Hausdorff and topological dimension for polynomial automorphisms of $\mathbb{C}^2$

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Abstract

Let $g$ be a polynomial automorphism of $\mathbb{C}^2$. We study the Hausdorff dimension and topological dimension of the Julia set of $g$. We show that when $g$ is a hyperbolic mapping, then the Hausdorff dimension of the Julia set is strictly greater than its topological dimension. Moreover, the Julia set cannot be locally connected. We also provide estimates for the dimension of the Julia sets in the general (not necessarily hyperbolic) case.

1 Introduction

Let $g$ be a polynomial automorphism of $\mathbb{C}^2$. We can associate with $g$ a dynamical degree $d$ which is a conjugacy invariant, see Section 2 for the definition. We are interested in nontrivial dynamics which can occur only for $d > 1$. We define $K^\pm$ as the set of points in $\mathbb{C}^2$ with bounded forward/backward orbits, $K = K^+ \cap K^-$, $J^\pm = \partial K^\pm$ and $J = J^+ \cap J^-$. We refer to $J^\pm$ as the positive/negative Julia set of $g$ and $J$ is the Julia set of $g$. The set $J^\pm$ is unbounded and connected, while $K$ and $J$ are compact. The “chaotic” dynamic (recurrent dynamics with sensitive dependence on the initial conditions in forward and backward time) can only occur on the Julia set $J$. The nontriviality of the dynamics of $g$ is reflected by the fact that $g|_K$ has positive topological entropy equal to $\log d$. Note that the complex Jacobian determinant $\det Dg$ is constant in $\mathbb{C}^2$. We will use $a = \det Dg$ as standard notation in this paper. We can restrict our considerations to the volume decreasing case ($|a| < 1$) and to the volume preserving case ($|a| = 1$), because otherwise we can consider $g^{-1}$. In this paper we will use $|a| \leq 1$ as a standing assumption.

An important and nontrivial feature of a dynamical system is the Hausdorff dimension.

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dimension of its closed invariant sets. For a polynomial map of \( \mathbb{C} \) this subject has been studied successfully, especially in the case of hyperbolic and parabolic maps, see [U] for a recent survey article. The central part of the theory is based on the so-called Bowen- Ruelle formula, which determines the Hausdorff dimension of the Julia set as the zero of the pressure function.

In the case of polynomial automorphisms of \( \mathbb{C}^2 \) the situation is more complicated because in contrast to polynomials of \( \mathbb{C} \), the derivative \( Dg(p) \) induces a contracting and an expanding direction in the tangent space. Nevertheless, when \( g \) is a hyperbolic mapping, we also have a version of the Bowen- Ruelle formula; it provides the Hausdorff dimension of the unstable/stable slice in terms of the zero of the unstable/stable pressure function. We will use this result to analyze the Hausdorff dimension of \( J \).

Let us now describe our results more precisely.

First, we consider the topological dimension of the Julia set of a hyperbolic mapping \( g \). In particular, we show that the topological dimension of the Julia set can be expressed in terms of an intersection formula, that is,

\[
\dim_{\text{top}} J = \dim_{\text{top}} J^+ + \dim_{\text{top}} J^- - 4.
\]  

(1.1)

The proof of this result is based on general features about hyperbolic sets. Note that the analogous intersection formula for the Hausdorff dimension also holds (see [Vo3]). As an immediate consequence the topological dimension of \( J \) can be at most 2. We also provide a criterion which implies that \( J \) has topological dimension at least 1. Finally, we show that \( J \) cannot be locally connected.

Second, we deal with the Hausdorff dimension of the Julia set of a hyperbolic mapping \( g \). Let \( P_{u/s} \) denote the unstable/stable pressure function of \( g \) (see Section 4 for the definition). Then the result of Verjovsky and Wu [VW] implies that \( t_{u/s} = \dim_H \mathcal{W}_{u/s}^u(p) \cap J \) is given by the unique solution of the Bowen-Ruelle formula

\[
P_{u/s}(t) = 0.
\]  

(1.2)

Moreover, \( \dim_H J = t^u + t^s \) and \( \dim_H J^\pm = t_{u/s}^u + 2 \), see [Vo3] and the references therein. We show that the pressure functions are related to each other in terms of the Jacobian determinant. As a consequence we obtain the inequality

\[
t^s \leq \frac{t^u \log d}{\log d - t^u \log |a|}.
\]  

(1.3)

In particular, \( t^s < t^u \) for \( |a| < 1 \) and \( t^u = t^s \) for \( |a| = 1 \). This implies that when \( |a| \leq d^{-1/2} \), then the stable slice \( \mathcal{W}_s^u(p) \cap J \) is totally disconnected. Therefore, \( \dim_{\text{top}} J = 2 \) can occur only for \( |a| \) being sufficiently close to 1. Combining these results with a perturbation argument yields.

Theorem 4.11. If \( g \) is hyperbolic, then \( \dim_{\text{top}} J < \dim_H J \).
This property is sometimes (especially in applied sciences) used for defining fractal sets.

Last, we consider the dimension of the Julia sets of a general (not necessarily hyperbolic) polynomial automorphism $g$. We show that when $g$ is not volume preserving, then the upper box-dimension of $K$ is strictly smaller than 4. We also obtain that if $|a|$ is small, then the Hausdorff dimension of $J^-$ is close to 2. This generalizes a result of Fornæss and Sibony [FS] about complex Hénon mappings which are small perturbations of hyperbolic quadratic polynomials in $\mathbb{C}$.

This paper is organized as follows. In Section 2 we present the basic definitions and notations. The results about the topological dimension of the Julia set are presented in Section 3. Section 4 is devoted to the analysis of the Hausdorff dimension of the Julia set. In particular, we show that the Hausdorff dimension of the Julia set always exceeds its topological dimension. In Section 5 we derive Hausdorff dimension estimates for general polynomial automorphisms of $\mathbb{C}^2$.

