Punctual characterization of the unitary flat bundle of weight 1 PVHS and application to families of curves

Víctor González-Alonso *1 and Sara Torelli †1

1Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover

September 8, 2021

Abstract

In this paper we consider the problem of pointwise determining the fibres of the flat unitary subbundle of a PVHS of weight one. Starting from the associated Higgs field, and assuming the base has dimension 1, we construct a family of (smooth but possibly non-holomorphic) morphisms of vector bundles with the property that the intersection of their kernels at a general point is the fibre of the flat subbundle. We explore the first one of these morphisms in the case of a geometric PVHS arising from a family of smooth projective curves, showing that it acts as the cup-product with some sort of “second-order Kodaira-Spencer class” which we introduce, and check in the case of a family of smooth plane curves that this additional condition is non-trivial.

1 Introduction

Consider a polarized variation of Hodge structures (PVHS) of weight one and rank 2g on a smooth complex manifold B, which in particular consists of a short exact sequence of holomorphic vector bundles on B

\[ 0 \rightarrow E = E^{1,0} \rightarrow \mathcal{H} = \mathbb{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}_B \rightarrow E^{0,1} \rightarrow 0 \]

together with a Gauss-Manin connection on \( \mathcal{H} \). The bundle \( E \) carries a natural (maximal) flat unitary subbundle \( \mathcal{U} \subseteq E \) which encodes many properties of the natural modular map \( B \rightarrow A_g \) and its relation to the (open) Torelli locus \( T_g \subseteq A_g \) of jacobian varieties.

In particular, it can be used to study the Coleman-Oort conjecture on the (non)-existence of Shimura curves generically contained in \( T_g \). For example, recently Lu and Zuo used some properties of \( \mathcal{U} \) in [LZ19] to prove that certain types of Shimura curves do not lie in \( T_g \). Moreover, in [CLZ16] [CLZ18] together

*gonzalez@math.uni-hannover.de, corresponding author
†torelli@math.uni-hannover.de
with Chen they proved that for too big or too small rank of $\mathcal{U}$, the map $B \to \mathcal{A}_g$ does not describe an open subset of a Shimura curve. In a slightly more general setting, the flat unitary bundle $\mathcal{U}$ has been proven a very useful tool in the recent work of the second author with Ghigi and Pirola [GPT19] to bound the dimension of the totally geodesic subvarieties of $\mathcal{T}_g \subseteq \mathcal{A}_g$.

In the geometric case, where the PVHS arises from a family of smooth projective complex curves of genus $g$ over a quasi-projective curve $B$, $\mathcal{U}$ arises from the second Fujita decomposition [Fuj78, CD17, CK19]. Its rank is an important invariant that can be used to study Xiao’s conjecture on the relative irregularity (see the recent papers of the authors in collaboration with Barja, Naranjo and Stoppino [BGAN18, GAST19]) and has also applications to the study of hyperelliptic fibrations [LZ17].

It is therefore very interesting to have tools to compute the rank of $\mathcal{U}$ as explicitly as possible. However, since evaluating the Gauß-Manin connection involves knowing the sections of $E$ in open subsets, no direct characterization of the fibres $\mathcal{U}_b \subseteq E_b$ depending only on the point $b$ is yet known, even if we assume $b \in B$ to be general. This makes the computation or even the estimation of the rank of $\mathcal{U}$ a difficult problem, which in the geometric case can be done directly only under special circumstances (e.g. for families of cyclic covers of $\mathbb{P}^1$, or if the family of curves is supported in relatively rigid divisors, as developed by the first author in [GA16]).

In the general case the first approach is by somehow “linearising” the Gauß-Manin connection on the fibres: considering the associated Higgs field $\theta: E \to E^{0,1} \otimes \Omega^1_B$, which by definition satisfies $\mathcal{U}_b \subseteq \ker \theta_b$ for any $b \in B$ (see (2.3) and Definition 2.1). The Higgs field is actually a homomorphism of holomorphic vector bundles, and $\theta_b$ depends only of infinitesimal data around the point $b$. For example, in the geometric case, $\theta_b$ is determined by the cup-product with the Kodaira-Spencer class of the infinitesimal deformation of the corresponding fibre. Thus studying $\ker \theta_b$ and in particular computing its rank is a simpler task, and in the geometric case several techniques have been developed by the authors in [BGAN18, GAST19].

Ghigi, Pirola and the second author prove that, if $B \subseteq \mathcal{A}_g$ contains one (real) geodesic curve, then $\mathcal{U} = \mathcal{K} := \ker \theta$. On the other hand, in the recent work [GAT20], the authors showed that $\mathcal{U}$ and $\mathcal{K}$ can be arbitrarily different also in the geometric case, so the results obtained by bounding the rank of $\mathcal{K}$ can be very far away from the actual rank of $\mathcal{U}$. Our aim in the present work is to construct new linear conditions on the fibres $E_b$ that define precisely $\mathcal{U}_b \subseteq E_b$, at least for a general $b \in B$.

The main results of the paper are the following two theorems.

**Theorem** (2.10). Let $\dim B = 1$. Then there are smooth morphisms of vector bundles

$$\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(g)}: E \to E^{0,1}$$

such that for any $\alpha \in \Gamma (E)$ it holds

$$\alpha \in \Gamma (\mathcal{U}) \iff \eta^{(1)} (\alpha) = \eta^{(2)} (\alpha) = \ldots = \eta^{(g)} (\alpha) = 0.$$
In particular we have

\[ \mathcal{U}_b \subseteq \bigcap_{k=1}^{g} \ker \eta^{(k)}_b \subseteq E_b \]

with equality for \( b \) in a dense Zariski-open subset of \( B \).

**Theorem (3.10).** Let \( f : \mathcal{C} \to B \) be a family of smooth projective curves \( C_b = f^{-1}(b) \) with \( \dim B = 1 \). For any \( b \in B \) let \( \mathcal{K}_b \subseteq E_b = H^0(\omega_{C_b}) \) be the fibre of \( \mathcal{K} \) on \( B \), and \( \mu_b \in H^1(T_{C_b}) \) the second-order Kodaira-Spencer class of \( C_b \subseteq \mathcal{C} \) (Definition 3.7). Let

\[ \hat{\mu}_b : H^0(\omega_{C_b}) \xrightarrow{\mu_b} H^1(\mathcal{O}_{C_b}) = E_b^\vee \to \mathcal{K}_b^\vee. \]

Then \( \mathcal{U}_b \subseteq \mathcal{K}_b \cap \ker \hat{\mu}_b \).

It is worth to note that the additional morphisms of vector bundles in Theorem 2.10 are constructed from the Higgs field \( \theta \) by applying the connection \( \nabla^T \) induced on \( \text{Hom}^{s}(E, E^{0,1}) \) by the Gauß-Manin connection. If the base \( B \) of the family is actually a submanifold of \( \mathcal{A}_g \) (or the Siegel upper-half space \( \mathbb{H}_g \)), this connection \( \nabla^T \) is the restriction to \( B \) of the Levi-Civita connection of the Siegel metric. Thus our construction hints again at a connection between the equality \( \mathcal{U} = \mathcal{K} \) and the existence of geodesics inside \( B \) with respect to the Siegel metric.

Theorem 3.10 is a partial characterization of \( \mathcal{U}_b \subseteq \mathcal{K}_b \) in the geometric case given by the cup-product with a cohomology class \( \mu_b \in H^1(T_{C_b}) \) arising from \( \eta^{(2)} \), which we call second-order Kodaira-spencer class of the fibre \( C_b \) in \( \mathcal{C} \) because it depends on the second-order infinitesimal neighbourhood.

The paper is organized as follows. In Section 2 we set up the main notations for PVHS, find an intermediate characterization of the sections of \( \mathcal{U} \) still depending on the Gauß-Manin connection (Theorem 2.5) and we then define the morphisms of vector bundles \( \eta^{(k)} \) (Definition 2.6) that eventually lead to the punctual characterization of \( \mathcal{U}_b \) in Theorem 2.10.

In Section 3 we adapt our set up to the geometric case and study deeper the homomorphism \( \eta^{(2)} \). We compute an explicit coordinate description that leads us to the Definition of the second-order Kodaira-Spencer class \( \mu \) in Definition 3.7. We finally show that, after projecting to \( \mathcal{K}_b^\vee \), both \( \mu \) and \( \eta^{(2)} \) impose the same condition on \( \mathcal{K}_b \) and obtain thus Theorem 3.10.

In the final section 4 we implement the definition of \( \mu \) for an arbitrary family of smooth plane curves, and show that \( \mu \) actually is in general independent of the first-order infinitesimal deformation (Theorem 4.4), hence the additional restriction of Theorem 3.10 is non-trivial.

**Acknowledgements:** The authors would like to thank P. Frediani, A. Ghigi, L. Stoppino and G.P. Pirola for very useful and enlightening conversations around the topic.
2 Characterizing the sections of the flat unitary subbundle

In this section we first of all set up the notation we will use for PVHS of weight 1 and we recall the definitions and some basic properties of two naturally associated objects: the flat unitary bundle $\mathcal{U}$ and the kernel sheaf $\mathcal{K}$. Then we move on to the study of such objects. More precisely we aim to get a better understanding of the natural inclusion $\mathcal{U} \subseteq \mathcal{K}$, which so far has been characterized only by conditions of local kind. We present here our new results that allow to cut out $\mathcal{U} \subseteq \mathcal{K}$ just by using linear punctual conditions.

We will denote by $B$ a smooth complex manifold, with holomorphic tangent and cotangent bundles $T_B$ and $\Omega^1_B$. When necessary we will denote by $T_{B,\mathbb{R}}$ and $T_{B,\mathbb{C}} := T_{B,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ the real and complexified tangent bundles. Soon we will assume that $\dim B = 1$, but the first definitions can be stated for any dimension. We identify any holomorphic vector bundle $F$ on $B$ with its sheaf of holomorphic sections. We will denote by $\mathcal{A}(F) = F \otimes_{\mathcal{O}_B} \mathcal{C}^\infty (B)$ the sheaf of smooth sections of $F$, and more generally by $\mathcal{A}^k(F)$ (resp. $\mathcal{A}^{p,q}(F)$) the sheaf of smooth $k$-forms (resp. $(p,q)$-forms) with values in $F$.

