Covariant Relativistic Non-Equilibrium Thermodynamics of Multi-Component Systems

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Abstract Non-equilibrium and equilibrium thermodynamics of an interacting component in a relativistic multi-component system is discussed covariantly by exploiting an entropy identity. The special case of the corresponding free component is considered. Equilibrium conditions and especially the multi-component Killing relation of the 4-temperature are discussed. Two axioms characterize the mixture: additivity of the energy momentum tensors and additivity of the 4-entropies of the components generating those of the mixture. The resulting quantities of a single component and of the mixture as a whole, energy, energy flux, momentum flux, stress tensor, entropy, entropy flux, supply and production are derived. Finally, a general relativistic 2-component mixture is discussed with respect to their gravitation generating energy-momentum tensors.

1 Introduction

The treatment of multi-component systems is often restricted to transport phenomena in chemically reacting systems, that means, the mixture consisting of different components is shortly described by 1-component quantities such as temperature, pressure and energy which are not retraced to the corresponding quantities of the several components of the multi-component system. That is the case in non-relativistic physics [1, 2, 3] as well as in relativistic physics [4, 5, 6, 7, 8]. In this paper, the single component as an interacting
member of the mixture is investigated. Thus, each component of the mixture is equipped with its own temperature, pressure, energy and mass density which all together generate the corresponding quantities of the mixture.

Considering a multi-component system, three items have to be distinguished: one component as a member of the multi-component system which interacts with all the other components of the system, the same component as a free 1-component system separated from the multi-component system and finally the multi-component system itself as a mixture which is composed of its components. Here, all three items are discussed in a covariant-relativistic framework. For finding out the entropy-flux, -supply, -production and -density, a special tool is used: the entropy identity which constrains the possibility of an arbitrary choice of these quantities [9, 10, 11, 12]. Following J. Meixner and J.U. Keller that entropy in non-equilibrium cannot be defined unequivocally [13, 14, 15, 16, 17], the entropy identity is only an (well set up) ansatz for constructing a non-equilibrium entropy and further corresponding quantities. This fact in mind, a specific entropy and the corresponding Gibbs and Gibbs-Duhem equations are derived. The definition of the rest mass flux densities, of the energy and momentum balances and of the corresponding balances of the spin tensor are taken into account as constraints in the entropy identity by introducing fields of Lagrange multipliers. The physical dimensions of these factors allow to determine their physical meaning.

Equilibrium is defined by equilibrium conditions which are divided into basic ones given by vanishing entropy-flux, -supply and -production and into supplementary ones such as vanishing diffusion flux, vanishing heat flux and zero rest mass production [11, 12]. The Killing relation of the 4-temperature concerning equilibrium is shortly discussed. Constitutive equations are out of scope of this paper.

The paper is organized as follows: After this introduction, the kinematics of a multi-component system is considered in the next two sections for introducing the mass flux and the diffusion flux densities. The energy-momentum tensor is decomposed into its (3+1)-split, and the entanglement of the energy and momentum balances are discussed, followed by non-equilibrium thermodynamics of an interacting component of the mixture and that of the corresponding free component. The equilibrium of both is considered. Thermodynamics of the mixture starts with two axioms: additivity of the energy momentum tensors and of the 4-entropies of the components resulting in those of the mixture. Entropy, entropy flux, -supply and -production are found out. The paper finishes discussing the gravitation generated by a special general-relativistic 2-component system: one component equipped with a symmetric energy-momentum tensor, the other one with a skew-symmetric energy-momentum tensor. A summary and an appendix are added.

2 Kinematics

2.1 The components

We consider a multi-component system consisting of \( Z \) components. The component index \( A \) runs from 1 to \( Z \). Each component has its own rest frame \( B^A \) in which the rest mass density \( \varrho^A \) is locally defined. These rest mass densities are relativistic invariants.
and therefore frame independent.\(^1\)

In general, the components have different 4-velocities: \(u^A_k\), \(A = 1, 2, \ldots, Z; k = 1, \ldots, 4\), which all are tensors of first order under Lorentz transformation. We now define the component mass flux density as a 4-tensor of first order and the component mass production term as a scalar

\[
N_k^A := \varrho^A u_k^A, \quad N^{A k}_{\ :k} = \Gamma^A. \tag{1}
\]

Here, \((1)_{2}\) is the mass balance equation of the \(A\)-component. Consequently, we introduce the basic fields of the components

\[
\{\varrho^A, \ u_k^A\}, \quad A = 1, 2, \ldots, Z. \tag{2}
\]

The mass production term has two reasons: an external one by mass supply and one internal one by chemical reactions

\[
\Gamma^A = (\text{ex}) \Gamma^A + (\text{in}) \Gamma^A. \tag{3}
\]

The external mass supply \((\text{ex}) \Gamma^A\) depends on the environment of the system, whereas \((\text{in}) \Gamma^A\) is determined by chemical reactions depending on the set of frame-independent stoichiometric equations which are discussed in Appendix 12.3.

\subsection{The mixture}

As each component, also the multi-component system has a mass density \(\varrho\) and a 4-velocity \(u_k\) which are determined by the partial quantities of the components. For deriving \(\varrho\) and \(u_k\), we apply the nearly self-evident

\[\text{Mixture Axiom: The balance equation of a mixture looks like the balance equation of an one-component system.}\]

Especially here, the mixture axiom is postulated for the balance equations of mass, energy-momentum and entropy. According to the mixture axiom, the mass balance of the mixture looks according to \((1)_{2}\)

\[
N_k^A = \Gamma, \quad \Gamma = 0, \tag{4}
\]

with vanishing total mass production, if the mass of the mixture is conserved.\(^2\)

Now the question arises: which quantities of the components of the mixture are additive? Obviously, neither the mass densities \(\varrho^A\) nor the 4-velocities \(u_k^A\) are additive quantities according to their definitions. Consequently, we demand in accordance with the mixture axiom that the mass flux densities are additive.\(^3\)

\[\text{Setting I:}\]

\[
\sum_A N_k^A := N_k := \varrho u_k = \sum_A \varrho^A u_k^A \quad \rightarrow \quad u_k = \sum_A \frac{\varrho^A}{\varrho} u_k^A. \tag{5}\]

\(^1\)more details in Appendix 12.1

\(^2\)the mixture as a closed system

\(^3\)The sign \(\bullet\) stands for a setting and := for a definition.
For the present, \( \varrho \) and \( u_k \) are unknown. Of course, they depend on the basic fields of the components \( \text{(2)} \). Contraction with \( u_k \) and use of \( \text{(3)}_{2,3} \) results in

\[
\varrho = \frac{1}{c^2} \sum_A \varrho^A u_k^A u^k = \frac{1}{c^2} N_k u_k^k = \frac{1}{c^2} N_k \varrho \rightarrow \varrho = \pm \frac{1}{c} \sqrt{N_k N^k}, \tag{6}
\]

or in more detail

\[
\varrho = \pm \frac{1}{c} \sqrt{\sum_{A,B} \varrho^A \varrho^B u_k^A u_{Bk}}. \tag{7}
\]

The mass density \( \varrho \) and the 4-velocity \( u_k \) of the mixture are expressed by those of the components according to \( \text{(7)} \) and \( \text{(5)}_4 \). According to \( \text{(5)}_4 \), the 4-velocity of the mixture is a weighted mean value of the 4-velocities of the components. For the mass density, we have according to \( \text{(6)}_1 \) also a with the Kluitenbergh factor \( f_A^4 \) weighted mean value of the mass density components \( \text{[18]} \)

\[
f^A := \frac{1}{c^2} u_k^A u^k > 0 \rightarrow \varrho = \sum_A f^A \varrho^A = \sum_A f^A (u_k^A, u^k) \varrho^A, \tag{8}
\]

resulting in the entanglement of \( \varrho \) and \( u_k \) which are not independent of each other

\[
\varrho = R(\varrho^A, u_k^A, u^k), \quad u_k = U_k(\varrho^A, u_k^A, \varrho). \tag{9}
\]

According to \( \text{(5)}_1 \) and \( \text{(1)}_2 \), we obtain the additivity of the mass production terms

\[
N^k_{;k} = \sum_A N^{Ak} = \sum_A \Gamma^A = \Gamma \rightarrow \sum_A \varrho \Gamma^A = \varrho \Gamma, \quad \sum_A \Gamma^A = 0 \tag{10}
\]

### 2.3 The diffusion flux

From \( \text{(5)}_3 \) and \( \text{(3)}_2 \) follows

\[
0 = \sum_A \varrho^A u_k^A - u_k \sum_A f^A \varrho^A = \sum_A \varrho^A (u_k^A - f^A u_k). \tag{11}
\]

Introducing the diffusion flux density using \( \text{(227)}_2 \)

\[
J^A_k := \varrho^A (u_k^A - f^A u_k) = N_k^A - \varrho^A f^A u_k \rightarrow \sum_A J^A_k =: J_k = 0, \tag{12}
\]

we obtain

\[
J^A_k u^k = \varrho^A (u_k^A u_k^k - f^A c^2) = 0, \tag{13}
\]

\[
J^A_k u^{Ak} = c^2 \varrho^A [1 - (f^A)^2] = c^2 \varrho^A w^A = w^A N_k^A u^{Ak}, \tag{14}
\]

\[
1 - w^A \geq 0. \tag{15}
\]

\(^4>0 \) results from the representation of the 4-velocities in components

\(^5\) Chemical reactions are mass conserving.
By introducing the projectors
\[ h^m_{Am} := \delta^m_l - \frac{1}{c^2} u^A u_l, \quad h^m_{Al} := \delta^m_l - \frac{1}{c^2} u^m u_l, \] (16)
we obtain the following properties of the diffusion flux density:
\[ J^{Am} h^k_m = J^{Ak} = N^{Am} h^k_m \] (17)
\[ J^{Am} h^m_{Ak} = \bar{\varrho}^A f^A (f^A u_{Ak} - u_k) \] (18)
\[ J^{Ak} = J^{Am} h^m_{Ak} + \bar{\varrho}^A u_{Ak} u^A = J^{Am} h^m_{Ak} + w^A N^{Ak} \] (19)
\[ J^{Ak} : k = (J^{Am} h^m_{Ak} : k) + (\bar{\varrho}^A w^A) : k u_{Ak} + \bar{\varrho}^A w^A u^A : k. \] (20)

According to (17), the diffusion flux density is that part of the mass flux density which is perpendicular to the 4-velocity of the mixture. The diffusion flux density vanishes in 1-component systems \((u_k^A \equiv u_k)\) according to \(f^A = f = 1\) and (12).

3 The Energy-Momentum Tensor
3.1 Free and interacting components
The energy-momentum tensor \(T^{Ak l}\) of the \(A\)-component consists of two parts
\[ T^{Ak l} = \bar{T}^{Ak l} + \sum_B W^{Ak l}_B, \quad W^{Ak l}_B = 0. \] (21)
Here, \(\bar{T}^{Ak l}\) is the energy-momentum tensor of the free \(A\)-component, that is the case, if there are no interactions between the \(A\)-component and the other ones. \(W^{Ak l}_B\) describes the interaction between the \(B\)- and the \(A\)-component. The interaction between the external environment and the \(A\)-component is given by the force density \(k^{Ak}\) which appears in the energy-momentum balance equation
\[ T^{Ak l} : k = k^{Ak} = \Omega^{Ak} + \frac{1}{c^2} u^{A l} u^A k^{Am}, \quad \Omega^{Ak} u^l = 0, \] (22)
and in the balance equations of
energy: \(u^A T^{Ak l} : k = u^A k^{Ak} =: \Omega^A, \) (23)
and momentum: \(h^m_{Am} T^{Ak l} : k = h^m_{Am} k^{Ak} =: \Omega^{Am}. \) (24)
Consequently, the interaction of the \(A\)-component with the other components of the mixture modifies the energy-momentum tensor of the free \(A\)-component. Additionally, its interaction with the environment shows up in the source of the energy-momentum balance. According to its definition, \(T^{Ak l}\) is the energy-momentum tensor of the "\(A\)-component in the mixture".
3.2 (3+1)-split

The (3+1)-split of the energy-momentum tensor of the $A$-component is

$$T^{Akl} = \frac{1}{c^2} e^A u^A u^Al + u^Ak p^A + \frac{1}{c^2} q^A u^A + t^{Akl}. \quad (25)$$

The (3+1)-components of the energy-momentum tensor are\footnote{the (3+1)-split is made by taking the physical meaning of (26) and (27) into account, see (33) to (36)}

$$e^A := \frac{1}{c^2} T^{Ajm} u^A_j u^A_m, \quad p^A_l := \frac{1}{c^2} h^A_{ml} T^{Ajm} u^A_j, \quad (26)$$

$$q^A_k := h^Ak T^{Ajm} u^A_j, \quad t^{Akl} := h^Ak T^{Ajm} h^A_{ml}, \quad (27)$$

$$q^A_k u^A_k = 0, \quad p^A_l u^A_l = 0, \quad t^{Akl} u^A_k = 0, \quad t^{Akl} u^A_l = 0. \quad (28)$$

The (3+1)-split of tensors is a usual tool in relativistic continuum physics. The (3+1)-components –generated by the split– have physical significance which originally is hidden in the unsplitted tensors. Thus, we generate by (3+1)-splitting the following covariant quantities of the $A$-component: the energy density $e^A$, the momentum flux density $p^A_l$, the energy flux density $q^A_k$, the stress tensor $t^{Akl}$.

The symmetric part of the energy-momentum tensor (25) is

$$T^{A(kl)} = \frac{1}{c^2} e^A u^A u^Al + \frac{1}{2c} u^Ak \left( cp^A + \frac{1}{c} q^A \right) + \frac{1}{2c} \left( cp^A + \frac{1}{c} q^A \right) u^A + t^{A(kl)}, \quad (29)$$

and its anti-symmetric part is

$$T^{A[kl]} = \frac{1}{2c} u^Ak \left( cp^A - \frac{1}{c} q^A \right) - \frac{1}{2c} \left( cp^A - \frac{1}{c} q^A \right) u^A + t^{A[kl]}. \quad (30)$$

The stress tensor is composed of the pressure $p^A > 0, \wedge A$, and the viscous tensor $\pi^{Akl}$

$$t^{Akl} = -p^A h^{Akl} + \pi^{Akl}, \quad t^{Ak}_k = -3p^A. \quad (31)$$

We now consider the physical dimensions of the introduced quantities\footnote{the bracket $[\ ]$ signifies the physical dimension of $\square$}. According to (16) and (8), we have

$$[h^A_m] = 1, \quad [f^A] = 1. \quad (32)$$

By taking (31), (32), and (25) into account we obtain

$$[t^{Akl}] = [p^A] = [\pi^{Akl}] = [e^A] = [q^A_k] = [p^A_l] m^3 s = [p^A_l] m^3 s. \quad (33)$$

Pressure $= [p^A] = \frac{N}{m^3} = \frac{Nm}{m^3}$ = energy density $= \frac{kg m 1}{m^2}$ = $kg m 1 m^2 s m^3 s$ = momentum flux density, \footnote{the bracket $[\ ]$ signifies the physical dimension of $\square$}

$$[q^A_k] = [e^A] = \frac{Nm}{m^3 s} = \text{energy flux density}, \quad (35)$$

$$[p^A_l] = kg m 1 s m^3 = \text{momentum density}. \quad (36)$$
The (3+1)-split of the energy-momentum tensor can be written in a more compact form

\[ T^{Akl} = \frac{1}{c^2} Q^{Ak} u^{Al} + \tau^{Akl}, \quad (37) \]

\[ u^l T^{Akl} =: Q^{Ak} = e^A u^A + q^{Ak}, \quad h^{m}_{l} T^{Akl} =: \tau^{Akl} = u^A p^{Am} + t^{Akm}. \quad (38) \]

The energy-momentum tensor (37) is that of the \( A \)-component in the mixture, that means as discussed in sect.3.1, the \( A \)-component is not a free system and the (3+1)-split-components \( e^A, q^{Ak}, p^{Am} \) and \( t^{Akl} \) include the internal interaction of the \( A \)-component with all the other ones.

### 3.3 Additivity

We now consider the equivalent-system composed of the \( Z \) components: that is the mixture which consists of these \( Z \) interacting components. Because this interaction is already taken into account by the (3+1)-split-components, the energy-momentum tensors of the components are additive without additional interaction terms. Consequently, the energy-momentum tensor \( T^{kl} \) of the mixture is

**Setting II:**

\[ T^{kl} := \frac{1}{c^2} Q^k u^l + \tau^{kl} = \sum_A T^{Akl} = \sum_A \left( \frac{1}{c^2} Q^{Ak} u^{Al} + \tau^{Akl} \right). \quad (39) \]

Multiplication with \( u_l \) results by use of (8) and (38) in

\[ Q^k = \sum_A \left( Q^{Ak} f^A + \tau^{Akl} u_l \right), \quad (40) \]

and by multiplication with \( h^m_l \), (39) results in

\[ \tau^{km} = \sum_A \left( Q^{Ak} g^{Am} + \tau^{Akl} h^m_l \right), \quad (41) \]

\[ g^{Am} := \frac{1}{c^2} u^A h^m_l = \frac{1}{c^2} (u^A m - f^A u^m) = \frac{1}{c^2} g^A J^{Am}. \quad (42) \]

For an 1-component system \( (u_k^A \equiv u_k) \), we obtain according to (42) \( g^{Am} = g^m = 0 \) taking \( f^A = f = 0 \) into account.

