EX-POST CORE, FINE CORE AND RATIONAL EXPECTATIONS EQUILIBRIUM ALLOCATIONS

ANUJ BHOWMIK AND JILING CAO

Abstract. This paper investigates the ex-post core and its relationships to the fine core and the set of rational expectations equilibrium allocations in an oligopolistic economy with asymmetric information, in which the set of agents consists of some large agents and a continuum of small agents and the space of states of nature is a general probability space. We show that under appropriate assumptions, the ex-post core is not empty and contains the set of rational expectations equilibrium allocations. We provide an example of a pure exchange continuum economy with asymmetric information and infinitely many states of nature, in which the ex-post core does not coincide with the set of rational expectations equilibrium allocations. We also show that when our economic model contains either no large agents or at least two large agents with the same characteristics, the fine core is contained in the ex-post core.

1. Introduction

In general equilibrium theory, the core and competitive equilibrium are two important solution concepts. For an exchange economy with complete information, the core and its relationship to the set of competitive allocations have been studied intensively in the literature (for a comprehensive survey, refer to [3]). In the past few decades, several alternative cooperative and non-cooperative equilibrium concepts have been proposed, in the context of asymmetric information economies. The core of an economy with asymmetric information was first considered by Wilson [26], where the concepts of coarse and fine core were proposed. The fine core presumes that agents can share their information when they form a coalition and an allocation is not in the fine core, if a coalition has some distribution of the total endowments of its members which gives to all of its members a better pay-off in an event which the coalition can jointly discern. In [27], Yannelis introduced the concept of private core, which is an analogue concept to the core for an economy with complete (and symmetric) information, and proved that under appropriate assumptions, the private core is always non-empty. In the definition of the private core, when a coalition blocks an allocation, each member in the coalition uses only his own private information. Furthermore, Ein et al. [13, 14] studied the notion of ex-post core, in the sense that an ex-post core allocation cannot be ex-post blocked by any coalition. On the other hand, Radner [23] introduced the notion of a (Bayesian) rational expectations equilibrium by imposing the Bayesian (subjective
expected utility) decision doctrine, in order to capture the information revealed by the market clearing price. The fact that a Bayesian rational expectations equilibrium does not exist universally motivates de Castro et al. [12] to introduce the concept of a maximin rational expectations equilibrium, by replacing the Bayesian decision-making approach of Radner with the maximin expected utility. A good survey article for the equilibrium concepts in asymmetric (or differential) information economies is [17].

For economies with complete information, Aumann [6] proved that competitive and core allocations coincide, provided that there is a continuum of traders. The existence of such allocations was studied by Aumann [7] and Hildenbrand [19]. Extensions of these results to economies with asymmetric information were made by Einy et al. [13, 15]. In [13], Einy et al. first established some representation results on the ex-post core and the set of rational expectations equilibrium allocations. Then, these representations results together with Aumann’s Core Equivalence Theorem enabled them to show that if the economy is atomless and the utility function of each agent is measurable with respect to his information, then the set of rational expectations equilibrium allocations coincides with the ex-post core. In [15], Einy et al. showed that, if an economy is irreducible, then a competitive (or Walrasian expectations) equilibrium exists and, moreover, the set of competitive equilibrium allocations coincides with the private core. However, to obtain these results, they allow for free disposal on the feasibility (market clearing) constraints. This was motivated by an example [15] of an economy with asymmetric information which has a competitive equilibrium with free disposal, but if the feasibility constraints are imposed with an equality, then the economy does not have a competitive equilibrium where prices of all contingent contracts for future delivery are non-negative. In a few years later, Angeloni and Martins-da-Rocha [4] proved that the results in [15] are still valid without free-disposal.

In the past few years, techniques have been developed by Bhowmik et al. in [10] to investigate the existence of rational expectations equilibrium in a general model of pure exchange economies. Moreover, Bhowmik and Cao [11] established a representation result for rational expectations equilibrium allocations in terms of the state-wise Walrasian allocations. As a rational expectation equilibrium allocation is an interim solution concept and it takes into account the information of all other agents through market price, Bhowmik and Cao [11] showed their result by assuming that each agent knows his initial endowment and utility. Such assumptions lead to a fact that the information revealed by prices play no role and thus, the Bayesian (maximin) rational expectation equilibrium allocations becomes almost the same as the state-wise Walrasian allocation.

Our aim of this paper is to apply the results and techniques developed in [10, 11] to the study of the ex-post core and its relationships to the fine core and the set of rational expectations equilibrium allocations. We consider an oligopolistic economy with asymmetric information in which the set of agents consists of some large agents and a continuum of small agents. The uncertainty is model by a general probability space of states of nature in which each agent is characterized by a state-dependent utility function, a random initial endowment, an information partition and a prior

\[But they are not the same as both \( T \) and \( \Omega \) are infinite, see Example 3.9 in this paper.]
belief. Firstly, we establish a result on the existence and characterization of the ex-post core, which can be regarded as an extension of the corresponding result in [13] to a framework with infinitely many states of the nature. The proof of this result relies on the measurability of Walrasian equilibrium correspondence with respect to the information structure in the economy (see Theorem 3.2). In the presence of the result in [11] and Aumann’s Core Equivalence Theorem, we conclude that Bayesian (maximin) rational expectation equilibrium allocations are contained the ex-post core. This is a version of the first fundamental theorem of social welfare for large economies with asymmetric information. However, contrary to the equivalence result for finitely many states of nature in [13], we provide an example of a continuum economy with asymmetric information and infinitely many states of nature, in which the ex-post core strictly contains all rational expectations equilibrium allocations. This means that the core-Walras equivalence can fail in a continuum economy with asymmetric information when it has infinitely many states of nature. Secondly, we show that under appropriate assumptions and the assumption that there are only finitely many different information structures and all information is the joint information of agents, the fine core is contained in the ex-post core. This extends the corresponding result in [14]. To obtain this result, following [18], we first associated an atomless economy with our oligopolistic economy so that all large agents are broken into a continuum of small agents with similar characteristics. The idea of the proof is as follows: if an allocation is not in the ex-post core of our original economy, it must not be a core allocation in some complete information economy and so in the corresponding complete information atomless economy. Vind’s theorem (see [25]) implies that an arbitrary large coalition can be chosen so that it discerns any state of nature. With the help of some other techniques, we are able to show that the allocation is blocked by a coalition that jointly has full information in our original economy and thus, it is not in the fine core.

The structure of the paper is as follows. Section 2 presents the theoretical framework and outlines the basic model. We also study several correspondences associated with our basic model. These correspondences form the major part of our tool kits. Section 3 investigates a representation of the ex-post core and its relationship with the set of rational expectations equilibrium allocations. Section 4 studies the relationship between the ex-post core and the fine core. Finally, we provide some concluding remarks in Section 5.

2. The Model and Associated Correspondences

In this section, we describe a basic model of a pure exchange mixed economy with asymmetric information.

2.1. The model. We consider a pure exchange economy $\mathcal{E}$ with asymmetric information. The exogenous uncertainty is described by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is a set denoting all possible states of nature, the $\sigma$-algebra $\mathcal{F}$ denotes possible events, and $\mathbb{P}$ is a complete probability measure. The space of agents is a measure space $(T, \Sigma, \mu)$ with a complete, finite and positive measure $\mu$, where $T$ is the set of agents, $\Sigma$ is the $\sigma$-algebra of measurable subsets of $T$ whose economic weights on the market are given by $\mu$. Since $\mu(T) < \infty$, a classical result in measure theory claims that $T$ can be decomposed into the union of two parts: one is atomless and the other contains at most countably many atoms, that is,
$T = T_0 \cup T_1$, where $T_0$ is the atomless part and $T_1$ is the union of at most countably many $\mu$-atoms, refer to \cite{21} p.155. Let $A = \{A_n : n \geq 1\}$ be the family of all atoms in $T_1$, i.e., $T_1 = \bigcup_{n \geq 1} A_n$. Agents in $T_0$ are called "small agents", who are un-influential agents (the price takers). According to a standard interpretation, we can think that each $A_n$ arises from a group of small identical agents that decide to join and to act on the market only together. As consequence of such agreements, no proper subcoalitions of the group are possible and then the group is identified with an atom of $\mu$. Agents in $T_1$ are called "large agents", who are influential ones (the oligopolies). With an abuse of notation, we shall identify $T_1$ with $\mathcal{A}$, i.e., $T_1 = \mathcal{A}$.

The commodity space is the $\ell$-dimensional Euclidean space $\mathbb{R}^{\ell}$. For $\lambda > 0$, $B(0, \lambda)$ denotes the ball in $\mathbb{R}^{\ell}$ centred at $0$ with radius $\lambda$. The partial order on $\mathbb{R}^{\ell}$ is denoted by $\leq$. More precisely, for any two vectors $x = (x_1, ..., x_\ell)$ and $y = (y_1, ..., y_\ell)$ in $\mathbb{R}^{\ell}$, we write $x \leq y$ (or $y \geq x$) if $x_k \leq y_k$ for all $1 \leq k \leq \ell$. Furthermore, we write $x < y$ (or $y > x$) when $x \leq y$ and $x \neq y$, and $x \ll y$ (or $y \gg x$) when $x_k < y_k$ for all $1 \leq k \leq \ell$. Let $\mathbb{R}_{\ell}^+ = \{x \in \mathbb{R}^{\ell} : x \geq 0\}$, and let $\mathbb{R}_{\ell}^{++} = \{x \in \mathbb{R}_{\ell}^+ : x \gg 0\}$. In each state, the consumption set for every agent $t$ in $T$ is $\mathbb{R}^{\ell}_t$. Each agent $t$ in $T$ is characterized by a quadruple $(\mathcal{F}_t, U(t, \cdot), a(t, \cdot), \mathbb{P}_t)$, where

(i) $\mathcal{F}_t$ is the $\sigma$-algebra generated by a measurable partition $\Pi_t$ of $\Omega$ representing the private information of agent $t$,

(ii) $U(t, \cdot, \cdot) : \Omega \times \mathbb{R}_{\ell}^{++} \rightarrow \mathbb{R}$ is the state-dependent utility function of agent $t$,

(iii) $a(t, \cdot) : \Omega \rightarrow \mathbb{R}_{\ell}^+$ is the state-dependent initial endowment of agent $t$, and

(iv) $\mathbb{P}_t$ is a probability measure on $\mathcal{F}$, giving the prior belief of agent $t$.

