On the Holonomy of the Coulomb Connection over 3-manifolds with Boundary

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Abstract

Narasimhan and Ramadas showed in [16] that the Gribov ambiguity was maximal for the product $SU(2)$ bundle over $S^3$. Specifically they showed that the holonomy group of the Coulomb connection is dense in the gauge group. Instead of base manifold $S^3$, we consider here a base manifold with a boundary. In this with-boundary case we must include boundary conditions on the connection forms. We will use the so-called conductor boundary conditions on connections. With these boundary conditions, we will first show that the space of connections is a $C^\infty$ Hilbert principal bundle with respect to the associated conductor gauge group. We will consider the holonomy of the Coulomb connection for this bundle. If the base manifold is an open subset of $\mathbb{R}^3$ and we use the product principal bundle, we will show that the holonomy group is again a dense subset of the gauge group.
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Chapter 1

Introduction

1.1 Some Background to Yang-Mills Theory

This thesis is concerned with classical Yang-Mills theory. Our first task is to understand the very basics of this theory. Physically, Yang-Mills theory models the strong nuclear force in the same way that Maxwell’s equations model electromagnetic force. To understand the mathematics, we will need to impose a lot of structure. In this introduction, however, we will try to minimize the mathematical structure to get to the main idea, and refer the reader to Section 2 of [2] for the rigorous definitions. Later we will carefully lay out the mathematics that we need.

Two basic concepts we will need are the Yang-Mills equation and the group of gauge transformations.

1.1.1 The Yang-Mills Equation

Consider the following situation: Let $M$ be a compact oriented 3-dimensional manifold, and let $P \to M$ be a principal bundle over $M$ with compact connected structure group $K$. Furthermore, assume that $K$ acts faithfully on a finite dimensional real (or complex) inner product space by isometries. Thus we may view $K$ as a compact matrix subgroup of $U(V)$, and therefore the structure of $P$ induces a vector bundle $E := P \times_K V \to M$ where $V = \mathbb{R}^n$ or $\mathbb{C}^n$ for some $n$.

Given an Ehresmann connection $\omega_A$ on the principal bundle $P$, we obtain a Koszul connection $\nabla_A$ on the bundle $E$ whose curvature form $R_A$ is a vector-valued two-form on $M$. The functional

$$\mathcal{YM}(\nabla_A) := \int_M |R_A|^2 dVol$$

(1.1)

is well-defined for a certain norm. Any connection that is a local minimum of the functional $\mathcal{YM}$ is called a Yang-Mills connection. One can show that $\nabla_A$ is a Yang-Mills connection if and only if $d^*_A R_A = 0$. The equation

$$d^*_A R_A = 0$$
is called the Yang-Mills equation. The Yang-Mills equation has a similar structure to Maxwell’s equations. The Bianchi identity is that $d_A R_A = 0$, so the Yang-Mills “equations” are

$$d_A R_A = 0, \quad d^*_A R_A = 0.$$ 

One can formulate Maxwell’s equations in such a way that solving Maxwell’s equations in the absence of a current is equivalent to finding a real-valued 2-form $\eta$ on Minkowski space $\mathbb{R}^4$ such that

$$d\eta = 0, \quad d^*\eta = 0.$$ 

(For more on this construction, see the appendix to Chapter 2 and 10.2.8 in [3]). What makes solving the Yang-Mills equations difficult is the fact that the exterior derivative $d_A$ depends on the connection $\nabla_A$, and when written in local coordinates, this introduces a non-linear term of degree 3. In Maxwell’s equations, the exterior derivative $d$ is independent of the form $\eta$, and the equations are linear.

### 1.1.2 The Gauge Group

There is a group $G$, called the gauge group, that acts on the set of connections and preserves the functional $\mathcal{YM}$. So if $\nabla_A$ is a Yang-Mills connection and $g \in G$, then $\nabla_A \cdot g$ is also a Yang-Mills connection.

The gauge group is fairly complicated to define in the general case, so here we will consider it locally. Consider the principal bundle $P = \bar{O} \times K \to \bar{O}$, where $O$ is an open subset of $\mathbb{R}^3$ with smooth boundary, and $K$ is a compact matrix group. Then we have the associated vector bundle $E = \bar{O} \times V \to \bar{O}$, where $V = \mathbb{R}^n$ or $\mathbb{C}^n$, depending on whether the matrix group is real or complex. A gauge transformation $g$ is a mapping $g : \bar{O} \to K$ and the set $G$ of gauge transformations is the gauge group. Given a section $\sigma$ of $E$ and $g \in G$, we get a new section $g \cdot \sigma$ by

$$(g \cdot \sigma)(x) = g(x)\sigma(x).$$ 

Given this left action on sections, we can get a right action on a connection $\nabla^A$ on the bundle $E$. Indeed, we define the connection $\nabla^A \cdot g$ as

$$(\nabla^A \cdot g)_X \sigma = g^{-1} \cdot \nabla^A_X (g \cdot \sigma)$$

for any vector $X \in T(O)$. It turns out that this action preserves connections that originally came from Ehresmann connections on the bundle $P$. So if $C$ is the space of connections on $P$, $G$ acts on the right of $C$. Also, one can now check that if the connection $\nabla_A$ satisfies the Yang-Mills equation, then so does $\nabla_A \cdot g$.

Thus, a natural object to consider is the quotient space $C/G$, where $C$ is the set of connections. With some required modifications, the mapping $C \to C/G$ is an infinite dimensional principal bundle when $M$ is a compact 3-manifold without boundary. This bundle has been studied extensively in this case. This thesis focuses on understanding this bundle when the underlying manifold $M$ has boundary.
1.2 Conductor Boundary Conditions

We will be concerned with boundary conditions on $p$-forms that have been dubbed "conductor boundary conditions" by Gross in [8]. We say a form $\omega$ satisfies conductor boundary conditions if

$$i^*(\omega) = 0$$

where $i : \partial M \to M$ is the inclusion map. In other words, $\omega(X_1 \wedge \ldots \wedge X_p) = 0$ if $X_1, \ldots, X_p$ are all tangent to the boundary. This is half of the "relative" boundary conditions given by Ray and Singer in [19], and the boundary conditions of the "Dirichlet problem" of Marini in [13].

We can extend the notion of conductor boundary conditions to connections. Given an Ehresmann connection $\omega_A$ on $P$, we consider the induced Koszul connection $\nabla_A$ on the vector bundle $E$. Given a fixed connection $A_0$ on $P$, we say that a connection $\nabla_A$ satisfies conductor boundary conditions with respect to $\nabla_{A_0}$ if the 1-form $\nabla_A - \nabla_{A_0}$ satisfies conductor boundary conditions. Such a $\nabla_A$ equals $\nabla_{A_0}$ on the boundary in tangential directions, giving us a Dirichlet-like boundary condition (hence the terminology found in [13]). If we restrict our view to connections satisfying conductor boundary conditions with respect to a fixed $A_0$, we must change the gauge group so that it preserves the boundary conditions. A gauge transformation $g \in G$ is a section of certain bundle over $M$ with fibers diffeomorphic to $K$. The conductor gauge group $G_{\text{con}}$ consists of those gauge transformations $g$ such that $g|_{\partial M} \equiv e$, where $e$ is the identity of $K$.

Note that the definition of $G_{\text{con}}$ does not depend on the fixed connection $\nabla_{A_0}$.

1.3 The Gribov Ambiguity and Holonomy of the Coulomb Connection

The Gribov ambiguity comes up in the following setting, whose description is taken from [20]. Physicists would like to compute a certain integral over $C$ of the form

$$\frac{\int_{C} e^{-|R_A|^2} \{D \} D A}{\int_{C} e^{-|R_A|^2} D A}.$$ 

This integral comes from the Feynman approach to quantum field theory. The problem with this integral is that the integrand in the numerator is constant on $G$-orbits while the orbits are expected to have infinite measure. Morally, the integral should be taken over $C/G$ and not $C$. So physicists do the following: instead of integrating over all of $C$, take a continuous section $\sigma : C/G \to C$ and integrate over $\sigma(C/G)$, with an appropriate Jacobian weight factor from the change of variables. They had a specific section in mind: The infinite dimensional principal bundle $C \to C/G$ has its own connection called the Coulomb connection with its horizontal subspaces given by

$$H_A = \{ \tau : \tau \text{ is a } \mathfrak{t}\text{-valued 1-form, } d_A^*\tau = 0 \}.$$ 

We can define $S_A = \{ A + \tau : \tau \in H_A \} \subseteq C$. The physicists conjectured the following:
Conjecture 1. Fix a connection $\nabla_A \in \mathcal{C}$. Then for every $\nabla_{A'} \in \mathcal{C}$, there exists a unique $g \in \mathcal{G}$ such that $\nabla_{A'} \cdot g \in S_A$.

We will see that locally this is true, meaning that given a small enough open set $O \subseteq \mathcal{C}$ about $\nabla_A$, for every $\nabla_{A'} \in O$ there exists a unique $g \in \mathcal{G}$ such that $\nabla_{A'} \in S_A$. However, Gribov showed in [7].

Theorem 2 (The Gribov Ambiguity). Suppose $M = S^4$, $P = S^4 \times SU(2) \to S^4$, and the $\nabla_A$ is the flat connection (i.e. the Ehresmann connection whose horizontal subspaces are the tangent space to $S^4$ in $P$). In particular, there exists a connection $\nabla_{A'} \neq \nabla_A$ in the $\mathcal{G}$-orbit of $\nabla_A$ such that $\nabla_{A'} \in S_A$.

The “ambiguity” here is that given an $\nabla_A$, there might be multiple connections $\nabla_{A'}$ and $\nabla_{A''}$ that are gauge equivalent to $\nabla_A$ and are both in $S_A$. Hence, $\nabla_A$’s “representative” in $S_A$ is ambiguous.

Note that if Conjecture 1 were true, then the bundle $\mathcal{C}$ would be isomorphic to $\mathcal{G} \times S_A$, and thus be a trivial bundle. So, more generally, one can ask if the bundle $\mathcal{C}$ is trivial, or equivalently ask if it allows any sections. Both [16] and [20] show that

Theorem 3 (Generalized Gribov Ambiguity). If $M = S^3$ or $S^4$, and $P = M \times SU(2) \to M$, then no continuous section $\sigma : \mathcal{C} / \mathcal{G} \to \mathcal{C}$ exists.\textsuperscript{2}

The “ambiguity” here is that you cannot continuously choose a representative of each equivalence class of $\mathcal{C} / \mathcal{G}$. Hence, the physicists’ idea of using a continuous section is mathematically impossible.

Narasimhan and Ramadas took the Gribov ambiguity a bit further for $M = S^3$ in the following sense. Given two connections $\nabla_A, \nabla_B \in \mathcal{C}$, they ask: how many points in the $\mathcal{G}$-orbit of $\nabla_B$ can be connected to $\nabla_A$ via horizontal paths with respect to the Coulomb connection? The more points in the orbit can be connected by horizontal paths, the more “ambiguous” the Coulomb connection is. The number of points that can be connected is the same as the number of elements in the holonomy group at $\nabla_A$ of the Coulomb connection, and thus this holonomy group becomes the main object of study. Narasimhan and Ramadas show in [16] that if a certain metric is put on $S^3$ and the principal bundle considered is the product bundle $S^3 \times SU(2) \to S^3$, then the holonomy group is dense in the connected component of the identity of $\mathcal{G}$. So they say that the ambiguity is maximal. This also has some ramifications to physicists as is described in the introduction to [16].

In this thesis, we address the question whether this maximal ambiguity holds if the base manifold is compact and with boundary (unlike $S^3$) and $K$ is any compact semisimple matrix group.

\textsuperscript{1}This is Singer’s formulation of Gribov’s result as found in [20].
\textsuperscript{2}Only [20] outlines the proof of the $M = S^4$ case.
1.4 Summary of Results

The aim of this thesis is to investigate the Gribov ambiguity when the base manifold $M$ is a compact 3-manifold with boundary, and the connections under consideration satisfy conductor boundary conditions with respect to a fixed connection $\nabla_A_0$. We first need to prove that the corresponding bundle $C \rightarrow C/G$ is a $C^\infty$ principal bundle. To this end, we will be using connection forms and gauge transformations of Sobolev classes $k$ and $k+1$ and denote them $C^k_{\text{con},A_0}$ and $G^{k+1}_{\text{con}}$, respectively. Using standard techniques employed in [1], [15] and [18], we will prove in Chapter 2 that

**Proposition 4.** Suppose $M$ is a compact 3-manifold with boundary, $P \rightarrow M$ is a principal bundle with a compact structure group $K$, and $k > 3/2 + 1$. Then $C^k_{\text{con},A_0}/G^{k+1}_{\text{con}}$ is a $C^\infty$ Hilbert manifold, and $C^k_{\text{con},A_0} \rightarrow C^k_{\text{con},A_0}/G^{k+1}_{\text{con}}$ is a principal bundle.

With this proposition, it now makes sense to consider the holonomy group $H^k_{\text{con},A_0}$ of the Coulomb connection. The infinite dimensional version of the Ambrose Singer theorem (see [12] for the statement of this theorem) tells us that $\text{Lie}(H^k_{\text{con},A_0})$ is generated by the image of the curvature form at certain points of the bundle. Narasimhan and Ramadas use a similar but weaker fact to show that the holonomy group is dense in the connected component of the gauge group; indeed, they show that for a particular point $\omega \in C$, the span of the image of the curvature of the Coulomb connection at $\omega$ is dense in the Lie algebra of the gauge group. This leads directly to their result. However, in Chapter 3 we prove that in our case the image of the curvature cannot be dense in the Lie algebra of the gauge group:

**Lemma 5.** Let $M$ be a compact 3-manifold with boundary and let $\nabla_A$ be a connection of Sobolev class $k$ for $k > 3/2 + 1$ that satisfies conductor boundary conditions. Define a set $\mathcal{L}_A$ as

$$\mathcal{L}_A = \text{Span}\{R_A(\alpha, \beta) : \alpha, \beta \in H_A\}.$$

There exists a bounded nonzero operator $T_A : \text{Lie}(G^{k+1}_{\text{con}}) \rightarrow L^2(\xi_P|_{\partial M})$ such that $\mathcal{L}_A \subseteq \ker(T_A)$.

Hence, the image of the curvature form at any fixed point does not linearly generate the entire Lie algebra of $G^{k+1}_{\text{con}}$.

We next specialize to the case where our principal bundle is the trivial bundle $P = \tilde{O} \times K \rightarrow \tilde{O}$, where $\tilde{O} \subset \mathbb{R}^3$ is an open subset with a smooth boundary. In this case, we can consider the flat connection. Again, this is the Ehresmann connection whose horizontal subspaces are tangent to $\tilde{O}$ in $P$. We denote the corresponding Koszul connection on $\tilde{O} \times V \rightarrow \tilde{O}$ as $\nabla_0$ or simply $d$. We then can show that the converse of the previous lemma holds for smooth functions if $\nabla_A = \nabla_0$.

**Lemma 6.** Suppose we restricted the map $T_0$ above to smooth sections. Then $\ker(T_0|_{C^\infty}) = \mathcal{L}_0 \cap C^\infty$. 
We next consider the Lie algebra that $\mathcal{L}_0$ generates.

**Lemma 7.** Let $g \in C^\infty(\partial M)$. Then there exists $f \in [\mathcal{L}_0 \cap C^\infty, \mathcal{L}_0 \cap C^\infty]$ such that $T_0(f) = g$.

Using basic linear algebra and an argument of [16], we prove

**Theorem 8.** Let $f \in \text{Lie}(\mathcal{G}_{\text{con}}^{k+1}) \cap C^\infty$. Then $f$ is in the Lie algebra generated by $\mathcal{L}_0$. Hence, $\mathcal{H}_{\text{con}, A_0}$ is dense in the connected component of the identity of $\mathcal{G}_{\text{con}}^{k+1}$.

Hence, in this special case, the maximal ambiguity of Narasimhan and Ramadas exists even when we are dealing with manifolds with boundary.
Chapter 2

The Bundle $C^k_{con,A_0}/G^k_{con}$

In this chapter we will prove that the mapping $C^k_{con,A_0} \rightarrow C^k_{con,A_0}/G^k_{con}$ is indeed a $C^\infty$ vector bundle for $k > 3/2 + 1$. This is a standard result, and versions of it have been proved in [1], [15], [16], and [18].

2.1 General Background and Notation

$M$ will denote a compact oriented 3-dimensional Riemannian manifold with boundary, and $P \rightarrow M$ will denote a principal bundle with a semisimple compact structure group $K$. Furthermore, we assume that $K$ acts faithfully on a finite dimensional real (or complex) inner product space by isometries, and thus we view $K$ as a compact matrix group and a subgroup of $O(V)$ (or $U(V)$, respectively). Auxiliary bundles also come into play. The natural matrix multiplication of $K$ on $V := \mathbb{R}^n$ (or $V := \mathbb{C}^n$) induces a vector bundle $E := P \times_K V$ (for the definition of these associated bundles, see Chapter 1.5 in [10]). $K$ also acts on itself and $\mathfrak{k}$ via the adjoint representation, and thus we have the corresponding bundles $K_P := P \times_K K$ and $\mathfrak{k}_P := P \times_K \mathfrak{k}$.

Note that $\mathfrak{k}_P$ is a vector bundle, while $K_P$ is not. However, both $\mathfrak{k}_P$ and $K_P$ are subbundles of the vector bundle $\text{End}(V)_P := P \times_K \text{End}(V)$, where again $K$ acts by the adjoint action.

Recall the exponential map $\exp : \mathfrak{k} \rightarrow K$. Since $\text{Ad}(k) \circ \exp = \exp \circ \text{Ad}(k)$, for any $k \in K$, we have an induced map $\exp : \mathfrak{k}_P \rightarrow K_P$.

As $\text{End}(V)$ acts on $V$ in an obvious way, fibers of $\text{End}(V)_P$ act on fibers of $E$. Indeed, let $(p,T)_K \in \text{End}(V)_P$ and $(p,v)_K \in E$ be equivalence classes over the same point $x \in M$. Then we define the action as

$$(p,T)_K \cdot (p,v)_K = (p,Tv)_K.$$ 

It is easy to check that this is well-defined, and this induces a bundle isomorphism between $\text{End}(V)_P$ and $\text{Hom}(E,E)$ over the identity. Viewing $\mathfrak{k}_P$ and $K_P$ as subbundles of $\text{End}(V)_P$, fibers of $\mathfrak{k}_P$ and $K_P$ also act on fibers of $E$. 

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Similar reasoning also tells us that given two elements \( \phi, \psi \) in the same fiber of \( kP \), we can make sense of the Lie bracket \([\phi, \psi]\). Indeed, if \( \phi = (p, \phi')_K \) and \( \psi = (p, \psi')_K \), then
\[
[(p, \phi')_K, (p, \psi')_K] = (p, [\phi', \psi'])_K
\]
is well-defined by the Jacobi identity.

A Koszul connection \( \nabla_A \) on \( E \) induces a Koszul connection also called \( \nabla_A^{\text{Hom}} \) on \( \text{Hom}(E, E) \) (see [2], [3] for more background on \( \nabla_A^{\text{Hom}} \)). Often, we will write \( \nabla_A \) for \( \nabla_A^{\text{Hom}} \) if it is clear we are using this induced connection. \( \nabla_A^{\text{Hom}} \) induces a connection on \( kP \), and allows us to calculate \( \nabla_A \varphi \) for \( \varphi \in \Gamma(K_P) \). Note that \( \nabla_A \varphi \) is not necessarily a section of \( K_P \), but a section of \( \text{Hom}(E, E) \).

Of special interest is the trivial bundle \( P := \bar{O} \times K \to \bar{O} \). In this case, the induced bundles \( E, kP, \) and \( K_P \) are also direct products of the appropriate sort. For example, \( E = \bar{O} \times V \to \bar{O} \). In this case, the flat connection on \( P \) is the Ehresmann connection whose kernel is the tangent bundle of \( \bar{O} \) embedded in the tangent bundle of \( \bar{O} \times K \). Using parallel transport, one can check that the Koszul connection \( \nabla_0 \) induced on the product bundle \( E \) is given by
\[
\nabla_0 \sigma = d\sigma, \quad \sigma \in \Gamma(E)
\]
where \( d\sigma \) is the standard exterior derivative. Thus we will often use \( \nabla_0 \) and \( d \) interchangeably and call them the flat connection on \( E \).

We are only concerned with connections \( \nabla_A \) on \( E \) that are induced by connections on \( P \). Such connections are called \( K \)-connections, and one can show that \( \nabla_A \) is a \( K \)-connection if and only if the local connection form is \( k \)-valued (see [2]).

As a vector bundle, we may equip \( kP \) with a metric. Any \( \text{Ad} \)-invariant inner product on \( k \) will induce a Riemannian metric on \( kP \). In particular, we can use the trace inner product \((A, B) = \text{tr}(A^* B)\) to induce a metric on \( kP \). We now view \( kP \) as equipped with metric induced by the trace inner product on \( k \). Similarly, we equip the bundle \( \text{End}(V)_P \) with the trace inner product.

For any vector bundle \( \xi \) over \( M \), we define the bundle \( \Omega^j(\xi) := \text{Hom}(\Lambda^j(M), \xi) \). We call the elements of \( \Omega^j(\xi) \) \( \xi \)-valued \( j \)-forms, and generally call them vector-valued forms. (It would perhaps be better to call them “vector-bundle valued forms,” but this is not the standard terminology). By convention, we have \( \Omega^0(\xi) := \Gamma(\xi) \), the sections of \( \xi \). Any connection on \( \xi \) induces a connection on \( \Omega^j(\xi) \) that involves the Levi-Civita connection on \( M \). See [3] for more about these forms and this connection.

There are certain operations we will like to define on forms. Given any \( \alpha \in \Omega^1(\mathfrak{k}_P) \) and \( \phi \in \Gamma(\mathfrak{k}_P) \), we define the 1-form \([\alpha, \phi]\) by
\[
[\alpha, \phi](X) = [\alpha(X), \phi],
\]
for any \( X \in TM \). Also, given any \( \alpha, \beta \in \Omega^1(\mathfrak{k}_P) \), we define the product \([\alpha \cdot \beta] \in \Gamma(\mathfrak{k}_P) \) in the following way: Suppose locally \( \alpha = \sum_i \alpha_i dx_i \), and \( \beta = \sum_i \beta_i dx_i \).
and the associated metric tensor is \{g_{ij}\}. Then, we set

$$[\alpha \cdot \beta] = \sum_{i,j} [\alpha_i, \beta_j] g^{ij}, \quad (2.1)$$

noting that \( <dx_i, dx_j> = g^{ij} \) and the matrix \((g^{ij})\) is inverse to \((g_{ij})\). One can verify that this globally defines \([\alpha \cdot \beta]\) as a section of \(\mathfrak{t}_P\).

