The analytical $O(a_s^4)$ expression for the polarized Bjorken sum rule in the miniMOM scheme and the consequences for the generalized Crewther relation

A. L. Kataev$^{1,2}$, V. S. Molokoedov$^{1,2,3}$

1 Institute for Nuclear Research of the Russian Academy of Sciences (INR), 60th October Anniversary Prospect, 7a, 117312 Moscow, Russia
2 Moscow Institute of Physics and Technology (MIPT), 141700, Dolgoprudny, Russia
3 Landau Institute for Theoretical Physics of the Academy of Sciences of Russia, 142432, Moscow Region, Russia

E-mail: kataev@ms2.inr.ac.ru, viktor.molokoedov@mail.ru

Abstract. The analytical $O(a_s^4)$ perturbative QCD expression for the flavour non-singlet contribution to the Bjorken polarized sum rule in the rather applicable at present gauge–dependent miniMOM scheme is obtained. For the considered three values of the gauge parameter, namely $\xi = 0$ (Landau gauge), $\xi = -1$ (anti–Feynman gauge) and $\xi = -3$ (Stefanis–Mikhailov gauge), the scheme-dependent coefficients are considerably smaller than the gauge-independent $\overline{\text{MS}}$ results. It is found that the fundamental property of the factorization of the QCD renormalization group $\beta$-function in the generalized Crewther relation, which is valid in the gauge-invariant $\overline{\text{MS}}$ scheme up to $O(a_s^4)$-level at least, is unexpectedly valid at the same level in the miniMOM-scheme for $\xi = 0$, and for $\xi = -1$ and $\xi = -3$ in part.

1. Introduction

It is known that the hadronic tensor, which enters into the definitions of differential cross-sections of deep inelastic scattering (DIS) processes of charged leptons on nucleons, contains the terms, that are measured in the DIS of polarized leptons on polarized nucleons, namely

\begin{equation}
W_{\mu\nu}(p, q, s) = \frac{1}{4\pi} \int d^4z \ e^{iqz} \langle p, s | [J^\dagger_\mu(z), J_\nu(0)]|p, s\rangle \tag{1}
\end{equation}

\begin{align*}
= \left( \frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) F_1(x, Q^2) &+ \frac{1}{(pq)} \left( p_\mu - \frac{(pq)_\mu}{q^2} q_\mu \right) \left( p_\nu - \frac{(pq)_\nu}{q^2} q_\nu \right) F_2(x, Q^2) \\
+ i\varepsilon_{\mu\nu\lambda\rho} \frac{q^\lambda}{(pq)} \left( s^\rho (g_1(x, Q^2) + g_2(x, Q^2)) - \frac{(sq)^\rho}{pq} g_2(x, Q^2) \right) &+ \ldots ,
\end{align*}

where $J^\mu(z) = \sum_f Q_f \bar{q}_f(z) \gamma^\mu q_f(z)$ is the quark electromagnetic current, $p$ is the four-momentum of nucleon and $s$ is its spin, $q$ is the transferred momentum with $Q^2 = -q^2$, $0 \leq x = Q^2/(2pq) \leq 1$ is the Bjorken variable. In this expression the structure functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$ are the ones which are extracted from differential cross-section of
DIS process of polarized charged leptons on polarized nucleons, while \( F_1(x, Q^2) \) and \( F_2(x, Q^2) \) characterize the DIS processes with unpolarized particles.

The important characteristic of DIS process of polarized charged leptons on polarized nucleons is the Bjorken polarized sum rule, defined as

\[
\int_0^1 \left( g_{1p}^l(x, Q^2) - g_{1n}^l(x, Q^2) \right) dx = \frac{1}{6} \left| \frac{g_A}{g_V} \right| C_{Bjp}(Q^2),
\]

where index \( l \) defines the polarized charged lepton (\( e \) or \( \mu \)) in the concrete different experiments.