We consider a polarized variation of Hodge structures (PVHS) of weight one $(\mathcal{V}_Z, E, Q)$. Here $\mathcal{V}_Z$ is a local system of free abelian groups of rank 2$g$, $Q: \mathcal{V}_Z \otimes_{\mathbb{Z}} \mathcal{V}_Z \to \mathbb{Z}$ is a polarization, i.e. a non-degenerate antisymmetric unimodular $\mathbb{Z}$-bilinear map, and $E = E^{1,0} \subseteq \mathcal{H} := \mathcal{V}_Z \otimes_{\mathbb{Z}} \mathcal{O}_B$ is a holomorphic subbundle of rank $g$ where the Hodge metric $h(u,v) = iQ_C(u, v)$ is positive definite. In particular it holds $E^\perp = \overline{E}$ with respect to $h$, and hence there is a decomposition $E \oplus \overline{E} = \mathcal{H}$ as $\mathcal{C}^\infty$ vector bundles on $B$. Moreover $Q$ (or rather $h$) induces a holomorphic $\mathbb{C}$-linear isomorphism $E^\vee \cong E^{0,1} := \mathcal{H}/E$, hence there is the short exact sequence of holomorphic vector bundles

$$0 \to E \overset{\iota_1}{\to} \mathcal{H} = \mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_B \overset{\pi_2}{\to} E^\vee \to 0,$$

and a the orthogonal direct sum decomposition $\mathcal{H} = E \oplus E^\vee$ corresponds to $\mathcal{C}^\infty$ morphisms of vector bundles $\iota_2: E^\vee \to \mathcal{H}$ and $\pi_1: \mathcal{H} \to E$ such that

$$\pi_1 \circ \iota_1 = \text{id}_E, \quad \pi_2 \circ \iota_2 = \text{id}_{E^\vee} \quad \text{and} \quad \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \text{id}_\mathcal{H}. \quad (2.2)$$

Being the vector bundle associated to a local system, $\mathcal{H}$ comes equipped with the Gauß-Manin connection

$$\nabla^{\text{GM}}: \mathcal{A}(\mathcal{H}) \to \mathcal{A}^1(\mathcal{H}) = \mathcal{A}^{1,0}(\mathcal{H}) \oplus \mathcal{A}^{0,1}(\mathcal{H}),$$

induced by the de Rham differential $d: \mathcal{C}^\infty (B) \to \mathcal{A}^1_B = \mathcal{A}^{1,0}_B \oplus \mathcal{A}^{0,1}_B$ on $B$. This connection is holomorphic, i.e. the $(0,1)$-part is $\overline{\partial}_H: \mathcal{A}(\mathcal{H}) \to \mathcal{A}^{0,1}(\mathcal{H})$. Hence holomorphic sections of $\mathcal{H}$ are mapped to holomorphic sections of $\mathcal{H} \otimes \Omega^1_B$. Thus we will often also refer to the restriction $\nabla^{\text{GM}}: \mathcal{H} \to \mathcal{H} \otimes \Omega^1_B$ as the Gauß-Manin connection of the PVHS.

The Higgs field of the PVHS is

$$\theta := \pi_2 \circ \nabla^{\text{GM}} \circ \iota_1: E \to E^\vee \otimes \Omega^1_B, \quad (2.3)$$
which is $O_B$-linear, i.e. a (holomorphic) morphism of vector bundles. Moreover $\theta$ is symmetric, in the sense that $\theta^\vee = \theta \otimes \text{id}_{T_B} : E \otimes T_B \to E^\vee$.

**Definition 2.1.** The Hodge bundle carries the following interesting subsheaves:

1. The flat unitary local system associated to the PVHS (2.1) is
   $$U := \text{ker} \left( \nabla^\text{GM} \circ \iota_1 \right) \subseteq \text{ker} \nabla^\text{GM} = V^\mathbb{C} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}.$$

2. The associated holomorphic vector bundle $U := U \otimes_{\mathbb{C}} O_B \subseteq E$ is the flat unitary subbundle of $E$.

3. The kernel subsheaf is $K := \ker \theta$, the kernel of the Higgs field.

Note that by construction, it holds $U \subseteq K$.

**Remark 2.2.** Some remark about the use of subbundle and subsheaf.

1. The flat subbundle $U$ is actually a vector subbundle, namely the fibres of $U$ have constant rank and inject in the fibres of $E$, hence the quotient $E/U$ is also a vector bundle.

2. On the contrary the sheaves $K$ and $E/K$ are a priori only torsion-free (as subsheaves of the locally free sheaves $E$ and $E^\vee \otimes \Omega^1_B$ respectively). When $\dim B = 1$, both $K$ and $E/K$ are locally free, hence $K \subseteq E$ is actually also a subbundle.

By definition, $K$ is the kernel of a morphism of locally free sheaves $\theta$, hence its fibre at a general point $b \in B$ is
$$K_b := K \otimes \mathbb{C} (b) = \ker \left( \theta_b : E_b \to E^\vee_b \otimes \Omega^1_B (b) \right).$$

In particular the rank of $K$ can be computed by studying the $\mathbb{C}$-linear map $\theta_b$ at one general point $b \in B$.

On the other hand, the Gauß-Manin connection is not $O_B$-linear, hence the stalk of $U$ (i.e. the fibre of $U$) at any point $b \in B$ depends a priori on the restriction of $E \subseteq \mathcal{H}$ to an open neighbourhood of $b$.

The main result of this section is to show that, if $\dim B = 1$ there is locally a family of $C^\infty$ symmetric morphisms of vector bundles $\eta^{(k)} : E \to E^\vee$ whose kernels define $U$ at a general point $b \in B$.

To do so, we need to consider various connections induced by $\nabla^\text{GM}$.

**Definition 2.3.** Denote $\nabla^E$ and $\nabla^{E^\vee}$ denote the connections on $E$ and $E^\vee$ induced by $\nabla^\text{GM}$ and the decomposition $\mathcal{H} = E \oplus E^\vee$, i.e.

$$\nabla^E = \pi_1 \circ \nabla^\text{GM} \circ \iota_1 : \mathcal{A} (E) \to \mathcal{A}^1 (E) \quad (2.4)$$

and

$$\nabla^{E^\vee} = \pi_2 \circ \nabla^\text{GM} \circ \iota_2 : \mathcal{A} (E^\vee) \to \mathcal{A}^1 (E^\vee). \quad (2.5)$$

Let also $T := \text{Hom}^s_{O_B} (E, E^\vee) = \text{Sym}^2 E^\vee$ and $\nabla^T : \mathcal{A} (T) \to \mathcal{A}^1 (T)$ the connection defined by

$$\left( \nabla^T_\chi \eta \right) (\alpha) = \nabla^E_\chi (\eta (\alpha)) - \eta \left( \nabla^E_\chi \alpha \right) \in \Gamma \left( \mathcal{A} (E^\vee) \right). \quad (2.6)$$
for any $\alpha \in \Gamma (\mathcal{A}(E)), \eta \in \Gamma (\mathcal{A}(T))$ and $X \in \mathcal{A}(T_{B, \mathbb{C}})$.

Note that, since $\pi_1$ and $\iota_2$ are just $\mathcal{C}^\infty$ morphisms of vector bundles, $\nabla^E, \nabla^{E^\vee}$ and $\nabla^T$ are not holomorphic connections. So in particular, even if $\alpha \in \Gamma (E)$ and $X \in \Gamma (T_B)$ are holomorphic, the section $\nabla^E_X \alpha \in \mathcal{A}(E)$ is not necessarily holomorphic.

The formula (2.6) defines $\nabla^T_X \eta$ as a morphism $E \to E^\vee$, and it is straightforward to check that it is symmetric if $\eta$ is.

**Remark 2.4.** By definition, a holomorphic section $\alpha \in \Gamma (E)$ is a section of $K$ if and only if $\nabla^{GM} \alpha \in \Gamma (E \otimes \Omega^1_B) \subseteq \Gamma (\mathcal{H} \otimes \Omega^1_B)$. This means that $\nabla^E \alpha = \nabla^{GM} \alpha \ \forall \alpha \in \Gamma (K).$ \hfill (2.7)

In particular $\nabla^E \alpha$ is holomorphic if $\alpha \in \Gamma (K)$.

Moreover, if $K \subseteq E$ is locally free (e.g. if $\dim B = 1$), the formula (2.7) holds also for smooth sections $\alpha \in \Gamma (\mathcal{A}(K))$.

Assume from now on that $B \subseteq \mathbb{C}$ is an open disk with coordinate $t$. Thus the tangent bundle $T_B$ is globally generated by $\frac{\partial}{\partial t}$, and contraction with $\frac{\partial}{\partial t}$ gives isomorphisms $\iota_{\frac{\partial}{\partial t}} : \Omega^1_B \cong \mathcal{O}_B$ and $\iota_{\frac{\partial}{\partial t}} : \mathcal{A}_B^1 \cong \mathcal{A}_B^0$.

For any connection $\nabla$ we denote by $\nabla_t = \nabla_{\frac{\partial}{\partial t}}$ the corresponding derivation in the direction of $\frac{\partial}{\partial t}$.

We show now a characterization of the sections of $\mathcal{U}$ using the Higgs field and the iterations of $\nabla^{GM}_t$.

This is the first step towards our punctual characterization of the sections of $\mathcal{U}$.

**Theorem 2.5.** Let $\alpha \in \Gamma (E)$. Then it holds

$$\alpha \in \Gamma (\mathcal{U}) \iff \alpha, \nabla^{GM}_t \alpha, (\nabla^{GM}_t)^2 \alpha, \ldots, (\nabla^{GM}_t)^{g-1} \alpha \in \Gamma (K).$$ \hfill (2.8)

or equivalently

$$\alpha \in \Gamma (\mathcal{U}) \iff \theta(\alpha) = \theta((\nabla^{GM}_t)^2 \alpha) = \ldots = \theta((\nabla^{GM}_t)^{g-1} \alpha) = 0.$$ \hfill (2.9)

**Proof.** One direction is clear: if $\sigma_1, \ldots, \sigma_r \in \Gamma (\mathcal{U})$ is a basis of flat sections and $\alpha = \sum_{i=1}^r \alpha \iota_{\frac{\partial}{\partial t}} \sigma_i \in \Gamma (\mathcal{U}) \subseteq \Gamma (K)$ for holomorphic functions $\alpha_1, \ldots, \alpha_r$, then $(\nabla^{GM}_t)^n \alpha = \sum_{i=1}^r \frac{\partial^n \alpha_i}{\partial t^n} \sigma_i \in \Gamma (\mathcal{U}) \subseteq \Gamma (K)$ or every $n \geq 0$.

For the other direction, for any $n \geq 0$ let $\mathcal{K}_n \subseteq E$ be the smallest subsheaf generated by

$$\mathcal{U}, \alpha, \nabla^{GM}_t \alpha, \ldots, (\nabla^{GM}_t)^n \alpha.$$ By definition it holds $\mathcal{U} \subseteq \mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \ldots \subseteq \mathcal{K}_n \subseteq \ldots \subseteq E$. Denote by $\mathcal{K}' = \bigcup_{n \geq 0} \mathcal{K}_n \subseteq E$ the smallest subsheaf containing all the $(\nabla^{GM}_t)^i \alpha, i \geq 0$. 


Note that if \( K'_n = K'_{n+1} \), then the chain stabilizes, i.e. \( K' = K'_m = K'_n \) for every \( m \geq n \). Indeed, suppose that \( (\nabla_{t}^{GM})^{n+1} \alpha \in \Gamma(K'_n) \), so that \( K'_{n+1} = K'_n \). Then there exist \( \sigma \in \Gamma(U) \) and \( \beta_1, \ldots, \beta_n \in \mathcal{O}_B(B) \) such that

\[
(\nabla_{t}^{GM})^{n+1} \alpha = \sigma + \sum_{i=0}^{n} \beta_i (\nabla_{t}^{GM})^i \alpha.
\]

Thus applying \( \nabla_{t}^{GM} \) again it holds

\[
(\nabla_{t}^{GM})^{n+2} \alpha = \nabla_{t}^{GM} \sigma + \sum_{i=0}^{n} \frac{\partial \beta_i}{\partial t} (\nabla_{t}^{GM})^i \alpha + \sum_{i=0}^{n} \beta_i (\nabla_{t}^{GM})^{i+1} \alpha
\]

and analogously \( (\nabla_{t}^{GM})^m \alpha \in \Gamma(K'_n) \) for any \( m \geq n \). Since at each step of the ascending chain before the stabilization the rank raises by 1 and \( \text{rk} E = g \), it is clear that \( K'_{g-1} = K' \).

Now by construction it holds \( \nabla_{t}^{GM} K' \subseteq K' \), and also \( K' \subseteq K \) by the hypothesis \( (\nabla_{t}^{GM})^i \alpha \in \Gamma(K) \) for every \( i = 0, 1, \ldots, g - 1 \). We can then repeat the argument in [GPT19, Lemma 3.1] with \( K' \) instead of \( K \) and show that \( K' = U \), hence in particular \( \alpha \in \Gamma(U) \).

Note that in (2.8) we could have written \( \nabla_{t}^{E} \) instead of \( \nabla_{t}^{GM} \) because of Remark 2.4.

