Instability of solitary waves of the generalized higher-order KP equation

Amin Esfahani

School of Mathematics and Computer Science, Damghan University, Damghan, Postal Code 36716-41167, Iran
E-mail: amin@impa.br and esfahani@du.ac.ir

Received 29 March 2010, in final form 27 December 2010
Published 3 February 2011
Online at stacks.iop.org/Non/24/833

Recommended by A I Neishtadt

Abstract
This paper deals with the generalized higher-order Kadomtsev–Petviashvili (KP) equation. The strong instability of solitary wave solutions of this equation will be proved.

Mathematics Subject Classification: 35B35, 35Q51, 35A15, 35B40

1. Introduction

This paper is concerned with the generalized Kadomtsev–Petviashvili (GKP) equation with higher-order dispersion in the main direction of propagation

\[
\begin{align*}
    u_t + \alpha u_{xxx} + \beta u_{xxxxx} + u^p u_x &= v_y, \\
    v_x &= u_y, \\
\end{align*}
\]

(1.1)

where \((x, y) \in \mathbb{R}^2, t \geq 0\), the constants \(\beta\) and \(\alpha\) are \(\pm 1\) and \(u(x, y, t)\) is a real valued function. Such an equation occurs naturally in the modelling of certain long dispersive waves (cf [1, 10, 11]). The ‘usual’ GKP equation (actually so-called ‘GKP-I’) corresponds to \(\beta = 0\), \(p > 0\) and \(\alpha = 1\); which is, in turn, a two-dimensional version of the generalized KdV (GKdV) equation. To some extent, equation (1.1) can also be considered as a two-dimensional extension of the generalized Kawahara equation (cf [13, 22])

\[ u_t + \alpha u_{xxx} + \beta u_{xxxxx} + u^p u_x = 0. \]

Mathematical interests are concentrated on the well-posedness for the initial value problem associated with (1.1) and the existence of the solitary wave solutions of (1.1). There are many rigorous results that have recently appeared concerning the problems of local existence for equation (1.1); see, for example, [5, 9, 14, 20, 21].
The issue of existence (and nonexistence) of the solitary waves of (1.1) was investigated by de Bouard and Saut [6,7]; however, the solitary waves of (1.1) were observed numerically, for \( p = 1 \), in [1,10,11]. We recall that a solitary wave solution of (1.1) is a localized solution of (1.1) of the form \( u(x - ct, y) \), where \( c > 0 \) is the speed of wave propagation; alternatively, it is a solution \( u \) of the equation

\[
-c u_x + \alpha u_{xxx} + \beta u_{xxxxx} + u^p u_x = v_y, \quad v_x = u_y.
\]  

(1.2)

The authors in [6] showed that the GKP equation, for \( 0 < p < 4 \), possesses a nontrivial solitary wave; and then they established the existence of the solitary wave solution of (1.2), for \( p > 0 \), \( \alpha > 0 \) and \( \beta < 0 \). Moreover, the regularity, symmetry and decay estimates of solitary waves for the GKP equation and equation (1.1) were studied in [6,7]. In [8], the authors also proved that the ground states of the GKP equation are nonlinearly unstable for \( 4/3 < p < 4 \), using a qualitative and variational method developed by Bona et al [3]. In particular, they used essentially the property of invariance of the problem (when \( \beta = 0 \)) under scaling \( c^{1/p} u(\sqrt{c} x, cy) \) [8]. Liu and Wang [18] used a method developed by Cazenave and Lions (cf [4]) and showed that the set of minimizers for a variational problem associated with the Euler–Lagrange equation and satisfied by a solitary wave of the GKP equation is not empty, for \( 0 < p < 4 \), and it is stable when \( 0 < p < 4/3 \). In [16,17], Liu proved that the solitary waves of the GKP equation are strongly unstable, when \( 2 < p < 4 \). In fact he showed that the solution of the GKP equation, for \( 2 < p < 4 \), blows up in finite time in some sense, provided it begins near the solitary wave solution of the GKP equation. The main ingredients used in [16,17] are based on solving several complicated minimization problems, constructing several invariant sets and using a virial-type identity.

Our main aim in this paper is to investigate the instability property of solitary wave solutions for (1.1). But in contrast to the GKP equation, equation (1.2) (and also (1.1)) does not possess any scaling invariance; so that the techniques used in [6,7,16–18] seem not to be applied. To overcome this difficulty, we will use several minimization problems with multiple constraints to construct some sets that are conserved under the flow generated by (1.1). When \( p \geq 4 \), we shall also apply the techniques used in [16,17] and show that solitary waves for (1.1) are unstable by the mechanism of blow-up. Unfortunately, we do not know of a stability/instability result of solitary waves of (1.1), when \( p < 4 \).

This paper is organized as follows. In section 2, we shall prove the existence of solitary waves for (1.1) by a different method from that used in [6]. Indeed we will apply a mountain-pass argument to prove the existence of the solitary waves. We also show that our solitary waves (of (1.2)) are the ground states, i.e. they have minimal action. Moreover, we will also give some variational properties of these solutions. In section 3, we use several minimization problems with multiple constraints to construct some invariant sets under the flow of (1.1). Section 4 is devoted to prove that the solutions of (1.1) with initial data in these invariant sets must blow up in a finite time. We shall also show the solitary waves of (1.1) are strongly unstable in the sense of definition 4.1.

We end this section by introducing some notations that will be used throughout this paper.

1.1. Notations

We shall denote by \( \hat{\varphi} \) the Fourier transform of \( \varphi \), defined as

\[
\hat{\varphi}(\xi) = \int_{\mathbb{R}^2} \varphi(\omega)e^{-i\omega \cdot \xi} \, d\omega.
\]
For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^2)$ the nonhomogeneous Sobolev space defined by
\[ H^s(\mathbb{R}^2) = \{ \varphi \in \mathcal{S}'(\mathbb{R}^2) : \| \varphi \|_{H^s(\mathbb{R}^2)} < \infty \}, \]
where
\[ \| \varphi \|_{H^s(\mathbb{R}^2)} = \left( \int (1 + |\xi|^2)^s \hat{\varphi}(\xi)^2 d\xi \right)^{1/2}, \]
and $\mathcal{S}'(\mathbb{R}^2)$ is the space of tempered distributions.

Let $\mathcal{X}$ be the closure of $\mathcal{C}_c^\infty(\mathbb{R}^2)$ with the norm
\[ \| u \|_{\mathcal{X}} = \| u \|_{L^2(\mathbb{R}^2)} = c \| \nabla u \|_{L^2(\mathbb{R}^2)} + \| u \|_{L^2(\mathbb{R}^2)}, \]
for $u \in \mathcal{X}$ and $u = \psi_x$, where $\psi \in L_{loc}^2(\mathbb{R}^2)$, $\forall 2 \leq q < \infty$; we also have $v = \psi_y \in L^2(\mathbb{R}^2)$ by a choice of $\psi \in L_{loc}^2(\mathbb{R}^2)$; so that
\[ \| u \|_{\mathcal{X}} = c \| \nabla u \|_{L^2(\mathbb{R}^2)} + \| u \|_{L^2(\mathbb{R}^2)} + \| u_x \|_{L^2(\mathbb{R}^2)} + \| u_{xx} \|_{L^2(\mathbb{R}^2)}. \]

Let
\[ X_s = \left\{ u \in H^s(\mathbb{R}^2) : (\xi^{-1} \hat{u}(\xi, \eta))^\vee \in H^s(\mathbb{R}^2) \right\}, \]
with the norm
\[ \| u \|_{X_s} = \| u \|_{H^s(\mathbb{R}^2)} + \left\| (\xi^{-1} \hat{u}(\xi, \eta))^\vee \right\|_{H^s(\mathbb{R}^2)}, \]
where ‘$\vee$’ is the Fourier inverse transform.

