D0-branes in an H-field Background and Noncommutative Geometry

Masahiro Anazawa

Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606-8502, Japan

anazawa@yukawa.kyoto-u.ac.jp

Abstract

It is known that if we compactify D0-branes on a torus with constant B-field, the resulting theory becomes SYM theory on a noncommutative dual torus. We discuss the extension to the case of an H-field background. In the case of a constant H-field on a three-torus, we derive the constraints to realize this compactification by considering the correspondence to string theory. We carry out this work as a first step to examine the possibility to describe transverse M5-branes in Matrix theory.

PACS: 11.25.-w; 11.25.Sq; 11.15.-q
Keywords: D-branes; Matrix Theory; Compactification; Noncommutative Geometry
1. Introduction

Since the proposal of Matrix theory [1] as a nonperturbative formulation of M-theory in the infinite momentum frame or DLCQ of M-theory [2], it has passed various kinds of consistency tests [3]. Because the action of Matrix theory is equivalent to a regularized action of supermembrane in the light cone gauge [4], the relation of membranes to Matrix theory is clear, and fundamental strings can also be described in Matrix theory [3].

As for M5-branes, the situation is more obscure [6, 8, 7]. In [8], the supersymmetry algebra was examined for finite $N$ Matrix theory, and various brane charge densities were identified. While the charge density of longitudinal M5-brane wrapping 11th direction really emerged as a central charge, that of transverse M5-brane wasn’t found. In [9], a matrix model describing open membranes was formulated in the light cone gauge, motivated by the idea to describe M5-branes as boundaries on which open membranes can attach. However, there was also difficulty in describing transverse M5-branes. These situations are very unsatisfactory from the viewpoint of 11-dimensional Lorentz symmetry. In general, brane charges are expected to merge in the large $N$ limit, where 11-dimensional Lorentz symmetry should be hold. Then we consider that there still remain possibilities to describe transverse M5-branes in Matrix theory, if we consider some particular large $N$ limit.

On the other hand, it was shown that the usual compactification of Matrix theory on tori [10, 7] can be extended to noncommutative cases [11]. In the usual compactification on $T^d$, the resulting theory becomes SYM theory on a dual torus. This can be extended to SYM theory on a noncommutative dual torus. There, the parameters of the noncommutativity were constant, and their physical interpretation was argued to be the fluxes of the three-index gauge field $C_{-IJ}$ [11]. This field corresponds to $B_{IJ}$ in type IIA string theory. This noncommutative nature was derived in [12]–[17] from various points of view.

Since this fact shows that we can deal with $C_{-IJ}$ background in Matrix theory, this leads us to an another possibility of a description of transverse M5-branes. If we succeed to find a way to describe the backgrounds corresponding to non-zero transverse M5-brane charge in Matrix theory, the resulting theory might describe transverse M5-branes. Since transverse M5-branes correspond to NS5-branes in type IIA string theory, for example, we need to consider a background satisfying

$$\frac{1}{2\pi\alpha'} \int_{S^3} H = 2\pi n.$$  \hfill (1.1)

To do this, we have to find a way to treat space dependent $B_{IJ}$.

\footnote{In [18], an extension to non-constant $B_{IJ}$ was discussed using deformation quantization theory. There, $B_{IJ}$ depends on the position of the dual compact space. However, we need $B_{IJ}$ depending on the position of the original compact space $T^3$ not but dual space.}
a first step in this direction, we will consider a background satisfying

$$\frac{1}{2\pi \alpha'} \oint_{T^3} H = 2\pi n, \quad (1.2)$$

because this is the simplest and nontrivial extension of the case in [11].

In this paper, we will consider a low energy system of D0-branes, and discuss how to realize the compactification on $T^3$ with the background $B_{IJ}$ satisfying eq. $$\text{(1.2)}$$. We assume that this compactification can be realized in Matrix theory with suitable constraints on the matrices similar to those in [10]. We will discuss what constraints we need from the correspondence to string theory in a similar way to [13]. We will find that the center of mass coordinates of D0-branes no longer decouple from the remaining degrees of freedom. We also discuss the relation to the ordinary noncommutative torus compactification.

