Abstract

We analyze the decomposition of tensor products between infinite dimensional (unitary) and finite-dimensional (non-unitary) representations of $SL(2,\mathbb{R})$. Using classical results on indefinite inner product spaces, we derive explicit decomposition formulae, true modulo a natural cohomological reduction, for the tensor products.

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1 Introduction

The representation theory of $SL(2,\mathbb{R})$ was developed in the references listed in [1]. Since $SL(2,\mathbb{R})$ is non-compact, its unitary irreducible representations are infinite-dimensional and fall into several classes: the principal discrete series, the principal continuous series and the exceptional
series. There is also a series of finite-dimensional representations which are non-unitary. For a recent example of the application of both these unitary and non-unitary representations in physics, see [2].

While an exhaustive analysis of the decomposition of tensor products between unitary (infinite dimensional) representations was carried out by the authors listed in reference [3], they did not address the coupling of unitary and non-unitary representations.

In this paper we analyze the decomposition of tensor products between infinite dimensional (unitary) and finite dimensional (non-unitary) representations of $SL(2, \mathbb{R})$. We show that in the cases where these tensor products are not completely reducible, there exists a cohomological reduction of the product representation with respect to a nilpotent operator constructed from the Casimir. On this cohomology, the product representations become completely reducible. Using classical results on indefinite inner product spaces, we derive explicit decomposition formulae for the tensor products.

While several authors have discussed aspects of the decomposition of tensor products of finite-dimensional and infinite-dimensional representations of Lie groups [4], to our knowledge the methods and results of the current paper are new.

## 2 Representations of $SL(2, \mathbb{R})$

In this section we summarize some basic facts regarding the representation theory of $SL(2, \mathbb{R})$ (see [1] and [2]).

The commutation relations of the $SL(2, \mathbb{R})$ Lie algebra are

$$[L_1, L_2] = -iL_0, \quad [L_2, L_0] = iL_1, \quad [L_0, L_1] = iL_2.$$ 

Given a representation of $SL(2, \mathbb{R})$, we can define ladder operators

$$L_- \equiv L_1 + iL_2, \quad L_+ \equiv L_1 - iL_2$$

satisfying

$$[L_+, L_-] = 2L_0, \quad [L_0, L_\pm] = \pm L_\pm.$$ 

The operator $L_-$ ($L_+$) increases (decreases) the $L_0$ eigenvalue $m$ (called the weight) by one unit. A state annihilated by $L_+$ is conventionally called
highest weight and a state annihilated by $L_-$ lowest weight. Representations of $SL(2, \mathbb{R})$ are constructed by starting from some eigenstate of $L_0$ and repeatedly acting on it with $L_{\pm}$. The chain of states obtained in this way does not necessarily terminate on a highest or lowest weight state.

The Casimir operator is

$$L^2 = -L_1^2 - L_2^2 + L_0^2 = -L_-L_+ + L_0(L_0 - 1) = -L_+L_- + L_0(L_0 + 1).$$

(2)

The unitary irreducible representations of $SL(2, \mathbb{R})$ are all infinite-dimensional and are labelled by the parameter $\epsilon$ defined through

$$L^2 = \epsilon(\epsilon - 1),$$

as well as by the parameter $\epsilon \in \{0, \frac{1}{2}\}$ which determines whether the spectrum of $L_0$ is integral or half-integral.

The following unitary irreducible representations exist:

The “principal continuous series” $\mathcal{C}_\epsilon^h$: Here $L^2 < -\frac{1}{4}$ and we can write $h = \frac{1}{2} + i\lambda$ where $0 < \lambda \in \mathbb{R}$. There is neither a lowest weight nor a highest weight state, and the weights are $m = \epsilon + n$ for $n \in \mathbb{Z}$.

The “supplementary (or exceptional) continuous series” $\mathcal{S}_h$: Here $-\frac{1}{4} < L^2 < 0$ and $0 < h < \frac{1}{2}$ is real. There is neither a lowest nor a highest weight state. Only the case $\epsilon = 0$ occurs and the weights are $m \in \mathbb{Z}$.

The “discrete series” $\mathcal{D}_h^{\pm}$: Here $-\frac{1}{4} \leq L^2$ and $2h \in \mathbb{N}$. The highest weight representation $\mathcal{D}_h^+$ has weights $h + n$ for integer $n \geq 0$ and the lowest weight representation $\mathcal{D}_h^-$ has weights $-h - n$ for integer $n \geq 0$.

In addition to the unitary representations, there exists a series of finite-dimensional representations $\mathcal{D}_h$. They are characterized by $h = 0, -\frac{1}{2}, -1, \ldots$, and the weights are $h, h + 1, \ldots, -h$. These representations are not unitary with respect to any positive definite inner product, but, as we shall see below, they are (pseudo-) unitary with respect to an indefinite inner product on the state space.
We start by analyzing the coupling of the discrete series $\mathcal{D}_h^+$ with the finite-dimensional representations $\mathcal{D}_h$ (the analysis for $\mathcal{D}_h^+$ is entirely analogous and will not be presented here).

## 3 The discrete series $\mathcal{D}_h^+$

The highest weight representation $\mathcal{D}_h^+$ in the discrete series is generated from the highest weight state $\phi_h^{(h)}$ satisfying $L_+ \phi_h^{(h)} = 0$, where $h \in \frac{1}{2}, 1, \frac{3}{2}, \ldots$, by repeatedly applying $L_-$. The ladder of states

$$\phi_h^{(h)} \xrightarrow{L_-} \phi_{h+1}^{(h)} \xrightarrow{L_-} \phi_{h+2}^{(h)} \xrightarrow{L_-} \cdots$$  \hspace{1cm} (3)

does not terminate. Taking $e_h^{(h)} \equiv \phi_h^{(h)}$ to have unit normalization, the normalized basis

$$e_{h+k}^{(j)} \equiv \frac{1}{\sqrt{k! (2h) (2h+1) \cdots (2h+k-1)}} \phi_{h+k}^{(h)}$$  \hspace{1cm} (4)

satisfies

$$\langle e_{h+k}^{(h)} | e_{h+m}^{(h)} \rangle = \delta_{km},$$  \hspace{1cm} (5)

and

$$L_- e_{m+1}^{(h)} = \{(-h + m + 1) (h + m)\}^{\frac{1}{2}} e_{m+1}^{(h)}, \quad m = h, h + 1, \ldots$$  \hspace{1cm} (6)

and

$$L_+ e_{m+1}^{(h)} = \{(-h + m + 1) (h + m)\}^{\frac{1}{2}} e_{m}^{(h)}, \quad m = h, h + 1, \ldots$$  \hspace{1cm} (7)

The representations $\mathcal{D}_h^-$ may be similarly constructed from a lowest weight state annihilated by $L_-$. 

