Discrete symmetry breaking and baryon currents in U(N) and SU(N) gauge theories

B. Lucini and A. Patella
Physics Department, Swansea University, Singleton Park, Swansea SA2 8PP, UK

In SU(N) gauge theories with fermions in the fundamental or in a two-index (either symmetric or antisymmetric) representation formulated on a manifold with at least one compact dimension with non-trivial holonomy the discrete symmetries C, P and T are broken at small enough size of the compact direction(s) for certain values of N. We show that for those N in the broken phase a non-zero baryon current wrapping in the compact direction exists, which provides a measurable observable for the breaking of C, P and T. We prove that in all cases where the current is absent there is no breaking of those discrete symmetries. This includes the limit \( N \to \infty \) of the SU(N) gauge theory with symmetric or antisymmetric fermions and U(N) gauge theory at any value of N. We then argue that the component of the baryon current in the compact direction is the physical order parameter for C, P and T breaking due to the breaking of Lorentz invariance.

PACS numbers: 11.30.Er, 12.38.Aw, 12.38.Bx, 11.30.Cp.

I. INTRODUCTION

An interesting problem is the realisation of charge conjugation (C), parity (P) and time reversal (T) symmetries for U(N) and SU(N) gauge theories with fermions in the fundamental, two-index symmetric or antisymmetric and adjoint representations. These are strongly interacting theories including QCD, supersymmetric theories, and candidates for mechanisms of electroweak symmetry breaking. Although their phenomenology can be very different from QCD, in the literature those theories are often referred to as QCD-like theories; in this paper, we will follow this terminology.

An important result in Quantum Field Theory is the CPT theorem, which states that every local theory, which is Lorentz-invariant and whose energy is bounded from below, is also invariant under the successive application of P, C and T. The Vafa-Witten theorem [1] proves the conservation of parity for QCD-like theories in infinite volume. As for charge conjugation, there is a wealth of experimental evidence that C-parity is preserved in QCD [2]. Conservation of C and P together with the CPT theorem implies the conservation of time reversal T. In the absence of a rigorous theorem, it is reasonable to assume that conservation of C holds for generic U(N) and SU(N) gauge theories with Nf fermion flavours transforming in the fundamental, two-index symmetric or antisymmetric and adjoint representations of the gauge group. Hence, it is widely believed that in those theories the vacuum is symmetric under C, P and T.

The situation is different on small volumes, where Lorentz invariance is explicitly broken. In particular, it was noticed in [3] that in U(N) and SU(N) gauge theories with fermions in the (anti)symmetric representation on a \( R^3 \times S^1 \) manifold where the compact \( S^1 \) is closed with periodic boundary conditions for fermions (and hence identifies a compact spatial dimension), for a sufficiently small radius of the \( S^1 \), the effective potential is minimised by values of the Wilson line wrapping the \( S^1 \) with non-vanishing imaginary part. By analogy with the thermal case, this Wilson line is also referred to as the Polyakov loop. In most of the cases, a non-real value for the Polyakov loop implies that not only the center symmetry (if any), but also P, C, T and CPT are spontaneously broken [3]. We will refer to all these symmetries as discrete symmetries.