The vanishing conditions in (2.9) are still not defined fibrewise. In order to achieve this, we introduce the following morphisms of vector bundles.

**Definition 2.6.** For any \( k \in \mathbb{N} \) let \( \eta^{(k)}: E \to E' \) be the smooth symmetric morphisms (i.e. sections of \( \mathcal{A}(T) \)) defined recursively by

- \( \eta^{(1)} := \int \nu \theta : E \to E' \), and
- \( \eta^{(k)} := \nabla_{t}^{k} (\eta^{(k-1)}) = (\nabla_{t}^{k})^{k-1} (\eta^{(1)}) \) for \( k \geq 2 \).

**Remark 2.7.** Note that \( \eta^{(1)} \) is holomorphic, but the subsequent \( \eta^{(k)} \) for \( k > 1 \) are in general only smooth sections of \( T \).

One can interpret them thus as morphisms of sheaves \( \eta^{(k)}: E \to \mathcal{C}^{\infty}(E') \), where as usual a holomorphic vector bundle is identified with its sheaf of holomorphic sections.

**Definition 2.8.** For \( r \geq 1 \) we define

1. the morphism of sheaves \( \theta^{(r)} = (\eta^{(1)}, \ldots, \eta^{(r)}): E \to \mathcal{C}^{\infty}(E')^{\oplus r} \)
2. the subshef \( K^{(r)} := \ker \theta^{(r)} \subseteq E \).

**Proposition 2.9.** For every \( r \geq 1 \), \( K^{(r)} \subseteq E \) is a holomorphic vector subbundle, i.e. it is a locally free \( \mathcal{O}_B \)-module and the quotient \( E/K^{(r)} \) is also locally free.
Proof. We proceed by inducion on \( r \). The case \( r = 1 \) is the statement of Remark 2.2 (2).

Now suppose that the claim is true for some \( r \geq 1 \) and we will show that it is also true for \( r + 1 \). Note that by definition, \( \mathcal{K}^{(r+1)} \) coincides with the kernel of the restriction

\[
\eta^{(r+1)}|_{\mathcal{K}^{(r)}} : \mathcal{K}^{(r)} \to E^\vee.
\]  

(2.10)

Thus it is enough to show that (2.10) is a holomorphic morphism of vector bundles. Indeed this would imply that both \( \mathcal{K}^{(r+1)} \subseteq \mathcal{K}^{(r)} \) and \( \mathcal{K}^{(r)}/\mathcal{K}^{(r+1)} \subseteq E^\vee \) are subsheaves of locally free sheaves over a curve, hence locally free themselves.

We thus need to show that, if \( \alpha \) is a holomorphic section of \( \mathcal{K}^{(r)} \), then \( \eta^{(r+1)}(\alpha) \) is a holomorphic section of \( E^\vee \). By definition of \( \eta^{(r+1)} \) and \( \mathcal{K}^{(r)} \) it holds

\[
\eta^{(r+1)}(\alpha) = \nabla_t^{E^\vee} \left( \eta^{(r)}(\alpha) \right) - \eta^{(r)} \left( \nabla_t^{E} (\alpha) \right) = -\eta^{(r)} \left( \nabla_t^{GM} (\alpha) \right).
\]

Note that in the last equality we have used \( \nabla_t^{E} (\alpha) = \nabla_t^{GM} (\alpha) \), which holds because \( \nabla_t^{E} = \nabla_t^{GM} \) on \( \mathcal{K} = \mathcal{K}^{(1)} \supseteq \mathcal{K}^{(r)} \).

It remains only to show that \( \eta^{(r)} \left( \nabla_t^{GM} (\alpha) \right) \) is holomorphic. Since \( \nabla_t^{GM} (\alpha) \) is holomorphic and \( \eta^{(r)} \) is holomorphic on \( \mathcal{K} = \mathcal{K}^{(1)} \supseteq \mathcal{K}^{(r)} \), it is enough to show that \( \nabla_t^{GM} \alpha \) is indeed a section of \( \mathcal{K}^{(r-1)} \). But this follows easily from

\[
\eta^{(s)} \left( \nabla_t^{E} \alpha \right) = \nabla_t^{E^\vee} \left( \eta^{(s)} (\alpha) \right) - \eta^{(s+1)}(\alpha) = 0
\]

for every \( s = 1, \ldots, r - 1 \). \( \Box \)

Theorem 2.10. Let \( \alpha \in \Gamma (E) \). Then it holds \( \mathcal{U} = \mathcal{K}^{(g)} \), i.e.

\[
\alpha \in \Gamma (\mathcal{U}) \iff \eta^{(1)}(\alpha) = \eta^{(2)}(\alpha) = \ldots = \eta^{(g)}(\alpha) = 0.
\]  

(2.11)

In particular we have

\[
\mathcal{U}_b \subseteq \bigcap_{k=1}^{g} \ker \eta^{(k)}_b \subseteq E_b
\]

(2.12)

with equality for \( b \) in a dense Zariski-open subset of \( B \).

Proof. By Theorem 2.5 after contracting with \( \frac{\partial}{\partial t} \) it is enough to show that the conditions

\[
\eta^{(1)}(\alpha) = \eta^{(1)} \left( \nabla_t^{GM} \alpha \right) = \eta^{(1)} \left( \left( \nabla_t^{GM} \right)^2 \alpha \right) = \ldots = \eta^{(1)} \left( \left( \nabla_t^{GM} \right)^n \alpha \right) = 0.
\]  

(2.13)

and

\[
\eta^{(1)}(\alpha) = \eta^{(2)}(\alpha) = \ldots = \eta^{(n)}(\alpha) = 0.
\]  

(2.14)

are equivalent for any \( n \in \mathbb{N} \). We will actually show that both (2.13) and (2.14) are equivalent to the condition

\[
\eta^{(i)} \left( \left( \nabla_t^{GM} \right)^j \alpha \right) = 0 \quad \forall \ i + j \leq n.
\]  

(2.15)
It is obvious that (2.15) implies (2.13) (setting \(i = 1\)) and (2.14) (setting \(j = 0\)).

Suppose now that (2.13) holds, so that (2.15) holds for \(i = 1\) and any \(j\). In particular we have \((\nabla_t^{GM})^j \alpha \in \Gamma (\mathcal{K})\) for every \(j\), and therefore \(\nabla_t^E \left( (\nabla_t^{GM})^j \alpha \right) = (\nabla_t^{GM})^{j+1} \alpha\) by (2.7).

We proceed by induction on \(i\), assuming that (2.15) holds for a fixed \(i \geq 1\) and any \(j \leq n - i\). Then the identity (2.6) implies

\[
\eta^{(i+1)} \left( (\nabla_t^{GM})^j \alpha \right) = \left( \nabla_t^T \eta^{(i)} \right) \left( (\nabla_t^{GM})^j \alpha \right) = \nabla_t^E \left( \eta^{(i)} \left( (\nabla_t^{GM})^j \alpha \right) \right) - \eta^{(i)} \left( (\nabla_t^E \left( (\nabla_t^{GM})^j \alpha \right) \right) = \nabla_t^E \left( \eta^{(i)} \left( (\nabla_t^{GM})^j \alpha \right) \right) - \eta^{(i)} \left( (\nabla_t^{GM})^{j+1} \alpha \right) = 0 - 0 = 0
\]

for any \(j \leq n - (i + 1)\), i.e. (2.15) holds also for \(i + 1\).

We finally show, by induction on \(j\), that (2.14) implies (2.15). Indeed (2.14) is the case \(j = 0\) of (2.15). Assuming now that (2.15) holds for a fixed \(j\) and every \(i \leq n - j\), the same computations as above show

\[
\eta^{(i)} \left( (\nabla_t^{GM})^{j+1} \alpha \right) = \nabla_t^E \left( \eta^{(i)} \left( (\nabla_t^{GM})^j \alpha \right) \right) - \eta^{(i+1)} \left( (\nabla_t^{GM})^j \alpha \right) = 0 - 0 = 0
\]

for any \(i \leq n - (j + 1)\). This means that (2.15) holds for \(j + 1\), which concludes the proof of (2.11).

The inclusions (2.12) and the equality on the first inclusion for general \(b\) follow from Proposition 2.9.

We close this section putting our constructions in the context of totally geodesic submanifolds of the Siegel upper-half space \(\mathbb{H}_g\) (and of the moduli space \(\mathcal{A}_g\)). Indeed by choosing a frame of \(V_Z\) we obtain (a lifting of) the period map \(\gamma: B \to \mathbb{H}_g\), with the property that \(T = \text{Hom}^s(E, E^{\vee}) = \gamma^* T_{\mathbb{H}_g}\) and the Higgs field can be interpreted as the differential of \(\gamma\)

\[
d\gamma: T_B \longrightarrow \text{Hom}^s \left( E, E^{\vee} \right).
\]

Moreover, the connection \(\nabla^T\) coincides with the pullback of the metric Levi-Civita connection of the Siegel metric in \(\mathbb{H}_g\).

Under these identifications, and assuming that \(\gamma\) is an embedding, we have that \(\eta^{(1)} = d\gamma \left( \frac{\partial}{\partial t} \right)\) is a non-vanishing tangent vector field along \(B \subseteq \mathbb{H}_g\). Moreover \(\eta^{(2)} = \nabla^T_t \eta^{(1)}\) is the covariant derivative of \(\eta^{(1)}\) along itself, and the \(\eta^{(k)}\) are the subsequent derivatives.

Recall from the introduction that in some recent works Chen, Lu and Zuo but also many other authors relate the rank of \(U\) with the property of \(B \subseteq \mathbb{H}_g\) being totally geodesic. Of particular interest for the present paper is the result by Ghigi, Pirola and the second author in [GPT19] that if \(B\) contains one (real) geodesic, then \(U = \mathcal{K}\). The proof is basically consequence of [GPT19, Lemma 3.1] which we apply here to prove Proposition 2.5 in a slightly more general version.
3 Families of curves and geometric VHS

In this section we consider the tools of the previous section in the case of a geometric PVHS of weight one, i.e. which arises from a family of smooth projective curves. We study in detail the action of the second additional vector field $\eta^{(2)}$ in terms of the coordinate expressions of the relative 1-forms. In particular, we define a cohomology class $\mu$ of the tangent bundle of each fibre, which acts almost like $\eta^{(2)}$ and also gives a second linear condition defining $U \subseteq K$. These classes $\mu$ depend on the second-order neighbourhood of the fibres and we call them second-order Kodaira-Spencer class of the deformations.

To be precise, we consider a smooth family of projective curves $f : C \to B$ over a smooth complex manifold $B$. We denote by $C_b = f^{-1}(b)$ the curves of the family, all of which have the same genus $g$. We have then the mutually dual short exact sequences of vector bundles on $C$

$$0 \rightarrow f^*\Omega^1_B \rightarrow \Omega^1_C \rightarrow \Omega^1_{C/B} \cong \omega_{C/B} \rightarrow 0 \quad (3.1)$$

$$0 \rightarrow T_{C/B} \rightarrow T_C \rightarrow f^*T_B \rightarrow 0. \quad (3.2)$$

The Hodge structures of the fibres $C_b$ form a geometric PVHS

$$0 \rightarrow f_*\omega_{C/B} \rightarrow R^1f_*\mathbb{Z}_C \otimes \mathbb{Z} O_B \rightarrow R^1f_*O_C \rightarrow 0,$$

i.e. we have $\nabla_B = R^1f_*\mathbb{Z}_C$, $E = f_*\omega_{C/B}$ and the polarization $Q$ is induced by the cup product on the stalks $H^1(C_b, \mathbb{Z})$.