We will often be looking at the difference between two \(K\)-connections, and the following will be useful in looking at such differences. If \(\nabla_{A_1}\) and \(\nabla_{A_2}\) are \(K\)-connections, using the local characterization of \(K\)-connections, one can show that the difference \(\nabla_{A_1} - \nabla_{A_2}\) is a \(\mathfrak{t}_P\)-valued 1-form. Furthermore, if we set \(\alpha := \nabla_{A_1} - \nabla_{A_2}\), we have for any \(\phi \in \Gamma(\mathfrak{t}_P)\)

$$((\nabla^\Hom_{A_1} - \nabla^\Hom_{A_2})(\phi)) = [\alpha, \phi]. \quad (2.2)$$

Similarly, if \(\beta \in \Omega^1(\mathfrak{t}_P)\), one can show that

$$((\nabla^\Hom_{A_1})^\ast - (\nabla^\Hom_{A_2})^\ast)(\beta) = -[\alpha \cdot \beta]. \quad (2.3)$$

The previous two equations are ubiquitous in what follows. On sections we have \(d_A = \nabla_A\) and on 1-forms we have \(d_A^* = (\nabla_A)^\ast\). We will use both notations interchangeably on these respective domains. The curvature \(R_A\) of a \(K\)-connection \(\nabla_A\) is a \(\mathfrak{t}_p\)-valued 2-form. All the facts asserted in this paragraph can be found in \([2]\).

Using \(\nabla^\Hom_A\), we can show that a \(K\)-connection \(\nabla^\Hom_A\) is compatible with the metric on \(\text{End}(V)_P\) induced by the trace inner product. If we look locally, we have \(\nabla^\Hom_A = d + [A, \cdot]\), where \(d\) is the flat connection with respect to a local coordinate system and \(A\) is a local \(\mathfrak{t}\)-valued 1-form. Then for a vector \(X\) and local sections \(S, T\) of \(\text{End}(V)_P\), we have by the bilinearity of the inner product

$$X \cdot <S, T> = <dS(X), T> + <S, dT(X)>. \quad \text{Since we are locally working with matrices, we note that}$$

$$<[A(X), S], T> + <S, [A(X), T]> = \text{tr}([A(X), S]^\ast T) + \text{tr}(S^\ast [A(X), T])$$

$$= \text{tr}([S^\ast, A(X)]^\ast T) + \text{tr}(S^\ast [A(X), T])$$

$$= \text{tr}(S^\ast A(X)^\ast T) - \text{tr}(A(X)^\ast S^\ast T) + \text{tr}(S^\ast A(X)T) - \text{tr}(S^\ast TA(X))$$

$$= (\text{tr}(S^\ast A(X)T) + \text{tr}(A(X)^\ast S^\ast T)) + \text{tr}(S^\ast A(X)T) - \text{tr}(S^\ast TA(X))$$

where in the second to last line, we used the fact that \(\mathfrak{t} \subseteq \mathfrak{so}(V)\) (or \(\mathfrak{su}(V)\)). Hence,

$$X \cdot <S, T> = <dS(X), T> + <S, dT(X)> \quad \text{Hence,}$$

$$X \cdot <S, T> = <dS(X), T> + <S, dT(X)>) +$$

$$<[A(X), S], T> + <S, [A(X), T]>$$

$$= <\nabla^\Hom_A S, T> + <S, \nabla^\Hom_A T>, \quad (2.4)$$
proving metric compatibility. Furthermore, a $K$-connection $\nabla_A$ on $E$ and the Levi-Civita connection on $M$ induce a connection on $\Omega^j(t_P)$ that is compatible with the induced metric on $\Omega^j(t_P)$.

2.2 Sobolev Spaces of Connections and Gauge Groups

We define Sobolev spaces of sections of vector bundles as Palais does in [17]. Using the notation of [17], the space $L^p_k(\xi)$ is the space of sections of $\xi$ with $k$ Sobolev derivatives under the $L^p$ norm, and $L^p_k(\xi)^0$ is the completion of $C^\infty(\xi|_{\text{int}(M)})$ in the $L^p_k$ norm. As usual, we define $H^k(\xi) := L^2_k(\xi)$, where the latter notation is what [17] uses. Also converting from Palais’s notation, we put $H^k_0(\xi) := L^2_k(\xi)^0$.

Palais uses a local approach to define these Sobolev spaces. However, it can be shown that given a smooth connection $\nabla_A$ on $\xi$, then the norm on $C^\infty$ sections $f$

$$\|f\| = \|f\|_{L^p} + \sum_{i=1}^k \int_M <\nabla^i_A f, \nabla^i_A f>^{p/2} dVol$$

induces an equivalent norm on $L^2_k(\xi)$, and hence has the same completion. We will use both this global as well as Palais’s local point of view of Sobolev spaces of sections.

Gross in [8] has defined conductor boundary conditions on Sobolev spaces, which we will denote $H^k_{\text{con}}(\Omega^j(\xi))$ for appropriate vector bundles $\xi$, where $\Omega^j(\xi) := \text{Hom}(\Lambda^j TM, \xi)$.

Specifically, we define the conductor Sobolev space $H^k_{\text{con}}(\Omega^j(\xi))$ for $k \geq 1$ as

$$H^k_{\text{con}}(\Omega^j(\xi)) := \{ \alpha \in H^k(\Omega^j(\xi)) : \iota^*(\alpha) = 0, \text{ where } \iota : \partial M \to M \text{ is the inclusion} \}.$$  

(2.4)

Since $k \geq 1$, $\alpha|_{\partial M}$ is defined in the trace sense, so $\iota^*(\alpha)$ is defined almost everywhere. Again, $\iota^*(\alpha) = 0$ is equivalent to saying that $\alpha$ vanishes on wedges of vectors $X_1 \wedge \ldots \wedge X_j$, where all $X_i$ are tangent to $\partial M$. For a 0-form $\sigma$ (i.e. a section $\sigma$), $\iota^*(\sigma) = 0$ if and only if $\sigma|_{\partial M} = 0$. Hence, we see that

$$H^k_{\text{con}}(\xi) = H^1_{\text{con}}(\xi) \cap H^k(\xi), \quad k \geq 1.$$  

(2.5)

In what follows we use $k > 3/2 + 1$ so we can use the multiplication theorem of Sobolev spaces (see Corollary 9.7 in [17]).

Since we will be using conductor boundary conditions, we need a fixed smooth connection $\nabla_{A_0}$. Set $C^k_{\text{con}, A_0} := \nabla_{A_0} + H^k_{\text{con}}(\Omega^j(t_P))$. Note that all the connections in $C^k_{\text{con}, A_0}$ will be equal to $\nabla_{A_0}$ in tangential directions on the boundary. Also $C^k_{\text{con}, A_0}$ is an affine space and thus seen to be a $C^\infty$-Hilbert manifold. We will call any connection $\nabla_A$ $C^\infty$-smooth if $\nabla_A - \nabla_{A_0}$ is a smooth

\[\text{[Footnote: Marini in [13] has also defined these boundary conditions, although she calls them Dirichlet boundary conditions.]}\]
section of \( \mathfrak{fp} \); in other words, \( \nabla_A \) is a Koszul connection in the usual Riemannian geometry sense.

**Proposition 9.** Suppose \( k - 1 > 3/2 \), and \( \nabla_A \in \mathcal{C}^{k}_{\text{con},A_0} \). Then for \( 1 \leq m \leq k + 1 \), we have

\[
\nabla_A : H^{m+1}(\mathfrak{fp}) \to H^m(\Omega^1(\mathfrak{fp})) \\
\nabla_A^* : H^{m+1}(\Omega^1(\mathfrak{fp})) \to H^m(\mathfrak{fp})
\]

are bounded linear transformations. Also, \( \nabla_A : H^{m+1}_{\text{con}}(\mathfrak{fp}) \to H^m_{\text{con}}(\Omega^1(\mathfrak{fp})) \) for \( m = k - 1 \) and \( m = k \).

**Proof.** Taking the global view of Sobolev spaces, we see that \( \nabla_{A_0} \) maps \( H^{m+1}(\mathfrak{fp}) \) to \( H^m(\Omega^1(\mathfrak{fp})) \) and \( \nabla_{A_0}^* \) maps \( H^{m+1}(\Omega^1(\mathfrak{fp})) \) to \( H^m(\mathfrak{fp}) \). For \( A \in \mathcal{C}^k_{\text{con},A_0} \), if \( 1 \leq m \leq k + 1 \), and \( f \in H^{m+1}(\mathfrak{fp}) \), we have

\[
\nabla_A f = \nabla_{A_0} f + [\nabla_A - \nabla_{A_0}, f] \in H^m(\Omega^1(\mathfrak{fp})) \tag{2.6}
\]

and that the mapping \( f \to \nabla_A f \) is bounded by the multiplication theorem of Sobolev spaces. Similarly, for \( A \in \mathcal{C}^k_{\text{con},A_0} \), and \( 1 \leq m \leq k + 1 \), and \( \alpha \in H^{m+1}(\Omega^1(\mathfrak{fp})) \) we have

\[
\nabla_A^* \alpha = \nabla_{A_0}^* \alpha - [\nabla_A - \nabla_{A_0}, \alpha] \in H^m(\mathfrak{fp}).
\]

As for the last assertion of the proposition, note that if \( f \in H^{m+1}_{\text{con}}(\xi) \), then \( f \in C^1(\xi) \) and \( f|_{\partial M} \equiv 0 \). Looking locally at a trivializing neighborhood at the boundary, we have \( \nabla_A = d + A \), where \( d \) is the flat connection with respect to the trivialization. Since \( A \in H^k_{\text{con}}(\Omega^1(\xi)) \), we have \( A \in C^1(\Omega^1(\xi)) \). Let \( X \) be a tangential direction on the boundary. Then since \( f \) is constantly 0 on \( \partial M \), \( df(X) = 0 \) on the boundary. Also, \( [A(X), f] = 0 \) on the boundary since \( f = 0 \) on the boundary. Hence, globally, \( v'(\nabla_A f) \equiv 0 \) on \( \partial M \). Hence by (2.6), we have \( \nabla_A f \in H^m_{\text{con}}(\Omega^1(\xi)) \).

We now move onto the gauge transformations. If \( P \) were a trivial bundle, then the sections of \( K_P \) would be the gauge transformations described in the introduction. Generally, the sections of \( K_P \) are the **gauge transformations**. The Sobolev regularity and boundary conditions we will need is set in the following definition:

**Definition 10.** Let \( \nabla_{A_0} \) be a fixed smooth \( K \)-connection on \( E \). Let \( g \in H^{k+1}(K_P) \), with \( g|_{\partial M} \equiv e \), where \( e \) is the identity on \( K \). Then we say that \( g \in G^{k+1}_{\text{con}} \).

The Sobolev space \( H^{k+1}(K_P) \) is defined as in (15) as the completion of smooth sections of \( K_P \) in the norm \( H^{k+1}(\operatorname{End}(V))_P \). This completion without the boundary conditions we will call \( G^{k+1} \), as it is called in (15).

**Proposition 11.** \( G^{k+1}_{\text{con}} \) is a closed topological subgroup of \( G^{k+1} \).
Hence, globally, \( \iota_1 \) acts on \( \mathcal{C}^k_{\text{con},A_0} \) on the right in the following way. Suppose that \( \eta \in H_{\text{con}}^k(\Omega^1(t_P)) \). Then the action is

\[
(\nabla_{A_0} + \eta) \cdot g = \nabla_{A_0} + g^{-1}\nabla_{A_0}^\text{Hom} g + \text{Ad}(g^{-1})\eta.
\]

(2.7)

Note that for \( (\nabla_{A_0} + \eta) \cdot g \) to remain in \( \mathcal{C}^k_{\text{con},A_0} \), we need to have \( g^{-1}\nabla_{A_0}^\text{Hom} g \) satisfy conductor boundary conditions. The following proposition shows that this is the case.

**Proposition 12.** Suppose \( g \in \mathcal{G}_{\text{con}}^k \). Then we have \( g^{-1}\nabla_{A_0} g \in H_{\text{con}}^k(\Omega^1(t_P)) \).

**Proof.** Let \( g \in \mathcal{G}_{\text{con}}^k \). Since \( g \in H_{\text{con}}^{k+1}(k_P) \), there exist smooth \( g_n \in H_{\text{con}}^{k+1}(k_P) \) such that \( g_n \to g \) in \( H_{\text{con}}^{k+1}(\text{End}(V)) \). It is shown in [15] that inversion is continuous on \( \mathcal{G}_{\text{con}}^{k} \). Hence, \( (g_n)^{-1} \to g^{-1} \) in \( H_{\text{con}}^{k+1}(\text{End}(V)) \). Since \( \nabla_{A_0} \) is a smooth \( K \)-connection, we see that \( g_n^{-1}\nabla_{A_0}^\text{Hom} g_n \in \Omega^1(t_P) \), and by the multiplication theorem,

\[
\|g^{-1}\nabla_{A_0}^\text{Hom} g - g_n^{-1}\nabla_{A_0}^\text{Hom} g_n\|_{H^k} \leq \|g^{-1}\nabla_{A_0}^\text{Hom} g - g_n^{-1}\nabla_{A_0}^\text{Hom} g\|_{H^k} + \|g_n^{-1}\nabla_{A_0}^\text{Hom} g - g^{-1}\nabla_{A_0}^\text{Hom} g\|_{H^k} \leq C\|g^{-1} - g_n^{-1}\|_{H^{k+1}}\|\nabla_{A_0}^\text{Hom} g\|_{H^k} + \|g_n^{-1}\|_{H^{k+1}}\|g_n - g\|_{H^{k+1}} \to 0.
\]

Thus, \( g^{-1}\nabla_{A_0}^\text{Hom} g \in H_{\text{con}}^k(\Omega^1(t_P)) \).

Locally, we have \( \nabla_{A_0} = d + A_0 \), where \( A_0 \) is a \( C^\infty \)-smooth \( k \)-valued 1-form. Let \( X \) be a tangential direction. Since \( g \equiv e \) on \( \partial M \), we have \( dg(X) = 0 \). Also on the boundary, \( [A_0(X),g] = [A_0(X),e] = 0 \), since \( e \) commutes with everything. Hence, globally, \( \iota^*(\nabla_{A_0}^\text{Hom} g) \equiv 0 \), and thus \( \iota^*(g^{-1}\nabla_{A_0}^\text{Hom} g) \equiv 0 \). This proves that \( g^{-1}\nabla_{A_0}^\text{Hom} g \in H_{\text{con}}^k(\xi) \), and thus \( g \in \mathcal{G}_{\text{con}}^{k+1} \), as desired.

We also note the following. If \( \nabla_A \in \nabla_{A_0} + H_{\text{con}}^k(\Omega^1(t_P)) \), \( \eta := \nabla_A - \nabla_{A_0} \), and \( g \in \mathcal{G}_{\text{con}}^{k+1} \), then

\[
g^{-1}\nabla_A g = g^{-1}\nabla_{A_0} g + g^{-1}([\eta,g]) \in H_{\text{con}}^k(\Omega^1(t_P))
\]

since \([\eta,g] \equiv 0\) on the boundary (\( g \equiv e \) on the boundary and thus commutes with everything). So \( \nabla_{A_0} \) in Proposition 12 can be replaced by any \( H^k \) Sobolev connection. In particular, it can be replaced by any smooth connection, and so \( \mathcal{G}_{\text{con}}^{k+1} \) does not depend on the choice of the smooth \( K \)-connection \( \nabla_{A_0} \).

Now we can state our desired result for this chapter:

**Theorem 13.** Let \( k > 3/2 + 1 \). The quotient space \( \mathcal{C}_{\text{con},A_0}/\mathcal{G}_{\text{con}}^{k+1} \) is a \( C^\infty \) Hilbert manifold, and \( \pi : \mathcal{C}_{\text{con},A_0} \to \mathcal{C}_{\text{con},A_0}/\mathcal{G}_{\text{con}}^{k+1} \) is a principal bundle with structure group \( \mathcal{G}_{\text{con}}^{k+1} \).
The proof will be a straightforward adaptation of the work of Atiya et al, Mitter et al, and Parker in [11], [15], and [18], respectively. Before we can prove it, however, will need some basic tools which we will lay out in the next section.

2.3 The Poincaré Inequality, Green Operators, and Other Necessities

Before we start proving Theorem 13, we need some facts about the Sobolev spaces we will be working with. Many of these results will also be quite basic to the next chapter. The most basic of these results is the following Sobolev-Poincaré inequality:

**Proposition 14.** Let $\nabla_A$ be a smooth connection on a vector bundle $\xi \to M$ compatible with the metric. Then there exists a $\kappa_p > 0$ such that for any $f \in L^1_p(\xi)^0$ with $1 < p < \dim(M)$, we have

$$\|f\|_{L^p} \leq \kappa_p \|\nabla_A f\|_{L^p}.$$  \hspace{1cm} (2.8)

where $\kappa_p$ does not depend on the connection, but does depend on $p$.

**Proof.** First we consider real-valued functions on $M$, i.e. sections of the trivial bundle $M \times \mathbb{R}$. We first want to prove that the only constant function in $L^1_p(M \times \mathbb{R})^0$ is the 0 function. So let $g$ be a constant function in $L^1_p(M \times \mathbb{R})^0$. Using Theorem 9.3 in [17], we can extend the restriction map to a continuous map $R : L^1_p(M \times \mathbb{R}) \to L^p(\partial M \times \mathbb{R})$. Since $g \in L^1_p(M \times \mathbb{R})^0$, we have $g|_{\partial M} \equiv 0$. But $g$ is constant, so $g \equiv 0$ everywhere, proving our assertion.

We can now use a standard argument in proving Poincaré inequalities. Variations can be found in the proof of Lemma 3.8 in [9], and in [5]. We want to show that there exists a $\kappa_p > 0$ such that for any $g \in L^1_p(M \times \mathbb{R})^0$ with $1 < p < \dim(M)$, we have

$$\|g\|_{L^p} \leq \kappa_p \|dg\|_{L^p}.$$ \hspace{1cm} (2.9)

Consider the set $\mathcal{H} := \{g \in L^1_p(M \times \mathbb{R})^0 : \|g\|_{L^p} = 1\}$. Note that we have just shown in the previous paragraph that no constant functions lie in $\mathcal{H}$. Set

$$C_p := \inf_{g \in \mathcal{H}} \|dg\|_{L^p}.$$  

We will show that $C_p > 0$, thereby proving (2.9) by setting $\kappa_p := (C_p)^{-1}$. Let $\{f_n\} \in \mathcal{H}$ attain the above infimum (in other words, $\lim_{n \to \infty} \|dg\|_{L^p} = C_p$). Note that $\{f_n\}$ is bounded in $L^p$ norm and $L^p$ is reflexive (since $p > 1$), so there exists a subsequence that we again call $\{f_n\}$ that has a weak limit $f$ in $L^1_p$. By the Rellich-Kondrakov compactness theorem, since $1 \leq p < \dim M$, $\{f_n\}$ converges strongly in $L^p$ to this same function $f$. The strong convergence shows that $f \in \mathcal{H}$, while the weak convergence shows that

$$\|df\|_{L^p} \leq \liminf_{n \to \infty} \|df_n\|_{L^p} = C_p.$$
Since $f \in H$, we know that $f$ is not constant, and thus $C_p = \|df\|_{L^p}$ is not 0. Hence, $C_p > 0$, as desired, proving (2.9).

Now let $f \in C^1_0(\xi|_M)$. Then the function $|f|$ is globally Lipschitz, so by Lemma 2.8 in [3] we have $|f| \in L^p_1(M \times \mathbb{R})$. Since $|f|$ is continuous and 0 on $\partial M$, Theorem 5.5.2 in [5] tells us that $|f| \in L^p_1(M \times \mathbb{R})^0$. Hence, (2.9) yields

$$\|f\|_{L^p} \leq \kappa_p \|df\|_{L^p} \leq \kappa_p \|\nabla_A f\|_{L^p}.$$  

The second inequality is Kato’s inequality. This inequality only requires that $\nabla_A$ is compatible with the metric. For a proof of this inequality, see [18]. Since $C^1_0(\xi|_M)$ is dense in $L^p_1(\xi)^0$, the proposition has been proven.

Remark 1. Note that while the connection $\nabla_A$ needs to be compatible with the metric to invoke Kato’s inequality, the $\kappa_p$ above does not depend on the connection $\nabla_A$. This allows us to let $\nabla_A$ be a $L^q$ connection for any $q \geq \text{dim}(M)$ and still have (2.9) hold: Let $\nabla_A \rightarrow \nabla_{A_\infty}$ in $L^q$ with $\nabla_{A_\infty}$ smooth. Let $p^*$ be the Sobolev conjugate to $p$. Invoking the boundary condition-free Sobolev inequality (found in [9]), we have $f \in L^{p^*}(\xi)$, and thus

$$\|f\|_{L^p} \leq \kappa_p \|\nabla_A f\|_{L^p} \leq \kappa_p \|\nabla_{A_\infty} f\|_{L^p} + \|\nabla_{A_\infty} f\|_{L^p} \leq \kappa_p \|\nabla_{A_\infty} f\|_{L^p} + \|\nabla_{A_\infty} f\|_{L^{p^*}(\xi)} \rightarrow \kappa_p \|\nabla_{A_\infty} f\|_{L^p},$$  

since $1/p = 1/p^* + 1/(\text{dim}(M))$. In particular, since $H^k(\xi P) \subset C^1(\xi P) \subset L^\infty(\xi P)$, then (2.9) holds for $\nabla_A \in C^k_{\text{con},A_0}$.

The Sobolev-Poincaré inequality immediately tells us

Corollary 15. The action of $G_{\text{con}}^{k+1}$ on $C^k_{\text{con},A_0}$ is free.

Proof. Suppose $\nabla_A + \eta \in \nabla_A + H^k_{\text{con}}(\Omega(\xi P)), g \in G_{\text{con}}^{k+1}$, and $(\nabla_A + \eta) \cdot g = \nabla_A + \eta$. By (2.9), this means that

$$(\nabla_{A_0}^{\text{Hom}} + \eta)g = \nabla_{A_0}^{\text{Hom}} g + [\eta, g] = 0.$$  

By (2.9), we have

$$\|g - e\|_{L^2} \leq \kappa_2 \|\nabla_{A_0}^{\text{Hom}} + \eta\|_{L^2} = 0$$  

Hence, since $g$ is continuous, $g \equiv e$ and thus the corollary is proven.