### 3.4 (3+1)-components of the mixture

Starting with (38), we obtain

\[ Q^{Ak} u^A_k = c^2 e^A, \quad \tau^{Akl} h^A_k = q^{Am}, \quad \tau^{Akl} h^A_j = t^{Ajm}. \quad (43) \]
According to (37) and (39), these relations are analogous for the mixture. Consequently, from (40) follows
\[ Q^k u_k := c^2 e = \sum_A \left( Q^A f^A u_k + \tau^{Akl} u_l u_k \right), \] (45)
resulting with (43) in the energy density of the mixture
\[ c^2 e = \sum_A \left( c^2 e^A (f^A)^2 + q^A f^A u_k + c^2 p^A f^A u_l + t^{Akl} u_l u_k \right). \] (46)

From (40) follows
\[ Q^k h^m_k := q^m = \sum_A \left( f^A Q^A k_h^m + h^m_k \tau^{Akl} u_l \right), \] (47)
resulting with (43) in the energy flux density of the mixture
\[ q^m = \sum_A \left( c^2 e^A f^A g^Am + q^A k_h^m + c^2 p^A f^A h^m_l + t^{Akl} h^m_l u_k \right). \] (48)

From (41) follows
\[ \tau^{km} u_k := c^2 p^m = \sum_A \left( Q^A k u_k g^Am + \tau^{Akl} h^m_k u_k \right), \] (49)
resulting with (44) in the momentum density of the mixture
\[ c^2 p^m = \sum_A \left( c^2 e^A f^A g^Am + q^A k u_k g^Am + c^2 p^A f^A h^m_l + t^{Akl} h^m_l u_k \right). \] (50)

And from (44) follows finally
\[ \tau^{km} h^i_k := t^{jm} = \sum_A \left( Q^A k g^Am h^j_k + \tau^{Akl} h^m_l h^i_j \right) \] (51)
which by taking (38) into account results in the stress tensor and the pressure of the mixture
\[ t^{jm} = \sum_A \left( c^2 e^A g^{Aj} g^Am + q^A k h^j_k g^Am + c^2 p^A f^A h^m_i + t^{Akl} h^m_i h^j_k \right), \] (52)
\[ p = -\frac{1}{3} t^{jm} h_{jm} = -\frac{1}{3} \sum_A \left( Q^A k g^Am h^j_k + \tau^{Akl} h^m_l h^i_j \right) h_{jm} = \]
\[ = -\frac{1}{3} \sum_A \left( \frac{1}{c^2} Q^A k u^A p^A h_{kp} + \tau^{Akl} h_{kl} \right) = \]
\[ = -\frac{1}{3} \sum_A \left( \frac{1}{c^2} \left( c^2 u^A + q^A k \right) u^A p^A h_{kp} + (u^A p^A + t^{Akl}) h_{kl} \right). \] (53)
The additivity of the energy-momentum tensors (39) results in (46), (48), (50) and (52), relations which express the (3+1)-components of the energy-momentum tensor of the mixture as those of the components and their velocities
\[
\left\{ e, q^k, p^k, t^{kl} \right\} = F\left( e^A, q^{Ak}, p^{Ak}, t^{Akkl}, u^{Ak}, q(q^A, u_k^A, u^k), u^k(q^A, u_k^A, q) \right),
\]
\[
T_{kl} = \frac{1}{c^2} e^k u^l + u^k p^l + \frac{1}{c^2} q^k u^l + t^{kl}, \quad t^{kl} = -ph^{kl} + \pi^{kl}.
\]
The 4-velocity \( u^k \) is given by (5).

The influence of the additivity of the energy-momentum tensors on the balance equations of energy and momentum is investigated in the next section.

4 Entanglement of Energy and Momentum Balances

If the energy-momentum tensors of the \( A \)-component and of the mixture are \( T^{Akl} \) and \( T_{kl} \), the energy and momentum balances are according to the mixture axiom by use of (23) and (24)
\[
\text{energy:} \quad u^A T^{Akl} ;_k = \Omega^A, \quad u^l T_{kl} ;_k = \Omega,
\]
\[
\text{momentum:} \quad h^A_m T^{Akl} ;_k = \Omega^m, \quad h^m_l T_{kl} ;_k = \Omega^m.
\]
The balances (56), and (57) follow from (23) and (24) by the mixture axiom. Here, \( \Omega^A \) and \( \Omega \) are the energy supplies, and \( \Omega^m \) and \( \Omega^m \) the momentum supplies of the \( A \)-component and of the mixture.

The (3+1)-split of the divergence of the energy-momentum tensor of the \( A \)-component results by use of (16) in
\[
\delta^{tm}_{l} T^{Akl} ;_k = T^{Akm} ;_k = h^A_m T^{Akl} ;_k + \frac{1}{c^2} u^A_m u^l T^{Akl} ;_k.
\]
If the component index \( A \) is cancelled in (58), we obtain the decomposition of the divergence of the energy-momentum tensor of the mixture. Taking (56) and (57) into account, these divergences can be written as
\[
T^{Akm} ;_k = \Omega^m + \frac{1}{c^2} u^A_m \Omega^A, \quad T^{km} ;_k = \Omega^m + \frac{1}{c^2} u^m m \Omega.
\]
The additivity of the energy-momentum tensors (39) results in the additivity of the force densities \( h^p_m \)
\[
k^m = \Omega^m + \frac{1}{c^2} u^m m \Omega = \sum_A \left( \Omega^A_m + \frac{1}{c^2} u^A_m \Omega^A \right) = \sum_A h^A_m.
\]
Taking (23), and (24) into account, we obtain by multiplication of (60) with \( u_m \), resp. with \( h^p_m \),
\[
\Omega = \sum_A \left( \Omega^A_m u_m + f^A \Omega^A \right), \quad \Omega^p = \sum_A \left( \Omega^A_m h^p_m + g^A_p \Omega^A \right).
\]
\(^8\)this is a strong argument for the validity of Setting II (39)
Inserting (23) and (24), we obtain in more detail

\[ u_l T^{kl}_{\cdot k} = \sum_A \left\{ (h_l^A u_m + f^A u_l^A) T^{Akl}_{\cdot k} \right\}, \quad (62) \]

\[ h_l^p T^{kl}_{\cdot k} = \sum_A \left\{ (h_l^A h_m^p + g^A u_l^A) T^{Akl}_{\cdot k} \right\}, \quad (63) \]

As (61) indicates, the additivity of the energy-momentum tensors causes that the supplies of energy and momentum are entangled, expressed by the inequalities

\[ \sum_A f^A \Omega^A \neq \Omega, \quad \sum_A \Omega^A h_m^p \neq \Omega^p. \quad (64) \]

Also if the total force density and the total momentum supply are zero,

\[ T^{kl}_{\cdot k} = 0 \quad \rightarrow \quad \Omega_{iso} = 0 \land \Omega_{iso}^m = 0, \quad (65) \]

we obtain according to (61)

\[ \sum_A \Omega^A_{iso} u_m = - \sum_A f^A \Omega_{iso}^A \neq 0, \quad (66) \]

\[ \sum_A \Omega^A_{iso} h_m^p = - \sum_A g^p \Omega_{iso}^A \neq 0. \quad (67) \]

As expected, the supplies of energy and momentum remain entangled in a system of vanishing total force and momentum densities. The entanglement vanishes for such isolated systems for which the force and momentum supplies for all \( A \)-components are zero.

5 The Spin Tensor

5.1 (3+1)-split

The (3+1)-split of the spin tensor of an \( A \)-component is defined by inserting (16) into

\[ S^{Akab} = S^{Ampq} s^{A}_{mpq} s^{A}_{pq}. \quad (68) \]

Introducing the following covariant abbreviations

\[ s^{Amj} := S^{Akab} u_k^A h_a^m h_b^j, \quad s^{Amji} := S^{Akab} h_k^A h_a^m h_b^i, \]

\[ \Xi^{Am} := S^{Akab} u_k^A u_a^A h_b^m, \quad \Xi^{Amj} := S^{Akab} h_k^A u_a^A h_b^i, \quad (69) \]

\[ (68) \] results in

\[ S^{Akab} = - S^{Akba} = \]

\[ = u^A a \left( \frac{1}{c^3} u^A \Xi^{Ab} + \frac{1}{c^2} \Xi^{Ab} \right) - u^B b \left( \frac{1}{c^3} u^A \Xi^{Ab} + \frac{1}{c^2} \Xi^{Ab} \right) + s^{Akab} + \frac{1}{c^2} u^A s^{Ab}. \quad (71) \]

By (62) and (70) are introduced: the spin density \( s^{ab} \), the spin density vector \( \Xi^{b} \), the couple stress \( s^{kab} \) and the spin stress \( \Xi^{b} \).
Analogously to (37) and (38), a more compact form of the spin tensors is

\[ S^{Akab} = 2u^{A[a}L^{Akb]} + M^{Akab}, \quad (72) \]

\[ L^{Akb} := \frac{1}{c^4}u^{Ak}\Xi^{Ab} + \frac{1}{c^2}\Xi^{Akb}, \quad (73) \]

Taking (69) and (70) into account, we obtain

\[ S^{Akab}u^A_a = c^2L^{Akb}, \quad S^{Akab}h^A_mh^A_n = M^{Akmn}, \quad (74) \]

expression which are needed for formulating the entropy identity below.

### 5.2 Additivity

Analogously to Setting II, we introduce the spin tensor of the mixture as the sum of the spin tensors of the \( A \)-components.

**Setting III:**

\[ S^{kab} := 2u^{[a}L^{b]} + M^{kab} = \sum_A S^{Akab} = \sum_A \left( 2u^{A[a}L^{Akb]} + M^{Akab} \right) = \sum_A \left\{ 2u^{A[a} \left( \frac{1}{c^4}u^{Ak}\Xi^{Ab]} + \frac{1}{c^2}\Xi^{Akb]} \right) + S^{Akab} + \frac{1}{c^2}u^{Ak}s^{Aab} \right\}. \quad (75) \]

According to the mixture axiom, the spin tensor of the mixture is defined by \( S^{kab}_{1} \) as spin of an 1-component system resulting from (72) with \( A \equiv 1 \rightarrow \text{blank} \).

### 5.3 \((3+1)\)-components of the mixture

From (75), we obtain by taking the mixture axiom and (74) into account

\[ S^{kab}u^A_a = c^2L^{Aab} = \sum_A \left( 2u^{A[a}L^{Akb]} + M^{Akab} \right) u^A_a, \quad (76) \]

\[ S^{kab}h^A_mh^A_n = M^{kmn} = \sum_A \left( 2u^{A[a}L^{Akb]} + M^{Akab} \right) h^A_mh^A_n. \quad (77) \]

The \((3+1)\)-components of the spin tensor of the mixture result from (69) and (70) using the mixture axiom

\[ s^{mj} = S^{kab}u_kh^m_a h^j_b, \quad s^{mj} = S^{kab}h^m_kh^j_a h^i_b, \quad (78) \]

\[ \Xi^m = S^{kab}u_ku_a h^m_b, \quad \Xi^m = S^{kab}h^m_ku_a h^j_b, \quad (79) \]

and by inserting (75), or (76) and (77).
5.4 Spin balance equation

If there exists an external angular momentum density

\[ m_{ab} = -m_{ba}, \quad (80) \]

a spin balance equation of each \( A \)-component and of the mixture has to be taken into account

\[ S^{Akab}_{:k} = \frac{1}{c^2} m^{Aab}, \quad S^{kab}_{:k} = \frac{1}{c^2} m^{ab}. \quad (81) \]

According to Setting III,

\[ \sum_A m^{Aab} = m^{ab} \quad (82) \]

the additivity of the partial angular momenta is valid.

6 Thermodynamics of Interacting Components

6.1 The entropy identity

Starting with the (3+1)-split of the entropy 4-vector and the entropy balance equation

\[ S^{Ak} = s^A u^{Aak} + s^{Ak} \rightarrow S^{Ak}_{:k} = \sigma^A + \varphi^A, \quad (83) \]

\[ s^A := \frac{1}{c^2} S^{Ak}_{:k}, \quad s^{Ak} := S^{Amh}_{am} \quad (84) \]

we have to define the following four quantities in accordance with the balance equations of mass (12), of energy (23), of momentum (24) and of spin (81): the entropy density \( s^A \), the entropy flux density \( s^{Ak} \), the entropy production \( \sigma^A \) and the entropy supply \( \varphi^A \). Because there is no unequivocal entropy [15] and consequently, also no unique entropy density, -flux, -production and -supply, we need a tool which helps to restrict the arbitrariness for defining entropies. Such a tool is the entropy identity [10, 11] which is generated by adding suitable zeros to the entropy (83) which are related to the balances which are taking into account. These zeros are generated by choosing the following expressions: \( N^{Ak} \), \( u^{A}_{ik} T^{Akl} \), \( h^{Am}_{i} T^{Akl} \), \( u^{A}_{i} S^{Akab} \), \( h^{Am}_{i} h^{Ab}_{b} S^{Akab} \). Consequently, the entropy identity is chosen according to (11),(38) and (74)

\[ S^{Ak} \equiv s^A u^{Aak} + s^{Ak} + \kappa^A \left( N^{Ak} - \varphi^A u^{Aak} \right) + \]

\[ + \lambda^A \left( u^{A}_{ik} T^{Akl} - e^{A} u^{Aak} - q^{Ak} \right) + \]

\[ + \lambda^A \left( h^{Am}_{i} T^{Akl} - u^{Aak} p^{Am} - t^{Akln} \right) + \]

\[ + \Lambda^A \left( u^{A}_{i} h^{Ab}_{b} S^{Akab} - \frac{1}{c^2} u^{Ak} \Xi^{Am} - \Xi^{Akln} \right) + \]

\[ + \Lambda^A \left( h^{Am}_{i} h^{Ab}_{b} S^{Akab} - S^{Akln} - \frac{1}{c^2} u^{Ak} S^{Amn} \right). \quad (85) \]

The fields of Lagrange multipliers \( \kappa^A \), \( \lambda^A \), \( \lambda^A_{m} \), \( \Lambda^A_{m} \) and \( \Lambda^A_{mn} \) are quantities whose physical meaning becomes clear in the course of the exploitation of the entropy identity.
Here, $\kappa^A$ and $\lambda^A$ are scalars, undefined for the present, and for the likewise arbitrary quantities $\lambda_{m}^{A}$, $\Lambda_{m}^{A}$ and $\Lambda_{ab}^{A}$, tensors of first and second order. An identification of these Lagrange multipliers is given below after the definitions of entropy flux density, entropy production density and supply in section 6.3.

The entropy identity (85) depends on the balances which are taken into consideration as constraints: the balances of mass, energy, momentum and spin. The electro-magnetic field and quantum fields are included, if the energy-momentum tensor and the spin tensor of these fields are inserted into (85).

Considering the third, the fourth and the fifth row of (85), we obtain that the velocity parts of $\lambda_{m}^{A}$, $\Lambda_{m}^{A}$ and $\Lambda_{mn}^{A}$ can be set to zero according to (69) and (70). The symmetric part of $\Lambda_{mn}^{A}$ does not contribute to the fifth row of (85) and therefore it is set to zero, too.

$$\lambda_{m}^{A} h_{l}^{Am} = \lambda_{l}^{A}, \quad \Lambda_{m}^{A} h_{b}^{Am} = \Lambda_{b}^{A}, \quad \Lambda_{mn}^{A} h_{a}^{Am} h_{b}^{An} = \Lambda_{ab}^{A} = -\Lambda_{ba}^{A}. \quad (86)$$

The entropy identity (85) becomes by rearranging

$$S^{Ak} \equiv u^{Ak} \left( s^{A} - \kappa^{A} g^{A} - \lambda^{A} e^{A} - \lambda_{m}^{A} p^{Am} - \Lambda_{m}^{A} \frac{1}{c^2} s^{Am} - \Lambda_{mn}^{A} \frac{1}{c^2} s^{Amn} \right) +$$

$$+ s^{Ak} + \kappa^{A} \nu^{Ak} + (\lambda^{A} u_{l}^{A} + \lambda_{l}^{A}) T^{Akl} - \lambda^{A} q^{Ak} - \lambda_{l}^{A} \Theta^{Akl} +$$

$$+ \left( \Lambda_{b}^{A} u_{a}^{A} + \Lambda_{ab}^{A} \right) S^{Akab} - \Lambda_{m}^{A} \Xi^{Ak} - \Lambda_{mn}^{A} \Xi^{Akm}. \quad (87)$$

This identity transforms into another one by differentiation and by taking the balance equations of mass (11), of energy-momentum (22), of spin (81) and of entropy (83) into account.

$$S^{Ak} \equiv \left[ u^{Ak} \left( s^{A} - \kappa^{A} g^{A} - \lambda^{A} e^{A} - \lambda_{m}^{A} p^{Am} - \Lambda_{m}^{A} \frac{1}{c^2} s^{Am} - \Lambda_{mn}^{A} \frac{1}{c^2} s^{Amn} \right) \right] \cdot k +$$

$$+ s^{Ak} \cdot k + \kappa^{A} \nu^{Ak} + \kappa^{A} (\text{ex})^{A} + (\text{in})^{A} \Gamma^{A} +$$

$$+ \left( \lambda^{A} u_{l}^{A} + \lambda_{l}^{A} \right) T^{Akl} + \left( \lambda^{A} u_{l}^{A} + \lambda_{l}^{A} \right) k^{A} -$$

$$- \left( \lambda_{l}^{A} \Theta^{Akl} \right) \cdot k +$$

$$+ \left( \Lambda_{b}^{A} u_{a}^{A} + \Lambda_{ab}^{A} \right) S^{Akab} + \left( \Lambda_{b}^{A} u_{a}^{A} + \Lambda_{ab}^{A} \right) \frac{1}{c^2} m^{Aab} -$$

$$- \left( \Lambda_{m}^{A} \Xi^{Akm} \right) \cdot k - \left( \Lambda_{mn}^{A} \Xi^{Akmn} \right) \cdot k = \sigma^{A} + \varphi^{A}. \quad (88)$$

Here, $\sigma^{A}$ is the entropy production and $\varphi^{A}$ the entropy supply of the $A$-component. The identity (85) changes into the entropy production, if $s^{A}$, $s^{Ak}$ and $\varphi^{A}$ are specified below.