The Higgs field is the connecting homomorphism

$$\theta : f_*\omega_{C/B} \rightarrow R^1f_*O_C \otimes \Omega^1_B$$

obtained by pushing forward the short exact sequence (3.1).

Moreover at any point $b \in B$, $\theta_b : H^0(C_b, \Omega^1_{C_b}) \to H^1(C_b, O_{C_b}) \otimes T^*_B$ is given by cup- and interior product with the Kodaira-Spencer map $KS_b : T_{B,b} \to H^1(C_b, T_{C_b})$. More explicitly, for every $v \in T_{B,b}$ the class $\xi_v := KS_b(v) \in H^1(T_{C_b})$ corresponds to the first order deformation of $C_b$ inside $C$ in the direction of $v$, and it holds

$$\theta_b(\alpha)(v) = \xi_v \cdot \alpha \in H^1(C_b, O_{C_b}). \quad (3.3)$$

In particular, for general $b \in B$ it holds

$$\mathcal{K} \otimes \mathbb{C}(b) = \bigcap_{v \in T_{B,b} \setminus \{0\}} \ker (\xi_v : H^0(C_b, \Omega^1_{C_b}) \to H^1(C_b, O_{C_b})). \quad (3.4)$$

The interpretation of the Higgs field as the connecting homomorphism of the push-forward of (3.1) gives the following interpretation of the local sections of $\mathcal{K}$: if $V \subseteq B$ is an open disk, then

$$\Gamma(V, \mathcal{K}) = \text{im} \left( \Gamma \left( f^{-1}(V), \Omega^1_C \right) \rightarrow \Gamma \left( f^{-1}(V), \omega_{C/B} \right) \right), \quad (3.5)$$
namely sections of $K$ over $V$ correspond to families of holomorphic 1-forms on the fibres $C_b$ (for $b \in V$) arising as restrictions of a common holomorphic 1-form on $f^{-1}(V)$. With this in mind, Pirola and the second author proved in [PT20] that

$$\Gamma (V, U) = \text{im} \left( \Gamma \left( f^{-1}(V), \Omega^1_{C,B} \right) \longrightarrow \Gamma \left( f^{-1}(V), \omega_{C/B} \right) \right),$$

(3.6)

where $\Omega^1_{C,d} := \ker (d: \Omega^1_C \to \Omega^2_C)$ is the sheaf of closed holomorphic 1-forms on $C$. In other words, the flat sections of $E$ correspond to families of 1-forms on the fibres arising as restriction of a common closed holomorphic 1-form on $f^{-1}(V)$.

The closedness condition still cannot be checked on a given fibre $C_b$ fibre, but only on an open neighbourhood. In order to characterize $U_b$ just from infinitesimal data of $C_b$ inside $C$ we need to understand the homomorphisms $\eta^{(k)}_b: H^0(\omega_{C_b}) \to H^1(\mathcal{O}_{C_b})$ induced by the $\eta^{(k)}$ from Definition 2.6 in this geometric case. In this paper we will focus in the case $k = 2$, i.e. the first additional linear condition. Although the $\eta^{(k)}$ were defined for $\dim B = 1$ with a global coordinate $t$, we will make some preliminary computations in a more general setting, where $B \subseteq C$ is an open ball.

Since the family is assumed to be smooth, around any point $P \in C$ we can find a function $x$ such that $(x, t_1, \ldots, t_r)$ is a coordinate system around $P$. This means, the map $f$ is given by $(x, t_1, \ldots, t_r) \mapsto (t_1, \ldots, t_r)$ and $x$ restricts to a local coordinate on the fibres $C_b$. In this setting $f^* \Omega^1_B$ is (globally) generated by $\{dt_1, \ldots, dt_r\}$ and $\Omega^1_C$ resp. $\omega_{C/B}$ are locally generated by $\{dx, dt_1, \ldots, dt_n\}$ resp. $\{dx\}$. If $(x', t_1, \ldots, t_n)$ is another coordinate system, it holds

$$dx' = \frac{\partial x'}{\partial x} dx + \sum_{i=1}^n \frac{\partial x'}{\partial t_i} dt_i$$

(3.7)

in $\Omega^1_C$, while in $\omega_{C/B}$ we only have

$$dx' = \frac{\partial x'}{\partial x} dx.$$ 

(3.8)

A holomorphic section $\alpha \in \Gamma (E) = \Gamma \left( \omega_{C/B} \right)$ can be interpreted as a holomorphic section (on $C$) of $\omega_{C/B}$. According to the previous discussion, $\alpha$ can be locally written as $a (x, t_1, \ldots, t_r) dx$ with a holomorphic coefficient function $a$. If $a' (x', t_1, \ldots, t_r) dx'$ is another local expression of $\alpha$, it holds then $a = a' \frac{\partial x'}{\partial x}$. The same holds for a smooth section $\alpha \in \Gamma (\mathcal{A}(E))$, with the only additional condition that the local smooth function $a$ should be holomorphic on $x$, i.e. $\frac{\partial a}{\partial x} = 0$.

From the exact sequence (3.1), given any smooth section $\alpha \in \Gamma (\mathcal{A}(E))$ we can find a smooth lifting $\tilde{\alpha} \in \Gamma (\mathcal{A}(\Omega^1_C))$, with local expression

$$\tilde{\alpha} = a (x, t_1, \ldots, t_r) dx + \sum_{i=1}^r c_i (x, t_1, \ldots, t_r) dt_i.$$ 

(3.9)
Even if $\alpha$ is holomorphic, so that the function $a$ is holomorphic by assumption, the additional coefficient functions $c_1, \ldots, c_r$ are a priori only smooth functions. Actually, they can be chosen to be holomorphic if and only if $\alpha$ is a (holomorphic) section of $K$, but not otherwise.

More generally, any smooth section $\alpha$ of $H = R^1 f_* \mathcal{Z}_C \otimes_{\mathcal{O}} \mathcal{O}_B$ can be represented by a 1-form $\tilde{\alpha} \in \mathcal{A}^1 (C)$ with the additional condition that $d\tilde{\alpha}|_{C_b} = 0$ for any $b \in B$, so that

$$\alpha (b) = \left[ \tilde{\alpha}|_{C_b} \right] \in H^1 (C_b, \mathbb{C}).$$

In the case of sections of $E$ the closedness condition $d\tilde{\alpha}|_{C_b} = 0$ is automatically satisfied.

In order to obtain a precise description of the action of $\eta^{(2)}$, we first need some explicit description of the connection $\nabla^T$, for which in turn we need some explicit expression for the Gauss-Manin connection. This is given by the Cartan-Lie formula (see [Voi07, Proposition 9.14])

$$\left( \nabla^G \alpha \right)_b = \left[ (\text{int}_\chi d\tilde{\alpha})|_{C_b} \right] \in H^1 (C_b, \mathbb{C}),$$

where

1. $X \in \Gamma (B, \mathcal{A} (T_{B, \mathbb{C}}))$ is any smooth (complex) vector field on $B$,
2. $\chi$ is a smooth vector field on $C$ that descends to $X$, i.e. a $C^\infty$ lifting of $X$ under the short exact sequence

$$0 \to T_{C/B, \mathbb{C}} \to T_{C, \mathbb{C}} \xrightarrow{df} f^* T_{B, \mathbb{C}} \to 0$$

(3.11)
3. $\alpha$ is any smooth section of $\mathcal{H}$, represented by a 1-form $\tilde{\alpha} \in \mathcal{A}^1 (S)$ with $d\tilde{\alpha}|_{C_b} = 0$ for any $b \in B$, so that $\alpha (b) = \left[ \tilde{\alpha}|_{C_b} \right] \in H^1 (C_b, \mathbb{C})$.
4. In particular, if $\alpha$ is a section of $E$, we can assume $\tilde{\alpha} \in \mathcal{A}^{1,0} (\mathcal{C})$.

In this setting, we look now for an explicit description of $\nabla^T$. It is enough for our purposes to understand the action of $\nabla^T$ on the homomorphisms $E \to E^\vee$ arising from the Higgs field $\theta$. For the sake of simplicity, we introduce the following notations.

**Definition 3.1.** For each $i = 1, \ldots, r$ let:

1. $\xi_i := t_i \theta : E \to E^\vee$, which are holomorphic homomorphisms of vector bundles, and
2. $\chi_i \in \mathcal{A} (T_C)$, a smooth vector field on $C$ descending to $\frac{\partial}{\partial t_i}$. In an open subset $V$ of $C$ with coordinates $(x, t_1, \ldots, t_r)$ it can be written as

$$\chi_i = Z_i \frac{\partial}{\partial x} + \frac{\partial}{\partial t_i}$$

(3.12)

for a certain smooth function $Z_i : V \to \mathbb{C}$.

For each $i, j = 1, \ldots, r$ set $\eta_{ij} := \nabla^T \frac{\partial}{\partial t_i} \xi_j$. 

12
For any holomorphic section $\alpha \in \Gamma(E)$ we will find a quite explicit coordinate description of

$$\eta_{ij}(\alpha) = \left( \nabla^T_{\alpha} \xi_i \right)(\alpha) = \nabla^E_{\alpha} (\xi_i(\alpha)) - \xi_i \left( \nabla^E_{\alpha} \alpha \right).$$

We present now a step by step computation, since we need to define some additional functions at some intermediate steps, and summarize the result into a formal statement at the end.

Let $\tilde{\alpha} \in \Gamma(C^\infty(\Omega^1_C)) = \Gamma(A^{1,0}(C))$ be a representative of $\alpha$ with local expression $adx + \sum_{k=1}^n c_k dt_k$. Chosen the vector fields $\chi_i$, there is an obvious choice $c_k = -aZ_k$ for every $k = 1, \ldots, r$, which we will use later on. But for the sake of simplicity of notation, we will keep $c_k$ for the moment.

By (3.10), for any $b \in B$, $\left( \nabla^E_{\xi_i} \alpha \right)_b$ is represented by $[\operatorname{int} \chi_i, d\tilde{\alpha} |_{C_b}] \in H^1(C_b, \mathbb{C})$.

We compute first

$$\beta_i := \operatorname{int}_{\chi_i} d\tilde{\alpha}$$

$$= \operatorname{int}_{\chi_i} \left( -\sum_k \left( \frac{\partial a}{\partial t_k} - \frac{\partial c_k}{\partial x} \right) dx \wedge dt_k + \sum_{k,l} \frac{\partial c_k}{\partial t_l} dt_l \wedge dt_k + \sum_k \frac{\partial c_k}{\partial x} dx \wedge dt_k + \sum_{k,l} \frac{\partial c_k}{\partial t_k} dt_l \wedge dt_k \right)$$

$$= \left( \frac{\partial a}{\partial t_i} - \frac{\partial c_i}{\partial x} \right) dx - \sum_k \left( \frac{\partial a}{\partial t_k} - \frac{\partial c_k}{\partial x} \right) Z_i dt_k + \sum_k \left( \frac{\partial c_k}{\partial t_i} - \frac{\partial c_i}{\partial t_k} \right) dt_k - \frac{\partial c_k}{\partial t_k} dt_k - \sum_k \frac{\partial c_i}{\partial t_k} dt_k$$

We now need to split $\nabla^E_{\xi_i} \alpha$ into its parts of type $(1,0)$ and $(0,1)$, namely $\nabla^E_{\xi_i} \alpha$ and $\xi_i(\alpha)$ (or more precisely $\nu_1 \left( \nabla^E_{\xi_i} \alpha \right)$ and $\nu_2 \left( \xi_i(\alpha) \right)$), for which we need to take the harmonic representatives of the $\left( \nabla^E_{\xi_i} \alpha \right)_b$.

**Definition 3.2.** For each $i = 1, \ldots, r$, let $\varphi_i = \varphi_i(\alpha) : C \to \mathbb{C}$ be a smooth function such that the restrictions $\left( \tilde{\beta}_i \right)_b := (\beta_i + d\varphi_i) |_{C_b}$ are harmonic for every $b \in B$.