2 Notation and Preliminaries

Let $g$ be a polynomial automorphism of $\mathbb{C}^2$. We denote by $\deg(g)$ the maximum of the algebraic degree of the components of $g$. The dynamical degree of $g$ is defined by

$$d = \lim_{n \to \infty} (\deg(g^n))^{1/n}$$

(see [JS2], [FM]). According to the Friedland-Milnor classification Theorem [FM], each polynomial automorphism of $\mathbb{C}^2$ is either conjugate to an elementary polynomial automorphism of $\mathbb{C}^2$ (with dynamical degree 1 and trivial dynamics) or to a finite composition of generalized Hénon mappings (with dynamical degree greater than 1 and nontrivial dynamics). A finite composition of generalized Hénon mappings is a mapping

$$g = g_1 \circ \cdots \circ g_m,$$  \hspace{1cm} (2.4)

where each $g_i$ has the form

$$g_i(z, w) = (w, P_i(w) + a_i z),$$  \hspace{1cm} (2.5)

$P_i$ is a complex polynomial of degree $d_i \geq 2$ and $a_i$ is a non-zero complex number. Note that the dynamical degree of $g$ is equal to $d = d_1 \cdot \cdots \cdot d_m$ and therefore coincides with $\deg(g)$. The complex Jacobian determinant of $g$ is equal to $a = a_1 \cdot \cdots \cdot a_m$. We denote by $\mathcal{H}_{(d_1, \ldots, d_m)}$ the space of mappings of the form (2.4). Each $g \in \mathcal{H}_{(d_1, \ldots, d_m)}$ depends on $k$ complex and therefore on $2k$ real variables for some positive integer $k$. Note that $k$ is determined by $d_1, \ldots, d_m$. For all $d \geq 2$ we define

$$\mathcal{H}_d = \bigcup \mathcal{H}_{(d_1, \ldots, d_m)}.$$
where the union is taken over all \((d_1, \ldots, d_m)\) with \(d_i \geq 2\) such that \(d = d_1 \cdot \cdots \cdot d_m\).

Since dynamical properties are invariant under conjugacy, each polynomial automorphism of \(\mathbb{C}^2\) with nontrivial dynamics is represented in \(\mathcal{H}_d\) for some positive integer \(d\). In the remainder of this paper we will restrict our attention to mappings in \(\mathcal{H}_d\).

We denote by \(Hyp_d\) the space of hyperbolic mappings in \(\mathcal{H}_d\). A mapping \(g \in \mathcal{H}_d\) is called hyperbolic if its Julia set \(J\) is a hyperbolic set of \(g\). That is, the tangent bundle of \(J\) allows an invariant splitting \(E = E^s \oplus E^u\) such that the tangent map is uniformly contracting on \(E^s\) and uniformly expanding on \(E^u\) (see \([KH]\) for further details). The results of \([BS1]\) imply that in the hyperbolic situation \(J\) has index 1; that is \(\dim_{\mathbb{C}} E^{s/u}_p = 1\) for all \(p \in J\). The nonwandering set of \(g\) is equal to the union of \(J\) with finitely many sinks. Furthermore \(g\) is an Axiom A diffeomorphism and \(J\) is a basic set of \(g\). The most important feature of hyperbolic sets is that we can associate with each point \(p\) the local unstable/stable manifold \(W^{u/s}_p(p)\), varying continuously with \(p\). We will also denote by \(W^{u/s}_p(p)\) the (global) unstable/stable manifold of \(p\). In the case of mappings in \(Hyp_d\) the (local) unstable/stable manifolds are in fact complex manifolds of complex dimension one. Finally we note that \(Hyp_d\) is open.

We will work both with the Hausdorff and the topological dimension of a set \(A\). For the Hausdorff dimension, denoted by \(\dim_H A\), see \([M]\); and for the topological dimension, denoted by \(\dim_{\text{top}} A\), see \([HW]\). For any \(A\) we have \(\dim_{\text{top}} A \leq \dim_H A\).

### 3 Topological dimension of \(J\)

In this section we establish some results about the topological dimension of the Julia set of a polynomial automorphism of \(\mathbb{C}^2\). In particular we show that in the hyperbolic case the topological dimension of the Julia set is given by an intersection formula, see Theorem 3.4. We also conclude that if \(g\) is hyperbolic, then the Julia set cannot be locally connected.

We start with a basic Lemma.

**Lemma 3.1** Let \(g \in Hyp_d\). Then \(t^{u/s}_{\text{top}} = \dim_{\text{top}} W^{u/s}_p(p) \cap J\) does not depend on \(p \in J\).

**Proof.** Without loss of generality we only consider the unstable slice. Let \(p, q \in J\) such that \(|p-q|\) is small. Then it is a general result about hyperbolic sets that the holonomy mapping maps \(W^u_p(p) \cap J\) homeomorphically to \(W^u_q(q) \cap J\). Thus \(t^{u}_{\text{top}}\) is locally constant. It is shown in \([BS1]\) that \(g|J\) is topologically transitive. We conclude that \(t^{u}_{\text{top}}\) is independent of \(p \in J\). \(\square\)

We refer to \(t^{u/s}_{\text{top}}\) as the topological dimension of the unstable/stable slice. The topological dimension of these slices is related to the topological dimension of \(J^{\pm}\).
Theorem 3.2 Let $g \in \text{Hyp}_d$. Then $\dim_{\text{top}} J^\pm = t_{\text{top}}^u + 2$.