Rearranging the entropy identity results in

$$S^{Ak} \equiv u^{Ak} \left( s^{A} - \kappa^{A} g^{A} - \lambda^{A} e^{A} - \lambda_{m}^{A} p^{Am} - \Lambda_{m}^{A} \frac{1}{c^2} s^{Am} - \Lambda_{mn}^{A} \frac{1}{c^2} s^{Amn} \right) +$$

$$+ u^{Ak} \left( s^{A} - \kappa^{A} g^{A} - \lambda^{A} e^{A} - \lambda_{m}^{A} p^{Am} - \Lambda_{m}^{A} \frac{1}{c^2} s^{Am} - \Lambda_{mn}^{A} \frac{1}{c^2} s^{Amn} \right) \cdot k +$$

$$+ \left( s^{Ak} - \lambda^{A} q^{Ak} - \lambda_{l}^{A} \Theta^{Akl} - \Lambda_{m}^{A} \Xi^{Akm} - \Lambda_{mn}^{A} \Xi^{Akmn} \right) \cdot k +$$

$$13$$
\[ + \kappa^{(ex)} A (\Gamma^A + \left( \lambda^A u^A_i + \lambda^A \right) k^A_l + \left( \Lambda^A u^A_n + \Lambda^A \right) \frac{1}{c^2} m^{Ab} + \\
+ \kappa^{A (im)} A k^A_l + \kappa^{A (im)} T^A + \left( \lambda^A u^A_i + \lambda^A \right) k^A_l T^A + \left( \Lambda^A u^A_n + \Lambda^A \right) S^{Akab} = \]
\[ = \sigma^A + \varphi^A. \] (89)

Now we look for terms of the fifth row of (89) which fit into the first three rows of (89). The shape of these terms is \([u^{Ak} \_k \_k \_k] \_k \) according to the first two rows of (89) and \([\Psi^{Ak} u^A_k = 0] \) according to the third row. None of the seven terms of the fourth and fifth row of (89) have this shape, but inserting the energy-momentum tensor and the spin tensor into the fifth row of (89) may generate such terms.

The third term of the fifth row of (89) becomes
\[(\lambda^A u^A_i) \_k T^A = \left( \lambda^A \_k u^A_i + \lambda^A u^A_{l,k} \right) \left( \frac{1}{c^2} e^A u^A_k u^A_l + u^A p^A + \frac{1}{c^2} q^A u^A + t^A \right) = \]
\[ = \lambda^A \_k u^A_k e^A + \lambda^A u^A_{l,k} u^A p^A + \lambda^A \_k q^A - p^A \lambda^A u^A_{l,k} + \lambda^A u^A_{l,k} n^A, \] (90)
\[ \lambda^A \_k T^A = \lambda^A \_k \left( \frac{1}{c^2} e^A u^A_k u^A_l + u^A p^A + \frac{1}{c^2} q^A u^A + t^A \right). \] (91)

Summing up (90) and (91) results in
\[ \left( \lambda^A u^A_i + \lambda^A \right) T^A = \lambda^A \_k \left( \frac{1}{c^2} e^A u^A_k u^A_l + u^A p^A + \frac{1}{c^2} q^A u^A + t^A \right) = \]
\[ - p^A \lambda^A u^A_{l,k} + \lambda^A \_k \left( \frac{1}{c^2} e^A u^A_k u^A_l + u^A p^A + \frac{1}{c^2} q^A u^A + t^A \right). \] (92)

Evidently, the term \(- p^A \lambda^A u^A_{l,k}\) belongs to the first row of (89). After having inserted the underlined term of (92), the first two rows of (89) become
\[ u^A \_k \left( s^A - \kappa^A g^A - \lambda^A e^A - p^A \lambda^A - \lambda^A m^A p^A - \Lambda^A \frac{1}{c^2} \Xi^A m^A - \Lambda^A \frac{1}{c^2} \Xi^A s^A \right) + \]
\[ + \left( s^A - \kappa^A g^A - \lambda^A e^A - p^A \lambda^A - \lambda^A m^A p^A - \Lambda^A \frac{1}{c^2} \Xi^A m^A - \Lambda^A \frac{1}{c^2} \Xi^A s^A \right) \cdot (p^A \lambda^A)^\ast = \]
\[ = \left[ u^A \_k \left( s^A - \kappa^A g^A - \lambda^A e^A - p^A \lambda^A - \lambda^A m^A p^A - \Lambda^A \frac{1}{c^2} \Xi^A m^A - \Lambda^A \frac{1}{c^2} \Xi^A s^A \right) \right] \_k + + (p^A \lambda^A)^\ast. \] (93)

Thus, a rearranging of the entropy identity (89) results by taking (93) into account
\[ S^{Ak} \_k \_k = \]
\[ \left[ u^A \_k \left( s^A - \kappa^A g^A - \lambda^A e^A - p^A \lambda^A - \lambda^A m^A p^A - \Lambda^A \frac{1}{c^2} \Xi^A m^A - \Lambda^A \frac{1}{c^2} \Xi^A s^A \right) \right] \_k + + (p^A \lambda^A)^\ast + \left( s^A - \kappa^A g^A - \lambda^A e^A - p^A \lambda^A - \lambda^A m^A p^A - \Lambda^A \frac{1}{c^2} \Xi^A m^A - \Lambda^A \frac{1}{c^2} \Xi^A s^A \right) \_k + \]

\(^9\) the signs  \(\boxed{\_k} \),  \(\boxed{\_k} \),  \(\boxed{\_k} \) and  \(\boxed{\_k} \) mark terms which are related to each other in the sequel

\(^{\_k} \) is the "component time derivative"  \(\boxed{\_k} A := \boxed{\_k} A u^A_k \)

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\[
\begin{align*}
&+ \kappa^{A(cx)\Gamma A} + \left( \lambda^A u^A_i + \lambda^A_i \right) k^A l + \left( \Lambda^A_b u^A_a + \Lambda^A_{ab} \right) \frac{1}{c^2} u^{Aab} + \\
&+ \kappa^{A \, \xi N k^A + \kappa^{A(\alpha)\xi A} + \left( \Lambda^A_b u^A_a + \Lambda^A_{ab} \right) S^{A k b} + \\
&+ \lambda^A_{a k} \left( q^{A k} + e^A u^{A k} \right) + \lambda^A u^A_{l k} \left( \pi^{A k l} + u^{A k} p^{A l} \right) + \\
&+ \lambda^A \left( \frac{1}{c^2} e^A u^{A l} + p^{A l} \right) + \lambda^A u^A_{l k} \left( \frac{1}{c^2} q^{A k} u^{A l} + l^{A k l} \right) = \sigma^A + \varphi^A. \quad (94)
\end{align*}
\]

The third term of the fourth row of (94) results in
\[
\begin{align*}
\left( \Lambda^A_b u^A_a + \Lambda^A_{ab} \right) S^{A k b} &= \\
&= \left( \Lambda^A_{b k} u^A_a + \Lambda^A_{a k} u^A_{a k} + \Lambda^A_{ab} \right) \left( u^{A a} L^{A k b} - u^{A b} L^{A k a} + M^{A k a b} \right). \quad (95)
\end{align*}
\]

If (86) and (73) are taken into account, these nine terms are:
\[
\begin{align*}
&\Lambda^A_{b k} c^2 L^{A k b} = \Lambda^A_{b k} \left( \frac{1}{c^2} u^{A k} \Xi^{A b} + \Xi^{A k b} \right), \quad (96) \\
&\Lambda^A_b u^A_{a k} u^{A a} L^{A k b} = 0, \quad (97) \\
&\Lambda^A_{a b k} u^A_{a b} L^{A k b} = \Lambda^A_{a b k} u^A_{a b} \left( \frac{1}{c^2} u^{A k} \Xi^{A b} + \frac{1}{c^2} \Xi^{A k b} \right), \quad (98) \\
&-\Lambda^A_{b k} u^A_{a b} L^{A k a} = -\Lambda^A_{b k} u^A_{a b} \left( \frac{1}{c^2} u^{A k} \Xi^{A b} + \frac{1}{c^2} \Xi^{A k a} \right) = 0, \quad (99) \\
&-\Lambda^A_{a b k} u^A_{a b} L^{A k a} = 0, \quad (100) \\
&-\Lambda^A_{a b k} u^A_{a b} M^{A k a b} = 0, \quad (101) \\
&\Lambda^A_{a b k} u^A_{a b} M^{A k a b} = \Lambda^A_{a b k} \left( S^{A k a b} + \frac{1}{c^2} u^{A k} S^{A a b} \right), \quad (102) \\
&\Lambda^A_{a b k} M^{A k a b} = \Lambda^A_{a b k} \left( S^{A k a b} + \frac{1}{c^2} u^{A k} S^{A a b} \right). \quad (103) \\
&\end{align*}
\]

Rearranging of (96) to (104) results in:
\[
\begin{align*}
&\Lambda^A_{a b k} \Xi^{A b} + \Lambda^A_{a k} \Xi^{A k b} - \Lambda^A_{b k} u^A_{a} \left( S^{A k a b} + \frac{1}{c^2} \Xi^{A a b} \right), \quad (105) \\
&\Lambda^A_{a b k} \Xi^{A b} + \Lambda^A_{a k} \Xi^{A k b} - \Lambda^A_{b k} u^A_{a} \left( S^{A b a k} + \frac{1}{c^2} \Xi^{A a k} \right), \quad (106)
\end{align*}
\]

A comparison of (105) and (106) with the first two rows of (94) demonstrates that a term which fits into these rows does not appear in (105) and (106). Thus by taking (73) into account, a rearranging of the entropy identity (89) results in
\[
S^{A k}_{\xi k} \equiv
\]

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Extended Thermodynamics is set as a non-equilibrium variable, even if the stress tensor is non-symmetric.

Valid according to (30)– the momentum density is replaced by energy flux density which in non-relativistic

We now define the entropy density, Gibbs and Gibbs-Duhem equations

6.2 Exploitation of the entropy identity

6.2.1 Entropy density, Gibbs and Gibbs-Duhem equations

We now define the entropy rest density \( s^A \) according to the first row of (107)

\[
\begin{align*}
\left[ u^A \left( S^A - \kappa^A q^A - \lambda^A e^A - p^A \lambda^A + \lambda^A B^A m - \Lambda^A_m \frac{1}{c^2} \Xi^A m - \Lambda^A_{mn} \frac{1}{c^2} s^{A mn} \right) \right]_{,k} + \\
(\tilde{T}^A \Lambda^A \Lambda^A \cdot) + \left( S^A_{,k} - \lambda^A q^A_{,k} - \lambda^A t^{A kl} - \Lambda^A_m \Xi^A_{m k} - \Lambda^A_{mn} s^{A mn} \right)_{,k} + \\
+ \kappa^A (c^2) \Gamma^A + \left( \lambda^A u^A_{,i} + \lambda^A_i \right) k^A l + \left( \Lambda^A_b u^A_{,a} + \Lambda^A_a \right) \frac{1}{c^2} T^A_{a b} + \\
+ \kappa^A A^A N^A + \kappa^A (c^2) \Gamma^A + \lambda^A_{,k} \left( q^A_{,k} + e^A_{,k} \right) + \lambda^A u^A_{,l} k \left( \tau^{A kl} + u^{A k} p^A l \right) + \\
\left( \frac{1}{c^2} e^A_{,l} u^A_{,l} + p^A l \right) + \lambda^A u^A_{,l} k \left( \frac{1}{c^2} q^A_{,l} u^A_{,l} + t^{A kl} \right) + \\
\left( \frac{1}{c^4} u^A_{,a} \Xi^A_{a b} + \frac{1}{c^2} A^A_{ab} \right) + \lambda^A_{a b k} \left( \frac{1}{c^2} u^A_{,a} \Xi^A_{a b} + s^{A ab} \right) = \\
\sigma^A + \varphi^A.
\end{align*}
\]

This entropy identity is incomplete: the multi-temperature relaxation is missing which is generated by the different partial temperatures of the components of the mixture. Because of lucidity, the treatment of multi-temperature relaxation is postponed and will be considered below in sect.6.4. In the next section, we now specify \( s^A, s^A_{,k}, \varphi^A \) and \( \sigma^A \).

6.2 Exploitation of the entropy identity

6.2.1 Entropy density, Gibbs and Gibbs-Duhem equations

We now define the entropy rest density \( s^A \) according to the first row of (107)

\[
\begin{align*}
\frac{S^A}{q} = \kappa^A q^A + \lambda^A e^A + p^A \lambda^A + \lambda^A p^A l + \lambda^A \frac{1}{c^2} \Xi^A m + \lambda^A \frac{1}{c^2} s^{A mn},
\end{align*}
\]

resulting in the specific entropy

\[
\begin{align*}
\frac{s^A}{q} = \kappa^A \frac{q^A}{q} + \lambda^A \frac{e^A}{q} + p^A \lambda^A \frac{1}{q} \frac{1}{q} + \lambda^A \frac{p^A l}{q} + \lambda^A \frac{1}{c^2} \frac{\Xi^A m}{q} + \lambda^A \frac{1}{c^2} \frac{s^{A mn}}{q}.
\end{align*}
\]

A non-equilibrium state space –which is spanned by the independent variables– contains beside \( q^A, e^A, \) the spin variables \( \Xi^A m \) and \( s^{A mn} \) and additionally \( p^A l \) which extends the state space in the sense of Extended Thermodynamics. Consequently, we choose the state space

\[
\begin{align*}
z^A = \left( c^A, \frac{1}{q}, \frac{c^A}{q}, \frac{p^A l}{q}, \frac{\Xi^A m}{q}, \frac{s^{A mn}}{q} \right), \quad c^A := \frac{\varphi^A}{q}.
\end{align*}
\]

\[\text{If the energy-momentum tensor is presupposed to be symmetric –consequently \( p^A l = (1/c^2) q^A \) is valid according to (20)– the momentum density is replaced by energy flux density which in non-relativistic Extended Thermodynamics is set as an non-equilibrium variable, even if the stress tensor is non-symmetric.}\]
The corresponding Gibbs equation according to (109) and (110) is

\[
\frac{\xi^A}{\varrho} = \kappa^A \frac{\dot{\xi}^A}{\varrho} + \lambda^A \left( \frac{e^A}{\varrho} \right) + p^A \lambda^A \left( \frac{1}{\varrho} \right) + \lambda_l^A \left( \frac{P^A_{\varrho l}}{\varrho} \right) + \\
+ \Lambda_m^A \frac{1}{c^2} \left( \Xi^{Amn}_m \right) + \Lambda_{mn}^A \frac{1}{c^2} \left( s^{Amn} \right) \tag{111}
\]

Differentiation of (109) results in the Gibbs-Duhem equation by taking (111) into account

\[
0 = \frac{\dot{\xi}^A}{\varrho} = \kappa^A \frac{e^A}{\varrho} + \lambda^A \frac{\dot{e}^A}{\varrho} + p^A \lambda^A \left( \frac{1}{\varrho} \right) + \lambda_l^A \left( \frac{P^A_{\varrho l}}{\varrho} \right) + \\
\frac{\dot{s}^{Amn}}{\varrho} = -\kappa^A \frac{\dot{S}^A}{\varrho} - \lambda^A \frac{\dot{S}^A}{\varrho} - \lambda_l^A \frac{\dot{P}^A_{\varrho l}}{\varrho} - \Lambda_m^A \frac{1}{c^2} \Xi^{Amn} - \lambda_{mn}^A \frac{1}{c^2} s^{Amn}. \tag{112}
\]

Taking (113) and (108) into account, the entropy identity (107) becomes

\[
S^{Ak,j} = -\kappa^A \frac{\dot{a}^A}{\varrho} - \frac{\dot{\Lambda}^A}{\varrho} - \lambda^A \frac{\dot{\Lambda}^A}{\varrho} + \Lambda_m^A \frac{1}{c^2} \Xi^{Amn} - \lambda_{mn}^A \frac{1}{c^2} s^{Amn} + \\
+ \left( s^{Ak} - \lambda^A q^{Ak} - \lambda^A t^{Akl} - \lambda_m^A \Xi^{Amn} - \lambda_{mn}^A s^{Amn} \right)_{Ak} + \\
+ \kappa^{(ex)} \Gamma^A + \left( \lambda^A u^A_{\varrho} + \lambda^A \right) k_{Ak} + \left( \Lambda^A u^A_{\varrho} + \Lambda^A \right) \frac{1}{c^2} m^{Aab} + \\
+ \kappa^{(ex)} \Gamma^A + \Lambda^A k_{Ak} N^{Ak} + \left( \lambda^A u^A_{\varrho} + \lambda^A \right) k_{Ak} + \lambda^A u^A_{Ak} \left( u^A_{Ak} + u^A_{P^A_{\varrho l}} \right) + \\
+ \lambda^A \left( \frac{1}{c^2} e^A u^A_{P^A_{\varrho l}} + \frac{\dot{\varrho}^A}{\varrho} \right) + \lambda^A \frac{1}{c^2} u^A_{Ak} \left( u^A_{Ak} + \frac{\dot{\varrho}^A}{\varrho} \right) + \\
+ \lambda^A \frac{1}{c^2} u^A_{Ak} \Xi^{Ab} + \lambda^A \frac{1}{c^2} s^{Aab} - \frac{1}{c^2} \frac{\dot{\varrho}^A}{\varrho} + \frac{1}{c^2} s^{Aab} + \\
+ \frac{\dot{\Lambda}^A}{\varrho} \left( 1 \right) - \frac{\dot{\Lambda}^A}{\varrho} + \frac{1}{c^2} s^{Aab} + \frac{\dot{\Lambda}^A}{\varrho} \left( \frac{1}{c^2} \right) + \lambda^A \left( \frac{1}{c^2} u^A_{Ak} \Xi^{Ab} \right) + s^{Aab} = \\
= \sigma^A + \varphi^A. \tag{114}
\]

The marked terms cancel each other.