We are now in a position to show that $G_{\text{con}}^{k+1}$ is a Hilbert Lie group.

Proposition 16. The exponential map takes $H_{\text{con}}^{k+1}(\xi P)$ into $G_{\text{con}}^{k+1}$ and is a local homeomorphism at 0.
Proof. In [15] it is shown that \( \exp \) is a \( C^\infty \) smooth map \( H^{k+1}(\mathfrak{t}_p) \to G^{k+1} \), as well as a local diffeomorphism, without boundary conditions. Hence, for us to prove the proposition, we need only to show that \( \exp \) maps \( H^{k+1}_{\text{con}}(\mathfrak{t}_p) \) into \( G^{k+1}_{\text{con}} \), and for a neighborhood \( U \) of the identity in \( G^{k+1} \), \( \exp^{-1} = \log \) maps \( U \) into \( H^{k+1}_{\text{con}}(\mathfrak{t}_p) \).

To prove the first assertion, let \( f \in H^{k+1}_{\text{con}}(\mathfrak{t}_p) \). Since \( k+1 > 3/2 + 2 \) we have \( f \in C^2(\mathfrak{t}_p) \) with \( f_{\partial M} = 0 \). Then if \( g := \exp(f) \), we have that \( g \in C^2(K_P) \subseteq C^2(\text{End}(V)_P) \), and \( g_{\partial M} = e \), where \( e \) is the identity element of \( K \). Hence, by Proposition 12 \( g \in G_{\text{con}}^{k+1} \).

To prove the second assertion, since \( \exp : H^{k+1}(\mathfrak{t}_p) \to G^{k+1} \) is a local diffeomorphism between the spaces without boundary conditions, we need only to show that for small \( \xi \in H^{k+1}(\mathfrak{t}_p) \), if \( g := \exp(\xi) \in G^{k+1}_{\text{con}} \), then \( \xi \in H^{k+1}_{\text{con}}(\mathfrak{t}_p) \). Note that \( \sup \| \xi \|_{H^{k+1}} \), we can use the fact that the “pointwise” map \( \exp : \mathfrak{t} \to K \) is local diffeomorphism at 0 to say that since \( g_{\partial M} = e \), we have \( \xi_{\partial M} = 0 \). Since \( \xi \in H^{k+1}(\mathfrak{t}_p) \), this vanishing on the boundary implies that \( \xi \in H^{k+1}_{\text{con}}(\mathfrak{t}_p) \).

\[ \square \]

Corollary 17. The group \( G^{k+1}_{\text{con}} \) is a Hilbert Lie group whose Lie algebra is identifiable with \( H^{k+1}_{\text{con}}(\mathfrak{t}_p) \). \( G^{k+1}_{\text{con}} \) acts smoothly on \( C^k_{\text{con,A}_0} \).

Proof. The previous proposition shows that \( G^{k+1}_{\text{con}} \) is a local Hilbert Lie group, using logarithmic coordinates, with its Lie algebra identified with \( H^{k+1}_{\text{con}}(\mathfrak{t}_p) \). Exactly as in the proof of Theorem 2.18 of [15] (which shows \( G^{k+1} \) is a Hilbert Lie group), we can transport these coordinates throughout \( G^{k+1}_{\text{con}} \), and coordinate changes are smooth. The details are in [15].

As for the smoothness of the action, it is shown in Proposition 3.12 of [15] that the action \( C^k \times G^{k+1} \to G^k \) is smooth. The inclusions \( C^k_{\text{con,A}_0} \to C^k \) and \( G^{k+1}_{\text{con}} \to G^{k+1} \) are also smooth, so the composition \( C^k_{\text{con,A}_0} \times G^{k+1}_{\text{con}} \to C^k \) is smooth. But we previously showed that action maps \( C^k_{\text{con,A}_0} \times G^{k+1}_{\text{con}} \) into \( C^k_{\text{con,A}_0} \) as desired. \[ \square \]

Next on our agenda is to prove a version of Stokes’ theorem. In what follows, \( M \) can be of any dimension.

Lemma 18. Let \( M \) be a compact oriented Riemannian manifold with boundary, and \( P \to M \) a principle \( K \)-bundle over \( M \). Let \( u \in \Omega^p(\mathfrak{t}_P) \) and \( v \in \Omega^{p+1}(\mathfrak{t}_P) \). For any smooth connection \( \nabla_A \), we have

\[
\int_M \langle d_A u, v \rangle - \langle u, d_A^* v \rangle = \int_{\partial M} \langle \nu^* \wedge u, v \rangle,
\]

where \( \nu \) is the outward pointing normal vector on the boundary, and \( \nu^* \) is the Hilbert space adjoint of \( \nu \) using the metric on \( M \).

In particular, if \( u \) satisfies conductor boundary conditions, then

\[
\int_M \langle d_A u, v \rangle = \int_M \langle u, d_A^* v \rangle.
\]  \hspace{1cm} (2.10)

To prove this we will prove a series of propositions.
Proposition 19. Let $M$ be a Riemannian manifold with dimension $n$, and let \( \{e_i\} \) be an orthonormal frame field for $U \subseteq M$. Let $E \to M$ be a vector bundle over $M$. Then for $u \in \Omega^p(E)$ and $v \in \Omega^{p+1}(E)$,

\[
X|_U = \sum_{i=1}^n <e_i^* \wedge u, v>e_i
\]
defines a global vector field on $M$.

Proof. Suppose \( \{e_j\} \) and \( \{f_j\} \) are two local orthonormal frame fields over the same open set. Then define the matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ by

\[
e_j = \sum_i a_{ij} f_i, \quad f_j = \sum_i b_{ij} e_i
\]

Then $B^{-1} = A$. Also, note that since $e^*_j(f_i) = <e_j, f_i> = b_{ji}$, we have

\[
e^*_j = \sum_i b_{ji} f^*_i.
\]

Thus, since $B^{-1} = A$,

\[
\sum_j <e^*_j \wedge u, v> = \sum_{i,j,k} <b_{ji} f^*_i \wedge u, v > a_{kj} f_k
\]

\[
= \sum_{i,k} <f^*_i \wedge u, v > f_k (\sum_j a_{kj} b_{ji})
\]

\[
= \sum_{i,k} <f^*_i \wedge u, v > f_k \delta_{ik}
\]

\[
= \sum_i <f^*_i \wedge u, v > f_i
\]

Thus, the definition of $X$ is frame independent, proving that $X$ is a global vector field.

An obvious approach to proving Lemma 18 is to use a the regular Stokes’ theorem, or a variation of it. The variation we will use is the divergence theorem for manifolds. This motivates the following proposition:

Proposition 20. If $X$ is the vector field defined by

\[
X|_U = \sum_k <e^*_k \wedge u, v>e_k,
\]

then

\[
< d_A u, v > = < u, d_A^* v > = \text{div}(X),
\]

where $\text{div}(X)$ is taken with respect to the oriented volume form.
Proof. We will modify Gross’ proof in [8] to our manifold case. Choose a point \( x \in M \) and let \( \{ e_j \} \) be a geodesic frame at that point. First suppose \( u \) and \( v \) are locally of the form \( u = \phi e_{j_1}^* \wedge \ldots \wedge e_{j_p}^* \) and \( v = \psi e_{k_1}^* \wedge \ldots \wedge e_{k_{p+1}}^* \), where \( \phi \) and \( \psi \) are local sections of our vector bundle \( \mathfrak{f}_p \). Let \( \mathcal{D} \) be the Levi-Civita connection on \( M \). Then note at \( x \) we have

\[
D_Y e_{k_1}^* \wedge \ldots \wedge e_{k_{p+1}}^* (e_{i_1} \wedge \ldots \wedge e_{l_{p+1}}) = Y(e_{k_1}^* \wedge \ldots \wedge e_{k_{p+1}}(e_{i_1} \wedge \ldots \wedge e_{l_{p+1}})) - \sum_\alpha e_{k_1}^* \wedge \ldots \wedge e_{k_{p+1}}(e_{i_1} \wedge \ldots \wedge D_Y e_{l_\alpha} \wedge \ldots \wedge e_{l_{p+1}}) = 0 - \sum_\alpha 0 = 0.
\]

Thus, we have at \( x \)

\[
d_A^* v = - \sum_j t_{e_j}((\nabla_A)_{e_j}(\psi e_{k_1}^* \wedge \ldots \wedge e_{k_{p+1}}))
\]

\[
= - \sum_j t_{e_j}((\nabla_A)_{e_j}(\psi e_{k_1}^* \wedge \ldots \wedge e_{k_{p+1}} + \psi D e_k e_{k_1}^* \wedge \ldots \wedge e_{k_{p+1}}))
\]

\[
= - \sum_j ((\nabla_A)_{e_j} \psi) t_{e_j}(e_{k_1}^* \wedge \ldots \wedge e_{k_{p+1}}).
\]

Now we aim to obtain a similar expression for \( d_A(u) \). We first note that at \( x \)

\[
d(e_k^*)(e_i \wedge e_j) = e_i(e_k^*(e_j)) - e_j(e_k^*(e_i)) - e_k^*[e_i, e_j]) = 0 - 0 = e_k^*(D_{e_i} e_j - D_{e_j} e_i)
\]

\[
= 0.
\]

Thus, at \( x \)

\[
d(e_{j_1}^* \wedge \ldots \wedge e_{j_p}^*) = 0.
\]

So, at \( x \)

\[
d_A u = (\nabla_A \mu) \wedge e_{j_1}^* \wedge \ldots \wedge e_{j_p}^* + \mu d(e_{j_1}^* \wedge \ldots \wedge e_{j_p}^*)
\]

\[
= \sum_j ((\nabla_A)_{e_j} \theta) e_j^* \wedge e_{j_1}^* \wedge \ldots \wedge e_{j_p}^*.
\]

Let’s define \( \mu = e_{j_1}^* \wedge e_{j_2}^* \wedge \ldots \wedge e_{j_p}^* \) and \( \omega = e_{k_1}^* \wedge \ldots \wedge e_{k_{p+1}}^* \). Then using our expression for \( d_A u \) and \( d_A^* v \), we can write at \( x \)

\[
<d_A u, v > - < d_A^* v, u > = \sum_j <(\nabla_A)_{e_j} \theta) e_j^* \wedge \mu, v > + < u, ((\nabla_A)_{e_j} \psi) t_{e_j} (\omega) >.
\]

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Note that since \( e^*_j \wedge \mu \) and \( \omega \) are part of the orthonormal frame \( \{ e^*_j \wedge \ldots \wedge e^*_{j+p+1} \} \) for the bundle \( \Omega^{p+1}(M) \), we have

\[
< fe^*_j \wedge \mu, g\omega > = \sum < (fe^*_j \wedge \mu)(e_1 \wedge \ldots \wedge e_{t+p+1}), g\omega(e_1 \wedge \ldots \wedge e_{t+p+1}) >_{\mathbb{T}_p} = \langle f, g \rangle_{\mathbb{T}_p} < e^*_j \wedge \mu, \omega >_{\Omega^{p+1}(M)}.
\]

We use the above and metric compatibility to get at

\[
<dA, v > - < u, d^*_A v > = \sum_j < (\nabla_A e^*_j, \phi, \psi) + < \phi, (\nabla_A)e^*_j, \psi >, e^*_j \wedge \mu, \omega > = \sum_j < \phi e^*_j \wedge \mu, \psi \omega > + \sum_j < \phi e^*_j \wedge (e^*_j \wedge \mu), \psi \omega >
\]

where we also used the fact that \( < e^*_j \wedge \mu, \omega > \) is constant. So we have proven the proposition for all \( u \) and \( v \) with the special form \( u = \phi e^*_i \wedge \ldots \wedge e^*_p \) and \( v = \psi e^*_k \wedge \ldots \wedge e^*_{k+p+1} \). However, if \( u \) and \( v \) are arbitrary, then locally \( u = \sum_i u_i \) and \( v = \sum_j v_j \), where \( u_i \) and \( v_j \) are of the above special form. So, in this general case, we have at \( x \)

\[
<dA, v > - < u, d^*_A v > = \sum_{i,j} < dA u_i, v_j > - < u_i, d^*_A v_j >
\]

\[
= \sum_k < dA u_i, v_j > = \sum_k < u_i, \iota_{e^*_k} (v_j) >
\]

However, since \( \{ e_k \} \) is a geodesic frame, this last sum is exactly the divergence of \( X \) at the point \( x \) (see Exercise 3.8a in [4]). So, we have

\[
( < dA, v > - < u, d^*_A v >)(x) = \text{div}(X)(x).
\]

But \( x \in M \) was arbitrary, so we have our proposition.

The final ingredient for Lemma \[ \Box \] is the divergence theorem for oriented Riemannian manifolds which states that

\[
\int_M \text{div}(X) = \int_{\partial M} < \nu, X >.
\]

Now, we can finish up proving the lemma.
Proposition 21. Let $\{e_i\}$ be an orthonormal frame including the boundary so that $\nu = e_1$. Then, for the vector field $X$ defined in Proposition 20 on $\partial M$ we have
\[ < \nu, X > = < e_1, \sum_i e_i^* \wedge u, v > = < e_1^* \wedge u, v > = < \nu^* \wedge u, v >. \]
Thus, we have
\[
\int_M (dA u, v) - < u, d_A^* v > = \int_M \text{div}(X)
= \int_{\partial M} < \nu, X >
= \int_{\partial M} < \nu^* \wedge u, v >,
\]
which proves our lemma.

Proof of Lemma 18 Let $\{e_i\}$ be an orthonormal frame including the boundary so that $\nu = e_1$. Then, for the vector field $X$ defined in Proposition 20 on $\partial M$ we have
\[ < \nu, X > = < e_1, \sum_i e_i^* \wedge u, v > = < e_1^* \wedge u, v > = < \nu^* \wedge u, v >. \]
Thus, we have
\[
\int_M (dA u, v) - < u, d_A^* v > = \int_M \text{div}(X)
= \int_{\partial M} < \nu, X >
= \int_{\partial M} < \nu^* \wedge u, v >,
\]
which proves our lemma.

Note that, like the Sobolev-Poincaré inequality, we can replace the smooth connection $\nabla_A$ with a Sobolev connection $\nabla_A$ in, for example, $C_{k,\mathrm{con},A_0}^k$, and the smooth forms $u$ and $v$ with $H^1$ forms.

It will be essential for all that follows to have Green operators associated with every connection $\nabla_A \in C_{k,\mathrm{con},A_0}^k$. More specifically, given a $K$-connection $\nabla_A \in C_{k,\mathrm{con},A_0}^k$, we can define the Laplacian $\Delta_A = d_A^* d_A : H_{k,\mathrm{con}}^{m+1}(t_P) \rightarrow H_{k,\mathrm{con}}^{m-1}(t_P)$ for $1 \leq m \leq k$. The regularity is correct by the following argument: Since $\nabla_{A_0}$ is a smooth connection, clearly $\Delta_{A_0}$ is a bounded map from $H^m$ into $H^{m-2}$. Suppose $h = \nabla_A - \nabla_{A_0} \in H_{k,\mathrm{con}}^0(t_P)$. Then for $f \in H^{m+1}$,
\[
\Delta_A f = \Delta_{A_0} f + [d_A^* h, f] - [h \cdot [h, f]] - 2[h \cdot d_A^* f]. \tag{2.11}
\]
So, we have (allowing $\| \cdot \|_i$ to denote the $H^i$ norm)
\[
\|\Delta_A f\|_{m-1} \leq \|\Delta_{A_0} f\|_{m-1} + \|d_A^* h, f\|_{m-1} \tag{2.12}
+ \|[h \cdot [h, f]] - 2[h \cdot d_A^* f]\|_{m-1} \tag{2.13}
\leq \|\Delta_{A_0} f\|_{m-1} + C(||d_A^* h||_{k-1} ||f||_{m-1}) \tag{2.14}
+ ||h||_{k} ||[h, f]||_{m-1} + 2||h||_{k} ||d_A^* f||_{m-1} \tag{2.15}
\leq \|\Delta_{A_0} f\|_{m-1} + C(||h||_{k} ||f||_{m-1}) \tag{2.16}
+ ||h||_{k} ||[f]||_{m-1} + 2||h||_{k} ||f||_{m} < \infty \tag{2.17}
\]
where we used the fact that $H^m$ is a $H^{k-1}$ module, which is the case since $k \geq m$ and $k-1 \geq 3/2$. Thus, $\Delta_A$ is bounded from $H_{k,\mathrm{con}}^{m+1}(t_P)$ to $H_{k,\mathrm{con}}^{m-1}(t_P)$. Furthermore, we have

Proposition 21. Let $\nabla_A \in C_{k,\mathrm{con},A_0}^k$ for $k-1 \geq 3/2$, and suppose $1 \leq m \leq k$. Then the mapping $\Delta_A : H_{k,\mathrm{con}}^{m+1}(t_P) \rightarrow H_{k,\mathrm{con}}^{m-1}(t_P)$ is an isomorphism. Furthermore, if $f \in H_{k,\mathrm{con}}^2(t_P)$ and $\Delta_A f \in H_{k,\mathrm{con}}^{m-1}(t_P)$, then $f \in H_{k,\mathrm{con}}^{m+1}(t_P)$ and $\Delta_A f$
\[
\|f\|_{H^{m+1}} \leq C(\|\Delta_A f\|_{H^{m-1}} + \|f\|_{H^0}). \tag{2.18}
\]
We set $G_A : = (\Delta_A)^{-1}$ and call it the Green operator.

**Proof.** This proof will be a modification of a couple of standard arguments. Our main sources will be [3] and [16]. There is probably a much cleaner way to prove this, but the author could not come up with one. Be forewarned: the following is nasty, brutish, and long.

First we note that if $\Delta_A f = 0$ for some $f \in H^{m+1}_{\text{con}}(\mathfrak{t}_P)$, then by Lemma [18]

$$0 = (f, \Delta_A f)_{L^2} = (d_A f, d_A f)_{L^2}.$$  

So $d_A f = 0$. By the remark following Proposition [14] since $f \in H^1_0(\mathfrak{t}_P)$, we have $f = 0$. So ker $\Delta_A = 0$. Most of the remainder of this proof will be showing $\Delta_A$ is onto.

We can define an bilinear form $B : H^1_0(\mathfrak{t}_P) \times H^1_0(\mathfrak{t}_P) \rightarrow \mathbb{R}$ as

$$B(u, v) := \int_M < d_A u, d_A v >.$$

Setting $h = \nabla_A - \nabla_{A_0}$ (and again allowing $\| \cdot \|_i$ to denote the $H^i$-norm), we have

$$|B(u, v)| \leq \|d_A u\|_0 \|d_A v\|_0 \leq (\|d_A u\|_0 + ||h, u||_0)(\|d_A v\|_0 + ||h, v||_0) \leq C(||u||_1 + ||h||_k||u||_0)(||v||_1 + ||h||_k||v||_0) \leq C||h||_k(||u||_1 + ||v||_1).$$

So $B$ is bounded. By Proposition [14] $B$ is also coercive. Let $f \in H^0(\mathfrak{t}_P)$ be arbitrary. Then by the Lax-Milgram theorem (see Theorem 1 in Section 6.2.1 in [3]), there exists a unique $u \in H^0_1(\mathfrak{t}_P)$ such that $B(u, v) = \int_M < f, v >$ for all $v \in H^1_0$. The idea is that “$\Delta_A u = f$” (and if one considers $\Delta_A u \in H^{-1}$, then the quotes can be removed). We now want to show that our solution $u$ is also in $H^2$. To do this we will follow the proofs of Theorems 1 and 3 in Section 6.3 of [3] closely.

We will show that $u \in H^2(\mathfrak{t}_P)$ by showing that $u \in H^2(\mathfrak{t}_P|_{V_k})$ for a finite cover $\{V_k\}$ of $M$. First, let’s take an interior trivializing coordinate neighborhood $U$ of $M$. Let $D$ denote the flat connection on $U$. Then we set $\alpha := \nabla_A - D \in H^1_{\text{con}}(\Omega^1(\mathfrak{t}_P|U))$. Let $\alpha = \sum \alpha_i dx_i$. We also introduce the difference quotient $D^h_j g$ as

$$D^h_j g(x) = \frac{u(x + he_j) - u(x)}{h},$$

where $e_k$ is the $k^{th}$ Euclidean direction vector. Difference quotients behave much like derivatives. For example, they is a Leibniz-type rule:

$$D^h_j (vw) = v^h D^h_j w + (D^h_j v)w,$$

where $v^h(x) = v(x + he_j)$. They also satisfy the integral equality

$$\int_U < f, D^{-h}_j g > dx = - \int_U < (D^h_j f), g > dx$$  \hfill (2.20)
for $g$ with compact support in $U$. Here, $dx$ is the Lebesgue measure on $\mathbb{R}^n$, and we need Lebesgue measure for (2.20) since we need the fact that it is translation invariant. If $d\text{Vol}$ is the volume measure induced by the metric of $M$, then $d\text{Vol} = adx$, where $a := \sqrt{\det (g_{ij})}$. Since both $a$ and $1/a$ are bounded on $U$, integrability under these two measures is equivalent.

