SUBGROUP S–COMMUTATIVITY DEGREE OF FINITE GROUPS

DANIELE ETTORE OTERA AND FRANCESCO G. RUSSO

ABSTRACT. The so–called subgroup commutativity degree \( sd(G) \) of a finite group \( G \) is the number of permuting subgroups \((H, K) \in L(G) \times L(G)\) where \( L(G) \) is the subgroup lattice of \( G \), divided by \( |L(G)|^2 \). It allows us to measure how \( G \) is far from the celebrated classification of quasihamiltonian groups of K. Iwasawa. Here we generalize \( sd(G) \), looking at suitable sublattices of \( L(G) \), and show some new lower bounds.

1. Introduction

All groups in the present paper are supposed to be finite. Given two subgroups \( H \) and \( K \) of a group \( G \), the product \( HK = \{hk \mid h \in H, k \in K\} \) is not always a subgroup of \( G \). \( H \) and \( K \) permute if \( HK = KH \), or equivalently, if \( HK \) is a subgroup of \( G \). \( H \) is said to be permutable (or quasinormal) in \( G \), if it permutes with every subgroup of \( G \). It is possible to strengthen this notion in various ways. \( H \) is \( S \)–permutable (or \( S \)–quasinormal) in \( G \), if \( H \) permutes with all Sylow subgroups of \( G \) (for all primes in the set \( \pi(G) \) of the prime divisors of \( |G| \)). Historically, O. Kegel introduced \( S \)–permutable subgroups in 1962, generalizing a well–known result of O. Ore of 1939, who proved that permutable subgroups are subnormal (see [9] [15] for details). Roughly speaking, this notion deals with subgroups which are permutable with maximal subgroups. Several authors investigated the topic in the successive years and we mention [1] [2] [14] [15] for our aims.

The subgroup lattice \( L(G) \) of a group \( G \) is the set of all subgroups of \( G \) and is a complete bounded lattice with respect to the set inclusion, having initial element the trivial subgroup \( \{1\} \) and final element \( G \) itself (see [8] [15]). Its binary operations \( \wedge, \vee \) are defined by \( X \wedge Y = X \cap Y, X \vee Y = (X \cup Y), \) for all \( X, Y \in L(G) \). Furthermore, \( L(G) \) is modular, if all the subgroups of \( G \) satisfy the modular law. \( G \) is modular, if \( L(G) \) is modular (see [15] Section 2.1]). This notion is important, because of the following concept. A group \( G \) is quasihamiltonian, if all its subgroups are permutable. By a result of K. Iwasawa [15] Theorem 2.4.14, quasihamiltonian groups are classified, but, at the same time, these groups are characterized to be nilpotent and modular (see [15] Exercise 3, p.87]).

Now we recall some terminology from [14], which will be useful in the rest of the paper. Any non–empty set of subgroups \( S(G) \) of \( G \) may be always regarded as a sublattice of \( L(G) \) having initial element \( \bigwedge_{S \in S(G)} S \) and final element \( \bigvee_{S \in S(G)} S \). The symbol \( S^\perp(G) \) denotes the set of all subgroups \( H \) of \( G \) which are permutable.
with all $S \in S(G)$ and it is easy to check that $S^\perp(G)$ is a sublattice of $L(G)$ (see \[\text{[14]}\] Section 1). There is a wide literature when we choose $S(G)$ to be equal to the sublattice $M(G)$ of all maximal subgroups of $G$, or to the sublattice $sn(G)$ of all subnormal subgroups of $G$, or also to the sublattice $n(G)$ of all normal subgroups of $G$. Consequently, $L^\perp(G)$ is the sublattice of all permutable subgroups of $G$, $M^\perp(G)$ that of the subgroups permutable with all maximal subgroups of $G$ and so on for $sn^\perp(G)$ and $n^\perp(G) = L(G)$. Immediately, the role of the operator $\perp$ appears to be very intriguing for the structure of $G$ and several authors investigated this aspect. For instance, $G$ is quasihamiltonian if and only if $L(G) = L^\perp(G)$.

In Section 2 we will describe a notion of probability on $L(G)$, beginning from groups in which the subgroups in $sn(G)$ permutes with those in $M(G)$. The generality of the methods (we follow \[\text{[3, 4, 6, 7, 10, 11, 12, 13, 17]}\]) may be translated in terms of arbitrary sublattices, satisfying a prescribed restriction. Section 3 shows some consequences on the size of $|L(G)|$.