Taking (12) and (10) into account, we consider

\[
0 = \kappa^A_{\varrho} N^{Ak} - \kappa^A_{\varrho} N^{Ak} = \kappa^A_{\varrho} \left( N^{Ak} - N^{Ak} \right) = \\
= \kappa^A_{\varrho} \left( J^{Ak} + \varrho^A f^A u^k - J^{Ak} - \varrho^A f^A u^k \right) = \kappa^A_{\varrho} \left( J^{Ak} - J^{Ak} \right) = \\
= \kappa^A_{\varrho} J^{Ak} - \kappa^A_{\varrho} \left( J^{Am} h_{Am}^{Ak} + w^A N^{Ak} \right) = \\
= \kappa^A_{\varrho} \left( J^{Ak} - w^A N^{Ak} \right) - \kappa^A_{\varrho} \left( J^{Am} h_{Am}^{Ak} \right) + \kappa^A \left( J^{Am} h_{Am}^{Ak} \right) \tag{115}
\]

This zero contains the diffusion flux which does not appear up to here in the entropy identity (85). That means, the diffusion is missing in (114), and we will not ignore the underbraced terms in (119), but we insert (119) into (114). Consequently, the entropy identity results in

\[
S^{Ak}_{:k} \equiv \left( S^{Ak} - \lambda^A q^{Ak} - \lambda_i^A t^{Ak} - \Lambda_m^{Ak} - \Lambda^{Ak}_{km} - \kappa^A \lambda^{Ak}_{mn} s^{Ak}_{mn} - \kappa^A J^{Am} h^{Ak}_{m} \right)_{;k}
\]

\[+ \kappa^{A(e)} \Gamma^A + \left( \lambda^A u^A + \lambda_i^A \right)_{k} + 1 \frac{m^{Ak}}{c^2} + \kappa^A \left( \frac{m^{Ak}}{c^2} \right)_{;k} + \kappa^A \left( \frac{m^{Ak}}{c^2} \right)_{;k}
\]

\[+ \lambda^A \lambda_i^A t^{Ak} + \lambda^A u^A_{:k} \left( \pi^{Ak} + u^{Ak} p^{Al} \right) + \lambda^A \lambda_i^A t^{Ak} + \lambda^A \lambda_i^A t^{Ak} \left( \frac{1}{c^2} u^{Ak} + \frac{1}{c^2} \right)
\]

\[+ \Lambda^{Ak}_{:k} - \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k} + \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k} + \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k}
\]

\[= \sigma^A + \varphi^A.
\]

(116)

We now specify the entropy flux density \( s^{Ak} \) and the entropy supply \( \varphi^A \) in the next section.

### 6.2.2 Entropy flux, –supply and –production

According to the first row of (116), we define the **entropy flux density**

\[
S^{Ak}_{:k} \equiv \left( S^{Ak} - \lambda^A q^{Ak} - \lambda_i^A t^{Ak} - \Lambda_m^{Ak} - \Lambda^{Ak}_{km} s^{Ak}_{mn} - \kappa^A J^{Am} h^{Ak}_{m} \right)_{;k}
\]

\[+ \kappa^{A(e)} \Gamma^A + \left( \lambda^A u^A + \lambda_i^A \right)_{k} + 1 \frac{m^{Ak}}{c^2} + \kappa^A \left( \frac{m^{Ak}}{c^2} \right)_{;k} + \kappa^A \left( \frac{m^{Ak}}{c^2} \right)_{;k}
\]

\[+ \lambda^A \lambda_i^A t^{Ak} + \lambda^A u^A_{:k} \left( \pi^{Ak} + u^{Ak} p^{Al} \right) + \lambda^A \lambda_i^A t^{Ak} + \lambda^A \lambda_i^A t^{Ak} \left( \frac{1}{c^2} u^{Ak} + \frac{1}{c^2} \right)
\]

\[+ \Lambda^{Ak}_{:k} - \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k} + \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k} + \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k}
\]

\[= \sigma^A + \varphi^A.
\]

We now split the entropy identity (116) into the entropy production and the entropy supply. For this end, we need a criterion to distinguish between entropy production and supply. Such a criterion is clear for discrete systems: a local isolation suppresses the entropy supply but not the entropy production. Isolation means: the second row in (116) vanishes, if the \( A \)-component is isolated from the exterior of the mixture. Consequently, we define the **entropy supply** as follows

**Setting VI:**

\[
\varphi^A \equiv \kappa^{A(e)} \Gamma^A + \left( \lambda^A u^A + \lambda_i^A \right)_{k} + 1 \frac{m^{Ak}}{c^2} + \kappa^A \left( \frac{m^{Ak}}{c^2} \right)_{;k} + \kappa^A \left( \frac{m^{Ak}}{c^2} \right)_{;k}
\]

\[+ \lambda^A \lambda_i^A t^{Ak} + \lambda^A u^A_{:k} \left( \pi^{Ak} + u^{Ak} p^{Al} \right) + \lambda^A \lambda_i^A t^{Ak} + \lambda^A \lambda_i^A t^{Ak} \left( \frac{1}{c^2} u^{Ak} + \frac{1}{c^2} \right)
\]

\[+ \Lambda^{Ak}_{:k} - \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k} + \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k} + \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k}
\]

\[= \kappa^{A(e)} \Gamma^A + \lambda^A q^{Ak} + 1 \frac{m^{Ak}}{c^2} + \lambda^A u^A_{:k} \left( \pi^{Ak} + u^{Ak} p^{Al} \right) + \lambda^A \lambda_i^A t^{Ak} + \lambda^A \lambda_i^A t^{Ak} \left( \frac{1}{c^2} u^{Ak} + \frac{1}{c^2} \right)
\]

\[+ \Lambda^{Ak}_{:k} - \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k} + \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k} + \Lambda^{Ak}_{:k} + \frac{1}{c^2} \left( \frac{1}{c^2} \right)_{;k}
\]

\[= \sigma^A + \varphi^A.
\]

(118)
\[ \begin{aligned} &+ \lambda_{,ik}^A \left( \frac{1}{c^2} e^A u^A k u^A l + \frac{1}{c^2} q^A k u^A l + l^A k \right) + \\
&+ \Lambda_{,kk}^A \Xi^A k k - \Lambda_{,a}^A \left( s^{A k b} k + \frac{1}{c^2} \Sigma^A a k b \right) + \\
&+ \Lambda_{,a b}^A \frac{1}{c^2} u^{A[a} \Xi^A b] + \lambda_{,a b}^A \left( \frac{1}{c^2} u^{A[a} \Xi^A b] + s^{A k b} \right). \end{aligned} \]  

(119)

As expected, the entropy production is composed of terms which are a product of "forces" and "fluxes" as in the non-relativistic case. The expressions \( s^A, s^{A k}, \varphi^A \) and \( \sigma^A \) contain Lagrange multipliers which are introduced for formulating the entropy identity playing up to here the role of place-holders. Their physical meaning is discussed in the next section.

### 6.3 Fields of Lagrange multipliers

From non-relativistic physics, we know the physical dimensions of the entropy density and the entropy flux density by taking (33) and (35) into account

\[ \begin{aligned} [s^A] &= [e^A] \frac{1}{K} = \frac{N m}{m^3} \frac{1}{K}, \\
[s^{A k}] &= [q^{A k}] \frac{1}{K} = \frac{N m m}{m^3 m} \frac{1}{s K}. \end{aligned} \]  

(120)

According to (108), we have the following equation of physical dimensions

\[ [s^A] = [\lambda^A] [e^A]. \]  

(121)

Taking (120) into account, we obtain

\[ \frac{N}{m^2} \frac{1}{K} = [\lambda^A] \frac{N}{m^2} \rightarrow [\lambda^A] = \frac{1}{K}. \]  

(122)

that means, \( \lambda^A \) is a reciprocal temperature belonging to the \( A \)-component. Therefore, we accept the following

Setting VII:

\[ \lambda^A = \frac{\nu^A}{\Theta^A}, \]  

(123)

with the partial temperature \( \Theta^A \) of the \( A \)-component and a scalar \( \nu^A \) which is suitably chosen below.

According to (117), we have the following equation of physical dimensions

\[ [s^{A k}] = [k^A] [J^{A m}] [h^A m]. \]  

(124)

Taking (120), (12) and (32) into account, we obtain

\[ \frac{N}{m s} \frac{1}{K} = [k^A] \frac{k g m}{m^3 s} \frac{1}{s} \rightarrow [k^A] = \frac{m^2}{s^2} \frac{1}{K}. \]  

(125)

---

12: The mass production \( \Gamma^A \) due to chemical reactions can be expressed by the time rate of the reaction velocity, see (280) in sect. 12.3.

13: This temperature is a non-equilibrium one, the contact temperature \( \Theta^A_{eq} = T, n A \).
We know from the non-relativistic Gibbs equation that the chemical potentials $\mu^A$ have the physical dimension of the specific energy $e^A/\varrho^A$

$$\begin{align*}
[\mu^A] &= \left[\frac{e^A}{\varrho^A}\right] = \frac{N m^3}{m^2 \text{kg}} = \frac{kg \ m \ m}{s^2 \ kg} = \frac{m^2}{s^2} = K[k^A].
\end{align*}$$

Consequently, we make the following choice by taking (126) into consideration

**Setting VIII:**

$$\kappa^A \cdot \mu^A \Theta^A = \mu^A \Theta^A.$$ (127)

According to the second term of (118) we have the following equation of physical dimensions

$$\lambda^{Ak} = \lambda u^{Ak} = \frac{1}{K} m s$$ (128)

that means, $\lambda^{Ak}$ is proportional to a velocity and at the same time perpendicular to $u^{Ak}$ according to (86). Consequently, only the velocity $u^m$ of the mixture remains for defining $\lambda^{Ak}$ in accordance with (86)

**Setting IX:**

$$\lambda^{Ak} \cdot \frac{1}{\Theta^A} u^m l^{Ak}.$$ (129)

We know from the non-relativistic continuum theory and from (22) and (33) the following connection of the physical dimensions$^{14}$

$$[k^A] m = \left[\frac{N m^3}{m^3}\right] = [m^{ab}] = \frac{1}{s} [s^{ab}] = \frac{N}{m^2}.$$ (130)

From the last term of (71) follows by taking (130) into account

$$[S^{Akab}] = \frac{s}{m} [s^{Aab}] = \frac{s^2}{m^3} N.$$ (131)

From the first term of the third row of (87) follows by use of (83) and (120)

$$[S^{Ak}] = \left[\Lambda^A a\right] \frac{m}{s} [S^{Akab}] = \left[\Lambda^A a\right] [S^{Akab}] = \frac{N m \ m}{m^3} \frac{1}{s} K,$$

and taking (131) into account, we obtain

$$[\Lambda^A a] = \frac{1}{K} \frac{m}{s^2}, \quad [\Lambda^A ab] = \frac{1}{K} \frac{m m}{s^2}.$$ (133)

In accordance with (80)$_{2,3}$ and analogously to (129), the relations (133) allow the following

**Setting X:**

$$\Lambda^A a \cdot \frac{u^A_a}{\Theta^A}, \quad \Lambda^A ab \cdot \frac{1}{\Theta^A} \cdot u^A_{[m^m]} l^{Am} l^{An} = \frac{1}{2 \Theta^A} (u^A_{m^m} u^A_n - u^A_{n^m} u^A_m) l^{Am} l^{An}.$$ (134)

$^{14}$angular momentum = spin density per time
Inserting the Lagrange multipliers into the expression of entropy density (119), of entropy flux density (117) and of entropy supply (118), we obtain by use of (69)

\[ s^A = \frac{1}{\Theta A} \left( \mu^A q^A + \nu^A (e^A + p^A) + u_m p^{Am} + \cdot u_m \frac{1}{c^2} \Xi^{Am} + u_{[a]} u_{[b]} \frac{1}{c^2} s^{Aab} \right), \]  

(135)

\[ s^{Ak} = \frac{1}{\Theta A} \left( \nu^A q^{Ak} + u_m t^{Ak} + \mu^A J^{Am} h^{Ak}_m + \cdot u_m \Xi^{Ak} + \cdot u_{[a]} s^{Aab} \right), \]  

(136)

\[ \phi^A = \frac{1}{\Theta A} \left\{ \mu^A (ex) \Gamma^A + \left( \nu^A u^A_t + u_m h^{Am}_t \right) k^{Al} + \right. \\
+ \left( \cdot u_{[a]} \cdot u_{[b]} \cdot h^{Am}_a \cdot h^{An}_b \cdot \frac{1}{c^2} m^{Aab} \right \}, \]  

(137)

The entropy production density (119) results by inserting the Lagrange multipliers (123), (127), (129) and (134)

\[ \sigma^A = \left( \frac{\mu^A}{\Theta A} \right) \cdot J^{Am} h^{Ak}_m + \mu^A \left[ (\eta A) \Gamma^A + \left( J^{Am} h^{Ak}_m \right) k^{Ak} + \right. \\
+ \left( \frac{\nu^A}{\Theta A} \right) q^{Ak} + \frac{\nu^A}{\Theta A} u^{Al} ; k \left( \Pi^{Ak} + u^{Ak} p^{Al} \right) + \right. \\
+ \frac{1}{\Theta A} u^{Am} \frac{1}{c^2} e^A u^{Ak} u^{Al} + \frac{1}{c^2} q^{Ak} u^{Al} + t^{Ak} l^{Al} + \right. \\
+ \frac{1}{\Theta A} \Gamma^{Akb} - \frac{u^{A}_{[a]} u^{A}_{[b]}}{c^2} \left( s^{Akb} ; k + \frac{1}{c^2} \Gamma^{Aab} \right) + \right. \\
+ \frac{1}{\Theta A} u^{A}_{[m]} u^{A}_{[n]} \frac{1}{c^2} u^{A[l} \frac{1}{c^2} u^{Akb] l} + \right. \\
+ \frac{1}{\Theta A} u^{A}_{[m]} u^{A}_{[n]} \frac{1}{c^2} u^{A[l} \frac{1}{c^2} u^{Akb] l} + s^{Aab}. \]  

(138)

The underbraced terms result in

\[ \cdot u^{A}_{[a]} u^{A}_{[b]} \cdot s^{Aab} = \cdot u^{A}_{[a]} u^{A}_{[b]} \cdot s^{Aab} = 0, \]  

(139)

that means, the spin density does not appear in the entropy production.

The first four terms of the entropy production describe the four classical reasons of irreversibility: diffusion, chemical reactions, heat conduction and internal friction with a by the momentum flux density modified non-equilibrium viscous tensor. The fifth term of (138) is typical for an interacting \( A \)-component as a part of the mixture because it contains the 4-velocity of the mixture \( u^m \). The same is true for the last two spin terms which vanish for 1-component systems. In any case, all spin terms of the fields (135) to (138) related to entropy vanish with the 4-acceleration.

Up to now, a further phenomenon of irreversibility was not taken into consideration: the multi-temperature relaxation which is discussed in the next section.

\[ \text{[15]which vanishes in equilibrium and for free 1-component systems, as we will see below} \]

21
6.4 Multi-temperature relaxation and the partial temperatures

Because the different components of the mixture have different partial (reciprocal) temperatures $\lambda_A, A = 1, 2, ..., Z$, a multi-temperature relaxation takes place which is an irreversible phenomenon. Consequently, multi-temperature relaxation has to be taken into account in the entropy identity by adding a suitable zero as done in (85).