Take open sets $V, W$ such that $\bar{V} \subseteq W \subseteq \bar{W} \subseteq U$, and choose $\zeta : U \to [0,1]$ so that $\zeta|_V \equiv 1$ and $\text{supp}(\zeta) \subseteq W$. Set $v := -D_j^{-h}(\zeta^2 D_j^h u)$. Note that by our choice of $\zeta$, we can extend $v$ by 0 and have $v \in H^1_0(kp)$. Hence, we have

$$\int_U <d_A u, d_A v > d\text{Vol} = \int_U <f, v > d\text{Vol}. \quad (2.21)$$

Set

$$A := \int_U <Du + [\alpha, u], Dv > d\text{Vol} = \int_U <d_A u, Dv > d\text{Vol}$$

$$B := \int_U <f, v > d\text{Vol} - \int_U <Du, [\alpha, v] > d\text{Vol} - \int_U <[\alpha, u], [\alpha, v] > d\text{Vol}$$

$$= \int_U <f, v > d\text{Vol} - \int_U <d_A u, [\alpha, v] > d\text{Vol}.$$

Then (2.21) is equivalent to saying $A = B$. Now

$$A = - \sum_{i,l} \int_U <u_{x_i} + [\alpha_i, u], D_j^{-h}(\zeta^2 D_j^h u)_{x_l} > g^i d\text{Vol}$$
$$= \sum_{i,l} \int_U <D_j^h(g^i a(u_{x_i} + [\alpha_i, u])), (\zeta^2 D_j^h u)_{x_l} > dx$$
$$= \sum_{i,l} \int_U (<(g^i a)^h D_j^h(u_{x_i}), \zeta^2 D_j^h(u_{x_l}) > dx +$$
$$\sum_{i,l} \int_U (<(g^i a)^h D_j^h([\alpha_i, u]), \zeta^2 D_j^h(u_{x_l}) > +$$
$$<(g^i a)^h D_j^h(u_{x_i} + [\alpha_i, u]), 2\zeta \zeta x_i D_j^h u > +$$
$$<D_j^l(g^i a)(u_{x_i} + [\alpha_i, u]), \zeta^2 D_j^h(u_{x_l}) > +$$
$$<D_j^l(g^i a)(u_{x_i} + [\alpha_i, u]), 2\zeta \zeta x_i D_j^h u > dx).$$

Let the first sum of the last line equal $A_1$, and the second sum equal $A_2$. Since the metric matrix $\{g_{ij}\}$ is positive and continuous, there exists a constant $\tilde{\theta}$ such that

$$A_1 > \frac{\tilde{\theta}}{2} \int_U a^h \zeta^2 \sum_i |D_j^h(u_{x_i})|^2 dx$$
$$\geq \frac{\theta}{2} \int_U \zeta^2 \sum_i |D_j^h(u_{x_i})|^2 dx,$$
where $\theta = \hat{\theta} \cdot \inf a$. As for $A_2$, we have by Cauchy-Schwartz,

$$|A_2| \leq C \int_U \zeta (|D_j^h \sum_i [\alpha_i, u]| \cdot |D_j^h \sum_i u_x| + |D_j^h \sum_i u_x| \cdot |D_j^h u| +$$

$$|D_j^h \sum_i [\alpha_i, u]| \cdot |D_j^h u| + |\sum_i [\alpha_i, u]| \cdot |D_j^h u| + |\sum_i u_x| \cdot |D_j^h u| +$$

$$|\sum_i [\alpha_i, u]| \cdot |D_j^h \sum_i u_x| + |\sum_i u_x| \cdot |D_j^h u| + |\sum_i [\alpha_i, u]| \cdot |D_j^h u|)dx.$$

By the Peter-Paul inequality, for any $\epsilon > 0$, we have

$$|A_2| \leq \epsilon \int_U \zeta |D_j^h \sum_i u_x|^2 dx +$$

$$\frac{4}{\epsilon} \int_W |\sum_i u_x|^2 + |D_j^h u|^2 + |\sum_i [\alpha_i, u]|^2 + |D_j^h \sum_i [\alpha_i, u]|^2 dx.$$

As in Theorem 3(i) of Section 5.8.2 in [5], we have

$$\int_W |D_j^h u|^2 dx \leq C \int_U |\sum_i u_x|^2 dx.$$

Letting $\epsilon = \theta/2$, we have

$$|A_2| \leq \frac{\theta}{2} \int_U \zeta \sum_i |D_j^h u_x|^2 dx + C \int_U \sum_i |u_x|^2 + |[\alpha_i, u]|^2 + |[\alpha_i, u]_x|^2 dx.$$

So in sum, we have

$$A = A_1 + A_2 \geq A_1 - A_2$$

$$\geq \frac{\theta}{2} \int_U \zeta^2 \sum_i |D_j^h (u_x)|^2 dx - C \int_U \sum_i (|u_x|^2 + |[\alpha_i, u]|^2 + |[\alpha_i, u]_x|^2) dx.$$

Now we turn to $B$. We first look at the $L^2$ norm of $v$:

$$\int_U |v|^2 dx = \int_W |D_j^{-h} (\zeta^2 D_j^h u)|^2 dx$$

$$\leq C \int_U |\sum_i (\zeta^2 D_j^h u)_x|^2 dx$$

$$\leq C \int_U \zeta |D_j^h u|^2 + \sum_i \zeta |D_j^h u_x|^2 dx$$

$$\leq C \int_U \sum_i |u_x|^2 + \sum_i \zeta |D_j^h u_x|^2 dx.$$
we have

\[ |B| \leq C \int_U ([|f| + [\alpha \cdot Du]| + ||[\alpha \cdot [\alpha, u]]|) \cdot |v| dx \]

\[ \leq \epsilon \int_U |v|^2 dx + C/\epsilon \int_U |[\alpha \cdot Du]|^2 + ||[\alpha \cdot [\alpha, u]]|^2 dx \]

\[ \leq C\epsilon \int_U \sum_i \zeta[D_i^h u_{xi}]^2 dx + C/\epsilon \int_U |f|^2 + ||[\alpha \cdot Du]|^2 + ||[\alpha \cdot [\alpha, u]]|^2 + \sum_i |u_{xi}|^2 dx. \]

So choosing \( \epsilon = \theta/(4C) \), we combine the A and B inequalities to obtain

\[ (\int_V \sum_i |D_j^h u_{x_i}|^2 dx)^{1/2} \leq (\int_W \sum_i \zeta[D_j^h u_{x_i}]^2 dx)^{1/2} \]

\[ \leq C||f||_0 + ||[\alpha \cdot Du]|_0 + ||[\alpha \cdot [\alpha, u]]|_0 + \sum_i ||u_{xi}||_0 + \sum_i (||u_{xi}||_0 + ||[\alpha_i, u]||_0 + \sum_i ||[\alpha_i, u]_{xi}||_0) \]

\[ \leq C||f||_k (||f||_0 + ||u||_1), \]

where we used the fact that \( H^1 \) is a \( H^k \) module in the last line, and \( || \cdot ||_k = || \cdot ||_{H^k(\mathbb{T},|\cdot|)} \). Note that

\[ \alpha = \nabla A - D = (\nabla A - \nabla A_0) + (\nabla A_0 - D), \]

where the latter connection is smooth. So, \( ||\alpha||_{H^k(\mathbb{T},|\cdot|)} \leq C + ||\nabla A - \nabla A_0||_{H^k(\mathbb{T})} \). So by Theorem 5.8(ii) of Section 5.8.2 in [3], we have that \( u \in H^2(\mathbb{T},|\cdot|) \) and

\[ ||u||_{H^2(\mathbb{T},|\cdot|)} \leq C(1 + ||\nabla A - \nabla A_0||_k)(||f||_0 + ||u||_1), \tag{2.22} \]

where we have switched back to \( || \cdot ||_k = || \cdot ||_{H^k(\mathbb{T})} \). Hence, we can replace \( V \) above with any open set \( V \subseteq M \) such that \( \bar{V} \subseteq M \) and have (2.22) hold. So, now we can understand \( \Delta Au \) as an function defined a.e. on \( M \), and may deduce from Lemma 18 that \( \Delta Au = f \) a.e. and all second derivatives exist as a.e. defined functions on \( M \) (see the remark after Theorem 1 in Section 6.3 in [3]). This observation is important for the following boundary considerations.

Let \( U \) be a neighborhood of the boundary such that \( U \) is the open unit ball \( B_0(1) \) intersected with the upper half plane \( \{x_3 \geq 0\} \), and \( \{x_3 = 0\} \cap U \) is the boundary portion of \( U \). We define another cut-off function \( \zeta : \mathbb{R}^3 \to [0, 1] \) such that \( \zeta|_{B_0(1/2)} \equiv 1 \) and \( \text{supp}(\zeta) \subseteq B_0(3/4) \). Set \( V := B_0(1/2) \). For \( j \in \{1, \ldots, n-1\} \), we define \( v := -D_j^{-h}(\zeta^2 D_j^h (u)) \). One can check that

\[ v(x) = -\frac{1}{h^2}(\zeta^2(x - he_j)(u(x) - u(x - he_j)) - \zeta^2(x)(u(x + he_j) - u(x))) \]
for \( x \in U \). Careful inspection above reveals that since \( u|_{\partial M} = 0 \) in the trace sense, and \( \zeta \) vanishes near the boundary of the ball, after extending by 0 we have \( v \) in \( H^0_0(t_P) \). So,

\[
\int_U \frac{d_A u, d_A v}{d_{Vol}} = \int_U <f, v > d_{Vol}.
\]

By analogous estimates as the interior case, we have

\[
\int_U \sum_i |D^h u_{x_i}|^2 dx \leq C(1 + ||\nabla_A - \nabla_{A_0}||_k)^2(||f||_0 + ||u||_1)^2,
\]

and so \( u_{x_j x_j} \in H^2(t_P|_V) \) for all \( i, j \) such that \( i + j < 2 \cdot 3 \). So the we need only consider \( u_{x_3 x_3} \). Again, set \( \alpha := \nabla_A - D \), where \( D \) is the flat connection. Let \( \Delta = D^* D \). Then, recalling (2.11), we have

\[
u_{x_3 x_3} = \Delta u - u_{x_1 x_1} - u_{x_2 x_2} = f - [D^* h, u] + [\alpha \cdot [\alpha, u]] + 2[\alpha \cdot Du] - u_{x_1 x_1} - u_{x_2 x_2},
\]

where equality is a.e. Since everything on the right hand side is in \( L^2(t_P|_V) \), so is \( u_{x_3 x_3} \). Hence, \( u \in H^2(t_P|_V) \).

Since we can cover \( M \) with finitely many interior and boundary neighborhoods, we see that we have \( u \in H^2(t_P) \), as we desired. In sum, we have shown that given any \( f \in L^2(t_P) \), there exists a \( u \in H^2_{con}(t_P) \) such that \( \Delta_A u = f \). Our next job is to show that if \( f \in H^{m-1}(t_P) \), then \( u \in H^{m+1} \) for \( 1 \leq m \leq k \). Here we can use previous results, making things much easier.

The following argument is analogous to the one put forth in Section 3 of [16]. Again suppose our \( f \) from above is actually in \( H^{m-1}(t_P) \). Take any trivializing neighborhood \( U \) of \( M \) (interior or including the boundary). Using our previous notation, by Theorem 5 of Section 6.3 in [5] and the inequality following (2.11), we have

\[
||u||_{H^{m+1}(t_P|_U)} \leq C(||\Delta u||_{m-1} + ||u||_0)
\]

\[
\leq C(||\Delta_A u||_{m-1} + ||\alpha||_{L^2}||u||_{m-1} + ||\alpha||_{L^2}||u||_{m-1} + 2||\alpha||_{L^2}||u||_m + ||u||_0)
\]

\[
\leq C(||\Delta_A u||_{m-1} + ||u||_m + ||u||_0).
\]

where again \( ||\cdot||_i = ||\cdot||_{H^i(t_P|_U)} \). Interpolating on Sobolev norms, we have for any constant \( \epsilon > 0 \)

\[
||u||_m \leq \epsilon ||u||_{m+1} + C(\epsilon)||u||_0.
\]

So, choosing an appropriately small \( \epsilon \), we have

\[
||u||_{H^{m+1}(t_P|_U)} \leq C(||\Delta_A u||_{m-1} + ||u||_0).
\]

Summing over a finite cover of \( U \)'s, we see that \( u \in H^{m+1}_{con}(t_P) \). Hence, \( \Delta_A : H^{m+1}_{con}(t_P) \rightarrow H^{m-1}(t_P) \) is onto. So by the Open Mapping Theorem, \( \Delta_A \) is an isomorphism. (One can also use compactness of \( H^{m+1} \) in \( L^2 \) to directly show that the inverse is bounded, but we’ve done enough hard work for this proposition).
2.4 Proof of Theorem 13

Armed with the Lie algebra for $G^{k+1}_{con}$ and Green operators $G_A$ for all $\nabla_A \in \mathcal{C}^{k+1}_{con,A_0}$, we can now start proving Theorem 13.

In our proof of Theorem 13, we need to have some sort of slice lemma. Informally, a “slice” is a chunk of the $C^{k+1}_{con,A_0}$ to which our base $C^{k+1}_{con,A_0}/G^{k+1}_{con}$ will be locally diffeomorphic. Our slices will be modelled on the horizontal subspaces $H_A$ at each $K$-connection $\nabla_A$. The horizontal subspace $H_A$ is defined as

$$H_A = \{ \eta \in H^k_{con}(\Omega^1(t_P)) : d_A^*\eta = 0 \}.$$  

For our slice lemma we need to have the following: for sufficiently small $\eta \in H^k_{con,A_0}(t_P)$, there exists a unique gauge transformation $g \in G^{k+1}_{con}$ such that $(\nabla_A + \eta)g - \nabla_A \in H_A$. We would like to employ the implicit function theorem to find such gauge transformations, but clearly $G^{k+1}_{con}$ is not a Banach space. Instead, we consider the gauge algebra $H^{k+1}_{con}(t_P)$ and use it for the implicit function theorem.

We first prove the so-called “local completeness” of our action. This proof is from [11].

**Proposition 22.** Suppose $\nabla_A \in \mathcal{C}^{k+1}_{con,A_0}$. Then there exists $\epsilon > 0$ so that if $0 < \epsilon_1 \leq \epsilon$ there exists an $\epsilon_2 > 0$ such that for $\eta \in H^k_{con}(\Omega^1(t_P))$ with $\|\eta\|_{H^k} < \epsilon_1$ there exists a unique $g \in G^{k+1}_{con}$ such that $\|g - \epsilon\|_{H^{k+1}} < \epsilon_2$ and $(\nabla_A + \eta)g - \nabla_A \in H_A$. In other words, $d_A^*((\nabla_A + \eta)g - \nabla_A) = 0$. Furthermore, $\epsilon_2$ can be made arbitrarily small by making $\epsilon_1$ sufficiently small.

**Proof.** Consider the map $F : H^{k+1}_{con}(t_P) \times H^k_{con}(t_P) \to H^{k-1}_{con}(t_P)$ given by

$$F(X, \eta) = d_A^* (\exp(-X)d_A\exp(X) + \text{Ad}(\exp(-X))\eta). \quad (2.23)$$

$F$ is just a composition of linear and bilinear maps, and the exponential map. Thus, it is a smooth map of Banach spaces. Also, $F(0,0) = 0$. To calculate the first partial derivative of $F$ at $(0,0)$, note that $G(X) := F(X,0) = d_A^* (\exp(-X)d_A\exp(X))$. With the Baker-Campbell-Dynkin-Hausdorff formula, one can show that

$$D(\exp)(X)\xi = \left( \sum_{k=0}^{\infty} \frac{\text{ad}^k(X)\xi}{(k+1)!} \right) \exp(X).$$

So $D(\exp)(0)\xi = \xi$. Thus, by Chain Rule and Proposition 14 in [11], we have

$$DG(0)\xi = \ d^*_A((D\exp)(0)\xi \cdot d_A\exp(0) + \exp(-0)d_A(D\exp(0))\eta) \quad (2.24)$$

$$= \ d^*_A d_A\xi. \quad (2.25)$$

To apply the implicit function theorem, we need to show that $d_A^*d_A = \Delta_A : H^{k+1}_{con}(t_P) \to H^{k-1}_{con}(t_P)$ is an isomorphism. This is exactly the statement of Proposition 21. So applying the implicit function theorem we have: There exists a $C^\infty$ mapping $X : N_A \cap H^{k}_{con}(t_P) \to N_0 \cap H^{k+1}_{con}(t_P)$, where $N_A$ is a
neighborhood of 0 in $H^k_{\text{con}}(\mathfrak{p}_P)$ and $N_0$ is a neighborhood of 0 in $H^{k+1}_{\text{con}}(\mathfrak{p}_P)$ such that $X(\eta)$ is the unique member of $N_0$ satisfying

$$d^*_A(\exp(-X(\eta))d_A \exp(X(\eta)) + \text{Ad}(\exp(-X(\eta)))\eta) = 0.$$ 

By Proposition 16 $\exp : H^{k+1}_{\text{con}} \to G^{k+1}_{\text{con}}$ is a local diffeomorphism at 0. So, given a small enough $\eta$, setting $g := \exp(X(\eta))$ gives us the statement of the proposition. This may require a shrinking of the neighborhood $N_A$, but this is possible since $X$ is a continuous map. This last part of the proposition also follows from this continuity.

Now we want so-called “local effectiveness” of our action. This proof is again found in [1] and, more directly [18] but with simplifications:

**Proposition 23.** Suppose $\nabla_A \in \kappa_{\text{con},A_\emptyset}^k$. Then there exists an $\delta > 0$ so that if $\|\eta_1\|_{H^k}, \|\eta_2\|_{H^k} < \delta$, $\eta_1, \eta_2 \in H_A$ and $\eta_1 \neq \eta_2$ there exists no nontrivial $g \in G_{\text{con}}$ such that $(\nabla_A + \eta_1) \cdot g = (\nabla_A + \eta_2)$.

**Proof.** Suppose $\eta_1, \eta_2 \in H_A$ and $(\nabla_A + \eta_1) \cdot g = (\nabla_A + \eta_2)$, and $\|\eta_1\|_{H^k} < \delta$, for some $\delta > 0$ that is yet to be determined. Set $\nabla_1 = \nabla_A + \eta_1$. The idea here is to show that $\|g - e\|_{H^{k+1}}$ is small if both $\|\eta_1\|_{H^k}$ and $\|\eta_2\|_{H^k}$ are small, allowing us to invoke the uniqueness statement of Proposition 22. We have

$$\eta_2 = g^{-1} \nabla_A g + \text{Ad}(g^{-1})\eta_1$$

$$= g^{-1} \nabla_1 g.$$  

(2.26) (2.27)

Our main goal will be to show that $\|g - e\|_{H^{k+1}}$ is controlled by $\|g^{-1} \nabla_1 g\|_{H^k}$. First note that since $K$ is compact, we have a constant $\Omega' \geq 1$ so that

$$\sup_{k \in K} |k| \leq \Omega' < \infty$$

where the norm is induced by the trace inner product. Hence $\sup_{g \in G^{k+1}_{\text{con}}} \|g\|_{L^\infty} \leq \Omega' \cdot \text{Vol}M =: \Omega$, since the norm on $\text{End}(V)_P$ is induced by the trace inner product. Note that since $\nabla_1 \in H^k$, we can apply (2.8) to obtain

$$\|g - e\|_{L^2} \leq \kappa_2 \|\nabla_1 (g - e)\|_{L^2} \leq \kappa_2 \Omega \|g^{-1} \nabla_1 g\|_{L^2}.$$  

(2.28)

By Proposition 21 (note we could replace $\mathfrak{p}_P$ with $\text{End}(V)_P$ and nothing would change in the proof of Proposition 21), since $\nabla_1 \in H^k$, we have

$$\|g - e\|_{H^{k+1}} \leq C(\|\nabla_1^k \nabla_1 g\|_{H^{k-1}} + \|g - e\|_{L^2}).$$  

(2.29)

At first glance, one might expect that the constant $C$ above depends on $\eta_1$. However, this is not the case. Indeed, if we assume that $\delta < 1$ and thus $\|\eta_1\|_{H^k} < 1$, we can use the inequality of Proposition 21 with the connection $\nabla_A$ to get

$$\|g - e\|_{H^{k+1}} \leq C_A(\|\nabla_A^k \nabla_A (g - e)\|_{H^{k-1}} + \|g - e\|_{L^2})$$  

(2.30)
The inequality (2.17) with $\nabla A_0$ replaced by $\nabla A + \eta_1$ tells us that
\[
\|\nabla A \nabla A (g - e)\|_{H^{k-1}} \leq \|\nabla_1^* \nabla_1 (g - e)\|_{H^{k-1}} + C(\|\eta_1\|_{H^k} \|g - e\|_{H^{k-1}} + \|\eta_1\|_{H^k} ^2 \|g - e\|_{H^k}) + \|\eta_1\|_{H^k} ^2 \|g - e\|_{H^k} \leq \|\nabla_1^* \nabla_1 (g - e)\|_{H^{k-1}} + C\|g - e\|_{H^k},
\]
where we used the fact that $\|\eta_1\|_{H^k} < 1$ on the last line. Plugging the above into (2.30) yields
\[
\|g - e\|_{H^{k+1}} \leq C_A(\|\nabla_1^* \nabla_1 (g - e)\|_{H^{k-1}} + C\|g - e\|_{H^k} + \|g - e\|_{L^2}).
\]
A standard Sobolev norm interpolation then yields (2.29) with a constant $C$ that depends only on the fact that $\|\eta_1\|_{H^k} < 1$.

From (2.29) we have
\[
\|g - e\|_{H^{k+1}} \leq C(\|g\|_{H^{k+1}} \|g^{-1} \nabla_1^* \nabla_1 g\|_{H^{k-1}} + \|g - e\|_{L^2}) \leq C(\|g\|_{H^{k+1}} + \|e\|_{H^k}) \|g^{-1} \nabla_1^* \nabla_1 g\|_{H^{k-1}} + \|g - e\|_{L^2}) \leq C(\|g - e\|_{H^{k+1}} + \|g^{-1} \nabla_1^* \nabla_1 g\|_{H^{k-1}} + \|g - e\|_{L^2}) \leq C(\|g - e\|_{H^{k+1}} + \|g^{-1} \nabla_1^* \nabla_1 g\|_{H^{k-1}} + \|g - e\|_{L^2}).
\]
Note that
\[
\|g^{-1} \nabla_1^* \nabla_1 g\|_{H^{k-1}} \leq \|\nabla_1^* (g^{-1} \nabla_1 g)\|_{H^{k-1}} + \|\nabla_1 g^{-1} \cdot \nabla_1 g\|_{H^{k-1}} \leq C(\|g^{-1} \nabla_1 g\|_{H^k} + \|\nabla_1 g^{-1} \|_{H^{k-1}} \|g^{-1} \nabla_1 g\|_{H^k}) \leq C(\|g^{-1} \nabla_1 g\|_{H^k} + \|g^{-1} \nabla_1 g\|_{H^k} ^2).
\]
In the above, we used the fact that $0 = \nabla_1 (g^{-1} g) = (\nabla_1 g^{-1}) g + g^{-1} \nabla_1 g$ and Proposition 4. Also, a priori the constant $C$ should depend on $\eta_1$. However, using reasoning similar to that which we used to show (2.29) tells us that $C$ depends only on the fact that $\|\eta_1\|_{H^k} < 1$. So assuming that $\delta < 1$, then by (2.27) we have $\|g^{-1} \nabla_1 g\|_{H^k} < 1$. So we can remove the $\|g^{-1} \nabla_1 g\|_{H^k} ^2$ term above. Using the above and interpolation of Sobolev norms for $\|g - e\|_{H^{k+1}}$, we continue our inequality of (2.35) with any $\epsilon_1 > 0$:
\[
\|g - e\|_{H^{k+1}} \leq C(\|g - e\|_{H^{k+1}} + 1) \|g^{-1} \nabla_1 g\|_{H^k} \leq \epsilon_1 \|g - e\|_{H^{k+1}} \|g^{-1} \nabla_1 g\|_{H^k} + C(\epsilon_1) \|g^{-1} \nabla_1 g\|_{H^k}
\]
Let’s now assume that $\delta < 1/2$, which implies $\|g^{-1} \nabla_1 g\|_{H^k} < 1/2$. So taking $\epsilon_1 = 1$, the above can be written as
\[
\|g - e\|_{H^{k+1}} \leq 2C(1) \delta.
\]
Thus, taking $1 > \epsilon > 0$ as in Proposition 4, we can choose
\[
\delta < \min(\epsilon/(2C(1)), 1/2, \epsilon).
\]
Then by (2.36), we have $\|g - e\|_{H^{k+1}} < \epsilon$, so the uniqueness statement of Proposition 2 applies. Since $\eta_1, \eta_2 \in H_A$, this uniqueness tells us that $g \equiv e$. □
Now we can show that $C^k_{\text{con},A_0}/G_{\text{con}}^{k+1}$ is a Hilbert manifold and prove local triviality of the bundle $C^k_{\text{con},A_0} \to C^k_{\text{con},A_0}/G_{\text{con}}^{k+1}$. Most of the following is exactly from [15] including most notation.