The structure functions \( g_{1p(n)}^l(x, Q^2) \) characterize the spin distribution of polarized quarks and gluons inside nucleons. The world average value of the ratio of axial and vector charges of neutron \( \beta \) decay is \( g_A/g_V = -1.2723 \pm 0.0023 \) [1]. The general theoretical expression for the Bjorken polarized sum rule contains massive-dependent corrections [2], [3] and non-perturbative high-twist \( O(1/Q^2) \)-corrections, discussed in [4]. However, in this work we will consider the massless PT expression for the coefficient function \( C_{Bjp}(Q^2) \) in the \( SU(N_c) \) colour group only, which is known at the \( O(a_s^4) \) level in the \( \overline{MS} \)-scheme from the results of [5], supplemented the singlet contribution, which appears first at the same level [6] with the numerically small analytical coefficient, evaluated in [7]. Therefore, in general the massless PT expression for \( C_{Bjp}(Q^2) \) can be written down as:

\[
C_{Bjp}(Q^2) = C_{Bjp}^{NS}(Q^2) + \sum_f Q_f C_{Bjp}^{SI}(Q^2),
\]

where

\[
C_{Bjp}^{NS}(Q^2) = 1 + \sum_{n=1}^\infty \tau_n^{NS} \pi_n^s(Q^2), \quad C_{Bjp}^{SI}(Q^2) = \sum_{n=4}^\infty \tau_n^{SI} \pi_n^s(Q^2),
\]

where \( \tau_s = \pi_s/\pi \) is the strong coupling constant determined in \( \overline{MS} \) scheme. Remind that the first, second and third PT contributions to \( C_{Bjp}^{NS}(Q^2) \) in the \( \overline{MS} \)-scheme were analytically calculated in the works [5], [9] and [10] respectively. Note also that for \( n_f = 3 \) the singlet contribution is identically equal to zero.

It is known that the scheme–dependent PT series for the renormalization–group (RG) invariant quantities are asymptotic ones. The transformation to the special schemes, which minimize the contributions of high-order PT corrections to physical quantities, is one of the theoretical approaches for applying them in phenomenology. In [11] it was proposed to achieve this goal by using the special momentum (MOM) subtraction scheme, which in QCD is gauge–dependent. It should be noted, that in the MOM–like schemes the coefficients of the QCD RG \( \beta \)-function are becoming gauge–dependent starting from the two-loop level [12]. The gauge-dependence of the coefficients of both RG-invariant quantities and QCD \( \beta \)-function in various MOM schemes makes the analysis of the PT expansions for the quantities, interesting from experimental point of view, more complicated. However, the interest to the studies of the PT approximations for various QCD characteristics of observable physical processes in the special gauge-dependent MOM schemes became actual again. It is related mainly to the formulation of minimal MOM (mMOM) scheme [13]. Its analog was used in studies of the Functional Renormalization Group Equations [14].

In Section II we remind the main ideas of the definition of mMOM scheme and present the explicit analytical gauge–dependent relation between the QCD coupling constants in the \( \overline{MS} \) and MOM schemes with gauge covariant linear parameter \( \xi \) determined in mMOM scheme. The new analytical results for the PT mMOM coefficients of the \( O(a_s^4) \) approximation of the Bjorken polarized sum rule are presented. Their numerical values, specified to the Landau gauge
ξ = 0, to the anti–Feynman gauge ξ = −1 and to the Stefanis–Mikhailov gauge ξ = −3, first used in QCD by Stefanis in the work of [15] (see also [15]) to untangle the special features of renormalizations of gauge–invariant definition of QCD quark correlator, formulated with the help of the Wilson lines, and independently applied later on by Mikhailov in [17] (see also [18]) to clarify the renormalization of the gluon fields by the renormalon chain contributions, are compared with the \( \overline{\text{MS}} \) scheme results.

In Section III we will clarify that the application of the anti–Feynman and Stefanis–Mikhailov gauges are essential for satisfying the fundamental property of the factorization of the gauge–dependent QCD \( \beta \)-function in the \( \mathcal{O}(a_s^3) \) approximation in mMOM scheme variant of the QCD generalized Crewther relation (GCR), discovered in [19] in the gauge–independent \( \overline{\text{MS}} \) scheme, which in the conformal symmetry limit reproduce well-known massless quark-parton result, obtained in [20]. We will also stress that the transformation to the mMOM scheme when the Landau gauge is chosen will not spoil the property of the factorization of the analytical expression for the QCD \( \beta \)-function in the \( SU(N_c) \) GCR at the fourth order of PT, studied in the \( \overline{\text{MS}} \) scheme in [1].

