Finite-length Lyapunov exponents and conductance for quasi-1D disordered solids

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Abstract

The transfer matrix method is applied to finite quasi-1D disordered samples attached to perfect leads. The model is described by structured band matrices with random and regular entries. We investigate numerically the level spacing distribution for finite-length Lyapunov exponents as well as the conductance and its fluctuations for different channel numbers and sample sizes. A comparison is made with theoretical predictions and with numerical results recently obtained with the scattering matrix approach. The role of the coupling and finite size effects is also discussed.
I. INTRODUCTION

Since the discovery of non-self-averaging conductance fluctuations in mesoscopic conductors \[1\]–\[3\], it has become necessary to reconsider various fundamental aspects of quantum electronic transport. In particular, the one-parameter scaling hypothesis \[4\] has been challenged \[5\] on the ground that the conductance \(G\) is not a self-averaging quantity, and thus it is necessary to consider its entire distribution rather than just the mean value. One of the approaches that provide the possibility of obtaining the conductance distribution is based on the theory of random matrices (for a recent review see, for example, \[6\]). This theory exploits the connection between the conductance and finite-length Lyapunov exponents \[7\] of the transmission matrices (see also \[8\] and references therein). Thus, in this approach, the main interest is in the properties of the Lyapunov spectra (all Lyapunov exponents ordered in an increasing/decreasing way).

In this paper we study quasi-1D disordered samples of finite size described by Band Random Matrices (BRM). Such matrices have been extensively investigated in applications to thin wires (see the review \[9\] and references therein) or, equivalently, to 1D Anderson-type systems with long-range random hopping. Scaling properties of the localization length of eigenfunctions are already well understood since some years \[10,9\]. More recently, the scaling properties of averaged Lyapunov exponents have been also established both for infinite \[11\] and finite \[12\] BRMs. In this paper, we are mainly interested in statistical properties of the Lyapunov exponents and thereby of the conductance as a function of the sample size and the number of transmission channels.

One of the results of this paper concerns the comparison with the predictions of the Random Matrix Theory (RMT). It is known that, in contrast to other characteristics like the spectral rigidity, the shape of the level spacing distribution \(P(s)\) is quite insensitive to deviations from the Wigner-Dyson surmise, i.e. the prediction of the RMT. Nevertheless, in spite of an overall good correspondence between the \(P(s)\) numerically determined and the theoretical expression, here we show that the normalized second moment of the distribution \(P(s)\) reveals clear deviation from the theoretical predictions when applying the RMT to the Lyapunov Spectra. Accordingly, this deviation can serve as a sensitive measure of the correspondence of the data to the RMT results. We attribute such a discrepancy to the existence of a non-random repulsion acting on the low Lyapunov exponents from the left side, due to the symmetry of the Lyapunov spectrum. Though the observed deviation is relatively small, it manifests itself in the value of the variance for the conductance fluctuations which we have studied numerically by making use of the Landauer formula expressed in terms of the Lyapunov spectra.

The outline of the paper is as follows. In the next Sec. II we describe the physical setup which permits us to study the properties of conductance of finite samples attached to the leads. First, we introduce the model in terms of Hamiltonian band matrices and derive the corresponding transfer matrices. Using these matrices, one can compute the conductance of quasi-1D finite samples through the Landauer formula. This is done by introducing a specific matrix, the eigenvalues of which represent the key ingredient of the conductance expression. At the beginning of Sec. III, we give a very short outlook of theoretical results about the connection between fluctuations of the conductance and of finite-length Lyapunov exponents. According to these findings, the statistical properties of Lyapunov exponents may
be approximately described by the RMT. In particular, the “level” spacing distribution is expected to follow the famous Wigner-Dyson expression. In this Section we compare these predictions with our numerical data obtained for the whole Lyapunov spectrum, finding clear deviations not only in the upper but also in the lower part of the spectrum, where a good correspondence with RMT was expected instead. A study of both the conductance and its fluctuations in different regimes, ranging from the very localized to the metallic one, is performed in Sec. IV. In particular, we have determined the dependence of the average conductance on the related scaling parameter when passing from the localized to metallic regime. Another quantity of interest was the universal conductance fluctuations for which we have checked the theoretical predictions. Detailed discussion of our results is presented in the last Section.

II. THE PHYSICAL SETUP

The simplest general model in the class of quasi-1D or 1D systems with long range hopping is represented by the Schrödinger equation with interactions described by Band Random Matrices. This ensemble is defined as a set of real symmetric matrices the entries of which are independent Gaussian variables with zero average and variance $\sigma_0^2 = 1 + \delta_{n,m}$ ($\delta_{n,m}$ is the Kronecker symbol) if $|n-m| \leq b$, and zero otherwise (i.e., there are $2b+1$ nonzero elements in any row). The parameter $b$ defines the hopping range between neighboring sites; in the quasi-1D interpretation, it is the number of transverse channels along a thin wire. In the study of the electronic conduction in disordered wires at zero temperature, one uses mainly one of the two theoretical approaches based on ideas of Thouless and Landauer respectively. In the Thouless approach, the wire is regarded as a closed system, and its conductance is defined through the sensitivity of eigenvalues to changes of the boundary conditions \[13–15\]. In the Landauer approach, one has to embed a disordered sample into a perfectly ordered lattice, i.e. to add perfect leads both to the left and to the right of the disordered sample. Thus, in contrast with the former approach, the latter one deals with an open system (for a comparison of the Thouless and Landauer conductance see, e.g., \[16,17\]).