Proof. Without loss of generality we show the result only for the unstable slice. Let $p \in J$. The Stable Manifold Theorem implies that there exists a continuous mapping

$$\Psi_p : \overline{W}_\epsilon^u(p) \cap J \times \overline{D}(0, \epsilon) \to \mathbb{C}^2$$

such that $\Psi_p(q, \overline{D}(0, \epsilon)) = \overline{W}_\epsilon^u(q)$ and that $\Psi_p(q, \cdot \cdot)$ is injective for all $q \in \overline{W}_\epsilon^u(p) \cap J$ (see [Vo1] for further details). Here $\overline{W}_\epsilon^{u/s}(p)$ and $\overline{D}(0, \epsilon)$ denote the closed local unstable/stable manifold of $p$ and the closed disk in $\mathbb{C}$ with center $0$ and radius $\epsilon$ respectively. Since $J$ is a hyperbolic set for $g$, the mapping $g|_J$ is expansive. Thus we can assure by making $\epsilon$ smaller if necessary that $\Psi_p$ is injective and therefore a homeomorphism onto its image. Therefore

$$\dim_{\text{top}} \left( \bigcup_{q \in \overline{W}_\epsilon^u(p) \cap J} \overline{W}_\epsilon^u(q) \right) = t_{\text{top}}^u + 2. \quad (3.6)$$

As in the proof of Theorem 4.1 of [Wo3], there exist $p_1, ..., p_n \in J$ and $\epsilon_1, ..., \epsilon_n > 0$ such that for

$$\epsilon = \min \left\{ \frac{\epsilon_1}{2}, ..., \frac{\epsilon_n}{2} \right\} \quad (3.7)$$

and for all $p \in J$ we have that $\overline{W}_\epsilon^u(p)$ is contained in $\overline{W}_{\epsilon_k}^s(q)$ for some $q \in \overline{W}_{\epsilon_k}^u(p_k) \cap J$ and some $k \in \{1, ..., n\}$. Note that this follows from the fact that $J$ is a hyperbolic set with a local product structure. We conclude that

$$\dim_{\text{top}} \left( \bigcup_{p \in J} \overline{W}_\epsilon^u(p) \right) = t_{\text{top}}^u + 2. \quad (3.8)$$

It is a result of Bedford and Smillie [BS1] that $W^s(J) = J^+$. Therefore, we can conclude by Proposition 3.10 of [Ex] that

$$\bigcup_{p \in J} W^s(p) = J^+. \quad (3.9)$$

On the other hand, we have

$$\bigcup_{n \in \mathbb{N}} g^{-n} \left( \bigcup_{p \in J} \overline{W}_\epsilon^s(p) \right) = \bigcup_{p \in J} W^s(p). \quad (3.10)$$

Hence

$$\dim_{\text{top}} J^+ = 2 + t_{\text{top}}^u \quad (3.11)$$
and the proof is complete. □

The reason for considering closed local unstable/stable manifolds in the proof of Theorem 3.2 is that some basic properties for the topological dimension only hold in general for closed sets (see [HW]).

**Corollary 3.3** Let \( g \in \text{Hyp}_d \). Then \( t^u_{top} \in \{0,1\} \).

**Proof.** A subset of \( \mathbb{R}^n \) has topological dimension \( n \) if and only if it has non-empty interior (see [HW]). This implies \( \dim_{top} J^k \leq 3 \). Therefore the result follows immediately from Theorem 3.2. □

**Remark.** Since the topological dimension of a set is a lower bound for its Hausdorff dimension, the result of Corollary 3.3 also follows from \( \dim H\text{u/s}(p) \cap J < 2 \) (see [Wo3]).

Let \( g \in \text{Hyp}_d \). By [BS1], the Julia set \( J \) has a local product structure; this implies

\[
\dim_{top} J = t^u_{top} + t^s_{top}. \tag{3.12}
\]

Therefore, Corollary 3.3 yields

\[
\dim_{top} J \in \{0, 1, 2\}. \tag{3.13}
\]

Similarly as it was done in [Wo3] for the Hausdorff dimension, Theorem 3.2 and equation (3.12) imply an intersection formula for the topological dimension of the Julia sets.

**Theorem 3.4** Let \( g \in \text{Hyp}_d \). Then

\[
\dim_{top} J = \dim_{top} J^+ + \dim_{top} J^- - 4. \tag{3.14}
\]

We refer to equation (3.14) as the intersection formula for the topological dimension.

As \( J \) is compact, its topological dimension exceeds 0 if and only if \( J \) is not totally disconnected. This implies that a mapping \( g \in \mathcal{H}_d \) with a connected Julia set \( J \) provides an example for \( \dim_{top} J > 0 \). In the following result we present another criterion in terms of the existence of particular Fatou components.

**Theorem 3.5** Let \( g \in \mathcal{H}_d \) and \(|a| < 1\). Let \( p \in J \) be a saddle point of \( g \). Assume there exists a periodic connected component \( C \) of \( \text{int} K^+ \) such that \( C \cap J^- \neq \emptyset \). Then \( \dim_{top} W^r_c(p) \cap J \geq 1 \).

**Proof.** Without loss of generality we assume that \( p \) is a fixed point and \( C \) has period 1. Since \( C \cap J^- \neq \emptyset \), we may follow by Proposition 7 of [BS2] that \( \partial C = J^+ \). We have \( W^s_c(p) \subset J^- \). This implies \( W^r_c(p) \cap J = W^u_c(p) \cap J^+ \). Together we obtain that

\[
W^r_c(p) = (W^u_c(p) \cap C) \cup (W^u_c(p) \cap J) \cup (W^u_c(p) \setminus \overline{C}). \tag{3.15}
\]
is a disjoint union. We will show that none of the sets in this union is empty. The set $W_u(p) \cap J$ is not empty because it contains $p$. Since $C$ is $g$-invariant and $W_u(p) = J^-$ (see [BS2]), we conclude that $W_u(p) \cap C = \emptyset$. We have $W_u(p) \cap C, W_u(p) \cap J \subset K$. This implies that
\[
\bigcup_{n \in \mathbb{N}} g^n(W_u(p) \cap C) \quad (3.16)
\]
is bounded. The unstable manifold $W_u(p)$ is a holomorphic copy of $C$ and is therefore unbounded. Thus $W_u(p) \setminus C \neq \emptyset$. The local unstable manifold $W_u(p)$ is a topological disk and therefore cannot be separated by a subset of topological dimension 0 (see [HW]). We conclude that $\dim_{\text{top}} W_u(p) \cap J \geq 1$. 

