ON THE SYMMETRIC HOMOLOGY OF ALGEBRAS

DISSertation

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* * * * *

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The theory of symmetric homology, in which the symmetric groups $\Sigma_k^{op}$, for $k \geq 0$, play the role that the cyclic groups do in cyclic homology, begins with the definition of the category $\Delta S$, containing the simplicial category $\Delta$ as subcategory. Symmetric homology of a unital algebra, $A$, over a commutative ground ring, $k$, is defined using derived functors and the symmetric bar construction of Fiedorowicz. If $A = k[G]$ is a group ring, then $HS_*(k[G])$ is related to stable homotopy theory. Two chain complexes that compute $HS_*(A)$ are constructed, both making use of a symmetric monoidal category $\Delta S_+$ containing $\Delta S$, which also permits homology operations to be defined on $HS_*(A)$. Two spectral sequences are found that aid in computing symmetric homology. In the second spectral sequence, the complex $Sym^p_*$ is constructed. This complex turns out to be isomorphic to the suspension of the cycle-free chessboard complex, $\Omega_{p+1}$, of Vrećica and Živaljević. Recent results on the connectivity of $\Omega_n$ imply finite-dimensionality of the symmetric homology groups of finite-dimensional algebras. Finally, an explicit partial resolution is presented, permitting the calculation of $HS_0(A)$ and $HS_1(A)$ for a finite-dimensional algebra $A$. 
To my Mother, Crystal, whose steadfast encouragement spurred me to begin this journey.

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PUBLICATIONS

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LIST OF SYMBOLS

#S = The number of elements in the set S.
N\mathcal{C} = The nerve of the category \mathcal{C} (as a simplicial set).
B\mathcal{C} = The geometric realization of the category \mathcal{C}. (i.e., \vert N\mathcal{C}\vert.)
E_sG = The standard resolution of the group G (as a simplicial set).
EG = Contractible space on which G acts.
Mor\mathcal{C} = The class of all morphisms of the category \mathcal{C}.
Mor_{\mathcal{C}}(X,Y) = The set of all morphisms in \mathcal{C} from X to Y.
Obj\mathcal{C} = The class of all objects of the category \mathcal{C}.
S_n = Symmetric group on the letters \{1, 2, \ldots, n\}.
\Sigma_n = Symmetric group on the letters \{0, 1, \ldots, n - 1\}.
\Delta = The simplicial category.
Sets = The category of sets and set maps.
SimpSets = The category of simplicial sets and simplicial set maps.
Mon = The category of monoids and monoid maps.
k-Mod = The category of left k-modules, for a ring k.
k-Alg = The category of k-algebras and algebra homomorphisms.
k-SimpMod = The category of simplicial left k-modules and k-linear chain maps.
k-Complexes = The category of complexes of k-modules and chain maps.
Cat = The category of small categories and functors.
I S_\lambda = The identity representation of a subgroup S_\lambda of S_n.
A S_\lambda = The alternating (sign) representation of a subgroup S_\lambda of S_n.
R \uparrow G = Representation of G induced from a representation of a subgroup.
A^\otimes n = The n-fold tensor product of the algebra A over its ground ring.
CHAPTER 1
PRELIMINARIES AND DEFINITIONS

1.1 The Category $\Delta S$

Denote by $[n]$ the ordered set $\{0, 1, \ldots, n\}$. The category $\Delta S$ has as objects, the sets $[n]$ for $n \geq 0$, and morphisms are pairs $(\phi, g)$, where $\phi : [n] \rightarrow [m]$ is a non-decreasing map of sets (i.e., a morphism in $\Delta$), and $g \in \Sigma^{op}_{n+1}$. The element $g$ represents an automorphism of $[n]$, and as a set map, takes $i \in [n]$ to $g^{-1}(i)$. Indeed, a morphism $(\phi, g) : [n] \rightarrow [m]$ of $\Delta S$ may be represented as a diagram:

\[
\begin{array}{c}
[n] \\
g \\
\downarrow \\
[n] \xrightarrow{\phi} [m]
\end{array}
\]

Equivalently, a morphism in $\Delta S$ is a morphism in $\Delta$ together with a total ordering of the domain $[n]$. Composition of morphisms is achieved as in [10], namely:

\[(\phi, g) \circ (\psi, h) := (\phi \cdot g^*(\psi), \psi^*(g) \cdot h),\]

where $g^*(\psi)$ is the morphism of $\Delta$ defined by sending the first $\#\psi^{-1}(g(0))$ points of $[n]$ to 0, the next $\#\psi^{-1}(g(1))$ points to 1, etc. Note, if $\#\psi^{-1}(g(i)) = 0$, then the point $i \in [m]$ is not hit. $\psi^*(g)$ is determined as follows: For each $i \in [m]$, $(g^*(\psi))^{-1}(g^{-1}(i))$ has the same number of elements as $\psi^{-1}(i)$ If these sets are non-empty, take the order-preserving bijection $(g^*(\psi))^{-1}(g^{-1}(i)) \rightarrow \psi^{-1}(i)$.

It is often helpful to represent morphisms of $\Delta S$ as diagrams of points and lines, indicating images of set maps. Using these diagrams, it is easy to see how $g^*(\psi)$ and $\psi^*(g)$ are related to $(\psi, g)$ (see Figure 1.1).
Figure 1.1: Morphisms of $\Delta S$: $g \cdot \psi = (g^*(\psi), \psi^*(g)) = (\psi^g, g^\psi)$

Remark 1. Observe that the properties of $g^*(\phi)$ and $\phi^*(g)$ stated in Prop. 1.6 of [10] are formally rather similar to the properties of exponents. Indeed, if we denote:

$$g^\phi := \phi^*(g), \quad \phi^g := g^*(\phi),$$

then Prop. 1.6 becomes:

**Proposition 2.** For $g, h \in G_n$ and $\phi, \psi \in \text{Mor}\Delta$, 

- $(1.h)'$ \quad $g^{\phi \psi} = (g^\phi)^\psi$
- $(1.v)'$ \quad $\phi^{gh} = (\phi^g)^h$
- $(2.h)'$ \quad $(\phi \psi)^g = \phi^g \psi^{(g^\phi)}$
- $(2.v)'$ \quad $(gh)^\phi = g^\phi h^{(\phi^g)}$
- $(3.h)'$ \quad $g^{\text{id}_n} = g, \quad 1^\phi = 1$
- $(3.v)'$ \quad $\phi^1 = \phi, \quad \text{id}_n^g = \text{id}_n$
In what follows, the exponent notation may be used interchangeably with the standard notation.

The above construction for $\Delta S$ shows that the family of groups $\{\Sigma_n\}_{n \geq 0}$ forms a crossed simplicial group in the sense of Def. 1.1 of [10]. The inclusion $\Delta \hookrightarrow \Delta S$ is given by $\phi \mapsto (\phi, 1)$, where 1 is the identity element of $\Sigma_{n+1}^{\text{op}}$, and $[n]$ is the domain of $\phi$. For each $n$, let $\tau_n$ be the $(n+1)$-cycle $(0, n, n-1, \ldots, 1) \in \Sigma_{n+1}^{\text{op}}$. Thus, the subgroup generated by $\tau_n$ is isomorphic to $(\mathbb{Z}/(n+1)\mathbb{Z})^{\text{op}} = \mathbb{Z}/(n+1)\mathbb{Z}$. We may define the category $\Delta C$ as the subcategory of $\Delta S$ consisting of all objects $[n]$ for $n \geq 0$, together with those morphisms $(\phi, g)$ of $\Delta S$ for which $g = \tau_n^i$ for some $i$ (cf. [15]). In this way, we get a natural chain of inclusions,

$$
\Delta \hookrightarrow \Delta C \hookrightarrow \Delta S
$$

An equivalent characterization of $\Delta S$ comes from Pirashvili (cf. [24]), as the category $\mathcal{F}(\text{as})$ of ‘non-commutative’ sets. The objects are sets $n := \{1, 2, \ldots, n\}$ for $n \geq 0$. By convention, $\emptyset$ is the empty set. A morphism in $\text{Mor}_{\mathcal{F}(\text{as})}(n, m)$ consists of a map (of sets) $f : n \to m$ together with a total ordering, $\Pi_j$, on $f^{-1}(j)$ for all $j \in m$. In such a case, denote by $\Pi$ the partial order generated by all $\Pi_j$. If $(f, \Pi) : n \to m$ and $(g, \Psi) : m \to p$, their composition will be $(gf, \Phi)$, where $\Phi_j$ is the total ordering on $(gf)^{-1}(j)$ (for all $j \in p$) induced by $\Pi$ and $\Psi$. Explicitly, for each pair $i_1, i_2 \in (gf)^{-1}(j)$, we have

$i_1 < i_2$ under $\Phi$ if and only if $[f(i_1) < f(i_2)$ under $\Psi]$ or $[f(i_1) = f(i_2)$ and $i_1 < i_2$ under $\Pi]$. For example, let $f : \emptyset \to 5$ be given by:

$$
f : \begin{cases}
1, 5, 8 & \mapsto 1 \\
2, 7 & \mapsto 2 \\
3, 9 & \mapsto 3 \\
4 & \mapsto 4 \\
6 & \mapsto 5
\end{cases}
$$

Let the preordering $\Pi$ on pre-image sets be defined by: $8 < 1 < 5$, $2 < 7$, and $9 < 3$. 

3
Let $g : 5 \rightarrow 3$ be given by:

$$
g : \begin{cases} 
1,2 \mapsto 2 \\
3,4 \mapsto 1 \\
5 \mapsto 3 
\end{cases}
$$

Let the preordering $\Psi$ on pre-image sets be defined by: $1 < 2$ and $3 < 4$.

Then, the composition $(g, \Psi)(f, \Pi) = (gf, \Phi)$ will consist of the map $gf$:

$$
gf : \begin{cases} 
1,2,5,7,8 \mapsto 2 \\
3,4,9 \mapsto 1 \\
6 \mapsto 3 
\end{cases}
$$

and the corresponding preordering $\Phi$, defined by: $9 < 3 < 4$ and $8 < 1 < 5 < 2 < 7$.

There is an obvious inclusion of categories, $\Delta S \hookrightarrow \mathcal{F}(as)$, taking $[n]$ to $n+1$, but there is no object of $\Delta S$ that maps to 0. It will be useful to define $\Delta S_+ \supset \Delta S$ which is isomorphic to $\mathcal{F}(as)$:

**Definition 3.** $\Delta S_+$ is the category consisting of all objects and morphisms of $\Delta S$, with the additional object $[-1]$, representing the empty set, and a unique morphism $\iota_n : [-1] \rightarrow [n]$ for each $n \geq -1$.

**Remark 4.** Pirashvili’s construction is a special case of a more general construction due to May and Thomason [22]. This construction associates to any topological operad $\{\mathcal{C}(n)\}_{n \geq 0}$ a topological category $\widehat{\mathcal{C}}$ together with a functor $\widehat{\mathcal{C}} \rightarrow \mathcal{F}$, where $\mathcal{F}$ is the category of finite sets, such that the inverse image of any function $f : m \rightarrow n$ is the space

$$
\prod_{i=1}^{n} \mathcal{C}(\#f^{-1}(i)).
$$

Composition in $\widehat{\mathcal{C}}$ is defined using the composition of the operad. May and Thomason refer to $\widehat{\mathcal{C}}$ as the *category of operators* associated to $\mathcal{C}$. They were interested in the case of an $E_\infty$ operad, but their construction evidently works for any operad. The category of operators associated to the discrete $A_\infty$ operad $Ass$, which parametrizes monoid structures, is precisely Pirashvili’s construction of $\mathcal{F}(as)$, i.e. $\Delta S_+$. (See Chapter 5 for more on operads.)
One very useful advantage in enlarging our category to $\Delta S$ to $\Delta S_+$ is the added structure inherent in $\Delta S_+$.

**Proposition 5.** $\Delta S_+$ is a permutative category.

**Proof.** Recall from Def. 4.1 of [20] that a permutative category is a category $\mathcal{C}$ with the following additional structure:

An associative bifunctor $\odot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,

A *unit* object $e \in \text{Obj}_\mathcal{C}$, which acts as two-sided identity for $\odot$.

A natural transformation $\gamma : \odot \to \odot T$, where $T : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the transposition functor $(A, B) \mapsto (B, A)$, such that

$\gamma^2 = \text{id}$,

$\gamma_{A,e} = \gamma_{e,A} = \text{Id}_A$, for all objects $A$,

and the following diagram is commutative for all objects $A, B, C$:

$$
\begin{array}{c}
A \odot B \odot C \\
\downarrow \text{Id} \odot \gamma_{B,C} \\
A \odot C \odot B \\
\end{array}
\xrightarrow{\gamma_{A \odot B,C}}
\begin{array}{c}
C \odot A \odot B \\
\downarrow \gamma_{A,C} \odot \text{Id} \\
C \odot B \odot A \\
\end{array}
$$

For $\Delta S_+$, let $\odot$ be the functor defined on objects by:

$$[n] \odot [m] := [n + m + 1], \quad \text{(disjoint union of sets)},$$

and for morphisms $(\phi, g) : [n] \to [n']$, $(\psi, h) : [m] \to [m']$,

$$(\phi, g) \odot (\psi, h) = (\eta, k) : [n + m + 1] \to [n' + m' + 1],$$

where

$$
\eta : i \mapsto \begin{cases} 
\phi(i), & \text{for } i = 0, \ldots n \\
\psi(i - n - 1) + (n' + 1), & \text{for } i = n + 1, \ldots n + m.
\end{cases}
$$

and

$$
k : i \mapsto \begin{cases} 
g^{-1}(i), & \text{for } i = 0, \ldots n \\
h^{-1}(i - n - 1) + (n' + 1), & \text{for } i = n + 1, \ldots n + m.
\end{cases}
$$
In short, \((\phi, g) \circ (\psi, h)\) is just the morphism \((\phi, g)\) acting on the first \(n+1\) points of \([n+m+1]\), and \((\psi, h)\) acting on the remaining points.

The unit object will be \([-1] = \emptyset\). \(\circ\) is clearly associative, and \([-1]\) acts as two-sided identity.

Finally, define \(\gamma_{n,m} : [n] \circ [m] \to [m] \circ [n]\) to be the identity on objects, and on morphisms to be precomposition with the block transposition \(\beta_{n,m} : [n + m + 1] \to [n + m + 1]\). That is, \(\beta(i) = i + m + 1\), if \(i \leq n\), and \(\beta(i) = i - n - 1\), if \(i > n\).

\(\gamma_{m,n} \gamma_{n,m} = \text{id}\), which is true since \(\beta_{m,n} \beta_{n,m} = \text{id}\), and \(\gamma_{m,-1}\) is precomposition by \(\beta_{m,-1}\), which is clearly the identity (similarly for \(\gamma_{-1,m}\)).

For \([n], [m], [p]\), we have the following commutative diagram:

\[
\begin{array}{ccc}
[n] \circ [m + p + 1] & \overset{=}{\longrightarrow} & [n + m + 1] \circ [p] \\
\downarrow \text{id} \circ \gamma & & \downarrow \gamma \circ \text{id} \\
[n] \circ [p + m + 1] & \overset{=}{\longrightarrow} & [n + p + 1] \circ [m] \\
\end{array}
\]

This diagram commutes because the block transposition \(\beta_{n+m,p}\) can be accomplished by first transposing the blocks \([n+1, \ldots, n+m+1]\) and \([n+m+2, \ldots, n+m+p+2]\) while keeping the block \([0, \ldots, n]\) fixed, then transposing the blocks \([0, \ldots, n]\) and \([n+1, \ldots, n+p+1]\) while keeping the block \([n+p+2, \ldots, n+p+m+2]\) fixed.

For the purposes of computation, a morphism \(\alpha : [n] \to [m]\) of \(\Delta S\) may be conveniently represented as a tensor product of monomials in the formal non-commuting variables \(\{x_0, x_1, \ldots, x_n\}\). Let \(\alpha = (\phi, g)\), with \(\phi \in \text{Mor}_\Delta([n], [m])\) and \(g \in \Sigma_{n+1}^{\text{op}}\). The tensor representation of \(\alpha\) will have \(m + 1\) tensor factors. Each \(x_i\) will occur exactly once, in the order \(x_{g(0)}, x_{g(1)}, \ldots, x_{g(n)}\). The \(i^{th}\) tensor factor consists of the product of \(\#\phi^{-1}(i - 1)\) variables, with the convention that the empty product will be denoted 1. Thus, the \(i^{th}\) tensor factor records the total ordering of \(\phi^{-1}(i)\). As an example, the tensor representation of the morphism depicted in Fig. 1.2 is \(x_1 x_0 \otimes x_3 x_4 \otimes 1 \otimes x_2\).
With this notation, the composition of two morphisms $\alpha = X_0 \otimes X_1 \otimes \ldots \otimes X_m : [n] \rightarrow [m]$ and $\beta = Y_1 \otimes Y_2 \otimes \ldots Y_n : [p] \rightarrow [n]$ is given by:

$$\alpha \beta = Z_0 \otimes Z_1 \otimes \ldots \otimes Z_m,$$

where $Z_i$ is determined by replacing each variable in the monomial $X_i = x_{i_1} \ldots x_{i_s}$ in $\alpha$ by the corresponding monomials $Y_{j_k}$ in $\beta$. So, $Z_i = Y_{j_1} \ldots Y_{j_s}$. Thus, for example,

$$x_4x_0 \otimes 1 \otimes x_2x_3 \otimes x_1 \circ x_1x_6x_0 \otimes x_7x_4 \otimes 1 \otimes x_3 \otimes x_2x_5 = x_2x_5x_1x_6x_0 \otimes 1 \otimes x_3 \otimes x_7x_4.$$
1.2 The Symmetric Bar Construction

**Definition 6.** Let $A$ be an associative, unital algebra over a commutative ground ring $k$. Following [9], define a (covariant) functor $B^\text{sym}_* A : \Delta S \rightarrow k\text{-Mod}$ by:

$$B^\text{sym}_n A := B^\text{sym}_* A[n] := A^{\otimes (n+1)}$$

$$B^\text{sym}_* A(\alpha) : (a_0 \otimes a_1 \otimes \ldots \otimes a_n) \mapsto \alpha(a_0, \ldots, a_n),$$

where $\alpha : [n] \rightarrow [m]$ is represented in tensor notation, and evaluation at $(a_0, \ldots, a_n)$ simply amounts to substituting each $a_i$ for $x_i$ and multiplying the resulting monomials in $A$. If the pre-image $\alpha^{-1}(i)$ is empty, then the unit of $A$ is inserted.

**Remark 7.** Fiedorowicz [9] defines the symmetric bar construction functor for morphisms $\alpha = (\phi, g)$, where $\phi \in \text{Mor}\Delta([n], [m])$ and $g \in \Sigma_{n+1}^\text{op}$, via

$$B^\text{sym}_* A(\phi)(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \left( \prod_{a_i \in \phi^{-1}(0)} a_i \right) \otimes \ldots \otimes \left( \prod_{a_i \in \phi^{-1}(n)} a_i \right)$$

$$B^\text{sym}_* A(g)(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_{g^{-1}(0)} \otimes a_{g^{-1}(1)} \otimes \ldots \otimes a_{g^{-1}(n)}$$

However, in order that this becomes consistent with earlier notation, we should require $B^\text{sym}_* A(g)$ to permute the tensor factors in the inverse sense:

$$B^\text{sym}_* A(g)(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_{g(0)} \otimes a_{g(1)} \otimes \ldots \otimes a_{g(n)}.$$

**Proposition 8.** The symmetric bar construction $B^\text{sym}_* A$ is natural in $A$.

*Proof.* If $f : A \rightarrow A'$ is a morphism of $k$-algebras (sending $1_A \mapsto 1_{A'}$), then there is a family of induced functors $B^\text{sym}_n f : B^\text{sym}_n A \rightarrow B^\text{sym}_n A'$ defined by

$$(B^\text{sym}_n f)(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = f(a_0) \otimes f(a_1) \otimes \ldots \otimes f(a_n)$$

It is easily verified that the square below commutes for each $\Delta S$ morphism $\phi : [n] \rightarrow [m]$.

$$\begin{array}{ccc}
[n] & \xrightarrow{\phi} & [m] \\
\downarrow & & \downarrow \\
A^{\otimes (n+1)} & \xrightarrow{f^{\otimes (n+1)}} & B^{\otimes (n+1)} \\
\downarrow & & \downarrow \\
B^\text{sym}_* A(\phi) & \rightarrow & B^\text{sym}_* B(\phi) \\
\downarrow & & \downarrow \\
A^{\otimes (m+1)} & \xrightarrow{f^{\otimes (m+1)}} & B^{\otimes (m+1)}
\end{array}$$
Note that $B_{\text{sym}}^* A$ can be regarded as a simplicial $k$-module (i.e., a functor $\Delta^{\text{op}} \to k\text{-Mod}$) via the chain of functors:

$$\Delta^{\text{op}} \hookrightarrow \Delta C^{\text{op}} \cong \Delta C \hookrightarrow \Delta S.$$ (1.1)

Here, the isomorphism $D : \Delta C^{\text{op}} \to \Delta C$ is the standard duality (see [15]), which is defined on generators by:

$$\begin{align*}
D(d_i) &= \sigma_i, \quad (0 \leq i \leq n - 1) \\
D(d_n) &= \sigma_0 \tau^{-1} \\
D(s_i) &= \delta_{i+1}, \quad (0 \leq i \leq n - 1) \\
D(t) &= \tau^{-1}
\end{align*}$$

Remark 9. Fiedorowicz showed in [9] that $B_{\text{sym}}^* A \circ D = B_{\text{cyc}}^* A$, the cyclic bar construction (cf. [15]). By duality of $\Delta C$, it is equivalent to use the functor $B_{\text{sym}}^* A$, restricted to morphisms of $\Delta C$ in order to do computations of $HC_* (A)$.

### 1.3 Definition of Symmetric Homology

**Definition 10.** For a category $\mathcal{C}$, a covariant functor $F : \mathcal{C} \to k\text{-Mod}$ will be called a $\mathcal{C}$-module. Similarly, a contravariant functor $G : \mathcal{C} \to k\text{-Mod}$ will be called a $\mathcal{C}^{\text{op}}$-module (since $G^{\text{op}} : \mathcal{C}^{\text{op}} \to k\text{-Mod}$ is covariant).

**Definition 11.** For a category $\mathcal{C}$, if $N$ is a $\mathcal{C}$-module and $M$ is a $\mathcal{C}^{\text{op}}$-module, define the tensor product (over $\mathcal{C}$) thus:

$$M \otimes_{\mathcal{C}} N := \bigoplus_{X \in \text{Obj} \mathcal{C}} M(X) \otimes_k N(X) / \approx,$$

where the equivalence $\approx$ is generated by the following: For every morphism $f \in \text{Mor}_{\mathcal{C}}(X, Y)$, and every $x \in N(X)$ and $y \in M(Y)$, we have $y \otimes f_*(x) \approx f^*(y) \otimes x$.

Note, MacLane defines the tensor product as a *coend*:

$$M \otimes_{\mathcal{C}} N := \int_X (MX) \otimes (NX),$$
where we consider \((M-) \otimes (N-)\) as a bifunctor \(\mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{L}\), for a given cocomplete category \(\mathcal{L}\) equipped with a functor \(\otimes : \mathcal{L} \times \mathcal{L} \to \mathcal{L}\) (see [16]). When \(\mathcal{L} = k\text{-Mod}\), this construction yields the same as that above.

Alternatively, consider \(k[\text{Mor}\mathcal{C}]\), the free \(k\)-module with basis \(\text{Mor}\mathcal{C}\). We may define a ring structure on \(k[\text{Mor}\mathcal{C}]\) by defining products of basis elements thus:

\[
  f \cdot g := \begin{cases} 
    f \circ g, & \text{if } f \text{ is composable with } g \\
    0, & \text{otherwise}
  \end{cases}
\]

Note, \(k[\text{Mor}\mathcal{C}]\) will in general not have a unit, but only local units. Indeed, for any finitely generated \(k\)-submodule, \(M\), with basis \(\{v_1, \ldots, v_t\}\), each \(v_i\) is the sum of only finitely many terms \(c_{ij}f_j\), with \(f_j \in \text{Mor}\mathcal{C}\). Let \(\{f_1, f_2, \ldots, f_n\}\) be the set of those morphisms that occur in any of the \(v_i\)'s. Then there is an element in \(k[\text{Mor}\mathcal{C}]\) that acts as a two-sided unit for any element of \(M\), namely: \(\sum \text{id}_X\), where the sum extends over all those \(X \in \text{Obj}\mathcal{C}\) that appear as domains or codomains of the \(f_i\)'s.

Now, the category of \(\mathcal{C}\)-modules (with natural transformations as morphisms) is equivalent to the category of left \(k[\text{Mor}\mathcal{C}]\)-modules. The correspondence is as follows: For a \(\mathcal{C}\)-module \(M\), let \(\overline{M}\) be the \(k\)-module \(\bigoplus_{X \in \text{Obj}\mathcal{C}} MX\). For a morphism \(f : X \to Y\) in \(\text{Mor}\mathcal{C}\) and a homogeneous \(x \in \overline{M}\) (i.e., \(x \in MW\) for some \(W \in \text{Obj}\mathcal{C}\)), put:

\[
  f.x := \begin{cases} 
    f_*(x), & \text{if } x \in MX \\
    0, & \text{otherwise}
  \end{cases}
\]

 Extend the action of \(f\) to arbitrary \(v \in \overline{M}\) by linearity. This formula provides a module structure, since if \(x \in MX\) and \(X \xrightarrow{g} Y \xrightarrow{f} Z\) is a chain of morphisms in \(\mathcal{C}\), we have

\[
  (f \cdot g).x = (fg).x = (fg)_*(x) = f_*(g_*(x)) = f.(g_*(x)) = f.(g.x).
\]

Similarly, the category of \(\mathcal{C}^{\text{op}}\)-modules is equivalent to the category of right \(k[\text{Mor}\mathcal{C}]\)-modules, with action \(y \cdot f = f^*(y)\) for any \(y \in Y\) and \(f \in \text{Mor}_\mathcal{C}(X,Y)\).
Under this equivalence, the tensor product $M \otimes_{\mathcal{C}} N$ construction is simply the standard tensor product $\overline{M} \otimes_{k[\text{Mor}\mathcal{C}]} \overline{N}$, and thus we can define the modules

$$\text{Tor}^d_n(M, N) := \text{Tor}^{k[\text{Mor}\mathcal{C}]}_n(M, N).$$

Note, it is also possible to define $\text{Tor}^d_n(M, N)$ directly as the derived functors of the categorical tensor product construction (see [10]).

The trivial $\mathcal{C}$-module, denoted by $\underline{k}$, is the functor $\mathcal{C} \to k\text{-Mod}$ which takes every object to $k$ and every morphism to id$_k$. Under the above equivalence, this becomes the trivial left $k[\text{Mor}\mathcal{C}]$-module $\underline{k} = \bigoplus_{X \in \text{Obj}\mathcal{C}} k$. We also denote by $\underline{k}$ the trivial $\mathcal{C}^{\text{op}}$-module.

**Definition 12.** The **symmetric homology** of an associative, unital $k$-algebra, $A$, is denoted $HS_n(A)$, and is defined as:

$$HS_n(A) := \text{Tor}^\Delta_S(\underline{k}, B^\text{sym}_n A)$$

**Remark 13.** Note, the existing literature based on the work of Connes, Loday and Quillen consistently defines the categorical tensor product in the reverse sense: $N \otimes_{\mathcal{C}} M$ is the direct sum of copies of $NX \otimes_k MX$ modded out by the equivalence $x \otimes f^*(y) \approx f_*(x) \otimes y$ for all $\mathcal{C}$-morphisms $f : X \to Y$. In this context, $N$ is covariant, while $M$ is contravariant. I chose to follow the convention of Pirashvili and Richter [25] in writing tensor products as $M \otimes_{\mathcal{C}} N$ so that the equivalence $\xi : \mathcal{C}\text{-Mod} \to k[\text{Mor}\mathcal{C}]\text{-Mod}$ passes to tensor products in a straightforward way: $\xi(M \otimes_{\mathcal{C}} N) = \xi(M) \otimes_{k[\text{Mor}\mathcal{C}]} \xi(N)$.

**Remark 14.** Since

$$\underline{k} \otimes_\Delta S M \cong \colim_{\Delta S} M,$$

for any $\Delta S$-module $M$, we can alternatively describe symmetric homology as derived functors of colim:

$$HS_i(A) = \colim_{\Delta S}^{(i)} B^\text{sym}_n A.$$  

(To see the relation with higher colimits, we need to tensor a projective resolution of $B^\text{sym}_n A$ with $\underline{k}$.)
1.4 The Standard Resolution

Let $\mathcal{C}$ be a category. Henceforth, we shall use interchangeably the notion of $\mathcal{C}$-module and $k[\text{Mor}\mathcal{C}]$-module, under the equivalence mentioned in section 1.3. The rank 1 free $\mathcal{C}$-module is $k[\text{Mor}\mathcal{C}]$, with the left action of composition of morphisms. Now as $k$-module, $k[\text{Mor}\mathcal{C}]$ decomposes into the direct sum,

$$k[\text{Mor}\mathcal{C}] = \bigoplus_{X \in \text{Obj}\mathcal{C}} \left( \bigoplus_{Y \in \text{Obj}\mathcal{C}} k[\text{Mor}\mathcal{C}(X,Y)] \right).$$

By abuse of notation, denote $\bigoplus_{Y \in \text{Obj}\mathcal{C}} k[\text{Mor}\mathcal{C}(X,Y)]$ by $k[\text{Mor}\mathcal{C}(X,-)]$. So there is a direct sum decomposition as left $\mathcal{C}$-module,

$$k[\text{Mor}\mathcal{C}] = \bigoplus_{X \in \text{Obj}\mathcal{C}} k[\text{Mor}\mathcal{C}(X,-)].$$

Thus, the submodules $k[\text{Mor}\mathcal{C}(X,-)]$ are projective left $\mathcal{C}$-modules.

Similarly, $k[\text{Mor}\mathcal{C}]$ is the rank 1 free right $\mathcal{C}$-module, with right action of pre-composition of morphisms, and as such, decomposes as:

$$k[\text{Mor}\mathcal{C}] = \bigoplus_{Y \in \text{Obj}\mathcal{C}} k[\text{Mor}\mathcal{C}(-,Y)].$$

Again, the notation $k[\text{Mor}\mathcal{C}(-,Y)]$ is shorthand for $\bigoplus_{X \in \text{Obj}\mathcal{C}} k[\text{Mor}\mathcal{C}(X,Y)]$. The submodules $k[\text{Mor}\mathcal{C}(-,Y)]$ are projective as right $\mathcal{C}$-modules.

It will also be important to note that $k[\text{Mor}\mathcal{C}(-,Y)] \otimes_{\mathcal{C}} N \cong N(Y)$ as $k$-module via the evaluation map $f \otimes y \mapsto f_*(y)$. Similarly, $M \otimes_{\mathcal{C}} k[\text{Mor}\mathcal{C}(X,-)] \cong M(X)$.

Following Quillen ([26], Section 1), we make the following definition:

**Definition 15.** Given a functor $F : \mathcal{B} \to \mathcal{C}$ and a fixed object $Y$ in $\mathcal{C}$, let $F/Y$ denote the category whose objects are pairs $(X, \phi)$ where $X$ is an object of $\mathcal{B}$ and $\phi : FX \to Y$ is a morphism in $\mathcal{C}$. A morphism from $(X, \phi)$ to $(X', \phi')$ is a morphism $\psi : X \to X'$ such that $\phi' \circ F\psi = \phi$. When $F$ is the identity functor on $\mathcal{C}$, this construction is called the over-category (objects over $Y$), and is denoted by $\mathcal{C}/Y$. 

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Dually, let \( Y \backslash F \) denote the category whose objects are pairs \((X, \phi)\) for \( X \) in \( \mathcal{B} \) and \( \phi : Y \to FX \) in \( \mathcal{C} \). Here, a morphism from \((X, \phi)\) to \((X', \phi')\) is a morphism \( \phi : X \to X' \) such that \( F\psi \circ \phi = \phi' \). When \( F \) is the identity functor, this is called the under-category (objects under \( Y \)), and is denoted by \( Y \backslash \mathcal{C} \).

Given a functor \( F : \mathcal{B} \to \mathcal{C} \) of small categories define a functor \((F/-) : \mathcal{C} \to \text{Cat}\) as follows: The object \( Y \) is sent to the category \( F/Y \). If \( \nu : Y \to Y' \) is a morphism in \( \mathcal{C} \), the functor \((F/\nu) : F/Y \to F/Y' \) is defined on objects by \((X, \phi) \mapsto (X, \nu\phi)\). For a morphism \( \psi : (X, \phi) \to (X', \phi') \) in \( F/Y \), \( \psi \) may also represent a morphism in \( F/Y' \), since \( \phi' F\psi = \phi = \nu \phi' F\psi = \nu \phi \) (as morphisms in \( \mathcal{C} \)).

Again, we may dualize to obtain a \textit{contravariant} functor \((- \backslash F) : \mathcal{C} \to \text{Cat} \), defined on objects by \( Y \mapsto Y \backslash F \), and if \( \nu : Y \to Y' \), then \((\nu \backslash F) \) is a functor \((Y' \backslash F) \to (Y \backslash F) \) which takes \((X, \phi) \) to \((X, \phi\nu)\).

Thus, \((F/-) \) is a \( \mathcal{C} \)-category, and \((- \backslash F) \) is a \( \mathcal{C}^{\text{op}} \)-category. In what follows, we shall assume \( F \) is the identity functor of \( \mathcal{C} \). As noted in [10], the nerve of \((\mathcal{C}/-\)) \) is a simplicial \( \mathcal{C} \)-set, and the complex \( L_* \), given by:

\[
L_n := k [N(\mathcal{C}/-)_n]
\]

is a resolution by projective \( \mathcal{C} \)-modules of the trivial \( \mathcal{C} \)-module, \( k \). Here, the boundary map is \( \partial := \sum_i (-1)^i d_i \), where the \( d_i \)'s come from the simplicial structure of the nerve of \((\mathcal{C}/-)\).

For the definition of \( HS_* (A) \), we shall be more interested in the dual construction, which yields a resolution by projective \( \mathcal{C}^{\text{op}} \)-modules of the trivial \( \mathcal{C}^{\text{op}} \)-module, \( k \). Explicitly, define the complex \( \overline{T}_* \) by:

\[
\overline{T}_n := k [N(- \backslash \mathcal{C})_n]
\]

\[
\overline{T}_n(\mathcal{C}) := k \left\{ \{ \mathcal{C} \xrightarrow{g} A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} A_n \} \right\}
\]

For completeness, we shall provide a proof of:

**Proposition 16.** \( \overline{T}_* \) is a resolution of \( k \) by projective \( \mathcal{C}^{\text{op}} \)-modules.
Proof. Fix $C \in \text{Obj} \mathcal{C}$. Let $\epsilon : T_0(C) \to k$ be the map defined on generators by

$$
\epsilon(C \to A_0) := 1_k.
$$

We shall show the complex

$$
k \xleftarrow{\epsilon} T_0(C) \xleftarrow{\partial} T_1(C) \xleftarrow{\partial} \ldots
$$

is chain homotopic to the 0 complex. Explicitly, define

$$
h_{-1} : 1 \mapsto (C \xrightarrow{id} C)
$$

$$
h_n : (C \to A_0 \to \ldots \to A_n) \mapsto (C \xrightarrow{id} C \to A_0 \to \ldots A_n), \quad \text{for } i \geq 0
$$

We have $\epsilon h_{-1}(1) = \epsilon(C \to C) = 1$, so $\epsilon h_{-1} = \text{id}$. Next, in degree 0,

$$
(\partial h_0 + h_{-1} \epsilon)(C \to A_0) = \partial(C \to C \to A_0) + h_{-1}(1)
$$

$$
= (C \to A_0) - (C \to C) + (C \to C)
$$

$$
= (C \to A_0).
$$

Finally, let $n \geq 1$.

$$
(\partial h_n + h_{n-1} \partial)(C \to A_0 \to \ldots \to A_n)
$$

$$
= \partial(C \to C \to A_0 \to \ldots \to A_n) + h_{n-1} \sum_{i=0}^{n} (-1)^i(C \to A_0 \to \ldots \hat{A}_i \ldots \to A_n),
$$

where $\hat{A}_i$ means to omit the object $A_i$ by composing the map with target $A_i$ with the map with source $A_i$.

$$
= (C \to A_0 \to \ldots A_n) + \sum_{i=0}^{n} (-1)^{i+1}(C \to C \to A_0 \to \ldots \hat{A}_i \ldots \to A_n) +
$$

$$
\sum_{i=0}^{n} (-1)^i(C \to C \to A_0 \to \ldots \hat{A}_i \ldots \to A_n)
$$

$$
= (C \to A_0 \to A_0 \to A_0 \to \ldots).
$$

Hence, $\partial h_n + h_{n-1} \partial = \text{id}$, and so $h$ determines a chain homotopy $0 \simeq \text{id}$, proving that the complex is contractible.
Next, we show that the $C^{\text{op}}$-module $L_n$ is projective. Indeed,

$$L_n = \bigoplus_{C_n} k [\text{Mor}_{C}(-, A_0)],$$

where the direct sum is indexed over the set $C_n$ of all chains $A_0 \to A_1 \to \ldots \to A_n$. As we have seen above, $k [\text{Mor}_{C}(-, A_0)]$ is projective as $C^{\text{op}}$-module, therefore $L_n$ is projective. \hfill \Box

Thus, we may compute $HS_*(A)$ as the homology groups of the following complex:

$$0 \leftarrow L_0 \otimes_{\Delta S} B_*^{\text{sym}} A \leftarrow L_1 \otimes_{\Delta S} B_*^{\text{sym}} A \leftarrow L_2 \otimes_{\Delta S} B_*^{\text{sym}} A \leftarrow \ldots$$

(1.2)

**Corollary 17.** For an associative, unital $k$-algebra $A$, $HS_*(A) = H_*(k[N(- \setminus \Delta S)] \otimes_{\Delta S} B_*^{\text{sym}} A; k)$

**Remark 18.** By remark 9, it is clear that the related complex $k[N(- \setminus \Delta C)] \otimes_{\Delta C} B_*^{\text{sym}} A$ computes $HC_*(A)$.

**Remark 19.** Observe that every element of $L_n \otimes_{\Delta S} B_*^{\text{sym}} A$ is equivalent to one in which the first morphism of the $L_n$ factor is an identity:

$$[p] \xrightarrow{\alpha} [q_0] \xrightarrow{\beta_1} [q_1] \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_n} [q_n] \otimes (y_0 \otimes \ldots \otimes y_p)$$

$$= \alpha^*([q_0] \xrightarrow{\text{id}_{[q_0]}} [q_0] \xrightarrow{\beta_1} [q_1] \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_n} [q_n] \otimes (y_0 \otimes \ldots \otimes y_p))$$

$$\approx [q_0] \xrightarrow{\text{id}_{[q_0]}} [q_0] \xrightarrow{\beta_1} [q_1] \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_n} [q_n] \otimes \alpha_*((y_0, \ldots, y_p))$$

Thus, we may consider $L_n \otimes_{\Delta S} B_*^{\text{sym}} A$ to be the $k$-module generated by the elements

$$\left\{ [q_0] \xrightarrow{\beta_1} [q_1] \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_n} [q_n] \otimes (y_0 \otimes \ldots \otimes y_{q_0}) \right\},$$

(1.3)

where the tensor product is now over $k$.

The face maps $d_0, d_1, \ldots, d_n$ are defined on generators by:

$$d_0 \left( [q_0] \xrightarrow{\beta_1} [q_1] \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_n} [q_n] \otimes (y_0 \otimes \ldots \otimes y_{q_0}) \right) = [q_1] \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_n} [q_n] \otimes \beta_1(y_0, \ldots, y_{q_0}),$$

$$d_j \left( [q_0] \xrightarrow{\beta_1} \ldots \xrightarrow{\beta_{j-1}} [q_{j-1}] \otimes (y_0 \otimes \ldots \otimes y_{q_0}) \right) =$$
\([q_0] \overset{\beta_1}{\to} \ldots \overset{\beta_{j-1}}{\to} [q_{j-1}] \overset{\beta_j + 1}{\to} [q_j] \to \ldots \to [q_n] \otimes (y_0 \otimes \ldots \otimes y_{q_0}), \quad (0 < j < n),\]
\[d_n \left( [q_0] \overset{\beta_1}{\to} \ldots \overset{\beta_n}{\to} [q_n] \otimes (y_0 \otimes \ldots \otimes y_{q_0}) \right) = [q_0] \overset{\beta_1}{\to} \ldots \overset{\beta_{n-1}}{\to} [q_{n-1}] \otimes (y_0 \otimes \ldots \otimes y_{q_0}).\]

We now have enough tools to compute \(H_* \Sigma(k)\). First, we need to show:

**Lemma 20.** \(N(\Delta S)\) is a contractible complex.

**Proof.** Define a functor \(\mathcal{F} : \Delta S \to \Delta S\) by

\[\mathcal{F} : [n] \mapsto [0] \odot [n],\]
\[\mathcal{F} : f \mapsto \text{id}_{[0]} \odot f,\]

using the multiplication \(\odot\) defined in section 1.1.

There is a natural transformation \(\text{id}_{\Delta S} \to \mathcal{F}\) given by the following commutative diagram for each \(f : [m] \to [n] \):

\[
\begin{array}{ccc}
[m] & \xrightarrow{f} & [n] \\
\downarrow{\delta_0} & & \downarrow{\delta_0} \\
[m+1] & \xrightarrow{\text{id} \odot f} & [n+1]
\end{array}
\]

Here, \(\delta_j^{(k)} : [k-1] \to [k]\) is the \(\Delta\) morphism that misses the point \(j \in [k]\).

Consider the constant functor \(\Delta S \overset{[0]}{\to} \Delta S\) that sends all objects to [0] and all morphisms to \(\text{id}_{[0]}\). There is a natural transformation \([0] \to \mathcal{F}\) given by the following commutative diagram for each \(f : [m] \to [n] \):

\[
\begin{array}{ccc}
[0] & \xrightarrow{\text{id}} & [0] \\
\downarrow{0_0} & & \downarrow{0_0} \\
[m+1] & \xrightarrow{\text{id} \odot f} & [n+1]
\end{array}
\]

Here, \(0_j^{(k)} : [0] \to [k]\) is the morphism that sends the point 0 to \(j \in [k]\).
Natural transformations induce homotopy equivalences (see [30] or Prop. 1.2 of [26]), so in particular, the identity map on $N(\Delta S)$ is homotopic to the map that sends $N(\Delta S)$ to the nerve of a trivial category. Thus, $N(\Delta S)$ is contractible.

**Corollary 21.** The symmetric homology of the ground ring $k$ is isomorphic to $k$, concentrated in degree $0$.

*Proof.* By Cor. 17 and Remark 19, $\text{HS}_\ast(k)$ is the homology of the chain complex generated (freely) over $k$ by the chains

$$\left\{ [q_0] \overset{\beta_1}{\to} [q_1] \overset{\beta_2}{\to} \ldots \overset{\beta_n}{\to} [q_n] \otimes (1 \otimes \ldots \otimes 1) \right\},$$

where $\beta_i \in \text{Mor}_{\Delta S}([q_{i-1}],[q_i])$. Each chain $[q_0] \to [q_1] \to \ldots \to [q_n] \otimes (1 \otimes \ldots \otimes 1)$ may be identified with the chain $[q_0] \to [q_1] \to \ldots \to [q_n]$ of $N(\Delta S)$, and clearly this defines a chain isomorphism to $N(\Delta S)$. The result now follows from Lemma 20.

**1.5 Tensor Algebras**

For a general $k$-algebra $A$, the standard resolution is often too difficult to work with. In the following chapters, we shall see some methods of reducing the size of the standard resolution. In order to prove the results of chapter 2, it is necessary to prove these results first for the special case of tensor algebras. Indeed, tensor algebra arguments are also key in the proof of Fiedorowicz’s Theorem (Thm. 1(1) of [9]) about the symmetric homology of group algebras.

Let $T : k\text{-Alg} \to k\text{-Alg}$ be the functor sending an algebra $A$ to the tensor algebra generated by $A$.

$$TA := k \oplus A \oplus A^\otimes 2 \oplus A^\otimes 3 \oplus \ldots$$

The functor $T$ takes an algebra homomorphism $A \xrightarrow{f} B$ to the induced homomorphism $Tf$ defined on generators by:

$$Tf(a_0 \otimes a_1 \otimes \ldots \otimes a_k) = f(a_0) \otimes f(a_1) \otimes \ldots \otimes f(a_k).$$
There is an algebra homomorphism $\theta : TA \rightarrow A$, defined by multiplying tensor factors:

$$\theta(a_0 \otimes a_1 \otimes \ldots \otimes a_k) := a_0a_1\cdots a_k.$$ 

In fact, $\theta$ defines a natural transformation $T \rightarrow \text{id}$, as can be verified by the following commutative diagram (valid for all $A \xrightarrow{f} B$ in $k$-$\text{Alg}$).

\[
\begin{array}{ccc}
TA & \xrightarrow{\theta_A} & A \\
\downarrow T_f & & \downarrow f \\
TB & \xrightarrow{\theta_B} & B
\end{array}
\]

We shall also make use of a $k$-module homomorphism $h$ sending the algebra $A$ identically onto the summand $A$ of $TA$. This map is a natural transformation from the forgetful functor $U : k$-$\text{Alg} \rightarrow k$-$\text{Mod}$ to the functor $UT$.

\[
\begin{array}{ccc}
UA & \xrightarrow{h_A} & UTA \\
\downarrow U_f & & \downarrow Uf \\
UB & \xrightarrow{h_B} & UTB
\end{array}
\]

Henceforth, context will make it clear whether we are working with algebras or underlying $k$-modules, and so the functor $U$ shall be omitted.