Our model is described by the time-dependent Schrödinger equation,

$$i \frac{dc_n(t)}{dt} = \sum_{m=n-b}^{n+b} H_{n,m}c_m,$$

where $c_n(t)$ is the probability amplitude for an electron to be at site $n$ and $H_{n,m}$ is a symmetric banded matrix with random entries for the disordered part (see below). The eigenvalues can be obtained by substituting the relation $c_n(t) = \exp(-iEt)\psi_n$ in Eq. (\[1\]) and solving the resulting equation for $\psi_n$,

$$\psi_{n+b} = \frac{1}{H_{n,n+b}} \left( E\psi_n - \sum_{m=n-b}^{n+b-1} H_{n,m}\psi_m \right).$$

By defining $x_n(i) \equiv \psi_{n+b-i}$, the above equation can be recasted in the form of a $2b$-dimensional linear map $T_n$. 


\[ x_{n+1}(1) = \frac{1}{H_{n,n+b}} \left( E x_n(b) - \sum_{j=1}^{2b} H_{n,n+b-j} x_n(j) \right) \]
\[ x_{n+1}(j) = x_n(j+1) \quad 1 < j \leq 2b \]  

(3)

which provides an alternative interpretation of the stationary Schrödinger equation. In this picture, an eigenstate of Eq. (1) can be treated as a “trajectory” evolving under the action of map (3), and its localization properties are determined by the Lyapunov exponents. In the previous paper \[1\], we have investigated the shape of the Lyapunov spectrum in the limit of infinitely extended disordered samples. Here, aiming at a more complete understanding of conductance properties, we choose to rely on the multichannel Landauer formula \[7\] for the conductance of finite samples. The physical set-up requires considering a disordered sample of length \( L \) in between two perfect leads. At variance with the standard Anderson model, where only nearest-neighbor couplings are included, here the long-range hopping terms allow some freedom in the structure of the ordered leads and, especially, in the connection of the leads with the disordered sample. As for the leads, a natural way is to assume a band structure in the ordered part (leads) with \( H_{n,m} = V, \) for \( |n|, |m| > L \) (for the sake of simplicity we put \( V = 1 \)). As for coupling matrix elements, we choose them in the same way as in the bulk, \( \text{random Gaussian entries with zero mean and variance } \sigma^2_0 = 1 \).

There is an immediate analogy with the Anderson problem on a stripe of width \( b \), where one deals with \( 2b \times 2b \) matrices, too. However, in our case the one-step matrix \( T_n \) defining map (3) is not symplectic and the determinant is not equal to 1, but to \((-1)^{2b+1}(-H_{b+1,1}/H_{b+1,2b+1})\). Yet, the total transfer matrix \( T_N = \prod_{n=1}^{N} T_n \), where \( N = L + 2b \), satisfies the following relation

\[ (T_N)^\dagger \Sigma T_N = \Sigma \]  

(4)

where \( T_N^\dagger \) denotes the Hermitian conjugate of \( T_N \) and the matrix \( \Sigma \) is defined as

\[ \Sigma = \begin{pmatrix} 0 & S \\ -S^t & 0 \end{pmatrix}. \]  

(5)

Here \( S \) is a lower triangular matrix such that \( S_{ij} = 1 \), for any \( 1 \leq j \leq i \leq b \) and \( S^t \) denotes the transposed operator. In fact, Eq. (4) means that \( T_N \) has a “generalized symplectic” structure \[13\].

The scattering properties of the sample can be better investigated by choosing an appropriate base, namely by decomposing the eigenfunctions in the plane waves

\[ \psi_n = \frac{e^{i \lambda p}}{\sqrt{2\pi}}, \]  

(6)

supported by the ordered lattice. The corresponding eigenvalues are

\[ E = 1 + 2 \cos p + ... + 2 \cos(bp) = \frac{\sin(\frac{2b+1}{2}p)}{\sin(\frac{b}{2})}, \]  

(7)

while the velocity of a wave packet centered around \( p \) is
\[
v(p) = \frac{dE}{dp} = \frac{2b + 1}{2} \cos\left(\frac{2b+1}{2}p\right) \sin\left(\frac{2b+1}{2}p\right) = 2b + 1 \cdot \frac{1}{2} \cos\left(2b + 1 + 2p\right) \sin\left(p \frac{1}{2}\right) - \sin\left(2b + 1 + 2p\right) \sin\left(p \frac{1}{2}\right) \cos\left(p \frac{1}{2}\right).
\] (8)

For simplicity, in what follows, we study the case \( E = 0 \) for which all transmission channels are open. Therefore, there are \( b \) pairs of open channels with opposite velocities, \( p_k = (-1)^k \frac{2\pi k}{2b + 1} \), \( 1 \leq |k| \leq b \). (9)

The matrix \( M \) connecting the amplitudes to the left with those to the right of the scatterer is
\[
\begin{pmatrix}
A^R \\
B^R
\end{pmatrix} = S^{-1} N^R S_0 \begin{pmatrix}
A^L \\
B^L
\end{pmatrix} = M \begin{pmatrix}
A^L \\
B^L
\end{pmatrix}
\] (10)

where the matrix \( S \) is the transformation from the real space to the momentum space, i.e. it provides the expansion in momentum components.

In order to determine the conductance, one needs to express the flux arising along the lattice. This requires a further similarity transformation
\[
F = \Gamma M \Gamma^{-1},
\] (11)

where the diagonal \( 2b \times 2b \) matrix \( \Gamma \) is defined as \( \Gamma_{i,j} = \delta_{i,j} \sqrt{v_i} \) for \( i \leq b \) and \( \Gamma_{i,j} = \delta_{i,j} \sqrt{v_{i-b}} \) for \( i > b \). The above transformation is equivalent to the normalization of the scattering matrix and it takes into account the fact that each open channel propagates with a different velocity. From Eq. (10) it is easily seen that the transformation (11) corresponds to the change of variables \( a_{i}^{L,R} = \sqrt{v_i} A_{i}^{L,R} \) and \( b_{i}^{L,R} = \sqrt{v_i} B_{i}^{L,R} \). The square of the modulus of the new variables gives the flux of particles which enter and exit the disordered sample.

It is easy now to verify that \( F \) conserves fluxes,
\[
F^\dagger \sigma_3 F = \sigma_3
\] (12)

where \( \sigma_3 \) is a generalized Pauli \( \sigma_z \) matrix, i.e.
\[
\sigma_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\] (13)

and \( 1 \) denotes a \( b \times b \) identity matrix. Condition (12) is crucial, since it corresponds to unitarity of scattering matrix.

By using the symplecticity of \( F \) and \( F^\dagger F \), and the relations between \( F \) and the scattering matrix, we find that the dimensionless conductance (measured in \( e^2/h \) units) is readily written as
\[
G = \text{Tr} \left( \frac{2}{F^\dagger F + (F^\dagger F)^{-1} + 2} \right) = \frac{b}{\sum_{i=1}^{b} \frac{1}{1 + \cosh(2N \gamma_i(b, N))}}
\] (14)

where the \( \gamma_i(b, N) \)'s are the Lyapunov exponents of the matrix \( F^\dagger F \).

