COUNTING CLOSED GEODESICS ON RIEMANNIAN MANIFOLDS

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Abstract. Associated with every closed oriented smooth manifold $M$, let $\mathcal{R}_M$ denote the space of all pairs $(L, g)$, where $g$ is a Riemannian metric on $M$ and $L$ is a real number which is not the length of any closed $g$-geodesics. A locally constant geodesic count function $\pi_M : \mathcal{R}_M \to \mathbb{Z}$ is defined which virtually counts the number of closed $g$-geodesics of length less than $L$ at $(L, g) \in \mathcal{R}_M$. One can further refine the count function by fixing the free homotopy type of the closed geodesics. When $g$ is negatively curved, $\pi_M(L, g)$ is precisely the number of prime closed $g$-geodesics which have length smaller than $L$.

1. Introduction

Closed geodesics on negatively curved Riemannian manifolds have been investigated for decades. Phillips and Sarnak obtained the asymptotics of the count function over manifolds with constant negative curvature [PS87]. Their correspondence between the count function and the topological entropy of the metric was generalized to manifolds with variable negative curvature by Margulis [Mar69]. The result was sharpened by Anantharaman [Ana00] and by Pollicott and Sharp [PS98], and extended to more general classes of flows [PS01]. There are serious obstacles when one tries to study the same problem over Riemannian manifolds with arbitrary curvature. In particular, the set $\mathcal{Z}_g(L)$ of prime closed geodesics of length at most $L$ is not necessarily finite for an arbitrary metric $g$. Even for nice Riemannian metrics $g$ (i.e. such that $\mathcal{Z}_g(L)$ is finite for all $L$) a satisfactory way of counting the elements of $\mathcal{Z}_g(L)$ (i.e. so that certain natural criteria are satisfied) is not discussed in the literature.

The set of smooth Riemannian metrics on a closed smooth oriented manifold $M$ is denoted by $\mathcal{G}(M)$ and is equipped with $C^\ell$ topology for some $\ell \in \mathbb{Z}^+$. For $g \in \mathcal{G}(M)$, let $\mathcal{L}_g \subset \mathbb{R}$ denote the length spectrum of $g$, consisting of lengths of closed geodesics, including multiple covers. $\mathcal{L}_g$ is closed and has zero Lebesgue measure. Let $\mathcal{R}_g$ denote the open and dense complement of $\mathcal{L}_g$ in $\mathbb{R}$ and

$$\mathcal{R}_M = \{(L, g) \in \mathbb{R} \times \mathcal{G}(M) \mid L \in \mathcal{R}_g\} \subset \mathbb{R} \times \mathcal{G}(M).$$

Denote the set of free homotopy classes of closed loops on the manifold $M$ by $\mathcal{C}_M$, which is equipped with a natural action of $\mathbb{Z}$. Let $\mathbb{Z}[C_M]$ denote the $\mathbb{Z}$-module generated by $C_M$, where the generator associated with $\alpha \in C_M$ is denoted by $h^\alpha$.

**Theorem 1.1.** There is a locally constant geodesic count function $\pi_M : \mathcal{R}_M \to \mathbb{Z}[C_M]$, which virtually counts the number of closed $g$-geodesics of length less than $L$ and representing $\alpha \in C_M$, as the coefficient of $h^\alpha$ in $\pi_M(L, g)$. If $g \in \mathcal{G}(M)$ is negatively curved then $\pi_M(L, g)$ is the sum of $h^\alpha$ such that the unique closed geodesic representing $\alpha$ has length less than $L$.

Given $(L, g) \in \mathcal{R}_M$, the moduli space $\mathcal{Z}_g(L)$ may contain an infinite number of $g$-geodesics, and such geodesics may even appear in families. Nevertheless, the main purpose of this paper is to assign a well-justified virtual contribution to such moduli spaces. In particular, when $g \in \mathcal{G}(M)$ is nice and $\gamma$ is a prime closed $g$-geodesic of length $\ell_\gamma$, we present a definition of the local contribution $n_d(\gamma, g)$ of the $d$-fold cover of $\gamma$ to the geodesic count $\pi_M(L, g)$ when $L > d\ell_\gamma$. 

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For a path \( \vec{g} = \{g_t\}_{t \in [a,b]} \) in \( \mathcal{G}(M) \), let \( \mathcal{Z}_g(L) \) denote the set of all pairs \((t,\gamma)\) where \( t \in [a,b] \) and \( \gamma \) is a prime closed \( g_t \)-geodesics of length \( L \). \( \vec{g} \) is called nice if \( \mathcal{Z}_g(L) \) is finite for all \( L \in \mathbb{R}^+ \). A generic path connecting two nice metrics is nice. When \( \vec{g} \) is nice and \( \gamma \) is a prime closed \( g_t \)-geodesic of length \( L \), our definition of the local contribution may be extended to define \( n_d(\gamma, \vec{g}, t) \); the local contribution of the \( d \)-fold cover of \( \gamma \) to the count \( \pi_d(\gamma, \vec{g}, t) \) of closed \( \vec{g} \)-geodesics of length \( dL \). For a nice metric \( g \in \mathcal{G}(M) \) we have \( n_d(\gamma, g) = n_d(\gamma, \{ \frac{1}{d} g \}_{t=1, d}) \).

**Theorem 1.2.** If \( g \in \mathcal{G}(M) \) is negatively curved in a neighborhood of a prime closed geodesic \( \gamma \) then \( n_1(\gamma, g) = 1 \) and \( n_d(\gamma, g) = 0 \) for \( d > 1 \). Moreover, for a nice path \( \vec{g} = \{g_t\}_{t \in [a,b]} \) in \( \mathcal{G}(M) \) which connects the nice metrics \( g_a \) and \( g_b \), and for \( L \notin \mathcal{L}_{g_a} \cup \mathcal{L}_{g_b} \), we have

\[
\pi_M(L, g_a) = \sum_{d=1}^{\infty} \sum_{\gamma \in \mathcal{Z}_{g_a}(L/d)} n_d(\gamma, g_a) \mathcal{H}^{d[\gamma]}
\]

and

\[
\pi_M(L, g_b) - \pi_M(L, g_a) = \sum_{d=1}^{\infty} \sum_{(t, \gamma) \in \mathcal{Z}_{g_a}(L/d)} n_d(\gamma, g_b, t) \mathcal{H}^{d[\gamma]}.\]

Associated with \( \gamma \in \mathcal{Z}_g(L) \) is a Fredholm operator \( \Psi_\gamma \). For generic \( g \), the sign \( \epsilon(\gamma) = n_1(\gamma, g) \) in \( \{-1, +1\} \) is determined by the spectral flow connecting \( \Psi_\gamma \) to \( \Psi_{\gamma'} \), the operator associated with a closed geodesic \( \gamma' \) inside a negatively curved manifold. For a path \( \vec{g} \) in \( \mathcal{G}(M) \), a sequence in \( \mathcal{Z}_g(L) \) may converge to the double cover of a geodesic \( \gamma \in \mathcal{Z}_g(L/2) \). This phenomenon is described by a local Kuranishi model, which describes how contributions from multiple covers of a prime geodesic should be included in the count function. The technical steps are closely related to Taubes’ construction of Gromov invariants for symplectic 4-manifolds [Tau90]. Moreover, techniques from author’s earlier work [Eft16] are used in the proof. Having a correct analysis of the generic case provides the required tools for defining \( n_d(\gamma, g), n_d(\gamma, g_{\vec{g}}, t) \) and \( \pi_M \) in the general case.

Let \( Z \) denote the zero section in the tangent bundle \( TM \) of the \( n \)-manifold \( M \) and \( \sim \) denote the equivalence relation which identifies any vector in \( TM \setminus Z \) with its non-zero multiples. The quotient \( S_M = (TM \setminus Z) / \sim \) is then a closed smooth oriented manifold of dimension \( 2n - 1 \), which may be identified as the unit tangent bundle of \( M \) in the presence of a Riemannian metric. Denote the space of smooth vector fields on \( S_M \) by \( \mathcal{V}(S_M) \). Every Riemannian metric \( g \in \mathcal{G}(M) \) corresponds to a vector field \( \zeta_g \in \mathcal{V}(S_M) \) and closed geodesics on the Riemannian manifold \((M, g)\) are in correspondence with closed orbits of the geodesic flow (on \( S_M \)) associated with the Riemannian metric \( g \) (and the vector field \( \zeta_g \)). Counting closed geodesics on \( M \) is then closely related to the study of periodic orbits of vector fields on \( S_M \) and some standard fixed point theorems for dynamical systems, as will be discussed in [Eft].

The paper is organized as follows. In Section 2 we construct the appropriate space of generic Riemannian metrics so that the required finiteness properties are satisfied for the corresponding closed geodesics. In Section 3 we study limits of sequences of geodesics, especially when they converge to a multiply covered closed geodesic. The concept of super-rigidity is introduced as a byproduct. The sign for super-rigid closed geodesics and the count function are introduced and studied in Section 4. The proof of Theorem 1.1 for generic paths of Riemannian metrics appears in Section 5. We then discuss the generalization of definitions and the main theorem to the case of arbitrary Riemannian metrics and paths of metrics connecting them in Section 6.

2. Modiuli space of closed geodesics

2.1. Moduli space of geodesics on Riemannian manifolds. Let \( M \) denote a smooth closed manifold of dimension \( n \). Every Riemannian metric \( g \) on \( M \) is a positive definite symmetric 2-tensor on the tangent space \( TM \) of \( M \). Correspondingly, for any fixed positive integer \( \ell \), let \( \mathcal{H} = \mathcal{H}^\ell(M) \)
denote the $C^\ell$ completion of the space of smooth symmetric 2-tensors $g \in \text{Hom}(TM \otimes TM, \mathbb{R})$ and $\mathcal{G} = \mathcal{G}^\ell(M)$ denote the subset of $\mathcal{H}$ which consists of positive definite tensors. Let us also fix the integers $k, p > 1$ and denote the space of immersions from $S^1$ to $M$ which are of class $W^{k,p}$ by $\mathcal{X} = \mathcal{X}_{k,p}(M)$. Given $g \in \mathcal{G}$, a map $\gamma$ in $\mathcal{X}$ is called a closed $g$-geodesic (or closed geodesic, for short) if it satisfies $\nabla \dot{\gamma} \dot{\gamma} = 0$, where $\dot{\gamma} = \frac{d\gamma}{dt}$ and $\nabla = \nabla^g$ denotes the Levi-Civita connection associated with $g$. If $\gamma$ is given by $(\gamma(t))_{t=1}^\ell$, in local coordinates, the equation $\nabla \dot{\gamma} \dot{\gamma} = 0$ is equivalent to
\[
\frac{d^2\gamma_i}{dt^2} + \Gamma^i_{jk}(\gamma) \frac{d\gamma_j}{dt} \frac{d\gamma_k}{dt} = 0.
\]
Here $\Gamma^i_{jk}$ denote the Christoffel symbols of the metric $g$ in local coordinates. Clearly, $S^1$ (as the group of rotations) acts on the space of all closed geodesics; for $\zeta \in S^1$ and $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to M$ a closed geodesic we define $\gamma_\zeta = \zeta \ast \gamma : S^1 \to M$ by $\gamma_\zeta(t) = \gamma(t + \zeta)$. Our goal in this paper is to count closed geodesics up to this rotation action, as will be discussed in the following sections.

The pull-back of the tangent space $TM$ under a geodesic $\gamma$ is a vector bundle $W = \gamma^*TM$ over $S^1$. Let $A^r(S^1, W)$ denote the vector space of $W$-valued $r$-forms on $S^1$. The induced connection is given by a $n \times n$ matrix product $\theta = (\theta^i_{\ i})_{i,j=1}^n$ of 1-forms and the associated Hodge operator $\ast = \ast_g$ may be used to define an inner product on $A^r(S^1, W)$, by $\langle \phi, \psi \rangle = \int_{S^1} \phi \wedge \ast(\psi)$. Define the linear map
\[
d_g : A^{r+1}(S^1, W) \to A^r(S^1, W), \quad d_g(\phi) = d\phi + \theta \wedge \phi = (d\phi^i + \theta^i_{\ ji} \wedge \phi^j).\]
Let $\delta_g : A^{r+1}(S^1, W) \to A^r(S^1, W)$ denote the adjoint operator defined by $\langle \delta_g(\phi), \psi \rangle = \langle \phi, d_g(\psi) \rangle$. In local coordinates $\theta^i_{\ ji} = \Gamma^i_{jk} \gamma^k dt$ and $(\delta_g \phi)^i = -d\phi^i + \Gamma^i_{jk} \gamma^j \phi^k$ for $\phi = (\phi^i)_i \in A^1(S^1, W)$. Correspondingly, we may define the Laplace operator $\Delta_g = -(d_g \circ \delta_g + \delta_g \circ d_g)$. Geodesics are classified as the curves $\gamma$ such that $\delta_g(d(\gamma)) = 0$, and are thus harmonic, c.f. [ES64]. For $\gamma \in \mathcal{X}$ we have
\begin{equation}
\frac{d}{dt} \|\dot{\gamma}\|_g^2 = \langle \dot{\gamma}, \delta_g(d(\gamma)) \rangle_g \quad \forall \ g \in \mathcal{G}.
\end{equation}
In particular, geodesics are parameterized proportionally to the arc-length and every somewhere injective geodesic is an embedding away from a finite set of transverse self-intersections. If for $s \in \mathbb{R}$ and $\gamma \in \mathcal{X}$ we have $s \dot{\gamma} + \delta_g(d(\gamma)) = 0$, then from (1) we obtain
\[
\frac{d}{dt} \|\dot{\gamma}\|_g^2 = \langle \dot{\gamma}, -s \dot{\gamma} \rangle_g = -s \|\dot{\gamma}\|_g^2 \quad \Rightarrow \quad \|\dot{\gamma}(t)\|_g^2 = Ae^{-st} \quad \forall \ t \in \mathbb{R}.
\]
For the right-hand-side to be periodic, it is necessary that $A = 0$ or $s = 0$. The former corresponds to the constant map, while the latter is the case for a geodesic. Theorem 2.1 guarantees that for a generic metric on $M$, the space of geodesics is a zero dimensional manifold.

