PHASES OF HIGHER DIMENSIONAL BLACK HOLES

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Abstract

We review some of the most striking properties of the phase diagrams of higher dimensional black holes in pure gravity. We focus on static black hole solutions with Kaluza-Klein asymptotics and stationary black hole solutions in flat Minkowski space. Both cases exhibit a rich pattern of interconnected phases and merger points with topology changing transitions. In the first case, the phase diagram includes uniform and non-uniform black strings, localized black holes and sequences of Kaluza-Klein bubbles. In the latter case, it includes Myers-Perry black holes, black rings, black saturns and pinched black holes.

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1. Introduction

Our purpose in this short review is to summarize some of the recent progress in the study of black hole solutions in higher dimensional classical general relativity. There are already several excellent reviews of complementary aspects of this constantly growing subject. These include [1, 2, 3] and [4, 5, 6]. Our goal here will be a more modest one – to provide an illustration of some of the key lessons in this field, outline some of the most interesting future directions and equip the reader with a first guide to the literature.

We will focus on stationary black holes solutions in pure Einstein gravity with simple Kaluza-Klein or flat Minkowski space asymptotics. This is a minimal setup, which one can envision extending in many different directions, e.g. change the asymptotics, add more fields, more charges, supersymmetry etc. Despite the apparent simplicity of our setup (indeed solutions in this setup will be classified by a small number of asymptotic charges), we will discover that by changing the spacetime dimension a rich pattern of interconnecting phases with different horizon topology and stability properties arises.

1.1. Why study gravity in higher dimensions?

Classical general relativity in four spacetime dimensions has an obvious motivation: we live in an observably four-dimensional world. Why then venture into a different
dimension? There are several good reasons for that:

(i) String theory contains gravity and typically requires more than four dimensions. In fact, spacetime dimension itself is a dynamical concept in string theory. String theory may not have proven itself as the true theory of the nature so far, but it has had considerable success in explaining the microscopic degrees of freedom of black holes. The first successful statistical counting of black hole entropy was performed for a five-dimensional black hole [7].

(ii) The AdS/CFT correspondence relates a gravitational theory on asymptotically AdS spaces in \(d + 1\) dimensions to a non-gravitational quantum field theory (QFT) in \(d\) dimensions. Black holes in this context are useful for the analysis of the finite temperature properties of the QFT.

(iii) If large extra dimensions [8, 9] are realized in nature, higher dimensional black holes may arise as important physical objects with observable signatures in accelerators or in the universe (for a review see [10]).

Besides any potential applications classical general relativity in higher dimensions has an interest of its own. As we change the number of spacetime dimensions we discover new features with no counterpart in lower dimensions. The appearance of some differences is perhaps expected. After all, by increasing the number of spacetime dimensions we increase the number of degrees of freedom of the graviton and it is natural to expect that the theory will become more versatile. What is interesting to distinguish are precisely those properties of the theory that are dimension-dependent.

1.2. Novel features

In the following sections, we will encounter various instances where the standard lore of pure gravity in four dimensions breaks down as the spacetime dimension increases. Two of the most salient new features that we will encounter are the following.

1.2.1. Black hole (non-)uniqueness and horizon topology

The solutions of the vacuum Einstein equations \(R_{\mu\nu} = 0\) are constrained by powerful uniqueness theorems in four dimensions. A black hole is uniquely specified by the ADM mass \(M\) and the angular momentum \(J\) measured at infinity [11, 12, 13, 14]. The unique stationary solution is the Kerr black hole

\[
\begin{align*}
    ds^2 & = -dt^2 + \frac{\mu r}{\Sigma} (dt + a \sin^2 \theta d\phi)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2, \quad (1a) \\
    \Sigma & = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - \mu r + a^2, \quad \mu = 2GM, \quad a = \frac{J}{M}. \quad (1b)
\end{align*}
\]
$G$ is Newton's constant. For $J = 0$ we recover the unique static black hole solution: the Schwarzschild black hole.

The generalization of the Kerr black hole solution in more than four dimensions is provided by the Myers-Perry solution [15]. For a certain range of mass and angular momentum this is not anymore the unique stationary solution. The first explicit example of this violation of a uniqueness theorem was found by Emparan and Reall [16] in five dimensions. The new solution that does not exist in four dimensions is a black ring, i.e. a black hole with horizon topology $S^2 \times S^1$.

The black ring solution brings forth the violation of another uniqueness theorem that holds in four dimensions: the horizon topology uniqueness theorem. In four dimensions (with flat space asymptotics), one can show on general grounds that the horizon topology is unique. Hawking’s horizon topology theorem [18, 19] states that the integrated Ricci scalar over the two-dimensional horizon should be a positive number. Hence, the horizon topology can only be a sphere. This is no longer the case in more than four dimensions. Generalizations of Hawking’s horizon topology theorem have been considered in [18, 19]. The resulting restrictions on topology turn out to be less restrictive in higher dimensions. For example, one can show that the horizon must be positive Yamabe type, i.e. it must be a manifold with an everywhere positive Ricci scalar. In five dimensions the only allowed horizon topologies seem to be $S^3$ and $S^2 \times S^1$. In six dimensions many more choices appear to be allowed by topology, e.g. $S^4$, $S^3 \times S^1$, or even $S^2 \times \Sigma_g$, where $\Sigma_g$ is a two-dimensional Riemann surface with genus $g$. In more than six dimensions even a partial classification of allowed horizon topologies is currently lacking.

1.2.2. Stability and transitions

Another intriguing feature of higher dimensional gravity is the ubiquitous appearance of horizon instabilities and the dynamical transition between phases with different entropy and horizon topology. An example that arises in various contexts is the Gregory-Laflamme (GL) instability [20, 21], a long-wavelength instability that arises when the horizon has extended directions. In that case, the unstable mode evolves towards breaking up the horizon into smaller pieces. The possibility of the formation of a naked singularity during this process points towards a violation of the Cosmic Censorship Hypothesis. A related feature is the presence of merger points. These are points in the black hole phase diagrams, where black holes with different horizon topology meet in a topology changing transition.

