_2 &= 0.179669 \text{ km}^{-1}, \\
H_2 &= -7.15654 \text{ km}.
\end{align*}
\]

Though it would be aesthetically preferable to employ exponential functions in our model, we refrain from using them to avoid potential instabilities in the libraries we used for the integration of ODEs.

B. Ionospheric model

The effective index of refraction for electromagnetic wave propagation the ionosphere is given by the Appleton–Hartree equation [74, 75]. We consider here an approximation which assumes a collisionless plasma, and signal frequencies \( \omega := 2\pi f \) much greater than the gyrofrequency \( \omega_g := |qB_E/m_e| \) with \( B_E \) being the Earth’s magnetic field, and \( q, m_e \) being the respective charge and mass of the electron. For GNSS signals, \( f \sim 10^9 \text{ Hz} \), and \( \omega_g/\omega \approx 3 \times 10^{-3} \), so this approximation is reasonable. The corrections to the effective index of refraction \( n_{\text{ion}} \) from the gyrofrequency are in fact proportional to \( \omega_g/\omega \), and depend on the angle between the direction of radio wave propagation and \( B_E \). If gyrofrequency corrections become important, non-geometrical corrections to the geodesic equation may be needed, in which case the Gordon metric alone does not suffice for characterizing the propagation of electromagnetic signals. However, one may nonetheless suppress such corrections with higher signal frequencies.

Under the assumptions in the preceding paragraph, the Appleton-Hartree formula for the ionospheric phase index of refraction \( n_{\text{ph}} \) may be approximated as:

\[
\Delta n_{\text{ph}} \approx -\frac{\omega_p^2}{\omega^2} = -\left(4.024 \times 10^{-17}\right) [N_e/\text{m}^3], \tag{48}
\]

where \( \Delta n_{\text{ph}} = n_{\text{ph}} - 1 \), and \( \omega_p^2 = q^2 N_e/2\epsilon_0 m_e \) is the squared plasma frequency, with \( \epsilon_0 \) being the vacuum permittivity, and \( N_e \) the electron density in \( \text{m}^{-3} \). Since the corrections from the gyrofrequency \( \omega_g \) are linear, we assume that the index \( \Delta n_{\text{ph}} \) can only be modeled up to a precision of 0.3% for signals in the GHz range. It should be mentioned that since \( n_{\text{ph}} < 1 \), \( n_{\text{ph}} \) can only be the index of refraction associated with the phase velocity. To obtain the index of refraction associated with the group velocity, one employs the dispersion relation \( \omega^2 = k^2 + \omega_p^2 \) for cold, collisionless plasmas [74] to obtain the following expression for the group index of refraction (with \( \Delta n_{\text{ion}} = n_{\text{ion}} - 1 \)):

\[
\Delta n_{\text{ion}} \approx -\Delta n_{\text{ph}}. \tag{49}
\]

The electron density \( N_e \) can be determined by measurement and modeling; the Global Positioning System employs the Klobuchar model [76] and the Galileo GNSS makes use of the NeQuick-G ionospheric model detailed in [77] (which is a revised version of the NeQuick model in [78]). In these models, the ionospheric profile is described in terms of the dimensionless Epstein function:

\[
Ep(h, h_c, B) := \frac{4 \exp \left(\frac{h-h_c}{B}\right)}{\exp \left(\frac{h-h_c}{B}\right) + 1}^2, \tag{50}
\]

which has the form of a line shape function. One may approximate the above with the pseudo-Epstein function:

\[
\tilde{E}p(h, h_c, B) := \frac{1}{16} \left\{ \left[ 1 + \left(\frac{h-h_c}{2B}\right)^2 \right]^{-1} + \left[ 1 + \left(\frac{h-h_c}{4B}\right)^2 \right]^{-2} \right\}
+ \left[ 1 + \left(\frac{h-h_c}{2B}\right)^2 \right]^{-3} + \left[ 1 + \left(\frac{h-h_c}{4B}\right)^2 \right]^{-4} \right\}^2 \tag{51}
\]

which differs from the Epstein function by roughly one part in \( 10^3 \); this suffices, since the approximation for the Appleton–Hartree equation is only valid to \( 3 \times 10^{-3} \) for GNSS signal frequencies.