2. Measure theory on subgroup lattices

The following notion has analogies with \[\text{[6, Definitions 2.1, 3.1, 4.1]}\] and \[\text{[12, Equation 1.1]}\] and will be treated as in \[\text{[3, 4, 6, 7, 11, 12, 13, 17]}\].

**Definition 2.1.** For a group $G$,

\[
spd(G) = \frac{\{(X,Y) \in sn(G) \times M(G) \mid XY = YX\}}{|sn(G)| \cdot |M(G)|},
\]

is the subgroup $S$-commutativity degree of $G$.

$0 < spd(G) \leq 1$ denotes the probability that a randomly picked pair $(X,Y) \in sn(G) \times M(G)$ is permuting, that is, $XY = YX$. \[\text{(2.1)}\] may be rewritten, introducing the function $\chi : sn(G) \times M(G) \to \{0,1\}$ defined by

\[
\chi(X,Y) = \begin{cases} 
1, & \text{if } XY = YX, \\
0, & \text{if } XY \neq YX,
\end{cases}
\]

in the following form

\[
spd(G) = \frac{1}{|sn(G)| \cdot |M(G)|} \sum_{(X,Y) \in sn(G) \times M(G)} \chi(X,Y).
\]

In Definition 2.1 and 2.3, we may replace $sn(G) \times M(G)$ with $S(G) \times T(G)$, where $S(G)$ and $T(G)$ are two arbitrary sublattices of $L(G)$. We have chosen $sn(G) \times M(G)$, because \[\text{[11, 12]}\] describe the structure of the groups in which the subnormal subgroups permute with all Sylow subgroups (called $PST$-groups). If $Syl(G)$ is the set of all Sylow subgroups of $G$, $Syl(G) \subseteq M(G)$ and this means that we have already a classification for a group $G$ such that $sn(G) \subseteq Syl(G)^\perp$.

\[\text{2.3}\] allows us to to treat the problem from the point of view of the measure theory on groups. A computational advantage may be found in a formula for $spd(G_1 \times G_2)$, where $G_1$ and $G_2$ are two given groups.

**Corollary 2.2.** Let $G_i$ be a family of groups of coprime orders for $i = 1,2,\ldots,k$. Then $spd(G_1 \times G_2 \times \ldots \times G_k) = spd(G_1) \cdot spd(G_2) \cdots spd(G_k)$.

The techniques of proof are straightforward applications of \[\text{2.3}\] and the details are omitted. However, it is good to note that Corollary 2.2 shows the stability with respect to forming direct products of $spd(G)$: this fact was proved in \[\text{[3, 4, 6, 7]}\].
in different contexts. Another basic property is to relate \( spd(G) \) to quotients and subgroups of \( G \).

Let \( G = NH \) for a normal subgroup \( N \) of \( G \) and a subgroup \( H \) of \( G \) isomorphic to \( G/N \) (briefly, \( H \simeq G/N \)). In general, it is easy to check that \( sn(G/N) \) is lattice isomorphic to \( sn(H) \) (briefly, \( sn(G/N) \sim sn(H) \)) and that \( M(G/N) \sim M(H) \). We will concentrate on some special classes of groups, satisfying

\[
(2.4) \quad (sn(G/N) \times M(G/N)) \sim (sn(H) \times M(H)) \subseteq sn(G) \times M(G)
\]

\( (2.4) \)–\( (2.5) \), jointly with \( (2.3) \), allow us to conclude

\[
(2.5) \quad sn(N) \times M(N) \subseteq sn(G) \times M(G).
\]

Let \( G = NH \) for a normal subgroup \( N \) of \( G \) and a subgroup \( H \) of \( G \) isomorphic to \( G/N \) (briefly, \( H \simeq G/N \)). In general, it is easy to check that \( sn(G/N) \) is lattice isomorphic to \( sn(H) \) (briefly, \( sn(G/N) \sim sn(H) \)) and that \( M(G/N) \sim M(H) \). We will concentrate on some special classes of groups, satisfying

\[
(2.4) \quad (sn(G/N) \times M(G/N)) \sim (sn(H) \times M(H)) \subseteq sn(G) \times M(G)
\]

