Computing and Testing Small Vertex Connectivity in Near-Linear Time and Queries

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Abstract

We present a new, simple, algorithm for the local vertex connectivity problem (localVC) introduced by Nanongkai et al. [STOC’19]. Roughly, given an undirected unweighted graph $G$, a seed vertex $x$, a target volume $\nu$, and a target separator size $k$, the goal of LocalVC is to remove $k$ vertices “near” $x$ (in terms of $\nu$) to disconnect the graph in “local time”, which depends only on parameters $\nu$ and $k$. Nanongkai et al. presented an $O(\nu^{1.5}k \text{polylog}(\nu k))$-time deterministic algorithm for this problem. In this paper, we present a simple randomized algorithm with running time $O(\nu k^2)$ and correctness probability $2/3$. Our algorithm is faster than the previous one when $k = O(\sqrt{\nu})$. We also can handle directed graphs and achieve $(1 + \epsilon)$-approximation with even faster running time.

Plugging our new localVC algorithm in the generic framework of Nanongkai et al. immediately lead to a randomized $\tilde{O}(m + nk^3)$-time algorithm for the classic $k$-vertex connectivity problem on undirected graphs. ($O(T)$ hides polylog($T$).) This is the first near-linear time algorithm for any $4 \leq k \leq \text{polylog } n$. Previously, linear-time algorithms were known only for $k \leq 3$ [Tarjan FOCS’71; Hopcroft, Tarjan SICOMP’73], despite a linear-time algorithm being postulated since 1974 in the book of Aho, Hopcroft and Ullman. Previous fastest algorithm for small $k$ takes $\tilde{O}(m + n^{4/3}k^{7/3})$ time [Nanongkai et al., STOC’19].

This work is inspired by the algorithm of Chechik et al. [SODA’17] for computing the maximal $k$-edge connected subgraphs. In turn, our algorithms lead to some improvements over the bounds of Chechik et al. Forster and Yang [arXiv’19] has independently developed local algorithms similar to ours, and showed that they lead to an $\tilde{O}(k^3/\epsilon)$ bound for testing $k$-edge and -vertex connectivity, resolving two long-standing open problems in property testing since the work of Goldreich and Ron [STOC’97] and Orenstein and Ron [Theor. Comput. Sci.’11]. Inspired by this, we use local approximation algorithms to obtain bounds that are near-linear in $k$, namely $\tilde{O}(k/\epsilon)$ and $\tilde{O}(k/\epsilon^2)$ for the bounded and unbounded degree cases, respectively. For testing $k$-edge connectivity for simple graphs, the bound can be improved to $\tilde{O}(\min(k/\epsilon, 1/\epsilon^2))$.

\textbf{Independent work:} Independently from our result, Forster and Yang [2019] present local algorithms similar to ours and observed faster algorithms for computing the vertex connectivity and the maximal $k$-edge connected subgraphs (with lower running time than ours for the second problem).\textsuperscript{1} They additionally observed that this leads to resolving two open problems in property testing, but did not observe an application to approximating the vertex connectivity. Our bounds for testing connectivities were inspired by their observation.

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\textsuperscript{1}On April 17, 2019, our result was announced in the TCS+ talk by Thatchaphol Saranurak (https://youtu.be/V1q2filhjk) and Forster and Yang announced their result at https://arxiv.org/abs/1904.08382.
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1 Introduction

Vertex connectivity is a basic graph-theoretic concept. It concerns the smallest vertex cut where a set $S$ of vertices is a vertex cut of a graph $G$ if its removal disconnects some vertex $u \notin S$ from another vertex $v \notin S$. (When $G$ is directed, this means that there is no directed path from $u$ to $v$ in the remaining graph.) The vertex connectivity of $G$, denoted by $\kappa_G$, is the size of the smallest vertex cut. The goal of the vertex connectivity problem is to compute $\kappa_G$ and the smallest vertex cut. In this paper, we present a new, simple, algorithm for the vertex connectivity problem. There has been a long line of research on this problem since at least five decades ago (e.g. [Kle69, Pod73, ET75, Eve75, Gal80, EH84, Mat87, BDD+82, LLW88, CT91, NI92, CR94, Hen97, HRG00, Gab06, CGK14]). (See Nanongkai et al. [NSY19] for a more comprehensive literature survey.) For the undirected case, Aho, Hopcroft and Ullman [AHU74, Problem 5.30] asked in their 1974 book for an $O(m)$-time algorithm for computing $\kappa_G$. Prior to our result, $O(m)$-time

Local Vertex Connectivity (LocalVC). This problem concerns finding a vertex cut “near” a given vertex $x$. More precisely, for any vertex $v \in V$, define $N(v)$ to the set of neighbors of $v$, $\deg(v) = |N(v)|$, $N(L) = (\bigcup_{v \in L} N(v)) \setminus L$, and $\vol(L) = \sum_{v \in L} \deg(v)$ (we call $\vol(L)$ the volume of $L$). Given a vertex $x$ and two integers $\nu$ and $k$, the LocalVC problem concerns the set $L \subseteq V$ such that

$$x \in L, \ N(L) \text{ is a vertex cut of size less than } k, \text{ and } \vol(L) \leq \nu.$$

In other words, we are interested in a small vertex cut $N(L)$ that is “near” $x$ in the sense that $L$ has small volume. An algorithm for this problem takes as input $x$, $k$, $\nu$, and a pointer to an adjacency-list representation of $G$, and either

- outputs that $L \subseteq V$ satisfying Equation (1) does not exist, or
- returns a vertex cut $S$ of size less than $k$.

Nanongkai et al. [NSY19] recently introduced the LocalVC problem and designed a deterministic algorithm that takes $O(\nu^{1.5}k \polylog(\nu k))$ time under mild conditions. In this paper, we present a simple randomized (Monte Carlo) algorithm that takes $O(\nu k^2)$ time under the same conditions.

Theorem 1.1 (Main Result). There is a randomized (Monte Carlo) algorithm that takes as input a vertex $x \in V$ of an $n$-vertex $m$-edge graph $G = (V, E)$ represented as adjacency lists, and integers $k < n/4$ and $\nu < m/(640k)$ and runs in $O(\nu k^2)$ time to output either

- the “$\perp$” symbol indicating that, with probability at least $1/2$, $L \subseteq V$ satisfying Equation (1) does not exist, or
- a vertex cut $S$ of size less than $k$.

Note that the error probability $1/2$ above can be made arbitrarily small by repeating the algorithm. Compared to the previous algorithm of Nanongkai et al., our algorithm is faster when $k \leq \sqrt{\nu}$. It is worth noting that one can also derive an $(\nu k^{O(k)})$-time algorithm from the techniques of Chechik et al. [CHI+17] and some slower algorithms in the context of property testing (e.g. [GR02, OR11, YI12, YI10]). Our algorithm is in fact very simple: it simply repeatedly finds a path starting at $x$ and ending at some random vertex. Our analysis is also very simple.

(Global) Vertex Connectivity. The main application of our result is efficient algorithms for the vertex connectivity problem. There has been a long line of research on this problem since at least five decades ago (e.g. [Kle69, Pod73, ET75, Eve75, Gal80, EH84, Mat87, BDD+82, LLW88, CT91, NI92, CR94, Hen97, HRG00, Gab06, CGK14]). (See Nanongkai et al. [NSY19] for a more comprehensive literature survey.) For the undirected case, Aho, Hopcroft and Ullman [AHU74, Problem 5.30] asked in their 1974 book for an $O(m)$-time algorithm for computing $\kappa_G$. Prior to our result, $O(m)$-time
Algorithms were known only when $\kappa_G \leq 3$, due to the classic results of Tarjan [Tar72] and Hopcroft and Tarjan [HT73]. In this paper, we present an algorithm that takes near-linear time whenever $\kappa_G = O(\text{polylog}(n))$. In this paper, we obtain the first algorithm in many decades that guarantees a near-linear time complexity for higher values of $\kappa_G$.

**Theorem 1.2.** There is a randomized algorithm that takes as input an undirected graph $G$ and, with high probability, in time $O(m + nk^3)$ outputs a vertex cut $S$ of size $k = \kappa_G$.\(^2\)

The above result is near-linear time whenever $k = O(\text{polylog}(n))$. By combining with previous results (e.g. [HRG00, LLW88]), the best running time for solving vertex connectivity is $O(m + \min\{nk^3, n^2k, n^2 + nk^\omega\})$. Prior to our work, the best running time for $k > 3$ was $O(m + \min\{n^{4/3}k^{7/3}, n^2k, n^2 + nk^\omega\})$ [NSY19, HRG00, LLW88]. In particular, we have an improved running time when $k \leq O(\sqrt{n})$.

This result is obtained essentially by plugging in our LocalVC algorithm to the recent framework of Nanongkai et al. [NSY19]. The overall algorithm is fairly simple: Let $L$ be such that $N(L)$ is the optimal vertex cut. We guess the values $\nu = \text{vol}(L)$ and $k = \kappa_G$, and run our LocalVC algorithm with parameters $\nu$ and $k$ on $n/\nu$ randomly-selected seed nodes $x$.

**Approximation Algorithms and Directed Graphs.** Results in Theorems 1.1 and 1.2 can be generalized to $(1 + \epsilon)$-approximation algorithms and to algorithms on directed graphs. The approximation guarantee means that the output vertex cut $S$ is of size less than $\lceil (1 + \epsilon)k \rceil$. The time complexity for LocalVC is $O(\nu k / \epsilon)$. This improves the $O(\nu^{1.5}/(\sqrt{k} \epsilon^{1.5}))$-time algorithm of Nanongkai et al. [NSY19] when $k \leq (\nu/\epsilon)^{1/3}$. For approximating $\kappa_G$, the time complexity is $O(\min\{mk/\epsilon, n^{2-o(1)}\sqrt{k}/\text{poly}(\epsilon)\})$ where $k = \kappa_G$.

Observe that the time complexities for exact algorithms in Theorems 1.1 and 1.2 can be obtained by setting $\epsilon = 1/(2k)$ and using the fact that for undirected graphs we can ensure in $O(m)$ time that $m = O(nk)$ [NI92].

**Maximal $k$-Edge Connected Subgraphs.** For any set of vertices $C \subseteq V$, its induced subgraph $G[C]$ is a maximal $k$-edge-connected subgraph of $G$ if $G[C]$ is a $k$-edge-connected graph and no superset of $C$ has this property.\(^3\) Chechik et al. [CHI17] presented deterministic algorithms that can compute all maximal $k$-edge-connected subgraphs in $k^{\mathcal{O}(k)}m\sqrt{n}\log(n)$ time on undirected graphs and $k^{\mathcal{O}(k)}m\sqrt{n}\log(n)$ time on directed graphs.

Our result is mainly inspired by a part of Chechik et al.’s algorithms which runs some algorithm as a subroutine. Note that it is not hard to adapt their techniques to solve LocalVC in $k^{\mathcal{O}(k)}\nu$ time. Our result is an improvement over this, and in turn implied an improved running time for computing the maximal $k$-edge-connected subgraphs. We improve the dependency on $k$ in Chechik et al.’s result from $k^{\mathcal{O}(k)}$ to $\text{poly}(k)$.

**Independent work by Forster and Yang [FY19].** Independently from this paper (see Footnote 1), Forster and Yang present results similar to the above-mentioned results, except that (i) they show additional steps that lead to a better time complexity for computing the maximal $k$-edge connected subgraphs on undirected graphs, namely $O(k^4n^{3/2}\log n + km\log^2 n)$, (ii) they show some applications in graph property testing, and (iii) they do not consider approximation algorithms. Inspire by their property testing results, and by using approximation LocalVC algorithms, we obtain new bounds for testing vertex- and edge-connectivity below.

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\(^2\)As usual, with high probability (w.h.p.) means with probability at least $1 - 1/n^c$ for arbitrary constant $c \geq 1$.

\(^3\)Recall that a graph is $k$-edge connected we we need to remove at least $k$ edges so that there is no path from some node $u$ to another node $v$. 
This paper connects to directed graphs can be tested in In the bounded-degree incident-list model, Testing Vertex- and Edge-Connectivity. The study of testing graph properties, initiated by Goldreich et al. [GGR98], concerns the number of queries made to answer a question about graph properties. In the (unbounded-degree) incident-lists model [GR02, OR11], it is assumed that there is a list $L_v$ of edges incident to each node $v$ (or lists of outgoing and incoming edges for directed graphs), and an algorithm can make a query $q(v, i)$ for the $i^{th}$ edge in the list $L_v$ (if $i$ is bigger than the list size, the algorithm receives a special symbol in return). For any $\epsilon > 0$, we say that an $m$-edge graph $G$ is $\epsilon$-far from having a property $P$ if the number of edge insertions and deletions to make $G$ satisfies $P$ is at least $\epsilon m$. Testing $k$-vertex connectivity is a problem where we want to distinguish between when $G$ is $k$-vertex connectivity and when it is $\epsilon$-far from having such property. Testing $k$-edge connectivity is defined analogously. It is assumed that the algorithm receives $n, \epsilon$, and $k$ in the beginning. We show the following.

