STATISTICS | RESEARCH ARTICLE

Some compactness results by elliptic operators

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Abstract: In this paper, we get two compactness results for complete manifolds by applying a (sub-) elliptic second-order differential operator on distance functions. The first is an extension of a theorem of Galloway and gets an upper estimate for the diameter of the manifold and the second is an extension of a theorem of Ambrose.

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1. Introduction

One of the most important and celebrated results in Riemannian geometry is the Myer’s compactness theorem (Myers, 1941). This major theorem and its generalizations have many applications (Alvarez et al., 2015; Ambrose, 1957; Frankel & Galloway, 1981; Galloway, 1981). It states if $M$ is a complete Riemannian manifold and its Ricci curvature is bounded below by $(n-1)\alpha > 0$, then $M$ is compact and its diameter satisfies $\text{diam}(M) \leq \frac{\alpha}{n-1}$, also by the same argument for the universal covering space, one can conclude that $M$ has finite first fundamental group. We recall two important generalizations of this theorem. The first is about Galloway’s theorem as follows.

Theorem 1.1. (Galloway, 1981)(Galloway)Let $M$ be a complete Riemannian manifold and for any unit vector field $X$, one has

$$\text{Ric}(X,X) \geq \alpha + \langle \nabla \varphi, X \rangle,$$

where $\alpha$ is a positive constant and $\varphi$ is any smooth function satisfying $|\varphi| \leq c$. Then $M$ is compact and its diameter is bounded from above by

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PUBLIC INTEREST STATEMENT

The compactness theorem by Myer’s and volume comparison theorem by Bishop-Gromov are essential tools in differential geometry and analysis on manifolds. In this paper, by using a elliptic second-order differential operator on distance functions, we give two compactness results for complete manifolds.

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Theorem 1.2. (Ambrose, 1957) Let $M$ be a complete Riemannian manifold and $p$ be a fixed point for which every geodesic $\gamma(t)$ emanating $p$ satisfies,
\[
\int_0^1 \text{Ric}(\gamma'(t), \gamma'(t)) = \infty.
\]

Then $M$ is compact.

The second important generalization, is a theorem of Ambrose. Namely,

**Theorem 1.3. (Wei & Willie, 2009)** Let $M$ be a complete Riemannian manifold and $f \in C^\infty(M)$. If $|f| \leq k$ and
\[
\text{Ric} + \text{Hess}f \geq (n - 1)H > 0,
\]

then $M$ is compact and its diameter satisfies,
\[
diam(M) \leq \frac{\pi}{\sqrt{H}} + \frac{4k}{(n - 1)\sqrt{H}}.
\]

This result was extended to the operator $\Delta_{\nu}u := \Delta u + \langle V, \nabla u \rangle$, where $V \in X(\mathbb{M})$ by J.Wu in (Wu, 2017). Also M. P. Cavalcante and et al. extended Theorems 1.1 and 1.2 for weighted manifolds as follows.

**Theorem 1.4. (Weighted Galloway theorem) (Cavalcante et al., 2015)** Let $M$ be a complete Riemannian weighted manifold, $V$ be a smooth vector field, and for any unit vector field $X$ on $M$ one has,
\[
\text{Ric}^V = \text{Ric}(X, X) + \frac{1}{2}(\Delta_{\nu}g)(X, X) - \frac{1}{k} V^* \otimes V^* \geq (n - 1)c + \langle V\varphi, X \rangle,
\]

where $k$ and $c$ are positive constant and $\varphi$ is a smooth function with $|\varphi| \leq c$. Then $M$ is compact and its diameter satisfies,
\[
diam(M) \leq \frac{\pi}{\sqrt{(n - 1)c}} \left( \frac{b}{\sqrt{(n - 1)c}} + \sqrt{\frac{b^2}{(n - 1)c} + (n - 1) + 4k} \right).
\]

**Theorem 1.5. (Weighted Ambrose theorem) (Cavalcante et al., 2015)** Let $M$ be a complete Riemannian weighted manifold, $p$ a fixed point and for each geodesic $\gamma(t)$ emanating from $p$ we have,
\[
\int_0^{\infty} \text{Ric}^V_{\gamma}(\gamma'(t), \gamma'(t))dt = \infty,
\]

where $\text{Ric}^V_{\gamma} = \text{Ric} + \text{Hess}f - \frac{1}{k} df \otimes df$ and $k \in (0, \infty)$. Then $M$ is compact.
In this paper, we generalize Theorems 1.1 and 1.2 by means of some kind of (sub-)elliptic operators and lower bound of the extended Ricci tensor $\text{Ric}(X,A) - \frac{1}{2}(\triangle g)(X,X)$ as follows, the first is an extension of Galloway’s theorem.

**Theorem 1.6.** Let $M$ be a complete Riemannian manifold, $A$ a $(1,1)$-self adjoint tensor field, $v \in \mathcal{X}(\mathcal{M})$ a smooth vector field and $\varphi$ a smooth function with $|\varphi| \leq K_0$. Fixed $p \in M$ and define $r(x) = \text{dist}(p,x)$. Assume $H>0$ be some constant and the following conditions are satisfied,

- for some constant $H>0$ and any unit vector field $X \in \mathcal{X}(\mathcal{M})$ we have,

$$\text{Ric}(X,A) - \frac{1}{2}(\triangle g)(X,X) \geq (n-1)\delta_n H|X|^2 + (\nabla \varphi,X),$$

- $\text{sec}(M) \geq -G(r(x))$ and $\lim_{t\to 0^+} t^2 \ln(f(t)) = 0$, where $G(t)$ and $f(t)$ are defined in Lemma 3.1,

