On one question of Shemetkov about composition formations

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Abstract. In this paper one construction of composition formations was introduced. This construction contains formations of quasinilpotent groups, c-supersoluble groups, groups defined by ranks of chief factors and some new classes of groups. A partial answer on a question of L. A. Shemetkov about the intersection of $\mathfrak{F}$-maximal subgroups and the $\mathfrak{F}$-hypercenter was given for these composition formations.

Keywords. Finite group; c-supersoluble group; quasinilpotent group, quasi-$\mathfrak{F}$-group; hereditary saturated formation; solubly saturated formation; $\mathfrak{F}$-maximal subgroup; $\mathfrak{F}$-hypercenter of a group.

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Introduction

All groups considered here will be finite. Through $G$, $p$ and $\mathfrak{X}$ we always denote here respectively a finite group, a prime and a class of groups.

One of the directions in the modern group theory is to construct classes of groups and study the properties of all groups in such class (formation, Fitting class, Schunk class and etc.)

Recall that a formation is a class $\mathfrak{X}$ of groups with the following properties: (a) every homomorphic image of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group, and (b) if $G/M$ and $G/N$ are $\mathfrak{X}$-groups, then also $G/(M \cap N) \in \mathfrak{X}$.

One of the main classes of formations is the class of local formations. Recall that a function of the form $f : \mathbb{P} \to \{\text{formations}\}$ is called a formation function and a formation $\mathfrak{F}$ is called local [1, IV, 3.1] if

$$\mathfrak{F} = \{G | G/C_G(H) \in f(p) \text{ for every } p \in \pi(H) \text{ and every chief factor } H \text{ of } G\}$$

for some formation function $f$. In this case $f$ is called a local definition of $\mathfrak{F}$. By the Gaschütz-Lubeseder-Schmid theorem, a formation is local if and only if it is non-empty and saturated, i.e. from $G/\Phi(G) \in \mathfrak{F}$ it follows that $G \in \mathfrak{F}$ where $\Phi(G)$ is the Frattini subgroup of $G$. The classes of all unit $\mathfrak{E}$, nilpotent $\mathfrak{N}$, metanilpotent $\mathfrak{N}_2$, supersoluble $\mathfrak{U}$ and soluble groups $\mathfrak{S}$ are examples of local formations.

The most applications of local formations are in the theory of soluble groups. Let us mention another interesting class of formations of soluble groups. Let $\overline{N}$ be a chief factor of $G$. Then $\overline{N} = N_1 \times \cdots \times N_n$ where $N_i$ are isomorphic simple groups. The number $n = r(\overline{N}, G)$ is the rank of $\overline{N}$ in $G$. A rank function $R$ [1, VII, 2.3] is a map which associates with each prime $p$ a set $R(p)$ of natural numbers. For each rank function let

$$\mathfrak{C}(R) = \{G \in \mathfrak{S} | \text{ for all } p \in \mathbb{P} \text{ each } p\text{-chief factor of } G \text{ has rank in } R(p)\}.$$

Note that $\mathfrak{C}(R)$ is a formation. Heineken [2] and Harman [3] characterized all rank functions $R$ for which $\mathfrak{C}(R)$ is local. Analogues questions for formations not of full characteristic were studied by Huppert [4], Kohler [5] and Harman [2]. Haberl and Heineken [6] characterized all rank functions $R$ for which $\mathfrak{C}(R)$ is a Fitting formation.

A function of the form $f : \mathbb{P} \cup \{0\} \to \{\text{formations}\}$ is called a composition definition. Recall [7, p. 4] that a formation $\mathfrak{F}$ is called composition or Baer-local if
\[ \mathfrak{F} = (G \mid G/G_{\mathfrak{F}} \in f(0) \text{ and } G/C_{G}(\mathcal{H}) \in f(p) \) for every abelian p-chief factor $\mathcal{H}$ of $G$ \]

for some composition definition $f$. A formation is composition (Baer-local) [11 IV, 4.17] if and only if it is solubly saturated, i.e. from $G/\Phi(G_{\mathfrak{F}}) \in \mathfrak{F}$ it follows that $G \in \mathfrak{F}$, where $G_{\mathfrak{F}}$ is the soluble radical of $G$.

Note that a local formation is a composition formation. The converse is false. An example of nonlocal composition formation is the class of all quasinilpotent groups $\mathfrak{H}$ that was introduced by Bender [9].

Recall that a chief factor $\mathcal{H}$ of $G$ is called $\mathfrak{X}$-central in $G$ provided $\mathcal{H} \times G/C_{G}(\mathcal{H}) \in \mathfrak{X}$ (see [8] p. 127–128), otherwise it is called $\mathfrak{X}$-eccentric. The symbol $Z_{\mathfrak{X}}(G)$ denotes the $\mathfrak{X}$-hypercenter of $G$, that is, the largest normal subgroup of $G$ such that every chief factor $\mathcal{H}$ of $G$ below it is $\mathfrak{X}$-central. If $\mathfrak{X} = \mathcal{H}$ is the class of all nilpotent groups, then $Z_{\mathfrak{H}}(G) = Z_{\infty}(G)$ is the hypercenter of $G$. If $\mathfrak{F}$ is a composition formation, then by [7, 1, 2.6]

\[ \mathfrak{F} = (G \mid Z_{\mathfrak{F}}(G) = G). \]

The general definition of a composition formation $\mathfrak{F}$ gives little information about the action of an $\mathfrak{F}$-group $G$ on its non-abelian chief factors. Therefore several families of composition formations were introduced by giving additional information about the action of an $\mathfrak{F}$-group on its non-abelian chief factors. For example, in [10, 11] Guo and Skiba introduced the class $\mathfrak{F}^{*}$ of quasi-$\mathfrak{F}$-groups for a saturated formation $\mathfrak{F}$:

\[ \mathfrak{F}^{*} = (G \mid \text{for every } \mathfrak{F}\text{-eccentric chief factor } \mathcal{H} \text{ and every } x \in G, x \text{ induces an inner automorphism on } \mathcal{H}). \]

If $\mathfrak{H} \subseteq \mathfrak{F}$ is a normally hereditary saturated formation, then $\mathfrak{F}^{*}$ is a normally hereditary solubly saturated formation by [10, Theorem 2.6].

Another example of a nonlocal composition formation is the class of all c-supersoluble groups that was introduced by Vedernikov in [12]. Recall that a group is called $c$-supersoluble (SC-group in the terminology of Robinson [13]) if every its chief factor is a simple group. The products of c-supersoluble groups were studied in [14, 15, 16, 17, 18]. According to [19], a group $G$ is called $\mathfrak{J}$c-supersoluble if every chief $\mathfrak{J}$-factor of $G$ is a simple group, where $\mathfrak{J}$ is a class of simple groups. In [19] the same idea was applied for some other classes of groups. The products and properties of such groups were studied in [19, 20].

In [21] the class $\mathfrak{F}_{ca}$ of ca-$\mathfrak{F}$-groups was introduced:

\[ \mathfrak{F}_{ca} = (G \mid \text{abelian chief factors of } G \text{ are } \mathfrak{F}\text{-central and other chief factors are simple groups}). \]

The class $\mathfrak{F}_{ca}$ was studied in [21, 22], where $\mathfrak{F}$ is a saturated formation. In particular it is a composition formation.

In this paper we generalize constructions of quasi-$\mathfrak{F}$-groups, ca-$\mathfrak{F}$-groups, $\mathfrak{J}$c-supersoluble groups and groups defined by the rank function in the sense of the following definition.

**Definition 1.** (1) A generalized rank function $\mathcal{R}$ is a map defined on direct products of isomorphic simple groups by

(a) $\mathcal{R}$ associates with each simple group $S$ a pair $\mathcal{R}(S) = (A_{\mathcal{R}}(S), B_{\mathcal{R}}(S))$ of possibly empty disjoint sets $A_{\mathcal{R}}(S)$ and $B_{\mathcal{R}}(S)$ of natural numbers.

(b) If $N$ is the direct products of simple isomorphic to $S$ groups, then $\mathcal{R}(N) = \mathcal{R}(S)$.

(2) Let $\mathcal{N}$ be a chief factor of $G$. We shall say that a generalized rank of $\mathcal{N}$ in $G$ lies in $\mathcal{R}(\mathcal{N})$ (briefly $gr(\mathcal{N}, G) \in \mathcal{R}(\mathcal{N})$) if $r(\mathcal{N}, G) \in A_{\mathcal{R}}(\mathcal{N})$ or $r(\mathcal{N}, G) \in B_{\mathcal{R}}(\mathcal{N})$ and if some $x \in G$ fixes a composition factor $\mathcal{H}/K$ of $\mathcal{N}$ (i.e. $\mathcal{H}/K = \mathcal{H}$ and $Kxe = K$), then $x$ induces an inner automorphism on it.