## 4 The finite-dimensional series $\mathcal{D}_h$

These representation $\mathcal{D}_h$, where $h \in 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots$, may be generated from a highest weight state $\phi_h^{(h)}$, where $h \in \{0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots\}$. The sequence of basis vectors

$$\phi_h^{(h)} \xrightarrow{L_-} \phi_{h+1}^{(h)} \xrightarrow{L_-} \cdots \xrightarrow{L_-} \phi_{-h}^{(h)}$$  \hspace{1cm} (8)
is finite.

These representations can be made (pseudo-) unitary by choosing an indefinite inner product on the state space. To understand this, note that in a (pseudo-) unitary representation, the ladder operator $L_+$ is adjoint to $L_-$, so that we can write the inner product of the state $\phi_{h+k}$ with itself as follows

$$\left\langle \phi_{h+k}^{(h)} \middle| \phi_{h+k}^{(h)} \right\rangle = \left\langle \left( L_- \right)^k \phi_h^{(h)} \middle| \left( L_- \right)^k \phi_h^{(h)} \right\rangle = \left\langle \phi_h^{(h)} \middle| \left( L_+ \right)^k \left( L_- \right)^k \phi_h^{(h)} \right\rangle.$$  

Commuting the $L_+$ operators to the right, this becomes

$$\left\langle \phi_{h+k}^{(h)} \middle| \phi_{h+k}^{(h)} \right\rangle = \left( \begin{array}{c} k! \end{array} \right) \left( \begin{array}{c} 2h \end{array} \right) \left( \begin{array}{c} 2h+1 \end{array} \right) \cdots \left( \begin{array}{c} 2h+k-1 \end{array} \right) \left\langle \phi_h^{(h)} \middle| \phi_h^{(h)} \right\rangle.$$  

Since $h < 0$, we see that squared norms of the sequence of states in (8) have alternating signs, implying that the inner product is indefinite. The normalized states

$$e_{h+k}^{(i)} \equiv \frac{1}{i^k \sqrt{k! \left| 2h \right| \left| 2h+1 \right| \cdots \left| 2h+k-1 \right|} \phi_{h+k}^{(h)},$$

have indefinite inner product

$$\left\langle e_{h+k}^{(h)} \middle| e_{h+m}^{(h)} \right\rangle = \left( -i \right)^k \delta_{km},$$

and the action of $L_\pm$ on this basis is given by

$$L_- e_{m}^{(h)} = i \left\{ (-h+m+1) \left| h+m \right| \right\}^{\frac{1}{2}} e_{m+1}^{(h)} \quad m = h, h+1, \ldots, -h$$

$$= \left\{ (-h+m+1) (h+m) \right\}^{\frac{1}{2}} e_{m+1}^{(h)}, \quad (-1)^{\frac{1}{2}} \equiv +i$$

and

$$L_+ e_{m+1}^{(h)} = i \left\{ (-h+m+1) \left| h+m \right| \right\}^{\frac{1}{2}} e_{m}^{(h)} \quad m+1 = h, h+1, \ldots, -h$$

$$= \left\{ (-h+m+1) (h+m) \right\}^{\frac{1}{2}} e_{m}^{(h)}, \quad (-1)^{\frac{1}{2}} \equiv +i.$$

These formulae are the continuation to negative $h$ of the corresponding formulae (6) and (7) for the discrete series.
5 Tensoring $D_{\frac{1}{2}}^+$ and $D_{-\frac{1}{2}}$

We now start our study of the coupling of the finite-dimensional representations of $SL(2, \mathbb{R})$ with the discrete series.

As a warmup example, we study the product of the $h = \frac{1}{2}$ discrete series representation with the $h = -\frac{1}{2}$ finite-dimensional representation. The product representation will turn out to be reducible but not completely reducible (the representation matrices are not fully decomposable), and to contain null (zero norm) states. We will identify a natural cohomological reduction procedure that makes the product decomposable and eliminates the null states. In this particular example, after applying the reduction, the product of the two representations will give the trivial representation. Schematically, indicating the cohomological reduction by an arrow, we have

$$D_{\frac{1}{2}}^+ \otimes D_{-\frac{1}{2}} \rightarrow D_0 \equiv 1.$$  

To start our analysis, note that there is a highest weight state in the product state space given by

$$|0\rangle \equiv e^{\left(\frac{1}{2}\right)}_{\frac{1}{2}} \otimes e^{\left(-\frac{1}{2}\right)}_{-\frac{1}{2}}.$$  

This state is annihilated by $L_+ \equiv L_+^{\left(\frac{1}{2}\right)} + L_+^{\left(-\frac{1}{2}\right)}$ and satisfies

$$L^2 |0\rangle = 0,$$  

like the trivial representation. However, in contrast with the trivial representation, $L_- |0\rangle$ is not zero. Still, one can verify that $(L_-)^k |0\rangle$ has zero norm with respect to the indefinite inner product on the product space for $k = 1, 2, 3, \ldots$. Concretely

$$L_-^n |0\rangle = \sqrt{n} (n-1)! |n\rangle,$$  

where the states

$$|n\rangle \equiv \sqrt{n} \left( e_{n+\frac{1}{2}}^{\left(\frac{1}{2}\right)} \otimes e_{-\frac{1}{2}}^{\left(-\frac{1}{2}\right)} + i e_{n-\frac{1}{2}}^{\left(\frac{1}{2}\right)} \otimes e_{\frac{1}{2}}^{\left(-\frac{1}{2}\right)} \right)$$  

are null.