Another interesting property of higher dimensional gravity is the appearance, in some cases, of a critical spacetime dimension, where some properties of the above patterns may change abruptly as we change the dimension.

Finally, various bounds that exist in four dimensions (like the Kerr bound $J \leq GM^2$) are lifted in higher dimensions. It is possible, although not exhibited in full generality, that a dynamical version of these bounds and a new version of black hole uniqueness is reinstated in higher dimensions by demanding stability [22]. The issue
of perturbative stability becomes more complicated when additional fields, besides the graviton, are present.

1.3. Solution generating techniques

The reasons that are responsible for the richness of higher dimensional gravity are also responsible for its complexity and our limited efficiency in uncovering its full breadth of solutions. Here are some of the methods that have been used successfully in this endeavor.

1.3.1. Exact methods

The complexity of the equations can be reduced, in general, by assuming certain symmetries. In such cases, a clever ansatz for the form of the solution, or a clever choice of the coordinate system may simplify the equations considerably. An example of this strategy is the Weyl ansatz \([23, 24]\) for static and stationary solutions with \(D - 2\) commuting Killing vectors in \(D\) dimensions. This method has been used successfully to find exact solutions in five and six-dimensional Kaluza-Klein (KK) spaces, and rotating black ring solutions in five dimensional asymptotically flat space.

In some cases, when an exact solution is known, many more can be generated with methods such as the inverse scattering method \([25, 26, 27, 28]\). This method has been extremely efficient for five-dimensional stationary black hole solutions in flat space. Unfortunately, this method is not as helpful for similar purposes in higher dimensions. For a further solution generating mechanism see \([29]\).

1.3.2. Approximate methods

In certain cases, we have no analytic tools to determine the exact solution, but we happen to know the solution is a certain regime of parameters. Then, one may attempt to find the solution with a perturbative expansion around the known limit. An example of such a method is the matched asymptotic expansion that has been employed successfully in \([30, 31, 32, 33, 34, 35]\). This method applies in problems that contain two (or more) widely separate scales. We will review its basic features in section 3.

1.3.3. Numerical work

When the analytical methods fail, the only way to proceed is by numerical methods. In the context of KK black holes these techniques have been successfully applied in \([36, 37, 38, 39, 40, 41]\) for non-uniform black strings and in \([42, 43, 44]\) for localized black holes (see section 2 for the definition of these objects). The combination of approximate (semi-exact) analytical methods and numerical analysis may be our best bet in generic situations where exact solutions seem out of reach.
1.4. A short outline

In this review we will focus on black hole solutions of the vacuum Einstein equations $R_{\mu\nu} = 0$ in $D \geq 4$ dimensions. Section 2 reviews the case of KK asymptotics. Section 3 reviews the case of stationary rotating black hole and black ring solutions in flat space with a single angular momentum. Several interesting open problems are summarized in the final section 4.

2. Kaluza-Klein black holes

Kaluza-Klein black holes will provide our first illustration of the novel features of higher dimensional gravity. Our definition of Kaluza-Klein black holes in $d+p$ dimensions will be the following: vacuum solutions of the pure gravity Einstein equations with at least one event horizon that asymptote at infinity to $d$-dimensional Minkowski spacetime times a $p$-torus ($M^d \times T^p$). For concreteness, we will mostly discuss here static and neutral solutions in the $p = 1$ (i.e. the $S^1$) case.

We will discover that certain aspects of the KK discussion in this section are instrumental in uncovering some of the crucial properties of rotating black holes and black rings in flat space in the next section.

A lengthy review of complementary aspects of the subject presented in this section can be found in [4, 5, 6].

2.1. The parameters

For any spacetime which asymptotes to $M^d \times S^1$ we can define the mass $M$ and tension $T$. These two asymptotic quantities can be used to parametrize the various phases of KK black holes in a $(\mu, n)$ phase diagram, which will be defined in a moment.

For any localized static object the mass and the tension can be determined from the asymptotics of the metric

$$g_{tt} \simeq -1 + \frac{c_t}{r^{d-3}}, \quad g_{zz} \simeq 1 + \frac{c_z}{r^{d-3}} \quad (2)$$

as $r \to \infty$. We are parametrizing the asymptotic spacetime $M^d \times S^1$ by the variables $t, x^1, ..., x^{d-1}$ and $z \sim z + L$. The radial variable $r = \sqrt{\sum_i (x^i)^2}$. In this notation, the mass $M$ and tension $T$ are given by

$$M = \frac{\Omega_{d-2} L}{16 \pi G} [(d-2)c_t - c_z], \quad T = \frac{\Omega_{d-2}}{16 \pi G} [c_t - (d-2)c_z] \quad (3)$$

where $\Omega_d = \frac{2 \pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$ is the volume of a unit $d$-sphere. For more details on the gravitational tension of black holes and branes we refer the reader to [47, 48, 49, 50, 51].

It will be convenient to define the following set of dimensionless quantities

$$\mu = \frac{16 \pi G}{L^{d-2}} M, \quad s = \frac{16 \pi G}{L^{d-1}} S, \quad n = \frac{T L}{M} \quad (4)$$
where $S$ is the entropy. For later purposes it will also be useful to define the quantities

$$
\ell = \mu^{-\frac{1}{D-3}}, \quad a_H = \mu^{-\frac{D-2}{D-3}} S
$$

where $D$ is the total spacetime dimension (here $D = d + 1$).

The program set forth in [45, 53] is to plot all phases of Kaluza-Klein black holes in a $(\mu, n)$ diagram. One can show on general grounds [45] that the relative tension $n$ should always lie within the range $0 \leq n \leq d - 2$. The upper bound follows from the Strong Energy Condition. The lower bound was found in [54, 55].

2.2. Black holes and strings on $\mathcal{M}^d \times S^1$

There are three main types of neutral and static black objects on $\mathcal{M}^d \times S^1$ with a single connected horizon and $SO(d - 1)$ symmetry – the uniform black string, the non-uniform black string and the localized black holes. The former two have an event horizon with topology $S^{d-2} \times S^1$ and the latter horizon topology $S^{d-1}$. All these phases are depicted for $d = 5$ in Fig. 1 which is based on the numerical results of Refs. [37, 44]. The most prominent features of this diagram are the following.