2. Three-torus with an H-field background

Let us consider the compactification of Matrix theory on a rectangular torus. For flat background, the Lagrangian of Matrix theory is given by the low energy Lagrangian of D0-branes in ten dimensions,

$$\mathcal{L} = \frac{1}{2g_s \sqrt{\alpha'}} \text{Tr} \left\{ (D_t X^I)^2 + \frac{1}{2(2\pi \alpha')^2} [X^I, X^J]^2 + \frac{i}{2\pi \alpha'} \Theta^T D_t \Theta - \frac{1}{(2\pi \alpha')^2} \Theta^T \Gamma_I [X^I, \Theta] \right\}, \quad (2.1)$$

where $I, J = 1, \cdots, 9$. Here, the appropriate $\alpha' \to 0$ limit specified in [19, 20] must be taken to make this system correspond to M-theory in 11 dimensions. As it is well known, for the usual compactification on $T^d$ with radii $R_i$, we should impose the quotient condition on the matrices [19, 20],

$$U_i^{-1} X^I U_i = X^I + 2\pi R_i \delta_{I,i} \quad i = 1, \cdots, d$$

$$U_i^{-1} \Theta U_i = \Theta, \quad (2.2)$$

where $U_i$ are unitary operators. This relation means that $X^i + 2\pi R_i$ is equivalent to $X^i$ up to a unitary transformation. From the consistency of these relations, $U_i U_j U_i^{-1} U_j^{-1}$ must commute to $X^I$ and $\Theta$, so we obtain [11]

$$U_i U_j = e^{2\pi i \theta_{ij}} U_j U_i. \quad (2.3)$$

For the case where $\theta_{ij}$ are constant, their physical interpretation is argued to be the fluxes of $B_{ij}$ integrated on the $T^2$ extending $x^i$ and $x^j$ directions [11]-[14].
In this paper we would like to consider a $T^3$ compactification with a non-zero $H_{123}$ background, which is topologically quantized as

$$T_2 \oint_{T^3} H = 2\pi n,$$  \hspace{1cm} (2.4)

where $T_2$ is $1/2\pi\alpha'$ and $n$ is an integer. In this case, $B_{ij}$ are not constant and depend on the position in $T^3$, so the above procedure of toroidal compactification cannot be applied. The quotient condition must be modified. In this paper we will discuss how to modify the usual quotient condition to describe the torus compactification with non-zero $H$ field. Throughout the paper, we mainly consider the case of a D0-brane compactified on $T^3$, because the extension to N D0-branes is straightforward.

Let us assume that $(x^1, x^2, x^3)$ directions are compactified on $T^3$ and we have constant $H_{123}$. For simplicity we take $B_{ij}$ as

$$B_{12}(x^3) = \frac{1}{(2\pi)^3 R_1 R_2 R_3} \frac{2\pi n}{T_2} x^3,$$  \hspace{1cm} (2.5)

and $B_{23} = B_{31} = 0$. This configuration is topologically nontrivial and the boundary condition of $B_{ij}$ is specified introducing a nontrivial gauge transformation

$$B(x^3 + 2\pi R_3) = B(x^3) + d\Lambda^{(1)},$$  \hspace{1cm} (2.6)

where $\Lambda^{(1)}$ is a 1-form field. To be specific, we take as

$$\left(\begin{array}{c}
\Lambda_1^{(1)} \\
\Lambda_2^{(1)}
\end{array}\right) = \left(\begin{array}{c}
0 \\
\frac{1}{(2\pi)^3 R_1 R_2} \frac{2\pi n}{T_2} (x^1 - y^1)
\end{array}\right),$$  \hspace{1cm} (2.7)

where for later convenience we have included $y^i$, which are position coordinates of the D0-brane on the $T^3$ defined up to $2\pi R_i$.

**Condition on string field**

Let us make sure that a D0-brane can really exist in the background $B_{ij}$. Because the $B_{ij}$ is defined using the gauge transformation (2.6), the matter field coupling to $B_{ij}$ must be defined by the corresponding gauge transformation. This matter field is a string field. In the covering space of $T^3$, that is $R^3$, there are infinite mirrors of the D0-brane, which form a lattice. Let us label the D0-branes as $(a, b, c)$, where $a, b, c$ are the integers which indicate the position of the D0-branes on the lattice. There are strings connecting any two D0-branes. For strings which starts from $(a, b, c)$ and ends at $(a', b', c')$, let us introduce a string field operator

$$\Phi_{(a,b,c)(a',b',c')}.$$  \hspace{1cm} (2.8)

This string field must satisfy the boundary condition corresponding to that of $B_{ij}$.
In general, the gauge transformation of string field or string wave functional can be decided by requiring the invariance of string transition amplitudes. Let us consider a open string whose end points are on general D-branes. The transition amplitude is written by

\[ \int [\mathcal{D}X \mathcal{D}\psi] \Psi^* (c_2, t_2) \Psi (c_1, t_1) e^{iS_{\text{string}}}, \]  

(2.9)

where \( \Phi(c_i, t_i) \) are string wave functionals for string paths \( c_i \). When \( B \) transforms as

\[ B \rightarrow B + \Lambda^{(1)}, \]  

(2.10)

the string wave functionals and 1-form gauge fields \( A \) on the D-branes have to transform as

\[ \Psi(c) \rightarrow \exp(-iT_2 \int_c \Lambda^{(1)}) \Psi(c) \]  

(2.11)

\[ A \rightarrow A - \Lambda^{(1)} \]  

(2.12)

to keep the amplitude invariant.