Proof of Theorem 13. Our quotient space $C^k_{\text{con},A_0}/G_{\text{con}}^{k+1}$ first needs a topology. We give it the quotient topology under the projection $\pi : C^k_{\text{con},A_0} \to C^k_{\text{con},A_0}/G_{\text{con}}^{k+1}$.

Fix a connection $\nabla_A \in C^k_{\text{con},A_0}$ and consider the mapping $F : H_{\text{con}}^k \times G_{\text{con}}^{k+1} \to H_{\text{con}}^k$ given by

$$F(\eta, g) = (\nabla_A + \eta) \cdot g - \nabla_A.$$  \hspace{1cm} (2.37)

Then $F$ is continuous and $F(0, e) = 0$. Hence, for a ball $B_3(A) := \{ \nabla_A + \eta : \| \eta \|_{H_k} < \epsilon \}$, there exists $\epsilon_1 > 0$ and $\epsilon_2 > 0$ so that for $\eta \in H_{\text{con}}^k$ and $g \in G_{\text{con}}^{k+1}$ such that $\| \eta \|_{H_k} < \epsilon_1$ and $\| g - e \|_{H_{k+1}} < \epsilon_2$, then $F(\eta, g) \in B_3(A)$. Set $\delta > 0$ to the $\delta$ in Proposition 25. Set $\epsilon_1 > 0$ so that it is less than $\min(\delta, \epsilon_1)$, the $\epsilon$ in Proposition 22 and so that the corresponding $\epsilon_2$ in Proposition 22 is less than $\epsilon_2$. Set $\pi : C^k_{\text{con},A_0} \to C^k_{\text{con},A_0}/G_{\text{con}}^{k+1}$ and $Q_A := \pi(B_3(A))$. Consider the restriction $\pi_A := \pi|_{S_A}$, where

$$S_A := \pi^{-1}(Q_A) \cap B_3(A) \cap (\nabla_A + H_A).$$

Clearly $\pi_A$ maps into $Q_A$. We now show that this mapping is onto. Given an equivalence class $[\nabla_{A'}] \in Q_A$, we can assume without loss of generality that $\nabla_{A'} \in B_3(A)$. By Proposition 25 there exists $g \in G_{\text{con}}^{k+1}$ with $\| g - e \|_{H_{k+1}} < \epsilon_2$ and $F(\nabla_{A'} - \nabla_A, g) \in H_A$. Since $\nabla_{A'} \in B_3(A)$ and $\| g - e \|_{H_{k+1}} < \epsilon_2 < \epsilon_2$, we see that

$$\nabla_{A'} \cdot g = \nabla_A + F(\nabla_{A'} - \nabla_A, g) \in B_3(A).$$

Hence, $\nabla_{A'} \cdot g \in S_A$ and $\pi_A(\nabla_{A'} \cdot g) = [A']$. So $\pi_A$ maps onto $Q_A$. Suppose $\nabla_{A_1}, \nabla_{A_2}$ are in the domain of $\pi_A$ and $\pi_A(\nabla_{A_1}) = \pi_A(\nabla_{A_2})$. Then there exists $g \in G_{\text{con}}^{k+1}$ such that $\nabla_{A_1} \cdot g = \nabla_{A_2}$. Since $\nabla_{A_1}, \nabla_{A_2} \in B_3(\epsilon)$, we can apply Proposition 25 to conclude that $\nabla_{A_1} = \nabla_{A_2}$. Hence, $\pi_A$ is injective. Since $Q_A$ has the quotient topology, the bijectivity of $\pi_A$ shows that it is a homeomorphism. We will call its inverse $\sigma_A : Q_A \to S_A$.

We get a Hilbert manifold chart $\phi_A : Q_A \to (S_A - \nabla_A) \subseteq H_A$ given by $\phi_A([\nabla_{A'}]) = \sigma_A([\nabla_{A'}]) - \nabla_A$. It is easy to see that $(S_A - \nabla_A)$ is an open subset of $H_A$. The next step is to show that coordinate changes are smooth. To this end, we define a map $g_A : \pi^{-1}(Q_A) \to G_{\text{con}}^{k+1}$ as follows: $g_A(\nabla_{A'})$ is the unique element of $G_{\text{con}}^{k+1}$ so that

$$\nabla_{A'} \cdot g_A(\nabla_{A'})^{-1} = \sigma_A([\nabla_{A'}]).$$  \hspace{1cm} (2.38)

$g_A(\nabla_{A'})$ exists and is unique by Corollary 15 and Propositions 25 and 26. If $[\nabla_{A'}] \in Q_{A_1} \cap Q_{A_2}$, then we compute from (2.38) that

$$\phi_{A_1}([\nabla_{A'}]) = \sigma_{A_1}([\nabla_{A'}]) - \nabla_{A_1} = \sigma_{A_1}([\sigma_{A_2}([\nabla_{A'}])] - \nabla_{A_1}$$  \hspace{1cm} (2.39)

$$\phi_{A_2}([\nabla_{A'}]) \cdot g_A(\nabla_{A'})^{-1} - \nabla_{A_1}$$  \hspace{1cm} (2.40)

$$= (\nabla_{A_2} + \phi_{A_2}([\nabla_{A'}])) \cdot g_A(\nabla_{A'}).$$  \hspace{1cm} (2.41)

$$g_A(\nabla_{A_1} + (\nabla_{A_2} - \nabla_{A_1} + \phi_{A_2}([\nabla_{A'}]))^{-1} - \nabla_{A_1}.$$  \hspace{1cm} (2.42)
Since we know that gauge transformations are smooth, we need only show that \( g_A \) is smooth for all \( \nabla_A \in C^k_{\text{con},A_0} \) to show that this coordinate change is smooth. This will come in the proof of local triviality of the quotient \( \pi C^k_{\text{con},A_0} \rightarrow C^k_{\text{con},A_0}/G^{k+1}_\text{con} \), which follows.

Note that when we show that the coordinate change is smooth, we will have a smooth map \( \Phi_{A_2} \circ \Phi_{A_1}^{-1} \) from an open subset of \( H_{A_1} \) to an open subset of \( H_{A_2} \). The first derivative of \( \Phi_{A_2} \circ \Phi_{A_1}^{-1} \) will thus provide an isomorphism from \( H_{A_1} \) to \( H_{A_2} \).

We want to show that a certain map \( \Phi_A : Q_A \times G^{k+1}_{\text{con}} \rightarrow \pi^{-1}(Q_A) \) is a smooth diffeomorphism. This map is given by \( \Phi_A((\nabla_{A'}), g) = \sigma_A((\nabla_{A'})) \cdot g \).

Since gauge transformations are smooth, we see that \( \Phi_A \) is smooth. Also, \( \Phi_A \) is a bijection with the inverse \( \Phi_A^{-1}(\nabla_{A'}) = (\pi(A'), \kappa(A)(\nabla_{A'})) \). So, if we can show that \( \Phi_A^{-1} \) is smooth, then \( g_A \) will also be smooth making our coordinate change map \( 2.42 \) smooth. To consider the smoothness of \( \Phi_A^{-1} \), we will look at \( \Phi_A \) at coordinates and show that \( \Phi_A \) is a local diffeomorphism at all points.

We know that \( Q_A \) is diffeomorphic to an open neighborhood \( S_A := S_A - \nabla_A \) in \( H_A \) (since we haven’t shown that \( C^k_{\text{con},A_0}/G^{k+1}_{\text{con}} \) is a manifold yet, to be correct we should replace \( Q_A \) in the domain of \( \Phi_A \) with \( S_A \), prove smoothness of the inverse which then gives us that \( C^k_{\text{con},A_0}/G^{k+1}_{\text{con}} \) is a manifold, and then replace \( S_A \) with \( Q_A \) to give local triviality. To avoid this extra confusing layer, we sweep this detail under the rug.) Given a fixed \( g \in G^{k+1}_{\text{con}} \), we have a neighborhood \( M_g \) of the form \( M_g = \{ \exp(\xi) \cdot g : \xi \in H^{k+1}_{\text{con}}(\mathfrak{h}), \|\xi\|_{H^{k+1}} < \epsilon \} \). The set \( V_\epsilon(0) = \{ \xi \in H^{k+1}_{\text{con}}(\mathfrak{h}), \|\xi\|_{H^{k+1}} < \epsilon \} \) then provides coordinates for \( M_g \). Finally, \( \pi^{-1}(Q_A) \) has coordinates under the mapping \( \nabla_{A'} \mapsto \nabla_{A'} - \nabla_A \). So, we can rewrite \( \Phi_A : S_A \times V_\epsilon(0) \rightarrow \pi^{-1}(Q_A) - \nabla_A \subseteq H^{k}_{\text{con}}(\mathfrak{h}) \) as

\[
\Phi_A(\tau, \xi) = g^{-1} \exp(-\xi) \nabla_A^\text{Hom}(\exp(\xi)g) + \text{Ad}(g^{-1} \exp(-\xi))(\tau) = g^{-1} \exp(-\xi)(\nabla_A^\text{Hom}(\exp(\xi))g + g^{-1} \exp(-\xi) \exp(\xi)(\nabla_A^\text{Hom}g) + \text{Ad}(g^{-1} \exp(-\xi))(\tau) = \text{Ad}(g^{-1})(\exp(-\xi)(\nabla_A^\text{Hom}(\exp(\xi)) + \text{Ad}(\exp(-\xi))(\tau)) + g^{-1} \nabla_A^\text{Hom}g.
\]

To use the inverse function theorem, we want to show that \( (\Phi_A)_* \) is invertible at all points. Fixing a \( g \in G^{k+1}_{\text{con}} \), and using the coordinates of \( M_g \), we need only to consider the invertibility of \( (\Phi_A)_*(\tau, 0) \). Since \( \Phi_A \) restricted to the first variable is affine, we have

\[
(\Phi_A)_*(\tau, 0)(\eta) = \text{Ad}(g^{-1})(\eta).
\]

By a calculation similar to \( 2.22 \), we have

\[
(\Phi_A)_*(\tau, 0)(h) = \text{Ad}(g^{-1})(\nabla_A^\text{Hom}h + -h\tau + \tau h) = \text{Ad}(g^{-1})(\nabla_A^\text{Hom}h),
\]

where \( \nabla_{A'} = \nabla_A + \tau \). Adding up our partial derivatives yields

\[
(\Phi_A)_*(\tau, 0)(\eta, h) = \text{Ad}(g^{-1})(\eta + \nabla_A^\text{Hom}h). \tag{2.43}
\]
To show \((\Phi_A)_{\ast}(\tau,0)\) is an isomorphism, we will first show that it has trivial kernel, and then show it is onto. Also, we will drop the “Hom” from \(\nabla^i_H\).

Suppose \((\Phi_A)_{\ast}(\tau,0)(\eta,h) = 0\). Since \(\nabla^i_A\eta = 0\), we have from (2.38)

\[
\Delta_A h + \nabla^i_A [\tau, h] = \nabla^i_A(\nabla_A h + [\tau, h]) = \nabla^i_A(\nabla_A h + \eta) = \text{Ad}(g)(\Phi_A)_{\ast}(\tau,0)(\eta,h) = 0.
\]

Applying the Green operator \(G_A\) to the above yields

\[
h + G_A \nabla^i_A [\tau, h] = 0.
\]

By the boundedness of \(G_A : H^{k-1}(\mathfrak{p}_P) \to H^{k+1}_{con}(\mathfrak{p}_P)\) and \(\nabla^i_A : H^k(\mathfrak{p}_P) \to H^{k-1}(\mathfrak{p}_P)\), we have

\[
\|h\|_{H^{k+1}} \leq C\|\nabla^i_A [\tau, h]\|_{H^{k-1}} \leq C\|[\tau, h]\|_{H^k} \leq C\|\tau\|_{H^k}\|h\|_{H^{k+1}}.
\]

For small enough \(\tau\), the above implies that \(h = 0\), which in turn implies \(\eta = 0\). Thus, for small enough \(\tau\), \(\ker(\Phi_A)_{\ast}(\tau,0)\) is 0.

Now we can move onto surjectivity. Define a map \(P_{A'} : H^k_{con}(\Omega^1(\mathfrak{p}_P)) \to H_{A'}\) as \(P_{A'}(\omega) = (1 - \nabla_A G_A \nabla^i_{A'})(\omega)\). We can rewrite (2.43) as

\[
(\Phi_A)_{\ast}(\tau,0)(\eta,h) = \text{Ad}(g^{-1})(\nabla_A (h + G_A \nabla^i_A \eta) + P_{A'}\eta).
\]

We have written \((\Phi_A)_{\ast}(\tau,0)\) in the form \((\Phi_A)_{\ast}(\tau,0) : \tilde{\mathcal{S}}_A \times V_c(0) \to \text{Ad}(g^{-1})(\text{Im}(\nabla_A') \oplus H_{A'}\}\). It is easy to see that \(\text{Im}(\nabla_A') \oplus H_{A'}\) is indeed a direct sum and that \(\text{Im}(\nabla_A') \oplus H_{A'} = H^k_{con}(\Omega^1(\mathfrak{p}_P))\) (see, for example, [8] and [10]). Note that since \(g \in \mathcal{G}^k_{con}\), \(\text{Ad}(g^{-1})\) maps \(H^k_{con}\) to itself isomorphically. Hence, to prove surjectivity, we must show that for every \(h_0 \in H^k_{con}\) and \(\eta_0 \in H_{A'}\), we have a (unique) solution to

\[
(\Phi_A)_{\ast}(\tau,0)(\eta,h) = \text{Ad}(g^{-1})(d_{A'}h_0 + \eta_0).
\]

Consider the function \(H : H^k_{con}(\mathfrak{p}_P) \oplus H_{A} \oplus \tilde{\mathcal{S}}_A \to H^k_{con}(\mathfrak{p}_P)\) given by

\[
H(\tau, \eta_0, \eta) = \eta - d_{A'} G_A [\tau \cdot \eta] - \eta_0.
\]

Note that \(H(0,0,0) = 0\), \(H\) is continuous and linear in the last two variables, and \(H_{-3(0,0,0)} = -\text{Identity}\). One can also show that \(H\) is \(C^1\) (see [8] for example). So the implicit function theorem says that there exists an \(\epsilon > 0\) so if \(\|\tau\|_{H^k},\|\eta_0\|_{H^k} < \epsilon\), there exists an \(\eta(\tau,\eta_0)\) such that \(H(\tau, \eta_0, \eta(\tau,\eta_0)) = 0\). Let \(\eta_0 \in H_{A}\) be arbitrary, and \(\tau \in H^k_{con}(\mathfrak{p}_P)\) so that \(\|\tau\|_{H^k} < \epsilon\). Choose \(N > 0\) so that \(\|(1/N)\eta_0\|_{H^k} < \epsilon\). Then using linearity in the last two variables we have

\[
H(\tau, \eta_0, N\eta(\tau,(1/N)\eta_0)) = 0.
\]

Since \(\eta := N\eta(\tau,(1/N)\eta_0) \in H_{A}\), we have

\[
\eta_0 = \eta - d_{A'} G_A d_{A'}^\ast \eta = P_{A'}(\eta).
\]
So we have a solution for $\eta_0$. Now set $h$ to

$$h := h_0 - G_A' d_A' \eta.$$

Since $h_0$ satisfies boundary conditions and $G_A'$ maps $H^{k-1}$ into $H_{con}^{k+1}(t_P)$, we have $h \in H_{con}^{k+1}(t_P)$. Furthermore

$$d_A' h_0 = d_A'(h + G_A' d_A' \eta).$$

Thus, by (2.44), we have found a solution to $(\Phi_A)_{*,(\tau,0)}(\eta, h) = \text{Ad}(g^{-1})(d_A' h_0 + \eta_0)$. Thus we have surjectivity for small $\tau$. Hence $\Phi_A$ is a local diffeomorphism at all points, and therefore a diffeomorphism. Local triviality is thus proven, and we have finally shown that $C_{con,A_0}^k/G_{con}^{k+1}$ is a Hilbert manifold and $C_{con,A_0}^k \rightarrow C_{con,A_0}^k/G_{con}^{k+1}$ is a principal bundle. \qed
Chapter 3

The Holonomy of the Coulomb Connection

Now that we know that the bundle $C^k_{\text{con},A_0} \rightarrow C^k_{\text{con},A_0}/G^{k+1}_{\text{con}}$ is a principal bundle, we can consider holonomy group. The connection we will consider is the called the Coulomb connection whose connection form at $\nabla_A$ is defined as $G_A d^*$. Then the corresponding horizontal at $\nabla_A$ is $H_A$. Recall the definition of $H_A$ as

$$H_A = \{ \alpha \in H^k_{\text{con}}(\Omega_1^1(kP)) : d^*_A \alpha = 0 \}.$$ 

This connection is natural in the sense that $H_A$ is the $L^2$ orthogonal complement to the vertical vectors at $\nabla_A$. Indeed, one can show that given $\gamma \in \text{Lie}(G^{k+1}_{\text{con}}) = H^{k+1}_{\text{con}}(\Omega_1^1(t_F))$, the fundamental vector field associated to $\gamma$ is $d_A \gamma$. Hence the vertical vectors are those vectors of the form $d_A \gamma$ for some $\gamma \in \text{Lie}(G^{k+1}_{\text{con}})$ (see [8] or [16]). By the same reasoning as the proof of Lemma 7.1 in [16], the Coulomb connection is indeed a connection on $C^k_{\text{con},A_0} \rightarrow C^k_{\text{con},A_0}/G^{k+1}_{\text{con}}$.

We begin our investigation of the holonomy group by considering the image of the curvature form $\Omega$ of the Coulomb connection. Let $R_A$ be the curvature form of the Coulomb connection at $\nabla_A$. By the same calculation in the proof of Lemma 7.2 in [16], we have

$$R_A(\alpha, \beta) = -2G_A([\alpha \cdot \beta]), \text{ for } \alpha, \beta \in H_A. \quad (3.1)$$

In this investigation, certain types of coordinates at the boundary are useful, and are the subject of the next section.

3.1 Coordinates at the Boundary

Consider the following system of coordinates at the boundary that satisfy the following:

A1. $\partial/\partial x_n$ is orthogonal to $\partial/\partial x_1, \ldots, \partial/\partial x_{n-1}$ on the boundary.
A2. \( \partial/\partial x_n \) has norm 1 everywhere.

A3. \( \partial/\partial x_n \) is the inward pointing normal vector on the boundary.

We describe such a coordinate system as Type A. Fortunately, this definition is not in vain, as such coordinates always exist:

**Proposition 24.** Let \( M \) be a Riemannian \( n \)-manifold with boundary. A coordinate system \( \{x_1, \ldots, x_n\} \) satisfying A1-A3 above exists around each point of the boundary of \( M \).

**Proof.** The following construction is based on [14] and [19], and [19] uses this type of coordinates. Let \( p \) be a point on the boundary. Take a chart on the set \( U \) near \( p \) with coordinates \((y_1, \ldots, y_n)\) and image in the upper half space so that \( y_n^{-1}(0) \cap U = \partial M \cap U \). Then the function \( u(y_1, \ldots, y_n) := y_n \) satisfies \( u^{-1}(0) = \partial M \cap U \). Let \( \nu \) be the inward pointing normal. For any point \( p' \in \partial M \), \( d_{p'}u(\nu) = c \cdot \partial/\partial y_n u = c > 0 \) where \( c = ||\partial/\partial y_n||^{-1} \). Let \( X \) be a local vector field that is dual to the 1-form \( du \), i.e. \( X = \text{grad } u \). Since \( u \) has no critical points, \( X \) never vanishes. So we may set \( Y = X/(||X||) \). Then, as in [14], we may consider the flow of \( Y \), denoted \( \Phi \). As in [19], \( \Phi : [0, \delta) \times \partial M \cap U \to U \) is a diffeomorphism for some \( \delta > 0 \). We now define coordinates via this diffeomorphism: Let \( x_i : \partial M \to \mathbb{R} \) be coordinates for \( \partial M \cap U \), with inverse \( \psi \). Define \( x_i : U \to \mathbb{R} \) as \( x_i(\Phi(t, q)) = x_i(q) \). and define \( x_n : U \to \mathbb{R} \) as \( x_n(\Phi(t, q)) = t \).