The lengths \( \gamma_i^{-1}(b, N) \) depend both on \( N \) and on the realization of the disorder, but the ergodic multiplicative theorem (13) ensures that, in the limit \( N \to \infty \), \( \gamma_i^{-1}(b, N) \) converges towards the “localization length” \( l_i^\infty \).
\[
\lim_{N \to \infty} \gamma_i^{-1}(b, N) = l_i^\infty,
\]
(15)
i.e. \(l_i^\infty\) is a self-averaging quantity which depends on the energy \(E\) and on the bandwidth \(b\) only. Moreover, still in the limit \(N \to \infty\), the contribution of the similarity transformations introduced to pass from the transfer matrix \(T_N\) to \(F\) becomes negligible, implying that \(\gamma_i(b, N)\) are equal to the logarithms of the eigenvalues of the matrix
\[
O = \lim_{L \to \infty} \left[ (T_N)^\dagger (T_N) \right]^{1/2(L+2b)},
\]
(16)
i.e. to the Lyapunov exponents of the product of the bare transfer matrices.

In the following we shall denote with \(\gamma_1(b, N) > \gamma_2(b, N) > \ldots > \gamma_i(b, N) > \ldots > \gamma_{2b}(b, N)\) the effective (“finite-length”) Lyapunov exponents [20] computed over a number \(L\) of iterations at energy \(E = 0\). Because of the symplectic structure of \(F\), the Lyapunov exponents \(\gamma_i(b, N)\) come in pairs with opposite values: for this reason, it is sufficient to compute only the positive exponents.

### III. LYAPUNOV EXPONENTS

The standard Landauer approach to quantum transport in a two-probe geometry allows us to express the conductance \(G\) in terms of \(b\) real positive Lyapunov exponents \(\gamma_i\), see Eq. (14). Thus, the knowledge of the joint probability distribution \(P(\{\gamma_i\})\) provides a complete statistical description of the conductance. Accordingly, we start this section by discussing the fluctuations of the Lyapunov spectra.

In the metallic regime, an expression for \(P(\{\gamma_i\})\) has been proposed on the basis of Random Matrix Theory [21–24]. Namely, it was conjectured that the matrix
\[
X = \frac{F^\dagger F + (F^\dagger F)^{-1} - 2}{4}
\]
(17)
has the typical structure of full random matrices. This conjecture is based on the observation that \(F\) results from the product of many independent random matrices so that its entries are random numbers all of the same order. According to the RMT, the joint probability distribution of the Lyapunov exponents \(\gamma_i\) is of the type
\[
P(\gamma_1, \gamma_2, \ldots, \gamma_b) = C_\beta \exp[-\beta H_N(\gamma_1, \gamma_2, \ldots, \gamma_b)]
\]
(18)
where \(C_\beta\) is a normalization constant, \(\beta = 1, 2\) or 4 is a symmetry parameter and \(H_N\) is the effective Hamiltonian
\[
H_N(\gamma_1, \gamma_2, \ldots, \gamma_b) = -\sum_{i<j}^b u(\gamma_i, \gamma_j) + \sum_{i=1}^b V(\gamma_i).
\]
(19)
Eq. (18) has the form of a Gibbs distribution at temperature \(T = 1/\beta\) for a fictitious system of classical particles on the semi straight-line \((0, \infty)\) under the external potential \(V(\gamma_i)\). As for the two-particle interaction, in conventional RMT it has the form
\[
u(\gamma_i, \gamma_j) = -\ln |\cosh(2N\gamma_i) - \cosh(2N\gamma_j)|.
\]
(20)
which corresponds to the logarithmic repulsion between the eigenvalues $\lambda_i = \cosh(2N\gamma_i)$. All microscopic parameters are contained in the function $V(\gamma)$, while the interaction potential $u$ is independent of them and has a geometrical origin (see the discussion in [6]). The first impression was that, if $V(\gamma)$ is suitably chosen, the distribution (18,19) with (20) provides an accurate description of transport properties in the metallic regime. However, recent developments [24] have shown that the expression (20) does not give the right value for the universal conductance fluctuations in the quasi-1D limit. The correct result is obtained if the two-body interaction potential (20) is replaced by (21) (see details in [6]),

$$u(\gamma_i, \gamma_j) = -\frac{1}{2} \ln |\cosh(2N\gamma_i) - \cosh(2N\gamma_j)| - \frac{1}{2} \ln |\gamma_i^2 - \gamma_j^2| - \ln(2N) .$$  

Although the above expression has been derived for the case $\beta = 2$ only, i.e. for broken time-reversal symmetry, there are indications that it is also valid for $\beta = 1, 4$.

Less is known about the properties of Lyapunov exponents in the localized regime and in the metal-insulator transition (see [8] for a brief review).

Below, we study numerically fluctuation properties of the Lyapunov exponents $\gamma_1$. Standard techniques for the direct computation of eigenvalues are affected by large inaccuracies due to the high probability of small denominators in Eq. (3). In order to avoid this problem, we have applied the algorithm originally developed in [26] for the infinite $N$–limit. The method consists in considering a formally infinite sequence of matrices all equal to $F F^\dagger$, which can be seen as “replicas” of the same disordered sample. The Lyapunov exponents of the corresponding product of matrices are then computed in a standard way by recursively applying the single matrices to $b$ independent vectors which are continuously orthonormalized (see also [27,12]). The advantage of this approach is that one can keep the accuracy under control by renormalizing the vectors also in the intermediate steps that correspond to the application of the various matrices composing $F$.

In practice the number of “replicas” has been fixed by imposing that the accuracy on the Lyapunov exponents is better than $10^{-4}$ for each disorder realization. Moreover, ensemble averages have been performed over sets of more than 2500 realizations.

In the metallic regime, the RMT predicts that level spacings for eigenvalues $\lambda_i$, and, therefore, the normalized differences

$$\delta_i = \frac{\gamma_{i+1} - \gamma_i}{\gamma_{i+1} - \gamma_i} \quad ; \quad 1 \leq i \leq b ; \quad \gamma_{b+1} = 0$$

of the Lyapunov exponents are distributed according to the Wigner-Dyson (WD) surmise [22,23]. This expectation has been numerically confirmed in the two- [22] and three-dimensional [27] Anderson model.