The quadruple $(\mathcal{F}_t, U(t, \cdot, \cdot), a(t, \cdot), \mathbb{P}_t)$ is sometimes known as characteristics of agent $t$. Two agents are said to be the same type if they have the same characteristics. Formally, the economy $\mathcal{E}$ can be expressed by

$$\mathcal{E} = \{(\Omega, \mathcal{F}, \mathbb{P}) ; \ (T, \Sigma, \mu); \ \mathbb{R}_{\ell}^+; \ (\mathcal{F}_t, U(t, \cdot, \cdot), a(t, \cdot), \mathbb{P}_t)_{t \in T}\}.$$ 

In the complete information Arrow-Debreu-McKenzie model, prices are vectors in $\mathbb{R}_{\ell}^+ \setminus \{0\}$. Following the standard treatment in the literature (e.g., see \cite{7}), price vectors are normalized so that their sum is $1$.

In this paper, we use the symbol $\Delta$ to denote the simplex of normalized price vectors, i.e.,

$$\Delta = \left\{ p \in \mathbb{R}_{\ell}^+ : \sum_{h=1}^{\ell} p^h = 1 \right\}. $$

Put $\Delta_+ = \Delta \cap \mathbb{R}_{\ell}^{++}$. Throughout the paper, $\Delta$ and $\Delta_+$ are equipped with the relative Euclidean topology. A price system of $\mathcal{E}$ is an $\mathcal{F}$-measurable function $\pi : \Omega \rightarrow \Delta$, where $\Delta$ is equipped with the Borel structure $\mathscr{B}(\Delta)$ generated by the relative Euclidean topology.

Let $\sigma(\pi)$ be the smallest $\sigma$-algebra contained in $\mathcal{F}$ and generated by a price system $\pi$. Intuitively, $\sigma(\pi)$ represents the information revealed by $\pi$. The combination of agent $t$’s private information $\mathcal{F}_t$ and the information revealed by the price system $\pi$ is given by the smallest $\sigma$-algebra $\mathcal{G}_t$ that contains both $\mathcal{F}_t$ and $\sigma(\pi)$. Formally, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\pi)$. For any $\omega \in \Omega$, let $\mathcal{G}_t(\omega)$ denote the smallest element of $\mathcal{G}_t$ that contains $\omega$.

As interpreted in \cite{12}, the economy $\mathcal{E}$ extends over three time periods: ex ante ($t = 0$), interim ($t = 1$) and ex post ($t = 2$). At $t = 0$, the state space, the
partitions, the structure of the economy and the price functional \( \pi : \Omega \to \Delta \) are common knowledge. This stage does not play any role in our analysis and it is assumed just for a matter of clarity. At \( \tau = 1 \), each individual learns his private information and the prevailing prices \( \pi(\omega) \), and thus learns \( \mathcal{F}_t \). With these in his mind, the agent plans how much he will consume \( x(\omega) \). However, his actual consumption may be contingent to the final state of the nature, which is not yet known by him. The individual agent only knows that one of the states \( \omega' \in \mathcal{F}_t(\omega) \) will be realized. Therefore, he needs to make sure that he will be able to pay his consumption plan \( x(\omega') \) for all \( \omega' \in \mathcal{F}_t(\omega) \). At \( \tau = 2 \), each individual agent \( t \in T \) receives and consumes his entitlement \( f_t(\omega) \).

Recall that a function \( u : \mathbb{R}^+_t \to \mathbb{R} \) is strictly increasing if \( u(x) < u(y) \) for any \( x, y \in \mathbb{R}^+_t \) with \( x < y \), and it is quasi concave if

\[
u(\alpha x + (1 - \alpha)y) \geq \min\{u(x), u(y)\}
\]

for any \( x, y \in \mathbb{R}^+_t \) with \( x \neq y \) and any \( 0 < \alpha < 1 \). In “\( \geq \)” in the above inequality is replaced with “\( \succ \)”, then \( u \) is called strictly quasi concave.

Throughout the paper, the following standard assumptions will be used. These assumptions are similar to those in 10 11.

\( (A_1) \) The initial endowment function \( a : (T, \Sigma, \mu) \times (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^+_t \) is \( \Sigma \otimes \mathcal{F} \)-measurable such that \( a(\cdot, \omega) \) is Bochner integrable and \( \int_T a(\cdot, \omega)d\mu \succ 0 \) for each \( \omega \in \Omega \).

\( (A_1') \) The initial endowment function \( a : (T, \Sigma, \mu) \times (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^+_t \) is \( \Sigma \otimes \mathcal{F} \)-measurable such that \( a(\cdot, \omega) \) is Bochner integrable and \( a(\cdot, \omega) \succ 0 \) \( \mu \)-a.e. on \( T \) for each \( \omega \in \Omega \).

\( (A_2) \) \( U(\cdot, \cdot, x) : (T, \Sigma, \mu) \times (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \) is \( \Sigma \otimes \mathcal{F} \)-measurable for all \( x \in \mathbb{R}^+_t \).

\( (A_3) \) For each \( (t, \omega) \in T \times \Omega \), \( U(t, \omega, \cdot) : \mathbb{R}^+_t \to \mathbb{R} \) is continuous and strictly increasing.

\( (A_4) \) For each \( (t, \omega) \in T \times \Omega \), \( U(t, \omega, \cdot) \) is strictly quasi-concave.

\( (A_4') \) For each \( (t, \omega) \in T_1 \times \Omega \), \( U(t, \omega, \cdot) \) is quasi-concave.

Here, we would like to add some comments on these assumptions. Note that the condition \( \int_T a(\cdot, \omega)d\mu \succ 0 \) for each \( \omega \in \Omega \) in \( (A_1) \) or \( \alpha(\cdot, \omega) \succ 0 \) \( \mu \)-a.e. on \( T \) for each \( \omega \in \Omega \) in \( (A_1') \), which implies that no commodity is totally absent from the market, has been commonly used for results on the existence of an equilibrium, for instance, see 10 11 13 14. The joint measurability of the initial endowment \( a \) in \( (A_1) \) and \( (A_1') \) has been used in 10 11 for general models of asymmetric information economies with infinitely many states of nature. The assumption \( (A_1') \) is stronger than \( (A_1) \) and is used in 8 9. Assumption \( (A_2) \) is equivalent to the measurability condition used in 6 7. Since then, it has been widely used in the literature, see 10 11 12 13 14. Although \( (A_1) \) and \( (A_2) \) are not used in Einvy et al. 14, \( U(\cdot, \cdot, x) \) and \( a(t, \cdot) \) are required to be \( \mathcal{F} \)-measurable for all \( (t, x) \in T \times \mathbb{R}^+_t \). Finally, \( (A_3) \), \( (A_4) \) and \( (A_4') \) impose properties on the agents’ utility functions. These assumptions have been quite commonly used in the literature.
A member $S$ of $\Sigma$ with $\mu(S) > 0$ is called a coalition of $\mathcal{E}$. Let $L_1(\mu, \mathbb{R}^\ell)$ denote the set of all equivalent classes of Bochner integrable functions from $T$ into $\mathbb{R}^\ell$. An assignment in $\mathcal{E}$ is a function $f : T \times \Omega \rightarrow \mathbb{R}^\ell$ such that for every $\omega \in \Omega$, $f(\cdot, \omega) \in L_1(\mu, \mathbb{R}^\ell)$, and for every $t \in T$, $f(t, \cdot)$ is $\mathcal{F}$-measurable. If an assignment $f$ is also feasible, i.e., for every $\omega \in \Omega$,\[
abla \int_T f(\cdot, \omega) d\mu = \int_T a(\cdot, \omega) d\mu,\]
then it is called an allocation. Note that under (A1), the initial endowment $a$ is an allocation in $\mathcal{E}$.

2.2. Correspondences Associated with $\mathcal{E}$. Following [10, Lemma 2], we define a function $\delta : \Delta_+ \rightarrow \mathbb{R}_{++}$ such that for each $p = (p^1, \cdots, p^\ell) \in \Delta_+$,
\[
\delta(p) = \min \{p^h : 1 \leq h \leq \ell\}.
\]
For any $(t, \omega, p) \in T \times \Omega \times \Delta_+$, let
\[
\gamma(t, \omega, p) = \frac{1}{\delta(p)} \sum_{h=1}^\ell a^h(t, \omega), \quad \text{and} \quad b(t, \omega, p) = \gamma(t, \omega, p) \mathbf{1},
\]
where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^\ell$. Define the correspondence $X : T \times \Omega \times \Delta_+ \supseteq \mathbb{R}^\ell_+$ by
\[
X(t, \omega, p) = \{x \in \mathbb{R}^\ell_+ : x \leq b(t, \omega, p)\}
\]
for all $(t, \omega, p) \in T \times \Omega \times \Delta_+$. The budget correspondence $B : T \times \Omega \times \Delta \supseteq \mathbb{R}^\ell_+$ is defined by
\[
B(t, \omega, p) = \{x \in \mathbb{R}^\ell_+ : \langle p, x \rangle \leq \langle p, a(t, \omega) \rangle\}
\]
for all $(t, \omega, p) \in T \times \Omega \times \Delta$. Note that $X$ and $B$ are non-empty, closed- and convex-valued such that $B(t, \omega, p) \subseteq X(t, \omega, p)$ for all $(t, \omega, p) \in T \times \Omega \times \Delta_+$. Furthermore, the compactness of $X(t, \omega, p)$ implies that $B(t, \omega, p)$ is compact for every $(t, \omega, p) \in T \times \Omega \times \Delta_+$.

Following [1], we say that a correspondence $F : (T, \Sigma, \mu) \supseteq \mathbb{R}^\ell$ is weakly $\Sigma$-measurable if
\[
F^{-1}(V) = \{t \in T : F(t) \cap V \neq \emptyset\} \in \Sigma
\]
for all open subset $V$ of $\mathbb{R}^\ell$. Wherever no confusion arises in the sequel, we shall omit $\Sigma$ in the definition of a weakly $\Sigma$-measurable correspondence. A function $f : (T, \Sigma, \mu) \rightarrow \mathbb{R}^\ell$ is called a measurable selection of $F$ if $f$ is $\Sigma$-measurable and $f(t) \in F(t) \mu$-a.e.

Lemma 2.1 ([5]). Let $F : (T, \Sigma, \mu) \supseteq \mathbb{R}^\ell$ be a correspondence. Then the following statements are equivalent:

(i) $F$ is weakly $\Sigma$-measurable.
(ii) $F$ has a measurable graph, that is, $\text{Gr}_F \in \Sigma \otimes \mathcal{B}(\mathbb{R}^\ell)$.
(iii) For every $x \in \mathbb{R}^\ell$, $\text{dist}(x, F(\cdot)) : T \rightarrow \mathbb{R}_+$ is $\Sigma$-measurable.

