Integral representation of some functions related to the Gamma function

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Abstract. We prove that the functions \( \Phi(x) = \frac{\Gamma(x + 1)^{1/x}}{x} (1 + \frac{1}{x})^x \) and \( \log \Phi(x) \) are Stieltjes transforms.

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1. Introduction and main results

In [11] the authors introduce a subclass of the completely monotonic functions which they call logarithmically completely monotonic, and the main result in [12] is that the function

\[
\Phi(x) = \frac{\Gamma(x + 1)^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x
\]

is logarithmically completely monotonic.

We characterize the class of logarithmically completely monotonic functions as the infinitely divisible completely monotonic functions studied by Horn in [14]. We prove that Stieltjes transforms (see (8) below) belong to this class and that \( \Phi \) and \( \log \Phi \) are both Stieltjes transforms. Each of these statements imply the result of [12]. The following explicit representations are obtained:

\[
\log \Phi(x) = \int_0^\infty \frac{\varphi(s)}{s + x} \, ds, \quad x > 0,
\]

where

\[
\varphi(s) = \begin{cases} 
1 - s & \text{if } 0 \leq s < 1 \\
1 - n/s & \text{if } n \leq s < n + 1, \ n = 1,2,\ldots 
\end{cases}
\]

and

\[
\Phi(x) = 1 + \int_0^\infty \frac{h(s)}{s + x} \, ds, \quad x > 0,
\]

with

\[
h(s) = \frac{1}{\pi} \frac{s^{s-1}}{|1 - s|^s \Gamma(1 - s)^{1/s}} \sin(\pi \varphi(s)), \ s \geq 0.
\]

Note that the density \( \varphi(s) \) takes its values in the interval \([0,1]\), and this is the clue to the fact that also \( \Phi \) is a Stieltjes transform. The density \( h \) is continuous on \([0,\infty]\) with \( h(0) = \exp(-\gamma) \), where \( \gamma \) is Euler’s constant, and \( h(n) = 0 \) for \( n \in \mathbb{N} \).

Recall that a function \( f : [0, \infty[ \to \mathbb{R} \) is said to be completely monotonic, if \( f \) has derivatives of all orders and satisfies

\[
(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all } \ x > 0 \quad \text{and} \quad n = 0,1,2,\ldots
\]
Bernstein’s Theorem, cf. [15, p. 161], states that \( f \) is completely monotonic if and only if
\[
f(x) = \int_0^\infty e^{-xs}d\mu(s),
\]
where \( \mu \) is a nonnegative measure on \([0, \infty)\) such that the integral converges for all \( x > 0 \). The set of completely monotonic functions is denoted \( \mathcal{C} \).

In [12] the authors call a function \( f : [0, \infty[ \to [0, \infty[ \) logarithmically completely monotonic if it is \( C^\infty \) and
\[
(-1)^k \log f(x) \geq 0, \quad \text{for } k = 1, 2, \ldots.
\]

If we denote the class of logarithmically completely monotonic functions by \( \mathcal{L} \), we have \( f \in \mathcal{L} \) if and only if \( f \) is a positive \( C^\infty \)-function such that \(-(\log f)' \in \mathcal{C}\).

The functions of class \( \mathcal{L} \) have been implicitly studied in [3], and Lemma 2.4(ii) in that paper can be stated as the inclusion \( \mathcal{L} \subset \mathcal{C} \), a fact also established in [11].

The class \( \mathcal{L} \) can be characterized in the following way, established by Horn [14, Theorem 4.4]:

**Theorem 1.1.** For a function \( f : [0, \infty[ \to [0, \infty[ \) the following are equivalent:

(i) \( f \in \mathcal{L} \)
(ii) \( f^\alpha \in \mathcal{C} \) for all \( \alpha > 0 \)
(iii) \( \sqrt[n]{f} \in \mathcal{C} \) for all \( n = 1, 2, \ldots \).

Another way of expressing the conditions of Theorem 1.1 is that the functions in \( \mathcal{L} \) are those completely monotonic functions for which the representing measure \( \mu \) in (8) is infinitely divisible in the convolution sense: For each \( n \in \mathbb{N} \) there exists a positive measure \( \nu \) on \([0, \infty[\) with \( n \)th convolution power equal to \( \mu \), viz. \( \nu^{*n} = \mu \).

By condition (ii) there exists a convolution semigroup \( (\mu_\alpha)_{\alpha > 0} \) of positive measures such that the Laplace transform of \( \mu_\alpha \) is \( f^\alpha \). Note that the convolution of any two positive measures on \([0, \infty[\) is well-defined and we have \( \mu_\alpha * \mu_\beta = \mu_{\alpha+\beta} \).

In the special case of \( f(0+) = 1 \) this is very classical: This is the description of infinitely divisible distributions in probability. Since there are probabilities which are not infinitely divisible we have \( \mathcal{C} \setminus \mathcal{L} \neq \emptyset \).

