Nonstandard representations of type $C$ affine Hecke algebra from $K$-operators

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September 10, 2014

Abstract

We construct nonstandard finite-dimensional representations of type $C$ affine Hecke algebra from the viewpoint of quantum integrable models. There exist two classes of nonstandard solutions to the Yang-Baxter equation called the Cremmer-Gervais and Jordanian $R$-matrices. These $R$-matrices also satisfy the Hecke-relation, thus can be used to construct nonstandard finite-dimensional representations of type $A$ affine Hecke algebra. We construct the corresponding nonstandard representations for type $C$ affine Hecke algebra by explicitly constructing solutions to the reflection equation under the Hecke relation. We achieve this by taking the finite-dimensional representations and deBaxterizing the $K$-operators acting on the infinite-dimensional function space, taking advantage of the fact that the Cremmer-Gervais and Jordanian $R$-matrices can be obtained from the $R$-operator.

Mathematics Subject Classification (2010). 20C08, 16T25, 14H70, 17B37.

Keywords. Affine Hecke algebra, Yang-Baxter equation, reflection equation, quantum integrable models.

1 Introduction

Affine Hecke algebras play special roles in mathematics and mathematical physics. They are not only important one of the most algebras in representation theory related to Yangians and quantum affine algebras for example, but also have various applications to other branches from mathematical physics, knot theory to the recent progress in categorification, geometric representation theory and so on (see [1, 2] for example for general treatments and applications of affine Hecke algebras).

From the point of view of mathematical physics, the relationship between affine hecke algebras and quantum integrable models are very intimate, in particular with the $R$-matrix which is the most basic object in quantum integrable models. Trigonometric $R$-matrices can be constructed from affine Hecke algebra of type $A$. In fact, the standard $R$-matrix of the quantum group $U_q(\widehat{sl}_2)$ [3, 4] without spectral parameter is nothing but a representation for a generator of the affine Hecke algebra. The braid relation which is one type of the defining relations among the generators for the affine Hecke algebra is nothing but the Yang-Baxter relation in quantum integrable models. The trigonometric $R$-matrix with spectral parameter can be constructed as a combination of a generator and a permutation operator. This construction can be generalized to other root systems.

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For the affine Hecke algebra of type $C$, the boundary condition of the corresponding quantum integrable models is modified from the standard periodic boundary condition to the open boundary condition. From the point of view of quantum integrable models, two of the generators of type $C$ affine Hecke algebra are nothing but the solutions to the reflection equation [5, 6], which is the boundary condition on quantum systems to ensure integrability of the model. There are many solutions to the reflection equation constructed today (see [7–10] for example).

In this paper, we construct nonstandard finite-dimensional representations of type $C$ affine Hecke algebra. Despite its extensive studies on various aspects, it seems that the problem of concretely realizing representations of affine Hecke algebra is not well investigated. We approach this problem by using the power of quantum integrable models. From the point of view of affine Hecke algebra, the interesting $R$-matrices are those satisfying the Hecke relation. Besides the standard $R$-matrix of the quantum group $U_q(s\hat{\mathfrak{l}}_2)$, there is a nonstandard one called the Cremmer-Gervais $R$-matrix. This $R$-matrix originally appeared in the context of the Toda field theory [11] as a constant $R$-matrix, and the Baxterized $R$-matrix was derived in [12]. The Cremmer-Gervais $R$-matrix satisfies the Hecke relation, thus can be served as a representation for type $A$ affine Hecke algebra. We call the representation as a nonstandard trigonometric representation since it comes from the nonstandard $R$-matrix.

In this paper, we construct nonstandard representations for type $C$ affine Hecke algebra. From the point of view of quantum integrable models, the problem of constructing nonstandard trigonometric representations for type $C$ is equivalent to finding solutions to the reflection equation of the Cremmer-Gervais $R$-matrix under the Hecke relation.

To achieve this, we first review how the Baxterized Cremmer-Gervais $R$-matrix was derived [12] It was derived by taking finite-dimensional representation of the trigonometric limit of the elliptic Shibukawa-Ueno $R$-operator [13], which is an infinite-dimensional $R$-operator acting on the space of functions. Taking the finite-dimensional representation of the elliptic Shibukawa-Ueno $R$-operator gives the Baxter-Belavin model, whose trigonometric limit is the standard $R$-matrix of $U_q(s\hat{\mathfrak{l}}_2)$. The nonstandard Cremmer-Gervais $R$-matrix is obtained in a different manner. We first take the trigonometric limit of the Shibukawa-Ueno $R$-operator. After that, we take the finite-dimensional representation in the trigonometric basis which gives the Cremmer-Gervais $R$-matrix (see also [14] for a similar construction).

By taking advantage of this fact, we construct solutions to the reflection equation of the Cremmer-Gervais $R$-matrix in the following way. We start from the elliptic $K$-operator by Hikami-Komori [16, 17] which are solutions to the reflection equation of the Shibukawa-Ueno $R$-operator. We first take the trigonometric limit of the elliptic $K$-operator to get the trigonometric $K$-operator. We next take the finite-dimensional representation in the trigonometric basis to get the Baxterized $K$-matrix. By construction, the $K$-matrix is the solution to the Baxterized $R$-matrix. To extract representations of the affine Hecke algebra, we need one more thing to do. Namely, one has to do the deBaxterization, i.e., extracting the constant deBaxterized $K$-matrix out of the spectral-parameter-dependent $K$-matrix by getting rid of the spectral parameter. We show from its construction that the $K$-matrix satisfy the constant reflection equation and the Hecke relation, which are exactly the defining relations the boundary generators of affine Hecke algebra of type $C$ must satisfy.

We also construct another type of nonstandard representation for a special case of affine Hecke algebra of type $C$ by considering a class of quantum integrable models of rational type. The rational Jordanian $R$-matrix and its associated $K$-matrix serve as representations for the generators. We also find the representations in the similar way. We first take furthermore the rational limit of the trigonometric $R$-operator and the $K$-operator, take finite-dimensional representations in the rational basis. Then we deBaxterize to get the $R$-matrix and its corresponding $K$-matrix.

In the next section, we state the main results about the nonstandard representations of type $C$ affine Hecke algebra, and give the outline of the proof. In section 3, we first review the Cremmer-Gervais $R$-matrix and its construction from the Shibukawa-Ueno $R$-operator by taking the finite-dimensional representation in the trigonometric basis and deBaxterizing. Then we apply the same degeneration procedure to construct solutions to the reflection equation to the Cremmer-Gervais $R$-matrix from the $K$-operator by Hikami-Komori, and deBaxterize to obtain the construct representations for the
boundary generators of type $C$ affine Hecke algebra, which gives the proof for one of the main results. In section 4, we apply a similar procedure to the rational case to give the other main results. Section 5 is devoted to the conclusion.

2 Type $C$ affine Hecke algebra and nonstandard representations

In this paper, we construct representations of the following type $C$ affine Hecke algebra.

**Definition 2.1** Type $C$ affine Hecke algebra $H_n = H_n(t, t_n, t_0)$ is defined as an algebra generated by $T_j$, $j = 0, \cdots, n$ satisfying the following relations

\begin{align}
(T_0 - t_0)(T_0 + t_0^{-1}) &= 0, \quad (2.1) \\
(T_j - t)(T_j + t^{-1}) &= 0, \quad 1 \leq j \leq n - 1, \quad (2.2) \\
(T_n - t_n)(T_n + t_n^{-1}) &= 0, \quad (2.3) \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0, \quad (2.4) \\
T_jT_{j+1}T_j &= T_{j+1}T_jT_{j+1}, \quad 1 \leq j \leq n - 2, \quad (2.5) \\
T_{n-1}T_nT_{n-1}T_n &= T_nT_{n-1}T_nT_{n-1}, \quad (2.6) \\
T_jT_k &= T_kT_j, \quad |j - k| \geq 2. \quad (2.7)
\end{align}

We give nonstandard representations for type $C$ affine Hecke algebra by extracting the generators from a class of trigonometric and rational quantum integrable models. To state the main theorem, we first fix notations. Let $V$ be an $N$-dimensional complex vector space. We denote the orthonormal basis of $V$ by $\{e_j, j = 0, \cdots, N - 1\}$. The matrix element $[A]^k_j$ of $A \in \text{End}(V)$ with respect to this basis is defined as

$$Ae_j = \sum_{k=0}^{N-1} c_k[A]^k_j. \quad (2.8)$$

The permutation matrix $P$ is defined as

$$P(x \otimes y) = y \otimes x \text{ for any } x, y \in V. \quad (2.9)$$

Let $V^\otimes n = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ be a tensor product of complex vector spaces. For a matrix $A \in V$, let us define a matrix $A_j \in \text{End}(V_1 \otimes \cdots \otimes V_n)$ as a matrix acting on the complex vector space $V_j$ as $A$, and acting on the remaining complex vector spaces $V_k \ (k \neq j)$ as an identity matrix. For a matrix $A_{jk} \in \text{End}(V_j \otimes V_k)$, we define $A_{jk}$ as

$$A_{jk} = A_{jk}P_{jk}. \quad (2.10)$$

**Definition 2.2** We define the $R$-matrix $R^{xz}(q, p) \in V \otimes V$ and the $K$-matrix $K^{xz}(r, s) \in V$ as

$$[R^{xz}(q, p)]^{kl}_{ij} = p^{2(j-k)} \times \begin{cases} 
q_i, & \text{for } i = k \geq j = l, \\
q_i^{-1}, & \text{for } i = k < j = l, \\
-q_i + q_i^{-1}, & \text{for } i < k < j, i + j = k + l, \\
-q_i - q_i^{-1}, & \text{for } j \leq k < i, i + j = k + l, \\
0, & \text{otherwise.}
\end{cases} \quad (2.11)$$

3
\[
[K^{\text{tr}}(r, s)]_j^k = s^{i-k} \times \begin{cases} 
-r^{-1}, & \text{for } j \leq k, j + k = N - 1, \\
r, & \text{for } j > k, j + k = N - 1, \\
r - r^{-1}, & \text{for } j \leq k < N - 1 - j, \\
r + r^{-1}, & \text{for } N - 1 - j < k < j, \\
0, & \text{otherwise.} 
\end{cases} 
\]

(2.12)

Using the matrices defined above, we have the following representation for type C affine Hecke algebra.