The following proposition is similar to [10] Proposition 4.1 and is a special case of [11] Lemma 2.

Proposition 2.2. Assume that an economy $\mathcal{E}$ satisfies (A1). Then $B$ is weakly $\Sigma \otimes \mathcal{F} \otimes \mathcal{B}(\Delta)$-measurable and $X$ is weakly $\Sigma \otimes \mathcal{F} \otimes \mathcal{B}(\Delta_+)$-measurable.
Furthermore, it is easy to see that

By (A) functions from $t, ω, p$ for all $(t, ω, p)$ and

Finally, it is easy to see that

By (A), for every $x \in C(t, ω, p)$ and $(t, ω, p) \in T \times Ω \times Δ$, $(p, x) \geq (p, a(t, ω))$.

Furthermore, it is easy to see that

$$B(t, ω, p) \cap (t, ω, p) = B(t, ω, p) \cap C(t, ω, p)$$

for all $(t, ω, p) \in T \times Ω \times Δ$. Note that under (A), $U(t, ω, \cdot)$ is continuous on the

The following proposition is similar to [10, Proposition 4.2].

**Proposition 2.3.** Assume that an economy $E$ satisfies (A1)-(A3). Then $C^X$ is weakly $Σ \otimes \mathcal{F} \otimes \mathcal{B}(Δ^+)$-measurable.

**Proof.** By Proposition 2.2, $B$ is weakly $Σ \otimes \mathcal{F} \otimes \mathcal{B}(Δ)$-measurable. Thus, by [11, Corollary 18.14], there exists a sequence $\{f_n : n \geq 1\}$ of $Σ \otimes \mathcal{F} \otimes \mathcal{B}(Δ)$-measurable functions from $T \times Ω \times \Delta$ to $R^+_4$ such that

$$B(t, ω, p) = \{f_n(t, ω, p) : n \geq 1\}$$

for all $(t, ω, p) \in T \times Ω \times Δ$.

For each $n \geq 1$, define $C_n : T \times Ω \times Δ \to R^+_4$ by letting

$$C_n(t, ω, p) = \{x \in R^+_4 : U(t, ω, x) \geq U(t, ω, f_n(t, ω, p))\},$$

and $ξ_n : T \times Ω \times Δ \times R^+_4 \to R$ by letting

$$ξ_n(t, ω, p, x) = U(t, ω, x) - U(t, ω, f_n(t, ω, p)).$$

Note that $ξ_n(\cdot, \cdot, x)$ is $Σ \otimes \mathcal{F} \otimes \mathcal{B}(Δ)$-measurable for all $x \in R^+_4$, and

$$C(t, ω, p) = \bigcap_n \{C_n(t, ω, p) : n \geq 1\}$$

for all $(t, ω, p) \in T \times Ω \times Δ$. It follows that for all $(t, ω, p) \in T \times Ω \times Δ$,

$$C^X(t, ω, p) = \bigcap_n \{C_n(t, ω, p) : n \geq 1\} \cap X(t, ω, p).$$

Applying an argument similar to that in Proposition 2.2, it can be shown that each $C_n$ is $Σ \otimes \mathcal{F} \otimes \mathcal{B}(Δ)$-measurable. Since $X$ is compact-valued, then $C^X$ is $Σ \otimes \mathcal{F} \otimes \mathcal{B}(Δ^+)$-measurable.

The idea of the next lemma is included in the proof of [10, Theorem 4.3]. For the sake of self-completeness of this paper, we extracted it here as a separate lemma with a complete proof.

**Lemma 2.4.** Assume that an economy $E$ satisfies (A1)-(A3). Let $\{p_n : n \geq 1\} \subseteq Δ^+$ converge to some $p \in Δ^+$. For each $(t, ω, p) \in T \times Ω \times Δ^+$,

$$C^X(t, ω, p) \subseteq \text{Li } C^X(t, ω, p_n).$$
Definition 3.1. Let \( f \) be an allocation in an economy \( \mathscr{E} \) with asymmetric information. For each \( \omega \in \Omega \), let \( \mathscr{E}(\omega) \) denote the complete information economy, given by

\[
\mathscr{E}(\omega) = \{(T, \Sigma, \mu): \mathbb{R}_+^T; (U(t, \omega, \cdot), a(t, \omega))_{t \in T}\}.
\]

The core of \( \mathscr{E}(\omega) \) is denoted by \( \mathbf{C}(\mathscr{E}(\omega)) \). The set of all Walrasian equilibria and all Walrasian equilibrium allocations of \( \mathscr{E}(\omega) \) are denoted by \( \text{WE}(\mathscr{E}(\omega)) \) and \( \text{WA}(\mathscr{E}(\omega)) \), respectively. Then, \( \mathbf{C}: \omega \mapsto \mathbf{C}(\mathscr{E}(\omega)) \) and \( \text{WE}: \omega \mapsto \text{WE}(\mathscr{E}(\omega)) \) define two correspondences.

3. The Ex-post Core and Rational Expectations

Equilibrium Allocations

In this section, we discuss the existence of an ex-post core allocation in our model and also the relationship between the ex-post core and the set of (Bayesian or maximin) rational expectations equilibrium allocations.

**Definition 3.1.** (\cite{I3}) Let \( f \) be an allocation in an economy \( \mathscr{E} \), let \( S \in \Sigma \) be a coalition. We say that \( f \) is ex-post blocked by \( S \) if there exist a state of nature \( \omega_0 \in \Omega \) and an assignment \( g \) such that

\[
(g(\cdot, \omega_0)d\mu = \int S a(\cdot, \omega_0)d\mu, \text{ and}
\]

\[
U(t, \omega, g(t, \omega_0)) \geq U(t, \omega_0, f(t, \omega_0)) \mu-a.e. \text{ on } S.
\]

In addition, an allocation \( f \) is called an ex-post core allocation if it cannot be ex-post blocked by any coalition. The ex-post core of \( \mathscr{E} \), denoted by \( \mathbf{C}(\mathscr{E}) \), is the set of all the ex-post core allocations of \( \mathscr{E} \).

The main result of this section is the following theorem on the ex-post core.
Theorem 3.2. Suppose that an economy $\mathcal{E}$ satisfies $(A_1)$-$(A_4)$. Then the ex-post core of $\mathcal{E}$ is not empty. Moreover,

$$C(\mathcal{E}) = \{ f : f \text{ is an allocation and } f(\omega, \cdot) \in C(\omega) \text{ for all } \omega \in \Omega \}.$$  

To provide a proof of Theorem 3.2, we need some preparation. First of all, the following result, which is a special case of the Kuratowski-Ryll-Nardzewski measurable selection theorem (refer to [1, 18.13]), will be needed.

Lemma 3.3. Let $F : T \mapsto \mathbb{R}^\ell$ be a weakly $\Sigma$-measurable correspondence such that $F(t)$ is non-empty and closed for all $t \in T$. Then $F$ admits a $\Sigma$-measurable selection.

Secondly, the following result on the weak measurability of WE is also needed for the proof of Theorem 3.2.

Theorem 3.4. Assume that an economy $\mathcal{E}$ satisfies $(A_1)$-$(A_3)$ and $(A_4')$. Then WE is weakly $\mathcal{F}$-measurable.

Proof. Note that under the given assumptions, $\text{WE}(\omega) \neq \emptyset$ for all $\omega \in \Omega$. Consider the correspondences $F : (\Omega, \mathcal{F}, \mathbb{P}) \Rightarrow L_1(\mu, \mathbb{R}^\ell)$, defined by

$$F(\omega) = \left\{ f \in L_1(\mu, \mathbb{R}^\ell) : \int_T f \, d\mu - \int_T a(\cdot, \omega) \, d\mu = 0 \right\}$$

and $G : (\Omega, \mathcal{F}, \mathbb{P}) \Rightarrow L_1(\mu, \mathbb{R}^\ell) \times \Delta$, defined by $G(\omega) = F(\omega) \times \Delta$. First of all, we claim that $F$ has a measurable graph, and thus $G$ also has a measurable graph. To see this, define a function $\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \times L_1(\mu, \mathbb{R}^\ell) \rightarrow \mathbb{R}^\ell$ by

$$\varphi(\omega, f) = \int_T f \, d\mu - \int_T a(\cdot, \omega) \, d\mu.$$  

Note that for every $f \in L_1(\mu, \mathbb{R}^\ell)$, $\varphi(\cdot, f)$ is $\mathcal{F}$-measurable and for every $\omega \in \Omega$, $\varphi(\omega, \cdot)$ is norm-continuous. Thus, $\varphi$ is $\mathcal{F} \otimes \mathcal{B}(L_1(\mu, \mathbb{R}^\ell))$-measurable. The conclusion follows from the fact $\text{Gr}_F = \varphi^{-1}(0)$.

Let $Q^n \cap \Delta_+ = R$, where $Q^n$ is the set of vectors in $\mathbb{R}^\ell$ with rational components. Note that $R$ is countable and dense in $\Delta$. For each $p \in R$, define a correspondence $H_p : (\Omega, \mathcal{F}, \mathbb{P}) \Rightarrow L_1(\mu, \mathbb{R}^\ell)$ by

$$H_p(\omega) = \mathcal{S}_{C^\infty(\omega, p)}$$

where $\mathcal{S}_{C^\infty(\omega, p)}$ is the set of integrable selections of $C^\infty(\omega, p)$. Fix a $p \in R$ and define a function $\zeta : L_1(\mu, \mathbb{R}^\ell) \times (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}_+$ by $\zeta(g, \omega) = \text{dist}(g, H_p(\omega))$. Furthermore, for each $g \in L_1(\mu, \mathbb{R}^\ell)$, let the function $\xi^g : (T, \Sigma, \mu) \times (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}_+$ be defined by

$$\xi^g(t, \omega) = \text{dist}(g(t), C^{\infty}(t, \omega, p)).$$

Claim 1. For each simple function $g \in L_1(\mu, \mathbb{R}^\ell)$, $\zeta(g, \omega) = \int_T \xi^g(\cdot, \omega) \, d\mu$ holds for all $\omega \in \Omega$.