In the case of the background we are considering, corresponding to eq.(2.6), the string field has to satisfy the boundary condition

\[ \phi(a,b,c+1)(a',b',c'+1) = \exp(-iT_2 \int_{\text{path}} \Lambda^{(1)}) \phi(a,b,c)(a',b',c') . \]  

(2.13)

Since we are considering the low energy limit, the path can be taken as the straight path connecting the two D0-branes. Substituting eq.(2.7) into eq.(2.13), we obtain

\[ \phi(a,b,c+1)(a',b',c'+1) = e^{i\pi n(a+a')(b-b')} \phi(a,b,c)(a',b',c') . \]  

(2.14)

For the translation in \( x^1 \) or \( x^2 \) direction, we don’t need any gauge transformation, then

\[ \phi(a+1,b,c)(a'+1,b',c') = \phi(a,b,c)(a',b',c') \]
\[ \phi(a,b+1,c)(a',b+1,c') = \phi(a,b,c)(a',b',c') \]  

(2.15)

must be satisfied. We can easily see that these conditions (2.14) and (2.15) are consistent each other. Therefore we can say that the background under consideration is really consistent. Note here that in order for the consistency to hold, \( n \) has to be quantized as an integer.

3. Constraints for the compactification

In this section we will consider how to realize the the system specified in the last section in Matrix theory. We assume that this system can be described by Matrix theory.
with some constraints similar to eq. (2.2). We will examine what constraint is necessary from the correspondence to string theory.

Naively off diagonal matrix elements of Matrix theory correspond to the components of the string field as

$$X^I_{(a,b,c)(a',b',c')} \rightarrow \phi^I_{(a,b,c)(a',b',c')} ,$$

but they are not exactly equivalent when background $B$ field exists. In [13], in the case of constant $B$ field on $T^2$, the authors showed that ordinary products between string fields must be replaced with the $\ast$-products. Essentially, the $\ast$-product is equivalent to the noncommutative relation (2.3). On the other hand, in the case of non-constant $B$ field, the same argument cannot be applied, but we will see that we can consider in a similar way.

**Interaction term**

In the Matrix theory action (2.1), there are products of four off diagonal matrix elements. Then, let us consider the product

$$I_4 = X^I_{(a_1,b_1,c_1)(a_2,b_2,c_2)} X^I_{(a_2,b_2,c_2)(a_3,b_3,c_3)} \cdots X^I_{(a_4,b_4,c_4)(a_1,b_1,c_1)} .$$

The corresponding four open strings form a closed path, and we interpret that this term describes an interaction between the four open strings (figure 1). Then this term should correspond to

$$I_4' = f \phi^I_{(a_1,b_1,c_1)(a_2,b_2,c_2)} \phi^I_{(a_2,b_2,c_2)(a_3,b_3,c_3)} \cdots \phi^I_{(a_4,b_4,c_4)(a_1,b_1,c_1)} ,$$

where we have included an unknown additional factor $f$ to represent the correction by the $B$ field. We will discuss this correction in the following.