**Theorem 1.3.** In the unbounded-degree incident-list model, $k$-vertex (where $k < n/4$) and $-edge connectivity for directed graphs can be tested in $O(k/\epsilon^2)$ queries with probability at least $2/3$. Further, $k$-edge connectivity for simple directed graphs can be tested in $O(k/\epsilon^2)$ queries.

In particular, our $O(k/\epsilon^2)$ bound is linear in $k$, and it can be independent of $k$ for testing $k$-edge connectivity on simple graphs. In the bounded-degree incident-list model, the maximum degree $d$ is assumed to be given to the algorithm and a graph is said to be $\epsilon$-far from a property $P$ if it needs at least $\epsilon d$ edge modifications to have such property. We show the following.

**Theorem 1.4.** In the bounded-degree incident-list model, $k$-vertex (where $k < n/4$) and $-edge connectivity for directed graphs can be tested in $O(k/\epsilon^2)$ queries with probability at least $2/3$. Further, $k$-edge connectivity for simple directed graphs can be tested in $O(k/\epsilon^2)$ queries.

It has been open for many years whether the bounds from [GR02, OR11, YI10, YI12] which are exponential in $k$ can be made polynomial (this was asked in e.g. [OR11]). Forster and Yang [FY19] answered this using the same result as our local algorithms. The dependence on $k$ in their bounds is at least $k^3$, even on bounded-degree graphs. We can improve the dependence on $k$ essentially by using approximation local algorithms.

Detailed comparisons: To precisely compare our bounds with the previous ones, note that there are two sub-models: (i) In the unbounded-degree incident-list model, previous work assumes that $d = m/n$ is known to the algorithm in the beginning. (ii) in the bounded-degree incident-list model, the maximum degree $d$ is assumed to be given to the algorithm and a graph is said to be $\epsilon$-far from a property $P$ if it needs at least $\epsilon d$ edge modifications to have such property. Our $O(k/\epsilon^2)$ bound can be generalized to $O(k^2/(\epsilon^2 d))$ bound in the unbounded-degree model. Similarly, our
\( \tilde{O}(\min\{k^2, 1/\epsilon^3\}) \) bound can be generalized to \( \tilde{O}(\min\{k^2/(\bar{d}e^2), k/(\bar{d}e^3)\}) \) bound in the unbounded-degree model.\(^4\) The bounds that are exponential in \( k \) by [OR11, YI10, YI12] are \( \tilde{O}(\frac{dk}{\epsilon})^{k+1} \) and \( \tilde{O}(\frac{dk}{\epsilon})^k d \) in the unbounded- and bounded-degree models, respectively, for testing both directed \( k \)-vertex and -edge connectivity. The bounds that are polynomial in \( k \) by Forster and Yang [FY19] are (i) \( \tilde{O}(k^5/(\epsilon d)^2) \) for \( k \)-vertex connectivity in the unbounded-degree model, (ii) \( \tilde{O}(k^4/(\epsilon d)^2) \) for \( k \)-edge connectivity in the unbounded-degree model, and (iii) \( \tilde{O}(k^3/\epsilon) \) for both \( k \)-vertex and -edge connectivity in the bounded-degree model. Table 1 details comparisons between our results, and those from [FY19].

2 Preliminaries

Let \( G = (V,E) \) be a directed graph. For any \( S,T \subseteq V \), let \( E(S,T) = \{(u,v) \mid u \in S, v \in T\} \). For each vertex \( u \), we let \( \text{deg}^\text{out}(u) \) denote the out-degree of \( u \) respectively. For a set \( S \subseteq V \), the out-volume of \( S \) is \( \text{vol}^\text{out}(S) = \sum_{u \in S} \text{deg}^\text{out} u \). A set of out-neighbors of \( S \) is \( N^\text{out}(S) = \{ v \mid (u,v) \in E(S,V-S) \} \). We can define in-degree \( \text{deg}^\text{in}(u) \), in-volume \( \text{vol}^\text{in}(S) \), and a set of in-neighbors \( N^\text{in}(S) \) analogously. We add a subscript \( G \) to the notation when it is not clear which graph we are referring to.

We say that \( (L,S,R) \) is a separation triple of \( G \) if \( L,S,R \) partition \( V \) where \( L,R \neq \emptyset \) and \( E(L,R) = \emptyset \). We also say that \( S \) is a vertex cut of \( G \) of size \( |S| \). \( S \) is an st-vertex cut if \( s \in L \) and \( t \in R \). We say that \( s \) and \( t \) is \( k \)-connected (or \( k \)-vertex-connected) if there is no st-vertex cut of size less than \( k \). \( G \) is \( k \)-connected if \( s \) and \( t \) is \( k \)-connected for every pair \( s,t \in V \).

3 Local edge connectivity

In this section, we show local algorithms for detecting an edge cut of size \( k \) and volume \( \nu \) containing some seed node in time \( O(\nu k^2) \). Both the algorithms and analysis are very simple.

**Theorem 3.1.** There is a randomized (Monte Carlo) algorithm that takes as input a vertex \( x \in V \) of an \( n \)-vertex \( m \)-edge graph \( G = (V,E) \) represented as adjacency lists, a volume parameter \( \nu \), a cut-size parameter \( k \geq 1 \), and an accuracy parameter \( \epsilon \in (0,1] \) where \( \nu < \epsilon m/8 \) and runs in \( O(\nu k^2/\epsilon) \) time and outputs either

- the “\( \perp \)” symbol indicating that, with probability \( 1/2 \), there is no \( S \ni x \) where \( |E(S,V-S)| < k \) and \( \text{vol}^\text{out}(S) \leq \nu \), or
- a set \( S \ni x \) where \( S \neq V \), \( |E(S,V-S)| < (1+\epsilon)k \) and \( \text{vol}^\text{out}(S) \leq 10\nu/\epsilon.\(^5\)"

By setting \( \epsilon < \frac{1}{k^2} \), we have that \( (1+\epsilon)k = k \). In particular, we obtain an algorithm for the exact problem:

**Corollary 3.2.** There is a randomized (Monte Carlo) algorithm that takes as input a vertex \( x \in V \) of an \( n \)-vertex \( m \)-edge graph \( G = (V,E) \) represented as adjacency lists, a volume parameter \( \nu \), and a cut-size parameter \( k \geq 1 \) where \( \nu < m/8k \) and runs in \( O(\nu k^2) \) time and outputs either

- the “\( \perp \)” symbol indicating that, with probability \( 1/2 \), there is no \( S \ni x \) where \( |E(S,V-S)| < k \) and \( \text{vol}^\text{out}(S) \leq \nu \), or
- a set \( S \ni x \) where \( S \neq V \), \( |E(S,V-S)| < k \) and \( \text{vol}^\text{out}(S) \leq 10\nu k \).
The algorithm for Theorem 3.1 in described in Algorithm 1. We start with the following important observation.

**Lemma 3.3.** Let $S \subseteq V$ be any set where $x \in S$. Let $P_{xy}$ be a path from $x$ to $y$. Suppose we reverse the direction of edges in $P_{xy}$. Then, we have $|E(S, V - S)|$ and $\text{vol}^{\text{out}}(S)$ are both decreased exactly by one if $y \notin S$. Otherwise, $|E(S, V - S)|$ and $\text{vol}^{\text{out}}(S)$ stay the same.

It is clear that running time of Algorithm 1 is $O(\nu/k \times O(\nu/\epsilon) = O(\nu k/\epsilon)$ because the DFS tree only requires $O(\nu/\epsilon)$ for visiting $O(\nu/\epsilon)$ edges. The two lemmas below imply the correctness of Theorem 3.1

**Lemma 3.4.** If a set $S$ is returned, then $S \ni x$, $S \neq V$, $|E(S, V - S)| < \lfloor (1 + \epsilon)k \rfloor$ and $\text{vol}^{\text{out}}(S) \leq 10\nu/\epsilon$.

**Proof.** If $S$ is returned, then the DFS tree $T$ get stuck at $S = V(T)$. That is, $|E(S, V - S)| = 0$ and $\text{vol}^{\text{out}}(S) \leq 8\nu/\epsilon$ at the end of the algorithm. Note that $x \in S$ and $S \neq V$ because $8\nu/\epsilon < m$. Observe that the algorithm has reversed strictly less than $\lfloor (1 + \epsilon)k \rfloor$ many paths $P_{xy}$, because the algorithm did not reverse a path in the iteration that $S$ is returned. So Lemma 3.3 implies that, initially, $|E(S, V - S)| < \lfloor (1 + \epsilon)k \rfloor$ and, $\text{vol}^{\text{out}}(S) < 8\nu/\epsilon + \lfloor (1 + \epsilon)k \rfloor \leq 10\nu/\epsilon$. \qed

**Lemma 3.5.** If $\bot$ is returned, then, with probability at least 1/2, there is no $S \ni x$ where $|E(S, V - S)| < k$ and $\text{vol}^{\text{out}}(S) \leq \nu$.

**Proof.** Suppose that such $S$ exists. We will show that $\bot$ is returned with probability less than 1/2. Suppose that no set $S'$ is returned before the last iteration. It suffices to show that at the beginning of the last iteration, $|E(S, V - S)| = 0$ with probability at least 1/2. If this is true, then the DFS tree $T$ in the last iteration will not be able to visit more than $\nu$ edges and so will return the set $V(T)$.

Let $k' = \lfloor (1 + \epsilon)k \rfloor - 1$ denote the number of iterations excluding the last one. Let $X_i$ be the random variable where $X_i = 1$ if the sampled edge $(y', y)$ in the $i$-th iteration of the algorithm is such that $y \in S$. Otherwise, $X_i = 0$. As $\text{vol}^{\text{out}}(S)$ never increases, observe that $E[X_i] \leq \frac{\text{vol}^{\text{out}}(S)}{|E_{\text{org}}|} \leq \frac{\nu}{8\nu/\epsilon} = \epsilon/8$ for each $i \leq k'$. Let $X = \sum_{i=1}^{k'} X_i$. We have $E[X] \leq \epsilon k'/8$ by linearity of expectation and $\Pr[X \leq \epsilon k'/4] \geq 1/2$ by Markov’s inequality. So $\Pr[X \leq \lfloor \epsilon k'/4 \rfloor] \geq 1/2$ as $X$ is integral.

Let $Y = k' - X$. Notice that $Y$ is the number of times before the last iteration where the algorithm samples $y \notin S$. We claim that $k' - \lfloor \epsilon k'/4 \rfloor \geq k - 1$ (see the proof at the end). Hence,
with probability at least 1/2, \( Y \geq k' - \lfloor \epsilon k'/4 \rfloor \geq k - 1 \geq |E(S, V - S)| \). By Lemma 3.3, if \( Y \geq |E(S, V - S)| \), then \( |E(S, V - S)| = 0 \) at the beginning of the last iteration. This concludes the proof.

**Claim 3.6.** \( k' - \lfloor \epsilon k'/4 \rfloor \geq k - 1 \) for \( \epsilon \in [0, 1] \)

**Proof.** If \( \epsilon < 4/k' \), then \( \lfloor \epsilon k'/4 \rfloor = 0 \), so \( k' - \lfloor \epsilon k'/4 \rfloor = \lfloor (1 + \epsilon)k \rfloor - 1 \geq k - 1 \). If \( \epsilon \geq 4/k' \), then

\[
\begin{align*}
k' - \lfloor \epsilon k'/4 \rfloor & \geq (1 - \epsilon/4)k' \\
& \geq (1 - \epsilon/4)((1 + \epsilon)k - 2) \\
& \geq (1 - \epsilon/4)(1 + \epsilon)k - 2 \\
& \geq (1 + \epsilon/2)k - 2 \\
& \geq k - 1.
\end{align*}
\]

where the last inequality is because \( \epsilon k/2 \geq 4/k' \cdot k/2 \geq 1 \) as \( k' \leq \lfloor (1 + \epsilon)k \rfloor \leq 2k \). \( \Box \)

### 4 Local vertex connectivity

In this section, we show the vertex cut variant of the local algorithms from Section 3.