**Theorem 2.1.** Let $M$ be a closed smooth manifold. With the above notation fixed, the space

\[
\mathcal{Z} = \mathcal{Z}_{k,p}(M) := \{(g, \gamma) \mid g \in \mathcal{G}^\ell(M) \quad \text{and} \quad \gamma \in \mathcal{X}_{k,p} \quad \text{is a somewhere injective} \quad g-\text{geodesic}\}
\]

is a separable $C^{\ell-k-1}$ Banach manifold. There is a Bair subset $\mathcal{G}^{\ell,*}(M) \subset \mathcal{G}^\ell(M)$ such that for every $g \in \mathcal{G}^{\ell,*}(M)$ the quotient of the one dimensional manifold $\mathcal{Z}_g = \{\gamma : S^1 \to M \mid (g, \gamma) \in \mathcal{Z}\}$ by the rotation action of $S^1$ is a zero dimensional manifold. Moreover, if $n \geq 3$ we may choose the Bair subset $\mathcal{G}^{\ell,*}(M) \subset \mathcal{G}^\ell(M)$ so that for every $g \in \mathcal{G}^{\ell,*}(M)$ all maps $\gamma \in \mathcal{Z}_g$ are $C^\ell$ embeddings.

**Proof.** Denote the Sobolev space $\Gamma^{k,p}(S^1, \gamma^*TM)$ of sections with values in $\gamma^*TM$ by $A^{k,p}_0(\gamma^*TM)$. Let $\mathcal{E} = \mathcal{E}^{\ell}_{k,p}$ denote the vector space over $\mathbb{R} \times \mathcal{G} \times \mathcal{X}$ such that over a point $(s, g, \gamma) \in \mathbb{R} \times \mathcal{G} \times \mathcal{X}$ the fiber $\mathcal{E}_{(s,g,\gamma)}$ is given by $A^{k-2,p}_0(\gamma^*TM)$. Define the section $\Phi : \mathbb{R} \times \mathcal{G} \times \mathcal{X} \to \mathcal{E}$ of the vector bundle $\mathcal{E} \to \mathbb{R} \times \mathcal{G} \times \mathcal{X}$ by
\[
\Phi(s, g, \gamma) = s \cdot \dot{\gamma} + \delta_g(d(\gamma)) \in \mathcal{E}_{(s,g,\gamma)} \quad \forall \ (s, g, \gamma) \in \mathbb{R} \times \mathcal{G} \times \mathcal{X}.
\]
Let $d\Phi$ denote the differential of $\Phi$. At a zero $\beta = (s, g, \gamma) \in \mathbb{R} \times G \times \mathcal{X}$ of $\Phi$, where we are forced to have $s = 0$, the tangent space to $E$ at $\Phi(\beta)$ naturally decomposes as
\[ T_{\Phi(\beta)}E = \mathbb{R} \oplus T_gG \oplus T_\gamma\mathcal{X} \oplus \mathcal{E}_\beta, \]
and we may thus project over $\mathcal{E}_\beta$ to obtain a map
\[ F = D_\beta\Phi : \mathbb{R} \oplus T_gG \oplus T_\gamma\mathcal{X} = \mathbb{R} \oplus H^{\ell-1}(M) \oplus A^0_{k,p}(\gamma^*TM) \to \mathcal{E}_\beta = A^0_{k-2,p}(\gamma^*TM). \]

Note that $\mathcal{Z}$ may be identified with $\Phi^{-1}(0) = \{0\} \times \mathcal{Z} \subset \mathbb{R} \times G \times \mathcal{X}$. The Sobolev spaces $A^0_{k,p}(\gamma^*TM)$ form a bundle $\mathcal{F} \to \mathbb{R} \times G \times \mathcal{X}$. Let us denote the restriction of $F$ to $A^0_{k,p}(\gamma^*TM)$ by $\Psi$, which gives a bundle homomorphism $\Psi : \mathcal{F} \to \mathcal{E}$. Over each point $\beta$ as above, $\Psi$ is a Fredholm operator of index zero. We will sometimes write $\Psi_{g,\gamma}$ or $\Psi(g,\gamma)$ to emphasize that the restriction of $\Psi$ to the fiber over the point $(g,\gamma)$ is considered. The adjoint of $\Psi$, is another Fredholm operator
\[ \Psi^* : A^0_{k,q}(\gamma^*TM) \to A^0_{k-2,q}(\gamma^*TM), \]
where $\frac{1}{p} + \frac{1}{q} = 1$. Every element in the cokernel of $F$ may be represented by an element $\zeta \in \text{Ker}(\Psi^*)$ which is orthogonal to the image of $F$. Let us assume that $\zeta \neq 0$ is such a section in $A^0_{k,q}(\gamma^*TM)$.

Choose a point $t \in S^1$ such that $\gamma$ is injective at $t$ and $\zeta(t)$ is non-zero. Choose the local coordinates around $z = \gamma(t)$ so that $\gamma$ is given by $(t, 0, ..., 0)$ and the matrix $g(\gamma(t))$ is given by the identity matrix. Moreover, we may assume that the Christoffel symbols $\Gamma^i_{jk}$ vanish along the path. Let $G \in \mathcal{H}$ be given by $(G_{ij})_{i,j=1}^m$ in these coordinates. We may then compute
\[ F(0, G, 0)^i = -2g^{im}(\partial_{\gamma}G_{mk} + \partial_kG_{mj} - \partial_mG_{jk} - G_{ml}\Gamma^l_{jk})\gamma^{ij}\hat{e}^k = (2\partial_{\gamma}G_{11} - 4\partial_1G_{11})\gamma(t). \]

Let us set $G_{ij} = 0$, except for $f = G_{1,1}$, which is assumed to be a bump function supported in a neighborhood of $z$. It follows that $F(0, G, 0)^i$ is given by $2\partial_i f$ for $i > 1$ and by $-2\partial_1 f$ for $i = 1$. The values of the bump functions $(\partial_i f) \circ \gamma$ in a neighborhood of $t$ on $S^1$ may be chosen arbitrarily for $i > 1$. Since $\zeta$ is orthogonal to $F(0, G, 0)$, it follows that $\zeta^i$ is zero for $i > 1$. In particular, if we set $\lambda(t) = f(\gamma(t))$, the inner product of $\zeta$ with $F(0, G, 0)$ is given by
\[ 0 = \langle F(0, G, 0), \zeta \rangle_g = \int_{t-\epsilon}^{t+\epsilon} \zeta^i(s)\lambda(s)ds = - \int_{t-\epsilon}^{t+\epsilon} \frac{d\zeta^1}{dt}(s)\lambda(s)ds. \]
Here $\epsilon > 0$ is chosen so that $\lambda$ is supported on $(t-\epsilon, t+\epsilon)$. In particular, it follows from the above discussion that $d\zeta^1/dt$ vanishes at $t$. Obviously, we can replace $t$ by a nearby value, and it is implied that $\zeta$ is a constant multiple of $\gamma$ is a neighborhood of $t$. Since the above observation is true for every point $t$ where $\gamma$ is injective, it follows that $\zeta$ is a constant multiple of $\gamma$ over $S^1$. But $\zeta$ is also orthogonal to $F(1, 0, 0) = \hat{\gamma}$. This last observation implies that $\zeta$ is zero, and $F$ is thus surjective. From the surjectivity of $F$ it follows that $\{0\} \times \mathcal{Z}$ is smoothly cut out and is thus a separable Banach manifold of class $C^{\ell-k-1}$, and that the projection map from $\mathcal{Z}$ to $G = G^\ell(M)$ is a Fredholm map of index 1. For $g$ in a subset $\mathcal{G}_1$ of $G$, which consists of the regular values of the projection map, $\mathcal{Z}_g$ is a 1-dimensional manifold, which comes with the rotation action of $S^1$. It follows that every such $\mathcal{Z}_g$ is a union of circles, and that the quotient $\tilde{\mathcal{Z}}_g = \mathcal{Z}_g/S^1$ is a zero-dimensional manifold. We will usually abuse the notation and identify $\mathcal{Z}$ and $\mathcal{Z}_g$ with their quotients under the action of $S^1$.

Similarly, we may consider the complement $S^* \subset S^1 \times S^1$ of the diagonal and let $\mathcal{X}^* \subset \mathcal{X} \times S^*$ denote the subset consisting of all triples $(\gamma, x, y)$ such that $\gamma(x) = \gamma(y)$. Clearly, $\mathcal{X}^*$ is a smooth Banach manifold. We may pull the bundle $\mathcal{E} \to \mathbb{R} \times G \times \mathcal{X}$ back to $\mathbb{R} \times G \times \mathcal{X} \times S^*$, and then restrict it to $\mathbb{R} \times G \times \mathcal{X} \times S^*$. The new bundle is still denoted by $\mathcal{E}$. The section $\Phi$ defined before induces a
section $\Phi^*: \mathbb{R} \times \mathcal{G} \times \mathcal{X}^* \to \mathcal{E}$. The above argument may be repeated to show that $\Phi^*$ is transverse to the zero section and thus

$$Z^* := \left\{ (s, g, \gamma, x, y) \mid (s, g, \gamma) \in Z \text{ and } \gamma(x) = \gamma(y) \right\}$$

is a Banach manifold which fibers over $\mathcal{G}$ via a Fredholm projection map of index $3 - m$. We denote the set of regular values of this second projection map by $\mathcal{G}_2$. We then set $\mathcal{G}^* = \mathcal{G}^{\ell, \ast}(M) = \mathcal{G}_1 \cap \mathcal{G}_2$. If $g \in \mathcal{G}^*$, then $Z_g$ is a zero dimensional manifold (since $g \in \mathcal{G}_1$). If $m \geq 3$, then all maps in $Z_g$ are injective, since otherwise, there are $x \neq y$ in $S^1$ and $\gamma \in Z_g$ such that $\gamma(x) = \gamma(y)$. But this implies that $(\gamma, x, y)$, along with its orbit under the rotation action of $S^1$, belongs to $Z_g^*$, while this latter space is a manifold of dimension $3 - m$ (and thus can not accommodate the 1-dimensional orbit of $(\gamma, x, y)$). This completes the proof.

**2.2. The energy foliation.** The energy functional is defined on $\mathcal{G}^{\ell} \times \mathcal{X}_{\kappa,p}$ by

$$E : \mathbb{R} \times \mathcal{G}^{\ell} \times \mathcal{X}_{\kappa,p} \to \mathbb{R}^+, \quad E(s, g, \gamma) := \int_{S^1} \|\dot{\gamma}\|^2_g.$$ 

In order to show that the pre-image

$$Z^\ell = Z_{k,p}^{0}(M) = E^{-1}(c) \subset Z = Z_{k,p}$$

is a sub-manifold, we need to show that the linearization $dE : T_{(s,g,\gamma)}Z \to \mathbb{R}$ is surjective for all $(s, g, \gamma) \in Z^\ell$. The tangent space to $Z$ at $\beta = (s, g, \gamma)$ is the kernel of the operator

$$F = D_\beta \Phi : \mathbb{R} \oplus T_g \mathcal{G} \oplus T_{\gamma} \mathcal{X} \to \mathcal{E}_\beta$$

from the proof of Theorem 2.1. Note that the Riemannian metric $g$ may be regarded as a tangent vector in $T_g \mathcal{G}$, and that $F(0, g, 0) = 0$. The easiest way to see this is to consider the path $\{\beta_r = (s, rg, \gamma)\}_{r \in (1-\epsilon, 1+\epsilon)}$ in $Z$, and note that $(0, g, 0)$ is the tangent to this path at $r = 1$. Direct computation implies that for every $H \in T_g \mathcal{G}$ we have

$$dE(0, H, 0) = \int_{S^1} H_{\gamma}(t)(\dot{\gamma}, \ddot{\gamma}) dt.$$

In particular, we have $dE(0, g, 0) = E(0, \gamma) > 0$, which means that $dE$ is surjective over $Z$.

The above observation implies that the level sets of the energy functional $E : Z \to \mathbb{R}$ are submanifolds of $Z$, which give the energy foliation of $Z$.

**2.3. Linearization of the differential.** Let us assume that $(M, g)$ is a Riemannian manifold as before and that $\gamma : \mathbb{R} \to M$ is a (possibly non-simple) geodesic. Choose a point on $\gamma$, which corresponds to $0 \in \mathbb{R}$, and choose normal coordinates $(x^1, ..., x^n)$ at $\gamma(0)$ so that $g_{ij}(\gamma(0)) = \delta_{ij}$ and $\Gamma^i_{jk}(\gamma(0)) = 0$. We simplify our notation by denoting $d\gamma/dt$ by $\dot{\gamma}$, and $d^2\gamma/dt^2$ by $\ddot{\gamma}$. Similarly, denote $d^2\gamma/dt^2$ by $\dot{\gamma}$ and $d^2\gamma/dt^2$ by $\ddot{\gamma}$. Choose $\zeta \in \Gamma(\mathbb{R}, \gamma^* TM)$, and represent $\zeta \circ \gamma$ as $\eta^i(t) \partial_i = \zeta^i(\gamma(t)) \partial_i$. A direct computation, using the fact that the Christoffel symbols vanish at $\gamma(0)$, implies that

$$\Psi(\zeta)(0) = D\Phi(\zeta)(0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} D\Phi(\exp_0(\epsilon \zeta))(0) = (\partial_j \Gamma^i_{kl}) \eta^j \dot{\gamma}^k \dot{\gamma}^l.$$