**Figure 1: Interaction of four open strings**

Let us consider a string world sheet $\Sigma$ whose boundary is given by the four open strings. When we move the D0-brane on $T^3$ a little in $x^3$ direction, that is $y^3 \rightarrow y^3 + \Delta y^3$,
the world sheet shifts as $\Sigma \to \Sigma_{\text{new}}$ keeping its shape. The value of the sting action evaluated on the world sheet changes as
\[ \Delta S_{\text{string}} = -T_2 \left[ \int_{\Sigma_{\text{new}}} B - \int_{\Sigma} B \right] = - \int_{\Sigma} dx^1 dx^2 (2\pi)^2 R_1 R_2 R_3 \Delta y^3, \] (3.4)
where we have substituted the explicit form of $B$. Here the important point is that $\Delta S_{\text{string}}$ is decided only by the boundary of $\Sigma$ because the integrand in eq.(3.4) is constant. Since $S_{\text{string}}$ changes by $\Delta S_{\text{string}}$, $I_4$ should have dependence on $y^3$ as
\[ I'_4(y^3 + \Delta y^3) = I'_4(y^3) e^{i\Delta S_{\text{string}}}, \] (3.5)
and the factor $f$ in eq.(3.3) should have this $y^3$ dependence. Eq.(3.4) is proportional to the area of the tetragon projected on $(x^1, x^2)$ plane from the tetragon decided by the four open strings, and we have
\[ \Delta S_{\text{string}} = -n \frac{\Delta y^3}{R_3} \frac{1}{2} \{(a_1 b_2 - a_2 b_1) + (a_2 b_3 - a_3 b_2) + (a_3 b_1 - a_1 b_3) + (a_1 b_1 - a_2 b_2)\}. \] (3.6)
From eqs.(3.5) and (3.6), we see that if $I'_4$ is written as
\[ I'_4(y^3) \sim e^{in\frac{\Delta y^3}{R_3} \frac{1}{2}(-a_1 b_2 + a_2 b_1)} \phi_{(1,1,1)}(a_2 b_2, c_2) \cdots e^{in\frac{\Delta y^3}{R_3} \frac{1}{2}(-a_4 b_1 + a_1 b_4)} \phi_{(4,4,4)}(a_1 b_1, c_1), \] (3.7)
it has the desired $y^3$ dependence. Since $I'_4$ should correspond to $I_4$, eq.(3.7) suggests that the correspondence between $X^I$ and $\phi^I$ must be modified as
\[ X^I_{(a,b,c)(a',b',c')} \leftrightarrow e^{i\frac{\Delta y^3}{R_3} \frac{1}{2}(-a b' + a' b)} \phi_{(a,b,c)(a',b',c')}. \] (3.8)
Here we have ambiguity in the additional factor $e^{i\frac{\Delta y^3}{R_3} \frac{1}{2}(-a b' + a' b)}$ in this relation. For example, if we replace $(-a b' + a' b)$ with $(a + a')(b - b')$ or $-(a - a')(b + b')$, $I_4$ still has the same $y^3$ dependence. In these replacements the individual matrix elements $X^I_{(a,b,c)(a',b',c')}$ change but the products like
\[ X^I_1(y^3)(a_1, b_1, c_1)(a_2, b_2, c_2) \cdots X^I_k(y^3)(a_k, b_k, c_k)(a_1, b_1, c_1) \] (3.9)
don’t change at all. Then we consider that these replacements don’t occur any physical difference. In addition we can replace $y^3$ with $y^3 + (\text{constant})$ in eq.(3.8). This shift corresponds to the change $B_{12} \to B_{12} + (\text{constant})$. From these considerations we adopt the relation (3.8). In section 5, we will have an argument which supports this relation.

Note that the product (3.8) produces a desired phase factor under the global shift of $y^3$, that is $c_j \to c_j + 1$ or $y^3 \to y^3 + 2\pi R_3$ in addition to the local shift of $y^3$. If we shift
each $c_j$ as $c_j \rightarrow c_j + 1$, the product produces a factor due to eq. (2.14). This factor is consistent with the change of the string action $S_{\text{string}}$ under this shift.

The important point of eq. (3.8) is that the center of mass coordinate $y^3$ of the D-brane doesn’t decouple from the remaining degrees of freedom any longer. In original Matrix theory this decoupling occurs, but this is not the case when an $H$ field background exists. This result is the main nontrivial feature of the compactification with an $H$ field background.

**Constraints**

Let us summarize the condition on the matrices for the $T^3$ compactification under consideration. $X^I$ must depend on $y^3$ as

$$X^I(y^3)_{(a,b,c)(a',b',c')} = e^{i \pi \frac{y^3}{3} (-ab' + a'b)} X^I(0)_{(a,b,c)(a',b',c')},$$

where $y^3 = X^3_{(a,b,c)(a,b,c)}$ (mod $2\pi R_3$).

The off diagonal matrix elements must satisfy the conditions

$$X^I(0)_{(a+1,b,c)(a+1,b',c')} = X^I(0)_{(a,b,c)(a',b',c')}$$
$$X^I(0)_{(a,b+1,c)(a,b+1,c')} = X^I(0)_{(a,b,c)(a',b',c')}$$
$$X^I(0)_{(a,b,c+1)(a,b,c+1)} = e^{i \pi n (a+a')(b-b')} X^I(0)_{(a,b,c)(a',b',c')}.$$  \hspace{1cm} (3.10) \hspace{1cm} \hspace{1cm} (3.11)

For the diagonal matrix elements, we have the conditions

$$X^I(0)_{(a+1,b,c)(a+1,b,c)} = X^I(0)_{(a,b,c)(a,b,c)} + 2\pi R_1 \delta I,1$$
$$X^I(0)_{(a,b+1,c)(a,b+1,c)} = X^I(0)_{(a,b,c)(a,b,c)} + 2\pi R_2 \delta I,2$$
$$X^I(0)_{(a,b,c+1)(a,b,c+1)} = X^I(0)_{(a,b,c)(a,b,c)} + 2\pi R_3 \delta I,3.$$  \hspace{1cm} (3.12)

Note here that the relation

$$X^I(y^3 + 2\pi R_3)_{(a,b,c)(a',b',c')} = X^I(y^3)_{(a,b,c)(a',b',c')}$$

is satisfied. As for the fermionic part $\Theta_{(a,b,c)(a',b',c')}$, we have conditions similar to eqs. (3.10) and (3.11) for both off-diagonal and diagonal matrix elements.