For simplicity, we construct a simple model for the electron density \( N_e \) consisting of a sum of pseudo-Epstein functions:

\[
N_e := \alpha_D \tilde{E}p(h, h_D, b_D) + \alpha_E \tilde{E}p(h, h_E, b_E)
+ \alpha_F \tilde{E}p(h, h_F, b_E) \right] \tag{52}
\]
The subscripts D, E, F on the parameters correspond to the respective ionospheric layers. Of course the precise profiles for the ionospheric layers are rather complicated and time-dependent, depending on the time of day and calendar date; in practice, such detailed profiles are provided by the aforementioned ionospheric models (see for instance [77] for a detailed description of the NeQuick-G model employed in the Galileo system). However, a simplified model for the ionospheric layers will suffice for demonstrating the viability of our algorithm. We choose the parameter values:

\[
\begin{align*}
\alpha_D &= 10^{12} \text{ m}^{-3} & h_D &= 75 \text{ km} & b_D &= 5 \text{ km} \\
\alpha_E &= 2.5 \times 10^{11} \text{ m}^{-3} & h_E &= 130 \text{ km} & b_E &= 30 \text{ km} \\
\alpha_F &= 10^{11} \text{ m}^{-3} & h_F &= 300 \text{ km} & b_F &= 50 \text{ km}.
\end{align*}
\]

(C. Perturbation model

To evaluate the potential accuracy of our algorithm, we will introduce perturbations to simulate the effect of uncertainties and errors in modeling the effective index of refraction. The perturbations we introduce are simple rescalings of the form:

\[
n_{\text{eff}} = 1 + \Delta n_{\text{atm}}(1 + \delta_1 \tilde{p}_1(h)) + \Delta n_{\text{ion}}(1 + \delta_2 \tilde{p}_2(h)),
\]

where \(\tilde{p}_A(h)\) (\(A \in \{1, 2\}\)) denotes a perturbation function \(-1 < \tilde{p}_A(h) < 1\) and \(\delta_1\) and \(\delta_2\) correspond to the respective fractional perturbations to \(\Delta n_{\text{atm}}\) and \(\Delta n_{\text{ion}}\). For the tests, we choose \(\tilde{p}_A(h)\) to have the form:

\[
\tilde{p}_A(h) = \sum_i \alpha_i \tilde{L}_s(h, h_0, \sigma, \sigma_i)
\]

where \(\alpha_i\) are coefficients and \(\tilde{L}_s(h, h_0, \sigma)\) represents a line shape function centered at \(h = h_0\) with a width \(\sigma\). We choose for the line shape function the following:

\[
\tilde{L}_s(h, h_0, \sigma) = \frac{\sigma^2}{\sigma^2 + (h - h_0)^2} \frac{\sigma^4}{\sigma^4 + (h - h_0)^4}
\]

which qualitatively resembles a Lorentzian function, but with a faster falloff. For the coefficients, we choose \(\alpha_i = +1\) (to maximize the refraction of the geodesics), and for \(\tilde{h}_0\) and \(\tilde{\sigma}\), we choose for \(p_1(h)\) the following:

\[
\begin{align*}
\tilde{h}_0 &= (0, 4, 8, 12, 16) \\
\tilde{\sigma} &= (2.0, 1.5, 1.8, 1.7, 1.5),
\end{align*}
\]

and for \(p_2(h)\)

\[
\begin{align*}
\tilde{h}_0 &= (150, 200, 250, 300, 350) \\
\tilde{\sigma} &= (21, 15, 18, 21, 10).
\end{align*}
\]

all in units of km.

We now discuss estimates for \(\delta_1\), which corresponds to the magnitude of fractional uncertainties in \(\Delta n_{\text{atm}}\) due to variations in humidity and measurement uncertainties in the temperature and pressure profiles near the surface of the Earth. Humidity variations contribute \(\sim 2 \times 10^{-4}\) in \(\delta_1\) (see [79] for a formula from which one may derive this estimate). Achievable uncertainties [80] of up to \(\delta P = 15\,\text{Pa}\) and \(\delta T = 0.2\,\text{C}\) at sea level correspond to a contribution of \(\sim 7 \times 10^{-4}\) in \(\delta_0\). After including humidity variations (with addition in quadrature), one arrives at an uncertainty of \(\sim 7.4 \times 10^{-4}\), which we round up to obtain \(\delta_1 = 10^{-3}\).