Studying the properties of U(N) and SU(N) gauge theories under C, P and T on compact spaces at various values of the sizes of the compact dimensions and with various topologies is important for several reasons. Firstly, understanding the behaviour of those theories under C, P and T is mandatory for establishing rigorous equivalences in the large N limit between some classes of observables in two different theories like orientifold planar equivalence [4, 5, 6] (for which conservation of C must be assumed [3, 7]) or volume independence of some observables [8, 9]. Then, a clear mapping of the phase structure of U(N) and SU(N) gauge theories with fermions in two-index representations provides useful guidance to lattice investigations of novel strong interactions as the underlying mechanism for electroweak symmetry breaking [10, 11, 12, 13, 14] and of orientifold planar equivalence [15, 16], which are still in their infancy. Last but not least, an SU(3) gauge theory with fermions in the antisymmetric representation, which is a possible formulation of QCD, also presents breaking of C, P and T. Hence, the problem of spontaneous breaking of those discrete symmetries in small volume could potentially affect QCD. Not surprisingly then, Ref. [3] has prompted several analytical [17, 18] and numerical [18, 19, 20] studies of the phase structure of QCD-like theories on manifold with compact directions (see also [21] and more recently [22], where the phases of gauge theories with fermions in the fundamental and in two-index representations are discussed in great detail). The numerical studies were mostly concerned with the restoration of C, P and T above a critical radius, which was investigated in detail in [19, 20]. In [18], it was shown that a physical manifestation of this symmetry breaking in QCD is the existence of a non-zero baryon current wrapping the \( S^1 \).

In this work, we further develop and extend the proposal of [18], showing that the phenomenon of a persistent
baryon current wrapping the $S^1$ is present in all cases (among the considered ones) in which there is $C$, $P$ and $T$ symmetry breaking. Vice versa, when this current is null, there is no $C$, $P$ and $T$ symmetry breaking. This is also true in the cases where the degenerate vacua are identified by a purely imaginary Polyakov loop: we shall show that in those cases the apparent breaking is due to an unphysical definition of $C$, $P$ and $T$ symmetries; when the physical operators are used, the conservation of those symmetries is manifest. The paper is organised as follows. The issue of spontaneous breaking of $C$, $P$ and $T$ symmetries for small compactification radius is briefly reviewed in Sect. III where we also define our notations. In Sect. IV further developing a recent suggestion of [22], the role of the Polyakov loop as an order parameter for the breaking of degenerate vacua, study under which conditions this current is different from zero and discuss its physical origin. In Sect. IV we derive the expression of the baryon current in the non-symmetric vacua, study under which conditions this current is different from zero and discuss its physical origin. Building on this result, in Sect. V we conclude that the baryon current is the physical order parameter for $C$, $P$ and $T$ symmetry breaking due to the breaking of Lorentz invariance.

II. BREAKING OF $C$, $P$ AND $T$ SYMMETRIES AT SMALL COMPACTIFICATION RADIUS

We consider a gauge theory defined on a $R^3 \times S^1$ manifold. We will discuss both the case of periodic boundary conditions (PBC) and antiperiodic boundary conditions (ABC) for the $S^1$, whose size will be denoted by $L$. For convenience, we regularise the $R^3$ in the infrared by formulating the theory on a box of size $L_R^3 \times L$, with $L_R \gg L$ and taking the limit $L_R \to \infty$ of the results. As we will see, $L_R$ disappears from the final formulae. Since our results are independent of the boundary conditions on the directions send to infinity, we don’t need to specify those boundary conditions.

The Lagrangian for a $U(N)$ or $SU(N)$ gauge theory with $N_f$ fermion flavours transforming in the representation $\mathcal{R}$ of the gauge group is given by

$$\mathcal{L} = -\frac{1}{2g^2} \text{Tr} (G_{\mu\nu}(x)G^{\mu\nu}(x)) + \sum_{i=1}^{N_f} \bar{\psi}_i(x) \left( i \partial \mathcal{R}[A] - m \right) \psi_i(x),$$

where $G_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}]$ and the subscript $\mathcal{R}[A]$ indicates the gauge field in the representation $\mathcal{R}$. In this work, we will consider fermions in the fundamental, in the two-index symmetric and in the two-index antisymmetric representations. Theories with fermions in the adjoint representation will not be investigated, since they do not present spontaneous breaking of $C$, $P$ and $T$ at small volume [22, 23, 24].