In order to characterize the repulsion between any two consecutive Lyapunov exponents, it is useful to introduce the normalized width $y$ of the “level” spacing distribution [27],

$$y(\delta_i) \equiv \frac{\sigma(\delta_i)}{\langle \delta_i \rangle}$$

where $\sigma$ denotes the corresponding standard deviation. For the WD-distribution, the parameter $y$ is equal to $y_{WD} = \sqrt{4/\pi - 1} \approx 0.523$. Values of $y < y_{WD}$ are interpreted as the signature of a stronger rigidity in the Lyapunov spectra. In particular, the limit $y \to 0$
corresponds to a $\delta$-distribution, i.e. to a “crystallization” of the spectrum [6]. For small values of $y$, the distribution of the spacings $\delta_i$ around their mean values is quite close to Gaussian; this situation occurs in the localized regime when the repulsion is very strong (see below). Notice that in the complete absence of repulsion (which corresponds to a Poisson distribution), $y$ is equal to 1 which thus represents an upper bound.

It is known that the degree of localization in finite samples can be effectively described by means of the scaling parameter

$$\Lambda = b^2 / N$$

which is connected to the $b^2$ proportionality of the localization length in infinite samples (see, e.g., [9, 10]). Numerical results for three different values of the scaling parameter $\Lambda = 10.29, 6.95, 6.27$ in the metallic regime are presented in Fig. 1 by open circles, diamonds and stars, respectively. The repulsion parameter $y$ is reported versus the quantity $\chi = i/b$, the standard parameter used in the representation of the whole Lyapunov spectra (see also [11,12]). As one can see, the value of $y$ is almost constant for $\chi$ larger than 0.2. Although, in average, it is very close to the theoretical prediction $y_{BW}$, there is a systematic difference, namely, $y > y_{BW}$. This is a clear indication of a disagreement with respect to RMT expectations. The only exception is a narrow region around $\chi = 1$ where $y$ is slightly smaller and thus more in agreement with the RMT.

Larger Lyapunov exponents, in the region $\chi \leq 0.2$, are characterized by a much larger deviation from the WD-value. In particular, for $\chi \to 0$, the value of $y$ decreases, revealing a much stronger repulsion than predicted by the Random Matrix Theory. This result is in contrast to previous studies for the 2D Anderson model [27,22] where it was found that the distribution for $\delta_i$ follows the WD-form for all Lyapunov exponents.

Now, we analyze the statistical properties of the lowest Lyapunov exponent $\gamma_b(b,N)$ for different values of the scaling parameter $\Lambda$. Numerical data shows that the distribution of $\gamma_b$ changes dramatically when $\Lambda$ decreases. For example, for $\Lambda \approx 0.6$, the distribution $P(\delta_b)$ turns out to be very different from the WD-form, and for $\Lambda \approx 0.2$ it looks like the Gaussian distribution around the mean value of $\delta_b$. In order to describe this transition quantitatively, we have fitted numerical data to the expression

$$P(\delta) = A\delta^\beta (1 + B\beta\delta) f(\beta) \exp\left( -\frac{\pi^2}{16} \beta^2 \delta^2 - \frac{\pi}{2} (1 - \beta/2)\delta \right)$$

where $f(\beta)$ is the fitting function, $f(\beta) = 2\beta(1 - \beta/2)/\beta - 0.16874$. The constants $A$ and $B$ are defined through the normalization conditions

$$\int P(s) \, ds = 1; \quad \int s \, P(s) \, ds = 1$$

while the free parameter $\beta$ is the effective repulsion taking values in $[0, \infty)$. The above phenomenological expression has been suggested in Ref. [28] (see [29] for details) in order to describe the level spacing distribution for quasi-1D disordered samples of finite size, in the dependence on the degree of localization. A similar distribution has been also used in the description of the spacing between real particles (on a ring) interacting with each other via 2D Coulomb forces. In that case, $\beta$ plays the role of the inverse temperature in the
thermodynamic equilibrium [30, 31]. For $\beta = 0$, the dependence $P(\delta)$ reduces to the Poisson distribution, while for $\beta = \infty$ it converges to a $\delta$-function centered at $\delta = 1$. Moreover, for $\beta = 1, 2, 4$ the expression (25) is very close to that one given by the RMT (it even turns out to be more accurate than the WD-surmise, see [29]).

The “inverse temperature” $\beta$ is a very appropriate indicator to characterize the transition from a totally uncorrelated Lyapunov spectrum ($\beta = 0$) to a “perfect crystal” ($\beta = \infty$). In fact, it can be effectively determined by fitting the data with expression (25). In order to emphasize the validity of our results, let us notice that the quality of the fit has been quantitatively checked by performing a $\chi^2$ test, that has been always brilliantly passed. In the metallic regime (e.g., for $\Lambda = 10.24$), the best fit of the numerical data for $\gamma_b$ gives $\beta = 1$ with a high accuracy, indicating that the minimum Lyapunov exponent is, indeed, characterized by the WD-distribution [23]. When moving towards the localized regime, where $\Lambda \ll 1$, the value of the repulsion parameter increases; for $\Lambda \approx 0.2$, we have $\beta \approx 3.95$. One can study the transition from one to the other regime by computing the repulsion parameter $\beta$ for different values of $b$ and $N$. The good data collapse observed in Fig. 2 strongly indicate that $\beta$ (for the minimum Lyapunov exponent) is a function of the scaling parameter $\Lambda = b^2/N$ only, a result which is in the spirit of previous studies [10, 9]. Moreover, one can observe that already for $\Lambda \geq 4$, the repulsion parameter reaches the lower limit $\beta = 1$ (we remind that, being the condition $b \ll N$ fulfilled, we are, for $\Lambda \gg 1$, in the metallic regime). For $\Lambda < 1$, the repulsion parameter starts increasing very fast and in the limit $\Lambda \to 0$ it diverges. In other words, in the extremely localized regime, the probability distribution for the lowest Lyapunov exponent is a $\delta$-like function, which can be approximated by a Gaussian. A similar effect has been observed in previous studies of 2D Anderson-type models [27, 32].

