Bounding the Porous Exponential Domination Number of Apollonian Networks *

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Abstract

Given a graph $G$ with vertex set $V$, a subset $S$ of $V$ is a dominating set if every vertex in $V$ is either in $S$ or adjacent to some vertex in $S$. The size of a smallest dominating set is called the domination number of $G$. We study a variant of domination called porous exponential domination in which each vertex $v$ of $V$ is assigned a weight by each vertex $s$ of $S$ that decreases exponentially as the distance between $v$ and $s$ increases. $S$ is a porous exponential dominating set for $G$ if all vertices in $S$ distribute to vertices in $G$ a total weight of at least 1. The porous exponential domination number of $G$ is the size of a smallest porous exponential dominating set. In this paper we compute bounds for the porous exponential domination number of special graphs known as Apollonian networks.

1 Introduction

Exponential domination was first introduced in [3] and further studied in [1]. Apollonian networks and their applications were independently introduced in [2] and [4], and further studied in [8] and [9]. We refer the reader to [5] and [6] for a comprehensive treatment of the topic of domination in graphs and its many variants. General graph theoretic notation and terminology may be found in [7]. Given a graph $G$, we denote its set of vertices by $V(G)$ and its set of edges by $E(G)$. The degree of a vertex $v$ in $G$ is denoted by $d_G(v)$. The distance in $G$ between vertices $x$ and $y$, denoted by $d_G(x,y)$, is defined to be the length of a shortest path in $G$ that joins $x$ and $y$, if such a path exists, and infinity otherwise. The diameter of $G$, denoted $diam(G)$, is the largest such distance: $diam(G) = \max\{d_G(x,y) \mid x, y \in V(G)\}$.

Let $G$ be a graph, $S \subseteq V(G)$, and $v \in V(G)$. The porous exponential domination weight of $S$ at $v$ is

$$w^*_S(v) = \sum_{u \in S} \frac{1}{2^{d_G(u,v)-1}}$$

and $S$ is a porous exponential dominating set for $G$ if $w^*_S(v) \geq 1$ for all $v \in V(G)$. The size of a smallest porous exponential dominating set for $G$ is the porous exponential domination number of $G$, and is denoted by $\gamma^*_e(G)$. These definitions were first introduced in [3], although that paper is primarily concerned with another variant, $\gamma_e(G)$, called the nonporous exponential domination number of $G$. The key difference between porous exponential domination and nonporous exponential domination is whether the distribution of weights from $S$ may “pass through” other vertices in $S$, as is evidenced by the slightly different definition of nonporous weight:

$$w_S(v) = \begin{cases} 
\sum_{u \in S} \frac{1}{2^{d_G(u,v)-1}} & \text{if } v \notin S \\
2 & \text{if } v \in S 
\end{cases}$$

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where \( f(u, v) \) is defined to be the length of a shortest path joining \( u \) and \( v \) in the subgraph induced by \( V(G) \setminus (S \setminus \{u\}) \) if such a path exists, and infinity otherwise. It is clear that \( \gamma^*_e(G) \leq \gamma_e(G) \).

Having defined porous exponential domination, we now define Apollonian networks. Let \( G_1 \) be a complete graph on three vertices and let \( U_1 = V(G_1) \). Let \( G_2 \) be a complete graph on four vertices such that \( U_1 \subseteq V(G_2) \), and let \( U_2 = V(G_2) \setminus V(G_1) \). For \( k > 2 \) we define \( G_k \) and \( U_k \) recursively by extending \( G_{k-1} \) and \( U_{k-1} \) as follows: for each \( u \in U_{k-1} \), and for each adjacent pair \( \{x, y\} \) of neighbors of \( u \) in \( G_{k-1} \), we create a new vertex \( v \in U_k \) that is adjacent to each of \( u, x, y \) in \( G_k \). (Consequently, \( u, v, x, \) and \( y \) are all pairwise adjacent in \( G_k \).) We call \( G_k \) the \( k \)th Apollonian network, and for \( 1 \leq j \leq k \), we call \( U_j \) the \( j \)th generation of vertices in \( G_k \). Note that \( V(G_k) = \bigcup_{j=1}^{k} U_j \) and \( U_k = V(G_k) \setminus V(G_{k-1}) \). This recursive process is more easily visualized by starting with a particular planar embedding of \( G_1 \) and obtaining \( G_k \) from \( G_{k-1} \) by adding a new vertex to each interior face and triangulating, as shown in Figures 1 through 4. We note, however, that our formal definition above does not depend upon the planar embedding.