Remarks.
Note that no hyperbolicity is required in Theorem 3.3. Bedford and Smillie classified in [BS2] recurrent connected components of $\text{int}K^+$ in the case $|a| < 1$. They showed that such a component $C$ is periodic and must either be a basin of attraction or the stable set of a rotational domain (Siegel disk, Herman ring), and that in all these cases $C \cap J^- \neq \emptyset$ holds. Therefore Theorem 3.3 applies. Note that Theorem 3.3 also holds in the case of a non-recurrent periodic connected component $C$ of $\text{int}K^+$ which satisfies $C \cap J^- \neq \emptyset$. If $g \in \text{Hyp}_d$, then by [BS1] the only possibility for a connected component of $\text{int}K^+$ is a basin of attraction.

Finally, we show that $J$ cannot be locally connected.

**Theorem 3.6** Let $g \in \text{Hyp}_d$. Then the Julia set $J$ is not locally connected.

**Proof.** Let us assume $J$ is locally connected and $p \in J$ is a saddle fixed point. The local product structure of $J$ implies that there exists a connected neighborhood $U \subset J$ of $p$ such that $U$ is homeomorphic to $(W_u(p) \cap J) \times (W^s(p) \cap J)$. Therefore each of the sets, $W_u(p) \cap J$, $W^s(p) \cap J$, is connected and has topological dimension equal to 1. It is shown in [BS1] that $g|_J$ is topologically mixing. This implies
\[
\bigcup_{n \in \mathbb{Z}} g^n(U) = J. \quad (3.17)
\]
We conclude that $J$ is connected, and therefore $g$ is an unstably connected mapping (see [BS6]). Thus, we may conclude by Corollary 5.9 of [BS7] that $W_u(p) \cap J$ is a Cantor set. This is a contradiction to $\dim_{\text{top}} W^s(p) \cap J = 1$ which proves the desired result. \[\square\]
4 Topological pressure, Bowen- Ruelle formula and fractal Julia sets

In this section we apply the Bowen- Ruelle formula to obtain information about the Hausdorff dimension of the Julia set of a hyperbolic polynomial automorphism of $\mathbb{C}^2$. In particular, we use properties of topological pressure to show that the Hausdorff dimensions of the unstable and stable slices are related to each other by the Jacobian determinant of the mapping (see Corollary 4.7). This relation is then applied in Theorem 4.11 to show that the Hausdorff dimension of the Julia set always exceeds its topological dimension.

First we recall the concept of topological pressure and some of its properties. The main references for the following are [R] and [Wa].

Let $(X, d)$ be a compact metric space and $T : X \to X$ a continuous mapping. For $n \in \mathbb{N}$ we define a new metric $d_n$ on $X$ given by $d_n(x, y) = \max_{i=0, \ldots, n-1} d(T^i(x), T^i(y))$. Let $\epsilon > 0$. A set $F \subset X$ is called $(n, \epsilon)$-separated with respect to $T$ if $d_n(x, y) < \epsilon$ implies $x = y$ for $x, y \in F$. For all $(n, \epsilon) \in \mathbb{N} \times \mathbb{R}^+$ let $F_n(\epsilon)$ be a maximal $(n, \epsilon)$-separated set with respect to $T$ (in the sense of inclusion). We denote by $C(X, \mathbb{R})$ the Banach space of all continuous functions from $X$ to $\mathbb{R}$. The topological pressure of $T$ is a mapping $P(T, \cdot)$ from $C(X, \mathbb{R})$ to $\mathbb{R} \cup \{\infty\}$ defined by

$$P(T, \varphi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left( \sum_{x \in F_n(\epsilon)} \exp \left( \sum_{i=0}^{n-1} \varphi \circ T^i(x) \right) \right).$$

(4.18)

If $T$ is expansive, then we can omit the limit $\epsilon \to 0$ and take the value for a fixed $\epsilon > 0$ with the property that $2\epsilon$ is a constant of expansivity for $T$. Note that $P(T, 0) = h_{top}(T)$, where $h_{top}(T)$ denotes the topological entropy of $T$. We define $M(X, T)$ to be the space of all $T$-invariant Borel probability measures on $X$. The variational principle provides the formula

$$P(T, \varphi) = \sup_{\mu \in M(X, T)} \left( h_\mu(T) + \int_X \varphi d\mu \right),$$

(4.19)

where $h_\mu(T)$ denotes the measure theoretic entropy of $T$ with respect to $\mu$. In the sequel we assume that $h_{top}(T) < \infty$. The topological pressure has the following properties.

i) The topological pressure is a convex function.

ii) If $\varphi$ is a strictly negative function, then the function $t \mapsto P(T, t\varphi)$ is strictly decreasing.

iii) For all $\varphi \in C(X, \mathbb{R})$ we have

$$h_{top}(T) + \min \varphi \leq P(T, \varphi) \leq h_{top}(T) + \max \varphi.$$
In many hyperbolic systems topological pressure is related to dimension. We will discuss this relation in the case of polynomial automorphisms of $\mathbb{C}^2$. Let $g \in Hyp_d$. We define

$$\phi^{u/s} : J \to \mathbb{R} \quad p \mapsto \log ||Dg(p)||_{E_{p}^{u/s}}||$$

and the unstable/stable pressure function

$$P^{u/s} : \mathbb{R} \to \mathbb{R} \quad t \mapsto P(g|J, \mp t\phi^{u/s}).$$

The following result due to Verjovsky and Wu [VW] is crucial for our further presentation.