Finally, for what concerns the properties of the other Lyapunov exponents in the localized regime, we can look at full circles, diamonds and triangles in Fig. 1. First, we can notice that for all $i$’s, including $i = b$, the quantity $y(\delta_i)$ decreases as we move towards the localized regime, leading to a Gaussian-like distribution. This is in agreement with the previous results for the minimum Lyapunov exponent $\gamma_b$ for which the same conclusion was reached by measuring directly the repulsion parameter $\beta$. Moreover, in comparison with the metallic regime, the convergence to the $\delta$-like distribution for the maximal part of the spectra becomes smoother. One should note that in the limit $N \to \infty$, the distribution $P(\delta_i)$ for $i < b$ was instead found to be Poissonian in the localized regime of the 2D Anderson model [27, 32]. This means that the quasi-1D limit which we consider here, is somewhat special. Namely, the eigenvalues of the transfer matrix $F^\dagger F$ are well separated, their number $b$ being fixed while their mean separation increases linearly with $N$. On the other hand, if we consider a two-dimensional square of size $N \times N$, then $b$ will also diverge with $N$ and the assumption of well separated eigenvalues is no longer valid.

IV. CONDUCTANCE

The present understanding of Anderson localization theory is mainly based on the one-parameter scaling. However, recent experimental results indicate that in order to understand the transport properties of disordered systems, one should study not only the mean expectation values but also the whole probability distribution.
One of the quantities that attracted a lot of research interests during last two decades, is the conductance. Theoretical and experimental results have shown (see references in [6]) that the variance of the conductance is $\mathcal{O}(1)$ (in $e^2/h$ units), independently of the average conductance, if the conductor is phase-coherent. A number of theoretical methods has been developed to compute the (universal) conductance fluctuations arising in the above mentioned (metallic) regime. Many numerical calculations have also been performed although, until recently, mostly in connection with the Thouless approach which assumes that the system is closed (on this subject, see [17]). On the other hand, very little efforts have been made to study open systems, taking into account the specific form of the coupling with the leads [16,18], as in our case. Yet, it is known that the conductance of a mesoscopic sample depends sensitively on the measurement geometry and on the way the leads are attached to the sample [33]. In this section we study the properties of the conductance $G$ in the quasi-1D model [1] and compare them with theoretical expectations and numerical results obtained in a slightly different model [16].

Our first interest is in the geometric average of the conductance $\langle \ln G \rangle$ in the metallic regime $1 \ll b \ll N$ (here, the brackets $\langle ... \rangle$ denote an average over different realizations of the random potential). As expected, the average conductance $\tilde{G} \equiv \exp (\langle \ln G \rangle)$ turns out to depend on $b$ and $N$ only through the scaling parameter $\Lambda = b^2/N$ (see Fig. 3, where we have reported the outcome of several simulations performed for different values of $N$ and $b$). The existence of a specific function $\tilde{G}(\Lambda)$ is equivalent to the standard assumption made in the scaling theory [4,34], i.e. to assuming that the logarithmic derivative of the conductance,

$$
\eta \equiv \frac{d \ln \tilde{G}}{d \ln N}
$$

is a function only of $\tilde{G}$. The representation of the numerical results in terms of $\eta$ and $\tilde{G}$ is given in Fig. 4.

From the inset of Fig. 3, it can be seen that in the metallic regime, i.e. for $\Lambda \gg 1$, $\tilde{G} \simeq a_1 \Lambda + a_0$, that is the conductance has the expected “ohmic” dependence on the sample length ($\Lambda \sim 1/N$). In the $\eta$ representation, this is tantamount to saying that $\eta \simeq -1 + a_0/\tilde{G}$ with $a_0 < 0$. In the localized regime, $\Lambda \ll 1$, Fig. 3 reveals an exponential decrease of the conductance with the chain length, $\tilde{G} \simeq b_0 \exp(-b_1/\Lambda)$, a behavior which corresponds to $\eta = \ln(\tilde{G}/b_0)$.

Let us now discuss some approximate theoretical expressions of $G$ and compare them with our numerical results. First of all, we want to derive a sufficiently accurate expression for the effective Lyapunov exponents in order to determine the conductance from the Landauer formula in the limit of large $b$ and $N$ but finite $\Lambda$. The extensive studies performed in Ref. [12] have suggested that

$$
\gamma_i(b,N) \approx \gamma_i(b,\infty) - \frac{\Lambda}{b^{1.7+i/b}},
$$

is an expression convincingly tested in the bulk of the spectrum. Moreover, it was shown that

$$
\gamma_i(b,\infty) \approx \frac{\Omega(\chi)}{b} + \frac{\omega(\chi)}{b^2}
$$

(29)
where \( \Omega(\chi) \) represents the asymptotic Lyapunov spectrum, while \( \omega(\chi) \) is the leading correction term. By substituting Eq. (29) in Eq. (28), one obtains
\[
\gamma_i(b, N) \approx \frac{\Omega(\chi)}{b} + \frac{\omega(\chi)}{b^2} - \frac{\Lambda}{b^{1.7 + \chi}}.
\]  
(30)
As the leading contribution to the conductance is given by the “small” Lyapunov exponents (i.e. \( \chi \approx 1 \)), it is convenient to expand \( \Omega(\chi) \) around \( \chi = 1 \). By retaining the leading terms (recall that \( \Omega(1) = 0 \)) one obtains
\[
\gamma_i(b, N) \approx \Omega_1 \frac{(b - i)}{b^2} + \Omega_2 \frac{(b - i)^2}{b^3} + \frac{\omega(1) - \Omega_1}{\Lambda} - \frac{\Lambda}{b^{0.7}}.
\]  
(31)
where \( \Omega_1 \) is the absolute value of the first of derivative of \( \Omega(\chi) \), while \( \Omega_2 \) is the second derivative.

From the Landauer expression (14), one can see that the argument of the hyperbolic cosines is
\[
\alpha_j \equiv 2\gamma_j(b, N)N \approx \frac{2\Omega_1}{\Lambda}j + \frac{2\Omega_2}{b\Lambda}(j - 1)^2 + \frac{2\omega(1) - \Omega_1}{\Lambda} - \frac{2\Lambda}{b^{0.7}}.
\]  
(32)
where \( j = b - i + 1 \) Accordingly, one can see that in the limit \( b, N \to \infty \) but finite \( \Lambda \), the second and the last term in the r.h.s. vanish and thus we neglect them.