These functions $\varphi_1, \ldots, \varphi_r$ exist by general theory and are well defined up to constants depending on the $t_i$’s. Then the components of type $(1,0)$ and $(0,1)$ of $\left( \tilde{\beta}_i \right)_b$ are also harmonic and define the Hodge decomposition of $\left( \nabla^E_{\xi_i} \alpha \right)_b$, i.e. $\nabla^E_{\xi_i} \alpha$ is represented by

$$\tilde{\beta}_i^{1,0} = \beta_i^{1,0} + \varphi_i = \left( \frac{\partial a}{\partial t_i} - \frac{\partial c_i}{\partial x} \right) dx - \sum_k \left( \frac{\partial a}{\partial t_k} - \frac{\partial c_k}{\partial x} \right) Z_i dt_k + \sum_k \left( \frac{\partial c_k}{\partial t_i} - \frac{\partial c_i}{\partial t_k} \right) dt_k + \frac{\partial \varphi_i}{\partial x} dx + \sum_k \frac{\partial \varphi_i}{\partial t_k} dt_k,$$

and

$$\tilde{\beta}_i^{0,1} = \beta_i^{0,1} + \varphi_i = -\frac{\partial c_i}{\partial x} dx - \sum_k \frac{\partial c_i}{\partial t_k} dt_k + \frac{\partial \varphi_i}{\partial x} dx + \sum_k \frac{\partial \varphi_i}{\partial t_k} dt_k,$$  \hspace{1cm} (3.13)

represents $\xi_i(\alpha)$ as a section of $\mathcal{H}$ by means of the splitting $\nu_2 : E^\vee \hookrightarrow \mathcal{H}$.

**Remark 3.3.** Indeed, $\xi_i(\alpha)$ as a section of $E^\vee$ can also be directly represented by $\beta_i^{0,1}$, i.e.

$$(\xi_i(\alpha))_b = \left[ -\frac{\partial c_i}{\partial x} dx |_{C_b} \right] = \left[ a \frac{\partial Z_i}{\partial x} dx |_{C_b} \right] \in H^1_\nu(\mathcal{O}_{C_b}),$$ \hspace{1cm} (3.15)
where the last equality follows by taking \( c_i = -aZ_i \), or by adding the global \( \bar{\partial} \)-exact \((0,1)\)-form \( \bar{\partial} (\text{int}_x, \bar{\alpha})|_{C_b} \) whose local expression is precisely \( \bar{\partial} (aZ_i + c_i)|_{C_b} \).

From the Dolbeault construction of the connecting homomorphism in cohomology of the exact sequence (3.2) it follows that the Kodaira-Spencer class \( KS_b \left( \frac{\partial}{\partial t_i} \right) \in H^1 (T_{C_b}) \) is represented in Dolbeault cohomology by the \((0,1)\)-form with values in \( T_{C_b} \) whose local expression is

\[
\frac{\partial Z_i}{\partial x} d\bar{\tau} \otimes \frac{\partial}{\partial x}|_{C_b}.
\]

Thus equation (3.15) recovers the formula (3.3), i.e. shows that \( (\xi)_b \) acts by cup- and interior product with \( KS_b \left( \frac{\partial}{\partial t_i} \right) \).

It remains now to apply (3.10) again, keeping just the terms in \( d\bar{\tau} \) (since every other term vanishes when restricting to \( C_b \) or when projecting to \( H^1_{\bar{\partial}} (O_{C_b}) \cong H^{0,1} (C_b) = E'_\chi \)), to compute

\[
\nabla_{\partial \chi_j}^E (\xi_i (\alpha))_b = \left[ \text{int}_x d\bar{\tau}^{(0,1)} \right]_{C_b} \left[ \begin{array}{c}
- \frac{\partial^2 c_i}{\partial t_j} Z_j - \frac{\partial^2 c_i}{\partial \partial t_j} + Z_i \frac{\partial^2 \varphi_i}{\partial \partial t_j} + \frac{\partial^2 \varphi_i}{\partial \partial t_j} \\
- \frac{\partial^2 c_i}{\partial \partial t_j} Z_j + \frac{\partial^2 \varphi_i}{\partial \partial t_j} + \chi_j \left( \frac{\partial \varphi_i}{\partial t_j} \right)
\end{array} \right]_{C_b} dt
\]

\[
= \left[ \begin{array}{c}
\left( \frac{\partial^2 c_i}{\partial \partial t_j} Z_j + \frac{\partial^2 c_i}{\partial \partial t_j} \left( \frac{\partial a}{\partial t_j} \frac{\partial c_j}{\partial x} \right) Z_i \right) - \frac{\partial}{\partial \partial \tau} \left( \frac{\partial c_j}{\partial t_i} - \frac{\partial c_i}{\partial t_j} \right) - Z_j \frac{\partial^2 \varphi_i}{\partial \partial \tau} - \frac{\partial^2 \varphi_i}{\partial \partial \tau} \\
\left( \frac{\partial^2 c_i}{\partial \partial t_j} Z_j - \frac{\partial^2 c_j}{\partial \partial t_i} Z_i + \frac{\partial a}{\partial t_j} \frac{\partial Z_i}{\partial \partial t_j} - \frac{\partial a}{\partial \partial t_j} - \frac{\partial c_j}{\partial \partial t_i} - \frac{\partial^2 c_j}{\partial \partial t_i} - \frac{\partial^2 c_i}{\partial \partial t_j} - \frac{\partial^2 c_i}{\partial \partial t_j} - \chi_j \left( \frac{\partial \varphi_i}{\partial \partial \tau} \right) \right)
\end{array} \right]_{C_b} dt
\]

and finally (note the exchange of indices \( i \) and \( j \) in the second summand)

\[
\nabla_{\partial \chi_j}^T (\xi_i) (\alpha)_b = \nabla_{\partial \chi_j}^E (\xi_i (\alpha))_b - \xi_i \left( \nabla_{\partial \tau}^E \alpha \right)_b
\]

\[
= \left[ \begin{array}{c}
- \frac{\partial^2 c_i}{\partial \partial t_j} Z_j - \frac{\partial^2 c_i}{\partial \partial t_j} + \chi_j \left( \frac{\partial \varphi_i}{\partial \partial \tau} \right) \\
- \frac{\partial^2 c_j}{\partial \partial t_j} Z_i + \frac{\partial^2 c_j}{\partial \partial t_j} \left( \frac{\partial a}{\partial t_i} \frac{\partial Z_i}{\partial \partial t_j} + \frac{\partial c_i}{\partial \partial t_j} + \chi_j \left( \frac{\partial \varphi_j}{\partial \partial \tau} \right) \right)
\end{array} \right]_{C_b} dt
\]

\[
= \left[ \begin{array}{c}
\left( \frac{\partial^2 c_i}{\partial \partial t_j} Z_j + \frac{\partial a}{\partial t_j} \frac{\partial \varphi_j}{\partial \partial \tau} + \frac{\partial c_i}{\partial \partial t_j} + \chi_j \left( \frac{\partial \varphi_j}{\partial \partial \tau} \right) \right) \\
\left( \frac{\partial^2 c_j}{\partial \partial t_i} Z_i + \frac{\partial a}{\partial t_i} \frac{\partial \varphi_i}{\partial \partial \tau} - \frac{\partial c_i}{\partial \partial t_i} + \chi_i \left( \frac{\partial \varphi_i}{\partial \partial \tau} \right) \right)
\end{array} \right]_{C_b} dt
\].
If we assume $c_i = -aZ_i$ and $c_j = -aZ_j$, the formula becomes:

$$
\left( \nabla_{\partial_j} \xi_i \right) (\alpha) = \left[ a \left( \frac{\partial^2 Z_j}{\partial x \partial x} Z_i - \frac{\partial Z_j}{\partial x} \frac{\partial Z_i}{\partial x} + \frac{\partial^2 Z_j}{\partial t \partial t} \right) + \chi_j \left( \frac{\partial \varphi_j}{\partial \xi} \right) + \chi_i \left( \frac{\partial \varphi_j}{\partial \xi} \right) \right] |_{C_b} \, d\xi.
$$

(3.16)

We can obtain a nicer formula if we change the representative in $H_\partial^1 (\mathcal{O}_{C_b})$ by adding the $\partial$-exact form $\partial (\chi_i (\varphi_j) + \chi_j (\varphi_i)) |_{C_b}$. In the chosen local coordinates we have

$$
\partial (\chi_i (\varphi_j)) |_{C_b} = \partial \left( Z_i \frac{\partial \varphi_j}{\partial x} + \frac{\partial \varphi_j}{\partial \xi} \right) = \left( \chi_i \left( \frac{\partial \varphi_j}{\partial \xi} \right) + \frac{\partial Z_i}{\partial \xi} \frac{\partial \varphi_j}{\partial x} \right) \, d\xi.
$$

Thus subtracting $\partial (\chi_i (\varphi_j) + \chi_j (\varphi_i)) |_{C_b}$ from (3.16) we obtain the following

**Theorem 3.4.** If $\alpha \in \Gamma (E)$ is locally given by $a (x, t_1, \ldots, t_r) \, dx$, then for every $i, j = 1, \ldots, r$

$$
\left( \nabla_{\partial_j} \xi_i \right) (\alpha) |_{C_b} = \left[ a \left( \frac{\partial^2 Z_j}{\partial x \partial x} Z_i + \frac{\partial^2 Z_j}{\partial \xi \partial \xi} \frac{\partial Z_i}{\partial \xi} \frac{\partial Z_i}{\partial \xi} - \frac{\partial Z_j}{\partial \xi} \frac{\partial Z_i}{\partial \xi} \right) - \frac{\partial Z_i}{\partial \xi} \frac{\partial \varphi_j}{\partial x} - \frac{\partial Z_j}{\partial \xi} \frac{\partial \varphi_i}{\partial x} \right] |_{C_b} \, d\xi.
$$

(3.17)

where the globally defined functions $\varphi_i, \varphi_j : C \to \mathbb{C}$ are as in Definition 3.2.

We restrict ourselves now to the case where $\dim B = 1$ and $B$ has a global coordinate $t$. In this case we have only one $\xi = \eta^{(1)} = \xi + \partial \theta$ and $\eta^{(2)} = \nabla^T \xi$, and we write $Z$ for the function such that the vector field lifting $\partial \theta$ can be written as $\chi = Z \partial_{\theta} + \partial \theta$. In this setting Theorem 3.4 immediately gives:

**Corollary 3.5.** If $\alpha \in \Gamma (E)$ is locally given by $a (x, t) \, dx$, then at every $b \in B$ it holds

$$
\left( \eta^{(2)} \right) (\alpha) |_{C_b} = \left[ a \left( Z \frac{\partial^2 Z}{\partial \xi \partial \xi} + \frac{\partial^2 Z}{\partial \xi \partial \xi} \frac{\partial Z}{\partial \xi} \frac{\partial Z}{\partial \xi} - \frac{\partial Z}{\partial \xi} \frac{\partial \varphi}{\partial x} - \frac{\partial Z}{\partial \xi} \frac{\partial \varphi}{\partial x} \right) |_{C_b} \, d\xi
$$

(3.18)

where $\varphi : C \to \mathbb{C}$ is a smooth function satisfying the analogous condition as in Definition 3.2.