By choosing a diagonal background gauge field along the $S^1$ (identified as the third direction)

$$A_{3}^{bg} = \left( \begin{array}{cc} \frac{v_1}{L} \\ \vdots \\ \frac{v_N}{L} \end{array} \right),$$

with the constraint $\sum_i v_i = 0 \text{ mod}(2\pi)$ if the gauge group is $SU(N)$, and integrating out the non-zero modes (after gauge-fixing), the effective potential $V$ for the eigenvalues $v_i$ is defined (we are keeping implicit the integration over fermions, ghosts and auxiliary fields):

$$e^{iLV_3(v_1, \ldots, v_N)} = \int_{\text{non-zero modes}} DA \ e^{i[S(A+A^{bg})+S_{GF}(A+A^{bg})]},$$

where $S_{GF}$ is the gauge-fixing action and $V_3 = L_R^3$. If $v_i^*$ is a set of values that minimise the effective potential, then the partition function (in the Minkowskian formalism) is given by

$$Z = e^{iLV_3(v_1^*, \ldots, v_N^*)}.$$

In the general case of degenerate vacua, the partition function does not depend on the vacuum (the minimum value of $V$ is always the same). On the contrary, the expectation value of Wilson line $W$ wrapping the $S^1$

$$W = P e^{i \int_0^L A_3(x) dx},$$

where $P$ is the path integral.
term that does not depend on which is one for the fundamental and two for the symmetric and antisymmetric two-index representations, up to a constraint on the phases in the case of an SU(N) or U(N) respect to a spatial plane orthogonal to the compact dimension),

value for the baryon current indicates the simultaneous breaking of these symmetries is conserved for the baryon current to be zero. As an instance in [22], which we follow closely in this section.

The effective potential \( V \) can be evaluated at one loop. The explicit derivation is standard and can be found for instance in [22], which we follow closely in this section. \( V \) can be split into

\[
V = V_{\text{Adj}} + V_R,
\]

where \( V_{\text{Adj}} \) is the contribution of the gauge (plus ghost) part of the action and \( V_R \) is the fermion contribution. The fermion contribution is given by

\[
V_R = \frac{2m^2N_f}{\pi^2L^2} \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^2} \text{Re} \text{Tr}_R W^n K_2(nLm),
\]

where \( \text{Re} \text{Tr}_R W^n \) indicates the real part of the trace of \( W^n \) in the representation \( R \) and \( K_2 \) is the order two modified Bessel function of the second kind. In Eq. (6), the sum is weighted by \((\pm 1)^n\) for ABC, while the plus sign refers to the PBC case.

The gauge contribution (which we do not need to write in detail) generates an attractive force for the eigenvalues. Therefore the minimum of the effective potential \( V \), Eq. (6) is obtained at the point in which all the \( v \) equal and this common value \( v^* \) minimises the term due to the fermions. Introducing the \( N \)-ality of the representation \( N_R \), which is one for the fundamental and two for the symmetric and antisymmetric two-index representations, up to a term that does not depend on \( v^* \), we get

\[
V_R = \frac{2m^2N_f}{\pi^2L^2} \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^2} \text{dim}_R \cos (N_R n v^*) K_2(nLm),
\]

dim \( R \) being the dimension of the representation. The minimisation of the effective potential must keep into account the constraint on the phases in the case of an SU(N) gauge group. The minima in the cases of interest are listed in Table I. The expectation value of the Wilson line is given by

\[
\langle \text{Tr}W \rangle = N e^{iv^*}.
\]

For ABC the system is in the usual thermal phase: the center symmetry (if any) is spontaneously broken, while \( P \), \( C \) and \( T \) are unbroken. However for PBC, also the symmetries \( P \), \( C \) and \( T \), defined in the standard way:

\[
\langle \text{Tr}W \rangle \rightarrow \langle \text{Tr}W^\dagger \rangle,
\]

are broken for fundamental fermions and SU(N) gauge group with \( N \) odd, or with fermions in the (anti)symmetric two-index representations and both SU(N) and U(N) gauge groups with any \( N \). We will discuss this definition of \( P \), \( C \) and \( T \) in Sect. IV.