**Theorem 4.1.** There is a randomized (Monte Carlo) algorithm that takes as input a vertex \( x \in V \) of an \( n \)-vertex \( m \)-edge graph \( G = (V, E) \) represented as adjacency lists, a volume parameter \( \nu \), a cut-size parameter \( k \), and an accuracy parameter \( \epsilon \in (0, 1] \) where \( \nu < m/640 \) and \( k \leq n/4 \), and runs in \( \mathcal{O}(\nu k/\epsilon) \) time and outputs either

- the “\( \perp \)” symbol indicating that, with probability 1/2, there is no separation triple \( (L, S, R) \) where \( L \ni x, |S| < k \) and \( \text{vol}^\text{out}(L) \leq \nu \), or
- a vertex cut of size less than \( \lfloor (1 + \epsilon)k \rfloor \).

By setting \( \epsilon < \frac{1}{k} \), we have that \( \lfloor (1 + \epsilon)k \rfloor = k \). In particular, we obtain an algorithm for the exact problem:

**Corollary 4.2.** There is a randomized (Monte Carlo) algorithm that takes as input a vertex \( x \in V \) of an \( n \)-vertex \( m \)-edge graph \( G = (V, E) \) represented as adjacency lists, a volume parameter \( \nu \), and a cut-size parameter \( k \) where \( \nu < m/640k \) and \( k \leq n/4 \), and runs in \( \mathcal{O}(\nu k^2) \) time and outputs either

- the “\( \perp \)” symbol indicating that, with probability 1/2, there is no separation triple \( (L, S, R) \) where \( L \ni x, |S| < k \) and \( \text{vol}^\text{out}(L) \leq \nu \), or
- a vertex cut of size less than \( k \).

To prove Theorem 4.1, in Section 4.1 we first reduce the problem to the edge version of the problem using the well-known reduction (e.g. [Eve75, HRG00]) and then in Section 4.2 we plug the algorithm from Theorem 3.1 into the reduction.

---

\(^6\)The main reason we choose the factor 8 in the number \( 8\nu/\epsilon \) of visited edges by the DFS is for simplifying the following inequalities.
Lemma 4.3. Let $(L, S, R)$ be a separation triple in $G$ where $S = N_{G}^\text{out}(L)$. Let $L' = \{v_{\text{in}}, v_{\text{out}} | v \in L\} \cup \{v_{\text{in}} | v \in S\}$ be a set of vertices in $G'$. Then $|E_{G'}(L', V' - L')| = |S|$ and $\vol_{G'}^{\text{out}}(L') \leq \vol_{G}^{\text{out}}(L)$.

Proof. As $S = N_{G}^\text{out}(L)$, we have $|E_{G'}(L', V' - L')| = |\{(v_{\text{in}}, v_{\text{out}}) | v \in S\}| = |S|$. Also, $\vol_{G'}^{\text{out}}(L') = \vol_{G}^{\text{out}}(L) + |L| + |S| \leq 2\vol_{G}^{\text{out}}(L)$ because every vertex in $G$ has out-degree at least 1 and $S = N_{G}^\text{out}(L)$. \hfill $\square$

Lemma 4.4. Let $L' \ni x$ be a set of vertices of $G'$. Then, there is a set of vertices $L$ in $G$ such that $\vol_{G}^{\text{out}}(L) \leq 2\vol_{G'}^{\text{out}}(L')$ and $|S| \leq |E_{G'}(L', V' - L')|$ where $S = N_{G}(L)$. Given $L'$, $L$ can be constructed in $O(\vol_{G'}^{\text{out}}(L'))$ time.

Let $R = V - (L \cup S)$. We have $R \neq \emptyset$, i.e. $(L, S, R)$ is a separation triple if $\vol_{G}^{\text{out}}(L') \leq m'/32$ and $|E_{G'}(L', V' - L')| \leq n/2$.

Proof. First of all, note that if there is $v$ where $v_{\text{out}} \in L'$ and $\deg_{G'}^{\text{out}}(v_{\text{out}}) \leq |E_{G'}(L', V' - L')|$, then we can return $L = \{v\}$ and $S = N_{G}^\text{out}(\{v\})$ and we are done. So from now, we assume that $\deg_{G'}^{\text{out}}(v_{\text{out}}) > |E_{G'}(L', V' - L')|$. By the structure of $G'$, observe that there are sets $S_1, S_2 \subseteq V$ be such that

$$E_{G'}(L', V' - L') = \{(v_{\text{in}}, v_{\text{out}}) | v \in S_1\} \cup \{(u_{\text{out}}, v_{\text{in}}) | v \in S_2\}$$

Let $L'_0 = L' \cup \{v_{\text{in}} | v \in S_2\}$. See Figure 1 for illustration. So there is a set $S \subseteq V$ where

$$E_{G'}(L'_0, V' - L'_0) = \{(v_{\text{in}}, v_{\text{out}}) | v \in S\}.$$
We have $|S| = |E_{G'}(L_0, V' - L_0)| \leq |E_{G'}(L', V' - L')|$ because for each $v \in S_2$, $\deg^{\text{out}}(v_{in}) \leq 1 \leq \deg^{\text{in}}(v_{in})$. Also, $\text{vol}^{\text{out}}_{G'}(L_0) \leq \text{vol}^{\text{out}}_{G'}(L') + |E_{G'}(L', V' - L')| \leq 2 \text{vol}^{\text{out}}_{G'}(L')$.

Let $L = \{v \mid v_{in}, v_{out} \in L_0\}$. Note that $L \cap S = \emptyset$. See Figure 1 for illustration. Observe that $x \in L$ because $x_{out} = x_{in}$. Moreover, $N_G(L) = S$ and $\text{vol}^{\text{out}}_{G'}(L) \leq \text{vol}^{\text{out}}_{G'}(L_0) \leq 2 \text{vol}^{\text{out}}_{G'}(L')$. $L$ be can constructed in time $O(|\{v_{out} \in L'\}|) = O(\text{vol}^{\text{out}}_{G'}(L'))$ because the minimum out-degree of vertices in $G$ is 1.

For the second statement, observe that $R = V - (L \cup S) = \{v \mid v_{in} \notin L_0\}$. Let $V'_{in} = \{v_{in} \in V'\} \cup \{x\}$ and $V'_{out} = \{v_{out} \in V'\} - \{x\}$. Let $k' = |E_{G'}(L_0', V' - L_0')|$. Suppose for contradiction that $R = \emptyset$. We claim that

$$|V' - L_0'| = |V'_{out} - L_0'| = k'.$$

This is because $L_0' \supseteq V'_{in}$, $V' - L_0' \subseteq V'_{out}$, and $E_{G'}(L_0', V' - L_0')$ only contains edges of the form $(v_{in}, v_{out})$. Now, there are two cases. If $m' \geq 4n'k'$, then we have

$$m' = \text{vol}^{\text{out}}_{G'}(L_0') + \text{vol}^{\text{out}}_{G'}(V' - L_0')$$

$$\leq 2 \text{vol}^{\text{out}}_{G'}(L') + n'\left|V'_{out} - L_0'\right|$$

$$\leq 2 \cdot m'/32 + n'k'$$

$$\leq m'/16 + m'/4 < m'$$

which is a contradiction. Otherwise, we have $m' < 4n'k'$. Note that $n' < 2n$ by the construction of $G'$ and so $m' < 8nk'$. Hence, we have

$$\text{vol}^{\text{out}}_{G'}(L_0') \geq |L_0' \cap V'_{out}|k' \geq (n - k')k' \geq nk'/2 > m'/16$$

which contradicts $\text{vol}^{\text{out}}_{G'}(L_0') \leq 2 \text{vol}^{\text{out}}_{G'}(L') \leq 2 \cdot m'/32$.

4.2 Proof of Theorem 4.1

Given an $n$-vertex $m$-edge $G = (V, E)$ represented as adjacency lists, a vertex $x \in V$ and parameters $\nu, k, \epsilon$ from Theorem 4.1 where $\nu \leq cm/640$ and $k \leq n/4$, we will work on the split graph $G'$ with $n'$-vertex $m'$-edge as described in Section 4.1. The adjacency lists of $G'$ can be created “on the fly”. Let LocalEC($x'$, $\nu'$, $k'$, $\epsilon'$) denote the algorithm from Theorem 4.1 with parameters $x'$, $\nu'$, $k'$, $\epsilon'$. We run LocalEC($x$, $2\nu$, $k$, $\epsilon$) on $G' = (V', E')$ in time $O(\nu k/\epsilon)$. Note that $2\nu \leq cm/8 \leq cm/8$ as required by Theorem 3.1.

If it returns $\perp$, then, with probability 1/2, there is no $L' \ni x$ where $|E_{G'}(L', V' - L')| < k$ and $\text{vol}^{\text{out}}_{G'}(L') \leq 2\nu$. By Lemma 4.3, with probability 1/2, there is no separation triple $(L, S, R)$ where $L \ni x$, $|S| < k$ and $\text{vol}^{\text{out}}(L) \leq \nu$.

Otherwise, if it returns $L'$ where $L' \ni x$ where $|E_{G'}(L', V' - L')| < [(1 + \epsilon)k]$ and $\text{vol}^{\text{out}}_{G'}(L') \leq 20\nu/\epsilon$. By Lemma 4.4, we can obtain in $O(\nu/\epsilon)$ time and two sets $L$ and $S = N_{G'}^\text{out}(L)$ where $|S| < [(1 + \epsilon)k]$ and $\text{vol}^{\text{out}}_{G'}(L) \leq 2 \text{vol}^{\text{out}}_{G'}(L') \leq 40\nu/\epsilon$. Let $R = V - L \cup S$. As $\text{vol}^{\text{out}}_{G'}(L') \leq 20\nu/\epsilon \leq m'/32$ and $[(1 + \epsilon)k] \leq n/2$, we have that $(L, S, R)$ is a separation triple by Lemma 4.4. That is, $S$ is a vertex cut.

5 Vertex connectivity

In this section, we show the first near-linear time algorithm for checking of $k$-connectivity for any $k = \tilde{O}(1)$ in both undirected and directed graphs.
Theorem 5.1. There is a randomized (Monte Carlo) algorithm that takes as input an undirected graph $G$ and a cut-size parameter $k$ and an accuracy parameter $\epsilon \in (0, 1)$, and in time $\tilde{O}(m + nk^2/\epsilon)$ either outputs a vertex cut of size less than $\lfloor (1 + \epsilon)k \rfloor$ or declares that $G$ is $k$-connected w.h.p. By setting $\epsilon < 1/k$, the same algorithm decides (exact) $k$-vertex-connectivity of $G$ in $\tilde{O}(m + nk^3)$ time.

By combining with the state-of-the-art algorithms for undirected graph, we obtain the following.

Corollary 5.2. There is a randomized (Monte Carlo) algorithm that takes as input an undirected graph $G$ and a cut-size parameter $k$ and an accuracy parameter $\epsilon \in (0, 1)$, and in time $\tilde{O}(m + \poly(1/\epsilon)\min\{nk^2, n^{5/3+o(1)}k^{2/3}, n^{3+o(1)}/k, n^\omega\})$ either outputs a vertex cut of size less than $\lfloor (1 + \epsilon)k \rfloor$ or declares that $G$ is $k$-connected w.h.p. For exact vertex connectivity, there is a randomized (Monte Carlo) algorithm for exact $k$-vertex-connectivity of $G$ in $\tilde{O}(m + \min\{nk^3, n^2k, n^\omega + nk^\omega\})$ time.

Proof. For approximate vertex connectivity, Nanongkai et al. [NSY19] (Theorem 1.2) present $\tilde{O}(m + \poly(1/\epsilon)\min\{k^{4/3}n^{1/3}, n^{5/3+o(1)}k^{2/3}, n^{3+o(1)}/k, n^\omega\})$-time algorithm. By Theorem 5.1, we have the $\tilde{O}(m + nk^2/\epsilon)$-time algorithm. Combining both algorithms, the term $k^{4/3}n^{1/3}$ is replaced by $nk^2$. For exact vertex connectivity, we combine the running time in Theorem 5.1 with the $\tilde{O}(m + nk^3)$-time algorithm, which is given by Henzinger, Rao and Gabow [HRG00], and Linial, Lovász and Wigderson [LLW88].

Now we present the new results for directed graph.