It would be nice if we could truncate the ladder of states generated by $L_-$ as soon as we reach a null state. The proper way of doing this
is by noticing that the Casimir operator \( Q \equiv L^2 \) is nilpotent. Therefore calculating its cohomology defines a natural reduction procedure on the state space. The physicist will notice the analogy of this construction with BRST reduction, where \( Q \) is analogous to a BRST operator (see appendix A).

To see that \( L^2 \) is nilpotent, note that in terms of the basis consisting of \(|n\rangle\) and the additional null states

\[
|\tilde{n}\rangle \equiv \frac{1}{2\sqrt{n}} \left( -e^{\frac{i}{n+\frac{1}{2}}} \otimes e^{-\frac{i}{n-\frac{1}{2}}} + i e^{\frac{i}{n-\frac{1}{2}}} \otimes e^{-\frac{i}{n-\frac{1}{2}}} \right),
\]

it is not hard to calculate
\[
L^2 |\tilde{n}\rangle = |n\rangle, \quad L^2 |n\rangle = 0,
\]
leading to the matrix representation

\[
L^2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}.
\]

In this basis, the inner product is represented by the matrix

\[
G \equiv \begin{pmatrix}
1 & -1 & -1 & \cdots \\
-1 & 1 & -1 & \cdots \\
-1 & -1 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots 
\end{pmatrix}
\]

It is easily checked that \((L^2)^2 = 0\) and that \(L^2\) is hermitian with respect to the inner product \(G\), as follows from

\[
L^2 = (L^2)^\dagger = G(L^2)^\dagger G,
\]
where the dagger denotes the adjoint with respect to the indefinite inner product \( \langle \cdot | \cdot \rangle \) on the state space and the + sign denotes the usual conjugate matrix.

The set of states \( \{ |n\rangle \} \) defined above forms an \( L^2 = 0 \) subspace which is closed under \( SL(2, R) \), but does not fall into any of the representations considered in section \( \text{II} \) unless we can get rid of the null states
\[
\{ |1\rangle , |2\rangle , |3\rangle , \ldots \}.
\]

In addition, since the complementary set \( |\bar{n}\rangle \) is not closed under \( SL(2, R) \), the full product representation is non-decomposable.

The following procedure gets rid of these pathologies in the product. We construct the cohomology \( \ker Q/\text{im} Q \) with respect to the operator \( Q = L^2 \). The cohomology consists of the single class \( [|0\rangle] \), and is a one-dimensional, positive definite Hilbert space. Since the generators \( L_i \) all commute with \( Q \), they can consistently be reduced to the cohomology. The induced operators \( [L_+] \), \( [L_-] \) and \( [L_0] \) on the quotient space are defined as
\[
\begin{align*}
[L_i] (|\phi\rangle) &= [L_i |\phi\rangle], \quad |\phi\rangle \in \ker Q.
\end{align*}
\]

and are indeed zero on the quotient space, corresponding to the trivial representation of \( SL(2, R) \).

6 General product representations

We now generalize the method of section \( \text{V} \) to the analysis of a general product representation of a member of the discrete series and a member of the finite-dimensional series.

In appendix \( \text{C} \) we prove the result, important in what follows, that the Casimir operator \( L_2 \) can be decomposed into Jordan blocks of dimension at most two (appendix \( \text{B} \) discusses properties of operators on indefinite inner product spaces). Since the one and two-dimensional Jordan blocks of \( L^2 \) are of the forms \( (\lambda_i) \) and
\[
\begin{pmatrix}
\lambda_i & 1 \\
0 & \lambda_i
\end{pmatrix},
\]
we can build a nilpotent cohomology operator in the product space of two arbitrary representations in terms of \( L^2 \) as the orthogonal direct sum
\[
Q = Q_{\lambda_1} \oplus Q_{\lambda_2} \oplus \cdots,
\]
where $Q_{\lambda_i}$ is defined on the principal vector subspace $V_{\lambda_i}$ belonging to the eigenvalue $\lambda_i$ of $L^2$ by

$$Q_{\lambda_i} = L^2|_{V_{\lambda_i}} - \lambda_i.$$  \hfill (19)

Since $Q_{\lambda_i}$ consists of one- and two-dimensional blocks of the forms (0) and

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$  \hfill (20)

$Q$ is nilpotent and is a valid cohomology operator.

Taking the cohomology of $Q$ discards any Jordan blocks of dimension two in the decomposition of $L^2$, so that all principal vectors of the reduced $[L^2]$ will be eigenvectors. Equivalently, $[L^2]$ can be decomposed into Jordan blocks of dimension 1.

Let us see how $Q$ affects the analysis of a general product representation. As in the example of section 5, negative $h$ representations may be present in the product in the form

\[
\begin{array}{cccccc}
\phi_h^- & L_- & \phi_h^+ & L_+ & \phi_h^+ & L_- \\
L_+ & L_+ & L_+ & L_+ & L_+ & L_+
\end{array}
\]

The boxes represent two-dimensional subspaces spanned by null vectors $\phi_{-h+n}^-$ and $\phi_{-h+n}^+$ on which $L^2$ has the Jordan normal form (17) and $Q$ has the form (20). We see that just as in the example of section 5, the ladder of states generated from the highest weight $\phi_h^+$ consists of null states for $m > -h$.

Because of the additional states to the right of $\phi_{-h}^+$, this representation is not in the original class of irreducible positive or negative spin representations that we started out with. In other words, the original class of irreducible representations is not closed under multiplication, unless we can get rid of the extra states.

Taking the cohomology with respect to $Q$ discards the boxed subspaces in the above diagram, and on the cohomology classes the representation has the familiar form

\[
\begin{array}{cccccc}
\phi_h^- & L_- & \phi_h^+ & L_+ & \phi_h^+ & L_- \\
L_+ & L_+ & L_+ & L_+ & L_+ & L_+
\end{array}
\]

9
Henceforth, when we analyze product representations, it will always be assumed that we are working in the cohomology with respect to the associated operator $Q$. This amounts to a redefinition of the product as the tensor product followed by the cohomological (BRST) reduction.

