Motion of Spin 1/2 Massive Particle in a Curved Spacetime

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Abstract
Quasi-classical picture of motion of spin 1/2 massive particle in a curved spacetime is built on base of simple Lagrangian model. The one is constructed due to analogy with Lagrangian of massive vector particle [6]. Equations of motion and spin propagation coincide with Papapetrou equations describing dynamic of classical spinning particle in a curved spacetime [2, 3].

Keywords Spin 1/2 massive particle; Dirac equation; Papapetrou equation.

1 Introduction
In terms of classical Lagrange formalism motion of structureless test particle in curved spacetime is described by simplest form of Lagrangian:

\[ \mathcal{L} = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle, \]

where \( \dot{x} = \dot{x}^i \partial_i \), \( \dot{x} = \frac{dx^i}{ds} \), \( x^i \) are coordinates of the particle and \( s \) is length along worldline of the particle. Euler-Lagrange equations for this Lagrangian lead to the geodesic equation.

In case of non scalar particles additional terms containing spin variables can be included into the Lagrangian. These internal variables change equations of motion of the particle due to spin-gravitational interaction. Frenkel was first who pointed to fact that spin changes trajectory of motion of particles in external field [1]. Motion of extended spinning particle in curved spacetime was studied by Papapetrou [2] and Dixon [1]. Similar problem was studied by Turakulov [3] by means of classic Hamiltonian formalism in approximation of spinning rigid body in tangent space. In the mentioned works it was shown that equation of motion differs from geodesic equation due to term which is contraction the curvature with spin and velocity. A number of attempts to
describe motion of quantum particles with spin on base of Lagrangian models were made for last eight decades [1]. However, a satisfactory description was not obtained [6]. Nevertheless, studies both Maxwell and Dirac equations point to the fact that equations of motion might include contraction spin with the curvature [4, 5]. In our recent work [8] a derivation of Papapetrou equations for photon on base of field variational principles was completed. In turn, an approach to derive the equations of motion for spin 1 massive and massless particles by means of classical Lagrange formalism are shown in our paper [6].

Particularly, it was shown in the work [6] that motion of massive vector particle of spin 1 in curved spacetime can be described by the Lagrangian:

$$\mathcal{L} = m < \dot{x}, \dot{x} > - \dot{A}^2 + m^2 \dot{A}^2,$$

where $\dot{A}$ is a vector field which is attached to worldline of the particle and orthogonal to $\dot{x}$. Components of the field expressed in local orthonormal frame are generalized coordinates which describe spin of the particle.

The goal of present work is to develop a Lagrangian approach for spin 1/2 massive particle. We consider the particle as quasi-classical. This means that motion of the particle is described not only by its coordinates $\{x^i\}$ in spacetime but also by internal spin variables specifying spin degrees of freedom of the particle in terms of quantum mechanics. It should be noted that spin variables are elements of suitable spinor spaces. In turn, determination of the spaces demands presence of orthonormal frame in considered domain of spacetime. Since particle is massive we introduce length parameter $s$ along worldline of the particle which plays role of time parameter in Lagrangian formalism. Thus, generalized velocities conjugated with coordinates $\{x^i\}$ $\dot{x}^i = dx^i/ds$, define timelike vector $\tilde{n}_0 = \dot{x}^i \partial_i = \dot{x}$ of unit length along the worldline:

$$\langle \tilde{n}_0, \tilde{n}_0 \rangle = 1.$$ 

There are vectors $\{\tilde{n}_\alpha\}$, $\alpha = 1, 2, 3$; orthogonal to $\tilde{n}_0$ such that $\langle \tilde{n}_\alpha, \tilde{n}_\beta \rangle = \eta_{\alpha\beta} = -\delta_{\alpha\beta}$. The vectors together with $\tilde{n}_0$ constitute comoving frame along the worldline. Since, by construction, $\dot{x}$ has no $\tilde{n}_\alpha$ components we call spacelike coframe $\{\tilde{n}_\alpha\}$ as rest frame of the particle. Besides, rest frame $\{\tilde{n}_\alpha\}$ is defined with accuracy up to arbitrary spatial rotation belonged to group $SO(3)$. Due to the fact that generalized coordinates and velocities of different nature must be independent spinor variables should be referred to rest frame $\{\tilde{n}_\alpha\}$ of space-like vectors. In other words, spinor variables are elements of linear spaces of representation of group $SO(3)$.