In various papers complete monotonicity for special functions has been established by proving the stronger statement that the function is a Stieltjes transform, i.e. is of the form
\[
f(x) = a + \int_0^\infty \frac{d\mu(s)}{s + x},
\]
where \( a \geq 0 \) and \( \mu \) is a nonnegative measure on \([0, \infty[\) satisfying
\[
\int_0^\infty \frac{1}{1 + s}d\mu(s) < \infty.
\]

See [2], [4], [5], [6], [8], [9].

The set of Stieltjes transforms will be denoted \( \mathcal{S} \). We clearly have \( \mathcal{S} \subset \mathcal{C} \). For more information about this class see [7].

**Theorem 1.2.** \( \mathcal{S} \setminus \{0\} \subset \mathcal{L} \).

**Theorem 1.3.** The functions
\[
\Phi(x) = \frac{\Gamma(x+1)^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x
\]
and
\[
\log \Phi(x) = \frac{\log \Gamma(x+1)}{x} - \log x + x \log \left(1 + \frac{1}{x}\right)
\]
are Stieltjes transforms with the representations [4] and [2].
**Remark 1.4.** The class $\mathcal{S}$ has the following stability properties: If $f \in \mathcal{S}, f \neq 0$ then $1/f(1/x)$ and $1/(xf(x))$ are again Stieltjes transforms, cf. [5]. Therefore the following functions belong to $\mathcal{S}$:

$$\frac{1}{[\Gamma(1+x)]^{1/x}(1+1/x)^x}, \quad \frac{1}{x[\Gamma(1+1/x)]^{1/x}(1+x)^{1/x}}, \quad \frac{1}{[\Gamma(1+1/x)]^{1/x}(1+x)^{1/x}}.$$ 

It was proved in [3] that $[\Gamma(1+1/x)]^{1/x} \in \mathcal{S}$, so also the following functions are Stieltjes transforms:

$$\frac{1}{\Gamma(1+x)} \frac{1}{x}, \quad \frac{1}{\Gamma(1+1/x)} \frac{1}{x(1+x)}, \quad \frac{1}{x\Gamma(1+1/x)} \frac{1}{x(1+x)}.$$ 

In [2] it was proved that $(1+1/x)^{-x} \in \mathcal{S}$. Therefore the function $\Phi$ given by (1) is a quotient of known Stieltjes transforms, but this does not imply that the function itself is a Stieltjes transform.

2. Proofs

For completeness we include a proof of Theorem 1.1.

**Proof.** "(i) $\Rightarrow (ii)". Since $f \in \mathcal{L}$ implies $f^\alpha \in \mathcal{L}$ for all $\alpha > 0$, it is enough to prove the inclusion $\mathcal{L} \subset \mathcal{C}$. Although this is done in [3] and [11] we include the easy proof. By assumption $-(\log f)' = -f'/f \in \mathcal{C}$, so in particular $-f' \geq 0$. Assume now that $(-1)^k f^{(k)} \geq 0$ for $k \leq n$. Then

$$(-1)^{n+1} f^{(n+1)} = (-1)^n (-\log f)' f^{(n)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k ((-\log f)' f^{(k)} (-1)^{n-k} f^{(n-k)}) \geq 0,$$

and (ii) follows by induction.

"(ii) $\Rightarrow (iii)"$ is obvious.

"(iii) $\Rightarrow (i)". If $f^{1/n} \in \mathcal{C}$ we have in particular $-f^{1/n}' = (-1/n) f^{-1+1/n} f' \in \mathcal{C}$. Multiplying by $n$ and letting $n \to \infty$ we see that the limit function $-f'/f$ belongs to $\mathcal{C}$, because $\mathcal{C}$ is closed under pointwise limits, cf. [7]. This establishes (i). □

**Proof of Theorem 1.2.** Let $f \in \mathcal{S}$ be non-zero, and let $\alpha > 0$. By Theorem 1.1 it is enough to prove that $f^\alpha \in \mathcal{C}$. Writing $\alpha = n + a$ with $n = 0, 1, \ldots$ and $0 \leq a < 1$ we have $f^\alpha = f^n f^a$, and using the stability of $\mathcal{C}$ under multiplication and that $f^n \in \mathcal{S}$, cf. [4], the assertion follows. □

**Proof of Theorem 1.3.** Using the expression (3) for $\varphi$ we find

$$\int_0^\infty \frac{\varphi(s)}{s + x} \, ds = \int_0^1 \frac{1 - s}{s + x} \, ds + \sum_{k=1}^\infty \int_k^{k+1} \frac{1 - k/s}{s + x} \, ds$$

$$= -1 + (x + 1) \log \left(1 + \frac{1}{x}\right)$$

$$+ \sum_{k=1}^\infty \left[ \left(1 + \frac{k}{x}\right) \log \left(1 + \frac{1}{x + k}\right) - \frac{k}{x} \log \left(1 + \frac{1}{k}\right) \right].$$

