A free central-limit theorem for dynamical systems.

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Abstract

The free central-limit theorem, a fundamental theorem in free probability, states that empirical averages of freely independent random variables are asymptotically semi-circular. We extend this theorem to general dynamical systems of operators that we define using a free random variable \( X \) and a group of \(*\)-automorphisms describing the evolution of \( X \). We introduce free mixing coefficients that measure how far a dynamical system is from being freely independent. Under conditions on those coefficients, we prove that the free central-limit theorem also holds for these processes and provide Berry-Essen bounds. We generalize this to triangular arrays and \( u \)-statistics. Finally we draw connections with classical probability and random matrix theory with a series of examples.

1 Introduction

The free central-limit theorem is a fundamental theorem of free probability. It states that the distribution of a sum of \( n \) freely independent operators is asymptotically semi-circular as \( n \) goes to infinity. This mirrors the classical central limit theorem with the condition of independence replaced by free independence (sometimes called freeness) and the Gaussian limit replaced by a semi-circular limit. More precisely, let \( \mathcal{A} \) be a Von-Neumann algebra with a normal faithful semi-definite trace \( \tau \) and adjoint operator \( * \). If \( X_1, \ldots, X_n \) are free copies of a self-adjoint free random variable \( X \) satisfying \( \tau(X) = 0 \) and \( \tau(X^2) = 1 \) then the free central-limit theorem guarantees that

\[
S_n := \frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow{d} \mu_{sc}.
\]

This theorem shares deep connections with the semi-circular law in random matrix theory for which it notably provides an alternative proof. However this result, and subsequent ones, rely on the assumption that \((X_1, \ldots, X_n)\) are free or “asymptotically” free. This is a strong assumption that might not be respected by many processes of interest such as sequences of dependent matrices. For example, if \((X^n_i)\) is a sequence of dependent random matrices with correlated Gaussian entries then the free central limit theorem does not apply to the study of the eigenvalue distribution of \( \frac{1}{\sqrt{n}} \sum_{i \leq n} X^n_i \). To be applicable, the free central-limit theorem would need to be generalized to weakly “freely dependent” sequences of free random variables. Another shortfall of the free central limit theorem
is that it applies exclusively to empirical averages and not to more complex quantities such as u-statistics of which \( \sum_{i,j \leq n} \Phi(X_i, X_j) \) (for a continuous function \( \Phi \)) is an example. This represents a disconnect with what is known for the classical central-limit theorem where the assumption of independence has been successfully relaxed for general dynamical systems \([10]\), and where the limit of u-statistics are extensively studied. This paper bridges this gap by generalizing the free central-limit theorem to empirical averages and u-statistics of dynamical systems. A non-commutative dynamical system is defined by a free random variable \( X \) and a group of \(*\)-automorphisms \((K_g)_{g \in G}\) describing the evolution of \( X \). Examples include stationary and quantum exchangeable sequences of free random variables. Our goal is to establish conditions under which empirical averages and u-statistics of \((X_g) := (K_g(X))\) are asymptotically semi-circular. In the classical setting, the central limit theorem is generalized to the dependent setting by defining \([4]\) mixing conditions that quantify how far a process is from being independent and imposing conditions on the speed of decay of those coefficients. Mirroring this, in section 2 we define “free mixing” coefficients that quantify how far from being (conditionally) free a process is. If those free mixing coefficients are small enough, then in section 2 we demonstrate that empirical averages of the dynamical system are asymptotically semi-circular with a radius that depends on the structure of the process. We extend this, under moment conditions, to unbounded operators as well as to triangular arrays, we also provide bounds of the type of Berry-Essen. Then, in section 3 we consider non-commutative u-statistics and under similar free mixing conditions prove that their limiting distribution is also asymptotically semi-circular. Finally, we illustrate the utility of this new notion of “free mixing” by a series of examples notably from random matrix theory in sections 2.3 and 3.3.

1.1 Related literature

The non-commutative law of large numbers \([7]\) states that if \( X_1, \ldots, X_n \) are freely independent and identically distributed random variables satisfying \( \tau(X_1) = 0 \) then under moment conditions we have \( \frac{1}{n} \sum_{i \leq n} X_i \rightarrow 0 \). This key result was greatly extended to stationary sequences of non-commutative operators \([18]\); as well as to general dynamical systems of operators \([11]\). We study, under free mixing conditions, the speed of convergence of those generalized laws of large numbers.

The free central limit-theorem, first introduced by Voiculescu \([28]\), is a second-order result that establishes a speed of convergence for the classical law of large numbers. It was later extended in \([8]\), under moment conditions, to free convolutions of unbounded operators, and by Speicher to the multivariate case in \([24]\). Moreover, it was generalized to operators that are conditionally free (also called ’free with amalgamation’) in \([23]\), as well as to operators that satisfy a slightly weaker notion of free independence in \([16]\). Just as in classical probability, Berry-Essen type bounds guarantee a speed of convergence for the free central-limit theorem. They were first established for sums of bounded free operators \([15]\) then later for free unbounded operators \([10]\), and finally to the multivariate and conditionally free case \([21]\).

Semi-circular limits have also been of great interest in the random matrix literature.
We define the \( \tau \) which notably satisfies spectral theorem if \( X \). Finally inspired by application in communication theory operator-valued matrices have been the object of increased interest \([6,21]\). In section 2.3 we prove, under general conditions, that the empirical average of dependent patterned random matrices is asymptotically semi-circular. The study of this class of random matrices has been motivated by applications in \([12]\). \([5]\) studied the spectral norm of those matrices by using ideas coming from free probability.

2 Main results

2.1 Definitions and notations

Let \( \mathcal{A} \) be a Von Neumann algebra with a normal faithful semi-finite trace \( \tau \) and adjoint operator \( \ast \). We write \( \pi(\mathcal{A}) \) its faithful representation on a Hilbert space \( \mathcal{H}_\lambda \), and for simplicity identify it with \( \mathcal{A} \). Throughout this paper, an important example will be \( \mathcal{A}_m := L_\infty \otimes M_m(\mathbb{C}) \), the algebra of random matrices of size \( m \) with essentially bounded entries. In this case, the trace is taken to be \( X \rightarrow \frac{1}{m} \mathbb{E}(\text{Tr}(X)) \). We say that an operator \( X \) is affiliated with \( \mathcal{A} \) if all its spectral projections belong to \( \mathcal{A} \). It is, in addition, \( \tau \)-measurable if for all \( \delta \) there is a projection \( p \) of \( \mathcal{H}_\lambda \) such that \( \tau(I-p) \leq \delta \) and \( p\mathcal{H} \subset \mathcal{D}(X) \) where \( \mathcal{D}(X) \) designates the domain of \( X \). We write \( \mathcal{A}_\tau \) the set of \( \tau \)-measurable operators, this forms a Hausdorff complete *-algebra.

Free random variable and non-commutative \( L_p \)-space A free random variable is an operator \( X \in \mathcal{A}_\tau \), for example a random matrix of size \( m \). We say that it is self-adjoint if it verifies \( X^* = X \). We denote by \( \sigma(X) \) the spectrum of \( X \). According to the spectral theorem if \( X \) is self-adjoint then there is a unique projection-valued measure \( P_X \) on \( \sigma(X) \) such that \( X = \int_{\sigma(X)} \lambda dP_X(\lambda) \). We call \( \mu_X := \tau \circ P_X \) the distribution of \( X \) which notably satisfies \( \tau(X^k) := \int_{\mathbb{R}} x^k d\mu_X(x) \).

We define the \( L_p \)-space and \( L_p \)-norm of \( \mathcal{A}_\tau \) in the following way:

\[
L_p(\mathcal{A}, \tau) := \left\{ X \in \mathcal{A}_\tau \mid \int_{-\infty}^{\infty} t^p d\mu_{|X|^2}(t) < \infty \right\}, \quad \|X\|_{\tau} = \left( \int_{-\infty}^{\infty} t^p d\mu_{|X|^2}(t) \right)^{\frac{1}{p}}.
\]

Similarly as in the classical setting, we can define a notion of multivariate distribution. The challenge however is that as free random variables do not necessarily commute we cannot rely on the spectral theorem. Instead, we remark that \( X, Y \in \mathcal{A} \) have the same distributions if and only if all their moments are equal. This is the notion we extend to the multivariate setting. We denote by \( \mathbb{C}(x_1, \ldots, x_k, x_1^*, \ldots, x_k^*) \) the algebra of non commutative *-polynomials in the formal random variables \( x_1, \ldots, x_k \). We say that \( (X_i) \in \mathcal{A}^N \) has the same distribution that \( (Y_i) \in \mathcal{A}^N \) if and only if for all integers \( k \in \mathbb{N} \) and for all polynomials \( P \in \mathbb{C}(x_1, \ldots, x_k, x_1^*, \ldots, x_k^*) \) we have:

\[
\tau(P(X_{i_1}, \ldots, X_{i_k}, X_{i_1}^*, \ldots, X_{i_k}^*)) = \tau(P(Y_{i_1}, \ldots, Y_{i_k}, Y_{i_1}^*, \ldots, Y_{i_k}^*));
\]
and we write \((X_i) \overset{d}{=} (Y_i)\). To extend this to elements of \(A_r\), we note that general elements of \(A_r\) do not necessarily have finite moments. We say instead that \((X_i) \in A_r^N\) has the same distribution that \((Y_i) \in A_r^N\) if there are sequences \((p_{i,n})\) and \((p_{i,n}')\) of projectors of \(H_\lambda\) satisfying:

- \(\tau(1 - p_{i,n}), \tau(1 - p_{i,n}') \xrightarrow{n \to \infty} 0\) for all \(i \in \mathbb{N}\);
- \((X_{i_1}p_{i_1,n}, \ldots, X_{i_k}p_{i_k,n}) \in A^k\), \((Y_{i_1}p_{i_1,n}', \ldots, Y_{i_k}p_{i_k,n}') \in A^k\) for all \(n \in \mathbb{N}\);
- \((X_{i_1}p_{i_1,n}, \ldots, X_{i_k}p_{i_k,n}) \overset{d}{=} (Y_{i_1}p_{i_1,n}', \ldots, Y_{i_k}p_{i_k,n}')\) for all \(n \in \mathbb{N}\).

Finally we say that a sequence \((a_n) \in A_r^\mathbb{N}\) converges almost everywhere (a.e) to \(a \in A_r\) if for all \(\epsilon > 0\) there is a projection of \(H_\lambda\) that we denote \(p_\epsilon\) such that (i) \(\tau(p_\epsilon) \geq 1 - \epsilon\); and (ii) \(\|(a_n - a)p_\epsilon\|_\infty \to 0\).

**L.s.c.H amenable group** Let \(G\) be a topological group. If its topology is locally compact, second countable and Haussdroff we say that the group is locally compact, second countable and Haussdroff \((l.c.s.c.H)\). For example if \(G\) is a countable group equipped with the discrete topology it is \(l.c.s.c.H\). We always equip \(G\) with its Borel \(\sigma\)-algebra \(B(G)\). For every \(l.c.s.c.H\) group we can find a measure \(|\cdot|\) on \(B(G)\) that is left-invariant:

\[
|gS| = |S|, \quad \forall g \in G \text{ and } S \in B(G).
\]

Such a measure is called a *Haar*-measure. It is unique up to a multiplicative constant. For example if \(G = \mathbb{R}\) the Lebesgue measure is a Haar measure or if \(G\) is a countable group the cardinality is also a Haar measure. In general we note that if \(K \subset G\) is a compact subset then its Haar measure is finite \(|K| < \infty\). Informally a Haar measure generalizes the notion of volume and the eq. (1) shows that a set can be shifted without changing its volume. In the following \(|\cdot|\) will always designate a Haar measure. Similarly as the volume is left invariant if \(G\) is \(l.c.s.c.H\) we can always find a metric \(d(\cdot, \cdot)\) on \(G\) that is left invariant:

\[
d(\cdot, \cdot) = d(g\cdot, g\cdot) \quad \forall g \in G.
\]

For example if \(G\) is a finitely generated nilpotent group then the word metric is left invariant or if \(G = \mathbb{R}\) is the group of reals then \(d : (x, y) \rightarrow |x - y|\) is left invariant. We denote \(B(\phi, t)\) the metric ball in \(G\) centered around \(\phi \in G\) and of radius \(t > 0\). We shorthand \(B(t) := B(e, t)\) where \(e\) designates the neutral element of \(G\). Finally we write \(d\) the Haussdroff distance induced by \(d\):

\[
d(G_1, G_2) := \min_{g_1 \in G_1, g_2 \in G_2} d(g_1, g_2).
\]

A group is said to be *amenable* if we can find a sequence \((A_n)\) of subsets of \(G\) that satisfy for all compact set \(K \subset G\):

\[
\frac{|A_n \cap KA_n|}{|A_n|} \xrightarrow{n \to \infty} 0.
\]
Such a sequence is called a Følner sequence. For example if $G = \mathbb{R}^r$ then the sequence of subsets $\left( (0, n]^r \right)$ is a Følner sequence. Similarly, if $G = S(N)$ is the group of permutations of $N$ and if we write $S_n \subset S(N)$ the subgroup of permutations that leave $[n, \infty)$ invariant then the sequence $(S_n)$ is a Følner sequence. We say that $(A_n)$ is tempered if

$$\left| \bigcup_{k<n} A_k^{-1} A_n \right| \leq c |A_n|$$

for some $c > 0$ and all $n \in \mathbb{N}$.

Not every Følner sequence is tempered but every l.c.s.c.H amenable group $G$ contains a Følner sequence that is also tempered. 

**Group indexed process and definition of $\mathcal{F}_G(\cdot)$** Let $X := (X_g)_{g \in G}$ be a sequence of free random variables indexed by a l.c.s.c.H amenable group $G$ with Følner sequence $(A_n)$. We call $(X_g)$ a non-commutative process indexed by $G$.

For all subsets $G \subset \mathbb{G}$ we denote $\mathcal{F}_G(X)$ the unital algebra generated by $\{X_g, g \in G\}$. For example if $G = \mathbb{Z}$ is the group of integers and $G = [i]$ then $\mathcal{F}_G(X)$ is the unital algebra generated by $\{X_l, 0 \leq l \leq i\}$.

**Invariant dynamical systems** A special case of non-commutative processes are non-commutative dynamical systems. They are defined by a free random variable $\tilde{X}$ and a group of *-automorphisms describing the evolution of $\tilde{X}$. More specifically, let $(K_g)_{g \in G}$ be a net of *-automorphisms of $A_r$ that satisfies

1. $\tau(K_g(a)) = \tau(a)$ for all $a \in L_1(A_r, \tau)$ \hspace{1cm} $(H_1)$.
2. $K_g \circ K_{g'} = K_{gg'}$ \hspace{1cm} $(H_2)$.

We define $X := (X_g)$ as the sequence of images of $\tilde{X}$: $X_g := K_g(\tilde{X})$. We call $X$ a dynamical system and say that $(K_g)$ defines a group action on $A_r$. Examples of dynamical systems include stationary fields for which we take $G$ to be $\mathbb{Z}^r$ or exchangeable sequences of free random variables for which we take $G$ to be $S(N)$. We note that the condition $(H_1)$ implies that the distribution of $X_g$ is the same for all $g \in G$. We say that $X$ is distributionally invariant under the action of $G$. Classically when we take $G = \mathbb{Z}^r$ we call such an invariant process stationary; and when we take $G = S(N)$ we call such a process exchangeable. The converse is also true: Any distributionally invariant process $(X_g)$ can alternatively be defined as a dynamical system.

**Proposition 1.** Let $\tilde{X} \in L_1(A_r, \tau)$ be a self adjoint operator and $(K_g)$ be a set of *-automorphisms satisfying conditions $(H_1)$-$(H_2)$). If we write $X_g = K_g(\tilde{X})$, then the process $X := (X_g)$ satisfies

$$(X_{g_1}, \ldots, X_{g_k}) \overset{d}{=} (X_{gg_1}, \ldots, X_{gg_k}), \hspace{1cm} \forall g, g_1, \ldots, g_k \in \mathbb{G}.$$ 

Conversely let $(Z_g) \in \mathcal{A}^G_r$ be a sequence indexed by the group $\mathbb{G}$. Denote $B \subset \mathcal{A}_r$ the unital sub-algebra generated by $(Z_g)$. If for all $g, g_1, \ldots, g_k \in \mathbb{G}$ we have $(Z_{g_1}, \ldots, Z_{g_k}) \overset{d}{=} (Z_{gg_1}, \ldots, Z_{gg_k})$, and if $g \rightarrow Z_g$ is continuous almost everywhere then there is a net $(K_g)$ of *-automorphisms of $B$ that verifies conditions $(H_1)$-$(H_2)$ and is such that $K_g(Z_{g'}) = Z_{gg'}$ for all $g, g' \in \mathbb{G}$. 


The invariant algebra is defined as
\[ \mathcal{F}^{\text{tail}}(X) := \{ a | a \in \mathcal{A}_r, \ K_g(a) = a, \ \forall g \in \mathcal{G} \} \subset \mathcal{A}_r. \]

There is a unique linear map \( E : \mathcal{A}_r \to \mathcal{F}^{\text{tail}}(X) \) that satisfies (i) \( E(aXb) = aE(X)b \) for all \( X \in \mathcal{A}_r \) and \( a, b \in \mathcal{F}^{\text{tail}}(X) \); and (ii) \( \tau(X) = \tau(E(X)) \) for all \( X \in \mathcal{A}_r \). We call \( E \) the non-commutative conditional expectation on \( \mathcal{F}^{\text{tail}}(X) \) (see \[20\] for more background).

Finally, we say that \( (X_g) \) is ergodic if for all \( A, B \in \mathcal{F}_G \) we have:
\[
\lim_{n \to \infty} \frac{1}{|A_n|} \int_{A_n} \tau(K_g(A)B) \, d|g| = \tau(A)\tau(B).
\]

When the dynamical system \( (X_g) \) is ergodic then the invariant-algebra \( \mathcal{F}^{\text{tail}}(X) \) is trivial.

**Ergodic theorem** In classical probability, if \( \mathcal{G} = \mathbb{Z} \) is the group of integers then the average of observations \( (Y_i) \) over \([n]\) is called an empirical average. Intuitively, the Følner sequence \( (A_n) \) plays the same role for \( \mathcal{G} \) than \([n]\) does for \( \mathbb{Z} \); it is an exhaustive and stable sequence of subsets. Therefore we call \( X^n := \frac{1}{|A_n|} \int_{A_n} X_g \, d|g| \) an empirical average. For example if \( \mathcal{G} = \mathbb{Z}^r \) and the Følner sequence is chosen to be \( A_n := \mathbb{Z}^r \) then we have \( X^n = \frac{1}{n} \sum_{i = 1}^{n} X_{i-1} \ldots X_0 \). Our goal is to study the asymptotic of this estimator. The ergodic theorem for dynamical systems \[11\] states that empirical averages of dynamical systems converge to their conditional expectation \( E(X) \).

**Theorem 1.** \[11\] Let \( \hat{X} \in L_1(\mathcal{A}, \tau) \) and \( (K_g) \) be a sequence of *-automorphisms respecting \( (H_1) - (H_2) \). Define \( X_g = K_g(\hat{X}) \) and choose \( (A_n) \) to be a tempered Følner sequence of \( \mathcal{G} \). Then empirical averages converge to their conditional expectation:
\[
\frac{1}{|A_n|} \int_{A_n} X_g \, d|g| \xrightarrow{L_1} E(X_e).
\]

This theorem is the generalization of the classical ergodic theorem established by E. Linderstrauss \[20\] to the non-commutative setting.

Our goal is to establish a speed of convergence for theorem \[1\] Mirroring how the central limit theorem can be extended to weakly dependent processes, we require that \( (X_g) \) is “not too far” from being free. We quantitate this through mixing coefficients.

**Free-mixing** coefficients We define free mixing coefficients that quantify how far \( (X_g) \) is from being freely independent. In classical probability, the dependence of a stationary sequence \( (Z_i) \) is quantified through strong-mixing coefficients, alternatively called \( \alpha \)-mixing coefficients. They are defined as
\[
\alpha(i) := \sup_{A \in \sigma(Z_{-\infty}^0)} \sup_{B \in \sigma(Z_{i,\infty})} \left| P(A, B) - P(A)P(B) \right|,
\]
where \( \sigma(Z_{-\infty}^0) \) and \( \sigma(Z_{i,\infty}) \) designate the sigma-fields of events generated by the observations \( \ldots, X_{-1}, X_0 \) and respectively by the observations \( X_i, X_{i+1}, \ldots \). The faster \( \alpha(i) \) decreases as a function of \( i \) the weaker the dependence between the observations \( (Z_i) \) is. The central-limit theorems has been extended to dependent sequences by enforcing conditions on the strong mixing coefficients \[9\]. This notion has been generalized to general
dynamical systems \((Z_g)\) by choosing a metric on the underlying group and upper-bounding the correlations between events depending on \(\{Z_{g_1}, Z_{g_2}\}\) and events depending on \(\{Z_g, g \in \Gamma\}\) for a subset \(\Gamma\) “far away” from \(g_1\) and \(g_2\). The free mixing coefficients we define resemble those strong-mixing coefficients. However due to the non-commutativity of the process \((X_g)\) we will need to control various alternating products of the type \(E(a_1 b_1 a_2 b_2 a_3)\) for \(a_1, a_2, a_3\) and \(b_1, b_2\) belonging to some “far away” algebras. We write \(C[b]\) the collection of sets that are “far away” from each other

\[
C[b] := \left\{ (G_1, G_2) \mid G_1, G_2 \subset \mathbb{G}, \text{s.t.} \bar{d}(G_1, G_2) \geq b \text{ and } \text{card}(G_1) \leq 2 \right\}.
\]

For a complex number \(\gamma \in \mathbb{C} \setminus \mathbb{R}\) we denote \(R_\gamma : A_r \to A\) the following function

\[
R_\gamma : A \to \text{Im}(\gamma) [A - \gamma 1_A]^{-1}.
\]

The function \(R_\gamma\) is called resolvent and plays a central role in functional calculus. We write

\[
\mathcal{H}_\lambda := \{ R_\gamma \mid \text{Im}(\gamma) > \lambda \}.
\]

If \((G_1, G_2) \in C[b]\) are “far away” from each other; and \(a_1, a_2, a_3 \in F^{G_1}(X)\) and \(b_1, b_2 \in F^{G_2}(X)\) then we hope that \(a_1, a_2\) and \(a_3\) are almost freely independent from \(b_1\) and \(b_2\). Our free mixing coefficients will capture this for specific choices of \(a_1, a_2, a_3\) and \(b_1, b_2\).

In this goal for \(a \in L_1(A, \tau)\) we write \(\overline{\mathfrak{m}} := A - E(a)\); we centralize and normalize \(X_g\) and define \(X_g^N := X_g - E[X_g]\). For \(f \in \mathcal{H}_\lambda\) we write \(f^K := f\left(\frac{1}{\sqrt{|K|}} \int_K X_g d|g|\right)\). The free mixing coefficients are defined as

\[
\mathbb{N}^{j,\lambda}[b|G] := \sup_{(g, g') \in C[b]} \sup_{f \in \mathcal{H}_\lambda} \max_{g \in F^{G}(X)} \left\{ \left\| E\left( f^G X_g^N \overline{f^G} X_{g'}^N f^G \right) \right\|_1 \right\}
\]

\[
\mathbb{N}^{j,\lambda}[b|G] := \sup_{(g, G \in \mathcal{A})} \sup_{f \in \mathcal{H}_\lambda} \max_{g \in F^{G}(X)} \left\{ \left\| E\left( f^{G} X_g^N E(f^G) X_{g'}^N f^G \right) \right\|_1 \right\}
\]

We call free mixing coefficients of \((X_g)\) the elements

\[
\mathbb{N}[b|G] = (\mathbb{N}^{j,\lambda}[b|G], \mathbb{N}^3[b|G]).
\]

We remark that if \((X_g)\) is free then for all \(b \geq 1\) we have \(\mathbb{N}[b|G] = (0, 0)\). This is also the case if \((X_g)\) is free with amalgamation (see proposition 3). We remark that if \((X_g)\) is ergodic then the conditional expectation is trivial: \(E(a) = \tau(a)1_A\) for all \(a \in A\).

Therefore as \(\tau\) is a normal trace then the expression of the free mixing coefficients can be simplified to control alternating products of size 4.

Let \(A, B\) be two independent random matrices in the Gaussian Unitary Ensemble \(GUE(m)\). It is well known that \(A\) and \(B\) are asymptotically free as the size of the matrices \(m\) goes to infinity [2]. We extend this beyond the Gaussian Unitary Ensemble to patterned
random matrices and dependent sequences of random matrices. We show under general conditions that “almost independent” random matrices are also “almost free”. In proposition [4] we show that under general conditions if \((X_{i,m})_{i \in \mathbb{Z}}\) is a stationary sequence of random matrices with Gaussian entries then we can bound its free-mixing coefficients in terms of its \(\alpha\)-mixing coefficients. We present here an example of this for block-independent random matrices.

**Example 2.1.** Let \((X_{i,m})_{i \in \mathbb{Z}}\) be a stationary sequence of self-adjoint random matrices of size \(m \times m\) with centered Gaussian entries. Write \((\alpha_m[\cdot])\) the strong-mixing coefficients of \((X_{i,m})_{i \in \mathbb{Z}}\). Let \(P_m = (P_{m1}^1, \ldots, P_{mN_m}^m)\) be a symmetric partition of \(\{m\}^2\). Suppose that for all indexes \((k_1, t_1) \in P_{j1,n}\) and indexes \((k_2, t_2) \in P_{j2,n}\) that belong to different elements of the partitions \(j_1 \neq j_2\), the entries \((X_{k_1, t_1}^{i,m})_{i \in \mathbb{Z}}\) are independent of \((X_{k_2, t_2}^{i,m})_{i \in \mathbb{Z}}\).

Assume that for all \(i, j \leq m\) we have \(\mathbb{E}(X_{i,j}^{1,m}) = m^{-1}\) then the following holds for all \(\lambda > 0\)

\[
\mathcal{N}_m^{i,m}[\{\mathbb{Z}\}] \leq \alpha_m[\mathbb{Z}]^{\frac{1}{m^2}} + \frac{\max_{k \leq N_m} \# P_{km}}{m}, \quad \mathcal{N}_m^{i,m}[\{\mathbb{Z}\}] \leq \alpha_m[\mathbb{Z}]^{\frac{1}{m^2}}.
\]

**Operator-valued radius and Stieltjes transform** We call a \(\mathcal{F}_{\text{tail}}(X)\)-valued radius a completely positive map from \(A\) into \(\mathcal{F}_{\text{tail}}(X)\). We remark that, as \(\mathcal{F}_{\text{tail}}(X)\) is not necessarily trivial, the mixing coefficients \((\mathcal{N}[\mathbb{G}])\) can be vanishing without the process \((X_g)\) being ergodic in which case the operator radius might be operator valued. We define the \(\mathcal{F}_{\text{tail}}(X)\)-valued Stieltjes transform of a self adjoint free random variable \(Y\) by:

\[
S_Y : \gamma \rightarrow E\left(\left[ Y - \gamma 1_A \right]^{-1}\right).
\]

We say that a self-adjoint free random variable \(Y\) follows an operator valued semi-circular law with radius \(\eta\) if its Stieltjes transform satisfies:

\[
\eta(S_Y(\gamma))S_Y(\gamma) + \gamma S_Y(\gamma) + 1_A = 0, \quad \forall \gamma \in \mathbb{C} \setminus \mathbb{R}
\]

\[
S_Y(\gamma) \sim -\frac{1}{\gamma} 1_A, \quad \text{as } \gamma \rightarrow \infty.
\] (4)

**2.2 Main result for empirical averages.**

Let \((\mathcal{A}_n)\) be a sequence of Von Neumann algebras with normal faithful and semidefinite trace \(\tau_n\). Choose \((G_n)\) to be a sequence of l.c.s.c.H amenable groups with Følner sequence \((A_{n,n})\). Let \(d_n(\cdot, \cdot)\) be a left-invariant distance over \(G_n\) and denote by \(B^n(\cdot, \cdot)\) the induced balls. Let \(X_n = (X_g^n)_{g \in G_n}\) be a dynamical system of self-adjoint free random variables. We denote by \(\mathcal{F}_{\text{tail}}(X_n)\) the tail-algebra of \((X_g^n)\) and write \(E_n\) the non-commutative conditional expectation on \(\mathcal{F}_{\text{tail}}(X_n)\). Our goal is to study the asymptotics of

\[
W_n := \frac{1}{\sqrt{|A_{n,n}|}} \int_{A_{n,n}} X_g^n - E_n(X_g^n) d|g|.
\]
In this goal, we write \( \mu_n \) the distribution of \( W_n \) and denote \( (\mathcal{N}_n[b|G_n]) \) the free mixing coefficients of \( (X_n^n) \). To control how fast those mixing coefficients decrease, we define

\[
R^n_\mu[b] := \sum_{k \geq b} \left( B^k_{k+1} \setminus B^k_k \right) \cap A_{n,n} \mathcal{N}_n^n[k|G_n].
\]

If the free mixing coefficients decrease fast enough we show that \( W_n \) is asymptotically semi-circular. More specifically, we make the following hypothesis: For all \( \lambda > 0 \)

\[
\limsup_{n \to \infty} \left\{ R^n_\mu[b] + |B^n_b|\mathcal{N}_n^n \lambda [b|G] \right\} \xrightarrow{b \to \infty} 0 \quad (H_{\text{mixing}})
\]

The distribution of \( W_n \) is compared to a semi-circular distribution with radius \( \eta_n \) defined as the following completely positive map

\[
\eta_n : a \to \int_{A_{n,n}} E_n(X^n_n aX^n_n)dg.
\]

We write \( S_n(\cdot) \) the operator-valued Stieljes transform of \( W_n \), and \( S_n^{sc}(\cdot) \) the Stieljes transform of the semi-circular operator \( Y^{sc,\eta_n} \) with radius \( \eta_n(\cdot) \).