Proof of Claim 1. Let $g \in L_1(\mu, \mathbb{R}^\ell)$ be a given simple measurable function. As $g$ is a step-function with finitely many values, it follows from Lemma 2.1 and Proposition 2.3 that $\xi^g$ is $\Sigma \otimes \mathcal{F}$-measurable. In addition, since

$$\xi^g(t, \omega) \leq \|g(t) - b(t, \omega, p)\|$$
for all \((t, \omega) \in T \times \Omega\), \(\xi^g(\cdot, \omega)\) is also integrably bounded and thus \(\xi^g(\cdot, \omega) \in L^1(\mu, \mathbb{R}^\ell)\) for all \(\omega \in \Omega\). It is easy to check that \(\int_T \xi^g(\cdot, \omega) d\mu \leq \zeta(g, \omega)\) for all \(\omega \in \Omega\). Suppose \(\int_T \xi^g(\cdot, \omega_0) d\mu < \zeta(g, \omega_0)\) holds for some \(\omega_0 \in \Omega\). Then, there is an \(\varepsilon > 0\) such that

\[
\int_T \xi^g(\cdot, \omega_0) d\mu + \varepsilon \mu(T) < \zeta(g, \omega_0).
\]

Next, we define \(A : T \Rightarrow \mathbb{R}^\ell\) and \(\alpha : T \times \mathbb{R}^\ell \rightarrow \mathbb{R}\) by

\[
A(t) = \{y \in C^\infty(t, \omega_0, p) : \|g(t) - y\| \leq \xi^g(t, \omega_0) + \varepsilon\}
\]

and

\[
\alpha(t, y) = \|g(t) - y\| - \xi^g(t, \omega_0).
\]

As done in the above, it can be shown that \(\alpha\) is \(\Sigma \otimes \mathcal{F}\)-measurable and thus \(\text{Gr}_A = \{(t, y) \in T \times \mathbb{R}^\ell : \alpha(t, y) \leq \varepsilon\} \cap \text{Gr}_{C^\infty(t, \omega_0, p)}\) is measurable. By Lemma 3.3, \(A\) has a measurable selection \(h : T \rightarrow \mathbb{R}^\ell\) satisfying

\[
\|g - h\|_{L_1} \leq \int_T \xi^g(\cdot, \omega_0) d\mu + \varepsilon \mu(T).
\]

As \(h \in H_p(\omega_0)\), we have

\[
\zeta(g, \omega_0) \leq \int_T \xi^g(\cdot, \omega_0) d\mu + \varepsilon \mu(T),
\]

which is a contradiction. \(\Box\)

**Claim 2.** For each function \(g \in L^1(\mu, \mathbb{R}^\ell)\), \(\zeta(g, \omega) = \int_T \xi^g(\cdot, \omega) d\mu\) holds for all \(\omega \in \Omega\), and thus \(H_p\) is weakly \(\mathcal{F}\)-measurable.

**Proof of Claim 2.** Let \(\{g_n : n \geq 1\}\) be a sequence of simple measurable functions converging to \(g\) in \(L^1(\mu, \mathbb{R}^\ell)\). By [1] Theorem 13.6, there is a subsequence \(\{g_{n_k} : k \geq 1\}\) of \(\{g_n : n \geq 1\}\) and a function \(h \in L^1(\mu, \mathbb{R}^\ell)\) such that \(|g_{n_k}| \leq h\) for all \(k \geq 1\) and \(\{g_{n_k} : k \geq 1\}\) converges pointwise to \(g\). Thus, \(\{\xi^{g_{n_k}}(t, \omega) : k \geq 1\}\) converges to \(\xi^g(t, \omega)\) for all \((t, \omega) \in T \times \Omega\). As

\[
\xi^{g_{n_k}}(t, \omega) \leq \|h(t) + b(t, \omega, p)\|
\]

we have \(\{\xi^{g_{n_k}}(\cdot, \omega) : k \geq 1\}\) is dominated by the integrable function \(h + b(\cdot, \omega, p)\). Hence, by the Lebesgue dominated convergence theorem, we have

\[
\lim_{k \to \infty} \int_T \xi^{g_{n_k}}(\cdot, \omega) d\mu = \int_T \xi^g(\cdot, \omega) d\mu
\]

for all \(\omega \in \Omega\). On the other hand, \(\{\zeta(g_{n_k}, \omega) : k \geq 1\}\) converges to \(\zeta(g, \omega)\) for every \(\omega \in \Omega\). Thus, we have

\[
\zeta(g, \omega) = \int_T \xi^g(\cdot, \omega) d\mu
\]

for all \(\omega \in \Omega\). Moreover, since each \(\zeta(g_{n_k}, \cdot)\) is \(\mathcal{F}\)-measurable, we have \(\zeta(g, \cdot)\) is \(\mathcal{F}\)-measurable. Therefore, \(H_p\) is weakly \(\mathcal{F}\)-measurable. \(\Box\)
Define a correspondence $H : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow L_1(\mu, \mathbb{R}^t) \times \Delta$ by letting

$$H(\omega) = \bigcup \{H_p(\omega) \times \{p\} : p \in R\}$$

for all $\omega \in \Omega$, where the closure operation is taken in the product topology on $L_1(\mu, \mathbb{R}^t) \times \Delta$ induced by the norm of $L_1(\mu, \mathbb{R}^t)$ and the norm of $\mathbb{R}^t$.

**Claim 3.** $\text{WE}(\omega) = H(\omega) \cap G(\omega)$ for all $\omega \in \Omega$.

**Proof of Claim 3.** Fix some $\omega \in \Omega$. Let $(f, p) \in \text{WE}(\omega)$. Clearly, $(f, p) \in G(\omega)$. By (A₃), we have $p \in \Delta_+$. It follows that $f(t) \in C^X(t, \omega, p)$, $\mu$-a.e. on $T$. Now, suppose that $\{p_n : n \geq 1\} \subseteq R$ is a sequence converging to $p$. By Claim 2,

$$\text{dist}(f, H_{p_n}(\omega)) = \int_T \eta_n d\mu,$$

where $\eta_n : T \rightarrow \mathbb{R}$ is defined by $\eta_n(t) = \text{dist}(f(t), C^X(t, \omega, p_n))$. By Lemma 2.4,

$$C^X(t, \omega, p) \subseteq \text{Li} C^X(t, \omega, p_n)$$

for all $t \in T$. Hence, for each $t \in T$ and each $n \geq 1$, we can choose some $f_n(t) \in C^X(t, \omega, p_n)$ such that $f_n(t) \rightarrow f(t)$, $\mu$-a.e on $T$. It follows that $\eta_n(t) \rightarrow 0$, $\mu$-a.e on $T$. Define

$$\beta = \inf \{(\delta(p_n) : n \geq 1) \cup \{\delta(p)\}\}.$$

Then $\beta > 0$ and for each $t \in T$, let

$$d(t) = \frac{1}{\beta} \sum_{h=1}^t a_h(t, \omega).$$

Note that

$$\eta_n(t) \leq \|f_n(t) - f(t)\| \leq 2\|d(t)\|,$$

$\mu$-a.e on $T$ for all $n \geq 1$. By the Lebesgue dominated convergence theorem again, we have

$$\text{dist}(f, H_{p_n}(\omega)) = \int_T \eta_n d\mu \rightarrow 0.$$

It follows that $(f, p) \in H(\omega)$.

Let $(f, p) \in H(\omega) \cap G(\omega)$ for an arbitrarily fixed $\omega \in \Omega$. Analogous to the proof of Claim 2, we can find a sequence $\{r_n : n \geq 1\} \subseteq R$ and $f_n \in H_{r_n}(\omega)$ such that $(f_n, r_n) \rightarrow (f, p)$ in $L_1(\mu, \mathbb{R}^t) \times \Delta$, as $n \rightarrow \infty$ and $\{f_n : n \geq 1\}$ pointwise converges to $f$. Since $p \in \Delta$, by (A₃), we must have $(p, \int_T a(\cdot, \omega) d\mu) > 0$. Put,

$$S = \{t \in T : \langle p, a(t, \omega) \rangle > 0\}.$$

Definitely, $S \in \Sigma$ and $\mu(S) > 0$. Define

$$A_n = \{t \in S : f_n(t) \notin C^X(t, \omega, r_n)\} \text{ and } A = \bigcup A_n.$$

Since $\mu(A_n) = 0$ for all $n \geq 1$, one must have $\mu(A) = 0$. Choose a $t \in S \setminus A$. If $f(t) \notin C(t, \omega, p)$ for some $t \in S \setminus A$, by (A₃), there must exist an element $y \in \mathbb{R}^t_+$ such that $\langle p, y \rangle < \langle p, a(t, \omega) \rangle$ and $U(t, \omega, y) > U(t, \omega, f(t))$. As a result, $\langle r_n, y \rangle < \langle r_n, a(t, \omega) \rangle$ and $U(t, \omega, y) > U(t, \omega, f_n(t))$ for sufficiently large $n$, which is a contradiction. Thus, $f(t) \in C(t, \omega, p)$ for all $t \in S \setminus A$. Since $U(t, \omega, \cdot)$ is strictly increasing, $\langle p, f(t) \rangle \geq \langle p, a(t, \omega) \rangle$ $\mu$-a.e. on $S$. Moreover, $\langle p, f(t) \rangle \geq 0 = \langle p, a(t, \omega) \rangle$ for all $t \in T \setminus S$. Hence, $\langle p, f(t) \rangle \geq \langle p, a(t, \omega) \rangle$ $\mu$-a.e. on $T$, which together with the feasibility of $f$ implies that $f(t) \in B(t, \omega, p)$ $\mu$-a.e. on $T$. If $\mu(T \setminus S) = 0$, then
\[(f, p) \in \mathbf{WE}(\omega).\] Otherwise, we first claim that \(p \in \Delta_+.\) If not, there is some \(z > 0\) such that \(\langle p, z \rangle = 0.\) Consequently, \(f(t) + z \in B(t, \omega, p)\) and
\[
U(t, \omega, f(t) + z) > U(t, \omega, f(t))
\]
for all \(t \in S \setminus A,\) which is a contradiction. So, \(B(t, \omega, p) = \{0\}\) and \(f(t) = 0\) for \(\mu\)-a.e. on \(T \setminus S.\) Thus, \((f, p) \in \mathbf{WE}(\omega).\)

By Claim 2, \(\text{Gr}_{\mathbf{WE}} = \text{Gr}_H \cap \text{Gr}_G.\) Since both \(\text{Gr}_H\) and \(\text{Gr}_G\) are measurable, \(\text{Gr}_{\mathbf{WE}}\) is measurable. Hence, \(\mathbf{WE}\) is weakly \(\mathcal{F}\)-measurable.

Now, we are ready to provide a proof of Theorem 3.2, as promised previously.

