Counterexamples Related to Commutators of Unbounded Operators

Mohammed Hichem Mortad

Abstract. The present paper is exclusively devoted to examples and counterexamples about commutators and self commutators of unbounded operators on a Hilbert space. As a bonus, we provide a simpler counterexample than McIntosh’s famous example obtained some while ago.

Mathematics Subject Classification. Primary 47B47; Secondary 47A05, 47B25.

Keywords. Commutators, self-commutators, bounded and unbounded operators.

1. Introduction

The formal commutator of two non necessarily bounded operators $A$ and $B$ is defined to be $AB - BA$. We have called it “formal” as, unlike the bounded case, $AB - BA = 0$ does not always imply the (strong) commutativity of $A$ and $B$ when say $A$ and $B$ are self-adjoint. The first such counterexample is due to Nelson [10]. Apparently, the first textbook to include it is [13]. The same example is developed in detail in [14], pp. 257–258. Perhaps the simplest counterexample is due to Schm¨udgen and may be found in e.g. [15].

The self commutator of a densely defined operator $T$ is defined to be $TT^* - T^*T$.

The main purpose of this paper is to exhibit examples and counterexamples to questions related to commutators and self-commutators as regards boundedness, closedness and self-adjointness.

Readers throughout the paper will observe how fascinating the use of matrices of unbounded operators helps to find such counterexamples. The same
approach has equally allowed us to find more interesting counterexamples on a different topic (see [9]).

We refer readers to [16] for properties and results about matrices of unbounded operators (see also [8] or [11]). For the general theory of unbounded operators, readers may wish to consult [15] or [17] (see also [12]). Finally, it is worth noticing that there is an extensive work on estimating the norm of commutators of some classes of bounded operators. We cite among others ([2–5]).

2. Main Counterexamples

We start with an auxiliary result which is also interesting in its own.

**Proposition 2.1.** There exists a densely defined unbounded and closed operator $B$ such that $B^2$ and $|B|B$ are bounded whereas $B|B|$ is unbounded and closed.

**Proof.** Let $H$ be a complex Hilbert space and let $A$ be an unbounded self-adjoint and positive operator with domain $\mathcal{D}(A) \subset H$ (for instance $Af(x) = e^{x^2}f(x)$ on its maximal domain on $L^2(\mathbb{R})$). Let

$$B = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

be defined on $H \oplus \mathcal{D}(A)$. Then $B$ is closed and as seen before $B^2 = 0$ on $H \oplus \mathcal{D}(A)$. Now,

$$|B| = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$$

and so $|B|B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, that is, $|B|B$ is bounded on $H \oplus \mathcal{D}(A)$. However,

$$B|B| = \begin{pmatrix} 0 & A^2 \\ 0 & 0 \end{pmatrix}$$

is clearly unbounded on $H \oplus \mathcal{D}(A^2)$.

**Proposition 2.2.** There are two densely defined unbounded and closed operators $B$ and $C$ such that $CB - BC$ is bounded and unclosed while $|C|B - B|C|$ is unbounded and closed.

**Proof.** Let $B$ be as in Proposition 2.1 and set $C = B$. Then $CB - BC = 0|_{\mathcal{D}(B^2)}$ is clearly bounded and unclosed. By a glance at Proposition 2.1 again, we easily see that

$$|B|B - B|B| = \begin{pmatrix} 0 & -A^2 \\ 0 & 0 \end{pmatrix}$$

and that it is closed (and unbounded) on $\mathcal{D}(|B|B - B|B|) = H \oplus \mathcal{D}(A^2)$.