**Theorem 2.** Let \( (X^n) \) be a sequence array of self-adjoint free random variables satisfying \( \sup_{n \in \mathbb{N}} \|X^n\|_3 < \infty \). Let \( (K_n^n) \) be a sequence of \( * \)-automorphisms satisfying conditions \((H_1) - (H_2)\). Write \( X^n_n := K^n_0(X^n) \) and denote by \( (\mathcal{N}_n[b|G_n]) \) the free mixing coefficients of \( (X^n_0) \); suppose that \((H_{\text{mixing}})\) holds. Let \( \gamma_{x,\nu} = x + i\nu \in \mathbb{C} \setminus \mathbb{R} \) be a complex number with \( \nu > 0 \). Set \( (b_n) \) to be a sequence of integers. There is a constant \( C \) not depending of \( X^n, G_n, n, b_n \) or \( \gamma_{x,\nu} \) such that if we write \( K = C (\|X^n_0\|^3 + \|Y^{sc,\eta_n}\|^3) \lor 1 \) we have

\[
\left\| S_n(\gamma_{x,\nu}) - S_n^{sc}(\gamma_{x,\nu}) \right\|_1 \leq K \frac{1}{\nu^2} \left( R^n_\mu[b_n] + |A_{n,n} \triangledown B_{b_n} A_{n,n}| \right) + |B^n_{b_n}| |\mathcal{N}_n^n[b_n|G_n]| + \frac{|B^n_{b_n}|^2}{\sqrt{|A_{n,n}| \nu^2}} \to 0.
\]

**Remark 1.** We note that the size of the Berry-Esseen bound depends on how fast the free mixing coefficients \( (\mathcal{N}_n[b|G_n]) \) decrease as a function of \( b \). Notably if \( \mathcal{N}_n[b|G_n] = 0 \) for all \( b > 0 \) then we obtain

\[
\left\| S_n(\gamma_{x,\nu}) - S_n^{sc}(\gamma_{x,\nu}) \right\|_1 = O\left( \frac{1}{\sqrt{|A_{n,n}| \nu^2}} \right).
\]

Note that in the case where \( G_n = \mathbb{Z} \), and \( (X^n) \) are freely independent then the upper-bound we obtain is of order \( O\left( \frac{1}{\sqrt{\text{Var}(x)}} \right) \). \( \text{[14]} \) shows that the optimal rate is indeed of order \( \frac{1}{\sqrt{n}} \); but the term \( \nu^{-4} \) is greater than what could be obtained by analytical methods (e.g. see \( \text{[15, 16]} \)). However, those rely strongly on the fact that the following equation holds when \( X \) and \( Y \) are freely independent: \( S^{-1}_{X+Y}(z) = S^{-1}_X(z) + S^{-1}_Y(z) + z^{-1} \). Those methods are not adaptable for free mixing processes \( (X) \).
2.3 Examples

In this section, we explore a few illustrative examples of dynamical systems \((X_g)\). We bound their free mixing coefficients and deduce that their asymptotic distribution is semi-circular.

2.3.1 Examples of general operators

Firstly we note that if \((X_i)\) is a freely independent sequence then its free mixing coefficients are null. However the reverse does not hold and null free mixing coefficients imply something weaker than freeness. Similarly [16] proved a free central limit theorem under the assumption that \((X_i)\) respect the following conditions:

\[(H'_1). \quad \tau(X_k Y_1) = \tau([X_k^2 - \tau(X_k^2)]Y_1) = 0 \quad \text{for all } Y_1 \in F_{N \setminus \{k\}}(X)\]

\[(H'_2). \quad \tau(X_k Y_1 X_k Y_2) = \tau(X_k^2) \tau(Y_1) \tau(Y_2) \quad \text{for all } Y_1, Y_2 \in F_{N \setminus \{k\}}(X).\]

Any process satisfying those conditions has null free mixing coefficients.

**Proposition 2.** Let \((X_i)\) be a sequence of identically distributed self-adjoint free random variables satisfying conditions \(H'_1\) and \(H'_2\). Let \((\mathcal{N}[\cdot|Z])\) denote the free mixing coefficients of \((X_i)\) then we have: \(\mathcal{N}[b|Z] = 0\) for all \(b \geq 1\) and \((X_i)\) is ergodic. Therefore \(\frac{1}{\sqrt{n}} \sum_{i \leq n} [X_i - \tau(X_i)]\) is asymptotically semi-circular with radius \(\tau(X_k^2)\).

Another important example is quantum exchangeable processes: those are processes whose distribution is invariant under the coaction of quantum permutations. These processes play in free probability an analogous role to the one exchangeable sequences have in classical probability. Indeed, [17] proved that \((X_i)\) is quantum exchangeable if and only if conditionally on its tail algebra it is freely independent and identically distributed. This notably implies that quantum exchangeable sequences are exchangeable but the reverse does not hold.

**Proposition 3.** Choose \(G = \mathbb{Z}\) and let \((X_i) \in \mathcal{A}^\mathbb{N}\) be a quantum exchangeable sequence of free self-adjoint random variables. We have

\[\mathcal{N}[b|\mathcal{S}(\mathbb{N})] = 0, \quad \forall b > 0.\]

This implies that \(\frac{1}{\sqrt{n}} \sum_{i \leq n} [X_i - E(X_i)]\) is asymptotically semi-circular with radius \(\eta : a \rightarrow E(X_1aX_1)\).

2.3.2 Examples of random matrices

An important class of examples are random matrices. Let \(\mathcal{A}_m\) be the set of random matrices of size \(m\) with essentially bounded entries. It forms a Von-Neumann algebra with \(\tau_m(\cdot) = \frac{1}{m} \mathbb{E}(\text{Tr}(\cdot))\) as the normal faithful trace. Finally for any real-valued random variable \(Y\) we write \(\|Y\|_{L_p}\) its \(L_p\) norm such as defined in classical probability. In this subsection we provide examples of free mixing coefficients for dynamical systems of random matrices.
In this section we consider patterned random matrices. Those are random matrices with patterned entries. To make this precise, let $N$ be a sequence of deterministic matrices. Centered Gaussian entries that are not necessarily identically distributed or independent.

We prove, under mixing and moment conditions, that the asymptotic distribution of $m \times m$ matrices has been studied in the setting of concentration in equalities [5] and discussed with independent diagonals, or sparse matrices etc. This class of patterned random matrices includes Gaussian random matrices with independent blocks, examples of such matrices include Gaussian random matrices with independent blocks, we consider in this subsection have the following form

\[
X^z,m = \sum_{s \leq N_m} Z_s z A^{s,m}.
\]

Random matrices with patterned entries

In this section we consider patterned random matrices. Those are random matrices with centered Gaussian entries that are not necessarily identically distributed or independent. The distributions of the entries is described through a sequence of deterministic matrices. To make this precise, let $N_m > 0$ be an integer and let $(A^{s,m})_{s \leq N_m}$ be a sequence of self-adjoint deterministic $m \times m$ matrices. We assume that $(A^{s,m})_{s \leq N_m}$ are orthogonal to each other meaning that $\text{Tr}(A^{s,m}A^{s',m}) = 0$ for all different choices of $s \neq s' \leq N_m$. Let $(Z_s)_{s \leq N_m}$ be a stationary random field of standard Gaussian vectors meaning that for all $z \in \mathbb{R}^r$ we have $(Z_1,z, \ldots, Z_{N_m},z) \sim N(0, Id)$. The random matrices $(X^z,m)$ we consider in this subsection have the following form

\[
\mathcal{V}_m(X^m)^2 := \sup_{\text{Tr}(|M|^2) \leq 1} \sum_{s \leq N_m} |\text{Tr}(A^{s,m}M)|^2;
\]

\[
\sigma_m(X^m)^2 := \|E((X^{1,m})^2)\|_\infty.
\]

We observe that under conditions on the size of $\mathcal{V}_m(X^m)$ and $\sigma_m(X^m)$ then we obtain that $\sum_{s \in [k_m]} X^{z,m}$ is asymptotically semi-circular.

**Proposition 4.** Let $(X^z,m)_{z \in \mathbb{Z}^d}$ be a stationary random field of random matrices of size $m \times m$ defined as in eq. (5). Write $\{\alpha_m[i]\}$ the strong mixing coefficients of $(X^z,m)$. Then, there is a constant $C$ that does not depend on $m$ or $b$ such that

\[
\mathcal{R}_m^b[b|Z^d] \leq C \left[ \alpha_m[b] \right. ^{\frac{2}{r+1}} + \left. \frac{\sigma_m(X^m)^2 \mathcal{V}_m(X^m)^2}{\sum_{s \leq N_m} \|A^{s,m}\|^2} \sum_{l \geq 0} t^{-l-1} \alpha_m[l] \right] ^{\frac{2}{r+1}}
\]

\[
\mathcal{R}_m^s[b|Z^d] \leq C \alpha_m[b] ^{\frac{2}{r+1}}
\]
Moreover assume that \( \sup_m l^{-1} \alpha_m[l] \xrightarrow{l \to \infty} 0 \) and if \( \frac{\sigma_m(X^m)}{\sqrt{\sum_{s \leq N_m} \|A^{s,m}\|^2}} \xrightarrow{m \to \infty} 0 \). Then for all \( \gamma \in \mathbb{C} \setminus \mathbb{R} \) the following holds
\[
\|S_m(\gamma) - S_m^{sc}(\gamma)\|_1 \to 0
\]
where \( S_m^{sc} \) is the Stieltjes transform of a semi-circular operator with radius \( \eta_m : A \to \sum_{s \in \mathbb{Z}^d} \frac{1}{m} \mathbb{E}(X^{1,m} A X^{s,m}) \).

Another important example is the class of jointly exchangeable arrays. Let \( X := (X_{i,j}) \) be a random array, we say that it is jointly exchangeable if for all permutations \( \pi \in \mathcal{S}(\mathbb{N}) \) we have: \( X \overset{d}{=} (X_{\pi(i),\pi(j)}). \) We write \( \sigma(\mathcal{S}(\mathbb{N})) \) the sigma-field generated by events \( A \) satisfying \( \mathbb{P}(X \in A) = \mathbb{P}((X_{\pi(i),\pi(j)}) \in A) \) for all permutations \( \pi \in \mathcal{S}(\mathbb{N}); \) and denote \( \mathbb{E}(|S(\mathbb{N})|) \) for the conditional expectation knowing \( \sigma(\mathcal{S}(\mathbb{N})). \)

We say that a random matrix \( Y \) of size \( n \times n \) is jointly exchangeable if there is a random exchangeable array \( X \) such that: \( Y := (X_{i,j})_{(i,j) \leq n}. \) An important example are adjacency matrices of exchangeable graphs \([14]\). We define \( M_{\text{tail}} \) to be the algebra generated by random invariant arrays:
\[
M_{\text{tail}} := \{ a | (a_{i,j}) = (a_{\pi(i),\pi(j)}) \quad \forall \pi \in \mathcal{S}(\mathbb{N}) \}.
\]

Finally we write \( M_{\text{tail}} : = \{ y \in L_{\infty} \otimes M_n(\mathbb{C}) \mid \exists a \in M_{\text{tail}} \text{ s.t } y = (a_{i,j})_{(i,j) \leq n} \} \) the set of matrices that can be obtained by truncating elements of \( M_{\text{tail}} \).

**Proposition 5.** Let \( (X^{1,m}) \) be a sequence of \( \sigma(\mathcal{S}(\mathbb{N}))-\)conditionally independent and identically distributed random matrices of size \( m \times m. \) We assume that they are self-adjoint and have centered entries: \( \mathbb{E}(X^{1,m} | \mathcal{S}(\mathbb{N})) = 0. \) Suppose that \( X^{1,m} \) is jointly exchangeable and that the entries \( (X^{1,m}_{i,j}) \) admit a second moment. We denote by \( (\mathcal{N}_m[i|\mathcal{S}(\mathbb{N})]) \) the free mixing coefficients of \( \frac{1}{\sqrt{m}} X^{1,m} \). Suppose that the following conditions hold
\[
\sup_m \max_{i \leq m} \frac{\|X^{1,m}_{i,j}\|_2}{\min_{i \leq m} \|X^{1,m}_{i,j}\|_2} < \infty \quad \text{and} \quad \sup_m \max_{i \leq m} \|X^{1,m}_{i,j}\|_2 < \infty.
\]
Then there is a constant \( C \) independent of \( m \) such that
\[
\mathcal{N}_m[b|Z] = 0, \quad \mathcal{N}_m^{i,j}[b|Z] \leq \frac{C}{\sqrt{m}} \quad \forall b \geq 1.
\]

We denote \( \eta_n \) the following mapping \( \eta_n : a \to E_m(X^{1,m} a X^{1,m}) \) then the distribution of \( \frac{1}{m} \sum_{i \leq m} X^{1,m} \) converges to a semi-circular distribution with radius \( \eta_n \).

### 3 Generalization to u-statistics

#### 3.1 Notations and definitions

In classical probability, u-statistics are a key quantity, and under general conditions their limiting distribution is well known to be Gaussian see e.g \([27]\). Let \( (Y_i)_{i \in \mathbb{Z}} \) be
We remark that if \( k \) dynamical system. In general, if \((X, \mathcal{A}, \mu)\) defining an action \((G, \mathcal{K})\) on \( A \) and \( \mathcal{D} \) be an integer and let \( h : X^k \rightarrow \mathbb{R} \) be a measurable function. We call the statistics \( s_n := \frac{1}{n^k} \sum_{i_1, \ldots, i_k \in [n]} h(Y_{i_1}, \ldots, Y_{i_k}) \) a u-statistics. For example, if we take \( k = 2 \) then the following quantity is a u-statistics \( \frac{1}{n^k} \sum_{i,j \leq n} h(Y_i, Y_j) \). Under general moment conditions and mixing conditions \[ \text{we know that} \ s_n \text{ is asymptotically Gaussian and converges at a rate of} \ n^{-1/2}. \text{We remark that if we define the process} \ Z := (Z_{g_1, \ldots, g_k} z_1, \ldots, z_k) \text{as} \ Z_k = Z_{g_1, \ldots, g_k} := h(Y_{g_1}, \ldots, Y_{g_k}) \text{then} \ s_n \text{ can be re-expressed as the following empirical average} \ s_n = \frac{1}{n^k} \sum_{z \in [n]^k} Z_z. \text{While the distribution of} \ Z \text{ is not invariant under the natural action of} \ Z^k; \ (j_1, \ldots, j_k) \cdot Z := (Z_{i_1+j_1, \ldots, i_k+j_k}); \text{it is however invariant under the induced joint action (or diagonal action);} \ (j_1, \ldots, j) \cdot Z. \text{In other words} \ s_n \text{ can be seen as an empirical average of a jointly invariant process} \ Z. \text{We use this insight to extend the definition of u-statistics to non-commutative processes.}

**Non-commutative u-statistics** Let \( G \) be a l.c.s.c.H amenable group with Følner sequence \((A_n)\) and let \( \mathcal{A} \) be a Von-Neumann algebra with normal faithful and semi-definite trace \( \tau \). Let \( k \in \mathbb{N} \) be an integer, we study the asymptotics of processes indexed by \( G^k \). We remark that the sequence \((A_n)\) is a Følner sequence of the amenable group \( G \). Let \( (X_g)_{g \in G^k} \) be a sequence of elements of \( \mathcal{A} \) indexed by \( G^k \), our goal is to study the asymptotic of the average \[ X_n := \frac{1}{|A_n|^k} \int_{A_n^k} X_g d\mu_g. \]

We observe that the form of \( X_n \) resembles the one of a u-statistics and indeed, under conditions on the distribution of \((X_g)_{g \in G^k}\), we will call \( X_n \) a non-commutative u-statistics. We denote by \( g := (g_1, \ldots, g_k) \) the elements of \( G^k \) and for all subset \( G \subset G \) we write \[ D_k(H) := \{ g \mid g \in G^k, \ g_i = g_j \ \forall i, j \leq k \} \]
and call \( D_k(G) \) the “diagonal” of \( G^k \). For example if \( G = \mathbb{Z} \) is the group of integers and \( k = 2 \) then the diagonal \( D(\mathbb{Z}^2) \) is the set \((i, i) \ i \in \mathbb{Z}) \). We say that \((X_g)\) is jointly invariant with respect to \( G^k \) if it satisfies:

\[ (H_3) \text{ For all} \ g \in D_k(G) \text{ and all} \ g_1, \ldots, g_d \in G^k, \text{ we have} \]

\[ (X_{g_1}, \ldots, X_{g_d}) \overset{d}{=} (X_{g_{d1}}, \ldots, X_{g_{dd}}). \]

We remark that if \( k = 1 \) then \((X_g)\) is jointly invariant if and only if it is an invariant dynamical system. In general, if \((X_g)\) is jointly invariant with respect to \( G^k \) then we call \( X^k \) a non-commutative u-statistics. The following proposition demonstrates that the classical notion of u-statistics can be embedded into this framework.

**Proposition 6.** Let \( \Phi : A^k \rightarrow A \) be a continuous function. Choose \( G \) to be a l.c.s.c.H group defining an action \((K_g')\) on \( A \). Let \( Y \in A \) be self adjoint, we write \( Y_g := K_g'(Y) \) and denote \( X_g := \Phi(Y_{g_1}, \ldots, Y_{g_k}) \) for all \( g \in G^k \).
The net \((X_g)\) is jointly invariant with respect to \(\mathbb{G}^k\) and the following is a \(u\)-statistic
\[
\frac{1}{\left|A^k_n\right|} \int_{A^k_n} \Phi(Y_{g_1}, \ldots, Y_{g_k})d|g| = \frac{1}{\left|A^k_n\right|} \int_{A^k_n} X_g d|g|.
\]

\(\mathcal{F}_G^d(\cdot)\) and tail algebra with respect to the diagonal action
Playing a similar role than the tail-algebra in the empirical case is the tail-algebra with respect to the diagonal action. To define it, we introduce \(\mathcal{F}_G^d(X) \subset \mathcal{A}_\tau\) the unital-subalgebra of \(\tau\)-measurable operators generated by \(\{X_g; g \in G\}\) where \(G \subset \mathbb{G}^k\). For all \(g \in \mathbb{G}^k\) we denote:
\[
\mathcal{F}^d, \text{tail}(g) := \bigcap_{i \in \mathbb{N}} \mathcal{F}^d_{D_k(G \setminus A^i)_g}(X).
\]
We remark that if \(k = 1\) then this corresponds to the notion of tail-algebra defined in section 2: \(\mathcal{F}^d, \text{tail}(g) = \mathcal{F}^{\text{tail}}(X)\) for all \(g \in \mathbb{G}^k\). However, when \(k > 1\) we know that \(D_k(G)\) is not \(\mathcal{F}^d, \text{tail}(g)\) generable. we denote this notion as \(D_k(G)\)

\[
\mathcal{F}^d, \text{tail}(g) := \bigcap_{i \in \mathbb{N}} \mathcal{F}^d_{D_k(G \setminus A^i)_g}(X).
\]
We set \(E_D(\cdot)\) to be the non-commutative conditional expectation on \(\mathcal{F}^{\text{tail}}_{D_k(G)}(X)\).

Free mixing coefficients for u-statistics
To prove that the limit of \(S_n\) is semi-circular, we define a new notion of free-mixing and explain how it relates to the previous one defined in eq. (2). In classical probability, if a stationary process \((X_i)\) is strongly-mixing then \(h(X_i, X_j)\) and \(h(X_k, X_l)\) become increasingly independent as the distance between \(\{i, j\}\) and \(\{k, l\}\) grows to infinity. We generalize this notion to the non-commutative setting. Let \(d_k(\cdot, \cdot)\) be the pseudo distance on \(\mathbb{G}^k\) defined as:
\[
d_k(g, g') := \min_{i,j \leq k} d(g_i, g'_j).
\]
We denote \(C_k(b)\) the induced ball of radius \(b\) around \(g\) and \(\bar{d}_k(\cdot, \cdot)\) the induced Hausdorff pseudo-distance. We write
\[
C_k(b) := \{(G_1, G_2); G_1, G_2 \subset \mathbb{G}^k, \bar{d}_k(G_1, G_2) \leq b, \text{ card}(G_1) \leq 2\}.
\]
For a complex number \(\gamma \in \mathbb{C} \setminus \mathbb{R}\) we denote \(R_\gamma : \mathcal{A}_\tau \rightarrow \mathcal{A}\) the following function
\[
R_\gamma : A \rightarrow \text{Im}(\gamma)[A - \gamma 1_A]^{-1}.
\]
We set \(H_\lambda := \{R_\gamma | \text{Im}(\gamma) > 0\} \subset \mathcal{A}\). For \(K \subset \mathbb{G}^k\) and all \(f \in H_\lambda\) we write \(f^K := f\left(\left.\frac{1}{\sqrt{|K|}} \int_K X_g d|g|\right.\right)\). For an operator \(a \in L_1(A, \tau)\) we write \(\overline{a} := a - E_D(a)\). Finally for a process \((X_g)\) we use the shorthand
\[
X^n := \frac{X_g - E_D(X_g)}{\|X_g\|_2}.
\]
We define the free global mixing coefficients of \((X_g)\) as \(\mathcal{R}^s[\cdot|\mathbb{G}^k] := (\mathcal{R}^s, \mathcal{R}^s_{\lambda}[\cdot|\mathbb{G}^k], \mathcal{R}^s_{\lambda}[\cdot|\mathbb{G}^k])\)
where we have:

$$\mathbb{N}^{s,b}[\mathbb{G}^k] := \sup_{\{g, G', G\} \in \mathbb{G}[b]} \sup_{Y_{1,3} \in \mathcal{F}_G(X)} \sup_{\max_i \|Y_i\| \leq 1} \left\| E_D \left( Y_1 X_N^G Y_2 X_N^G Y_3 \right) \right\|_1$$

$$\mathbb{N}^{s,\lambda}[\mathbb{G}^k] := \sup_{\{g, G', G\} \in \mathbb{G}[b]} \sup_{\max \lambda} \max_{f \in H_k} \left\{ \left\| E_D \left( f^{\mathbb{G}} X_N^G \overline{f^{\mathbb{G}}} X_N^G f^{\mathbb{G}} \right) \right\|_1, \left\| E_D \left( f^{\mathbb{G}} X_N^G E(f^{\mathbb{G}}) X_N^G f^{\mathbb{G}} \right) \right\|_1 \right\}$$

We observe that $\mathbb{N}^{s}[-\mathbb{G}^k]$ is very similar to the free mixing coefficients defined in eq. (2) where the group $\mathbb{G}$ has been replaced by $\mathbb{G}^k$ and where the metric has been replaced by a pseudo distance. Indeed when $k = 1$ then the two type of free mixing coefficients concur

$$\mathbb{N}^{s}[-\mathbb{G}] = \mathbb{N}[-\mathbb{G}]$$

### 3.2 Main results for u-statistics

Let $(A^n)$ be a sequence of Von Neumann algebras with normal faithful semi-definite trace $(\tau_n)$. Define $(G_n)$ to be a sequence of l.c.s.c.H amenable groups with Følner sequence $(A_{i,n})$. Let metric $d_n(\cdot, \cdot)$ be a left-invariant metric on $G_n$ and denote by $B^n(g, b)$ the ball of radius $b$ around the element $g \in G_n$. We shorthand $B^n(b) := B^n(e, b)$. Let $(k_n)$ be a sequence of integers and choose $(X^n_G)$ to be a sequence of self-adjoint free random variables indexed by $G_n^k$. We denote $E_{D_n}$ the non-commutative conditional expectation on $\mathcal{D}(G_n^k)$ (X^n) and denote $(\mathbb{N}^{s}[-\mathbb{G}^k])$ the free global mixing coefficients of $(X^n_G)$. We define $\eta_n$ to be the following completely positive map

$$\eta_n : A \rightarrow \frac{1}{|A_{n,n}|^{2k_n-1}} \int_{A_{n,n}} E_{D_n} \left( X^n_G a X^n_G \right) d|a| d|\cdot|.$$  

Our goal is to study the asymptotic of

$$W_n := \frac{1}{|A_{n,n}|^{k_n-2}} \int_{A_{n,n}} X^n_G - E_{D_n}(X^n_G) d|g|$$

and we write $\mu_n$ the distribution of $W_n$. We prove that $\mu_n$ converges to a semi-circular law. Finally we write $S_n(\cdot)$ the operator-valued Stieljes transform of $W_n$, and $S_n^{sc}(\cdot)$ the Stieljes transform of the semi-circular operator $Y^{sc,\eta_n}$ with radius $\eta_n(\cdot)$.

**Theorem 3.** Let $(k_n)$ be a sequence of integers; and let $(X^n_G)$ be a triangular array of self-adjoint free random variables. Suppose that

i. $\sup_{n \in \mathbb{N}} \sup_{g \in G_n^k} \|X^n_G\|_3 < \infty$

ii. $\sum_{b \geq 0} \left| B^n_{k+1} \setminus B^n_k \right| A_{n,n} \|\mathbb{N}^{s,b}[\mathbb{G}^k] \| < \infty$
Then there is a constant $K = O\left(\sup_{\theta \in \mathbb{G}^k_n} \left\|X_{\theta}^n\right\|^3_3 + \|Y^{sc, \eta_n}\|^3_3\right) \land 1$ such that for all sequences $(b_n)$ the following holds

$$\left\|S_n^{sc}(\gamma_{x,v}) - S_n^{sc}(\gamma_{x,v})\right\| \leq K \left(\frac{k_n^4}{\sqrt{\lambda_{A_{n,n}}}} + \frac{k_n^3}{\nu^4} \left[\mathcal{R}_n^*[b_n] + |B^n_{b_n}|^{n,j}\nu[b_n|\mathbb{G}^k_n]\right]\right),$$

where $\mathcal{R}_n^*[b] := \sum_{k \geq b} \left|B^n_{k+1} \setminus B^n_k\right| \Lambda_{n,n}|k|^{\mathbb{G}^k_n}.$

We note that the candidate radius $\eta_n$ is more complex than one would expect as it involves all pairs $(g, g')$ in $A^{2k_n^2}_{n,n}$. As $(X_{\theta}^n)$ is not invariant under the full action of $\mathbb{G}^k_n$, $\eta_n$ cannot be directly simplified. This contrasts with classical probability where the exact form of the asymptotic variance of u-statistics is known. For example if $(X_i)$ is a stationary sequence and $h$ is symmetric in its coordinates, under mixing conditions we know that $\sum_{i,j \leq n} h(X_i, X_j)$ is asymptotically normal with variance: $\sigma^2 := 4 \sum_k \text{cov}(h(X_{1,k}), h(X_{k,1}))$, where we wrote $h(X_{1,k}) := \lim_{t \to \infty} \frac{1}{t} \sum_{k \in [l]} h(X_{1,k}, X_k)$. To bridge this gap, we prove that $\eta_n$ can indeed be further simplified if some additional mixing conditions are also imposed. To make those more intuitive we remember that if $(X_i)$ is a stationary sequence then the distribution of $h(X_0, X_j)$ and $h(X_0, X_j)$ become more similar as $j$ and $l$ get further away from 0. In this paper, we call this behavior marginal mixing. We generalize it to the non-commutative setting. In this goal, we define the following set

$$\mathcal{C}_n^{[i]} := \left\{ (g^1, g^2, g^3) \in \mathbb{G}^k_n \mid d((g^1_i, g^2_i, g^3_i), (g^1_i, g^2_i)) \geq b & g^3_j = g^3_j, \forall i \neq j \right\},$$

where we have defined $[g^1_i, g^2_i)^i := \{g^1_i \mid l \neq i\} \bigcup \{g^2_i \mid l \leq k_n\} \subset \mathbb{G}_n$. We call $(\mathbb{N}_n^m[b], \mathbb{G}^k_n)$ the free marginal mixing coefficients of $(X_g)$ and define them as:

$$\mathbb{N}_n^m[b|\mathbb{G}^k_n] := \sup_{(g^1, g^2, g^3) \in \mathcal{C}_n^{[i]}} \sup_{Y \in \mathcal{F}_{\text{tail}}^{\mathbb{T}}(\mathbb{G}^k_n)(X^n)} \{\left\|E_{D_n}\left[X_{g^1}^n - X_{g^2}^n\right]Y \left[X_{g^1}^n - X_{g^2}^n\right]\right\}_1 \leq 1\}

If the free marginal mixing coefficients decay fast enough then $\eta_n$ converges to the following radius:

$$\bar{\eta}_n : a \to \int_{A_{n,n}} E_n\left(\hat{X}_{j,g}^n \cdot a\hat{X}_{j,g}^n\right) d[g],$$

where $\hat{X}_{j,g}^n := \lim_{p \to \infty} \frac{1}{A_{p,n}} \int_{\mathbb{I}_p(i,g)} X^g_{i} d[g]$ and $\mathbb{I}_p(i,g) = \{\theta \mid \theta \in A_{p,n}, \text{s.t.} \theta_i = g\}$. Denote by $S_{n}(\cdot)$ the operator-valued Stieljes transform of $W_n$, and $S_{n}^{sc}(\cdot)$ the Stieljes transform of the operator valued semi-circular operator $Y^{sc, \eta_n \ast}$ with radius $\bar{\eta}_n(\cdot)$.
Theorem 4. Let \((k_n)\) be a sequence of integers; and let \((X^n_g)\) be a triangular array of self-adjoint free random variables that are jointly invariant with respect to \(G_n^{k_n}\). Suppose that

i. \(X^n_g \in L_3(A^n_o, \tau)\) and \(E_{D^n}(X^n_g) = 0\) for all \(g \in G_n^{k_n}\)

ii. \(\sum_{b \geq 0} \|B^n_{k+1} \setminus B^n_k \cap A_{n,n} \| N_n^{r,s} [g \mid G_n^{k_n}] < \infty\)

iii. \(T^n_m := \sum_{k \geq 0} \|B^n_{k+1} \setminus B^n_k \cap A_{n,n} \| N^n_m[k \mid G_n^{k_n}] = o(|A_n|)\).