For uncertainties in the ionosphere, we consider several values for \(\delta_2\), the magnitude of fractional uncertainties in \(\Delta n_{\text{ion}}\), which is determined by uncertainties in the ionospheric electron density. One might expect to model the ionospheric electron density to an accuracy of a few percent, as uncertainties of <10% in the total electron content (the electron density integrated along a path) can in principle be achieved [81]. In our tests, we will consider values up to \(\delta_2 = 0.10\), which corresponds to an uncertainty of 10% in the magnitude of fractional uncertainties in \(\Delta n_{\text{ion}}\).

VI. CODE TEST AND BENCHMARKS

A. Test description

The tests of the squirrel.jl code are performed in a manner similar to the comparison tests in Sec. IV A of the cereal.jl code with Kerr-Schild geodesics. In particular, we generated a set of \(10^3\) target points \(X_{\text{tar}}\) on the WGS-84 reference ellipsoid and initial data for a spray of \(N \geq 5\) null geodesics from each of the target points (with the exception of the benchmark tests, which include \(N = 4\) emission points). We consider up to 6 emission points in our tests since GNSS satellite constellations are typically designed with the requirement that six satellites are in view at any given time [82, 83]. We then integrate each geodesic to a radial coordinate value of \(\sim 26.5 \times 10^9\,\text{km}\), the endpoints of which are then used as inputs for the locator functions in the squirrel.jl code. The output of the squirrel.jl code is then compared with the target points \(X_{\text{tar}}\).

The effective geometry is described by the Gordon metric (38) with the effective index of refraction given by Eqs. (43), (46), (48) and (49). For the “background” space-time geometry, we use the weak field metric (39), which incorporates gravitational effects.

All test calculations were performed with double floating point precision (Float64) to reduce execution time, though squirrel.jl is written to accommodate extended precision calculations (DoubleFloats [48], for instance). For the generation of test cases, the tolerance (both relative and absolute) for the ODE solvers is chosen to be \(10^{-14}\), and a high order integrator is employed, in particular the 9th order AutoVern9 in OrdinaryDiffEq.jl.
(as opposed to the 7th order AutoVern7 integrator used in the locator functions in the squirrel.jl code). A basic validation test was performed for the generation of test cases, where the outputs have been compared for different tolerances; in all cases, relative differences were on the order of machine precision. For the location code, the (user specified) tolerances are chosen to be $10^{-10}$ to reduce execution time.

B. Test results

Test results for $n = 5$ emission points are presented in Figs. 5 through 12. The vertical errors correspond to errors projected in the direction orthogonal to the WGS-84 reference ellipsoid, and the horizontal errors correspond to errors projected along directions tangent to the WGS-84 ellipsoid. For the Kerr metric, the squirrel.jl code can achieve in most cases submillimeter accuracy, demonstrating a potential for extreme precision in vacuum environments; the methods presented in the squirrel.jl
code may be ideal for relativistic location in space navigation. When atmospheric and ionospheric effects are included, the squirrel.jl code yields errors on the order of a centimeter, as illustrated in Figs. 7 and 8.

Figures 9 through 12 illustrate positioning errors when including uncertainties in the determination of the atmospheric and ionospheric index of refraction. These tests were performed using the perturbation model described in the preceding section; the test cases were generated with the unperturbed metric, and for the tests themselves, the perturbed metric is used in the locator functions of the squirrel.jl code (which take the metric functions as an input). Errors resulting from an uncertainty of 1% in the determination of the ionospheric refractive index are illustrated in Figs. 9 and 10; this is the best result one can realistically expect to achieve with
the approximation (48) for GNSS signal frequencies of \(\sim 1\) GHz (but we reiterate that higher signal frequencies can achieve improved accuracy with the same approximation). Even then, the horizontal positioning errors are for the most part confined to less than \(\sim 10\) cm to a 95% confidence level, while the vertical errors exhibit a systematic shift of \(\sim 20\) cm, which corresponds to the fact that the perturbations to the index of refraction in our perturbation model are positive, and that the index of refraction profiles vary primarily in the radial direction.