## 7 Characters

The analysis of the decomposition of product representations may be greatly simplified by reformulating it as an algebraic problem in terms of characters.

We would like the expression for the character of a representation to be invariant under the above BRST reduction to the cohomology of $Q$. Since the discarded Jordan subspaces have zero metric signature, a definition for the character of a group element $U$ that will ignore these blocks is

$$
\chi(U) = \sum_n \text{sig}(V_{\lambda_n}) \lambda_n
$$

where $V_{\lambda_n}$ is the principal vector subspace corresponding to the eigenvalue $\lambda_n$ of $U$ and $\text{sig}(\cdot)$ denotes the signature. Since the signature of a subspace is invariant under unitary transformations $V$ [5], this gives a basis invariant expression invariant under conjugation $U \rightarrow VUV^{-1}$.

At this point we just warn the reader that this definition needs modification in the infinite-dimensional case, where characters may only exist in the distributional sense. We will discuss this issue in more detail below.

Using the following properties of the signature

\begin{align}
\text{sig}(V \otimes W) &= \text{sig}V \cdot \text{sig}W, \\
\text{sig}(V \oplus W) &= \text{sig}V + \text{sig}W,
\end{align}

it follows that the characters satisfy the following important algebraic properties

\begin{align}
\chi(U_1 \otimes U_2) &= \chi(U_1) \cdot \chi(U_2), \\
\chi(U_1 \oplus U_2) &= \chi(U_1) + \chi(U_2),
\end{align}

(where the right hand side of the first formula may be invalid in certain infinite-dimensional cases when the product of the distributions $\chi(U_1)$ and
\( \chi(U_2) \) may be undefined). These are exactly the properties that make the characters useful for analyzing the decomposition of a product representation into a direct sum of irreducible representations. The first property ensures that the character of a product representation is simply the product of the characters of the individual representations. In other words,

\[
\chi^{R_1 \otimes R_2} = \chi^{R_1} \chi^{R_2}.
\]

The second property then ensures that if \( R_1 \otimes R_2 = \sum_i n_i R_i \), then

\[
\chi^{R_1 \otimes R_2} = \sum_i n_i \chi^{R_i},
\]

where the weight \( n_i \) denotes the degeneracy of the representation \( R_i \) in the product. In the indefinite metric case, the weight \( n_i \) will be negative if the inner product on \( R_i \) is of opposite sign to the usual conventions as in (5), or equivalently, if the lowest weight state has negative norm squared. This is due to the inclusion of the signature in the definition (22) above.

We now discuss some issues specific to infinite-dimensional representations. Note that the definition (22) presupposes that the spectrum of the representation matrix \( U \) is discrete and that the sum in (22) converges. For our infinite-dimensional representations, this will not be true. However, in this case, one can still define the characters of suitable smeared versions of the operators \( U \) (see [6]). The characters become distributions defined on the group manifold.

The group \( SL(2, \mathbb{R}) \) has three distinct families of conjugacy classes. These are the elliptic elements \( \mathcal{E} \), indexed by the continuous parameter \( \theta \), of which a typical element is given by \( e^{i \theta L_0} \), and two families \( \mathcal{H}_\pm \) of hyperbolic elements, each indexed by a continuous parameter \( \sigma \) and with typical element given by \( \pm e^{i \sigma L_1} \).

The characters of the discrete series representation \( \mathcal{D}_h^+ \) were obtained in [6]: The character of the elliptic element \( e^{i \theta L_0} \) is

\[
\chi_{\mathcal{E}^{(h)}}(\theta) = \frac{e^{ih\theta}}{1 - e^{i\theta}} = \frac{-e^{i(h - \frac{1}{2})\theta}}{2i \sin \frac{1}{2} \theta}.
\]
and that of the hyperbolic elements is
\[
\chi^{(h)}_{H\pm}(\sigma) = \pm \chi^{(h)}_E(\theta = i\sigma) = \pm e^{-(h-\frac{1}{2})\sigma} \frac{1}{2 \sinh \frac{1}{2} \sigma}.
\] (28)

We now compute the characters of the finite dimensional representations, which we denote by \(\chi^{(h)}\) for \(h \leq 0\). Since these representations have the indefinite inner product (11), our definition (22) does not reduce to the ordinary trace, but rather weighs the eigenvalues according to the metric signature of the corresponding eigenspace. We find, for \(h < 0\)
\[
\chi^{(h)}_E(\theta) = e^{ih\theta} (1 - e^{i\theta} + e^{2i\theta} - \cdots + (-)^{2h} e^{-2ih\theta})
\]
\[
= \begin{cases} 
\frac{\cos \left(h - \frac{1}{2}\right) \theta}{\cos \frac{1}{2} \theta} & \text{if } 2h \text{ even,} \\
\frac{i \sin \left(h - \frac{1}{2}\right) \theta}{\cos \frac{1}{2} \theta} & \text{if } 2h \text{ odd.}
\end{cases}
\] (29)

The characters of the hyperbolic elements are
\[
\chi^{(h)}_{H\pm}(\sigma) = \pm \chi^{(h)}_E(\theta = i\sigma) = \begin{cases} 
\pm \cosh \left(h - \frac{1}{2}\right) \sigma & \text{if } 2h \text{ even,} \\
\pm \sinh \left(h - \frac{1}{2}\right) \sigma & \text{if } 2h \text{ odd.}
\end{cases}
\] (30)

8 General products of discrete and finite series representations

The analysis of the product of discrete and finite series can now be carried out by calculating the algebra of the characters. This is most easily done by expanding the characters in powers of \(e^{i\theta}\) or \(e^\sigma\) as in the first line of (29), performing the multiplication and collecting terms.