Then there is a constant \(K = O(\sup_{g \in G^n} \|X^n_g\|_3^3 + \|Y^{sc,\eta_n}\|_3^3)\) such that for all sequences \((b_n)\) the following holds

\[
\left\| S_n(\gamma_{x,\nu}) - S_n^{sc}(\gamma_{x,\nu}) \right\|_1 \leq K \left[ \frac{|A_{n,n} \setminus B^n_{b_0} \cap A_{n,n}|}{\nu^3 |A_{n,n}|} + \frac{k_n^4 |B^n_{b_0}|^2}{\sqrt{|A_{n,n}|} \nu^4} + \frac{k_n^2}{\nu^3} \left[ \mathcal{R}_n^s[b_n \mid B^n_{b_0}] + |B^n_{b_0}| N^n_m[k \mid G_n^{k_n}] \right] \right]
\]

where we set \(\mathcal{R}_n^s[b] := \sum_{k \geq b} \|B^n_{k+1} \setminus B^n_k \cap A_{n,n} \| N_n^{r,s} [k \mid G_n^{k_n}]\).

3.3 Examples

In this section we present examples of u-statistics, bound their free mixing coefficients and prove that they are asymptotically semi-circular.

3.4 Examples for general operators

We denote \(M(A)\) the algebra of arrays \(x := (x_{i,j})_{i,j \in \mathbb{Z}}\) with entries in \(A\). An important subclass is the set of arrays \(X = (X_{i,j})\) whose distribution is invariant under the joint action of \(\mathbb{Z}: (z, z) \cdot X := (X_{z+i,z+j})\) and is quantum exchangeable on disjoint coordinates. To make this precise, for every \(k > 0\) we say that \((\{z^1, \ldots, z^j\}) := (I_j) \subset \prod(\mathbb{Z}^2)^k\) is a sequence of coordinate-disjoint sets if for all \(j_1 \neq j_2\) the following holds

\[
\{z^1_{1,1}, z^1_{1,2}, \ldots, z^j_{1,1}, z^j_{1,2}\} \cap \{z^1_{2,1}, z^1_{2,2}, \ldots, z^j_{2,1}, z^j_{2,2}\} = \emptyset.
\]

For all \(I \subset (\mathbb{Z}^2)^k\), we write \(\sigma_I\) the unital algebra generated by \((X_z)_{z \in I}\). We say that \(X\) is quantum exchangeable on disjoint coordinates if for all \(k \geq 1\), all sequences \((I_j) \subset (\mathbb{Z}^2)^k\) and all sequences of free random variables \((Y_j) \subset \prod_{j} \sigma_{I_j}\) the sequence \((Y_j)\) is quantum-exchangeable.

We study the free mixing coefficients of such an element \(X\).
In this subsection we consider deterministic stationary sequence of standard Gaussian vectors meaning that for all \( a, b > 0 \), \( \lambda > 0 \).

Therefore \( \frac{1}{n^2} \sum_{k=1}^{n} Z_k^2 \) is asymptotically semi-circular with a radius satisfying \( \eta(a) = 4\mathbb{E}_{Z^2}(X_{1,2}aX_{1,3}) \).

Secondly we bound the free global mixing coefficients of the classical u-statistics presented in proposition \( \text{[6]} \). Those will depend on the properties of the free mixing properties of the underlying process \((Y_k)\) through a slightly more complex notion of free mixing than the one defined in eq. \( \text{(2)} \). In this goal, for \( a \in \mathcal{A} \) we write \( \tau := a - E_D(a) \) and we define

\[
\mathbb{N}'[b|G] := \sup_{(G_1,G_2,G) \in C[b]} \sup_{|G_1||G_2| \leq k} \max \left\{ \frac{\| E(Y_1Y_2Z_k) \|_1}{\| E(Y_1Y_2Z_k) \|_1} \right\}.
\]

For simplicity we suppose that \( \Phi \in C(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \) is a polynomial. A similar result holds for any function that can be successfully approximated by polynomials.

**Proposition 8.** Let \( (\mathcal{A}, \tau, *) \) be a Von-Neumann algebra. Let \( \Phi : \mathcal{A}^k \to \mathcal{A} \) be a polynomial. Let \( (K_g') \) be a net of \( * \)-automorphisms from \( \mathcal{A} \) into itself. We choose \( Y \in \mathcal{A} \) to be a self-adjoint free random variable and write \( Y_g := K_g'(Y) \). Define \( Z_g := \Phi(Y_{g1}, \ldots, Y_{gk}) \). Let \( \mathbb{N}'[\cdot|G^k] \) denote the free global mixing coefficients of \( \left( Z_g \right) \). We have:

\[
\mathbb{N}'[b|G^k] \leq \sup_{g \in G^k} \frac{\| Z_g \|_\infty^2}{\| Z_g \|_2^2} \quad \mathbb{N}'[b|G], \quad \mathbb{N}'[b|G^k] \leq \sup_{g \in G^k} \frac{\| Z_g \|_\infty^2}{\| Z_g \|_2^2} \quad \mathbb{N}'[b|G], \quad \forall b \in \mathbb{N}.
\]

Therefore if \((Y_g)\) is a freely independent sequence then \( \mathbb{N}'[b|G^k] = \mathbb{N}'[b|G^k] = 0 \) for all \( b > 0 \).

**3.5 Examples of u-statistics for random matrices with patterned entries**

In this subsection we consider \( u \)-statistics that are functions of patterned matrices. In this goal let \( N_m > 0 \) be an integer and let \( (A^{s,m})_{s \leq N_m} \) be a sequence of self-adjoint deterministic \( m \times m \) matrices. We assume that \( (A^{s,m})_{s \leq N_m} \) are orthogonal to each other: \( Tr(A^{s,m}A^{s',m}) = 0 \) for all different choices of \( s \neq s' \leq N_m \). Let \( \left( (Z_{s,i})_{s \leq N_m} \right)_{i \in \mathbb{Z}} \) be a stationary sequence of standard Gaussian vectors meaning that for all \( i \in \mathbb{Z} \) we have
that \((Z_{1,i}, \ldots, Z_{N,m,i}) \sim N(0, Id)\) is a standard Gaussian vector. We define \((Y^{i,m})\) to be random matrices that have the following form

\[
Y^{i,m} = \sum_{s \leq N_{m}} Z_{s,i} A^{s,m}.
\]

In proposition 4 we showed that the empirical average of \((Y^{i,m})\) is asymptotically semi-circular. In this subsection, we prove that in general u-statistics of \((Y^{i,m})\) are also asymptotically semi-circular. Using the same notations than in [8] we write

\[
\begin{align*}
\mathcal{V}_{m}(Y^{m})^2 &:= \sup_{\|M\|^2 \leq 1} \left|\text{Tr}(A^{s,m} M)\right|^2; \\
\sigma_{m}(Y^{m})^2 &:= \|E((Y^{1,m})^2)\|_{\infty}.
\end{align*}
\]

Moreover for all \(p \geq 1\) we also write

\[
\begin{align*}
s_{m,p} &:= \max_{i,j \leq m} \left( \sum_{s \leq N_{m}} |A_{i,j}^{s,m}| \right)^p \wedge 1 \\
\tilde{s}_{m,p} &:= \sup_{1 \leq l \leq p} \left( \frac{1}{m^{1-\epsilon/2}} \sum_{i,j} \left( \left( \sum_{s \leq N_{m}} (A_{i,j}^{s,m})^2 \right)^{l(1+\frac{\epsilon}{2})} + \sum_{s \leq N_{m}} (A_{i,j}^{s,m})^{l(2+\epsilon)} \right) \right)^{\frac{2}{2+\epsilon}}.
\end{align*}
\]

Let \(p \in \mathbb{N}\), we will consider polynomials of degree \(p\). Choose \(\Phi_{m} : M_{m}(\mathbb{C}) \times M_{m}(\mathbb{C}) \rightarrow M_{m}(\mathbb{C})\) to be a polynomial of degree \(p\) that is symmetric: \(\Phi_{m}(A, B) = \Phi_{m}(B, A)\) and is such that \(\Phi_{m}(A, B)^* = \Phi_{m}(A^*, B^*)\) for all \(A, B \in M_{m}(\mathbb{C})\). We define

\[
X^{(i,j),m} := \Phi_{m}(Y^{i,m}, Y^{j,m}).
\]

In this section, we prove that \(\sum_{i,j \leq m} X^{(i,j),m}\) is asymptotically semi-circular.

**Proposition 9.** Let \((Y^{i,m})\) be a stationary sequence of matrices of size \(m \times m\) that are defined as described in eq. (7). Let \((\Phi_{m})\) be a sequence of symmetric polynomials of degree \(p\) that are such that \(\Phi_{m}(A, B)^* = \Phi_{m}(A^*, B^*)\) for all \(A, B \in M_{m}(\mathbb{C})\). We write for all \(z \in \mathbb{Z}^2\)

\[
X^{z,m} := \Phi_{m}(Y^{z_{1,m}}, Y^{z_{2,m}}).
\]

We suppose that for all \(i, j \leq m\) we have \(E(X^{i,j,m}) = 0\). Define \((\alpha_{m}(\cdot))\) to be the \(\alpha\)-mixing coefficients of \((Y^{i,m})\). Moreover we denote by \((N^{*}_{m,\cdot}([Z^2]))\) the free global mixing coefficients of \((X^{z,m})\). Suppose that there is an \(\epsilon > 0\) such that

\[
\sup_{n} \sum_{b \geq 0} \alpha_{n}(b) \frac{2}{2+\epsilon} < \infty.
\]

Then the following holds

\[
\begin{align*}
N^{*,\lambda}_{n}[b|Z^2] &\lesssim \frac{1}{\inf_{z \in \mathbb{Z}^2} \|X^{z,m}\|_{2}^{2}} \left( \alpha_{m}(b) \frac{2}{2+\epsilon} \tilde{s}_{m,p} + \sigma_{m}(Y^{m})^2 \mathcal{V}_{m}(Y^{m})^2 s_{m,p-1}(1 + \log(N_{m}))^{2(p-1)} \right) \\
N^{*,s}_{n}[b|Z^2] &\lesssim \frac{1}{\inf_{z \in \mathbb{Z}^2} \|X^{z,m}\|_{2}^{2}} \alpha_{m}(b) \frac{2}{2+\epsilon} \tilde{s}_{m,p}.
\end{align*}
\]
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A Proof of theorem 2

We prove the following proposition that directly implies theorem 2.
Proposition 10. Let \((X^n_g)\) be a triangular array of self-adjoint free random variables invariant under \(G_n\). Suppose that \((X^n_g)\) satisfies all the condition of theorem 2. Let \(S_n(\cdot)\) denote the (operator-valued) Stieltjes transform of \(\hat{\mu}_n\), and \(S^{sc}_n(\cdot)\) the Stieltjes transform of the operator-valued semi-circular operator \(Y^{sc,n}\) with radius \(\eta_n(\cdot)\). The following upper bound holds

\[
\left\| S_n(\gamma_{x,\nu}) - S^{sc}_n(\gamma_{x,\nu}) \right\|_1 \\
\leq \frac{3R_n^3[b_n]}{\nu^3} + \frac{|A_n\Delta B^n_{b_n}A_n|}{|A_{n,n}|} + 2|B^n_{b_n}|N^{\nu,\nu}_n[b_n|G_n] \\
+ \frac{9}{\sqrt{|A_{n,n}|}} \left[ B^n_{2b_0} \right] \frac{3}{3} \frac{|X^n|}{3} + |B^n_0| \frac{2}{2} \frac{Y^{sc,n_0}}{2} + \frac{3}{3} \frac{|A_n\Delta B^n_{b_n}A_n|}{|A_{n,n}|}.
\]

where \(\gamma_{x,\nu} = x + iv\).

Proof. Our goal is to upper-bound \(\left\| S_n(\gamma_{x,\nu}) - S^{sc}_n(\gamma_{x,\nu}) \right\|_1\). To do this we interpolate between those two quantities by defining a function \(g: [0,1] \rightarrow \mathcal{F}^{\text{tail}}(X)\) that satisfies:

\[
\|g(1) - g(0)\|_1 = \left\| S_n(\gamma_{x,\nu}) - S^{sc}_n(\gamma_{x,\nu}) \right\|_1;
\]

and subsequently prove that \(g\) is differentiable and has bounded derivatives.

We first note that we can suppose without loss of generality that \(E_n(X^n_g) = 0\) for all \(g \in G_n\) and do so. For all operators \(Y\) we write the resolvent as \(R(Y, \gamma) = [Y - \gamma 1_A]^{-1}\); and define the following averages

\[
W^n_{g,b} := \frac{1}{\sqrt{|A_{n,n}|}} \int_{A_{n,n} \setminus B^n(g,b)} X^n_{g'}d|g'| \quad b > 0, \ g \in A^n.
\]

Let \((Y^n_g)\) be free copies of \(Y^{sc,n_0}\); and write \(Y^n := \frac{1}{\sqrt{|A_{n,n}|}} \int_{A_{n,n}} Y^n_gd|g|\). We remark that \(Y^{sc,n_0}\) has the same distribution than \(Y^n\). For all \(t \in [0,1]\) we define the following interpolating processes

\[
W_n(t) := \sqrt{t}W_n + \sqrt{1-t}Y^n, \quad W^n_{g,b}(t) := \sqrt{t}W^n_{g,b} + \sqrt{1-t}Y^n \\
W^{sc,n_0}(t) := \sqrt{t}W_n + \frac{\sqrt{1-t}}{\sqrt{|A_{n,n}|}} \int_{A_{n,n} \setminus B^n(g,0)} Y^n_gd|g|.
\]

We note that \(W_n(0) = Y^n\) and that \(W_n(1) = W_n\). We also remark that if group \(G_n\) is discrete then \(A_{n,n} \setminus B^n(g,0) = \{g\}\) which implies that \(W^{sc,n_0}(t) \neq W_n(t)\).

Finally for simplicity we use the following shorthand notations:

\[
R^n_{g,t} := R(W_n(t), \gamma_{x,\nu}), \quad R^n_{g,t} := R(W^n_{g,b}(t), \gamma_{x,\nu}) \\
R^{sc,n_0}_{g,t} := R(W^{sc,n_0}(t), \gamma_{x,\nu}), \quad S^n_{g,t} := E_n(R^n_{g,t}), \quad S^{sc,n_0}_{g,t} := E_n(R^{sc,n_0}_{g,t}).
\]
We define the function $g : [0, 1] \rightarrow \mathcal{F}^{tail}(X)$ as $g(t) = E_n(R_n^t)$. As $W_n(t)$ interpolates $Y^n$ and $W_n$ we have:

$$g(0) = S_n^{sc}(\gamma_{x,\nu}), \quad g(1) = S_n(\gamma_{x,\nu}).$$

We prove that $g$ is Gateaux-differentiable. For all $\epsilon > 0$ and all $t \in [0, 1]$ such that $t + \epsilon \in [0, 1]$ we have:

$$g(t + \epsilon) - g(t) = E_n \left( R_n^{t+\epsilon} - R_n^t \right)$$

$$\quad \equiv E_n \left( R_n^{t+\epsilon} \left[ W_n(t) - W_n(t + \epsilon) \right] R_n^t \right)$$

$$\quad = E_n \left( R_n^{t+\epsilon} \left[ \sqrt{t} - \sqrt{t + \epsilon} \right] W_n R_n^t \right)$$

$$\quad + E_n \left( R_n^{t+\epsilon} \left[ \sqrt{1 - t} - \sqrt{1 - (t + \epsilon)} \right] Y_n^{sc, \eta_n} R_n^t \right),$$

where to get (a) we exploited the fact that for all $Y, Y^* \in \mathcal{A}$, the following identity holds:

$$R(Y, \gamma_{x,\nu}) - R(Y^*, \gamma_{x,\nu}) = R(Y, \gamma_{x,\nu})(Y^* - Y)R(Y^*, \gamma_{x,\nu}).$$

This implies that $g$ is Gateaux-differentiable and satisfies

$$g'(t) = -E_n \left( R_n^t \left[ \frac{W_n}{2\sqrt{t}} - \frac{Y_n^{sc, \eta_n}}{2\sqrt{1 - t}} \right] R_n^t \right).$$

Therefore to upper-bound $\|S_n(\gamma_{x,\nu}) - S_n^{sc}(\gamma_{x,\nu})\|_1$ it is sufficient to bound $\left\| \int_0^1 g'(t) dt \right\|_1$.

To do so, we first remark that for all $t \in (0, 1)$ we have

$$E_n \left[ R_n^t \frac{W_n}{\sqrt{t}} R_n^t \right] \equiv \frac{1}{\sqrt{|A_{n,n}|}} \int_{A_{n,n}} E_n \left[ R_n^t \frac{X_n}{\sqrt{t}} R_n^t \right] d|g|$$

$$\quad = \frac{1}{\sqrt{|A_{n,n}|}} \int_{A_{n,n}} E_n \left[ R_n^t \frac{X_n}{\sqrt{t}} \left[ R_n^t - R_{bn}^{g,t} \right] \right] d|g|$$

$$\quad + \frac{1}{\sqrt{|A_{n,n}|}} \int_{A_{n,n}} E_n \left[ R_n^t - R_{bn}^{g,t} \right] \frac{X_n}{\sqrt{t}} \frac{X_n}{R_{bn}^{g,t}} d|g|$$

$$\quad + \frac{1}{\sqrt{|A_{n,n}|}} \int_{A_{n,n}} E_n \left[ R_{bn}^{g,t} \frac{X_n}{\sqrt{t}} R_{bn}^{g,t} \right] d|g|$$

$$\quad \equiv \frac{-1}{\sqrt{|A_{n,n}|}} \int_{A_{n,n}} E_n \left[ R_n^t \left[ W_n - W_{g,bn}^n \right] R_{bn}^{g,t} X_n^t \frac{R_n^t}{R_{bn}^{g,t}} \right] d|g|$$

$$\quad - \frac{1}{\sqrt{|A_{n,n}|}} \int_{A_{n,n}} E_n \left[ R_{bn}^{g,t} X_n^t \left[ W_n - W_{g,bn}^n \right] R_{bn}^{g,t} \right] d|g|$$

$$\quad + \frac{1}{\sqrt{|A_{n,n}|}} \int_{A_{n,n}} E_n \left[ R_{bn}^{g,t} \frac{X_n}{\sqrt{t}} R_{bn}^{g,t} \right] d|g|$$

$$\quad = A_1^{\gamma_{x,\nu}} + A_2^{\gamma_{x,\nu}} + A_3^{\gamma_{x,\nu}}$$

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where (a) comes from the linearity of the functional $E_n$ and where to get (b) we used eq. (10). The rest of proof consist in: (i) proving that $\|a_3^{\gamma,n}\|_1 \to 0$ and (ii) in re-expressing $a_1^{\gamma,n}$ and $a_2^{\gamma,n}$.

We start by proving that $\|a_3^{\gamma,n}\|_1 \to 0$. Using the definition of the free mixing coefficients ($N_n^e[\cdot|\mathcal{G}_n]$) we have:

\[
\|E_n \left[ R_{bn}^{g,t} \frac{X^n}{\sqrt{t}} R_{bn}^{g,t} \right] \|_1 \\
\leq \sum_{b \geq b_n} \left\| E_n \left[ R_{bn}^{g,t} \frac{X^n}{\sqrt{t}} [R_{bn}^{g,t} - R_{bn}^{g,t}] \right] \right\|_1 \\
+ \sum_{b \geq b_n} \left\| E_n \left[ [R_{bn}^{g,t} - R_{bn}^{g,t}] \frac{X^n}{\sqrt{t}} R_{bn}^{g,t} \right] \right\|_1 \\
\overset{(a)}{\leq} \sum_{b \geq b_n} \left\| E_n \left[ R_{bn}^{g,t} X^n R_{bn}^{g,t} [W_{bn}^{g,b+1} - W_{bn}^{g,b}] R_{bn}^{g,t} \right] \right\|_1 \\
+ \sum_{b \geq b_n} \left\| E_n \left[ R_{bn}^{g,t} [W_{bn}^{g,b+1} - W_{bn}^{g,b}] R_{bn}^{g,t} X^n R_{bn}^{g,t} \right] \right\|_1 \\
\leq \frac{1}{\sqrt{|A_{n,n}|}} \sum_{b \geq b_n} \int_{B^n(b+1,1) \setminus B^n(b,1)} \left\| E_n \left[ R_{bn}^{g,t} X^n R_{bn}^{g,t} X^n R_{bn}^{g,t} \right] \right\|_1 \, dg' \\
+ \frac{1}{\sqrt{|A_{n,n}|}} \sum_{b \geq b_n} \int_{B^n(b+1,1) \setminus B^n(b,1)} \left\| E_n \left[ R_{bn}^{g,t} X^n R_{bn}^{g,t} X^n R_{bn}^{g,t} \right] \right\|_1 \, dg'
\]

where (a) comes from the triangular inequality.

Therefore we obtain that

\[
\|E_n \left[ R_{bn}^{g,t} \frac{X^n}{\sqrt{t}} R_{bn}^{g,t} \right] \|_1 \\
\overset{(a)}{\leq} \frac{2}{\sqrt{|A_{n,n}|}} \left\| X^n \right\|_2 \sum_{b \geq b_n} \left\| R_{v,b}^{n}(t) \right\|_2 \left\| R_{v,b+1}^{n}(t) \right\|_\infty \| B_{bn}^{n} \setminus B_{bn}^{n} \| B_n^e[\cdot|\mathcal{G}_n].
\]

and where to get (a) we used the definition of $N_n^e[\cdot|\mathcal{G}_n]$ in eq. (2) coupled with the fact that $E_n(X^n) = 0$ and lemma 5. Therefore by taking the average over $g \in A_{n,n}$ we obtain that

\[
\left| a_3^{\gamma,n} \right| \leq \frac{2 \left\| X^n \right\|_2 \| B_n^e \|_{\mathcal{B} \mathcal{N}}[|\mathcal{G}_n]}{\nu^3} \to 0.
\]

The next step consists in re-expressing $a_1^{\gamma,n}$ and $a_2^{\gamma,n}$ in terms of $R_{2bn}^{g,t}$ and $X^n$. Using
eq. (10) for all $g \in A_{n,n}$ we have

$$\left\| E_n \left[ R^t_n - R^g,_{2b_n} t \left[ W_n - W_{g,b_n}^{n} \right] R^g,_{2b_n} X^t_n \right] \right\|_1 \leq \left\| E_n \left[ R^t_n \left[ W_n - W_{g,2b_n}^{n} \right] R^g,_{2b_n} t \left[ W_n - W_{g,b_n}^{n} \right] \right] \right\|_1 \leq \frac{|B_{2b_n}^{n}|^2}{|A_{n,n}|^2 \nu^4} \left\| X^t_n \right\|_3^3 \tag{12}$$

where (a) comes from the H"older-inequality combined with the fact that $|B(g, 2b_n)^n| \leq |B_{2b_n}^{n}|$. Similarly, the following also holds:

$$\left\| E_n \left[ R^g,_{2b_n} t \left[ W_n - W_{g,b_n}^{n} \right] R^g,_{2b_n} t X^t_n \right] \right\|_1 \leq \frac{|B_{2b_n}^{n}|^2}{|A_{n,n}|^2 \nu^4} \left\| X^t_n \right\|_3^3 \; \tag{13}$$

as well as

$$\left\| E_n \left[ R^g,_{2b_n} t \left[ W_n - W_{g,b_n}^{n} \right] \right] \right\|_1 \leq \frac{|B_{2b_n}^{n}|^2}{|A_{n,n}|^2 \nu^4} \left\| X^t_n \right\|_3^3 \; \tag{14}$$

Therefore by averaging over $g \in A_{n,n}$ the results of eq. (12), eq. (13) and eq. (14) we obtain that:

$$\left| a_1^{r,\nu} + \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \left| \int_{B(g,b_n)} E_n \left[ R^g,_{2b_n} t X^t_n R^g,_{2b_n} t X^t_n \right] \right| \right| \leq \frac{3|B_{2b_n}^{n}|^2}{\sqrt{|A_{n,n}| \nu^4}} \left\| X^t_n \right\|_3^3 \tag{15}$$

and similarly we obtain that:

$$\left| a_2^{r,\nu} + \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \left| \int_{B(g,b_n)} E_n \left[ R^g,_{2b_n} t X^t_n R^g,_{2b_n} t X^t_n \right] \right| \right| \leq \frac{3|B_{2b_n}^{n}|^2}{\sqrt{|A_{n,n}| \nu^4}} \left\| X^t_n \right\|_3^3 \tag{16}$$

Therefore combined together eq. (11), eq. (15) and eq. (16) imply that

$$\left\| g(1) - g(0) \right\|_1 \leq \int_0^1 \left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \left| \int_{B(g,b_n)} E_n \left[ R^g,_{2b_n} t X^t_n R^g,_{2b_n} t X^t_n \right] \right| \right| \leq \frac{6|B_{2b_n}^{n}|^2}{\sqrt{|A_{n,n}| \nu^4}} \left\| X^t_n \right\|_3^3 + \frac{2\left\| X^t_n \right\|_2^2 R^p_n |b_n|}{\nu^3}.$$
Moreover by exploiting the free independence of \((Y^n_g)\) we obtain for all \(t \in (0,1)\) that:

\[
\left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} E_n \left[ R_{g,t}^{n,\eta} Y_g^n R_{g,t}^{n,\bar{\eta}} Y_g^n R_{g,t}^{n,\eta \bar{\eta} + \eta} \right] d|g| - E_n \left[ R_n^t Y^{\eta,\bar{\eta} + \eta} \right] \right\|_1 \leq \frac{6|B^n_0|^2}{\sqrt{|A_{n,n}|} |\nu|^4} \left\| Y^{\eta,\bar{\eta} + \eta} \right\|_3^3.
\]

We remark that if the group \(G_n\) is discrete then \(|B^n_0| = 1\).