**Theorem 2.1 (Nonstandard trigonometric representation)** Let \( \hat{T}_j, j = 0 \cdots, n \) be the following matrices acting on the spin module \( V^\otimes n \)

\[
\hat{T}_0 = K_1^{\text{tr}}(t_0, s_0), \\
\hat{T}_j = \hat{K}_{j,j+1}^{\text{tr}}(t, p), \quad j = 1, \cdots, n - 1, \\
\hat{T}_n = K_n^{\text{tr}}(t_n, s_n). 
\]

(2.14)

The map \( \rho \) defined as

\[
\rho(T_j) = \hat{T}_j, \quad j = 0, \cdots, n, 
\]

(2.17)

is a representation map \( \rho : H_n(t, t_n, t_0) \rightarrow \text{End}(V^\otimes n) \). Namely, \( \{\hat{T}_j, j = 0, \cdots, n\} \) gives a representation for type C affine Hecke algebra \( H_n(t, t_n, t_0) \). ■

We call this representation a nonstandard trigonometric representation of type C affine Hecke algebra. The term "nonstandard trigonometric" means that this representation differs from the finite-dimensional representations constructed in for example, based on the standard trigonometric \( R \)-matrix of the quantum group \( U_q(\hat{\mathfrak{sl}}_2) \) by Drinfeld and Jimbo, and its associated \( K \)-matrix. The term "nonstandard trigonometric" means that this representation comes from the nonstandard trigonometric \( R \)-matrix called the Cremmer-Gervais \( R \)-matrix and its associated \( K \)-matrix defined in (2.11) and (2.12) which we show in this paper.

For the case when the parameters of the type C affine Hecke algebra is special \( t = t_0 = t_n = 1 \), we can also construct another nonstandard representation.

**Definition 2.3** We define the \( R \)-matrix \( R^{\text{ra}}(\kappa, h) \in V \otimes V \) and the \( K \)-matrix \( K^{\text{ra}}(\nu, g) \in V \) as

\[
[R^{\text{ra}}(\kappa, h)]_{i,j}^{k,l} = (-1)^{j-i}h^{i+j-k-l} \left\{ \begin{array}{c} i \\ k \\ j \\ l \end{array} \right\} - \frac{\kappa}{h} \sum_m (-1)^{m-k} \left\{ \begin{array}{c} i \\ m \\ j \end{array} \right\} \left\{ \begin{array}{c} m-k-1 \\ l \end{array} \right\} \epsilon(j, m, k), 
\]

where \( \epsilon(i, j, k) \) is defined as

\[
\epsilon(i, j, k) = \begin{cases} 1, & \text{for } i \leq k < j, \\
-1, & \text{for } j \leq k < i, \\
0, & \text{otherwise.} 
\end{cases} 
\]

(2.19)

\[
[K^{\text{ra}}(\nu, g)]_j^k = (-1)^j \left\{ \begin{array}{c} j \\ k \end{array} \right\} g^{j-k} + 2\nu \sum_{0 \leq l < j} (-1)^{j-l} \left\{ \begin{array}{c} j-l-1 \\ k-l \end{array} \right\} g^{j-k-1}. 
\]

(2.20)

The following serves as another representation for type C affine Hecke algebra when \( t = t_0 = t_n = 1 \).
Theorem 2.2 (Nonstandard rational representation) Let $\hat{T}_j$, $j = 0, \cdots, n$ be the following matrices acting on the spin module $V^\otimes n$

\begin{align*}
\hat{T}_0 &= K_1^{TA}(v_0, g_0), \\
\hat{T}_j &= R_{j,j+1}^{\alpha}(\kappa, h), \quad j = 1, \cdots, n-1, \\
\hat{T}_n &= K_n^{TA}(v_n, g_n).
\end{align*}

The map $\rho$ defined as

$$\rho(T_j) = \hat{T}_j, \quad j = 0, \cdots, n,$$

is a representation map $\rho : H_n(1, 1, 1) \rightarrow \text{End}(V^\otimes n)$. Namely, $\{\hat{T}_j, \quad j = 0, \cdots, n\}$ gives a representation for type C affine Hecke algebra $H_n(1, 1, 1)$. \hfill \Box

We call the above representation as “nonstandard rational” representation for type C affine Hecke algebra since the generators are coming from a class of rational solutions (2.18) to the Yang-Baxter relation called the Jordanian $R$-matrix, and its associated $K$-matrix (2.20) which satisfies the reflection relation.

3 Outline of the proof

In this section, we give the proof of the main theorems. We explain the procedures of the degeneration and deBaxterization to extract the nonstandard representations from quantum integrable models. We give the outline in this section, and give the detailed lemmas and propositions essential to construct the trigonometric and rational representations in sections 4 and 5, respectively.

3.1 Trigonometric representation

We construct a representation of type C affine Hecke algebra by extracting the generators from a class of trigonometric quantum integrable models. In quantum integrable models, the $R$-matrix $R(\lambda)$ satisfying the Yang-Baxter relation

$$R_{12}(\lambda)R_{13}(\lambda + \mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda + \mu)R_{12}(\lambda),$$

is the fundamental object. The parameter $\lambda$ of $R(\lambda)$ is called the spectral parameter, and is important to treat quantum integrable models. However, to construct representations of affine Hecke algebras, we want to get rid of it. If an $R$-matrix $R(\lambda)$ satisfying the Yang-Baxter relation (3.1) can be decomposed using the permutation $P$ and the $\lambda$-independent $\hat{R}$ as

$$R(\lambda) = f(\lambda)(P + g(\lambda)\hat{R}), \quad g(\lambda) = \frac{e^{-2\pi i\lambda} - 1}{q - q^{-1}},$$

and the $\lambda$-independent $\hat{R}$ satisfies the Hecke relation

$$(\hat{R} - q)(\hat{R} + q^{-1}) = 0,$$

where $\hat{R} = RP$, then one can show that $\lambda$-independent $\hat{R}$ satisfies the Yang-Baxter relation without spectral parameter

$$R_{12}R_{13}R_{24} = R_{23}R_{13}R_{12},$$

which is equivalent to the braid relation for $\hat{R} = RP$

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}.$$
This shows that the $\check{R}_{j,j+1}$, $j = 1, \cdots, n - 1$ can be used as a representation for the generators $T_j$, $j = 1, \cdots, n - 1$. The braid relation
\[
\check{R}_{j,j+1}\check{R}_{j+1,j+2}\check{R}_{j,j+1} = \check{R}_{j+1,j+2}\check{R}_{j,j+1}\check{R}_{j+1,j+2}, \quad 1 \leq j \leq n - 2,
\]
corresponds to a defining relation (2.5) for the type $C$ affine Hecke algebra, and the Hecke relation (2.2) can be represented by the Hecke relation for the matrix $R$ under the identification of parameters $q = t$
\[
(\check{R}_{j,j+1} - q)(\check{R}_{j,j+1} + q^{-1}) = 0.
\]
We refer to this procedure to construct representations from quantum integrable models as deBaxterization. The Cremmer-Gervais $R$-matrix realizes a deBaxterization procedure to construct representations for type $A$ affine Hecke algebra, which we explain later.

If once a representation for $T_j$, $j = 1, \cdots, n - 1$ is constructed, the remaining thing is to construct the representations for the generators $T_0, T_n$ which satisfy the defining relations (2.1), (2.3), (2.4) and (2.6). The representation can also be constructed from objects of quantum integrable models. The quantum integrability under the reflecting boundary condition is ensured by the reflection relation
\[
R_{21}(\lambda_1 - \lambda_2)K_1(\lambda_1)R_{12}(\lambda_1 + \lambda_2)K_2(\lambda_2) = K_2(\lambda_2)R_{21}(\lambda_1 + \lambda_2)K_1(\lambda_1)R_{12}(\lambda_1 - \lambda_2),
\]
where $R_{21}(\lambda) = p_{12}R_{12}(\lambda)p_{12}$. The representations for generators $T_0$ and $T_n$ can be constructed from the $K$-matrix $K(\lambda)$ satisfying the reflection relation (3.8) in the following way.