Theorem 5.3. There is a randomized (Monte Carlo) algorithm that takes as input a directed graph $G$ and a cut-size parameter $k$ and an accuracy parameter $\epsilon \in (0, 1)$, and in time $\tilde{O}(m + \poly(1/\epsilon)\min\{mk^2, n^2\sqrt{k}/\epsilon, n^{7/3+o(1)}k^{1/6}, n^{3+o(1)}/k, n^\omega\})$ either outputs a vertex cut of size less than $\lfloor (1 + \epsilon)k \rfloor$ or declares that $G$ is $k$-connected w.h.p. For exact vertex connectivity, there is a randomized (Monte Carlo) algorithm for exact $k$-vertex-connectivity of $G$ in $\tilde{O}(\min\{mk^2, k^3n + k^{3/2}m^{1/2}n\})$ time.

Similarly, by combining with the state-of-the-art algorithms for directed graph, we obtain the following.

Corollary 5.4. There is a randomized (Monte Carlo) algorithm that takes as input an undirected graph $G$ and a cut-size parameter $k$ and an accuracy parameter $\epsilon \in (0, 1)$, and in time $\tilde{O}(\poly(1/\epsilon)\min\{mk, mn^{2/3+o(1)}k^{1/3}, k^{1/2}n^{2+o(1)}, n^{7/3+o(1)}/k^{1/6}, n^{3+o(1)}/k, n^\omega\})$ either outputs a vertex cut of size less than $\lfloor (1 + \epsilon)k \rfloor$ or declares that $G$ is $k$-connected w.h.p. For exact vertex connectivity, there is a randomized (Monte Carlo) algorithm for exact $k$-vertex-connectivity of $G$ in $\tilde{O}(\min\{mk^2, k^3n + k^{3/2}m^{1/2}n, mn, n^\omega + nk^\omega\})$ time.

Proof. For approximate vertex connectivity, Nanongkai et al. [NSY19] (Theorem 1.2) present $\tilde{O}(\poly(1/\epsilon)\min\{m^{4/3}, mn^{2/3}k^{1/2}, mn^{2/3+o(1)}k^{1/3}, n^{7/3+o(1)}k^{1/6}, n^{3+o(1)}/k, n^\omega\})$-time algorithm. By Theorem 5.3, we have the $\tilde{O}(\min\{mk^2, k^3n + k^{3/2}m^{1/2}n\})$-time algorithm. Combining both algorithms, the terms $m^{4/3}$ and $mn^{2/3}k^{1/2}$ are subsumed. For exact vertex connectivity, we combine the running time in Theorem 5.3 with the $\tilde{O}(\min\{mn, n^\omega + nk^\omega\})$-time algorithm, which is given by Henzinger, Rao and Gabow [HRG00], and by Cheriyan and Reif [CR94].

To prove Theorem 5.1 and Theorem 5.3, we will apply our framework [NSY19] for reducing the vertex connectivity problem to the local vertex connectivity problem. To describe the reduction, let $T_{pair}(m, n, k, \epsilon, p)$ be the time required for, given vertices $s$ and $t$, either finding a st-vertex cut of size less than $\lfloor (1 + \epsilon)k \rfloor$ or declaring that $s$ and $t$ is $k$-connected with probability at least
Let $T_{\text{local}}(\nu, k, \epsilon, p) = O(mk)$ be the time for solving correctly probability at least $1 - p$ the local vertex connectivity problem from Theorem 4.1 when a volume parameter is $\nu$, the cut-size parameter is $k$, and the accuracy parameter is $\epsilon$.

**Lemma 5.5 ([NSY19] Lemma 5.14, 5.15).** There is a randomized (Monte Carlo) algorithm that takes as input a graph $G$, a cut-size parameter $k$, and an accuracy parameter $\epsilon > 0$, and runs in time in one of these expressions

\[
\hat{O}(m/\nu) \cdot (\text{time for solving correctly probability at least } \nu) + T_{\text{local}}(\nu, k, \epsilon, 1/poly(n)) \tag{2}
\]

\[
\hat{O}(n/\nu) \cdot (\text{time for solving correctly probability at least } \nu) + T_{\text{local}}(\nu^2 + \nu k, k, \epsilon, 1/poly(n)) \tag{3}
\]

where $\nu \leq m$, and $\nu \leq n$ are optimizing parameters that can be chosen, and either outputs a vertex cut of size less than $\lfloor (1+\epsilon)k \rfloor$ or declares that $G$ is $k$-connected w.h.p.

For completeness, we give a simple proof sketch of Equation (2) which is used for our algorithm for undirected graphs. The idea for other equations is similar and also simple.

**Proof sketch.** Suppose that $G$ is not $k$-connected. It suffices to show an algorithm that outputs a vertex cut of size less than $\lfloor (1+\epsilon)k \rfloor$ w.h.p. By considering both $G$ and its reverse graph (where the direction of each edge is reversed), there exists w.l.o.g. a separation triple $(L, S, R)$ where $\text{vol}^{\text{out}}(L) \leq \text{vol}^{\text{out}}(R)$. There are two cases.

Suppose $\text{vol}^{\text{out}}(L) \geq \nu$. By sampling $\hat{O}(m/\nu)$ pairs of edges $e = (x, x')$ and $f = (y, y')$, there exists w.h.p. a pair $(e, f)$ where $x \in L$ and $y \in R$. For such pair $(x, y)$, if we check $x$ and $y$ is $k$-connected in time $T_{\text{pair}}(m, n, k, \epsilon, 1/poly(n))$, we must obtain an $xy$-vertex cut of size less than $\lfloor (1+\epsilon)k \rfloor$. So, if we check this for each pair $(x, y)$ and we will obtain the cut w.h.p.

Suppose $\text{vol}^{\text{out}}(L) \leq \nu$. Suppose further that $\text{vol}^{\text{out}}(L) \in (2^{i-1}, 2^i]$. By sampling $\hat{O}(m/2^i)$ pair of edges $e = (x, x')$, there exists w.h.p. a edge $e$ where $x \in L$. For such vertex $x$, if we check the local vertex connectivity in time $T_{\text{local}}(2^i, k, \epsilon, 1/poly(n))$, then the algorithm must return a vertex cut of size less than $\lfloor (1+\epsilon)k \rfloor$. So, if we check this for each pair $(x, y)$ and we will obtain the cut w.h.p.

To conclude, the running time in the first case is $\hat{O}(m/\nu) \cdot T_{\text{pair}}(m, n, k, \epsilon, 1/poly(n))$. For the second case, we try all $O(\log n)$ many $2^i$, each of which case takes $\hat{O}(m/2^i) \cdot T_{\text{local}}(2^i, k, \epsilon, 1/poly(n)) = \hat{O}(m/\nu) \cdot T_{\text{local}}(\nu, k, \epsilon, 1/poly(n))$ assuming that $T_{\text{local}}(\nu, k, \epsilon, 1/poly(n)) \geq \nu$. This complete the proof of the running time. For the correctness, if $G$ is not $k$-connected, we must obtain a desired vertex cut of size $\lfloor (1+\epsilon)k \rfloor$ w.h.p. So if we do not find any cut, we declare that $G$ is $k$-connected w.h.p. \( \square \)

### 5.1 Undirected graphs

Here, we prove Theorem 5.1. First, it suffices to show an algorithm with $\hat{O}(mk/\epsilon)$ time. Indeed, by using the sparsification algorithm by Nagamochi and Ibaraki [NI92], we can sparsify an undirected graph in linear time so that $m = O(nk)$ and $k$-connectivity is preserved. By this preprocessing, the total running time is $O(m) + \hat{O}((nk)k/\epsilon)) = \hat{O}(m + nk^2/\epsilon)$ as desired. Next, we assume that $k \leq \min\{n/4, 5\delta\}$ where $\delta$ is the minimum out-degree of $G$. If $k > 5\delta$, then it is $G$ is clearly not $k$-connected and the out-neighbor of the vertex with minimum out-degree is a vertex cut of size less than $k$. If $k > n/4$, then we can invoke the algorithm by Henzinger, Rao and Gabow [HRG00] for solving the problem exactly in time $O(mn) = O(mk)$.

Now, we have $T_{\text{pair}}(m, n, k, \epsilon, p) = O(mk)$ by Ford-Fulkerson algorithm. By repeating the algorithm from Theorem 4.1 $O(\log \frac{1}{p})$ times for boosting success probability, $T_{\text{local}}(\nu, k, \epsilon, p) = O(\nu k e^{-1} \log \frac{1}{p})$. We choose $\nu = O(em)$ as required by Theorem 4.1 and also $k \leq \min\{n/4, 5\delta\}$. Applying Lemma 5.5 (Equation (2)), we obtain an algorithm for Theorem 5.1 with running time

\[
\hat{O}(m/em) \times O(mk + (em)k e^{-1} \log n) = \hat{O}(mk/\epsilon).
\]
5.2 Directed graphs

Here, we prove Theorem 5.3. We again assume that \( k \leq \min\{n/4, 5\delta\} \) using the same reasoning as in the undirected case. We first show how to obtain the claimed time bound for the approximate problem. Note that the \( \tilde{O}(mk/\epsilon) \)-time algorithm follows by the same argument as in the undirected case, because both Ford-Fulkerson algorithm and the local algorithm from Theorem 4.1 work as well in directed graphs.

Next, we show an approximate algorithm with running time \( \tilde{O}(\text{poly}(1/\epsilon)n^{2+o(1)}\sqrt{k}) \). We assume \( k \leq n^{2/3} \) (for \( k \geq n^{2/3} \), we use state-of-the-art \( \tilde{O}(\text{poly}(1/\epsilon)n^{3+o(1)}/k) \)-time algorithm by [NSY19]). We have \( T_{\text{local}}(\nu, k, \epsilon, p) = O(\nu k \epsilon^{-1} \log \frac{1}{p}) \) by Theorem 4.1 and \( T_{\text{pair}}(m, n, k, \epsilon, 1/\text{poly } n) = \tilde{O}(\text{poly}(1/\epsilon)n^{2+o(1)}) \) using the recent result for \( (1+\epsilon)- \) approximating the minimum \( st \)-vertex cut by Chuzhoy and Khanna [CK19]. By choosing \( \pi = n/\sqrt{k} \) for Lemma 5.5 (Equation (3)), we obtain an algorithm with running time

\[
\tilde{O}(n/\pi) \cdot (n^{2+o(1)} \text{poly}(1/\epsilon) + (\pi^2 k + \pi k^2)/\epsilon) = \tilde{O}(\sqrt{k} \text{poly}(1/\epsilon)) \cdot (n^{2+o(1)} + n^2 + nk^{1.5})
= \tilde{O}(n^{2+o(1)} \sqrt{k} \text{poly}(1/\epsilon)).
\]

Next, we show how to obtain the time bound for the exact problem. First, observe that we can obtain a \( \tilde{O}(mk^2) \)-time exact algorithm from the \( \tilde{O}(mk/\epsilon) \)-time approximate algorithm by setting \( \epsilon < 1/k \). It remains to show an algorithm with the running time \( \tilde{O}(k^3 n + k^{3/2} m^{1/2} n) \).

By Corollary 4.2, there is an exact algorithm for local vertex connectivity with running time \( T_{\text{local}}(\nu, k, 1/2k, p) = O(\nu k^2 \log \frac{1}{p}) \). Also, we have \( T_{\text{pair}}(m, n, k, \epsilon, p) = O(mk) \) by Ford-Fulkerson algorithm. By choosing \( \pi = O(\sqrt{m/k}) \) in Lemma 5.5 (Equation (3)), we obtain an algorithm with running time

\[
\tilde{O}(n/\pi) \cdot (mk + (\pi^2 + \pi k)k^2) = \tilde{O}(n/\pi) \cdot (mk + (m/k + \sqrt{mk})k^2)
= \tilde{O}(n\sqrt{k/m} \cdot (mk + k^{2.5} \sqrt{m})
= \tilde{O}(k^{3/2} m^{1/2} n + k^3 n).
\]

Note that \( \pi^2 + \pi k = O(m/k) \) as required by Corollary 4.2.

6 Property Testing

In this section, we show property testing algorithms for distinguishing between a graph that is \( k \)-edge/\( k \)-vertex connected and a graph that is \( \epsilon \)-far from having such property with constant probability for both unbounded-degree and bounded-degree incident-list model. Recall that for any \( \epsilon > 0 \), a directed graph \( G \) is \( \epsilon \)-far from having a property \( P \) if at least \( \epsilon m \) edge modifications are needed to make \( G \) satisfy property \( P \). We assume that \( \tilde{d} = m/n \) is known to the algorithm at the beginning.