These \( \{x_1, \ldots, x_n\} \) are coordinates on \( U \). Now, note that

\[
\frac{d}{dt}(u \circ \Phi_t(q))|_{t=t_0} = du(X/(||X||))(\Phi_{t_0}(q)) \\
= (X, X/(||X||))(\Phi_{t_0}(q)) = ||X||(\Phi_{t_0}(q)).
\]

Hence, since \( u \circ \Phi_0(q) = u(q) = 0 \), we have

\[
u(\Phi_t(q)) = \int_0^t ||X||(\Phi_s(q))ds.
\]

Using the above, we see that on the boundary,

\[
du \left( \frac{\partial}{\partial x_i} \right) = 0 = dx_n \left( \frac{\partial}{\partial x_i} \right), \text{ for } i < n
\]

\[
du \left( \frac{\partial}{\partial x_n} \right) = ||X|| = ||X||dx_n \left( \frac{\partial}{\partial x_n} \right).
\]

Hence,

\[
du|_{\partial M} = ||X|| \cdot dx_n|_{\partial M}.
\]
Now we can start showing that our properties are satisfied. For a function \( f \) on \( U \)
\[
\frac{\partial}{\partial x_n} f = \frac{d}{dt} f \circ \Phi(t, \psi(x_1, \ldots, x_{n-1})) = \frac{X}{||X||} \cdot f.
\]
Thus, \( \frac{\partial}{\partial x_n} = \frac{X}{||X||} \). In particular, \( ||\frac{\partial}{\partial x_n}|| = 1 \), satisfying property A2 above. For \( i < n \), on the boundary we have by \( 3.2 \)
\[
0 = \frac{d}{dt}(x_i(q)) = \frac{d}{dt}(x_i(\Phi(t, q))) = \frac{1}{||X||} dx_i(\text{grad } u)
\]
\[
= \frac{1}{||X||} du(\text{grad } x_i)
\]
\[
= dx_n(\text{grad } x_i) = (\text{grad } x_n, \text{grad } x_i)
\]
\[
= g^{in}.
\]
Since \( \{g^{ij}\} \) is a symmetric matrix, this implies that \( g_{in} = 0 \) on the boundary, proving property A1 above. As for property A3, since \( g_{in} = 0 \), \( \frac{\partial}{\partial x_n} \) is normal to the boundary. So it is either inward or outward pointing. Since \( (x_1, \ldots, x_n) \) is a chart on the upper half plane, we have that \( \frac{\partial}{\partial x_n} \) is inward pointing by definition, completing the proof. \( \square \)

We will also have the occasion to use a slightly different type of coordinates on the boundary. If in the proof of Proposition 24 we instead let \( Y = X/||\text{grad } X||^2 \) where \( X = \text{grad } u \), and let \( \Psi \) be the corresponding flow, then
\[
\frac{d}{dt}(u \circ \Psi(t, q)) = du(X/||X||^2) = (X, X/||X||) = 1.
\]
So \( u(\Psi(t, q)) = t \). So we can let \( y_n = u \) and \( y_i(\Psi(t, q)) := y_i(q) \) \( i = 1, \ldots, n-1 \), where \( (y_1, \ldots, y_{n-1}) \) is a chart on the boundary. Then following reasoning similar to the proof of Proposition 24 gives us coordinates that satisfy

B1. \( \partial/\partial y_n \) is orthogonal to \( \partial/\partial y_1, \ldots, \partial/\partial y_{n-1} \) everywhere.

B2. \( \partial/\partial y_n \) is a positive (perhaps nonconstant) multiple of the inward pointing normal.

The last condition follows from the fact that \( du(\nu) > 0 \) on the boundary. We creatively describe such coordinates as Type B. In what follows, we will use \( \{x_1, x_2, x_3\} \) to denote Type A coordinates and \( g_{ij} \) to denote the associated metric tensor. For Type B coordinates, we use \( \{y_1, y_2, y_3\} \) and \( \{h_{ij}\} \).

Also, if \( h_{ij} \) is the metric tensor of a Type B coordinate system, note that by condition B1 we have
\[
h_{33} = \frac{1}{h^{33}} = \frac{1}{||\text{grad } u||^2}.
\]
3.2 Mean Curvature

It turns out that the mean curvature of the boundary comes into play in our characterization of the image of the curvature form. We give a brief explanation of mean curvature and calculate it using Type A and Type B coordinates. We use $\mathcal{H}$ for this background. Let $f : S \rightarrow M$ be an immersed submanifold, and let $\nabla$ be the Levi-Civita connection on $M$. The second fundamental form $B$ is a mapping $B : T_pS \times T_pS \rightarrow (T_pS)^\perp$ given by

$$B(x, y) := (\nabla_x Y)^N,$$

where $Y$ is any local extension of $y$, and $Z^N$ is the normal component of a vector $Z \in T_pM$ with respect to $S$. While not immediately apparent, it one can verify that $B$ is well-defined, symmetric, and a bilinear mapping of $C^\infty(S)$ modules (see [4]). Given a fixed $\eta \in (T_pS)^\perp$, the Riesz representation theorem gives us a mapping $S_\eta : T_pS \rightarrow T_pS$ satisfying

$$(S_\eta(x), y) = (B(x, y), \eta).$$

One can show

$$S_\eta(x) = -(\nabla_x \tilde{\eta})^T,$$

where $\tilde{\eta}$ is a local extension of $\eta$, and $Z^T$ is the tangent component of a vector $Z \in T_pM$.

The trace of this operator $S_\eta$ is important. $S$ is called minimal if $\text{tr}(S_\eta) = 0$ for all $\eta \in (T_pS)^\perp$ and $p \in S$. If $S$ is an oriented hypersurface and $\eta \in (T_pS)^\perp$ has norm 1 and is pointing in the direction corresponding to the orientation, then

$$H := \frac{1}{\dim(S)} \text{tr}(S_\eta)$$

is called the mean curvature of $f$. For us, the relevant immersion is $\iota : \partial M \rightarrow M$ and the normal vector will be the outward pointing normal which we denote $\nu$ (so the inward pointing normal is still $\nu$).

A certain quantity will come up often when working with Type A coordinates on our 3-manifold $M$. Let $\{g_{ij}\}$ be the metric tensor in a Type A coordinate system, and let $a = \sqrt{\det(g_{ij})}$. Note that $a$ never vanishes. Then we can consider the function on the boundary

$$\tau(x) = \frac{\partial a}{\partial x_3}(x) \cdot \frac{1}{a(x)}.$$

Two natural questions now enter one’s mind: is this $\tau$ globally well-defined, and what does this have to do with mean curvature?

**Proposition 25.** Consider the immersion $\iota : \partial M \rightarrow M$. Then the mean curvature $H$ satisfies

$$H = \frac{1}{2} \tau.$$
Since mean curvature is globally defined (on \( \partial M \)), so is \( \tau \).

**Proof.** Let \( \{ x_1, x_2, x_3 \} \) be Type A coordinates at the boundary, and let \( g_{ij} \) and \( \Gamma^m_{ij} \) be the corresponding metric tensor and Christoffel symbols, respectively. Since the connection we are considering is the Levi-Civita connection, we have

\[
\Gamma^m_{ij} = \frac{1}{2} \sum_k \left( \frac{\partial}{\partial x^i} (g_{jk}) + \frac{\partial}{\partial x^j} (g_{ik}) - \frac{\partial}{\partial x^k} (g_{ij}) \right) g^{km}.
\]

(see, for example, [4]). By our choice of coordinate system, we have \( g_{13} = g_{23} = g^{13} = g^{23} = 0 \) on the boundary and \( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \) are tangent to the boundary. So on the boundary,

\[
\Gamma^1_{13} = \frac{1}{2} \sum_k \left( \frac{\partial}{\partial x_1} (g_{3k}) + \frac{\partial}{\partial x_3} (g_{1k}) - \frac{\partial}{\partial x_k} (g_{13}) \right) g^{k1} \quad (3.9)
\]

\[
= \frac{1}{2} \left( \frac{\partial}{\partial x_1} (g_{31}) + \frac{\partial}{\partial x_3} (g_{11}) - \frac{\partial}{\partial x_1} (g_{13}) \right) g^{11} + \frac{1}{2} \left( \frac{\partial}{\partial x_1} (g_{32}) + \frac{\partial}{\partial x_3} (g_{12}) - \frac{\partial}{\partial x_1} (g_{13}) \right) g^{21} \quad (3.10)
\]

\[
= \frac{1}{2} \left( \frac{\partial}{\partial x_3} (g_{11}) g^{11} + \frac{\partial}{\partial x_3} (g_{12}) g^{21} \right) \quad (3.11)
\]

A similar calculation yields

\[
\Gamma^2_{23} = \frac{1}{2} \left( \frac{\partial}{\partial x_3} (g_{11}) g^{12} + \frac{\partial}{\partial x_3} (g_{22}) g^{22} \right) \quad (3.13)
\]

Now, on the boundary, note that

\[
g^{11} = \frac{g_{22}}{\det(g_{ij})}, \quad g^{22} = \frac{g_{11}}{\det(g_{ij})}, \quad g^{12} = g^{21} = \frac{-g_{12}}{\det(g_{ij})} \quad (3.14)
\]

Using a Laplace expansion on the bottom row of \( (g_{ij}) \), we also have

\[
\det(g_{ij}) = g_{31} \cdot \begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{vmatrix} - g_{32} \cdot \begin{vmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{vmatrix} + g_{33} \cdot \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}.
\]

On the boundary,

\[
g_{31} = \begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{vmatrix} = g_{32} = \begin{vmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{vmatrix} = 0.
\]

Since \( g_{33} \equiv 1 \) everywhere, we have by product rule

\[
\frac{\partial}{\partial x_3} \det(g_{ij}) = \frac{\partial}{\partial x_3} \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \text{ on } \partial M. \quad (3.15)
\]

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Combining (3.12), (3.13), (3.14), and (3.15), we have
\[
\Gamma_{13}^1 + \Gamma_{23}^2 = \frac{1}{2 \det(g_{ij})} \left( \frac{\partial}{\partial x_3} (g_{11} g_{22} + \frac{\partial}{\partial x_3} (g_{22} g_{11} - 2 \frac{\partial}{\partial x_3} (g_{12} g_{12})) \right)
\]
\[
= \frac{1}{2 \det(g_{ij})} \frac{\partial}{\partial x_3} \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix}
\]
\[
= \frac{1}{2 \det(g_{ij})} \frac{\partial}{\partial x_3} (\det(g_{ij}))
\]
\[
= \tau.
\]
Hence,
\[
H = \frac{1}{2} \text{tr} \left( S_{\nu} \right) = -\frac{1}{2} \left( (\nabla_{\nu_x} (\nu_{\gamma_x} (-\frac{\partial}{\partial x_3}))_1 + (\nabla_{\nu_x} (\nu_{\gamma_x} (-\frac{\partial}{\partial x_3}))_2 \right)
\]
\[
= \frac{1}{2} (\Gamma_{13}^1 + \Gamma_{23}^2)
\]
\[
= \frac{1}{2} \tau,
\]
as desired. □

We can also write \( \tau \) in terms of Type B coordinates:

**Lemma 26.** Let \( \{y_1, y_2, y_3\} \) be Type B coordinates. Let \( \{h_{ij}\} \) be the associated metric tensor and let \( c := \sqrt{\det(h_{ij})} \). Then

\[
2\tau = 2\sqrt{h_{33}} \cdot d(\sqrt{h_{33}}^{-1}(\nu)) + 2 \frac{dc(\nu)}{c}.
\]

**Proof.** Let \( \{y_1, y_2, y_3\} \) be Type B coordinates. Also consider Type A coordinates \( \{x_1, x_2, x_3\} \) constructed with \( u = y_3 \). Then \( x_i \equiv y_i \) on the boundary for \( i = 1, 2, 3 \). In particular, this means that

\[
b_{ij} := \frac{\partial x_i}{\partial y_j} = \delta_{ij} \text{ for } i = 1, 2, 3, j = 1, 2,
\]
where \( B \) is the derivative matrix of the coordinate change. Also, on the boundary we have

\[
\frac{\partial}{\partial y_3} = (\sqrt{h_{33}})\nu = \sqrt{h_{33}} \frac{\partial}{\partial x_3}.
\]

So \( b_{33} = \sqrt{h_{33}} \delta_{33} \). So \( B \) is determined on \( \partial M \). Also, on the boundary for \( i, j = 1, 2 \)

\[
0 = \frac{\partial}{\partial y_i} (b_{3j}) = \frac{\partial}{\partial y_i} \left( \frac{\partial x_j}{\partial y_3} \right)
\]
\[
= \frac{\partial}{\partial y_3} \left( \frac{\partial x_j}{\partial y_i} \right) = \frac{\partial}{\partial y_3} (b_{ji}).
\]
So on the boundary
\[ \frac{\partial}{\partial x^3} (b_{ij}) \text{ for } i, j = 1, 2. \]

Hence, on the boundary
\[ \det(B) = b_{33} = \sqrt{h_{33}}, \]

and
\[ \frac{\partial}{\partial x^3} (\det(B)) = \frac{\partial}{\partial x^3} \left( \begin{vmatrix} b_{31} & b_{12} & b_{13} \\ b_{22} & b_{12} & b_{23} \\ b_{21} & b_{21} & b_{23} \end{vmatrix} - b_{32} \begin{vmatrix} b_{11} & b_{13} \\ b_{21} & b_{23} \end{vmatrix} + b_{33} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \right) \]
\[ = \frac{\partial}{\partial x^3} (b_{33}) = \frac{\partial}{\partial x^3} (\sqrt{h_{33}}). \]

Let \( g_{ij} \) be the metric tensor of the \( x_i \)'s and \( a := \sqrt{\det(g_{ij})} \). Then
\[ 2\tau = 2 \frac{\partial a}{\partial x^3} a = \frac{\partial}{\partial x^3} (\det(g_{ij})) \frac{1}{\det(g_{ij})} \]
\[ = \frac{\partial}{\partial x^3} (\det(B^{-1})^2 \det(h_{ij})) \frac{1}{\det(B^{-1})^2 \det(h_{ij})} \]
\[ = (2 \det(B^{-1}) \frac{\partial}{\partial x^3} (\det(B^{-1})) \det(h_{ij}) + \det(B^{-1})^2 \frac{\partial}{\partial x^3} (\det(h_{ij})) \frac{1}{\det(B^{-1})^2 \det(h_{ij})} \]
\[ = 2\sqrt{h_{33}} \frac{\partial}{\partial x^3} (\sqrt{h_{33}}^{-1}) + \frac{\partial}{\partial x^3} (\det(h_{ij})) \frac{1}{\det(h_{ij})}. \]

Since
\[ \frac{\partial}{\partial x^3} (\det(h_{ij})) \frac{1}{\det(h_{ij})} = 2 \frac{\partial c}{\partial x^3} \frac{1}{c} \]
and \( \frac{\partial}{\partial x^3} = \nu \), we have the result. \( \square \)

### 3.3 The Image of the Curvature Form

We will use this \( \tau \) to prove the following lemma, which relates to the image of the curvature form.

**Lemma 27.** Suppose \( M \) is a 3-manifold with boundary, \( k + 1 > 3/2 \), \( \alpha, \beta \in H_{conc}^{k+1}(t_P) \cap H_A \text{ and } \nabla_A \in C_{conc,A}^k \). Then
\[ d_A[\alpha \cdot \beta][\nu] = -2\tau[\alpha \cdot \beta] \text{ on } \partial M, \tag{3.16} \]
where \( \nu \) is the normal inward pointing vector field.

Since \( k + 1 > 3/2 \), note that \( [\alpha \cdot \beta] \) is \( C^1 \), and thus \( d_A[\alpha \cdot \beta] \) is continuous. Hence, the above equality is true not just in the trace sense, but as an equality of two continuous functions.
Proof. We will use Type A coordinates \((x_1, x_2, x_3)\), and assume that the vector bundle \(\mathfrak{P}\) is also trivialized in this neighborhood. Recall that the metric tensor in this coordinate system has the feature that \(g_{33} = \delta_{33}\) on the boundary, and \(g_{33} = 1\) everywhere. Thus, \(g^{33} = \delta_{33}\) also on the boundary. Also, \(\frac{\partial}{\partial x_3}\) is the inward pointing normal vector on the boundary. Take \(\alpha, \beta\) as above and define \(\alpha_i, \beta_i\) so that \(\alpha = \sum_{i=1}^{3} \alpha_i dx_i\) and \(\beta = \sum_{i=1}^{3} \beta_i dx_i\). Since we are assuming \(\mathfrak{P}\) has a fixed trivialization in our neighborhood, we can view the \(\alpha_i\) and \(\beta_i\) as \(\mathfrak{k}\)-valued functions. Also, since \(\frac{\partial}{\partial x_3}\) is the inward pointing normal vector and \(\alpha, \beta\) satisfy conductor boundary conditions, we have

\[
\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \text{ on } \partial M. \tag{3.17}
\]

Let \(d\) be the flat connection with respect to our fixed trivialization of \(\mathfrak{P}\) and define a \(\mathfrak{k}\)-valued 1-form \(A\) so that \(d_A = d + A\). Define \(A_i\) so that \(A = \sum_i A_i dx_i\). On this coordinate patch, we have

\[
[\alpha \cdot \beta] = \sum_{j,k=1}^{3} [\alpha_j, \beta_k](dx_j \cdot dx_k) = \sum_{j,k=1}^{3} [\alpha_j, \beta_k]g^{jk}. \tag{3.18}
\]

Taking the derivative \(d_A\) yields

\[
d_A([\alpha \cdot \beta]) = \sum_{j,k=1}^{3} d_A([\alpha_j, \beta_k]g^{jk})
\]

\[
= \sum_{j,k=1}^{3} [d_A(\alpha_j), \beta_k]g^{jk} + [\alpha_j, d_A(\beta_k)]g^{jk} + [\alpha_j, \beta_k]d(g^{jk})
\]

If both \(j, k < 3\), then \(\alpha_j = \beta_k = 0\), and hence

\[
[d_A(\alpha_j), \beta_k]g^{jk} + [\alpha_j, d_A(\beta_k)]g^{jk} + [\alpha_j, \beta_k]d(g^{jk}) = 0 \text{ on } \partial M. \tag{3.19}
\]

Suppose \(j = 3\) and \(k < 3\). Then \(\beta_k = 0\) on \(\partial M\) by (3.17), and \(g^{3k} = 0\) on \(\partial M\) since we are using Type A coordinates. Thus, (3.19) holds in this case also. Similarly, if \(j < 3\) and \(k = 3\) then \(\alpha_j = 0\) and \(g^{3j} = 0\) and thus (3.19) holds. In sum, we have

\[
d_A([\alpha \cdot \beta])|_{\partial M} = [d_A \alpha_3, \beta_3] + [\alpha_3, d_A \beta_3] + [\alpha_3, \beta_3]d(g^{33}). \tag{3.20}
\]

Using the adjoint matrix, we see that

\[
g^{33} = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \over det(g_{ij})
\]

Combining the fact that \(\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = det(g_{ij})\) on \(\partial M\) and (3.15) yields

\[
\frac{\partial g^{33}}{\partial x_3} = \left(\frac{\partial}{\partial x_3} \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}\right) \frac{det(g_{ij}) - \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}(\frac{\partial}{\partial x_3} det(g_{ij}))}{det(g_{ij})^2}
\]

\[
= 0 \text{ on } \partial M. \tag{3.21}
\]

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Hence, we have
\[ d_A([\alpha \cdot \beta])|_{\partial M} \left( \frac{\partial}{\partial x_3} \right) = [d_A \xi_3(\frac{\partial}{\partial x_3}), \beta_3] + [\alpha_3, d_A \beta_3(\frac{\partial}{\partial x_3})]. \] (3.23)

We will leave \( d_A([\alpha \cdot \beta])|_{\partial M} \) for the moment and investigate what \( d^*_A \alpha = d^*_A \beta = 0 \) means in our coordinate system. We will calculate \( d^* \) by using the Hodge star operator. One can verify that
\[ *dx_j = a(g^{j1} dx_2 \wedge dx_3 + g^{j2} dx_3 \wedge dx_1 + g^{j3} dx_1 \wedge dx_2), \]
where \( a = \sqrt{\text{det}(g_{ij})} = 1/\sqrt{\text{det}(g^{ij})} \). Using this, we calculate
\[ -d^* \alpha = *d \ast (\alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dx_3) \]
\[ = *d \left( a \left( \sum_j \alpha_j (g^{j1} dx_2 \wedge dx_3 + g^{j2} dx_3 \wedge dx_1 + g^{j3} dx_1 \wedge dx_2) \right) \right) \]
\[ = *d(a \left( \sum_j \alpha_j g^{j1} dx_2 \wedge dx_3 + \alpha_j g^{j2} dx_3 \wedge dx_1 + \alpha_j g^{j3} dx_1 \wedge dx_2 \right)) \]
\[ + \left( \sum_j \frac{\partial}{\partial x_1} (\alpha_j g^{ji}) + \frac{\partial}{\partial x_2} (\alpha_j g^{j2}) + \frac{\partial}{\partial x_3} (\alpha_j g^{j3}) \right) \ast (adx_1 \wedge dx_2 \wedge dx_3) \]
\[ = *d(a \left( \sum_j \alpha_j g^{j1} dx_2 \wedge dx_3 + \alpha_j g^{j2} dx_3 \wedge dx_1 + \alpha_j g^{j3} dx_1 \wedge dx_2 \right)) \]
\[ + \left( \sum_{i,j} \frac{\partial}{\partial x_i} (\alpha_j g^{ji}) \right). \]

By (3.17) and the fact that \( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \) are tangent at the boundary, we have that for \( i = 1, 2, j = 1, 2 \)
\[ \frac{\partial}{\partial x_i} (\alpha_j g^{ji})|_{\partial M} = (\frac{\partial}{\partial x_i} (\alpha_j g^{ji}) + \alpha_j \frac{\partial}{\partial x_i} (g^{ji}))|_{\partial M} \]
\[ = 0 \cdot g^{ji} + 0 \cdot \frac{\partial}{\partial x_i} (g^{ji}) = 0. \]

Also, since \( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \) are tangent to the boundary, we also have for \( i = 1, 2 \)
\[ \frac{\partial}{\partial x_i} (\alpha_3 g^{3i})|_{\partial M} = (\frac{\partial}{\partial x_i} (\alpha_3 g^{3i}) + \alpha_3 \frac{\partial}{\partial x_i} (g^{3i}))|_{\partial M} \]
\[ = \frac{\partial}{\partial x_i} (\alpha_3) \cdot 0 + \alpha_3 \cdot 0 = 0. \]
Finally, for \( j = 1, 2 \):
\[
\frac{\partial}{\partial x_3} (\alpha_j g^{j3})|_{\partial M} = \left( \frac{\partial}{\partial x_3} (\alpha_j g^{j3} + \alpha_j \frac{\partial}{\partial x_3} (g^{j3})) \right)|_{\partial M} \\
= \frac{\partial}{\partial x_3} (\alpha_j) \cdot 0 + 0 \cdot \frac{\partial}{\partial x_3} (g^{j3}) = 0.
\]

Hence, \(-d^*\alpha\) on the boundary reduces to
\[
-d^*\alpha|_{\partial M} = \frac{\partial}{\partial x_3} (g^{33}\alpha_3) + *(da)(g^{33}\alpha_3dx_1 \wedge dx_2).
\]

We showed that \( \partial/\partial x_3 (g^{33}) = 0 \) in (3.21). Thus, we obtain
\[
-d^*\alpha|_{\partial M} = \frac{\partial}{\partial x_3} (\alpha_3) + \frac{1}{a} \frac{\partial a}{\partial x_3} (\alpha_3).
\]

Since \( d^*_A\alpha = d^*\alpha - [A \cdot \alpha] \), we have,
\[
-d^*_A\alpha|_{\partial M} = \frac{\partial \alpha_3}{\partial x_3} + \frac{1}{a} \frac{\partial a}{\partial x_3} (\alpha_3) - [\alpha \cdot A] \tag{3.24}
\]
\[
= \frac{\partial \alpha_3}{\partial x_3} + \frac{1}{a} \frac{\partial a}{\partial x_3} (\alpha_3) + [A_3, \alpha_3], \tag{3.25}
\]

where we used (3.17) and (3.18) (replacing \( \beta \) with \( A \)) in the last line. Of course, an analogous statement holds for \( \beta \) replacing \( \alpha \).