Therefore,

$$\int_0^\infty \frac{\varphi(s)}{s + x} \, ds = \log \Phi(x)$$
if and only if
\[
\log \Gamma(x + 1) = x(\log(1 + x) - 1) + \sum_{k=1}^{\infty} \left[ (k + x) \log \left(1 + \frac{1}{x + k}\right) - k \log \left(1 + \frac{1}{k}\right) \right]
\]
for \(x \geq 0\). Both sides vanish for \(x = 0\), and they have the same derivative \(\psi(x + 1)\), where \(\psi\) is the digamma function. This follows easily by the classical formula
\[
\psi(x) = \log x + \sum_{k=0}^{\infty} \left[ \log \left(1 + \frac{1}{x + k}\right) - \frac{1}{x + k} \right]
\]
cf. [8, 8.362(2)]. This shows that \(\log \Phi\) is a Stieltjes transform with the representation (2). In particular \(\Phi\) is completely monotonic with the limit 1 at infinity.

To see that a function \(f\) is a Stieltjes transform we will use the characterization of these functions via complex analysis, see [1] p. 127 or [5]. It is necessary and sufficient that \(f\) has a holomorphic extension to the cut plane \(A = \mathbb{C} \setminus (-\infty, 0]\), and satisfies \(\text{Im } f(z) \leq 0\) for \(\text{Im } z > 0\) and \(f(x) \geq 0\) for \(x > 0\). For a Stieltjes transform \(f\) given by (8) we have \(a = \lim_{x \to \infty} f(x)\), and the measure \(\mu\) is the limit in the vague topology of
\[
-\frac{1}{\pi} \text{Im } f(-x + iy) \, dx
\]
as \(y \to 0^+\).

We clearly have \(\Phi(x) > 0\) for \(x > 0\) and the holomorphic extension of \(\Phi\) is given by \(\Phi(z) = \exp(\log \Phi(z))\), where \(\log \Phi(z)\) is the holomorphic extension obtained by the representation (2). This can also be described in the following way: For \(z \in A\) we let \(\log \Gamma(z)\) be the unique holomorphic branch, which is real for \(x > 0\), and we let \(\text{Log}\) denote the principal logarithm. Then
\[
\frac{\log \Gamma(z + 1)}{z} - \text{Log } z + z \log \left(1 + \frac{1}{z}\right)
\]
is a holomorphic branch of \(\log \Phi(z)\) in \(A\), and since it agrees with \(\log \Phi(x)\) for \(x > 0\), we have
\[
\log \Phi(z) = \frac{\log \Gamma(z + 1)}{z} - \text{Log } z + z \log \left(1 + \frac{1}{z}\right) = \int_0^\infty \frac{\varphi(s)}{s + z} \, ds, \quad z \in A.
\]
(11)

For \(z = x + iy, y > 0\) we get
\[
\text{Im } \log \Phi(x + iy) = -\int_0^\infty \frac{\varphi(s)y}{(s + x)^2 + y^2} \, ds,
\]
and since \(0 \leq \varphi(s) \leq 1\) for \(s \geq 0\) we get
\[
\text{Im } \log \Phi(x + iy) \in ]-\pi, 0[, \quad x \geq 0, \, y \to 0^+
\]
hence
\[
\text{Im } \Phi(x + iy) = |\Phi(x + iy)| \sin(\text{Im } \log \Phi(x + iy)) < 0,
\]
which shows that \(\Phi\) is a Stieltjes transform. For \(x \geq 0, y \to 0^+\) we further get
\[
-\frac{1}{\pi} \text{Im } \Phi(-x + iy) \to h(x) := \frac{1}{\pi} |\Phi(-x)| \sin(\pi \varphi(x)), \quad x \geq 0
\]
which is is a continuous nonnegative function on \([0, \infty[\). Therefore the convergence is uniform for \(x\) in compact subsets of \([0, \infty[\), so \(h\) is the density of the representing measure as a Stieltjes transform.

This shows the following integral representation of \(\Phi\):
\[
\Phi(z) = 1 + \int_0^\infty \frac{h(s)}{s + z} \, ds, \quad z \in A
\]
(13)
where
\[
h(s) = \frac{1}{\pi} \frac{s^{s-1}}{|1 - s|^s |\Gamma(1 - s)|^{1/s}} \sin(\pi \varphi(s)), \quad s \geq 0.
\]
Remark 2.1. There is a close relationship between Stieltjes transforms and Pick functions. For the latter see [1] and [10]. It is possible to find the integral representation (2) of \( \log \Phi(x) \) using integral representations of the three terms of (10). Here \( \log(z) \) and \( z \log(1 + 1/z) \) are Pick functions for which the integral representations are easily obtained. The integral representation of the Pick function \( \log \Gamma(z+1)/z \) was found in [9]. The author first found the density (3) in this way, but once \( \phi \) is found the present direct approach seems easier.

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