- $|f^2| \leq K_2$,

- $|V| \leq K_6$,

- $|\text{div}A| \leq K_3$,

- $K_4 \leq \text{tr}(A) \leq K_5$,

- for some smooth function $K_1 : \mathbb{R} \to \mathbb{R}$, we have

$$\left|\langle X, T^2(\triangledown X)\rangle\right| \leq \frac{1}{n} K_1(r(x))|X|^2, \forall X \in \mathcal{X}(\mathcal{M}),$$

such that

$$\int_0^{\infty} K_1(t) \frac{\gamma(t)}{|H|} dt \leq K_7,$$

where

$$K_1(t) = \begin{cases} \sin^2 \left(\sqrt{\gamma(t)}\right) K_1(t); & t \leq \frac{\pi}{\sqrt{\gamma(t)}}, \\ K_1(t); & t \geq \frac{\pi}{\sqrt{\gamma(t)}}. \end{cases}$$

for some constants $K_2, K_3, K_6, K_7$. Then $M$ is compact, its fundamental group is finite and its diameter satisfies,

$$\text{diam}(M) \leq \frac{\pi}{\sqrt{H}} + \frac{1}{\delta_n(n-1)H} \left( \frac{K_6}{(n-1)} + K_3 + K_7 + 2\sqrt{H(K_2 + K_5)} \right).$$

The second result is an extension of theorem of Ambrose.

**Theorem 1.7.** Let $M$ be a complete Riemannian manifold, $A$ a $(1,1)$-self adjoint tensor field, $p \in M$ be a fixed point and define $r(x) = \text{dist}(p,x)$, assume the following conditions are satisfied,

- there exist some constant $M$ such that $|\nabla f^2 + \nabla \text{tr}(A) - \text{div}A|<M$,

- $\text{sec}(M) \geq -G(r(x))$ and $\lim_{t\to 0^+} t^2 \ln(f(t)) = 0$, where $G(t)$ and $f(t)$ are defined in Lemma 3.1,

- $\left|\langle X, T^2(\triangledown X)\rangle\right| \leq \frac{1}{n} K_1(r(x))|X|^2, \forall X \in \mathcal{X}(\mathcal{M})$, where $K_1(r)$ is some function, such that

$$\int_1^{\infty} K_1(t) \frac{\gamma(t)}{|H|} dt < \infty,$$

- along every geodesic $\gamma(t)$ emanating from $p$ one has $\lim_{t\to +\infty} \int_1^t \text{Ric}(\gamma'(t), A\gamma'(t)) dt = \infty$, then $M$ is compact and its fundamental group is finite.
2. Preliminaries
In this section, we present the preliminaries. Throughout the paper \( M = (M, \langle , \rangle) \) is a complete Riemannian manifold. First, we give some definitions.

**Definition 2.1.** A self-adjoint operator \( A \) on \( M \) is a \((1, 1)\)-tensor field with the following property,
\[
\forall X, Y \in \mathfrak{X}(\mathbb{M}), \langle \mathcal{A}X, Y \rangle = \langle X, \mathcal{A}Y \rangle.
\]

Now we define bounded operator \( A \) as follows.

**Definition 2.2.** Let \( A \) be a self-adjoint positive definite operator on \( M \), \( A \) is called bounded if there are constants \( \delta_0 > 0 \) such that \( 0 \leq \langle X, AX \rangle < \delta \), for any unit vector field \( \forall X, Y \in \mathfrak{X}(\mathbb{M}), \langle \mathcal{A}X, Y \rangle = \langle X, \mathcal{A}Y \rangle \).

By the following definition, we give some notations about second-order differential operator \( L \) on a manifold \( M \) with \( L1 = 0 \). In fact a second-order differential \( L \) with \( L1 = 0 \) can be written as
\[
Lu = \Delta_A u + X.u = \text{div}(A\nabla u) + Y.u,
\]
where \( X, Y \) are some suitable vector fields and \( X = \text{div} + Y.\)

**Definition 2.3.** Let \( A \) be a self-adjoint operator on \( M, X \in \mathfrak{X}(\mathbb{M}) \) and \( u \in \mathcal{C}^\infty(M) \), We define
\[
a) \quad L_{\mathcal{A}X}(u) := \text{div}(A\nabla u) + \langle X, \nabla u \rangle,
\]
\[
b) \quad \Delta_L (u) := \sum_i (\nabla^2 u)_i A e_i,
\]
\[
c) \quad \Delta_{\mathcal{A}X} (u) := \Delta(u) + \langle \nabla u, X \rangle.
\]

**Definition 2.4.** Let \( A \) be a \((1, 1)\)-tensor field on \( M \). Define \( T^A \) as,
\[
T^A(X, Y) := \langle \nabla_X A Y - (\nabla_Y A) X \rangle.
\]

It is clear that \( T^A \) is a \((2, 1)\) tensor field.

**Example 2.5.** If \( A \) is the shape operator of a hypersurface \( \Sigma^n \subset M^{n+1} \) then
\[
T^A(Y, X) = (R(Y, X) N)^T,
\]
where \( R \) is the curvature tensor of \( M \) and \( N \) is a unit normal vector field on \( \Sigma^n \subset M^{n+1} \).

We recall the following extended Bochner formula from (Alencar et al., 2015; Gomes & Miranda, 2018) to prove Theorem 2.10 which is the main tools to get the compactness results.

**Proposition 2.6.** Let \( A \) be a self-adjoint operator on \( M \), then,
\[
\frac{1}{2} \Delta_L (|\nabla u|^2) = tr(A \circ \text{hess}^2(u)) + \langle \nabla u, \nabla (\Delta_L u) \rangle - \Delta_L (\nabla u) + \text{Ric}_A (\nabla u, \nabla u),
\]
where \( \text{Ric}_A \) is defined in (Fatemi & Azami, 2018).