The uniform black string and Gregory-Laflamme instabilities. The metric of the uniform black string solution is

$$
\begin{align*}
\frac{ds^2}{f} &= -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_{d-2}^2 + dz^2, \\
f &= 1 - \frac{r_0^{d-3}}{r^{d-3}},
\end{align*}
$$

Fig. 1. Black hole and string phases for $d = 5$ in a $(\mu, n)$ phase diagram. The horizontal (red) line at $n = 1/(d - 2) = 1/3$ is the uniform string branch. The (blue) branch emanating at the Gregory-Laflamme point is the non-uniform string branch. It was obtained numerically in [37]. The (purple) branch starting at the point $(\mu, n) = (0, 0)$ is the localized black hole branch. It was obtained numerically in [44]. The numerical data suggest that the non-uniform and localized branches meet at a horizon topology changing merger point. Figure reprinted from Ref. [6].
where $d\Omega_{d-2}$ is the metric element of a $(d-2)$-dimensional unit sphere. This metric is simply the direct sum of the $d$-dimensional Schwarzschild-Tangherlini static black hole with an extra flat $z$ direction. This phase has constant relative tension $n = \frac{1}{d-2}$.

A crucial feature of the uniform black string solution is the Gregory-Laflamme instability – a long-wavelength instability [20, 21] associated to a mode that propagates along the direction of the string (the $z$ direction) and grows exponentially with time. The instability occurs for any $\mu < \mu_{GL}$. For $\mu > \mu_{GL}$ the string is believed to be classically stable. For $\mu = \mu_{GL}$ there is a marginal, static mode which signals the emergence of a new branch of black string solutions that are non-uniform along the circle. An analytic formula for the Gregory-Laflamme mass $\mu_{GL}$ is not known. Numerical estimates can be found in [20, 36, 38] and are summarized in [6]. A large $d$ analytical expression was derived in [56].

The non-uniform black string, the localized black hole and a merger. The new non-uniform branch emanating at $\mu_{GL}$ (the blue line in Fig. 1) has the same horizon topology as the uniform black string $S^{d-2} \times S^1$, but a smaller relative tension $n < \frac{1}{d-2}$. The analytic form of the non-uniform black string is not known. Perturbatively around $\mu_{GL}$ one finds that the relative tension behaves as

$$n(\mu) = \frac{1}{d-2} - \gamma(\mu - \mu_{GL}) + O((\mu - \mu_{GL})^2).$$

(7)

The qualitative behavior of the non-uniform branch depends on the sign of $\gamma$. Surprisingly, it turns out [38] that there is a critical dimension $D = d + 1 = 14$ where $\gamma$ changes sign. $\gamma$ is positive for $d \leq 12$ and negative for $d \geq 13$.

The phase diagram in Fig. 1 includes an additional branch (the purple branch) that comes off the origin $(\mu, n) = (0, 0)$. This is the branch of a black hole localized in the $S^1$. It has horizon topology $S^{d-1}$. The existence of such solutions is intuitively clear. In the small mass limit $\mu \ll 1$ and to leading order, this is simply a $(d + 1)$-dimensional Schwarzschild black hole. The presence of the periodic boundary conditions in $z$ deform the finite $\mu$ solution away from Schwarzschild. The exact analytic form of this branch is also unknown. It is possible, however, to analyze the metric of small black holes analytically by using the method of asymptotic expansions [30, 31, 33]. This method sets up a perturbative expansion around the small mass limit $\mu \ll 1$. A similar approach in the next section will teach us about higher dimensional black rings in an ultra-spinning regime.

Fig. 1 suggests that the above two branches meet at a topology changing transition point. The scenario of such a transition was suggested early on by Kol [57]. More details about the physics of the merger point can be found in [57, 4].

Other phases. There are several other phases that supplement the $(\mu, n)$ phase diagram in Fig. 1. These include:

- Copies. In [53] it was argued that one can generate new solutions by copying solutions on the circle several times following an idea in [58]. For any positive integer $k$ the new solution has mass and relative tension $\tilde{\mu} = \frac{\mu}{k}$, $\tilde{n} = n$. More details can
be found in [6] and the original references therein.

- **Kaluza-Klein bubbles.** All solutions in Fig. 1 lie within the range $0 \leq n \leq \frac{1}{d-2}$. Solutions in the range $\frac{1}{d-2} < n \leq d-2$ were argued to exist in [59] (see also [23, 60]). They comprise of the static Kaluza-Klein bubble and bubble-black hole sequences. Further details on these solutions can be found in [6] and the original references therein.

- **Multi-black hole solutions.** Multi-black hole solutions were considered recently in Ref. [34] using the method of asymptotic expansions.

### 2.3. A note on black membranes and black strings on $\mathcal{M}^d \times T^2$

There is a similar pattern of phases in other Kaluza-Klein spaces, e.g. $\mathcal{M}^d \times T^p$. Clearly new possibilities arise as we increase the dimension of the compact manifold. An example that will be relevant for the discussion of the next section is the case of the torus ($p = 2$). Black membrane phases that are non-uniform in one of the directions of the two-torus can be imported straightforwardly from the simpler $p = 1$
case that we described above. Such phases were considered in [35]. In Fig. 2, which is taken from Ref. [35], we plot the rescaled entropy \(a_H\) in terms of the rescaled mass parameter \(\ell\) (see the definition (5)) for the uniform black membrane phase (dotted line), the non-uniform black membrane phase (solid line), the localized black string phase (dashed line) and the \(k = 2\) copies of the latter two.

3. Phases of neutral rotating black holes and rings in flat space

In this section we will summarize what is known about the phases of neutral rotating black holes in the \(D\)-dimensional Minkowski space \(\mathcal{M}^D\). Again we are considering pure gravity, so we are solving the vacuum Einstein equations \(R_{\mu\nu} = 0\). In \(D\) dimensions a black hole can rotate in \(\frac{D-1}{2}\) independent directions. For simplicity, we will consider the case of black holes with a single non-vanishing angular momentum, i.e. black holes rotating on a single plane. Sometimes, it will be convenient to denote the total dimension as \(D = n + 4\).