(3) With each generalized rank function $\mathcal{R}$ and a class of groups $\mathcal{X}$ we associate a class

\[ \mathcal{X}(\mathcal{R}) = (G \mid \mathcal{H} \not\in \mathcal{X} \text{ and } gr(\mathcal{H}, G) \in \mathcal{R}(\mathcal{H}) \) for every $\mathcal{X}$-eccentric chief factor $\mathcal{H}$ of $G) \].
Example 1. A lot of above mentioned formations can be described with the help of our construction:

1. Let $\mathcal{E} = (1)$. Assume that $\mathcal{R}(H) = (\{1\}, \emptyset)$ if $H$ is abelian and $\mathcal{R}(H) = (\emptyset, \emptyset)$ otherwise. Then $\mathcal{E}(\mathcal{R}) = \mathcal{U}$.

2. If $\mathcal{R}(H) \equiv (\{1\}, \emptyset)$, then $\mathcal{E}(\mathcal{R})$ is the class $\mathcal{U}_c$ of all $c$-supersoluble groups.

3. Let $\mathfrak{J}$ be a class of simple groups. If $\mathcal{R}(H) \equiv (\{1\}, \emptyset)$ for $H \in \mathfrak{J}$ and $\mathcal{R}(H) = (\mathbb{N}, \emptyset)$ otherwise, then $\mathcal{E}(\mathcal{R})$ is the class of all $\mathfrak{J}c$-supersoluble groups.

4. Assume that $\mathcal{R}(H) = (A_{\mathcal{R}}(H), \emptyset)$ if $H$ is abelian and $\mathcal{R}(H) = (\emptyset, \emptyset)$ otherwise. Then $\mathcal{R}$ is a rank function.

5. Let $\mathcal{R}(H) = (\emptyset, \{1\})$ if $H$ is abelian and $\mathcal{R}(H) = (\emptyset, \emptyset)$ otherwise. Then $\mathcal{E}(\mathcal{R}) = \mathcal{R}^*$.

6. If $\mathcal{R}(H) \equiv (\emptyset, \{1\})$, then $\mathcal{E}(\mathcal{R}) = \mathcal{R}^{**}$.

7. Assume that $\mathcal{R}(H) = (\emptyset, \{1\})$ if $H$ is abelian and $\mathcal{R}(H) = (\{1\}, \emptyset)$ otherwise. Then $\mathcal{E}(\mathcal{R}) = \mathcal{R}_{ca}$.

8. Let $\mathfrak{N} \subseteq \mathfrak{J}$ be a normally hereditary saturated formation. If $\mathcal{R}(H) \equiv (\emptyset, \{1\})$, then $\mathfrak{N}(\mathcal{R}) = \mathfrak{J}^*$ (see the proof of Corollary 1.3).

9. Let $\mathfrak{F} \subseteq \mathfrak{S}$ be a normally hereditary saturated formation, $\mathcal{R}(H) = (\emptyset, \emptyset)$ for abelian $H \not\in \mathfrak{F}$ and $\mathcal{R}(H) = (\{1\}, \emptyset)$ for non-abelian $H$. Then $\mathfrak{F}(\mathcal{R}) = \mathfrak{F}_{ca}$.

Question 1. For what non-empty normally hereditary solubly saturated formations $\mathfrak{X}$ do the equality $\text{Int}_{\mathfrak{X}}(G) = Z_{\mathfrak{X}}(G)$ hold for every group $G$?

The solution to this question for hereditary saturated formations was obtained by Skiba in [24, 25] (for the soluble case, see also Beidleman and Heineken [26]) and for the class of all quasi-$\mathfrak{F}$-groups, where $\mathfrak{F}$ is a hereditary saturated formation, was given in [27]. In particular, the intersection of maximal quasinilpotent subgroups is the quasinilpotent hypercenter. The aim of this paper is to give the answer on this question for $\mathfrak{F}(\mathcal{R})$.

Preliminaries

The notation and terminology agree with the books [1, 7]. We refer the reader to these books for the results on formations. Recall that $G^{\mathfrak{F}}$ is the $\mathfrak{F}$-residual of $G$ for a formation $\mathfrak{F}$; $G_{\mathfrak{F}}$ is the soluble radical of $G$; $\tilde{F}(G)$ is defined by $\tilde{F}(G)/\Phi(G) = \text{Soc}(G/\Phi(G))$; $\pi(G)$ is the set of all prime divisors of $G$; $\pi(\mathfrak{X}) = \bigcup_{G \in \mathfrak{X}} \pi(G)$; $\mathcal{R}_p\mathfrak{F} = (G | G/O_p(G) \in \mathfrak{F})$ is a formation for a formation $\mathfrak{F}$; $G$ is called $s$-critical for $\mathfrak{X}$ if all proper subgroups of $G$ are $\mathfrak{X}$-groups and $G \not\in \mathfrak{X}$; $\text{Aut}G$, $\text{Inn}G$ and $\text{Out}G$ are respectively groups of all, inner and outer automorphisms of $G$; $E\mathfrak{X}$ is the class of groups all whose composition factors are $\mathfrak{X}$-groups; $N \wr S_n$ is the natural wreath product of $N$ and the symmetric group $S_n$ of degree $n$. 

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1 Main Results

1.1 The canonical composition definition of $\mathfrak{F}(\mathcal{R})$

Recall that any nonempty composition formation $\mathfrak{F}$ has an unique composition definition $F$ such that $F(p) = \mathfrak{F}(p) F(p) \subseteq \mathfrak{F}$ for all primes $p$ and $F(0) = \mathfrak{F}$ (see [21, 1.6]). In this case $F$ is called the canonical composition definition of $\mathfrak{F}$. We shall say that a generalized rank function $\mathcal{R}$ is (resp. very) good if for any simple group $S$ holds:

(a) from $a \in A_\mathcal{R}(S)$ it follows that $b \in A_\mathcal{R}(S)$ for any natural $b/a$ (resp. $b \leq a$);

(b) from $a \in B_\mathcal{R}(S)$ it follows that $b \in A_\mathcal{R}(S) \cup B_\mathcal{R}(S)$ (resp. $b \in B_\mathcal{R}(S)$) for any natural $b/a$ (resp. $b \leq a$).

Theorem 1. Let $\mathfrak{F} \subseteq \mathfrak{F}$ be a composition formation with the canonical composition definition $F$ and $\mathcal{R}$ be a generalized rank function. Then

(1) $\mathfrak{F}(\mathcal{R})$ is a composition formation with the canonical composition definition $F_\mathcal{R}$ such that $F_\mathcal{R}(0) = \mathfrak{F}(\mathcal{R})$ and $F_\mathcal{R}(p) = F(p)$ for all $p \in \mathbb{P}$.

(2) If $\mathfrak{F}$ is normally hereditary and $\mathcal{R}$ is good, then $\mathfrak{F}(\mathcal{R})$ is normally hereditary.

Corollary 1.1 ([14, Theorem 1]). $\mathcal{U}_c$ is a composition formation with the canonical composition definition $h$ such that $h(p) = \mathfrak{F}(\mathcal{U}_c(p - 1))$ for every prime $p$ and $h(0) = \mathcal{U}_c$.

In [28] the class $w\mathcal{U}$ of widely supersoluble groups was introduced. It is a hereditary saturated formation of soluble groups. Recall [18] that a group is called widely $c$-supersoluble if its abelian chief factors are $w\mathcal{U}$-central and other chief factors are simple groups.

Corollary 1.2 ([18, Theorem A]). The class $\mathcal{U}_{cw}$ of widely $c$-supersoluble groups is a normally hereditary composition formation with the canonical composition definition $h$ such that $h(p) = \mathfrak{F}(w\mathcal{U} \cap \mathfrak{F}(p - 1))$ for every prime $p$ and $h(0) = \mathcal{U}_{cw}$.

Corollary 1.3 ([10, Theorem 2.6]). For every saturated formation $\mathfrak{F}$ containing all nilpotent groups with the canonical local definition $F$, the formation $\mathfrak{F}^*$ is composition with the canonical composition definition $F^*$ where $F^*(p) = F(p)$ for every prime $p$ and $F^*(0) = \mathfrak{F}^*$. Moreover, if the formation $\mathfrak{F}$ is normally hereditary, then $\mathfrak{F}^*$ is also normally hereditary.

In the proof of Theorem 1 we will need the following lemmas:

Lemma 1. Let $H/K$ and $M/N$ be $G$-isomorphic chief factors of $G$.

(a) Then they have the same generalized rank.

(b) [1, 1.14] $H/K \rtimes G/C_G(H/K) \simeq M/N \rtimes G/C_G(M/N)$.