\( (2.4) \)–\( (2.5) \), jointly with \( (2.3) \), allow us to conclude

\[
(2.6) \quad \sum_{(X,Y) \in sn(G) \times M(G)} \chi(X,Y) \geq \sum_{(X,Y) \in sn(N) \times M(N)} \chi(X,Y);
\]

\[
(2.7) \quad \sum_{(X,Y) \in sn(G) \times M(G)} \chi(X/N, Y/N) = \sum_{(Z,T) \in M(H) \times M(H)} \chi(Z,T)
\]

and consequently

\[
(2.8) \quad 2|sn(G)| \cdot |M(G)| \cdot spd(G) \geq \sum_{(X,Y) \in sn(N) \times M(N)} \chi(X,Y) + \sum_{(Z,T) \in M(H) \times M(H)} \chi(Z,T).
\]

\[
= |sn(N)| \cdot |M(N)| \cdot spd(N) + |sn(G/N)| \cdot |M(G/N)| \cdot spd(G/N).
\]

Similar techniques have been used by Tănăsceanu [17] in order to study the subgroup commutativity degree

\[
(2.9) \quad sd(G) = \frac{\left| \{(X,Y) \in L(G)^2 \mid XY = YX \} \right|}{|L(G)|^2} = \frac{1}{|L(G)|^2} \sum_{(x,y) \in L(G)^2} \chi(x,y).
\]

[17] can be seen as a natural extension, to the context of the lattice theory, of the concept of commutativity degree

\[
(2.10) \quad d(G) = \frac{\left| \{(x,y) \in G^2 \mid xy = yx \} \right|}{|G|^2} = \frac{1}{|G|^2} \sum_{x \in G} |C_G(x)|,
\]

where \( C_G(x) = \{ g \in G \mid gx = xg \} \). There are several contributions on \( d(G) \) in [3 [4 [6 [7 [10 [11 [12 [13]. The main strategy of investigation is to begin with the case of equality at 1 and then describe the situation, when we leave this extremal case. Upper and lower bounds will measure the distance from known classes of groups. For instance, \( d(G) = 1 \) if and only if \( G \) is abelian; \( sd(G) = 1 \) if and only if \( L(G) = L(G)^\perp \). Therefore the next are milestones for the rest of the paper.

**Corollary 2.3.** In a group \( G \) we have \( spd(G) = 1 \) if and only if \( sn(G) \subseteq M^+(G) \) or \( M(G) \subseteq sn^+(G) \).

**Proof.** It follows from the above considerations. \( \square \)

**Corollary 2.4.** If \( G \) is a nilpotent group, then \( spd(G) = 1 \).

**Proof.** Application of Corollary 2.3 noting that \( M(G) \subseteq n(G) \subseteq sn^+(G) \). \( \square \)

**Corollary 2.5.** In a group \( G \) we have \( \frac{|sn(G)| \cdot |M(G)|}{|L(G)|^2} \cdot spd(G) \leq sd(G) \) and the equality holds if and only if \( sn(G) = M(G) = L(G) \).
Proof. Since \( \text{sn}(G) \times M(G) \subseteq L(G)^2 \), \( \{(X, Y) \in \text{sn}(G) \times M(G) \mid XY = YX \} \subseteq \{(X, Y) \in L(G)^2 \mid XY = YX \} \) and then

\[
|\text{sn}(G)| |M(G)| \text{spd}(G) = |\{(X, Y) \in \text{sn}(G) \times M(G) \mid XY = YX \}|
\]

\[
\leq |\{(X, Y) \in L(G)^2 \mid XY = YX \}| = |L(G)|^2 \text{sd}(G)
\]

from which the inequality follows. The rest is clear. \( \square \)

Corollary 2.4 clarifies the situation for nilpotent groups. Then we proceed to study solvable groups. Unfortunately, these cannot be described as in [17 Proposition 2.4]: Different techniques are necessary. We recall that an abelian group \( A \) of order \( n = p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m} \), for suitable powers of \( p_1, p_2, \ldots, p_m \in \pi(A) \), has a canonical decomposition of the form \( A \cong A_1 \times A_2 \times \ldots \times A_m \), where \( n_1, \ldots, n_m \) are positive integers and \( A_1, A_2, \ldots, A_m \) are the prime factors. It is well-known that \( |L(A)| = |L(A_1)| \cdot |L(A_2)| \cdot \ldots \cdot |L(A_m)| \). In case \( p = p_1 = p_2 = \ldots = p_m \) [18 Proposition 3.2] shows that the number of maximal subgroups of the elementary abelian \( p \)-group \( \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p^{\alpha_k}} \) is equal to \( \frac{p^k - 1}{p - 1} \), for suitable integers \( 1 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k \) and \( k \geq 1 \).