**Theorem 4.1** Let $g \in Hyp_d$ and $p \in J$. Then $t^{u/s} = \dim_H W_{p}^{u/s} \cap J$ does not depend on $p \in J$. Furthermore $t^{u/s}$ is given by the unique solution of

$$P^{u/s}(t) = 0.$$ (4.21)

Equation (4.21) is called Bowen-Ruelle formula. We refer to $t^{u/s}$ as the Hausdorff dimension of the unstable/stable slice. Theorem 4.1 can be applied to derive rough bounds for the Hausdorff dimension of $J$. In comparison to equation (4.21) the dynamical meaning of these bounds is more transparent.

**Theorem 4.2** Let $g \in Hyp_d$. We define

$$\underline{s} = \lim_{n \to \infty} \frac{1}{n} \log \left( \max \{ ||Dg^n(p)||_{E_{p}^{u}} : p \in J \} \right)$$

$$\overline{s} = \lim_{n \to \infty} \frac{1}{n} \log \left( \min \{ ||Dg^n(p)||_{E_{p}^{s}} : p \in J \} \right).$$

Then

$$\left( \frac{1}{\overline{s}} + \frac{1}{\overline{s} - \log |a|} \right) \log d \leq \dim_H J \leq \left( \frac{1}{\underline{s}} + \frac{1}{\underline{s} - \log |a|} \right) \log d.$$ (4.22)

**Proof.** Since $E_{p}^{u/s}$ is of complex dimension one, we have

$$||Dg^n(p)||_{E_{p}^{u/s}} = \prod_{i=0}^{n-1} ||Dg(g^i(p))||_{E_{g^i(p)}^{u/s}} ||$$ (4.23)

for all $n \in \mathbb{N}$ and all $p \in J$. Therefore, the limits defining $\underline{s}$ and $\overline{s}$ exist (see [Vo] for further details). Since $g$ is hyperbolic, we have $\underline{s} > 0$. Let us consider the upper bound in inequality (4.22). For $n \in \mathbb{N}$ we define

$$\phi^u_n = \frac{1}{n} \log ||Dg^n||_{E_u^n}||.$$

It follows from equation (4.23) that

$$\int \phi^u d\mu = \int \phi^u_n d\mu$$ (4.24)
for all $n \in \mathbb{N}$ and all $\mu \in M(J, g|J)$ (see [Wo3]). We conclude by equation (4.19) and Theorem 4.1 that
\[ P(g|J, -t^n \phi_n^u) = 0 \]  
for all $n \in \mathbb{N}$. Let us recall that $h_{top}(g|J) = \log d$ (see [BS3]). Therefore equation (4.20) yields
\[ \log d - tu \geq \log \left( \min \{|Dg^n(p)|_{E_p} : p \in J\} \right) \geq 0. \]  
Taking the limit implies $tu \leq \log d$.
In [Wo1] we have shown that
\[ s - \log |a| = \lim_{n \to \infty} \frac{1}{n} \log \left( \min \{|Dg^{-n}(p)|_{E_p} : p \in J\} \right). \]  
Therefore the same argumentation as above applied to $g^{-1}$ yields $t^s \leq \frac{\log d}{s - \log |a|}$.
Applying
\[ dim_H J = t^u + t^s \]  
completes the proof for the upper bound in inequality (1.23). Analogously we obtain the proof for the lower bound in inequality (1.22).

Remark. Friedland and Ochs provided in [FO] a formula for the Hausdorff dimension of the Julia set in terms of entropies and Lyapunov exponents of invariant probability measures (see also [Wo3]). This result can also be applied to derive lower and upper bounds for the Hausdorff dimension of $J$ similar to those in equation (4.22).

Now we introduce a natural generalization of bidisks in $\mathbb{C}^2$.

**Definition 4.3** Let $E_1, E_2 \subset \mathbb{C}^2$ be complex lines through the origin such that $\mathbb{C}^2 = E_1 \oplus E_2$. Let $p \in \mathbb{C}^2$ and $r = (r_1, r_2)$, where $r_1, r_2 > 0$. We define the $(E_1, E_2)$-bidisk with center $p$ and radius $r$ by
\[ P_{E_1, E_2}(p, r_1, r_2) = \{ p + e_1 + e_2 : e_1 \in E_1, e_2 \in E_2, |e_1| < r_1, |e_2| < r_2 \} \]  
Furthermore $P_{E_1, E_2}(r_1, r_2) = P_{E_1, E_2}(0, r_1, r_2)$.

We need the following Lemma.

**Lemma 4.4** Let $g \in Hyp_d$. Then there exist $C_1, C_2 > 0$ such that for all $p \in J$ and all $r_1, r_2 > 0$ we have
\[ C_1 r_1^2 r_2^2 \leq vol(P_{E_1^p, E_2^p}(r_1, r_2)) \leq C_2 r_1^2 r_2^2. \]  
Proof. Let $(e_p^*)_{p \in J}$ and $(e_p^u)_{p \in J}$ be continuous families in $\mathbb{C}^2$ such that $e_p^* \in E_1^p, e_p^u \in E_2^p$ and $|e_p^*|, |e_p^u| = 1$ for all $p \in J$. For $p \in J$ we define a linear mapping
\[ A_p : \mathbb{C}^2 \to \mathbb{C}^2 \quad (z_1, z_2) \mapsto z_1 e_p^* + z_2 e_p^u. \]  
for all $n \in \mathbb{N}$.
are well-defined and $0 < C_1 \leq C_2 < \infty$.