2. The Bjorken polarized sum rule in mMOM scheme
The mMOM scheme was introduced in [13] and was initially formulated as the easiest way to satisfy the common MOM schemes property of the non-renormalization of the gluon–ghost–anti-ghost vertex in the Landau gauge \( \xi = 0 \) thanks to the equality between the renormalization constant of this vertex in mMOM and \( \overline{\text{MS}} \) schemes [13]:

\[
Z_{cg}^{\text{mMOM}} = Z_{cg}^{\overline{\text{MS}}}. \tag{5}
\]

Taking into account this relation (5) and the MOM–like renormalization conditions for gluon and ghost propagators, one can obtain the following relations between strong coupling constant \( a_s^{\text{mMOM}} \equiv a_s \) in mMOM scheme and \( a_s^{\overline{\text{MS}}} \equiv \overline{a}_s \) and between gauge parameters \( \xi^{\text{mMOM}} \equiv \xi \) and \( \xi^{\overline{\text{MS}}} \equiv \overline{\xi} \) [13, 21, 22]:

\[
a_s(\mu^2) = \frac{\overline{a}_s(\mu^2)}{(1 + \Delta(\mu^2))(1 + \Omega(\mu^2))^2}, \quad \xi(\mu^2) = \overline{\xi}(\mu^2)(1 + \Delta(\mu^2)), \tag{6}
\]

where gluon and ghost self-energies functions \( \Delta(q^2) \) and \( \Omega(q^2) \) are evaluated in the \( \overline{\text{MS}} \) scheme and defined by the higher order corrections to the corresponding propagators. In the arbitrary \( SU(N_c) \) colour group their explicit form was analytically calculated in [23] at \( \mathcal{O}(a_s^3) \) level and in [22] in the fourth order PT approximation. Note that these \( \overline{\text{MS}} \) results for self-energies \( \Delta(q^2) \) and \( \Omega(q^2) \) depend on the gauge parameter starting from one- and two–loop level correspondingly. The expressions (6) allow us to get the following explicit analytical three-loop approximation of the function \( \overline{a}_s(a_s, \xi) \):

\[
\overline{a}_s = a_s \left( 1 + \delta_1 a_s + \delta_2 a_s^2 + \delta_3 a_s^3 + \mathcal{O}(a_s^4) \right), \tag{7}
\]

\[
\delta_1 = \left( -\frac{169}{144} - \frac{1}{8} \xi - \frac{1}{16} \xi^2 \right) C_A + \frac{5}{9} T_F n_f, \tag{8}
\]

\[
\delta_2 = \left[ -\frac{18941}{20736} + \frac{39}{128} \zeta_3 + \left( \frac{889}{2304} - \frac{11}{64} \zeta_3 \right) \xi + \left( \frac{203}{2304} + \frac{3}{128} \zeta_3 \right) \xi^2 - \frac{3}{256} \xi^3 \right] C_A \tag{9}
\]

\[
+ \left[ -\frac{107}{648} + \frac{\zeta_3}{2} - \frac{5}{36} \xi - \frac{5}{72} \xi^2 \right] C_A T_F n_f + \left[ \frac{55}{48} - \zeta_3 \right] C_F T_F n_f + \frac{25}{81} T_F^2 n_f^2, \tag{9}
\]
\[
\delta_3 = \left[ -\frac{1935757}{2985984} + \frac{7495}{18432} \zeta_3 + \frac{7805}{12288} \zeta_5 + \left( \frac{4877}{36864} - \frac{611}{1536} \zeta_3 + \frac{295}{1024} \zeta_5 \right) \xi \right] C_A^3 + \left[ \frac{233}{1536} \zeta_3 + \frac{5}{3072} \zeta_5 \right] C_F^2 T_{Fnf}^2 \\
+ \left[ \frac{5}{2304} C_F^2 T_{Fnf}^2 \right] + \frac{125}{729} T_{Fnf}^3,
\]

where \( C_F \) and \( C_A \) are the Casimir operators, \( T_F \) is the Dynkin index, \( n_f \) is the number of active flavours.