**Solutions to the constraints**

Let us consider the general solution to the conditions (3.10) ~ (3.11). Using the conditions, any off diagonal elements can be expressed as

$$X^I(y^3)_{(a,b,c)(a',b',c')} = e^{i \pi (a-a')(b-b') c'} e^{i \pi \frac{y^3}{3} [-a-a'+b-b'a]} X^I(0)_{(a-a',b-b',c-c')}(0,0,0).$$  \hspace{1cm} (3.13)
Here, we used $e^{in\pi(a-a')(b-b')c'}$ instead of $e^{in\pi(a+a')(b-b')c'}$. Let us represent $X^I$ as operators on a space of functions

$$v(\xi) = \sum_{a',b',c'} v(a',b',c') e^{ia\xi_1/\Sigma_1} e^{ib\xi_2/\Sigma_2} e^{ic\xi_3/\Sigma_3},$$

(3.15)

where $v(a',b',c')$ is a vector on which the original matrices $X^I$ operate, and $\Sigma_i = \alpha'/R_i$ are the radii of the “dual torus”. The general solution to the conditions (3.10) $\sim$ (3.11) can be expressed as

$$X^I(\xi; u^3) = (2\pi\alpha') \left( -i\partial^I + A^I(\xi; u^3) \right), \quad i = 1, 2, 3$$

(3.16)

$$A^I(\xi; u^3) = \sum_{p,q,r} A^I_{(p,q,r)} e^{ip\xi_1/\Sigma_1} e^{iq\xi_2/\Sigma_2} e^{ir\xi_3/\Sigma_3} e^{n\pi p\Sigma_3\partial_1} e^{n\pi q\Sigma_3\partial_2} e^{n\pi r\Sigma_3\partial_3} (-p\Sigma_2\partial_2 + q\Sigma_1\partial_1).$$

(3.17)

Here $u^3$ is related to $y^3$ by

$$u^3 = y^3/\alpha' = 2\pi A^3_{(0,0,0)},$$

(3.18)

As for $X^a(\xi; u^3)$ ($a = 4, \cdots, 9$) and $\Theta(\xi; u^3)$, they are expressed in the same form as eq.(3.17).

4. Action

So far we have considered the interaction terms in Matrix theory. Let us now consider the kinetic term

$$\text{Tr}[(D_t X^I)^2].$$

(4.1)

Since $X^I$ has time dependence through $y^3(t)$ as well as usual one, we have

$$\frac{d}{dt} X^I(t; y^3(t)) = \frac{\partial}{\partial t} X^I(t; y^3(t))\bigg|_{y^3} + y^3 \frac{\partial}{\partial y^3} X^I(t; y^3(t)).$$

(4.2)

However, due to the second term in eq.(4.2) and eq.(3.10),

$$\frac{d}{dt} X^I(t; y^3(t))_{(a,b,c)(a',b',c')} \frac{d}{dt} X_I(t; y^3(t))_{(a',b',c')(a,b,c)}$$

(4.3)

turns out to be not invariant under the shift $a, a' \rightarrow a + 1, a' + 1$. We have to construct a theory which is invariant under the shifts like this because we would like to obtain a theory compactified on $T^3$. On the other hand if we replace $\frac{d}{dt} X^I$ with $\frac{\partial}{\partial t} X^I|_{y^3}$ in eq.(4.3), the corresponding product turns out to be invariant under these shifts. This fact suggests that we should interpret $\dot{X}^I$ as $\frac{\partial}{\partial t} X^I|_{y^3}$. In the original Matrix theory Lagrangian in flat space, $\frac{d}{dt} X^I$ and $\frac{\partial}{\partial t} X^I|_{y^3}$ are equivalent, then we consider that there is no contradiction in this interpretation.

\footnote{We have a space dependent $B$ field background, so there is not a T-duality between $T^3 \leftrightarrow \tilde{T}^3$ from the viewpoint of string theory.}
Before we write down the resulting action explicitly, let us briefly comment on the generalization to the case where there are \( N \) D0-branes on \( T^3 \). In this case we should interpret \( y^3 \) as the center of mass coordinate of the \( N \) D0-branes. Each matrix element \( X^I_{(a,b,c),(a',b',c')} \) is generalized to a matrix \( X^I_{(a,b,c),(a',b',c')} \), where \( k, l = 1, \cdots, N \). For each \((k,l)\) we have the same conditions as eqs. (3.10) \( \sim \) (3.12).

