Recovering MOND from extended metric theories of gravity

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We show that the Modified Newtonian Dynamics (MOND) regime can be fully recovered as the weak-field limit of a particular theory of gravity formulated in the metric approach. This is possible when Milgrom’s acceleration constant is taken as a fundamental quantity which couples to the theory in a very consistent manner. As a consequence, the scale invariance of the gravitational interaction is naturally broken. In this sense, Newtonian gravity is the weak-field limit of general relativity and MOND is the weak-field limit of that particular extended theory of gravity. We also prove that a Noether’s symmetry approach to the problem yields a conserved quantity coherent with this relativistic MONDian extension.

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I. INTRODUCTION

Milgrom [1, 2] developed a non-relativistic theory of gravity in order to explain observed flat rotation curves of spiral galaxies. This Modified Newtonian Dynamics (MOND) theory has proved useful in explaining a great variety of astronomical phenomena without requiring the presence of a dark matter component (see e.g. [3, 4] and references therein).

As explained by an extended Newtonian approach to gravity, Mendoza et al. [5] showed that the key feature of MOND is the introduction of a fundamental acceleration $a_0$ in the theory, which on itself at the non-relativistic level, makes gravity scale dependent.

Through the years, finding the relativistic extension of MOND has become a big challenge. The most successful attempt was proposed by Bekenstein [6] who formulated a Tensor-Vector-Scalar (TeVeS) relativistic theory of MOND. This approach requires tensor, vector and scalar fields to achieve a self-consistent description. However, the many cumbersome mathematical complications that TeVeS present are evident. Furthermore, it cannot reproduce crucial astrophysical phenomena (see e.g. [7]).

On the other hand, since extended metric $f(R)$ theories of gravity have proved very successful on a wide variety of cosmological scenarios [see e.g. 8, 11, and references therein] it is natural to seek a relativistic generalisation of MOND in this direction, which has not deserved too much attention since Soussa and Woddard [13] developed a no-go theorem which prevented all metric $f(R)$ theories of gravity to become relativistic candidates for MOND. However, Mendoza and Rosas-Guevara [14] showed counterexamples of this no-go theorem disproving its general validity. Furthermore, the works by Capozziello et al. [15, 17] and Sobouti [18] made clear that particular $f(R)$ models are capable of explaining phenomena usually ascribed to MOND. Up to now, the challenge has been to formulate a suitable $f(R)$ theory able to converge to standard MOND in the non-relativistic regime.

In this article, we show how a particular $f(R)$ metric theory of gravity, derived from first principles, is capable of reproducing MOND when its non-relativistic regime is reached. To do so, in Section II we set the foundations of a metric theory of gravity with the use of correct dimensional quantities. Then, in Section III we solve the problem of a point mass source on a static space-time and find, at first order of approximation, the particular form of the function $f(R)$. In Section IV the integration constants of the theory are fixed by solving the same problem in the formalism of metric perturbations. In Section V we show that the existence of a Noether symmetry confirms the appearance of a characteristic scale of the problem in the MONDian regime. Finally in Section VI we comment on the obtained results.

II. DIMENSIONAL GROUNDING

Let us assume that a point mass $M$ located at the origin of coordinates generates a relativistic gravitational field in the MONDian regime and that a metric formalism describes the field equations.

This problem is characterised by the following quantities: the speed of light in vacuum $c$, the mass $M$ of the central object generating the gravitational field, Newton’s constant of gravity $G$ and Milgrom’s acceleration constant $a_0$. With these parameters, two “fundamental lengths” can be built:
The gravitational radius \( r_g \) is a length that appears once relativistic effects are introduced on a theory of gravity. The mass-length scale \( l_M \), as described in [3], is a characteristic length which appears on a gravitational theory when MONDian effects are to be taken into account (for consistency we note here that a third length, \( \lambda := l_M^2/r_g^2 = c^2/a_0 \), that does not contain the mass, can be constructed as a combination of the previous two).