For any positive numbers $a$ and $b$, the notation $a \lesssim b$ means that there exists a positive constant $C$ such that $a \leq Cb$.

Throughout this paper, we consider the case $\beta = -\alpha = -1$, $c > 0$ and $p = k/m$ where $m$ is any odd integer and $k$ is any even integer so that $\int_{\mathbb{R}^2} u_{p+2} \, dx \, dy = \| u \|_{L^{p+2}(\mathbb{R}^2)}^p$.

## 2. Ground states and variational characterizations

In this section, we give a brief proof of the existence of localized solitary waves of equation (1.1); i.e. the nontrivial solutions of (1.2). We will also give some variational properties of these solutions.

First we state the result of the existence of solitary waves of (1.2).

**Theorem 2.1.** Let $p > 0$ be arbitrary. Then (1.2) admits a nontrivial solitary wave solution $u \in \mathcal{X}$.

**Sketch of the proof.** We adapt to the arguments in [25]. We consider the minimax value
\[ d = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} S(\gamma(t)), \]
where
\[ S(u) = E(u) + cF(u), \quad \Gamma = \{ \gamma \in C([0, 1], \mathcal{X}) : \gamma(0) = 0, \; S(\gamma(1)) < 0 \}, \]
\[ E(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( u_x^2 + u_{xx}^2 + v^2 \right) \, dx \, dy - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^2} u^{p+2} \, dx \, dy, \]
\[ F(u) = \frac{1}{2} \int_{\mathbb{R}^2} u^2 \, dx \, dy, \]
and $v$ is defined as in (1.2); and then use the mountain-pass lemma without the Palais–Smale condition (cf [2]) by employing the embedding (see [6])
\[ \mathcal{X} \hookrightarrow L^q(\mathbb{R}^2), \quad 2 \leq q < \infty; \]
and the following compact embedding:
\[ \mathcal{X} \hookrightarrow L^q(\mathbb{R}^2), \quad 2 \leq q < \infty. \] (2.5)

**Remark 2.2.** Similar to [6], using the Pohozaev-type identities, one can prove that no solitary wave of (1.1) can exist if one of the following cases occurs:

(i) \( \beta \alpha = 1 \) and \( (p - 4)\alpha > 0 \),
(ii) \( \beta \alpha = 1, (4 - p)\alpha > 0 \), when \( c \) is sufficiently large,
(iii) \( \beta = -\alpha = 1 \).

**Remark 2.3.** It is noteworthy that the quantities \( E \) and \( F \) are formally conserved by solutions of (1.1), i.e.
\[ E(u(t)) = E(u(0)) \quad \text{and} \quad F(u(t)) = F(u(0)). \]

Next, we are going to obtain some variational properties of our solitary waves which are important in our instability analysis. Let \( I(u) := \langle S'(u), u \rangle \),
\[ d' := \inf \sup_{u \in \mathcal{X}, t > 0} S(tu), \] (2.6)
where
\[ \mathcal{X} = \left\{ u \in \mathcal{X}, G(u) := \int_{\mathbb{R}^2} u^{p+2} \, dx \, dy > 0 \right\}; \] (2.7)
and
\[ m := \inf\{S(u); \, u \in \mathcal{N}\}, \] (2.8)
where
\[ \mathcal{N} = \{ u \in \mathcal{X}; \, u \neq 0, \, I(u) = 0 \}. \] (2.9)

**Lemma 2.4.** There is a nontrivial minimizer for (2.6) and (2.8) which is a solitary wave solution of (1.2). Furthermore \( d' = m = d \).

**Remark 2.5.** It is worth noting that for every \( u \in \mathcal{X} \), there exists a unique \( t = t_u \) such that \( tu \in \mathcal{N} \),
\[ S(tu) = \max_{t > 0} S(\tau u) \]
and \( t \) depends continuously only on \( u \in \mathcal{X} \). In fact, it is easy to see that the function
\[ \frac{d}{dt} S(\tau u) = I(\tau u) = \tau^2 \left( \|u\|_{\mathcal{X}}^2 - \frac{\tau^p}{p+1} \int_{\mathbb{R}^2} u^{p+2} \, dx \, dy \right) \]
vanishes at only one point \( t = t_u > 0 \). Thus \( S \) is positive on \( \mathcal{N} \). Since \( S(0) = 0 \), we see that \( t \) is a point of maximum for \( S(\tau u) \).

To prove lemma 2.4, we need the following helpful lemma.

**Lemma 2.6.** Let \( \{u_k\}_{k \in \mathbb{N}} \) be a bounded sequence in \( \mathcal{X} \). Suppose that there exists \( r > 0 \) such that
\[ \sup_{\zeta \in \mathbb{R}^2} \|u_k\|_{L^2(B_r(\zeta))} \to 0, \] (2.10)
as \( k \to \infty \), where \( B_r(\zeta) \) is the ball of radius \( r \) centred at \( \zeta \in \mathbb{R}^2 \). Then \( u_k \to 0 \) in \( L^p(\mathbb{R}^2) \) for \( 2 < p < \infty \).

**Proof.** Let \( 2 < p < \infty \). By the Hölder inequality and (2.4), there holds
\[ \|u\|_{L^p(B_r(\zeta))} \lesssim \|u\|_{L^2(B_r(\zeta))} \|u\|_{L^p(B_r(\zeta))} \lesssim \|u\|_{L^2(B_r(\zeta))}. \]
Now we cover \( \mathbb{R}^2 \) by balls of radius \( r \) in such a way that each point of \( \mathbb{R}^2 \) is contained in at most 3 balls, then we obtain
\[
\|u\|_{L^p(B_r(\zeta))} \lesssim \sup_{\zeta \in \mathbb{R}^2} \|u\|_{L^p(\mathbb{R}^2)}.
\] (2.11)

Hence we conclude that \( u_k \to 0 \) in \( L^p(\mathbb{R}^2) \), for \( 2 < p < \infty \), by combining (2.10) and (2.11).

The proof of lemma 2.4 follows.

**Proof of lemma 2.4.** First, we prove that there is a nontrivial minimizer for (2.8) which is a solution of (1.2).

By (2.4), one can easily observe that there exist \( \epsilon_1, \epsilon_2 > 0 \) such that for every nontrivial function \( u \in \mathcal{N} \), we have \( \|u\|_\mathcal{X} \geq \epsilon_1 \) and \( S(u) \geq \epsilon_2 \).