In order to compare the properties of different phases a common scale needs to be introduced. Classical general relativity with Minkowski space asymptotics does not possess an intrinsic scale, hence we will take our reference scale to be one of the physical parameters of our solutions. We choose to take the mass and define the dimensionless quantities

\[ j^{n+1} \propto \frac{J^{n+1}}{GM^{n+2}}, \quad a_H^{n+1} \propto \frac{A^{n+1}}{(GM)^{n+2}} \]  

where \(J\) is the angular momentum and \(A\) the horizon area. Convenient proportionality constants can be found in [35]. We will plot different phases in a “phase diagram” of the rescaled horizon area \(a_H\) versus rescaled angular momentum \(j\).

To show clearly how dimension affects the qualitative features of rotating black holes we will consider in turn the phase diagrams in four, five and higher dimensions. The material in this section is heavily based on Ref. [35] where the reader can find a more detailed discussion.

3.1. The “simplicity” of four dimensions

As we reviewed in the introduction, vacuum black hole solutions of pure gravity are severely constrained by uniqueness theorems. This is amply visible in the phase diagram of Fig. 3 - there is a single branch, that of the Kerr black hole. The well known analytic form of the Kerr black hole metric (1) allows the explicit computation of the horizon area

\[ A = 8\pi G \left( GM^2 + \sqrt{G^2 M^4 - J^2} \right). \]

The branch terminates at a maximum value of the angular momentum \(J_* = GM^2\), the Kerr bound. At \(J_*\) the black hole has finite horizon area.

Another feature of four dimensions is the absence of stationary multi-black hole solutions. We will soon see that this is another property unique to four dimensions.
Fig. 3. A plot of the rescaled entropy $a_H$ versus rescaled angular momentum $j$ in four dimensions for rotating black hole solutions. There is a single branch – the Kerr black hole – ending at the Kerr bound value of the angular momentum $j^*$.

Fig. 4. A plot of the rescaled entropy $a_H$ versus rescaled angular momentum squared $j^2$ in five dimensions for rotating black hole solutions. Three branches are plotted: the Myers-Perry black hole branch (green), the black ring branch (more entropic black plus red lines) and the black saturn branch (less entropic black plus red lines). For the black ring/saturn the black line represents the fat black ring/saturn and the red line the thin black ring/saturn. It is possible that these phases exhaust the allowed phases of rotating black holes with a single angular momentum in thermal equilibrium.

3.2. The “finesse” of five dimensions

The phase diagram of five-dimensional neutral black holes with a single angular momentum is depicted in Fig. 4. It is possible [61] that the phases depicted in Fig. 4 exhaust the allowed phases of rotating black holes with a single angular momentum in thermal equilibrium (see, however, [62]). Work related to the classification of the five-dimensional phases can be found in [63, 64]. The main properties of the phases in Fig. 4 are the following.

The green line extending from $J = 0$ (the Schwarzschild-Tangerlini black hole) to $J = J_\ast = \sqrt{\frac{32G^2}{27\pi}} M^3$ (a five-dimensional version of the Kerr bound) represents the
Myers-Perry (MP) branch. This branch is the generalization of the Kerr black hole in five dimensions. Hence, it represents a rotating black hole with horizon topology $S^3$. The metric is known analytically for the Myers-Perry black hole [15]. The presence of a maximal angular momentum is common in four and five dimensions, however, in five dimensions the maximally spinning black hole has zero horizon area, i.e. it exhibits a naked singularity.

Besides the MP branch, Fig. 4 includes two more phases: the black ring [16] and the black saturn phases [65]. Both of them have two components – one depicted by a thick black line (commonly known as the fat black ring/saturn), and another depicted by a thin red line that goes off to infinite angular momentum (the latter is known as the thin black ring/saturn). The black ring branch is the most entropic phase at large angular momentum. It represents a rotating black hole solution with a single connected horizon of topology $S^2 \times S^1$. The black saturn branch has a disconnected horizon with two components: a central horizon with topology $S^3$ and an outer horizon with topology $S^2 \times S^1$. This is an object in static equilibrium that comprises of a central rotating MP black hole and an outer rotating black ring. The two disconnected horizons are in thermodynamic equilibrium, i.e. they have the same temperature and horizon angular velocity.

The neutral black rings and saturns that we consider here have several types of instabilities. A complete classical stability analysis is complicated and has not been performed for these objects, however, in certain regimes it is known that these objects exhibit instabilities. For example, an ultra-spinning thin black ring is well-approximated by a boosted black string and boosted black strings are known to suffer from Gregory-Laflamme instabilities [16, 66]. A fat black ring is also expected to suffer from instabilities under radial perturbations [67]. It is unclear whether there is an intermediate regime of angular momentum where the black ring is stable. Similarly, black saturns are expected to suffer from instabilities. Some of them are instabilities of the outer black ring, others are instabilities special to the black saturn. For instance, moving the central MP black hole off the center is expected to destabilize the saturn.

More phases can be added to the phase diagram in Fig. 4 as soon as we relinquish the assumption of thermodynamic equilibrium. Then, one can argue for the existence of continuous families of multi-black hole solutions that cover any point in the strip with $0 \leq j < \infty$ and $0 < a_H < a_{H,\text{max}}$ ($a_{H,\text{max}}$ is the area of a Schwarzschild black hole) [61]. At any non-zero $j$, the maximum area is achieved by black Saturns having an almost static central black hole that carries most of the mass, and a very thin and long black ring that carries most of the angular momentum.

The presence of all these new phases demonstrates clearly the lessons we outlined in the introduction. The uniqueness theorems that were so powerful in four dimensions are now explicitly violated and the new phases exhibit novel properties, e.g. new horizon topologies, stationary multi-black hole configurations etc.
Fig. 5. Area vs spin for fixed mass, $a_H(j)$, in seven dimensions. The thick curve is the exact result for the MP black hole. The vertical dotted line intersects this curve at the inflection point $j_{\text{mem}} = 2^{1/4}/\sqrt{3}, a_H = \sqrt{2}$. It signals the onset at larger $j$ of membrane-like behavior for MP black holes. The thin curve represents thin black rings which have been analyzed via the method of asymptotic expansions at large $j$. Figure reprinted from Ref. [35].