The final Lagrangian can be obtained under the replacements

\[
\text{Tr} \rightarrow \frac{1}{(2\pi)^3 \Sigma_1 \Sigma_2 \Sigma_3} \int d^3 \xi \\
X^I, \Theta \rightarrow X^I(\xi, t; u^3), \Theta(\xi, t; u^3) \\
\dot{X}^I, \dot{\Theta} \rightarrow \frac{\partial}{\partial t} X^I(\xi, t; u^3) |_{u^3}, \frac{\partial}{\partial t} \Theta(\xi, t; u^3) |_{u^3}
\]

in the original Matrix theory Lagrangian (2.1). We obtain

\[
S = \frac{1}{g_{YM}^2} \int dt d^3 \xi \text{Tr} \left\{ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} \frac{1}{(2\pi \alpha')^2} (D_\mu X^a)^2 + \frac{1}{4} \frac{1}{(2\pi \alpha')^3} [X^a, X^b]^2 \\
+ \frac{i}{2} \frac{1}{(2\pi \alpha')^3} \Theta^T \Gamma^\mu D_\mu \Theta - \frac{1}{2} \frac{1}{(2\pi \alpha')^4} \Theta^T \Gamma^\alpha [X^a, \Theta] \right\},
\]

where the Yang-Mills coupling on the dual torus is given by \( g_{YM}^2 = 2\pi g_s \Sigma_1 \Sigma_2 \Sigma_3 \alpha'^{3/2} \), and \( \mu, \nu = 0, \cdots, 3 \) and

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu] \\
D_\mu X^a = \partial_\mu X^a + i [A_\mu, X^a], \quad D_\mu \Theta = \partial_\mu \Theta + i [A_\mu, \Theta].
\]

The action (4.3) appears to be the ordinary SYM action on the dual torus. However, the fields \( A_\mu, X^a \) and \( \Theta \) are not ordinary \( N \times N \) matrices. Each element of the matrices isn’t a function but an operator which is written as

\[
[A^I(\xi; u^3)]_{k l} = \sum_{p,q,r} \left[ A^I_{(p,q,r)} \right]_{k l} e^{iq\xi_1/\Sigma_1} e^{ip\xi_2/\Sigma_2} e^{ir\xi_3/\Sigma_3} e^{n_1 pq \Sigma_3 \partial_3} e^{n_2 pq \Sigma_3 u^3 (\alpha, \Sigma_3 u^3)}
\]

where \( [A^I_{(p,q,r)}]_{k l} = [A^I_{(-p,-q,-r)}]^*_{k l} \) and

\[
u^3 = 2\pi \frac{1}{N} \text{Tr} [A^3_{(0,0,0)}] = \frac{1}{(2\pi)^3 \Sigma_1 \Sigma_2 \Sigma_3} \int d^3 \xi \frac{1}{N} \text{Tr} [A^3(\xi; u^3)].
\]

Then, the action (4.3) includes a nontrivial interaction between the U(1) part of the gauge field \( u^3 \) and the remaining SU(N) part.

5. Connection with noncommutative geometry

In this section we will make clear the connection of the constraints obtained in section 4 with the noncommutative algebra in the literature\footnote{\text{For example, see} \cite{21, 22, 23, 24}.} of noncommutative torus compactification. To do this, we will reformulate the quotient condition into a form similar to eqs. (2.2) \( \sim \) (2.3).
For a while, let us concentrate our attention only on \((a,b)\) indices of the matrices \(X^I(y^3)\), and consider the trace with respect to \((a,b)\) indices

\[
\sum_{a_i,b_i} X^{I_1}(y^3)(a_1,b_1,c_1)(a_2,b_2,c_2)X^{I_2}(y^3)(a_3,b_3,c_3) \cdots X^{I_k}(y^3)(a_k,b_k,c_k)(a_1,b_1,c_1) \equiv \text{Tr} \left( X^{I_1}(y^3)X^{I_2}(y^3) \cdots X^{I_k}(y^3) \right)
\]  

(5.1)

By bringing together the \(y^3\) dependence of all \(X^I(y^3)\), the trace can be written as

\[
\sum_{a_i,b_i} X^{I_1}(0)(a_1,b_1,c_1)(a_2,b_2,c_2) \cdots X^{I_k}(0)(a_k,b_k,c_k)(a_1,b_1,c_1) e^{-in\frac{y^3}{R_3} \Delta}
\]  