Now let \( \{u_k\} \subset \mathcal{N} \) be a minimizing sequence of (2.8). Then we obtain \( \|u_k\|_\mathcal{X} \geq \epsilon_1 \) and
\[
S(u_k) = \frac{p}{2(p+1)(p+2)} \|u_k\|_\mathcal{X}^2.
\]

So that \( \{u_k\} \) is bounded in \( \mathcal{X} \). To show that there is a convergent subsequence, with a limit \( u \in \mathcal{X} \), similar to [6], we use the concentration-compactness lemma [15], applied to the sequence
\[
\rho_k = cu_k^2 + (u_k)_x^2 + (u_k)_y^2 + v_k^2.
\]

First, using lemma 2.6, the evanescence case is excluded. To rule out the dichotomy case, one shows that
\[
m < m_\sigma := \inf \left\{ S(u) - \frac{1}{2} I(u); \ I(u) = \sigma \right\},
\]
for all \( \sigma < 0 \). Now if the dichotomy would occur, i.e. \( u_k \) splits into a sum \( u_k^1 + u_k^2 \) and the distance of the supports of these functions tends to +\( \infty \), then one shows that \( I(u_k^1) \to \sigma \), \( I(u_k^2) \to -\sigma \), \( \sigma \in \mathbb{R} \) and \( m \geq m_\sigma + m_{-\sigma} > m \) which is a contradiction. Therefore, the sequence \( u_k \) concentrates and the limit \( u \) satisfies \( I(u) \leq 0 \). The case \( I(u) < 0 \) can be excluded by the same reason as above; and we obtain that \( u \in \mathcal{X} \) is a minimizer for (2.8).

Now since \( u \) is a minimizer for (2.8), there exists a Lagrange multiplier \( \theta \) such that \( S'(u) = \theta I(u) \). Since \( S'(u), u \) = 0 and
\[
\langle I'(u), u \rangle = \|u\|_\mathcal{X}^2 - \frac{1}{p+1} \int_{\mathbb{R}^2} u^{p+2} dx \, dy \leq -C\|u\|_\mathcal{X}^2 < 0,
\]
we claim that \( \theta = 0 \), i.e. \( u \) is a solution of (1.2). Now we prove the identity \( d = m = d' \).

First, it is easily seen that if \( u \in \mathcal{N} \), then \( I(u) = 0 \) and \( u \in \mathcal{X} \). On the other hand, if \( u \in \mathcal{X} \), then \( tu \in \mathcal{X} \), for any \( t > 0 \). Hence by remark 2.5 and the definitions of \( m \) and \( d' \), we see that \( m = d' \).

Now we show that \( d \geq m \). Since \( u^{p+2} \) is superquadratic at zero, we see that \( I(u) > 0 \) in a neighbourhood of the origin, except of zero. Hence \( I(\gamma(t)) > 0 \), for \( \gamma \in \Gamma \) and for some \( t > 0 \). Now for \( u \in \mathcal{X} \), we have
\[
2S(u) = \|u\|_\mathcal{X}^2 - \frac{2}{(p+1)(p+2)} G(u) > I(u).
\]

Hence \( I(\gamma(t)) < 0 \), therefore \( \gamma(t) \) crosses \( \mathcal{N} \) and this implies that \( d \geq m \).

Finally we prove that \( d' \geq d \). Let \( u \in \mathcal{X} \), then we have \( (tu)^{p+2} \geq Ct^\mu \), for some \( C, \mu > 0 \), if \( t > 0 \) is large enough. This implies that \( S(tu) < 0 \) for every \( u \in \mathcal{X} \) for sufficiently large \( t > 0 \). Hence the half-axis \( \{tu; \ t > 0\} \) generates, in a natural way, an element of \( \Gamma \). This leads us to the inequality \( d \leq d' \) \( \square \).
Remark 2.7. By lemma 2.4, one can observe that \( \mathcal{N} \neq \emptyset \) and \( \mathcal{N} \) is a manifold. Moreover, one can easily see that there exists \( C > 0 \) such that for any \( u \in \mathcal{N} \), \( \|u\|_\mathcal{N} \geq C > 0 \). Furthermore, for any \( u \in \mathcal{X} \) with \( I(u) < 0 \), it is deduced from remark 2.5 that there exists a unique \( t \in (0, 1) \) such that \( tu \in \mathcal{N} \). More precisely,

\[
t = \left( \frac{(p + 1)I(u)}{G(u)} + 1 \right)^{1/p}.
\]

The following theorem characterizes the minimax value \( m \). Let

\[
\mathcal{D}(u) := \frac{p}{2(p + 1)(p + 2)} G(u).
\]

Theorem 2.8. Let \( u \in \mathcal{X}, u \neq 0 \). Then the following statements are equivalent:

(i) \( u \) is a ground state of (1.2),

(ii) \( I(u) = 0 \) and \( \mathcal{D}(u) = m = \inf \{ \mathcal{D}(w); w \in \mathcal{N} \} \),

(iii) \( \mathcal{D}(u) = m \) and \( I(u) = 0 = \inf \{ I(w); w \in \mathcal{X}, \mathcal{D}(w) = m \} \),

(iv) \( m = S(u) = \inf \{ S(w); w \in \mathcal{N} \} \) and \( I(u) = 0 \).

Proof. By definition, it is obvious to see that (i) \( \Leftrightarrow \) (iv).

Implication (ii) \( \Rightarrow \) (i) is proved in lemma 2.4.

(ii) \( \Rightarrow \) (iii): if \( u \) satisfies (ii), \( I(u) = 0 \). Assume that there exists \( w \in \mathcal{X} \) such that \( \mathcal{D}(w) = m \) and \( I(w) < 0 \). Then \( \mathcal{D}(w) > 0 \) and there exists \( t_0 \in (0, 1) \) such that \( I(t_0 w) = 0 \). This contradicts \( \mathcal{D}(t_0 w) = \mathcal{D}(w) = m \).

(iii) \( \Rightarrow \) (ii): let \( u \in \mathcal{X} \) satisfying (iii). Then \( \mathcal{D}(u) = m \) and \( I(u) = 0 \). Assume that there exists \( w \in \mathcal{X} \) such that \( I(w) = 0 \) and \( \mathcal{D}(w) < m \). Thus there exists \( t_0 > 1 \) such that \( \mathcal{D}(t_0 w) = m \). Now we have \( I(t_0 w) < 0 \); and this contradicts (iii).

(iv) \( \Rightarrow \) (ii): assume that (iv) holds. Then there is a Lagrange multiplier \( \theta \) such that \( S'(u) = \theta I'(u) \). Similar to lemma 2.4, we obtain that \( \theta = 0 \); so that \( S'(u) = 0 \). Now if \( w \in \mathcal{X} \), with \( w \neq 0 \), is a solution of (1.2), then \( w \in \mathcal{N} \) and \( S'(w) = 0 \). Therefore, the definition of \( m \) implies that \( S(u) \leq S(w) \).

Finally we state another characterization of the minimax value \( m \).