We now revisit (3.23) and plug in (3.25):
\[
d_A([\alpha_3, \beta_3])|_{\partial M} = [d_A \alpha_3, \beta_3](\partial/\partial x_3) + [\alpha_3, d_A \beta_3](\partial/\partial x_3)
\]
\[
= \frac{\partial \alpha_3}{\partial x_3} + [A_3, \alpha_3], \beta_3]
\]
\[
= \left[ \alpha_3, \frac{\partial \beta_3}{\partial x_3} \right] + [\alpha_3, [A_3, \beta_3]]
\]
\[
= -\frac{1}{a} \frac{\partial a}{\partial x_3} (\alpha_3) + [A_3, \alpha_3] + d^*_A\alpha_3, \beta_3] + [A_3, [\alpha_3, \beta_3]]
\]
\[
= -\frac{2}{a} \frac{\partial a}{\partial x_3} (\alpha_3) + [d^*_A\alpha_3, \beta_3] + [\alpha_3, d^*_A\beta_3]
\]
\[
= -2\tau [\alpha \cdot \beta]|_{\partial M} - [d^*_A\alpha_3, \beta_3] - [\alpha_3, d^*_A\beta_3]
\]
\[
= -2\tau [\alpha \cdot \beta]|_{\partial M}.
\]

where we again used (3.18) on the second to last line, as well as the fact that \( \alpha, \beta \in H_A \). The lemma is thus proven. \( \square \)
Inspired by the previous result, we define a linear map
\[ T_A : \text{Lie}(\mathcal{G}_{\text{con}}^{k+1}) \to L^2(\mathcal{t}_P|_{\partial M}) \]
given by
\[ T_A(f) = d_A \Delta_A f + 2\tau A f. \] (3.26)
Counting derivatives (note that \( \text{Lie}(\mathcal{G}_{\text{con}}^{k+1}) = H_{\text{con}}^{k+1}(\mathcal{t}_P) \)), and using Theorem 9.3 in [17], we see that \( T_A \) is well-defined and bounded. Define a set \( \mathcal{L}_A \subseteq \text{Lie}(\mathcal{G}_{\text{con}}^{k+1}) \) as
\[ \mathcal{L}_A := \text{Span}\{\mathcal{R}_A(\alpha \cdot \beta) : \alpha, \beta \in H_A\} \] (3.27)
The previous lemma yields

**Corollary 28.** The set \( \mathcal{L}_A \) is contained in \( \ker(T_A) \). In particular, since \( T_A \) is not identically 0, we have that \( \overline{\mathcal{L}_A} \) is a proper subset of \( \text{Lie}(\mathcal{G}_{\text{con}}^{k+1}) \).

**Proof.** Let \( g \in \mathcal{L}_A \). By Lemma 27 since
\[ \mathcal{R}_A(\alpha \cdot \beta) = -2G_A[\alpha \cdot \beta], \]
\( \Delta_A f \) satisfies
\[ T_A(g) = d_A(\Delta_A g)(\nu) + 2\tau(\Delta_A g) = 0, \]
proving the corollary. \qed

This corollary shows that the image of the curvature form can never be dense in the gauge algebra, unlike the case in [16].

### 3.4 A Partial Converse of Lemma 27

The natural question now is whether the converse of Lemma 27 holds. A quick argument shows that it cannot.

**Lemma 29.** The converse to Lemma 27 does not hold. More specifically, there exists \( f \in \ker(T_A) - \mathcal{L}_A \) if the connection \( \nabla_A \in \mathcal{C}_{\text{con},A_0}^k \) also lies in \( \mathcal{C}_{\text{con},A_0}^{k+1} \).

**Proof.** Suppose \( \nabla_A \in \mathcal{C}_{\text{con},A_0}^{k+1} \). Note that \( \mathcal{L}_A \subseteq H_{\text{con}}^{k+2}(\mathcal{t}_P) \) since \( \nabla_A \in \mathcal{C}_{\text{con},A_0}^{k+1} \), and thus \( G_A : H^k(\mathcal{t}_P) \to H_{\text{con}}^{k+2}(\mathcal{t}_P) \) exists by Proposition 21. However, the domain of \( T_A \) is \( \text{Lie}(\mathcal{G}_{\text{con}}^{k+1}) = H_{\text{con}}^{k+1}(\mathcal{t}_P) \). This disparity in regularity will sink the converse as follows. We first construct \( f \in H^{k-1}(\mathcal{t}_P) \) that is not in \( H^k(\mathcal{t}_P) \) and is 0 in a neighborhood of the boundary: Take an open subsets \( U \subseteq W \subseteq M \) such that \( U \cap W \subseteq \partial M \) and take \( \tilde{f} \in H^{k-1}(\mathcal{t}_P|_U) - H^k(\mathcal{t}_P|_U) \) and \( \tilde{f} \in H^{k-1}(W) \).

Take a smooth function \( \zeta : M \to [0,1] \) such that \( \zeta|_U \equiv 1 \) and \( \text{supp}(\zeta) \subseteq W \). Then \( f := \zeta \cdot \tilde{f} \in H^{k-1}(\mathcal{t}_P|_U) \), and we can extend \( f \) by 0 to have \( f \in H^{k-1}(\mathcal{t}_P) \).

By the equivalence in Section 4 of [17], since \( f|_U = \tilde{f} \notin H^k(\mathcal{t}_P|_U) \) we have \( f \notin H^k(\mathcal{t}_P) \). However, we do have that \( f \) is 0 in a neighborhood of the boundary. Set \( g := G_A f \). Then \( g \in H_{\text{con}}^{k+1}(\mathcal{t}_P) = \text{Lie}(\mathcal{G}_{\text{con}}^{k+1}) \). If \( g \in H^{k+2}(\mathcal{t}_P) \), then \( f = \Delta_A g \in H^{k} \) which is a contradiction. So \( g \in H_{\text{con}}^{k+1}(\mathcal{t}_P) - H_{\text{con}}^{k+2}(\mathcal{t}_P) \), and \( \Delta_A g = f \) vanishes in a neighborhood of \( \partial M \), so \( d_A(\Delta_A f)(\nu) = 0 = 2\tau(\Delta_A f) \).

So, \( g \in \ker(T_A) \), but not in \( H^{k+2}(\mathcal{t}_P) \) and thus not in \( \mathcal{L}_A \), as desired. \qed
In the previous lemma, regularity considerations sunk the converse. However, if we took regularity out of the equation, perhaps the converse would hold. In other words, perhaps we have
\[ \ker T_A \cap C^\infty = \mathcal{L}_A \cap C^\infty. \]
So that the Green operator \( G_A \) maps smooth functions to smooth functions, we also want the connection \( \nabla_A \) to be \( C^\infty \). In this setting, the converse does for a specific set up. Namely, if \( P \) is the trivial bundle \( \bar{O} \times K \rightarrow \bar{O} \) for a bounded open set \( O \subseteq \mathbb{R}^3 \) with smooth boundary with and the base connection \( \nabla_{A_0} \) is the flat connection. In this set up, \( K_P \) is isomorphic to \( \bar{O} \times K \rightarrow \bar{O} \). So we can view gauge transformations \( g \) as \( K \)-valued functions on \( \bar{O} \), gauge algebra elements \( \psi \) as \( \mathfrak{k} \)-valued functions, and \( \mathfrak{k}_P \)-valued forms as \( \mathfrak{k} \)-valued forms.

As in Chapter 2, we denote the flat connection as \( \nabla_0 \). This means we should denote exterior differentiation by \( d_0 \), but since \( \nabla_0 = d \) (as asserted in Chapter 2), we will instead simply use \( d \) without a subscript. Similarly, we denote \( d_0^* \) by simply \( d^* \).

To do prove the converse of Lemma 27 in this case, we proceed locally. We first consider interior neighborhoods. For this, we prove a lemma originally outlined by L. Gross.

**Lemma 30.** Let \((a,b)^3\) be a cube in \( \mathbb{R}^3 \), and let \( \Psi : (a,b)^3 \rightarrow \mathfrak{k} \) be smooth and have compact support. Then \( \Psi \in \text{Span}\{[\alpha \cdot \beta] : \alpha, \beta \in C^\infty_c((a,b)^3 \otimes \mathfrak{k}), d^* \alpha = d^* \beta = 0\} \).

**Proof.** Since \( \mathfrak{k} = [\mathfrak{k}, \mathfrak{k}] \) by semisimplicity, without loss of generality we can assume \( \Psi(x,y,z) = \psi(x,y,z) \cdot \left[ A, B \right] \), where \( A, B \) are fixed elements of \( \mathfrak{k} \) and \( \psi \in C^\infty_c((a,b)^3) \). Choose \( c, d, i, j, k, l \in \mathbb{R} \) so that \( \text{supp}(\psi) \subseteq (c,d)^3 \) and \( a < j < k < i < c < d < l < b \). We can then find a function \( h : (a,b)^3 \rightarrow \mathbb{R} \) so that
1. \( h \) is smooth
2. \( h|_{(c,d)^3} \equiv \psi \)
3. \( \text{supp}(h) \subset (j,d)^3 \)
4. \( \int_j^d h(x,s,z)ds = 0 \) for any fixed \( x, z \).
5. \( h(x,y,z) = 0 \) if \( (x,z) \notin (c,d)^2 \).
6. \( h|_{y \in (i,c)} \equiv 0 \).

Specifically, define
\[
\eta(t) = \begin{cases} 
C \exp\left(\frac{1}{1-(\frac{1}{x}+\frac{1}{y})(t-\frac{i+x+y}{2})^2}\right) & \text{if } t \in (k,i) \\
0 & \text{if else}
\end{cases}
\]
where $C$ is chosen so the integral of $\eta$ is 1. Let $I(x, z) := \int_I^d \psi(x, s, z)ds$, and finally define $h(x, y, z) := -I(x, z)\eta(y) + \psi(x, y, z)$. One can check that $h$ satisfies the above properties.

Define $F : (a, b)^3 \to \mathbb{R}$ as $F(x, y, z) = \int_a^y h(x, s, z)ds$. Then $F$ is smooth and $\text{supp}(f) \subset (j, d)^3$ by Properties 3 and 4 above. Also, it is clear that $F_y = h$. We now construct another function $G : (a, b)^3 \to \mathbb{R}$. Define $G$ as $G(x, y, z) = \phi(x)v(y, z)$, where $\phi : (a, b) \to \mathbb{R}$ is constructed so $\phi \in C_c^\infty(a, b)$ and $\phi|_{[c, d]}(x) = x$, and $v : (a, b)^2 \to \mathbb{R}$ is constructed so $v|_{[c, d]} \equiv 1$, and $\text{supp}(v) \subset (i, l)^2$. Then $G_x|_{[c, d]} = 1$ and has compact support. Let us consider $\Theta(x, y, z) := F_y(x, y, z) \cdot G_x(x, y, z) = h(x, y, z) \cdot G_x(x, y, z)$. We will show that $\Theta(x, y, z) = \psi(x, y, z)$ by looking at it in cases:

First, if $(x, z) \notin (c, d)^2$, then by property 4 we have $h(x, y, z) = 0 = \psi(x, y, z)$. So for our next cases we can assume $(x, z) \in (c, d)^2$, and thus $G_x(x, y, z) = v(y, z)$. If $x \in (a, i]$, then $v(y, z) = 0$ since $\text{supp}(v) \subset (i, l)^2$. Hence, $\Theta(x, y, z) = 0 = \psi(x, y, z)$. If $y \in (i, c)$, then $h(x, y, z) = 0$ by property 5 and so $\Theta(x, y, z) = 0 = \psi(x, y, z)$. If $x \in (i, c] \cap [c, d]$, then $\Theta(x, y, z) = \psi(x, y, z) = 1 = \psi(x, y, z)$. And finally, if $x \in (d, b)$, then $h(x, y, z) = 0$ so $\psi(x, y, z)$, hence in all cases, $\Theta(x, y, z) = \psi(x, y, z)$.

Now define 2-forms $\omega_1, \omega_2$ as $\omega_1 = -F \cdot Ady \wedge dz$ and $\omega_2 = G \cdot Bdz \wedge dx$. Let $\alpha := d^*\omega_1$ and $\beta := d^*\omega_2$. Since $(d^*)^2 = 0$, we have $d^*\alpha = d^*\beta = 0$. Now,

\[
\alpha = d^*(\omega_1) = *d*(\omega_1) = *d(-F \cdot Adx)
= *(F_g \cdot Adx \wedge dy - F_z \cdot Ady \wedge dz)B = -F_z \cdot Ady - F_y \cdot Adz.
\]

Similarly,

\[
\beta = *d(G \cdot Bdy) = *(G_x Bdx \wedge dy - G_z Bdy \wedge dz) = -G_z \cdot Bdy + G_x \cdot Bdz
\]

Since $\Theta = \psi$, 

\[
[\alpha, \beta] = [0, G_z \cdot B] + [F_z \cdot A, 0] + [F_y A, G_z B] = \psi[A, B] = \Psi,
\]

as desired.

We extend this result to any domain $O$.

**Lemma 31.** Let $O \subset \mathbb{R}^3$ be a bounded open set. Then $C_c^\infty(O \otimes \mathfrak{t}) = \text{Span}\{[\alpha, \beta] : \alpha, \beta \in C_c^\infty(\Lambda^1(O \otimes \mathfrak{t}))\} = \{d^*\alpha = d^*\beta = 0\}$.

**Proof.** Let $\Psi \in C_c^\infty(O \otimes \mathfrak{t})$ be arbitrary. Let $\{C_k\}_{k=1}^n$ be a finite family of open cubes that cover the support of $\Psi$ and are contained in $O$. Let $\{\lambda_k\}$ be a partition of unity subordinate to the cover $\{C_k\}$. Then the function $\lambda_k \cdot \Psi$ lies in $C_c^\infty(C_k \otimes \mathfrak{t})$. By the previous lemma, there exists sequences $\{\alpha_i\}_{i=1}^n, \{\beta_i\}_{i=1}^n$ such that each $\alpha_i, \beta_i \in C_c^\infty(\Lambda^1(C_k \otimes \mathfrak{t}))$, $d^*\alpha_i = d^*\beta_i = 0$ and $\lambda_k \cdot \Phi = \sum_{i=1}^n [\alpha_i, \beta_i]$. Extending the $\alpha$’s and $\beta$’s by zero, we have $\alpha_i, \beta_i \in C_c^\infty(\Lambda^1(O \otimes \mathfrak{t}))$, $d^*\alpha = d^*\beta = 0$, and $\lambda_k \cdot \Phi = \sum_{i=1}^n [\alpha_i, \beta_i]$ on $0$. Thus, $\lambda_k \cdot \Phi \in \text{Span}\{[\alpha, \beta] : \alpha, \beta \in C_c^\infty(\Lambda^1(O \otimes \mathfrak{t}))\}$, $d^*\alpha = d^*\beta = 0$. So, $\Psi = \sum_{k=1}^n (\lambda_k \cdot \Phi) \in \text{Span}\{[\alpha, \beta] : \alpha, \beta \in C_c^\infty(\Lambda^1(O \otimes \mathfrak{t}))\}$, $d^*\alpha = d^*\beta = 0$, as desired.

\[\square\]
We now look at neighborhoods of the boundary of $O$ and see if all the smooth $\Psi$ that satisfy the boundary condition of Lemma 27 are in the desired span. We see that this is so.

**Lemma 32.** Let $O \subset \mathbb{R}^3$ be open and bounded, and let $U$ be a neighborhood of $O$ that includes the boundary, admits the Type B coordinates $\{y_1, y_2, y_3\}$, and is a cube under these coordinates. Let $\Psi : U \rightarrow \mathfrak{t}$ be smooth, have compact support, and

$$d\Psi(\nu) = -2\tau\Psi$$ on $\partial O \cap U$. \hfill (3.28)

Then $\Psi \in \text{Span}\{[\alpha \cdot \beta] : \alpha, \beta \in C^\infty_c(U \otimes \mathfrak{t}) ; d^* \alpha = d^* \beta = 0; \alpha, \beta \text{ satisfy CBC}\}$.

In the preceding lemma and in what follows, a smooth 1-form $\alpha$ satisfies conductor boundary conditions (or CBC for short) if $\alpha \in H^1_{\text{loc}}(\mathfrak{t}_p)$ as well as being smooth. This is equivalent to saying that $\iota^*(\alpha) = 0$, where $\iota : \partial M \rightarrow M$ is the inclusion, or saying the tangential component of $\alpha$ on the boundary is 0. Also, viewing $U$ as the cube $(0, 1) \times (0, 1) \times [0, 1)$, a function $f \in C^\infty_c(U)$ has its support contained in $(\epsilon, 1 - \epsilon) \times (\epsilon, 1 - \epsilon) \times [0, 1 - \epsilon)$ for some $\epsilon > 0$. The point is that it need not vanish on the boundary $\{y_3 = 0\}$.

**Proof of Lemma 32.** Let $\{v_i\}$ be basis of $\mathfrak{t}$. Then we can write $\Psi = \sum \psi_i \cdot v_i$. Since the basis elements are independent, by (3.28) we have that $d\psi_i(\nu) = -2\tau\psi_i$. Since $\mathfrak{t}$ is semisimple, each basis element $v_i$ can be written as a sum of commutators $v_i = \sum_{j=1}^{\infty} [f^i_j, g^j_i]$. Hence, we can write $\Psi$ as

$$\Psi = \sum_{i} \sum_{j=1}^{\infty} \psi_i [f^i_j, g^j_i].$$

So without loss of generality we can assume $\Psi = \psi \cdot [A, B]$, where $A, B$ are fixed elements of $\mathfrak{t}$ and $\psi \in C^\infty_c(U)$ and $d\psi(\nu) = -2\tau\psi$.

Coordinatize $U$ using Type B coordinates $(y_1, y_2, y_3)$ under which the domain is a cube. Then, without loss of generality, $U = (0, \delta) \times (0, \delta) \times [0, \delta)$ under these coordinates. Here, let $a := \sqrt{\det(h_{ij})}$, where $h_{ij}$ is the metric tensor of our chart.

Choose $c, d, i, j, k, l \in \mathbb{R}$ so that $\text{supp}(\psi) \subset (c, d) \times (c, d) \times [0, \delta)$ and $0 < j < k < i < c < d < l < \delta$. We can define a function $h$ similar to the $h$ in Lemma 30. This time we set $h(y_1, y_2, y_3) = -I(y_1, y_3)\eta(y_2) + h^{33}(y_1, y_2, y_3)a(y_1, y_2, y_3)^2\psi(y_1, y_2, y_3)$, where $\eta$ is the same bump function from Lemma 30 and

$$I(y_1, y_3) = \int_{c}^{d} h^{33}(y_1, s, y_3)a(y_1, s, y_3)^2\psi(y_1, s, y_3)ds.$$ 

Then $h$ has the following properties, analogous to the previous case in Lemma 30:

1. $h$ is smooth
2. $h|_{[c,d] \times [c,d] \times [0,d]} \equiv h^{33}(y_1, y_2, y_3)a(y_1, y_2, y_3)^2\psi|_{[c,d] \times [c,d] \times [0,d]}$.

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3. \( \text{supp}(h) \subset (c, d) \times (j, d) \times [0, d) \),

4. \( \int_0^{y_2} h(y_1, s, y_3) ds = 0 \) for any fixed \( y_1, y_3 \) and \( y_2 \geq d \) or \( y_2 \leq j \),

5. \( h|_{(i, c)} \equiv 0 \).

\( h \) has an additional property. Using Lemma 26, we have

\[
\frac{\partial (h^{33} a^2 \cdot \psi)}{\partial y_3} = 2\sqrt{h^{33}} \frac{\partial \sqrt{h^{33}}}{\partial y_3} a^2 \psi + h^{33} 2a \cdot \frac{\partial a}{\partial y_3} \cdot \psi + h^{33} a^2 \cdot \frac{\partial \psi}{\partial y_3} \tag{3.29}
\]

\[
= h^{33} a^2 \left[ 2 \left( \frac{1}{\sqrt{h^{33}}} \frac{\partial \sqrt{h^{33}}}{\partial y_3} a^2 + \frac{1}{a} \cdot \frac{\partial a}{\partial y_3} \right) \psi + \frac{\partial \psi}{\partial y_3} \right] \tag{3.30}
\]

\[
= 0 \text{ on } \partial O \cap U. \tag{3.31}
\]

Hence, differentiating under the integral yields

\[
\frac{\partial h}{\partial y_3} = 0 \text{ on } \partial O \cap U. \tag{3.32}
\]

Define \( F : U \to \mathbb{R} \) as \( F(y_1, y_2, y_3) = \int_0^{y_2} h(y_1, s, y_3) ds \). Then \( F \) is smooth and \( \text{supp}(F) \subset (c, d) \times (j, d) \times [0, d) \) be properties \( 3 \) and \( 4 \) above. Also, it is clear that \( F_2 = h \). Also, by \( \text{Lemma 32} \), differentiating under the integral sign yields

\[
\frac{\partial F}{\partial y_3} = 0 \text{ on } \partial O \cap U. \tag{3.33}
\]

We now construct another function \( G : [0, \delta]^3 \to \mathbb{R} \) which is completely analogous to the \( G \) in Lemma \( 36 \). Define \( G \) as \( G(y_1, y_2, y_3) = v_1(y_1) v_2(y_2) v_3(y_3) \), where \( v_i : [0, 1] \to \mathbb{R} \) is constructed as follows: \( v_1 \in C_c^\infty([0, \delta]) \), \( v_1|_{(c, d)}(x) = x \), and \( \text{supp}(v_1) \subset (i, l); v_2 \in C_c^\infty([0, \delta]), v_2|_{(c, d)} = 1 \), and \( \text{supp}(v_2) \subset (i, l) \); \( v_3 \in C_c^\infty([0, \delta]), v_3|_{[0, d)} = 1 \), and \( \text{supp}(v_3) \subset [0, l] \). Then \( G_1|_{(c, d) \times (c, d) \times [0, d)} = 1 \), and has compact support in \( U \). Let us again consider \( \Theta(y_1, y_2, y_3) := F_2(y_1, y_2, y_3) \cdot G_1(y_1, y_2, y_3) = h(y_1, y_2, y_3) \cdot G_1(y_1, y_2, y_3) \), and show that \( \Theta = h^{33} a^2 \psi \) by looking at it in cases:

First, if \( (y_1, y_3) \notin (c, d) \times [0, d) \), then by property \( 4 \) we have \( h(y_1, y_2, y_3) = 0 = h^{33}(y_1, y_2, y_3) a(y_1, y_2, y_3) \psi(y_1, y_2, y_3) \). So now we can assume \( (y_1, y_3) \in (c, d) \times [0, d) \), and thus \( G_1(y_1, y_2, y_3) = v_2(y_2) \). If \( y_2 \in (0, l) \), then \( v_2(y_2) = 0 \) since \( \text{supp}(v_2) \subset (i, l) \). Hence, \( \Theta(y_1, y_2, y_3) = 0 = h^{33}(y_1, y_2, y_3) a(y_1, y_2, y_3) \psi(y_1, y_2, y_3) \).