Obviously $P_{C \times \{0\}, \{0\} \times C}(r_1, r_2)$ is mapped bijectively by $A_p$ to $P_{E_p, E_p}(r_1, r_2)$. Let $r_1, r_2 > 0$ and $p \in J$. Then

$$C_1 r_1^2 r_2^2 = C_1 \pi^{-2} \text{vol}(P_{C \times \{0\}, \{0\} \times C}(r_1, r_2))$$
$$= C_1 \pi^{-2} |\det A_p|^{-2} \text{vol}(P_{E_p, E_p}(r_1, r_2))$$
$$\leq \text{vol}(P_{E_p, E_p}(r_1, r_1))$$
$$= |\det A_p|^{-2} \text{vol}(P_{C \times \{0\}, \{0\} \times C}(r_1, r_1))$$
$$\leq C_2 r_1^2 r_2^2.$$  \hfill (4.30)

This completes the proof. \hfill $\Box$

As an immediate consequence we obtain the following.

**Lemma 4.5** Let $g \in \text{Hyp_d}$. Then there exists $C_1, C_2 > 0$ such that

$$C_1 |a|^n \leq ||Dg^n|_{E_p}| \cdot ||Dg^n|_{E_p}| \leq C_2 |a|^n$$

for all $p \in J$ and all $n \in \mathbb{N}$.

**Proof.** We have

$$Dg^n(p)(P_{E_p, E_p}(1, 1)) = P_{E_p, E_p}(1, 1)(||Dg^n|_{E_p}|, ||Dg^n|_{E_p}|).$$  \hfill (4.32)

Therefore the result follows from Lemma 4.4. \hfill $\Box$

Next we show that the unstable and stable pressure functions are related to each other by the Jacobian determinant of the mapping $g$.

**Proposition 4.6** Let $g \in \text{Hyp_d}$ and $t \geq 0$. Then $P^n(t) = P^s(t) - t \log |a|$.

**Proof.** Let $\varepsilon > 0$ such that $2\varepsilon$ is a constant of expansivity for $g|_J$. Let $C_1, C_2$ be the constants in Lemma 4.4. For all $n \in \mathbb{N}$ let $F_n(\varepsilon)$ be a maximal $(n, \varepsilon)$-separated set with respect to $g|_J$. Let $t \geq 0$. Then

$$\frac{1}{n} \log \left( \sum_{p \in F_n(\varepsilon)} \exp \left( \sum_{i=0}^{n-1} -t \phi^i \circ g^i(p) \right) \right)$$
$$= \frac{1}{n} \log \left( \sum_{p \in F_n(\varepsilon)} ||Dg^n(p)||_{E_p}^{-t} \right)$$
$$\leq \frac{1}{n} \log \left( \sum_{p \in F_n(\varepsilon)} \left( C_1 |a|^n ||Dg^n(p)||_{E_p}^{-1} \right)^{-t} \right)$$
$$\leq \frac{1}{n} \log \left( C_1^{-t} |a|^{-tn} \sum_{p \in F_n(\varepsilon)} ||Dg^n(p)||_{E_p}^{-t} \right)$$
$$= \frac{1}{n} \log C_1 - t \log |a| + \frac{1}{n} \log \left( \sum_{p \in F_n(\varepsilon)} \exp \left( \sum_{i=0}^{n-1} t \phi^i \circ g^i(p) \right) \right).$$
Taking the upper limit implies \( P^u(t) \leq P^s(t) - t \log |a| \). The opposite inequality follows analogously. \( \square \)

Remark. Note that this result is essentially based on the facts that the stable and unstable spaces are of complex dimension one and that \( g \) has constant Jacobian determinant.

Proposition 4.6 implies an inequality which relates the Hausdorff dimensions of the unstable and stable slice.

**Corollary 4.7** Let \( g \in \text{Hyp}_d \). Then

\[
t^s \leq \frac{t^u \log d}{\log d - t^u \log |a|}.
\]  

(4.34)

In particular, \( t^s < t^u \) for \( |a| < 1 \), and \( t^s = t^u \) for \( |a| = 1 \).

**Proof.** Let us recall our standing assumption \( |a| \leq 1 \). It is well-known that \( P^u(0) = P^s(0) = h_{top}(g|J) = \log d \). On the other hand, we deduce from Proposition 4.6 that \( P^s(t^u) = t^u \log |a| \). Since \( t \mapsto P^s(t) \) is a convex function, its graph lies below the line segment joining \((0, \log d)\) and \((t^u, t^u \log |a|)\). Therefore application of Theorem 4.1 implies equation (4.34). The rest is trivial. \( \square \)

**Corollary 4.8** Let \( g \in \text{Hyp}_d \). Then

i) If \( |a| < 1 \), then

\[
dim_H J^- \leq \frac{t^u \log d}{\log d - t^u \log |a|} + 2 < dim_H J^+.
\]  

(4.35)

ii) If \( |a| = 1 \), then \( dim_H J^+ = dim_H J^- \).

**Proof.** We showed in \( \text{Wo3} \) that \( dim_H J^\pm = t^{u/s} + 2 \). Therefore, the result follows from Corollary 4.7. \( \square \)

Inequality (4.34) can also be applied to provide a criterion for occurrence of a totally disconnected stable slice.

**Corollary 4.9** Let \( C \) be a connected component of \( \text{Hyp}_d \). Assume there exists \( g \in C \) with \( |a| \leq d^{-1/2} \). Then the stable slice of every \( f \in C \) is a Cantor set.

**Proof.** The stable slice is a Cantor set if and only if its topological dimension is equal to 0. By the stability of hyperbolic sets we conclude that the stable slice has constant topological dimension in \( C \). Therefore, it is sufficient to show the result for \( g \). We have \( t^u < 2 \) (see \( \text{Wo3} \)). Thus equation (4.34) implies

\[
t^s < \frac{2 \log d}{\log d - 2 \log |a|}.
\]  

(4.36)
From $|a| \leq d^{-1/2}$ we conclude that $t^* < 1$. \qed

Remark. In particular we have shown that if $g \in Hyp_d$ and $dim_{top} J = 2$, then $|a|$ has to be sufficiently close to 1. Let us mention that the stable slice of $g \in Hyp_d$ is also totally disconnected under the assumption that $J$ is connected (see [BS7]).