As discussed by Mendoza et al. [3], in the non-relativistic regime, a test particle located at the radial coordinate \( r \) from the origin will obey the MONDian dynamics when \( l_M/r \ll 1 \). When \( l_M/r \gg 1 \) the gravitational field is Newtonian. As such, when relativistic effects are taken into account for the gravitational field, then standard general relativity should be recovered in the limit \( l_M/r \gg 1 \), and a relativistic version of MOND should be obtained when \( l_M/r \ll 1 \). This shows that the pursuit of a complete metric description leads one to consider the scale-dependence of gravity.

The length scales presented in equation (1) must somehow appear in a relativistic theory of gravity which accepts the fundamental nature of the constant \( a_0 \). For example, in the metric formalism, a generalised Hilbert action \( S_H \) can be written in the following way:

\[
S_H = -\frac{c^3}{16\pi G L_M^2} \int f(\chi)\sqrt{-g} \, d^4x, \tag{2}
\]

which slightly differs from its traditional form (see e.g. [3, 19])

\[
S_H = -\frac{c^3}{16\pi G} \int f(R)\sqrt{-g} \, d^4x, \tag{3}
\]

since we have introduced the following dimensionless quantity:

\[
\chi := L_M^2 R, \tag{4}
\]

where \( R \) is Ricci's scalar and \( L_M \) defines a length fixed by the parameters of the theory. The explicit form of the length \( L_M \) has to be obtained once a certain known limit of the theory is taken, usually a non-relativistic limit. Note that the definition of \( \chi \) gives a correct dimensional character to the action (2), something that is not completely clear in all previous works dealing with a metric description of the gravitational field. For \( f(\chi) = \chi \) the standard Einstein-Hilbert action is obtained.

On the other hand, the matter action has its usual form,

\[
S_m = -\frac{1}{2c} \int \mathcal{L}_m \sqrt{-g} \, d^4x, \tag{5}
\]

with \( \mathcal{L}_m \) the Lagrangian density of the system. The null variations of the complete action, i.e. \( \delta (S_H + S_m) = 0 \), yield the following field equations:

\[
f'(\chi) \chi_{\mu\nu} - \frac{1}{2} f(\chi) g_{\mu\nu} - L_M^2 (\nabla_\mu \nabla_\nu g - g_{\mu\nu} \Delta) f'(\chi) = \frac{8\pi G L_M^2}{c^4} T_{\mu\nu}, \tag{6}
\]

where the dimensionless Ricci tensor \( \chi_{\mu\nu} \) is given by:

\[
\chi_{\mu\nu} := L_M^2 R_{\mu\nu}, \tag{7}
\]

and \( R_{\mu\nu} \) is the standard Ricci tensor. The Laplace-Beltrami operator has been written as \( \Delta := \nabla^\alpha \nabla_\alpha \) and the prime denotes derivative with respect to its argument. The energy-momentum tensor \( T_{\mu\nu} \) is defined through the following standard relation:

\[
\delta S_m = -\frac{1}{2} \int \nabla_\mu \nabla_\nu g \delta g^{\mu\nu} \, d^4x, \tag{8}
\]

where \( \delta g^{\mu\nu} \) follows a \((+,−,−,−)\) signature for the metric \( g_{\mu\nu} \) and use Einstein’s summation convention over repeated indices.

The trace of equation (8) is:

\[
f'(\chi) \chi - 2f(\chi) + 3L_M^2 \Delta f(\chi) = \frac{8\pi G L_M^2}{c^4} T, \tag{9}
\]

where \( T := T_0^0 \).

Since we are only interested on the gravitational field produced by a point mass source located at the origin, then the mass density \( \rho \) is given by

\[
\rho = M \delta(r), \tag{10}
\]

where \( \delta(r) \) represents the three dimensional Dirac delta function. With this, it follows that the only non-zero component of the energy-momentum tensor is given by

\[
T_{00} = \rho c^2 = c^2 M \delta(r). \tag{11}
\]

A point mass distribution generates a stationary spherically symmetric space-time and so, the trace equation (8) contains all the relevant information relating the field equations. In what follows we assume a power law form for the function \( f(\chi) \), i.e.

\[
f(\chi) = \chi^b. \tag{12}
\]

