Dynamical Zero Modes and Criticality in Continuous Light Cone
Quantization of $\Phi^4_{1+1}$

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Critical behaviour of the 2D scalar field theory in the LC framework is reviewed. The notion of dynamical zero modes is introduced and shown to lead to a non trivial covariant dispersion relation only for Continuous LC Quantization (CLCQ). The critical exponent $\eta$ is found to be governed by the behaviour of the infinite volume limit under conformal transformations properties preserving the local LC structure. The $\beta$-function is calculated exactly and found non-analytic, with a critical exponent $\omega = 2$, in agreement with the conformal field theory prediction of Calabrese et al.

1. Introduction

It is by now well established that the infrared behavior of the $\Phi^4_{1+1}$ LC field theory is controlled by a constrained zero mode of the field operator, which is the LC signature of a nontrivial vacuum. This constrained zero mode appears if the dimensionless coupling $g$ is stronger than a critical coupling $g_{cr}$. For $g > g_{cr}$ the order parameter $\langle \Phi \rangle$ becomes nonzero. The characteristics of the resulting second order phase transition are those of a mean field theory. The constrained zero mode being a static quantity it is not able to furnish information on the dynamics of the phase transition as for example: fluctuations of the order parameter, long range correlation functions, dispersion relations for the fluctuating fields etc.... All these phenomena are conditioned by “Dynamical Zero Modes” which we shall describe and study in the following. In doing this we shall make an important simplification: Since the dynamics of the phase transition is governed by long range fluctuations whose amplitudes are small close to the transition point, we can linearize the equations of motion of the fluctuating fields.

2. Field Decomposition

The Lagrangien $L = \frac{1}{2} (\partial^+ \Phi) (\partial^- \Phi) - \frac{1}{2} m^2 \Phi^2 - \frac{1}{4!} \Phi^4$ leads to the equation of motion (EQM) $\partial^+ \partial^- \Phi - m^2 \Phi - \frac{1}{4!} \Phi^3 = 0$.

We decompose the field $\Phi(x)$ into a classical field $\Phi_c(x)$ and a quantum field $\Psi(x)$: $\Phi(x) = \Phi_c(x) + \Psi(x)$. The classical field acts as a kind of background for the quantum field: it is tantamount to an inhomogenous medium in which the quantum field evolves. This implies a generalization of the Haag series for the field $\Psi(x)$:

$$\Psi(x) = \varphi_0(x) + \int dx_1^+ dx_2^- \left[ : \varphi_0(x_1) \varphi_0(x_2) : + g_2 \left( x_1^+ - x^- , x_2^- - x^- ; x \right) \right] + \psi_3(x) + ....$$

with $x = (x^-, x^+)$. The second term on the r.h.s. of Eq. (2.1) contains two free-field operators $\varphi_0(x)$ and will be abbreviated by $\psi_2(x)$, the next one $\psi_3(x)$ contains three free-field operators, etc. The free-field $\varphi_0(x)$ has the Fock-expansion:

$$\varphi_0(x) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} f(k^+)$$

$$\left[ a(k^+) e^{i k x} + a^+(k^+) e^{-i k x} \right] ,$$
with \( \{a(k^+_1), a^+(k^+_2)\} = \delta(k^+_1 - k^+_2) \) and \( k.x = k^-x^+ + k^+x^-, k^- = \frac{2\pi}{\lambda} (m^2 > 0) \). The properties of the test function \( f(k^+) \) are discussed in \( \#3 \) : its effect is to render the integral finite at both ends of the spectrum in \( k^+ \) and the parameter entering any possible form of \( f(k^+) \) is related in an essential way to the final renormalization procedure of the theory.

Some remarks are in order:

1. The contributions \( \psi_2 \) and \( \psi_3 \) are of different character: while \( \psi_3 \) has perturbative contributions, \( \psi_2 \) is purely nonperturbative and does arise only in the presence of constrained or dynamical zero modes.