Errors from an uncertainty of 10% in the ionospheric profile are illustrated in Figs. 11 and 12, which increases the horizontal errors to \(\sim 1\) m and the systematic shift in the vertical errors to \(\sim 2\) m. Upon comparison with the single frequency errors reported in the latest Galileo quarterly report [84] for the first three months of 2021, we note that the RMS and 95% confidence level values are somewhat comparable to the performance of Galileo (RMS and 95% CL \(\sim 1\) m), albeit for a smaller sample size in our case. Of course, the results presented here do not take into account other GNSS errors, such as multipath, satellite timing and ephemeris errors, the latter two of which have been addressed in [20, 35–38]. In the Galileo error budget, such errors [referred to as Signal in Space Errors (SISE)] are on the order of half a meter [84], and for single-frequency users, are smaller in magnitude than the error contributions from ionospheric and tropospheric effects. To compare the errors presented in this article with those of [84], one should add in quadrature a \(\sim 0.5\) m contribution from SISE; even with such a correction to the RMS and 95% CL values presented in Fig. 11 for horizontal positioning, our corrected errors (0.776 m [RMS] and 1.06 m [95%CL]) remain smaller than those reported in [84], and with fewer errors \(\geq 20\) m in proportion. This indicates a potential for improved performance, even with an uncertainty\(^5\) of 10% in the determination of the ionospheric free electron density and for the stated uncertainties in the determination of atmospheric parameters in the lower troposphere. Moreover, reduced uncertainties in the determination of the ionospheric electron density profile to the 1% level can reduce the 95% CL errors by a factor of 10, to roughly a decimeter.

One can obtain improved accuracy with additional emission points. With \(n = 6\) emission points, the largest errors are significantly reduced, as indicated in the figures 13 through 16 for the cases with 1% and 10% fractional uncertainty in the ionospheric electron density \(N_e\). These two cases are chosen since they form the boundary cases for the accuracy that one might expect to be achievable with the squirrel.jl code at GNSS signal frequencies of \(\sim 1\) GHz. In both cases, we find a significant reduction in the number of large errors for \(n = 6\) emission points. There were no errors above the stated thresholds for the \(n = 5\) cases (2 m for the 1% case and 20 m for the 10% case); for the \(n = 6\) cases, we find only one sample with an error \(> 1\) m threshold for a 1% uncertainty in \(N_e\), and 7 samples with horizontal errors \(> 5\) m a 10% uncertainty in \(N_e\). The RMS and 95% confidence level errors are roughly the same for the vertical errors (owing to the systematic shift that the perturbations introduce), but are reduced by a factor of 2 for the horizontal errors. This result indicates that in principle, the inclusion of additional emission points can significantly reduce the number of large errors in the squirrel.jl code.

### C. Benchmarks

Some basic benchmarks have been performed for various situations; the results are displayed in Table I. The results we report were obtained on standard desktop computer with an Intel i5-7500 processor, and with

\[^5\] An interesting question worth investigating (left for future work) is whether one can construct simple, high-accuracy ionospheric models which reduce this uncertainty—see [81, 85].

![Graph of horizontal positioning errors](image-url)

**TABLE I.** Benchmarks performed with squirrel.jl. \(N\) is the number of emission points. Tests were performed with the Kerr-Schild metric, the unperturbed Gordon metric, and the perturbed Gordon metric corresponding to a 10% uncertainty in the ionospheric profile.

| \(N\) | Kerr-Schild | Gordon | Pert. Gordon |
|------|-------------|--------|--------------|
| 4    | 27 ms       | 193 ms | 223 ms       |
| 5    | 101 ms      | 553 ms | 588 ms       |
| 6    | 358 ms      | 1.74 s | 1.95 s       |

![Graph of horizontal positioning errors](image-url)
four threads enabled. The benchmarks were performed for three geometries: the Kerr-Schild metric, the unperturbed Gordon metric (representing atmospheric and ionospheric effects), and the perturbed Gordon metric corresponding to a 10\% uncertainty in the ionospheric profile. We consider three cases, with $N = 4, 5, 6$ emission points. We note that the $N = 4$ Kerr-Schild case has an execution time comparable to that reported in [37, 38] for a Schwarzschild location method. In the case of $N = 4$ emission points, the flat spacetime methods of [40, 41] and Sec. II B return two guesses due to the bifurcation problem, so the squirrel algorithm is applied twice (one for each guess) for $N = 4$ emission points. This is seen in the fact that the execution time for $N = 4$ is longer than one might expect from the number of combinations $C(5, 4) = 5$, which is supported by the execution times for the Gordon and perturbed Gordon cases.\(^6\) Comparing the $N = 5$ and $N = 6$ cases, we find a scaling roughly consistent with the number of combinations $C(5, 4) = 5, C(6, 4) = 15$, which suggests an increase in computational complexity by a factor of three.