III. ORDER OF MAGNITUDE APPROACH

Let us first analyse the problem described in the previous Section by performing an order of magnitude approximation of the trace equation (8). Under these circumstances, \( \chi \rho / \chi \approx 1/\chi \), \( \Delta \approx -1/r^2 \) and the mass density \( \rho \approx M/r^3 \). This approximation implies that equation (8) takes the following form:
\[ \chi^b (b - 2) - 3b L_M^2 \chi^{(b-1)}_{r^2} \approx \frac{8\pi GM L_M^2}{c^2 r^3}. \] (12)

Note that the second term on the left-hand side of equation (12) is much greater than the first term when the following condition is satisfied:

\[ Rr^2 \approx \frac{3b}{2 - b}. \] (13)

At the same order of approximation, Ricci’s scalar \( R \approx \kappa = R_c^{-2} \), where \( \kappa \) is the Gaussian curvature of space and \( R_c \) its radius of curvature and so, relation (13) essentially means that

\[ R_c \gg r. \] (14)

In other words, the second term on the left-hand side of equation (12) dominates the first one when the local radius of curvature of space is much greater than the characteristic length \( r \). This should occur in the weak-field regime, where MONDian effects are expected. For a metric description of gravity, this limit must correspond to the relativistic regime of MOND. In this article we will only deal with this approximation. At the end of the current Section we show an equivalent relation to inequality (14) which has a more physical meaning.

Under assumption (14), equation (12) takes the following form:

\[ R^{(b-1)} \approx -\frac{8\pi GM}{3bc^2 r^2 L_M^{2(b-1)}}. \] (15)

We now recall the well known relation followed by the Ricci scalar at second order of approximation at the non-relativistic level [20]:

\[ R = -\frac{2}{c^2} \nabla^2 \phi = +\frac{2}{c^2} \nabla \cdot a, \] (16)

where the negative gradients of the gravitational potential \( \phi \) provide the acceleration \( a := -\nabla \phi \) felt by a test particle on a non-relativistic gravitational field. At order of magnitude, equation (16) can be approximated as

\[ R \approx -\frac{2\phi}{c^2 r^2} \approx \frac{2a}{c^2 r}. \] (17)

Substitution of this last equation on relation (15) gives

\[ a \approx -\frac{c^2 r}{2L_M^2} \left( \frac{8\pi GM}{3bc^2 r} \right)^{1/(b-1)} \approx -c^{(b-4)/(b-1)} r^{(b-2)/(b-1)} L_M^{-2} (GM)^{1/(b-1)}. \] (18)

This last equation converges to a MOND-like acceleration \( a \propto 1/r \) if \( b-2 = -(b-1) \), i.e. when

\[ b = 3/2. \] (19)

Also, at the lowest order of approximation, in the extreme non-relativistic limit, the velocity of light \( c \) should not appear on equation (15) and so, the only possibility is that \( L_M \) depends on a power of \( c \), i.e.

\[ L_M^{-2} \propto c^{(4-2b)/(b-1)} = c^2, \quad \text{and so,} \quad L_M \propto c^{-1}. \] (20)

As discussed in Section II, the length \( L_M \) must be constructed by fundamental parameters describing the theory of gravity and so, let us assume that

\[ L_M = \zeta \frac{\ell_{\text{g}}^{\alpha} \ell_{\text{M}}^{\beta}}{c}, \quad \text{with} \quad \alpha + \beta = 1, \] (21)

where the constant of proportionality \( \zeta \) is a dimensionless number of order one that will be formally obtained in Section IV. Substituting equation (21) and the value obtained in (19) into relation (20) it then follows that

\[ \alpha = \beta = 1/2, \quad \text{i.e.} \quad L_M \approx \ell_{\text{g}}^{1/2} L_{\text{M}}^{1/2}. \] (22)

If we now substitute this last result and relation (19) in equation (18) it follows that

\[ a \approx -\frac{(a_0 GM)^{1/2}}{r}, \] (23)

which is the traditional form of MOND in spherical symmetry (see e.g. [3] and references therein). Also, the results of equation (23) in (17) mean that

\[ R \approx \frac{r_g}{L_M} \frac{1}{r^2}, \] (24)

and so, inequality (14) is equivalent to

\[ L_M \gg r_g. \] (25)

The regime imposed by equation (25) is precisely the one for which MONDian effects should appear in a relativistic theory of gravity. This is an expected generalisation of the results presented by Mendoza et al. [5] in the weak field limit regime for which \( l_M \ll r \) and so, combining this with equation (25) yields \( r \gg L_M \gg r_g \). In this connection, we also note that Newton’s theory of gravity is recovered in the limit \( L_M \gg r \gg r_g \).
IV. WEAK FIELD LIMIT APPROACH

We now use the trace (3) to the lowest order of perturbation. Results of this perturbation in the Newtonian limit for other metric theories of gravity have been reported by [22, 23]. However, since we are interested in the lowest order of approximation in the MONDian regime, we expect different results.