The following cases occur:
When $h_1 > 0$ (discrete series), $h_2 < 0$ (finite series) and $h_1 > |h_2|$, we have for the elliptic elements
\[
\chi^{(h_1)} \chi^{(h_2)} = \chi^{(h_1+h_2)} - \chi^{(h_1+h_2+1)} + \cdots \pm \chi^{(h_1-h_2)}. \tag{31}
\]
Since we saw that the characters of the hyperbolic elements are related to those of the elliptic elements by the analytic continuation $\chi_{H_\pm}^{(h)}(\sigma) = \pm \chi^{(h_1)}(\theta = i\sigma)$, it trivially follows that the same relation is satisfied by the hyperbolic characters. Therefore
\[
\mathcal{D}_{h_1}^+ \otimes \mathcal{D}_{h_2} \rightarrow \sum_{h=|h_1+h_2|}^{h_1-h_2} (-)^{h_1+h_2-h} \mathcal{D}_h^+, \tag{32}
\]
where the arrow indicates cohomological reduction with respect to the operator $Q$.

Note that this equation (32) weighs representations with both positive and negative signs, which contain information about the signature of the inner product on the corresponding subspaces. More precisely, one should read
\[
R_1 - R_2 \equiv R_1 \oplus (-R_2),
\]
where $-R_2$ denotes the representation $R_2$ but with opposite signature inner product.

When $h_1 > 0$ (discrete series), $h_2 < 0$ (finite series), $h_1 \leq |h_2|$ and $h_1 + h_2$ is integral, then
\[
\chi^{(h_1)} \chi^{(h_2)} = \chi^{(h_1+h_2)} + \chi^{(h_1+h_2+1)} + \cdots + \chi^{(0)} + \chi^{(-h_1-h_2+2)} - \chi^{(-h_1-h_2+3)} + \cdots \pm \chi^{(h_1-h_2)}, \tag{33}
\]
where we have suppressed the subscript indicating elliptic/hyperbolic. It follows that
\[
\mathcal{D}_{h_1}^+ \otimes \mathcal{D}_{h_2} \rightarrow \sum_{h=|h_1+h_2|}^{0} D_h + \sum_{h=-|h_1-h_2|+2}^{h_1-h_2} (-)^{h_1+h_2} \mathcal{D}_h^+. \tag{34}
\]
When \( h_1 > 0 \) (discrete series), \( h_2 < 0 \) (finite series), \( h_1 \leq |h_2| \) and \( h_1 + h_2 \) is half-integral, then
\[
\chi^{(h_1)} \chi^{(h_2)} = \chi^{(h_1+h_2)} + \chi^{(h_1+h_2+1)} + \ldots + \chi^{(-\frac{1}{2})} + \chi^{(\frac{1}{2})} - \chi^{(-h_1-h_2+2)} + \chi^{(-h_1-h_2+3)} - \ldots \pm \chi^{(h_1-h_2)}.
\]
(35)

It follows that
\[
\mathcal{D}^+_{h_1} \otimes \mathcal{D}^+_{h_2} \rightarrow \sum_{h=h_1+h_2}^{\frac{-1}{2}} D^+_h + \sum_{h=-h_1-h_2+2}^{h_1-h_2} (-)^{h+h_1+h_2+1} D^+_h.
\]
(36)

Finally, we calculate the coupling of finite series representations among themselves. When \( h_1 \leq 0 \) and \( h_2 \leq 0 \) (both finite series representations) and \( |h_2| \leq |h_1| \), then, again by expanding in powers of \( e \) and collecting terms, we find
\[
\chi^{(h_1)} \chi^{(h_2)} = \chi^{(h_1+h_2)} - \chi^{(h_1+h_2+1)} + \ldots \pm \chi^{(h_1-h_2)}.
\]
(37)

It follows that
\[
\mathcal{D}_{h_1} \otimes \mathcal{D}_{h_2} = \sum_{h=h_1+h_2}^{-|h_1-h_2|} (-)^{h+h_2-h} D_h.
\]
(38)

In this case, no cohomological reduction is needed to obtain the right hand side. This decomposition is similar to what occurs in the representation theory of \( SU(2) \). This is as expected, since the finite dimensional representations of \( SU(2) \) and \( SL(2, \mathbb{R}) \) are simply related by a rotation of two of the generators by \( i \) (and renaming \( -h = j \)). However, unlike the \( SU(2) \) case, the equation (38) weighs representations with both positive and negative signs, which contain information about the signature of the inner product on the corresponding subspaces.

For completeness, we list the product decomposition of two discrete representations
\[
\mathcal{D}^+_{h_1} \otimes \mathcal{D}^+_{h_2} = \sum_{h=h_1+h_2}^{\infty} D^+_h.
\]
(39)
9 Associativity

Let us denote by \( R_1 \tilde{\otimes} R_2 \) the above cohomological reduction of \( R_1 \otimes R_2 \) with respect to the operator \( Q \). The operation \( \tilde{\otimes} \) is not associative, as can be seen from a simple counterexample

\[
D^+_{\frac{1}{2}} \tilde{\otimes} (D^+_{\frac{1}{2}} \tilde{\otimes} D^-_{\frac{1}{2}}) = D^+_{\frac{1}{2}} \tilde{\otimes} 1 = D^+_{\frac{1}{2}},
\]

where we have used (34), while

\[
(D^+_{\frac{1}{2}} \tilde{\otimes} D^+_{\frac{1}{2}}) \tilde{\otimes} D^-_{\frac{1}{2}} = (D^+_{\frac{1}{2}} \oplus D^+_{\frac{1}{2}} \oplus \cdots) \tilde{\otimes} D^-_{\frac{1}{2}}
\]

where we have used (39) and (32).

From the example it is, however, obvious that the product can be made associative if we make the identification \( R \oplus (-R) \sim 0 \), where we remind the reader that \(-R\) denotes the representation \( R \) with inner product of opposite signature. In other words, we take the quotient with respect to sums of representations of the form \( R \oplus (-R) \). The definition (22) of the characters is obviously invariant with respect to this quotient, and the reduction of \( \tilde{\otimes} \) with respect to the quotient is associative.