Theorem 2.9. Let \( s \geq 5 \) and \( \mathcal{L} = \{ u \in X_s; u \neq 0, I(u) = 0 \} \). Then

\[
\inf_{u \in \mathcal{L}} S(u) = m.
\]

Proof. Let \( \ell = \inf_{u \in \mathcal{L}} S(u) \). Then clearly, \( \ell \geq m \). To see \( \ell \leq m \), it suffices to prove that

\[
S(u^*) \geq \inf_{u \in \mathcal{L}} S(u) - \epsilon,
\]

for any \( \epsilon > 0 \) and any \( u^* \in \mathcal{N} \). Since \( X_s \) is dense in \( \mathcal{X} \), we can find a sequence \( \{ u_n \} \subset X_s \) such that

\[
u^* = u_n + w_n
\]

with \( w_n \to 0 \) in \( \mathcal{X} \) as \( n \to \infty \). Since \( u^* \neq 0, I(u^*) = 0 \) and \( w_n \to 0 \) in \( \mathcal{X} \), we know that

\[
\lim_{n \to \infty} G(u_n) \neq 0 \quad \text{and} \quad \lim_{n \to \infty} I(u_n) = 0.
\]

Similar to remark 2.7, it is deduced that there exists a sequence \( \{ \theta_n \} \subset \mathbb{R} \) such that \( \theta_n u_n \in \mathcal{L} \) and \( \lim_{n \to \infty} \theta_n = 1 \).

Now by writing \( u^* = \theta_n u_n + (1 - \theta_n)u_0 + w_n \), comparing \( S(u^*) \) with \( S(\theta_n u_n) \) and using the facts \( 1 - \theta_n \to 0 \) and \( w_n \to 0 \) (strongly) in \( \mathcal{X} \), as \( n \to \infty \), we deduce

\[
\|u^*\|^2_\mathcal{X} \geq \|\theta_n u_n\|^2_\mathcal{X} - \frac{2\epsilon}{3}.
\]
and
\[ G(u^*) \leq G(\partial_n u_n) + \frac{\epsilon (p + 1)(p + 2)}{3}. \]

Therefore,
\[ S(u^*) \geq S(\partial_n u_n) - \epsilon \geq \inf_{u \in L} S(u) - \epsilon. \]

This completes the proof. \(\square\)

3. Invariant sets

In this section, we will use the minimization problems defined in section 2 to construct several sets that are invariant under the flow generated by (1.1). First we state the following local existence and uniqueness theorem that can be obtained along the lines of arguments of [19, theorems 3.1 and 3.3], [23, theorem 2.1] and [24, theorem 1.1]. This result is sufficient for our purposes. It is noteworthy that one may also prove the following theorem by applying Kato’s theory [12].

**Theorem 3.1.** Let \( u_0 \in X_s, s \geq 5 \). Then there exists \( T > 0 \) such that (1.1) has a unique solution \( u(t) \) with \( u(0) = u_0 \) satisfying
\[ u \in C([0, T); X_s) \cap C^1([0, T); H^{s-5}(\mathbb{R}^2)), \quad \partial_x^{-1} u_y \in C([0, T); H^{s-1}(\mathbb{R}^2)). \]

and if \( \partial_x^{-2}(u_0)_{yy} \in L^2(\mathbb{R}^2), \) one has \( u_t \in L^\infty((0, T); X_0), \) \( \partial_x^{-1} u_{yt} \in L^\infty((0, T); H^{-1}(\mathbb{R}^2)). \)

Moreover \( |y| u \in L^\infty((0, T); L^2(\mathbb{R}^2)), \) if \( |y| u_0 \in L^2(\mathbb{R}^2). \) Furthermore, we have \( F(u(t)) = F(u_0) \) and \( E(u(t)) = E(u_0). \)

Next we define the sets
\[ \mathcal{R} = \{ u \in X_s; u \neq 0, I(u) < 0, \mathcal{R}(u) = 0 \} \]
and
\[ \mathcal{Q} = \{ u \in X_s; u \neq 0, I(u) < 0, \mathcal{Q}(u) = 0 \}, \]
where
\[ \mathcal{R}(u) = \int_{\mathbb{R}^2} \left( \frac{1}{2} u_x^2 + u_x^2 \right) dx \ dy - \frac{p}{2(p + 1)(p + 2)} \int_{\mathbb{R}^2} u^{p+2} dx \ dy \] (3.1)
and
\[ \mathcal{Q}(u) = \int_{\mathbb{R}^2} \left( \frac{1}{2} u_x^2 + u_x^2 + v^2 \right) dx \ dy - \frac{p}{(p + 1)(p + 2)} \int_{\mathbb{R}^2} u^{p+2} dx \ dy. \] (3.2)

The following lemma gives a relation between \( m \) and above minimax values.

**Lemma 3.2.** If \( p \geq 4 \), then \( d_{\mathcal{R}} \geq m \) and \( d_{\mathcal{Q}} \geq m \), where
\[ d_{\mathcal{R}} = \inf_{u \in \mathcal{R}} S(u) \quad \text{and} \quad d_{\mathcal{Q}} = \inf_{u \in \mathcal{Q}} S(u). \]

**Proof.** First we prove the first part of the lemma. For any \( u \in \mathcal{R} \), by theorem 2.9, we want to find \( w \in \mathcal{Q} \) such that \( S(u) \geq S(w) \). We note that for any \( u \in \mathcal{R} \), \( I(u) < 0 \) and \( \mathcal{R}(u) = 0 \). For any \( \lambda > 0 \), denote \( u_\lambda(x, y) = \lambda u(\lambda x, \lambda y) \). A straightforward calculation reveals that
\[ I(u_\lambda) = \int_{\mathbb{R}^2} \left( c u^2 + \lambda^2 u_x^2 + \lambda^4 u_{xx}^2 + v^2 \right) dx \ dy - \frac{\lambda^p}{p + 1} \int_{\mathbb{R}^2} u^{p+2} dx \ dy. \]
Thus it is easy to see that
\[ I(u_\lambda) \rightarrow \int_{\mathbb{R}^2} (cu^2 + v^2) \, dx \, dy > 0, \]
as \( \lambda \rightarrow 0 \) and
\[ I(u_\mu) \rightarrow I(u) < 0, \]
as \( \lambda \rightarrow 1 \); so that we can find \( \mu \in (0, 1) \) such that \( I(u_\mu) = 0 \), i.e. \( u_\mu \in \mathcal{Z} \). Therefore \( S(u_\mu) \geq m \). Now we show that \( S(u) \geq S(u_\mu) \), for any \( \lambda \in (0, 1) \). Indeed,
\[ S(u_\lambda) = \frac{1}{2} \int_{\mathbb{R}^2} \left( cu^2 + \lambda^2 u^4 + \lambda^4 u_{xx}^2 + v^2 \right) \, dx \, dy - \frac{\l(\lambda)}{(p+1)(p+2)} \int_{\mathbb{R}^2} u^{p+2} \, dx \, dy \]
yields that
\[ S(u) - S(u_\lambda) = \int_{\mathbb{R}^2} \left[ (1 - \lambda^2) u^2 + \frac{1}{2} \lambda^4 u_{xx}^2 \right] \, dx \, dy - \frac{1 - \lambda^2}{(p+1)(p+2)} \int_{\mathbb{R}^2} u^{p+2} \, dx \, dy. \]
Since \( R(u) = 0 \), we have
\[ S(u) - S(u_\lambda) = \int_{\mathbb{R}^2} \left[ g_1(\lambda) u^2 + g_2(\lambda) u_{xx}^2 \right] \, dx \, dy, \]
where
\[ g_1(\lambda) = \frac{1}{2} (1 - \lambda^2) - \frac{1}{p} (1 - \lambda^4), \quad \text{and} \quad g_2(\lambda) = \frac{1}{2} (1 - \lambda^4) - \frac{1}{p} (1 - \lambda^2). \]
Note that \( g_1(\lambda) \rightarrow \frac{1}{2} - 1/p \geq 0 \) as \( \lambda \rightarrow 0 \), \( g_1(\lambda) \rightarrow 0 \) as \( \lambda \rightarrow 1 \) and \( g_1(\lambda) < 0 \) for \( \lambda \in (0, 1) \); thus \( g_1(\lambda) \geq 0 \) for \( \lambda \in (0, 1) \). The same argument proves that \( g_2(\lambda) \geq 0 \) for \( \lambda \in (0, 1) \). Thus \( S(u) \geq S(u_\lambda) \) for any \( \lambda \in (0, 1) \). In particular, \( S(u) \geq S(u_\mu) \). By setting \( w = u_\mu \), we conclude the proof of first part of the lemma.