\section{Summary and Discussion}

In this article, we have described and demonstrated a new method for relativistic location in slightly curved, but otherwise generic spacetime geometries. Though such methods may be of primary interest for high precision space navigation in regions beyond the ionosphere, we have demonstrated, by way of simple analogue gravity models, that our method can nonetheless be used to incorporate atmospheric and ionospheric effects in terrestrial positioning. Though one might regard such effects as ancillary from a purely general relativistic perspective,

\(^6\) The discrepancy in the Kerr-Schild case may be due to overhead related to the different methods employed by the \texttt{squirrel} code between the $N = 4$ and $N = 5$ cases.
we argue that they are still of fundamental importance in the sense that the placement of emission coordinates near the surface of the Earth will depend on the knowledge of the profile for the effective atmospheric and ionospheric refractive index.

The methods we have described and implemented make use of state of the art automatic differentiation and ODE libraries available in the Julia language, which permit the efficient evaluation of the derivatives of numerical solutions of the geodesic equation performed with respect to initial data. Combined with a quasi-Newton root-finding algorithm, we have demonstrated that our methods can, with guesses provided by the flat spacetime relativistic location formula of [21], accurately and efficiently compute the intersection point of future pointing null cones from a set of spacelike separated emission points. In particular, our implementation, the squirrel.jl code [56], can with 5 emission points achieve submillimeter accuracy for terrestrial positioning (satellite orbits at ∼ 26.5 × 10^3 km, target point at surface of Earth) in a vacuum Kerr-Schild metric. When atmospheric and ionospheric effects are included by way of the Gordon a vacuum Kerr-Schild metric. When atmospheric and ionospheric free electron density profile, and to less than 10 cm for a 1% uncertainty, with an execution time of less than 1 s on a desktop computer for five emission points.

Our test results indicate that the relativistic location algorithm implemented in the squirrel.jl code can achieve extreme precision for space navigation in vacuum regions beyond the ionosphere; we will describe in detail the applications of our methods to deep space navigation elsewhere. Our tests also indicate that implementations of our method have the potential for performance comparable to or exceeding the single-frequency performance of Galileo, assuming that the local atmospheric properties of the troposphere are known to typical measurement uncertainties and the ionospheric free electron profile is known to an uncertainty of 10% or less. Alternatively, the methods presented in this article may perhaps be of interest as an additional method for atmospheric and ionospheric tomography (see [86] and references therein for an overview of methods in GNSS tomography).

The results we have presented here are complementary to those of [20, 35–38], which addresses the problem of incorporating general relativistic effects in the determination of satellite ephemerides. A more comprehensive collection of methods for relativistic positioning will require at the minimum a relativistic location algorithm and an algorithm for determining satellite ephemerides, with the goal of developing methods capable of reducing the largest contributions to GNSS error budgets.