Proof. Let $\alpha : H/K \to M/N$ be a $G$-isomorphism. Since $H/K$ and $M/N$ are isomorphic groups, they have the same rank. Assume that $x \in G$ fixes a composition factor $A/B$ of $H/K$ and induces an inner automorphism $\alpha aB$ on it. Note that $\alpha(A/B)$ is a composition factor of $M/N$. From $\alpha(A/B) = \alpha(A/B^z) = \alpha(A/B)$ it follows that $x$ fixes $\alpha(A/B)$ and it is straightforward to check that $x$ induces an inner automorphism $\alpha(aB)$ on it. Since $\alpha^{-1}$ is also $G$-isomorphism, we see that the generalized ranks of $H/K$ and $M/N$ are the same.

Lemma 2 ([1, 1.15]). Let $\overline{H}$ be a chief factor of $G$. Then

(1) If $\mathfrak{F}$ is a composition formation and $F$ is its canonical composition definition, then $\overline{H}$ is $\mathfrak{F}$-central if and only if $G/C_G(\overline{H}) \subseteq F(p)$ for all $p \in \pi(\overline{H})$ in the case when $\overline{H}$ is abelian, and $G/C_G(\overline{H}) \subseteq \mathfrak{F}$ when $\overline{H}$ is non-abelian.

(2) If $\mathfrak{F}$ is a local formation and $F$ is its canonical local definition, then $\overline{H}$ is $\mathfrak{F}$-central if and only if $G/C_G(\overline{H}) \subseteq F(p)$ for all $p \in \pi(\overline{H})$.

Lemma 3 ([1, 2.6]). Let $\mathfrak{F}$ be a solubly saturated formation. Then $\mathfrak{F} = (G | G = Z_2(G))$. 


Recall that $C^p(G)$ is the intersection of all abelian $p$-chief factors of $G$ ($C^p(G) = G$ if $G$ has no such chief factors). Let $f$ be a composition definition of a composition formation $\mathfrak{F}$. It is known that $\mathfrak{F} = (G:G_{\mathfrak{F}} \in f(0))$ and $G/C^p(G) \in f(p)$ for all $p \in \pi(G)$ such that $G$ has an abelian $p$-chief factor).

**Lemma 5.** Let a normal subgroup $N$ of $G$ be a direct product of isomorphic simple non-abelian groups. Then $N$ is a direct product of minimal normal subgroups of $G$.

**Proof of Theorem**

1. From (b) and (c) of the Isomorphism Theorems [1] 2.1A] and Lemma [1] it follows that $\mathfrak{X}(\mathcal{R})$ is a formation for any class of groups $\mathfrak{X}$. So, $\mathfrak{F}(\mathcal{R})$ is a formation. Let $\mathfrak{H} = CLF(F_\mathcal{R})$.

Assume $\mathfrak{H} \setminus \mathfrak{F}(\mathcal{R}) \neq \emptyset$. Let chose a minimal order group $G$ from $\mathfrak{H} \setminus \mathfrak{F}(\mathcal{R})$. Since $\mathfrak{F}(\mathcal{R})$ is a formation, $G$ has an unique minimal normal subgroup $N$ and $G/N \in \mathfrak{F}(\mathcal{R})$.

Suppose that $N$ is abelian. Then it is a $p$-group. Since $N$ is $\mathfrak{H}$-central in $G$ by Lemma [3] $G/C_N(N) \in F_\mathcal{R}(p)$ by Lemma [2] From $F_\mathcal{R}(p) = F(p)$ and Lemma [2] it follows that $N$ is an $\mathfrak{F}$-central chief factor of $G$. Hence $G \in \mathfrak{F}(\mathcal{R})$, a contradiction.

So $N$ is non-abelian. Note that $G_{\mathfrak{F}} \leq C_G(N)$ by [2] 1, 1.5. Hence $G \simeq G/C_G(N) \in F_\mathcal{R}(0) = \mathfrak{F}(\mathcal{R})$, the contradiction. Thus $\mathfrak{H} \subseteq \mathfrak{F}(\mathcal{R})$.

Assume $\mathfrak{F}(\mathcal{R}) \setminus \mathfrak{H} \neq \emptyset$. Let chose a minimal order group $G$ from $\mathfrak{F}(\mathcal{R}) \setminus \mathfrak{H}$. Since $\mathfrak{H}$ is a formation, $G$ has an unique minimal normal subgroup $N$ and $G/N \in \mathfrak{H}$.

If $N$ is abelian, then $G/C_G(N) \in F(p)$ for some $p$ by Lemmas [2] and [3] From $F_\mathcal{R}(p) = F(p)$ and Lemma [2] it follows that $N$ is $\mathfrak{H}$-central in $G$. So $G \in \mathfrak{H}$, a contradiction.

Hence $N$ is non-abelian. It means that $G_{\mathfrak{F}} \simeq 1$. Therefore $G/G_{\mathfrak{F}} \simeq G \in \mathfrak{F}(\mathcal{R}) = F_\mathcal{R}(0)$.

Note that $N \leq C^p(G)$ for all primes $p$. So $C^p(G)/N = C^p(G/N)$. From $G/N \in \mathfrak{H}$ it follows that $G/C^p(G) \simeq (G/N)/C^p(G/N) \in F_\mathcal{R}(p)$ for any $p$ such that $G$ has an abelian chief $p$-factor. Therefore $G \in \mathfrak{H}$, the contradiction. So $\mathfrak{F}(\mathcal{R}) \subseteq \mathfrak{H}$. Thus $\mathfrak{F}(\mathcal{R}) = \mathfrak{H}$.

2. Let $F$ be the canonical composition definition of $\mathfrak{F}$. $G$ be an $\mathfrak{F}$-group and $1 = N_0 \leq N_1 \leq \cdots \leq N_n = N \leq G$ be the part of chief series of $G$ below $N$. Let $H/K$ be a chief factor of $N$ such that $N_i = K \leq H \leq N_i$ for some $i$.

If $N_i/N_{i-1} \not\in \mathfrak{F}$, then it is a non-abelian group. According to Lemma [2] $N_i/N_{i-1}$ is a direct product of minimal normal subgroups of $N/N_i$. Let $L/N_{i-1}$ be one of them and $L_1/N_{i-1}$ be a direct simple factor of it. Note that $r(N_i/N_{i-1}, G) = [G : N_G(L_1/N_{i-1})], N_G(L_1/N_{i-1}) \cap N = N_L(L_1/N_{i-1})$ and $[G : N_G(L_1/N_{i-1})]$ is a divisor of $[G : N_G(L_1/N_{i-1})]$ by [23] §1, Lemma 1. It means that $r(L/N_{i-1}, N)$ divides $r(N_i/N_{i-1}, G)$ and every composition factor of $L/N_{i-1}$ is a composition factor of $N_i/N_{i-1}$. Since $\mathcal{R}$ is a good generalized rank function, $gr(L/N_{i-1}, N) \in \mathcal{R}(L/N_{i-1})$ for any chief factor $L/N_{i-1}$ of $N$ between $N_i/N_{i-1}$ and $N_i$.

If $N_i/N_{i-1} \in \mathfrak{F}$, then it is $\mathfrak{F}$-central in $G$. Note that $H/K \in \mathfrak{F}$.

Assume that $N_i/N_{i-1}$ is abelian. Then $G/C_G(N_i/N_{i-1}) \in F(p)$ for some $p$ by Lemma [2]. Note that $F(p)$ is a normally hereditary formation by [1] IV, 3.16. Since

$$NC_G(N_i/N_{i-1})/C_G(N_i/N_{i-1}) \leq G/C_G(N_i/N_{i-1}),$$

we see that

$$NC_G(N_i/N_{i-1})/C_G(N_i/N_{i-1}) \simeq N/C_N(N_i/N_{i-1}) \in F(p).$$

From $C_N(N_i/N_{i-1}) \leq C_N(H/K)$ it follows that $N/C_N(H/K)$ is a quotient group of $N/C_N(N_i/N_{i-1})$. Thus $N/C_N(H/K) \in F(p)$. Now $H/K$ is an $\mathfrak{F}$-central chief factor of $N$ by Lemma [2].
Assume that $N_i/N_{i-1}$ is non-abelian. Then $G/C_G(N_i/N_{i-1}) \in \mathfrak{F}$ by Lemma 2. Hence $NC_G(N_i/N_{i-1})/C_G(N_i/N_{i-1}) \in \mathfrak{F}$. By analogy $N/C_N(H/K) \in \mathfrak{F}$. So $H/K$ is an $\mathfrak{F}$-central chief factor of $N$ by Lemma 2.