The proof of the second part of the lemma is similar. Let \( u \in \mathcal{Z} \). Then \( I(u) < 0 \) and \( Q(u) = 0 \). Now denote
\[ u_\lambda(x, y) = \lambda^2 u(\lambda x, \lambda^3 y), \]
then
\[ I(u_\lambda) = \int_{\mathbb{R}^2} \left[ cu^2 + \lambda^2 u^2 + \lambda^4 u_{xx}^2 + \lambda^4 v^2 \right] \, dx \, dy = \frac{\lambda^{2p}}{p+1} \int_{\mathbb{R}^2} u^{p+2} \, dx \, dy. \]
It is readily seen that
\[ I(u_\lambda) \rightarrow \|u\|_{L^2(\mathbb{R}^2)}^2 > 0, \]
as \( \lambda \rightarrow 0 \), and \( I(u_\mu) \rightarrow I(u) < 0 \), as \( \lambda \rightarrow 1 \). Therefore, we can find \( \mu \in (0, 1) \) such that \( I(u_\mu) = 0 \), i.e. \( u_\mu \in \mathcal{Z} \). Hence \( S(u_\mu) \geq m \).

Now we prove that \( S(u) \geq S(u_\mu) \), for any \( \lambda \in (0, 1) \). Indeed, from
\[ S(u_\lambda) = \frac{1}{2} \int_{\mathbb{R}^2} \left[ cu^2 + \lambda^2 u^2 + \lambda^4 u_{xx}^2 + \lambda^4 v^2 \right] \, dx \, dy - \frac{\lambda^{2p}}{(p+1)(p+2)} \int_{\mathbb{R}^2} u^{p+2} \, dx \, dy \]
and \( Q(u) = 0 \), we obtain
\[ S(u) - S(u_\lambda) = \int_{\mathbb{R}^2} \left[ g_1(\lambda) u^2 + g_2(\lambda) \left( u_{xx}^2 + v^2 \right) \right] \, dx \, dy, \]
where
\[ g_1(\lambda) = \frac{1}{2} (1 - \lambda^2) - \frac{1}{2p} (1 - \lambda^2p), \quad \text{and} \quad g_2(\lambda) = \frac{1}{2} (1 - \lambda^4) - \frac{1}{p} (1 - \lambda^2p). \]
An argument similar to the first part of the lemma shows that $g_1(\lambda) \geq 0$ and $g_2(\lambda) \geq 0$, for $\lambda \in (0, 1)$. Hence $S(u) \geq S(u_\mu)$ for any $\lambda \in (0, 1)$. In particular, $S(u) \geq S(u_\mu)$. By setting $w = u_\mu$, we complete the proof. □

Now similar to [16, 17], we define the following sets:

$$
\mathcal{D}_1 = \{u \in X_i; \ S(u) < m, \ I(u) < 0, \ Q(u) < 0\}; \\
\mathcal{D}_2 = \{u \in X_i; \ S(u) < m, \ I(u) < 0, \ Q(u) > 0\}; \\
\mathcal{R}_1 = \{u \in X_i; \ S(u) < m, \ I(u) < 0, \ R(u) > 0\}; \\
\mathcal{R}_2 = \{u \in X_i; \ S(u) < m, \ I(u) < 0, \ R(u) < 0\}.
$$

We show the invariance of these sets under flow generated by (1.1). The following lemma is crucial to obtain the blow-up result.

**Lemma 3.3.** Suppose that $s \geq 5$, $p \geq 4$ and $c > 0$. Let $u_0$ be the initial data such that the corresponding solution $u(t)$ of (1.1) is in $C([0, T); X_i)$ for some $T > 0$ and satisfies $E(u(t)) = E(u_0)$ and $F(u(t)) = F(u_0)$ for $0 \leq t < T$. Then

(i) $u_0 \in \mathcal{R}_i$ implies that $u(t) \in \mathcal{R}_i$, $\forall t \in [0, T)$, $i = 1, 2$;  
(ii) $u_0 \in \mathcal{D}_i$ implies that $u(t) \in \mathcal{D}_i$, $\forall t \in [0, T)$, $i = 1, 2$.

**Proof.** Here we only consider the invariance of $\mathcal{D}_1$ under the flow of (1.1). The invariance of $\mathcal{D}_2$, $\mathcal{R}_1$ and $\mathcal{R}_2$ can be analogously proved.

Let $u_0 \in \mathcal{D}_1$. Then by theorem 3.1, we know that if $S(u_0) < m$, then $S(u(t)) = S(u_0) < m$. We prove that $I(u(t)) < 0$, for all $t \in [0, T)$. The argument is made by contradiction. Suppose that, from the continuity, there exists $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$. Then since $u(t_0) \neq 0$, it is deduced that $S(u(t_0)) \geq m$. This contradicts with $S(u(t)) < m$, for all $t \in (0, T)$. Therefore,

$$
I(u(t)) < 0,
$$

for all $t \in [0, T)$.

Finally, we show that $Q(u(t)) < 0$, for all $t \in [0, T)$. We will argue by contradiction and suppose that, from the continuity, there exists $t_0 \in (0, T)$ such that $Q(u(t_0)) = 0$. Since we have proved that $I(u(t)) < 0$, for all $t \in (0, T)$, we have $u(t_0) \in \mathcal{D}_2$. So that $S(u(t_0)) \geq d_\mathcal{D} \geq m$; which contradicts with $S(u(t)) < m$, for all $t \in (0, T)$. Thus, $Q(u(t)) < 0$, for all $t \in [0, T)$. □

**Remark 3.4.** Similar to [16], one can easily show that if $u_0 \in \mathcal{R}_1 \cap \mathcal{D}_1$ then it follows from remark 2.3 that the solution $u(t)$ of (1.1), corresponding to the initial data $u(0) = u_0$, satisfies $\|u(t)\|_x \leq C(\|u_0\|_x, \varphi)$, for a solitary wave $\varphi$ of (1.2). Consequently, the solution $u(t)$ of (1.1) is bounded globally for $t \geq 0$; and the blow-up cannot occur in finite time in $\mathcal{D}_i$.

However, we do have a blow-up solution which is only due to the transverse dispersion; we will state it in the next section.