Denote by $\mathcal{Y}_s A$, the complex $k[N(- \setminus \Delta S)] \otimes_{\Delta S} B_{s}^{\text{sym}} A$ of Cor. 17.

**Proposition 22.** The assignment $A \mapsto \mathcal{Y}_s A$ is functorial.

**Proof.** We have to say what happens to morphisms. If $f : A \rightarrow B$ is a morphism of $k$-algebras (sending $1_A \mapsto 1_B$), then there is an induced chain map

$$\text{id} \otimes B_{s}^{\text{sym}} f : k[N(- \setminus \Delta S)] \otimes_{\Delta S} B_{s}^{\text{sym}} A \rightarrow k[N(- \setminus \Delta S)] \otimes_{\Delta S} B_{s}^{\text{sym}} B,$$
defined on $k$-chains by:

$$[n] \to [p_0] \to [p_1] \to \cdots \to [p_k] \otimes (a_0 \otimes \cdots \otimes a_n) \mapsto [n] \to [p_0] \to [p_1] \to \cdots \to [p_k] \otimes (f(a_0) \otimes \cdots \otimes f(a_n))$$

$B_{sym}^\ast f$ is a natural transformation by Prop. 8.

For a general $k$-algebra $A$, resolve $A$ by tensor algebras. The resulting long exact sequence may be regarded as a $k$-complex, where $TA$ is regarded as degree 0.

$$0 \leftarrow A \xrightarrow{\theta_A} TA \xrightarrow{\theta_1} T^2A \xrightarrow{\theta_2} \cdots$$

(1.4)

The maps $\theta_n$ for $n \geq 1$ are defined in terms of face maps:

$$\theta_n := \sum_{i=0}^{n} (-1)^i T^{n-i} \theta T_i A.$$

(1.5)

In Section 2.1, we shall need to use an important property of the maps $\theta_n$:

**Proposition 23.** $\theta_n$ defines a natural transformation $T^{p+1} \to T^p$.

**Proof.** The map $\theta_n$ depends on the algebra $A$. When we wish to distinguish which algebra is associated with $\theta_n$, we shall use the notation $\theta_n(A)$. Now, let $f : A \to B$ be any unital map of algebras. Consider the diagram below:

$$
\begin{array}{ccc}
T^{n+1}A & \xrightarrow{\theta_n(A)} & T^n A \\
\downarrow\ T^{n+1} f & & \downarrow T^n f \\
T^{n+1} B & \xrightarrow{\theta_n(B)} & T^n B
\end{array}
$$

We must show this diagram commutes. Now, $\theta_n(A) = \sum_{i=0}^{n} (-1)^i T^{n-i} \theta T_i A$. Let $0 \leq i \leq n$, and consider the following diagram:

$$
\begin{array}{ccc}
T(T^i A) & \xrightarrow{\theta_{T^i A}} & T^i A \\
\downarrow T(T^i f) & & \downarrow T^i f \\
T(T^i B) & \xrightarrow{\theta_{T^i B}} & T^i B
\end{array}
$$
This diagram commutes by naturality of $\theta$. Now, apply the functor $T^{n-i}$ to each object and functor to get the corresponding commutative diagram for the $i^{th}$ face map of $\theta_n$.

\[
\begin{array}{c}
T^{n+1}A \xrightarrow{T^{n-i}\theta_{T^iA}} T^iA \\
\downarrow T^{n+1}f \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
the squares of the bigraded module anti-commute, introduce the sign \((-1)^p\) on each vertical differential. Call the resulting double complex \(\mathcal{T}_{s,*}\).

\[
\begin{array}{cccccc}
& & & & & \\
& \downarrow & \downarrow & \downarrow & \downarrow & \\
& Y_2TA & Y_2T^2A & Y_2T^3A & \cdots & \\
& d_2 & -d_2 & d_2 & & \\
& \downarrow & \downarrow & \downarrow & \downarrow & \\
& Y_1TA & Y_1T^2A & Y_1T^3A & \cdots & \\
& d_1 & -d_1 & d_1 & & \\
& \downarrow & \downarrow & \downarrow & \downarrow & \\
& Y_0TA & Y_0T^2A & Y_0T^3A & \cdots & \\
\end{array}
\]

Consider a second double complex, \(\mathcal{A}_{s,*}\), consisting of the complex \(Y_sA\) as the 0\(^{th}\) column, and 0 in all positive columns.

\[
\begin{array}{cccccc}
& & & & & \\
& \downarrow & \downarrow & \downarrow & \downarrow & \\
& Y_2A & 0 & 0 & \cdots & \\
& d_2 & & & & \\
& \downarrow & \downarrow & \downarrow & \downarrow & \\
& Y_1A & 0 & 0 & \cdots & \\
& d_1 & & & & \\
& \downarrow & \downarrow & \downarrow & \downarrow & \\
& Y_0A & 0 & 0 & \cdots & \\
\end{array}
\]

**Theorem** 25. There is a map of double complexes, \(\Theta_{s,*}: \mathcal{T}_{s,*} \to \mathcal{A}_{s,*}\) inducing isomorphism in homology

\[
H_*(\text{Tot}(\mathcal{T}); k) \to H_*(\text{Tot}(\mathcal{A}); k)
\]
Proof. The map $\Theta_{*,*}$ is defined as:

$$
\Theta_{p,q} := \begin{cases} 
\mathcal{Y}_q \theta_A, & p = 0 \\
0, & p > 0 
\end{cases}
$$

This map is easily verified to be a map of double complexes, since most components of $\mathcal{A}_{*,*}$ are 0. On the $0^{th}$ column, we just verify that $d_q(A) \circ \mathcal{Y}_q \theta_A = \mathcal{Y}_{q-1} \theta_A \circ d_q(T A)$, but this follows since $\mathcal{Y}_*$ is functorial ($\mathcal{Y}_* \theta_A$ is a chain map). The isomorphism follows from the exactness of the sequence (1.6).

Remark 26. Observe that

$$
H_*(\text{Tot}(\mathcal{A}); k) = H_*(\mathcal{Y}_* A; k) = HS_*(A).
$$

This permits the computation of symmetric homology of any given algebra $A$ in terms of tensor algebras:

Corollary 27. For an associative, unital $k$-algebra $A$,

$$
HS_*(A) \cong H_*(\text{Tot}(\mathcal{T}); k),
$$

where $\mathcal{T}_{*,*}$ is the double complex $\{\mathcal{Y}_p T_q^{p+1} A\}_{p,q \geq 0}$.

The following lemma shows why it is advantageous to work with tensor algebras.

Lemma 28. For a unital, associative $k$-algebra $A$, there is an isomorphism of $k$-complexes:

$$
\mathcal{Y}_* TA \cong \bigoplus_{n \geq -1} Y_n, \quad (1.9)
$$

where

$$
Y_n = \begin{cases} 
k [N(\Delta S)], & n = -1 \\
k [N([n] \backslash \Delta S)] \otimes_{k \Sigma_{n+1}^{op}} A^{\otimes (n+1)}, & n \geq 0 
\end{cases}
$$

Moreover, the differential respects the direct sum decomposition.
Proof. Any generator of $\mathcal{Y}TA$ has the form
\[ [p] \overset{\alpha}{\rightarrow} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes u, \]
where
\[ u = \left( \bigotimes_{a \in A_0} a \right) \otimes \left( \bigotimes_{a \in A_1} a \right) \otimes \ldots \otimes \left( \bigotimes_{a \in A_p} a \right), \]
and $A_0, A_1, \ldots, A_p$ are finite ordered lists of elements of $A$. Indeed, each $A_j$ may be thought of as an element of $A^{m_j}$ (set product). If $A_j = \emptyset$, then $m_j = 0$, and we use the convention that an empty tensor product is equal to $1 \in k$. We say that the corresponding tensor factor is trivial. (Caution, $1_A \in A$ is not considered trivial, since it has degree 1 in the tensor algebra.) Now, let $m = (\sum m_j) - 1$. We shall use the convention that $A^0 = \emptyset$. Let
\[ \pi : A^{m_0} \times A^{m_1} \times \ldots \times A^{m_p} \longrightarrow A^{m+1} \]
be the evident isomorphism. Let $A_m = \pi(A_0, A_1, \ldots, A_p)$.

Case 1. If $u$ is non-trivial (i.e., $A_m \neq \emptyset$), then construct the element
\[ u' = \bigotimes_{a \in A_m} a \]
Next, construct a $\Delta$-morphism $\zeta_u : [m] \rightarrow [p]$ as follows: $\zeta_u$ sends the point $0, 1, \ldots, m_0 - 1$ identically onto 0, then sends the the points $m_0, m_0 + 1, \ldots, m_0 + m_1 - 1$ onto 1, etc. It should be clear that $(\zeta_u)_*(u') = u$. An example will clarify. Suppose
\[ u = (a_0 \otimes a_1) \otimes 1 \otimes (a_2 \otimes a_3 \otimes a_4) \in (A \otimes A) \oplus k \oplus (A \otimes A \otimes A). \]
Then
\[ u' = a_0 \otimes a_1 \otimes a_2 \otimes a_3 \otimes a_4, \]
and $\zeta_u : [4] \rightarrow [2]$ has preimages: $\zeta_u^{-1}(0) = \{0, 1\}, \zeta_u^{-1}(1) = \emptyset, \zeta_u^{-1}(2) = \{2, 3, 4\}$ (or, in tensor notation, $\zeta_u = x_0 x_1 \otimes 1 \otimes x_2 x_3 x_4$). Note, the elements $a_i$ need not be distinct.

Then under the $\Delta S$-equivalence,
\[ [p] \overset{\alpha}{\rightarrow} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes u = [p] \overset{\alpha}{\rightarrow} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes (\zeta_u)_*(u') \]
The assignment is well-defined with respect to the $\Delta S$-equivalence since the total number of non-trivial tensor factors in $u$ is the same as the total number of non-trivial tensor factors in $\phi_s(u)$ for any $\phi \in \text{Mor}\Delta S$. It is this property of tensor algebras that is essential in making the proof work.

Note that the only equivalence that persists after rewriting the generators is invariance under the symmetric group action:

$$[m] \xrightarrow{\alpha\sigma} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes u' \approx [m] \xrightarrow{\alpha} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes \sigma(u'), \text{ for } \sigma \in \Sigma_{m+1}^{\text{op}}$$

This shows that any such non-trivial element in $\mathcal{Y}_s T A$ may be written uniquely as an element of

$$k [N([m] \setminus \Delta S)] \otimes_{k^{\Sigma_{m+1}^{\text{op}}}} A^{\otimes(m+1)}.$$

**Case 2.** If $u$ is trivial (i.e., $A_m = \emptyset$), then

$$[p] \xrightarrow{\alpha} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes u = [p] \xrightarrow{\alpha} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes 1_k^{\otimes(p+1)}.$$

This element is equivalent to:

$$[q_0] \xrightarrow{id} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes 1_k^{\otimes(q_0+1)},$$

and so this element can be identified with $[q_0] \rightarrow \ldots \rightarrow [q_n] \in k [N(\Delta S)]$.

Thus, the isomorphism (1.9) is proven. Note, the fact that total number of non-trivial tensor factors is preserved under $\Delta S$ morphisms also proves that the differential respects the direct sum decomposition. $\square$

### 1.6 Symmetric Homology with Coefficients

Following the conventions for Hochschild and cyclic homology in Loday [15], when we need to indicate explicitly the ground ring $k$ over which we compute symmetric homology of $A$, we shall use the notation:

$$HS_s(A \mid k)$$
Furthermore, since the notion “ΔS-module” does not explicitly state the ground ring, we shall use the bulkier “ΔS-module over k” when the context is ambiguous.

If $\mathcal{Y}_s$ is a complex that computes symmetric homology of the algebra $A$ over $k$, we may make the following definition:

**Definition 29.** The symmetric homology of $A$ over $k$, with coefficients in a left $k$-module $M$ is

$$HS_s(A; M) := H_s(\mathcal{Y}_s \otimes_k M)$$

Note, this definition is independent of the particular choice of complex $\mathcal{Y}_s$, so we shall generally use the complex $\mathcal{Y}_s A = k[N(\Delta S)] \otimes_{\Delta S} B^{sym}_s A$ of Cor. 17 in this section.

**Proposition 30.** If $M$ is flat over $k$, then

$$HS_s(A; M) \cong HS_s(A) \otimes_k M$$

*Proof.* Since $M$ is $k$-flat, the functor $- \otimes_k M$ is exact, and so commutes with homology functors. In particular,

$$H_n(\mathcal{Y}_s A \otimes_k M) \cong H_n(\mathcal{Y}_s A) \otimes_k M$$

□

**Corollary 31.** For any $\mathbb{Z}$-algebra $A$, $HS_n(A; \mathbb{Q}) \cong HS_n(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

**Lemma 32.** i. If $A$ is a flat $k$-algebra, then $\mathcal{Y}_n A$ is flat for each $n$.

ii. If $A$ is a projective $k$-algebra, then $\mathcal{Y}_n A$ is projective for each $n$.

*Proof.* By Remark 19 we may identify:

$$\mathcal{Y}_n A \cong \bigoplus_{m \geq 0} \left( k[N([m] \setminus \Delta S)_{n-1}] \otimes_k A^{\otimes(m+1)} \right).$$

Note, $k[N([m] \setminus \Delta S)_{n-1}]$ is free, so if $A$ is flat, then $\mathcal{Y}_n A$ is a direct sum of modules that are tensor products of free with flat modules, hence $\mathcal{Y}_n A$ is flat. Similarly, if $A$ is projective, $\mathcal{Y}_n A$ is also, since tensor products and direct sums of projectives are projective. □
Proposition 33. If $B$ is a commutative $k$-algebra, then there is an isomorphism

$$HS_*(A \otimes_k B \mid B) \cong HS_*(A; B)$$

Proof. Here, we are viewing $A \otimes_k B$ as a $B$-algebra via the inclusion $B \cong 1_A \otimes_k B \hookrightarrow A \otimes_k B$. Observe, there is an isomorphism

$$(A \otimes_k B) \otimes_B (A \otimes_k B) \cong A \otimes_k A \otimes_k (B \otimes_B B) \cong (A \otimes_k A) \otimes_k B.$$ 

Iterating this for $n$-fold tensors of $A \otimes_k B$,

$$\underbrace{(A \otimes_k B) \otimes_B \cdots \otimes_B (A \otimes_k B)}_n \cong \underbrace{A \otimes_k \cdots \otimes_k A}_{n} \otimes_k B$$

This shows that the $\Delta S$-module over $B$, $B_{sym}^*(A \otimes_k B)$ is isomorphic as $k$-module to $(B_{sym}^* A) \otimes_k B$ over $k$. The proposition now follows essentially by definition. Let $\mathcal{T}_*$ be the resolution of $k$ by projective $\Delta S^{op}$-modules (over $k$) given by $\mathcal{T}_* = k[N(- \setminus \Delta S)]$. Then, if we take tensor products (over $k$) with the algebra $B$, we obtain

$$\mathcal{T}_* \otimes_k B \cong B[N(- \setminus \Delta S)],$$

which is a projective resolution of the trivial $\Delta S^{op}$-module over $B$, $B$. Thus,

$$HS_*(A \otimes_k B \mid B) = H_* \left((\mathcal{T}_* \otimes_k B) \otimes_{B[\text{Mor}\Delta S]} B_{sym}^*(A \otimes_k B); B \right) \quad (1.10)$$

On the chain level, there are isomorphisms:

$$(\mathcal{T}_* \otimes_k B) \otimes_{B[\text{Mor}\Delta S]} B_{sym}^*(A \otimes_k B) \cong (\mathcal{T}_* \otimes_k B) \otimes_{B[\text{Mor}\Delta S]} (B_{sym}^* A \otimes_k B)$$

$$\cong (\mathcal{T}_* \otimes_k [\text{Mor}\Delta S] B_{sym}^* A) \otimes_k B \quad (1.11)$$

The complex (1.11) computes $HS_*(A; B)$ by definition. □

Remark 34. Since $HS_*(A \mid k) = HS_*(A \otimes_k k \mid k)$, Prop. 33 allows us to identify $HS_*(A \mid k)$ with $HS_*(A; k)$. 

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The construction $HS_*(A; -)$ is a covariant functor, as is immediately seen on the chain level. Moreover, Prop. 22 implies $HS_*(-; X)$ is a covariant functor for any left $k$-module, $X$.

**Proposition 35.** Suppose $0 \to X \to Y \to Z \to 0$ is a short exact sequence of left $k$-modules, and suppose $A$ is a flat $k$-algebra. Then there is an induced long exact sequence in symmetric homology:

$$\ldots \to HS_n(A; X) \to HS_n(A; Y) \to HS_n(A; Z) \to HS_{n-1}(A; X) \to \ldots$$

(1.12)

Moreover, a map of short exact sequences, $(\alpha, \beta, \gamma)$, as in the diagram below, induces a map of the corresponding long exact sequences (commutative ladder)

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

$$0 \longrightarrow X' \longrightarrow Y' \longrightarrow Z' \longrightarrow 0$$

(1.13)

**Proof.** By Lemma 32, the hypothesis $A$ is flat implies that the following is an exact sequence of chain complexes:

$$0 \to \mathcal{Y}_*A \otimes_k X \to \mathcal{Y}_*A \otimes_k Y \to \mathcal{Y}_*A \otimes_k Z \to 0.$$ 

This induces a long exact sequence in homology

$$\ldots \to H_n(\mathcal{Y}_*A \otimes_k X) \to H_n(\mathcal{Y}_*A \otimes_k Y) \to H_n(\mathcal{Y}_*A \otimes_k Z) \to H_{n-1}(\mathcal{Y}_*A \otimes_k X) \to \ldots$$

as required.

Now let $(\alpha, \beta, \gamma)$ be a morphism of short exact sequences, as in diagram (1.13). Consider
the diagram,

\[
\begin{array}{ccc}
\vdots & \vdots \\
HS_n(A; X) & \xrightarrow{\alpha^*} & HS_n(A; X') \\
\downarrow & & \downarrow \\
HS_n(A; Y) & \xrightarrow{\beta^*} & HS_n(A; Y') \\
\downarrow & & \downarrow \\
HS_n(A; Z) & \xrightarrow{\gamma^*} & HS_n(A; Z') \\
\partial & & \partial' \\
\downarrow & & \downarrow \\
HS_{n-1}(A; X) & \xrightarrow{\alpha^*} & HS_{n-1}(A; X') \\
\vdots & \vdots \\
\end{array}
\]

Since \(HS_n(A; -)\) is functorial, the upper two squares of the diagram commute. Commutativity of the lower square follows from the naturality of the connecting homomorphism in the snake lemma.

\[\square\]

Remark 36. Any family of additive covariant functors \(\{T_n\}\) between two abelian categories is said to be a long exact sequence of functors if it takes short exact sequences to long exact sequences such as (1.12) and morphisms of short exact sequences to commutative ladders of long exact sequences such as (1.14). See [7], Definition 1.1 and also [23], section 12.1. The content of Prop. 35 is that for \(A\) flat, \(\{HS_n(A; -)\}_{n \in \mathbb{Z}}\) is a long exact sequence of functors.

We now state the Universal Coefficient Theorem for symmetric homology.

Theorem 37. If \(A\) is a flat \(k\)-algebra, and \(B\) is a commutative \(k\)-algebra, then there is a spectral sequence with

\[
E_2^{p,q} := \text{Tor}_p^k(HS_q(A \mid k), B) \Rightarrow HS_*(A; B).
\]
Proof. Let $T_q : k\text{-Mod} \to k\text{-Mod}$ be the functor $HS_q(A; -)$. Observe, since $A$ is flat, $\{T_q\}$ is a long exact sequence of additive covariant functors (Rmk. 36 and Prop. 35); $T_q = 0$ for sufficiently small $q$ (indeed, for $q < 0$); and $T_q$ commutes with arbitrary direct sums, since tensoring and taking homology always commutes with direct sums. Hence, by the Universal Coefficient Theorem of Dold (2.12 of [7]. See also McCleary [23], Thm. 12.11), there is a spectral sequence with

$$E_2^{p,q} := \text{Tor}_p^k(T_q(k), B) \Rightarrow T_*(B).$$

As an immediate consequence, we have the following result.

**Corollary 38.** If $f : A \to A'$ is a $k$-algebra map between flat algebras which induces an isomorphism in symmetric homology, $HS_*(A) \xrightarrow{\cong} HS_*(A')$, then for a commutative $k$-algebra $B$, the map $f \otimes \text{id}_B$ induces an isomorphism $HS_*(A; B) \xrightarrow{\cong} HS_*(A'; B)$.

Under stronger hypotheses, the universal coefficient spectral sequence reduces to short exact sequences. Recall some notions of ring theory (c.f. the article Homological Algebra: Categories of Modules (200:K), Vol. 1, pp. 755-757 of [12]). A commutative ring $k$ is said to have global dimension $\leq n$ if for all $k$-modules $X$ and $Y$, Ext$^m_k(X, Y) = 0$ for $m > n$. $k$ is said to have weak global dimension $\leq n$ if for all $k$-modules $X$ and $Y$, Tor$^k_m(X, Y) = 0$ for $m > n$. Note, the weak global dimension of a ring is less than or equal to its global dimension, with equality holding for Noetherian rings but not in general. A ring is said to be hereditary if all submodules of projective modules are projective, and this is equivalent to the global dimension of the ring being no greater than 1.

**Theorem 39.** If $k$ has weak global dimension $\leq 1$, then the spectral sequence of Thm. 37 reduces to short exact sequences,

$$0 \longrightarrow HS_n(A \mid k) \otimes_k B \longrightarrow HS_n(A; B) \longrightarrow \text{Tor}_1^k(HS_{n-1}(A \mid k), B) \longrightarrow 0. \quad (1.15)$$

Moreover, if $k$ is hereditary and and $A$ is projective over $k$, then these sequences split (unnaturally).
Proof. Assume first that $k$ has weak global dimension $\leq 1$. So $\text{Tor}_p^k(T_q(k), B) = 0$ for all $p > 1$. Following Dold’s argument (Corollary 2.13 of [7]), we obtain the required exact sequences,

$$0 \rightarrow T_n(k) \otimes_k B \rightarrow T_n(B) \rightarrow \text{Tor}_1^k(T_{n-1}(k), B) \rightarrow 0.$$ 

Assume further that $k$ is hereditary and $A$ is projective. Then by Lemma 32, $\mathcal{Y}_n A$ is projective for each $n$. Theorem 8.22 of Rotman [29] then gives us the desired splitting. 

Remark 40. The proof given above also proves UCT for cyclic homology. A partial result along these lines exists in Loday ([15], 2.1.16). There, he shows $HC_*(A \mid k) \otimes_k K \cong HC_*(A \mid K)$ and $HH_*(A \mid k) \otimes_k K \cong HH_*(A \mid K)$ in the case that $K$ is a localization of $k$, and $A$ is a $K$-module, flat over $k$. I am not aware of a statement of UCT for cyclic or Hochschild homology in its full generality in the literature.

For the remainder of this section, we shall obtain a converse to Cor. 38 in the case $k = \mathbb{Z}$.

Theorem 41. Let $f : A \rightarrow A'$ be an algebra map between torsion-free $\mathbb{Z}$-algebras. Suppose for $B = \mathbb{Q}$ and $B = \mathbb{Z}/p\mathbb{Z}$ for any prime $p$, the map $f \otimes \text{id}_B$ induces an isomorphism $HS_*(A; B) \rightarrow HS_*(A'; B)$. Then $f$ also induces an isomorphism $HS_*(A) \cong HS_*(A')$.

First, note that Prop. 35 allows one to construct the Bockstein homomorphisms

$$\beta_n : HS_n(A; Z) \rightarrow HS_{n-1}(A; X)$$

associated to a short exact sequence of $k$-modules, $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$, as long as $A$ is flat over $k$. These Bocksteins are natural in the following sense:

Lemma 42. Suppose $f : A \rightarrow A'$ is a map of flat $k$-algebras. and $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is a short exact sequence of left $k$-modules. Then the following diagram is commutative for each $n$:

$$\begin{array}{ccc}
HS_n(A; Z) & \xrightarrow{\beta} & HS_{n-1}(A; X) \\
\downarrow f_* & & \downarrow f_* \\
HS_n(A'; Z) & \xrightarrow{\beta'} & HS_{n-1}(A'; X)
\end{array}$$
Moreover if the induced map \( f_* : HS_*(A; W) \to HS_*(A'; W) \) is an isomorphism for any two of \( W = X, W = Y, W = Z \), then it is an isomorphism for the third.

**Proof.** A and \( A' \) flat imply both sequences of complexes are exact:

\[
0 \to \mathcal{Y}_*A \otimes_k X \to \mathcal{Y}_*A \otimes_k Y \to \mathcal{Y}_*A \otimes_k Z \to 0.
\]

\[
0 \to \mathcal{Y}_*A' \otimes_k X \to \mathcal{Y}_*A' \otimes_k Y \to \mathcal{Y}_*A' \otimes_k Z \to 0.
\]

The map \( \mathcal{Y}_*A \to \mathcal{Y}_*A' \) induces a map of short exact sequences, hence induces a commutative ladder of long exact sequences of homology groups. In particular, the squares involving the boundary maps (Bocksteins) must commute.

Now, assuming further that \( f_* \) induces isomorphisms \( HS_*(A; W) \to HS_*(A'; W) \) for any two of \( W = X, W = Y, W = Z \), let \( V \) be the third module. The 5-lemma implies isomorphisms \( HS_n(A; V) \xrightarrow{\cong} HS_n(A'; V) \) for each \( n \).

We shall now proceed with the proof of Thm. 41. All tensor products will be over \( \mathbb{Z} \) for the rest of this section.

**Proof.** Let \( A \) and \( A' \) be torsion-free \( \mathbb{Z} \)-modules. Over \( \mathbb{Z} \), torsion-free implies flat. Let \( f : A \to A' \) be an algebra map inducing isomorphism in symmetric homology with coefficients in \( \mathbb{Q} \) and also in \( \mathbb{Z}/p\mathbb{Z} \) for any prime \( p \). For \( m \geq 2 \), there is a short exact sequence,

\[
0 \longrightarrow \mathbb{Z}/p^{m-1}\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^m\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.
\]

Consider first the case \( m = 2 \). Since \( HS_*(A; \mathbb{Z}/p\mathbb{Z}) \to HS_*(A'; \mathbb{Z}/p\mathbb{Z}) \) is an isomorphism, Lemma 42 implies the induced map is an isomorphism for the middle term:

\[
f_* : HS_*(A; \mathbb{Z}/p^2\mathbb{Z}) \xrightarrow{\cong} HS_*(A'; \mathbb{Z}/p^2\mathbb{Z}) \tag{1.16}
\]

(Note, all maps induced by \( f \) on symmetric homology will be denoted by \( f_* \).)

For the inductive step, fix \( m > 2 \) and suppose \( f \) induces an isomorphism in symmetric homology, \( f_* : HS_*(A; \mathbb{Z}/p^{m-1}\mathbb{Z}) \xrightarrow{\cong} HS_*(A'; \mathbb{Z}/p^{m-1}\mathbb{Z}) \). Again, Lemma 42 implies the induced map is an isomorphism on the middle term.

\[
f_* : HS_*(A; \mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\cong} HS_*(A'; \mathbb{Z}/p^m\mathbb{Z}) \tag{1.17}
\]
Denote $\mathbb{Z}/p^\infty \mathbb{Z} := \lim \downarrow \mathbb{Z}/p^m \mathbb{Z}$. Note, this is a direct limit in the sense that it is a colimit over a directed system. The direct limit functor is exact (Prop. 5.3 of [32]), so the maps $H_{S_n}(A; \mathbb{Z}/p^\infty \mathbb{Z}) \to H_{S_n}(A'; \mathbb{Z}/p^\infty \mathbb{Z})$ induced by $f$ are isomorphisms, given by the chain of isomorphisms below:

$$\begin{align*}
H_{S_n}(A; \mathbb{Z}/p^\infty \mathbb{Z}) &\cong H_n(\lim \downarrow \mathfrak{Y}A \otimes \mathbb{Z}/p^m \mathbb{Z}) \\
&\cong \lim \downarrow H_* (\mathfrak{Y}A \otimes \mathbb{Z}/p^m \mathbb{Z}) \\
&\cong \lim \downarrow H_* (\mathfrak{Y}A' \otimes \mathbb{Z}/p^m \mathbb{Z}) \\
&\cong H_{S_n}(A'; \mathbb{Z}/p^\infty \mathbb{Z})
\end{align*}$$

(Note, $f_*$ here stands for $\lim \downarrow H_n(\mathfrak{Y}f \otimes \text{id})$.)

Finally, consider the short exact sequence of abelian groups,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \bigoplus_{p \text{ prime}} \mathbb{Z}/p^\infty \mathbb{Z} \longrightarrow 0$$

The isomorphism $f_* : H_{S_*}(A; \mathbb{Z}/p^\infty \mathbb{Z}) \to H_{S_*}(A'; \mathbb{Z}/p^\infty \mathbb{Z})$ passes to direct sums, giving isomorphisms for each $n$,

$$f_* : H_{S_n} \left( A; \bigoplus_p \mathbb{Z}/p^\infty \mathbb{Z} \right) \cong H_{S_n} \left( A'; \bigoplus_p \mathbb{Z}/p^\infty \mathbb{Z} \right).$$

Together with the assumption that $H_{S_*}(A; \mathbb{Q}) \to H_{S_*}(A'; \mathbb{Q})$ is an isomorphism, another appeal to Lemma 42 gives the required isomorphism in symmetric homology induced by $f$:

$$f_* : H_{S_n}(A) \xrightarrow{\cong} H_{S_n}(A')$$

Remark 43. Theorem 41 may be useful for determining integral symmetric homology, since rational computations are generally simpler (see Section 3.2), and computations mod $p$ may be made easier due to the presence of additional structure, namely homology operations (see Chapter 5).

Finally, we state a result along the lines of McCleary [23], Thm. 10.3. Denote the torsion submodule of the graded module $H_*$ by $\tau(H_*)$. 

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Theorem 44. Suppose $A$ is free of finite rank over $\mathbb{Z}$. Then there is a singly-graded spectral sequence with

$$E_1^1 := H_{*}(A; \mathbb{Z}/p\mathbb{Z}) \Rightarrow H_{*}(A)/\tau(H_{*}(A)) \otimes \mathbb{Z}/p\mathbb{Z},$$

with differential map $d^1 = \beta$, the standard Bockstein map associated to $0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0$. Moreover, the convergence is strong.

The proof McCleary gives on p. 459 carries over to our case intact. All that is required for this proof is that each $H_n(\mathbb{Y}_* A)$ be a finitely-generated abelian group. The hypothesis that $A$ is finitely-generated, coupled with a result of Chapter 4, namely Cor. 98, guarantees this. Note, over $\mathbb{Z}$, free of finite rank is equivalent to flat and finitely-generated.

Theorem 44 is a version of the Bockstein spectral sequence for symmetric homology.

1.7 Symmetric Homology of Monoid Algebras

The symmetric homology for the case of a monoid algebra $A = k[M]$ has been studied by Fiedorowicz in [9]. In the most general formulation (Prop. 1.3 of [9]), we have:

Theorem 45.

$$H_{*}(k[M]) \cong H_{*}(B(C_{\infty}, C_1, M); k),$$

where $C_1$ is the little 1-cubes monad, $C_{\infty}$ is the little $\infty$-cubes monad, and $B(C_{\infty}, C_1, M)$ is May’s functorial 2-sided bar construction (see [19]).

The proof makes use of a variant of the symmetric bar construction:

Definition 46. Let $M$ be a monoid. Define a functor $B_{*}^{sym} M : \Delta S \rightarrow \text{Sets}$ by:

$$B_{n}^{sym} M := B_{*}^{sym} M[n] := M^{n+1}, \text{ (set product)}$$

$$B_{*}^{sym} M(\alpha) : (m_0, \ldots, m_n) \mapsto \alpha(m_0, \ldots, m_n), \quad \text{for } \alpha \in \text{Mor}\Delta S.$$
Definition 47. For a $\mathcal{C}$-set $X$ and $\mathcal{C}^{\text{op}}$-set $Y$, define the $\mathcal{C}$-equivariant set product:
\[
Y \times_{\mathcal{C}} X := \left( \prod_{C \in \text{Obj} \mathcal{C}} Y(C) \times X(C) \right) / \approx,
\]
where the equivalence $\approx$ is generated by the following: For every morphism $f \in \text{Mor}_{\mathcal{C}}(C, D)$, and every $x \in X(C)$ and $y \in Y(D)$, we have $(y, f^*(x)) \approx (f^*(y), x)$.

Note that $B_{\text{sym}}^* M$ is a $\Delta S$-set, and also a simplicial set, via the chain of functors in section 1.2. Let $\mathcal{X}_* := N(\Delta S)^{\text{op}} \times_{\Delta S} B_{\text{sym}}^* M$.

Proposition 48. $\mathcal{X}_*$ is a simplicial set whose homology computes $HS_*(k[M])$.

Proof. It is clear that $\mathcal{X}_*$ is a simplicial set. The standard construction for finding the homology of a simplicial set is to create the complex $k[\mathcal{X}_*]$, with face maps induced by the face maps of $\mathcal{X}_*$. Since $M$ is a $k$-basis for $k[M]$, $B_{\text{sym}}^* M$ acts as a $k$-basis for $B_{\text{sym}}^* k[M]$.

Then, observe that $k[N(\Delta S)^{\text{op}} \times_{\Delta S} B_{\text{sym}}^* M] = k[N(\Delta S)] \otimes_{\Delta S} B_{\text{sym}}^* k[M]$.  

If $M = JX_+$ is a free monoid on a generating set $X$, then $k[M] = T(X)$, the (free) tensor algebra over $k$ on the set $X$. In this case, we have the following:

Lemma 49. 
\[
HS_* (T(X)) \cong H_* \left( \prod_{n \geq -1} \bar{X}_n; k \right),
\]
where
\[
\bar{X}_n = \begin{cases} 
N(\Delta S), & n = -1 \\
N([n] \setminus \Delta S) \times_{\Sigma_{n+1}^{\text{op}}} X^{n+1}, & n \geq 0
\end{cases}
\]

Proof. This is a consequence of Lemma 28 when the tensor algebra is free, generated by $X = \{x_i \mid i \in \mathcal{X}\}$. By the lemma, we obtain a decomposition
\[
\mathcal{Y}_* T(X) \cong k[N(\Delta S)] \oplus \left( \bigoplus_{n \geq 0} k[N([n] \setminus \Delta S)] \otimes_{\Sigma_{n+1}^{\text{op}}} k[X^{n+1}] \right),
\]
\[
\cong k \left[ N(\Delta S) \amalg \prod_{n \geq 0} N([n] \setminus \Delta S) \times_{\Sigma_{n+1}^{\text{op}}} X^{n+1} \right],
\]

computing $HS_* (T(X))$.  

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Remark 50. This proves Thm. 45 in the special case that $M$ is a free monoid.

If $M$ is a group, $\Gamma$, then Fiedorowicz [9] found:

**Theorem 51.**

$$HS_\ast(k[\Gamma]) \cong H_\ast(\Omega\Omega S^\infty(B\Gamma); k)$$

This final formulation shows in particular that $HS_\ast$ is a non-trivial theory. While it is true that $H_\ast(\Omega S^\infty(X)) = H_\ast(QX)$ is well understood, the same cannot be said of the homology of $\Omega\Omega S^\infty X$. Indeed, May states that $H_\ast(QX)$ may be regarded as the free allowable Hopf algebra with conjugation over the Dyer-Lashof algebra and dual of the Steenrod algebra (See [5], preface to chapter 1, and also Lemma 4.10). Cohen and Peterson [6] found the homology of $\Omega\Omega S^\infty X$, where $X = S^0$, the zero-sphere, but there is little hope of extending this result to arbitrary $X$ using the same methods.

We shall have more to say about $HS_1(k[\Gamma])$ in section 6.4.
CHAPTER 2
ALTERNATIVE RESOLUTIONS

2.1 Symmetric Homology Using $\Delta S_+$

In this section, we shall show that replacing $\Delta S$ by $\Delta S_+$ in an appropriate way does not affect the computation of $H_{S_*}$.

Definition 52. For an associative, unital algebra, $A$, over a commutative ground ring $k$, define a functor $B_{*}^{sym+}A : \Delta S_+ \rightarrow k$-Mod by:

\[
\begin{cases}
B_{n}^{sym+}A := B_{*}^{sym}A[n] := A \otimes (n+1) \\
B_{-1}^{sym+}A := k,
\end{cases}
\]

\[
\begin{cases}
B_{*}^{sym+}A(\alpha) : (a_0 \otimes a_1 \otimes \ldots \otimes a_n) \mapsto \alpha(a_0, \ldots, a_n), \quad \text{for } \alpha \in \text{Mor} \Delta S, \\
B_{*}^{sym+}A(\iota_n) : \lambda \mapsto \lambda(1_A \otimes \ldots \otimes 1_A), \quad (\lambda \in k).
\end{cases}
\]

Consider the functor $\mathcal{Y}_{*}^{+} : k$-Alg $\rightarrow k$-complexes given by:

\[
\mathcal{Y}_{*}^{+}A := k[N(- \setminus \Delta S_+)] \otimes \Delta S_+ B_{*}^{sym+}A. \quad (2.1)
\]

\[
\mathcal{Y}_{*}^{+}f = \text{id} \otimes B_{*}^{sym+}f
\]

The functoriality of $\mathcal{Y}_{*}^{+}$ depends on the naturality of $B_{*}^{sym+}f$, which follows from the naturality of $B_{*}^{sym}f$ in most cases. The only case to check is on a morphism $[-1] \xrightarrow{\iota_m} [m]$. For the object $[-1]$, $B_{-1}^{sym+}f$ will be the identity map of $k$. 

\[
\begin{array}{ccc}
\begin{array}{c}
\xrightarrow{\text{id}} \xrightarrow{\text{id}} \\
B_{*}^{sym}A(\iota_m) \xrightarrow{A \otimes (m+1)} B_{*}^{sym}B(\iota_m)
\end{array}
\end{array}
\]
The commutativity of this diagram is clear, since $f(1) = 1$, and $B_*^{sym}A(t_m)$, $B_*^{sym}B(t_m)$ are simply the unit maps.

Note, the differential of $\mathcal{Y}_+^+ A$ will be denoted $d_*(A)$. As before, when the context is clear, the differential will simply be denoted $d_*$.

Our goal is to prove the following:

**Theorem 53.** For an associative, unital $k$-algebra $A$,

$$HS_*(A) = H_* (\mathcal{Y}_+^+ A; k)$$

As the preliminary step, we shall prove the theorem in the special case of tensor algebras. We shall need an analogue of Lemma 28 for $\Delta S_+$. 

**Lemma 54.** For a unital, associative $k$-algebra $A$, there is an isomorphism of $k$-complexes:

$$\mathcal{Y}_+^+ TA \cong \bigoplus_{n \geq -1} Y_+^n,$$  \hspace{1cm} (2.2)

where

$$Y_+^n = \begin{cases}
  k [N(\Delta S_+)], & n = -1 \\
  k [N([n] \setminus \Delta S_+)] \otimes_{k \Sigma_{n+1}} A^{\otimes (n+1)}, & n \geq 0
\end{cases}$$

Moreover, the differential respects the direct sum decomposition.

**Proof.** The proof follows verbatim as the proof of Lemma 28, only with $\Delta S$ replaced with $\Delta S_+$ throughout. \hfill \Box

**Lemma 55.** There is a chain map $J_A : \mathcal{Y}_+ A \to \mathcal{Y}_+^+ A$, which is natural in $A$.

**Proof.** First observe that the the inclusion $\Delta S \hookrightarrow \Delta S_+$ induces an inclusion of nerves:

$$N(- \setminus \Delta S) \hookrightarrow N(- \setminus \Delta S_+),$$

which in turn induces the chain map

$$k [N(- \setminus \Delta S)] \otimes_{\Delta S} B_*^{sym} A \to k [N(- \setminus \Delta S_+)] \otimes_{\Delta S} B_*^{sym} A$$
\( k[N(- \Delta S_+)] \) is a right \( \Delta S \)-module as well as a right \( \Delta S_+ \)-module. Similarly, \( B^{sym+} A \) is both a left \( \Delta S \)-module and a left \( \Delta S_+ \)-module. There is a natural transformation \( B^{sym} A \rightarrow B^{sym+} A \), again induced by inclusion of categories \( \Delta S \hookrightarrow \Delta S_+ \), hence there is a chain map

\[
k[N(- \Delta S_+)] \otimes_{\Delta S} B^{sym} A \rightarrow k[N(- \Delta S_+)] \otimes_{\Delta S} B^{sym+} A.
\]

Finally, pass to tensors over \( \Delta S_+ \):

\[
k[N(- \Delta S_+)] \otimes_{\Delta S} B^{sym+} A \rightarrow k[N(- \Delta S_+)] \otimes_{\Delta S_+} B^{sym+} A.
\]

The composition gives a chain map \( J_A : \mathcal{Y}_* A \rightarrow \mathcal{Y}_*^+ A \). We must verify that \( J \) is a natural transformation \( \mathcal{Y}_* \rightarrow \mathcal{Y}_*^+ \). Suppose \( f : A \rightarrow B \) is a unital algebra map.

\[
\begin{array}{ccc}
\mathcal{Y}_* A & \xrightarrow{J_A} & \mathcal{Y}_*^+ A \\
\mathcal{Y}_* f & & \mathcal{Y}_*^+ f \\
\mathcal{Y}_* B & \xrightarrow{J_B} & \mathcal{Y}_*^+ B
\end{array}
\]

Let \( y = [p] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes (a_0 \otimes \ldots \otimes a_p) \) be an \( n \)-chain of \( k[N(- \Delta S)] \otimes_{\Delta S} B^{sym} A \). Then \( J_A(y) \) has the same form in \( k[N(- \Delta S_+)] \otimes_{\Delta S_+} B^{sym+} A \). So,

\[
\mathcal{Y}_*^+ f (J_A(y)) = [p] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes (f(a_0) \otimes \ldots \otimes f(a_p)) = J_B (\mathcal{Y}_* f(y))
\]

Our goal now will be to show the following:

**Theorem 56.** For a unital, associative \( k \)-algebra \( A \), the chain map

\[
J_A : \mathcal{Y}_* A \rightarrow \mathcal{Y}_*^+ A
\]

induces an isomorphism on homology

\[
H_* (\mathcal{Y}_* A; k) \xrightarrow{\cong} H_* (\mathcal{Y}_*^+ A; k)
\]

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Lemma 57. For a unital, associative $k$-algebra $A$, the chain map

$$J_{TA} : \mathcal{Y}_*TA \longrightarrow \mathcal{Y}_+^*TA$$

induces an isomorphism in homology, hence

$$HS_*(TA) = H_*(\mathcal{Y}_+^*TA; k)$$

Proof.

$$HS_*(TA) = H_*(\mathcal{Y}_*TA; k), \quad \text{by definition.}$$

There is a commutative square of complexes:

$$\begin{array}{ccc}
\mathcal{Y}_*TA & \xrightarrow{J_{TA}} & \mathcal{Y}_+^*TA \\
\cong & & \cong \\
\bigoplus_{n \geq -1} Y_n & \xrightarrow{j_*} & \bigoplus_{n \geq -1} Y_n^+
\end{array}$$

The isomorphisms on the left and right follow from Lemmas 28 and 54. The map $j_*$ defined as follows. For $n = -1$,

$$j_* : k [N(\Delta S)] \rightarrow k [N(\Delta S_+)] \quad (2.3)$$

is induced by inclusion of categories $\Delta S \hookrightarrow \Delta S_+$. For $n \geq 0$,

$$j_* : k [N([n] \setminus \Delta S)] \otimes_{k \Sigma_{n+1}} A^{\otimes(n+1)} \longrightarrow k [N([n] \setminus \Delta S_+)] \otimes_{k \Sigma_{n+1}} A^{\otimes(n+1)} \quad (2.4)$$

is again induced by inclusion of categories. Observe that $N(\Delta S_+)$ is contractible, since $[-1] \in \text{Obj}(\Delta S_+)$ is initial. Thus by Lemma 20, the map $j_*$ of (2.3) is a homotopy equivalence. Now, for $n \geq 0$, there is equality $N([n] \setminus \Delta S_+) = N([n] \setminus \Delta S)$, since there are no morphisms $[n] \rightarrow [-1]$ for $n \geq 0$, so (2.4), and therefore $j_*$, is a homotopy equivalence. This implies that $J_{TA}$ is must also be a homotopy equivalence. □
Remark 58. Observe, this lemma provides our first major departure from the theory of cyclic homology. The proof above would not work over the categories $\Delta C$ and $\Delta C_+$, as $N(\Delta C)$ is not contractible.

Consider a double complex $T_{*,*}^+$, the analogue of complex (1.7) for $\Delta S_+$.

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
Y_2^+ TA \leftrightarrow Y_2^+ T^2 A \leftrightarrow Y_2^+ T^3 A \leftarrow \cdots \\
d_2 \\
Y_1^+ TA \leftrightarrow Y_1^+ T^2 A \leftrightarrow Y_1^+ T^3 A \leftarrow \cdots \\
d_1 \\
Y_0^+ TA \leftrightarrow Y_0^+ T^2 A \leftrightarrow Y_0^+ T^3 A \leftarrow \cdots \\
\end{array}
\]

(2.5)

The maps $\theta_1, \theta_2, \ldots$ are defined by formula (1.5).