III. DERIVATION OF THE CURRENT

The Noether 4-current associated to the baryon number is

\[
j^\mu(x) = \bar{\psi} \gamma^\mu \psi(x).
\]

We showed in [18] (see also [25, 26]) that the component \( j^3(x) \) along the compact dimension is the local order parameter for the breaking of \( P \), \( C \) and \( T \) in the case of SU(N) with fermions in the fundamental representation. The existence of a non-null baryon current wrapping the \( S^1 \) is a physical consequence of the non-invariance of the degenerate ground states of the system under \( C \), \( P \) and \( T \) symmetries. Here we want to generalize this result to SU(N) or U(N) with fermions in the two-index symmetric and antisymmetric representations. We stress that a non-zero value for the baryon current indicates the simultaneous breaking of \( P \) (and in particular the reflection symmetry with respect to a spatial plane orthogonal to the compact dimension), \( C \) and \( T \). In other words it is enough that one of these symmetries is conserved for the baryon current to be zero.

The vacuum expectation value of the baryon current can be defined by adding a source term in the action:

\[
\langle j^3 \rangle = -\frac{i}{ZV_3N_f} \frac{d}{d\alpha} \bigg|_{\alpha=0} \int \mathcal{D}A \mathcal{D} \bar{\psi} \mathcal{D} \psi \exp \left\{ iS_{YM}[A] + i \int \sum_{l=1}^{N_f} \bar{\psi}_l(x) \gamma^\nu \left( i\partial_\nu - \mathcal{R}[A_\nu] - m + \frac{\alpha}{L} \delta^3 \right) \psi_l(x) \right\}.
\]
The previous relationship shows that \( \langle j^3 \rangle \neq 0 \) if \( N_R v^* \neq k \pi / 2 \), with \( k \) integer. Hence at sufficiently small \( L \) such that the one-loop calculation is reliable, a baryon current is generated only for gauge group \( SU(N) \) and fermions either in the fundamental representation with PBC for odd values of \( N \) or in the (anti)symmetric two-index representation and PBC for values of \( N \) that are not multiple of 4 (see summary in Table I). A lattice calculation in \( SU(3) \) gauge theory with fermions in the fundamental representation showed the persistence of this current beyond the perturbative regime [18].

A quick inspection of the vacua shows that at large-\( N \) the \( SU(N) \) and \( U(N) \) gauge groups coincide, and therefore the baryon current vanishes.

A simple argument can elucidate the physical origin of the baryon current. We start from the thermal effective potential \( V^{ABC}(v; \mu) \) in the presence of a chemical potential \( \mu \) (3 is now the thermal direction). Even if the argument proposed here is completely non-perturbative, we quote the one-loop potential from [22]:

\[
V_R^{ABC}(v; \mu) = \frac{m^2N_f}{\pi^2L^2} \sum_{n=1}^\infty \frac{(-1)^n}{n^2} \left( e^{n\mu L}Tr_RW^{+n} + e^{-n\mu L}Tr_RW^n \right) K_2(nLm) .
\]  

For a generic representation \( \mathcal{R} \) with \( N \)-ality equal to \( N_R \neq 0 \) the source term can be absorbed by a shift of the gauge field along the compact dimension (thanks to the invariance of the YM action under a constant shift of the gauge field):

\[
A_3 \rightarrow A_3 + \frac{\alpha}{N_R L} .
\]

This argument leads to the exact formula which relates the baryon current to the effective potential of the eigenvalues of the Wilson line:

\[
\langle j^3 \rangle = \left. \frac{L}{N_f} \frac{d}{d\alpha} \right|_{\alpha=0} V \left( v_1^* + \frac{\alpha}{N_R}, \ldots, v_N^* + \frac{\alpha}{N_R} \right) .
\]