Before stating our main results, we record a few elementary facts based upon our construction of \( G_k \) and observation of small cases:

**Remark 1.1.** \( |U_1| = 3, |U_k| = 3^{k-2} \) for \( k > 1 \), and \( |V(G_k)| = \sum_{j=1}^{k} U_j = 3 + \sum_{j=0}^{k-2} 3^j = \frac{3^{k-1} + 5}{2} \).

**Remark 1.2.** \( |E(G_k)| = 3 + \sum_{j=2}^{k} 3|U_j| = 3 + \sum_{j=2}^{k} 3^{j-1} = \frac{3^k + 3}{2} \).

**Remark 1.3.** Since every vertex in \( V(G_3) \) is adjacent to the single vertex in \( U_2 \), we know that \( \gamma^*_e(G_3) = 1 \).

**Remark 1.4.** Let \( S \) be any pair of vertices from \( V(G_2) \). Since every vertex in \( V(G_5) \) is adjacent to at least one of the vertices in \( V(G_2) \) and every pair of vertices in \( V(G_2) \) is adjacent, we know that every vertex of \( V(G_5) \) is within distance 2 of both vertices in \( S \) and therefore \( \gamma^*_e(G_5) = 2 \). (See Figure 5.)

We further invite the reader to verify our observations and computations for the order, diameter, and porous exponential domination number of \( G_k \) for \( k \leq 7 \), as presented in Table 1 below.
Figure 5: $G_5$

Table 1: Observations for $G_k$, $k \leq 7$

| $k$ | $|V(G_k)|$ | $|E(G_k)|$ | $diam(G_k)$ | $\gamma'_e(G_k)$ |
|-----|------------|------------|-------------|-----------------|
| 1   | 3          | 3          | 1           | 1               |
| 2   | 4          | 6          | 1           | 1               |
| 3   | 7          | 15         | 2           | 1               |
| 4   | 16         | 42         | 3           | 2               |
| 5   | 43         | 123        | 3           | 2               |
| 6   | 124        | 366        | 4           | 3               |
| 7   | 367        | 1095       | 5           | 3               |
2 Main Results

In Remark 1.4 we compute \( \gamma_\text{e}(G_5) = 2 \) by observation, but as \( k \) increases, the number of vertices increases exponentially and \( \gamma_\text{e} \) becomes increasingly difficult to compute by brute force. Thus, our main results in this paper are upper and lower bounds for \( \gamma_\text{e}(G_k) \). For all \( k \geq 6 \) we show that \( U_{k-3} \) is a porous exponential dominating set for \( G_k \), which proves the following:

**Theorem 2.1.** For \( k \geq 6 \), \( \gamma_\text{e}(G_k) \leq 3^{k-5} \).

We can improve upon this bound for \( k \geq 11 \) by constructing a porous exponential dominating set using all of the vertices of a smaller Apollonian network rather than just a generation. In particular, we dominate \( G_k \) with \( V(G_{k-7}) \) and prove the following:

**Theorem 2.2.** For \( k \geq 10 \), \( \gamma_\text{e}(G_k) \leq \frac{3^{k-5}+5}{2} \).

To establish a lower bound, we apply a theorem from [3] that bounds \( \gamma_\text{e}(G) \) from below in terms of \( \text{diam}(G) \). In order to do this, we compute \( \text{diam}(G_k) \) for all \( k \). This establishes the following:

**Theorem 2.3.** For all \( k \in \mathbb{N} \), \( \gamma_\text{e}(G_k) \geq \left\lceil \frac{2k+5}{12} \right\rceil \).