Therefore, we are left with a linear expression in \( j \), with two coefficients to be determined, namely \( \Omega_1 \) and \( \omega(1) \). From the analytic knowledge of the minimum Lyapunov exponent \( (j = 1) \), we have that \( \omega(1) = 3/2 \). Moreover, previous simulations for the Lyapunov spectra have clearly indicated that the slope is \( \Omega_1 = 1.5 \). As a consequence, we are left with
\[
\alpha_j = 3j/\Lambda,
\]  
(33)
which, upon substitution into Eq. (14) implies
\[
G = \sum_{j=1}^{\infty} \frac{2}{1 + \cosh(3j/\Lambda)},
\]  
(34)
where, in view of the limit \( b \to \infty \), the sum is extended to infinity.

In the metallic regime, the conductance expression can be written in a compact form by transforming the sum into an integral,
\[
G = \frac{2\Lambda}{3} \int_{3/\Lambda}^{\infty} \frac{dy}{1 + \cosh y},
\]  
(35)
where we have introduced the variable \( y = 3j/\Lambda \). The solution of the integral finally yields,
\[
G = \frac{4\Lambda}{3\exp(3/\Lambda) + 1}.
\]  
(36)
In the limit \( \Lambda \gg 1 \) one gets
\[
G = a_1 \Lambda + a_0 = \frac{b_0}{N} - 1
\]  
(37)
with \( a_1 = 2/3 \) and \( a_0 = -1 \). The first term in (37) coincides with the theoretical result according to which the leading term for the conductance in the metallic regime is the ratio of the localization length \( l_b^\infty \) to the size \( N \) of the sample. One should stress that the finite size corrections are neglected in this term (see, e.g., [4]). A more important correction is given by the term \( a_0 \) and is related to weak localization effects. The meaning of this quantum correction is that the back scattering is larger in comparison to the classical result and, therefore, the conductance is smaller than the contribution given by the first term. In the theory (see the review [6] and references therein), the value of \( a_0 \) is known to be \( a_0 = -1/3 \) [36]. It is, however, interesting to note that a simple estimate of this correction [37] gives the value \( a_0 = -1 \) as in our expression (37).

In the metallic regime, the best fit of our numerical data with the expression (37) gives the values \( a_1 \approx 0.74 \) and \( a_0 \approx -0.41 \). While the value of \( a_1 \) is close to the theoretical prediction, the correction term \( a_0 \) is somehow different. A discussion about the correspondence between numerics and theory is given below.

We now go to consider the opposite limit of strong localization, \( \Lambda \ll 1 \). In this case, the sum (34) is dominated by the first contribution,

\[
G = b_0 \exp \left( -\frac{b_1}{\Lambda} \right) \tag{38}
\]

with \( b_0 = 2 \) and \( b_1 = 3 \). Let us first discuss the leading dependence given by the term \( b_1 \). One should first stress that the coefficient \( b_1 \) is different for different definitions of the average. Specifically, if one treat \( G \) as the average conductance \( \langle G \rangle \), the value of \( b_1 \) was found to be \( b_1 = 3/4 \). For the geometric average \( \tilde{G} \), instead, (also known as the “typical conductance”), the theoretical prediction is \( b_1 = 3 \) (see [38,6]). Therefore, the expression (38), which refers to the latter case, gives the correct exponential dependence.

Less is known about the correction to the exponential dependence which in (38) is absorbed in the term \( b_0 \). To the best of our knowledge, this correction is unknown for \( \tilde{G} \), the quantity of our numerical interest. At the same time, for the average conductance \( \langle G \rangle \) the correction has the form \( b_0 \sim \Lambda^{3/2} \) [38]. One can suggest, however, that for the geometric conductance \( \tilde{G} \) the correction has the form \( b_0 \sim \Lambda^\nu \) with some value \( \nu \geq 1 \). Our data do not allow to estimate the correction to the exponential dependence. The best fit to the expression (38) with \( b_1 \) as a constant, gives the following values, \( b_1 \approx 2.9 \) and \( b_0 \approx 5.58 \). One can see that the value of \( b_1 \) is quite close to the theoretical one, \( b_1 = 3 \).

A comparison of the theoretical expression (34) with the data in a broad interval of the localization parameter \( \Lambda \) is presented in Fig. 3. Considering that no free parameters are present in the theoretical formula, the overall agreement represents a nice confirmation of the various approximations behind Eq. (34). Nonetheless a small but clear deviation can be seen in the intermediate region \( \Lambda \sim 1 \). It is, therefore, natural and important to understand whether the disagreement is to be ascribed to the misjudgement of some correction term, or to finiteness of \( b \) and \( N \) in the simulations. Moreover, it is important to recall that conductance fluctuations are not taken into account in the derivation of Eq. (34), since it is expressed in terms of the mean Lyapunov exponents. This analysis goes beyond the scope of the present paper.

One can also describe our data for the conductance \( \tilde{G} \) by making use the phenomenological expression suggested in [17].
$$\tilde{G} \equiv \exp \langle \ln G \rangle = \frac{c_1 + c_2 \exp(-c_4/\Lambda)}{\exp(c_3/\Lambda) - 1}. \quad (39)$$

with four parameters $c_i$. By fitting directly all the parameters, one finds an overall excellent correspondence with our data, but a careful inspection of the behavior in the metallic and localized regimes is less convincing. On the other hand, one can determine the parameters $c_i$, in such limit cases by exploiting the correspondence between $c_i$ and $a_0$, $a_1$, $b_0$, $b_1$, see Eqs. (37) and (38),

$$a_1 = \frac{c_1 + c_2}{c_3}; \quad a_0 = -\frac{c_2 c_4}{c_3} - \frac{c_1 + c_2}{2}; \quad b_1 = c_3; \quad b_0 = c_1 \quad (40)$$

Unexpectedly, the resulting dependence (39) with these (different) values of $c_i$, turns out to be practically indistinguishable from the analytical expression (34). Therefore, the dependence (39) has no advantage in comparison with Eq. (34). It should be noted that in [16], where the same problem has been studied with a different method (scattering matrix approach) and different boundary conditions (the coupling matrix elements have been chosen to be constant as in the leads, instead of random as in the bulk), the parameters $c_i$ and the behavior in the metallic regime are slightly different. At the moment, we cannot give a convincing explanation of this discrepancy. In fact, theoretically, the influence of the coupling is still not understood, while, numerically, the quality of the data is not such to allow drawing a definite about the influence of the coupling.