Under the assumption of spherical symmetry for a static space-time, with a diagonal metric given by

\[ g_{00} = 1 + \frac{2\phi}{c^2}, \quad g_{11} = -1, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta, \]  

then

\[ \Delta f'(\chi) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial f'(\chi)}{\partial x^\nu} \right), \]

\[ = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f'(\chi)}{\partial r} \right) = -\nabla^2 f'(\chi), \]  

at the lowest order of perturbation in \( \phi/c^2 \). In this case, when condition (14) or equivalently equation (25) is satisfied, the trace (3) of the field equations is given by:

\[ -3\nabla^2 f'(\chi) = \frac{8\pi G}{c^2} \rho. \]  

Note that this equation can also be obtained by performing a direct perturbation of (8). This is so because in the weak field limit, the field equations (11) are studied at orders of powers of \( c^{-2} \). At the lowest zeroth-order of the perturbation, the Ricci scalar \((0) R = 0\) everywhere and so, it describes a flat space-time. At the next second perturbation order \( O(\chi) = O(L_M^2) + O((2) R)\), where \((2) R\) represents Ricci’s scalar at the second order of the perturbation. Since \( O(L_M^2) = 1 \) according to relation (20), then \( O(\chi) = 4 \). If now we use the power law relation (11) with (19) and perturb the trace (8), it follows that the first two terms on the right-hand side are of order \( O(\chi^{3/2}) = 6 \), while the remaining two terms are of order \( O(L_M^2 \chi^{1/2}) = 4 \) and so, it follows that the trace (8) at its lowest non-zero perturbation order is exactly relation (25).

Equation (28) can be integrated straightforward using the standard Poisson equation of Newtonian gravity:

\[ \nabla^2 \phi_N = 4\pi G \rho, \]  

for the Newtonian potential \( \phi_N \). Substitution of this relation on (28) yields

\[ \nabla^2 \left( f'(\chi) + \frac{2}{3c^2} \phi_N \right) = 0. \]  

The trivial solution of the previous Laplace-like equation occurs when the argument of the Laplacian is equal to zero, and so

\[ f'(\chi) = -\frac{2}{3c^2} \phi_N. \]  

Substitution of equations (11), (19), (21), and (22) in relation (31) leads to

\[ R = \left( \frac{4}{9} \right) \zeta^{-2} \left( \frac{r_g}{l_M} \right) \frac{1}{r^2}, \]  

where we have used the fact that for a point mass \( M \), the Newtonian potential is given by:

\[ \phi_N = -G \frac{M}{r}. \]  

We now substitute relation (16) into equation (32) to obtain:

\[ \frac{2}{c^2} \nabla \cdot a = \left( \frac{4}{9} \right) \zeta^{-2} \left( \frac{r_g}{l_M} \right) \frac{1}{r^2}. \]  

Integrating this equation over a spherical volume of radius \( r \) and using Gauss’s theorem on the left-hand side, we obtain

\[ a = -\left( \frac{2\sqrt{2}}{9\zeta} \right)^2 \left( \frac{c^2 r_g}{l_M} \right) \frac{1}{r}. \]  

and so, by choosing

\[ \zeta = \frac{2\sqrt{2}}{9}, \]  

we reach the MOND acceleration limit

\[ a = -\left( \frac{a_0 GM}{r} \right)^{1/2}, \]  

at the lowest perturbation order of the theory.