Before we can prove these theorems, we need some basic results about Apollonian networks.

3 Apollonian Networks

All of the vertices in \( G_2 \) are adjacent to each other, but for larger values of \( k \), the adjacencies are more restrictive. Recall that \( x \) is a neighbor of \( y \) in \( G \) if \( x \) is adjacent to \( y \) in \( G \), and the set of \( y \)’s neighbors in \( G \) is the neighborhood of \( y \) in \( G \), denoted \( N_G(y) \).

**Lemma 3.1.** For all \( k \geq 2 \), and for every vertex \( v \) in \( U_k \),

(i) \( v \) has no neighbor in \( U_k \)

(ii) \( v \) has a neighbor in \( U_{k-1} \)

(iii) \( v \) has exactly 3 distinct neighbors in \( V(G_{k-1}) \) and these vertices are also pairwise adjacent.

(iv) For all \( r < k \) and for all \( u \in U_r \), if \( u \) is adjacent to \( v \) then \( |N_{G_k}(u) \cap N_{G_k}(v)| = 2 \).

(v) if \( r < k \) and \( v \) has more than one neighbor in \( U_r \), then \( r = 1 \)

*Proof.* Parts (i), (ii), and (iii) follow directly from the construction of \( G_k \) because when a new vertex \( v \) is added to \( U_k \), it is made adjacent to a vertex \( u \) of \( U_{k-1} \) and two of \( u \)’s neighbors in \( V(G_{k-1}) \), say \( n_1 \) and \( n_2 \). By part (iii), if one of \( v \)’s neighbors is \( u \), then the other two are neighbors of both \( u \) and \( v \), and (iv) follows. We prove (v) by contradiction. Suppose that \( 1 < r < k \) and two of \( u \), \( n_1 \), and \( n_2 \) are in \( U_r \). We know that \( u \in U_{k-1} \), so if \( k-1 - r > 1 \) then \( n_1, n_2 \notin U_r \) by part (i). If \( 1 < r < k-1 \) then it must be that \( n_1 \) and \( n_2 \) are the two vertices in \( U_r \). But by the construction of \( G_{k-1} \), all three of \( u \)'s neighbors in \( G_{k-2} \) (including \( n_1 \) and \( n_2 \)) must be adjacent. This contradicts (i) for \( n_1 \in U_r \) since \( r > 1 \).

**Corollary 3.2.** For all \( k \geq 4 \) and for every vertex \( v \in U_k \), \( v \) has at least one neighbor in \( V(G_{k-3}) \).

*Proof.* By Lemma 3.1 part (iii), \( v \) has exactly 3 distinct neighbors in \( V(G_{k-1}) \), and these vertices are also pairwise adjacent. Denote these vertices by \( n_1 \), \( n_2 \), and \( n_3 \), and suppose that \( n_1 \in U_r \), \( n_2 \in U_s \), and \( n_3 \in U_t \), where \( r \leq s \leq t \leq k-3 \). Since \( k \geq 4 \), then \( k-3 \geq 1 \) and if \( r > k-3 \) then by pigeonhole principle, two of \( r \), \( s \), and \( t \) must be equal which contradicts Lemma 3.1 part (i). Therefore \( r \leq k-3 \) and \( n_1 \in V(G_{k-3}) \).