2. The constrained zero mode is obtained from projection of \( \psi_2 \) onto the vacuum sector.

For the following we need the detailed structure of \( \psi_2(x) \):

\[
\psi_2(x) = \int_0^\infty \frac{dk^+dk^+}{\sqrt{k_1 k_2}} f(k_1^+)f(k_2^+)
\]

\[
\{g^{+\pm}_{2}(k_1, k_2; x)[a^+(k_1^+)a^+(k_2^+)+a^-(k_1^+)+a^-(k_2^+)]
\]

\[
+G_2(k_1, -k_2; x)a^+(k_1^+)a(k_2^+)e^{\mp\frac{i}{\lambda}(k_1 + k_2).x}
\]

The \( x \)-dependence in the amplitudes \( g^{+\pm}_{2} \) and \( G_2 = g^{+\pm} + g^{-\pm} \) is induced by a non-translationally invariant classical background field \( \Phi_c(x) \). It comes in addition to the free-field dependencies - present in the exponential functions - as a non-perturbative phenomenon.

3. Dynamical Zero Mode (DZM) in \( \psi_2 \)

We consider the case of a periodic classical background field in the long wavelength limit (which we need for the linearization of the equations of motion):

\[
\Phi_{c,\text{periodic}}(x) = \tilde{\Phi}_c(k) e^{\mp\frac{i}{\lambda}k.x}
\]

Due to the linearization assumption in \( \Phi(k) \) the background field imposes a periodic variation in \( x \) on \( g^{+\pm}_{2} \) and \( G_2 \) via the equations of motion. Therefore the following form is assumed for the amplitude of the DZM mode:

\[
G_{2,\text{periodic}}(k_1^+, -k_2^+; x) = G_2^0(k_1^+, -k_2^+) e^{\mp\frac{i}{\lambda}k.x}
\]

The resulting form of zero mode, which is noted \( \tilde{\Omega}_k \), is obtained from Eq.\((4.3)\) by integration of \( \psi_2 \) over \( x^- \). It takes the general form

\[
\tilde{\Omega}_k = \int dk_1^+ C_1(k_1^+, k^+)a^+(k_1^+ + k^+)a(k_1^+),
\]

where the coefficient \( C_1(k_1^+, k^+) \) is related to \( G_2^0(k_1^+, -k_1^+ - k^+) \). For \( k^+ = 0 \) the usual zero mode component of \( \psi_2 \) is obtained \[\#1\]. \( \tilde{\Omega}_k \) is diagonal in the number of particles but it can transfer momentum \( \pm k^+ \), though it is translationally invariant. This property is a consequence of motion in a periodic background field.

4. Solution of the Coupled Equations for \( \Phi_c, g^{+\pm}_{2} \) and \( G_2 \).

Only linear terms in \( \Phi_c(x) \) are kept in the EQM which, after projection onto the vacuum, one and two-particles states, gives a coupled system for \( \Phi_c(x), G_2 \) and \( g^{+\pm}_{2} \). We only give the general structure for the Fourier components, \( \Phi_c, \tilde{g}_{2}^{+\pm} \) and \( \tilde{G}_2 \), with a renormalized squared-mass \( \mu^2 = m^2 + \frac{\lambda}{8\pi} \int_0^\infty \frac{dk^+f^2(k^+)}{k^+} \)

\[
\Delta^{-1}_{\Phi_c} \tilde{\Phi}_c + K_{\Phi} \otimes \tilde{G}_{2} + \tilde{g}_{2}^{+\pm} = 0,
\]

\[
\Delta^{-1}_{G_2} \tilde{G}_2 + K^{++} \otimes \tilde{G}_2 + \tilde{g}_{2}^{+\pm} = g \tilde{\Phi}_c,
\]

\[
\Delta^{-1}_{g^{+\pm}_{2}} \tilde{g}_{2}^{+\pm} + K^{++} \otimes \tilde{G}_2 + \tilde{g}_{2}^{+\pm} = g \tilde{\Phi}_c.
\]

Here the \( \Delta^{-1} \)'s are inverse propagators for the fields on which they act and the \( K \)'s are interaction kernels. \( \Phi_c(k^+, k^-) \) is the driving term for \( \tilde{G}_2 \) and \( \tilde{g}_{2}^{+\pm} \) which are therefore truly nonperturbative quantities. However one has to distinguish between the two cases \( k^+ = 0 \) and \( k^+ \neq 0 \). The first case yields the equation for critical coupling and the second the covariant dispersion relation.