If $K(\lambda)$ satisfying the reflection relation (3.8) can be decomposed using the identity and the $\lambda$-independent $K$ as
\[
K(\lambda) = a(\lambda)(I + b(\lambda)K), \quad b(\lambda) = e^{-4\pi i \lambda} - 1
\]
and the $\lambda$-independent $K$ satisfies the Hecke relation
\[
(K - r)(K + r^{-1}) = 0,
\]
then $K$ satisfies the reflection equations
\[
\check{R}_{12}K_1\check{R}_{12}K_1 = K_1\check{R}_{12}K_1\check{R}_{12},
\]
\[
\check{R}_{n-1,n}K_n\check{R}_{n-1,n}K_n = K_n\check{R}_{n-1,n}K_n\check{R}_{n-1,n},
\]
which can be identified as representations for the defining relations (2.4) and (2.6). The Hecke relations
\[
(K_1 - r_0)(K_1 + r_0^{-1}) = 0,
\]
\[
(K_n - r_n)(K_n + r_n^{-1}) = 0,
\]
can be identified with the Hecke relations for $T_0$ and $T_n$ (2.1) and (2.3) under the identification of boundary parameters $r_0 = t_0$ and $r_n = t_n$. Thus, the $K$-matrices $K_1$ and $K_n$ serves as representations for $T_0$ and $T_n$ respectively. We call this procedure to construct representations of $T_0$ and $T_n$ from the reflection equation (3.8) as deBaxterization for boundary.

The above is a description of a general procedure to construct representations of type $C$ affine Hecke algebra $H_n(t, t_n, t_0)$ from quantum integrable models by deBaxterization to get rid of the spectral parameters. This procedure applies if the extracted $\lambda$-independent $R$-matrix $R$ and $K$-matrix $K$ satisfy the Hecke relations. Now we construct a representation from a class of trigonometric quantum integrable models. We start from the Cremmer-Gervais $R$-matrix $R^{\text{tr}}(q,p)$ [11, 18] which is a nonstandard representation of the quantum group $U_q(S\hat{sl}_2)$.

\[
[R^{\text{tr}}(q,p)]_{ij}^{kl} = p^{2(j-k)} \times \begin{cases} 
q, & \text{for } i = k \geq j = l, \\
q^{-1}, & \text{for } i = k < j = l, \\
-q+q^{-1}, & \text{for } i < k < j, i+j = k+l, \\
q-q^{-1}, & \text{for } j \leq k < i, i+j = k+l, \\
0, & \text{otherwise.}
\end{cases}
\]
This is nothing but the $R$-matrix in Definition 2.2. The Cremmer-Gervais $R$-matrix $R^{tr}(q,p)$ satisfies the Yang-Baxter relation and the Hecke relation,

$$(R^{tr}(q,p) - q)(R^{tr}(q,p) + q^{-1}) = 0,$$  \hspace{1cm} (3.16)

thus can be used for representations for the generators $T_1, \cdots, T_{n-1}$ as $T_j = R^{tr}_{j,j+1}(t,p), j = 1, \cdots, n-1$. The remaining step is to construct representations for the generators $T_0$ and $T_n$. This corresponds to finding solutions to the reflection equations under the Hecke relation. We achieve this by the following degeneration scheme [12]. First, note there is a degeneration scheme to the Cremmer-Gervais $R$-matrix $R^{tr}(q,p)$ from the Shibukawa-Ueno $R$-operator $R^{ell}(\lambda)$, which is an infinite-dimensional $R$-operator acting on the function space. This scheme can be summarized in the following diagram.

$$R^{ell}(\lambda) \longrightarrow R^{tr}(\lambda) \longrightarrow R^{tr}(\lambda,q,p) \longrightarrow R^{tr}(q,p).$$  \hspace{1cm} (3.17)

We start from the elliptic $R$-operator $R^{ell}(\lambda)$ [13]. The first thing to do is to take the trigonometric limit of the $R$-operator. Next, you twist the $R$-operator and take the finite-dimensional representation in the trigonometric basis to get the spectral parameter-dependent Cremmer-Gervais $R$-matrix $R^{tr}(\lambda,q,p)$ from the infinite-dimensional $R$-operator. Finally, one deBaxterizes $R^{tr}(\lambda,q,p)$ to obtain $R^{tr}(q,p)$. Note that the trigonometric $R$-matrix we consider here is different from the standard $R$-matrix. You should not reverse the order of the degeneration. From the Shibukawa-Ueno $R$-operator, one obtains the standard $R$-matrix by first taking the finite-dimensional representation in the elliptic basis, and then taking the trigonometric limit, but not by first taking the trigonometric limit of the Shibukawa-Ueno $R$-operator and then taking its finite-dimensional representation in the trigonometric basis. What you get in this case is the Cremmer-Gervais matrix $R^{tr}(\lambda,q,p)$.

Taking advantage of this fact of the degeneration and deBaxterization scheme to obtain the $R$-matrix $R^{tr}(\lambda,q,p)$, we apply the same degeneration scheme to find solutions to the reflection relations corresponding to the $R$-matrix $R^{tr}(\lambda,q,p)$. The scheme is given as follows.

$$K^{ell}(\lambda) \longrightarrow K^{tr}(\lambda) \longrightarrow K^{tr}(\lambda,r,s) \longrightarrow K^{tr}(r,s).$$  \hspace{1cm} (3.18)

We start from the elliptic $K$-operator $K^{ell}(\lambda)$ by Hikami and Komori [16,17], which corresponds to the solution to the reflection equation of the Shibukawa-Ueno $R$-operator $R^{ell}(\lambda)$. First, we take the trigonometric limit $K^{tr}(\lambda)$ of the elliptic $K$-operator $K^{ell}(\lambda)$. Next we twist the $K$-operator and take the finite-dimensional representation in the trigonometric basis to obtain $K^{tr}(\lambda,r,s)$. Finally, we deBaxterize $K^{tr}(\lambda,r,s)$ to get $K^{tr}(r,s)$, whose explicit matrix elements are given by

$$[K^{tr}(r,s)]^k_j = s^{j-k} \begin{cases} -r^{-1}, & \text{for } j \leq k, \ j+k = N-1, \\
-1, & \text{for } j > k, \ j+k = N-1, \\
1, & \text{for } j \leq k < N-1 - j, \\
r - r^{-1}, & \text{for } N-1 - j < k < j, \\
0, & \text{otherwise}. \end{cases}$$  \hspace{1cm} (3.19)

which is the matrix given in Definition 2.2. One can show that this $K$-matrix satisfies the Hecke relation

$$(K^{tr}(r,s) - r)(K^{tr}(r,s) + r^{-1}) = 0,$$  \hspace{1cm} (3.20)

and thus can be served as representations for $T_0$ and $T_n$ as $T_0 = K^{tr}_{1,0}(t_0,s_0)$ and $T_n = K^{tr}_{n,n}(t_n,s_n)$. One can easily see the remaining relation (2.7) holds by looking at which spaces the $R$-matrices and the $K$-matrices act on nontrivially. This ends the proof that the matrices given in Definition 2.2 satisfy all the defining relations for the affine Hecke algebra of type $C$, and one can use them as representations for the generators, thus proving Theorem 2.1. The details of the calculation are given in the next section.
3.2 Rational representation

For the special case \( t = t_0 = t_1 = 1 \) of type \( C \) affine Hecke algebra \( H_n(t, t_0, t_1) \), one can obtain another representation for the algebra from the rational Jordanian \( R \)-matrix and its corresponding \( K \)-matrix. Since the degeneration [12] and deBaxterization scheme is similar, let us point out the differences. On the deBaxterization scheme, one replaces decomposition of the \( R \)-matrix \( R(\lambda) \) (3.2) and \( K \)-matrix \( K(\lambda) \) (3.9) by

\[
R(\lambda) = f(\lambda)(P + g\lambda R), \quad (3.21)
\]
\[
K(\lambda) = a(\lambda)(I + b\lambda K), \quad (3.22)
\]

and the Hecke relations in (3.3) and (3.10) by

\[
\hat{R}^2 - 1 = 0, \quad (3.23)
\]
\[
K^2 - 1 = 0. \quad (3.24)
\]

The diagram to get the rational Jordanian \( R \)-matrix \( R^{ra}(\kappa, h) \) from the Shibukawa-Ueno \( R \)-operator \( R^{el}(\lambda) \) is changed as

\[
R^{el}(\lambda) \rightarrow R^{tr}(\lambda) \rightarrow R^{ra}(\lambda) \rightarrow R^{ra}(\lambda, \kappa, h) \rightarrow R^{ra}(\kappa, h). \quad (3.25)
\]

The difference from the procedure to obtain the trigonometric Cremmer-Gervais \( R \)-matrix is that we degenerate furthermore the infinite-dimensional \( R \)-operator from the trigonometric one \( R^{el}(\lambda) \) to the rational one \( R^{ra}(\lambda) \). Then we take the finite-dimensional representation of the \( R \)-operator to get \( R^{ra}(\lambda, \kappa, h) \), and deBaxterize it to obtain \( R^{ra}(\kappa, h) \).

Correspondingly, the \( K \)-matrix for the reflection equation of the Jordanian \( R \)-matrix can be obtained in the following procedure:

\[
K^{el}(\lambda) \rightarrow K^{tr}(\lambda) \rightarrow K^{ra}(\lambda) \rightarrow K^{ra}(\lambda, \nu, g) \rightarrow K^{ra}(\nu, g). \quad (3.26)
\]

First, we degenerate the trigonometric Hikami-Komori \( K \)-operator furthermore to the rational \( K \)-operator. Then we take the finite-dimensional representation of the rational \( K \)-matrix \( K^{ra}(\lambda, \nu, g) \), and deBaxterize to obtain the rational \( K \)-matrix without spectral parameter \( K^{ra}(\nu, g) \). The explicit forms of the rational Jordanian \( R \)-matrix and \( K \)-matrix are given in Definition 2.3, and can be served as a representation for \( H_n(1, 1, 1) \). The details are given in the section 5.