Given \( k \in \mathbb{N}, r < k, \) and \( v \in U_r \), define \( P_k(v) = \{ \{x, y\} \mid x \in U_k \text{ and } v, x, \text{ and } y \text{ are pairwise adjacent}\} \). This is the set of pairs of vertices, at least one of which is from the \( k \)th generation, that form triangles with \( v \) in \( G_k \), the very same triangles that will anchor the \((k+1)\)st generation of vertices. By the construction of \( G_{k+1} \), there is a one-to-one correspondence between \( P_k(v) \) and the \((k+1)\)st generation neighbors of \( v \). It follows that \( |P_k(v)| = |N_{G_{k+1}}(V) \cap U_{k+1}| \), in other words the number of \((k+1)\)st generation neighbors of \( v \). The next lemma states that the number of such neighbors doubles with every generation.

**Lemma 3.3.** For all \( k \in \mathbb{N}, \) for all \( r \leq k, \) and for all \( v \in U_r, \) \( |P_{k+1}(v)| = 2|P_k(v)| \).
Proof. By the construction of $G_{k+1}$, there is a one-to-one correspondence between $P_k(v)$ and the $(k + 1)$st generation neighbors of $v$. It follows that the members of $P_{k+1}(v)$ are precisely the pairs $\{z, x\}$ and $\{z, y\}$ where $z \in U_{k+1} \cap N_{G_{k+1}}(v)$ and $\{x, y\} \in P_k(v)$.

**Corollary 3.4.** For all $k \in \mathbb{N}$, for all $r \leq k$, and for all $v \in U_r$,

$$|P_k(v)| = \begin{cases} 3(2^{k-r}) & \text{when } r > 1 \\ 2^{k-1} & \text{when } r = 1 \end{cases}$$

Proof. We proceed by induction on $k$. If $k = 1$ then $r = 1$, then indeed for all $v \in U_1$, $|P_1(v)| = 1 = 2^{1-1}$. If $k > 1$ and $r = 1$ then, by Lemma 3.3, $|P_k(v)| = 2|P_{k-1}(v)| = 2(2^{k-2}) = 2^{k-1}$ by inductive hypothesis. If $k = 2$ and $r = 2$ then for the single vertex $v \in U_2$, $|P_2(v)| = 3 = 3(2^{1-1})$. If $k > 2$ and $r > 1$ then, by Lemma 3.3, $|P_k(v)| = 2|P_{k-1}(v)| = 2(3(2^{(k-1)-r})) = 3(2^{k-r})$ by inductive hypothesis.

**Corollary 3.5.** For all $k \geq 2$, and for all $v \in V(G_{k-1})$, $v$ has a neighbor in $U_k$.

Proof. By the construction of $G_k$ there is a one-to-one correspondence between $P_{k-1}(v)$ and the $k$th generation neighbors of $v$. By Corollary 3.4 $|P_{k-1}(v)|$ is nonnegative, and therefore $v$ has a neighbor in $U_k$.

**Corollary 3.6.** For all $k \geq 2$, and for all $v \in V(G_k) \setminus U_{k-1}$, $v$ has a neighbor in $U_{k-1}$.

Proof. If $v \in U_k$ then the result follows immediately from Lemma 3.1 part (ii). If $v \in U_r$, where $r \leq k - 2$ then the result follows from Corollary 3.5.

**Lemma 3.7.** For all $k \in \mathbb{N}$, for all $r \leq k$, and for all $v \in U_r$,

$$d_{G_k}(v) = \begin{cases} |P_k(v)| & \text{when } r > 1 \\ |P_k(v)| + 1 & \text{when } r = 1 \end{cases}$$

Proof. By the construction of $G_{k+1}$, there is a one-to-one correspondence between $P_k(v)$ and the $(k + 1)$st generation neighbors of $v$. It follows that for all $k \in \mathbb{N}$, for all $r \leq k$, and for all $v \in U_r$,