Proof of Theorem 3.2 First of all, for the sake of convenience, we put
\[
X = \{f : f \text{ is an allocation and } f(\cdot, \omega) \in \mathbf{C}(\mathcal{E}(\omega)) \text{ for all } \omega \in \Omega\}.
\]

It is easy to see that \(\mathbf{WE}\) is non-empty closed-valued. Then, following Theorem 3.1 \(\mathbf{WE}\) is weakly \(\Sigma\)-measurable. By Lemma 3.3 \(\mathbf{WE}\) has a measurable selection \(\omega \mapsto \{f(\cdot, \omega), \pi(\omega)\}\). Note that \(\pi(\omega) \in \Delta_+\) for all \(\omega \in \Omega.\) Under assumption (A1), \(B(t, \omega, \pi(\omega)) \cap \mathcal{C}_F(t, \omega, \pi(\omega))\) is singleton for all \((t, \omega) \in T \times \Omega.\) Let \(g : T \times \Omega \to \mathbb{R}_+^\ell\) be the function defined by
\[
g(t, \omega) = B(t, \omega, \pi(\omega)) \cap \mathcal{C}_F(t, \omega, \pi(\omega))
\]
for all \((t, \omega) \in T \times \Omega.\) By Proposition 2.2 and Proposition 2.3, \(g\) is \(\Sigma \otimes \mathcal{F}\)-measurable. Hence, \(g(\cdot, \cdot)\) is \(\mathcal{F}\)-measurable for all \(t \in T.\) For all \(\omega \in \Omega, g(\cdot, \omega) = g(\cdot, \omega)\mu\)-a.e., which implies that \((g(\cdot, \omega), \pi(\omega)) \in \mathbf{WE}(\omega)\) for all \(\omega \in \Omega.\) It follows that \(g(\cdot, \omega) \in \mathbf{C}(\mathcal{E}(\omega))\) for all \(\omega \in \Omega.\) Hence, \(X \neq \emptyset.\) Then, it is straightforward to see that \(X \subseteq \mathbf{C}(\mathcal{E}).\)

To show that \(\mathbf{C}(\mathcal{E}) \subseteq X,\) we suppose that there exists an \(f \in \mathbf{C}(\mathcal{E}) \setminus X.\) Then there exists a state \(\omega_0 \in \Omega\) such that \(f(\cdot, \omega_0) \notin \mathbf{C}(\mathcal{E}(\omega_0)).\) This means that \(f\) is blocked in \(\mathcal{E}(\omega_0)\) by some coalition \(S.\) Therefore, there exists an assignment \(g : T \to \mathbb{R}_+^\ell\) in \(\mathcal{E}(\omega_0)\) such that
\[(i) \int_S g d\mu = \int_S a(\cdot, \omega_0) d\mu, \text{ and}
(ii) U(t, \omega_0, g(t)) > U(t, \omega_0, f(t, \omega_0)) \mu\text{-a.e. on } S.
\]

Define
\[
\Omega_0 = \big\{\omega \in \Omega : \int_S g d\mu = \int_S a(\cdot, \omega) d\mu\big\}.
\]

Obviously, \(\omega_0 \in \Omega_0\) and \(\Omega_0 \in \mathcal{F}.\) Define a function \(h : T \times \Omega \to \mathbb{R}_+^\ell\) by
\[
h(t, \omega) = \begin{cases} g(t), & \text{if } (t, \omega) \in S \times \Omega_0; \\ a(t, \omega), & \text{otherwise}. \end{cases}
\]

Then, it can be readily checked that \(h\) is an assignment in \(\mathcal{E}\) such that
\[(iii) \int_S h(\cdot, \omega_0) d\mu = \int_S a(\cdot, \omega_0) d\mu, \text{ and}
(iv) U(t, \omega_0, h(t, \omega_0)) > U(t, \omega_0, f(t, \omega_0)) \mu\text{-a.e. on } S.
\]

This means that \(f\) is ex-post blocked by \(S\) (via an assignment \(h\) at the state \(\omega_0),\) which contradicts with the fact of \(f \in \mathbf{C}(\mathcal{E}).\) \qed
Next, we discuss the consequences of Theorems 3.2 and 3.4. We need to introduce two competitive equilibrium concepts in the economy model $\mathcal{E}$ discussed in Subsection 2.1: maximin rational expectations equilibrium and Bayesian rational expectations equilibrium. Given an agent $t \in T$, a state of nature $\omega \in \Omega$ and a price system $\pi : \Omega \to \Delta$, let $B_{REE}(t, \omega, \pi)$ be defined by

$$B_{REE}(t, \omega, \pi) = \{ x \in (\mathbb{R}_{+}^L)^\Omega : x(\omega') \in B(t, \omega', \pi(\omega')) \text{ for all } \omega' \in \mathcal{G}_t(\omega) \}.$$  

The maximin utility of each agent $t \in T$ with respect to $\mathcal{G}_t$ at $x : \Omega \to \mathbb{R}_{+}^L$ in state $\omega \in \Omega$, denoted by $U_{REE}(t, \omega, x)$, is defined by

$$U_{REE}(t, \omega, x) = \inf \{ U(t, \omega', x(\omega')) : \omega' \in \mathcal{G}_t(\omega) \}.$$  

**Definition 3.5** ([12]). Given an allocation $f$ and a price system $\pi$ in an economy $\mathcal{E}$, the pair $(f, \pi)$ is called a maximin rational expectations equilibrium (abbreviated as maximin REE) of $\mathcal{E}$ if $f(t, \omega) \in B(t, \omega, \pi(\omega))$ and $f(t, \cdot)$ maximizes $U_{REE}(t, \omega, \cdot)$ on $B_{REE}(t, \omega, \pi)$ for all $(t, \omega) \in T \times \Omega$. In this case, $f$ is called a maximin rational expectations allocation, and the set of such allocations is denoted by $MREE(\mathcal{E})$.

Define $L_{REE}^t$ by

$$L_{REE}^t = \{ x \in (\mathbb{R}_{+}^L)^\Omega : x \text{ is } \mathcal{G}_t\text{-measurable} \}.$$  

For a given $x \in L_{REE}^t$, recall that the Bayesian expected utility of agent $t$ with respect to $\mathcal{G}_t$ at $x$ is given by $E_t[U(t, \cdot, x)|\mathcal{G}_t]$.

**Definition 3.6** ([2 ([23])]. Given an allocation $f$ and a price system $\pi$ in an economy $\mathcal{E}$, the pair $(f, \pi)$ is called a Bayesian rational expectations equilibrium (abbreviated as Bayesian REE) of $\mathcal{E}$ if

(i) for each $t \in T$, $f(t, \cdot)$ is $\mathcal{G}_t$-measurable;
(ii) for all $(t, \omega) \in T \times \Omega$, $f(t, \omega) \in B(t, \omega, \pi(\omega))$;
(iii) for all $(t, \omega) \in T \times \Omega$,

$$E_t[U(t, \cdot, f(t, \cdot))|\mathcal{G}_t](\omega) = \max_{x \in B_{REE}(t, \omega, \pi)} E_t[U(t, \cdot, x)|\mathcal{G}_t](\omega).$$  

In this case, $f$ is called a rational expectations allocation, and the set of such allocations is denoted by $REE(\mathcal{E})$.

As a corollary of Theorem 3.4, we can retrieve the result on the existence of a maximin REE or Bayesian REE obtained in [11].

**Proposition 3.7.** If an economy $\mathcal{E}$ satisfies assumptions $(A_1)$-$(A_4)$, then we have $MREE(\mathcal{E}) = REE(\mathcal{E}) \neq \emptyset$.

**Proof.** Note that under the assumption $WE(\omega) \neq \emptyset$ for all $\omega \in \Omega$. From the proof of Theorem 3.4, we can see that $WE$ is closed-valued and weakly $\Sigma$-measurable, thus it has a $\mathcal{F}$-measurable selection $\omega \mapsto (f(\omega), \pi(\omega))$. Then, it is easy to verify that the pair $(g, \pi)$ defined in the proof of Theorem 3.2 is a maximin rational expectations equilibrium of $\mathcal{E}$. \(\square\)

As a consequence of Theorem 3.2 and [11, Corollary 1], we deduce the following version of the first fundamental theorem of social welfare for our model.

**Corollary 3.8.** Suppose that an economy $\mathcal{E}$ satisfies $(A_1)$-$(A_4)$. Then, we have $MREE(\mathcal{E}) = REE(\mathcal{E}) \subseteq C(\mathcal{E})$. 

Next, we provide an example of a continuum economy \( \mathcal{E} \) with asymmetric information and infinitely many states of nature in which the ex-post core \( \mathbf{C}(\mathcal{E}) \) can strictly contain \( \text{REE}(\mathcal{E}) \).

**Example 3.9.** Consider an economy \( \mathcal{E} \) defined by

\[
\mathcal{E} = \{(\Omega, \mathcal{F}, \mathbb{P}), (T, \Sigma, \mu) ; \mathbb{R}_+^T ; (\mathcal{F}_t, U(t, \cdot, \cdot), a(t, \cdot), \mathbb{P}_t)_{t \in T}\},
\]

where \( T = \Omega = [0, 1] \), \( \Sigma \) and \( \mathcal{F} \) are the Borel \( \sigma \)-algebra on \([0, 1]\), \( \mu \) and \( \mathbb{P} \) are the Lebesgue probability measure. The commodity space is \( \mathbb{R}^2 \). Let \( \mathcal{F}_t \) and \( \mathbb{P}_t \) be arbitrary information partition and the prior belief of agent \( t \in T \). The utility and the initial endowment of each agent are given by \( U(t, \omega, x) = \sqrt{x_1} + \sqrt{x_2} \) and

\[
a(t, \omega) = \begin{cases} (1, 2), & \text{if } (t, \omega) \in [0, \frac{1}{2}] \times \Omega; \\ (3, 2), & \text{if } (t, \omega) \in \left(\frac{1}{2}, 1 \right] \times \Omega,
\end{cases}
\]

respectively. Then \( \mathcal{E}(\omega) = \mathcal{E}(\omega') \) for all \( \omega, \omega' \in \Omega \). Furthermore, it can be easily checked that \((A_1)-(A_4)\) are satisfied.