In order to obtain values of the coefficients \( c_i^{NS} \) of the flavour non-singlet Bjorken function in the mMOM scheme we use the known four-loop MS results for \( C_{Bjp}^{NS} \) from \( [5] \) (the corresponding coefficients are denoted as \( \bar{c}_i^{NS} \)), the relation \( [7] \) and the renorm–invariant property of the physical quantity \( C_{Bjp}^{NS} \). The relations between the coefficients in mMOM and MS schemes read:

\[
c_1^{NS} = \bar{c}_1^{NS}, \quad c_2^{NS} = \bar{c}_2^{NS} + \delta_1 \bar{c}_1^{NS}, \quad c_3^{NS} = \bar{c}_3^{NS} + 2\delta_1 \bar{c}_2^{NS} + \delta_2 \bar{c}_1^{NS}, \quad c_4^{NS} = \bar{c}_4^{NS} + 3\delta_1 \bar{c}_3^{NS} + (2\delta_2 + \delta_3) \bar{c}_2^{NS} + \delta_3 \bar{c}_1^{NS},
\]

Using these relations we find the explicit form of coefficients of the Bjorken function in the mMOM scheme at the \( \mathcal{O}(a_s^3) \) level:

\[
c_1^{NS} = -\frac{3}{4} C_F, \quad c_2^{NS} = \frac{21}{32} C_F^2 + \left( -\frac{107}{192} + \frac{3}{32} \xi + \frac{3}{64} \xi^2 \right) C_F C_A + \frac{1}{12} C_F T_{Fnf}, \quad c_3^{NS} = -\frac{3}{128} C_F + \left[ \frac{1415}{2304} - \frac{11}{12} \zeta_3 - \frac{21}{256} \xi - \frac{21}{256} \xi^2 \right] C_F^2 C_A + \left[ -\frac{13}{36} + \frac{\zeta_3}{3} \right] C_F^2 T_{Fnf}^2 + \left[ \frac{13}{18} + \frac{3}{8} \xi - \frac{1}{48} \xi^2 \right] C_F C_A T_{Fnf} - \frac{5}{24} C_F T_{Fnf}^2 + \left[ -\frac{20585}{9216} \right] C_F C_A, \quad c_4^{NS} = \frac{117}{512} \zeta_3 + \frac{55}{24} \zeta_5 + \left\{ \frac{215}{3072} \frac{33}{256} \right\} \xi + \left\{ \frac{349}{3072} - \frac{9}{512} \zeta_3 \right\} \xi^2 + \frac{9}{1024} \xi^3 \right\} C_F C_A.
\]
\[
\frac{d_{\text{NS}}^{abcd; \text{pbcde}}}{N_c} \left[ -\frac{3}{16} + \frac{\zeta_3}{4} + \frac{5}{4} \zeta_5 \right] + n_f \frac{d_{\text{NS}}^{abcd; \text{pbcde}}}{N_c} \left[ \frac{13}{16} + \frac{3}{5} \zeta_5 \right] + n_f \frac{d_{\text{NS}}^{abcd; \text{pbcde}}}{N_c} \left[ \frac{3}{4} + \frac{5}{4} \zeta_5 \right] + n_f \frac{d_{\text{NS}}^{abcd; \text{pbcde}}}{N_c} \left[ \frac{13}{16} + \frac{3}{5} \zeta_5 \right]
\]

Here in expression (16) \( d_{\text{NS}}^{abcd; \text{pbcde}} = \text{Tr}(t^a t^b t^c t^d)/2 \) and \( d_{\text{NS}}^{abcd; \text{pbcde}} = \text{Tr}(C^a t^b C^c t^d)/2 \) are the total symmetric tensors with generators \( t^a \) of the fundamental representation and the adjoint representation \( C^a \) of the Lie algebra of the \( SU(N_c) \) group.