The spaces are constructed as follows. Pauli matrices $\{\hat{\sigma}^\alpha\}$ referred to rest frame are introduced. The matrices generate local Clifford algebras referred to the frame. Besides, local Clifford algebra introduced this way specifies two local spinor spaces attached to the worldline. These spaces are two spaces of spinor representations of the group $SO(3)$ which are isomorphic to each other under Hermitian conjugation. In our approach elements of the spaces $\psi^1, \psi$ play role of generalized coordinates which describe spin degrees of freedom of the particle.
The desired Lagrangian should depend on generalized coordinates $\{x^i\}$, $\{\psi^\dagger, \psi\}$ and their derivatives over $s$. At the same time the Lagrangian should contain covariant derivatives of spinor variables $\psi^\dagger$ and $\psi$. Moreover, the Lagrangian must contain term with $\langle \dot{x}, \dot{x} \rangle$ which yields left hand side (LHS) of geodesic equation. Euler-Lagrange equations for spinor variables are expected to yield reduced form of Dirac equation for wave propagating along the worldline of the particle. In the limiting case of zero gravitation the equation coincides with Dirac equation formulated in comoving frame of plane spinor wave of positive energy. All these requirements determine the form of the Lagrangian describing motion of massive spin $1/2$ particle in curved spacetime. Thus, Euler-Lagrange equations are reduced to equations describing motion of the spin along the particle worldline and the worldline shape. The equations obtained this way become identical to Papapetrou equations for classic spinning particle [3].

2 Lagrange formalism for massive particle of spin $1/2$

In order to describe spin variables of the Lagrangian we supplement timelike unit vector $\vec{n}_0 = \dot{x}$ tangent to worldline of the particle by spatial orthonormal frame $\{\vec{n}_\alpha\}$ whose vectors are orthogonal to $\vec{n}_0$. Frame $\{\vec{n}_\alpha\}$ is a rest frame of the particle. Spacelike vectors $\vec{n}_\alpha$ together with $\dot{x}$ constitute orthonormal comoving frame along the worldline. We denote covector comoving frame as $\{\nu^\alpha\}$, so $\{\nu^\alpha\}$ is covector rest frame dual to vector frame $\{\vec{n}_\alpha\}$ in tangent subspace orthogonal to $\dot{x}$. Then we introduce Pauli matrices $\hat{\sigma}^1, \hat{\sigma}^2, \hat{\sigma}^3$ referred to the covector rest frame. The matrices are constant in chosen frame and obey anticommutation rules as follows:

$$\{\hat{\sigma}^\alpha, \hat{\sigma}^\beta\} = -2\eta^{\alpha\beta},$$

where

$$\eta^{ab} = \langle \nu^a, \nu^b \rangle = \text{diag}(1, -1, -1, -1) = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right).$$

Algebraic span of Pauli matrices yields local sample of Clifford algebra in each point of the worldline. Union of the local Clifford algebras constitute fibre bundle of Clifford algebra along the worldline.

Invertible elements $R$ of Clifford algebra such that

$$R^{-1} = R^\dagger,$$

where $R^\dagger$ stands for Hermitian conjugated matrix, constitute $Spin(3)$ group. There is an endomorphism $R : SO(3) \rightarrow Spin(3)$ defined by formula:

$$R_L \hat{\sigma}^a R_L^{-1} = L_{b\cdot}^a \hat{\sigma}^b, \quad (L_{b\cdot}^a) \in SO(3),$$

so that each element of $L \in SO(3)$ is covered twice [9] by elements $\pm R_L \in Spin(3)$. 

3
Elements of local Clifford algebra are operators on two local spinor spaces referred to considered local frame on the worldline. The spaces are local linear spaces of representation of group $\text{Spin}(3)$ and $\text{SO}(3)$. Elements of the local spaces $\psi \in S$ and $\psi^\dagger \in S^\dagger$ are $2 \times 1$ and $1 \times 2$ complex matrices accordingly. This way element $L$ of group of spatial rotations $\text{SO}(3)$ acts on spaces of representation of the group as follows:

$$' \psi = R_L \psi, \quad ' \psi^\dagger = \psi^\dagger R_L^{-1}, \quad \psi \in S, \psi^\dagger \in S^\dagger.$$  

(3)

Union of the local spinor spaces constitute spinor fibre bundle on the worldline.