For every \( p = (p_1, p_2) \in \Delta_+ \), the demand of agent \( t \) in each state is given by

\[
D(t, \omega, p) = \left\{ \begin{array}{ll}
\left( \frac{p_2(1+p_2)}{p_1}, \frac{p_1(1+p_2)}{p_2} \right), & \text{if } (t, \omega) \in [0, \frac{1}{2}] \times \Omega; \\
\left( \frac{p_2(2+p_2)}{p_1}, \frac{p_1(2+p_2)}{p_2} \right), & \text{if } (t, \omega) \in \left(\frac{1}{2}, 1 \right] \times \Omega.
\end{array} \right.
\]

By the market-clearing condition, we can show that the equilibrium price is \( p_0 = \left(\frac{1}{2}, \frac{1}{2}\right) \). Thus, \( D(t, \omega, p_0) = \left(\frac{5}{4}, \frac{5}{4}\right) \) for all \( (t, \omega) \in [0, \frac{1}{2}] \times \Omega \) and \( D(t, \omega, p_0) = \left(\frac{5}{4}, \frac{5}{4}\right) \) for all \( (t, \omega) \in \left(\frac{1}{2}, 1 \right] \times \Omega \). For each \( \omega \in \Omega \), let \( h : T \to \mathbb{R}_+^2 \) be an allocation in \( \mathcal{E}(\omega) \) defined by

\[
h(t) = \left\{ \begin{array}{ll}
\left( \frac{3}{2}, \frac{3}{2} \right), & \text{if } t \in [0, \frac{1}{2}]; \\
\left( \frac{5}{2}, \frac{5}{2} \right), & \text{if } t \in \left(\frac{1}{2}, 1 \right].
\end{array} \right.
\]

Take a subset \( A \) of \( \Omega \) with \( A \notin \mathcal{F} \), and consider \( f : T \times \Omega \to \mathbb{R}_+^2 \), defined by

\[
f(t, \omega) = \left\{ \begin{array}{ll}
(1, 1), & \text{if } t \in A \text{ and } t = \omega; \\
h(t), & \text{otherwise}.
\end{array} \right.
\]

(i) It is clear that \( f \) is feasible.

(ii) For each \( t \in T \), \( f(t, \cdot) \) is \( \mathcal{F} \)-measurable. To see this, we first choose \( t \in A \). In this case, \( f(t, \omega) = h(t) \) for all \( \omega \in \Omega \) with \( \omega \neq t \); and \( f(t, \omega) = (1, 1) \) if \( \omega = t \). Now, take \( t \in T \setminus A \), then \( f(t, \omega) = h(t) \) for all \( \omega \in \Omega \). Thus, in both cases, \( f(t, \cdot) \) is \( \mathcal{F} \)-measurable.

(iii) It is clear that for each \( \omega \in \Omega \), \( f(\cdot, \omega) \) is \( \mu \)-integrable.

Since \( (f(\cdot, \omega), p_0) \in \text{WE}(\omega) \) for each \( \omega \in \Omega \), we conclude that \( f \in \mathbf{C}(\mathcal{E}(\omega)) \) and thus \( f \in \mathbf{C}(\mathcal{E}) \). Now, consider two mappings \( g_1 : t \mapsto (t,t) \) and \( g_2 : (t,t) \mapsto t \) defined by \( g_1(t) = (t,t) \) and \( g_2(t,t) = t \), respectively. It can be readily checked that

\[
(g_2 \circ f \circ g_1)(t) = \begin{cases} 1, & \text{if } t \in A; \\
e(t), & \text{otherwise},
\end{cases}
\]

where \( e(t) = \frac{3}{4} \) if \( t \in [0, \frac{1}{2}] \); and \( e(t) = \frac{5}{4} \) if \( t \in \left(\frac{1}{2}, 1 \right] \). Since \( A \notin \mathcal{F} \), then \( g_2 \circ f \circ g_1 \) is not \( \mathcal{F} \)-measurable. It follows that \( f \) is not \( \Sigma \otimes \mathcal{F} \)-measurable. By Corollary 1 in \([11]\), \( \text{REE}(\mathcal{E}) \neq \emptyset \) and \( f \notin \text{REE}(\mathcal{E}) \).
4. The Ex-post Core and the Fine Core

In this section, we study the relationship between the ex-post core and the fine core in a mixed economy with asymmetric information. We show that under appropriate assumptions, the fine core is contained in the ex-post core (see Theorem 4.6). This extends a result of Einy et al. in [14]. To achieve this goal, we use a standard approach, which embeds the original mixed economy $\mathcal{E}$ into the auxiliary atomless economy $\mathcal{E}^*$ obtained by splitting each large agent into a continuum of small agents of the same type.

4.1. Interpretation via associated continuum economies. We define an atomless economy $\mathcal{E}^*$ associated with $\mathcal{E}$. Let $(T_1^*, \Sigma_1^*, \mu_1^*)$ be an atomless, complete and positive measure space such that $T_0 \cap T_1^* = \emptyset$, where each agent $A_n$ one-to-one corresponds to a measurable subset $A_n^*$ of $T_1^*$ with $\mu^*(A_n^*) = \mu(A_n)$ and $T_1^* = \bigcup\{A_n^*: n \geq 1\}$. One can think that $T_1^*$ is constructed as follows: Partition the interval $[\mu(T_0), \mu(T)]$, which is identified with $T_1^*$, as the disjoint union of the intervals $A_n^*$ given by $A_n^* = [\mu(T_0), \mu(T_0) + \mu(A_1)], \ldots, and$

$$A_n^* = \left[\mu(T_0) + \mu\left(\bigcup_{i=1}^{n-1} A_i\right), \mu(T_0) + \mu\left(\bigcup_{i=1}^{n} A_i\right)\right], \ldots.$$

Define $T^* = T_0 \cup T_1^*$ with the $\sigma$-algebra

$$\Sigma^* = \Sigma_{T_0} \oplus \Sigma_{T_1}^* = \{A \cup B : A \cap B = \emptyset, A \in \Sigma_{T_0}, B \in \Sigma_{T_1}^*\}$$

and the measure $\mu^* : \Sigma^* \to \mathbb{R}_+$ such that for each $C \in \Sigma^*$,

$$\mu^*(C) = \mu_{T_0}(C \cap T_0) + \mu_{T_1}^*(C \cap T_1^*).$$

Following [9, 22], the space of agents of $\mathcal{E}^*$ is $(T^*, \Sigma^*, \mu^*)$. In addition, in $\mathcal{E}^*$, the space of states of nature and the consumption set for each agent $t \in T^*$ at each state $\omega \in \Omega$ are still $(\Omega, \mathcal{F}, \mathcal{P})$ and $\mathbb{R}_+$, respectively. Finally, the characteristics $(\mathcal{F}^*_t, U^*(t, \cdot, \cdot), a^*(t, \cdot, \cdot), \mathcal{P}^*_t)$ of each agent $t \in T^*$ in $\mathcal{E}^*$ are defined as follows:

$$\mathcal{F}^*_t = \left\{\begin{array}{ll}
\mathcal{F}_t, & \text{if } t \in T_0; \\
\mathcal{F}_{A_n^*}, & \text{if } t \in A_n^*;
\end{array}\right.$$ 

$$U^*(t, \omega, \cdot) = \left\{\begin{array}{ll}
U(t, \omega, \cdot), & \text{if } (t, \omega) \in T_0 \times \Omega; \\
U(A_n, \omega, \cdot), & \text{if } (t, \omega) \in A_n^* \times \Omega,
\end{array}\right.$$ 

$$a^*(t, \omega) = \left\{\begin{array}{ll}
a(t, \omega), & \text{if } (t, \omega) \in T_0 \times \Omega; \\
a(A_n, \omega), & \text{if } (t, \omega) \in A_n^* \times \Omega,
\end{array}\right.$$ 

and

$$\mathcal{P}^*_t = \left\{\begin{array}{ll}
\mathcal{P}_t, & \text{if } t \in T_0; \\
\mathcal{P}_{A_n^*}, & \text{if } t \in A_n^*.
\end{array}\right.$$ 

For each $\omega \in \Omega$, we can define an atomless and deterministic economy $\mathcal{E}^*(\omega)$ associated with $\mathcal{E}(\omega)$ as

$$\mathcal{E}^*(\omega) = \{(T^*, \Sigma^*, \mu^*); \mathbb{R}_+^t; (U^*(t, \omega, \cdot), a^*(t, \omega))_{t \in T^*}\}.$$

Similar to that of $\mathcal{E}$, we call a member $S$ of $\Sigma^*$ with $\mu^*(S) > 0$ a coalition of $\mathcal{E}^*$. Given a coalition $S \in \Sigma^*$ of $\mathcal{E}^*$, let $\Sigma^*_S = \{A \in \Sigma^* : A \subseteq S\}$.

The following lemma is a particular case of [9, Lemma 3.6].
Lemma 4.1 ([9]). Given $\omega \in \Omega$, if $f \in L_1(\mu^*, \mathbb{R}^\ell)$ and $S, R$ are two coalitions of $\mathcal{E}^*(\omega)$ such that $\mu^*(S \cap R) > 0$, then

$$H = \text{cl}\left\{ \mu^*(B), \int_B f d\mu^* : B \in \Sigma_S^* \right\}$$

is a convex subset of $\mathbb{R} \times \mathbb{R}^\ell$. Moreover, for any $0 < \delta < 1$, there is a sequence $\{C_n : n \geq 1\} \subseteq \Sigma_S^*$ of coalitions in $\mathcal{E}^*$ such that $\mu^*(C_n \cap R) = \delta \mu^*(S \cap R)$ for all $n \geq 1$ and

$$\lim_{n \to \infty} \int_{C_n} f(\cdot, \omega)d\mu^* = \delta \int_S f(\cdot, \omega)d\mu^*.$$

Lemma 4.2. Assume that an economy $\mathcal{E}$ satisfies $(A'_1), (A_2)$ and $(A_3)$. Let $\omega \in \Omega$ be a state of nature. If an allocation $f^*$ in $\mathcal{E}^*(\omega)$ is blocked by a coalition $S \subseteq T^*$, then for any $0 < \varepsilon \leq \mu^*(S \cap T_1^*)$, there exist a coalition $R^*$ such that

$$R^* \subseteq \bigcup \{ S^i : i \in \mathcal{P}(S) \},$$

$$\mu^*(R^* \cap T_1^*) = \varepsilon \text{ and } \mu^*(R^* \cap S^i) > 0 \text{ for all } i \in \mathcal{P}(S),$$

and an allocation $g^*$ in $\mathcal{E}^*(\omega)$ such that $f^*$ is blocked by $R^*$ via $g^*$ in $\mathcal{E}^*(\omega)$.