**Lemma 2.6.** Let \( N = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \) be a non-trivial normal abelian subgroup of \( G \) with \( 0 \leq \alpha_1 + \alpha_2 \) and \( 1 \leq \alpha_1 \leq \alpha_2 \) such that \( G/N \) is of prime order and (2.4)–(2.5) are satisfied. Then

\[
\text{spd}(G) \geq \frac{f(p, \alpha_1, \alpha_2)}{2 |\text{sn}(G)| |M(G)|},
\]

where \( f(p, \alpha_1, \alpha_2) = \frac{1}{p^2 - 2p + 1} \left( (\alpha_2 - \alpha_1 + 1)p^{\alpha_1 + \alpha_2 + 2} - (\alpha_2 - \alpha_1 - 1)p^{\alpha_1 + 1} - (\alpha_1 + \alpha_2 - 1)p^2 - (\alpha_1 + \alpha_2 + 11)p + (\alpha_1 + \alpha_2 + 5) \right) \) is a polynomial function depending only on \( N \).

**Proof.** We note that \( G = HN \), where \( G/N \cong H \) is of prime order, so that it is meaningful to formulate the conditions in (2.4) and (2.5), requiring that they are satisfied. From (2.3),

\[
(2.11) \quad \text{spd}(G) \geq \frac{|\text{sn}(N)| |M(N)| \text{spd}(N) + |\text{sn}(G/N)| |M(G/N)| \text{spd}(G/N)}{2 |\text{sn}(G)| |M(G)|}
\]

\[
|\text{sn}(G/N)| = |M(G/N)| = 2, \text{ by a counting argument, and } \text{spd}(N) = \text{spd}(G/N) = 1, \text{ by Corollary 2.4, then (2.12)}
\]

\[
(2.12) \quad \frac{|\text{sn}(N)| |M(N)|}{2 |\text{sn}(G)| |M(G)|} + \frac{4}{2 |\text{sn}(G)| |M(G)|} = \frac{1}{2 |\text{sn}(G)| |M(G)|} (|\text{sn}(N)| |M(N)| + 4)
\]

[18] Theorem 3.3 implies \( |\text{sn}(N)| = |L(N)| = \frac{1}{(p - 1)^2} [(\alpha_2 - \alpha_1 + 1)p^{\alpha_1 + 2} - (\alpha_2 - \alpha_1 - 1)p^{\alpha_1 + 1} - (\alpha_1 + \alpha_2 + 3)p + (\alpha_1 + \alpha_2 + 1)] \), and, as noted above, \( |M(N)| = \frac{p^2 - 1}{p - 1} = p + 1 \), hence

\[
(2.13) \quad = \frac{1}{2 |\text{sn}(G)| |M(G)|} \left( \left( \frac{p + 1}{p - 1} \right)^2 ((\alpha_2 - \alpha_1 + 1)p^{\alpha_1 + 2} - (\alpha_2 - \alpha_1 - 1)p^{\alpha_1 + 1} - (\alpha_1 + \alpha_2 + 3)p + (\alpha_1 + \alpha_2 + 1) \right) + 4 \right)
\]

in order to write better the above expression we introduce the coefficients

\[
C_1 = \alpha_2 - \alpha_1 + 1; \; C_2 = \alpha_2 - \alpha_1 - 1; \; C_3 = \alpha_1 + \alpha_2 + 3; \; C_4 = \alpha_1 + \alpha_2 + 1
\]
and then we get
\[
= \frac{1}{2} \frac{1}{|\text{sn}(G)| \cdot |\text{M}(G)|} \left( \frac{1}{(p - 1)^2} (C_1 p^{\alpha_1 + 3} + (C_1 - C_2) p^{\alpha_1 + 2} - C_2 p^{\alpha_1 + 1}
\right.
\]
\[+(4 - C_3) p^2 - (8 + C_3) p + (4 + C_4)) \right).
\]

Developing the computations in the brackets, we get the polynomial \( f(p, \alpha_1, \alpha_2) \).
\( \square \)

**Lemma 2.7.** Let \( N = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \) be a non-trivial normal subgroup of \( G \) with \( 0 \leq \alpha_1 + \alpha_2 \) and \( 1 \leq \alpha_1 \leq \alpha_2 \) such that \( G/N \) is of prime order. Then
\[
sd(G) \geq \frac{g(p, \alpha_1, \alpha_2)}{2 |L(G)|^2},
\]
where \( g(p, \alpha_1, \alpha_2) = \frac{1}{(p - 1)^2} \left( (\alpha_2 - \alpha_1 + 1) p^{\alpha_1 + 2} - (\alpha_2 - \alpha_1 - 1) p^{\alpha_1 + 1} - (\alpha_1 + \alpha_2 + 3) p + (\alpha_1 + \alpha_2 + 1) \right)^2 + 4 \) is a polynomial function depending only on \( N \).