**Remark 3.5.** It is easy to see that if $u$ is a solitary wave of (1.2) and $p \geq 2$, then the energy $E(u) > 0$. More precisely, the Pohozaev-type identities (5.8) and (5.9) in [6] give

$$
\int_{\mathbb{R}^2} (u_x^2 + 2u_{xx}^2) \, dx \, dy = 2 \int_{\mathbb{R}^2} v^2 \, dx \, dy = \frac{p}{(p + 1)(p + 2)} \int_{\mathbb{R}^2} u^{p+1} \, dx \, dy.
$$

(3.3)
Hence (3.3) and the definition of $E$ in (2.2) yield

$$E(u) = \left(\frac{3}{4} - \frac{1}{p}\right) \int_{\mathbb{R}^2} u_x^2 \, dx \, dy + \left(1 - \frac{2}{p}\right) \int_{\mathbb{R}^2} u_{xx}^2 \, dx \, dy > 0.$$  

**Remark 3.6.** If $p \geq 4$ and $u_0 \in \mathcal{D}$, then $E(u_0) > 0$. Indeed, since $\mathcal{R}(u_0) > 0$, we obtain

$$E(u_0) > \int_{\mathbb{R}^2} \left[\left(\frac{1}{2} - \frac{1}{p}\right) \left(\partial_t u_0^2 + \frac{1}{2} - \frac{2}{p}\right) \left(\partial_x^2 u_0^2 + \frac{1}{2} u_0^2\right)\right] \, dx \, dy > 0.$$  

### 4. Blow-up and instability

In this section, we are going to prove the strong instability of solitary waves of (1.1), by the mechanism of blow-up. First, it is worth recalling the definition of the strong instability.

**Definition 4.1.** We say that the solitary wave $\varphi$ is strongly unstable if for any $\delta > 0$ there exists $u_0 \in X_\delta$ ($s \geq 5$), close to $\varphi$ in $\mathcal{D}$ with $\|u_0 - \varphi\|_X < \delta$, such that the solution $u(t)$ of (1.1) with initial data $u(0) = u_0$ blows up in finite time.

Now, we are ready to state our blow-up result (see remark 3.4).

**Theorem 4.2.** If $p \geq 4$, $u_0 \in \mathcal{D}$ and $\nu u_0 \in L^2(\mathbb{R}^2)$, then the solution $u(t)$ of (1.1), corresponding to the initial data $u(0) = u_0$, blows up in finite time. More precisely, there exists $0 < \tau < \infty$ such that

$$\lim_{t \to \tau^-} \|u_0(t)\|_{L^2(\mathbb{R}^2)} = +\infty.$$  

**Proof.** For any $u_0 \in \mathcal{D}$ and $\nu u_0 \in L^2(\mathbb{R}^2)$, we have $u(t) \in \mathcal{D}$, by lemma 3.3. Thus $S(u(t)) < m$, $I(u(t)) < 0$, $Q(u(t)) < 0$ and $\mathcal{R}(u(t)) > 0$. For any $\lambda > 0$, we denote

$$u_\lambda(x, y) = \lambda^2 u(\lambda x, \lambda y).$$  

Then similar to the proof of lemma 3.2, it is straightforward to verify that

$$I(u_\lambda(t)) \to c \|u(t)\|_{L^2(\mathbb{R}^2)}^2 > 0,$$

as $\lambda \to 0$, and

$$I(u_\lambda(t)) \to I(u(t)) < 0,$$

as $\lambda \to 1$. Hence there exists $\mu \in (0, 1)$ such that $I(u_\mu(t)) = 0$ and $I(u_0(t)) < 0$ for any $\lambda \in (0, 1]$.

Now since $Q(u(t)) < 0$, one of the following cases occurs: $Q(u_\lambda(t)) < 0$ for all $\lambda \in [\mu, 1]$, $Q(u_\lambda(t)) = 0$ for some $\nu \in (\mu, 1)$, or $Q(u_\lambda(t)) = 0$.

If $Q(u_\lambda(t)) < 0$ for all $\lambda \in [\mu, 1]$, or $Q(u_\lambda(t)) = 0$, then we have $I(u_\mu(t)) = 0$, $Q(u_\nu(t)) \leq 0$ and $S(u_\nu(t)) \geq 0$. Moreover, a straightforward calculation reveals that

$$S(u(t)) - S(u_\mu(t)) = \frac{1}{2} \int_{\mathbb{R}^2} \left[\left(1 - \mu^2\right) u_{xx}^2(t) + \left(1 - \mu^2\right) \left(u_{xx}^2 + v^2(t)\right)\right] \, dx \, dy$$

$$\geq \frac{1}{2p} \int_{\mathbb{R}^2} \left[\left(1 - \mu^2\right) u_{xx}^2(t) + \left(1 - \mu^2\right) \left(u_{xx}^2 + v^2(t)\right)\right] \, dx \, dy$$

$$= \frac{1}{p} \left(Q(u(t)) - Q(u_\mu(t))\right) \geq \frac{1}{p} Q(u(t)).$$
If there is \( \nu \in (\mu, 1) \) such that \( Q(u_\nu(t)) = 0 \), then it follows that \( I(u_\nu(t)) < 0 \) and \( Q(u_\nu(t)) = 0 \), i.e. \( u_\nu(t) \in \mathcal{D} \). Thus \( S(u_\nu(t)) \geq d_{\mathcal{D}} \geq m \); and a similar computation yields
\[
S(u(t)) - S(u_\nu(t)) \geq \frac{1}{p} (Q(u(t)) - Q(u_\nu(t))) \geq \frac{1}{p} Q(u(t)).
\]
Therefore in all cases, we have
\[
Q(u(t)) \leq p(S(u_0) - m) = -\varrho < 0.
\]

Note that, similar to [19], one can prove the following virial-type identity:
\[
\mathcal{I}''(t) = 8 \int_{\mathbb{R}^2} \left[ v^2(t) - \frac{p}{2(p+1)(p+2)} u^{p+2}(t) \right] dx dy = 8 (Q(u(t)) - R(u(t))) ,
\]
where
\[
\mathcal{I}(t) = \int_{\mathbb{R}^2} y^2 u^2(t) dx dy.
\]
On the other hand, \( R(u(t)) > 0 \), since \( u_0 \in \mathcal{R}_1 \). Thus, we find that there exists \( t_0 > 0 \) such that \( \mathcal{I}(t_0) = 0 \). The invariance of \( F \), the Weyl–Heisenberg inequality and remark 3.4 imply that there exists a blow-up time \( \tau \in (0, t_0] \) such that
\[
\lim_{t \to \tau^-} \left\| u_\nu(t) \right\|_{L^2(\mathbb{R}^2)} = +\infty;
\]
and the proof is complete. \( \square \)

Now our instability theorem reads as follows.

**Theorem 4.3 (Instability).** Let \( p \geq 4 \). Suppose that \( \varphi \) is a solitary wave solution of (1.2) with \( c > 0 \). Then \( \varphi \) is strongly unstable in the sense of definition 4.1.

To prove theorem 4.3, we need the following lemmas.