**Theorem 6.1.** For unbounded-degree model, there is a property testing algorithm for \( k \)-edge (\( k \)-vertex where \( k < n/4 \)) connectivity with correct probability at least 2/3 that uses \( \tilde{O}(k^2/(\epsilon^2 \tilde{d})) \) queries (same for \( k \)-vertex) and runs in \( \tilde{O}(k^2/(\epsilon^{11/3} \tilde{d})) \) time (\( \tilde{O}(k^2/(\epsilon^{2.5} \tilde{d})) \) time for \( k \)-vertex). If \( \tilde{d} \) is unknown, then there is a similar algorithm that uses \( \tilde{O}(k/\epsilon^2) \) queries (same for \( k \)-vertex), and runs in \( \tilde{O}(k/\epsilon^{8/3}) \) time (\( \tilde{O}(k/\epsilon^{2.5}) \) time for \( k \)-vertex). If \( G \) is simple, then the same algorithm for testing \( k \)-edge-connectivity queries at most \( \tilde{O}(\min\{k^2/(\tilde{d}^2), k/(\tilde{d}^3)\}) \) (or \( \tilde{O}(\min\{k/\epsilon^2, 1/\epsilon^3\}) \) edges if \( \tilde{d} \) is unknown), and runs in \( \tilde{O}(1/(\epsilon^{14/3} \tilde{d})) \) (or \( \tilde{O}(1/\epsilon^{11/3}) \) if \( \tilde{d} \) is unknown).
For bounded-degree model, we assume that $d$ is known in the beginning.

**Theorem 6.2.** For bounded-degree model, there is a property testing algorithm for $k$-edge (k-vertex where $k < n/4$) connectivity with correct probability at least $2/3$ that uses $\tilde{O}(k/\epsilon)$ queries (same for $k$-vertex) and runs in $\tilde{O}(k/\epsilon^{8/3})$ time ($\tilde{O}(k/\epsilon^{1.5})$ time for $k$-vertex). If $G$ is simple, then the same algorithm for testing $k$-edge-connectivity queries at most $\tilde{O}(\min\{k/\epsilon, 1/\epsilon^2\})$.

We prove Theorem 6.1 using properties of $\epsilon$-far from being $k$-edge/vertex connected from [OR11] and [FJ99] along with a variant of approximate LocalEC in Section 6.1, and approximate LocalVC in Section 6.3.

### 6.1 Testing $k$-Edge-Connectivity: Unbounded-Degree Model

In this section, we prove Theorem 6.1 for testing $k$-edge-connectivity. The key tool for our property testing algorithm is approximate local edge connectivity in a suitable form for the application to property testing. We can derive the following gap version of LocalEC in [NSY19] by essentially setting $\epsilon = \text{gap}/k$.

**Lemma 6.3** (Implicit in [NSY19]). There is a randomized (Monte Carlo) algorithm that takes as input a vertex $x \in V$ of an $n$-vertex $m$-edge directed graph $G = (V,E)$ represented as incidence lists, a volume parameter $\nu$, a cut-size parameter $k \geq 1$, and “gap” parameter $\text{gap} \in (0,k)$ where $\nu < \text{gap} \cdot m/(8k)$, queries at most $\tilde{O}(\nu k/\text{gap})$ edges, runs in $\tilde{O}((\nu/\text{gap})^{5/3}k)$ time, and

- if there is a vertex-set $S$ such that $S \ni x$, $\text{vol}^{\text{out}}(S) \leq \nu$, and $|E(S,V-S)| < k - \text{gap}$, then it returns an edge-cut of size less than $k$,
- if there is no vertex-set $S$ such that $S \ni x$, $\text{vol}^{\text{out}}(S) \leq \nu$, and $|E(S,V-S)| < k$, then it returns the symbol $\perp$.

We present an algorithm for testing $k$-edge-connectivity assuming Lemma 6.3.

**Algorithm.**

1. Sample $\Theta(\frac{1}{\epsilon})$ vertices uniformly.
2. If any of the sampled vertex has degree less than $k$, returns the corresponding edge-cut.
3. Sample $\Theta(\frac{k \log k}{\epsilon d})$ vertices uniformly (if $\bar{d}$ is unknown, then we sample $\Theta(\frac{\log k}{\epsilon})$ instead).
4. For each sampled vertex $x$, and for $i \in \{0,1,\ldots, \lceil \log_2 k \rceil\}$,
   (a) let $\nu = 2^{i+2}\epsilon^{-1}\lceil \log_2 k \rceil$, and $\text{gap} = 2^i - 1$.
   (b) run GapLocalEC$(x,\nu,k,\text{gap})$ on both $G$ and $G^R$ where $G^R$ is $G$ with reversed edges.
5. Return an edge-cut of size less than $k$ if any execution of GapLocalEC returns a cut. Otherwise, declare that $G$ is $k$-edge-connected.

**Query and Time Complexity.** We first show that the number of edge queries is at most $\tilde{O}(k^2/(\epsilon^2 \bar{d}))$. For each sampled vertex $x$ and $i \in \{0,1,\ldots, \lceil \log_2 k \rceil\}$, by Lemma 6.3, GapLocalEC queries $\tilde{O}(\nu k/\text{gap}) = \tilde{O}(k/\epsilon)$ edges. The result follows from we repeat $\log_2 k$ times per sample, and we sample $\tilde{O}(k \log k/(\epsilon d))$ times. Next, we show that the running time is $\tilde{O}(k^2/(\epsilon^{11/3} \bar{d}))$. This follows from the same argument, but we use the running time for GapLocalEC instead of edge-query complexity. If $\bar{d}$ is unknown, we can remove the term $k/\bar{d}$ from above since we sample $\Theta(\frac{\log k}{\epsilon})$ vertices instead.
Correctness. If \( G \) is \( k \)-edge-connected, the algorithm above never returns an edge-cut. We show that if \( G \) is \( \epsilon \)-far from being \( k \)-edge-connected, then the algorithm outputs an edge-cut of size less than \( k \) with constant probability. We start with simple observation.

Lemma 6.4. If \( m < nk/4 \), then with constant probability, the algorithm outputs an edge-cut of size less than \( k \) at step 2.

Proof. Suppose \( m < nk/4 \). There are at most \( n/2 \) nodes with out-degree at least \( k \). Hence, there are at least \( n/2 \) nodes of degree less than \( k \). In this case, we can sample \( O(1) \) time where each sampled node \( x \) we check \( \text{deg}^\text{out}(x) < k \) to get \( k \)-edge-cut with constant probability.

From now we assume that

\[
m \geq nk/4. 
\]

Next, we state important properties when \( G \) is \( \epsilon \)-far from being \( k \)-edge-connected. For any non-empty subset \( X \subset V \), let \( d^\text{out}(X) = |E(X,V-X)| \), and \( d^\text{in}(X) = |E(V-X,X)| \).

Theorem 6.5 ([OR11] Corollary 8). A directed graph \( G = (V,E) \) is \( \epsilon \)-far from being \( k \)-edge-connected (for \( k \geq 1 \) if and only if there exists a family of disjoint subsets \( \{X_1,\ldots,X_t\} \) of vertices for which either \( \sum_{i}(k-d^\text{out}(X_i)) > \epsilon m \) or \( \sum_{i}(k-d^\text{in}(X_i)) > \epsilon m \).

Let \( F := \{X_1,\ldots,X_t\} \) as in Theorem 6.5. We assume without loss of generality that

\[
\sum_{i}(k-d^\text{out}(X_i)) > \epsilon m. 
\]

Let \( C_{-1} = \{X \in F: k \leq d^\text{out}(X)\} \). For \( i \in \{0,1,\ldots,\lfloor \log_2 k \rfloor \} \), let \( C_i = \{X \in F: k - d^\text{out}(X) \in [2^i,2^{i+1})\} \). Note that

\[
2^i \leq k, \text{ for any } i \in \{0,\ldots,\lfloor \log_2 k \rfloor \} 
\]

and

\[
F = \bigcup_{i=-1}^{\lfloor \log_2 k \rfloor} C_i 
\]

where \( \bigcup \) is the disjoint union. Let \( C_{i,\text{big}} = \{X \in C_i: \text{vol}^\text{out}(X) \geq 2^{i+2} \epsilon^{-1}(\lfloor \log_2 k \rfloor + 1)\} \), and \( C_{i,\text{small}} = C_i - C_{i,\text{big}} \). The following lemma is the key for the algorithm’s correctness.

Lemma 6.6. There is \( i \) such that \( |C_{i,\text{small}}| \geq \epsilon d/4k(\lfloor \log_2 k \rfloor + 1) \). If \( d \) is unknown, we have \( |C_{i,\text{small}}| \geq \epsilon n/(16(\lfloor \log_2 k \rfloor + 1)) \) instead.

We show that Lemma 6.6 implies the correctness of the algorithm. By sampling \( O(k \log k/(\epsilon d)) \) many vertices (or \( O(\log k/e) \) if \( d \) is unknown), we get an event where a sampled vertex belongs to some vertex set \( X \in C_{i,\text{small}} \) with constant probability (since \( C_{i,\text{small}} \) contains disjoint sets). We run GapLocalEC for every \( i \in \{0,1,\ldots,\lfloor \log_2 k \rfloor \} \) using \( \nu = 2^{i+2} \epsilon^{-1}\lfloor \log_2 k \rfloor \), and \( \text{gap} = 2^i - 1 \); also, there exists \( i \) such that \( |C_{i,\text{small}}| \geq \epsilon d/(4k(\lfloor \log_2 k \rfloor + 1)) \) (or \( \epsilon n/(16(\lfloor \log_2 k \rfloor + 1)) \)) if \( d \) is unknown) by Lemma 6.6. Therefore, by Lemma 6.3 GapLocalEC outputs an edge-cut of size less than \( k \) with constant probability.
Proof of Lemma 6.6. We show that there is $i > 0$ such that $|C_i| > \epsilon m/(2^i([\log_2 k] + 1))$. First, we show that there is $i > 0$ such that

$$\sum_{X \in C_i} (k - d^\text{out}(X)) > \epsilon m/(\lfloor \log_2 k \rfloor + 1). \tag{8}$$

Suppose otherwise that for every $i$, $\sum_{X \in C_i} (k - d^\text{out}(X)) \leq \epsilon m/(\lfloor \log_2 k \rfloor + 1)$. We have $\sum_{X \in \mathcal{C}} (k - d^\text{out}(X)) \leq \sum_{i=0}^{\lfloor \log_2 k \rfloor} \sum_{X \in C_i} (k - d^\text{out}(X)) \leq \epsilon m$. However, this contradicts Equation (5) as in Theorem 6.5. Second, we claim that for any $i$,

$$|C_i|^{2i+1} \geq \sum_{X \in C_i} (k - d^\text{out}(X)).$$

This follows trivially from that each element $X$ in the set $C_i$, $k - d^\text{out}(X) \leq 2^{i+1}$. For $i$ that satisfies Equation (8) we have

$$|C_i| \geq \sum_{X \in C_i} (k - d^\text{out}(X))/2^{i+1} > \epsilon m/(2^{i+1}([\log_2 k] + 1)). \tag{9}$$

Recall that $C_{i,\text{big}} = \{X \in C_i: \text{vol}^\text{out}(X) \geq 2^{i+2}\epsilon^{-1}([\log_2 k] + 1)\}$, and $C_{i,\text{small}} = C_i - C_{i,\text{big}}$. We show that $|C_{i,\text{big}}| < |C_i|/2$. Therefore, for $i$ that satisfies Equation (9) we have

$$2|C_{i,\text{big}}| \leq \sum_{X \in C_{i,\text{big}}} \text{vol}^\text{out}(X)/(2^{i+1}\epsilon^{-1}([\log_2 k] + 1)) \leq \epsilon m/(2^{i+1}([\log_2 k] + 1)) < |C_i|. \tag{9}$$

The first inequality is because the term $\text{vol}^\text{out}(X)/(\gamma\epsilon^{-1}([\log_2 k] + 1)) \geq 2$ for each $X \in C_{i,\text{big}}$, we have $\sum_{X \in C_{i,\text{big}}} \text{vol}^\text{out}(X)/(2^{i+1}\epsilon^{-1}([\log_2 k] + 1)) \geq 2|C_{i,\text{big}}|$. The second inequality is because elements in $C_{i,\text{big}}$ are disjoint and thus $\sum_{X \in C_{i,\text{big}}} \text{vol}^\text{out}(X) \leq m$. The final inequality follows from Equation (9).

