Capture into Resonance and Escape from it in a Forced Nonlinear Pendulum

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Abstract—We study the dynamics of a nonlinear pendulum under a periodic force with small amplitude and slowly decreasing frequency. It is well known that when the frequency of the external force passes through the value of the frequency of the unperturbed pendulum’s oscillations, the pendulum can be captured into resonance. The captured pendulum oscillates in such a way that the resonance is preserved, and the amplitude of the oscillations accordingly grows. We consider this problem in the frames of a standard Hamiltonian approach to resonant phenomena in slow-fast Hamiltonian systems developed earlier, and evaluate the probability of capture into resonance. If the system passes through resonance at small enough initial amplitudes of the pendulum, the capture occurs with necessity (so-called autoresonance). In general, the probability of capture varies between one and zero, depending on the initial amplitude. We demonstrate that a pendulum captured at small values of its amplitude escapes from resonance in the domain of oscillations close to the separatrix of the pendulum, and evaluate the amplitude of the oscillations at the escape.

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Dedicated to Professor Alain Chenciner on his 70th birthday

1. INTRODUCTION

A pendulum under the action of an external force is an important and ubiquitous model in various areas of nonlinear dynamics. One of the interesting phenomena important in applications is capture of the pendulum’s oscillations into resonance with the oscillations of the external force. Consider first the pendulum initially at rest, and the external force of small amplitude \( \varepsilon \) and frequency equal to the frequency of the pendulum’s linear oscillations \( \omega_0 \). It is known that such a force can increase the pendulum’s amplitude up to a value of order \( \varepsilon^{1/3} \). At larger amplitudes the dependence of the pendulum’s frequency on the amplitude (i.e., the pendulum’s nonlinearity) results in breakup of the resonance. However, if the frequency of the external force is not constant but slowly decreasing with time at a rate \( \delta \), the so-called autoresonance phenomenon occurs (see, e.g., [1] and the references therein). When the frequency of the force passes through the value \( \omega_0 \), the pendulum is captured into resonance with the force, and the amplitude of its oscillations increases in such a way that the pendulum stays in the resonance.

This phenomenon has been studied in numerous works starting from the pioneering papers of V.I. Veksler and E.M. McMillan [2, 3] where a crucial role of the autoresonance in particle accelerators was demonstrated. However, it is methodically important and interesting to consider
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this phenomenon using the standard and well-developed Hamiltonian approach to resonant phenomena in nonlinear systems [4]. We do this in the present paper. We show that automatic capture into resonance (capture with probability one) occurs not only for zero initial amplitude of the pendulum, but also for small non-zero initial amplitudes. We also consider the case of larger initial amplitudes and describe the capture into resonance and evaluate its probability. A captured pendulum escapes from resonance at a certain final amplitude of oscillations. We describe this phenomenon and find the energy of the pendulum at the escape.

We study mostly the important “adiabatic” case, when the variation rate of the external frequency is much smaller than the amplitude of the external force: \( \delta \ll \varepsilon \). However, some conclusions are drawn also for the case when \( \delta \sim \varepsilon \). In particular, we show that capture into resonance is impossible, if \( \varepsilon \) is smaller than a certain threshold that depends on the value of \( \delta \). This agrees with the so-called threshold phenomenon (see [1]).

2. CAPTURE INTO RESONANCE AT SMALL INITIAL VALUES OF THE AMPLITUDE

We start at the Hamiltonian of a pendulum under the action of a time-periodic external forcing:

\[
H = \frac{P^2}{2} - \omega_0^2 \cos Q + \varepsilon \cos \psi \cdot Q. \quad (2.1)
\]

Here \((P, Q)\) are the canonically conjugate momentum and the coordinate, \(\omega_0\) is the frequency of linear oscillations, \(\varepsilon \ll 1\) and \(\psi\) are the amplitude and the phase of the external forcing, respectively. Assume that \(\dot{\psi} = \omega(\delta t)\) is a slowly varying frequency of the external forcing, \(0 < \delta \ll 1\).

A standard way to study this system near the 1:1 resonance is to introduce the action-angle variables of the unperturbed pendulum as a new pair of canonical variables, average the Hamiltonian near the resonance, and expand it into series. This approach is implemented in Section 3 of this paper. However, when the initial amplitude of the pendulum is zero or small, of order \(\varepsilon^{1/3}\), it is easier to apply a different method. Namely, one can expand the cosine in (2.1) and use symplectic polar coordinates instead of the exact action-angle variables. Below in this section we use this latter approach. The results obtained by the two methods at small values of the initial amplitude asymptotically agree with each other.

Thus, in this section we study the case of small amplitude of the pendulum’s oscillations: \(|Q| \ll 1\). Expanding \(\cos Q\) into series and omitting a constant, we find in the main approximation

\[
H = \frac{P^2}{2} + \omega_0^2 Q^2 - \omega_0^2 Q^4 - \varepsilon \cos \psi \cdot \sqrt{2\rho/\omega_0} \sin \gamma - \rho \omega. \quad (2.2)
\]

Introduce new canonical variables (so-called symplectic polar coordinates) \(\rho\) and \(\phi\):

\[
Q = \sqrt{2\rho/\omega_0} \sin \phi, \quad P = \sqrt{2\rho\omega_0} \cos \phi. \quad (2.3)
\]

In terms of the new variables the Hamiltonian takes the form

\[
H = \rho \omega_0 - \frac{\rho^2}{6} \sin^4 \phi + \varepsilon \cos \psi \cdot \sqrt{2\rho/\omega_0} \sin \phi. \quad (2.4)
\]

We consider the situation where the system is close to the 1:1 resonance, i.e., when \(\dot{\phi} \approx \dot{\psi}\). Introduce the resonance phase \(\gamma = \phi - \psi\) as a new variable. To do this, make a canonical change of variables \((\rho, \phi) \mapsto (\tilde{\rho}, \gamma)\) defined by the generating function \(W(\tilde{\rho}, \gamma) = \tilde{\rho}(\phi - \psi)\). In the new variables, the Hamiltonian is \(\tilde{H} = H + \partial W/\partial t = H - \rho \omega\). One can average over the fast phase and obtain the Hamiltonian averaged near the resonance (we omit tildes):

\[
\tilde{H} = \rho \omega_0 - \frac{\rho^2}{16} + \frac{1}{2} \varepsilon \sqrt{2\rho/\omega_0} \sin \gamma - \rho \omega. \quad (2.5)
\]

Introduce another pair of canonical variables \((x, y)\):

\[
x = \sqrt{2\rho} \sin \gamma, \quad y = \sqrt{2\rho} \cos \gamma. \quad (2.6)
\]