To study the energy behavior of the obtained by us four-loop PT expression for the Bjorken polarized sum rule in the mMOM scheme it is necessary to take into account the gauge-dependent expression of the RG \( \beta \)-function in this scheme at the four-loop level, analytically evaluated in [21], and to define the corresponding QCD scale parameter \( \Lambda_{\text{mMOM}} \). Using the analytical expressions for the next-to-leading order coefficients of \( C_{BJS}^{\text{NS}} \) in the \( \overline{\text{MS}} \) and mMOM schemes and the concept of the corresponding effective scale parameter [23, 24], we obtain the following gauge- and flavour-dependent relation:

\[
\Lambda_{\text{mMOM}}(\xi, n_f) = \Lambda_{\overline{\text{MS}}} \cdot \exp \left( \frac{(169 + 18 \xi + 9 \xi^2) C_A - 80 T_{F n_f}}{264 C_A - 96 T_{F n_f}} \right)
\]

It will be shown in the next Section that the values of the gauge parameters \( \xi = -3, -1 \) are highlighted by the presence of factorization of the RG \( \beta \) function in mMOM scheme variant of the \( O(a_s^3) \) approximation of the fundamental Generalized Crewther relation for the product of non-singlet Adler and Bjorken functions, while in the Landau gauge \( \xi = 0 \) this property is true at \( O(a_s^3) \) order. In view of this it is interesting to consider the asymptotic behavior of the PT series for the obtained by us \( O(a_s^3) \) approximations of the flavour non-singlet Bjorken polarized sum rule in mMOM in these three theoretically prominent gauges with the well-known \( \overline{\text{MS}} \) scheme results. This will be done in Table 1.
The flavour NS Bjorken function \( C_{Bjp}^{NS} \) in \( \overline{\text{MS}} \) and mMOM schemes. Therefore, the convergence of the PT series is much better in mMOM scheme. Decreasing both in the dependent coefficients, related to the mMOM scheme, are considerably smaller than the ones, obtained in \( \overline{\text{MS}} \) scheme. Note also that unlike \( \overline{\text{MS}} \) expressions for Bjorken function, the series calculated in mMOM scheme as a rule do not obey sign constant behavior character. It is also interesting to note, that on the contrary to the absolute values of the coefficients the \( O(\pi_s^3) \) expressions are decreasing both in the \( \overline{\text{MS}} \) and mMOM-schemes with increasing number of quarks flavors from \( n_f=3 \) to \( n_f=4, 5 \). The similar feature is manifesting itself in the process of applications of mMOM-scheme to the analysis of \( O(a_s^4) \)-approximations to other physical quantities (see e.g \([26, 27, 28]\)). These facts became noticeable starting from the three–loop level.

It may be of interest to compare these numerical results with the ones obtained in the process of recent analysis of the PT approximations for the Bjorken polarized sum rule \([29]\) where the mMOM was also used.

| \( n_f \) | \( \xi \) | The flavour NS Bjorken function \( C_{Bjp}^{NS} \) in \( \overline{\text{MS}} \) and mMOM schemes |
|-------|-------|-----------------------------------------------------------------------------------|
| 3     | —     | \( 1 - \pi_s - 3.583 \pi_s^2 - 20.2153 \pi_s^3 - 175.74950 \pi_s^4 \) |
|       | 0     | \( 1 - a_s - 0.896 a_s^2 + 1.4262 a_s^3 - 22.96225 a_s^4 \) |
|       | -1    | \( 1 - a_s - 1.083 a_s^2 + 0.2312 a_s^3 - 31.54404 a_s^4 \) |
|       | -3    | \( 1 - a_s - 0.333 a_s^2 - 1.0317 a_s^3 - 44.09174 a_s^4 \) |
| 4     | —     | \( 1 - \pi_s - 3.250 \pi_s^2 - 13.8503 \pi_s^3 - 102.40202 \pi_s^4 \) |
|       | 0     | \( 1 - a_s - 0.840 a_s^2 + 3.0375 a_s^3 - 12.34185 a_s^4 \) |
|       | -1    | \( 1 - a_s - 1.028 a_s^2 + 1.8633 a_s^3 - 18.49192 a_s^4 \) |
|       | -3    | \( 1 - a_s - 0.278 a_s^2 + 0.5170 a_s^3 - 31.54819 a_s^4 \) |
| 5     | —     | \( 1 - \pi_s - 2.917 \pi_s^2 - 7.8402 \pi_s^3 - 41.95977 \pi_s^4 \) |
|       | 0     | \( 1 - a_s - 0.785 a_s^2 + 4.5099 a_s^3 - 3.61660 a_s^4 \) |
|       | -1    | \( 1 - a_s - 0.972 a_s^2 + 3.3566 a_s^3 - 7.57175 a_s^4 \) |
|       | -3    | \( 1 - a_s - 0.222 a_s^2 + 1.9269 a_s^3 - 21.14145 a_s^4 \) |