10 Coupling of finite-dimensional and continuous series representations

We will start by investigating the decomposition of products of the form \( C_h \otimes D_{h'} \) of a principal series and a finite-dimensional series representation. Here \( h = \frac{1}{2} + is \) for \( s > 0 \) real, and \( h' = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots \)

The first case we consider is

\[
C_h^e \otimes D_{-\frac{1}{2}}.
\]

With respect to the (non-normalized) bases \( |f^h_m\rangle \) for \( h = 1+is, -\frac{1}{2} \), in which

\[
L_- |f^h_m\rangle = (m + 1 - h) |f^h_{m+1}\rangle
\]

\[
L_+ |f^h_{m+1}\rangle = (m + h) |f^h_m\rangle,
\]
the Casimir decomposes into blocks of the form

\[
L^2 \rightarrow \begin{pmatrix}
  h(h - 1) - m - \frac{1}{4} & (m + 1 - h) \\
  -(m + h) & h(h - 1) + m + \frac{3}{4}
\end{pmatrix}
\]

Diagonalizing, we find the eigenvalues of \(L^2\) to be independent of \(m\) and given by \(\tilde{h}(\tilde{h} - 1)\) for \(\tilde{h} = h \pm \frac{1}{2}\). In other words, multiplication by \(D_{-\frac{1}{2}}\) takes us from a continuous series representation with \(h = \frac{1}{2} + is\) to a direct sum of two representations with \(\tilde{h} = (\frac{1}{2} + is) \pm \frac{1}{2}\).

However, this is not a true decomposition since the inner product cannot be diagonalized simultaneously with \(L^2\). Indeed, it is a property of pseudo-hermitian operators such as \(L^2\) that any complex eigenvalues come in conjugate pairs [5] and that the corresponding eigenspaces are null, and are not orthogonal but rather dual, meaning that with respect to the eigenbasis in each block the inner product takes the form

\[
\begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix}
\]

To denote this situation we write

\[
C^\epsilon_h \otimes D_{-\frac{1}{2}} = C^\epsilon_{h-\frac{1}{2}} \# C^\epsilon_{h+\frac{1}{2}}
\]

(41)

where the representations \(C^\epsilon_{h\pm\frac{1}{2}}\) with \(h = \frac{1}{2} + is \pm \frac{1}{2}\) each has degenerate (null) inner product (which explains their absence from the traditional taxonomy of section 2).

To calculate the coupling of \(C^\epsilon_h\) with an arbitrary finite representation \(D_h\), we note that the latter can all be built up from products of \(D_{-\frac{1}{2}}\). Indeed, from (38) we obtain the recursive relations \((h = -1, -\frac{3}{2}, \ldots)\)

\[
D_h = D_{h+\frac{1}{2}} \otimes D_{-\frac{1}{2}} + D_{h+1}
\]

(42)

modulo the identification \(R \oplus (-R) \sim 0\) discussed in the previous section. Together with (41), this allows us to obtain the decomposition of an arbitrary product. For example, using

\[
D_{-1} = D_{-\frac{1}{2}} \otimes D_{-\frac{1}{2}} + D_0
\]
we get, for \( h = \frac{1}{2} + is \),
\[
C_h^e \otimes D_{-1} = C_h^e \otimes D_{-\frac{1}{2}} \otimes D_{-\frac{1}{2}} + C_h^e \\
= \left( C_{h+\frac{1}{2}}^e \# C_{h-\frac{1}{2}}^e \right) \otimes D_{-\frac{1}{2}} + C_h^e \\
= C_{h+1}^e \# C_{h-1}^e + C_h^e \# C_h^e + C_h^e \\
= C_{h+1}^e \# C_{h-1}^e + C_h^e
\]
where we have used \( C_h^e \# C_h^e = C_h^e - C_h^e \sim 0 \), by diagonalization of the metric \((40)\).

Continuing in this vein, we obtain the product decompositions
\[
C_{h_1}^e \otimes D_{h_2} = \sum_{k=0}^{\lfloor h_2 - \frac{1}{2} \rfloor} c_{2|h_2|,k} \left( C_{h_1+h_2+k}^e \# C_{h_1-h_2-k}^e \right)
\]
for \( h_2 = -\frac{1}{2}, -\frac{3}{2}, \ldots \), and
\[
C_{h_1}^e \otimes D_{h_2} = \sum_{k=0}^{\lfloor h_2 \rfloor} c_{2|h_2|,k} \left( C_{h_1+h_2+k}^e \# C_{h_1-h_2-k}^e \right) + C_{h_1}^e
\]
for \( h_2 = 0, -1, -2, \ldots \). In these formulae, the nonzero entries in the table of coefficients \( c_{ij} \) are
\[
(c_{ij}) = \begin{pmatrix}
1 \\
1 \\
1 & 1 \\
1 & 3 \\
1 & 5 & 1 \\
1 & 7 & 9 \\
1 & 9 & 21 & 1 \\
1 & 11 & 37 & 31 & \ldots \\
\vdots
\end{pmatrix}
\]
Here each entry is generated from the sum of the three entries arranged in an L-shape above it. More formally, they are generated from the recursion
\[
c_{i+1,j+1} = c_{i,j+1} + c_{i,j} + c_{i-1,j} \\
c_{0,j} = 1 \\
c_{2i,i} = 1
\]
which reminds one of the Fibonacci sequence. In fact, the actual Fibonacci sequence makes an appearance in the next result:

We now consider the decomposition of products of the supplementary continuous series $S_h$, $0 < h < \frac{1}{2}$ with the finite representations $D_{h'}$. In fact, the allowed range of $h$ is a priori $0 < h < 1$, but representations indexed by $h$ and $1 - h$ are isomorphic, allowing us to restrict attention to half the range. As above, tensoring with $D_{-\frac{1}{2}}$ gives a direct sum of representations $S_{h + \frac{1}{2}}$ and $S_{h - \frac{1}{2}}$. The first of these is isomorphic to $S_{\frac{1}{2} - h'}$, while the second lies outside the allowed range for unitary $S_h$ and once again denotes a representation with degenerate inner product. Since the eigenvalues of $L^2$ are distinct and not conjugate, the corresponding eigenspaces are orthogonal, with the metric now taking the form

$$
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
$$

(48)

In contrast to the previous situation, the inner product factorizes and the representations are now independent. If, in addition to the identification $R \oplus (-R) \sim 0$ of the previous section, we identify $R \oplus N \sim R$ whenever the inner product factorizes and $N$ is degenerate, we get the identity

$$S_h \otimes D_{-\frac{1}{2}} \rightarrow S_{\frac{1}{2} - h}.$$

Generating $D_{-n}$ by repeated tensoring with $D_{-\frac{1}{2}}$ as in (42), we can now calculate the general product decomposition

$$S_h \otimes D_{-k} \rightarrow \text{fib}_k S_{r^k(h)}.$$

(49)

where $r^k \equiv r \circ r \circ \cdots \circ r$ denotes $k$ applications of the reflection $h \mapsto \frac{1}{2} - h$ and $\text{fib}_k$ denotes the Fibonacci sequence.