Moreover, we observe that to finish upper bounding \(\|g(1) - g(0)\|_1\) we need to compare

\[
\frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,\bar{g})} E_n \left[ R_{g,t}^{n,\eta} Y_g^n R_{g,t}^{n,\eta} Y_g^n R_{g,t}^{n,\eta} \right] d|g|d|g|
\]

with

\[
\frac{1}{|A_{n,n}|} \int_{A_{n,n}} E_n \left[ R_{g,t}^{n,\eta} Y_g^n R_{g,t}^{n,\eta} Y_g^n R_{g,t}^{n,\eta} \right] d|g|.
\]

To do so we introduce the following notation for all \(g, g' \in G_n\)

\[
\eta_{g,g'} : a \to E_n \left[ X_g^n a X_{g'}^n \right].
\]

Using the triangular inequality we obtain that

\[
\left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,\bar{g})} E_n \left[ R_{g,t}^{n,\eta} Y_g^n R_{g,t}^{n,\eta} Y_g^n R_{g,t}^{n,\eta} \right] d|g|d|g| - E_n \left( R_n^t \eta_n (S^n_{n,t}) R_n^t \right) \right\|_1 \\
\leq \left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,\bar{g})} E_n \left[ R_{g,t}^{n,\eta} Y_g^n R_{g,t}^{n,\eta} Y_g^n R_{g,t}^{n,\eta} \right] d|g|d|g| - E_n \left( R_n^t \eta_n (S^n_{n,t}) R_n^t \right) \right\|_1 \\
+ \left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,\bar{g})} E_n \left( R_n^t \eta_n \eta_{g,g'} (S^n_{n,t}) R_n^t \right) d|g|d|g| \right\|_1 \\
+ \left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,\bar{g})} E_n (R_n^t \eta_n \eta_{g,g'} (S^n_{n,t}) R_n^t) d|g|d|g| - E_n \left( R_n^t \eta_n (S^n_{n,t}) R_n^t \right) \right\|_1 \\
\leq (c_1^{\gamma_{n,t}}) + (c_2^{\gamma_{n,t}}) + (c_3^{\gamma_{n,t}})
\]

We bound each term successively. The first term is bounded using the definition of \((N^n_{\eta}[::G_n])\), and the later terms are bounded using the triangular inequality and eq. (10).
We focus first on the term $c_1^{\gamma_{x',\nu}}$. By definition of $\mathcal{N}_n^{(x,\nu)}[\cdot|G_n]$, we obtain that

$$
\left| (c_1^{\gamma_{x',\nu}}) \right| \leq \left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,b_n)} E_n \left[ R_{2b_n}^{g,t} X^n g, \nu R_{2b_n}^{g,t} \right] - E_n \left[ R_{2b_n}^{g,t} \eta_g \eta_{g'}(S_{g,t}^{2b_n}) R_{2b_n}^{g,t} \right] d[g'] d|g| \right\|_1
$$

$$
\leq \left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,b_n)} E_n \left[ R_{2b_n}^{g,t} X^n g, \nu R_{2b_n}^{g,t} \right] d[g'] d|g| \right\|_1
$$

$$
+ \left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,b_n)} E_n \left[ R_{2b_n}^{g,t} X^n g, \nu R_{2b_n}^{g,t} \right] d[g'] d|g| \right\|_1
$$

$$
\leq \left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,b_n)} E_n \left[ R_{2b_n}^{g,t} X^n g, \nu R_{2b_n}^{g,t} \right] d[g'] d|g| \right\|_1
$$

$$
+ \left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,b_n)} E_n \left[ R_{2b_n}^{g,t} X^n g, \nu R_{2b_n}^{g,t} \right] d[g'] d|g| \right\|_1
$$

$$
\leq 2 \left\| X^n \right\|^2_{2} |B^n_{b_n}|^{\frac{3}{2}} \left\| \mathcal{N}_n^{(x,\nu)}[\cdot|\mathcal{G}_n] \right\|_1
$$

$$
\leq 2 \left\| X^n \right\|^2_{2} |B^n_{b_n}|^{\frac{3}{2}} \left\| \mathcal{N}_n^{(x,\nu)}[\cdot|\mathcal{G}_n] \right\|_1 .
$$

(17)

To bound $(c_2^{\gamma_{x',\nu}})$ we use the triangular inequality and obtain that

$$
\left\| E_n \left[ R_{2b_n}^{g,t} \eta_g \eta_{g'}(S_{g,t}^{2b_n}) R_{2b_n}^{g,t} \right] - E_n \left[ R_{2b_n}^{g,t} \eta_g \eta_{g'}(S_{g,t}^{2b_n}) R_{2b_n}^{g,t} \right] \right\|_1
$$

$$
\leq \left\| E_n \left[ \left[ R_{2b_n}^{g,t} - R_{2b_n}^{g,t} \right] \eta_g \eta_{g'}(S_{g,t}^{2b_n}) R_{2b_n}^{g,t} \right] \right\|_1 + \left\| E_n \left[ R_{2b_n}^{g,t} \eta_g \eta_{g'}(S_{g,t}^{2b_n}) - \eta_g \eta_{g'}(S_{g,t}^{2b_n}) R_{2b_n}^{g,t} \right] \right\|_1
$$

$$
\leq \left\| R_{2b_n}^{g,t} (t) \left[ W_n - W_{g,2b_n}(t) \right] R_{2b_n}^{g,t} \eta_g \eta_{g'}(S_{g,t}^{2b_n}) R_{2b_n}^{g,t} \right\|_1
$$

$$
+ \left\| R_{2b_n}^{g,t} E_n \left( X g, E_n \left[ R_{2b_n}^{g,t} (t) \right] \left[ W_n - W_{g,2b_n}(t) \right] R_{2b_n}^{g,t} \right) X g, \nu R_{2b_n}^{g,t} \right\|_1
$$

$$
+ \left\| R_{2b_n}^{g,t} \eta_g \eta_{g'}(S_{g,t}^{2b_n}) R_{2b_n}^{g,t} \left[ W_n - W_{g,2b_n}(t) \right] R_{2b_n}^{g,t} \right\|_1
$$

$$
\leq \frac{3|B^n_{b_n}|}{\sqrt{|A_{n,n}|}^{1/4}} \left\| X^n \right\|^3_{3}. \tag{18}
$$

where to get (a) we used eq. (10); and (b) was obtained by using the Cauchy-Schwarz inequality. Therefore by averaging over $g \in A_{n,n}$ and $g' \in B^n(g,b_n)$ we obtain that

$$
(c_2^{\gamma_{x',\nu}}) \leq \frac{3|B^n_{b_n}|^2}{\sqrt{|A_{n,n}|}^{1/4}} \left\| X^n \right\|^3_{3}. \tag{18}
$$

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Finally by the triangular inequality we remark that

\[
|c^3_{x,v}| \leq \left| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,b_n)} E_n \left( R^t_n \eta_{g,e}(S^0_n,t) R^t_n \right) d|g'| d|g| - E_n \left( R^t_n \eta_{n}(S^0_n,t) R^t_n \right) \right|_1
\]

\[
\leq \left| \frac{1}{|A_{n,n}|} \int_{A_{n,n}} \int_{B^n(g,b_n)} E_n \left( R^t_n \eta_{g,e}(S^0_n,t) R^t_n \right) d|g'| d|g| - \int_{B^n(c,b_n)} E_n \left( R^t_n \eta_{g,e}(S^0_n,t) R^t_n \right) d|g| \right|_1
\]

\[
+ \left| \int_{B^n(c,b_n)} E_n \left( R^t_n \eta_{g,e}(S^0_n,t) R^t_n \right) d|g| - \int_{A_{n,n}} E_n \left( R^t_n \eta_{g,e}(S^0_n,t) R^t_n \right) d|g| \right|_1
\]

\[
\leq \left\| X_e \right\|_2^2 \frac{|A_n \triangle B^0_n A_n|}{\nu^3 |A_{n,n}|} + \left\| X_e \right\|_2^2 R^n_{\eta}[b_n].
\]

(19)

where to obtain (a) we exploited the Cauchy-Schwarz inequality. Therefore by combining eq. (17), eq. (18) and eq. (19) we obtain that

\[
\left\| g(1) - g(0) \right\|_1 \leq \int_0^1 \left| S^0_n,t \eta_n(S^0_n,t) S^0_n - \frac{1}{|A_{n,n}|} \int_{A_{n,n}} E_n \left[ R^{sc,\eta_n} R^{sc,\eta_n} R^{sc,\eta_n} \right] d|g| \right| dt
\]

\[
+ \left\| X_e \right\|_2^2 \frac{3R^n_{\eta}[b_n]}{\nu^3 |A_{n,n}|} + \frac{|A_n \triangle B^0_n A_n|}{|A_{n,n}|} + 2|B^0_n| |N^{j,\nu}_n[b_n]|_G
\]

\[
+ \frac{1}{\sqrt{|A_{n,n}| \nu^3}} \left[ 9|B^0_n|^2 \left\| X_e \right\|_3^3 + 6|B^0_n|^2 \left\| Y^{sc,\eta_n} \right\|_3^3 \right].
\]

Moreover following similar arguments we prove that

\[
\int_0^1 \left| E_n \left( R^t_n \eta_{n}(S^0_n,t) R^t_n \right) - \frac{1}{|A_{n,n}|} \int_{A_{n,n}} E_n \left[ R^{sc,\eta_n} R^{sc,\eta_n} R^{sc,\eta_n} \right] d|g| \right| dt
\]

\[
\leq \frac{3 \left\| Y^{sc,\eta_n} \right\|_3^3}{\sqrt{|A_{n,n}| \nu^3}} + \frac{\left\| Y^{sc,\eta_n} \right\|_2^3}{|A_{n,n}| \nu^4} |A_n \triangle B^0_n A_n|.
\]

(21)

Therefore by combining eq. (20) and eq. (21) we finally obtain that:

\[
\left\| g(1) - g(0) \right\|_1 \leq \left\| X_e \right\|_2^2 \frac{3R^n_{\eta}[b_n]}{\nu^3 |A_{n,n}|} + \frac{|A_n \triangle B^0_n A_n|}{|A_{n,n}|} + 2|B^0_n| |N^{j,\nu}_n[b_n]|_G
\]

\[
+ \frac{1}{\sqrt{|A_{n,n}| \nu^3}} \left[ 9|B^0_n|^2 \left\| X_e \right\|_3^3 + 6|B^0_n|^2 \left\| Y^{sc,\eta_n} \right\|_3^3 \right] + \frac{\left\| Y^{sc,\eta_n} \right\|_2^3}{\nu^4} |A_n \triangle B^0_n A_n|.
\]
B Proof of theorem \(3\) and theorem \(4\)

We first prove an intermediary result that we use to deduce theorem \(3\) and theorem \(4\). In this goal given a completely positive function: \(\eta : A \rightarrow \mathcal{F}_{\text{tail}}^{\text{D}(G_{kn})}(X^n)\) we write

\[
\left\| \eta(\cdot) - \eta_n(\cdot) \right\|_{\text{op}} \ := \sup_{a \in \mathcal{F}_{\text{tail}}^{\text{D}(G_{kn})}} \left\| \eta(a) - \eta_n(a) \right\|_1.
\]

**Proposition 11.** Let \((X^n)\) be a triangle array of free random variables satisfying all the conditions of theorem \(3\). Let \((\eta'_n)\) be a sequence of completely positive maps. Let \(S_n(\cdot)\) denote the (operator-valued) Stieltjes transform of \(\frac{1}{|A_{kn}|^{kn-\frac{3}{2}}} \int_{A_{kn}^n} X_g d|g|\), and \(S_n^{sc}(\cdot)\) the Stieltjes transform of the operator-valued semi-circular operator \(Y^{sc,\eta'_n}\) with radius \(\eta'_n(\cdot)\). The following upper bound holds

\[
\left\| S_n(\gamma_x,\nu) - S_n^{sc}(\gamma_x,\nu) \right\|_1 \leq \frac{12|B_0|^2 |Y^{sc,\eta'_n}|_3^3 + 18k_1^4 |B_{20}|^2 \sup_{g \in G_{kn}} \|X_g\|_3^3 + 3k_1^2 \sup_{g \in G_{kn}} \left\| X_n\right\|_2^{2} R_n^{1}[b_n]}{\nu^3} \]

\[
+ \frac{1}{\nu^3} \left\| \eta'_n(\cdot) - \eta_n(\cdot) \right\|_{\text{op}} \cdot \left\| \frac{2k_1^2 \sup_{g \in G_{kn}} \left\| X_n\right\|_2^{2} |B_{20}|^{n,\gamma_j,\nu} |b_n|^{G_{kn}}}{\nu^3} \right\|_2^{3} + \frac{2k_1^2 \sup_{g \in G_{kn}} \left\| X_n\right\|_2^{2} |B_{20}|^{n,\gamma_j,\nu} |b_n|^{G_{kn}}}. \]

where \(\gamma_x,\nu = x + i\nu\).

**Proof.** The proof of proposition \(11\) as the proof of proposition \(10\) adapts the Linderberg method; and to do so it creates an operator \(W'_n(t)\) that interpolates between \(W_n\) and \(Y^{sc,\eta'_n}\).

In this goal we introduce some notations. For all operators \(W\) and \(\gamma \in \mathbb{C} \setminus \mathbb{R}\) we write \(R(W,\gamma) = |W - \gamma 1_A|^{-1}\). Moreover we define

\[
W_n := \frac{1}{|A_{kn}|^{kn-\frac{3}{2}}} \int_{A_{kn}^n} X_g^n d|g|. \]

Let \(Y^n\) be a free operator-valued semi-circular operator with radius \(\eta'_n\). Let \((Y^n_g)\) be free copies of \(Y^n\). We write \(Y^{sc,\eta'_n} := \frac{1}{\sqrt{|A_{kn}|}} \int_{A_{kn}^n} Y^n_g d|g|\). We note that \(Y^{sc,\eta'_n}\) is defined as an empirical average of \((Y^n_g)\) when \(g\) varies over \(A_{kn}\) not \(A_{kn}^n\). For all \(t \in [0, 1]\), we define the following interpolating processes:

\[
W_n(t) := \sqrt{t} W_n + \sqrt{1 - t} Y^{sc,\eta'_n} \]

\[
W_g^{sc,\eta'_n}(t) := \sqrt{t} W_n + \frac{\sqrt{1 - t}}{\sqrt{|A_{kn}|}} \int_{A_{kn}^n \setminus B^n(g,0)} Y^n_g d|g|. \]
Moreover to be able to handle the fact \( (X^n_{\mathcal{g}}) \) is not free we introduce some slightly modified \( W_n(t) \). For all \( \mathcal{g}, \mathcal{g}' \in \mathbb{G}^{k_n} \) and all \( b > 0 \) we write

\[
W_{\mathcal{g},b}^n(t) := W_n(t) - \frac{1}{|A_{n,n}|^{k_n}} \int_{B_{k_n}(\mathcal{g},b)^n} X^n_{\mathcal{g}} d|\mathcal{g}'|
\]

\[
W_{\mathcal{g},\mathcal{g}',b}^n(t) := W_n(t) - \frac{\sqrt{t}}{|A_{n,n}|^{k_n}} \int_{B_{k_n}(\mathcal{g},2b)^n \cup B_{k_n}(\mathcal{g}',2b)^n} X^n_{\mathcal{g}} d|\mathcal{g}|.
\]

As in the proof of proposition \([10]\) \( W_n(t) \) interpolates between \( W_n \) and \( Y^{sc,n_0'} \). For simplicity we use the following shorthand notations:

\[
R^t_n := R(W_n(t), \gamma_{x,\nu}), \quad P^t_{\mathcal{g}} := R(W^n_{\mathcal{g},b}(t), \gamma_{x,\nu}), \quad P_{\mathcal{g}}^{\mathcal{g}',t} := R(W^n_{\mathcal{g},\mathcal{g}',b}(t), \gamma_{x,\nu}).
\]

Denote \( g : t \to E_{D_n}(R^t_n) \), we remark that

\[
g(1) - g(0) = S_n(\gamma_{x,\nu}) - S^{sc}_{n}(\gamma_{x,\nu}).
\]

Therefore the objective is to upper-bound \( \|g(1) - g(0)\|_1 \). In this goal, similarly than in the proof of proposition \([10]\) we notice that \( g \) is Gateaux-differentiable and that its derivative respects

\[
g'(t) = -E_{D_n}(R^t_n \left[ \frac{W_n}{2\sqrt{t}} - \frac{Y^{sc,n_0'}}{2\sqrt{1-t}} \right] R^t_n).
\]

The key of the proof consists in upper-bounding \( |g'(t)| \). Firstly we note for all \( t \in (0,1) \) that we have:

\[
E_{D_n}\left[ R^t_n \frac{W_n}{\sqrt{t}} R^t_n \right] = \frac{1}{|A_{n,n}|^{k_n}} \int_{A_{n,n}} E_{D_n}\left[ R^t_n \frac{X^n_{\mathcal{g}} R^t_n}{\sqrt{t}} \right] d|\mathcal{g}|
\]

\[
\quad = \frac{1}{|A_{n,n}|^{k_n}} \int_{A_{n,n}} E_{D_n}\left[ R^t_n \frac{X^n_{\mathcal{g}} R^t_n}{\sqrt{t}} \right] d|\mathcal{g}|
\]

\[
\quad = \frac{1}{|A_{n,n}|^{k_n}} \int_{A_{n,n}} E_{D_n}\left[ R^t_n \frac{X^n_{\mathcal{g}} R^t_n}{\sqrt{t}} \right] d|\mathcal{g}|
\]

\[
\quad = a_1^{\gamma_{x,\nu}} + a_2^{\gamma_{x,\nu}} + a_3^{\gamma_{x,\nu}}
\]

where (a) comes from the linearity of the functional \( E_{D_n} \) and (b) from eq. \([10]\). The rest of proof consists in proving that: (i) \( |a_3^{\gamma_{x,\nu}}| \to 0 \) and (ii) on re-expressing \( a_1^{\gamma_{x,\nu}} \) and \( a_2^{\gamma_{x,\nu}} \).
We start by proving that \( |a^3_{\gamma,\nu}| \to 0 \). Using the definition of the free global mixing coefficients \( (\rho_{n}^{n}[:|G_{n}^{k}]) \) we have:

\[
\begin{align*}
|a^3_{\gamma,\nu}| &\leq \left\| E_{D_n} \left[ \frac{X_n^{g}}{\sqrt{t}} R_{b_{n}}^{g,t} - R_{b_{n}}^{g,t} \right] \right\|_1 \\
\leq (a) &\sum_{b \geq b_{n}} \left\| E_{D_n} \left[ R_{b_{n}}^{g,t} \frac{X_n^{g}}{\sqrt{t}} \left[ R_{b_{n}}^{g,t} - R_{b_{n}}^{g,t} \right] \right] \right\|_1 \\
&\quad + \sum_{b \geq b_{n}} \left\| E_{D_n} \left[ R_{b_{n}}^{g,t} - R_{b_{n}}^{g,t} \right] \frac{X_n^{g}}{\sqrt{t}} R_{b_{n}}^{g,t} \right\|_1 \\
\leq (b) &\sum_{b \geq b_{n}} \left\| E_{D_n} \left[ \frac{R_{b_{n}}^{g,t} X_n^{g}}{\sqrt{t}} \left[ W_{n,b_{n}+1}^{g,b} - W_{n,b}^{g,b} \right] R_{b_{n}}^{g,t} \right] \right\|_1 \\
&\quad + \sum_{b \geq b_{n}} \left\| E_{D_n} \left[ R_{b_{n}+1}^{g,t} \left[ W_{n,b_{n}+1}^{g,b} - W_{n,b}^{g,b} \right] R_{b_{n}+1}^{g,t} \right] \right\|_1 \\
\leq (c) &\frac{2k_{n}^2}{\sqrt{\text{Vol}^{n}A_{n,n}^{k} |G_{n}^{k}|}} \sup_{g \in G_{n}^{k}} \left\| X_n^{g} \right\|_2^2 \sum_{b \geq b_{n}} \left| B_{b_{n}+1}^{n} \setminus B_{b_{n}}^{n} \right| n_{n}^{\gamma,\nu} [b_{n} |G_{n}^{k}|].
\end{align*}
\]

where (a) comes from the triangular inequality and (b) from eq. (110). While (c) was obtained using successively the definition of \( (\rho_{n}^{n}[:|G_{n}^{k}]) \), the fact that \( E_{D_n} (X_n^{g}) = 0 \) as well as the fact that \( |B_{k_{n}}^{n} (g, b + 1) \setminus B_{k_{n}}^{n} (g, b)| \leq k_{n}^2 |B_{b_{n}+1}^{n} \setminus B_{b_{n}}^{n} | n_{n}^{k-1} |A_{n,n}|. \)

Therefore by taking the average over \( g \in A_{n,n}^{k} \) we obtain that

\[
|a^3_{\gamma,\nu}| \leq \frac{2k_{n}^2}{\nu^2} \sup_{g \in G_{n}^{k}} \left\| X_n^{g} \right\|_2^2 \mathcal{R}^{n}_{n} [b_{n}].
\]

The next step consists in re-expressing \( a^3_{\gamma,\nu} \) as an alternative quantity that is easier to bound. Using eq. (110) we remark for all \( g, g' \in A_{n,n}^{k} \) that

\[
\begin{align*}
\left\| E_{D_n} \left[ \frac{X_n^{g}}{\sqrt{t}} R_{b_{n}}^{g,t} X_n^{g} R_{b_{n}}^{g,t} - X_n^{g} R_{b_{n}}^{g,t} X_n^{g} R_{b_{n}}^{g,t} \right] \right\|_1 \\
\leq \left\| E_{D_n} \left[ R_{b_{n}}^{g,t} \left[ W_{n}(t) - W_{g,g',2b_{n}}(t) \right] R_{b_{n}}^{g,t} \right] \right\|_1 \\
&\leq (a) \frac{2k_{n}^2}{\nu^2} \sup_{g \in G_{n}^{k}} \left\| X_n^{g} \right\|_3^3.
\end{align*}
\]

where (a) comes from the Hölder-inequality coupled with the fact that

\[
|B_{k_{n}}^{n} (g', 2b_{n}) \cup B_{k_{n}}^{n} (g, 2b_{n})| \leq 2k_{n}^2 |B_{2b_{n}}^{n} | n_{n}^{k-1} |A_{n,n}|. \]

Similarly we can prove that:

\[
\left\| E_{D_n} \left[ R_{b_{n}}^{g,t} X_n^{g} R_{b_{n}}^{g,t} X_n^{g} \right] \right\|_1 \leq \frac{2k_{n}^2}{\nu^4} \sup_{g \in G_{n}^{k}} \left\| X_n^{g} \right\|_3^3;
\]

as well as:

\[
\left\| E_{D_n} \left[ R_{b_{n}}^{g,t} - R_{b_{n}}^{g,t} \right] X_n^{g} R_{b_{n}}^{g,t} X_n^{g} \right\|_1 \leq \frac{2k_{n}^2}{\nu^4} \sup_{g \in G_{n}^{k}} \left\| X_n^{g} \right\|_3^3.
\]
Therefore by combining eq. (23), eq. (24) and eq. (25) and averaging over \( g \in A_{n,n}^{k_n} \) and \( g' \in B_{k_n}^n(g, b_n) \) we obtain that

\[
\begin{align*}
&\left| a_1^{\gamma_x,\nu} + \frac{\sqrt{t}}{|A_{n,n}|^{2k_n - 1}} \int_{A_{n,n}} \int_{B_{k_n}^n(g, b_n)} E_D_n \left[ R_n^{g, g', t} X_n^{\gamma_x, \nu} R_n^{g', t} \right] d|g'| |d|g| \right| \\
&\leq \frac{6k_n^4 |B_{2b_n}^n|^2}{\sqrt{|A_{n,n}|^{2k_n - 1}}} \sup_{g \in G_{k_n}} \left\| X_n^{\gamma_x, \nu} \right\|_3^3 \tag{26}
\end{align*}
\]

where to get (a) we used the fact that: \( B_{k_n}^n(g, 2b_n) \leq k_n^2 |B_{b_n}^n| |A_{n,n}|^{k_n - 1} \). Similarly we have:

\[
\begin{align*}
&\left| a_2^{\gamma_x,\nu} + \frac{\sqrt{t}}{|A_{n,n}|^{2k_n - 1}} \int_{A_{n,n}} \int_{B_{k_n}^n(g, b_n)} E_D_n \left[ R_n^{g, g', t} X_n^{\gamma_x, \nu} R_n^{g', t} \right] d|g'| |d|g| \right| \\
&\leq \frac{6k_n^4 |B_{2b_n}^n|^2}{\sqrt{|A_{n,n}|^{2k_n - 1}}} \sup_{g \in G_{k_n}} \left\| X_n^{\gamma_x, \nu} \right\|_3^3 \tag{27}
\end{align*}
\]

Therefore using eq. (22), eq. (26) and eq. (27) we get that:

\[
\begin{align*}
&\left\| g(1) - g(0) \right\|_1 \\
&\leq \int_0^1 \left\| \frac{1}{|A_{n,n}|^{2k_n - 1}} \int_{A_{n,n}} \int_{B_{k_n}^n(g, b_n)} E_D_n \left[ R_n^{g, g', t} X_n^{\gamma_x, \nu} R_n^{g', t} \right] d|g'| |d|g| \\
&\quad - E_D_n \left[ R_n^{t, Y_n^{\gamma_x, \nu}} \right] \right\|_1 \right| dt + \frac{12k_n^4 |B_{2b_n}^n|^2}{\sqrt{|A_{n,n}|^{2k_n - 1}}} \sup_{g \in G_{k_n}} \left\| X_n^{\gamma_x, \nu} \right\|_3^3 \\
&\quad + \frac{2k_n^2 \sup_{g \in G_{k_n}} \left\| X_n^{\gamma_x, \nu} \right\|_2^2 \left| B_n^t \right|}{\nu^4} \\
&\quad \tag{28}
\end{align*}
\]

Moreover if we define the following shorthand notation \( R_{g,t}^{sc,\eta_n} = R(W_n^{sc,\eta}(t), \gamma_x, \nu) \) then by exploiting the free independence of \( (Y_n^{\gamma_x, \nu}) \) we obtain for all \( t \in (0, 1) \) that:

\[
\begin{align*}
&\left\| \frac{1}{|A_{n,n}|^{2k_n - 1}} \int_{A_{n,n}} E_D_n \left[ R_n^{g, g', t} Y_n^{\gamma_x, \nu} R_n^{g', t} \right] d|g| - E_D_n \left[ R_n^{t, Y_n^{\gamma_x, \nu}} \right] \right\|_1 \\
&\leq \frac{6 |B_0|^2}{\sqrt{|A_{n,n}|^{2k_n - 1}}} \left\| Y_n^{\gamma_x, \nu} \right\|_3^3 \tag{29}
\end{align*}
\]
By combining eq. (29) and eq. (28) we therefore have:

\[
\left\| g(1) - g(0) \right\|_1 \leq \frac{1}{\left| A_{n,n} \right|^{2k_n - 1}} \int_{A_{n,n}} \int_{T_{2bm}(g,b_n)} E_{D_n} \left[ R_{2bm}^g \eta_n R_{2bm}^g \eta_n R_{n}^g \eta_n R_{n}^g \eta_n \right] \left| \frac{d|g'|}{d|g|} \right| \left| d|g'| \right| \left| d|g| \right|
- \frac{1}{\left| A_{n,n} \right|} \int_{A_{n,n}} E_{D_n} \left[ R_{g} \eta \right] Y_{g} R_{g} \eta \left| Y_{g} R_{g} \eta \right| \left| d|g| \right| \left| d|g| \right| + \frac{12c_{k_n}^4 |B_2|^2}{\left| A_{n,n} \right|^{2k_n} \left| B_0 \right|^{2k_n}} \sup_{g \in \mathcal{G}_{2m}} \left| X_n^e \right|_3^3
+ \frac{2c_{k_n}^2 \sup_{g \in \mathcal{G}_{2m}} \left| X_n^e \right|_2^2 R_{n}^e \left| b_n \right|}{\left| B_0 \right|^{2k_n}} + \frac{6|B_0|^2}{\left| A_{n,n} \right|^{2k_n} \left| B_0 \right|^{2k_n}} \left| Y_{n} \right|_3^3
\]

(30)

Therefore to get the desired result it is sufficient to compare

\[
\frac{1}{\left| A_{n,n} \right|^{2k_n - 1}} \int_{A_{n,n}} \int_{T_{2bm}(g,b_n)} E_{D_n} \left[ R_{2bm}^g \eta_n R_{2bm}^g \eta_n R_{n}^g \eta_n R_{n}^g \eta_n \right] \left| \frac{d|g'|}{d|g|} \right| \left| d|g'| \right| \left| d|g| \right|
\]

with

\[
\frac{1}{\left| A_{n,n} \right|} \int_{A_{n,n}} E_{D_n} \left[ R_{g} \eta \right] Y_{g} R_{g} \eta \left| Y_{g} R_{g} \eta \right| \left| d|g| \right| \left| d|g| \right| + \frac{12c_{k_n}^4 |B_2|^2}{\left| A_{n,n} \right|^{2k_n} \left| B_0 \right|^{2k_n}} \sup_{g \in \mathcal{G}_{2m}} \left| X_n^e \right|_3^3
+ \frac{2c_{k_n}^2 \sup_{g \in \mathcal{G}_{2m}} \left| X_n^e \right|_2^2 R_{n}^e \left| b_n \right|}{\left| B_0 \right|^{2k_n}} + \frac{6|B_0|^2}{\left| A_{n,n} \right|^{2k_n} \left| B_0 \right|^{2k_n}} \left| Y_{n} \right|_3^3
\]

In this goal we introduce some shorthand notations. For all \( t \in [0, 1] \) we denote:

\[ \eta \in \mathcal{G} : a \to E_{D_n} \left[ X_n^e a X_n^e \right], \quad S_{2bm}^0 : = E_{D_n} \left[ R(W_{2bm}^g(t), \gamma_{x,v}) \right], \quad S_n^0 : = E_{D_n} \left[ R_n^t \right]. \]

Using the triangular inequality we remark that:

\[
\left\| \frac{1}{\left| A_{n,n} \right|^{2k_n - 1}} \int_{A_{n,n}} \int_{T_{2bm}(g,b_n)} E_{D_n} \left[ R_{2bm}^g \eta_n R_{2bm}^g \eta_n R_{n}^g \eta_n R_{n}^g \eta_n \right] \left| \frac{d|g'|}{d|g|} \right| \left| d|g'| \right| \left| d|g| \right| - E_{D_n} \left[ R_n^t \eta_n(S_n^0 R_n^t) \right] \right\|_1 \leq \left\| \frac{1}{\left| A_{n,n} \right|^{2k_n - 1}} \int_{A_{n,n}} \int_{T_{2bm}(g,b_n)} E_{D_n} \left[ R_{2bm}^g \eta_n R_{2bm}^g \eta_n R_{n}^g \eta_n R_{n}^g \eta_n \right] \left| \frac{d|g'|}{d|g|} \right| \left| d|g'| \right| \left| d|g| \right|
- E_{D_n} \left[ R_{2bm}^g \eta_n \eta_n (S_{2bm}^0) R_{2bm}^g \eta_n \eta_n \right] \right\|_1
+ \left\| \frac{1}{\left| A_{n,n} \right|^{2k_n - 1}} \int_{A_{n,n}} \int_{T_{2bm}(g,b_n)} E_{D_n} \left[ R_{2bm}^g \eta_n \eta_n (S_{2bm}^0) R_{2bm}^g \eta_n \eta_n \right] - E_{D_n} \left[ R_n^t \eta_n \eta_n (S_n^0) R_n^t \right] \right\|_1
+ \left\| \frac{1}{\left| A_{n,n} \right|^{2k_n - 1}} \int_{A_{n,n}} \int_{T_{2bm}(g,b_n)} E_{D_n} \left[ R_n^t \eta_n \eta_n (S_n^0) R_n^t \right] \left| \frac{d|g'|}{d|g|} \right| \left| d|g'| \right| \left| d|g| \right| - E_{D_n} \left[ R_n^t \eta_n \eta_n (S_n^0) R_n^t \right] \right\|_1
\leq (c_1^7 x, v) + (c_2^7 x, v) + (c_3^7 x, v)
\]