Table 1. The behaviour of the PT series for \( C_{Bjp}^{NS} \) at the \( O(a_s^4) \) level for \( SU(3) \) QCD with \( n_f = 3, 4, 5 \) active flavours in the \( \overline{\text{MS}} \) and mMOM schemes for three values of the gauge parameter \( \xi = 0, -1, -3 \).

These results demonstrate that for all three used gauges the numerical values of the gauge-dependent coefficients, related to the mMOM scheme, are considerably smaller than the ones, obtained in \( \overline{\text{MS}} \) scheme. Therefore, the convergence of the PT series is much better in mMOM scheme. Note also that unlike \( \overline{\text{MS}} \) \( O(\pi_s^3) \) expressions for Bjorken function, the series calculated in mMOM scheme as a rule do not obey sign constant behavior character. It is also interesting to note, that on the contrary to the absolute values of the coefficients the \( O(\pi_s^3) \) expressions are decreasing both in the \( \overline{\text{MS}} \) and mMOM-schemes with increasing number of quarks flavors from \( n_f=3 \) to \( n_f=4, 5 \). The similar feature is manifesting itself in the process of applications of mMOM-scheme to the analysis of \( O(a_s^4) \)-approximations to other physical quantities (see e.g \([26, 27, 28]\)). These facts became noticeable starting from the three–loop level.

It may be of interest to compare these numerical results with the ones obtained in the process of recent analysis of the PT approximations for the Bjorken polarized sum rule \([29]\) where the mMOM was also used.

3. The generalized Crewther relation in mMOM scheme

One of the most well-known manifestations of the consequences of the conformal symmetry in massless quark-parton model is the existence of the Crewther relation \([20]\):

\[
D^{NS}C_{Bjp}^{NS} \bigg|_{\xi=0} = 1 .
\]
The unity in the l.h.s of this equation corresponds to the normalized massless quark–parton result, obtained from the application of the OPE approach to the one-loop axial-vector-vector (AVV) triangle diagram, which defines $\pi^0 \to \gamma\gamma$ decay amplitude. In equation (15) the physical quantity $D^{NS}$ denotes the Born approximation of the Adler function that characterizes the process of one-photon electron–positron annihilation into hadrons and $C^{NS}_{Bj}$ is the normalized massless Born approximation of the theoretical expression for the Bjorken polarized sum rule, defined in (2) and (3). However, in realistic QCD the conformal symmetry is broken. This leads to the existence of the discovered in [19] generalized Crewther relation, which in $\overline{MS}$ scheme can be written down as:

$$D^{NS}(\pi_s)C^{NS}_{Bj}(\pi_s) = 1 + \Delta_{csb}(\pi_s).$$  \hspace{1cm} (19)