Let us assume that $\nabla$ is the Levi-Civita connection associated with the metric $g$ on the manifold $M$. Our first observation is that

$$\nabla_\gamma \nabla_\zeta \zeta = \gamma^k \dot{\gamma}^l \eta^j \partial_k \Gamma^i_{jl} + \dot{\gamma}^k \dot{\gamma}^l \partial_k \partial_j \zeta^i + \partial_j \zeta^i \dot{\gamma}^j = \gamma^k \dot{\gamma}^l \eta^j \partial_k \Gamma^i_{jl} + \dot{\eta}^i$$

$$\nabla_\zeta \nabla_\dot{\gamma} \zeta = \zeta^k(\partial_k \dot{\gamma}^j)(\partial_j \zeta^i) + \dot{\zeta}^k \dot{\gamma}^j(\partial_j \partial_k \zeta^i) + \zeta^k \dot{\gamma}^l \dot{\gamma}^j \partial_k \Gamma^i_{jl}$$

$$\nabla_\gamma \nabla_\dot{\gamma} \zeta = \gamma^k(\partial_k \dot{\gamma}^j)(\partial_j \zeta^i) + \dot{\gamma}^k \dot{\gamma}^j(\partial_j \partial_k \zeta^i) + \dot{\gamma}^k \dot{\gamma}^j \dot{\zeta}^i \partial_k \Gamma^i_{jl}$$

$$\nabla_{\zeta, \gamma} \dot{\gamma} = \zeta^k(\partial_k \dot{\gamma}^j)(\partial_j \zeta^i) - \dot{\gamma}^k(\partial_k \zeta^j)(\partial_j \dot{\gamma}^i).$$
From the above four equations we find that at $t = 0$:

$$
\nabla_\gamma \nabla_\gamma \zeta + R(\zeta, \dot{\gamma}) \dot{\gamma} = \nabla_\gamma \nabla_\gamma \zeta + \nabla_\zeta \nabla_\gamma \dot{\gamma} - \nabla_\gamma \nabla \zeta \dot{\gamma} - \nabla_{[\zeta, \gamma]} \dot{\gamma}
$$

$$
= \left( \zeta^k \dot{\gamma}^i \partial_k \Gamma_{j,i}^l + \eta^i \right) + \left( \zeta^k (\partial_j \dot{\gamma}^i)(\partial_k \zeta^i) + \zeta^i \dot{\gamma}^j (\partial_j \partial_k \zeta^i) + \zeta^k \dot{\gamma}^i \dot{\gamma}^j \partial_k \Gamma_{j,i}^l \right)
$$

$$
- \left( \dot{\gamma}^k (\partial_j \gamma^i)(\partial_k \zeta^i) + \dot{\gamma}^i \zeta^j (\partial_j \partial_k \dot{\gamma}^i) + \dot{\gamma}^j \zeta^i (\partial_j \partial_k \dot{\gamma}^i) \right)
$$

$$
= \eta^i + \partial_k \zeta^i \partial_k \gamma^j \partial_k \Gamma_{j,i}^l |_{l=0}
$$

(3)

Here $R = R_g$ is the Riemann curvature, defined by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$. Combining this computation with Equation (2) we find

$$
\Psi(\zeta) = \Psi_{(g, \gamma)}(\zeta) = \nabla_\gamma \nabla_\gamma \zeta + R(\zeta, \dot{\gamma}) \dot{\gamma}.
$$

(4)

If $\zeta = \lambda \dot{\gamma}$, $\lambda : S^1 \to \mathbb{R}$ is a real valued function and $\gamma$ is parametrized by arc length, we find

$$
\Psi(\zeta) = \nabla_\gamma \left( \lambda \dot{\gamma} + \lambda \nabla \dot{\gamma} \right) + \lambda R(\dot{\gamma}, \dot{\gamma}) \dot{\gamma} = \ddot{\lambda} \dot{\gamma}.
$$

This computation implies that the images of the tangent sections (to $\gamma$) under $D\Phi$ are again tangent to $\gamma$. We will use this computation of the operator $\Psi$ in our upcoming discussions.

3. Compactness and rigidity for closed geodesics

3.1. Limits of closed geodesics. For a closed curve $\gamma : S^1 \to M$ we denote the energy $\|\dot{\gamma}\|^2_g$ of $\gamma$ by $E(\gamma) = E_g(\gamma)$, and let $\tau_g(\gamma) = -\delta_g(\dot{\gamma})$ denote the tension field. If $\gamma$ is a geodesic, $E(\gamma) = \ell(\gamma)^2$ where $\ell(\gamma) = \ell_g(\gamma)$ denotes the length of $\gamma$. Given a metric $g \in \mathcal{G}^\ell$ and an upper bound $E = L^2$ on the energy of geodesics (i.e. upper bound $L$ on the lengths of geodesics), let us define

$$
\mathcal{Z}^\ell_g(L) = \{ \gamma : S^1 \to M \mid (g, \gamma) \in \mathcal{Z}^\ell_g \text{ and } \ell(\gamma) \leq L \}.
$$

We will focus on the case where $k = p = q = 2$ to simplify the discussion. It is also assumed that $\ell > 3$. The space $\mathcal{Z}^\ell_g(L)$ is almost compact, by standard arguments which have been available for decades. The following theorem is proved by small modification to proof of [Kli78 Theorem 1.4.7].

Theorem 3.1. Let us assume that a metric $g \in \mathcal{G}^\ell(M)$ is given on a closed oriented smooth manifold $M$. Suppose that a sequence $\{ \gamma_n : S^1 \to M \}^\infty_{n=1}$ in $\mathcal{X}^\ell_2(M)$ is given such that

- the sequence $\{ E_g(\gamma_n) \}^\infty_{n=1}$ of energies is bounded above by $E$,
- the sequence $\{ \|\tau_g(\gamma_n)\|_2 \}^\infty_{n=1}$ of tensions tends to zero, where $\| \cdot \|_2$ denotes the $L^2$ norm.

Then $\{ \gamma_n \}^\infty_{n=1}$ has limit points in $\mathcal{X}^\ell_2(M)$ and any such limit point $\gamma$ is a $g$-geodesic, i.e. $\tau_g(\gamma) = 0$.

Theorem 3.1 implies that any limit point of a sequence in $\mathcal{Z}^\ell_g(L)$ is a map $\gamma \in \mathcal{X}^\ell_2(M)$ which satisfies $\tau_g(\gamma) = 0$, and is thus $C^\ell$. The only possible problem is that $\gamma$ may fail to be somewhere injective or to be an immersion. If $\gamma$ fails to be an immersion, it follows that $\gamma$ is constant, at some point $x \in M$. We may then choose some local coordinates on an open set $U$ containing $x$ and assume that for large values of $n$, the image of $\gamma_n$ is in $U$. We may thus write $\gamma_n(t) = (\gamma_n^i(t))_{i=1}^m$ for $t \in [0, 2\pi]$, and

$$
\frac{d^2 \gamma_n^i}{dt^2} + \Gamma^i_{jk} \frac{d \gamma_n^j}{dt} \frac{d \gamma_n^k}{dt} = 0.
$$

Since the integral of $d^2 \gamma_n^i/dt$ over the interval $[0, 2\pi]$ is zero, we can find $s_i \in [0, 2\pi]$ so that $d \gamma_n^i/dt(s_i) = 0$. Set $\epsilon_n$ equal to the supremum of $|d \gamma^i/dt|$ over $[0, 2\pi]$ and let $\epsilon_n$ denote the maximum
of $\epsilon_n^i$. It is clear that $\epsilon_n$ converge to zero as $n$ goes to infinity. We can then compute
\[
\left| \frac{d\gamma_n^i}{dt}(s) \right| = \int_{s_i}^{s} \frac{d\gamma_n^i}{dt^2} dt \leq 2\pi \epsilon_n^i \sup_{y \in U} \left| \Gamma_{jk}(y) \right| \leq 2\pi (m\epsilon_n)^2 C_U
\]
\[\Rightarrow \epsilon_n^i \leq 2\pi m^2 C_U (\epsilon_n)^2, \quad i = 1, \ldots, m \quad \text{i.e.} \quad \epsilon_n \geq \frac{1}{2\pi m^2 C_U}.
\]
Here $C_U$ is a positive constant which is equal to the supremum of $|\Gamma_{jk}|$ over $U$. The above contradiction rules out the possibility that a sequence in $Z_g^\ell(L)$ converges to the constant map.

If the non-constant limit curve $\gamma$ is not somewhere injective, the uniqueness of the solutions for differential equations implies that $\gamma$ is of the form $\gamma = \gamma' \circ \varphi_d$, where $\gamma' : S^1 \to M$ is a somewhere injective immersion and $\varphi_d : S^1 \to S^1$ is a $d$-sheeted covering map, given by restricting the map $z \mapsto z^d$ (on the complex plane) to the unit circle. It follows that $\gamma'$ is a geodesic of class $W^{2,2}$.

Let us denote the normal bundle of $\gamma'$ in $M$ by $N_{\gamma'}$. The linearized operator
\[\Psi = \Psi_{(g,\gamma)} : A^0_{2,2}(\gamma^*TM) \to A^0_{0,2}(\gamma^*TM)\]
may be composed with projection map from $\gamma^*TM$ to $N'_\gamma$ to give a map to $A^0_{0,2}(\varphi^*_d N'_\gamma)$. Since the tangent sections are mapped to tangent sections by $\Psi$, it follows that this latter map factors through a linear operator
\[\Psi^\perp = \Psi^\perp_{(g,\gamma)} : A^0_{2,2}(\varphi^*_d N'_\gamma) \to A^0_{0,2}(\varphi^*_d N'_\gamma).
\]

The convergence of a subsequence of the sequence $\{\gamma_n\}_n$ gives a non-zero section $\xi \in A^0_{2,2}(\varphi^*_d N'_\gamma)$ which is in the kernel of the linear operator $\Psi^\perp$. If $g \in G^{2,\infty}(M)$ (i.e. if $g$ is generic) it follows that $\xi$ can not be of the form $\varphi^*_d \xi'$ for a section $\xi' \in A^0_{2,2}(N'_\gamma)$. However, to rule out the general case, we need a stronger form of regularity for the metric $g$. The situation we face here is quite similar to what happens in Gromov-Witten theory, when a sequence of somewhere injective $J$-holomorphic curves converge to a $J$-holomorphic curve which is multiply covered. Inspired with the terminology used in that context, we make the following definition.

**Definition 3.2.** We call $\gamma \in Z_g^\ell$ a rigid geodesic if the kernel of the linearized operator
\[\Psi^\perp_{(g,\gamma)} : A^0_{2,2}(N_\gamma) \to A^0_{0,2}(N_\gamma)\]
is trivial. If $d$ is a positive integer, $\gamma$ is called $d$-rigid if the kernel of
\[\Psi^\perp_{(g,\gamma \circ \varphi_d)} : A^0_{2,2}(\varphi^*_d N_\gamma) \to A^0_{0,2}(\varphi^*_d N_\gamma)\]
is trivial. Here $\varphi_d : S^1 \to S^1$ is the $d$-sheeted covering induced by the map $z \mapsto z^d$ over the complex plane. We call $\gamma$ super rigid if it is $d$-rigid for every positive integer $d$.

The possibility of having a sequence $\{\gamma_n\}_{n=1}^\infty$ in $Z_g^\ell(L)$ is thus ruled out if we choose $g$ so that all curves in $Z_g^\ell(L/d)$ are $d$-rigid for every positive integer $d$. In order to show that such a choice is possible, we use the technique introduced in [Eft16]. In fact, we would like to avoid the triples $(g, \gamma, \xi)$, where $g$ is a Riemannian metric in $G^\ell$, $\gamma \in Z_g^\ell$ is a $g$-geodesic and $\xi$ is in the kernel of
\[\Psi^\perp_{(g,\gamma \circ \varphi_d)} : A^0_{2,2}(\varphi^*_d N_\gamma) \to A^0_{0,2}(\varphi^*_d N_\gamma).
\]

Every such triple will be called a bad triple. We will study the moduli space of bad triples which form the walls in the moduli space of all closed geodesics.
3.2. Rigidity for geodesics associated with generic metrics. Given \((g, \gamma) \in \mathcal{Z} = \mathcal{Z}^\ell\), we set
\[ \mathcal{F}_{(g, \gamma)}^\perp := A_{2,2}^0(\phi_{dN, \gamma}) \]
and by putting these vector spaces together, we construct a bundle \(\mathcal{F}^\perp \rightarrow \mathcal{Z}\). The linear operators \(\Psi_{(g, \gamma) \circ \varphi_d}\) give a bundle homomorphism \(\Psi^\perp : \mathcal{F}^\perp \rightarrow \mathcal{E}^\perp\), where \(\mathcal{E}^\perp \rightarrow \mathcal{Z}\) is defined as by setting
\[ \mathcal{E}_{(g, \gamma)}^\perp := A_{0,2}^0(\phi_{dN, \gamma}). \]
The group \(\mathbb{Z}/d\) acts on the fibers of both \(\mathcal{E}^\perp\) and \(\mathcal{F}^\perp\) as follows. Let \(\psi_d : S^1 \rightarrow S^1\) denote the rotation by \(2\pi/d\). Clearly \((\psi_d)^d\) is the identity and for \(\xi \in A_{\ast, 2}^0(\phi_{dN, \gamma})\), the section \(\psi_d^\ast \xi\) belongs to the same vector space. Let \(R(\mathbb{Z}/d)\) denote the group ring of \(\mathbb{Z}/d\) which may be identified with \(R_d = \mathbb{R}[t]/\langle t^d \rangle\). The elements of \(R_d\) may be represented as \(\sum_{i=0}^{d-1} a_i t^i\). Associated with each irreducible real representation \(\rho\) of \(\mathbb{Z}/d\), there is a maximal ideal \(I_{\rho}\) of the group-ring \(R_d\) which has index 1 or 2 in \(R_d\). Let us define
\[ A_{0,2}^0(\phi_{dN, \gamma}, \rho) := \{ \xi \in A_{\ast, 2}^0(\phi_{dN, \gamma}) | \sum_{i=0}^{d-1} a_i(\psi_d)^i \xi = 0, \quad \forall \sum_{i=0}^{d-1} a_i t^i \in I_{\rho} \}. \]
Correspondingly, the fibers of \(\mathcal{E}^\perp\) and \(\mathcal{F}^\perp\) are decomposed in correspondence with the irreducible real representations of \(\mathbb{Z}/d\), i.e. \(\mathcal{E}^\perp = \bigoplus_{\rho} \mathcal{E}_{\rho}\) and \(\mathcal{F}^\perp = \bigoplus_{\rho} \mathcal{F}_{\rho}\) where the fibers of \(\mathcal{E}_{\rho}\) and \(\mathcal{F}_{\rho}\) at \((g, \gamma)\) are given by
\[ A_{0,2}^0(\phi_{dN, \gamma}, \rho) \quad \text{and} \quad A_{2,2}^0(\phi_{dN, \gamma}, \rho), \]
respectively. Clearly, the linear map \(\Psi_{(g, \gamma) \circ \varphi_d}\) behaves well with respect to these decompositions, and we obtain the homomorphism \(\Psi_{\rho} : \mathcal{F}_{\rho} \rightarrow \mathcal{E}_{\rho}\).