The term \( \Delta_L (\nabla u) \) in (2.1) is very complicated and depends on the algebraic and analytic properties of the tensor field \( A \). So we try to simplify it to get the better estimates. First, by the following Lemmas, we show some relations about the second covariant derivative of the tensor field \( A \).
Lemma 2.7. Let $A$ be a $(1,1)$-self-adjoint tensor field on $M$ and $X, Y, Z \in \mathfrak{X}(\mathcal{M})$, then

a) $\left(\nabla^2 A\right)(X, Y, Z) = \left(\nabla^2 A\right)(Y, Z, X) + R(Z, Y)(AX) - A(R(Z, Y)X),$

b) $\left(\nabla^2 A\right)(X, Y, Z) - \left(\nabla^2 A\right)(Y, X, Z) = \left(\nabla^2 T^A\right)(Y, X).

Proof. For part (a) we have,

$$\nabla^2 A(X, Y, Z) = \nabla_2(\nabla A)(X, Y, Z) = \nabla_2(\nabla A)(X, Y)$$

$$= \nabla_2(\nabla A)(X, Y, Z) = \nabla_2(\nabla A)(X, Y)$$

$$= \nabla_2(\nabla A)(X, Y) - \nabla_2(\nabla A)(X, Y)$$

$$= \nabla_2(\nabla A)(X, Y) - \nabla_2(\nabla A)(X, Y)$$

Similarly,

$$\nabla^2 A(X, Z, Y) = \nabla_2(\nabla A)(X, Z, Y) = \nabla_2(\nabla A)(X, Z) - \nabla_2(\nabla A)(X, Z)$$

Thus

$$\nabla^2 A(X, Y, Z) - \nabla^2 A(X, Z, Y) = \nabla_2(\nabla A)(X, Y) - \nabla_2(\nabla A)(X, Z)$$

$$= \nabla_2(\nabla A)(X, Y) - \nabla_2(\nabla A)(X, Z)$$

For part (b), by definition of $T$, we have

$$\nabla^2 A(X, Y, Z) = \nabla_2(\nabla A)(X, Y, Z) = \nabla_2(\nabla A)(X, Y, Z) + T^A(Y, X)$$

$\nabla^2 A(X, Z, Y) = \nabla_2(\nabla A)(X, Z, Y) = \nabla_2(\nabla A)(X, Z, Y)$

$$= \nabla_2(\nabla A)(X, Z, Y) - \nabla_2(\nabla A)(X, Z, Y)$$

$$= \nabla_2(\nabla A)(X, Z, Y) - \nabla_2(\nabla A)(X, Z, Y)$$

Lemma 2.8. Let $A$ be a $(1,1)$ – self-adjoint tensor field on $M$, then

$$(\Delta A)(X, X) = \langle \nabla^2 \text{div} A, X \rangle - \text{Ric}_A(X, X) + \text{Ric}(X, AX) + \langle \nabla^* T^A(X), X \rangle,$$

where $\nabla^*$ is adjoint of $\nabla$ and

$\nabla^* T^A(X) = \sum_i (\nabla_e T^A)(e_i, X).$

Proof. For simplicity let $\{e_i\}$ be an orthonormal local frame field in a normal neighborhood of $p$ such that with $\nabla_e e_j = 0$ at $p$. At $p$ Lemma 2.7 implies,

$$\langle \Delta A, X, X \rangle = \sum_i \langle \nabla^2 A, X, e_i, e_i \rangle = \sum_i \langle \nabla^2 A, e_i, e_i, X \rangle + \langle \nabla^* T^A, X \rangle.$$

So by Lemma 2.7, part (a) we have

$$\langle \Delta A, X, X \rangle = \sum_i \langle \nabla^2 A, e_i, e_i, X \rangle + \langle \nabla^* T^A, X \rangle$$

$$= \sum_i \langle \nabla^2 A, e_i, e_i, X \rangle + \langle \nabla^* T^A, X \rangle$$

Now, we ready to simplify the term $\Delta \nabla^* A\mu$ in (1).
Proposition 2.9. Let $A$ be a $(1,1)$-self-adjoint tensor field on $M$ and $u \in C^\infty(M)$, then

\[
\Delta_{(v,v,A)} u = \nabla u \nabla \text{tr}(A) - \langle (\nabla v_i \text{div} A), \nabla u \rangle + \sum_i \langle e_i, T^A(\nabla u, \nabla u) \rangle \\
+ \sum_i \langle T^A(\nabla u, \nabla u), e_i \rangle + \text{Ric}_A(\nabla u, \nabla u) - \text{Ric}(\nabla u, A \nabla u).
\]  

(2.2)

Proof. Let $A$ be a $(1,1)$-tensor field, then

\[
\Delta_{(v,v,A)} u = \sum_i \langle \nabla e_i (\nabla v_i A) \nabla u \rangle + \sum_i \langle \nabla e_i \nabla u, T^A(\nabla u, e_i) \rangle
\]

\[
= \sum_i \langle \nabla e_i (\nabla v_i A) \nabla u \rangle - \sum_i \langle \nabla e_i (\nabla v_i A^2) \nabla u \rangle \\
- \sum_i \langle \nabla e_i \nabla v_i (\nabla v_i A) \nabla u \rangle + \sum_i \langle \nabla e_i \nabla v_i u, T^A(\nabla u, e_i) \rangle
\]

\[
= \sum_i \langle \nabla e_i (\nabla v_i A) e_i \rangle + \sum_i \langle \nabla e_i T^A(\nabla u, e_i) \rangle - \langle \nabla v_i (\Delta A) \nabla u \rangle
\]

\[
- \Delta_{(v,v,A)} u + 2 \sum_i \langle \nabla e_i \nabla v_i u, T^A(\nabla u, e_i) \rangle.
\]

