The Finite Matroid-based Valuation Conjecture is False

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A generative description for gross substitute valuations.

Valuation:

\[ u : 2^{[n]} \to \mathbb{R}, u(\emptyset) = 0, S \subseteq T \Rightarrow u(S) \leq u(T). \]

Take \( \text{conv}\{ (a, u(a)) : a \in \{0, 1\}^n \} \) and project down: get the regular subdivision \( \Delta_u \) of \([0, 1]^n\).

**Definition.** \( u \) is GS iff edges of \( \Delta_u \) have directions in \( \{ e_i - e_j, e_i \} \).

**Important subclass:** OXS valuations.  
(max-bipartite matching)

\[ u(I) = \max\{ \text{match of } I \text{ to RHS} \} \]

\[
\begin{array}{c}
1 & \xrightarrow{a} & A \\
& \xleftarrow{a'} & \\
2 & \xrightarrow{b} & B \\
\end{array}
\]

\[ u(1) = a, \quad u(2) = \max(a', b) \]

\[ u(\{1, 2\}) = \max(a + b, a') \]

\[ ^a\text{Equivalent to Kelso-Crawford via theorems of Murota and Tomizawa / Fujishige / Danilov-Koshevoy-Lang / Gelfand-Goresky-MacPherson-Serganova} \]
A generative description for gross substitute valuations.

**Valuation:**

\[ u : 2^n \rightarrow \mathbb{R}, \ u(\emptyset) = 0, \ S \subseteq T \Rightarrow u(S) \leq u(T). \]

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**Important subclass: OXS valuations.** (max-bipartite matching)

\[ u(I) = \max\{ \text{match of } I \text{ to RHS} \} \]

\[
\begin{align*}
1 & \xrightarrow{a} A & u(1) = a, u(2) = \max(a', b) \\
2 & \xrightarrow{b} B & u(\{1, 2\}) = \max(a + b, a')
\end{align*}
\]

\(^a\)Equivalent to Kelso-Crawford via theorems of Murota and Tomizawa / Fujishige / Danilov-Koshevoy-Lang / Gelfand-Goresky-MacPherson-Serganova

**Theorem.** (Lehmann-Lehmann-Nissan, 2006)\(^a\)

\( OXS \subsetneq GS \subsetneq \text{submodular valuations.} \)

**OXS and submodulars have generative descriptions:**

**OXS = merging of unit demands:**

\[
\begin{align*}
1 & \xrightarrow{a} A & 1 & \xrightarrow{a} A \\
2 & \xrightarrow{b} B & 2 & \xrightarrow{b} B
\end{align*}
\]

**Generative descriptions are useful! Is there a generative description for GS?**

\(^a\)Lehmann, Lehmann and Nissan: Combinatorial auctions with decreasing marginal utilities
Recap: **Theorem.** (Lehmann-Lehmann-Nissan 2006)

\[ OXS \subsetneq GS \subsetneq SM. \]

**OXS** = (unit demand, merging)

\[
\begin{array}{c}
1 \quad a \\
2 \quad b \\
\end{array}
\quad A
\quad =
\quad \begin{array}{c}
1 \quad a \\
2 \quad b \\
\end{array}
\quad A
\quad \ast
\quad \begin{array}{c}
1 \\
2 \\
\end{array}
\quad B
\]

Can start from **OXS** and make it bigger.

**Hatfield-Milgrom idea.** Add **endowment**, another operation that preserves **OXS**.

**EAV** = (unit demand, (merging, endowment))

**EAV Conjecture.** \(^a\) \(EAV = GS\).

\(^a\)Hatfield and Milgrom, Matching with contracts, 2005
Recap: **Theorem.** (Lehmann-Lehmann-Nissan 2006)

\[ \text{OXS} \subseteq \text{GS} \subseteq \text{SM}. \]

OXS = (unit demand, merging)

\[
\begin{array}{ccc}
1 & \xrightarrow{a} & A \\
2 & \xrightarrow{b} & B
\end{array}
\]

Can start from OXS and make it bigger.