**Corollary 4.10** Let $g \in Hyp_d$ and $dim_{top} J \geq 1$. Then

$$dim_H J \geq 1 + t^* > 1. \tag{4.37}$$

**Proof.** i) $t^u_{top} = 1$: Since the topological dimension of a set is a lower bound of its Hausdorff dimension, we obtain $t^u \geq 1$. We have $t^s > 0$. Therefore equation (4.37) follows from $dim_H J = t^u + t^s$.

ii) $t^s_{top} = 1$: Corollary 4.7 implies $1 = t^s_{top} \leq t^u$ and therefore equation (4.37) follows as in i). \qed

Note that in view of Theorem 3.5, Corollary 4.10 is in particular applicable if $g \in Hyp_d$ has an attracting periodic point.

Finally, we combine our observations to obtain the main result of this section.

**Theorem 4.11** Let $g \in Hyp_d$. Then $dim_{top} J < dim_H J$.

**Proof.** Without loss of generality $|a| \leq 1$.

If $dim_{top} J = 0$, the inequality follows from the fact that the Hausdorff dimension of $J$ is strictly positive (see [VW], [BS3] and [Wo1]). If $dim_{top} J = 1$, the estimate follows from Corollary 4.10.

It remains to consider the case $dim_{top} J = 2$.

If $|a| < 1$, then Corollary 4.3, Corollary 4.7 and equation (3.12) imply

$$dim_H J = t^u + t^s = 2t^s + (t^u - t^s) \geq 2t^s + (t^u - t^s) = 2 + (t^u - t^s) > 2. \tag{4.38}$$

To complete the proof we have to consider the case $dim_{top} J = 2$ and $|a| = 1$.

Assume there exists $g \in Hyp_d$ such that $dim_H J = 2 = dim_{top} J$ and $|a| = 1$.

This implies that $t^u = t^u_{top} = t^s = t^s_{top} = 1$. Since $g \in Hyp_d$, there exist generalized Hénon mappings $g_1, ..., g_m$ such that $g = g_1 \circ ... \circ g_m$. Each $g_i$ has the form

$$g_i(z, w) = (w, P_i(w) + a_i z),$$

where $P_i$ is a complex polynomial of degree at least 2 and $a_i$ is a non-zero complex number. Note that $a = \prod_{i=1}^m a_i$. For a fixed $0 < \delta < 1$ let us consider $(g_{\epsilon})_{\epsilon \in (-\delta, \delta)} \subset H_d$ defined by

$$g_{\epsilon} = g_{1, \epsilon} \circ g_2 \circ ... \circ g_m,$$

where

$$g_{1, \epsilon}(z, w) = (w, P_1(w) + (1 - \epsilon)a_1 z).$$

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Note that \( \det Dg_\epsilon = (1 - \epsilon)a \) and \( g_0 = g \). Let \( C \) denote the connected component of \( Hyp_d \) containing \( g \). Since \( Hyp_d \) is open we can assure by making \( \delta \) smaller if necessary that \( g_\epsilon \in C \) for \( |\epsilon| < \delta \). This implies that \( (g_\epsilon)_{\epsilon \in (\delta, \bar{\delta})} \) is a real analytic family in \( Hyp_d \). Thus it follows from a result of [VW] that \( t^u_\epsilon \) and \( t^s_\epsilon \) are real analytic functions of \( \epsilon \). The topological dimension of the unstable/stable slice is constant in \( C \). This implies

\[
t^u_\epsilon, t^s_\epsilon \geq 1 \tag{4.39}
\]

for all \( |\epsilon| < \delta \). Since \( t^s_\epsilon \) is real analytic we can write

\[
t^s_\epsilon = 1 + \alpha \epsilon + \alpha^2 \epsilon^2 + O(|\epsilon|^3). \tag{4.40}
\]

Therefore we conclude from equation (4.39) that \( \alpha_1 = 0 \). Making \( \delta \) smaller if necessary, there exists a positive constant \( \alpha \) such that

\[
t^s_\epsilon \leq 1 + \alpha |\epsilon|^2 \tag{4.41}
\]

for \( |\epsilon| < \delta \). Let us now restrict our attention to the case \( \epsilon \in (0, \delta) \). Thus \( \det Dg_\epsilon = 1 - \epsilon < 1 \), that is, \( g_\epsilon \) is a volume decreasing mapping. Application of Corollary 4.7 and an elementary calculation yield

\[
t^s_\epsilon \leq \frac{t^u_\epsilon \log d}{\log d - t^s_\epsilon \log(1 - \epsilon)} \leq \frac{(1 + \alpha \epsilon^2) \log d}{\log d - (1 + \alpha \epsilon^2) \log(1 - \epsilon)} \leq \frac{\log d + \alpha \epsilon^2 \log d}{\log d + \epsilon(1 + \alpha \epsilon^2)} \leq \frac{\log d + \alpha \epsilon^2 \log d}{\log d + \epsilon}. \tag{4.42}
\]

Therefore, if \( \epsilon \) is sufficiently small, then \( t^s_\epsilon < 1 \). But this is a contradiction to equation (4.39) and the proof is complete. □

5 The general case

In this section we present some results about the dimension of the Julia sets of a general polynomial automorphism of \( \mathbb{C}^2 \). I.e., \( g \) is not assumed to be necessarily hyperbolic.

Let \( g \in \mathcal{H}_d \) and \( V \subset \mathbb{C}^2 \) compact such that \( K \subset \text{int}V \) and \( g^{\pm 1}(J^\pm \cap V) \subset J^\pm \cap V \). It is shown in [BS1] that a closed bidisk \( V \) of sufficiently large radius satisfies this property. We define

\[
s^\pm = \lim_{n \to \infty} \frac{1}{n} \log \left( \max \{ ||Dg^{\pm n}(p)|| : p \in J^\pm \cap V \} \right). \tag{5.43}
\]
The submultiplicativity of the operator norm guarantees the existence of the limit defining \( s^\pm \). Since all norms in \( \mathbb{C}^n \) are equivalent, the value of \( s^\pm \) is independent of the norm.