We bound each term successively. The first term is bounded using the definition of \( (N_{n,j}^t(s) \mid \mathcal{G}_n) \), and the later terms are a consequence of the triangular inequality and eq. (10). We focus first on the term \( c_1^7 x, v \). Using the definition of the free global mixing
where to get (a) we used eq. (10) and Cauchy-Schwarz inequality. Therefore by averaging coefficients, we obtain that

\[ |(c_1^{x,v})| \]

\[ \leq \| A_{n,n} \|_{2k_n-1} \int_{A_{n,n}} \int_{B_{k_n}(g, b_n)} E_{D_n} [B_{2b_n}^{g,k'} \times^{t} X_{k}^{n} R_{2b_n}^{g,k'} X_{k}^{n} B_{2b_n}^{g,k'}] \]

\[ \leq \| A_{n,n} \|_{2k_n-1} \int_{A_{n,n}} \int_{B_{k_n}(g, b_n)} E_{D_n} [B_{2b_n}^{g,k'} \times^{t} X_{k}^{n} R_{2b_n}^{g,k'} X_{k}^{n} B_{2b_n}^{g,k'}] \]

\[ + \| A_{n,n} \|_{2k_n-1} \int_{A_{n,n}} \int_{B_{k_n}(g, b_n)} E_{D_n} [R_{2b_n}^{g,k'} X_{k}^{n} R_{2b_n}^{g,k'} X_{k}^{n} B_{2b_n}^{g,k'}] |d| |d| \]

\[ \leq 2 \sup_{g \in G^{k_n}} \| X_{g}^{n} \|_{2}^{2} \int_{A_{n,n}} \int_{B_{k_n}(g, b_n)} \eta_{x,v}^{n} [d((g, g'), A_{n,n} \setminus B_{k_n}(g, 2b_n)) |G^{k_n}|] |d| |d| \]

\[ \leq 2k_n \sup_{g \in G^{k_n}} \| X_{g}^{n} \|_{2}^{2} \frac{B_{2b_n}^{n} |k_{n}^{x,v}[b_{n}|G^{n}]|}{\nu^3} \]

The next step consists in bounding \( c_2^{x,v} \). Using the triangular inequality we obtain that:

\[ \| E_{D_n} (R_{2b_n}^{g,k'} \times^{t} X_{k}^{n} R_{2b_n}^{g,k'}) - E_{D_n} (R_{n}^{t} \eta_{g,k'} (S_{n,t}^{0} R_{n}^{t}) \|_{1} \]

\[ \leq \| R_{2b_n}^{g,k'} - R_{n}^{t} \|_{1} \eta_{g,k'} (S_{n,t}^{0} R_{n}^{t}) \|_{1} + \| R_{n}^{t} \eta_{g,k'} (S_{n,t}^{0} R_{n}^{t}) - R_{n}^{t} \|_{1} \]

\[ \leq \frac{6k_n^2 |B_{2b_n}^{n}|}{\nu^3} \sup_{g \in G^{k_n}} \| X_{g}^{n} \|_{3}^{2} \]

where to get (a) we used eq. (10) and Cauchy-Schwarz inequality. Therefore by averaging over \( g \in A_{n,n} \) and \( g' \in B_{k_n}(g', b_n) \) this implies that

\[ \langle c_2^{x,v} \rangle \leq \frac{6 k_n^3 |B_{2b_n}^{n}|}{\nu^3} \sup_{g \in G^{k_n}} \| X_{g}^{n} \|_{3}^{3} \]

where to get (a) we used the fact that \( |B_{k_n}(g, 2b_n)| \leq k_n^2 |B_{2b_n}^{n}| A_{n,n} |k_n-1|. The next
step is to bound \((c_3^{x,v})\). Using the triangular inequality we have
\[(c_3^{x,v})\]
\[
\begin{align*}
&\leq \left\| \frac{1}{|A_{n,n}|^{2k_n-1}} \int_{A_{n,n}^{n-1} \times B_{n_n}(g,b_n)} E_{\mathcal{D}_n} \left( R_{n}^t \eta_n^t \mathbb{S}_n^0(R_{n}^t \eta_n^t) \right) d|g'| d|g| - E_{\mathcal{D}_n} \left( R_{n}^t \eta_n^t \mathbb{S}_n^0(R_{n}^t \eta_n^t) \right) \right\|_{1} \\
&\leq \left\| \frac{1}{|A_{n,n}|^{2k_n-1}} \int_{A_{n,n}^{n-1} \times B_{n_n}(g,b_n)} E_{\mathcal{D}_n} \left( R_{n}^t \eta_n^t \mathbb{S}_n^0(R_{n}^t \eta_n^t) \right) d|g'| d|g| \right\|_{1} \\
&\quad + \frac{1}{\nu^3} \left\| \eta_n^t(\cdot) - \eta_n(\cdot) \left|_{\mathcal{A}^{\text{tail}}_{\mathcal{D}(c_{n,n})}} \right\|_{\text{op}} \\
&\leq k_n^2 \sup_{g \in \mathcal{G}_{n,n}} \left\| X_n^g \right\|_{2}^2 \mathcal{R}_{n}[b_n] + \frac{1}{\nu^3} \left\| \eta_n^t(\cdot) - \eta_n(\cdot) \left|_{\mathcal{A}^{\text{tail}}_{\mathcal{D}(c_{n,n})}} \right\|_{\text{op}}.
\end{align*}
\]
where to obtain (a) we exploited the Cauchy-Schwarz inequality. This implies that
\[
\left\| g(1) - g(0) \right\|_{1} \\
\leq \int_{0}^{1} \left\| S_{n,t}^0 \eta_n^t \mathbb{S}_n^0 - \frac{1}{|A_{n,n}|} \int_{A_{n,n}} E_{\mathcal{D}_n} \left[ R_{g,t}^s \eta_n^t \mathbb{S}_n^0(R_{g,t}^s \eta_n^t) \right] d|g| \right\|_{1} dt \\
+ \frac{12 k_n^4 |B_{2b_n}|^2}{|A_{n,n}|^{2k_n}} \sup_{g \in \mathcal{G}_{n,n}} \left\| X_n^g \right\|_{2}^2 + \frac{18 k_n^4 |B_{2b_n}|^2}{|A_{n,n}|^{2k_n}} \frac{\sup_{g \in \mathcal{G}_{n,n}} \left\| X_n^g \right\|_{2}^2 \mathcal{R}_{n}[b_n]}{\nu^3} \\
+ \frac{6 k_n^4 |B_{2b_n}|^2}{|A_{n,n}|^{2k_n}} \frac{\sup_{g \in \mathcal{G}_{n,n}} \left\| X_n^g \right\|_{2}^2 + \frac{12 k_n^4 |B_{2b_n}|^2}{|A_{n,n}|^{2k_n}} \frac{\sup_{g \in \mathcal{G}_{n,n}} \left\| X_n^g \right\|_{2}^2 |B_{2b_n}|}{\nu^3}}{\nu^3}.
\]
Moreover similarly we can prove that
\[
\int_{0}^{1} \left\| S_{n,t}^0 \eta_n^t \mathbb{S}_n^0 - \frac{1}{|A_{n,n}|} \int_{A_{n,n}} E_{\mathcal{D}_n} \left[ R_{g,t}^s \eta_n^t \mathbb{S}_n^0(R_{g,t}^s \eta_n^t) \right] d|g| \right\|_{1} dt \\
\leq \frac{6 |B_{0}|^2}{|A_{n,n}|^{2k_n}} \left\| Y_{\mathcal{S},\mathcal{N}}^n \right\|_{3}^3.
\]
Therefore we finally obtain that:
\[
\left\| g(1) - g(0) \right\|_{1} \\
\leq \frac{1}{\sqrt{|A_{n,n}|}} \left[ 12 |B_{0}|^2 \left\| Y_{\mathcal{S},\mathcal{N}}^n \right\|_{3}^3 + 18 k_n^4 |B_{2b_n}|^2 \sup_{g \in \mathcal{G}_{n,n}} \left\| X_n^g \right\|_{2}^2 \right] + \\
\quad + \frac{1}{\nu^3} \left\| \eta_n^t(\cdot) - \eta_n(\cdot) \left|_{\mathcal{A}^{\text{tail}}_{\mathcal{D}(c_{n,n})}} \right\|_{\text{op}} \\
+ \frac{k_n^2 \sup_{g \in \mathcal{G}_{n,n}} \left\| X_n^g \right\|_{2}^2}{\nu^3} \left[ 2 |B_{2b_n}| \left\| X_n^g \right\|_{2}^2 + \mathcal{R}_{n}[b_n] \right].
\]
Proof of theorem 3. We prove that proposition 10 implies that theorem 3 holds.

Proof. Taking \( \eta_n'(\cdot) := \eta_n(\cdot) \) gives the desired result. □

We prove that proposition 10 implies that theorem 4 holds.

**Proposition 12.** Let \((X_g^n)\) be a triangle array of free random variables satisfying all the conditions of theorem 2. Let \(S_n(\cdot)\) denote the (operator-valued) Stieltjes transform of
\[
\frac{1}{|A_{n,n}|^{k_n - \frac{1}{2}}} \int_{A_{n,n}} X_g d|g|,
\]
and \(S_n^{sc,\eta_n}(\cdot)\) the Stieltjes transform of the operator valued semicircular operator \(Y^{sc,\eta_n}\) with radius \(\tilde{\eta}_n(\cdot)\). The following upper bound holds
\[
\left\| S_n(\gamma_{x,\nu}) - S_n^{sc,\eta_n}(\gamma_{x,\nu}) \right\|_1 \leq \frac{1}{|A_{n,n}|^{\nu^3}} \left[ k_n^4 |B_{2n_n}^n| \sup_{g \in \mathbb{G}_{\nu n}} \|X_g^n\|_2 \|T_m^n + 2|A_{n,n} \triangle B_{2n_n}^n A_{n,n}| \sup_{g \in \mathbb{G}_{\nu n}} \|X_g^n\|_2 + \|Y^{sc,\eta_n}\|_2^2 \right] \]
\[
+ \frac{1}{\sqrt{|A_{n,n}|^{\nu^3}}} \left[ 12 |B_0^n|^2 \|Y^{sc,\eta_n}\|_3^3 + 18 k_n^4 |B_{2n_n}^n|^2 \sup_{g \in \mathbb{G}_{\nu n}} \|X_g^n\|_2^3 \|\gamma_{x,\nu}\|_3^3 \right] \max \left\{ 5 k_n^2 \sup_{g \in \mathbb{G}_{\nu n}} \|X_g^n\|_2^2 \|\gamma_{x,\nu}\|_3^3 \right\} \frac{|\nu|^4}{\nu^3} \left[ 3 k_n^2 \sup_{g \in \mathbb{G}_{\nu n}} \|X_g^n\|_2^2 \|\gamma_{x,\nu}\|_3^3 \right] \frac{R_n^s(b_n)}{\nu^3} \]
\[
+ \frac{k_n^2 \sup_{g \in \mathbb{G}_{\nu n}} \|X_g^n\|_2^2 |B_{2n_n}^n| \|\gamma_{x,\nu}\|_3^3 \left[ B_{2n_n}^n \right] \|\gamma_{x,\nu}\|_3^3 \right\} \frac{R_n^s(b_n)}{\nu^3} \]
where \(\gamma_{x,\nu} = x + i \nu\).

Proof. We choose \(\eta_n'(\cdot) = \tilde{\eta}_n(\cdot)\). As \((X_g^n)\) satisfies all the conditions of theorem 4 it also satisfies the conditions of theorem 3. Therefore using proposition 11 we know that
\[
\left\| S_n(\gamma_{x,\nu}) - S_n^{sc,\eta_n}(\gamma_{x,\nu}) \right\|_1 \leq \frac{12 |B_0^n|^2 \|Y^{sc,\eta_n}\|_3^3 + 18 k_n^4 |B_{2n_n}^n|^2 \sup_{g \in \mathbb{G}_{\nu n}} \|X_g^n\|_2^3 + \frac{3 k_n^2 \sup_{g \in \mathbb{G}_{\nu n}} \|X_g^n\|_2^2 \|\gamma_{x,\nu}\|_3^3}{\nu^3} \right\} \frac{R_n^s(b_n)}{\nu^3} \]
\[
+ \frac{k_n^2 \sup_{g \in \mathbb{G}_{\nu n}} \|X_g^n\|_2^2 |B_{2n_n}^n| \|\gamma_{x,\nu}\|_3^3 \left[ B_{2n_n}^n \right] \|\gamma_{x,\nu}\|_3^3 \right\} \frac{R_n^s(b_n)}{\nu^3} \]
where \(\eta_n'(\cdot) = \tilde{\eta}_n(\cdot)\). Let \(a \in \mathcal{F}_{\text{tail}}^{\text{tail}}(\mathcal{G}_{\nu n}) (X^n)\) be an operator verifying \(\|a\|_{\infty} \leq 1\), we remark that
\[
\eta_n(a) = \frac{1}{|A_{n,n}^{2k_n-1}|} \int_{A_{n,n}^{2k_n}} \eta_{g,g'}(a) d|g| d|g'|.
\]
To prove the desired result we firstly propose a simpler form for \(\eta_{g,g'}(\cdot)\) when \(g' \in B_{2n_n}^n(g)\). We write:
\[
\mathcal{I}_{b_n}(g,g') := \{(i,j) | d(g_i, g'_j) \leq b_n\}.
\]
Let $g_1, g_2, g_3, g_4 \in \mathbb{G}^k_n$ be elements such that $\{(i, j)\} \subset \mathcal{I}_{h_n}(g_1, g_2)$ and such that $g_i = g_i^1$ and $g_i^j = g_i^4$. We want to compare $\eta_{g_1, g_2}^j(\cdot)$ with $\eta_{g_1^3, g_4^j}(\cdot)$. In this goal, we build $(g^1_1)$ and $(g^2_1)$ to be interpolating sequences between $(g_1, g_2)$ and $(g_3, g_4)$. We define them as

$$
g^1_m := \begin{cases} g^3_m & \text{if } m \leq 1 \\ g_m & \text{otherwise} \end{cases} \quad \text{and} \quad g^2_m := \begin{cases} g^4_m & \text{if } m \leq 1 \\ g_m & \text{otherwise} \end{cases}
$$

By the triangular inequality and the definition of free marginal mixing coefficients, the following holds

$$\left\| \eta_{g_1^1, g_2^j}(a) - \eta_{g_1^3, g_4^j}(a) \right\|_1 \leq \sum_{i=1}^{k_n} \left\| \eta_{g_1^1, g_2^j}(a) - \eta_{g_1^3, g_4^j}(a) \right\|_1 \leq \sup_{g \in \mathbb{G}^k_n} \|X^n_g\|_2 \sum_{l=1}^{k_n} \mathbb{N}_n \left[ d\left(\{g^1_l, g^3_l\}, [g^1_l, g^2_l]^l \bigcup [g^3_l, g^4_l]^l\right) \right] \mathcal{G}^{k_n} + \mathbb{N}_n \left[ d\left(\{g^2_l, g^4_l\}, [g^2_l, g^1_l]^l \bigcup [g^4_l, g^3_l]^l\right) \right] \mathcal{G}^{k_n}$$

Therefore using the definition of $\bar{X}^n_{i, g_1^1}$ and $\bar{X}^n_{j, g_2^j}$ we obtain that:

$$\left\| \eta_{g_1, g_2}(a) - E_{D_n}\left( \bar{X}^n_{i, g_1^1} a \bar{X}^n_{j, g_2^j} \right) \right\|_1 \leq 2 \sup_{g \in \mathbb{G}^k_n} \|X^n_g\|_2 \sum_{l=1}^{k_n} \mathbb{N}_n \left[ d\left(\{g^1_l, g^3_l\}, [g^1_l, g^2_l]^l \bigcup [g^3_l, g^4_l]^l\right) \right] \mathcal{G}^{k_n} + \mathbb{N}_n \left[ d\left(\{g^2_l, g^4_l\}, [g^2_l, g^1_l]^l \bigcup [g^4_l, g^3_l]^l\right) \right] \mathcal{G}^{k_n}.$$
Therefore using the triangular inequality we obtain that:

\[
\left\| \frac{1}{|A_{n,n}|^{2k-1}} \int_{A_{n,n}^{b_n}}^{A_{n,n}} \int_{B_{n}(g,b_n)}^{B_{n}(g,b_n)} \eta_{g,g'}(a)d|g'|d|g| - \eta_n(a) \right\|_1 \\
\leq \sum_{i,j \leq k_n} \left\| \frac{1}{|A_{n,n}|^{2k-1}} \int_{A_{n,n}^{b_n}}^{A_{n,n}} \int_{B_{n}(g,b_n)}^{B_{n}(g,b_n)} \eta_{g,g'}(a) - E_{D_n}(\tilde{X}_{i,g}^n a \tilde{X}_{j,g'}^n) d|g'|d|g| \right\|_1 \\
+ \sum_{i,j \leq k_n} \left\| \frac{1}{|A_{n,n}|} \int_{A_{n,n} \times B_n(g,b_n)} E_{D_n}(\tilde{X}_{i,g}^n a \tilde{X}_{j,g'}^n) d|g'|d|g| - \int_{B_n(e,b_n)} E_{D_n}(\tilde{X}_{i,g}^n a \tilde{X}_{j,g'}^n) d|g| \right\|_1 \\
+ \| \int_{A_{n,n} \setminus B_n(e,b_n)} E_{D_n}(\tilde{X}_{i,g}^n a \tilde{X}_{j,g'}^n) d|g| \|_1 \\
\leq \frac{4k_n^2 |B_{n}| \sup_{g \in \mathbb{G}^k_n} \| X_g^n \|_2^2 \int_{T_n} + \sup_{g \in \mathbb{G}^k_n} \| X_g^n \|_2^2 \frac{|A_{n,n} \Delta B_{b_n} A_{n,n}|}{|A_{n,n}|}} + k_n^2 \sup_{g \in \mathbb{G}^k_n} \| X_g^n \|_2^2 R_n^s[b_n].
\]

where to obtain (a) we exploited the Cauchy-Schwarz inequality. Moreover using the definition of $N^{s,s}[-;G_k^n]$ we obtain that

\[
\left\| \frac{1}{|A_{n,n}|^{2k-1}} \int_{A_{n,n}^{b_n}}^{A_{n,n}} \int_{B_{n}(g,b_n)}^{B_{n}(g,b_n)} \eta_{g,g'}(a)d|g'|d|g| \right\|_1 \\
\leq \sum_{b \geq b_n} N^{s,s}[b;G_k^n] \sup_{g \in \mathbb{G}^k_n} \| X_g^n \|_2^2 \int_{A_{n,n}^{b_n}}^{A_{n,n}} \int_{A_{n,n}^{b_n} \setminus B_{n}(g,b_n)} \mathbb{I}(d(g,g') \in [b, b+1)) d|g'|d|g| \\
\leq k_n^2 \sup_{g \in \mathbb{G}^k_n} \| X_g^n \|_2^2 R_n^s[b_n].
\]

As $a \in \mathcal{A}_{D_{n}(G_k^n)}^{\text{tail}}$ is arbitrary we have:

\[
\left\| \eta_n(\cdot) - \eta_n(\cdot) \right\|_{\text{op}} \leq \sup_{g \in \mathbb{G}^k_n} \| X_g^n \|_2^2 \left[ k_n^2 |B_{b_n}| \sup_{g \in \mathbb{G}^k_n} \| X_g^n \|_2^2 T_n + |A_{n,n} \Delta B_{b_n} A_{n,n}| \right] + \frac{2k_n^2 \sup_{g \in \mathbb{G}^k_n} \| X_g^n \|_2^2 R_n^s[b_n]}{
u^4}.
\]
Therefore we finally obtain that:
\[
\|g(1) - g(0)\|_1 \leq \frac{1}{|A_{n,n}|} \left[ k_n^2 |B_{b,n}| \sup_{g \in \mathbb{G}^k} \|X_n^{g}\|_2 T_{b,n}^m + 2 |A_{n,n} \triangle B_{b,n} A_{n,n}| \left[ \sup_{g \in \mathbb{G}^k} \|X_n^{g}\|_2 + \|Y^{sc,n}\|_2 \right] \right] \\
+ \frac{12 |B_0|^2 \|Y^{sc,n}\|_3^3 + 18 k_n^4 |B_{2b,n}|^2 \sup_{g \in \mathbb{G}^k} \|X_n^{g}\|_3^3 + 5 k_n^2 \sup_{g \in \mathbb{G}^k} \|X_n^{g}\|_2^2 \tau_n[b_n]}{\sqrt{|A_{n,n}|} \nu^4} \\
+ \frac{k_n^2 \sup_{g \in \mathbb{G}^k} \|X_n^{g}\|_2^2 |B_{b,n}| \tau_n^s |j_p^p[b_n]| G_n}{\nu^4}.
\]

\[\square\]

\section*{C Preliminary lemmas for theorems 2 to 4}

\textbf{Lemma 5.} Choose \( g, g' \in G \) and \( G \subset \mathbb{G}_n \) to be such that \( \bar{d}_n(\{g\}, G \cup \{g'\}) \geq b \). For all \( Y_1, Y_2, Y_3 \in \mathcal{F}_G(X^n) \) the following holds
\[
\left\| E_n \left[ Y_1 X_n^{g} Y_2 X_n^{g} Y_3 \right] \right\|_1 \leq \tau_n \leq [b|G_n].
\]

\textit{Proof.} Choose \( g, g' \in G \) and \( G \subset \mathbb{G} \) to be such that \( \bar{d}_n(\{g\}, G \cup \{g'\}) \geq b \). Let \( Y_1, Y_2, Y_3 \in \mathcal{F}_G(X^n) \). Firstly we remark that by definition of the noncommutative conditional expectation we have
\[
\left\| E_n \left[ Y_1 X_n^{g} Y_2 X_n^{g} Y_3 \right] \right\|_1 = \sup_{a \in \mathcal{F}^{\text{tail}}(X^n)} \tau \left( a Y_1 X_n^{g} Y_2 X_n^{g} Y_3 \right).
\]
Using the fact that \( \tau \) is tracial we observe that for all \( a \in \mathcal{F}^{\text{tail}}(X^n) \) such that \( \|a\|_\infty \leq 1 \) then we have
\[
\tau \left( a Y_1 X_n^{g} Y_2 X_n^{g} Y_3 \right) = \tau \left( Y_3 X_n^{g} Y_2 a Y_1 X_n^{g} \right) \leq \tau_n \leq [b|G_n]
\]
where to obtain (a) we used the fact that \( \|Y_3\|_\infty, \|Y_2 a Y_1\|_\infty \leq 1 \). \[\square\]

\textbf{Lemma 6.} Choose \( g, g' \in G^{k_n} \) and \( G \subset \mathbb{G} \) to be such that \( \bar{d}_n(\{g\}, G \cup \{g'\}) \geq b \). For all \( Y_1, Y_2, Y_3 \in \mathcal{F}_G(X^n) \) the following holds
\[
\left\| E_n \left[ Y_1 X_n^{g} Y_2 X_n^{g} Y_3 \right] \right\|_1 \leq \tau_n \leq [b|G_n]
\]

\textit{Proof.} The proof follows in the same fashion than the proof of lemma 5 \[\square\]
C.1 Preliminary results to proposition 9 and proposition 4

In this section we present a known result (see e.g. [4]), that will be used in the proof of proposition 9 and proposition 4.

Let \(Y := (Y_x)_{x \in \mathbb{Z}^d}\) be a stationary random field with entries taking value in a Borel space \(\mathcal{Y}\). Write \((\alpha[b])\) the strong mixing coefficients of \(Y\). Let \(f : \mathcal{Y}^{k_1} \times \mathcal{Y}^{k_2} \to \mathbb{R}\) be a measurable function.

For all finite subset \(Z := \{z_1, \ldots, z_{|Z|}\} \subset \mathbb{Z}^d\) we write \(Z \cdot Y := (Y_{z_1}, \ldots, Y_{z_{|Z|}})\). We denote \(\tilde{Y} := (\tilde{Y}_x)\) an independent copy of \(Y\).

**Lemma 7.** Fix \(l \in \mathbb{N}\). Select any subsets \(Z_1, Z_2 \subset \mathbb{Z}^d\) of respective size \(k\) and \(2\) that \(\min_{x \in Z_1, x' \in Z_2} \min_{j \leq d} |z_j - z'_j| \geq l\). Then the following holds:

\[
\|E(f(Z_1 \cdot Y, Z_2 \cdot Y)) - E(f(Z_1 \cdot Y, Z_2 \cdot \tilde{Y}))\| \leq 4C \alpha(l) \frac{2+\delta}{2},
\]

where \(C = \|f(Z_1 \cdot Y, Z_2 \cdot Y) - f(Z_1 \cdot Y, Z_2 \cdot \tilde{Y})\|_{L_1(\mathbb{Z}^d)}\).

**Proof.** Abbreviate \(\Delta h(Y) := f(Z_1 \cdot Y, Z_2 \cdot Y) - f(Z_1 \cdot Y, Z_2 \cdot \tilde{Y})\).

We first consider the case \(\|\Delta h\|_\infty < \infty\), and then the general case.

**Case 1:** \(\|\Delta h\|_\infty < \infty\).

Fix \(\delta > 0\). Then there is \(N_\delta \in \mathbb{N}\) sets \((A_i, B_i)_{i \leq N_\delta}\) and coefficients \(c_1, \ldots, c_{N_\delta}\) with \(|c_i| \leq \|\Delta h\|_\infty\) such that the approximation

\[
\Delta h^*(Y) := \sum_{i=1}^{N_\delta} c_i (\mathbb{I}(Z_1 \cdot Y \in A_i)(\mathbb{I}(Z_2 \cdot Y \in B_i) - \mathbb{I}(Z_2 \cdot \tilde{Y} \in B_i))
\]

satisfies \(\|\Delta h(Y) - \Delta h^*(Y)\|_\infty \leq \delta\). Moreover we have,

\[
|E(\Delta h^*(Y))| \leq \sum_{i=1}^{N_\delta} |c_i| |E(\mathbb{I}(Z_1 \cdot Y \in A_i)(\mathbb{I}(Z_2 \cdot Y \in B_i) - \mathbb{I}(Z_2 \cdot \tilde{Y} \in B_i)))|
\]

\[
\leq 2\|\Delta h\|_\infty \alpha(l),
\]

where the second inequality follows from the definition of the \(\alpha\)-mixing coefficients and by the triangle inequality. Since \(\delta\) may be arbitrarily small, we have

\[
|E[f(Z_1 \cdot Y, Z_2 \cdot Y) - f(Z_1 \cdot Y, Z_2 \cdot \tilde{Y})]| \leq 2\|\Delta h\|_\infty \alpha(l).
\]

**Case 2:** \(\|\Delta h\|_\infty\) not bounded. With no loss of generality, we can suppose that \(\|\Delta h\|_{1+\frac{\delta}{2}} \leq 1\).

For \(r \in \mathbb{R}\), define \(\Delta h_r := \Delta h(\Delta h \leq r)\) and \(\overline{\Delta h_r} := \Delta h - \Delta h_r\). By Hölder’s inequality we have,

\[
|E[f(Z_1 \cdot Y, Z_2 \cdot Y) - f(Z_1 \cdot Y, Z_2 \cdot \tilde{Y})]| \leq |E(\Delta h_r)| + |E(\overline{\Delta h_r})|
\]

\[
\leq 2r\alpha(l) + 2r^{-\frac{\delta}{2}}.
\]

The result follows for \(r = \alpha(l) \frac{2+\delta}{2}\). \(\square\)
D Proof of proposition 1

Proof. We first establish that if \((K_g)\) defines a group action on \(A_+\) then for all \(g, g_1, \ldots, g_k \in \mathbb{G}\) the following holds \((X_{g_1}, \ldots, X_{g_k}) \overset{d}{=} (X_{gg_1}, \ldots, X_{gg_k})\). In this goal we firstly remark that as \(K_g(\cdot)\) is a \(*\)-automorphism if \((X_{g_1}, \ldots, X_{g_k}) \in A^k\) then for all \(P \in \mathbb{C} < x_1, \ldots, x_k, x_1', \ldots, x_k' >\) we have
\[
P(X_{gg_1}, \ldots, X_{gg_k}, X_{g_1}, \ldots, X_{g_k}) = K_g \left( P(X_{g_1}, \ldots, X_{g_k}, X_{g_1}^*, \ldots, X_{g_k}^*) \right).
\]
Therefore we have
\[
\tau \left( P(X_{gg_1}, \ldots, X_{gg_k}, X_{g_1}, \ldots, X_{g_k}) \right) = \tau \left( P(X_{g_1}, \ldots, X_{g_k}, X_{g_1}^*, \ldots, X_{g_k}^*) \right).
\]
Which implies the equality in distribution. If \((X_{g_1}, \ldots, X_{g_k})\) does not belong to \(A^k\), then by definition of \(A_+\) we know that there is a sequence \((p^n_1, \ldots, p^n_k)\) of projectors of \(H_\lambda\) such that:

\[
\begin{align*}
&\bullet (X_{g_1}p_1^n, \ldots, X_{g_k}p_k^n) \in A^k; \\
&\bullet \max_{i \in [k]} \tau(1 - p^n_i) \to 0.
\end{align*}
\]

As \((X_{g_1}p_1^n, \ldots, X_{g_k}p_k^n) \in A\), using the result we just proved the following holds
\[
(X_{g_1}p_1^n, \ldots, X_{g_k}p_k^n) \overset{d}{=} (K_g(X_{g_1}p_1^n), \ldots, K_g(X_{g_k}p_k^n)).
\]
As \(K_g\) is a \(*\)-automorphism the following holds
\[
K_g\left( X_{g_1}p_1^n, \ldots, X_{g_k}p_k^n \right) = \left( K_g(X_{g_1})K_g(p_1^n), \ldots, K_g(X_{g_k})K_g(p_k^n) \right) = \left( X_{gg_1}K_g(p_1^n), \ldots, X_{gg_k}K_g(p_k^n) \right).
\]
Using hypothesis \(H_2\) we have \(\tau(1 - K_g(p_i^n)) = \tau(1 - p_i^n)\); which implies that \(\max_{i \leq k} \tau(1 - K_g(p_i^n)) \to 0\). Moreover we observe that \(K_g(p_i^n)\) is a projector as it satisfies \(K_g(p_i^n)^2 = K_g(p_i^n \times p_i^n) = K_g(p_i^n)\). Therefore using the definition of multivariate distributions we obtain that the first claim of proposition \(1\) holds.