**Theorem 3.3.** Let us fix the positive integer \(d\) and the irreducible real representation \(\rho\) of \(\mathbb{Z}/d\mathbb{Z}\). For every closed oriented smooth manifold \(M\) the space
\[ W = W_{g, d, \rho} := \{ (g, \gamma, \xi) \mid (g, \gamma) \in \mathcal{Z}^\ell(M) \quad \text{and} \quad 0 \neq \xi \in A_{0,2}^0(\phi_{dN, \gamma}, \rho) \cap \text{Ker}(\Psi_{\gamma}) \} \]
is a separable \(C^{\ell-k-1}\) Banach manifold, which fibers over \(G^\ell\) and for \(g\) in a Bair subset \(G_{d, \rho}^\ell \subset G^\ell\),
\[ W_g = W_{g, d, \rho} = \{ \gamma : S^1 \rightarrow M \mid (g, \gamma, \xi) \in W \} = \emptyset. \]

**Proof.** The differential of \(\Psi_{\rho} : \mathcal{F}_{\rho} \rightarrow \mathcal{E}_{\rho}\) at a zero \((g, \gamma, \xi)\) may be projected over the fiber of \(\mathcal{E}_{\rho}\) through \((g, \gamma)\) to give a linear operator
\[ D\Psi_{\rho} : T_{(g, \gamma)} \mathcal{Z} \oplus A_{2,2}^0(\phi_{dN, \gamma}, \rho) \rightarrow A_{0,2}^0(\phi_{dN, \gamma}, \rho). \]
In order to show the first claim of the theorem, it suffices to show that this linear operator is surjective. First, note that the restriction of \(D\Psi_{\rho}\) to \(A_{2,2}^0(\phi_{dN, \gamma}, \rho)\) defines a Fredholm operator
\[ F = F_{g, \gamma, \rho} : A_{2,2}^0(\phi_{dN, \gamma}, \rho) \rightarrow A_{0,2}^0(\phi_{dN, \gamma}, \rho). \]
If \(F^*\) denotes the adjoint operator associated with \(F\), every element in the cokernel of \(D\Psi_{\rho}\) is represented by some \(\zeta \in \text{Ker}(F^*)\) which is orthogonal to the image of \(D\Psi_{\rho}\).

The image of the representation \(\rho\) is either \(\mathbb{R}\) or \(\mathbb{C}\). As discussed in [Eft16, Lemma 4.4], the latter case is the more difficult one, and we will present the proof in this case. If the image of the representation \(\rho\) is in \(\mathbb{C}\), it follows that the maximal ideal \(I_{\rho}\) has index 2 in \(\mathbb{R}_d\). Let us denote the image of the generator \(\psi_d\) of \(\mathbb{Z}/d\) under the representation \(\rho\) by \(\lambda \in S^1 - \{ \pm 1 \}\), which is a \(d\)-th root of unity. The sections \(\xi\) and \(\zeta\) will then have twin sections, denoted by \(\xi'\) and \(\zeta'\) respectively, such that
\[
\begin{pmatrix}
\psi_d^* \xi \\
\psi_d^* \zeta'
\end{pmatrix}
= 
\begin{pmatrix}
\text{Re}(\lambda) & \text{Im}(\lambda) \\
-\text{Im}(\lambda) & \text{Re}(\lambda)
\end{pmatrix}
\begin{pmatrix}
\xi \\
\zeta'
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\psi_d^* \xi \\
\psi_d^* \zeta'
\end{pmatrix}
= 
\begin{pmatrix}
\text{Re}(\lambda) & \text{Im}(\lambda) \\
-\text{Im}(\lambda) & \text{Re}(\lambda)
\end{pmatrix}
\begin{pmatrix}
\xi \\
\zeta'
\end{pmatrix}.
\]
Note that $\zeta'$ is also in the cokernel of $F$, and that it is also orthogonal to the image of $D\Psi_\rho$. In fact,

$$
(D\Psi_\rho(\tau), \psi_d^*\zeta) = (D\Psi_\rho(\psi_d^*\tau), \zeta) = 0,
$$

where $\psi_d = \psi_d^{-1} = \psi_d^{d-1}$ is the inverse of $\psi_d$, $\tau = (\tau_1, \tau_2) \in T_{(\varphi, \gamma)}Z \oplus A^0_{2, 2}(\varphi_n^*N, \rho)$ is an arbitrary section, and $\psi_d^*\tau = (\tau_1, \psi_d^*\tau_2)$ is its pull-back under $\psi_d$ which is again in $T_{(\varphi, \gamma)}Z \oplus A^0_{2, 2}(\varphi_n^*N, \rho)$.

As in the proof of Theorem 2.1, let us assume that $t \in S^1$ is fixed and choose the local coordinates around $z = \gamma(\varphi_d(t))$ so that $\gamma$ is given by $(t, 0, ..., 0)$ and the matrix $g(\gamma(\varphi_d(t)))$ is given by the identity matrix. As before, the Christoffel symbols $\Gamma_{ij}^k$ vanish along the path. Let $G \in \mathcal{H}$ be given by $(G_{ij})_{i,j=1}^m$ in these coordinates, where $G_{ij} = 0$, except for $f = G_{1,1}$, which is assumed to be a function supported in a neighborhood of $z$. Furthermore, let us assume that $f$ and $df$ vanish along $\gamma$. It then follows from the proof of Theorem 2.1 that $(G, 0)$ belongs to the tangent space $T_{(\varphi, \gamma)}Z$ for any such function $f$. To compute the image of the vector under $F$, we use the computation of the linear operator $\Psi(\xi)$ from Equation 3. Taking the differential of $\Psi$ in the direction of the vector $(G, 0)$, it corresponds to changing the Christoffel symbols in the local computation of Equation 3 while keeping $\gamma$ and $\xi$ unchanged. If we use $G$ to perturb the metric, the corresponding perturbation in the Christoffel symbols along the geodesic $\gamma$ is described by changing $\partial_\rho \Gamma^i_{jk}$ by $\partial_\rho \Delta^i_{jk}$, where $\Delta^i_{1i} = \partial_i f$, $\Delta^i_{11} = -\partial_i t$ for $i > 1$ and the rest of $\Delta^i_{jk}$ are zero. Replacing in Equation 3 we obtain

$$(D\Psi_\rho(G, 0, 0))^i = \xi^k \gamma^j \partial_k \Delta^i_{jk} = -\xi^k \partial_k \partial_i f.$$  

However, note that unlike the proof of Theorem 2.1, the differential will not be trivial away from an interval $J_t = (t - \epsilon, t + \epsilon)$ around $t$. In fact, the support of the differential is over $\cup_{i=0}^{d-1} \psi_d^*(J_t)$. It follows that for every such perturbation $G = G_f$,

$$
0 = \int_{S^1} \langle D\Psi_\rho(G_f, 0, 0), \zeta \rangle_g dt = -\int_{J_t} \sum_{i=0}^{d-1} \xi_i^k \xi_i^j \partial_k \partial_j f,
$$

where $\xi_i^k$ and $\xi_i^j$ denote the $k$-th and $j$-th coordinates of $(\psi_d^*)^i \xi$ and $(\psi_d^*)^i \zeta$, respectively. Similarly,

$$
0 = \int_{S^1} \langle D\Psi_\rho(G_f, 0, 0), \zeta' \rangle_g dt = -\int_{J_t} \sum_{i=0}^{d-1} \xi_i^k (\zeta')^j \partial_k \partial_j f.
$$

By varying $f$ we conclude that for every point $t$ and every symmetric matrix $H = (H_{jk})_{j,k=2}^m$,

$$
0 = \sum_{i=0}^{d-1} \sum_{j,k=2}^m \xi_i^k(t) H_{jk}(\zeta'_{ij}) + \sqrt{-1}(\zeta'_{ij}) = \sum_{i=0}^{d-1} \sum_{j,k=2}^m \xi_i^k(t) H_{jk}(\zeta^j + \sqrt{-1}(\zeta')^j)\lambda^i.
$$

If $\zeta$ is non-zero, this implies that

$$
\sum_{i=0}^{d-1} \xi_i(t) \text{Re}(\lambda^i) = \sum_{i=0}^{d-1} \xi_i(t) \text{Im}(\lambda^i) = 0 \quad \Rightarrow \quad \sum_{i=0}^{d-1} \text{Re}(\lambda^i) t^i, \sum_{i=0}^{d-1} \text{Im}(\lambda^i) t^i \in I_\rho.
$$

The last conclusion follows since $I_\rho$ is maximal. Since $\zeta$ and $\zeta'$ belong to $A^0_{2, 2}(\varphi_n^*N, \rho)$, we have

$$
\sum_{i=0}^{d-1} \text{Re}(\lambda^i) \zeta_i = \sum_{i=0}^{d-1} \text{Re}(\lambda^i) \zeta'_i = 0 \quad \Rightarrow \quad 0 = \sum_{i=0}^{d-1} \text{Re}(\lambda^i)(\zeta_i + \sqrt{-1}\zeta'_i) = \sum_{i=0}^{d-1} \text{Re}(\lambda^i)\lambda^i(\zeta + \sqrt{-1}\zeta'),
$$

$$
0 = \left(\sum_{i=0}^{d-1} (\text{Re}(\lambda^i))^2\right) \zeta - \left(\sum_{i=0}^{d-1} \text{Re}(\lambda^i)\text{Im}(\lambda^i)\right) \zeta'.
$$

Since $\lambda \neq \pm 1$, it follows that $\sum_{i=0}^{d-1} \text{Re}(\lambda^i)\text{Im}(\lambda^i) = \text{Im}(\lambda^{2d-1}/(\lambda^2 - 1)) = 0$. Since the coefficient of $\zeta$ in the above expression is positive, it follows that $\zeta = 0$, proving the surjectivity of $D\Psi_\rho$. 
From surjectivity of $D\Psi_\rho$ it follows that the zero locus of $\Psi_\rho$ is transversely cut out and $W$ is thus a separable $C^{\ell-k-1}$ Banach manifold. Again, since the symbol of $\Psi$ is self-adjoint, it follows that the index of the projection map from $W$ to $G^\ell$ is zero. If $G^\ell_{d,\rho}$ denotes the intersection of $G^\ell_{d,\star}$ with the regular values of this latter projection map, it follows that for every $g \in G^\ell_{d,\star}$ the fiber $W^g_\rho$ is a zero dimensional manifold. Nevertheless, if $(g, \gamma, \xi) \in W^g_\rho$ then $(g, \gamma, r\xi) \in W^g_\rho$ for every non-zero real number $r \in \mathbb{R}$. It follows from this contradiction that for $g \in G^\ell_{d,\star}$ the fiber $W^g_\rho$ is in fact empty. This completes the proof of the theorem. \hfill $\square$

Given the real number $L > 0$ and the integer $\ell > 2$ we let $G^\ell_{d,\star}(M, L) \subset G^\ell(M)$ denote the set of metrics $g$ such that all $g$-geodesics of length less than $L$ are $d$-rigid. The above theorem implies that $G^\ell_{d,\star}(M, L)$ is dense in $G^\ell(M)$. It also follows from fixing the upper bound $L^2$ on the energy of closed geodesics that $G^\ell_{d,\star}(M, L)$ is open in $G^\ell(M)$. We then set

$$G^\ell_{d,\star}(M) := \bigcap_{k=1}^d G^\ell_{k,\star}(M, d) \subset G^\ell(M)$$

Once again, $G^\ell_{d,\star}$ is an open and dense subset of $G^\ell(M)$. Moreover, note that we have the inclusions

$$\cdots \subset G^\ell_{d+1}(M) \subset G^\ell_d(M) \subset G^\ell_{d-1}(M) \subset \cdots$$

In particular, the Bair subset $G^{\ell,\star\star}(M) := \bigcap_d G^\ell_{d,\star}(M)$ of $G^\ell(M)$ consists of the metrics $g$ such that all closed $g$-geodesics are super-rigid.

3.3. Regularity and smooth Riemannian metrics. We have already observed that the subset $G^{\ell,\star\star}(M)$ of super nice metrics of class $C^\ell$ is a Bair subset of $G^\ell(M)$. We would now like to turn our attention to smooth metrics. The passage from the study of $C^\ell$ metrics to smooth metrics is completely standard. For $\ell \in \{2, 3, \ldots, \infty\}$ let $G^{\ell,d}(M)$ denote the set of all metrics $g \in G^\ell(M)$ such that every (non-constant) closed $g$-geodesic $\gamma$ with $kl^d(\gamma) \leq d$ is $k$-rigid, where $k \in \mathbb{Z}^+$ is arbitrary.

**Lemma 3.4.** The subset $G^{\ell,d}(M)$ of $G^\ell(M)$ is open in $C^\ell$ topology, for $\ell \in \{2, 3, \ldots, \infty\}$.

**Proof.** Suppose otherwise that $g \in G^{\ell,d}(M)$ is not an interior point, and that $\{g_n\}_{n=1}^\infty$ is a sequence in $G^\ell(M) \setminus G^{\ell,d}(M)$ converging to $g$. Correspondingly, we have a sequence $\{\gamma_n\}_{n=1}^\infty$, where $\gamma_n$ is a non-constant $g_n$-geodesic, $k_n l^d_\rho(\gamma_n) < d$ and $\gamma_n$ is not $k_n$-rigid. In particular, $E_{g_n}(\gamma_n) \leq d^2$. After passing to a subsequence, Theorem 3.1 guarantees that the sequence $\{\gamma_n\}_{n=1}^\infty$ converges to a closed non-constant $g$-geodesic $\gamma$, which is a $d$-sheeted covering of a somewhat injective closed $g$-geodesic $\gamma'$. The sequence $k_n$ is thus bounded above, and after passing to a subsequence, we can assume that all $k_n$ are equal to the same value $k_0$. As a consequence, $k d^\ell(\gamma') = kl^d_\rho(\gamma) \leq d$. Since $\gamma_n$ is not $k_n$-rigid, it follows that $\gamma'$ is not $kd^\ell$-rigid. This contradicts the assumption that $g \in G^{\ell,d}(M)$. \hfill $\square$

For $\ell < \infty$, it is clear that $G^{\ell,\star\star} = \bigcap_d G^{\ell,d}(M)$, and it is thus implied that $G^{\ell,d}$ is both open and dense in the $C^\ell$ topology (as a subset of $G^\ell(M)$). On the other hand,

$$G^{\infty,d}(M) = G^\infty(M) \cap G^{\ell,d}(M),$$

and this means that $G^{\infty,d}(M)$ is a dense subset of $G^\infty(M)$ in $C^\ell$ topology. This observation implies the following corollary.