**Hatfield-Milgrom idea.** Add endowment, another operation that preserves OXS.

EAV = (unit demand, (merging, endowment))

**EAV Conjecture.**\(^a\) \( \text{EAV} \subseteq \text{GS} \).

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**Theorem.** (Ostrovsky and Paes Leme)\(^a\)

EAVs are strongly exchangeable; GS may not. In particular, \( \text{EAV} \subsetneq \text{GS} \).

**Ostrovsky-Leme idea.** Start with a bigger generating set

Weighted matroid rank \( \rho^w : 2^{[n]} \to \mathbb{R} : \)

\[ \rho^w(S) = \max_{I \subseteq S} \sum_{i \in I} w_i. \]

\( \text{MBV}_{m,n} = (\{\rho^w : \text{ground set at most } [m]\}, (\text{merging, endowment}) ) \)

\[ \rho^w \in \text{GS}_n \Rightarrow \text{MBV}_{m,n} \subseteq \text{GS}_n. \]

Best possible case: \( m = n \). The only case where \( \{\rho^w\} \subset \text{GS}_n. \)

**Finite MBV conjecture.**\(^a\) For each \( n \), there exists \( m(n) \) s.t. \( \text{MBV}_{m(n),n} = \text{GS}_n. \)

\(^a\)Gross substitutes and endowed assignment valuations, 2015
Could the MBV conjecture be true?

Recap: OXS = (unit demand, merging)

\[
\begin{align*}
1 \xrightarrow{a} A & \quad 1 \xrightarrow{a} A \\
2 \xrightarrow{b} B & \quad \ast \quad 2 \xrightarrow{b} B
\end{align*}
\]

EAV = (unit demand, (merging, endowment)).

\[
MBV_{m,n} = \{\rho^w : \text{ground set at most } [m]\},
\]

(merging, endowment))

**Finite MBV conjecture.** For each \( n, \exists m(n) \) s.t.

\[
MBV_{m(n),n} = GS_n.
\]

---

**Snapshots**

![He loves me...](image1)

He loves me...

![He loves me not...](image2)

He loves me not...
Could the MBV conjecture be true?

Recap: OXS = (unit demand, merging)

\[
\begin{align*}
1 & \overset{a}{\longrightarrow} A \\
2 & \overset{b}{\longrightarrow} B
\end{align*}
\]

\[
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Finite MBV conjecture.\(^a\) For each \(n\), \(\exists m(n)\) s.t. 

\[
MBV_{m(n),n} = GS_n.
\]

\[\text{Why it could be true.} \]

Gross substitutes = generalization of matroid ranks

1. \(GS_n\) contains all matroid ranks on ground set \(S \subseteq [n]\).

2. Operations that preserve GS are generalizations of matroid operations.

merging = matroid union; endowment = contraction

3. Good intuition on why EAV fails.

OXS generalizes transversals. EAV generalizes gammoids. Not all matroids are gammoids \(\Rightarrow\) EAV fails.

4. MBV contains all weighted matroid ranks \(\Rightarrow\) big enough...?

Snapshots

He loves me...
He loves me not...
Unfortunately...

Best possible case $m(n) = n$ is not possible.

**Theorem.** (T. 2019). For $m(n) = n$, $n \geq 4 \iff MBV_{n,n} \subsetneq GS_n$. 
Why the MBV fails for $m(n) = n$

$MBV_{m,n} = \{ \rho^w : \text{ground set at most } [m] \}$, 
(merging, endowment)

**Finite MBV conjecture.**
For each $n$, $\exists m(n)$ s.t. 
$MBV_{m(n),n} = GS_n$.

**Theorem.** (T. 2019). For $m(n) = n$, 
$n \geq 4 \iff MBV_{n,n} \subsetneq GS_n$

---

Why the MBV fails for $m(n) = n$

1. Endowment+merging cannot create irreducible valuations $u : 2^{[n]} \to \mathbb{R}$, 
   
   $u = v \ast w, v, w \in GS_n \Rightarrow u = v \text{ or } u = w$.