**Lemma 2.3.** There are two unbounded and self-adjoint operators $A$ and $B$ such that $\mathcal{D}(A) = \mathcal{D}(B)$, $A^2 - B^2$ is bounded but $AB - BA$ is unbounded.
Proof. Let $T$ be any unbounded and self-adjoint operator with domain $\mathcal{D}(T) \subset H$ where $H$ is a Hilbert space. Next, define $A = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$ and $B = \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix}$ and so $\mathcal{D}(A) = \mathcal{D}(B) = \mathcal{D}(T) \oplus \mathcal{D}(T)$. Hence $A^2 = B^2 = \begin{pmatrix} T^2 & 0 \\ 0 & T^2 \end{pmatrix}$ and so $A^2 - B^2$ is bounded (on $\mathcal{D}(A^2)$) whilst $A^2 - B^2$ is bounded (on $\mathcal{D}(A^2)$) whilst

$\begin{pmatrix} 0 & -2T^2 \\ 2T^2 & 0 \end{pmatrix}$

is obviously unbounded.

We already know that there exist two unbounded self-adjoint operators $A$ and $B$ such that $AB - BA$ is bounded (on its domain) while $|A|B - B|A|$ is unbounded. The first (and apparently the only) counterexample is due to McIntosh in [7] who answered a question raised by the great T. Kato. The example we are about to give here is new and simpler than McIntosh’s. Moreover, in our case both $AB - BA$ and $|A|B - B|A|$ are even closed.

Proposition 2.4. There exist two unbounded and self-adjoint operators $A$ and $B$ such that $AB - BA$ is bounded (and closed) whilst $|A|B - B|A|$ is unbounded (and also closed).

The counterexample is based on the following recently obtained result:

Lemma 2.5 ([1]). There are unbounded self-adjoint positive operators $A$ and $B$ (explicitly defined on $L^2(\mathbb{R})$) such that

$\mathcal{D}(A^{-1}B) = \mathcal{D}(BA^{-1}) = \{0\}$

(where $A^{-1}$ and $B^{-1}$ are not bounded).

Now, we give the promised counterexample:

Proof. Let $R, S, T$ be three self-adjoint operators on a Hilbert space $H$ (to be precised below) with domains $\mathcal{D}(R)$, $\mathcal{D}(S)$ and $\mathcal{D}(T)$ respectively. Assume also that $S$ is positive. Now, define on $H \oplus H$ the operators

$A = \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix}$ and $B = \begin{pmatrix} T & 0 \\ 0 & R \end{pmatrix}$

with domains $\mathcal{D}(A) = \mathcal{D}(S) \oplus \mathcal{D}(S)$ and $\mathcal{D}(B) = \mathcal{D}(T) \oplus \mathcal{D}(R)$ respectively. Hence

$AB - BA = \begin{pmatrix} 0 & SR - TS \\ ST - RS & 0 \end{pmatrix}$.

Since $|A| = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$, it follows that

$|A|B - B|A| = \begin{pmatrix} ST - TS & 0 \\ 0 & SR - RS \end{pmatrix}$.
To obtain the appropriate operators, we now choose two unbounded self-adjoint positive operators $C$ and $D$ such that
\[ \mathcal{D}(CD) = \mathcal{D}(DC) = \{0_{L^2(\mathbb{R})}\} \]
(just like in Lemma 2.5). Next, define
\[
S = \begin{pmatrix} C & 0 \\ 0 & 2C \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}
\]
(and so $S$ is self-adjoint and positive with domain $D(S) = D(C) \oplus D(C)$). Clearly,
\[
ST = \begin{pmatrix} 0 & CD \\ 2CD & 0 \end{pmatrix} \quad \text{and} \quad TS = \begin{pmatrix} 0 & 2DC \\ DC & 0 \end{pmatrix}
\]
Hence
\[
\mathcal{D}(ST) = \mathcal{D}(TS) = \{0_{L^2(\mathbb{R})^2}\}.
\]
This says that $AB - BA$ is bounded. In fact, $AB - BA$ is trivially bounded as it is only defined on $\{0\}$ (the zero of $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$) and $AB - BA$ is therefore closed.