**Corollary 3.5.** For every $\ell \in \{2, 3, \ldots\}$, $G^{\infty,d}(M)$ is an open and dense subset of $G^\infty(M)$ in $C^\ell$ topology. In particular, $G^{\infty,\star\star}(M) = \bigcap_d G^{\infty,d}(M)$ is a Bair subset of $G^\infty(M)$ in $C^\ell$ topology.
4. Counting closed geodesics on Riemannian manifolds

4.1. Wall-crossing for the sign of linearized operator. Fix a Riemannian metric $g$ on the smooth oriented closed manifold $M$. Let us assume that $\gamma : S^1 \rightarrow M$ is a geodesic, and that $t$ denotes the parameter on $S^1$. The tangent bundle of $M$ may be trivialized along $\gamma$ as $TM|_{\gamma(S^1)} = S^1 \times \mathbb{R}^n$, such that the first component denotes the direction of $\gamma$, and such that the standard frame on $\mathbb{R}^n$ is orthonormal with respect to the metric on $M$. The linear operator $\Psi_{(g,\gamma)}$ at $\gamma$ may be expressed, in these local coordinates, as

$$\Psi_{(g,\gamma)}(\zeta^i e_i) = \left( \frac{d^2}{dt^2} \zeta^i + A^i_j(t) \frac{d}{dt} \zeta^j + B^i_j(t) \zeta^j \right) e_i$$

Since $\Psi_{(g,\gamma)}(e_1) = \Psi_{(g,\gamma)}(\dot{\gamma}) = 0$, it follows that $B^i_j(t) = 0$ for all $i$ and all $t$. The pair of matrices $(A, B)$ are then the sections of a trivial bundle of rank $n(2n - 1)$ over $S^1$. Let $\Gamma_o(\gamma^*TM)$ and $\Gamma'_o(\gamma^*TM)$ denote the quotients of $A^i_{B,j}(\gamma^*TM)$ and $A^i_{B,j}(\gamma^*TM)$ by $\dot{\gamma}$, respectively. For generic $g$, the main theorem of the previous section implies that for every geodesic $\gamma$ the induced map

$$\Psi_{(g,\gamma)} : \Gamma_o(\gamma^*TM) \rightarrow \Gamma'_o(\gamma^*TM)$$

is a bijection, with trivial kernel and cokernel. To every such geodesic we would like to assign a sign $\epsilon(\gamma)$, which depends on the section $(A, B)$ as will be described below.

Let us denote the space of all sections $(A, B)$ as above by $A$. Clearly, $A$ is contractible. Every section $(A, B)$ corresponds to a Fredholm operator $L_{A,B}$ of index 0 which is defined by

$$L_{A,B} : \Gamma_o(\mathbb{R}^n) \rightarrow \Gamma_o(\mathbb{R}^n) \quad L_{A,B}(\zeta^i e_i) := \left( \frac{d^2}{dt^2} \zeta^i + A^i_j(t) \frac{d}{dt} \zeta^j + B^i_j(t) \zeta^j \right) e_i.$$ 

Since $B^i_j = 0$, the above operator is well-defined. The determinant line bundle of $L = \{L_{A,B}\}_{(A,B) \in A}$ is a trivial line bundle over $A$. Let $A^r \subset A$ denote the subspace of $A$ consisting of the sections $(A, B)$ such that the kernel of $L_{A,B}$ is of rank $r$.

**Lemma 4.1.** Each $A^r$ is an (open) analytic sub-variety of $A$ of codimension at least $r$.

**Proof.** Let us fix a point $(A, B) \in A^r$. The intersection of $A^r$ with an open neighborhood $U$ of $(A, B)$ in $A$ is then identified with the zero set of an analytic function

$$\Psi^r : U \subset A \rightarrow \text{Hom}(|\text{Ker}(L_{A,B})|, \text{Coker}(L_{A,B})).$$

The first derivative of $\Psi^r$ may be computed explicitly

$$d\Psi^r(a, b)(\zeta^i e_i) = (a^i_j \frac{d}{dt} \zeta^j + b^i_j \zeta^j) e_i \quad \forall (a, b) \in T_{(A,B)}U.$$ 

We show the claim for $r = 1$. The proof of the general case is similar, and follows the standard methods. When $r = 1$, the kernel is generated by a single section $\zeta = \zeta^1 e_1$, which is non-zero away from finitely many points. Choose the generator $\eta \in \text{Ker}(L_{A,B}^*) \simeq \text{Coker}(L_{A,B})$ so that it is orthogonal to the image of $L_{A,B}$. Here $L_{A,B}^*$ denotes the adjoint operator associated with $L_{A,B}$. If $d\Psi^1(a, b) = 0$ for all $a, b$, it follows that $\eta$ is orthogonal to all vectors of the form $a^i_j d\zeta^j / dt = \dot{a}^i_j \zeta^j$ and $b^i_j \zeta^j$. If $\eta$ is non-zero and $\zeta$ is not a multiple of $e_1$ at some point, the inner product of $\eta$ and $b^i_j \zeta^j$ may be made non-zero by choosing an appropriate vector $b$. Since $\eta$ is non-zero away from finitely many points, it follows that $\zeta = \zeta^1 e_1$. Since $\zeta$ is not a constant multiple of $e_1$, it follows that $d\zeta / dt$ is non-zero away from finitely many points. Thus we may choose $a$ so that $a^i_j d\zeta^j / dt$ is not orthogonal to $\eta$. From this contradiction, it follows that $d\Psi^1$ is non-zero. Thus $A^1$ is an analytic sub-variety of codimension 1. For the general case, we need to push the above argument further and show that $d\Psi^r$ is of rank at least $r$ if the rank of $\text{Ker}(L_{A,B})$ is $r$.  

\[\square\]
The above lemma implies that the closure of $A^1$ (which we call the walls) cuts $A^0 \subset A$ into a number of chambers. If we fix a positive generator of the line (in the determinant line bundle) which passes through one point in $A^0$, it is then possible to choose a generator for the lines through all points of $A^0$, so that the sign of the generator changes when we cross the walls (i.e. $A^1$). Comparing this generator with the trivialization of the determinant line bundle, we obtain an associated sign for each chamber. These signs are well-defined up to an overall multiplication by $-1$.

4.2. The sign associated with closed geodesics. In order to fix the choice of the sign function of the previous subsection, independent of the choice of the coordinates, we need the following definition.

**Definition 4.2.** A point $(A, B) \in A$ is called negatively curved if
\[
\int_{S^1} \langle L_{A,B}(\zeta), \zeta \rangle \, dt < 0 \quad \forall \ 0 \neq \zeta \in \Gamma_\epsilon(\mathbb{R}^n).
\]

The negatively curved points in $A$ form a convex subset, and are thus connected. Moreover, it is clear from the definition that the set of negatively curved points, which is non-empty, is included in $A^0$. As a result, the set of negatively curved points determine (a connected convex subset of) a distinguished chamber in $A$, which will be called the negatively curved chamber. We choose the sign function so that the sign associated with the negatively curved chamber is $+1$. With this value fixed, associated with every $(A, B) \in A^0$ we obtain a sign $\epsilon(A, B) = \epsilon(L_{A,B}) \in \{+1, -1\}$.

**Definition 4.3.** Let us assume that $(M,g)$ is a Riemannian manifold and that $g \in G^{l,*}(M)$ is a generic metric. If $\gamma : S^1 \to M$ is a closed $g$-geodesic, then $\epsilon(\gamma) \in \{+1, -1\}$ is the sign $\epsilon(\Psi_{(g,\gamma)})$ of the linear operator $\Psi$ at $(g, \gamma)$. Similarly, if $\gamma$ is $d$-rigid for some positive integer $d$, we define $\epsilon_d(\gamma) \in \{+1, -1\}$ to be the sign $\epsilon(\Psi_{(g, \gamma \circ \varphi_d)})$ of the linear operator $\Psi$ at $(g, \gamma \circ \varphi_d)$. We let $\delta(\gamma) = (\epsilon_2(\gamma) - \epsilon_1(\gamma))/2$. The $g$-geodesic $\gamma$ is called definite if $\delta(\gamma) = 0$ and is called indefinite if $\delta(\gamma) \neq 0$.

Denote the set of (free) homotopy classes of closed loops in $M$ by $C_M = \Omega(S^1, M)$. There is a natural action of $\mathbb{Z}$ on $C_M$, where $n \cdot \alpha$ corresponds to $n$ times traversing $\alpha$ (for $n \in \mathbb{Z}$ and $\alpha \in C_M$). For $\alpha \in C_M$ and $n \in \mathbb{Z}$, we may thus talk about the class $\alpha/n$, which is empty if $\alpha$ is not of the form $n \cdot \beta$ for some $\beta \in C_M$. The free homotopy class of a closed loop $\gamma : S^1 \to M$ is denoted by $[\gamma] \in C_M$.

If $g$ is a $C^\ell$ metric on $M$ and $\gamma : S^1 \to M$ is of class $W^{2,2}$, it is implied that $\gamma$ is of class $C^{\ell+1}$. In particular, there is an associated homotopy class $[\gamma] \in C_M$. If $g \in G^{\ell,**}(M)$ is a generic metric, we may count the closed $g$-geodesics with sign. For technical reasons which will become clear in the upcoming section, it is more appropriate to consider contributions from the closed geodesics that are multiply covered. More precisely, we make the following definition of the count functions $\pi_g$.

**Definition 4.4.** For every $\alpha \in C_M$, $L \in \mathbb{R}^+$ and $g \in G^{\ell,**}(M)$, let $Z^\ell_g(L, \alpha)$ denote the subset of $Z^\ell_g(L)$ which consists of geodesics representing $\alpha$. Define the geodesic count functions associated with the metric $g$ by
\[
\pi_g(L, \alpha) = \sum_{\gamma \in Z^\ell_g(L, \alpha)} \epsilon(\gamma) + \sum_{\gamma \in Z^\ell_g(L/2, \alpha/2)} \delta(\gamma) \quad \text{and} \quad \pi_g(L) = \sum_{\alpha \in C_M} \pi_g(L, \alpha) h^\alpha.
\]

The meaning of this definition is that multiple covers of definite geodesics do not contribute to our count function, while the double covers of indefinite geodesics have non-trivial contribution.

**Proposition 4.5.** Let us assume that $(M,g)$ is a negatively curved manifold. Then $g \in G^{\ell,*}(M)$. Moreover, every closed $g$-geodesic $\gamma$ is super-rigid and we have $\epsilon(\gamma) = 1$ and $\delta(\gamma) = 0$. 
Proof. Let \((M, g)\) be a Riemannian manifold as before and \(\gamma : S^1 \to M\) be a (possibly non-simple) geodesic. Choose a vector field \(\xi\) along \(\gamma\) and compute \(\Psi_{(g, \gamma)}(\xi)\) using Equation (4) to find

\[
\langle \Psi_{(g, \gamma)}(\xi), \xi \rangle_g = \int_{S^1} \langle R(\xi, \dot{\gamma}) \dot{\gamma}, \dot{\gamma} \rangle_g \, dt - \frac{1}{2} \int_{S^1} \|\nabla_\gamma \dot{\gamma}\|^2_g \, dt
\]

Since the manifold \((M, g)\) is negatively curved, both terms in the final integral are non-positive. The integral is thus negative, unless \(\xi\) is a multiple of \(\dot{\gamma}\) at each point (so that the corresponding curvature term is trivial) and its length remains constant (so that it remains constant relative to \(\dot{\gamma}\), and the length integral becomes zero, as well). Thus, the kernel of \(\Psi_{(g, \gamma)}\) consists of the constant multiples of \(\dot{\gamma}\). This observation proves that every closed \(g\)-geodesic on \(M\) is super-rigid and that \(g\) is generic. It is also clear from Equation (7) that the sign of every closed geodesic is positive. Repeating the argument for the double-covers of closed geodesics completes the proof.

\[
\text{Proof. } \gamma : S^1 \to M \text{ be a closed geodesic and } \xi \text{ a vector field along } \gamma. \text{ As before,}
\]

\[
\langle \Psi_{(g, \gamma)}(\xi), \xi \rangle_g = \int_{S^1} \langle R(\xi, \dot{\gamma}) \dot{\gamma}, \dot{\gamma} \rangle_g \, dt - \frac{1}{2} \int_{S^1} \|\nabla_\gamma \dot{\gamma}\|^2_g \, dt \leq 0.
\]

Let \(L = L_{A, B}\) denote a negatively curved element of \(A\). Then \(\Psi_{(g, \gamma)} + sL\) belongs to the negatively curved chamber in \(A\) for every \(s > 0\). Since \(g\) is generic, \(\Psi_{(g, \gamma)}\) is an interior point to one of the chambers, and the existence of the path \(\{\Psi_{(g, \gamma)} + sL\}_{s>0}\) implies that the chamber containing \(\Psi_{(g, \gamma)}\) is in fact the negatively curved chamber. In particular, \(\epsilon(\gamma) = +1\). Again, repeating the argument for the double covers of closed geodesics completes the proof.

**Proposition 4.6.** Let \(g \in G^{\ell, *}(M)\) be chosen so that the sectional curvatures are everywhere non-positive. Then for every closed \(g\)-geodesic \(\gamma\), we have \(\epsilon(\gamma) = +1\) and \(\delta(\gamma) = 0\).