4 Trigonometric representation

In this and the next sections, we give details of the proof outlined in the last section to construct representations. In this section, we consider the nonstandard trigonometric representations of type \( C \) affine Hecke algebra. The term trigonometric comes from the fact that the representation comes from quantum integrable models of trigonometric type. For completeness, we first review the degeneration scheme [12] from the Shibukawa-Ueno \( R \)-operator [13] to the trigonometric Cremmer-Gervais \( R \)-matrix. Then we give the details of obtaining solutions to the reflection equation from the Hikami-Komori \( K \)-operator. We also compare the obtained solution with our former result on the full solution space for \( N = 3 \) [19].

4.1 Cremmer-Gervais \( R \)-matrix from Shibukawa-Ueno \( R \)-operator

In this section, we review how the Cremmer-Gervais \( R \)-matrix is extracted from the Shibukawa-Ueno \( R \)-operator.
Definition 4.1 [13] Let $\mathcal{M}$ be a space of meromorphic functions of $\zeta$ on $\mathbb{C}^n$. The Shibukawa-Ueno $R$-operator $R_{jk}^{\text{ell}}(\lambda) \in \text{End}(\mathcal{M})$ is defined as

$$R_{jk}^{\text{ell}}(\lambda) = \sigma_\lambda(\zeta_j - \zeta_k; \tau) s_{jk} - \sigma_\kappa(\zeta_j - \zeta_k; \tau),$$

where

$$\sigma_\nu(\zeta; \tau) = \frac{\theta_1(\zeta + \nu; \tau) \theta_1(0; \tau)}{\theta_1(\zeta; \tau) \theta_1(\nu; \tau)}, \quad \theta_1(\zeta; \tau) = -\theta \left[ \frac{1}{\tau} \right](\zeta; \tau),$$

and

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right](\zeta; \tau) = \sum_{m \in \mathbb{Z}} \exp(\pi i \tau(m + a)^2 + 2\pi i(m + a)(\zeta + b)).$$

(4.4)

The Shibukawa-Ueno $R$-operator can be twisted [15] using the shift operator

$$(T_j(\mu)f)(\zeta_1, \cdots, \zeta_j, \cdots, \zeta_n) = f(\zeta_1, \cdots, \zeta_j + \mu, \cdots, \zeta_n),$$

as

$$\tilde{R}_{jk}^{\text{ell}}(\lambda) := T_j(-\beta) T_k(\beta) R_{jk}^{\text{ell}}(\lambda) T_j(-\beta) T_k(\beta).$$

Theorem 4.1 [13] The Shibukawa-Ueno $R$-operator $\tilde{R}_{jk}^{\text{ell}}(\lambda)$ satisfies the Yang-Baxter equation

$$\tilde{R}_{12}(\lambda) \tilde{R}_{13}(\lambda + \mu) \tilde{R}_{23}(\mu) = \tilde{R}_{23}(\lambda) \tilde{R}_{13}(\lambda + \mu) \tilde{R}_{12}(\mu).$$

The action of $\tilde{R}_{12}^{\text{ell}}(\lambda)$ on $f(\zeta_1, \zeta_2)$ is explicitly given as

$$\tilde{R}_{12}^{\text{ell}}(\lambda)f(\zeta_1, \zeta_2) = \sigma_\lambda(\zeta_1 - \zeta_2 - 2\beta; \tau)f(\zeta_2, \zeta_1) - \sigma_\kappa(\zeta_1 - \zeta_2; \tau)f(\zeta_1 - 2\beta, \zeta_2 + 2\beta).$$

(4.8)

The Cremmer-Gervais $R$-matrix can be obtained from the Shibukawa-Ueno $R$-operator as follows. First, one takes the trigonometric limit of the elliptic $R$-operator

$$\tilde{R}_{12}^{\text{tr}}(\lambda) := (2\pi i)^{-1} \lim_{\tau \to \infty} \tilde{R}_{12}^{\text{ell}}(\lambda).$$

(4.9)

The action of the trigonometric $R$-operator on the function $f(\zeta_1, \zeta_2)$ is given by

$$\tilde{R}_{12}^{\text{tr}}(\lambda)f(\zeta_1, \zeta_2) = \frac{zw_1 - z^{-1} p^2 w_2}{(z - z^{-1})(w_1 - p^2 w_2)} f(\zeta_2, \zeta_1) - \frac{qw_1 - q^{-1} p^2 w_2}{(q - q^{-1})(w_1 - p^2 w_2)} f(\zeta_1 - 2\beta, \zeta_2 + 2\beta).$$

(4.10)

where we have defined $w_j = e^{2\pi i \zeta_j}$, $z = e^{\pi i \lambda}$, $p = e^{2\pi i \beta}$, $q = e^{\pi i \kappa}$. Next, one takes the finite-dimensional representation. We restrict the space of functions $\mathcal{M}$ to the finite-dimensional subspace $V_N^{\text{tr}} \otimes V_N^{\text{tr}}$ where

$$V_N^{\text{tr}} = \text{Span}\left\{ \phi_k(\zeta) = e^{\pi i (2k - N + 1)\zeta} \middle| k = 0, 1, \cdots, N - 1 \right\}.$$}

(4.11)

Calculating the matrix elements of the trigonometric $R$-operator explicitly using $\phi_k(\zeta)$, $k = 0, 1, \cdots, N - 1$ as the basis, one gets [12]

$$\tilde{R}_{12}^{\text{tr}}(\lambda) \phi_i(\zeta_1) \phi_j(\zeta_2) = \sum_{k,l=0}^{N-1} [R^{\text{tr}}(\lambda, q, p)]_{ij}^{kl} \phi_k(\zeta_1) \phi_l(\zeta_2),$$

(4.12)
Lemma 4.1

Let an

Note that

We now apply this lemma. One finds the

Since this

relations (2.4) and (2.2) of type C affine Hecke algebra, thus we can use the Cremmer-Gervais R-matrix multiplied by the permutation matrix \( \tilde{R} \), a representation for the generators \( T_1, \cdots, T_{n-1} \) of the affine Hecke algebra of type C:

\[
T_j = \tilde{R}_j^t(t, p), \quad j = 1, \cdots, n - 1.
\]
4.2 $K$-matrix from Hikami-Komori $K$-operator

We now apply the same degeneration scheme to find solutions to the reflection equation of the $R$-matrix $R_{tr}(q,p)$. We start from the elliptic $K$-operator by Hikami and Komori.

**Definition 4.2** [16, 17] Let $\mathcal{M}$ be a space of meromorphic functions of $\zeta$ on $\mathbb{C}^n$. The Hikami-Komori $K$-operator $\mathcal{K}^\ell_j(\lambda) \in \text{End}(\mathcal{M})$ is defined as

$$\mathcal{K}^\ell_j(\lambda) = \sigma_\nu(\zeta; \tau)s_j - \sigma_{2\lambda}(\zeta; \tau),$$

where

$$(s_j f)(\zeta_1, \cdots, \zeta_j, \cdots, \zeta_n) = f(\zeta_1, \cdots, -\zeta_j, \cdots, \zeta_n).$$

The Hikami-Komori $K$-operator can be twisted using the shift operator as

$$\mathcal{K}^\ell_j(\lambda) := T_j(-\gamma)\mathcal{K}^\ell_j(\lambda)T_j(\gamma).$$

**Theorem 4.2** [16, 17] The Hikami-Komori $K$-operator $\mathcal{R}^\ell_j(\lambda)$ is a solution to the reflection equation of the Shibukawa-Ueno $R$-operator

$$\mathcal{R}^\ell_{12}(\lambda - \mu)\mathcal{R}^\ell_1(\lambda + \mu)\mathcal{R}^\ell_2(\mu) = \mathcal{R}^\ell_2(\mu)\mathcal{R}^\ell_{12}(\lambda + \mu)\mathcal{R}^\ell_1(\lambda - \mu).$$

The action of $\mathcal{K}^\ell_j(\lambda)$ on $f(\zeta)$ is explicitly given as

$$\mathcal{K}^\ell_j(\lambda) f(\zeta) = \sigma_\nu(\zeta - \gamma, \tau)f(\zeta + 2\gamma) - \sigma_{2\lambda}(\zeta - \gamma, \tau)f(\zeta).$$

Now we calculate the $K$-matrix corresponding to the Cremmer-Gervais $R$-matrix from the Hikami-Komori $K$-operator, following the same line as the previous subsection. First, one takes the trigonometric limit of the elliptic $K$-operator

$$\mathcal{K}^{tr}_j(\lambda) := (2\pi i)^{-1}\lim_{\tau \to \infty} \mathcal{K}^\ell_j(\lambda).$$

The action of the untwisted ($\gamma = 0$) trigonometric $K$-operator on the function $f(\zeta)$ is given by

$$\mathcal{K}^{tr}_j(\lambda) f(\zeta) = \frac{r - wr^{-1}}{(r - r^{-1})(w - 1)}f(\zeta) - \frac{wz^2 - z^{-2}}{(z^2 - z^{-2})(w - 1)}f(\zeta),$$

where we have defined $w = e^{2\pi i \ell}$, $z = e^{\pi i \lambda}$, $r = e^{-\pi iv}$. Next, one takes the finite-dimensional representation. We restrict the space of functions $\mathcal{M}$ to the finite-dimensional subspace $V^N$. Calculating explicitly the matrix elements of the trigonometric $K$-operator using $\phi_k(\zeta)$, $k = 0, 1, \cdots, N - 1$ as the basis, one gets the following:

**Proposition 4.1** The matrix elements $[\mathcal{K}^{tr}(\lambda, r, s)]^k_j$ of the trigonometric $K$-operator $\mathcal{K}^{tr}_j(\lambda)$ in the basis $\phi_k(\zeta)$, $k = 0, 1, \cdots, N - 1$

$$\mathcal{K}^{tr}_j(\lambda) \phi_j(\zeta) = \sum_{k=0}^{N-1} [\mathcal{K}^{tr}_j(\lambda, r, s)]^k_j \phi_k(\zeta),$$

is given by

$$[\mathcal{K}^{tr}(\lambda, r, s)]^k_j = s^{j-k} \times \begin{cases} (r^{-1}z^{-2} - rz^2)/(r - r^{-1})(z^2 - z^{-2}), & \text{for } j = k = (N-1)/2, \\ -z^{2\text{sgn}(2j-N+1)}/(z^2 - z^{-2}), & \text{for } j = k \neq (N-1)/2, \\ -r^{\text{sgn}(2j-N+1)}/(r - r^{-1}), & \text{for } k = N - 1 - j, j \neq (N-1)/2, \\ \text{sgn}(N - 1 - 2j), & \text{for } \min(j, N - 1 - j) < k < \max(j, N - 1 - j), \\ 0, & \text{otherwise.} \end{cases}$$

\[\text{\blacksquare}\]
Proof.

We first consider the case with no twist $\gamma = 0$. The case with general $\gamma$ can be obtained from $\gamma = 0$ in a simple way.

We act $K^{tr}(\lambda)$ on $\phi_j(\zeta)$

$$K^{tr}(\lambda)\phi_j(\zeta) = \frac{r - wr^{-1}}{(r - r^{-1})(w - 1)} e^{\pi i (N - 2j - 1) \zeta} - \frac{wz^2 - z^{-2}}{(z^2 - z^{-2})(w - 1)} e^{\pi i (2j - N + 1) \zeta}$$

$$= e^{\pi i (1 - N) \zeta} \left\{ \frac{r - wr^{-1}}{(r - r^{-1})(w - 1)} w^{N - 1 - j} - \frac{wz^2 - z^{-2}}{(z^2 - z^{-2})(w - 1)} w^j \right\}$$

$$= e^{\pi i (1 - N) \zeta} \frac{(z^2 - z^{-2})(r - wr^{-1}) w^{N - 1 - j} - (r - r^{-1})(wz^2 - z^{-2}) w^j}{w - 1}.$$

(4.31)

We reorganize the second factor in the last line into polynomials in $w$ as follows.

$$\frac{(z^2 - z^{-2})(r - wr^{-1}) w^{N - 1 - j} - (r - r^{-1})(wz^2 - z^{-2}) w^j}{w - 1}$$

$$= \frac{1}{w - 1} \begin{vmatrix} w^{N - 1 - j} & w^j \\ (r - r^{-1})(wz^2 - z^{-2}) & (z^2 - z^{-2})(r - wr^{-1}) \end{vmatrix}$$

$$= \frac{1}{w - 1} \begin{vmatrix} w^{N - 1 - j} - w^j \\ (w - 1)(rz^2 - r^{-1} z^{-2}) & (z^2 - z^{-2})(r - wr^{-1}) \end{vmatrix}$$

$$= (z^2 - z^{-2})(r - wr^{-1}) \frac{w^{N - 1 - j} - w^j}{w - 1} - w^j (rz^2 - r^{-1} z^{-2})$$

$$= (z^2 - z^{-2})(r - wr^{-1}) \sum_l \epsilon(j, N - 1 - j, l) w^j - w^j (rz^2 - r^{-1} z^{-2})$$

$$= (z^2 - z^{-2}) \sum_l \{r\epsilon(j, N - 1 - j, l) w^j - r^{-1} \epsilon(j, N - 1 - j, l) w^{l+1} \} - w^j (rz^2 - r^{-1} z^{-2})$$

$$= (z^2 - z^{-2}) \sum_l \{r\epsilon(j, N - 1 - j, l) - r^{-1} \epsilon(j, N - 1 - j, l - 1) \} w^j - w^j (rz^2 - r^{-1} z^{-2}).$$

(4.32)

The sum in the last line can be explicitly calculated using the definition of $\epsilon(i, j, k)$

$$\epsilon(i, j, k) = \begin{cases} 1, & \text{for } i \leq k < j, \\ -1, & \text{for } j \leq k < i, \\ 0, & \text{otherwise}, \end{cases}$$

(3.33)

as

$$r\epsilon(j, N - 1 - j, l) - r^{-1} \epsilon(j, N - 1 - j, l - 1)$$

$$= \begin{cases} (r - r^{-1}) \text{sgn}(N - 1 - 2j), & \text{for } \text{min}(j, N - 1 - j) < l < \max(j, N - 1 - j), \\ r \text{sgn}(N - 1 - 2j), & \text{for } l = j \neq N - 1 - j, \\ -r \text{sgn}(2j - N + 1), & \text{for } l = N - 1 - j \neq j, \\ 0, & \text{otherwise}. \end{cases}$$

(4.34)

Combining (4.31), (4.32) and (4.34) gives the proof of the proposition for the case with no twist $\gamma = 0$. The case with nonzero twist $\gamma$ can be included through a simple relation. Since the action of the shift operator on the trigonometric basis is diagonal

$$T(\gamma)\phi_j(\zeta) = e^{\pi ir(2k - N + 1)} \phi_j(\zeta),$$

(4.35)

one has the following simple relation for the matrix elements between the twisted and nontwisted $K$-matrices $\tilde{K}^{tr}(\lambda) = T(-\gamma)K^{tr}(\lambda)T(\gamma)$

$$[K^{tr}(\lambda, r, s)]^k_j = s^{j-k}[K^{tr}(\lambda, r)]^k_j.$$

(4.36)
where $s = e^{2\pi i \gamma}$, which concludes the proof including twist.

\begin{proposition}
The matrix $K^{tr}(\lambda, r, s)$ is a solution to the reflection equation of the Cremmer-Gervais trigonometric R-matrix $R^{tr}(\lambda, r, s)$.
\end{proposition}

\begin{proof}
This follows from the fact that the $K$-matrix is constructed as a degeneration of the Hikami-Komori $K$-operator, which is a solution to the reflection equation of the Shibukawa-Ueno $R$-operator.

So far, we found a solution $K^{tr}(\lambda, r, s)$ to the reflection equation of the Cremmer-Gervais $R$-matrix $R^{tr}(\lambda, q, p)$ with spectral parameter

$$R_{12}(\lambda - \mu)K_{1}(\lambda)R_{21}(\lambda + \mu)K_{2}(\mu) = K_{2}(\mu)R_{12}(\lambda + \mu)K_{1}(\lambda)R_{21}(\lambda - \mu). \quad (4.37)$$

To extract the representations for the generators $T_{0}$ and $T_{n}$, we use the following lemma.

\begin{lemma}
Let $R(\lambda)$ be an $R$-matrix which satisfies the properties in Lemma 4.1. If the corresponding $K$-matrix $K(\lambda)$ satisfying the reflection equation (4.37) can be decomposed as

$$K(\lambda) = a(\lambda)(I + b(\lambda)K), \quad b(\lambda) = \frac{e^{-4\pi i \lambda} - 1}{r - r^{-1}}, \quad a(\lambda) \neq 0, \quad (4.38)$$

and the $\lambda$-independent $K$ satisfies the Hecke relation

$$(K - r)(K + r^{-1}) = 0, \quad (4.39)$$

then the $\lambda$-independent $K$ satisfies the reflection equations without spectral parameter

$$R_{12}K_{1}\hat{R}_{12}K_{1} = K_{1}\hat{R}_{12}K_{1}\hat{R}_{12}, \quad (4.40)$$

$$\hat{R}_{12}K_{2}\hat{R}_{12}K_{2} = K_{2}\hat{R}_{12}K_{2}\hat{R}_{12}. \quad (4.41)$$

\end{lemma}

\begin{proof}
Multiplying the both sides of the spectral parameter dependent reflection equation (4.37) by the permutation operators $P$ and inserting the decomposition relations (4.15), (4.38) and Hecke relations (4.16), (4.39) into it, one finds the coefficients of the terms $K\hat{R}$ and $\hat{R}K$ have the following form:

$$(b(\mu) + b(\lambda))g(\lambda - \mu) + (b(\mu) - b(\lambda))g(\lambda + \mu) + (q - q^{-1})b(\mu)g(\lambda + \mu) + (r - r^{-1})b(\lambda)b(\mu)g(\lambda - \mu). \quad (4.42)$$

We can show by explicit calculation that this becomes zero. Cancelling out these vanishing terms, the remaining equations are nothing but the reflection equations without spectral parameters (4.40), (4.41).