The next goal is to prove that the second part of proposition \(1\) holds. Let \(\{g_i\} \subset \mathbb{G}\) be a dense countable subset of \(\mathbb{G}\). For all \(g' \in \mathbb{G}\) there are sequences \((p^n_{g_i})\) and \((p'^n_{g_i})\) of projectors of \(H_\lambda\) such that

\[
\begin{align*}
&\bullet \tau(1 - p'^n_{g_i}), \tau(1 - p^n_{g_i}) \to 0 \text{ for all } i \in \mathbb{N}; \\
&\bullet (Z_{g_1}p'^n_{g_1}, \ldots, Z_{g_k}p'^n_{g_k}) \overset{d}{=} (Z_{g'g_1}p'^n_{g_1}, \ldots, Z_{g'g_k}p'^n_{g_k}) \text{ for all } k \in \mathbb{N}.
\end{align*}
\]

Then according to [22] theorem 4.10 we know that there is \(K_{g'}\) a \(*\)-automorphism of \(B\) such that \(K_{g'}(Z_{g_i}p^n_{g_i}) = Z_{g'g_i}p'^n_{g_i}\) for all \(i \leq n\) and all \(n \in \mathbb{N}\). As \(K_{g'}\) is a \(*\)-automorphism we have that it is a.e continuous:
\[
K_{g'}(Z_{g_i}) = \lim_{n \to \infty} K_{g'}(Z_{g_i}p^n_{g_i}) = \lim_{n \to \infty} Z_{g'g_i}p'^n_{g_i} = Z_{g'g_i}.
\]
Therefore we obtain that \( \forall i \in \mathbb{N} \) we have \( K_{g'}(Z_{g}) = Z_{g'g_i} \). Using the density of the subset \( \{g_i\} \) we get that \( K_{g'} \) satisfies \( H_1 \) and is such that: \( K_{g'}(Z_g) = Z_{g'g} \) for all \( g \in \mathcal{G} \). This defines a net \( (K_g) \) of \(*\)-automorphisms that satisfy property \((H_2)\) on \( \mathcal{B} \).

\[ \square \]

### E  Proof of proposition \([\text{G}]\)

**Proof.** To prove the desired result we need to check that the net \((X_g)\) verifies condition \((H_3)\). For ease of notation we write \( Z_g := (Y_{g_1}, \ldots, Y_{g_k}) \) for all \( g \in \mathcal{G}^k \). We note that for all \( g^1, \ldots, g^z \in \mathcal{G}^k \) we have:

\[
(Z_{g^1}, \ldots, Z_{g^z}) \overset{d}{=} (Z_{g'g^1}, \ldots, Z_{g'g^z}), \quad \forall g' \in D_k(\mathcal{G}).
\]

Therefore if \( \Phi \in \mathbb{C}(x_1, \ldots, x_k, x_1^*, \ldots, x_k^*) > \) was a polynomial then using the definition of multivariate distributions we would obtain that

\[
(X_{g^1}, \ldots, X_{g^z}) \overset{d}{=} (X_{g'g^1}, \ldots, X_{g'g^z}), \quad \forall g' \in D_k(\mathcal{G}).
\]

In general, by the Stone-Weierstrass theorem as \( \Phi(\cdot) \) is continuous there is a sequence of polynomials \((\Phi_n : \mathcal{A}^k \to \mathcal{A}) \in \mathbb{C}(x_1, \ldots, x_k, x_1^*, \ldots, x_k^*)^N \) such that \( \|\Phi_n(Y_{g_1}, \ldots, Y_{g_k}) - X_g\|_\infty \to 0 \) for all \( g \in \mathcal{G}^k \). This implies that condition \((H_3)\) holds.

\[ \square \]

### F  Proof of proposition \([\text{S}]\)

**Proof.** Let \( g \in \mathcal{G}^k \) we shorthand \( S(g) := \{g_1, \ldots, g_k\} \). As \( \Phi \) is a polynomial we have \( \Phi(Y_{g_1}, \ldots, Y_{g_k}) \in \mathcal{F}_{S(g)}(Y) \). Moreover, using the definition of \( \mathcal{F}_{D(\mathcal{G}^k)}^{\text{tail}} \) one can also prove that \( E_{D}(\cdot) = E(\cdot) \). The results of proposition \([\text{S}]\) are then following directly from the definition of \( \mathcal{N}^k[\cdot|\mathcal{G}] \).

\[ \square \]

### G  Proof of example \([2.1]\) and proposition \([4]\)

We note that example \([2.1]\) is a consequence of proposition \([4]\) Therefore we only present the proof for the latter.

### G.1  Proof of proposition \([4]\)

**Proof.** We remark that we can suppose without loss of generality that the entries of the random matrices \((X^z,m)\) have mean 0 and do so in the following. The first step of the proof consists on establishing the form of \( E_m \) (the non-commutative conditional expectation on \( \mathcal{F}_{Z^z,m}^{\text{tail}}(X^m) \)). We will then use this to upper-bound the free mixing coefficients.

We remark that as \((X^z,m)\) is a stationary random field then according to proposition \([1]\) we can find \((K_{z,m})\) a sequence of \(*\)-automorphisms of \( \mathcal{F}_{Z^z}(X^m) \) such that \( X^z,m = \)
for all \( z \in \mathbb{Z}^r \). Let \( K \subset \mathbb{Z}^r \) be an arbitrary subset and choose an element \( F \in \mathcal{F}_K(X^m) \) then according to theorem 1 we know that:

\[
E_m(F) = \lim_{n \to \infty} \frac{1}{\|n\|^\gamma} \sum_{z \in [n]^r} K_z(F).
\]

Moreover as \( \alpha_m(i) \xrightarrow{i \to \infty} 0 \) then we know that \((X_z^m)\) is an ergodic random field of matrices. Therefore by Linderstrauss pointwise ergodic theorem [20] we know that

\[
\lim_{n \to \infty} \frac{1}{\|n\|^\gamma} \sum_{z \in [n]^r} K_z(F) = (E(F_{i,j})).
\]

Therefore for any element \( F \) in \( \mathcal{F}_{\mathbb{Z}^r}(X^m) \) the non commutative conditional expectation of \( F \) is given by:

\[
E_m(F) = (E(F_{i,j}))_{i,j \leq m}.
\] (31)

Let \( \lambda > 0 \) be a real. We first upper-bound \( \mathbb{R}^\lambda \cdot [\mathbb{Z}^d] \). In this goal, let \( z, z' \in \mathbb{Z}^r \) and \( K \subset \mathbb{Z}^r \) be such that

\[
\min_{z^* \in K} \min_{j \leq d} (|z_j - z^*_j|, |z'_j - z^*_j|) \geq b.
\]

Choose \( \gamma \in \mathbb{C} \) with \( \text{Im}(\gamma) > \lambda \) and write \( f : x \to \text{Im}(\gamma) \cdot [x - \gamma \text{Id}]^{-1} \). We shorthand \( f^K := f\left(\frac{1}{|K|} \sum_{z \in K} X_z^m\right) \) as well as \( f^K := f^K - E_m(f^K) \). The proof will work in two stages: Firstly we show that \( \left\| E(f^K X_z^m f^K X_{z'}^m f^K) \right\|_1 \) is small and then we establish that \( \left\| E(f^K X_z^m E_m(f^K) X_{z'}^m f^K) \right\|_1 \) is also small.

Firstly we remark that by definition of the \( L_1 \) norm we have

\[
\left\| E_m(f^K X_z^m f^K X_{z'}^m f^K) \right\|_1 \leq \sup_{a \in \mathcal{F}_{\mathbb{Z}^r}^{\text{tail}}(X^m)} \tau \left( a f^K X_z^m f^K X_{z'}^m f^K \right)
\]

Choose \( a \in \mathcal{F}_{\mathbb{Z}^r}^{\text{tail}}(X^m) \) such that \( \|a\|_\infty \leq 1 \). We note that by eq. (31) that \( a \) is a deterministic matrix. Let \((\tilde{X}_z^m, \tilde{X}_{z'}^m)\) be a copy of \((X_z^m, X_{z'}^m)\) that is independent
from \((X_{s,m}^z)\). Then by definition of the strong-mixing coefficients we have:

\[
\tau\left( a f^K X_{s,m}^z, fK \tilde{X}_{s,m}^z \right) - \tau\left( a f^K \tilde{X}_{s,m}^z, fK \tilde{X}_{s,m}^z \right) \\
\leq \frac{1}{m} \left| \mathbb{E}\left( \text{Tr}(a f^K X_{s,m}^z fK \tilde{X}_{s,m}^z) \right) - \mathbb{E}\left( \text{Tr}(a f^K \tilde{X}_{s,m}^z fK \tilde{X}_{s,m}^z) \right) \right| \\
\leq \frac{8}{m} \alpha_m[\mathfrak{b}] \frac{1}{2^+} \sqrt{\mathbb{E}\left( \text{Tr}((X_{s,m}^z)^2) \right)} \parallel X_{s,m}^1 \parallel_{L^{2+}}^2 \\
\leq \frac{8}{m} \alpha_m[\mathfrak{b}] \frac{1}{2^+} \sqrt{\mathbb{E}\left( \text{Tr}((X_{s,m}^2)^2) \right)} \parallel X_{s,m}^1 \parallel_{L^{2+}}^2 \\
\leq \frac{8}{m} \alpha_m[\mathfrak{b}] \frac{1}{2^+} \left( \sum_{s \leq N_m} \text{Tr}\left( (A_{s,m}^z)^2 \right) Z_{s,1}^2 \right)^{\frac{1}{2^+}} (\sum_{s \leq N_m} \parallel A_{s,m}^z \parallel_2^2) \\
\leq 8 \parallel Z \parallel_{2^+} \alpha_m[\mathfrak{b}] \frac{1}{2^+} \left( \sum_{s \leq N_m} \parallel A_{s,m}^z \parallel_2^2 \right) \\
\tag{32}
\]

where \(Z\) designates a standard normal random variable and where (a) is a consequence of the equation \(\tau_m(\cdot) = \frac{1}{m} \mathbb{E}(\text{Tr}(\cdot)); \) and where to get (b) we used lemma \[\] and the fact that for any two matrices \(A, B \in M_m(\mathbb{C})\) we have \(\frac{1}{m} \text{Tr}(f^K a f^K A f^K B) \leq \parallel A \parallel_2 \parallel B \parallel_2\). To obtain (c) we used the fact that the matrices \((A_{s,m}^z)\) are orthogonal and where finally to obtain (d) we used the fact that the following inequality holds by Jensen inequality

\[
\left( \sum_{s \leq N_m} \text{Tr}\left( (A_{s,m}^z)^2 \right) Z_{s,1}^2 \right)^{\frac{1}{2^+}} \leq \left( \sum_{s \leq N_m} \text{Tr}\left( (A_{s,m}^z)^2 \right) \right)^{\frac{1}{2^+}} \left( \sum_{s \leq N_m} \text{Tr}\left( (A_{s,m}^z)^2 Z_{s,1}^2 \right) \right)^{\frac{1}{2^+}}
\]

Moreover if we write \(\rho_{s,z}^z = \text{Cov}(Z_{s,z}, Z_{s,z}^z)\) we observe that

\[
\tau\left( a f^K \tilde{X}_{s,m}^z, fK \tilde{X}_{s,m}^z \right) \\
\leq \tau\left( fK a f^K \tilde{X}_{s,m}^z, fK \tilde{X}_{s,m}^z \right) \\
\leq \frac{1}{m} \sum_{i_1 \leq i_4 \leq m} \mathbb{E}\left( (fK a f^K)_{i_1,i_2} \tilde{X}_{i_2,i_3} X_{i_3,i_4} \tilde{X}_{i_4,i_1} \right) \\
\leq \frac{1}{m} \sum_{i_1 \leq i_4 \leq m} \mathbb{E}\left( X_{i_2,i_3} X_{i_3,i_4} \text{Cov}(fK a f^K, fK a f^K)_{i_1,i_2} \right) \\
\leq \frac{1}{m} \sum_{i_1 \leq i_4 \leq m} \sum_{s \leq N_m} A_{s,m}^z X_{s,i_3} X_{s,i_4} \rho_{s,z}^z \text{Cov}(fK a f^K, fK a f^K)_{i_1,i_2} \right)
\]

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where to obtain (a) we used the independence of \((\tilde{X}^{s,m})\) and \((X^{s,m})\), and where (b) is consequence from the fact that by definition of \(f^K\) we have
\[
E\left( (f^K a f^K)_{i_1, i_2} f^K_{j_1, j_2} \right) = E\left( (f^K a f^K)_{i_1, i_2} \right) - E\left( (f^K a f^K)_{j_1, j_2} \right).
\]

Moreover we define
\[
1 \sqrt{|K|} \sum_{z \in K} x^{z,m} = \sum_{s \leq N_m} Z^{K} A^{s,m}.
\]

We remark that the function \(g_1\) and \(g_2\) are differentiable. Indeed for all \(x, x' \in \mathbb{R}^N\) and all \(\epsilon > 0\) we have
\[
\epsilon \left[ \sum_{s \leq N_m} x_s A^{s,m} - \gamma I d \right]^{-1} \left[ \sum_{s \leq N_m} (x_s + \epsilon x'_s) A^{s,m} - \gamma I d \right]^{-1} - \left[ \sum_{s \leq N_m} x_s A^{s,m} - \gamma I d \right]^{-1} \left[ \sum_{s \leq N_m} (x_s + \epsilon x'_s) A^{s,m} - \gamma I d \right]^{-1}
\]
and
\[
\epsilon \left[ \sum_{s \leq N_m} x_s A^{s,m} - \gamma I d \right]^{-1} a \left[ \sum_{s \leq N_m} x_s A^{s,m} - \gamma I d \right]^{-1} - \left[ \sum_{s \leq N_m} x_s A^{s,m} - \gamma I d \right]^{-1} \left[ \sum_{s \leq N_m} (x_s + \epsilon x'_s) A^{s,m} - \gamma I d \right]^{-1} a \left[ \sum_{s \leq N_m} (x_s + \epsilon x'_s) A^{s,m} - \gamma I d \right]^{-1}
\]

To bound \(\text{cov}(\tilde{Z}^{K}_{i_1, i_2}, f^K a f^K)_{i_1, i_2}\) we will exploit the heat equation. To do so we let \((Z^K_i)\) be a sequence of Gaussian vectors with diagonal variance-covariance matrix \(\Sigma^K\) given by \(\Sigma^K_{j,j} = \sigma^2_{K,j}\) where \(\sigma^2_{K,j} = \text{var}(\frac{1}{\sqrt{|K|}} \sum_{z \in K} Z_{i,z})\). We remark that
\[
\frac{1}{\sqrt{|K|}} \sum_{z \in K} x^{z,m} d = \sum_{s \leq N_m} Z^{K} A^{s,m}.
\]
Therefore the gradient of $g_1$ is given by
\[
\nabla_t g_1(x) = g_1(x)A^{1,m}g_1(x).
\]

Similarly we remark that the gradient of $g_2$ is given by
\[
\nabla_t g_2(x) = g_1(x)A^{1,m}g_2(x) + g_2(x)A^{1,m}g_1(x).
\]

We define $Z_{i,z}(t) = tZ_{i,z} + \sqrt{1-t^2}Z_{i,z}$ where $(\tilde{Z}_{i,z})$ is an independent copy of $(Z_{i,z})$. For any subset $K \subset \mathbb{Z}^c$ we write
\[
Z_s^K(t) = \frac{1}{\sqrt{|K|}} \sum_{s \in K} Z_{s,z}(t).
\]

Using the heat equation [19] [section 5.5] we know that
\[
\text{cov}\left(t_{i,s}, U^K_{m} A^K_{i_1,i_2}\right) = \text{Im}(\gamma)^3 \int_0^1 \mathbb{E}\left(\langle \nabla g_1(Z^K)_{i_1,i_2}, \nabla g_2(Z^K(t))_{i_1,i_2}\rangle\right) dt
\]
\[
= \text{Im}(\gamma)^3 \int_0^1 \sum_{s' \leq N_m} \sigma_{K,s}^2 \mathbb{E}\left(\langle \nabla s' g_1(Z^K)_{i_1,i_2}, \nabla s' g_2(Z^K(t))_{i_1,i_2}\rangle\right) dt
\]
\[
= \text{Im}(\gamma)^3 \int_0^1 \sum_{s' \leq N_m} \sigma_{K,s}^2 \mathbb{E}\left((g_1(Z^K)A^{s',m}g_1(Z^K))_{i_1,i_2}\right) dt
\]
\[
+ \text{Im}(\gamma)^3 \int_0^1 \sum_{s' \leq N_m} \sigma_{K,s}^2 \mathbb{E}\left((g_1(Z^K)A^{s',m}g_1(Z^K)) A^{s',m}g_1(Z^K(t))_{i_1,i_2}\right) dt
\]

Therefore we directly obtain that
\[
\left| \tau(q^{K_1} \tilde{X}^{a,m} f^K_1 \tilde{X}^{a',m} f^K_2) \right|
\]
\[
\leq \frac{\text{Im}(\gamma)^3}{m} \int_0^1 \left| \sum_{s,s' \leq N_m} \sigma_{K,s}' \rho_{s,s'}^a \mathbb{E}\left(\text{Tr}\left[g_1(Z^K(t))A^{s',m}g_1(Z^K)A^{s,m}g_1(Z^K(t))A^{s',m}g_1(Z^K)A^{s,m}\right]\right)\right| dt
\]
\[
+ \frac{\text{Im}(\gamma)^3}{m} \int_0^1 \left| \sum_{s,s' \leq N_m} \sigma_{K,s}' \rho_{s,s'}^a \mathbb{E}\left(\text{Tr}\left[g_2(Z^K(t))A^{s',m}g_1(Z^K)A^{s,m}g_1(Z^K)A^{s',m}g_1(Z^K)A^{s,m}\right]\right)\right| dt
\]
\[
\leq \frac{\text{Im}(\gamma)^3}{m} \int_0^1 \left| \sum_{s,s' \leq N_m} \sigma_{K,s}' \rho_{s,s'}^a \mathbb{E}\left(\text{Tr}\left[\sum_{s,s' \leq N_m} \sigma_{K,s}\rho_{s,s'}^a g_1(Z^K(t))A^{s',m}g_2(Z^K(t))A^{s,m}g_1(Z^K)A^{s',m}g_1(Z^K(t))A^{s,m}\right]\right)\right| dt
\]
\[
+ \frac{\text{Im}(\gamma)^3}{m} \int_0^1 \left| \sum_{s,s' \leq N_m} \sigma_{K,s}' \rho_{s,s'}^a \mathbb{E}\left(\text{Tr}\left[\sum_{s,s' \leq N_m} \sigma_{K,s}\rho_{s,s'}^a g_1(Z^K(t))A^{s',m}g_1(Z^K(t))A^{s,m}g_1(Z^K)A^{s',m}g_1(Z^K(t))A^{s,m}\right]\right)\right| dt
\]
\[
\leq 2\text{Im}(\gamma)^{-1} \sup_{Y_1,Y_2,Y_3} \left| \sum_{s,s' \leq N_m} \sigma_{K,s}' \rho_{s,s'}^a A^{s,m}Y_1 A^{s',m}Y_2 A^{s,m}Y_3 A^{s',m}\right|_1
\]
\[
\leq 2\text{Im}(\gamma)^{-1} \sup_{U_1,U_2,U_3 \in U(m)} \left| \sum_{s,s' \leq N_m} \sigma_{K,s}' \rho_{s,s'}^a A^{s,m}U_1 A^{s',m}U_2 A^{s,m}U_3 A^{s',m}\right|_1
\]

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where in (a) we remarked that $\|g_1\|_\infty, \|g_2\|_\infty \leq \text{Im}(\gamma)^{-1}$ and where to obtain (b) we used the fact that every matrix $\|Y\|_\infty \leq 1$ can be written as convex combination of unitary matrix. We then remark that for any choice of $U_1, U_2, U_3$ we have

$$\left\| \sum_{s,s' \leq N_m} \sigma_{K,s}^2 \rho_{z,s}^2 A^{s,m} U_1 A^{s',m} U_2 A^{s',m} U_3 A^{s',m} \right\|_1$$

$$= \left\| \mathbb{E} \left( \tilde{X}^{z,m} U_1 S_K U_2 \tilde{X}^{z',m} U_3 S_K \right) \right\|_1$$

Therefore the value of $\left\| \sum_{s,s' \leq N_m} \sigma_{K,s}^2 \rho_{z,s}^2 A^{s,m} U_1 A^{s',m} U_2 A^{s',m} U_3 A^{s',m} \right\|_1$ only depends on the distribution of $(\tilde{X}^{z,m}, \tilde{X}^{z',m})$ and $S_K$. We remark that $S_K$ can be seen as an $m^2$ dimensional Gaussian vector with the variance-covariance matrix $\Sigma$. It follows that if we write $(C_i)$ the (unnormalized) orthogonal eigenvectors of $\Sigma$ and if $(Z_i)$ is a sequence of i.i.d standard normal random variables then we observe that $S_K = \sum_{i \leq m^2} Z_i C_i$. This directly implies that for all

$$\left\| \sum_{s,s' \leq N_m} \sigma_{K,s}^2 \rho_{z,s}^2 A^{s,m} U_1 A^{s',m} U_2 A^{s',m} U_3 A^{s',m} \right\|_1$$

$$= \left\| \mathbb{E} \left( \tilde{X}^{z,m} U_1 S_K U_2 \tilde{X}^{z',m} U_3 S_K \right) \right\|_1$$

$$= \left\| \sum_{s \leq N_m, s' \leq m^2} \rho_{z,s}^2 A^{s,m} U_1 C_s U_2 A^{s',m} U_3 C_{s'} \right\|_1.$$ 

Now note that by definition we have $\text{Tr}(C_i C_j) = 0$ if $i \neq j$ are different. Therefore we can note that for any matrix $Y$ we have $\sum_{i \leq m^2} \frac{1}{\|C_i\|_{HS}} C_i Y C_i = \text{Tr}(Y) \text{Id}$ (see lemma 4.8 in [44]). Moreover we remark that

$$\left\| \sum_{s \leq N_m, s' \leq m^2} \rho_{z,s}^2 A^{s,m} U_1 C_s U_2 A^{s',m} U_3 C_{s'} \right\|_1$$

$$\leq \sup_{\|x\|, \|y\| \leq 1} \left( \sum_{s' \leq m^2} \|C_{s'} x\|^2 \right)^{1/2} \left( \sum_{s \leq N_m} \|C_s\|^2 \right)^{1/2} \left( \sum_{s \leq N_m} \|C_s\|^2 \right)^{1/2}.$$ 

Note that we have

$$\sum_{s' \leq m^2} \left\| \sum_{s \leq N_m} \rho_{z,s}^2 A^{s,m} U_3 A^{s',m} U_2 C_{s'} U_1 A^{s',m} \right\|^2$$

$$= \sum_{s' \leq m^2} \sum_{s_1, s_2 \leq N_m} \rho_{z,s}^2 \rho_{z,s'}^2 y^* U_3 A^{s_1,m} U_2 C_{s'} U_1 A^{s_1,m} A^{s_2,m} U_1 C_{s'} U_2 A^{s_2,m} U_3 y$$

$$\leq \max_i \|C_i\|_{HS}^2 \sum_{s_1, s_2 \leq N_m} \rho_{z,s}^2 \rho_{z,s'}^2 y^* U_3 A^{s_1,m} A^{s_2,m} U_3 y \text{Tr} \left( A^{s_1,m} A^{s_2,m} \right)$$

$$(a) = \max_i \|C_i\|_{HS}^2 \sum_{s \leq N_m} \left( \rho_{z,s}^2 \right)^2 y^* U_3 A^{s,m} A^{s,m} U_3 y \text{Tr} \left( A^{s,m} \right)^2$$

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where to obtain (b) we used the fact that the matrices \((A^{s,m})\) are orthogonal. Therefore we obtain that
\[
\sup_{\|y\|\leq 1} \left| \sum_{s \leq N} (\rho_{s,x,x'})^2 y^* U_{s} A^{s,m} A^{s,m} U_3 y \text{Tr}((A^{s,m})^2) \right|
\leq \sup_{s \leq N} \text{Tr}((A^{s,m})^2) \sum_{s \leq N} (\rho_{s,x,x'})^2 \|A^{s,m} y\|_2^2
\leq \sup_{s \leq N} \text{Tr}((A^{s,m})^2) \sum_{s \leq N} \|A^{s,m} y\|_2^2
\leq \text{V}_m(X^m)^2 \sigma_m(X^m)^2.
\]
where to get the last inequality we used the fact that by definition we have
\[
\text{V}_m(X^m)^2 = \sup_{\|B\| \leq 1} \sum_{i \leq N} \text{Tr}(A^{i,m} B^2),
\]
and the fact that if we define \(M_s = \frac{A^{s,m}}{\sqrt{\text{Tr}((A^{s,m})^2)}}\) then we remark that \(\text{Tr}(M_s^2) = 1\) and \(\sum_{j \leq N} \text{Tr}(A^{j,m} M_s)^2 = \text{Tr}((A^{s,m})^2)\). Moreover, we remark that we also have:
\[
\max_{i \leq m^2} \|C_i\|_{\text{HS}} \leq \sup_{\|B\| \leq 1} \sum_{s \leq N} \sigma_{K,s}^2 \|\text{Tr}(A^{s,m} B^2)\|
\leq \max_{s \leq N} \sigma_{K,s}^2 \sup_{\|B\| \leq 1} \sum_{s \leq N} \|\text{Tr}(A^{s,m} B^2)\|^2
\leq \max_{s \leq N} \sigma_{K,s}^2 \text{V}_m(X^m)^2.
\]
Moreover according to lemma [7] for all \(s \leq N_m\) we have
\[
\sigma_{K,s}^2 \lesssim \|Z\|_{2+\epsilon}^2 \sum_{b \leq \infty} rb^{f-1} \alpha_m(b) \frac{1}{s^{\gamma^*}}.
\]
This directly implies that
\[
\max_{i \leq m^2} \|C_i\|_{\text{HS}}^2 \lesssim \text{V}_m(X^m)^2 \|Z\|_{2+\epsilon}^2 \sum_{b \leq \infty} rb^{f-1} \alpha_m(b) \frac{1}{s^{\gamma^*}}.
\]
Finally we also remark that
\[
\sup_{\|x\| \leq 1} \sum_{s' \leq m^2} \|C_{s',x}\|^2 = \|E(S^2_K)\|_{\infty}
= \| \sum_{s \leq N_m} \sigma_{K,s}^2 (A^{i,s})^2 \|_{\infty} \lesssim \max_{s \leq N_m} \sigma_{K,s}^2 \sum_{s \leq N_m} (A^{i,s})^2 \|_{\infty}.
\]
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Moreover according to lemma\textsuperscript{7} for all $s \leq N_m$ we have
\[\sigma_{K,s}^2 \lesssim \|Z\|_{2+\epsilon}^2 \sum_{b \leq \infty} rb^{-1} \alpha_m[b] \frac{\sigma_m(X^m)^2}{\epsilon} \]
Therefore we obtain that
\[
\sup_{\|x\| \leq 1} \sum_{s' \leq m^2} \|C_{s',x}\|^2 \lesssim \|Z\|_{2+\epsilon}^2 \sum_{b \leq \infty} rb^{-1} \alpha_m[b] \frac{\sigma_m(X^m)^2}{\epsilon} \]
Therefore we obtain that
\[
\left| \tau(af^K \tilde{X}^{z,m}, f^K \tilde{X}'^{z',m}, f^K) \right| \lesssim \sum_{b \geq 0} b^{r-1} \alpha_m[b] \frac{\sigma_m(X^m)^2}{\epsilon} \sum \epsilon \right]
\]
Combined with eq.\textsuperscript{32} we obtain that
\[
\left| \tau(af^K X^{z,m}, f^K X'^{z',m}, f^K) \right| \lesssim \sum_{s \leq N_m} \left\| A_{s,m} \right\|_2 \alpha_m[b] \frac{\sigma_m(X^m)^2}{\epsilon} \sum \epsilon \right]
\]
Now we bound $\left\| E_m(\overline{f^K X^{z,m} E_m(f^K) X'^{z',m}, f^K}) \right\|_1$. In this goal we write
\[A^{z,z'} := X^{z,m} E_m(f^K) X'^{z',m} \]
Firstly we remark that by definition of the $L_1$ norm we have
\[
\left\| E_m(\overline{f^K X^{z,m} E_m(f^K) X'^{z',m}, f^K}) \right\|_1 \leq \sup_{a \in F_{\text{tail}}^{\text{eq}}(X^m)} \tau(af^K \tilde{A}^{z,z'}, f^K). \]
Choose $a \in F_{\text{tail}}^{\text{eq}}(X^m)$ such that $\|a\|_{\infty} \leq 1$. We note that by eq.\textsuperscript{51} that $a$ is a deterministic matrix. Let $(\tilde{X}^{z,m}, \tilde{X}'^{z',m})$ be a copy of $(X^{z,m}, X'^{z',m})$ that is independent from $(X^{z,m})$ and write $\tilde{A}^{z,z'} := X^{z,m} E_m(f^K) \tilde{X}'^{z',m}$. Then by definition of the strong-
mixing coefficients and by lemma 7 we have:

\[
\left| \tau(af^K A^z A^x f^K - af^K \tilde{A}^z A^x f^K) \right|
\]

\[
\leq \left| \frac{1}{m} \sum_{i_1,i_3 \leq m} \text{cov}(A^z_{i_2,i_3}, (af^K)_i) \right|
\]

\[
\leq \left| \frac{1}{m} \E \left( \text{Tr}(af^K X^z A^x f^K) - \frac{1}{m} \E \left( \text{Tr}(af^K \tilde{X}^z A^x f^K) \right) \right) \right|
\]

\[
\leq \frac{8}{m} \alpha_m [b] \frac{1}{2} \left| \sum_{s \leq m} T r((A^s)^2) Z_{s,2}^2 \right|_{L_{2+}}^2
\]

\[
\leq 8 \| Z \|_2^2 + \alpha_m [b] \frac{1}{2} \left( \sum_{s \leq m} \| A^s \|_2^2 \right)
\]

where \( Z \) designates a standard normal random variable and where (a) is a consequence of the equation \( \tau(\cdot) = \frac{1}{m} \E(\text{Tr}(\cdot)) \); and where to get (b) we used lemma 7 and the fact that \( \| af^K \|_\infty, \| f^K \|_\infty \leq 1 \). Moreover, we remark that

\[
\tau(af^K A^z A^x f^K - af^K \tilde{A}^z A^x f^K) = \tau(af^K A^z A^x f^K)
\]

Finally we remark that

\[
\| X^1, m \|_2^2 = \frac{1}{m} \sum_{i,j \leq m} \E((X^1)_{i,j}^2)
\]

\[
= \frac{1}{m} \sum_{s \leq m} \sum_{i,j \leq m} (A^s_{i,j})^2
\]

\[
= \sum_{s \leq m} \| A^s \|_2^2.
\]

This directly implies that

\[
\alpha_m^b[Z] \lesssim \| Z \|_2 \frac{1}{2} \alpha_m [b] \frac{1}{2} + \frac{1}{m} \| A^s \|_2^2 \sum_{l \geq 0} t^{l-1} \alpha_m [l] \frac{1}{2} \sigma_m(X^m)^2 Y_m(X^m)^2.
\]

Therefore we remark that there is a constant \( C \) that is independent from \( m \) and \( b \) such that:

\[
\alpha_m^b[Z] \leq C \left[ \alpha_m [b] \frac{1}{2} + \frac{\sigma_m(X^m)^2 Y_m(X^m)^2}{\sum_{s \leq m} \| A^s \|_2^2} \sum_{l \geq 0} t^{l-1} \alpha_m [l] \frac{1}{2} \right].
\]
We now move on to bounding $\mathbb{R}_m^s |Z^r|$. Let $z, z' \in \mathbb{Z}^r$ and $K \subset \mathbb{Z}^r$ be such that
\[
\min_{z' \in K \cup \{z\}} d(z', z) \geq b.
\]
Choose $Y_1, Y_2, Y_3 \in F_{\mathbb{Z}^r}^{\text{tail}}(X^m)$ be operators satisfying $\|Y_1\|_\infty, \|Y_2\|_\infty, \|Y_3\|_\infty \leq 1$. Firstly we remark that by definition of the $L_1$ norm we have
\[
\|E_m(Y_1 X^z,m Y_2 X^{z'},m Y_3)\|_1 \leq \sup_{a \in F_{\mathbb{Z}^r}^{\text{tail}}(X^m)} \|a\|_\infty \leq 1.
\]
Choose $a \in F_{\mathbb{Z}^r}^{\text{tail}}(X^m)$ such that $\|a\|_\infty \leq 1$. We note that by eq. (31) that $a$ is a deterministic matrix.