Note

\[
\sum_i \langle \nabla e_i T^A(e_i, \nabla u) \rangle + 2 \sum_i \langle \nabla e_i \nabla u, T^A(\nabla u, e_i) \rangle
\]

\[
= \sum_i \langle \nabla e_i \nabla u, T^A(e_i, \nabla u) \rangle + \sum_i \langle \nabla u, (\nabla v_i T^A)(e_i, \nabla u) \rangle \\
+ \sum_i \langle \nabla u, T^A(e_i, \nabla e_i) \rangle + 2 \sum_i \langle \nabla e_i \nabla u, T^A(\nabla u, e_i) \rangle
\]

\[
= \sum_i \langle \nabla u, (\nabla v_i T^A)(e_i, \nabla u) \rangle + \sum_i \langle \nabla u, T^A(e_i, \nabla e_i) \rangle
\]

\[
= \langle \nabla u, (\nabla v_i T^A)(\nabla u) \rangle + \sum_i \langle \nabla u, T^A(e_i, \nabla e_i) \rangle + \sum_i \langle \nabla v_i \nabla u, T^A(\nabla u, e_i) \rangle
\]

\[
= \langle \nabla u, (\nabla v_i T^A)(\nabla u) \rangle + \sum_i \langle e_i, T^A(\nabla u, \nabla e_i) \rangle.
\]

In other words,

\[
\Delta_{(v,v,A)} u = \langle \nabla u, \text{div}(\nabla v_i A) \rangle - \langle \nabla u, (\Delta A) \nabla u \rangle + \langle \nabla u, (\nabla v_i T^A)(\nabla u) \rangle
\]

\[+ \sum_i \langle e_i, T^A(\nabla u, \nabla e_i) \rangle.
\]

But,

\[
\langle \nabla u, \text{div}(\nabla v_i A) \rangle = \sum_i \langle \nabla u, (\nabla v_i (\nabla v_i A)) e_i \rangle = \sum_i \langle (\nabla v_i (\nabla v_i A)) \nabla u, e_i \rangle
\]
\[
\sum_i \left( \langle \nabla_V (\nabla_V A) \rangle e_i + T^{(\nabla_V A)}(e_i, \nabla u), e_i \rangle \right)
\]

\[
= \nabla u. \nabla u. \text{tr}(A) + \sum_i \left( T^{(\nabla_V A)}(e_i, \nabla u), e_i \right).
\]

\[
\Delta_{(\nabla_V A)} u = \nabla u. \nabla u. \text{tr}(A) - \langle \nabla u, (\nabla u)^2 \rangle + \langle \nabla u, (\nabla u) \rangle + \langle \nabla u, (\nabla u) \rangle
\]

Finally the result concludes by Lemma 2.8.

Here is another extension of Bochner- formula, which we use it as one of the mail tools to get the compactness results.

**Theorem 2.10.** (type-Bochner formula)Let \( X, Y, Z \in \mathfrak{X}(\mathcal{M}) \) and \( A \) be a \((1,1)\)-self-adjoint tensor field on \( M \), then for any smooth function \( u \) we have,

\[
\frac{1}{2} \Delta_A (\nabla u)^2 = \text{tr}(A \circ \text{hess}^2 u) + \langle \nabla u, \nabla (\Delta_A u) \rangle - \nabla u. \nabla u. \text{tr}(A) + \langle \nabla u. \nabla A \rangle. \nabla u
\]

\[
- \sum_i \left( e_i, T^A (\nabla u, e_i, \nabla u) \right) - \sum_i \left( T^{(\nabla_V A)}(e_i, \nabla u), e_i \right) + \text{Ric}(\nabla u, A \nabla u).
\]

Finally, the result follows, by considering

\[
\nabla u. \langle V, \nabla u \rangle - \frac{1}{2} \langle \text{Lap}(V), \nabla u, \nabla u \rangle
\]

\[
= \langle \nabla_V V, \nabla u \rangle + \langle V, \nabla_V \nabla u \rangle - \frac{1}{2} \langle V, \nabla u, \nabla u \rangle + \langle [V, \nabla u], \nabla u \rangle
\]

\[
= \langle \nabla_V V, \nabla u \rangle + \langle V, \nabla_V \nabla u \rangle - \frac{1}{2} \langle V, \nabla u, \nabla u \rangle + \langle V, \nabla u - \nabla_V V, \nabla u \rangle
\]

\[
= \langle \nabla_V V, \nabla u \rangle + \langle V, \nabla_V \nabla u \rangle - \frac{1}{2} \langle V, \nabla u, \nabla u \rangle + \langle V, \nabla_V u, \nabla u \rangle - \langle V, \nabla_V V, \nabla u \rangle
\]

\[
= \langle V, \nabla_V \nabla u \rangle - \frac{1}{2} \langle V, \nabla V, \nabla u \rangle + \langle V, \nabla_V u, \nabla u \rangle
\]

\[
= 2 \text{Hess}(V, \nabla u) - \frac{1}{2} \langle V, \nabla V, \nabla u \rangle = \text{Hess}(V, \nabla u).
\]
3. Extended Laplacian comparison theorem

In this section we shall extend the mean curvature comparison theorem by some (sub-) elliptic operators, so we apply Theorem 2.10 to the distance function \( r(x) = \text{dist}(p, x) \), where \( p \) is a fixed point. First, we need some estimate for terms \( \sum_i \langle e_i, T^A(\partial_r, \nabla_x e_i) \rangle \) and \( \sum_i \langle (\nabla^{X + A})_i(\partial_r, e_i), e_i \rangle \).