The formulation of the theory described so far implies that, as soon as we relax the hypothesis for which the gravitational action is strictly the one described by standard general relativity, new characteristic lengths have to be considered. In the following Section, we show that such scales coherently appear as constants of motion of the problem.
V. NOETHER’S SYMMETRIES

The above results are based on the fact that we are assuming the power law relation \( f(R) \propto R^{3/2} \) for the gravitational action. In particular, for \( f(R) \propto R^{3/2} \) the MOND acceleration regime is recovered. This particular theory also admits exact cosmological solutions where a matter dominated era evolves towards the accelerated universe observed today \( [24] \) and recovers the observed dynamics at astrophysical scales \( [25] \). Also, this solution is particularly relevant since its conformal transformation is exactly invertible, as shown in \( [26] \). Noether’s symmetries give rise to conserved quantities that are directly related to characteristic length scales \( [27] \).

In order to develop a Noether’s approach to the problem, let us consider \( f(\chi) \) gravity in static spherical symmetry, following the same ideas as the ones exposed in Section \( [11] \). The spherical point–like \( f(\chi) \) Lagrangian can be obtained by imposing the spherical symmetry directly into the action \( [2] \). As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number. The technique is based on the choice of a suitable Lagrange multiplier defined by assuming the known explicit form of Ricci’s scalar \( \bar{R} \) \( [27] \).

The static spherically symmetric metric can be expressed as
\[
ds^2 = A(r)c^2 dt^2 - B(r)dr^2 - C(r)d\Omega^2, \tag{38}\]
where \( d\Omega^2 := d\theta^2 + \sin^2 \theta d\varphi^2 \) is the angular displacement and \( C(r) := r^2 \). Since we are interested in the weak-field limit approach of the \( f(\chi) \) theory, let us assume \( B(r) = 1 \).

The point–like \( f(\chi) \) Lagrangian \( L \) is obtained by rewriting the action \( [2] \) as
\[
S = -\frac{c^3}{16\pi G \bar{M}} \int \frac{f(\chi) - \lambda(\chi - \bar{\chi})}{\sqrt{-g}} d^4x, \tag{39}\]
where \( \lambda \) is a Lagrange multiplier and \( \bar{\chi} = \bar{L}_M \bar{R} \), for the known Ricci scalar \( \bar{R} \) expressed in terms of the metric \( [35] \) with \( B(r) = 1 \):
\[
\bar{R} = \frac{A''}{A} + \frac{2C''}{C} + \frac{A' C'}{AC} - \frac{A'^2}{2A^2} - \frac{C'^2}{2C^2} - \frac{2}{C}. \tag{40}\]

In the previous equation, the prime denotes the derivative with respect to the radial coordinate \( r \). Variations of the action with respect to \( \chi \) give the explicit form of the Lagrange multiplier:
\[
\lambda = \frac{df(\chi)}{d\chi} := f_\chi. \tag{41}\]

Substituting this result in the action \( [39] \) and eliminating the boundary terms (cf. \( [27] \)), the point–like Lagrangian is obtained:
\[
L = -\frac{L_M^2}{\sqrt{A}} \left[ \frac{Af_\chi C'^2 + f_\chi A'C' + C f_{\chi \chi} A'\chi'}{2C} + 2Af_{\chi \chi} C'\chi' \right]. \tag{42}\]

Note that this Lagrangian is canonical since only the generalised positions, \( \mathbf{q} = (A, C, \chi) \), and their generalised velocities \( \dot{\mathbf{q}} = (A', C', \chi') \), appear explicitly.

In the MONDian regime, where equation \( [14] \) holds, the last two terms on the right-hand side of equation \( [11] \) are of order \( Cf \approx C\chi^{3/2} \) and so, both are much smaller than \( L_M^2 f_\chi \). This statement is also true at the lowest non-zero order of perturbation, since all terms on the right-hand side of equation \( [11] \) are of order 4, except the last two which are of order 6. With this, it follows that in the MOND regime the Lagrangian \( [11] \) can be written as
\[
L = -\frac{L_M^2}{\sqrt{A}} \left[ \frac{Af_\chi C'^2 + f_\chi A'C' + C f_{\chi \chi} A'\chi'}{2C} + 2Af_{\chi \chi} C'\chi' + 2A \right]. \tag{43}\]

We now search for symmetries related to the cyclic variables and then reduce the dynamics. According to Noether’s theorem, the existence of symmetry properties for the dynamics described by the Lagrangian implies the existence of conserved quantities \( [28–30] \). In principle, this approach allows to select particular \( f(\chi) \) gravity models compatible with spherical symmetry.