The fact that in (3.18) we can group many terms with a common factor $a$ is not casual, since actually these terms come from a well-defined $(0, 1)$-form with values in $T_{C_b}$

**Lemma 3.6.** For fixed $t = b$, the expression

$$
\left( Z \frac{\partial^2 Z}{\partial \xi \partial \xi} + \frac{\partial^2 Z}{\partial \xi \partial \xi} \frac{\partial Z}{\partial \xi} \frac{\partial Z}{\partial \xi} - \frac{\partial Z}{\partial \xi} \frac{\partial \varphi}{\partial x} - \frac{\partial Z}{\partial \xi} \frac{\partial \varphi}{\partial x} \right) d\xi \otimes \frac{\partial}{\partial x}
$$

(3.19)

defines a global $(0, 1)$-form with coefficients on $T_{C_b}, \mu_b \in \Gamma \left( C_b, A^{0,1} (T_{C_b}) \right)$.

**Proof.** Let $V, V' \subseteq C$ be two open subsets with coordinates $(x, t)$ and $(x', t')$, where $t = t'$ is the coordinate giving the fibration $f : C \to B$. Note that although $t = t'$, the vector fields $\frac{\partial}{\partial t}$ on $V$ and $\frac{\partial}{\partial t'}$ on $V'$ are different on $V \cap V'$, since the are defined depending on the remaining coordinates $x$ resp. $x'$. Actually, on $V \cap V'$ there are the following relations:

$$
\frac{\partial}{\partial x} = g \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial t} = h \frac{\partial}{\partial t'} + \frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial x'} = \frac{1}{g} \frac{\partial}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial t'} = -\frac{h}{g} \frac{\partial}{\partial x} + \frac{\partial}{\partial t'}
$$

(3.20)
where \( g = \frac{\partial f}{\partial x} \) and \( h = \frac{\partial f}{\partial t} \). Analogously it holds \( dx' = \frac{\partial f'}{\partial x} dx + \frac{\partial f'}{\partial t} dt \), and thus \( dx'|_{C_b} = \frac{\partial f'}{\partial x} dx|_{C_b} \) for every \( b \in B \). Note that \( g \neq 0 \) at every point (otherwise \((x, t) \mapsto (x', t')\) would not be a change of coordinates) and also
\[
\frac{\partial g}{\partial t} = \frac{\partial^2 x'}{\partial x \partial t} = \frac{\partial h}{\partial x}. \tag{3.21}
\]

Let also \( \chi = Z\frac{\partial}{\partial x} + \frac{\partial}{\partial t} = Z'\frac{\partial}{\partial x'} + \frac{\partial}{\partial t} \) be the local expressions of \( \chi \) in \( V \cap V' \) with respect to both systems of coordinates. From (3.20) it follows that
\[
Z' = Zg + h. \tag{3.22}
\]

From (3.22), using that \( g \) and \( h \) are holomorphic and conjugating (3.20), we obtain
\[
\frac{\partial Z'}{\partial x'} = g\frac{\partial Z}{\partial x} = \frac{g \partial Z}{g} \frac{\partial Z}{g},
\]
and thus using (3.20) and (3.22) again
\[
Z'\frac{\partial^2 Z'}{\partial x' \partial x'} = \frac{Zg + h}{g} \frac{\partial}{\partial x'} \left( \frac{g \partial Z}{g} \frac{\partial Z}{g} \right) = \frac{Zg + h}{g} \left( g\frac{\partial^2 Z}{\partial x \partial x} + \frac{\partial g}{\partial x} \frac{\partial Z}{g} \right), \tag{3.23}
\]
\[
\frac{\partial^2 Z'}{\partial t' \partial x'} = -\frac{h}{g} \frac{\partial}{\partial x'} \left( \frac{g \partial Z}{g} \frac{\partial Z}{g} \right) + \frac{1}{g} \frac{\partial}{\partial t'} \left( \frac{g \partial Z}{g} \frac{\partial Z}{g} \right), \tag{3.24}
\]
\[
= -\frac{h}{g} \left( g\frac{\partial^2 Z}{\partial x \partial x} + \frac{\partial g}{\partial x} \frac{\partial Z}{g} \right) + \frac{1}{g} \left( \frac{\partial g}{\partial t} \frac{\partial Z}{g} + \frac{\partial^2 Z}{\partial x \partial t} \right), \tag{3.25}
\]
\[
\frac{\partial Z'}{\partial x'} \frac{\partial Z'}{\partial x'} = \frac{1}{g} \frac{\partial(Zg + h)}{\partial x} \frac{\partial Z}{g} = \frac{1}{g} \frac{\partial Z}{\partial x} \left( \frac{\partial Z}{g} + \frac{\partial h}{\partial x} \right). \tag{3.26}
\]

Summing up and cancelling repeated terms using (3.21), we obtain
\[
Z'\frac{\partial^2 Z'}{\partial x' \partial x'} + \frac{\partial^2 Z'}{\partial t' \partial x'} - \frac{\partial Z'}{\partial x'} \frac{\partial Z'}{\partial x'} = \frac{g}{g} \left( Z\frac{\partial^2 Z}{\partial x \partial x} + \frac{\partial^2 Z}{\partial t \partial x} - \frac{\partial Z}{\partial x} \frac{\partial Z}{\partial x} \right). \tag{3.27}
\]

Combining this with the fact that \( dx' \otimes \frac{\partial}{\partial x'}|_{C_b} = \eta dx \otimes \frac{1}{g} \frac{\partial}{\partial x}|_{C_b} \), it turns out that
\[
\left( Z'\frac{\partial^2 Z'}{\partial x' \partial x'} + \frac{\partial^2 Z'}{\partial t' \partial x'} - \frac{\partial Z'}{\partial x'} \frac{\partial Z'}{\partial x'} \right) dx' \otimes \frac{\partial}{\partial x'}|_{C_b} = \left( Z\frac{\partial^2 Z}{\partial x \partial x} + \frac{\partial^2 Z}{\partial t \partial x} - \frac{\partial Z}{\partial x} \frac{\partial Z}{\partial x} \right) dx \otimes \frac{\partial}{\partial x}|_{C_b}.
\]

Thus the expression defining \( \mu \) does not depend on the choice of coordinates \((x, t)\), as wanted.

\[\square\]

**Definition 3.7.** We call the class \([\mu_b] \in H^1_{\bar{\partial}}(T_{C_b})\) second-order Kodaira-Spencer class of \( C_b \subseteq C \).

**Remark 3.8.** Here some comments on the classes \( \mu_b \).

1. There is the following very good motivation to the name: as already mentioned in Remark 3.5, the first-order Kodaira-Spencer class of \( C_b \subseteq C \) can be computed as \([\partial \chi]|_{C_b} \in H^1_{\bar{\partial}}(T_{C_b})\), which depends only on the first-order infinitesimal neighbourhood of \( C_b \) in \( C \). Thus \( \mu \) depends only on the second-order infinitesimal neighbourhood of \( C_b \) in \( C \), because of the term \( \frac{\partial^2 Z}{\partial x \partial x} \).
2. The classes $\mu_b$ have been defined pointwise. Although their definition suggest that they depend smoothly on $b \in B$ (or even possibly holomorphically, as the classes $\xi_b$ do), it is not yet clear to us how to prove this. Indeed the first-order Kodaira-Spencer classes $\xi_b$ can be gathered into a holomorphic section of $R^1 f_* T_C/B$ given by the connecting homomorphism of pushing-forward the short exact sequence $(3.2)$. It would be interesting to obtain such a description of the classes $\mu_b$.

3. Recall from the end of Section 2 that in the case the induced period map $\gamma: B \rightarrow M_g \rightarrow A_g$ (or one lifting to the Siegel upper-half space $\mathbb{H}_g$) is an embedding, we can think of $\eta^{(2)}$ as the covariant derivative of the tangent vector field to $B$ along itself. We can then decompose $\eta^{(2)} = \eta^{(2)}_T + \eta^{(2)}_N$, where $\eta^{(2)}_T$ is tangent to $M_g$, and $\eta^{(2)}_N$ is normal. Since $T_{[C_b]} M_g = H^1 (T_{C_b})$, the $\mu_b$ also form a (possibly not even continuous) vector field along $B$ tangent to $M_g$, and it would be very interesting to compare both vector fields $\mu$ and $\eta^{(2)}_T$.

We will now focus our attention on the class $\eta^{(2)} (\alpha)_b - \mu_b \cdot \alpha_b \in H^1 (\mathcal{O}_{C_b}) \cong H^0 (\omega_{C_b}) \vee$, the remaining term of $(3.18)$, locally represented by $-2 \left( \frac{\partial \varphi}{\partial x} \frac{\partial Z}{\partial x} \wedge \frac{dx}{\partial x} \right)_{C_b}$. More precisely, we consider its action on any form $\alpha'_b \in K_{b}$.

**Lemma 3.9.** For any $\alpha'_b \in \ker (\xi_b): H^0 (\omega_{C_b}) \rightarrow H^1 (\mathcal{O}_{C_b})$ it holds

\[
\left( \eta^{(2)} (\alpha)_b - \mu_b \cdot \alpha_b \right) \cdot \alpha'_b = 0 \in H^1 (\omega_{C_b}) \cong \mathbb{C}.
\]

**Proof.** Let $\alpha'_b$ be locally represented by $a (x) dx$, and recall that $\xi_b$ is represented locally by $\left( \frac{\partial Z}{\partial x} \frac{dx}{\partial x} \right)_{C_b}$. Considering Dolbeaut cohomology, it holds $\xi_b \cdot \alpha'_b = 0$ if and only if there is a global smooth function $\gamma: C_b \rightarrow \mathbb{C}$ such that $\overline{\partial} \gamma = \left( \frac{\partial Z}{\partial x} \frac{dx}{\partial x} \right)_{C_b}$. In this case we can write

\[
\left( \eta^{(2)} (\alpha)_b - \mu_b \cdot \alpha_b \right) \cdot \alpha'_b = -2 \left( \frac{\partial \varphi}{\partial x} \frac{\partial Z}{\partial x} \frac{dx}{\partial x} \wedge dx \right) = 2 \left[ \partial \varphi \wedge \overline{\partial} \gamma \right] \in H^1 (\omega_{C_b}) \cong H^2_{\partial R} (C_b, \mathbb{C})
\]

Now it is clear that $2 \partial \varphi \wedge \overline{\partial} \gamma = d (\varphi \overline{\partial} \gamma - \gamma \partial \varphi)$ is $d$-exact, and the claim follows.

□

With Lemma 3.9 at hand, we can now state the main result of this section regarding additional conditions defining the fibres of $\mathcal{U}$.

**Theorem 3.10.** For any $b \in B$ let $\mathcal{K}_b \subseteq E_b = H^0 (\omega_{C_b})$ be the fibre of $\mathcal{K}$ on $B$, and $\mu_b \in H^1 (T_{C_b})$ the second-order Kodaira-Spencer class of $C_b \subseteq C$. Let

\[
\widehat{\mu}_b: H^0 (\omega_{C_b}) \xrightarrow{\mu_b} H^1 (\mathcal{O}_{C_b}) = E_b' \rightarrow \mathcal{K}_b'.
\]

Then $\mathcal{U}_b \subseteq \mathcal{K}_b \cap \ker \widehat{\mu}_b$.

**Proof.** Let $\xi_b \in H^1 (T_{C_b})$ be the first-order Kodaira-Spencer class at $b$. In general we have $\mathcal{K}_b \subseteq \ker (\xi_b)$. Hence we can apply Lemma 3.9 to show that $\widehat{\mu}_b$ coincides with the composition

\[
H^0 (\omega_{C_b}) \xrightarrow{\eta^{(1)}_b} H^1 (\mathcal{O}_{C_b}) = E_b' \rightarrow \mathcal{K}_b'.
\]
Finally from Theorem 2.10 we have
\[ U_b \subseteq K_b \cap \ker \eta_b^{(1)} \subseteq K_b \cap \ker \hat{\mu}_b. \]

4 Families of plane curves

In this section we consider the special case of a smooth family of plane curves over a disk \( 0 \in B \subset \mathbb{C} \). We find a more explicit formula for \( \mu \) in terms of the defining polynomials and in particular we see that at a general point \( \mu_b \) and \( \xi_b \) can be different, hence \( \mu \) induces a non-trivial additional condition.