In order that $|A||B - |A||$ be unbounded, it suffices then to exhibit a self-adjoint $R$ such that $SR - RS$ is unbounded. Consider $R = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ where $I$ is the usual identity operator (hence $R$ is bounded and self-adjoint). Therefore,
\[
SR = \begin{pmatrix} 0 & C \\ 2C & 0 \end{pmatrix}, \quad RS = \begin{pmatrix} 0 & 2C \\ C & 0 \end{pmatrix}
\]
and
\[
SR - RS = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}
\]
which is patently unbounded, as coveted. \hfill \Box

Now, we consider the case of self-commutators. First, we show that the self-commutator of a densely defined closed operator may only be defined at 0.

**Proposition 2.6.** There exists a densely defined and closed operator $T$ such that
\[ \mathcal{D}(TT^*) \cap \mathcal{D}(T^*T) = \{0\} \] and hence
\[ \mathcal{D}(TT^* - T^*T) = \{0\}. \]

**Proof.** Consider two unbounded and self-adjoint operators $A$ and $B$ which obey $\mathcal{D}(A) \cap \mathcal{D}(B) = \{0\}$ (see e.g. [6]). Hence obviously $\mathcal{D}(A^2) \cap \mathcal{D}(B^2) = \{0\}$.

Now, set $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ and so $T^* = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$. Therefore
\[
TT^* = \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix} \quad \text{and} \quad T^*T = \begin{pmatrix} B^2 & 0 \\ 0 & A^2 \end{pmatrix}.
\]
Thus
\[ D(TT^*) \cap D(T^*T) = D(TT^* - T^*T) = \{(0, 0)\}, \]
as needed. □

In the next two counterexamples, we show that \( TT^* - T^*T \) may be bounded but \( |T||T^*| - |T^*||T| \) may be not, and vice versa.

**Proposition 2.7.** There exists a densely defined closed operator \( T \) such that \( TT^* - T^*T \) is unbounded but \( |T||T^*| - |T^*||T| \) is bounded (even zero on some domain!).

**Proof.** Let \( A \) be an unbounded, self-adjoint and positive operator with domain \( D(A) \subset H \). Define \( T \) with domain \( D(T) = D(A) \oplus H \) by
\[
T = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad \text{and so} \quad T^* = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}.
\]
Hence
\[
TT^* = \begin{pmatrix} 0 & 0 \\ 0 & A^2 \end{pmatrix} \quad \text{and} \quad T^*T = \begin{pmatrix} A^2 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Also,
\[
|T||T^*| = \begin{pmatrix} 0 & |D(A)| \\ 0 & |D(A)| \end{pmatrix} \quad \text{and} \quad |T^*||T| = \begin{pmatrix} 0 & |D(A)| \\ 0 & |D(A)| \end{pmatrix}.
\]
Thus,
\[
TT^* - T^*T = \begin{pmatrix} -A^2 & 0 \\ 0 & A^2 \end{pmatrix}
\]
is clearly unbounded (and self-adjoint) whereas \( |T||T^*| - |T^*||T| \) is bounded (in fact, it is the zero operator on \( D(A) \oplus D(A) \)). □

**Proposition 2.8.** There exists a densely defined and closed operator \( T \) such that \( TT^* - T^*T \) is bounded whereas \( |T||T^*| - |T^*||T| \) is unbounded.

**Proof.** Let \( A \) and \( B \) be two self-adjoint and positive operators with domains \( D(A) \) and \( D(B) \). Set \( T = \begin{pmatrix} 0 & \sqrt{A} \\ \sqrt{B} & 0 \end{pmatrix} \). Hence
\[
TT^* - T^*T = \begin{pmatrix} A - B & 0 \\ 0 & B - A \end{pmatrix}
\]
and
\[
|T||T^*| - |T^*||T| = \begin{pmatrix} \sqrt{B}\sqrt{A} - \sqrt{A}\sqrt{B} & 0 \\ 0 & \sqrt{A}\sqrt{B} - \sqrt{B}\sqrt{A} \end{pmatrix}.
\]
To get the desired counterexample, it suffices to have \( D(A) \cap D(B) = \{0\} \) and so \( TT^* - T^*T \) becomes trivially bounded, and at the same time, the same
pair must make $\sqrt{A}\sqrt{B} - \sqrt{B}\sqrt{A}$ unbounded (and thus $|T||T^*| - |T^*||T|$ too becomes unbounded). One possible choice is to consider $A$ defined on $L^2(\mathbb{R})$