**Proof.** We note that \( G = HN \), where \( G/N \cong H \) is of prime order. [2.4] is in this case \( L(G/N)^2 \sim L(H)^2 \subseteq L(G)^2 \) and is always satisfied. Analogously, [2.5] becomes \( L(N)^2 \subseteq L(G)^2 \) and is satisfied, too. Then (2.8) becomes
\[
(2.14) \quad sd(G) \geq \frac{|L(N)|^2 sd(N) + |L(G/N)|^2 spd(G/N)}{2 |L(G)|^2}
\]
and, from the assumptions, \( |L(G/N)| = 2 \), \( sd(G/N) = sd(N) = 1 \) but again [18 Theorem 3.3] implies \( |L(N)|^2 = \frac{1}{(p - 1)^2} \left( (\alpha_2 - \alpha_1 + 1) p^{\alpha_1 + 2} - (\alpha_2 - \alpha_1 - 1) p^{\alpha_1 + 1} - (\alpha_1 + \alpha_2 + 3) p + (\alpha_1 + \alpha_2 + 1) \right)^2 \). Therefore
\[
(2.15) \quad \geq \frac{1}{2 |L(G)|^2} \left( \frac{1}{(p - 1)^2} \left( (\alpha_2 - \alpha_1 + 1) p^{\alpha_1 + 2} - (\alpha_2 - \alpha_1 - 1) p^{\alpha_1 + 1}
\right.
\]
\[-(\alpha_1 + \alpha_2 + 3) p + (\alpha_1 + \alpha_2 + 1) \right)^2 + 4),
\]
where it appears clear what is the polynomial function \( g(p, \alpha_1, \alpha_2) \), which we are looking for. \( \square \)

As usual \( Fit(G) \) denotes the Fitting subgroup of \( G \).

**Theorem 2.8.** Let \( G \) be a solvable group in which \( C = C_{G}(Fit(G)) = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \), for \( 0 \leq \alpha_1 + \alpha_2 \), \( 1 \leq \alpha_1 \leq \alpha_2 \), \( p \) a prime and \( |G : C| \) a prime.

(i) If (2.4) (2.5) are satisfied, then \( spd(G) \geq \frac{f(p, \alpha_1, \alpha_2)}{2 |\text{sn}(G)| \cdot |\text{M}(G)|} \), where \( f(p, \alpha_1, \alpha_2) \) is a polynomial function depending only on \( C \).

(ii) \( sd(G) \geq \frac{g(p, \alpha_1, \alpha_2)}{2 |L(G)|^2} \), where \( g(p, \alpha_1, \alpha_2) \) is a polynomial function depending only on \( C \).

**Proof.** Since \( G \) is solvable, it is well-known that \( C \) is an abelian normal subgroup of \( G \). Then our position is correct in assuming \( C = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \), with \( 0 \leq \alpha_1 + \alpha_2 \), \( 1 \leq \alpha_1 \leq \alpha_2 \), \( p \) prime and \( G/C \) of prime order. Now (i) is an application of Lemma [2.6] and (ii) of Lemma [2.7]. \( \square \)
The lower bound in Lemma 2.7 for $sd(G)$ is more precise than the following bound, which was the first to be presented in literature.

**Corollary 2.9** (See [17, Corollary 2.6].) A group $G$ possessing a normal abelian subgroup of prime index has $|L(G)|^2 sd(G) \geq |L(N)|^2 + 2|L(N)| + 1$.

A different restriction is obtained when we multiply up (2.6)–(2.7).