We start fixing some notations and recalling general facts about the tangent and normal bundles of smooth plane curves. Recall that the tangent bundle of \( P^2 \) fits into the Euler exact sequence
\[ 0 \to \mathcal{O}_{P^2} \to \bigoplus_{i=0}^{2} \mathcal{O}_{P^2}(1) \frac{\partial}{\partial Y_i} \to \mathcal{T}_{P^2} \to 0, \]
where the first map is given by the Euler relation \( 1 \mapsto Y_0 \frac{\partial}{\partial Y_0} + Y_1 \frac{\partial}{\partial Y_1} + Y_2 \frac{\partial}{\partial Y_2} \). In particular \( \mathcal{T}_{P^2} \) is globally generated by the sections \( Y_j \frac{\partial}{\partial Y_i} \) with \( i, j = 0, 1, 2 \). Moreover on the standard open set \( U_j = \{ [a_0 : a_1 : a_2] \in \mathbb{P}^2 \mid a_j \neq 0 \} \cong \mathbb{C}^2 \) with affine coordinates \( y_{i/j} := \frac{Y_j}{Y_i} \) (\( i \neq j \)) we have the identification \( \frac{\partial}{\partial y_{i/j}} = Y_j \frac{\partial}{\partial Y_i} \).

Let \( F(Y_0, Y_1, Y_2) \in \mathbb{C}[Y_0, Y_1, Y_2] \) be a homogeneous polynomial of degree \( d \) defining a smooth projective curve
\[ C = V(F) = \{ [a_0 : a_1 : a_2] \mid F(a_0, a_1, a_2) = 0 \} \subseteq \mathbb{P}^2, \]
i.e. such that
\[ V(F_0, F_1, F_2) = \{ [a_0 : a_1 : a_2] \in \mathbb{P}^2 \mid F_0(a_0, a_1, a_2) = F_1(a_0, a_1, a_2) = F_2(a_0, a_1, a_2) = 0 \} = \emptyset, \]
where for \( i = 0, 1, 2 \) we denote by \( F_i = \frac{\partial F}{\partial Y_i} \).

The normal bundle satisfies \( N_{C/P^2} \cong \mathcal{O}_C(d) \), and fits in the exact sequence
\[ 0 \to \mathcal{T}_C \to \mathcal{T}_{P^2|C} \to \mathcal{O}_C(d) \to 0, \tag{4.1} \]
where \( \nu \) is given by
\[ \frac{\partial}{\partial y_{i/j}|_C} = Y_j \frac{\partial}{\partial Y_i|_C} \mapsto Y_j \frac{\partial F}{\partial Y_i|_C} = Y_j F_i|_C \in H^0(C, \mathcal{O}_C(d)). \]

We now consider the cover of \( C \) by the open subsets
\[ V_{i/j} := \{ [a_0 : a_1 : a_2] \in C \mid a_j \neq 0, F_k(a_0, a_1, a_2) \neq 0 \} \subseteq C \]
where \( k \) denotes the remaining index, i.e. \( \{ i, j, k \} = \{ 0, 1, 2 \} \). The implicit function theorem applied to \( V_{i/j} \subseteq U_j \cong \mathbb{C}^2 \) implies:
1. \( x_{i/j} := (y_{i/j})_{|V_{i/j}} \) is a local coordinate for \( C \) around every point \( P \in V_{i/j} \)

2. \( T_{C|V_{i/j}} \) is generated by

\[
\frac{\partial}{\partial x_{i/j}} := \left( \frac{\partial}{\partial y_{i/j}} - \frac{F_i}{F_k} \frac{\partial}{\partial y_{k/j}} \right)_{|C} = Y_j \left( \frac{\partial}{\partial Y_i} - \frac{F_i}{F_k} \frac{\partial}{\partial Y_k} \right)_{|C}, \tag{4.2}
\]

3. and \( N_{C/P^2|V_{i/j}} \) is generated by \( \frac{\partial}{\partial y_{k/j}} \) on \( C \).

Note that, although \( x_{i/j} \) is in general not a global coordinate on \( V_{i/j} \), the associated derivation at every \( P \in V_{i/j} \) defines the vector field (4.2) defined on the whole \( V_{i/j} \).

The first connecting homomorphism of the long exact sequence of cohomology of (4.1) gives a linear map

\[
\delta: H^0 (N_{C/P^2}) \cong \mathbb{C} [Y_0, Y_1, Y_2] / (F) \rightarrow H^1 (T_C), \tag{4.3}
\]

whose kernel is the degree \( d \) part of the jacobian ideal \( J := (F) \subseteq \mathbb{C} [Y_0, Y_1, Y_2] \).

If \( G \in \mathbb{C} [Y_0, Y_1, Y_2] / (F) \) represents a global section \( G|_C \) of \( H^0 (N_{C/P^2}) \), we want to give a more explicit description of \( \delta G|_C \) in terms of Dolbeaut cohomology. To this aim note first that on \( V_{i/j} \) \( G|_C \) can be lifted to \( G \frac{Y_j}{F_k} \frac{\partial}{\partial y_{k/j}} \in H^0 (V_{i/j}, T_{P^2|C}) \). Thus if \( \{ \rho_{i/j} \}_{i \neq j} \) is a \( C^\infty \) partition of unity such that \( \text{supp} \rho_{i/j} \subseteq V_{i/j} \) for every \( i \neq j \), we can lift \( G|_C \) to the \( C^\infty \) section

\[
\sigma = \sum_{i,j} \rho_{i/j} G \frac{\partial}{Y_j F_k \partial y_{k/j}} \in \mathcal{A} (T_{P^2|C}),
\]

and thus \( \delta G|_C \) is represented by

\[
\bar{\partial} \sigma = \sum_{i,j} \bar{\partial} \rho_{i/j} G \frac{\partial}{Y_j F_k \partial y_{k/j}} = \sum_{i,j} \bar{\partial} \rho_{i/j} G \frac{\partial}{F_k \partial Y_k |_C} \in \mathcal{A}^{0,1} (T_{P^2|C}). \tag{4.4}
\]

It can be quickly shown that \( \bar{\partial} \sigma \in \mathcal{A}^{0,1} (T_C) \) because

\[
\nu (\bar{\partial} \sigma) = \sum_{i,j} \bar{\partial} \rho_{i/j} G \frac{\partial F}{F_k \partial Y_k |_C} = G \bar{\partial} \left( \sum_{i,j} \rho_{i/j} \right) = 0.
\]

However we would like to have a more explicit description of \( \bar{\partial} \sigma \) in terms of the generator \( \frac{\partial}{\partial x_{i/j}} \) of \( T_C \) on \( V_{i/j} \). For this we set

\[
\rho_{ij} = \rho_{i/j} + \rho_{j/i}. \tag{4.5}
\]

**Lemma 4.1.** The class \( \delta G|_C \in H^1 (T_C) \) is represented by the form given in the open subset \( V_{i/j} \) as

\[
\bar{\partial} \sigma = G \left( \frac{1}{F_i} \bar{\partial} p_{k,j} - \frac{Y_i}{Y_j F_j} \bar{\partial} p_{i,k} \right) \frac{\partial}{\partial x_{i/j}}.
\]
Proof. For the sake of simplicity we consider the case $i = 0, j = 2$. It is enough to show that

$$
\sum_{i \neq j} \frac{\partial \rho_{i,j}}{F_k} \frac{\partial}{\partial Y_k}|_C = \frac{1}{Y_2} \left( \frac{1}{F_0} \partial \rho_{1,2} - \frac{Y_0}{Y_2 F_2} \partial \rho_{0,1} \right) \frac{\partial}{\partial x_{0/2}},
$$

or grouping the terms with same $k$, that

$$
\sum_{i < j} \frac{\partial \rho_{i,j}}{F_k} \frac{\partial}{\partial Y_k}|_C = \frac{1}{Y_2} \left( \frac{1}{F_0} \partial \rho_{1,2} - \frac{Y_0}{Y_2 F_2} \partial \rho_{0,1} \right) \frac{\partial}{\partial x_{0/2}}.
$$

The Euler relation gives, $\frac{\partial}{\partial t_2} = \frac{1}{Y_2} \left( Y_0 \frac{\partial}{\partial t_0} + Y_1 \frac{\partial}{\partial t_1} \right)$, so that

$$
\sum_{i < j} \frac{\partial \rho_{i,j}}{F_k} \frac{\partial}{\partial Y_k}|_C = \frac{\partial \rho_{0,1}}{F_2} \frac{\partial}{\partial Y_2}|_C + \frac{\partial \rho_{0,2}}{F_1} \frac{\partial}{\partial Y_1}|_C + \frac{\partial \rho_{1,2}}{F_0} \frac{\partial}{\partial Y_0}|_C.
$$

Using now that $\frac{\partial}{\partial x_{0/2}} = Y_2 \left( \frac{\partial}{\partial t_0} - \frac{F_0}{F_1} \frac{\partial}{\partial t_1} \right)|_C$ and $\rho_{0,2} + \rho_{1,2} = 1 - \rho_{0,1}$, we can write

$$
\sum_{i < j} \frac{\partial \rho_{i,j}}{F_k} \frac{\partial}{\partial Y_k}|_C = \left( \frac{\partial \rho_{1,2}}{F_0} - \frac{Y_0 \partial \rho_{0,1}}{Y_2 F_2} \right) \frac{\partial}{\partial x_{0/2}} - \partial \rho_{0,1} \left( \frac{1}{F_1} + \frac{Y_1}{Y_2 F_2} + \frac{Y_0 F_0}{Y_2 F_1 F_2} \right) \frac{\partial}{\partial Y_1}|_C
$$

where in the last equality the Euler identity $Y_2 F_2 + Y_1 F_1 + Y_0 F_0 = dF$ has been used to prove the vanishing of the last summand. 

We come now to families. Let $B \subseteq \mathbb{C}$ be the unit disk and $R = \mathcal{O}_C (B)$ the ring of holomorphic functions on $B$. A family of plane curves of degree $d$ over $B$ is defined by $F (Y_0, Y_1, Y_2, T) \in R [Y_0, Y_1, Y_2]|_d$, a homogeneous polynomial of degree $d$ on $Y_0, Y_1, Y_2$, and where $T$ denotes the coordinate of $B$. The family is then defined by

$$
S = \{ ([a_0 : a_1 : a_2], b) \in \mathbb{P}^2 \times B \mid F (a_0, a_1, a_2, b) = 0 \} \subseteq \mathbb{P} \times B,
$$

with $\pi : S \to B$ given by the projection onto the second factor.

We assume moreover that for every $b \in B$ the polynomial $F (Y_0, Y_1, Y_2, b) \in \mathbb{C} [Y_0, Y_1, Y_2]$ defines a smooth curve $C_b$ in $\mathbb{P}^2 \times \{ b \}$, i.e

$$
\{ ([a_0 : a_1 : a_2], b) \in \mathbb{P}^2 \times B \mid F_0 (a_0, a_1, a_2, b) = F_1 (a_0, a_1, a_2, b) = F_2 (a_0, a_1, a_2, b) = 0 \} = \emptyset,
$$

20
In analogy with the previous notation, we set \( F_T = \frac{\partial F}{\partial t} \), \( F_{ij} = \frac{\partial F_i}{\partial y_j} \), \( F_{Tij} = \frac{\partial^2 F}{\partial y_i \partial y_j} \), \( F_{Tii} = \frac{\partial^2 F}{\partial y_i^2} \), and \( F_{TT} = \frac{\partial^2 F}{\partial t^2} \).