Proof. If $\varepsilon = \mu^*(S \cap T_1^*)$, there is nothing to prove. So, let $0 < \varepsilon < \mu^*(S \cap T_1^*)$. By the techniques in [9 Lemma 3.5], we can find a function $h^* : T^* \to \mathbb{R}^\ell$ such that

$$U^*(t, \omega, h^*(t)) > U^*(t, \omega, f^*(t)), \mu\text{-a.e on } S$$

and

$$\int_S (a^*(\cdot, \omega) - h^*(\cdot))d\mu^* \gg 0.$$
4.2. The ex-post core and the fine core. In this subsection, we will present and prove our main result of this section. We assume that our economy $\mathcal{E}$ only admits finitely many information structures. More precisely, we assume that each agent’s information partition is a member of $\{\mathcal{D}_1, \cdots, \mathcal{D}_n\}$. For any $1 \leq i \leq n$ and any coalition $S$, let $S^i = \{t \in S : \Pi_t = \mathcal{D}_i\}$ and
\[
\mathcal{P}(S) = \{i : \mu(S^i) > 0, 1 \leq i \leq n\}.
\]

The symbol $\bigvee\{\mathcal{D}_i : i \in \mathcal{P}(S)\}$ is used to denote the $\sigma$-algebra on $\Omega$, which is generated by the common refinement of members of $\{\mathcal{D}_i : i \in \mathcal{P}(S)\}$. We will need the following two additional assumptions.

$\text{(A}_5\text{)}$ $\mu(T^i) > 0$ for all $1 \leq i \leq n$, $T = \bigcup\{T^i : 1 \leq i \leq n\}$ and $\bigvee_{i=1}^n \mathcal{D}_i = \mathcal{F}$.

$\text{(A}_6\text{)}$ All large agents in $\mathcal{E}$ are of the same type, i.e., having the same characteristics.

Following [26], an information structure for a coalition $S$ in an economy $\mathcal{E}$ is a family $\{\mathcal{G}_t : t \in S\}$ of $\sigma$-algebras on $\Omega$ such that $\mathcal{G}_t \subseteq \mathcal{F}$ for all $t \in S$ and $\{t \in S : \mathcal{G}_t = \mathcal{F}\} \subseteq \Sigma$ for any $\sigma$-algebra $\mathcal{F}$ on $\Omega$ with $\mathcal{G} \subseteq \mathcal{F}$. A communication system for a coalition $S$ is an information structure $\{\mathcal{G}_t : t \in S\}$ for $S$ such that
\[
\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \bigvee\{\mathcal{D}_i : i \in \mathcal{P}(S)\}, \mu\text{-a.e. on } S.
\]

A communication system $\{\mathcal{G}_t : t \in S\}$ for a coalition $S$ is called a full communication system if $\mathcal{G}_t = \bigvee\{\mathcal{D}_i : i \in \mathcal{P}(S)\}$, $\mu$-a.e. on $S$.

**Definition 4.3** ([26]). An allocation $f$ in $\mathcal{E}$ is said to be fine blocked by a coalition $S$ in an economy $\mathcal{E}$ if there are an allocation $g$ in $\mathcal{E}$, a communication system $\{\mathcal{G}_t : t \in S\}$ for $S$ and a nonempty event $\Omega_0 \in \bigcap_{t \in S} \mathcal{G}_t$ such that for all $\omega \in \Omega_0$,
\[
\int_S g(\cdot, \omega)d\mu = \int_S a(\cdot, \omega)d\mu
\]
and
\[
\mathbb{E}_t[U(t, \cdot, g(t, \cdot))|\mathcal{G}_t](\omega) > \mathbb{E}_t[U(t, \cdot, f(t, \cdot))|\mathcal{G}_t](\omega), \mu\text{-a.e. on } S.
\]

The fine core of $\mathcal{E}$, denoted by $\mathcal{C}^{\text{fine}}(\mathcal{E})$, is the set of all allocations that cannot be fine blocked by any coalition in $\mathcal{E}$.

Let $\mathcal{E} \neq \emptyset$. For any allocation $f$ in $\mathcal{E}$, let $\bar{f} : T \times \Omega \to \mathbb{R}^+_1$ be an allocation in $\mathcal{E}$ defined by
\[
\bar{f}(t, \omega) = \begin{cases} 
    f(t, \omega), & \text{if } (t, \omega) \in T_0 \times \Omega; \\
    \frac{1}{\mu(T_1)} \int_{T_1} f(\cdot, \omega)d\mu, & \text{if } (t, \omega) \in T_1 \times \Omega.
\end{cases}
\]

**Theorem 4.4.** Assume that an economy $\mathcal{E}$ satisfies $(A_1')$, $(A_2)$-(A$_3$), $(A_4')$ and (A$_5$)-(A$_6$). If $f \in \mathcal{C}^{\text{fine}}(\mathcal{E})$ and $\mathcal{E} \neq \emptyset$, then
\[
U(t, \omega, \bar{f}(t, \omega)) = U(t, \omega, f(t, \omega)), \mu\text{-a.e. on } T
\]
and for all $\omega \in \Omega$.

**Proof.** Firstly, we show that for all $(t, \omega) \in T_1 \times \Omega,$
\[
U(t, \omega, f(t, \omega)) \geq U(t, \omega, \bar{f}(t, \omega)).
\]
Suppose the contrary. There exist a state $\omega_0 \in \Omega$ and a coalition $S \subseteq T_1$ such that
\[
U(t, \omega_0, \bar{f}(t, \omega_0)) > U(t, \omega_0, f(t, \omega_0))
\]
for all $t \in S$. For a sequence $\{r_m : m \geq 1\} \subseteq (0, 1)$ converging to 1, the function
$$\zeta_m : S \to \mathbb{R}_+^t,$$
defined by
$$\zeta_m(t) = U(t, \omega_0, r_m \tilde{f}(t, \omega_0)) - U(t, \omega_0, f(t, \omega_0)),$$
is $\Sigma_S$-measurable. For each $m \geq 1$, put
$$S_m = \{t \in S : \zeta_m(t) > 0\}.$$
As $S = \bigcup_{m \geq 1} S_m$, then $\mu(S_{m_0}) > 0$ for some $m_0 \geq 1$. Put
$$z_0 = -\left(1 - \frac{r_{m_0}\mu(S_{m_0})}{\mu(T_1)}\right) \int_{T_1} a(\cdot, \omega_0)d\mu.$$
Choose an $\varepsilon > 0$ with $z_0 + B(0, 2\varepsilon) \subseteq -\mathbb{R}_+^t$. For each $i \in \mathcal{P}(T_0)$ and $R \in \Sigma_{T_0^i}$, let
$$b_i(R) = \int_R (f(\cdot, \omega_0) - a(\cdot, \omega_0))d\mu - \frac{r_{m_0}\mu(S_{m_0})}{\mu(T_1)} \int_{T_0^i} (f(\cdot, \omega_0) - a(\cdot, \omega_0))d\mu.$$
Applying Lemma 4.1 with $\delta = \frac{r_{m_0}\mu(S_{m_0})}{\mu(T_1)}$, we can get a coalition $R_i$ in $\mathcal{C}$ with $R_i \subseteq T_0^i$ such that $b_i(R_i) \in B \left(0, \frac{r_{m_0}\mu(S_{m_0})}{\mu(T_1)}\right)$. Put
$$R_0 = \bigcup \{R_i : i \in \mathcal{P}(T_0)\}.$$
Let $E = R_0 \cup S_{m_0}$. Then, by (A_5) and (A_6), we have
$$\bigwedge \{Q_t : i \in \mathcal{P}(E)\} = \mathcal{F}.$$
Pick an $x \in B(0, \varepsilon) \cap \mathbb{R}_+^t$ and define $g : T \to \mathbb{R}_+^t$ by
$$g(t) = \begin{cases} f(t, \omega_0) + \frac{r_{m_0}\mu(S_{m_0})}{\mu(T_1)} & \text{if } t \in R_0; \\ r_{m_0}\tilde{f}(t, \omega_0) & \text{if } t \in S_{m_0}; \\ f(t, \omega_0) & \text{otherwise.} \end{cases}$$
Then,
$$U(t, \omega_0, g(t)) > U(t, \omega_0, f(t, \omega_0)), \text{ } \mu\text{-a.e. on } E.$$
Furthermore,
$$\int_E g d\mu = \int_{R_0} f(\cdot, \omega_0)d\mu + \frac{r_{m_0}\mu(S_{m_0})}{\mu(T_1)} \int_{T_1} f(\cdot, \omega_0)d\mu + x.$$
Using the fact that
$$\int_{T_1} (f(\cdot, \omega_0) - a(\cdot, \omega_0))d\mu = -\int_{T_0} (f(\cdot, \omega_0) - a(\cdot, \omega_0))d\mu,$$
we can easily verify that for all $\omega \in \Omega$,
$$-z_0 + \int_E (g(\cdot) - a(\cdot, \omega_0))d\mu = \sum_{i \in \mathcal{P}(T_0)} b_i(R_i) + x \in B(0, 2\varepsilon).$$
It follows that
$$d = \int_E a(\cdot, \omega_0) - \int_E g d\mu \gg 0.$$
Then, the function $h : E \to \mathbb{R}_+^t$ defined by $h(t) = g(t) + \frac{d}{\mu(E)}$ for all $t \in T$, satisfies
$$U(t, \omega_0, h(t)) > U(t, \omega_0, f(t, \omega_0)), \text{ } \mu\text{-a.e. on } E.
Define
\[ \Omega_0 = \bigcap \{ \mathcal{D}_i(\omega_0) : i \in \mathcal{P}(E) \}, \]
where \( \mathcal{D}_i(\omega_0) \) is the atom in \( \mathcal{D}_i \) containing \( \omega_0 \). Note that the set
\[ A_t = \{ \omega \in \Omega : U(t, \omega, h(t)) > U(t, \omega, f(t, \omega)) \} \]
is \( \mathcal{F} \)-measurable and \( \omega_0 \in A_t \) for all \( t \in E \). Consequently, by (A5) and (A6), \( \Omega_0 \subseteq A_t \) for all \( t \in E \). Since the map \( \omega \mapsto \int_E a(\cdot, \omega) \mu \) is \( \mathcal{F} \)-measurable, we have
\[ \Omega_0 \subseteq \left\{ \omega \in \Omega : \int_E h \mu = \int_E a(\cdot, \omega) \mu \right\}. \]
Define another function \( y : T \times \Omega \to \mathbb{R}_+^l \) by
\[ y(t, \omega) = \begin{cases} h(t), & \text{if } (t, \omega) \in E \times \Omega_0; \\ a(t, \omega), & \text{otherwise}. \end{cases} \]
Note that \( y \) is an allocation. Thus, we have
\[ \mathbb{E}_t \left[ U(t, \cdot, f(t, \cdot)) \bigvee \{ \mathcal{D}_i : i \in \mathcal{P}(E) \} \right] = U(t, \cdot, f(t, \cdot)) \]
and
\[ \mathbb{E}_t \left[ U(t, \cdot, y(t, \cdot)) \bigvee \{ \mathcal{D}_i : i \in \mathcal{P}(E) \} \right] = U(t, \cdot, y(t, \cdot)). \]
Furthermore, for all \( \omega \in \Omega_0 \), we have
\[ U(t, \omega, y(t, \omega)) > U(t, \omega, f(t, \omega)), \mu\text{-a.e. on } E, \]
which implies that \( f \) is fine blocked by \( E \) via \( y \). This contradicts with the assumption that \( f \in \mathcal{C}^{fine}(\mathcal{E}) \). Hence,
\[ U(t, \omega, f(t, \omega)) \geq U(t, \omega, \bar{f}(t, \omega)) \]
for all \( (t, \omega) \in T_1 \times \Omega \).