4.1. Critical coupling \( k^+ = 0 \)

The starting point is the analysis of Ref \[\#1\]. With \( C(q) = C_1(q, 0) \) in Eq.\((3.3)\) the equations determining the critical coupling read
\[ C(q_1)[q_1+g] + g\phi_0 + 2g \int_0^\infty \frac{dk_2}{k_2} f^2(k_1) \{ \tilde{g}^{++}_2(k_1, q_1) + \frac{1}{4} \{ \tilde{G}_2(k_1, -q_1) + \tilde{G}_2(q_1, -k_1) \} \} = 0, \quad (4.4) \]

\[ \phi_0 + \frac{g}{6} \int \frac{dk_1}{k_1} f^2(k_1) \{ f^2(k_1) C(k_1) + \int \frac{dk_2}{k_2} f^2(k_2) \{ 4 \tilde{g}^{++}_2(k_1, q_1) + \tilde{G}_2(k_1, -k_2) \} \} = 0, \quad (4.5) \]

where \( g = \frac{\lambda}{4\pi^2} \) is the dimensionless coupling.

The order parameter is \( \phi_0 \equiv \tilde{\Phi}_c(0) \), the constant value of the classical field. The simplest approximations to these equations neglects the terms in \( \tilde{g}^{++}_2 \) and \( \tilde{G}_2 \). Eq. (4.4) gives then

\[ C^{(0)}(g, q) = -\frac{g\phi_0}{q_1 + g}, \quad (4.6) \]

For \( \phi_0 \neq 0 \) Eq. (4.5) determines the critical coupling \( g_{cr}^{(0)} \) after proper renormalization [1]. With (4.6) Eq. (4.4) becomes

\[ \phi_0 \left[ 1 - \frac{g_{cr}^{(0)}}{g_{cr}^{(0)}} \right] \log(g_{cr}^{(0)}) = 0 \quad \text{i.e.} \quad g_{cr}^{(0)} = 4.19. \quad (4.7) \]

To extend this simple analysis an iterative procedure is used first to study the properties of the solutions to the system of Eqs. (4.2-4.3). The zero mode contributions in the kernels \( K \) of these equations are isolated. This fixes the initial source functions \( \tilde{g}_2^{(0)} \) and \( \tilde{G}_2^{(0)} \) in the iterative procedure. It is found that the iterated solutions keep the same shape as the initial ones and that, order by order, the changes occur only in a re-definition of the coupling \( g \) present in \( C^{(0)}(g, q) \) of Eq. (4.6). This final effective coupling \( g_f \) results from the resummation of a geometric series in \(-gx(g)\sqrt{3}/9\) as

\[ g_f = \frac{g}{1 + gx(g)\sqrt{3}/9}, \quad (4.8) \]

with \( x(g) = 1 + \frac{g}{12} \) for \( 0 \leq g \leq 7 \). The approximate solution \( \tilde{g}_2^{++} \) finally reads

\[ \tilde{g}_2^{++}(q_1, q_2) = \frac{g}{(q_1^2 + q_2^2 + 2q_1q_2)} \left\{ C^{(0)}(g_f, q_1) f^2(q_1) + C^{(0)}(g_f, q_2) f^2(q_2) + 2\phi_0 \right\}, \quad (4.9) \]

while one has simply \( \tilde{G}_2(q_1, -q_2) = \theta(q_1 - q_2) \tilde{g}_2^{++}(q_1, -q_2) \). Going back to Eq. (4.3) and after renormalisation the equation for \( g_{cr} \) is found as

\[ 1 - \frac{g_{cr}}{6}(1 + \frac{2}{9}g_{cr}\pi\sqrt{3}) \ln(g_{cr}) + \frac{1}{27}g_{cr}^2\pi\sqrt{3}\ln[1 + \frac{1}{9}g_{cr}\pi\sqrt{3}(1 + \frac{g_{cr}}{12})] = 0. \quad (4.10) \]