We extend the previous notation for the open subsets \( U_i = \{ ([a_0 : a_1 : a_2], b) \in \mathbb{P}^2 \times B \mid a_i \neq 0 \} \subseteq \mathbb{P}^2 \times B \) and \( V_{i/j} = \{ ([a_0 : a_1 : a_2], b) \in S \mid a_j \neq 0, F_k (a_0, a_1, a_2, b) \neq 0 \} \).

The implicit function theorem shows that

1. \( x_{i/j} := \frac{Y_i}{Y_j} \big|_{V_{i/j}} \) and \( t_{i/j} := T_{V_{i/j}} \) are local coordinates around every \( P \in V_{i/j} \), and the projection \( \pi : S \to B \) is given by \( t_{i/j} \).

2. \( T_{S|V_{i/j}} \) is generated by

\[
\frac{\partial}{\partial x_{i/j}} := \frac{\partial}{\partial y_j} - \frac{F_i}{F_k} \frac{\partial}{\partial Y_k} \big|_S = Y_j \left( \frac{\partial}{\partial Y_i} - \frac{F_i}{F_k} \frac{\partial}{\partial Y_k} \right) \big|_S, \tag{4.6}
\]

and

\[
\frac{\partial}{\partial t_{i/j}} := \frac{\partial}{\partial T} - \frac{F_T}{Y_j} \frac{\partial}{\partial Y_k} \big|_S = \left( \frac{\partial}{\partial T} - \frac{F_T}{F_k} \frac{\partial}{\partial Y_k} \right) \big|_S. \tag{4.7}
\]

3. For each \( b \in B \), \( T_{C_b|V_{i/j}} \) and \( N_{C_b|S|V_{i/j}} \) are generated by \( \frac{\partial}{\partial x_{i/j}} \big|_{C_b} \) and \( \frac{\partial}{\partial t_{i/j}} \big|_{C_b} \) respectively.

Considering again a partition of unity \( \{ \rho_{i/j} \}_{i \neq j} \) with \( \text{supp} \rho_{i/j} \subseteq V_{i/j} \), we can construct a \( C^\infty \) vector field \( \chi \) on \( S \) such that \( d\pi(\chi) = \frac{\partial}{\partial t} \) as

\[
\chi = \sum_{i \neq j} \rho_{i/j} \frac{\partial}{\partial t_{i/j}} = \sum_{i \neq j} \rho_{i/j} \left( \frac{\partial}{\partial T} - \frac{F_T}{Y_j F_k} \frac{\partial}{\partial Y_k} \right) \big|_S. \tag{4.8}
\]

In order to compute the local expressions of \( \mu \) in the \( V_{i/j} \) with this choice of \( \chi \), we first need to compute the local expressions of \( \chi \), i.e. the functions \( Z_{i/j} \) such that \( \chi|_{V_{i/j}} = Z_{i/j} \frac{\partial}{\partial x_{i/j}} + \frac{\partial}{\partial t_{i/j}} \). As in the case of one curve, we set \( \rho_{i,j} = \rho_{i/j} + \rho_{j/i} \) for \( i \neq j \).

**Lemma 4.2.** For any \( i \neq j \) it holds \( \chi|_{V_{i,j}} = Z_{i,j} \frac{\partial}{\partial x_{i,j}} + \frac{\partial}{\partial t_{i,j}} \) with

\[
Z_{i,j} = \frac{F_T}{Y_j} \left( \frac{\rho_{k,j}}{F_i} - \frac{Y_i}{Y_j} \frac{\rho_{i,k}}{F_j} \right). \tag{4.9}
\]

**Proof.** As in the proof of Lemma 4.1 for the sake of simplicity we show the case \( i = 0, j = 2 \).

From (4.8) we have

\[
\chi = \sum_{i \neq j} \rho_{i/j} \frac{\partial}{\partial t_{i/j}} = \sum_{i \neq j} \rho_{i/j} \left( \frac{\partial}{\partial T} - \frac{F_T}{Y_j F_k} \frac{\partial}{\partial Y_k} \right) \big|_S = \frac{\partial}{\partial T} \big|_S F_T \left( \frac{\rho_{1,2}}{F_0} \frac{\partial}{\partial Y_0} + \frac{\rho_{0,2}}{F_1} \frac{\partial}{\partial Y_1} + \frac{\rho_{0,1}}{F_2} \frac{\partial}{\partial Y_2} \right) \big|_S.
\]
Using the Euler relation $\frac{\partial}{\partial y_2} = \frac{-1}{2} \left( Y_0 \frac{\partial}{\partial y_0} + Y_1 \frac{\partial}{\partial y_1} \right)$ and grouping conveniently, we obtain

$$
\chi = \frac{\partial}{\partial T} |_S - F_T \left[ \left( \frac{\rho_{1,2}}{F_0} - \frac{Y_0 \rho_{0,1}}{Y_2 F_2} \frac{\partial}{\partial Y_0} \right) \frac{\partial}{\partial y_0} + \left( \frac{\rho_{0,2}}{F_1} - \frac{Y_1 \rho_{0,1}}{Y_2 F_2} \frac{\partial}{\partial Y_2} \right) \frac{\partial}{\partial y_1} \right] |_S
$$

$$
= \left( \frac{\partial}{\partial T} - \frac{F_T}{F_1} \frac{\partial}{\partial y_1} \right) |_S - F_T \left( \frac{\rho_{1,2}}{F_0} - \frac{Y_0 \rho_{0,1}}{Y_2 F_2} \frac{\partial}{\partial Y_0} \right) \frac{\partial}{\partial y_0} + F_T \left[ \frac{1}{F_1} - \frac{Y_0}{F_0} \left( \frac{\rho_{1,2}}{F_0} - \frac{Y_0 \rho_{0,1}}{Y_2 F_2} \right) - \left( \frac{\rho_{0,2}}{F_1} - \frac{Y_1 \rho_{0,1}}{Y_2 F_2} \right) \frac{\partial}{\partial y_1} \right] |_S
$$

Thus it just remains to show that the last summand vanishes, which follows again using the identities $1 - \rho_{1,2} - \rho_{0,2} = \rho_{0,1}$ and $Y_0 F_0 + Y_1 F_1 + Y_2 F_2 = dF$, since

$$
\left[ \frac{1}{F_1} - \frac{Y_0}{F_0} \left( \frac{\rho_{1,2}}{F_0} - \frac{Y_0 \rho_{0,1}}{Y_2 F_2} \right) - \left( \frac{\rho_{0,2}}{F_1} - \frac{Y_1 \rho_{0,1}}{Y_2 F_2} \right) \right] |_S = \left[ \frac{1 - \rho_{1,2} - \rho_{0,2}}{F_1} + \frac{Y_0 \rho_{0,1}}{Y_2 F_2} + \frac{Y_1 \rho_{0,1}}{Y_2 F_2} \right] |_S
$$

$$
= \rho_{0,1} \frac{Y_2 F_2 + Y_0 F_0 + Y_1 F_1}{Y_2 F_1 F_2} |_S = 0
$$

\[\blacksquare\]

**Remark 4.3.** Note that at any $b \in B$ the Kodaira-Spencer class $KS_b \left( \frac{\partial}{\partial T} \right)$ is represented by $\overline{\partial} \chi|_{C_b}$, which on $V_{i,j}$ is given by

$$
\overline{\partial} Z_{i,j} \frac{\partial}{\partial x_{i,j}} |_{C_b} = F_T \left( \frac{\overline{\partial} \rho_{k,j}}{Y_j} \left( \frac{\overline{\partial} \rho_{i,k}}{Y_i} - \frac{Y_i}{Y_j} \overline{\partial} \rho_{i,k} \right) \frac{\partial}{\partial x_{i,j}} \right) |_{C_b}.
$$

Lemma 4.1 shows then that $KS_b \left( \frac{\partial}{\partial T} \right) = \delta F_T|_{C_b}$ is given by the first derivative of the defining equation on the parameter $T$, as expected.

We are now ready to compute the local expression of $\mu_b$ for any $b \in B$.

**Theorem 4.4.** For any $b \in B$, the form $\mu_b \in A^{0,1} (T_{C_b})$ is represented on $V_{0/2}$ by an expression

$$
\left( \frac{F_{TT}}{Y_2} \left[ \frac{\overline{\partial} \rho_{1,2}}{F_0} - \frac{Y_0 \overline{\partial} \rho_{0,1}}{Y_2 F_2} \right] + A \right) \frac{\partial}{\partial x_{0/2}}, \tag{4.10}
$$

where the function $A$ depends only on first and second derivatives of $F$ different from $F_{TT}$.

**Proof.** Expand the expression (3.19) with $Z$ as in (4.9) with $i = 0$ and $j = 2$. \[\blacksquare\]

Thus the second-order Kodaira-Spencer class depends explicitly on $\delta F_{TT}|_{C_b}$, and other terms involving the first-order Kodaira-Spencer class and other derivatives of the defining equation. In any case, this direct dependancy on $F_{TT}$ shows that $\mu_b$ is in general independent of the first-order Kodaira-Spencer class, hence the condition obtained in Theorem 3.10 is non-trivial.
References

[BGAN18] Miguel Ángel Barja, Víctor González-Alonso, and Juan Carlos Naranjo. Xiao’s conjecture for general fibred surfaces. *J. Reine Angew. Math.*, 739:297–308, 2018.

[CD17] Fabrizio Catanese and Michael Dettweiler. Answer to a question by Fujita on Variation of Hodge Structures. *Adv. Stud. in Pure Math.*, 74-04(04):73–102, 2017.

[CK19] Fabrizio Catanese and Yujiro Kawamata. Fujita decomposition over higher dimensional base. *Eur. J. Math.*, 5(3):720–728, 2019.

[CLZ16] Ke Chen, Xin Lu, and Kang Zuo. On the Oort conjecture for Shimura varieties of unitary and orthogonal types. *Compos. Math.*, 152(5):889–917, 2016.

[CLZ18] Ke Chen, Xin Lu, and Kang Zuo. The Oort conjecture for Shimura curves of small unitary rank. *Communications in Mathematics and Statistics*, 6(3):249–268, 2018.

[Fuj78] Takao Fujita. The sheaf of relative canonical forms of a Kähler fiber space over a curve. *Proc. Japan Acad. Ser. A Math. Sci.*, 54(7):183–184, 1978.

[GA16] Víctor González-Alonso. On deformations of curves supported on rigid divisors. *Ann. Mat. Pura Appl. (4)*, 195(1):111–132, 2016.

[GAST19] Víctor González-Alonso, Lidia Stoppino, and Sara Torelli. On the rank of the flat unitary summand of the Hodge bundle. *Trans. Amer. Math. Soc.*, 372(12):8663–8677, 2019.

[GAT20] Víctor González-Alonso and Sara Torelli. Families of curves with Higgs field of arbitrarily large kernel. appeared online in *bull. london math. soc.*, 2020.

[GPT19] Alessandro Ghigi, Gian Pietro Pirola, and Sara Torelli. Totally geodesic subvarieties in the moduli space of curves. appeared online in *commun. contemp. math.*, 2019.

[LZ17] Xin Lu and Kang Zuo. On the slope of hyperelliptic fibrations with positive relative irregularity. *Trans. Amer. Math. Soc.*, 369(2):909–934, 2017.

[LZ19] Xin Lu and Kang Zuo. The Oort conjecture on Shimura curves in the Torelli locus of curves. *J. Math. Pures Appl. (9)*, 123:41–77, 2019.

[PT20] Gian Pietro Pirola and Sara Torelli. Massey products and Fujita decompositions on fibrations of curves. *Collect. Math.*, 71(1):39–61, 2020.

[Voi07] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.