**Proposition 2.10.** Let $N$ be a normal subgroup of a group $G = NH$ satisfying (2.4) and (2.5). Then

$$spd(G) \geq \frac{1}{|\mathrm{sn}(G)||\mathrm{M}(G)|} \sum_{(X,Y)\in\mathrm{sn}(N)\times\mathrm{M}(N)} \chi(X,Y) \chi(Z,T).$$

**Proof.** From (2.6)–(2.7) and the Cauchy inequality for numerical series,

$$|\mathrm{sn}(G)|^2 |\mathrm{M}(G)|^2 spd(G)^2 \geq \sum_{(X,Y)\in\mathrm{sn}(N)\times\mathrm{M}(N)} \chi(X,Y) \cdot \sum_{(Z,T)\in\mathrm{sn}(H)\times\mathrm{M}(H)} \chi(Z,T)$$

$$\geq \sum_{(X,Y)\in\mathrm{sn}(N)\times\mathrm{M}(N)} \chi(X,Y) \chi(Z,T).$$

All are positive quantities and then, extracting the square root, the result follows. 

The next result answers in a certain sense to [17, Problem 4.1].

**Corollary 2.11.** Let $N$ be a normal subgroup of a group $G = NH$. Then

$$sd(G) \geq \frac{1}{|L(G)|^2} \sum_{(X,Y)\in\mathrm{L}(N)^2} \chi(X,Y) \chi(Z,T).$$

**Proof.** Firstly, we note that the corresponding versions of (2.4) and (2.5) for $sd(G)$ are always satisfied. Then we argue as in Proposition 2.10. 

3. Applications and final considerations

The symmetric group on 3 elements $S_3 = \mathbb{Z}_2 \ltimes \mathbb{Z}_3 = \langle a, b \mid a^3 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ has $sd(S_3) = \frac{5}{6}$ (see [17, p.2510]), is metabelian and satisfies the description in Theorem 2.8 since it is an example of a primitive group of affine type (see [2]). This group was the origin of our investigation. In fact, a primitive group $P$ of affine type is a semidirect product with normal factor $Fit(P)$. Furthermore, $Fit(P)$ turns out to be elementary abelian and $C_P(Fit(P)) = Fit(P)$. This means that Theorem 2.8 gives a good description for the subgroup commutativity degree and for the subgroup S–commutativity degree of such groups. While [11671111213] show that we may classify a group, when restrictions on $d(G)$ are given, the problem is still open for $sd(G)$ and $spd(G)$. We illustrate one case, involving $sd(G)$. This is to justify the interest in Section 2 in the new bounds.

**Corollary 3.1.** A metabelian group $G$ with $|G|$ and $|G/G'|$ of prime orders is cyclic, whenever the bound in Corollary 2.9 is achieved with $sd(G) = \frac{5}{6}$. 

Proof. We begin from \(|L(G)|^2 \geq 6 = |L(N)|^2 + 2|L(N)| + 1\), which becomes \(|L(G)|^2 \geq 4 + 4 + 1 = 9\), then \(2 \leq |L(G)| = \sqrt{\frac{96}{7}} = \sqrt{13.3} < 4\). This implies either \(|L(G)| = 2\) or \(|L(G)| = 3\). In the first case, \(G\) is cyclic of prime order. In the second case, \(G\) is lattice isomorphic to \(C_p^2\) for a suitable prime \(p\). In both cases \(G\) is cyclic. \(\square\)

The control of \(|L(G)|\) was the main ingredient in the previous proof. Unfortunately, formulas for the growth of \(L(G)\) are hard to find and \([16]\) helps our investigations. The Môbius number of \(L(G)\) is a number which allows us to control the size of \(|L(G)|\). In case of a symmetric group \(S_n\), it is denoted by \(\mu(1, S_n)\) and was conjectured to be \((-1)^{n-1} (|\text{Aut}(S_n)|/2)\) for all \(n > 1\) (see \([16]\) p.1). For \(n \leq 11\), this was proved by H. Pahlings. Recent progresses are summarized below.

**Theorem 3.2** (See \([16]\), Theorems 1.6, 1.8, 1.10).

(i) Let \(p\) be a prime. Then \(\mu(1, S_p) = (-1)^{p-1} \frac{p!}{2} \).

(ii) Let \(n = 2p\) and \(p\) be an odd prime. Then

\[
\mu(1, S_n) = \begin{cases} 
-n!, & \text{if } n - 1 \text{ is prime and } p \equiv 3 \mod 4, \\
\frac{n!}{2}, & \text{if } n = 2, \\
-\frac{n!}{2}, & \text{otherwise}.
\end{cases}
\]

(iii) Let \(n = 2^a\) for an integer \(a \geq 1\). Then \(\mu(1, S_n) = -\frac{2^a}{2} \).