Consider a second double complex, $A_{*,*}^+$, the analogue of complex (1.8) for $\Delta S_+$. It consists of the complex $Y_s^+ A$ as the $0^{th}$ column, and 0 in all positive columns.

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
Y_2^+ A \leftarrow 0 \leftarrow 0 \leftarrow \cdots \\
d_2 \\
Y_1^+ A \leftarrow 0 \leftarrow 0 \leftarrow \cdots \\
d_1 \\
Y_0^+ A \leftarrow 0 \leftarrow 0 \leftarrow \cdots \\
\end{array}
\]

(2.6)
We may think of each double complex construction as a functor:

\[ A \mapsto \mathcal{A}_{*,*}(A) \]
\[ A \mapsto \mathcal{T}_{*,*}(A) \]
\[ A \mapsto \mathcal{A}^+_{*,*}(A) \]
\[ A \mapsto \mathcal{T}^+_{*,*}(A) \]

Each functor takes unital morphisms of algebras to maps of double complexes in the obvious way – for example if \( f : A \to B \), then the induced map \( \mathcal{T}_{*,*}(A) \to \mathcal{T}_{*,*}(B) \) is defined on the \((p,q)\)-component by the map \( \mathcal{Y}_q T^{p+1} f \). The induced map commutes with vertical differentials of \( \mathcal{A}_{*,*} \) and \( \mathcal{T}_{*,*} \) (resp., \( \mathcal{A}^+_{*,*} \) and \( \mathcal{T}^+_{*,*} \)) by naturality of \( \mathcal{Y}_* \) (resp. \( \mathcal{Y}^+_* \)), and it commutes with the horizontal differentials of \( \mathcal{T}_{*,*} \) and \( \mathcal{T}^+_{*,*} \) by naturality of \( \theta_n \) (see Prop. 23).

The map \( J \) induces a natural transformation \( J_{*,*} : \mathcal{A}_{*,*} \to \mathcal{A}^+_{*,*} \), defined by

\[
J_{p,*}(A) = \begin{cases} 
J_A : \mathcal{Y}_* A \to \mathcal{Y}^+_* A, & p = 0 \\
0, & p > 0 
\end{cases}
\]

Define a map of bigraded modules, \( K_{*,*}(A) : \mathcal{T}_{*,*}(A) \to \mathcal{T}^+_{*,*}(A) \) by:

\[
K_{p,*}(A) = J_{T^{p+1} A} : \mathcal{Y}_* T^{p+1} A \to \mathcal{Y}^+_* T^{p+1} A
\]

Now, \( K_{*,*}(A) \) commutes with the vertical differentials because each \( J_{T^{p+1} A} \) is a chain map. \( K_{*,*}(A) \) commutes with the horizontal differentials because of naturality of \( J \). Finally, \( K_{*,*} \) defines a natural transformation \( \mathcal{T}_{*,*} \to \mathcal{T}^+_{*,*} \), again by naturality of \( J \).

Recall by Thm 25, there is a map of double complexes,

\[
\Theta_{*,*}(A) : \mathcal{T}_{*,*}(A) \to \mathcal{A}_{*,*}(A)
\]
inducing an isomorphism in homology of the total complexes. Observe that $\Theta_{*,*}$ provides a natural transformation $\mathcal{T}_{*,*} \to \mathcal{A}_{*,*}$, since $\Theta_{*,*}(A)$ is defined in terms of $\theta_A$, which is natural in $A$. We shall need the analogous statement for the double complexes $\mathcal{T}^+_{*,*}$ and $\mathcal{A}^+_{*,*}$.

**Theorem 59.** For any unital associative algebra, $A$, there is a map of double complexes, $\Theta^+_{*,*}(A) : \mathcal{T}^+_{*,*}(A) \to \mathcal{A}^+_{*,*}(A)$ inducing isomorphism in homology

$$H_* (\operatorname{Tot}(\mathcal{T}^+(A)); k) \to H_* (\operatorname{Tot}(\mathcal{A}^+(A)); k)$$

Moreover, $\Theta^+_{*,*}$ provides a natural transformation $\mathcal{T}^+_{*,*} \to \mathcal{A}^+_{*,*}$.

**Proof.** The map $\Theta^+_{*,*}(A)$ is defined as:

$$\Theta^+_{p,q}(A) := \begin{cases} \mathcal{V}^+_q \theta_A, & p = 0 \\ 0, & p > 0 \end{cases}$$

This map is a map of double complexes by functoriality of $\mathcal{V}^+_q$, and the isomorphism follows from the exactness of the sequence (1.6). Naturality of $\Theta_{*,*}$ follows from naturality of $\theta$. \(\square\)

**Lemma 60.** The following diagram of functors and transformations is commutative.

$$\begin{array}{ccc} \mathcal{T}_{*,*} & \xrightarrow{\Theta_{*,*}} & \mathcal{A}_{*,*} \\ \downarrow K_{*,*} & & \downarrow J_{*,*} \\ \mathcal{T}^+_{*,*} & \xrightarrow{\Theta^+_{*,*}} & \mathcal{A}^+_{*,*} \end{array} \quad (2.7)$$

**Proof.** It suffices to fix an algebra $A$ and examine only the $(p,q)$-components.

$$\begin{array}{ccc} \mathcal{T}_{p,q}(A) & \xrightarrow{\Theta_{p,q}(A)} & \mathcal{A}_{p,q}(A) \\ \downarrow K_{p,q}(A) & & \downarrow J_{p,q}(A) \\ \mathcal{T}^+_{p,q}(A) & \xrightarrow{\Theta^+_{p,q}(A)} & \mathcal{A}^+_{p,q}(A) \end{array} \quad (2.8)$$
If \( p > 0 \), then the right hand side of (2.8) is trivial, so we may assume \( p = 0 \). In this case, diagram (2.8) becomes:

\[
\begin{array}{ccc}
\mathcal{Y}_q T A & \xrightarrow{\mathcal{Y}_q \theta A} & \mathcal{Y}_q A \\
\downarrow & & \downarrow \\
\mathcal{Y}_q^+ T A & \xrightarrow{\mathcal{Y}_q^+ \theta A} & \mathcal{Y}_q^+ A
\end{array}
\]  

(2.9)

This diagram commutes because of naturality of \( J \).

To any double complex \( \mathcal{B}_{s,*} \) over \( k \), we may associate two spectral sequences: \( (E_1, \mathcal{B})_{s,*} \), obtained by first taking vertical homology, then horizontal; and \( (E_{II}, \mathcal{B})_{s,*} \), obtained by first taking horizontal homology, then vertical. In the case that \( \mathcal{B}_{s,*} \) lies entirely within the first quadrant, both spectral sequences converge to \( H_* (\text{Tot}(\mathcal{B}); k) \) (See [23], Section 2.4). Maps of double complexes induce maps of spectral sequences, \( E_I \) and \( E_{II} \), respectively.

Fix the algebra \( A \), and consider the commutative diagram of spectral sequences induced by diagram (2.7). The induced maps will be indicated by an overline, and explicit mention of the algebra \( A \) is suppressed for brevity of notation.

\[
\begin{array}{ccc}
E_{II} \mathcal{T} & \xrightarrow{\overline{\mathcal{T}}} & E_{II} \mathcal{A} \\
\downarrow & & \downarrow \\
E_{II} \mathcal{T}^+ & \xrightarrow{\Theta^+} & E_{II} \mathcal{A}^+
\end{array}
\]  

(2.10)

Now, by Thm. 25 and Thm. 59, we know that \( \Theta_{s,*} \) and \( \Theta_{s,*}^+ \) induce isomorphisms on total homology, so \( \overline{\Theta} \) and \( \overline{\Theta}^+ \) also induce isomorphisms on the limit term of the spectral sequences.

In fact, both \( \overline{\Theta}^r \) and \( \overline{\Theta}^{r^\prime} \) are isomorphisms \( (E_{II} \mathcal{T})^r \to (E_{II} \mathcal{A})^r \) for \( r \geq 1 \). This is because taking horizontal homology of \( \mathcal{T}_{s,*} \) (resp. \( \mathcal{T}_{s,*}^+ \)) kills all components in positive columns, leaving only the 0\(^{th}\) column, which is chain-isomorphic to the 0\(^{th}\) column of \( \mathcal{A}_{s,*} \) (resp. \( \mathcal{A}_{s,*}^+ \)). On the other hand, taking horizontal homology of \( \mathcal{A}_{s,*} \) (resp. \( \mathcal{A}_{s,*}^+ \)) does not change that double complex.
Consider a second diagram of spectral sequences, with induced maps indicated by a hat.

\[
\begin{array}{c}
E_1T \xrightarrow{\hat{\theta}} E_1A \\
\downarrow \hat{K} \quad \downarrow \hat{j} \\
E_1T^+ \xrightarrow{\hat{\Theta}^+} E_1A^+ \\
\end{array}
\]

(2.11)

Now the map \( \hat{K} \) induces an isomorphism on the limit terms of the sequences \( E_1T \) and \( E_1T^+ \) as a result of Lemma 57. As before, \( \hat{K}^r \) is an isomorphism for \( r \geq 1 \).

Now, since \( H_* (\text{Tot}(A); k) = H_* (\mathcal{Y}_* A; k) \) and \( H_* (\text{Tot}(A^+); k) = H_* (\mathcal{Y}_*^+ A; k) \), we can put together a chain of isomorphisms

\[
H_* (\mathcal{Y}_* A; k) \cong (E_{11}\mathcal{A})_*^\infty \xleftarrow{\cong} (E_{11}\mathcal{T})_*^\infty \cong (E_1T)_*^\infty \xrightarrow{\hat{K}_*^\infty \cong} (E_1T^+)_*^\infty \\
\cong (E_{11}\mathcal{A}^+)_*^\infty \xrightarrow{\cong} (E_{11}\mathcal{T}^+)_*^\infty \cong H_* (\mathcal{Y}_*^+ A; k)
\]

(2.12)

Commutativity of Diagram (2.7) ensures that the composition of maps in Diagram 2.12 is the map induced by \( J_A \), hence proving Thm. 56.

As a direct consequence, \( HS_* (A) \cong H_* (\mathcal{Y}_*^+ A; k) \), proving Thm. 53.

### 2.2 The Category \( \text{Epi}\Delta S \) and a Smaller Resolution

The complex (1.2) is an extremely large and unwieldy for computation. Fortunately, when the algebra \( A \) is equipped with an augmentation, \( \epsilon : A \to k \), (1.2) is homotopic to a much smaller subcomplex. Let \( I \) be the augmentation ideal, and let \( \eta : k \to A \) be the unit map.

Since \( \epsilon \eta = \text{id}_k \), the following short exact sequence splits (as \( k \)-modules):

\[
0 \to I \to A \xrightarrow{\epsilon} k \to 0,
\]

and every \( x \in A \) can be written uniquely as \( x = a + \eta(\lambda) \) for some \( a \in I \), \( \lambda \in k \). This property will allow \( B_{n}^{sym} A \) to be decomposed in a useful way.
Definition 61. Suppose $J \subseteq [n]$. Define

$$B_{n,J}A := B_0 \otimes B_1 \otimes \ldots \otimes B_n,$$

where $B_j = \begin{cases} I & \text{if } j \in J \\ \eta(k) & \text{if } j \notin J \end{cases}$.

Define $B_{-1,\emptyset}A = k$.

Lemma 62. For each $n \geq -1$, there is a direct sum decomposition of $k$-modules

$$B_{n}^{\text{sym}}A \cong \bigoplus_{J \subseteq [n]} B_{n,J}A.$$

Proof. The splitting of the unit map $\eta$ implies that $A \cong \eta(k) \oplus I$ as $k$-modules. So, for $n \geq 0$,

$$B_{n}^{\text{sym}}A = (\eta(k) \oplus I)^{\otimes (n+1)} \cong \bigoplus_{J \subseteq [n]} B_{n,J}A.$$

For $n = -1$, $B_{-1}^{\text{sym}}A = k = B_{n,\emptyset}A$. (Recall, $[-1] = \emptyset$).

Definition 63. A basic tensor is any tensor $w_0 \otimes w_1 \otimes \ldots \otimes w_n$, where each $w_j$ is in $I$ or is equal to the unit of $A$. Call a tensor factor $w_j$ trivial if it is the unit of $A$; otherwise, the factor is called non-trivial. If all factors of a basic tensor are trivial, then the tensor is called trivial, otherwise non-trivial.

It will become convenient to include the object $[-1]$ in $\Delta$. Let $\Delta_+$ be the category with objects $[-1], [0], [1], [2], \ldots$, and morphisms are all those of $\Delta$ together with $\iota_n : [-1] \to [n]$ for $n \geq -1$. $\Delta_+$ may be thought of as the subcategory of $\Delta S_+$ consisting of all non-decreasing set maps.

For a basic tensor $Y \in B_{n}^{\text{sym}}A$, we shall define a map $\delta_Y \in \text{Mor} \Delta_+$ as follows: If $Y$ is trivial, let $\delta_Y = \iota_n$. Otherwise, $Y$ has $\overline{\pi} + 1$ non-trivial factors for some $\overline{\pi} \geq 0$. Define $\delta_Y : [\overline{\pi}] \to [n]$ to be the unique injective map that sends each point $0, 1, \ldots, \overline{\pi}$ to a point $p \in [n]$ such that $Y$ is non-trivial at factor $p$. Let $\overline{Y}$ be the tensor obtained from $Y$ by omitting all trivial factors if such exist, or $\overline{Y} := 1$ if $Y$ is trivial. Note, $\overline{Y}$ is the unique basic tensor such that $(\delta_Y)_*(\overline{Y}) = Y$. 

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Proposition 64. Any chain $[q] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y \in k[N(\Delta S_{+})] \otimes_{\Delta S_{+}} B_{*}^{sym,+} A$, where $Y$ is a basic tensor, is equivalent to a chain $[\overline{q}] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes \overline{Y}$, where either $\overline{Y}$ has no trivial factors or $\overline{Y} = 1$ and $\overline{q} = -1$.

Proof. Let $\delta_Y$ and $\overline{Y}$ be defined as above, and let $[\overline{q}]$ be the source of $\delta_Y$.

$$[q] \overset{\phi}{\rightarrow} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y = [q] \overset{\phi}{\rightarrow} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes (\delta_Y)_*(\overline{Y})$$

$$\approx [\overline{q}] \overset{\phi_{\overline{Y}}}{\rightarrow} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes \overline{Y}$$

Next, we turn our attention to the morphisms in the chain. Our goal is to reduce to those chains that involve only epimorphisms.

Definition 65. Let $\mathcal{C}$ be a category. The category $\text{Epi}\mathcal{C}$ (resp. $\text{Mono}\mathcal{C}$) is the subcategory of $\mathcal{C}$ consisting of the same objects as $\mathcal{C}$ and only those morphisms $f \in \text{Mor}\mathcal{C}$ that are epimorphisms (resp. monomorphisms).

The morphisms of $\text{Epi}\mathcal{C}$ from $X$ to $Y$ may be denoted $\text{Epi}_\phi(X,Y)$. Similarly, the morphisms of $\text{Mono}\mathcal{C}$ from $X$ to $Y$ may be denoted $\text{Mono}_\phi(X,Y)$.

Note, a morphism $\alpha = (\phi, g) \in \text{Mor}\Delta S_{+}$ is epic, resp. monic, if and only if $\phi$ is epic, resp. monic, as morphism in $\Delta_{+}$.

Proposition 66. Any morphism $\alpha \in \text{Mor}\Delta S_{+}$ decomposes uniquely as $(\eta, \text{id}) \circ \gamma$, where $\gamma \in \text{Mor}(\text{Epi}\Delta S_{+})$ and $\eta \in \text{Mor}(\text{Mono}\Delta_{+})$.

Proof. Suppose $\alpha$ has source $[-1]$ and target $[n]$. Then $\alpha = \iota_n$ is the only possibility, and this decomposes as $\iota_n \circ \text{id}_{[-1]}$. Now suppose the source of $\alpha$ is $[p]$ for some $p \geq 0$. Write $\alpha = (\phi, g)$, with $\phi \in \text{Mor}\Delta$ and $g \in \Sigma_{p+1}^{op}$. We shall decompose $\phi$ as follows: For $\phi : [p] \rightarrow [r]$, suppose $\phi$ hits $q + 1$ distinct points in $[r]$. Then $\pi : [p] \rightarrow [q]$ is induced by $\phi$ by maintaining the order of the points hit. $\eta$ is the obvious order-preserving monomorphism $[q] \rightarrow [r]$ so that $\eta \pi = \phi$ as morphisms in $\Delta$. To get the required decomposition in $\Delta S$, use: $\alpha = (\eta, \text{id}) \circ (\pi, g)$. 

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Now, if $(\xi, \text{id}) \circ (\psi, h)$ is also a decomposition, with $\xi$ monic and $\psi$ epic, then

$$(\xi, \text{id}) \circ (\psi, h) = (\eta, \text{id}) \circ (\pi, g)$$

$$(\xi, \text{id}) \circ (\psi, g^{-1}h) = (\eta, \text{id}) \circ (\phi, \text{id})$$

$$(\xi \psi, g^{-1}h) = (\eta \phi, \text{id}),$$

proving $g = h$. Uniqueness will then follow from uniqueness of such decompositions entirely within the category $\Delta$. The latter follows from Theorem B.2 of [15], since any monomorphism (resp. epimorphism) of $\Delta$ can be built up (uniquely) as compositions of $\delta_i$ (resp. $\sigma_i$).

Explicitly, if $\alpha = X_0 \otimes X_1 \otimes \ldots \otimes X_m : [n] \to [m]$, with $X_i \neq 1$ for $i = j_0, j_1, \ldots j_k$, then $\text{im}(\alpha)$ is isomorphic to the object $[k]$. The surjection onto $[k]$ is $X_{j_0} \otimes \ldots \otimes X_{j_k}$. The $\Delta$ injection $[k] \hookrightarrow [m]$ is $Z_0 \otimes \ldots \otimes Z_m$, where $Z_i = 1$ if $i \neq j_0, j_1, \ldots j_k$ and for $i = j_0, \ldots, j_k$, the monomials $Z_i$ are the symbols $x_0, x_1, \ldots, x_k$, in that order. For example,

$$(x_2 x_3 \otimes 1 \otimes x_1 \otimes 1 \otimes x_0 = x_0 \otimes 1 \otimes x_1 \otimes 1 \otimes x_2 \circ x_2 x_3 \otimes x_1 \otimes x_0)$$

When morphisms are not labelled, we shall write:

$$[p] \to [r] = [p] \mapsto \text{im}([p] \to [r]) \hookrightarrow [r].$$

For any $p \geq -1$, if $[p] \xrightarrow{\beta} [r_1] \xrightarrow{\alpha} [r_2]$, then there is an induced map

$$\text{im}([p] \to [r_1]) \xrightarrow{\pi} \text{im}([p] \to [r_2])$$

making the diagram commute:

$$\begin{equation}
\begin{array}{ccc}
[r_1] & \xrightarrow{\alpha} & [r_2] \\
\downarrow{\eta_1} & \nearrow{\beta} & \searrow{\alpha \beta} \\
[p] & \nearrow{\pi_1} & \downarrow{\pi_2} \\
\downarrow{\pi} & \nearrow{\pi_\alpha} & \downarrow{\pi_\beta} \\
\text{im}([p] \to [r_1]) & \xrightarrow{\pi} & \text{im}([p] \to [r_2]) \\
\end{array}
\end{equation}$$

(2.13)
\( \pi \) is the epimorphism induced from the map \( \alpha \eta_1 \). Furthermore, for morphisms

\[
[p] \to [r_1] \overset{\alpha_1}{\rightarrow} [r_2] \overset{\alpha_2}{\rightarrow} [r_3],
\]

we have:

\[
\alpha_2 \alpha_1 = \alpha_2 \circ \alpha_1,
\]

i.e., the epimorphism construction is a functor \(([p] \setminus \Delta S_+) \to ([p] \setminus \text{Epi} \Delta S_+)\).

Define a variant of the symmetric bar construction:

**Definition 67.** \( B^{sym+}_\ast I : \text{Epi} \Delta S_+ \to \text{k-Mod} \) is the functor defined by:

\[
\begin{cases}
B^{sym+}_n I := I^\otimes(n+1), & n \geq 0, \\
B^{sym+}_{-1} I := k,
\end{cases}
\]

\( B^{sym+}_\ast I(\alpha) : (a_0 \otimes a_1 \otimes \ldots \otimes a_n) \mapsto \alpha(a_0, \ldots, a_n), \) for \( \alpha \in \text{Mor}(\text{Epi} \Delta S_+) \)

This definition makes sense, since the only epimorphism with source \([-1]\) is \( i_{-1} = \text{id}_{[-1]} \), sending \( B^{sym+}_{-1} I = k \) identically to itself. Since \( \alpha \in \text{Epi} \Delta S_+([p],[q]) \) for \( p \geq 0 \), there is no need for a unit element in \( I \). Furthermore, any product of elements of the ideal \( I \) must also be in \( I \).

Consider the simplicial \( k \)-module:

\[
\mathcal{Y}^{epi}_\ast A := k[N(- \setminus \text{Epi} \Delta S_+)] \otimes_{\text{Epi} \Delta S_+} B^{sym+}_\ast I
\]

There is an obvious inclusion,

\( f : \mathcal{Y}^{epi}_\ast A \to \mathcal{Y}^+\ast A \)

Define a chain map, \( g \), in the opposite direction as follows. First, by prop. 64 and observations above, we only need to define \( g \) on the chains \([q] \to [q_0] \to \ldots \to [q_n] \otimes Y \) where \( Y \in B^{sym+}_\ast I \) already. In this case, define:

\( g([q] \to [q_0] \to \ldots \to [q_n] \otimes Y) \)
\[
\begin{aligned}
&= \begin{cases}
[-1] \to [-1] \to \ldots \to [-1] \otimes 1, & \text{Y trivial}, \\
[q] \to \im([q] \to [q_0]) \to \im([q] \to [q_1]) \to \ldots \to \im([q] \to [q_n]) \otimes Y, & \text{Y non-trivial}
\end{cases}
\end{aligned}
\]

I claim \(g\) is well-defined. Indeed, if \(Y \in B^\text{sym} q I\) is trivial, then \(q\) must be \(-1\). If \(\psi : [-1] \to [q']\) is any morphism of \(\Delta S^+_q\), then we know \(\psi = \iota_{q'}\), and \((\iota_{q'})_*(Y)\) is still a trivial tensor. We have equivalent chains:

\[[−1] → [q_0] → ... → [q_n] ⊗ 1 ≈ [q'] → [q_0] → ... → [q_n] ⊗ 1^\otimes(q'+1)\]

Applying \(g\) to the chain on the left results in a chain of identity maps,

\[[−1] → [−1] → ... → [−1] ⊗ Y.\]

In order to apply \(g\) to the chain on the right, it must be put into the correct form. Since \(1^\otimes(q'+1)\) is trivial, we must use \(\delta = \iota_{q'}\) to rewrite the chain. But what results is exactly the chain on the left, so \(g\) is well-defined in this case.

Suppose now that \(q ≥ 0\) and \(Y \in B^\text{sym} q I\). Let \(\psi : [q] → [q']\) be any morphism of \(\Delta S^+_q\). Since \(q ≥ 0\), \(\psi ∈ \text{Mor} \Delta S\). We have equivalent chains:

\[[q] \overset{φψ}{\to} [q_0] \overset{α_1}{\to} ... \overset{α_n}{\to} [q_n] ⊗ Y \approx [q'] \overset{φ}{\to} [q_0] \overset{α_1}{\to} ... \overset{α_n}{\to} [q_n] ⊗ ψ_*(Y).\]

Applying \(g\) on the left hand side yields

\[[q] \overset{φψ}{\to} \im(φψ) \to ... \to \im([q] → [q_n]) ⊗ Y, \quad (2.15)\]

Consider the chain on the right hand. If \(ψ\) happens to be an epimorphism, then \(ψ_*(Y) ∈ B^\text{sym} q I\), and we may apply \(g\) directly to this chain. In this case, we get:

\[[q'] \overset{φ}{\to} \im(φ) \to ... \to \im([q'] → [q_n]) ⊗ ψ_*(Y) \quad (2.16)\]

Now, since \(ψ\) is epic, \(\im(φψ) = \im(φ)\). Moreover, \(\im(α_k ... α_1 φ ψ) = \im(α_k ... α_1 φ)\) for each \(k = 1, 2, ... , n\), and the induced morphisms are equal:

\((\im([q'] → [q_k]) → \im([q'] → [q_{k+1}])) = (\im([q] → [q_k]) → \im([q] → [q_{k+1}]))\)
Hence, the chain (2.16) is equal to:

\[ q' \xrightarrow{\phi} \text{im}(q) \xrightarrow{[q_0]} \ldots \xrightarrow{[q_n]} \text{im}(q) \xrightarrow{\psi} Y \]

\[
\approx [q] \xrightarrow{\phi \circ \psi} \text{im}(q) \xrightarrow{[q_0]} \ldots \xrightarrow{[q_n]} \text{im}(q) \xrightarrow{\psi} Y.
\]

Thus, since \( \psi = \overline{\psi} = \phi \circ \overline{\psi} \), the chains (2.15) and (2.16) are equivalent.

Suppose now that \( \psi \) is not epimorphic. Use Prop. 66 to write \( \psi = \pi \eta \) for \( \pi \in \text{Epi}\Delta S_+ \) and \( \eta \in \text{Mono}\Delta_+ \). By the previous, it is clear that we may assume \( \pi = \text{id} \), so that \( \psi \) is a monomorphism of \( \Delta_+ \). In this case, \( \psi_*(Y) \) may have trivial tensor factors. Now, \( g \) is only defined for chains in which the factor \( Y \in B_{n}^{\text{sym} + A} \) is a basic tensor having no trivial factors, so we must use Prop. 64 to rewrite the chain as:

\[
[q] \xrightarrow{\phi \circ \psi} [q_0] \xrightarrow{\ldots} [q_n] \otimes \psi_*(Y).
\]

Since \( Y \) is in \( B_{q}^{\text{sym} + I} \) and \( \psi \) is a monomorphism of \( \Delta \), we have \( \overline{\psi}_*(Y) = Y \), and \( \delta_{\psi_*(Y)} = \psi \), by uniqueness of the decomposition. Thus, when we apply \( g \) to this chain, we must apply it to the equivalent chain:

\[
[q] \xrightarrow{\phi \circ \psi} [q_0] \xrightarrow{\ldots} [q_n] \otimes Y.
\]

This shows that \( g \) is well-defined.

Now, \( gf = \text{id} \), since if \( [p] \rightarrow [r] \) is in \( \text{Mor}(\text{Epi}\Delta S) \), then the epimorphism construction \( [p] \rightarrow \text{im}(r) \rightarrow [r] \) is just the original morphism.

**Proposition 68.** \( fg \simeq \text{id} \).

**Proof.** In what follows, we assume \( Y \) is a basic tensor in \( B_{q}^{\text{sym} + I} \). Define a presimplicial homotopy \( h \) from \( fg \) to \( \text{id} \) as follows:

\[
h_{j}^{(n)}([q] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y) := [q] \xrightarrow{\phi \circ \psi} [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y.
\]

\( h_{j} \) is well-defined by the functorial properties of the epimorphism construction.
Suppose $0 \leq i < j \leq n$. We have $d_i h_j = h_{j-1} d_i$, since $d_i$ on the right hand side reduces the number of nodes to the left of $[q_j]$ by one. We also use the functoriality of the epimorphism construction here.

Suppose $1 \leq j + 1 < i \leq n + 1$. $d_i h_j = h_{j-1} d_i$, since $h_j$ on the left hand side shifts all nodes to the right of (and including) $[q_j]$ to the right by one.

Suppose $0 < i \leq n$. First apply $d_i h_i$ to an arbitrary chain.

$$
\begin{array}{l}
[q] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y \\
\downarrow \quad h_i \\
[q] \rightarrow \text{im}([q] \rightarrow [q_0]) \rightarrow \ldots \rightarrow \text{im}([q] \rightarrow [q_{i-1}]) \rightarrow \text{im}([q] \rightarrow [q_i]) \hookrightarrow [q_i] \rightarrow \ldots \rightarrow [q_n] \otimes Y \\
\downarrow \quad d_i \\
[q] \rightarrow \text{im}([q] \rightarrow [q_0]) \rightarrow \ldots \rightarrow \text{im}([q] \rightarrow [q_{i-1}]) \rightarrow [q_i] \rightarrow \ldots \rightarrow [q_n] \otimes Y
\end{array}
$$

Apply $d_i h_{i-1}$ to the same chain.

$$
\begin{array}{l}
[q] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y \\
\downarrow \quad h_{i-1} \\
[q] \rightarrow \text{im}([q] \rightarrow [q_0]) \rightarrow \ldots \rightarrow \text{im}([q] \rightarrow [q_{i-1}]) \hookrightarrow [q_i] \rightarrow \ldots \rightarrow [q_n] \otimes Y \\
\downarrow \quad d_i \\
[q] \rightarrow \text{im}([q] \rightarrow [q_0]) \rightarrow \ldots \rightarrow \text{im}([q] \rightarrow [q_{i-1}]) \rightarrow [q_i] \rightarrow \ldots \rightarrow [q_n] \otimes Y
\end{array}
$$

The fact that the composition of $\text{im}([q] \rightarrow [q_{i-1}]) \rightarrow [q_{i-1}] \rightarrow [q_i]$ is equal to the composition of $\text{im}([q] \rightarrow [q_{i-1}]) \rightarrow \text{im}([q] \rightarrow [q_i]) \rightarrow [q_i]$ follows from the commutativity of the outside square of diagram 2.13. Thus, $d_i h_i = d_i h_{i-1}$.

Finally,

$$d_0 h_0([q] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y) = d_0([q] \rightarrow \text{im}([q] \rightarrow [q_0]) \hookrightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y) = [q] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y,$$

and

$$d_{n+1} h_n([q] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y)$$
\[ d_{n+1}([q] \rightarrow \text{im}([q] \rightarrow [q_0]) \rightarrow \ldots \rightarrow \text{im}([q] \rightarrow [q_n]) \leftarrow [q_n] \otimes Y) \]
\[ = [q] \rightarrow \text{im}([q] \rightarrow [q_0]) \rightarrow \ldots \rightarrow \text{im}([q] \rightarrow [q_n]) \otimes Y \]
\[ = g([q] \rightarrow [q_0] \rightarrow \ldots \rightarrow [q_n] \otimes Y) \]

Hence, id \simeq fg, as required. \qed

**Proposition 69.** If A has augmentation ideal I, then

\[ HS_*(A) = H_* \left( \mathcal{Y}_*^{\text{epi}} A; k \right) = H_* \left( k[N(- \setminus \text{Epi\Delta S}_+)] \otimes_{\text{Epi\Delta S}_+} B_*^{\text{sym}_+} I; k \right). \]

**Proof.** The complex (2.14) has been shown to be chain homotopy equivalent to the complex \( \mathcal{Y}_*^{\text{epi}} A \), which by Thm. 53, computes \( HS_*(A) \). \qed

**Remark 70.** The condition that A have an augmentation ideal may be lifted (as Richter conjectures), if it can be shown that \( N(\text{Epi\Delta S}) \) is contractible. As partial progress along these lines, it can be shown that \( N(\text{Epi\Delta S}) \) is simply-connected.
CHAPTER 3
A SPECTRAL SEQUENCE FOR $HS_*(A)$

3.1 Filtering by Number of Strict Epimorphisms

In this chapter, fix a unital associative algebra $A$ over commutative ground ring $k$. We also assume $A$ comes equipped with an augmentation, and denote the augmentation ideal by $I$. Let $\mathcal{Y}^{\text{epi}*}_A$ be the complex (2.14). Since $A$ is fixed, it will suffice to use the notation $\mathcal{Y}^{\text{epi}*}_A$ in place of $\mathcal{Y}^{\text{epi}}_A$. As we have seen above in Section 2.2, $H_*(\mathcal{Y}^{\text{epi}*}_A) = HS_*(A)$.

Consider a filtration of $\mathcal{Y}^{\text{epi}*}_A$ by number of strict epimorphisms, or jumps:

$$\mathcal{F}_p \mathcal{Y}^{\text{epi}*}_q \quad \text{is generated by}$$

$$\{[m_0] \to [m_1] \to \ldots \to [m_q] \otimes Y, \text{where } m_{i-1} > m_i \text{ for no more than } p \text{ distinct values of } i \}.$$

The face maps of $\mathcal{Y}^{\text{epi}}_A$ only delete morphisms or compose morphisms, so this filtration is compatible with the differential of $\mathcal{Y}^{\text{epi}}_A$. The filtration quotients are easily described:

$$E^0_{p,q} := \mathcal{F}_p \mathcal{Y}^{\text{epi}*}_q / \mathcal{F}_{p-1} \mathcal{Y}^{\text{epi}*}_q \quad \text{is generated by}$$

$$\{[m_0] \to [m_1] \to \ldots \to [m_q] \otimes Y, \text{where } m_{i-1} > m_i \text{ for exactly } p \text{ distinct values of } i \}.$$

The induced differential on $E^0_{p,q}$ is of bidegree $(0, -1)$, so we may form a spectral sequence with $E^1_{p,q} = H_{p+q}(E^0_{p,*})$ (cf. [23], [31]).

**Lemma 71.** There are chain maps (one for each $p$):

$$E^0_{p,*} \to \bigoplus_{m_0 > \ldots > m_p} \left(I^{\otimes (m_0+1)} \otimes k \left[ \prod_{i=1}^p \text{Epi}_{\Delta^+}([m_{i-1}], [m_i]) \right] \otimes k_{\Sigma_{m+p+1}} E_* \Sigma_{m+p+1} \right),$$

inducing isomorphisms in homology:

$$E^1_{p,q} \cong \bigoplus_{m_0 > \ldots > m_p} H_q \left( \Sigma^{op}_{m+p+1} I^{\otimes (m_0+1)} \otimes k \left[ \prod_{i=1}^p \text{Epi}_{\Delta^+}([m_{i-1}], [m_i]) \right] \right).$$

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We use the convention that $I^\otimes 0 = k$, and $\Sigma_0 \cong 1$, the trivial group. Note, we avoid the notation $B_m^{sym+} I$ here so as not to confuse the $k$-module structure of $I^\otimes (m+1)$ with any irrelevant additional structure.

We will begin by defining two related chain complexes:

Denote by $B(m_0, ..., m_p)$, the chain complex:

$$\bigoplus \left( [m_0] \cong \cdots \cong [m_0] \smallsetminus [m_1] \cong \cdots \cong [m_{p-1}] \smallsetminus [m_p] \otimes I^\otimes (m_0+1) \right),$$

where the sum extends over all such chains that begin with 0 or more isomorphisms of $[m_0]$, followed by a strict epimorphism $[m_0] \twoheadrightarrow [m_1]$, followed by 0 or more isomorphisms of $[m_1]$, followed by a strict epimorphism $[m_1] \twoheadrightarrow [m_2]$, etc., and the last morphism must be a strict epimorphism $[m_{p-1}] \twoheadrightarrow [m_p]$. $B(m_0, ..., m_p)$ is a subcomplex of $E^{0}_{p,*}$ with the same induced differential, and there is a $\Sigma_{m_{p+1}}$-action given by postcomposition of $g$, regarded as an automorphism of $[m_p]$.

Denote by $M(m_0, ..., m_p)$, the chain complex consisting of 0 in degrees different from $p$, and

$$M_p^{(m_0,...,m_p)} := I^\otimes (m_0+1) \otimes k \left[ \prod_{i=1}^{p} \text{Epi}_{\Delta_{i+}}([m_{i-1}], [m_i]) \right],$$

the coefficient group that shows up in Lemma 71. This complex has trivial differential.

Now, $B_p^{(m_0,...,m_p)}$ is generated by elements of the form

$$[m_0] \smallsetminus [m_1] \smallsetminus \cdots \smallsetminus [m_p] \otimes Y,$$

where the chain consists entirely of strict epimorphisms of $\Delta S_+$. Observe that

$$B_p^{(m_0,...,m_p)} = k[\text{Epi}_{\Delta S_+}([m_0],[m_1])] \otimes \cdots \otimes k[\text{Epi}_{\Delta S_+}([m_{p-1}],[m_p])] \otimes I^\otimes (m_0+1)$$

$$\cong I^\otimes (m_0+1) \otimes k[\text{Epi}_{\Delta S_+}([m_0],[m_1])] \otimes \cdots \otimes k[\text{Epi}_{\Delta S_+}([m_{p-1}],[m_p])]$$

as $k$-module. Now, each $k[\text{Epi}_{\Delta S_+}([m],[n])]$ is a $(k\Sigma_{n+1})$-$(k\Sigma_{m+1})$-bimodule. View $\sigma \in \Sigma_{n+1} = \text{Aut}_{\Delta S_+}([n])$ and $\tau \in \Sigma_{m+1} = \text{Aut}_{\Delta S_+}([m])$ as automorphisms. Then the action of $\sigma$, resp., $\tau$, is by postcomposition, resp., precomposition. The bimodule structure, $$(\sigma, \phi) \tau = \phi \tau \sigma \phi^{-1},$$
σ.(ϕ.τ), follows easily from associativity of composition in ΔS+, (σ ∘ ϕ) ∘ τ = σ ∘ (ϕ ∘ τ). We shall use the equivalent interpretation of k[EpiΔS+(m, [n])] as (kΣ_{m+1})-(kΣ_{n+1})-bimodule.

Explicitly, an element of EpiΔS+(m, [n]) is a pair (ψ, g), with ψ ∈ EpiΔ+(m, [n]) and g ∈ Σ_{m+1}^{op}, so for τ ∈ Σ_{m+1} and σ ∈ Σ_{n+1},

\[(ψ, g).σ = (id, σ) ∘ (ψ, g) = (ψ^σ, σ^ϕ \circ g) = (ψ^σ, gσ^ψ),\]
\[τ.(ψ, g) = (ψ, g) ∘ (id, τ) = (ψ, g ∘ τ) = (ψ, τg).\]

Also, since B_{sym}^{+} I is a ΔS_{+}-module, we may view it as a right ΔS_{+}^{op}-module, hence B_{m_0}^{sym} I = I^{⊗(m_0+1)} is a right (kΣ_{m_0+1})-module.

With this in mind, (3.1) becomes a (kΣ_{m_p+1})-module, where the action is the right action of kΣ_{m_p+1} on the last tensor factor by postcomposition, and the isomorphism given above respects this action.

Consider the k-module:

\[M := I^{⊗(m_0+1)} \otimes_{kG_0} k[EpiΔS_+(m_0, [m_1])] \otimes_{kG_1} \ldots \otimes_{kG_{p-1}} k[EpiΔS_+(m_{p-1}, [m_p])]\]

(3.2)

where G_i is the group Σ_{m_i+1}. I claim that M is isomorphic to \(\mathcal{M}_p^{(m_0, ..., m_p)}\) as k-module. Indeed, any element

\[Y \otimes (ψ_1, g_1) \otimes \ldots \otimes (ψ_p, g_p)\]

in M is equivalent to one in which all g_i are identities by writing (ψ_p, g_p) = g_p.(ψ_p, id) then commuting g_p over the tensor to the left and iterating this process to the leftmost tensor factor. Thus, we may write the element uniquely as

\[Z \otimes φ_1 \otimes \ldots \otimes φ_p,\]

where all tensors are now over k, and all morphisms are in EpiΔS_+.

This isomorphism also allows us to view \(\mathcal{M}_p^{(m_0, ..., m_p)}\) as a (Σ_{m_p+1}^{op})-module. The action is defined as the right action of Σ_{m_p+1} on the tensor factor k[EpiΔS_+(m_{p-1}, [m_p])]. We then use the isomorphism to express this action in terms of \(\mathcal{M}_p^{(m_0, ..., m_p)}\).
Let $\gamma_*$ be a chain map $B_{(m_0, \ldots, m_p)}^* \to M_{(m_0, \ldots, m_p)}^*$ defined as the zero map in degrees different from $p$, and the canonical map

$$I \otimes_{(m_0+1)} k\left[\text{Epi}_{\Delta S_+}([m_0], [m_1])\right] \otimes \ldots \otimes k\left[\text{Epi}_{\Delta S_+}([m_{p-1}], [m_p])\right] \to$$

$$I^{\otimes (m_0+1)} \otimes_{kG_0} k\left[\text{Epi}_{\Delta S_+}([m_0], [m_1])\right] \otimes_{kG_1} \ldots \otimes_{kG_p} k\left[\text{Epi}_{\Delta S_+}([m_{p-1}], [m_p])\right],$$

in degree $p$. $\gamma_*$ is $\Sigma_{m_{p+1}}$-equivariant due to an elementary property of bimodules:

**Proposition 72.** Suppose $R$ and $S$ are $k$-algebras, $A$ is a right $S$-module, and $B$ is an $S$-$R$-bimodule, then the canonical map $A \otimes_k B \to A \otimes_S B$ is a map of right $R$-modules.

Our aim now is to show that $\gamma_*$ induces an isomorphism in homology.

**Proposition 73.** $\gamma_*$ induces an isomorphism

$$H_\ast(B_{(m_0, \ldots, m_p)}^*) \to H_\ast(M_{(m_0, \ldots, m_p)}^*)$$

**Proof.** We shall prove this by induction on $p$.

Suppose $p = 0$. Observe that

$$B_n = \begin{cases} I^{\otimes (m_0+1)}, & n = 0 \\ 0, & n > 0. \end{cases}$$

Moreover, $\gamma_*$ is the identity $B_{(m_0)}^* \to M_{(m_0)}^*$.

Next, for the induction step, we assume $\gamma_*: B_{(m_0, \ldots, m_{p-1})}^* \to M_{(m_0, \ldots, m_{p-1})}^*$ induces an isomorphism in homology for any string of $p$ numbers $m_0 > m_1 > \ldots > m_{p-1}$. Now assume $m_p < m_{p-1}$.

Let $G = \Sigma_{m_{p-1}+1}$. As graded $k$-module, there is a degree-preserving isomorphism:

$$\theta_* : B_{(m_0, \ldots, m_{p-1})}^* \otimes_{kG} E_* G \otimes k\left[G \times \text{Epi}_{\Delta_+}([m_{p-1}], [m_p])\right] \to B_{(m_0, \ldots, m_{p-1}, m_p)}^* \text{ (3.3)}$$

where the degree of an element $u \otimes (g_0, \ldots, g_n) \otimes (g, \phi)$ is defined recursively (Note, all elements of $B_n^{(m_0)}$ are of degree 0):

$$\text{deg} (u \otimes (g_0, \ldots, g_n) \otimes (g, \phi)) := \text{deg} (u) + n + 1.$$
Here, we are using the resolution \( E_*G \) of \( k \) as \( kG \)-module defined by \( E_nG = k[\prod^{n+1} G] \),
with \( G \)-action \( g.(g_0, g_1, \ldots, g_n) = (gg_0, g_1, \ldots, g_n) \), and face maps
\[
\partial_i(g_0, g_1, \ldots, g_n) = \begin{cases} (g_0, \ldots, g_i g_{i+1}, \ldots, g_n), & 0 \leq i < n \\ (g_0, g_1, \ldots, g_{n-1}), & i = n \end{cases}
\]
\( \theta_* \) is defined on generators by:
\[
\theta_* : u \otimes (g_0, g_1, \ldots, g_n) \otimes (g, \phi) \mapsto
\]
\[
(u.g_0) \ast [m_{p-1}] \xrightarrow{g_1} [m_{p-1}] \xrightarrow{g_2} \ldots \xrightarrow{g_n} [m_{p-1}] \xrightarrow{(\phi, g)} [m_p],
\]
where \( u.g_0 \) is the right action defined above for \( D^{(m_0, \ldots, m_p)}_* \), and \( v \ast [n] \rightarrow \ldots \rightarrow [m] \) is the concatenation of chains (\( v \) must have final target \([n]\)). \( \theta_* \) is well-defined since for \( h \in G \),
\[
u.h \otimes (g_0, g_1, \ldots, g_n) \otimes (g, \phi) \xrightarrow{\theta_*}
\]
\[
((u.h).g_0) \ast [m_{p-1}] \xrightarrow{g_1} [m_{p-1}] \xrightarrow{g_2} \ldots \xrightarrow{g_n} [m_{p-1}] \xrightarrow{(\phi, g)} [m_p],
\]
while on the other hand,
\[
u \otimes h.(g_0, g_1, \ldots, g_n) \otimes (g, \phi) = u \otimes (h g_0, g_1, \ldots, g_n) \otimes (g, \phi) \xrightarrow{\theta_*}
\]
\[
(u.(h g_0)) \ast [m_{p-1}] \xrightarrow{g_1} [m_{p-1}] \xrightarrow{g_2} \ldots \xrightarrow{g_n} [m_{p-1}] \xrightarrow{(\phi, g)} [m_p],
\]
If we define a right action of \( \Sigma_{m_{p+1}} \) on \( k[\Sigma \times \text{Epi}_{\Delta_i}([m_{p-1}], [m_p])] \) via
\[
(g, \phi).h \mapsto (gh^{\phi}, \phi^h),
\]
then \( \theta_* \) is a map of right \( (k \Sigma_{m_{p+1}}) \)-modules, since the action defined above simply amounts to post-composition of the morphism \( (\phi, g) \) with \( h \).
\( \theta_* \) has a two-sided \( \Sigma_{m_{p+1}}^{\text{op}} \)-equivariant inverse, defined by:
\[
u \ast [m_{p-1}] \xrightarrow{g_1} [m_{p-1}] \xrightarrow{g_2} \ldots \xrightarrow{g_n} [m_{p-1}] \xrightarrow{(\phi, g)} [m_p]
\]
\[
\mapsto u \otimes (\text{id}, g_1, g_2, \ldots, g_n) \otimes (g, \phi).
\]
Observe that $\mathcal{B}_*^{(m_0,\ldots,m_{p-1})} \otimes_{kG} E_* G \otimes k[G \times \text{Epi}_{\Delta^+}([m_{p-1}],[m_p])]$ is a tensor product of two chain complexes, and thus a chain complex in its own right. The differential is given by:

$$\partial (u \otimes (g_0, \ldots, g_n) \otimes (g, \phi)) = \partial (u) \otimes (g_0, \ldots, g_n) \otimes (g, \phi) + (-1)^{\text{deg}(u)} u \otimes \partial ((g_0, \ldots, g_n) \otimes (g, \phi)).$$

Note, the $n^{th}$ face map of $E_* G \otimes k[G \times \text{Epi}_{\Delta^+}([m_{p-1}],[m_p])]$ is defined by:

$$\partial_n ((g_0, \ldots, g_n) \otimes (g, \phi)) = (g_0, \ldots, g_{n-1}) \otimes (g_n g, \phi).$$

We shall verify that $\theta_*$ is a chain map with respect to this differential.