$$Af(x) = e^{\frac{x^2}{4}} f(x)$$

with domain $\mathcal{D}(A) = \{f \in L^2(\mathbb{R}) : e^{\frac{x^2}{4}} f \in L^2(\mathbb{R})\}$. Then $A$ is self-adjoint and positive (also boundedly invertible). Setting $B := \mathcal{F}^* A \mathcal{F}$, where $\mathcal{F}$ denotes the usual $L^2(\mathbb{R})$-Fourier transform, we then know from [6] that

$$\mathcal{D}(A) \cap \mathcal{D}(B) = \{0\}.$$

The unique positive self-adjoint square root of $A$ is given by

$$\sqrt{A} f(x) = e^{\frac{x^2}{4}} f(x)$$

with domain $\mathcal{D}(\sqrt{A}) = \{f \in L^2(\mathbb{R}) : e^{\frac{x^2}{4}} f \in L^2(\mathbb{R})\}$. From the functional calculus, we equally know that $\sqrt{B} := \mathcal{F}^* \sqrt{A} \mathcal{F}.$

There remains to check that $\sqrt{B}\sqrt{A} - \sqrt{A}\sqrt{B}$ is unbounded. A way of seeing that is to consider the action of this operator on some Gaussian functions. More precisely, let $f_0(x) = e^{-ax^2}$ where $a > 0$ is yet to be determined. Then $f_0 \in L^2(\mathbb{R})$ (for all $a > 0$) as

$$\|f_0\|_2^2 = \sqrt{\frac{\pi}{2a}}.$$

Now, it may be checked that

$$\mathcal{F}^* \sqrt{A} \mathcal{F} \sqrt{A} f_0(x) = \frac{2}{\sqrt{5 - 4a}} e^{-\frac{5a-1}{4a} x^2},$$

a calculation valid for $1/4 < a < 5/4$. Similarly, we find that

$$\sqrt{A} \mathcal{F}^* \sqrt{A} \mathcal{F} f_0(x) = \frac{1}{\sqrt{1-a}} e^{-\frac{1-5a}{4(a-1)} x^2}$$

valid for the values of $a$ in $(1/5, 1)$. Therefore, for $1/4 < a < 1$ we have

$$(\sqrt{B}\sqrt{A} - \sqrt{A}\sqrt{B}) f_0(x) = \frac{2}{\sqrt{5 - 4a}} e^{-\frac{4a-1}{5a} x^2} - \frac{1}{\sqrt{1-a}} e^{-\frac{1-5a}{4(a-1)} x^2}.$$

Thus,

$$\|(\sqrt{B}\sqrt{A} - \sqrt{A}\sqrt{B}) f_0\|_2^2 = \frac{4\sqrt{\pi}}{5 - 4a} \frac{1}{\sqrt{\frac{2(4a-1)}{5-4a}}} - \frac{4\sqrt{\pi}}{\sqrt{5 - 4a \sqrt{1-a}}} \times \frac{1}{\sqrt{\frac{1-5a}{4(a-1)} + \frac{4a-1}{5-4a}}}$$