A heat transfer $H^{AB}$ between two components of the mixture – $A$ and $B$ – takes place by multi-temperature relaxation, if the corresponding temperatures of the components are different from each other. Consequently, the entropy exchange between these two components is

\[ G^{AB} := H^{AB} \left( \frac{1}{\Theta_A} - \frac{1}{\Theta_B} \right), \quad H^{AB} = -H^{BA}, \quad \dot{H}^{AB} = 0. \]  
(140)

Here $H^{AB}$ is an energy density and $G^{AB}$ an entropy density according to (121).

\[ [H^{AB}] = [e^A], \quad [G^{AB}] = \left[ e^A \right] \frac{1}{K} = [s^A]. \]  
(141)

For the $A$-component, this results according to (140) in

\[ H^A := \sum_B H^{AB}, \quad \sum_B H^{AB} = 0, \quad \sum_A H^A = 0, \]  
(142)

\[ G^A := \sum_B H^{AB} \left( \frac{1}{\Theta_A} - \frac{1}{\Theta_B} \right), \]  
(143)

\[ \sum_A G^A = \sum_{AB} H^{AB} \left( \frac{1}{\Theta_A} - \frac{1}{\Theta_B} \right) \neq 0, \text{ if } \Theta^A \neq \Theta^B. \]  
(144)

The entropy exchange of the $A$-component according to multi-temperature exchange (143) and (140), has now to be introduced into the entropy identity (107). Because $G^A$ has the same physical dimension as $s^A$ according to (141), it fits into the first row of (107). Therefore we add the zero

\[ 0 = -\left[ u^{Ak} G^A \right]_{,k} + \sum_B \left[ u^{Ak} H^{AB} \left( \frac{1}{\Theta_A} - \frac{1}{\Theta_B} \right) \right]_{,k} \]  
(145)

to (107). Taking (140) into account and inserting

\[ \sum_B \left[ u^{Ak} H^{AB} \left( \frac{1}{\Theta_A} - \frac{1}{\Theta_B} \right) \right]_{,k} = u^{Ak} \dot{G}^A + \sum_B H^{AB} \left( \frac{1}{\Theta_A} - \frac{1}{\Theta_B} \right)^*, \]  
(146)

into (145)

\[ 0 = -\left[ u^{Ak} G^A \right]_{,k} + u^{Ak} \dot{G}^A + \sum_B H^{AB} \left( \frac{1}{\Theta_A} - \frac{1}{\Theta_B} \right)^*, \]  
(147)

\[ ^{16}\text{do not take multi-temperature relaxation for the heat conduction which is caused by a temperature gradient} \]
we obtain three additional terms which can be directly introduced into the entropy identity without defining an additional Lagrange multiplier. According to (107), the three terms of (147) are attached as follows

\[ -G^A \rightarrow \text{entropy density}, \quad \mu^A u^{Ak} G^A \rightarrow \text{entropy supply}, \]

\[
\sum_B \sum_{AB} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right)^* \rightarrow \text{entropy production density},
\]

Introducing these terms as demonstrated in sect. 6.1 into the entropy identity (116), the entropy density (135) becomes

\[ s^A = \frac{1}{\Theta^A} \left( \mu^A g^A + \nu^A (e^A + p^A) + u_m p^{Am} + \frac{1}{c_2} \{ \Sigma^A + u[\mu, \nu, \Theta, u, e, p, \Pi, \Xi, s, G, \pi] \} \right) + G^A, \quad \text{(151)} \]

and the state space (110) is extended by \( G^A/\varrho \)

\[ z^A = \left( c^A, \frac{e^A}{\varrho}, \frac{p^A}{\varrho}, \frac{\Xi^A}{\varrho}, \frac{s^{Amn}}{\varrho}, \frac{G^A}{\varrho} \right), \quad c^A := \frac{\varrho^A}{\varrho}. \quad \text{(152)} \]

and consequently the Gibbs equation (111) by \( (G^A/\varrho)^* \). The Gibbs-Duhem equation (112) is untouched by including the multi-temperature relaxation.

According to (149) the entropy supply (137) results in

\[ \varphi^A = \frac{1}{\Theta^A} \left\{ \mu^{A(\text{ex})} \Gamma^A + \left( \nu^A u^A + u_m h^A \right) \right\} + \sum_B \sum_{AB} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right)^* + \left( \frac{\mu^A}{\Theta^A} \right)_{;k} + \left( \frac{\nu^A}{\Theta^A} \right)_{;k} + \left( \frac{1}{\Theta^A} \right)_{;k} + \sum_B H^{AB} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right)^* + \sum_B \sum_{AB} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right)^* \right\} + u^{Ak} G^A, \quad \text{(153)} \]

and the entropy production density (138) becomes by (150)

\[ \sigma^A = \left( \frac{\mu^A}{\Theta^A} \right)_{;k} J^{Am} h^A_{mk} + \frac{\mu^A}{\Theta^A} \left( \frac{\nu^A}{\Theta^A} \right)_{;k} (J^{Am} h^A_{mk}) + \sum_B H^{AB} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right)^* + \sum_B \sum_{AB} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right)^* \right\} + \left( \frac{\nu^A}{\Theta^A} \right)_{;k} \Xi^{Akk} - \left( \frac{\mu^A}{\Theta^A} \right)_{;k} \Xi^{Akk} + \left( \frac{1}{\Theta^A} \right)_{;k} u_A \left[ \frac{1}{c} u^A \Xi^{Akk} \right] + \left( \frac{1}{\Theta^A} \right)_{;k} \left( \frac{1}{c} u_A \Xi^{Akk} \right) + \left( \frac{1}{\Theta^A} \right)_{;k} \left( \frac{1}{c} u_A \Xi^{Akk} \right) + \left( \frac{1}{\Theta^A} \right)_{;k} \left( \frac{1}{c} u_A \Xi^{Akk} \right). \quad \text{(154)} \]

The ten terms of the entropy production density (154) have as already discussed after (138), the following meaning:
• diffusion: \((\mu^A/\Theta^A)_{,k}J_A^m h_A^{m,k}\),
• by diffusion modified chemical reaction: \((\mu^A/\Theta^A)\left((\text{in})\Gamma^A + (J_A^m h_A^{m,k})_{,k}\right)\),
• heat conduction: \((\nu^A/\Theta^A)_{,k}q_A^{Ak}\),
• multi-component modified internal friction: \((\nu^A/\Theta^A)u_A^A_{,k}\left(\pi^{Ak} + q_A^{Ak}\right)\),
• multi-component interaction:\((u_m^m h_m^{m,l}/\Theta^A)_{,k}\left(\frac{1}{c^2}e_A^{Ak} u_A^{Al} + \frac{1}{c^2}q_A^{Ak} u_A^{Al} + L^{Ak}\right)\),
• multi-temperature relaxation: \(\sum_B H_{AB}^{AB}\left((1/\Theta^A) - (1/\Theta^B)\right)\).

The transition from the interacting \(^A\)-component to the free 1-component system is considered in sect.7 and that to the mixture in sect.8. All quantities introduced up to here are non-equilibrium ones, because we did not consider equilibrium conditions up to now. This will be done in the next section.

### 6.6 Equilibrium

#### 6.6.1 Equilibrium conditions

Equilibrium is defined by equilibrium conditions which are divided into basic and supplementary ones [11, 12]. The basic equilibrium conditions are given by vanishing entropy production, vanishing entropy flux density and vanishing entropy supply\(^{18}\):

\[\sigma_{eq}^A \equiv 0 \land \xi_{eq}^{Ak} \equiv 0 \land \varphi_{eq}^A \equiv 0.\]  

\(^{17}\)This term vanishes in equilibrium and for 1-component systems: see sect.7

\(^{18}\)The sign \(\equiv\) stands for a setting which implements an equilibrium condition.
A first supplementary equilibrium condition is the vanishing of all diffusion flux densities. According to (12)\textsubscript{1}, we obtain
\[ J_k^{A\text{eq}} = 0 \quad \Longrightarrow \quad u_k^{A\text{eq}} = f_{eq}^A u_k^{eq} \quad \Longrightarrow \quad c^2 = f_{eq}^A u_k^{eq} u_k^{A\text{eq}}. \quad (158) \]

Taking (8)\textsubscript{1} into account, (158)\textsubscript{3} results in
\[ (f_{eq}^A)^2 = 1 \quad \Longrightarrow \quad f_{eq}^A = \pm 1. \quad (159) \]

Consequently, we have to demand beyond (158)\textsubscript{1} the supplementary equilibrium condition that the mass densities are additive in equilibrium. We obtain according to (8)\textsubscript{2} and (14)\textsubscript{2}
\[ \varrho_{eq} = \sum_A \varrho_{eq}^A \longrightarrow f_{eq}^A = 1 \quad \longrightarrow \quad w_{eq}^A = 0. \quad (160) \]

Taking (158)\textsubscript{2} and (129) into account, (160)\textsubscript{2} yields
\[ u_k^{A\text{eq}} = u_k^{eq} \quad \longrightarrow \quad \lambda_{eq}^{Ak} = 0, \quad g_{eq}^{Am} = 0. \quad (161) \]

Further supplementary equilibrium conditions are given by vanishing covariant time derivatives, except that of the four-velocity:
\[ \Pi_{eq}^l = 0, \quad \Pi \neq u^l, \quad (162) \]
that means $u_{eq}^l$ is in general not zero in equilibrium. Consequently, the time derivatives of all expressions which contain the 4-velocity must be calculated separately, as we will see below.

According to (162)\textsubscript{1}, we obtain
\[ \varrho_{eq}^A = 0, \quad \left( \frac{\nu_{eq}^A}{\Theta_{eq}^A} \right)^{\star} = 0, \quad (163) \]
and the (3+1)-components of the energy-momentum tensor, (26) and (27), satisfy
\[ \dot{e}_{eq}^A = 0, \quad \dot{p}_{eq}^{Al} = 0, \quad \dot{q}_{eq}^{Ak} = 0, \quad \dot{p}_{eq}^A = 0, \quad \dot{\pi}_{eq}^{Akl} = 0. \quad (164) \]

Starting with (8)\textsubscript{1}, we have
\[ \dot{f}_{eq}^A = \frac{1}{c^2} \left( \dot{u}_{eq}^A u_m^{eq} u_m^{eq} + u_m^{eq} \dot{u}_m^{eq} \right). \quad (165) \]

Taking (161)\textsubscript{1} into account, this results in
\[ \dot{f}_{eq}^A = 0 \quad \longrightarrow \quad \dot{w}_{eq}^A = 0. \quad (166) \]

In equilibrium, we have according to (161)\textsubscript{1} and (16)
\[ h_{eq}^{Am} = h_{eq}^{\mu \nu}, \quad (167) \]
and according to (129) resulting in

\[ \lambda^{\text{Aeq}}_{i,k} = \left( \frac{1}{\Theta^A_{\text{m}h^A_{\text{m}}}} \right)^{\text{eq}} : k = 0. \] (168)

Despite of \( \Lambda^{\text{Aeq}}_a \neq 0 \) according to (134) and (162), the time derivatives of the Lagrange multipliers vanish in equilibrium, and according to (111) and (112), Gibbs and Gibbs–Duhem equations are identically satisfied in equilibrium, if the "shift of the time derivative"

\[ \lambda^A_m \frac{\Xi^A_m}{\varrho} = \left( \frac{u^A_m \Xi^A_m}{\varrho} \right) \cdot - \lambda^A_m \left( \frac{\Xi^A_m}{\varrho} \right) \cdot = 0 - \lambda^A_m \left( \frac{\Xi^A_m}{\varrho} \right) \cdot \] (169)

is taken into account.

Another supplementary equilibrium condition is the vanishing of the mass production terms in (10)

\[ (\text{ex}) \Gamma^A_{\text{eq}} \doteq 0 \land (\text{in}) \Gamma^A_{\text{eq}} \doteq 0 \rightarrow \Gamma^A_{\text{eq}} = 0 \] (170)

Thus, we obtain from (12), (163) and (170)

\[ \varrho^A_{\text{eq}} u^{Ak} + \varrho^A_{\text{eq}} u^{Ak} : k = \Gamma^A \rightarrow u^{Ak} : k = 0. \] (171)

The equilibrium temperature is characterized by vanishing multi-temperature relaxation

\[ \Theta^A_{\text{eq}} \doteq \Theta^B_{\text{eq}} \doteq \Theta^C_{\text{eq}} \doteq \ldots =: \Theta_{\text{eq}} \rightarrow G^A = 0, \land A. \] (172)

Often one can find in literature [25] the case of equilibrium of multi-temperature relaxation: although out of equilibrium, only one temperature is considered in multi-component systems. This case is realistic, if the relaxation of multi-temperature relaxation to equilibrium is remarkably faster than that of the other non-equilibrium variables [26].

Taking (161)₁, (26)₂ and (172) into account, the entropy density (151) becomes in equilibrium using the shift of the time derivative

\[ s^A_{\text{eq}} = \frac{1}{\Theta^A_{\text{eq}}} \left( \mu^A_{\text{eq}} \varrho^A_{\text{eq}} + \nu^A_{\text{eq}} (e^A_{\text{eq}} + p^A_{\text{eq}}) \right). \] (173)

Beyond the usual expression for the entropy density in thermostatics\(^{19}\), it includes an acceleration dependent spin term. The energy density and the pressure are here defined by the (3+1)-decomposition [25] of the energy-momentum tensor. The chemical potential is as well as the temperature introduced as a Lagrange multiplier.

Taking (161)₁, (158)₁ and (172) into account, the entropy flux density (136) vanishes in equilibrium, resulting in

\[ 0 = \nu^A_{\text{eq}} q^A_{\text{eq}} + u^A_{\text{eq}} \Xi^A_{\text{eq}} \rightarrow q^A_{\text{eq}} = 0, \] (174)

using the shift of the time derivative according to (169) and (162).

\(^{19}\)Below, we will see that \( \nu^A_{\text{eq}} = 1 \)
Finally the entropy supply (153) results in

\[ 0 = \nu_{eq}^{A} u_{Aeq}^{k} k_{eq} + u_{b}^{Aeq} u_{Aeq}^{A} \frac{1}{c^2} \mu_{eq}^{A}, \]

that means, the power exchange caused by the force density and by the angular momentum density vanishes in equilibrium.

The entropy production (154) has to vanish in equilibrium according to the basic equilibrium condition (157). Taking (170), (158), (174), (164), (161) and (172) into account and using (162), (154) results in

\[ 0 = \nu_{eq}^{A} u_{Aeq}^{k} \pi_{eq}^{Akl} + \left( \frac{\mu_{b}^{Aeq}}{\Theta_{eq}} \right)_{jk} \mu_{eq}^{A} \Theta^{A} \pi_{eq}^{Akl} + \left( \frac{u_{b}^{Aeq}}{\Theta_{eq}} u_{Aeq}^{A} \left( s_{Akab} ; k \right) \right)_{eq}. \]  

\[ \text{(176)} \]

The third term of the second row of (154) vanishes by shift of the time derivative. In equilibrium, spin terms appear in the vanishing power exchange (175) and in the spin modified internal friction (176).

As demonstrated, equilibrium of an \( A \)-component in the mixture is described by three basic equilibrium conditions (157) and six supplementary ones: (158), (160), (162), (170), (172). Often, we can find in the literature [27, 28] equilibrium conditions which are different from those postulated here. The reason for that is, that entropy production and supply and the entropy flux as starting-points for the basic equilibrium conditions differ from the expressions (135) to (138). Such different equilibrium conditions and their derivations are considered in the next two sections.