We have also performed a detailed analysis of the distribution $P(G)$ in various regimes. In particular, it was found that the shape of the distribution is controlled by the same scaling parameter $\Lambda$. In the localized region, the distribution is approximately log-normal (see Fig. 5). The variance of $\ln G$ decreases upon increasing $\Lambda$ i.e. moving towards the metallic regime and in the localized regime satisfies the following relation

$$\text{var}(\ln G) = -A_0 \ln G - B_0, \quad (41)$$

where $A_0 \simeq 1.98 \pm 0.03$ and $B_0 \simeq 3 \pm 0.4$ (see Fig. 6). The value of the slope is fully consistent with theoretical expectations and with the numerical results obtained in [16]. However, let us recall that in the computation of Thouless conductance, it was found that $A_0 \simeq 1$ [15]. This means that fluctuations properties of Landauer and Thouless conductance are different (see also [17]). The value of the constant $B_0$ is instead slightly different from what found in [16]. This should be again a consequence of the different connection between the disordered sample and the leads.

Moving towards the metallic regime, the shape of the distribution $P(G)$ becomes normal (see Fig. 7). This result is in perfect agreement with the theoretical predictions for quasi-1D conductors [22].

The last problem that we have addressed in this paper is related to the so-called Universal Conductance Fluctuations. As is known (see, for example, the review [6]), the correct value of the variance of the dimensionless conductance is $\text{var}(G) = 2/15$. At the same time, direct application of the standard Random Matrix Theory to transmission matrices in the metallic regime gives $\text{var}(G) = 1/8$ [4]. The difference is small but crucial, it indicates that the repulsion between the transmission eigenvalues is not correctly described by the logarithmic two-body interaction, which is the core of the Random Matrix Theory.
this sense, it is important to study directly in numerical experiments how the fluctuations of the conductance depend on the degree of localization. We have performed simulations for different values of the scaling parameter $\Lambda$. Our data for the variance are reported in Fig. 8. As one can see, for $\Lambda$ still approximately equal to 4.5, the variance remains close to 1/8, the value predicted by the RMT. However, a further increase of the scaling parameter up to $\Lambda \approx 7.4$ leads the fluctuations to approach $2/15$, the value expected from either diagrammatic calculations or the diffusion-equation approach for transfer matrices (for the references see [6]). One should stress that it is important to satisfy the condition $b \ll N$ in order to distinguish the metallic regime from the ballistic one. The latter, highly non-generic, regime is reflected in the points in Fig. 8 which deviate from above when this condition is violated. The data clearly indicates that the variance reaches the limit value $2/15$ very slowly, an effect revealed also by a different approach, based on the scattering matrix determined through the numerical solution of the Lippman-Schwinger equation, see [16].

V. DISCUSSION

In this paper we have studied the statistical properties of the Lyapunov spectra for quasi-1D disordered samples of finite size in the connection with electric conductance and its fluctuations. The key point for our study is represented by Landauer expression, giving the conductance in terms of the Lyapunov exponents of a suitable transfer matrix. The mathematical model we have used is the band matrix ensemble with random entries for the disordered part.

The central question we have numerically investigated, is how statistical properties of the Lyapunov exponents are reflected in the properties of the conductance. The main theory which gives quite a good description of the properties of the conductance, is the standard Random Matrix Theory (RMT) applied to transfer matrices. The conjecture that the RMT is a good mathematical tool for a theoretical approach, is based on the observation that different channels are strongly but irregularly coupled. This approach has led to many interesting theoretical results [6] which have found quite a good support in several numerical experiments. Nonetheless, some small deviations can be found with respect to the RMT predictions. The most famous concerns the value of the normalized variance of the conductance fluctuations which is $2/15$ instead of $1/8$. The difference is small but important for the theoretical implications about the universal properties of the conductance fluctuations.

The origin of the above discrepancy has been attributed to different structure of the Lyapunov spectra compared to the spectrum of truly random matrices. For instance, in the RMT, the eigenvalues occupy the infinite line (for infinite random matrices with a fixed variance of the matrix elements), while the Lyapunov spectrum of the product of transfer matrices is symmetric with respect to the origin, a circumstance yielding an additional (non-random) repulsion between negative and positive “levels”, which is absent in completely random matrices. On the other hand, in the diffusive regime, given the large number of channels, such an effect may be neglected. Indeed, earlier numerical experiments have shown quite a good agreement with the RMT prediction; for example, the level spacing distribution was found to be well described by the famous Wigner-Dyson (WD) surmise (for the lowest Lyapunov exponent in the diffusive regime).
In our study, we have paid special attention to the whole Lyapunov spectrum, performing some qualitative tests aimed at emphasizing deviations of the level statistics from the Wigner-Dyson form. By making use of the normalized variance of the level spacing distribution, we have found that the statistical properties of the lowest Lyapunov exponent (as well as of the second one) are, indeed, described very well by the WD form. However, we have also detected a clear deviation from the RMT predictions for all the other exponents. In particular, it was found that the repulsion is weaker than that predicted by the RMT. This fact is quite unexpected since one might conjecture that the strongest influence of the negative part of the spectrum occurs for the lowest Lyapunov exponent, not for the bulk. Indeed, the lowest Lyapunov exponent has a “random” repulsion from one side only; on the negative side, the repulsion is of a different “regular” nature, due to the symmetry. However, it is precisely the lowest Lyapunov exponent to exhibit the best agreement with the random matrix approach (see also [23]). Thus we are lead to conjecture that besides the symmetry, a further feature must be present which contributes to differentiate a transfer matrix from purely random one. Anyway, the deviations observed in the bulk of the Lyapunov spectrum represent a further more detailed evidence of the failure of the RMT which has to be added to the difference between $1/8$ and $2/15$ for the conductance fluctuation (see also discussion and references in [6]).

To study the level spacing distribution in the general case (when the distribution does not coincide with the WD form), we have used the phenomenological expression suggested in [28] (see details in [29]) for the description of the so-called intermediate statistics which occurs due to strong localization effects. As long as the effective repulsion ranges between $0$ (Poissonian) and $\beta = 1$ (WD-distribution for the Gaussian Orthogonal Ensemble, GOE), the expression could be compared with the Brody-distribution [39]. However, in the present case, the effective repulsion is larger ($\beta > 1$), even diverging in the limit of strong localization, where the distribution becomes an increasingly narrow Gaussian (a feature already discussed in the literature [22]). Our analytical expression for the level spacing distribution allows to extract the repulsion parameter and to determine its dependence on the degree of localization (see Fig. 2). In analogy to other applications [31,28], the effective repulsion turns out to depend only on the scaling parameter $b^2/N$.