**Proof.** Let \(\gamma : S^1 \to M\) be a closed geodesic and \(\xi\) a vector field along \(\gamma\). As before,

\[
\langle \Psi_{(g, \gamma)}(\xi), \xi \rangle_g = \int_{S^1} \langle R(\xi, \dot{\gamma}) \dot{\gamma}, \dot{\gamma} \rangle_g \, dt - \frac{1}{2} \int_{S^1} \|\nabla_\gamma \dot{\gamma}\|^2_g \, dt \leq 0.
\]

Let \(L = L_{A, B}\) denote a negatively curved element of \(A\). Then \(\Psi_{(g, \gamma)} + sL\) belongs to the negatively curved chamber in \(A\) for every \(s > 0\). Since \(g\) is generic, \(\Psi_{(g, \gamma)}\) is an interior point to one of the chambers, and the existence of the path \(\{\Psi_{(g, \gamma)} + sL\}_{s>0}\) implies that the chamber containing \(\Psi_{(g, \gamma)}\) is in fact the negatively curved chamber. In particular, \(\epsilon(\gamma) = +1\). Again, repeating the argument for the double covers of closed geodesics completes the proof.

### 5. Paths of metrics and invariance of geodesic counts

#### 5.1. Cobordisms between moduli spaces of closed geodesics

Let \(g_0, g_1 \in G^{\ell, **}(M)\) denote a pair of generic Riemannian metrics on a closed oriented smooth manifold \(M\). Define the spaces \(\mathcal{X} = X_{2, 2}(M)\), \(\mathcal{G} = G^{\ell}(M)\) and \(\mathcal{Z} = Z_{2, 2}^{\ell}(M)\) as before. Let

\[
\mathcal{P} = \mathcal{P}_{g_0, g_1} = \left\{ \vec{g} : [0, 1] \to G^{\ell}(M) \mid \vec{g}(0) = g_0 \text{ and } \vec{g}(1) = g_1 \right\}
\]

denote the space of (smooth) paths of metrics which start from \(g_0\) and end at \(g_1\). We may then pull the bundle \(\mathcal{E}\) from \(\mathcal{X}\) back over \([0, 1] \times \mathcal{P} \times \mathcal{X}\), and define the section \(\Phi\) of the new bundle by \(\Phi(t, \vec{g}, \gamma) := \delta_{\vec{g}(t)}(d\gamma)\). Subsequently we may define

\[
\mathcal{Z}_{g_0, g_1} = Z_{g_0, g_1}^{\ell}(M) = \left\{ (t, \vec{g}, \gamma) \mid \vec{g} \in \mathcal{P}, \ t \in [0, 1] \text{ and } (\vec{g}(t), \gamma) \in \mathcal{Z} \right\}
\]

If \(\vec{g} \in \mathcal{P}_{g_0, g_1}^{\ell}(M)\) is a path of metrics, by a \(\vec{g}\)-geodesic we mean a \(\vec{g}(t)\)-geodesic for some \(t \in [0, 1]\). An argument similar to the one presented in the proof of Theorem 2.1 implies the following theorem.

**Theorem 5.1.** For \(g_0, g_1 \in G^{\ell, **}(M)\), the space \(Z_{g_0, g_1}^{\ell}(M)\) is a separable \(C^{\ell-3}\) Banach manifold, which fibers over \(\mathcal{P}_{g_0, g_1}^{\ell}(M)\) via a Fredholm projection map of index 1. There is a Bair subset \(\mathcal{P}_{g_0, g_1}^{\ell, *}(M) \subset \mathcal{P}_{g_0, g_1}^{\ell, *}(M)\) such that for every \(\vec{g} \in \mathcal{P}_{g_0, g_1}^{\ell, *}(M)\), the quotient of

\[
\mathcal{Z}_{\vec{g}}^{\ell}(M) = \left\{ (t, \gamma) \mid (t, \vec{g}, \gamma) \in Z_{g_0, g_1}^{\ell}(M) \right\}
\]

by the action of \(S^1\) is a 1-manifold. \(\mathcal{P}^{\ell, *}_{g_0, g_1}\) may be chosen so that for all \(\vec{g} \in \mathcal{P}^{\ell, *}_{g_0, g_1}\), the quotient of

\[
\mathcal{Z}_{\vec{g}}^{\ell}(L) = \left\{ (t, \gamma) \mid (t, \vec{g}, \gamma) \in Z_{g_0, g_1}^{\ell}(M) \right\}
\]
by the action of $S^1$ is a 0-manifold for every $L$ not in a closed subset $\mathcal{L}_g \subset \mathbb{R}$ with zero Lebesgue measure. If $n > 3$ we may also assume that for all $\bar{g} \in \mathcal{P}_{g_0,g_1}^{\ell,*,\ast}$ and $(t, \gamma) \in Z^\ell_g(M)$, $\gamma$ is an embedding.

**Proof.** The transversality arguments presented in the proof of Theorem 2.1 may be used here without any major changes to prove the above theorem. The only part which requires a separate argument is the claim about the structure of $Z^\ell_g(L)$. For this last claim, we need to consider the map defined over the quotient of $Z^\ell_g(M)$ by the rotation action of $S^1$, which sends a pair $(t, \gamma)$ to $l_{\bar{g}^\delta_t}(\gamma) \in \mathbb{R}^+$. The singular values of this energy map form a closed subset of $\mathbb{R}^+$ with zero Lebesgue measure, which may be denoted by $\tilde{E}_g \subset \mathbb{R}^+$.

Again, we usually abuse the notation and identify $Z^\ell_{g_0,g_1}(M)$ and $Z^\ell_g(M)$ (for $\bar{g} \in \mathcal{P}_{g_0,g_1}^{\ell,*,\ast}$) with their quotients under the rotation action of $S^1$. In particular, we treat $Z^\ell_g(M)$ as a 1-manifold.

**5.2. The invariance of geodesic count function.** Based on Theorem 5.1, we prove the following theorem about the invariance of the count functions $\pi_{\bar{g}^\delta_t}(L, \alpha)$ along a path $\bar{g}$ in $\mathcal{P}_{g_0,g_1}^{\ell,*,\ast}$, and for $L \in \mathbb{R}^+ \setminus \mathcal{L}_g$. The map $\Pi : Z^\ell_g \to [0, 1] \times \mathbb{R}^+$, which sends a $g_\ell$-geodesic $\gamma$ of length $l$ to $(t, l)$, is an immersion near $Z^\ell_g(L)$, and every $\gamma \in Z^\ell_g(L)$ comes with a vector $(t(\gamma), l(\gamma))$ tangent to the image of $\Pi$ in $[0, 1] \times [0, L]$, which points out of the aforementioned rectangle, see Figure 1. We let $\epsilon'(\gamma) = +1$ if $t(\gamma) > 0$ and $\epsilon'(\gamma) = -1$ if $t(\gamma) < 0$ (we can assume that for generic $g$, $t(\gamma) \neq 0$ for all boundary points). We may then set $\bar{\epsilon}(\gamma) = \epsilon(\gamma)\epsilon'(\gamma)$ and $\bar{\delta}(\gamma) = \delta(\gamma)\epsilon'(\gamma)$. For $\alpha \in \mathcal{C}_M$ let

$$
\bar{\pi}_g(L, \alpha) = \sum_{\gamma \in Z^\ell_g} \bar{\epsilon}(\gamma) + \sum_{\gamma \in Z^\ell_g/2,\alpha/2} \bar{\delta}(\gamma) \quad \text{and} \quad \bar{\pi}_g(L) = \sum_{\alpha \in \mathcal{C}_M} \pi_g(L, \alpha)h^\alpha.
$$

**Theorem 5.2.** Fix the Riemannian metrics $g_0, g_1 \in \mathcal{G}^{\ell,**}(M)$ over the closed smooth oriented manifold $M$. There is a Bair subset $\mathcal{P}_{g_0,g_1}^{\ell,**}(M)$ of $\mathcal{P}_{g_0,g_1}^{\ell,*,\ast}(M)$ such that for every $\bar{g} \in \mathcal{P}_{g_0,g_1}^{\ell,**}(M)$, every $L$ outside a closed subset $\mathcal{L}_g \subset \mathbb{R}^+$ with zero measure, and every free homotopy class $\alpha \in \mathcal{C}_M$,

$$
\pi_{g_1}(L, \alpha) - \pi_{g_0}(L, \alpha) = \bar{\pi}_g(L, \alpha).
$$

**Figure 1.** For generic $\bar{g}$, $Z^\ell_g([0, L])$ is a 1-manifold and its image in $[0, 1] \times [0, L]$ under $\Pi$ is used to define $\epsilon' : Z^\ell_g(L) \to \{\pm 1\}$. The limit points inside the rectangle correspond to the double covers of geodesics in $Z^\ell_g([0, L/2])$, as illustrated.
Proof. The proof is closely related to the proofs of [Tau96 Lemma 5.10] and [Tau96 Lemma 5.11], as we will see below. Set $g_t = \bar{g}(t)$ and consider the subset of $Z^t_{g_t}(M)$ defined by

$$Z^t_{g_t}([0, L]) = \left\{ (t, \gamma) \in Z^t_{g_t}(M) \mid \ell_{g_t}(\gamma) \leq L \right\}.$$ 

The compactness results of Section 3 imply that $Z^t_{g_t}([0, L])$ is a compact 1-manifold with boundary. The boundary components of this manifold are of four types:

- $g_0$-boundary points which are in correspondence with the points in $Z^t_{g_0}(L)$.
- $g_1$-boundary points which are in correspondence with the points in $Z^t_{g_1}(L)$.
- $L$-boundary points which are in correspondence with $(t, \gamma) \in Z^t_{g_t}(L)$.
- The internal boundary points. These boundary points are in correspondence with the convergence of a sequence in $Z^t_{g_t}([0, L])$ to the $d$-fold cover of a closed geodesic in $Z^t_{g_n}([0, L/d])$ for some $d > 0$ and some $t \in (0, 1)$.

Replacing the Bair subset $G^{t,*}(M)$ with a smaller Bair subset of $G^t(M)$ we can assume that for every $g \in G^{t,*}(M)$ there are distinct real values

$$0 < t_1 < t_2 < \cdots < t_{N_L} < 1, \quad 0 < r_1 < r_2 \cdots < r_{N_L/2} \quad \text{and} \quad 0 < s_1 < s_2 < \cdots < s_{N_I} < 1$$

and the corresponding $\bar{g}$-geodesics $\gamma_{t_i}$, $\gamma_{r_i}$ and $\gamma_{s_i}$ such that $\{(t_i, \gamma_{t_i})\}_{i=1}^{N_L}$ are all the $L$-boundary points of $Z^t_{g_t}([0, L])$, $\{(r_i, \gamma_{r_i})\}_{i=1}^{N_{L/2}}$ are all the $(L/2)$-boundary points of $Z^t_{g_t}([0, L/2])$ such that $\gamma_{t_i}$ is indefinite, and that $\{(s_i, \gamma_{s_i})\}_{i=1}^{N_I}$ are all the internal boundary points of $Z^t_{g_t}([0, L])$.

Let us first study the internal boundary points. The strategy is completely similar to the strategy used in the proof of Theorem 3.3. We will thus omit the details and will only outline our approach.

Let $t \in \{s_1, \cdots , s_{N_I}\}$ and $\gamma = \gamma_{t_i}$.

The convergence of a sequence $(g^n, \gamma^n)$ to $(g_t, \gamma \circ \varphi_d)$ gives a sections $(G, \xi)$, where $G \in T_g \mathcal{G}^t(M)$ and $\xi \in A^0_{d,2}(\varphi_d^*N\gamma)$, such that

$$d\Phi^\perp_{(g_t, \gamma \circ \varphi_d)}(G, 0) + \Psi^\perp_{(g_t, \gamma \circ \varphi_d)}(\xi) = 0.$$ 

Moreover, $\xi$ is not invariant under the action of $\psi_d$. If $\bar{g}$ is generic, the operator $d\Phi^\perp_{(g_t, \gamma)}$ is surjective, and has one-dimensional kernel. The kernel is either generated by an element of the form $(G, \xi')$ or by an element of the form $(0, \xi')$, for some $\xi' \in A^0_{d,2}(N\gamma)$ and $G = \frac{d}{dx} \bar{g}|t$. In the first case, $\xi - \varphi_d^*\xi' \in \text{Ker}(\Psi^\perp_{(g_t, \gamma \circ \varphi_d)})$, while in the second case, $\varphi_d^*\xi' \in \text{Ker}(\Psi^\perp_{(g_t, \gamma \circ \varphi_d)})$. In either case, we obtain a point in $W^t_{g_t, d, \rho}(M)$. For a generic path $\bar{g}$ the quotient of the moduli space

$$W^t_{g_t, d, \rho}(M) = \left\{ (t, \gamma, \xi) \mid (g_t, \gamma, \xi) \in W^t_{d, \rho}(M) \right\}$$

by the rotation action of $S^1$ is a 1-dimensional manifold. If the irreducible representation $\rho$ is such that $\rho(\psi_d) \neq \pm 1$, this means that $W^t_{g_t, d, \rho}(M)$ is empty, since having $(t, \gamma, \xi)$ in this space means that the 2-dimensional vector space spanned by $(t, \gamma, (\psi^k_d)^*\xi)$ is also in $W^t_{g_t, d, \rho}(M)$. This contradiction implies that the only relevant representations are the representations $\rho$ with $\rho(\psi_d) = \pm 1$.

From the above discussion, either $d = 2$ and the kernel of $\Psi_{(g_t, \gamma \circ \varphi_d)}$ contains a non-trivial element $\xi$ with $\psi^*_d \xi = -\xi$, or the kernel of $\Psi_{(g_t, \gamma)}$ is non-trivial. We may refine the Bair set $G^{t,*}(M)$ so that for $\bar{g} \in G^{t,*}(M)$ and $(t, \gamma) \in Z^t_{g_t}(M)$ only one of the above two possibilities can happen. Furthermore, in both cases, we may assume that the image of $G = \frac{d}{dt} \bar{g}|t$ under $d\Phi$ is non-trivial in the cokernel of $\Psi$.

Let us first assume that the kernel of $\Psi_{(g_t, \gamma)}$ is non-trivial and is generated by $\xi'$. In this case, the only non-trivial element in the kernel of $\Psi_{g_t, \gamma \circ \varphi_d}$ is $\varphi_d^*\xi'$ and its cokernel is thus 1-dimensional as well.
Since the image of \( G \) under \( d\Phi \) is non-trivial in this cokernel, the moduli space of closed \( \bar{g} \)-geodesics near \( \gamma \circ \varphi_d \) (after taking the quotient by the rotation action of \( S^1 \)) is modeled on \( \text{Ker}(\Psi_{(g, \gamma \circ \varphi_d)}) \). Similarly, the moduli space of closed \( \bar{g} \)-geodesics near \( \gamma \) (after taking the quotient by the rotation action of \( S^1 \)) is modeled on \( \text{Ker}(\Psi_{(g, \gamma)}) \). Thus, every \( \bar{g} \)-geodesic \( \theta \) near \( \gamma \circ \varphi_d \) is of the form \( \theta' \circ \varphi_d \) for a \( \bar{g} \)-geodesic \( \theta' \) near \( \gamma \). This contradicts our initial assumption that \( \gamma \) is multiply covered by the limit of a sequence of closed \( g \)-geodesics.