**Lemma 4.4.** There exist the constants \( \lambda > 0 \) and \( \mu > 0 \) such that for \( w_{\lambda, \mu}(x, y) = \lambda \varphi(x, \mu y) \),
\[
S(w_{\lambda, \mu}) < m, \quad I(w_{\lambda, \mu}) < 0, \quad Q(w_{\lambda, \mu}) < 0 \quad \text{and} \quad R(w_{\lambda, \mu}) > 0. \quad (4.1)
\]

**Proof.** Let \( \varphi \) be a solitary wave solution of (1.2) with \( \varphi_y = \psi_x \) defined as in section 1.1. First, by a straightforward computation (see remark 3.5), one has
\[
I(\varphi) = Q(\varphi) = R(\varphi) = 0. \quad (4.2)
\]
Reciprocally,
\[
S(w_{\lambda, \mu}) = \frac{\lambda^2}{2\mu} \int_{\mathbb{R}^2} \left[ c\varphi^2 + \varphi_x^2 + \varphi_{xx}^2 + \mu^2 \psi^2 - \frac{2\lambda p \varphi^{p+2}}{(p+1)(p+2)} \right] dx dy, \quad (4.3)
\]
\[
I(w_{\lambda, \mu}) = \frac{\lambda^2}{\mu} \int_{\mathbb{R}^2} \left[ c\varphi^2 + \varphi_x^2 + \varphi_{xx}^2 + \mu^2 \psi^2 - \frac{\lambda p \varphi^{p+2}}{p+1} \right] dx dy, \quad (4.4)
\]
\[
Q(w_{\lambda, \mu}) = \frac{\lambda^2}{\mu} \int_{\mathbb{R}^2} \left[ \frac{1}{2} \varphi^2 + \varphi_x^2 + \varphi_{xx}^2 \right] dx dy - \frac{p\lambda^p \varphi^{p+2}}{(p+1)(p+2)}, \quad (4.5)
\]
\[
R(w_{\lambda, \mu}) = \frac{\lambda^2}{\mu} \int_{\mathbb{R}^2} \left[ \frac{1}{2} \varphi^2 + \varphi_x^2 \right] dx dy - \frac{p\lambda^p \varphi^{p+2}}{2(p+1)(p+2)}. \quad (4.6)
\]
Using (4.2), it is straightforward to verify that the conditions in (4.1) are equivalent to conditions
\[ R(w_{\lambda, \mu}) = g_1(\lambda, \mu) \int_{\mathbb{R}^2} \left( \frac{1}{2} \phi_x^2 + \phi_{xx}^2 \right) \, dx \, dy > 0, \]  
(4.7)
\[ Q(w_{\lambda, \mu}) = g_2(\lambda, \mu) \int_{\mathbb{R}^2} \left( \frac{1}{2} \phi_x^2 + \phi_{xx}^2 \right) \, dx \, dy < 0, \]  
(4.8)
\[ I(w_{\lambda, \mu}) = g_3(\lambda, \mu) \int_{\mathbb{R}^2} \left( \frac{1}{2} \phi_x^2 + \phi_{xx}^2 \right) \, dx \, dy < 0, \]  
(4.9)
and
\[ S(w_{\lambda, \mu}) = g_4(\lambda, \mu) \int_{\mathbb{R}^2} \left( \frac{1}{2} \phi_x^2 + \phi_{xx}^2 \right) \, dx \, dy \leq \int_{\mathbb{R}^2} \left( \frac{1}{2} \phi_x^2 + \phi_{xx}^2 \right) \, dx \, dy, \]  
(4.10)
where
\[ g_1(\lambda, \mu) = \frac{\lambda^2}{\mu} - \frac{\lambda^{p+2}}{\mu}, \]
\[ g_2(\lambda, \mu) = \frac{\lambda^2}{\mu} + \lambda^2 \mu - \frac{2\lambda^{p+2}}{\mu}, \]
\[ g_3(\lambda, \mu) = \frac{\lambda^2 \mu}{2} + \frac{(p + 4)\lambda^2}{p \mu} - \frac{2(2p + 2)\lambda^{p+2}}{p \mu}, \]
\[ g_4(\lambda, \mu) = \frac{\lambda^2 \mu}{2} + \frac{(p + 4)\lambda^2}{2p \mu} - \frac{2\lambda^{p+2}}{p \mu}. \]
Hence conditions (4.7)–(4.10) are equivalent to
\[ \lambda^p < 1, \]  
(4.11)
\[ 2\lambda^p > 1 + \mu^2, \]  
(4.12)
\[ 2(p + 2)\lambda^p > 4 + p(1 + \mu^2), \]  
(4.13)
\[ \frac{\lambda^2 \mu}{2} + \frac{(p + 4)\lambda^2}{2p \mu} - \frac{2\lambda^{p+2}}{p \mu} < 1. \]  
(4.14)
Now by taking \( \mu^2 = 1 - \epsilon \) and \( \lambda^p \in (\lambda^*, 1) \), where \( \epsilon > 0 \) is small enough and
\[ \lambda^* = \max \left\{ 1 - \frac{\epsilon}{2}, 1 - \frac{p\epsilon}{2(p + 2)} \right\}, \]
(4.1) thus holds for \( w_{\lambda, \mu} \); and the proof is complete. \( \square \)

**Lemma 4.5.** For any \( u, w \in \mathcal{X} \) with \( \|u\|_{\mathcal{X}} \leq C \) and \( \|w\|_{\mathcal{X}} \leq 1 \), there exist the constants \( C_i > 0, i = 1, 2, 3, 4, \) independent of \( w \) such that
\[ Q(u + w) < Q(u) + C_1 \|w\|_{\mathcal{X}}, \]  
(4.15)
\[ R(u) < R(u + w) + C_2 \|w\|_{\mathcal{X}}, \]  
(4.16)
\[ S(u + w) < S(u) + C_3 \|w\|_{\mathcal{X}}, \]  
(4.17)
and
\[ I(u + w) < I(u) + C_4 \|w\|_{\mathcal{X}}. \]  
(4.18)

**Proof.** We prove (4.15) and (4.16). One can prove (4.17) and (4.18) in the same vein.

First recalling the definition of \( p \) (in section 1.1), then using the inequality
\[ |u|^{p+2} + |w|^{p+2} - |u + w|^{p+2} \lesssim |w| |u|^{p+1} + |w|^{p+1} |u|, \]
and the embedding (2.4), it follows that

$$\int_{\mathbb{R}^2} \left( |u|^{p+2} - |u + w|^{p+2} \right) \, dx \, dy \lesssim \|w\|_\mathcal{X}^{p+2} + \left( \|u\|_\mathcal{X} \|w\|^{p+1}_\mathcal{X} + \|u\|^{p+1}_\mathcal{X} \|w\|_\mathcal{X} \right) \lesssim \|w\|_\mathcal{X} .$$

Thus, using

$$\int_{\mathbb{R}^2} \left( \partial_i^x (u + w) \right)^2 \, dx \, dy \lesssim \|w\|_\mathcal{X} + \|w\|^2_\mathcal{X} + \int_{\mathbb{R}^2} \left( \partial_i^x u \right)^2 \, dx \, dy$$

$$\lesssim \|w\|_\mathcal{X} + \int_{\mathbb{R}^2} \left( \partial_i^x u \right)^2 \, dx \, dy, \quad i = 1, 2$$

and

$$\int_{\mathbb{R}^2} \left( \partial_i^{-1} (u + w) \right)^2 \, dx \, dy \lesssim \|w\|_\mathcal{X} + \|w\|^2_\mathcal{X} + \int_{\mathbb{R}^2} \left( \partial_i^{-1} u \right)^2 \, dx \, dy$$

$$\lesssim \|w\|_\mathcal{X} + \int_{\mathbb{R}^2} \left( \partial_i^{-1} u \right)^2 \, dx \, dy,$$

it is deduced that

$$Q(u + w) - Q(u) \lesssim \|w\|_\mathcal{X} + \int_{\mathbb{R}^2} \left( |u|^{p+2} - |u + w|^{p+2} \right) \, dx \, dy \lesssim \|w\|_\mathcal{X} .$$

This proves (4.15).