**Proposition 5.1** The value of \( s^\pm \) is independent of the choice of \( V \).

*Proof.* We have \( W^u(K) = K^+ \) and \( W^s(K) = K^- \) (see [BS1]). Therefore, the proof follows by elementary arguments (see [Wo1] for details). \( \Box \)

For a bounded set \( A \subset \mathbb{R}^n \) we denote \( \overline{\dim}_B A \) to be the upper box-dimension of \( A \) (see [M] for the definition). We have the inequality \( \dim_H A \leq \overline{\dim}_B A \) and equality for sufficiently regular sets \( A \). We will now consider the case when \( g \) is volume decreasing. In this situation it was observed in [FM] that \( K^- \) has Lebesgue measure zero and therefore \( K^- = J^- \). The following theorem provides an even stronger result.

**Theorem 5.2** Let \( g \in \mathcal{H}_d \) and \( |a| < 1 \). Then

\[
\overline{\dim}_B K^- \cap V \leq 4 - \frac{2 \log (|a|^{-1})}{s^-} < 4. \tag{5.44}
\]

*Proof.* Note that the real Jacobian determinant of \( g^{-1} \) as a mapping of \( \mathbb{R}^d = \mathbb{C}^2 \) is equal to \( |a|^{-2} \). Therefore, the result follows immediately from Theorem 1.1 of [Wo2]. \( \Box \)

Remarks.

i) Since \( W^u(K) = K^- \) we can define an exhaustion \( V_k = g^k(V \cap K^-) \) of \( K^- \). This implies that the upper bound in inequality (5.44) is also an upper bound for the Hausdorff dimension of \( K^- \).

ii) In particular Theorem 5.2 implies that, if \( g \in \mathcal{H}_d \) is not volume preserving, then the upper box-dimension of \( K^- \) is strictly smaller than 4.

We can apply Theorem 5.2 to show that if the modulus of the Jacobian determinant is small, then the Hausdorff dimension of \( K^- \) is close to 2. Note that for a hyperbolic mapping this result already follows from equation (4.35).

Let \( (P_{c_1})_{c_1 \in C_1}, ..., (P_{c_m})_{c_m \in C_m} \) be families of complex polynomials of fixed degree \( d \) \( \geq 2 \) such that \( C = C_1 \times ... \times C_m \) is a compact subset of \( \mathbb{C}^k \) for some \( k \in \mathbb{N} \). For \( a = (a_1, ..., a_m) \in (D(0,1) \setminus \{0\})^m \) and for \( c \in C \) we set \( g_{a,c} = g_{a_1,c_1} \circ ... \circ g_{a_m,c_m} \), where \( g_{a,c}(z, w) = (w, P_c(w) + a_i z) \). In the following we use the notation \( |a| = |a_1| \cdots |a_m| \). This implies that \( |\det Dg_{a,c}| = |a| \) for all \( (a, c) \in (D(0,1) \setminus \{0\})^m \times C \).

**Proposition 5.3** For all \( \epsilon > 0 \) there exists \( a_0 > 0 \) such that for all \( 0 < |a| < a_0 \) and all \( c \in C \) we have \( \dim_H K_{a,c}^- < 2 + \epsilon \).

*Proof.* The compactness of \( C \) implies that there exists \( V \subset \mathbb{C}^2 \) compact such that for all \( (a, c) \in (D(0,1) \setminus \{0\})^m \times C \) we have \( K_{a,c} \subset int V \) and \( g_{a,c}^{-1}(K_{a,c}^- \cap V) \subset K_{a,c}^- \cap V \). Therefore an elementary calculation implies that there exists \( \alpha > 0 \) such that

\[
\max \{ ||Dg_{a,c}^{-1}(p)|| : p \in K_{a,c}^- \cap V \} \leq \alpha |a|^{-1}. \tag{5.45}
\]
for all \((a, c) \in (D(0,1) \setminus \{0\})^m \times C\). The result follows now immediately from Theorem 5.2.

Remark. For\næss and Sibony [FS] considered complex Hénon mappings which are in a particular sense close to a hyperbolic quadratic polynomial in \(C\). They showed the hyperbolicity of these mappings and that the Hausdorff dimension of \(J^-\) is close to 2. Proposition 5.3 can be considered as a generalization of this result even to the case of nonhyperbolic mappings.

Finally we apply the Hölder continuity of the Green function to obtain a lower bound for the Hausdorff dimension of \(J^\pm\). Let \(g \in \mathcal{H}_d\). Let us recall the following definition introduced by Hubbard:

\[
G^\pm(p) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |g^{\pm n}(p)|.
\]

It is shown in [BS1] that \(G^\pm\) is continuous and plurisubharmonic on \(C^2\), and pluriharmonic on \(C^2 \setminus J^\pm\). The function \(G^\pm\) is the Green function of the sets \(K^\pm\) and \(J^\pm\), and is called Green function of \(g^{\pm 1}\).

Fornaess and Sibony showed in [FS] that the Green function \(G^\pm\) is in fact Hölder continuous and that the corresponding Hölder exponent plus 2 provides a lower bound for the Hausdorff dimension of \(J^\pm\). In [Wo1] we improved the estimate for the Hölder exponent and derived the following result.

**Theorem 5.4** Let \(g \in \mathcal{H}_d\) and \(s^\pm\) as in equation (5.43). Then for all \(s^\pm_0 > s^\pm\) the Green function \(G^\pm\) is Hölder continuous with Hölder exponent \(\frac{\log d}{s^\pm_0}\) on every compact subset of \(C^2\). Furthermore

\[
\dim H J^\pm \geq 2 + \frac{\log d}{s^\pm}.
\]

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