Thus every chief $\mathfrak{F}$-factor of $N$ is $\mathfrak{F}$-central and $gr(H, N) \in R(H)$ for other chief factors $H$ of $N$ by Jordan-Hölder theorem. Thus $N \in \mathfrak{F}(R)$. \hfill \Box

**Proof of Corollaries 1.1, 1.2 and 1.3.** Recall that every local formation is composition. It is known that if $F$ is the canonical local definition of a local formation $\mathfrak{F}$, then $D$ is the canonical composition definition of $\mathfrak{F}$ where $D(0) = \mathfrak{F}$ and $F(p) = D(p)$ for all prime $p$.

Let $R(H) \equiv \{(1), 0\}$. Note that $R$ is good. Recall that the classes of all supersoluble and widely supersoluble groups are hereditary local formations with the canonical local definitions $F(p) = \mathfrak{S}_p \mathfrak{A}(p-1)$ and $D(p) = \mathfrak{S}_p(G| G \in w \mathfrak{A} \cap \mathfrak{M}_p A(p-1))$ (see [18, Lemma 3.2]) for every prime $p$ respectively. Note that $\mathfrak{M}_{w} = \mathfrak{M}(R)$ and $\mathfrak{M}_{w} = w \mathfrak{M}(R)$. Now Corollaries 1.1 and 1.2 directly follows from Theorem 1.

Let $R(H) \equiv \{(0, 1)\}$ and $\mathfrak{F}$ be a hereditary local formation. Again $R$ is good. Let $\overline{H} \in \mathfrak{F}$ be an $\mathfrak{F}$-eccentric chief factor of an $\mathfrak{F}^*$-group $G$. Note that $r(H, G) = 1$ and $\overline{H} \times G/C_G(H)$ is a quotient of $\overline{H} \times \overline{H} \in \mathfrak{F}$. Thus, $\mathfrak{F}$ is $\mathfrak{F}$-central in $G$, a contradiction. It means that $\mathfrak{F}^* = \mathfrak{F}(R)$. Now Corollary 1.3 directly follows from Theorem 1. \hfill \Box

### 1.2 The structure of an $\mathfrak{F}(R)$-group

The aim of this subsection is to obtain the characterization of an $\mathfrak{F}(R)$-group.

**Definition 2.** Let $Z(G, R, \mathfrak{F}, n)$ be the greatest $G$-invariant subgroup of $G$ such that $\overline{H} \not\in \mathfrak{F}$, $r(H, G) > n$ and $gr(H, G) \in R(H)$ for every its $G$-composition $\mathfrak{F}$-eccentric in $G$ factor $H$.

Let $C$ be a set and $R$ be a generalized rank function. We say that $R(\overline{H}) \subseteq C$ if $A_R(\overline{H}) \cup B_R(\overline{H}) \subseteq C$. By $R(\overline{H}) \cap C$ we mean $(A_R(\overline{H}) \cap C, B_R(\overline{H}) \cap C)$.

**Remark 1.** (1) Let $N$ and $M$ be normal subgroups of $G$. According to (b) of the Isomorphism Theorems [11, 2.1A] every $G$-composition factor of $NM$ is $G$-isomorphic to a $G$-composition factor of $N$ or $M$. Hence $Z(G, R, \mathfrak{F}, n)$ exists in every group by Lemma 1.

(2) It is clear that $G \in \mathfrak{F}(R)$ iff $G = Z(G, R, \mathfrak{F}, 0)$.

(3) If $R(S) \subseteq [0, 1]$ for every simple group $S$, then $Z(G, R, \mathfrak{F}, n) = Z_{\mathfrak{F}}(G)$ for $n > 1$.

**Theorem 2.** Let $\mathfrak{F}$ be a solubly saturated formation containing all nilpotent groups such that $\mathfrak{F}$ contains every composition factor of every $\mathfrak{F}$-group and $R$ be a generalized rank function. Then the following statements are equivalent:

1. $G$ is an $\mathfrak{F}(R)$-group.

2. Let $Z = Z(G, R, \mathfrak{F}, 4)$. Then $gr(N/Z, G) \in R(N/Z) \cap [1, 4]$ for every minimal normal subgroup $N/Z$ of $G/Z$ and $(G/Z)/Soc(G/Z)$ is a soluble $\mathfrak{F}$-group.

3. The following holds:
   (a) $G_{\mathfrak{F}} = G_{E\mathfrak{F}}$
   
   (b) If $N \leq G$ and $N \leq G_{\mathfrak{F}}$, then $(G_{\mathfrak{F}}/N)_{E\mathfrak{F}} = Z(G_{\mathfrak{F}}/N)$.

   (c) Let $n$ be the least number such that there is a simple non-$\mathfrak{F}$-section in $S_{n+1}$ and $T = G_{\mathfrak{F}} \cap Z(G, R, \mathfrak{F}, n)$. Then $G_{\mathfrak{F}}/T \leq Soc(G/T)$ and $N/T \not\in \mathfrak{F}$, $r(N/T, G) \leq n$ and $gr(N/T, G) \in R(N/T)$ for every minimal normal subgroup $N/T$ of $G/T$ from $G_{\mathfrak{F}}/T$.

Recall [11, p. 13] that a group is called *semisimple* provided it is either identity or the direct product of some simple non-abelian groups.

**Corollary 2.1 ([11 X, 13.6]).** A group $G$ is quasinilpotent if and only if $G/Z_{\infty}(G)$ is semisimple.
Corollary 2.2 ([19] Theorem 2.8]). Let $\mathfrak{F}$ be a normally hereditary saturated formation containing all nilpotent groups. A group $G$ is a quasi-$\mathfrak{F}$-group if and only if $G/Z_\mathfrak{F}(G)$ is semisimple.

Corollary 2.3 ([22] Theorem A]). Let $\mathfrak{H} \subseteq \mathfrak{F}$ be a saturated formation of soluble groups. Then $G \in \mathfrak{F}_{\text{sa}}$ if and only if $G^\mathfrak{F} = G^\mathfrak{H}$, $Z(G^\mathfrak{F}) \leq Z_\mathfrak{F}(G)$ and $G^\mathfrak{F}/Z(G^\mathfrak{F})$ is a direct product of $G$-invariant simple non-abelian groups.

Corollary 2.4 ([18] Proposition 2.4]). A group $G$ is $c$-supersoluble if and only if there is a perfect normal subgroup $D$ such that $G/D$ is supersoluble, $D/Z(G)$ is a direct product of $G$-invariant simple groups, and $Z(D)$ is supersolubly embedded in $G$.

Corollary 2.5 ([18] Theorem B]). A group $G$ is widely $c$-supersoluble if and only if $G_{\text{rel}} = G^\mathfrak{F}$, $Z(G_{\text{rel}}) \leq Z_\mathfrak{F}(G)$ and $G_{\text{rel}}/Z(G_{\text{rel}})$ is a direct product of $G$-invariant simple non-abelian groups.

Proof of Theorem 2. (1) $\Rightarrow$ (2) Let $G \in \mathfrak{F}(\mathfrak{R})$, $Z = Z(G, \mathfrak{R}, \mathfrak{F}, 4)$ and $K/Z = \mathfrak{K} = \text{Soc}(G/Z)$. From the definition of $\mathfrak{F}(\mathfrak{R})$ it follows that $\mathfrak{G}/\mathfrak{K}$ does not have minimal normal $\mathfrak{F}$-subgroups. Note that $r(K, \mathfrak{G}) \leq 4$ for every minimal normal subgroup $K_i$ of $\mathfrak{G}$ ($i = 1, \ldots, n$) by the definition of $Z(G, \mathfrak{R}, \mathfrak{F}, 4)$. Note that $\mathfrak{K}_i = \mathfrak{K}_{i,1} \times \cdots \times \mathfrak{K}_{i,k}$ is the direct product of isomorphic simple groups and $1 \leq k \leq 4$. Hence $\text{Aut}(\mathfrak{K}_i) \simeq \text{Aut}(\mathfrak{K}_{i,1}) \times S_k$ by [31] 1.1.20. Note that $S_k$ is soluble and $\text{Out}(\mathfrak{K}_{i,1})$ is soluble by Schreier conjecture. It means that $\text{Out}(\mathfrak{K}_i)$ is soluble.