3.3. The abyss of higher dimensions

Five dimensions are special for several reasons. First of all, in five dimensions the complexity of the Einstein equations is still under considerable control. Symmetries and the solution generating techniques of section 1.3 allow for a fairly complete picture of the allowed phases. This is not the case for six dimensions and higher, where semi-quantitative methods are currently the major source of information. The still illusive discovery of the exact metric of a higher dimensional black ring is a characteristic example of the increasing complexity of higher dimensional gravity. A nice general discussion on the difficulties of higher dimensional gravity can be found in the introductory section of Ref. [2].

The special nature of five dimensions is qualitatively apparent also in some of the key properties of the Myers-Perry solutions, which are known analytically in any dimension $D$. What controls the key qualitative properties of the MP solution is a competition between the centrifugal repulsion $\frac{J^2}{M^2 r^2}$, which is dimension independent, and the gravitational attraction $-\frac{G M}{r^{D-2}}$, which is dimension dependent. For $D = 4$ the gravitational attraction is the contribution that dominates at large distances. For $D > 5$ it is the centrifugal repulsion. For $D = 5$ the two contributions are comparable.

The Myers-Perry curve is depicted as a solid black line in Fig. 5 (for $D = 7$). The most notable new feature of the $D > 5$ MP curves is the absence of a Kerr bound. In fact, the curve changes its Kerr-like character at large enough angular momentum.
The point where this change takes place can be approximated by the inflection point \( j_{\text{mem}} \) where \( \frac{\partial^2 a_H}{\partial j^2} = 0 \). It has been argued [68] that the ultra-spinning regime of the Myers-Perry black hole is captured by a static black membrane solution. The horizon spreads out along the plane of rotation with a size proportional to the angular momentum. We will return to this point in a moment.

### 3.4. Forging the black ring

Besides the Myers-Perry black holes it is natural to expect on the basis of the five-dimensional example that black rings are also part of the \( D \)-dimensional phase diagram. In the absence of exact black ring solutions in arbitrary dimensions, Ref. [35] proceeded to construct approximate black ring solutions around an ultra-spinning regime using the method of asymptotic expansions.

Black rings in any dimension \( D \) are objects of horizon topology \( S^{D-3} \times S^1 \). They are characterized by two dimensionful parameters: \( r_0 \) (roughly the fatness of the horizon) and \( R \) (the \( S^1 \) radius of the horizon). The physical intuition behind a black ring is the simple intuitive picture that a black ring arises from a straight black string by bending the string and spinning it to balance the circular ring. This picture becomes more precise in the ultra-spinning regime.

In the regime of large angular momentum the ring becomes long and thin and the two scales \( r_0, R \) become widely separated, i.e. \( r_0/R \ll 1 \). In this case, a black ring is well approximated by a boosted black string. This is an object with horizon topology \( S^{D-3} \times \mathbb{R} \) and metric (recall \( n = D - 4 \geq 1 \))

\[
\alpha \text{ is a boost parameter whose precise value we will determine in a moment.}
\]

The basic idea behind the method of matched asymptotic expansions is to setup a perturbative scheme where we solve the equations of motion independently in the near-horizon and far-away regions of the black object and then match the two expansions in the overlap region which is large when \( r_0 \ll R \). The near-horizon expansion is an expansion in powers of \( r/R \) around the boosted black string solution (10), whereas in the far-away region we expand in powers of \( r_0/r \).

Ref. [35] performed this expansion to first order and found an explicit solution which describes, to this order, a thin black ring solution in arbitrary dimension. In the process, one finds that regularity of the solution forces a single value of the boost parameter \( \alpha \) in (10). This is

\[
\sinh^2 \alpha = \frac{1}{n} \iff R = \frac{n + 2}{\sqrt{n+1}} \frac{J}{M} .
\]  

The second equality between the different parameters of the boosted black string expresses the balancing condition of the ring. This condition is equivalent to the
vanishing of the tension of the boosted black string, i.e. $T_{zz} = 0$. This is a special case of the Carter equations \[ K^{\rho}_{\mu\nu} T^\mu_{\nu} = 0 \]

which follow from the conservation law of the stress-energy momentum tensor. $K^{\rho}_{\mu\nu}$ is the second fundamental tensor, an object that extends the notion of extrinsic curvature to submanifolds of co-dimension larger than one.

The method of asymptotic expansions and the Carter equations (12) are also a useful tool in the exploration of black objects with more general horizon topologies \[70\]. Recently, these tools have been used successfully to extend the above analysis to black rings in (A)dS \[72\].

3.5. **Towards a complete phase diagram**

The above results demonstrate the existence of a thin black ring in arbitrary dimensions. We plotted this new phase as a thin line in Fig. 5. Notice that the black ring is more entropic than the MP black hole in the ultra-spinning regime.

A nice picture for the rest of the phase diagram follows from the ensuing observation. We noticed above that the MP black hole is approximated well beyond the critical spin $j_{\text{mem}}$ by a black square torus with size $L \sim \frac{1}{\sqrt{M}}$ and $S^{D-4}$ size $r_0$. In section 2 we reviewed the basic features of the phase diagram of black strings and

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*An early report on the results of \[70\] can be found in \[71\].
branes. Hence, the analogy between ultra-spinning MP black holes and black membranes suggests the picture summarized in Fig. 6. The pinching of the horizon of a non-uniform black membrane to the horizon of a localized black string corresponds to the pinching of the horizon of an ultra-spinning MP black hole to the horizon of black ring. Thus, importing the black membrane phase diagram in Fig. 2 to the phase diagram of rotating black holes in $D$ dimensions suggests the completion of Fig. 5 that appears in Fig. 7 (in this analogy associate the KK mass $\mu$ with the MP spin $j$). The analog of the Gregory-Laflamme mode on the MP branch signals the existence of a pinched black hole which merges with the thin black ring branch.