Lemma 3.1. Let \( M \) be a complete Reimannian manifold and \( p \) be a fixed point, \( r(x) = \text{dist}(p, x) \). Assume the radial sectional curvature of \( M \) satisfies \( \text{sec}_{rad} M \geq -G(r(x)) \) and there is some smooth function \( K_1 : \mathbb{R} \rightarrow \mathbb{R} \), which satisfies

\[
|\langle X, T^A(\partial_r, X) \rangle||X| \leq \frac{1}{n} K_1(r(x))|X|^2, \quad \forall X \in (M),
\]

then we have the following estimates,

\[
\sum_i \langle e_i, T^A(\partial_r, \nabla_x e_i) \rangle \leq K_1(r(x)) \left( \frac{f'(r(x))}{f(r(x))} \right),
\]

where

\[
\begin{align*}
f'' - Gf &= 0, \\
f(0) &= 0, \quad f'(0) = 1,
\end{align*}
\]

(3.1)

where \( G : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function.

**Proof.** By \( \text{sec}_{rad} M \geq -G(r(x)) \) we have the following estimate for \( \text{Hessr}(X, X) := \langle \nabla_x \nabla r, X \rangle \), (see (Pigola et al., 2008))

\[ \text{Hessr}(X, X) \leq \frac{f'(r)}{f(r)} \left( |X|^2 - \langle \partial_r, X \rangle^2 \right). \]

So for the local orthonormal vector field \( \{e_i\} \) which diagonalize \( \text{hessr}(X) := \nabla_x \nabla r \) we have, \( h_{ii} := \text{Hessr}(e_i, e_i) \)

\[
\sum_i \langle e_i, T^A(\partial_r, \nabla_x e_i) \rangle = \sum_i h_{ii}(e_i, T^A(\partial_r, e_i)) \leq n \max_i |h_{ii}| \max_i \langle e_i, T^A(\partial_r, e_i) \rangle \]

\[
\leq K_1(r(x)) \left( \frac{f'(r)}{f(r)} \right).
\]

To approximate \( \sum_i \langle (\nabla^{X + A})_i(\partial_r, e_i), e_i \rangle \), we give the following definition.

**Definition 3.2.** Let \( A \) be a \((1, 1)\) - tensor field on a complete Riemannian manifold \( M^n \), \( p \in M \) be a fixed point and \( r(x) := \text{dist}(p, x) \) be the distance function, we define \( f^A : M \rightarrow \mathbb{R} \) with the following property,

\[
\sum_i \langle T^{(X + A)}_i(\partial_r, e_i), e_i \rangle \leq \text{Hess}^A(\partial_r, \partial_r).
\]

**Remark 3.3.** Here, we give some description and example about \( f^A \). Note that, when \( \nabla^2 A = 0 \), then \( f^A = 0 \). In fact \( f^A \) is not unique and it depends on the Algebraic and Analytic properties of the tensor field \( A \). In general, let,

\[
K(r) := \max_{X, Y, M, \|X\| = 1, \|Y\| = 1, \|r\| = r} \sum_i \langle T^{(X + A)}_i(e_i, X), e_i \rangle |X|,
\]

where the radial sectional curvature satisfies \( \text{sec}_{rad} \leq -G \) and \( f \) be the solution of differential equation (3.1), then \( f^A \) can be the solution of the following differential inequality,
\[ K(r) \leq \left( f^r + \frac{f}{f} (f^r) \right) \bigg( (f^r) \bigg) > 0. \]

Here we get an extension of mean curvature comparison theorem.

**Theorem 3.4 (Extended mean curvature comparison)** Let \( M \) be a complete Riemannian manifold, \( A \) a \((1,1)\)-self-adjoint tensor field, \( V \in \Xi(\Omega) \) a smooth vector field and \( \varphi \) a smooth function with \( |\varphi| \leq K_0 \). Fixed \( p \in M \) and define \( r(x) = \text{dist}(p, x) \). Assume \( H > 0 \) be some constant and the following conditions are satisfied,

a) for any unit vector field \( X \in \Xi(\Omega) \) we have,
\[
\text{Ric}(V, AX) - \frac{1}{2} (\mathcal{L}_V g)(X, X) \geq (n - 1) \delta_n H |X|^2 + \langle \nabla \varphi, X \rangle.
\]

b) \( K_n \leq \text{tr}(A) \leq K_S \),

c) \( |f^r| \leq K_2 \) for some constant \( K_2 \)

d) \( \text{sec}_\Omega M \geq -G(r(x)) \) and \( \lim_{t \to 0} t^2 \ln(f(t)) = 0 \), where \( G(t) \) and \( f(t) \) are defined in Lemma 3.1.

e) \( |\text{div} A| \leq K_3 \),

f) \( |V| \leq K_6 \).

Then along any minimal geodesic segment from \( x_0 \) we have,

a) for \( r \leq \frac{1}{\sqrt{2}n} \),
\[
L_{A, V} (f^r + \text{tr}(A)) r \leq \delta_n \left( 1 + \frac{n^2 + 2K_2 - 2K_0}{\delta_n n^4} \right) (\Delta_H r) + \left( \frac{K_0}{n^2 - 1} + K_3 + 2K_0 \right)
+ \frac{1}{n^2 - 1} \int_0^r \sin^2(t) \frac{f(t)}{f(t)} \, dt.
\]

b) for \( \frac{1}{\sqrt{2}n} \leq r \leq \frac{1}{2\sqrt{2}n} \),
\[
L_{A, V} (f^r + \text{tr}(A)) r \leq \delta_n \left( 1 + \frac{n^2 + 2K_2 - 2K_0}{\delta_n n^4} \right) (\Delta_H r) + \left( \frac{1}{n^2 - 1} K_6 + K_3 + 2K_0 \right)
+ \frac{1}{n^2 - 1} \int_0^r \sin^2(t) \frac{f(t)}{f(t)} \, dt.
\]