$$d_{G_{k+1}}(v) = d_{G_k}(v) + |P_k(v)|.$$ 

We now prove the lemma by induction on $k$. If $k = 1$ then $r = 1$ and $d_{G_1}(v) = 2 = 1 + 1 = |P_1(v)| + 1$. If $k > 1$ and $r = 1$ then $d_{G_k}(v) = d_{G_{k-1}}(v) + |P_{k-1}(v)| = |P_{k-1}(v)| + 1 + |P_{k-1}(v)| = 2|P_{k-1}(v)| + 1 = |P_k(v)| + 1$ by inductive hypothesis and Lemma 3.3. If $k = 2$ and $r = 2$ then for the single vertex $v \in U_2$, $d_{G_2}(v) = 3 = |P_1(v)|$. If $k > 2$ and $r > 1$ then $d_{G_k}(v) = d_{G_{k-1}}(v) + |P_{k-1}(v)| = |P_{k-1}(v)| + |P_{k-1}(v)| = 2|P_{k-1}(v)| = |P_k(v)|$ by inductive hypothesis and Lemma 3.3.

**Corollary 3.8.** For all $k \in \mathbb{N}$, for all $r \leq k$, and for all $v \in U_r$,

$$d_{G_k}(v) = \begin{cases} 3(2^{k-r}) & \text{when } r > 1 \\ 2^{k-1} + 1 & \text{when } r = 1 \end{cases}$$

Proof. This follows immediately from Corollary 3.4 and Lemma 3.7.

### 4 Upper Bounds for $\gamma_e^*$

In [3] the nonporous exponential dominating number of $G$, denoted $\gamma_e(G)$, is defined and the following theorem is proved:

**Theorem 4.1.** (Dankelmann, et al) If $G$ is a connected graph of order $n$, then $\gamma_e(G) \leq \frac{2}{5}(n + 2)$.

This theorem, together with Remark 1.1 and the fact that $\gamma_e^*(G) \leq \gamma_e(G)$, immediately establishes the following corollary:

**Corollary 4.2.** For all $k \in \mathbb{N}$, $\gamma_e(G_k) \leq \frac{3^{k-1} + 9}{5}$.
The recursive nature of our construction of $G_k$ makes it clear that for, $k > 1$, $G_k$ can be conceived as a union of three copies of $G_{k-1}$. More precisely, if we consider the three triangles in $G_2$ that include the vertex in $U_2$, each could be the first generation of a copy of $G_{k-1}$. Together, these three copies of $G_{k-1}$ comprise a copy of $G_k$. This perspective is also discussed in [9]. The following lemma follows immediately from this construction.

**Lemma 4.3.** For all $k \in \mathbb{N}$, $\gamma_\nu^*(G_{k+1}) \leq 3\gamma_\nu^*(G_k)$.

**Corollary 4.4.** For $k \geq 5$, $\gamma_\nu^*(G_k) \leq 2(3^{k-5})$.

**Proof.** By induction on $k$. If $k = 5$ then the result follows immediately from Remark [1.4]. If $k > 5$ then by Lemma 4.3, $\gamma_\nu^*(G_k) \leq 3\gamma_\nu^*(G_{k-1}) = 3(2(3^{(k-1)-5})) = 2(3^{k-5})$ by inductive hypothesis. □

We now establish a better upper bound by proving Theorem 2.1.

**Theorem 2.1.** Suppose $k \geq 10$. Let $S = U_{k-3}$ and compute $w^*_S(v)$ for all $v \in V(G_k)$.

1. **Case 1:** Suppose $v \in V(G_{k-4})$. By Corollary 3.5, $v$ has a neighbor in $S$ and $w^*_S(v) \geq 1$.

2. **Case 2:** Suppose $v \in U_{k-3}$. Then $v \in S$ and $w^*_S(v) \geq 2$.

3. **Case 3:** Suppose $v \in U_{k-2}$. By Corollary 3.6, $v$ has a neighbor in $S$ and $w^*_S(v) \geq 1$.

**Proof.** Suppose $k \geq 10$. Let $S = V(G_{k-7})$ and compute $w^*_S(v)$ for all $v \in V(G_k)$.

1. **Case 1:** Suppose $v \in U_j$, $j \leq k - 4$. Then by Corollary 3.2, either $v \in S$ or $v$ has a neighbor in $S$. In both cases, $w^*_S(v) \geq 1$.