6.6.2 Killing relation of the 4-temperature

Starting with (90) \( A \), we now consider the following relations

\[ (\lambda^{A} ; _{k} u_{l}^{A} + \lambda^{A} u_{l;k}^{A}) \frac{1}{c^{2}} \epsilon^{A} u^{Ak} u^{Al} = \lambda^{A} \epsilon_{A}, \]

\[ (\lambda^{A} ; _{k} u_{l}^{A} + \lambda^{A} u_{l;k}^{A}) u^{Ak} p^{Al} = -\lambda^{A} u_{l}^{A} p^{Al}, \]

\[ (\lambda^{A} ; _{k} u_{l}^{A} + \lambda^{A} u_{l;k}^{A}) \frac{1}{c^{2}} q^{Ak} u^{Al} = \lambda^{A} q^{Ak}, \]

\[ -(\lambda^{A} ; _{k} u_{l}^{A} + \lambda^{A} u_{l;k}^{A}) p^{A} h^{Ak}, \]

\[ (\lambda^{A} ; _{k} u_{l}^{A} + \lambda^{A} u_{l;k}^{A}) \pi^{Akl} = \lambda^{A} u_{l;k}^{A} \pi^{Akl}. \]

\[ \text{(181)} \]

Taking (178), (179) and (181) into account, we obtain from (90) \( A \)

\[ (\lambda^{A} u_{l}^{A})_{,k} \left( T^{Akl} \frac{1}{c^{2}} \epsilon^{A} u^{Ak} u^{Al} + p^{A} h^{Ak} \right) = \\
= -\lambda^{A} u_{l}^{A} p^{Al} + \lambda^{A} q^{Ak} + \lambda^{A} u_{l;k}^{A} \pi^{Akl}. \]

\[ \text{(182)} \]

Replacing the second row of (154) by the LHS of (182) yields the entropy production of vanishing multi-temperature relaxation and vanishing spin by taking (123) into account without spin:

\[ \sigma_{0}^{A} = (\lambda^{A} \mu^{A})_{,k} J^{Am} h^{Ak} + \lambda^{A} \mu^{A} \left( (in) \Gamma^{A} - (J^{Am} h^{Ak})_{,k} \right) + \]

27
\[ \begin{align*}
&+ (\lambda^A u^A_l)_;k \left( T^{Akl} - \frac{1}{c^2} \epsilon^A u^A_k u^A_l + p^A h^{Akl} \right) + \\
&+ \left( \frac{1}{\Theta_A} u^m h^A_{ml} \right)_;k \left( \frac{1}{c^2} \epsilon^A u^A_k u^A_l + \frac{1}{c^2} q^A u^A_l + t^{Akl} \right). 
\end{align*} \]  
(183)

Evident is that
\[ \begin{align*}
(\lambda^A u^A_l)_;k \left( T^{Akl} - \frac{1}{c^2} \epsilon^A u^A_k u^A_l + p^A h^{Akl} \right) &= 0 \quad (184)
\end{align*} \]

is not a sufficient condition for equilibrium because the equilibrium conditions \((170)_2, (158)_1\) and \((161)_1\) are not necessarily satisfied and the entropy production \((183)\) does not vanish. If the energy-momentum tensor is symmetric, \((184)\) results in
\[ T^{Akl} = T^{Alk} : \left[ (\lambda^A u^A_l)_;k + (\lambda^A u^A_k)_;l \right] \left( T^{Akl} - \frac{1}{c^2} \epsilon^A u^A_k u^A_l + p^A h^{Akl} \right) = 0, \]  
(185)
an expression which as well as \((184)\) is not sufficient for equilibrium. Consequently, the \textit{Killing relation of the 4-temperature} \(\lambda^A u^A_l\)
\[ \left[ (\lambda^A u^A_l)_;k + (\lambda^A u^A_k)_;l \right] = 0 \]  
(186)
is also not sufficient for equilibrium.\(^{20}\)

If equilibrium is presupposed, the equilibrium conditions \((170)_2, (158)_1\) and \((161)_1\) are satisfied, the entropy production vanishes and
\[ T^{Akl} = T^{Alk} : \left[ (\lambda^A u^A_l)_;k + (\lambda^A u^A_k)_;l \right] \left( T^{Akl} - \frac{1}{c^2} \epsilon^A u^A_k u^A_l + p^A h^{Akl} \right)^{eq} = 0, \]  
(187)
\[ T^{eq} = T^{eq} : \left[ (\lambda^A u^A_l)_;k + (\lambda^A u^A_k)_;l \right] \left( T^{Akl} - \frac{1}{c^2} \epsilon^A u^A_k u^A_l + p^A h^{Akl} \right)^{eq} = 0 \]  
(188)
are necessary conditions\(^{21}\) for equilibrium according to \((183)\), if the spin is ignored. If the spin is taken into account, \((187)\) results by use of the fifth row of \((154)\) in
\[ (\lambda^A u^A_l)_;k \left( T^{Akl} - \frac{1}{c^2} \epsilon^A u^A_k u^A_l + p^A h^{Akl} \right)^{eq} = \]
\[ = - \frac{u^A_b}{\Theta^A} \frac{u^{Aeq}_{Ak} \Theta_{eq}}{u^{Aeq}_{Ak} \Theta_{eq} s^{Akb}_{;k}}. \]  
(189)

There are different possibilities to satisfy \((187)\) and \((188)\) which are discussed in the next section.

### 6.6.3 The gradient of the 4-temperature

The necessary condition for equilibrium ignoring spin \((187)\) can be differently satisfied generating different types of equilibria. There are three possibilities:

If equilibrium exists, one of the following three conditions is valid:

\(^{20}\)a fact which is well-known \(^{12}\)

\(^{21}\)but as discussed, not sufficient conditions
\[(\lambda^A u^A_{i,j})_{eq} = 0 \rightarrow \lambda^A_{eq} u^A_{eq} + \lambda^A_{eq} u^A_{i,k} = 0, \quad (190)\]

\[T_{eq}^{Ak} = \frac{1}{c^2} e^A_{eq} u^A_{eq} - p^A_{eq} h_{eq}, \quad (191)\]

\[(\lambda^A u^A_{i,j})_{eq} \neq 0 \land T_{eq}^{Ak} \neq \frac{1}{c^2} e^A_{eq} u^A_{eq} - p^A_{eq} h_{eq} \], and (187) is valid. (192)

Multiplication of (190) with \(u^A_{eq}\) results in
\[
\lambda^A_{eq} = 0 \land u^A_{eq} = 0, \quad (193)\]

that means, (190) represents an intensified equilibrium because additionally to the usual equilibrium conditions mentioned in sect.6.6.1, (193) is valid.

If (191) is valid, the equilibrium exists in a perfect material whose entropy production is zero. If the considered material is not perfect and if the equilibrium is not intensified, (192) is valid, and the question arises, whether (187) can be valid under these constraints.

To answer this question, we consider (177) to (181) in equilibrium. According to the equilibrium conditions, we obtain
\[
(\lambda^A u^A_{i,j})_{eq} = 0 \rightarrow \lambda^A_{eq} u^A_{eq} + \lambda^A_{eq} u^A_{i,k} = 0, \quad (194)\]

\[(\lambda^A u^A_{i,l} + \lambda^A u^A_{i,k})_{eq} \frac{1}{c^2} e^A_{eq} u^A_{eq} = 0, \quad (195)\]

\[(\lambda^A u^A_{i,l} + \lambda^A u^A_{i,k})_{eq} \frac{1}{c^2} e^A_{eq} u^A_{eq} = 0, \quad (196)\]

\[-(\lambda^A u^A_{i,l} + \lambda^A u^A_{i,k})_{eq} p^A_{eq} h_{eq} = 0, \quad (197)\]

\[(\lambda^A u^A_{i,l} + \lambda^A u^A_{i,k})_{eq} a^A_{eq} h_{eq} = 0. \quad (198)\]

Summing up (194) to (198) yields
\[
(\lambda^A u^A_{i,l})_{eq} T_{eq}^{Ak} = 0. \quad (199)\]

Consequently, (187) is satisfied because each of the three terms vanishes for its own, thus being compatible with (192). If an \(^A\)-component of a mixture is in equilibrium, two types of equilibria can occur: one in an arbitrary material showing the usual equilibrium conditions and another one which has beyond the the usual equilibrium conditions vanishing temperature gradient and vanishing 4-velocity gradient according to (193).

Evident is that an 1-component system which does not interact with other components is as a special case included in the theory of an \(^A\)-component in the mixture. This case is discussed in the next section.

7 Special Case: 1-Component System

7.1 Entropy flux, -supply and -density

An 1-component system\(^{22}\) can be described by setting equal all component indices of a multi-component system
\[
A, B, C, ..., Z \rightarrow 0, \quad (200)\]

\(^{22}\)that is not a mixture which is a multi-component system by definition
and for shortness, we omit this common index 0. Then the basic fields of an 1-component system are according to (2)

rest mass density and 4-velocity: $\{\varrho, u_k\}$. (201)

The equations (3) of Setting I change into identities. According to (81), (12), (14), (22) and (81), we have

$$f = 1, \quad J_k = 0, \quad w = 0, \quad T^{kl} = k^l \quad S^{kab}_{\ :k} = \frac{1}{c^2} m^{ab}. \quad (202)$$

The Lagrange multipliers become according to (123), (127), (129) and (134)

$$\lambda = \frac{\nu}{\Theta}, \quad \kappa = \frac{\mu}{\Theta}, \quad \lambda^k = 0, \quad \Lambda_a = \frac{u_a}{\Theta} \quad \Lambda_{ab} = 0. \quad (203)$$

The entropy density (135) and the state space (110) are as in equilibrium of the $A$-component (173)

$$s = \frac{1}{\Theta} \left( \mu \varrho + \nu(e + p) + u_m \frac{1}{c^2} \Xi^m \right), \quad z = (\varrho, e, \Xi^m). \quad (204)$$

The entropy flux (136), the entropy supply (137) and the entropy production (154) are by taking (182) into account

$$s^k = \frac{1}{\Theta} (\nu q^k + \dot{u}_m \Xi^{km}), \quad \varphi = \frac{1}{\Theta} \left( \mu \Gamma + \nu u_l k^l + \mu u_a \frac{1}{c^2} m^{ab} \right), \quad (205)$$

$$\sigma = \left( \frac{\nu}{\Theta} u_l \right)_{\ :k} \left( T^{kl} - \frac{1}{c^2} e u_k u^l + p h^{kl} \right) + \left( \frac{\mu}{\Theta} \right)_{\ :k} \Xi^{kb} - \frac{\mu}{\Theta} u_a s^{kab}_{\ :k}. \quad (206)$$

According to sect.3.4, the (3+1)-components of the mixture change into the corresponding quantities of the 1-component system. The necessary equilibrium conditions of an 1-component system are equal to those of an $A$-component in the mixture (24).

7.2 Equilibrium and reversibility

Vanishing entropy production out of equilibrium

$$\sigma^{rev} = 0, \quad \left( s^k \neq 0 \lor \varphi \neq 0 \right) \quad (207)$$

belongs to reversible processes and vice versa (29). According to (206),

$$\left( \frac{\nu}{\Theta} u_l \right)^{rev}_{\ :k} \left( T^{kl} - \frac{1}{c^2} e u_k u^l + p h^{kl} \right)^{rev} =$$

$$= -\left( \frac{\mu}{\Theta} \right)^{rev}_{\ :k} \Xi^{kb} + \left( \frac{\mu}{\Theta} u_a \right)^{rev} \left( s^{kab}_{\ :k} \right)^{rev}. \quad (208)$$

23 There are no chemical reactions in 1-component systems.

24 In equilibrium: "all cats are grey"
is sufficient and necessary for vanishing entropy production in 1-component systems. But concerning equilibrium, (208) is as well as (189) only necessary but not sufficient for it. Thus, all results of sect 6.6.3 change into those of an 1-component system, if the component index $A$ is omitted, $eq$ is replaced by $rev$, equilibrium is not presupposed and the generated expressions belong to reversible processes and vice versa.

Comparing (208) with (189) and ignoring the spin, the derivative of the 4-temperature and the Killing relation of the 4-temperature

$$\lambda_{kl}^{rev} = 0,$$

or

$$\lambda_{kl}^{rev} \cdot (\lambda_{kl}^{rev} + \lambda_{kl}^{rev}) = 0 \quad (209)$$

are rather conditions for reversible processes in 1-component systems because the entropy production is enforced to be zero without existing equilibrium. Independently of the 4-temperature, we obtain the well-known fact [30] that according to (206) all processes of perfect materials are reversible in 1-component systems without spin

$$T_{per} := \frac{1}{c^2} e u_l^k u^l - p h_{kl}^{rev} \rightarrow \sigma_{per} = \left(\frac{u_b}{\Theta}\right)_{;k}^{\Xi_{kb}} - \left(\frac{u_a}{\Theta}\right)_{;k}^{S_{kab}}, \quad (210)$$

8 Thermodynamics of a Mixture

According to the mixture axiom in sect 6.2, the balance equations of a mixture look like those of an 1-component system. But a mixture as a whole behaves differently from the interacting $A$-component in the mixture and also differently from an 1-component system which both were discussed in sect 6 and sect 7. Because the interaction between the components is still existing in the mixture, the diffusion fluxes and also the multi-temperature relaxation do not vanish as in 1-component systems. Because component indices $A$ do not appear in the description of mixtures, they are summed up in contrast to 1-component systems for which they vanish. The Settings I to XI enforce the mixture axiom resulting in

mass balance:

$$N^{Ak}_{;k} = \Gamma^A \rightarrow N^{k}_{;k} = \Gamma, \quad (211)$$

energy balance:

$$\dot{u}^A_{l} T^{Ak}_{;k} = \Omega^A \rightarrow \dot{u} T^{k}_{;k} = \Omega, \quad (212)$$

momentum balance:

$$h^{Am}_{l} T^{Ak}_{;k} = \Omega^{Am} \rightarrow h^{m}_{l} T^{k}_{;k} = \Omega^{m}, \quad (213)$$

spin balance:

$$S^{Akab}_{;k} = \frac{1}{c^2} m^{Aab} \rightarrow S^{kab}_{;k} = \frac{1}{c^2} m^{ab}. \quad (214)$$

The Settings I to XI are concerned with the balance equations (211) to (214), with the entropy density, the entropy flux density, the entropy supply, with the Lagrange multipliers and the multi-temperature relaxation. Obviously, we need an additional setting concerning the entropy of the mixture which will be formulated in the next section.

8.1 Additivity of 4-entropies

8.1.1 Entropy density and -flux

Like the additivity of the mass flux densities (5), the energy-momentum tensors (39) and the spin tensors (75) of the $A$-components, we demand that also of the 4-entropies
are additive

Setting XII:

\[ S^k = \sum_A S^{Ak}. \]  

(215)

Consequently, we obtain from (156) by use of (19) and (38)

\[ S^k = \sum_A \left\{ \frac{\mu^A}{\Theta^A} \left( (1 - w^A) N^{Ak} + J^{Ak} \right) + \frac{1}{\Theta^A} \left( \nu^A Q^{Ak} + u_m \tau^{Ak_m} \right) + \frac{\nu^A}{\Theta^A} p^A u^{Ak} + \right. \]

\[ \left. + \frac{1}{\Theta^A} u^A_m \left( \frac{1}{c^2} \Xi^{Am} u^{Ak} + \Xi^{Ak_m} \right) + \frac{1}{\Theta^A} u^A_m u^A_\bar{m} \left( \frac{1}{c^2} s^{Aab} u^{Ak} + s^{Akab} \right) + G^A u^{Ak} \right\}. \]  

(216)

According to (84), we obtain the entropy density and the entropy flux density of the mixture by use of (13), (81), (17) and (42)

\[ S^k u^k = c^2 s = \sum_A \left\{ \frac{\mu^A}{\Theta^A} \left( (1 - w^A) g^A c^2 f^A + \frac{1}{\Theta^A} \left( \nu^A Q^{Ak} + u_p \tau^{Ak_p} \right) \right) u^k + \right. \]

\[ \left. + \left( \frac{\nu^A}{\Theta^A} p^A + G^A \right) c^2 f^A + \frac{1}{\Theta^A} u^A_m \left( \Xi^{Am} f^A + \Xi^{Ak_m} u^k \right) + \right. \]

\[ \left. + \frac{1}{\Theta^A} u^A_m u^A_\bar{m} \left( s^{Aab} f^A + s^{Akab} u^k \right) \right\}, \]  

(217)

\[ S^k h^m_k = s^m = \sum_A \left\{ \frac{\mu^A}{\Theta^A} f^{Am} + \frac{1}{\Theta^A} \left( \nu^A Q^{Ak} + u_p \tau^{Ak_p} \right) h^m_k \right. \]

\[ + \left( \frac{\nu^A}{\Theta^A} p^A + G^A \right) c^2 g^{Am} + \frac{1}{\Theta^A} u^A_p \left( \Xi^{Ap} g^{Am} + \Xi^{Ak_p} h^m_k \right) + \right. \]

\[ \left. + \frac{1}{\Theta^A} u^A_m u^A_\bar{m} \left( \frac{1}{c^2} s^{Aab} g^{Am} + s^{Akab} h^m_k \right) \right\}. \]  

(218)

Taking (40) into consideration, we introduce by comparing with (217) and (218) the

Setting XIII:

\[ \nu^A \equiv f^A. \]  

(219)

With this setting, the expressions of the entropy density and the entropy flux of the mixture correspond to those which are generated by the additivity of the energy-momentum tensors: (15) to (52).

Finally, we obtain the entropy and entropy flux densities of the mixture

\[ s = \sum_A \left\{ \frac{\mu^A}{\Theta^A} \left( (1 - w^A) g^A f^A + \frac{1}{c^2} \frac{1}{\Theta^A} \left( f^A Q^{Ak} + u_p \tau^{Ak_p} \right) \right) u^k + \right. \]

\[ \left. + \left( \frac{\mu^A}{\Theta^A} p^A + G^A \right) f^A + \frac{1}{\Theta^A} u^A_m \frac{1}{c^2} \left( \Xi^{Am} f^A + \Xi^{Ak_m} u^k \right) + \right. \]

\[ \left. + \frac{1}{\Theta^A} u^A_m u^A_\bar{m} \frac{1}{c^2} \left( s^{Aab} f^A + s^{Akab} u^k \right) \right\}, \]  

(220)
\[ s^m = \sum_A \left\{ \frac{\mu^A}{\Theta^A} J^{Am} + \frac{1}{\Theta^A} \left( f^A Q^{Ak} + u_p T^{Ak} \right) h_k^m \right\} + \left( \frac{f^A}{\Theta^A} p^A + C^A \right) c^2 g^{Am} + \frac{1}{\Theta^A} \cdot u_p \left( \Xi^A g^{Am} + \Xi^{Ak} h_k^m \right) + \frac{1}{\Theta^A} \cdot u^A_\nu \left( \frac{1}{c^2} s^{Ab} g^{Am} + s^{Ak} h_k^m \right) \cdot \left( f^A \Theta^A \right) q^{Ak} + \frac{f^A}{\Theta^A} u^A_{l,k} \left( \Pi^{Ak} + u^{Ak} p^A \right) + \frac{1}{\Theta^A} u^A_{l,m} \left( \frac{1}{c^2} s^{Ab} u^{Ak} + \frac{1}{c^2} q^{Ak} u^A + t^{Ak} \right) \right\}. \]

These expressions of the entropy and entropy flux densities of the mixture are direct results of Setting XII \( (215) \). They will be considered in sect \( 8.2 \).