Since $\mathfrak{K} \subseteq \mathfrak{F}$, $\text{Out}(\mathfrak{G}) \simeq 1$. Hence $\mathfrak{K} = \hat{\text{F}}(\mathfrak{G})$. Recall that $\mathfrak{K} = \mathfrak{K}_1 \times \cdots \times \mathfrak{K}_n$. Note that every element $xZ$ induces an automorphism $\alpha_{x,i}$ on $\mathfrak{K}_i$ for $i = 1, \ldots, n$. Let 

$$\varphi : xZ \to (\alpha_{x,1}, \ldots, \alpha_{x,n})$$

It is clear that $\varphi(xZ)\varphi(yZ) = \varphi(xyZ)$. Also note that if $\varphi(xZ) = \varphi(yZ)$, then $y^{-1}xZ$ acts trivially on every $\mathfrak{K}_i$. According to [30] §7.11 $C_G(\hat{\text{F}}(\mathfrak{G})) \subseteq \hat{\text{F}}(\mathfrak{G})$. So

$$y^{-1}xZ \in \cap_{i=1}^n C_{\hat{\text{F}}(\mathfrak{G})}(\mathfrak{K}_i) = C_{\hat{\text{F}}(\mathfrak{G})}(\mathfrak{K}) = C_{\hat{\text{F}}(\mathfrak{G})} \subseteq \hat{\text{F}}(\mathfrak{G}) = \mathfrak{K}.$$

Hence $y^{-1}xZ = 1Z$. Now $yZ = xZ$ and $\varphi$ is injective. Hence $\varphi$ is the monomorphism from $\mathfrak{G}$ to $\text{Out}(\mathfrak{K}_1) \times \cdots \times \text{Out}(\mathfrak{K}_n)$. Note that $\varphi(\mathfrak{K}) = \text{Inn}(\mathfrak{K}_1) \times \cdots \times \text{Inn}(\mathfrak{K}_n)$. It is straightforward to check that

$$(\text{Out}(\mathfrak{K}_1) \times \cdots \times \text{Out}(\mathfrak{K}_n))/\text{Inn}(\mathfrak{K}_1) \times \cdots \times \text{Inn}(\mathfrak{K}_n)) \simeq \text{Out}(\mathfrak{K}_1) \times \cdots \times \text{Out}(\mathfrak{K}_n)$$

Now $G/K \simeq \mathfrak{G}/\mathfrak{K} \simeq \varphi(\mathfrak{G})/\varphi(\mathfrak{K})$ can be viewed as subgroup of $\text{Out}(\mathfrak{K}_1) \times \cdots \times \text{Out}(\mathfrak{K}_n) \in \mathfrak{F}$. Hence every chief factor of $G$ above $K$ is soluble and, hence, $\mathfrak{F}$-central in $G$.

Theorem 2. (2), (3) $\Rightarrow$ (1) From (2) or (c) of (3) it follows that a group $G$ has a chief series such that $\mathfrak{H} \not\subseteq \mathfrak{F}$ and $gr(\mathfrak{H}, G) \in \mathfrak{R}(\mathfrak{H})$ for every its $\mathfrak{F}$-eccentric chief factor $\mathfrak{H}$. By Jordan-Hölder theorem and Lemma [1] it follows that every chief series of $G$ has this property. Thus $G \in \mathfrak{F}$.

(1) $\Rightarrow$ (3) Assume now that $G \in \mathfrak{F}(\mathfrak{R})$. (a) Note that every chief factor of $G$ above $G^E\mathfrak{F}$ is an $\mathfrak{F}$-group and hence $\mathfrak{F}$-central in $G$ by the definition of $\mathfrak{F}(\mathfrak{R})$. So $Z_\mathfrak{F}(G/G^E\mathfrak{F}) = G/G^E\mathfrak{F}$. Therefore $G/G^E\mathfrak{F} \in \mathfrak{F}$ and $G^E\mathfrak{F} \leq G^E\mathfrak{F}$. From $\mathfrak{F} \subseteq E\mathfrak{F}$ it follows that $G^E\mathfrak{F} \leq G^E\mathfrak{F}$. Thus $G^E\mathfrak{F} = G^E\mathfrak{F}$.

(b) Note that $H_{E\mathfrak{F}} \leq Z_\mathfrak{F}(H)$ for every $\mathfrak{F}(\mathfrak{R})$-group $H$. By [32] Corollary 2.3.1, $G^E\mathfrak{F} \leq C_G(Z_\mathfrak{F}(G))$. Hence $G^E\mathfrak{F} \cap Z_\mathfrak{F}(G) = Z(G^E\mathfrak{F})$. So if $N \leq G$ and $N \leq G^E\mathfrak{F}$, then $(G^E\mathfrak{F}/N)_{E\mathfrak{F}} = Z(G^E\mathfrak{F}/N)$.

(c) Let $n$ be the least number such that there is a simple non-$\mathfrak{F}$-group in $S_{n+1}$ and $T = G^E\mathfrak{F} \cap Z(G, \mathfrak{R}, \mathfrak{F}, n)$. Let $N/T = N$ be a minimal normal subgroup of $\mathfrak{G}/G/T$ that lies in $\mathfrak{G}/G/T$. From the definition of $Z(G, \mathfrak{R}, \mathfrak{F}, n)$ it follows that $N/T \not\subseteq \mathfrak{F}$ and $r(N, G) \leq n$. Note that $\mathfrak{G}/G \simeq 1$ by $\mathfrak{R} \subseteq \mathfrak{F}$. Hence $\text{Inn}(\mathfrak{G}) \simeq 1$. From Lemma [3] it follows that $\text{Soc}(\mathfrak{G}) \cap \mathfrak{G} = \text{Soc}(\mathfrak{G}) = \hat{\mathfrak{G}}(\mathfrak{G})$. Note that $N = N_1 \times \cdots \times N_k$ is a direct product of isomorphic simple non-abelian groups. Recall that $k \leq n$.

By [31] 1.1.20 $\text{Aut}(N) \simeq \text{Aut}(\overline{N}) \simeq S_k$. Now every subgroup of $\text{Aut}(N)/N$ belongs $E\mathfrak{F}$ by the definition of $n$, Schreier conjecture and $\mathfrak{F} \subseteq E\mathfrak{F}$. The same arguments as in (1) $\Rightarrow$ (2) shows that $G^E\mathfrak{F}/\hat{\mathfrak{G}}(\mathfrak{G}) \in E\mathfrak{F}$. From $G^E\mathfrak{F} = G^E\mathfrak{F}$ it follows that $G^E\mathfrak{F}/\hat{\mathfrak{G}}(\mathfrak{G}) = G^E\mathfrak{F}/(\text{Soc}(\mathfrak{G}) \cap \mathfrak{G}) \simeq 1$.  □
Proof of Corollaries 2.1, 2.2, 2.3, 2.4 and 2.5 Note that Corollary 2.1 directly follows from Corollary 2.2.

Let $\mathcal{R}(H) \equiv (\emptyset, \{1\})$ and $G \in \mathfrak{F}^*$. Then $\mathfrak{F}(\mathcal{R}) = \mathfrak{F}^*$ and $Z = Z(G, \mathcal{R}, \mathfrak{F}, 4) = Z_{\mathfrak{F}}(G)$. According to (1) of the proof of Theorem 2, $(G/Z)/\text{Soc}(G/Z)$ is isomorphic to a subgroup of the outer automorphisms group of $\text{Soc}(G/Z)$ induced by $G$. Note that in this case $G$ induces inner automorphism groups on every minimal normal subgroup of $G/Z$. It means that $G/Z \simeq 1$. Thus Corollary 2.2 is proved.

Let $\mathfrak{F}$ be a hereditary saturated formation, $\mathcal{R} \subset \mathfrak{F} \subset \mathfrak{S}$ and $\mathcal{R}(H) \equiv (\{1\}, \emptyset)$. Then $\mathfrak{F}(\mathcal{R}) = \mathfrak{F}_{\text{cn}}$. Note that $E\mathfrak{F} = \mathfrak{S}$. Let $n$ be the least number such that there is a simple non-$\mathfrak{F}$-section in $S_{n+1}$. It is clear that $n = 4$. So $Z(G, \mathcal{R}, \mathfrak{F}, n) = Z_{\mathfrak{F}}(G)$. Now $G^{\mathfrak{S}} \cap Z(G, \mathcal{R}, \mathfrak{F}, n) = G^{\mathfrak{S}} \cap Z_{\mathfrak{F}}(G) \leq (G^{\mathfrak{S}})_{E\mathfrak{F}}$. Thus Corollary 2.3 directly follows from (3) of Theorem 2.

Note that the class of all widely $c$-supersoluble groups $\mathcal{U}_{\text{cw}} = (w\mathcal{U})_{\text{ca}}$. So Corollary 2.5 directly follows from Corollary 2.3.

Recall that a normal subgroup $H$ of a group $G$ is supersoluble embedded iff $H \leq Z_n(G)$; $E\mathcal{U} = \mathfrak{S}$ and $(G^{\mathfrak{g}}) = G^{\mathfrak{S}}$. Now Corollary 2.4 directly follows from Corollary 2.3.