The one-loop value of the baryon current can be computed by plugging the one-loop effective potential \( \mathcal{V} \) into \( \langle j^3 \rangle \), with the shift \( \alpha \) implemented as \( W \rightarrow \exp \left( i \frac{\alpha}{N_R} \right) W \):

\[
\langle j^3 \rangle = \left. \frac{2m^2}{\pi^2 L} \frac{d}{d\alpha} \right|_{\alpha=0} \sum_{n=1}^\infty \frac{(-1)^n}{n^2} Re \left( e^{i\alpha n} Tr_RW^n \right) K_2(nLm) =
\]

\[
= - \frac{2m^2}{\pi^2 L} \sum_{n=1}^\infty \frac{(-1)^n}{n} Im TrRW^n K_2(nLm) =
\]

\[
= - \frac{2m^2}{\pi^2 L} \sum_{n=1}^\infty \frac{(-1)^n}{n} \sin(N_R n v^*) K_2(nLm) .
\]

The previous relationship shows that \( \langle j^3 \rangle \neq 0 \) if \( N_R v^* \neq k \pi / 2 \), with \( k \) integer. Hence at sufficiently small \( L \) such that the one-loop calculation is reliable, a baryon current is generated only for gauge group \( SU(N) \) and fermions either in the fundamental representation with PBC for odd values of \( N \) or in the (anti)symmetric two-index representation and PBC for values of \( N \) that are not multiple of 4 (see summary in Table I). A lattice calculation in \( SU(3) \) gauge theory with fermions in the fundamental representation showed the persistence of this current beyond the perturbative regime [18].

A quick inspection of the vacua shows that at large-\( N \) the \( SU(N) \) and \( U(N) \) gauge groups coincide, and therefore the baryon current vanishes.
The effective potential for PBC is obtained at imaginary chemical potential \( \mu = i \pi / L \). This can be easily understood by the one-loop formula (17), noticing that in this case the imaginary chemical potential gives an extra \((-1)^n\) factor in the sum. From a non-perturbative point of view, if \( \psi \) is the fermion field with ABC, then the fermion field \( \psi' \) with PBC is given by the following change of variables in the functional integral:

\[
\psi'(x) = e^{\frac{i \mu}{L}} \psi(x).
\] (18)

The extra phase generates in the Euclidean Lagrangian the term \( \frac{i \pi}{L} \bar{\psi} \gamma^3 \psi \), which is exactly an imaginary chemical potential.

Moreover, also the source \( \alpha \) couples to the operator \( i \bar{\psi} \gamma^3 \psi \) (which is Hermitean in the Euclidean). The effective potential for PBC fermions in presence of this external source is:

\[
V^{PBC} \left( v + \frac{\alpha}{N_R}; \mu \right) = V^{ABC} \left( v; \mu + \frac{\pi - \alpha}{L} \right).
\] (19)

Here we are using that the effective potential is analytical in the chemical potential, i.e. it depends on the complex chemical potential \( \mu \) but not on its complex conjugate \( \bar{\mu} \). This fact is clear for the one-loop formula (17), and it is discussed in (27) for the non-perturbative case.

The baryon current for PBC fermions is therefore computed:

\[
\langle j^3 \rangle_{PBC} = \frac{L}{N_f} \frac{d}{d \alpha} \left|_{\alpha=0} \right. V^{PBC} \left( v^* + \frac{\alpha}{N_R}; 0 \right) = -\frac{i}{N_f} \frac{d}{d \mu} \left|_{\mu=i\pi/L} \right. V^{ABC} \left( v^*; \mu \right).
\] (20)

The baryon current along a spatial direction for PBC fermions is mapped into the baryon density for a thermal system (ABC) with an imaginary chemical potential. The properties of the latter system have been firstly studied in (27) and are one of the key aspects of contemporary lattice investigations of systems at finite baryon density \([28, 29]\). In particular, it is well-known that an imaginary chemical potential \( i \pi / L \) induces a finite baryon density in the SU(3) theory. For U(\( N \)) gauge theories the baryon number is zero, since it is the Noether charge of the local U(1) gauge symmetry, and the imaginary chemical potential can be gauged away.