Image of an infinitesimal rotation $L = 1 - \varepsilon \in \text{SO}(3)$ is:

$$R_{1 - \varepsilon} = \hat{1} + \delta Q = \hat{1} + 1/4 \varepsilon_{\alpha\beta} \hat{\sigma}^\alpha \hat{\sigma}^\beta.$$  

(4)

The infinitesimal transformation rotates elements of the rest frame:

$$\delta \nu^\alpha = -\varepsilon_{\beta \gamma} \nu^\beta.$$  

(5)

Accordingly (3) the rotation initiates a transformation of spinors:

$$\delta \psi = 1/8 \varepsilon_{\beta \gamma} [\hat{\sigma}^\beta, \hat{\sigma}^\gamma] \psi, \quad \delta \psi^\dagger = -1/8 \varepsilon_{\beta \gamma} \psi^\dagger [\hat{\sigma}^\beta, \hat{\sigma}^\gamma],$$  

(6)

under which due to (2) Pauli matrices rotates as follows:

$$' \hat{\sigma}^\alpha = R \hat{\sigma}^\alpha R^{-1}, \quad \delta \hat{\sigma}^\alpha = [\delta Q, \hat{\sigma}^\alpha];$$

$$\delta \hat{\sigma}^\alpha = -\varepsilon_{\beta \gamma} \hat{\sigma}^\beta = 1/4 \varepsilon_{\beta \gamma} \left[\hat{\sigma}^\beta \hat{\sigma}^\gamma, \hat{\sigma}^\alpha\right].$$

It is seen that the rotation coincides with rotation of components of contravariant vector with accuracy up to opposite sign. Thus, if we take into account both of the transformations Pauli matrices become invariant as it is accepted in field theory \[10\].

State of the particle is described by its coordinates $\{x^i\}$ in space time, spinor variables $\{\psi, \psi^\dagger\}$ which are elements of spinor fibre bundles on the worldline and their derivatives $\dot{x}^i = \frac{dx^i}{ds}$ and $\frac{d\psi}{ds}, \frac{d\psi^\dagger}{ds}$ over length $s$ along the worldline. Rest frame rotates under motion of the particle:

$$\dot{\nu}^\alpha = -\omega^\alpha_\beta (\dot{x}) \nu^\beta,$$

where angular velocities are given by values of Cartan’ rotation 1-forms $\omega^\alpha_\beta = \gamma_{\alpha\beta}^\gamma \nu^\gamma$ on vector $\dot{x}$. So do spinor variables referred to the frame. Their transformations are given by equations (3) where $\varepsilon_{\beta \gamma} = \gamma_{\beta \gamma}^\alpha$. Account of the transformations are taken by covariant derivatives of spinor variables along the worldline:

$$\psi = \frac{d\psi}{ds} + \frac{1}{4} \gamma_{b\delta \varepsilon} \dot{x}^b \hat{\sigma}^\delta \hat{\sigma}^\varepsilon \psi,$$

$$\dot{\psi} = \frac{d\psi^\dagger}{ds} - \frac{1}{4} \gamma_{b\delta \varepsilon} \dot{x}^b \psi^\dagger \hat{\sigma}^\delta \hat{\sigma}^\varepsilon.$$  

(7)
Besides, total covariant derivatives (with taking account of spinor transformation and rotation of vector indexes) of Pauli matrices are zero.