For the same $i$, since $|C_{i,\text{big}}| < |C_i|/2$, we have

$$|C_{i,\text{small}}| \geq |C_i|/2 \geq \epsilon m/(2^{i+2}([\log_2 k] + 1)) \geq \epsilon nd/(4k([\log_2 k] + 1)). \tag{10}$$

The last inequality follows from $m = nd$, and Equation (6). If $\bar{d}$ is unknown, by Equation (4), the last inequality becomes $\epsilon m/(2^{i+2}([\log_2 k] + 1)) \geq \epsilon nk/(16k[\log_2 k]) = \epsilon n/(16([\log_2 k] + 1))$. This follows from Equation (6) and Equation (4). \hfill \Box

An improved bound for a simple graph. The same algorithm gives an improved bound when $G$ is simple. If $\epsilon \geq 4/k$, the algorithm queries at most $O(k^2/(\epsilon^2\bar{d})) = O(1/\epsilon^4\bar{d})$ edges (and $O(1/\epsilon^3)$ edges if $\bar{d}$ is unknown). Now, we assume $\epsilon > 4/k$, we show that there are $\Omega(\epsilon nk/\bar{d})$ (or $\Omega(\epsilon n)$ if $\bar{d}$ is unknown) many vertices with degree less than $k$.

Lemma 6.7. If $\epsilon > 4/k$, $G$ is simple, and $\epsilon$-far from being $k$-edge-connected, then there exist at least $\epsilon nk/2$ vertices ($\epsilon nk/(8k)$ vertices if $\bar{d}$ is unknown) with degree less than $k$.

Lemma 6.7 immediately yields the correctness of the algorithm as number of singleton with degree less than $k$ is at least $\epsilon nk/2$ vertices ($\epsilon nk/(8k)$ vertices if $\bar{d}$ is unknown), and we sample $\Theta(k/(\epsilon \bar{d}))$ (or $\Theta(1/\epsilon)$ vertices if $\bar{d}$ is unknown) at step 1 and 2 to check if each sampled vertex has degree less than $k$. Next, we prove Lemma 6.7.
Proof of Lemma 6.7. Let \( \mathcal{C} = \{ X : k - d^\text{out}(X) \geq 1 \} \). We claim that
\[
|\mathcal{C}| > \epsilon m/k.
\] (11)
This follows from
\[
|\mathcal{C}| k \geq \sum_{X \in \mathcal{C}} (k - d^\text{out}(X)) \geq \sum_{X \in \mathcal{F}} (k - d^\text{out}(X)) > \epsilon m.
\]
The first inequality follows from each term \( k - d^\text{out}(X) \) is at most \( k \), and there are \( |\mathcal{C}| \) terms. The second inequality follows from each \( X \in \mathcal{F} \setminus \mathcal{C} \), \( k - d^\text{out}(X) \leq 0 \). The third inequality follows from Equation (5).

Let \( \mathcal{C}_{\text{big}} = \{ X \in \mathcal{C} : \text{vol}^\text{out}(X) \geq 2k/\epsilon \} \), and \( \mathcal{C}_{\text{small}} = \mathcal{C} - \mathcal{C}_{\text{big}} \). We claim that
\[
|\mathcal{C}_{\text{small}}| > \epsilon n/8.
\] (12)
First, we show that
\[
|\mathcal{C}_{\text{big}}| < |\mathcal{C}|/2.
\] (13)
This follows from
\[
|\mathcal{C}_{\text{big}}| \leq \sum_{X \in \mathcal{C}_{\text{big}}} \text{vol}^\text{out}(X)/(2k\epsilon^{-1}) \leq \epsilon m/(2k) < |\mathcal{C}|/2.
\]
The first inequality follows from the fact that for each \( X \in \mathcal{C}_{\text{big}} \), \( \text{vol}^\text{out}(X)/(2k\epsilon^{-1}) \geq 1 \). Hence, \( \sum_{X \in \mathcal{C}_{\text{big}}} \text{vol}^\text{out}(X)/(2k\epsilon^{-1}) \geq |\mathcal{C}_{\text{big}}| \). The second inequality follows from the fact that \( \mathcal{C}_{\text{big}} \) contains disjoint sets, and \( \sum_{X \in \mathcal{C}_{\text{big}}} \text{vol}^\text{out}(X) \leq \text{vol}^\text{out}(V) = m \). The last inequality follows from Equation (11).

Next, we have
\[
|\mathcal{C}_{\text{small}}| \geq |\mathcal{C}|/2 \geq \epsilon m/(2k) \geq \epsilon (nd)/(2k) \geq \epsilon (nd)/2.
\] (14)
The first inequality follows from Equation (13) and that \( \mathcal{C}_{\text{small}} = \mathcal{C} - \mathcal{C}_{\text{big}} \). The second inequality follows from Equation (11). The third inequality follows from \( m = nd \). If \( \bar{d} \) is unknown, the last part of Equation (14) becomes \( m/(2k) \geq \epsilon (nk)/(8k) \geq \epsilon n/8 \). This follows from Equation (4).

It suffices to show that, for each \( X \in \mathcal{C}_{\text{small}} \), the average degree of vertices in \( X \), which is \( \frac{\text{vol}^\text{out}(X)}{|X|} \), is less than \( k \). If this is true, then there exists node \( x \in X \) where \( \text{deg} x < k \). Since the sets in \( \mathcal{C} \) are disjoint, each set \( X \in \mathcal{C} \) contains a vertex with degree less than \( k \), and \( |\mathcal{C}_{\text{small}}| > \epsilon n/8 \) (by Equation (12)), we have that the number of singleton vertex with degree less than \( k \) is \( > \epsilon n/8 \), and we are done.

Now, fix \( X \in \mathcal{C}_{\text{small}} \) and we want to show that \( \frac{\text{vol}^\text{out}(X)}{|X|} < k \). Consider three cases. If \( |X| = 1 \), then \( \frac{\text{vol}^\text{out}(X)}{|X|} = d^\text{out}(X) < k \). Next, if \( |X| \geq 2/\epsilon \), then \( \frac{\text{vol}^\text{out}(X)}{|X|} < \frac{2k/\epsilon}{2/\epsilon} = k \) as \( X \in \mathcal{C}_{\text{small}} \). In the last case, we have \( 2 \leq |X| < 2/\epsilon \leq k/2 \). Note that \( \text{vol}^\text{out}(X) \leq d^\text{out}(X) + |X|^2 \) because the graph is simple. So,
\[
\frac{\text{vol}(X)}{|X|} \leq \frac{d(X) + |X|^2}{|X|} < \frac{k}{|X|} + |X| < \frac{k}{2} + \frac{k}{2} = k.
\]

\[
6.2 \text{ Testing } k\text{-Edge-Connectivity: Bounded-Degree Model}
\]
In this section, we prove Theorem 6.1 for testing \( k \)-edge-connectivity for bounded degree model. In this model, we know the maximum out-degree \( d \). We assume that \( G \) is \( d \)-regular, meaning that every vertex has degree \( d \). If \( G \) is not \( d \)-regular, we can “treat” \( G \) as if it is \( d \)-regular as follows. For
any list $L_v$ of size less than $d$, and $i \in [|L_v|, d]$, we ensure that query$(v,i)$ returns a self-loop edge (i.e., an edge $(v,v)$).

**Edge-sampling procedure.** The key property of a $d$-regular graph is that we can sample edge uniformly as follows. We first sample a vertex $x \in V$. Then, we make query$(x,i)$ where $i$ is an integer sampled uniformly from $[1,d]$.

**Proposition 6.8.** For any edge $e \in E$, the probability that $e$ is sampled from the edge-sampling procedure is $1/m$.

**Proof.** Fix any edge $e \in E$. The edge $e$ belongs to some list $L_v$. Therefore, the probability that $e$ is queried according to edge-sampling procedure is

$$
P(e \text{ is queried}) = P(e \text{ is queried} \mid L_v \text{ is sampled})P(L_v \text{ is sampled}) + 
P(e \text{ is queried} \mid L_v \text{ is not sampled})P(L_v \text{ is not sampled})
$$

$$
= P(e \text{ is queried} \mid L_v \text{ is sampled})P(L_v \text{ is sampled})
$$

$$
= (1/d)(1/n) = 1/m. 
$$

We present an algorithm for testing $k$-edge-connectivity for bounded-degree model assuming Lemma 6.3.

**Algorithm.**

1. Sample $\Theta(\frac{1}{\epsilon})$ vertices uniformly.

2. If any of the sampled vertex has degree less than $k$, returns the corresponding edge-cut.

3. For each $i \in \{0, \ldots, \lfloor \log_2 k \rfloor \}$, and for each $j \in \{0, \ldots, \lfloor \log_2 \eta_i \rfloor \}$ where $\eta_i = 2^{i+2}\epsilon^{-1}\lfloor \log_2 k \rfloor$,

(a) Sample $\Theta(\frac{\lfloor \log_2 k \rfloor \lfloor \log_2 \eta_i \rfloor}{\epsilon^{2^{j-i}}}) = \tilde{\Theta}(\frac{1}{\epsilon^{2^{j-i}}})$ edges uniformly.

(b) let $\nu = 2^{j+1}$, and $\text{gap} = 2^i - 1$.

(c) run GapLocalEC$(x, \nu, k, \text{gap})$ on both $G$ and $G^R$ where $G^R$ is $G$ with reversed edges, and $x$ is a vertex from the sampled edge of the form $(x,y)$.

4. Return an edge-cut of size less than $k$ if any execution of GapLocalEC returns a cut. Otherwise, declare that $G$ is $k$-edge-connected.

**Query and Time Complexity.** We first show that the number of edge queries is at most $\tilde{O}(k/\epsilon)$. For each vertex $x$ from the sampled edge $(x,y)$ and for each $(i,j)$ pair in loops, by Lemma 6.3, GapLocalEC queries $\tilde{O}(\nu k / \text{gap}) = \tilde{O}(2^{j-i} k)$ edges, and we sample $\tilde{O}(1/(\epsilon 2^{j-i}))$ times. Therefore, by repeating $\tilde{O}(1)$ time, the total edge queries is at most $\tilde{O}(k/\epsilon)$.

Next, we show that the running time is $\tilde{O}(k/\epsilon^{8/3})$. This follows from the same argument, but we use the running time for GapLocalEC instead of edge-query complexity. For each iteration, the running time is $\tilde{O}((\nu / \text{gap})^{5/3} k \cdot 1/(\epsilon 2^{j-i})) = \tilde{O}((2^{j-i})^{2/3} k / \epsilon) = \tilde{O}(k/\epsilon^{8/3})$. The last inequality follows because by definition $2^j \leq 2^{j+2} \epsilon^{-1} \lfloor \log_2 k \rfloor$.

**Correctness.** If $G$ is $k$-edge-connected, then the algorithm never returns any edge-cut, and we are done. Suppose $G$ is $\epsilon$-far from being $k$-edge-connected, then we show that the algorithm outputs an edge-cut of size less than $k$ with constant probability. Since $G$ is $d$-regular, we have $d = d$. Therefore,
we can use results from Section 6.1. Let \( \mathcal{F} := \{X_1, \ldots, X_t\} \) as in Theorem 6.5. We assume without loss of generality that
\[
\sum_i (k - d^{\text{out}}(X_i)) > \epsilon m.
\] (15)

Let \( \mathcal{C}_{-1} = \{X \in \mathcal{F}: k \leq d^{\text{out}}(X)\} \). For \( i \in \{0, 1, \ldots, \lceil \log_2 k \rceil\} \), let \( \mathcal{C}_i = \{X \in \mathcal{F}: k - d^{\text{out}}(X) \in [2^i, 2^{i+1})\} \). Let \( \mathcal{C}_{i, \text{big}} = \{X \in \mathcal{C}_i: \text{vol}^{\text{out}}(X) \geq 2^{i+2}(\lceil \log_2 k \rceil + 1)/\epsilon\} \), and \( \mathcal{C}_{i, \text{small}} = \mathcal{C}_i - \mathcal{C}_{i, \text{big}} \). By Lemma 6.6, there is \( i \) such that
\[
|\mathcal{C}_{i, \text{small}}| \geq \epsilon nd/(4k(\lceil \log_2 k \rceil + 1)) = \epsilon m/(4k(\lceil \log_2 k \rceil + 1)).
\] (16)

This last inequality follows since \( nd = nd = m \). We fix \( i \) as in Equation (16). Let \( \eta_i = 2^{i+2}\epsilon^{-1}\lceil \log_2 k \rceil \).