**Proof of Theorem 3.4.** We are inspired by the proof of (Fatemi & Azami, 2018). By Lemma 3.1 and Theorem 2.10, we get the following differential inequality,

\[
0 \geq \frac{(\Delta_H r)^2}{(n - 1) \delta_n} + \partial r. \Delta_H r - \partial r. \partial r. \text{tr}(A) + \partial r. \langle \text{div} A, r \rangle - K_1 (\frac{f'(r)}{f(r)}) - \partial r. \partial r. f^r
+ \text{Ric}(\partial r, A \partial r) - \frac{1}{2} (\mathcal{L}_V g)(\partial r, \partial r).
\]

Let \( y(t) \) be a minimal geodesic through the point \( x_0 \). Then,
\[
0 \geq \frac{(\Delta_H r)^2}{(n - 1) \delta_n} + (\Delta_H r') - (\text{tr}(A) + f^r(t)) \bigg( (\text{tr}(A) + f^r(t))' (t) + \langle \text{div} A, y'(t) \rangle' - K_1 (\frac{f'(t)}{f(t)}) - \partial r. \partial r. f^r
+ (n - 1) \delta_n H + \varphi'(r)).
\]

On the space form \( M^n_0 \) with constant sectional curvature \( H \), we have
\[
\left( \frac{\Delta_H r'}{\delta_n} - \Delta_H r \right) \leq - \frac{(\Delta_H r')(n - 1) \delta_n}{(n - 1) \delta_n} - \frac{(\Delta_H r')^2}{n - 1} + \frac{1}{\delta_n} \left( (f^r + \text{tr}(A))' (t) - \langle \text{div} A, y'(t) \rangle' + \varphi'(t) \right)
\]

(3.2)
\[
\begin{align*}
+ \frac{1}{\delta_n} K_1(t) \frac{f'(t)}{f(t)}.
\end{align*}
\]

Formula (3.3) gives,
\[
\begin{align*}
\left( sn_n^0(r) \left( \frac{\Delta_n v_f}{\delta_n} - \Delta_d r \right) \right)' \leq 2 sn_n^0(r) sn_n^0(r) \left( \frac{\Delta_a v_f}{\delta_n} - \Delta_d r \right) + sn_n^0(r) \left( \frac{\Delta_a v_f}{\delta_n} - \Delta_d r \right)'.
\end{align*}
\]
\[
\begin{align*}
\leq sn_n^0(r) \left( \frac{2 \Delta_d r}{(n-1)} \left( \frac{\Delta_a v_f}{\delta_n} - \Delta_d r \right) - \left( \frac{(\Delta_d r)^2}{(n-1)} \delta_n^2 - \frac{(\Delta_d r)^2}{n-1} \right) \right) + \frac{sn_n^0(r)}{\delta_n} \left[ (f^a + tr(A))' (t) - (\text{div} A, y'(t))' + \phi'(t) \right]
\end{align*}
\]
\[
\begin{align*}
+ \frac{sn_n^0(r)}{\delta_n} K_1(t) \frac{f'(t)}{f(t)}
\end{align*}
\]
\[
\begin{align*}
= - sn_n^0(r) \left( \frac{\Delta_a}{\delta_n} - \Delta_d r \right) ^2 + \frac{sn_n^0(r)}{\delta_n(n-1)} \langle V, y'(t) \rangle + \frac{sn_n^0(r)}{\delta_n} [ (f^a + tr(A))' (t) + \frac{sn_n^0(r)}{\delta_n} (f^0 r) ]
\end{align*}
\]
\[
\begin{align*}
+ \frac{sn_n^0(r)}{\delta_n} \left[ - (\text{div} A, y'(t))' + \phi'(t) + K_1(t) \frac{f'(t)}{f(t)} \right].
\end{align*}
\]