Let $u \otimes (g_0, \ldots, g_n) \otimes (g, \phi)$ be a chain with $\text{deg}(u) = p$. Denote by $\partial_i$, $(0 \leq i \leq p + n + 1)$, the $i^{th}$ face map in either chain complex. If $i < p$, then clearly $\partial_i \theta_* = \theta_* \partial_i$, since this face map acts only on $u$.

Suppose now that $i = p$, and let $(\psi, h)$ be the final morphism in the chain $u$.

$$(\ldots \to [m_{p-2}] \xrightarrow{(\psi, h)} [m_{p-1}]) \otimes (g_0, g_1, \ldots, g_n) \otimes (g, \phi)$$

$$\xrightarrow{\partial_p} (\ldots \to [m_{p-2}] \xrightarrow{(\psi, h)} [m_{p-1}]) \otimes (g_0 g_1, g_2, \ldots, g_n) \otimes (g, \phi)$$

$$\xrightarrow{\theta_*} (\ldots \to [m_{p-2}] \xrightarrow{(\psi, h), (g_0 g_1)} [m_{p-1}] \xrightarrow{g_2} \ldots \xrightarrow{g_n} [m_{p-1}] \xrightarrow{(\phi, g)} [m_p]).$$

On the other hand,

$$(\ldots \to [m_{p-2}] \xrightarrow{(\psi, h)} [m_{p-1}]) \otimes (g_0, g_1, \ldots, g_n) \otimes (g, \phi)$$

$$\xrightarrow{\theta_*} (\ldots \to [m_{p-2}] \xrightarrow{(\psi, h), g_0} [m_{p-1}] \xrightarrow{g_1} \ldots \xrightarrow{g_n} [m_{p-1}] \xrightarrow{(\phi, g)} [m_p]),$$

$$\xrightarrow{\partial_p} (\ldots \to [m_{p-2}] \xrightarrow{(\psi, h), g_0, g_1} [m_{p-1}] \xrightarrow{g_2} \ldots \xrightarrow{g_n} [m_{p-1}] \xrightarrow{(\phi, g)} [m_p]).$$

Next, suppose $i = p + j$ for some $1 \leq j < n$. In this case, $\partial_i$ has the effect of combining $g_i$ and $g_{i+1}$ into $g_i g_{i+1}$, for either chain, so clearly $\theta_* \partial_i = \partial_i \theta_*$. Finally, for $i = p + n$,

$$\theta_* \partial_{p+n} (u \otimes (g_0, \ldots, g_n) \otimes (g, \phi))$$

$$= \theta_* (u \otimes (g_0, \ldots, g_{n-1}) \otimes (g_n g, \phi))$$

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while

$$\partial_{p+n}\theta_* (u \otimes (g_0, \ldots, g_n) \otimes (g, \phi))$$

$$= \partial_{p+n} \left( (u.g_0) \ast [m_{p-1}] \xrightarrow{g_1} \ldots \xrightarrow{g_{n-1}} [m_{p-1}] \xrightarrow{(\phi, g_n g)} [m_p] \right)$$

$$= (u.g_0) \ast [m_{p-1}] \xrightarrow{g_1} \ldots \xrightarrow{g_{n-1}} [m_{p-1}] \xrightarrow{g_n (\phi, g_n g)} [m_p]$$

Hence, the map $\theta_*$ is a chain isomorphism.

The next step in this proof is to prove a chain homotopy equivalence,

$$\mathcal{B}^{(m_0, \ldots, m_{p-1})} \otimes_k G \otimes k[G \times \operatorname{Epi}_{\Delta^+}([m_{p-1}], [m_p])]$$

$$\cong \mathcal{B}_*^{(m_0, \ldots, m_{p-1})} \otimes k[E_{\Delta^+}([m_{p-1}], [m_p])]$$

To that end, we shall define chain maps $F_*$ and $G_*$ between the two complexes. Let

$$\mathcal{U}_* := \mathcal{B}_*^{(m_0, \ldots, m_{p-1})}, \text{ and } S := E_{\Delta^+}([m_{p-1}], [m_p]).$$

Define

$$F_* : \mathcal{U}_* \otimes_k G \otimes k[G \times S] \longrightarrow \mathcal{U}_* \otimes k[S],$$

$$F_*(u \otimes (g_0) \otimes (g, \phi)) := u.(g_0 g) \otimes \phi,$$

$$F_*(u \otimes (g_0, \ldots, g_n) \otimes (g, \phi)) := 0, \quad \text{if } n > 0.$$
Next, let

\[ G_* : \mathcal{U}_* \otimes k[S] \to \mathcal{U}_* \otimes_{kG} E_* G \otimes k[G \times S] \]

be the composite

\[ \mathcal{U}_* \otimes k[S] \overset{j}{\to} \mathcal{U}_* \otimes_{kG} G \otimes k[S] = \mathcal{U}_* \otimes_{kG} E_0 G \otimes k[S] \]

\[ \overset{inc}{\to} \mathcal{U}_* \otimes_{kG} E_* G \otimes k[G \times S], \]

where \( j \) is induced by the map sending a generator \( \phi \in S \) to \((id, \phi) \in G \times S\), and \( inc \) is induced by the inclusion \( E_0 G \hookrightarrow E_* G \). Observe,

\[ F_* G_*(u \otimes \phi) = F_*(u \otimes (id) \otimes (id, \phi)) = (u.id) \otimes \phi. \]

Thus, \( F_* G_* \) is the identity. I claim \( G_* F_* \simeq \text{id} \).

The desired homotopy will be given by:

\[ h_* : u \otimes (g_0, \ldots, g_n) \otimes (g, \phi) \mapsto (-1)^{\deg(u)+n} u \otimes (g_0, \ldots, g_n, g) \otimes (id, \phi). \]

First, observe:

\[ G_* F_*(u \otimes (g_0) \otimes (g, \phi)) = G_*(u.(g_0g) \otimes \phi) = u.(g_0g) \otimes (id) \otimes (id, \phi), \]

\[ G_* F_*(u \otimes (g_0, \ldots, g_n) \otimes (g, \phi)) = G_*(0) = 0, \quad \text{for } n > 0. \]

Hence, there are two cases we must explore. For \( n = 0 \),

\[ h \partial(u \otimes (g_0) \otimes (g, \phi)) = h(\partial u \otimes (g_0) \otimes (g, \phi)) \]

\[ = (-1)^{\deg(u)-1} \partial u \otimes (g_0, g) \otimes (id, \phi) \]

On the other hand,

\[ \partial h(u \otimes (g_0) \otimes (g, \phi)) = \partial((-1)^{\deg(u)} u \otimes (g_0, g) \otimes (id, \phi)) \]

\[ = (-1)^{\deg(u)} \partial u \otimes (g_0, g) \otimes (id, \phi) + (-1)^{2\deg(u)} u \otimes (g_0g) \otimes (id, \phi) - (-1)^{2\deg(u)} u \otimes (g_0) \otimes (g, \phi) \]
\[ (-1)^{\deg(u)} \partial u \otimes (g_0, g) \otimes (\text{id}, \phi) + u \otimes (g_0 g) \otimes (\text{id}, \phi) - u \otimes (g_0, g, \phi) \]

So,

\[ (h \partial + \partial h) \left( u \otimes (g_0) \otimes (g, \phi) \right) = u.(g_0g) \otimes (\text{id}) \otimes (\text{id}, \phi) - u \otimes (g_0) \otimes (g, \phi) \]

\[ = (G_* F_* - \text{id}) \left( u \otimes (g_0) \otimes (g, \phi) \right). \]

The case \( n > 0 \) is handled similarly:

\[ u \otimes (g_0, \ldots, g_n) \otimes (g, \phi) \xrightarrow{\partial} \]

\[ \partial u \otimes (g_0, \ldots, g_n) \otimes (g, \phi) + \]

\[ (-1)^{\deg(u)} \left[ \sum_{j=0}^{n-1} (\partial u \otimes (g_0, \ldots, g_j g_{j+1}, \ldots, g_n) \otimes (g, \phi)) + \right] \]

\[ \left( (-1)^n u \otimes (g_0, \ldots, g_{n-1}) \otimes (g_n g, \phi) \right) \]

\[ \xrightarrow{h} (-1)^{\deg(u) + n-1} \partial u \otimes (g_0, \ldots, g_n, g) \otimes (\text{id}, \phi) + \]

\[ (-1)^{2\deg(u) + n-1} \sum_{j=0}^{n-1} \left( (-1)^j u \otimes (g_0, \ldots, g_j g_{j+1}, \ldots, g_n, g) \otimes (\text{id}, \phi) \right) + \]

\[ (-1)^{2\deg(u) + 2n-1} u \otimes (g_0, \ldots, g_{n-1}, g_n g) \otimes (\text{id}, \phi) \]

\[ = -(-1)^{\deg(u) + n} \partial u \otimes (g_0, \ldots, g_n, g) \otimes (\text{id}, \phi) + \]

\[ \sum_{j=0}^{n-1} \left( (-1)^{j+n-1} u \otimes (g_0, \ldots, g_j g_{j+1}, \ldots, g_n, g) \otimes (\text{id}, \phi) \right) + \]

\[ (-1)^n u \otimes (g_0, \ldots, g_{n-1}, g_n g) \otimes (\text{id}, \phi). \]  

(3.4)

On the other hand,

\[ u \otimes (g_0, \ldots, g_n) \otimes (g, \phi) \xrightarrow{h} (-1)^{\deg(u) + n} u \otimes (g_0, \ldots, g_n, g) \otimes (\text{id}, \phi) \]

\[ \xrightarrow{\partial} (-1)^{\deg(u) + n} \left[ \partial u \otimes (g_0, \ldots, g_n, g) \otimes (\text{id}, \phi) + \right] \]
\[
\sum_{j=0}^{n-1} \left( (-1)^{\deg(u)+j} u \otimes (g_0, \ldots, g_j, g_{j+1}, \ldots, g_n, g) \otimes (id, \phi) \right) + \\
(-1)^{\deg(u)+n} u \otimes (g_0, \ldots, g_{n-1}, g_n g) \otimes (id, \phi) + (-1)^{\deg(u)+n+1} u \otimes (g_0, \ldots, g_n) \otimes (g, \phi) \right] \\
= (-1)^{\deg(u)+n} \partial u \otimes (g_0, \ldots, g_n, g) \otimes (id, \phi) + \\
\sum_{j=0}^{n-1} \left( (-1)^{j+n} u \otimes (g_0, \ldots, g_j, g_{j+1}, \ldots, g_n, g) \otimes (id, \phi) \right) + \\
u \otimes (g_0, \ldots, g_{n-1}, g_n g) \otimes (id, \phi) - u \otimes (g_0, \ldots, g_n) \otimes (g, \phi). \tag{3.5}
\]

Now, adding eq. (3.5) to eq. (3.4) yields \((-1)^{u} \otimes (g_0, \ldots, g_n) \otimes (g, \phi)\), proving the relation

\[ h \partial + \partial h = G \ast F \ast - id, \quad \text{for } n > 0. \]

To complete the proof, simply observe that every map in the following is either a chain isomorphism or a homotopy equivalence (each of which is also \(\Sigma_{m+1}\)-equivariant):
We must verify that this composition is indeed the map $\gamma_*$. Denote by $\gamma'_*$, the composition defined by (3.6). If $u \in \mathcal{B}^{(m_0,\ldots,m_p)}_*$ has degree greater than $p$, then there is some isomorphism $g$ showing up in the chain. If $g$ is an isomorphism of any $[m_i]$ for $i < p - 1$, then $\gamma'_*(u) = 0$ since then the tensor factor of $u$ in $\mathcal{B}^{(m_0,\ldots,m_p-1)}_*$ would have degree greater than $p - 1$, hence $\gamma_*$ would send this element to 0 in $\mathcal{M}^{(m_0,\ldots,m_p)}_*$. If, on the other hand, $g$ is an isomorphism of $[m_{p-1}]$, then $F_*\theta_*^{-1}(u) = 0$, since there would be a factor in $E_* G$ of degree greater than 0. Thus, $\gamma'_*(u) = 0$ for any $u$ of degree different from $p$. 

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Now, if $u$ is of degree $p$,

$$u = Y \otimes (\psi_1, g_1) \otimes (\psi_2, g_2) \otimes \ldots \otimes (\psi_p, g_p)$$

$\theta^{-1}$ $\mapsto \left[ Y \otimes (\psi_1, g_1) \otimes \ldots \otimes (\psi_{p-1}, g_{p-1}) \right] \otimes (\text{id}) \otimes (g_p, \psi_p)$

$F_2$ $\mapsto \left[ Y \otimes (\psi_1, g_1) \otimes \ldots \otimes (\psi_{p-1}, g_{p-1}) \otimes g_p \right] \otimes \psi_p$

It should be clear that applying $\gamma_*$ to the $B^{(m_0, \ldots, m_{p-1})}$-factor of the tensor product would have the same effect as $\gamma_*$ on the original chain, $u$. $\square$

Now, we may prove Lemma 71. Let $G = \Sigma_{m_p+1}$. Observe,

$$E_{p,q}^0 \cong \bigoplus_{m_0 > \ldots > m_p} \bigoplus_{s+t=q} B^{(m_0, \ldots, m_p)} \otimes_{kG} E_* G,$$

with differential corresponding exactly to the vertical differential defined for $E^0$. Note, the outer direct sum respects the differential $d^0$, so the $E^1$ term given by:

$$E_{p,q}^1 = H_{p+q}(E_{p,*}^0) \cong \bigoplus_{m_0 > \ldots > m_p} H_{p+q}(B^{(m_0, \ldots, m_p)} \otimes_{kG} E_* G),$$  \hspace{1cm} (3.7)

where we view $B^{(m_0, \ldots, m_p)} \otimes_{kG} E_* G$ as a double complex. In what follows, let $(m_0, \ldots, m_p)$ be fixed. In order to take the homology of the double complex, we set up another spectral sequence. From the discussion above, the total differential is given by

$$\partial_{\text{total}} = d^v + d^h,$$

where

$$d^v(u \otimes (g_0, \ldots, g_t)) := \partial_B(u) \otimes (g_0, \ldots, g_t),$$

and

$$d^h(u \otimes (g_0, \ldots, g_t)) := (-1)^{\text{deg}(u)} u \otimes \partial_E(g_0, \ldots, g_t),$$

where $\partial_B$ and $\partial_E$ are the differentials previously mentioned for $B^{(m_0, \ldots, m_p)}$ and $E_* G$, respectively. Thus, there is a spectral sequence $\{\overline{E}_{*,*}^*, d^{v}\}$ with

$$\overline{E}_2^2 \cong H_{*,*} \left( H \left( B^{(m_0, \ldots, m_p)} \otimes_{kG} E_* G, d^h \right), d^v \right),$$

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in the notation of McCleary (see [23], Thm 3.10). Since this is a first quadrant spectral sequence, it must converge to \( H_*(\mathcal{B}_{s}^{(m_0,\ldots,m_p)} \otimes_{kG} E_*) \). Let us examine what happens after taking the horizontal differential. Let \( t \) be fixed:

\[
\bar{E}^1_{s,t} = H_*(\mathcal{B}_{s}^{(m_0,\ldots,m_p)} \otimes_{kG} E_t, d^h)
\]

\[
\cong H_*(\mathcal{B}_{s}^{(m_0,\ldots,m_p)} \otimes_{kG} E_t,
\]

since \( E_t \) is flat as left \( kG \)-module (in fact, \( E_t \) is free). Then, by Prop. 73,

\[
\bar{E}^1_{s,t} \cong H_*(\mathcal{M}_{s}^{(m_0,\ldots,m_p)} \otimes_{kG} E_t,
\]

\[
= \begin{cases} 
I^{\otimes (m_0+1)} \otimes k \left[ \prod_{i=1}^{p} \text{Epi}_{\Delta_+}([m_{i-1}],[m_i]) \right] \otimes_{kG} E_t, & \text{in degree } p \\
0, & \text{in degrees different from } p
\end{cases}
\]

So, the only groups that survive are concentrated in column \( p \). Taking the vertical differential now amounts to obtaining the \( G^{\text{op}} \)-equivariant homology of

\[
I^{\otimes (m_0+1)} \otimes k \left[ \prod_{i=1}^{p} \text{Epi}_{\Delta_+}([m_{i-1}],[m_i]) \right],
\]

so

\[
\bar{E}^2_{p,t} \cong H_t \left( G^{\text{op}}; I^{\otimes (m_0+1)} \otimes k \left[ \prod_{i=1}^{p} \text{Epi}_{\Delta_+}([m_{i-1}],[m_i]) \right] \right).
\]

Since \( \bar{E}^2_{s,t} = 0 \) for \( s \neq p \), the sequence collapses here. Thus,

\[
H_{p+q}(\mathcal{B}_{s}^{(m_0,\ldots,m_p)} \otimes_{kG} E_* G) \cong H_q \left( G^{\text{op}}; I^{\otimes (m_0+1)} \otimes k \left[ \prod_{i=1}^{p} \text{Epi}_{\Delta_+}([m_{i-1}],[m_i]) \right] \right)
\]

Putting this information back into eq. (3.7), we obtain the desired isomorphism:

\[
E^1_{p,q} \cong \bigoplus_{m_0,\ldots,m_p} H_q \left( G^{\text{op}}; I^{\otimes (m_0+1)} \otimes k \left[ \prod_{i=1}^{p} \text{Epi}_{\Delta_+}([m_{i-1}],[m_i]) \right] \right).
\]

A final piece of information needed in order to use Lemma 71 for computation is a description of the horizontal differential \( d^1_{p,q} \) on \( E^1_{p,q} \). This map is induced from the differential \( d \) on \( \mathcal{B}_s \), and reduces the filtration degree by 1. Thus, it is the sum of face maps that combine strict epimorphisms.
Let
\[ [u] \in \bigoplus H_q \left( \sum_{m_{p+1}}^{\text{op}} \otimes I^{\otimes (m_0+1)} \otimes k \left[ \prod_{i=1}^{p} \text{Epi}_{\Delta_i}([m_{i-1}],[m_i]) \right] \right) \]
be represented by a chain:
\[ u = Y \otimes (\phi_1, \phi_2, \ldots, \phi_p) \otimes (g_0, \ldots, g_q). \]

Then, the face maps are defined by:
\[ \partial_0(u) = (\phi_1)_*(Y) \otimes (\phi_2, \ldots, \phi_p) \otimes (g_0, \ldots, g_q), \]
\[ \partial_i(u) = Y \otimes (\phi_1, \ldots, \phi_{i+1}\phi_i, \ldots, \phi_p) \otimes (g_0, \ldots, g_q), \quad \text{for } 0 < i < p, \]

The last face map has the effect of removing the morphism \( \phi_p \) by iteratively commuting it past any group elements to the right of it.
\[ \partial_p(u) = Y \otimes (\phi_1, \ldots, \phi_{p-1}) \otimes (g'_0, \ldots, g'_q), \]

where
\[ g'_i = g_i^{\phi_0\phi_1\cdots\phi_{i-1}}. \]

Note that \( \partial_p \) involves a change of group from \( \Sigma_{m_p} \) to \( \Sigma_{m_{p-1}} \).

**Proposition 74.** The spectral sequence \( E_{p,q}^r \) above collapses at \( r = 2 \).

**Proof.** This proof relies on the fact that the differential \( d \) on \( \mathcal{Y}_s \) cannot reduce the filtration degree by more than 1. Explicitly, we shall show that \( d^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r \) is trivial for \( r \geq 2 \).

\( d^r \) is induced by \( d \) in the following way. Let \( Z_p^r = \{ x \in \mathcal{F}_p \mathcal{Y}_s \mid d(x) \in \mathcal{F}_{p-r} \mathcal{Y}_s \} \). Then \( E_{p,s}^r = Z_p^r / (Z_p^{r-1} + dZ_{p+r-1}^{r-1}) \). Now, \( d \) maps
\[ Z_p^r \rightarrow Z_{p-r}^r \]
and
\[ Z_{p-1}^{r-1} + dZ_{p+r-1}^{r-1} \rightarrow dZ_{p-1}^{r-1}. \]
Hence, there is an induced map $\bar{d}$ making the square below commute. $d'$ is obtained as the composition of $\bar{d}$ with a projection onto $E_{p-r,*}^r$.

\[
\begin{array}{c}
Z_p^r \xrightarrow{d} Z_{p-r}^r \\
\pi_1 \downarrow \quad \quad \quad \pi_2 \downarrow \\
Z_p^r / (Z_{p-1}^r + dZ_{p+r-1}^r) \xrightarrow{\bar{d}} Z_{p-r}^r / dZ_{p-1}^r \\
\cong \quad \quad \quad \cong \\
E_{p,*}^r \xrightarrow{d'} Z_{p-r}^r / (Z_{p-r-1}^r + dZ_{p-1}^r) \xrightarrow{\pi'} E_{p-r,*}^r
\end{array}
\]

In our case, $x \in Z_p^r$ is a sum of the form

\[
x = \sum_{q \geq 0} a_i \left(Y \otimes (f_1, f_2, \ldots, f_q)\right),
\]

where $a_i \neq 0$ for only finitely many $i$, and the sum extends over all symbols $Y \otimes (f_1, f_2, \ldots, f_q)$ with $Y \in B_{sym}^{\text{in}}, I$, $f_j \in \text{EpiS}_+$ composable maps, and at most $p$ of the $f_j$ maps are strict epimorphisms. The image of $x$ under $\pi_1$ looks like

\[
\pi_1(x) = \sum_{q \geq 0} a_i [Y \otimes (f_1, f_2, \ldots, f_q)],
\]

where exactly $p$ of the $f_j$ maps are strictly epic. There are, of course, other relations present as well – those arising from modding out by $dZ_{p+r-1}^r$. Consider, $\bar{d}\pi_1(x)$. This should be the result of lifting $\pi_1(x)$ to a representative in $Z_p^r$, then applying $\pi_2 \circ d$. One such representative is:

\[
y = \sum_{q \geq 0} a_i \left(Y \otimes (f_1, f_2, \ldots, f_q)\right),
\]

in which each symbol $Y \otimes (f_1, f_2, \ldots, f_q)$ has exactly $p$ strict epimorphisms. Now, $d(y)$ is a sum

\[
d(y) = \sum_{q \geq 0} b_i \left(Z \otimes (g_1, g_2, \ldots, g_{q-1})\right),
\]
where each symbol $Z \otimes (g_1, g_2, \ldots, g_{q-1})$ has either $p$ or $p - 1$ strict epimorphisms, since $d$ only combines two morphisms at a time. Thus, if $r \geq 2$, then $d(y) \in Z_{p-r} \Rightarrow d(y) = 0$. But then, $d\pi_1(x) = \pi_2d(y) = 0$, and $d^r = \pi'^d$ is the zero map. \qed

3.2 Implications in Characteristic 0

In this section, we shall assume that $k$ is a field of characteristic 0. Then for any finite group $G$ and $kG$-module $M$, $H_q(G, M) = 0$ for all $q > 0$ (see [3], for example). Thus, by Lemma 71, the $E^1$ term of the spectral sequence is concentrated in row 0, and

$$E^1_{p,0} \cong \bigoplus_{m_0 > \ldots > m_p} (I \otimes (m_0 + 1) \otimes k \prod_{i=1}^p \text{Epi}_{\Delta_+}([m_{i-1}], [m_i])) / \Sigma_{op} \mu_{p+1},$$

that is, the group of co-invariants of the coefficient group, under the right-action of $\Sigma_{m_p+1}$. Since $E^1$ is concentrated on a row, the spectral sequence collapses at this term. Hence for the $k$-algebra $A$, with augmentation ideal $I$,

$$HS_*(A) = H_*(\bigoplus_{p \geq 0} \bigoplus_{m_0 > \ldots > m_p} (I \otimes (m_0 + 1) \otimes k \prod_{i=1}^p \text{Epi}_{\Delta_+}([m_{i-1}], [m_i])) / \Sigma_{op} \mu_{p+1}, \ d^1). \ (3.8)$$

This complex is still rather unwieldy as the $E^1$ term is infinitely generated in each degree. In the next chapter, we shall see another spectral sequence that is more computationally useful.
CHAPTER 4
A SECOND SPECTRAL SEQUENCE

4.1 Filtering by Degree

Again, we shall assume $A$ is a $k$-algebra equipped with an augmentation, and whose augmentation ideal is $I$. Assume further that $I$ is free as $k$-module, with countable basis $X$. Let $\mathcal{Y}_*^{epi}$ be the complex 2.14, with differential $d = \sum(-1)^i \partial_i$. Denote by $B_*^{sym} I$, the restriction of $B_*^{sym} I$ to $\Delta S$:

$$B_n^{sym} I := I^{\otimes(n+1)}, \text{ for } n \geq 0.$$  

Let

$$\mathcal{Y}_* = \bigoplus_{q \geq 0} \bigoplus_{m_0 \geq \ldots \geq m_q} k[m_0] \rightarrow [m_1] \rightarrow \ldots \rightarrow [m_q] \otimes B_{m_0}^{sym} I. \quad (4.1)$$

Observe that there is a splitting of $\mathcal{Y}_*^{epi}$ as:

$$\mathcal{Y}_*^{epi} \cong \mathcal{Y}_* \oplus k[N(*)],$$

where $*$ is the trivial subcategory of Epi$\Delta S_+$ consisting of the object $[-1]$ and morphism $id_{[-1]}$. The fact that $I$ is an ideal ensures that this splitting passes to homology. Hence, we have:

$$HS_*(A) \cong H_*(\mathcal{Y}_*) \oplus k_0,$$

where $k_0$ is the graded $k$-module consisting of $k$ concentrated in degree 0.

**Definition 75.** The reduced symmetric homology of $A$ is defined,

$$\tilde{HS}_*(A) := H_*(\mathcal{Y}_*)$$
Now, since \( I = k[X] \) as \( k \)-module, \( B^{\text{sym}}_n I = k[X]^\otimes (n+1) \). Thus, as \( k \)-module, \( \widetilde{\mathcal{Y}} \) is generated by elements of the form:

\[
([m_0] \to [m_1] \to \ldots \to [m_p]) \otimes (x_0 \otimes x_1 \otimes \ldots \otimes x_{m_0}), \quad x_i \in X.
\]

Using an isomorphism analogous to that of eq. 3.1, we may write:

\[
\widetilde{\mathcal{Y}} = \bigoplus_{m_0 \geq 0} \bigoplus_{m_0 \geq \ldots \geq m_q} B^{\text{sym}}_m I \otimes k \prod_{i=1}^q \text{Epi}_{\Delta S}([m_{i-1}], [m_i]).
\]

\[
\cong \bigoplus_{m_0 \geq 0} \bigoplus_{m_0 \geq \ldots \geq m_q} k[X]^\otimes (m_0+1) \otimes k \prod_{i=1}^q \text{Epi}_{\Delta S}([m_{i-1}], [m_i]).
\]

The face maps are given explicitly below:

\[
\partial_0(Y \otimes f_1 \otimes f_2 \otimes \ldots \otimes f_q) = f_1(Y) \otimes f_2 \otimes \ldots \otimes f_q;
\]

\[
\partial_i(Y \otimes f_1 \otimes \ldots \otimes f_q) = Y \otimes f_1 \otimes \ldots \otimes (f_{i+1}f_i) \otimes \ldots \otimes f_q;
\]

\[
\partial_q(Y \otimes f_1 \otimes \ldots \otimes f_{q-1} \otimes f_q) = Y \otimes f_1 \otimes \ldots \otimes f_{q-1}.
\]

Consider a filtration \( \mathcal{G}_* \) of \( \widetilde{\mathcal{Y}} \) by degree of \( Y \in B^{\text{sym}}_n I \):

\[
\mathcal{G}_p \mathcal{Y}_q = \bigoplus_{p \geq m_0 \geq 0} \bigoplus_{m_0 \geq \ldots \geq m_q} k[X]^\otimes (m_0+1) \otimes k \prod_{i=1}^q \text{Epi}_{\Delta S}([m_{i-1}], [m_i]).
\]

The face maps \( \partial_i \) for \( i > 0 \) do not affect the degree of \( Y \in k[X]^\otimes (m_0+1) \). Only \( \partial_0 \) needs to be checked. Since all morphisms are epic, \( \partial_0 \) can only reduce the degree of \( Y \). Thus, \( \mathcal{G}_* \) is compatible with the differential \( d \). The filtration quotients are:

\[
E^0_{p,q} = \bigoplus_{p=m_0 \geq \ldots \geq m_q} k[X]^\otimes (p+1) \otimes k \prod_{i=1}^q \text{Epi}_{\Delta S}([m_{i-1}], [m_i]).
\]

The induced differential, \( d^0 \) on \( E^0 \) differs from \( d \) only when \( m_0 > m_1 \). Indeed,

\[
d^0 = \begin{cases} 
  d, & m_0 = m_1, \\
  d - \partial_0, & m_0 > m_1.
\end{cases}
\]
$E^0$ splits into a direct sum based on the product of $x_i$’s in $(x_0, \ldots, x_p) \in X^{p+1}$. For $u \in X^{p+1}$, let $C_u$ be the set of all distinct permutations of $u$. Then,

$$E^0_{p,q} = \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} \left( \bigoplus_{p=m_0 \geq \ldots \geq m_q \in C_u} w \otimes k \left[ \prod_{i=1}^{q} \text{Epi}_{\Delta S}([m_{i-1}], [m_i]) \right] \right).$$

Before proceeding with the main theorem of this chapter, we must define four related categories $\tilde{S}_p$, $\tilde{S}'_p$, $S_p$, and $S'_p$. In the definitions that follow, let $\{z_0, z_1, z_2, \ldots x_p\}$ be a set of formal non-commuting variables.

**Definition 76.** $\tilde{S}_p$ is the category with objects formal tensor products $Z_0 \otimes \ldots \otimes Z_s$, where each $Z_i$ is a non-empty product of $z_i$’s, and every one of $z_0, z_1, \ldots, z_p$ occurs exactly once in the tensor product. There is a unique morphism

$$Z_0 \otimes \ldots \otimes Z_s \rightarrow Z'_0 \otimes \ldots \otimes Z'_t,$$

if and only if the tensor factors of the latter are products of the factors of the former in some order. In such a case, there is a unique $\beta \in \text{Epi}_{\Delta S}$ so that $\beta_s(Z_0 \otimes \ldots \otimes Z_s) = Z'_0 \otimes \ldots \otimes Z'_t$.

$\tilde{S}_p$ has initial objects $\sigma(z_0 \otimes z_1 \otimes \ldots \otimes z_p)$, for $\sigma \in \Sigma_{p+1}^{op}$, so $N\tilde{S}_p$ is a contractible complex. Let $\tilde{S}'_p$ be the full subcategory of $\tilde{S}_p$ with all objects $\sigma(z_0 \otimes \ldots \otimes z_p)$ deleted.

Let $S_p$ be a skeletal category equivalent to $\tilde{S}_p$. In fact, we make make $S_p$ the quotient category, identifying each object $Z_0 \otimes \ldots \otimes Z_s$ with any permutation of its tensor factors, and identifying morphisms $\phi$ and $\psi$ if their source and target are equivalent. This category has nerve $NS_p$ homotopy-equivalent to $N\tilde{S}_p$ (see Prop. 2.1 in [30], for example). Now, $S_p$ is a poset with unique initial object, $z_0 \otimes \ldots \otimes z_p$. Let $S'_p$ be the full subcategory (subposet) of $S_p$ obtained by deleting the object $z_0 \otimes \ldots \otimes z_p$. Clearly, $S'_p$ is a skeletal category equivalent to $\tilde{S}'_p$.

**Theorem 77.** There is spectral sequence converging weakly to $\tilde{H}S_*(A)$ with

$$E^1_{p,q} \cong \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_{p+q}(EG_u \ltimes G_u \left| NS_p/NS'_p \right.; k),$$

where $G_u$ is the isotropy subgroup of the orbit $u \in X^{p+1}/\Sigma_{p+1}$.
Recall, for a group \( G \), right \( G \)-space \( X \), and left \( G \)-space \( Y \), \( X \ltimes_G Y \) denotes the equivariant half-smash product. If \( \ast \) is a chosen basepoint for \( Y \) having trivial \( G \)-action, then

\[
X \ltimes_G Y := (X \times G Y)/(X \times G \ast) = X \times Y/\approx,
\]

with equivalence relation defined by \((x.g, y) \approx (x.g.y)\) and \((x, \ast) \approx (x', \ast)\) for all \( x, x' \in X \), \( y \in Y \) and \( g \in G \) (cf. [21]). In our case, \( X \) is of the form \( EG \), with canonical underlying complex \( E^\ast G \). In chapter 3, we took \( E^\ast G \) to have a left \( G \)-action, \( g.(g_0, g_1, \ldots, g_n) = (gg_0, g_1, \ldots, g_n) \), but \( E^\ast G \) also has a right \( G \)-action, \( (g_0, g_1, \ldots, g_n).g = (g_0, g_1, \ldots, g_ng) \).

It is this latter action that we shall use in the definitions of \( EG_u \ltimes_G u|N\tilde{S}_p/N\tilde{S}'_p\) and \( EG_u \ltimes_G u|N\tilde{S}_p/N\tilde{S}'_p\).

Observe, both \( N\tilde{S}_p \) and \( N\tilde{S}'_p \) carry a left \( \Sigma_{p+1} \)-action (hence also a \( G_u \)-action) given by viewing \( \sigma \in \Sigma_{p+1} \) as the \( \Delta S \)-isomorphism \( \sigma \in \Sigma_{p+1}^\text{op} \), then pre-composing with the \( Z_i \)'s (regarded as morphisms written in tensor notation). The result of the composition should be expressed in tensor notation in order to be consistent.

\[
\sigma.(Z_0 \xrightarrow{\phi_1} Z_1 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_q} Z_q) := Z_0\sigma \xrightarrow{\phi_1} Z_1\sigma \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_q} Z_q\sigma.
\]

Define for each \( u \in X^{p+1}/\Sigma_{p+1} \), the following subcomplex of \( E^0_{p,q} \):

\[
\mathcal{M}_u := \bigoplus_{p=m_0 \geq \ldots \geq m_q} \bigoplus_{w \in C_u} w \otimes k \left[ \prod_{i=1}^q \text{Epi}_{\Delta S}([m_{i-1}], [m_i]) \right]
\]

Lemma 78. There is a chain-isomorphism

\[(N\tilde{S}_p/N\tilde{S}'_p)/G_u \cong \mathcal{M}_u\]

Proof. The forward map is given on generators by:

\[(Z_0 \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_q} Z_q) \mapsto Z_0(u) \otimes \phi_1 \otimes \ldots \otimes \phi_q.\]

This map is well-defined, since if \( g \in G_u \), then

\[g.(Z_0 \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_q} Z_q) = (Z_0g \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_q} Z_qg) \mapsto \]
$$Z_0 \overline{g}(u) \otimes \phi_1 \otimes \ldots \otimes \phi_q = Z_0(u) \otimes \phi_1 \otimes \ldots \otimes \phi_q$$

For the opposite direction, we begin with a generator of the form

$$w \otimes \phi_1 \otimes \ldots \otimes \phi_q, \quad w \in C_u. \quad (4.2)$$

Let $\tau \in \Sigma_{p+1}$ so that $w = \overline{\tau}(u)$. Then the image of 4.2 is defined to be

$$\tau \mapsto \phi_1 \overline{\tau} \mapsto \ldots \mapsto \phi_q \ldots \mapsto \phi_1 \overline{\tau}. \quad (4.3)$$

We must check that this definition does not depend on choice of $\tau$. Indeed, if $w = \overline{\sigma}(u)$ also, then $u = \sigma^{-1}(u)$, hence $\tau \sigma^{-1} \in G_u$. Thus,

$$\sigma \mapsto \phi_1 \sigma \mapsto \ldots \mapsto \phi_q \ldots \mapsto \phi_1 \sigma \mapsto \tau \sigma^{-1} \cdot \overline{\sigma} = \tau \mapsto \phi_1 \overline{\tau} \mapsto \ldots \mapsto \phi_q \ldots \mapsto \phi_1 \overline{\tau}.$$ 

The maps are clearly inverse to one another. All that remains is to verify that these are chain maps. For $i > 0$, the face maps $\partial_i$ simply compose the maps $\phi_{i+1}$ and $\phi_i$ in either chain complex, so only the zeroth face map needs to be checked. First, for the forward map, assume $\phi_1$ is an isomorphism.

$$(Z_0 \mapsto \ldots \mapsto Z_q) \mapsto Z_0(u) \otimes \phi_1 \otimes \ldots \otimes \phi_q \mapsto \phi_1 Z_0(u) \otimes \phi_2 \otimes \ldots \otimes \phi_q,$$

while

$$(Z_0 \mapsto \ldots \mapsto Z_q) \mapsto (Z_1 \mapsto \ldots \mapsto Z_q) \mapsto Z_1(u) \otimes \phi_2 \otimes \ldots \otimes \phi_q.$$

The two results agree since $\phi_1 Z_0 = Z_1$. If $\phi_1$ is not an isomorphism, then it must be a strict epimorphism, and so $Z_0(u) \otimes \phi_1 \otimes \ldots \otimes \phi_q = 0$ in $E^0$. On the other hand, the chain $Z_1 \to \ldots \to Z_q$ is in $N\tilde{S}'_p$, hence should also be identified with 0.

In the reverse direction, assume $w = \overline{\tau}(u)$ as above, and let $\phi_1$ be an isomorphism.

$$w \otimes \phi_1 \otimes \ldots \otimes \phi_q \mapsto (\overline{\tau} \mapsto \phi_1 \overline{\tau} \mapsto \ldots \mapsto \phi_q \ldots \mapsto \phi_1 \overline{\tau}) \mapsto (\phi_1 \overline{\tau} \mapsto \phi_2 \phi_1 \overline{\tau} \mapsto \ldots \mapsto \phi_q \ldots \mapsto \phi_1 \overline{\tau}),$$

while

$$w \otimes \phi_1 \otimes \ldots \otimes \phi_q \mapsto (\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_q \mapsto \phi_1(w) \otimes \phi_2 \otimes \ldots \otimes \phi_q \mapsto (\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_q \mapsto \phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_q).$$
The rightmost expression results from the observation that if \( w = \varphi(u) \), then \( \phi_1(w) = \phi_1 \varphi(u) \).

Now, if \( \phi_1 \) is a strict epimorphism, then both results are 0 for similar reasons as above. \( \square \)

Using this lemma, we identify \( \mathbb{M}_u \) with the orbit complex \( (\tilde{N}\tilde{S}_p/\tilde{N}\tilde{S}_p')/G_u \). Now, the complex \( \tilde{N}\tilde{S}_p/\tilde{N}\tilde{S}_p' \) is a free \( G_u \)-complex, so we have an isomorphism:

\[
H_*( (\tilde{N}\tilde{S}_p/\tilde{N}\tilde{S}_p')/G_u ) \cong H_*^{G_u}(\tilde{N}\tilde{S}_p/\tilde{N}\tilde{S}_p'),
\]

(i.e., \( G_u \)-equivariant homology. See [3] for details). Then, by definition,

\[
H_*^{G_u}(\tilde{N}\tilde{S}_p/\tilde{N}\tilde{S}_p') = H_*(G_u, \tilde{N}\tilde{S}_p/\tilde{N}\tilde{S}_p'),
\]

which may be computed using the free resolution, \( E_*G_u \) of \( k \) as right \( G_u \)-module. The resulting complex \( k[E_*G_u] \otimes_{kG_u} k[\tilde{N}\tilde{S}_p]/k[\tilde{N}\tilde{S}_p'] \) is a double complex isomorphic to the quotient of two double complexes, namely:

\[
(k[E_*G_u] \otimes_{kG_u} k[\tilde{N}\tilde{S}_p])/(k[E_*G_u] \otimes_{kG_u} k[\tilde{N}\tilde{S}_p'])
\]

\[
\cong k[ (E_*G_u \times_{G_u} \tilde{N}\tilde{S}_p)/(E_*G_u \times_{G_u} \tilde{N}\tilde{S}_p')].
\]

This last complex may be identified with the simplicial complex of the space

\[
(EG_u \times_{G_u} |\tilde{N}\tilde{S}_p|)/(EG_u \times_{G_u} |\tilde{N}\tilde{S}_p'|)
\]

\[
\cong EG_u \times_{G_u} |\tilde{N}\tilde{S}_p/\tilde{N}\tilde{S}_p'|.
\]

The last piece of the puzzle involves simplifying the spaces \( |\tilde{N}\tilde{S}_p/\tilde{N}\tilde{S}_p'| \). Since \( S \) is a skeletal subcategory of \( \tilde{S} \), there is an equivalence of categories \( \tilde{S} \simeq S \), inducing a homotopy equivalence of complexes (hence also of spaces) \( |\tilde{N}\tilde{S}| \simeq |NS| \). Note that \( NS \) inherits a \( G_u \)-action from \( N\tilde{S} \), and the map \( \tilde{S} \to S \) is \( G_u \)-equivariant. Consider the fibration

\[
X \to EG \times_G X \to BG
\]

associated to a group \( G \) and path-connected \( G \)-space \( X \). The resulting homotopy sequence breaks up into isomorphisms \( 0 \to \pi_i(X) \cong \pi_i(EG \times_G X) \to 0 \) for \( i \geq 2 \) and a short exact
sequence $0 \to \pi_1(X) \to \pi_1(EG \times_G X) \to G \to 0$. If there is a $G$-equivariant $f : X \to Y$ for a path-connected $G$-space $Y$, then for $i \geq 2$, we have isomorphisms

$$\pi_i(EG \times_G X) \leftarrow \pi_i(X) \xrightarrow{f_*} \pi_i(Y) \to \pi_i(EG \times_G Y),$$

and a diagram corresponding to $i = 1$:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(EG \times_G X) & \longrightarrow & G & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_1(Y) & \longrightarrow & \pi_1(EG \times_G Y) & \longrightarrow & G & \longrightarrow & 0 \\
& & & & & & & & \\
& & & & & & (\text{id} \times f)_* & & \\
\end{array}
\]

Thus, there is a weak equivalence $EG \times_G X \to EG \times_G Y$. In our case, we wish to obtain weak equivalences:

$$EG_u \times_{G_u} |N\tilde{S}_p| \to EG_u \times_{G_u} |N\tilde{S}_p|$$

and

$$EG_u \times_{G_u} |N\tilde{S}_p'| \to EG_u \times_{G_u} |N\tilde{S}_p'|,$$

inducing a weak equivalence

$$EG_u \times_{G_u} |N\tilde{S}_p' / N\tilde{S}_p'| \to EG_u \times_{G_u} |N\tilde{S}_p / N\tilde{S}_p'|.$$

This will follow as long as the spaces $|N\tilde{S}_p'|$ and $|N\tilde{S}_p'|$ are path-connected. (Note, $|N\tilde{S}_p|$ and $|N\tilde{S}_p|$ are path-connected because they are contractible). In fact, we need only check $|N\tilde{S}_p'|$, since this space is homotopy-equivalent to $|N\tilde{S}_p'|$.