### IV. PHYSICAL DEFINITION OF \( C, P \) AND \( T \) SYMMETRIES

In Sect. [11] we have discussed the argument of [3] about the spontaneous breaking of \( C, P \) and \( T \) symmetries in QCD-like theories on manifolds with non-trivial holonomy and the proposal of using the trace of the Wilson loop wrapping the \( S^1 \) as an order parameter for this breaking. In Sect. [11] we have shown that in most cases in which the Polyakov loop has a non-zero imaginary part a baryon current wrapping the short circle is generated. However for SU(\( N \)) theories with \( N \) multiple of 4 and U(\( N \)) theories for arbitrary value of \( N \) with the fermions in the (anti)symmetric two-index representation, the current is zero but the imaginary part of \( \langle \text{TrW} \rangle \) is different from zero (see summary in Table [4]). One is led to conclude that the baryon current is not a good order parameter for \( C, P \) and \( T \) breaking.

However the definition of \( C, P \) and \( T \) on the Hilbert space is not uniquely determined. In the easy case of QED, all these transformations are defined up to a phase. For generic phases for the gauge field, these transformations are not symmetries (i.e. they do not commute with the Hamiltonian). However since the transition probabilities are independent of these phases, it is enough to find one set of values for which the operators on the Hilbert space are symmetries, in order to conclude that the physics is invariant under \( C, P \) and \( T \).

For the cases in Table [4] in which the baryon current is zero but the Wilson line has a non-zero imaginary part, we propose the following interpretation: the breaking of \( C, P \) and \( T \) is only apparent and is due to a bad definition (we will refer to it as the "naive" definition) of the corresponding operators on the Hilbert space. In all these cases, an alternative definition (we will refer to it as the "physical" definition) is possible. We will show that all the local observables transform in the same way, independently of the definition. In particular the physical operators still commute with the Hamiltonian, and define unbroken symmetries (unlike the naive operators). As a consequence, the baryon current wrapping the \( S^1 \) is shown to be the order parameter for \( C, P \) and \( T \) symmetry breaking, in all the considered cases.

As suggested in [22], in either SU(\( N \)) theories with \( N \) multiple of 4 or U(\( N \)) theories for arbitrary value of \( N \), with fermions in the (anti)symmetric two-index representation, the free energy is independent of the (periodic or antiperiodic) boundary conditions for the fermions. This statement can be extended to all gauge invariant observables, which are local in the sense that they depend on the elementary fields in a region not wrapping around the compact dimension. In fact, the two vacua corresponding respectively to PBC and APB for the fermions are connected by a
large gauge transformation $\Omega(x^3)$, which is periodic in the compact dimension up to a phase:

$$\Omega(x^3 + L) = \exp \left( \frac{i \pi}{2} \right) \Omega(x^3). \quad (21)$$

Notice that the $e^{i \pi/2}$ is always an element of the gauge group if this group is $U(N)$; in the case of $SU(N)$, the phase is an element of the group only if $N$ is a multiple of 4. In those cases, the action is invariant under the transformation $\Omega$ but, however, PBC fermions $\psi$ in the (anti)symmetric two-index representation are mapped into ABC fermions $\psi'$:

$$\psi'(x^3 + L) = R[\Omega(x^3 + L)]\psi(x^3 + L) = \exp (i\pi) R[\Omega(x^3)]\psi(x^3) = -\psi'(x^3). \quad (22)$$

We call $G$ the unitary operator, implementing the large gauge transformation $\Omega$ on the Hilbert space:

$$|0\rangle_{PBC} = G|0\rangle_{ABC}. \quad (23)$$

If $O$ is a local gauge-invariant observable (like for instance the baryon current), it is clearly not sensitive to the periodicity violation in the definition of $G$, therefore $OG_O^\dagger = O$ and, as anticipated:

$$\langle 0|O|0\rangle_{ABC} = \langle 0|OG_O^\dagger|0\rangle_{PBC} = \langle 0|O|0\rangle_{PBC}. \quad (24)$$