2. **Case 2:** Suppose $v \in U_j$, $k - 3 \leq j \leq k - 2$. If $v$ has a neighbor in $S$, then $w^*_S(v) \geq 1$. Otherwise, by Corollary 3.2, $v$ has a neighbor in $S$. By Corollary 3.2, $n$ has at least two neighbors in $S$. Therefore, $v \in S$ is within distance 2 of at least two distinct vertices of $S$, and $w^*_S(v) \geq 1$.

3. **Case 3:** Suppose $v \in U_{k-1}$. If $v$ has a neighbor in $S$, then $w^*_S(v) \geq 1$. Otherwise, by Corollary 3.2, $v$ has a neighbor in $S$. By Corollary 3.2, $n$ has at least two neighbors in $S$. Therefore, $v \in S$ is within distance 2 of at least two distinct vertices of $S$, and $w^*_S(v) \geq 1$.

4. **Case 4:** Suppose $v \in U_k$. If $v$ has a neighbor in $S$, then $w^*_S(v) \geq 1$. Otherwise, by Corollary 3.2, $v$ has a neighbor in $S$. By Corollary 3.2, $n$ has at least two neighbors in $S$. Therefore, $v \in S$ is within distance 2 of at least two distinct vertices of $S$, and $w^*_S(v) \geq 1$.

We have shown that $S$ is a porous exponential dominating set for $G_k$. By Remark 1.1, $|S| = 3^{k-5}$, and therefore $\gamma_\nu^*(G_k) \leq 3^{k-5}$.
5 Lower Bound for $\gamma_e^*$

Recall that for a connected graph $G$, the diameter of $G$, denoted $\text{diam}(G)$, is the largest possible distance between a pair of vertices in $G$. In [3], the nonporous exponential domination number of $G$, denoted $\gamma_e(G)$, is defined and the following theorem is proven:

**Theorem 5.1.** (Danelkamm, et al) If $G$ is a connected graph, then $\gamma_e(G) \geq \left\lceil \frac{\text{diam}(G)+2}{3} \right\rceil$.

In fact, the proof of this result in [3] is sufficient to establish the following lemma:

**Lemma 5.2.** If $G$ is a connected graph, then $\gamma_e^*(G) \geq \left\lceil \frac{\text{diam}(G)+2}{3} \right\rceil$.

We now compute $\text{diam}(G_k)$ for every Apollonian network $G_k$.

**Lemma 5.3.** For all $k \in \mathbb{N}$, $\text{diam}(G_{k+3}) \leq \text{diam}(G_k) + 2$.

**Proof.** Suppose $x,y \in V(G_{k+3})$ and $d_{G_{k+3}}(x,y) = \text{diam}(G_{k+3})$. By Lemma 3.1 and Corollary 3.2, we know that $x$ and $y$ have neighbors $u$ and $v$, respectively, in $V(G_k)$. It follows that

$$\text{diam}(G_{k+3}) = d_{G_{k+3}}(x,y) \leq d_{G_k}(u,v) + 2 \leq \text{diam}(G_k) + 2.$$ 

\[ \square \]

**Corollary 5.4.** For all $k \in \mathbb{N}$, $\text{diam}(G_k) \leq \left\lceil \frac{4k+1}{3} \right\rceil$.

**Proof.** We proceed by induction on $k$ and show that $\text{diam}(G_k) \leq \frac{4k+1}{3}$. For $k = 1, 2, 3$, it is easy to verify that $\text{diam}(G_k) = 1, 1, 2$, respectively, and establish the desired result. For $k > 3$, by Lemma 5.3 $\text{diam}(G_k) \leq \text{diam}(G_{k-3}) + 2 \leq \frac{4(k-3)+1}{3} + 2 = \frac{4k+1}{3}$, by inductive hypothesis. Since $\text{diam}(G_k)$ is an integer, the result follows.