We are thus left with the case where \( d = 2 \) and the kernel of \( \Psi_{(g, \gamma \circ \varphi_2)} \) is generated by a non-trivial element \( \xi \) with \( \psi^*_d \xi = -\xi \). If the image of \( G \) under \( d\Phi \) is trivial in the cokernel of \( \Psi_{(g, \gamma \circ \varphi_2)} \), we obtain a point \((t, \bar{g}, \gamma, \xi)\) in the moduli space
\[
\begin{align*}
\left\{ (s, \bar{g}, \gamma, \xi) \in P_{g_0, g_1}^\ell(M), & \quad (s, \gamma) \in Z^\ell_g(M), \\
\text{Ker}(\Psi^\perp_{(g, \gamma \circ \varphi_2)}) & = \{ \xi \} \quad \text{and} \quad d\Phi^\perp_{g_s, \gamma}(\frac{d}{dt}\bar{g}|_s) = 0 \in \text{Coker}(\Psi^\perp_{(g, \gamma \circ \varphi_2)})
\right\},
\end{align*}
\]
where \( \rho \) denotes the representation corresponding to \(-1\). One can show, as before, that this moduli space is a Banach manifold and which inherits an action of \( S^1 \times \mathbb{R} \). The quotient of the moduli space under this action fibers over \( P_{g_0, g_1}^\ell(M) \) via a Fredholm map of index \(-1\). In particular, by passing to a subset \( P_{g_0, g_1}^{\ell, \ast} \subset P_{g_0, g_1}^\ell \) which is of second category in \( P_{g_0, g_1}^\ell(M) \), we can assume that for \( \bar{g} \in P_{g_0, g_1}^{\ell, \ast} \) the fiber of the above moduli space over \( \bar{g} \) is empty. This observation allows us to further assume that the image of \( G = \frac{d}{dt}\bar{g}|_t \) under \( d\Phi \) is non-trivial in the cokernel of \( \Psi_{(g, \gamma \circ \varphi_2)} \).

Let \( q : Z^\ell_g(M) \to [0, 1] \) denote the projection over the interval \([0, 1]\). In the aforementioned remaining case, all the requirements needed to use the argument of \cite{Tau96} Lemma 5.10 are satisfied.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{geodesics.png}
\caption{A sequence of geodesics in \( Z^\ell_g([0, L], \alpha) \) may converge to the double cover of a geodesic \( \gamma \in Z^\ell_g([0, L/2], \alpha/2) \). The pair \((\epsilon, \delta)\) associated with the geodesics in a neighborhood \( \tau(t - \epsilon, t + \epsilon) \) of \( \gamma \) in \( Z^\ell_g([0, L/2], \alpha/2) \) and the geodesics in \( \tau_2(I) \subset Z^\ell_g([0, L], \alpha) \) which are close to \( \gamma \circ \varphi_2 \) follows one of the 4 illustrated patterns.}
\end{figure}
Then there is an interval $I$, which is either $(t - \epsilon, t)$ or $(t, t + \epsilon)$, and the maps $\tau_1 : (-\epsilon, \epsilon) \to Z^d_{\bar{g}}(M)$ and $\tau_2 : I \to Z^d_{\bar{g}}(M)$ which give a local model for $Z^d_{\bar{g}}(M)$ via the following properties:

- $q \circ \tau_1$ and $q \circ \tau_2$ are the identity maps of $(t - \epsilon, t + \epsilon)$ and $I$, respectively.
- $\tau_1(t) = (t, \gamma)$ and $(t, \gamma \circ \varphi_2)$ is the limit of $\tau_2(s)$ as $s$ approaches $t$. Moreover, $(t, \gamma \circ \varphi_2)$ is not the limit of any sequence in $Z^d_{\bar{g}}(M) \setminus \tau_2(I)$.
- The kernel of $\Psi_{(\tau_1(s), \rho)}$ is trivial for $\rho = \pm 1$ and $s \neq t$. For $s = t$ the kernel is trivial for $\rho = 1$, while it is 1-dimensional for $\rho = -1$.

Let $\tau_1(s) = (s, \gamma_s)$ and $\tau_2(s) = (s, \gamma'_s)$. Then we have $\epsilon_1(\gamma'_s) = \epsilon_2(\gamma'_s) = -\epsilon_2(\gamma_s)$ for $s \in I$. For $s \in (t - \epsilon, t + \epsilon) \setminus I$ we have $\epsilon_1(\gamma_s) = \epsilon_1(\gamma - \epsilon)$ and $\epsilon_2(\gamma_s) = -\epsilon_2(\gamma - \epsilon)$. In particular, $\gamma'_s$ is definite, while precisely one of $\gamma_s$ and $\gamma'_s$ is definite and the other one is indefinite, see Figure 2.

If $t$ is a critical value for the projection map $q$, there is an interval $I$, which is either $(t - \epsilon, t)$ or $(t, t + \epsilon)$, and the maps $\lambda^\pm : I \to Z^d_{\bar{g}}(M)$ such that the following conditions are satisfied (compare with [Tau96] Lemma 5.9):

- $q \circ \lambda^+$ and $q \circ \lambda^-$ are the identity map of $I$.
- $\lambda^\pm(s) = (s, \gamma^\pm_s)$ and $\gamma^\pm_s$ converge to the same curve $(t, \gamma_t) \in Z^d_{\bar{g}}(M)$. Moreover, $(t, \gamma_t)$ is not the limit of any sequence in $Z^d_{\bar{g}}(M) \setminus (\lambda^+(I) \cup \lambda^-(I))$.
- The kernel of $\Psi_{(\lambda^+(s), \rho)}$ is trivial for $s \in I$ and $\rho = \pm 1$, while the kernel of $\Psi_{(t, \gamma, \rho)}$ is 1-dimensional and the kernel of $\Psi_{(t, \gamma, \rho)}$ is trivial for $\rho = -1$.

These observations imply that $\epsilon(\gamma^+_s) = -\epsilon(\gamma^-_s)$ and $\delta(\gamma^+_s) = -\delta(\gamma^-_s)$.

The above study, implies that $\pi_{g_s}(L, \alpha)$ (and thus $\pi_{g_s}(L)$) remains constant for $s$ belonging to an interval $(t - \epsilon, t + \epsilon)$ around $t \in \{s_1, \ldots, s_{N_L}\}$, see Figure 2. Moreover, it follows that $\pi_{g_s}(L)$ is locally constant on $[0, 1] - \{t_1, \ldots, t_{N_L}, r_1, \ldots, r_{N_L/2}\}$. When we pass a value $t_i$ which corresponds to some $L$-boundary point $(t_i, \gamma_{t_i})$, the value of $\pi_{g_s}(L)$ changes by $\epsilon'(\gamma_{t_i})\epsilon(\gamma_{t_i})$ and when we pass a value $r_i$ which corresponds to a $(L/2)$-boundary point $(r_i, \gamma_{r_i})$, the value of $\pi_{g_s}(L)$ changes by $\epsilon'(\gamma_{r_i})\delta(\gamma_{r_i})$. The claim of the theorem follows immediately, if we restrict the attention to (not necessarily prime) closed geodesics representing a particular free homotopy class $\alpha \in C_M$.

We may then use the discussion of Subsection 3.3 and prove the following theorem.

**Theorem 5.3.** Fix the Riemannian metrics $g_0, g_1 \in G^\infty,\ast(M)$ over the closed smooth manifold $M$. There is a subset $\mathcal{P}^\infty,\ast (M) \setminus \mathcal{P}^\infty,\ast (M)$, which is Baire in $C^d$ topology for $d \in \mathbb{Z}_{\geq 3}$, such that for every $\bar{g} = \{g_t\}_{t \in [0, 1]} \in \mathcal{P}^\infty,\ast (M)$, every $L$ outside a closed subset $\mathcal{L}_{\bar{g}}$ of $\mathbb{R}^+$ with zero measure, and every free homotopy class $\alpha \in C_M$ we have

$$\pi_{g_1}(L, \alpha) - \pi_{g_0}(L, \alpha) = \bar{g}(L, \alpha).$$

6. Non-generic Riemannian metrics and counting geodesics

6.1. Contribution of isolated geodesics. Let $g$ be a smooth metric on the closed manifold $M$.

**Definition 6.1.** A closed $g$-geodesic $\gamma : S^1 \to M$ is called an isolated embedded geodesic if $\gamma$ is an embedding and for every integer $d > 0$ there is a tubular neighborhood $U_d$ of the image of $\gamma$ such that $U_d$ does not contain any other $g$-geodesics, which are isotopic in $U_d$ to the $k$-fold cover of $\gamma$ for $k \leq d$. An isolated $g$-geodesic is an immersed closed $g$-geodesic which satisfies a similar criteria.

Associated with any isolated geodesic, we may define a generating function which encodes the contribution of $\gamma$ and its multiple covers to the geodesic count function associated with $g$, as follows. Consider a sequence $\{g_n\}_n$ of metrics in $\mathcal{G}^\infty,\ast$ which converges to $g$. We then have

$$Z_{g_n}(U_d) = Z_{g_n}(U_{d}, d^+) \cup Z_{g_n}(U_{d}, d),$$
where $Z_{g_n}(U_d, k)$ consists of $g_n$-geodesics in $U_d$ which are isotopic to the l-cover of $\gamma$ for $l \leq k$, and $Z_{g_n}(U_d, k^+) = Z_{g_n}(U_d) \setminus Z_{g_n}(U_d, k)$. Having fixed $d$, as $n$ goes to infinity, the lengths of the curves in $Z_{g_n}(U_d, d^+)$ becomes larger than $d\ell(\gamma)$. In fact, for every $\epsilon > 0$ and $n$ sufficiently large, $\pi_{g_n}(d\ell(\gamma) + \epsilon)$ corresponds to the count of geodesics in $Z_{g_n}(U_d, d)$. Moreover, for $m$ and $n$ sufficiently large, and $\tilde{g}_{n,m}$ a generic path connecting them which stays sufficiently close to $g$, no $\tilde{g}_{n,m}$-geodesic has length between $d\ell(\gamma) + \frac{1}{2}$ and $(d + 1)\ell(\gamma) - \frac{1}{2}$. In particular, $\pi_{g_n}(d\ell(\gamma) + \epsilon)$ is zero, implying $\pi_{g_n}(d\ell(\gamma) + \epsilon) = \pi_{g_n}(d\ell(\gamma) + \epsilon)$. Let us denote this stabilized count function by $\pi_{g_n}^0(\gamma, d)$, and let

$$n_d(\gamma) = n_d(\gamma, g) = \pi_{g_n}^0(\gamma, d) - \pi_{g_n}^0(\gamma, d - 1).$$

The above discussion implies that $n_d(\gamma, g)$ does not depend on the choice of $U_d$ or the sequence $\{g_n\}_n$.

**Definition 6.2.** Having fixed the above notation, $n_d(\gamma, g)$ is called the virtual contribution of the $d$-fold cover of $\gamma$ to the geodesic count function $\pi_g$. The contribution generating function $f_\gamma$ and the contribution polynomials $f_L^\gamma$ for $L \in \mathbb{R}^+$ associated with the isolated $g$-geodesic $\gamma$ are defined by

$$f_\gamma(z) = f_{M, g, \gamma}(z) = \sum_{d=1}^\infty n_d(\gamma, g)z^d$$

and

$$f_L^\gamma(z) = f_{M, g, \gamma, L}(z) = \sum_{d: d\ell(\gamma) \leq L} n_d(\gamma, g)z^d,$$

respectively. In particular, the polynomials $f_L^\gamma$ converge to $f_\gamma$ as $L$ goes to infinity.

**Definition 6.3.** Given a Riemannian metric $g$ on the smooth closed oriented manifold $M$ the subset

$$L_g = \{d\ell(\gamma) | d \in \mathbb{Z}^+ \text{ and } \gamma \in Z_g(M)\},$$

of the real numbers is called the length spectrum of the metric $g$. The metric $g$ is called a nice metric if its length spectrum is a discrete subset of $\mathbb{R}$ and there are finitely many closed geodesics of length $L$ for every $L \in L_g$.

Note that the length spectrum $L_g$ is always closed. Moreover, for each $c \in \mathbb{R}$, the intersection $L_g \cap (-\infty, c]$ is in correspondence with the set of critical values of an energy functional

$$E_c : U_c \subset M^{k_c} \to \mathbb{R}^+,$$

where the integer $k_c$ and the open subset $U_c \subset M^{k_c}$ depend on the metric $g$ and the value $c \in \mathbb{R}$, see [Mil63, Theorem 16.2]. In particular, the length spectrum of $g$ has zero measure as a subset of $\mathbb{R}$ and the complement $R_g = \mathbb{R} \setminus L_g$ is thus open and dense. If the manifold $M$ and the Riemannian metric $g$ are real analytic, then the length spectrum $L_g$ is discrete by [SS11, Proposition 1.2].