The existence of the unitary operator $G$ in particular implies that the thermodynamics and the spectra corresponding to different boundary conditions are exactly the same. This clearly suggests that the spontaneous breaking of $P$, $C$ and $T$ is only apparent and is due to a wrong implementation of the physical symmetries as (anti)unitary operators on the Hilbert space. The naive definitions

$$P_0 : A_\mu(t, x) \rightarrow g^{\mu\nu}A_\nu(t, -x) \quad (25)$$

$$\psi(t, x) \rightarrow \gamma_0 \psi(t, -x), \quad (26)$$

$$C_0 : A_\mu(t, x) \rightarrow -A_\mu(t, x)^T, \quad (27)$$

$$\psi(t, x) \rightarrow i\gamma_0 \gamma_2 \psi(t, x)^T, \quad (28)$$

$$T_0 : A_\mu(t, x) \rightarrow g_{\mu\nu}A_\nu(-t, x) \quad (29)$$

$$\psi(t, x) \rightarrow -i\gamma_5 \gamma_0 \gamma_2 \psi(-t, x) \quad (30)$$

lead to unbroken symmetries in the case of ABC. For PBC the naive operators should be replaced by:

$$P = GP_0G^\dagger, \quad (31)$$

$$C = GC_0G^\dagger, \quad (32)$$

$$T = GT_0G^\dagger. \quad (33)$$

On local gauge-invariant observables, these operators and the naive ones acts in the same way; for instance $CO^\dagger C_0 = C_0OC_0^\dagger$. However the physical operators define unbroken symmetries for fermions with PBC:

$$C|0\rangle_{PBC} = GC_0|0\rangle_{ABC} = |0\rangle_{PBC}. \quad (34)$$

and similarly for $P$ and $T$. One can actually think that the physical and naive operators represent different and independent symmetries, some of which are unbroken and some others are not. This is not true, since all the naive operators can be written as a product of the corresponding physical operator and a gauge transformation periodic up to a phase $\exp(i\pi)$, which is an element of the center $Z_2$. In the case of the charge conjugation for instance $C_0 = CG_2$. This argument reveals that the broken symmetry is always and only the center symmetry.

V. CONCLUSIONS

In this paper, we have shown that in SU($N$) gauge theories with fermions in the fundamental, two-index antisymmetric and two-index symmetric representations, whenever the breaking of Lorentz invariance due to the existence of a small $S^1$ induces the simultaneous breaking of $C$, $P$ and $T$ symmetries, a non-zero baryon current is induced that wrap the compact direction. Contrary to the imaginary part of the Wilson loop wrapping the $S^1$, in the case in which there are fermions in a two-index representation, the baryon current is not sensitive to the breaking of the $Z_2$ symmetry. Hence, this baryon current is the true order parameter for the symmetry breaking. It would be interesting to study what happens for other gauge groups. For instance, if the gauge group is SO($N$), the theory is real and there should not be any breaking of $C$, $P$ and $T$. Accordingly, we would expect a null baryon current in that case. Theories with gauge group SO($N$) and Sp($N$) and gauge theories based on exceptional groups are currently under investigation. The results will be reported in a future publication.
We thank A. Armoni and J. Myers for helpful discussion on the physical meaning of $C$-parity breaking. This work has been partially supported by STFC under contracts PP/E007228/1 and ST/G000506/1. B.L. is supported by the Royal Society.

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[30] Note that contrary to what stated in [3], a purely imaginary Polyakov loop does not imply breaking of $P$, $C$ and $T$ symmetries, as firstly noted in [22]. This issue will be discussed in detail in this paper.