1.3 On one question of L. A. Shemetkov

The main result of this section that makes the contribution to the solution of Question 1 is:

**Theorem 3.** Let $\mathfrak{F}$ be a hereditary saturated formation containing all nilpotent groups, $m$ be a natural number with $\mathfrak{S}_{\{\mathcal{R} \mid \mathcal{R} \leq \mathfrak{F}\}} \subset \mathfrak{F}$; $\mathcal{R}$ be a very good generalized rank function such that $\mathcal{R}(N) \subset [0, m]$ for any simple group $N$. Then the following statements are equivalent:

1. $Z_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{F}}(G)$ holds for every group $G$ and
   $$\bigcup_{n=1}^{m}(\text{Out}(G) \cap S_n \mid G \not\in \mathfrak{F} \text{ is a simple group and } n \in A_{\mathcal{R}}(G)) \subset \mathfrak{F}.$$

2. $Z_{\mathfrak{F}(\mathcal{R})}(G) = \text{Int}_{\mathfrak{F}(\mathcal{R})}(G)$ holds for every group $G$.

**Corollary 3.1** ([27 Theorem 1]). Let $\mathfrak{F}$ be a hereditary saturated formation containing all nilpotent groups. Then $\text{Int}_{\mathfrak{F}}(G) = Z_{\mathfrak{F}}(G)$ holds for every group $G$ if and only if $\text{Int}_{\mathfrak{F}^*}(G) = Z_{\mathfrak{F}^*}(G)$ holds for every group $G$.

**Corollary 3.2.** Let $\mathcal{R}$ be a very good generalized rank function. Then $Z_{\mathcal{R}(\mathcal{R})}(G) = \text{Int}_{\mathcal{R}(\mathcal{R})}(G)$ holds for every group $G$ if and only if for any simple non-abelian group $N$ holds:

1. $\mathcal{R}(N) \subset [0, 2]$;
2. if $1 \in A_{\mathcal{R}}(N)$, then $\text{Out}(N)$ is nilpotent;
3. if $2 \in A_{\mathcal{R}}(N)$, then $\text{Out}(N)$ is a 2-group.

**Example 2.** Let $\mathfrak{F}_1$ (resp. $\mathfrak{F}_2$) be a class of groups whose abelian chief factors are central and non-abelian chief factors are arbitrary (resp. are directs products of at most 2 alternating groups). Then $Z_{\mathfrak{F}_1}(G) = \text{Int}_{\mathfrak{F}_1}(G)$ and $Z_{\mathfrak{F}_2}(G) = \text{Int}_{\mathfrak{F}_2}(G)$ hold for every group $G$ and there exist groups $G_1$ and $G_2$ with $Z_{\mathcal{R}_{\text{cn}}}(G_1) \neq \text{Int}_{\mathcal{R}_{\text{cn}}}(G_1)$ and $Z_{\mathfrak{F}_1}(G_2) \neq \text{Int}_{\mathfrak{F}_1}(G_2)$.

It is important to mention that if $Z_{\mathcal{R}(\mathcal{R})}(G) = \text{Int}_{\mathcal{R}(\mathcal{R})}(G)$ holds for every group $G$, then $\mathcal{R}$ is bounded.

**Theorem 4.** Let $\mathfrak{F} \neq \mathfrak{S}$ be a hereditary saturated formation containing all nilpotent groups and $\mathcal{R}$ be a very good generalized rank function.

1. Assume that $Z_{\mathfrak{F}(\mathcal{R})}(G) = \text{Int}_{\mathfrak{F}(\mathcal{R})}(G)$ holds for every group $G$. Let
   $$C_1 = \min_{G \in \mathcal{M}(\mathfrak{F})} \max_{\text{F}(G) = \text{F}(G)} |M| - 1.$$

Then $\mathcal{R}(S) \subset [0, C_1]$ for every simple group $S \not\in \mathfrak{F}$.
(2) Let
\[ C_2 = \max \{ m \in \mathbb{N} \mid \mathcal{G}_{\{q \in \mathbb{P} \mid q \leq m\}} \subseteq \mathcal{B} \} . \]
If \( \mathcal{R}(S) \subseteq [0, C_2] \) for every simple group \( S \notin \mathcal{B} \), then \( gr(\overline{H}, G) \in \mathcal{R}(\overline{H}) \) for every \( G \)-composition factor \( \overline{H} \notin \mathcal{B} \) below \( \text{Int}_{\mathcal{B}(\mathcal{R})}(G) \).

Recall [31, 3.4.5] that every solubly saturated formation \( \mathcal{B} \) contains the greatest saturated subformation \( \mathcal{B}_1 \) with respect to set inclusion.

**Theorem 5** ([31, 3.4.5]). Let \( F \) be the canonical composition definition of a non-empty solubly saturated formation \( \mathcal{B} \). Then \( F \) is a local definition of \( \mathcal{B}_1 \), where \( F(p) = F(p) \) for all \( p \in \mathbb{P} \).

The following result directly follows from Theorems [1] and [5].

**Proposition 1.** Let \( \mathcal{B} \) be a local formation containing all nilpotent groups and \( R \) be a generalized rank function. Then \( \mathcal{B}(\mathcal{R}) \subseteq \mathcal{B} \).

In the view of Theorem [5], A.F. Vasil’ev asked if it is possible to reduce Question [1] for solubly saturated formations to the case of saturated formations. Recall that \( D_0 \mathcal{X} \) is the class of groups which are the direct products \( \mathcal{X} \)-groups. Partial answer on A.F. Vasil’ev’s question is given in Theorem 6.

**Theorem 6.** Let \( F \) be the canonical composition definition of a non-empty solubly saturated formation \( \mathcal{B} \). Assume that \( F(p) \subseteq \mathcal{B}_1 \) for all \( p \in \mathbb{P} \) and \( \mathcal{B}_1 \) is hereditary.

1. Assume that \( \text{Int}_{\mathcal{B}_1}(G) = Z_{\mathcal{B}_1}(G) \) holds for every group \( G \). Let
\[ \mathcal{B}_1 = \{ S \mid \text{S is a simple group} \} \]
for every \( \mathcal{B}_1 \)-central chief \( D_0(S) \)-factor is \( \mathcal{B}_1 \)-central.

Then every chief \( D_0 \mathcal{B}_1 \)-factor of \( G \) below \( \text{Int}_{\mathcal{B}_1}(G) \) is \( \mathcal{B}_1 \)-central in \( G \).

2. ([27]) Let \( \mathcal{B}_1 \) be the canonical composition definition of a non-empty \( \mathcal{B} \)-maximal subgroup of \( G \). Assume that \( \text{Int}_{\mathcal{B}_1}(G) = Z_{\mathcal{B}_1}(G) \) holds for every group \( G \), then \( \text{Int}_{\mathcal{B}_1}(G) = Z_{\mathcal{B}_1}(G) \) holds for every group \( G \).

**Proof of Theorem 6.** From \( F(p) = \mathfrak{N}_p F(p) \subseteq \mathcal{B}_1 \) and Theorem [5], it follows that if we restrict \( F \) to \( \mathbb{P} \), then we obtain the canonical local definition of \( \mathcal{B}_1 \).

1. Let \( \overline{H} = H/K \) be a chief \( D_0 \mathcal{B}_1 \)-factor of \( G \) below \( \text{Int}_{\mathcal{B}_1}(G) \).

(a) If \( \overline{H} \) is abelian, then \( MC_G(\overline{H})/C_G(\overline{H}) \in F(p) \) for every \( \mathcal{B}_1 \)-maximal subgroup \( M \) of \( G \).

(b) If \( \overline{H} \) is non-abelian, then \( MC_G(\overline{H})/C_G(\overline{H}) \in F(p) \) for every \( \mathcal{B}_1 \)-maximal subgroup \( M \) of \( G \).

Let \( \mathcal{B}_1 \) be a maximal subgroup of \( G \). By Lemma [3], \( \overline{H} = \overline{H}_1 \times \cdots \times \overline{H}_n \) is a direct product of minimal normal subgroups \( \overline{H}_i \) of \( M = \overline{H}/K \). Hence \( \overline{H}_i \) is \( \mathcal{B}_1 \)-central in \( \overline{H} \) for all \( i = 1, \ldots, n \). Note that \( H_i/H_{i-1} \) is an \( \mathcal{B}_1 \)-central chief factor of \( M \) for all \( i = 1, \ldots, n \). So \( M/C_M(H_i/H_{i-1}) \in F(p) \) for all \( i = 1, \ldots, n \). Therefore \( M/C_M(\overline{H}_i) \in \mathfrak{N}_p F(p) = F(p) \) by [14, Lemma 1]. Now
\[ MC_G(\overline{H}_i)/C_G(\overline{H}_i) \simeq M/C_M(\overline{H}_i) \in F(p) \]
for every \( \mathcal{B}_1 \)-maximal subgroup \( M \) of \( G \).

(c) Assume that \( \mathcal{B}_1 \)-maximal subgroup \( M \) of \( G \).