\[ \square \]

**Lemma 5.5.** For all $k \in \mathbb{N}$ there exists $x, y \in U_k$ such that $d_{G_k}(x,y) = \text{diam}(G_k)$.

**Proof.** First, observe that the statement is true for $k = 1$, so we may assume $k \geq 2$. Let $u, v \in V(G_k)$ such that $d_{G_k}(u,v) = \text{diam}(G_k)$. If $u \in U_k$ then let $x = u$. Otherwise, by Corollary 3.3, there exists $x \in U_k$ such that $x$ is adjacent to $u$. If $v \in U_k$ then let $y = v$. Otherwise, by Corollary 3.3, there exists $y \in U_k$ such that $y$ is adjacent to $v$. Let $P$ be a shortest path joining $x$ and $y$. Let $w_1$ be the vertex adjacent to $x$ in $P$, and $w_2$ be the vertex adjacent to $y$ in $P$. By Lemma 3.1 part (iii), $u$ is adjacent to $w_1$ and $v$ is adjacent to $w_2$. Define $Q$ to be the path formed by replacing $x$ and $y$ in $P$ with $u$ and $v$. Then the length of $Q$ is the same as the length of $P$. Since $d_{G_k}(u,v) = \text{diam}(G_k)$, this shows that the length of $P$ is at least $\text{diam}(G_k)$. Since $P$ is a shortest path joining $x$ and $y$, $d_{G_k}(x,y) = \text{diam}(G_k)$.

\[ \square \]

**Lemma 5.6.** For all $k \in \mathbb{N}$, $\text{diam}(G_{k+3}) \geq \text{diam}(G_k) + 2$.

**Proof.** The result is easily seen to be true for $k = 1$, so we may assume that $k \geq 2$. (See Figure 4 and Table 1) By Lemma 5.5 let $u, v \in V(G_k)$ such that $d_{G_k}(u,v) = \text{diam}(G_k)$. By Lemma 3.1 any path joining $u$ and $v$ must include vertices from $V(G_{k-1})$. By Corollary 3.3, $u$ has a neighbor in $U_{k+1}$, and by the construction of $G_{k+2}$, $u$ and $v$ have a common neighbor in $U_{k+2}$. By the construction of $G_{k+3}$, $u$, $w_1$, and $w_2$ have a common neighbor in $U_{k+3}$. By Lemma 3.1, $u$, $w_1$, and $w_2$ are the only neighbors of $x$ in $G_{k+3}$. Therefore, $u$ has a neighbor $y \in U_{k+3}$ such that $N_{G_{k+3}}(x) \cap V(G_{k-1}) = \emptyset$. An analogous argument shows that $v$ has a neighbor $z \in U_{k+3}$ such that $N_{G_{k+3}}(y) \cap V(G_{k-1}) = \emptyset$. Note that any path joining $x$ and $y$ must include vertices from $V(G_{k-1})$ because otherwise we could construct a path joining $x$ and $v$ without such vertices, which contradicts our earlier claim to the contrary.

Let $P$ be a shortest path $x, w_1, w_2, \ldots, w_m, y$ joining $x$ and $y$. Choose $i$ as small as possible and $j$ as large as possible such that $w_i, w_j \in V(G_{k-1}) \cap V(P)$. Since the only neighbors of $x$ are $u$, $w_1$, and $w_2$ then $w_i$ is adjacent to at least one of these. By the construction of $G_{k+1}$ and $G_{k+2}$, $u$ is adjacent to all of the neighbors of $u_1$ and $u_2$ in $V(G_{k-1})$, and therefore $u$ is adjacent to $w_i$. Analogously, $v$ is adjacent to $w_j$. Let $Q$ be the path $u, w_i, w_{i+1}, \ldots, w_{j-1}, w_j, v$ joining $u$ and $v$. Since $N_{G_{k+3}}(x) \cap V(G_{k-1}) = \emptyset$ and $N_{G_{k+3}}(y) \cap V(G_{k-1}) = \emptyset$, the length of $P$ is at least 2 more than the length of $Q$. It follows that the length of $P$ is at least $\text{diam}(G_k) + 2$. Since $P$ is a shortest length path joining $x$ and $y$, $\text{diam}(G_{k+3}) \geq \text{diam}(G_k) + 2$.