For \( j \in \{0, 1, \ldots, \lceil \log_2 \eta_i \rceil\} \), let \( \mathcal{C}_{i, \text{small}, j} = \{X \in \mathcal{C}_{i, \text{small}}: \text{vol}^{\text{out}}(X) \in [2^j, 2^{j+1})\} \).

**Lemma 6.9.** For \( i \) that satisfies Equation (16), there is \( j \) such that \( \sum_{X \in \mathcal{C}_{i, \text{small}, j}} \text{vol}^{\text{out}}(X) \geq \epsilon m2^{j-i}/(4(\lceil \log_2 k \rceil + 1)(\lceil \log_2 \eta_i \rceil + 1)) \).

We show that Lemma 6.9 implies the correctness. By sampling \( \Theta(\lceil \log_2 k \rceil \lceil \log_2 \eta_i \rceil) = \tilde{\Theta}(2^{\eta_i}) \) edges, we get an event where a sampled edge \((u, v)\) has \( u \in X \) for some \( X \in \mathcal{C}_{i, \text{small}, j} \) with constant probability (since \( \mathcal{C}_{i, \text{small}, j} \) contains disjoint elements). For each \((i, j) \in \{0, 1, \ldots, \lceil \log_2 \eta_i \rceil\} \times \{0, \ldots, \lceil \log_2 \eta_i \rceil\} \), we run GapLocalEC with \( \nu = 2^{j+1} \), and gap = \( 2^i - 1 \); also, there exists \((i, j)\) such that \( \sum_{X \in \mathcal{C}_{i, \text{small}, j}} \text{vol}^{\text{out}}(X) \geq \epsilon m2^{j-i}/(4(\lceil \log_2 k \rceil + 1)(\lceil \log_2 \eta_i \rceil + 1)) \) by Lemma 6.9. Therefore, by Lemma 6.3, GapLocalEC outputs an edge-cut of size less than \( k \) with constant probability.

**Proof of Lemma 6.9.** We claim that there is \( j \) such that
\[
|\mathcal{C}_{i, \text{small}, j}| \geq |\mathcal{C}_{i, \text{small}}|/(\lceil \log_2 \eta_i \rceil + 1) \] (17)

Suppose otherwise. We have for all \( j \in \{0, \ldots, \lceil \log_2 \eta_i \rceil\} \), \( |\mathcal{C}_{i, \text{small}, j}| < |\mathcal{C}_{i, \text{small}}|/(\lceil \log_2 \eta_i \rceil + 1) \). Therefore, \( \sum_{j \in \{0, \ldots, \lceil \log_2 \eta_i \rceil\}} |\mathcal{C}_{i, \text{small}, j}| < |\mathcal{C}_{i, \text{small}}| \), a contradiction. Now, for the same \( j \), we have
\[
\sum_{X \in \mathcal{C}_{i, \text{small}, j}} \text{vol}^{\text{out}}(X) \geq |\mathcal{C}_{i, \text{small}, j}|2^j
\geq |\mathcal{C}_{i, \text{small}}|2^j/(\lceil \log_2 \eta_i \rceil + 1)
\geq \epsilon m2^j/(4(\lceil \log_2 k \rceil + 1)(\lceil \log_2 \eta_i \rceil + 1))
\geq \epsilon m2^{j-i}/(4(\lceil \log_2 k \rceil + 1)(\lceil \log_2 \eta_i \rceil + 1))
\] (6)

The first inequality is because the set \( \mathcal{C}_{i, \text{small}, j} \) contains disjoint elements, and that \( \text{vol}^{\text{out}}(X) \geq 2^j \) by definition.

**An improved bound for a simple graph.** The same algorithm gives an improved bound, \( \tilde{O}(\min\{k/\epsilon, 1/\epsilon^2\}) \) queries, when \( G \) is simple. If \( \epsilon \geq 4/k \), then the algorithm queries at most \( \tilde{O}(k/\epsilon) = \tilde{O}(1/\epsilon^2) \) edges. Otherwise, \( \epsilon > 4/k \), by Lemma 6.7, there are \( \Omega(\epsilon n) \) many vertices with degree less than \( k \), and this implies that the algorithm outputs an edge-cut of size less than \( k \) at step 2.
6.3 Testing k-Vertex-Connectivity: Unbounded-Degree Model

In this section, we prove Theorem 6.1 for testing k-vertex-connectivity. The key tool for our property testing algorithm is approximate local vertex connectivity in a suitable form for the application to property testing. We can derive the following gap version of LocalVC in [NSY19] by essentially setting $\epsilon = \text{gap}/k$.

**Lemma 6.10** ([NSY19] Theorem 4.1). There is a randomized (Monte Carlo) algorithm that takes as input a vertex $x \in V$ of an n-vertex $m$-edge directed graph $G = (V,E)$ represented as incidence-lists with minimum out-degree $\delta \geq 1$, a volume parameter $\nu$, a cut-size parameter $k$, and “gap” parameter $\text{gap} \in (0,k)$ where $\nu < \text{gap} \cdot m/(640k)$, and $k \leq n/4$, that queries at most $\tilde{O}(\nu k/\text{gap})$ edges, runs in $\tilde{O}((\nu/\text{gap})^{1.5} k)$ time and

- if there is a separation triple $(L,S,R)$ where $L \ni x, |S| < k - \text{gap}, \text{vol}^{\text{out}}(L) \leq \nu$ and either $\min_{v \in L}\{\text{deg}^{\text{out}}(v)\} < k$ or $|S| < k - \text{gap}$, then it returns a vertex-cut of size less than $k$,
- if there is no separation triple $(L,S,R)$ where $L \ni x, |S| < k, \text{vol}^{\text{out}}(L) \leq \nu$, then it returns the symbol $\perp$.

We present an algorithm for testing k-vertex-connectivity assuming Lemma 6.10, and analysis.

**Algorithm.**

1. Sample $\Theta(1)$ vertices uniformly.
2. If any of the sampled vertex $x$ has out-degree less than $k$, returns $N(x)$.
3. Sample $\Theta(k \log k/(\epsilon \bar{d}))$ vertices uniformly (if $\bar{d}$ is unknown, sample $\Theta(\log k/\epsilon)$ vertices instead).
4. For each sampled vertex $x$, and for $i \in \{0,\ldots,\log_2 k\}$,
   - (a) let $\nu = 2^{i+3}[\log_2 k]/\epsilon$, and $\text{gap} = 2^i - 1$.
   - (b) run GapLocalVC$(x, \nu, k, \text{gap})$ on both $G$ and $G^R$ where $G^R$ is the same graph with reversed edges.
5. Return a vertex-cut of size less than $k$ if any execution of GapLocalVC returns a vertex-cut. Otherwise, declare that $G$ is k-vertex-connected.

**Query and Time Complexity.** We first show that the number of edge queries is at most $\tilde{O}(k^2/(\epsilon^2 \bar{d}))$. For each sampled vertex $x$ and $i \in \{1,\ldots,\log_2 k\}$, by Lemma 6.10, GapLocalVC queries $O(\nu k/\text{gap}) = \tilde{O}(k/\epsilon)$ edges. The result follows from we repeat $\lceil \log_2 k \rceil$ times per sample, and we sample $O(k \log k/(\epsilon \bar{d}))$ times. Next, we show that the running time is $\tilde{O}(k^2/(\epsilon^{2.5} \bar{d}))$. This follows from the same argument, but we use the running time for GapLocalVC instead of edge-query complexity.

**Correctness.** If $G$ is $k$-vertex-connected, it is clear that the GapLocalVC never returns any vertex-cut. We show that if $G$ is $\epsilon$-far from $k$-vertex-connected, then the algorithm outputs a vertex-cut of size less than $k$ with constant probability. We start with simple observation.

**Lemma 6.11.** If $m < nk/4$, then with constant probability, the algorithm outputs an vertex-cut of size less than $k$ at step 2.

**Proof.** Suppose $m < nk/4$. There are at most $n/2$ nodes with out-degree at least $k$. Hence, there are at least $n/2$ nodes of degree less than $k$. In this case, we can sample $O(1)$ time where each sampled node $x$ we check $|N(x)| < k$ to get $k$-vertex-cut with constant probability. $\square$
From now we assume that
\[ m \geq nk/4. \]  

We start with important properties when \( G \) is \( \epsilon \)-far from \( k \)-vertex-connected. We say that two separation triples \((L,S,R)\) and \((L',S',R')\) are independent if \( L \cap L' = \emptyset \) or \( R \cap R' = \emptyset \).

**Theorem 6.12** ([OR11] Corollary 17). If a directed graph \( G = (V,E) \) is \( \epsilon \)-far from being \( k \)-vertex-connected, then there exists a set \( F' \) of pairwise independent separation triples\(^7\) such that \( \sum_{(L,S,R) \in F'} \max\{k - |S|, 0\} > em \).

Let \( \mathcal{F} \) be a family of pairwise independent separation triples of \( G \) such that \( p(\mathcal{F}) := \sum_{(L,S,R) \in \mathcal{F}} \max\{k - |S|, 0\} \) is maximized. By Theorem 6.12, we have \( \sum_{(L,S,R) \in \mathcal{F}} \max\{k - |S|, 0\} > em \).

We say that a left-partition \( L \) of a separation triple \((L,S,R)\) is small if \( |L| \leq |R| \). Similarly, a right-partition \( R \) is small if \(|R| \leq |L| \).

**Lemma 6.13** ([FJ99] Lemma 7). The small left-partitions\(^8\) in \( \mathcal{F} \) are pairwise disjoint, and the small right-partitions in \( \mathcal{F} \) are pairwise disjoint.

Let \( \mathcal{F}_L \) be the set of separation triples with small left-partitions in \( \mathcal{F} \), and \( \mathcal{F}_R \) be the set of separation triples with small-right partitions in \( \mathcal{F} \). By Theorem 6.12, we have that \( \max\{p(\mathcal{F}_L), p(\mathcal{F}_R)\} > em/2 \). We assume without loss of generality that
\[ p(\mathcal{F}_L) > em/2. \]  

Let \( \mathcal{C}_{-1} = \{(L,S,R) \in \mathcal{F}_L : k \leq |S|\} \). For \( i \in \{0, \ldots, \lfloor \log_2 k \rfloor\} \), let \( \mathcal{C}_i = \{(L,S,R) \in \mathcal{F}_L : k - |S| \in [2^i, 2^{i+1})\} \). Let \( \mathcal{C}_{i,\text{big}} = \{(L,S,R) \in \mathcal{C}_i : \text{vol}^{\text{out}}(L) \geq 2^{i+3}e^{-1}(\lfloor \log_2 k \rfloor + 1)\} \), and \( \mathcal{C}_{i,\text{small}} = \mathcal{C}_i - \mathcal{C}_{i,\text{big}} \). The following lemma is the key for the algorithm’s correctness.

**Lemma 6.14.** There is \( i \) such that \( |\mathcal{C}_{i,\text{small}}| > en/(8k(\lfloor \log_2 k \rfloor + 1)) \). If \( \bar{d} \) is unknown, then there is \( i \) such that \( |\mathcal{C}_{i,\text{small}}| \geq en/(32(\lfloor \log_2 k \rfloor + 1)) \).

We show that Lemma 6.14 implies the correctness of the algorithm. By sampling \( \Theta(k \log k/(\epsilon \bar{d})) \) many nodes (or \( \Theta(\log k/(\epsilon)) \) vertices if \( \bar{d} \) is unknown), we get an event where \( x \) belongs to some vertex set \( L \) in separation triple \((L,S,R) \in \mathcal{C}_{i,\text{small}}\) with constant probability (this follows since \( \mathcal{C}_{i,\text{small}} \) contains pairwise disjoint small left-partitions by Lemma 6.13). We run GapLocalVC for every \( i \in \{0, 1, \ldots, \lfloor \log_2 k \rfloor\} \), and there exists \( i \) such that \( |\mathcal{C}_{i,\text{small}}| \geq en/(8k(\lfloor \log_2 k \rfloor + 1)) \) (or \( |\mathcal{C}_{i,\text{small}}| \geq en/(32(\lfloor \log_2 k \rfloor + 1)) \)) by Lemma 6.14. Therefore, by Lemma 6.10, GapLocalVC outputs a vertex-cut of size less than \( k \) with constant probability.