which gives \( g_{cr} = 4.78 \). For comparison with other studies using the convention of Parisi et al [2] this has to be translated [1] into the reduced coupling unit \( r \) which is just 1 at the perturbative one loop level. To the above values of \( g_{cr}^{(0)} \) and \( g_{cr} \) corresponds the values \( r = 1.5 \) and \( r = 1.71 \). This is in agreement with recent estimates from high-temperature \((r = 1.754) [3] \) and strong coupling expansions \((r = 1.746) [3] \) and Monte Carlo simulations \((r = 1.71) [4] \). A recent high precision estimate \( r \) of \( g_4 \) for the 2D Ising model leads to \( r = \frac{244}{85} = 1.7534 \). However the RG-improved fifth order perturbative result of Ref. [4] is \( r_5 = 1.837(30) \). For a discussion of this result see Ref. [10].

### 4.2. Dispersion relation \((k^+ \neq 0)\)

The DZM of Eq. (4.3) is now explicitely isolated in Eq. (4.2). A generalization of Eq. (4.4) for the coefficient \( C_1(q_i^+, k^+) \) follows: upon neglecting the integral contributions in \( \tilde{g}_2^{++} \) and \( \tilde{G}_2 \), it reads simply

\[ \{ 1 + \frac{g}{2\pi^2} \left[ \frac{1}{q_i^2} + q_i^2 + k^2 \right] \} C_1(q_i^+, k^+ \right) \]

\[ + \frac{g\phi_0(k)}{Vq_i^+(q_i^2 + k^2)} = 0, \quad (4.11) \]

where \( V \) is the volume of the invariant measure \([dx^-]\). Its specific analysis is discussed below. The solution is then

\[ C_1(q_i^+, k^+) = \frac{\frac{g\phi_0(k)}{Vq_i^+(q_i^2 + k^2)} + \frac{2}{9}(2q_i^2 + k^2) \sqrt{k^2 + k^2 + 2q_i^2} \sqrt{k^2 + k^2 + 2q_i^2}}{\sqrt{k^2 + k^2 + 2q_i^2} \sqrt{k^2 + k^2 + 2q_i^2}} \quad (4.12) \]

With this expression for \( C_1 \) used in Eq. (4.11) the dispersion relation becomes

\[ \mu^2 - k^2 \phi_0(k) + \frac{1}{2\pi^2} \int_0^\infty dk_1^+ \left[ \frac{f^2(k_1^+)}{\sqrt{k_1^2 + k_1^2 + 2q_i^2}} \sqrt{k_1^2 + k_1^2 + 2q_i^2} \right] \]

\[ C_1(k_i^+, k^+) \] + non-zero modes terms = 0. \quad (4.13)
Given this form the question of covariance of the DZM contribution immediately arises. This issue can be clarified with the help of the 2nd order perturbative contribution to the self-energy (sunset diagram). Indeed its explicit form written in terms of light-cone momenta is not evidently covariant but becomes so if momenta are expressed in units of the external momentum $k^+$. Following the same path, in Eq. (4.12), with $q^+_i$ expressed in units of $k^+$, the volume $V$, which has dimension of length, must also be written in units of $\frac{1}{k^+}$, i.e. $V$ is just $\frac{1}{k^+}$ times a function of Lorentz scalars. What is its argument? The external source function $\Phi_c(x)$ actually serves to probe characteristic distances of the system. For a periodic background field the interaction provokes a dispersion which induces another length scale, the energy flow scale, which is related to the off-shellness of the process. It is measured in terms of $k^-$, say $\frac{k^+}{k^+}$. Hence in effect $V \propto \phi(x)\Phi_c(x)(1-\alpha) = \frac{k^+}{k^+}w(\mu^2/k^+)^{1-\alpha}$. Since $\alpha$ is arbitrary any scalar function of $\frac{k^+}{k^+}$ is legitimate. Thus

$$V = \frac{1}{k^+}v\left(\frac{k^+}{\mu^2}\right) \equiv \frac{k^-}{\mu^2}w(\mu^2/k^+)^{1-\alpha},$$