The analogy with Fig. 2 suggests the existence of further GL-like points along the MP branch. From each of these points a pinched black hole with more and more pinches comes off. The second GL point gives rise to a MP black hole that develops a circular pinch, which then grows deeper until the merger with a black Saturn configuration in thermal equilibrium (see Fig. 8).

There is a possibility that a second kind of black Saturn, also in thermal equilibrium, exists in $D \geq 6$, which would not have a counterpart in five dimensions. Indeed, in our phase diagrams, we fix the total $M$ and $J$ of the black Saturn, but the mass and spin of its two constituents are not fixed separately. Instead, under thermal equilibrium we demand only that the temperature and angular velocity of the black ring and the central black hole are equal. So, besides the black Saturn with a small, round central black hole, that we have discussed above, it may be possible to have another one with a large, pancaked central black hole. With the assumption that these pancaked black Saturns exist, Ref. [35] proposed a picture
Fig. 8. Proposal for the phase diagram of thermal equilibrium phases in $D \geq 6$. The solid lines and figures have significant arguments in their favor, while the dashed lines and figures might not exist and admit conceivable, but more complicated, alternatives. Reprinted from Ref. [35].

where the pancaked black Saturn branch persists to infinite angular momentum and in between exhibits GL-like zero modes from which new branches of pinched black Saturn solutions emerge. A possible completion of the phase diagram where these solutions merge with the above-mentioned pinched MP black holes is depicted in Fig. 8. A more detailed discussion of this proposal and its underlying assumptions can be found in Ref. [35]. The more recent work [70] suggests that the phase of pancaked black Saturns is unlikely and hence that this part of the phase diagram in Fig. 8 needs to be modified.

4. Open problems

There are many aspects of higher dimensional black holes that we did not discuss in this short review. For a more complete discussion of complementary aspects of higher dimensional black holes, e.g. more examples and applications, we recommend the Refs. [1, 2, 3, 4, 5, 6]. We would like to conclude with a brief list of interesting open problems related to the issues covered in this review. More can be found in the aforementioned references.

**Beyond the ultra-spinning regime.** It would be interesting to find a direct construction of the entire black ring phase and of the other branches proposed in Fig. 5. Because of the complexity of the problem, numerical methods appear to be the most promising approach.

**Classical stability.** For most of the phases described in this review a classical stability analysis has not been performed explicitly. For some of them it is known
in a suitable regime that the black hole is unstable. For example, thin black rings and pancaked MP black holes in the ultra-spinning regime are well-approximated by black strings and membranes, which are known to suffer from Gregory-Laflamme instabilities \[10, 66, 67, 68\]. However, this instability may switch off at lower spin \( j \sim O(1) \). Other cases where it would be nice to know if there are unstable modes include the non-uniform black string and the localized black hole for KK black holes and the rotating pinched black holes in flat space.

Many features of stability change when the system includes more fields beside the metric. It would be interesting to find general \( D \) situations where the stability properties of the neutral solutions in this review are improved.

**More fields, other asymptotics.** Semi-quantitative methods, like the method of asymptotic expansions, are useful in uncovering the properties of black holes in more general situations with more fields, more charges and other asymptotics. Black rings in AdS or dS spacetimes have been studied recently in this way in \[72\]. We point out that exact five-dimensional black ring solutions with charges and dipoles have been constructed in \[73, 74, 75\]. The existence of small supersymmetric black rings in \( D \geq 5 \) was argued in \[76\]. Neutral and charged black strings in AdS have been discussed in \[77, 78, 79, 80\].

**More angular momenta.** The phase diagrams proposed in Ref. \[35\] are referring to black holes with a single angular momentum. One may try to extend these ideas to black rings with horizon \( S^{n+1} \times S^1 \) rotating not only along the \( S^1 \), but also along the \( S^{n+1} \). Ultra-spinning along the extra directions may lead to additional pinches which presumably connect to phases with horizon topology \( S^n \times S^1 \times S^1 \). In five dimensions exact solutions of doubly rotating black rings have been considered in \[81\]. Extremal multi-spinning black rings in diverse dimensions in pure gravity were discussed in \[82\].

**Blackfolds – other horizon topologies.** The basic physical intuition that a process of bending and spinning black strings gives rise to black rings may be applied to more general black \( p \)-branes to obtain black holes with more exotic horizon topologies – black objects we will generally refer to as blackfolds. The method of asymptotic expansions and the balancing condition \[12\] are useful constructive tools in the exploration of such objects. Progress in this direction has been achieved recently in \[70\].

**Plasma balls and rings.** For black holes in AdS there is also another more indirect approach based on the AdS/CFT correspondence. In the original Ref. \[83\] a correspondence was setup between stationary, axially symmetric, spinning configurations of fluid in \( \mathcal{N} = 4 \) super-Yang-Mills compactified to \( d = 3 \) on a Scherk-Schwarz circle and large rotating black holes and black rings in AdS. The phase diagram of the rotating fluid exhibits many of the qualitative features of MP black holes and black rings in asymptotically flat space that we discussed in this review. Higher dimensional generalizations of this setup provide predictions for black holes
in compactified $AdS_D$ with $D > 5$. Further work in this direction has appeared in [84, 85, 86, 87]. There is a fruitful interplay between this method and more direct approaches to black holes in AdS, which deserves further study.