### 8.1.2 Entropy supply and production density

From \( (215) \) and \( (83) \) follows the entropy balance equation of the mixture

\[ S^k;_k = \sum_A S^{Ak;}_k = \sum_A \left( \sigma^A + \varphi^A \right) = \varpi + \varphi, \]  

satisfying the mixture axiom. Accepting the additivity of the entropy supplies of the \( A \)-component\(^ \text{25} \)

Setting XIV:

\[ \varphi = \sum_A \varphi^A, \]  

we obtain from \( (222) \) the additivity of the entropy productions of the \( A \)-components

\[ \varpi = \sum_A \sigma^A. \]  

The entropy supply of the mixture follows from \( (153), (223) \) and \( (219) \)

\[ \varphi = \sum_A \left\{ \frac{1}{\Theta^A} \left[ \mu^A (ex) \Gamma^A + \left( f^A u^A + u_m h^{Am} \right) k_{Al} + \left( u^A_{b,a} + u^A_{m,n} h^{Am} h_{b}^{An} \right) \frac{1}{c^2} m_{Ab} \right] + u^{Ak} G^A \right\}. \]

The entropy production of the mixture follows from \( (154), (224) \) and \( (219) \)

\[ \varpi = \sum_A \left\{ \left( \frac{\mu^A}{\Theta^A} \right)_k J^{Am} h^{Am}_m + \frac{\mu^A}{\Theta^A} \left[ \left( in \right) \Gamma^A + \left( J^{Am} h^{Am}_m \right)_k \right] + \left( f^A \right) q^{Ak} + \frac{f^A}{\Theta^A} u^A_{l,k} \left( \Pi^{Ak} + u^{Ak} p^A \right) + \frac{1}{\Theta^A} u^A_{l,m} \left( \frac{1}{c^2} e^A u^{Ak} u^A + \frac{1}{c^2} q^{Ak} u^A + t^{Ak} \right) \right\}. \]

\(^{25}\)Supplies are caused by external influences, productions by internal ones. That the reason why they do not mix up.
\[
+ \sum_B H^{AB} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right) \cdot \\
+ \left( \frac{u_B^A}{\Theta^A} \right) \Xi^{Abk} - \frac{u_B^A}{\Theta^A} u_a^{Akab} \cdot k + \\
+ \left( \frac{1}{\Theta^A} \cdot u^A_{[m u_m]} h_a^{Am} h_b^{An} \right) \cdot \frac{1}{c^2} u^A[a \Xi^{Ab}] + \\
+ \left( \frac{1}{\Theta^A} \cdot u^A_{[m u_m]} h_a^{Am} h_b^{An} \right) \cdot \frac{1}{c^2} u^A[a \Xi^{Ab}] + s^{Akab} \right). \tag{226}
\]

### 8.2 Partial and mixture temperatures

We now consider the positive term in the second row of the entropy density \((220)\) by which a mixture temperature \(\Theta^m\) can be defined. This mixture temperature is only a formal quantity because it is not evident that a thermometer exists which measures \(\Theta^m\): the partial temperatures are internal contact variables \([31]\) measured by thermometers which are selective for the temperature \(\Theta^A\) of the corresponding \(A\)-component. Evident is, the measured mixture temperature is a certain mean value of the partial temperatures of the components of the mixture \([32, 33, 34]\), but this measured mean value may depend on the individual thermometer and may be different from \(\Theta^m\), that means, the measured temperature is not unequivocal. Different definitions of the mixture temperature can be found in literature \([35]\). But a unique mixture temperature –independent of thermometer selectivities or arbitrary definitions– is given in the case of multi-temperature relaxation equilibrium \([172]\). This case is often silently presupposed in literature, if only one temperature is used in multi-component non-equilibrium systems. Only this sure case is considered in the sequel.

We now introduce the mixture quantities \(e^m\) and \(q^m\) to the entropy density \(s\) and the entropy flux density \(s^m\). According to \((45)\) and \((47)\), we obtain

\[
\frac{1}{c^2} \sum_A \frac{1}{\Theta^A} \left( Q^{Ak} f^A + u_p \tau^{Ak} \right) u_k = \frac{e}{\Theta^m}, \quad \sum_A \frac{1}{\Theta^A} \left( Q^{Ak} f^A + u_p \tau^{Ak} \right) h_k^m = \frac{q^m}{\Theta^m}. \tag{228}
\]

Taking \((227)\) and \((228)\) into account, \((220)\) and \((221)\) result in

\[
s = \frac{e}{\Theta^m} + \sum_A \left\{ \frac{\mu^A}{\Theta^A} \left( 1 - w^A \right) g^A f^A + \frac{1}{c^2} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^m} \right) \left( f^A Q^{Ak} + u_p \tau^{Ak} \right) u_k + \\
+ \left( \frac{f^A}{\Theta^m} \Theta^A + G^A \right) f^A + \frac{1}{\Theta^A} u_m^A \frac{1}{c^2} (\Xi^{Am} f^A + \Xi^{Akm} u_k) + \\
+ \frac{1}{\Theta^A} u_{[m u_m]} h_a^{Am} h_b^{An} \cdot \frac{1}{c^2} \left( s^{Aab} f^A + s^{Akab} u_k \right) \right\}, \tag{229}
\]

\[
s^m = \frac{q^m}{\Theta^m} + \sum_A \left\{ \frac{\mu^A}{\Theta^A} f^{Am} + \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^m} \right) \left( f^A Q^{Ak} + u_p \tau^{Ak} \right) h_k^m + \\
\]

34
\[ s = \frac{e}{\Theta} + \frac{1}{\Theta} \sum_A \left\{ \mu^A \left( 1 - w^A \right) g^A f^A + (f^A)^2 p^A + \right. \\
+ \frac{1}{c^2} \sum_A \left( \mu^A \right) \left( s^{Aab} g^A + s^{Akab} h_k^n \right) + \frac{1}{c^2} \sum_A \left( \mu^A \right) \left( s^{Aab} g^A + s^{Akab} h_k^n \right) \right\}, \quad (231) \\

The first term in the sum of (231) can be exploited by use of the mean value theorem according to (15) and \([3, 1, 2]\):

\[ \sum_A \mu^A (1 - w^A) g^A f^A = \mu \sum_A (1 - w^A) f^A g^A = \mu \sum_A (f^A)^3 g^A \quad (233) \]

Consequently, the chemical potential of the mixture is

\[ \mu := \sum_A \mu^A \left( 1 - w^A \right) g^A f^A / \sum_B (1 - w^B) g^B f^B, \quad (234) \]

and the entropy density of the mixture (231) yields

\[ s = \frac{1}{\Theta} e + \frac{1}{\Theta} \mu \sum_A (f^A)^3 g^A + \frac{1}{\Theta} \sum_A p^A (f^A)^2 + \\
+ \sum_A \left\{ \mu^A \frac{1}{c^2} \left( s^{Aab} g^A + s^{Akab} h_k^n \right) + \frac{1}{c^2} \sum_A \left( s^{Aab} g^A + s^{Akab} h_k^n \right) \right\}. \quad (235) \]

The entropy density (235) of the mixture in multi-temperature equilibrium is similarly constructed, but different from the expression (204) of an 1-component system: there are the energy-, the mass-, the pressure and the spin-term.
8.3.2 Entropy production and -supply

The entropy supply (225) results in
\[ \phi = \sum_A \left( \mu_A^{(ex)} \Gamma^A + \left( f^A u^A_i + u_m h^{Am}_i \right) k^A l + \left( \cdot u^A_m u^{Am}_m + \cdot u_m u^{Am}_m h^{Am}_b \right) \frac{1}{c^2} m^{Ab} \right). \] (236)

The entropy production density (226) becomes
\[ \sigma = \sum_A \left\{ \left( \frac{\mu_A}{\Theta} \right) J^{Am} h^{Ak}_m + \frac{\mu_A}{\Theta} \left[ (in) \Gamma^A + (J^{Am} h^{Ak}_m) \right] + \left( \frac{f^A}{\Theta} \right) q^{Ak}_k + \frac{f^A}{\Theta} u^{A}_l k \left( \pi^{Ak}_{kl} + u^{Ak} P^A \right) + \left( \frac{1}{\Theta} \right) u^{A}_{ml} \left( \frac{1}{c^2} e^A u^{Ak} u^{Al} + \frac{1}{c^2} q^{Ak} u^{Al} + t^{Ak} \right) + \left( \frac{u^A_a}{\Theta} \right) \xi^{Ab} - \left( \frac{u^A_a}{\Theta} \right) S^{Ab};k + \left( \frac{u^A_a}{\Theta} \right) u^{Am}_m h^{Ak}_b \left( \frac{1}{c^2} \frac{e^A}{u^{Ak}} + s^{Akab} \right) \right\}. \] (237)

The meaning of each individual term of (237) was already discussed with regard to the \( A \)-component according to (154).

Entropy density (235), entropy flux density (232), entropy supply (236), entropy production density (237) and chemical potential (234) of the mixture are represented by sums of quantities of the \( A \)-components. As expected, the \( (3+1) \)-components of the energy-momentum tensor cannot represent the mentioned thermodynamical quantities because diffusion fluxes, chemical potentials and temperature are not included in the energy-momentum tensor. From them, only the energy density \( e \) and the energy flux density \( q^m \) of the mixture appear in entropy and entropy flux densities.

A temperature \( \Theta \) of the mixture can be defined independently of multi-temperature equilibrium. According to (227), \( 1/\Theta \) is a weighted mean value of the reciprocal partial temperatures of the components arranged with the partial pressures, a construction which seems very special. As already mentioned in sect.8.2, a mixture temperature is not well defined because it depends on the component sensitivity of a thermometer.

8.4 Total equilibrium

Evident is that the equilibrium conditions of a mixture follow from those of the \( A \)-components which we considered in sect.6.6.1. Consequently, a demand of additional equilibrium conditions for mixtures is not necessary. Presupposing the equilibrium conditions of an \( A \)-component (discussed in sect.6.6.1) and multi-temperature equilibrium, we start with
the repetition
\[
\begin{align*}
f_{eq}^A &= 1, \quad w_{eq}^A = 0, \quad u_{eq}^A = u_{eq}^A, \quad g_{eq}^A = 0, \quad \Gamma_{eq}^A = \Gamma_{eq}^A = 0, \quad (ex)\Gamma_{eq}^A = (m)\Gamma_{eq}^A = 0, \\
e_{eq} &= \sum_A e_{eq}^A, \quad q_{eq}^A = 0, \quad p_{eq}^A = \sum_A p_{eq}^A, \quad t_{eq}^{jm} = \sum_A t_{eq}^{jm}. 
\end{align*}
\]

(238)

(239)

Taking (238) and (239) into account, the entropy density (235) of the mixture in equilibrium and the entropy flux density (232) result in
\[
s_{eq} = 1 + \sum_A u_{eq}^A, \quad q_{eq}^A = 0, \quad t_{eq}^{jm} = \sum_A t_{eq}^{jm}.
\]

(239)

if the the shifting of the time derivative (169), (162), (69) and (70) are applied. Interesting is, that the spin terms cancel in equilibrium.

The entropy supply (236) results in equilibrium
\[
\begin{align*}
s_{eq} &= \frac{1}{\Theta} s_{eq} + \frac{1}{\Theta} \mu_{eq} + \frac{1}{\Theta} \sum_A p_{eq} + 0, \\
\sum_A A_{eq}^A &= 0
\end{align*}
\]

(240)

according to (175).

The entropy production density (237) results in equilibrium
\[
\begin{align*}
\varphi_{eq} &= \frac{1}{\Theta} \sum_A \left\{ u_{eq}^A l_{eq}^A + u_{eq}^A u_{eq}^A \frac{1}{\Theta} m_{eq}^A \right\} = 0 \\
\sum_A A_{eq}^A &= 0
\end{align*}
\]

(241)

according to (176).

The vanishing entropy supply of the mixture (241) is satisfied by the equilibrium conditions (170), (175) and (167). The vanishing entropy production of the mixture (242) is satisfied by the equilibrium conditions (170), (158), (174), (176), (162) and (167).

From (4) follows in equilibrium analogously to (171)
\[
uf_{eq}^A = 0.
\]

(243)

8.5 (3+1)-entropy-components and spin

If the spin is taken into consideration, acceleration terms appear in the entropy density and production, (235) and (237), and in the entropy flux density and supply, (232) and (236). The four components (69) and (70) of the spin are differently distributed over the (3+1)-components of the entropy:

- the entropy density (151) of an A-component depends on the spin density \( s^{Aab} \) and on the spin density vector \( \Xi^{Am} \), whereas the entropy density of the mixture (220) depends on the four spin quantities (69) and (70). In 1-component systems, the entropy density (201) depends only on the spin vector \( \Xi^{m} \). In equilibrium, the entropy density is for all cases independent of the spin, (173) and (240).
• the *entropy flux density* \( (136) \) of an \( A \)-component depends on the couple stress \( s^{Akab} \) and on the spin stress \( \Xi^{Ak}_{km} \), whereas the entropy flux density of the mixture \( (232) \) depends on the four spin quantities \( (69) \) and \( (70) \). In 1-component systems, the entropy flux density \( (205)_1 \) depends only on the spin vector \( \Xi^m \). In equilibrium, the entropy flux density \( (173) \) and \( (240)_2 \) vanishes and induces \( q^{Ak}_{eq} = 0 \).

• the *entropy supply* of an \( A \)-component \( (153) \) is as well independent of the spin as for the mixture \( (236) \) and for an 1-component system \( (205)_2 \). The entropy supply vanishes in equilibrium, and a connection between the force density \( k^{Al}_{eq} \) and the angular momentum density \( m^{Aab}_{eq} \) is established, \( (175) \) and \( (241) \).

• the *entropy production density* \( (154) \) and \( (237) \) does not depend on the spin density \( s^{Aab} \) for an \( A \)-component and for the mixture, but a dependence upon the three other \((3+1)\)-spin-components exists. In 1-component systems, the entropy production density \( (206) \) depends on the spin stress \( \Xi^{kb} \) and on the couple stress \( s^{kab} \). The entropy production density vanishes in equilibrium, and a connection between the viscosity tensor \( \pi^{Ak}_{kl} \) and the spin stress and the couple stress is established, \( (176) \) and \( (242) \).

9 Balances, Constitutive Equations and the 2nd Law

Up to here, a special material was not taken into account: all considered relations are valid independently of the material which is described by constitutive equations supplementing the balance equations. Especially, the entropy productions \( (154) \) of the \( A \)-component and \( (237) \) of the mixture are not specified for particular materials. There are different possibilities for introducing constitutive equations\(^{27}\). Because constitutive equations are not in the center of our considerations, we restrict ourselves on the easiest ansatz which only serves for elucidation of the problem: Balance equations are generally valid for all materials, that means, they cannot be solved without choosing a special material characterized by constitutive equations which inserted into the balance equations transform these into a system of solvable differential equations for the wanted fields.

The entropy production of the \( A \)-component \( (154) \) is a sum of two-piece products whose factors are so-called ”fluxes” and ”forces”. According to \( (154) \), the ten fluxes are

\[
\mathcal{Y}^A = \left\{ J^{Amh}h^{Ak}_m, \left[ (in) \Gamma^A - \left( J^{Amh}h^{Ak}_m \right)_{jk} \right], \left( q^{Ak}, \left( \mu^{Akl} + u^{Akp}Al \right), \right. \right.
\]
\[
\left. \left( \frac{1}{c^2}e^A u^{Ak}u^{Al} + \frac{1}{c^2}q^{Ak}u^{Al} + t^{Akl} \right), H^{AB}, u^{Aab}, \right. \]
\[
\left. \frac{1}{c^2} u^{A[a}u^{A]}Ab, \left( \frac{1}{c^2} u^{A[a}u^{A]}Ab + s^{Aab} \right) \right\}, \quad (244)
\]

and the corresponding ten forces are

\[
\mathcal{X}^A = \left\{ \left( \frac{\mu^A}{\Theta^A} \right)_{jk}, \frac{\mu^A}{\Theta^A}, \left( f^A_{ik} \right)_{jk}, \frac{f^A_{ik}u^A_{ik}}{\Theta^A} \right\},
\]

\(^{27}\text{as an ansatz, or better by construction procedures} \[37, 38\] \]
The entropy production density \( \sigma^A \) of an \( A \)-component can be written as a scalar product of forces and fluxes

\[
\sigma^A = \mathcal{Y}^A \cdot \mathcal{X}^A,
\]

a relation which is valid independently of the material in consideration. The material is described by the dependence of the fluxes on the forces, by the constitutive equations

\[
\mathcal{Y}^A = F^A(\mathcal{X}^A)
\]

which have to be introduced into the expression of the partial entropy production density \( \sigma^A \) resulting in the entropy production density of the mixture by

\[
\sigma = \sum_A F^A(\mathcal{X}^A) \cdot \mathcal{X}^A \geq 0.
\]

The inequality is caused by the Second Law which states that the entropy production of the mixture is not negative after having inserted the constitutive equations into the general expression. Consequently, the Second Law represents a constraint for the constitutive equations, and it makes no sense to take the Second Law into consideration before the constitutive equations are inserted. The entropy production of sub-systems – here the \( A \)-components – is not necessarily positive semi-definite.