**Theorem 6.4.** Let us assume that $g$ is a nice Riemannian metric on $M$. Then all $g$-geodesics are isolated and for every sequence $\{g_n\}_n$ in $G_{M, g}^{\infty, \ast, \ast}$ which converge to $g$ and every $L \in R_g$ we have

$$\lim_{n \to \infty} \pi_{g_n}(L) = \sum_{d=1}^\infty \sum_{\gamma \in Z_g(L/d)} n_d(\gamma, g)h^d[\gamma].$$

**Proof.** Fix $L \in R_g$ and a homotopy class $\alpha \in C_M$. There are then only finitely many closed $g$-geodesics $\gamma$ such that $d[\gamma] = \alpha$, while $d\ell(\gamma) \leq L$, for some positive integer $d$. The set

$$Z_{M, g}(L, \alpha) = \bigcup_d Z_{M, g}(L/d, \alpha/d)$$

is then finite. Since no geodesic in $Z_{M, g}(L, \alpha)$ has length equal to $L$, we may choose $\epsilon > 0$ so that $Z_{M, g}(L, \alpha) = Z_{M, g}(L \pm \epsilon, \alpha)$, and choose $n$ sufficiently large so that every curve in $Z_{M, g_n}(L, \alpha)$ is $\epsilon$-close to (the multiple cover) of a curve in $Z_{M, g}(L, \alpha)$. Thus, associated with every curve $\gamma \in Z_{M, g_n}(L, \alpha)$ which contributes to the left-hand-side of Equation 5 we obtain a curve $\gamma'$ in $Z_{M, g}(L/d, \alpha/d)$, so that $\gamma$ is $\epsilon$-close to the $d$-sheeted cover of $\gamma'$. Moreover, the contribution of $\gamma$ to $n_d(\gamma, g)$ is equal to its contribution of to $\pi_{g_n}(L, \alpha)$. This completes the proof of the theorem. \[\square\]
Let us now assume that \( \mathcal{g} = \{ g_t \}_{t \in [0,1]} \) is a path of Riemannian metrics in \( \mathcal{G}_{g_0, g_1} \) and that \((t, \gamma) \in Z_g^f(L)\) is a \( g_t \)-geodesic of length \( L \) for some \( t \). We call \( \gamma \) an isolated geodesic in \( Z_g^f(L) \) if \( \gamma \) and its multiple covers are not the limit points of other closed geodesics in \( \bigcup_{d=1}^{\infty} Z_g^f(dL) \). The discussion of this subsection then assigns a well-defined virtual contribution \( n_d(\gamma, \bar{g}, t) \) of the \( d \)-cover of \( \gamma \) to the geodesic count function \( \pi_g(dL) \).

Let us further assume that \( g_0 \) and \( g_1 \) are in \( \mathcal{G}^{\ell,**} \) and that \( Z_g^f(L/d) \) consists of isolated points in the above sense for all \( d \in \mathbb{Z}^+ \). Then we may choose a sequence \( \{ \bar{g}_n \}_{n} \) of paths in \( \mathcal{G}_{g_0, g_1}^{\ell,**} \) which converges to \( \bar{g} \). For \( L \notin \mathcal{L}_{g_0} \cup \mathcal{L}_{g_1} \), Theorem 5.3 implies that \( \pi_{g_n}(L) \) remains constant (equal to \( \pi_{g_1}(L) - \pi_{g_0}(L) \)) as \( n \) goes to infinity. Moreover, the discussion of this subsection implies that this common value becomes equal to

\[
\sum_{d=1}^{\infty} \sum_{(t, \gamma) \in Z_g(L/d)} n_d(\gamma, \bar{g}, t) \ell^{d[\gamma]} = \pi_{g_1}(L) - \pi_{g_0}(L).
\]

We may thus set \( \pi_{\bar{g}}(L) \) equal to the above common value.

### 6.2. Geodesic count function for an arbitrary Riemannian metrics

Let \( g \) be an arbitrary Riemannian metric on a smooth closed oriented maniold \( M \) and \( \mathcal{R}_g \) denote the complement of the set \( \mathcal{L}_g \) of geodesic lengths.

**Lemma 6.5.** If \( L \in \mathcal{R}_g \), there is an open neighborhood \( \mathcal{U}_L = \mathcal{U}_{L,g} \) of \( g \) in \( \mathcal{G}^{\ell}(M) \) (in \( C^{\ell} \) topology) such that for every \( g' \in \mathcal{U}_L \) we have \( L \in \mathcal{R}_{g'} \).

**Proof.** Suppose otherwise that there is a sequence \( \{ g_n \}_{n} \) of Riemannian metrics on \( M \) which converge to \( g \) such that for each \( g_n \) there is a closed \( g_n \)-geodesic \( \gamma_n \) (possibly not prime) which has length \( L \). A subsequence of \( \gamma_n \) would then converge to a closed \( g \)-geodesic (possibly not prime) which has length \( L \). This contradiction implies that for \( g' \) in an open set \( \mathcal{U}_L \) around \( g \) (in \( C^{\ell} \) topology), \( L \in \mathcal{R}_{g'} \). \( \square \)

If \( g_0, g_1 \in \mathcal{U}_L \cap \mathcal{G}^{\infty,**} \) and \( \bar{g} = \{ g_t \}_{t \in [0,1]} \) is a generic path in \( \mathcal{U}_L \) which connects them, then \( \pi_{\bar{g}}(L) = 0 \), since \( Z_{\bar{g}}(L) \) is empty. In particular, \( \pi_{g_0}(L) = \pi_{g_1}(L) \) by Theorem 5.3. Let us set

\[
\mathcal{L}_M = \{(L, g) \in \mathbb{R} \times \mathcal{G}(M) \mid L \in \mathcal{L}_g \} \quad \text{and} \quad \mathcal{R}_M = (\mathbb{R} \times \mathcal{G}(M)) \setminus \mathcal{L}_M.
\]

We equip \( \mathcal{G}(M) \) with \( C^{\ell} \) topology (for a sufficiently large value of \( \ell \), e.g. \( \ell > 4 \)), while \( \mathcal{R}_M \) inherits its topology from the product topology on \( \mathbb{R} \times \mathcal{G}(M) \).

**Definition 6.6.** Define the geodesic count function \( \pi_M : \mathcal{R}_M \to \mathbb{Z}[\mathcal{C}_M] \) associated with the smooth closed oriented manifold \( M \) by setting \( \pi_M(L, g) = \pi_{g'}(L) \), where \( g' \in \mathcal{U}_L \cap \mathcal{G}^{\infty,**} \) is arbitrary.

The following theorem summarizes our investigations in this paper.

**Theorem 6.7.** Given a smooth closed oriented manifold \( M \), the associated geodesic count function \( \pi_M : \mathcal{R}_M \to \mathbb{Z}[\mathcal{C}_M] \) is locally constant. Moreover, if \( g \in \mathcal{G}(M) \) is nice we have

\[
\pi_M(L, g) = \sum_{d=1}^{\infty} \sum_{\gamma \in Z_g(L/d)} n_d(\gamma, g) \ell^{d[\gamma]}.
\]

If \( \bar{g} = \{ g_t \}_{t \in [0,1]} \) is a path in \( \mathcal{G}(M) \) and \( L \notin \mathcal{L}_{g_0} \cup \mathcal{L}_{g_1} \) is a real number so that \( Z_{\bar{g}}(L) \) is finite then

\[
\pi_M(L, g_1) - \pi_M(L, g_0) = \sum_{d=1}^{\infty} \sum_{(t, \gamma) \in Z_{\bar{g}}(L/d)} n_d(\gamma, \bar{g}, t) \ell^{d[\gamma]}.
\]
Proof. Only the last part requires a proof. Let \( \bar{g}' = \{g'_t\}_t \) be a generic path in \( \mathcal{P}^\infty_{g_0,g_1} \), which connects \( g'_0 \in \mathcal{U}_{L,g_0} \cap \mathcal{C}^\infty_{g_0,g_1} \) to \( g'_1 \in \mathcal{U}_{L,g_1} \cap \mathcal{C}^\infty_{g_0,g_1} \) and is sufficiently close to \( \bar{g} \). Then we have \( \pi_M(L,g_0) = \pi_M(L,g'_0), \pi_M(L,g_1) = \pi_M(L,g'_1) \), and \( \bar{g}' \) may be used to define all local contributions \( n_d(\gamma, \bar{g}, t) \) for \( (t, \gamma) \in \mathcal{Z}_{\bar{g}}(L) \). Theorem 5.3 implies the last equality in the statement of the theorem.

For an arbitrary Riemannian metric \( g \in \mathcal{G}(M) \) and \( L_0, L_1 \in \mathcal{R}_g \) with \( L_0 < L_1 \), the moduli space
\[
\mathcal{Z}_g([L_0,L_1]) = \mathcal{Z}_g(L_1) \setminus \mathcal{Z}_g(L_0)
\]
may consist of infinitely many closed geodesics, and may even contain continuous families of closed geodesics of the same length. Moreover, the interval \([L_0,L_1]\) may contain limit points from \( \mathcal{L}_g \). Theorem 6.7 may be interpreted as regarding \( \pi_M(L_1,g) - \pi_M(L_0,g) \) as the virtual contribution of \( \mathcal{Z}_g([L_0,L_1]) \) to the geometric count function. This virtual contribution is particularly interesting when \([L_0,L_1]\) contains a single value \( L \) from \( \mathcal{L}_g \), while closed \( g \)-geodesics of length \( L \) are not isolated. In this case, the aforementioned virtual contribution (i.e. the difference \( \pi_M(L_1,g) - \pi_M(L_0,g) \)) may be regarded as the virtual number of closed \( g \)-geodesics of length \( L \).

6.3. The marked length spectrum associated with Riemannian manifolds. If \( g \in \mathcal{G}(M) \) is a negatively curved Riemannian metric on the smooth closed oriented manifold \( M \), one may define a function \( \ell_g : \mathcal{C}_M \to \mathbb{R}^+ \) by setting \( \ell_g(\alpha) \) equal to the length of the unique closed \( g \)-geodesic representing the free homotopy class \( \alpha \in \mathcal{C}_M \). The function \( \ell_g \) is called the marked length spectrum associated with the Riemannian metric \( g \). Clearly, \( \ell_g \) is completely determined by \( \pi_g = \pi_M(g,\cdot) \), and vice versa, \( \pi_g \) is determined \( \ell_g \). In other words, \( \pi_g \) carries the same information as \( \ell_g \). For an arbitrary Riemannian metric \( g \in \mathcal{G}(M) \), one may thus regard
\[
\pi_g = \pi_M(g,\cdot) : \mathcal{R}_g \to \mathbb{Z}[\mathcal{C}_M]
\]
as a generalization of the marked length spectrum. If the geodesic flow associated with \( g \in \mathcal{G}(M) \) is Anasov, then \( \mathcal{L}_g \) is discrete. One main (and old) problem in Riemannian geometry is to study whether the length spectrum \( \mathcal{L}_g \) determines the metric \( g \) in this case. First counterexamples of different Riemannian metrics with constant negative curvature which share the same length spectrum were constructed by Vignéras [Vig80]. Nevertheless, the following (relatively long-standing) conjecture was first formulated by Burns and Katok [BK85, Problem 3.1]:

Conjecture 6.8. If both \( g_0, g_1 \in \mathcal{G}(M) \) have negative sectional curvatures, and \( \pi_{g_0} = \pi_{g_1} \) (and in particular \( \mathcal{L}_{g_0} = \mathcal{L}_{g_1} \)) then there exists a smooth diffeomorphism \( f : M \to M \) with \( g_1 = f^* g_0 \).

Although the conjecture has remained unsettled for almost 35 years, a proof for the case of negatively curved surfaces was first presented by Otal [Ota90], and independently by Croke [Cro90] (see [Wil14] for lecture notes on the proof). Guillarmou and Lefeuvre have recently proved a local version of Conjecture 6.8 for Anasov Riemannian metrics which are not positively curved [GL19]. Namely, they show that if \( g_0 \in \mathcal{G}(M) \) has non-positive sectional curvatures and Anasov geodesic flow, then for every metric \( g_1 \in \mathcal{G}(M) \) with \( \pi_{g_0} = \pi_{g_1} \) which is sufficiently close to \( g_0 \) in \( C^\ell \) norm (with \( \ell \) sufficiently large), we have \( g_1 = f^* g_0 \).

One can investigate whether this local version of the conjecture of Burns and Katok (i.e. Guillarmou-Lefeuvre theorem) is true for arbitrary Riemannian metrics \( g \in \mathcal{G}(M) \), if
\[
\pi_g = \pi_M(\cdot,g) : \mathcal{R}_g \to \mathbb{Z}[\mathcal{C}_M]
\]
is used as a replacement for the marked length spectrum. More precisely, we ask:

Question 6.1. Which Riemannian metrics \( g \in \mathcal{G}(M) \) have rigid marked length spectrum, in the sense that there is a neighborhood \( B_g \subset \mathcal{G}(M) \) of \( g \) in \( C^\ell \) topology (for \( \ell \) sufficiently large) such that for every \( g' \in B_g \) if \( \mathcal{R}_g = \mathcal{R}_{g'} \) and \( \pi_g = \pi_{g'} \) then \( g' = f^* g \) for some diffeomorphism \( f : M \to M \)?
Of course, it makes sense to start with the above question for a generic Riemannian metric \( g \in \mathcal{G}^{\infty, \ast \ast}(M) \), as the length spectrum \( \mathcal{L}_g \subset \mathbb{R} \) is discrete in this case. It is also interesting to investigate the above question in examples of non-standard Riemannian metrics on \( S^2 \) which share the same length spectrum with the standard metric of constant positive curvature, e.g. for Zoll metrics where the geodesic flow is of period \( 2\pi \) (c.f. [Gui76]).

Another natural question is the asymptotic behavior of the function \( \pi_g = \pi_M(\cdot, g) \). As discussed in the introduction, the growth of \( \pi_g \) for a negatively curved metric on a surface is related to the topological entropy of the metric [PS87, Mar69, PS98]. Over the sphere \( S^2 \), it is known that the growth of the number \( N_g(L) \) of closed \( g \)-geodesics of length at most \( L \) satisfies the following inequality (KT72, Mos77, Hin84) and [Hin93]:

\[
\liminf_{L \to \infty} \frac{N_g(L)}{(L/ \log(L))} > 0.
\]

Nevertheless, \( \pi_g(L) \) is typically different from \( N_g(L) \), and in particular, it is always finite. Since for every Riemannian metric \( g \in \mathcal{G}(M) \) the length spectrum \( \mathcal{L}_g \subset \mathbb{R} \) is closed and has zero Lebesgue measure, it makes sense to ask about the asymptotic behavior of \( \pi_g(L) \) as \( L \) grows large. In particular, when \( M \) is a surface of genus \( h \) and \( g \in \mathcal{G}(M) \) is an arbitrary Riemannian metric, the asymptotic behavior of \( \pi_g \) and its potential relation to the topological entropy of \( g \) is a natural question to ask for \( h > 1 \), while for \( h = 0 \) one is lead to compare \( \pi_g(L) \) with \( L/ \log(L) \).

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