Let \(\mu(1, G) \in \{\mu(1, S_p), \mu(1, S_n)\}\), being \(\mu(1, S_p)\) and \(\mu(1, S_n)\) the values in Theorem 3.2 under the given restrictions on \(n\) and \(p\). In a certain sense, the following result specifies our considerations on \(S_n\), when we have an arbitrary primitive group of affine type for which the subgroup lattice is growing as \(S_n\).

**Corollary 3.3.** Under the assumptions of Theorem 2.8, let \(G\) be a solvable group such that \(|L(G)| = \mu(1, G)|G| = \mu(1, G)\). Then \(sd(G) \geq \frac{g(p, \alpha_1, \alpha_2)}{2 \mu(1, G)^2}\), where \(g(p, \alpha_2, \alpha_2)\) is a polynomial function depending only on \(C_G(\text{Fit}(G))\).

**Acknowledgement**

The second author thanks D.E. Otera and the Université Paris-Sud 11 for the hospitality during the month of May 2010, in which the significant part of the present project has been done. We are also grateful to some colleagues, who detected two fundamental problems in the original version of the manuscript, allowing us to enlarge our perspectives of study in the present version.

**References**

1. R.K. Agrawal, Finite groups whose subnormal subgroups permute with all Sylow subgroups, *Proc. Amer. Math. Soc.* 47 (1975), 77–83.
2. J.C. Beidleman and H. Heineken, Finite soluble groups whose subnormal subgroups permute with certain classes of subgroups *J. Group Theory* 6 (2003), 139–158.
3. K. Chiti, M.R.R. Moghaddam and A.R. Salemkar, \(n\)-isoclinism classes and \(n\)-nilpotency degree of finite groups, *Algebra Collo.* 12 (2005), 225–261.
4. A.K. Das and R.K. Nath, On a lower bound of commutativity degree, *Rend. Circ. Mat. Palermo* 59 (2010), 137–142.
5. J.D. Dixon and B. Mortimer, *Permutation Groups*, Springer, Berlin, 1996.
6. A. Erfanian, F. Lescot and R. Rezaei, On the relative commutativity degree of a subgroup of a finite group, *Comm. Algebra* 35 (2007), 4183–4197.
7. A. Erfanian, R. Rezaei and F.G. Russo, Relative \(n\)-isoclinism classes and relative \(n\)-th nilpotency degree of finite groups, e-print, Cornell University Library, [arXiv:1003.2306](http://arxiv.org/abs/1003.2306).
[8] G. Grätzer, *General Lattice Theory*, 2nd edition, Birkhäuser, 2003, Basel.
[9] J. Lennox and S. Stonehewer, *Permutable subgroups*, Oxford University Press, Oxford, 1993.
[10] H. Mohammadzadeh, A.R. Salemkar and H. Tavallaee, A remark on the commuting probability in finite groups, *South East Bull. Math. Sci.* 34 (2010), 755–763.
[11] P. Niroomand and R. Rezaei, On the exterior degree of finite groups, *Comm. Algebra* 39 (2011), 335–343.
[12] R. Rezaei and F.G. Russo, $n$-th relative nilpotency degree and relative $n$-isoclinism classes, *Carpathian J. Math.*, to appear.
[13] F.G. Russo, The generalized commutativity degree in a finite group, *Acta Univ. Apulensis Math. Inform* 18 (2009), 161–167.
[14] P. Schmid, Subgroups permutable with all Sylow subgroups, *J. Algebra* 207 (1998), 285–293.
[15] R. Schmidt, *Subgroup Lattices of Groups*, de Gruyter, Berlin, 1994.
[16] J. Shareshian, On the Möbius number of the subgroup lattice of the symmetric group, *J. Comb. Theory Ser. A* 78 (1997), 236–267.
[17] M. Tărnăuceanu, Subgroup commutativity degrees of finite groups, *J. Algebra* 321 (2009), 2508–2520.
[18] M. Tărnăuceanu, An arithmetic method of counting the subgroups of a finite abelian group, *Bull. Math. Soc. Sci. Math. Roumanie* 53 (2010), 373–386.

Département de Mathématique, Université Paris-Sud 11, Batiment 425, Faculté de Science d’Orsay, F-91405, Orsay Cedex, France

E-mail address: daniele.otera@math-psud.fr

Department of Mathematics, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy.

E-mail address: francescog.russo@yahoo.com