11 Products of two continuous series representations

The product decomposition of two continuous series representations is known and may be found in the references [3]. Here we point out an issue
that should be addressed when combining those results with the formalism of the current paper. We have from reference [3]

\[ C_h^0 \otimes D^+_\frac{1}{2} = \sum_{k=\frac{1}{2}}^{\infty} D^+_k + \int_{\frac{1}{2}}^{\frac{1}{2} + i \infty} dh' C_{h'}^0. \]

We note the curious fact that the right hand side does not depend on \( h \). Consequently, neither could its product with the finite-dimensional series representation \( D_{-\frac{1}{2}} \)

\[(C_h^0 \otimes D^+_\frac{1}{2}) \tilde{\otimes} D_{-\frac{1}{2}}.\]

On the other hand,

\[ C_h^0 \otimes (D^+_\frac{1}{2} \tilde{\otimes} D_{-\frac{1}{2}}) = C_h^0 \otimes 1 = C_h^0, \]

which does depend on \( h \). In other words, \( \otimes \) does not associate over \( \tilde{\otimes} \).

It would be interesting to investigate whether the definition of \( \tilde{\otimes} \) can be extended to the case where both arguments are in the continuous series, in such a way that associativity is regained. Since this question is likely to need technology beyond the scope of this paper [3], we defer it to future work.

### 12 Analytic continuation and \( SU(2) \)

It is easily verified that that the finite-dimensional representations of the group \( SL(2, \mathbb{R}) \) may be related to the finite-dimensional representations of \( SU(2) \) via the transformation \( L_- = iJ_+ \), \( L_+ = iJ_- \), \( L_0 = J_3 \) and \( h = -j \).

When we apply this transformation to the infinite-dimensional discrete series generators of \( SL(2, \mathbb{R}) \), we obtain an irreducible set of infinite-dimensional operators that formally satisfies the \( su(2) \) Lie algebra. These generators cannot be exponentiated to give a representation of the full \( SU(2) \) group, since there exists a theorem stating that any continuous irreducible representation of a compact Lie group is finite-dimensional. However, the generators can be exponentiated on a neighborhood of the identity, where they form so-called “analytic” representations of \( SU(2) \) as defined by Segal [7]. In [8] and [9], we studied the recoupling theory of these infinite-dimensional analytic representations of \( SU(2) \). The results of the present paper are a straightforward adaptation of the methods used in [9].
In fact, the decomposition formulae (32), (34), (36), (38) and (39) correspond up to relative signs to the corresponding formulae involving the finite-dimensional (continuous) and infinite-dimensional negative spin (analytic) representations of $SU(2)$ studied in [9].

13 Conclusion

In this paper we analyzed the decomposition of tensor products of the infinite dimensional (unitary) and the finite dimensional (non-unitary) representations of $SL(2, \mathbb{R})$. Using classical results on indefinite inner product spaces, combined with cohomological methods, we were able to derive explicit decomposition formulae, true modulo a well-defined cohomological reduction, for the tensor products.

As explained in section 11 it would be interesting to revisit the existing results on tensor products between two continuous series representations to better understand whether an associative extension of $\otimes$ can be defined.

It would also be interesting to determine whether some of these results can be generalized to other non-compact groups.

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Appendices

A BRST cohomology

In the BRST formalism [10], the analysis of physical states and operators is carried out in terms of an operator $Q$ that is hermitian and nilpotent. In other words,

$$Q\dagger = Q, \quad Q^2 = 0.$$
States are called physical if they satisfy
\[ Q |\phi\rangle = 0, \]
and are regarded as equivalent if they differ by a \( Q \)-exact state. In other words,
\[ |\phi\rangle \sim |\phi\rangle + Q |\chi\rangle, \]
where \(|\chi\rangle\) is an arbitrary state. More formally, physical states are elements of the cohomology of \( Q \), defined as the quotient vector space
\[ \ker Q / \text{im} Q \]
with elements
\[ [|\phi\rangle] \equiv |\phi\rangle + \text{im} Q. \]
The inner product on this quotient space may be defined in terms of the original inner product by noting that all elements of \( \text{im} Q \) are orthogonal to all elements of \( \ker Q \), so that the induced inner product defined on equivalence classes in the cohomology by
\[ \langle \phi + \text{im} Q | \phi' + \text{im} Q \rangle \equiv \langle \phi | \phi' \rangle, \quad \phi, \phi' \in \ker Q \]
is well defined.

A hermitian operator \( A \) is regarded as physical if \( [A, Q] = 0 \). This ensures that \( A \) leaves \( \text{im} Q \) invariant, so that the reduced operator \( [A] \) defined on the cohomology classes by
\[ [A] [|\phi\rangle] = [A |\phi\rangle] \quad (50) \]
is well-defined.

### B Operators on indefinite inner product spaces

We review a few general facts regarding hermitian operators on indefinite inner product spaces, also known as pseudo-hermitian operators [5].

It is important to be aware that not all results that are valid for positive definite spaces are valid when the inner product is not positive definite. For example, not all pseudo-hermitian operators are diagonalizable. A good counterexample is precisely the operator \( Q = J^2 \) above. In addition,
not all eigenvalues are necessarily real. In particular, a pseudo-hermitian operator may have complex eigenvalues that come in conjugate pairs.