The term $\Delta_{csb}(\pi_s)$ in (19) is appearing starting from the second order of PT. In the related to dimensional regularization gauge–invariant $\overline{MS}$ scheme, which is commonly used in the multiloop QCD calculations, its expression was first written down in [19] at the $O(\pi_s^3)$ level in the following factorized form

$$\Delta_{csb} = \left(\frac{\beta(\pi_s)}{a_s}\right)\sum_{i\geq 1} K_i \overline{a}_s^i$$  \hspace{1cm} (20)

and confirmed at the $O(\pi_s^4)$ order in [2]. Here coefficient functions $K_i$ depend on monomials of $SU(N_c)$ group structures $C_F$, $C_A$ and $T_{F-n_f}$. The RG $\beta(\pi_s)$-function is defined as

$$\beta(\pi_s) = \mu^2 \frac{d\beta(\pi_s)}{d\mu^2} = -\sum_{i\geq 0} \beta_i \pi_s^{i+2},$$  \hspace{1cm} (21)

In this work we will need to know its three-loop approximation only, which was computed in the $\overline{MS}$ scheme in [30] and confirmed later on in [31]. The corresponding expression of the $\beta$ function in the $mMOM$ scheme can be found in analytical form in [21], [22] and depends on $\xi$ beginning from two–loop level.

It is important now to understand whether the fundamental property of factorization of the conformal anomaly term ($\beta(\pi_s)/\pi_s$) in (20) is fulfilled in the gauge–invariant $\overline{MS}$–like schemes only. To study this problem we will extend the consideration of the representation (20) to the gauge–dependent mMOM scheme.

Regardless of the used renormalization schemes the physical quantities $D^{NS}$ and $C^{NS}_{Bj}$ in (19) obey the property of the renormalization invariance. Using this property and omitting the details of considerations we obtain the following transition relations for the $K_i$ terms in (20) from $\overline{MS}$ scheme to any other renormalization scheme AS:

$$K_{1}^{AS} = K_{1}^{\overline{MS}}, \quad K_{2}^{AS} = K_{2}^{\overline{MS}} + \left(\frac{\beta_{1}^{\overline{MS}} - \beta_{1}^{AS}}{\beta_{0}} + 2\delta_{1}^{AS}\right)K_{1}^{\overline{MS}},$$  \hspace{1cm} (22)

$$K_{3}^{AS} = K_{3}^{\overline{MS}} + \left(\frac{\beta_{1}^{\overline{MS}} - \beta_{1}^{AS}}{\beta_{0}} + 3\delta_{1}^{AS}\right)K_{2}^{\overline{MS}} +$$

$$+ \left(\frac{\beta_{2}^{\overline{MS}} - \beta_{2}^{AS}}{\beta_{0}} + 3\beta_{1}^{\overline{MS}} - 2\delta_{1}^{AS} - \frac{\beta_{1}^{\overline{MS}} - \beta_{1}^{AS}}{\beta_{0}} \cdot \frac{\beta_{1}^{AS} - \beta_{1}^{AS}}{\beta_{0}} + 2\delta_{2}^{AS} + (\delta_{1}^{AS})^2\right)K_{1}^{\overline{MS}}$$  \hspace{1cm} (23)

We emphasize that if all the fractions included in these expressions are explicitly divided by QCD $\beta_0$–factors, then the property of factorization of the conformal symmetry breaking factor ($\beta(a_s^{AS})/a_s^{AS}$) in AS scheme will be valid at the $O(a_s^{AS^3})$ at least. The terms $\delta_{1}^{AS}$ in (22), (23) are the AS–analogies of the $\delta_1$ terms in (7). If we fix mMOM scheme instead of AS scheme, we obtain that at $O(a_s^2)$ level the factorization of $\beta(a_s)/a_s$ function for $K_2$ term is possible for three distinguished values of the gauge parameter $\xi$ only, namely for $\xi = 0$ (Landau gauge),
for $\xi = -1$ (anti–Feynman gauge) and for $\xi = -3$ \cite{15}, \cite{17} (Stefanis–Mikhailov gauge). At the $O(a_s^3)$ approximation the $\beta(a_s)/a_s$ factorization property is valid for the Landau gauge only and the partial factorization holds for anti–Feynman and Stefanis–Mikhailov gauges (when the factorization condition is imposed, then for one of the six possible color monomials the concrete coefficient is not determined). Thus we conclude that total factorization of the conformal anomaly $\beta(a_s)/a_s$ in the GCR is also possible for gauge–dependent renormalization schemes. Therefore we rule out the gauge invariance as a cause of the factorization property. Theoretical reasons of these our findings are yet unclear to us. The detailed description of the outlined in this talk our studies are in progress.