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**References**

1. R. Emparan and H. S. Reall, “Black rings,” *Class. Quant. Grav.* **23** (2006) R169, hep-th/0608012.
2. R. Emparan and H. S. Reall, “Black Holes in Higher Dimensions,” [0801.3471](https://arxiv.org/abs/0801.3471).
3. N. A. Obers, “Black Holes in Higher-Dimensional Gravity,” [0802.0519](https://arxiv.org/abs/0802.0519).
4. B. Kol, “The phase transition between caged black holes and black strings: A review,” *Phys. Rept.* **422** (2006) 119–165, hep-th/0411240.
5. T. Harmark and N. A. Obers, “Phases of Kaluza-Klein black holes: A brief review,” hep-th/0503020.
6. T. Harmark, V. Niarchos, and N. A. Obers, “Instabilities of black strings and branes,” *Class. Quant. Grav.* **24** (2007) R1–R90, hep-th/0701022.
7. A. Strominger and C. Vafa, “Microscopic Origin of the Bekenstein-Hawking Entropy,” *Phys. Lett.* **B379** (1996) 99–104, hep-th/9601029.
8. N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, “The hierarchy problem and new dimensions at a millimeter,” *Phys. Lett.* **B429** (1998) 263–272, hep-ph/9803315.
9. I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, “New dimensions at a millimeter to a Fermi and superstrings at a TeV,” *Phys. Lett.* **B436** (1998) 257–263, hep-ph/9804398.
10. P. Kanti, “Black holes in theories with large extra dimensions: A review,” *Int. J. Mod. Phys.* **A19** (2004) 4899–4951, hep-ph/0402168.
11. W. Israel, “Event horizons in static vacuum space-times,” *Phys. Rev.* **164** (1967) 1776–1779.
12. B. Carter, “Axisymmetric black hole has only two degrees of freedom,” *Phys. Rev. Lett.* **26** (1971) 331–333.
13. S. W. Hawking, “Black holes in General Relativity,” *Commun. Math. Phys.* **25** (1972) 152–166.
14. D. C. Robinson, “Uniqueness of the Kerr black hole,” *Phys. Rev. Lett.* **34** (1975) 905–906.
15. R. C. Myers and M. J. Perry, “Black holes in higher dimensional space-times,” *Ann. Phys.* **172** (1986) 304.
16. R. Emparan and H. S. Reall, “A rotating black ring in five dimensions,” *Phys. Rev. Lett.* **88** (2002) 101101, hep-th/0110260.
17. S. W. Hawking and G. F. R. Ellis, “The Large scale structure of space-time.”
Cambridge University Press, Cambridge, 1973.
18. C. Helfgott, Y. Oz, and Y. Yanay, “On the topology of black hole event horizons in higher dimensions,” *JHEP* **02** (2006) 025, [hep-th/0509013](hep-th/0509013).
19. G. J. Galloway and R. Schoen, “A generalization of Hawking’s black hole topology theorem to higher dimensions,” *Commun. Math. Phys.* **266** (2006) 571–576, [gr-qc/0509107](gr-qc/0509107).
20. R. Gregory and R. Laflamme, “Black strings and $p$-branes are unstable,” *Phys. Rev. Lett.* **70** (1993) 2837–2840, [hep-th/9301052](hep-th/9301052).
21. B. Kol, “Speculative generalization of black hole uniqueness to higher dimensions,” [hep-th/0208056](hep-th/0208056).
22. R. Emparan and H. S. Reall, “Generalized Weyl solutions,” *Phys. Rev. D* **65** (2002) 084025, [hep-th/0110258](hep-th/0110258).
23. T. Harmark, “Stationary and axisymmetric solutions of higher-dimensional general relativity,” *Phys. Rev. D* **70** (2004) 124002, [hep-th/0408141](hep-th/0408141).
24. V. A. Belinsky and V. E. Zakharov, “Integration of the Einstein equations by the inverse scattering problem technique and the calculation of the exact soliton solutions,” *Sov. Phys. JETP* **48** (1978) 985–994.
25. V. A. Belinsky and V. E. Zakharov, “Stationary gravitational solitons with axial symmetry,” *Sov. Phys. JETP* **50** (1979) 1.
26. V. Belinski and E. Verdaguer, “Gravitational solitons.” Cambridge, UK: Univ. Pr. (2001) 258 p.
27. O. J. C. Dias, T. Harmark, R. C. Myers, and N. A. Obers, “Multi-black hole configurations on the cylinder,” *JHEP* **06** (2004) 053, [hep-th/0406002](hep-th/0406002).
28. D. Karasik, C. Sahabandu, P. Suranyi, and L. C. R. Wijewardhana, “Analytic approximation to 5 dimensional black holes with one compact dimension,” *Phys. Rev. D* **71** (2005) 024024, [hep-th/0410078](hep-th/0410078).
29. D. Gorbonos and B. Kol, “Matched asymptotic expansion for caged black holes: Regularization of the post-Newtonian order,” *Class. Quant. Grav.* **22** (2005) 3935–3960, [hep-th/0505009](hep-th/0505009).
30. E. Sorkin, “A critical dimension in the black-string phase transition,” *Phys. Rev. Lett.* **93** (2004) 031601, [hep-th/0402216](hep-th/0402216).
39. B. Kleihaus, J. Kunz, and E. Radu, “New nonuniform black string solutions,” JHEP 06 (2006) 016, hep-th/0603119.

40. E. Sorkin, “Non-uniform black strings in various dimensions,” Phys. Rev. D74 (2006) 104027, gr-qc/0608115.

41. B. Kleihaus and J. Kunz, “Interior of Nonuniform Black Strings,” 0710.1726.

42. E. Sorkin, B. Kol, and T. Piran, “Caged black holes: Black holes in compactified spacetimes. II: 5d numerical implementation,” Phys. Rev. D69 (2004) 064032, hep-th/0310096.

43. H. Kudoh and T. Wiseman, “Properties of Kaluza-Klein black holes,” Prog. Theor. Phys. 111 (2004) 475–507, hep-th/0310104.

44. H. Kudoh and T. Wiseman, “Connecting black holes and black strings,” Phys. Rev. Lett. 94 (2005) 161102, hep-th/0409111.

45. T. Harmark and N. A. Obers, “New phase diagram for black holes and strings on cylinders,” Class. Quantum Grav. 21 (2004) 1709–1724, hep-th/0309116.

46. B. Kol, E. Sorkin, and T. Piran, “Caged black holes: Black holes in compactified spacetimes. I: Theory,” Phys. Rev. D69 (2004) 064031, hep-th/0309190.