For our purposes, we will restrict consideration to pseudo-hermitian operators with real eigenvalues, of which $L^2$ will be the relevant example. The domain of such an operator $A$ can always be decomposed into a direct sum of pairwise orthogonal subspaces, in each of which we can choose a basis such that $A$ has the so-called Jordan normal block form

$$
\begin{pmatrix}
\lambda & 1 \\
\lambda & 1 \\
\ddots & \ddots \\
& \lambda \\
\end{pmatrix}
$$

and the inner product has the form

$$
\pm
\begin{pmatrix}
1 \\
& 1 \\
& & 1 \\
& & & 1 \\
\end{pmatrix}.
$$

Notice that only the first vector in the subspace is an eigenvector of $A$ with eigenvalue $\lambda$. If the Jordan block has dimension larger than 1, this eigenvector is null. The vectors $v$ in this subspace are called principal vectors belonging to the eigenvalue $\lambda$ and satisfy

$$(A - \lambda)^m v = 0$$

for some integer $m \geq 1$. The sequence of vectors $v_i$ spanning this subspace satisfying

$$Av_i = \lambda v_i + v_{i-1}$$

is called a Jordan chain.

### C  Jordan decomposition of $L^2$

In this appendix we prove that, in an arbitrary highest weight representation, $L^2$ can be decomposed into Jordan blocks of dimension at most two.

First, note that, since $L^2$ commutes with $L_0$, we can decompose $L^2$ into Jordan blocks in each eigenspace of $L_z$. Consider such a Jordan block on
a subspace $V_m$ consisting of principal vectors of $L^2$ belonging to an eigenvalue $h(h - 1)$ and with $L_0$ eigenvalue $m$. Taking any vector $v$ in $V_i$, by applying $L_+$ to it repeatedly we will eventually obtain zero, since by assumption our representation is highest weight, so that the spectrum of weights of $L_0$ is bounded from below. This procedure gives a highest weight state $L_+^k v$, and we can use (2) to obtain $h(h - 1)$ in terms of the $L_0$ eigenvalue $m - k$ of this highest weight state. Since $m$ is integer or half-integer, the possible values of $h$ are also integer or half-integer, either positive or negative. In the following, we shall take the positive solution $h > 0$.

Now consider the sequence of subspaces

$$V_m \xrightarrow{L_+} V_{m-1} \xrightarrow{L_+} V_{m-2} \xrightarrow{L_+} \cdots .$$

Since $L^2$ commutes with $L_+$, we see that $L^2$ takes each of the subspaces $V_i$ to itself. Furthermore, as long as $i \not\in \{-h + 1, h\}$, $L_+$ cannot change the dimension of these subspaces since that would imply that $\ker L_-$ is not empty, so there would be highest weight states at values of $i$ inconsistent with $j$. In other words, the dimensions of the above sequence of spaces $V_i$ can at most jump at $i \in \{-h + 1, h\}$. As a corollary, taking into account the fact that the spectrum of $L_0$ is bounded from below, the above sequence terminates at either $i = -h + 1$ or $i = h$.

Furthermore, $L^2$ consists of a single Jordan block on each of the subspaces $V_i$. By assumption, this is true for the first element $V_m$ of the sequence. In general, assume that $L^2$ consists of a single Jordan block on $V_i$ and consider $V_{i-1}$. Since from (2) we have that $L_-V_{i-1} = L_-L_+V_i = (-L^2 + L_0(L_0 - 1)V_i$, we see that $L_-V_{i-1} \subseteq V_i$. Now if $L^2$ were to consist of more than one Jordan block on $V_{i-1}$, each of these blocks would contain an eigenvector of $L^2$. Since $L_-$ commutes with $L^2$, all these eigenvectors will be taken by $L_-$ to eigenvectors in $V_i$, of which there is only one by assumption. Therefore $L_-$ is not one to one, its kernel on $V_{i-1}$ is nontrivial, and there is a highest weight state in $V_{i-1}$, which is inconsistent with $j$ unless $i \in \{-h + 1, h\}$. This proves the assertion when $i \not\in \{-h + 1, h\}$.

Now consider the case $i = h$. The case $i = -h + 1$ is similar. Suppose again that $L^2$ consisted of more than one Jordan block, and therefore more than one eigenvector, on $V_{h-1}$. This would imply, by the above argument, that the operator $L_-$ had nontrivial kernel on $V_{h-1}$. Choose $\tilde{v}_{h-1} \in V_h$ such that $L_-\tilde{v}_{h-1} = 0$. Since $V_{h-1} = L_+V_h$, there is a $v_h \in V_h$ such that
\[ \tilde{v}_{h-1} = L_+ \tilde{v}_h. \] Then \( 0 = L_- L_+ \tilde{v}_h = (-L^2 + L_0(L_0 - 1))\tilde{v}_h, \) which implies that \( \tilde{v}_h \) is an eigenvector of \( L^2 \) and therefore is proportional to the unique eigenvector \( v_h \) in \( V_h \). Now note that \( L_- v_{h-1} \) cannot be zero for more than one eigenvector in \( V_{h-1} \), because if that were the case, then by the above argument there would be more than one linearly independent eigenvector of \( L^2 \) in \( V_h \). Therefore we can find a \( v_{h-1} \) such that \( L_- v_{h-1} = v_h \). Then

\[ L_+ L_- v_{h-1} = (-L^2 + L_0(L_0 + 1))v_{h-1} = [-h(h-1) + (h-1)h]v_{h-1} = 0, \]

or \( L_+ v_h = 0 \). But we had \( 0 \neq \tilde{v}_{h-1} = L_+ \tilde{v}_h \), and \( \tilde{v}_h \propto v_h \), which is a contradiction. This proves the assertion when \( i = h \).

We have proved that \( L^2 \) consists of a single Jordan block on each \( V_i \). This means that each \( V_i \) contains one and only one eigenvector. Consequently, since elements in the kernel of \( L_+ \) are automatically eigenvectors, the dimension of the \( V_i \) can be reduced by at most one at each of the two transition points \( V_h \xrightarrow{L_+} V_{h-1} \) and \( V_{-h+1} \xrightarrow{L_+} V_{-h} \). Since the sequence terminates at either \( V_h \) or \( V_{-h+1} \), the initial space \( V_m \) can be at most two-dimensional. This completes the proof that the Jordan blocks of \( L^2 \) are at most two-dimensional.

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