**Lemma 79.** For $p > 2$, $|N\tilde{S}_p'|$ is path-connected.

**Proof.** Assume $p > 2$ and let $W_0 := z_0z_1 \otimes z_2 \otimes \ldots \otimes z_p$. This represents a vertex of $N\tilde{S}_p'$. Suppose $W = Z_0 \otimes \ldots \otimes Z_i'z_0z_1Z_i'' \otimes \ldots \otimes Z_s$. Then there is a morphism $W_0 \to W$, hence an
edge between $W_0$ and $W$. Next, suppose $W = Z_0 \otimes \ldots \otimes Z_i z_0 Z_i' z_1 Z_i'' \otimes \ldots \otimes Z_s$. There is a path:

$$Z_0 \otimes \ldots \otimes Z_i z_0 Z_i' z_1 Z_i'' \otimes \ldots \otimes Z_s$$

$$\downarrow$$

$$Z_0 Z_1 \ldots Z_i z_0 Z_i' z_1 Z_i'' \ldots Z_s$$

$$\uparrow$$

$$Z_0 Z_1 \ldots Z_i' Z_i'' z_0 Z_i' z_1 Z_i'' \ldots Z_s$$

$$\downarrow$$

$$Z_0 Z_1 \ldots Z_i' Z_i'' Z_i' z_1 Z_i'' \ldots Z_s$$

$$\uparrow$$

$$Z_0 Z_1 \ldots Z_i'' Z_i' Z_i'' \otimes Z_i' Z_i'' \ldots Z_s$$

$$\downarrow$$

$$Z_0 z_1 Z_i'' \ldots Z_s \otimes Z_0 Z_1 \ldots Z_i' Z_i''$$

$$\uparrow$$

$$W_0$$

Similarly, if $W = Z_0 \otimes \ldots \otimes Z_i' z_1 Z_i'' Z_i' z_0 Z_i'' \otimes \ldots \otimes Z_s$, there is a path to $W_0$. Finally, if $W = Z_0 \otimes \ldots \otimes Z_s$ with $z_0$ occurring in $Z_i$ and $z_1$ occurring in $Z_j$ for $i \neq j$, there is an edge to some $W'$ in which $Z_i Z_j$ occurs, and thus a path to $W_0$. \hfill \Box

The above discussion coupled with Lemma 78 produces the required isomorphism in homology, hence proving Thm. 77 for $p > 2$:

$$E_{p,q}^1 = \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_{p+q}(\mathcal{M}_u) \cong \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_{p+q} \left( EG_u \times_{G_u} |\tilde{N}\tilde{S}_p/\tilde{N}\tilde{S}'_p|; k \right)$$

$$\cong \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_{p+q} \left( EG_u \times_{G_u} |N S_p/N S'_p|; k \right).$$

The cases $p = 0, 1$ and 2 are handled individually:

Observe that $|N \tilde{S}'_0|$ and $|N S'_0|$ are empty spaces, since $\tilde{S}_0'$ has no objects.

$$EG_u \times_{G_u} |N \tilde{S}'_0| = EG_u \times_{G_u} |N S'_0| = \emptyset.$$
Furthermore, any group $G_u$ must be trivial. Thus,

$$H_q \left( EG_u \ltimes_{G_u} |\tilde{N}\tilde{S}_0/\tilde{N}\tilde{S}'_0|; k \right) = H_q \left( |\tilde{N}\tilde{S}_0|; k \right)$$

$$\cong H_q (|N\tilde{S}_0|; k) = H_q \left( EG_u \ltimes_{G_u} |N\tilde{S}_p/N\tilde{S}'_p|, k \right),$$

completing the theorem for $p = 0$.

Next, since $|\tilde{N}\tilde{S}'_1|$ is homeomorphic to $|N\tilde{S}'_1|$, each space consisting of the two discrete points $z_0z_1$ and $z_1z_0$, the theorem is true for $p = 1$ as well.

For $p = 2$, observe that $|\tilde{N}\tilde{S}'_2|$ has two connected components, $\tilde{U}_1$ and $\tilde{U}_2$ that are interchanged by any odd permutation $\sigma \in \Sigma_3$. Similarly, $|N\tilde{S}'_2|$ consists of two connected components, $U_1$ and $U_2$, interchanged by any odd permutation of $\Sigma_3$. Now, restricted to the alternating group, $A\Sigma_3$, we certainly have weak equivalences for any subgroup $H_u \subseteq A\Sigma_3$,

$$EH_u \times_{H_u} \tilde{U}_1 \cong EH_u \times_{H_u} U_1,$$

$$EH_u \times_{H_u} \tilde{U}_2 \cong EH_u \times_{H_u} U_2.$$

The action of an odd permutation induces equivariant homeomorphisms

$$\tilde{U}_1 \cong \tilde{U}_2,$$

$$U_1 \cong U_2,$$

and so if we have a subgroup $G_u \subseteq \Sigma_3$ generated by $H_u \subseteq A\Sigma_3$ and a transposition, then the two connected components are identified in an $A\Sigma_3$-equivariant manner. Thus, if $G_u$ contains a transposition,

$$EG_u \times_{G_u} |\tilde{N}\tilde{S}'_2| \cong EH_u \times_{H_u} \tilde{U}_1 \cong EH_u \times_{H_u} U_1 \cong EG_u \times_{G_u} |N\tilde{S}'_2|.$$

This completes the case $p = 2$ and the proof of Thm. 77.

**Corollary 80.** If the augmentation ideal of $A$ satisfies $I^2 = 0$, then

$$HS_n(A) \cong \bigoplus_{p \geq 0} \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_n(EG_u \ltimes_{G_u} N\tilde{S}_p/N\tilde{S}'_p; k).$$
Proof. This follows from consideration of the original $E^0$ term of the spectral sequence. $E^0$ is generated by chains $Y \otimes \phi_0 \otimes \ldots \otimes \phi_n$, with induced differential $d^0$, agreeing with the differential $d$ of $\mathcal{F}_*$ when $\phi_0$ is an isomorphism. When $\phi_0$ is a strict epimorphism, however, $d^0 = d - \partial_0$. But if $\phi_0$ is strictly epic, then

$$\partial_0(Y \otimes \phi_0 \otimes \ldots \otimes \phi_n) = (\phi_0)_*(Y) \otimes \phi_1 \otimes \ldots \otimes \phi_n = 0,$$

since $(\phi_0)_*(Y)$ would have at least one tensor factor that is the product of two or more elements of $I$, hence, $d^0$ agrees with $d$ in the case that $\phi_0$ is strictly epic. But if $d^0$ agrees with $d$ for all elements, the spectral sequence collapses at level 1. \hfill \Box

4.2 The complex $Sym_+^{(p)}$

Note, for $p > 0$, there are homotopy equivalences

$$|NS_p/NS'_p| \simeq |NS_p| \vee S|NS'_p| \simeq S|NS'_p|,$$

since $|NS_p|$ is contractible. $|NS_p|$ is a disjoint union of $(p + 1)!$ $p$-cubes, identified along certain faces. Geometric analysis of $S|NS'_p|$, however, seems quite difficult. Fortunately, there is an even smaller chain complex that computes the homology of $|NS_p/NS'_p|$.

**Definition 81.** Let $p \geq 0$ and impose an equivalence relation on $k[Epi_{\Delta S}([p], [q])]$ generated by:

$$Z_0 \otimes \ldots \otimes Z_i \otimes Z_{i+1} \otimes \ldots \otimes Z_q \approx (-1)^{ab}Z_0 \otimes \ldots \otimes Z_{i+1} \otimes Z_i \otimes \ldots \otimes Z_q,$$

where $Z_0 \otimes \ldots \otimes Z_q$ is a morphism expressed in tensor notation, and $a = \deg(Z_i) := |Z_i| - 1$, $b = \deg(Z_{i+1}) := |Z_{i+1}| - 1$. Here, $\deg(Z)$ is one less than the number of factors of the monomial $Z$. Indeed, if $Z = z_{i_0} z_{i_1} \ldots z_{i_s}$, then $\deg(Z) = s$.

The complex $Sym_+^{(p)}$ is then defined by:

$$Sym_+^{(p)} := k[Epi_{\Delta S}([p], [p - i])] / \sim,$$  (4.4)
The face maps will be defined recursively. On monomials,

\[
\partial_i(z_{j_0} \cdots z_{j_s}) = \begin{cases} 
0, & i < 0, \\
z_{j_0} \cdots z_{i-1} \otimes z_{i+1} \cdots z_{j_s}, & 0 \leq i < s, \\
0, & i \geq s.
\end{cases}
\]  

(4.5)

Then, extend \(\partial_i\) to tensor products via:

\[
\partial_i(W \otimes V) = \partial_i(W) \otimes V + W \otimes \partial_{i-deg(W)}(V).
\]  

(4.6)

In the above formula, \(W\) and \(V\) are formal tensors in \(k[\text{Epi}_{\Delta S}(p, q)]\), and

\[
\text{deg}(W) = \text{deg}(W_0 \otimes \cdots \otimes W_t) := \sum_{k=0}^t \text{deg}(W_k).
\]

The boundary map \(\text{Sym}_n^{(p)} \to \text{Sym}_{n-1}^{(p)}\) is then

\[
d_n = \sum_{i=0}^n (-1)^i \partial_i = \sum_{i=0}^{n-1} (-1)^i \partial_i
\]

Remark 82. The result of applying \(\partial_i\) on any formal tensor will result in only a single formal tensor, since in eq. 4.6, at most one of the two terms will be non-zero).

Remark 83. There is an action \(\Sigma_{p+1} \times \text{Sym}_i^{(p)} \to \text{Sym}_i^{(p)}\), given by permuting the formal indeterminates \(z_i\). Furthermore, this action is compatible with the differential.

Lemma 84. \(\text{Sym}_n^{(p)}\) is chain-homotopy equivalent to \(k[\text{NS}_p]/k[\text{NS}_p']\).

Proof. Let \(v_0\) represent the common initial vertex of the \(p\)-cubes making up \(\text{NS}_p\). Then, as cell-complex, \(\text{NS}_p\) consists of \(v_0\) together with all corners of the various \(p\)-cubes, together with \(i\)-cells for each \(i\)-face of the cubes. Thus, \(\text{NS}_p\) consists of \((p+1)!\) \(p\)-cells with attaching maps \(\partial I^p \to (\text{NS}_p)^{p-1}\) defined according to the face maps for \(\text{NS}_p\) given above. Presently, I shall provide an explicit construction.

Label each top-dimensional cell with the permutation induced on \(\{0, 1, \ldots, p\}\) by the final vertex, \(z_{i_0} z_{i_1} \ldots z_{i_p}\). On a given \(p\)-cell, for each vertex \(Z_0 \otimes \cdots \otimes Z_s\), there is an ordering of the tensor factors so that \(Z_0 \otimes \cdots \otimes Z_s \to z_{i_0} z_{i_1} \ldots z_{i_p}\) preserves the order of formal indeterminates \(z_i\). Rewrite each vertex of this \(p\)-cell in this order. Now, any chain

\[
(z_{i_0} \otimes z_{i_1} \otimes \cdots \otimes z_{i_p}) \to \ldots \to z_{i_0} z_{i_1} \ldots z_{i_p}
\]
is obtained by choosing the order in which to combine the factors. In fact, the $p$-chains are in bijection with the elements of $S_p$. A given permutation $\{1, 2, \ldots, p\} \mapsto \{j_1, j_2, \ldots, j_p\}$ will represent the chain obtained by first combining $z_{j_0} \otimes z_{j_1}$ into $z_{j_0} z_{j_1}$, then combining $z_{j_1} \otimes z_{j_2}$ into $z_{j_1} z_{j_2}$. In effect, we “erase” the tensor product symbol between $z_{j_r - 1}$ and $z_{j_r}$ for each $j_r$ in the list above.

We shall declare that the natural order of combining the factors will be the one that always combines the last two:

$$ (z_{i_0} \otimes \ldots \otimes z_{i_{p-1}} \otimes z_{i_p}) \rightarrow (z_{i_0} \otimes \ldots \otimes z_{i_{p-1}} z_{i_p}) \rightarrow (z_{i_0} \otimes z_{i_{p-2}} z_{i_{p-1}} z_{i_p}) \rightarrow \ldots $$

This corresponds to a permutation $\rho := \{1, \ldots, p\} \mapsto \{p, p - 1, \ldots, 2, 1\}$, and this chain will be regarded as positive. A chain $C_\sigma$, corresponding to another permutation, $\sigma$, will be regarded as positive or negative depending on the sign of the permutation $\sigma \rho^{-1}$. Finally, the entire $p$-cell should be identified with the sum

$$ \sum_{\sigma \in S_p} \text{sign}(\sigma \rho^{-1}) C_\sigma $$

It is this sign convention that permits the inner faces of the cube to cancel appropriately in the boundary maps. Thus we have a map on the top-dimensional chains:

$$ \theta_p : \text{Sym}_p^{(p)} \rightarrow \left( k[NS_p]/k[NS'_p] \right)_p. $$

Define $\theta_*$ for arbitrary $k$-cells by sending $Z_0 \otimes \ldots \otimes Z_{p-k}$ to the sum of $k$-length chains with source $z_0 \otimes \ldots \otimes z_p$ and target $Z_0 \otimes \ldots \otimes Z_{p-k}$ with signs determined by the natural order of erasing tensor product symbols of $z_0 \otimes \ldots \otimes z_p$, excluding those tensor product symbols that never get erased.

The following example should clarify the point. $W = z_3 z_0 \otimes z_1 \otimes z_2 z_4$ is a 2-cell of $\text{Sym}_4^{(4)}$. $W$ is obtained from $z_0 \otimes z_1 \otimes z_2 \otimes z_3 \otimes z_4 = z_3 \otimes z_0 \otimes z_1 \otimes z_2 \otimes z_4$ by combining factors in some order. There are only 2 erasable tensor product symbols in this example. The natural order (last to first) corresponds to the chain:

$$ z_3 \otimes z_0 \otimes z_1 \otimes z_2 \otimes z_4 \rightarrow z_3 \otimes z_0 \otimes z_1 \otimes z_2 z_4 \rightarrow z_3 z_0 \otimes z_1 \otimes z_2 z_4. $$
So, this chain shows up in $\theta_*(W)$ with positive sign, whereas the chain

$$z_3 \otimes z_0 \otimes z_1 \otimes z_2 \otimes z_4 \rightarrow z_3z_0 \otimes z_1 \otimes z_2 \otimes z_4 \rightarrow z_3z_0 \otimes z_1 \otimes z_2z_4$$

shows up with a negative sign.

Now, $\theta_*$ is easily seen to be a chain map $Sym_*(p) \rightarrow k[N\mathbb{S}_p]/k[N\mathbb{S}'_p]$. Geometrically, $\theta_*$ has the effect of subdividing a cell-complex (defined with cubical cells) into a simplicial space. □

As an example, consider $|N\mathbb{S}_2|$. There are 6 2-cells, each represented by a copy of $I^2$. The 2-cell labelled by the permutation $\{0, 1, 2\} \mapsto \{1, 0, 2\}$ consists of the chains

$$z_1 \otimes z_0 \otimes z_2 \rightarrow z_1 \otimes z_0 \otimes z_2 \rightarrow z_1z_0z_2$$

and

$$-(z_1 \otimes z_0 \otimes z_2 \rightarrow z_1z_0 \otimes z_2 \rightarrow z_1z_0z_2).$$

Hence, the boundary is the sum of 1-chains:

$$(z_1 \otimes z_0z_2 \rightarrow z_1z_0z_2) - (z_1 \otimes z_0 \otimes z_2 \rightarrow z_1z_0z_2) + (z_1 \otimes z_0 \otimes z_2 \rightarrow z_1z_0z_2)$$

$$-(z_1z_0 \otimes z_2 \rightarrow z_1z_0z_2) + (z_1 \otimes z_0 \otimes z_2 \rightarrow z_1z_0z_2) - (z_1 \otimes z_0 \otimes z_2 \rightarrow z_1z_0 \otimes z_2)$$

$$= (z_1 \otimes z_0 \otimes z_2 \rightarrow z_1z_0z_2) + (z_1 \otimes z_0 \otimes z_2 \rightarrow z_1 \otimes z_0z_2)$$

$$-(z_1z_0 \otimes z_2 \rightarrow z_1z_0z_2) - (z_1 \otimes z_0 \otimes z_2 \rightarrow z_1z_0 \otimes z_2).$$

These 1-chains correspond to the 4 edges of the square.

Thus, in our example this 2-cell of $|N\mathbb{S}_p|$ will correspond to $z_1z_0z_2 \in Sym_2^{(2)}$, and its boundary in $|N\mathbb{S}_p/N\mathbb{S}'_p|$ will consist of the two edges adjacent to $z_0 \otimes z_1 \otimes z_2$ with appropriate signs:

$$(z_0 \otimes z_1 \otimes z_2 \rightarrow z_1 \otimes z_0z_2) - (z_0 \otimes z_1 \otimes z_2 \rightarrow z_1z_0 \otimes z_2).$$

The corresponding boundary in $Sym_1^{(2)}$ will be: $(z_1 \otimes z_0z_2) - (z_1z_0 \otimes z_2)$, matching with the differential already defined on $Sym_*(p)$. See Figs. 4.1 and 4.2.

Now, with one piece of new notation, we may re-interpret Thm. 77.
Figure 4.1: $|NS_2|$ consists of six squares, grouped into two hexagons that share a common center vertex.

**Definition 85.** Let $G$ be a group. Let $k_0$ be the chain complex consisting of $k$ concentrated in degree 0, with trivial $G$-action. If $X_*$ is a right $G$-complex, $Y_*$ is a left $G$-complex with $k_0 \hookrightarrow Y_*$ as a $G$-subcomplex, then define the *equivariant half-smash tensor product* of the two complexes:

$$X_* \otimes_G Y_* := (X_* \otimes kG Y_*) / (X_* \otimes kG k_0)$$

**Corollary 86.** There is spectral sequence converging weakly to $\tilde{H}S_* (A)$ with

$$E_{p,q}^1 \cong \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_{p+q} (E_* G_u \otimes_{G_u} Sym_*^{(p)} ; k) ,$$

where $G_u$ is the isotropy subgroup of the orbit $u \in X^{p+1}/\Sigma_{p+1}$.

### 4.3 Algebra Structure of $Sym_*$

We may consider $Sym_* := \bigoplus_{p \geq 0} Sym_*^{(p)}$ as a bigraded differential algebra, where $bideg(W) = (p + 1, i)$ for $W \in Sym_*^{(p)}$. The product

$$\boxtimes : Sym_*^{(p)} \otimes Sym_*^{(q)} \rightarrow Sym_*^{(p+q+1)}$$
Figure 4.2: $\text{Sym}^{(2)} \simeq N\mathcal{S}_2/N\mathcal{S}'_2$. The center of each hexagon is $z_0 \otimes z_1 \otimes z_2$.

is defined by:

$$ W \boxtimes V := W \otimes V', $$

where $V'$ is obtained from $V$ by replacing each formal indeterminate $z_r$ by $z_{r+p+1}$ for $0 \leq r \leq q$. Eq. 4.6 then implies:

$$ d(W \boxtimes V) = d(W) \boxtimes V + (-1)^{\text{bideg}(W)} W \boxtimes d(V), $$

(4.7)

where $\text{bideg}(W)$ is the second component of $\text{bideg}(W)$.

**Proposition 87.** The product $\boxtimes$ is defined on the level of homology. Furthermore, this product (on both the chain level and homology level) is skew commutative in a twisted sense:

$$ W \boxtimes V = (-1)^{ij} \tau(V \boxtimes W), $$

where $\text{bideg}(W) = (p+1, i)$, $\text{bideg}(V) = (q+1, j)$, and $\tau$ is the permutation sending

$$ \{0, 1, \ldots, q, q+1, q+2, \ldots, p+q, p+q+1\} \mapsto \{p+1, p+2, \ldots, p+q+1, 0, 1, \ldots, p-1, p\} $$

In fact, $\tau$ is nothing more than the block transformation $\beta_{q,p}$ defined in section 1.1.
Proof. By Eq. 4.7, the product of two cycles is again a cycle, and the product of a cycle with a boundary is a boundary, hence there is an induced product on homology.

Now, suppose $W \in \text{Sym}_i^{(p)}$ and $V \in \text{Sym}_j^{(p)}$. So,

$$W = Y_0 \otimes Y_1 \otimes \ldots \otimes Y_{p-i},$$

$$V = Z_0 \otimes Z_1 \otimes \ldots \otimes Z_{q-j}.$$

$$V \otimes W = V \otimes W' = (-1)^\alpha W' \otimes V, \quad (4.8)$$

where $W'$ is related to $W$ by replacing each $z_r$ by $z_{r+q+1}$. The exponent $\alpha$ is determined by the relations in $\text{Sym}_i^{(p+q+1)}$:

$$\alpha = \left[ \text{deg}(Z_0) + \ldots + \text{deg}(Z_{q-j}) \right] \left[ \text{deg}(Y_0) + \ldots + \text{deg}(Y_{p-i}) \right]$$

$$= \text{deg}(V) \text{deg}(W).$$

Observe that $\text{deg}(W)$ is exactly $i$ and $\text{deg}(V) = j$. Indeed, $W \in \text{Epi}_\Delta([p], [p-i])$, so there are exactly $i$ distinct positions where one could insert a tensor product symbol. That is, there are $i$ cut points in $W$. Since $\text{deg}(W) = \sum \text{deg}(Y_k)$, and $\text{deg}(Y_k)$ is one less than the number of factors in $Y_k$, it follows that $\text{deg}(Y_k)$ is exactly the number of cut points in $Y_k$. Hence, $\text{deg}(W) = i$.

Next, apply the block transformation $\tau$ to eq. 4.8 to obtain

$$\tau(V \otimes W) = (-1)^\alpha \tau(W' \otimes V) = (-1)^\alpha W \otimes V' = (-1)^\alpha W \otimes V,$$

where $V'$ is obtained by replacing $z_r$ by $z_{r+p+1}$ in $V$. \hfill \Box

4.4 Computer Calculations

In principle, the homology of $\text{Sym}_*^{(p)}$ may be found by using a computer. In fact, we have the following results up to $p = 7$:
**Theorem 88.** For $0 \leq p \leq 7$, the groups $H_\ast(Sym_\ast^{(p)})$ are free abelian and have Poincaré polynomials $P_p(t) := P\left(H_\ast(Sym_\ast^{(p)}); t\right)$:

\[
\begin{align*}
P_0(t) &= 1, \\
P_1(t) &= t, \\
P_2(t) &= t + 2t^2, \\
P_3(t) &= 7t^2 + 6t^3, \\
P_4(t) &= 43t^3 + 24t^4, \\
P_5(t) &= t^3 + 272t^4 + 120t^5, \\
P_6(t) &= 36t^4 + 1847t^5 + 720t^6, \\
P_7(t) &= 829t^5 + 13710t^6 + 5040t^7.
\end{align*}
\]

**Proof.** These computations were performed using scripts written for the computer algebra systems \textsc{GAP} [11] and \textsc{Octave} [8]. See Appendix A for more detail about the scripts used to calculate these polynomials. \hfill \Box

### 4.5 Representation Theory of $H_\ast(Sym_\ast^{(p)})$

By remark 83, the groups $H_i(Sym_\ast^{(p)}; k)$ are $k\Sigma_{p+1}$-modules, so it seems natural to investigate the irreducible representations comprising these modules.

**Proposition 89.** Let $C_{p+1} \hookrightarrow \Sigma_{p+1}$ be the cyclic group of order $p + 1$, embedded into the symmetric group as the subgroup generated by the permutation $\tau_p := (0, p, p-1, \ldots, 1)$. Then there is a $\Sigma_{p+1}$-isomorphism:

\[
H_p(Sym_\ast^{(p)}) \cong AC_{p+1} \uparrow \Sigma_{p+1},
\]

i.e., the alternating representation of the cyclic group, induced up to the symmetric group.

**Note,** for $p$ even, $AC_{p+1}$ coincides with the trivial representation $IC_{p+1}$.  

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Moreover, \( H_p(\text{Sym}_p) \) is generated by the elements \( \sigma(b_p) \), for the distinct cosets \( \sigma C_{p+1} \), where \( b_p \) is the element:

\[
b_p := \sum_{j=0}^{p} (-1)^{jp} \tau_p^j (z_0 z_1 \ldots z_p).
\]

**Proof.** Let \( w \) be a general element of \( \text{Sym}_p \).

\[
w = \sum_{\sigma \in \Sigma_{p+1}} c_{\sigma} \sigma(z_0 z_1 \ldots z_p),
\]

where \( c_{\sigma} \) are constants in \( k \). \( H_p(\text{Sym}_p) \) consists of those \( w \) such that \( d(w) = 0 \). That is,

\[
0 = \sum_{\sigma \in \Sigma_{p+1}} c_{\sigma} \sigma(z_0 \ldots z_i \otimes z_{i+1} \ldots z_p)
\]

\[
= \sum_{\sigma \in \Sigma_{p+1}} \sum_{i=0}^{p-1} (-1)^i c_{\sigma} \sigma(z_0 \ldots z_i \otimes z_{i+1} \ldots z_p).
\]

(4.9)

Now for fixed \( \sigma \), the terms corresponding to \( \sigma(z_0 \ldots z_i \otimes z_{i+1} \ldots z_p) \) occur in pairs in the above formula. The obvious term of the pair is

\[
(-1)^i c_{\sigma} \sigma(z_0 \ldots z_i \otimes z_{i+1} \ldots z_p).
\]

Not so obviously, the second term of the pair is

\[
(-1)^{(p-i-1)i} (-1)^{p-i-1} c_p \rho(z_0 \ldots z_{p-i-1} \otimes z_{p-i} \ldots z_p),
\]

where \( \rho = \sigma \tau_p^{p-i} \). Thus, if \( d(w) = 0 \), then

\[
(-1)^i c_{\sigma} + (-1)^{(p-i-1)(i+1)} c_\rho = 0,
\]

\[
c_\rho = (-1)^{(p-i-1)(i+1)+i} c_{\sigma} = (-1)^{(p-i)(i+1)} c_{\sigma}.
\]

Set \( j = p - i \), so that

\[
c_\rho = (-1)^{j(p-j+1)} c_{\sigma} = (-1)^{jp-j^2 + j} c_{\sigma} = (-1)^{jp} c_{\sigma}.
\]

This proves that the only restrictions on the coefficients \( c_{\sigma} \) are that the absolute values of coefficients corresponding to \( \sigma, \sigma \tau_p, \sigma \tau_p^2, \ldots \) must be the same, and their corresponding signs
in \( w \) alternate if and only if \( p \) is odd; otherwise, they have the same signs. Clearly, the elements \( \sigma(b_p) \) for distinct cosets \( \sigma C_{p+1} \) represents an independent set of generators over \( k \) for \( H_p(Sym_\ast^{(p)}) \).

Observe that \( b_p \) is invariant under the action of \( sign(\tau_p) \tau_p \), and so \( b_p \) generates an alternating representation \( AC_{p+1} \) over \( k \). Induced up to \( \Sigma_{p+1} \), we obtain the representation \( AC_{p+1} \uparrow \Sigma_{p+1} \) of dimension \( (p+1)!/(p+1) = p! \), generated by the elements \( \sigma(b_p) \) as in the proposition. \( \Box \)

**Definition 90.** For a given proper partition \( \lambda = [\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_s] \) of the \( p+1 \) integers \( \{0, 1, \ldots, p\} \), an element \( W \) of \( Sym_\ast^{(p)} \) will designated as type \( \lambda \) if it equivalent to \( \pm(Y_0 \otimes Y_1 \otimes Y_2 \otimes \ldots \otimes Y_s) \) with \( deg(Y_i) = \lambda_i - 1 \). That is, each \( Y_i \) has \( \lambda_i \) factors.

The notation \( Sym_\lambda^{(p)} \) or \( Sym_\lambda \) will denote the \( k \)-submodule of \( Sym_{p-s}^{(p)} \) generated by all elements of type \( \lambda \).

In what follows, \( |\lambda| \) will refer to the number of components of \( \lambda \). The action of \( \Sigma_{p+1} \) leaves \( Sym_\lambda \) invariant for any given \( \lambda \), so the there is a decomposition

\[
Sym_{p-s}^{(p)} = \bigoplus_{\lambda \vdash (p+1), |\lambda| = s+1} Sym_\lambda
\]

as \( k\Sigma_{p+1} \)-module.

**Proposition 91.** For a given proper partition \( \lambda \vdash (p+1) \),

(a) \( Sym_\lambda \) contains exactly one alternating representation \( A\Sigma_{p+1} \) iff \( \lambda \) contains no repeated components.

(b) \( Sym_\lambda \) contains exactly one trivial representation \( I\Sigma_{p+1} \) iff \( \lambda \) contains no repeated even components.

**Proof.** \( Sym_\lambda \) is a quotient of the regular representation, since it is the image of the \( \Sigma_{p+1} \)-map

\[
\pi_\lambda : k\Sigma_{p+1} \rightarrow Sym_\lambda
\]

\[
\sigma \mapsto \psi_\lambda(\sigma),
\]

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where $\sigma \in \Sigma_{p+1}$ is the $\Delta S$ automorphism of $[p]$ corresponding to $\sigma$ and $\psi_\lambda$ is a $\Delta$ morphism $[p] \to [|\lambda|]$ that sends the points $0, \ldots, \lambda_0 - 1$ to $0$, the points $\lambda_0, \ldots, \lambda_0 + \lambda_1 - 1$ to $1$, and so on. Hence, there can be at most 1 copy of $A\Sigma_{p+1}$ and at most 1 copy of $I\Sigma_{p+1}$ in $Sym_\lambda$.

Let $W$ be the “standard” element of $Sym_\lambda$. That is, the indeterminates $z_i$ occur in $W$ in numerical order. $A\Sigma_{p+1}$ exists in $Sym_\lambda$ iff the element

$$V = \sum_{\sigma \in \Sigma_{p+1}} \text{sign}(\sigma)\sigma(W)$$

is non-zero.

Suppose that some component of $\lambda$ is repeated, say $\lambda_i = \lambda_{i+1} = \ell$. If $W = Y_0 \otimes Y_1 \otimes \ldots \otimes Y_s$, then $\text{deg}(Y_i) = \text{deg}(Y_{i+1}) = \ell - 1$. Now, we know that

$$W = (-1)^{\text{deg}(Y_i)\text{deg}(Y_{i+1})}Y_0 \otimes \ldots \otimes Y_{i+1} \otimes Y_i \otimes \ldots Y_s$$

$$= (-1)^{\ell^2 - 2\ell + 1}\alpha(W)$$

$$= -(-1)^{\ell}\alpha(W),$$

for the permutation $\alpha \in \Sigma_{p+1}$ that exchanges the indices of indeterminates in $Y_i$ with those in $Y_{i+1}$ in an order-preserving way. In $V$, the term $\alpha(W)$ shows up with sign $\text{sign}(\alpha) = (-1)^{\ell^2} = (-1)^{\ell}$, thus cancelling with $W$. Hence, $V = 0$, and no alternating representation exists.

If, on the other hand, no component of $\lambda$ is repeated, then no term $W$ can be equivalent to $\pm\alpha(W)$ for $\alpha \neq \text{id}$, so $V$ survives as the generator of $A\Sigma_{p+1}$ in $Sym_\lambda$.

A similar analysis applies for trivial representations. This time, we examine

$$U = \sum_{\sigma \in \Sigma_{p+1}} \sigma(W),$$

which would be a generator for $I\Sigma_{p+1}$ if it were non-zero.

As before, if there is a repeated component, $\lambda_i = \lambda_{i+1} = \ell$, then

$$W = (-1)^{\ell - 1}\alpha(W).$$
However, this time, $W$ cancels with $\alpha(W)$ only if $\ell - 1$ is odd. That is, $|\lambda_i| = |\lambda_{i+1}|$ is even. If $\ell - 1$ is even, or if all $\lambda_i$ are distinct, then the element $U$ must be non-zero.

**Proposition 92.** $H_i(Sym_{p+1}^\epsilon)$ contains an alternating representation for each partition $\lambda \vdash (p + 1)$ with $|\lambda| = p - i$ such that no component of $\lambda$ is repeated.

**Proof.** This proposition will follow from the fact that $d(V) = 0$ for any generator $V$ of an alternating representation in $Sym_\lambda$. Then, by Schur’s Lemma, the alternating representation must survive at the homology level.

Let $V$ be the generator mentioned above,

$$V = \sum_{\sigma \in \Sigma_{p+1}} \text{sign}(\sigma)\sigma(W).$$

$d(V)$ consists of terms $\partial_j(\sigma(W)) = \sigma(\partial_j(W))$ along with appropriate signs.

For a given, $j$, write

$$\partial_j(W) = (-1)^{a+\ell}Y_0 \otimes \ldots \otimes Y_1\{0, \ldots, \ell\} \otimes Y_i\{\ell + 1, \ldots, m\} \otimes \ldots \otimes Y_s,$$

where if $Y = z_{i_0}z_{i_1} \ldots z_{i_r}$, then the notation $Y\{s, \ldots, t\}$ refers to the monomial $z_{i_s}z_{i_{s+1}} \ldots z_{i_t}$, assuming $0 \leq s \leq t \leq r$. In the above expression, $a = \text{deg}(Y_0) + \ldots + \text{deg}(Y_{i-1})$.

Now, we may use the relations in $Sym_\ast$ to rewrite eq. 4.10 as

$$(-1)^{a+\ell+\ell(m-\ell-1)}Y_0 \otimes \ldots \otimes Y_1\{\ell + 1, \ldots, m\} \otimes Y_i\{0, \ldots, \ell\} \otimes \ldots \otimes Y_s.$$ (4.11)

Let $\alpha$ be the block permutation that relabels indices thus:

$$(-1)^{a+m\ell-\ell^2}\alpha(Y_0 \otimes \ldots \otimes Y_i\{0, \ldots, m - \ell - 1\} \otimes Y_i\{m - \ell, \ldots, m\} \otimes \ldots \otimes Y_s)$$ (4.12)

Now, The above tensor product also occurs in $\partial_{j'}(\text{sign}(\alpha)\alpha(W))$ for some $j'$. This term looks like:

$$\text{sign}(\alpha)(-1)^{a+\ell}Y_0 \otimes \ldots \otimes Y_i\{0, \ldots, m - \ell - 1\} \otimes Y_i\{m - \ell, \ldots, m\} \otimes \ldots \otimes Y_s$$ (4.13)

$$= (-1)^{(m-\ell)(\ell+1)+a+m-\ell-1}\alpha(Y_0 \otimes \ldots \otimes Y_i\{0, \ldots, m - \ell - 1\} \otimes Y_i\{m - \ell, \ldots, m\} \otimes \ldots \otimes Y_s)$$ (4.14)
\[ = (-1)^{m\ell - \ell^2 + a - 1} \alpha (Y_0 \otimes \ldots \otimes Y_{\ell} \{0, \ldots, m - \ell - 1\} \otimes Y_{\ell} \{m - \ell, \ldots, m\} \otimes \ldots \otimes Y_s) \quad (4.15) \]

Compare the sign of eq. 4.15 with that of eq. 4.12. I claim the signs are opposite, which would mean the two terms cancel each other in \(d(V)\). Indeed, all that we must show is that

\[
(a + m\ell - \ell^2) + (m\ell - \ell^2 + a - 1) \equiv 1 \mod 2,
\]

which is obviously true.

By Proposition 92, it is clear that if \(p + 1\) is a triangular number – i.e., \(p + 1\) is of the form \(r(r + 1)/2\) for some positive integer \(r\), then the lowest dimension in which an alternating representation may occur is \(p + 1 - r\), corresponding to the partition \(\lambda = [r, r - 1, \ldots, 2, 1]\).

A little algebra yields the following statement for any \(p\):

**Corollary 93.** \(H_i(Sym^{(p)}_*)\) contains an alternating representation in degree \(p + 1 - r\), where

\[
r = \lfloor \sqrt{2p + 9}/4 - 1/2 \rfloor.
\]

Moreover, there are no alternating representations present for \(i \leq p - r\).

**Proof.** Simply solve \(p + 1 = r(r + 1)/2\) for \(r\), and note that the increase in \(r\) occurs exactly when \(p\) hits the next triangular number.

There is not much known about the other irreducible representations occurring in the homology groups of \(Sym^{(p)}_*\), however computational evidence shows that \(H_i(Sym^{(p)}_*)\) contains no trivial representation, \(I\Sigma_{p+1}\), for \(i \leq p - r\) (\(r\) as in the conjecture above) up to \(p = 50\).

### 4.6 Connectivity of \(Sym^{(p)}_*\)

Quite recently, Vrečica and Živaljević [34] observed that the complex \(Sym^{(p)}_*\) is isomorphic to the suspension of the cycle-free chessboard complex \(\Omega_{p+1}^+\) (in fact, the isomorphism takes the form \(k[\Omega_{p+1}^+] \to Sym^{(p)}_*\), where \(\Omega_{p+1}^+\) is the augmented complex).
The $m$-chains of the complex $\Omega_n$ are generated by lists

$$L = \{(i_0, j_0), (i_1, j_1), \ldots, (i_m, j_m)\},$$

where $1 \leq i_0 < i_1 < \ldots < i_m \leq n$, all $1 \leq j_s \leq n$ are distinct integers, and the list $L$ is cycle-free. It may be easier to say what it means for $L$ not to be cycle free: $L$ is not cycle-free if there exists a subset $L_c \subseteq L$ and re-ordering of $L_c$ so that

$$L_c = \{(\ell_0, \ell_1), (\ell_1, \ell_2), \ldots, (\ell_{t-1}, \ell_t), (\ell_t, \ell_0)\}.$$ 

The differential of $\Omega_n$ is defined on generators by:

$$d\left(\{(i_0, j_0), \ldots, (i_m, j_m)\}\right) := \sum_{s=0}^{m} (-1)^s \{(i_0, j_0), \ldots, (i_{s-1}, j_{s-1}), (i_{s+1}, j_{s+1}), \ldots, (i_m, j_m)\}.$$ 

For completeness, an explicit isomorphism shall be provided:

**Proposition 94.** Let $\Omega_n^+$ denote the augmented cycle-free $(n \times n)$-chessboard complex, where the unique $(-1)$-chain is represented by the empty $n \times n$ chessboard, and the boundary map on 0-chains takes a vertex to the unique $(-1)$-chain. For each $p \geq 0$, there is a chain isomorphism

$$\omega_* : k[S\Omega_{p+1}^+] \rightarrow Sym_*^{(p)}$$

**Proof.** Note that we may define $m$-chains of $\Omega_{p+1}$ as cycle-free lists

$$L = \{(i_0, j_0), (i_1, j_1), \ldots, (i_m, j_m)\},$$

with no requirement on the order of $\{i_0, i_1, \ldots, i_m\}$, under the equivalence relation:

$$\{(i_{\sigma^{-1}(0)}, j_{\sigma^{-1}(0)}), \ldots, (i_{\sigma^{-1}(m)}, j_{\sigma^{-1}(m)})\} \approx \text{sign}(\sigma)\{(i_0, j_0), \ldots, (i_m, j_m)\},$$

for $\sigma \in \Sigma_{m+1}$. 

Suppose $L$ is an $(m + 1)$-chain of $S\Omega_{p+1}^+$ (*i.e.* an $m$-chain of $\Omega_{p+1}^+$). Call a subset $L' \subseteq L$ a queue if there is a reordering of $L'$ such that

$$L' = \{((\ell_0, \ell_1), (\ell_1, \ell_2), \ldots, (\ell_{t-1}, \ell_t)\},$$
and $L'$ is called a maximal queue if it is not properly contained in any other queue. Since $L$ is supposed to be cycle-free, we can partition $L$ into some number of maximal queues, $L'_1, L'_2, \ldots, L'_q$. Let $\sigma$ be a permutation representing the re-ordering of $L$ into maximal ordered queues.

$$L \approx \text{sign}(\sigma)\{(\ell_0^{(1)}, \ell_1^{(1)}), (\ell_1^{(1)}, \ell_2^{(1)}), \ldots, (\ell_{t_{i_1}}^{(1)}, \ell_{t_{i_1}}^{(1)}), \ldots, (\ell_0^{(q)}, \ell_1^{(q)}), (\ell_{t_{i_1}}^{(q)}, \ell_{t_{i_2}}^{(q)}), \ldots, (\ell_{t_{i_{q-1}}}^{(q)}, \ell_{t_{i_q}}^{(q)})\}$$

Each maximal ordered queue will correspond to a monomial of formal indeterminates $z_i$.

The correspondence is as follows:

$$\{(\ell_0, \ell_1), (\ell_1, \ell_2), \ldots, (\ell_{t-1}, \ell_t)\} \mapsto z_{i_0-1}z_{i_1-1} \cdots z_{t-1}. \quad (4.16)$$

For each maximal ordered queue, $L'_s$, denote the monomial obtained by formula (4.16) by $Z_s$.

Let $k_1, k_2, \ldots, k_u$ be the numbers in $\{0, 1, 2, \ldots, p\}$ such that $k_r + 1$ does not appear in any pair $(i_s, j_s) \in L$.

Now we may define $\omega_*$ on $L = L'_1 \cup L'_2 \cup \ldots \cup L'_q$.

$$\omega_{m+1}(L) := Z_1 \otimes Z_2 \otimes \ldots \otimes Z_q \otimes z_{k_1} \otimes z_{k_2} \otimes \ldots \otimes z_{k_u}.$$

Observe, if $L = \emptyset$ is the $(-1)$-chain of $\Omega_{p+1}^+$, then there are no maximal queues in $L$, and so

$$\omega_0(\emptyset) = z_0 \otimes z_1 \otimes \ldots \otimes z_p.$$

$\omega_*$ is a (well-defined) chain map due to the equivalence relations present in $Sym_*(p)$ (See formulas (4.4), (4.5), and (4.6)). To see that $\omega_*$ is an isomorphism, it suffices to exhibit an inverse. To each monomial $Z = z_{i_0}z_{i_1} \cdots z_{i_t}$ with $t > 0$, there is an associated ordered queue $L' = \{(i_0 + 1, i_1 + 1), (i_1 + 1, i_2 + 1), \ldots (i_{t-1} + 1, i_t + 1)\}$. If the monomial is a singleton, $Z = z_{i_0}$, the associated ordered queue will be the empty set. Now, given a generator $Z_1 \otimes Z_2 \otimes \ldots \otimes Z_q \in Sym_*(p)$, map it to the list $L := L'_1 \cup L'_2 \cup \ldots \cup L'_q$, preserving the original order of indices.

\[\square\]

**Theorem 95.** $Sym_*(p)$ is $\left\lfloor \frac{2}{3}(p-1) \right\rfloor$-connected.
Proof. See Thm. 10 of [34].

This remarkable fact yields the following useful corollaries:

**Corollary 96.** The spectral sequences of Thm. 77 and Cor. 86 converge strongly to $\tilde{H}S_*^*(A)$.

*Proof.* This relies on the fact that the connectivity of the complexes $Sym_*^{(p)}$ is a non-decreasing function of $p$. Fix $n \geq 0$, and consider the component of $E^1$ residing at position $p, q$ for $p + q = n$,

$$
\bigoplus_u H_n(E_u \otimes G_u Sym_*^{(p)}).
$$

A priori, the induced differentials whose sources are $E^1_{p,q}, E^2_{p,q}, E^3_{p,q}, \ldots$ will have as targets certain subquotients of

$$
\bigoplus_u H_{n-1}(E_u \otimes G_u Sym_*^{(p+1)}), \bigoplus_u H_{n-1}(E_u \otimes G_u Sym_*^{(p+2)}),
$$

$$
\bigoplus_u H_{n-1}(E_u \otimes G_u Sym_*^{(p+3)}), \ldots
$$

Now, if $n - 1 < \lfloor (2/3)(p + k - 1) \rfloor$ for some $k \geq 0$, then for $K > k$, we have

$$
H_{n-1}(Sym_*^{(p+K)}) = 0,
$$

hence also

$$
H_{n-1}(E_u \otimes G_u Sym_*^{(p+K)}) = 0,
$$

using the fibration mentioned in the proof of Thm. 77 and the Hurewicz Theorem. Thus, the induced differential $d^k$ is zero for all $k \geq K$.

On the other hand, the induced differentials whose targets are $E^1_{p,q}, E^2_{p,q}, E^3_{p,q}, \ldots$ must be zero after stage $p$, since there are no non-zero components with $p < 0$. 

*Corollary 97.* For each $i \geq 0$, there is a positive integer $N_i$ so that if $p \geq N_i$, there is an isomorphism

$$
H_i(\mathcal{G}_p \mathcal{Y}_*^i) \cong \tilde{H}S_i(A).
$$
Corollary 98. If $A$ is finitely-generated over a Noetherian ground ring $k$, then $H_{S_\ast}(A)$ is finitely-generated over $k$ in each degree.

Proof. Examination of the $E^1$ term shows that the $n^{th}$ reduced symmetric homology group of $A$ is a subquotient of:

$$\bigoplus_{p \geq 0} \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_n(E_u \otimes G_u Sym^{(p)}_u; k)$$

Each $H_n(E_u \otimes G_u Sym^{(p)}_u; k)$ is a finite-dimensional $k$-module. The inner sum is finite as long as $X$ is finite. Thm. 95 shows the outer sum is finite as well.

The bounds on connectivity are conjectured to be tight. This is certainly true for $p \equiv 1 \pmod{3}$, based on Thm. 16 of [34]. Corollary 12 of the same paper establishes the following result:

Either $H_{2k}(Sym^{(3k-1)}_u) \neq 0$ or $H_{2k}(Sym^{(3k)}_u) \neq 0$.

For $k \leq 2$, both statements are true. When the latter condition is true, this gives a tight bound on connectivity for $p \equiv 0 \pmod{3}$. When the former is true, there is not enough information for a tight bound, since we are more interested in proving that $H_{2k-1}(Sym^{(3k-1)}_u)$ is non-zero, since for $k = 1, 2$, we have computed the integral homology:

$$H_1(Sym^{(2)}_u) = \mathbb{Z} \quad \text{and} \quad H_3(Sym^{(5)}_u) = \mathbb{Z}.$$  

4.7 Filtering $Sym^{(p)}_u$ by partition types

In section 4.5, we saw that $Sym^{(n)}_u$ decomposes over $k\Sigma_{n+1}$ as a direct sum of the submodules $Sym_\lambda$ for partitions $\lambda \vdash (n+1)$. Filter $Sym^{(n)}_u$ by the size of the largest component of the partition.

$$\mathcal{F}_p Sym^{(n)}_u := \bigoplus_{\lambda \vdash (n+1), |\lambda| = n+1-(p+q), \lambda_0 \leq p+1} Sym_\lambda.$$
where $\lambda = [\lambda_0, \lambda_1, \ldots, \lambda_{n-q}]$, is written so that $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-q}$. The differential of $Sym^{(n)}$ respects this filtering, since it can only reduce the size of partition components. With respect to this filtering, we have an $E^0$ term for a spectral sequence:

$$E^0_{p,q} \cong \bigoplus_{\lambda \vdash (n+1), |\lambda| = n+1-(p+q), \lambda_0 = p+1} Sym_{\lambda}.$$ 

The vertical differential $d^0$ is induced from $d$ by keeping only those terms of $d(W)$ that share largest component with $W$. 