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[1] Neil Ashby, “Relativity and the global positioning system,” Physics Today 55, 41–47 (2002).
[2] Neil Ashby, “Relativity in the Global Positioning System,” Living Rev. Rel. 6, 1 (2003).
[3] Peter JG Teunissen and Oliver Montenbruck, Springer handbook of global navigation satellite systems, Vol. 1 (Springer, 2017).
[4] Anthony Flores, “Global positioning system wing (gpsw) systems engineering & integration, interface specification is-gps-200n,” https://www.gps.gov/technical/icwg/IS-GPS-200E.pdf (2010), [Online; accessed 15-December-2021].
[5] Bartolome Coll, “Relativistic positioning systems,” AIP Conf. Proc. 841, 277–284 (2006), arXiv:gr-qc/0601110.
[6] Bartolome Coll and Jose Marla Pozo, “Relativistic positioning systems: The Emission coordinates,” Class. Quant. Grav. 23, 7395–7416 (2006), arXiv:gr-qc/0606044.
[7] Etienne Augé, Jacques Dumarchez, and Jean Tran Thanh Van, “Gravitational waves and experimental gravity,” in The XLVIth Recontres de Moriond and GPhyS Colloquium (2011).
[8] J. F. Pascual-Sanchez, “Introducing Relativity in Global Navigation Satellite Systems,” Annalen Phys. 16, 258–273 (1997), arXiv:gr-qc/0507121.
[9] Bartolomé Coll, “Epistemic relativity: An experimental approach to physics,” Fundam. Theor. Phys. 196, 291–315 (2019), arXiv:1712.05712 [gr-qc].
[10] Neus Puchades and Diego Sáez, “Approaches to relativistic positioning around Earth and error estimations,” Adv. Space Res. 57, 499–508 (2016), arXiv:1601.05712 [gr-qc].
[11] Neus Puchades and Diego Sáez, “Relativistic Positioning Systems: Numerical Simulations,” Acta Futura 7, 103–110 (2013), arXiv:1404.1000 [gr-qc].
[12] Neus Puchades and Diego Saez, “Relativistic positioning: Four-dimensional numerical approach in Minkowski space-time,” Astrophys. Space Sci. 341, 631–643 (2012), arXiv:1112.6054 [gr-qc].
[13] Bartolome Coll, Joan Josep Ferrando, and Juan Antonio Morales-Lladosa, “Positioning in a flat two-dimensional space-time: the delay master equation,” Phys. Rev. D 82, 084038 (2010), arXiv:1005.0477 [gr-qc].
[14] Bartolome Coll, Joan Josep Ferrando, and Juan Anto-
pects of the Galileo Programme (2009) arXiv:0912.4418 [gr-qc].
[52] Simon David Burton, Numerical Experimentation in Differential Geometry, Ph.D. thesis, The Australian National University (2007).
[53] C. G. Broyden, “A class of methods for solving nonlinear simultaneous equations,” Mathematics of Computation 19, 577–593 (1965).
[54] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, Numerical Recipes 3rd Edition: The Art of Scientific Computing (Cambridge University Press, 2007).
[55] Pierre Christian and Chi-Kwan Chan, “FANTASY: User-Friendly Symplectic Geodesic Integrator for Arbitrary Metrics with Automatic Differentiation,” Astrophys. J. 909, 67 (2021), arXiv:2010.02237 [gr-qc].
[56] Christopher Rackauckas and Qing Nie, “Differential equations in julia,” https://github.com/justincfeng/squirrel.jl (2021).
[57] Pierre Christian and Chi-Kwan Chan, “FANTASY: User-Friendly Symplectic Geodesic Integrator for Arbitrary Metrics with Automatic Differentiation,” Astrophys. J. 909, 67 (2021), arXiv:2010.02237 [gr-qc].
[58] James H Verner, “Numerically optimal runge–kutta pairs with interpolants,” Numerical Algorithms 53, 383–396 (2010).
[59] Christopher Rackauckas and Qing Nie, “Differentialequations.jl – a performant and feature-rich ecosystem for solving differential equations in julia,” The Journal of Open Research Software 7:1, e151 (2021).
[60] G Di Marzo, “Rodas5 (4)-méthodes de rosenbrock d’ordre 5 (4) adaptées aux problemes différentiels-algébriques,” MSc Mathematics Thesis (1993).
[61] J. Revels, M. Lubin, and T. Papamarkou, “Forward-mode automatic differentiation in Julia,” arXiv:1607.07892 [cs.MS] (2016).
[62] James H Verner, “Numerically optimal runge–kutta pairs with interpolants,” Numerical Algorithms 53, 383–396 (2010).
[63] Christopher Rackauckas and Qing Nie, “Differentialequations.jl – a performant and feature-rich ecosystem for solving differential equations in julia,” The Journal of Open Research Software 7:1, e151 (2021).
[64] G Di Marzo, “Rodas5 (4)-méthodes de rosenbrock d’ordre 5 (4) adaptées aux problemes différentiels-algébriques,” MSc Mathematics Thesis (1993).
[65] J. C. Feng, F. Hejda, and S. Carloni, “squirrel.jl,” https://github.com/justincfeng/squirrel.jl (2021).
[66] Christopher Rackauckas and Qing Nie, “Differentialequations.jl – a performant and feature-rich ecosystem for solving differential equations in julia,” The Journal of Open Research Software 7:1, e151 (2021).
[67] Pierre Christian and Chi-Kwan Chan, “FANTASY: User-Friendly Symplectic Geodesic Integrator for Arbitrary Metrics with Automatic Differentiation,” Astrophys. J. 909, 67 (2021), arXiv:2010.02237 [gr-qc].