$$+ \frac{\sqrt{\pi}}{(1-a)\sqrt{\frac{1-5a}{2(a-1)}}}$$

where $1/4 < a < 1$. Upon sending $a \to 1/4$ say, we readily see that the quantity $\|(\sqrt{B}\sqrt{A} - \sqrt{A}\sqrt{B}) f_0\|_2^2/\|f_0\|_2$ goes to infinity, making $\sqrt{B}\sqrt{A} - \sqrt{A}\sqrt{B}$ unbounded, as wished. □
Remark. The pair of $A$ and $B$ which appeared in the previous proof provides also a counterexample for a problem related to the topic of the paper. Indeed, since $\mathcal{D}(A) \cap \mathcal{D}(B) = \{0\}$, it follows that $D(AB - BA) = \{0\}$ whereby making $AB - BA$ trivially bounded. According, we have two unbounded self-adjoint boundedly invertible $A$ and $B$ such that $AB - BA$ is bounded whilst $\sqrt{B}\sqrt{A} - \sqrt{A}\sqrt{B}$ is unbounded.

Conjecture

Inspired by the pair $S$ and $R$ which appeared in Proposition 2.4, we propose the following conjecture:

Conjecture 2.9. For any unbounded self-adjoint operator $A$, there is always a bounded and self-adjoint operator $B$ such that $AB - BA$ is unbounded.

Acknowledgements

The author wishes to thank the referee for all his/her remarks and suggestions which have been necessary to improve some parts of the paper.

References

[1] Dehimi, S., Mortad, M.H.: Chernoff like counterexamples related to unbounded operators. Kyushu J. Math. (to appear)
[2] Fong, C.K.: Norm estimates related to self-commutators. Linear Algebra Appl. 74, 151–156 (1986)
[3] Kittaneh, F.: Inequalities for commutators of positive operators. J. Funct. Anal. 250(1), 132–143 (2007)
[4] Kittaneh, F.: Norm inequalities for commutators of positive operators and applications. Math. Z 258(4), 845–849 (2008)
[5] Kittaneh, F.: Norm inequalities for commutators of self-adjoint operators. Integral Equ. Oper. Theory 62(1), 129–135 (2008)
[6] Kosaki, H.: On intersections of domains of unbounded positive operators. Kyushu J. Math. 60(1), 3–25 (2006)
[7] McIntosh, A.: Counterexample to a question on commutators. Proc. Am. Math. Soc. 29, 337–340 (1971)
[8] Möller, M., Szafraniec, F.H.: Adjoints and formal adjoints of matrices of unbounded operators. Proc. Am. Math. Soc. 136(6), 2165–2176 (2008)
[9] Mortad, M.H.: On the triviality of domains of powers and adjoints of closed operators. Acta Sci. Math. (Szeged) (to appear)
[10] Nelson, E.: Analytic vectors. Ann. Math. 70, 572–615 (1959)
[11] Öta, S., Schmüdgen, K.: Some selfadjoint $2 \times 2$ operator matrices associated with closed operators. Integral Equ. Oper. Theory 45(4), 475–484 (2003)
[12] Putnam, C.R.: Commutation Properties of Hilbert Space Operators and Related Topics. Springer-Verlag, New York (1967)
[13] Reed, M., Simon, B.: Methods of Modern Mathematical Physics, Vol. 1: Functional Analysis. Academic Press, Cambridge (1972)
[14] Schmüdgen, K.: Unbounded Operator Algebras and Representation Theory. Operator Theory: Advances and Applications, vol. 37. Birkhäuser Verlag, Basel (1990)
[15] Schmüdgen, K.: Unbounded Self-Adjoint Operators on Hilbert Space, vol. 2. Springer, Berlin (2012)
[16] Tretter, Ch.: Spectral Theory of Block Operator Matrices and Applications. Imperial College Press, London (2008)
[17] Weidmann, J.: Linear Operators in Hilbert Spaces. Springer, Berlin (1980)

Mohammed Hichem Mortad
Department of Mathematics
University of Oran 1
Ahmed Ben Bella
B.P. 1524
El Menouar
Oran 31000
Algeria

and

Oran
Algeria
e-mail: mhmortad@gmail.com;
mortad.hichem@univ-oran1.dz

Received: February 5, 2019.
Accepted: September 13, 2019.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.