47. T. Harmark and N. A. Obers, “General definition of gravitational tension,” JHEP 05 (2004) 043, hep-th/0403103.

48. T. Harmark and N. A. Obers, “New phases of near-extremal branes on a circle,” JHEP 09 (2004) 022, hep-th/0407094.

49. R. C. Myers, “Stress tensors and Casimir energies in the AdS/CFT correspondence,” Phys. Rev. D60 (1999) 046002, hep-th/9903103.

50. J. H. Traschen and D. Fox, “Tension perturbations of black brane spacetimes,” Class. Quant. Grav. 21 (2004) 289–306, gr-qc/0103106.

51. P. K. Townsend and M. Zamaklar, “The first law of black brane mechanics,” Class. Quant. Grav. 18 (2001) 5269–5286, hep-th/0107228.

52. D. Kastor and J. Traschen, “Stresses and strains in the first law for Kaluza-Klein black holes,” JHEP 09 (2004) 022, hep-th/0607051.

53. T. Harmark and N. A. Obers, “Phase structure of black holes and strings on cylinders,” Nucl. Phys. B684 (2004) 183–208, hep-th/0309230.

54. J. H. Traschen, “A positivity theorem for gravitational tension in brane spacetimes,” Class. Quant. Grav. 21 (2004) 1343–1350, hep-th/0308173.

55. T. Shiromizu, D. Ida, and S. Tomizawa, “Kinematical bound in asymptotically translationally invariant spacetimes,” Phys. Rev. D69 (2004) 027503, gr-qc/0309061.

56. B. Kol and E. Sorkin, “On black-brane instability in an arbitrary dimension,” Class. Quant. Grav. 21 (2004) 4793–4804, gr-qc/0407058.

57. B. Kol, “Topology change in general relativity and the black-hole black-string transition,” hep-th/0206220.

58. G. T. Horowitz, “Playing with black strings,” hep-th/0205069.

59. H. Elvang, T. Harmark, and N. A. Obers, “Sequences of bubbles and holes: New phases of Kaluza-Klein black holes,” JHEP 01 (2005) 003, hep-th/0407050.

60. H. Elvang and G. T. Horowitz, “When black holes meet Kaluza-Klein bubbles,” Phys. Rev. D67 (2003) 044015, hep-th/0210303.

61. H. Elvang, R. Emparan, and P. Figueras, “Phases of five-dimensional black holes,” JHEP 05 (2007) 056, hep-th/0702111.

62. J. Evslin and C. Krishnan, “Metastable Black Saturns,” 0804.4575.

63. Y. Morisawa and D. Ida, “A boundary value problem for the five-dimensional rotating black holes,” Phys. Rev. D69 (2004) 124005, gr-qc/0401100.

64. S. Hollands and S. Yazadjiev, “Uniqueness theorem for 5-dimensional black holes.”
with two axial killing fields,” arXiv:0707.2775 [gr-qc].

65. H. Elvang and P. Figueras, “Black saturn,” JHEP 05 (2007) 050, hep-th/0701035.

66. J. L. Hovebo and R. C. Myers, “Black rings, boosted strings and Gregory-Laflamme,” Phys. Rev. D73 (2006) 084013, hep-th/0601079.

67. H. Elvang, R. Emparan, and A. Virmani, “Dynamics and stability of black rings,” JHEP 12 (2006) 074, hep-th/0608076.

68. R. Emparan and R. C. Myers, “Instability of ultra-spinning black holes,” JHEP 09 (2003) 025, hep-th/0308056.

69. B. Carter, “Essentials of classical brane dynamics,” Int. J. Theor. Phys. 40 (2001) 2099–2130, gr-qc/0012036.

70. R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, “Blackfolds,” to appear.

71. R. Emparan, “New phases of black holes in higher dimensions,” talk given at “Gravitational Thermodynamics and the Quantum Nature of Space Time,” ICMS, Edinburgh, June 16, 2008.

72. M. M. Caldarelli, R. Emparan, and M. J. Rodriguez, “Black Rings in (Anti)-deSitter space,” 0806.1954.

73. H. Elvang, “A charged rotating black ring,” hep-th/0305247.

74. H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, “A supersymmetric black ring,” Phys. Rev. Lett. 93 (2004) 211302, hep-th/0407065.

75. R. Emparan, “Rotating circular strings, and infinite non-uniqueness of black rings,” JHEP 03 (2004) 064, hep-th/0402149.

76. A. Dabholkar, N. Iizuka, A. Iqubal, A. Sen, and M. Shigemori, “Spinning strings as small black rings,” JHEP 04 (2007) 017, hep-th/0611166.

77. Y. Brihaye, T. Delsate and E. Radu, “On the stability of AdS black strings,” Phys. Lett. B 662 (2008) 264, 0710.4034.

78. T. Delsate, “Perturbative non uniform black strings in AdS6,” Phys. Lett. B 663, 118 (2008), 0802.1392.

79. Y. Brihaye and T. Delsate, “Charged-Rotating Black Holes in Higher-dimensional (A)DS-Gravity,” 0806.1583.

80. T. Delsate, “New stable phase of non uniform black strings in AdS4,” 0808.2752.

81. A. A. Pomeransky and R. A. Sen’kov, “Black ring with two angular momenta,” hep-th/0612005.

82. P. Figueras, H. K. Kunduri, J. Lucietti and M. Rangamani, “Extremal vacuum black holes in higher dimensions,” Phys. Rev. D 78 (2008) 044042, 0803.2998.

83. S. Lahiri and S. Minwalla, “Plasmarings as dual black rings,” arXiv:0705.3404 [hep-th].

84. S. Bhattacharyya, S. Lahiri, R. Loganayagam, and S. Minwalla, “Large rotating AdS black holes from fluid mechanics,” 0708.1770.

85. S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, “Nonlinear Fluid Dynamics from Gravity,” JHEP 02 (2008) 045, 0712.2456.

86. S. Bhattacharyya et al., “Forced Fluid Dynamics from Gravity,” 0806.0006.

87. S. Bhardwaj and J. Bhattacharya, “Thermodynamics of Plasmarballs and Plasmarings in 3+1 Dimensions,” 0806.1897.