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CHAPTER 5

$E_\infty$-STRUCTURE AND HOMOLOGY OPERATIONS

5.1 Definitions

In this chapter, homology operations are defined for $HS_*(A)$, following May [18]. Let $Y^+A$ be the simplicial $k$-module of section 2.1. The key is to show that $Y^+A$ admits the structure of $E_\infty$-algebra. This will be accomplished using various guises of the Barratt-Eccles operad. While our final goal is to produce an action on the level of $k$-complexes, we must induce the structure from the level of categories and through simplicial $k$-modules. Finally, we may use Definitions 2.1 and 2.2 of [18] to define homology operations at the level $k$-complexes.

Remark 99. Throughout this chapter, we shall fix $S_n$ to be the symmetric group on the letters $\{1, 2, \ldots, n\}$, given by permutations $\sigma$ that act on the left of lists of size $n$. i.e.,

$$\sigma.(i_1, i_2, \ldots, i_n) = (i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \ldots, i_{\sigma^{-1}(n)}) ,$$

so that $(\sigma \tau).L = \sigma.(\tau.L)$ for all $\sigma, \tau \in S_n$, and an $n$-element list, $L$.

In this chapter, we make use of operad structures in various categories, so for completeness, formal definitions of symmetric monoidal category as well as operad, operad-algebra, and operad-module will be given below.

Definition 100. A category $\mathcal{C}$ is symmetric monoidal if there is a bifunctor $\odot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ together with:

1. A natural isomorphism,

$$a : \odot(id_{\mathcal{C}} \times \odot) \to \odot(\odot \times id_{\mathcal{C}})$$
satisfying the MacLane pentagon condition (commutativity of the following diagram for all objects $A, B, C, D$).

2. A unit object $e \in \text{Obj}_C$, together with natural isomorphisms

$$\ell : e \circ \text{id}_C \to \text{id}_C$$

$$r : \text{id}_C \circ e \to \text{id}_C$$

making the following diagram commute for all objects $A$ and $B$:

3. A natural transformation $s : \circ \to \circ T$, where $T : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the transposition functor $(A, B) \mapsto (B, A)$, such that $s^2 = \text{id}$ and $s$ satisfies the hexagon identity (commutativity of the following diagram for all objects $A, B, C$).
Observe that if $a_{A,B,C}$, $\ell_A$ and $r_A$ are identity morphisms, then this definition reduces to that of a permutative category.

Let $S$ denote the symmetric groupoid as category. For our purposes, we may label the objects by $0, 1, 2, \ldots$, and the only morphisms of $S$ are automorphisms, $\text{Aut}_S(n) := S_n$, the symmetric group on $n$ letters.

**Definition 101.** Suppose $C$ is a symmetric monoidal category, with unit $e$. An operad $\mathcal{P}$ in the category $C$ is a functor

$$\mathcal{P} : S^{\text{op}} \to C,$$

with $\mathcal{P}(0) = e$, together with the following data:

1. Morphisms $\gamma_{k,j_1,\ldots,j_k} : \mathcal{P}(k) \circ \mathcal{P}(j_1) \circ \ldots \circ \mathcal{P}(j_k) \to \mathcal{P}(j)$, where $j = \sum j_s$. For brevity, we denote these morphisms simply by $\gamma$. The morphisms $\gamma$ should satisfy the following associativity condition. The diagram below is commutative for all $k \geq 0$, $j_s \geq 0$, $i_r \geq 0$.

Here, $T$ is a map that permutes the components of the product in the specified way, using the symmetric transformation $s$ of $C$. Coherence of $s$ guarantees that this is a well-defined map.

**Associativity:**

$$\mathcal{P}(k) \circ \bigcirc_{s=1}^k \mathcal{P}(j_s) \circ \bigcirc_{r=1}^j \mathcal{P}(i_r) \xrightarrow{T} \mathcal{P}(k) \circ \bigcirc_{s=1}^k \left( \mathcal{P}(j_s) \circ \bigcirc_{r=j_1+\ldots+j_s-1+1}^{j_1+\ldots+j_s} \mathcal{P}(i_r) \right)$$

$$\gamma \circ \text{id} \circ j$$

$$\mathcal{P}(j) \circ \bigcirc_{r=1}^j \mathcal{P}(i_r)$$

$$\mathcal{P}(k) \circ \bigcirc_{s=1}^k \mathcal{P}\left( \sum_{r=1}^{j_1+\ldots+j_s} i_r \right)$$

$$\gamma$$

$$\mathcal{P}\left( \sum_{r=1}^{j_1+\ldots+j_k} i_r \right)$$
2. A Unit morphism $\eta: e \to \mathcal{P}(1)$ making the following diagrams commute:

**Left Unit Condition:**

\[
\begin{array}{ccc}
  e \odot \mathcal{P}(j) & \xrightarrow{\eta \odot \text{id}} & \mathcal{P}(1) \odot \mathcal{P}(j) \\
  \downarrow{\ell} & & \downarrow{\gamma} \\
  \mathcal{P}(j) & & \mathcal{P}(j)
\end{array}
\]

**Right Unit Condition:**

\[
\begin{array}{ccc}
  \mathcal{P}(j) \odot e \odot j & \xrightarrow{\text{id} \odot \eta \odot j} & \mathcal{P}(j) \odot \mathcal{P}(1) \odot j \\
  \downarrow{r^j} & & \downarrow{\gamma} \\
  \mathcal{P}(j) & & \mathcal{P}(j)
\end{array}
\]

Here, $r^j$ is the *iterated* right unit map defined recursively (for an object $A$ of $\mathcal{C}$):

\[
r^j_A := \begin{cases} 
  r_A, & j = 1 \\
  r^{j-1}_A \left( r_A \odot \text{id}^{\odot (j-1)} \right), & j > 1
\end{cases}
\]

3. The right action of $S_n$ on $\mathcal{P}(n)$ for each $n$ must satisfy the following *equivariance conditions*. Both diagrams below are commutative for all $k \geq 0$, $j_s \geq 0$, $(j = \sum j_s)$, $\sigma \in S^{\text{op}}_k$, and $\tau_s \in S^{\text{op}}_{j_s}$. Here, $T_\sigma$ is a morphism that permutes the components of the product in the specified way, using the symmetric transformation $s$. $\sigma\{j_1, \ldots, j_k\}$ denotes the permutation of $j$ letters which permutes the $k$ blocks of letters (of sizes $j_1, j_2, \ldots, j_k$) according to $\sigma$, and $\tau_1 \oplus \ldots \oplus \tau_k$ denotes the image of $(\tau_1, \ldots, \tau_k)$ under the evident inclusion $S^{\text{op}}_{j_1} \times \ldots \times S^{\text{op}}_{j_k} \hookrightarrow S^{\text{op}}_j$. 

\[
\begin{array}{ccc}
  \mathcal{P}(j) \odot e \odot j & \xrightarrow{T_\sigma \odot \eta \odot j} & \mathcal{P}(j) \odot (\mathcal{P}(1) \odot j) \\
  \downarrow{r^j} & & \downarrow{\gamma} \\
  \mathcal{P}(j) & & \mathcal{P}(j)
\end{array}
\]
Equivariance Condition A:

\[
P(k) \otimes \bigoplus_{s=1}^{k} P(j_s) \xrightarrow{id \otimes T_{\sigma}} P(k) \otimes \bigoplus_{s=1}^{k} P(j_{\sigma^{-1}(s)})
\]

\[
\xrightarrow{\sigma \otimes id \otimes k}
\]

\[
P(k) \otimes \bigoplus_{s=1}^{k} P(j_s) \xrightarrow{\gamma} P(j)
\]

Equivariance Condition B:

\[
P(k) \otimes \bigoplus_{s=1}^{k} P(j_s) \xrightarrow{\gamma} P(j)
\]

\[
\xrightarrow{id \otimes \tau_1 \otimes \ldots \otimes \tau_k}
\]

\[
P(k) \otimes \bigoplus_{s=1}^{k} P(j_s) \xrightarrow{\gamma} P(j)
\]

Definition 102. For a symmetric monoidal category \( \mathcal{C} \) with product \( \odot \) and an operad \( \mathcal{P} \) over \( \mathcal{C} \), a \( \mathcal{P} \)-algebra structure on an object \( X \) in \( \mathcal{C} \) is defined by a family of maps

\[
\chi : \mathcal{P}(n) \otimes_{S_n} X^\otimes n \rightarrow X,
\]

which are compatible with the multiplication, unit maps, and equivariance conditions of \( \mathcal{P} \).

Note, the symbol \( \otimes_{S_n} \) denotes an internal equivariance condition:

\[
\chi(\pi, \sigma \odot x_1 \odot \ldots \odot x_n) = \chi(\pi \odot x_{\sigma^{-1}(1)} \odot \ldots \odot x_{\sigma^{-1}(n)})
\]

If \( X \) is a \( \mathcal{P} \)-algebra, we will say that \( \mathcal{P} \) acts on \( X \).
**Definition 103.** Let $\mathcal{P}$ be an operad over the symmetric monoidal category $\mathcal{C}$, and let $\mathcal{M}$ be a functor $S^\text{op} \to \mathcal{C}$. A (left) $\mathcal{P}$-module structure on $\mathcal{M}$ is a collection of structure maps,

$$\mu : \mathcal{P}(n) \odot \mathcal{M}(j_1) \odot \ldots \odot \mathcal{M}(j_n) \to \mathcal{M}(j_1 + \ldots + j_n),$$

satisfying the evident compatibility relations with the operad multiplication of $\mathcal{P}$. For the precise definition, see [13].

In the course of this chapter, it shall become necessary to induce structures up from small categories to simplicial sets, then to simplicial $k$-modules, and finally to $k$-complexes. Each of these categories is symmetric monoidal. For notational convenience, all operads, operad-algebras, and operad-modules will carry a subscript denoting the ambient category over which the structure is defined:

| Category               | Sym. Mon. Product | Notation  |
|------------------------|-------------------|-----------|
| Small categories       | Cat               | $\times$ (product of categories) | $\mathcal{P}_{\text{cat}}$ |
| Simplicial sets        | SimpSet           | $\times$ (degree-wise set product) | $\mathcal{P}_{\text{ss}}$ |
| Simplicial $k$-modules | $k$-SimpMod       | $\hat{\otimes}$ (degree-wise tensor product) | $\mathcal{P}_{\text{sm}}$ |
| $k$-complexes          | $k$-Complexes     | $\otimes$ (tensor prod. of chain complexes) | $\mathcal{P}_{\text{ch}}$ |

**Remark 104.** The notation $\hat{\otimes}$, appearing in Richter [28], is useful for indicating degree-wise tensoring of graded modules:

$$(A_\ast \hat{\otimes} B_\ast)_n := A_n \otimes_k B_n,$$

as opposed to the standard tensor product (over $k$) of complexes:

$$(A_\ast \otimes B_\ast)_n := \bigoplus_{p+q=n} A_p \otimes_k B_q$$

Furthermore, we are interested in certain functors from one category to the next in the list. These functors preserve the symmetric monoidal structure in a sense we will make
precise in Section 5.2 – hence, it will follow that these functors send operads to operads, operad-modules to operad-modules, and operad-algebras to operad-algebras.

\[
\begin{array}{c}
\text{(Cat, } \times) \\
\downarrow N \quad \text{(Nerve of categories)} \\
\text{(SimpSet, } \times) \\
\downarrow k[-] \quad \text{(k-linearization)} \\
\text{(k-SimpMod, } \hat{\otimes}) \\
\downarrow N \quad \text{(Normalization functor)} \\
\text{(k-Complexes, } \otimes)
\end{array}
\]

Remark 105. Note, the normalization functor \(N\) is one direction of the Dold-Kan correspondence between simplicial modules and complexes.

Remark 106. The ultimate goal of this chapter is to construct an \(E_\infty\) structure on the chain complex associated with \(\mathcal{M}^+A\), i.e. an action by an \(E_\infty\)-operad. While we could define the notion of \(E_\infty\)-operad over general categories, it would require extra structure on the ambient symmetric monoidal category \((\mathcal{C}, \odot)\) – which the examples above possess. To avoid needless technicalities, we shall instead define versions of the Barratt-Eccles operad over each of our ambient categories, and take for granted that they are all \(E_\infty\)-operads. For a more general discussion of \(E_\infty\)-operads and algebras in the category of chain complexes, see [17].

Definition 107. \(\mathcal{D}_{\text{cat}}\) is the operad \(\{\mathcal{D}_{\text{cat}}(m)\}\) in the category \(\text{Cat}\), where \(\mathcal{D}_{\text{cat}}(0) = \ast\), \(\mathcal{D}_{\text{cat}}(m)\) is the category whose objects are the elements of \(S_m\), and for each pair of objects, \(\sigma, \tau\), we have \(\text{Mor}(\sigma, \tau) = \{\tau \sigma^{-1}\}\). The structure map (multiplication) \(\delta\) in \(\mathcal{D}_{\text{cat}}\) is a functor defined on objects by:

\[
\delta : \mathcal{D}_{\text{cat}}(m) \times \mathcal{D}_{\text{cat}}(k_1) \times \ldots \times \mathcal{D}_{\text{cat}}(k_m) \longrightarrow \mathcal{D}_{\text{cat}}(k), \quad \text{where } k = \sum k_i
\]

\[
(\sigma, \tau_1, \ldots, \tau_m) \mapsto \sigma \{k_1, \ldots, k_m\}(\tau_1 \oplus \ldots \oplus \tau_m)
\]
Here, $\tau_1 \oplus \ldots \oplus \tau_m \in S_k$ and $\sigma\{k_1, \ldots, k_m\} \in S_k$ are defined as in Definition 101. The functor takes the unique morphism

$$(\sigma, \tau_1, \ldots, \tau_m) \to (\rho, \psi_1, \ldots, \psi_m)$$

to the unique morphism

$$\sigma\{k_1, \ldots, k_m\}(\tau_1 \oplus \ldots \oplus \tau_m) \to \rho\{k_1, \ldots, k_m\}(\psi_1 \oplus \ldots \oplus \psi_m).$$

The action of $S_m^{\text{op}}$ on objects of $\mathcal{D}_{\text{cat}}(m)$ is given by right multiplication.

Remark 108. We are following the notation of May for our operad $\mathcal{D}_{\text{cat}}$ (See [20], Lemmas 4.3, 4.8). May’s notation for $\mathcal{D}_{\text{cat}}(m)$ is $\tilde{\Sigma}_m$, and he defines the related operad $\mathcal{D}$ over the category of spaces, as the geometric realization of the nerve of $\tilde{\Sigma}$. The nerve of $\mathcal{D}_{\text{cat}}$ is generally known in the literature as the Barratt-Eccles operad (See [1], where the notation for $N\mathcal{D}_{\text{cat}}$ is $\Gamma$).

5.2 Operad-Module Structure

In order to best define the $E_\infty$ structure of $\mathcal{Y}_s^+A$, we will begin with an operad-module structure over the category of small categories, then induce this structure up to the category of $k$-complexes.

Definition 109. Define for each $m \geq 0$, a category,

$$\mathcal{X}_{\text{cat}}(m) := [m - 1] \setminus \Delta S_+ = \underline{m} \setminus \mathcal{F}(\text{as}),$$

(See Section 1.1 for a definition of $\mathcal{F}(\text{as})$.)

Identifying $\Delta S_+$ with $\mathcal{F}(\text{as})$, we see that the morphism $(\phi, g)$ of $\Delta S_+$ consists of the set map $\phi$, precomposed with $g^{-1}$ in order to indicate the total ordering on all preimage sets. Thus, precomposition with symmetric group elements defines a right $S_m$-action on objects of $\underline{m} \setminus \mathcal{F}(\text{as})$. When writing morphisms of $\mathcal{F}(\text{as})$, we may avoid confusion by writing the
automorphisms as elements of the symmetric group rather than its opposite group, with the understanding that $g \in \text{Mor}\mathcal{F}(\text{as})$ corresponds to $g^{-1} \in \text{Mor}\Delta S_4$.

$$\mathcal{K}_{\text{cat}}(m) \times S_m \to \mathcal{K}_{\text{cat}}(m)$$

$$(\phi, g). h := (\phi, gh)$$

Let $m, j_1, j_2, \ldots, j_m \geq 0$, and let $j = \sum j_s$. We shall define a family of functors,

$$\mu = \mu_{m, j_1, \ldots, j_m} : D_{\text{cat}}(m) \times \prod_{s=1}^{m} \mathcal{K}_{\text{cat}}(j_s) \to \mathcal{K}_{\text{cat}}(j). \quad (5.1)$$

Assume that the pairs of morphisms $f_i, g_i$ ($1 \leq i \leq m$) have specified sources and targets:

$$j_{i} \xrightarrow{f_i} p_i \xrightarrow{g_i} q_i.$$ 

$\mu$ is defined on objects by:

$$\mu(\sigma, f_1, f_2, \ldots, f_m) := \sigma\{p_1, p_2, \ldots, p_m\}(f_1 \circ f_2 \circ \ldots \circ f_m). \quad (5.2)$$

For compactness of notation, denote

$$\underline{m} := \{1, \ldots, m\},$$

as ordered list, along with a left $S_m$ action,

$$\tau\underline{m} := \{\tau^{-1}(1), \ldots, \tau^{-1}(m)\}.$$ 

Then for any permutation, $\tau\underline{m}$, of the ordered list $\underline{m}$, and any list of $m$ numbers, $\{j_1, \ldots, j_m\}$, denote

$$\hat{j}_{\tau\underline{m}} := \{j_{\tau^{-1}(1)}, \ldots, j_{\tau^{-1}(m)}\}.$$ 

Furthermore, if $f_1, f_2, \ldots, f_m \in \text{Mor}\mathcal{F}(\text{as})$, denote

$$f_{\tau\underline{m}} := f_{\tau^{-1}(1)} \circ \ldots \circ f_{\tau^{-1}(m)}$$

so in particular, we may write:

$$\mu(\sigma, f_1, f_2, \ldots, f_m) = \sigma\{\underline{m}\} f_{\underline{m}}^\circ$$

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Using this notation, define $\mu$ on morphisms by:

\[
(\sigma, f_1, \ldots, f_m) \xrightarrow{\mu} \sigma\{p_m\} f_m^\odot
\]

\[
(\tau, g_1 f_1, \ldots, g_m f_m) \xrightarrow{\mu} \tau\{q_m\} (g_m f_m)^\odot
\]

(5.3)

It is useful to note three properties of the block permutations and symmetric monoidal product in the category $\coprod_{m \geq 0} \mathcal{K}_{\text{cat}}(m)$.

**Proposition 110.**

1. For $\sigma, \tau \in S_m$, and non-negative $p_1, p_2, \ldots, p_m$,

\[
(\sigma \tau)\{p_m\} = \sigma\{p_m\} \tau\{p_m\},
\]

2. For $\sigma \in S_m$, and morphisms $g_i \in \text{Mor}_{\mathcal{F}(as)}(p_i, q_i)$, (1 ≤ $i$ ≤ $m$),

\[
\sigma\{q_m\} g_m^\odot = g_m^\odot \sigma\{p_m\},
\]

3. For morphisms $j_i \xrightarrow{f_i} p_i \xrightarrow{g_i} q_i$,

\[
(g_m f_m)^\odot = g_m^\odot f_m^\odot
\]

**Proof.** Property 1 is a standard composition property of block permutations. See [17], p. 41, for example. Property 2 expresses the fact that it does not matter whether we apply the morphisms $g_i$ to blocks first, then permute those blocks, or permute the blocks first, then apply $g_i$ to the corresponding permuted block. Finally, property 3 is a result of functoriality of the product $\odot$.

Using these properties, it is straightforward to verify that $\mu$ is a functor. We shall show that $\mu$ defines a $\mathcal{D}_{\text{cat}}$-module structure on $\mathcal{K}_{\text{cat}}$.

First observe that if $(\mathcal{B}, \odot)$ is a permutative category, then $\mathcal{B}$ admits the structure of $E_\infty$-algebra. We may express this structure using the $E_\infty$-operad, $\mathcal{D}_{\text{cat}}$. The structure map is a family of functors

\[
\theta : \mathcal{D}_{\text{cat}}(m) \times \mathcal{B}^m \to \mathcal{B},
\]

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given on objects $C_i \in \text{Obj} \mathcal{B}$ by:

$$\theta(\sigma, C_1, \ldots, C_m) := C_{\sigma^{-1}(1)} \circ \cdots \circ C_{\sigma^{-1}(m)},$$

and on morphisms $(\sigma, C_1, \ldots, C_m) \to (\tau, D_1, \ldots, D_m)$ by:

$$(\sigma, C_1, \ldots, C_m) \xrightarrow{\theta} C_{\sigma^{-1}(1)} \circ \cdots \circ C_{\sigma^{-1}(m)}$$

\[ \downarrow \tau^{-1} \times f_1 \times \cdots \times f_m \]

\[ \Downarrow f_{\sigma^{-1}(1)} \circ \cdots \circ f_{\sigma^{-1}(m)} \]

$$(\tau, D_1, \ldots, D_m) \xrightarrow{\theta} D_{\tau^{-1}(1)} \circ \cdots \circ D_{\tau^{-1}(m)}$$

We just need to verify that $\theta$ satisfies the expected equivariance condition. Recall, the required condition is

$$\theta(\sigma \tau, C_1, \ldots, C_m) = \theta(\sigma, C_{\tau^{-1}(1)}, \ldots, C_{\tau^{-1}(m)})$$

This is easily verified on objects, as the left-hand side evaluates as:

$$\theta(\sigma \tau, C_1, \ldots, C_m) = C_{(\sigma \tau)^{-1}(1)} \circ \cdots \circ C_{(\sigma \tau)^{-1}(m)},$$

while the right-hand side evaluates as:

$$\theta(\sigma, C_{\tau^{-1}(1)}, \ldots, C_{\tau^{-1}(m)}) = C_{\tau^{-1}(\sigma^{-1}(1))} \circ \cdots \circ C_{\tau^{-1}(\sigma^{-1}(m))}$$

$$= C_{(\sigma \tau)^{-1}(1)} \circ \cdots \circ C_{(\sigma \tau)^{-1}(m)}.$$

Now, let $M$ be a monoid with unit, 1. Let $\mathcal{X}_s^+ M := N(\Delta \setminus \Delta \setminus S_+) \times_{\Delta \setminus S_+} B_{s_{\text{sym+}}}^s M$. This is analogous to the construction $\mathcal{X}_s$ of section 1.7.

**Proposition 111.** $\mathcal{X}_s^+ M$ is the nerve of a permutative category.

**Proof.** Consider a category $\mathcal{T}M$ whose objects are the elements of $\coprod_{n \geq 0} M^n$, where $M^0$ is understood to be the set consisting of the empty tuple, $\{()\}$. Morphisms of $\mathcal{T}M$ consist
of the morphisms of $\Delta S_+$, reinterpreted as follows: A morphism $f : [p] \to [q]$ in $\Delta S$ will be considered a morphism $(m_0, m_1, \ldots, m_p) \to f(m_0, m_1, \ldots, m_p) \in M^{q+1}$. The unique morphism $\iota_n$ will be considered a morphism $(1, 1, \ldots, 1) \in M^{n+1}$. The nerve of $\mathcal{M}$ consists of chains, 

$$(m_0, \ldots, m_n) \xrightarrow{f_1} f_1(m_0, \ldots, m_n) \xrightarrow{f_2} \ldots \xrightarrow{f_i} f_i \ldots f_1(m_0, \ldots, m_n)$$

This chain can be rewritten uniquely as an element of $M^n$ together with a chain in $N\Delta S$:

$$\left([n] \xrightarrow{f_1} [n_1] \xrightarrow{f_2} \ldots \xrightarrow{f_i} [n_i], (m_0, \ldots, m_n)\right),$$

which in turn is uniquely identified with an element of $\mathcal{M}^+ M$:

$$\left([n] \xrightarrow{id} [n] \xrightarrow{f_1} [n_1] \xrightarrow{f_2} \ldots \xrightarrow{f_i} [n_i], (m_0, \ldots, m_n)\right)$$

Clearly, since any element of $\mathcal{M}^+ M$ may be written so that the first morphism of the chain component is the identity, $N(\mathcal{M})$ can be identified with $\mathcal{M}^+ M$.

Now, we show that $\mathcal{M}$ is permutative. Define the product on objects:

$$(m_0, \ldots, m_p) \odot (n_0, \ldots, n_q) := (m_0, \ldots, m_p, n_0, \ldots, n_q),$$

and for morphisms $f, g \in \text{Mor} \mathcal{M}$, simply use $f \odot g$ as defined for $\Delta S_+$ in section 1.1. Associativity is strict, since it is induced by the associativity of $\odot$ in $\Delta S_+$. There is also a strict unit, the empty tuple, $()$.

The natural transposition (i.e., $\gamma : \odot \to \odot T$ of the definition given in Prop 5) is defined on objects by:

$$\gamma : (m_0, \ldots, m_p) \odot (n_0, \ldots, n_q) \to (n_0, \ldots, n_q) \odot (m_0, \ldots, m_p)$$

$$\gamma = \beta_{p,q} \quad (\text{the block transformation morphism of section 1.1}).$$

Suppose we have a morphism in the product category,

$$(f, g) : ((m_0, \ldots, m_p), (n_0, \ldots, n_q)) \to ((m'_0, \ldots, m'_s), (n'_0, \ldots, n'_t))$$
Then it is easy to verify that the map $\gamma$ defined above makes the following diagram commutative (showing $\gamma$ is a natural transformation).

\[
\begin{array}{ccc}
(m_0, \ldots, m_p, n_0, \ldots, n_q) & \xrightarrow{\gamma = \beta_{p,q}} & (n_0, \ldots, n_q, m_0, \ldots, m_p) \\
\downarrow f \circ g & & \downarrow g \circ f \\
(m'_0, \ldots, m'_s, n'_0, \ldots, n'_t) & \xrightarrow{\gamma = \beta_{s,t}} & (n'_0, \ldots, n'_t, m'_0, \ldots, m'_s)
\end{array}
\]

The coherence conditions are satisfied for $\gamma$ in the same way as in $\Delta S_+$. \hfill \square

In particular, given any monoid $M$, the category $\mathcal{J}M$ constructed in the proof of Prop. 111 has the structure of $E_\infty$-algebra.

**Lemma 112.** $\mathcal{H}_{\text{cat}}$ has the structure of a $\mathcal{D}_{\text{cat}}$-module.

**Proof.** Let $X = \{x_i\}_{i \geq 1}$ be a countable set of formal independent indeterminates, and $J(X_+)$ the free monoid on the set $X_+ := X \cup \{1\}$. Since the category $\mathcal{J}J(X_+)$ is permutative, there is an $E_\infty$-algebra structure

\[
\theta : \mathcal{D}_{\text{cat}}(m) \times [\mathcal{J}J(X_+)]^m \to \mathcal{J}J(X_+)
\]

We can identify:

\[
\prod_{m \geq 0} (\mathcal{H}_{\text{cat}}(m) \times S_m X^m) = \mathcal{J}J(X_+)
\]

via the map

\[
(f, x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \mapsto f(x_{i_1}, x_{i_2}, \ldots, x_{i_m}).
\]

The fact that $J(X_+)$ is the free monoid on $X_+$ ensures that there is a map in the inverse direction, well-defined up to action of the symmetric group $S_m$. Note, the action of $S_m$ on $X^m$ is by permutation of the components $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$.

Next, let $m, j_1, j_2, \ldots, j_m \geq 0$, and let $j = \sum j_s$. Furthermore, let

\[
X_s = (x_{j_1+j_2+\ldots+j_{s-1}+1}, \ldots, x_{j_1+j_2+\ldots+j_s}).
\]

Define a functor $\alpha = \alpha_{m,j_1,\ldots,j_m}$:

\[
\alpha : \mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^m \mathcal{H}_{\text{cat}}(j_s) \to \mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^m \left( \mathcal{H}_{\text{cat}}(j_s) \times S_{j_s} X^{j_s} \right)
\]
\[ \alpha (\sigma, f_1, f_2, \ldots, f_m) = \left( \sigma, \prod_{s=1}^{m} (f_s, X_s) \right). \]

\(\alpha\) takes a morphism
\[ \tau \sigma^{-1} \times g_1 \times \ldots \times g_m \in \text{Mor} \left( \mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^{m} \mathcal{K}_{\text{cat}}(j_s) \right) \]
to the morphism
\[ \tau \sigma^{-1} \times (g_1 \times \text{id}_{j_1}) \times \ldots \times (g_m \times \text{id}_{j_m}) \in \text{Mor} \left( \mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^{m} \left( \mathcal{K}_{\text{cat}}(j_s) \times s_{j_s} X_{j_s} \right) \right). \]

Let \(\text{inc}\) be the inclusion of categories:
\[ \text{inc} : \mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^{m} \left( \mathcal{K}_{\text{cat}}(j_s) \times s_{j_s} X_{j_s} \right) \rightarrow \mathcal{D}_{\text{cat}}(m) \times \left[ \prod_{i \geq 0} \left( \mathcal{K}_{\text{cat}}(i) \times s_i X_i \right) \right]^m, \]
induced by the evident inclusion of for each \(s\):
\[ \mathcal{K}_{\text{cat}}(j_s) \times s_{j_s} X_{j_s} \hookrightarrow \prod_{i \geq 0} \left( \mathcal{K}_{\text{cat}}(i) \times s_i X_i \right) \]

Let \(\alpha_0\) be the functor
\[ \alpha_0 : \mathcal{K}_{\text{cat}}(j) \rightarrow \mathcal{K}_{\text{cat}}(j) \times s_j X^j \]
\[ \alpha_0(f) = (f, x_1, x_2, \ldots, x_j), \]
and \(\text{inc}_0\) be the inclusion \(\mathcal{K}_{\text{cat}}(j) \times s_j X^j \hookrightarrow \prod_{i \geq 0} \left( \mathcal{K}_{\text{cat}}(i) \times s_i X_i \right)\).

Next, consider the following diagram. The top row is the map \(\mu\) of (5.1), and the bottom row is the operad-algebra structure map for \(\mathcal{J}(X_{+})\).

\[
\begin{array}{ccc}
\mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^{m} \mathcal{K}_{\text{cat}}(j_s) & \xrightarrow{\mu} & \mathcal{K}_{\text{cat}}(j) \\
\downarrow \alpha & & \downarrow \alpha_0 \\
\mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^{m} \left( \mathcal{K}_{\text{cat}}(j_s) \times s_{j_s} X_{j_s} \right) & \xrightarrow{\text{inc}} & \mathcal{K}_{\text{cat}}(j) \times s_j X^j \\
\downarrow \text{inc} & & \downarrow \text{inc}_0 \\
\mathcal{D}_{\text{cat}}(m) \times \left[ \prod_{i \geq 0} \left( \mathcal{K}_{\text{cat}}(i) \times s_i X_i \right) \right]^m & \xrightarrow{\theta} & \prod_{i \geq 0} \left( \mathcal{K}_{\text{cat}}(i) \times s_i X^i \right)
\end{array}
\]

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I claim that this diagram commutes. Let \( w := (\sigma, f_1, \ldots, f_m) \in \mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^m \mathcal{K}_{\text{cat}}(j_s) \) be arbitrary. Following the left-hand column of the diagram, we obtain the element

\[
\alpha(w) = \left( \sigma, \prod_{s=1}^m (f_s, X_s) \right). \tag{5.5}
\]

It is important to note that the list of all \( x_r \) that occur in expression (5.5) is exactly \( \{x_1, x_2, \ldots, x_j\} \) with no repeats, up to permutations by \( S_{j_1} \times \cdots \times S_{j_m} \). Thus, after applying \( \theta \), the result is an element of the form:

\[
\theta \alpha(w) = (F, x_1, x_2, \ldots, x_j), \tag{5.6}
\]

up to permutations by \( S_j \). Thus, \( \theta \alpha(w) \) is in the image of \( \alpha_0 \), say \( \theta \alpha(w) = \alpha_0(v) \). All that remains is to show that \( \mu(w) = v \). Let us examine closely what the morphism \( F \) in formula (5.6) must be. \( \alpha(w) \) is identified with the element

\[
\left( \sigma, \prod_{s=1}^m f_s(X_s) \right)
\]

of \( \mathcal{D}_{\text{cat}}(m) \times [\mathcal{T}J(X_+)]^m \), and \( \theta \) sends this element to

\[
\theta \alpha(w) = \bigcirc_{s=1}^m f_{\sigma^{-1}(s)}(X_{\sigma^{-1}(s)}). \tag{5.7}
\]

\( \theta \alpha(w) \) is interpreted in \( \mathcal{K}_{\text{cat}}(j) \times_{S_j} X^j \) as follows: Begin with the tuple \( (x_1, x_2, \ldots, x_j) \). This tuple is divided into blocks, \( (X_1, \ldots, X_m) \). Apply \( f_1 \odot \cdots \odot f_m \) to obtain

\[
f_1(X_1) \odot \cdots \odot f_m(X_m).
\]

Finally, apply the block permutation \( \sigma \{p_1, \ldots, p_m\} \) to get the correct order in the result (see Fig. 5.1). This shows that \( F = \sigma \{p_1, \ldots, p_m\}(f_1 \odot \cdots \odot f_m) \), as required. Now that we have the diagram (5.4), it is straightforward to show that \( \mu \) satisfies the associativity condition for an operad-module structure map. Essentially, associativity is induced by the associativity condition of the algebra structure map \( \theta \). All that remains is to verify the unit and equivariance conditions.
\begin{align*}
\bullet x_1 & \rightarrow \bullet y_1 & = f_1(X_1) & \quad f_{\sigma^{-1}(1)}(X_{\sigma^{-1}(1)}) \\
\bullet x_2 & \rightarrow \bullet y_{p_1} & = f_2(X_2) & \quad f_{\sigma^{-1}(2)}(X_{\sigma^{-1}(2)}) \\
\vdots & \vdots & \vdots & \vdots \\
\bullet x_{j_1} & \rightarrow \bullet y_{p_1+1} & = f_m(X_m) & \quad f_{\sigma^{-1}(m)}(X_{\sigma^{-1}(m)}) \\
\bullet x_{j_1+1} & \rightarrow \bullet y_{p_{m-1}+1} & & \circ \\
\vdots & \vdots & \vdots & \vdots \\
\bullet x_{j_{m-1}+1} & \rightarrow \bullet y_{p_m} & & \circ \\
\bullet x_{j_m} & \rightarrow \bullet y_{p_m+1} & & \circ \\
\end{align*}

\begin{align*}
& \circ f_1 \circ \ldots \circ f_m \\
& \sigma\{p_1, \ldots, p_m\}
\end{align*}

Figure 5.1: $\theta_{\alpha}(\sigma, f_1, \ldots, f_m)$, interpreted as an object of $\mathcal{K}_{\text{cat}}(j) \times S_j X^j$
It is trivial to verify the unit condition on the level of objects. The unit object of $\mathcal{D}_{\text{cat}}(1)$ is the identity of $S_1$. According to formula (5.3), we obtain the following diagram for morphisms $f : j \rightarrow p$ and $g : p \rightarrow q$. Clearly, the right-hand column is identical to the morphism $f \circ g$.

\[
\begin{array}{ccc}
(id_{S_1}, f) & \xrightarrow{\mu} & (id_{S_p})f \\
\downarrow id_{S_1} \times g & & \downarrow (id_{S_q})g \\
(id_{S_1}, gf) & \xrightarrow{\mu} & (id_{S_q})gf
\end{array}
\]

(Note, there is no corresponding right unit condition in an operad-module structure.)

Now, specify the right-action of $\rho \in S_\mathcal{J}$ on $\mathcal{K}_{\text{cat}}(j)$ as precomposition by $\rho$. That is, $f.\rho := f\rho$ for $f \in j \setminus \mathcal{J}(\mathcal{J})$. A routine check verifies the equivariance on the level of objects. Let $f_i \in \mathcal{K}_{\text{cat}}(j_i)$ have specified source and target:

\[
\begin{array}{c}
\mathcal{J}p_i \rightarrow j_i
\end{array}
\]

Equivariance A:

\[
\begin{array}{ccc}
(\sigma, f_1, \ldots, f_m) & \xrightarrow{id \times T_\tau} & (\sigma, f_{\tau^{-1}(1)}, \ldots, f_{\tau^{-1}(m)}) \\
\downarrow \tau \times id^m & & \downarrow \mu \\
(\sigma \tau, f_1, \ldots, f_m) & & (\sigma \tau \{p_m\}) f_m^{\circ} \\
\downarrow \mu & & \downarrow \tau \{j_m\} \\
(\sigma \tau \{p_m\} f_m^{\circ}) & = & (\sigma \tau \{p_m\} \tau \{p_m\} f_m^{\circ}) = (\sigma \tau \{p_m\} f_m^{\circ} \tau \{j_m\})
\end{array}
\]

Equivariance B:

\[
\begin{array}{ccc}
(\sigma, f_1, \ldots, f_m) & \xrightarrow{\mu} & \sigma \{p_m\} f_m^{\circ} \\
\downarrow \text{id} \times \tau_1 \times \ldots \times \tau_m & & \downarrow \tau_1 \oplus \ldots \oplus \tau_m \\
(\sigma, f_1 \tau_1, \ldots, f_m \tau_m) & \xrightarrow{\mu} & \sigma \{p_m\} (f_m \tau_m)^{\circ} = \sigma \{p_m\} f_m^{\circ} \tau_m^{\circ}
\end{array}
\]
Remark 113. It turns out that \( \mathcal{K}_{\text{cat}} \) is in fact a pseudo-operad. Recall from [17] that a pseudo-operad is a ‘non-unitary’ operad. That is, there are multiplication maps that satisfy operad associativity, and actions by the symmetric groups that satisfy operad equivariance conditions, but there is no requirement concerning a left or right unit map. The multiplication is defined as the composition:

\[
\mathcal{K}_{\text{cat}}(m) \times \prod_{s=1}^{m} \mathcal{K}_{\text{cat}}(j_s) \xrightarrow{\pi \times \text{id}^f} \mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^{m} \mathcal{K}_{\text{cat}}(j_s) \xrightarrow{\mu} \mathcal{K}_{\text{cat}}(j_1 + \ldots + j_m),
\]

where \( \pi : \mathcal{K}_{\text{cat}}(m) \to \mathcal{D}_{\text{cat}}(m) \) is the projection functor defined as isolating the group element (automorphism) of an \( \mathcal{F}(as) \) morphism:

\[
(\phi, g) \mapsto g
\]

Indeed, \( \pi \) defines a covariant isomorphism of the subcategory \( \text{Aut}(m \setminus \mathcal{F}(as)) \) onto \( \mathcal{D}_{\text{cat}}(m) \).

Now that we have a \( \mathcal{D}_{\text{cat}} \)-module structure on \( \mathcal{K}_{\text{cat}} \), we shall proceed in steps to induce this structure to an analogous operad-module structure on the level of \( k \)-complexes. First, we shall require the definition of \textit{lax symmetric monoidal functor}. The following appears in [33], as well as [27].

\textbf{Definition 114.} Let \( \mathcal{C} \), resp. \( \mathcal{C}' \), be a symmetric monoidal category with multiplication \( \odot \), resp. \( \boxdot \). Denote the associativity maps in \( \mathcal{C} \), resp. \( \mathcal{C}' \) by \( a \), resp. \( a' \), and the commutation maps by \( s \), resp. \( s' \). A functor \( F : \mathcal{C} \to \mathcal{C}' \) is a lax symmetric monoidal functor if there are natural maps

\[
\begin{align*}
F : FA \boxdot FB & \to F(A \odot B) \\
\end{align*}
\]

such that the following diagrams are commutative:

\[
\begin{align*}
FA \boxdot (FB \boxdot FC) & \xrightarrow{\text{id} \boxdot f} FA \boxdot F(B \odot C) \xrightarrow{f} F(A \odot (B \odot C)) \\
(FA \boxdot FB) \boxdot FC & \xrightarrow{f \boxdot \text{id}} F(A \odot B) \boxdot FC \xrightarrow{f} F((A \odot B) \odot C)
\end{align*}
\]
If the transformation $f$ is a natural isomorphism, then the functor $F$ is called *strong symmetric monoidal*.

Observe that the functor $N : \textbf{Cat} \to \textbf{Simpset}$ is strong symmetric monoidal with associated natural map, $S_s$:

$$S_s : N \mathcal{A}_{cat} \times N \mathcal{B}_{cat} \longrightarrow N (\mathcal{A}_{cat} \times \mathcal{B}_{cat}) ,$$

defined on $n$-chains:

$$\left( A_0 \overset{f_1}{\rightarrow} A_1 \overset{f_2}{\rightarrow} \cdots \overset{f_n}{\rightarrow} A_n, \ B_0 \overset{g_1}{\rightarrow} B_1 \overset{g_2}{\rightarrow} \cdots \overset{g_n}{\rightarrow} B_n \right) \overset{S_s}{\mapsto} \left( A_0, B_0 \overset{f_1 \times g_1}{\rightarrow} \cdots \overset{f_n \times g_n}{\rightarrow} (A_n, B_n) \right)$$

The $k$-linearization functor, $k[-] : \textbf{SimpSet} \to k\textbf{-SimpMod}$ is also strong symmetric monoidal, with associated natural map,

$$k[\mathcal{A}_{ss}] \otimes k[\mathcal{B}_{ss}] \longrightarrow k[\mathcal{A}_{ss} \times \mathcal{B}_{ss}],$$

defined degree-wise:

$$k[(\mathcal{A}_{ss})_n] \otimes k[(\mathcal{B}_{ss})_n] \xrightarrow{\cong} k[(\mathcal{A}_{ss})_n \times (\mathcal{B}_{ss})_n] = k[(\mathcal{A}_{ss} \times \mathcal{B}_{ss})_n].$$

Finally, the normalization functor, $N : k\textbf{-SimpMod} \to k\textbf{-Complexes}$ is lax symmetric monoidal, with associated natural map $f$ being the Eilenberg-Zilber shuffle map (see [28]).

$$Sh : N \mathcal{A}_{sm} \otimes N \mathcal{B}_{sm} \longrightarrow N (\mathcal{A}_{sm} \hat{\otimes} \mathcal{B}_{sm})$$

Now, define the versions of $\mathcal{D}_{cat}$ and $\mathcal{K}_{cat}$ over the various symmetric monoidal categories we are considering:

$$\mathcal{D}_{ss} = N \mathcal{D}_{cat} \quad \mathcal{K}_{ss} = N \mathcal{K}_{cat}$$
$$\mathcal{D}_{sm} = k[\mathcal{D}_{ss}] \quad \mathcal{K}_{sm} = k[\mathcal{K}_{ss}]$$
$$\mathcal{D}_{ch} = N \mathcal{D}_{sm} \quad \mathcal{K}_{ch} = N \mathcal{K}_{sm}$$
Lemma 115. Let \((\mathcal{C}, \odot, e)\) and \((\mathcal{C}', \Box, e')\) be symmetric monoidal categories, and \(F : \mathcal{C} \to \mathcal{C}'\) a lax symmetric monoidal functor with associated natural transformation \(f\) such that \(e' = Fe\).

1. If \(\mathcal{P}\) is an operad over \(\mathcal{C}\), then \(F\mathcal{P}\) is an operad over \(\mathcal{C}'\).

2. If \(\mathcal{P}\) is an operad and \(\mathcal{M}\) is a \(\mathcal{P}\)-module over \(\mathcal{C}\), then \(F\mathcal{M}\) is an \(F\mathcal{P}\)-module over \(\mathcal{C}'\).

3. If \(\mathcal{P}\) is an operad over \(\mathcal{C}\) and \(Z \in \text{Obj}\mathcal{C}\) is a \(\mathcal{P}\)-algebra, then \(FZ\) is an \(F\mathcal{P}\)-algebra over \(\mathcal{C}'\).

Proof. Note that properties (5.8) and (5.9) imply that the associativity map \(a'\) and symmetry map \(s'\) of \(\mathcal{C}'\) may be viewed as induced by the associativity map \(a\) and symmetry map \(s\) of \(\mathcal{C}\). That is, all symmetric monoidal structure is carried by \(F\) from \(\mathcal{C}\) to \(\mathcal{C}'\).

Denote by \(f^m\) the natural transformation induced by \(f\) on \(m + 1\) components:

\[f^m : FA_0 \Box FA_1 \Box \ldots \Box FA_m \to F(A_0 \odot A_1 \odot \ldots \odot A_m).
\]

Technically, we should write parentheses to indicate associativity in the source and target of \(f^m\), but property (5.8) of the functor and the MacLane Pentagon diagram of Def. 100 makes this unnecessary.