In the second part of the paper, we have investigated various properties of the conductance in a wide range of the localization parameter $\Lambda$. In particular, we have numerically computed the dependence of the geometric (“typical”) conductance $\tilde{G}$ on $\Lambda$, from the very localized to the diffusive regime. One should recall theoretical results on the dependence of the conductance on $\Lambda$ are available only in the limit cases of a strong localization and in the metallic regime.

Starting from the Landauer expression and taking into account the main properties of the Lyapunov spectra [11,12], we have obtained an analytical expression for the conductance with no free parameters. Evaluation of this expression in the two opposite (metallic and localized) limits have shown that this expression gives, in general, the correct behavior in comparison with some theoretical predictions. In particular, in the metallic regime the leading term is exactly the same as predicted by the theory, although the weak localization correction is slightly different.

A clear (but not very strong) difference has been found in the intermediate region $\Lambda \approx 1$. The source of this discrepancy is still not clear, in any case, one should emphasize that
in this region, both the coupling to the leads and finite size effects can play a significant role (we would like also to stress that there is no analytical prediction in this region for \( \tilde{G} \)). Moreover, we would like to note that the question of the influence of the coupling on the statistical properties of the conductance is not theoretically understood even for the relatively simple case of quasi-1D geometry. Recent numerical experiments \[40\] with the same model \[1\] have shown that by changing the degree of coupling, the structure of scattering states changes dramatically. In our present study, we have used the so-called “matching” principle according to which the variance of coupling elements is the same as that one in the bulk (see also \[12\]); in this case the influence of the coupling is expected to be negligible theoretical predictions can be used.

Finally, our results for the conductance fluctuations are in excellent agreement with the theoretical predictions. In particular, the distribution of the conductance has been found to be log-normal in the localized regime and of Gaussian in the diffusive regime. Furthermore, in the localized regime, the proportionality coefficient between the variance of the logarithm of the conductance and the logarithm of the conductance itself, is very close to the theoretical value 2. It is interesting to note, that when determining the conductance according to the Thouless definition via the curvature of the levels of a closed systems, this value was instead found to be 1 \[13\]. This fact indicates that fluctuation properties of the curvature in the localized regime cannot be associated with that of conductance fluctuations (see also \[17\]).

In the metallic regime, we have carefully studied the universal conductance fluctuations. One should stress that, numerically, this problem is far from trivial, since the sample length \( N \) has to be larger than band size \( b \) (defining the mean free path) but smaller that \( b^2 \) (\( N \ll b^2 \), where \( b^2 \) is a measure of the localization length in infinite samples). On the other hand, one also needs a sufficient statistics in order to discriminate between \( 1/8 \) and \( 2/15 \) for the value of the variance of the normalized conductance. We have performed an extensive numerical analysis to overcome the above mentioned computational difficulties. The main result we have obtained is the observation of a slow convergence to the theoretical value \( 2/15 \) upon increasing the localization parameter \( \Lambda \). Nonetheless, we have also found a “pre-diffusive” regime, \( \Lambda \approx 4.5 \) where the variance stays very close to \( 1/8 \). This is an indirect indication that many Lyapunov exponents come into play, besides the lowest one, which have fluctuations slightly different from that given by conventional RMT. Thus, both the universal conductance fluctuations and the origin of the difference between \( 1/8 \) and \( 2/15 \), predicted theoretically in \[24\], are nicely confirmed by our numerical study of the quasi-1D model.

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FIGURES

FIG. 1. Parameter $y$ (see Eq. (23)) for different values of the scaling parameter $\Lambda = b^2 / N$ versus the normalized number $\chi = i / b$ of the Lyapunov exponent. The horizontal line corresponds to the Wigner-Dyson surmise.

FIG. 2. Effective repulsion parameter $\beta$ is plotted versus $\Lambda$ for the minimum Lyapunov exponent $\gamma_b$.

FIG. 3. Average logarithm of the conductance versus the logarithm of the scaling parameter $\Lambda$ for different values of the parameters $N, b$ with $N = L + 2b$. The solid curve is the expression (31). In the inset the geometrical average of conductance versus $\Lambda$ is given for the metallic regime ($l_\infty \sim b^2 \gg N$). The best least square fit gives $y = ax + b$ with $a \simeq 0.75$ and $b \simeq -0.6$.

FIG. 4. Scaling function $\eta$. The horizontal line gives the asymptotic limit $\eta = 1$ for $G \gg 1$. Another line is used as a guide for the asymptotic slope of $\eta$ in the localized regime. In the inset we present the data for $\eta$ in the $G >> 1$ limit. Straight line corresponds to the best least square fit (see in the text).

FIG. 5. Log-normal distribution of the conductance in the localized regime.

FIG. 6. Variance of the logarithm of the conductance against the average logarithm of the conductance in the localized regime $\Lambda \ll 1$. The fit shows that the variance of the logarithm is two times larger than the mean value of the logarithm of conductance.

FIG. 7. Normal distribution of the conductance in the metallic regime $\Lambda \gg 1$.

FIG. 8. Variance of the logarithm of conductance versus the logarithm of the scaling parameter $\Lambda$. The low straight line is $1/8$, the upper is $2/15$. The error bars are of the order of the symbols. For all cases, we have averaged over more than 2500 different realizations of the disorder.
Fig. 1 Kottos et. al.

\( L=100, \ b=44 \)
\( L=150, \ b=40 \)
\( L=175, \ b=40 \)
\( L=1400, \ b=30 \)
\( L=300, \ b=10 \)
\( L=1200, \ b=15 \)
Fig. 2 Kottos et al.
Fig. 3 Kottos et al.
Fig. 4 Kottos et al.
\[ \langle \ln G \rangle = -9.735 \]

\[ \text{var}(\ln G) = 15.7 \]

L = 900

b = 15

Fig. 5 Kottos et al.
Fig. 7 Kottos et. al.

\[ \langle G \rangle = 4.15 \]

\[ \text{var}(G) = 0.13 \]
Fig. 8 Kottos et al.