4. Acknowledgments
The authors are grateful to S.V. Mikhailov for constructive remarks. One of us (A.K) would like to thank organizers of XVIth International Workshop on High Energy Spin Physics (DSPIN-17) and A.V. Efremov personally for hospitality in Dubna. His work on studies of the theoretical features of the generalized Crewther relation in QCD is supported by the Russian Science Foundation Grant No. 14-22-00161. The work of V.M related to the calculation and analysis of the non-singlet Bjorken function in mMOM scheme is supported by the Russian Science Foundation grant No. 16-12-10151.

5. References
[1] C. Patrignani et al. [Particle Data Group], Chin. Phys. C 40 (2016) no.10, 100001.
[2] O. V. Teryaev and O. L. Veretin, hep-ph/9602362.
[3] J. Blumlein and W. L. van Neerven, Phys. Lett. B 450 (1999) 417.
[4] A. L. Kataev, Mod. Phys. Lett. A 20 (2005) 2007.
[5] P. A. Baikov, K. G. Chetyrkin and J. H. Kuhn, Phys. Rev. Lett. 104 (2010) 132004.
[6] S. A. Larin, Phys. Lett. B 723 (2013) 348.
[7] P. A. Baikov, K. G. Chetyrkin and J. H. Kuhn, Nucl. Part. Phys. Proc. 261-262 (2015) 3.
[8] J. Kodaira, S. Matsuda, T. Muta, K. Sasaki and T. Uematsu, Phys. Rev. D 20 (1979) 627.
[9] S. G. Gorishny and S. A. Larin, Phys. Lett. B 172 (1986) 109.
[10] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 259 (1991) 345.
[11] W. Celmaster and R. J. Gonsalves, Phys. Rev. Lett. 42 (1979) 1435.
[12] O. V. Tarasov and D. V. Shirkov, Sov. J. Nucl. Phys. 51 (1990) 877 [Yad. Fiz. 51 (1990) 1380].
[13] L. von Smekal, K. Maltman and A. Sternbeck, Phys. Lett. B 681 (2009) 336.
[14] J. M. Pawlowski, D. F. Litim, S. Nedelko and L. von Smekal, Phys. Rev. Lett. 93 (2004) 152002.
[15] N. G. Stefanis, Nuovo Cim. A 83 (1984) 205.
[16] N. G. Stefanis, Acta Phys. Polon. Supp. 6 (2013) 71.
[17] S. V. Mikhailov, Phys. Lett. B 431 (1998) 387.
[18] S. V. Mikhailov, Phys. Rev. D 62 (2000) 034002.
[19] D. J. Broadhurst and A. L. Kataev, Phys. Lett. B 315 (1993) 179.
[20] R. J. Crewther, Phys. Rev. Lett. 28 (1972) 1421.
[21] J. A. Gracey, J. Phys. A 46 (2013) 225403.
[22] B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, JHEP 1706 (2017) 040.
[23] K. G. Chetyrkin and A. Retey, [hep-ph/0007088].
[24] N. V. Krasnikov, Nucl. Phys. B 192 (1981) 497.
[25] G. Grunberg, Phys. Lett. 95B (1980) 70 Erratum: [Phys. Lett. 110B (1982) 501].
[26] J. A. Gracey, Phys. Rev. D 90 (2014) no.9, 094026.
[27] A. L. Kataev and V. S. Molokoedov, Phys. Rev. D 92 (2015) no.5, 054008.
[28] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, JHEP 1708 (2017) 113.
[29] C. Ayala, G. Cvetic, A. V. Kotikov and B. G. Shaikhatdenov, arXiv:1708.06284 [hep-ph].
[30] O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, Phys. Lett. 93B (1980) 429.
[31] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 303 (1993) 334.