We now apply this lemma to extract the constant $K$-matrix.

\begin{proposition}
The $K$-matrix $K^{tr}(r, s)$ whose matrix elements are explicitly given by

$$[K^{tr}(r, s)]_{j}^{k} = s^{-\lambda + \kappa} \times \begin{cases} 
- r^{-1}, & \text{for } j \leq k, \ j + k = N - 1, \\
- r, & \text{for } j > k, \ j + k = N - 1, \\
r - r^{-1}, & \text{for } j \leq k < N - 1 - j, \\
r + r^{-1}, & \text{for } N - 1 - j < k < j, \\
0, & \text{otherwise},
\end{cases} \quad (4.43)$$

is a solution to the constant reflection equation of the Cremmer-Gervais $R$-matrix $R^{tr}(p, q)$.
\end{proposition}
The $K$-matrix $K^{\text{tr}}(\lambda, r, s)$ satisfying the reflection equation (4.37) can be decomposed as

$$K^{\text{tr}}(\lambda, r, s) = e^{2\pi i \lambda}(r^{-1} - r) \left( I + \frac{e^{-4\pi i \lambda} - 1}{r - r^{-1}} K^{\text{tr}}(r, s) \right), \quad (4.44)$$

with the $\lambda$-dependent $K$-matrix $K^{\text{tr}}(r, s)$ explicitly given by (4.43). To apply Lemma 4.2, one also needs to show that $K^{\text{tr}}(r, s)$ satisfies the Hecke relation (4.39). This follows by comparing the expression

$$K^{\text{tr}}(\lambda, r, s) K^{\text{tr}}(-\lambda, r, s) = \{(r - r^{-1})^2 - (e^{2\pi i \lambda} - e^{-2\pi i \lambda})^2\} I, \quad (4.45)$$

which can be calculated using the operator expression for $K^{\text{tr}}(\lambda, r, s)$, and comparing with another expression

$$K^{\text{tr}}(\lambda, r, s) K^{\text{tr}}(-\lambda, r, s) = (r - r^{-1})^2 I + (e^{2\pi i \lambda} - e^{-2\pi i \lambda})^2 (r - r^{-1}) K - K^2, \quad (4.46)$$

obtained from the decomposition (4.44).

We have shown all the conditions the $K$-matrices must satisfy to apply Lemma 4.2, and the proposition follows from the lemma. \hfill $\square$

Lemma 4.2 and Proposition 4.3 shows that the $K$-matrix $K^{\text{tr}}(\lambda, r, s)$ satisfies the reflection relations (4.40), (4.41) and the Hecke relation (4.39) which can be identified with the defining relations (2.4), (2.6), (2.1) and (2.3) of type $C$ affine Hecke algebra, one obtains a representation for $T_0$ and $T_n$ in terms of the $K$-matrix:

$$T_0 = K^{\text{tr}}_0(t_0, s_0), \quad (4.47)$$
$$T_n = K^{\text{tr}}_n(t_n, s_n). \quad (4.48)$$

The representation for the generators in terms of the constant Cremmer-Gervais $R$-matrix and its associated $K$-matrix (4.21), (4.47) and (4.48) proves Theorem 2.1.

### 4.3 Another representation

One can obtain another representation starting from another elliptic $K$-operator [17]

$$K_j(\lambda) = \sigma_0(\zeta; \tau) s_j - \sigma_2(\zeta; \tau), \quad (4.49)$$

where

$$\sigma_\nu(\zeta; \tau) = \frac{\theta_2(\zeta + \nu; \tau) \theta_1'(0; \tau)}{\theta_2(\zeta; \tau) \theta_1(\nu; \tau)}, \quad \theta_2(\zeta; \tau) = \theta \left[ \begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right], \quad (4.50)$$

which is another solution satisfying the reflection equation of the Shibukawa-Ueno $R$-operator. We just present the results for the explicit matrix elements of the finite-dimensional $K$-matrix for $N$ odd which can be obtained in the same line as the previous subsection.

**Proposition 4.4** The following $K$-matrices $\overline{K}^{\text{tr}}(\lambda, r, s)$ and $\overline{K}^{\text{tr}}(r, s)$ is a solution to the reflection equation of the Cremmer-Gervais $R$-matrix with and without spectral parameter, respectively.

$$\left[ \overline{K}^{\text{tr}}(\lambda, r, s) \right]_{j}^{k} = s^{j-k} \times \begin{cases} (r^{-1} z^{-2} - r z^2)/(r - r^{-1})(z^2 - z^{-2}), & \text{for } j = k = (N - 1)/2, \\
-2 \text{sgn}(2j - N + 1)/(z^2 - z^{-2}), & \text{for } j = k \neq (N - 1)/2, \\
-2 \text{sgn}(2j - N + 1)/(r - r^{-1}), & \text{for } k = N - 1 - j, \ j \neq (N - 1)/2, \\
(-1)^{N-1-j-k} \text{sgn}(N - 1 - 2j), & \text{for } \min(j, N - 1 - j) < k < \max(j, N - 1 - j), \\
0, & \text{otherwise}. \end{cases}, \quad (4.51)$$

14
\[ \mathbf{K}'(r,s)^k_j = s^{j-k} \times \begin{cases} -r^{-1}, & \text{for } j \leq k, j + k = N - 1, \\ -r, & \text{for } j > k, j + k = N - 1, \\ (-1)^{N-1-j-k}\text{sgn}(N-1-2j)(r-r^{-1}), & \text{for } j \leq k < N - 1 - j, \\ 0, & \text{for } N - 1 - j < k < j, \\ (-1)^{N-1-j-k}\text{sgn}(N-1-2j)(r-r^{-1}), & \text{otherwse.} \end{cases} \] (4.52)

\section{4.4 \( N = 3 \)}

Let us compare the \( K \)-matrix obtained as a result of the degeneration and deBaxterization from the elliptic \( K \)-operator with the full solution of the reflection equation in the case \( N = 3 \). One finds that the full constant \( K \)-matrix is given by

\[ \mathbf{K}' = \begin{pmatrix} d_1 + d_5 & 0 & d_3 \\ d_4 & d_5 & d_6 \\ -d_7 & 0 & 0 \end{pmatrix}, \] (4.53)

where the solution manifold \( S \) of the parameters \( d_j, j = 1, 3, 4, 5, 6, 7 \) is given by the Segre threefold

\[ S = \{(d_1, d_3, d_4, d_5, d_6, d_7) \in \mathbb{P}^2(\mathbb{C}) \mid d_1d_5 - d_3d_7 = 0, d_1d_6 - d_3d_4 = 0, d_4d_5 - d_6d_7 = 0\}. \] (4.54)

The points in the Segre threefold can be parameterized by \( \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \) via the map

\[ \psi : \mathcal{U} = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \rightarrow S, \]

\[ \psi((D_1, D_2) \times (E_1, E_2, E_3)) = (D_1E_1, D_2E_1, D_1E_3, D_2E_2, D_2E_3, D_1E_2). \] (4.55)

The full constant \( K \)-matrix satisfies the generalized Hecke relation

\[ (\mathbf{K}')^2 - (d_1 + d_5)\mathbf{K}' + d_3d_7 = 0. \] (4.57)

The \( K \)-matrix obtained from the elliptic \( K \)-operator for \( N = 3 \)

\[ \mathbf{K}'(r,s) = \begin{pmatrix} r - r^{-1} & 0 & -s^2r \\ s^{-1}(r - r^{-1}) & -r^{-1} & s(r^{-1} - r) \\ -s^{-2}r & 0 & 0 \end{pmatrix}, \] (4.58)

lives on a submanifold \( \mathcal{V} \) of the projective space \( \mathcal{U} \) parametrizing the Segre threefold \( S \)

\[ \mathcal{V} = \{(s^{-1}, s) \times (-rs, -r^{-1}s^{-1}, r^{-1} - r) \in \mathcal{U}\} = \{(D_1, D_2) \times (E_1, E_2, E_3) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \mid D_1D_2 + 1 = 0, E_1E_2 - 1 = 0, D_1E_1 + D_2E_2 + E_3 = 0\}. \] (4.59)

We make some comments. The full rational constant \( K \)-matrix can be Baxterized to give the spectral parameter dependent \( K \)-matrix

\[ \mathbf{K}'(z) = dz^2 + (d_1 + d_5)z^4 + (1 - z^4)\mathbf{K}' = \begin{pmatrix} d_1 + d_5 + dz^2 & 0 & d_3(1 - z^4) \\ d_4(1 - z^4) & d_5 + dz^2 + d_1z^4 & d_6(1 - z^4) \\ -d_7(1 - z^4) & 0 & dz^2 + (d_1 + d_5)z^4 \end{pmatrix}. \] (4.60)

Including the parameter \( d \) which appears in the Baxterization, the solution manifold \( \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \) is lifted up to \( \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \). However, this \( K \)-matrix is not the full solution to the reflection equation of the Cremmer-Gervais \( R \)-matrix with spectral parameter. There is another solution whose solution manifold is parametrized by \( \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C}) \) which does not seem to be obtained at least from a simple Baxterization scheme. This phenomena is not observed in the standard \( R \)-matrix of \( U_q(\mathfrak{sl}_3) \) \cite{9}. See \cite{19} for more details about the full solution space of \( N = 3 \) Cremmer-Gervais \( R \)-matrix.
5 Rational representation

For the special case \( t = t_0 = t_n = 1 \) of the type \( C \) affine Hecke algebra, one can construct another nonstandard representation from quantum integrable models. We call this as nonstandard rational representation since this comes from a class of quantum integrable models of rational type. As again, we first review how the rational Jordanian \( R \)-matrix [12] is obtained from the \( R \)-operator. Then we construct the corresponding \( K \)-matrix.