There are different methods for exploiting the dissipation inequality which are beyond this paper because special materials are here out of scope.

10 Special Case: General Relativity Theory

10.1 Extended Belinfante/Rosenfeld procedure

The basic equations of General-Covariant Continuum Thermodynamics (GCCT) of a mixture (sect. 8) contain covariant derivatives depending on the geometry of the space-time in which the physical processes occur. Here, the pseudo-Riemannian space of General Relativity Theory (GRT) is chosen as a special case.

In GRT, as a consequence of Einstein’s equations

\[
R^{ab} - \frac{1}{2} g^{ab} R = \kappa \Theta^{ab} \quad \Rightarrow \quad \Theta^{ab} = \Theta^{ba}, \quad \Theta^{a b : b} = 0,
\]

the gravitation generating energy-momentum tensor \( \Theta^{ab} \) has to be symmetric and divergence-free (\( R^{ab} \) is the Ricci tensor, \( g^{ab} \) the metric, \( R = R^{m}_{\ m} \)). According to and the
energy-momentum tensor of the mixture $T_{kl}$ may be neither symmetric nor divergence-free. The same is true for spin divergence $S_{kab;}^k$. Consequently, both tensors cannot serve as gravitation generating tensors in Einstein’s equations, and the question arises: how can the balance equations (249) be incorporated into the general-covariant framework of GRT? The answer to that question has been proved by the following extended Belinfante/Rosenfeld procedure whose special relativistic version is well known since a long time [41, 42, 43]. The general relativistic version is as follows

**Proposition [44]:** The general-covariant Belinfante/Rosenfeld procedure generates a symmetric and divergence-free tensor

$$\Theta_{ab} := T_{ab} - \frac{1}{2} \left[ S_{kab} + S_{abk} + S_{bak} \right]_{;k},$$

(251)

if the GCCT balances (249) and the Mathisson-Papapetrou equations (252) and (253)

$$\frac{1}{c^2} m_{ab} \overset{\text{MP}}{=} \frac{1}{2} \left[ S_{kab} + S_{abk} + S_{bak} \right]_{;k} = -\frac{1}{2} R_{bklm} S^{klm},$$

(252)

(253)

($R_{bklm}$ is the curvature tensor) are valid as necessary constraints for the force density and the angular momentum.

The Mathisson-Papapetrou equations are general-covariant including the special-relativistic case which is characterized by replacing the covariant derivatives by commuting partial ones and by $R_{bklm} \equiv 0$.

Inserting (252) into (251) results in

$$\Theta_{ab} = T_{(ab)} - \frac{1}{2} \left[ S_{abc} + S_{bac} \right]_{;c} = T_{(ab)} - S_{(ab)c;}^c,$$

(254)

a tensor which is symmetric and divergence-free according to (251) and (253).

The general-covariant Belinfante/Rosenfeld procedure transforms by use of the symmetric spin divergence $S_{(ab)c;}^c$ the not necessary divergence-free symmetric part of the energy-momentum tensor $T_{(ab)}$ into a symmetric and divergence-free tensor $\Theta_{ab}$. Or in other words: the energy-momentum tensor $T_{ab}$ (not necessary symmetric and divergence-free) is tranformed into the mutant $\Theta_{ab}$ (symmetric and divergence-free).

The decisive step for connecting GRT and GCCT is the following usually used

**Setting XV:**

$$\Theta_{ab} \overset{\text{Setting XV}}{=} \Theta_{ab}.$$  

(255)

The mutant which is created by the Belinfante/Rosenfeld procedure is the gravitation generating energy-momentum tensor of Einstein’s equation (250). According to (249), the mixture (and not single components) determines the geometry.

---

28 This name was coined by H.-H. von Borzeszkowski
10.2 Example: 2-component plain-ghost mixture

10.2.1 The plain component

The balance equations defining the plain component (P) are according to (1), (22) and (81)

\[ N^P_k = \varrho^P u^P_k, \quad N^{Pk}_k = 0, \quad (256) \]
\[ T_P^{[ab]} = 0, \quad T_P^{(ab)} : a = 0 \equiv k^b_P, \quad (257) \]
\[ S^{abc}_P = 0, \quad S^{kab}_P : k = 0 \equiv \frac{1}{c^2} m^{ab}_P. \quad (258) \]

By definition, the plain component is characterized by a symmetric and divergence-free energy-momentum tensor and vanishing spin. According to (252) and (253), the Mathisson-Papapetrou equations are satisfied by (256) to (258), so that the plain component of cause fits into GRT. If the plain component is regarded as an 1-component system, the gravitation generating energy-momentum tensor is as expected according to (254) to (255)

\[ \Theta^{ab}_P = T_P^{(ab)}. \quad (259) \]

The situation changes, if the plain component is regarded as a mixture component of a 2-component mixture whose second component is introduced in the next section.

10.2.2 The ghost component

The balance equations defining the ghost component (G) are

\[ N^G_k = g^G u^G_k = 0, \quad (260) \]
\[ T_G^{(ab)} = 0, \quad T_G^{[ab]} : a = k^b_G, \quad (261) \]
\[ S^{kab}_G : k = \frac{1}{c^2} m^{ab}_G, \quad S^{(ab)k}_G : ; a = 0. \quad (262) \]

By definition, the ghost component is characterized by vanishing mass density. The Mathisson-Papapetrou equations demand

\[ \frac{1}{c^2} m^{ab}_G ; a = 2k^b_G, \quad (263) \]

a dependence between force and angular momentum densities. If the ghost component is regarded as an 1-component system, the gravitation generating energy-momentum tensor is according to (254) to (255)

\[ \Theta^{ab}_G = - S^{(ab)k}_G : k. \quad (264) \]

Surprising is, that a "non-material" system such as the ghost component has a gravitational effect.

Plain and ghost components form a 2-component mixture which is discussed in the next section.

\[ ^{29} \text{and not as a mixture component} \]
\[ ^{30} \text{no "normal" material (} g^G = 0), \text{ therefore the name "ghost"} \]
10.2.3 The plain-ghost mixture

According to the settings (5), (39) and (75), we obtain for the mixture by taking (256) and (260), (257) and (261) into account

\[ N_k = N_k^P + N_k^G = \varphi u_k^P, \]
\[ T^{ab} = T^{(ab)}_P + T^{(ab)}_G \quad \rightarrow \quad T^{(ab)} = T^{(ab)}_P, \quad T^{[a} = T^{[a}_G, \]
\[ T^{ab}_{:a} = T^{[ab]}_{:a}, \]
\[ S^{abc} = S^{abc}_P + S^{abc}_G \quad \rightarrow \quad S^{(abc)}_{:c} = S^{(abc)}_{:c}, \]
\[ S^{kab}_{:c} = S^{kab}_{:c} + S^{kab}_{:c} = \frac{1}{c^2} m^{ab}_G. \]

For fitting the GCCT plain-ghost mixture into the GRT, the Mathisson-Papapetrou equations (252) and (253) have to be satisfied for the mixture. Taking (258) and (267) into account, (252) results in

\[ \frac{1}{c^2} m^{ab}_G = 2 T^{[ab]}_G. \]

According to (257) and (267), (265) yields (263).

The gravitation generating energy-momentum tensor of the plain-ghost mixture is according to (254), (266) and (268)

\[ \Theta^{ab} = T^{(ab)} - S^{(ab)}_{:c} = T^{(ab)}_P - S^{(ab)}_{:c} = \Theta^{ab}_P + \Theta^{ab}_G \]

which is different from (259) and (264).

10.3 "Dark matter" as a ghost component?

The previous statements allow to discuss the following (strange) situation: An observer takes the plain-ghost mixture for an 1-component plain mixture because the ghost component is "dark": no additional mass density and no spin can be detected from the point of view of the plain component. Erroneously, this observer supposes that (256) to (258) are valid, but in fact, (265) to (269) are true. Observations of gravitational effects yield that the gravitation generating energy-momentum tensor \( \Theta^{ab}_P \) (259) does not describe the observed gravitation because the ghost component of the plain-ghost mixture is invisible for the observer according to (261). This situation remembers lively the search for "dark matter" which should correct the energy-momentum tensor of the "visible matter". If the observer speculates that the "dark matter" is a matter-free and spin-equipped object according to (260) to (262), the gravitation generating energy-momentum tensor \( \Theta^{ab} \) describes the gravitation correctly in contrast to \( \Theta^{ab}_P \).

One question arises: Does a ghost component exit in nature and what is its physical essence?
11 Summary

A multi-component system is formed by its components which are characterized by own individual quantities, such as velocity, density, chemical potential, stress tensor, temperature, heat flux and entropy flux densities, entropy production and supply and further items. All these individual quantities determine those of the multi-component system which is described as a mixture. Individual temperatures of the components result in multi-temperature relaxation towards the corresponding equilibrium generating a common temperature of all components and the mixture. A temperature of the mixture in multi-temperature relaxation non-equilibrium depends on the used thermometer and cannot be defined unequivocally.

Starting out with the rest mass densities of the components of the multi-component system, the mass flux densities of the components are defined by introducing their different 4-velocities. The mixture of the components is characterized by several settings. The first one is the additivity of the component’s mass flux densities to the mass flux density of the mixture. In combination with the mixture axiom, this setting allows to define mass density and 4-velocity of the mixture and the diffusion fluxes of the components. The non-symmetric energy-momentum tensor of one component interacting with the mixture is introduced, and its (3+1)-split together with the component’s mass and diffusion flux densities are generating the entropy identity [11]. The exploitation of the entropy identity requires additional settings: the entropy density, flux and supply. These settings result in physical interpretations of entropy density, flux and supply. The entropy production follows from the entropy identity which restricts possible arbitrariness of defining.

By use of the entropy identity, Lagrange multipliers are introduced concerning the constraints taken into consideration. These are temperature, chemical potential and an additional non-equilibrium variable which characterizes the considered component to be a part of the mixture. Beside the classical irreversible processes – diffusion, chemical reactions, heat conduction and friction – an additional irreversible process – multi-temperature relaxation – appears due to the embedding of the considered component into the mixture. Different from the classical case, the mass production term, the heat flux density and the viscous tensor are modified by so-called effective quantities.

Equilibrium is defined by equilibrium conditions which are divided into necessary and supplementary ones [11, 12, 36]. The necessary equilibrium conditions are given by vanishing entropy production, vanishing entropy flux density and vanishing entropy supply. Supplementary equilibrium conditions are: vanishing diffusion flux densities, vanishing component time derivatives [11] and vanishing of the mass production terms. Presupposing these equilibrium conditions, we obtain: all components have the same 4-velocity, all heat flux densities are zero, the power as well as the divergence of the 4-velocity of each component vanish, and the viscous tensor is perpendicular to the velocity gradient.

The corresponding free component is defined by undistinguishable component indices [32]. This 1-component system represents the easiest classical case serving as a test, if the interacting component in the mixture is correctly described. The vanishing of the entropy production in equilibrium is shortly investigated: the so-called Killing relation of

\[31\text{except that of the 4-velocity}\]

\[32\text{that is not the mixture which is a multi-component system}\]
the vector of 4-temperature is neither a necessary nor a sufficient condition for equilibrium. Also the statement that materials are perfect in equilibrium cannot be confirmed.

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12 Appendices

12.1 Rest mass densities

Consider two frames, \( B^A \) and \( B^B \). \( B^A \) is the rest frame of the \( A \)-component and \( B^B \) that of the \( B \)-component. The corresponding rest mass densities are

\[
\text{rest mass/rest volume: } \rho^A := \frac{m_A^0}{V_A^0}, \quad \rho^B := \frac{m_B^0}{V_B^0}.
\]

(272)

For comparing the rest mass densities, we have to choose the rest volumes to be equal

\[
V_A^0 = V_B^0 = \ldots = V_0.
\]

(273)

By definition, the rest mass densities do not depend on the frame, that means, the rest mass densities are relativistic invariants which should not be confused with the measured densities in non-resting frames

\[
\text{\( B^B \): } \rho_B^A = \frac{\rho_A^A}{1 - v_{AB}^2/c^2}, \quad \text{\( B^A \): } \rho_A^B = \frac{\rho_B^B}{1 - v_{BA}^2/c^2}, \quad v_{AB} = -v_{BA}.
\]

(274)

Here \( \rho_A^A \) is the density of the \( A \)-component in the rest frame of the \( B \)-component, and \( v_{AB} \) is the translational 3-velocity of \( B^A \) in the frame \( B^B \). These densities are out of scope in this paper.

We now consider \( \text{(8)}_1 \) in the rest frame \( B_O \) of the mixture which is defined by \( u^k_{O} = (0, 0, 0, c) \). Consequently, we obtain

\[
f_O^A = \frac{1}{c^2} u_{4O}^A c = \frac{1}{c} \frac{c}{\sqrt{1 - v_{AO}^2/c^2}}.
\]

(275)

Inserting (275) into \( \text{(8)}_2 \) results in the mass density of the mixture in its rest frame

\[
\rho_O = \sum_A f_O^A \rho_O^A = \sum_A \frac{1}{\sqrt{1 - v_{AO}^2/c^2}} \frac{\rho_A^A}{1 - v_{AO}^2/c^2} = \sum_A \frac{\rho_A^A}{(1 - v_{AO}^2/c^2)^{3/2}}.
\]

(276)

The same result is obtained, if \( \text{(8)}_3 \) is written down for the rest system of the mixture.

12.2 Example: Uniform component velocities

If there exists a common rest frame \( B^0 \) for all \( A \)-components

\[
u^A_k = u^0_k, \quad \forall A.
\]

(277)
According to (5), we obtain
\[ \rho u_k = u_k^0 \sum_A \theta^A \Rightarrow \rho c^2 = u_k^0 \sum_A \theta^A \land \rho u_k u_k^0 = c^2 \sum_A \theta^A, \] (278)
and with (8) follows
\[ \rho c^2 = c^2 f^0 \sum_A \theta^A \land \rho c^2 f^0 = c^2 \sum_A \theta^A \Rightarrow (f^0)^2 = 1, \] (279)
resulting in
\[ f^0 = \pm 1. \] (280)
We obtain from (8)
\[ \rho = f^0 \sum_a \theta^A = \pm \sum_a \theta^A \Rightarrow f^0 = +1, \] (281)
and taking (278) into account
\[ u_k = u_k^0. \] (282)
As expected, the 4-velocity of the mixture is identical with the uniform component velocities.

12.3 Stoichiometric equations

The system of the relativistic stoichiometric equations runs as follows
\[ \sum_A \nu^A_\alpha M^A_0 = 0, \] (283)
component index: \(A = 1, 2, \ldots, Z,\) reaction index: \(\alpha = 1, 2, \ldots, \Omega.\)

The stoichiometric coefficients \(\nu^A_\alpha\) are scalars, and the partial rest mole mass \(M^A_0\) is defined using the scalar mole number \(n^A\) and the mole concentration \(\xi^A\) of the \(A\)-component
\[ M^A_0 := \frac{m^A_0}{n^A} = \frac{V_0}{n^A} \theta^A = \frac{\theta^A}{\xi^A}, \quad \xi^A := \frac{n^A}{V_0} \] (284)
according to (272) and (273). The stoichiometric coefficients \(\nu^A_\alpha\) are determined by the partial rest mole masses \(M^A_0, A = 1, 2, \ldots, Z,\) before and after the \(\alpha\)th reaction.

The time derivative of the mole number is determined by the reaction velocities \(\xi_\alpha\)
\[ \dot{n}^A = \sum_\alpha \nu^A_\alpha \dot{\xi}_\alpha. \] (285)
Multiplication with \(M^A_0\) results by use of (283) in
\[ M^A_0 \dot{n}^A = \sum_\alpha \nu^A_\alpha M^A_0 \dot{\xi}_\alpha \Rightarrow \sum_A M^A_0 \dot{n}^A = 0. \] (286)
Starting out with the physical dimensions

\[
[\nu_A^A] = \text{mol}, \quad [n] = \text{mol}, \quad [M_0^A] = \frac{kg}{\text{mol}}, \quad [\xi^A] = \frac{\text{mol}}{m^3}, \quad [\xi_\alpha] = \frac{1}{s},
\]

that of the mass production term in the first row of (138) is evidently

\[
[[^\text{(in)} \Gamma^A]] = \frac{kg}{m^3s}
\]

A comparison with

\[
[M_0^A n^A] = \frac{kg}{s}
\]

shows that \(^{(\text{in})} \Gamma^A\) is the density which belongs to the mass production (286). Because according to (273), all rest mass densities are referred to the relativistic invariant \(V_0\), we obtain from (289) and (288) with (286)

\[
^{(\text{in})} \Gamma^A = \frac{1}{V_0} \sum_\alpha \nu_\alpha^A M_0^A \xi_\alpha \rightarrow \sum_A ^{(\text{in})} \Gamma^A = 0.
\]

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