Let $X^z,m$ be a copy of $X^z,m$ that is independent from $(X^z,m)$. Then by definition of the strong-mixing coefficients and lemma 7 we have:
\[
\left\lvert \tau \left( a Y_1 X^z,m Y_2 X^{z'},m Y_3 \right) - \tau \left( a X^z,m Y_2 X^{z'},m Y_3 \right) \right\rvert \leq \frac{1}{m} \sum_{1 \leq i_1, i_2 \leq m} \text{cov} \left( X_{i_1,i_2}^{z,m}, X_{i_3,i_4}^{z',m} (Y_3, a Y_1)_{i_1,i_2} (Y_2)_{i_3,i_4} \right) \lesssim \alpha_m [b] \|Z\|_2^2 + \varepsilon \sum_{s \leq N_m} \left\lVert A^s,m \right\rVert_2^2.
\]
Moreover we remark that $\tau \left( a Y_1 X^z,m Y_2 X^{z'},m Y_3 \right) = 0$. This directly implies that
\[
\left\lVert E_m \left( Y_1 X^z,m Y_2 X^{z'},m Y_3 \right) \right\rVert \lesssim \alpha_m [b] \|Z\|_2^2 + \varepsilon \sum_{s \leq N_m} \left\lVert A^s,m \right\rVert_2^2.
\]

H Proof of proposition 5

Proof. We remark that we can suppose without loss of generality that the entries of the random matrices $(X^z,m)$ have a conditional mean of 0 and do so in the following.

We remark that for all jointly invariant random matrices $X$, $E_m(X)$ is a $m \times m$ random matrix whose diagonal entries are all equal to $\mathbb{E}(X_{1,1} | S(N))$ and whose non-diagonal entries are all equal to $\mathbb{E}(X_{1,2} | S(N))$. For ease of notation, we will write $F_m$ the set of all $m \times m$ matrices that are $\sigma(S(N))$ measurable and have respectively all the diagonal terms and non-diagonal terms equal almost surely.

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We start by bounding $\mathbb{E}_m[|Z|]$. Let $K \subset \mathbb{N}\setminus\{1, 2\}$ and choose $Y_1, Y_2, Y_3 \in \mathcal{F}(X^m)$ be such that $\|Y_1\|_{\infty}, \|Y_2\|_{\infty}, \|Y_3\|_{\infty} \leq 1$. By definition of the non-commutative conditional expectation we know that
\[
\tau\left(E_m \left(Y_1^{1:m}X^{2, m}_2Y_3\right)\right) \\
\leq \sup_{a \in F_m} \tau\left(aY_1^{1:m}X^{2, m}_2Y_3\right)
\]
Let $a \in F_m$ be such that $\|a\|_{\infty} \leq 1$. By definition of $F_m$ we know that $a$ is $\sigma(S(N))$ measurable. Using the conditional independence of $X^{1,m}$ and $(X^{2,m}, Y_1, Y_2, Y_3)$ we obtain that:
\[
\tau\left(aY_1^{1:m}X^{2, m}_2Y_3\right) = \tau\left(aY_1E_m(X^{1,m})Y_2X^{2, m}_3\right) = 0.
\]
Therefore we obtain that $\mathbb{E}_m[1|Z] = 0$. For all $\lambda > 0$, we now want to bound $\mathbb{E}_m^{(\lambda)}[|Z|$. In this goal let $z_1, z_2 \in \mathbb{Z}$ and let $K \subset \mathbb{Z}\setminus\{z_1, z_2\}$. Choose $\gamma \in \mathbb{C}$ with $\Im(\gamma) > \lambda$ and write $f: x \rightarrow \Im(\gamma)(|x| - \gamma \Id)^{-1}$. We shorthand $f^k := f\left(\frac{1}{\sqrt{|K|}}\sum_{z \in K} X^{z,m}\right)$ as well as $\overline{f^k} := f^k - E_m(f^k)$. We first prove that $\|E_m(f^k X^{z_1,m}f^k X^{z_2,m} f^k)\|_1$ is small. In this goal, let $a \in F_m$ be such that $\|a\|_{\infty} \leq 1$ then we have
\[
\tau\left(a f^k X^{z_1,m}_{\sqrt{m}} f^k X^{z_2,m}_{\sqrt{m}} f^k\right) \\
= \frac{1}{m^2} \sum_{i_1,i_4 \leq n} \mathbb{E}\left[\left((f^k a f^k)_{i_1,i_2}X^{z_1,m}_{i_2,i_3}(\overline{f^k})_{i_3,i_4}X^{z_2,m}_{i_4,i_1}S(N)\right)\right] \\
= \frac{1}{m^2} \sum_{i_1,i_4 \leq n} \mathbb{E}\left[\mathbb{E}\left(X^{z_1,m}_{i_2,i_3}X^{z_2,m}_{i_4,i_1}\bigg|S(N)\right)\mathbb{E}\left((f^k a f^k)_{i_1,i_2}(\overline{f^k})_{i_3,i_4}S(N)\right)\right] \\
\leq \left(\frac{2}{m^2}\right) \sum_{i_1,i_3 \leq m} \mathbb{E}\left[\mathbb{E}\left(X^{z_1,m}_{i_2,i_3}X^{z_2,m}_{i_2,i_1}\bigg|S(N)\right)\mathbb{E}\left((f^k a f^k)_{i_1,i_2}(\overline{f^k})_{i_3,i_4}S(N)\right)\right] \\
+ \left(\frac{1}{m^2}\right) \sum_{i_2,i_4 \leq m} \mathbb{E}\left[\mathbb{E}\left(X^{z_1,m}_{i_2,i_3}X^{z_2,m}_{i_4,i_2}\bigg|S(N)\right)\mathbb{E}\left((f^k a f^k)_{i_1,i_2}(\overline{f^k})_{i_3,i_4}S(N)\right)\right] \\
+ \left(\frac{1}{m^2}\right) \sum_{i_2,i_4 \leq m} \mathbb{E}\left[\mathbb{E}\left(X^{z_1,m}_{i_2,i_3}X^{z_2,m}_{i_4,i_2}\bigg|S(N)\right)\mathbb{E}\left((f^k a f^k)_{i_1,i_2}(\overline{f^k})_{i_3,i_4}S(N)\right)\right]
\]
where (a) comes from the conditional independence of $X^{z_1,m}_{i,j}$ and $X^{z_2,m}_{k,j}$ when the indexes are distinct $\{k, j\} \cap \{l, i\} = \emptyset$ (which is a consequence of the joint exchangeability of the entries). Firstly we can remark that as $\|f^k a f^k\|_{\infty}, \|f^k\|_{\infty} \leq 1$ then we have
\[
\frac{1}{m^2} \sum_{i_1,i_3 \leq m} \mathbb{E}\left[\mathbb{E}\left(X^{z_1,m}_{i_2,i_3}X^{z_2,m}_{i_2,i_1}\bigg|S(N)\right)\mathbb{E}\left((f^k a f^k)_{i_1,i_2}(\overline{f^k})_{i_3,i_4}S(N)\right)\right] \\
\leq \frac{1}{m^2} \sup_{i,j \leq m} \|X^{z,m}_{i,j}\|_2.
\]
Moreover we remark that the following decomposition holds:

\[
\frac{1}{m^2} \sum_{i_2, i_4 \leq m} \mathbb{E} \left[ \mathbb{E} \left( X_{i_2, i_3}^{z_1, m} X_{i_4, i_2}^{z_2, m} \mid S(N) \right) \mathbb{E} \left( (f^K a f^K)_{i_3, i_4} (f^K)_{i_2, i_2} \mid S(N) \right) \right] \\
= \frac{1}{m^2} \sum_{i_2 \leq m} \sum_{i_3 \neq i_2} \mathbb{E} \left[ \mathbb{E} \left( X_{i_2, i_3}^{z_1, m} X_{i_4, i_2}^{z_2, m} \mid S(N) \right) \mathbb{E} \left( (f^K a f^K)_{i_3, i_4} (f^K)_{i_2, i_2} \mid S(N) \right) \right] \\
+ \frac{1}{m^2} \sum_{i_2 \leq m} \sum_{i_3, i_4 \leq m} \mathbb{E} \left[ \mathbb{E} \left( X_{i_2, i_3}^{z_1, m} X_{i_4, i_2}^{z_2, m} \mid S(N) \right) \mathbb{E} \left( (f^K a f^K)_{i_3, i_4} (f^K)_{i_2, i_2} \mid S(N) \right) \right].
\]

We will bound each term successively. Firstly we remark that

\[
\sum_{i_2 \leq m} \sum_{i_3 \neq i_2} \mathbb{E} \left[ \mathbb{E} \left( X_{i_2, i_3}^{z_1, m} X_{i_4, i_2}^{z_2, m} \mid S(N) \right) \mathbb{E} \left( (f^K a f^K)_{i_3, i_4} (f^K)_{i_2, i_2} \mid S(N) \right) \right] \\
= \mathbb{E} \left[ \mathbb{E} \left( X_{1, 2}^{z_1, m} X_{1, 3}^{z_2, m} \mid S(N) \right) \operatorname{cov} \left( \sum_{i_3, i_4 \leq m} (f^K a f^K)_{i_3, i_4}, \sum_{i_2 \leq m} f^K_{i_2, i_2} \mid S(N) \right) \right] \\
- \mathbb{E} \left[ \mathbb{E} \left( X_{1, 2}^{z_1, m} X_{1, 3}^{z_2, m} \mid S(N) \right) \sum_{i_2 \leq m} \sum_{i_3, i_4 \leq m} \operatorname{cov} \left( \sum_{i_2 \leq m} f^K_{i_2, i_2}, (f^K a f^K)_{i_3, i_4} \mid S(N) \right) \right].
\]

We bound each term successively. We write \( S_K := \frac{1}{\sqrt{|K||m|}} \sum_{z \in K} X^{z, m} \) and for all \( i, j \leq m \) we write \( \tau_{i,j} \) the permutation that permutes \( \{i; j\} \) and leaves all the other indexes invariant. We define \( X^{z, m, i,j} := (X_{\tau_{i,j}(1), \tau_{i,j}(2)}^{z, m})_{i_1, i_2 \leq m} \) and set \( S_{z, i,j} := \frac{1}{\sqrt{|K||m|}} \sum_{z \in K} X^{z, m, i,j} \).

By Effron Stein inequality we remark that

\[
\mathbb{E} \left[ \text{var} \left( \sum_{i_2 \leq m} f^K_{i_2, i_2} \mid S(N) \right) \right] \\
\leq \sum_{j \leq m} \mathbb{E} \left[ \left( \sum_{i_2 \leq m} f^K_{i_2, i_2} - \sum_{i_2 \leq m} f^K_{i_2, i_2} (S_{z, K}^{1, m+1}) \right)^2 \mid S(N) \right] \\
\leq m \mathbb{E} \left[ \left( \text{Tr} \left( f^K (S_{z, K}^{1, m+1}) \right) \right)^2 \right] \\
\leq m \Im(\gamma)^{-1} \mathbb{E} \left[ \left( \text{Tr} \left( f^K (S_{z, K}^{1, m+1}) \right) \right)^2 \right] \\
\leq 2 m \Im(\gamma)^{-1} \mathbb{E} \left[ \left( \text{Tr} \left( (S_{z, K}^{1, m+1}) \right) \right)^2 \right] \\
\leq 4 m \Im(\gamma)^{-1} \mathbb{E} \left( \sum_{i \leq m} (S_{z, K}^{i, i} - S_{z, K}^{i, i+1})^2 \right) \\
\leq 4 \Im(\gamma)^{-1} \mathbb{E} \left( \sum_{i \leq m} \left( \frac{1}{|K|} \sum_{z \in K} X_{i, 1}^{z, m} - X_{i, m+1}^{z, m} \right)^2 \right) \\
\leq 16 m \Im(\gamma)^{-1} \sup_{i,j \leq m} \|X_{i, j}^{1, m}\|_2^2
\]
where \((a_1)\) is a consequence of the Effron-Stein inequality, where \((a_2)\) is a consequence of the exchangeability and where to get \((a_3)\) we used the fact that by definition \(f^K\) is a resolvent. To obtain \((a_4)\) we used the fact that if we write \((\lambda_i^K)\) the \((\text{random})\) eigenvalues of \(f^K\) then we have \(\max_j |\lambda_j^K| \leq 1\). Finally \((a_5)\) is a consequence of the fact that the rank of \(|S_K - S_{K,m+1}^m|\) is less than 2.

Therefore by Cauchy-Schwarz we obtain that
\[
\left| \mathbb{E} \left[ \mathbb{E}(X_{1,2}^{1,m} X_{1,3}^{1,m} | S(N)) \right] \right|
\leq \sup_{i,j \leq m} \|X_{i,j}^{1,m}\| \sqrt{\mathbb{E} \left[ \var\left( \sum_{i_2 \leq m} f^K_{i_2,i_2} \right) \right] \mathbb{E} \left[ \left( \sum_{i_3,i_4 \leq m} (f^K a f^K)_{i_3,i_4} \right) \right]}
\leq 4(m)^{3/2} \sqrt{\text{Im}(\gamma)^{-1}} \sup_{i,j \leq m} \|X_{i,j}^{1,m}\|^{3/2}
\]

where to get \((a)\) we used the fact that \(\|f^K a f^K\|_\infty \leq 1\).

In addition we observe that
\[
\left| \mathbb{E} \left[ \mathbb{E}(X_{1,2}^{1,m} X_{1,3}^{1,m} | S(N)) \right] \right|
\leq \left| \sum_{i_2 \leq m} \sum_{i_3,i_4 \leq m} \mathbb{E} \left[ \mathbb{E}(X_{1,2}^{1,m} X_{1,3}^{1,m} | S(N)) \right] \mathbb{E} \left[ (f^K a f^K)_{i_3,i_4} \right] \mathbb{E} \left[ (f^K a f^K)_{i_2,i_2} | S(N) \right] \right|
\leq 4m \sup_{i,j \leq m} \|X_{i,j}^{1,m}\|^{2}.
\]

Similarly by exploiting the fact that \(\|f^K\|_\infty, \|f^K a f^K\|_\infty \leq 1\) then we have
\[
\sum_{i_2 \leq m} \sum_{i_3,i_4 \leq m} \mathbb{E} \left[ \mathbb{E}(X_{i_2,i_3}^{1,m} X_{i_4,i_2}^{1,m} | S(N)) \right] \mathbb{E} \left[ (f^K a f^K)_{i_3,i_4} \right] \mathbb{E} \left[ (f^K a f^K)_{i_2,i_2} | S(N) \right]
\leq \left| \sum_{i_2 \leq m} \sum_{i_3 \leq m} \sum_{i_4 \leq m} \mathbb{E} \left[ \mathbb{E}(X_{i_2,i_3}^{1,m} X_{i_4,i_2}^{1,m} | S(N)) \right] \mathbb{E} \left[ (f^K a f^K)_{i_3,i_4} \right] \mathbb{E} \left[ (f^K a f^K)_{i_2,i_2} | S(N) \right] \right|
\leq 4m \sup_{i,j \leq m} \|X_{i,j}^{1,m}\|^{2}.
\]

Therefore we obtain that
\[
\frac{1}{m^2} \sum_{i_2,i_4 \leq m} \mathbb{E} \left[ \mathbb{E}(X_{i_2,i_3}^{1,m} X_{i_4,i_2}^{1,m} | S(N)) \right] \mathbb{E} \left[ (f^K a f^K)_{i_3,i_4} \right] \mathbb{E} \left[ (f^K a f^K)_{i_2,i_2} | S(N) \right]
\leq \frac{4}{m^2} \sup_{i,j \leq m} \|X_{i,j}^{1,m}\|^{3} + \frac{8}{m} \sup_{i,j \leq m} \|X_{i,j}^{1,m}\|^{2}.
\]

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Similarly we can prove that
\[
\frac{1}{m^2} \sum_{i_2 \leq m} \mathbb{E} \left[ \mathbb{E} \left( X_{i_2, i_3}^{1, m} X_{i_4, i_5}^{1, m} | S(\mathcal{N}) \right) \mathbb{E} \left( (fK a fK)^{i_2, i_3} (fK)^{i_4, i_5} | S(\mathcal{N}) \right) \right] \\
\leq \frac{4}{\sqrt{m}} \|X_{i,j}^{1, m}\|_2^2 + \frac{8}{m} \sup_{i,j \leq m} \|X_{i,j}^{1, m}\|_2^2.
\]

We now want to prove that if we write \(A^{z_1, z_2} := X^{z_1, m} E_m(fK) X^{z_2, m}\) and shorthand \(A^{z_1, z_2} := A^{z_1, z_2} - E_m(A^{z_1, z_2})\) then the following is small \(\|E_m(fK A^{z_1, z_2} fK)\|_1\). To see this we first remark that \(A^{z_1, z_2} = E_m(A^{z_1, z_2} | S(\mathcal{N}))\) and note that \(A^{z_1, z_2}\) is conditionally independent (on \(\sigma(S(\mathcal{N}))\)) from \(fK\). Therefore we directly obtain that
\[
\|E_m(fK A^{z_1, z_2} fK)\|_1 = 0.
\]

Therefore as we have assumed that \(\sup_m \frac{\sup_{i,j \leq m} \|X_{i,j}^{1, m}\|_2}{\min_{i,j \leq m} \|X_{i,j}^{1, m}\|_2} < \infty\) we obtain that there is a constant \(C\) such that
\[
\mathbb{R}_n^{\lambda}[1|Z] \leq C \left[ \frac{\sup_{i,j \leq m} \|X_{i,j}^{1, m}\|_2}{\sqrt{m}} + \frac{1}{m} \right].
\]

\[\square\]

I Proof of proposition 9

Proof. We remark that we can suppose without loss of generality that the entries of the random matrices \((X^{z,m})\) have mean 0 and do so in the following. We denote \(E_{D_m}\) the non-commutative conditional expectation on \(\mathcal{F}^{\text{tail}}_{D(\mathbb{Z})}(Y^{m})\). We remark that for any \(F \in \mathcal{F}_{G}^{\delta}(Y^{m})\), we have
\[
E_{D_m}(F) = (\mathbb{E}(F_{i,j}))_{i,j \leq m}.
\]

Let \(\lambda > 0\). We first upper-bound \(\mathbb{R}^{\star,j, \lambda}[\cdot|Z^2]\). In this goal, let \(z, z' \in \mathbb{Z}^2\) and \(K \subset \mathbb{Z}^2\) be such that
\[
\min_{z \in K} \min_{j \leq 2} (|z_j - z'_j|, |z'_j - z_j'|) \geq b.
\]

Choose \(\gamma \in \mathbb{C}\) with \(\text{Im}(\gamma) > \lambda\) and write \(f : x \rightarrow \text{Im}(\gamma) [x - \gamma \text{Id}]^{-1}\). We shorthand \(fK := f \left( \frac{1}{\sqrt{|K|}} \sum_{z \in K} X_{z,m} \right)\) as well as \(fK := fK - E_{D_m}(fK)\). The proof will work in two stages: Firstly we show that \(\|E_{D_m}(fK X_{z,m} fK X_{z',m} fK)\|_1\) is small and then we establish that \(\|E_{D_m}(fK X_{z,m} fK X_{z',m} fK)\|_1\) is also small. In this goal, we remark that by definition of the \(L_1\) norm we have
\[
\|E_{D_m}(fK X_{z,m} fK X_{z',m} fK)\|_1 \leq \sup_{a \in \mathcal{F}^{\text{tail}}_{D(\mathbb{Z})}(Y^{m})} \tau \left( a fK X_{z,m} fK X_{z',m} fK \right)
\]

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Choose \( a \in \mathcal{F}^{\text{tail}}_{D(\mathbb{Z})}(Y^m) \) such that \( \|a\|_{\infty} \leq 1 \). We note that by eq. (35) that \( a \) is a deterministic matrix. Let \( \tilde{X}^{z,m}, \tilde{X}'^{z,m} \) be a copy of \( (X^{z,m}, X'^{z,m}) \) that is independent from \( (X^{z,m}) \). Then by definition of the strong-mixing coefficients and lemma 7, we have
\[
|\tau(a f^K X^{z,m} f^K X'^{z,m} f^K) - \tau(a f^K \tilde{X}^{z,m} f^K \tilde{X}'^{z,m} f^K)| \\
\leq \frac{1}{m} E \left( \left| \text{Tr}(a f^K X^{z,m} f^K X'^{z,m} f^K) - \text{Tr}(a f^K \tilde{X}^{z,m} f^K \tilde{X}'^{z,m} f^K) \right| \right) \\
\leq \frac{1}{m} \alpha_m(b) \sum_{l=1}^{m} \left( \| \text{Tr}(a f^K X^{z,m} f^K X'^{z,m} f^K) \|_{1+\frac{1}{l}} + \| \text{Tr}(a f^K \tilde{X}^{z,m} f^K \tilde{X}'^{z,m} f^K) \|_{1+\frac{1}{l}} \right)
\]
where (a) is a consequence of the equation \( \tau_m(\cdot) = \frac{1}{m} E(\text{Tr}(\cdot)) \); (b) comes from lemma 7 and where to get (c) we used the fact that for any matrix \( A, B \in M_m(\mathbb{C}) \) by Hölder inequality the following holds
\[
\frac{1}{m} |\text{Tr}(f^K a f^K A f^K B)| \leq 2 \|A\|_2 \|B\|_2
\]
Moreover we observe that by Jensen inequality we have:
\[
E(\text{Tr}((X^{z,m})^2)^{1+\frac{1}{l}}) \leq m^l E \left( \sum_{i,j \leq m} (X_{i,j}^{z,m})^{2+\epsilon} \right) \\
\leq \sup_{l=1}^{m} \sup_{\epsilon \in \mathbb{R}} \sum_{i,j \leq m} E \left( (Y_{i,j}^{z,m})^{2+\epsilon} \right)
\]
where to get (a) we used the fact that \( \Phi \) is a polynomial of degree \( p \). As \( (Z_{s,z}) \) are independent normal then using Rosenthal inequality we obtain that for all \( l \leq p \)
\[
E \left( (Y_{i,j}^{z,m})^{l(2+\epsilon)} \right) \leq \left( \sum_{s \leq N_m} (A_{i,j}^{s,m})^{2(l(2+\epsilon))} \right) + \sum_{s \leq N_m} (A_{i,j}^{s,m})^{l(2+\epsilon)}.
\]
Therefore we obtain that
\[
|\tau(a f^K X^{z,m} f^K X'^{z,m} f^K) - \tau(a f^K \tilde{X}^{z,m} f^K \tilde{X}'^{z,m} f^K)| \\
\leq \alpha_m(b) \sup_{l \leq p} \left( \frac{1}{m^{1-\epsilon/2}} \sum_{i,j} \left( \sum_{s \leq N_m} (A_{i,j}^{s,m})^{2(l(2+\epsilon))} \right) + \sum_{s \leq N_m} (A_{i,j}^{s,m})^{l(2+\epsilon)} \right)^{\frac{1}{2+\epsilon}}
\]
(37)
Moreover we observe that
\[
\left| \tau\left(af^K X^{z,m}f^K X^{z',m}f^K\right) \right|
\leq \left| \tau\left(f^K_0 af^K X^{z,m}f^K X^{z',m}\right) \right|
\leq \frac{1}{m} \sum_{i_1,i_2 \leq m} E\left(\left(f^K_0 af^K\right)_{i_1,i_2} X^{z,m}_{i_1,i_3} X^{z',m}_{i_4,i_1}\right)
\leq \left(\frac{a}{m}\right) \sum_{i_1,i_2 \leq m} E\left(X^{z,m}_{i_2,i_3} X^{z',m}_{i_4,i_1}\right) E\left(\left(f^K_0 af^K\right)_{i_1,i_2} f^K_{i_3,i_4}\right)
\leq \left(\frac{b}{m}\right) \sum_{i_1,i_2 \leq m} E\left(X^{z,m}_{i_2,i_3} X^{z',m}_{i_4,i_1}\right) \text{cov}\left(f^K_{i_3,i_4}, \left(f^K_0 af^K\right)_{i_1,i_2}\right)
\]

where to obtain (a) we used the independence of \((\tilde{X}^{z,m})\) and \((X^{z,m})\), and where (b) is consequence from the fact that by definition of \(f^K\) we have
\[
E\left(\left(f^K_0 af^K\right)_{i_1,i_2} f^K_{i_3,i_4}\right)
= E\left(\left(f^K_0 af^K\right)_{i_1,i_2} f^K_{i_3,i_4}\right) - E\left(\left(f^K_0 af^K\right)_{i_1,i_2} f^K_{i_3,i_4}\right)
= \text{cov}\left(f^K_{i_3,i_4}, \left(f^K_0 af^K\right)_{i_1,i_2}\right).
\]

To bound \(\text{cov}\left(f^K_{i_3,i_4}, \left(f^K_0 af^K\right)_{i_1,i_2}\right)\) we will exploit the heat equation. To do so, we define
\[
g_1 : x \to \frac{1}{|K|^{3/2}} \sum_{i,j \in K} \Phi_m\left(\sum_{s \leq N_m} x_{s,i} A^{s,m}, \sum_{s \leq N_m} x_{s,j} A^{s,m}\right) - \gamma \text{Id}\]
and
\[
g_2 : x \to g_1(x)ag_1(x).
\]
For ease of notation, we will write \(Z^K := (Z_{s,i})_{s \leq N_m, i \in K}\) and let \((Z'_{s,i})\) be an independent copy of \((Z_{s,i})\). We define \(Z_{s,i}(t) = \sqrt{t}Z_{s,i} + \sqrt{1-t^2}Z'_{s,i}\) and \(Z^K(t) := (Z_{s,i}(t))_{s \leq N_m, i \in K}\). Moreover we also write \(Y_{s,i}(t) := \sum_{s \leq N_m} Z_{s,i}(t) A^{s,m}\). As \(\Phi_m(\cdot, \cdot)\) is a polynomial, there is a polynomial \(\Phi'_m(\cdot, \cdot, \cdot)\) of degree 1 in its last coordinate such that such that
\[
\frac{\Phi_m(A + H, B) - \Phi'_m(A, B, H)}{\|H\|} \rightarrow 0, \quad \forall A, B, H \in M_m(\mathbb{C}). \tag{38}
\]
As \(\Phi'_m(A, B, H)\) is of degree 1 in \(H\) there is an integer \(M \in \mathbb{N}\) and polynomials \((P_{1,l}(\cdot, \cdot))\) and \((P_{2,l}(\cdot, \cdot))\) such that:
\[
\Phi'_m(A, B, H) = \sum_{l \leq M} P_{1,l}(A, B) H P_{2,l}(A, B).
\]