5.1 Jordanian \( R \)-matrix

The Jordanian \( R \)-matrix can be obtained from the Shibukawa-Ueno \( R \)-operator as follows. First, we furthermore degenerate the trigonometric \( R \)-operator. Namely, we take the rational limit, replacing functions \( \sin(\pi i \zeta) \) of \( \zeta \) to \( \zeta \).

\[
\overline{R}^{ra}_{12}(\lambda) := \overline{R}^{ra}_{12}(\lambda)|_{\sin(\pi i \zeta) \rightarrow \zeta}.
\]

(5.1)

The action of the rational \( R \)-operator on the function \( f(\zeta_1, \zeta_2) \) is given by

\[
\overline{R}^{ra}_{12}(\lambda) f(\zeta_1, \zeta_2) = \frac{\zeta_1 - \zeta_2 - 2\beta + \lambda}{(\zeta_1 - \zeta_2 - 2\beta)\lambda} f(\zeta_2, \zeta_1) - \frac{\zeta_1 - \zeta_2 - 2\beta + \kappa}{(\zeta_1 - \zeta_2 - 2\beta)\kappa} f(\zeta_1 - 2\beta, \zeta_2 + 2\beta).
\]

(5.2)

Next, one takes the finite-dimensional representation. We restrict the space of functions \( \mathcal{M} \) to the finite-dimensional subspace \( V^a_N \otimes V^a_N \) where

\[
V^a_N = \text{Span}\{ \psi_k(\zeta) = \zeta^k \mid k = 0, 1, \cdots, N - 1 \}.
\]

(5.3)

Calculating the matrix elements of the rational \( R \)-operator explicitly using \( \psi_k(\zeta) \), \( k = 0, 1, \cdots, N - 1 \) as the basis, one gets [12, 18]

\[
\overline{R}^{ra}_{12}(\lambda) \psi_i(\zeta_1) \psi_j(\zeta_2) = \sum_{k,l=0}^{N-1} [R^{ra}(\lambda, \kappa, h)]_{ij}^{kl} \psi_k(\zeta_1) \psi_l(\zeta_2),
\]

(5.4)

with

\[
[R^{ra}(\lambda, \kappa, h)]^{kl}_{ij} = \frac{1}{\chi} \delta_{il} \delta_{jk} + (-1)^{j-l} h^{i+j-k-l} \left\{ \begin{array}{ccc} i & j & k \end{array} \right\} - \frac{\kappa}{h} \sum_m (-1)^{m-k} \left\{ \begin{array}{ccc} i & m & k \end{array} \right\} \left\{ \begin{array}{ccc} j & m-k-1 & l \end{array} \right\} \right\}.
\]

(5.5)

where \( h = -2\beta \). The \( R \)-matrix \( R^{ra}(\lambda, \kappa, h) \) is called the Jordanian \( R \)-matrix. From the construction from the Shibukawa-Ueno \( R \)-operator, the Jordanian \( R \)-matrix satisfies the Yang-Baxter relation

\[
R_{12}(\lambda) R_{13}(\lambda + \mu) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda + \mu) R_{12}(\lambda).
\]

(5.6)

We now apply the following lemma to extract the constant \( R \)-matrix.

Lemma 5.1 Let an \( R \)-matrix \( R(\lambda) \) satisfying the Yang-Baxter relation (5.6) can be decomposed as

\[
R(\lambda) = f(\lambda)(P + g\lambda R), \quad f(\lambda) \neq 0,
\]

(5.7)

and the \( \lambda \)-independent \( R \) satisfies the degenerate Hecke relation

\[
\tilde{R}^2 - I = 0,
\]

(5.8)

where \( \tilde{R} = RP \). The \( \lambda \)-independent \( R \) satisfies the Yang-Baxter relation without spectral parameter

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\]

(5.9)
We now apply this lemma. One can easily see the $R$-matrix $R^\alpha(\lambda, \kappa, h)$ (5.5) can be decomposed as

$$R^\alpha(\lambda, \kappa, h) = \frac{1}{\lambda} (P + \lambda R^\alpha(\kappa, h)), \quad (5.10)$$

with the $\lambda$-independent $R$-matrix given by

$$\left[R^\alpha(\kappa, h)\right]_{ij}^{kl} = (-1)^{l-i} h^{i+j-k-l} \left\{ \binom{i}{k} \binom{j}{l} - \frac{1}{h} \sum_m (-1)^{m-k} \binom{j+m-k-1}{l} \epsilon(j, m, k) \right\}. \quad (5.11)$$

Since this $R$-matrix can be shown to satisfy the degenerate Hecke relation (5.8), we can apply the lemma and the $R$-matrix $R^\alpha(\kappa, h)$ satisfy the Yang-Baxter relation (5.9) which is equivalent to the braid relation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \quad (5.12)$$

This braid relation (5.12) together with the Hecke relation (5.8) can be identified with the defining relations (2.4) and (2.2) of type $C$ affine Hecke algebra, thus we can use the Jordanian $R$-matrix multiplied by the permutation matrix $\hat{R}_{12}$ as a representation for the generators $T_1, \cdots, T_{n-1}$ of the special case $t = t_0 = t_n = 1$ of the affine Hecke algebra of type $C$:

$$T_j = \hat{R}_{12}^\alpha(\kappa, h), \quad j = 1, \cdots, n-1. \quad (5.13)$$

### 5.2 Jordanian $K$-matrix

Now we calculate the $K$-matrix corresponding to the Jordanian $R$-matrix from the Hikami-Komori $K$-operator. First, one takes furthermore the rational limit of the trigonometric $K$-operator $\tilde{K}^\alpha(\lambda)$:

$$\tilde{K}^\alpha(\lambda) := \tilde{K}^\alpha(\lambda)|_{\sin(\pi i \zeta) \to \zeta}. \quad (5.14)$$

The action of the rational $K$-operator on the function $f(\zeta)$ is given by

$$\tilde{K}^\alpha(\lambda)f(\zeta) = \frac{\zeta + g/2 + \nu}{(\zeta + g/2)^\nu} f(-\zeta - g) - \frac{\zeta + g/2 + 2\lambda}{2(\zeta + g/2)\lambda} f(\zeta), \quad (5.15)$$

where $g = -2\gamma$. Next, one takes the finite-dimensional representation. We restrict the space of functions $\mathcal{M}$ to the finite-dimensional subspace $V_N^\alpha$. The matrix elements of the rational $K$-operator using $\psi_k(\zeta)$, $k = 0, 1, \cdots, N - 1$ can be calculated explicitly as the basis. One gets the following:

**Proposition 5.1** The matrix elements $[K^\alpha(\lambda, \nu, g)]_j^k$ of the rational $K$-operator $\tilde{K}^\alpha(\lambda)$ in the basis $\psi_k(\zeta)$, $k = 0, 1, \cdots, N - 1$ is given by

$$[K^\alpha(\lambda, \nu, g)]_j^k = \frac{1}{2\lambda} \delta_{jk} + \frac{(-1)^j}{\nu} \binom{j}{k} g^{j-k} + 2 \sum_{0 \leq l < j} (-1)^{j-l} \binom{j-l-1}{k-l} g^{j-k-1}. \quad (5.17)$$
We act $\tilde{K}^{ra}(\lambda)$ on $\psi_j(\zeta)$

\[
\tilde{K}^{ra}(\lambda)\psi_j(\zeta) = -\frac{\zeta^j}{2\lambda} + \frac{1}{\nu}(-\zeta - k)^j + \frac{\{(\zeta + k/2) - k/2\}^j - \{(\zeta + k/2) - k/2\}^j}{\zeta + k/2}
\]

\[
= -\frac{\zeta^j}{2\lambda} + \frac{1}{\nu} \sum_k (-1)^j \left( \begin{array}{c} j \\ k \end{array} \right) \gamma^{j-k} \zeta^k + 2 \sum_{k,0 \leq l \leq j} (-1)^{j-l} \left( \begin{array}{c} j - l \leq j \\ k - l \end{array} \right) \gamma^{j-k-1} \zeta^k. \tag{5.18}
\]

The following follows from the construction procedure from the $K$-operator.

**Proposition 5.2** The matrix $K^{ra}(\lambda, \nu, g)$ is a solution to the reflection equation of the Jordanian rational $R$-matrix $R^{ra}(\lambda, \kappa, h)$.

**Proof.** This follows from the fact that the $K$-matrix is constructed as a degeneration of the Hikami-Komori $K$-operator, which is a solution to the reflection equation of the Shibuoka-Ueno $R$-operator.

So far, we found a solution $K^{ra}(\lambda, \nu, g)$ to the reflection equation of the Jordanian $R$-matrix $R^{ra}(\lambda, \kappa, h)$ with spectral parameter

\[
R_{12}(\lambda - \mu)K_1(\lambda)R_{21}(\lambda + \mu)K_2(\mu) = K_2(\mu)R_{12}(\lambda + \mu)K_1(\lambda)R_{21}(\lambda - \mu). \tag{5.19}
\]

To extract the representations for the generators $T_0$ and $T_n$, we use the following lemma.