\[ \square \]
Together, Lemma 5.3 and Lemma 5.6 imply the following result which was stated in [8] with greater generality but without a complete proof.

**Corollary 5.7.** For all \( k \in \mathbb{N} \), \( \text{diam}(G_{k+3}) = \text{diam}(G_k) + 2 \).

**Corollary 5.8.** For all \( k \in \mathbb{N} \), \( \text{diam}(G_k) \geq \lceil \frac{2k-1}{3} \rceil \).

**Proof.** We proceed by induction on \( k \) and show that \( \text{diam}(G_k) \geq \frac{2k-1}{3} \). For \( k = 1, 2, 3 \), it is easy to verify that \( \text{diam}(G_k) = 1, 1, 2 \), respectively, and establish the desired result. For \( k > 3 \), by Lemma 5.6, \( \text{diam}(G_k) \geq \text{diam}(G_{k-3}) + 2 \geq \frac{2(k-3)-1}{3} + 2 = \frac{2k-1}{3} \), by inductive hypothesis. Since \( \text{diam}(G_k) \) is an integer, the result follows.

**Corollary 5.9.** For all \( k \in \mathbb{N} \), \( \text{diam}(G_k) = \lceil \frac{2k-1}{3} \rceil \).

**Proof.** This result follows easily from Corollary 5.4, Corollary 5.8, and the fact that \( \lfloor \frac{2k+1}{3} \rfloor = \lceil \frac{2k+1}{3} \rceil \), which the reader can easily check by cases \( k \equiv 0, 1, 2 \pmod{3} \).

We can now prove Theorem 2.3:

**Proof.** By Corollary 5.9, \( \text{diam}(G_k) = \lceil \frac{2k-1}{3} \rceil \geq \frac{2k-1}{3} \), and therefore \( \text{diam}(G_k)+2 \geq \frac{2k+5}{12} \). By Lemma 5.2, \( \gamma^*_e(G_k) \geq \left\lfloor \frac{\text{diam}(G_k)+2}{4} \right\rfloor \geq \left\lfloor \frac{2k+5}{12} \right\rfloor \).

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### References

[1] M. Anderson, R. Brigham, J. Carrington, R. Vitray, J. Yellen, On Exponential Domination of \( C_m \times C_n \), *J. Graphs. Combin.* 6 No. 3 (2009) 341-351.

[2] J. Andrade, H. Herrmann, R. Andrade, L. Silva, Simultaneously Scale-Free, Small World, Euclidean, Space Filling, and with Matching Graphs, *Phys. Rev. Lett.* 94(1) (2005) 18702.

[3] P. Dankelmann, D. Day, D. Erwin, S. Mukwembi, H. Swart, Domination with Exponential Decay, *Discrete Math* 309(19) (2009) 5877-5883.

[4] J.P.K. Doye and C. Massen, Characterizing the network topology of the energy landscapes of atomic clusters *J. Chem. Phys.* 122 (2005) 84105.

[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[7] D. B. West, *Introduction to Graph Theory, 2nd ed.*, Prentice Hall, 2000.

[8] Z. Zhang, F. Comellas, G. Fertin, L. Rong, High Dimensional Apollonian Networks, *J. Phys. A: Math. Gen.* 39 (2006) 1811.

[9] Z. Zhang, B. Wu, and F. Comellas, The Number of Spanning Trees in Apollonian Networks, *Discrete Applied Mathematics*. (to appear)