Lagrangian of the particle is covariant under internal transformations of the rest frames. Hence derivatives \( m/2 \psi^\dagger \psi < \dot{x}, \dot{x} > \) which yields geodesic equation and an addend providing validity of the reduced Dirac equation. There are also terms including derivatives of spinor fields and term proportional to \( m\psi^\dagger \psi \). Due to fact that Dirac equation is of first order partial differential equation the Lagrangian is to be linear over the derivatives of spinor variables. Analysis shows that to obey such the requirement we should accept the form of the Lagrangian:

\[
\mathcal{L} = \frac{m}{2} \psi^\dagger \psi < \dot{x}, \dot{x} > + \frac{i}{\hbar} \left( \psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi \right) + \frac{m}{2} \psi^\dagger \psi. \tag{8}
\]

It must be kept in mind that the Lagrangian is function of generalized coordinates \( x^i, \dot{x}^i \) and their velocities \( \frac{dx^i}{ds}, \frac{d\dot{x}^i}{ds} \). At the same time covariant form of the Lagrangian includes derivatives represented in orthonormal frame. Due to this we recall formulas of transformations between the frames:

\[
\partial / \partial x^i = n^a_i \partial / \partial x^a, \quad n^a_i \delta^b_a = \delta^i_b, \tag{9}
\]

\[
\dot{x}^a = n^a_i \dot{x}^i, \quad \dot{x}^i = n^i_a \dot{x}^a, \quad n^a_i n^b_a = \delta^i_b.
\]

### 3 Euler-Lagrange equations for spinor variables

Due to \( \Psi = \partial \mathcal{L} / \partial (d\psi^\dagger / ds) = +i\hbar \psi / 2, \quad \Psi^\dagger = \partial \mathcal{L} / \partial (d\psi / ds) = -i\hbar \psi^\dagger / 2 \)

Euler Lagrange equations for the considered generalized coordinates read:

\[
\frac{d}{ds} \psi = \partial \mathcal{L} / \partial \psi^\dagger, \quad \frac{d}{ds} \psi^\dagger = \partial \mathcal{L} / \partial \psi. \tag{10}
\]

Straightforward calculation of the right hand side (RHS) of the above equations gives:

\[
\partial \mathcal{L} / \partial \psi = m\psi - \frac{i\hbar}{2} \dot{\psi} - \frac{i\hbar}{2} \frac{1}{4} \gamma^{\delta \sigma} \dot{x}^b \bar{\sigma}^b \sigma^\delta \sigma^\sigma \psi,
\]

\[
\partial \mathcal{L} / \partial \psi^\dagger = m\psi^\dagger + \frac{i\hbar}{2} \dot{\psi}^\dagger - \frac{i\hbar}{2} \frac{1}{4} \gamma^{\delta \sigma} \dot{x}^b \bar{\sigma}^b \psi^\dagger \sigma^\delta \sigma^\sigma.
\]

Now it is seen that the RHS of the equations \( \text{[10]} \) completes ordinary derivatives of spinor variables in the LHS up to covariant derivatives. This way Euler-Lagrange equations for \( \psi^\dagger, \psi \) generalized coordinates become:

\[
i\hbar \dot{\psi} = m\psi, \quad i\hbar \dot{\psi}^\dagger = -m\psi^\dagger. \tag{11}
\]

The equations coincide with reduced form of Dirac equations for free motion of particle with positive energy in flat spacetime \( \text{[11]} \).
4 Generalized momentum conjugated with $x^i$ and conservation of spin

Due to the definition $p_i = \partial L / \partial \dot{x}^i$. However it is convenient to operate with generalized momenta expressed in orthonormal frame: $p_a = n^i \dot{x}^i$. Differentiating (8) over $\dot{x}^a$ we obtain:

$$ p_a = \frac{\partial L}{\partial \dot{x}^a} = m \psi \psi^\dagger \eta_{ab} \dot{x}^b - \frac{i \hbar}{2} \frac{1}{2} \gamma_{abc} \psi \psi^\dagger \sigma^b \sigma^c \psi. $$

We define spin of the particle as:

$$ S^{\delta \epsilon} = -\frac{i \hbar}{4} \psi \psi^\dagger \sigma^{[\delta} \sigma^{\epsilon]} \psi, \quad (12) $$

where $[,]$ stands for commutator of the Pauli matrices. It is seen that RHS of the above equation can be represented as $\hbar/2 \epsilon_{\delta \epsilon \zeta} \psi \psi^\dagger \sigma^\zeta$, where $\epsilon_{\delta \epsilon \zeta}$ is Levi-Civita symbol for the 3-space. In terms of quantum mechanics the expression can be interpreted as averaged value of operator of spin $\hbar \sigma^\zeta / 2$ \[11\] in state described by wave function $\psi$ in the tangent rest space. Moreover, it can be shown that definition (12) accords to formula for 0-component of current of spin derived from Noether theorem in field theory \[12\]. The spin is element of space which is tensor product of two copies of tangent rest space. Thus, it has no 0-component and we can decide that condition of orthogonality of the spin to velocity is satisfied: $\dot{x}^b S_b^{\gamma} \equiv 0$. After that we can represent expression for the generalized momentum $p_a$ as follows:

$$ p_a = \pi_a + \frac{1}{2} \gamma_{abc} S^{\delta \epsilon} \quad, \quad (13) $$

where $\pi_a = m \psi \psi^\dagger \eta_{ab} \dot{x}^b$ is part of the momentum including generalized velocities over $x^i$ coordinates.

According to the equation (11) straightforward calculation of covariant derivative of spin (12) gives:

$$ \frac{D S^{\delta \epsilon}}{ds} = \frac{d S^{\delta \epsilon}}{ds} + \dot{x}^a (\gamma_{a \gamma} S^{\gamma \epsilon} + \gamma_{a \gamma} S^{\delta \gamma}) \equiv 0, \quad (14) $$

where we took account of the fact that total covariant derivatives of Pauli matrices are zero.

5 Euler-Lagrange equations for $x^i$ variables

Euler-Lagrange equations for $x^i$ variables read:

$$ dp_i / ds = \partial L / \partial \dot{x}^i. $$

It is convenient to rewrite the above equation in orthonormal frame due to (9):

$$ n_i \frac{d}{ds} (n^b p_b) = n^i \partial L / \partial x^i. \quad (15) $$
After differentiation and expression velocities \( \dot{x}^k \) via its components in orthonormal frame the LHS of (15) becomes:

\[
\dot{n}_a n_i k n_c \dot{x}^c p_b + \frac{d p_a}{d s}.
\]  

(16)

where \( (),_k \) means differentiation over \( x^k \) variable. Calculation the RHS of (15) gives:

\[
n_i^a \left[ m \phi^i \psi \eta_{bc} n_k^b \dot{x}^d \dot{x}^c + \frac{i h}{2} \left( \gamma_{b \delta \epsilon, i} \dot{x}^b + \gamma_{b \delta \epsilon} n_c^b n_i^k \dot{x}^c \right) \psi^i \psi^\delta \psi \right] =
\]

\[
= m \phi^i \psi \eta_{bc} n_k^b \dot{x}^d \dot{x}^c + \frac{1}{2} \left( \gamma_{b \delta \epsilon, i} \dot{x}^b + \gamma_{b \delta \epsilon} n_c^b n_i^k \dot{x}^c \right) S^\delta \epsilon =
\]

\[
= n_k^b n_c^k \dot{x}^c \left( \pi^b + \frac{1}{2} \gamma_{b \delta \epsilon} S^\delta \epsilon \right) + \frac{1}{2} \gamma_{b \delta \epsilon, i} \dot{x}^b S^\delta \epsilon
\]  

(17)

Equating (16) to (17) we obtain:

\[
\frac{d}{ds} p_a + n_i^b \left( n_i^a n_c^k - n_k^a n_c^i \right) \dot{x}^c = \frac{1}{2} n_a^i \gamma_{b \delta \epsilon, i} \dot{x}^b S^\delta \epsilon.
\]

(18)

But due to Cartan' first structure equation it is easy to see that:

\[
n_i^b \left( n_i^a n_c^k - n^k n_c^i \right) = \gamma_{ac, b} - \gamma_{ca, b}.
\]

This gives us equation as follows:

\[
\frac{D}{ds} p_a + \gamma_{ac, b} \dot{x}^c p_b = \frac{1}{2} \gamma_{b \delta \epsilon, a} \dot{x}^b S^\delta \epsilon.
\]  

(19)