(a) If \( \overline{H} \) is abelian, then \( MC_G(\overline{H})/C_G(\overline{H}) \in F(p) \) for every \( \mathcal{B}_1 \)-maximal subgroup \( M \) of \( G \).

If \( \overline{H} \) is non-abelian, then it is a direct product of isomorphic non-abelian simple groups.

Let \( M \) be an \( \mathcal{B}_1 \)-maximal subgroup of \( G \). By Lemma [3], \( \overline{H} = \overline{H}_1 \times \cdots \times \overline{H}_n \) is a direct product of minimal normal subgroups \( \overline{H}_i \) of \( M = \overline{H}/K \). Now \( \overline{H}_i \) is \( \mathcal{B}_1 \)-central in \( \overline{H} \) for all \( i = 1, \ldots, n \). Hence \( \overline{H}_i \) is \( \mathcal{B}_1 \)-central in \( \overline{H} \) for all \( i = 1, \ldots, n \) by the definition of \( \mathcal{B}_1 \). Therefore \( M/C_M(\overline{H}_i) \in F(p) \) for all \( p \in \pi(\overline{H}_i) \) by Lemma [2]. Note that \( C_M(\overline{H}_i) = \cap_{i=1}^n C_M(\overline{H}_i) \). Since \( F(p) \) is a formation,
\[ M/\cap_{i=1}^n C_M(\overline{H}_i) = M/C_M(\overline{H}_i) \in F(p) \]
for all \( p \in \pi(\overline{H}) \). It means that \( MC_G(\overline{H})/C_G(\overline{H}) \simeq M/C_M(\overline{H}_i) \in F(p) \) for every \( p \in \pi(\overline{H}_i) \).
Let $Q/C_G(H)$ be an $\mathfrak{S}$-maximal subgroup of $G/C_G(H)$. Then there exists an $\mathfrak{S}$-maximal subgroup $N$ of $G$ with $NC_G(H)/C_G(H) = Q/C_G(H)$ by [1, 5.7]. From $\mathfrak{S} \subseteq \mathfrak{B}$ it follows that there exists an $\mathfrak{B}$-maximal subgroup $L$ of $G$ with $N \leq L$. So

$$Q/C_G(H) \leq LC_G(H)/C_G(H) \in F(p) \text{ for all } p \in \pi(H)$$

by (a) and (b). Since $F(p)$ is hereditary by [1 IV, 3.16], $Q/C_G(H) \in F(p)$. It means that all $\mathfrak{B}$-maximal subgroups of $G/C_G(H)$ are $F(p)$-groups. Hence all $\mathfrak{B}$-subgroups of $G/C_G(H)$ are $F(p)$-groups.

(d) $\overline{H}$ is $\mathfrak{S}$-central in $G$.

Assume now that $\overline{H}$ is not $\mathfrak{S}$-central in $G$. So $G/C_G(H) \not\in F(p)$ for some $p \in \pi(H)$ by Lemma 2. It means that $G/C_G(H)$ contains an $s$-critical for $F(p)$ subgroup $S/C_G(H)$. Since $\mathrm{Int}_\mathfrak{S}(G) = Z_{\mathfrak{S}}(G)$ holds for every group $G$, $S/C_G(H) \in \mathfrak{S}$ by [24 Theorem A]. Therefore $S/C_G(H) \in F(p)$ by (c), a contradiction. Thus $\overline{H}$ is $\mathfrak{S}$-central in $G$.

**Proof of Theorem 4.**

(1) Since $Z_{\mathfrak{S}}(\mathfrak{R}) = \mathrm{Int}_{\mathfrak{S}}(\mathfrak{R})$ holds for every group $G$, we see that $Z_{\mathfrak{S}}(G) = \mathrm{Int}_{\mathfrak{S}}(G)$ holds for every group $G$ by Theorem 4 and Proposition 4.

(a) There is $G \in M(\mathfrak{F})$ with $F(G) = \tilde{F}(G)$.

From $\mathfrak{F} \neq \mathfrak{G}$ it follows that there exist $F(p)$-critical groups for some $p$. Let $N$ be the minimal order group among them. Then $O_p(N) = 1$ and $N$ has the unique minimal normal subgroup. Note that $N \in \mathfrak{F}$ by [24 Theorem A]. There exists a simple $\mathfrak{F}_p$-module $M$ which is faithful for $N$ by [1, 10.3B]. Let $G = M \times N$. Note that $M = F(G) = \tilde{F}(G)$ and $G \not\in \mathfrak{F}$. From $F(p) = \mathfrak{M}_p F(p)$ it follows that $G \in M(\mathfrak{S})$.

(b) If $G \in M(\mathfrak{F})$ with $F(G) = \tilde{F}(G)$, then $G \not\in \mathfrak{F}(\mathfrak{R})$ for any generalized rank function $\mathfrak{R}$.

Since $\mathfrak{F}$ is saturated, we see that $G/\Phi(G) \in M(\mathfrak{F})$. Note that $G/\Phi(G)$ has a unique minimal normal subgroup $N/\Phi(G)$ and $N/\Phi(G)$ is an abelian $\mathfrak{F}$-eccentric chief factor of $G$. From $\mathfrak{R} \subseteq \mathfrak{F}$ and the definition of $\mathfrak{F}(\mathfrak{R})$ it follows that $G \not\in \mathfrak{F}(\mathfrak{R})$ for any generalized rank function $\mathfrak{R}$.

(c) Let $G \in M(\mathfrak{S})$ with $F(G) = \tilde{F}(G)$, $m$ be the greatest number among orders of $\mathfrak{S}$-maximal subgroups of $G$, $S$ be a simple group with $m \in \mathfrak{R}(S)$ and $T = S_{\text{reg}} G$. Then $Z_{\mathfrak{F}(\mathfrak{R})}(T) \neq \mathrm{Int}_{\mathfrak{F}(\mathfrak{R})}(T)$.

Let $N$ be the base of $T$ and $M$ be an $\mathfrak{S}(\mathfrak{R})$-maximal subgroup of $T$. Note that $MN/N \in \mathfrak{F}(\mathfrak{R})$ is isomorphic to some subgroup of $G \not\in \mathfrak{F}(\mathfrak{R})$. Let $K$ be a maximal subgroup of $G$. Then $K \in \mathfrak{S}(\mathfrak{R})$. Let $\mathfrak{S}$ be the property of wreath product it follows that $N$ is the direct product of $|G : K|$ minimal normal subgroups of rank $|K|$ of $NK$ and if some element on $NK$ fixes a composition factor of some of these subgroups, then it induces an inner automorphism on it. It means that $NK \in \mathfrak{F}(\mathfrak{R})$. Hence $N \leq \mathrm{Int}_{\mathfrak{R}(\mathfrak{R})}(T)$. Assume that $N \leq Z_{\mathfrak{F}(\mathfrak{R})}(T)$. So $T/C_T(N) \in \mathfrak{F}(\mathfrak{R})$. Then $G \not\in \mathfrak{F}(\mathfrak{R})$ is isomorphic to a quotient of $T/C_T(N)$, a contradiction. Hence $N$ is an $\mathfrak{F}(\mathfrak{R})$-eccentric chief factor of $T$.

(d) The final step.

From (c) it follows that if

$\min_{G \in M(\mathfrak{S}) \text{ with } F(G) = \tilde{F}(G) \text{ is a maximal subgroup of } G} |M| = \mathfrak{R}(S) \text{ for some simple group } S \not\in \mathfrak{S}$,

then there is a group $T$ with $Z_{\mathfrak{F}(\mathfrak{R})}(T) \neq \mathrm{Int}_{\mathfrak{F}(\mathfrak{R})}(T)$. The contradiction to $Z_{\mathfrak{F}(\mathfrak{R})}(G) = \mathrm{Int}_{\mathfrak{F}(\mathfrak{R})}(G)$ holds for every group $G$.

(2) Let $n = \{q \in \mathbb{P} \mid q \leq C_2 \}$ and $\overline{H} = H/K \not\in \mathfrak{S}$ be a $G$-composition factor below $\mathrm{Int}_{\mathfrak{F}(\mathfrak{R})}(G)$. Then $\overline{H} = H_1 \times \cdots \times H_n$ is the direct product of isomorphic simple groups. Let $T/K = T = \cap_{i=1}^n N_i(H_i)$.

Let $M$ be an $\mathfrak{F}(\mathfrak{R})$-maximal subgroup of $G$. Then $H/K \leq M/K = M \in \mathfrak{F}(\mathfrak{R})$. Since $\overline{H}$ is non-abelian it is the direct product of minimal normal subgroups of $M$ by Lemma 3. Let $N_{i,j} = N_{i,j} \times \cdots \times N_{i,k}$ be one of them and $T_i = \cap_{i=1}^n N_i(H_i)$. Then $M/T_i \in \mathfrak{F}(\mathfrak{R})$. From $k \leq C_2$ it follows that $\overline{M}/T_i \in \mathfrak{S}(\mathfrak{R})$. Since $\mathfrak{S}_n$ is a formation, we see that $\overline{M}/T_i \simeq M/T_i \cap_i T_i \in \mathfrak{S}_n$. 