Note \( \lim_{r \to 0} sn_n^0(r) \left( \frac{\Delta_a v_f}{\delta_n} - \Delta_d r \right) = 0 \). So integration with respect to \( r \) concludes,
\[
\begin{align*}
\frac{1}{\delta_n} sn_n^0(r) (\Delta_a v_f) \leq \frac{1}{\delta_n(n-1)} \int_0^r (sn_n^0(t))' (V, y'(t)) dt + \frac{1}{\delta_n} \int_0^r sn_n^0(t) (f^a(t))' dt
\end{align*}
\]
\[
\begin{align*}
+ \frac{1}{\delta_n} \int_0^r sn_n^0(t) \left[ (tr(A)(t))' - (\text{div} A, y'(t))' + \phi'(t) + \left( K_1(t) \frac{f'(t)}{f(t)} \right) \right] dt
\end{align*}
\]
\[
\begin{align*}
= sn_n^0(r) (\Delta_d r) + \frac{1}{\delta_n(n-1)} \int_0^r (sn_n^0(t))' (V, y'(t)) dt
\end{align*}
\]
\[
\begin{align*}
+ \frac{1}{\delta_n} sn_n^0(r) (f^a(r) + tr(A)(r))' + \frac{1}{\delta_n} sn_n^0(r) (\phi(r) - (\text{div} A, \partial t) r)
\end{align*}
\]
\[
\begin{align*}
+ \frac{1}{\delta_n} \int_0^r \left( sn_n^0(t) (K_1(t) \frac{f'(t)}{f(t)} ) dt - \frac{1}{\delta_n} \int_0^r (sn_n^0(t))' (f^a(t) + tr(A)(t))' dt
\end{align*}
\]
\[
\begin{align*}
- \frac{1}{\delta_n} \int_0^r (sn_n^0(t))' (\phi(t) - (\text{div} A, y'(t))) dt.
\end{align*}
\]
By definition one has,
\[
\frac{1}{\delta_n} sn^2_3(r) L_{A,V}(\alpha+tr(A)) r \leq sn^2_3(r)(\Delta ur) + \frac{1}{\delta_n(n-1)} \int_0^r (sn^2_3(t))' (V,y'(t)) dt
\]
\[+ \frac{1}{\delta_n} sn^2_3(r) \varphi(r) + \frac{1}{\delta_n} \int_0^r sn^2_3(t)K_3(t) \frac{f'(t)}{f(t)} dt
\]
\[- \frac{1}{\delta_n} \int_0^r (sn^2_3(t))' (f^A(t) + tr(A)(t))' dt
\]
\[- \frac{1}{\delta_n} \int_0^r (sn^2_3(t))' (\varphi(t) - \langle div A, y'(t) \rangle) dt,
\]
note \((sn^2_3(t))' \geq 0\), hence
\[
\int_0^r (sn^2_3(t))' (\text{div} A, y'(t)) dt \leq K_3 \int_0^r (sn^2_3(t))' dt = K_3 sn^2_3(r).
\]
and
\[
\int_0^r (sn^2_3(t))' (V,y'(t)) dt \leq K_6 \int_0^r (sn^2_3(t))' dt = K_6 sn^2_3(r).
\]
Therefore the integration by parts implies,
\[
\frac{1}{\delta_n} sn^2_3(r) L_{A,V}(\alpha+tr(A)) r \leq sn^2_3(r)(\Delta ur) + \frac{1}{\delta_n(n-1)} K_6 + K_3 + 2K_0 \) sn^2_3(r)
\]
\[+ \frac{1}{\delta_n} \int_0^r sn^2_3(t)K_3(t) \frac{f'(t)}{f(t)} dt
\]
\[- \frac{1}{\delta_n} (f^A(r) + tr(A)(r)) (sn^2_3(r))'
\]
\[+ \frac{1}{\delta_n} \int_0^r (sn^2_3(t))' (f^A(t) + tr(A)(t)) dt.
\]
For proof (a) we have inequality \(r \leq \frac{\delta_n}{\delta_n(n-1)}\), then (3.4) concludes,
\[
\frac{1}{\delta_n} sn^2_3(r) L_{A,V}(\alpha+tr(A)) r \leq sn^2_3(r)(\Delta ur) + \frac{1}{\delta_n(n-1)} K_6 + K_3 + 2K_0 \) sn^2_3(r)
\]
\[+ \frac{1}{\delta_n} \int_0^r sn^2_3(t)K_3(t) \frac{f'(t)}{f(t)} dt + \frac{2K_2 + K_5 - K_4}{\delta_n} (sn^2_3(r))'.
\]
We know \((sn^2_3(r))' = \frac{2}{\delta_n(n-1)}(\Delta ur)(sn^2_3(r))\) so
\[
L_{A,V}(\alpha+tr(A)) r \leq \delta_n \left(1 + \frac{4K_2 + 2K_5 - 2K_4}{\delta_n(n-1)}\right)(\Delta ur) + \left(\frac{K_6}{(n-1)} + K_3 + 2K_0\right)
\]
\[+ \frac{1}{sn^2_3(r)} \int_0^r sn^2_3(t)K_3(t) \frac{f'(t)}{f(t)} dt.
\]
For proof (b), we have
\[
\int_0^r (sn^2_3(t))' (f^A(t) + tr(A)) dt \leq (K_2 + K_5) \int_0^r (sn^2_3(t))' dt - K_2 \int_0^r (sn^2_3(t))' dt.
\]
\[ + K_4 \int_0^1 (\sin^2 \theta_0(t))^{\gamma} \, dt \]
\[ = \frac{4K_2 + 2(K_5 - K_4)}{\sqrt{H}} + (K_4 - K_2)\sin(2r). \]

Notice that
\[ \frac{1}{\sin^2 \theta_0(r)} \left( \frac{4K_2 + 2(K_5 - K_4)}{\sqrt{H}} + (K_4 - K_2)\sin(2r) \right) = \left( \frac{8K_2 + 4(K_5 - K_4)}{(n-1)\sin(2\sqrt{H}r)} + \frac{2(K_4 - K_2)}{n-1} \right)(\Delta_{Hr}) \]

Therefore,
\[ L_{A.V.\cdot V_l(A^* + tr(A))}f \leq \delta \|r \|_r \left( 1 + \frac{8K_2 + 4(K_5 - K_4)}{(n-1)\sin(2\sqrt{H}r)} + \frac{1}{(n-1)K_6 + K_3 + 2K_0} \right)(\Delta_{Hr}) + \frac{1}{\sin^2 \theta_0(r)} \int_0^1 \sin^2(\sqrt{H}t)K_1(t) \frac{f'(t)}{f(t)} \, dt \]

(3.5)

Remark 3.5. Note \( L_{A.V.\cdot V_l(A^* + tr(A))} = \Delta_{X_r} \) for \( X := \text{divA} + \nabla (f^4 + tr(A)) \). So by Lemma 3.8 in (Fatemi & Azami, 2018) the inequality (3.5) is valid in barrier sense.

4. Proofs of theorems 1.6 and 1.7

Now we prove the Theorem 1.6 by using the so called excess functions. We recall for the point \( p, q \in M \) the excess function is defined by \( e_{p,q}(x) = d(p, x) + d(q, x) - d(p, q) \). For the proof, we use the idea in (Wei & Willie, 2009; Wu, 2017). By adapting their approach we obtain the compactness result using the extended mean curvature Theorem 3.4 for the elliptic differential operator \( L_{A.V.\cdot V_l(A^* + tr(A))} \) to the excess function.