(4.14)

with $v(z) = zw(1/z)$. The infinite volume limit is then performed such that $(k^+ \rightarrow 0, k^- \rightarrow \infty)$ with $k^2$ fixed. $C_1(q^+_i, k^+)$ of Eq. (4.12) is therefore finite in the infinite volume limit and the DZM contribution to the dispersion relation can be written $I_{DZM}(k^2)\phi_c(k)$ with $I_{DZM}(k^2)$ given as

$$I_{DZM}(k^2) = -\frac{\lambda g}{24\pi}\int_0^\infty dx \frac{1}{v(x)(1+x)\ln\left(\frac{g+v(\frac{k^2}{\mu^2})+h(g,v)}{g+v(\frac{k^2}{\mu^2})-h(g,v)}\right)},$$

(4.15)

with $h(g,v) = \sqrt{g^2 + v^2(\frac{k^2}{\mu^2})}$. The dispersion relation becomes simply

$$k^2 - \mu^2 - I_{DZM}(k^2) = 0.$$  

(4.16)

In the limit $k^2 \rightarrow 0$ it should reproduce the constraint (after renormalization)

$$\theta_3 = \frac{\phi_c}{\mu^2}(1 - \frac{g}{6} \ln(g)) = 0.$$  

(4.17)

The comparaison of these last two relations shows that $\lim_{k^2 \rightarrow 0} v(\frac{k^2}{\mu^2})$ should be zero. For $v(\frac{k^2}{\mu^2}) \ll g$ the following expansion holds

$$\frac{g}{\mu^2}I_{DZM}(k^2) = \frac{g}{\mu^2} \left[\ln(g) - \ln(\frac{g}{\mu}) + O(v^2 \ln(v))\right].$$

In this expression the divergent part is given by the $\ln(v)$ term and is taken care of by renormalisation. If $\lim_{k^2 \rightarrow 0} v(\frac{k^2}{\mu^2}) \propto (k^2)^\alpha$ with $\alpha < 1$ the leading term in the dispersion relation is the one linear in $v$. Hence the dependence of $v$ on $k^2$ determines completely the dispersion relation and therefore the critical exponent $\eta$. The scaling behaviour would just be a pure volume effect. However the precise expression of $I_{DZM}(k^2)$ depends on the power of the scalar field interaction term $\Phi^{2k}$, through the form of the constraint determining $C_1(q^+_i, k^+)$. The form of $v(\frac{k^2}{\mu^2})$ is dictated by conformal transformations preserving the local light cone structure. It is well known that critical exponents of the 2D Ising model just come out from conformal invariance of the underlying fermionic field theory.

It is quite instructive to make a comparison with the would be situation in DLCQ. Instead of the integral in Eq. (4.15) the following sum is obtained

$$S(m, g) = \sum_{n=1}^{\infty} \frac{1}{n(n+m) + \frac{1}{2}(2n+m)},$$

(4.18)

where $m$ is the mode number of the external $k^+ = \frac{2\pi m}{L}$. The sum is finite and yields different discrete results for each possible value of $m = 0, 1, 2, \ldots$. It is boost invariant and independent of the size $L$. However the limit $k^2 \rightarrow 0$, which is necessary for the extraction of $\eta$, cannot be performed. One might think to repair this by an analytic prolongation to the domain $0 < m < 1$. But this has to be done keeping $k^2$ fixed. Moreover, if $k^+(m)$ is not fixed anymore by periodic boundary conditions, boost invariance is lost and has to be restored in one way or another. But this goes beyond the DLCQ framework, raising questions of consistency. On
the contrary CLCQ permits a consistent analysis in the limit $k^2 \to 0$.

5. $\beta$-function and critical exponent $\omega$

To determine the $\beta$-function the constraint is considered as a prescription for the calculation of the critical mass (cf Eq.(4.17)). $M^2(g, \Lambda)$ is just $\mu^2$ times the left hand side of Eq.(4.10), with $\mu^2 = \lambda/(4\pi g)$. One has

$$\beta(g) = M \left[ \frac{\partial M}{\partial g} \right]^{-1}(\lambda, \Lambda) = -2g \frac{N(g)}{D(g)} \quad (5.1)$$

with

$$N(g) = [1 - \frac{g_4}{6} (1 + g \frac{2\pi}{\sqrt{3}} \ln(g)) + g^2 \frac{\pi}{2\sqrt{3}} \ln(1 + g \frac{2\pi}{\sqrt{3}} (1 + \frac{g}{12})) \bigl(1 + g \frac{2\pi}{\sqrt{3}} (1 + \frac{g}{12})\bigr)]$$