**Proof of Lemma 6.14.** We show that there is \( i > 0 \) such that
\[ |\mathcal{C}_i| > em/(2^{i+2}(\lfloor \log_2 k \rfloor + 1)). \]  

First, we show that there is \( i > 0 \) such that
\[ \sum_{(L,S,R) \in \mathcal{C}_i} (k - |S|) > em/(2(\lfloor \log_2 k \rfloor + 1)). \]

---

\(^7\)We use the term separation triple \((L,S,R)\) instead of the term one-way pair \((L,R)\) used by [OR11] for notational consistency in our paper. These terms are equivalent in that there is no edge from \( L \) to \( R \) and our \( S \) is their \( V - (L \cup R) \).

\(^8\)In [FJ99], they use the term one-way pair \((T,H)\), and define a tail \( T \) of a pair \((T,H)\) if small if \(|T| \leq |H| \). Similarly, they define a head \( H \) of a pair \((T,H)\) to be small if \(|H| \leq |T| \). We only rephrase from “tail” to left-partition, and “head” to right-partition.
Suppose otherwise that for every \( i \), \( \sum_{(L,S,R) \in C_i} (k - |S|) \leq \epsilon m / (2(|\log_2 k| + 1)) \). We have \( \sum_{(L,S,R) \in F_L} (\max \{k - |S|, 0\}) = \sum_{i=0}^{\lfloor \log_2 k \rfloor} \sum_{(L,S,R) \in C_i} (k - |S|) = \sum_{i=0}^{\lfloor \log_2 k \rfloor} \sum_{(L,S,R) \in C_i} (k - |S|) \leq \epsilon m / 2. \) However, this contradicts Equation (19). Second, we show that for any \( i \),

\[
|C_i|^{2^{i+1}} \geq \sum_{(L,S,R) \in C_i} (k - |S|). \tag{22}
\]

This follows trivially from that each \((L, S, R)\) in the set \( C_i \), \( k - |S| \leq 2^{i+1} \). Therefore, for \( i \) that satisfies Equation (21), we have

\[
|C_i|^{(22)} \sum_{(L,S,R) \in C_i} (k - |S|)/2^{i+1} > \epsilon m / (2^{i+2}(\lfloor \log_2 k \rfloor + 1)). \tag{23}
\]

Recall that \( C_{i,\text{big}} = \{(L, S, R) \in C_i : \text{vol}^{\text{out}}(L) \geq 2^{i+3}(|\log_2 k| + 1)/\epsilon\} \), and \( C_{i,\text{small}} = C_i - C_{i,\text{big}} \). We claim that for \( i \) that satisfies Equation (23), \( |C_{i,\text{big}}| < |C_i|/2 \). Indeed, we have

\[
4|C_{i,\text{big}}| \leq \sum_{(L,S,R) \in C_{i,\text{big}}} \text{vol}^{\text{out}}(L)/(\gamma \epsilon^{-1}(|\log_2 k| + 1)) \leq m / (\gamma \epsilon^{-1}(|\log_2 k| + 1)) \leq 2|C_i|. \tag{20}
\]

The first inequality follows because \( \text{vol}^{\text{out}}(L)/(2^{i+1} \epsilon^{-1}(|\log_2 k| + 1)) \geq 4 \) for each \((L, S, R) \in C_{i,\text{big}}\). The second inequality follows since left-partitions in \( C_{i,\text{big}} \) are disjoint, and \( \sum_{(L,S,R) \in C_{i,\text{big}}} \text{vol}^{\text{out}}(L) \leq m \).

Next, we have

\[
|C_{i,\text{small}}| \geq |C_i|/2 > \epsilon m / (2^{i+3}(|\log_2 k| + 1)) \geq m d / (8k(|\log_2 k| + 1)). \tag{24}
\]

The first inequality follows because \( |C_{i,\text{big}}| < |C_i|/2 \), and \( |C_i| = |C_{i,\text{big}}| + |C_{i,\text{small}}| \). The last inequality follows because \( m = nd \), and \( 2^i \leq k \). If \( d \) is unknown, the last inequality of Equation (24) becomes \( \epsilon m / (2^{i+3}(|\log_2 k| + 1)) \geq \epsilon n / (32k(|\log_2 k| + 1)) = \epsilon n / (32(|\log_2 k| + 1)). \) \( \square \)

### 6.4 Testing \( k \)-Vertex-Connectivity: Bounded-Degree Model

In this section, we prove Theorem 6.1 for testing \( k \)-vertex-connectivity for bounded degree model. By the same argument in Section 6.2, we assume that \( G \) is \( d \)-regular, and thus we can sample edge uniformly by Proposition 6.8.

We present an algorithm for testing \( k \)-edge-connectivity for bounded-degree model assuming Lemma 6.10.

**Algorithm.**

1. Sample \( \Theta(1) \) vertices uniformly.
2. If any of the sampled vertex has degree less than \( k \), returns its out-neighbors.
3. For each \( i \in \{0, \ldots, \lfloor \log_2 k \rfloor\} \), and for each \( j \in \{0, \ldots, \lfloor \log_2 \eta_i \rfloor\} \) where \( \eta_i = 2^{i+2} \epsilon^{-1} \lfloor \log_2 k \rfloor \),
   - (a) Sample \( \Theta((\lfloor \log_2 k \rfloor \lfloor \log_2 \eta_i \rfloor)/\epsilon^{2(2^i - 1)}) \) edges uniformly.
   - (b) let \( \nu = 2^{i+1} \), and \( \text{gap} = 2^i - 1 \).
   - (c) run \( \text{GapLocalVC}(x, \nu, k, \text{gap}) \) on both \( G \) and \( G^R \) where \( G^R \) is \( G \) with reversed edges, and \( x \) is a vertex from the sampled edges of the form \((x, y)\).
4. Return a vertex-cut of size less than \( k \) if any execution of \( \text{GapLocalVC} \) returns a vertex-cut. Otherwise, declare that \( G \) is \( k \)-vertex-connected.

**Query and Time Complexity.** We first show that the number of edge queries is at most \( \tilde{O}(k/\epsilon) \).

For each vertex \( x \) from the sampled edge \((x,y)\) and for each \((i,j)\) pair in loops, by Lemma 6.10, \( \text{GapLocalVC} \) queries \( \tilde{O}(\nu k/\text{gap}) = \tilde{O}(2^{j-i}k) \) edges, and we sample \( \tilde{O}(1/(\epsilon 2^{j-i})) \) times. Therefore, by repeating \( \tilde{O}(1) \) iterations, the total edge queries is at most \( \tilde{O}(k/\epsilon) \).

Next, we show that the running time is \( \tilde{O}(k/\epsilon^{1.5}) \). This follows from the same argument, but we use the running time for \( \text{GapLocalEC} \) instead of edge-query complexity. For each iteration, the running time is \( \tilde{O}((\nu/\text{gap})^{1.5} k \cdot 1/(\epsilon 2^{j-i})) = \tilde{O}((2^{j-i} k)/(\epsilon 2^{j-i})) = \tilde{O}(k/\epsilon^{1.5}) \). The last inequality follows because by definition \( 2^j \leq 2^{i+2} \epsilon^{-1} \log_2 k \).

**Correctness.** If \( G \) is \( k \)-vertex-connected, then the algorithm never returns any vertex-cut, and we are done. Suppose \( G \) is \( \epsilon \)-far from being \( k \)-vertex-connected, then we show that the algorithm outputs a vertex-cut of size less than \( k \) with constant probability. Since \( G \) is \( d \)-regular, we have \( \tilde{d} = d \). Therefore, we can use results from Section 6.3. Let \( \mathcal{F}_L \) be the set of separation triples with small left-partitions in \( \mathcal{F} \), and \( \mathcal{F}_R \) be the set of separation triples with small-right partitions in \( \mathcal{F} \). By Theorem 6.12, we have that \( \max\{p(\mathcal{F}_L), p(\mathcal{F}_R)\} > \epsilon m/2 \). We assume without loss of generality that

\[
p(\mathcal{F}_L) > \epsilon m/2. \tag{25}
\]

Let \( \mathcal{C}_{-1} = \{(L, S, R) \in \mathcal{F}_L: k \leq |S|\} \). For \( i \in \{0, \ldots, \lfloor \log_2 k \rfloor\} \), let \( \mathcal{C}_i = \{(L, S, R) \in \mathcal{F}_L: k - |S| \in [2^i, 2^{i+1})\} \). Let \( \mathcal{C}_i,\text{big} = \{(L, S, R) \in \mathcal{C}_i: \text{vol}^{\text{out}}(L) \geq 2^{i+3} \epsilon^{-1} \lfloor \log_2 k \rfloor\} \), and \( \mathcal{C}_i,\text{small} = \mathcal{C}_i - \mathcal{C}_i,\text{big} \). By Lemma 6.14, there is \( i \) such that

\[
|\mathcal{C}_i,\text{small}| > \epsilon nd/(8(\lfloor \log_2 k \rfloor + 1)) = \epsilon m/(8(\lfloor \log_2 k \rfloor + 1)). \tag{26}
\]

The last inequality follows since \( nd = m \). We fix \( i \) as in Equation (26). Let \( \eta_i = 2^{i+3} \epsilon^{-1} \lfloor \log_2 k \rfloor \).

For \( j \in \{0, \ldots, \lfloor \log_2 \eta_i \rfloor\} \), let \( \mathcal{C}_{i,\text{small},j} = \{(L, S, R) \in \mathcal{C}_{i,\text{small}}: \text{vol}^{\text{out}}(L) \in [2^j, 2^{j+1})\} \).

**Lemma 6.15.** For \( i \) that satisfies Equation (26), there is \( j \) such that \( \sum_{(L,S,R) \in \mathcal{C}_{i,\text{small},j}} \text{vol}^{\text{out}}(L) \geq \epsilon m 2^{j-i}/(8(\lfloor \log_2 k \rfloor + 1)(\lfloor \log_2 \eta_i \rfloor + 1)) \).



We show that Lemma 6.15 implies the correctness. By sampling \( \Theta((\lfloor \log_2 k \rfloor, \lfloor \log_2 \eta_i \rfloor)) = \tilde{O}(1/\epsilon^{2^{i+1}}) \) edges, we get an event where a sampled edge \((u,v)\) has \( u \in L \) for some \( L \) from a separation triple \( (L,S,R) \in \mathcal{C}_{i,\text{small},j} \) with constant probability (since \( \mathcal{C}_{i,\text{small}} \) contains pairwise disjoint small left-partitions by Lemma 6.13). For each \((i,j)\) \( \in \{0,1,\ldots,\lfloor \log_2 k \rfloor\} \times \{0,\ldots,\lfloor \log_2 \eta_i \rfloor\} \), we run \( \text{GapLocalVC} \) with \( \nu = 2^{j+1} \), and \( \text{gap} = 2^i - 1 \); also, there exists \((i,j)\) such that \( \sum_{X \in \mathcal{C}_{i,\text{small},j}} \text{vol}^{\text{out}}(X) \geq \epsilon m 2^{j-i}/(8(\lfloor \log_2 k \rfloor + 1)(\lfloor \log_2 \eta_i \rfloor + 1)) \) by Lemma 6.15. Therefore, by Lemma 6.10, \( \text{GapLocalVC} \) outputs a vertex-cut of size less than \( k \) with constant probability.

**Proof of Lemma 6.15.** We claim that there is \( j \) such that

\[
|\mathcal{C}_{i,\text{small},j}| \geq |\mathcal{C}_{i,\text{small}}|/(\lfloor \log_2 \eta_i \rfloor + 1) \tag{27}
\]

Suppose otherwise. We have for all \( j \in \{0,\ldots,\lfloor \log_2 \eta_i \rfloor\} \), \( |\mathcal{C}_{i,\text{small},j}| < |\mathcal{C}_{i,\text{small}}|/(\lfloor \log_2 \eta_i \rfloor + 1) \).
Therefore, \( \sum_{j \in \{0, \ldots, \lfloor \log_2 \eta_i \rfloor \}} |C_{i, \text{small},j}| < |C_{i, \text{small}}| \), a contradiction. Now, for the same \( j \), we have

\[
\sum_{(L,S,R) \in C_{i, \text{small},j}} \text{vol}^{\text{out}}(L) \geq |C_{i, \text{small},j}| 2^j
\]

\[
\geq |C_{i, \text{small}}| 2^j / (\lfloor \log_2 \eta_i \rfloor + 1)
\]

\[
\geq c m 2^j / (8k (\lfloor \log_2 k \rfloor + 1) (\lfloor \log_2 \eta_i \rfloor + 1))
\]

The first inequality is because the small left-partitions in \( C_{i, \text{small},j} \) are pairwise disjoint by Lemma 6.13, and that \( \text{vol}^{\text{out}}(L) \geq 2^j \) by definition. The last inequality follows since \( 2^i \leq k \) by definition. 

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