(5.2)

where

\[
\Delta = \frac{1}{2} \{(a_1 b_2 - a_2 b_1) + (a_2 b_3 - a_3 b_2) + \cdots + (a_k b_1 - a_1 b_k)\}
\]  

(5.3)

and \(\Delta\) is equivalent to the area of the k-polygon decided by the k vertices \((a_i, b_i)\) (figure 2). Let us change the way to calculate the area and divide the k-polygon into triangles as in figure 2. Then \(\Delta\) turns into a sum of the areas of the triangles, and can be written as

\[
\Delta = \frac{1}{2} \left\{(a_1 - a_2, b_1 - b_2) \times (a_2 - a_3, b_2 - b_3) + (a_1 - a_3, b_1 - b_3) \times (a_3 - a_4, b_3 - b_4) \right.
\]

\[
\left. + (a_1 - a_4, b_1 - b_4) \times (a_4 - a_5, b_4 - b_5) + \cdots \right\},
\]  

(5.4)

where \(\times\) denotes the operation of the outer product.

![Figure 2: Division of k-polygon into triangles](image)

Now let us introduce \(*\)-product by the relation

\[
X^{I_1} \ast X^{I_2} = \sum_{a_2,b_2} e^{-i\pi\theta(y^3)|(a_1-a_2,b_1-b_2)\times(a_2-a_3,b_2-b_3)|} X^{I_1}_{(a_1,b_1,c_1)(a_2,b_2,c_2)}X^{I_2}_{(a_2,b_2,c_2)(a_3,b_3,c_3)}
\]  

(5.5)

where

\[
2\pi\theta(y^3) = \frac{ny^3}{R_3} = T_2 \int_{T^2(x^3=y^3)} B.
\]  

(5.6)
This product is defined only with respect to \((a,b)\) indices and we don’t carry out the summation with respect to \(c\) indices, so that we have used the notation \(\ast\) instead of usual \(*\). The \(\ast\)-product is almost same as the ordinary \(*\)-product. However, the parameter \(\theta(y^3)\) is not just a constant but it is dependent on the dynamical variable \(y^3\). Using the \(\ast\)-product, it is not difficult to show that the trace eq.(5.2) can be written as

\[
\text{Tr} \left( X^I_1(y^3) X^I_2(y^3) \cdots X^I_k(y^3) \right) = \sum_{a_1, b_1} X^I_1(0) \ast X^I_2(0) \ast \cdots \ast X^I_k(0) .
\] (5.7)

Note that all the \(y^3\) dependence on the left hand side has turned into the \(y^3\) dependence in the definition of the \(\ast\)-product on the right hand side. In other words, if we use the \(y^3\)-dependent \(\ast\)-products instead of the usual products, we can think \(X^I\) independent of \(y^3\).

Since it is known that the effect of the \(\ast\)-product is equivalent to the algebra of \(\tilde{U}_i\) satisfying the noncommutative relation

\[
\tilde{U}_1 \tilde{U}_2 = e^{-i2\pi\theta(y^3)} \tilde{U}_2 \tilde{U}_1 \quad \tilde{U}_1 \tilde{U}_3 = \tilde{U}_3 \tilde{U}_1 , \quad \tilde{U}_2 \tilde{U}_3 = \tilde{U}_3 \tilde{U}_2 ,
\] (5.8)

the constraints obtained in section 3 can be put into a form similar to eq.(2.2). Let us follow [24], and rewrite the quotient condition. Introducing the operators \(\partial_i\) defined by

\[
[\partial_i, \tilde{U}_j] = i\tilde{U}_j \delta_{ij} ,
\] (5.9)

we define the unitary operators \(U_i\) by

\[
U_1 = \tilde{U}_1 e^{2\pi\theta(y^3)\partial_2} , \quad U_2 = \tilde{U}_2 e^{-2\pi\theta(y^3)\partial_1} , \quad U_3 = \tilde{U}_3 .
\] (5.10)

Then \(U_i\) and \(\tilde{U}_j\) commute each other, and \(U_i\) satisfy the noncommutative relation

\[
U_1 U_2 = e^{i2\pi\theta(y^3)} U_2 U_1 \quad U_1 U_3 = U_3 U_1 , \quad U_2 U_3 = U_3 U_2 .
\] (5.11)

Using \(U_i\), the quotient condition corresponding to eqs. (3.10) \sim (3.12) can be written as

\[
U_1^{-1} X^I U_1 = X^I + 2\pi R_1 \delta_{I,1} , \quad U_2^{-1} X^I U_2 = X^I + 2\pi R_2 \delta_{I,2} , \quad U_3^{-1} X^I U_3 = e^{-i\pi n \tilde{\partial}_1} X^I + 2\pi R_3 \delta_{I,3} .
\] (5.12)