   Analogue of irreducible matroid ranks.

2. Let 
   
   $G_n = \{ v : 2^{[n]} \to \mathbb{R}, v \in GS_n, v \text{ irreducible } \}$.

   Then 
   
   finite MBV conjecture is true $\Rightarrow G_n \subseteq \{ \rho^w \}$.

3. Counter-example: construct a family of irreducibles $C_n$ outside of weighted matroid ranks:

   $C_n \subset G_n \text{ but } C_n \cap \{ \rho^w \} = \emptyset$. 

Recipe 1. $C_n = \text{partition valuations}$. 

Simple family indexed by set partitions of $[n]$. 

(+) Works for $n \geq 4$
(+) direct proof
(-) little insight.
Recipe 2: geometric construction of irreducibles

Geometry of merging. If \( F \in \Delta_{u \ast v} \), then 
\[
F = (F^u + F^v) \cap [0, 1]^n \quad \text{for some} \quad F^u \in \Delta_u, \\
F^v \in \Delta_v.
\]

\( M \)-irreducible polytopes are obstructions to reducibility:
\[
P = (P^1 + P^2) \cap [0, 1]^n \Rightarrow P = P^1 \quad \text{or} \quad P = P^2.
\]

Lemma. If \( \Delta_u \) has a full-dimensional \( M \)-irreducible face \( F \), then \( u \) is irreducible.

Proposition. For \( P \subset [0, 1]^n \) is \( M \uparrow \), define
\[
\rho_P : 2^{[n]} \to \mathbb{R}
\]
\[
\rho_P(I) = \max\{x_I : x \in P\}.
\]

If \( \rho_P \) is the rank function of an irreducible matroid, then \( P \) is \( M \)-irreducible.

Bonus. \( \rho^w \) is irreducible in \( MBV_{n,n} \) iff the matroid is irreducible. So \( MBV_{n,n} \) is generated by weighted rank of irreducible matroids.
Recipe 2: geometric construction of irreducibles

Geometry of merging. If \( F \in \Delta_{uv} \), then
\[
F = (F^u + F^v) \cap [0, 1]^n
\]
for some \( F^u \in \Delta_u \), \( F^v \in \Delta_v \).

\( M \)-irreducible polytopes are obstructions to reducibility:
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Recipe for irreducible valuations.

a. \( \rho^w \) = weighted rank of irreducible matroid of rank \( \geq 2 \) (smallest one is \( M(K_4), n = 6 \))
b. Modify \( \Delta_{\rho^w} \): split the independence polytope
\[
v = \rho^w + c \cdot (1 - 1_\emptyset)
\]
c. \( v \in GS_n, \Delta_v \) has an \( M \)-irreducible face, \( \Rightarrow v \) is irreducible.
Summary.

\[ OXS = (\text{unit demand, merging}) \]

\[
\begin{align*}
1 & \xrightarrow{a} A \\
2 & \xrightarrow{b} B
\end{align*}
\]

\[ MBV_{m,n} = (\{\rho^w : \text{ground set at most } [m]\}, \text{(merging, endowment)}) \]

\[ EAV = (\text{unit demand, (merging, endowment)}) \]

We have

\[ OXS_n \subset EAV_n \subset MBV_{n,n} \subset GS_n \subset SM_n. \]

What’s next?

- More operations?
- Consider \( m(n) > n? \)
- Characterize all irreducibles?
- A different approach: matroid rank sums?

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Thanks to: Rakesh Vohra, Renato Paes Leme, Kazuo Murota, and two anonymous referees.
Summary.

$OXS = (\text{unit demand, merging})$

\[ \begin{array}{c}
1 \xrightarrow{\alpha} A \\
2 \xrightarrow{\beta} B
\end{array} \]

$EAV = (\text{unit demand, (merging, endowment)})$.

$MBV_{m,n} = (\{\rho^w : \text{ground set at most } [m]\},$

(merging, endowment))

We have

$OXS_n \subset EAV_n \subset MBV_{n,n} \subset GS_n \subset SM_n$. 

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