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We write
\[ \hat{P}_{1,l}^{K,i}(t) = \frac{1}{|K|} \sum_{j \in K} P_{1,l}(Y_{i,m}^{j}(t), Y_{j,m}^{j}(t)) \]
and
\[ \hat{P}_{2,l}^{K,i}(t) = \frac{1}{|K|} \sum_{j \in K} P_{2,l}(Y_{i,m}^{j}(t), Y_{j,m}^{j}(t)). \]
For ease of notation when \( t = 0 \) we shorthand \( \hat{P}_{1,l}^{K,i} := \hat{P}_{1,l}^{K,i}(0) \) and \( \hat{P}_{2,l}^{K,i} := \hat{P}_{2,l}^{K,i}(0) \). We remark that for all \( (x_{s,i}) \) and \( (h_{s,i}) \) the following holds
\[ g_{1}(x + h) - g_{1}(x) = g_{1}(x) \left[ \frac{1}{|K|^{3/2}} \sum_{i,j \in K} \Phi_{m} \left( \sum_{s \leq N_{m}} (x_{s,i} + h_{s,i})A_{s,m}^{s,m}, (x_{s,j} + h_{s,j})A_{s,m}^{s,m} \right) \right] - \frac{1}{|K|^{3/2}} \sum_{i,j \in K} \Phi_{m} \left( \sum_{s \leq N_{m}} x_{s,i}A_{s,m}^{s,m}, x_{s,j}A_{s,m}^{s,m} \right) \]
Therefore using the symmetry of \( \Phi_{m}(\cdot, \cdot) \) we remark that \( g_{1} \) is differentiable and that
\[ \nabla_{i,s} g_{1}(Z^{K}) = \frac{2}{|K|^{1/2}} g_{1}(Z^{K}) \left[ \sum_{l \leq M} \hat{P}_{1,l}^{K,i} A_{s,m}^{s,m} \hat{P}_{2,l}^{K,i}(t) \right] g_{1}(Z^{K}) \]
In the same way, we remark that the function \( g_{2} \) is also differentiable and that the following identity holds
\[ \nabla_{i,s} g_{2}(Z^{K}(t)) = \frac{2}{|K|^{1/2}} g_{2}(Z^{K}(t)) \left[ \sum_{l \leq M} \hat{P}_{1,l}^{K,i} A_{s,m}^{s,m} \hat{P}_{2,l}^{K,i}(t) \right] g_{1}(Z^{K}(t)) \]
\[ + \frac{2}{|K|^{1/2}} g_{1}(Z^{K}(t)) \left[ \sum_{l \leq M} \hat{P}_{1,l}^{K,i} A_{s,m}^{s,m} \hat{P}_{2,l}^{K,i}(t) \right] g_{2}(Z^{K}(t)). \]
We write \( \rho_{1,2,s'} = \text{Cov}(Z_{s',1}, Z_{s',2}) \). Using the heat equation [19] section 5.5] we know that
\[ \text{cov} \left( f_{K_{1},i}^{K_{1},i}, (e^{K_{1}}A_{i}^{K_{1}})_{i,1,2} \right) \]
\[ = \text{Im}(\gamma)^{3} \int_{0}^{1} \sum_{1 \leq l, 1 \leq K} \sum_{s' \leq N_{m}} \rho_{1,2,s'} E \left[ \nabla_{s',1} g_{1}(Z^{K})_{1,i,1} \nabla_{s',2} g_{2}(Z^{K}(t))_{1,i,2} \right] \]
\[ = \frac{4}{K} \text{Im}(\gamma)^{3} \int_{0}^{1} \sum_{s' \leq N_{m}} \sum_{l, 1 \leq K} \rho_{1,2,s'} E \left[ g_{1}(Z^{K}) \left( \sum_{l \leq M} \hat{P}_{1,l}^{K,i} A_{s'}^{s',m} \hat{P}_{2,l}^{K,i}(t) \right) g_{1}(Z^{K}(t))_{1,i,1} \right] \]
\[ + \frac{4}{K} \text{Im}(\gamma)^{3} \int_{0}^{1} \sum_{s' \leq N_{m}} \sum_{l, 1 \leq K} \rho_{1,2,s'} E \left[ g_{2}(Z^{K}(t)) \left( \sum_{l \leq M} \hat{P}_{1,l}^{K,i} A_{s'}^{s',m} \hat{P}_{2,l}^{K,i}(t) \right) g_{1}(Z^{K}(t))_{1,i,2} \right] dt. \]
We can also use the heat equation \[19\] [section 5.5] to reexpress \(\text{cov}(X_{12,i_1}^z, X_{14,i_1}^z, X_{14,i_1}^z, X_{12,i_1}^z)\). In this goal for all \(i, j \in \mathbb{N}\) we can write

\[
H_{i,j} : x \rightarrow \Phi_m \left( \sum_{s \leq N_m} x_{s,i} A^{s,m}, \sum_{s \leq N_m} x_{s,j} A^{s,m} \right).
\]

We remark that \(H_{i,1}(Z) = X(z_1, z_2)^m\) and \(H_{z_1, z_2}(Z) = X(z_1, z_2)^m\). We also note that \(H_{i,j}\) is differentiable and for all \(l \in \mathbb{N}\) we have

\[
\nabla_{l,z} H_{i,j}(Z) = \begin{cases} \sum_{l \leq M} P_{1,l}(Y, z)^m P_{2,l}(Y, z)^m & \text{if } l = i \\ \sum_{l \leq M} P_{1,l}(Y, z)^m P_{2,l}(Y, z)^m & \text{if } l = j \\ 0 & \text{otherwise.} \end{cases}
\]

We write \(\sum_{l \leq M} P_{1,l}^{k_1:k_2:k_3}(t) A^{s,m} P_{2,l}^{k_1:k_2:k_3}(t) := \nabla_{k_1,k_2} H_{k_1,k_2}(Z(t))\). Using the heat equation we also obtain that

\[
\text{Cov} \left[ X_{12,i_1}^z, X_{14,i_1}^z, X_{14,i_1}^z, X_{12,i_1}^z \right] = \int_0^1 \sum_{s \leq N_m} \sum_{l_1 \in \{z_1, z_2\}} \sum_{l_2 \in \{z_1', z_2'\}} \rho_{l_1,l_2,s} \mathbb{E} \left( \nabla_{l_1,l_1} \left( H_{z_1, z_1}'(Z(t)) \nabla_{z_1, z_1}'(Z(t)) \right) dt \right)
\]

We will designate by \(\tilde{P}_{1,l}^{z_1,z_2,m}(t) A^{s,m} \tilde{P}_{2,l}^{z_1,z_2,m}(t)\) independent (from \(X_{12}^z\)) copies of respectively \(P_{1,l}^{k_1:k_2:k_3}\) and \(P_{2,l}^{k_1:k_2:k_3}(t)\). For a matrix \(A \in M_n(\mathbb{C})\) we write \(\sigma(A) := \|A\|_\infty\) the spectral norm of \(A\). Using eq. (40) and eq. (39) we obtain that

\[
\left| \tau \left( a_j^f K \tilde{X}_{z_1, z_2, m}^{fK} \tilde{X}_{z_1', z_2', m}^{fK} \right) \right| \lesssim \text{Im}(\gamma) \int_{[0,1]^2} \sum_{s \leq N_m} \sum_{l_1 \in \{z_1, z_2\}} \sum_{l_2 \in \{z_1', z_2'\}} \frac{\left| \rho_{l_1,l_2,s} \right|}{|K|} \mathbb{E} \left( \text{Tr} \left( \mathbb{E} \left[ g_2(Z(t)) A^{s,m} \tilde{P}_{2,l}^{k_1,k_2}(t) g_1(Z(t)) \right] \right) \right)
\]

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We remark that \( g_i(Z^k(t)) \frac{\dot{P}_{1,l}^{K,i}}{\sigma(g_i(Z^k(t)) \frac{\dot{P}_{1,l}^{K,i}}{\sigma(z^k)}} \) is almost surely a matrix with a spectral radius of less than 1. Similar statements can be made for the other random matrices. We, therefore, obtain that

\[
\begin{align*}
\left| \tau \left( a_f^{K} \bar{X}_{z_1} \sigma(\bar{X}_{z_2} m f K) \right) \right| &\lesssim 2M^4 \text{Im(}\gamma)^{-1} \sup_{l \leq M} \left\| \sigma(\hat{P}_{1,1}^{K,k}) \sigma(\hat{P}_{2,1}^{K,k}) \right\|_{4}^2 \sup_{l \leq M} \left\| \sigma(\hat{P}_{1,1}^{K,k}) \sigma(\hat{P}_{2,1}^{K,k}) \right\|_{4}^2 \\
&\times \sup_{Y_{1,4}} \left\| \sum_{s,t \leq N_m} \sum_{l_1,l_2 \in K} \frac{\rho_{l_1,2} \rho_{l_2,4,s}^{(K)}}{K} A^{s,m} Y_{1,4}^{s,m} Y_{2}^{s,m} Y_{3}^{s,m} Y_{4}^{s,m} \right\|_{1}
\end{align*}
\]

where to obtain (a) we used the fact that every matrix \( \|Y\|_{\infty} \leq 1 \) can be written as a convex combination of unitary matrices. We bound each term successively. Firstly we remark for any matrix \( A \in M_n(\mathbb{R}) \) we have that \( \sigma(A) \leq \max_{i,j} |A_{i,j}|. \) Moreover as the polynomials \( (P_{1,l})_{l \leq M} \) and \( (P_{2,l})_{l \leq M} \) are all of degree \( p - 1 \) for all \( l \leq M \) and all \( i \in K \) we have

\[
\max(\sigma(\hat{P}_{1,l}^{K,i}), \sigma(\hat{P}_{2,l}^{K,i})) \lesssim \left[ \sup_{s \leq N_m} |Z_{s,i}|^{p-1} \wedge 1 \right] + \frac{1}{K} \sum_{j \in K} \sup_{s \leq N_m} |Z_{s,j}|^{p-1} \wedge 1.
\]

For ease of notation we write \( s_{m,p-1} := \max_{i,j} \left( \sum_{s \leq N_m} |A_{i,j}^{s,m}| \right)^{p-1} \wedge 1. \) Moreover similarly for all \( z^* \in \{z, z'\} \) and all \( k \in \{z^*_1, z^*_2\} \) we also have

\[
\max(\sigma(\hat{P}_{1,1}^{K,k}), \sigma(\hat{P}_{2,1}^{K,k})) \lesssim \left( \sup_{s \leq N_m} |Z_{s,k}|^{p-1} \wedge 1 \right) s_{m,p-1}.
\]

Moreover we know that \( (Z_{s,i}) \) are i.i.d standard normal random variables therefore for every \( i \in \mathbb{N} \) the following holds

\[
E(\max_{s \leq N_m} |Z_{s,i}|^{p-1}) \lesssim \sqrt{1 + \log(N_m)}^{p-1}.
\]
Therefore using Cauchy-Swartz inequality we obtain that
\[
2M^1 \sup_{l \leq M} \sigma(P_{1,l}^k) \sigma(P_{2,l}^k) \leq \sup_{l \leq M, k \in K} \sigma(P_{1,l}^k \sigma(P_{2,l}^k) \leq \sup_{l \leq M, \mathbf{z} \in \{\mathbf{z}, \mathbf{z}'\}} \sigma(P_{1,l}^k \sigma(P_{2,l}^k)
\]
\[
\leq s_{m,p-1}(1 + \log(N_m))^{2(p-1)}.
\]

We now move on to upper-bounding
\[
\sup_{U_{1:4} \in U(m)} \left\| \sum_{s,s' \leq N_m} \sum_{l_1,l_2 \in K} \frac{|\rho_{1:2,s'}| |\rho_{3:4,s}|}{|K|} A^{s,m} U_1 A^{s',m} U_2 A^{s,m} U_3 A^{s',m} U_4 \right\|_1.
\]

In this goal, we write \( T_{K,s'}^2 := \sum_{l_1,l_2 \in K} \frac{|\rho_{1:2,s'}|}{|K|} \) and denote \( \tilde{\rho}_s := \sum_{l_1 \in \{\mathbf{z}, \mathbf{z}'\}} |\rho_{3:4,s}| \). Let \( (N_i) \) be a sequence of standard normal random variables and denote \( D_K := \sum_{s \leq N_m} T_{K,s} N_s A^{s,m} \).

We then remark that for any choice of \( U_1, U_2, U_3, U_4 \) we have
\[
= \left\| \sum_{s,s' \leq N_m} \sum_{l_1,l_2 \in K} \frac{|\rho_{1:2,s'}| |\rho_{3:4,s}|}{|K|} A^{s,m} U_1 A^{s',m} U_2 A^{s,m} U_3 A^{s',m} U_4 \right\|_1.
\]
\[
\leq \left\| \sum_{s,s' \leq N_m} \tilde{\rho}_s \mathbb{E} \left( A^{s,m} U_1 D_K U_2 A^{s',m} U_3 D_K U_4 \right) \right\|_1.
\]

Therefore the value of \( \left\| \sum_{s,s' \leq N_m} T_{K,s'}^2 \tilde{\rho}_s A^{s,m} U_1 A^{s',m} U_2 A^{s,m} U_3 A^{s',m} U_4 \right\|_1 \) only depends on the distribution of \( D_K \) and \( (A^{s,m}) \). We remark that \( D_K \) can be seen as a \( m^2 \) dimensional Gaussian vector with the variance-covariance matrix \( \Sigma \). It follows that if we write \( C_i \) the (unnormalized) orthogonal eigenvectors of \( \Sigma \) and if \( (Z_i) \) is a sequence of i.i.d standard normal random variables then we observe that \( D_K \overset{d}{=} \sum_{i \leq m^2} Z_i C_i \). This directly implies that the following holds
\[
= \left\| \sum_{s,s' \leq N_m} \sum_{l_1,l_2 \in K} \frac{|\rho_{1:2,s'}| |\rho_{3:4,s}|}{|K|} A^{s,m} U_1 A^{s',m} U_2 A^{s,m} U_3 A^{s',m} U_4 \right\|_1.
\]
\[
\leq \left\| \sum_{s \leq N_m, s' \leq m^2} \tilde{\rho}_s A^{s,m} U_1 C_{s'} U_2 A^{s',m} U_3 C_{s'} U_4 \right\|_1.
\]

Now note that by definition we have \( \text{Tr}(C_i C_j) = 0 \) if \( i \neq j \) are differently valued. Therefore we can note that for any matrix \( Y \) we have
\[
\sum_{i \leq m^2} \frac{1}{\|C_i\|^2_{HS}} C_i Y C_i = \text{Tr}(Y) \text{Id} \quad (41)
\]
(see lemma 4.8 in [5]). Moreover we remark that
\[
\left\| \sum_{s \leq N, s' \leq m^2} \tilde{\rho}^s A^{s,m} U_1 C_s' U_2 A^{s,m} C_s U_4 \right\|_1 \\
\leq \sup_{\|x\|,\|y\| \leq 1} \left( \sum_{s \leq N, s' \leq m^2} \tilde{\rho}_s \left\langle U_3 C_s' U_4 x, A^{s,m} U_2^* C_s^* A^{s,m} U_1^* y \right\rangle \right) \\
\leq \sup_{\|x\|,\|y\| \leq 1} \left( \sum_{s' \leq m^2} \left( \sum_{s \leq N} \tilde{\rho}^s A^{s,m} U_2^* C_s^* A^{s,m} U_1^* \right)^2 \right)^{1/2}. 
\]

Note that the following holds
\[
\sum_{s' \leq m^2} \left\| \sum_{s \leq N} \tilde{\rho}^s A^{s,m} U_2^* C_s^* A^{s,m} U_1^* \right\|^2 \\
= \sum_{s' \leq m^2} \sum_{s_1, s_2 \leq N} \tilde{\rho}^{s_1} \tilde{\rho}^{s_2} y^*_1 A^{s_1,m} C_{s_1'} U_2 A^{s_2,m} U_2^* C_{s_2'} A^{s_2,m} U_1^* \\
\overset{(a)}{\leq} \max_i \|C_i\|_{HS}^2 \sum_{s_1, s_2 \leq N} \tilde{\rho}^{s_1} \tilde{\rho}^{s_2} y^*_1 A^{s_1,m} A^{s_2,m} U_1^* y Tr \left( A^{s_1,m} A^{s_2,m} \right) \\
\overset{(b)}{=} \max_i \|C_i\|_{HS}^2 \sum_{s \leq N} (\tilde{\rho}^s)^2 y^*_1 A^{s,m} A^{s,m} U_1^* y Tr \left( (A^{s,m})^2 \right) 
\]
where to obtain (a) we used eq. (11) and where to obtain (b) we used the fact that the matrices \((A^{s,m})\) are orthogonal. Therefore we obtain that
\[
\sup_{\|y\| \leq 1} \left\| \sum_{s \leq N} (\tilde{\rho}^s)^2 y^*_1 A^{s,m} A^{s,m} U_1^* y Tr \left( (A^{s,m})^2 \right) \right\|_1 \\
= \sup_{\|y\| \leq 1} \left( \sum_{s \leq N} (\tilde{\rho}^s)^2 y^*(A^{s,m})^2 y Tr \left( (A^{s,m})^2 \right) \right) \\
\leq \sup_{s \leq N} Tr((A^{s,m})^2) \sup_{\|y\| \leq 1} \sum_{s \leq N} (\tilde{\rho}^s)^2 \|A^{s,m} y\|_2^2 \\
\leq (\tilde{\rho}_s)^2 \sup_{s \leq N} Tr((A^{s,m})^2) \sup_{\|y\| \leq 1} \sum_{s \leq N} \|A^{s,m} y\|_2^2 \\
\leq V_m (Y^m)^2 \sigma_m (Y^m)^2. 
\]
where to get the last inequality we used the fact that \( \sup_{s \leq N} \tilde{\rho}_s \leq 4 \) and that by definition we have
\[
V_m (Y^m)^2 = \sup_{\|M\|^2 \leq 1} \sum_{i \leq N} Tr(A^{i,m} M)^2, 
\]
and the fact that if we define \( M_s = \frac{A^{s,m}}{\sqrt{Tr((A^{s,m})^2)}} \) then we have \( Tr(M_s^2) = 1 \) and
\[
\sum_{j \leq N_m} Tr(A^{j,m}M_s)^2 = Tr((A^{s,m})^2).
\]
Moreover, we remark that we also have:

\[
\max_{i \leq m^2} \|C_i\|_{HS}^2 \leq \sup_{ Tr(|M|^2) \leq 1} \sum_{s \leq N_m} T^2_{K,s} |Tr(A^{s,m}M)|^2 \leq \max_{s \leq N_m} T^2_{K,s} \sup_{ Tr(|M|^2) \leq 1} \sum_{s \leq N_m} |Tr(A^{s,m}M)|^2 = \max_{s \leq N_m} T^2_{K,s} \mathcal{V}_m(Y^m)^2.
\]
Moreover according to lemma 7 for all \( s \leq N_m \) we have

\[
T^2_{K,s} \lesssim \|Z\|_{2+\epsilon}^2 \sum_{b \leq \infty} \alpha_m[b] \frac{1}{2+\epsilon}
\]
This directly implies that

\[
\max_{i \leq m^2} \|C_i\|_{HS}^2 \leq \mathcal{V}_m(Y^m)^2 \|Z\|_{2+\epsilon}^2 \sum_{b \leq \infty} \alpha_m[b] \frac{1}{2+\epsilon}.
\]
Finally we also remark that

\[
\sup_{ \|x\| \leq 1} \sum_{s' \leq m^2} \|U_3 C_{s'} U_4 x\|^2 = \sup_{ \|x\| \leq 1} \sum_{s' \leq m^2} \|C_{s'} x\|^2 = \| \sum_{s \leq N_m} T^2_{K,s} (A^{s,s})^2 \|_{\infty} \leq \max_{s \leq N_m} T^2_{K,s} \| \sum_{s \leq N_m} (A^{K,s})^2 \|_{\infty}.
\]
Moreover according to lemma 7 for all \( s \leq N_m \) we have

\[
T^2_{K,s} \lesssim \|Z\|_{2+\epsilon}^2 \sum_{b \leq \infty} \alpha_m[b] \frac{1}{2+\epsilon}
\]
Therefore we obtain that

\[
\sup_{ \|x\| \leq 1} \sum_{s' \leq m^2} \|C_{s'} x\|^2 \leq \|Z\|_{2+\epsilon}^2 \sum_{b \leq \infty} \alpha_m[b] \frac{1}{2+\epsilon} \sigma_m(Y^m)^2.
\]
Therefore we obtain that there is a constant \( K \) that does not depend on \( m \) such that

\[
\left| \tau \left( a_j K \hat{X}^{z,m} \hat{K} \hat{X}^{z',m} r K \right) \right| \leq K \sum_{l \geq 0} \alpha_m[l] \frac{1}{2+\epsilon} \sigma_m(Y^m)^2 \mathcal{V}_m(Y^m)^2 s_{m,p-1} \log(N_m) \right)^{2(p-1)}.
\]
Combined with eq. 37, we obtain that
\[
\left| \tau \left( a f^K X^{z,m}_m f^K X^{z',m} f^K \right) \right| \\
\lesssim \sum_{l \geq 0} \alpha_m[l] \frac{1}{2} m^{1/2} \sigma_m (Y^m)^2 \nu_m (Y^m)^2 s_{n-1} \left( 1 + \log(N_m) \right)^2 (p-1) \\
+ \alpha_m[b] \sup_{1 \leq l \leq p} \left( \left( \sum_{i,j} \left( A_{i,j}^{s,m} \right)^2 \right)^{(l+\epsilon)} + \sum_{s \leq N_m} (A_{i,j}^{s,m})^2 \right) \frac{2}{2^{\epsilon + \epsilon}}
\]

Now we bound \( \| E_{D_m} \left( f^K X^{z,m}_m E_{D_m}(f^K) X^{z',m} f^K \right) \|_1 \). In this goal we write
\[
A^{z,z'} := X^{z,m}_m E_{D_m}(f^K) X^{z',m}
\]
Firstly we remark that by definition of the \( L_1 \) norm we have
\[
\| E_{D_m} \left( f^K X^{z,m}_m E_{D_m}(f^K) X^{z',m} f^K \right) \|_1 \\
\leq \sup_{\|a\|_\infty \leq 1} \tau \left( a f^K A^{z,z'} f^K \right).
\]
Choose \( a \in F^{l_{\text{tail}}} \mathcal{D}(\mathbb{Z})^2 \) \( X^m \) such that \( \|a\|_\infty \leq 1 \). We note that by eq. 55 that \( a \) is a deterministic matrix. Let \( \left( \hat{X}^{z,m}_m, \hat{X}^{z',m}_m \right) \) be a copy of \( (X^{z,m}_m, X^{z',m}_m) \) that is independent from \( (X^{z,m}_m) \) and write \( \hat{A}^{z,z'} := X^{z,m}_m E_{D_m}(f^K) X^{z',m}_m \). Then by definition of the strong-mixing coefficients we have that there is constant \( C > 0 \) that does not depend on \( m \) such that
\[
\tau \left( a f^K A^{z,z'} f^K - a f^K \hat{A}^{z,z'} f^K \right) \\
\lesssim \frac{1}{m} \sum_{i_1, i_3 \leq m} \text{cov} \left( A_{i_2, i_3}^{z,z'} \left( af^K \right)_{i_1, i_3} f^K \right) \\
\lesssim \frac{8}{m} \alpha_m[b] \frac{1}{\varepsilon} \left( \sqrt{\text{Tr}(X^{z,m}_m)^2} \right) \left( \sqrt{\text{Tr}(X^{z',m}_m)^2} \right) \frac{2}{2^{\epsilon + \epsilon}},
\]
where (a) is a consequence of the equation \( \tau_{m}(\cdot) = \frac{1}{m} \mathbb{E}(\text{Tr}(\cdot)) \); and where to get (b) we used lemma 7 and the fact that \( \|af^K\|_\infty, \|f^K\|_\infty \leq 1 \). Moreover, we remark that
\[
\tau \left( a f^K A^{z,z'} f^K \right) \overset{(a)}{=} \frac{1}{m} \sum_{i_1, i_3 \leq m} \mathbb{E} \left( (af^K)_{i_1, i_3} f^K \right) \left[ \mathbb{E}(A_{i_2, i_3}^{z,z'}) - \mathbb{E}(A_{i_2, i_3}^{z,z'}) \right] \\
= 0
\]
where (a) is a consequence of the independence of \( \hat{A}^{z,z'} \) and \( f^K \).
This directly implies that

\[
\kappa^{s,i,j}_{m}(b|z_{2}^{2}) \lesssim \sum_{l \geq 0} \alpha_{m}[l] \frac{1}{l^{\frac{1}{2}+}} \sigma_{m}(Y^{m})^{2} \nu_{m}(Y^{m})^{2} \frac{1}{m^{p-1}} (1 + \log(N_{m}))^{2(p-1)} + \alpha_{m}[b] \frac{1}{l^{\frac{1}{2}+}} \sup_{1 \leq l \leq p} \left( \frac{1}{m^{1-\epsilon/2}} \sum_{i,j} \left( \left( \sum_{s \leq N_{m}} \left( A_{i,j}^{s,m} \right)^{2} \right)^{(l(1+\epsilon/2))} + \sum_{s \leq N_{m}} \left( A_{i,j}^{s,m} \right)^{(l(2+\epsilon))} \right) \right) \right)^{\frac{1}{l^{\frac{1}{2}+}}}
\]

We now move on to bounding \( \kappa^{s,i,j}_{m}[|Z^{2}|] \). Let \( z, z' \in \mathbb{Z} \) and \( K \subset \mathbb{Z} \) be such that

\[
\min_{x \in K \cup \{z'\}} \min_{j,l} d(z_{j,l}, z'_{j,l}) \geq b.
\]

Choose \( Y_{1}, Y_{2}, Y_{3} \in \mathcal{F}^{\text{tail}}_{D_{m}(\mathbb{Z}^{2})}(Y^{m}) \) be operators satisfying \( \|Y_{1}\|_{\infty}, \|Y_{2}\|_{\infty}, \|Y_{3}\|_{\infty} \leq 1 \). Firstly we remark that by definition of the \( L_{1} \) norm we have

\[
\|E_{D_{m}}(Y_{1}X^{z,m}Y_{2}X^{z',m}Y_{3})\|_{1} \leq \sup_{a \in \mathcal{F}^{\text{tail}}_{D_{m}(\mathbb{Z}^{2})}(Y^{m})} \tau(aY_{1}X^{z,m}Y_{2}X^{z',m}Y_{3})
\]

Choose \( a \in \mathcal{F}^{\text{tail}}_{D_{m}(\mathbb{Z}^{2})}(Y^{m}) \) such that \( \|a\|_{\infty} \leq 1 \). We note that by eq. (35) that \( a \) is a deterministic matrix.

Let \( \tilde{X}^{z,m} \) be a copy of \( X^{z,m} \) that is independent from \( (X^{z,m}) \). Then by definition of the strong-mixing coefficients we have:

\[
\left| \tau(aY_{1}X^{z,m}Y_{2}X^{z',m}Y_{3}) - \tau(aY_{1}\tilde{X}^{z,m}Y_{2}X^{z',m}Y_{3}) \right| \leq \frac{1}{m} \left| \mathbb{E}\left( \text{Tr}(aY_{1}X^{z,m}Y_{2}X^{z',m}Y_{3}) \right) - \mathbb{E}\left( \text{Tr}(aY_{1}\tilde{X}^{z,m}Y_{2}X^{z',m}Y_{3}) \right) \right| \lesssim \frac{\alpha_{m}[b]}{m} \left[ \left| \text{Tr}(aY_{1}X^{z,m}Y_{2}X^{z',m}Y_{3}) \right|_{1+\epsilon/2} + \text{Tr}(aY_{1}\tilde{X}^{z,m}Y_{2}X^{z',m}Y_{3}) \right]_{1+\epsilon/2}
\]

\[
\leq \alpha_{m}[b] \left( \frac{1}{m^{1-\epsilon/2}} \sum_{i,j} \left( \left( \sum_{s \leq N_{m}} \left( A_{i,j}^{s,m} \right)^{2} \right)^{(l(1+\epsilon/2))} + \sum_{s \leq N_{m}} \left( A_{i,j}^{s,m} \right)^{(l(2+\epsilon))} \right) \right) \right)^{\frac{1}{l^{\frac{1}{2}+}}}
\]

where to get (a) we used lemma 7. Moreover we remark that \( \tau(aY_{1}\tilde{X}^{z,m}Y_{2}X^{z',m}Y_{3}) = 0 \). This directly implies that

\[
\left| E_{D_{m}}(Y_{1}X^{z,m}Y_{2}X^{z',m}Y_{3}) \right| \lesssim \alpha_{m}[b] \left( \frac{1}{m^{1-\epsilon/2}} \sum_{i,j} \left( \left( \sum_{s \leq N_{m}} \left( A_{i,j}^{s,m} \right)^{2} \right)^{(l(1+\epsilon/2))} + \sum_{s \leq N_{m}} \left( A_{i,j}^{s,m} \right)^{(l(2+\epsilon))} \right) \right) \right)^{\frac{1}{l^{\frac{1}{2}+}}}
\]

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This directly implies that

\[ R^{**}[b[Z^2]] \lesssim \alpha_m [b] \frac{1}{m^{1-\epsilon/2}} \sup_{1 \leq l \leq p} \left( \frac{1}{m^{1-\epsilon/2}} \sum_{i,j} \left( \sum_{s \leq N_m} (A_{i,j}^{s,m})^2 \right)^{(1+\frac{2}{l})} + \sum_{s \leq N_m} (A_{i,j}^{s,m})^{(2+\epsilon)} \right)^{\frac{2}{1+\epsilon}} \]

(43)