Here, \(y^3 = \frac{1}{N} \text{Tr} (X^3) \pmod{2\pi R_3}\), and we introduced the operators \(\bar{\partial}_i\) which are defined by

\[
\left[ \bar{\partial}_i \bar{U}_1^{p_1} \bar{U}_2^{p_2} \bar{U}_3^{p_3} \right] = (ip_i)^{p_1} \bar{U}_1^{p_1} \bar{U}_2^{p_2} \bar{U}_3^{p_3} .
\] (5.13)

\(^4\)These \(\bar{\partial}_i\) don’t correspond to \(\partial_i\) in eq.(3.16). In this section we change the normalization to make clear the correspondence, for example, to [24].
In the third equation of (5.12) we have introduced the operation \[ e^{-\pi n \partial_1 \partial_2 \cdots} \], so as to incorporate the effect of the gauge transformation of the string field, eq.(2.14). So the quotient condition eq.(5.12) is a little different from that of \( T^d \) compactification with constant B-field, eq.(2.2).

In units where \( 2\pi \alpha' = 1 \), the general solution to eq.(5.12) corresponding to the trivial bundle is given by

\[
X^i = -2\pi i R_i \partial_i + [e^{-\pi n \partial_1 \partial_2 \partial_3} A^i (\tilde{U}_j)] ,
\]

\[
X^a = [e^{-\pi n \partial_1 \partial_2 \partial_3} X'^a (\tilde{U}_j)] ,
\]

where \( A^i (\tilde{U}_j) \) and \( X'^a (\tilde{U}_j) \) are arbitrary power functions of \( \tilde{U}_j \). In order to make clear the correspondence of the description in this section to that in section 3, let us compare eq.(5.14) with eqs.(3.16) and (3.17). The effect of the factor \( e^{\frac{\pi}{2} \Sigma_3 u_3 (\Sigma_2 \partial_2 + \Sigma_1 \partial_1)} \) in eq.(3.17) is realized through the \( y^3 \)-dependent noncommutative relation (5.8). But the factor \( e^{\pi pq \Sigma_3 \partial_3} \) in eq.(3.17) still remains and it has turned into the operation \( [e^{-\pi n \partial_1 \partial_2 \partial_3} \cdots] \) in eq.(5.14).

The quotient condition (5.12) can also be applied to the case of N D0-branes. Although we don’t discuss solutions corresponding to twisted bundles explicitly, we expect that they also exist as in \([23, 24]\).

6. Conclusion

In this paper, we considered D0-branes compactified on \( T^3 \) with an H-field background. We assumed that this system can be described by Matrix theory with some appropriate constraints on the matrices. Then we examined what constraints we should impose in order for the resulting theory to be consistent with string theory. We obtained eqs.(3.10) \sim (3.12) as the constraints. We put the constraints into a form where the noncommutative relation is apparent, eqs. (5.11) and (5.12).

The resulting theory is not ordinary super Yang-Mills theory on a noncommutative three-torus as in the case of constant background B-field. In the case of an H-field background, the parameter of noncommutativity \( \theta \) is not a constant, but it becomes \( y^3 \)-dependent and dynamical. Thus, in general, the center of mass coordinates of the D0-branes don’t decouple from the remaining degrees of freedom any longer. We also have to introduce an additional operation \( [e^{-\pi n \partial_1 \partial_2 \partial_3} \cdots] \) in the gauge field on the dual torus, eq.(5.14), in order to incorporate the effect of the gauge transformation of the string field, eq.(2.14). These are the main differences from the compactification with constant B-field.

We investigated this system as a first step to examine the possibility to describe transverse M5-branes in Matrix theory. Transverse M5-branes correspond to NS5-branes in type IIA string theory. Does this system have a corresponding 5-brane charge? Let us
consider the relation of the background we discussed and NS-5branes. For example, for
\( n = 1 \) case the background corresponds to the configuration where oppositely charged
two NS5-branes are infinitely away from each other in one of the directions transverse to
\( T^3 \). Thus the total brane charge is zero, and we can’t expect nonzero brane charge. We
consider, however, that it will be very interesting to examine whether the discussion in
this paper can be extended to the case of a background with nonzero 5-brane charge.

It would also be interesting to examine the relation of the present formalism to the
matrix theory in weak background fields [25] and supermembrane theory in an arbitrary
SUGRA background [26].

**Acknowledgements**

We would like to thank H. Kunitomo for helpful discussions and careful reading of the
manuscript. We are grateful Prof. M. Ninomiya for warmhearted encouragement. This
work was supported in part by JSPS Research Fellowship for Young Scientists and the
Grant-in-Aid for Scientific Research (8970) from the Ministry of Education, Science and
Culture.

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