If \( y_2 \in (i, l) \), then \( h(y_1, y_2, y_3) = 0 \) by property \( 5 \) and so \( \Theta(y_1, y_2, y_3) = 0 = h^{33}(y_1, y_2, y_3) a(y_1, y_2, y_3) \psi(y_1, y_2, y_3) \). If \( y_2 \in [c, d] \), then \( h(y_1, y_2, y_3) \in [c, d]^2 \times [0, d) \). So

\[
\Theta(y_1, y_2, y_3) = h^{33}(y_1, y_2, y_3) a(y_1, y_2, y_3) \psi(y_1, y_2, y_3) \cdot 1
\]

\[
= h^{33}(y_1, y_2, y_3) a(y_1, y_2, y_3) \psi(y_1, y_2, y_3).
\]

And finally, if \( y_2 \in (d, \delta) \), then by property \( 6 \)

\[
h(y_1, y_2, y_3) = 0 = h^{33}(y_1, y_2, y_3) a^2(y_1, y_2, y_3) \psi(y_1, y_2, y_3).
\]
Hence, in all cases, $\Theta(y_1, y_2, y_3) = h^{33}(y_1, y_2, y_3)a^2(y_1, y_2, y_3)\psi(y_1, y_2, y_3)$.

Now define 2-forms $\omega_1, \omega_2$ as 

$$\omega_1 = -F \cdot A(\ast^{-1}(dy_1))$$

and

$$\omega_2 = G \cdot B(\ast^{-1}(dy_2)).$$

Let $\alpha := d^* \omega_1$ and $\beta := d^* \omega_2$. Since $(d^*)^2 = 0$, we have $d^* \alpha = d^* \beta = 0$. Let $h(i, j)$ be the $i, j$ minor of the inverse metric tensor matrix $h^{ij}$. Also, given distinct $j, k \in \{1, 2, 3\}$, let $i(j, k)$ be the number in $\{1, 2, 3\}$ that is neither $j$ nor $k$. Also, define sgn$(j, k, i(j, k))$ as $+/-1$, whichever satisfies the equality

$$dy_1 \wedge dy_2 \wedge dy_3 = \text{sgn}(j, k, i(j, k))dy_j \wedge dy_k \wedge dy_{i(i, j, k)}.$$

Then one can check that

$$\ast(dy_j \wedge dy_k) = \text{sgn}(j, k, i(j, k))a \cdot \sum_l (-1)^{|i(j, k)|}h(i(j, k), l)dy_l. \quad (3.34)$$

So we calculate

$$\alpha = d^*(\omega_1) = \ast d*(\omega_1) = \ast d(-F \cdot A dy_1)$$

$$= \ast(F_2 Ady_1 \wedge dy_2 - F_3 Ady_3 \wedge dy_1)$$

$$= (a \cdot \sum_i (-1)^{i+1}(F_2 h(3, i) + F_3 h(2, i))dy_i)A.$$

Note that in our Type B coordinates we have $h(3, 1) = h(3, 2) = 0$ always. So since $F_3 = 0$ on the boundary by [48], $\alpha$ satisfies CBC. Similarly,

$$\beta = \ast d(G \cdot B dy_2) = \ast(G_1 B dy_1 \wedge dy_2 - G_3 B dy_2 \wedge dy_3)$$

$$= (a \cdot \sum_j (-1)^{j+1}(G_1 h(3, j) - G_3 h(1, j))dy_j)B.$$

Since $v_3(y_3)$ is constant for $y_3 \in [0, d]$, we have $G_3(0, 1) \times (0, 1) \times [0, d] = 0$. This and the fact that $h(3, 1) = h(3, 2) = 0$ show that $\beta$ satisfies CBC.

To calculate $[\alpha \cdot \beta]$, we first note that by Laplace expansions of determinants, we have

$$\sum_j (-1)^{i+j}h^k jh(i, j) = \det(h^{ij})\delta_{ik}. \quad (3.35)$$

Indeed, if $i = k$ then the above sum is the Laplace expansion along the $k^{th}$ row of $h^{ij}$. If $i$ and $k$ are distinct, then the sum is a determinant of a matrix with a repeated row, and thus equal to 0. So using the above and the fact that

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\[ h^{ij} = h^{ji}, \text{ we have} \]

\[
[a \cdot \beta] = a^2 \left( \sum_{i,j} (-1)^{i+j} (F_2h(3, i) + F_3g(2, i))(G_1g(3, j) - G_3g(1, i))h^{ij} \right) [A, B]
\]

\[
= a^2 \left( \sum_{i,j} (-1)^{i+j} (F_2G_1h(3, j)h^{ji}h(3, i) - F_2G_3h(1, j)h^{ji}h(3, i) + F_3G_1h(3, j)h^{ji}h(2, i) - F_3G_3h(1, j)h^{ji}h(2, i)) \right) [A, B]
\]

\[
= a^2 \det(h^{ij}) \sum_j (F_2G_1h(3, j) - F_2G_3h(1, j)) \delta_{j2} + (F_3G_1h(3, j) - F_3G_3h(1, j)) \delta_{j2}] [A, B]
\]

\[
= (F_2G_1h(3, 3) - F_2G_3h(1, 3) + F_3G_1h(3, 2) - F_3G_3h(1, 2)) [A, B]
\]

\[
= (F_2G_1h(3, 3) - F_3G_3h(1, 2)) [A, B].
\]

Since \( v_3(y_3) \) is constant on \([0, d]\), we have \( G_3 | (0, 1) \times (0, 1) \times [0, d] = 0 \). Since \( \supp(F) \subset (c, d) \times (j, d) \times [0, d] \), we have \( F_3 | (0, 1) \times (0, 1) \times [d, 1] = 0 \). Hence, \( F_3G_3 = 0 \). So, continuing the above, we have

\[
[a \cdot \beta] = (F_2G_1h(3, 3)) [A, B] = (F_2G_1 - \frac{\det(h^{ij})}{h^{33}}) [A, B] = (h^{33} a^2 \psi \frac{\det(h^{ij})}{h^{33}}) [A, B] = (\det(h_{ij}) \det(h^{ij}) \psi)[A, B] = \psi [A, B],
\]

as desired. \( \square \)

We now extend this to a global result, and prove our main lemma.

**Lemma 33.** Let \( O \subset \mathbb{R}^3 \) be a bounded open set. Let \( f \in C^\infty(O \otimes \mathfrak{t}) \). Then

\[ df(\nu) = -2\tau f \text{ on } \partial O \quad (3.36) \]

if and only if

\[ f \in \text{Span}\{[a \cdot \beta] : a, \beta \in C^\infty_c(A^1(O \otimes \mathfrak{t})), d^*\alpha = d^*\beta = 0, \alpha, \beta \text{ satisfy CBC}\}. \]

**Proof.** The backward direction has already been shown in Lemma 27. For the forward direction, suppose \( f \) satisfies \( df(\nu) = -2\tau f \text{ on } \partial O \). There exists a finite cover \( \{U_k\}_{k=0}^m \) of \( O \) that satisfies the following: \( \{U_k\}_{k=1}^m \) covers the boundary and each \( U_k \) for \( k \geq 1 \) is a cube in Type A coordinates, and there is a partition of unity \( \{\lambda_k\}_{k=0}^m \) subordinate to \( \{U_k\}_{k=0}^m \) so that \( d\lambda_k(\nu) = 0 \) on the boundary.

Indeed, cover \( \partial O \) with finite Type A coordinate neighborhoods \( \{U_k\}_{k=1}^m \) where \( U_k = (-\delta, \delta)^2 \times [0, \delta] \). Let \( W_k := (-\delta/4, \delta/4)^2 \times [0, \delta/4] \), and \( V_k := (-\delta/2, \delta/2)^2 \times [0, \delta/2] \). Choose a smooth \( \gamma_k : (-\delta, \delta)^2 \to \mathbb{R} \) so that \( \gamma_k|_{(-\delta/4, \delta/4)} \equiv 1 \) and \( \supp(\gamma_k) \subset [-\delta/2, \delta/2] \). Take a smooth \( \eta : [0, \delta] \to \mathbb{R} \) so that \( \eta|_{[0, \delta/4]} \equiv 1 \) and \( \supp(\eta) \subset [0, \delta/2] \). Then set \( \gamma_k : U_k \to \mathbb{R} \) as

\[ \gamma_k(x_1, x_2, x_3) = \tilde{\gamma}_k(x_1, x_2) \eta(x_3). \]
Then $\gamma_k|_{W_k} \equiv 1$ and $\text{supp}(\gamma_k) \subset V_k$. Now, take open sets $W_0$ and $U_0$ of 0 so that $W_0 \subset U_0$, $U_0 \subset O$, and $\{W_k\}_{k=0}^n$ covers $O$. Take a smooth function $\gamma_0 : O \to \mathbb{R}$ such that $\gamma_0|_{W_0} \equiv 1$ and $\text{supp}\gamma_0 \subset U_0$. Set $\gamma := \sum_{k=0}^n \gamma_k$, and let $\lambda_k := \gamma_k/\gamma$. Then $\{\lambda_k\}$ is a partition of unity with respect to $\{U_k\}_{k=0}^n$. Also, on the support of $\gamma_k$ for $k > 0$,

$$d\gamma_k(\nu) = \frac{\partial}{\partial x_3}(\eta(x_3))\gamma_k(x_1, x_2) = 0.$$ 

Hence, $d\gamma_k(\nu) = 0$. For $k = 0$, $\gamma_k$ vanishes in a neighborhood of the boundary, so $d\gamma_k(\nu) = 0$ for all $k$. Hence, $d\gamma(\nu) = 0$. So, for $k > 0$,

$$d\lambda_k(\nu) := \frac{d\gamma_k(\nu)\gamma - \gamma_k d\gamma(\nu)}{\gamma} = 0,$$

as we desired.

With such a partition of unity, we have $d(\lambda_kf)(\nu) = -2\tau \lambda_kf$ on $\partial O$. So, by Lemmas 31 and 32 there exists $\{\alpha_i\}_{i=1}^n, \{\beta_i\}_{i=1}^n$ such that each $\alpha_i, \beta_i \in C_\infty^2(\Lambda^1(U_k \otimes \mathfrak{t}))$, $d^*\alpha_i = d^*\beta_i = 0$, $\alpha, \beta$ satisfy CBC, and $\lambda_k \cdot f = \sum_{i=1}^n [\alpha_i \cdot \beta_i]$ on $U_k$. Extending the $\alpha_i$’s and $\beta_i$’s by zero, we have $\alpha_i, \beta_i \in C_\infty^2(\Lambda^1(O \otimes \mathfrak{t}))$, $d^*\alpha = d^*\beta = 0$, $\alpha, \beta$ satisfy CBC, and $\lambda_k \cdot f = \sum_{i=1}^n [\alpha_i \cdot \beta_i]$ on $O$. Thus, $\lambda_k \cdot f \in \text{Span}\{[\alpha \cdot \beta] : \alpha, \beta \in C_\infty^2(\Lambda^1(O \otimes \mathfrak{t}))$, $d^*\alpha = d^*\beta = 0, \alpha, \beta$ satisfy CBC$. So, $f = \sum_{k=1}^m (\lambda_k \cdot f) \in \text{Span}\{[\alpha \cdot \beta] : \alpha, \beta \in C_\infty^2(\Lambda^1(O \otimes \mathfrak{t}))$, $d^*\alpha = d^*\beta = 0, \alpha, \beta$ satisfy CBC$, as desired.

Recasting this with our operator $T_A$, we have

**Corollary 34.** Suppose $P = \tilde{O} \times K \to \tilde{O}$ with the flat connection $\nabla_0$ as the base connection. Let $g \in \text{Lie}(G_{can}^{k+1})$ be smooth. Then

$$g \in \ker(T_0) \text{ if and only if } f \in \mathcal{L}_0.$$ 

**Proof.** Set $f = \Delta g$, and apply Lemma 33 to $f$. \hfill \Box

### 3.5 The Generation of the Smooth Gauge Algebra

In this section we will use brackets of the image of the curvature form to get every smooth function in $\text{Lie}(G_{can}^{k+1})$ for the special case $P = \tilde{O} \times K \to \tilde{O}$. The main tool will be Lemma 33. The first thing we must do is see how changes when we introduce brackets. More specifically, note that if $g \in \mathcal{L}_0$, then Lemma 33 says that

$$d(\Delta g)(\nu) = -2\tau \Delta g.$$  

(3.37)

We want to know how changes if $g$ above is replaced by $[g_1, g_2]$, for $g_i \in \mathcal{L}_0$. Indeed, we have
Lemma 35. Suppose $g_1, g_2 \in \mathcal{L}_0$. Then we have

$$d(\Delta([g_1, g_2]))(\nu) = -2\tau \Delta[g_1, g_2] + 3[\Delta g_1, dg_2(\nu)] + 3[dg_1(\nu), \Delta g_2]. \quad (3.38)$$

Proof. First note that

$$\Delta([g_1, g_2]) = \Delta g_1 + [g_1, \Delta g_2] - 2[dg_1 \cdot dg_2].$$

So we have

$$d(\Delta([g_1, g_2]))(\nu) = d(\Delta g_1 + [g_1, \Delta g_2] - 2[dg_1 \cdot dg_2])(\nu) \quad (3.39)$$

$$= [d(\Delta g_1)(\nu), g_2] + [\Delta g_1, dg_2(\nu)] + [dg_1(\nu), \Delta g_2] \quad (3.40)$$

$$+ [g_1, d(\Delta g_2)(\nu)] - 2d([dg_1 \cdot dg_2])(\nu). \quad (3.41)$$

By Lemma 36, we have

$$d(\Delta g_1)(\nu) = -2\tau \Delta g_1. \quad (3.42)$$

Examining the proof of Lemma 27, we see that if $\alpha, \beta \in H^k_{\text{con}}(\mathfrak{k}_P)$ but are not necessarily horizontal, then we generally have

$$d_A([\alpha \cdot \beta])(\nu) = -2\tau [\alpha \cdot \beta] - [d^*_A \alpha, \beta(\nu)] - [\alpha(\nu), d^*_A \beta]. \quad (3.43)$$

The above yields

$$-2d([dg_1 \cdot dg_2])(\nu) = -2(-2\tau [dg_1 \cdot dg_2] - [\Delta g_1, dg_2(\nu)] - [dg_1(\nu), \Delta g_2]). \quad (3.44)$$

Plugging in (3.42) and (3.43) into (3.41), we have

$$d(\Delta([g_1, g_2]))(\nu) = -2\tau [\Delta g_1, g_2] + [\Delta g_1, dg_2(\nu)] + [dg_1(\nu), \Delta g_2] - 2\tau [g_1, \Delta g_2]$$

$$- 2(-2\tau [dg_1 \cdot dg_2] - [\Delta g_1, dg_2(\nu)] - [dg_1(\nu), \Delta g_2])$$

$$= -2\tau ([\Delta g_1, g_2] + [g_1, \Delta g_2] - 2[dg_1 \cdot dg_2]) + 3[\Delta g_1, dg_2(\nu)]$$

$$+ 3[dg_1(\nu), \Delta g_2]$$

$$= -2\tau \Delta([g_1, g_2]) + 3[\Delta g_1, dg_2(\nu)] + 3[dg_1(\nu), \Delta g_2],$$

as desired. \hfill \Box

We will now show that the new term in Lemma 35 is actually very general.

Lemma 36. Let $F$ be a smooth $\mathfrak{k}$-valued function on $\partial O$. Then there exists smooth $g_i, h_i \in \mathcal{L}_0$ such that

$$d(\Delta(\sum_i [g_i, h_i]))(\nu) + 2\tau \Delta(\sum_i [g_i, h_i]) = F.$$ 

Proof. Since $\mathfrak{k}$ is semi-simple, there exists $A_i, B_i, C_i \in \mathfrak{k}$ such that

$$F = \sum_i f_i([A_i, B_i], C_i)$$
for some real valued smooth functions $f_i$. So, without loss of generality, assume that $F = f[[A, B], C]$ for some $A, B, C \in \mathfrak{t}$.

Take any non-negative, nonzero, real-valued $\phi \in C_0^\infty(O)$. By the Strong Minimum principle, we have $G\phi > 0$ in $O$. Thus, we can apply Lemma 3.4 of [6] to get

$$\frac{\partial(G\phi)}{\partial \nu} < 0.$$ 

In particular, $d(G\phi)(\nu)$ never vanishes. We set $h := G\phi \cdot C$. Since $\Delta h = \phi \cdot C$ has compact support, $h \in L_0$ by Lemma 33.

Let $\{U_k\}_{k=0}^m$ be an open cover of $O$ such that $\{U_k\}_{k=1}^m$ covers $\partial O$ and $U_k$ are cubes in Type A coordinates for $k \geq 1$. Let $\{\lambda_k\}_{k=1}^m$ be the corresponding partition of unity for the cover $\{U_k \cap \partial O\}_{k=1}^m$ of the boundary. We set $f_k := \lambda_k \cdot \frac{f}{3d(G\phi)(\nu)}$.

In the cube of $U_k$, suppose the $x_3$ interval is $[0, a]$. Choose a $C^\infty$ function $\eta : [0, a] \to [0, 1]$ such that $\eta|_{[0, a/4]} \equiv 1$ and $\text{supp}(\eta) \subseteq ([0, a/2])$. We can extend $f_k$ to a function $\tilde{f}$ on $U_k$ by

$$\tilde{f}(x_1, x_2, x_3) = f_k(x_1, x_2)\eta(x_3) \exp(-2\tau(x_1, x_2) x_3).$$

Note that the support of $\tilde{f}$ lies in $U_k$, so $\tilde{f}$ is a function on all of $\bar{O}$. On $U_k$, we have

$$d\tilde{f}_k(\nu) = \frac{\partial}{\partial x_3}|_{x_3=0} f_k(x_1, x_2)\eta(x_3) \exp(-2\tau(x_1, x_2) x_3)$$

$$= -2\tau(x_1, x_2) f_k(x_1, x_2)\eta(x_3) \exp(-2\tau(x_1, x_2) x_3)$$

$$= -2\tau \tilde{f}_k.$$

By Lemma 33, the above shows that $G\tilde{f}_k[A, B] \in L_0$. Let $g = \sum_{k=1}^m G\tilde{f}_k[A, B]$. We now verify that $g$ and $h$ were well-chosen. By Lemma 33 and since $\Delta h|_{\partial O} \equiv 0$,

$$d(\Delta([g, h]))(\nu) + 2\tau \Delta[g, h] = 3\Delta g, dh(\nu) + 3\Delta g, dh(\nu)$$

$$= 3\Delta g, dh(\nu)$$

$$= 3\sum_k f_k[A, B], dG\phi(\nu) \cdot C$$

$$= 3\left(\sum_k \lambda_k \cdot \frac{f}{3dG\phi(\nu)}[A, B], dG\phi(\nu)C\right)$$

$$= f[[A, B], C] = F,$$

proving the lemma.

We are now at the point where we can prove our main theorem. Let $\mathcal{F}$ be the Lie algebra generated by $\text{Span}(\text{Im}(\mathcal{R}_0))$. 

\[52\]
Theorem 37. Suppose our principal bundle is $\bar{O} \times K \to \bar{O}$, where $O \subseteq \mathbb{R}^3$ is open and bounded. Suppose $g \in \text{Lie}(G^{k+1}_{\text{con}})$ and is $C^\infty$. Then $g \in \mathcal{F}$.

Proof. Let $g \in \text{Lie}(G^{k+1}_{\text{con}}) \cap C^\infty$. Recall our linear map $T_0 : C^\infty(O, \mathfrak{t}) \to C^\infty(\partial O, \mathfrak{t})$ by

$$T_0(f) = d(\Delta f)(\nu) + 2\tau \Delta f.$$ 

Set $u := T_0(g)$. By Lemma 36 there exists a smooth function $f \in \mathcal{F}$ such that $T_0(f) = u$. Since $T_0$ is linear, we have that $T_0(g - f) = 0$. By Lemma 33 we know that $g - f \in \text{Span}(\text{Im}(R_0)) \subseteq \mathcal{F}$. Hence, $g = f + (g - f) \in \mathcal{F}$, as we desired. \[\square\]

The above theorem gives us our main result.

Corollary 38. Suppose our principal bundle is $\bar{O} \times K \to \bar{O}$, where $O \subseteq \mathbb{R}^3$ is open and bounded, and suppose $\nabla A_0 = \nabla 0$. The holonomy group $H^{k}_{\text{con},0}(\nabla 0)$ with base point $\nabla 0$ of the Coulomb connection of the associated bundle $C^{k}_{\text{con},A_0} \to C^{k}_{\text{con},A_0}/G^{k+1}_{\text{con}}$ is dense in the connected component of the identity of $G^{k+1}_{\text{con}}$.

Before we prove this corollary, we should mention what we mean by “holonomy group.” We define $H^k_{\text{con},0}(\nabla 0)$ the the same way it would be defined in finite dimensions. That is $g \in H^k_{\text{con},0}(\nabla 0)$ if and only if $\nabla 0 \cdot g$ can be connected to $\nabla 0$ by a horizontal path in $C^k_{\text{con},A_0}$. It has been shown that with this definition, $H^k_{\text{con},0}(\nabla 0)$ is a Banach Lie group, and the restricted holonomy group $(H^k_{\text{con},0})^0(\nabla 0)$ is also a Banach Lie group (for the statement of this theorem, see [21]).

Proof. This follows directly from Lemma 7.6 and Proposition 7.7 in [16]. Specifically, Lemma 7.6 and the beginning of the proof of Proposition 7.7 of [16] imply that every element of $\mathcal{F}$ is the tangent vector to a curve in $(H^k_{\text{con},0})^0(\nabla 0)$. Then Proposition 7.7 of [16] tells us that $(H^k_{\text{con},0})^0(\nabla 0)$ is dense in the connected component of $G^{k+1}_{\text{con}}$ since $\mathcal{F}$ is dense in $\text{Lie}(G^{k+1}_{\text{con}})$, completing the proof. \[\square\]
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