Let \(\mathcal{P}\) have structure map \(\gamma\). Define the structure map \(\gamma'\) for \(F\mathcal{P}\):

\[\gamma' := F\gamma \circ f^m : F\mathcal{P}(m) \Box F\mathcal{P}(j_1) \Box \ldots \Box F\mathcal{P}(j_m) \to F\mathcal{P}(j_1 + \ldots + j_m).
\]

Let \(e\) (resp. \(e'\)) be the unit object of \(\mathcal{C}\) (resp. \(\mathcal{C}'\)). Then \(e' = Fe\). If \(\mathcal{P}\) has unit map \(\eta : e \to \mathcal{P}(1)\), then define the unit map of \(F\mathcal{P}\) by

\[\eta' := F\eta : e' \to F\mathcal{P}(1).
\]

The action of \(S_n\) on \(F\mathcal{P}\) is defined by viewing \(F\mathcal{P}\) as a functor \(S^\text{op} \to \mathcal{C}'\):

\[S^\text{op} \xrightarrow{\mathcal{P}} \mathcal{C} \xrightarrow{F} \mathcal{C}'
\]

To verify that the proposed structure on \(F\mathcal{P}\) defines an operad is straightforward, but the required commutative diagrams do not fit on one page!

Assertions 2 and 3 are proved similarly.
Corollary 116. \( \mu \) induces a multiplication map \( \tilde{\mu}_* \) on the level of chain complexes, making \( K_{\text{ch}} \) into a \( D_{\text{ch}} \)-module.

\[
\tilde{\mu}_*: D_{\text{ch}}(m) \otimes K_{\text{ch}}(j_1) \otimes \cdots \otimes K_{\text{ch}}(j_m) \to K_{\text{ch}}(j_1 + \cdots + j_m).
\]

Proof. Since \( K_{\text{cat}} \) is a \( D_{\text{cat}} \)-module, and the functors \( N, k[-] \) and \( N \) are symmetric monoidal (the first two in the strong sense, the third in the lax sense), this result follows immediately from Lemma 115. \( \square \)

5.3 \( E_\infty \)-Algebra Structure

In this section we use the operad-module structure defined in the previous section to induce a related operad-algebra structure.

Definition 117. Suppose \((C, \odot)\) is a cocomplete symmetric monoidal category. We say \( C \odot \) distributes over colimits, or is distributive, if the natural map

\[
\colim_{i \in I} (B \odot C_i) \longrightarrow B \odot \colim_{i \in I} C_i
\]

is an isomorphism.

Remark 118. All of the ambient categories we consider in this chapter, \( \text{Cat} \), \( \text{SimpSet} \), \( k\text{-SimpMod} \), and \( k\text{-Complexes} \), are cocomplete and distributive.

Lemma 119. Suppose \((C, \odot)\) is a cocomplete distributive symmetric monoidal category, \( \mathcal{P} \) is an operad over \( C \), \( \mathcal{L} \) is a left \( \mathcal{P} \)-module, and \( Z \in \text{Obj} C \). Then

\[
\mathcal{L}(Z) := \coprod_{m \geq 0} \mathcal{L}(m) \odot_{S_m} Z^\otimes m
\]

admits the structure of a \( \mathcal{P} \)-algebra.

Remark 120. The notation \( \mathcal{L}(Z) \) appears in Kapranov and Manin [13], and is also present in [17] as the Schur functor of an operad ([17], Def 1.24), \( S_{\mathcal{F}}(Z) \).
Proof. What we are looking for is an equivariant map

\[ \mathcal{P}(\mathcal{L}(Z)) \to \mathcal{L}(Z). \]

That is, a map

\[ \prod_{n \geq 0} \mathcal{P}(n) \circ_{S_n} \left[ \prod_{m \geq 0} \mathcal{L}(m) \circ_{S_m} Z^\odot m \right] \circ_n \to \prod_{m \geq 0} \mathcal{L}(m) \circ_{S_m} Z^\odot m, \]

satisfying the required associativity conditions for an operad-algebra structure.

Observe that the equivariant product \( \circ_{S_m} \) may be constructed as the coequalizer of the maps corresponding to the \( S_m \)-action. That is,

\[ \mathcal{L}(m) \circ_{S_m} Z^\odot m = \text{coequalizer} \left\{ \sigma^{-1} \circ \sigma \right\}, \]

where \( \sigma^{-1} \circ \sigma : \mathcal{L}(m) \odot Z^\odot m \to \mathcal{L}(m) \odot Z^\odot m \) is given by right action of \( \sigma^{-1} \) on \( \mathcal{L}(m) \) and by permutation of the factors of \( Z^\odot m \) by \( \sigma \) (See [17], formula (1.11)). Thus, \( \mathcal{L}(Z) \) may be expressed as a (small) colimit. Since we presuppose that \( \odot \) distributes over all small colimits, it suffices to fix an integer \( n \) as well as \( n \) integers \( m_1, m_2, \ldots, m_n \), and examine the following diagram:

\[ \begin{array}{c}
\mathcal{P}(n) \odot \left( \left[ \mathcal{L}(m_1) \odot Z^\odot m_1 \right] \odot \ldots \odot \left[ \mathcal{L}(m_n) \odot Z^\odot m_n \right] \right) \\
\downarrow T \\
\mathcal{P}(n) \odot \left( \mathcal{L}(m_1) \odot \ldots \odot \mathcal{L}(m_n) \right) \odot \left( Z^\odot m_1 \odot \ldots \odot Z^\odot m_n \right) \\
\downarrow \mu \odot \text{id} \\
\mathcal{L}(m_1 + \ldots + m_n) \odot \left( Z^\odot m_1 \odot \ldots \odot Z^\odot m_n \right) \\
\downarrow a \\
\mathcal{L}(m_1 + \ldots + m_n) \odot Z^\odot (m_1 + \ldots + m_n)
\end{array} \]

In this diagram, \( T \) is the evident shuffling of components so that the components \( \mathcal{L}(m_i) \) are grouped together, \( \mu \) is the operad-module structure map for \( \mathcal{L} \), and \( a \) stands for the various associativity maps that are required to obtain the final form. This composition defines a family of maps

\[ \eta : \mathcal{P}(n) \odot \bigodot_{i=1}^{n} \left[ \mathcal{L}(m_i) \odot Z^\odot m_i \right] \to \mathcal{L}(m_1 + \ldots + m_n) \odot Z^\odot (m_1 + \ldots + m_n). \]
The maps $\eta$ pass to $S_m$-equivalence classes, producing a family of maps:

$$\eta : \mathcal{P}(n) \odot \bigodot_{i=1}^{n} [\mathcal{L}(m_i) \odot_{S_{m_i}} Z \odot_{m_i}] \longrightarrow \mathcal{L}(m_1 + \ldots + m_n) \odot_{S_{m_1 \times \ldots \times S_{m_n}}} Z \odot_{m_1 + \ldots + m_n}.$$

Let $p$ be the evident projection map for a right $S_{m_1 + \ldots + m_n}$-object $M$ and left $S_{m_1 + \ldots + m_n}$-object $N$,

$$M \odot_{S_{m_1 \times \ldots \times S_{m_n}}} N \xrightarrow{p} M \odot_{S_{m_1 + \ldots + m_n}} N.$$

Define the family of maps, $\chi := p \circ \eta$,

$$\chi : \mathcal{P}(n) \odot \bigodot_{i=1}^{n} [\mathcal{L}(m_i) \odot_{S_{m_i}} Z \odot_{m_i}] \longrightarrow \mathcal{L}(m_1 + \ldots + m_n) \odot_{S_{m_1 + \ldots + m_n}} Z \odot_{m_1 + \ldots + m_n}.$$

That is, we have a structure map, $\chi$, defined for each $n$:

$$\chi : \mathcal{P}(n) \odot [\mathcal{L}(Z)] \odot_{n} \longrightarrow \mathcal{L}(Z).$$

Since $\mathcal{L}$ is a left $\mathcal{P}$-module, $\chi$ is compatible with the multiplication maps of $\mathcal{P}$. Equivariance follows from external equivariance conditions on $\mathcal{L}$ as $\mathcal{P}$-module, together with the internal equivariance relations present in

$$\mathcal{L}(m_1 + \ldots + m_n) \odot_{S_{m_1 + \ldots + m_n}} Z \odot_{m_1 + \ldots + m_n},$$

inducing the required operad-algebra structure map

$$\chi : \mathcal{P}(n) \odot_{S_{n}} [\mathcal{L}(Z)] \odot_{n} \longrightarrow \mathcal{L}(Z).$$

\[\square\]

Let $A$ be an associative, unital algebra over $k$. Let $\mathcal{Y}_+ A = k[N(- \setminus \Delta S_+)] \otimes_{\Delta S_+} B_{sym}^+ A$ be the complex from section 2.1, regarded as simplicial $k$-module. Observe, we may identify:

$$k[N(- \setminus \Delta S_+)] \otimes_{\text{Aut} \Delta S_+} B_{sym}^+ A = \bigoplus_{n \geq 0} \mathcal{K}_{mn}(n) \otimes_{S_{n}} A \overset{\otimes}{\longrightarrow} \mathcal{K}_{mn}(A).$$

(Note, $A$ is regarded as a trivial simplicial object, with all faces and degeneracies being identities.)
Lemma 121. $\mathcal{K}_{\text{sm}}(A)$ has the structure of an $E_\infty$-algebra over the category of simplicial $k$-modules,

$$\chi : D_{\text{sm}}(\mathcal{K}_{\text{sm}}(A)) \longrightarrow \mathcal{K}_{\text{sm}}(A)$$

Proof. This follows immediately from Lemma 119 and the fact that $k\text{-SimpMod}$ is cocomplete and distributive. \hfill \Box

Remark 122. The fact that $\mathcal{K}_{\text{cat}}$ is a pseudo-operad (See Remark 113) implies that $\mathcal{K}_{\text{ss}}$ and $\mathcal{K}_{\text{sm}}$ are also pseudo-operads (c.f. Lemma 115). Now, the properties of the Schur functor do not depend on existence of a right unit map for $\mathcal{K}_{\text{sm}}$, so we could conclude immediately that $S^* \mathcal{K}_{\text{ss}}(A) = \mathcal{K}_{\text{sm}}(A)$ is a ‘pseudo-operad’-algebra over $\mathcal{K}_{\text{sm}}$, however the preceding proof requires a bit less machinery.

Lemma 123. The $D_{\text{sm}}$-algebra structure on $\mathcal{K}_{\text{sm}}(A)$ induces a quotient $D_{\text{sm}}$-algebra structure on $\mathcal{Y}^+A$,

$$\overline{\chi} : D_{\text{sm}}(\mathcal{Y}^+A) \longrightarrow \mathcal{Y}^+A$$

That is, $\mathcal{Y}^+A$ is an $E_\infty$-algebra over the category of simplicial $k$-modules.

Proof. We must verify that the structure map $\chi$ from Lemma 121 is well-defined on equivalence classes in $\mathcal{Y}^+A$. Let $\overline{\chi}$ be defined by applying $\chi$ to a representative, so that we obtain a map

$$\overline{\chi} : D_{\text{sm}}(n) \hat{\otimes} S_n (\mathcal{Y}^+A)^{\hat{\otimes} n} \rightarrow \mathcal{Y}^+A.$$ 

It suffices to check $\overline{\chi}$ is well-defined on 0-chains. Let $f_i, g_i, 1 \leq i \leq n$, be morphisms of $\mathcal{F}(as)$ with specified sources and targets:

$$m_i \xrightarrow{f_i} p_i \xrightarrow{g_i} q_i.$$ 

Let $V_i$ be a simple tensor of $A^{\otimes m_i}$, that is, $V_i := a_1 \otimes a_2 \otimes \ldots \otimes a_{m_i}$ for some $a_s \in A$. Let $\sigma \in S_n$. Consider the 0-chain of $D_{\text{sm}}(n) \hat{\otimes} S_n (\mathcal{Y}^+A)^{\hat{\otimes} n}$:

$$\sigma \otimes (g_1 f_1 \otimes V_1) \otimes \ldots \otimes (g_n f_n \otimes V_n). \quad (5.10)$$
The map $\chi$ sends this to:

$$
\sigma\{q_n\}(g_n f_n) \otimes V_n^\otimes,
$$

where $V_n^\otimes := V_1 \otimes \ldots \otimes V_n \in A^{\otimes (m_1 + \ldots + m_n)}$.

On the other hand, the element (5.10) is equivalent under Mor $(\mathcal{F}(as))$-equivariance to:

$$
\sigma \hat{\otimes} (g_1 \otimes f_1(V_1)) \hat{\otimes} \ldots \hat{\otimes} (g_n \otimes f_n(V_n)),
$$

(5.11)

and $\chi$ sends this to:

$$
\sigma\{q_n\}g_n^\otimes [f_n(V_n)]^\otimes
$$

$$
= \sigma\{q_n\}g_n^\otimes [(f_n)^\otimes] (V_n^\otimes)
$$

$$
\approx \sigma\{q_n\}g_n^\otimes f_n^\otimes V_n^\otimes
$$

$$
= \sigma\{q_n\}(g_n f_n)^\otimes \otimes V_n^\otimes.
$$

This proves $\chi$ is well defined, and so $\mathcal{Y}_*^+ A$ admits the structure of an $E_\infty$-algebra over the category of simplicial $k$-modules. □

**Theorem 124.** The $\mathcal{D}_{sm}$-algebra structure on $\mathcal{Y}_*^+ A$ induces a $\mathcal{D}_{ch}$-algebra structure on $N\mathcal{Y}_*^+ A$ (as $k$-complex).

**Proof.** Again since the normalization functor $N$ is lax symmetric monoidal, the operad-algebra structure map of $\mathcal{Y}_*^+ A$ induces an operad algebra structure map over $k$-complexes (by Lemma 115):

$$
\widetilde{\chi} : \mathcal{D}_{ch}(n) \otimes_{S_n} (N\mathcal{Y}_*^+ A)^{\otimes n} \to N\mathcal{Y}_*^+ A.
$$

□

**Corollary 125.** $HS_*(A)$ admits a Pontryagin product, giving it the structure of graded commutative and associative algebra.

**Proof.** The product is induced on the chain level by first choosing any 0-chain, $c$, in $\mathcal{D}_{ch}(2)$ corresponding to the generator of $1 \in H_0(\mathcal{D}_{ch}(2)) = k$, and then taking the composite,

$$
k \otimes N\mathcal{Y}_*^+ A \otimes N\mathcal{Y}_*^+ A \to \mathcal{D}_{ch}(2) \otimes N\mathcal{Y}_*^+ A \otimes N\mathcal{Y}_*^+ A
$$

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\[ \to \mathcal{D}_{\text{ch}}(2) \otimes_{k[S_2]} \mathbb{N}\mathcal{Y}^+ A \otimes \mathbb{N}\mathcal{Y}^+ A \to \mathbb{N}\mathcal{Y}^+ A \]

That is, for homology classes \([x]\) and \([y]\) in \(H_*^s(A)\), the product is defined by

\[ [x] \cdot [y] := [\tilde{\chi}(c \otimes x \otimes y)] \quad (5.12) \]

Now, the choice of \(c\) does not matter, since \(\mathcal{D}_{\text{ch}}(2)\) is contractible. Indeed, since each \(\mathcal{D}_{\text{ch}}(p)\) is contractible, Thm. 124 shows that \(\mathbb{N}\mathcal{Y}^+ A\) is a \textit{homotopy-everything} complex, analogous to the homotopy-everything spaces of Boardman and Vogt [2]. Thus, the product (5.12) is associative and commutative in the graded sense on the level of homology (see also May [19], p. 3).

\[ \square \]

Remark 126. The Pontryagin product of Cor. 125 is directly related to the algebra structure on the complexes \(\text{Sym}^*(p)\) of Chapter 4. Indeed, if \(A\) has augmentation ideal \(I\) which is free and has countable rank over \(k\), and \(I^2 = 0\), then by Cor. 80, the spectral sequence would collapse at the \(E_1\) stage, giving:

\[ H_*^s(A) \cong \bigoplus_{p \geq 0} \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_n(EG_u \ltimes_{G_u} \text{Sym}^*(p); k), \]

and the product structure of \(\text{Sym}^*(p)\) may be viewed as a restriction of the algebra structure of \(H_*^s(A)\) to the free orbits.

5.4 Homology Operations

Recall, for a commutative ring, \(k\), and a cyclic group \(\pi\) of order \(p\), there is a periodic resolution of \(k\) by free \(k\pi\)-modules (cf [18], [3]):

**Definition 127.** Let \(\tau\) be a generator of \(\pi\). Let \(N = 1 + \tau + \ldots + \tau^{p-1}\). Define a \(k\pi\)-complex, \((W, d)\) by:

- \(W_i\) is a the free \(k\pi\)-module on the generator \(e_i\), for each \(i \geq 0\), with differential:

\[
\begin{align*}
    d(e_{2i+1}) &= (\tau - 1)e_{2i} \\
    d(e_{2i}) &= Ne_{2i-1}
\end{align*}
\]
$W$ also has the structure of a $k\pi$-coalgebra, with augmentation $\epsilon$ and coproduct $\psi$:

$$\epsilon(\tau^j e_0) = 1$$

$$\begin{align*}
\psi(e_{2i+1}) &= \sum_{j+k=i} e_{2j} \otimes e_{2k+1} + \sum_{j+k=i} e_{2j+1} \otimes \tau e_{2k} \\
\psi(e_{2i}) &= \sum_{j+k=i} e_{2j} \otimes e_{2k} + \sum_{j+k=i-1} \left( \sum_{0 \leq r < s < p} \tau^r e_{2j+1} \otimes \tau^s e_{2k+1} \right)
\end{align*}$$

In what follows, we shall specialize $p$ prime and to $k = \mathbb{Z}/p\mathbb{Z}$ (as a ring). Let $\pi = C_p$ (as group), and denote by $W$ the standard resolution of $k$ by $k\pi$-modules, as in definition 127.

Recall, $\mathcal{D}_{ch}(p)$ is a contractible $k$-complex on which $S_p$ acts freely. Embed $\pi \hookrightarrow S_p$ by $\tau \mapsto (1,p,p-1,\ldots,2)$. Clearly $\pi$ acts freely on $\mathcal{D}_{ch}(p)$ as well. Thus, there exists a homotopy equivalence $\xi : W \to \mathcal{D}_{ch}(p)$.

Observe that the complex $N\mathcal{Y}^+ A$ computes $H_*S_*(A)$, since it is defined as the quotient of $\mathcal{Y}^+ A$ by degeneracies. By Thm. 124, $N\mathcal{Y}^+ A$, has the structure of $E_\infty$-algebra, so by results of May, if $x \in H_*(N\mathcal{Y}^+ A) = H_*S_*(A)$, then $e_i \otimes x^{\otimes p}$ is a well-defined element of $H_*(W \otimes_{k\pi} (N\mathcal{Y}^+ A)^{\otimes p})$, where $e_i$ is the distinguished generator of $W_i$. We then use the homotopy equivalence $\xi : W \to \mathcal{D}_{ch}(p)$ and $\mathcal{D}_{ch}$-algebra structure of $N\mathcal{Y}^+ A$ to produce the required map. Define $\kappa$ as the composition:

$$H_*\left( W \otimes_{k\pi} (N\mathcal{Y}^+ A)^{\otimes p} \right) \xrightarrow{H(\xi \otimes \text{id}^{(p)})} H_*\left( \mathcal{D}_{ch}(p) \otimes_{k\pi} (N\mathcal{Y}^+ A)^{\otimes p} \right) \xrightarrow{H(\chi)} H_*(N\mathcal{Y}^+ A)$$

This gives a way of defining homology (Steenrod) operations on $H_*S_*(A)$. Following definition 2.2 of [18], first define the maps $D_i$. For $x \in HS_q(A)$ and $i \geq 0$, define

$$D_i(x) := \kappa(e_i \otimes x^{\otimes p}) \in HS_{pq+i}(A).$$

**Definition 128.** If $p = 2$, define:

$$P_s : HS_q(A) \to HS_{q+s}(A)$$

$$P_s(x) = \begin{cases} 
0 & \text{if } s < q \\
D_{s-q}(x) & \text{if } s \geq q
\end{cases}$$
If $p > 2$ (i.e., an odd prime), let

$$\nu(q) = (-1)^{s+\frac{q-1}{2}(p-1)}\left(\frac{D-1}{2}\right)^q,$$

and define:

$$P_s : HS_q(A) \to HS_{q+2s(p-1)}(A)$$

$$P_s(x) = \begin{cases} 
0 & \text{if } 2s < q \\
\nu(q)D_{(2s-q)(p-1)}(x) & \text{if } 2s \geq q 
\end{cases}$$

$$\beta P_s : HS_q(A) \to HS_{q+2s(p-1)-1}(A)$$

$$\beta P_s(x) = \begin{cases} 
0 & \text{if } 2s \leq q \\
\nu(q)D_{(2s-q)(p-1)-1}(x) & \text{if } 2s > q 
\end{cases}$$

Note, the definition of $\nu(q)$ given here differs from that given in [18] by the sign $(-1)^s$ in order that all constants be collected into the term $\nu(q)$. 

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CHAPTER 6
LOW-DEGREE SYMMETRIC HOMOLOGY

6.1 Partial Resolution

As before, $k$ is a commutative ground ring. In this chapter, we find an explicit partial resolution of $k$ by projective $\Delta S^{op}$-modules, allowing the computation of $HS_0(A)$ and $HS_1(A)$ for a unital associative $k$-algebra $A$.

**Theorem 129.** $HS_i(A)$ for $i = 0, 1$ is the homology of the following partial chain complex

$$0 \leftarrow A \xleftarrow{\partial_1} A \otimes A \otimes A \xleftarrow{\partial_2} (A \otimes A \otimes A \otimes A) \oplus A,$$

where

$$\partial_1 : a \otimes b \otimes c \mapsto abc - cba,$$

$$\partial_2 : \begin{cases} 
    a \otimes b \otimes c \otimes d & \mapsto ab \otimes c \otimes d + d \otimes ca \otimes b + bca \otimes 1 \otimes d + d \otimes bc \otimes a, \\
    a & \mapsto 1 \otimes a \otimes 1.
\end{cases}$$

The proof will proceed in stages from the lemmas below.

**Lemma 130.** For each $n \geq 0$,

$$0 \leftarrow k \xleftarrow{\epsilon} k[\text{Mor}_{\Delta S}([n],[0])] \xleftarrow{\rho} k[\text{Mor}_{\Delta S}([n],[2])],$$

is exact, where $\epsilon$ is defined by $\epsilon(\phi) = 1$ for any morphism $\phi : [n] \rightarrow [0]$, and $\rho$ is defined by $\rho(\psi) = (x_0x_1x_2) \circ \psi - (x_2x_1x_0) \circ \psi$ for any morphism $\psi : [n] \rightarrow [2]$. Note, $x_0x_1x_2$ and $x_2x_1x_0$ are $\Delta S$ morphisms $[2] \rightarrow [0]$ written in tensor notation (see section 1.1).

**Proof.** Clearly, $\epsilon$ is surjective. Now, $\epsilon \rho = 0$, since $\rho(\psi)$ consists of two morphisms with opposite signs. Let $\phi_0 = x_0x_1 \ldots x_n : [n] \rightarrow [0]$. The kernel of $\epsilon$ is spanned by elements
φ−φ₀ for φ ∈ MorΔ₅([n], [0]). So, it suffices to show that the sub-module of $k[\text{Mor}_ΔS([n],[0])]$ generated by $(x₀x₁x₂)ψ − (x₂x₁x₀)ψ$ (for ψ : [n] → [2]) contains all of the elements φ − φ₀. In other words, it suffices to find a sequence

$$\phi =: \phiₖ, \phiₖ₋₁, \ldots, \phi₂, \phi₁, \phi₀$$

so that each φᵢ is obtained from φᵢ₊₁ by reversing the order of 3 blocks, XYZ → ZYX. Note, X, Y, or Z can be empty. Let $\phi = xᵢ₀xᵢ₁...xᵢₙ$. If $\phi = \phi₀$, we may stop here. Otherwise, we produce a sequence ending in $\phi₀$ by way of two types of rearrangements.

**Type I:**

$$xᵢ₀xᵢ₁...xᵢₙ \rightsquigarrow xᵢₙxᵢ₀xᵢ₁...xᵢₙ₋₁.$$  

**Type II:**

$$xᵢ₀xᵢ₁...xᵢₖ₋₁xᵢₖxᵢₖ₊₁...xᵢₙ \rightsquigarrow xᵢₖ₊₁...xᵢₙxᵢₖxᵢᵢ₀xᵢ₁...xᵢₖ₋₁.$$  

In fact, it will be sufficient to use a more specialized version of the Type II rearrangement. **Type II’:**

$$xᵢ₀xᵢ₁...xᵢₖ₋₁xᵢₖxₖ₊₁...xₙ \rightsquigarrow xₖ₊₁...xₙxᵢₖxᵢᵢ₀xᵢ₁...xᵢₖ₋₁,$$

where $iₖ \neq k$.

Beginning with $\phi$, perform Type I rearrangements until the final variable is $xₙ$. For convenience of notation, let this new monomial be $xⱼ₀xⱼ₁...xⱼₙ$. Of course, $jₙ = n$. If $jₖ = k$ for all $k = 0, 1, \ldots, n$, then we are done. Otherwise, there will be a number $k$ such that $jₖ \neq k$ but $jₖ₊₁ = k₊₁, \ldots, jₙ = n$. Perform a Type II’ rearrangement with $jₖ$ as pivot, followed by enough Type I rearrangements to make the final variable $xₙ$ again. The net result of such a combination is that the ending block $xₖ₊₁xₖ₊₂...xₙ$ remains fixed while the beginning block $xⱼ₀xⱼ₁...xⱼₖ$ becomes $xⱼₖxⱼ₀...xⱼₖ₋₁$. It is clear that applying this combination repeatedly will finally obtain a monomial $xⱼ₀xⱼ₁...xⱼₖ₋₁xₖxₖ₊₁...xₙ$. After a finite number of steps, we finally obtain $\phi₀$. □

Let $𝔫ₙ = \{xᵢ₀xᵢ₁...xᵢₖ₋₁ \otimes xᵢₖ \otimes xᵢₖ₊₁xᵢₖ₊₂...xₙ : k ≥ 1, iₖ \neq k\}$. $k[𝔫ₙ]$ is a free submodule of $k[\text{Mor}_ΔS([n],[2])]$ of size $(n+1)!−1$. This count is obtained by observing that
\\{x_{i_0} \ldots x_{i_{k-1}} \otimes x_{i_k} \otimes x_{k+1} \ldots x_n : k = c, i_k \neq k\} \text{ has exactly } c \cdot c! = (c + 1)! - c! \text{ elements, }

then adding the telescoping sum from } c = 1 \text{ to } n.

**Corollary 131.** When restricted to } k[B_n], \text{ the map } \rho \text{ of Lemma 130 is surjective onto the kernel of } \epsilon.

**Proof.** In the proof of Lemma 130, the Type I rearrangements correspond to the image of elements \(x_{i_0} \ldots x_{i_{n-1}} \otimes x_{i_n} \otimes 1\). Note, we did not need \(i_n = n\) in any such rearrangement. The Type II' rearrangements correspond to the image of elements \(x_{i_0} \ldots x_{i_{k-1}} \otimes x_{i_k} \otimes x_{k+1} \ldots x_n\), for \(k \geq 1\) and \(i_k \neq k\).

**Lemma 132.** \(#\text{Mor}_{\Delta S}([n], [m]) = \frac{(m+n+1)!}{m!}\), \text{ so } k[\text{Mor}_{\Delta S}([n], [m])] \text{ is a free } k\text{-module of rank } (m+n+1)!/m!.

**Proof.** A morphism } \phi : [n] \to [m] \text{ of } \Delta S \text{ is nothing more than an assignment of } n+1 \text{ objects into } m+1 \text{ compartments, along with a total ordering of the original } n+1 \text{ objects, hence:}

\[
#\text{Mor}_{\Delta S}([n], [m]) = \binom{m+n+1}{m}(n+1)! = \frac{(m+n+1)!}{m!}.
\]

**Lemma 133.** \(\rho|_{k[B_n]} \text{ is an isomorphism } k[B_n] \cong \ker \epsilon\).

**Proof.** Since the rank of } k[\text{Mor}_{\Delta S}([n], [m])] \text{ is } (n+1)!, \text{ the rank of the kernel of } \epsilon \text{ is } (n+1)! - 1. \text{ The isomorphism then follows from Corollary 131.}

**Lemma 134.** The relations of the form:

\[
XY \otimes Z \otimes W + W \otimes ZX \otimes Y + YZX \otimes 1 \otimes W + W \otimes YZ \otimes X \approx 0 \quad (6.1)
\]

and\[
1 \otimes X \otimes 1 \approx 0 \quad (6.2)
\]

\text{collapse } k[\text{Mor}_{\Delta S}([n], [2])] \text{ onto } k[B_n].
Proof. This proof proceeds in multiple steps.

Step 1.

\[ X \otimes 1 \otimes 1 \approx 1 \otimes X \otimes 1 \approx 1 \otimes 1 \otimes X \approx 0. \] (6.3)

\[ 1 \otimes X \otimes 1 \approx 0 \] proceeds directly from Eq. 6.2. Letting \( X = Y = W = 1 \) in Eq. 6.1 yields

\[ 3(1 \otimes Z \otimes 1) + Z \otimes 1 \otimes 1 \approx 0 \Rightarrow Z \otimes 1 \otimes 1 \approx 0. \]

Then, \( X = Z = W = 1 \) in Eq. 6.1 produces

\[ 2(Y \otimes 1 \otimes 1) + 1 \otimes 1 \otimes Y + 1 \otimes Y \otimes 1 \approx 0 \Rightarrow 1 \otimes 1 \otimes Y \approx 0. \]

Step 2.

\[ 1 \otimes X \otimes Y + 1 \otimes Y \otimes X \approx 0. \] (6.4)

Let \( Z = W = 1 \) in Eq. 6.1. Then

\[ XY \otimes 1 \otimes 1 + 1 \otimes X \otimes Y + YX \otimes 1 \otimes 1 + 1 \otimes Y \otimes X \approx 0 \]

\[ \Rightarrow 1 \otimes X \otimes Y + 1 \otimes Y \otimes X \approx 0, \quad \text{by step 1.} \]

Step 3. [Degeneracy Relations]

\[ X \otimes Y \otimes 1 \approx X \otimes 1 \otimes Y \approx 1 \otimes X \otimes Y. \] (6.5)

Let \( X = W = 1 \) in Eq. 6.1.

\[ Y \otimes Z \otimes 1 + 1 \otimes Z \otimes Y + YZ \otimes 1 \otimes 1 + 1 \otimes YZ \otimes 1 \approx 0 \]

\[ \Rightarrow Y \otimes Z \otimes 1 + 1 \otimes Z \otimes Y \approx 0, \quad \text{by step 1.} \]

\[ \Rightarrow Y \otimes Z \otimes 1 - 1 \otimes Y \otimes Z \approx 0, \quad \text{by step 2.} \] (6.6)

Next, let \( X = Y = 1 \) in Eq. 6.1.

\[ 1 \otimes Z \otimes W + W \otimes Z \otimes 1 + Z \otimes 1 \otimes W + W \otimes Z \otimes 1 \approx 0 \]

\[ \Rightarrow 1 \otimes Z \otimes W + 2(1 \otimes W \otimes Z) + Z \otimes 1 \otimes W \approx 0, \quad \text{by Eq. 6.6,} \]
\[
1 \otimes Z \otimes W - 2(1 \otimes Z \otimes W) + Z \otimes 1 \otimes W \approx 0, \quad \text{by step 2,}
\]
\[
Z \otimes 1 \otimes W - 1 \otimes Z \otimes W \approx 0.
\]

**Step 4. [Sign Relation]**

\[
X \otimes Y \otimes Z + Z \otimes Y \otimes X \approx 0
\]

Let \( Y = 1 \) in Eq. 6.1.

\[
X \otimes Z \otimes W + W \otimes ZX \otimes 1 + ZX \otimes 1 \otimes W + W \otimes Z \otimes X \approx 0,
\]
\[
\Rightarrow X \otimes Z \otimes W + 1 \otimes W \otimes ZX + 1 \otimes ZX \otimes W + W \otimes Z \otimes X \approx 0, \quad \text{by step 3,}
\]
\[
\Rightarrow X \otimes Z \otimes W + W \otimes Z \otimes X \approx 0, \quad \text{by step 2.}
\]

**Step 5. [Hochschild Relation]**

\[
XY \otimes Z \otimes 1 - X \otimes YZ \otimes 1 + ZX \otimes Y \otimes 1 \approx 0
\]

Let \( W = 1 \) in Eq. 6.1.

\[
XY \otimes Z \otimes 1 + 1 \otimes ZX \otimes Y + YZX \otimes 1 \otimes 1 + 1 \otimes YX \otimes X \approx 0,
\]
\[
\Rightarrow XY \otimes Z \otimes 1 + ZX \otimes Y \otimes 1 + 0 - X \otimes YX \otimes 1 \approx 0, \quad \text{by steps 1, 3 and 4.}
\]

**Step 6. [Cyclic Relation]**

\[
\sum_{j=0}^{n} \tau_n^j (x_{i_0}x_{i_1} \ldots x_{i_{n-1}} \otimes x_i \otimes 1) \approx 0,
\]

where \( \tau_n \in \Sigma_{n+1} \) is the \((n + 1)\)-cycle \((0, n, n - 1, \ldots, 2, 1)\), which acts by permuting the indices. For \( n = 0 \), there are no such relations (indeed, no relations at all). For \( n = 1 \), the cyclic relation takes the form \( x_0 \otimes x_1 \otimes 1 + x_1 \otimes x_0 \otimes 1 \approx 0 \), which follows from degeneracy and sign relations.

Assume now that \( n \geq 2 \). For each \( k = 1, 2, \ldots, n - 1 \), define:

\[
\begin{align*}
A_k &:= x_{i_0}x_{i_1} \ldots x_{i_{k-1}}, \\
B_k &:= x_{i_k}, \\
C_k &:= x_{i_{k+1}} \ldots x_{i_n}.
\end{align*}
\]
By the Hochschild relation,
\[ 0 \approx \sum_{k=1}^{n-1} (A_k B_k \otimes C_k \otimes 1 - A_k \otimes B_k C_k \otimes 1 + C_k A_k \otimes B_k \otimes 1). \]

But for \( k \leq n - 2 \),
\[ A_k B_k \otimes C_k \otimes 1 = A_{k+1} \otimes B_{k+1} C_{k+1} \otimes 1 = x_{i_0} \ldots x_{i_k} \otimes x_{i_{k+1}} \ldots x_{i_n} \otimes 1. \]

Thus, after some cancellation:
\[ 0 \approx -A_1 \otimes B_1 C_1 \otimes 1 + A_{n-1} B_{n-1} \otimes C_{n-1} \otimes 1 + \sum_{k=1}^{n-1} C_k A_k \otimes B_k \otimes 1 \]
\[ = -(x_{i_0} \otimes x_{i_1} \ldots x_{i_n} \otimes 1) + (x_{i_0} \ldots x_{i_{n-1}} \otimes x_{i_n} \otimes 1) + \sum_{k=1}^{n-1} x_{i_{k+1}} \ldots x_{i_n} x_{i_0} \ldots x_{i_{k-1}} \otimes x_{i_k} \otimes 1 \]
\[ = (x_{i_0} \ldots x_{i_{n-1}} \otimes x_{i_n} \otimes 1) + (x_{i_1} \ldots x_{i_n} \otimes x_{i_0} \otimes 1) + \sum_{k=1}^{n-1} x_{i_{k+1}} \ldots x_{i_n} x_{i_0} \ldots x_{i_{k-1}} \otimes x_{i_k} \otimes 1, \]
by sign and degeneracy relations. This last expression is precisely the relation Eq. 6.9.

**Step 7.**

Every element of the form \( X \otimes Y \otimes 1 \) is equivalent to a linear combination of elements of \( \mathcal{B}_n \).

To prove this, we shall induct on the size of \( Y \). Suppose \( Y \) consists of a single variable. That is, \( X \otimes Y \otimes 1 = x_{i_0} \ldots x_{i_{n-1}} \otimes x_{i_n} \otimes 1 \). Now, if \( i_n \neq n \), we are done. Otherwise, we use the cyclic relation to write
\[ x_{i_0} \ldots x_{i_{n-1}} \otimes x_{i_n} \otimes 1 \approx -\sum_{j=1}^{n} \tau_n^j (x_{i_0} \ldots x_{i_{n-1}} \otimes x_{i_n} \otimes 1). \]

Now suppose \( k \geq 1 \) and any element \( Z \otimes W \otimes 1 \) with \( |W| = k \) is equivalent to an element of \( k[\mathcal{B}_n] \). Consider \( X \otimes Y \otimes 1 = x_{i_0} \ldots x_{i_{n-k-1}} \otimes x_{i_{n-k}} \ldots x_{i_n} \otimes 1 \). Let
\[
\begin{cases}
  A_k := x_{i_0} x_{i_1} \ldots x_{i_{k-1}}, \\
  B_k := x_{i_{n-k}} \ldots x_{i_{n-1}}, \\
  C_k := x_{i_n}.
\end{cases}
\]
Then, by the Hochschild relation,

\[ X \otimes Y \otimes 1 = A_k \otimes B_k C_k \otimes 1 \approx A_k B_k \otimes C_k \otimes 1 + C_k A_k \otimes B_k \otimes 1. \]

But since \(|C_k| = 1\) and \(|B_k| = k\), this last expression is equivalent to an element of \(k[\mathcal{B}_n]\).

**Step 8. [Modified Cyclic Relation]**

\[ \sum_{j=0}^{k} \tau_k^j (x_{i_0} x_{i_1} \ldots x_{i_{k-1}} \otimes x_{i_k} \otimes x_{i_{k+1}} \ldots x_{i_n}) \approx 0 \pmod{k[\{A \otimes B \otimes 1\}].} \quad (6.10) \]

Note, the \((k + 1)\)-cycle \(\tau_k\) permutes the indices \(i_0, i_1, \ldots, i_k\), and fixes the rest.

First, we show that \(X \otimes Y \otimes W + Y \otimes X \otimes W \approx 0 \pmod{k[\{A \otimes B \otimes 1\}].}\) Indeed, if we let \(Z = 1\) in Eq. 6.1, then

\[ XY \otimes 1 \otimes W + W \otimes X \otimes Y + YX \otimes 1 \otimes W + W \otimes Y \otimes X \approx 0, \]

\[ \Rightarrow -(X \otimes Y \otimes W + Y \otimes X \otimes W) + XY \otimes 1 \otimes W + YX \otimes 1 \otimes W \approx 0, \quad \text{by step 4}, \]

\[ \Rightarrow X \otimes Y \otimes W + Y \otimes X \otimes W \approx XY \otimes 1 \otimes W + YX \otimes 1 \otimes W, \quad \text{by step 3}. \quad (6.11) \]

Now, we have \(X \otimes Y \otimes W + Y \otimes X \otimes W \approx 0 \pmod{k[\{A \otimes B \otimes 1\}].}\) Note that this last expression can be used to prove the modified cyclic relation for \(k = 1\).

Next, rewrite Eq. 6.1:

\[ XY \otimes Z \otimes W + W \otimes ZX \otimes Y + YZX \otimes 1 \otimes W + W \otimes YZ \otimes X \approx 0, \]

\[ \Rightarrow XY \otimes Z \otimes W - Y \otimes ZX \otimes W + YZX \otimes W \otimes 1 - X \otimes YZ \otimes W \approx 0, \quad \text{by steps 3 and 4}, \]

\[ \Rightarrow XY \otimes Z \otimes W + ZX \otimes Y \otimes W + YZX \otimes W \otimes 1 - X \otimes YZ \otimes W \approx 0 \pmod{k[\{A \otimes B \otimes 1\}].}, \]

using the relation \(X \otimes Y \otimes W + Y \otimes X \otimes W \approx 0 \pmod{k[\{A \otimes B \otimes 1\}].}\) proven above.

\[ \Rightarrow XY \otimes Z \otimes W - X \otimes YZ \otimes W + ZX \otimes Y \otimes W \approx 0 \pmod{k[\{A \otimes B \otimes 1\}].}. \quad (6.12) \]

Eq. 6.12 is a modified Hochschild relation, and we can use it in the same way we used the Hochschild relation in step 6. Assume \(k \geq 2\), and define for \(j = 1, 2, \ldots k - 1:\)

\[
\begin{align*}
A_j &:= x_{i_0} x_{i_1} \ldots x_{i_{j-1}}, \\
B_j &:= x_{i_j}, \\
C_j &:= x_{i_{j+1}} \ldots x_{i_k}.
\end{align*}
\]
Using the modified Hochschild relation together with the observation that for \( j \leq k - 2 \),

\[
A_j B_j \otimes C_j \otimes W = A_{j+1} \otimes B_{j+1} C_{j+1} \otimes W,
\]

we finally arrive at the sum:

\[
0 \equiv -A_1 \otimes B_1 C_1 \otimes W + A_{k-1} B_{k-1} \otimes C_{k-1} \otimes W + \sum_{j=1}^{k-1} C_j A_j \otimes B_j \otimes W \pmod {k\{A \otimes B \otimes 1\}}
\]

\[
\equiv -(x_{i_0} \otimes x_{i_1} \ldots x_{i_n} \otimes W) + (x_{i_0} \ldots x_{i_{n-1}} \otimes x_{i_n} \otimes W) + \sum_{k=1}^{n-1} x_{i_{k+1}} \ldots x_{i_n} x_{i_0} \ldots x_{i_{k-1}} \otimes x_{i_k} \otimes W
\]

\[
\equiv (x_{i_1} \ldots x_{i_n} \otimes x_{i_0} \otimes W) + (x_{i_0} \ldots x_{i_{n-1}} \otimes x_{i_n} \otimes W) + \sum_{k=1}^{n-1} x_{i_{k+1}} \ldots x_{i_n} x_{i_0} \ldots x_{i_{k-1}} \otimes x_{i_k} \otimes W.
\]

**Step 9.**

Every element of the form \( X \otimes Y \otimes x_n \) is equivalent to an element of \( k[\mathcal{B}_n] \).

We shall use the modified cyclic relation and modified Hochschild relation in a similar way as cyclic and Hochschild relations were used in step 7. Again we induct on the size of \( Y \). If \( |Y| = 1 \), then

\[
X \otimes Y \otimes x_n = x_{i_0} \ldots x_{i_{n-2}} \otimes x_{i_{n-1}} \otimes x_n.
\]

If \( i_{n-1} \neq n - 1 \), then we are done. Otherwise, use the modified cyclic relation to re-express \( X \otimes Y \otimes x_n \) as a sum of elements of \( k[\mathcal{B}_n] \), modulo \( k\{A \otimes B \otimes 1\} \). Of course, by step 7, all elements \( A \otimes B \otimes 1 \) are also in \( k[\mathcal{B}_n] \).

Next, suppose \( k \geq 1 \) and any element \( Z \otimes W \otimes x_n \) with \( |W| = k \) is equivalent to an element of \( k[\mathcal{B}_n] \). Consider \( X \otimes Y \otimes x_n = x_{i_0} \ldots x_{i_{n-k-2}} \otimes x_{i_{n-k-1}} \ldots x_{i_{n-1}} \otimes x_n \). Using the modified Hochschild relation with

\[
\begin{align*}
A_k &:= x_{i_0} x_{i_1} \ldots x_{i_{n-k-2}}, \\
B_k &:= x_{i_{n-k-1}} \ldots x_{i_{n-2}}, \\
C_k &:= x_{i_{n-1}}.
\end{align*}
\]

we obtain:

\[
X \otimes Y \otimes x_n = A_k \otimes B_k C_k \otimes x_n \approx A_k B_k \otimes C_k \otimes x_n + C_k A_k \otimes B_k \otimes x_n \pmod {k\{A \otimes B \otimes 1\}}.
\]

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But since \(|C_k| = 1\) and \(|B_k| = k\), this last expression is equivalent to an element of \(k[\mathcal{B}_n]\).

**Step 10.**

Every element of \(k[\text{Mor}_{\Delta S}([n],[2])]\) is equivalent to a linear combination of elements from the set

\[
\mathcal{C}_n := \{X \otimes x_i \otimes 1 \mid i \neq n\} \cup \{X \otimes x_{i-1} \otimes x_n \mid i-1 \neq n-1\} \cup \{X \otimes Y \otimes Z x_n \mid |Z| \geq 1\} \quad (6.13)
\]

Note, the \(k\)-module generated by \(\mathcal{C}_n\) contains \(k[\mathcal{B}_n]\).

Let \(X \otimes Y \otimes Z\) be an arbitrary element of \(k[\text{Mor}_{\Delta S}([n],[2])]\). If \(|X| = 0\), \(|Y| = 0\), or \(|Z| = 0\), then the degeneracy relations imply that this element is equivalent to an element of the form \(X' \otimes Y' \otimes 1\). Step 7 implies this element is equivalent to one in \(k[\mathcal{B}_n]\), hence also in \(k[\mathcal{C}_n]\).

Suppose now that \(|X|, |Y|, |Z| \geq 1\). If \(x_n\) occurs in \(X\), use the relation \(X \otimes Y \otimes W \approx -Y \otimes X \otimes W \pmod{k[\{A \otimes B \otimes 1\}]}\) to ensure that \(x_n\) occurs in the middle factor. If \(x_n\) occurs in the \(Z\), use the sign relation and the above relation to put \(x_n\) into the middle factor.

In any case, it suffices to assume our element has the form:

\[
X \otimes U x_n V \otimes Z.
\]

By the modified Hochschild relation,

\[
X \otimes U x_n V \otimes Z \approx X U x_n \otimes V \otimes Z + V X \otimes U x_n \otimes Z, \quad \text{(mod } k[\{A \otimes B \otimes 1\}]),
\]

\[
\approx -Z \otimes V \otimes X U x_n + Z \otimes V X \otimes U x_n.
\]

The first term is certainly in \(k[\mathcal{C}_n]\), since \(|X| \geq 1\). If \(|U| > 0\), the second term also lies in \(k[\mathcal{C}_n]\). If, however, \(|U| = 0\), then use step 9 to re-express \(Z \otimes V X \otimes x_n\) as an element of \(k[\mathcal{B}_n]\).