10
From $\mathfrak{H} \subseteq \mathfrak{F}$ it follows that every element of $G$ lies in some $\mathfrak{F}(\mathcal{R})$-maximal subgroup of $G$. Hence $\overline{G}/T \cong G/T \in \mathfrak{S}_n$ and this group is isomorphic to a transitive group of permutations on $\{\overline{F}_1, \ldots, \overline{F}_n\}$. According to [1, 1.5.7] there is a $\pi$-subgroup $L$ of $G$ with $LT = G$. Note that $L \in \mathfrak{F}$. Hence there is an $\mathfrak{F}(\mathcal{R})$-maximal subgroup $Q$ of $G$ with $L \subseteq Q$. So $LH \subseteq Q$. Now $H/K$ is a minimal normal subgroup of $Q/K$. Therefore $m = r(\overline{G}, Q) \in \mathcal{R}(\overline{G})$.

If $m \in A_{\mathcal{R}}(\overline{H})$, then we are done. Assume that $m \in B_{\mathcal{R}}(\overline{H})$. Now for every $x \in G$, there is a $\mathfrak{F}(\mathcal{R})$-maximal subgroup $P$ of $G$ with $x \in Q$. So $H/K$ is a direct product of minimal normal subgroups of $Q/K$. Since $\mathcal{R}$ is a very good generalized rank function, if $x$ fixes a composition factor of $\overline{H}$, then it induces an inner automorphism on it. Thus $\text{gr}(\overline{H}, G) \in \mathcal{R}(\overline{H})$. $\square$

**Proposition 2.** Let $\mathfrak{F}$ be a hereditary saturated formation containing all nilpotent groups and $\mathcal{R}$ be a very good generalized rank function. Then $Z_{\mathfrak{F}(\mathcal{R})}(G) \leq \text{Int}_{\mathfrak{F}(\mathcal{R})}(G)$.

**Proof.** Let $\mathfrak{F}$ be a hereditary saturated formation with the canonical local definition $F$, $M$ be an $\mathfrak{F}(\mathcal{R})$-maximal subgroup of $G$ and $N = MZ_{\mathfrak{F}(\mathcal{R})}(G)$. Let show that $N \in \mathfrak{F}(\mathcal{R})$. It is sufficient to show that $H/K \notin \mathfrak{F}$ and $\text{gr}(H/K, G) \in \mathcal{R}(H/K)$ for every $\mathfrak{F}$-eccentric chief factor $H/K$ of $N$ below $Z_{\mathfrak{F}(\mathcal{R})}(G)$.

Let $1 = Z_0 \leq Z_1 \leq \cdots \leq Z_n = Z_{\mathfrak{F}(\mathcal{R})}(G)$ be a chief series of $G$ below $Z_{\mathfrak{F}(\mathcal{R})}(G)$. Then we may assume that $Z_{i-1} \leq K \leq H \leq Z_i$ for some $i$ by the Jordan-H"{o}lder theorem. Note that

\[
(Z_i/Z_{i-1}) \rtimes G/C_G(Z_i/Z_{i-1}) \in \mathfrak{F}(\mathcal{R}).
\]

Hence if $Z_i/Z_{i-1} \notin \mathfrak{F}$, then $\text{gr}(Z_i/Z_{i-1}, G) \in \mathcal{R}(Z_i/Z_{i-1})$ and $Z_i/Z_{i-1}$ is non-abelian. Note that every composition factor of $H/K$ is a composition factor of $Z_i/Z_{i-1}$ and $\mathcal{R}$ is a very good generalized rank function. Hence $\text{gr}(H/K, N) \in \mathcal{R}(H/K)$.

If $Z_i/Z_{i-1} \in \mathfrak{F}$, then it is an $\mathfrak{F}(\mathcal{R})$-central chief factor of $(Z_i/Z_{i-1}) \rtimes G/C_G(Z_i/Z_{i-1})$. In this case $G/C_G(Z_i/Z_{i-1}) \in F(p)$ for all $p \in \pi(Z_i/Z_{i-1})$ by Lemma 2. Since $F(p)$ is hereditary by [I] IV, 3.16

\[
NC_G(Z_i/Z_{i-1})/C_G(Z_i/Z_{i-1}) \cong N/C_N(Z_i/Z_{i-1}) \in F(p)
\]

for all $p \in \pi(Z_i/Z_{i-1})$. Note that $N/C_N(H/K)$ is a quotient group of $N/C_N(Z_i/Z_{i-1})$. Thus $H/K$ is an $\mathfrak{F}(\mathcal{R})$-central chief factor of $N$ by Lemma 2.

Hence $N \in \mathfrak{F}(\mathcal{R})$. So $N = MZ_{\mathfrak{F}(\mathcal{R})}(G) = M$. Therefore $Z_{\mathfrak{F}(\mathcal{R})}(G) \leq M$ for every $\mathfrak{F}(\mathcal{R})$-maximal subgroup $M$ of $G$. $\square$

**Proof of Theorem 3.** (1) $\Rightarrow$ (2). Suppose that $Z_{\mathfrak{F}}(G) = \text{Int}_{\mathfrak{F}}(G)$ holds for every group $G$ and

\[
\bigcup_{n=1}^{\infty} (\text{Out}(G) \wr S_n \mid G \notin \mathfrak{F} \text{ is a simple group and } n \in A_{\mathcal{R}(G)}) \subseteq \mathfrak{F}.
\]

Let show that $Z_{\mathfrak{F}(\mathcal{R})}(G) = Z_{\mathfrak{F}(\mathcal{R})}(G)$ also holds for every group $G$. Let $H = H/K$ be a chief factor of $G$ below $\text{Int}_{\mathfrak{F}(\mathcal{R})}(G)$ and $\overline{G} = G/K$. Note that $G/C_G(H) \cong G/C_G(\overline{H})$.

(a) If $\overline{H} \in \mathfrak{F}$, then it is $\mathfrak{F}(\mathcal{R})$-central in $G$.

Directly follows from Theorem 3 (3) of Definition 1 and $\mathfrak{F} \subseteq \mathfrak{F}(\mathcal{R})$.

(b) If $\overline{H} \notin \mathfrak{F}$, then $\text{gr}(\overline{H}, G) \in \mathcal{R}(\overline{H})$.

Directly follows from (2) of Theorem 4.

(c) If $\overline{H} \notin \mathfrak{F}$ and $n = r(\overline{H}, G) \in A_{\mathcal{R}(\overline{G})}$, then $\overline{H}$ is $\mathfrak{F}(\mathcal{R})$-central in $G$.

Recall that $\overline{G}/C_{\mathfrak{F}(\mathcal{R})}(\overline{H}) \overline{H}$ is isomorphic to a subgroup of $\text{Out}(\overline{H})$ and $\overline{H} = \overline{H}_1 \times \cdots \times \overline{H}_n$ is the direct product of isomorphic simple groups. So $\overline{G}/C_{\mathfrak{F}(\mathcal{R})}(\overline{H}) \overline{H}$ is isomorphic to a subgroup of $\text{Out}(\overline{H}_1) \wr S_n \in \mathfrak{F}$. Since $\mathfrak{F}$ is hereditary, $\overline{G}/C_{\mathfrak{F}(\mathcal{R})}(\overline{H}) \overline{H} \in \mathfrak{F}$. Now $G/C_G(\overline{H}) \in \mathfrak{F}(\mathcal{R})$ by Jordan-H"{o}lder theorem and the definition of $\mathfrak{F}(\mathcal{R})$. So $\overline{H}$ is $\mathfrak{F}(\mathcal{R})$-central in $G$ by Lemma 2.

(d) If $\overline{H} \notin \mathfrak{F}$ and $n = r(\overline{H}, G) \in B_{\mathcal{R}(\overline{G})}$, then $\overline{H}$ is $\mathfrak{F}(\mathcal{R})$-central in $G$.

Now $\overline{H} = \overline{H}_1 \times \cdots \times \overline{H}_n$ is the direct product of isomorphic simple groups and every element of $G$ that fixes some $\overline{H}_i$ induces an inner automorphism on it. Hence $\overline{H}C_G(\overline{H}) = \bigcap_{i=1}^{n} N_{G}(\overline{H}_i)$. According to [III] 1.1.40(6) $\overline{G}/C_{\mathfrak{F}(\mathcal{R})}(\overline{H}) \overline{H}$ is isomorphic to a subgroup of $S_n$. From $\mathfrak{S}_n \subseteq \mathfrak{S}_n \subseteq$
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