**Lemma 5.2** Let $R(\lambda)$ be an $R$-matrix which satisfies the properties in Lemma 5.1. If the corresponding $K$-matrix $K(\lambda)$ satisfying the reflection equation (5.19) can be decomposed as

\[
K(\lambda) = a(\lambda)(I + b\lambda K), \ a(\lambda) \neq 0, \tag{5.20}
\]

and the $\lambda$-independent $K$ satisfies the Hecke relation

\[
K^2 - I = 0, \tag{5.21}
\]

then the $\lambda$-independent $K$ satisfies the reflection equations without spectral parameter

\[
\hat{R}_{12}K_1\hat{R}_{12}K_1 = K_1\hat{R}_{12}K_1\hat{R}_{12}, \tag{5.22}
\]

\[
\hat{R}_{12}K_2\hat{R}_{12}K_2 = K_2\hat{R}_{12}K_2\hat{R}_{12}. \tag{5.23}
\]

**Proof.** Multiplying both sides of the spectral parameter dependent reflection equation (5.19) by the permutation operators $P$ and inserting the decomposition relations (5.7), (5.20) and Hecke relations (5.8), (5.21) into it, one finds the coefficients of the terms $K\hat{R}$ and $\hat{R}K$ have the following form

\[
bg(\lambda + \mu)(\lambda - \mu) + bg(\mu - \lambda)(\lambda + \mu), \tag{5.24}
\]

which is obviously zero. The remaining equations are nothing but the reflection equations without spectral parameters (5.22), (5.23).

We now apply this lemma to extract the constant $K$-matrix.

**Proposition 5.3** The $K$-matrix $K^{ra}(\nu, g)$ whose matrix elements are explicitly given by

\[
[K^{ra}(\nu, g)]_{ij}^k = (-1)^j \left( \begin{array}{c} j \\ k \end{array} \right) \gamma^{i-k} + 2\nu \sum_{0 \leq l \leq j} (-1)^{i-l} \left( \begin{array}{c} j - l \leq j \\ k - l \end{array} \right) \gamma^{i-k-1} \tag{5.25}
\]

is a solution to the constant reflection equation of the Jordanian $R$-matrix $R^{ra}(\nu, g)$.
Proof.
The $K$-matrix $K^\text{ra}(\lambda, \nu, g)$ satisfying the reflection equation (5.19) can be decomposed as

$$K^\text{ra}(\lambda, \nu, g) = -\frac{1}{2\lambda}\left( I - \frac{2\lambda}{\nu} K^\text{ra}(\nu, g) \right),$$

(5.26)

with the $\lambda$-dependent $K$-matrix $K^\text{ra}(\nu, g)$ explicitly given by (5.25). The Hecke relation (5.21) which one needs to apply Lemma 5.2 follows by comparing the expression

$$K^\text{ra}(\lambda, \nu, g) K^\text{ra}(-\lambda, \nu, g) = \left( \frac{1}{4\lambda^2} + \frac{1}{\nu^2} \right) I,$$

(5.27)

which can be calculated using the operator expression for $K^\text{ra}(\lambda, \nu, g)$, and comparing with another expression

$$K^\text{ra}(\lambda, \nu, g) K^\text{ra}(-\lambda, \nu, g) = \frac{1}{4\lambda^2} I + \frac{1}{\nu^2} K^2,$$

(5.28)

obtained from the decomposition (5.26).

We have shown all the conditions the $K$-matrices should satisfy to apply Lemma 5.2, and the proposition follows.

Lemma 5.2 and Proposition 5.3 shows that the $K$-matrix $K^\text{ra}(\nu, g)$ satisfies the reflection relations (5.22), (5.23) and the Hecke relation (5.21) which can be identified with the defining relations (2.4), (2.6), (2.1) and (2.3) of type $C$ affine Hecke algebra, one obtains a representation for $T_0$ and $T_n$ in terms of the $K$-matrix:

$$T_0 = K^\text{ra}_1(\nu_0, g_0),$$

(5.29)

$$T_n = K^\text{ra}_n(\nu_n, g_n).$$

(5.30)

The representation for the generators in terms of the constant Cremmer-Gervais $R$-matrix and its associated $K$-matrix (5.13), (5.29) and (5.30) proves Theorem 2.2. Namely, one has a representation for the affine Hecke algebra $H_n(1, 1, 1)$.

5.3 $N = 3$

For $N = 3$, we find the full $K$-matrix of the constant reflection equation for the Jordanian $K$-matrix is given as

$$K^\text{ra} = \begin{pmatrix} c_1 & c_2 & c_3 \\ 0 & c_5 & c_6 \\ 0 & 0 & c_9 \end{pmatrix}.$$

(5.31)

Here the parameters $c_1, c_2, c_3, c_5, c_6, c_9$ live on the following solution manifold

$$c_2c_6 + c_3(c_1 - c_5) = 0,$$

(5.32)

$$c_2(c_1 - c_6) = 0,$$

(5.33)

$$(c_1 - c_3)(c_1 - c_9) = 0.$$

(5.34)

The full constant $K$-matrix satisfies the generalized Hecke relation

$$(K^\text{ra})^2 - (c_5 + c_9) K^\text{ra} + c_5c_9 = 0.$$

(5.35)

Examining the solution manifold, one finds the constant $K$-matrix can be furthermore divided into two types

$$K^\text{ra}_1 = \begin{pmatrix} c_1 & c_2 & \alpha c_2 \\ 0 & c_5 & \alpha(c_5 - c_1) \\ 0 & 0 & c_1 \end{pmatrix}, \quad \alpha \in \mathbb{C},$$

(5.36)
and
\[ K^{ra}_{II} = \begin{pmatrix} c_1 & 0 & c_3 \\ 0 & c_1 & c_6 \\ 0 & 0 & c_9 \end{pmatrix} \quad (5.37) \]

The solution manifold \( \mathcal{B} \) of the first solution \( K^{ra}_{II} \) is the projective space \( \mathbb{P}^3(\mathbb{C}) \)
\[ \mathcal{A} = \{(c_1, c_2, c_5, \alpha) \in \mathbb{P}^3(\mathbb{C}) \} \quad (5.38) \]

The solution obtained as a degeneration from the elliptic \( K \)-operator
\[ K^{ra} = \begin{pmatrix} 1 & -\nu - 2g & g^2 + 2g\nu \\ 0 & -1 & 2g \\ 0 & 0 & 1 \end{pmatrix} \quad (5.39) \]

multiplied by an overall factor lives on a hyperplane \( \mathcal{B} \) of the solution manifold \( \mathcal{A} \) of the first solution \( K^{ra}_{II} \)
\[ \mathcal{B} = \{(c_1, c_2, c_5, \alpha) \in \mathbb{P}^3(\mathbb{C}) \mid c_5 = -c_1 \} \quad (5.40) \]

The solution obtained as a degeneration and deBaxterization from the \( K \)-operator can construct representations only for the special case \( H_n(1,1,1) \). On the other hand, the Hecke relation (5.35) shows that the full solution can construct representations for \( H_n(1,\ell_0,\ell_n) \). We finally remark that the full constant \( K \)-matrix can be Baxterized to give the spectral parameter-dependent \( K \)-matrix
\[ K^{ra}(\lambda) = c - \lambda(c_5 + c_9) + 2\lambda K^{ra} \]
\[ = \begin{pmatrix} c + \lambda(2c_1 - c_5 - c_9) & 2\lambda c_2 & 2\lambda c_3 \\ 0 & c + \lambda(c_5 - c_9) & 2\lambda c_6 \\ 0 & 0 & c + \lambda(c_9 - c_5) \end{pmatrix} \quad (5.41) \]

6 Discussion

In this paper, we constructed explicit nonstandard representations of type \( C \) affine Hecke algebra. Concretely realizing representations of affine Hecke algebra is not an easy problem. We can approach this problem by using the power of quantum integrable models. For type \( C \) affine Hecke algebra, we achieved this by using two classes of quantum integrable models under the reflecting boundary condition. The nonstandard Cremmer-Gervais \( R \)-matrix serves as representations for the generators of type \( A \) affine Hecke algebra since it satisfies the Hecke relation as well as the Yang-Baxter relation. To construct nonstandard representations for type \( C \) is equivalent to finding solutions of reflection equation under the Hecke relation (see [20, 21] for standard representations of type \( C \) affine Hecke algebra based on standard \( R \) and \( K \)-matrices).

We constructed them by taking appropriate degeneration and deBaxterization of the Hikami-Komori elliptic \( K \)-operator. We also constructed another representation for a special case of type \( C \) affine Hecke algebra from the rational Jordanian \( R \)-matrix and its corresponding \( K \)-matrix, also achieved by the degeneration and deBaxterization scheme from the \( R \)-operator and the \( K \)-operator. The degeneration procedure in [12] and in this paper seems to show a systematic way of constructing representations of affine Hecke algebras not found yet. Finding nonstandard representations do not have to seem recipes. However, starting from the infinite-dimensional operators and taking finite-dimensional representations can yield nonstandard representations. Showing relations at the level of operators are much easier than at the level of finite-dimensional representations. It may be worth investigating affine Hecke algebras associated with other root systems in this way for example. It may be worthwhile to investigate affine Hecke algebras of type \( C \) from the point of view of boundary quantum group [22]. It may also be interesting to use the results in this paper to formulate and study boundary (type \( C \)) analogue of the nilpotency indices of the \( R \)-matrices (type \( A \) [23, 24], or to relate other integrable systems such as the classical top (see [25] for example of relating nonstandard \( R \)-matrix with integrable tops).
Acknowledgments

This work was partially supported by grants-in-aid for Scientific Research (C) No. 24540393 and for Young Scientists (B) No. 25800223.

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