Suppose that there are a state \( \omega_* \in \Omega \) and a coalition \( D \subseteq T_1 \) such that
\[ U(t, \omega_*, f(t, \omega_*)) > U(t, \omega_*, \bar{f}(t, \omega_*)) \]
for all \( t \in D \). By Jensen’s inequality,
\[ U \left( t, \omega_*, \int_D \frac{f(\cdot, \omega_*)}{\mu(D)} \mu \right) > \int_D \frac{U(t, \cdot, f(t, \cdot))}{\mu(D)} \mu \]
and
\[ \left. U \left( t, \omega_*, \int_{T_1 \setminus D} \frac{f(\cdot, \omega_*)}{\mu(T_1 \setminus D)} \mu \right) \right|_{D} \geq U \left( t, \omega_*, \bar{f}(t, \omega_*) \right). \]
Let \( \delta = \frac{\mu(D)}{\mu(T_1)} \). Since
\[ \bar{f}(t, \omega_*) = \delta \int_D \frac{f(\cdot, \omega_*)}{\mu(D)} \mu + (1 - \delta) \int_{T_1 \setminus D} \frac{f(\cdot, \omega_*)}{\mu(T_1 \setminus D)} \mu, \]
then
\[ U(t, \omega_*, \bar{f}(t, \omega_*)) \geq U(t, \omega_*, \bar{f}(t, \omega_*)). \]
This is a contradiction, which implies that
\[ U(t, \omega, \bar{f}(t, \omega)) = U(t, \omega, f(t, \omega)) \]
for all \( (t, \omega) \in T \times \Omega. \) \( \square \)
Corollary 4.5. Assume that an economy $\mathcal{E}$ satisfies $(A'_1)$, $(A_2)$-$(A_3)$, $(A'_4)$ and $(A_5)$-$(A_6)$. Then $f \in \mathcal{C}(\mathcal{E})$ if and only if $\bar{f} \in \mathcal{C}(\mathcal{E})$.

The following theorem is an extension of [14, Theorem 3.1] to a mixed economy.

Theorem 4.6. Assume that an economy $\mathcal{E}$ satisfies $(A'_1)$, $(A_2)$-$(A_3)$, $(A'_4)$ and $(A_5)$-$(A_6)$. If either $|\mathcal{A}| \geq 2$ or $\mathcal{A} = \emptyset$, then $\mathcal{C}^{fine}(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E})$.

Proof. First, we assume $|\mathcal{A}| \geq 2$ and $f \in \mathcal{C}^{fine}(\mathcal{E})$. If $f \notin \mathcal{C}(\mathcal{E})$, then, by Corollary 4.3, $\bar{f} \notin \mathcal{C}(\mathcal{E})$. By Theorem 4.2, there is an $\omega_0 \in \Omega$ such that $\bar{f} \notin \mathcal{C}(\mathcal{E}(\omega_0))$.

Next, we consider an allocation $f^* : T^* \to \mathbb{R}^+_0$ in $\mathcal{E}(\omega_0)$ defined by

$$
\bar{f}^*(t) = \begin{cases} f(t, \omega_0), & \text{if } t \in T_0; \\ 1_{\mu(T_1)} \int_{T_1} f(\cdot, \omega) d\mu, & \text{if } t \in T_1^*.
\end{cases}
$$

It is clear that $\bar{f}^* \notin \mathcal{C}(\mathcal{E}(\omega_0))$. Choose an arbitrary $A_{n_0} \in \mathcal{A}$ and let $\mu(A_{n_0}) = \varepsilon > 0$. By Vind’s theorem (see [8, Theorem 3.1] or [25]), $\bar{f}^*$ is blocked by a coalition $S$ in $\mathcal{E}(\omega_0)$, which can be chosen such that $\mu^*(S) = \mu(T_0) + \varepsilon$, if

$$
\mu^*(T_1^* \setminus A_{n_0}^*) = \min\{\mu(T_0^i) : i \in \mathcal{P}(T_0)\},
$$

and

$$
\mu^*(S) > \mu^*(T_1^*) - \min\{\mu(T_0^i) : i \in \mathcal{P}(T_0)\},
$$

otherwise. In either case, it can be checked that $\mu^*(S \cap T_1^*) \geq \varepsilon$ and $\mathcal{P}(S) = \{1, 2, \cdots, n\}$. By Lemma 4.2, we can have a coalition $E^*$ in $\mathcal{E}(\omega_0)$ with

$$
\mathcal{P}(E^*) = \mathcal{P}(S) \text{ and } \mu^*(E^* \cap T_1^*) = \varepsilon, \text{ which blocks } \bar{f}^* \text{ via } h^* \in \mathcal{E}(\omega_0).
$$

Consider a coalition $E$ in $\mathcal{E}$ defined by $E = (E^* \cap T_0^i) \cup A_{n_0}$. Then, $\mathcal{P}(E) = \{1, 2, \cdots, n\}$. Now, we consider a function $h : E \to \mathbb{R}^+_0$ defined by

$$
h(t) = \begin{cases} h^*(t), & \text{if } t \in E^* \cap T_0; \\ \varepsilon \int_{E^* \cap T_1^i} h^* d\mu^*, & \text{otherwise}.
\end{cases}
$$

Obviously,

$$
U(t, \omega_0, h(t)) > U(t, \omega_0, \bar{f}(t, \omega_0)), \mu\text{-a.e. on } E^* \cap T_0.
$$

By Jensen’s inequality, if $t \in A_{n_0}$, we have

$$
U(t, \omega_0, h(t)) > U(t, \omega_0, \bar{f}(t, \omega_0)).
$$

Moreover,

$$
\int_E h d\mu = \int_E a(\cdot, \omega_0) d\mu.
$$

Similar to that in Theorem 4.3, we can define $\Omega_0$ and an allocation $y : T \times \Omega \to \mathbb{R}^+_0$ in $\mathcal{E}$ such that

$$
y(t, \omega) = \begin{cases} h(t), & \text{if } (t, \omega) \in E \times \Omega_0; \\ a(t, \omega), & \text{otherwise}.
\end{cases}
$$

Note that

$$
\mathbb{E}_t \left[ U(t, \cdot, f(t, \cdot)) \right] = U(t, \cdot, f(t, \cdot))
$$
and
\[ \mathbb{E}_t \left[ U(t, \cdot, y(t, \cdot)) \right] \bigg\| \mathcal{D}_i : i \in \mathcal{P}(E) \bigg\| = \mathcal{E}_t \left[ U(t, \cdot, y(t, \cdot)) \right]. \]
Thus, \( f \) is fine blocked by \( E \) via \( y \). This is a contradiction.

In case that \( \mathcal{A} = \emptyset \), \( f \in \mathcal{C}^{\text{fine}}(\mathcal{E}) \) but \( f \notin \mathcal{C}(\mathcal{E}) \), an argument similar to the previous case can be applied. The major difference is that in this case, the blocking coalition \( E \) can be chosen such that
\[ \mu(E) > \mu(T_0) - \min\{ \mu(T_i) : i \in \mathcal{P}(T_0) \}. \]
The rest part of the proof is almost identical with that of the previous case. \( \square \)

Applying the core-Walras equivalence theorem in [18, 24], we have the following corollary.

**Corollary 4.7.** Assume that an economy \( \mathcal{E} \) satisfies \((A'_1)\), \((A_2)-(A_3)\), \((A'_4)\) and \((A_5)-(A_6)\). If \( f \in \mathcal{C}^{\text{fine}}(\mathcal{E}) \), then \( f(\omega, \cdot) \in \mathcal{W}_A(\omega) \) for every \( \omega \in \Omega \).

### 5. Concluding Remarks

A considerable amount of research work on different types of core and equilibrium concepts in economies with asymmetric information can be found in the literature. In particular, attempts in extending the classical equivalence of competitive equilibrium allocations and core allocations in a standard complete information economy have been made. For instance, the reader can refer to [14, 15, 16, 20]. In this paper, we focus our study on the ex-post core and its relationships to the fine core and the set of rational expectations equilibrium allocations, in two major parts.

The first part of the paper concerns the relationship between the ex-post core and the set of rational expectations equilibrium allocations. For our economic model, we apply a variety of techniques from Set-Valued Analysis to establish a representation result on the ex-post core (see Theorem 3.2). In an early paper [11], Bhowmik and Cao established a similar representation result for the set of rational expectations equilibrium allocations. These two representation results imply that for our model of asymmetric information economies, rational expectations equilibrium allocations are contained in the ex-post core (see Corollary 3.8).

To our knowledge, the idea of representing the ex-post core (resp. the set of rational expectations equilibrium allocations) by selections from the core (resp. competitive equilibrium) correspondence of the associated family of complete information economies is from [13]. The fundamental difference between [13] and this paper is that economies in [13] are assumed to have only finitely many states of nature, while economies in this paper are allowed to have infinitely many states of nature. Representation results in [13], together with Aumann’s Core Equivalence Theorem, imply that if the economy is atomless and the utility function of each trader is measurable with respect to his information field, then the set of rational expectations equilibrium allocations coincides with the ex-post core. However, this generally does not hold, when an atomless asymmetric information economy has infinitely many states of nature (see Example 3.9).

The second part of this paper emphasizes on the relationship between the fine core and the ex-post core in oligopolistic economies. We show that under standard assumptions and the assumption that there are only finitely many different information structures and all information is the joint information of agents, the
fine core is contained in the ex-post core, if an economy is either atomless or has at least two large agents with the same characteristics (see Theorem 4.6). This result can be regarded as an extension of the corresponding result in [14], where economies are assumed to be atomless only and have only finitely many states of nature. It would be interesting to know if the conclusion of Theorem 4.6 still holds for a mixed economy with only one large agent or with two large agents having different characteristics.

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Department of Economics, Shiv Nadar University, NH91, Tehsil Dadri, Gautam Budh
Dha Nagar, Uttar Pradesh 201314, India
E-mail address: anujbhowmik09@gmail.com

Department of Mathematical Sciences, School of Engineering, Computer and Mathematical Sciences, Auckland University of Technology, Private Bag 92006, Auckland
1142, New Zealand
E-mail address: jiling.cao@aut.ac.nz