Since we expect to obtain an equation whose LHS coincides with geodesic equation generalized momentum \( p_a \) in the (13) should be presented in explicit form given by (13). Leaving only \( D \pi_a / ds \) at LHS of (18) we after some simple derivations can rewrite the equation as:

\[
\frac{D}{ds} \pi_a = - \frac{1}{2} \frac{d}{ds} \left[ \gamma_{b \delta \epsilon} S^\delta \epsilon \right] + \frac{1}{2} \left[ \gamma_{ba, \epsilon} \dot{x}^c - \gamma_{ab, \epsilon} \dot{x}^c + \gamma_{b \delta \epsilon, a} \right] \dot{x}^b S^\delta \epsilon =
\]

\[
= \frac{1}{2} \gamma_{a b, \epsilon} \frac{d}{ds} S^\delta \epsilon + \frac{1}{2} \left[ \gamma_{b \delta \epsilon, a} d \dot{x}^b \right] (\dot{n}_a, \dot{n}_b) \dot{x}^b S^\delta \epsilon
\]  

(19)

Now equation (14) allows us to exclude \( d S^\delta \epsilon / ds \) from the (19). After this has been done the RHS of (19) turns:

\[
\frac{1}{2} \dot{x}^b \left[ \omega_{b \epsilon} \wedge \omega_{c \epsilon} (\dot{n}_a, \dot{n}_b) + d \omega_{b \epsilon} (\dot{n}_a, \dot{n}_b) \right] S^\delta \epsilon = \frac{1}{2} \dot{x}^b \Omega_{b \epsilon} (\dot{n}_a, \dot{n}_b) S^\delta \epsilon,
\]
where
\[
\Omega^d_c = d\omega^d_c + \omega^d_c \wedge \omega^e_c = 1/2 R^d_{c\,ab} \nu^a \wedge \nu^b
\]
is 2-form of curvature. This way Euler-Lagrange equations for generalized co-
ordinates \(x^i\) become identically with equations:
\[
D\pi_a/ds = 1/2 R_{\delta\epsilon ab} \dot{x}^b S^{\delta\epsilon}.
\]
Substituting \(\pi_a = m\psi^\dagger \eta_{ab} \dot{x}^b\) into the above equation and reminding that the
coefficient at \(\eta_{ab} \dot{x}^b\) is constant we can rewrite the equation as follows:
\[
m\psi^\dagger \psi D\dot{x}^a \frac{ds}{\dot{s}} = 1/2 R_{\delta\epsilon \,a\,b} \dot{x}^b S^{\delta\epsilon}.
\]
It is seen that (14) and (20) constitute set of equations of motion of massive
particle with spin \(s = 1/2\) which coincides with system of Papapetrou equations
for motion of classical spinning particle in curved space-time as they presented
in work [3].

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References

[1] A. Frydryszak, Lagrangian Models of Particles with Spin: the First Seventy
Years. arXive:hep-th/9601020 v.1, 6 Jan (1996)
[2] Papapetrou A, Proc. R. Soc. A209, p248 (1951)
[3] Turakulov Z Ya, Classical Mechanics of Spinning Particle in a Curved
Space. arXive:dg-ga/9703008 v.1, 14 March (1997)
[4] P.D. Mannheim, arXive:gr-qc/9810087, p21
[5] A.S. Eddington, The Mathematical theory of Relativity, Cambridge Univ.
Press (1965)
[6] Turakulov Z Ya, Safonova M, Motion of a Vector Particle in a Curved
Space-Time. I. Lagrangian Approach. Mod Phys Lett A18 (2003) 579
[7] Turakulov Z Ya, Safonova M, Motion of a Vector Particle in a Curved
Space-Time. II. First-Order Correction to a Geodesic in a Schwarzschild
background. Mod Phys Lett A 20 (2005) 2785

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[8] Turakulov Z. Ya, Muminov A. T, *Electromagnetic field with constraints and Papapetrou equation*. Zeitschrift fur Naturforschung 61a, 146 (2006)

[9] M. Berg, C. DeWitt-Morette, Sh. Gwo and E. Kramer, *The Pin Groups in Physics: C,P and T*, arXive:math-ph/0012006 (2000).

[10] Seminaire Arthur Besse 1978/79 *Geometrie Riemannienne en Dimension 4* CEDIC/FERNAND NATHAN Paris (1981)

[11] Messiah A. *Quantum Mechanics*. vol.1,2 New York: J. Wiley & Sons (1958)

[12] Bogoliubov N N and Shirkov D V, *Introduction to the Theory of Quantized Fields*. New York: Wiley-Interscience (1959)