A conserved quantity exists if the Lie derivative of the Lagrangian \( [12] \) along the vector field \( \mathbf{X} \) vanishes:
\[
\mathcal{L}_\mathbf{X} L = \alpha_i \nabla_{\mathbf{q}_i} L + \alpha'_i \nabla_{\dot{\mathbf{q}}_i} L = 0, \tag{44}\]
for \( i = 1, 2, 3 \), in the configuration space \( (A, C, \chi) \). Solving equation \( [43] \) means to find out the functions \( \alpha_i \) which constitute the Noether vector \( \alpha_i \). However, the relation \( [43] \) implicitly depends on the form of \( f(\chi) \) and then, by solving it, we also get \( f(\chi) \) models compatible with spherical symmetry. On the other hand, by fixing the \( f(\chi) \) function, we can explicitly solve \( [43] \). In principle, the same procedure can be worked out any time Noether’s symmetries are identified \( [31] \).

The general form of the Noether vector is given by the solution of the Killing equations for the components \( \alpha_i \) of the vector \( \alpha \) in a flat space-time (cf. \( [32] \)):
\[
\alpha_1 = k_1 A + p_1, \\
\alpha_2 = k_2 C + p_2, \\
\alpha_3 = k_3 \chi + p_3, \tag{45}\]
with \( k_i \) and \( p_i \) constants. Using the power law \( [11] \) and the general solution \( [14] \) in equation \( [43] \), it follows that
where $k$ is an integration constant and we have assumed in the calculations that $b \neq 1$, $\chi' \neq 0$ and $2AC'' + A'C \neq 0$ in order to obtain solutions different from general relativity (cf. Section III). For this case, the related constant of motion, $\Sigma_0$, is given by

$$\Sigma_0 = \alpha_i \nabla q_i \mathcal{L} = L_b^k b(b - 1)kA^{-1/2}C\chi^{-2}2(b - 1)A\chi' - A'\chi].$$  \tag{46}$$

In general relativity, where $b = 1$, Noether’s symmetry approach gives $\Sigma_0 = 2r_k$, which is exactly the Schwarzschild radius (cf. Section III). On the other hand, in the MONDian regime, where equation (46) is valid for $b = 3/2$, $C(r) = r^2$ and at the lowest order of perturbation $A(r) = 1 + 2\phi/c^2$, the related constant of motion is given by

$$\Sigma_0 = \frac{3}{2}k^2l_M. \tag{47}$$

The key point in this relation is that it includes two characteristic lengths, namely the mass length-scale $l_M$ and the gravitational radius $r_g$. This is coherent with the results discussed on the previous Sections, since both characteristic lengths must appear on a correct relativistic metric theory of MOND.

VI. DISCUSSION

We have shown that a metric theory of gravity $f(\chi) = \chi^{3/2}$, where the dimensionless Ricci scalar $\chi$ is given by equation (1), converges to MOND in the non-relativistic regime. We note that a previous attempt in this direction was made in the B.Sc. thesis of Rosas-Guevara where a metric $f(R) \propto R^{3/2}$ theory was proposed to account for relativistic effects of MOND. However, the correct approximation (29) was never introduced on that analysis and therefore, the MONDian limit was not achieved on that work.

We have also shown that the appearance of a new characteristic length related to MOND’s acceleration is coherent with Noether’s symmetries related to the problem.

The metric theory of gravity presented here is by no means a complete description at all scales of gravitation. It only deals with the MONDian regime of gravity, i.e. when equation (25) is valid. In other words, our description breaks the scale invariance of gravity in a more general way than the one described in [2].

The mass dependence of $\chi$ means that the mass needs to appear on Hilbert’s action [2]. This is traditionally not the case, since that action is thought to be purely a function of the geometry of space-time due to the presence of mass and energy. However, it was Sobouti [18] who first encountered this peculiarity in the Hilbert action when dealing with a metric generalisation of MOND. Following his remarks [18], one should not be surprised if some of the commonly accepted notions, even at the fundamental level of the action, require generalisations and re-thinking. An extended metric theory of gravity goes beyond the traditional general relativity ideas and in this way, we probably need to change our standard view of its fundamental principles.

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