and

$$D(g) = (1 + \frac{g_4}{6}) \bigl(1 + g \frac{2\pi}{\sqrt{3}} (1 + \frac{g}{12})\bigr) +$$

$$\frac{g^2 \pi}{27} \bigl(1 - \frac{g^2 \pi}{108} (1 + \frac{g}{12})\bigr) + \frac{g^4 \pi}{27} \bigl[1 + \frac{g^2 \pi}{9} (1 + \frac{g}{12})\bigr]$$

$$\ln\left(\frac{g^2}{(1 + g \frac{2\pi}{\sqrt{3}} (1 + \frac{g}{12}))}\right)$$

From standard renormalization group analysis one must have

$$\lim_{g\to 0} \beta(g) = -(4 - D)g + O(g^2) \quad (5.2)$$

which is indeed the case for \(5.1\). To lowest order in the Haag expansion the corresponding $\beta$-function is just

$$\beta_0(g) = -2g \frac{(1 - \frac{g_4}{6} \ln(g))}{(1 + \frac{g}{12})}. \quad (5.3)$$

The two functions $\beta(g)$ and $\beta_0(g)$ are plotted in Fig.1. In both cases the exponent $\omega = \beta'(g_{cr}) = 2$ exactly, in agreement with the recent analysis of Ref.(10). In particular these authors emphasize that at $D = 2$ perturbative estimates of the critical value of $g$ and of $\omega$ are not reliable due to strong nonanalytic corrections to the $\beta$-function.

6. Conclusion

Critical behaviour in LC quantization is a key issue since it is the domain of non-perturbative physics, an alleged benefit of this particular quantization scheme. However the problem of scaling behaviour (e.g. critical exponents) has so far attracted little interest as compared to mechanisms of symmetry breaking. Whereas discretized LC techniques can still be useful for symmetry breaking studies, we have shown here that a full understanding of the physics of zero modes combined with continuous non-compact field dynamics is mandatory to treat covariantly the low $k^+$ region determining the scaling behaviour. Conformal transformations preserving the local light cone structure are then found to govern precisely the scaling law. Yet it is a puzzling problem to relate directly the CLCQ approach to $2D$ conformal field theory where critical exponents are known to be exactly determined by conformal symmetry properties. Finally the $\beta$ function, although in keeping with RG analysis at small coupling, presents interesting non-analytic properties and
leads to a critical exponent $\omega = 2$ in agreement with recent non-perturbative analysis.

REFERENCES

1. P. Grangé, P. Ullrich, E. Werner, Phys. Rev. D57 (1998) 4981.
2. G. Parisi, J.Stat.Phys. 23, 49 (1980), Nuovo Cimento A21, 179 (1974).
3. P. Butera, M. Comi, Phys. Rev B54, 15828, (1996).
4. A. Pelissetto, E. Vicari, Nucl. Phys. B519, 626, (1998).
5. M. Campostrini, A. Pelissetto, P. Rossi, E. Vicari, Nucl. Phys. B459, 207, (1996).
6. J.K. Kim, A. Patrascioiu, Phys. Rev D47, 2588, (1993).
7. J.K. Kim, Phys. Lett. D345, 469, (1995).
8. M. Caselle, M. Hasebusch, A. Pelissetto, E. Vicari, J. Phys. A33, 8171, (2000).
9. E.V. Orlov, A.I. Sokolov, preprint hep-th/0003140 ; Fiz. Tver. Tela 42, 11, 2087, (2000).
10. P. Calabrese, M. Caselle, A. Celli, A. Pelissetto, E. Vicari, J. Phys. A33, 8155, (2000).
11. S. Salmons, P. Grangé, E. Werner, to be published.
12. Y. Grandati, ”Introduction to Conformal Field Theory” Ann. Phys.Fr 17, 3, 159, (1992) ; P. Ginsparg, ”Applied Conformal Field Theory”, Les Houches Session XLIX, E. Brézin and J. Zinn-Justin eds, Elsevier, New York, (1989).