Proof of Theorem 1.6. Let \( p, q \) are two points in \( M \) with \( \text{dist}(p, q) \geq \frac{r}{2} \sqrt{n} \). Define \( B := \text{dist}(p, q) - \frac{r}{2} \sqrt{n} \), \( r_1(x) := \text{dist}(p, x) \) and \( r_2(x) := \text{dist}(q, x) \). Let \( e_{p,q}(x) \) be the excess function associated to the points \( p, q \). By triangle inequality, we have \( e_{p,q}(x) \geq 0 \) and \( e_{p,q}(y(t)) = 0 \), where \( y \) is the minimal geodesic joining \( p, q \). Hence by Remark 3.4

\[ L_{A.V.\cdot V_l(A^* + tr(A))}e_{p,q}(y(t)) \geq 0 \]

in the barrier sense. Let \( y_1 = y\left(\frac{r}{2} \sqrt{n}\right) \) and \( y_2 = y\left(\frac{B + \frac{r}{2} \sqrt{n}}{2} \right) \). So \( r_i(y_i) = \frac{r}{2} \sqrt{n}, i = 1, 2 \). Theorem 3.4 (a) concludes that

\[ L_{A.V.\cdot V_l(A^* + tr(A))}(r_i(y_i)) \leq \left( \frac{K_6}{(n-1)} + K_3 + 2K_0 \right) + \sqrt{H} \delta_n(4K_2 + 2(K_5 - K_4)) \]
\[ + \int_0^{\frac{r}{2} \sqrt{n}} \sin^2(\sqrt{H}t)K_1(t) \frac{f'(t)}{f(t)} \, dt. \]

(4.1)

Also, By integration with (3.2), we get

\[ L_{A.V.\cdot V_l(A^* + tr(A))}f_2(y_2) \leq L_{A.V.\cdot V_l(A^* + tr(A))}f_1(y_2) - B(n-1)\delta_nH + K_0 + \int_{\frac{r}{2} \sqrt{n}}^{\frac{B + \frac{r}{2} \sqrt{n}}{2}} K_1(t) \frac{f'(t)}{f(t)} \, dt. \]

(4.2)

So by (4.1) and (4.2), we have

\[ 0 \leq L_{A.V.\cdot V_l(A^* + tr(A))}(e_{p,q})(y_2) = L_{A.V.\cdot V_l(A^* + tr(A))}(f_2(y_2) + f_1(y_2)) \]
\[ \leq \left( \frac{K_6}{(n-1)} + K_3 + 3K_0 \right) + \sqrt{H} \delta_n(4K_2 + 2(K_5 - K_4) + K_7 - B(n-1)\delta_nH, \]

thus
The finiteness of its fundamental group can be proved by the similar argument in (Fatemi & Azami, 2018).

Finally, we prove Theorem 1.2, the proof is an adaptations of (Cavalcante et al., 2015; Wraith, 2006).

Proof of Theorem 1.7. Assume the contrary, i.e. the manifold $M$ is not compact. So there is a ray $\gamma(t)$ emanating from the fixed point $x_0$. The geodesic $\gamma(t)$ is minimal, so it has no conjugate point, thus $\Delta_\gamma t$ is smooth along $\gamma(t)$. Along the geodesic $\gamma(t)$ we know

\[
\text{Ric}(\gamma'(t), A\gamma'(t)) \leq -\frac{(\Delta_\gamma t)^2}{(n-1)\delta_n} - (\Delta_\gamma t)' + (\text{tr}(A))' + (f^3(t))'' - (\text{div} A, \gamma'(t))'
\]

By integration along the geodesic $\gamma(t)$, we have

\[
\lim_{t \to \infty} \int_1^t \text{Ric}(\gamma'(t), A\gamma'(t))dt \leq \lim_{t \to \infty} \int_1^t \left\{ -\frac{1}{(n-1)\delta_n} (\Delta_\gamma t)^2 + (\Delta_\gamma t)'ight\}dt
\]

\[
+ \lim_{t \to \infty} \int_1^t \left\{ (f^3(t))'' + (\nabla\text{tr}(A) - \text{div} A, \gamma'(t))'ight\}dt
\]

\[
+ \lim_{t \to \infty} \int_1^t K_\delta(t) f'(t) dt
\]

\[
\leq \lim_{t \to \infty} \int_1^t -\left\{ -\frac{1}{(n-1)\delta_n} (\Delta_\gamma t)^2 + (\Delta_\gamma t)'ight\}dt + M.
\]

For simplicity Let $-\Delta_\gamma t = f(r)$, by smoothness of $\Delta_\gamma t$ on the geodesic $\gamma$, $f(r)$ is smooth for $r>0$. From the assumption, we have

\[
\lim_{t \to \infty} \int_1^t \left\{ f'(t) - \frac{1}{(n-1)\delta_n} f^2(t) \right\}dt = \infty.
\]

So, \( \lim_{r \to \infty} f(r) = \infty \) and

\[
\lim_{r \to \infty} \left[ f(r) - \int_1^r f^2(t)dt \right] = \infty. \tag{4.3}
\]

By (4.3), there exists some $r_2 > 1$ such that $f(r) - \int_1^r f^2(t)dt > 10$ for all $r > r_2$. Inductively, define $r_{n+1} = r_n + 10^{-1}-n$. Note that when $f(r) \geq n$, for $r \geq r_{n-1}$, then for $r \geq r_{n+1}$, $f(r) \geq (r_0 - r_{n-1})\alpha r^2$. By induction, we have $f(r) \geq 10^n$, $\forall r \geq r_n$, so $\lim_{n \to \infty} f(r_n) = \infty$. But $\lim_{n \to \infty} r_n = r_\infty \leq r_2 + 10/9$, which is a contradiction with the smoothness of $f(r)$ on $(0, \infty)$. 

\[
B \leq \frac{1}{(n-1)\delta_n H} \left( \frac{K_\delta}{(n-1)} + 3K_0 + K_7 + 2\sqrt{H} \delta_\gamma (2K_2 + K_5 - K_6) \right).
\]

and

\[
dist(p, q) \leq \frac{\pi}{\sqrt{\delta_h}} + \frac{1}{\delta_n(n-1)H} \left( \frac{K_\delta}{(n-1)} + 3K_0 + K_7 + 2\sqrt{H} \delta_\gamma (2K_2 + K_5 - K_6) \right).
\]
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