Observe that Step 10 proves Lemma 134 for \(n = 0, 1, 2\), since in these cases, any elements that fall within the set \(\{X \otimes Y \otimes Z x_n \mid |Z| \geq 1\}\) must have either \(|X| = 0\) or \(|Y| = 0\), hence are equivalent via the degeneracy relation to elements of the form \(A \otimes B \otimes 1\). In what follows, assume \(n \geq 3\).
Step 11.

\[ W \otimes Z \otimes X x_n \approx W \otimes x_n Z \otimes X \quad \text{(mod } k[\{A \otimes B \otimes 1\} \cup \{A \otimes B \otimes x_n\}]) \quad \text{(6.14)} \]

Let \( Y = x_n \) in Eq. 6.1.

\[ X x_n \otimes Z \otimes W + W \otimes ZX \otimes x_n + x_n ZX \otimes 1 \otimes W + W \otimes x_n Z \otimes X \approx 0, \]

\[ \Rightarrow W \otimes Z \otimes X x_n \approx W \otimes ZX \otimes x_n + x_n ZX \otimes W \otimes 1 + W \otimes x_n Z \otimes X, \]

\[ \approx W \otimes x_n Z \otimes X \quad \text{(mod } k[\{A \otimes B \otimes 1\} \cup \{A \otimes B \otimes x_n\}]) \].

Step 12.

Every element of \( k[\text{Mor}_{\Delta S}([n],[2])] \) is equivalent to a linear combination of elements from the set

\[ \mathcal{D}_n := \{X \otimes x_{i-2} \otimes x_{n-1} x_n \mid i_{n-2} \neq n-2\} \cup \{X \otimes Y \otimes Z x_{n-1} x_n \mid |Z| \geq 1\}, \quad \text{(6.15)} \]

modulo \( k[\{A \otimes B \otimes 1\} \cup \{A \otimes B \otimes x_n\}] \).

Let \( X \otimes Y \otimes Z \) be an arbitrary element of \( k[\text{Mor}_{\Delta S}([n],[2])] \). Locate \( x_{n-1} \) and use the techniques of Step 10 to re-express \( X \otimes Y \otimes Z \) as a linear combination of terms of the form:

\[ W_j \otimes Z_j \otimes X_j x_{n-1}, \]

modulo \( k[\{A \otimes B \otimes 1\}] \). Now, for each \( j \), we will want to re-express \( W_j \otimes Z_j \otimes X_j \) as linear combinations of vectors in which \( x_n \) occurs only in the second factor. If \( x_n \) occurs in \( W_j \), then we just use the modified cyclic relation:

\[ W_j \otimes Z_j \otimes X_j x_{n-1} \approx -Z_j \otimes W_j \otimes X_j x_{n-1} \quad \text{(mod } k[\{A \otimes B \otimes 1\}]). \]

If \( x_n \) occurs in \( X_j \), then first use Eq. 6.1 with \( Y = x_{n-1} \):

\[ X x_{n-1} \otimes Z \otimes W + W \otimes ZX \otimes x_{n-1} + x_{n-1} ZX \otimes 1 \otimes W + W \otimes x_{n-1} Z \otimes X \approx 0, \]

\[ \Rightarrow W \otimes Z \otimes X x_{n-1} \approx W \otimes ZX \otimes x_{n-1} + x_{n-1} ZX \otimes W \otimes 1 + W \otimes x_{n-1} Z \otimes X, \]

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\[ \approx W \otimes ZX \otimes x_{n-1} + W \otimes x_{n-1}Z \otimes X \pmod{k \{ \{A \otimes B \otimes 1\} \}}, \]
\[ \approx W \otimes ZX \otimes x_{n-1} + Wx_{n-1} \otimes Z \otimes X + ZW \otimes x_{n-1} \otimes X \pmod{k \{ \{A \otimes B \otimes 1\} \}}, \]
\[ \approx W \otimes ZX \otimes x_{n-1} + Z \otimes X \otimes Wx_{n-1} - ZW \otimes X \otimes x_{n-1}, \pmod{k \{ \{A \otimes B \otimes 1\} \}}. \]

Thus, we can express our original element \( X \otimes Y \otimes Z \) as a linear combination of elements of the form:
\[ X' \otimes U'x_nV' \otimes Z'x_{n-1}, \]
modulo \( k \{ \{A \otimes B \otimes 1\} \} \). Use the modified Hochschild relation to obtain
\[ X'U'x_nV' \otimes Z'x_{n-1} \approx X'U'x_nV' \otimes Z'x_{n-1} + x_nV'X' \otimes U' \otimes Z'x_{n-1} \pmod{k \{ \{A \otimes B \otimes 1\} \}}. \]
\[ \approx X'U' \otimes x_nV' \otimes Z'x_{n-1} - U' \otimes x_nV'X' \otimes Z'x_{n-1} \pmod{k \{ \{A \otimes B \otimes 1\} \}}. \]
\[ \approx X'U' \otimes V' \otimes Z'x_{n-1}x_n - U' \otimes V'X' \otimes Z'x_{n-1}x_n \pmod{k \{ \{A \otimes B \otimes 1\} \cup \{A \otimes B \otimes x_n\} \}}, \]
by step 11. If \( |Z'| \geq 1 \), then we are done, otherwise, we have elements of the form \( X'' \otimes Y'' \otimes x_{n-1}x_n \). Use an induction argument analogous to that in step 9 to re-express this type of element as a linear combination of elements of the form:
\[ \approx U \otimes x_{i_{n-2}} \otimes x_{n-1}x_n, \quad i_{n-2} \neq n-2, \pmod{k \{ \{A \otimes B \otimes 1\} \}}. \]

**Step 13.**

Every element of \( k \{ \text{Mor}_{\Delta S}([n], [2]) \} \) is equivalent to an element of \( k[\mathcal{B}_n] \).

We shall use an iterative re-writing procedure. First of all, define
\[ \mathcal{B}_n^i := \{ A \otimes x_{i_{n-j}} \otimes x_{n-j+1} \ldots x_n \mid i_{n-j} \neq n-j \}, \]
\[ \mathcal{C}_n^j := \{ A \otimes B \otimes Cx_{n-j+1} \ldots x_n \mid |C| \geq 1 \}. \]

Now clearly, \( \mathcal{B}_n = \bigcup_{j=0}^{n-1} \mathcal{B}_n^j \).

By steps 10 and 12, we can reduce any arbitrary element \( X \otimes Y \otimes Z \) to linear combinations of elements in \( \mathcal{B}_n^0 \cup \mathcal{B}_n^1 \cup \mathcal{B}_n^2 \cup \mathcal{C}_n^2 \). Suppose now that we have reduced elements to linear
combinations from $B^0_n \cup B^1_n \cup \ldots \cup B^j_n \cup C^j_n$, for some $j \geq 2$. I claim any element of $C^j_n$ can be re-expressed as a linear combination from $B^0_n \cup B^1_n \cup \ldots \cup B^{j+1}_n \cup C^{j+1}_n$.

Let $X \otimes Y \otimes Zx_{n-j+1} \ldots x_n$, with $|Z| \geq 1$. Let $w := x_{n-j+1} \ldots x_n$. We may now think of $X \otimes Y \otimes Zw$ as consisting of the variables $x_0, x_1, \ldots, x_{n-j}, w$, hence, by step 12, we may re-express this element as a linear combination of elements from the set

$$\{X \otimes x_{i_{n-j-1}} \otimes x_{n-j}w \mid i_{n-j-1} \neq n-j-1\} \cup \{X \otimes Y \otimes Zx_{n-j}w \mid |Z| \geq 1\},$$

modulo $k[\{A \otimes B \otimes 1\} \cup \{A \otimes B \otimes w\}]$. Of course, this implies the element may written as a linear combination of elements from $B^{j+1}_n \cup C^{j+1}_n$, modulo $k[\{A \otimes B \otimes 1\} \cup \{A \otimes B \otimes x_{n-j+1} \ldots x_n\}]$. Since $\{A \otimes B \otimes x_{n-j+1}x_{n-j+2} \ldots x_n\} \subseteq C^{j-1}_n$, the inductive hypothesis ensures that $\{A \otimes B \otimes x_{n-j+1} \ldots x_n\} \subseteq B^0_n \cup \ldots \cup B^j_n$. This completes the inductive step.

After a finite number of iterations, then, we can re-express any element $X \otimes Y \otimes Z$ as a linear combination from the set $B^0_n \cup \ldots B^{n-1}_n \cup C^{n-1}_n = B_n \cup C^{n-1}_n$. But $C^{n-1}_n = \{A \otimes B \otimes Cx_2 \ldots x_n \mid |C| \geq 1\}$. Any element from this set has either $|A| = 0$ or $|B| = 0$, therefore is equivalent to an element from $k[B_n]$ already.

Corollary 135. If $\frac{1}{2} \in k$, then the four-term relation

$$XY \otimes Z \otimes W + W \otimes ZX \otimes Y + YZX \otimes 1 \otimes W + W \otimes YZ \otimes X \approx 0 \quad (6.16)$$

is sufficient to collapse $k[\text{Mor}_{\Delta S}([n], [2])]$ onto $k[B_n]$.

Proof. We only need to modify step 1 of the previous proof:

Step 1'.

$$X \otimes 1 \otimes 1 \approx 1 \otimes X \otimes 1 \approx 1 \otimes 1 \otimes X \approx 0.$$

Letting three variables at a time equal 1 in Eq. 6.16,

$$2(1 \otimes 1 \otimes W) + 2(W \otimes 1 \otimes 1) \approx 0, \quad \text{when } X = Y = Z = 1. \quad (6.17)$$

$$3(1 \otimes Z \otimes 1) + Z \otimes 1 \otimes 1 \approx 0, \quad \text{when } X = Y = W = 1. \quad (6.18)$$
\[2(Y \otimes 1 \otimes 1) + 1 \otimes Y \otimes 1 + 1 \otimes 1 \otimes Y \approx 0, \quad \text{when } X = Z = W = 1. \quad (6.19)\]

Now, replace \( W \) with \( X \) in Eq. 6.17, \( Z \) with \( X \) in Eq. 6.18, and \( Y \) with \( X \) in Eq. 6.19. Then

\[-3[2(1 \otimes 1 \otimes X) + 2(X \otimes 1 \otimes 1)] - 2[3(1 \otimes X \otimes 1) + X \otimes 1 \otimes 1] + 6[2(X \otimes 1 \otimes 1) + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X] \]

\[= 4(X \otimes 1 \otimes 1).\]

\[\Rightarrow X \otimes 1 \otimes 1 \approx 0,\]

as long as \( 2 \) is invertible in \( k \). Next, by Eq. 6.17, \( 2(1 \otimes 1 \otimes X) \approx 0 \Rightarrow 1 \otimes 1 \otimes X \approx 0 \). Finally, Eq. 6.19 gives \( 1 \otimes X \otimes 1 \approx 0. \)

Lemma 134 together with Lemma 133 and Lemma 130 show the following sequence is exact:

\[0 \leftarrow k \leftarrow k[\text{Mor}_{\Delta S}([n], [0])] \overset{\rho}{\leftarrow} k[\text{Mor}_{\Delta S}([n], [2])]
\]

\[\overset{(\alpha, \beta)}{\leftarrow} k[\text{Mor}_{\Delta S}([n], [3])] \oplus k[\text{Mor}_{\Delta S}([n], [0])], \quad (6.20)\]

where \( \alpha : k[\text{Mor}_{\Delta S}([n], [3])] \rightarrow k[\text{Mor}_{\Delta S}([n], [2])] \) is induced by

\[x_0x_1 \otimes x_2 \otimes x_3 + x_3 \otimes x_2x_0 \otimes x_1 + x_1x_2x_0 \otimes 1 \otimes x_3 + x_3 \otimes x_1x_2 \otimes x_0,\]

and \( \beta : k[\text{Mor}_{\Delta S}([n], [0])] \rightarrow k[\text{Mor}_{\Delta S}([n], [2])] \) is induced by \( 1 \otimes x_0 \otimes 1 \). This holds for all \( n \geq 0 \), so the following is a partial resolution of \( k \) by projective \( \Delta S^{op} \)-modules:

\[0 \leftarrow k \leftarrow k[\text{Mor}_{\Delta S}(-, [0])] \overset{\rho^*}{\leftarrow} k[\text{Mor}_{\Delta S}(-, [2])] \overset{(\alpha^*, \beta^*)}{\leftarrow} k[\text{Mor}_{\Delta S}(-, [3])] \oplus k[\text{Mor}_{\Delta S}(-, [0])]
\]

Hence, we may compute \( HS_0(A) \) and \( HS_1(A) \) as the homology groups of the following complex:

\[0 \leftarrow k[\text{Mor}_{\Delta S}(-, [0])] \otimes_{\Delta S} B^*_{\text{sym}} A \overset{\otimes \text{id}}{\leftarrow} k[\text{Mor}_{\Delta S}(-, [2])] \otimes_{\Delta S} B^*_{\text{sym}} A \overset{(\alpha, \beta) \otimes \text{id}}{\leftarrow}
\]

\[\left( k[\text{Mor}_{\Delta S}(-, [3])] \oplus k[\text{Mor}_{\Delta S}(-, [0])] \right) \otimes_{\Delta S} B^*_{\text{sym}} A.
\]

This complex is isomorphic to the one from Thm. 129, via the evaluation map

\[k[\text{Mor}_{\Delta S}(-, [p])] \otimes_{\Delta S} B^*_{\text{sym}} A \overset{\approx}{\rightarrow} B^*_{p \text{sym}} A.
\]
6.2 Low-degree Computations of $HS_\ast(A)$

**Theorem 136.** For a unital associative algebra $A$ over commutative ground ring $k$,

$$HS_0(A) \cong A/([A, A]),$$

where $([A, A])$ is the ideal generated by the commutator submodule $[A, A]$.

*Proof.* By Thm. 129, $HS_0(A) \cong A/k[\{abc - cba\}]$ as $k$-module. But $k[\{abc - cba\}]$ is an ideal of $A$, since if $x \in A$, then

$$xabc - xbca = (x)(a)(bc) - (bc)(a)(x) + (cba)(x) - (x)(cba) \in k[\{abc - cba\}].$$

Clearly $([A, A]) \subseteq k[\{abc - cba\}]$, and $k[\{abc - cba\}] \subseteq ([A, A])$ since

$$abc - cba = a(bc - cb) + a(cb) - (cb)a.$$

\[\square\]

**Corollary 137.** If $A$ is commutative, then $HS_0(A) \cong A$.

Note that Theorem 136 implies that symmetric homology does not preserve Morita equivalence, since for $n > 1$,

$$HS_0(M_n(A)) = M_n(A)/([M_n(A), M_n(A)]) = 0.$$

This implies $HS_\ast(M_n(A)) = 0$, since for any $x \in HS_\ast(M_n(A))$, $x = 1 \cdot x = 0 \cdot x = 0$, via the Pontryagin product of Cor. 125, while in general $HS_0(A) = A/([A, A]) \neq 0$.

By working with the complex in Thm. 129, an explicit formula for the product $HS_0(A) \otimes HS_1(A) \rightarrow HS_1(A)$ can be determined.

**Proposition 138.** For a unital associative algebra $A$ over commutative ground ring $k$, $HS_1(A)$ is a left $HS_0(A)$-module, via

$$A/([A, A]) \otimes HS_1(A) \rightarrow HS_1(A)$$
Here, elements of $HS_1(A)$ are represented as equivalence classes of elements in $A \otimes A \otimes A$, via the complex from Thm. 129.

Moreover, there is a right module structure

\[ HS_1(A) \otimes HS_0(A) \longrightarrow HS_1(A) \]

\[ [x \otimes y \otimes z] \otimes [a] \mapsto [xa \otimes y \otimes z] - [x \otimes ya \otimes z] + [x \otimes y \otimes za], \]

and the two actions are equal.

**Proof.** Define the products on the chain level. For $a \in A$ and $x \otimes y \otimes z \in A \otimes A \otimes A$, put

\[ a.(x \otimes y \otimes z) := ax \otimes y \otimes z - x \otimes ya \otimes z + x \otimes y \otimes az \]  \hspace{1cm} (6.21)

\[ (x \otimes y \otimes z).a := xa \otimes y \otimes z - x \otimes ay \otimes z + x \otimes y \otimes za \]  \hspace{1cm} (6.22)

There are a few details to verify:

1. Formula (6.21) gives an action

\[ A \otimes HS_1(A) \rightarrow HS_1(A) \]

The product is unital, since

\[ 1.[x \otimes y \otimes z] = [x \otimes y \otimes z] - [x \otimes y \otimes z] + [x \otimes y \otimes z] = [x \otimes y \otimes z]. \]

Similarly, Formula (6.22) is a unital product.

Associativity follows by examining the following expression (on the level of chains):

\[ (ab).(x \otimes y \otimes z) - a.(b.(x \otimes y \otimes z)) \]

\[ = (abx \otimes y \otimes z - x \otimes yab \otimes z + x \otimes y \otimes abz) - \begin{pmatrix} abx \otimes y \otimes z - bx \otimes ya \otimes z + bx \otimes y \otimes az \\ -ax \otimes yb \otimes z + x \otimes yba \otimes z - x \otimes yb \otimes az \\ +ax \otimes y \otimes bz - x \otimes ya \otimes bz + x \otimes y \otimes abz \end{pmatrix} \]

\[ = -x \otimes yab \otimes z + bx \otimes ya \otimes z - bx \otimes y \otimes az + ax \otimes yb \otimes z - x \otimes yba \otimes z \]
\[ +x \otimes yb \otimes az - ax \otimes y \otimes bz + x \otimes ya \otimes bz. \quad (6.23) \]

We may view the variables \( a, b, x, y \) and \( z \) in Eq. (6.23) as formal variables, and hence the expression itself may be regarded as chain in \( k \langle \text{Mor}_{\Delta S}([4],[2]) \rangle \) of the complex (6.20). Now, the differential \( \rho \) of complex (6.20) applied to Eq. (6.23) results in:

\[
\begin{pmatrix}
-xyabz + bxyaz - axybz - xybax + xyzab + xaybz - xyba - xyb - xabz \\
+zyabx - zyabx + azbyx - zybx + zyba - abyx + bzyx - bzyx
\end{pmatrix}.
\quad (6.24)
\]

All terms cancel, showing that Eq. (6.23) is in the kernel of \( \rho \). Exactness of complex (6.20) implies that Eq. (6.23) is in the image of \((\alpha, \beta)\). Thus, there is a 2-chain \( C \in A^\otimes 4 \oplus A \) such that \( \partial_2(C) = (ab) \cdot (x \otimes y \otimes z) - a \cdot (b \cdot (x \otimes y \otimes z)) \), and on the level of homology,

\[ (ab) \cdot [x \otimes y \otimes z] = a \cdot (b \cdot [x \otimes y \otimes z]) \]

The associativity of Formula (6.22) is proven in the same way.

2. Formula (6.21) induces an action

\[ HS_0(A) \otimes HS_1(A) \rightarrow HS_1(A) \]

It is sufficient to show that if \( u \) is a 1-cycle, then \( (ab - ba) \cdot u \) is a boundary for any \( a, b \in A \). Consider the following element.

\[ w := (ab - ba) \cdot (x \otimes y \otimes z) - (a \otimes b \otimes 1) \cdot \partial_1 (x \otimes y \otimes z) = (ab - ba) \cdot (x \otimes y \otimes z) - (a \otimes b \otimes 1) \cdot (xyz - zyx) \]

After expanding and canceling a pair of terms,

\[
w = \begin{pmatrix}
abx \otimes y \otimes z - x \otimes yab \otimes z + x \otimes y \otimes abz \\
-bax \otimes y \otimes z + x \otimes yba \otimes z - x \otimes y \otimes baz \\
-axyz \otimes b \otimes 1 + a \otimes xyzb \otimes 1 + azyx \otimes b \otimes 1 - a \otimes zyxb \otimes 1
\end{pmatrix}.
\]

Consider all variables as formal, so \( w \in k \langle \text{Mor}_{\Delta S}([4],[2]) \rangle \). A routine verification shows that \( \rho(w) = 0 \), hence by exactness, \( w \) is a boundary. Now, any 1-cycle \( u \) is a \( k \)-linear combination of elements of the form \( x \otimes y \otimes z \), so the chain

\[ (ab - ba) \cdot u - (a \otimes b \otimes 1) \cdot \partial_1 (u) = (ab - ba) \cdot u \]

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is a boundary.

We show that Formula (6.22) induces an action

$$HS_0(A) \otimes HS_1(A) \rightarrow HS_1(A)$$

in the analogous way, using

$$v := (x \otimes y \otimes z). (ab - ba) - \partial_1(x \otimes y \otimes z). (a \otimes b \otimes 1)$$

Lastly, we prove that the product structures are equal. Consider the following element.

$$t := a. (x \otimes y \otimes z) - (x \otimes y \otimes z). a - a \otimes \partial_1(x \otimes y \otimes z) \otimes 1$$

$$= a. (x \otimes y \otimes z) - (x \otimes y \otimes z). a - a \otimes xyz \otimes 1 + a \otimes zyx \otimes 1$$

It can be verified that $\rho(t) = 0$, proving $t \in k[\text{Mor}_\Delta S([3], [2])]$ is a boundary. If $u$ is a 1-cycle, then

$$a.u - u.a - a \otimes \partial_1(u) \otimes 1 = a.u - u.a$$

is a boundary, which shows that $a.[u] = [u].a$ in $HS_1(A)$.

Note, we expect the product structure given above to agree with the Pontryagin product, but this has not been verified yet due to time constraints.

Using GAP, we have made the following explicit computations of degree 1 integral symmetric homology. The $HS_0(A)$-module structure is also displayed where known.
| $A$          | $HS_1(A \mid \mathbb{Z})$ | $HS_0(A)$-module structure |
|-------------|--------------------------|-----------------------------|
| $\mathbb{Z}[t]/(t^2)$ | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | Generated by $u$ with $2u = 0$ |
| $\mathbb{Z}[t]/(t^3)$ | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | Generated by $u$ with $2u = 0$ and $t^2u = 0$ |
| $\mathbb{Z}[t]/(t^4)$ | $(\mathbb{Z}/2\mathbb{Z})^4$ | Generated by $u$ with $2u = 0$ |
| $\mathbb{Z}[t]/(t^5)$ | $(\mathbb{Z}/2\mathbb{Z})^4$ | |
| $\mathbb{Z}[t]/(t^6)$ | $(\mathbb{Z}/2\mathbb{Z})^6$ | |
| $\mathbb{Z}[C_2]$ | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | Generated by $u$ with $2u = 0$ |
| $\mathbb{Z}[C_3]$ | 0                        | |
| $\mathbb{Z}[C_4]$ | $(\mathbb{Z}/2\mathbb{Z})^4$ | Generated by $u$ with $2u = 0$ |
| $\mathbb{Z}[C_5]$ | 0                        | |
| $\mathbb{Z}[C_6]$ | $(\mathbb{Z}/2\mathbb{Z})^6$ | |

Based on these calculations, we conjecture:

**Conjecture 139.**

$$HS_1(k[t]/(t^n)) = \begin{cases} 
(k/2k)^n, & \text{if } n \geq 0 \text{ is even.} \\
(k/2k)^{n-1} & \text{if } n \geq 1 \text{ is odd.}
\end{cases}$$

**Remark 140.** The computations of $HS_1(\mathbb{Z}[C_n])$ are consistent with those of Brown and Loday [4]. See section 6.4 for a more detailed treatment of $HS_1$ for group rings.

Additionally, $HS_1$ has been computed for the following examples. These computations were done using **GAP** in some cases and in others, **Fermat** [14] computations on sparse matrices were used in conjunction with the **GAP** scripts. (*e.g.* when the algebra has dimension greater than 6 over $\mathbb{Z}$).
6.3 Splittings of the Partial Resolution

Under certain circumstances, the partial complex in Thm.129 splits as a direct sum of smaller complexes. This observation becomes increasingly important as the dimension of the algebra increases. Indeed, some of the computations of the previous section were done using splittings.

**Definition 141.** For a commutative $k$-algebra $A$ and $u \in A$, define the $k$-modules:

$$(A^{\otimes n})_u := \{a_1 \otimes a_2 \otimes \ldots \otimes a_n \in A^{\otimes n} \mid a_1 a_2 \cdots a_n = u\}$$

**Proposition 142.** If $A = k[M]$ for a commutative monoid $M$, then the complex in Thm.129 splits as a direct sum of complexes

$$0 \leftarrow (A)_u \xleftarrow{\partial_1} (A \otimes A \otimes A)_u \xleftarrow{\partial_2} (A \otimes A \otimes A \otimes A)_u \oplus (A)_u, \quad (6.25)$$
where $u$ ranges over the elements of $M$. For each $u$, the homology groups of Eq. (6.25) will be called the $u$-layered symmetric homology of $A$, denoted $HS_i(A)_u$. Thus, for $i = 0, 1$, we have:

$$HS_i(A) \cong \bigoplus_{u \in M} HS_i(A)_u.$$ 

**Proof.** Since $M$ is a commutative monoid, there are direct sum decompositions as $k$-module:

$$A^\otimes n = \bigoplus_{u \in M} (A^\otimes n)_u.$$ 

The maps $\partial_1$ and $\partial_2$ preserve the products of tensor factors, so the inclusions $(A^\otimes n)_u \hookrightarrow A^\otimes n$ induce maps of complexes, hence the complex itself splits as a direct sum. \qed

We may use layers to investigate the symmetric homology of $k[t]$. This algebra is monoidal, generated by the monoid $\{1, t, t^2, t^3, \ldots\}$. Now, the $t^m$-layer symmetric homology of $k[t]$ will be the same as the $t^m$-layer symmetric homology of $k[M_{m+1}^n]$, where $M_{q}^p$ denotes the cyclic monoid generated by a variable $s$ with the property that $s^p = s^q$. Using this observation and subsequent computation, we conjecture:

**Conjecture 143.**

$$HS_1(k[t])_{tm} = \begin{cases} 
0 & m = 0, 1 \\
\frac{k}{2k} & m \geq 2
\end{cases}$$

This conjecture has been verified up to $m = 18$, in the case $k = \mathbb{Z}$.

### 6.4 2-torsion in $HS_1$

The occurrence of 2-torsion in $HS_1(A)$ for the examples considered in sections 6.2 and 6.3 comes as no surprise, based on Thm. 51. First consider the following chain of isomorphisms:

$$\pi_2^s(B\Gamma) = \pi_2(\Omega^\infty S^\infty(B\Gamma)) \cong \pi_1(\Omega\Omega^\infty S^\infty(B\Gamma))$$

$$\cong \pi_1(\Omega_0\Omega^\infty S^\infty(B\Gamma)) \xrightarrow{h} H_1(\Omega_0\Omega^\infty S^\infty(B\Gamma)).$$

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Here, $\Omega_0 \Omega^\infty S^\infty(B\Gamma)$ denotes the component of the constant loop, and $h$ is the Hurewicz homomorphism, which is an isomorphism since $\Omega_0 \Omega^\infty S^\infty(B\Gamma)$ is path-connected and $\pi_1$ is abelian. On the other hand, by Thm. 51,

$$HS_1(k[\Gamma]) \cong H_1(\Omega_0 \Omega^\infty S^\infty(B\Gamma); k) \cong H_1(\Omega_0 \Omega^\infty S^\infty(B\Gamma)) \otimes k.$$  

(All tensor products will be over $\mathbb{Z}$ in this section.) Now $\Omega_0 \Omega^\infty S^\infty(B\Gamma)$ consists of isomorphic copies of $\Omega_0 \Omega^\infty S^\infty(B\Gamma)$, one for each element of $\Gamma/\langle\Gamma,\Gamma\rangle$, so we may write

$$H_1(\Omega_0 \Omega^\infty S^\infty(B\Gamma)) \otimes k \cong H_1(\Omega_0 \Omega^\infty S^\infty(B\Gamma)) \otimes k[\Gamma/\langle\Gamma,\Gamma\rangle].$$

Thus, we obtain the result:

**Corollary 144.** If $\Gamma$ is a group, then

$$HS_1(k[\Gamma]) \cong \pi_2^s(B\Gamma) \otimes k[\Gamma/\langle\Gamma,\Gamma\rangle].$$

Now, by results of Brown and Loday [4], if $\Gamma$ is abelian, then $\pi_2^s(B\Gamma)$ is the reduced tensor square. That is,

$$\pi_2^s(B\Gamma) = \Gamma \bar{\otimes} \Gamma = (\Gamma \otimes \Gamma) / \approx,$$

where $g \otimes h \approx -h \otimes g$ for all $g, h \in \Gamma$. (This construction is notated with multiplicative group action in [4], since they deal with the more general case of non-abelian groups.) So in particular, if $\Gamma = C_n$, the cyclic group of order $n$, we have

$$\pi_2^s(BC_n) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \text{ even.} \\ 0 & n \text{ odd.} \end{cases}$$

**Corollary 145.**

$$HS_1(k[C_n]) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^n & n \text{ even.} \\ 0 & n \text{ odd.} \end{cases}$$

**Proof.** The result follows from Cor. 144, as $k[C_n/\langle C_n, C_n \rangle] \cong k[C_n] \cong k^n$, as $k$-module. \(\square\)
6.5 Relations to Cyclic Homology

The relation between the symmetric bar construction and the cyclic bar construction arising from the chain of inclusions (1.1) gives rise to a natural map

$$HC_*(A) \to HS_*(A)$$

Indeed, by remark 18, we may define cyclic homology via:

$$HC_*(A) = \text{Tor}^\Delta_*(k, B^\text{sym}_* A).$$

Using the partial complex of Thm. 129, and an analogous one for computing cyclic homology (c.f. [15], p. 59), the map (6.26) for degrees 0 and 1 is induced by the following partial chain map:

\[
\begin{array}{ccccccccc}
0 & & A & & A \otimes A & & A^{\otimes 3} \oplus A \\
\downarrow \gamma_0 = \text{id} & & \downarrow \gamma_1 & & \downarrow \gamma_2 \\
0 & & A & & A^{\otimes 3} & & A^{\otimes 4} \oplus A
\end{array}
\]

In this diagram, the boundary maps in the upper row are defined as follows:

$$\partial^C_1 : a \otimes b \mapsto ab - ba$$

$$\partial^C_2 : \begin{cases} 
  a \otimes b \otimes c & \mapsto ab \otimes c - a \otimes bc + ca \otimes b \\
  a & \mapsto 1 \otimes a - a \otimes 1
\end{cases}$$

The boundary maps in the lower row are defined as in Thm. 129.

$$\partial^S_1 : a \otimes b \otimes c \mapsto abc - cba$$

$$\partial^S_2 : \begin{cases} 
  a \otimes b \otimes c \otimes d & \mapsto ab \otimes c \otimes d - d \otimes ca \otimes b + bca \otimes 1 \otimes d + d \otimes bc \otimes a \\
  a & \mapsto 1 \otimes a \otimes 1
\end{cases}$$

The partial chain map is given in degree 1 by $\gamma_1(a \otimes b) := a \otimes b \otimes 1$. In degree 2, $\gamma_2$ is defined on the summand $A^{\otimes 3}$ via

$$a \otimes b \otimes c \mapsto (a \otimes b \otimes c \otimes 1 - 1 \otimes a \otimes bc \otimes 1 + 1 \otimes ca \otimes b \otimes 1 + 1 \otimes 1 \otimes abc \otimes 1 - b \otimes ca \otimes 1 \otimes 1) - 2abc - cab,$$

and on the summand $A$ via

$$a \mapsto (-1 \otimes 1 \otimes a \otimes 1) + (4a).$$
APPENDIX A

SCRIPTS USED FOR COMPUTER CALCULATIONS

The computer algebra systems GAP, Octave and Fermat were used to verify proposed theorems and also to obtain some concrete computations of symmetric homology for some small algebras. Here is a link to a tar-file of the scripts that were created in the course of writing this dissertation as well as the \LaTeX source of this dissertation.

http://arxiv.org/e-print/0807.4521v1/

The tar-file contains the following files:

- **Basic.g** - Some elementary functions, necessary for some functions in DeltaS.g

- **HomAlg.g** - Homological Algebra functions, such as computation of homology groups for chain complexes.

- **Fermat.g** - Functions necessary to invoke Fermat for fast sparse matrix computations.

- **fermattogap, gaptofermat** - Auxiliary text files for use when invoking Fermat from GAP.

- **DeltaS.g** - This is the main repository of scripts used to compute various quantities associated with the category $\Delta S$, including $HS_1(A)$ for finite-dimensional algebras $A$.

In order to use the functions of DeltaS.g, simply copy the above files into the working directory (such as `/gap/), invoke GAP, then read in DeltaS.g at the prompt. The dependent modules will automatically be loaded (hence they must be present in the same directory as
DeltaS.g. Note, most of the computations involving homology require substantial memory to run. I recommend calling GAP with the command line option "-o mem", where mem is the amount of memory to be allocated to this instance of GAP. All computations done in this dissertation can be accomplished by allocating 20 gigabytes of memory. The following provides a few examples of using the functions of DeltaS.g

[ault@math gap]$ gap -o 20g

gap> Read("DeltaS.g");
gap>

gap> ## Number of morphisms [6] --> [4]
gap> SizeDeltaS( 6, 4 );
1663200

gap>

gap> ## Generate the set of morphisms of Delta S, [2] --> [2]
gap> EnumerateDeltaS( 2, 2 );

[ [ [ 0, 1, 2 ], [ ], [ ] ], [ [ 0, 2, 1 ], [ ], [ ] ],
  [ [ 1, 0, 2 ], [ ], [ ] ], [ [ 1, 2, 0 ], [ ], [ ] ],
  [ [ 2, 0, 1 ], [ ], [ ] ], [ [ 2, 1, 0 ], [ ], [ ] ],
  [ [ 0, 1 ], [ 2 ], [ ] ], [ [ 0, 2 ], [ 1 ], [ ] ],
  [ [ 1, 0 ], [ 2 ], [ ] ], [ [ 1, 2 ], [ 0 ], [ ] ],
  [ [ 2, 0 ], [ 1 ], [ ] ], [ [ 2, 1 ], [ 0 ], [ ] ],
  [ [ 0, 1 ], [ ], [ 2 ] ], [ [ 0, 2 ], [ ], [ 1 ] ],
  [ [ 1, 0 ], [ ], [ 2 ] ], [ [ 1, 2 ], [ ], [ 0 ] ],
  [ [ 2, 0 ], [ ], [ 1 ] ], [ [ 2, 1 ], [ ], [ 0 ] ],
  [ [ 0 ], [ 1, 2 ], [ ] ], [ [ 0 ], [ 2, 1 ], [ ] ],
  [ [ 1 ], [ 0, 2 ], [ ] ], [ [ 1 ], [ 2, 0 ], [ ] ],
  [ [ 2 ], [ 0, 1 ], [ ] ], [ [ 2 ], [ 1, 0 ], [ ] ],]
\[
\begin{array}{ccc}
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[ [ 1 ], [ 2 ], [ 0 ] ], & [ [ 2 ], [ 0 ], [ 1 ] ], & [ [ 2 ], [ 1 ], [ 0 ] ], \\
[ [ 0 ], [ ], [ 1, 2 ] ], & [ [ 0 ], [ ], [ 2, 1 ] ], & [ [ 1 ], [ ], [ 2, 0 ] ], \\
[ [ 1 ], [ ], [ 0, 2 ] ], & [ [ 1 ], [ ], [ 2, 0 ] ], & [ [ 2 ], [ ], [ 1, 0 ] ], \\
[ [ 2 ], [ ], [ 0, 1 ] ], & [ [ 2 ], [ ], [ 1, 0 ] ], & [ [ ], [ 0, 1, 2 ], [ ] ], \\
[ [ ], [ 0, 2, 1 ], [ ] ], & [ [ ], [ 0, 2, 1 ], [ ] ], & [ [ ], [ 1, 0, 2 ], [ ] ], \\
[ [ ], [ 1, 0, 2 ], [ ] ], & [ [ ], [ 1, 0, 2 ], [ ] ], & [ [ ], [ 1, 2, 0 ], [ ] ], \\
[ [ ], [ 1, 0, 2 ], [ ] ], & [ [ ], [ 1, 2, 0 ], [ ] ], & [ [ ], [ 2, 0, 1 ], [ ] ], \\
[ [ ], [ 2, 0, 1 ], [ ] ], & [ [ ], [ 2, 0, 1 ], [ ] ], & [ [ ], [ 2, 1, 0 ], [ ] ], \\
[ [ ], [ 2, 0, 1 ], [ ] ], & [ [ ], [ 2, 1, 0 ], [ ] ], & [ [ ], [ 0, 1 ], [ 2 ] ], \\
[ [ ], [ 0, 1 ], [ 2 ] ], & [ [ ], [ 0, 1 ], [ 2 ] ], & [ [ ], [ 0, 2 ], [ 1 ] ], \\
[ [ ], [ 0, 2 ], [ 1 ] ], & [ [ ], [ 0, 2 ], [ 1 ] ], & [ [ ], [ 1, 0 ], [ 2 ] ], \\
[ [ ], [ 1, 0 ], [ 2 ] ], & [ [ ], [ 1, 0 ], [ 2 ] ], & [ [ ], [ 1, 2 ], [ 0 ] ], \\
[ [ ], [ 1, 2 ], [ 0 ] ], & [ [ ], [ 1, 2 ], [ 0 ] ], & [ [ ], [ 2, 0 ], [ 1 ] ], \\
[ [ ], [ 2, 0 ], [ 1 ] ], & [ [ ], [ 2, 0 ], [ 1 ] ], & [ [ ], [ 2, 1 ], [ 0 ] ], \\
[ [ ], [ 2, 1 ], [ 0 ] ], & [ [ ], [ 2, 1 ], [ 0 ] ], & [ [ ], [ 0 ], [ 1, 2 ] ], \\
[ [ ], [ 0 ], [ 1, 2 ] ], & [ [ ], [ 0 ], [ 1, 2 ] ], & [ [ ], [ 0 ], [ 2, 1 ] ], \\
[ [ ], [ 0 ], [ 2, 1 ] ], & [ [ ], [ 0 ], [ 2, 1 ] ], & [ [ ], [ 1, 0, 2 ], [ ] ], \\
[ [ ], [ 1, 0, 2 ], [ ] ], & [ [ ], [ 1, 0, 2 ], [ ] ], & [ [ ], [ 1, 2, 0 ], [ ] ], \\
[ [ ], [ 1, 2, 0 ], [ ] ], & [ [ ], [ 1, 2, 0 ], [ ] ], & [ [ ], [ 2, 0, 1 ], [ ] ], \\
[ [ ], [ 2, 0, 1 ], [ ] ], & [ [ ], [ 2, 0, 1 ], [ ] ], & [ [ ], [ 2, 1, 0 ], [ ] ], \\
[ [ ], [ 2, 1, 0 ], [ ] ]
\end{array}
\]

gap>

\texttt{gap> \#\# Generate only the epimorphisms [2] \textarrow{} [2]}

\texttt{gap> EnumerateDeltaS( 2, 2 : epi );}

\texttt{[ [ [ 0 ], [ 1 ], [ 2 ] ], [ [ 0 ], [ 2 ], [ 1 ] ], [ [ 1 ], [ 0 ], [ 2 ] ], [ [ 1 ], [ 2 ], [ 0 ] ], [ [ 2 ], [ 0 ], [ 1 ] ], [ [ 2 ], [ 1 ], [ 0 ] ] ]}

\texttt{gap>}

\texttt{gap> \#\# Compose two morphisms of Delta S.}

\texttt{gap> a := Random(EnumerateDeltaS(4,3));}

\texttt{[ [ 0 ], [ 2, 4, 1 ], [ ], [ 3 ] ]}
gap> b := Random(EnumerateDeltaS(3,2));
[ [ ], [ 3, 0, 2 ], [ 1 ] ]
gap> MultDeltaS(b, a);
[ [ ], [ 3, 0 ], [ 2, 4, 1 ] ]
gap> MultDeltaS(a, b);
Maps incomposeable
[ ]
gap>

gap> ## Examples of using morphisms of Delta S to act on simple tensors

gap> A := TruncPolyAlg([3,2]);
<algebra of dimension 6 over Rationals>

gap> ## TruncPolyAlg is defined in Basic.g

gap> ## TruncPolyAlg([i_1, i_2, ..., i_n]) is generated by

gap> ## x_1, x_2, ..., x_n, under the relation (x_j)^{(i_j)} = 0.

gap> g := GeneratorsOfLeftModule(A);
[ X^[ 0, 0 ], X^[ 0, 1 ], X^[ 1, 0 ], X^[ 1, 1 ], X^[ 2, 0 ], X^[ 2, 1 ] ]
gap> x := g[2]; y := g[3];
X^[ 0, 1 ]
X^[ 1, 0 ]

gap> v := [ x*y, 1, y^2 ];

gap> ## v represents the simple tensor xy \otimes 1 \otimes y^2.
[ X^[ 1, 1 ], 1, X^[ 2, 0 ] ]
gap> ActByDeltaS( v, [[2], [], [0], [1]] );
[ X^[ 2, 0 ], 1, X^[ 1, 1 ], 1 ]
gap> ActByDeltaS( v, [[2], [0,1]] );
[ X^[ 2, 0 ], X^[ 1, 1 ] ]
gap> ActByDeltaS( v, [[2,0], [1]] );
gap> ## Symmetric monoidal product on DeltaS_+
gap> a := Random(EnumerateDeltaS(4,2));
[ [ ], [ 2, 1, 0 ], [ 3, 4 ] ]

gap> b := Random(EnumerateDeltaS(3,3));
[ [ ], [ ], [ ], [ 1, 3, 2, 0 ] ]

gap> MonoidProductDeltaS(a, b);
[ [ ], [ 2, 1, 0 ], [ 3, 4 ], [ ], [ ], [ ], [ 6, 8, 7, 5 ] ]

gap> MonoidProductDeltaS(b, a);
[ [ ], [ ], [ ], [ 1, 3, 2, 0 ], [ ], [ 6, 5, 4 ], [ 7, 8 ] ]

gap> MonoidProductDeltaS(a, []);
[ [ ], [ 2, 1, 0 ], [ 3, 4 ] ]

gap> ## Symmetric Homology of the algebra A, in degrees 0 and 1.

gap> SymHomUnitalAlg(A);
[ [ 0, 0, 0, 0, 0, 0 ], [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 6, 0, 0 ] ]

gap> ## '0' represents a factor of Z, while a non-zero p represents

gap> ## a factor of Z/pZ.

gap> ## Using layers to compute symmetric homology

gap> C2 := CyclicGroup(2);
<pc group of size 2 with 1 generators>

gap> A := GroupRing(Rationals, DirectProduct(C2, C2));
<algebra-with-one over Rationals, with 2 generators>

gap> ## First, a direct computation without layers:

gap> SymHomUnitalAlg(A);
[ [ 0, 0, 0, 0 ], [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ] ]
gap> ## Next, compute $\text{HS}_0(A)_u$ and $\text{HS}_1(A)_u$ for each generator $u$.
gap> g := GeneratorsOfLeftModule(A);
[ (1)*<identity> of ..., (1)*f2, (1)*f1, (1)*f1*f2 ]
gap> SymHomUnitalAlgLayered(A, g[1]);
[ [ 0 ], [ 2, 2, 2 ] ]
gap> SymHomUnitalAlgLayered(A, g[2]);
[ [ 0 ], [ 2, 2, 2 ] ]
gap> SymHomUnitalAlgLayered(A, g[3]);
[ [ 0 ], [ 2, 2, 2 ] ]
gap> SymHomUnitalAlgLayered(A, g[4]);
[ [ 0 ], [ 2, 2, 2 ] ]
gap> ## Computing $\text{HS}_1(Z[t])$ by layers:
gap> SymHomFreeMonoid(0,10);

$\text{HS}_1(k[t])_{t^0}$ : [ ]
$\text{HS}_1(k[t])_{t^1}$ : [ ]
$\text{HS}_1(k[t])_{t^2}$ : [ 2 ]
$\text{HS}_1(k[t])_{t^3}$ : [ 2 ]
$\text{HS}_1(k[t])_{t^4}$ : [ 2 ]
$\text{HS}_1(k[t])_{t^5}$ : [ 2 ]
$\text{HS}_1(k[t])_{t^6}$ : [ 2 ]
$\text{HS}_1(k[t])_{t^7}$ : [ 2 ]
$\text{HS}_1(k[t])_{t^8}$ : [ 2 ]
$\text{HS}_1(k[t])_{t^9}$ : [ 2 ]
$\text{HS}_1(k[t])_{t^{10}}$ : [ 2 ]
gap> ## Poincare polynomial of $\text{Sym}^*\{p\}$ for small $p$.

gap> ## There is a check for torsion, using a call to Fermat
gap> ## to find Smith Normal Form of the differential matrices.
gap> PoincarePolynomialSymComplex(2);
C_0 Dimension: 1
C_1 Dimension: 6
C_2 Dimension: 6
D_1
SNF(D_1)
D_2
SNF(D_2)
2*t^2+t

gap> PoincarePolynomialSymComplex(5);
C_0 Dimension: 1
C_1 Dimension: 30
C_2 Dimension: 300
C_3 Dimension: 1200
C_4 Dimension: 1800
C_5 Dimension: 720
D_1
SNF(D_1)
D_2
SNF(D_2)
D_3
SNF(D_3)
D_4
SNF(D_4)
D_5
SNF(D_5)
120 \cdot t^5 + 272 \cdot t^4 + t^3
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