To prove (4.16), using again the embedding (2.4) and the inequality

$$|u + w|^{p+2} - |u|^{p+2} \lesssim |w| |u|^{p+1} + |w| |u|^{p+2},$$

we obtain

$$\int_{\mathbb{R}^2} (u + w)^{p+2} \, dx \, dy \lesssim \|w\|_\mathcal{X} + \int_{\mathbb{R}^2} u^{p+2} \, dx \, dy. \quad (4.19)$$

As above, equation (4.19) and

$$\int_{\mathbb{R}^2} \left[ \left( \partial_i^x u \right)^2 - \left( \partial_i^x (u + w) \right)^2 \right] \, dx \, dy \lesssim \|w\|_\mathcal{X}, \quad i = 1, 2,$$

imply that

$$\mathcal{R}(u) - \mathcal{R}(u + w) \lesssim \|w\|_\mathcal{X} .$$

Finally, here is the proof of theorem 4.3.

**Proof of theorem 4.3.** For any $\delta > 0$, let $\epsilon > 0$ be sufficiently small; and $\mu, \lambda$ and $w_{\lambda, \mu}$ be as in lemma 4.4 such that

$$\|w_{\lambda, \mu} - \varphi\|_\mathcal{X} < \delta / 2.$$

Since $X_\delta$ is dense in $\mathcal{X}$, we find $u_0 \in X_\delta$ such that

$$\|u_0 - w_{\lambda, \mu}\|_\mathcal{X} < \min \left\{ 1, \frac{\delta}{2}, \frac{\sigma}{\sum_{i=1}^{4} C_i} \right\},$$

where $C_i, i = 1, 2, 3, 4$, are as in lemma 4.5 and

$$\sigma = \min \left\{ m - S(u_{\lambda, \mu}), -I(u_{\lambda, \mu}), \mathcal{R}(u_{\lambda, \mu}), -\mathcal{Q}(u_{\lambda, \mu}), \frac{\delta}{2} \right\} .$$

Therefore

$$\|u_0 - \varphi\|_\mathcal{X} \leq \|u_0 - w_{\lambda, \mu}\|_\mathcal{X} + \|w_{\lambda, \mu} - \varphi\|_\mathcal{X} < \delta;$$

and lemmas 4.4 and 4.5 imply that $u_0 \in \mathcal{A}_1 \cap \mathcal{L}_1$. Since $\delta$ is arbitrary, then theorem 4.2 implies that the solution $u(t)$ of (1.1) with the initial data $u(0) = u_0$ blows up in a finite time.

The proof of theorem 4.3 is complete. □
Acknowledgment

The author would like to thank the referee for many valuable comments.

References

[1] Abramyan L A and Stepanyants Y A 1985 The structure of two-dimensional solitons in media with anomalously small dispersion Sov. Phys.—JETP 61 963–6
[2] Ambrosetti A and Rabinowitz P H 1973 Dual variational methods in critical point theory and applications J. Funct. Anal. 14 349–81
[3] Bona J L, Souganidis P E and Strauss W A 1987 Stability and instability of solitary waves of Korteweg–de Vries type Proc. R. Soc. Lond. Ser. A 411 395–412
[4] Cazenave T and Lions P L 1982 Orbital stability of standing waves for some nonlinear Schrödinger equations Commun. Math. Phys. 85 549–61
[5] Chen W, Li J and Miao C 2008 On the low regularity of the fifth order Kadomtsev–Petviashvili I equation J. Diff. Eqns 245 3433–69
[6] de Bouard A and Saut J C 1997 Solitary waves of generalized Kadomtsev–Petviashvili equations Ann. Inst. H. Poincaré Anal. Non Linéaire 14 211–36
[7] de Bouard A and Saut J C 1997 Symmetries and decay of the generalized Kadomtsev–Petviashvili solitary waves Siam J. Math. Anal. 28 1064–85
[8] de Bouard A and Saut J C 1996 Remarks on the stability of generalized KP solitary waves Contemp. Math. 200 75–84
[9] Isaza P, Lópeze J and Mejía J 2006 Cauchy problem for the fifth order Kadomtsev–Petviashvili (KPII) equation J. Math. Pures Appl. 90 (9) 338–52
[10] Karpman V I and Belashov V Y 1991 Dynamics of two-dimensional solitons in weakly dispersive media Phys. Lett. A 154 131–39
[11] Karpman V I and Belashov V Y 1991 Evolution of three-dimensional nonlinear pulses in weakly dispersive media Phys. Lett. A 154 140–44
[12] Kato T 1975 Quasi-Linear Equations of Evolution, with Applications to Partial Differential Equations (Springer Lecture Notes in Mathematics vol 448) (Berlin: Springer) pp 25–70
[13] Kawahara R 1972 Oscillatory solitary waves in dispersive media J. Phys. Soc. Japan 33 260–64
[14] Li J and Xiao J 2008 Well-posedness of the fifth order Kadomtsev–Petviashvili I equation in anisotropic Sobolev spaces with nonnegative indices J. Math. Pures Appl. 90 9) 338–52
[15] Lions P L 1984 The concentration-compactness principle in the calculus of variations. The locally compact case: 1 Ann. Inst. H. Poincaré, Anal. Non linéaire 1 109–45
Lions P L 1984 The concentration-compactness principle in the calculus of variations. The locally compact case: 2 Ann. Inst. H. Poincaré, Anal. Non linéaire 4 223–83
[16] Liu Y 2001 Blow up and instability of solitary waves to a generalized Kadomtsev–Petviashvili equation Trans. Am. Math. Soc. 353 191–208
[17] Liu Y 2002 Strong instability of solitary-wave solutions to a Kadomtsev–Petviashvili equation in three Dimensions J. Diff. Eqns 180 153–70
[18] Liu Y and Wang X P 1997 Nonlinear stability of solitary waves of a generalized Kadomtsev–Petviashvili equation Commun. Math. Phys. 183 253–66
[19] Saut J C 1993 Remarks on the generalized Kadomtsev–Petviashvili equations Indiana Math. J. 42 1011–26
[20] Saut J C and Tzvetkov N 1999 The Cauchy problem for higher-order KP equations J. Diff. Eqns 153 196–222
[21] Saut J C and Tzvetkov N 2000 The Cauchy problem for the fifth order KP equations J. Math. Pures Appl. 79 307–38
[22] Schneider G and Wayne C E 2002 The rigorous approximation of long-wavelength capillary-gravity waves Arch. Ration. Mech. Anal. 162 247–85
[23] Tom M M 2000 Some Generalizations of the Kadomtsev–Petviashvili Equations J. Math. Anal. Appl. 243 64–84
[24] Ukai S 1989 Local solutions of the Kadomtsev–Petviashvili equations J. Fac. Sci. Univ. Tokyo Sect. 1 A 36 193–209
[25] Willem M 1996 Minimax Theorems (Boston: Birkhäuser)